A New Approach to the Yukawa Puzzle

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Abstract

We do a systematic analysis of the question of calculability of CKM matrix elements in terms of quark mass ratios, within the framework of the hypothesis of universality of strength for Yukawa couplings (USY), where all Yukawa couplings have equal moduli, and the flavor dependence is only in their phases. We use the fact that the limit $m_u = m_d = 0$ is specially simple in USY, to construct the various ansätze. It is shown that the experimentally observed CKM matrix can be obtained within USY ansätze corresponding to simple relations among phases of Yukawa couplings. Within USY, one finds a natural explanation why Cabibbo mixing is significantly larger than the other CKM mixings. In the most successful of the USY ansätze, one obtains in leading order: $|V_{us}| = \sqrt{m_d/m_s}$; $|V_{cb}| = \sqrt{2}(m_s/m_b)$; $|V_{ub}| = (1/\sqrt{2})\sqrt{m_d m_s/m_b^2}$; $|V_{td}| = 3 |V_{ub}|$. We study the behavior of this USY ansatz under the renormalization group.
1. Introduction

In the standard model (SM), the flavor structure of Yukawa interactions is not constrained by any symmetry, thus leading to arbitrary Yukawa couplings consisting of three 3x3 complex matrices. This arbitrariness is used to fit the lepton and quark masses, as well as the four physical parameters of the Cabibbo-Kobayashi-Maskawa (CKM) matrix.

Finding a deeper insight into the pattern of fermion masses and mixings, is one of the major outstanding problems in particle physics. In the past, there have been various attempts at relating the pattern of CKM mixings to the quark mass ratios \[1\]. In most schemes, one assumes that some of the Yukawa matrix elements vanish, which leads to testable relations between quark masses and the elements of the CKM matrix. Recently, a classification has been done \[2\] of all Hermitian matrices with "textures zeros", which conform to our present knowledge on quark masses and CKM matrix elements.

In this paper, we will analyse in a systematic way the question of calculability of the CKM matrix elements within the framework \[3\] of universality of strength for Yukawa couplings (USY). In USY all the Yukawa couplings of the quarks have equal moduli, but differ in their phases. A physical motivation for the USY hypothesis may be found in the following observations:

(i) The Yukawa interactions are the only couplings of the SM which can be complex. All other couplings are constrained to be real by hermiticity.

(ii) Most of the arbitrary parameters of the SM arise precisely from the Yukawa couplings.

If one assumes that the above two features are somehow related, one is naturally led to the USY hypothesis, i.e. to the idea that the arbitrariness of Yukawa couplings results from the fact that they can be complex, with flavor-dependent phases, but universal strength.

In our study of the problem of calculability of the CKM matrix elements within the USY framework we will start by considering the limit where the first generation of quarks is massless. It has been previously pointed out \[4\] that within USY all solutions leading to \(m_u = m_d = 0\), can be classified. In this paper, we construct various new ansätze based on USY by considering the different ways of departing from this limit which lead to calculability of CKM matrix elements in terms of quark mass ratios. The ansätze correspond to simple relations among the USY phases. In the search for these relations among phases, we will use as guiding principles, on the one hand simplicity and on the other hand some of the main features of the experimentally observed pattern of CKM mixings, namely the fact that \(|V_{us}|\) is of order \((m_d/m_s)^{1/2}\), while \(|V_{cb}|\) is of order \((m_s/m_b)\).

We offer a generic argument how these relations can be obtained and show that they naturally arise within the USY framework.

The paper is organized as follows: in the next section, we will set our notation and characterize the parameter space of the Yukawa couplings within USY. In section 3, we will address in a systematic way the question of calculability of CKM matrix within the USY framework. In section 4 we confront the various ansätze with the experimental value of quark masses and mixings. A study of the behavior under the renormalization
group of a particularly successful USY ansatz is presented in section 5. In section 6, we compare the analysis of the present paper with the texture zeros approach to the Yukawa puzzle. Our conclusions are presented in section 7. In the appendix, we show some special features of USY. In particular, we point out that within USY, a successful prediction for $|V_{us}|$ requires three generations. Indeed we show that for two generations, USY leads either to an arbitrary non-calculable Cabibbo angle or to an unrealistic relation $|\theta_C| = \frac{m_d}{m_s} \pm \frac{m_u}{m_c}$.

2. USY parameter space

For completeness and in order to settle our notation, we describe the parameter space of Yukawa couplings in the USY framework. We assume that there are two Higgs doublets $\Phi_u, \Phi_d$ which give mass to the up and down quarks respectively, through Yukawa couplings of universal strength. All the flavor dependence is contained in the phases of Yukawa couplings and therefore the up and down quark masses have the form:

$$[M_u]_{ij} = c_u \exp[i\phi^u_{ij}] \quad [M_d]_{ij} = c_d \exp[i\phi^d_{ij}]$$

(2.1)

with $c_u = \lambda v_u$, $c_d = \lambda v_d$, where $\lambda$ denotes the universal strength of Yukawa couplings and $v_u = \langle \Phi_u \rangle$, $v_d = \langle \Phi_d \rangle$. One can eliminate some of the phases appearing in Eq.(2.1) by making weak-basis transformations of the type:

$$M_u \rightarrow M'_u = K^u_L \cdot M_u \cdot K^u_R$$

$$M_d \rightarrow M'_d = K^d_L \cdot M_d \cdot K^d_R$$

(2.2)

where $K_L, K^u_{R,d}$ are diagonal unitary matrices, i.e. of the form $\text{diag}(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3})$. Obviously the transformations of Eq.(2.2) keep the USY form of Yukawa couplings. In general it is not possible to obtain $M'_u, M'_d$ Hermitian, while maintaining the USY form. Actually it can be readily verified that the Hermitian USY matrices lead to unrealistic mass matrices. However, it can be easily seen that, by making the weak-basis transformations of Eq.(2.2), one can transform any USY matrices $M_u, M_d$ given by Eq.(2.1), into the form:

$$M_u = c_u \begin{pmatrix} e^{ip_u} & e^{ir_u} & 1 \\ e^{iq_u} & 1 & e^{it_u} \\ 1 & 1 & 1 \end{pmatrix} \quad M_d = c_d K^\dagger \begin{pmatrix} e^{ip_d} & e^{ir_d} & 1 \\ e^{iq_d} & 1 & e^{it_d} \\ 1 & 1 & 1 \end{pmatrix} \cdot K$$

(2.3)

where $K = \text{diag}(1, e^{i\alpha_1}, e^{i\alpha_2})$. It is clear from Eq.(2.3) that the phases $\alpha_i$ will affect the CKM matrix but not the quark mass spectrum which depends only on $p_u, q_u; r_u; t_u$. It is useful to introduce dimensionless Hermitian matrices defined by

$$H_u = \frac{1}{3c_u^2} M_u M_u^\dagger \quad H_d = \frac{1}{3c_d^2} M_d M_d^\dagger$$

(2.4)

and the related pure phase mass matrices
\[ M_{\text{phase}}^u = \begin{pmatrix} e^{ip_u} & e^{ir_u} & 1 \\ e^{iq_u} & 1 & e^{it_u} \\ 1 & 1 & 1 \end{pmatrix} \quad M_{\text{phase}}^d = \begin{pmatrix} e^{ip_d} & e^{ir_d} & 1 \\ e^{iq_d} & 1 & e^{it_d} \\ 1 & 1 & 1 \end{pmatrix} \] (2.5)

In these pure phase matrices, the constants \( c_{u,d} \) and the diagonal unitary matrix \( K \) are not included in order to simplify the presentation. The eigenvalues \( \lambda_{u,d} \) of \( H_{u,d} \) are also dimensionless, and related to the squared quark masses \( m_i^2 \) by
\[ \lambda_i = \frac{3m_i^2}{m_1^2 + m_2^2 + m_3^2}. \]

The coefficients of the characteristic equations are given by the trace \( tr(H) \), the second invariant \( \chi(H) \), and the determinant \( \delta(H) \) of the matrices \( H \)
\[
tr H \equiv \lambda_1 + \lambda_2 + \lambda_3 = 3
\]
\[
\chi(H) \equiv \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = \frac{1}{8}[\sin^2(\frac{p}{2}) + \sin^2(\frac{q}{2}) + \sin^2(\frac{r}{2}) + \sin^2(\frac{t}{2}) + \sin^2(\frac{p+q}{2}) + \sin^2(\frac{p+r}{2}) + \sin^2(\frac{p+t}{2}) + \sin^2(\frac{q+r}{2})]
\] (2.6)
\[
\delta(H) \equiv \lambda_1 \lambda_2 \lambda_3 = \left| det(M_{\text{phase}}) \right|^2 = \frac{1}{27} \left| e^{ip} + e^{iq} + e^{i(r+t)} - e^{i(p+t)} - e^{i(q+r)} - 1 \right|^2 = \frac{1}{27} |A + B|^2
\]
where
\[
A = -(1 - e^{ip})(1 - e^{it}) \quad B = e^{it}(e^{i(q-t)} - 1)(1 - e^{ir}) \] (2.7)

The quark mass hierarchy leads to the constraint \( \chi(H) \ll 1 \) and since \( \chi(H) \) is the sum of positive definite quantities, the modulus of each one of the phases \( p, q, r \) and \( t \) has to be small, at most of order \( \frac{m_2}{m_3} \).

3. **Calculability of \( V_{CKM} \)**

It has been shown \[3\], \[3\] that within the USY hypothesis, one can correctly fit the observed pattern of CKM matrix elements, as well as the value of quark masses. However, without any further assumptions, the USY hypothesis has the disadvantage of containing too many free parameters. In this section, we will make a systematic study of ansätze based on USY, leading to calculability of the CKM matrix elements, in terms of quark masses. In order to achieve this, each one of the quark mass matrices \( M_u, M_d \) should depend only on the over-all constants \( c_u, c_d \) and two phases. One will then have a total of six parameters in \( M_u, M_d \) which will be fixed by the value of the six quark masses. As a result the CKM matrix will be a function of quark mass ratios with no free parameters.
In our search for these ansätze, we will start by considering the limit $m_u = m_d = 0$. In the USY framework, this limit is specially interesting since it is possible to find exactly all solutions \[4\] leading to $\det(M_{u,d}) = 0$. We divide these solutions into two classes:

\[
\begin{align*}
\text{Class-I} & \quad \begin{cases} 
  a) & \ p = 0, \ t = q \\
  b) & \ t = 0, \ r = 0 \\
  c) & \ r = p, \ q = 0
\end{cases} \\
\text{Class-II} & \quad \begin{cases} 
  a) & \ p = 0, \ r = 0 \\
  b) & \ q = 0, \ t = 0 \\
  c) & \ r = p - q, \ t = -r
\end{cases}
\end{align*}
\] (3.1)

where in each solution, the omitted parameters are arbitrary, only constrained by $\chi(H)$, given in Eq.(2.6).

The solutions of Eqs.(3.1) can be readily understood from Eqs.(2.6), where the determinant of the pure phase matrix, $\det(M_{\text{phase}})$, is written as the sum of two complex numbers $A, B$. Solutions Ic) and IIc) correspond to having $A = -B$, while the other solutions correspond to $A = B = 0$. In order to see the distinct physical implications of these two classes of solutions, it is useful to consider the limit $K = 1$ (i.e. $\alpha_1 = \alpha_2 = 0$), where $K$ has been defined in Eq.(2.3). We recall that the two phases $\alpha_1, \alpha_2$ do not affect the quark mass spectrum, only entering in the CKM matrix. For $K = 1$, and in the limit $m_u = m_d = 0$, solutions of Class-II cannot generate a realistic CKM matrix, while solutions of Class-I can generate a realistic CKM matrix even in the limit of massless $m_u, m_d$.

In the search for a viable ansatz based on USY, we will consider that the masses for the first generation are generated through a small deviation of the limit $\det(M_u) = \det(M_d) = 0$. We will do all calculations exactly, without using perturbation theory. The fact that in USY this limit is characterized by two conditions on the phases, suggests that the generation of mass for the first generation can be obtained through the relaxation of one of these conditions. Following this suggestion, one finds the following cases\[\footnote{We have not included the case \{r = p - q\}, since it leads to unrealistic predictions for $V_{CKM}$.}:

\[
1) \ \{p = 0\}; \quad 2) \ \{t = 0\}; \quad 3) \ \{t = q\}; \quad 4) \ \{t = -r\}
\] (3.2)

One can find another set of physically equivalent cases by making the interchange $(p, r) \leftrightarrow (q, t)$. For definiteness, we will consider next the case $\{p = 0\}$. From Eq.(2.6), one obtains for the determinant of the pure phase matrix $M_{\text{phase}}$:

\[
\det(M_{\text{phase}}) = e^{it}(e^{i(q-t)} - 1)(1 - e^{ir})
\] (3.3)

It follows then that:

\[
|\det(M_{\text{phase}})| = 4 \left| \sin\left(\frac{q - t}{2}\right) \sin\left(\frac{r}{2}\right) \right|
\] (3.4)

At this stage, taking $K = 1$, we have in each of the original full mass matrices $M_u, M_d$ four parameters, namely $(c_u, q_u, r_u, t_u)$ and $(c_d, q_d, r_d, t_d)$. In order to achieve full calculability of the CKM matrix (i.e. having $V_{CKM}$ entirely expressed in terms of quark masses ratios, with no free parameters), each one of the matrices $M_u, M_d$ should contain
only three parameters. Therefore we need an extra relation, among the phases \( q, r, t \). In looking for such a relation, ”simplicity” will be our guiding principle. In particular we will search for relations among \( q, r \) and \( t \) such that \( \det(M^{\text{phase}}) \) depends only on one phase-parameter. Later, we will present a heuristic argument in favor of this scenario. Following this suggestion and taking into account Eq.\((3.4)\), we set

\[
|q - t| = |r|
\]  

thus obtaining:

\[
|\det(M^{\text{phase}})| = 4 \sin^2\left(\frac{r}{2}\right)
\]  

or equivalently

\[
\sin^2\left(\frac{r}{2}\right) = \frac{3}{4}\sqrt{3} \delta
\]

where we used the relation \(|\det(M^{\text{phase}})| = 3\sqrt{3} \delta\) from Eq.\((2.6)\). If we now insert the relations \( p = 0 \), \( (q - t) = r \) in the expression for \( \chi \) given by Eq.\((2.6)\), we get:

\[
\chi(H) = \frac{4}{9} \left[ \sin^2\left(\frac{q - r}{2}\right) + \sin^2\left(\frac{q + r}{2}\right) + 2 \sin^2\left(\frac{q}{2}\right) + 4 \sin^2\left(\frac{r}{2}\right) \right]
\]

Using the identity \( \sin^2\left(\frac{q - r}{2}\right) + \sin^2\left(\frac{q + r}{2}\right) = 2 \sin^2\left(\frac{q}{2}\right) + 2 \sin^2\left(\frac{r}{2}\right) - 4 \sin^2\left(\frac{q}{2}\right) \sin^2\left(\frac{r}{2}\right) \) we can write:

\[
\chi(H) = \frac{8}{9} \left[ 2 \sin^2\left(\frac{q}{2}\right) + 3 \sin^2\left(\frac{r}{2}\right) - 2 \sin^2\left(\frac{q}{2}\right) \sin^2\left(\frac{r}{2}\right) \right]
\]

Using Eqs.\((3.7)\) and \((3.9)\), one finally obtains:

\[
\sin^2\left(\frac{q}{2}\right) = \frac{9}{16} \chi - \frac{9}{8} \sqrt{3} \delta
\]

It is worth summarizing what we have accomplished so far. By studying the limit \( \det(M_{u,d}) = 0 \), we motivated an ansatz where \( p = 0 \), \( t = q - r \), which led to the following results:

(i) Each one of the mass matrices \( M_u, M_d \) depends on three parameters \( \{c_{u,d}, q_{u,d}, r_{u,d}\} \).

(ii) The parameters \( c_{u,d} \) are overall constants which are fixed by the sum of quark squared masses through the relation \( c^2 = \frac{1}{9}(m_1^2 + m_2^2 + m_3^2) \), while \( |r| \) and \( |q| \) are fixed by Eqs.\((3.7)\) and \((3.10)\), respectively.

So far, we have been only concerned with the quark mass spectrum. Next, we turn to the CKM matrix and present the previously mentioned heuristic argument justifying why the situation described above is potentially useful to obtain a calculable and physically interesting CKM matrix. The argument goes as follows: let us consider an ansatz, not necessarily based on USY, leading to mass matrices which depend only on two parameters \( \rho_1, \rho_2 \), such that, at least to leading order, \( |\rho_1|^2 \propto |\det(M^0)| \) while \( |\rho_2|^2 \propto \chi(H^0) \), where
$M^o$ and $H^o = M^o M^{o\dagger}$ are the dimensionless matrices defined by $M^o = M/\sqrt{tr(H)}$. Then, if the mixing $|V_{12}|$ between the first and second generations is proportional to the ratio $|\rho_1/\rho_2|$ one would have:

$$|V_{12}| \propto \frac{|\rho_1|}{|\rho_2|} \propto \sqrt{\frac{|det(M^o)|}{\chi(H^o)}} \quad (3.11)$$

Taking into account that the definition of $M^o$ implies that $|det(M^o)| \propto \frac{m_1 m_2 m_3}{(m_1^2 + m_2^2 + m_3^2)^{3/2}}$ while $\chi(H^o) \propto \frac{m_1^4 + m_2^4 + m_3^4}{(m_1^2 + m_2^2 + m_3^2)^2}$ one obtains in leading order:

$$|V_{12}| \propto \sqrt{\frac{m_1}{m_2}} \quad (3.12)$$

The above qualitative argument will be used as a guideline to do a systematic search for ansätze within the USY hypothesis which have the feature of leading to $|V_{12}| \propto \sqrt{\frac{m_1}{m_2}}$ while $|V_{23}| \propto \frac{m_2}{m_3}$. Next, we analyse in some detail two examples of ansätze constructed following the guidelines described above.

**Ansatz I:** $\{p = 0, \ t = q - r\}$

This is the ansatz we have just constructed, leading to mass matrices of the form:

$$M_{u,d} = c_{u,d} \begin{pmatrix}
1 & e^{ir} & 1 \\
e^{iq} & 1 & e^{(q-r)} \\
1 & 1 & 1
\end{pmatrix} \quad (3.13)$$

In order to diagonalize the Hermitian matrices $H_u, H_d$ of Eq.(2.4) and to obtain the CKM matrix, it is useful to make first a change of weak basis:

$$H_u \rightarrow H'_u = F_\dagger \cdot H_u \cdot F \quad ; \quad H_d \rightarrow H'_d = F_\dagger \cdot H_d \cdot F \quad (3.14)$$

where $F$ is given by:

$$F = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix} \quad (3.15)$$

Through this weak-basis transformation we change from a ”democratic” to a ”heavy” basis. The matrices $H'_u, H'_d$ are diagonalized by unitary transformations:

$$U'_u \cdot H'_u \cdot U_u = D_u \quad ; \quad U'_d \cdot H'_d \cdot U_d = D_d \quad (3.16)$$

where $D = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$. The CKM matrix is then given by: $V_{CKM} = U'_u U'_d$. Since for this ansatz $\sin^2(q/2)$ and $\sin^2(r/2)$ can be expressed in terms of quark mass ratios as in Eqs.(3.7) and (3.10), an exact analytical solution to the eigenvalue equation:
\[(H' - \lambda_i \mathbb{I}) \vec{v}_i = 0\] (3.17)
can be found. One obtains in leading order:

\[|U_{d12}| = \sqrt{\frac{m_1}{m_s}} \quad |U_{d23}| = \sqrt{2} \frac{m_s}{m_b} \quad |U_{d13}| = \frac{|U_{d12}|}{|U_{d23}|/2} = 1 \sqrt{\frac{2}{2}} \frac{m_s}{m_2} \quad |U_{d31}| = 3 |U_{d13}| \] (3.18)

Of course, entirely analogous expressions hold for \(U_{ij}^u\). The values of \(U_{ij}^u, U_{ij}^d\) given by Eq.(3.18) can be obtained in a much simpler way, by noting that from Eq.(3.16) one gets:

\[H'_{23} = U_{21} U_{31}^* \lambda_1 + U_{22} U_{32}^* \lambda_2 + U_{23} U_{33}^* \lambda_3 \approx U_{23} U_{33}^* \lambda_3 \] (3.19)

On the other hand, \(H'_{23}\) is readily evaluated from Eqs.(3.13) and (3.14), and one obtains in leading order:

\[|H'_{23}| = 2 \frac{\sqrt{2}}{3} |q| \] (3.20)

Taking into account that \(\lambda_3 = 3 m_3^2/(m_1^2 + m_2^2 + m_3^2) \approx 3\) and \(|U_{33}| \approx 1\) it follows from Eqs.(3.15) and (3.20) that in leading order:

\[|U_{23}| = \frac{1}{3} |H'_{23}| = \frac{2 \sqrt{2}}{9} |q| = \sqrt{2} \frac{m_2}{m_3} \] (3.21)

where we used \(|q| \approx (9/2)(m_2/m_3)\) from Eq.(3.10). Similarly, from \(H'_{13}\), written as:

\[H'_{13} = U_{11} U_{31}^* \lambda_1 + U_{12} U_{32}^* \lambda_2 + U_{13} U_{33}^* \lambda_3 \approx U_{13} U_{33}^* \lambda_3 \] (3.22)

and taking into account from Eqs.(3.13) and (3.14) that

\[|H'_{13}| = \frac{1}{\sqrt{6}} |r| \] (3.23)

it follows that:

\[|U_{13}| = \frac{1}{3} |H'_{13}| = \frac{1}{3 \sqrt{6}} |r| = \frac{1}{\sqrt{2}} \sqrt{\frac{m_1 m_2}{m_3^2}} \] (3.24)

where we used the fact that Eq.(3.7) implies \(r \approx 3 \sqrt{3} \sqrt{m_1 m_2/m_3^2}\).

In an analogous way one can derive the leading order value of \(|U_{12}|\) and then using unitarity obtain \(|U_{31}|\). These values agree with those presented in Eq.(3.18).

A qualitative understanding of these predictions for \(V_{CKM}\) can be obtained by viewing this ansatz as a small perturbation of the zero mass solution Class-Ia) of the equation \(det(M) = 0\) corresponding to mass matrices of the form:
\[ M = c \begin{pmatrix} 1 & e^{ir} & 1 \\ e^{iq} & 1 & e^{iq} \\ 1 & 1 & 1 \end{pmatrix} \] (3.25)

Since in this limit \( m_u = m_d = 0 \), it is straightforward to find the mass eigenstates. After making the weak-basis transformation of Eq.(3.14) one obtains for small \( |r_d/q_d| \), in leading order:

\[
|U_{12}^d| = \sqrt{\frac{3}{2}} \frac{|r_d|}{q_d} \quad |U_{23}^d| = \sqrt{2} \frac{m_s}{m_b} \\
|U_{13}^d| = |U_{12}^d||U_{23}^d|/2 \quad |U_{31}^d| = 3|U_{13}^d| \]
(3.26)

This shows that already in the limit \( m_u = m_d = 0 \), the main features of the Ansatz-I are manifest: there is a clear distinction between the mixing of the second and third generations and the mixing of the first and second generations. While \( |U_{23}| \) is proportional to a small parameter \( |q| \), which then leads to the prediction \( |U_{23}| = \sqrt{2}(m_s/m_b) \), \( |U_{12}| \) is proportional to the ratio of two small parameters \( |r| \) and \( |q| \). Thus, one finds a natural explanation why \( |U_{12}| \) is much larger than the other mixings. The Ansatz-I of Eq.(3.13) can thus be viewed as a small perturbation of the mass matrix of Eq.(3.25) whose main effect, is generating mass for the first family and fixing the ratio \( |r/q| \) to the value \( |r/q| \approx \frac{2}{\sqrt{3}} \sqrt{\frac{m_s}{m_u}} \), thus leading to the successful prediction \( |U_{12}| = \sqrt{\frac{m_s}{m_u}} \).

We find remarkable the occurrence of the numerical factor \( 2/\sqrt{3} \) in \( |r/q| \), which just cancels with the factor \( \sqrt{3}/2 \) in Eq.(3.26).

**Ansatz II:** \{ \( t = 0, r = -q \) \}

So far we have constructed only one ansatz, following the general procedure described in the beginning of this section. We will construct a second ansatz, by putting \( t = 0 \) in Eq.(2.3) and noting that in this case one has:

\[ \text{det}(M_{\text{phase}}) = -(1 - e^{ir})(1 - e^{iq}) \] (3.27)

Following our heuristic argument that \( |\text{det}(M)| \) should depend only of one parameter, we put \( |r| = |q| \), in particular \( r = -q \), to obtain:

\[ |\text{det}(M_{\text{phase}})| = 4 \sin^2\left(\frac{\chi}{2}\right) \]
(3.28)

\[ \chi(H) = \frac{9}{8}[2 \sin^2(\frac{\chi}{2}) + 3 \sin^2(\frac{\chi}{2}) - 2 \sin^2(\frac{\chi}{2}) \sin^2(\frac{\chi}{2})] \]
and thus

\[ \sin^2\left(\frac{r}{2}\right) = \frac{3}{4} \sqrt{3}\delta \quad , \quad \sin^2\left(\frac{p}{2}\right) = \frac{9}{16} \chi - \frac{9}{8} \sqrt{3}\delta \]
(3.29)

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As in the previous ansatz, $M_{u,d}$ depend each only on three parameters $c_{u,d}, p_{u,d}$ and $r_{u,d}$, which are fixed in terms of quark mass ratios through Eq. (3.29). An exact solution for the quark eigenstates can be readily found and one obtains in leading order:

$$
|U_{12}| = \sqrt{m_1/m_2} \quad |U_{23}| = \frac{1}{\sqrt{2}} \left[ 1 - \sqrt{3} \sqrt{m_1/m_2} \right] ^{\frac{m_3}{m_2}} \quad |U_{31}| = \frac{\sqrt{3}}{2} \frac{m_1}{m_3} \quad |U_{13}| = \frac{1}{\sqrt{2}} \sqrt{\frac{m_1 m_2}{m_3}}
$$

(3.30)

By now, it should be clear the procedure to construct ansätze based on USY, whose main features are predicting $U_{12} \propto \sqrt{m_1/m_2}$ while $|U_{23}| \propto m_2/m_3$. In Table 1, we present the various ansätze, together with their predictions for $V_{CKM}$, in leading order.

4. Confronting with experiment

In the previous section, we have done a systematic search for ansätze based on USY which can lead to calculability of the CKM matrix. Our starting point was looking for ansätze which correctly predict the values of $|V_{12}|$ and $|V_{23}|$. The emphasis on these two matrix elements is justified on a number of grounds. On the one hand, these are the two best measured off-diagonal elements of the CKM matrix. On the other hand, for three generations, unitarity of the CKM matrix leads to the following exact relations:

$$
|V_{21}|^2 = |V_{12}|^2 - \epsilon \\
|V_{32}|^2 = |V_{23}|^2 - \epsilon
$$

(4.1)

where $\epsilon = |V_{31}|^2 - |V_{13}|^2$. Due to the smallness of $\epsilon$ compared to $|V_{12}|^2$ and $|V_{23}|^2$, one has to a good approximation $|V_{21}| \cong |V_{12}|$ and $|V_{32}| \cong |V_{23}|$. Therefore, a good prediction for $|V_{12}|^2$, $|V_{23}|^2$ implies also a good prediction for $|V_{21}|^2$, $|V_{32}|^2$.

Before making a detailed comparison of the predictions of the various ansätze with the experimental results, the following point is in order. The predictions for $|V_{ij}|$ presented in Table 1, were obtained with the assumption $\alpha_1 = \alpha_2 = 0$. We recall that these are the phases which enter into the most general parametrization of USY in Eq. (2.3) and which do not affect the quark mass spectrum but enter in the CKM matrix. In the limit of vanishing $\alpha_i$, the various USY ansätze presented in Table 1 lead to full calculability of $V_{CKM}$, in terms of quark mass ratios with no free parameters. In the limit $\alpha_i = 0$, only Ansatz-I can correctly predict all elements of $V_{CKM}$. However, it is clear that the requirement of full calculability (i.e. no free parameters) is not necessary. Actually, Ansatz-I is rather unique, since it is the only ansatz which correctly predicts $V_{CKM}$, without free parameters. Indeed, to our knowledge, none of the numerous ansätze proposed in the literature predict $V_{CKM}$, without free parameters. For example, the well known Fritzsch ansatz [6] predicts $V_{CKM}$ in terms of quark mass ratios and two arbitrary phases. These two phases are entirely analogous to the $\alpha_i$ phases which appear in the USY framework. Analogous arbitrary phases also appear in all the texture zero ansätze which have been recently classified [2].
Next, we give two examples where the values of $|V_{CKM}|$ are correctly predicted. The first example corresponds to Ansatz-I, where the CKM matrix is correctly predicted in terms of quark mass ratios, without free parameters. The second example corresponds to Ansatz-II, where $V_{CKM}$ is predicted in terms of quark mass ratios and one free parameter (the phase $\alpha_2$ is set to $\alpha_2 = 0.0093$).

Input: $m_t^{\text{physical}} = 174 \, GeV$ and

\[
\begin{bmatrix}
m_u(1 \, GeV) &= 1.0 \, MeV \\
m_c(1 \, GeV) &= 1.35 \, GeV \\
m_d(1 \, GeV) &= 6.5 \, MeV \\
m_s(1 \, GeV) &= 165 \, MeV \\
m_b(1 \, GeV) &= 5.4 \, GeV
\end{bmatrix}_{\text{Ansatz-I}},
\begin{bmatrix}
m_u(1 \, GeV) &= 1.0 \, MeV \\
m_c(1 \, GeV) &= 1.4 \, GeV \\
m_d(1 \, GeV) &= 8.3 \, MeV \\
m_s(1 \, GeV) &= 210 \, MeV \\
m_b(1 \, GeV) &= 5.1 \, GeV
\end{bmatrix}_{\text{Ansatz-II}}
\]

Output: $V_{CKM} = \begin{bmatrix} 0.9752 & 0.2213 & 0.0032 \\ 0.2210 & 0.9745 & 0.0391 \\ 0.0117 & 0.0375 & 0.9992 \end{bmatrix}_{\text{Ansatz-I}}, \begin{bmatrix} 0.9754 & 0.2206 & 0.0030 \\ 0.2205 & 0.9747 & 0.0377 \\ 0.0005 & 0.0374 & 0.9993 \end{bmatrix}_{\text{Ansatz-II}}$ (4.2)

It is clear that in both ansätze, one obtains a good fit for the experimentally observed $V_{CKM}$. The most salient difference between the two ansätze, is the fact that Ansatz-II requires a larger value for $m_s$ than Ansatz-I. This reflects the fact that Ansatz-I predicts in leading order $|V_{cb}| = \sqrt{2} \frac{m_u}{m_b}$, while Ansatz-II predicts $|V_{cb}| = \frac{1}{\sqrt{2}} \frac{m_u}{m_b}$ in leading order.

So far, we have only presented the predictions of our ansätze for the moduli of $V_{CKM}$. Note that at present, with the exception of the CP violating parameter $\epsilon$, all experimental results only measure or put bounds on the moduli of $V_{CKM}$. In our ansätze one can readily evaluate $J \equiv \text{Im}(V_{12}V_{23}^*V_{13}^*V_{23})$ which measures the strength of CP violation. One obtains $J(\text{Ansatz-I}) = 1.8 \times 10^{-7}$, $J(\text{Ansatz-II}) = 0.8 \times 10^{-6}$. These values are smaller than what is required to account for the experimental value of $\epsilon$. Note however that in most extensions of the SM there are new contributions to $\epsilon$. This is true for example in the minimal supersymmetric standard model.

5. Renormalization Group Analysis

In the previous section, we have tacitly assumed that our ansätze are implemented at 1 GeV. It is often advocated that one should look for a fundamental theory of flavor at the unification scale. In this section, we will analyse how our results for Ansatz-I change if we implement our ansätze at the unification scale. For that, we need to study the renormalization group evolution of the CKM matrix and the quark masses. The renormalization group equations (RGE) have been derived in a variety of models. We will use the RGE for the Yukawa couplings in the case of the SM with two Higgs doublets.
More precisely, we will use the approximate equations \[7\] for the diagonalized quark Yukawa coupling ratios \(Q^i_q\):

\[
16\pi^2 \frac{d \log Q^i_q}{d\tau} = a_q\lambda_t^2 + b_q\lambda_b^2 \quad ; \quad q = t, b \quad ; \quad i = 1, 2
\]

where \(\tau = \log(\mu/M)\), \(Q^1_q = \lambda_u/\lambda_t\), \(Q^2_q = \lambda_c/\lambda_t\), \(a_t = b_b = 3/2\), \(a_b = b_t = 1/2\) and the \(\lambda_j\) are the diagonalized Yukawa couplings.

The evolution of \(\lambda_t\) and \(\lambda_b\) has to be calculated from the original RGE of the Yukawa coupling matrices. However these reduce to

\[
16\pi^2 \frac{d \log \lambda_t}{d\tau} = C \lambda_t^2 - 8 g_3^2 - \frac{9}{4} g_2^2 - \frac{17}{12} g_1^2
\]

in the approximation where only one Yukawa coupling is dominant \[8\]. We will consider two possibilities. The first is assuming strict universality, thus having in the Lagrangean prior to symmetry breaking, only one universal coupling constant \(\lambda\) as indicated in Eq.(2.1). This leads to \(\lambda_b = \lambda_t\) and due to the fact that \(m_t \gg m_b\) this of course requires \(v_u \gg v_d\). The other possibility is assuming that the overall strength of Yukawa couplings is different in the up and down quark sectors, leading to \(\lambda_b \ll \lambda_t\). These two possibilities lead to different values for \(C\) in Eq.(5.2)

\[
C = 5 \quad (\lambda_b = \lambda_t) \quad ; \quad C = \frac{9}{2} \quad (\lambda_b \ll \lambda_t)
\]

In the study of the RGE for \(V_{CKM}\), we will use a Wolfenstein-like parametrization \[9\] defined in the following way:

\[
V_{12} = \lambda; \quad V_{23} = A\lambda^2; \quad V_{13} = A\mu\lambda^3\exp(i\phi)
\]

\(V_{11}, V_{22}, V_{33}, \) real positive

The advantage of this parametrization is that all parameters are simply and exactly related to measurable quantities, since the following definitions hold: \(\lambda = |V_{12}|, A = |V_{23}/V_{12}^2|, \mu = |V_{13}/(V_{12}V_{23})|\) and \(\phi = \arg(V_{13}V_{22}V_{12}^*V_{23}^*)\). The only parameter with relevant evolution is \(A\) whose RGE is given by:

\[
16\pi^2 \frac{d \log(A)}{d\tau} = -a \lambda_t^2
\]

where \(a = 1\) when \(\lambda_b = \lambda_t\) and \(a = 1/2\) when \(\lambda_b \ll \lambda_t\).

We also need the RGE for \(SU(3) \otimes SU(2) \otimes U(1)\) gauge coupling constants for the case of the SM with two Higgs doublets,

\[
\frac{d \alpha_i}{d\tau} = -b_i \alpha_i^2; \quad \alpha_i = \frac{g_i^2}{4\pi} \quad ; \quad i = 3, 2, 1
\]

where \(b_{03} = 11 - 2n_f/3\), \(b_{02} = 7 - 2n_f/3\), \(b_{01} = -\frac{1}{3} - 10n_f/9\) and \(n_f\) denotes number of flavors.
In order to make the integration of all RGE, Eqs. (5.1), (5.2) and (5.5), we have to start at some energy. As we implement our ansatz at $M_X = 10^{16}$ GeV, the most logical assumption would be to choose $M = M_X$ in $\tau = \log(\mu/M)$. In order to achieve this we have to know the quark masses and gauge coupling constants at this energy. For the light quarks $u, d, s, c, b$ we use the known values at 1 GeV and at a first step use the QCD running mass equations to calculate the masses at 180 GeV, and then the Eq. (5.1) to obtain the quark ratios at $M_X$. For the gauge coupling constants we use the values at 180 GeV and Eq. (5.6) to obtain their values at $M_X$.

Knowing the quark mass ratios at $M_X$, we implement our ansatz at this scale. This means that we calculate the corresponding $V_{CKM}$ using Eqs. (3.7) and (3.10) for the phases $(r, q)$ and the diagonalization formulas. We use then Eq. (5.5) and the parametrization described to evaluate $V_{CKM}$ down to 1 GeV.

Input:

$$m_u(1 \text{ GeV}) = 1.0 \text{ MeV} \quad m_d(1 \text{ GeV}) = 6.5 \text{ MeV} \quad \alpha_1(180 \text{ GeV}) = 0.010$$

$$m_c(1 \text{ GeV}) = 1.35 \text{ GeV} \quad m_s(1 \text{ GeV}) = 165 \text{ MeV} \quad \alpha_2(180 \text{ GeV}) = 0.033 \quad (5.7)$$

$$m_b(1 \text{ GeV}) = 5.4 \text{ GeV} \quad \alpha_3(180 \text{ GeV}) = 0.107$$

The physical top mass is chosen to be 174 GeV.

Output: $V_{CKM} =$

$$\begin{pmatrix}
0.9752 & 0.2212 & 0.0030 \\
0.2209 & 0.9745 & 0.0378 \\
0.0113 & 0.0362 & 0.9993
\end{pmatrix}_{\lambda_b = \lambda_t} \quad \begin{pmatrix}
0.9752 & 0.2212 & 0.0033 \\
0.2209 & 0.9745 & 0.0395 \\
0.0120 & 0.0378 & 0.9992
\end{pmatrix}_{\lambda_b \ll \lambda_t} \quad (5.8)$$

Comparing the results of Eq. (4.2) with those of Eq. (5.8), it is clear that the same input of quark masses at 1 GeV, imposing our ansatz at 1 GeV or at $M_X$, leads to similar predictions for $V_{CKM}$. The only appreciable difference is in $V_{cb}$, when we take $\lambda_b = \lambda_t$.

6. Comparison with texture zero approach

In this section, we will address the question of whether there are some common features between the USY ansätze which we have constructed and some of the texture structures classified in Ref. [2]. More precisely, one may ask whether some of the USY ansätze predict texture zeros. We will not do an exhaustive study of the above question which is rendered specially difficult due to the enormous freedom one has of making weak-basis transformations which change the structure of Yukawa couplings but do not alter their physical content.

For definiteness, let us consider the following USY ansatz $^3$

$^3$This is a slight variant of the ansatz of Eq. (3.13), which leads to the same predictions for $V_{CKM}$ and it is more convenient for the analysis that follows
\[
M_{u,d} = c_{u,d} \left( \begin{array}{ccc}
1 & e^{ir} & 1 \\
e^{iq} & 1 & e^{i(q+r)} \\
1 & 1 & 1 
\end{array} \right)_{u,d} (6.1)
\]

We make now the following weak-basis transformations:

\[
M_d \rightarrow M'_u = F^\dagger \cdot M_d \cdot K_u \cdot F
\]

\[
M_d \rightarrow M'_d = F^\dagger \cdot M_d \cdot K_d \cdot F (6.2)
\]

where \(K_u = \text{diag}(1, e^{iq}, 1)\), \(K_d = \text{diag}(1, e^{iq}, 1)\) and \(F\) is given by Eq.(3.15). In the new basis the mass matrices have the form:

\[
M'_{u,d} = 3c_{u,d} \left( \begin{array}{cccc}
0 & A & \frac{A}{\sqrt{2}} \\ -A & B & C \\ -\frac{A}{\sqrt{2}} & C & D 
\end{array} \right)_{u,d} (6.3)
\]

where

\[
A = \frac{1}{3\sqrt{3}} e^{iq} [e^{ir} - 1] \cong \frac{i}{3\sqrt{3}} (r)
\]

\[
B = \frac{2}{9} [1 - e^{i(q+r)}] \cong -\frac{2i}{9} (q + r) (6.4)
\]

\[
C = \frac{-4 + 3e^{iq} + e^{i(q+r)}}{9\sqrt{2}} \cong \frac{i}{9\sqrt{2}} [2q + r]
\]

\[
D = \frac{4 + 3e^{iq} + 2e^{i(q+r)}}{9} \cong 1
\]

Note that \(M'_u, M'_d\) have the same structure and both exhibit zeros in the (1,1) element. It is interesting to note that all the texture zero structures classified in Ref.[2] also have zeros in the (1,1) position. Note that \(M'_{u,d}\) are written in the so called heavy-basis, in the sense that after factoring out the over-all constant \(3c_{u,d}\), the moduli of all the elements of \(M'_{u,d}\) are much smaller than 1, except the element (3,3). At this stage one may ask whether it is possible to make further weak-basis transformations, leading to other texture zeros. Since for \(M'_{u,d}\) the following relations hold:

\[
(M'_{u,d})_{12} = \sqrt{2} (M'_{u,d})_{13} \quad (M'_{u,d})_{31} = \sqrt{2} (M'_{u,d})_{21} (6.5)
\]

one may be tempted to make the weak-basis transformation \footnote{This weak-basis transformation was pointed out to us by Daniel Felizardo and João Seixas}:

\[
M'_{u,d} \rightarrow M''_{u,d} = O^T \cdot M'_{u,d} \cdot O (6.6)
\]

where
\[ O = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{pmatrix} \quad \theta = \arctan \left( \frac{1}{\sqrt{2}} \right) \] (6.7)

The above weak-basis transformation will indeed lead to zeros in the elements (1,3), (3,1) of \( M'_{u,d} \) while maintaining the zero in (1,1). However the matrices \( M'_{u,d} \) are no longer written in a heavy-basis and thus a comparison with the texture zero structures of Ref.\[2] loses its meaning.

To conclude, in the case of the USY ansatz of Eq.(6.1), one has in the heavy-basis, a texture zero in the element (1,1) in both \( M'_u \) and \( M'_d \). This USY ansatz is further characterized by a simple relation among some of the matrix elements, given by Eq.(6.5)

7. Conclusions

The idea that the flavor structure of Yukawa couplings is all contained in their phases is intriguing. The limit \( m_u = m_d = 0 \) is specially interesting in USY, since all solutions correspond to simple choices for the USY phases and can be readily classified. We have explored this fact to make a systematic search for calculability of the CKM matrix elements in terms of quark ratios. It was pointed out that within USY a natural explanation is found for the mixing between the first two generations being significantly larger than other CKM mixings. The ansätze we presented have a highly predictive power, since Ansatz-I predicts \( V_{CKM} \) in terms of quark mass ratios with no free parameters and Ansatz-II predicts \( V_{CKM} \) with only one free parameter. The fact that the experimentally observed CKM matrix can be accommodated within USY ansätze corresponding to simple relations among the phases, makes the USY hypothesis specially appealing.
Appendix

In this appendix we show some rather peculiar features of USY. In part (a), we consider the USY hypothesis for the case of two generations and analyse the question of calculability of the Cabibbo angle in terms quark mass ratios. Following the approach used for three generations, we will show that for two generations USY leads either to an arbitrary non-calculable Cabibbo angle or to the unrealistic relation

$$|\theta_C| = \frac{m_d}{m_s} \pm \frac{m_u}{m_c}.$$  

In part (b), we will show that the experimentally observed quark masses together with the USY hypothesis, necessarily imply $V_{CKM} \neq I$, independently of all USY parameters.

a) USY in two generations

For the case of two generations, an appropriate weak basis transformation with diagonal unitary phase matrices, analogues of Eq.(2.2), transforms the up and down quark mass matrices without loss of generality, into

$$M_u = c_u \begin{pmatrix} e^{ip_u} & 1 \\ 1 & 1 \end{pmatrix}, \quad M_d = c_d \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \cdot \begin{pmatrix} e^{ip_d} & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} \tag{A1}$$

It is clear that the phase $\alpha$ does not affect the quark spectrum. In an attempt to reach calculability of the Cabibbo angle and following our approach for three generations, we set the phase $\alpha$ to zero. The $M_{u,d}$ mass matrix is then of the form

$$M = c \begin{pmatrix} e^{ip} & 1 \\ 1 & 1 \end{pmatrix} \tag{A2}$$

We have the following mass spectrum

$$m_u^2 = 4c_u^2 \sin^2\left(\frac{p_u}{4}\right), \quad m_d^2 = 4c_d^2 \sin^2\left(\frac{p_d}{4}\right),$$

$$m_c^2 = 4c_u^2 \cos^2\left(\frac{p_u}{4}\right), \quad m_s^2 = 4c_d^2 \cos^2\left(\frac{p_d}{4}\right) \tag{A3}$$

These relations fix the phases $p_u$ and $p_d$ in terms of the quark mass ratios,

$$\tan^2\left(\frac{p_u}{4}\right) = \frac{m_u^2}{m_c^2}, \quad \tan^2\left(\frac{p_d}{4}\right) = \frac{m_d^2}{m_s^2} \tag{A4}$$

The Cabibbo angle is obtained by diagonalizing the mass matrix $H = MM^\dagger$, through the unitary matrix

$$U = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i\frac{p}{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{A5}$$

The $V_{CKM}$ matrix is given by:

$$V_{CKM} = \begin{pmatrix} \cos \theta_C & -\sin \theta_C \\ \sin \theta_C & \cos \theta_C \end{pmatrix} \tag{A6}$$
where we have eliminated unphysical phases in $V_{CKM}$. The Cabibbo angle $\theta_C$ is given in leading order by

$$|\theta_C| = \left| \frac{p_d}{4} - \frac{p_u}{4} \right|$$  \hspace{1cm} (A7)

Combining this equation with Eq. (A4), we find in leading order

$$|\theta_C| = \frac{m_d}{m_s} \pm \frac{m_u}{m_c}$$  \hspace{1cm} (A8)

the signs depend on the square roots which have to be calculated from Eq. (A4).

In connection with this result we make the following observations:

i) If the diagonal matrix $K = \text{diag}(1, e^{i\alpha})$ in Eq. (A1) is present with $\alpha$ arbitrary, then the relation of Eq. (A8) is lost. In this case there is no correlation between the masses and the Cabibbo angle.

ii) The fact that for two generations one obtains the $|\theta_C| \approx m_d/m_s$ instead of $|\theta_C| \approx \sqrt{m_d/m_s}$ can be viewed in the context of the USY hypothesis as an indication that there are more than two generations. We find quite intriguing the fact that in USY, the successful relation $|V_{us}| = \sqrt{m_d/m_s}$ naturally arises for three generations.

b) **Quark spectrum and $V_{CKM} \neq I$ in USY**

We will show that in three generations USY one necessarily has $V_{CKM} \neq I$, since $V_{CKM} = I$ would imply an unrealistic quark mass spectrum.

Note that $V_{CKM} = I$ implies

$$[H_u, H_d] = 0$$  \hspace{1cm} (A9)

where $H_u$ and $H_d$ are defined as in Eq. (2.4). Writing for all practical purposes $H_{u,d}$ in the form

$$H_u = \begin{pmatrix} 1 & x_1 e^{-i\alpha_1} & x_2 e^{-i\alpha_2} \\ x_1 e^{i\alpha_1} & 1 & x_3 e^{-i\alpha_3} \\ x_2 e^{i\alpha_2} & x_3 e^{i\alpha_3} & 1 \end{pmatrix}$$  \hspace{1cm} (A10)

one concludes from Eq. (A9), that

$$\frac{x_i^u}{x_i^d} = \frac{x_j^u}{x_j^d}, \quad \alpha_i^u = \alpha_i^d, \quad i, j = 1, 2, 3$$  \hspace{1cm} (A11)

By inserting these relations into the characteristic equations of $H$ for the determinant, $\delta$, and second invariant, $\chi$,

$$\delta(H) = \delta = 1 + 2x_1 x_2 x_3 \cos(\alpha_1 + \alpha_3 - \alpha_2) - x_1^2 - x_2^2 - x_3^2$$  \hspace{1cm} (A12)

$$\chi(H) = \chi = 3 - x_1^2 - x_2^2 - x_3^2$$

---

4This was mentioned in Ref.[3]
one finds

\[
\frac{x_1^u x_2^u x_3^u}{x_1^d x_2^d x_3^d} = \frac{1 - \chi_u - \delta_u}{1 - \chi_d - \delta_d}
\]  \hspace{1cm} (A13)

\[
\left(\frac{x_1^u}{x_1^d}\right)^2 \left[ 1 + \left(\frac{x_2^u}{x_1^d}\right)^2 + \left(\frac{x_3^u}{x_1^d}\right)^2 \right] = \frac{1}{1 - \chi_u^3}
\] \hspace{1cm} (A14)

Finally, combining Eqs. (A11), (A13) and (A14), one gets an exact relation between the masses of the up and down sector,

\[
\frac{(1 - \chi_u^3)^\frac{3}{2}}{1 - \chi_u - \delta_u} = \frac{(1 - \chi_d^3)^\frac{3}{2}}{1 - \chi_d - \delta_d}
\] \hspace{1cm} (A15)

Calculating the Taylor-MacLaurin series on both sides of this equation, yields in first order, \(\chi_u^2 = \chi_d^2\), thus implying that, \(m_t = m_c \frac{m_u}{m_s}\), which is in clear disagreement with the experimental value of the top quark mass.
Table 1

|   |   |   |
|---|---|---|
| 1) | \( p = 0 \) | \( s = q + r \) |
|   | \( r \rightarrow \delta \) | \( q + r \rightarrow \chi \) |
| \( |V_{12}| = \sqrt{\frac{m_2}{m_3}} \) | \( |V'_{13}| = \frac{\sqrt{2}}{2} \sqrt{\frac{m_1 m_2}{m_3}} \) | \( |V_{23}| = \sqrt{\frac{2 m_2}{m_3}} \) |
| 2) | \( s = 0 \) | \( r = -q \) |
|   | \( q \rightarrow \delta \) | \( p \rightarrow \chi \) |
| \( |V_{12}| = \sqrt{\frac{m_2}{m_3}} \) | \( |V'_{13}| = \frac{\sqrt{2}}{2} \sqrt{\frac{m_1 m_2}{m_3}} \) | \( |V_{23}| = \frac{m_2}{m_3} (1 - \sqrt{3} \sqrt{\frac{m_1}{m_2}}) \) |
| 3) | \( s = q \) | \( q = -p \) |
|   | \( p \rightarrow \delta \) | \( r - p \rightarrow \chi \) |
| \( |V_{12}| = \sqrt{\frac{m_2}{m_3}} \) | \( |V'_{13}| = \frac{\sqrt{2}}{2} \sqrt{\frac{m_1 m_2}{m_3}} \) | \( |V_{23}| = \frac{m_2}{m_3} (1 + 2 \sqrt{3} \sqrt{\frac{m_1}{m_2}}) \) |
| 4a) | \( s = -r \) | \( r = \frac{p - q}{2} \) |
|   | \( \frac{p - q}{2} \rightarrow \delta \) | \( \frac{p + q}{2} \rightarrow \chi \) |
| \( |V_{12}| = \sqrt{\frac{m_2}{m_3}} \) | \( |V'_{13}| = \frac{\sqrt{2}}{2} \sqrt{\frac{m_1 m_2}{m_3}} \) | \( |V_{23}| = \frac{m_2}{m_3} \) |
| 4b) | \( s = -r \) | \( q = -p \) |
|   | \( r \rightarrow \delta \) | \( p \rightarrow \chi \) |
| \( |V_{12}| = \sqrt{\frac{m_2}{m_3}} \) | \( |V'_{13}| = \frac{\sqrt{2}}{2} \sqrt{\frac{m_1 m_2}{m_3}} \) | \( |V_{23}| = \frac{m_2}{m_3} \) |

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