Of overlapping Cantor sets and earthquakes: Analysis of the discrete Chakrabarti-Stinchcombe model

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Abstract

We report an exact analysis of a discrete form of the Chakrabarti-Stinchcombe model for earthquakes [Physica A 270, 27 (1999)] which considers a pair of dynamically overlapping finite generations of the Cantor set as a prototype of geological faults. In this model the n-th generation of the Cantor set shifts on its replica in discrete steps of the length of a line segment in that generation and periodic boundary conditions are assumed. We determine the general form of time sequences for the constant magnitude overlaps and hence obtain the complete time-series of overlaps by the superposition of these sequences for all overlap magnitudes. From the time-series we derive the exact frequency distribution of the overlap magnitudes. The corresponding probability distribution of the logarithm of overlap magnitudes for the n-th generation is found to assume the form of the binomial distribution for n Bernoulli trials with probability 1/3 for the success of each trial. For an arbitrary pair of consecutive overlaps in the time-series where the magnitude of the earlier overlap is known, we find that the magnitude of the later overlap can be determined with a definite probability; the conditional probability for each possible magnitude of the later overlap follows the binomial distribution for k Bernoulli trials with probability 1/2 for the success of each trial and the number k is determined by the magnitude of the earlier overlap. Though this model does not produce the Gutenberg-Richter law for earthquakes, our results indicate that the fractal structure of faults admits a probabilistic prediction of earthquake magnitudes.

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1 Introduction

Earthquakes are outcomes of fault dynamics in the lithosphere. A geological fault is comprised of two rock surfaces in contact, created by a fracture in the rock layers. The two sides of the fault are in slow relative motion which causes the surfaces to slide. However, owing to friction the surfaces tend to stick and stress develops in the regions of contact. When the accumulated stress exceeds the resistance due to friction, the fault surfaces slip. The potential energy of the strain is thereby released, causing an earthquake. The slip is eventually stopped by friction and stress development resumes. Strain continues to develop till the fault surfaces slip again. This intermittent stick-slip process is the essential feature of fault dynamics. The overall distribution of earthquakes, including main shocks, foreshocks and aftershocks, is given by the Gutenberg-Richter law [1, 2]:

\[
\log_{10} N_r(M > M) = a - b M \tag{1}
\]

where, \(N_r(M > M)\) denotes the number (or, the frequency) of earthquakes of magnitudes \(M\) that are greater than a certain value \(M\). The constant \(a\) represents the total number of earthquakes of all magnitudes: \(a = \log_{10} N_r(M > 0)\) and the value of the coefficient \(b \approx 1\) is presumed to be universal. In an alternative form, the Gutenberg-Richter law is expressed as a relation for the number (or, the frequency) of earthquakes in which the energy released \(E\) is greater than a certain value \(E\):

\[
N_r(E > E) \sim E^{-b/\beta} \tag{2}
\]

where \(\beta \approx 3/2\) is the coefficient in the energy-magnitude relation [3].

One class of models for simulating earthquakes is based on the collective motion of an assembly of connected elements that are driven slowly, of which the block-spring model due to Burridge and Knopoff [5] is the prototype. The Burridge-Knopoff model and its variants [6, 7] have the stick-slip dynamics necessary to produce earthquakes. The underlying principle in this class of models is self-organized criticality [8].

Another class of models for simulating earthquakes is based on overlapping fractals. These models are motivated by the observation that a fault surface is a fractal object [9] [10] [11] [12] [13]. Consequently a fault may be viewed as a pair of overlapping fractals. Fractional Brownian profiles have been commonly used as models of fault surfaces [14] [15] [16]. In that case the dynamics of a fault is represented by one Brownian profile drifting on another and each intersection of the two profiles corresponds to an earthquake [16]. However the simplest possible model of a fault – from the fractal point of view – was proposed by Chakrabarti and Stinchcombe [17]. This model is a schematic representation of a fault by a pair of dynamically overlapping Cantor sets. It is not realistic but, as a system of overlapping fractals, it has the essential feature. Since the Cantor set is a fractal with a simple construction procedure, it allows us to study in detail the statistics of the overlap of one fractal object on another.

In this paper we study the outcome of discrete dynamics in the Chakrabarti-Stinchcombe model. Our first aim is to construct the time-series of overlaps of the \(n\)-th generation of the regular Cantor set on its replica with periodic boundary conditions, where one set shifts on the other in discrete steps. We use a finite generation of the Cantor set since self-similarity in natural objects extends
only over a finite range. In this model the overlap magnitudes and their logarithms correspond to the energies released and the magnitudes of earthquakes respectively. Our second aim is to obtain the statistics of the overlap magnitudes. From the time-series we derive the exact frequency distribution of the overlap magnitudes. For the \( n \)-th generation of the Cantor set the probability distribution of the logarithm of overlap magnitudes is found to assume the form of a binomial distribution for \( n \) Bernoulli trials and not an exponential law or a power law of the forms expressed in Eqs. (1) and (2). As the Cantor set is approached through successive generations, the most probable overlap magnitude is found to approach the cube-root of the maximum overlap magnitude. Our third aim is to determine whether it is possible to predict the occurrence of an overlap of a particular magnitude after a certain interval of time from an overlap that is known to have occurred. For this purpose we apply the theory of conditional probability. We consider the simplest case, that of two consecutive overlaps in the time-series, where the magnitude of the earlier overlap is preassigned and using this we calculate the probabilities of all possible magnitudes of the later overlap. Each conditional probability is found to follow a binomial distribution for a certain number of Bernoulli trials that is determined by the magnitude of the earlier overlap. The asymptotic form of the most probable magnitude of the later overlap is found to be inversely proportional to the square-root of the magnitude of the earlier overlap. We use the expression for the conditional probability to study three important cases: where the magnitude of the later overlap is less than that of the earlier one, where it is the converse of the previous case and where the magnitude of the two overlaps are equal. On the basis of the fractal overlap analogy between a pair of fault surfaces and a pair of overlapping Cantor sets, our results indicate that, from the knowledge of earthquakes recorded in the past it is possible to predict, with a probability, the magnitude of earthquakes at least for a short term in the future.

2 The model: Two dynamically overlapping Cantor sets

We consider a discrete form of the model of fault dynamics proposed by Chakrabarti and Stinchcombe [17]. The construction of the model requires two identical generations of the regular Cantor set. The procedure of constructing a regular Cantor set begins with a line segment of unit length, called the ‘initiator’. This line segment is divided into three equal parts and the middle part is removed to obtain the first generation; that serves as the ‘generator’ of the Cantor set [18]. The procedure is repeated for each of the two line segments of the first generation to obtain the second generation and so on. Therefore the \( n \)-th generation of the Cantor set is a finite set of \( 2^n \) line segments, each of length \( 1/3^n \). If this procedure is repeated an infinite number of times the remainder set of discrete points is called the regular Cantor set. In the rest of this paper the term Cantor set will always mean the regular Cantor set that we have just described. It is an exact self-similar fractal of dimension \( \log_3 2 \). The construction of the first few generations is shown in Fig. 1. Since it is practically impossible to generate the regular Cantor set exactly, we use the set of line segments obtained after a finite number of generations. It is reasonable to work with a finite generation of the
Cantor set since in all naturally occurring fractals the notion of self-similarity only applies between a lower and an upper cutoff length [18].

The model is defined in terms of two dynamically overlapping fractals which are represented by the \( n \)-th generation of the Cantor set and its replica. The sequences of line segments in the two finite sets of the \( n \)-th generation are considered as a schematic representation of the fractal profiles of the two surfaces of the fault. The magnitude of the overlap of one fault surface on the other is measured in this model as the number of line segments of one set that overlap exactly on line segments of the other set and we denote it by \( Y_n \) for the \( n \)-th generation sets. The motion of the fault surfaces is simulated by shifting one of the sets relative to the other in the following manner. We assume that initially the \( n \)-th generation of the Cantor set and its replica overlap completely, i.e., every line segment of one set overlaps exactly on the corresponding line segment of the other set. Then we shift one of the two finite sets relative to the other in one particular direction in uniform discrete steps of length \( 1/3^n \), that is the length of a line segment in the \( n \)-th generation. The length of a step is chosen to be \( 1/3^n \) in order to ensure that each line segment of one set either overlaps exactly on one of the other set or does not overlap at all. The overlap magnitude \( Y_n \) after every step is given by the number of overlapping segments. Consequently it is convenient to define time \( t \) for this dynamical process as a discrete variable and measure its value as the number of uni-directional steps by which one set has shifted from its initial position relative to the other. The time-series of overlaps is given by the sequence \( \{Y_n(t)\} \). Due to the discrete nature of the shifting process the magnitude of the overlap has discrete values. Further the structure of the \( n \)-th generation allows the number of overlapping segments to be in powers of 2 only: \( Y_n = 2^{n-k} \), \( k = 0, \ldots, n \). We shall write the magnitude of overlap as \( 2^{n-k} \) in order to indicate the generation \( n \) to which it belongs. We use periodic boundary conditions which produces a periodicity in the time-series. In Fig. we plot the time-series of overlap for the first few generations of the Cantor set. These plots show that the time-series has the appearance of a self-affine profile.

3 Construction of the time-series

In the following we construct the time-series of overlaps \( Y_n(t) \) of the \( n \)-th generation of the Cantor set on its replica by determining all of its structural features. We assume that periodic boundary conditions are assigned to the finite sets. The structural features of the time-series consist of a periodicity, a symmetry and a set of distinct temporal sequences of overlap magnitudes. The most important are the sequences of constant magnitude overlaps. The complete time-series is derived by superposing the constant overlap sequences of all possible overlap magnitudes.

(i) Since the initial condition is chosen as the complete overlap of the two sets, the initial overlap is maximum and its magnitude is given by:

\[
Y_n(0) = \max Y_n = 2^n.
\]  

(ii) Since the length of each line segment in the \( n \)-th generation of the Cantor set is \( 1/3^n \) and the boundary conditions are periodic, the time-series of the
overlap for the generation $n$ repeats itself after every $3^n$ time-steps:

$$Y_n(t) = Y_n(t + 3^n).$$  \hfill (4)

Subsequently whenever we shall refer to the variable $t$ and its values it will mean

$$t \equiv t \mod 3^n,$$  \hfill (5)

such that we shall study the time-series within the period $0 \leq t < 3^n$.

(iii) Since the structure of all generations of the Cantor set is symmetric about its center and the boundary conditions are periodic, the time-series of the overlap in the period $0 \leq t < 3^n$ (i.e., in all periods of the kind $k3^n \leq t < (k+1)3^n$, $k = 0, 1, 2, \ldots$) has a symmetric form:

$$Y_n(t) = Y_n(3^n - t), \quad 0 \leq t < 3^n. \hfill (6)$$

Therefore, for every sequence of overlaps occurring at time-steps $\{t\}$ within a period of the time-series, we have a complementary sequence of overlaps occurring at the time-steps $\{3^n - t\}$ within the same period.

(iv) After the first $3^{n-1}$ time-steps of shifting one $n$-th generation set relative to the other, the overlapping region is similar to the generation $n - 1$ of the Cantor set. Consequently the next $3^{n-1}$ time-steps forms a period of the time-series of overlap of two sets of generation $n - 1$. Within this period of $3^{n-1}$ time-steps, after the first $3^{n-2}$ time-steps the overlapping region is similar to the generation $n - 2$ of the Cantor set and therefore the next $3^{n-2}$ time-steps form a period of the time-series of overlap of two sets of generation $n - 2$. This process occurs recursively and therefore a period of the time-series of the generation-$n$ overlap is a nested structure of the periods of the time-series of the overlaps of all the preceding generations, for example, after $3^{n-1} + 3^{n-2} + \cdots + 3^{n-k}$ time-steps the overlapping region is similar to generation $n - k$ of the Cantor set, for which, following Eq. (6), the magnitude of overlap is given by:

$$Y_n \left( \sum_{r=1}^{k} 3^{n-r} \right) = 2^{n-k}, \quad k = 1, \ldots, n. \hfill (7)$$

The sequence of overlaps generated by Eq. (7) and the complementary sequence obtained by the symmetry property of Eq. (6), along with the initial condition (Eq. 3), forms the skeleton of the entire time-series (illustrated in part (a) of Fig. 2). The self-affine profile of the time-series observed in Fig. 3 is due to this property of the overlap. However it does not provide any more detail of the time-series.

(v) If the generation-$n$ of a Cantor set is shifted by a single time-step relative to another, the resulting overlap is just a unit line segment – owing to the periodic boundary conditions the last line segment of the former set overlaps on the first line segment of the latter set. This overlapping segment is similar to the initiator of the Cantor set (i.e., generation zero) with respect to the ratio $1/3^n$. Similarly, after a shift by three time-steps, the last two line segments of the former set overlaps on first two line segments of the latter set and the overlapping region is similar to the first generation of the Cantor set with respect to the ratio $1/3^{n-1}$. In general, owing to the periodic boundary conditions, a shift of $3^k$ time-steps ($0 \leq k \leq n$) from the initial position results in the
overlapping of the last \(2^k\) line segments of one set on the first \(2^k\) segments of the other set and the overlapping region is similar to the generation-\(k\) of the Cantor set with respect to the ratio \(1/3^{n-k}\). Therefore, beginning with a unit overlap resulting after a unit time-step, a sequence of shifts (time-steps) that are in geometric progression with common ratio \(3\) produce a sequence of overlaps with magnitudes that are in geometric progression with common ratio \(2\):

\[
Y_n \left(3^k\right) = 2^k; \quad k = 0, \ldots, n. \tag{8}
\]

This structural feature is illustrated in part (b) of Fig. 2. The same geometric progression of overlap magnitudes also occur for a geometric progression of time-steps beginning with a unit overlap after the fifth time-step:

\[
Y_n \left(5 \times 3^k\right) = 2^k; \quad k = 0, \ldots, n. \tag{9}
\]

For each of the sequences given by Eqs. (8) and (9) the complementary sequence is obtained by the symmetry property of Eq. (6).

(vi) Since the \(n\)-th generation of the Cantor set contains \(2^n\) line segments that are arranged by the generator in a self-similar manner, the magnitude of the overlaps, beginning from the maximum, form a geometric progression of descending powers of 2. As the magnitude of the maximum overlap is \(2^n\) (observed at \(t = 0\)), the magnitude of the next largest overlap is \(2^{n-1}\), i.e., a half of the maximum. A pair of nearest line segments form a doublet and the generation-\(n\) of the Cantor set has \(2^{n-1}\) such doublets. Within a doublet, each of the two line segments are two steps away from the other. Therefore, if one of the sets is shifted from its initial position by two time-steps relative to the other, only one of the segments of every doublet of the former set will overlap on the other segment of the corresponding doublet of the latter set, thus resulting in an overlap of magnitude \(2^{n-1}\). An overlap of \(2^{n-1}\) also occurs if we consider quartets formed of pairs of nearest doublets and shift one set from its initial position by \(2 \times 3\) time-steps relative to the other. Similarly, in the case of octets formed of pairs of nearest quartets, a shift of \(2 \times 3^2\) time-steps is required to produce an overlap of \(2^{n-1}\). Considering pairs of blocks of \(2^{r_1}\) nearest segments \((r_1 \leq n - 1)\), an overlap of magnitude \(2^{n-1}\) occurs for a shift of \(2 \times 3^{r_1}\) time-steps:

\[
Y_n \left(2 \times 3^{r_1}\right) = 2^{n-1}; \quad r_1 = 0, \ldots, n - 1. \tag{10}
\]

The complementary sequence is obtained as usual by the symmetry property of Eq. (6). An illustration of this constant overlap sequence is given in part (c) of Fig. 2.

Because of the self-similar structure of the Cantor set the line of argument used to derive Eq. (10) will work recursively for overlaps of all consecutive magnitudes. For example, the next overlap magnitude is \(2^{n-2}\), i.e., a quarter of the maximum. For each time-step \(t_1\) at which an overlap of \(2^{n-1}\) segments occur, there are two subsequences of overlaps of \(2^{n-2}\) segments that are mutually symmetric with respect to \(t_1\); one of the subsequences precedes \(t_1\), the other succeeds \(t_1\). Therefore the sequence of \(t\) values at which an overlap of \(2^{n-2}\) segments occurs is determined by the sum of two terms, one from each of two geometric progressions, one nested within the other:

\[
Y_n \left(2 \left[3^{r_1} \pm 3^{r_2}\right]\right) = 2^{n-2}; \tag{11}
\]
The first term belongs to the geometric progression that determines the sequence of \( t \) values appearing in equation Eq. (10) for the overlaps of \( 2^{n-1} \) segments, while the second term belongs to a geometric progression nested within the first. The symmetry property of Eq. (6) provides the complementary sequence.

In general the sequence of time-step values at which an overlap of \( 2^{n-k} \) segments (\( 1 \leq k \leq n \)) occurs is determined by the sum of \( k \) terms, one from each of \( k \) geometric progressions, nested in succession:

\[
Y_n (2 [3^{r_1} \pm 3^{r_2} \pm \cdots \pm 3^{r_{k-1}} \pm 3^{r_k}]) = 2^{n-k}; \quad (12)
\]

\[
k = 1, \ldots, n;
\]

\[
r_1 = k - 1, \ldots, n - 1;
\]

\[
r_2 = k - 2, \ldots, r_1 - 1;
\]

\[
\vdots
\]

\[
r_{k-1} = 1, \ldots, r_{k-2} - 2;
\]

\[
r_k = 0, \ldots, r_{k-1} - 1.
\]

For each value of \( k \) in the above equation there is a complementary sequence due to the symmetry property of Eq. (6). Eq. (12) along with Eqs. (3), (4) and (6) determine the entire time-series of overlaps of the \( n \)-th generation of the Cantor set on its replica. Assuming that the initial overlap is given by Eq. (3), the time-series is the superposition of the sequences of constant magnitude overlaps given by Eq. (12) for all possible overlap magnitudes.

Now we consider the special case of unit overlaps, i.e., when the overlap is only on a unit segment. We shall write the magnitude of a unit overlap for the \( n \)-th generation of the Cantor set as \( 2^{n-n} \) in order to indicate the generation index explicitly. A unit overlap occurs when \( k = n \) in Eq. (12). The sequence of \( t \) values at which these occur is given by:

\[
Y_n (2 [3^{r_1} \pm 3^{r_2} \pm \cdots \pm 3^{r_{k-1}} \pm 3^{r_k}]) = 2^{n-n} = 1. \quad (13)
\]

The above equation shows \( 2^{n-1} \) occurrences of the unit overlap. The same number of unit overlaps also occur in the complementary sequence obtained by using Eq. (6). Therefore, in a period of the time-series for the \( n \)-th generation, there are altogether \( 2^n \) unit overlaps.

4 Analysis of the time-series

The analysis of the complete time-series determined by Eqs. (3), (4), (6) and Eq. (12) is carried out in two parts. First we systematically derive the exact number of overlaps of any magnitude \( Y_n \) in a period of the time-series. Next we apply probability theory. The probability of occurrence of an overlap of magnitude \( Y_n \) after any arbitrary time-step is obtained directly from its frequency. We then derive the conditional probability of the occurrence of an overlap of magnitude \( Y'_n \) if it is given that the preceding overlap in the time-series has magnitude \( Y_n \).
4.1 Frequency of overlap magnitudes

In the following $\text{Nr}(Y_n)$ denotes the frequency of overlaps of magnitude $Y_n$, i.e., the number of times an overlap of $Y_n$ segments occurs in a period of the time-series for $n$-th generation of the Cantor set. From the sequences defined by Eqs. (3), (10) and (11) and their complementary parts, we get respectively:

$$\text{Nr}(2^n) = 1,$$

$$\text{Nr}(2^{n-1}) = 2n$$

and

$$\text{Nr}(2^{n-2}) = 2 \sum_{r_1=1}^{n-1} 2r_1 = 2n(n-1).$$

Similarly we can calculate the frequency of overlaps of all magnitudes. For example,

$$\text{Nr}(2^{n-3}) = 2 \sum_{r_1=2}^{n-1} \sum_{r_2=1}^{r_1-1} 2r_2 = \frac{4}{3} n(n-1)(n-2),$$

$$\text{Nr}(2^{n-4}) = 2 \sum_{r_1=3}^{n-1} \sum_{r_2=2}^{r_1-1} \sum_{r_3=1}^{r_2-1} 2r_3 = \frac{2}{3} n(n-1)(n-2)(n-3),$$

$$\text{Nr}(2^{n-5}) = 2 \sum_{r_1=4}^{n-1} \sum_{r_2=3}^{r_1-1} \sum_{r_3=2}^{r_2-1} \sum_{r_4=1}^{r_3-1} 2r_4 = \frac{4}{15} n(n-1)(n-2)(n-3)(n-4).$$

In general, from Eq. (12) the frequency of overlaps of magnitude $Y_n = 2^{n-k}$ is given by:

$$\text{Nr}(2^{n-k}) = 2 \sum_{r_1=k-1}^{n-1} \sum_{r_2=k-2}^{r_1-1} \cdots \sum_{r_{k-1}=1}^{r_{k-2}-1} 2r_{k-1} = C_k n(n-1)(n-2) \cdots (n-k+1)$$

$$= C_k \frac{n!}{(n-k)!}.$$  

The value of the constant $C_k$ is determined from the case of unit overlaps in the following way. In the above equation we keep the index $k$ constant and choose the generation index $n = k$. As a result we get the frequency of unit overlaps for the $k$-th generation:

$$\text{Nr}(2^{k-k}) = C_k k!.$$
We have followed the notation explained before Eq. (13) that the magnitude of the unit overlap for the \( k \)-th generation of the Cantor set is written as \( 2^{k-k} \) in order to indicate the generation index. On the other hand, we get the frequency of the unit overlap for the \( k \)-th generation from the sequence defined in Eq. (13) by replacing the index \( n \) by \( k \):

\[
N_r (2^{k-k}) = 2^k. \tag{22}
\]

Comparing Eqs. (21) and (22) we get the value of the constant:

\[
C_k = \frac{2^k}{k!}. \tag{23}
\]

Now we have the complete formula for the frequency of overlaps of magnitude \( Y_n = 2^{n-k} \) in the time-series for the \( n \)-th generation of the Cantor set:

\[
N_r (2^{n-k}) = \frac{2^k}{k!} \frac{n!}{(n-k)!} = 2^k \binom{n}{k}, \quad k = 0, \ldots, n \tag{24}
\]

where \( \binom{n}{k} \) denotes the binomial coefficient. The frequency distribution \( N_r(Y_n) \) obtained in Eq. (24) above for the overlap magnitudes \( Y_n = 2^{n-k}, k = 0, \ldots, n \), is exact. It is our central result and all subsequent results in this paper will be derived from it. It differs from the earlier claims \cite{17,19} that the frequency distribution follows a power law; possible reason for this difference will be mentioned in the discussion.

Using the binomial theorem, we have for the entire period:

\[
\sum_{k=0}^{n} N_r (2^{n-k}) = \sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n, \tag{25}
\]

since there are \( 3^n \) time-steps in a period of the time-series. From Eq. (24) we can calculate the cumulative frequency of overlaps with magnitudes that are greater than a preassigned value, say, \( 2^{n-k} \), by a partial sum of binomial terms:

\[
N_r (Y_n \geq 2^{n-k}) = \sum_{r=0}^{k} N_r (2^{n-r}) = \sum_{r=0}^{k} 2^r \binom{n}{r}, \tag{26}
\]

which does not have a closed form. Fig. 4 shows the frequency distribution and cumulative distribution of overlap magnitudes for three consecutive generations. For large overlap magnitudes, which is realized for large values of \( n \) and \( k \ll n \), Eq. (24) reduces to the following asymptotic form:

\[
N_r (2^{n-k}) \sim \frac{2^k n^k}{k!}. \tag{27}
\]

This is evident from the expressions for \( k = 0, \ldots, 5 \) in Eqs. (13) - (19). Using the notations \( Y_n = 2^{n-k} \) and \( \max Y_n = 2^n \), Eq. (24) for the \( n \)-th generation may be written as:

\[
N_r (Y_n) \sim \left( \frac{\max Y_n}{Y_n} \right)^{1+\log_2 n} \frac{1}{(\log_2 (\max Y_n/Y_n))!}. \tag{28}
\]
which shows that the factor $1/Y_n$, which was the one obtained in [17], is overwhelmed by the factor $1/Y_n \log_2 n$ in the expression for $N_r(Y_n)$ for large $n$ and small $k$.

4.2 Prediction probability of overlap magnitudes

We now analyze the time-series of overlap from the point of view of probability theory. The treatment by probability theory is necessary to determine whether the occurrence of an overlap of a certain magnitude can be predicted from the knowledge of the magnitude of an overlap that has already occurred. We continue to consider the case of the $n$-th generation of the Cantor set overlapping on its replica. Let $Pr(Y_n)$ denote the probability that after any arbitrary time-step $t$ we get an overlap of $Y_n$ segments. For the general case, $Y_n = 2^{n-k}$, $k = 0, \ldots, n$, it is given by:

$$Pr \left( 2^{n-k} \right) = \frac{N_r(2^{n-k})}{\sum_{k=0}^{n} N_r(2^{n-k})}$$

$$= \frac{2^k}{3^n} \binom{n}{k} \left( \frac{1}{3} \right)^{n-k} \left( \frac{2}{3} \right)^k.$$ (29)

The final expression in the above equation has the form of the binomial distribution for $n$ Bernoulli trials [20] with probability $1/3$ for the success of each trial and therefore $Pr(Y_n)$ stands for the probability of the case where the number of successes is given by $\log_2 Y_n$. Since $Pr \left( 2^{n-k} \right)$ is maximum for $k = \lfloor 2(n+1)/3 \rfloor$ (and also for $k = 2(n+1)/3 - 1$ when $n+1$ is a multiple of 3), the most probable overlap magnitude in general is given by:

$$\hat{Y}_n = 2^{n-\lfloor 2(n+1)/3 \rfloor} \quad (30)$$

where the floor function $\lfloor x \rfloor$ is defined as the greatest integer less than or equal to $x$ [21]. Since $\max Y_n = 2^n$, we get the following asymptotic relation for large values of the generation index $n$:

$$\hat{Y}_n \sim (\max Y_n)^{1/3}. \quad (31)$$

Next we consider the conditional probability that an overlap of magnitude $Y_n'$ occurs after the time-step $t+1$ if it is known that an overlap of magnitude $Y_n$ has occurred after the previous time-step $t$, for any arbitrary $t$. The conditional probability is given by:

$$Pr \left( Y_n' \mid Y_n \right) = \frac{N_r(Y_n, Y_n')}{N_r(Y_n)} \quad (32)$$

where $N_r(Y_n, Y_n')$ is the number of ordered pairs of consecutive overlaps $(Y_n, Y_n')$ occurring in a period of the time-series for the $n$-th generation. It follows from Eq. (12) that overlaps of magnitude $2^{n-k}$ in the time-series are immediately succeeded (i.e., after the next time-step) by overlaps of magnitude $2^r$, $0 \leq r \leq k$. 

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only and never by an overlap of magnitude greater than $2^k$. For $Y_n = 2^{n-k}$ and $Y_n' = 2^r$, $r = 0, \ldots, k$ we have:

$$
Nr \left( 2^{n-k}, 2^r \right) = \binom{n}{k} \binom{k}{r}.
$$

Therefore, from Eqs. (24), (32) and (33) we get:

$$
Pr \left( 2^r \mid 2^{n-k} \right) = \frac{1}{2^k} \binom{k}{r}; \quad k = 0, \ldots, n
$$

$$
= \binom{k}{r} \left( \frac{1}{2} \right)^r \left( \frac{1}{2} \right)^{k-r}.
$$

Here we find that the conditional probability follows the binomial distribution for $k$ Bernoulli trials with probability $1/2$ for the success of each trial. Eq. (34) shows that the expression of $Pr(2^r \mid 2^{n-k})$ is independent of the generation index $n$. Therefore, for fixed $k$ and $r$, it has the same value for all generations $n$, provided that $0 \leq r \leq k \leq n$. Since the conditional probability $Pr(2^r \mid 2^{n-k})$ is maximum when $r = \lfloor (k+1)/2 \rfloor$ (and also for $r = (k+1)/2 - 1$ when $k+1$ is a multiple of 2), the most probable overlap magnitude $Y_n'$ to occur next to an overlap of magnitude $Y_n = 2^{n-k}$ is given by:

$$
\hat{Y}_n' = 2^\lfloor (k+1)/2 \rfloor.
$$

For large values of the generation index $n$ and large $k$, $k \leq n$, from the above equation we get the following asymptotic relation:

$$
\hat{Y}_n' \sim \sqrt{\max \{ Y_n \}}.
$$

We consider three applications of Eq. (34) assuming in each case that $Y_n = 2^{n-k}$ and $Y_n'$ is the overlap magnitude occurring next to $Y_n$ in the time-series. First, we find that

$$
Pr(Y_n' \leq Y_n \mid Y_n) = \sum_{r=0}^{k} Pr \left( 2^r \mid 2^{n-k} \right)
$$

$$
= 1 \quad \text{for } 0 \leq k \leq \frac{n}{2}.
$$

This implies that, for $k \leq n/2$, an overlap of magnitude $Y_n = 2^{n-k}$ is always followed by an overlap of equal or less magnitude $Y_n'$. Consequently the case of an overlap of magnitude greater than that of the preceding one (i.e., $Y_n' > Y_n$) appears only for $k > n/2$. Second, we have

$$
Pr(Y_n' \geq Y_n \mid Y_n) = \sum_{r=n-k}^{k} Pr \left( 2^r \mid 2^{n-k} \right)
$$

$$
= \frac{1}{2^k} \sum_{r=n-k}^{k} \binom{k}{r} \quad \text{for } \frac{n}{2} \leq k \leq n.
$$

Since the final expression in the above equation involves a partial sum of binomial coefficients, it does not have a closed form, except for the trivial case $k = n$. 11
and a few other special cases. Therefore it must be calculated numerically for specific values of \( n \) and \( k \). Third, if the magnitude of overlap after a certain time-step is \( Y_n = 2^{n-k} \), the probability that an overlap of the same magnitude also occurs after the next time-step is given by:

\[
\Pr (Y'_n = Y_n | Y_n) = \Pr (2^{n-k} | 2^{n-k}) = \begin{cases} 
0, & 0 \leq k < n/2 \\
\frac{1}{2^n} \left( \frac{k}{n-k} \right), & n/2 \leq k \leq n.
\end{cases}
\] (39)

In this way we can derive the probability of various cases where \( Y'_n \) is specifically related to \( Y_n \).

5 Discussion

In this paper we have reported the construction and analysis of the time-series of overlaps for the discrete Chakrabarti-Stinchcombe model where the \( n \)-th generation of the Cantor set shifts on its replica, with periodic boundary conditions, in discrete uniform steps. The \( n \)-th generation of the Cantor set is effectively a fractal for lengths between 1 and \( 1/3^n \) (or, between 1 and \( 3^n \), if the length of a line segment in the \( n \)-th generation is taken as the unit) and therefore it has no characteristic scale within this range; this is also reflected in the structure of the time-series within a period. However the frequencies and the corresponding probabilities of the binary logarithm of overlap magnitudes follow a binomial distribution, which has a characteristic scale given by the most probable overlap magnitude. The existence of a most probable value in the frequency distribution creates the possibility of predicting overlap magnitudes in the time-series. We have shown the utility of this simple model in predicting an event from the knowledge of the preceding event. Though it is believed that earthquakes cannot be predicted [22], the analysis presented in the previous section indicates that the overlapping fractal structure of faults ought to admit probabilistic predictions of earthquake magnitudes. It is left to be explored how the probability of predicting an overlap magnitude in this model increases with the width of the preceding interval of time-steps within which all the overlap magnitudes are known.

The Chakrabarti-Stinchcombe model, proposed for studying the mechanism of earthquakes, is by far the simplest model containing the rudiments of the overlapping structure of a geological fault. The authors of the original model [17] analysed, using renormalization group argument, the case of the Cantor set shifting continuously over its replica with open boundary conditions and reported that the density of the overlap magnitudes \( Y \), in the infinite generation limit \( (n \to \infty) \), was in the form of a continuous power law: \( \rho(Y) \propto 1/Y \). However the discrete form of the model defined for finite generations of the Cantor set with periodic boundary conditions – the subject of this paper – has a binomial distribution of the binary logarithm of overlap magnitudes (Eq. 24 or Eq. 29). If, instead of periodic boundary conditions, open boundary conditions are used, the time-series will be of finite duration, consisting of \( 3^n \) time-steps; with the initial condition given by the position of maximum overlap at \( t = 0 \), as stated in Eq. 4, there will be no overlapping line segments of the two \( n \)-th generation sets for all odd values of \( t \) and the frequency of each overlap magnitude,
except the maximum, will be only half of the value given by Eq. (24). For large
generation indices the asymptotic form of the binomial distribution is given by
the continuous normal approximation [20], but it never acquires the form of a
power law. The difference in the results of Ref. [17] and this paper is apparently
due to the difference in the nature of the shifting process which is continuous in
the former and discrete in the latter (with the size of each discrete step being
equal to the length of the line segment in the finite generation). The authors
of Ref. [17] also report that their result is valid for all types of Cantor sets and
in fractals of higher dimensions, e.g., in the case of two overlapping Sierpinsky
carpets. Though the discrete forms of these cases are yet to be studied, it has
been reported in [19] that for two spanning clusters at the percolation threshold
on a square lattice – where each cluster is a random fractal embedded in two
dimensions – the overlap magnitudes follow a normal distribution.

Finally we consider a variant of the Chakrabarti-Stinchcombe model defined
as the $n$-th generation of the Cantor set overlapping on its complement in the
unit line segment; the complement of the $n$-th generation set is obtained by
replacing each line segment (of length $1/3^n$) in the latter by an empty segment
and each empty segment by a line segment. The overlap magnitudes in this
variant model are given by $Y_n = 2^n - 2^k$, $k = 0, \ldots, n$. The time-series of
overlaps can be directly derived from Eq. (12) by replacing each overlap mag-
nitude $2^n - 2^k$ in the time-series of the original model with an overlap magnitude
$2^n - 2^{n-k}$. Consequently the frequency distribution of overlap magnitudes is
given by $\text{Nr}(2^n - 2^k) = 2^{n-k} \binom{n}{k}$. The variant appears to be more realis-
tic than the original model since each of the two parts of a fractured rock is
complementary to the other. However, in both cases the frequency of overlap
magnitudes are found to follow binomial distributions.

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Figure 1: The ‘initiator’ ($n = 0$) and the first four generations in the construction of the regular Cantor set. The first generation ($n = 1$) provides the ‘generator’. This figure illustrates the process of constructing successive generations described in the text.
Figure 2: Three important features of the time-series are illustrated for the first four generations ($n = 1, 2, 3, 4$) of the Cantor set. The features within a period of each of the first three generations appear successively nested within a period of the fourth. (a) The skeleton of the time-series. It is the nested structure of the maximum overlaps for all the generations. (b) A sequence of overlaps whose magnitudes are in geometric progression with common ratio 2 which occur after the time-steps whose values are in geometric progression with common ratio 3. (c) The sequence of overlaps of a constant magnitude that is half the maximum, shown for a period for all four generations.
Figure 3: A period of the time-series of overlap magnitudes for the $n$-th generation of the Cantor set overlapping on its replica, for the first four generations ($n = 1, 2, 3$ and $4$) according to Eqs. (3), (4), (6) and (12). It shows that a period of the time-series for each generation is nested within a period of the time-series for the next generation. Owing to this recurrence, the sequence of overlap magnitudes in a period of the time-series for the Cantor set ($n = \infty$) forms a self-affine profile.
Figure 4: (a) The frequency distribution of the overlap magnitudes $Y_n = 2^{n-k}$, $k = 0, \ldots, n$ for three consecutive generations $n = 11, 12, 13$ according to Eq. (24). The points have been joined by lines only to make out the asymmetric Gaussian appearance of the distributions. (b) The cumulative distribution of overlap magnitudes for the three consecutive generations shown in part (a), calculated by using Eq. (26). The cumulative distribution for each generation has been scaled by the period of the corresponding time-series.