Gerstenhaber algebra and Deligne’s conjecture on Tate-Hochschild cohomology
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Abstract

Using non-commutative differential forms, we construct a complex called singular Hochschild cochain complex for any associative algebra over a field. The cohomology of this complex is isomorphic to the Tate-Hochschild cohomology in the sense of Buchweitz. By a natural action of the cellular chain operad of the spineless cacti operad, introduced by R. Kaufmann, on the singular Hochschild cochain complex, we provide a proof of the Deligne’s conjecture for this complex. More concretely, the complex is an algebra over the (dg) operad of chains of the little 2-discs operad. By this action, we also obtain that the singular Hochschild cochain complex has a $B_\infty$-algebra structure and its cohomology ring is a Gerstenhaber algebra.

Inspired by the original definition of Tate cohomology for finite groups, we define a generalized Tate-Hochschild complex with the Hochschild chains in negative degrees and the Hochschild cochains in non-negative degrees. There is a natural embedding of this complex into the singular Hochschild cochain complex. In the case of a self-injective algebra, this embedding becomes a quasi-isomorphism. In particular, for a symmetric algebra, this allows us to show that the Tate-Hochschild cohomology ring, equipped with the Gerstenhaber algebra structure, is a Batalin-Vilkovisky algebra.

Keywords. Tate-Hochschild cohomology, Gerstenhaber algebra, Batalin-Vilkovisky algebra, Deligne’s conjecture, $B_\infty$-algebra

1 Introduction

Hochschild cohomology, introduced by Hochschild [Hoc] in 1945, is a cohomology theory for associative algebras. Motivated by Eilenberg-MacLane’s approach to the cohomology theory of groups, Hochschild introduced a cochain complex $C^*(A, M)$ for an associative algebra $A$ and an $A$-$A$-bimodule $M$. The Hochschild cohomology groups (with
coefficients in $M$) of $A$, denoted by $\text{HH}^*(A, M)$, are defined as the cohomology groups of $C^*(A, M)$. Recall that $\text{HH}^i(A, M)$ is isomorphic to the space of morphisms from $A$ to $s^i M$ in the bounded derived category $\mathcal{D}^b(A \otimes A^{\text{op}})$ of $A$-$A$-bimodules, where $s^i$ is the $i$-th shift functor for $i \in \mathbb{Z}$.

Later in the 1960s, Gerstenhaber [Ger63] found that there is a rich algebraic structure on $C^*(A, A)$ when studying the deformation theory of associative algebras. There is a cup product, which makes $C^*(A, A)$ into a differential graded (dg) associative algebra. This cup product has a remarkable property that it is not commutative on $C^*(A, A)$ but graded commutative up to homotopy. He also constructed a differential graded (dg) Lie algebra (of degree $-1$) structure on $C^*(A, A)$. The induced Lie bracket on $\text{HH}^*(A, A)$ satisfies the graded Leibniz rule with respect to the cup product. Nowadays we call $\text{HH}^*(A, A)$, together with the Lie bracket (called Gerstenhaber bracket) and cup product, a Gerstenhaber algebra (cf. Theorem 2.1). Moreover, Gerstenhaber showed that the dg Lie algebra $C^{*+1}(A, A)$ controls the deformation theory of $A$.

Recall that the little 2-discs operad is a topological operad whose space in arity $n$ is the topological space of standard embeddings (i.e. translations composed with dilations) of the disjoint union of $n$ discs into a standard disc. Cohen [Coh] in 1973 found that if a topological space $X$ is an algebra over the little 2-discs operad, then its singular homology $H_*(X)$ is a Gerstenhaber algebra. In 1993, Deligne asked whether the Hochschild cochain complex $C^*(A, A)$ of an associative algebra $A$ has a natural action of the little 2-discs operad. This is the original Deligne’s conjecture for Hochschild cochain complexes, which has been proved by several researches using different chain models of the little 2-discs operad (cf. [Tam, Kon, KoSo, Vor, McSm, Kau07a]). We also refer to [Kon] for its connection with Kontsevich’s deformation quantization theorem.

On the other hand, in the 1980s, Buchweitz in an unpublished manuscript [Buc] provided a general framework for Tate cohomology of Gorenstein algebras. To do this, he introduced the notion of stable derived category as the Verdier quotient of the bounded derived category by the full subcategory consisting of compact objects. This notion is also known as the singularity category, rediscovered by Orlov [Ori] in the study of homological mirror symmetry. Under Buchweitz’s framework, for any Noetherian algebra $A$ (not necessarily commutative), it is very natural to define the Tate-Hochschild cohomology groups as the morphism spaces from $A$ to $s^i A$ ($i \in \mathbb{Z}$) in the singularity category $\mathcal{D}_{sg}(A \otimes A^{\text{op}})$ of finitely generated $A \otimes A^{\text{op}}$-modules. The Tate-Hochschild cohomology has been investigated by a few authors (cf. [BeJo, EuSc, Ngu]) only in the case of Frobenius algebras. We remark that Tate cohomology is also implicitly Tate-Vogel’s cohomology [Goi] and exposed in [AvVe]. The Tate-Hochschild cohomology is also called singular Hochschild cohomology in [RiWa, Wan].

This paper attempts to provide a more complete picture of Tate-Hochschild cohomol-
ogy by describing richer algebraic structures as it was done for Hochschild cohomology. These algebraic structures might shed new light on the study of Tate-Hochschild cohomology not only in algebra but also in other fields such as noncommutative geometry, symplectic geometry, operad theory, and string topology.

We start with constructing a complex $C^{*}_{\text{sg}}(A, A)$, called singular Hochschild cochain complex, for any associative algebra $A$, which calculates the Tate-Hochschild cohomology of $A$. It is a colimit of Hochschild cochain complexes $C^{*}(A, \Omega^{p}_{\text{nc}}(A))$ with coefficients in the non-commutative differential forms $\Omega^{p}_{\text{nc}}(A)$ (concentrated in degree $-p \in \mathbb{Z}_{\leq 0}$) along natural embeddings $\theta_{p} : C^{*}(A, \Omega^{p}_{\text{nc}}(A)) \hookrightarrow C^{*}(A, \Omega^{p+1}_{\text{nc}}(A))$, $f \mapsto f \otimes \text{id}_{\Lambda^{p+1}}$ (cf. Definition 3.2). In other words, $C^{*}_{\text{sg}}(A, A)$ has a filtration of cochain complexes

$$0 \subset C^{*}(A, A) \subset C^{*}(A, \Omega^{1}_{\text{nc}}(A)) \subset \cdots \subset C^{*}(A, \Omega^{p}_{\text{nc}}(A)) \subset \cdots \subset C^{*}_{\text{sg}}(A, A).$$

This yields a natural map from $\text{HH}^{*}(A, A)$ to the cohomology, denoted by $\text{HH}^{*}_{\text{sg}}(A, A)$, of $C^{*}_{\text{sg}}(A, A)$. Moreover, this map coincides with the one induced by the quotient functor from $\mathcal{D}^{b}(A \otimes A^{\text{op}})$ to the singularity category $\mathcal{D}_{\text{sg}}(A \otimes A^{\text{op}})$ (cf. Theorem 3.6).

A natural question is whether the Gerstenhaber algebra structure on $\text{HH}^{*}(A, A)$ can be extended to $\text{HH}^{*}_{\text{sg}}(A, A)$. We give an affirmative answer to this question by an explicit construction of a cup product and Lie bracket (of degree -1) on $C^{*}_{\text{sg}}(A, A)$, which makes $\text{HH}^{*}_{\text{sg}}(A, A)$ into a Gerstenhaber algebra (cf. Corollary 5.3). We further show that the natural map from $\text{HH}^{*}(A, A)$ to $\text{HH}^{*}_{\text{sg}}(A, A)$ is a morphism of Gerstenhaber algebras.

In the series of papers [Kau05, Kau07a, Kau08], the author introduced the (topological) operad $\mathcal{C}^{\text{act}}$ of spineless cacti. He proved that the cellular chain (dg) operad $CC_{\ast}(\mathcal{C}^{\text{act}})$ is equivalent to the operad of chains of the little 2-discs operad (cf. [Kau07a] Theorem 3.11). There is a natural action of $CC_{\ast}(\mathcal{C}^{\text{act}})$ on $C^{*}(A, A)$. Recall that the brace operations (cf. Definition 5.4) on $C^{*}(A, A)$, due to Kadeishvili [Kad] and Getzler [Get94], play a crucial role in almost all existing proofs of the Deligne’s conjecture. The brace operations, together with the cup product, endow $C^{*}(A, A)$ with a $B_{\infty}$-algebra structure (cf. [Vor] Theorem 3.1). Let $\mathcal{B}^{\text{brace}}$ be the dg suboperad of the endomorphism operad $\text{Endop}(C^{*}(A, A))$, generated by the cup product and the brace operations. From [Kau07a Proposition 4.9] it follows that $\mathcal{B}^{\text{brace}}$ is isomorphic to $CC_{\ast}(\mathcal{C}^{\text{act}})$. In this paper, we will show that the action of $CC_{\ast}(\mathcal{C}^{\text{act}})$ on $C^{*}(A, A)$ can be naturally extended to $C^{*}_{\text{sg}}(A, A)$. As a consequence, we obtain that $C^{*}_{\text{sg}}(A, A)$ is a $B_{\infty}$-algebra with a $B_{\infty}$-subalgebra $C^{*}(A, A)$ (cf. Theorem 5.1) and that the Deligne’s conjecture holds for $C^{*}_{\text{sg}}(A, A)$ (cf. Theorem 5.2).

Motivated by the original definition of Tate cohomology for finite groups, we construct another unbounded complex $\mathcal{D}^{*}(A, A)$, called generalized Tate-Hochschild complex, for any associative algebra $A$:

$$\mathcal{D}^{*}(A, A) : \cdots \overset{b_{1}}{\rightarrow} C_{1}(A, A^{\vee}) \overset{b_{2}}{\rightarrow} C_{2}(A, A^{\vee}) \overset{\mu}{\rightarrow} A \overset{\delta_{0}}{\rightarrow} C_{1}(A, A) \overset{\delta_{1}}{\rightarrow} \cdots,$$
where $A^\vee := \text{Hom}_{A \otimes A^{op}}(A, A \otimes A^{op})$ and the differential $\mu$ is given by $\mu (\sum_i x_i \otimes y_i) = \sum_i x_i y_i$. In general, this complex does not calculate the Tate-Hochschild cohomology, but however there exists a natural embedding of $\mathcal{D}^*(A, A)$ into $C^*_{\text{sg}}(A, A)$. Moreover, this embedding becomes a quasi-isomorphism if $A$ is a self-injective algebra over a field. In particular, when $A$ is symmetric, this allows us to prove that the Tate-Hochschild cohomology, equipped with the Gerstenhaber algebra structure, is a Batalin-Vilkovisky (BV) algebra. The BV differential operator is induced by the Connes’ $B$ operator and its dual on $\mathcal{D}^*(A, A)$ (cf. Theorem 6.17). Inspired by the cyclic Deligne’s conjecture [Kau08], it is natural to ask whether $C^*_{\text{sg}}(A, A)$ (or equivalently, $\mathcal{D}^*(A, A)$) is an algebra over the framed little 2-discs operad if $A$ is a symmetric algebra.

**Related and future works:** It follows from [LoVa05] that the Hochschild cohomology of a dg algebra is isomorphic to the Hochschild cohomology of dg enhancements of its derived category. Inspired by this fact, it is interesting to study the relationship between the Tate-Hochschild cohomology and the Hochschild cohomology of dg enhancements of its singularity category. This problem is closely related to the uniqueness (up to quasi-equivalences) of dg enhancements of a singularity category since two quasi-equivalent dg categories have the same Hochschild cohomology (cf. [Kel, Toe]).

Let $(R, m)$ be a regular local ring. Suppose that $w \in m$ be a non-zero element such that the hypersurface $\text{Spec}(R/w)$ has an isolated singularity at $m$. From [Dyc, Corollary 6.4] it follows that the Hochschild cohomology of the 2-periodic dg category of matrix factorizations $\text{MF}_Z(R, w)$ is isomorphic to the Jacobian algebra $R/(\partial_1 w, \cdots, \partial_n w)$ in even degree and vanishes in odd degree. This is a $\mathbb{Z}/2\mathbb{Z}$-graded version of Hochschild cohomology. But $\text{HH}^*_{\text{sg}}(R/w, R/w)$ is isomorphic to the Tyurina algebra $R/(w, \partial_1 w, \cdots, \partial_n w)$ in each even degree. Thus in general, $\text{HH}^*_{\text{sg}}(R/w, R/w)$ is not isomorphic to the Hochschild cohomology of the dg enhancement $\text{MF}_Z(R, w)$ of $\mathcal{D}_{\text{sg}}(R/w)$ after translating the $\mathbb{Z}/2\mathbb{Z}$-graded version to $\mathbb{Z}$-graded one. On the contrary, let $Q$ be a finite quiver (not necessarily acyclic) without sources or sinks. Denote by $A_Q$ the radical square zero algebra $kQ/(Q_2)$, where $Q_2$ is the set of paths of length 2. We show [ChLiWa] that $\text{HH}^*_{\text{sg}}(A_Q, A_Q)$ is isomorphic to the Hochschild cohomology of the dg category $\mathcal{K}_{\text{ac}}(A_Q-\text{Inj})^c$, the full subcategory of compact objects in the dg category of acyclic complexes of injective $A_Q$-modules, of $\mathcal{D}_{\text{sg}}(A_Q)$. It is known that $\mathcal{K}_{\text{ac}}(A_Q-\text{Inj})^c$ is a dg enhancement of $\mathcal{D}_{\text{sg}}(A_Q)$ (cf. [Kra]).

In order to understand the relevance of the algebraic structures discussed in this paper in other fields such as symplectic geometry and string topology, we generalize our constructions to the dg case in [RiWa]. As an application, we provide a rational homotopy invariant of topological spaces. More explicitly, suppose that $X$ is a topological space and $C^*(X)$ is its rational singular cochain complex, which is clearly a dg algebra. We show that the singular Hochschild cochain complex $C^*_{\text{sg}}(C^*(X), C^*(X))$ gives a rational homotopy invariant of $X$. We also obtain that $\text{HH}^*_{\text{sg}}(C^*(M), C^*(M))$ of a simply-
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connected closed manifold $M$ is isomorphic to the Rabinowitz-Floer homology of the unit disc cotangent bundle $D(T^*M) \subset T^*M$ with the canonical symplectic structure (cf. [RiWa] Theorem 7.1 and [CiFrOa] Theorem 1.10). Inspired by the open-closed and closed-open string maps in symplectic geometry (cf. [Sei]), it is interesting to wonder whether this isomorphism lifts to the chain level from a geometric point of view.

In deformation theory, there is a general guiding principle that every deformation problem in characteristic zero is governed by a dg Lie algebra, due to numerous researchers such as Deligne, Grothendieck, Drinfeld, and Kontsevich. Recently, this principle is formulated by Lurie ([Lur]) via the language of ∞-categories. To the best of the author’s knowledge, it is still unclear which deformation problem the dg Lie algebra $\mathcal{C}^*_{sg}(A,A)$ controls. Motivated by the works [LoVd06, KeLo] on the deformation theory of abelian categories and triangulated categories, it is expected that $\mathcal{C}^*_{sg}(A,A)$ is related to the deformation theory of the singularity category $\mathcal{D}_{sg}(A)$.

Throughout this paper, we fix a field $k$. For simplifying the notation, we always write $\otimes$ instead of $\otimes_k$ and write $\text{Hom}$ instead of $\text{Hom}_k$, when no confusion may occur. For simplicity, we write $s\alpha_{i,j} := s\alpha_i \otimes s\alpha_{i+1} \otimes \cdots \otimes s\alpha_j \in (sA)^{\otimes j-i+1}$ for $0 \leq i \leq j$, where $s$ is the shift functor in the category of complexes.

2 Preliminaries

2.1 Hochschild homology and cohomology

2.1.1 Normalized bar resolution

Let $A$ be a unital associative algebra over a field $k$. Let $\overline{A}$ be the quotient $k$-module $A/(k \cdot 1)$ of $A$ by the $k$-scalar multiplies of the unit. Denote by $s\overline{A}$ the graded $A$-module concentrated in degree $-1$, namely, $(s\overline{A})^{-1} = \overline{A}$. The normalized bar resolution $\text{Bar}_{s}(A)$ is the complex of $A$-$A$-bimodules with $\text{Bar}_p(A) = A \otimes s\overline{A}^{\otimes p} \otimes A$ for $p \in \mathbb{Z}_{\geq 0}$ and the differentials

$$d_p(a_0 \otimes s\overline{a}_{1,p} \otimes a_{p+1}) := \sum_{i=0}^{p} (-1)^i a_0 \otimes s\overline{a}_{1,i-1} \otimes s\overline{a}_{i+1} \otimes s\overline{a}_{i+2,p} \otimes a_{p+1}.$$  

Here we remark that the term corresponding to $i = 0$ in the above formula should be $a_0 a_1 \otimes s\overline{a}_{2,p+1}$ and similarly the term of $i = p$ is $a_0 \otimes s\overline{a}_{1,p-1} \otimes a_p a_{p+1}$. In order to shorten the formula, we write the sum in such uniform way when no confusion may occur. It is well-known (cf. e.g. [Lod, Zim]) that $\text{Bar}_{s}(A)$ is a projective bimodule resolution of $A$.

2.1.2 Definitions of Hochschild (co)-homology

Let $A$ be an associative algebra over a field $k$ and $M$ be a graded $A$-$A$-bimodule. The normalized Hochschild cochain complex $C^*(A,M)$ with coefficients in $M$ is obtained
by applying the functor $\text{Hom}_{A \otimes A^{op}}(-, M)$ to the normalized bar resolution $\text{Bar}_*(A)$ and then using the canonical isomorphisms $\text{Hom}_{A \otimes A^{op}}(A \otimes sA^p \otimes A, M) \cong \text{Hom}(sA^p, M)$. Therefore, $C^*(A, M)$ is the following complex

$$\cdots \rightarrow C^{-1}(A, M) \rightarrow C^0(A, M) \xrightarrow{\delta^0} \cdots \rightarrow C^{p-1}(A, M) \xrightarrow{\delta^{p-1}} C^p(A, M) \xrightarrow{\delta^p} \cdots$$

where $C^p(A, M) := \prod_{i \in \mathbb{Z}_{\geq 0}} \text{Hom}((sA)^{\otimes i}, M)^p$ for $p \in \mathbb{Z}$ and

$$\text{Hom}((sA)^{\otimes i}, M)^p := \{ f \in \text{Hom}((sA)^{\otimes i}, M) \mid f \text{ is graded of degree } p \}.$$

Here we recall that $sA$ is a graded $k$-module concentrated in degree $-1$. The differential is given by

$$(−1)^{p+1}\delta^p(f)(s\bar{a}_{1,i+1}) = a_1 f(s\bar{a}_{2,i+1}) + \sum_{j=1}^i (-1)^j f(s\bar{a}_{1,j-1} \otimes s\bar{a}_{j+1} \otimes s\bar{a}_{j+2,i+1})$$

$$+ (-1)^{i+1} f(s\bar{a}_{1,i})a_{i+1},$$

for $f \in C^p(A, M)$. The $p$-th hochschild cohomology of $A$ with coefficients in $M$, denoted by $\text{HH}^p(A, M)$, is defined as the cohomology group $\frac{\text{Ker}(\delta^p)}{\text{Im}(\delta^{p-1})}$ of $C^*(A, M)$. In particular, we call $\text{HH}^*(A, A)$ the Hochschild cohomology ring of $A$, and $C^*(A, A)$ the Hochschild cochain complex of $A$.

Similarly, applying the functor $M \otimes_{A \otimes A^{op}} - \rightarrow \text{Bar}_*(A)$ and then using isomorphisms $M \otimes_{A \otimes A^{op}} (A \otimes sA^{\otimes p} \otimes A) \cong M \otimes sA^{\otimes p}$, we obtain the normalized Hochschild chain complex $C_*(A, M)$ of $A$ with coefficients in $M$, with $C_p(A, M) = \bigoplus_{i \in \mathbb{Z}_{\geq 0}} (M \otimes (sA)^{\otimes i})^p$. Here $m \otimes \bar{s}_{1,i} \in C_p(A, M)$ if and only if $|m| - i = p$, where by $|m|$ we mean the degree of $m$. The differential $b_p : C_p(A, M) \rightarrow C_{p-1}(A, M)$ is given by

$$b_p(m \otimes \bar{s}_{1,i}) = (-1)^m ma_1 \otimes \bar{s}_{2,i} +$$

$$\sum_{j=1}^{i-1} (-1)^{j+|m|} m \otimes \bar{s}_{1,j-1} \otimes \bar{s}_{j+1} \otimes \bar{s}_{j+2,i} + (-1)^p a_1 m \otimes \bar{s}_{1,i-1}.$$

The $p$-th Hochschild homology of $A$ with coefficients in $M$, denoted by $\text{HH}_p(A, M)$, is defined as the homology group $\frac{\text{Ker}(b_p)}{\text{Im}(b_{p+1})}$ of $C_*(A, M)$.

Since $\text{Bar}_*(A)$ is a projective bimodule resolution of $A$, it follows that $\text{HH}^p(A, M) \cong \text{Ext}_A^p(A^{op}, M)$ and $\text{HH}_p(A, M) \cong \text{Tor}_A^p(A^{op}, M)$. We have $\text{HH}_0(A, M) \cong \text{Tor}_A^0(A^{op}, M)$.

2.2 Gerstenhaber and Batalin-Vilkovisky algebras

In the 1960s when Gerstenhaber [Ger63] studied the deformation theory of algebras, he found that there is a rich structure on the Hochschild cochain complex $C^*(A, A)$. Besides the graded $k$-module structure, it has a differential graded (dg) associative algebra structure with the cup product $(g \cup f)(s\bar{a}_{1,m+n}) = f(s\bar{a}_{1,m})g(s\bar{a}_{m+1,m+n})$, for $f \in C^m(A, A)$.
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and \( g \in C^n(A, A) \). This cup product has a remarkable property: it is not (graded) commutative in \( C^*(A, A) \), but the induced product on \( \text{HH}^*(A, A) \) is graded commutative. There is also a differential graded (dg) Lie algebra structure on \( C^{*+1}(A, A) \) with the Gerstenhaber bracket defined as follows: for \( f \in C^m(A, A) \) and \( g \in C^n(A, A) \),

\[
[f, g] := f \circ g - (-1)^{(m-1)(n-1)} g \circ f
\]

where

\[
f \circ g(s_{a_1,m+n-1}) := \sum_{i=1}^{m} (-1)^{(i-1)(n-1)} f(s_{a_1,i-1} \otimes g(s_{a_1,i+n-1}) \otimes s_{i+n,m+n-1}).
\]

Furthermore, the induced Gerstenhaber bracket in \( \text{HH}^*(A, A) \) satisfies the graded Leibniz rule with respect to the cup product. In summary, Gerstenhaber proved the following result.

**Theorem 2.1 ([Ger63]).** The Hochschild cohomology ring \( \text{HH}^*(A, A) \) is a “Gerstenhaber algebra” in the following sense:

(i) \( (\text{HH}^*(A, A), \cup) \) is a graded commutative algebra with the unit \( 1 \in \text{HH}^0(A, A) \),

(ii) \( (\text{HH}^{*+1}(A, A), [\cdot, \cdot]) \) is a graded Lie algebra,

(iii) The operations \( \cup \) and \( [\cdot, \cdot] \) are compatible through the (graded) Leibniz rule,

\[
[f, g \cup h] = [f, g] \cup h + (-1)^{(m-1)n} g \cup [f, h],
\]

where \( f \in C^m(A, A) \) and \( g \in C^n(A, A) \).

**Remark 2.2.** In general, we call a graded \( k \)-module \( G = \bigoplus_i G^i \), equipped with two operations \( (\cup, [\cdot, \cdot]) \) satisfying the above conditions (i), (ii) and (iii), Gerstenhaber algebra. A nontrivial example is the Batalin-Vilkovisky algebra, motivated by quantum field theory.

**Definition 2.3.** Let \( V^* = \bigoplus_n V^n \) be a graded commutative (associative) algebra. We say that \( V^* \), equipped with a differential \( \Delta : V^* \rightarrow V^{*-1} \) of degree \(-1\), is a Batalin-Vilkovisky (BV) algebra if the following conditions hold,

1. \( \Delta(1) = 0 \) and \( \Delta^2 = 0 \),

2. for any \( a \in V^m, b \in V^n \) and \( c \in V^* \),

\[
\Delta(abc) = \Delta(ab)c + (-1)^m a \Delta(bc) + (-1)^n b \Delta(ac)
- \Delta(a)bc - (-1)^m a \Delta(b)c - (-1)^m+n ab \Delta(c).
\]
To each BV algebra, one can associate a graded Lie bracket $[\cdot, \cdot]$ as the obstruction of $\Delta$ being a (graded) derivation with respect to the multiplication of $V^\bullet$. Explicitly, $[a, b] := (-1)^m (\Delta(ab) - \Delta(a)b - (-1)^m a\Delta(b))$ for $a \in V^m$ and $b \in V^n$. This is called BV identity of $V^\bullet$. It follows from [Get94, Proposition 1.2] that the graded commutative algebra $V^\bullet$, endowed with this Lie bracket $[\cdot, \cdot]$, is a Gerstenhaber algebra.

Almost all the examples of BV algebras in literature (cf. e.g. [ChSu, Get94]) are strongly inspired by quantum field theory and string theory. A typical example is the Hochschild cohomology $\text{HH}^*(A, A)$ of a symmetric algebra $A$.

**Theorem 2.4** ([Tra, Men, Kau07b]). Let $A$ be a symmetric algebra. Then $(\text{HH}^*(A, A), \cup, [\cdot, \cdot])$ is a BV algebra whose BV operator $\Delta$ is the dual of the Connes’ $B$ operator.

Recall that a finite-dimensional algebra $A$ is symmetric if there is an associative, symmetric and non-degenerate bilinear form $\langle \cdot, \cdot \rangle : A \times A \to k$. More explicitly, $\langle ab, c \rangle = \langle a, bc \rangle = \langle ba, c \rangle$ for all $a, b, c \in A$, and the map $A \to D(A), a \mapsto \langle a, - \rangle$ from $A$ to the $k$-linear dual $D(A)$ is an isomorphism. Note that the pairing $\langle \cdot, \cdot \rangle$ on $A$ induces a graded pairing (still denoted by $\langle \cdot, \cdot \rangle$),

$$\langle \cdot, \cdot \rangle : C^*(A, A) \times C_*(A, A) \to k \quad (2.1)$$

defined by $\langle f, a_0 \otimes s\bar{a}_{1,m} \rangle := \langle a_0, f(s\bar{a}_{1,m}) \rangle$, for any $f \in C^m(A, A)$ and $a_0 \otimes s\bar{a}_{1,m} \in C_m(A, A)$. The BV operator $\Delta$ on $\text{HH}^*(A, A)$ is determined by $(-1)^m \langle \Delta(f)(s\bar{a}_{1,m}), a_0 \rangle = \langle B(a_0 \otimes s\bar{a}_{1,m}), f \rangle$, where $B$ is the Connes’ $B$ operator defined by

$$B(a_0 \otimes s\bar{a}_{1,m}) = \sum_{i=1}^{m+1} (-1)^{m+1} s\bar{a}_{i,m} \otimes s\bar{a}_0 \otimes s\bar{a}_{1,i-1}. \quad (2.2)$$

### 2.3 Noncommutative differential forms

There are several ways to define noncommutative differential forms of an associative $k$-algebra (not necessarily commutative). In the following, let us recall two of the (equivalent) definitions appeared in [CuQu, Gin].

The first definition is originally due to Cuntz-Quillen [CuQu]. Let $A$ be an $k$-algebra. The noncommutative differential forms of $A$ is the graded $k$-module $\Omega^\bullet_{\text{nc}}(A) := \bigoplus_{m \in \mathbb{Z}_{\geq 0}} A \otimes s\bar{A}^{\otimes m}$. There is a product on $\Omega^\bullet_{\text{nc}}(A)$ defined by

$$(a_0 \otimes s\bar{a}_{1,m})(a_{m+1} \otimes s\bar{a}_{m+2,m+n+1}) := \sum_{i=0}^{m} (-1)^{m-i} a_0 \otimes s\bar{a}_{1,i-1} \otimes s\bar{a}_{i+1} \otimes s\bar{a}_{i+2,m+n+1},$$

where $a_0 \otimes s\bar{a}_{1,m} \in A \otimes s\bar{A}^{\otimes m}$ and $a_{m+1} \otimes s\bar{a}_{m+2,m+n+1} \in A \otimes s\bar{A}^{\otimes n}$. It is clear that this product gives rise to a (graded) $A$-$A$-bimodule structure on $\Omega^\bullet_{\text{nc}}(A)$. The left action is
given by the multiplication of $A$ and the right action (denoted by $\bullet$) is by

$$(a_0 \otimes s \bar{a}_{1,n}) \bullet a_{n+1} = \sum_{i=0}^{n} (-1)^{n-i} a_0 \otimes s \bar{a}_{1,i-1} \otimes s \bar{a}_i a_{i+1} \otimes s \bar{a}_{i+2,n+1},$$

for any $a_{n+1} \in A$ and $a_0 \otimes \bar{a}_{1,n} \in A \otimes s \bar{A}^{\otimes n}$. There is a natural isomorphism of $A$-$A$-bimodules $\Omega^p_{nc}(A) \otimes_A \Omega^q_{nc}(A) \cong \Omega^{p+q}_{nc}(A)$. The following lemma will be used frequently throughout this paper.

**Lemma 2.5.** For any $r, s \in \mathbb{Z}_{>0}$, the following identity holds in $\Omega_{nc}^{r+s}(A)$.

$$(a_0 \otimes s \bar{a}_{1,r+s-1}) \bullet a_{r+s} = (-1)^{s-1}(a_0 \otimes s \bar{a}_{1,r}) \bullet a_{r+1} \otimes s \bar{a}_{r+2,r+s} + \sum_{i=1}^{s-1} (-1)^{s+i-1} a_0 \otimes s \bar{a}_{1,r+i-1} \otimes s \bar{a}_{r+i} a_{r+i+1} \otimes s \bar{a}_{r+i+2,r+s}. $$

**Proof.** This follows from a straightforward computation.

The other (equivalent) definition is as follows. For any $p \in \mathbb{Z}_{\geq 0}$, denote by $\Omega^p_{sy}(A)$ the cokernel $\text{Coker}(\text{Bar}_{p+1}(A) \xrightarrow{d_p} \text{Bar}_p(A))$ in the normalized bar resolution $\text{Bar}_*(A)$ (cf. Section 2.1.1). Clearly, $\Omega^p_{sy}(A)$ is a graded $A$-$A$-bimodule concentrated in degree $-p$. Observe that $\Omega^0_{sy}(A) \cong \Omega^0_{nc}(A) = A$. Generally, we have the following result.

**Lemma 2.6.** There is a natural isomorphism $\alpha : \Omega^\bullet_{sy}(A) \to \Omega^\bullet_{nc}(A)$ of graded $A$-$A$-bimodules.

**Proof.** For $p \in \mathbb{Z}_{\geq 0}$, we define the morphism $\alpha_p : \Omega^p_{sy}(A) \to \Omega^p_{nc}(A)$ to be the composition $\Omega^p_{sy}(A) \hookrightarrow \text{Bar}_{p-1}(A) = A \otimes s \bar{A}^{\otimes p-1} \otimes A \xrightarrow{id_A \otimes \text{id}_{s \bar{A}^{\otimes p-1}} \otimes \pi} A \otimes s \bar{A}^{\otimes p}$, where $\pi : A \to s \bar{A}$ is the canonical projection (of degree 1) and $id_V$ is the identity morphism of a $k$-module $V$. It is straightforward that $\alpha_p$ is invertible with the inverse $\alpha_p^{-1} : A \otimes s \bar{A}^{\otimes p} \xrightarrow{d_p} A \otimes s \bar{A}^{\otimes p} \otimes A \xrightarrow{d_p} \Omega^p_{sy}(A)$, where the first morphism $\tau_p$ sends $x$ to $(-1)^p x \otimes 1$. Thus it remains to show that $\alpha_p$ is an $A$-$A$-bimodule homomorphism. Indeed, we have that

$$\alpha_p^{-1}(a_0(a_1 \otimes s \bar{a}_{2,p}) \bullet a_{p+1})$$

$$= \sum_{i=1}^{p} (-1)^{i}d_p(a_0a_1 \otimes s \bar{a}_{2,i-1} \otimes s \bar{a}_{i+1} \otimes s \bar{a}_{i+2,p+1} \otimes 1)$$

$$= (-1)^pd_p(a_0a_1 \otimes s \bar{a}_{2,p} \otimes a_{p+1}) + d_p \circ d_{p+1}(a_0a_1 \otimes s \bar{a}_{2,p+1} \otimes 1)$$

$$= a_0\alpha_p^{-1}(a_1 \otimes s \bar{a}_{2,p})a_{p+1},$$

where the last identity follows from $d^2 = 0$. This proves the lemma. 

Based on Lemma 2.6, we will identify $\Omega^\bullet_{sy}(A)$ with $\Omega^\bullet_{nc}(A)$ as graded $A$-$A$-bimodules.
3 Tate-Hochschild cohomology

Let $k$ be a field. We construct a cochain complex, called singular Hochschild cochain complex, for any associative $k$-algebra $A$. The $i$-th cohomology group is isomorphic to the morphism spaces from $A$ to $s^iA$ in the singularity category $\mathcal{D}_{sg}(A \otimes A^{op})$.

3.1 Singular Hochschild cochain complex

Recall that $\Omega^p_{nc}(A)$ is a graded $A$-$A$-bimodule concentrated in degree $-p$. We consider a family, indexed by $p \in \mathbb{Z}_{\geq 0}$, of Hochschild cochain complexes $C^*(A, \Omega^p_{nc}(A))$. For any $p \in \mathbb{Z}_{\geq 0}$, we define an embedding of cochain complexes (of degree zero),

$$\theta_p : C^*(A, \Omega^p_{nc}(A)) \hookrightarrow C^*(A, \Omega^{p+1}_{nc}(A)), f \mapsto f \otimes \text{id}_{sA}.$$ 

Here we recall that $C^m(A, \Omega^p_{nc}(A)) = \text{Hom}((sA)^{\otimes m+p}, A \otimes (sA)^{\otimes p})$ for $m \in \mathbb{Z}$.

**Lemma 3.1.** $\theta_p \circ \delta = \delta \circ \theta_p$.

**Proof.** For $f \in C^m(A, \Omega^p_{nc}(A))$ and $n := m + p$, we have that

$$(-1)^{m+1}(\theta_p \circ \delta)(f)(s\alpha_{1,n+2}) = a_1 f(s\alpha_{2,n+1}) \otimes s\alpha_{n+2} + \sum_{i=1}^{n} (-1)^i f(s\alpha_{1,i-1} \otimes s\alpha_i \alpha_{i+1} \otimes s\alpha_{i+2,n+1}) \otimes s\alpha_{n+2} + (-1)^{n+1} f(s\alpha_{1,n}) \otimes s\alpha_{n+1 \alpha_{n+2}} + (-1)^{n+2} (f(s\alpha_{1,n}) \otimes s\alpha_{n+1}) \triangleright a_{n+2} = (-1)^{m+1} \delta(\theta_p(f))(s\alpha_{1,n+2}),$$

where we used Lemma 2.5 in the first identity. This proves $\theta_p \circ \delta = \delta \circ \theta_p$. 

**Definition 3.2.** Let $A$ be an associative $k$-algebra. Then the singular Hochschild cochain complex of $A$, denoted by $C^*_{sg}(A, A)$, is defined as the colimit of the inductive system in the category of cochain complexes of $k$-modules,

$$0 \hookrightarrow C^*(A, A) \xrightarrow{\theta_0} C^*(A, \Omega^1_{nc}(A)) \xrightarrow{\theta_1} \cdots \xrightarrow{\theta_{p-1}} C^*(A, \Omega^p_{nc}(A)) \xrightarrow{\theta_p} \cdots.$$ 

Namely, $C^*_{sg}(A, A) := \text{colim}_{p \in \mathbb{Z}_{\geq 0}} C^*(A, \Omega^p_{nc}(A))$. Its cohomology groups are denoted by $\text{HH}^*_{sg}(A, A)$.

**Remark 3.3.** Since the map $\theta_p$ is injective for any $p \in \mathbb{Z}_{\geq 0}$, there is a (bounded below) filtration of cochain complexes of $C^*_{sg}(A, A)$,

$$0 \subset C^*(A, A) \subset \cdots \subset C^*(A, \Omega^p_{nc}(A)) \subset \cdots \subset C^*_{sg}(A, A).$$

This yields a natural map $\rho : \text{HH}^*(A, A) \rightarrow \text{HH}^*_{sg}(A, A)$. In the following, we will see that $\rho$ is in fact a morphism of Gerstenhaber algebras (cf. Corollary 5.3). In [RiWa],
Gerstenhaber algebra and Deligne’s conjecture on Tate-Hochschild cohomology

Figure 1: Two types of graphic presentations of $f \in C^{m-p}(A, A \otimes s\overline{A}^{\otimes p})$. The left one is treelike presentation and the right one is cactus-like presentation.

we generalized the definition of $C^*_sg(A, A)$ to any dg associative algebra $A$, in order to understand the relevance of the algebraic structures discussed in this paper in symplectic geometry and string topology.

In [JoSt] the authors formalize the use of graphs in tensor categories. Morphisms in a tensor category are presented by graphs, and operations (e.g. compositions and tensor products) on morphisms are presented by operations (e.g. gratings and unions) on graphs. Since the category $k$-mod of $k$-modules, which we are basically working on in this paper, is particularly a tensor category with the tensor product $\otimes_k$, we will use graphs to present morphisms and operations in $k$-mod. For more details on graph theory, one may refer to [JoSt, Kau05, Kau07a].

Figure 1 illustrates two types (tree-like and cactus-like) of graphic presentations of $f \in C^{m-p}(A, A \otimes (s\overline{A})^{\otimes p})$. The tree-like presentation is the usual graphic presentation of morphisms in tensor categories used in [JoSt]. We read the graph from top to bottom and left to right. The inputs $(s\overline{A})^{\otimes m}$ are ordered from left to right at the top, while the outputs $A \otimes (s\overline{A})^{\otimes p}$ are ordered in the same way at the bottom. We use the color blue to distinguish the special output $A$. The orientations of edges are from top to bottom. To study the $B_\infty$-algebra structure on $C^*_sg(A, A)$ (cf. Section 5.2), we also need to use the cactus-like presentation. An element $f \in C^{m-p}(A, \Omega_{inc}^p(A))$ is presented as follows.

Process 3.4. First, the image of $0 \in \mathbb{R}$ in $S^1 := \mathbb{R}/\mathbb{Z}$ is decorated by a blue dot (cf. Figure 1). We call the image zero point of $S^1$. The blue radius pointing towards the dot represents the special output $A$. Then the inputs $(s\overline{A})^{\otimes m}$ are indicated by $m$ (black) radii on the left semicircle pointing towards the center of $S^1$ in clockwise, and finally the other outputs are indicated by $p$ radii on the right semicircle pointing outwards the center of $S^1$ in counterclockwise.

By Definition 3.2, two elements $f \in C^m(A, \Omega_{inc}^{p_1}(A))$ and $g \in C^m(A, \Omega_{inc}^{p_2}(A))$ for $p_1 \geq$...
The representatives of $f \in C_{sg}^{m-p}(A, A)$. The vertical lines on the upper right treelike graph represent identities $\text{id}_{sA}$ of $sA$. Similarly, the identities are also represented by horizontal chords on the lower right circle. $p_2$ represent the same element $[f] = [g]$ in $C_{sg}^{m}(A, A)$ if and only if $f = g \otimes \text{id}_{sA}^{(p_2-p_1)}$, as depicted in Figure 2. This allows us to add (or remove) some vertical lines on the right of the tree-like graph or chords on the circle. In the following sections, we will see that the two types of graphic presentations have different advantages. It is easier to write down the corresponding morphisms from tree-like presentations, while it is more convenient to construct operations on $C_{sg}^*(A, A)$ using cactus-like presentations.

### 3.2 Relationship with singularity category

In this section, we fix a (both left and right) Noetherian algebra $A$ over a field $k$. Let $\mathcal{D}^b(A)$ be the bounded derived category of finitely generated left $A$-modules. Let $\text{Perf}(A)$ denote the full subcategory consisting of those complexes which are quasi-isomorphic to bounded complexes of finitely generated projective $A$-modules. Then the singularity category $\mathcal{D}_{sg}(A)$ is defined as the Verdier quotient of the triangulated category $\mathcal{D}^b(A)$ by $\text{Perf}(A)$.

**Remark 3.5.** The notion of singularity category was introduced by Buchweitz in an unpublished manuscript [Buc]. He proved that the singularity category $\mathcal{D}_{sg}(A)$ is triangle equivalent to the stable category $\text{MCM}(A)$ of maximal Cohen-Macaulay modules when
the algebra $A$ is Gorenstein. Later, Orlov [Orl] independently rediscovered a global version of singularity category motivated by homological mirror symmetry.

Buchweitz in the same manuscript, provided a general framework for Tate cohomology. Let $M, N$ be two modules over a Gorenstein algebra $S$. The $i$-th Tate cohomology group of $M$ with values in $N$ is defined as $\text{Hom}_{\mathcal{D}(S)}(M, s^i N)$. In loc. cit. Buchweitz denoted it by $\text{Ext}^i_S(M, N)$. Clearly, this notion generalizes the Tate cohomology of finite groups. Under this framework, it is natural to define Tate-Hochschild cohomology groups as $\text{Ext}^*_A(A, A)$ for a Noetherian algebra $A$, compared with Hochschild cohomology.

**Theorem 3.6.** Let $A$ be a Noetherian $k$-algebra. Then there exists a natural isomorphism $\Phi : \HH^*_A(A, A) \to \Ext^*_A(A, A)$.

**Proof.** First, let us fix an integer $m \in \mathbb{Z}$. From the fact that the colimit commutes with the cohomology functor in the category of cochain complexes, it follows that

$$\HH^m_{sg}(A, A) \cong \colim_{H^m(\theta_p)} \HH^m(A, \Omega^p_{nc}(A)). \quad (3.1)$$

We observe that the map $H^m(\theta_p) : \HH^m(A, \Omega^p_{nc}(A)) \to \HH^m(A, \Omega^p_{nc}+1(A))$ coincides with the connecting morphisms in the long exact sequence

$$\cdots \to \HH^m(A, \text{Bar}_p(A)) \to \HH^m(A, \Omega^p_{nc}(A)) \to \HH^{m+1}(A, s^{-1}\Omega^p_{nc}+1(A)) \to \cdots$$

induced by the short exact sequence $0 \to s^{-1}\Omega^p_{nc}+1(A) \to \text{Bar}_p(A) \to \Omega^p_{nc}(A) \to 0$. Here, we identify $\Omega^p_{nc}(A)$ with $\Omega^p_{sy}(A)$ by Lemma 2.6. Note that there is a natural isomorphism between $\HH^{m+1}(A, s^{-1}\Omega^p_{nc}(A))$ and $\HH^{m+1}(A, \Omega^p_{nc}(A))$.

From [Bel, Corollary 3.3] and [Buc], it follows that

$$\Ext^m_{A \otimes A^{op}}(A, A) \cong \colim_{\theta'_p} \text{Hom}_{A \otimes A^{op}}(s^{-p-m}\Omega^p_{sy}(A), s^{-p}\Omega^p_{sy}(A)) \quad (3.2)$$

where $\text{Hom}_{A \otimes A^{op}}$ represents the morphism spaces in the stable category $A \otimes A^{op}$-mod of $A$-$A$-bimodules; the map $\theta'_p$ is induced by the fact that $A \otimes A^{op}$-mod is a left triangulated category with left shift functor the syzygy functor $\Omega^1_{sy}$. Combining the isomorphisms (3.1) and (3.2), it is sufficient to show that $\colim_{\theta_p} \text{Hom}_{A \otimes A^{op}}(s^{-p-m}\Omega^p_{sy}(A), s^{-p}\Omega^p_{sy}(A))$. First, let us define a morphism between them. Note that there is a canonical map

$$\Phi_{m,p} : \HH^m(A, \Omega^p_{nc}(A)) \to \text{Hom}_{A \otimes A^{op}}(s^{-p-m}\Omega^p_{sy}(A), s^{-p}\Omega^p_{sy}(A))$$

since any cocycle $f \in C^m(A, \Omega^p_{nc}(A))$ can be represented by an $A$-$A$-bimodule morphism $\overline{f} : \Omega^{m+p}_{sy}(A) \to \Omega^p_{sy}(A)$ and any coboundary factors through (see Diagram 3.3) the
projective $A$-$A$-bimodule $\text{Bar}_{m+p-1}(A)$.

$\xymatrix{ \text{Bar}_{m+p+1}(A) \ar[d]_{d_{m+p+1}} & \text{Bar}_{m+p}(A) \ar[d]_{d_{m+p}} \ar[r]_{f} & \Omega_{\text{sy}}^{p}(A) \ar[d] \ar[r] & \text{Bar}_{m+p-1}(A) }$ (3.3)

We observe that both of the two maps $H^m(\theta_p)$ and $\theta'_p$ (cf. (3.1) and (3.2)) correspond to the same lifting from the bottom horizontal maps $f$ to the top horizontal maps $\hat{f}$:

$\xymatrix{ \Omega_{\text{sy}}^{m+p+1}(A) \ar[r]^{\hat{f}} \ar[d] & \Omega_{\text{sy}}^{p+1}(A) \ar[d] \ar[r] & \text{Bar}_{m+p}(A) \ar[d] \ar[r] & \text{Bar}_{m+p-1}(A) \ar[d] \\
\Omega_{\text{sy}}^{m+p}(A) \ar[r]_{f} & \Omega_{\text{sy}}^{p}(A) \ar[r] & \text{Bar}_{p}(A) \ar[r] & \text{Bar}_{m+p-1}(A) }$ (3.4)

where $\hat{f}$ is given by $\hat{f}(a_0 \otimes s\overline{\alpha}_{1,m+p}) = f(a_0 \otimes \overline{\alpha}_{1,m+p-1}) \otimes s\overline{\alpha}_{m+p}$. Again we use the identification of $\Omega_{\text{sy}}^{p}(A)$ with $\Omega_{\text{nc}}^{p}(A)$ by Lemma 2.6. Therefore we get that the maps $\Phi_{m,s}$ are compatible with the colimit constructions and then we have a canonical map

$\Phi_{m} : \colim_{\theta_p} \text{HH}^m(A, \Omega_{\text{nc}}^{p}(A)) \to \colim_{\theta'_p} \text{Hom}_{A \otimes A^{\text{op}}}(s^{-p-m}\Omega_{\text{sy}}^{p+m}(A), s^{-p}\Omega_{\text{sy}}^{p}(A))$.

Claim that $\Phi_{m}$ is surjective. Indeed, assuming

$f \in \colim_{\theta'_p} \text{Hom}_{A \otimes A^{\text{op}}}(s^{-p-m}\Omega_{\text{sy}}^{p+m}(A), s^{-p}\Omega_{\text{sy}}^{p}(A))$,

then there exists $p_0 \in \mathbb{Z}_{\geq 0}$ such that $f$ can be represented by a certain element $f' \in \text{Hom}_{A \otimes A^{\text{op}}}(\Omega_{\text{sy}}^{m+p_0}(A), \Omega_{\text{sy}}^{p_0}(A))$. Thus we obtain a Hochschild cocyle $\alpha := f' \circ d_{m+p_0} \in \text{Hom}_{A \otimes A^{\text{op}}}(\text{Bar}_{m+p_0}(A), \Omega_{\text{sy}}^{p_0}(A))$. By the definition of $\Phi_{m}$, we have $\Phi_{m}(\alpha) = f$. This proves that $\Phi_{m}$ is surjective.

It remains to show that $\Phi_{m}$ is injective. Suppose $\beta \in \colim_{H^m(\theta_p)} \text{HH}^m(A, \Omega_{\text{nc}}^{p}(A))$ such that $\Phi_{m}(\beta) = 0$. Since $\Phi_{m}$ is surjective, $\beta$ can be represented by an element $\beta' \in \text{HH}^m(A, \Omega_{\text{sy}}^{p}(A))$ for some $p_0$ such that $\Phi_{m,p_0}(\beta') = 0$. This means that $\Phi_{m,p_0}(\beta')$ factors through the differential $d_{p_0} : \text{Bar}_{p_0}(A) \to \Omega_{\text{sy}}^{p_0}(A)$.

$\xymatrix{ \Omega_{\text{sy}}^{m+p_0}(A) \ar[r]_{\Phi_{m,p_0}(\beta')} & \Omega_{\text{sy}}^{p_0}(A) \ar[d]^{d_{p_0}} \\
& \text{Bar}_{p_0}(A) }$ (3.5)
Consider the long exact sequence
\[
\cdots \rightarrow \HH^m(A, \text{Bar}_{p_0}(A)) \xrightarrow{d_{p_0}^*} \HH^m(A, \Omega^p_{nc}(A)) \xrightarrow{H^m(\theta_{p_0})} \HH^{m+1}(A, s^{-1}\Omega_{p_0}^{p+1}) \rightarrow \cdots
\]
From Diagram (3.5), it follows that \([\sigma] \in \HH^m(A, \text{Bar}_{p_0}(A))\) and
d_{p_0}^*([\sigma]) = \beta' \in \HH^m(A, \Omega^p_{nc}(A)) \].
Then \(0 = H^m(\theta_{p_0})d_{p_0}^*([\sigma]) = H^m(\theta_{p_0})(\beta')\), thus \(\beta = 0 \in \text{colim} \HH^m(A, \Omega^p_{nc}(A))\).
Therefore \(\Phi_m\) is injective. This proves the theorem.

\textbf{Remark 3.7.} From the proof, we have the following commutative diagram.
\[
\begin{array}{ccc}
\text{Ext}^*_{A \otimes A^{op}}(A, A) & \xrightarrow{\rho'} & \text{Ext}^*_{A \otimes A^{op}}(A, A) \\
\cong & & \Phi_* \\
\HH^*(A, A) & \xrightarrow{\rho} & \HH^*_{sg}(A, A)
\end{array}
\]
where \(\rho'\) is induced by the quotient functor from the bounded derived category \(\mathcal{D}^b(A \otimes A^{op})\) to the singularity category \(\mathcal{D}_{sg}(A \otimes A^{op})\).

\section{Gerstenhaber algebra structure}

In this and the next section, we will prove that there is a Gerstenhaber algebra structure on \(\HH^*_{sg}(A, A)\) to make the natural map \(\rho : \HH^*(A, A) \rightarrow \HH^*_{sg}(A, A)\) into a morphism of Gerstenhaber algebras.

\subsection{Cup product}

For any \(m, n, p, q \in \mathbb{Z}_{\geq 0}\), the \textit{cup product}
\[
\cup : C^{m-p}(A, \Omega^p_{nc}(A)) \otimes C^{n-q}(A, \Omega^q_{nc}(A)) \rightarrow C^{m+n-p-q}(A, \Omega^{p+q}_{nc}(A))
\]
is defined by the following formula,
\[
f \cup g := (\mu \otimes \id_{sA}^{\otimes p+q}) \left( \id_A \otimes f \otimes \id_{sA}^{\otimes q} \right) \left( g \otimes \id_{sA}^{\otimes m} \right),
\]
for any \(f \in C^{m-p}(A, \Omega^p_{nc}(A))\) and \(g \in C^{n-q}(A, \Omega^q_{nc}(A))\). Here \(\id_{sA}\) is the identity morphism of \(sA\). When \(p = q = 0\), we recover the cup product on \(C^*(A, A)\) (cf. Section 2.2). The cup product can be depicted by the treelike or cactus-like presentation (cf. Figure 3).

\textbf{Lemma 4.1.} For any \(f \in C^{m-p}(A, \Omega^p_{nc}(A))\) and \(g \in C^{n-q}(A, \Omega^q_{nc}(A))\), we have
\[
\delta(f \cup g) = \delta(f) \cup g + (-1)^{m-p} f \cup \delta(g).
\]
Figure 3: Cup product $g \cup f$ in $C^*_{sg}(A, A)$ for $f \in C^{m-p}_{sg}(A, A), g \in C^{n-q}_{sg}(A, A)$. For simplicity, the orientation arrows (from top to bottom) in the tree-like presentation are omitted. In the cactus-like presentation, by the blue arrows connecting blue radii, we mean the multiplication of $A$.

**Proof.** Without loss of generality, we may assume that $m \geq q$. Then we have
\[
(-1)^\epsilon \left( \delta(f \cup g) - (-1)^{m-p} f \cup \delta(g) \right) (s\alpha_{1,m+n+1})
\]
\[
= \sum_{i=n+1}^{m+n} \sum_j (-1)^j c^i_0 \left( s\alpha_{i+1,m+n-q} \otimes \cdots \otimes s\alpha_{i+2,m+n+1} \right)
+ (-1)^n \sum_j \left( \mu \otimes \text{id}_{sA}^{p+q} \right) \left( \text{id}_{sA} \otimes f \otimes \text{id}_{sA}^{p+q} \right) \left( \left( c^j_0 \otimes s\alpha_{1,q} \right) \downarrow a_{n+1} \otimes s\alpha_{n+2,m+n+1} \right)
+ (-1)^{m+n+1} \sum_j \left( c^j_0 \left( s\alpha_{i+1,m+n-q} \otimes s\alpha_{m-q+1,m+n} \right) \downarrow a_{m+n+1} \right)
\]

where $\epsilon = m+n-p-q+1$ and $g(s\alpha_{1,n}) := \sum_j c^j_0 \otimes s\alpha_{1,q}$. Then it follows from Lemma 2.5 that the right hand side of the above identity equals to $(-1)^\epsilon \delta(f) \cup g(s\alpha_{1,m+n+1})$. Therefore, $\delta(f \cup g) = \delta(f) \cup g + (-1)^{m-p} f \cup \delta(g)$. \hfill \qed

Since $\theta_p(f) \cup g = f \cup \theta_q(g) = \theta_{p+q}(f \cup g)$, the cup product (still denoted by $\cup$) is well-defined on $C^*_{sg}(A, A)$.

**Proposition 4.2.** The complex $(C^*_{sg}(A, A), \delta)$, equipped with the cup product $\cup$, is a dg (unital associative) algebra.

**Proof.** Since the cup product is associative, this proposition follows from Lemma 4.1. \hfill \qed

### 4.2 Lie bracket

For any $m, n, p, q \in \mathbb{Z}_{\geq 0}$, we define the **Lie bracket**
\[
[\cdot, \cdot] : C^{m-p}(A, \Omega^p_{nc}(A)) \otimes C^{n-q}(A, \Omega^q_{nc}(A)) \to C^{m+n-p-q-1}(A, \Omega^{p+q}_{nc}(A)). \tag{4.2}
\]
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Figure 4: The treelike and cactus-like presentations of $f \circ_i g$ for $1 \leq i \leq m$. For simplicity, the projection $\pi : A \rightarrow sA$ is omitted in the cactus-like presentation.

as follows. For any $f \in C^{m-p}(A, \Omega^p_{nc}(A))$ and $g \in C^{n-q}(A, \Omega^q_{nc}(A))$, denote

$$f \circ_i g := \begin{cases} (f \otimes \text{id}_{sA}^\otimes)(\text{id}_{sA}^{\otimes i-1} \otimes g \otimes \text{id}_{sA}^{\otimes m-i}) & \text{for } 1 \leq i \leq m, \\ (\text{id}_{A} \otimes \text{id}_{sA}^{\otimes i-1} \otimes g \otimes \text{id}_{sA}^{\otimes p+i})(f \otimes \text{id}_{sA}^{\otimes m-i}) & \text{for } -p \leq i \leq -1, \end{cases}$$

where $g := (\pi \otimes \text{id}_{sA}^\otimes)g$ and $\pi : A \rightarrow sA$ is the canonical projection of degree $-1$. We set

$$f \circ g := \sum_{i=1}^{m} (-1)^{(n-q-1)(i-1)} f \circ_i g - \sum_{i=-p}^{-1} (-1)^{(n-q-1)(i-m-p-1)} f \circ_{-i} g.$$ 

The Lie bracket is given by $[f, g] := f \circ g - (-1)^{(m-p-1)(n-q-1)} g \circ f$. When $p = q = 0$, the Lie bracket $[\cdot, \cdot]$ coincides with the classical Gerstenhaber bracket (cf. Section 2.2) on $C^*(A, A)$. Since $\theta_p(f) \circ g = f \circ \theta_q(g) = \theta_{p+q}(f \circ g)$, the circle product is well-defined on $C^*_{\text{sg}}(A, A)$ and so is the Lie bracket $[\cdot, \cdot]$. Figures 4 and 5 illustrate the graphic presentations of the circle product. Clearly, $[\cdot, \cdot]$ is graded skew-symmetric:

$$[f, g] = -(-1)^{|f|-1(|g|-1)} [g, f].$$

We observe that the differential $\delta$ of $C^*_{\text{sg}}(A, A)$ can be expressed by the Lie bracket $[\cdot, \cdot]$ and the multiplication $\mu$ of $A$, namely,

$$\delta(f) = [\mu, f], \quad (4.3)$$

for any $f \in C^*_{\text{sg}}(A, A)$.

Remark 4.3. Readers may note that the multiplication $\mu$ is not in $C^2(A, A)$. Recall that we have a natural projection $\pi : A \rightarrow \overline{A}$. We take a $k$-linear split injection $\nu : \overline{A} \hookrightarrow A$ such that $\pi \nu = \text{id}_A$. Denote $\overline{\mu} := \nu(\mu \otimes \nu)$ in $C^2(A, A)$. Then we have $\delta(f) = [\overline{\mu}, f]$ since $\delta(f)$ is independent on the choice of the split injection $\nu$. By abuse of notation, we write $\delta(f) = [\mu, f]$.
Firstly, let us denote the left hand side of the above identity by $f$. Here $m$.

From a straightforward computation, we get

$$\forall (r, q) : B_r(\Delta, i) = \sum_{i=1}^{m+1} B_i^{<0}(f, g).$$

Secondly, we deal with the first term $\sum_{i=1}^{m+1} B_i^{>0}(f, g)$. For $0 \leq i \leq m$, set

$$C_i^{>0}(f, g) := \left(\mu \otimes \text{id}_{sA}^{>0+q}\right)\left(\text{id}_{sA} \otimes f \otimes \text{id}_{sA}^{>q}\right)\left((1 \otimes s\bar{\sigma}_{i+1, i} \otimes g(\bar{\sigma}_{i+1, i+n}) \otimes \bar{\sigma}_{i+n+1, m+n}\right).$$

From a straightforward computation, we get $B_i^{>0}(f, g) = C_i^{>0}(f, g) - C_{i-1}^{>0}(f, g)$ for $1 \leq i \leq m$. Since $B_{m+1}^{>0}(f, g) = (-1)^{ms}\delta(f) \circ_{m+1} g$, we have

$$\sum_{i=1}^{m+1} B_i^{>0}(f, g) = (-1)^{ms}\delta(f) \circ_{m+1} g + C_{m+1}^{>0}(f, g) - C_{0}^{>0}(f, g). \quad (4.4)$$

Finally, we need to simplify the second term $\sum_{i=1}^{p} B_{<-i}^{<0}(f, g)$. For $0 \leq i \leq p$, we set

$$C_i^{<0}(f, g) := (-1)^{(s-1)(i-r-1)+r-1-1}
\sum_j \left(\bigotimes_j \bar{\sigma}_{i+1, i} \otimes \text{id}_{sA}^{>0+q-i}\right)\left(s\bar{\sigma}_{i+1, i+p} \otimes s\bar{\sigma}_{m+1, m+n}\right).$$
where \( f(s\pi_{1,m}) := \sum_j c_j^i \otimes \pi_{1,p} \). For \( 1 \leq i \leq p \), we have \( B_{\leq i}^0(f,g) = C_{i-1}^0(f,g) - C_{i-1}^{<0}(f,g) \). Thus we get
\[
\sum_{i=1}^p B_{\leq i}^0(f,g) = C_{p}^{<0}(f,g) - C_{0}^{<0}(f,g).
\] (4.5)

Since \( C_{0}^{>0}(f,g) = (-1)^{r-1}(f \cup g)(s\pi_{1,m+n}) \) and \( C_{0}^{<0}(f,g) = (-1)^{s(r-1)}(g \cup f)(s\pi_{1,m+n}) \), combining (4.4) and (4.5), we obtain
\[
B(f,g) = (-1)^{r-1}(f \cup g - (-1)^{(s-1)(r-1)} g \cup f)(s\pi_{1,m+n})
\]
\[
+ (-1)^{ms} \delta(f) \circ_{m+1} g + C_{m}^{>0}(f,g) - C_{p}^{<0}(f,g).
\] (4.6)

From (4.6), it is enough to verify \((-1)^{ms} \delta(f) \circ_{m+1} g = C_{p}^{<0}(f,g) - C_{m}^{>0}(f,g)\). This identity follows from a straightforward computation. This proves the proposition.

**Corollary 4.5.** The cup product \( \cup \) in \( \text{HH}_{sg}^* (A, A) \) is graded commutative.

**Proof.** It is a direct consequence of Proposition 4.4.

**Remark 4.6.** Recall that the usual cup product (we denote it by \( \cup \) in this paper)
\[
\cup : C^m (A, N) \otimes C^m (A, M) \rightarrow C^{m+n} (A, M \otimes_A N)
\]
is given by \( f \cup g(s\pi_{1,m+n}) = g(s\pi_{1,n}) \otimes_A f(s\pi_{n+1,m+n}) \) for any two \( A\)-\( A \)-bimodules \( M \) and \( N \). In particular, we have
\[
\cup : C^m (A, \Omega^p_{nc} (A)) \otimes C^m (A, \Omega^q_{nc} (A)) \rightarrow C^{m+n} (A, \Omega^{p+q}_{nc} (A))
\]
by the canonical isomorphism \( \Omega^p_{nc} (A) \otimes_A \Omega^q_{nc} (A) \cong \Omega^{p+q}_{nc} (A) \). Generally, \( f \cup g \) is not equal to \( f \cup g \) in \( C^{m+n} (A, \Omega^{p+q}_{nc} (A)) \). Since \( \cup \) is not compatible with the maps \( \theta_p \), it is not well-defined in \( \text{HH}_{sg}^* (A, A) \). In this sense, the cup product \( \cup \) may be more interesting than the usual one \( \cup' \).

**Proposition 4.7.** Let \( A \) be a Noetherian \( k \)-algebra. Then the map \( \Phi_* : \text{HH}_{sg}^* (A, A) \rightarrow \text{Ext}_{A^p \otimes A^p}^* (A, A) \) (cf. Theorem 3.6) is an isomorphism of graded algebras, where the algebra structure on \( \text{Ext}_{A^p \otimes A^p}^* (A, A) \) is given by the Yoneda product.

**Proof.** For any \( f \in C^{m-p} (A, \Omega^p (A)) \) and \( g \in C^{n-q} (A, \Omega^q (A)) \), by a similar computation as in the proof of Proposition 4.4, we have
\[
f \cup g - f \cup' g = \sum_{i=1}^q (-1)^{(m-p-1)(i-1)} \delta(g \circ_{-i} f - \delta(g) \circ_{-i} f - (-1)^{n-q-1} g \circ_{-i} \delta(f)).
\]

Thus \( \cup \) coincides with \( \cup' \) at the cohomology level. Therefore, this proposition follows from the fact that the usual cup product \( \cup' \) corresponds to the Yoneda product.
Proposition 4.8. \((C^*_{sg}(A, A), \delta, \{\cdot, \cdot\})\) is a dg Lie algebra of degree \(-1\).

Proof. It follows from Proposition 4.4 that \(\{\cdot, \cdot\}\) is compatible with the differential \(\delta\). Namely, we have
\[
\delta([f, g]) = (\delta)_{|f|-1} [f, \delta(g)] + [\delta(f), g]
\]
for any \(f, g \in C^*_sg(A, A)\).

It is sufficient to verify the Jacobi identity,
\[
(-1)^{|f|-1}|h|-1 [f, [g, h]] + (-1)^{|f|-1}|g|-1 [g, [h, f]] + (-1)^{|g|-1}|h|-1 [h, [f, g]] = 0,
\]
where we recall that \(|f|\) is the degree of \(f\). Note that to verify the Jacobi identity is equivalent to verify the so-called pre-Lie identity (cf. [Ger63]),
\[
f \circ (g \circ h) - (f \circ g) \circ h = (-1)^{|g|-1}|h|-1 (f \circ (h \circ g) - (f \circ h) \circ g).
\]

From Theorem 5.1 and the identity (5.2) in the following, we have
\[
(f \circ g) \circ h - f \circ (g \circ h) = (-1)^{|g|-1}|h|-1 f \{g, h\} + f \{h, g\},
\]
where \(f \{g, h\}\) is the brace operation on \(C^*_sg(A, A)\). Roughly speaking, the summands of \(f \{g, h\}\) consist of tree-like graphs with three vertices \(f, g\) and \(h\) such that the special output (i.e. the blue output) is given by \(f\), and the level of \(g\) is higher than the level of \(h\) (cf. Figure 6). This yields the pre-Lie identity.

\section{5 \(B_\infty\)-algebra and Deligne's conjecture on \(C^*_sg(A, A)\)}

Throughout this section, we fix an associative algebra \(A\) over a field \(k\). The aim of this section is to prove the following two results.

Theorem 5.1. There is a \(B_\infty\)-algebra structure on \(C^*_sg(A, A)\) such that the normalized Hochschild cochain complex \(C^*(A, A)\) is a \(B_\infty\)-subalgebra.
Theorem 5.2. The complex $C^*_sg(A, A)$ is an algebra over the operad of chains of the little 2-disc operad. Equivalently, the Deligne’s conjecture holds for $C^*_sg(A, A)$.

Combining Propositions 4.2 and 4.8 and Theorem 5.1, we obtain the following result.

Corollary 5.3. Let $A$ be a $k$-algebra. Then the Tate-Hochschild cohomology $\text{HH}^*_sg(A, A)$, equipped with the cup product $\cup$ and Lie bracket $[\cdot, \cdot]$, is a Gerstenhaber algebra. Moreover, the natural map $\rho : \text{HH}^*(A, A) \to \text{HH}^*_sg(A, A)$ (cf. Remark 3.3) is a morphism of Gerstenhaber algebras.

Throughout this section, we consider the opposite cup product $f \cup^\text{op} g := (-1)^{|f||g|} g \cup f$ on $C^*_sg(A, A)$. Since $\cup$ is graded commutative on $\text{HH}^*_sg(A, A)$, we have $\cup = \cup^\text{op}$.

5.1 $B_{\infty}$-algebras

The brace operations on $C^*(A, A)$, described by Kadeishvili [Kad] and Getzler [Get93], are a natural generalization of the Gerstenhaber circle product $\circ$ (cf. Section 2.2).

Definition 5.4. For $f \in C^m(A, A)$ and $g_i \in C^n(A, A)$ where $i = 1, \cdots, k$, the brace operation is defined as

$$f\{g_1, \cdots, g_k\}(s\bar{a}_{1,N}) = \sum_{1 \leq i_1 \leq \cdots \leq i_k \leq m\atop i_j + n_j \leq i_{j+1}} (-1)^{\epsilon} f(s\bar{a}_{1,i_1}, g_i(s\bar{a}_{i_1+1,i_1+n_1}), \cdots, s\bar{a}_{i_k}, g_k(s\bar{a}_{i_k+1,i_k+n_k}), \cdots, s\bar{a}_N)$$

where $\epsilon := \sum_{j=1}^k (n_j - 1)i_j$ and $N = m + \sum_{i=1}^k n_k - k$. Recall that $\bar{g}_i := \pi \circ g_i$ where $\pi : A \to sA$ is the natural projection of degree $-1$.

Obviously, the brace operation $f\{g_1, \cdots, g_k\}$ is of degree $-k$ and $f\{g_1\} = f \circ g_1$.

Definition 5.5 ([Bau]). A $B_{\infty}$-algebra structure on a graded vector space $V := \bigoplus_{n \in \mathbb{Z}} V^n$ is the structure of a dg bialgebra on the tensor coalgebra $(T(sV) := \bigoplus (sV)^{\otimes p}, \Delta)$ such that the element $1 \in k = (sV)^{\otimes 0}$ is the unit of $T(sV)$. Here $\Delta : T(sV) \to T(sV) \otimes T(sV)$ is defined by

$$\Delta(sa_1 \otimes \cdots \otimes sa_p) = \sum_{i=0}^p (sa_1 \otimes \cdots \otimes sa_i) \otimes (sa_{i+1} \otimes \cdots \otimes sa_p).$$

Since the tensor coalgebra is cofree and both the differential $D : T(sV) \to T(sV)$ and the product $m : T(sV) \otimes T(sV) \to T(sV)$ are compatible with the coproduct, they are determined by a collection of $k$-linear maps $D_p : (sV)^{\otimes p} \to sV$ of degree 1 and $m_{p,q} : (sV)^{\otimes p} \otimes (sV)^{\otimes q} \to (sV)^{\otimes p+q}$ of degree zero for $p, q \in \mathbb{Z}_{\geq 0}$, subject to some relations (called $B_{\infty}$-relations) [Vor, Section 2.2].
On \( C^*(A, A) \), we take \( D_1 = \delta, D_2 = \cup^{op} \) and \( D_p = 0 \) for \( p \neq 1, 2 \). Let \( m_{1,0} = m_{0,1} = \text{id} \) and \( m_{1,q} \) be the brace operation. For other \( p, q \), we set \( m_{p,q} = 0 \). Then this collection \((D_p, m_{p,q})\) defines a \( B_\infty \)-algebra structure on \( C^*(A, A) \) (cf. [Vor, Theorem 3.1]). In this case, the \( B_\infty \)-algebra relations are simplified as follows.

1. \((C^*(A, A), D_1, D_2)\) is a dg associative algebra.

2. Higher pre-Jacobi identities.

\[
x \{ y_{1,m} \} \{ z_{1,n} \} = \sum_{0 \leq i_1 \leq \cdots \leq i_m \leq n} (-1)^{\epsilon} x \{ z_{1,i_1}, y_1 \{ z_{i_1+1}, \cdots, z_{i_m}, y_m \{ z_{i_m+1}, \cdots, z_n \} \}
\]

where \( \epsilon := \sum_{p=1}^{m} \left( (|y_p| - 1) \sum_{q=1}^{i_p} (|z_q| - 1) \right) \).

3. Distributivity.

\[
(x_1 \cup x_2) \{ y_{1,n} \} = \sum_{k=0}^{n} (-1)^{|x_2|} \sum_{p=1}^{k} \sum_{q=0}^{n-p} (|y_p| - 1) (x_1 \{ y_{1,k} \} \cdot (x_2 \{ y_{k+1,n} \}),
\]

4. Higher homotopies.

\[
\delta(x \{ y_{1,l} \}) = (-1)^{|x|(|y_1| - 1)} y_1 \cdot (x \{ y_{2,l} \}) + (-1)^{\epsilon_{l-1}} (x \{ y_{1,l-1} \}) \cdot y_l
\]

\[
= \delta(x \{ y_{1,l} \}) - \sum_{i=1}^{l-1} (-1)^{\epsilon_i} x \{ y_{1,i}, \delta(y_{i+1}), y_{i+2,l} \}
\]

\[
- \sum_{i=1}^{l-2} (-1)^{\epsilon_{i+1}+1} x \{ y_{1,i}, y_{i+1}, y_{i+2}, y_{i+3,l} \},
\]

where \( \epsilon_i := |x| + \sum_{p=1}^{i} (|y_p| - 1) \). For simplicity, we denote \( x \cdot y := x \cup^{op} y \).

**Remark 5.6.** Conversely, a collection of \( k \)-linear maps \((D_p, m_{p,q})\) on a graded space \( V \) with \( D_p = 0, p \neq 1, 2 \) and \( m_{p,q} = 0, p > 1 \), satisfying the above relations (1)-(4), defines a \( B_\infty \)-algebra structure on \( V \). For more details on brace operations and \( B_\infty \)-algebras, one may refer to [Vor, Kau07a, Kell, MaShSt].

### 5.2 Brace operations on \( C^*_{sg}(A, A) \)

In this section, we will extend brace operations on \( C^*(A, A) \) to \( C^*_{sg}(A, A) \), using the cactus-like presentations. We prove that the brace operations, with the opposite cup product \( \cup^{op} \), define a \( B_\infty \)-algebra structure on \( C^*_{sg}(A, A) \).

Fix \( k \in \mathbb{Z}_{\geq 1} \). Let \( f' \in C_{sg}^{m'}(A, A) \) and \( g_i' \in C_{sg}^{n_i'}(A, A) \) for \( m', n_i' \in \mathbb{Z} \) and \( i = 1, 2, \cdots, k \). Take representatives \( f \in C^{m-p}(A, \Omega_{\text{uc}}^p) \) of \( f' \) and \( g_i \in C_{sg}^{m_i-q_i}(A, \Omega_{\text{uc}}^{q_i}) \) of \( g_i' \),
respectively. Here we have \( m - p = m' \) and \( n_i - q_i = n'_i \) for \( i = 1, \ldots, k \). The brace operation \( f' \{ g'_1, \ldots, g'_k \} \) is defined as

\[
f' \{ g'_1, \ldots, g'_k \} = \sum_{0 \leq j \leq k} (-1)^{\epsilon} B_{(l_1, \ldots, l_{k-j})}^{(i_1, \ldots, i_j)}(f; g_1, \ldots, g_k),
\]

where \( B_{(l_1, \ldots, l_{k-j})}^{(i_1, \ldots, i_j)}(f; g_1, \ldots, g_k) \) is illustrated in Figure 8, and

\[
\epsilon := \sum_{r=1}^{j} (n'_r - 1)(i_r - r + n'_1 + n'_2 + \cdots + n'_{r-1}) + k - j
\]

\[
\quad + \sum_{r=1}^{k-j} (n'_{r+j} - 1)(l_{k-j+1-r} + m' + n'_1 + \cdots + n'_{r+j-1} + r + j).
\]

Let us describe the summand \( B_{(l_1, \ldots, l_{k-j})}^{(i_1, \ldots, i_j)}(f; g_1, \ldots, g_k) \) in detail. Firstly, we fix an integer array \( (i_1, \ldots, i_j; l_1, \ldots, l_{k-j}) \), where \( 0 \leq j \leq k, 1 \leq i_1 < \cdots < i_j \leq m \) and \( 1 \leq l_1 \leq l_2 \leq \cdots \leq l_{k-j} \leq p \). Secondly, we use the cell on the left in Figure 7. We put \( f \) into the circle 1 and \( g_i \) into the circle \( i + 1 \), respectively. The inputs and outputs are then placed according to Process 3.4 described in Section 3.1, as shown in Figure 8. For each \( 1 \leq r \leq j \), the zero point (i.e. blue dot) of the circle \( g_r \) is connected with the \( i_r \)-th radius in the left semi-circle of \( f \) via a red curve. For each \( 1 \leq r \leq k - j \), the zero point of the circle \( g_{j+r} \) is connected with the open arc between the \( (l_{k-j+r+1} - 1) \)-th and \( l_{k-j+r+1} \)-th radii in the right semi-circle via a red curve. Thirdly, we need to identify some inputs with outputs. For each \( 1 \leq r \leq j \), add a dashed arrow from the zero point of \( g_r \) to the \( i_r \)-th radius. Starting from the global zero point (i.e. the zero point of \( f \)), walk clockwise along the red path (i.e. the outside circles and the red curves) and record the inputs and outputs (including the special outputs of \( g_i \)) in order as a sequence. When an input is found closely behind an output in this sequence, we call this pair out-in. Let us define a process.

**Process 5.7.** Once the pair out-in appears in the sequence, we add a dashed arrow from the corresponding output to input in the graph. Delete this pair and renew the sequence. Then repeat the above operations until no pair out-in left.
After applying this Process, we obtain a final sequence with all inputs preceding all outputs. Finally, we translate the updated cactus-like graph into a treelike graph by putting the inputs (in the final sequence) on the top and outputs on the bottom (e.g. Figure 9). We therefore get the $k$-linear map $B_{(i_1, \cdots, i_k-j)}^{(j_1, \cdots, j_k)}(f; g_1, \cdots, g_k)$ from $sA \otimes_t sA$ to $A \otimes_t sA$, where $s$ and $t$ are the numbers of the input and output in the final sequence, respectively.

From Lemma 5.11 below, it follows that the brace operation $f' \{g_1', \cdots, g_k'\}$ is well-defined, namely, it does not depend on the choice of representatives $f, g_1, \cdots, g_k$.

**Remark 5.8.** If $f \in C^m(A, A)$ and $g_i \in C^m_i(A, A)$ for $1 \leq i \leq k$, we recover the original brace operation $f \{g_1, \cdots, g_k\}$ on $C^*(A, A)$ (cf. Definition 5.4). The cup product $\cup^{op}$ on $C^*_sg(A, A)$ can be interpreted as

$$f \cup^{op} g = \mu\{f, g\} \quad (5.6)$$

for $f, g \in C^*_sg(A, A)$, where $\mu$ is the multiplication of $A$ (cf. Remark 4.3).

**Proof of Theorem 5.1.** From Proposition 4.2 and Remark 5.6 it is sufficient to verify the identities (5.2)-(5.4) for $C^*_sg(A, A)$. From (4.3) and (5.6) it follows that (5.4) is a special case of the identity (5.3). Hence it remains to check (5.2) and (5.3) for $C^*_sg(A, A)$. The verifications can be done directly by graphic presentations. This proves the theorem. 

---

**Figure 8:** A summand $B_{(i_1, \cdots, i_k-j)}^{(j_1, \cdots, j_k)}(f; g_1, \cdots, g_k)$ in the brace operation $f \{g_1, \cdots, g_k\}$. The diagram illustrates the graphical representation of the brace operation.
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5.3 An action of $CC_*(\mathcal{C} act)$ on $C^*_sg(A, A)$

In this section we will generalize the brace action to any cell in the cellular chain model $CC_*(\mathcal{C} act)$.

In the series of papers [Kau05, Kau07b, Kau08], the author introduced the (topological) operad $\mathcal{C} act$ of spineless cacti. He constructed a natural action of the cellular chain model $CC_*(\mathcal{C} act)$ on $C^*(A, A)$. Let $Brace$ be the dg suboperad of the endomorphism operad $\text{Endop}(C^*(A, A))$, generated by the cup product and brace operations on $C^*(A, A)$. The author proved that $CC_*(\mathcal{C} act)$ is isomorphic to $Brace$ (cf. [Kau07a, Proposition 4.9]), and equivalent to the operad of chains of the little 2-discs operad (cf. [Kau07a, Theorem 3.11]). As a conclusion, he provided a proof of Deligne’s conjecture for $C^*(A, A)$.

From the above analysis, any cell in $CC_*(\mathcal{C} act)$ induces an action on $C^*(A, A)$. For instance, the cell on the left in Figure 7 corresponds to the brace operation of degree $-k$ while the cell on the right corresponds to the (opposite) cup product $\cup^{op}$. More generally,
any cell in $CC_*(\mathcal{C} act)$ can be represented by a cactus-like graph as shown in Figure 10. Explicitly, the zero point of a circle is indicated by the blue dot. By a (red) curve connecting two circles, we mean that the two circles intersect at the endpoints of the curve (called the intersection point). Note that at least one endpoint of each curve should coincide with the zero point. In other words, we do not allow that two circles intersect at non-zero points. We allow that three or more circles intersect at one common point. For a cell, there is only one global zero point (called root) which may or may not be an intersection point. In fact, the root is the only zero point which is not necessary to be an intersection point. If the root indeed is an intersection point, the other endpoint(s) of the red curve(s) must be zero point(s). Note that the degree (or dimension) of the cell equals the number of the intersection points (except the global zero point).

Remark 5.9. The cactus-like presentations of cells in $CC_*(\mathcal{C} act)$ described above are slightly different from the ones in the original papers [Kau05, Kau07a]. We use red curves to indicate the intersection points. This modification could make it more convenient to define the action of a cell on $C^*_{sg}(A, A)$. For more details on $\mathcal{C} act$ and $CC_*(\mathcal{C} act)$, refer to [Kau05, Kau07a].

Let us now generalize the brace operations to any cell in $CC_*(\mathcal{C} act)$. Let $\tau$ be a cell in $CC_*(\mathcal{C} act(k))$ of degree $l \in \mathbb{Z}_{\geq 0}$ (e.g. Figure 10). Take any $k$ elements $f'_i \in C^m_{sg}(A, A)$ for $1 \leq i \leq k$, the action of $\tau$ on $f'_1 \otimes \cdots \otimes f'_k$, denoted by $\tau(f'_1, \cdots, f'_k)$, is defined as follows.

First step: Choose a representative $f_i \in C^{m_i-p_i}(A, \Omega_{nc}^{p_i}(A))$ for each $f'_i$, where $m_i - p_i = m'_i$ for $1 \leq i \leq k$. We put $f_i$ into the $i$-th circle of $\tau$ according to Process 3.4.
Recall that there is only one nonzero endpoint of the red curves at any intersection point (except the global zero point). We need to fix the type of an intersection point by moving the nonzero endpoint of the red curves along the circle so that it either coincides with the $j$-th radius (i.e. input, $1 \leq j \leq m_i$) of $f_i$, or located in the open arc between the $(l - 1)$-th and $l$-th radii (i.e. outputs, $1 \leq l \leq p_i$) of $f_i$. Accordingly, we say that the intersection point has type $j$ or $-l$. If an intersection point contains more than one red curves, we multiply in order all the corresponding special outputs (i.e. blue radii), and then get a new output. We stress that when moving the endpoints of curves along a fixed circle, the order (starting from the zero point in clockwise) of the intersection points must be preserved.

Once the types of all intersection points are fixed, we arrange them as an integer array, labelled by curves. This sequence is called a type of $\tau$. Denote the set of all intersection points (except the global zero point) by $I(\tau)$. Clearly, a type is a map from $I(\tau)$ to $\mathbb{Z}^l$, where $l$ is the degree of $\tau$. We denoted by $\mathcal{S}(\tau)(f_1, \cdots, f_k)$ the set of all types of $\tau$ associated with $f_1 \otimes \cdots \otimes f_k$.

**Second step:** For any fixed type $\Phi \in \mathcal{S}(\tau)(f_1, \cdots, f_k)$, we need to add dashed arrows from inputs to outputs. Starting from the global zero point, walk along the red path (i.e. outside circles and the red curves); record the inputs and outputs, except those already connected by dashed arrows, as a sequence. We apply Process 5.7 to get a final cactus-like graph and a sequence in which all inputs precede outputs.

**Third step:** By translating the above cactus-like graph into the treelike graph, we get a $k$-linear map $\tau(\Phi; f_1, \cdots, f_k) : sA^{\otimes s} \to A \otimes sA^{\otimes t}$, where $s$ and $t$ are the numbers of inputs and outputs in the final sequence, respectively. It is clear that $s - t = \sum_{i=1}^{k} m'_i - l$.

Therefore, we have the following definition.
Definition 5.10. The action of $\tau \in CC_l(C_{act}(k))$ on $f_1' \otimes \cdots \otimes f_k'$ is defined as

$$
\tau(f_1', \cdots, f_k') := \sum_{\Phi \in \mathcal{T}(\tau)(f_1, \cdots, f_k)} (-1)^{\epsilon(\Phi)} \tau(\Phi; f_1, \cdots, f_k),
$$

where the sign $(-1)^{\epsilon(\Phi)}$ is determined by signs in brace operations since $\tau$ is generated by cells corresponding to the cup product and brace operations.

Lemma 5.11. $\tau(f_1', \cdots, f_k')$ is well-defined, namely, it does not depend on the choice of representatives $f_1, \cdots, f_k$.

Proof. Since the action $\tau$ on $f_1, \cdots, f_k$ can be written as the (opposite) cup product and compositions of brace operations, it is sufficient to prove that $\cup^{op}$ and brace operations are independent of the choice of representatives. From Proposition 4.2, it follows that the cup product $\cup^{op}$ is well-defined on $C^*_{sg}(A,A)$. Thus it remains to check the following identities on $C^*_{sg}(A,A)$,

$$
\begin{align*}
 f\{g_1, \cdots, g_k\} &= (f \otimes \text{id}_{sA})\{g_1, \cdots, g_k\} \\
 f\{g_1, \cdots, g_k\} &= f\{g_1, \cdots, g_i \otimes \text{id}_{sA}, \cdots, g_k\}
\end{align*}
$$

where $1 \leq i \leq k$ and $f \in C^{m-p}(A, \Omega^p_{nc}(A)), g_i \in C^{m-q_i}(A, \Omega^q_{nc}(A))$. Let us check the first identity. Observe that all the terms on the left hand side are cancelled out by terms on the right hand side. We need to cancel out the remaining terms on the right hand side. Note that the cactus-like presentation of each remaining term has the following property: there is a red curve connecting with the $(m+1)$-th input or the open arc between the $p$-th and $(p+1)$-th outputs of $f$. Assume that the circle $g_j$ intersects with $f$ at the $(m+1)$-th input via a red curve (cf. the left graph in Figure 12), this term will cancel with the one
whose cactus-like presentation is obtained by just moving the red chord into the open arc between \(p\)-th and \((p+1)\)-th output of \(f\) (cf. the right graph in Figure 12). In this way, all the remaining terms cancel out. This verifies the first identity. The second identity can be verified by the same argument. This proves the lemma.

\[ \delta( \cdots ) = \quad + \sum_{i=2}^{k} \delta( \cdots ) \quad + \quad \cdots \]

**Figure 13:** The differential in \(CC^*(\mathcal{C}_{\text{act}})\).

Proof of Theorem 5.2 Since \(CC^*_s(\mathcal{C}_{\text{act}})\) is equivalent to the operad of chains of the little 2-discs operad (cf. [Kau07a, Proposition 4.9]), it is sufficient to prove that the action of \(CC^*_s(\mathcal{C}_{\text{act}})\) (cf. Definition 5.10) induces a morphism of dg operads \(\varphi : CC^*_s(\mathcal{C}_{\text{act}}) \to \text{End}(C^*_{\text{sg}}(A, A))\). It is not difficult to show that \(\varphi\) is compatible with the compositions. Let us prove that \(\varphi\) is compatible with the differentials. Since \(CC^*_s(\mathcal{C}_{\text{act}})\) is generated by the cells as shown in Figure 7, it is sufficient to check \(\varphi(\delta(\tau)) = \delta(\varphi(\tau))\), where \(\tau\) is the cell corresponding to the brace operation. From Figure 13 it follows that to prove the above identity is equivalent to prove (5.4). This proves the theorem.

\[ \delta( \cdots ) = \quad + \sum_{i=2}^{k} \delta( \cdots ) \quad + \quad \cdots \]

\[ \delta( \cdots ) = \quad + \sum_{i=2}^{k} \delta( \cdots ) \quad + \quad \cdots \]

6 An application to self-injective algebras

6.1 Generalized Tate-Hochschild complex

Before the case of self-injective algebras, let us start with a more general setting. Let \(A\) be an associative algebra over a field \(k\). Denote by \(A^\vee := \text{Hom}_{A \otimes A^{\text{op}}}(A, A \otimes A^{\text{op}})\). It is clear that \(A^\vee\) is isomorphic to the zeroth Hochschild cohomology \(\text{HH}^0(A, A \otimes A^{\text{op}})\). Thus we have

\[ A^\vee \cong \left\{ \sum_i x_i \otimes y_i \in A \otimes A \mid \sum_i ax_i \otimes y_i = \sum_i x_i \otimes y_ia, \text{ for any } a \in A \right\}, \]

where the isomorphism sends \(\alpha \in A^\vee\) to \(\alpha(1)\). Note that \(A^\vee\) has an \(A\)-\(A\)-bimodule structure: For any \(\sum_i x_i \otimes y_i \in A^\vee\) and \(a, b \in A\), the action is \(a \cdot (\sum_i x_i \otimes y_i) \cdot b := \sum_i x_i b \otimes ay_i\).
Recall that \((C_{s}(A,A^{\vee}),b)\) is the Hochschid chain complex with coefficients in \(A^{\vee}\). We now construct an unbounded complex

\[
\mathcal{D}^{*}(A,A) : \cdots \xrightarrow{b_{2}} C_{1}(A,A^{\vee}) \xrightarrow{b_{1}} A^{\vee} \xrightarrow{\mu} A \xrightarrow{\delta^{i}_{1}} C_{1}(A,A) \xrightarrow{\delta^{i}_{1}} \cdots
\]

(6.1)

where \(C_{i}(A,A)\) is in degree \(i\) and \(\mu : A^{\vee} \rightarrow A\) is given by the multiplication of \(A\). Let us denote the cohomology of \(\mathcal{D}^{*}(A,A)\) by \(\text{TH}^{*}(A,A)\).

**Lemma 6.1.** There is a natural embedding of complexes \(i^{*} : \mathcal{D}^{*}(A,A) \hookrightarrow C^{*}_{\text{sg}}(A,A)\).

**Proof.** For \(i \geq 0\), it is known from Definition 3.2 that \(C_{i}^{*}(A,A)\) is a subspace of \(C_{\text{sg}}^{i}(A,A)\). For \(i < 0\), we define a map \(i^{i} : C_{i-1}(A,A^{\vee}) \rightarrow C^{i}_{\text{sg}}(A,A)\) by the following formula

\[
i^{i}(\alpha) := \sum_{j} x_{j} \otimes \bar{a}_{1,-i-1} \otimes y_{j} \in \Omega_{\text{nc}}^{i}(A) \subset C^{i}_{\text{sg}}(A,A),
\]

where \(\alpha = \left(\sum_{j} x_{j} \otimes y_{j}\right) \otimes \bar{a}_{1,-i-1}\). Then we need to check \(i^{*} \circ \partial = \partial^{*} \circ i^{*}\). For \(i \geq 0\), it is clear that \(i^{i+1} \circ \partial^{i} = \delta^{i} \circ i^{i}\). For \(i < 0\), we claim \(i^{0} \circ i^{i+1} \circ \partial^{i} = \delta^{i} \circ i^{i}\). Indeed,

\[
(\theta_{0} \circ i^{i+1} \circ \partial^{i}(\alpha))(\bar{b}) = (-1)^{i-1} \sum_{j} (x_{j} \otimes \bar{a}_{1,-i-1}) \triangledown y_{j} \otimes \bar{b}
= (-1)^{i} \sum_{j} (x_{j} \otimes \bar{a}_{1,-i-1} \otimes y_{j}) \triangledown \bar{b} - (-1)^{i} x_{j} \otimes \bar{a}_{1,-i-1} \otimes y_{j} \bar{b}
= \delta^{i} \circ i^{i}(\alpha)(\bar{b}),
\]

where the second identity follows from Lemma 2.5 and the third one follows from \(\sum_{j} x_{j} \otimes y_{j} \bar{b} = \sum_{j} b_{x_{j}} \otimes y_{j}\). This proves the lemma. \(\square\)

### 6.2 \(*\)-product on \(\mathcal{D}^{*}(A,A)\)

Let \(A\) be an associative algebra (not necessarily, self-injective) over a field \(k\). We construct a product (of degree zero), called \(*\)-product, on \(\mathcal{D}^{*}(A,A)\)

\[
* : \mathcal{D}^{*}(A,A) \otimes \mathcal{D}^{*}(A,A) \rightarrow \mathcal{D}^{*}(A,A),
\]

(6.2)

which extends the cup product on \(C^{*}(A,A)\) and the cap product between \(C^{*}(A,A)\) and \(C_{*}(A,A^{\vee})\).

(i) For \(p, q \geq 0\), define \(* : C_{p}(A,A^{\vee}) \otimes C_{q}(A,A^{\vee}) \rightarrow C_{p+q+1}(A,A^{\vee})\) by

\[
\alpha \ast \beta = \sum_{i,j} (x_{i}'x_{i} \otimes y_{j}') \otimes s_{\bar{a}_{1,p}} \otimes s_{\bar{b}_{1,q}},
\]

where \(\alpha = (\sum_{i} x_{i} \otimes y_{i}) \otimes s_{\bar{a}_{1,p}}\) and \(\beta = \left(\sum_{j} x_{j}' \otimes y_{j}'\right) \otimes s_{\bar{b}_{1,q}}.\)
(ii) For \( m, p \in \mathbb{Z}_{\geq 0} \) such that \( p \geq m \), define \( \star : C_p(A, A^\vee) \otimes C^m(A, A) \to C_{p-m}(A, A^\vee) \) as the usual cap product. Namely, for \( f \in C^m(A, A) \) and \( \alpha = (\sum_i x_i \otimes y_i) \otimes s\bar{a}_{1,p} \), we have
\[
\alpha \star f := \sum_i (x_i \otimes f(s\bar{a}_{p-m+1,p})y_i) \otimes s\bar{a}_{1,p-m}.
\]
Similarly, we define \( \star : C^m(A, A) \otimes C_p(A, A^\vee) \to C^{m-p}_{-1}(A, A) \) by
\[
f \star \alpha := \sum_i (x_i f(s\bar{a}_{1,m}) \otimes y_i) \otimes s\bar{a}_{m+1,p}.
\]

(iii) For \( m, p \in \mathbb{Z}_{\geq 0} \) such that \( p < m \), define \( \star : C^m(A, A) \otimes C_p(A, A^\vee) \to C^{m-p-1}(A, A) \) by the following formula,
\[
f \star \alpha(s\bar{b}_{1,m-p-1}) := \sum_i f(s\bar{b}_{1,m-p-1} \otimes s\bar{x}_i \otimes s\bar{a}_{1,p})y_i.
\]
Similarly, \( \star : C_p(A, A^\vee) \otimes C^m(A, A) \to C^{m-p-1}(A, A) \) is defined by
\[
\alpha \star f(s\bar{b}_{1,m-p-1}) = \sum_i x_if(s\bar{a}_{1,p} \otimes s\bar{b}_i \otimes s\bar{b}_{1,m-p-1}).
\]

(iv) For \( m, n \in \mathbb{Z}_{\geq 0} \), we define \( f \star g := f \cup g \), for \( f \in C^m(A, A) \) and \( g \in C^n(A, A) \).

**Lemma 6.2.** The \( \star \)-product is compatible with the differential \( \partial \) in \( \mathcal{D}^*(A, A) \). As a result, it induces a well-defined product (still denoted by \( \star \)) on the cohomology \( TH^*(A, A) \).

**Proof.** This follows from straightforward computations. \( \square \)

**Remark 6.3.** In general, the \( \star \)-product restricted to the complex \( C_*(A, A^\vee) \) is not a chain map since \( \partial(\alpha \star \beta) \neq \pm \alpha \star \partial(\beta) \) if \( \alpha \in C_0(A, A^\vee) \). In order to make it well-defined, we have to extend the \( \star \)-product from \( C_*(A, A^\vee) \) to \( \mathcal{D}^*(A, A) \). Assume that \( A \) is a commutative symmetric algebra. The \( \star \)-product restricted to \( C_{\geq 0}(A, A) \) coincides with the so-called Abbaspour product (see [Abb] Theorem 6.1) motivated by certain operations in string topology. For more details and further investigation, one may refer to [RiWa].

**Remark 6.4.** In general, the \( \star \)-product on \( \mathcal{D}^*(A, A) \) is **not** associative although it is well-known that the associativity holds when restricted to either \( \mathcal{D}^{\geq 0}(A, A) \) or \( \mathcal{D}^{<0}(A, A) \). For instance, let \( \alpha := (\sum_i x_i \otimes y_i) \otimes s\bar{a}_{1,p}, \beta := (\sum_j x'_j \otimes y'_j) \otimes s\bar{b}_{1,q} \), and \( f \in C^m(A, A) \), where \( p, q > m > 0 \). We have
\[
(\alpha \star f) \star \beta - \alpha \star (f \star \beta) = \partial m_3(\alpha, f, \beta) - m_3(\partial(\alpha), f, \beta) - (-1)^{p-1}m_3(\alpha, \partial(f), \beta) - (-1)^{m-p-1}m_3(\alpha, f, \partial(\beta))
\]
where
\[
m_3(\alpha, f, \beta) = \sum_{i,j} \sum_{k=1}^{m} (-1)^{(m-1)(m-k-1)} (x_i' \otimes y_i') \otimes s\bar{\alpha}_{1,p-m+k} \otimes f(s\alpha_{p-m+k+1,p} \otimes s\bar{y}_i \otimes s\bar{b}_{1,k-1}) \otimes s\bar{b}_{k,p}.
\]

This means that the associativity holds up to homotopy. From this point of view, it might be interesting to ask whether this extends to an \(A_\infty\)-algebra structure with \((\partial, *, m_3, \cdots)\) on \(D^*(A, A)\). In [RiWa, Proposition 6.5], we give an affirmative answer to this question in the case where \(A\) is a (dg) symmetric algebra. For general cases, further investigations are needed.

**Proposition 6.5.** The *-product is graded commutative and associative on \(TH^*(A, A)\).

**Proof.** Let us first verify the graded commutativity in the following cases.

1. For \(\alpha \in C_p(A, A^\vee)\) and \(\beta \in C_q(A, A^\vee)\), denote
\[
\beta \bullet \alpha := \sum_{i,j} \sum_{k=1}^{p+1} (-1)^{qk} (x_i \otimes y_i) \otimes s\bar{\alpha}_{1,k-1} \otimes s\bar{x}_j \otimes s\bar{b}_{1,q} \otimes s\bar{y}_j \otimes s\bar{\alpha}_{k,p}.
\]
Then we have
\[
\partial(\beta \bullet \alpha) = (-1)^q \beta \star \alpha + \sum_{i,j} \sum_{k=1}^{p+1} \sum_{l=0}^{k-2} (-1)^{qk+l} (x_i \otimes y_i) \otimes s\bar{\alpha}_{1,l-1} \otimes s\bar{a}_l \bar{a}_{l+1}
\]
\[
\otimes s\bar{a}_{l+2,k-1} \otimes s\bar{x}_j \otimes s\bar{b}_{1,q} \otimes s\bar{y}_j \otimes s\bar{\alpha}_{k,p} + \sum_{i,j} \sum_{k=1}^{p+1} \sum_{l=0}^{p} (-1)^{q(k-1)+l}
\]
\[
(x_i \otimes y_i) \otimes s\bar{a}_{1,k-1} \otimes s\bar{x}_j \otimes s\bar{b}_{1,q} \otimes s\bar{y}_j \otimes s\bar{a}_{k,l-1} \otimes s\bar{a}_l \bar{a}_{l+1} \otimes s\bar{a}_{l+2,p}
\]
\[
-(-1)^{(p+1)(q+1)+q} \alpha \star \beta + \partial(\beta) \bullet \alpha
\]
\[
= (-1)^q (\beta \star \alpha - (-1)^{(p+1)(q+1)} \alpha \star \beta) + (-1)^q \beta \bullet \partial(\alpha) + \partial(\beta) \bullet \alpha.
\]
Thus on \(TH^*(A, A)\), we have \(\beta \star \alpha - (-1)^{(p-1)(q-1)} \alpha \star \beta = 0\).

2. For \(m, p \in \mathbb{Z}_{\geq 0}\) such that \(p \geq m - 1\), denote
\[
f \bullet \alpha := \sum_{i} \sum_{k=1}^{p-m+1} (-1)^{(m-1)k} (x_i \otimes y_i) \otimes s\bar{\alpha}_{1,k-1} \otimes f(s\alpha_{k,k+m-1}) \otimes s\bar{\alpha}_{k+m,p}.
\]
Then we have
\[
\partial(f \bullet \alpha) = (-1)^{m-1} (f \star \alpha - (-1)^{m(p-1)} \alpha \star f) + \partial(f) \bullet \alpha + (-1)^{m-1} f \bullet \partial(\alpha).
\]
Thus on \(TH^*(A, A)\), we have \(f \star \alpha - (-1)^{m(p-1)} \alpha \star f = 0\).
3. For \(m, p \in \mathbb{Z}_{\geq 0}\) such that \(p \leq m\), denote

\[
(f \cdot \alpha)(s\overline{b}_{1,m-p}) := \sum_{i} \sum_{k=1}^{m-p+1} (-1)^{pk+m-1} f(s\overline{b}_{1,k-1} \otimes sx_{i} \otimes \overline{s}_{a_{1,p}} \otimes s\overline{b}_{k,m-p}).
\]

By a similar computation, we have \(f \star \alpha - (-1)^{(p-1)m} \alpha \star f = 0\) on \(\text{TH}^*(A, A)\).

It remains to verify the associativity. Since \(\star\) is graded commutative on \(\text{TH}^*(A, A)\), it is enough to verify \((x \star y) \star z = x \star (y \star z)\) for \(x, y \in H^*(\mathcal{D}^{<0}(A, A))\) and for \(x, y \in H^*(\mathcal{D}^{\geq 0}(A, A))\). But for these two cases, from a direct computation it follows that the above identity already holds on \(\mathcal{D}^*(A, A)\). This proves the proposition.

\[\text{Corollary 6.6.} \ \overline{\iota}^* : \text{TH}^*(A, A) \to \text{HH}^*_{\text{sg}}(A, A) \text{ is a morphism of graded algebras.}\]

\[\text{Proof.} \ \text{Observe that} \ \iota^* \ \text{is compatible with the products} \ \star \ \text{and} \ \cup \ \text{at the cochain level. Thus the result follows from Proposition 6.5.}\]

\[\text{Remark 6.7.} \ \text{In general, the morphism} \ \overline{\iota}^* \ \text{is not an isomorphism. For instance, consider the radical square zero algebra} \ A = kQ/\langle Q_2 \rangle \ \text{of the quiver} \ Q \ \text{with only one vertex and two loops. We prove in [Wan, Section 5] that} \ \text{HH}^*_{\text{sg}}(A, A) \ \text{is of infinite dimension in each degree, while} \ \text{TH}^*(A, A) \ \text{is of finite dimension in each degree. Nevertheless, in the following section, we will prove that} \ \overline{\iota}^* \ \text{is an isomorphism if} \ A \ \text{is a self-injective algebra.}\]

### 6.3 The case of self-injective algebras

In this section, we fix a finite dimensional self-injective algebra \(A\) over a field \(k\). Recall that \(A\) is \textit{self-injective} if \(A\) itself is injective as a left (or equivalently, right) \(A\)-module. Clearly, symmetric algebras are naturally self-injective. Self-injective algebras play an important role in representation theory, mainly due to the fact that their stable module categories have a natural triangulated structure (cf. e.g. [Zim, Section 5.1.4]). Moreover, we have the following result.

\[\text{Theorem 6.8 ([Ric Theorem 2.1]).} \ \text{Let} \ A \ \text{be a self-injective algebra. Then the canonical functor} \ F_A : A\text{-mod} \to \mathcal{D}_{\text{sg}}(A) \ \text{is an equivalence between triangulated categories.}\]

Since \(A\) is self-injective, so is \(A \otimes A^{\text{op}}\). Thus from Theorem 6.8, it follows that there is an equivalence \(F_{A \otimes A^{\text{op}}} : (A \otimes A^{\text{op}})\text{-mod} \to \mathcal{D}_{\text{sg}}(A \otimes A^{\text{op}})\) of triangulated categories. In particular, it induces an isomorphism

\[
\text{Hom}_{A \otimes A^{\text{op}}}(A, \Omega^p_{\text{sg}}(A)) \cong \text{Ext}^p_{A \otimes A^{\text{op}}}(A, A).
\]

(6.4)

Based on this isomorphism, we prove the following result.

\[\text{Proposition 6.9.} \ \text{The embedding} \ \iota^* : \mathcal{D}^*(A, A) \hookrightarrow C^*_{\text{sg}}(A, A) \ \text{is a quasi-isomorphism.}\]
6.4 The case of symmetric algebras

From now on, we fix a symmetric algebra \( (A, \langle \cdot, \cdot \rangle) \) over a field \( k \). Recall that there is a natural isomorphism \( A \cong A^\vee, x \mapsto \sum_\lambda e_\lambda x \otimes e^\lambda \) of \( A \)-\( A \)-bimodules with inverse \( A^\vee \to A, \sum_i x_i \otimes y_i \mapsto \sum_i \langle y_i, 1 \rangle x_i \), where \( \{e_\lambda\} \) is a basis of \( A \) and \( \{e^\lambda\} \) is its dual basis with respect to the pairing \( \langle \cdot, \cdot \rangle \) (cf. [Bro]). Under this isomorphism, \( D^*(A, A) \) is naturally isomorphic to the following complex (still denoted by \( D^*(A, A) \)),

\[
D^*(A, A) := (\cdots \xrightarrow{b_2} C_1(A, A) \xrightarrow{b_1} C_0(A, A) \xrightarrow{\tau} C^0(A, A) \xrightarrow{d_0} C^1(A, A) \xrightarrow{d_1} \cdots),
\]

where \( \tau(x) := \sum_\lambda e_\lambda x e^\lambda \). One may easily write down the star product \( * \) (cf. Section 6.2) on the new complex \( D^*(A, A) \).

\[\text{Proof.}\] First we note that \( \iota^p \), for \( p \in \mathbb{Z}_{\geq 0} \), induces an isomorphism at the cohomology level from Theorem 6.8 and the proof of Theorem 3.6. Let us prove that this also holds for \( p \in \mathbb{Z}_{\leq 0} \). Indeed, we note that \( D^p(A, A) \cong \text{HH}^0(A, \text{Bar}_{-p-1}(A)) \) and the differential \( \partial^p : D^p(A, A) \to D^{p+1}(A, A) \) coincides with the differential \( d_{-p-1} := \text{HH}^0(A, d_{-p-1}) : \text{HH}^0(A, \text{Bar}_{-p-1}(A)) \to \text{HH}^0(A, \text{Bar}_{-p-2}(A)) \). Hence we have the following commutative diagram,

\[
\begin{array}{c}
\cdots \to D^p(A, A) \xrightarrow{\partial^p} D^{p+1}(A, A) \to \cdots \to D^{-1}(A, A) \to \text{HH}^0(A, A) \\
\text{ident.} \quad \text{ident.} \quad \text{ident.} \quad \text{ident.} \quad \text{ident.} \\
\cdots \to \text{HH}^0(\text{Bar}_{-p-1}) \xrightarrow{-d_{-p-1}} \text{HH}^0(\text{Bar}_{-p-2}) \to \cdots \to \text{HH}^0(\text{Bar}_0) \to \text{HH}^0(A, A) \\
\end{array}
\]

(6.5)

where for simplicity we write \( \text{HH}^0(A, \text{Bar}_{-p}) \) as \( \text{HH}^0(\text{Bar}_p) \). Observe that the \( p \)-th cohomology of the lower complex is isomorphic to \( \text{Hom}_{A \otimes A^\text{op}}(A, \Omega_{sy}^p(A)) \). Thus we have an isomorphism between \( H^p(D^*(A, A)) \) and \( \text{Hom}_{A \otimes A^\text{op}}(A, \Omega_{sy}^p(A)) \). Therefore, from (6.4) and Theorem 3.6, \( \iota^p \) induces an isomorphism in cohomology. This proves the proposition. \( \square \)

Remark 6.10. This proposition shows that \( \text{HH}^*_\text{sg}(A, A) \) can be computed by \( D^*(A, A) \) if \( A \) is a self-injective algebra. Thus we have

\[
\text{HH}^i_{\text{sg}}(A, A) \cong \begin{cases} 
\text{HH}^i(A, A) & \text{for } i > 0, \\
\text{HH}_{-i-1}(A, A^\vee) & \text{for } i < -1 
\end{cases}
\]

and for \( i = -1, 0 \), we have an exact sequence,

\[
0 \to \text{HH}_{-1}^\text{sg}(A, A) \to A^\vee \otimes_{A \otimes A^\text{op}} A \xrightarrow{\tau} \text{HH}^0(A, A) \to \text{HH}^0_{\text{sg}}(A, A) \to 0.
\]

In fact, this result is a special case of [Buc Corollary 6.4.1] since self-injective algebras are naturally Gorenstein. Hence the quasi-isomorphism \( \iota^* \) is viewed as a lifting of Buchweitz’s result to the cochain level.
6.4.1 Lie bracket on $D^*(A,A)$

Recall that there is a non-degenerate pairing $\langle \cdot, \cdot \rangle$ on $D^*(A,A)$ (cf. (2.1)),

$$\langle f, \alpha \rangle = \langle \alpha, f \rangle = \begin{cases} \langle f(s\alpha_{1,m}), a_0 \rangle & \text{if } m = n, \\ 0 & \text{Otherwise,} \end{cases}$$

where $f \in C^m(A,A)$ and $\alpha := a_0 \otimes s\alpha_{1,n} \in C_n(A,A)$.

**Lemma 6.11.** (i) For $x, y \in D^*(A,A)$, we have $\langle \partial(x), y \rangle = (-1)^{|x|-1} \langle x, \partial(y) \rangle$.

(ii) For $x, y, z \in D^*(A,A)$, we have $\langle x \ast y, z \rangle = \langle x, y \ast z \rangle$.

**Proof.** This follows from a straightforward computation. \qed

We now define a Lie bracket $\{ \cdot, \cdot \}$ (of degree -1) on $D^*(A,A)$ in the following cases.

(i) For $p, q \in \mathbb{Z}_{\geq 0}$, define $\{ \cdot, \cdot \} : C_p(A,A) \otimes C_q(A,A) \rightarrow C_{p+q+2}(A,A)$ as $\{ \alpha, \beta \} := \alpha \ast \beta - (-1)^{pq} \beta \ast \alpha$, where

$$\beta \ast \alpha := \sum_{i=1}^{p+1} (-1)^{qi} a_0 \otimes s\alpha_{1,i-1} \otimes s\beta_{1,q} \otimes se_\lambda b_0 \otimes s\beta_{1,q} \otimes se_\lambda \otimes s\alpha_{i,p}.$$ 

(ii) For $f, g \in C^*(A,A)$, define $\{ f, g \}$ to be the classical Gerstenhaber bracket $[f, g]$.

(iii) For $m, p \in \mathbb{Z}_{\geq 0}$ such that $p \geq m - 1$, define $\{ \cdot, \cdot \} : C_p(A,A) \otimes C^m(A,A) \rightarrow C_{p-m+1}(A,A)$ as $\{ \{ \alpha, f \}, g \} := (-1)^{m-1} \langle \alpha, [f, g] \rangle$, for all $g \in C^{p-m+1}(A,A)$. Since the pairing is non-degenerate, the above identity uniquely determines the Lie bracket $\{ \alpha, f \}$. Similarly, we define $\{ f, \alpha \}$ by $\{ \{ f, \alpha \}, g \} := (-1)^{m-1} \langle \alpha, [g, f] \rangle$.

(iv) For $p \leq m - 2$, the bracket $\{ \cdot, \cdot \} : C_p(A,A) \otimes C^m(A,A) \rightarrow C^{m-p-2}(A,A)$ is uniquely determined by $\{ f, \alpha \} = (-1)^p \langle f, \{ \alpha, \beta \} \rangle$. Similarly, $\{ \alpha, f \}$ determines the Lie bracket $\{ \alpha, f \}$.

It is clear that $\{ \cdot, \cdot \}$ is graded skew-symmetric.

**Lemma 6.12.** For $x, y \in D^*(A,A)$, we have $\partial(\{ x, y \}) = \{ \partial(x), y \} + (-1)^{|x|-1} \{ x, \partial(y) \}$.

**Proof.** This follows from Lemma 6.11 and (6.3). \qed

**Lemma 6.13.** Let $\alpha := a_0 \otimes s\alpha_{1,p} \in C_p(A,A)$ and $f \in C^m(A,A)$. Then

1. If $p \geq m - 1$, we have

$$\begin{align*}
\{ \alpha, f \} &= - \sum_{i=1}^{p-m+1} (-1)^{(m-1)(p+i)} a_0 \otimes s\alpha_{1,i-1} \otimes s\tilde{f}(s\alpha_{i+m-1}) \otimes s\alpha_{i+m,p} \\
&\quad + \sum_{\lambda} \sum_{i=1}^{m} (-1)^{(i-1)(p-m)+m-1} a_0, f\left(s\alpha_{1,i-1} \otimes s\alpha_\lambda \otimes s\alpha_{i+p-m-1,p}\right) e_\lambda \otimes s\alpha_{i+p-m}.
\end{align*}$$
2. if \( p \leq m - 2 \), we have

\[
\{ \alpha, f \}(s_{b_1,r}) = - \sum_{i=1}^{m-p-1} (-1)^{(m-i)} f(s_{b_1,i-1} \otimes s e_{\lambda a_0} \otimes s e_{\lambda} \otimes s b_{i,r}) + \\
\sum_{\lambda,\mu=1}^{p+1} (-1)^{ri+p} (a_0, f(s_{a_1,i-1} \otimes s e_{\lambda e_{\mu}} \otimes s b_{1,r} \otimes s e_{\lambda} \otimes s a_{i,p}) \langle a_0, f(s_{e_{\lambda} \otimes s b_{1,r}}) \rangle e^\mu
\]

where \( r := m - p - 2 \).

**Proof.** This follows from straightforward computations. \(\square\)

**Remark 6.14.** We stress that the Jacobi identity does not hold on \( D^*(A, A) \), although it does indeed when all three elements are restricted to either \( D^<0(A, A) \) or \( D^>0(A, A) \) (through a direct computation). Nevertheless, it follows from Proposition 6.19 below that the Jacobi identity holds on the cohomology level since the Lie bracket \( \{\cdot, \cdot\} \) coincides with \( [\cdot, \cdot] \) on \( \text{HH}_{sg}(A, A) \). As a subsequent investigation, we prove in [RiWa] that the Lie bracket \( \{\cdot, \cdot\} \) can be extended to an \( L_\infty \)-algebra structure on \( D^*(A, A) \).

**Remark 6.15.** The Lie bracket \( \{\cdot, \cdot\} \) restricted to \( C_{>0}(A, A) \) coincides with the bracket constructed in [Abb] Theorem 6.1, if \( A \) is a commutative symmetric algebra. Moreover, Abbaspour proved in loc. cit. that the homology \( H_*(C_{>0}(A, A)) \), endowed with the star-product and Lie bracket \( \{\cdot, \cdot\} \), is a BV algebra (without unit) whose BV operator is the Connes’ B operator. Namely,

\[
\{ \alpha, \beta \} = (-1)^p (B(\alpha \star \beta) - B(\alpha) \star \beta - (-1)^{p-1} \alpha \star B(\beta)), \quad (6.6)
\]

for any \( \alpha \in H^p(C_{>0}(A, A)) \) and \( \beta \in H^q(C_{>0}(A, A)) \), where \( p, q \in \mathbb{Z}_{>0} \). In the following, we will prove that this identity also holds on \( H^{<0}(D^*(A, A)) \) for a (not necessarily commutative) symmetric algebra \( A \).

**Proposition 6.16.** Let \( A \) be a symmetric \( k \)-algebra. Then \( H^{<0}(D^*(A, A)) \), equipped with the star-product and Lie bracket \( \{\cdot, \cdot\} \), is a BV algebra (with unit) whose BV operator is the Connes’ B operator.

**Proof.** The Jacobi identity for \( \{\cdot, \cdot\} \) on \( D^{<0}(A, A) \) can be verified by a direct computation. Let us prove the BV identity \( (6.6) \). Since the proof is completely analogous to the one of [Abb] Theorem 6.1 if \( \alpha, \beta \in H^{<2}(D^*(A, A)) \), we omit this proof here. Thus it is sufficient to consider the cases where at least one of \( \alpha, \beta \) is either in \( H^{-1}(D^*(A, A)) \) or \( H^0(D^*(A, A)) \). We define the operator \( B|_{D^0(A,A)} = 0 \).

1. If \( \alpha, \beta \in H^0(D^*(A, A)) \), then \( \{ \alpha, \beta \} = 0 \). So the BV identity holds.
2. If only \( \alpha \in H^0(\mathcal{D}^\ast(A, A)) \), then we have \( \{\{\alpha, \beta\}, f\} = (-1)^q \langle \beta, \{f, \alpha\}\rangle \) for all \( f \in C^{q+1}(A, A) \). Thus the BV identity for \( \{\alpha, \beta\} \) follows from that for \( [f, \alpha] \) on \( \text{HH}^\ast(A, A) \) (cf. Section 2.2).

3. If \( \alpha \in H^{-1}(\mathcal{D}^\ast(A, A)) \), we write \( \alpha := a_0 \in C_0(A, A) \). Then we have \( \partial(a_0) = \sum \lambda e_\lambda a_0 e^\lambda = 0 \). Observe that
\[
\partial(H_1(\alpha, \beta)) = \beta \bullet \alpha + B_1(\alpha \ast \beta) - B(\alpha) \ast \beta
\]
where \( H_1(\alpha, \beta) = \sum \lambda 1 \otimes s\overline{a}_0 \otimes e_\lambda b_0 \otimes s\overline{b}_{1,q} \otimes s e^\lambda \). Here \( B(\alpha \ast \beta) = B_1(\alpha \ast \beta) + B_2(\alpha \ast \beta) \) and \( B_1(\alpha \ast \beta) = (-1)^q s\alpha_0 e_\lambda b_0 \otimes s\overline{b}_{1,q} \otimes e^\lambda \). Therefore, it remains to verify the identity \( \alpha \bullet \beta = B_2(\alpha, \beta) + \alpha \ast B(\beta) \) in \( H^{-q-3}(\mathcal{D}^\ast(A, A)) \).

Let us construct a homotopy
\[
H_2(\alpha, \beta) = \sum \lambda \sum_{0 \leq j, i \leq q} (-1)^{(j-1)q} 1 \otimes s\overline{b}_{j+1,i} \otimes s e_\lambda a_0 \otimes s e^\lambda \otimes s\overline{b}_{i+1,q} \otimes s e_\lambda b_0.
\]
Substituting \( \partial(\alpha) = 0 \) and \( \partial(\beta) = 0 \) into \( \partial(H_2(\alpha, \beta)) \), we get three terms \(-\alpha \bullet \beta, B_2(\alpha, \beta)\) and \( \alpha \ast B(\beta) \), which correspond to the terms when \( j = 0, i = q \) and \( j = i \), respectively. Checking the sign, we have
\[
\partial(H_2(\alpha, \beta)) = -\alpha \bullet \beta + B_2(\alpha, \beta) + \alpha \ast B(\beta).
\]
This verifies the BV identity. It remains to prove the Leibniz rule. Recall that the embedding \( \iota : \mathcal{D}^\ast(A, A) \hookrightarrow C^\ast_{\text{sg}}(A, A) \) is a quasi-isomorphism. We note that \( \iota(\{\alpha, \beta\}) = [\iota(\alpha), \iota(\beta)] \) for any \( \alpha, \beta \in \mathcal{D}^{\leq 0}(A, A) \). Thus the Leibniz rule for \( H^{\leq 0}(\mathcal{D}^\ast(A, A)) \) is deduced from that for \( \text{HH}^\ast_{\text{sg}}(A, A) \) (cf. Corollary 5.3). This proves the proposition. \( \square \)

6.4.2 BV algebra structure

From Theorem 2.4 and Proposition 6.16, it follows that both \( \text{HH}^\ast_{\text{sg}}(A, A) \) and \( \text{HH}^{\leq 0}_{\text{sg}}(A, A) \) have a BV algebra structure. The aim of this section is to prove the following result.

**Theorem 6.17.** Let \( A \) be a symmetric \( k \)-algebra. Then the Tate-Hochschild cohomology \( \text{HH}^\ast_{\text{sg}}(A, A) \), equipped with the cup product \( \cup \) and Lie bracket \( [\cdot, \cdot] \) (cf. Section 4), is a BV algebra whose BV operator \( \Delta^\ast \) is defined on \( \mathcal{D}^\ast(A, A) \) by
\[
\Delta^i := \begin{cases} 
\Delta^i & \text{for } i > 0, \\
0 & \text{for } i = 0, \\
-B_{-i-1} & \text{for } i \leq -1,
\end{cases}
\]
where \( \Delta \) is determined by
\[
(-1)^{m-1}\langle \Delta(f)(s\overline{a}_{1,m-1}), a_0 \rangle = \langle B(a_0 \otimes s\overline{a}_{1,m-1}), f \rangle.
\]
Lemma 6.18. For any $\alpha \in \text{TH}^*(A, A)$ and $\beta \in \text{TH}^*(A, A)$, we have

$$\{\alpha, \beta\} = (-1)^{|\alpha|}(\tilde{\Delta}(\alpha \ast \beta) - \tilde{\Delta}(\alpha) \ast \beta - (-1)^{|\alpha|}\alpha \ast \tilde{\Delta}(\beta)),$$

Proof. It follows from Theorem 2.4 and Proposition 6.16 that the BV identity for $\{\alpha, \beta\}$ holds in the case $\alpha, \beta \in \text{TH}^{\geq 0}(A, A)$ or $\alpha, \beta \in \text{TH}^{\leq 0}(A, A)$. As for other cases, let $\alpha \in C_p(A, A)$ and $f \in C^m(A, A)$ such that $p \geq m - 1$. By definition, we have

$$\langle \{\alpha, f\}, g \rangle = (-1)^{m-1}\langle \alpha, [f, g] \rangle \quad (6.7)$$

for all $g \in C^{p-m+1}(A, A)$. From Theorem 2.4, it follows that

$$[f, g] = (-1)^m\left(\tilde{\Delta}(f \cup g) - \tilde{\Delta}(f) \cup g - (-1)^m f \cup \tilde{\Delta}(g)\right).$$

Substituting this into (6.7), we get

$$\langle \{\alpha, f\}, g \rangle = (-1)^{m-1}\langle \alpha, [f, g] \rangle = -\langle \alpha, \tilde{\Delta}(f \cup g) - \tilde{\Delta}(f) \cup g - (-1)^m f \cup \tilde{\Delta}(g) \rangle = \langle (-1)^{p-1}\left(\tilde{\Delta}(\alpha \cup f) - \tilde{\Delta}(\alpha) \cup f - (-1)^{p-1}\alpha \cup \tilde{\Delta}(f)\right), g \rangle.$$

Thus, we have $\{\alpha, f\} = (-1)^{p-1}\left(\tilde{\Delta}(\alpha \cup f) - \tilde{\Delta}(\alpha) \cup f - (-1)^{p-1}\alpha \cup \tilde{\Delta}(f)\right)$. By the same argument, we obtain the BV identity for $p < m - 1$. This proves the lemma.

Proposition 6.19. The isomorphism $\tilde{\iota}^* : \text{TH}^*(A, A) \cong \text{HH}^*_{sg}(A, A)$ is compatible with Lie brackets. Namely, $[\tilde{\iota}^*(\alpha), \tilde{\iota}^*(\beta)] = \tilde{\iota}^*(\{\alpha, \beta\})$ for $\alpha, \beta \in \text{TH}^*(A, A)$. In particular, the Jacobi identity for the Lie bracket $\{\cdot, \cdot\}$ holds on $\text{TH}^*(A, A)$.

Proof. Clearly, the identity $[\iota^*(\alpha), \iota^*(\beta)] = \iota^*(\{\alpha, \beta\})$ holds on $C^*_{sg}(A, A)$ for the two cases where $\alpha, \beta \in D^{\geq 0}(A, A)$ or $\alpha, \beta \in D^{\leq 0}(A, A)$. It remains to prove that

$$[\tilde{\iota}^*(\alpha), \tilde{\iota}^*(f)] = \tilde{\iota}^*(\{\alpha, f\}) \quad (6.8)$$

for $\alpha \in \text{TH}^{-p-1}(A, A))$ and $f \in \text{TH}^m(A, A)$, where $p, m \in \mathbb{Z}_{\geq 0}$. We need to consider the following two cases.

1. If $p \geq m - 1$, to prove the identity (6.8) is equivalent to prove the following commutative diagram.

$$\begin{array}{ccc}
\text{TH}^m(A, A) \otimes \text{TH}^{p-m-2}(A, A) & \xrightarrow{\{\cdot, \cdot\}} & \text{TH}^{m-p-2}(A, A) \\
\downarrow_{\kappa_{m,1} \otimes \kappa_{p-1,m+2}} \cong & & \cong \downarrow_{\kappa_{m-2,p+2}} \\
\text{HH}^m(A, \Omega^1_{nc}(A)) \otimes \text{HH}^{p-m-2}(A, \Omega_{nc}^{p+2}(A)) & \xrightarrow{[\cdot, \cdot]} & \text{HH}^m(A, \Omega_{nc}^{p+2}(A))
\end{array}$$
Here the map $\kappa_{r,s} : D^r(A, A) \to C^r(A, \Omega^{s}_{nc}(A))$ is defined as

$$
\kappa_{r,s} := \begin{cases} 
\theta_{r-s, r} \circ \cdots \circ \theta_0 \circ \iota^r & \text{if } r - s \geq 0, \\
\theta_{s-r, r} \circ \cdots \circ \theta_{0, r} \circ \iota^r & \text{if } r - s < 0,
\end{cases}
$$

where we recall that $\theta_{s,r} : C^r(A, \Omega^{s}_{nc}(A)) \to C^r(A, \Omega^{s+1}_{nc}(A))$ sends $f$ to $f \otimes \text{id}_{sA}$. Write $\alpha := a_0 \otimes s\bar{a}_{1,p} \in C_p(A, A)$. For any $s\bar{b}_{1,m+1} \in (s\bar{A})^{\otimes m+1}$, we have

$$
[k_{m,1}(f), k_{-p-1,p+2}(\alpha)](s\bar{b}_{1,m+1})
= \sum_{\lambda} \sum_{i=1}^{p+1} (-1)^{p(i-1)} f(s\bar{b}_{1,i-1} \otimes s\bar{a}_{i-1} \otimes s\bar{a}_{1,m-i}) \otimes s\bar{a}_{m-i+1,p} \otimes s\bar{e}_{\lambda} \otimes s\bar{b}_{i,m+1}
+ \sum_{\lambda} \sum_{i=p-m+2}^{p+1} (-1)^{(m-1)i} e_{\lambda} a_0 \otimes s\bar{a}_{1,i-1} \otimes s\bar{f}(s\bar{a}_{i,i+m-1}) \otimes s\bar{a}_{i+m,p} \otimes s\bar{e}_{\lambda} \otimes s\bar{b}_{1,m+1} + \sum_{\lambda} \sum_{i=p-m+2}^{p+1} (-1)^{(m-1)i} e_{\lambda} a_0 \otimes s\bar{a}_{1,i-1} \otimes s\bar{f}(s\bar{a}_{i,p} \otimes s\bar{e}_{\lambda} \otimes s\bar{b}_{1,m+1} + s\bar{b}_{m-p+i-1,m+1}.
$$

and

$$
\kappa_{m-3,p+2}(\{f, \alpha\})(s\bar{b}_{1,m+1})
= \sum_{\lambda} \sum_{i=1}^{p+1} (-1)^{(m-1)i} e_{\lambda} a_0 \otimes s\bar{a}_{1,i-1} \otimes s\bar{f}(s\bar{a}_{i,i+m-1}) \otimes s\bar{a}_{i+m,p} \otimes s\bar{e}_{\lambda} \otimes s\bar{b}_{1,m+1} - \sum_{\lambda,\mu} \sum_{i=1}^{m} (-1)^{(m-i)(p-m)+1} \langle a_0, f(s\bar{a}_{1,i-1} \otimes s\bar{e}_{\lambda} \otimes s\bar{a}_{i+p-m+1,p}) \rangle e_{\mu} e_{\lambda} \otimes s\bar{a}_{i,i+p-m} \otimes s\bar{e}_{\mu} \otimes s\bar{b}_{1,m+1},
$$

where $\{e_{\lambda}\}$ is a basis of $A$ and $\{e_{\lambda}\}$ is the dual basis with respect to $\langle \cdot, \cdot \rangle$. Comparing the above two identities, we get

$$
([k_{m,1}(f), k_{-p-1,p+2}(\alpha)] - \kappa_{m-3,p+2}(\{f, \alpha\}))(s\bar{b}_{1,m+1})
= \sum_{\lambda} \sum_{i=1}^{m} (-1)^{p(i-1)} f(s\bar{b}_{1,i-1} \otimes s\bar{a}_{i-1} \otimes s\bar{a}_{1,m-i}) \otimes s\bar{a}_{m-i+1,p} \otimes s\bar{e}_{\lambda} \otimes s\bar{b}_{i,m+1}
+ \sum_{\lambda} \sum_{i=p-m+2}^{p+1} (-1)^{(m-1)i} e_{\lambda} a_0 \otimes s\bar{a}_{1,i-1} \otimes s\bar{f}(s\bar{a}_{i,p} \otimes s\bar{e}_{\lambda} \otimes s\bar{b}_{1,m-p+i-2}) \otimes s\bar{b}_{m-p+i-1,m+1} - \sum_{\lambda,\mu} \sum_{i=1}^{m} (-1)^{(m-i)(p-m)+1} \langle a_0, f(s\bar{a}_{1,i-1} \otimes s\bar{e}_{\lambda} \otimes s\bar{a}_{i+p-m+1,p}) \rangle e_{\mu} e_{\lambda} \otimes s\bar{a}_{i,i+p-m} \otimes s\bar{e}_{\mu} \otimes s\bar{b}_{1,m+1}.
$$

(6.9)
For any $1 \leq k \leq m$, let us denote
\[ B_{k-1}(f, \alpha)(s\overline{b}_{1,m+1}) := \sum_{\lambda} (-1)^{p(k-1)} f(s\overline{b}_{1,k-1} \otimes s\overline{\epsilon}_{\lambda}a_0 \otimes s\overline{a}_{m-k+1,p} \otimes s\overline{\epsilon} \otimes s\overline{b}_{k,m+1} \right. \\
+ \sum_{\lambda} (-1)^{(m-1)(p-k-1)} e_\lambda a_0 \otimes s\overline{a}_{1,p'} \otimes s\overline{f}(s\overline{a}_{p'+1,p} \otimes s\overline{\epsilon} \otimes s\overline{b}_{1,k-1} \otimes s\overline{b}_{k,m+1},
\]
where $p' := p - m + k$. For any $0 \leq k \leq m - 1$, denote
\[ C_k(f, \alpha)(s\overline{b}_{1,m+1}) := \sum_{\lambda, \mu} \sum_{i=k+1}^{m} (-1)^{(m-i+k)(p-m)+m} \langle a_0, f(s\overline{a}_{1,i-k-1} \otimes s\overline{\epsilon}_{\lambda} \otimes s\overline{b}_{k,m+1} \rangle \\
\otimes s\overline{a}_{i+p-m+1,p}) \rangle e_\lambda \otimes s\overline{a}_{i-k,i+p-m} \otimes s\overline{\epsilon} \otimes s\overline{b}_{1,m+1}.
\]
In particular, we have
\[ C_0(f, \alpha)(s\overline{b}_{1,m+1}) = \sum_{\lambda, \mu} \sum_{i=1}^{m} (-1)^{(m-i)p-m} \langle a_0, f(s\overline{a}_{1,i-1} \otimes s\overline{\epsilon}_{\lambda} \otimes s\overline{a}_{i+p-m+1,p}) \rangle e_\lambda \otimes s\overline{a}_{i-k,i+p-m} \otimes s\overline{\epsilon} \otimes s\overline{b}_{1,m+1}
\]
and
\[ [\kappa_{p,1}(f), \kappa_{-p-1,2}(\alpha)] - \kappa_{m-p-2,2}(\{f, \alpha\}) = \sum_{k=1}^{m} B_{k-1}(f, \alpha) - C_0(f, \alpha). \tag{6.10} \]

For any $1 \leq k \leq m - 1$, set
\[ H_k(f, \alpha)(s\overline{b}_{1,m}) := \sum_{\lambda, \mu} \sum_{i=k+1}^{m} (-1)^{(m-i-k)(p-m)+p+k} \langle a_0, f(s\overline{a}_{1,i-k-1} \otimes s\overline{\epsilon}_{\lambda} \otimes s\overline{b}_{1,k-1} \right. \\
\otimes s\overline{\epsilon} \otimes s\overline{b}_{k,m+1} \rangle e_\lambda \otimes s\overline{a}_{i-k,i+p-m} \otimes s\overline{\epsilon} \otimes s\overline{b}_{k,m+1}.
\]
We claim that $\delta(H_k(f, \alpha)) = C_k(f, \alpha) - C_{k-1}(f, \alpha) + B'_{k-1}(f, \alpha)$, where
\[ B'_{k-1}(f, \alpha)(s\overline{b}_{1,m+1}) := \sum_{\lambda} (-1)^{p(m-1)+m-k} f(s\overline{b}_{1,k-1} \otimes s\overline{\epsilon}_{\lambda} \otimes s\overline{a}_{p-m+k+1,p}) a_0 \otimes s\overline{a}_{1,p-m+k} \otimes s\overline{\epsilon} \otimes s\overline{b}_{k,m+1} \\
- (-1)^{k(p-m)+p} \sum_{\lambda} e_\lambda \otimes s\overline{a}_{m-k+1,p} \otimes a_0 f(s\overline{a}_{1,m-k} \otimes s\overline{\epsilon} \otimes s\overline{b}_{1,k-1} \otimes s\overline{b}_{k,m+1}. \]
Indeed, $B'_{k-1}(f, \alpha)$ appears when $i = k + 1$ and $i = m$ in $\delta(H_k(f, \alpha))$, using $\partial(f) = 0$. We note that the remaining terms in $\delta(H_k(f, \alpha))$ are cancelled by $C_k(f, \alpha) - C_{k-1}(f, \alpha)$, using $\partial(f) = \partial(\alpha) = 0$. This proves the claim. Therefore, we have

$$\sum_{k=1}^{m-1} \delta(H_k(f, \alpha)) = C_{m-1}(f, \alpha) - C_0(f, \alpha) + \sum_{k=1}^{m-1} B'_{k-1}(f, \alpha).$$

Substituting this identity into (6.10), we obtain

$$[\kappa_{m,1}(f), \kappa_{p-1,p+2}(\alpha)] - \kappa_{m-p-2,p+2}(f, \alpha) = \sum_{k=1}^{m} (B_{k-1}(f, \alpha) - B'_{k-1}(f, \alpha)),$$

where $B'_{m-1}(f, \alpha) := C_{m-1}(f, \alpha)$ for simplicity. Thus it is sufficient to verify

$$\sum_{k=1}^{m} (B_{k-1}(f, \alpha) - B'_{k-1}(f, \alpha)) = 0.$$

For $1 \leq k \leq m$, denote

$$H'_{k-1}(f, \alpha)(s \bar{b}_{1,m}) = \sum_{\lambda} \sum_{i=1}^{p-m+k+1} (-1)^{p(k+i-1)+m-k-1} f(s \bar{b}_{1,k-1} \otimes s \bar{e}_{\alpha} \otimes s \bar{e}_i \otimes s \bar{e}_{m+i-k-1}) \otimes s \bar{e}_{m+i-k,p} \otimes s \bar{e}_0 \otimes s \bar{e}_{1,i-1} \otimes s \bar{e}_{k,m} + \sum_{\lambda} \sum_{i=1}^{p-m+k+1} (-1)^{\epsilon_i} e_{\lambda} \otimes s \bar{e}_{p-i+2,p} \otimes s \bar{e}_0 \otimes s \bar{e}_{1,p-i+k+m+1} \otimes s \bar{e}_f(s \bar{e}_{p-i+k+m+2,p-i+1} \otimes s \bar{e}_{\lambda} \otimes s \bar{e}_{b_{1,k-1}}) \otimes s \bar{b}_{k,m},$$

where $\epsilon_i = p(i - 1) + (m - 1)(p - k - 1)$. Then we have

$$\sum_{k=1}^{m} \delta(H'_{k-1}(f, \alpha)) = \sum_{k=1}^{m} (B_{k-1}(f, \alpha) - B'_{k-1}(f, \alpha)),$$

since we have the term $B_{k-1}(f, \alpha)$ when $i = 1$, and $B'_{k-1}(f, \alpha)$ when $i = p - m + k + 1$. All other terms in $\sum_{k=1}^{m} \delta(H'_{k-1}(f, \alpha))$ are cancelled by $\partial(f) = 0 = \partial(\alpha)$. Therefore the identity (6.8) is verified in this case.

2. If $p \leq m - 2$, we need to prove the following commutative diagram

$$\begin{array}{ccc}
\text{TH}^m(A, A) \otimes \text{TH}^{-p-1}(A, A) & \xrightarrow{\{\cdot, \cdot\}} & \text{TH}^{m-p-2}(A, A) \\
\text{id} \otimes \kappa_{p-1,p+2} & \cong & \kappa_{m-p-2,p+2} \\
\text{HH}^m(A, A) \otimes \text{HH}^{-p-1}(A, \Omega^{p+2}_{nc})(A) & \xrightarrow{\{\cdot, \cdot\}} & \text{HH}^{m-p-2}(A, \Omega^{p+2}_{nc}(A)).
\end{array}$$
By an analogous computation as in Case 1, we get that

\[
(f, \kappa_{-p-2, p+2}(\alpha)) = \kappa_{m-p-2, p+2}(\{f, \alpha\})(s\vec{b}_{1,m})
\]

\[
= \sum_{\lambda} \sum_{i=m-p}^{m} (-1)^{p(i-1)} f(s\vec{b}_{1,i-1} \otimes s\vec{e}_{\lambda}a_{0} \otimes s\vec{a}_{1,m-i}) \otimes s\vec{a}_{m-i+1,p} \otimes s\vec{e}_{\lambda} \otimes s\vec{b}_{i,m}
\]

\[
+ \sum_{\lambda} \sum_{i=1}^{p+1} (-1)^{(m-1)i} e_{\lambda} a_{0} \otimes s\vec{a}_{1,i-1} \otimes s\vec{f}(s\vec{a}_{i,p} \otimes s\vec{e}_{\lambda} \otimes s\vec{b}_{1,p'}) \otimes s\vec{b}_{p'-1,m}
\]

\[
- \sum_{\lambda, \mu} \sum_{i=1}^{p+1} (-1)^{(m-p)i+mp} \langle a_{0}, f(s\vec{a}_{1,i-1} \otimes s\vec{e}_{\lambda} \otimes s\vec{b}_{1,m-p-2} \otimes s\vec{e}_{\lambda} \otimes s\vec{a}_{i,p}) \rangle
\]

\[
e^{\mu} \otimes s\vec{b}_{m-p-1,m},
\]

where \( p' := m - p + i - 2 \). For \( 0 \leq k \leq p \), we denote

\[
B_{k}(f, \alpha)(s\vec{b}_{1,m})
\]

\[
= \sum_{\lambda} (-1)^{(m-k)} f(s\vec{b}_{1,m-p+k-1} \otimes s\vec{e}_{\lambda}a_{0} \otimes s\vec{a}_{1,p-k}) \otimes s\vec{a}_{p-k+1,p} \otimes s\vec{e}_{\lambda}
\]

\[
\otimes s\vec{b}_{m-p+k,m} + \sum_{\lambda} (-1)^{(m-1)(k-1)} e_{\lambda} a_{0} \otimes s\vec{a}_{1,k} \otimes s\vec{f}(s\vec{a}_{k+1,p} \otimes s\vec{e}_{\lambda}
\]

\[
\otimes s\vec{b}_{1,m-p+k-1}) \otimes s\vec{b}_{m-p+k,m}.
\]

For \( 0 \leq k \leq p \), we denote

\[
C_{k}(f, \alpha)(s\vec{b}_{1,m})
\]

\[
= \sum_{\lambda, \mu} \sum_{i=1}^{p+1-k} (-1)^{(m-p)(k+i)+mp} \langle a_{0}, f(s\vec{a}_{i,i-1} \otimes s\vec{e}_{\mu} \otimes s\vec{b}_{1,m-p+k-2} \otimes s\vec{e}_{\lambda}
\]

\[
\otimes s\vec{a}_{i+k,p}) \rangle (e^{\mu} \otimes s\vec{a}_{i,i+k-1}) \otimes e_{\lambda} \otimes s\vec{b}_{m-p+k-1,m}.
\]

It follows from (6.11) that

\[
[f, \kappa_{-p-1, p+2}(\alpha)] - \kappa_{m-p-2, p+2}(\{f, \alpha\}) = \sum_{k=0}^{p} B_{k}(f, \alpha) - C_{0}(f, \alpha).
\]

(6.12)

For \( 0 \leq k \leq p \), set

\[
H_{k}(f, \alpha)(s\vec{b}_{1,m-1})
\]

\[
= \sum_{\lambda, \mu} \sum_{i=1}^{p+1-k} (-1)^{(m-p)(i-1)+mp} \langle a_{0}, f(s\vec{a}_{i,i-1} \otimes s\vec{e}_{\mu} \otimes s\vec{b}_{1,m-p+k-2} \otimes s\vec{e}_{\lambda}
\]

\[
\otimes s\vec{a}_{i+k,p}) \rangle e^{\mu} \otimes s\vec{a}_{i,i+k-1} \otimes s\vec{e}_{\lambda} \otimes s\vec{b}_{m-p+k-1,m-1}.
\]
We claim that \( \delta(H_k(f, \alpha)) = C_{k+1}(f, \alpha) - C_k(f, \alpha) + B_k'(f, \alpha) \), where

\[
\begin{align*}
B_k'(f, \alpha)(s\overline{b}_1) &= \sum_{\lambda} (-1)^{mp+k} f(s\overline{b}_{1,m-p+k-1} \otimes s\overline{c}_\lambda \otimes s\overline{a}_{k+1,p}) a_0 \otimes s\overline{a}_{1,k} \otimes se\lambda \\
& \quad \otimes s\overline{b}_{m-p+k,m} - \sum_{\lambda} (-1)^{(m-p)k+m} e_\lambda \otimes s\overline{a}_{p+1-k,p} \otimes s\overline{a}_0 (s\overline{a}_{1,p} \otimes se\lambda) \\
& \quad \otimes s\overline{b}_{1,m-p+k-1} \otimes s\overline{b}_{m-p+k,m}
\end{align*}
\]

and \( C_{p+1}(f, \alpha) := 0 \). Indeed, the term \( B_k'(f, \alpha) \) appears when \( i = 1 \) and \( i = p + 1 - k \) by \( \partial(f) = 0 \) in \( \delta(H_k(f, \alpha)) \). The remaining terms are exactly equal to \( C_{k+1}(f, \alpha) - C_k(f, \alpha) \). Therefore, we have

\[
\sum_{k=0}^{p} \delta(H_k(f, \alpha)) = -C_0 + \sum_{k=0}^{p} B_k'(f, \alpha).
\]

Substituting this identity into (6.12), we get

\[
[f, \kappa_{-p+1,p+2}(\alpha)] - \kappa_{m-p+2,p+2}(\{f, \alpha\}) = \sum_{k=0}^{p} (B_k(f, \alpha) - B_k'(f, \alpha)).
\] (6.13)

For \( 0 \leq k \leq p \), denote

\[
\begin{align*}
H_k'(f, \alpha)(s\overline{b}_1) &= \sum_{\lambda} \sum_{j=1}^{k+1} (-1)^{p(m-k-j-1)+k-1} f(s\overline{b}_{1,p'} \otimes s\overline{c}_\lambda \otimes s\overline{a}_{j,j+p-k-1}) \otimes s\overline{a}_{j+p-k,p} \otimes s\overline{a}_0 \\
& \quad \otimes s\overline{a}_{1,j-1} \otimes se\lambda \otimes s\overline{b}_{p'+1,m-1} + \sum_{\lambda} \sum_{j=1}^{k+1} (-1)^{p(j-1)+(m-1)(k-1)} e_\lambda \otimes s\overline{a}_{j+1,p} \\
& \quad \otimes s\overline{a}_0 \otimes s\overline{a}_{1,k+j-1} \otimes s\overline{f}(s\overline{a}_{k-j+2,p-j+1} \otimes se\lambda \otimes s\overline{b}_{1,p'}) \otimes s\overline{b}_{p'+1,m-1}.
\end{align*}
\]

where \( p' = m - p + k - 1 \). Then using \( \partial(f) = 0 = \partial(\alpha) \), we have

\[
\sum_{k=0}^{p} \delta(H_k'(f, \alpha)) = \sum_{k=0}^{p} (B_k(f, \alpha) - B_k'(f, \alpha))
\]

since \( B_k(f, \alpha) \) appears in \( \delta(H_k'(f, \alpha)) \) when \( j = 1 \), and \( B_k'(f, \alpha) \) appears when \( j = k + 1 \). All other terms are cancelled by \( \partial(f) = 0 = \partial(\alpha) \).

This proves that \( \overline{\iota}^\ast \) is compatible with Lie brackets. 

\[\square\]

**Proof of Theorem 6.17** It follows from Proposition 6.19 that the Jacobi identity for \( \{\cdot, \cdot\} \) holds on \( \text{TH}^\ast(A, A) \), thus \( \text{TH}^\ast(A, A), \ast, \{\cdot, \cdot\} \) is a Gerstenhaber algebra. Then Lemma 6.18 infers that \( \text{TH}^\ast(A, A), \ast, \{\cdot, \cdot\}, \hat{\Delta} \) is a BV algebra. Since \( \overline{\iota}^\ast \) is compatible with products and Lie brackets, we obtain that \( \text{HH}^\ast_{sg}(A, A), \cup, [\cdot, \cdot], \overline{\iota} \circ \hat{\Delta} \circ \overline{\iota}^{-1} \) is a BV algebra and clearly, \( \overline{\iota}^\ast \) is an isomorphism of BV algebras. This proves the theorem. 

\[\square\]
Remark 6.20. It would be interesting to wonder whether the cyclic Deligne’s conjecture holds for $D^*(A, A)$, namely, whether $D^*(A, A)$ is an algebra over the frame little 2-discs operad (cf. e.g. [MaShSt, Kau08]). This conjecture would yield the result of Theorem 6.17 from the operadic point of view.

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