A Non-Interactive Quantum Bit Commitment Scheme that Exploits the Computational Hardness of Quantum State Distinction

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Abstract: We propose an efficient quantum protocol performing quantum bit commitment, which is a simple cryptographic primitive involved with two parties, called a committer and a verifier. Our protocol is non-interactive, uses no supplemental shared information, and achieves computational concealing and statistical binding under a natural complexity-theoretic assumption. An earlier protocol in the literature relies on the existence of an efficient quantum one-way function. Our protocol, on the contrary, exploits a seemingly weaker assumption on computational difficulty of distinguishing two specific ensembles of reduced quantum states. This assumption is guaranteed by, for example, computational hardness of solving the graph automorphism problem efficiently on a quantum computer.

Keywords. quantum bit commitment, quantum computation, distinction problem, graph automorphism, computational concealing, statistical binding

1 Introduction

Bit commitment is a fundamental cryptographic primitive between two parties and its schemes have been applied to build other useful cryptographic protocols, including secure coin flipping, zero-knowledge proofs, secure multiparty computation, signature schemes, and secret sharing. A protocol for the bit commitment demands the following two security notions: concealing and binding. In a committing phase, Alice ( Committer ) first commits a bit and sends Bob ( verifier ) its encrypted information, from which Bob cannot decipher her bit. In an opening phase, she reveals her bit; however, Bob can detect her wrongdoing if she presents the bit different from what she had committed in the earlier phase.

A quantum key distribution scheme [1] is well-known to be unconditionally secure, whereas it is proven that no quantum bit commitment scheme achieves unconditional security [9, 13]. Chailloux and Kerenidis [3] recently argued that no protocol for quantum bit commitment achieves a cheating probability of less than 0.739. These facts immediately prompt us to seek a reasonable means to build practically durable protocols for quantum bit commitment. Technological limitations of current quantum device, on one hand, have been used to design feasible protocols in, e.g., [6, 7]. Dumais, Mayers, and Salvail [8], on the other hand, used a computationally difficult problem to construct a (non-interactive) protocol for quantum bit commitment. Their protocol requires the total communication cost of $O(n)$ qubits, where $n$ is a security parameter, and the security of the protocol relies on the existence of quantum one-way permutation (namely, a function that permutes a given set of strings with the one-way property that the function is easily computed but is hard to be inverted). In particular, the binding condition for the protocol is proven as follows. If the condition does not hold, then Alice must have a strategy to deceive Bob. Using her strategy, we can efficiently invert a given quantum one-way permutation on a quantum computer, leading to a contradiction. This protocol was later extended by Tanaka [14] to quantum string commitment using an additional technique of quantum fingerprinting to reduce the communication cost between Alice and Bob. Recently, Koshi and Odaira [12] reduced this assumption to the existence of quantum one-way functions.

Whether a quantum one-way permutation exists is still an open question and it seems quite difficult to settle down the question. Naturally, we can ask if a use of quantum one-way permutation can be replaced by any other (seemingly weaker) computational assumption. In this paper, we look for other means to construct a quantum bit commitment protocol; in particular, we are interested in a computational problem of distinguishing between two ensembles of quantum states. This type of problem has been used to guarantee the security of quantum protocols. Chailloux, Kerenidis, and Rosgen [4] drew from a slightly technical assumption of $\text{QSZK} \not\subset \text{QMA}$ which is not known to be true, a conclusion that a scheme for “auxiliary-input” quantum bit commitment (which allows Alice and Bob to apply the same POVM operations during the committing phase) exists. The purpose of this paper is to present a new scheme for quantum bit commitment with no such auxiliary inputs.

In 2005, Kawachi, Koshi, Nishimura, and Yamakami [11] devised two special ensembles of (reduced) quantum states and, from these ensembles, they built a quantum public-key cryptosystem whose security relies on a computational assumption that the ensembles are hard to distinguish efficiently. These ensembles posses....

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†This statement means that there exists a quantum statistical zero-knowledge proof system that cannot be expressed as a form of quantum Merlin-Arthur proof system.
quite useful properties (stated in Section 3.3), which are interesting on its own light and have been sought for other applications. As such an application, we actually use the two ensembles to build the aforementioned new scheme for quantum bit commitment. Our scheme, given in Section 3.3, is non-interactive, uses no auxiliary inputs, and achieves computational concealing and statistical binding at communication cost of $O(n \log n)$ qubits. Those security conditions of our scheme follows from an assumption on computational hardness of distinguishing the two ensembles. Note that, if this hardness assumption fails to hold, then, for example, we can efficiently solve on a quantum computer a classical problem, known as the graph automorphism problem (GA), in which we are asked to determine whether a given undirected graph is isomorphic to itself. This problem is not yet known to be either polynomial-time solvable or NP-complete. More importantly, our scheme has a concrete, explicit description, independent of the correctness of the assumption, and potentially it might be applied to other fields.

The computationally concealing condition for our scheme follows directly from the indistinguishability of two encrypted quantum states produced for two different committed bits 0 and 1. The statistical binding condition is met by an application of state partitioning, which is a means to partition a given quantum state into two specific orthogonal states. The details of the security conditions will be given in Section 4.

2 Main Theorem

Throughout this paper, we will work with various finite dimensional Hilbert spaces. For instance, $\mathcal{H}_2$ denotes the 2-dimensional Hilbert space spanned by a binary basis $\{|0\rangle, |1\rangle\}$; that is, $\mathcal{H}_2 = \text{span}\{|0\rangle, |1\rangle\}$. We use Dirac’s ket notation $|\phi\rangle$ to express quantum states and $\langle \phi |$ for the conjugate transpose of $|\phi\rangle$. The notation $\langle \phi | \psi \rangle$ expresses the inner product of $|\phi\rangle$ and $|\psi\rangle$. The norm of $|\phi\rangle$ is given as $\| |\phi\rangle \| = \sqrt{\langle \phi | \psi \rangle}$. An orthogonal measurement (or von Neumann measurement) of a quantum state is described by a set of orthogonal projections acting on a given Hilbert space.

We will use informal term of quantum algorithms to describe transformations of quantum bits (or qubits) throughout this paper. A quantum algorithm has been often modeled by a mechanical device of quantum Turing machine [2] or quantum circuit families [13]. We are particularly interested in quantum algorithms that terminate within a polynomial number of steps with respect to the size of input instances. We call such algorithms polynomial-time algorithms.

We choose the following definition for the computationally concealing condition, because this captures a more intuitive notion of Bob being unable to gaining any significant amount of information out of Alice. As we will show in Lemma 4.2 this definition comes from the indistinguishability between two quantum states sent from Alice.

In a committing phase, Alice encodes her committed bit $a$ into a quantum state and sends its (possibly reduced) state $\chi_a$ to Bob. We demand the following security against Bob.

**Definition 2.1** (computational concealing) A non-interactive quantum bit commitment scheme is computationally concealing if, for any positive polynomial $p$, there is no polynomial-time quantum algorithm that outputs $a$ from instance $\chi_a$ with error probability at least $1/2 + 1/p(n)$ for any $n \in \mathbb{N}^+$.

The security against Alice requires the following notion of statistical binding. In the committing phase, Alice starts with $|0\rangle$. She applies a quantum transformation $U_1$ and sends a subsystem $\mathcal{H}_{\text{commit}}$. At the beginning of an opening phase (or a revealing phase), Alice applies $U_2^{(a)}$, where $a \in \{0, 1\}$, if she wants to convince Bob that her committed bit is $a$. Alice’s cheating strategy is modeled by a triplet $U = (U_1, U_2^{(0)}, U_2^{(1)})$. Let $T_a^{(U)}(n)$ be the probability that Bob convinces himself that $a$ is truly a committed bit, provided that Bob faithfully follows the scheme.

**Definition 2.2** (statistical binding) A non-interactive quantum bit commitment scheme is statistically binding if there exists a negligible function $\varepsilon(n)$ such that, for any cheating strategy $U = (U_1, U_2^{(0)}, U_2^{(1)})$ of Alice, $T_0^{(U)}(n) + T_1^{(U)}(n) \leq 1 + \varepsilon(n)$ holds for every length $n \in \mathbb{N}^+$.

Our main theorem concerns the notion of indistinguishable ensembles of quantum states which are generated efficiently. First, we introduce the necessary terminology to explain the statement.

It is time to introduce extra notions and notations. Let $\mathbb{N}^+$ denote the set of all positive integers and set $N = \{0\} \cup \mathbb{N}^+$. An ensemble $\{\rho(n)\}_{n \in \mathbb{N}^+}$ of (reduced) quantum states is said to be efficiently generated if there exist two polynomially-bounded functions $q, \ell : \mathbb{N}^+ \to \mathbb{N}^+$ and a polynomial-time quantum algorithm

\begin{equation}
\end{equation}

\footnote{A function $f : \mathbb{N}^+ \to \mathbb{N}$ is polynomially bounded if there exists a (positive) polynomial $p$ for which $f(n) \leq p(n)$ for all $n \in \mathbb{N}$.}
\( \mathcal{A} \) such that, on every input \(|1^n\rangle|0\) (\( n \in \mathbb{N}^+ \)), (1) \( \mathcal{A} \) generates \(|1^n\rangle|\Phi \rangle \) of \( q(n) \) qubits and (2) \( \rho(n) \) is obtained by tracing out the first \( \ell(n) \) qubits of \( \Phi \); in notation, \( \rho(n) = \text{tr}_{\ell(n)}(|\Phi\rangle \langle \Phi|) \). Let \( \{\rho(n)\}_{n \in \mathbb{N}^+} \) and \( \{\chi(n)\}_{n \in \mathbb{N}^+} \) be two ensembles of (reduced) quantum states. We say that a quantum algorithm \( \mathcal{A} \) distinguishes between \( \{\rho(n)\}_{n \in \mathbb{N}^+} \) and \( \{\chi(n)\}_{n \in \mathbb{N}^+} \) with advantage \( \delta(n) \) if, for every \( n \in \mathbb{N}^+ \), it holds that

\[
\delta(n) = |\text{Prob}[\mathcal{A}(1^n, \rho(n)) = 1] - \text{Prob}[\mathcal{A}(1^n, \chi(n)) = 1]|.
\]

For each permutation \( \pi \in \mathcal{S}_n \), we define the quantum state \( |\phi^{(\pi)}(\sigma)\rangle \) as

\[
|\phi^{(\pi)}(\sigma)\rangle = \frac{1}{\sqrt{2}}(|\sigma\rangle + (-1)^{i}\sigma\pi\rangle).
\]

For each permutation \( \pi \) in \( K_n \), we partition \( S_n \) into two subsets \( \tilde{S}_0 \) and \( \tilde{S}_1 \), which satisfy the following condition: for every index \( a \in \{0, 1\} \) and for all elements \( \sigma \in S_n, \sigma \in \tilde{S}_a \) implies \( \sigma\pi \in \tilde{S}_{1-a} \). Let \( S_n^{(\pi)} \) denote one of these subsets of \( S_n \) that contains \( \pi \). Notice that \( S_n^{(\pi)} \) is uniquely determined from \( \pi \). It is easily seen that the set \( \mathcal{B}^{(\pi)} = \{|\phi^{(\pi)}(\sigma)\rangle \mid \sigma \in S_n^{(\pi)}, s \in \{0, 1\}\} \) forms a computational basis for the Hilbert space \( \mathcal{H}_{S_n} = \text{span}\{|\sigma\rangle \mid \sigma \in S_n\} \).

In what follows, we fix \( n \in \mathbb{N}^+ \) and \( \pi \in K_n \). For each bit \( s \in \{0, 1\} \), we define the quantum state

\[
\rho^{(\pi)}(n) = \frac{1}{|S_n|} \sum_{\sigma \in S_n} |\phi^{(\pi)}(\sigma)\rangle \langle \phi^{(\pi)}(\sigma)|, \quad n \in \mathbb{N}^+.
\]
Notice that the quantum states $\rho^{(\pi)}_t(n)$ and $\rho^{(\pi)}_r(n)$ are respectively denoted $\rho^{(\pi)}_t(n)$ and $\rho^{(\pi)}_r(n)$ in $\mathbb{H}$. For convenience, we use the notation $\Phi_s^{(\pi)}(n)$ to denote a pure quantum state $\frac{1}{\sqrt{|S_n|}} \sum_{\sigma \in S_n} |\sigma\rangle\langle\sigma|^{(\pi)}$, which is a purification of $\rho^{(\pi)}_t(n)$, because $\rho^{(\pi)}_t(n)$ coincides with the partial trace $\text{tr}_1((\Phi_s^{(\pi)}(n))|\Phi_s^{(\pi)}(n))$, where $\text{tr}_1$ is the partial trace over the first register (that is, the operator tracing out the first register). It is also useful to note that $\sum_{\sigma \in S_n} |\sigma\rangle\langle\sigma|^{(\pi)} = \sum_{\sigma \in S_n} |\sigma\rangle\langle\sigma|^{(\pi)}|\sigma\rangle\langle\sigma|^{(\pi)}$; thus, it holds that $\text{tr}_1((\Phi_s^{(\pi)}(n))|\Phi_s^{(\pi)}(n)) = \text{tr}_1((\Phi_s^{(\pi)}(n))|\Phi_s^{(\pi)}(n))$

To make our notation simple, we hereafter omit “$n$” whenever “$n$” is clear from the context.

Note that any quantum state $|\gamma\rangle$ in $\mathcal{H}_{S_n}$ can be expressed as $\sum_{a \in \{0,1\}} \sum_{\sigma \in S_n} \alpha_{a,\sigma}|\phi^{(\pi)}_{a,\sigma}\rangle$ for any fixed permutation $\pi \in K_n$. Basic properties of $|\phi^{(\pi)}_{a,\sigma}\rangle$ and $|\Phi_s^{(\pi)}\rangle$ are summarized in the following lemma. In the lemma, we conveniently use two notations “∧” and “∨” to mean the logical connectives “AND” and “OR”, respectively.

**Lemma 3.1** Let $n \in \mathbb{N}^+$, $s \in \{0,1\}$, $\pi \in K_n$, and $\sigma, \tau \in S_n$.

1. $|\phi^{(\pi)}_{1,\sigma}\rangle = (-1)^{s}|\phi^{(\pi)}_{0,\sigma}\rangle$.
2. $\langle \phi^{(\pi)}_{1,\sigma}|\phi^{(\pi)}_{0,\tau}\rangle = 0$.
3. $\langle \phi^{(\pi)}_{1,\sigma}|\phi^{(\pi)}_{0,\tau}\rangle = 1$ if $\sigma = \tau$; $(-1)^s$ if $\sigma = \tau^\pi$; 0 otherwise.
4. $\langle \phi^{(\pi)}_{1,\sigma}|\varphi^{(\pi)}_{0}\rangle = 1$ if $\pi = \kappa \land (\sigma = \tau \lor \sigma = \tau^\kappa)$; $\frac{1}{2}$ if $\pi \neq \kappa \land (\sigma = \tau \lor \sigma = \tau^\kappa); 0$ otherwise.
5. $\langle \phi^{(\pi)}_{1,\sigma}|\varphi^{(\pi)}_{0}\rangle = \frac{1}{2}$ if $\pi = \kappa \land (\sigma = \tau \lor \sigma = \tau^\kappa); -\frac{1}{2}$ if $\pi \neq \kappa \land (\sigma = \tau \lor \sigma = \tau^\kappa); 0$ otherwise.

**Proof.** (1) Since $\sigma \pi = \sigma$, it follows that $\sqrt{2}|\phi^{(\pi)}_{1,\sigma}\rangle = |\sigma\pi\rangle + (-1)^s|\sigma\rangle = (-1)^s(|\sigma\rangle + (-1)^s|\sigma\pi\rangle) = \sqrt{2}|\phi^{(\pi)}_{1,\sigma}\rangle$. This implies that $|\phi^{(\pi)}_{a,\sigma}\rangle = (-1)^s|\phi^{(\pi)}_{a,\sigma}\rangle$, leading to the desired consequence.

(2) For simplicity, we write $P$ for $\langle \phi^{(\pi)}_{0,\sigma}|\phi^{(\pi)}_{0,\tau}\rangle$. Note that $\pi \in K_n$ implies $\sigma \pi \neq \sigma$ because $\sigma \pi \pi = \sigma$ is equivalent to $\pi = \text{id}$. Since $2P = \langle \sigma| + \langle \sigma\pi|\rangle - \langle \sigma\pi|\rangle = \langle \sigma| + \langle \sigma\pi|\rangle - \langle \sigma|\pi\rangle$, $2P = 0$.

(3) Consider the case of $s = 0$. Let $P = \langle \phi^{(\pi)}_{0,\sigma}|\phi^{(\pi)}_{0,\tau}\rangle$. Note that $2P = 2\langle \sigma|\tau\rangle + \langle \sigma|\tau\rangle = \langle \sigma|\tau\rangle$. Since $\pi \neq \text{id}$, $\langle \sigma\pi|\tau\rangle = 0$ implies $\langle \sigma\pi|\tau\rangle = 0$ and also $\langle \sigma|\tau\rangle = 0$ implies $\langle \sigma|\tau\rangle = 0$. Thus, if either $\tau \neq \sigma$ or $\tau = \sigma$, we have $2P = 0$; otherwise, $2P = 0$. The other case of $s = 1$ is similarly handled.

(4) By setting $P = \langle \phi^{(\pi)}_{0,\sigma}|\phi^{(\pi)}_{0,\tau}\rangle$, we obtain $2P = \langle \sigma|\tau\rangle + \langle \sigma|\tau\rangle = \langle \sigma|\tau\rangle. \if\pi = \kappa, then (3) \else (4) \fi$ implies the desired result. Now, assume that $\pi \neq \kappa$. If $\sigma = \tau$, then $\langle \sigma|\tau\rangle = 0$ and the other terms in the above expansion of $2P$ are all zeros, because $\pi, \kappa \neq \text{id}$ and $\pi \neq \kappa$. From these results follows $2P = 0$.

(5) If $\tau = \tau^\kappa$, then $\langle \sigma|\tau\rangle = 1$ and the other terms are zeros; thus, we obtain $2P = 0$. The remaining case of $\sigma = \tau\pi$ is similarly handled. When all the above-mentioned cases fail, no terms in the expansion of $2P$ are 1. Therefore, we conclude that $2P = 0$.

(6) Let $P = \langle \phi^{(\pi)}_{0,\sigma}|\phi^{(\pi)}_{0,\tau}\rangle$. Note that $2P = \langle \sigma|\tau\rangle - \langle \sigma|\tau\rangle + \langle \sigma|\tau\rangle - \langle \sigma|\tau\rangle$. If $\pi = \kappa$, then (2) implies $2P = 0$. In what follows, we assume that $\pi \neq \kappa$. If $\sigma = \tau$, then it follows that $2P = \langle \sigma|\tau\rangle = 1$ because $\pi, \kappa \neq \text{id}$ and $\pi \neq \kappa$. Thus, we obtain $2P = 0$. Similarly, if $\pi = \tau\kappa$, we obtain $2P = \langle \sigma|\tau\rangle = 1$. From $\sigma = \tau$, it follows that $2P = -\langle \sigma|\tau\rangle = -1$. Finally, we note that $\sigma = \tau\kappa$ implies $2P = -\langle \sigma|\tau\rangle = -1$.

We give another useful property of $|\phi^{(\pi)}_{a,\sigma}\rangle$. This property will play an important role in Section 5.1.

**Lemma 3.2** For fixed $\pi \in K_n$ and $\sigma \in S_n$, it holds that $|\phi^{(\pi)}_{a,\kappa}\rangle = \frac{1}{|K_{\pi}|} \sum_{\kappa \in K_n} (|\phi^{(\pi)}_{0,\kappa}\rangle - |\phi^{(\pi)}_{a,\kappa}\rangle)$.

**Proof.** Fix $\pi \in K_n$ and $\sigma \in S_n$. Note that the value $\sqrt{2} \sum_{\kappa \in K_n} (|\phi^{(\pi)}_{0,\kappa}\rangle - |\phi^{(\pi)}_{a,\kappa}\rangle)$ is $\sum_{\kappa \in K_n} (|\sigma\kappa\rangle - |\sigma\kappa\rangle) = \sum_{\kappa \in K_n} (|\sigma\rangle - |\sigma\rangle)$. Let us consider the function $f : K_n \rightarrow K_n \cup \{\text{id}\}$ defined as $f(\kappa) = \pi\kappa$. This function satisfies the following three properties: (i) $f$ is one-to-one, (ii) $f(\pi) = \text{id}$, and (iii) there is no element $\kappa \in K_n$ satisfying $f(\kappa) = \pi$. We conclude that $f$ is a bijection on the restricted domain $K_n - \{\pi\}$. This fact implies that $\sum_{\kappa \in K_n} |\sigma\rangle = \sum_{\kappa \in K_n} |\sigma\kappa\rangle = |\sigma\rangle$. Overall, $\sqrt{2} \sum_{\kappa \in K_n} (|\phi^{(\pi)}_{0,\kappa}\rangle - |\phi^{(\pi)}_{a,\kappa}\rangle) = |\kappa\rangle - |\kappa\rangle + (|\sigma\rangle - |\sigma\rangle)$, which equals $|\kappa\rangle - |\kappa\rangle$. In other words, $\sqrt{2} (|K_n| - 1) (|\sigma\rangle - |\sigma\rangle)$. From this equality, the lemma follows immediately.

Hereafter, we will give two quantum procedures, which are useful in the description of our quantum bit commitment scheme in Section 3.3. First, we introduce several useful unitary operations. The **Hadamard transform** $H$ acts on the system $\mathcal{H}_2$ as $H|s\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^s|1\rangle)$ for every bit $s \in \{0,1\}$. The **controlled-$\pi$**...
operator $C_\pi$ acts on $\mathcal{H}_2 \otimes \mathcal{H}_{S_n}$ as $C_\pi |a\rangle = |a\rangle |\sigma\pi\rangle$ if $a = 1$, and $|a\rangle |\sigma\rangle$ otherwise. The controlled-NOT$_{id}$ operator $CNOT_{id}$ acts on $\mathcal{H}_2 \otimes \mathcal{H}_{S_n}$ as $CNOT_{id}|a\rangle |\sigma\rangle = (NOT(|a\rangle)) |\sigma\rangle$ if $\sigma = id$, and $|a\rangle |\sigma\rangle$ otherwise. Moreover, let $U_{sgn}$ denote a unitary operator mapping $|\sigma\rangle$ to $(-1)^{sgn(\sigma)} |\sigma\rangle$, where $\sigma \in S_3$ and $sgn(\sigma)$ is $1$ ($0$, resp.) if $\sigma$ is an even (odd, resp.) permutation in $S_n$. The controlled-SAWP operator $C_{\text{swap}}^{(1,2)}$ (with $1 \leq i < j \leq n$) exchanges the contents of the $i$th and $j$th registers among $n$ registers; that is, $C_{\text{swap}}^{(1,2)} |s_1\rangle \cdots |s_i\rangle \cdots |s_j\rangle \cdots |s_n\rangle = |s_1\rangle \cdots |s_j\rangle \cdots |s_i\rangle \cdots |s_n\rangle$.

We will present two useful quantum transforms.

[Procedure 1] The following procedure $P_1$ can generate the quantum state $\mathcal{H} = |0\rangle |\sigma\rangle |\phi(\pi)\rangle$ in the system $\mathcal{H}_2 \otimes \mathcal{H}_{S_n}^{(1)} \otimes \mathcal{H}_{S_n}^{(2)} \otimes \mathcal{H}_{S_n}^{(3)}$ from $|0\rangle |\sigma\rangle |\pi\rangle |\mid id\rangle$ if $\pi \neq id$. To $|\pi\rangle |\mid id\rangle$ in the system $\mathcal{H}_{S_n}^{(2)} \otimes \mathcal{H}_2 \otimes \mathcal{H}_{S_n}^{(3)}$, we first apply two operators $I \otimes H \otimes I$ and $I \otimes C_\pi$, where $H$ is the Hadamard transform. This process generates a quantum state $\frac{1}{\sqrt{2}} |\pi\rangle |\mid id\rangle + |\pi\rangle |\mid id\rangle)$. Since $\pi \neq id$, we apply $I \otimes CNOT_{id}$ and generate the state $\frac{1}{\sqrt{2}} |\pi\rangle |\mid id\rangle + |\pi\rangle |\pi\rangle)$. Now, consider $\frac{1}{\sqrt{2}} |\sigma\rangle |\mid id\rangle + |\pi\rangle |\pi\rangle) \in \mathcal{H}_{S_n}^{(2)} \otimes \mathcal{H}_{S_n}^{(3)}$. Multiply $\sigma$ in the first register from the left to the second register, generating $\frac{1}{\sqrt{2}} |\sigma\rangle |\sigma\pi\rangle |\pi\rangle)$. In the end, we obtain $|0\rangle |\sigma\rangle |\pi\rangle |\phi(\pi)\rangle$ in $\mathcal{H}$.

Similarly, we can generate $|0\rangle |\pi\rangle |\Phi_0(\pi)\rangle$ from $|0\rangle |\pi\rangle |\mid id\rangle$ in $\mathcal{H}_2 \otimes \mathcal{H}_{S_n} \otimes \mathcal{H}_{S_n} \otimes \mathcal{H}_{S_n}$ by running the following procedure $P_1$. After generating the state $\frac{1}{\sqrt{2}} |\pi\rangle |\mid id\rangle + |\pi\rangle |\pi\rangle)$. Now, consider $\frac{1}{\sqrt{2}} |\sigma\rangle |\mid id\rangle + |\pi\rangle |\pi\rangle) \in \mathcal{H}_{S_n} \otimes \mathcal{H}_{S_n}$. Multiply each $\sigma$ in the first register to the content of the second register. We then obtain $\frac{1}{\sqrt{2}} |\sigma\rangle |\phi(\pi)\rangle$, which is $|\Phi_0(\pi)\rangle$.

[Procedure 2] There is a simple procedure $P_2$ that transforms $|\phi(\pi)\rangle$ to $|\sigma(\pi)\rangle$ without knowing $(s, \pi)$ as follows. Initially, we have $|\phi(\pi)\rangle$. We apply $U_{sgn} \otimes U_{sgn}$ to $|\phi(\pi)\rangle$. The resulted quantum state is $\frac{1}{\sqrt{2}} (-1)^{sgn(\sigma)} |\sigma\rangle |\phi(\pi)\rangle$ if $\pi$ is an odd permutation, this state equals $\frac{1}{\sqrt{2}} (-1)^{2sgn(\sigma)} |\sigma\rangle |(-1)^{s+1} |\phi(\pi)\rangle$, which is exactly $|\phi(\pi)\rangle$. If we apply $I \otimes P_2$ to $|\Phi(\pi)\rangle$, then we immediately obtain $|\Phi(\pi)\rangle$.

### 3.2 State Partitioning

Our quantum bit commitment protocol in Section 3.3 requires a method to “partition” a given quantum state in the system $\mathcal{H}_{S_n}$ into two orthogonal states $\chi_0$ and $\chi_1$ that satisfy an extra property. A basic idea of state partitioning is inspired by a trapdoor property of [11] Theorem 2.1. Let $n \in \mathbb{N}^+$ and $\chi$ be any reduced state in $\mathcal{H}_{S_n}$. This state can be expressed as $\chi = \chi_0 + \chi_1$, where $\chi_s = \sum_{\sigma \in S_n} p_\sigma |\phi(\pi)\rangle |\phi(\pi)\rangle$ for each bit $s \in \{0, 1\}$.

#### State Partition Algorithm: $C_{SPA}$

- **(S1)** Take an instance of the form $\chi = |\pi\rangle |\sigma\rangle$ in a system $\mathcal{H}_{S_n} \otimes \mathcal{H}_{S_n}$. Prepare $|0\rangle |0\rangle \otimes \chi$ in $\mathcal{H} = \mathcal{H}_2 \otimes \mathcal{H}_{S_n} \otimes \mathcal{H}_{S_n}$.
- **(S2)** Apply $H \otimes I^{\otimes 2}$. Since the second register of $\mathcal{H}$ contains $\pi$, we can freely use the controlled-$\pi$ operator $C_\pi$. Here, we apply $C_{\text{swap}}^{(1,2)}(C_\pi \otimes I)C_{\text{swap}}^{(1,2)}$. Finally, apply $H \otimes I^{\otimes 2}$.
- **(S3)** The state $|0\rangle |0\rangle \otimes \chi$ changes into $|0\rangle |0\rangle \otimes |\pi\rangle |\sigma\rangle + |1\rangle |1\rangle \otimes |\pi\rangle |\phi(\pi)\rangle \otimes \chi_1$. When we observe the first register, we find $0$ (resp., $1$) with probability exactly $\frac{1}{2}$.

Here, we will briefly discuss the correctness of the above algorithm. Let $\chi$ be given at Step (S1). Assume that $\chi = \sum_{\sigma \in S_n} \sum_{s \in \{0, 1\}} p_{\sigma,s} |\phi(\pi)\rangle |\phi(\pi)\rangle$. We introduce a new system $\mathcal{H} \otimes \mathcal{H}' \otimes \mathcal{H}_{S_n}$ be $|\Phi\rangle = \sum_{\sigma \in S_n} \sum_{s \in \{0, 1\}} \sqrt{p_{\sigma,s}} |s\rangle |\phi(\pi)\rangle$. Note that $\chi = \text{tr}_1(|\Phi\rangle |\Phi\rangle)$. For each fixed $s$, we write $|\psi_s\rangle = \sum_{\sigma \in S_n} \sqrt{p_{\sigma,s}} |s\rangle |\phi(\pi)\rangle$. Note that $|\Phi\rangle = |\psi_0\rangle + |\psi_1\rangle$, where $\chi_s = \text{tr}_1(|\psi_s\rangle |\psi_s\rangle) = \sum_{\sigma \in S_n} \sqrt{p_{\sigma,s}} |\phi(\pi)\rangle |\phi(\pi)\rangle$ for each $s$.

Initially, we have a state $|\Phi'\rangle = |0\rangle |\pi\rangle |\Phi\rangle$, which equals $\sum_{\sigma \in S_n} \sum_{s \in \{0, 1\}} \sqrt{p_{\sigma,s}} |0\rangle |\pi\rangle |\phi(\pi)\rangle$. Since we work on a purified state, it is convenient to expand $C_\pi$ to $C'_\pi$ as follows. Let $C'_\pi = C_{\text{swap}}^{(1,2)}C_{\text{swap}}^{(3,4)}I \otimes C_\pi \otimes I^{\otimes 2}C_{\text{swap}}^{(1,2)}C_{\text{swap}}^{(3,4)}$. Step (S2) produces $|\Phi''\rangle = (H \otimes I^{\otimes 4})C'_\pi(H \otimes I^{\otimes 4})|\Phi'\rangle$. By a direct calculation, the
quantum state \((H \otimes I^{\otimes 4})C_\pi \Phi \otimes I^{\otimes 4})||0||\psi_s\rangle\) equals

\[
\frac{1}{2} \sum_{\sigma \in S_n} \sqrt{p_{\sigma,a}}||0||\pi||\sigma||\sigma\rangle \left( |\phi^0_{\sigma,a}|| \phi^0_{\sigma,a}\rangle \right) + \frac{1}{2} \sum_{\sigma \in S_n} \sqrt{p_{\sigma,1}}||0||\pi||\sigma||\sigma\rangle \left( |\phi^0_{\sigma,1}|| \phi^0_{\sigma,1}\rangle \right),
\]

which coincides with \(|s||\pi||\psi_s\rangle\) because Lemma 3.1(3) implies \(|\phi^0_{\sigma,a}\rangle = (-1)^s |\phi^0_{\sigma,a}\rangle\). Therefore, we conclude that \(|\Phi^0\rangle = ||0||\pi||\psi_0\rangle + ||1||\pi||\psi_1\rangle\).

Next, we want to examine the behavior of \(C_{SPA}\) on instance \(|\kappa||\kappa\rangle \otimes \chi\) given in Step (S1), where \(\kappa\) is different from \(\pi\). In what follows, let \(\kappa\) be any element in \(K_n - \{\pi\}\). Recall that \(|\Phi\rangle = \sum_{\sigma \in S_n} \sqrt{p_{\sigma}}\langle \sigma||\phi_{\sigma,a}\rangle\). Here, the quantum algorithm starts with \(|\Phi^0\rangle = ||0||\kappa||\Phi\rangle\).

\(\) Following Step (S2), we calculate \((H \otimes I^{\otimes 4})C_\pi \Phi \otimes I^{\otimes 4})||0||\pi||\psi_0\rangle\). Note that, since the content of the second register is \(\kappa\), we apply \(C_\kappa\) instead of \(C_\pi\).

The algorithm \(C_{SPA}\) produces a quantum state

\[
\sum_{a \in \{0, 1\}} \frac{1}{2\sqrt{2}} \sum_{\sigma \in S_n} \sqrt{p_{\sigma,a}}||0||\pi||\sigma||\sigma\rangle \left( |\phi^0_{\sigma,a}|| \phi^0_{\sigma,a}\rangle \right).
\]

This is equivalent to

\[
\sum_{a \in \{0, 1\}} \frac{1}{2} \sum_{\sigma \in S_n} \sqrt{p_{\sigma,a}}||0||\pi||\sigma||\sigma\rangle \left( |\phi^0_{\sigma,a}|| \phi^0_{\sigma,a}\rangle \right) + \left( |\phi^0_{\sigma,1}|| \phi^0_{\sigma,1}\rangle \right).
\]

When \(a = 0\) and \(s = 0\), we obtain \(\frac{1}{2} \sum_{\sigma \in S_n} \sqrt{p_{\sigma,a}}||0||\pi||\sigma||\sigma\rangle \left( |\phi^0_{\sigma,a}|| \phi^0_{\sigma,a}\rangle \right) + \left( |\phi^0_{\sigma,1}|| \phi^0_{\sigma,1}\rangle \right)\), where \(|\phi^0_{\sigma,0}\rangle|\phi^0_{\sigma,0}\rangle = 0\) by Lemma 3.1(3), the norm of this state is

\[
\left| \frac{1}{2} \sum_{\sigma \in S_n} \sqrt{p_{\sigma,a}}||0||\pi||\sigma||\sigma\rangle \left( |\phi^0_{\sigma,a}|| \phi^0_{\sigma,a}\rangle \right) \right|^2 = \frac{1}{4} \sum_{\sigma \in S_n} p_{\sigma,0} \left( 2 + 2 |\phi^0_{\sigma,0}\rangle\langle \phi^0_{\sigma,0}|| \phi^0_{\sigma,0}\rangle \langle \phi^0_{\sigma,0}|| \phi^0_{\sigma,0}\rangle \right) = \frac{1}{2} \sum_{\sigma} p_{\sigma,0}.
\]

Similarly, when \(a = 0\) and \(s = 1\), the state \(\frac{1}{2} \sum_{\sigma \in S_n} \sqrt{p_{\sigma,a}}||0||\pi||\sigma||\sigma\rangle \left( |\phi^0_{\sigma,1}|| \phi^0_{\sigma,1}\rangle \right)\) has norm \(\frac{1}{2} \sum_{\sigma} p_{\sigma,1}\).

By combining those values, we conclude that the probability of observing 0 in the first register is \(\frac{1}{2} \sum_{\sigma} p_{\sigma,0} + \frac{1}{2} \sum_{\sigma} p_{\sigma,1} = \frac{1}{2}\).

A similar argument proves that we observe 1 in the first register with probability exactly \(\frac{1}{2}\).

### 3.3 A New Protocol

We will present a new quantum bit commitment protocol. We use the following quantum system between Alice (committer) and Bob (verifier): \(H_{\text{all}} = H_{A,\text{private}} \otimes H_{\text{bit}} \otimes H_{\text{open}} \otimes H_{\text{commit}} \otimes H_{B,\text{private}}\), where \(H_{A,\text{private}}\) is a system that is used only by Alice, \(H_{\text{open}}\) holds a secret key produced by Alice, \(H_{\text{bit}}\) is a 1-qubit system for a committed bit by Alice, \(H_{\text{commit}}\) is used to produce an encrypted information regarding a committed bit, and \(H_{B,\text{private}}\) is a system used only by Bob. Different from \(H_{A,\text{private}}\) and \(H_{B,\text{private}}\), the systems \(H_{\text{bit}}, H_{\text{open}},\) and \(H_{\text{commit}}\) are accessed interchangeably by Alice and Bob at specific points during an execution of the protocol.

Consider the following bit commitment scheme between Alice and Bob. Let \(n\) be the security parameter on which Alice and Bob initially agree.

We intend to include the description of the ownerships of each system that makes up \(H_{\text{all}}\). Moreover, we write \(H_{\text{open}}\) for the system \(H_{\text{open1}} \otimes H_{\text{open2}}\).

**Comitting Phase:**

(C1) Initially, Alice owns the system \(H_A^{(C1)} = H_{A,\text{private}} \otimes H_{\text{bit}} \otimes H_{\text{open}} \otimes H_{\text{commit}}\) and Bob owns \(H_B^{(C1)} = H_{B,\text{private}}\). Starting with \(|0\rangle\) in \(H_{\text{all}}\), she randomly chooses her secret key \(\pi \in K_n\) in \(H_{\text{open2}}\).

(C2) She prepares \(|0||id\rangle\) in \(H_{\text{open1}} \otimes H_{\text{commit}}\) and generates \(|\Phi^{(C)}\rangle\) as described in Section 3.1.

(C3) Let \(a\) be a bit that Alice wants to commit. She produces \(|a\rangle\) in \(H_{\text{bit}}\). She then transforms \(|\Phi^{(C)}\rangle\) in \(H_{\text{open1}} \otimes H_{\text{commit}}\) into \(|\Phi^{(C)}_{a}\rangle\) by applying \(P_a\) when \(a = 1\).

(C4) She sends the system \(H_{\text{commit}}\) to Bob. Bob then receives the reduced state \(\rho^{(C)}\), which is called a commitment state. Bob should protect it from decoherence until the opening phase. In the end, Alice’s system becomes \(H_A^{(C4)} = H_{A,\text{private}} \otimes H_{\text{bit}} \otimes H_{\text{open}}\) and Bob’s becomes \(H_B^{(C4)} = H_{\text{commit}} \otimes H_{B,\text{private}}\).
In the following opening phase, Alice reveals her secret bit $a$. Bob then checks whether it is actually the bit committed by her in the committing phase.

**Opening Phase (or Revealing Phase):**

(R1) Alice’s current system is $\mathcal{H}_A^{(R1)} = \mathcal{H}_{A, private} \otimes \mathcal{H}_{bit} \otimes \mathcal{H}_{open}$ and Bob’s system is $\mathcal{H}_B^{(R1)} = \mathcal{H}_{commit} \otimes \mathcal{H}_{B, private}$. Alice sends the system $\mathcal{H}_{bit} \otimes \mathcal{H}_{open}$ to Bob.

(R2) Alice now owns the system $\mathcal{H}_A^{(R2)} = \mathcal{H}_{A, private}$ and Bob owns $\mathcal{H}_B^{(R2)} = \mathcal{H}_{open} \otimes \mathcal{H}_{bit} \otimes \mathcal{H}_{commit} \otimes \mathcal{H}_{B, private}$. Bob measures the two registers $\mathcal{H}_{bit} \otimes \mathcal{H}_{open}$ on the computational basis $\{0, 1\} \otimes S_n$. Assume that he obtains $(a, \pi)$ after the measurement. If $\pi \not\in K_n$, then Bob declares that Alice tries to deceive him. In what follows, we assume that $\pi \in K_n$.

(R3) Assume that, in the previous committing phase, Bob had received a reduced state $\chi$ in $\mathcal{H}_{commit}$ from Alice. Bob runs the state partition algorithm $C_{SPA}$ on input $|0\rangle|0\rangle \otimes \chi$ in $\mathcal{H}_{B, private} \otimes \mathcal{H}_{commit}$, provided that $\mathcal{H}_{B, private}$ is a 2-dimensional system.

(R4) Measure the system $\mathcal{H}_{B, private}$. If the outcome of the measurement is not $a$, then Bob declares that Alice tries to deceive him.

(R5) Whenever $a = 1$, first apply $P_2$ to change $|\Phi_1^{(\pi)}\rangle$ to $|\Phi_0^{(\pi)}\rangle$. Apply $\tilde{P}_1^{-1}$ (which is given in Section 3.1) to $\mathcal{H}_2 \otimes \mathcal{H}_{open} \otimes \mathcal{H}_{commit}$. Measure the system $\mathcal{H}_{open1} \otimes \mathcal{H}_{commit}$ in state $|0\rangle|id\rangle$. If $(0, id)$ is observed, then Bob accepts $a$ as Alice’s committed bit. Otherwise, Bob declares that Alice tries to deceive him.

In Step (R2), Bob does not observe the subsystem $\mathcal{H}_{open1}$ because, otherwise, the entanglement between $\mathcal{H}_{open1}$ and $\mathcal{H}_{commit}$ could be destroyed and Steps (R3) and (R5) might not work properly.

In the subsequent section, we will analyze the above protocol in details.

## 4 Security Analysis of the Scheme

We will examine the security of the quantum bit commitment protocol given in Section 3.3. We will show that our protocol is computationally concealing in Section 4.1. This is a direct consequence of [11]. A more complex analysis is required to show the statistically binding condition in 4.2.

### 4.1 Computationally Concealing Condition

The concealing condition for bit commitment requires that Bob cannot retrieve any information on $a$ during a committing phase after honest Alice commits $a$ and sends a quantum state associated with $a$. Intuitively, this condition is satisfied because Bob does not know $\gamma$, which locks the information on $a$ inside the quantum state, and thus there may be no way for Bob to obtain the information on $a$ with probability higher than a given parameter.

In our scheme, the notion of computational concealing, given in Section 2, is rephrased as follows. Our quantum bit commitment scheme is *computationally concealing* if, for any positive polynomial $p$, there is no polynomial-time quantum algorithm that outputs $a$ from $\rho_a$ with error probability at least $1/2 + 1/p(n)$ for any $n \in \mathbb{N}^+$. We will show that our protocol achieves the above computational concealing condition under the assumption that GA is hard to solve efficiently on a quantum computer.

**Theorem 4.1** Let $n$ be an agreed security parameter. If no polynomial-time quantum algorithm solves GA with non-negligible probability, then our scheme satisfies the computational concealing condition.

Theorem 4.1 follows from the lemma below.

**Lemma 4.2** Let $k \in \mathbb{N}^+$. If no polynomial-time quantum algorithm solves GA with error probability at least $2^{-n}$, where $n$ is the vertex set size of an input graph, then Bob cannot distinguish between $\{\rho_0^{(\pi)}(n)^{\otimes k}\}_n \in \mathbb{N}^+$ and $\{\rho_1^{(\pi)}(n)^{\otimes k}\}_\mathbb{N}^+$ with advantage at least $1/p(n)$ for any positive polynomial $p$. 


Before proving Lemma 4.2 we give the proof of Theorem 4.1.

**Proof of Theorem 4.1.** Assume that there is an efficient quantum algorithm $A$ that produces $a$ from $\rho^{(\pi)}$ with probability at least $1/2 + 1/p(n)$ for a certain fixed positive polynomial $p$. Let us consider the following quantum algorithm $B$: on input $\chi \in \{\rho_0^{(\pi)}, \rho_1^{(\pi)}\}$, run $A$ and obtain a bit, say, $a$. By the property of $A$, it follows that $\text{Prob}[B(1^n, \rho_0^{(\pi)}) = a] \geq 1/2 + 1/p(n)$ for each bit $a \in \{0, 1\}$. Thus, we obtain $|\text{Prob}[B(1^n, \rho_0^{(\pi)}) = 1] - \text{Prob}[B(1^n, \rho_1^{(\pi)}) = 1]| \geq 2/p(n)$. Hence, we can distinguish between $\rho_0^{(\pi)}$ and $\rho_1^{(\pi)}$ with advantage at least $2/p(n)$. By Lemma 4.2 we conclude that $GA$ is polynomial-time solvable on a quantum computer with non-negligible probability. □

To prove Lemma 4.2 we recall the computational distinction problem QSCD$_{ff}$ introduced by Kawachi et al. [11]. Since we need only a restricted form of this problem, we re-formulate this problem in the following fashion. Let $k$ be a fixed constant in $\mathbb{N}^+$. 

**k-Quantum State Computational Distinction Problem k-QSCD$_{ff}$ (weaker version):**

- **INSTANCE:** a string $1^n$ and a $k$-fold quantum state $\rho^{\otimes k}$ with $\rho \in \{\rho_0^{(\pi)}(n), \rho_1^{(\pi)}(n)\}$ for a certain fixed (but hidden) permutation $\pi \in K_n$, depending only on $n$, where $n \in \mathbb{N}^+$. 
- **OUTPUT:** YES, if $\rho = \rho_0^{(\pi)}(n)$; NO, otherwise.

For convenience, we say that a quantum algorithm $A$ **solves** $k$-QSCD$_{ff}$ with advantage $p(n)$ if $A$ distinguishes between $\{\rho_0^{(\pi)}(n)^{\otimes k}\}_{n \in \mathbb{N}^+}$ and $\{\rho_1^{(\pi)}(n)^{\otimes k}\}_{n \in \mathbb{N}^+}$ with advantage $p(n)$. Moreover, we say that a quantum algorithm $A$ **solves** $k$-QSCD$_{ff}$ with **average advantage** $p$ on length $n$ if the average, over all $\pi \in K_n$ chosen uniformly at random, of the advantage with which $A$ distinguishes between $\{\rho_0^{(\pi)}(n)^{\otimes k}\}_{n \in \mathbb{N}^+}$ and $\{\rho_1^{(\pi)}(n)^{\otimes k}\}_{n \in \mathbb{N}^+}$ is exactly $p$. We note that, by combining [11] Theorem 2.2 and [11] Theorem 2.5, the following holds.

**Lemma 4.3** [11] Let $k \in \mathbb{N}^+$. If a quantum algorithm $A$ solves $k$-QSCD$_{ff}$ with average advantage at least $1/p(n)$ for a certain positive polynomial $p$, then there exists a quantum algorithm that solves GA for infinitely-many lengths with probability at least $1 - 2e^{-n}$, where $e$ is the base of natural logarithms and $n$ refers to the vertex set size of an input graph of $GA$.

At last, we return to the proof of Lemma 4.2.

**Proof of Lemma 4.2.** We will show the contrapositive of the lemma. Let $k \in \mathbb{N}^+$. Assume that there are a positive polynomial $p$ and a polynomial-time quantum algorithm $A$ that distinguishes between $\rho_0^{(\pi)}(n)^{\otimes k}$ and $\rho_0^{(\pi)}(n)^{\otimes k}$ with advantage at least $1/p(n)$. In other words, we can solve $k$-QSCD$_{ff}$ in polynomial time with advantage at least $1/p(n)$. Lemma 4.3 therefore implies that $GA$ is solvable for infinitely-many input lengths $n$ on an appropriate quantum computer in polynomial time with error probability at most $2e^{-n}$, which is bounded from above by $2^{-n}$. □

### 4.2 Statistically Binding Condition

The binding condition for classical bit commitment requires that adversarial Alice cannot cheat Bob simply by revealing a different bit $a'$ together with a different key $\pi'$ to Bob. For quantum bit commitment, Dumais et al. [8] formulated a condition for a quantum bit commitment scheme to be **statistically binding** in the case of non-interactive schemes. Other definitions for binding condition are found in, e.g., [5].

Conventionally, we say that Alice **unveils** $a$ (with probability $p$) if, in the opening phase, Bob observes $a$ and convinces himself that this is truly a committed bit (with probability $p$).

Here, we cope with a general adversary model, proposed in [8], which describes adversarial Alice’s attack $U$ as a triplet $(U_1, U_2^{(0)}, U_2^{(1)})$ of unitary operators.

1. At the beginning of the committing phase, adversarial Alice starts with the initial state $|0\rangle$ in her system $H_{a,\pi}^{(C1)} = H_{\text{A, private}} \otimes H_{\text{bit}} \otimes H_{\text{open}} \otimes H_{\text{commit}}$. Instead of taking Steps (C1)–(C3), she applies the unitary operator $U_1$ to $|0\rangle$ in $H_{a,\pi}^{(C1)}$ and generates a quantum state $|\eta^{(C1)}\rangle = U_1|0\rangle = \sum_{a \in \{0,1\}} \sum_{\sigma \in S_n} \sum_{\pi \in K_n} |a,\pi,\sigma\rangle |\pi\rangle |\gamma_{a,\pi,\sigma}\rangle$ with $\sum_{a,\pi,\sigma} \| |\xi_{a,\pi,\sigma}\rangle \|^2 = 1$. Since $|\gamma_{a,\pi,\sigma}\rangle \in H_{S_n}$ =
state that Bob receives from adversarial Alice is of the form \( \chi \), span \( \{ \phi_{\pi}(\tau) \mid \tau \in S_n(\pi), s \in \{0, 1\} \} \), it holds that \( |\gamma_{a,\pi,\sigma} \rangle = \sum_{\tau,s} a_{\tau,s} |\phi_{\tau,s}(\pi)\rangle \) for an appropriate set \( \{a_{\tau,s}(\pi,\sigma)\}_{\tau,s} \). Hence, we obtain

\[
|\eta^{(C1)}\rangle = \sum_{\pi \in K_n, \tau \in S_n(\pi)} \sum_{a, s \in \{0, 1\}} |\chi^{(\tau,s)}(\pi,a)\rangle |a\rangle |\pi\rangle |\sigma\rangle |\phi_{\tau,s}(\pi)\rangle,
\]

where \( |\chi^{(\tau,s)}(\pi,a)\rangle = a_{\tau,s}(\pi,\sigma) |\xi_{a,\pi,\sigma}\rangle \). At Step (C4), she sends the system \( H_{\text{commit}} \) to Bob. The quantum state that Bob receives from adversarial Alice is of the form \( \chi \). Given Alice’s cheating strategy \( U \mid \pi, \sigma \), instead of Steps (R2)–(R5), Bob simply performs \( M_{\text{open}}(\pi,\sigma) \) measurement operator acting on \( \sum_{a, \pi, \sigma} |\phi_{\pi,a}(\sigma)\rangle |\phi_{\pi,a}(\sigma)\rangle \). If there exists a negligible function \( \varepsilon(n) \) such that, for any cheating strategy \( U = (U_1, U_0(0), U_1(1)) \) of Alice, \( T_0^{(H)}(n) + T_1^{(H)}(n) \leq 1 + \varepsilon(n) \) holds for every length \( n \in \mathbb{N}^+ \).

We will show that our protocol achieves the statistical binding condition.

**Theorem 4.4** If no polynomial-time quantum algorithm solves GA with non-negligible probability, then our quantum bit commitment scheme is statistically binding.

To prove this theorem, we first introduce a new problem, called the hidden permutation search problem, which is closely related to the indistinguishability between \( \rho_0^{(\pi)} \) and \( \rho_1^{(\pi)} \).

**Hidden Permutation Search Problem HPSP:**

- Instance: a string \( 1^n \) with \( n \in \mathbb{N}^+ \) and a quantum state \( \rho_0^{(\pi)}(n) \) with a certain hidden permutation \( \pi \in K_n \).
- Output: \( \pi \).

We say that a quantum algorithm \( A \) solves HPSP with average probability \( p \) on length \( n \), if, over all permutations \( \pi \in K_n \) chosen uniformly at random, \( A \) takes instance \((1^n, \rho_0^{(\pi)}(n))\) and outputs \( \pi \) with probability exactly \( p \).

**Lemma 4.5** Assume that there exist a cheating strategy \( U \) of Alice and a positive polynomial \( p \) satisfying \( T_0^{(H)}(n) + T_1^{(H)}(n) \geq 1 + 1/p(n) \) for infinitely-many lengths \( n \). Then, there exist a positive polynomial \( q \) and a polynomial-time quantum algorithm that solves HPSP with average probability at least \( 1/q(n) \) for infinitely-many lengths \( n \).

Because the proof of Lemma 4.5 requires a special treatment, we will give it in the next section.
Lemma 4.6 If there are a positive polynomial \( p \) and a polynomial-time quantum algorithm that solves HPSP with average probability at least \( 1/p(n) \) for infinitely-many \( n \), then there are a positive polynomial \( q \) and a polynomial-time quantum algorithm that solves 2-QSCD\( _\Pi \) with average advantage at least \( 1/q(n) \) for infinitely-many lengths \( n \).

Proof. Let \( p \) be a positive polynomial and let \( A \) be a polynomial-time quantum algorithm that solves HPSP with probability at least \( 1/p(n) \). Let \( \rho \otimes \rho \) with \( \rho \in \{ \rho_0^{(n)}, \rho_1^{(n)} \} \) be an instance of 2-QSCD\( _\Pi \), where \( \pi \) is an unknown permutation in \( K_n \). Let \( s \in \{ 0, 1 \} \) and assume that \( \rho = \rho_0^{(n)} \). Now, our task is to determine whether \( s = 0 \) or \( s = 1 \).

Let \( A_0 \) be a polynomial-time quantum algorithm that generates \( \pi \) from input \( (1^n, \rho_0^{(n)}) \) with probability, say, \( \gamma_n \), which is at least \( 1/p(n) \). Let \( A_1 \) be a quantum algorithm that takes input \( \rho_1^{(n)} \), transforms \( \rho_1^{(n)} \) to \( \rho_0^{(n)} \) by running \( P_2 \), and finally apply \( A_0 \). Obviously, \( A_1 \) outputs \( \pi \) with the same probability as \( A_0 \) with \( \rho_0^{(n)} \). Now, let us consider input \( \rho_1^{(n)} \otimes \rho_1^{(n)} \) with unknown values \( s \) and \( \pi \).

Using the first state \( \rho = \rho_s^{(n)} \), we obtain \( \pi \) as follows. Consider a purification \( |\Phi\rangle = |\Phi_s^{(n)}\rangle \) of \( \rho \). Starting with \( |0\rangle|\Phi\rangle \), apply \( H \otimes I \) to obtain \( \frac{1}{\sqrt{2}}(|0\rangle|\Phi\rangle + |1\rangle|\Phi\rangle) \). We apply \( A_0 \) and \( A_1 \) separately to generate \( \frac{1}{\sqrt{2}}(|0\rangle \otimes (I \otimes A_0)|\Phi\rangle + |1\rangle \otimes (I \otimes A_1)|\Phi\rangle) \). It follows that \( \sum_{\kappa \in K_n} |0\rangle|\kappa\rangle|\xi_{s,0}\rangle + \sum_{\kappa \in K_n} |1\rangle|\kappa\rangle|\xi_{s,1}\rangle \). By our assumption, we obtain \( |||\xi_{s,\pi}\rangle||^2 = \gamma_n \) for every \( s \in \{ 0, 1 \} \).

Next, we apply \( C_{SPA} \) to the second register and the second input state \( \rho_s^{(n)} \); that is, \( |\kappa\rangle|\kappa\rangle \otimes \rho_s^{(n)} \). If \( \kappa \) is different from \( \pi \), then after running \( C_{SPA} \), we observe 0 and 1 with equal probability, as we have argued in Section 4.2.

Therefore, the probability that we correctly obtain \( s \) is exactly \( \frac{1}{2}(1 - \gamma_n) + \gamma_n = \frac{1}{2} + \frac{\gamma_n}{2} \). Calling the entire quantum algorithm by \( B \), we have just proven that \( \text{Prob}[B(1^n, \rho_0^{(n)} \otimes \rho_0^{(n)}) = s] = \frac{1}{2} + \frac{\gamma_n}{2} \). From this equation, it follows that

\[
\left| \text{Prob}[B(1^n, \rho_0^{(n)} \otimes \rho_0^{(n)}) = 1] - \text{Prob}[B(1^n, \rho_1^{(n)} \otimes \rho_1^{(n)}) = 1] \right| = \left| \frac{1}{2} - \frac{\gamma_n}{2} - \frac{1}{2} + \frac{\gamma_n}{2} \right| = \gamma_n.
\]

Since \( \gamma_n \geq 1/p(n) \), the lemma follows. \( \square \)

Using the above lemmas, we can prove Theorem 4.3.

Proof of Theorem 4.3 We want to show the contrapositive of the theorem. First, assume that there exist a positive polynomial \( p \) and Alice’s cheating strategy \( \mathcal{U} = (U_1, U_2^{(0)}, I) \) such that \( T_0^{(n)}(U_1) + T_1^{(n)}(U_2^{(0)}) \geq 1 + 1/p(n) \) for infinitely-many lengths \( n \). By Lemmas 4.5 and 4.6, we conclude that 2-QSCD\( _\Pi \) can be solved by a certain polynomial-time quantum algorithm with average advantage at least \( 1/q(n) \) for a certain polynomial \( q \). By Lemma 4.3 GA must be solved on a quantum computer in polynomial time with average probability at least \( 1/r(n) \) for a certain polynomial \( r \). \( \square \)

5 Quantum Algorithm for HPSP

In the previous section, we have left Lemma 4.5 unproven. Here, we will give its missing proof by constructing an appropriate quantum algorithm that solves HPSP with non-negligible probability, provided that the statistical binding condition does not hold.

Recall that adversarial Alice takes \( (U_1, U_2^{(0)}, U_2^{(1)}) \) as her cheating strategy. To simplify our analysis, as in [8], we replace \( (U_1, U_2^{(0)}, U_2^{(1)}) \) by \( (\tilde{U}_1, \tilde{U}_2^{(0)}, I) \), where \( \tilde{U}_1 = (U_2^{(1)} \otimes I_{\text{commit}})U_1 \) and \( \tilde{U}_2^{(0)} = U_2^{(0)}(U_2^{(1)})^\dagger \), without changing the probability that Alice successfully cheats Bob. For convenience, hereafter, we write \( U_1 \) and \( U_2^{(0)} \) (without “tilde”) for \( \tilde{U}_1 \) and \( \tilde{U}_2^{(0)} \), respectively, and deal only with \( \mathcal{U} = (U_1, U_2^{(0)}, I) \) as adversarial Alice’s cheating strategy, where \( U_2^{(1)} = I \).

5.1 Distillation Algorithm

We will present an important subroutine that makes up of the quantum algorithm that solves HPSP in Section 5.2. Recall that adversarial Alice is now taking the cheating strategy \( \mathcal{U} = (U_1, U_2^{(0)}, I) \), while Bob faithfully follows the protocol.
Let us recall from Eq. (2) that $|\eta(C_1)|$ is of the form $\sum_{\pi \in K_n} \sum_{\pi, \sigma \in S_n} |\xi_{\pi, \sigma}^{(1,1)}| |\pi| |\sigma| |\phi_{\pi, \sigma}^{(1)}|$, which is obtained by an application of $U_1 \otimes I$ to $|0\rangle$ in the entire system $H_{all} = H_A \otimes H_B$. Note that $T^{(1)}_{I,1}(n) = \| (I \otimes M_1)|\eta(C_1)\|_2^2$ since $U^{(1)}_{I} = I$. For convenience, we define $|\eta_{\text{per}}(C_1)|$ to be the normalized state of $(I \otimes M_1)|\eta(C_1)$; that is, $|\eta_{\text{per}}(C_1)| = \sqrt{T^{(1)}_{I,1}(n)}(I \otimes M_1)|\eta(C_1)$. This is an ideal quantum state for adversarial Alice because, from this state, Alice unkeys 1 with certainty. By the definition of $M_1$, we can assume that $|\eta_{\text{per}}(C_1)|$ has the form $\sum_{\pi \in K_n} |\xi_{\pi, \pi}^{(1)}| |\pi| |\phi_{1}^{(1)}|$ with $\sum_{\pi \in K_n} |\xi_{\pi, \pi}^{(1)}|_2^2 = 1$.

Now, we will demonstrate how to implement $I \otimes M_1$ algorithmically and distill $|\eta_{\text{per}}(C_1)|$ from $|\eta(C_1)|$ using the measurement in the computational basis.

**Distillation Algorithm $A_{\text{dist}}$:**

(D1) Prepare an additional register in state $|0\rangle$ in $H_2$. Given $|0\rangle|\eta(C_1)|$ in $H_2 \otimes H_A^{(1)}$, we focus on a term $|\xi_{\pi, \pi}^{(1)}| |\pi| |\phi_{\pi, \pi}^{(1)}|$ in the subsystem $H_A^{(1)}$. We measure the second register in state $|1\rangle$. The state collapses to $|\xi_{\pi, \pi}^{(1)}| |\pi| |\phi_{\pi, \pi}^{(1)}|$.

(D2) Transform $|0\rangle|\pi| |\phi_{\pi, \pi}^{(1)}|$ in $H_2 \otimes H_{open} \otimes H_{\text{commit}}$ to $|\pi| |\phi_{\pi, \pi}^{(1)}|$ by applying $P_{SPA}$ given in Section 3.2. Measure the first register in state $|1\rangle$. We then obtain $|\xi_{\pi, \pi}^{(1)}| |\pi| |\phi_{\pi, \pi}^{(1)}|$.

(D3) Change $|\phi_{\pi, \pi}^{(1)}|$ to $|\phi_{\pi, 0}^{(1)}|$ by applying $P_2$. Prepare $|\phi_{\pi, 0}^{(1)}|$ and transform $|\pi| |\phi_{\pi, 0}^{(1)}|$ in $H_{open} \otimes H_{\pi} \otimes H_{\text{commit}}$ to $|\pi| |\pi| |\phi_{\pi, 0}^{(1)}|$ by applying $P_{\pi}^{-1}$. We then obtain $|\xi_{\pi, 1}^{(1)}| |\pi| |\phi_{\pi, 0}^{(1)}|$. Measure the fourth register in state $|\sigma\rangle$ to obtain $|\xi_{\pi, 1}^{(1)}| |\pi| |\phi_{\pi, 0}^{(1)}| |\phi_{\sigma, 0}^{(1)}|$.

(D4) Apply $P_1$ to $|\sigma| |\pi| |\phi_{\pi, 0}^{(1)}|$ in $H_{open} \otimes H_{\text{commit}}$ to obtain $|\pi| |\phi_{\sigma, 0}^{(1)}|$.

It is not difficult to see that the above algorithm $A_{\text{dist}}$ transforms $|\eta(C_1)|$ into a quantum state $|\eta_{\text{per}}(C_1)|$, with probability $T_{I,1}(n) = \| (I \otimes M_1)|\eta(C_1)|\|_2^2$.

In the following argument, we are focused on $|\eta_{\text{per}}(C_1)|$. Now, we fix a permutation $\pi' \in K_n$, which is a hidden permutation of an instance $\rho_{\pi'}^{(1)}$ of HPSP. Note that the Hilbert space span$\{|\phi_{\sigma, 0}^{(1)}| \mid \sigma \in S_n\}$ is determined by a basis $\mathcal{B}_{\pi'}^{(1)} = \{ |\phi_{\sigma, 0}^{(1)}| \mid \sigma \in S_n^{(\pi')}\}$, where $S_n^{(\pi')} = \{ \sigma \in S_n \mid \sigma \pi' = \pi' \sigma \}$. First, we want to measure $H_{\text{commit}}$ in states $|\phi_{\sigma, 0}^{(1)}>$ for an arbitrary permutation $\sigma \in S_n^{(\pi')}$. This is formally done by a measurement operator $M_{\pi'} = \sum_{\sigma \in S_n^{(\pi')}} M_{\text{commit}}^{(0, \pi', \sigma)}$, which projects a quantum state in $H_{\text{commit}}$ onto $|\phi_{\sigma, 0}^{(1)}>$'s. Letting $|\eta(\pi')\rangle = (I \otimes M_{\pi'})|\eta_{\text{per}}(C_1)|$, we want to determine an actual form of $|\eta(\pi')\rangle$. For brevity, we set $\omega_n = \frac{|K_n|!}{\sqrt{|S_n|(|K_n| - 1)!}}$.

**Lemma 5.1** For each fixed $\pi' \in K_n$, $|\eta(\pi')\rangle = \omega_n \sum_{\sigma \in S_n} \sum_{\pi \in K_n} |\xi_{\pi, \pi}^{(1)}| |\pi| |\phi_{\pi, \pi}^{(1)}| |\phi_{\sigma, 0}^{(1)}|.$

**Proof.** As the first step, we intend to express $|\eta_{\text{per}}(C_1)|$ in terms of $|\phi_{\sigma, 0}^{(1)}>$. Recall that $|\eta_{\text{per}}(C_1)| = \frac{1}{\sqrt{|S_n|}} \sum_{\sigma, \pi \in S_n} |\xi_{\pi, \pi}^{(1)}| |\pi| |\phi_{\pi, \pi}^{(1)}|$. For convenience, let $|\Theta_{\pi, \pi}\rangle = \frac{1}{\sqrt{|S_n|}} |\xi_{\pi, \pi}^{(1)}| |\pi| |\phi_{\pi, \pi}^{(1)}|$. Since $|\eta_{\text{per}}(C_1)| = \sum_{\pi \in K_n} |\Theta_{\pi, \pi}\rangle$ $|\pi| |\phi_{\pi, \pi}^{(1)}|$, $|\eta_{\text{per}}(C_1)|$ can be expressed as $\sum_{\pi, \sigma} |\Theta_{\pi, \sigma}\rangle |\phi_{\pi, \sigma}^{(1)}|$. Since $|\phi_{\pi, \sigma}^{(1)}| = \delta \sum_{\kappa \in K_n} |\phi_{\pi, \sigma}^{(k)}| |\phi_{\pi, \sigma}^{(k)}|$. By Lemma 3.2, $|\eta_{\text{per}}(C_1)|$ is written as $\delta \sum_{\kappa \in K_n} \sum_{\pi, \sigma} |\Theta_{\pi, \sigma}\rangle |\phi_{\pi, \sigma}^{(k)}|$. Since $\sum_{\pi, \sigma} |\Theta_{\pi, \sigma}\rangle |\phi_{\pi, \sigma}^{(k)}|$ equals $\sum_{\pi, \sigma} |\Theta_{\pi, \sigma}\rangle |\phi_{\pi, \sigma}^{(k)}|$, the state $|\eta_{\text{per}}(C_1)|$ is further written as

$$|\eta_{\text{per}}(C_1)| = \delta \sum_{\kappa \in K_n} \sum_{\pi, \sigma} |\Theta_{\pi, \sigma}\rangle |\phi_{\pi, \sigma}^{(k)}| = \frac{\delta}{|S_n|} \sum_{\kappa} \sum_{\pi, \sigma} |\xi_{\pi, \pi}^{(1)}| |\pi| |\phi_{\pi, \pi}^{(1)}| |\phi_{\sigma, 0}^{(k)}|,$$ (3)

since $|\Theta_{\pi, \pi}\rangle = \frac{1}{\sqrt{|S_n|}} |\xi_{\pi, \pi}^{(1)}| |\pi| |\phi_{\pi, \pi}^{(1)}|$. Obviously, if $\sigma \in S_n^{(\pi')}$, then $M_{\pi'}|\phi_{\sigma, 0}^{(1)}| = |\phi_{\sigma, 0}^{(1)}|$. If $\sigma \in S_n^{(\pi')}$, we obtain $M_{\pi'}|\phi_{\sigma, 0}^{(1)}| = |\phi_{\sigma, 0}^{(1)}|$. Let $\kappa \in K_n - \{\pi'\}$. For any $\sigma \in S_n^{(\pi')}$, since $|\phi_{\sigma, 0}^{(k)}| |\phi_{\sigma, 0}^{(k)}| = \frac{1}{|S_n|}$ by Lemma
In the subsequent subsection, we will explain how to solve HPSP efficiently on a quantum computer.

For convenience, we denote by $|\psi_{\sigma,\tau}\rangle = |\phi^0_{\sigma,0}\rangle$ when $\sigma, \tau$ are given only a reduced state $|\psi_{\sigma,\tau}\rangle$. Thus, it follows that $M_{\sigma,\tau} = |\phi^0_{\sigma,0}\rangle$. Overall, it holds that $M_{\sigma,\tau} = |\phi^0_{\sigma,0}\rangle$ for all permutations $\sigma \in S_n$.

Since $|\eta(\pi')\rangle = (I \otimes M_{\pi'})|\psi_{\text{perf}}\rangle$, from Eq. (3), $|\eta(\pi')\rangle$ is expressed as

$$|\eta(\pi')\rangle = \frac{\delta^2}{\sqrt{|S_n|}} \left[ \frac{\sum_{\pi} |\xi_{\pi}|1|\pi|\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle + \frac{1}{2} \sum_{\pi} |\xi_{\pi}|1|\pi|\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle} \right]$$

$$= \frac{\delta^2}{\sqrt{|S_n|}} \left[ \sum_{\pi} |\xi_{\pi}|1|\pi|\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle + \frac{1}{2} \sum_{\pi} |\xi_{\pi}|1|\pi|\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle \right]$$

$$= \frac{\delta^2}{\sqrt{|S_n|}} \left[ |K_n| \sum_{\pi} |\xi_{\pi}|1|\pi|\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle \right]$$

Therefore, we obtain $|\eta(\pi')\rangle = \frac{|K_n|\delta^2}{\sqrt{|S_n|(|K_n| - 1)}} \sum_{\pi} |\xi_{\pi}|1|\pi|\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle$ by the definition of $\delta$.

For convenience, we denote by $|\eta_{\text{norm}}(\pi')\rangle$ the normalized state of $|\eta(\pi')\rangle$, i.e., $|\eta_{\text{norm}}(\pi')\rangle = \frac{1}{||\eta(\pi')||} |\eta(\pi')\rangle$.

**Lemma 5.2** $|\eta_{\text{norm}}(\pi')\rangle = \omega_n^2 \sum_{\sigma \in S_n} \sum_{\pi \in K_n} |\xi_{\pi}|1|\pi|\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle$, where $\omega_n = \frac{1}{\sqrt{|S_n||K_n|}}$.

**Proof.** We want to estimate the value $||\eta(\pi')||$. First, we claim that $||\eta(\pi')||^2 = \delta^2 |K_n|(|K_n| + 1)^2$. If this claim is true, then $|\eta_{\text{norm}}(\pi')\rangle$ is written as

$$|\eta_{\text{norm}}(\pi')\rangle = \left[ \sum_{\pi \in K_n} \sum_{\sigma \in S_n} |\eta_{\text{norm}}(\pi')\rangle \right] = \frac{|K_n| + 1}{\sqrt{|S_n||K_n|}} \left[ \frac{1}{|S_n|} \sum_{\pi \in K_n} |\xi_{\pi}|1|\pi|\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle \right]$$

The aforementioned claim (***) will be proven as follows. First, we note that $||\eta(\pi')||^2$ equals $\omega_n^2 \sum_{\sigma,\tau} \sum_{\pi} ||\xi_{\pi,\pi'}||^2 (|\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle)\langle\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle$, which is $\omega_n^2 \sum_{\pi} ||\xi_{\pi,\pi'}||^2 |\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle\langle\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle$. By Lemma 3.1, $|\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle = 1$ if $\tau = \sigma; -1$ if $\tau = \sigma; 0$ otherwise. Moreover, $|\phi_{\pi,0}'\rangle\langle\phi_{\pi,1}'\rangle = 1$ if $\tau = \sigma$ or $\tau = \sigma'; 0$ otherwise. Thus, it follows that $||\eta_{\text{norm}}(\pi')||^2 = \omega_n^2 \sum_{\pi \in K_n} ||\xi_{\pi,\pi'}||^2 |\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle\langle\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle = \omega_n^2 \sum_{\pi \in K_n} ||\xi_{\pi,\pi'}||^2 \cdot |S_n|$. Since $\sum_{\pi \in K_n} ||\xi_{\pi,\pi'}||^2 = 1 - |||\xi_{\pi,\pi'}||^2\rangle$, we obtain $||\eta_{\text{norm}}(\pi')||^2 = \omega_n^2 |S_n| |1 - |||\xi_{\pi,\pi'}||^2\rangle|^2\rangle$. By the definition of $\omega_n$, the lemma follows.

In the subsequent subsection, we will explain how to solve HPSP efficiently on a quantum computer.

### 5.2 HPSP Algorithm

To solve HPSP, we first generate a quantum state $|\eta_{\text{norm}}(\pi')\rangle$ with an appropriate probability, apply $U_2^{(0)}$, and finally measure selected qubits. The following quantum algorithm $\mathcal{A}_{HPSP}$ behaves exactly as described. In what follows, we tend to drop superscript ‘U’ from $T_{\mathcal{A}}^{(k)}$ for brevity.

**HPSP Algorithm** $\mathcal{A}_{HPSP}$:

1. Assume that we are given a quantum state $\rho = \rho_{\pi'}^{(\pi')}$ with an unknown permutation $\pi' \in K_n$. We consider its purification of the form $\Phi_0^{(\pi')} = \frac{1}{\sqrt{|S_n|}} \sum_{\pi \in S_n} |\pi\rangle\langle\pi'| \phi_{\pi,0}' \rangle$. Since we are given only a reduced state $\rho$, we assume that we are allowed to manipulate only the first register of $|\Phi_0^{(\pi')\rangle}$. Starting with $|0\rangle$, we apply $U_1 \otimes I$ and then run $\mathcal{A}_{\text{div}}$. We then obtain $\sqrt{T_1(n)}|\eta^{(C1)}\rangle$; that is, $
abla = \sum_{\sigma,\tau} \sum_{\pi \in K_n} |\xi_{\pi}|1|\pi|\sigma|\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle$.

2. Transform $|\pi\rangle|\sigma|\phi_{\pi,1}'\rangle$ into $|\pi\rangle|\sigma|\Pi|d\rangle$ by running $P_2$ and $P_1^{-1}$. Now, we obtain

$$\frac{\sqrt{T_1(n)}}{|S_n|} \sum_{\sigma,\tau} |\xi_{\pi}|1|\pi|\sigma|\phi_{\pi,1}'\rangle|\phi_{\pi,0}'\rangle \otimes \sum_{\tau} |\tau\rangle|\phi_{\pi,0}'\rangle.$$
(M3) Swap two registers $|\sigma\rangle|\text{id}\rangle$ and $|\tau\rangle|\phi_{\text{CM}}(\pi')\rangle$ to obtain $\sqrt{T_1(n)} \sum_{\tau,\pi} |\xi_{\tau,\pi}\rangle |\pi\rangle |\phi_{\text{CM}}(\pi')\rangle \otimes \frac{1}{\sqrt{|S_n|}} \sum_{\sigma} |\sigma\rangle |\text{id}\rangle$.

(M4) Transform $|\pi\rangle|\tau\rangle$ into $|\pi\rangle |\phi_{\text{CM}}(\pi')\rangle$ by applying $P_1$. Moreover, transform $\frac{1}{\sqrt{|S_n|}} \sum_{\sigma} |\sigma\rangle |\text{id}\rangle$ into $|\text{id}\rangle|\text{id}\rangle$.

The current state is now of the from $\sqrt{T_1(n)} |1\rangle |\pi\rangle |\phi_{\text{CM}}(\pi')\rangle \otimes \frac{1}{\sqrt{|S_n|}} \sum_{\sigma} |\sigma\rangle |\text{id}\rangle$, which equals $\sqrt{T_1(n)} |1\rangle |\pi\rangle |\phi_{\text{CM}}(\pi')\rangle |\phi_{\text{CM}}(\pi')\rangle \otimes |\text{id}\rangle|\text{id}\rangle$ by Lemma 5.2.

(M5) Apply $U_2(0) \otimes I$ to the subsystem $H_A^{(R_1)} \otimes H_B^{(R_1)}$.

(M6) Measure $H_{\text{bit}} \otimes H_{\text{open}}$. Whenever we observe $(a, \pi)$, if $a \neq 0$, then reject. Otherwise, output $\pi$.

To complete the proof of Lemma 5.5, it suffices to show that the success probability $p_{\pi'}$ of obtaining $\pi'$ from $\rho_0(\pi)$ by running $\mathcal{A}_{\text{HPSP}}$, over all $\pi \in K_n$, chosen uniformly at random, is at least $\frac{1}{3p(n)^2}$. This statement follows from two separate claims. The first claim below makes a bridge between the probability $p_{\pi'}$ and the state $(I \otimes M_0)(U_2(0) \otimes I)(I \otimes M_1)|\eta(\text{C}1)\rangle$.

**Claim 1** For any fixed $\pi' \in K_n$, the success probability $p_{\pi'}$ of obtaining $\pi'$ from $\rho_0(\pi)$ by running $\mathcal{A}_{\text{HPSP}}$ is at least $2(1 - \frac{1}{|K_n|})^2 \| (I \otimes M_0)(U_2(0) \otimes I)(I \otimes M_1)|\eta(\text{C}1)\rangle \|^2$.

**Proof.** Recall that $\tilde{M}_{\pi'} = \sum_{\delta \in \mathcal{C}_{\pi'}} \rho_{\text{CM}}(a\pi')$ and $M_0 \equiv \sum_{\pi \in K_n} M_{\text{bit}}(a) \otimes M_{\text{open}}(a,\pi) \otimes M_{\text{mix}}(\pi)$ for each index $a \in \{0, 1\}$. Since two operators $I \otimes \tilde{M}_{\pi'}$ and $U_2(0) \otimes I$ are commutative, it follows that

$$(I \otimes \tilde{M}_{\pi'})(U_2(0) \otimes I)|\eta(\text{C}1)\rangle = (U_2(0) \otimes I)(I \otimes \tilde{M}_{\pi'})|\eta(\text{C}1)\rangle = (U_2(0) \otimes I)|\eta(\pi')\rangle.$$  

Since $I \otimes M_0 = (I \otimes M_0)(I \otimes \tilde{M}_{\pi'})$, we obtain

$$(I \otimes M_0)(U_2(0) \otimes I)|\eta(\text{C}1)\rangle = (I \otimes M_0)|\eta(\text{C}1)\rangle = (I \otimes M_0)(U_2(0) \otimes I)|\eta(\pi')\rangle.$$  

Through Steps (M1)–(M4), we generate $\sqrt{T_1(n)} |1\rangle |\pi\rangle |\phi_{\text{CM}}(\pi')\rangle \otimes \frac{1}{\sqrt{|S_n|}} \sum_{\sigma} |\sigma\rangle |\text{id}\rangle$. From Steps (M5)–(M6), the success probability $p_{\pi'}$ for a fixed $\pi'$ is exactly $\| \sqrt{T_1(n)} |1\rangle |\pi\rangle |\phi_{\text{CM}}(\pi')\rangle \otimes \frac{1}{\sqrt{|S_n|}} \sum_{\sigma} |\sigma\rangle |\text{id}\rangle \| \|^2$. By the proof of Lemma 5.2 it holds that $|\eta(\text{C}1)\rangle = \frac{\sqrt{T_1(n)|\phi(\pi')\rangle \langle \phi(\pi')|}}{\sqrt{\| |\phi(\pi')\rangle \|^2}}$. Thus, $p_{\pi'}$ equals $\| \sqrt{T_1(n)} |1\rangle |\pi\rangle |\phi_{\text{CM}}(\pi')\rangle \otimes \frac{1}{\sqrt{|S_n|}} \sum_{\sigma} |\sigma\rangle |\text{id}\rangle \| \|^2$. By combining Claims 1 and 2, we obtain the desired consequence that the success probability of obtaining $\pi'$ from $\rho_0(\pi)$ by running $\mathcal{A}_{\text{HPSP}}$ is at least $2(1 - \frac{1}{|K_n|})^2 \cdot \frac{1}{3p(n)^2} \geq 2 \cdot \frac{1}{3p(n)^2} \geq \frac{2}{3p(n)^2}$ for any number $n \geq 3$.

### 6 A Brief Discussion on Our Protocol

Through Sections 3–5, we have shown that our quantum bit commitment scheme achieves computational concealing and statistical binding at communication cost of $O(n \log n)$, where $n$ is a security parameter, since Alice sends the information on $(a, \sigma, \pi, |\phi_{\text{CM}}(\pi')\rangle)$ to Bob during the two phases and the permutations $\sigma$ and $\pi$ are expressed using $O(n \log n)$ bits. Although our protocol requires a weaker assumption than that of [8], 13
the communication cost is larger. For a practical application, it is better to reduce the communication cost as in the case of, for example, [14].

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