I. Introduction

In the great number of topological insulators (TI) and topological superconductors (TSC) discovered or proposed [1–9], the transition from topologically trivial to nontrivial phases can in general be triggered by varying all kinds of energy parameters, e.g. chemical potential [10–13], hopping [14], interface coupling [15–24], etc. The canonical way of identifying the topological phase transition is to first consult the dimension and symmetry class [25, 26] of the system to see if topologically nontrivial phases are possible, and if so then seek for the value of the driving energy parameter at which the spectral gap closes at certain momenta, since topological phase transition necessarily involves band inversion. Ideally, the topological invariant jumps discretely when the system undergoes a topological phase transition, therefore it seems rather ambiguous to identify any asymptotic critical behavior from the topological invariant itself.

Drawing analogy from the Landau order parameter paradigm, a profound question is whether there exists a universal scaling scheme that can judge the topological phase transitions driven by any energy parameter in any dimension and symmetry class, and if the scaling procedure is by any means different from Kadanoff’s scaling theory [27] since the systems under question possess no Landau order parameter. Moreover, it would be of tremendous usage if the scaling scheme also renormalizes the driving energy parameter. The proposal related to entanglement entropy first shed a light on this issue [28–30]. On the other hand, since a great number of TIs and TSCs are characterized by the topological invariants that are often calculated from the momentum space integration of a certain integrand function that represents the curvature of the many-body state, it is natural to suspect that a scaling scheme that renormalizes the integrand function can be used to judge topology. Depending on the system, the integrand function may be Berry curvature [31–33], Berry connection [34], or any other quantity whose integration gives the topological invariant. For the sake of a general discussion, hereafter the integrand function is referred to as the curvature function, and the topological invariant calculated from the integration of the curvature function is referred to as the winding number (to draw analogy to the number of knots in a closed string, and...
to distinguish it from other means of calculating topological invariants). If a scaling scheme for the curvature function exists and gives a RG flow of the driving energy parameter that judges the topological phase transition, the following criteria are expected: (1) the fixed point in the parameter space of the driving energy parameter has a particular configuration of curvature function that is invariant under the scaling procedure. (2) any point in the parameter space flows into some fixed point while preserving the winding number during the flow. (3) Different fixed points correspond to different winding numbers.

In this article, we propose a scaling procedure that satisfies the aforementioned criteria for inversion-symmetric systems. The motivation comes from a simple observation: to know the number of knots that a messy string contains, one can either integrate the curvature along the string (integrating the curvature function to get winding number), or simply stretch the string until the knots become obvious (the proposed scaling procedure). A single equation is proposed to implement this knot-tying principle in TI and TSC: let \( F(\textbf{k}, \Gamma) \) be the curvature function at momentum \( \textbf{k} \) from which the winding number in \( d \)-dimension \( \mathbb{C} = \int d^d\textbf{k} F(\textbf{k}, \Gamma) \) is calculated, with \( \Gamma \) the driving energy parameter that controls the topology. Given a \( \Gamma \) in the parameter space, we seek for a new \( \Gamma' \) that satisfies

\[
F(\textbf{k}_0, \Gamma') = F(\textbf{k}_0 + \delta\textbf{k}, \Gamma)
\]

where \( \textbf{k}_0 \) is a high symmetry point, and \( \delta\textbf{k} \) is a small deviation away from it satisfying \( F(\textbf{k}_0 + \delta\textbf{k}, \Gamma) = F(\textbf{k}_0 - \delta\textbf{k}, \Gamma) \). As we shall see in the examples below, the operation of equation (1) in inversion-symmetric TI or TSC correctly captures the topological phase transition driven by any energy parameter. The fixed point is reached by iteratively solving \( \Gamma \), and by considering a small deviation \( \delta\textbf{k} \), one obtains the RG equation for \( \Gamma \) in the form of differential equation. The curvature function gradually evolves into a fixed point configuration that is invariant under this procedure, analogous to a string with its knots tight and cannot be stretched anymore. Moreover, for a great variety of models, an asymptotic universal critical behavior of the curvature function near the gap-closing momenta is revealed despite the system in general has no Landau order parameter or any short range correlation. The critical behavior may be verified by ultracold atoms in optical lattices in the case that the curvature function is the Berry curvature.

II. Deviation-reduction mechanism

To explain the mechanism behind equation (1) and demonstrate its analogy to knot-tying, consider a inversion-symmetric TI or TSC defined on a \( d \)-dimensional cubic lattice. If the system at \( \Gamma \) and at the fixed point \( \Gamma' \) have the same topology, then the curvature function at \( \Gamma \) can be expanded by

\[
F(\textbf{k}, \Gamma) = F_f(\textbf{k}, \Gamma') + F_i(\textbf{k}, \Gamma)
\]

\[
= F_f(\textbf{k}, \Gamma') + \sum_{m_1} \sum_{m_2} \ldots \sum_{m_d} \lambda_{m_1, \ldots, m_d} \left( \prod_{i=1}^d \cos m_i k_i \right).
\]

(2)

where \( F_f(\textbf{k}, \Gamma') \) is the fixed point curvature function that satisfies \( F_f(\textbf{k}_0, \Gamma') = F_f(\textbf{k}_0 + \delta\textbf{k}, \Gamma') \), i.e. it is invariant under the operation of equation (1), and \( F_i(\textbf{k}, \Gamma) \) is the deviation away from this fixed point configuration at \( \Gamma \), which has Fourier component \( \lambda_{m_1, \ldots, m_d} \). The expansion of \( F_f(\textbf{k}, \Gamma) \) is sound because it must not contribute to the integration of winding number

\[
\mathcal{C} = \int d^d\textbf{k} F_f(\textbf{k}, \Gamma') = \int d^d\textbf{k} F_f(\textbf{k}, \Gamma'),
\]

(3)

such that the system at \( \Gamma \) and at \( \Gamma' \) have the same topology. Suppose we choose a particular high symmetry point such as \( \textbf{k}_0 = (0, \pi, 0, \ldots, \pi) \), and a small displacement away from it along \( j \)th coordinate \( \delta\textbf{k} = (0, 0, \ldots, \delta k_j, 0, 0) \). Note we apply the operation of equation (1) by using equation (2). Expanding around \( \textbf{k}_0 \) gives

\[
F_f(\textbf{k}_0, \Gamma') - F_i(\textbf{k}_0, \Gamma') = -\frac{1}{2} m_j^2 \delta k_j^2 \sum_{m_1} \sum_{m_2} \ldots \sum_{m_d} \text{sgn}(m_{m_1}) \lambda_{m_1, \ldots, m_d} \tag{4}
\]

This means the deviation at \( \textbf{k}_0 \) changes under this operation according to the slope of the deviation at \( \textbf{k}_0 + \delta\textbf{k} \). If the initial value \( \Gamma \) gives \( F_f(\textbf{k}_0, \Gamma') > 0 \) (\(< 0 \)), i.e. a positive (negative) deviation at \( \textbf{k}_0 \), then \( F_f(\textbf{k}, \Gamma') \) must curve down (curve up) as moving from \( \textbf{k}_0 \) to \( \textbf{k}_0 + \delta\textbf{k} \) in order to conserve the winding number, so \( \partial_{\delta\textbf{k}} F_f(\textbf{k}, \Gamma')|_{\textbf{k}=\textbf{k}_0+\delta\textbf{k}} < 0 \) (\(> 0 \)) and consequently \( |F_f(\textbf{k}_0, \Gamma')| \) is reduced under this scaling procedure. Thus continuously applying equation (1) makes \( F_f(\textbf{k}, \Gamma') \) approaching zero and \( F(\textbf{k}, \Gamma) \) approaching \( F_f(\textbf{k}, \Gamma') \). In other words, the principle behind equation (1) is that the deviation of the curvature function from its fixed point configuration is gradually reduced under this scaling procedure. As we shall see below, in \( d = 1 \) systems, this is synonymous to stretching a messy string to reduce its messiness until the knots are tight. In general, there could be a situation that \( F_f(\textbf{k}_0, \Gamma') > 0 \) and \( \partial_{\delta\textbf{k}} F_f(\textbf{k}_0, \Gamma')|_{\textbf{k}=\textbf{k}_0+\delta\textbf{k}} > 0 \) at a given \( \Gamma \) so \( F_f(\textbf{k}_0, \Gamma') \) increases at the beginning of the scaling process, but \( |F_f(\textbf{k}_0, \Gamma')| \) must eventually be reduced because the scaling process still leads to \( \partial_{\delta\textbf{k}} F_f(\textbf{k}, \Gamma')|_{\textbf{k}=\textbf{k}_0+\delta\textbf{k}} < 0 \) in order to conserve winding number.

Expanding equation (1) in both \( \textbf{d} \Gamma' = \Gamma' - \Gamma \) and \( \delta k_j^2 \equiv \text{d} \Gamma \) yields the RG equation. In the leading order,

\[
\frac{d\Gamma'}{d\Gamma} = \frac{1}{2} \frac{\partial^2 F_f(\textbf{k}, \Gamma')|_{\textbf{k}=\textbf{k}_0}}{\partial k_j^2} F_f(\textbf{k}_0, \Gamma').
\]

(6)

This explains that the proper scaling parameter should be defined as quadratic in the displacement \( \delta k_j^2 \equiv \text{d} \Gamma \), whereas the \( \text{d}k_j = \text{d} \Gamma \) in equation (5) is merely introduced to demonstrate
the deviation-reduction mechanism. Since the RG equation involves only one variable $\Gamma$, it is always possible to map the RG flow into the motion of an overdamped particle in a conservative potential [35, 36]. Equation (2), together with the fact that at critical point $\Gamma$, the curvature function diverges at the gap-closing momentum which is at a high symmetry point $k_0$ if the system is inversion-symmetric [37], indicate that the curvature function near the gap-closing point $k_0$ can be expanded by

$$F(k_0 + \delta k, \Gamma) = \frac{F(k_0, \Gamma)}{1 \pm \xi^2 |\delta k|^2}$$

(7)

As shown below, $\xi$ explicitly characterizes the scale invariance at the fixed point or critical point. Note that although equation (7) is similar to the Ornstein–Zernike form of correlation function [38], $\xi$ does not represent the correlation length of short range fluctuations which is obviously absent in T1 and TSC. Instead, it is a length scale defined from the curvature function near the gap-closing point that signals the scale invariance under the proposed scaling scheme. Alternatively, one may use its inverse $\kappa = 1/\xi$ as a momentum scale to characterize the scale invariance, which works equally well.

Two remarks are made before we move to concrete examples. Firstly, as in any RG procedure, this scaling scheme itself does not give a meaning to the fixed point, i.e. it does not tell us whether the fixed point is topologically trivial or nontrivial, which may nevertheless be clarified by direct calculation of the curvature function at gap-closing point which stops changing), as indicated in figure 1(c), both have different meaning than those in Kadanoff dimensional continuum [41], the low energy sector of which has the generic Dirac form.

$$\delta t' - \delta t = \delta t, \quad (\delta k)^2 = dl,$$

and using $\partial_{\delta t} \varphi_k(\delta t) = \partial_{\delta t} \arctan(\text{Im} h_k^2/\text{Re} h_k^2)$ one obtains the RG equation at the leading order

$$\frac{d\delta t}{dl} = \frac{\delta t}{4} \left(1 - \frac{\delta t^2}{l^2}\right) \text{ if } k_0 = 0,$$

$$\frac{d\delta t}{dl} = \frac{l^2}{4\delta t} \left(1 - \frac{\delta t^2}{l^2}\right) \text{ if } k_0 = \pi,$$

(11)

both reproduce the correct critical points at $\delta t_0 = 0$ and fixed points at $\delta t_f = \pm t$, as indicated by the RG flow in figure 1. Through directly calculating the winding number, the $\delta t_f = t$ fixed point is topologically trivial and the $\delta t_f = -t$ fixed point is nontrivial. The example shown in figure 1(a) (red line) clearly demonstrates the deviation-reduction mechanism of equation (5), where the initial value $\Gamma = \delta t$ gives $F(k_0, \Gamma) > 0$ and $\partial_{\delta t} F(k, \Gamma)|_{k = k_0 + \delta k} < 0$ such that $F(k_0, \Gamma)$ is gradually reduced to zero under the operation of equation (1). The choice of different $k_0$ means different $F(k_0, \Gamma)$ to start with, hence the speed of converging to the fixed point configuration is also different, as reflected in the two equations in equation (11). Note that the phase transition occurs when the gap $k_0 = \pi$ closes, but the scaling scheme works in this model even if one chooses $k_0 = 0$ that is not the gap-closing point.

Alternatively, one can introduce the ratio $\gamma = (t - \delta t)/(t + \delta t)$ and discuss the topological phase transition upon tuning $\gamma$. Using equation (1) with $k_0 = 0$ leads to

$$\frac{d\gamma}{dl} = \beta(\gamma) = \frac{\gamma - 1}{2(\gamma + 1)}$$

(12)

which well reproduces the two critical points at $\gamma = \pm 1$. Thus the proposed scaling scheme is valid whether the system is parametrized by $\delta t$ or $t$.

From equation (7) we obtain the $\xi$ deduced from the gap-closing point $k_0 = \pi$

$$\xi = \left|\frac{1}{4\delta t} + \frac{1}{\delta t^2}\right|^{1/2}$$

(13)

As shown in figure 1, $\xi = \infty$ at the critical point $\delta t = 0$, and $\xi = 0$ at the topologically nontrivial fixed point $\delta t = -t$, both signature the scale invariance [40]. We emphasize that the scaling scheme here is a procedure akin to knot-tying, and scale-invariance means that the curvature function converges to a configuration analogous to a string with all its knots tight (curvature function at gap-closing point stops changing), as indicated in figure 1(c), both have different meaning than those in Kadanoff’s scaling theory. The topologically trivial fixed point $\delta t = t$ has $\xi = 11/\pi^2$, meaning that while the amplitude of $\partial_{\delta t} \varphi_k(\delta t)$ is approaching zero everywhere, its functional form is approaching the first harmonic $\cos k$.

III. Applications

III.A. Su–Schrieffer–Heeger (SSH) model

To demonstrate the scaling scheme in $d = 1$, we start from the spinless SSH model [14] with periodic boundary condition (PBC), described by the Hamiltonian

$$H = \sum_i (t + \delta t) c_i^+ c_i + (t - \delta t) c_{i+1}^+ c_i + v c_i$$

(8)

The topology of this model is judged by the winding number $C$ of the operator $h_k(\delta t) = (t + \delta t) + (t - \delta t) e^{i \delta t} |h_k(\delta t)| e^{-i \phi_k(\delta t)}$ in the complex space when one goes through the entire Brillouin zone (BZ). Defining $q_k = h_k/|h_k| = e^{-i \phi_k(\delta t)}$, the winding number is calculated from the curvature function which is the Berry connection [34] $\partial_{\delta t} \varphi_k(\delta t)$ in this case,

$$C = \frac{1}{2\pi} \oint d\delta t q_k^{-1} \partial_{\delta t} q_k = \frac{1}{2\pi} \int d\delta t \partial_{\delta t} \varphi_k(\delta t).$$

(9)

We now search for a new $\delta t'$ by applying equation (1)

$$\partial_{\delta t} \varphi_k(\delta t')|_{k = k_0} = \partial_{\delta t} \varphi_k(\delta t)|_{k = k_0 + \delta k}$$

(10)

III.B. Chern insulators in a continuum

To demonstrate the feasibility of equation (1) in higher dimensions, we consider spinless Chern insulators in the $d$-dimensional continuum [41], the low energy sector of which has the generic Dirac form
Figure 1. (a) The proposed scaling process, equation (1), applied to the topologically nontrivial phase of the SSH model with the choice $k_0 = \pi$. Given an initial $\delta t$ and the corresponding Berry connection $\partial_k \phi_k(\delta t)$ (red line), we find a new $\delta t'$ by demanding $\partial_k \phi_k(\delta t')$ at $k_0 + \delta k$ to be equal to that of $\partial_k \phi_k(\delta t)$ at $k_0$, as indicated by the dash line. This procedure reduces the deviation away from and leads to the fixed point configuration $\phi_k$ expressed as a vector field in the complex space for each configuration is indicated by colored arrows. (b) The length scale $\xi$ defined in equation (7) as a function of $\delta t$. (c) The RG flow of $\delta t$. When joining the $\phi_k$ arrows in (a) at a particular $\delta t$ head to tail in sequence to form a string, this scaling procedure resembles stretching the string to reveal whether it has a knot, as indicated by color arrows.

Figure 2. (a) Berry curvature of the topologically nontrivial phase of $d = 2$ Chern insulators in the continuum, which has the form of equation (16), at evenly spaced values of $0.1 \leq M \leq 1.0$. All the lines evolve to the fixed point configuration labeled by the blue line under the proposed scaling procedure. (b) The length scale $\xi$ defined in equation (7) diverges at $M_c = 0$ and vanishes at $M_f = (d + 1)/(2d + 4)B$. We set $B = 1$ in these plots. (c) The RG flow of $M$. 
One sees that choosing a particular \( \phi_\sigma \) and the lower sign \( -\) at high symmetry points for the four phases of \( \mathcal{M} \), writing the energy parameter to be renormalized \( \phi_\sigma \) and \( k \) while keeping \( \phi_\sigma \) in (a). (b) The signs of \( \delta \phi \) at high symmetry points \( \mathbf{k}_0 \) with the same \( \delta \mathbf{k} = (\delta k_x, 0) \). One sees that choosing a particular \( \mathbf{k}_0 \) captures the transition caused by gap-closing at this point, in accordance with the sign change of \( \delta \phi \) in (a).

\[
H(\mathbf{k}) = \sum_i d_i(\mathbf{k})I_i
\]

where \( I_i \) satisfy the Clifford algebra. The curvature function from which the winding number is calculated, take \( d = 2 \) and \( d = 4 \) as examples, is

\[
F = \varepsilon^{abc} \partial_a \partial_b \partial_e \partial_c \epsilon_
u \quad \text{for } d = 2
\]

\[
F = \varepsilon^{abcd} \partial_a \partial_b \partial_e \partial_d \epsilon_
u \quad \text{for } d = 4
\]

which has the generic form

\[
F(\mathbf{k}, M) = \frac{M + Bk^2}{\alpha(k^2 + (M - Bk^2)^2)^{d+1/2}}
\]

where the prefactor \( \alpha \) depends on symmetry and dimension but is unimportant for our argument. Using equation (1) with the only high symmetry point \( \mathbf{k}_0 = (0, 0) \) and \( \delta \mathbf{k} = (\delta k_x, 0) \), writing the energy parameter to be renormalized as \( M' = M = -\frac{dM}{dl} \) while keeping \( B \) constant, and defining \( |\delta \mathbf{k}|^2 = dl \), the leading order RG equation is

\[
\frac{dM}{dl} = \left( \frac{d+1}{2d} \right) \frac{1}{M} - \left( \frac{d+2}{d} \right) B
\]

which has generically a critical point at \( M_c = 0 \), and the two fixed points \( M_f = (d+1)/(2d+4)B \) and \( M_f = -\infty \) assuming \( B > 0 \). The length scale calculated from equation (7)

\[
\xi = 2dM - \frac{(d+2)B}{M} \left( \frac{1}{M} \right)^{1/2}
\]

vanishes at the fixed points and diverges at the critical point. The results for \( d = 2 \) spinless Chern insulators, which have \( d_1 = k_x \), \( d_2 = k_y \), and \( d_3 = M - Bk^2 \) are shown in figure 2, where the curvature function is the Berry curvature [41]. In contrast to the knot-tying picture in \( d = 1 \), the scaling procedure in \( d = 2 \) is a process to stretch the skyrmion texture of \( d(\mathbf{k}) \) in equation (14) without changing the skyrmion number [41] until the curvature function flattens to second order at the gap-closing point (blue line in figure 2(a)).

Two systems of similar kind are Haldane’s \( d = 2 \) graphene model [1] and Kane–Mele model [2, 3]. Consider the spinless Haldane model whose expansion around \( \mathbf{K} \) and \( \mathbf{K}' \) points of the reciprocal space of the hexagonal lattice is described by the Hamiltonian [41]

\[
\mathcal{H}(\mathbf{k}_0 + \delta \mathbf{k}) = -\frac{3}{2} c \cos \phi + \frac{3}{2} d (\delta k_x, \delta k_y, \delta k_z) + \left( M \pm 3 \sqrt{3} \sin \phi \right) \sigma_z
\]

where upper sign is for \( \mathbf{k}_0 = \mathbf{K} \) and the lower sign \( \mathbf{k}_0 = \mathbf{K}' \). Applying equations (15) and (1) yields

\[
\frac{dM}{dl} = \frac{3}{4} \frac{(3\ell^2/2)^2}{M^2 + 3\sqrt{3} \sin \phi}.
\]

\[
\xi = \left( \frac{3\ell^2/2}{M^2 + 3\sqrt{3} \sin \phi} \right)^{1/2},
\]

correctly reproducing the critical points at \( M_c = \pm 3 \sqrt{3} \sin \phi \).

III.C. Chern insulators on a cubic lattice

To discuss the Chern insulators on a \( d \)-dimensional cubic lattice, we consider equation (14) in \( d = 2 \) and \( d = 4 \) with components [41]

\[
\mathbf{d} = \left( \sin k_x, \sin k_y, \ldots, \sin k_d, M - 2B \left( \frac{d}{d} \cos k_i \right) \right).
\]

The \( 2^d \) high symmetry points in the first quartet \( \mathbf{k}_0 = (k_0, k_0, ..., k_0) \) consist of each \( k_0 \) being either 0 or \( \pi \). The curvature function expanded around each \( \mathbf{k}_0 \) takes the form
\[ F(\mathbf{k}_0 + \delta \mathbf{k}, M) = \frac{(-1)^N [(M - 4BN_c) + \Xi(\delta \mathbf{k})] }{[\delta \mathbf{k}]^2 + (M - 4BN_c) - \Xi(\delta \mathbf{k})^2} \quad \cdot \]
\[ \Xi(\delta \mathbf{k}) = B \sum_{i\in \pi} \delta k_i^2 - \sum_{i\in \pi} \delta k_i^2 \quad \quad \quad (22) \]

where \( \sum_{i\in \pi} (\sum_{i\in \pi}) \) denotes summation over \( \delta k_i \) at which \( k_0 = 0 (k_0 = \pi) \), and \( N_i \) is the number of \( \pi \)'s in \( \{k_0\} \). Applying equation (1) with \( \delta \mathbf{k} = \delta \mathbf{\hat{s}}, \delta \mathbf{\hat{c}} \) along one particular coordinate \( \mathbf{s} \) yields
\[ \frac{dM}{dl} = \frac{1}{2} \left( \frac{d+1}{M - 4BN_c} \right) \mp \frac{d+2}{d} B, \quad (23) \]
and hence the generic fixed points \( M_f = 4BN_c \pm (d + 1)/(2d + 4)B \) and \( M_f = \mp \infty \) when choosing a particular \( k_0 \). The length scale \( \xi \) defined from the gap-closing point satisfies
\[ \xi = \left[ \left( \frac{d+1}{2} \right) \frac{1}{(M - 4BN_c)^2} \mp \frac{d+2}{d} B \right]^{1/2} \quad \quad \quad (24) \]

The top sign in each \( \pm \) or \( \mp \) corresponds to the case when the component of \( k_0 \) in the scaling direction \( \delta \mathbf{k} = \delta \mathbf{\hat{s}}, \delta \mathbf{\hat{c}} \) is \( k_0 = 0 \), and the bottom sign is when \( k_0 = \pi \). Because \( N_i \) takes any integer value from 0 to \( d \), generically there are \( d + 1 \) critical points located at \( M_c = 4BN_c \). The phase transition at a particular \( M \), takes place when the gap closes [37, 41] at one set of \( k_0 \)'s that have the same \( N_i \), in accordance with applying equation (1) at these \( k_0 \)'s to capture the change of curvature function near them, as shown for \( d = 2 \) in figure 3. The chiral \( p \)-wave superconductors in the continuum and in the lattice [25] have the same generic form of curvature function as the Chern insulators [41] (with the replacement \( M \rightarrow \) chemical potential and \( B \rightarrow 1/2 \times \) effective mass), and hence practically the same critical points and critical behavior. We remark that the SSH model [42] and Haldane’s graphene model [43] have been realized by ultracold atoms in optical lattices, where the systems can be driven close to the critical point. The predicted critical behavior of \( \xi \) can be verified in the systems where the curvature function is the Berry curvature, which may be measurable by detecting the anomalous velocity associated with the Berry curvature [43] or momentum space interferometry techniques [44, 45].

IV. Conclusions

In summary, we present a scaling procedure to judge topological phase transitions driven by any energy parameter. The procedure is valid for inversion-symmetric models in any dimension and symmetry class provided the topological invariant is calculated from the integration of a certain curvature function. Our formalism reveals that the concept of scaling in topologically ordered systems falls into a completely different realm than Kadanoff’s scaling theory in the Landau order parameter paradigm. Based on a simple knot-tying picture, the scaling procedure renormalizes the curvature function while keeping the winding number intact, through which the RG flow of the driving energy parameter \( \Gamma \) is obtained, as well as the RG equation, equation (6), under an infinitesimal operation. In essence, the scaling procedure uses the divergence of second derivative of the curvature function at the gap-closing momentum to find the critical point, and uses the flattening of the second derivative to find the fixed point, both achieved under a single operation of equation (1). A length scale defined from the Berry curvature near the gap-closing momentum shows an asymptotic universal critical behavior that diverges at the critical point in first power and vanishes at the nontrivial fixed point in square root if it is finite
\[ \xi \propto \left| \Gamma - \Gamma_f \right|^{1/2} \left| \Gamma - \Gamma_c \right|^{-1} \quad \text{if} \quad \Gamma_f \neq \pm \infty \]
\[ \xi \propto \left| \Gamma - \Gamma_c \right|^{-1} \quad \text{if} \quad \Gamma_f = \pm \infty \quad \quad \quad (25) \]

for a variety of inversion-symmetric systems examined. Applications to a broader class of models, such as those driven by interactions or inversion-asymmetric models, will be subject to future investigations.

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