ONE-DIMENSIONAL FERMIONS AS TWO-DIMENSIONAL DROPLETS
VIA CHERN-SIMONS THEORY

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ABSTRACT

Based on the observation that a particle motion in one dimension maps to a two-dimensional motion of a charged particle in a uniform magnetic field, constrained in the lowest Landau level, we formulate a system of one-dimensional nonrelativistic fermions by using a Chern-Simons field theory in 2+1 dimensions. Using a hydrodynamical formulation we obtain a two-dimensional droplet picture of one-dimensional fermions. The dynamics involved is that of the boundary between a uniform density of particles and vortices. We use the sharp boundary approximation. In order to test our approach we apply it to a system of fermions in a harmonic oscillator potential. In the case of well separated boundaries we derive the one-dimensional collective field Hamiltonian. Symmetries of the theory are also discussed as properties of curves in two dimensions.

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I. Introduction

Nonrelativistic fermions in one space dimension have recently attracted a lot of attention. It has been shown [1] that they describe the singlet sector of the one-dimensional hermitian matrix model, which in the double scaling limit describes the $c = 1$ string model [2]. The collective field theory method [3] is a powerful approach to the large $N$ matrix model, and it has been successfully applied to the string problem [4]. It provides a bosonic field theoretic description of the model and it can be viewed as a bosonization of apparent relativistic fermions near the Fermi surface [5] or simply as a bosonization of nonrelativistic fermions [6]. In spite of its success, the method used, namely semiclassical perturbation, is questionable, since strictly speaking it is not applicable near the turning points.

A rather different approach is the phase space description of Polchinski [7] and the recent developments [8] related to it. Another result, suggestive of the underlying two-dimensional nature of the problem, is the emergence of the algebra of area preserving diffeomorphisms in this context [9] which also appears in the study of fractional quantum Hall effect [10]. There should be a close mathematical relationship between these physically different problems, namely two-dimensional quantum gravity coupled to a scalar field and fractional quantum Hall effect. We like to pursue this point further in this paper.

Our approach is based on the observation that a system of fermions in one-space dimension is equivalent to a system of fermions constrained in the lowest Landau level. The two-dimensional configuration space of the latter maps to the phase space of the former. This is described in section II.

Bosonization in two space dimensions is achieved by a singular gauge transformation [11] or equivalently by introducing a Chern-Simons gauge field [12]. Thus we use a model of a Schrödinger Bose field coupled to a Chern-Simons gauge field. The lowest Landau level condition is imposed by considering (nonrelativistic) zero mass charged fermions in a uniform magnetic field. Since in the Schrödinger theory the mass appears in the denominator of the kinetic energy term, we first linearize
it by using an auxiliary field and then set the mass to zero. In effect we obtain a set of second class constraints describing the lowest Landau level condition. At this stage one may use the machinery of the Dirac theory of constraints, but we choose a simpler intuitive approach. We first solve the constraint equation classically in order to get fewer unconstrained collective variables, and then develop a canonical formalism using them. We extensively use the technique developed in the collective field approach to fractional quantum Hall effect [13], which is after all a hydrodynamical description of fermions. These issues are discussed in section III.

In section IV we apply our method to a system of free fermions in a harmonic oscillator potential; this simple example illustrates the technique and also leads to a more concrete physical picture. In this analysis it becomes obvious that the relevant collective variables are those of a boundary between a uniform density of particles and vortices. The system becomes equivalent to a droplet in two dimensions, and the excitations of the system are due to surface oscillations.

In section V we extend the theory to the case where the particle density field occupies a region in the shape of an infinite strip with the upper and lower boundaries well separated. We obtain the collective field Hamiltonian of [3] and [4] in a similar manner as in [7]. This is a good representation of the system provided the potential prevents the two boundaries from touching. The separation of boundaries is achieved in the harmonic oscillator potential, but not in the inverse harmonic oscillator potential with small chemical potential.

The final section is devoted to discussions in which we also include symmetries and a space-time description of the theory.

II. The Relation Between Particles in One Space Dimension and Particles in Two Space Dimensions in Strong Magnetic Field

Let us consider a particle in a potential in one space dimension. The Hamilto-
\[ H = \frac{1}{2m} p^2 + v(x). \] (2.1)

For the specific case of harmonic oscillator, which we shall investigate extensively below, the potential is given by
\[ v(x) = \frac{m}{2} \omega^2 x^2. \] (2.2)

In quantum theory, \( p \) is a differential operator given by \(-i \frac{\partial}{\partial x}\).

Next let us consider a massless charged particle in a uniform magnetic field in two space dimensions. The Lagrangian is given by
\[ L = A \cdot \dot{x} - A_0(x, y). \] (2.3)

In Landau gauge the vector potential \( A \) is given by
\[ A \equiv (yB, 0), \quad \text{with} \quad B > 0 \] (2.4)

so that the first term of \( L \) is
\[ A \cdot \dot{x} = B y \dot{x}. \] (2.5)

Since the canonical momentum of \( x \) is
\[ p_x = B y, \] (2.6)

the Hamiltonian of the system can be written as
\[ H = A_0(x, \frac{1}{B} p_x). \] (2.7)

Thus the one dimensional particle system is equivalent to a two dimensional massless charged particle system under uniform magnetic field provided
\[ A_0(x, y) = \frac{B^2}{2m} y^2 + v(x). \] (2.8)

In the following section we shall exploit this observation for systems of many fermions in one space dimension and develop a theory of Fermi liquid by using the
field theory of Chern-Simons gauge field coupled to a nonrelativistic Schrödinger Bose field in two space dimensions.

III. Chern-Simons Field Theory for Massless Fermions

Let us first consider a system of massive (mass $m_0$) fermions in electromagnetic field in two space dimensions. The Hamiltonian is given by

$$H = \frac{1}{2m_0} \sum_{a=1}^{N} (\Pi_a)^2 + \sum_{a} A_0(x_a)$$

$$= \frac{1}{2m_0} \sum_{a=1}^{N} (\Pi^x_a + i\Pi^y_a)(\Pi^x_a - i\Pi^y_a) + \sum_{a} A_0(x_a) + \frac{B}{2m_0}N, \quad (3.1)$$

where

$$\Pi^i = p_i - A^i(x) = -i \frac{\partial}{\partial x^i} - A^i(x), \quad i = x, y \quad (3.2)$$

We use the metric $(+,-,-)$. We bosonize the above quantum mechanical system by using a singular gauge transformation $[12]$. As a result we obtain an effective bosonic Hamiltonian. Its form is the same as in (3.1) except that $\Pi^i$ is now given by

$$\Pi^i_a = p_{ia} - A^i(x_a) - a^i(x_a), \quad (3.3)$$

where $a^i(x_a)$ is

$$a^i(x_a) = -\epsilon^{ij} \sum_{b \neq a} \frac{(x_a - x_b)^j}{|x_a - x_b|^2}. \quad (3.4)$$

The effective bosonic system can be described in a second quantized language in terms of a Bose Schrödinger wave field $\psi$ $[6]$. We further introduce a Chern-Simons gauge field as an auxiliary field to ensure (3.4). We obtain the following Lagrangian density for the original fermionic system

$$\mathcal{L} = \bar{\psi} \Pi_0 \psi - \frac{1}{2m_0} \bar{\psi}(\Pi^x + i\Pi^y)(\Pi^x - i\Pi^y)\psi + \mathcal{L}_{CS}, \quad (3.5)$$
where
\[ \Pi_0 = i\partial_0 - A_0 - a_0, \quad L_{CS} = -\frac{1}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho. \] (3.6)

Next we linearize the kinetic energy term by using auxiliary fields \( \lambda, \bar{\lambda} \):
\[ L = \bar{\psi}\Pi_0\psi - \bar{\psi}(\Pi^x + i\Pi^y)\lambda - \bar{\lambda}(\Pi^x - i\Pi^y)\psi + L_{CS} + 2m_0|\lambda|^2 \] (3.7)
and then set \( m_0 = 0 \). The auxiliary field \( \lambda \) plays the role of a Lagrange multiplier.

Next we change variables from \( \psi, \bar{\psi} \) to \( \rho, \theta \) through
\[ \psi = \sqrt{\rho} e^{i\theta}. \] (3.8)
We obtain
\[ L = b_\mu(\partial^\mu \theta + A^\mu + a^\mu - \frac{1}{2} \epsilon^{\mu\nu\rho} \partial_\nu \ln \rho) + L_{CS}, \] (3.9)
where
\[ b^0 = -\rho, \quad \frac{1}{2}(b_x + ib_y) = \bar{\lambda}\psi. \] (3.10)
The phase \( \theta \) can have a regular part and a singular part:
\[ \theta = \theta_{\text{reg}} + \theta_s. \] (3.11)
The singular part is due to a vortex configuration:
\[ \partial^\mu \theta_s = v^\mu, \quad \nabla \times \mathbf{v} = 2\pi \rho_V, \quad \nabla \cdot \mathbf{v} = 0 \] (3.12)
where \( \rho_V \) is the density of vortices. If we set
\[ b^i = \epsilon^{ij} \partial_j h + \partial^i g \] (3.13)
and insert this into (3.9) we see that \( a_0, h \) and \( g \) act as Lagrange multipliers. We
obtain

\[ L = -\rho(\dot{\theta}_{\text{reg}} + v_0 + A_0) + \frac{1}{4\pi} \epsilon^{ij} a_i \dot{a}_j \]  
(3.14)

along with the conditions imposed by the Lagrange multipliers

\[ \nabla \times a = 2\pi \rho \]
\[ \nabla \times v + \nabla \times A + \nabla \times a - \frac{1}{2} \nabla^2 \ln \rho = 0 \]
\[ \nabla^2 \theta_{\text{reg}} + \nabla \cdot a = 0 \]  
(3.15)

Using the fact that \( a_i \) can be written as

\[ a_i = \frac{1}{\nabla^2} (2\pi \epsilon_{ij} \partial_j \rho - \partial_i \nabla \cdot a) \]  
(3.16)

we notice that the first and last term in (3.14) cancel each other because of (3.15). Therefore, the dynamics of the system is simply described by

\[ L = -(v_0 + A_0)\rho \]  
(3.17)

with the subsidiary condition (notice that \( \nabla \times A = -B, \ B > 0 \))

\[ B + \frac{1}{2} \nabla^2 \ln \rho - 2\pi \rho V - 2\pi \rho = 0. \]  
(3.18)

This subsidiary condition is the same as the constraint equation for electrons being in the lowest Landau level in the fractional quantum Hall system [13].

Notice that

\[ -\int d^2 x v_0 \rho = \int \int \rho(x) 2\pi G(x - x') \dot{\rho} V(x') d^2 x d^2 x' \]  
(3.19)

where \( G \) is the Green’s function satisfying

\[ \epsilon^{ij} \partial_i \partial_j G(x - x') = \delta^2(x - x'), \quad \nabla^2 G = 0. \]  
(3.20)

An explicit form of \( G \) is given by

\[ G(x) = \frac{\theta}{2\pi} = \frac{\text{Im} \ln z}{2\pi}, \]  
(3.21)

We first solve the subsidiary condition (3.18) and then develop a canonical
formalism.

The uniform field configuration

$$\rho = \frac{B}{2\pi}, \quad \rho V = 0$$  \hspace{1cm} (3.22)

is the simplest field configuration for which the subsidiary condition (3.18) is satisfied. This corresponds to a uniform distribution in the phase space of one-dimensional fermions. It is obviously unphysical since $A_0$ given by (2.8) makes the second term in the Lagrangian (3.17), i.e. the energy, infinite.\footnote{This state is perfectly physical for the fractional quantum Hall system. It corresponds to the case where all the lowest Landau levels are completely filled.} One should select solutions of the subsidiary condition in such a way that the energy $\int d^2 x A_0 \rho$ stays in the neighborhood of minimum. Thus the canonical procedure suggested earlier depends on the form of $A_0$. We shall discuss it in detail for the case of a harmonic oscillator potential (2.2) in the following section.

\section*{IV. Fermions in Harmonic Oscillator Potential}

Here we study a system of $N$ fermions in a harmonic oscillator potential. The single particle Hamiltonian is given by (2.1) and (2.2). The system consists of $N$ fermions occupying the equally spaced energy levels. The ground state and the excited states are easily obtained by the standard independent particle picture. It is known that in the large $N$ limit the low frequency excitations are classified by using a set of Bose oscillators with frequency $\omega_n = n\omega$, \hspace{1cm} (n = 1, 2, \ldots, ). We shall describe this simple known problem by using the method described in the previous section.

Let us choose

$$B = m\omega$$  \hspace{1cm} (4.1)

in (2.8) and combine it with (2.2). Then we obtain

$$A_0 = \frac{1}{2}m\omega^2(x^2 + y^2) - \mu$$
$$\frac{1}{2} m \omega^2 ((x^2 + y^2) - r_0^2), \quad (4.2)$$

where $\mu$ is a chemical potential and we parametrize it as in the second line. It is quite obvious that the lowest energy configuration is the one where $\rho$ takes the maximum value inside a circle of radius $r_0$ and the minimum value outside.

We notice that in general the constraint equation (3.18) is the equation for vortices. If one appropriately scales the coordinates and then sets $4\pi \rho = e^{2f_1}$, one obtains

$$-\Delta f_1 + \frac{1}{2}(e^{2f_1} - 1) = -2\pi \rho_V. \quad (4.3)$$

Along with the requirement of singlevaluedness and finiteness of $\psi$, one finds [14] that (4.3) admits vortex solutions if $\rho_V$ is given by

$$\rho_V = \sum_k n_k \delta^{(2)}(x - x_k), \quad (4.4)$$

where $x_k$’s are vortex positions and the vorticities, $n_k$’s, are positive integers. The particle density $\rho$ is zero at the position of vortex.

Coming back to the harmonic oscillator potential problem, in order that the energy stays in the neighborhood of the minimum, we must choose configurations with almost no vortices inside the circle and infinitely many, densely populated outside. Below we shall use, for solutions of the subsidiary condition (3.16), the sharp boundary approximation where $\rho = B/2\pi$ inside and zero outside the boundary and respectively $\rho_V = B/2\pi$ outside and zero inside the boundary. We parametrize the boundary by $\delta r(\theta)$ and treat it as a dynamical variable.

$$\rho(x) = \frac{B}{2\pi} \theta(-r + r_0 + \delta r(\theta))$$
$$\approx \frac{B}{2\pi} (\theta(-r + r_0) + \delta r(\theta) \delta(r - r_0) - \frac{(\delta r)^2}{2} \delta'(r - r_0) + \cdots) \quad (4.5)$$

$$\rho_V(x) = \frac{B}{2\pi} \theta(r - r_0 - \delta r(\theta))$$
\[ \approx \frac{B}{2\pi} (\theta (r - r_0) - \delta r(\theta) \delta (r - r_0) + \frac{(\delta r)^2}{2} \delta'(r - r_0) + \cdots) \quad (4.6) \]

Since the particle number \( N \) is proportional to \( r_0^2 \), this expansion is a large \( N \) expansion.

For the first term in the Lagrangian (3.17) we obtain

\[ -\int \rho v_0 = -r_0^2 \left( \frac{B}{2\pi} \right)^2 \int \int \delta r(\theta) Im \ln(e^{i\theta} - e^{i\theta'}) \delta \dot{r}(\theta') d\theta d\theta', \quad (4.7) \]

keeping up to the quadratic term in \( \delta r \) (higher order terms are with higher power of \( r_0^{-1} \)).

Expanding \( \delta r \) as

\[ \delta r(\theta) = \frac{\sqrt{2}}{r_0 B} \sum_{n > 0} \sqrt{n} (q_n \sin n\theta + p_n \cos n\theta) \quad (4.8) \]

and noticing that

\[ Im \ln(e^{i\theta} - e^{i\theta'}) = \theta - \sum_{n > 0} \frac{1}{n} (\sin n\theta' \cos n\theta - \cos n\theta' \sin n\theta) \quad (4.9) \]

we obtain that

\[ -\int \rho v_0 = \sum_{n > 0} p_n \dot{q}_n \quad (4.10) \]

up to a total derivative, and

\[ \int A_0 \rho = \sum_{n > 0} n\omega \frac{1}{2} (p_n^2 + q_n^2) \quad (4.11) \]

up to a constant. Thus the system is equivalent to an ensemble of harmonic oscillators. We obtain the collective excitation spectrum

\[ \omega_n = n\omega, \quad (4.12) \]

which we anticipated.
V. Reduction To A Field Theory In One Space Dimension

Guided by the above analysis, we formulate, in this section, the theory for a general case in which the particle density is non-zero within two well separated boundaries. We take the following expression as an approximate solution of the subsidiary condition (3.18):

$$\rho(x, y, t) = \frac{B}{2\pi} \theta(y_+(x, t) - y)\theta(y - y_-(x, t))$$  \hspace{1cm} (5.1)

We assume that the time variation of \(y_\pm\) is much smaller than the difference \(y_+ - y_-\) and set

$$y_\pm(x, t) = y_\pm^0(x) + \delta y_\pm(x, t), \quad |\delta y_\pm(x, t)| \ll y_\pm^0(x) - y_\pm^0(x)$$  \hspace{1cm} (5.2)

We then expand \(\rho\) and \(\rho V\) as

\[
\rho \approx \frac{B}{2\pi} \left( \theta(y_+^0 - y)\theta(y - y_-^0) + \delta y_+ \delta(y - y_-^0) - \delta y_- \delta(y - y_+^0) + \cdots \right)
\]

\[
\rho V \approx \frac{B}{2\pi} \left( \theta(y - y_+^0) + \theta(y_0^0 - y) - \delta y_+ \delta(y - y_+^0) + \delta y_- \delta(y - y_-^0) + \cdots \right)
\]  \hspace{1cm} (5.3)

and insert the above expressions into the first term in the Lagrangian. We obtain

$$-\int \rho v_0 = -\left(\frac{B}{2\pi}\right)^2 \int dx dx' (\delta y_+(x)\delta \dot{y}_+(x') Im \ln(x + iy_+^0(x) - x' - iy_+^0(x')) + \delta y_-(x)\delta \dot{y}_-(x') Im \ln(x + iy_+^0(x) - x' - iy_-^0(x')))$$  \hspace{1cm} (5.4)

The cross term has been omitted in the above expression since it can be shown to have a form of total derivative in time.

The Green’s function is in general written as

$$Im \ln(x + iy_\pm^0(x) - x' - iy_\pm^0(x')) = \pm \pi \theta(x - x') + f_\pm(x, x'),$$  \hspace{1cm} (5.5)

where \(f_\pm(x, x')\) is symmetric under the interchange of \(x\) and \(x'\) and therefore it does not contribute to (5.4). In deriving (5.5) we took into account the fact that
the prime coordinate corresponds to the vortices, therefore it lies above the upper boundary and below the lower boundary. Thus (5.4) is now given by

\[- \int \rho v_0 = - \left( \frac{B}{2\pi} \right)^2 \pi \int dx dx' \left( \delta y_+(x) \delta \dot{y}_+(x') - \delta y_-(x) \delta \dot{y}_-(x') \right) \theta(x - x'). \quad (5.6)\]

The normal mode expansion is given by

\[\delta y_{\pm}(x) = \int_{0}^{\infty} dk \left( \delta y^R_{\pm}(k) \cos kx + \delta y^I_{\pm}(k) \sin kx \right). \quad (5.7)\]

Using that \( \theta(x) = \frac{1}{2\pi i} \int du \frac{e^{iux}}{u-ie} \), we obtain

\[- \int \rho v_0 = - \frac{B^2}{2} \int_{0}^{\infty} dk \frac{\delta y^I_{\pm}(k) \delta \dot{y}^R_{\pm}(k) - \delta y^I_{\pm}(k) \delta \dot{y}^R_{\pm}(k)}{k}. \quad (5.8)\]

where we discarded again a total time derivative term. Thus the quantum commutation relations are given by

\[[\delta y^R_{\pm}(k), \delta y^I_{\pm}(k')] = \mp i \frac{2k}{B^2} \delta(k - k'), \quad (5.9)\]

accordingly by

\[[y_{\pm}(x), y_{\pm}(x')] = \mp i \frac{2\pi}{B^2} \delta'(x - x'). \quad (5.10)\]

The Hamiltonian is given by

\[\int \rho A_0 = \int \rho \left( \frac{B^2 y^2}{2} + v(x) - \mu \right) d^2 r = \frac{B}{2\pi} \int dx \left( \frac{B^2}{6} (y^3_+ - y^3_-) + (v(x) - \mu) (y_+(x) - y_-(x)) \right). \quad (5.11)\]
where we set $m = 1$. Let us define

$$y_\pm(x) = \frac{1}{B}(\pm \pi \phi(x) + \partial \Pi(x))$$

(5.12)

where $\phi$ and $\Pi$ satisfy the standard canonical commutation relation. In terms of these new variables the Hamiltonian is given by

$$\int \rho A_0 = \int dx \left( \frac{1}{2} \phi (\partial \Pi)^2 + \frac{\pi^2}{6} \phi^4 + (\nu(x) - \mu)\phi \right)$$

(5.13)

which is precisely the collective field Hamiltonian [3,4].

VI. Discussions

We have formulated one-dimensional nonrelativistic fermions by using Chern-Simons field theory. After converting the field variables into the hydrodynamical collective variables and eliminating the constraint, we reached a droplet picture of one-dimensional fermions. It is a system of incompressible liquid in two dimensions. The dynamical degrees of freedom are the fluctuations of the boundary between a uniform density of particles and vortices. These are similar to the edge excitations described in [15].

In section V we obtained the collective field Hamiltonian for the case of a boundary curve that does not fluctuate widely, so it can be described as a single valued function of $x$, and furthermore it does not touch itself. Obviously this should be modified for general cases. In the case that the curve touches itself one has to take into account possible rearrangement of the boundaries.

These issues are most relevant in the case of an inverted harmonic oscillator potential with small chemical potential. This corresponds to the $c = 1$ string model with strong coupling [16]. It is clear from Fig.1 that rearrangement of the boundary is necessary when one considers boundary oscillations near the turning points.
We should mention that the one-dimensional collective field Hamiltonian derived here differs from the one presented in [6], where we directly bosonized the one-dimensional fermions, by a subleading in $N$ order term, which contains derivatives of the particle density. We believe that the sharp boundary approximation used is responsible for the absence of this term. Further in the case of the inverted harmonic oscillator potential with small chemical potential, appropriate treatment of the surface rearrangement in the region of the turning points will produce the effect of the extra derivative term in the collective field Hamiltonian, since they both express a tunneling effect.

In general in order to accommodate a possible rearrangement of the boundaries we follow the surface motion in space-time and consider a connected region $\Omega$ in Minkowski space, in which the liquid flows. We choose $\Omega$ as shown in Fig.2 to include fission and fusion of droplets. The action is given by

$$S = -\int_{\Omega} (\nu_0 + A_0) \rho. \tag{6.1}$$

This is essentially (3.17), but here $\rho(x, t)$ is 1 (we set $B/2\pi = 1$) inside $\Omega$ and zero outside and $\rho_V$ is the other way around. The boundary of $\Omega$ is specified by

$$x^\mu = X^\mu(\sigma, \tau). \tag{6.2}$$

Thus $S$ is a functional of $X^\mu(\sigma, \tau)$. Based on this action, in principle, one can discuss the quantum mechanical motion of surfaces. We hope we can come back to this problem in future.

We shall now consider briefly the symmetry of the system. It is obvious that reparametrization of the surface is a symmetry. The issue is whether this symmetry is relevant to the dynamics of surfaces. Since the functional dependence of $X^\mu(\sigma, \tau)$ is implicit in the action (6.1), we look at the classical equation of motion. For this purpose we consider a droplet with a simple surface and parametrize it by setting $\tau = t$, where $t$ is time. The boundary of the droplet at $t$ is given by a two
dimensional closed curve \( \mathbf{r}(s, t) \). In this discussion we use \( \mathbf{r}(s) \) instead of \( \mathbf{X}(\sigma) \) in order to conform to the notation of [17]. The density \( \rho(x, t) \) is expressed as

\[
\rho(x, t) = \int_{\Omega(t)} d\mathbf{x}' \delta(\mathbf{x} - \mathbf{x}'),
\]

and its variation with respect to a boundary change is given by

\[
\delta \rho(x, t) = \oint ds \mathbf{r} \times \mathbf{r}_s \delta(\mathbf{x} - \mathbf{r}(s, t)),
\]

and a similar expression for \( \rho_V \). The variation of action (6.1) is then given by

\[
\delta S = \int dt \oint ds \oint ds' \mathbf{r}_s(s, t) \times \delta \mathbf{r}(s, t)
\]

\[
\left( \text{Im} \ln(z_+(s) - z_-(s')) - \text{Im} \ln(z_+(s') - z_-(s)) \right) \mathbf{r}_s(s', t) \times \mathbf{r}_t(s', t)
\]

\[
+ \int dt \oint ds \mathbf{r}_s(s, t) \times \delta \mathbf{r}(s, t) A_0(\mathbf{r}(s, t)),
\]

where \( z_+(s), z_-(s) \) are points infinitesimally close to the boundary \( z(s) \), inside and outside respectively. Since

\[
\text{Im} \ln(z_+(s) - z_-(s')) - \text{Im} \ln(z_+(s') - z_-(s)) = \pi \epsilon(s - s') \mod(2\pi)
\]

where

\[
\epsilon(s - s') = \begin{cases} +1 & \text{for } s - s' > 0 \\ -1 & \text{for } s - s' < 0 \end{cases}
\]

we obtain the following equation of motion:

\[
2\pi \mathbf{r}_t \times \mathbf{r}_s = \frac{\partial}{\partial s} A_0(\mathbf{r}(s, t)).
\]

This equation is invariant under time dependent reparametrization of \( s \). It is amusing to notice that the enclosed area is a constant of motion as a consequence of (6.7), since

\[
\frac{\partial}{\partial t} \int d\mathbf{x} \rho(\mathbf{x}, t) = \oint ds \mathbf{r}_t \times \mathbf{r}_s = \frac{1}{2\pi} \oint ds \frac{\partial}{\partial s} A_0(\mathbf{r}(s, t)) = 0.
\]

After all it merely states the fermion number conservation.
In a recent paper, Goldstein and Petrich [17] discuss the relation between the KdV hierarchy and closed curve dynamics of the form

\[ \mathbf{r}_t = U \mathbf{n} + W \mathbf{t}, \quad (6.9) \]

where \( \mathbf{n} \) and \( \mathbf{t} \) are normal and tangential unit vectors respectively. By a suitable reparametrization of \( s \) one can set \( \mathbf{t} = \mathbf{r}_s \). Thus, the dynamics of our curve corresponds to that of

\[ U = \frac{1}{2\pi} \frac{\partial}{\partial s} A_0(\mathbf{r}), \]

a total derivative in \( s \), and arbitrary \( W \). Since \( W \) is arbitrary, this includes a much wider class than the KdV.

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FIGURES

FIG.1. Rearrangement of the boundary.

FIG.2. Fission and fusion of droplets.