GEOMETRICAL ORDER–OF–MAGNITUDE ESTIMATES FOR SPATIAL CURVATURE IN REALISTIC MODELS OF THE UNIVERSE

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Abstract

The thoughts expressed in this article are based on remarks made by Jürgen Ehlers at the Albert–Einstein–Institut, Golm, Germany in July 2007. The main objective of this article is to demonstrate, in terms of plausible order–of–magnitude estimates for geometrical scalars, the relevance of spatial curvature in realistic models of the Universe that describe the dynamics of structure formation since the epoch of matter–radiation decoupling. We introduce these estimates with a commentary on the use of a quasi–Newtonian metric form in this context.

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In Memoriam: Jürgen Ehlers (1929–2008)

1 Introduction

In July 2007, Jürgen Ehlers gave a talk at the Albert–Einstein–Institut, Golm, Germany on the relevance of spatial curvature in models of structure formation in the Universe since the epoch of matter–radiation decoupling. This article aims to make his comments publicly available, after providing a contextual setting by first commenting on the more usual approach to these issues in structure formation studies.

The so–called \textit{longitudinal gauge} is often employed in the study of scalar perturbations at a Friedmannian background cosmology, and is considered a preferred frame because it offers an explicit Newtonian limit; cf. Ref. \cite{31}.\textsuperscript{1} While its local foundations are unambiguous (see, e.g., Refs. \cite{2,25,22}), its global use is less clear; but its status is such that it has been elevated to a paradigm: \textit{the dynamics of the inhomogeneous, real Universe can be described globally, from the largest scales down to the scales

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\textsuperscript{1}This limit obtains the standard Eulerian formulation of the equations of Newtonian cosmology. An alternative Lagrangian formulation of these equations is presented in Ref. \cite{10}. In the Lagrangian representation of the equations of relativistic cosmology the natural Newtonian limit is obtained in the matter–comoving frame; cf. Sec. 4.2.1 of Ref. \cite{4}.
where spatial curvature effects become significant, by a single quasi–Newtonian metric form. This is most lively reflected in articles that deal with the possible impact of inhomogeneities on expansion properties of the Universe (the so–called “backreaction problem”), a topic that — especially recently — is often discussed in the quasi–Newtonian setting (see, e.g., Refs. [16, 17, 26, 18, 28] and many others).

In the longitudinal gauge, fixed in relation with a $3 + 1$ decomposition of the cosmological spacetime manifold according to Arnowitt, Deser and Misner [1], the lapse function and the spatial metric are specified so as to provide a “perturbed Newtonian setting”. The metric form for the physical spacetime is set to be

$$h_g = -N^2(t, x^k) \, dt \otimes dt + g_{ij}(t, x^k) \, dx^i \otimes dx^j,$$

where the lapse function $N$ and the spatial metric coefficients $g_{ij}$ of a family of spacelike 3–surfaces $\{t = \text{constant}\}$ orthogonal to an irrotational and shearfree timelike reference congruence $n = N^{-1} \partial_t$ are given by

$$N^2 = \ell_0^2 a^2(t) [1 + 2\Phi(t, x^k)] , \quad g_{ij} = \ell_0^2 a^2(t) [1 - 2\Psi(t, x^k)] \, \gamma_{ij},$$

implying the vanishing of each of the spatial Cotton–York tensor, $C_{ij}(g) = 0$, and the magnetic Weyl curvature, $H_{ab}(n) = 0$, and so ensuring the strict absence of gravitational radiation (cf. Ref. [15]). Here, $a(t)$ denotes the dimensionless scale factor of a spatially homogeneous and isotropic solution of Einstein’s field equations, contained in the metric form (2) for $0 = \Phi = \Psi$, and $\gamma_{ij}$ is a spatial metric of constant curvature, i.e., $R(\gamma) = \text{constant}$. Backreaction effects are not taken into account. Frequently, the simplifying choice

$$\gamma_{ij} = \delta_{ij}$$

(so that $R(\gamma) = 0$ holds true) is made. The constant $\ell_0$ represents the unit of the physical dimension [length], $t$ is the dimensionless (conformal) local coordinate time, and the dimensionless $x^i$ (being local coordinates in the tangent spaces at any spatial position in the 3–surfaces in standard general relativity) are here considered as “background coordinates”, i.e., the inhomogeneous metric perturbations encoded in the functions $\Phi$ and $\Psi$ are, in the framework of gauge–invariant cosmological perturbation theory, considered as functions of globally defined coordinates. In this framework, $\Phi$ and $\Psi$ correspond to Bardeen’s gauge–invariant potentials for scalar metric perturbations [2]. For a perfect fluid energy–momentum tensor, upon neglecting terms quadratic in peculiar velocities, they can be set equal to each other:

$$\Phi = \Psi$$

(cf. Ref. [25, p 223]).

The above metric setting comes with a list of restrictions that have to be imposed on the perturbation function $\Psi(t, x^i)$; in the standard literature, this list comprises the following conditions:

$$|\Psi| \ll 1 , \quad \left| \frac{\partial \Psi}{\partial t} \right|^2 \ll \frac{1}{a^2} \gamma^{ij} D_i \Psi D_j \Psi , \quad (\gamma^{ij} D_i \Psi D_j \Psi)^2 \ll \gamma^{ik} \gamma^{jl} D_i D_j \Psi D_k D_l \Psi ;$$

the operator $D_i$ denotes the covariant derivative associated with the constant curvature spatial metric $\gamma_{ij}$. Note that the last inequality, which is a necessary condition for perturbations in the spatial metric to be small, only compares the sizes of two specific spatial curvature terms; it does not say that spatial curvature has to be small per se.

Among the many careful papers that do include the above list, we select the recent paper by Ishibashi and Wald [20]. On p 238 of their work these authors assert that “the metric (1)–(3) appears to very accurately describe our Universe on all scales, except in the immediate vicinity of black holes and neutron stars”, and they continue: “The basis for this assertion is simply that the FLRW metric appears to provide a very accurate description of all phenomena observed on large scales, whereas Newtonian gravity appears to provide an accurate description of all phenomena observed on small scales.”
They also sketch a typical cosmologically relevant energy–momentum tensor that assumes the form of a homogeneous perfect fluid for “homogeneously distributed matter” and an inhomogeneous continuum of “dust” that, this latter, could be approximated as

\[ T^{(m)} = \rho(t, x^i) \, dt \otimes dt \, , \]  

on the assumption of small (non–relativistic) peculiar velocities. In general, in cosmological perturbation theory, peculiar velocities are being taken into account, e.g., to first order of smallness.

We take this paper as an example of the aforementioned paradigm. We do not fully enter the issue raised in it related to the recent discussion on whether backreaction effects may account for the dark energy problem of the standard model of cosmology or not (we refer the reader to Ref. [23], and especially to the recent paper [24], where it is also commented on the applicability of the quasi–Newtonian metric form, and the review papers [13, 29, 4]).

However, let us add a remark that shows that it is, from a fully relativistic point of view, implausible that the physical spacetime metric in the form of Eqs. (1)–(3) can be carried to the point of describing a realistic model of the Universe, if we employ this metric as an approximate solution to Einstein’s field equations. While it is true that this metric can provide a good local description of either a perturbed expanding Universe or a quasi–static domain of local matter condensations, it is not at all clear to what extent it can represent both simultaneously. The specific issue is: “How large a domain in space and time can be covered by such a ‘global’ coordinate system in a realistic model representing both the dynamical expanding Universe and imbedded local large–scale voids?” This is a crucial issue in cosmology.

The strong conclusions that are advanced, e.g., in the paper by Ishibashi and Wald [20], can, of course, be tested on the grounds of a realistic evaluation of this metric ansatz as an approximate solution to the field equations of general relativity, irrespective of a particular framework from which this metric ansatz has been derived. We point out here that this metric form and the accompanying list of restrictions does not contain any condition that forces us to choose a particular 4–velocity field, say, \( u \) of a fluid continuum evolving in this spacetime, except it is implicit that peculiar velocities will be small. Potential danger is associated with the form of the energy–momentum tensor (6) for an inhomogeneous dust continuum, taken from Ref. [20, Eq. (4)]: if, in the chosen time slicing of the cosmological spacetime manifold, we take this form of the energy–momentum tensor literally, i.e., ignore the approximation sign which is supposed to imply quantitatively negligible peculiar velocities, then there is strictly no relation of the quasi–Newtonian metric form to inhomogeneities (cf. Ref. [15, p 3566]) and, e.g., questions such as the backreaction of inhomogeneities cannot even be addressed. We would simply be looking at a spatially homogeneous solution in an odd coordinate system, which makes the metric “look” spatially inhomogeneous in terms of the perturbation function \( \Psi(t, x^i) \).

Indeed, if we choose a fluid 4–velocity field \( u \) normal to the spacelike 3–surfaces defined by this metric form (which is equivalent to setting peculiar velocities exactly to zero), then this implies a shear–free fluid motion (see, e.g., Ref. [15]). It then follows that the fluid in this spacetime must be spatially homogeneous “in most cases”. More precisely, if the energy–momentum tensor only represents a dust matter source, then shearfree solutions always describe a spatially homogeneous continuum; see Theorem 1 and Corollary 1 on p 1210 in the paper by Collins and Wainwright [6]. In the case of perfect fluid sources, there are some inhomogeneous solutions that are, however, of no obvious cosmological relevance; see, e.g., Refs. [6, 7, 32]. Already this remark makes clear that a thoughtless application of the quasi–Newtonian metric form can quickly run into trouble: when the approximation made ignores the peculiar velocity terms in the field equations, it is in danger of running into effects related to these restrictions applying to the corresponding exact solutions.

A warning against the assumption that a Newtonian limit of this kind is without problems in the cosmological context is the following: it is an exact theorem that shearfree dust solutions of Einstein’s field equations cannot both expand and rotate, i.e.,

\[ \sigma = 0 \, , \quad p = 0 \quad \Rightarrow \quad \theta \omega = 0 \, ; \]  

(7)
see Ref. [11]. However, shearfree solutions of the corresponding Newtonian equations do exist where this is not true: they can both expand and rotate; cf. Ref. [27]. Consequently, the Newtonian limit is singular. Consider a sequence \( \text{GRT}(i)_{\sigma=0} \) of relativistic shearfree dust solutions with a limiting solution \( \text{GRT}(0)_{\sigma=0} \) that constitutes the Newtonian limit of this sequence. The latter solution will necessarily satisfy Eqs. (7) because every solution \( \text{GRT}(i)_{\sigma=0} \) in the sequence does so. The corresponding exact Newtonian solution \( \text{NGT}(0)_{\sigma=0} \) will therefore also necessarily satisfy Eqs. (7). But the Newtonian solutions \( \text{NGT}(j)_{\sigma=0} \) that do not satisfy Eqs. (7) are clearly not obtainable as limits of any sequence of relativistic solutions \( \text{GRT}(j)_{\sigma=0} \). Assuming Einstein’s field equations represent the genuine theory of gravitational interactions in the physical Universe, with solutions of the Newtonian equations an acceptable approximation to relativistic solutions under suitable circumstances, this result tells us that not all Newtonian solutions are indeed such acceptable approximations.

2 Notes of Jürgen Ehlers’ remarks

We wish to emphasise that the subsequent notes shall not be understood as a “photographic reproduction” of Jürgen Ehlers’ original blackboard writings, given during a talk on Thu, July 26, 2007 when two of us (TB and HvE) were present, but rather that the essence of his remarks is being truthfully reproduced and summarised.

The starting point of Jürgen’s considerations for gravitating physical systems was the central assumption that Einstein’s general theory of relativity constitutes the “correct” theory of gravitational interactions on the scales of the solar system, stars, neutron stars and black holes, on which this theory has been reliably tested to remarkable accuracy. It is our recollection that Jürgen was very cautious here and specifically related his comments to tested scales only. However, at the end of his talk he pointed out that an extrapolation to cosmological scales of the matter he had raised was conceivable and so could be a natural related investigation. The intention of the argument he gave was to illustrate the fact that, even if metrical perturbations for a gravitating system of the above mentioned scales are small in magnitude, the derivatives of such perturbations, the second–order ones in particular, can be physically significant.

2.1 Metric level

We henceforth consider a domain \( \mathcal{D} \) of a given spacetime manifold \( \mathcal{M} \). On \( \mathcal{D} \) we decompose the physical spacetime metric \( ^4g \) into a leading–order term \( ^0g \) and a term of small relative deviations \( h \), the latter referred to as perturbations in \( ^4g \):

\[
^4g = ^0g + h
\]

\[
\approx \mathcal{O}(1) + \mathcal{O}(\varepsilon) ;
\]

\( h \) is assumed to be of first order in an appropriate dimensionless smallness parameter \( \varepsilon \) with \( |\varepsilon| \ll 1 \).\(^2\)

With appreciable changes experienced in \( ^0g \) we associate a macrosopic characteristic spacetime scale \( D \), while, analogously, with appreciable changes in \( h \) we associate a microsopic characteristic spacetime scale \( d \). The scale ratio

\[
\frac{D}{d}
\]

thus constitutes a dimensionless physical quantity of special interest for order–of–magnitude estimates in respect to leading–order physical effects in two–scale gravitating systems of the kind outlined.

\(^2\)For reasons of notational ease we will subsequently drop the superscript “4” from \( ^4g \).
2.2 Connection level

At the level of the spacetime connection the decomposition of Eq. (8) suggests that
\[
\Gamma(g) = g^{-1} \partial g \\
\approx (g^{-1} + h)(\partial g + \partial h) \\
\approx (1 + \varepsilon) \frac{1}{D} \left(1 + \varepsilon \frac{D}{d}\right) \\
\approx \frac{1}{D} \left(1 + \varepsilon \frac{D}{d} + \ldots\right) \approx \Gamma + \frac{1}{\Gamma} .
\] (10)

2.3 Curvature level

Einstein’s field equations of gravitational interactions are formulated at the level of spacetime curvature. It is at this level that we notice/observe the characteristic dynamical features of relativistic gravitational physics. For the spacetime curvature we obtain
\[
R(g) = \partial \Gamma + \Gamma \partial = g^{-1} \partial^2 g + (g^{-1} \partial g)^2 \\
\approx (1 + \varepsilon) \frac{1}{D^2} \left[1 + \varepsilon \left(\frac{D}{d}\right)^2\right] + (1 + \varepsilon)^2 \frac{1}{D^2} \left(1 + \varepsilon \frac{D}{d}\right)^2 \\
\approx \frac{1}{D^2} \left[1 + \varepsilon \left(\frac{D}{d}\right)^2 + \ldots\right] \approx \frac{0}{\Gamma} + \frac{1}{\Gamma} .
\] (11)

We note that, even when deviations in the metrical amplitude itself are small, resultant deviations in the spacetime curvature may become significant, depending on the specific value of the scale ratio \(D/d\). A significant influence on the spacetime curvature will therefore arise provided that
\[
\varepsilon \left(\frac{D}{d}\right)^2 \approx \mathcal{O}(1) ,
\] (12)
or, in a \(\log(D/d)–\log(\varepsilon)\) representation, whenever
\[
\log \left(\frac{D}{d}\right) \approx -\frac{1}{2} \log(\varepsilon) .
\] (13)

We conclude that the application of a strictly Newtonian or quasi–Newtonian description of gravitational interactions to gravitating systems with two characteristic spacetime scales becomes justified when for this system at least the two constraints (i) \(\varepsilon \ll 1\) and (ii) \(\varepsilon(D/d)^2 \ll 1\) are simultaneously satisfied. Order–of–magnitude considerations of this kind are relevant for all gravitating systems for which characteristic spacetime scales \(D\) and \(d\) can be identified.

Jürgen, confining himself to spatial considerations within the scheme outlined above, presented a simple numerical example for the solar system, where general relativity is well established. He chose \(\varepsilon \approx 4.23 \times 10^{-6}\) for the magnitude of deviations in the spatial metric from a Euclidian geometry, \(D \approx 150 \times 10^6\) km for the radius of the Earth’s orbit around the Sun, and \(d \approx 6.95 \times 10^5\) km for the radius of the Sun to find \(\varepsilon(D/d)^2 \approx 0.20\). Jürgen remarked that analogous order–of–magnitude estimates are helpful indicators for the relevance of curvature effects, too, in gravitating systems of the scales of galaxies, clusters of galaxies, and the entire Universe itself. Taking the liberty to extrapolate the validity of general relativity (unmodified) all the way up to the scales of the observable Universe (and thus ignoring the subtleties and intricacies associated with an averaging approach to cosmology which would be more appropriate here; cf. Refs. [12], [4] or [33]), we now turn to following Jürgen’s suggestion and work out the numerical details for the astrophysical and cosmological systems that he had mentioned.
3 Numerical examples

We will adapt Jürgen’s argument to spatial geometries and estimate the order–of–magnitude of perturbations in the spatial metric and the spatial curvature in the context of structures observed in the matter distribution at different scales up to the scale of the observable Universe (see also Refs. [13] and [5]); the required reduction of 4–D considerations to 3–D considerations can be obtained in terms of a “thin sandwich approach” as employed, e.g., in the appendix of Ref. [3]. To this end we project typical space–time length scales for respective astrophysical and cosmological gravitating systems into the present–day spacelike 3–surface $S_0 : \{t = t_0\}$ of observers located on Earth; for these observers (which are in relative motion to the cosmological Hubble flow) the length scales we thus consider constitute instantaneous proper spatial distances (see Refs. [9] or [14] for technical details). Here we choose the macroscopic scale $D$ to represent the size of the gravitating system, which we take to be its physical diameter, while we choose the microscopic scale $d$ to represent the smoothing scale, i.e., the scale below which we neglect the influence of inhomogeneities in the matter distribution and the geometry. In general $d$ will be the physical diameter of a gravitating substructure of the system in question, with $D/d \gg 1$. The working hypothesis of our investigations shall be the frequently encountered assumption that overall the geometry of space within gravitating systems at astrophysical and cosmological scales is flat and that deviations from the Euclidian geometry can be modelled in terms of small perturbations. We emphasise that the idea is to see what realistic order–of–magnitude estimates can be obtained for the contribution of representative structures in the distribution of matter to the curvature of space at different scales.

For a specific two–scale gravitating system, we will evaluate an order–of–magnitude estimate for the following two characteristic dimensionless parameters:

(i) the ratio between the Schwarzschild and the proper physical radii of the system, i.e.,

\[ \varepsilon := \frac{4GM}{c^2D}, \]

(note that in the cosmological context, where $M$ typically scales as $D^3$, cf. Sec. 12.1 of Ref. [30], the parameter $\varepsilon$ scales as $D^2$), and

(ii) the spatial curvature perturbation

\[ \varepsilon \left(\frac{D}{d}\right)^2. \]

Besides the Earth’s orbit around the Sun (A1), we select as further representative gravitating systems at astrophysical scales a typical galaxy (A2) and a typical cluster of galaxies (A3). A galaxy we assume to contain 100 billion solar mass stars within a spatial domain of diameter 100000 ly, while we model a cluster of galaxies as containing 1000 galaxies (of 100 billion solar mass stars each) within a spatial domain of diameter 5 Mpc.\(^3\) In both of these cases, we are deliberately confining ourselves to considerations of luminous (baryonic) matter only. The effects of a considerable factor (possibly 10, or larger still) due to additional components of non–luminous matter in the mass content $M$ on the parameters $\varepsilon$ and $\varepsilon(D/d)^2$ can be easily traced.

For the examples C1 to C3 of gravitating systems at cosmological scales, we obtain the information on the mass content $M$ of a sphere of radius $D/2$ in Euclidian space from the value

\[ \rho_m \approx 2.58 \times 10^{-27} \text{ kg m}^{-3} \]

for an average (baryonic and dark matter) mass density, which derives from the WMAP five–year data results $\Omega_m h^2 \approx 0.14$ and $h \approx 0.71$ (so $\Omega_m \approx 0.27$) taken from the work by Hinshaw et al [19].

\(^3\)A nice starting point for obtaining realistic values for the masses $M$ and diameters $D$ and $d$ is the convenient online encyclopedia en.wikipedia.org. See also the comprehensive summary by Cox [8].
cosmic void example C1 we assume that the underdensity is balanced by a corresponding overdensity associated with the enveloping wall structure so that the value of $\rho_m$ in Eq. (16) can be employed without modification. Conservative values for the scales $D$ and $d$ were selected for examples C1 and C2, which represent typical cases of large–scale structures. We consider the particular choice of $300h^{-1}\text{Mpc}$ for the scale of statistical homogeneity in the observable Universe as a conservative lower limit. By definition there are no matter structures beyond this scale. Therefore, in the example C3 of the present–day Hubble sphere we only evaluate the parameter $\varepsilon$. It turns out that with $D = 2r_{H_0} = 2c/H_0$ and $M$ computed from $\rho_m$ of Eq. (16), in this case $\varepsilon$ becomes identical to the matter density parameter $\Omega_m$. We ask the reader to choose their own set of realistic values for $M$, $D$ and $d$ in order to obtain further estimates for $\varepsilon$ and $\varepsilon(D/d)^2$.

| Gravitating system / Smoothing scale | Mass $M$ | Diameters $D$ and $d$ | $D/d$ | $\varepsilon$ | $\varepsilon(D/d)^2$ |
|-----------------------------------|---------|----------------------|-------|--------------|---------------------|
| A1: Earth’s orbit / Sun           | $\approx M_\odot$ (1.99 x 10^{30} kg) | $300 \times 10^9$ km $1.39 \times 10^6$ km | 216 | $4.24 \times 10^{-6}$ | 0.20 |
| A2: Galaxy / Open star cluster    | $\approx 10^{11} M_\odot$ (1.99 x 10^{41} kg) | 100000 ly $30$ ly | 3333 | $6.23 \times 10^{-7}$ | 6.92 |
| A3: Cluster of galaxies / Galaxy  | $\approx 10^{14} M_\odot$ (1.99 x 10^{44} kg) | 5 Mpc $0.03$ Mpc | 167 | $3.82 \times 10^{-6}$ | 0.11 |
| C1: Void / Wall                  | $\approx (1/6)\pi\rho_m D^3$ (2.98 x 10^{45} kg) | $30h^{-1}$ Mpc $3h^{-1}$ Mpc | 10 | $6.78 \times 10^{-6}$ | $6.78 \times 10^{-4}$ |
| C2: Homogeneity scale / Supercluster | $\approx (1/6)\pi\rho_m D^3$ (2.98 x 10^{48} kg) | $300h^{-1}$ Mpc $30h^{-1}$ Mpc | 10 | $6.78 \times 10^{-4}$ | $6.78 \times 10^{-2}$ |
| C3: Hubble sphere / —            | $\approx (1/6)\pi\rho_m D^3$ (2.38 x 10^{52} kg) | — | — | 0.27 | — |

Table 1: Order–of–magnitude estimates for spatial metric and spatial curvature perturbations in six representative cases of astrophysical and cosmological gravitating systems with two characteristic length scales. For the systems A2 and A3, we are considering luminous (baryonic) masses only. The masses in examples C1 to C3 were computed from $\rho_m$ (baryonic and dark matter) given in Eq. (16) on the assumption that each of them is contained within a sphere of radius $D/2$ in flat space. The single relevant parameter to characterise the system C3 is the ratio between its Schwarzschild radius and its physical radius, $\varepsilon$. In this case it does correspond to the matter density parameter $\Omega_m$. The reader is invited to play with her/his own set of numbers for $M$, $D$ and $d$ to obtain further estimates.

The results of our order–of–magnitude estimates are summarised in Table 1. It is immediately evident that the two constraints $\varepsilon \ll 1$ and $\varepsilon(D/d)^2 \ll 1$, which would justify the application of a strictly Newtonian or quasi–Newtonian discription of gravitational interactions, appear to be simultaneously satisfied only for the cosmological examples C1 and C2.

However, the cosmic void example C1, which we included for illustrative purposes, does not really constitute a playground for perturbation theory in Euclidian space. In this context (negative) spatial curvature is a zeroth–order effect. In general relativity, an effective negative spatial curvature is implied for a void–dominated cosmological model, i.e., a non–flat average spatial curvature distribution, while the examples above assume a flat average spatial curvature distribution. The order of magnitude of spatial curvature is best estimated from the Gauß (Hamiltonian, or energy) constraint amongst Einstein’s field

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The determination of this scale depends on the statistical measure of inhomogeneity employed. Often a lower value is quoted that is, however, based on weak statistical measures like the two–point correlation function. Inhomogeneities mirrored by morphological differences show up in higher–order correlations of the distribution that can be captured by calulating the Minkowski Functionals; see, e.g., Kerscher et al [21].
equations (see, e.g., Ref. [5]), which shows that the physical contribution by spatial curvature can be compensated only by a cosmological constant, and this on a single chosen scale only.

In the homogeneity scale example C2, the order of magnitude of the spatial curvature perturbation $\varepsilon (D/d)^2$ we find is close to 10%. Less conservative values for the scale ratio $D/d$ (with $D$ fixed) would lead to larger values still, indicating that also in this context perturbation theory as the sole basis of a dynamical description of structure formation does not appear to be without its problems of consistency.

Most striking is the violation of the second (curvature) constraint in the astrophysical systems A2 and A3, bearing in mind in these cases our restriction to luminous baryonic matter, and a further, potentially large factor in the mass content $M$. For such systems, usually, the notion of dark matter is being invoked in order to account for quantitative deviations from the Newtonian expectations.

We think that these order–of–magnitude estimates provide a strong call for a proper relativistic treatment of the underlying gravitational physics in these systems; spatial curvature is an inherently relativistic phenomenon, unknown to the Newtonian theory. The claim on the validity of a quasi–Newtonian metric according to Eqs. (1)–(5) to describe gravitational physics on all scales in the observable Universe, apart from black holes and neutron stars, is thus seriously called into question.

4 Significance of spatial curvature in cosmology

We are now going to address an order–of–magnitude estimate for the spatial curvature in spatially inhomogeneous cosmological models in the context of a quasi–Newtonian description. However, we will not look at spatial averages including backreaction effects (the reader may find such estimates in Ref. [5]), but rather evaluate the spatial Ricci curvature scalar directly from the quasi–Newtonian spatial metric ansatz of Eq. (2), with $\gamma_{ij}$ a spatial metric of constant curvature, i.e., $R(\gamma) = \text{constant}$. The Christoffel connection symbols for the quasi–Newtonian spatial metric are given by

$$\Gamma^j_{ik}(g) = \Gamma^j_{ik}(\gamma) - \frac{1}{1 - 2\Psi} \left[ \partial_i \Psi \delta^j_k + \partial_k \Psi \delta^j_i - \partial_t \Psi \gamma^{jl} \gamma_{lk} \right],$$

(17)

while the resultant spatial Ricci curvature scalar is

$$R(g) = \frac{1}{\ell_0 a^2(t)(1 - 2\Psi)} \left[ R(\gamma) + \frac{4\gamma^{ij} D_i D_j \Psi}{1 - 2\Psi} + \frac{6\gamma^{ij} D_i \Psi D_j \Psi}{(1 - 2\Psi)^2} \right].$$

(18)

It is standard to introduce the normalisation $R(\gamma) = 6k$, where $k \in \{-1, 0, +1\}$. Employing the correspondences

$$\ell_0 a(t) \leftrightarrow D, \quad \Psi \leftrightarrow \varepsilon, \quad \text{and} \quad D_i \leftrightarrow (D/d),$$

(19)

we find that

$$R(g) \approx \frac{1}{D^2} \left[ O(k) + O \left( \varepsilon \frac{D^2}{d^2} \right) + O \left( \varepsilon^2 \frac{D^2}{d^2} \right) \right],$$

(20)

or, to first order in $\varepsilon < 1$,

$$R(g) \approx \frac{1}{D^2} \left[ k + \varepsilon \left( \frac{D}{d} \right)^2 + \ldots \right].$$

(21)

this result is perfectly in line with Jürgen’s estimate of Eq. (11).

Our next step is to compare the sizes of the perturbation terms in the quasi–Newtonian spatial Ricci curvature scalar (18), the spatial Laplacian (relative to $\gamma_{ij}$) of the perturbation function $\Psi$ and the squared

As pointed out in Section 1, neglecting in the quasi–Newtonian framework peculiar velocities altogether leads to a hypersurface homogeneous solution of Einstein’s field equations, where in the chosen time slicing the perturbation function $\Psi(t, x')$ is spatially constant so that all spatial gradients of this function vanish. By a suitable reparametrisation of the time coordinate $t$, the function $\Psi(t, x')$ can then be set equal to zero for any arbitrary time interval.
spatial gradient of $\Psi$, with each other on the basis of our estimates for the cosmological examples C1 and C2 displayed in Table 1. We recall that for reasons of simplicity our investigations in Section 3 were grounded on the standard assumption that the geometry of space is Euclidian and perturbation theory can be employed to accurately model deviations thereof. For cases C1 and C2 we now turn to consider the restrictions of Eq. (5), which were imposed on spatial metric and spatial curvature perturbations in Eq. (2) of Ref. [20]. With the correspondences of Eq. (19), we need to check whether the inequalities

$$|\Psi| \ll 1, \quad |\gamma^{ij} D_i \Psi D_j \Psi|^2 \ll |\gamma^{ij} D_i D_j \Psi|^2 \quad \Leftrightarrow \quad |\varepsilon| \ll 1, \quad \varepsilon^4 (D/d)^4 \ll \varepsilon^2 (D/d)^4 \quad (22)$$

hold. Our results for this consideration are displayed in Table 2.

| Gravitating system / Smoothing scale | $\varepsilon$ | $\varepsilon^4 (D/d)^4$ | $\varepsilon^2 (D/d)^4$ |
|-------------------------------------|----------------|----------------|----------------|
| C1: Void / Wall                     | $6.78 \times 10^{-6}$ | $2.11 \times 10^{-17}$ | $4.60 \times 10^{-7}$ |
| C2: Homogeneity scale / Supercluster| $6.78 \times 10^{-4}$ | $2.11 \times 10^{-9}$ | $4.60 \times 10^{-3}$ |

Table 2: Comparison of order–of–magnitude estimates for spatial metric perturbations $|\Psi|$ and squared spatial curvature perturbations $|\gamma^{ij} D_i \Psi D_j \Psi|^2$ and $|\gamma^{ij} D_i D_j \Psi|^2$ for the cosmological cases C1 and C2 considered in Table 1.

We find that the quasi–Newtonian restrictions according to Eq. (22) are satisfied in both cases. However, as we argued in Section 3, an exclusively perturbative approach to modelling the dynamics of structure formation processes for cosmological gravitating systems like C1 and C2 is questionable: a large–scale cosmic void can never be considered just a perturbation of a flat background space, while for the system C2 the spatial curvature effect of the matter structure $\varepsilon (D/d)^2$ can easily surpass the 10% level (cf. our respective estimate of Section 3) for scale ratios $D/d$ (with $D$ fixed) only slightly larger than 10. Moreover, as soon as spatial curvature becomes dynamically significant in the cosmological context (and, of course, at smaller scales), it is a generic feature that the local spatial coordinate system used to describe the dynamics will inevitably break down at a finite proper distance from the origin. In consequence, a proper (effective or average) relativistic treatment of the underlying gravitational interactions appears to be the appropriate one, implying in particular that the validity of a quasi–Newtonian spacetime metric, with its associated globally defined coordinate system, at most scales in the observable Universe is very doubtful indeed. In contrast to the quasi–Newtonian treatments, a suitably averaged model can be considered as a “background”, and coordinates may be introduced at this “background”; but here after effectively smoothing out local coordinate singularities that may appear on small scales. We emphasise that local Ricci curvature singularities appearing in spatially inhomogeneous models are often the result of an oversimplified matter model such as a hydrodynamical description of matter, and are not necessarily artifacts of a certain temporal gauge choice. It ought to be a natural aspiration of “precision cosmology” to be aware of and take into account all of these matters.

Lastly, we emphasise that our geometrical order–of–magnitude estimates for present–day spatial curvature effects remain essentially unchanged when employing a temporal gauge alternative to the Eulerian viewpoint of the longitudinal gauge — the simultaneously synchronous and Hubble–flow–comoving (or matter–comoving) temporal gauge for an irrotational dust fluid source with 4–velocity field $u$, which is also frequently employed in the description of structure formation in the late Universe.

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