POTENTIAL OPERATORS ASSOCIATED WITH HANKEL AND HANKEL-DUNKL TRANSFORMS

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Abstract. We study Riesz and Bessel potentials in the settings of Hankel transform, modified Hankel transform and Hankel-Dunkl transform. We prove sharp or qualitatively sharp pointwise estimates of the corresponding potential kernels. Then we characterize those $1 \leq p, q \leq \infty$, for which the potential operators satisfy $L^p - L^q$ estimates. In case of the Riesz potentials, we also characterize those $1 \leq p, q \leq \infty$, for which two-weight $L^p - L^q$ estimates, with power weights involved, hold. As a special case of our results, we obtain a full characterization of two power-weight $L^p - L^q$ bounds for the classical Riesz potentials in the radial case. This complements an old result of Rubin and its recent reinvestigations by De Nápoli, Drelichman and Durán, and Duoandikoetxea.

1. Introduction

In their seminal article [25], Muckenhoupt and Stein outlined a program of development of harmonic analysis in the framework of the modified Hankel transform $H_\alpha$. This context emerges naturally in connection with radial analysis in Euclidean spaces. Indeed, it is well known that the Fourier transform of a radial function in $\mathbb{R}^n$, $n \geq 1$, reduces directly to the modified Hankel transform of order $\alpha = n/2 - 1$. Moreover, the radial part of the standard Laplacian in $\mathbb{R}^n$ is the Bessel operator $\frac{d^2}{dz^2} + \frac{2\alpha + 1}{z} \frac{d}{dz}$, $\alpha = n/2 - 1$, which is the natural ‘Laplacian’ in harmonic analysis associated with $H_\alpha$. In [25, Section 16], among other results, an analogue of the celebrated Hardy-Littlewood-Sobolev theorem was stated for fractional integrals (Riesz potentials) corresponding to $H_\alpha$.

Recent years brought a growing interest in harmonic analysis related to Hankel transforms/Bessel operators. For instance, Betancor, Harboure, Nowak and Viviani [9] delivered a thorough study of mapping properties of maximal operators, Riesz transforms and Littlewood-Paley-Stein type square functions in the settings of modified and non-modified Hankel transforms. This, as well as many earlier results (see the references in [9]), was done in dimension one. Recently, harmonic analysis in the context of Bessel operators was developed in higher dimensions, see Betancor, Castro and Curâbelo [6, 7], Betancor, Castro and Nowak [8], and Castro and Szarek [13]. More recently, in a similar spirit Castro and Szarek [14] investigated fundamental harmonic analysis operators in a wider Hankel-Dunkl setting. The latter situation is a special and the most explicit case of a general framework based on the Dunkl Laplacian and the Dunkl transform, when the underlying Coxeter group is isomorphic to $\mathbb{Z}_2^\infty$. For more details on the Dunkl theory we refer to the survey article [30].

In this paper we study Riesz and Bessel potentials associated with Hankel and Hankel-Dunkl transforms in dimension one. We prove sharp pointwise estimates of the corresponding Riesz potential kernels (Theorems 2.1 and 2.14) and qualitatively sharp pointwise estimates for the related Bessel potential kernels (Theorems 2.7 and 2.18). This enables us to characterize those $1 \leq p, q \leq \infty$, for which the potential operators satisfy $L^p - L^q$ estimates, see Theorem 2.2, Corollary 2.11 and Theorem 2.16 for the

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results on the Riesz potentials, and Theorems 2.8, 2.12 and 2.19 for the results on the Bessel potentials. Moreover, in case of the Riesz potentials, we also determine those $1 \leq p, q \leq \infty$, for which two power-weighted $L^p - L^q$ bounds hold, see Theorems 2.5, 2.10 and 2.15. All these results are Hankel or Hankel-Dunkl counterparts of a series of recent sharp results concerning potential operators in several classic settings related to discrete orthogonal expansions: Hermite function expansions [28], Jacobi and Fourier-Bessel expansions [26], and Laguerre function and Dunkl-Laguerre expansions [29]. The frameworks studied in this paper correspond to continuous orthogonal expansions, and the general approach elaborated in the above mentioned articles applies here as well. However, the present $L^p - L^q$ results have somewhat different flavor.

An interesting by-product of our results is an alternative proof of a radial analogue of the celebrated two power-weighted $L^p - L^q$ estimates for the classical Riesz potentials due to Stein and Weiss [34]. Such an analogue was obtained by Rubin [32] in the eighties of the last century. Rubin’s result, being apparently overlooked, was recently reinvestigated and refined by De Nápoli, Drelichman and Durán [15], and Duoandikoetxea [17]. In fact, Corollary 2.6 below slightly extends the above mentioned results, see the related comments following the statement.

Crucial aspects of our results are their sharpness and completeness. The latter means, in particular, that in each of the contexts we treat the full admissible range of the associated parameter of type, which in the Hankel-Dunkl setting manifests in including an ‘exotic’ case of negative multiplicity functions. Some parts of our results were obtained earlier, by various authors, which is always commented in the relevant places according to our best knowledge. In this connection, we mention again the article of Muckenhoupt and Stein [25], and the works of Gadjev and Aliev [18] where Riesz and Bessel potentials in the context of the modified Hankel transform were investigated, Thangavelu and Xu [36] where Riesz potentials for the Dunkl transform were introduced and studied, Hassani, Mustapha and Sifi [21] where the subject was continued, Betancor, Martínez and Rodríguez-Mesa [10] where Riesz potentials for the Hankel transform were considered, and Ben Salem and Tonalari [5] where Bessel potentials for the Dunkl transform were studied. We note that there is a very wide variety of papers and results pertaining to potential operators in numerous settings. For instance, Anker [2] investigated Riesz and Bessel potentials in the framework of non-compact symmetric spaces and his analysis, like ours, was based on sharp pointwise estimates of the corresponding kernels.

The Riesz and Bessel potentials we study, defined as integral operators, are naturally connected with negative powers of the underlying ‘Laplacians’ defined spectrally. This can easily be seen in case of the Bessel potentials, but the issue is more delicate for the Riesz potentials, see Propositions 2.3, 2.10 and 2.17. Consequently, our results can be used to obtain $L^p - L^q$ bounded extensions of negative powers of Bessel operators and the one-dimensional Dunkl Laplacian. Note also that our precise description of the potential kernels enables further research, including quite natural questions of more general weighted inequalities, weak and restricted weak type estimates, etc. Finally, we remark that acquaintance with the Dunkl theory in its general form is not necessary to follow the part of the paper related to the Hankel-Dunkl transform.

The paper is organized as follows. In Section 2 we introduce the three settings investigated and state the main results. Section 3 is devoted to deriving sharp or qualitatively sharp estimates of the relevant potential kernels. In Section 4 we prove $L^p - L^q$ bounds for the Riesz and Bessel potential operators.

**Notation.** Throughout the paper we use a standard notation consistent with that from [26, 28, 29]. In particular, $X \lesssim Y$ indicates that $X \leq CY$ with a positive constant $C$ independent of significant quantities. We shall write $X \simeq Y$ when simultaneously $X \lesssim Y$ and $Y \lesssim X$. Furthermore, $X \simeq Y \exp(-cZ)$ means that there exist positive constants $C, c_1, c_2$, independent of significant quantities, such that

$$C^{-1}Y \exp(-c_1 Z) \leq X \leq CY \exp(-c_2 Z).$$
In a number of places we will use natural and self-explanatory generalizations of the “$\sim\sim$” relation, for instance, in connection with certain integrals involving exponential factors. In such cases the exact meaning will be clear from the context. By convention, “$\sim\sim$” is understood as “$\sim$” whenever no exponential factors are involved. The symbols “$\vee$” and “$\wedge$” mean the operations of taking maximum and minimum, respectively.

We treat positive kernels and integrals as expressions valued in the extended half-line $[0, \infty]$. Similar remark concerns expressions occurring in various estimates, with the natural limiting interpretations like, for instance, $(0^+)^\beta = \infty$ when $\beta < 0$.

For definitions and terminology related to $L^p$ and weak $L^p$ spaces see, for instance, [24, Chapter 1]. Given a non-negative weight $w$, by $L^p(w^p d\mu)$ we understand the weighted $L^p$ space with respect to a measure $\mu$. This means that $f \in L^p(w^p d\mu)$ if and only if $wf \in L^p(d\mu)$. The latter allows us to abuse slightly the notation by admitting also $p = \infty$. Thus, by convention, $L^\infty(w^\infty d\mu)$ consists of all measurable functions $f$ such that $wf$ is essentially bounded. Given $1 \leq p \leq \infty$, we denote by $p'$ its conjugate exponent, $1/p + 1/p' = 1$.

For an easy distinction between the settings of modified and non-modified Hankel transform we use calligraphic letters when denoting objects within the latter context. For objects in the Hankel-Dunkl setting we use the blackboard bold font, e.g. $L_\alpha$, $\mathbb{H}_\alpha$.

2. Preliminaries and statement of results

Let $\alpha > -1$. Define the measure
\[ d\mu_\alpha(x) = x^{2\alpha + 1} \, dx \]
on $\mathbb{R}_+ = (0, \infty)$ and the functions
\[ \phi_\alpha(u) = u^{-\alpha} J_\alpha(u) \quad \text{and} \quad \varphi_\alpha(u) = \sqrt{u} J_\alpha(u), \quad u > 0, \]
where $J_\alpha$ denotes the Bessel function of the first kind and order $\alpha$. The modified Hankel transform $H_\alpha$ and the (non-modified) Hankel transform $\mathcal{H}_\alpha$ are given by
\[ H_\alpha f(x) = \int_0^\infty \phi_\alpha(xy) f(y) \, d\mu_\alpha(y) \quad \text{and} \quad \mathcal{H}_\alpha f(x) = \int_0^\infty \varphi_\alpha(xy) f(y) \, dy, \quad x > 0, \]
for appropriate functions $f$ on $\mathbb{R}_+$. It is well known that $H_\alpha \circ H_\alpha = \text{Id}$ and $\mathcal{H}_\alpha \circ \mathcal{H}_\alpha = \text{Id}$ on $C_c^\infty(\mathbb{R}_+)$. Moreover,
\[ \|H_\alpha f\|_{L^2(\mathbb{R}_+, d\mu_\alpha)} = \|f\|_{L^2(\mathbb{R}_+, d\mu_\alpha)} \quad \text{and} \quad \|\mathcal{H}_\alpha f\|_{L^2(\mathbb{R}_+, dx)} = \|f\|_{L^2(\mathbb{R}_+, dx)} \]
for $f \in C_c^\infty(\mathbb{R}_+)$. Thus $H_\alpha$ and $\mathcal{H}_\alpha$ extend uniquely to isometric isomorphisms on $L^2(\mathbb{R}_+, d\mu_\alpha)$ and $L^2(\mathbb{R}_+, dx)$, respectively. These extensions are denoted by still the same symbols, even though they do not express via the integral formulas in general. We believe that this will not lead to a confusion, and the exact meaning of $H_\alpha$ and $\mathcal{H}_\alpha$ will always be clear from the context.

Intimately connected with the Hankel transforms are the Bessel operators
\[ L_\alpha = -\frac{d^2}{dx^2} - \frac{2\alpha + 1}{x} \frac{d}{dx} \quad \text{and} \quad \mathcal{L}_\alpha = -\frac{d^2}{dx^2} - \frac{1/4 - \alpha^2}{x^2}. \]
Considered initially on $C_c^2(\mathbb{R}_+)$, they are symmetric and positive in $L^2(\mathbb{R}_+, d\mu_\alpha)$ or in $L^2(\mathbb{R}_+, dx)$, respectively. The standard self-adjoint extensions of $L_\alpha$ and $\mathcal{L}_\alpha$ (denoted here by the same symbols) are given in terms of $H_\alpha$ and $\mathcal{H}_\alpha$, respectively. More precisely, we have (see [11, Section 4])
\[ L_\alpha f = H_\alpha(y^2 H_\alpha f), \quad \text{Dom} L_\alpha = \{ f \in L^2(\mathbb{R}_+, d\mu_\alpha) : y^2 H_\alpha f \in L^2(\mathbb{R}_+, d\mu_\alpha) \}, \]
and similarly in case of $\mathcal{L}_\alpha$ and $\mathcal{H}_\alpha$. Note that
\[ H_\alpha(L_\alpha f)(x) = x^2 H_\alpha f(x) \quad \text{and} \quad \mathcal{H}_\alpha(\mathcal{L}_\alpha f)(x) = x^2 \mathcal{H}_\alpha f(x), \quad \text{a.a.} \; x > 0, \]
for \( f \in \text{Dom} \, L_\alpha \) or \( f \in \text{Dom} \, L_\alpha \), respectively. These identities are true for all \( x > 0 \) when \( f \in C^2_\infty(\mathbb{R}+) \). The latter fact may be easily verified directly, since \( \phi_\alpha \) and \( \varphi_\alpha \) express eigenfunctions of \( L_\alpha \) and \( L_\alpha \),

\[
(1) \quad L_\alpha \phi_\alpha(xy) = y^2 \phi_\alpha(xy) \quad \text{and} \quad L_\alpha \varphi_\alpha(xy) = y^2 \varphi_\alpha(xy), \quad x, y > 0
\]

(here \( L_\alpha \) and \( L_\alpha \) are the differential operators applied in the \( x \) variable) and \( J_\alpha \) admits the well known asymptotics

\[
(2) \quad J_\alpha(u) \simeq u^\alpha, \quad u \to 0^+, \quad \text{and} \quad J_\alpha(u) = \mathcal{O}(u^{-1/2}), \quad u \to \infty.
\]

The settings of \( L_\alpha \) and \( L_\alpha \) are intertwined by the unitary isomorphism

\[
M_{\alpha+1/2}: L^2(\mathbb{R}+, d\mu_\alpha) \to L^2(\mathbb{R}+, dx), \quad \text{where} \quad M_\alpha f(x) = x^\alpha f(x), \quad x > 0;
\]

in particular, for \( \alpha = -1/2 \) the two contexts coincide. Thus we have \( H_\alpha \circ M_{\alpha+1/2} = M_{\alpha+1/2} \circ H_\alpha \) in \( L^2(\mathbb{R}+, d\mu_\alpha) \) and \( L_\alpha \circ M_{\alpha+1/2} = M_{\alpha+1/2} \circ L_\alpha \) on \( \text{Dom} \, L_\alpha \), and similarly for other objects, for instance fractional integrals investigated in this paper.

For \( \alpha > -1 \), we also consider the Hankel-Dunkl transform (cf. [10])

\[
(3) \quad \mathbb{H}_\alpha f(x) = \int_{-\infty}^{\infty} \psi_\alpha(xy) f(y) dw_\alpha(y), \quad x \in \mathbb{R}.
\]

Here

\[
\psi_\alpha(u) = \frac{1}{2} \left[ \phi_\alpha(|u|) + iu\phi_{\alpha+1}(|u|) \right], \quad u \in \mathbb{R}
\]

(the value \( \psi_\alpha(0) \) is understood in a limiting sense) and \( w_\alpha \) is the measure on \( \mathbb{R} \) given by

\[
dw_\alpha(x) = |x|^{2\alpha+1} dx.
\]

The integral in (3) converges for decent \( f \), in particular for \( f \in C^\infty_\infty(\mathbb{R} \setminus \{0\}) \). It is known that \( (\mathbb{H}_\alpha \circ \mathbb{H}_\alpha) f(x) = f(-x), \quad f \in C^\infty_c(\mathbb{R} \setminus \{0\}) \), and \( \mathbb{H}_\alpha \) extends to an isometry on \( L^2(\mathbb{R}, dw_\alpha) \) (we denote this extension by the same symbol). For \( \alpha \geq -1/2 \) this follows from the general Dunkl theory, see [23] Theorem 4.26; the full range \( \alpha > -1 \) is treated in [27] Proposition 1.3.

The one-dimensional Dunkl Laplacian

\[
L_\alpha f(x) = L_\alpha f(x) + (\alpha + 1/2) \frac{f(x) - f(-x)}{x^2}, \quad x \in \mathbb{R},
\]

considered initially on \( C^2_\infty(\mathbb{R} \setminus \{0\}) \), is symmetric and positive in \( L^2(\mathbb{R}, dw_\alpha) \). A natural self-adjoint extension of \( L_\alpha \) is given by

\[
\mathbb{L}_\alpha f = \mathbb{H}_\alpha^{-1}(y^2 \mathbb{H}_\alpha f), \quad \text{Dom} \, \mathbb{L}_\alpha = \{ f \in L^2(\mathbb{R}, dw_\alpha) : y^2 \mathbb{H}_\alpha f \in L^2(\mathbb{R}, dw_\alpha) \}.
\]

Clearly, \( \mathbb{H}_\alpha(\mathbb{L}_\alpha f)(x) = x^2 \mathbb{H}_\alpha f(x) \) for \( f \in \text{Dom} \, \mathbb{L}_\alpha \) and a.a. \( x \in \mathbb{R} \). This identity holds for all \( x \in \mathbb{R} \) when \( f \in C^2_\infty(\mathbb{R} \setminus \{0\}) \), as can be verified with the aid of (2) and the relation

\[
\mathbb{L}_\alpha \psi_\alpha(xy) = y^2 \psi(xy), \quad x, y \in \mathbb{R},
\]

where \( \mathbb{L}_\alpha \) is the differential-difference operator applied in the \( x \) variable.

The setting of \( \mathbb{H}_\alpha \) and \( \mathbb{L}_\alpha \) was discussed in numerous papers, see for instance [31], where the transform was called the generalized Hankel transform, or more recent articles [36, 14] and references therein. The parameter \( \alpha \) represents the so-called multiplicity function, which is non-negative if and only if \( \alpha \geq -1/2 \). The value \( \alpha = -1/2 \) corresponds to the trivial multiplicity function, and in this case we recover the classical setting of the Fourier transform and the Euclidean Laplacian on \( \mathbb{R} \). When restricted to even functions, \( \mathbb{H}_\alpha \) and \( \mathbb{L}_\alpha \) reduce to \( H_\alpha \) and \( L_\alpha \), and the Hankel-Dunkl setting coincides with the framework of the modified Hankel transform.
2.1. The setting of the modified Hankel transform. The integral kernel \( \{W_t^\alpha\}_{t>0} \) of the Hankel semigroup \( \{\exp(-tL_\alpha)\} \) is given by

\[
W_t^\alpha(x, y) = \int_0^\infty e^{-u^2t} \phi_\alpha(xu) \phi_\alpha(yu) d\mu_\alpha(u), \quad x, y > 0.
\]

The last integral can be computed, in fact we have

\[
W_t^\alpha(x, y) = \frac{1}{2t} \exp \left(-\frac{x^2 + y^2}{4t}\right)(xy)^{-\alpha} I_\alpha \left(\frac{xy}{2t}\right), \quad x, y > 0,
\]

where \( I_\alpha \) denotes the Bessel function of the second kind of order \( \alpha \). The function \( I_\alpha \) is strictly positive on \((0, \infty)\) and satisfies the well known asymptotics

\[
I_\alpha(u) \simeq z^\alpha, \quad u \to 0^+, \quad \text{and} \quad I_\alpha(u) \simeq u^{-1/2} e^u, \quad u \to \infty.
\]

Given \( \sigma > 0 \), consider the negative power \( (L_\alpha)^{-\sigma} \) defined in \( L^2(d\mu_\alpha) \) by means of the spectral theorem. From the form in which the spectral resolution of \( L_\alpha \) is defined in terms of \( H_\alpha \) (see [11, Section 4]) it follows that

\[
(L_\alpha)^{-\sigma} f = H_\alpha(x^{-2\sigma} H_\alpha f), \quad f \in \text{Dom}(L_\alpha)^{-\sigma},
\]

where

\[
\text{Dom}(L_\alpha)^{-\sigma} = \{ f \in L^2(d\mu_\alpha) : x^{-2\sigma} H_\alpha f \in L^2(d\mu_\alpha) \}.
\]

Taking into account the formal identity

\[
(L_\alpha)^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty \exp(-tL_\alpha)\, t^{\sigma-1} dt,
\]

it is natural to introduce the potential kernel

\[
K^{\alpha,\sigma}(x, y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty W_t^\alpha(x, y)\, t^{\sigma-1} dt,
\]

and to consider the corresponding potential operator

\[
I^{\alpha,\sigma} f(x) = \int_0^\infty K^{\alpha,\sigma}(x, y) f(y) d\mu_\alpha(y), \quad x > 0,
\]

with its natural domain \( \text{Dom} I^{\alpha,\sigma} \) consisting of those functions \( f \) for which the above integral converges \( x \)-a.e. We will see in a moment that the integral defining \( K^{\alpha,\sigma}(x, y) \) converges for all \( x \neq y \) when \( \sigma < \alpha + 1 \), and diverges for all \( x, y > 0 \) when \( \sigma \geq \alpha + 1 \). Also \( K^{\alpha,\sigma}(x, y) > 0 \) and \( K^{\alpha,\sigma}(x, y) = K^{\alpha,\sigma}(y, x) \) for all \( x, y > 0 \). Moreover, the heat kernel, and so the potential kernel, satisfy the homogeneity properties

\[
W_t^\alpha(x, y) = r^{2(\alpha+1)} W_{r^2t}(rx, ry), \quad K^{\alpha,\sigma}(x, y) = r^{2(\alpha+1-\sigma)} K^{\alpha,\sigma}(rx, ry), \quad x, y, r > 0,
\]

that result in the following homogeneity of the potential operator:

\[
I^{\alpha,\sigma}(f_r) = r^{-2\sigma} (I^{\alpha,\sigma} f)_r, \quad r > 0.
\]

Here \( f_r(x) = f(rx), r > 0 \), denotes the dilation of a function \( f \).

The exact behavior of \( K^{\alpha,\sigma}(x, y) \) is described in the following.

**Theorem 2.1.** Let \( \alpha > -1 \). If \( 0 < \sigma < \alpha + 1 \), then

\[
K^{\alpha,\sigma}(x, y) \simeq (x + y)^{-2\alpha-1} \begin{cases} \frac{|x - y|^{2\alpha-1}}{\log \frac{2(x+y)}{|x-y|}}, & \sigma < 1/2, \\ log \frac{2(x+y)}{|x-y|}, & \sigma = 1/2, \\ (x + y)^{2\alpha-1}, & \sigma > 1/2, \end{cases}
\]

uniformly in \( x, y > 0 \). If \( \sigma \geq \alpha + 1 \), then \( K^{\alpha,\sigma}(x, y) = \infty \) for all \( x, y > 0 \).

This result enables us to characterize \( L^p - L^q \) boundedness of the potential operator \( I^{\alpha,\sigma} \).

**Theorem 2.2.** Let \( \alpha > -1 \) and \( 0 < \sigma < \alpha + 1 \). Assume that \( 1 \leq p, q \leq \infty \). Then

(i) \( L^p(d\mu_\alpha) \subset \text{Dom} I^{\alpha,\sigma} \) if and only if \( p < \frac{2\alpha + 1}{\sigma} \).
(ii) $I^{\alpha,\sigma}$ is bounded from $L^p(d\mu_{\alpha})$ to $L^q(d\mu_{\alpha})$ if and only if
\[ \frac{1}{q} = \frac{1}{p} - \frac{\sigma}{\alpha + 1} \quad \text{and} \quad 1 < p < \frac{\alpha + 1}{\sigma} \quad \text{and} \quad \alpha \geq -1/2; \]

(iii) $I^{\alpha,\sigma}$ is bounded from $L^1(d\mu_{\alpha})$ to weak $L^q(d\mu_{\alpha})$ for $q = \frac{\alpha + 1}{\alpha + 1 - \sigma}$ if and only if $\alpha \geq -1/2$.

In the case $\alpha \geq -1/2$ the sufficiency part of Theorem 2.2 (ii) was known earlier, see [25, Section 16(j)] and the comments closing Section 2.1. Apart from that, the result seems to be new.

We now explain the way in which $I^{\alpha,\sigma}$ and $(L_\alpha)^{-\sigma}$ are connected. Since the issue is delicate, our approach will be slightly pedantic. As test functions we shall use the space $H_\alpha(C_c^\infty)$ which is dense in $L^2(d\mu_{\alpha})$; here $C_c^\infty = C_c^\infty(\mathbb{R})$. The inclusion $H_\alpha(C_c^\infty) \subset L^2(d\mu_{\alpha})$ and the density follow from the fact that $H_\alpha$ extends to an isometry on $L^2(d\mu_{\alpha})$. But in fact more can be said about $H_\alpha(C_c^\infty)$. Given $g \in C_c^\infty$, $H_\alpha g$ is continuous on $\mathbb{R}$, and
\begin{equation}
H_\alpha g(x) = O(1), \quad x \to 0^+ \quad \text{and} \quad H_\alpha g(x) = O(x^{-k}), \quad x \to \infty
\end{equation}
for each fixed $k \in \mathbb{N}$. The first of these relations is a simple consequence of [2]. The second one can be verified by applying $H_\alpha$ to $(L_\alpha)^{\bar{\sigma}} g$ and then using the symmetry of $L_\alpha$. [1], and again [2]. Hence the inclusion $H_\alpha(C_c^\infty) \subset L^p(d\mu_{\alpha})$, $1 \leq p \leq \infty$, follows. In particular, by Theorem 2.2 (i), $H_\alpha(C_c^\infty) \subset \text{Dom } I^{\alpha,\sigma}$, $0 < \sigma < \alpha + 1$. The question of density of $H_\alpha(C_c^\infty)$ in $L^p(d\mu_{\alpha})$ spaces is more involved. In case $\alpha \geq -1/2$ and $1 < p < \infty$ such density follows from [25, Theorem 4.7].

The next result shows that $I^{\alpha,\sigma}$ and $(L_\alpha)^{-\sigma}$ coincide on $H_\alpha(C_c^\infty)$. Combined with the comments above and the $L^p - L^q$ results for $I^{\alpha,\sigma}$ proved in this paper, it can be used to obtain $L^p - L^q$ bounded extensions of negative powers of $L_\alpha$.

**Proposition 2.3.** Let $\alpha > -1$ and $0 < \sigma < \alpha + 1$. For every $f \in H_\alpha(C_c^\infty)$ we have
\[ I^{\alpha,\sigma} f(x) = (L_\alpha)^{-\sigma} f(x), \quad \text{a.a. } x > 0. \]

**Proof.** Let $f = H_\alpha g$, $g \in C_c^\infty$. It was just explained that $f \in \text{Dom } I^{\alpha,\sigma}$. To check that also $f \in \text{Dom } (L_\alpha)^{-\sigma}$, that is $(\cdot)^{-2\sigma} H_\alpha f \in L^2(d\mu_{\alpha})$, note that $f \in L^1(d\mu_{\alpha})$, hence $g = H_\alpha f$ and, consequently, the desired property follows. Moreover,
\[ (L_\alpha)^{-\sigma} f(x) = H_\alpha((\cdot)^{-2\sigma} g)(x) = \int_0^\infty \phi_\alpha(xy) y^{-2\sigma} g(y) d\mu_{\alpha}(y), \quad \text{a.a. } x > 0. \]

On the other hand, using the definitions of $I^{\alpha,\sigma}$ and $K^{\alpha,\sigma}$, and then interchanging the order of integration (this is easily seen to be legitimate) gives
\begin{equation}
\Gamma(\sigma) I^{\alpha,\sigma} f(x) = \int_0^\infty \int_0^\infty W_t^\alpha(x,y) f(y) d\mu_{\alpha}(y) t^{\sigma - 1} dt.
\end{equation}

We now focus on the inner integral with a fixed $t > 0$. Using the definition of $W_t^\alpha(x,y)$ and then changing the order of integrals (which is justified with the aid of [2] and [10]) we get
\[ \int_0^\infty W_t^\alpha(x,y) f(y) d\mu_{\alpha}(y) = \int_0^\infty \int_0^\infty e^{-u^2 t} \phi_\alpha(xu) \phi_\alpha(yu) f(y) d\mu_{\alpha}(y) d\mu_{\alpha}(u) = \int_0^\infty e^{-u^2 t} \phi_\alpha(xu) H_\alpha f(u) d\mu_{\alpha}(u). \]

Coming back to the integral in (7), this gives us
\[ \Gamma(\sigma) I^{\alpha,\sigma} f(x) = \int_0^\infty \int_0^\infty e^{-u^2 t} t^{\sigma - 1} \phi_\alpha(xu) g(u) d\mu_{\alpha}(u) dt. \]

Since the support of $g$ is separated from 0 and $\infty$, we can once again interchange the order of integration and evaluate first the integral in $t$, which is precisely $\Gamma(\sigma) u^{-2\sigma}$. The conclusion follows. \[ \square \]
Recall the following classical result of E. M. Stein and G. Weiss [34, Theorem B'] concerning two-weight $L^p - L^q$ estimates, with power weights involved, for the Euclidean fractional integral (Riesz)

$$I^\sigma f(x) = \int_{\mathbb{R}^n} \frac{f(y) \, dy}{|x - y|^{n-2\sigma}}, \quad x \in \mathbb{R}^n, \quad 0 < \sigma < n/2.$$ 

**Theorem 2.4** (Stein & Weiss). Let $n \geq 1$ and $0 < \sigma < n/2$. Let $a, b \in \mathbb{R}$ and assume that $1 \leq p \leq q < \infty$. If $a < n/p'$, $b < n/q$, $a + b \geq 0$, and $\frac{1}{q} = \frac{1}{p} + \frac{a+b-2\sigma}{n}$, then

$$\| |x|^{-b} I^\sigma f \|_{L^q(\mathbb{R}^n, dx)} \lesssim \| |x|^a f \|_{L^p(\mathbb{R}^n, dx)}$$

uniformly in $f \in L^p(\mathbb{R}^n, |x|^p dx)$.

In this paper we obtain the following sharp analogue of Theorem 2.4 in the context of the modified Hankel transform.

**Theorem 2.5.** Let $\alpha > -1$ and $0 < \sigma < \alpha + 1$. Let $a, b \in \mathbb{R}$ and assume that $1 \leq p, q \leq \infty$.

(i) $L^p(x^a \, d\mu_\sigma) \subset \text{Dom} I^{\alpha, \sigma}$ if and only if

$$2\sigma - \frac{2a + 2}{p} < a < \frac{2a + 2}{p'} \quad (\text{both when } p = 1).$$

(ii) The estimate

$$\| |x|^{-b} I^{\alpha, \sigma} f \|_{L^q(d\mu_\sigma)} \lesssim \| |x|^a f \|_{L^p(d\mu_\sigma)}$$

holds uniformly in $f \in L^p(x^a \, d\mu_\sigma)$ if and only if the following conditions are satisfied:

(a) $p \leq q$,
(b) $\frac{1}{q} = \frac{1}{p} + \frac{a+b-2\sigma}{2a+2}$,
(c) $a < \frac{2a+2}{p}$ \quad ($\leq$ when $p = q' = 1$),
(d) $b < \frac{2a+2}{q}$ \quad ($\leq$ when $p = q' = 1$),
(e) $\frac{1}{q} \geq \frac{1}{p} - 2\sigma$ \quad ($> \text{ when } p = 1 \text{ or } q = \infty$).

Notice that (e) is superfluous when $\sigma > 1/2$, and in case $\sigma = 1/2$ it is equivalent to $(p, q) \neq (1, \infty)$. Moreover, because of (b), condition (e) in Theorem 2.5 may be replaced by

$$(e') \quad a + b \geq (2a+1)(\frac{1}{p} - \frac{1}{q}) \quad (> \text{ when } p = 1 \text{ or } q = \infty).$$

Finally, observe that (b) is simply forced by the homogeneity [5].

It is well known that for $\alpha = n/2 - 1$ the setting of the modified Hankel transform corresponds to the radial framework on $\mathbb{R}^n$, $n \geq 1$. In particular, if $f$ is a radial function on $\mathbb{R}^n$, $f(x) = f_0(|x|)$, then $-\Delta f(x) = (L_\alpha f_0)(|x|)$ and $I^\sigma f$ and $I^{\alpha, \sigma} f_0(| \cdot |)$ coincide up to a constant factor independent of $f$. Moreover, integration of $f$ in $\mathbb{R}^n$ with respect to Lebesgue measure reduces to integration of $f_0$ against $d\mu_\sigma$. These standard facts together with Theorem 2.5 specified to $\alpha = n/2 - 1$ lead to the following sharp variant of Theorem 2.5 for radially symmetric functions.

**Corollary 2.6.** Let $n \geq 1$ and $0 < \sigma < n/2$. Let $a, b \in \mathbb{R}$ and assume that $1 \leq p, q \leq \infty$. The estimate

$$\| |x|^{-b} I^\sigma f \|_{L^q(\mathbb{R}^n, dx)} \lesssim \| |x|^a f \|_{L^p(\mathbb{R}^n, dx)}$$

holds uniformly in all radial functions $f \in L^p(\mathbb{R}^n, |x|^p dx)$ if and only if the following conditions hold:

(a) $p \leq q$,
(b) $\frac{1}{q} = \frac{1}{p} + \frac{a+b-2\sigma}{n}$,
(c) $a < \frac{p}{p'}$ \quad ($\leq$ when $p = q' = 1$),
(d) $b < \frac{p}{q}$ \quad ($\leq$ when $p = q' = 1$),
(e) $a + b \geq (n-1)(\frac{1}{q} - \frac{1}{p})$ \quad ($> \text{ when } p = 1 \text{ or } q = \infty$).
Under the assumption $1 < p \leq q < \infty$ the sufficiency part of Corollary 2.6 was proved by Rubin [32] Theorem 3] already in 1983. This remarkable result was overlooked and rediscovered recently by De Nápoli, Drelichman and Durán [15, Theorem 1.2]; see also [15, Section 5] for an interesting application to weighted imbedding theorems. Previous partial results in the same direction can be found in [19, 22, 37], see also the comments in [15, Section 1]. As for the necessity part, under the assumption $1 \leq p \leq q < \infty$ this question has recently been studied by Duandikkoetxea [17, Theorem 5.1]; see also [15, Remark 4.2]. Our result completes the previous efforts by including $q = \infty$ and also by proving the necessity of the condition $p \leq q$.

Given $\sigma > 0$, we also consider analogues of the classical Bessel potentials, $(I + L_\sigma)^{-\sigma}$. These are well defined spectrally on the whole $L^2(d\mu_\alpha)$, and can be extended to more general functions by means of an integral representation. Since the integral kernel of the semigroup generated by $-(I + L_\sigma)$ is \{exp$(-t)W_t^\alpha\}_{t>0}$, we introduce the potential operator

$$J^{\alpha,\sigma}f(x) = \int_0^\infty H^{\alpha,\sigma}(x,y)f(y)\,d\mu_\alpha(y), \quad x > 0,$$

where

$$H^{\alpha,\sigma}(x,y) = \frac{1}{1(\sigma)}\int_0^\infty e^{-t}W_t^\alpha(x,y)t^{\sigma-1}\,dt.$$  

Clearly, $0 < H^{\alpha,\sigma}(x,y) < K^{\alpha,\sigma}(x,y)$ for all $x, y > 0$. This implies, in particular, that $J^{\alpha,\sigma}$ inherits positive $L^p - L^q$ mapping properties of $I^{\alpha,\sigma}$, and the same is true for weak type estimates. An analogous remark pertains to Bessel and Riesz potentials in the two other settings of this paper.

The next result provides qualitatively sharp description of the behavior of $H^{\alpha,\sigma}(x,y)$.

**Theorem 2.7.** Let $\alpha > -1$ and let $\sigma > 0$. The following estimates hold uniformly in $x, y > 0$.

(i) If $x + y \leq 1$, then

$$H^{\alpha,\sigma}(x,y) \asymp \chi_{\{\sigma > \alpha + 1\}} \log \frac{1}{x+y} + (x+y)^{-2\alpha-1} \left\{ \begin{array}{ll} |x-y|^{2\sigma-1}, & \sigma < 1/2, \\
\log \frac{2(x+y)}{|x-y|}, & \sigma = 1/2, \\
(x+y)^{2\sigma-1}, & \sigma > 1/2. \end{array} \right.$$  

(ii) If $x + y > 1$, then

$$H^{\alpha,\sigma}(x,y) \asymp (x+y)^{-2\alpha-1}\exp\left\{-c|x-y|\right\} \left\{ \begin{array}{ll} |x-y|^{2\sigma-1}, & \sigma < 1/2, \\
1 + \log^+ \frac{1}{|x-y|}, & \sigma = 1/2, \\
1, & \sigma > 1/2. \end{array} \right.$$  

Thus, among other things, we see that the kernel $H^{\alpha,\sigma}(x,y)$ behaves in an essentially different way depending on whether $(x,y)$ is close to the origin of $\mathbb{R}^2$ or far from it. Moreover, the local behavior (i) is exactly the same as that of the Riesz potential kernel associated with Laguerre function expansions of convolution type, see [29, Theorem 2.1 (i)]. Furthermore, for $\sigma < \alpha + 1$ we have $H^{\alpha,\sigma}(x,y) \simeq K^{\alpha,\sigma}(x,y)$ when $x$ and $y$ stay bounded.

The description of $H^{\alpha,\sigma}(x,y)$ from Theorem 2.7 enables a direct analysis of the potential operator $J^{\alpha,\sigma}$. In particular, it allows us to characterize those $1 \leq p, q \leq \infty$, for which $J^{\alpha,\sigma}$ is $L^p - L^q$ bounded. Notice that the statement below implicitly contains the fact that $L^p(d\mu_\alpha) \subset \text{Dom} J^{\alpha,\sigma}$ for all $1 \leq p \leq \infty$. Moreover, specified to $p = 2$ it allows one to check that $J^{\alpha,\sigma}$ coincides in $L^2(d\mu_\alpha)$ with the negative power $(I + L_\sigma)^{-\sigma}$ defined spectrally.

**Theorem 2.8.** Let $\alpha > -1$, $\sigma > 0$ and $1 \leq p, q \leq \infty$.

(a) If $\alpha \geq -1/2$, then $J^{\alpha,\sigma}$ is bounded from $L^p(d\mu_\alpha)$ to $L^q(d\mu_\alpha)$ if and only if

$$\frac{1}{p} - \frac{\sigma}{\alpha + 1} \leq \frac{1}{q} \leq \frac{1}{p} \quad \text{and} \quad \left( \frac{1}{p}, \frac{1}{q} \right) \notin \left\{ \left( \frac{\sigma}{\alpha + 1}, 0 \right), \left( 1, 1 - \frac{\sigma}{\alpha + 1} \right) \right\}.$$
Proposition 2.9. Let $H$ be a Hankel transform. Moreover, the fact that $R$ this structure is inherited from the Euclidean convolution on the modified Hankel transform can be transmitted to the Hankel transform case. Setting is intertwined with the previous one. Thus some of the results obtained in the framework of the non-modified Hankel transform. The related Riesz potential kernel is

$$K^{\alpha,\sigma}(x) = \frac{1}{\Gamma(\sigma)} \int_0^\infty W_\alpha^\sigma(x,y)t^\sigma-1 dt = (xy)^\alpha+1/2 K^{\alpha,\sigma}(x,y),$$

and the corresponding Riesz potential operator is

$$T^{\alpha,\sigma} f(x) = \int_0^\infty K^{\alpha,\sigma}(x,y)f(y) dy = x^{-\alpha+1/2} T^{\alpha,\sigma} ((\cdot)^{-\alpha-1/2} f)(x), \quad x > 0.$$ 

In view of the above connection between the Hankel potential kernels, Theorem 2.4 gives sharp estimates for $K^{\alpha,\sigma}(x,y)$ as well. It is therefore clear that the natural domains of $T^{\alpha,\sigma}$ and $I^{\alpha,\sigma}$ satisfy

$$\text{Dom } T^{\alpha,\sigma} = M_{\alpha+1/2}(\text{Dom } I^{\alpha,\sigma}).$$

The corresponding homogeneity properties of the heat kernel and the potential kernel read as

$$W_\alpha^\sigma(x,y) = rW_\alpha^\sigma(rx,ry), \quad K^{\alpha,\sigma}(x,y) = r^{1-2\alpha} K^{\alpha,\sigma}(rx,ry), \quad x, y, r > 0,$$

and lead to homogeneity of the potential operator,

$$T^{\alpha,\sigma} f(r) = r^{-2\alpha} (T^{\alpha,\sigma} f)(r), \quad r > 0.$$ 

We define the negative power $(L_\alpha)^{-\sigma}$ and its domain $\text{Dom } (L_\alpha)^{-\sigma}$ by replacing $H_\alpha$ and $L^2(dx)$ by $\mathcal{H}_\alpha$ and $L^2(dx)$, respectively, in the corresponding definitions in the modified Hankel transform setting. It is immediate to check that $(L_\alpha)^{-\sigma}$ and $(L_\alpha)^{-\sigma}$ are also intertwined in the sense that $\text{Dom } (L_\alpha)^{-\sigma} = M_{\alpha+1/2}(\text{Dom } (L_\alpha)^{-\sigma})$, and

$$(L_\alpha)^{-\sigma} f = M_{\alpha+1/2}(L_\alpha)^{-\sigma}(M_{-\alpha-1/2} f), \quad f \in \text{Dom } (L_\alpha)^{-\sigma}.$$ 

Moreover, the fact that $H_\alpha(C^\infty_c) \subset \text{Dom } I^{\alpha,\sigma}$ together with (11) show that $H_\alpha(C^\infty_c) \subset \text{Dom } T^{\alpha,\sigma}$ and $H_\alpha(C^\infty_c)$ is a dense subspace of $L^2(dx)$. These facts combined with Proposition 2.3 justify the following.

Proposition 2.9. Let $\alpha > -1$ and $0 < \sigma < 1/2$. For every $f \in \mathcal{H}_\alpha(C^\infty_c)$ we have

$$T^{\alpha,\sigma} f(x) = (L_\alpha)^{-\sigma} f(x), \quad a.a. \ x > 0.$$
Relation (3) between $\mathcal{I}^{\alpha,\sigma}$ and $I^{\alpha,\sigma}$ and Theorem 2.9 allow us to obtain in a straightforward manner a characterization of weighted $L^p - L^q$ boundedness of $\mathcal{I}^{\alpha,\sigma}$.

**Theorem 2.10.** Let $\alpha > -1$ and $0 < \sigma < \alpha + 1$. Let $a, b \in \mathbb{R}$ and assume that $1 \leq p, q \leq \infty$.

(i) $L^p(x^{ap}dx) \subset \text{Dom} \mathcal{I}^{\alpha,\sigma}$ if and only if
$$2\sigma - \frac{1}{p} - \alpha - \frac{1}{2} < a < \frac{1}{p'} + \alpha + \frac{1}{2} \quad (\text{both} \leq \text{when} \ p = 1).$$

(ii) The estimate
$$\|x^{-b}\mathcal{I}^{\alpha,\sigma}f\|_{L^q(dx)} \lesssim \|x^a f\|_{L^p(dx)}$$
holds uniformly in $f \in L^p(x^{ap}dx)$ if and only if the following conditions are satisfied:

(a) $p \leq q$,
(b) $\frac{1}{q} = \frac{1}{p} + a + b - 2\sigma$,
(c) $a < \frac{1}{p} + \alpha + \frac{1}{2}$ ($\leq$ when $p = q' = 1$),
(d) $b < \frac{1}{q} + \alpha + \frac{1}{2}$ ($\leq$ when $p = q' = 1$),
(e) $a + b \geq 0$ ($> \text{when} \ p = 1 \ or \ q = \infty$).

Let us distinguish the special case of Theorem 2.10 when no weights are involved, i.e. $a = b = 0$.

**Corollary 2.11.** Let $\alpha > -1$ and $0 < \sigma < \alpha + 1$. Assume that $1 \leq p, q \leq \infty$.

(i) If $\alpha \geq -1/2$, then $L^p(dx) \subset \text{Dom} \mathcal{I}^{\alpha,\sigma}$ if and only if $\frac{1}{p} > 2\sigma - \alpha - 1/2$ ($\geq$ if $p = 1$), and $\mathcal{I}^{\alpha,\sigma}$ is bounded from $L^p(dx)$ to $L^q(dx)$ if and only if
$$\frac{1}{q} = \frac{1}{p} - 2\sigma \quad \text{and} \quad p > 1 \quad \text{and} \quad q < \infty.$$

(ii) If $\alpha < -1/2$, then $L^p(dx) \subset \text{Dom} \mathcal{I}^{\alpha,\sigma}$ if and only if $\alpha + 3/2 > \frac{1}{p} > 2\sigma - \alpha - 1/2$ (both $\geq$ if $p = 1$), and $\mathcal{I}^{\alpha,\sigma}$ is bounded from $L^p(dx)$ to $L^q(dx)$ if and only if
$$\frac{1}{q} = \frac{1}{p} - 2\sigma \quad \text{and} \quad \frac{1}{p'} > -\alpha - \frac{1}{2} \quad \text{and} \quad \frac{1}{q} > -\alpha - \frac{1}{2}.$$

Notice that, in view of Corollary 2.11 $\mathcal{I}^{\alpha,\sigma}$ cannot be $L^p - L^q$ bounded when $\sigma \geq 1/2$. The $L^p - L^q$ boundedness from Corollary 2.11 (i) is known earlier, see [10, Theorem 1.3] where the argument was based on the estimate
$$W_t^\alpha(x, y) \leq C_{\alpha}W_t(x - y), \quad x, y, t > 0,$$
or rather its Poisson kernel analogue; here and elsewhere $W_t(u)$ is the Gauss-Weierstrass kernel on $\mathbb{R}$,
$$W_t(u) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{u^2}{4t} \right).$$

The quoted result contains also the weak type $(1, \frac{1}{1-2\sigma})$ estimate for $\mathcal{I}^{\alpha,\sigma}$, in case $\alpha \geq -1/2$ and $\sigma < 1/2$. Our present results show that the following slightly stronger statement is true: given $\sigma < 1/2$, the operator $\mathcal{I}^{\alpha,\sigma}$ is bounded from $L^1(dx)$ to weak $L^{1/(1-2\sigma)}(dx)$ if and only if $\alpha \geq -1/2$. Indeed, for $\alpha \geq -1/2$ the kernel $K^{\alpha,\sigma}(x, y)$ is controlled by $K^{-1/2,\sigma}(x, y) = K^{-1/2,\sigma}(x, y)$, as easily verified with the aid of the asymptotics [3]. Then $\mathcal{I}^{\alpha,\sigma}$ is controlled by $I^{-1/2,\sigma}$, and since $d\mu_{-1/2}(x) = dx$, the weak type of $\mathcal{I}^{\alpha,\sigma}$ follows from Theorem 2.2 (iii) specified to $\alpha = -1/2$. On the other hand, $\mathcal{I}^{\alpha,\sigma}$ is not even defined on $L^1(dx)$ when $\alpha < -1/2$, see Corollary 2.11 (ii).

For $\sigma > 0$, consider the Bessel potentials
$$\mathcal{J}^{\alpha,\sigma}f(x) = \int_0^\infty \mathcal{H}^{\alpha,\sigma}(x, y)f(y)dy, \quad x > 0,$$
where
$$\mathcal{H}^{\alpha,\sigma}(x, y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t}W_t^\alpha(x, y)t^{\sigma-1}dt = (xy)^{\alpha+1/2}H^{\alpha,\sigma}(x, y).$$
Sharp description of the behavior of $\mathcal{H}^{\alpha,\sigma}(x,y)$ follows immediately from Theorem 2.12. Then it is not hard to see that for $\alpha < -1/2$ the inclusion $L^p(dx) \subset \text{Dom} J^{\alpha,\sigma}$ can hold only if $p > 2/(2\alpha + 3)$.

The next result gives a complete and sharp description of $L^p - L^q$ boundedness of $J^{\alpha,\sigma}$. It reveals that for $\alpha \geq -1/2$, $J^{\alpha,\sigma}$ behaves exactly like $J^{-1/2,\sigma}$, and thus like the classical Bessel potential $J^{-1/2,\sigma}$ (see Theorem 2.19 below). On the other hand, for $\alpha < -1/2$ the $L^p - L^q$ behavior of $J^{\alpha,\sigma}$ is more subtle, and partially this is caused by the restriction on $p$ mentioned above.

**Theorem 2.12.** Let $\alpha > -1$, $\sigma > 0$ and $1 \leq p, q \leq \infty$.

(a) If $\alpha \geq -1/2$, then $J^{\alpha,\sigma}$ is bounded from $L^p(dx)$ to $L^q(dx)$ if and only if

\[
\frac{1}{p} - 2\sigma \leq \frac{1}{q} \leq \frac{1}{p} \quad \text{and} \quad \left(\frac{1}{p}, \frac{1}{q}\right) \notin \{(2\sigma,0), (1,1-2\sigma)\}.
\]

(b) If $\alpha < -1/2$ and $p > 2/(2\alpha + 3)$, then $J^{\alpha,\sigma}$ is bounded from $L^p(dx)$ to $L^q(dx)$ if and only if

\[
p = q \quad \text{and} \quad \frac{1}{q} > -\alpha - \frac{1}{2}.
\]

Notice that Theorem 2.12 contains implicitly the inclusions $L^p(dx) \subset \text{Dom} J^{\alpha,\sigma}$ for all $1 \leq p \leq \infty$ in case $\alpha \geq -1/2$, and for $p > \frac{2}{2\alpha+3}$ in case $\alpha < -1/2$. Moreover, Theorem 2.12 specified to $p = 2$ allows one to verify that $J^{\alpha,\sigma}$ coincides in $L^2(dx)$ with the negative power $(I + L_\alpha)^{-\sigma}$ defined spectrally.

**Remark 2.13.** For the two specific values $\alpha = \pm 1/2$, the theory presented in Section 2.2 takes a simpler form due to elementary expressions for the Bessel and the modified Bessel functions,

\[
J_{\pm 1/2}(u) = \sqrt{\frac{2}{\pi u}} \begin{cases} 
\sin u, & \text{if } \varphi_{\pm 1/2}(u) = \sqrt{\frac{2}{\pi}} \begin{cases} 
\sin u, & \text{if } x,y > 0, \\
\cos u, & \text{if } x,y < 0,
\end{cases} \\
\cosh u, & \text{if } x,y < 0,
\end{cases}
I_{\pm 1/2}(u) = \sqrt{\frac{2}{\pi u}} \begin{cases} 
\sinh u, & \text{if } x,y > 0, \\
\cosh u, & \text{if } x,y < 0.
\end{cases}
\]

We have

\[
\varphi_{\pm 1/2}(u) = \sqrt{\frac{2}{\pi}} \begin{cases} 
\sin u, & \text{if } x,y > 0, \\
\cos u, & \text{if } x,y < 0,
\end{cases}
\]

and therefore $\mathcal{H}_{\pm 1/2}$ is the sine or the cosine transform on $\mathbb{R}_+$, respectively. Moreover,

\[
W_{\ell}^{\pm 1/2}(x,y) = W_{\ell}(x-y) \mp W_{\ell}(x+y).
\]

Consequently, for $0 < \sigma < \pm 1/2 + 1$ and $x,y > 0$,

\[
K^{\pm 1/2,\sigma}(x,y) = c_\sigma \left(|x-y|^{2\sigma-1} \mp |x+y|^{2\sigma-1}\right), \quad I^{\pm 1/2,\sigma} f(x) = c_\sigma I^\sigma \hat{f}_\pm(x),
\]

where $I^\sigma$ denotes the Euclidean potential operator in dimension one, and $\hat{f}_\pm$ are the even and odd, respectively, extensions of $f$ to $\mathbb{R}$.

### 2.3. The setting of the Hankel-Dunkl transform.

The integral kernel of the Hankel-Dunkl semigroup $\{\exp(-tL_\alpha)\}$,

\[
W_{\ell}^\sigma(x,y) = \int_{\mathbb{R}} e^{-u^2 t} \overline{\psi_\sigma(xu)} \psi_\sigma(yu) \, dw_\sigma(u), \quad x,y \in \mathbb{R},
\]

is related to the Hankel heat kernels $W_{\ell}^\sigma$ and $W_{\ell}^{\sigma+1}$ through the identity

\[
W_{\ell}^\sigma(x,y) = \frac{1}{2} \left[ W_{\ell}^\sigma(|x|,|y|) + xy W_{\ell}^{\sigma+1}(|x|,|y|) \right], \quad x,y \in \mathbb{R}
\]

(the values of $W_{\ell}^\sigma(x,y)$ at $x = 0$ or $y = 0$ are understood in a limiting sense). Consequently, the associated Riesz potential kernel,

\[
K^{\alpha,\sigma}(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty W_{\ell}^\sigma(x,y) t^{\sigma-1} \, dt,
\]

and the corresponding potential operator,

\[
\Gamma^{\alpha,\sigma} f(x) = \int_\mathbb{R} K^{\alpha,\sigma}(x,y) f(y) \, dw_\sigma(y), \quad x \in \mathbb{R},
\]
Theorem 2.15. Let $x, y \in \mathbb{R}$ uniformly in $\mathcal{R}$ suggest that, similarly as in the Dunkl-Laguerre setting considered in [29], the potential kernel also has $\alpha$ where $\Phi^{(1)}(14)$

Theorem 2.14. Let $x, y \in \mathbb{R}$ uniformly in $\mathcal{R}$ show that $W^\alpha_0(x, y)$ is the function on the real line given by

$$ W^\alpha_0(x, y) = \frac{1}{2}(2t)^{-\alpha-1} \exp \left( -\frac{x^2 + y^2}{4t} \right) \Phi_\alpha \left( \frac{xy}{2t} \right), $$

where $\Phi_\alpha$ is the function on the real line given by

$$ \Phi_\alpha(u) = \frac{I_\alpha(u)}{u^\alpha} + \frac{I_{\alpha+1}(u)}{u^{\alpha+1}}, $$

with proper interpretation of the ratios when $u \leq 0$; see e.g. [30]. An analysis similar to that performed in [29] Section 3.2 shows that $W^\alpha_0(x, y)$ takes both positive and negative values when $\alpha < -1/2$. This suggests that, similarly as in the Dunkl-Laguerre setting considered in [29], the potential kernel also has this property and hence cannot be sharply estimated in the spirit of Theorem 2.11 when $\alpha < -1/2$. Thus in the next result we restrict to $\alpha \geq -1/2$, i.e. to non-negative multiplicity functions.

**Theorem 2.14.** Let $\alpha \geq -1/2$ and let $0 < \sigma < \alpha + 1$. Then

$$ K^{\alpha,\sigma}(x, y) \simeq (|x| + |y|)^{-2\alpha-1} \begin{cases} |x-y|^{2\sigma-1}, & \sigma < 1/2, \\ \log \frac{2(|x|+|y|)}{|x-y|}, & \sigma = 1/2, \\ (|x| + |y|)^{2\sigma-1}, & \sigma > 1/2, \end{cases} $$

uniformly in $x, y \in \mathbb{R}$.

The analogue of Theorem 2.14 in the present setting is the following.

**Theorem 2.15.** Let $\alpha > -1$ and $0 < \sigma < \alpha + 1$. Let $a, b \in \mathbb{R}$ and assume that $1 \leq p, q \leq \infty$.

(i) $L^p(|x|^p \, dw_\alpha) \subset \text{Dom} \, K^{\alpha,\sigma}$ if and only if

$$ 2\sigma - \frac{2\alpha + 2}{p} < a < \frac{2\alpha + 2}{p^\alpha} \quad (\text{both when } p = 1). $$
Proposition 2.17. The following analogue of Propositions 2.3 and 2.9 holds.

Theorem 2.18. Let $\alpha > -1$ and $0 < \sigma < \alpha + 1$. Assume that $1 \leq p, q \leq \infty$. Then

(i) $L^p(dw_\alpha) \subset \text{Dom} \mathbb{H}^{\alpha,\sigma}$ if and only if $p < \frac{2a+2}{\alpha}$;

(ii) $\mathbb{H}^{\alpha,\sigma}$ is bounded from $L^p(dw_\alpha)$ to $L^q(dw_\alpha)$ if and only if

$$\frac{1}{q} = \frac{1}{p} - \frac{\sigma}{\alpha + 1} \quad \text{and} \quad 1 < p < \frac{\alpha + 1}{\sigma} \quad \text{and} \quad \alpha \geq -1/2;$$

(iii) $\mathbb{H}^{\alpha,\sigma}$ is bounded from $L^1(dw_\alpha)$ to weak $L^q(dw_\alpha)$ for $q = \frac{\alpha + 1}{\alpha + 1 - \sigma}$ if and only if $\alpha \geq -1/2$.

For $\alpha \geq -1/2$ items (ii) and (iii) of Theorem 2.16 are essentially contained in the multi-dimensional results [36] Proposition 4.2 and Theorem 4.3. Similar results in a general setting of the Dunkl transform and an arbitrary group of reflections can be found in [21].

The next result delivers qualitatively similar results in the modified Hankel transform setting, simply by replacing $H_\alpha$ and $L^2(d\mu_\alpha)$ by $H_{\alpha,\beta}$ and $L^2(dw_\alpha)$, respectively, in the relevant definitions in Section 2.1. The following analogue of Propositions 2.3 and 2.4 holds.

Proposition 2.17. Let $\alpha > -1$ and $0 < \sigma < \alpha + 1$. For every $f \in \mathbb{H}_\alpha(C^\infty(\mathbb{R} \setminus \{0\}))$ we have

$$\mathbb{H}^{\alpha,\sigma} f(x) = (\mathbb{H}_\alpha)^{\sigma} f(x), \quad \text{a.a.} \ x \in \mathbb{R}.$$ 

For $\sigma > 0$, we consider also the Bessel potentials

$$\mathbb{H}^{\alpha,\sigma} f(x) = \int_\mathbb{R} \mathbb{H}^{\alpha,\sigma}(x,y) f(y) \, dw_\alpha(y), \quad x \in \mathbb{R},$$

with

$$\mathbb{H}^{\alpha,\sigma}(x,y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t \mathbb{H}^{\alpha,\sigma}_t} f(y) t^{\sigma - 1} dt = \frac{1}{2} \left[ H^{\alpha,\sigma}(|x|,|y|) + xy H^{\alpha+1,\sigma}(|x|,|y|) \right], \quad x,y \in \mathbb{R}.$$ 

Similarly to (13) and (14), we have

$$|\mathbb{H}^{\alpha,\sigma}(x,y)| \lesssim H^{\alpha,\sigma}(|x|,|y|), \quad x,y \in \mathbb{R},$$

and

$$\mathbb{H}^{\alpha,\sigma}(x,y) \asymp H^{\alpha,\sigma}(x,y), \quad x,y > 0.$$ 

These relations, together with Theorem 2.8 enable a characterization of $L^p - L^q$ boundedness of $\mathbb{H}^{\alpha,\sigma}$. Nevertheless, it is interesting to find the exact behavior of $\mathbb{H}^{\alpha,\sigma}(x,y)$. The next result delivers qualitatively sharp estimates of this kernel in case $\alpha > -1/2$ (the case of a positive multiplicity function). In case $\alpha < -1/2$ the kernel should be expected to take both positive and negative values.

Theorem 2.18. Let $\alpha > -1/2$ and let $\sigma > 0$. The following bounds hold uniformly in $x, y \in \mathbb{R}$.

(A) Assume that $xy \geq 0$, i.e. $x$ and $y$ have the same sign.

(Ai) If $|x| + |y| \leq 1$, then

$$\mathbb{H}^{\alpha,\sigma}(x,y) \asymp \chi_{\{\sigma > \alpha + 1\}} + \chi_{\{\sigma = \alpha + 1\}} \frac{1}{|x| + |y|}.$$
\[ + (|x| + |y|)^{-2\alpha - 1} \begin{cases} |x - y|^{2\sigma - 1}, & \sigma < 1/2, \\ \log \frac{2(|x| + |y|)}{|x - y|}, & \sigma = 1/2, \\ (|x| + |y|)^{2\sigma - 1}, & \sigma > 1/2. \end{cases} \]

(Aii) If \(|x| + |y| > 1\), then
\[H^{\alpha, \sigma}(x, y) \simeq (|x| + |y|)^{-2\alpha - 1} \exp \left(-c|x - y|\right) \begin{cases} |x - y|^{2\sigma - 1}, & \sigma < 1/2, \\ 1 + \log^+ \frac{1}{|x - y|}, & \sigma = 1/2, \\ 1, & \sigma > 1/2. \end{cases} \]

(B) Assume that \(xy < 0\), i.e. \(x\) and \(y\) have opposite signs.
(Bi) If \(|x| + |y| \leq 1\), then
\[H^{\alpha, \sigma}(x, y) \simeq \chi_{\{\sigma > \alpha + 1\}} + \chi_{\{\sigma = \alpha + 1\}} \log \frac{1}{|x| + |y|} + (|x| + |y|)^{2\sigma - 2\alpha - 2}.\]

(Bii) If \(|x| + |y| > 1\), then
\[H^{\alpha, \sigma}(x, y) \simeq (|x| + |y|)^{-2\alpha - 1} \exp \left(-c|x + y|\right)(|x| + |y|)^{-2}.\]

For \(\alpha = -1/2\) the kernel \(H^{\alpha, \sigma}(x, y)\) corresponds to the classical one-dimensional Bessel potential. For the sake of completeness, recall that
\[H^{-1/2, \sigma}(x, y) \simeq \exp \left(-c|x - y|\right) \begin{cases} |x - y|^{2\sigma - 1}, & \sigma < 1/2, \\ 1 + \log^+ \frac{1}{|x - y|}, & \sigma = 1/2, \\ 1, & \sigma > 1/2, \end{cases} \]

uniformly in \(x, y \in \mathbb{R}\). Actually, there is an explicit formula for \(H^{-1/2, \sigma}(x, y)\) involving Macdonald’s function, which leads to more precise asymptotics than the above, even in the multi-dimensional case; see \([4] p. 416–417\). Notice that \(H^{-1/2, \sigma}(x, y)\) has an exponential decay along the line \(y = -x\), which is not the case of \(H^{\alpha, \sigma}(x, y)\) when \(\alpha > -1/2\), cf. the comments following \([29]\) Theorem 2.4.

Finally, we establish a sharp description of \(L^p - L^q\) boundedness of \(J^{\alpha, \sigma}\), which happens to coincide with that for \(J^{\alpha, \sigma}\).

**Theorem 2.19.** Let \(\alpha > -1\), \(\sigma > 0\) and \(1 \leq p, q \leq \infty\).

(a) If \(\alpha \geq -1/2\), then \(J^{\alpha, \sigma}\) is bounded from \(L^p(d\omega_a)\) to \(L^q(d\omega_a)\) if and only if
\[
\frac{1}{p} - \frac{\sigma}{\alpha + 1} \leq \frac{1}{q} \leq \frac{1}{p} \quad \text{and} \quad \left(\frac{1}{p}, \frac{1}{q}\right) \notin \left\{\left(\frac{\sigma}{\alpha + 1}, 0\right), \left(1, 1 - \frac{\sigma}{\alpha + 1}\right)\right\}.
\]

(b) If \(\alpha < -1/2\), then \(J^{\alpha, \sigma}\) is bounded from \(L^p(d\omega_a)\) to \(L^q(d\omega_a)\) if and only if \(p = q\).

Note that this result in the classical case \(\alpha = -1/2\) was known earlier, see \([4]\) p. 470. The sufficiency part of (a) is partially contained in \([30]\) Theorems 4.5 and 4.6. Apart from that the theorem is new. Note also that Theorem 2.19 specified to \(p = 2\) allows one to ensure that \(J^{\alpha, \sigma}\) coincides in \(L^2(d\omega_a)\) with the negative power \((I + \delta_a)^{-\sigma}\) defined spectrally.

The definitions of the Riesz and Bessel potentials in the Hankel-Dunkl setting considered in this paper in the framework of \(\mathbb{R}\) with the reflection group isomorphic to \(Z_2\) are in the case \(\alpha \geq -1/2\) consistent (up to multiplicative constants) with those investigated in the literature in a general framework of an arbitrary finite reflection group in \(\mathbb{R}^d\). See the papers \([30, 21, 5]\).

To explain this, look only at the Bessel potentials since for the Riesz potentials one can argue similarly (merely by neglecting the factor \(e^{-t}\) in the relevant places). To keep our explanation concise we follow the notation from \([3]\) changing only the character \(\alpha\) to \(2\alpha\) in order to avoid a notational collision; the reader may also consult the survey \([30]\) for necessary details.
For a fixed reflection group on \( \mathbb{R}^d \), let \( \gamma, \tau_y, *_\gamma, w_\gamma \) denote respectively: an index associated to a multiplicity function, a Dunkl-type generalized translation and convolution, and a weight function. The Bessel-Dunkl potential operator is then defined as

\[
J_{\gamma}^\tau f = b_{\gamma, \tau} * f,
\]

where

\[
b_{\gamma, \tau}(x) = \frac{c_{\gamma, \tau}}{\Gamma(\sigma)} \int_0^\infty e^{-t \gamma - \sqrt{t}} \exp\left(-\frac{|x|^2}{4t}\right) t^{\sigma-1} dt,
\]

see \cite{[5]} (3.1) and (3.2). (Note that evaluating the latter integral with the factor \( e^{-t} \) removed results in \( |x|^{2\gamma - 2\sigma} \) times a constant, which is the generalized convolution kernel appearing in the definition of the Riesz potential of order \( 2\sigma \), see \cite{[36]}.) In fact, see \cite{[5]} (3.6), \( J_{\gamma}^\tau \) is an integral operator with the kernel

\[
J_{\gamma}^\tau(x, y) = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-|t\gamma - \sqrt{|t|}|} \exp\left(-\frac{|x|^2}{4|t|}\right) t^{\sigma-1} dt,
\]

where \( F_i^\gamma(t) = (2t)^{-\gamma - \frac{d}{2}} \exp\left(-\frac{|u|^2}{4t}\right) \) is, up to a multiplicative constant, the modified Gauss kernel and

\[
\tau_{-y}(F_i^\gamma)(x) = c_{\gamma} \int_{\mathbb{R}^d} e^{-|t|\gamma} E_{\gamma}(ix, u) E_{\gamma}(iy, u) w_{\gamma}(u) du,
\]

is the Dunkl-type heat kernel, see \cite{[36]} p. 123; \( E_{\gamma}(\cdot, \cdot) \) denotes here the Dunkl kernel.

Coming back to our specific case of \( \mathbb{R} \) where the parameter \( \alpha \) represents the multiplicity function, it may be seen that \( E_{\gamma}(ix, u) \) and \( E_{\gamma}(iy, u) \) appearing above are, up to a multiplicative constant, equal to \( \psi_\alpha(xu) \) and \( \overline{\psi}_\alpha(yu) \), respectively; see \cite{[36]} Example 2.1 or \cite{[27]}. This explains the consistence indicated above.

### 3. Estimates of the potential kernels

In this section we prove Theorems \ref{thm:main} and \ref{thm:main}. We begin with a technical result that provides sharp description of the integral \( E_{\gamma}(T, S) \) defined below. This is essentially \cite{[28]} Lemma 2.3, see also \cite{[29]} Lemma 3.2. Let

\[
E_{\gamma}(T, S) = \int_0^1 t^A \exp\left(-Tt^{-1} - St\right) dt, \quad 0 \leq T, S < \infty.
\]

**Lemma 3.1** (\cite{[28]} Lemma 2.3). Let \( A \in \mathbb{R} \). Then

\[
E_{\gamma}(T, S) \simeq \exp\left(-c_{\gamma} \sqrt{T(T \vee S)}\right) \begin{cases} T^{A+1}, & A < -1, \\ 1 + \log^+ \frac{1}{T(T \vee S)}, & A = -1, \\ (S \vee 1)^{-A-1}, & A > -1, \end{cases}
\]

uniformly in \( T, S \geq 0 \).

#### 3.1. Estimates of the Hankel potential kernels.

**Proof of Theorem \ref{thm:main}** Using the standard asymptotics \cite{[4]} we find that

\[
W_i^{\gamma}(x, y) \simeq \begin{cases} t^{-\alpha - 1} \exp\left(-\frac{x^2 + y^2}{4t}\right), & xy \leq t, \\ (xy)^{-\alpha - 1/2} \exp\left(-\frac{(x-y)^2}{4t}\right), & xy > t. \end{cases}
\]

Consequently,

\[
K_{\alpha, \gamma}(x, y) \simeq (xy)^{-\alpha - 1/2} \int_0^xy t^{\sigma - 3/2} \exp\left(-\frac{(x-y)^2}{4t}\right) dt + \int_{xy}^\infty t^{\sigma - 2} \exp\left(-\frac{x^2 + y^2}{4t}\right) dt
\]

\[
\equiv I_0 + I_\infty.
\]
Changing the variables of integrations $t \mapsto xyt$ and $t \mapsto xy/t$, respectively, we arrive at

\[ I_0 = (xy)^{\sigma-a-1} E_{\sigma-3/2} \left( \frac{(x-y)^2}{4xy}, 0 \right), \quad I_\infty = (xy)^{\sigma-a-1} E_{\alpha-\sigma} \left( 0, \frac{x^2+y^2}{4xy} \right); \]

notice that, in view of Lemma 3.1, $I_\infty = \infty$ when $\sigma \geq \alpha + 1$. Now applying Lemma 3.1 twice we get

\[ I_\infty + I_0 \simeq (x+y)^{2\sigma-2\alpha-2} \exp \left( -\frac{(x-y)^2}{xy} \right) \left\{ \begin{array}{ll}
|x-y|^{2\sigma-1}, & \sigma < 1/2, \\
1 + \log \sqrt{\frac{x+y}{x-y}}, & \sigma = 1/2, \\
(x+y)^{2\sigma-1}, & \sigma > 1/2.
\end{array} \right. \]

To proceed, we consider two cases. If $(x-y)^2 \leq xy$, then $xy \simeq (x+y)^2$. So in this case

\[ K^{\alpha,\sigma}(x,y) \simeq (x+y)^{2\sigma-2\alpha-2} + (x+y)^{-\alpha-1/2} \exp \left( -\frac{(x-y)^2}{xy} \right) \left\{ \begin{array}{ll}
|x-y|^{2\sigma-1}, & \sigma < 1/2, \\
1, & \sigma = 1/2, \\
(x+y)^{2\sigma-1}, & \sigma > 1/2.
\end{array} \right. \]

Since the second term on the right-hand side above is the dominating one, the desired bounds follow.

In the opposite case, when $(x-y)^2 > xy$, we observe that $x$ and $y$ are non-comparable in the sense that either $y < C^{-1}x$ or $y > Cx$ for a fixed $C > 1$. For symmetry reasons, we may assume that $y < x$. Then the bounds we must verify take the form $K^{\alpha,\sigma}(x,y) \simeq x^{2\sigma-2\alpha-2}$. On the other hand, we know that

\[ K^{\alpha,\sigma}(x,y) \simeq x^{2\sigma-2\alpha-2} + (xy)^{-\alpha-1/2} \exp \left( -\frac{x}{y} \right) \left\{ \begin{array}{ll}
x^{2\sigma-1}, & \sigma < 1/2, \\
1, & \sigma = 1/2, \\
(xy)^{\sigma-1/2}, & \sigma > 1/2.
\end{array} \right. \]

This relation remains true after multiplying the exponential by an arbitrary power of $x/y$. Therefore we see that the first term dominates in the above sum, and the conclusion follows. \(\square\)

**Remark 3.2.** Simpler tools than Lemma 3.1 are sufficient for the proof of Theorem 2.7; see [28, Lemma 2.1 and 2.2]. However, we decided to use Lemma 3.1 since it allows for more compact notation and the proof.

**Remark 3.3.** Theorem 2.7 can be proved in another way, via expressing $K^{\alpha,\sigma}(x,y)$ by the Hankel-Poisson kernel $P_t^\alpha(x,y)$, that is the integral kernel of the semigroup generated by means of the square root of $L_\alpha$. We have

\[ K^{\alpha,\sigma}(x,y) = \frac{1}{\Gamma(2\sigma)} \int_0^\infty P_t^\alpha(x,y) t^{2\sigma-1} dt. \]

Using sharp estimates for $P_t^\alpha(x,y)$ from [9, Theorem 6.1] and estimating the emerging integral with the aid of [26, Lemma 3.1] leads to the bounds asserted in Theorem 2.7. Nevertheless, our approach based on the Hankel heat kernel is more direct.

**Proof of Theorem 2.7.** Using (17) we write

\[ H^{\alpha,\sigma}(x,y) \simeq (xy)^{-\alpha-1/2} \int_0^{xy} e^{-t} \exp \left( -\frac{(x-y)^2}{4t} \right) t^{3/2} dt + \int_{xy}^\infty e^{-t} \exp \left( -\frac{x^2+y^2}{4t} \right) t^{\sigma-2} dt \]

\[ \equiv I_0 + I_\infty. \]

Changing the variables of integrations $t \mapsto xyt$ and $t \mapsto xy/t$, respectively, we get

\[ I_0 = (xy)^{\sigma-a-1} E_{\sigma-3/2} \left( \frac{(x-y)^2}{4xy}, xy \right), \quad I_\infty = (xy)^{\sigma-a-1} E_{\alpha-\sigma} \left( xy, \frac{x^2+y^2}{4xy} \right). \]
Applying now Lemma 3.1 twice we arrive at the estimates

\[ I_0 \simeq (xy)^{-\alpha/2} \exp \left( -c \left[ |x-y| \lor \left( \frac{(x-y)^2}{xy} \right) \right] \right) \begin{cases} |x-y|^{2\sigma-1}, & \sigma < 1/2, \\ 1 + \log^+ \frac{1}{|x-y|\sqrt{1+\frac{y}{x^2}}}, & \sigma = 1/2, \\ (1 \land xy)^{\sigma-1/2}, & \sigma > 1/2 \end{cases} \]

and

\[ I_\infty \simeq \exp \left( -c[(x+y) \lor xy] \right) \begin{cases} (x+y)^{2\sigma-2\alpha-2}, & \sigma < \alpha + 1, \\ 1 + \log^+ \frac{1}{(x+y)\lor xy}, & \sigma = \alpha + 1, \\ 1, & \sigma > \alpha + 1. \end{cases} \]

These bounds will be used in the sequel repeatedly, without further mention.

We first show (i) and thus assume to this end that \( x+y \leq 1 \). We will inspect the cases of comparable and non-comparable \( x \) and \( y \). For symmetry reasons, below we may always assume that \( x \geq y \).

**Case 1: \( y \leq x \leq 2y \).** We must show that

\[ I_0 + I_\infty \simeq \chi_{\{\sigma > \alpha + 1\}} + \chi_{\{\sigma = \alpha + 1\}} \log \frac{1}{x} + x^{-2\alpha-1} \begin{cases} |x-y|^{2\sigma-1}, & \sigma < 1/2, \\ 1 + \log^+ \frac{x}{|x-y|}, & \sigma = 1/2, \\ x^{2\sigma-1}, & \sigma > 1/2. \end{cases} \]

Since \( y \simeq x < 1 \), it is easily seen that

\[ I_\infty \simeq \chi_{\{\sigma > \alpha + 1\}} + \chi_{\{\sigma = \alpha + 1\}} \left( 1 + \log \frac{1}{x} \right) + \chi_{\{\sigma < \alpha + 1\}} x^{2\sigma-2\alpha-2} \]

\[ \simeq \chi_{\{\sigma > \alpha + 1\}} + \chi_{\{\sigma = \alpha + 1\}} \log \frac{1}{x} + x^{2\sigma-2\alpha-2}. \]

As for \( I_0 \), we observe that for \( x \) and \( y \) under consideration

\[ |x-y| \lor \left( \frac{(x-y)^2}{xy} \right) \simeq |x-y| \lor \left( 1 - \frac{y}{x} \right)^2 \leq 1 \]

and

\[ 1 + \log^+ \frac{1}{|x-y| \lor \left( \frac{(x-y)^2}{xy} \right)} \simeq 1 + \left( \log \frac{x}{|x-y|} \lor \left( \log \frac{x^2}{(x-y)^2} \right) \right) \simeq 1 + \log \frac{x}{|x-y|}. \]

Consequently,

\[ I_0 \simeq x^{-2\alpha-1} \begin{cases} |x-y|^{2\sigma-1}, & \sigma < 1/2, \\ 1 + \log \frac{x}{|x-y|}, & \sigma = 1/2, \\ x^{2\sigma-1}, & \sigma > 1/2. \end{cases} \]

Combining the above bounds of \( I_0 \) and \( I_\infty \) we get (18).

**Case 2: \( x > 2y \).** Now the desired estimates take the form

\[ I_0 + I_\infty \simeq \chi_{\{\sigma > \alpha + 1\}} + \chi_{\{\sigma = \alpha + 1\}} \log \frac{1}{x} + x^{2\sigma-2\alpha-2}. \]

As in the previous case, \( I_\infty \) is comparable to the expression in (19), so it suffices to check that \( I_0 \) is controlled by \( I_\infty \). Observe that for \( 2y < x < 1 \)

\[ I_0 \simeq (xy)^{-\alpha/2} \exp \left( -\frac{x}{y} \right) \begin{cases} x^{2\sigma-1}, & \sigma \leq 1/2, \\ (xy)^{\sigma-1/2}, & \sigma > 1/2. \end{cases} \]

This relation remains true if the right-hand side is multiplied by an arbitrary power of \( x/y \), since the ratio is at least 2. Therefore

\[ I_0 \simeq \exp \left( -\frac{x}{y} \right) x^{2\sigma-2\alpha-2} \]

and the conclusion follows.
We pass to proving (ii), so from now on we consider \( x + y > 1 \). Again, we may and do assume that \( x \geq y \) and distinguish the cases of comparable and non-comparable arguments.

**Case 1:** \( y \leq x \leq 2y \). We aim at showing that

\[
I_0 + I_\infty \simeq x^{-2\alpha-1} \exp \left( -c|x-y| \right) \begin{cases} |x-y|^{2\sigma-1}, & \sigma < 1/2, \\ 1 + \log^+ \frac{1}{|x-y|}, & \sigma = 1/2, \\ 1, & \sigma > 1/2. \end{cases}
\]

We have

\[
I_\infty \simeq \exp \left( -cx^2 \right) \begin{cases} x^{2\sigma-2\alpha-2}, & \sigma < \alpha + 1, \\ 1, & \sigma \geq \alpha + 1, \end{cases}
\]

so \( I_\infty \simeq \exp(-cx^2) \), which is controlled by the right-hand side in (20). On the other hand, for \( x \) and \( y \) under consideration

\[
|x-y| \vee \frac{(x-y)^2}{xy} \simeq |x-y|,
\]

therefore \( I_0 \) is comparable, in the sense of "\( \simeq \)"., with the right-hand side of (20). Now (20) follows.

**Case 2:** \( x > 2y \). In this case the bounds to be verified are simply

\[
I_0 + I_\infty \simeq \exp(-cx).
\]

We have, for \( x \) and \( y \) satisfying \( x + y > 1 \) and \( x > 2y \),

\[
I_0 \simeq (xy)^{-\alpha-1/2} \exp \left( -cx[1 \lor y^{-1}] \right) \begin{cases} x^{2\sigma-1}, & \sigma \leq 1/2, \\ (1 \land xy)^{\sigma-1/2}, & \sigma > 1/2 \end{cases}
\]

and

\[
I_\infty \simeq \exp \left( -cx[1 \lor y] \right) \begin{cases} x^{2\sigma-2\alpha-2}, & \sigma < \alpha + 1, \\ 1, & \sigma \geq \alpha + 1. \end{cases}
\]

If \( y \geq 1 \), then it is easily seen that \( I_0 \simeq \exp(-cx) \) and \( I_\infty \simeq \exp(-cxy) \), so (21) follows. On the other hand, if \( y < 1 \) then \( I_\infty \simeq \exp(-cx) \) and

\[
I_0 \simeq (xy)^{-\alpha-1/2} \exp \left( -\frac{cx}{y} \right) \begin{cases} x^{2\sigma-1}, & \sigma \leq 1/2, \\ (1 \land xy)^{\sigma-1/2}, & \sigma > 1/2. \end{cases}
\]

Multiplying the right-hand side above by \((x/y)^{-\alpha-1/2}\) (this does not change the estimates) and using the bounds \( x < x/y \) and \( 1 \land xy \leq 1 \) we find that \( I_0 \) is in fact controlled by the right-hand side in (21). The conclusion again follows.

The proof of Theorem [2.7] is complete.

### 3.2. Estimates of the Hankel-Dunkl potential kernels

We first focus our attention on the Dunkl heat kernel \( W_\alpha^t(x,y) \). Recall that this kernel is given by means of the auxiliary function \( \Phi_\alpha \). In [29, Section 3.2] it was shown that

\[
\Phi_\alpha(u) \simeq u^{-\alpha} I_\alpha(u), \quad u \geq 0,
\]

for \( \alpha > -1 \) (here the value of the right-hand side at \( u = 0 \) is understood in a limiting sense), and when \( \alpha > -1/2 \)

\[
\Phi_\alpha(u) \simeq |u|^{-\alpha} I_\alpha(|u|) (1 \land |u|^{-1}), \quad u < 0.
\]

Moreover, \( \Phi_\alpha \) is negative in case \( \alpha < -1/2 \) and \( u < 0 \), provided that \( |u| \) is sufficiently large. Note the particular explicit case \( \Phi_{-1/2}(u) = \sqrt{2/\pi} \exp(u) \) corresponding to the classical one-dimensional Gauss-Weierstrass kernel \( W_{t^{-1/2}}(x,y) = W_t(x-y) \).

From the above properties of \( \Phi_\alpha \) we conclude that \( W_\alpha^t(x,y) \) attains negative values when \( \alpha < -1/2 \), \( xy < 0 \) and \( |xy|/t \) is sufficiently large. Furthermore, with the aid of the standard asymptotics [4] for \( I_\alpha \), we also get the following sharp and explicit description of \( W_\alpha^t(x,y) \).
Proposition 3.4. Let \( \alpha > -\frac{1}{2} \). The following estimates hold uniformly in \( x, y \in \mathbb{R} \) and \( t > 0 \).

(a) If \( xy \geq 0 \), then
\[
W^\alpha_t(x, y) \simeq \begin{cases} 
 t^{-\alpha-1} \exp \left( -\frac{x^2 + y^2}{4t} \right), & xy \leq t, \\
 \frac{1}{\sqrt{t}} \exp \left( -\frac{(x-y)^2}{4t} \right), & xy > t.
\end{cases}
\]

(b) If \( xy < 0 \), then
\[
W^\alpha_t(x, y) \simeq \begin{cases} 
 t^{-\alpha-1} \exp \left( -\frac{x^2 + y^2}{4t} \right), & |xy| \leq t, \\
 |xy|^{-\alpha-3/2} t \exp \left( -\frac{(|x|-|y|)^2}{4t} \right), & |xy| > t.
\end{cases}
\]

Item (a) will not be needed in the sequel, nevertheless we state it for the sake of completeness. We are now in a position to prove Theorem 2.14.

Proof of Theorem 2.14. Since \( \mathbb{K}^{-1/2,\sigma}(x, y) \) is the classical one-dimensional Riesz potential kernel, we look at \( \alpha > -\frac{1}{2} \). Consider first \( xy \geq 0 \). Since \( \mathbb{K}^{\alpha,\sigma}(x, y) = \mathbb{K}^{\alpha,\sigma}(-x, -y) \), we may assume that \( x, y \geq 0 \). If \( x, y > 0 \), then we easily get the desired estimates by means of (13) and Theorem 2.1. If \( x = 0 \) or \( y = 0 \), then (14) still holds, with a limiting understanding of the values of \( \mathbb{K}^{\alpha,\sigma}(x, y) \) and, implicitly, \( W^\alpha_t(x, y) \). Tracing the proof of Theorem 2.1, one ensures that the asserted bounds for \( \mathbb{K}^{\alpha,\sigma}(x, y) \) remain true for all \( x, y \geq 0 \). Hence the conclusion again follows.

Now assume that \( xy < 0 \). By Proposition 3.4 (b) we have
\[
\mathbb{K}^{\alpha,\sigma}(x, y) \simeq \mathbb{K}^{\alpha,\sigma}(|x|, |y|),
\]
where, for \( \tilde{x}, \tilde{y} > 0 \),
\[
\mathbb{K}^{\alpha,\sigma}(\tilde{x}, \tilde{y}) = (\tilde{x}\tilde{y})^{-\alpha-3/2} \int_0^{\tilde{y}} t^{-\alpha-1/2} \exp \left( -\frac{(\tilde{x} - \tilde{y})^2}{4t} \right) dt + \int_0^\infty t^{-\alpha-2} \exp \left( -\frac{\tilde{x}^2 + \tilde{y}^2}{4t} \right) dt
\]
\[
\equiv I_0 + I_\infty.
\]
Here \( I_\infty \) agrees with \( I_\infty \) in the proof of Theorem 2.1 and
\[
I_0 = (\tilde{x}\tilde{y})^{-\alpha-1} E_{\alpha-1/2} \left( \frac{(\tilde{x} - \tilde{y})^2}{4\tilde{x}\tilde{y}}, 0 \right).
\]
Proceeding as in the proof of Theorem 2.1, we infer that \( \widetilde{\mathbb{K}}^{\alpha,\sigma}(\tilde{x}, \tilde{y}) \simeq (\tilde{x} + \tilde{y})^{2\sigma-2\alpha-2} \). This, in view of (22) and the relation \(|x| + |y| = |x - y|\) valid when \( xy < 0 \), finishes the proof. \( \square \)

To prove Theorem 2.18 it is necessary to obtain good estimates of the kernel
\[
\widetilde{H}^{\alpha,\sigma}(x, y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-t} W^\alpha_t(x, -y) t^{\alpha-1} dt, \quad x, y > 0.
\]

Lemma 3.5. Let \( \alpha > -\frac{1}{2} \). The following estimates hold uniformly in \( x, y > 0 \).

(i) If \( x + y \leq 1 \), then
\[
\widetilde{H}^{\alpha,\sigma}(x, y) \simeq \chi_{\{\sigma > \alpha + 1\}} + \chi_{\{\sigma = \alpha + 1\}} \log \frac{1}{x+y} + (x+y)^{2\sigma-2\alpha-2}.
\]

(ii) If \( x + y > 1 \), then
\[
\widetilde{H}^{\alpha,\sigma}(x, y) \simeq (x+y)^{-2\alpha-1} \exp \left( -c|x-y| \right) (x+y)^{-2}.
\]

Proof. In view of Proposition 3.4 (b),
\[
\widetilde{H}^{\alpha,\sigma}(x, y) \simeq (xy)^{-\alpha-3/2} \int_0^{xy} t^{-\alpha-1/2} \exp \left( -\frac{(x-y)^2}{4t} \right) t^{\alpha-1/2} dt + \int_{xy}^\infty e^{-t} \exp \left( -\frac{x^2 + y^2}{4t} \right) t^{\alpha-2} dt
\]
\[
\equiv I_0 + I_\infty.
\]
Here \( I_\infty \) agrees with \( I_\infty \) from the proof of Theorem 2.7 and
\[
I_0 = (xy)^{\sigma-\alpha-1} E_{\sigma-1/2} \left( \frac{(x-y)^2}{4xy}, xy \right).
\]
Proceeding as in the proof of Theorem 2.7, we arrive at the desired conclusion. Details are left to the reader. \( \square \)

**Proof of Theorem 2.18.** The reasoning is analogous to that in the proof of Theorem 2.14. Here instead of (14) and Theorem 2.1 one uses (16) and Theorem 2.7, respectively. The relevant estimate for the case of arguments having opposite signs is provided by Lemma 3.5. \( \square \)

4. \( L^p - L^q \) estimates

In this section we prove all the \( L^p - L^q \) results in the three settings investigated.

4.1. \( L^p - L^q \) estimates in the setting of the modified Hankel transform. It is convenient to prove Theorem 2.5 first. In the proof we will need the following characterization of two power-weight \( L^p - L^q \) inequalities for the Hardy operator and its dual.

**Lemma 4.1.** Let \( A, B \in \mathbb{R} \) and let \( 1 \leq p, q \leq \infty \).

(a) The estimate
\[
\left\| x^B \int_0^x h(y) \, dy \right\|_{L^q(\mathbb{R}^+, dx)} \lesssim \left\| x^A h \right\|_{L^p(\mathbb{R}^+, dx)}
\]
holds uniformly in \( h \in L^p(\mathbb{R}^+, x^A dx) \) if and only if \( p \leq q \) and \( A - \frac{1}{p} = B + \frac{1}{q} \) and \( A < \frac{1}{p} \) (\( \leq \) in case \( p = q' = 1 \)).

(b) The estimate
\[
\left\| x^B \int_x^\infty h(y) \, dy \right\|_{L^q(\mathbb{R}^+, dx)} \lesssim \left\| x^A h \right\|_{L^p(\mathbb{R}^+, dx)}
\]
holds uniformly in \( h \in L^p(\mathbb{R}^+, x^A dx) \) if and only if \( p \leq q \) and \( A - \frac{1}{p} = B + \frac{1}{q} \) and \( B > -\frac{1}{q} \) (\( \geq \) in case \( p = q' = 1 \)).

**Proof.** The case \( p \leq q \) is contained in [12 Theorems 1 and 2] specified to power weights. On the other hand, it seems to be well known, at least as a folklore, that the estimates in (a) and (b) do not hold when \( q < p \). In (a) the case \( q < p < \infty \) can be easily concluded, for instance, from [33 Theorem 2.4] (see also references given there). The analogous fact in (b) follows by duality. For \( q < p = \infty \) a direct counterexample of \( h(y) = y^{-A} \) does the job. \( \square \)

Another tool we shall use is Young’s inequality in the context of the multiplicative group \( G = (\mathbb{R}^+, \frac{dx}{x}) \) equipped with the natural convolution \( f \ast g(x) = \int_{\mathbb{R}^+} f(y)g(y^{-1}x) \frac{dy}{y} \), see e.g. [20 Theorem 1.2.12]. Note that an extension of Young’s inequality to weak type spaces (cf. [20 Theorem 1.4.24]), in the context of \( G \) or \((\mathbb{R} \setminus \{0\}, \frac{dx}{|x|})\), was one of the main tools in [12].

**Lemma 4.2** (Young’s inequality). Let \( 1 \leq p, q, r \leq \infty \) satisfy \( \frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{r} \). Then for any \( f \in L^p(G) \) and \( g \in L^q(G) \) we have \( f \ast g \in L^r(G) \) and
\[
\|f \ast g\|_{L^r(G)} \leq \|g\|_{L^q(G)} \|f\|_{L^p(G)}.
\]

We are now prepared to prove Theorem 2.5.

**Proof of Theorem 2.7.** By Theorem 2.3 we have
\[
K_{\alpha,\sigma} (x, y) \simeq (x + y)^{2\sigma-2\alpha-2} + \chi_{(x/2<y<2x)} y^{-2\alpha-1} \left[ \chi_{\{\sigma<1/2\}} |x-y|^{2\sigma-1} + \chi_{\{\sigma=1/2\}} \log \frac{2(x+y)}{|x-y|} \right],
\]
uniformly in \( x, y > 0 \). Thus we can write the estimates
\[
(23) \quad T^{\alpha,\sigma} f \simeq H_0 f + H_\infty f + \chi_{\{\sigma<1/2\}} T f + \chi_{\{\sigma=1/2\}} S f, \quad f \geq 0,
\]
where the relevant operators are defined as follows:

\[ H_0 f(x) = x^{2\sigma - 2\alpha - 2} \int_0^x f(y) \, dy \mu_\alpha(y), \quad H_\infty f(x) = \int_x^\infty y^{2\sigma - 2\alpha - 2} f(y) \, dy \mu_\alpha(y), \]

\[ T f(x) = \int_{x/2}^{2x} |x - y|^{2\sigma - 1} f(y) \, dy, \quad S f(x) = \int_{x/2}^{2x} \log \frac{2(x + y)}{|x - y|} f(y) \, dy. \]

Clearly, if each term of the right-hand side in (23) is well defined (i.e. the defining integrals converge for a.a. \( x > 0 \)) for a fixed, not necessarily non-negative \( f \), then so is \( I^{\alpha, \sigma} \). On the other hand, if any of these terms is not well defined for an \( f \geq 0 \), then neither is \( I^{\alpha, \sigma} f \). Similar implications pertain to weighted \( L^p - L^q \) mapping properties.

We first prove (i). Let \( f \in L^p(x^\alpha \, dy \mu_\alpha) \). By means of Hölder’s inequality it is straightforward to verify that \( H_0 f \) is well defined when \( a < \frac{2\alpha+2}{p} \) (≤ if \( p = 1 \)) and \( H_\infty f \) is well defined when \( a > 2\sigma - \frac{2\alpha+2}{p} \) (≥ if \( p = 1 \)). These conditions are sharp in the sense that if \( a \) is beyond the indicated ranges, then there exists a function \( g \in L^p(x^\alpha \, dy \mu_\alpha) \) such that \( H_0 g(x) = \infty, x > 0 \), or \( H_\infty g(x) = \infty, x > 0 \), respectively. Essentially, the simplest examples of such \( g \) are the following. In case of \( H_0 \), \( g(y) = \chi_{(y < 1)} y^{-2\alpha - 2} \) when \( a > \frac{2\alpha+2}{p} \) (≥ if \( p = \infty \)), and \( g(y) = \chi_{(y < 1)} y^{-2\alpha - 2} / \log \frac{x}{\alpha} \) when \( a = \frac{2\alpha+2}{p} \) and \( 1 < p < \infty \). In case of \( H_\infty \), \( g(y) = \chi_{(y > 2)} y^{-2\sigma} \) when \( a < 2\sigma - \frac{2\alpha+2}{p} \) (≤ if \( p = \infty \)), and \( g(y) = \chi_{(y > 2)} y^{-2\sigma} / \log y \) when \( a = 2\sigma - \frac{2\alpha+2}{p} \) and \( 1 < p < \infty \). Altogether, this shows that condition (8) is necessary and sufficient for the sum \( H_0 f + H_\infty f \) to be well defined.

Now it suffices to ensure that \( T f \) and \( S f \) are well defined under (8). But even more is true, since in fact no restrictions on \( p \) and \( a \) are needed. Indeed, let \( 0 \leq f \in L^p(x^a \, dy \mu_\alpha) \) with arbitrary \( a \in \mathbb{R} \) and \( 1 \leq p \leq \infty \). Consider \( \tilde{f} = f \chi_{(1/n, \infty)} \) \( n \) large and fixed. Since \( T f(x) = T f(x) \) for \( x \in (2/n, n/2) \), it is enough to check that \( T \tilde{f} \) is well defined, and similarly in case of \( S \). Observe that \( \tilde{f} \in L^p(\mathbb{R}_+, \, dx) \) and since its support is bounded, also \( \tilde{f} \in L^1(\mathbb{R}_+, \, dx) \). Then using the Fubini-Tonelli theorem we get

\[ \|T \tilde{f}\|_{L^1(\mathbb{R}_+, \, dx)} = \int_0^\infty \int_{y/2}^{2y} \|x - y\|^{2\sigma - 1} \, dx \tilde{f}(y) \, dy \simeq \int_0^\infty y^{2\sigma} \tilde{f}(y) \, dy \lesssim \| \tilde{f} \|_{L^1(\mathbb{R}_+, \, dx)} < \infty. \]

The case of \( S \) is analogous, a simple computation leads to

\[ \|S \tilde{f}\|_{L^1(\mathbb{R}_+, \, dx)} = \int_0^\infty \int_{y/2}^{2y} \log \frac{2(x + y)}{|x - y|} \, dx \tilde{f}(y) \, dy \simeq \int_0^\infty y \tilde{f}(y) \, dy < \infty. \]

The conclusion follows.

We pass to proving (ii). We will analyze separately each of the four terms on the right-hand side of (23). Altogether, this will imply (ii). Since all the considered operators are non-negative, below we may and always do assume that \( f \geq 0 \).

**Analysis of \( H_0 \).** Substituting \( f(y) = y^{-2\alpha - 1} h(y) \) we see that the estimate

\[ \|x^{-b} H_0 f\|_{L^q(dy \mu_\alpha)} \lesssim \|x^a f\|_{L^p(dy \mu_\alpha)} \]

is equivalent to

\[ \left\| x^{-b + (2\alpha+1)/q} \int_0^x h(y) \, dy \right\|_{L^p(\mathbb{R}_+, \, dx)} \lesssim \left\| x^{(2\alpha+1)/p - 2\alpha - 1} h \right\|_{L^p(\mathbb{R}_+, \, dx)}. \]

Applying now Lemma (21) (a) (specified to \( A = a - \frac{2\alpha+1}{p} \) and \( B = -b + \frac{2\alpha+1}{p} + 2\sigma - 2\alpha - 2 \)) we conclude that this holds if and only if \( p \leq q \) and \( \frac{1}{q} = \frac{1}{p} + \frac{b + 2\alpha + 2}{2\alpha + 2} \) and \( a < \frac{2\alpha+2}{p} \) (≤ in case \( p = q' = 1 \)). The latter three are precisely conditions (a), (b) and (c) of the theorem.

**Analysis of \( H_\infty \).** Substituting \( f(y) = y^{1-2\alpha} h(y) \) we can write the estimate

\[ \|x^{-b} H_\infty f\|_{L^q(dy \mu_\alpha)} \lesssim \|x^a f\|_{L^p(dy \mu_\alpha)} \]
in the equivalent form
\[
\left\| x^{-b+2(\alpha+1)/q} \int_x^\infty h(y) \, dy \right\|_{L^q(\mathbb{R}_+,dx)} \lesssim \left\| x^{a+2(\alpha+1)/p+1-2\sigma} h \right\|_{L^p(\mathbb{R}_+,dx)}.
\]

Using Lemma 4.1 (b) (with \( A = a + \frac{2\alpha+1}{p} + 1 - 2\sigma \) and \( B = -b + \frac{2\alpha+1}{q} \)) we infer that this holds if and only if \( p \leq q \) and \( \frac{1}{q} = \frac{1}{p} + \frac{2\alpha+2-2\sigma}{2\alpha+2} \) and \( b < \frac{2\alpha+2}{q} \) (\( \leq \) in case \( p = q' = 1 \)). These are conditions (a), (b) and (d) of the theorem.

**Analysis of \( T \) in case \( \sigma < 1/2 \).** Here we may assume that (a)-(d) are satisfied. Observe that, in view of condition (b),
\[
a + \frac{2\alpha+1}{p} + b - \frac{2\alpha+1}{q} = 2\sigma + \frac{1}{q} - \frac{1}{p}.
\]
Thus, letting \( f(y) = g(y)y^{-a-(2\alpha+1)/p} \), we see that the estimate
\[
\left\| x^{-b} T f \right\|_{L^q(d\mu,\gamma)} \lesssim \left\| x^a f \right\|_{L^p(d\mu,\gamma)}
\]
can be restated as
\[
(24) \quad \left\| T \left( y^{-2\sigma-1/q+1/p} g \right) \right\|_{L^q(\mathbb{R}_+,dx)} \lesssim \left\| g \right\|_{L^p(\mathbb{R}_+,dx)}.
\]

We claim that condition (e) is necessary for (24) to hold. Indeed, assume that \( \frac{1}{q} < \frac{1}{p} - 2\sigma \) and take \( g(y) = \chi(1/2,1)(y) (1-y)^\gamma \) with \( \gamma = -\frac{1}{p} + \varepsilon \), where \( 0 < \varepsilon < \frac{1}{p} - 2\sigma - \frac{1}{q} \). Then the right-hand side of (24) is finite. On the other hand, for \( x \in (1,3/2) \)
\[
\int_{x/2}^{2x} |x - y|^{2\sigma-1} \chi(1/2,1)(y) (1-y)^\gamma y^{-2\sigma-1/q+1/p} \, dy \gtrsim \int_{3/4}^{1} |x - y|^{2\sigma-1}(1-y)^\gamma \, dy
\]
\[
= (x-1)^{2\sigma+\gamma} \int_{0}^{1/(4(x-1))} (1+u)^{2\sigma-1} u^\gamma \, du.
\]
Since here \( 1/(4(x-1)) > 1/2 \), the last integral is larger than a positive constant and so the left-hand side in (24) is larger than the constant times
\[
\left\| \chi(1,3/2)(x) \right\|_{L^\infty(dx)} (x-1)^{2\sigma+\gamma}.
\]
But this expression is infinite since \( 2\sigma + \gamma < -\frac{1}{2} \), so (24) does not hold. When \( p = 1 \) and \( \frac{1}{q} = 1 - 2\sigma \), the counterexample of \( g(y) = \chi(1/2,1)(y) (1-y)^{-2\sigma}/\log^2 \frac{2}{1-y} \) shows in a similar manner that \( T \) is not bounded from \( L^{1/2\sigma}(\mathbb{R}_+,dx) \) to \( L^{\infty}(\mathbb{R}_+,dx) \), and then, by duality, neither bounded from \( L^1(\mathbb{R}_+,dx) \) to \( L^{1/(1-2\sigma)}(\mathbb{R}_+,dx) \). The claim follows.

Next, we prove that condition (e) is sufficient for (24). When \( \frac{1}{q} = \frac{1}{p} - 2\sigma \), this is an obvious consequence of the classical Hardy-Littlewood-Sobolev theorem in dimension one. So it remains to verify (24) in case \( \frac{1}{q} > \frac{1}{p} - 2\sigma \). For this purpose we let \( f(x) = F(x)x^{-1/p} \) and write (24) as
\[
(25) \quad \left\| \int_{x/2}^{2x} y^{-1} \frac{y^{-1}x - 1}{2\sigma-1} F(y) \, dy \right\|_{L^q(\mathbb{R}_+,\frac{dx}{x})} \lesssim \| F \|_{L^p(\mathbb{R}_+,\frac{dx}{x})}.
\]
This is precisely \( L^p - L^q \) estimate for the convolution operator on the multiplicative group \((\mathbb{R}_+,\frac{dx}{x})\) given by the convolution kernel
\[
K(u) = \chi(1,1/2)(u) |u - 1|^{2\sigma-1}, \quad u > 0.
\]
Notice that \( K \in L^r(\mathbb{R}_+,\frac{dx}{x}) \) if (and only if) \( \frac{1}{r} > 1 - 2\sigma \). Now, with the aid of Lemma 4.1 we readily arrive at the desired conclusion.

**Analysis of \( S \) in case \( \sigma = 1/2 \).** When \( \sigma = 1/2 \) condition (e) says that \( (p,q) \neq (1,\infty) \). Assuming that, the estimate we must prove is equivalent to
\[
\left\| \int_{x/2}^{2x} \log \frac{2(y^{-1}x + 1)}{|y^{-1}x - 1|} F(y) \, dy \right\|_{L^q(\mathbb{R}_+,\frac{dx}{x})} \lesssim \| F \|_{L^p(\mathbb{R}_+,\frac{dx}{x})}.
\]
Now the relevant convolution kernel is
\[ K(u) = \chi_{(1/2, 2)}(u) \log \frac{2(u + 1)}{|u - 1|}, \quad u > 0. \]
Since \( K \in L^r(\mathbb{R}_+, \frac{dx}{x}) \) for all \( 1 \leq r < \infty \), the conclusion follows by Lemma 4.2.

On the other hand, the analogous well-known result for the classical one-dimensional Riesz potential. Thus we may assume we show that

Proof of Theorem 2.2 (iii). We turn to proving Theorem 2.2. Items (i) and (ii) follow immediately from Theorem 2.5 specified to \( a = b = 0 \), so it remains to prove (iii).

**Proof of Theorem 2.2 (iii).** Assume that \( \alpha \geq -1/2 \) and fix \( q = \frac{\alpha + 1}{\alpha + 1 - \sigma} \). We will prove that \( I^{\alpha, \sigma} \) is of weak type \((1, q)\). Because of (25), it suffices to show the weak type \((1, q)\) of the operators \( H_0 + H_\infty, T \) in case \( \sigma < 1/2 \) and \( S \) in case \( \sigma = 1/2 \).

Treatment of \( H_0 + H_\infty \) is straightforward. We have
\[
(H_0 + H_\infty) f(x) \lesssim \int_0^\infty (x + y)^{2\sigma - 2\alpha - 2} f(y) \, d\mu_\alpha(y), \quad x > 0,
\]
uniformly in \( f \geq 0 \). Since \( \|(x + \cdot)^{2\sigma - 2\alpha - 2}\|_{L^\infty(\mathbb{R}_+)} \approx x^{2\sigma - 2\alpha - 2}, x > 0 \), we get
\[
\|(H_0 + H_\infty) f(x)\| \lesssim x^{2\sigma - 2\alpha - 2} \|f\|_{L^1(d\mu_\alpha)}, \quad x > 0.
\]
As easily verified, the function \( x \mapsto x^{2\sigma - 2\alpha - 2} \) belongs to weak \( L^q(d\mu_\alpha) \) and the desired conclusion follows.

Next, let \( \sigma < 1/2 \) and consider the operator \( T \). In case \( \alpha = -1/2 \) the weak type \((1, \frac{1}{1-2\sigma})\) follows from the analogous well-known result for the classical one-dimensional Riesz potential. Thus we may assume \( \alpha > -1/2 \). We claim that \( T \) is even bounded from \( L^1(d\mu_\alpha) \) to \( L^q(d\mu_\alpha) \), \( q = \frac{\alpha + 1}{\alpha + 1 - \sigma} \). To prove the claim, we show that \( T \) is bounded from \( L^q(d\mu_\alpha), q' = \frac{\alpha + 1}{\sigma + 1} \), to \( L^\infty(d\mu_\alpha) \). This is enough, because the kernel of \( T \) is symmetric and \( L^1(d\mu_\alpha) \subset (L^1(d\mu_\alpha))^* = (L^\infty(d\mu_\alpha))^* \). By Hölder's inequality and the change of variable \( y/x = u \) we obtain
\[
|Tf(x)| \lesssim \|f\|_{L^q(d\mu_\alpha)} \left( \int_{x/2}^{2x} y^{-(2\alpha + 1)/q'} |x - y|^{2\sigma - 1} \, dy \right)^{1/q}.
\]
Since \( (2\sigma - 1)q > -1 \) when \( \alpha > -1/2 \), the last integral is finite. We see that \(|Tf(x)| \lesssim \|f\|_{L^q(d\mu_\alpha)}, \ x > 0 \), and the claim follows.

Finally, consider the operator \( S \) in case \( \sigma = 1/2 \). We will check that \( S \) is even bounded from \( L^1(d\mu_\alpha) \) to \( L^q(d\mu_\alpha) \), \( q = \frac{\alpha + 1}{\alpha + 1 - \sigma} \). Proceeding as in case of \( T \) above we arrive at
\[
|Sf(x)| \lesssim \|f\|_{L^q(d\mu_\alpha)} \left( \int_{x/2}^{2x} y^{-(2\alpha + 1)/q'} \log \frac{2(x + y)}{|x - y|} \, dy \right)^{1/q}.
\]
Here the last integral is finite, so \(|Sf(x)| \lesssim \|f\|_{L^q(d\mu_\alpha)}, \ x > 0 \), and the conclusion follows.
To finish the proof, we need to ensure that \( I^{\alpha,\sigma} \) is not of weak type \((1, \frac{\alpha+1}{\alpha-1})\) when \( \alpha < -1/2 \). This follows by an *au contraire* argument. If \( I^{\alpha,\sigma} \) were of weak type \((1, \frac{\alpha+1}{\alpha-1})\), then by duality (see the proof of [28] Theorem 3.1 or item (C) in the beginning of [26] Section 4) it would be of restricted weak type \((\frac{\alpha-1}{\alpha+1}, \infty)\) and then, by interpolation, bounded from \( L^p(d\mu_\alpha) \) to \( L^q(d\mu_\alpha) \) for \( p \) and \( q \) satisfying \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{\alpha-1} \) and \( 1 < p < \frac{\alpha+1}{\alpha-1} \). But this contradicts the necessity part of item (ii).

Passing to the proof of Theorem 2.8, it is easily seen that it follows from Lemmas 4.3 and 4.4 below that describe sharply \( L^p - L^q \) behavior of two auxiliary operators (with non-negative kernels) into which \( J^{\alpha,\sigma} \) splits naturally. These operators are interesting in their own right, so the lemmas provide slightly more information than actually needed to conclude Theorem 2.8.

We split \( J^{\alpha,\sigma} \) according to the kernel splitting

\[
J^{\alpha,\sigma}(x, y) = H^{\alpha,\sigma}(x, y) + \chi_{\{x \neq 0\}} H^{\alpha,\sigma}(x, y) \equiv H_0^{\alpha,\sigma}(x, y) + H^{\alpha,\sigma}_\infty(x, y)
\]

and denote the resulting integral operators by \( J_0^{\alpha,\sigma} \) and \( J^{\alpha,\sigma}_\infty \), respectively.

**Lemma 4.3** ([29] Lemma 4.1 and [26]). Let \( \alpha > -1, \sigma > 0 \) and \( 1 \leq p, q \leq \infty \). Set \( \delta := (-1/2) \vee \alpha + 1 \). Then \( J_0^{\alpha,\sigma} \) is bounded from \( L^p(d\mu_\alpha) \) to \( L^q(d\mu_\alpha) \) if and only if

\[
\frac{1}{p} - \frac{\sigma}{\delta} \leq \frac{1}{q} \quad \text{and} \quad \left( \frac{1}{p}, \frac{1}{q} \right) \notin \left\{ \left( \frac{\sigma}{\delta}, 0 \right), \left( 1, 1 - \frac{\sigma}{\delta} \right) \right\}.
\]

**Lemma 4.4.** Let \( \alpha > -1, \sigma > 0 \) and \( 1 \leq p, q \leq \infty \). Then \( J^{\alpha,\sigma}_\infty \) is bounded from \( L^p(d\mu_\alpha) \) to \( L^q(d\mu_\alpha) \) if and only if \( \alpha < -1/2 \) and \( p = q \) or \( \alpha \geq -1/2 \) and

\[
\frac{1}{p} - 2\sigma \leq \frac{1}{q} \leq \frac{1}{p} \quad \text{and} \quad \left( \frac{1}{p}, \frac{1}{q} \right) \notin \left\{ \left( 2\sigma, 0 \right), \left( 1, 1 - 2\sigma \right) \right\}.
\]

In view of Theorem 2.4, the behavior of \( H_0^{\alpha,\sigma}(x, y) \) is exactly the same as the behavior of the analogous kernel in the setting of Laguerre function expansions of convolution type that was considered in [29] Section 4.1]. Consequently, Lemma 4.3 coincides with [29] Lemma 4.1 and hence it follows from the results of Nowak and Roncal [26], as commented in [29] Section 4.1]. To prove Lemma 4.4 we will need the following technical result.

**Lemma 4.5.** Let \( \alpha > -1 \) and \( \sigma > 0 \). Then the estimates

\[
\| H^{\alpha,\sigma}_\infty(x, \cdot) \|_{L^p(d\mu_\alpha)} \asymp (1 \vee x)^{(2\alpha+1)(1/p-1)}, \quad x > 0,
\]

hold for \( 1 \leq p \leq \infty \) when \( \sigma > 1/2 \) and for \( 1 \leq p < \frac{1}{1-\sigma} \) when \( \sigma \leq 1/2 \). Moreover, for \( \sigma \leq 1/2 \) and \( \frac{1}{1-2\sigma} \leq p \leq \infty \), we have

\[
\| H^{\alpha,\sigma}_\infty(x, \cdot) \|_{L^p(d\mu_\alpha)} = \infty, \quad x > 4.
\]

Actually, only \( (25) \) will be used in the sequel. However, we include also \( (27) \) to show that \( (26) \) is optimal in the sense of the ranges of admissible parameters.

**Proof of Lemma 4.5.** By Theorem 2.7, \( H^{\alpha,\sigma}_\infty(x, y) \) satisfies the estimates of Theorem 2.7 (ii) outside the square \( 0 < x, y \leq 2 \), and vanishes inside this square. Therefore, it is convenient to consider separately the cases \( \sigma < 1/2, \sigma = 1/2 \) and \( \sigma > 1/2 \). In what follows we treat the case \( \sigma < 1/2 \) leaving a similar analysis for the remaining cases to the reader. We also observe that considering \( 0 < x < 1 \) and \( x > 4 \) is enough for the proof of \( (26) \), see the proof of [29] Lemma 4.3].

Let \( \sigma < 1/2 \). In view of Theorem 2.7 (ii),

\[
H^{\alpha,\sigma}_\infty(x, y) \asymp \chi_{\{x \neq 0\}}(x + y)^{-2\alpha-1}|x - y|^{2\sigma-1} \exp \left( -c|x - y| \right).
\]

Hence, if \( 0 < x < 1 \), then

\[
\int_2^\infty H^{\alpha,\sigma}_\infty(x, y)^p y^{2\alpha+1} dy \asymp \int_2^\infty y^{(2\alpha+1)(1-p)+(2\sigma-1)p} \exp \left( -c By \right) dy \asymp 1
\]
for \( p < \infty \), and
\[
\text{ess sup}_{y > 2} H_{\infty}^{\alpha, \sigma}(x, y) \simeq \sup_{y > 2} y^{2\sigma - 2\alpha - 2} \exp(-cy) \simeq 1.
\]
Thus (20) for \( x < 1 \) follows. If \( x > 4 \), then for \( p < \frac{1}{1 - 2\sigma} \) and for the decisive interval \((x/2, 3x/2)\) we have
\[
\int_{x/2}^{3x/2} H_{\infty}^{\alpha, \sigma}(x, y) y^{2\alpha + 1} dy \simeq x^{(2\alpha + 1)(1 - p)} \int_{x/2}^{3x/2} \exp(-cy) |x - y|^{(2\sigma - 1)p} dy
\]
\[
\simeq x^{(2\alpha + 1)(1 - p)} \int_{0}^{x/2} \exp(-c\alpha) u^{(2\sigma - 1)p} du
\]
\[
\simeq x^{(2\alpha + 1)(1 - p)}.
\]
Notice that the assumption imposed on \( p \) guarantees convergence of the last integral. Checking that the relevant integrals over \((0, x/2)\) and \((3x/2, \infty)\) are controlled by \( x^{(2\alpha + 1)(1 - p)} \) is straightforward. Now (20) follows.

If \( \frac{1}{1 - 2\sigma} \leq p < \infty \), then the above argument leads also to (27). Finally, we have
\[
\|H_{\infty}^{\alpha, \sigma}(x, \cdot)\|_{\infty} \geq \text{ess sup}_{x/2 < y < 3x/2} H_{\infty}^{\alpha, \sigma}(x, y) \simeq x^{-2\alpha - 1} \text{ess sup}_{x/2 < y < 3x/2} \exp(-c|x - y|) |x - y|^{2\sigma - 1} = \infty,
\]
which justifies (27) for \( p = \infty \).

**Proof of Lemma 4.5** The structure of the proof is as follows. The upper estimate of Lemma 4.5 readily enables us to establish \( L^1 - L^q \) boundedness of \( J_{\infty}^{\alpha, \sigma} \) for the relevant \( q \). This, together with a duality argument based on the symmetry of the kernel, \( H_{\infty}^{\alpha, \sigma}(x, y) = H_{\infty}^{\alpha, \sigma}(y, x) \), and the Riesz-Thorin interpolation theorem, gives \( L^p - L^q \) bounds for \( p \) and \( q \) satisfying \( p = q \) in case \( \alpha < -1/2 \) or
\[
\frac{1}{p} - 2\sigma < \frac{1}{q} < \frac{1}{p}
\]
when \( \alpha \geq -1/2 \). The case when \( \alpha \geq -1/2 \) and \( \frac{1}{q} = \frac{1}{p} - 2\sigma \) and \( 2\sigma < \frac{1}{p} < 1 \) follows from Theorem 2.2 (ii) and the fact that \( H_{\infty}^{\alpha, \sigma}(x, y) \) is dominated pointwise by \( K^{\alpha, \sigma}(x, y) \). Finally, the lack of \( L^p - L^q \) boundedness for the relevant \( p \) and \( q \) will be shown by indicating explicit counterexamples. To simplify the notation, in what follows \( \| \cdot \|_p \) denotes the norm in the Lebesgue space \( L^p(\mathbb{R}_+, d\mu_\alpha) \).

The \( L^1 - L^q \) boundedness of \( J_{\infty}^{\alpha, \sigma} \) holds for
\[
q \in \left\{ \left[ 1, \infty \right] \text{, } \sigma > 1/2, \text{ or } q = 1, \right\}, \quad \sigma \leq 1/2,
\]
when \( \alpha \geq -1/2 \) or \(-1 < \alpha < -1/2 \), respectively. Indeed, by Minkowski’s integral inequality (naturally extended to the case \( q = \infty \)), we get
\[
\|J_{\infty}^{\alpha, \sigma} \|_{L^q} \leq \|f \|_1 \|H_{\infty}^{\alpha, \sigma}(\cdot, y)\|_{L^q}
\]
and the assertion follows provided that \( \|H_{\infty}^{\alpha, \sigma}(\cdot, y)\|_{L\infty} < \infty \) (the outer norms are always taken with respect to the \( y \) variable). For \( q = \infty \) this is the case if \( \alpha \geq -1/2 \) and \( \sigma > 1/2 \), since then, by Lemma 4.5
\[
\|H_{\infty}^{\alpha, \sigma}(\cdot, y)\|_{L\infty} \leq \sup_{y > 0} (1 \vee y)^{-2(\alpha + 1)} < \infty.
\]
On the other hand, for \( 1 \leq q < \infty \) in case \( \sigma > 1/2 \), or for \( 1 \leq q < \frac{1}{1 - 2\sigma} \) in case \( \sigma \leq 1/2 \) (so that Lemma 4.5 can be applied),
\[
\|H_{\infty}^{\alpha, \sigma}(\cdot, y)\|_{L\infty} \leq \sup_{y > 0} (1 \vee y)^{(2\alpha + 1)/(q - 1)} < \infty,
\]
provided that \((2\alpha + 1)(\frac{1}{q} - 1) \leq 0 \), and this happens if \( q \) satisfies the imposed restrictions.
We now use the fact that, due to the symmetry of the kernel and a duality argument, $L^p - L^q$ boundedness of $J^\alpha_\infty$ for some $1 \leq p, q < \infty$ implies $L^q - L^p$ boundedness of $J^\alpha_\infty$. This allows us to infer from the results already obtained that $J^\alpha_\infty$ is $L^p - L^\infty$ bounded provided that
\[ p \in \begin{cases} 
(1, \infty], & \sigma > 1/2, \\
(1/2, \infty], & \sigma \leq 1/2,
\end{cases} \text{ or } p = \infty,\]
when $\alpha \geq -1/2$ or $-1 < \alpha < -1/2$, respectively. Applying the Riesz-Thorin interpolation theorem we conclude $L^p - L^q$ boundedness of $J^\alpha_\infty$ in all the relevant cases.

Passing to the negative results, we must verify the following items.

(a) $J^\alpha_\infty$ is not $L^p - L^q$ bounded when $p > q$.
(b) $J^\alpha_\infty$ is not $L^p - L^q$ bounded when $p < q$ and $\alpha < -1/2$.
(c) $J^\alpha_\infty$ is not $L^p - L^q$ bounded when $1/2 < p(n + 1)/q < 2\alpha + 1$ and $\alpha < 1/2$ and $\alpha \geq -1/2$.
(d) $J^\alpha_\infty$ is not $L^p - L^q$ bounded for $\left(\frac{1}{p}, \frac{1}{q}\right) \in \{(2\alpha, 0), (1, 1 - 2\alpha)\}$ when $\sigma \leq 1/2$ and $\alpha \geq -1/2$.

To show (a), consider first $p = \infty$. Then, by Lemma 4.5,
\[ \|J^\alpha_\infty 1\|_q \simeq \int_0^\infty d\mu_\alpha(x) = \infty, \]
hence $J^\alpha_\infty$ is not $L^\infty - L^q$ bounded. To treat the case $p < \infty$, take $f(y) = \chi_{(y > 2)}y^{-\xi}$, where $\xi > 0$ satisfies $\frac{2(\alpha + 1)}{p} < \xi \leq \frac{2(\alpha + 1)}{q}$. Then $f \in L^p(d\mu_\alpha)$. We claim that $J^\alpha_\infty f \notin L^q(d\mu_\alpha)$. Indeed, using the lower bound from Theorem 2.7 (ii), we obtain
\[ J^\alpha_\infty f(x) \geq f(x) \int_{x/2}^x H^\alpha_\infty(x, y) \, d\mu_\alpha(y) \]
\[ \gtrsim f(x) \int_{x/2}^x \exp(-c(x - y)) \begin{cases} 
(x - y)^{2\sigma - 1}, & \sigma < 1/2 \\
1 + \log^+ \frac{1}{x-y}, & \sigma = 1/2 \\
1, & \sigma > 1/2
\end{cases} \, dy \]
\[ \simeq f(x), \]
where the last relation follows by the change of variable $x - y = u$. Since $f \notin L^q(d\mu_\alpha)$, the claim follows.

To justify (b), consider $f_n = \chi_{(n, n+1)}$ with $n$ large. Then $\|f_n\|_p \simeq n(2\alpha + 1)/p$. Moreover, by Theorem 2.7 (ii), for $x \in (n, n + 1)$
\[ J^\alpha_\infty f_n(x) \gtrsim \int_n^{n+1} \exp(-c(x - y)) \begin{cases} 
|x - y|^{2\sigma - 1}, & \sigma < 1/2 \\
1 + \log^+ \frac{1}{|x-y|}, & \sigma = 1/2 \\
1, & \sigma > 1/2
\end{cases} \, dy \simeq 1. \]
This implies that
\[ \|J^\alpha_\infty f_n\|_q \gtrsim \left( \int_n^{n+1} d\mu_\alpha(x) \right)^{1/q} \simeq n(2\alpha + 1)/q. \]
Now, if $J^\alpha_\infty$ were $L^p - L^q$ bounded, then we would have
\[ n(2\alpha + 1)/q \lesssim \|J^\alpha_\infty f_n\|_q \lesssim \|f_n\|_p \simeq n(2\alpha + 1)/p, \quad n \to \infty. \]
But this is not possible since $2\alpha + 1 < 0$ and $p < q$.

Items (c) and (d) are proved by means of Theorem 2.7 (ii) and the counterexamples presented in connection with items (b) and (c) in the proof of [29, Lemma 4.2]. We leave further details to interested readers. \qed
4.2. \( L^p - L^q \) estimates in the setting of the non-modified Hankel transform. Let us first check that Theorem 2.10 follows in a straightforward manner from Theorem 2.5.

**Proof of Theorem 2.10** By means of (10) we see that

\[
L^p(x^{\alpha+1/2}dx) \subset \Dom J^\alpha \sigma \quad \text{if and only if} \quad L^p(x^{\alpha+1/2-(2\alpha+1)/p}d\mu_\alpha) \subset \Dom I^\alpha \sigma.
\]

Thus (i) follows from Theorem 2.5 (i).

To show (ii), notice that, in view of (9), the estimate in question is equivalent to the bound

\[
\| x^{-b+\alpha+1/2-(2\alpha+1)/q} f^\alpha \sigma g \|_{L^q(d\mu_\alpha)} \lesssim \| x^{\alpha+1/2-(2\alpha+1)/p} g \|_{L^p(d\mu_\alpha)}
\]

for all \( g \in L^p(x^{\alpha+1/2-(2\alpha+1)/p}d\mu_\alpha) \). This combined with Theorem 2.5 (ii) (with \((e)\) replaced by \((e')\)) gives the desired conclusion.

Similarly to Theorem 2.8 Theorem 2.12 follows readily from the two lemmas, stated below, describing \( L^p - L^q \) behavior of two auxiliary operators with non-negative kernels which \( J^\alpha \sigma \) splits into. More precisely, we split the operator \( J^\alpha \sigma \) according to the kernel splitting

\[
H^\alpha \sigma(x, y) = \chi_{\{x \leq 2, y \leq 2\}} H_0^\alpha \sigma(x, y) + \chi_{\{x \geq y > 2\}} H^\alpha \sigma(x, y) \equiv H_0^\alpha \sigma(x, y) + H_\infty^\alpha \sigma(x, y)
\]

and denote the resulting integral operators by \( J_0^\alpha \sigma \) and \( J_\infty^\alpha \sigma \), respectively.

**Lemma 4.6** ([29] Lemma 4.4 and [26]). Let \( \alpha > -1, \sigma > 0 \) and \( 1 \leq p, q \leq \infty \).

(a) \( \text{If} \ \alpha \geq -1/2, \text{then} \ J_0^\alpha \sigma \text{ is bounded from} \ L^p(dx) \text{ to} \ L^q(dx) \text{ if and only if} \)

\[
\frac{1}{p} - 2\sigma \leq \frac{1}{q} \quad \text{and} \quad \left( \frac{1}{p}, \frac{1}{q} \right) \notin \{(2\sigma, 0), (1, 1 - 2\sigma)\}.
\]

(b) \( \text{Let} \ \alpha < -1/2, \text{ then} \ L^p(dx) \subset \Dom J_0^\alpha \sigma \text{ if and only if} \ p > 2/(2\alpha + 3). \ \text{In this case} \ J_0^\alpha \sigma \text{ is bounded from} \ L^p(dx) \text{ to} \ L^q(dx) \text{ if and only if} \)

\[
\frac{1}{p} - 2\sigma \leq \frac{1}{q} \quad \text{and} \quad \frac{1}{q} > -\alpha - \frac{1}{2}.
\]

**Lemma 4.7.** Let \( \alpha > -1, \sigma > 0 \) and \( 1 \leq p, q \leq \infty \). Then \( J_\infty^\alpha \sigma \) satisfies the positive \( L^p - L^q \) mapping properties stated in Theorem 2.12 for \( J^\alpha \sigma \), but is neither bounded from \( L^p(dx) \) to \( L^q(dx) \) when \( p > q \), nor when \( p < q \) in case \( \alpha < -1/2 \).

By Theorem 2.7 and (11), the kernel \( H_0^\alpha \sigma(x, y) \) behaves precisely in the same way as its analogue in the setting of Laguerre function expansions of Hermite type studied in [29] Section 4.2. Thus Lemma 4.6 coincides with [29] Lemma 4.4. The latter is a consequence of the results in [29], see the related comments in [29] Section 4.2. To prove Lemma 4.7 we will mostly appeal to the results obtained in the modified Hankel transform setting. Essentially, only the case \( \alpha < -1/2 \) requires new arguments.

**Proof of Lemma 4.7** In view of (11) and Theorem 2.7 (ii), for \( \alpha \geq -1/2 \) the kernel \( H_\infty^\alpha \sigma(x, y) \) is controlled by \( H_{-1/2}^\alpha \sigma(x, y) \). Thus \( J_\infty^\alpha \sigma \) inherits the \( L^p - L^q \) boundedness of \( J_{-1/2}^\alpha \sigma \) (note that \( d\mu_{-1/2} \) is the Lebesgue measure). This together with Lemma 4.4 gives the positive results of the lemma in case \( \alpha \geq -1/2 \).

Next observe that for any \( \alpha > -1 \), the two above mentioned kernels are comparable if the arguments are, see (4),

\[
H_\infty^\alpha \sigma(x, y) \sim H_{-1/2}^\alpha \sigma(x, y), \quad x/2 < y < 2x.
\]

So to prove the required negative result we can use the counterexamples from (a) and (b) of the proof of Lemma 4.4 since they involve only comparable arguments of the kernel (in connection with the case \( p = \infty \), notice that \( J_1 \sigma 1(x) \gtrsim 1, x > 4 \), by (28) and the proof of Lemma 4.5).
Proposition 4.8. Lemma 4.5. The proof is very similar to that of Lemma 4.5.

\[ f \]

Further, by the symmetry of the kernel, \( f \)

\[ f \]

\[ f \]

\[ f \]

\[ f \]

\[ f \]

\[ f \]

We will show that \( U^1 \) and \( U^2 \) are \( L^p \)-bounded for \( p \) satisfying

\[ -\alpha - \frac{1}{2} < \frac{1}{p} < \alpha + \frac{3}{2} \]

This will finish the proof.

By (11), Theorem 2.7 (ii) and Hölder’s inequality,

\[ |U^1 f(x)| \lesssim x^{-\alpha - 1/2} \exp(-cx) \begin{cases} x^{2\sigma - 1}, & \sigma < 1/2 \\ 1 + \log + \frac{1}{x}, & \sigma = 1/2 \\ 1, & \sigma > 1/2 \end{cases} \|\chi_{\{y < x\}} y^{\alpha + 1/2}\|_p \|f\|_p. \]

Since \( p > \frac{2}{\sigma + 3} \), the \( L^{p'} \) norm here is finite and comparable to \( x^{\alpha + 3/2 - 1/p} \). Then we get

\[ |U^1 f(x)| \lesssim g(x) \|f\|_p, \quad x > 0, \]

where

\[ g(x) = x^{1 - 1/p} \exp(-cx) \begin{cases} x^{2\sigma - 1}, & \sigma < 1/2 \\ 1 + \log + \frac{1}{x}, & \sigma = 1/2 \\ 1, & \sigma > 1/2. \end{cases} \]

Since \( g \in L^p \), we see that \( U^1 \) is \( L^p \)-bounded.

Considering \( U^2 \), we recall that it is the dual of \( U^1 \) and use the already proved result for \( U^1 \). \( \square \)

Finally, for the sake of completeness and, perhaps, reader’s curiosity, we formulate an analogue of Lemma 4.3. The proof is very similar to that of Lemma 4.3.

Proposition 4.8. Let \( \alpha > -1, \sigma > 0 \) and \( 1 \leq p \leq \infty \). Then the estimates

\[ \|\mathcal{H}_{\infty}^{\alpha, \sigma}(x, \cdot)\|_p \approx \begin{cases} x^{\alpha + 1/2}, & x \leq 1 \\ 1, & x > 1 \end{cases} \]

hold provided that \( p \) satisfies \( \frac{1}{p} > 1 - 2\sigma \) and, in addition, \( \frac{1}{p} > -\alpha - 1/2 \) in case \( \alpha < -1/2 \).

Moreover, for the remaining \( p \) we have

\[ \|\mathcal{H}_{\infty}^{\alpha, \sigma}(x, \cdot)\|_p = \infty, \quad x > 4. \]

4.3. \( L^p - L^q \) estimates in the setting of the Hankel-Dunkl transform. We will argue along the lines of the proof of [28, Theorem 2.6]. The following notation will be useful. For a function \( f \) on \( \mathbb{R} \), define \( f_+ \) and \( f_- \) as functions on \( \mathbb{R}_+ \) given by \( f_{\pm}(x) = f(\pm x), x > 0 \). In a similar way, let \( K_+^{\alpha, \sigma} \) and \( K_-^{\alpha, \sigma} \) be the kernels on \( \mathbb{R}_+ \times \mathbb{R}_+ \) determined by \( K_{\pm}^{\alpha, \sigma}(x, y) = K_{\pm}^{\alpha, \sigma}(x, \pm y), x, y > 0 \). Denote the corresponding integral operators related to the measure space \( (\mathbb{R}_+, d\mu_\alpha) \) by \( \mathcal{I}_{\pm}^{\alpha, \sigma} \), respectively.

It is clear that for any fixed \( 1 \leq p \leq \infty \),

\[ \|f\|_{L^p(d\mu_\alpha)} \approx \|f_+\|_{L^p(d\mu_\alpha)} + \|f_-\|_{L^p(d\mu_\alpha)}. \]

Further, by the symmetry of the kernel, \( K^{\alpha, \sigma}(-x, y) = K^{\alpha, \sigma}(x, -y) \), and the symmetry of \( w_\alpha \),

\[ (\mathcal{I}_{\pm}^{\alpha, \sigma} f)_{\pm} = I_+^{\alpha, \sigma}(f_+) + I_-^{\alpha, \sigma}(f_-). \]
Proof of Theorem 2.15 We first prove the sufficiency in (i). Take \( a \) satisfying the relevant condition and assume that \( f \in L^p(|x|^\alpha d\mu_a) \). Then \( f_+ \in L^p(|x|^\alpha d\mu_+ a) \). Since, in view of (13), the kernels \( K_{\pm}^\alpha(x,y) \) are controlled by \( K^\alpha(x,y) \), from Theorem 2.5 (i) it follows that \( f_+, f_- \in \text{Dom} \mathbb{H}_\pm^\alpha \). Now (29) gives the desired conclusion.

To show the necessity in (i), consider \( f \) such that \( f_- \equiv 0 \). Then \( (\mathbb{H}_+^\alpha f)_+ = \mathbb{H}_+^\alpha(f_+) \). We see that if \( f \in \text{Dom} \mathbb{H}_+^\alpha \), then \( f_+ \in \text{Dom} \mathbb{H}_+^\alpha \), and so \( f_+ \in \text{Dom} \mathbb{H}_+^\alpha \), because of (14). This means that if \( L^p(|x|^\alpha d\mu_a) \subset \text{Dom} \mathbb{H}_+^\alpha \), then \( L^p(|x|^\alpha d\mu_a) \subset \text{Dom} \mathbb{H}_+^\alpha \), and so the condition in question must be satisfied in virtue of Theorem 2.16 (i).

Proving (ii), assume first that (a)-(e) are satisfied. Observe that, because of (13) and Theorem 2.5 (ii), the operators \( \mathbb{H}_+^\alpha \) satisfy
\[
\|x^{-b} \mathbb{H}_+^\alpha f\|_{L^p(d\mu_a)} \lesssim \|x^a f\|_{L^p(d\mu_a)}
\]
uniformly in \( g \in L^p(|x|^\alpha d\mu_a) \). Therefore, with (29) in mind, we can write
\[
\|x^{-b} \mathbb{H}_+^\alpha f\|_{L^p(d\mu_a)} \simeq \|x^{-b} \mathbb{H}_+^\alpha f\|_{L^p(d\mu_a)} + \|x^{-b} (\mathbb{H}_+^\alpha f)_-\|_{L^p(d\mu_a)} \\
\leq \|x^{-b} \mathbb{H}_+^\alpha f\|_{L^p(d\mu_a)} + \|x^{-b} \mathbb{H}_+^\alpha f\|_{L^p(d\mu_a)} \\
+ \|x^{-b} \mathbb{H}_+^\alpha f\|_{L^p(d\mu_a)} + \|x^{-b} \mathbb{H}_+^\alpha f\|_{L^p(d\mu_a)} \\
\lesssim \|x^a f\|_{L^p(d\mu_a)} + \|x^a f\|_{L^p(d\mu_a)} \\
\simeq \|x^a f\|_{L^p(d\mu_a)}.
\]

Finally, to show the necessity part in (ii) we consider \( f \) such that \( f_- \equiv 0 \). Then, in view of (14), \( \mathbb{H}_+^\alpha(f_+) \simeq I^\alpha(f_+) \). Therefore, by Theorem 2.4 (ii), conditions (a)-(e) are necessary for \( \mathbb{H}_+^\alpha \) to satisfy (30). But such an estimate is implied by the analogous one for \( I^\alpha \), because \( \|x^{-b} I^\alpha(f_+)\|_{L^p(d\mu_a)} = \|x^{-b} (I^\alpha f)_+\|_{L^p(d\mu_a)} \lesssim \|x^{-b} \mathbb{H}_+^\alpha f\|_{L^p(d\mu_a)} \) and \( \|x^a f\|_{L^p(d\mu_a)} = \|x^a f_+\|_{L^p(d\mu_a)} \). The conclusion follows.

Proof of Theorem 2.10 Items (i) and (ii) are special cases of the corresponding parts in Theorem 2.15. To show (iii) we use Theorem 2.2 (iii) and arguments similar to those from the proof of Theorem 2.15. Details are left to the reader.

Proof of Theorem 2.19 The reasoning relies on arguments analogous to those from the proof of Theorem 2.15 combined with (13), (16), Theorem 2.8 and an analogue of (29). We omit the details.

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