Stringy Hodge numbers of strictly canonical nondegenerate singularities

Jan Schepers∗

Dedicated to Joost van Hamel

Abstract

We describe a class of isolated nondegenerate hypersurface singularities that give a polynomial contribution to Batyrev’s stringy $E$-function. These singularities are obtained by imposing a natural condition on the facets of the Newton polyhedron, and they are strictly canonical. We prove that Batyrev’s conjecture concerning the nonnegativity of stringy Hodge numbers is true for complete varieties with such singularities, under some additional hypotheses on the defining polynomials (e.g. convenient or weighted homogeneous). The proof uses combinatorics on lattice polytopes. The results form a strong generalisation of previously obtained results for Brieskorn singularities.

1 Introduction

1.1. Batyrev defined now more than a decade ago the stringy $E$-function for complex algebraic varieties with log terminal singularities [Ba]. It is a rational function in two variables $u, v$ if the singularities are Gorenstein and canonical. Batyrev made moreover the following fascinating conjecture: if the stringy $E$-function of a projective variety is a polynomial $\sum_{p,q} a_{p,q} u^p v^q$ then the stringy Hodge numbers $h_{st}^{p,q} := (-1)^{p+q} a_{p,q}$ are nonnegative. This conjecture is motivated by the fact that stringy Hodge numbers share many other properties with usual Hodge numbers of smooth projective varieties.

1.2. It is known that the stringy $E$-function of toric varieties with Gorenstein (and hence canonical) singularities is a polynomial and that Batyrev’s conjecture is true for such complete varieties ([Ba] and [MP]). The same remarks apply to varieties with Gorenstein quotient singularities. Yasuda relates the stringy Hodge numbers in that case with orbifold cohomology [Ya]. Together

∗Supported by VICI grant 639.033.402 from the Netherlands Organisation for Scientific Research (NWO). During the completion of this paper, the author was a Postdoctoral Fellow of the Research Foundation - Flanders (FWO).

Address: Jan Schepers, Katholieke Universiteit Leuven, Department of Mathematics, Celestijnenlaan 200B, 3001 Leuven, Belgium.

E-mail: janschepers1@gmail.com.
with Veys we proved in [SV1] Batyrev’s conjecture in full generality for three-folds and also for a class of isolated singularities in dimension \( \geq 4 \) (see Theorem 3.9 for the precise statement). The disadvantage of that theorem in higher dimension is that the conditions demanded for the singularities will prevent the stringy \( E \)-function in many examples from being a polynomial.

1.3. In this paper we focus our attention on nondegenerate hypersurface singularities. This is a very computable class of singularities due to the connection to the toric world and hence they often serve as a testing ground for open problems. They are defined using the Newton polyhedron \( \Gamma(f) \) associated to their equation \( f = 0 \). This Newton polyhedron induces a decomposition \( \Delta_f \) of the first orthant of the dual space into cones, called the first Varchenko subdivision in [Ste]. We study isolated nondegenerate singularities with a natural extra condition on the 1-dimensional cones of \( \Delta_f \), i.e. the cones of \( \Delta_f \) associated to codimension 1 faces (called facets) of \( \Gamma(f) \). We call the subdivision \( \Delta_f \) then crepant (see Definition 4.8 and Remark 4.9 (1)). These singularities are strictly canonical and that means that they are part of the ‘worst’ untreated case in the aforementioned theorem of [SV1], but on the other hand they are in a sense the best chance for obtaining a polynomial stringy \( E \)-function (Remark 5.4 (4)). The following theorem is our main result. It gives a strong generalisation of the results from [SV2] about Brieskorn singularities. See Proposition 4.11 and Theorem 5.3.

1.4. **Theorem.** Let \( V \) be an algebraic variety whose singularities are analytically isomorphic to isolated nondegenerate singularities with crepant first Varchenko subdivision. Then the stringy \( E \)-function of \( V \) is a polynomial. If \( V \) is complete and if the defining polynomial \( f \) of each of the singularities is convenient (i.e. \( f \) contains a nonzero term \( a_i x_i^{b_i} \) for each variable \( x_i \)) or if the set of compact facets of \( \Gamma(f) \) has a unique maximal element (e.g. \( f \) weighted homogeneous) then the stringy Hodge numbers of \( V \) are nonnegative.

1.5. The proof of this theorem mainly uses combinatorics of lattice polytopes: we will see that the ingredients to compute the contribution of the singularity to the stringy \( E \)-function can be expressed in terms of lattice polytopes (Theorem 4.14). The condition of having a crepant first Varchenko subdivision is crucial for this, together with the formula for the Hodge-Deligne polynomial of a nondegenerate hypersurface in the torus from [BB] and the formula for the contribution of the singularity itself from [SV2].

1.6. This paper is organised as follows. In Section 2 we gather all the combinatorial definitions that we need and we prove a few useful lemma’s. That section is self-contained and can be read separately from the rest of the paper. In Section 3 we review Batyrev’s definitions of the stringy \( E \)-function and the stringy Hodge numbers. We recall the basic facts about nondegenerate singularities in Section 4. There we also define nondegenerate singularities with crepant first Varchenko subdivision and we prove that the contribution of such singularities to the stringy \( E \)-function is a polynomial. Finally, in Section 5 we prove the
Acknowledgements. Part of this work was carried out during a stay at the Institut des Hautes Études Scientifiques (IHÉS). I am very grateful that I was given the possibility to work there. I also want to thank Ann Lemahieu and Wim Veys for helpful discussions.

2 Combinatorial preliminaries

2.1. In this section we summarise some combinatorial aspects of Eulerian posets and lattice polytopes that are used later.

2.2. Let $P$ be a finite poset (i.e. partially ordered set). If $x, y \in P$ and $x \leq y$, then the interval $[x, y]$ is the set $\{z \in P \mid x \leq z \leq y\}$. The dual $P^*$ of $P$ is obtained by taking the same underlying set with inverted partial order relation. We assume that $P$ has a minimal element $\hat{0}$ and a maximal element $\hat{1}$ and that every maximal chain $\hat{0} = x_0 < x_1 < \cdots < x_d = \hat{1}$ has the same length $d$. In that case there exists a unique rank function $\rho : P \to \{0, \ldots, d\}$ such that $\rho(x)$ equals the length of a saturated chain in the interval $[\hat{0}, x]$. One calls $P$ then graded of rank $d$. If every nontrivial interval in $P$ has the same number of elements of even and odd rank, then $P$ is called Eulerian. There is an equivalent formulation in terms of the M"obius function. This function $\mu$ is defined on pairs $(x, y) \in P \times P$ with $x \leq y$ in the following inductive way:

$$
\mu(x, x) = 1, \quad \text{for all } x \in P,
$$

$$
\mu(x, y) = - \sum_{x \leq z < y} \mu(x, z), \quad \text{for } x < y.
$$

A finite graded poset is then Eulerian if and only if $\mu(x, y) = (-1)^{\rho(y) - \rho(x)}$ for all $x \leq y$. It is easy to see that an interval in an Eulerian poset is again Eulerian and that the dual $P^*$ of an Eulerian poset $P$ is Eulerian as well.

2.3. Definition. Let $P$ be an Eulerian poset of rank $d$. Define $g(P, t), h(P, t) \in \mathbb{Z}[t]$ by the following recursive rules:

$$
g(P, t) = h(P, t) = 1 \text{ if } d = 0,
$$

$$
h(P, t) = \sum_{0 < x \leq 1} (t - 1)^{\rho(x) - 1} g([x, 1], t) \text{ if } d > 0,
$$

$$
g(P, t) = \tau_{<d/2}((1 - t)h(P, t)) \text{ if } d > 0,
$$

where $\tau_r : \mathbb{Z}[t] \to \mathbb{Z}[t]$ is the truncation operator defined by

$$
\tau_r \left( \sum_i a_i t^i \right) = \sum_{i < r} a_i t^i.
$$

This definition was given by Stanley [St1, §2]. In fact the polynomials defined by Stanley for $P$, are in our notation $g(P^*, t)$ and $h(P^*, t)$. See also [BB] Def.
2.4. For $d > 0$, $\deg h(\mathcal{P}, t) = d - 1$ and $\deg g(\mathcal{P}, t) \leq (d - 1)/2$.

2.4. These polynomials have the following properties, with $\mathcal{P}$ an Eulerian poset of rank $d > 0$:

1. $h(\mathcal{P}, t) = t^{d-1}h(\mathcal{P}, t^{-1})$,

2. $\sum_{\hat{0} \leq x \leq \hat{1}} g(\hat{0}, x, t) g([x, \hat{1}]^*, t)(-1)^{\rho(\hat{1})-\rho(x)} = 0$ and also

$$\sum_{\hat{0} \leq x \leq \hat{1}} (-1)^{\rho(x)-\rho(\hat{0})} g(\hat{0}, x, t) g([x, \hat{1}], t) = 0.$$ 

The first one is proved in [St1, Thm. 2.4] and the second one, called Stanley’s convolution property, in [St2, Cor. 8.3].

2.5. Let $P$ be a lattice polytope in $\mathbb{R}^n$ (i.e. the convex hull of a finite number of points with vertices in $\mathbb{Z}^n$). The dimension of $P$ is the dimension of the smallest affine subspace of $\mathbb{R}^n$ containing $P$. We also allow $P = \emptyset$ as a polytope of dimension $-1$. A face of a lattice polytope $P$ is any intersection of $P$ with a hyperplane $H$ in $\mathbb{R}^n$ such that $P$ is completely contained in one of the two closed halfspaces determined by $H$. A facet is a face of codimension 1. The empty set and $P$ itself are also considered as faces of $P$, but are called improper faces. For faces $F, F'$ of $P$ we write $F \leq F'$ if $F \subseteq F'$. In this way the set of faces of $P$ becomes an Eulerian poset determined by $H$. We denote this poset by $\mathcal{P}(P)$. The dual poset $\mathcal{P}(P)^*$ is also of the form $\mathcal{P}(Q)$ for a lattice polytope $Q$. For such posets we have the following properties:

1. $g(\mathcal{P}(P), t)$ and $h(\mathcal{P}(P), t)$ have nonnegative coefficients,

2. $g(\mathcal{P}(P), t) = 1$ if and only if $P$ is a simplex.

The proof of the first statement uses the connection with toric varieties and their intersection cohomology, see Theorem 3.1 and Corollary 3.2 from [St3, p.122].

2.6. Braden and MacPherson define relative $g$-polynomials by proving the following [BMP, Prop. 2] for arbitrary polytopes (they do not restrict to polytopes with integer vertices).

Proposition. There is a unique family of polynomials $g(P, F, t) \in \mathbb{Z}[t]$ associated to a polytope $P$ and a face $F$ of $P$, satisfying the following relation: for all $P, F$, we have

$$\sum_{F \leq E \leq P} g(E, F, t) g([E, P]^*, t) = g(\mathcal{P}(P)^*, t).$$ 

We remark that Braden and MacPherson use the same definition as Stanley for $g$-polynomials, so compared to their formula we have to use dual posets at the appropriate places. But our notation $g(P, F, t)$ corresponds to theirs. Relative $g$-polynomials have the following properties:
(1) the coefficients of \( g(P,F,t) \) are nonnegative if \( P \) is a lattice polytope (again proved using intersection cohomology, see [BMP] Thm. 4),

(2) \( g(P,P,t) = g(P(P)^*, t) \) and if \( P \neq \emptyset \) then \( g(P,\emptyset, t) = 0 \).

2.7. If \( G \) is a face of \( P \) then it is not hard to see that there exists a lattice polytope \( P/G \) whose poset of faces equals \([G,P]\) (by a slight adaptation of the construction in the introduction of [BMP]). Let \( F \) be a face of \( P \) that contains \( G \). Then \( P/G \) has a face corresponding to \( F \) that we denote by \( F/G \). Formula (1) applied for \( P/G \) and \( F/G \) can be written as

\[
\sum_{F \leq E \leq P} g(E/G, F/G, t) g([E, P]^*, t) = g([G, P]^*, t).
\]

Below we will use the notation \( g([G,E],[G,F],t) \) for \( g(E/G,F/G,t) \).

2.8. Let \( P \) be a lattice polytope of dimension \( d \geq 0 \). Denote by \( f_P(m) \) the number of lattice points \( |mP \cap \mathbb{Z}^n| \) for \( m \in \mathbb{Z}_{>0} \). It is well known that the so-called Ehrhart generating series \( 1 + \sum_{m \geq 0} f_P(m) t^m \) can be written in the form

\[
h_P^*(t) = \frac{1}{(1-t)^{d+1}},
\]

where \( h_P^*(t) \) is a polynomial of degree \( s \leq d \) with nonnegative integer coefficients. Moreover, \( l = d + 1 - s \) is the smallest integer such that \( tP \) contains a lattice point in its relative interior.

2.9. Let \( p \) be a vertex of a positive dimensional polytope \( P \). The closed star neighbourhood \( \text{star}_{\partial P}(p) \) of \( p \) in the boundary \( \partial P \) of \( P \) is the following set of faces:

\[
\{F \text{ face of } P \mid F \text{ is a face of a proper face } Q \text{ of } P \text{ with } p \in Q\}.
\]

The group \( AGL(n,\mathbb{Z}) \) consists of affine transformations \( A \) of \( \mathbb{R}^n \) such that \( A(\mathbb{Z}^n) = \mathbb{Z}^n \). Two lattice polytopes \( P \) and \( Q \) in \( \mathbb{R}^n \) are called isomorphic if there exists such an affine transformation \( A \) with \( A(P) = Q \). In that case we clearly have \( h_P^*(t) = h_Q^*(t) \). Let \( Q \) be a lattice polytope in \( \mathbb{R}^{n-1} \). We define the standard lattice pyramid \( \Pi(Q) \) over \( Q \) as the convex hull of \( Q \times \{0\} \) and \((0, \ldots, 0, 1) \) in \( \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R} \). A polytope \( P \) in \( \mathbb{R}^n \) is called a lattice pyramid over a facet \( F \) of \( P \) if \( P \) is isomorphic to a standard lattice pyramid \( \Pi(Q) \) such that \( F \) corresponds to \( Q \) under this isomorphism. It is not hard to prove that then \( h_P^*(t) = h_F^*(t) \). This generalises in the following lemma.

2.10. Lemma. Let \( P \) be a lattice polytope in \( \mathbb{R}^n \) of dimension \( >0 \) and \( p \) a vertex of \( P \). Assume that for all facets \( F \) of \( P \) not containing \( p \), the convex hull of \( F \) and \( p \) is a lattice pyramid over \( F \). Then

\[
h_P^*(t) = \sum_{\substack{G \text{ proper face} \\ G \notin \text{star}_{\partial P}(p)}} h_Q^*(t) (t - 1)^{\dim P - \dim G - 1}.
\]
Proof. We may assume that the vertex \( p \) lies at the origin of \( \mathbb{R}^n \) and hence we can consider it as a vertex of all multiples \( mP \) of \( P \). First we look at the poset \( Q = \{ G \leq P \mid G \notin \text{star}_{\partial P}(p) \} \). Note that every interval in \( Q \) is Eulerian and hence \( \mu_Q(F, G) = (-1)^{\dim G - \dim F} \) for all \( F \leq G \) in \( Q \). For \( G \in Q \) we denote the convex hull of \( G \) and \( p \) by \((G, p)\). We define the function

\[
g_m : Q \to \mathbb{Z}_{\geq 0} : G \mapsto |(m(G, p) \setminus \{p\}) \cap \mathbb{Z}^n|
\]

and the function \( f_m : Q \to \mathbb{Z}_{\geq 0} \) that for \( G \in Q \) counts the number of integer points in \((m(G, p) \setminus \{p\}) \) that is not contained in any \( m(F, p) \) with \( F \in Q \) and \( F < G \). Then

\[
g_m(G) = \sum_{\substack{F \in Q \\colon F \leq G \\setminus \{p\}}} f_m(F)
\]

and hence we can apply the Möbius inversion formula [St3, Prop. 3.7.1] to \( Q, f_m, g_m \) to conclude that

\[
0 = f_m(P) = \sum_{G \in Q} (-1)^{\dim P - \dim G} |(m(G, p) \setminus \{p\}) \cap \mathbb{Z}^n|.
\]

Thus

\[
|(mP \setminus \{p\}) \cap \mathbb{Z}^n| = \sum_{\substack{G \text{ proper face} \\colon G \notin \text{star}_{\partial P}(p)}} (-1)^{\dim P - \dim G - 1} |(m(G, p) \setminus \{p\}) \cap \mathbb{Z}^n|. \quad (*)
\]

Secondly, \( Q' = \text{star}_{\partial P}(p) \cup \{P\} \) is a finite graded poset with \( \emptyset = \hat{0} \) and \( P = \hat{1} \), where each interval \([x, y]\) with \( y \neq \hat{1} \) is Eulerian. So by definition of the Möbius function we have

\[
\mu_Q'(\hat{0}, \hat{1}) = -\sum_{G \in \text{star}_{\partial P}(p)} (-1)^{\dim G + 1}.
\]

By [St3] Prop. 3.8.8] this equals the reduced topological Euler characteristic of the space

\[
\bigcup_{G \in \text{star}_{\partial P}(p)} G.
\]

This space is contractible to \( \{p\} \) and hence its reduced Euler characteristic is 0. Because \( P(P) \) is Eulerian, we deduce that

\[
1 = \sum_{\substack{G \text{ proper face} \\colon G \notin \text{star}_{\partial P}(p)}} (-1)^{\dim P - \dim G - 1}. \quad (**)
\]

Adding (*) and (**) gives

\[
|mP \cap \mathbb{Z}^n| = \sum_{\substack{G \text{ proper face} \\colon G \notin \text{star}_{\partial P}(p)}} (-1)^{\dim P - \dim G - 1} |m(G, p) \cap \mathbb{Z}^n|.
\]
So the Ehrhart series of $P$ equals
\[ \sum_{G \text{ proper face } G \in \text{star}_{\partial P}(p)} (-1)^{\dim P - \dim G - 1} \left( 1 + \sum_{m \in \mathbb{Z}_{>0}} |m(G, p) \cap \mathbb{Z}^n| t^m \right). \]

Multiplying by $(1 - t)^{\dim P + 1}$ one gets
\[ h^*_P(t) = \sum_{G \text{ proper face } G \in \text{star}_{\partial P}(p)} h^*_G(t) \left( -1 \right)^{\dim P - \dim G - 1} (1 - t)^{\dim P - \dim G - 1}. \]

\[ \blacksquare \]

2.11. Definition [BM, Def. 5.3]. Let $P$ be a lattice polytope. We define the polynomial $\tilde{S}(P, t) \in \mathbb{Z}[t]$ by the formula
\[ \tilde{S}(P, t) = \sum_{\emptyset \leq F \leq P} (-1)^{\dim P - \dim F} h^*_F(t) g([F, P], t), \]

where we sum over all faces of $P$, with $h^*_F(t)$ the $h^*$-polynomial of the lattice polytope $F$ (with $h^*_\emptyset(t) = 1$) and with $g([F, P], t)$ the $g$-polynomial of the interval $[F, P]$ in the Eulerian poset $\mathcal{P}(P)$. Note that $\tilde{S}(\emptyset, t) = 1$ and $\tilde{S}(P, t) = 0$ if $\dim P = 0$.

2.12. The polynomial $\tilde{S}(P, t)$ has the following properties.

1. $\deg(\tilde{S}(P, t)) \leq \dim P$.
2. $\tilde{S}(P, 0) = 0$ if $\dim P \geq 0$ since a $h^*$- and a $g$-polynomial always have constant coefficient 1 and since $P$ has an equal number of even- and odd-dimensional faces.
3. The coefficients of $\tilde{S}(P, t)$ are nonnegative. In [BM, Prop. 5.5] they are interpreted as the dimensions of the pieces of the pure Hodge structure on the lowest weight part of the middle cohomology of a nondegenerate affine hypersurface in the maximal torus of the toric variety associated to $P$.
4. For instance from this description one has the reciprocity law $\tilde{S}(P, t) = t^{\dim P + 1} \tilde{S}(P, t^{-1})$ [BM, Rem. 5.4].

For more information on this polynomial we refer to [BN, §4].

2.13. We will need an extension of the definition of the $\tilde{S}$-polynomial. An order ideal $\mathcal{I}$ in a poset $\mathcal{P}$ is a subset for which $x \in \mathcal{I}$ and $y \leq x$ imply $y \in \mathcal{I}$.

Definition. Let $P$ be a lattice polytope and $\mathcal{I} \subsetneq \mathcal{P}(P)$ be an order ideal of $\mathcal{P}(P)$. We define the polynomial $\tilde{S}(P, \mathcal{I}, t) \in \mathbb{Z}[t]$ by the formula
\[ \tilde{S}(P, \mathcal{I}, t) = \sum_{\emptyset \leq F \leq P} (-1)^{\dim P - \dim F} h^*_F(t) g([F, P], t). \]

Note that
(1) \( \deg \tilde{S}(P, \mathcal{I}, t) \leq \dim P \),
(2) \( \tilde{S}(P, \emptyset, t) = \tilde{S}(P, t) \),
(3) \( \tilde{S}(P, \{\emptyset\}, t) = \tilde{S}(P, t) + (-1)^{\dim P} g(\mathcal{P}(P), t) \). In particular, if \( P \) is odd-dimensional then the constant coefficient of \( \tilde{S}(P, \{\emptyset\}, t) \) is \(-1\).

Hence in general there can be negative coefficients in \( \tilde{S}(P, \mathcal{I}, t) \). For now, we study the special case where \( \mathcal{I} = \{ F \leq P \mid F \not\supseteq Q \} \) for a fixed face \( Q \) of \( P \). We denote \( \tilde{S}(P, \mathcal{I}, t) \) then by \( \tilde{S}(P, Q, t) \) and we show in Corollary 2.15 that \( \tilde{S}(P, Q, t) \) has nonnegative coefficients.

2.14. Proposition. Let \( P \) be a lattice polytope and \( Q' \leq Q \) be faces of \( P \). For a face \( F \) of \( P \) denote by \( F \lor Q \) the unique smallest face of \( P \) containing \( F \) and \( Q \). Then

\[
\tilde{S}(P, Q, t) = \tilde{S}(P, Q', t) + \sum_{Q' \leq F < P \atop Q \not\leq F} (-1)^{\dim P - \dim F - 1} h_F(t) g([F, P], t) \tilde{S}(F, Q', t).
\]

Proof. We work by induction on \( \dim P - \dim Q' \). The case \( P = Q = Q' \) is trivial. So assume \( \dim P - \dim Q' > 0 \). By definition

\[
\tilde{S}(P, Q, t) = \tilde{S}(P, Q', t) + \sum_{Q' \leq F < P \atop Q \not\leq F} (-1)^{\dim P - \dim F - 1} h_F(t) g([F, P], t).
\]

By Stanley’s convolution property we have

\[
g([F, P], t) = \sum_{F \leq G < P \atop Q \leq G} (-1)^{\dim P - \dim G - 1} g([F, G], t) g([G, P]^*, t)
\]

\[
+ \sum_{F \leq G' < P \atop Q \leq G'} (-1)^{\dim P - \dim G' - 1} g([F, G'], t) g([G', P]^*, t).
\]

We put this in (2) and exchange the sums to find

\[
\tilde{S}(P, Q, t) = \tilde{S}(P, Q', t) + \sum_{Q' \leq G < P \atop Q \not\leq G} g([G, P]^*, t) \tilde{S}(G, Q', t)
\]

\[
+ \sum_{Q \leq G' < P} g([G'^*, P], t) (\tilde{S}(G', Q', t) - \tilde{S}(G', Q, t)).
\]

By the induction hypothesis this becomes

\[
= \tilde{S}(P, Q', t) + \sum_{Q' \leq G < P \atop Q \not\leq G} g([G, P]^*, t) \tilde{S}(G, Q', t)
\]

\[
- \sum_{Q \leq G' < P} g([G'^*, P], t) \sum_{E:Q \leq G' \atop Q \not\leq E} g([E, G'], [E, E \lor Q], t) \tilde{S}(E, Q', t)
\]
\[ \tilde{S}(P, Q', t) + \sum_{Q' \leq G < P \atop Q \notin G} g([G, P]^*, t) \tilde{S}(G, Q', t) - \sum_{Q \leq G' < P} g([G', P]^*, t) \sum_{Q' \leq E < G'} g([E, G'], [E, E \vee Q], t) \tilde{S}(E, Q', t) \]

\[ = \tilde{S}(P, Q', t) + \sum_{Q' \leq G < P \atop Q \notin G} g([G, P]^*, t) \tilde{S}(G, Q', t) \]

\[ - \sum_{Q' \leq E < P \atop Q \notin E} \tilde{S}(E, Q', t) \sum_{E \vee Q \leq G' < P} g([E, G'], [E, E \vee Q], t) g([G', P]^*, t) \]

\[ = \tilde{S}(P, Q', t) + \sum_{Q' \leq F < P \atop Q \notin F} g([F, P], [F, F \vee Q], t) \tilde{S}(F, Q', t), \]

where the last step uses the formula for the relative \( g \)-polynomial of 2.7. \( \blacksquare \)

2.15. Corollary. If \( Q' \leq Q \) are faces of a lattice polytope \( P \) then

\[ \tilde{S}(P, Q', t) \leq \tilde{S}(P, Q, t) \]

(i.e. the inequality holds coefficientwise). In particular, \( \tilde{S}(P, Q, t) \) has nonnegative coefficients.

Proof. Note that the second statement follows immediately since we can take \( Q' = \emptyset \) and use that \( \tilde{S}(P, t) \) has nonnegative coefficients. The first statement is easily proved using induction on \( \dim P \), Proposition 2.14 and the nonnegativity of the coefficients of the relative \( g \)-polynomials. \( \blacksquare \)

2.16. For arbitrary order ideals \( I \subsetneq P(P) \) of the face poset of a polytope \( P \) we have the following recursion formula.

Proposition.

\[ \tilde{S}(P, I, t) = h_P^*(t) - \sum_{\emptyset \leq F' \subset P \atop F' \notin I} \tilde{S}(F, \mathcal{I} \cap P(F), t) g([F, P]^*, t). \]

Proof. The right hand side equals

\[ h_P^*(t) - \sum_{\emptyset \leq F' \subset P \atop F' \notin I} \sum_{\emptyset \leq F'' \subset F' \atop F'' \notin I} (-1)^{\dim F - \dim F''} h_{F''}^*(t) g([F'', F], t) g([F, P]^*, t) \]

\[ = h_P^*(t) - \sum_{\emptyset \leq F' \subset P \atop F' \notin I} h_{F'}^*(t) \sum_{F' \leq P} (-1)^{\dim F - \dim F'} g([F', F], t) g([F, P]^*, t) \]

\[ = h_P^*(t) + \sum_{\emptyset \leq F' \subset P \atop F' \notin I} (-1)^{\dim F - \dim F'} h_{F'}^*(t) g([F', P], t) \]

\[ = \tilde{S}(P, I, t), \]
where we used Stanley’s convolution property for the $g$-polynomial.

3 Stringy Hodge numbers

3.1. In this section we review Batyrev’s definition of stringy Hodge numbers, generalising usual Hodge numbers of smooth projective varieties.

3.2. Let $X$ be a reduced but not necessarily irreducible complex algebraic variety of dimension $d$. The Hodge-Deligne polynomial of $X$ is defined as

$$H(X; u, v) := \sum_{i=0}^{2d} \sum_{p,q=0}^{d} (-1)^i h^{p,q}(H^i_c(X, \mathbb{C})) u^p v^q \in \mathbb{Z}[u, v],$$

where $h^{p,q}(H^i_c(X, \mathbb{C}))$ denotes the dimension of the $H^{p,q}$-component of the natural mixed Hodge structure on $H^i_c(X, \mathbb{C})$. For a nonreduced variety $X$ we put $H(X; u, v) := H(X_{\text{red}}; u, v)$. For a smooth projective variety $X$, the coefficient of $u^p v^q$ in $H(X; u, v)$ is modulo the factor $(-1)^{p+q}$ simply the Hodge number $h^{p,q}(X)$. The Hodge-Deligne polynomial is a generalised Euler characteristic:

1. if $Y$ is a Zariski-closed subvariety of $X$, then $H(X; u, v) = H(Y; u, v) + H(X \setminus Y; u, v),$

2. for a product $X \times X'$ one has $H(X \times X'; u, v) = H(X; u, v) \cdot H(X'; u, v).$

3.3. Let $Y$ from now on be a normal irreducible variety. It is called $\mathbb{Q}$-Gorenstein if a multiple $rK_Y$ of the canonical class $K_Y$ is Cartier ($r \in \mathbb{Z}_{>0}$) and Gorenstein if $K_Y$ itself is Cartier. For example a normal hypersurface in a smooth variety is Gorenstein. Let $f : X \to Y$ be a log resolution of $Y$. This means that $f$ is a proper birational morphism from a smooth variety $X$, such that the exceptional locus $D$ of $f$ is a divisor with smooth components and normal crossings. Denote the irreducible components of $D$ by $D_i$, where $i$ lives in a finite index set $I$. For a $\mathbb{Q}$-Gorenstein $Y$ we have a linear equivalence

$$rK_X \equiv f^*(rK_Y) + \sum_{i \in I} b_i D_i,$$

with $b_i \in \mathbb{Z}$ uniquely determined. One divides this formally by $r$ and calls the rational number $\alpha_i := b_i/r$ the discrepancy coefficient of $D_i$. The variety $Y$ is called log terminal, canonical or terminal if all $\alpha_i > -1, \geq 0$ or $> 0$ respectively. These definitions do not depend on the chosen log resolution and intuitively speaking these classes of singularities are rather ‘mild’. If $Y$ is canonical but not terminal it is called strictly canonical.

3.4. Let $Y$ be log terminal. Choose a log resolution $f : X \to Y$ and use the same notations as above. For a subset $J$ of $I$ we set $D_J = \bigcap_{j \in J} D_j$ (so $D_\emptyset = X$) and $D_J^o = D_J \setminus \bigcup_{i \in I \setminus J} D_i$. The varieties $D_J^o$ give a natural stratification of $X$. 

10
**Definition** [Ba Def. 3.1]. The stringy $E$-function of $Y$ is defined as

$$E_{st}(Y; u, v) := \sum_{J \subseteq I} H(D_J^2; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1},$$

where $a_j$ is the discrepancy coefficient of $D_j$ and where the product over $j$ has to be interpreted as 1 if $J = \emptyset$.

Batyrev used motivic integration to prove that this formula does not depend on the chosen resolution [Ba Thm. 3.4].

3.5. The following remarks are in order:

1. If $Y$ is Gorenstein, then $E_{st}(Y; u, v)$ is a rational function. It lives then in $\mathbb{Z}[[u, v]] \cap \mathbb{Q}(u, v)$.

2. If $Y$ is smooth, then $E_{st}(Y; u, v) = H(Y; u, v)$. More generally, if $Y$ has a crepant resolution (i.e. a log resolution $f : X \to Y$ such that all discrepancy coefficients are 0), then $E_{st}(Y; u, v) = H(X; u, v)$.

3. We can choose the log resolution $f : X \to Y$ such that it is an isomorphism when restricted to the inverse image of the nonsingular part of $Y$. In particular, using such a log resolution, we see that if $Y$ has only an isolated singularity at a point $y$, then we can write

$$E_{st}(Y; u, v) = H(Y \setminus \{y\}; u, v) + \sum_{\emptyset \neq J \subseteq I} H(D_J^2; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1}.$$ 

We call $E_{st}(Y; u, v) - H(Y \setminus \{y\}; u, v)$ the local contribution of the isolated singularity and denote it by $E_{st,y}(Y; u, v)$.

3.6. For a projective variety $Y$ of dimension $d$ Batyrev [Ba Thm. 3.7] proved that

1. $E_{st}(Y; u, v) = (uv)^d E_{st}(Y; u^{-1}, v^{-1})$,

2. $E_{st}(Y; 0, 0) = 1$.

Note that this generalises the relations $h^{p,q}(Y) = h^{q,p}(Y) = h^{d-p,d-q}(Y)$ and $h^{0,0}(Y) = 1$ for a smooth projective $Y$.

3.7. Let $Y$ now be a projective variety of dimension $d$ with Gorenstein canonical singularities. Assume that $E_{st}(Y; u, v)$ is a polynomial $\sum_{p,q} a_{p,q} u^p v^q$. Batyrev then defines the stringy Hodge numbers of $Y$ as $h_{st}^{p,q}(Y) = (-1)^{p+q} a_{p,q}$ [Ba Def. 3.8]. By 3.5 (2) and 3.6 one has

1. stringy Hodge numbers $h_{st}^{p,q}(Y)$ can only be nonzero for $0 \leq p \leq d$ and $0 \leq q \leq d$,

2. for smooth projective varieties stringy Hodge numbers are equal to usual Hodge numbers,
(3) $h^{q,p}_s(Y) = h^{q,p}_st(Y) = h^{d-p,d-q}_s(Y) = h^{d-p,d-q}_st(Y)$ and $h^{0,0}_s(Y) = 1$.

The following intriguing question is however still open.

3.8. Conjecture [Ba, Conj. 3.10]. Stringy Hodge numbers are nonnegative.

We remark that it is not clear when to expect a polynomial stringy $E$-function. It is true for Gorenstein toric varieties [Ba, Prop. 4.4] and the stringy Hodge numbers (or in that case better stringy Betti numbers) are nonnegative for complete Gorenstein toric varieties [MP, Thm. 1.2]. For varieties with Gorenstein quotient singularities the stringy $E$-function is also polynomial. Yasuda showed that the stringy Hodge numbers for such complete varieties coincide with the orbifold cohomology Hodge numbers [Ya, Rem. 1.4 (2)]. In the next section we describe a natural class of isolated strictly canonical nondegenerate hypersurface singularities that also have a polynomial stringy $E$-function. In Section 5 we prove that Batyrev’s conjecture holds for complete varieties with such singularities, under the additional hypothesis of Theorem 1.4.

Batyrev’s conjecture is easy for surfaces. Indeed, canonical surface singularities are classified: it are precisely the so-called $A$-$D$-$E$ singularities. One knows that these singularities admit a crepant resolution and hence Batyrev’s conjecture for surfaces follows from 3.5 (2). In higher dimension there is the following theorem [SV1, Thm. 3.1 and Cor. 3.4].

3.9. Theorem.

(1) For threefolds, Batyrev’s conjecture is true in full generality.

(2) Let $Y$ be a projective variety of dimension $d \geq 4$ with at most isolated Gorenstein canonical singularities and with polynomial stringy $E$-function. Assume that $Y$ has a log resolution $f : X \to Y$ such that all discrepancy coefficients of irreducible exceptional components are $> \left\lfloor \frac{d-4}{2} \right\rfloor$. Then the stringy Hodge numbers of $Y$ are nonnegative.

4 Nondegenerate singularities

4.1. In this section we recall the definition of nondegenerate hypersurface singularities and we explain how to compute their stringy $E$-function. We describe a natural class of isolated strictly canonical nondegenerate singularities that give rise to a polynomial stringy $E$-function. We conclude by giving a concrete formula for this contribution to the stringy $E$-function.

4.2. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a polynomial with $f(0) = 0$. We denote the hypersurface $\{f = 0\} \subset \mathbb{C}^n$ by $X_f$. We write $f = \sum_{m \in \mathbb{Z}_{\geq 0}^n} a_m x_m$ where $x_m = x_1^{m_1} \cdots x_n^{m_n}$. The Newton polyhedron $\Gamma(f)$ of $f$ is the convex hull in $\mathbb{R}^n$ of

$$\bigcup_{\substack{m \in \mathbb{Z}_{\geq 0}^n \setminus \{0\} \setminus a_m \neq 0}} m + (\mathbb{R}_{\geq 0})^n.$$
A face of $\Gamma(f)$ is defined as any nonempty intersection of $\Gamma(f)$ with a hyperplane $H$ such that $\Gamma(f)$ is completely contained in one of the two closed halfspaces determined by $H$. This is similar to the definition of a face of a polytope, but now we do not consider the empty set as a face and hence $\Gamma(f)$ is the only improper face of $\Gamma(f)$. For a face $\tau$ of $\Gamma(f)$ we write $f_\tau$ for the polynomial $\sum_{m \in \tau \cap (\mathbb{Z}_{\geq 0})^n} a_m x^m$. One calls $f$ nondegenerate with respect to its Newton polyhedron if the equation $f_\tau = 0$ defines a smooth subvariety of $(\mathbb{C}^*)^n$ for every compact face $\tau$ of $\Gamma(f)$.

4.3. From the Newton polyhedron of a polynomial $f$ one gets a partition of $(\mathbb{R}_{\geq 0})^n$ into cones (where $(\mathbb{R}_{\geq 0})^n$ should be considered as the first orthant of the space dual to the surrounding space of $\Gamma(f)$). This goes as follows. For a vector $v \in (\mathbb{R}_{\geq 0})^n$ set $m_f(v) = \inf_{w \in \Gamma(f)} \{v \cdot w\}$, where $\cdot$ is the standard inner product. In fact this infimum is attained and hence it is a minimum. The first meet locus $F(v)$ of $v$ is defined as

$$F(v) := \{w \in \Gamma(f) \mid v \cdot w = m_f(v)\}.$$  

This is a face of $\Gamma(f)$ and it is a compact face if and only if $v \in (\mathbb{R}_{> 0})^n$. For a face $\tau$ of $\Gamma(f)$ we can then define the cone $\delta_\tau$ associated to $\tau$ by

$$\delta_\tau := \{v \in (\mathbb{R}_{\geq 0})^n \mid F(v) = \tau\}.$$  

These cones form a partition of $(\mathbb{R}_{\geq 0})^n$ and their closures are pointed rational convex polyhedral cones with vertex at the origin, forming a fan $\Delta_f$. Following [Ste, §5] we call this fan $\Delta_f$ the first Varchenko subdivision. For a nondegenerate $f$ such that the origin is an isolated singularity of $X_f$, this construction gives the first step in a toric resolution of $(X_f, 0)$ [Va, §9, 10]. More precisely, $\Delta_f$ can be subdivided to a fan $\Delta'$ consisting of unimodular cones (i.e. simplicial cones that can be generated by a part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$). Then the proper birational toric map from the toric variety $X(\Delta')$, associated to $\Delta'$, to $\mathbb{C}^n$ is an embedded resolution of singularities of $(X_f, 0)$.

4.4. Let $g = \sum_{m \in \mathbb{Z}^n} b_m x^m \in \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$ be a Laurent polynomial. The Newton polytope $P$ of $g$ is the convex hull in $\mathbb{R}^n$ of those $m \in \mathbb{Z}^n$ with $b_m \neq 0$. It is a lattice polytope. For a nonempty face $F$ of $P$ we write $g_F$ for the Laurent polynomial $\sum_{m \in F \cap \mathbb{Z}^n} b_m x^m$. One calls $g$ nondegenerate with respect to its Newton polytope if the equation $g_F = 0$ defines a smooth subvariety in $(\mathbb{C}^*)^n$ for every nonempty face $F$ of $P$.

4.5. Let $P$ be a lattice polytope in $\mathbb{R}^n$ of maximal dimension. Let $g$ be a Laurent polynomial with $P$ as Newton polytope and assume that $g$ is nondegenerate with respect to $P$. Batyrev and Borisov derived a formula for the Hodge-Deligne polynomial of the hypersurface $Y_g := \{g = 0\} \subset (\mathbb{C}^*)^n$ [BB, Thm. 3.18]. In the proof of Proposition 5.5 of [BM] this formula is rewritten as follows.
Theorem. Using the notations of Section 2, \( H(Y_g; u, v) \) equals
\[
\frac{1}{uv} \left( (uv - 1)^{\dim P} + (-1)^{\dim P+1} \sum_{\emptyset \leq F \leq P} u^{\dim F+1} \tilde{S}(F, u^{-1}v) g([F, P]^*, uv) \right).
\]

4.6. Now let \( f \in \mathbb{C}[x_1, \ldots, x_n] \) be an irreducible polynomial with \( f(0) = 0 \) that is nondegenerate with respect to its Newton polyhedron. Assume that the hypersurface \( X_f \) has only an isolated singularity at \( 0 \). For a vector \( v = (v_1, \ldots, v_n) \in \mathbb{R}^n \) set \( \sigma(v) := v_1 + \cdots + v_n. \)

Proposition.

1. \( X_f \) is canonical \( \iff \) for all primitive vectors \( v \in (\mathbb{Z}_{\geq 0})^n \) we have \( \sigma(v) - m_f(v) \geq 1 \) (primitive means that gcd\((v_1, \ldots, v_n) = 1\)).

2. \( X_f \) is terminal \( \iff \) for all primitive vectors \( v \in (\mathbb{Z}_{\geq 0})^n \) different from the standard basis vectors \( e_i \) \((i = 1, \ldots, n)\) we have \( \sigma(v) - m_f(v) > 1. \)

Proof. This is essentially Theorem 4.6 of [Re2], but two remarks are in order. Firstly, the phrase ‘different from the standard basis vectors’ is missing in the formulation of that theorem but it should be there. Secondly, we have to explain that the above conditions for being canonical or terminal are not only necessary but also sufficient in the nondegenerate case. Let \( \Delta' \) be a fan as in 4.3. Let \( \delta \) be a cone of maximal dimension of \( \Delta' \) generated by integer vectors \( \delta^1 = (\delta_1^1, \ldots, \delta_1^n), \ldots, \delta^n = (\delta_1^n, \ldots, \delta_n^n) \) that form a \( \mathbb{Z} \)-basis for \( \mathbb{Z}^n. \) The affine toric variety \( X(\delta) \) associated to \( \delta \) is isomorphic to \( \mathbb{C}^n \) and \( X(\Delta') \) is covered by the open sets of the form \( X(\delta). \) The proper birational toric map \( h : X(\Delta') \to \mathbb{C}^n \) is locally given by
\[
h : X(\delta) \cong \mathbb{C}^n \to \mathbb{C}^n : (y_1, \ldots, y_n) \mapsto (x_1 = \prod_i y_i^{\delta_1^i}, \ldots, x_n = \prod_i y_i^{\delta_n^i})
\]
and the total inverse image of \( X_f \) on \( X(\delta) \) is given by an equation
\[
y_1^{m_f(\delta_1^i)} \cdots y_n^{m_f(\delta_n^i)} f_\delta(y_1, \ldots, y_n) = 0,
\]
with \( f_\delta(0, \ldots, 0) \neq 0 \) [Ya, §10]. Since \( f \) is nondegenerate, \( h \) gives an embedded resolution of singularities of \( X_f. \) Let \( X_f' \) be the proper transform of \( X_f \) under \( h. \) The discrepancy coefficient of an exceptional component of the induced log resolution \( h : X_f' \to X_f \) can be computed from the embedded resolution using the adjunction formula (for details see for instance the proof of Proposition 2.3 of [SV2]). It equals \( \sigma(\delta^i) - m_f(\delta^i) - 1 \) for an exceptional component whose intersection with \( X(\delta) \) is nonempty and lies in \( \{y_i = 0\}. \)

4.7. Assume now that \( X_f \) is canonical and has only an isolated singularity at the origin. The local contribution \( E_{st,0}(X_f; u, v) \) of the singularity to the stringy \( E \)-function of \( X_f \) can be computed by the following result ([SV2, Cor. 3.2], essentially work of Denef and Hoornaert [DH]).
Proposition.

\[ E_{st,0}(X_f; u, v) = \sum_{\text{compact faces } \tau \text{ of } \Gamma(f)} H(N_\tau; u, v) T_{\delta_\tau}(f, uv), \]

where \( N_\tau \) is the subvariety of \((\mathbb{C}^*)^n \) given by \( \{ f_\tau = 0 \} \) and where \( T_{\delta_\tau}(f, t) \) is the power series \( \sum_{v \in (\mathbb{Z}_{>0})^n \cap \delta_\tau} t^{m_f(v)-\sigma(v)} \) (one can show that this power series belongs to \( \mathbb{Q}(t) \)).

4.8. We are ready to define the singularities that we study during the rest of this paper.

Definition. Let \( f \in \mathbb{C}[x_1, \ldots, x_n] \) be nondegenerate with respect to its Newton polyhedron and let \( X_f \) have an isolated singularity at \( 0 \). We call the first Varchenko subdivision \( \Delta_f \) crepant if all primitive integer generators \( \delta^i \) of 1-dimensional cones of \( \Delta_f \) satisfy \( \sigma(\delta^i) - m_f(\delta^i) = 1 \).

4.9. Remark.

(1) Let \( \Delta' \) be a subdivision of \( \Delta_f \) in unimodular cones. As explained above, this subdivision gives an embedded resolution \( X(\Delta') \to \mathbb{C}^n \) of \( X_f \), inducing a log resolution. If \( \Delta_f \) is crepant then the exceptional components of this log resolution coming from 1-dimensional cones of \( \Delta_f \) all have discrepancy coefficient 0, so this explains the name.

(2) Note that \( \sigma - m_f \) is a linear function when restricted to a cone \( \delta \) of \( \Delta_f \). So if \( \delta \) is of maximal dimension then \( (\sigma - m_f)|_{\delta} \) gives rise to an element \( n_{\delta} \) of the dual \( \text{Hom}(\mathbb{Z}^n, \mathbb{Z}) \) of \( \mathbb{Z}^n \). If \( \Delta_f \) is crepant, then using these \( n_{\delta} \) we see that every cone \( \delta \) of maximal dimension is a Gorenstein cone (see [BN, Def. 1.8]).

(3) Proposition 4.6 shows that these singularities are strictly canonical: any primitive integer vector \( v \) is a linear combination of primitive generators of a cone of \( \Delta_f \) with nonnegative rational coefficients and hence \( \sigma(v) - m_f(v) \geq 1 \).

4.10. Example. In the following table we investigate the condition of having a crepant first Varchenko subdivision for the standard equations of the canonical or \( A-D-E \) surface singularities. In the second column we give the primitive generators of the 1-dimensional cones of \( \Delta_f \) different from the standard basis vectors.

| Singularity | Primitive generators | \( \Delta_f \) crepant? |
|-------------|----------------------|-----------------------|
| \( A_n : x^{n+1} + y^2 + z^2 = 0 \)\((n \geq 1)\) | \( n \) even : \((2, n+1, n+1)\) | no |
|             | \( n \) odd : \((1, \frac{n+1}{2}, \frac{n+1}{2})\) | yes |
| \( D_n : x^{n-1} + xy^2 + z^2 = 0 \)\((n \geq 4)\) | \((2, n-2, n-1), (2, 0, 1)\) | yes |
| \( E_6 \): \( x^4 + y^3 + z^2 = 0 \) | \((3, 4, 6)\) | yes |
| \( E_7 \): \( x^3 + xy^3 + z^2 = 0 \) | \((6, 4, 9), (2, 0, 1)\) | yes |
| \( E_8 \): \( x^5 + y^3 + z^2 = 0 \) | \((6, 10, 15)\) | yes |

4.11. **Proposition.** Let \( f \in \mathbb{C}[x_1, \ldots, x_n] \) be nondegenerate with respect to its Newton polyhedron and let \( X_f \) have an isolated singularity at \( 0 \). If the first Varchenko subdivision is crepant, then the local contribution \( E_{st,0}(X_f; u, v) \) is a polynomial.

**Proof.** We use Proposition 4.7. First we rewrite \( T_{\delta_\tau}(f, uv) \). Let \( \tau \) be a compact face of \( \Gamma(f) \), \( \delta_\tau \) the associated cone and \( \overline{\delta_\tau} \) its closure. Let \( P_\tau \) be the convex hull in \( \mathbb{R}^n \) of the primitive integer generators of the extreme rays of \( \overline{\delta_\tau} \). Let \( \tau' \) be a vertex of \( \tau \). Then \( \overline{\delta_{\tau'}} \) is a cone of maximal dimension of \( \Delta_f \). Let \( n_{\delta_{\tau'}} \) be as in Remark 4.9 (2). Then \( P_{\tau'} \) is the so-called support polytope of the Gorenstein cone \( \overline{\delta_{\tau'}} \) (i.e. all points where \( n_{\delta_{\tau'}} \) takes value 1), and \( P_\tau = P_{\tau'} \cap \overline{\delta_{\tau'}} \). We look at \( n_{\delta_{\tau'}} \) as a degree function on \( P_{\tau'} \) and on \( P_\tau \) (it obviously does not depend on the choice of the vertex). Then we have

\[
T_{\delta_\tau}(f, uv) = \sum_{v \in (\mathbb{Z}_{>0})^n \cap \delta_\tau} (uv)^{-n_{\delta_{\tau'}}(v)}.
\]

By Stanley’s reciprocity law (formulated in [St3, Thm. 4.6.14] for solutions of linear homogeneous diophantine equations, but allowing inequalities is no problem by the remark on p.222 of [St3]) this equals

\[
(-1)^{\dim \delta_\tau} \sum_{v \in (\mathbb{Z}_{\geq 0})^n \cap \overline{\delta_{\tau'}}} (uv)^{n_{\delta_{\tau'}}(v)}.
\]

This can be rewritten as

\[
(-1)^{\dim \delta_\tau} \sum_{i \in \mathbb{Z}_{\geq 0}} |(iP_\tau) \cap \mathbb{Z}^n| (uv)^i = \frac{h_{P_\tau}^*(uv)}{(uv - 1)^{\dim \delta_\tau}}
\]

by definition of the \( h^* \)-polynomial of the lattice polytope \( P_\tau \) of dimension \( \dim \delta_\tau - 1 \). Using Proposition 4.7 it suffices now to show that \( \frac{H(N_\tau; uv)}{(uv - 1)^{\dim \delta_\tau}} \) is a polynomial for each compact face \( \tau \). First we divide the equation \( f_\tau \) by one of the monomials appearing in it (this corresponds to moving one of the vertices of \( \tau \) to the origin). We get a Laurent polynomial \( f_\tau \) and \( N_\tau \cong \{(f_\tau = 0) \subset (\mathbb{C}^*)^n\} \). Then we use a coordinate change

\[
y_j = \prod_{i=1}^{n} x_i^{t_{i,j}}, \text{ where } j = 1, \ldots, n \text{ and } T = (t_{i,j}) \in GL_n(\mathbb{Z}),
\]

on \( (\mathbb{C}^*)^n \) as in Lemma 5.8 of [DH] to write

\[
\tilde{f}_\tau(x_1, \ldots, x_n) = h_\tau(y_1, \ldots, y_{\dim \tau})
\]
for a (nonunique) Laurent polynomial $h_\tau$. Then

$$N_\tau \cong (\mathbb{C}^*)^{n-\dim \tau} \times \{ h_\tau = 0 \} \subset (\mathbb{C}^*)^{\dim \tau}.$$  

Since $H(\mathbb{C}^*; u, v) = uv - 1$ and $n - \dim \tau = \dim \delta_\tau$ we conclude that  

$$\frac{H(N_\tau; u, v)}{(uv-1)^{\dim \tau}}$$

is a polynomial.

### 4.12. Example.

The previous proposition is in general not true for isolated strictly canonical nondegenerate singularities that do not have a crepant first Varchenko subdivision. Consider the polynomial $f = x_1^5 + x_2^3 + x_3^2 + x_4^3$.

Proposition 4.3 from [Re1] shows that the singularity $(X_f, 0)$ is canonical and Proposition 4.6 applied with $v = (1, 1, 1, 1)$ shows that it is strictly canonical. Using the combinatorial procedure of [SV2, Section 4] one finds the expression

$$
(\frac{uv}{uv})^4 - u^4v^3 - u^3v^4 + 3(\frac{uv}{uv})^3 - 2u^3v^2 - 2u^2v^3 + 4(\frac{uv}{uv})^2 - u^2v - uv^2 + 2uv + 1
$$

for $E_{st,0}(X_f; u, v)$.

### 4.13. In the proof of Proposition 4.11 we associated a lattice polytope $P_\tau$ to a compact face $\tau$ of the Newton polyhedron $\Gamma(f)$ of $f$. Now we define the lattice polytope $P_0$ as the convex hull of all the polytopes $P_\tau$ and the origin in $\mathbb{R}^{\geq 0}$. So $P_0$ is a kind of fundamental domain of the crepant first Varchenko subdivision $\Delta_f$. Denote the set of compact faces of $\Gamma(f)$ by $P_f$. We get an inclusion-reversing bijective correspondence between $P_f \cup \{ \emptyset \}$ and the faces of $P_0$ that are not contained in a coordinate hyperplane. Using these notations and the notations of Section 2 we can summarise the results so far in the following theorem.

### 4.14. Theorem. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be nondegenerate with respect to its Newton polyhedron and let $X_f$ have an isolated singularity at $0$. If the first Varchenko subdivision is crepant, then the local contribution $E_{st,0}(X_f; u, v)$ is given by

$$
\frac{1}{uv} \sum_{\mu \in P_f \cup \{ \emptyset \}} (-u)^{\dim \mu + 1} \tilde{S}(\mu, u^{-1}v) \tilde{S}(P_\mu, st(P_0 \setminus 0) \cap P(P_\mu); uv).
$$

**Proof.** In the proof of Proposition 4.11 we showed that

$$N_\tau \cong (\mathbb{C}^*)^{n-\dim \tau} \times \{ h_\tau = 0 \} \subset (\mathbb{C}^*)^{\dim \tau}$$

for a Laurent polynomial $h_\tau$ in variables $y_1, \ldots, y_{\dim \tau}$. Since $f$ is nondegenerate with respect to its Newton polyhedron, $h_\tau$ is nondegenerate with respect to its Newton polytope in $\mathbb{Z}^{\dim \tau}$. If we consider $\tau$ as a lattice polytope as well, then this Newton polytope is by construction isomorphic to $\tau$. Combining Proposition 4.7, the proof of Proposition 4.11 and Theorem 4.5 we get

$$E_{st,0}(X_f; u, v) = \sum_{\tau \in P_f} \frac{h_\tau^{P_0}(uv)}{uv} \left( (uv - 1)^{\dim \tau} + (-1)^{\dim \tau} A(\tau; u, v) \right).$$
where

\[ A(\tau; u, v) = \sum_{\emptyset \leq \mu \leq \tau} u^{\dim \mu + 1} \tilde{S}(\mu, u^{-1} v) g([\mu, \tau]^*, uv). \]

We split \( E_{st,0}(X_f; u, v) \) as \( A_1(uv) + A_2(u, v) \) with

\[ A_1(uv) := \sum_{\tau \in P_f} \frac{h^*_P(\tau)}{uv} \left( (uv - 1)^{\dim \tau} + (-1)^{\dim \tau + 1} g([\emptyset, \tau]^*, uv) \right), \]

\[ A_2(u, v) := \sum_{\tau \in P_f} (-1)^{\dim \tau + 1} \frac{h^*_P(\tau)}{uv} \overline{A}(\tau; u, v), \]

where

\[ \overline{A}(\tau; u, v) = \sum_{\emptyset < \mu \leq \tau} u^{\dim \mu + 1} \tilde{S}(\mu, u^{-1} v) g([\mu, \tau]^*, uv). \]

We remark that \( A_1(uv) \) and \( A_2(u, v) \) are both polynomials: the constant coefficient of a \( g \)-polynomial is always 1, so it follows immediately that \( A_1(uv) \) is a polynomial. And \( u^{\dim \mu + 1} \tilde{S}(\mu, u^{-1} v) \) is a homogeneous polynomial in \( u, v \) of degree \( \dim \mu + 1 \) without terms of the form \( cu^{\dim \mu + 1} \) or \( cv^{\dim \mu + 1} \) by 2.12. This shows that \( A_2(u, v) \) is a polynomial.

Note that \( P_{\emptyset} \) with chosen vertex \( 0 \) satisfies the conditions of Lemma 2.10. Using that lemma and Definition 2.13 we rewrite \( A_1(uv) \) as

\[ \frac{1}{uv} \left( h^*_P(\emptyset) + \tilde{S}(P_{\emptyset}, star_{\partial P_{\emptyset}(0)}, uv) - h^*_P(\emptyset) \right) = \frac{\tilde{S}(P_{\emptyset}, star_{\partial P_{\emptyset}(0)}, uv)}{uv}. \] (3)

By exchanging the sums we find that \( A_2(u, v) \) equals

\[ \frac{1}{uv} \sum_{\mu \in P_f} (-u)^{\dim \mu + 1} \tilde{S}(\mu, u^{-1} v) \sum_{\tau \in P_f, \tau \geq \mu} (-1)^{\dim \tau - \dim \mu} h^*_P(\tau) g([\mu, \tau]^*, uv) \]

\[ = \frac{1}{uv} \sum_{\mu \in P_f} (-u)^{\dim \mu + 1} \tilde{S}(\mu, u^{-1} v) \tilde{S}(P_\mu, star_{\partial P_\mu(0)} \cap P(\mu), uv). \] (4)

Adding formulae (3) and (4) ends the proof of the theorem. ■

5 Nonnegativity of stringy Hodge numbers

5.1. In the previous section we defined a class of isolated nondegenerate singularities that give a polynomial contribution to the stringy E-function. Now we will study Batyrev’s conjecture about the nonnegativity of the stringy Hodge numbers for these singularities. First we formulate a lemma that allows us to draw (global) conclusions about the stringy Hodge numbers from the local contributions of the singularities.
5.2. Lemma. Let $Y$ be a complete variety of dimension $d$ with at most isolated Gorenstein canonical singularities and with a polynomial stringy $E$-function. Assume that the local contribution of the singularities to the stringy $E$-function is $\sum_{i,j} c_{i,j} u^i v^j$ with $(-1)^{i+j} c_{i,j} \geq 0$ for $i + j \geq d$. Then the stringy Hodge numbers of $Y$ are nonnegative.

Proof. This is a generalisation of Lemma 4.4 from [SV2]. The proof given there proves in fact exactly this generalisation. ■

5.3. Theorem. Let $Y$ be a complete variety of dimension $d$ with isolated singularities, such that each singularity is analytically isomorphic to a nondegenerate hypersurface singularity $(X_f, 0)$ with crepant first Varchenko subdivision. Assume in addition that each defining polynomial $f \in \mathbb{C}[x_1, \ldots, x_{d+1}]$ of an occurring nondegenerate singularity satisfies

1. $f$ is convenient (i.e. $f$ contains a nonzero term $a_i x_i^{b_i}$ for each variable $x_i$),

2. or $\Gamma(f)$ has a unique maximal compact face.

Then the stringy Hodge numbers of $Y$ are nonnegative.

5.4. Remark.

(1) One should remark that these singularities in general do not allow a crepant resolution (see 3.5 (2)), as Example 4.3 from [Sch] shows. Indeed, in that example the stringy Hodge numbers do not satisfy the Hard Lefschetz property and hence they cannot be Hodge numbers of a smooth projective variety.

(2) The second condition on $f$ holds for example if $f$ is weighted homogeneous.

(3) Theorem 5.3 and Proposition 4.11 form a strong generalisation of most of the results on Brieskorn singularities from [SV2] Section 4.

(4) It is interesting to compare this result with Theorem 3.9. From the point of view of that theorem, strictly canonical singularities are the worst untreated case. On the other hand, strictly canonical singularities should provide most of the examples of polynomial stringy $E$-functions since the a priori denominator of the stringy $E$-function becomes worse if the discrepancy coefficients are bigger. These are two good reasons to study strictly canonical singularities in this context.

Proof of Theorem 5.3. Let $y \in Y$ be a singular point of $Y$, analytically isomorphic to the hypersurface singularity $(X_f, 0)$ for a nondegenerate polynomial $f \in \mathbb{C}[x_1, \ldots, x_{d+1}]$ with crepant first Varchenko subdivision. We will show that the local contribution $E_{st,y}(Y; u, v) = \sum_{i,j} c_{i,j} u^i v^j$ satisfies $(-1)^{i+j} c_{i,j} \geq 0$ for $i + j \geq d$ under one of the extra conditions from the theorem on $f$. Then we can apply Lemma 5.2 to deduce the theorem.
We use the formula from Theorem 4.14. Note that \( \frac{1}{w^i}(-u)^{\dim \mu + 1} \tilde{S}(\mu, u^{-1}v) \) is a homogeneous polynomial of degree \( \dim \mu - 1 \) by 2.12 if \( \mu \neq \emptyset \) (it is 0 if \( \dim \mu = 0 \)). The sign of its coefficients simply depends on the parity of the degree. If \( \mu = \emptyset \) this expression equals \( \frac{1}{w^i} \), but recall from the proof of Theorem 4.14 that \( \frac{\tilde{S}(P_0, star_{P_0}(0), uv)}{w^i} \) is a polynomial. Hence it suffices to show that for all \( \mu \in P_f \cup \{ \emptyset \} \) the expression

\[ \tau_{>(d-\dim \mu)/2} \tilde{S}(P_\mu, T_\mu, t) \]

has nonnegative coefficients, where we wrote \( T_\mu \) for \( star_{\partial P_0}(0) \cap P(\mu) \) and where \( \tau_{>} \) denotes the truncation operator as in Definition 2.3.

1. First assume that \( f \) is convenient. We will prove by descending induction on \( \dim \mu \) that

\[ \tau_{>(d-\dim \mu)/2} \tilde{S}(P_\mu, T_\mu, t) = \tau_{>(d-\dim \mu)/2} \tilde{S}(P_\mu, t). \tag{5} \]

If \( \dim \mu = d \) then \( P_\mu \) is a vertex and the equality (5) is trivial. Assume now that \( \dim \mu < d \). By Proposition 2.16 we have that \( \tau_{>(d-\dim \mu)/2} \tilde{S}(P_\mu, T_\mu, t) \) equals

\[
\tau_{>(d-\dim \mu)/2} h^*_P(t) - \sum_{\nu \in P_f} \sum_{\nu > \mu} \tau_{=i} g(\{[\mu, \nu], t) \cdot \tau_{>(d-\dim \mu)/2-i} \tilde{S}(P_\nu, T_\nu, t),
\]

where \( \tau_{\nu, \mu} \) is the maximal degree \( \frac{\dim \mu - \dim \mu - 1}{2} \) of \( g([\mu, \nu], t) \) and where \( \tau_{=i} \) selects the term of degree \( i \). By induction, this becomes

\[
\tau_{>(d-\dim \mu)/2} h^*_P(t) - \sum_{\nu \in P_f} \sum_{\nu > \mu} \tau_{\nu, \mu} g([\mu, \nu], t) \cdot \tau_{>(d-\dim \mu)/2-i} \tilde{S}(P_\nu, t)
\]

\[ = \tau_{>(d-\dim \mu)/2} \left( h^*_P(t) - \sum_{\nu \in P_f} g([\mu, \nu], t) \tilde{S}(P_\nu, t) \right). \tag{6} \]

Note that \( P_0 \) has no vertices in the coordinate hyperplanes apart from the origin and the standard basis vectors \( e_i \), since \( f \) is convenient. Hence all nonempty faces of \( P_0 \) that are contained in the order ideal \( star_{P_0}(0) \) are unimodular simplices and hence their \( \tilde{S} \)-polynomial is zero. So (6) equals

\[
\tau_{>(d-\dim \mu)/2} \left( h^*_P(t) - \sum_{\emptyset < F < P_\mu} g([F, P_\mu]^*, t) \tilde{S}(F, t) \right). \tag{7}
\]

Moreover, \( \deg g([\emptyset, P_\mu]^*, t) \leq (d - \dim \mu)/2 \) and hence (7) equals

\[
\tau_{>(d-\dim \mu)/2} \left( h^*_P(t) - \sum_{\emptyset < F < P_\mu} g([F, P_\mu]^*, t) \tilde{S}(F, t) \right)
\]

But by Proposition 2.16 again, this is \( \tau_{>(d-\dim \mu)/2} \tilde{S}(P_\mu, t) \) and this has nonnegative coefficients.
2. Secondly, assume that $\Gamma(f)$ has a unique maximal compact face $\nu$. Then for all $\mu \in \mathcal{P}_f \cup \{\emptyset\}$ we have

$$\tilde{S}(P_\mu, \text{star}_{\partial \mathcal{P}_f}(0) \cap \mathcal{P}(P_\mu), t) = \tilde{S}(P_\mu, P_\nu, t)$$

and Corollary 2.15 shows immediately that this has nonnegative coefficients.

This ends the proof of Theorem 5.3. ■

5.5. Example. The above proof shows that if $\Gamma(f)$ has a unique compact face, then all signs of the local contribution $E_{st,0}(X_f; u, v) = \sum_{i,j} c_{i,j} u^i v^j$ are ‘right’ in the sense that $(-1)^{i+j} c_{i,j} \geq 0$ for all $i, j$. Now we give a concrete example where the local contribution of a convenient nondegenerate singularity with crepant first Varchenko subdivision does have ‘wrong’ signs in low degree. Put

$$f = x_1^2 + x_2^{12} + x_3^{12} + x_4^{12} + x_5^{12} + x_6^{12} + (x_5 x_6)^3 \in \mathbb{C}[x_1, \ldots, x_6].$$

The Newton polyhedron $\Gamma(f)$ has two compact facets (i.e. faces of codimension 1). One of them has vertices coming from the monomials

$$x_1^2, x_2^{12}, x_3^{12}, x_4^{12}, x_5^{12}, (x_5 x_6)^3$$

and for the other one just replace $x_5^{12}$ by $x_6^{12}$. In particular, considered as lattice polytopes all compact faces are simplices. The 1-dimensional cones of the first Varchenko subdivision $\Delta_f$ are generated by a standard basis vector or by

$$(6, 1, 1, 1, 1, 3) \text{ or } (6, 1, 1, 1, 3, 1).$$

It is easy to check that $\Delta_f$ is indeed crepant. To compute the local contribution $E_{st,0}(X_f; u, v)$ one can use the formula from Theorem 4.14 together with the recursion formula from Proposition 2.16. Note that all the involved $q$-polynomials are 1 since the compact faces of the Newton polyhedron are all simplices (2.5 (2)). To compute the necessary $h^*$-polynomials we used the computer program Normaliz 2.0 by Bruns and Ichim [31]. After a rather long computation one finds

$$E_{st,0}(X_f; u, v) = 3(uv)^4 + 4(uv)^3 - 111 u^3 v^2 - 111 u^2 v^3 + 750 u^3 v + 3495 (uv)^2 + 750 uv^3 + 111 u^2 v + 111 uv^2 + 3uv + 1$$

and thus the sign of $u^i v^j$ is not equal to $(-1)^{i+j}$ for $i + j = 3$. The idea of this example is the following, using notations as above. Consider the compact face $\tau$ equal to the intersection of the two compact facets. Its associated polytope $P_\tau$ has $(6, 1, 1, 1, 1, 3)$ and $(6, 1, 1, 1, 3, 1)$ as vertices. It is 1-dimensional and it has one lattice point in its relative interior. So its $h^*$-polynomial equals $1 + t$ and hence $\tilde{S}(P_\tau, I_\tau, uv) = uv - 1$. This $-1$ gives the wrong signs. It has such a big influence because there are only 5 nonzero terms in the sum of the formula of Theorem 4.14 and the 4 others come from odd-dimensional faces (including the empty set).
5.5. Conclusive remarks.

(1) I think that Theorem 5.3 is valid for all isolated nondegenerate singularities with crepant first Varchenko subdivision. To generalise the proof of Theorem 5.3 one would need that
\[ \tau > \left( d - \dim \mu \right) / 2 \tilde{S}(P_\mu, star_{\partial P}(0) \cap \mathcal{P}(P_\mu), t) \]
has nonnegative coefficients, which I could only prove in the two cases of the theorem. In general, I would guess that the inequality
\[ \tau > \left( d - \dim \mu \right) / 2 \tilde{S}(P_\mu, star_{\partial P}(0) \cap \mathcal{P}(P_\mu), t) \geq \tau > \left( d - \dim \mu \right) / 2 \tilde{S}(P_\mu, t) \]
holds coefficientwise. Note that we proved an equality here for the first case of the theorem. The inequality holds for the second case of the theorem by Proposition 2.14 and by the nonnegativity of the coefficients of relative \( g \)-polynomials.

(2) Note that there is a big similarity between the formula of Theorem 4.14 and the formula for the global stringy \( E \)-function of a nondegenerate Calabi-Yau hypersurface in the toric variety associated to a reflexive polytope [BM Thm. 7.2]. It is interesting to compare this also with the combinatorial definition of the stringy \( E \)-function of a Gorenstein lattice polytope by Batyrev and Nill [BN Def. 4.8]. It is unclear to me what this similarity might mean or suggest. We mention that Batyrev and Nill also formulate an interesting conjecture about their combinatorial stringy \( E \)-function [BN Conj. 4.10]

References

[Ba] V. V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, Proc. Taniguchi Symposium 1997, In 'Integrable Systems and Algebraic Geometry, Kobe/Kyoto 1997', World Sci. Publ. (1999), 1-32.

[BB] V. V. Batyrev and L. A. Borisov, *Mirror duality and string-theoretic Hodge numbers*, Invent. Math. 126 (1996), 183-203.

[BN] V. V. Batyrev and B. Nill, *Combinatorial aspects of mirror symmetry*, in 'Integer points in polyhedra - geometry, number theory, representation theory, algebra, optimization, statistics', Contemp. Math. 452 (2008), 35-66.

[BM] L. A. Borisov and A. R. Mavlyutov, *String cohomology of Calabi-Yau hypersurfaces via mirror symmetry*, Adv. Math. 180 (2003), 355-390.

[BMP] T. Braden and R. MacPherson, *Intersection homology of toric varieties and a conjecture of Kalai*, Comment. Math. Helv. 74 (1999), 442-455.

[BI] W. Bruns and B. Ichim, *Normaliz 2.0*, available on http://www.mathematik.uni-osnabrueck.de/normaliz/index.html.

[DH] J. Denef and K. Hoornaert, *Newton polyhedra and Igusa’s local zeta function*, J. Number Theory 89 (2001), 31-64.

[MP] M. Mustaţă and S. Payne, *Ehrhart polynomials and stringy Betti numbers*, Math. Ann. 333 (2005), 787-795.
[Re1] M. Reid, *Canonical 3-folds*, in ‘Journées de géométrie algébrique d’Angers 1979’, Sijthoff & Noordhoff (1980), 273-310.

[Re2] M. Reid, *Young person’s guide to canonical singularities*, Algebraic Geometry Bowdoin 1985, Proc. Sympos. Pure Math., Vol. 46 Part 1 (1987), 345-414.

[Sch] J. Schepers, *On the Hard Lefschetz property of stringy Hodge numbers*, J. Algebra 321 (2009), 394-403.

[SV1] J. Schepers and W. Veys, *Stringy Hodge numbers for a class of isolated singularities and for threefolds*, Int. Math. Res. Not., Vol. 2007, article ID rnm016, 14 pages.

[SV2] J. Schepers and W. Veys, *Stringy E-functions of hypersurfaces and of Brieskorn singularities*, to appear in Adv. Geom.

[St1] R. Stanley, *Generalized H-Vectors, Intersection Cohomology of Toric Varieties, and Related Results*, Commutative Algebra and Combinatorics (Kyoto, 1985), Adv. Stud. Pure Math. 11 (1987), 187-213.

[St2] R. Stanley, *Subdivisions and local h-vectors*, J. Amer. Math. Soc. 5 (1992), 805-851.

[St3] R. Stanley, *Enumerative Combinatorics, Volume 1*, Cambridge Studies in Advanced Mathematics 49, Cambridge University Press (1997).

[Ste] D. A. Stepanov, *Combinatorial structure of exceptional sets in resolutions of singularities*, arXiv:math/0611903v1 [math.AG].

[Va] A. N. Varchenko, *Zeta-Function of Monodromy and Newton’s Diagram*, Invent. Math. 37 (1976), 253-262.

[Ta] T. Yasuda, *Twisted jets, motivic measures and orbifold cohomology*, Compos. Math. 240 (2004), 396-422.