A modified variational principle for gravity in the modified Weyl geometry

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Abstract
The usual interpretation of the Weyl geometry is modified in two senses. First, both the additive Weyl connection and its variation are treated as (1, 2) tensors under the action of the Weyl covariant derivative. Second, a modified covariant derivative operator is introduced which still preserves the tensor structure of the theory. With its help, the Riemann tensor in the Weyl geometry can be written in a more compact form. We justify this modification in detail from several aspects and obtain some insights along the way. By introducing some new transformation rules for the variation of tensors under the action of the Weyl covariant derivative, we find a Weyl version of the Palatini identity for the Riemann tensor. To derive the energy–momentum tensor and equations of motion for gravity in the Weyl geometry, one naturally applies this identity at first, and then converts the variation of the additive Weyl connection to those of the metric tensor and Weyl gauge field. We also discuss possible connections to the current literature on the Weyl-invariant extension of massive gravity and the variational principles in $f(R)$ gravity.

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1. Introduction

In search of ‘a true infinitesimal geometry’ which can only compare lengths at one infinitesimally close point, Weyl proposed that a spacetime manifold should only be equipped with a class of conformally equivalent metrics rather than a definite metric as in general relativity [1]. Once the metric is given a scale or ‘gauge’ freedom like this, the Riemannian geometry is generalized to the so-called Weyl geometry.

In one conformal class, a change of calibration or ‘gauge’ for the metric is accompanied by a transformation of the corresponding differential 1-form. Interpreting the latter as an electromagnetic field, and demanding that the physics is ‘gauge’ invariant, Weyl arrived at a gauge principle to unify the electromagnetism and general relativity. A decade later, Weyl
updated this principle to its modern form [2]. As he conjectured, this eventually led to the establishment of gauge theory which unified ‘electricity and matter’. (For reviews, see [3, 4].)

The relation between the Weyl invariance and conformal invariance has been investigated for many years. In particular, the authors of [5] suggested a correspondence between the Weyl invariance in the curved space and conformal invariance in flat space. The Brans–Dicke theory of gravity [6, 7] was also found to be relevant in this context. For recent discussions of this topic using the language of the Weyl geometry, see [8–11]. Other related works on the Weyl invariance are [12–17]. For some recent research on the Weyl geometry from the mathematical point of view, see e.g. [18–20].

It has been a popular idea to give mass to the graviton. These massive gravities have been under intense investigation in recent years. The topologically massive gravity [21] was connected to conformal field theory through the holographic principle [22, 23]. Some new theories were proposed including new massive gravity [24] and nonlinear massive gravity [25]. Weyl gravity has also attracted a lot of attention since the pieces of work in [26, 27]. Inspired by all these developments, some authors have studied the Weyl-invariant extension of new massive gravity [28], Dirac–Born–Infeld-type gravity [29], general higher curvature gravity theories [30] and topologically massive gravity [31].

In the Palatini approach to $f(R)$ gravity (see e.g. [32]), the field equation of the connection acquires a form which mostly resembles the Weyl connection in the Weyl integrable geometry (with the gauge field $W_\mu = \partial_\mu \phi$, $\phi = \frac{1}{2} \ln \partial^\rho f(R)$). Both metric and Palatini $f(R)$ gravities have been related to the Brans–Dicke theory via conformal transformations [33]. These facts may be connected with the pieces of work on the Weyl geometry mentioned above [9, 10]. Recently, based on [34–37], a biconnection variational principle has been proposed in [38] and applied to $f(R)$ gravity. If we set the second connection there to be the additive part of the Weyl connection and introduce a scalar field, the corresponding generalized action would have some resemblance with the Weyl-invariant extension of the Einstein–Hilbert action. These issues may deserve further research.

Compared to the case in the Riemannian geometry, the Riemann tensor in the Weyl geometry acquires an additive part which has a complicated dependence on the Weyl gauge field. When the action of a Weyl-invariant massive gravity or modified gravity is given, one typically needs to use the variational principle to find the energy–momentum tensor and the equation of motion for the Weyl gauge field. To do this, the Riemann tensor (in the Weyl geometry) has to be inserted first. The subsequent calculation of variation usually becomes formidable, and one may have to resort to software. It is desirable to have a method to track the procedure. In particular, a Weyl version of the Palatini identity would be much helpful.

We find that this object could be partially accomplished if one treats the additive Weyl connection and its variation as regular tensors, and introduces sensible transformation rules for them under the action of the Weyl covariant derivative. Interestingly, a modified derivative operator is found to be relevant. In contrast with the Riemannian geometry, such a modification is not only possible but also preserves Leibniz’s rule (product rule) even when the metric tensor is involved. With its help, the Riemann tensor can be derived in an alternative way. On the other hand, to find a Weyl version of the Palatini identity, we introduce some new transformation rules for the Weyl covariant derivative. Although they still have no analogues in the Riemannian geometry as one expected, the expressions are quite natural and similar to those in general relativity. This makes them very easy to understand. Along with the exploration, we also reinterpret some peculiarities of the Weyl geometry through the comparison with the Riemannian geometry.

The organization of this paper is as follows. In the following section, we review the relevant basics of the Weyl geometry and explain our idea in some detail. In section 3, we
take a close look at the peculiar behavior of the Weyl covariant derivative, and find that it still allows a modification which is forbidden in the Riemannian case. In section 4, using the transformation rule of tensors under the action of the modified Weyl covariant derivative, we obtain some formulas for the additive Weyl connection. A new expression for the Riemann tensor is given in section 5, and with its help we find new ways to derive the curvature tensors. Section 6 deals with the variational problem for gravity in the Weyl geometry. After clarifying some technical issues, we finally arrive at a Palatini-like identity for the Riemann tensor. This may provide a useful technique to find the energy–momentum tensor and the equations of motion. The conclusion and discussion can be found in section 7.

Our conventions are mostly those of [39] adapted to the Weyl geometry.

2. Motivation

Although the concept of the Weyl geometry is based on Weyl transformations, it suffices for our purposes to regard it as a generalization of the Riemannian geometry with a new connection. With the convention in [3] and denoting the Weyl gauge field by $W^\lambda_{\mu}$. The torsion-free Weyl connection is defined as follows:

\[ \tilde{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} + W^\lambda_{\mu\nu}, \] (1)

\[ \Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_{\mu} g_{\nu\rho} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu}), \] (2)

\[ W^\lambda_{\mu\nu} = g^{\lambda\rho} (g_{\rho\mu} W^\nu_{\nu} + g_{\rho\nu} W^\mu_{\mu} - g_{\mu\nu} W^\rho_{\rho}) \]
\[ = \delta^\lambda_\mu W^\nu_{\nu} + \delta^\lambda_\nu W^\mu_{\mu} - g_{\mu\nu} W^\lambda_{\lambda}. \] (3)

Here $\Gamma^{\lambda}_{\mu\nu}$ is the Christoffel connection (Levi-Civita connection) of the Riemannian geometry. For the expediency of exposition, we will refer to $W^\lambda_{\mu\nu}$ defined in equation (3) as the additive Weyl connection.

One can define the corresponding Weyl covariant derivative as

\[ \tilde{\nabla} T_{\mu\nu}^{a_1...a_n} = \nabla T_{\mu\nu}^{a_1...a_n} + W^a_{\mu a_1} T_{\mu\nu}^{a_2...a_n} + \cdots - W^a_{\nu a_1} T_{\mu\nu}^{a_2...a_n} + \cdots, \] (4)

where $\nabla$ is the usual (Riemannian) covariant derivative operator defined with the Christoffel connection (2). (To conform with the literature, we will not introduce an abbreviation such as WCD for the Weyl covariant derivative.) The analogue of the metricity (or metric-compatibility) condition is

\[ \tilde{\nabla}_\rho g_{\mu\nu} = -2W^\rho_{\mu\rho}, \] (5)

which is usually taken as the definition of the Weyl geometry. Note that some papers use different conventions from us where both the additive Weyl connection and the (non-)metricity condition may not be the same as here.

What interests us the most is the Weyl-invariant Riemann tensor:

\[ \tilde{R}^\rho_{\sigma\mu\nu} = \partial_\mu \tilde{\Gamma}^\rho_{\nu\sigma} - \partial_\nu \tilde{\Gamma}^\rho_{\mu\sigma} + \tilde{\Gamma}^\lambda_{\mu\nu} \tilde{\Gamma}^\rho_{\rho\sigma} - \tilde{\Gamma}^\rho_{\nu\sigma} \tilde{\Gamma}^\lambda_{\mu\rho}. \] (6)

More explicitly, we have

\[ \tilde{R}^\rho_{\sigma\mu\nu} = R^\rho_{\sigma\mu\nu} + \tilde{R}^\rho_{\sigma\mu\nu}, \] (7)

\[ \tilde{R}^\rho_{\sigma\mu\nu} = \nabla_\mu W^\rho_{\nu\sigma} - \nabla_\nu W^\rho_{\mu\sigma} + W^\rho_{\mu\lambda} W^\lambda_{\nu\sigma} - W^\rho_{\nu\lambda} W^\lambda_{\mu\sigma}. \] (8)

We will refer to $\tilde{R}^\rho_{\sigma\mu\nu}$ as an additive Riemann tensor. If one inserts the definition of the additive Weyl connection (3) in the above equation, the expression would become a little lengthy. (For the detailed formula, see equation (60) in section 5.)
When studying gravity theories in the Weyl geometry, one usually needs to deal with the tedious calculations involving the curvature tensors. One motivation of our work is to develop a useful toolkit for this. It turns out that our modification also leads to some interesting observations about the Weyl geometry. With the comparison with the Riemannian geometry in mind, all our procedures and techniques are actually quite easy to understand for general relativity practitioners.

Our first observation is that the tensor property of the additive Weyl connection (3) can be promoted from the Riemannian geometry to the Weyl geometry. From the point of view of the Riemannian geometry, it is of course a plain fact that the additive Weyl connection is a (1, 2) tensor, and it obeys the basic transformation rule under the action of the Riemannian covariant derivative. One may wonder what happens if it is treated as a (1, 2) tensor in the Weyl geometry. The behavior of the additive Weyl connection under general coordinate transformations does not concern us; instead, we will concentrate on the action of the Weyl covariant derivative (4) on it.

Following this suggestion, we have
\[
\tilde{\nabla}_\mu W^\rho_{\nu\sigma} = \nabla_\mu W^\rho_{\nu\sigma} + W^\rho_{\mu\lambda} W^\lambda_{\nu\sigma} - W^\lambda_{\mu\nu} W^\rho_{\lambda\sigma} - W^\lambda_{\mu\sigma} W^\rho_{\nu\lambda}.
\]

Then the additive Riemann tensor (8) becomes
\[
\tilde{\hat{\mathcal{R}}}^\rho_{\sigma\mu\nu} = \tilde{\nabla}_\mu W^\rho_{\nu\sigma} - \tilde{\nabla}_\nu W^\rho_{\mu\sigma} - W^\rho_{\mu\lambda} W^\lambda_{\nu\sigma} + W^\rho_{\nu\lambda} W^\lambda_{\mu\sigma},
\]

which is even harder to handle. However, if we introduce a modified Weyl covariant derivative as follows,
\[
\tilde{\nabla}_\rho T^{a_1...a_k}_{b_1...b_l} = \nabla_\rho T^{a_1...a_k}_{b_1...b_l} + \frac{1}{2} W^a_{\rho\lambda} T^{a_1...a_k}_{b_1...b_l} + \cdots - \frac{1}{2} W^a_{\rho b_k} T^{a_1...a_k}_{b_1...b_l} - \cdots,
\]

the expression turns out to be more compact
\[
\tilde{\hat{\mathcal{R}}}^\rho_{\sigma\mu\nu} = \tilde{\nabla}_\rho W^\rho_{\nu\sigma} - \tilde{\nabla}_\sigma W^\rho_{\nu\mu}.
\]

In this case, we have a modified (non-)metricity condition
\[
\tilde{\hat{\nabla}}_\rho g^\mu\nu = - W^\rho_{\mu\nu}.
\]

This will not be an issue if one notices that just like the Riemannian geometry, the Weyl geometry is also completely determined by the corresponding Riemann tensor.

Let us recall the Palatini identity in the Riemannian geometry (or more properly, in general relativity):
\[
\delta \hat{\mathcal{R}}^\rho_{\sigma\mu\nu} = \nabla_\mu \delta \Gamma^\rho_{\sigma\nu} - \nabla_\nu \delta \Gamma^\rho_{\sigma\mu}.
\]

One may be tempted to interpret the additive Weyl connection as a special variation of the Christoffel connection, i.e.
\[
\delta \Gamma^\rho_{\sigma\nu} \rightarrow W^\rho_{\mu\nu}, \quad \delta \hat{\mathcal{R}}^\rho_{\sigma\mu\nu} \rightarrow \tilde{\hat{\mathcal{R}}}^\rho_{\sigma\mu\nu}.
\]

However, to pass from equation (14) to equation (12), one still needs to introduce a new Weyl covariant derivative defined in equation (11). This interesting fact is also a sign that our choice may be the unique and nontrivial one.

Although this is a pleasant result, for the modified Weyl covariant derivative to be sensible, one needs to demonstrate that under its action the tensor structure is still preserved. Since the metric-compatibility condition fails in the Weyl geometry, one must make sure that the new definition is consistent with Leibniz’s rule (product rule) especially when the metric tensor is involved. This also guarantees that the two facets of the additive Weyl connection will not lead to any contradiction: besides the new character as a (1, 2) tensor (in the Weyl geometry), it is still a combination of Weyl gauge fields defined in equation (3). The detailed discussion can be found in the following two sections. We will also show that this modification has no analogue in the Riemannian geometry, and it is not equivalent to the case where one rescales the definition of the additive Weyl connection.
The second observation is related to the variational problem for gravity in the Weyl geometry. Inspired by the above idea, we find that if the variation of the additive Weyl connection is also treated as a \((1, 2)\) tensor, one would have a Weyl version of the Palatini identity. The explicit formula is

\[
\delta \tilde{R}^\rho_{\sigma\mu\nu} = (\tilde{\nabla}_\mu \delta + \delta \tilde{\nabla}_\mu)W^\rho_{\sigma} - (\tilde{\nabla}_\nu \delta + \delta \tilde{\nabla}_\nu)W^\rho_{\mu\sigma}.
\] (15)

This modified variational approach turns out to be quite transparent. However, there are some important issues we would like to elaborate on in section 6. The explanation of the transformation rule and the delta-like operator \(\delta \tilde{\nabla}_\mu\) can also be found there. While the usual approach deals with the explicit expression of the Riemann tensor directly, with the above two observations combined together, we arrive at a systematic and equivalent method for the variational problem.

At the end of this section, we would like to collect some formulas about the additive Weyl connection:

\[
W^\lambda_{\mu\lambda} = nW^\mu,
\] (16)

\[
g^{\mu\nu}W^\lambda_{\mu\nu} = -(n - 2)W^\lambda,
\] (17)

\[
W^\nu_{\mu\lambda}W^\lambda_{\mu
u} = \delta^\nu_wW^2,
\] (18)

\[
W^\mu_{\nu\lambda}W^\lambda_{\mu\nu} = 2W^\mu_{\nu}W^\nu - g_{\mu\nu}W^2.
\] (19)

\[
\nabla^\lambda W^\lambda_{\mu\nu} = \nabla^\mu W^\nu + \nabla^\nu W^\mu - g_{\mu\nu} \nabla \cdot W.
\] (20)

Here \(n\) denotes the dimension of the differentiable manifold, i.e. the spacetime dimension, and \(\nabla \cdot W \equiv \nabla_\mu W^\mu\). The derivations of these formulas are elementary. Although they are usually not explicitly spelled out in the literature, we will find them very useful for our exposition.

3. The peculiar behavior of Weyl covariant derivatives

In contrast with the Riemann–Cartan geometry which has a non-metricity tensor \(Q_{\rho\mu\nu} \equiv -\nabla_\rho g_{\mu\nu} = 0\), this metric-compatibility condition fails in the Weyl geometry. So the operation of raising and lowering of indices no longer commutes with the covariant derivative.

Because of this, the modified Weyl covariant derivative (11) shares a peculiarity with the usual one (4):

\[
\hat{\nabla}_\mu V^\mu \neq \hat{\nabla}^\mu V_\mu,
\] (21)

where \(V^\mu\) or \(V_\mu\) is a general vector. This fact can be shown as follows. For the LHS, we have

\[
\hat{\nabla}_\mu V^\mu = \nabla \cdot V + \frac{1}{2}W^\mu_{\mu\lambda}V^\lambda \\
= \nabla \cdot V + \frac{1}{2}nW \cdot V.
\] (22)

Here equation (16) has been used, and \(W \cdot V \equiv W^\mu_{\mu\lambda}V^\lambda\).

For the RHS, we have

\[
\hat{\nabla}^\mu V_\mu \equiv g^{\mu\nu}\hat{\nabla}_\nu V_\mu \\
= g^{\mu\nu}(\nabla_\nu V_\mu - \frac{1}{2}W^\nu_{\nu\mu}V_\mu) \\
= \nabla \cdot V + \frac{1}{2}(n - 2)W \cdot V.
\] (23)

where equation (17) has been used. Another way to calculate it is

\[
\hat{\nabla}^\mu V_\mu = \nabla \cdot V - \frac{1}{2}W^\mu_{\mu\lambda}V^\lambda \\
= \nabla \cdot V - \frac{1}{2}(g^{\mu\nu}W^\nu_{\mu} + \delta^\mu_{\mu}W^\mu - \delta^\mu_{\mu}W^\mu)V^\lambda \\
= \nabla \cdot V + \frac{1}{2}(n - 2)W \cdot V.
\] (24)
which involves an unfamiliar additive Weyl connection (see equation \((41)\)). Now, one naturally obtains the statement \((21)\). This also implies that we cannot define a symbol like \(\tilde{\nabla} \cdot V\). Although the derivations here may seem simple, we give the details just to show how our new Weyl covariant derivative works.

Note that the nonequality \((21)\) is actually a special case of the following general fact:

\[
\tilde{\nabla}^\rho V_\rho = g^{\rho\nu} \tilde{\nabla}_\nu V_\nu \neq \tilde{\nabla}_\nu (g^{\mu\nu} V_\mu).
\]

In spite of the above peculiarity, if we treat the metric as a usual tensor and stick to the basic transformation rule, all the calculations would still give sensible results. This is because the Weyl covariant derivatives preserve Leibniz’s rule even when the metric tensor is involved. When taking the action of them on the multiplication of any two (or more than two) tensors, one could use Leibniz’s rule first to expand it and then apply the transformation rule corresponding to the derivative operators, or reverse the order. These two ways of expansions should be consistent with each other.

To see this point more clearly, let us introduce a generalized Weyl covariant derivative as follows:

\[
\nabla_\rho T_{\alpha_1\cdots\alpha_k} = \nabla_\rho T_{\alpha_1\cdots\alpha_k} + sW^\rho_{\rho\mu} T_{\alpha_1\cdots\alpha_k} + \cdots = tW^\rho_{\rho\mu} T_{\alpha_1\cdots\alpha_k} - \cdots.
\]

Here the upper and lower ‘covariant weights’ \(s\) and \(t\) are just some numbers. Note that

\[
\tilde{\nabla}_\rho \equiv \nabla_\rho (s = t = 1), \quad \nabla_\rho \equiv \nabla_\rho (s = t = \frac{1}{2}).
\]

For ordinary tensors, the transformation rule \((26)\) always preserves Leibniz’s rule. This can be seen by expanding e.g. \(\nabla_\rho (V^\mu S_{\nu\tau})\) in two ways. More explicitly, we have

\[
\nabla_\rho (V^\mu S_{\nu\tau}) = \nabla_\mu V^\rho S_{\nu\tau} + V^\mu \nabla_\rho S_{\nu\tau} = \left(\nabla_\mu V^\rho + sW^\rho_{\rho\mu} V^\lambda\right) S_{\nu\tau} + V^\mu \left(\nabla_\rho S_{\nu\tau} - tW^\rho_{\rho\mu} S_{\nu\tau} - tW^\rho_{\rho\mu} S_{\nu\tau}\right).
\]

On the other hand,

\[
\nabla_\rho (V^\mu S_{\nu\tau}) = \nabla_\rho (V^\mu S_{\nu\tau}) + sW^\rho_{\rho\mu} (V^\lambda S_{\nu\tau}) - tW^\rho_{\rho\mu} (V^\nu S_{\nu\tau}) - tW^\rho_{\rho\mu} (V^\nu S_{\nu\tau}).
\]

These two equations are obviously equal to each other.

When the metric tensor is involved, one should be more careful. This is essentially because it is a special tensor, we use to raise or lower tensor indices. It suffices to look at a typical example such as

\[
\nabla_\rho (g_{\mu\nu} V^\nu) = \nabla_\rho g_{\mu\nu} V^\nu + g_{\mu\nu} \nabla_\rho V^\nu
\]

\[
= -2W^\rho_{\rho\mu} V^\nu + g_{\mu\nu} (\nabla_\rho V^\nu + sW^\nu_{\rho\nu} V^\lambda)
\]

\[
= -2W^\rho_{\rho\mu} V^\nu + \nabla_\rho V^\mu - V^\nu \nabla_\rho g_{\mu\nu} + s(g_{\mu\nu} W \cdot V + W^\mu_{\nu} V^\rho - W^\nu_{\nu} V^\rho)
\]

\[
= \nabla_\rho V^\mu - V^\nu \nabla_\rho g_{\mu\nu} + (s - 2t) W^\nu_{\nu} V^\rho - sW^\mu_{\nu} V^\rho + s g_{\mu\nu} W \cdot V.
\]

On the other hand, we have

\[
\nabla_\rho V^\mu = \nabla_\rho V^\mu - tW^\rho_{\rho\mu} V^\lambda
\]

\[
= \nabla_\rho V^\mu - t(W^\mu_{\nu} V^\nu + W^\nu_{\nu} V^\rho - g_{\rho\mu} W \cdot V).
\]

Since here \(\nabla_\rho g_{\mu\nu} = 0\), for these two equations to be equal to each other, a necessary condition is \(s = t\). Both the usual and modified Weyl covariant derivatives satisfy this condition. If we further require the additive Riemann tensor to take a compact form like equation \((12)\), one is naturally led back to our modified derivative operator:

\[
\nabla_\rho \equiv \nabla_\rho (s = t = \frac{1}{2}).
\]

Although the Weyl connection itself is not metric-compatible, one can show that any connection which is not metric-compatible cannot be promoted to the Weyl geometry. Accordingly, there will be no sensible definition of the Weyl covariant derivative to give
the non-metricity condition which is the basic definition of the Weyl geometry. In other words, a geometry (space) which is not Riemann–Cartan cannot be promoted to the Weyl geometry. This can be easily seen by looking at equations (30) and (31), with $\nabla_{\rho} g_{\mu \nu} = 0 \rightarrow \nabla_{\rho} g_{\mu \nu} \neq 0$. It is interesting that one can obtain this expected result through such an elementary way.

Note that in the definition of the modified Weyl covariant derivative (11), we do not rescale the operator $\hat{\nabla}$ simultaneously. This is related to the following fact. In the Riemannian geometry, there is no sensible definition of a generalized covariant derivative. Suppose one is given such a definition as

$$\nabla_{\rho} T^{a_1 \ldots a_n}_{b_1 \ldots b_m} = \partial_{\rho} T^{a_1 \ldots a_n}_{b_1 \ldots b_m} + \frac{s}{\Gamma^1} g^{a_1 \ldots a_n}_{\rho \lambda} T^{b_1 \ldots b_m}_{\lambda a_2 \ldots a_n} - \frac{t}{\Gamma^1} g^{a_1 \ldots a_n}_{\rho \lambda} T^{b_1 \ldots b_m}_{\lambda b_2 \ldots b_m} - \ldots$$

Take its action on the metric tensor and use the expression of the Christoffel connection given in equation (2), then we will have

$$\nabla_{\rho} g_{\mu \nu} = \partial_{\rho} g_{\mu \nu} - \frac{t}{\Gamma^1} (g^{a_1 \ldots a_n}_{\rho \lambda} g^{a_2 \ldots a_n}_{\lambda \mu} + g^{a_1 \ldots a_n}_{\rho \lambda} g^{a_2 \ldots a_n}_{\lambda \nu}) = (1 - t) \partial_{\rho} g_{\mu \nu},$$

$$\nabla_{\rho} g_{\mu \nu} = \partial_{\rho} g_{\mu \nu} + \frac{s}{\Gamma^1} (g^{a_1 \ldots a_n}_{\rho \lambda} g^{a_2 \ldots a_n}_{\lambda \mu} + g^{a_1 \ldots a_n}_{\rho \lambda} g^{a_2 \ldots a_n}_{\lambda \nu}) = \partial_{\rho} g_{\mu \nu} + \frac{1}{2} s [\partial_{\rho} g^{a_1 \ldots a_n}_{a_1 \ldots a_n} - g^{a_1 \ldots a_n}_{\sigma \lambda} \partial_{\rho} (g^{a_1 \ldots a_n}_{\sigma \mu} g^{a_2 \ldots a_n}_{\lambda \nu} + g^{a_1 \ldots a_n}_{\sigma \mu} g^{a_2 \ldots a_n}_{\lambda \nu})] = \partial_{\rho} g_{\mu \nu} + \frac{1}{2} s \partial_{\rho} g^{a_1 \ldots a_n}_{a_1 \ldots a_n}.$$  

(34)

For the metricity condition $\nabla_{\rho} g_{\mu \nu} = \nabla_{\rho} g_{\mu \nu} = 0$ to be preserved, the only choice is $s = t = 1$.

Due to the symmetric property of the additive Weyl connection, the operators $\hat{\nabla}_{\rho}$ and $\tilde{\nabla}_{\rho}$ actually have other good behaviors. For example, one finds that

$$\hat{\nabla}_{\mu} V^{2m} = \tilde{\nabla}_{\mu} V^{2m} = \nabla_{\mu} V^{2m}.$$

(35)

If we define the Weyl gauge field strengths as usual

$$F_{\mu \nu} = \nabla_{\mu} W_{\nu} - \nabla_{\nu} W_{\mu},$$

$$\hat{F}_{\mu \nu} = \hat{\nabla}_{\mu} W_{\nu} - \hat{\nabla}_{\nu} W_{\mu},$$

$$\tilde{F}_{\mu \nu} = \tilde{\nabla}_{\mu} W_{\nu} - \tilde{\nabla}_{\nu} W_{\mu},$$

one can also check the following fact:

$$\hat{F}_{\mu \nu} = \tilde{F}_{\mu \nu} = F_{\mu \nu}.$$  

(39)

4. Additive Weyl connection as a tensor in the Weyl geometry

The central task of this section is to check the action of the modified Weyl covariant derivative (11) on the additive Weyl connection (3) and to show that it does not spoil the tensor property of the latter. The procedure and formulas here are the basis of the following section where we will focus on the curvature tensors in the (modified) Weyl geometry.
4.1. A preliminary discussion

It is a trivial fact that the additive Weyl connection is a \((1,2)\) tensor in the Riemannian geometry. What we would like to find is its behavior under the action of the usual or modified Weyl covariant derivative. Before doing this, we need to clarify two special issues.

Firstly, since we do not introduce a Weyl-rescaled metric tensor like 
\[ g'_{\mu\nu} = e^{2\omega}g_{\mu\nu} \]
here, one can raise or lower the indices of the additive Weyl connection with the same metric tensor as in the Riemannian geometry. Specifically, we can define
\[ W_{\lambda,\mu\nu} = g_{\lambda\mu}W_{\nu} + g_{\lambda\nu}W_{\mu} - g_{\mu\nu}W_{\lambda}, \]
\[ W^{\lambda}_{\mu\nu} = g^{\lambda\mu}W_{\nu} + \delta^{\lambda}_{\nu}W_{\mu} - \delta^{\lambda}_{\mu}W_{\nu}, \]
\[ W^{\lambda}_{\mu} = \delta^{\lambda}_{\mu}W_{\nu} + g_{\mu\nu}W_{\lambda} - \delta^{\lambda}_{\nu}W_{\mu}, \]
and others. (We have added a comma in equation (40) to stress its symmetric property.) They all obey the regular transformation rule under the action of the Riemannian covariant derivative. However, these tensor properties of \( W^{\lambda}_{\mu\nu} \) have nothing to do with the Weyl geometry per se.

Secondly, our modification by introducing a new Weyl covariant derivative (11) is not equivalent to the case where one rescales the definition of the additive Weyl connection (3) while retaining the usual Weyl covariant derivative (4). Suppose we have a new additive Weyl connection as follows:
\[ W'_{\lambda,\mu\nu} = g_{\lambda\mu}W'_{\nu} + g_{\lambda\nu}W'_{\mu} - g_{\mu\nu}W'_{\lambda}, \]
\[ W'^{\lambda}_{\mu\nu} = g^{\lambda\mu}W'_{\nu} + \delta^{\lambda}_{\nu}W'_{\mu} - \delta^{\lambda}_{\mu}W'_{\nu}, \]
Demanding the corresponding Weyl covariant derivative to behave as the usual one in (4), one has
\[ \nabla'_{\rho}V^{\mu} = \nabla_{\rho}V^{\mu} + W'_{\mu\rho\lambda}V^{\lambda}, \]
\[ \nabla'_{\rho}g_{\mu\nu} = \nabla_{\rho}g_{\mu\nu} - W'_{\rho\mu\lambda}g_{\nu\lambda} - W'_{\rho\nu\lambda}g_{\mu\lambda} = -W_{\rho}g_{\mu\nu}, \]
which are the same as our modified case. Nevertheless, for the additive Riemann tensor (8), one would have
\[ \hat{R}^{\rho}_{\sigma\mu\nu} = \nabla_{\mu}W'_{\nu\rho} - \nabla_{\nu}W'_{\mu\rho} + W'_{\mu\rho\lambda}W'_{\nu\lambda} - W'_{\nu\rho\lambda}W'_{\mu\lambda} = \frac{1}{2}(\nabla_{\mu}W'_{\nu\rho} - \nabla_{\nu}W'_{\mu\rho}) + \frac{1}{2}(W'_{\mu\rho\lambda}W'_{\nu\lambda} - W'_{\nu\rho\lambda}W'_{\mu\lambda}). \]
This is clearly different from our case. One could also try to rescale the Weyl gauge field while retaining the definition of the additive Weyl connection. This would still change the value of the Riemann tensor.

4.2. Some formulas involving the additive Weyl connection

Before proceeding further, let us encapsulate our modification of the Weyl geometry as follows. The Weyl connection is kept unchanged as in equations (1)–(3), while we introduce a modified Weyl covariant derivative in equation (11). Accordingly, the non-metricity condition is modified to equation (13).
Using the definition in equation (11), we can easily find the following basic formulas for the Weyl gauge field:

\[
\nabla_\rho W^\mu = \nabla_\rho W^\mu + \frac{1}{2} W^\rho_{\mu\lambda} W^\lambda
\]

\[
= \nabla_\rho W^\mu + \frac{1}{2} \delta^\rho_\mu W^2,
\]

(48)

\[
\nabla_\rho W_{\mu\nu} = \nabla_\rho W_{\mu\nu} - \frac{1}{2} W^\rho_{\mu\nu} W_\rho
\]

\[
= (\nabla_\rho - W_\rho)_{\mu\nu} + \frac{1}{2} g_{\rho\nu} W^2,
\]

(49)

where equations (18) and (19) have been used. The other two useful equations are

\[
\nabla_\rho W^\rho = \nabla_\cdot W + \frac{1}{2} n W^2,
\]

(50)

\[
\nabla^\rho W_\rho = \nabla_\cdot W + \frac{1}{2} (n-2) W^2.
\]

(51)

Note that similar equations also appeared in section 3.

Now, we can study the behavior of the additive Weyl connection \( W^\lambda_{\mu\nu} \) under the action of our modified Weyl covariant derivative. Inserting the definition of \( W^\lambda_{\mu\nu} \), we have

\[
\nabla_\rho W^\lambda_{\mu\nu} = \nabla_\rho \left( \delta^\lambda_\rho W_\nu + \delta^\lambda_\rho W_\mu - g_{\mu\nu} W^\lambda \right)
\]

\[
= \delta^\lambda_\rho \nabla_\rho W_\nu + \delta^\lambda_\rho \nabla_\rho W_\mu - g_{\mu\nu} \nabla_\rho W^\lambda + \frac{1}{2} g_{\mu\nu} W_\rho W^\lambda,
\]

(52)

where the modified non-metricity condition (13) has been used.

With the basic formulas (48) and (49) at hand, one can expand the above equation as

\[
\nabla_\rho W^\lambda_{\mu\nu} = \delta^\lambda_\rho \left[ (\nabla_\rho - W_\rho) W_\nu + \frac{1}{2} g_{\rho\nu} W^2 \right] + \delta^\lambda_\rho \left[ (\nabla_\rho - W_\rho) W_\mu + \frac{1}{2} g_{\rho\mu} W^2 \right] - g_{\mu\nu} \left( \nabla_\rho W^\lambda + \frac{1}{2} \delta^\rho_\lambda W^2 \right) + g_{\mu\nu} W_\rho W^\lambda
\]

\[
= (\nabla_\rho - W_\rho) W^\lambda_{\mu\nu} + \frac{1}{2} \left( \delta^\lambda_\rho g_{\rho\nu} + \delta^\lambda_\rho g_{\rho\mu} - \delta^\rho_\lambda g_{\mu\nu} \right) W^2,
\]

(53)

This is an important equation which will be used to obtain the explicit expressions for curvature tensors in section 5.

Since we have asserted that \( W^\lambda_{\mu\nu} \) can be treated as a \((1, 2)\) tensor in the Weyl geometry, one should check this statement by doing the calculation in another way. Using the transformation rule in equation (11), we have

\[
\nabla_\rho W^\lambda_{\mu\nu} = \nabla_\rho W_{\mu\nu}^\lambda + \frac{1}{2} W^\lambda_{\rho\sigma} W_{\mu\nu}^\sigma - \frac{1}{2} W^\lambda_{\mu\sigma} W_{\rho\nu}^\sigma - \frac{1}{2} W^\lambda_{\mu\nu} W_{\rho\sigma}.
\]

(54)

Inserting the following equation:

\[
W^\lambda_{\mu\nu} W^\sigma_{\rho\sigma} = \delta^\lambda_\rho W_{\mu\nu} W^\sigma_{\rho\sigma} + W_{\mu\nu} W^\sigma_{\rho\mu} - g_{\rho\sigma} W^\lambda_{\mu\nu} W_{\rho\sigma},
\]

(55)

one is led to a quite lengthy calculation

\[
\nabla_\rho W^\lambda_{\mu\nu} = \nabla_\rho W_{\mu\nu} + \frac{1}{2} [W^\lambda_{\rho\sigma} W_{\mu\nu} - (\rho \leftrightarrow \nu) - (\rho \leftrightarrow \mu)]
\]

\[
\]

\[
= \nabla_\rho W_{\mu\nu} + \frac{1}{2} [\delta^\lambda_\rho W_{\nu\sigma} W_{\mu\nu} - (\nu \leftrightarrow \nu) - (\rho \leftrightarrow \mu)]
\]

\[
+ \frac{1}{2} W_{\rho\sigma} W_{\mu\nu} - (\rho \leftrightarrow \nu) - (\rho \leftrightarrow \mu)]
\]

\[
- \frac{1}{2} g_{\mu\nu} W^\lambda W^\sigma_{\rho\sigma} - (\rho \leftrightarrow \nu) - (\rho \leftrightarrow \mu)]
\]

\[
= \nabla_\rho W_{\mu\nu} + \frac{1}{2} [2\delta^\lambda_\rho W_{\nu\sigma} W_{\mu\nu} - 2W_{\rho\sigma} (\delta^\lambda_\rho W_{\nu} + \delta^\lambda_\rho W_{\nu}) + (\delta^\lambda_\rho g_{\rho\nu} + \delta^\lambda_\rho g_{\rho\nu} - \delta^\lambda_\rho g_{\rho\nu}) W^2] + \frac{1}{2} [2\delta^\lambda_\rho W_{\mu\nu} + W^\lambda (g_{\rho\sigma} W_{\nu} + g_{\rho\sigma} W_{\mu} - g_{\rho\sigma} W_{\nu}) - \frac{1}{2} W_{\rho\sigma} W_{\nu} - \frac{1}{2} W_{\rho\sigma} W_{\mu} - 3 g_{\rho\sigma} W_{\nu}]
\]

\[
= (\nabla_\rho - W_\rho) W_{\mu\nu} + \frac{1}{2} (\delta^\lambda_\rho g_{\rho\nu} + \delta^\lambda_\rho g_{\rho\nu} - \delta^\lambda_\rho g_{\rho\nu}) W^2.
\]

(56)

Thus, these two procedures are consistent with each other.

Let us rewrite this important formula below:

\[
\nabla_\rho W^\lambda_{\mu\nu} = (\nabla_\rho - W_\rho) W_{\mu\nu} + \frac{1}{2} (\delta^\lambda_\rho g_{\rho\nu} + \delta^\lambda_\rho g_{\rho\nu} - \delta^\lambda_\rho g_{\rho\nu}) W^2.
\]

(57)

With its help, we will derive the expressions of curvature tensors in the following section.
5. Curvature tensors in the modified Weyl geometry

When a modified Weyl covariant derivative is introduced as in equation (11), we can rephrase the additive Riemann tensor (8) in a more compact form

\[ \widehat{R}^{\rho}_{\sigma \mu \nu} = \nabla_{\rho} W_{\sigma \mu \nu} - \nabla_{\sigma} W^{\rho}_{\mu \nu}. \]  

(58)

This can be seen by recalling the basic equation in (54). In what follows, we will use this new formula to obtain the explicit expressions for the Riemann tensor, Ricci tensor and Ricci scalar in the (modified) Weyl geometry successively. The results can also be checked to agree with the usual approach.

1. Riemann tensor. Inserting equation (52) in the new formula (58), we immediately arrive at

\[ \widehat{R}^{\rho}_{\sigma \mu \nu} = \delta^{\rho}_{\sigma} \widehat{F}^{\mu \nu} - 2 \delta^{\rho}_{\mu} \nabla_{\nu} W_{\sigma} + 2 (g_{\sigma \mu \nu} W_{\rho} - g_{\sigma \rho \nu} W_{\mu} + g_{\rho \sigma \mu} W_{\nu}) W^{\rho}. \]  

(59)

Our convention here is \( A_{[\mu \nu]} \equiv \frac{1}{2} (A_{\mu \nu} - A_{\nu \mu}) \). The field strength \( \widehat{F}_{\mu \nu} \) is defined as in equation (38).

Using the basic formulas equations (48) and (49), and noticing the fact \( \widehat{F}_{\mu \nu} = F_{\mu \nu} \) (see equation (39)), we reproduce the well-known Weyl-invariant Riemann tensor:

\[ \widehat{R}^{\rho}_{\sigma \mu \nu} = R^{\rho}_{\sigma \mu \nu} = R^{\rho}_{\sigma \mu \nu} + \widehat{R}^{\rho}_{\sigma \mu \nu}, \]

\[ \widehat{R}^{\rho}_{\sigma \mu \nu} = \delta^{\rho}_{\sigma} F^{\mu \nu} - 2 \delta^{\rho}_{\mu} \nabla_{\nu} W_{\sigma} + 2 g_{\sigma \mu \nu} W^{\rho} - 2 W_{[\mu \nu]} W^{\rho} + 2 g_{\rho \sigma \mu} W_{\nu} W^{\nu} + 2 g_{\rho \sigma \nu} W_{\mu} W^{\mu}. \]  

(60)

We can also use equation (57) instead and insert it into equation (58) to arrive at

\[ \widehat{R}^{\rho}_{\sigma \mu \nu} = \nabla_{\rho} W^{\rho}_{\sigma \mu \nu} - \nabla_{\rho} W^{\rho}_{\sigma \mu \nu} - W_{\rho} W^{\rho}_{\sigma \mu \nu} + W_{\rho} W^{\rho}_{\sigma \mu \nu} + (\delta^{\rho}_{\rho} g_{\mu \nu} - \delta^{\rho}_{\mu} g_{\rho \nu}) W^{\rho}. \]  

(61)

With the definition of \( W^{k}_{\mu \nu} \), one expands the first four terms as

\[ \nabla_{\rho} W^{\rho}_{\sigma \mu \nu} - \nabla_{\rho} W^{\rho}_{\sigma \mu \nu} = \delta^{\rho}_{\rho} F^{\mu \nu} - 2 \delta^{\rho}_{\mu} \nabla_{\nu} W_{\sigma} + 2 g_{\sigma \mu \nu} W^{\rho} - \delta^{\rho}_{\mu} g_{\rho \nu} W^{\rho}. \]  

(62)

\[ W_{\rho} W^{\rho}_{\sigma \mu \nu} - W_{\rho} W^{\rho}_{\sigma \mu \nu} = 2 W_{[\mu \nu]} W^{\rho} + 2 g_{\rho \sigma \mu} W_{\nu} W^{\nu}. \]  

(63)

Finally, we still obtain the result in equation (60).

2. Ricci tensor. Instead of contracting the indices in the explicit formula for the Riemann tensor, let us use the key formula (58) directly to find the Weyl-invariant Ricci tensor. Now, we have

\[ \widehat{R}_{\sigma \nu} = \nabla_{\nu} W^{\rho}_{\sigma \rho} - \nabla_{\rho} W^{\rho}_{\sigma \rho} = \nabla_{\nu} W^{\rho}_{\sigma \rho} - (n - 2) \nabla_{\rho} W_{\sigma} - g_{\sigma \nu} (\nabla_{\rho} W^{\rho} - W^{2}). \]  

(64)

Using the basic equations (48) and (49) again, we obtain the final expression

\[ \widehat{R}_{\sigma \nu} = R_{\sigma \nu} + \widehat{R}_{\sigma \nu}, \]

\[ \widehat{R}_{\sigma \nu} = F_{\sigma \nu} - (n - 2) [(\nabla_{\nu} W_{\sigma} + g_{\sigma \nu} W^{2} - g_{\sigma \nu} \nabla \cdot W)] - g_{\sigma \nu} \nabla \cdot W. \]  

(65)

There is still another way to perform the calculation, i.e. using formula (57). Contracting the indices there, we have

\[ \nabla_{\nu} W^{\rho}_{\sigma \rho} = \nabla_{\nu} W^{\rho}_{\sigma \rho} - W_{\rho} W^{\rho}_{\sigma \rho} - \frac{1}{2} (n - 2) g_{\sigma \nu} W^{2} = \frac{1}{2} (n - 2) g_{\sigma \nu} W^{2} - W_{\rho} W^{\rho}_{\sigma \rho} \]  

(66)

where equations (19) and (20) have been used. Subtracting it by another term

\[ \nabla_{\nu} W^{\rho}_{\sigma \rho} = n (\nabla_{\nu} W_{\sigma}) W_{\rho} - \frac{1}{2} g_{\sigma \nu} W^{2}, \]  

(67)

one arrives at the result in equation (65) again.
Note that the Ricci tensor in equation (65) is not symmetric. It is customary to split it into two parts as
\[ \hat{\mathbf{R}}_{\sigma\nu} = \hat{\mathbf{R}}_{[\sigma\nu]} + \hat{\mathbf{R}}_{(\sigma\nu)}, \]  
(68)

\[ \hat{\mathbf{R}}_{[\sigma\nu]} = F_{\sigma\nu}, \]  
(69)

\[ \hat{\mathbf{R}}_{(\sigma\nu)} = - (n - 2)[(\nabla_\nu - W_\nu)W_\sigma + g_{\sigma\nu}W^2] - g_{\sigma\nu}\nabla \cdot W. \]  
(70)

(3) Ricci scalar. Using the definition of Ricci scalar in the Weyl geometry and equation (64), we have
\[ \hat{\mathbf{R}} \equiv g^{\rho\sigma}\hat{\mathbf{R}}_{\sigma\nu} = - (n - 2)\nabla^\rho W_\rho - n(\nabla_\rho W^\rho - W^2). \]  
(71)

Inserting equations (50) and (51) into the above equation, we arrive at
\[ \hat{\mathbf{R}} = R + \hat{\mathbf{R}}, \]  
\[ \hat{\mathbf{R}} = - 2(n - 1)\nabla \cdot W - (n - 1)(n - 2)W^2. \]  
(72)

Note that the Ricci scalar here is not Weyl-invariant; under the transformation \( g'_{\mu\nu} = e^{2\omega}g_{\mu\nu} \), it behaves like \( \hat{\mathbf{R}}' = e^{-2\omega}\hat{\mathbf{R}} \). We need to introduce a compensating scalar field to construct a Weyl-invariant extension of the Einstein–Hilbert action
\[ S = \int d^nx\sqrt{-g}(\mathbf{F}^2\mathbf{R} + \cdots). \]  
(73)

The Einstein tensor has the following form:
\[ \hat{\mathbf{G}}_{\mu\nu} = \mathbf{G}_{\mu\nu} + \hat{\mathbf{G}}_{\mu\nu}, \]  
\[ \hat{\mathbf{G}}_{\mu\nu} = \hat{\mathbf{R}}_{(\mu\nu)} - \frac{1}{2}g_{\mu\nu}\hat{\mathbf{R}} = (n - 2)[-(\nabla_\nu - W_\nu)W_\mu + g_{\mu\nu}[(\nabla \cdot W + \frac{1}{2}(n - 3)W^2)]. \]  
(74)

All the above curvature tensors we have obtained in the modified Weyl geometry (equations (59), (64) and (71)) are consistent with the usual approach. (One may compare our results with e.g. [28].) This means that we are not introducing a new geometry here, but only modify the Weyl geometry in some sense.

This can also be seen from the following fact. The Ricci and Bianchi identities are still the same as in the usual Weyl geometry. More explicitly, one has
\[ [\hat{\nabla}_\mu, \hat{\nabla}_\nu]V^\rho = \hat{\mathbf{R}}^\rho_{\sigma\mu\nu}V^\sigma, \]  
(75)

\[ \hat{\mathbf{R}}^\rho_{[\sigma\mu\nu]} = 0, \]  
(76)

\[ \hat{\nabla}_\rho(\hat{\mathbf{R}}^\rho_{\sigma\mu\nu})_{\mu\nu} = 0. \]  
(77)

One should not be tempted to write \([\hat{\nabla}_\mu, \hat{\nabla}_\nu]V^\rho \equiv \hat{\mathbf{R}}^\rho_{\sigma\mu\nu}V^\sigma, [\hat{\nabla}_\mu, \hat{\nabla}_\nu]V^\rho \equiv \hat{\mathbf{R}}^\rho_{\sigma\mu\nu}V^\sigma, \) etc. Here we use the symbol ‘\( \equiv \)’ to stress that these are questionable formulas. All in all, we actually keep the mathematical structure of the Weyl geometry intact.
6. A modified variational principle

6.1. Main obstacles

Given a gravity action, one usually needs to vary it with respect to the metric to obtain
the energy–momentum tensor. As for its Weyl-invariant extension, one also needs to find
the equation of motion for the Weyl gauge field through variation. A key formula for this
variational principle is the Palatini identity in the Riemannian geometry (or more properly, in
general relativity):

\[ \delta R^\rho_{\sigma \mu \nu} = \nabla_\mu \delta \Gamma^\rho_{\sigma \nu} - \nabla_\nu \delta \Gamma^\rho_{\mu \sigma}. \]  

(78)

It is desirable to have a Weyl version of this identity for the additive Riemann tensor. Although
we have already obtained a compact form for it in equation (58), to find an analogous formula
for its variation turns out to a quite nontrivial job. This is because of the following problem:

\[ \delta \hat{R}^\rho_{\sigma \mu \nu} \equiv \delta \hat{\nabla}^\mu W^\rho_{\sigma \nu} - \delta \hat{\nabla}^\nu W^\rho_{\mu \sigma} \neq \hat{\nabla}^\mu (\delta \rho \delta W_{\sigma}^{\nu} + \cdots) \]

(79)

Here we use the symbol ‘⊜’ to stress that an unverified transformation rule has been applied.
The problem remains even we choose another procedure as

\[ \hat{\nabla}^\mu \delta W_{\sigma}^{\rho} - \hat{\nabla}^\nu \delta W_{\mu}^{\rho} \]

(80)

In the above calculations, we vary \( W^{\rho}_{\sigma \nu} \) with respect to \( g_{\gamma \delta} \) and \( W_{\tau} \) at the same time. The situation
is unchanged when one does the variations individually. Also, in the practical calculation, one
usually uses the integration by parts to write \( \Phi^2 \nabla^\rho g_{\mu \nu} \) as \( -\nabla^\rho \Phi^2 g_{\mu \nu} \). These technical issues
do not concern us here.

We conclude that the symbol of variation does not commute with the Weyl covariant
derivative operator(s). The above results can be checked by hand. Since the calculations are
very lengthy and non-illuminating, we will not report them here.

Things become more interesting when one finds that even the Riemannian covariant
derivative does not commute with the symbol of variation when acting on the additive Weyl
connection:

\[ \delta \nabla^\rho W_{\nu \sigma} \neq \nabla^\rho \delta W_{\nu \sigma}. \]  

(81)

As we will show, all these problems can be solved if we notice a peculiarity inherited from the
Riemannian geometry, and define a sensible transformation rule for the variation of general
tensors in the Weyl geometry. Since the usual Weyl covariant derivative becomes more relevant
here rather than our modified one, the following mnemonic is useful: in the language of
equations (26) and (27), their ‘covariant weights’ are 1 and 1/2, respectively.

6.2. Basic definitions

The phenomenon in (81) can be easily explained if one notices the following proposition: in
Riemann–Cartan geometry, when the involved composite tensor has an explicit dependence on
the metric, the symbol of variation does not commute with the Riemannian covariant
derivative. This is because of a very simple fact

\[ [\delta, \nabla_\rho] g_{\mu \nu} = -\nabla_\rho \delta g_{\mu \nu}. \]  

(82)

where the metricity condition \( \nabla_\rho g_{\mu \nu} = 0 \) has been used. In other words, we have

\[ [\delta, \nabla_\rho] (\cdots + g_{\mu \nu} S^{\mu \nu}_{\eta \zeta} \cdots) \neq 0. \]  

(83)
Note that the tensor $S^\mu_{\lambda
u}$: here should have no common indices with $g_{\mu\nu}$. Otherwise, one would have a rather unpleasant result: $[\delta, \nabla_\rho] V_\mu \equiv 0$, while $[\delta, \nabla_\rho] (g_{\mu\nu} V^\nu) \neq 0$.

We then assume that any tensor and its variation obey exactly the same transformation rule under the action of the usual Weyl covariant derivative as follows:

$$\tilde{\nabla}_\rho \delta T^{a_i \cdots}_{b_i \cdots} = \nabla_\rho \delta T^{a_i \cdots}_{b_i \cdots} + W^{a_i}_{\rho\lambda} \delta T^{\lambda a_2 \cdots}_{b_i \cdots} + \cdots - W^{a_i}_{\rho b_i} \delta T^{a_2 \cdots}_{b_i \cdots} - \cdots .$$  \hspace{1cm} (84)

It should be noted that this definition has no analogue in the Riemannian geometry (or more properly, in general relativity). This means that one cannot define a similar transformation rule such as

$$\nabla_\rho \delta T^{a_i \cdots}_{b_i \cdots} \equiv \partial_\rho \delta T^{a_i \cdots}_{b_i \cdots} + \Gamma^{a_i}_{\rho\lambda} \delta T^{\lambda a_2 \cdots}_{b_i \cdots} + \cdots - \Gamma^{a_i}_{\rho b_i} \delta T^{a_2 \cdots}_{b_i \cdots} - \cdots .$$  \hspace{1cm} (85)

It would be a disaster for general relativists, since we always write $\delta \nabla_\rho T^{a_i \cdots}_{b_i \cdots} = \nabla_\rho \delta T^{a_i \cdots}_{b_i \cdots}$ for a single tensor. For the LHS, we have

$$\delta \nabla_\rho T^{a_i \cdots}_{b_i \cdots} = \partial_\rho \delta T^{a_i \cdots}_{b_i \cdots} + \delta (\Gamma^{a_i}_{\rho\lambda} T^{\lambda a_2 \cdots}_{b_i \cdots}) + \cdots - \delta (\Gamma^{a_i}_{\rho b_i} T^{a_2 \cdots}_{b_i \cdots}) = \cdots .$$  \hspace{1cm} (86)

These two equations are clearly not equal to each other.

However, when it comes to the additive Weyl connection, our definition indeed has strong resemblance with the transformation rule for the variation of the Christoffel connection

$$\tilde{\nabla}_\rho \delta W^\mu_{\rho\nu} = \nabla_\rho \delta W^\mu_{\rho\nu} + W^\lambda_{\rho\mu} \delta W^\sigma_{\rho\nu} - W^\sigma_{\rho\nu} \delta W^\lambda_{\rho\mu} - W^\mu_{\rho\nu} \delta W^\lambda_{\rho\mu} .$$  \hspace{1cm} (87)

$$\nabla_\rho \delta \Gamma^\mu_{\rho\nu} = \partial_\rho \delta \Gamma^\mu_{\rho\nu} + \Gamma^\nu_{\rho\sigma} \delta \Gamma^\mu_{\rho\sigma} - \Gamma^\sigma_{\rho\nu} \delta \Gamma^\mu_{\rho\sigma} - \Gamma^\mu_{\rho\nu} \delta \Gamma^\nu_{\rho\sigma} .$$  \hspace{1cm} (88)

While the Christoffel connection $\Gamma^\mu_{\rho\nu}$ itself is not a tensor and cannot be given a sensible transformation rule under the action of the covariant derivative, its variation $C^\mu_{\rho\nu} \equiv \delta \Gamma^\mu_{\rho\nu}$ is a genuine tensor. So equation (88) is not a manifestation of the questionable rule in equation (85). In contrast, we treat both the additive Weyl connection $W^\lambda_{\mu\nu}$ and its variation $\delta W^\lambda_{\mu\nu}$ as (1, 2) tensors in the Weyl geometry, and demand them to obey exactly the same transformation rule (see equations (4) and (84)).

In view of the fact that the symbol of variation does not commute with the Weyl covariant derivative, we introduce another definition

$$[\delta, \tilde{\nabla}_\rho] T^{a_i \cdots}_{b_i \cdots} = \delta W^{a_i}_{\rho\lambda} T^{\lambda a_2 \cdots}_{b_i \cdots} + \cdots - \delta W^{a_i}_{\rho b_i} T^{a_2 \cdots}_{b_i \cdots} - \cdots + [\delta, \nabla_\rho] T^{a_i \cdots}_{b_i \cdots} .$$  \hspace{1cm} (89)

It also has no analogue in the Riemannian geometry. By the inspection of equations (85) and (86), one may be tempted to introduce a similar definition:

$$[\delta, \nabla_\rho] T^{a_i \cdots}_{b_i \cdots} \equiv \delta \Gamma^{a_i}_{\rho\lambda} T^{\lambda a_2 \cdots}_{b_i \cdots} + \cdots - \delta \Gamma^{a_i}_{\rho b_i} T^{a_2 \cdots}_{b_i \cdots} - \cdots .$$  \hspace{1cm} (90)

However, with equations (85) and (90) combined together, all the calculations involving variation in general relativity would still be substantially changed. The mathematical structure there may be spoiled because of this. One can also check that our definitions have no conflict with the Riemannian geometry by simply setting $W^\lambda_{\mu\nu} \rightarrow 0$ in them. In other words, we do not change anything about the Riemannian covariant derivative in this work.

With the above definitions (84) and (89) at hand, we finally arrive at the following identity for the variation of the Riemann tensor in the Weyl geometry:

$$\delta \tilde{R}^\rho_{\sigma\mu\nu} = \delta R^\rho_{\sigma\mu\nu} + \tilde{\delta} \tilde{R}^\rho_{\sigma\mu\nu} ,$$

$$\delta \tilde{R}^\rho_{\sigma\mu\nu} = (\nabla_\rho \delta + [\delta, \nabla_\rho]) W^\rho_{\sigma\nu} - (\tilde{\nabla}_\rho \delta + [\delta, \tilde{\nabla}_\rho]) W^\rho_{\sigma\nu} .$$  \hspace{1cm} (91)

This identity is not complicated as it appears, since $\delta \nabla_\rho \equiv [\delta, \nabla_\rho]$ is just a delta-like operator, and is always trivial unless acting on the metric tensor. In other words, all we need is

$$[\delta, \nabla_\rho] W^\rho_{\sigma\nu} = \nabla_\rho \delta g_{\sigma\nu} W^\rho .$$  \hspace{1cm} (92)
One may try to modify definition (84) to the following unfamiliar form:

\[ \widetilde{\nabla}_\rho \delta T^{\mu \nu}_{\rho \delta} = \delta \nabla_\rho T^{\mu \nu}_{\rho \delta} + W^\rho_{\mu \delta} \delta T^{\mu \nu}_{\rho \delta} + \ldots - W^\rho_{\mu \delta} \delta T^{\mu \nu}_{\rho \delta} - \ldots. \]  

(93)

Then we will have a more pleasant result:

\[ \delta \widetilde{R}^\rho_{\sigma \mu \nu} = \widetilde{\nabla}_\mu \delta W^\rho_{\sigma \nu} - \widetilde{\nabla}_\nu \delta W^\rho_{\sigma \mu}. \]  

(94)

Also, we can totally forget about the peculiarity concerning the operator \( \delta \nabla_\rho \equiv [\delta, \nabla_\rho] \), which is just an unfortunate heritage from the Riemannian geometry.

### 6.3. The variation of the Ricci tensor

In the following, we will use the Weyl version of the Palatini identity (91) to find the variation of the additive Ricci tensor. Contracting the indices there and using equation (92), we have

\[ \delta \widetilde{R}_{\sigma \nu} = (\widetilde{\nabla}_\rho \delta + [\delta, \nabla_\rho]) W^\rho_{\sigma \nu} - (\widetilde{\nabla}_\nu \delta + [\delta, \nabla_\nu]) W^\rho_{\sigma \rho}. \]

(95)

Note that due to the remark under equation (83), \( 0 = [\delta, \nabla_\rho] W^\rho_{\sigma \nu} \neq \nabla_\rho \delta g_{\rho \nu} W^\rho_{\sigma \rho} \). If one uses equation (94) instead (as we recommend), then this peculiarity can be totally dismissed.

The first term in the above equation can be expanded as

\[ \widetilde{\nabla}_\rho \delta W^\rho_{\sigma \nu} = \widetilde{\nabla}_\rho (\delta^\rho_{\rho} W^\rho_{\sigma \nu} + \delta^\rho_{\sigma} W^\rho_{\nu \rho} - g_{\rho \sigma} \delta W^\rho_{\nu} - W^\rho_{\rho \nu} \delta g_{\rho \sigma}) \]

\[ = \widetilde{\nabla}_\rho \delta W^\rho_{\sigma \nu} + \nabla_\sigma \delta W^\rho_{\nu \rho} - \widetilde{\nabla}_\rho g_{\sigma \rho} \delta W^\rho_{\nu} - g_{\rho \sigma} \nu \delta W^\rho_{\nu} - \widetilde{\nabla}_\rho W^\rho_{\mu \nu} \delta g_{\rho \sigma} - W^\rho_{\rho \nu} \delta g_{\rho \sigma} - [\nabla \cdot W + W \cdot \nabla + (n - 2) W^2] \delta g_{\rho \sigma}. \]  

(96)

Here we have used the following basic formulas:

\[ \widetilde{\nabla}_\rho \delta W^\rho_{\sigma \nu} = \nabla_\rho \delta W^\rho_{\sigma \nu} - W^\rho_{\rho \sigma} \delta W^\rho_{\nu}, \]  

(97)

\[ W^\rho_{\rho \sigma} \delta g_{\rho \sigma} = (W \cdot \nabla - 2 W^2) \delta g_{\rho \sigma}. \]  

(99)

There is of course another way to perform the calculation in equation (96) which is a little lengthy

\[ W^\rho_{\rho \sigma} \delta g_{\rho \sigma} = \nabla_\rho \delta W^\rho_{\sigma \nu} + W^\rho_{\rho \sigma} \delta W^\rho_{\nu \rho} - W^\rho_{\rho \sigma} \delta W^\rho_{\nu \rho} - W^\rho_{\rho \sigma} \delta W^\rho_{\nu \rho} \]

\[ = \nabla_\rho \delta W^\rho_{\sigma \nu} + \nabla_\sigma \delta W^\rho_{\nu \rho} - g_{\rho \sigma} \nu \delta W^\rho_{\nu} - \nabla_\rho \cdot W + W \cdot \nabla \delta g_{\rho \sigma} + n(W_{\sigma} \delta W^\rho_{\nu} + W_{\rho} \delta W^\rho_{\nu} - g_{\rho \sigma} W_{\rho} \delta W^\rho_{\nu} - W^2 \delta g_{\rho \sigma}) \]

\[ - (n W_{\rho \sigma} \delta W^\rho_{\nu} + W^2 \delta W^\rho_{\rho \sigma} - g_{\rho \sigma} W_{\rho} \delta W^\rho_{\nu} - W^2 \delta g_{\rho \sigma}) \]

\[ - (W^2 \delta W^\rho_{\rho \sigma} + n W_{\sigma} \delta W^\rho_{\nu} - g_{\rho \sigma} W_{\rho} \delta W^\rho_{\nu} - W^2 \delta g_{\rho \sigma}) \]

\[ = \nabla_\rho \delta W^\rho_{\sigma \nu} + \nabla_\sigma \delta W^\rho_{\nu \rho} - 2(W_{\sigma} \delta W^\rho_{\nu} + W_{\rho} \delta W^\rho_{\nu} - g_{\rho \sigma} \nu \delta W^\rho_{\nu} + (n - 2) W^2 \delta g_{\rho \sigma} \]

\[ - [\nabla \cdot W + W \cdot \nabla + (n - 2) W^2] \delta g_{\rho \sigma}. \]  

(100)

The result is the same as expected.

Inserting equation (96) or equation (100) into equation (95), the variation of the additive Ricci tensor is obtained as follows:

\[ \delta \widetilde{R}_{\sigma \nu} = \nabla_\rho \delta W^\rho_{\sigma \nu} - (n - 1) \nabla_\rho \delta W^\rho_{\sigma \nu} + (n - 2) (W_{\sigma} \delta W^\rho_{\nu} + W_{\rho} \delta W^\rho_{\sigma}) \]

\[ - g_{\rho \sigma} \nu [\nabla_\rho + (n - 2) W^2] \delta W^\rho_{\nu} - [\nabla \cdot W + (n - 2) W^2] \delta g_{\rho \sigma}. \]  

(101)
From this, one can obtain another useful equation
\[ g^{\sigma \nu} \delta \hat{R}_{\sigma \nu} = -2(n-1)(\nabla_{\rho} + (n-2) W_{\rho}) \delta W^{\rho} - g^{\sigma \nu} (\nabla \cdot W + (n-2) W^{2}) \delta g_{\sigma \nu}. \]  
(102)

Note that the above derivations require no knowledge of the explicit form of the Weyl-invariant Ricci tensor as in equation (65) or equation (64).

Analogous definitions as in equations (84) and (89) can also be given for the modified Weyl covariant derivative. To check this, let us start from the additive Ricci tensor in equation (64) and take its variation. Then we will have
\[ \delta \hat{R}_{\sigma \nu} = \delta \hat{R}_{\sigma \nu} - (n-2) \delta \nabla_{\nu} W_{\sigma} - (\nabla_{\sigma} W^{\rho} - W^{2}) \delta g_{\sigma \nu} - g_{\sigma \nu} \delta \nabla_{\rho} W^{\rho} + 2 g_{\sigma \nu} W_{\rho} \delta W^{\rho} \]
\[ = \nabla_{\nu} \delta W_{\sigma} - \nabla_{\sigma} \delta W_{\nu} - (n-2) (\nabla_{\sigma} W^{\rho} - \frac{1}{2} \delta W^{\rho}_{\sigma \nu} W_{\rho}) - (\nabla \cdot W + \frac{1}{2} n W^{2} - W^{2}) \delta g_{\sigma \nu} \]
\[ - g_{\sigma \nu} (\nabla_{\rho} \delta W^{\rho} + \frac{1}{2} \delta W^{\rho}_{\rho \lambda} W^{\lambda}) + 2 g_{\sigma \nu} W_{\rho} \delta W^{\rho} \]
\[ = \nabla_{\nu} \delta W_{\sigma} - \nabla_{\sigma} \delta W_{\nu} - (n-2) (\nabla_{\sigma} W^{\rho} - \frac{1}{2} W^{\rho}_{\nu \sigma} W_{\rho} - \frac{1}{2} \delta W^{\rho}_{\sigma \nu} W_{\rho}) \]
\[ - (\nabla \cdot W + \frac{1}{2} n W^{2} - W^{2}) \delta g_{\sigma \nu} - g_{\sigma \nu} (\nabla_{\rho} \delta W^{\rho} + \frac{1}{2} W^{\rho}_{\rho \lambda} \delta W^{\lambda} + \frac{1}{2} \delta W^{\rho}_{\rho \lambda} W^{\lambda}) \]
\[ + 2 g_{\sigma \nu} W_{\rho} \delta W^{\rho}. \]  
(103)

Using the equation (see equation (19))
\[ \delta (W^{\rho}_{\nu \sigma} W_{\rho}) = 2 (W_{\nu} \delta W_{\sigma} + W_{\sigma} \delta W_{\nu} - g_{\nu \sigma} W_{\rho} \delta W^{\rho} - \frac{1}{2} W^{2} \delta g_{\nu \sigma}), \]  
(104)
and collecting all the terms, one still arrives at the result in equation (101).

Although all these procedures give the consistent results, in practical calculation we choose the following way: use the Weyl version of the Palatini identity in equation (91) and insert the definition of the additive Weyl connection, and then apply the transformation rule in equation (84) to expand the expression as in equation (96). The final result is the same as the usual approach which takes the variation of the explicit expressions of curvature tensors directly.

To apply our method to the Weyl-invariant extension of higher curvature gravity theories or other complicated situations (see [28–31]), one still needs more work. Other issues may also become relevant. We find it more suitable to return to this whole topic in another occasion.

7. Conclusion

Our main results can be summarized as follows. If we treat both the additive Weyl connection and its variation as ordinary (1, 2) tensors, and demand them to obey similar transformation rules under the action of Weyl covariant derivatives, then the Riemann tensor and its variation could be written in compact forms.

In addition to the usual approach which deals with the calculations involving curvature tensors by using the explicit expressions of the latter directly, now we have two new ways to obtain exactly the same results. One is free to choose between the following alternative procedures: (i) use the (covariant) transformation rule of the additive Weyl connection as a (1, 2) tensor and then insert the definition of the latter; (ii) insert its definition at first and then use the transformation rule of the Weyl gauge field as a vector.

For the convenience of the reader, we list the basic formulas below.

(i) A generalized Weyl covariant derivative is defined as
\[ \nabla_{\rho} T^{a_{1} \ldots a_{n}}_{b_{1} \ldots b_{m}} = \nabla_{\rho} T^{a_{1} \ldots a_{n}}_{b_{1} \ldots b_{m}} + s W^{a_{1}}_{\rho \beta_{1}} T^{a_{2} \ldots a_{n}}_{b_{1} \beta_{2} \ldots b_{m}} + \ldots + t W^{a_{1}}_{\rho \beta_{1} \beta_{2}} T^{a_{2} \ldots a_{n}}_{\beta_{3} \ldots b_{m}} + \ldots \]  
(105)

Two necessary conditions for it to be sensible are \( \nabla_{\rho} g_{\mu \nu} = 0 \) and \( s = t \). The usual Weyl covariant derivative and the modified one have 'covariant weights' 1 and 1/2, respectively: \( \nabla_{\rho} \equiv \nabla_{\rho}(s = t = 1) \), \( \nabla_{\rho} \equiv \nabla_{\rho}(s = t = \frac{1}{2}) \).
(i) Any tensor and its variation obey similar transformation rules under the action of the usual Weyl covariant derivative. This means that we have the following definition:

\[ \tilde{\nabla}_\rho \delta T^a_{b_1\ldots b_{i-1}b_{i+1}\ldots} = \nabla_\rho \delta T^a_{b_1\ldots b_{i-1}b_{i+1}\ldots} + W^a_{\rho\mu} \delta T^\mu_{b_1\ldots b_{i-1}b_{i+1}\ldots} + \cdots - W^a_{\rho b_1\ldots b_{i-1}b_{i+1}\ldots} \delta T^b_1 \cdots = \cdots . \]  

As remarked in the text, this definition has no analogue in the Riemannian geometry.

(iii) The symbol of variation and the Weyl covariant derivative do not commute, but we can consistently define

\[ ([\delta, \tilde{\nabla}_\rho] - [\delta, \nabla_\rho])T^a_{b_1\ldots b_{i-1}b_{i+1}\ldots} = \delta W^a_{\rho b_1\ldots b_{i-1}b_{i+1}\ldots} + \cdots - W^a_{\rho b_1\ldots b_{i-1}b_{i+1}\ldots} \delta T^b_1 \cdots = \cdots . \]

Here \([\delta, \nabla_\rho]\) is just a delta-like operator. (One can define \(\delta \nabla_\rho \equiv [\delta, \nabla_\rho]\).) It is always trivial except when acting on a composite tensor which has an explicit dependence on the metric. This is because of a simple fact:

\[ [\delta, \nabla_\rho]g_{\mu\nu} = -\nabla_\rho \delta g_{\mu\nu}. \]

(iv) The Riemann tensor and its variation in the Weyl geometry can be written in the following compact forms:

\[ \tilde{R}^\rho_{\sigma\mu\nu} = R^\rho_{\sigma\mu\nu} + \tilde{R}^\rho_{\sigma\mu\nu}, \]

\[ \check{R}^\rho_{\sigma\mu\nu} = \tilde{\nabla}_\rho W^\sigma_{\mu\nu} - \tilde{\nabla}_\nu W^\sigma_{\mu\rho}, \]

\[ \delta \tilde{R}^\rho_{\sigma\mu\nu} = \delta R^\rho_{\sigma\mu\nu} + \delta \tilde{R}^\rho_{\sigma\mu\nu}, \]

\[ \delta \check{R}^\rho_{\sigma\mu\nu} = (\tilde{\nabla}_\rho \delta + [\delta, \nabla_\rho])W^\sigma_{\mu\nu} - (\tilde{\nabla}_\nu \delta + [\delta, \nabla_\nu])W^\rho_{\mu\sigma}. \]

Here the definition of the additive Weyl connection is \(W^\lambda_{\mu\nu} = \delta^\lambda_\mu W^\sigma_{\nu\sigma} + \delta^\lambda_\nu W^\sigma_{\mu\sigma} - g_{\mu\nu} W^\lambda\). With equations (110) and (112) combined together, we arrive at a new and systematic approach to deal with all the calculations involving curvature tensors for gravity theories in the Weyl geometry.

Some comments are in order here. (i) We systematically absorb the Christoffel connection into the Riemannian covariant derivative, so most of our results can be easily extended to the case with torsion, i.e. the Riemann–Cartan–Weyl geometry. In that case, the Riemann tensor would have an extra term \(2\Gamma^\lambda_{\mu\nu} W^\rho_{\lambda\sigma}\). (ii) In our new definitions, nothing about the Riemannian covariant derivative is changed. Even the peculiarity concerning the delta-like operator is just a plain fact, although it may be overlooked by some authors. (iii) In the Riemannian geometry, we could not define a generalized covariant derivative because of the metricity condition. Although in the Weyl geometry this condition is relaxed, it does not mean that one can define an arbitrary covariant derivative. (iv) In the process of our exploration, we have encountered some general propositions. Although they may be commonplace to the literature, it is still interesting to interpret them in new ways.

As we have remarked in the introduction, the field equation of the connection in Palatini \(f(R)\) gravity is much like the Weyl connection in the Weyl integrable geometry (see e.g. [32]). The Palatini formalism used there also has some resemblance with the situation in the Weyl geometry. So it would be interesting to find more connections between these two areas. However, one should be aware of the differences there, e.g. the works on variational principles in \(f(R)\) gravity naturally took the non-metricity tensor and torsion into consideration. The biconnection variational principle proposed in [38] is also interesting. Nevertheless, if the second connection is replaced by the additive Weyl connection, the difference tensor used there will not be directly applicable to the case of the Weyl geometry. On the other hand, the corresponding action is still not in exactly the same form as the Weyl-invariant extension of
the Einstein–Hilbert action. In spite of these problems, it seems that the relation between $f(R)$ gravity and the Brans–Dicke theory could be rephrased in the language of the Weyl geometry. All these issues may need more investigation to be clarified.

To apply the Weyl version of the Palatini identity to explicit examples as in [28–31], one has to deal with the operation of raising and lowering of indices, the mixing of the Riemann tensor in the Riemannian geometry and its additive extension, and other technical issues. The nonsymmetric property of the Ricci tensor in the Weyl geometry may also lead to new problems. It would be nice to work the details out and see if any new technique is needed. One could also consider the Weyl-invariant extension of actions involving the covariant derivatives of curvature tensors which may have not been studied in the literature. Finally, from the phenomenological point of view the Weyl integrable case deserves more attention.

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