Belief revision in Institutions: A relaxation-based approach

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Abstract

Belief revision of knowledge bases represented by a set of sentences in a given logic has been extensively studied but for specific logics, mainly propositional, but also recently Horn and description logics. Here, we propose to generalize this operation from a model-theoretic point of view, by defining revision in a categorical abstract model theory known under the name of theory of institutions. In this framework, we generalize to any institution the characterization of the well known AGM postulates given by Katsuno and Mendelzon for propositional logic in terms of minimal change with respect to an ordering among interpretations. Moreover, we study how to define revision, satisfying the AGM postulates, from relaxation notions that have been first introduced in description logics to define dissimilarity measures between concepts, and the consequence of which is to relax the set of models of the old belief until it becomes consistent with the new pieces of knowledge. The proposed general framework can be instantiated in different logics such as propositional, description and Horn logics.

Keywords: Abstract belief revision, relaxation, AGM theory, theory of institutions

1 Introduction

Belief change is a field of knowledge representation that has received much attention in recent years. It is defined by three change operations, expansion, contraction and revision, that make evolve an agent belief with a new acquired knowledge. Belief expansion consists on adding new knowledge without checking consistency, whilst both contraction and revision consist on removing and adding, respectively, but consistently new knowledge. When knowledge bases are logical theories, i.e. a set of sentences in a given logic, these changes are governed by a set of postulates that have been set for the first time by Alchourròn, Gardenfors and Makinson [1], and since known as the AGM theory. Although defined in the abstract framework of logics given by Tarski [21], postulates of the AGM theory make strong assumptions on considered logics. Indeed, these latter (so-called Tarskian logics) have to be closed under the standard propositional connectors in \{\&, \lor, \neg, \Rightarrow\}, to be compact (i.e. property entailment depends on a finite set of axioms), and to satisfy the deduction theorem (i.e. entailment and implication are equivalent).
While compactness is a standard property of logics, to be closed under the standard propositional connectors is more questionable. Indeed, many non-classical logics such as description logics, equational logic or Horn clause logic widely used for various modern applications in computing science, do not satisfy such a constraint. Many works have then been proposed to study belief change in such non-classical logics. In this direction, we can cite Ribeiro & al.’s work in [18] that studies contraction at the abstract level of Tarskian logics. On the contrary, the adaptation of AGM postulates for revision for non-classical logics has been studied but for specific logics, mainly around description logics [13, 15, 16, 22] and Horn logics [6, 24]. The reason is revision can be abstractly defined in terms of expansion and retraction following the Levi identity but this requires the use of negation which rules out non-classical logics [17].

Moreover, the AGM theory gives no minimality criteria on removed or changed formulae, and then on the set of models of knowledge bases. Recently, both for contraction and revision, propositions of generalisation of AGM theory have been proposed in the framework of Tarskian Logics considering minimality criteria on removed formulae [18, 17]. The aim was to study contraction and revision for a larger family of logics containing non-classical ones such as description logics and Horn logics. On the contrary, up to our knowledge, generalisation of AGM theory with minimality criteria on the set of models of knowledge bases have never been proposed. The reason is semantics is not explicit in the abstract framework of logics defined by Tarski.

We then propose here to generalise AGM revision but in a categorical abstract model theory, the theory of institutions [11], which formalises the intuitive notion of logical system, including syntax, semantics and the satisfaction relation. Then, we propose to generalise to any institution the approach developed in [12] for the propositional logic and in [14] for the description logics. In this abstract framework, we will also show how to define revision operators from relaxation notion that have been introduced in description logics to define dissimilarity measures between concepts [9, 10] and the consequence of which is to relax the set of models of the old belief until it becomes consistent with the new pieces of knowledge.

The paper is organised as follows. Section 2 reviews some concepts, notations and terminology about institutions which are used by this work. In Section 3 we adapt AGM theory in the framework of institutions, and then give an abstract model-theoretic rewriting of AGM postulates. We then show in Section 3.2 that any revision operators satisfying such postulates accomplishes an update with minimal change to the set of models of knowledge bases. In Section 3.3 we introduce a general framework of relaxation-based revision operators and show that our revision operators lead to faithful assignment and then satisfy AGM postulates. In Section 4 we illustrate our abstract approach by providing revision operators in different logics, mainly non-classical logics such as Horn logics and description logics. Finally, Section 5 is dedicated to related works.

2 Institutions

The theory of institutions [11] is a categorical abstract model theory which emerges in computing science studies of software specification and semantics, in the context of the population explosion of logics there, with the ambition of doing as much as possible at the level of abstraction independent of commitment to any particular logic. Now institutions have become a common tool in the area of formal specification, in fact its most fundamental mathematical structure.

2.1 Basic definitions and examples

Definition 2.1 (Institution) An institution \( \mathcal{I} = (\text{Sig}, \text{Sen}, \text{Mod}, \models) \) consists of

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• a category $\Sigma$, objects of which are called signatures,
• a functor $\text{Sen} : \Sigma \to \text{Set}$ giving for each signature a set, elements of which are called sentences,
• a contravariant functor $\text{Mod} : \Sigma^{\text{op}} \to \text{Cat}$ giving for each signature a category, objects and arrows of which are called $\Sigma$-models and $\Sigma$-morphisms respectively, and
• a $\Sigma$-indexed family of relations $\models_\Sigma \subseteq \text{Mod}(\Sigma) \times \text{Sen}(\Sigma)$ called satisfaction relation,

such that the following property holds:

$\forall \sigma : \Sigma \to \Sigma', \forall M' \in \text{Mod}(\Sigma'), \forall \varphi \in \text{Sen}(\Sigma), \quad M' \models_\Sigma \text{Sen}(\sigma)(\varphi) \iff \text{Mod}(\sigma)(M') \models_\Sigma \varphi$

This goes a step beyond Tarski’s classic “semantic definition of truth” [20] and also generalises Barwise’s “Translation Axiom” [2]. Moreover, it is fundamental that sentences translate in the same direction as the change of notation, whereas models translate in the opposite direction (think of signature enrichment and model reduction). This is the reason for the functor $\text{Mod}$ in Definition 2.1 below to be contravariant. For the sake of generalisation, signatures are simply defined as objects of a category and sentences built over a signature are simply required to form a set. All other contingencies such as inductive definition of sentences are not considered. Similarly, models are simply seen as objects of a category, i.e. no particular structure is imposed on them.

Example 2.2 The following examples of institutions are of particular importance for computer science. Many other examples can be found in the literature (e.g. [11, 19]).

Propositional Logic (PL) Signatures and signature morphisms are sets of propositional variables and functions between them respectively.

Given a signature $\Sigma$, the set of $\Sigma$-sentences is the least set of sentences finitely built over propositional variables in $\Sigma$ and Boolean connectives in $\{\neg, \lor\}$. Given a signature morphism $\sigma : \Sigma \to \Sigma'$, $\text{Sen}(\sigma)$ translates $\Sigma$-formulae to $\Sigma'$-formulae by renaming propositional variables according to $\sigma$.

Given a signature $\Sigma$, the category of $\Sigma$-models is the category of mappings $\nu : \Sigma \to \{0, 1\}$ with identities as morphisms. Given a signature morphism $\sigma : \Sigma \to \Sigma'$, the forgetful functor $\text{Mod}(\sigma)$ maps a $\Sigma'$-model $\nu'$ to the $\Sigma$-model $\nu = \nu' \circ \sigma$.

Finally, satisfaction is the usual propositional satisfaction.

Horn Logic (HCL) An Horn clause for a signature $\Sigma$ in PL is a $\Sigma$-sentence of the form $\Gamma \Rightarrow \alpha$ where $\Gamma$ is a finite conjunction of propositional variables and $\alpha$ is a propositional variable. The institution of Horn clause logic is the sub-institution of PL whose signatures and models are those of PL and sentences are restricted to the conjunction of Horn clauses.

Many-sorted First Order Logic (FOL) Signatures are triples $(S, F, P)$ where $S$ is a set of sorts, and $F$ and $P$ are sets of function and predicate names respectively, both with arities in $S^* \times S$ and $S^+$ respectively. Signature morphisms $\sigma : (S, F, P) \to (S', F', P')$ consist of three functions between sets of sorts, sets of functions and sets of predicates respectively, the last two preserving arities.

Given a signature $\Sigma = (S, F, P)$, the $\Sigma$-atoms are of two possible forms: $t_1 = t_2$ where

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1 $\{0, 1\}$ are the usual truth-values.
2 $S^+$ is the set of all non-empty sequences of elements in $S$ and $S^* = S^+ \cup \{\epsilon\}$ where $\epsilon$ denotes the empty sequence.
Modal First Order Logic with global satisfaction (MFOL)

Equational Logic (EQL)

aka. quantified modal logic

Given a signature $\Sigma = (S, F, P)$, a $\Sigma$-model $M$ is a family $M = (M_s)_{s \in S}$ of sets (one for every $s \in S$), each one equipped with a function $f^M : M_{s_1} \times \ldots \times M_{s_n} \rightarrow M_s$ for every $f : s_1 \times \ldots \times s_n \rightarrow s \in F$ and with a $n$-ary relation $p^M \subseteq M_{s_1} \times \ldots \times M_{s_n}$ for every $p : s_1 \times \ldots \times s_n \in P$. Given a signature morphism $\sigma : \Sigma = (S, F, P) \rightarrow \Sigma' = (S', F', P')$ and a $\Sigma'$-model $M'$, $\text{Mod}(\sigma)(M')$ is the $\Sigma$-model $M$ defined for every $s \in S$ by $M_s = M'_s$, and for every function name $f \in F$ and predicate name $p \in P$, by $f^M = \sigma(f)^M$ and $p^M = \sigma(p)^M$.

Finally, satisfaction is the usual first-order satisfaction.

As for the institution PL, we can consider the sub-institution FHCL of FOL whose signatures and models are those of FOL and sentences are restricted to the conjunction of universal Horn sentences (i.e. formulae of the form $\Gamma \Rightarrow \alpha$ where $\Gamma$ is a finite conjunction of $\Sigma$-atoms and $\alpha$ is a $\Sigma$-atom).

Equational Logic (EQL)

An algebraic signature $(S, F)$ simply is a FOL signature without predicate symbols. The institution of equational logic is the sub-institution of FOL whose signatures and models are algebraic signatures and algebras respectively, and sentences are restricted to equations.

Rewriting Logic (RWL)

Given an algebraic signature $\Sigma = (S, F)$, $\Sigma$-sentences are formulae of the form $\varphi : t_1 \rightarrow t_1 \land \ldots \land t_n \rightarrow t_n \Rightarrow t \rightarrow t'$ where $t_i, t'_i \in T_F(X)$, $s_i \in S$, and $t, t' \in T_F(X)$. Models of rewriting logic are preorder models, i.e. given a signature $\Sigma = (S, F)$, $\text{Mod}(\Sigma)$ is the category of $\Sigma$-algebras $A$ such that for every $s \in S$, $As$ is equipped with a preorder $\geq$. Hence, $A \models \varphi$ if and only if for every variable interpretation $\nu : X \rightarrow A$, if each $\nu(t_i)^A \geq \nu(t'_i)^A$ then $\nu(t)^A \geq \nu(t')^A$ where $\varphi^A : T_F(A) \rightarrow A$ is the mapping inductively defined by: $f(t_1, \ldots, t_n)^A = f^A(t_1^A, \ldots, t_n^A)$.

Modal First Order Logic with global satisfaction (MFOL)

The category of signatures is the category of FOL signatures.

Given a FOL signature $\Sigma = (S, F, P)$, $\Sigma$-axioms are of the form $p(t_1, \ldots, t_n)$ and the set of $\Sigma$-formulae is the least set of formulae built over the set of $\Sigma$-axioms by finitely applying Boolean connectives in $\{\land, \lor\}$ and the quantifier $\forall$ and the modality $\square$.

Given a signature $\Sigma = (S, F, P)$, a $\Sigma$-model $(W, R)$, called Kripke frame, consists of a family $W = (W_i)_{i \in I}$ of $\Sigma$-models in FOL (the possible worlds) such that for every $i, j \in I$ and $s \in S$, and an "accessibility" relation $R \subseteq I \times I$. Given a signature morphism $\sigma : \Sigma = (S, F, P) \rightarrow \Sigma' = (S', F', P')$ and a $(S', F', P')$-model $(W', R')$, $\text{Mod}(\sigma)((W', R'))$ is the $(S, F, P)$-model $(W, R)$ defined for every $i \in I$ by $W^i = \text{Mod}(\sigma)(W'_i)$ and by $R = R'$. A $\Sigma$-sentence $\varphi$ is said to be satisfied by a $\Sigma$-model $(W, R)$, noted $(W, R) \models_\Sigma \varphi$, if for every $i \in I$ we have $(W_i, R'_i) \models_{\Sigma'} \varphi$, where $\models_{\Sigma'}$ is inductively defined on the structure of $\varphi$ as follows:

- $\text{atoms}$, Boolean connectives and quantifier are handled as in FOL,
- $(W, R) \models_i \square \varphi$ when $(W, R) \models_j \varphi$ for every $j \in I$ such that $i R j$.

Modal propositional logic (MPL) is the sub-institution of MFOL whose signatures are restricted to empty sets of sorts and function names and only 0-ary predicate names.

\[ T_F(X) \] is the term algebra of sort $s$ built over $F$ with sorted variables in a given set $X$.

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Modal First Order Logic with local satisfaction (LMFOL)  
Signatures and sentences are MFO signatures and MFO sentences. 
Given a signature $\Sigma = (S, F, P)$, a $\Sigma$-model is a pointed Kripke frame $(W = (W^i)_{i \in I}, R, W')$ where $j \in I$. The satisfaction of a $\Sigma$-sentence $\varphi$ by a $\Sigma$-model $(W, R, W')$, noted $(W, R, W') \models_{\Sigma} \varphi$, is defined by:

$$(W, R, W') \models_{\Sigma} \varphi \iff (W, R, W') \models_{\Sigma} \varphi.$$  

LMFOL with infinite disjunction and conjunction (LMFOL)  
This institution extends LMFOL to sentences of the form $\bigwedge \Phi$ and $\bigvee \Phi$ where $\Phi$ is a set (possibly infinite) of $\Sigma$-sentences. 
Given a pointed Kripke frame $(W, R, W')$,

- $(W, R, W') \models_{\Sigma} \bigwedge \Phi \iff \forall \varphi \in \Phi, (W, R, W') \models_{\Sigma} \varphi$
- $(W, R, W') \models_{\Sigma} \bigvee \Phi \iff \exists \varphi \in \Phi, (W, R, W') \models_{\Sigma} \varphi$

Description logic DL  
Signatures are triples $(N_C, N_R, I)$ where $N_C$, $N_R$ and $I$ are nonempty pairwise disjoint sets such that elements in $N_C$, $N_R$ and $I$ are concept names, role names and individuals, respectively. 
Signature morphisms $\sigma : (S, F, P) \to (S', F', P')$ consist of three functions between sets of concept names, sets of role names and sets of individuals respectively. 
Given a signature $\Sigma \in \text{Sign}$, $\text{Sen}(\Sigma)$ contains all the sentences of the form $C \subseteq D, x : C$ and $(x, y) : r$ where $x, y \in I, r \in N_R$ and $C$ is a concept inductively defined from $N_C$ and binary and unary operators in $\{\wedge, \vee, \exists r, \forall x, \exists y\}$ respectively. 

Given a signature $\Sigma \in \text{Sign}$, a $\Sigma$-model $\mathcal{O}$ is a set $\Delta^\Sigma$ equipped for every concept name $c \in N_C$ with a set $\sigma^\Sigma(c) \subseteq \Delta^\Sigma$, for every relation name $r \in N_R$ with a binary relation $\sigma^\Sigma(r) \subseteq \Delta^\Sigma \times \Delta^\Sigma$, and for every individual with a value $\sigma^\Sigma(x) \in \Delta^\Sigma$. A morphism $\mu : \mathcal{O} \to \mathcal{O}'$ is a mapping $\mu : \Delta^\Sigma \to \Delta'^\Sigma$ such that for every $c \in N_C$, every $r \in N_R$ and every $x \in I$, $\mu(c^\Sigma) \subseteq c'^\Sigma$, $\mu(r^\Sigma) \subseteq r'^\Sigma$ and $\mu(x^\Sigma) = x'^\Sigma$. Given a signature morphism $\sigma : \Sigma \to \Sigma'$, for every $\Sigma$-model $\mathcal{O}$, $\text{Mod}(\sigma)(\mathcal{O}')$ is the $\Sigma'$-model $\mathcal{O}'$ such that $\Delta'^\Sigma = \sigma(\Delta^\Sigma)$, for every $c \in N_C$, $c'^\Sigma = \sigma(c)^\Sigma$, for every $r \in N_R$, $r'^\Sigma = \sigma(r)^\Sigma$, and for every $x \in I$, $x'^\Sigma = \sigma(x)^\Sigma$. 

Given a $\Sigma$-model $\mathcal{O}$ and a context $C$ over the signature $\Sigma$, the evaluation of $C$ in $\mathcal{O}$ is inductively defined on the structure of $C$ as follows:

- if $C = c$ with $c \in N_C$, then $C^\mathcal{O} = c^\Sigma$;
- if $C = C' \cup D'$ (resp. $C = C' \cap D'$), then $C^\mathcal{O} = C'^\mathcal{O} \cup D'^\mathcal{O}$ (resp. $C^\mathcal{O} = C'^\mathcal{O} \cap D'^\mathcal{O}$);
- if $C = C'^\mathcal{O}$, then $C^\mathcal{O} = \Delta^\Sigma \setminus C'^\mathcal{O}$;
- if $C = \forall r.C'$, then $C^\mathcal{O} = \{x \in \Delta^\Sigma \mid \forall y \in \Delta^\Sigma, (x, y) \in r^\Sigma \Rightarrow y \in C'^\mathcal{O}\}$;
- if $C = \exists r.C'$, then $C^\mathcal{O} = \{x \in \Delta^\Sigma \mid \exists y \in \Delta^\Sigma, (x, y) \in r^\Sigma \wedge y \in C'^\mathcal{O}\}$.

The satisfaction relation $\models_\mathcal{O}$ is then defined:

- $\mathcal{O} \models_\mathcal{O} C \subseteq D$ iff $C^\mathcal{O} \subseteq D^\mathcal{O}$;
- $\mathcal{O} \models_\mathcal{O} x : C$ iff $x^\mathcal{O} \in C^\mathcal{O}$;
- $\mathcal{O} \models_\mathcal{O} (x, y) : r$ iff $(x^\mathcal{O}, y^\mathcal{O}) \in r^\mathcal{O}$.

\(O\) as "ontology".
2.2 Knowledge base and theories in institutions

Let us now consider a fixed but arbitrary institution $\mathcal{I} = (\text{Sig}, \text{Sen}, \text{Mod}, \models)$ such that Sig is cocomplete (i.e. it has both pushout and coproduct)\[1\]. This latter property will allow us to consider every time in the following sets of sentences over a same signature.

**Notation 2.3** Let $\Sigma \in [\text{Sig}]$ be a signature and $T$ be a set of $\Sigma$-sentences.

- $\text{Mod}(T)$ is the full sub-category of $\text{Mod}(\Sigma)$ whose objects are models of $T$.
- $Cn(T) = \{ \phi \in \text{Sen}(\Sigma) \mid \forall \mathcal{M} \in \text{Mod}(T), \mathcal{M} \models \phi \}$ is the set of so-called semantic consequences of $T$.

- Let $\mathcal{M} \subseteq \text{Mod}(\Sigma)$. Let us note $\mathcal{M}^* = \{ \phi \in \text{Sen}(\Sigma) \mid \forall \mathcal{M} \in \mathcal{M}, \mathcal{M} \models \phi \}$. Therefore, we have for every $T \subseteq \text{Sen}(\Sigma)$, $Cn(T) = \text{Mod}(T)^*$. When $\mathcal{M}$ is restricted to a model $\mathcal{M}$, $\mathcal{M}^*$ will be equivalently noted $\mathcal{M}^*$.

These two functions denoted respectively $\text{Mod}(\cdot)$ and $\cdot^*$ form what is know as a Galois connection in that they satisfy the following properties: $T, T' \subseteq \text{Sen}(\Sigma)$ and $\mathcal{M}, \mathcal{M}' \subseteq \text{Mod}(\Sigma)$ (see \[8\])

1. $T \subseteq T' \implies \text{Mod}(T') \subseteq \text{Mod}(T)$
2. $\mathcal{M} \subseteq \mathcal{M}' \implies \mathcal{M}^* \subseteq \mathcal{M}^*$
3. $T \subseteq \text{Mod}(T)^*$
4. $\mathcal{M} \subseteq \text{Mod}(\mathcal{M}^*)$

**Definition 2.4 (Knowledge base and theory)** A knowledge base $T$ is a set of $\Sigma$-sentences (i.e. $T \subseteq \text{Sen}(\Sigma)$)\[9\]. A knowledge base $T$ is said to be a theory if, and only if $T = \text{Cn}(T)$.

A theory $T$ is finitely representable if there exists a finite set $T' \subseteq \text{Sen}(\Sigma)$ such that $T = \text{Cn}(T')$.

**Proposition 2.5** For every institution $\mathcal{I}$, we have:

**Inclusion** $\forall T \subseteq \text{Sen}(\Sigma), T \subseteq \text{Cn}(T)$;

**Iteration** $\forall T \subseteq \text{Sen}(\Sigma), \text{Cn}(T) = \text{Cn}(\text{Cn}(T))$;

**Monotonicity** $\forall T, T' \subseteq \text{Sen}(\Sigma), T \subseteq T' \implies \text{Cn}(T) \subseteq \text{Cn}(T')$

**Proof** Inclusion and iteration are obvious properties of the mapping $\text{Cn}$ by definition\[10\].

Suppose $T \subseteq T'$. By the first property of the Galois connection above between $\text{Mod}(\cdot)$ and $\text{Cn}(\cdot)$, we have that $\text{Mod}(T') \subseteq \text{Mod}(T)$. Let $\phi \in \text{Cn}(T)$. Therefore, we have for every $\mathcal{M} \in \text{Mod}(T')$, that $\mathcal{M} \models \phi$, and then $\phi \in \text{Cn}(T')$.

Hence, institutions are Tarskian according to the definition of logics given by Tarski under which a logic is a pair $(\mathcal{L}, \text{Cn})$ where $\mathcal{L}$ is a set of expressions (formule) and $\text{Cn} : \mathcal{P}(\mathcal{L}) \to \mathcal{P}(\mathcal{L})$ is a mapping that satisfies the inclusion, iteration and monotonicity properties\[21\]. Indeed, from any institution $\mathcal{I}$ we can define the following Tarskian logic $(\mathcal{L}, \text{Cn})$ where

\[1\] All the examples of institutions given in the paper are cocomplete because signatures are set-based.

\[9\] Usually, in the framework of institutions, the set of semantic consequences of a theory $T$ is noted $T^\bullet$.

\[10\] In the framework of institutions, knowledge bases are also called presentations (cf. \[8\]).

\[10\] Inclusion is Property 3. of the Galois Connection above.
\[ \mathcal{L} = \bigcup_{\Sigma \in \text{Sign}} \text{Sen}(\Sigma); \]

- Given a set \( T \subseteq \mathcal{L} \), as \( \mathcal{I} \) is cocomplete, there exists a signature \( \Sigma \) such that \( T \subseteq \text{Sen}(\Sigma) \). Then, \( \text{Cn}(T) = \{ \varphi \in \text{Sen}(\Sigma) \mid \forall \mathcal{M} \in \text{Mod}(T), \mathcal{M} \models \varphi \} \).

Classically, consistency of a theory \( T \) is defined as \( \text{Mod}(T) = \emptyset \). The problem of such definition of consistency is its significance depends on the actual institution \( \mathcal{I} \). Hence, such consistency is significant for \text{FOL}, while in \text{EQL} and \text{FHCL} it is a trivial property since each set of sentences is consistent because \( \text{Mod}(T) \) always contains the trivial model. Here, to be more appropriate with our purpose to define revision for the largest family of logics, we propose a more general definition of consistency the meaning of which is there is at least a sentence which is not a semantic consequence.

**Definition 2.6 (Consistency)** \( T \subseteq \text{Sen}(\Sigma) \) is consistent if \( \text{Cn}(T) \neq \text{Sen}(\Sigma) \).

**Proposition 2.7** For every \( T \subseteq \text{Sen}(\Sigma) \), \( T \) is consistent if, and only if \( \text{Mod}(T) \setminus \{ \mathcal{M} \in \text{Mod}(\Sigma) \mid \mathcal{M}^* = \text{Sen}(\Sigma) \} \neq \emptyset \).

**Proof** Let us assume that \( \text{Mod}(T) \setminus \{ \mathcal{M} \in \text{Mod}(\Sigma) \mid \mathcal{M}^* = \text{Sen}(\Sigma) \} = \emptyset \). Therefore, by definition of \( \text{Cn}(T) \), this means that the only \( \Sigma \)-models that satisfy \( T \) are \( \mathcal{M} \) such that \( \mathcal{M}^* = \text{Sen}(\Sigma) \) (if there exist). Hence, we have \( \text{Cn}(T) = \text{Sen}(\Sigma) \).

Let us assume that \( \text{Cn}(T) = \text{Sen}(\Sigma) \). This means that every \( \Sigma \)-model \( \mathcal{M} \) such that \( \mathcal{M}^* \neq \text{Sen}(\Sigma) \) does not belong to \( \text{Mod}(T) \). \( \square \)

## 3 AGM postulates for revision in institutions

### 3.1 AGM postulates

AGM postulates for knowledge bases revision in institutions are easily adaptable. Given two knowledge bases \( T, T' \subseteq \text{Sen}(\Sigma) \), \( T \circ T' \) denotes the revision of \( T \) by \( T' \), that is, \( T \circ T' \) is obtained by adding consistently new knowledge \( T' \) to the old knowledge base \( T \). \( T \circ T' \) cannot be defined as \( T \cup T' \) because nothing ensures that \( T \cup T' \) is consistent. The revision operator has then to change minimally \( T \) so that \( T \circ T' \) is consistent. This is what the AGM postulates ensure.

- **(G1)** If \( T' \) is consistent, then so is \( T \circ T' \).
- **(G2)** \( \text{Mod}(T \circ T') \subseteq \text{Mod}(T') \).
- **(G3)** if \( T \cup T' = T \cup T' \) is consistent, then \( T \circ T' = T \cup T' \).
- **(G4)** if \( \text{Cn}(T') \subseteq \text{Cn}(T'') \), then \( \text{Mod}(T \circ T'') \subseteq \text{Mod}(T \circ T') \) (and then if \( \text{Cn}(T') = \text{Cn}(T'') \), then \( \text{Mod}(T \circ T') = \text{Mod}(T \circ T'') \)).
- **(G4-bis)** if \( \text{Cn}(T') \subseteq \text{Cn}(T'') \), then \( \text{Mod}(T' \circ T) \subseteq \text{Mod}(T' \circ T') \) (and then if \( \text{Cn}(T') = \text{Cn}(T'') \), then \( \text{Mod}(T' \circ T) = \text{Mod}(T' \circ T') \)).
- **(G5)** if \( T \cup T' \cup T'' \) is consistent, then \( T \circ (T' \cup T'') = (T \circ T') \cup T'' \).

Classically, Postulates (G4) and (G4-bis) are grouped into a same postulate

**G4** if \( \text{Cn}(T_1) = \text{Cn}(T'_1) \) and \( \text{Cn}(T_2) = \text{Cn}(T'_2) \), then \( \text{Mod}(T_1 \circ T_2) = \text{Mod}(T'_1 \circ T'_2) \)

\(^{11}\)By definition, for every \( \varphi \in T \), there exists a signature \( \Sigma_\varphi \) such that \( \varphi \in \text{Sen}(\Sigma_\varphi) \). Then, \( \Sigma \) is the coproduct of the diagram built over the signatures in \( \{ \Sigma_\varphi \mid \varphi \in T \} \).
Obviously, (G4) and (G4-bis) imply (G4'). The interest of breaking Postulate (G4') into Postulates (G4) and (G4-bis) is that in the proof of Theorem 3.2 we will mainly refer to Postulate (G4) to show the first implication.

Intuitively, any revision operator $\circ$ satisfying the five postulates above, effects minimal change, that is the models of $T \circ T'$ are the models of $T$ that are closest to models of $Mod(T')$ according to some metric distance for measuring the “distance” between models. This is what will be shown in the next section by showing a correspondence between AGM Postulates and orders over models.

### 3.2 Orders and AGM postulates

Let $\Sigma$ be a signature. Let $\mathbb{M} \subseteq Mod(\Sigma)$. A **pre-order** $\preceq$ over $\mathbb{M}$ is a reflexive and transitive binary relation over $\mathbb{M}$. We define $\prec$ as $\mathcal{M} \prec \mathcal{M}'$ if, and only if $\mathcal{M} \preceq \mathcal{M}'$ and $\mathcal{M}' \not\preceq \mathcal{M}$. $\preceq$ is **total** if for every $\mathcal{M}, \mathcal{M}' \in Mod(\Sigma)$, $\mathcal{M} \preceq \mathcal{M}'$ or $\mathcal{M}' \preceq \mathcal{M}$. $\preceq$ is **well-founded** if any infinite descending sequence of models $(\mathcal{M}_i)_{i \in \mathbb{N}}$ is stationary, i.e. there exists $j \in \mathbb{N}$ such that for every $k \geq j$, both $\mathcal{M}_k \preceq \mathcal{M}_{k+1}$ and $\mathcal{M}_{k+1} \preceq \mathcal{M}_k$. We define $\text{Min} (\mathbb{M}, \preceq) = \{ \mathcal{M} \in \mathbb{M} | \forall \mathcal{M}' \in \mathbb{M}, \mathcal{M}' \not\preceq \mathcal{M} \}$.

**Definition 3.1 (Faithful assignment)** Let $T \subseteq \text{Sen}(\Sigma)$ be a knowledge base. Let $\preceq_T \subseteq Mod(\Sigma) \times Mod(\Sigma)$ be a total pre-order. $\preceq_T$ is a faithful assignment (FA) if the following three conditions are satisfied:

1. if $\mathcal{M}, \mathcal{M}' \in Mod(T)$, $\mathcal{M} \not\preceq_T \mathcal{M}'$.
2. for every $\mathcal{M} \in Mod(T)$ and every $\mathcal{M}' \in Mod(\Sigma) \setminus Mod(T)$, $\mathcal{M} \prec_T \mathcal{M}'$.
3. for every $T' \subseteq \text{Sen}(\Sigma)$, if $Mod(T) \subseteq Mod(T')$, then $\preceq_T \subseteq \preceq_{T'}$ (and then, if $Mod(T) = Mod(T')$, then $\preceq_T = \preceq_{T'}$).

The property for $\preceq_T$ to be well-founded is not required in the paper [12] that introduces for the first time this notion of faithful assignment. The reason is revision is studied in the framework of propositional logic whose the set of interpretations for finite signatures is also finite, and then obviously, faithful pre-orders between interpretations are well-founded. More astonishing, this property is also not imposed in the paper [12] that studies revision in description logics. The consequence is Theorem 1 in [13] that makes a correspondence between AGM postulates and minimal models according to FA pre-orders, does not hold because of the condition (G3) in [13] ((G1) here). This leads to the following representation theorem in our framework.

**Theorem 3.2** Let $\circ$ be a revision operator. Then,

1. if $\circ$ satisfies Postulates (G1)-(G5), then there exists a FA for any knowledge base $T$ such that $Mod(T \circ T') = \text{Min}(Mod(T') \setminus \{ \mathcal{M} \in Mod(\Sigma) | \mathcal{M}^* = \text{Sen}(\Sigma) \}, \preceq_T$).
2. If there exists a FA for any knowledge base which is further well-founded such that $Mod(T \circ T') = \text{Min}(Mod(T') \setminus \{ \mathcal{M} \in Mod(\Sigma) | \mathcal{M}^* = \text{Sen}(\Sigma) \}, \preceq_T$, then $\circ$ satisfies Postulates (G1)-(G5).

**Proof** The proof draws heavily from that given in the paper [12].

1. Let us suppose that $\circ$ satisfies (G1)-(G5). For every knowledge base $T$, let us define the binary relation $\preceq_T \subseteq Mod(T) \times Mod(T)$ by: For all $\mathcal{M}, \mathcal{M}' \in Mod(\Sigma)$

$$\mathcal{M} \preceq_T \mathcal{M}' \text{ iff } \begin{cases} \text{ either } \mathcal{M} \in Mod(T) \\ \text{ or } \mathcal{M} \in Mod(T \circ M^*) \end{cases}$$
Let us first show that $\preceq_T$ is a total pre-order.

**Reflexivity** By (G2), $\text{Mod}(T \circ M^*) \subseteq \text{Mod}(M^*)$. Therefore, we have that $M \in \text{Mod}(T \circ M^*)$.

**Transitivity** Let $M \preceq_T M'$ and $M' \preceq_T M''$. Three cases have to be considered:

- **Case 1.** $M \in \text{Mod}(T)$. By definition of $\preceq_T$, we then have $M \preceq_T M''$.

- **Case 2.** $M \not\in \text{Mod}(T)$ and $M' \in \text{Mod}(T)$. By definition of $\preceq_T$, $M \in \text{Mod}(T \circ M')$. By (G2), we also have that $M \in \text{Mod}(M'')$, and then $M \in \text{Mod}(T)$ which contradicts our assumptions. Therefore, this case is impossible.

- **Case 3.** $M \not\in \text{Mod}(T)$ and $M' \not\in \text{Mod}(T)$. By definition of $\preceq_T$, we have:
  - $M \in \text{Mod}(T \circ M'^*)$
  - $M' \in \text{Mod}(T \circ M''')$

  By the fact that $M, M' \not\in \text{Mod}(T)$, by (G2), we have that $M''' \subseteq M' \subseteq M'$. Thus, by (G4), $\text{Mod}(T \circ M'') \subseteq \text{Mod}(T \circ M''')$. We then have that $M \in \text{Mod}(T \circ M''')$, and then $M \preceq_T M''$.

**Total** For every $M, M' \in \text{Mod}(\Sigma)$, by (G1), $\text{Mod}(T \circ M'') \not= \emptyset$, and then $\preceq_T$ is total.

Next, we show that $\preceq_T$ is a FA. The first condition easily follows from the definition of $\preceq_T$, and the third from Postulates (G4) and (G4-bis). Let us assume that $M \in \text{Mod}(T)$ and $M' \not\in \text{Mod}(T)$. Let us assume further that $M \in \text{Mod}(T \circ M')$. By (G2), this means that $M' \in \text{Mod}(M^*)$, and then $M' \in \text{Mod}(T)$ what is a contradiction with our hypothesis. Therefore, we have $M' \not\preceq_T M$.

It remains to show that $\text{Mod}(T \circ T') = \text{Min}(\text{Mod}(T'), \preceq_T)$. If $T'$ is inconsistent, by (G2), so is $T \circ T'$ and then by Proposition [4], $\text{Mod}(T \circ T') = \emptyset = \text{Min}(\text{Mod}(T') \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}, \preceq_T)$.

Let us assume that $T'$ is consistent.

- $\text{Mod}(T \circ T') \subseteq \text{Min}(\text{Mod}(T'), \preceq_T)$. Let $M \in \text{Mod}(T \circ T')$. Let us assume that $M \not\in \text{Min}(\text{Mod}(T'), \preceq_T)$. By (G2), $M \in \text{Mod}(T')$. By hypothesis, there exists $M' \in \text{Mod}(T') \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}$ such that $M' \prec_T M$.

  Here, two cases have to be considered:

  (a) $M' \in \text{Mod}(T')$. As $M' \in \text{Mod}(T') \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}$, then $T \cup T'$ is consistent, and then by (G3), $T \circ T' = T \cup T'$. Thus, $M \in \text{Mod}(T)$, and then $M \not\preceq_T M'$ what is a contradiction.

(b) $M' \not\in \text{Mod}(T')$. By definition of $\preceq_T$, this means that $M' \in \text{Mod}(T \circ M')$. As $M' \in \text{Mod}(T')$, $\text{Cn}(T' \cup M'^*) = M'^*$. By (G5), we can write $T \circ (T' \cup M'^*) = (T \circ T') \cup M'^*$. Thus, by (G4) we have that $\text{Mod}(T \circ T') \cap \text{Mod}(M'^*) = \text{Mod}(T \circ M'^*)$. By the hypothesis that $M' \prec_T M$, we deduce that $M \not\in \text{Mod}(T \circ M'^*)$, and then $M \not\in \text{Mod}(T \circ T')$ what is a contradiction.

- $\text{Min}(\text{Mod}(T'), \preceq_T) \subseteq \text{Mod}(T \circ T')$. Let $M \in \text{Min}(\text{Mod}(T'), \preceq_T) \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}$, $\preceq_T$. Let us assume that $M \not\in \text{Mod}(T \circ T')$. As $T'$ is consistent, by (G1) and (G2), there is $M' \in \text{Mod}(T \circ T')$. As $M \in \text{Mod}(T') \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}$, $\text{Mod}(T' \cup M'^*) = \text{Mod}(M'^*)$. By (G2) and (G5), we can write $\text{Mod}(T \circ T') \cap \text{Mod}(M^*) = \text{Mod}(T \circ M^*)$, and then $M' \in \text{Mod}(T \circ M^*)$. Hence, we have that $M' \not\preceq_T M$. In the same way, we
have that $\text{Mod}(T \cup T') \cap \text{Mod}(M^*) = \text{Mod}(T \cup M^*)$. As $M \not\subseteq \text{Mod}(T \cup M^*)$, we have that $M \not\subseteq \text{Mod}(T')$, and then $M \not\subseteq \text{Mod}(T)$. Therefore, $M \not\subseteq \text{Mod}(T \cup T')$ what is a contradiction.

2. Let us suppose that there exists a FA which is further well-founded for any knowledge base $T$ such that $\text{Mod}(T \cup T') = \text{Min}(\text{Mod}(T'))$, $\preceq_T$.

**(G1)** Let $T'$ be a consistent knowledge base. Therefore, we have $\text{Mod}(T') \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\} \neq \emptyset$, and then $\preceq_T$ being well-founded, we also have $\text{Min}(\text{Mod}(T')) \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}, \preceq_T \neq \emptyset$, whence we conclude that $T \cup T'$ is consistent.

**(G2)** Let $M \in \text{Mod}(T \cup T')$. By definition, $M \in \text{Min}(\text{Mod}(T')) \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\} = \emptyset$. This means that $M = \emptyset$ what is a contradiction.

**(G3)** Suppose that $T \cup T'$ is consistent. Let us show that $\text{Mod}(T \cup T) = \text{Mod}(T \cup T')$.

- $\text{Mod}(T \cup T) \subseteq \text{Mod}(T \cup T')$. Let $M \in \text{Mod}(T \cup T')$. By definition, $M \in \text{Min}(\text{Mod}(T')) \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}, \preceq_T$. Hence, we have that $M \in \text{Mod}(T)$. Let us suppose now that $M \not\subseteq \text{Mod}(T)$. As $T$ is consistent, $\text{Mod}(T) \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}, \preceq_T = \emptyset$ by Proposition 2.7. Therefore, for every $M' \in \text{Mod}(T)$ such that $M^* \not\subseteq \text{Sen}(\Sigma)$, $M' \preceq_T M$ what is a contradiction.

- $\text{Mod}(T \cup T') \subseteq \text{Min}(\text{Mod}(T')) \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}, \preceq_T$. Let $M \in \text{Mod}(T \cup T')$ such that $M \not\subseteq \text{Mod}(T)$. By hypothesis, there exists $M' \in \text{Mod}(T) \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}$ such that $M' \preceq_T M$. As $\preceq_T$ is a FA, by the first condition of FA, we also have that $M \preceq_T M'$ what is a contradiction.

**(G4)** Let us suppose that $\text{Cn}(T) \subseteq \text{Cn}(T')$. We then have that $\text{Mod}(T') \subseteq \text{Mod}(T')$. Hence, $\text{Min}(\text{Mod}(T')) \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}, \preceq_T \subseteq \text{Min}(\text{Mod}(T')) \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}, \preceq_T$.

**(G4-bis)** Let us suppose that $\text{Cn}(T') \subseteq \text{Cn}(T)$. We then have that $\text{Mod}(T') \subseteq \text{Mod}(T')$. Therefore, by the third property of FA, we have $\preceq_T \subseteq \preceq_T$, and then $\text{Min}(\text{Mod}(T') \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}, \preceq_T = \text{Min}(\text{Mod}(T') \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}, \preceq_T$.

**(G5)** Let us suppose that $T \cup T' \cup T''$ is consistent. Let us show that $\text{Mod}(T \cup T' \cup T'') = \text{Mod}(T \cup T' \cup T'')$.

(a) $\text{Mod}(T \cup T' \cup T'') \subseteq \text{Mod}(T \cup T' \cup T'')$. Let $M \in \text{Mod}(T \cup T' \cup T'')$. By definition, $M \in \text{Min}(\text{Mod}(T \cup T' \cup T'')) \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}, \preceq_T$, and then $M \in \text{Mod}(T)$ and $M \in \text{Mod}(T'')$. Let us assume that $\text{Min}(\text{Mod}(T') \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}, \preceq_T \cap \text{Mod}(T'')$. As $T \cup T' \cup T''$ is consistent, there exists $M' \in \text{Min}(\text{Mod}(T') \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}, \preceq_T \cap \text{Mod}(T'')$. By hypothesis, $M' \preceq_T M$. Now, $M' \in \text{Mod}(T' \cup T'')$, and then we also have that $M \preceq_T M'$ what is a contradiction.

(b) $\text{Mod}(T \cup T' \cup T'') \subseteq \text{Mod}(T \cup T' \cup T'')$. Let $M \in \text{Mod}(T \cup T' \cup T'') \cap \text{Mod}(T'')$. Let us assume that $\text{Min}(\text{Mod}(T' \cup T'')) \setminus \{M \in \text{Mod}(\Sigma) \mid M^* = \text{Sen}(\Sigma)\}, \preceq_T$. This means that there exists $M' \in \text{Mod}(T' \cup T'')$ such that $M'' \not\subseteq \text{Sen}(\Sigma)$ and $M' \preceq_T M$. But, we also have that $M' \in \text{Mod}(T')$, and then $M \preceq_T M'$ what is a contradiction.
3.3 Relaxation and AGM postulates

Relaxation have been introduced in [9, 10] in the framework of description logics with the aim of definition dissimilarity between concepts. Here, we propose to generalise this notion in the framework of institutions.

**Definition 3.3 (Relaxation)** Given a signature $\Sigma \in \text{Sign}$. A $\Sigma$-relaxation is a mapping $\rho : \text{Sen}(\Sigma) \to \text{Sen}(\Sigma)$ satisfying:

- **Extensivity** $\forall \varphi \in \text{Sen}(\Sigma), \text{Mod}(\varphi) \subseteq \text{Mod}(\rho(\varphi))$.
- **Exhaustivity** $\exists k \in \mathbb{N}, \text{Mod}(\rho^k(\varphi)) = \text{Mod}(\Sigma)$.

The interest of relaxations is they give rise to revision operators.

**Definition 3.4 (Revision based in relaxation)** Let $\rho$ be a $\Sigma$-relaxation. We define the revision operator $\circ : \mathcal{P}(\text{Sen}(\Sigma)) \times \mathcal{P}(\text{Sen}(\Sigma)) \to \mathcal{P}(\text{Sen}(\Sigma))$ as follows: $\rho^k = \underbrace{\rho \circ \cdots \circ \rho}_{k \text{ times}}$

where $K = \{k_\varphi \in \mathbb{N} \mid \varphi \in T_1\}$, $T_1 = T_1 \bigcup T_2$ and $\rho^K(T_1) = \{\rho^{k_\varphi}(\varphi) \mid k_\varphi \in K, \varphi \in T_1\}$, and such that:

- $T_1 \circ T_2$ is consistent
- for every $K' = \{k'_\varphi \in \mathbb{N} \mid \varphi \in T_1\}$ such that $\rho^{K'}(T_1) \subseteq T_1' \cup T_2'$ is consistent, we have: $\forall \varphi \in T_1', k_\varphi \leq k'_\varphi$ (minimality on the number of applying the relaxation)
- Minimality criteria on the partition $T_1' \bigcup T_1''$
  - $\text{Cn}(T_1' \cup T_2) = \text{Sen}(\Sigma)$
  - $\text{Cn}(T_1'' \cup T_2) \neq \text{Sen}(\Sigma)$
  - $\forall T, T_1'' \subseteq T \subseteq T_1, \text{Cn}(T \cup T_2) = \text{Sen}(\Sigma)$

Partitioning $T_1$ into $T_1'$ and $T_1''$ is not unique. Indeed, other partitioning could also agree. The only constraint is $T_1''$ is of maximal size, i.e. adding any formula of $T_1'$ to $T_1''$ leads to inconsistency when adding to $T_2$. This is what is expressed by the three properties in Definition 3.4.

**Theorem 3.5** From any $\Sigma$-relaxation $\rho$ we can define for every knowledge base $T \subseteq \text{Sen}(\Sigma)$ the binary relation $\preceq_T \subseteq \text{Mod}(\Sigma) \times \text{Mod}(\Sigma)$ as follows: Let us note $K_T$ any set $\{k_\varphi \mid \varphi \in T\}$

- $\mathcal{M} \preceq_T \mathcal{M}'$ if $\forall K_T, \mathcal{M}, \mathcal{M}' \in \text{Mod}(\rho^{K_T}(T)) \Rightarrow \mathcal{M} \in \text{Mod}(\rho^{K_T}(T))$

Then, $\preceq_T$ is a well-founded FA such that for every $T' \subseteq \text{Sen}(\Sigma)$, $\text{Mod}(T \circ T') = \text{Min}(\text{Mod}(T')) \setminus \{\mathcal{M} \in \text{Mod}(\Sigma) \mid \mathcal{M}'' = \text{Sen}(\Sigma), \preceq_T\}$.

**Proof** By construction, $\preceq_T$ is obviously a pre-order. Well-foundness follows from exhaustivity. Minimal elements are all models of $\text{Mod}(\rho^{K_T}(T))$ such that $K_T$ is the set containing the least natural number $k_\varphi$ such that $\rho^{K_T}(T)$ is consistent (by exhaustivity, such a set exists). The first condition of FA is obvious by construction. Let $\mathcal{M} \in \text{Mod}(T)$ and $\mathcal{M}' \in \text{Mod}(\Sigma) \setminus \text{Mod}(T)$. 

11
By definition, for every \( n \in \mathbb{N} \), \( M \in \text{Mod}(\rho^n(T)) \). Moreover, by both conditions of relaxations, there exists \( k \in \mathbb{N} \) such that \( M' \in \text{Mod}(\rho^k(T)) \) and for every \( k' < k \), \( M' \not\in \text{Mod}(\rho^{k'}(T)) \). Hence, \( M' \not\in \text{Mod}(T) \).

It remains to show that \( \text{Mod}(T \circ T') = \text{Min}(\text{Mod}(T') \setminus \{ M \in \text{Mod}(\Sigma) \mid M = \text{Sen}(\Sigma) \}) \). To simplify the proof, we suppose that \( T \circ T' = \rho^k(T) \cup T' \) (i.e. if \( T = T_1 \bigcap T_2 \), then \( T_2 = \emptyset \)), the more general case where \( T_2 \neq \emptyset \) being able to be easily obtained from this more simple case.

- \( \text{Mod}(T \circ T') \subseteq \text{Min}(\text{Mod}(T') \setminus \{ M \in \text{Mod}(\Sigma) \mid M = \text{Sen}(\Sigma) \}) \). Let \( M \in \text{Mod}(T \circ T') \). By definition of \( \circ \), there exists \( K = \{ k_\varphi \mid \varphi \in T \} \) such that \( M \in \text{Mod}(\rho^k(T) \cup T') \), and for every \( K' = \{ k_\varphi' \mid \varphi' \in T \} \) such that there exists \( \varphi' \in T \) with \( k_\varphi' < k_\varphi \), \( \rho^{K'}(T) \cup T' \) is inconsistent. Therefore, for every \( M' \in \text{Mod}(\rho^{K'}(T) \cup T') \), we have that \( \forall \varphi \in T, k_\varphi' \geq k_\varphi \). By extensively, we can deduce that \( M \not\preceq_T M' \), whence we can conclude that \( M \in \text{Min}(\text{Mod}(T') \setminus \{ M \in \text{Mod}(\Sigma) \mid M = \text{Sen}(\Sigma) \}) \).

- \( \text{Min}(\text{Mod}(T') \setminus \{ M \in \text{Mod}(\Sigma) \mid M = \text{Sen}(\Sigma) \}) \subseteq \text{Mod}(T \circ T') \). Let \( M \in \text{Min}(\text{Mod}(T') \setminus \{ M \in \text{Mod}(\Sigma) \mid M = \text{Sen}(\Sigma) \}) \). By definition of \( \circ \), there exists a least \( K \) such that \( \rho^K(T) \cup T' \) is consistent and \( T \circ T' = \rho^K(T) \cup T' \). As \( M \in \text{Min}(\text{Mod}(T') \setminus \{ M \in \text{Mod}(\Sigma) \mid M = \text{Sen}(\Sigma) \}) \), this means for every \( M' \in \rho^K(T) \) that \( M \not\preceq_T M' \), and then \( M \in \rho^K(T) \), whence we can conclude that \( M \in \text{Mod}(T \circ T') \).

\[ \square \]

**Corollary 3.6** The revision \( \circ \) of Definition 3.4 satisfies (G1)-(G5).

**Proof** This directly follows from Theorems 3.2 and 3.5. \[ \square \]

## 4 Applications

In this section, we illustrate our general approach by defining revision operators based on relaxation mapping for the logics \( \text{PL}, \ HCL, \ FOL \) and \( \text{DL} \).

### 4.1 Revision in PL

Here, drawing inspiration from Bloch & al.’s works in [3][4] on Morpho-Logics, we define relaxation mappings through dilation standard in mathematical morphology [3]. It is well-known in PL that knowing a formula is equivalent to knowing the set of its models. Hence, we can identify any propositional formula \( \varphi \) with the set of its interpretations \( \text{Mod}(\varphi) \). To define relaxation mapping in PL, we will apply set-theoretic morphological operations. First, let us recall basic definitions of dilation in mathematical morphology. Let \( X \) and \( B \) be two subsets of \( \mathbb{R}^n \). The dilation of \( X \) by \( B \), denoted respectively by \( D_B(X) \) is defined as follows:

\[
D_B(X) = \{ x \in \mathbb{R}^n \mid B_x \cap X \neq \emptyset \}
\]

where \( B_x \) denotes the translation of \( B \) at \( x \).

In PL, this leads to the following dilation of a formula \( \varphi \in \text{Sen}(\Sigma) \):

\[
\text{Mod}(D_B(\varphi)) = \{ \nu \in \text{Mod}(\Sigma) \mid B_x \cap \text{Mod}(\varphi) \neq \emptyset \}
\]
where $B_\nu$ contains all the models that satisfy some relationship with $\nu$. The relationship standardly used is a discrete distance $\delta$ between models, and the most commonly used is the Hamming distance $d_H$ where $d_H(\nu, \nu')$ for two propositional models over a same signature $\Sigma$ is the number of propositional symbols that are instantiated differently in $\nu$ and $\nu'$. From any distance $\delta$ between models, a distance from models to a formula is derived as follows: $d(\nu, \phi) = \min_{\nu' \models \phi} \delta(\nu, \nu')$. In this case, we can rewrite the dilation of formula as follows:

$$\text{Mod}(D_B(\phi)) = \{ \nu \mid d(\nu, \phi) \leq 1 \}$$

This consists on using the distance ball radius 1 as structuring elements. To ensure the exhaustivity condition to our relaxation, we need to add a condition on distances, the betweenness property.

**Definition 4.1 (Betweenness property)** Let $\delta$ be a discrete distance over a set $S$. $\delta$ has the betweenness property if for all $x, y \in S$ and all $k \in \{0, 1, \ldots, \delta(x, y)\}$, there exists $z \in S$ such that $\delta(x, z) = k$ and $\delta(z, y) = \delta(x, y) - k$.

The Hamming distance trivially satisfies the betweenness property. The interest for our purpose of this property is it allows from any model to reach any other ones, and then ensuring the exhaustivity property of relaxation.

**Proposition 4.2** $D_B$ is a relaxation when it is applied to formulæ $\phi \in \text{Sen}(\Sigma)$ such that $\Sigma$ is a finite set, and it is based on models distance that satisfies the betweenness property.

**Proof** It is extensive. Indeed, for every $\phi$ and for every model $\nu \in \text{Mod}(\phi)$, we have that $d(\nu, \phi) = 0$, and then $\phi \models D_B(\phi)$. Exhaustivity results on the fact that considered signatures are finite set and the betweenness property.

### 4.2 Revision in HCL

Many works have focused on belief revision involving propositional Horn formulas (cf. [7] to have an overview on these works). Here, we propose to extend relaxations that we have defined in the framework of PL to deal with the Horn fragment of propositional theories. First, let us introduce some notions that we will later use.

**Definition 4.3 (Model intersection)** Given a propositional signature $\Sigma$ and two $\Sigma$-models $\nu, \nu' : \Sigma \to \{0, 1\}$, we note $\nu \cap \nu' : \Sigma \to \{0, 1\}$ the $\Sigma$-model defined by: $p \mapsto \begin{cases} 1 & \text{if } \nu(p) = \nu'(p) = 1 \\ 0 & \text{sinon} \end{cases}$

Given a set of $\Sigma$-models $S$, we note

$$cl_{\cap}(S) = S \cup \{ \nu \cap \nu' \mid \nu, \nu' \in S \}$$

$cl_{\cap}(S)$ is then the closure of $S$ under intersection of positive atoms.

It is well-known that for any set $S$ closed under intersection of positive atoms, there exists a Horn sentence $\phi$ that defines $S$ (i.e. $\text{Mod}(\phi) = S$). Given a models distance $\delta$, we then define the relaxation mapping $\rho$ as follows: For every Horn formula $\phi$, $\rho(\phi)$ is any Horn formula $\phi'$ such that $\text{Mod}(\phi') = cl_{\cap}(\text{Mod}(D_B(\phi)))$ (by the previous property, we know that such a formula $\phi'$ exists).

**Proposition 4.4** With the same conditions than in Proposition 4.2, the mapping $\rho$ is a relaxation.

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12Hence, dilation of formulæ would be able to be also defined by using distance ball radius $n$ as structuring elements [4].
4.3 Revision in FOL

A trivial way to define a relaxation in FOL is to map any formula to a tautology. A less trivial and more interesting relaxation is to change universal quantifiers to existential ones. Indeed, given a formula \( \varphi \) of the form \( \forall x. \psi \). If \( \varphi \) is not consistent with a given theory \( T \), \( \exists x. \psi \) may be consistent with \( T \) (If it cannot be consistent for all values, it can be for some of them). In the following we suppose that given a signature \( \rho \), every formula \( \varphi \in \text{Sen}(\Sigma) \) is a disjunction of formulæ in prefix form (i.e. \( \varphi \) is of the form \( \bigvee_j Q_1^j x_1^j \ldots Q_n^j x_n^j. \psi_j \) where each \( Q_i^j \in \{ \forall, \exists \} \)). Let us define the relaxation \( \rho \) as follows: we give a tautology \( \tau \).

- \( \rho(\tau) = \tau \)
- \( \rho(\exists x_1 \ldots \exists x_n. \varphi) = \tau. \)
- Let \( \varphi = Q_1 x_1 \ldots Q_n x_n. \psi \) be a formula such that the set \( E_\varphi = \{ i \mid 1 \leq i \leq n, Q_i = \forall \} \neq \emptyset \). Then, \( \rho(Q_1 x_1 \ldots Q_n x_n. \varphi) = \bigvee_{i \in E_\varphi} \varphi_i \) where \( \varphi_i = Q_1^i x_1^i \ldots Q_n^i x_n^i. \psi \) such that for every \( j \neq i, 1 \leq j \leq n \), \( Q_j^i = Q_j \) and \( Q_j^i = \exists. \)
- \( \rho(\bigvee_j Q_1^j x_1^j \ldots Q_n^j x_n^j. \psi) = \bigvee_j \rho(Q_1^j x_1^j \ldots Q_n^j x_n^j. \psi). \)

Proposition 4.5 \( \rho \) is a relaxation.

Proof It is obviously extensive, and exhaustivity results on the fact that in a finite number of steps, we always reach the tautology \( \tau \). \( \square \)

4.4 Revision in DL

As already explained in Section 3.3, relaxation has been introduced in [9][10]. It has been first defined over concepts, and then extended to formulæ. In [9][10], concept relaxation is defined as follows:

Definition 4.6 (Concept relaxation) Given a signature \( \Sigma = (N_C, N_R, I) \), we note \( C(\Sigma) \) the set of concepts over \( \Sigma \). A concept relaxation \( \rho : C(\Sigma) \to C(\Sigma) \) is mapping that satisfies:

1. \( C \subseteq \rho(C) \)
2. \( \exists k \in \mathbb{N}, \top \subseteq \rho^k(C) \)

A trivial concept relaxation is the operation \( \rho_\top \) that maps every concept \( C \) to \( \top \). Other non-trivial concrete concept relaxations such as the one that changes universal quantifiers to existential ones as in FOL \( \bar{\rho} \) can be found in [9][10].

From any relaxation concept \( \rho \), we can define the relaxation mapping on formulæ that we also note \( \rho \) as follows: we suppose that any signature \( \Sigma = (N_C, N_R, I) \) always contains in \( N_R \) a relation name \( \top \) the meaning of which is in any model \( O, r^\top = \Delta^O \times \Delta^O \)

- \( \rho(a : C) = a : \rho(C) \)
- \( \rho((a, b) : r) = (a, b) : r^\top \)
- \( \rho(C \subseteq D) = C \subseteq \rho(D) \)

Proposition 4.7 From any concept relaxation \( \rho \), its extension to formulæ is a relaxation within the meaning of Definition 4.6.

Proof This directly results from both properties of Definition 4.6. \( \square \)

\(^{13}\)In this case, concept relaxation associates to every concept of the form \( \exists r_1 \ldots \exists r_n, D \) either \( \top \) or \( \rho'(D) \) where \( \rho' \) is a relaxation that can be related to relaxations as defined for PL since applied to propositional concepts.
5 Related work

Recently a first proposition of generalisation of AGM revision has been proposed in the framework of Tarskian Logics considering minimality criteria on removed formulæ [17] following previous works of same authors for contraction [18]. Representation results that make a correspondence between a large family of logics containing non-classical logics such as DL and HCL and AGM postulates for revision with such minimality criteria have then been obtained. Here, the proposed generalisation also gives similar representation theorems (cf. Theorem 3.2) but for a different minimality criteria. Indeed, we showed in Section 3.2 that revision operators satisfying Postulates (G1)-(G5) are precisely the ones that accomplish an update with minimal change to the set of models of knowledge bases generalising to any institution the approach developed in [12] for the logic PL and [14] for DL. However, our revision operator based on relaxation also has a minimality criteria on removed formulæ. Indeed, in [17] in addition to the standard AGM postulates for revision, the authors to express their minimality criteria on removed formulæ introduced the postulate (Relevance) that can be written in our framework as follows:

\[(\text{Relevance}) \text{ if } \varphi \in T \setminus (T \circ T'), \text{ then there exists } X, T \cap (T \circ T') \subseteq X \subseteq T, \text{ such that } Cn(X \cup T') \neq \text{Sen}(\Sigma) \text{ and } Cn(X \cup \{\varphi\} \cup T') = \text{Sen}(\Sigma)\]

And, we have the following result.

**Proposition 5.1** The revision operator of Definition 3.4 satisfies (Relevance). 

**Proof** Let \( \varphi \in T_1 \setminus (T_1 \circ T_2) \). By definition, this means that \( \varphi \in T_1' \). Therefore, it is sufficient to set \( X = T_1'' \). By the last condition of Definition 3.4 the minimality criteria on removed formulæ of (Relevance) is obviously satisfied. \( \square \)

6 Conclusion

We provided a generalization of belief revision from a model-theoretic point of view, by defining this operator in a categorical abstract model theory known under the name of theory of institutions. In this framework, we then generalized to any institution the characterization of the AGM postulates given by Katsuno and Mendelzon for propositional logic in terms of minimal change with respect to an ordering among interpretations. Finally, we studied how to define revision, satisfying the AGM postulates, from relaxation notions, and illustrated our approach to both classical and non-classical logics. Future works will concern the study of our generalization to other non-classical logics such as first-order Horn logics or equational logics.

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