COVERS OF D-TYPE ARTIN GROUPS

MEIRAV AMRAM\(^1\), ROBERT SHWARTZ\(^1\) AND MINA TEICHER

ABSTRACT. We study certain quotients of Generalized Artin groups which have natural map onto D-type Artin groups. In particular the Generalized Artin group \(A(T)\) defined by a signed graph \(T\). Then we find a certain quotient \(G(T)\) according to the graph \(T\), which have a natural map onto \(A(D_n)\) too. We prove that \(G(T)\) is isomorphic to a semidirect product of a group \(K\), with the Artin group \(A(D_n)\), where \(K\) depends only on the number of cycles and on the number of edges of the graph \(T\).

1. Introduction

Coxeter and Artin group are used in many domains in mathematics, such as dealing with reflections, symmetries, Lie Algebras classifications of finite simple groups, computations in algebraic geometry and in many other aspects.

The structures of Coxeter and Artin groups are very interesting since these groups are defined in a very easy way, in terms of generators and relations. These groups can be described easily by diagrams which are called Dynkin diagrams, and the groups has interesting properties in terms of group theory, like the cancelation property (see [4], [7]).

The goal of this paper is to find a structure for certain quotients of Artin groups which has natural map onto the finite type simply laced Artin group \(A(D_n)\), such that through that structure the word problem is soluble in that certain quotient. The terminate goal of the series of papers which we describe, is to find general structures for certain quotients of Coxeter and Artin groups, in a way that it will be easy to solve the word problem through natural maps onto Coxeter or Artin groups.

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The motivation of this paper comes from algebraic geometry. We take a projective surface $X$, with a generic map of degree $n$ to $\mathbb{CP}^2$, and $S$ is its branch curve in $\mathbb{CP}^2$. There is a natural map from the fundamental group $\pi_1(\mathbb{CP}^2 - S)$ to Artin’s braid group $Br_n$. The kernel of $\pi_1(\mathbb{CP}^2 - S) \to Br_n$ is a quotient of a map which was described in [3] in details. There is generalization of some important properties of Artin’s Braid groups to D-type, and other finite type Artin groups in [1]. Hence it is interesting to look on the structure of kernels of maps onto finite type Artin groups too as generalization of the kernel $\pi_1(\mathbb{CP}^2 - S) \to Br_n$.

This paper is the fourth in a series of papers which study certain quotients of Coxeter or Artin groups, which are defined by a graph $T$. In this section we shall describe the main results of those papers and the configuration of the current paper.

The first paper in the series, ”Coxeter covers of the symmetric group” [8], describes the structure of a certain quotient of a Coxeter group $C(T)$, which is defined by a graph $T$ and has a natural map onto the symmetric group $S_n$. The main theorem of [8] is that there exists a certain quotient $C_Y(T)$ of $C(T)$. This $C_Y(T)$ is isomorphic to $A_{t,n} \rtimes S_n$, where $A_{t,n}$ is a well defined group whose only invariants are $t$ (the number of cycles of $T$) and $n$ (the number of vertices of $T$). Since the word problem is solvable in $A_{t,n}$, it is also solvable in $C_Y(T)$.

The second paper ”Coxeter covers of the classical Coxeter groups” [2] generalizes the results of [8]. This paper deals with a wider class of Coxeter groups that we can map onto $B_n$ or $D_n$ (the classical Coxeter groups). In [2], the graph $T$ is generalized to a signed graph in which every edge is labeled either by $+1$ or by $-1$, and which may include loops. Similar signed graphs were introduced in [5], ”Signed graphs, root lattices and Coxeter groups”. The main theorem of [2] is as follows: there is a ceratin quotient $C_Y(T)$ of $C(T)$, which is isomorphic to $A_{t,n} \rtimes D_n$ or $A_{t,n} \rtimes B_n$, depending whether $T$ contains loops, or not.

The third paper [3], ”Artin covers of the braid group”, generalizes Coxeter covers to Artin covers. The graph $T$ defines an Artin group $A(T)$ (which means that the generators are not necessarily involutions). The main theorem of [3] is that there exists a quotient $G(T)$ of $A(T)$ that is isomorphic to $K_{t,n} \rtimes Br_n$, where $t$ is the number of cycles in $T$ and $n$ is the number of vertices of $T$. Since the word problem is solvable in $K_{t,n}$, it is solvable in $G(T)$ as well.
Our paper contains a combined generalization of [2] and [3] to Artin groups that have natural maps onto the simply laced finite type Artin group $A(D_n)$. According to [6], $A(D_n) \simeq F^{n-1} \rtimes Br_n$ and $A(B_n) \simeq F^n \rtimes Br_n$. In Coxeter groups, it is known that $D_n \leq B_n$, but $A(D_n)$ is not embeddable into $A(B_n)$. Therefore it is impossible to combine Artin covers $A(B_n)$ and $A(D_n)$ as done in [2]. Hence, in this paper, we deal only with covers of $A(D_n)$.

This paper is in the spirit of the paper "Artin covers of the braid groups" ([3]) and we assume the definitions, theorems and results given there, although this paper is a generalization of [3] and deals with a wider class of groups, e.g., groups whose Dynkin diagrams contain subgraphs of the form

(e.g., $A(D_n))$.

The paper is divided as follows. In Section 2, we define the group that we obtain from the signed graph $T$. Then we define $A_Y(T)$ as a quotient of $A(T)$, where the configuration of the subgraphs from which arise the relations of $A_Y(T)$ are the same as in [3]. We recall the basic properties of $A_Y(T)$. In Section 3 we consider a graph $T$ that contains only one cycle and only one anti-cycle. Then, following [3], we define $G(T)$ as a quotient of $A_Y(T)$. Using the properties of $A_Y(T)$, we prove that $G(T) \simeq K \rtimes A(D_n)$, where the structure of $K$ is very similar to the structure defined in [3][Section 6]. In Section 4 we prove that every signed graph $T$ is equivalent to a signed graph $T'(m)$ of the form

Then we prove that $G(T) \simeq K_{m,n} \rtimes A(D_n)$, where $n$ is the number of vertices of $T$ and $m + 1$ is the number of cycles and anti-cycles in $T$ ($T$ must include at least one anti-cycle). Our result is a generalization of the main theorem in [3].
Although the structure of $G(T)$ is very similar to the structure of $G(T)$ in [3], using signed graphs and mapping onto $A(D_n)$ instead of mapping onto $Br_n$ allows us to deal with a much wider class of simply laced Artin groups than those in [3].

2. The quotient $A_Y(T)$

Definition 1. We call a weighted graph $T$ "a signed graph" if every edge of $T$ contained in a cycle is signed by $+1$ or by $-1$.

For example,

![Figure 2](image)

Here the edges that are not contained in any cycle are not signed, and we may assume that the signs of all such edges are $+1$.

We denote by $s(e)$ the sign of the edge $e$. Let $A(T)$ be the generalized Artin group that corresponds to the graph $T$ (i.e., $A(T)$ is generated by the edges of $T$). In [3], each edge in a graph is considered as a positive signed edge. In this paper, there is a generalization to signed graphs, where each edge can be signed by $+$ or $-$. Therefore, the relations in $A(T)$ in this paper are:

- $u_1 < u_1, u_2 > = 1$ if $u_1$ and $u_2$ meet in a vertex
- $[u_1, u_2] = 1$ if $u_1$ and $u_2$ are disjoint
- There is no relation between $u_1$ and $u_2$ if $u_1$ and $u_2$ connect the same two vertices and $s(u_1) = s(u_2)$
In the case of a cycle with odd number of negative signs, we have an additional relation: \([u_1^{-1} \ldots u_{n-2}^{-1}u_{n-1}u_{n-2} \ldots u_1, u_n] = 1\). We call such a cycle an anti-cycle, (see [2]).

Note that an anti-cycle of length two has the form \(-u_1 u_2\), where \(u_1\) and \(u_2\) are two edges that connect the same two vertices but are signed differently. Hence, the induced relation is \([u_1, u_2] = 1\).

**Remark 2.** The graph \( \\
\) represents the finite type Artin group \( A(D_n) \).

**Definition 3.** Let \( T \) be a planar graph, \( A_Y(T) \) is the quotient of \( A(T) \) by the following relations (similar to the relations in [3] with an additional case):

1. \([w^{-1}uw, v] = 1\) if \( u, v, w \) as in \( u \)

2. \(< w^{-1}uw, v > = 1\) if \( u, v, w \) as in \( u \)

3. \([w^{-1}uw, v^{-1}w] = 1\) if \( u, v, w, x \) as in \( u \)

4. \(< w^{-1}uw, v^{-1}w > = 1\) if \( u, v, w, x \) as in \( u \)

Now we define virtual edges.
Definition 4. Let $x$ and $y$ be paths in a signed graph $T$, such that $x$ and $y$ intersect in no more than one point. Then we define a path $x \cdot y$, called a virtual edge, as:

1. $x \cdot y = y$ if $x$ and $y$ do not intersect;
2. if $x$ and $y$ intersect in one vertex.

The sign of $x \cdot y$ is $+1$ in the case when $x$ and $y$ have the same sign (both $+1$ or both $-1$). The sign of $x \cdot y$ is $-1$ if $x$ and $y$ are signed differently (one of them $+1$ and the other $-1$).

We note that the definition of $x \cdot y$ is similar to Definition 3.5 in [3], but we have also introduced signs for virtual edges.

Definition 5. Let $T$ be a planar graph. We define $\hat{T}$ as a graph with the same vertices as those of $T$. The edges of $\hat{T}$ are either actual or virtual, and for every ordered pair of edges $x, y \in T$, we have the virtual edge $x \cdot y$ in $\hat{T}$ with the corresponding sign. (See [3], Definition 3.7).

Theorem 6. Let $T$ be a graph. There is a well defined map $A(\hat{T}) \to A_Y(T)$, that maps each actual edge $x \in \hat{T}$ to $x \in T$ and each virtual edge $x \cdot y$ to $x^{-1}yx$.

Proof. The proof is the same as the proof of Theorem 3.8 of [3].

Remark 7. If $T$ is a simple graph, or if any two edges that connect the same two vertices are signed differently, then

$$A(T) \to A_Y(T) \to Br_n \cong A(S_n)$$

if there are no anti-cycles (i.e. $T$ is necessarily a simple graph), see [3][Remark 3.3].

And

$$A(T) \to A_Y(T) \to A(D_n)$$

if there exists at least one anti-cycle in $T$. 

\[\]

\[\]
3. Graphs with a single cycle

Let $T$ be a planar graph and let $x$ and $y$ be two edges in $T$ that have a common vertex $q$.

![Figure 4](image)

Let $x'$ be a path connecting $p$ and $r$; then (as in Section 8 of [3]) we can define a graph $T'$ where we replace $x$ by $x'$ and the sign of $x'$ is the product of the signs of $x$ and $y$, i.e., if $x$ and $y$ have the same sign then the sign of $x'$ is $+1$, and if $x$ and $y$ have the different signs then the sign of $x'$ is $-1$.

The map $x' \mapsto x'^{-1}yx$ defines an isomorphism between $A_Y(T')$ and $A_Y(T)$, as described in Section 8 of [3].

Now we classify the graphs $T$ that include a single cycle connected by a path to an anti-cycle or graphs that include only two anti-cycles connected by a path.

**Theorem 8.** Let $T$ be an anti-cycle of length $n$. Then $T$ is equivalent to a graph $T'$ that contains an anti-cycle of length two connected by a path.

![Figure 5](image)

**Proof.** Let $T$ be an anti-cycle with edges: $\sigma_1, \sigma_2, \ldots, \sigma_n$ such that $\prod_{i=1}^n s(\sigma_i) = -1$.

Let $\sigma_1$ be $\sigma_2^{-1} \sigma_3^{-1} \ldots \sigma_{n-1}^{-1} \sigma_n \sigma_{n-1} \ldots \sigma_2$ which we get from the triangulation property. Then $s(\sigma_1) = \prod_{i=2}^n s(\sigma_i)$ and since $\prod_{i=1}^n s(\sigma_i) = -1$, $s(\sigma_1) = -s(\sigma_1)$. Hence we get $T'$ which includes the anti-cycle of length two $\sigma_1$ and $\sigma_1$, where they are connected by a path. Since the only edges of $T$ that are involved in a cycle or in an anti-cycle are $\sigma_1$ and $\sigma_1$, we can omit the signs from $\sigma_2, \ldots, \sigma_{n-1}$.

\[\square\]

**Corollary 9.** All the anti-cycles of length $n$ are equivalent, and $A_Y(T) \simeq A(D_n)$. 
Proof. By Theorem 8, every anti-cycle of length \( n \) is equivalent to an anti-cycle of length two connected to a path. All the anti-cycles are equivalent to the same graph \( T' \). Since \( A_Y(T') \simeq A(D_n) \) for every equivalent graph \( T \), \( A_Y(T) \simeq A_Y(T') \simeq A(D_n) \). 

\[ \square \]

**Theorem 10.** Let \( T \) be a graph consisting of two anti-cycles connected to a path. Then \( T \) is equivalent to \( T' \), where \( T' \) is a cycle with an additional negatively signed edge connected to two adjacent vertices of the cycle.

![Figure 6](image)

**Proof.** By Theorem 8, \( T \) is equivalent to \( T'' \), where \( T'' \) consists of two anti-cycles of length two connected by a path. Let \( \sigma_1, \sigma_1^- \) and \( \sigma_n, \sigma_n^- \) be the edges of the two anti-cycles in \( T'' \) and \( \sigma_2, \ldots, \sigma_{n-1} \) be the edges of the path connecting them. By the triangulation property, there is a graph \( T' \) that is equivalent to \( T'' \), such that \( \sigma_n^- \) is replaced by \( u = \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{n-1}^{-1} \sigma_n \sigma_{n-1} \cdots \sigma_2 \sigma_1 \).

\[ \square \]

**Theorem 11.** Let \( T \) be a graph consisting of an anti-cycle \( C \) connected to a cycle by a path. Then \( T \) is equivalent to \( T' \), where \( T' \) is a cycle with an additional negatively signed edge, which is connected to two adjacent vertices of the cycle.

![Figure 7](image)

**Proof.** By the same proof as in Lemma 9.3 in [3], we have Figure 8.
Then using Theorem 10, the anti-cycle $C$ is equivalent to an anti-cycle of length two connected by a path. Then, combining this with the cycle, we get $T'$ as in Figure 9.

\[ \square \]

**Corollary 12.** Every graph $T$ with a path connecting either a cycle with an anti-cycle or two anti-cycles is equivalent to $T'$ of a form:

\[ \Rightarrow \]

**Proposition 13.** The length of the cycle in $T'$ is one less than the number of edges in $T$.

\[ \text{Proof.} \] $T'$ contains a cycle and an additional edge signed by $-1$. Hence, the length of the cycle in $T'$ is one less than the length of the number of edges in $T'$. Since we get $T'$ from $T$ by triangulation, and triangulation preserves the number of the edges, the proposition follows.

\[ \square \]

Now we define the group $G(T)$ for a graph $T$, where $T$ consists of either two anti-cycles connected by a path or an anti-cycle connected to a cycle by a path.

By Theorems 10 and 11, $T$ is equivalent to $T'$, where $T'$ is a cycle and an additional edge signed by $-1$ that is connected to two adjacent vertices in the cycle in $T'$, as in Figure 10. Hence, $A_Y(T') \simeq A_Y(T)$.

The edges of $T'$ are labelled by $\sigma_i$, $1 \leq i \leq n - 1$, $\sigma_1$ and $u$, the generators of $A(T')$.

We denote by $\alpha$ the element

\[ \alpha = L(\sigma_1, \ldots, \sigma_{n-1}) = \sigma_1^{-1} \sigma_2^{-1} \ldots \sigma_{n-2}^{-1} \sigma_{n-1} \sigma_{n-2} \ldots \sigma_2 \sigma_1. \]
Similarly to [3][Section 6], we define the path $\alpha$ as a directed path with starting point at the vertex $v_1$ and ending point at the vertex $v_n$. We notice that the path $\sigma_i$ is a path with starting point at $v_i$ and ending point at $v_{i+1}$, where $\sigma_i$ is the path with opposite direction to $\sigma_i$, which means that starting point of $\sigma_i$ is $v_{i+1}$, and the ending point of $\sigma_i$ is $v_i$.

Now we define signed paths for the action of $A(D_n)$ in $T'$. $\sigma_{-1}$ is the negatively signed path with starting point $v_1$ and ending point $v_2$, and similarly, $\sigma_{-1}$ is the negatively signed path with starting point $v_2$ and ending point $v_1$.

By Definition 3, $A_Y(T')$ is the group generated by $\sigma_1, \ldots, \sigma_{n-1}, \sigma_{-1}$ and $u$ with the relations

$[\sigma_i, \sigma_j] = 1$ for $|i - j| > 1$

$[u, \sigma_i] = 1$ for $2 \leq i \leq n - 2$

$[\sigma_{-1}, \sigma_j] = 1$ for $j \neq 2$

$< \sigma_i, \sigma_{i+1} >= 1$ for $1 \leq i \leq n - 2$

$< \sigma_{-1}, \sigma_2 >= 1$

$< \sigma_1, u >= 1$

$< \sigma_{n-1}, u >= 1$

$[\sigma_{-1}^{-1} u \sigma_1, \sigma_{-1}^{-1} \sigma_2 \sigma_{-1}] = 1$

$[\sigma_{-1}^{-1} u \sigma_1, \sigma_{-1}^{-1} \sigma_2 \sigma_{-1}] = 1$.

Note that $A_Y(T') = \langle A(D_n), u \rangle$, where $A(D_n)$ is the parabolic subgroup of $A_Y(T')$ generated by $\sigma_i$, $1 \leq i \leq n - 1$ and $\sigma_{-1}$.

As in [3][Section 6], we define $\alpha$ to be $x_{\alpha} = u \alpha^{-1}$, and $x_{\alpha} = \alpha^{-1} x_{\alpha} \alpha = \alpha^{-1} u$. The stabilizer of $\alpha$ in the action of $A(D_n)$ is generated by $\sigma_2, \ldots, \sigma_{n-2}, \sigma_{n-1} \sigma_{-1}^{-1} \alpha \sigma_{n-1}^{-1}, \alpha^2$ as in [3][Remark 6.2], and moreover, $\sigma_{n-1} \sigma_{-1}^{-1} \alpha \sigma_1 \sigma_{n-1}^{-1}, \sigma_1 \sigma_{-1}^{-1} \alpha \sigma_1 \sigma_{-1}^{-1}$ and $\sigma_1 \sigma_{-1}^{-1} \alpha \sigma_1 \sigma_{-1}^{-1}$ (three elements that involve $\sigma_{-1}$).

In the spirit of [3][Section 6], we define $G(T') = \langle A(D_n), x_{\alpha} \rangle$ with the relations

$< \sigma_1, x_{\alpha} \alpha >= 1$,

$< \sigma_{-1}, x_{\alpha} \alpha >= 1$,

$< \sigma_{n-1}, x_{\alpha} \alpha >= 1$,

$[x_{\alpha}, \sigma_i] = 1$, for $2 \leq i \leq n - 2$

which hold also in $A(T')$, and five additional relations concerning the stabilizer of $\alpha$:

$[x_{\alpha}, \alpha^2] = 1$.
Proposition 14. \([u, σ_{n-1}^{-1}ασ_1σ_{n-1}] = 1, \)
\([u, σ_1^{-1}ασ_1σ_1^{-1}] = 1, \)
\([u, σ_1^{-1}ασ_1σ_1^{-1}] = 1, \)
\([u, σ_{n-1}^{-1}ασ_1σ_{n-1}] = 1. \)

Since \(σ_i, 1 ≤ i ≤ n - 1\) and \(σ_1\) are conjugates to \(α\), we can easily define \(x_σ\) and \(x_σ^{-1}\) as they are defined in [3][Section 8].

Proof. Since \(x_σ = uα^{-1}\), the relation \([x_σ, σ_{n-1}^{-1}ασ_1σ_{n-1}] = 1\) implies \([uα^{-1}, σ_{n-1}^{-1}ασ_1σ_{n-1}] = 1\).

Now, \([ασ_{n-1}^{-1}, ασ_1α^{-1}] = 1\), which is equivalent to \([σ_1^{-1}ασ_{n-1}, σ_1^{-1}ασ_1] = 1\) which implies \([α, σ_{n-1}^{-1}ασ_1σ_{n-1}]\). Then we get \([u, σ_{n-1}^{-1}ασ_1σ_{n-1}] = 1\). The proof for the other three commutation relations is very similar.

\(\square\)

Proposition 15. \(x_σ = u(σ_i)σ_1^{-1}\) where \(u(σ_i) = L(σ_{i+1}, . . . , σ_{n-1}, u, σ_1, . . . , σ_{i-1})\)

and \(u(σ_i) = L(σ_2, . . . , σ_{n-1}, σ_1^{-1}σ_1uα_1^{-1}σ_1)\)

Proof. Since \(α = L(σ_1, . . . , σ_{n-1}) = σ_{n-1} . . . σ_2σ_1σ_2^{-1} . . . σ_{n-1}^{-1}, \)
\(σ_1 = σ_2^{-1} . . . σ_{n-1}^{-1}ασ_{n-1} . . . σ_2. \) It thus follows that \(σ_1\) is the conjugate of \(α\) by the element \(σ_{n-1} . . . σ_2. \) Hence \(u(σ_i) = σ_2^{-1} . . . σ_{n-1}^{-1}uα_1σ_{n-1} . . . σ_2, \) which is by definition \(L(σ_2, . . . , σ_{n-1}, u). \)

Now assume, by induction on \(i\), that \(u(σ_i) = L(σ_{i+1}, . . . , σ_{n-1}, u, σ_1, . . . , σ_{i-1}). \) Since
\(σ_{i+1} = σ_iσ_{i+1}σ_1σ_{i+1}^{-1}σ_i^{-1}, \) we get that \(u(σ_{i+1}) = σ_iσ_{i+1}L(σ_{i+1}, . . . , σ_{n-1}, u, σ_1, . . . , σ_{i-1})σ_{i+1}^{-1} =
= σ_iσ_{i+1}σ_1σ_{i+1}^{-1}σ_i^{-1} . . . σ_{n-1}^{-1}u^{-1}σ_1^{-1} . . . σ_{i-2}σ_{i-1}σ_{i-2} . . . σ_1uα_1σ_{n-1} . . . σ_{i+2}σ_{i+1}σ_{i+1}^{-1}σ_i^{-1} =
= σ_{i+2}σ_{i+1}σ_1σ_{i+1}^{-1}σ_i^{-1} . . . σ_{i-2}σ_{i+1}σ_{i-1}σ_{i-2} . . . σ_1uα_1σ_{n-1} . . . σ_{i+2} =
= σ_{i+2}σ_{i+1}σ_1σ_{i+1}^{-1}σ_i^{-1} . . . σ_{i-2}σ_{i+1}σ_1σ_{i-1}σ_{i-2} . . . σ_1uα_1σ_{n-1} . . . σ_{i+2} =
= L(σ_{i+2}, . . . , σ_{n-1}, u, σ_1, . . . , σ_i). \)

Now we prove the expression for \(u(σ_i). \) \(σ_1 = σ_2^{-1}σ_1^{-1}σ_2σ_1σ_2, \) hence \(u(σ_i) = σ_2^{-1}σ_1^{-1}u(σ_2)σ_1σ_2 = σ_2^{-1}σ_1^{-1}L(σ_3, . . . , σ_{n-1}, u, σ_1)σ_1σ_2 =
= σ_2^{-1}σ_1^{-1}σ_3^{-1} . . . σ_{n-1}^{-1}u^{-1}σ_1σ_{n-1} . . . σ_3σ_1σ_2 = σ_2^{-1}σ_3^{-1} . . . σ_{n-1}^{-1}σ_1^{-1}u^{-1}σ_1μ_1σ_{n-1} . . . σ_3σ_2 =
= \).
\[ = \sigma_2^{-1}\sigma_3^{-1}\ldots\sigma_{n-1}^{-1}(\sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1)\sigma_{n-1}\ldots\sigma_3\sigma_2 = L(\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, \sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1). \]

\[ \square \]

Now we prove that \( x_{\vec{\sigma}_i} \) and \( x_{\vec{\sigma}_1} \) satisfy similar relations as in [3][Section 6].

**Proposition 16.**

(1) \( [x_{\vec{\sigma}_i}, \sigma_j] = 1 \) for \( |i - j| > 1 \)

(2) \( [x_{\vec{\sigma}_i}, x_{\vec{\sigma}_j}] = 1 \) for \( |i - j| > 1 \)

(3) \( [x_{\vec{\sigma}_1}, \sigma_j] = 1 \) for \( j \neq 2 \)

(4) \( [x_{\vec{\sigma}_j}, \sigma_1] = 1 \) for \( j \neq 2 \)

(5) \( [x_{\vec{\sigma}_1}, x_{\vec{\sigma}_j}] = 1 \) for \( j \neq 2 \)

**Proof.** The proof of (1) and (2) is in [3].

We now prove (3). First we prove \( [x_{\vec{\sigma}_1}, \sigma_j] = 1 \) for \( j \geq 4 \). Since \( x_{\vec{\sigma}_1} = u^{(\vec{\sigma}_1)}\bar{\sigma}_1^{-1} \), and \([\sigma_1^{-1}, \sigma_j] = 1\), it is enough to prove that \([u^{(\sigma_1)}, \sigma_j] = 1\).

\[ [u^{(\sigma_1)}, \sigma_j] = [L(\sigma_2, \sigma_3, \ldots, \sigma_{n-1}, \sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1), \sigma_j] = \]

\[ = [\sigma_2^{-1}\sigma_3^{-1}\ldots\sigma_{n-1}^{-1}\sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1\sigma_{n-1}\ldots\sigma_3\sigma_2, \sigma_j] = \]

\[ = [\sigma_j^{-1}\sigma_j^{-1}\ldots\sigma_{n-1}^{-1}\sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1\sigma_{n-1}\ldots\sigma_j\sigma_{j-1}, \sigma_j] = \]

\[ = [\sigma_j^{-1}\ldots\sigma_{n-1}^{-1}\sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1\sigma_{n-1}\ldots\sigma_j, \sigma_{j-1}\sigma_{j-1}^{-1}] = \]

\[ = [\sigma_j^{-1}\ldots\sigma_{n-1}^{-1}\sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1\sigma_{n-1}\ldots\sigma_j, \sigma_{j-1}\sigma_{j-1}^{-1}\sigma_j] = \]

\[ = [\sigma_j^{-1}\ldots\sigma_{n-1}^{-1}\sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1\sigma_{n-1}\ldots\sigma_{j+1}, \sigma_{j-1}] = 1, \]

since \( 3 \leq j - 1 \leq n - 2 \).

Now we prove that \([u^{(\sigma_1)}, \sigma_1] = 1\).

\[ [u^{(\sigma_1)}, \sigma_j] = \]

\[ = [\sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1, \sigma_{n-1}\ldots\sigma_3\sigma_2, \sigma_1] = \]

\[ = [\sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1, \alpha] = [u, \sigma_1^{-1}\alpha\sigma_1\sigma_1^{-1}] = 1 \] by Proposition 14.

Now we prove that \([u^{(\sigma_1)}, \sigma_3] = 1\).

\[ [u^{(\sigma_1)}, \sigma_3] = [\sigma_2^{-1}\sigma_3^{-1}\ldots\sigma_{n-1}^{-1}\sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1\alpha, \sigma_3] = \]

\[ = [\sigma_2^{-1}\ldots\sigma_{n-1}^{-1}\sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1\sigma_{n-1}\ldots\sigma_3\sigma_2, \sigma_3] = \]

\[ = [\sigma_4^{-1}\ldots\sigma_{n-1}^{-1}\sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1\sigma_{n-1}\ldots\sigma_4, \sigma_2] = \]

\[ = [\sigma_1^{-1}\sigma_1u\sigma_1^{-1}\sigma_1, \sigma_2] = [\sigma_1^{-1}u\sigma_1, \sigma_1^{-1}\sigma_1^{-1}] = 1 \]

according to Relation (3) in Definition 3 (see Figure 11). Hence, case (3) is proved.
Proposition 17. 

(1) $\sigma_i x_{\sigma_{i+1}} \sigma_i^{-1} = x_{\sigma_{i+1}} x_{\sigma_i} \sigma_i^{-1}, 1 \leq i \leq n - 2$

(2) $\sigma_i^{-1} x_{\sigma_{i-1}} \sigma_i = x_{\sigma_{i-1}} x_{\sigma_i} \sigma_i^{-1}, 1 \leq i \leq n - 2$

(3) $\sigma_2 x_{\sigma_1} \sigma_2^{-1} = x_{\sigma_2} \sigma_1$
(4) \( \sigma_2^{-1} x_{\sigma_1} \sigma_2 = \sigma_1 x_{\sigma_2} \sigma_1^{-1} \)

Proof. (1) and (2) have been proved in [3][Section 6]; hence, we prove (3) and (4).

Proof of (3).

\[
\sigma_2 x_{\sigma_1} \sigma_2^{-1} = \sigma_2 u^{(\sigma_1)} \sigma_1^{-1} \sigma_2^{-1} = (\sigma_2 u^{(\sigma_1)} \sigma_2^{-1}) (\sigma_2 \sigma_1^{-1} \sigma_2^{-1}) = \\
= (\sigma_2 (\sigma_2^{-1} \sigma_1^{-1} \sigma_2 ) u^{(\sigma_1)} (\sigma_2^{-1} \sigma_1^{-1} \sigma_2)) (\sigma_2 \sigma_1^{-1} \sigma_2^{-1} \sigma_1) = \\
= (\sigma_1^{-1} \sigma_2 u^{(\sigma_1)} \sigma_1^{-1} \sigma_1^{-1} \sigma_2) (\sigma_1^{-1} \sigma_1^{-1} \sigma_2) = \sigma_1^{-1} u^{(\sigma_2)} \sigma_1^{-1} \sigma_1^{-1} \sigma_1^{-1} \sigma_1^{-1} \sigma_1 = \\
= (\sigma_1^{-1} u^{(\sigma_2)} \sigma_1^{-1} \sigma_1^{-1} \sigma_1^{-1} \sigma_1^{-1} \sigma_1) = \sigma_1^{-1} u^{(\sigma_2)} \sigma_1^{-1} \sigma_1^{-1} \sigma_1^{-1} \sigma_1 = \sigma_1^{-1} x_{\sigma_2} \sigma_1.
\]

Proof of (4).

We want to prove that \( \sigma_2^{-1} x_{\sigma_1} \sigma_2 = \sigma_1 x_{\sigma_2} \sigma_1^{-1} \). It is enough to prove that

\[
\sigma_2^{-1} u^{(\sigma_1)} \sigma_2 = \sigma_1 u^{(\sigma_2)} \sigma_1^{-1}, \quad \text{since} \quad \sigma_2^{-1} \sigma_1^{-1} \sigma_2 = \sigma_1 \sigma_2^{-1} \sigma_1^{-1}, \quad \text{and} \quad x_{\sigma_1} = u^{(\sigma_1)} \sigma_1^{-1},
\]

where we know that \( u^{(\sigma_1)} (\sigma_2^{-1} \sigma_1^{-1} \sigma_2) u^{(\sigma_1)} (\sigma_2^{-1} \sigma_1^{-1} \sigma_2) = \sigma_1 \sigma_2^{-1} \sigma_1^{-1} \sigma_2 \). Thus, from

\[
\sigma_2 u^{(\sigma_1)} \sigma_2^{-1} = \sigma_1^{-1} u^{(\sigma_2)} \sigma_1 \quad \text{(see (1))}, \quad \text{we conclude that} \quad u^{(\sigma_1)} = \sigma_2^{-1} \sigma_1^{-1} u^{(\sigma_2)} \sigma_1 \sigma_2. \quad \text{Hence, we need to prove} \quad \sigma_2^{-2} \sigma_1^{-1} u^{(\sigma_2)} \sigma_1 \sigma_2 \sigma_2^{-2} = \sigma_1^{-1} u^{(\sigma_2)} \sigma_1. \quad \text{Since} \quad < \sigma_1, u^{(\sigma_2)} > = 1, \text{this is equivalent to} \quad \sigma_2^{-2} u^{(\sigma_2)} \sigma_1 (u^{(\sigma_2)})^{-1} \sigma_2 = (u^{(\sigma_2)})^{-1} \sigma_1 (u^{(\sigma_2)})^{-1}, \text{which is equivalent to proving} \quad
\]

\[
(u^{(\sigma_2)} u^{(\sigma_2)} u^{(\sigma_2)} \sigma_1 (u^{(\sigma_2)})^{-1} \sigma_2)^{-1} (u^{(\sigma_2)})^{-1} = \sigma_1. \quad \text{Since} \quad [\sigma_2, u^{(\sigma_2)}] = 1 \quad (\text{[3][Section 6]}), \text{it is equivalent to prove that} \quad (u^{(\sigma_2)})^{-1} \sigma_1 (u^{(\sigma_2)})^{-1} = \sigma_1. \quad \text{But} \quad [x_{\sigma_1}, x_{\sigma_i}] = z. \quad \text{[3][Section 6]} \text{implies that} \quad [x_{\sigma_1}, x_{\sigma_i}] = (u^{(\sigma_i)})^{-1} \sigma_1 = z \quad \text{for each} \quad 1 \leq i \leq n - 1. \quad \text{Hence,} \quad (u^{(\sigma_2)})^{-1} \sigma_1 = (u^{(\sigma_1)})^{-1} \sigma_1^{-1}. \quad \text{Hence the equation we want to prove can be written as} \quad (u^{(\sigma_1)})^{-1} \sigma_1 = (u^{(\sigma_1)})^{-1} \sigma_1^{-1} = \sigma_1. \quad \text{By Proposition 16,} \quad [u^{(\sigma_1)}, \sigma_1] = 1, \text{and} \quad [\sigma_1, \sigma_1] = 1, \text{we get the result and this completes the proof of (4).}
\]

\[\square\]

Proposition 18. \( [x_{\sigma_i}, x_{\sigma_{i+1}}] = [x_{\sigma_1}, x_{\sigma_2}] = z, \text{where} \quad z^2 = 1 \quad \text{and} \quad z \in C(G(T')) \).

Proof. By [3][Proposition 6.8], \( [x_{\sigma_i}, x_{\sigma_{i+1}}] = [x_{\sigma_j}, x_{\sigma_{j+1}}] \)

for each \( 1 \leq i \leq n - 2, 1 \leq j \leq n - 2. \)

Let \( G'(T') \) be the subgroup of \( G(T') \) generated by \( \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \) and \( u \). By Propositions 16 and 17 : \( [\sigma_1, u^{(\sigma_i)}] = 1 \) for \( i \neq 2 \) and \( \sigma_1 u^{(\sigma_2)} \sigma_1^{-1} = \sigma_2^{-1} u^{(\sigma_1)} \sigma_2, \sigma_1^{-1} u^{(\sigma_2)} \sigma_1 = \sigma_2 u^{(\sigma_1)} \sigma_2^{-1}. \)

Hence, \( G'(T') \) is isomorphic to the group \( G \) that is defined in [3][Section 6] as a quotient of \( A(T^{(1)}) \), where \( T^{(1)} \) is a single cycle.

The map \( \varphi : G'(T') \to G \) is an isomorphism, where \( \varphi(\sigma_1) = \sigma_1, \varphi(\sigma_i) = \sigma_i \)

for \( 2 \leq i \leq n - 1 \) and \( \varphi(x_{\sigma_1}) = x_{\sigma_1}. \) Since \( [x_{\sigma_1}, x_{\sigma_2}] = [x_{\sigma_1}, x_{\sigma_{i+1}}] = z \) in \( G \), by
the isomorphism: \( [x_{\sigma_1}, x_{\sigma_2}] = [x_{\sigma_1}, x_{\sigma_{i+1}}] \) in \( G'(T') \). Hence \( [x_{\sigma_1}, x_{\sigma_2}] = [x_{\sigma_1}, x_{\sigma_{i+1}}] \) in \( G(T') \). Since \( [x_{\sigma_1}, x_{\sigma_2}] \) is a central element in \( G'(T') \) and \( [x_{\sigma_i}, x_{\sigma_{i+1}}] = [x_{\sigma_1}, x_{\sigma_2}], \) \( [[x_{\sigma_i}, x_{\sigma_{i+1}}], \sigma_1] = 1 \), and from [3][Section 6], \( [[x_{\sigma_i}, x_{\sigma_{i+1}}], \sigma_i] = 1 \) for \( 1 \leq i \leq n-1 \), we have \( [x_{\sigma_i}, x_{\sigma_{i+1}}] = [x_{\sigma_1}, x_{\sigma_2}] = z \), where \( z \in C(G(T')) \) and \( z^2 = 1 \).

\[ \square \]

**Proposition 19.** \( x_{\sigma_i} = zx_{\sigma_i} \) for \( 1 \leq i \leq n-1 \) and \( x_{\sigma_1} = zx_{\sigma_1} \).

**Proof.** \( x_{\sigma_i} = zx_{\sigma_i} \) is proved in [3][Proposition 6.8]. And \( x_{\sigma_1} = zx_{\sigma_1} \) can also be deduced from [3][Proposition 6.8] by considering \( G'(T') \), which was defined in the proof of Proposition 18.

\[ \square \]

**Theorem 20.** \( G(T') \cong K \rtimes A(D_n) \), where \( K \) is the group generated by \( x_{\sigma_i}, 1 \leq i \leq n-1 \) and \( x_{\sigma_1} \) with the relations
\[
[x_{\sigma_i}, x_{\sigma_j}] = 1, \ |i - j| > 1,
[x_{\sigma_1}, x_{\sigma_j}] = 1, \ j \neq 2,
[x_{\sigma_i}, x_{\sigma_j}] = z, \ |i - j| = 1,
[x_{\sigma_1}, x_{\sigma_2}] = z,
\]
\( z \) is a central element and \( z^2 = 1 \).

**Proof.** The subgroup generated by \( \sigma_1, \ldots, \sigma_{n-1} \) and \( \sigma_1 \) is \( A(D_n) \). Then using Propositions 16-19, we get the result from the same argument as in [3][Theorem 6.11].

\[ \square \]

4. The general case

**Theorem 21.** Every graph \( T \) that includes at least one anti-cycle is equivalent to a graph \( T^{(m)} \), where \( T^{(m)} \) consists of \( m \) cycles including the edges \( \sigma_1, \ldots, \sigma_{n-1}, u_i \) for \( 1 \leq i \leq m \) and a negatively signed edge \( \sigma_1 \) that connects the vertices \( v_1 \) and \( v_2 \) (see Figure 12).

**Remark:** Here \( m+1 \) is the number of the cycles and anti-cycles in \( T \), i.e., \( m+1 \) is the number of cycles in \( \bar{T} \), where \( \bar{T} \) is the graph obtained from \( T \) by omitting the signs.

**Proof.** The proof is by induction on \( m \). In the case \( m = 0 \), \( T \) contains an anti-cycle. Hence the number of negative signs in \( T \) is odd. By Theorem 9, \( \bar{T} \) is equivalent by triangulation to
a cycle since $\tilde{T}$ contains just one cycle. Since triangulation can only convert an anti-cycle to another anti-cycle and not to a cycle, $T$ is equivalent to an anti-cycle connected to a path, which is $T^{n(0)}$.

Assume by induction that for $m \leq k$ the theorem holds. Then $T$, with $k + 1$ cycles and anti-cycles, is equivalent to $T^{n(m)}$. Now assume $m = k + 1$. If we consider only $k$ cycles (i.e., the subgraph $\tilde{T}$ obtained by omitting one edge from one of the cycles or one of the anti-cycles of $T$), $\tilde{T}$ is equivalent to $T^{n(k)}$ by the induction hypothesis. Since triangulation preserves the number of the edges of $T$, $T$ contains one more edge $e$ which does not appear in $T^{n(k)}$. The edge $e$ forms one more cycle or one more anti-cycle, since triangulation preserves the number of cycles. Hence, $T$ is equivalent to the graph $T^{n(k),e}$ (see Figure 13), where the edge $e$ connects two vertices $v_i$ and $v_j$.

Then we look at the subgraph $T^{n(0),e}$ of $T^{n(k),e}$ that contains the edges $\sigma_i$, $1 \leq i \leq n - 1$, $\sigma_1$ and $e$ (i.e. $T^{n(0),e}$ we get from $T^{n(k),e}$ by omitting $u_i$ for $1 \leq i \leq k$). By Theorems 10 and 11, $T^{n(0),e}$ is equivalent to $T^{n(1)}$. Hence, $T^{n(k),e}$ is equivalent to $T^{n(k+1)}$ (adding the edges $u_i$ to $T^{n(1)}$).

\[ \square \]

**Proposition 22.** In $A_Y(T^{n(m)})$, the following relation holds for $1 \leq i < j \leq m$:

1. $\langle \sigma_1 u_i \sigma_1^{-1}, u_j \rangle = 1$
(2) \(< σ₁uᵢσ⁻¹₁, uᵢ >= 1
(3) \(< σₙ₋₁uᵢσ⁻¹ₙ₋₁, uᵢ >= 1
(4) \([σ₁uᵢσ⁻¹₁, σₙ₋₁uᵢσ⁻¹ₙ₋₁] = 1
(5) \([σ₁uᵢσ⁻¹₁, σₙ₋₁uᵢσ⁻¹ₙ₋₁] = 1
(6) \([σ₁uᵢσ⁻¹₁, σᵧuᵢσ⁻¹ᵧ] = 1
(7) \([σ₁uᵢσ⁻¹₁, σ₁uᵢσ⁻¹₁] = 1

Proof. The proof is derived directly from the definition of \(A_y(Tⁿ(m))\), see Figure 13. □

In the spirit of [3], we define \(x^{(j)}ᵦ = uᵢα⁻¹\) and \(x^{(j)}α = α⁻¹uᵢ\) for \(1 ≤ j ≤ m\).

Definition 23. Let \(G(Tⁿ(m))\) be a quotient of \(A_y(Tⁿ(m))\) by the following relations:
\[x^{(j)}ᵦ, σ₋₁ᵦασ⁻¹₁ = 1,\]
\[x^{(j)}ᵦ, σ₁ασ⁻¹₁ = 1,\]
\[x^{(j)}ᵦ, σ₁ασ⁻¹₁ = 1,\]
\[x^{(j)}ᵦ, σ₋₁ᵦασ⁻¹₁ = 1,\]
\[x^{(j)}ᵦ, α² = 1, \text{ for } 1 ≤ j ≤ m.\]

Theorem 24. \(G(Tⁿ(m)) \simeq K^{(m,n)} \rtimes A(D_n)\), where \(K^{(m,n)}\) is the group generated by \(x^{(i)}ᵦ\) and \(x^{(i)}ᵦ\), \(1 ≤ i ≤ m, 1 ≤ j ≤ n - 1\) with the following relations:
\[x^{(i)}ᵦx^{(j)}ᵦ = 1, \text{ } |k - l| > 1,\]
\[x^{(i)}ᵦx^{(j)}ᵦ = 1, \text{ } l ≠ 2,\]
\[x^{(i)}ᵦx^{(j)}ᵦ = zᵦ, \text{ } |k - l| = 1,\]
\[x^{(i)}ᵦx^{(j)}ᵦ = zᵦ,\]
\[x^{(i)}ᵦx^{(j)}ᵦ = [x^{(i)}ᵦ, x^{(j)}ᵦ] = [x^{(i)}ᵦ, x^{(j)}ᵦ], \text{ } |k - l| = 1, |p - q| = 1,\]
\[zᵦ² = 1.\]

Proof. The relations in \(K^{(m,n)}\) that do not involve \(x₋₁ᵦ\) were proved in [3][Section 10]. Now look at the subgroup \(Kᵀᵐⁿ\) of \(K^{(m,n)}\), where \(Kᵀᵐⁿ\) is generated by \(x₋₁ᵦ\) and \(x₋ᵦ\), where \(2 ≤ l ≤ n - 1\) (i.e., without the generator \(x₋₁ᵦ\)). Then \(Kᵀᵐⁿ\) is isomorphic to \(\tilde{K}ᵀᵐⁿ\), where \(\tilde{K}ᵀᵐⁿ\) is the group \(K^{(m,n)}\) from [3][Section 10], where \(ϕ(x^{(i)}ᵦ) = x^{(i)}ᵦ\), \(ϕ(x^{(i)}ᵦ) = x^{(i)}ᵦ\). Hence, by [3][Section 10], \([x^{(i)}ᵦ, x^{(j)}ᵦ] = 1\) for \(p ≥ 3\) and \([x^{(i)}ᵦ, x^{(j)}ᵦ] = [x^{(i)}ᵦ, x^{(j)}ᵦ] = 1\) for \(|p - q| = 1\). It remains to prove the relation between \(x^{(i)}ᵦ\) and \(x^{(i)}ᵦ\). By Proposition 22 ((5) and (6)), \([σ₁uᵢσ⁻¹₁, σ₋₁ᵦuᵦ] = 1\) implies that \([σ₁uᵦα⁻¹ᵦσ⁻¹₁, σ₋₁ᵦuᵦα⁻¹ᵦ] = 1\),
since \([\sigma_1\alpha\sigma_1^{-1}, \sigma_1\alpha\sigma_1^{-1}] = 1\). Hence \([\sigma_1x_{\alpha}^{(i)}\sigma_1^{-1}, \sigma_1x_{\alpha}^{(j)}\sigma_1^{-1}] = 1\). Then, from Proposition 17, 
\([\alpha^{-1}x_{\sigma_1}^{(i)}\alpha, \alpha^{-1}x_{\sigma_1}^{(j)}\alpha] = 1\) for each \(i\) and \(j\). This completes the proof. \[\qed\]

Note that by similar arguments as in [3], the word problem is soluble in \(K(m,n)\) and soluble in \(A(D_n)\) too, we get

**Corollary 25.** The word problem is soluble in \(G(T)\).

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Meirav Amram, Robert Shwartz, Mina Teicher, Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel

E-mail address: meirav,shwartz,teicher@macs.biu.ac.il