Minimal obstructions to \((s, 1)\)-polarity in cographs

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Abstract

Let \(k, l\) be non negative integers. A graph \(G\) is \((k, l)\)-polar if its vertex set admits a partition \((A, B)\) such that \(A\) induces a complete multipartite graph with at most \(k\) parts, and \(B\) induces a disjoint union of at most \(l\) cliques with no other edges. A graph is a cograph if it does not contain \(P_4\) as an induced subgraph.

It is known that \((k, l)\)-polar cographs can be characterized through a finite family of forbidden induced subgraphs, for any fixed choice of \(k\) and \(l\). The problem of determining the exact members of such family for \(k = 2 = l\) was posted by Ekim, Mahadev and de Werra, and recently solved by Hell, Linhares-Sales and the second author of this paper. So far, complete lists of such forbidden induced subgraphs are known for \(0 \leq k, l \leq 2\); notice that, in particular, \((1, 1)\)-polar graphs are precisely split graphs.

In this paper we focus on this problem for \((s, 1)\)-polar cographs. As our main result, we provide a recursive complete characterization of the forbidden induced subgraphs for \((s, 1)\)-polar cographs, for every non negative integer \(s\). Additionaly, we show that cographs having an \((s, 1)\)-partition for some integer \(s\) (here \(s\) is not fixed) can be characterized by forbidding a family of four graphs.

Keywords: Polar graph, cograph, forbidden sugraph characterization, monopolar graph, matrix partition, generalized colouring

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1 Introduction

All graphs in this paper are considered to be finite and simple. We refer the reader to [1] for basic terminology and notation. In particular, we use \(P_k\) and \(C_k\) to denote the path

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and cycle on \( k \) vertices, respectively.

Cographs were introduced by Corneil, Lerchs and Stewart Burlingham in [4]. A graph is a complement reducible graph, or \textit{cograph}, if it can be constructed using the following rules.

- \( K_1 \) is a cograph.
- If \( G \) is a cograph, then its complement \( \overline{G} \) is also a cograph.
- If \( G \) and \( H \) are cographs, then the disjoint union \( G + H \) is also a cograph.

In [4], seven characterizations of this family were presented; in this work we will extensively use two very well known of these. A graph is a cograph if and only if it is \( P_4 \)-free (it does not contain \( P_4 \) as an induced subgraph), if and only if the complement of any of its nontrivial connected subgraphs is disconnected.

In 1990, Peter Damaschke proved that the class of cographs is well quasi-ordered by the induced subgraph relation [6]; in other words, every hereditary property of graphs can be characterized by a finite family of forbidden induced subgraphs. Thus, finding the family of minimal forbidden induced subgraphs characterizing a given hereditary property in the class of cographs comes as a natural problem. The knowledge of such families has two obvious consequences, first, analyzing the structure of the members of one of this families (for example, fixing a certain value of a parameter which the hereditary property depends on) may shed some light on the general problem. Also, the members of these families are no-certificates for the associated decision problem. Consider for example a generalized colouring problem (partition the set of vertices of a graph into \( k \) subsets such that each part has some hereditary property), if we know the complete list of minimal forbidden induced subgraphs, an algorithm could be designed to receive a cograph \( G \), decide if it has a generalized colouring of the desired type, and return either the colouring of \( G \) (a yes-certificate) or one of the forbidden induced subgraphs (a no-certificate). Such an algorithm is known as a \textit{certifying algorithm}, and if the validity of the certificates can be verified efficiently (faster than the original algorithm), having a certifying algorithm makes it possible to verify the correctness of its implementations.

In the present work, we will focus on polar partitions. A \textit{polar partition} of a graph \( G \) is a partition of the vertices of \( G \) into parts \( A \) and \( B \) in such a way that the subgraph induced by \( A \) is a complete multipartite graph and the subgraph induced by \( B \) is the complement of a complete multipartite graph. A graph \( G \) is \textit{polar} if it admits a polar partition, and is \((s,k)\)-\textit{polar} if it admits a polar partition \((A,B)\) in which \( A \) has at most \( s \) parts and \( B \) at most \( k \) parts. When \( s = 1 \), an \((s,k)\)-polar graph (partition) is called a \textit{monopolar} graph (partition). Clearly, for any fixed non negative integers \( s \) and \( k \), having an \((s,k)\)-partition is a hereditary property, and thus, as we have already mentioned, \((s,k)\)-polar cographs can be characterized by a finite family of forbidden induced subgraphs. A \textit{cograph minimal} \((s,k)\)-\textit{polar obstruction} is a cograph which is not \((s,k)\)-polar, but such that every proper induced subgraph is. A \textit{cograph} \((s,k)\)-\textit{polar obstruction} is simply a cograph which is not \((s,k)\)-polar.

Polar graphs have received considerable attention in the literature since Chernyak and Chernyak proved in [5] that their recognition problem is \textit{\textbf{NP}}-complete. Surprisingly, Farrugia proved in [8] that the problem remains \textit{\textbf{NP}}-complete even for monopolar graphs, and
Churchley and Huang proved in [3], that monopolar recognition remains \( \mathcal{NP} \)-complete even when restricted to triangle-free graphs. Regarding these two problems, the class of claw-free graphs is interesting, it distinguishes monopolarity, which is polynomial time recognizable, from polarity, which is \( \mathcal{NP} \)-complete. \[3\]

We think that it is worth noticing that polar partitions are a particular case of a more general kind of partition problems, namely, matrix partitions. The concept of a matrix partition unifies many interesting graph partition problems. Given a symmetric \( n \times n \) matrix \( M \), with entries in \( \{0, 1, *\} \), an \( M \)-partition of a graph \( G \) is a partition \([V_1, \ldots, V_n]\) of \( V(G) \) such that, for every \( i, j \in \{1, \ldots, n\} \),

- \( V_i \) is completely adjacent to \( V_j \) if \( M_{ij} = 1 \),
- \( V_i \) is completely non-adjacent to \( V_j \) if \( M_{ij} = 0 \),
- There are no restrictions if \( M_{ij} = * \).

It follows from the definition that, in particular, if \( M_{ii} = 0 \) (\( M_{ii} = 1 \)), then \( V_i \) is a stable set (\( V_i \) is a clique). The \( M \)-partition problem asks whether or not an input graph \( G \) admits an \( M \)-partition. See [13] for a survey on the subject. It is easy to see that an \((s, k)\)-polar partition of \( G \) is a matrix partition in which the matrix \( M \) has \( s + k \) rows and columns, the principal submatrix induced by the first \( s \) rows is obtained from an identity matrix by exchanging 0’s and 1’s, the principal submatrix induced by the last \( k \) rows is an identity matrix, and all other entries are *\]. Therefore, it follows from [10], that for any fixed \( s \) and \( k \), the class of \((s, k)\)-polar graphs can be recognized in polynomial time. Feder, Hell and Hochstättler proved in [11] that if \( M \) is a matrix where all the off-diagonal entries of the principal submatrix with zeroes on the diagonal are equal to \( a \), all the off-diagonal entries of the principal submatrix with only ones on the diagonal are equal to \( b \) and all the remaining entries of \( M \) are equal to \( c \), with \( a, b, c \in \{0, 1, *\} \), then every cograph minimal \( M \)-obstruction has at most \((k + 1)(\ell + 1)\) vertices.

For very small values of \( s \) and \( k \) the minimal \((s, k)\)-polar obstructions are well known; a graph is \((0, k)\)-polar if and only if it is a disjoint union of at most \( k \)-cliques, it is \((s, 0)\)-polar if and only if it is a complete \( s \)-partite graph, and it is \((1, 1)\)-polar if and only if it is a split graph. It was shown by Foldes and Hammer [12] that a graph is split if and only if it is \( \{2K_2, C_4, C_5\} \)-free; it is folklore that a graph is a disjoint union of at most \( k \)-cliques if and only if its independence number is at most \( k \) and it is \( P_3 \)-free, which by complementation implies that a graph is a complete \( s \)-partite graph if and only if it is \( \{K_{s+1}, K_1 + K_2\} \)-free.

For cographs, Ekim, Mahadev and de Werra proved in [7] that there are only eight co-
graph minimal polar obstructions, and sixteen cograph minimal \((s, k)\)-polar obstructions when \( \min\{s, k\} = 1 \). \[3\]. In the same paper, they proposed the problem of finding a characterization of \((2, 2)\)-polar cographs; this problem was solved by Hell, Hernández-Cruz and Linhares-Sales in [14], where they proved that there are 48 cograph minimal \((2, 2)\)-polar ob-

\[1\] As it is usual in graph theory, we do not require every part of the partition to be non-empty.
structions. The exhaustive list of nine cograph minimal (2,1)-polar obstructions was found by Bravo, Nogueira, Protti and Vianna. [2].

In this work, we show that there are precisely four cograph minimal monopolar obstructions (see Figure 1), and provide a recursive characterization for cograph minimal (s,1)-polar obstructions. By taking complements it trivial to obtain analogous results for (1,k)-polar cographs.

We will denote the complement of \( G \) by \( \overline{G} \). We say that a component of \( H \) is trivial or an isolated vertex if it is isomorphic to \( K_1 \). A k-cluster is the complement of a complete k-partite graph, i.e., a disjoint union of k-cliques without any other edges.

Given graphs \( G \) and \( H \), the disjoint union of \( G \) and \( H \) is denoted by \( G + H \), and the join of \( G \) and \( H \) is denoted by \( G \oplus H \). Thus, the sum of \( n \) disjoint copies of \( G \) is denoted by \( nG \), and for disjoint graphs \( G_1, \ldots, G_k \), their disjoint union is denoted as \( \sum_{i=1}^{k} G_k \).

The rest of the paper is organized as follows. In Section 2, we prove some technical lemmas that will be used in Section 3 to prove our main results. In Section 4, a brief asymptotic estimation of the number of cograph minimal (s,1)-obstructions is given. Conclusions and future lines of work are presented in Section 5.

2 Preliminary results

We begin this section by characterizing graphs that are cograph minimal (s,1)-obstructions for every integer \( s \), with \( s \geq 2 \). We will call such obstructions essential. First, notice that if \( G \) is an (s,1)-polar graph with polar partition \( (A, B) \), then \( B \) is just a clique, and \( G[V - B] = G[A] \), is a complete multipartite graph. On the other hand, if \( G \) is a graph containing a clique \( K \) such that \( G[V - K] \) is a complete multipartite graph, then clearly \( (V - K, K) \) is an (s,1)-polar partition of \( G \). This simple observation is contained in the following remark.

Remark 1. Let \( G \) be a cograph. If for every clique \( K \) of \( G \), the induced subgraph \( G[V - K] \) contains \( P_3 \) as an induced subgraph, then \( G \) is not an (s,1)-polar cograph for any integer \( s \), \( s \geq 2 \).

Now, we can show the existence of some essential cograph essential (s,1)-polar obstructions.

Lemma 2. The graphs \( K_1 + 2K_2, \overline{K_2} + C_4, 2P_3 \) and \( K_1 + (P_3 \oplus K_2) \) depicted in Figure 1 are cograph minimal (s,1)-polar obstructions for every integer \( s \), \( s \geq 2 \).

Proof. It is evident that all the graphs shown in Figure 1 are cographs. By a simple exploration taking into consideration Remark 1, it is routine to verify that none of these graphs is an (s,1)-polar cograph for any positive integer \( s \). Furthermore it is easy to verify that in each of these graphs the deletion of any vertex results in a (2,1)-polar cograph, so all of them are cograph minimal (s,1)-polar obstructions for any integer \( s \) greater than or equal to 2. \( \square \)
Notice that all essential obstructions are disconnected, and, since they are small graphs, it is not hard to imagine that they will prevent larger disconnected minimal obstructions to exist. Our next lemma concretes this intuitive idea, showing that disconnected cograph minimal \((s,1)\)-polar obstructions have at most two components, except for \(K_1 + 2K_2\) and \(K_2 + C_4\); some additional restrictions on the structure of such minimal obstructions are also obtained.

**Lemma 3.** Let \(s\) be an integer, \(s \geq 2\). Then every cograph minimal \((s,1)\)-polar obstruction different from \(K_1 + 2K_2\) and \(K_2 + C_4\) has at most two connected components.

Moreover, if a cograph minimal \((s,1)\)-polar obstruction has two connected components and it is neither \(2P_3\) nor \(2K_{s+1}\), then one of its components is \(K_1\) or \(K_2\), and its other component is not a complete graph.

**Proof.** Let \(G\) be a cograph minimal \((s,1)\)-polar obstruction with at least three connected components. Observe that since \(G\) is not a split graph, \(G\) contains \(C_4\) or \(2K_2\) as an induced subgraph. In the former case, since \(G\) has at least three connected components, \(G\) contains \(K_2 + C_4\) as an induced subgraph. For the latter case, again, noting that \(G\) has at least three components leads to conclude that \(G\) contains \(K_1 + 2K_2\) as an induced subgraph. By the minimality of \(G\), the previous observations imply that \(G\) is isomorphic to \(K_2 + C_4\) or \(K_1 + 2K_2\). So we have that every cograph minimal \((s,1)\)-polar obstruction isomorphic to neither \(2P_3\) nor \(2K_{s+1}\) has at most two connected components.

Now, suppose that \(G\) is a cograph minimal \((s,1)\)-polar obstruction isomorphic to neither \(2P_3\) nor \(2K_{s+1}\), and with two connected components. Note that since \(G\) is \(2P_3\)-free, at least one of the components of \(G\) is a complete graph. If both components of \(G\) are complete graphs, then both of them must have at least \(s + 1\) vertices, otherwise \(G\) would be \((s,1)\)-polar; but in this case \(G\) should be isomorphic to \(2K_{s+1}\). Thus, we may assume that one component of \(G\) is a complete graph and the other one is not.

Finally, suppose for a contradiction that the complete component of \(G\) has three or more vertices, and let \(v\) be one of these vertices. By the minimality of \(G\) we have that \(G - v\) admits an \((s,1)\)-polar partition \((A,B)\). If \(B\) is contained in the complete component of \(G - v\), then \((A,B \cup \{v\})\) is an \((s,1)\)-polar partition of \(G\), a contradiction. Hence, \(B\) is contained in the non-complete component of \(G\). Clearly, \((G - v) - B\) contains \(P_3\) as an induced subgraph, and
thus, it cannot be covered by $A$, contradicting the choice of $(A, B)$ as an $(s, 1)$-polar partition of $G - v$. Since the contradiction arises from assuming that the complete component of $G$ has at least three vertices, then it should have at most two vertices.

So, it follows from the previous lemma that we can assume that every disconnected cograph minimal $(s, 1)$-obstruction contains either an isolated vertex or a component isomorphic to $K_2$. The following two lemmas describe the structure of the cograph minimal $(s, 1)$-obstructions with two components, other than the essential obstructions and $2K_{s+1}$. It is a bit surprising that for any integer $s$ greater than or equal to 2, there are only two such obstructions.

**Lemma 4.** Let $s$ be an integer, $s \geq 2$, and let $H$ be a connected cograph such that $G = H + K_2$ is a cograph minimal $(s, 1)$-polar obstruction. Then $G$ is isomorphic to $K_2 + (\overline{K_2} \oplus K_s)$.

**Proof.** Since $G$ is a cograph $(1, s)$-polar obstruction, $H$ is not a complete $s$-partite graph, so $H$ has $K_{s+1}$ or $P_3$ as induced subgraph. Nevertheless, if $H$ has $P_3$ as an induced subgraph, then $K_1 + 2K_2$ is an induced subgraph of $G$, contradicting the minimality of $G$. Therefore $H$ has $K_{s+1}$ as induced subgraph.

Let $K$ be a subset of $V(H)$ such that $H[K] \cong K_{s+1}$. Since $H$ is $P_3$-free, each vertex of $H$ that is not in $K$ is adjacent to every vertex in $K$, except maybe to one of them. Moreover, since $G$ is a $(s, 1)$-polar obstruction, $H$ is not a complete graph, and in consequence there is a vertex $v$ of $H$ that is non-adjacent to at least one vertex in $K$. Note that $H[K \cup \{v\}]$ is isomorphic to $\overline{K_2} \oplus K_s$, and hence, $G$ has $K_2 + (\overline{K_2} \oplus K_s)$ as an induced subgraph. But it is easy to verify that $K_2 + (\overline{K_2} \oplus K_s)$ is a cograph minimal $(1, s)$-polar obstruction, so, from the minimality of $G$ we have that $G$ is isomorphic to $K_2 + (\overline{K_2} \oplus K_s)$.

**Lemma 5.** Let $s$ be an integer, $s \geq 2$, and let $H$ be a connected cograph such that $G = H + K_1$ is a cograph minimal $(s, 1)$-polar obstruction non isomorphic to $K_2 + (\overline{K_2} \oplus P_3)$. Then $G$ is isomorphic to $K_1 + (C_4 \oplus K_{s-1})$.

**Proof.** Since $G$ is not a split graph, $G$ have $2K_2$ or $C_4$ as an induced subgraph, and evidently these subgraphs must be induced subgraphs of $H$. Nevertheless, if $H$ have $2K_2$ as induced subgraph, then $G$ contains $K_1 + 2K_2$ as an induced subgraph, and by the minimality of $G$, it must be isomorphic to $K_1 + 2K_2$, contradicting that $G$ has only two connected components. So there is a subset $C$ of the vertex set of $H$ that induces a $C_4$.

Let $v$ a vertex of $H$ that is not in $C$, which must exist, or else $G$ would be $(2, 1)$-polar. Then $v$ must be adjacent to some vertex of $C$, otherwise $G$ would have $\overline{K_2} + C_4$ as induced subgraph, which is not possible. On the other hand, since $G$ is a cograph, $v$ cannot be adjacent to exactly one vertex of $C$ nor can be adjacent to exactly two adjacent vertices of $C$. Furthermore, if $v$ is adjacent to three vertices of $C$, then $C \cup \{v\}$ induces $\overline{K_2} \oplus P_3$, and therefore $G$ has $K_2 + (\overline{K_2} \oplus P_3)$ as an induced subgraph, which by the minimality of $G$ implies that $G$ is isomorphic to $K_2 + (\overline{K_2} \oplus P_3)$, but we are assuming that $G$ is not. So we have that every vertex of $H$ that is not in $C$ must be adjacent to every vertex of $C$, or must be adjacent to exactly a pair of non adjacent vertices of $C$.
Let $D$ be the graph induced by the subset of vertices of $H$ that are not in $C$ but such that are adjacent to every vertex in $C$. Notice that if $D$ were a complete $(s-2)$-partite graph, then $H$ would be a complete $s$-partite graph, and therefore $G$ would be a $(s,1)$-polar graph. Thus, since we are assuming that $G$ is a $(s,1)$-polar obstruction, $D$ cannot be a complete $(s-2)$-partite graph, and in consequence $D$ has $P_3$ or $K_{s-1}$ as an induced subgraph.

Nevertheless, we claim that $D$ is a $P_3$-free graph. Otherwise, if $D$ has $P_3$ as an induced subgraph, then, together with any two non adjacent vertex of $C$ this would induce a $K_2 \oplus P_3$, which cannot occur. Then $D$ has $K_{s-1}$ as an induced subgraph, and hence $G$ have $K_1 + (C_4 \oplus K_{s-1})$ as induced subgraph. But $K_1 + (C_4 \oplus K_{s-1})$ is a cograph minimal $(s,1)$-polar obstruction, so $G$ is isomorphic to $K_1 + (C_4 \oplus K_{s-1})$.

So far, we have characterized all disconnected cograph minimal $(s,1)$-polar obstructions, which are a constant number for any choice of $s$. Taking into account that the number of minimal $(s,0)$-polar obstructions is two, regardless of the choice of $s$, it would seem possible to have a constant number of cograph minimal $(s,1)$-polar obstructions, we would only need to show that the number of such connected obstructions is a constant independent of $s$. Unfortunately, this will not be the case. It is easy to verify that a cograph $G$ is a minimal $(s,1)$-polar obstruction if and only if $G$ is a minimal $(1,s)$-polar obstruction. Thus, in order to characterize the connected cograph minimal $(s,1)$-polar obstructions, we will study their complements, the disconnected cograph minimal $(1,k)$-polar obstructions.

**Lemma 6.** Let $k$ be a nonnegative integer, and let $G$ be a cograph minimal $(1,k)$-polar obstruction. Then every component of $G$ is nontrivial, and if $G$ is not isomorphic to $(k+1)K_2$ then $G$ has at most $k$ components.

**Proof.** Suppose for a contradiction that $G$ has an isolated vertex $v$. Since $G$ is a cograph minimal $(1,k)$-polar obstruction, $G - v$ admits a $(1,k)$-polar partition $(A,B)$, but in such case $(A \cup \{v\}, B)$ is a $(1,k)$-polar partition of $G$, contradicting the minimality of $G$. Thus, we conclude that every component of $G$ has at least two vertices.

On the other hand, $H = (k+1)K_2$ is a cograph $(1,k)$-polar obstruction, because every $k$-cluster $K$ of $H$ intersect at most $k$ components of $H$, and therefore $H - K$ is a nonempty graph. Furthermore for every vertex $v \in V(H)$, $H - v$ is isomorphic to $kK_2 + K_1$, which is clearly a $(1,k)$-polar cograph, so $H$ is a cograph minimal $(1,k)$-polar obstruction.

Finally, if $G$ has more than $k$ components, since none of them is an isolated vertex, $G$ has $(k+1)K_2$ as an induced subgraph, so that $G \cong (k+1)K_2$. Thus, if $G \not\cong (k+1)K_2$, then $G$ has at most $k$ components.

**3 Main results**

In this section we will obtain a recursive characterization of disconnected cograph minimal $(1,k)$-polar obstructions to achieve our goal of characterizing all cograph minimal $(s,1)$-polar obstructions. We begin by describing a construction of a cograph minimal $(1,k)$-polar obstruction as a disjoint union of smaller minimal polar obstructions.
Lemma 7. Let \( t \) be an integer, \( t \geq 2 \), and for each \( i \in \{1, \ldots, t\} \), let \( G_i \) be a connected cograph minimal \((1, k_i)\)-polar obstruction that is a \((1, k_i + 1)\)-polar graph. Then, for \( m = t - 1 + \sum_{i=1}^{t} k_i \), the graph \( G = G_1 + \ldots + G_t \) is a cograph minimal \((1, m)\)-polar obstruction that is a \((1, m + 1)\)-polar graph.

Proof. Let \( G_1, \ldots, G_t \) and \( G \) be as in the hypothesis. We first prove by means of a contradiction that \( G \) is a cograph \((1, m)\)-polar obstruction. Suppose that \( G \) admits a \((1, m)\)-polar partition \((A, B)\), and define for each \( i \in \{1, 2, \ldots, t\} \) the sets \( A_i = V(G_i) \cap A \) and \( B_i = V(G_i) \cap B \). Note that every component of \( G[B] \) is contained in a component of \( G \). Denote the number of components of \( G_i[B_i] \) by \( l_i \); if \( k_i < l_i \) for every \( i \in \{1, \ldots, t\} \), we would have \( m + 1 = \sum_{i=1}^{t} (k_i + 1) \leq \sum_{i=1}^{t} l_i = m \), a contradiction. Hence, there is \( j \in \{1, \ldots, t\} \) such that \( l_j \leq k_j \). Nevertheless, we have that \( G_j \) is a cograph \((1, k_j)\)-polar obstruction and \((A_j, B_j)\) is a \((1, l_j)\)-polar partition of \( G_j \), so that \( k_i < l_i \) for every \( i \in \{1, \ldots, t\} \), contradicting our previous argument. Since the contradiction arises from assuming that \( G \) is a \((1, m)\)-polar cograph, we conclude that \( G \) is a cograph \((1, m)\)-polar obstruction.

Now we prove that \( G \) is minimal. If \( v \in V(G) \), then \( v \in V(G_j) \) for some \( j \in \{1, \ldots, t\} \), say, without loss of generality, for \( j = 1 \). Since \( G_1 \) is a cograph minimal \((1, k_1)\)-polar obstruction, the graph \( G_1 - v \) admits a \((1, k_1)\)-polar partition \((A_1, B_1)\), and since by hypothesis \( G_i \) is a \((1, k_i + 1)\)-cograph for each \( i \in \{2, 3, \ldots, t\} \), we have that \( G_i \) admits a \((1, k_i + 1)\)-polar partition \((A_i, B_i)\). Therefore, \( G - v \) is a \((1, m)\)-polar cograph with \((A, B)\) a \((1, m)\)-polar partition of \( G \), and therefore \( G \) is a \((1, m)\)-polar cograph.

Finally, since for each \( i \in \{1, 2, \ldots, t\} \) the graph \( G_i \) admits a \((1, k_i + 1)\)-polar partition \((A_i, B_i)\), then \( (\bigcup_{i=1}^{t} A_i, \bigcup_{i=1}^{t} B_i) \) is a \((1, m + 1)\)-polar partition of \( G \), and therefore \( G \) is a \((1, m \pm 1)\)-cograph.

Our goal is to prove that the cographs described in Lemma 7 are the only disconnected cograph minimal \((1, k)\)-polar obstructions. In order to achieve this we need the following technical, yet simple, result.

Lemma 8. Let \( t \) be an integer, \( t \geq 2 \), and for each \( i \in \{1, \ldots, t\} \), let \( G_i \) be a connected cograph minimal \((1, k_i)\)-polar obstruction that is a \((1, k_i + 1)\)-polar graph. Then, for \( m = t - 1 + \sum_{i=1}^{t} k_i \) and for any non negative integer \( \mu \), \( \mu < m \), \( G \) is not a cograph minimal \((1, \mu)\)-polar obstruction.

Proof. By considering the different cases in the characterization of disconnected cograph minimal \((s, 1)\)-polar obstructions, it is not hard to verify that any connected cograph minimal \((1, s)\)-polar obstruction \( G \) that is \((1, s + 1)\)-polar contains, for any non negative integer \( \sigma \) such that \( \sigma < s \), a proper induced subgraph \( G' \) that is both, a cograph minimal \((1, \sigma)\)-polar obstruction and a \((1, \sigma + 1)\)-polar graph.

Let \( \mu \) be a positive integer such that \( \mu < m \), and let \( s_1, \ldots, s_t \) be integers such that, for \( i \in \{1, \ldots, t\} \), \( 0 \leq s_i \leq k_i \) and \( \mu = t - 1 + \sum_{i=1}^{t} s_i \). By the choice of \( \mu \), \( s_i < k_i \) for at least one \( i \in \{1, \ldots, t\} \). For each \( i \in \{1, \ldots, t\} \), if \( s_i < k_i \) let \( H_i \) be a proper induced subgraph of \( G_i \) that is both, a cograph minimal \((1, s_i)\)-polar obstruction and a \((1, s_i + 1)\)-polar graph,
otherwise let \( H_i = G_i \). Then, by Lemma 7, \( H = H_1 + \cdots + H_t \) is a cograph minimal \((1, \mu)\)-polar obstruction that is a proper induced subgraph of \( G \), and therefore \( G \) is not a cograph minimal \((1, \mu)\)-polar obstruction.

We conclude the analysis of the disconnected cograph minimal \((1, k)\)-polar obstructions by showing that the cographs described in Lemma 7 are the only ones.

**Lemma 9.** Let \( G \) be a disconnected cograph minimal \((1, k)\)-polar obstruction with components \( G_1, \ldots, G_t \). Then, there exist non negative integers \( k_1, \ldots, k_t \) such that for each \( i \in \{1, \ldots, t\} \), \( G_i \) is a connected cograph minimal \((1, k_i)\)-polar obstruction that is a \((1, k_i + 1)\)-polar cograph, and \( \sum_{i=1}^t k_i = k - t + 1 \).

**Proof.** Since \( G \) is a cograph minimal \((1, k)\)-polar obstruction we have that, for each \( i \in \{1, \ldots, t\} \), the component \( G_i \) of \( G \) is a \((1, k)\)-polar graph. For each \( i \in \{1, \ldots, t\} \) and each \( v \in V(G_i) \), let \( k_v \) be the minimum non negative integer such that \( G_i - v \) is a \((1, k_v)\)-polar graph, and let \( k_i \) be the maximum of \( k_v \) on all the vertices \( v \) of \( G_i \), that is, \( k_i = \max\{k_v: v \in V(G_i)\} \). Note that for each \( i \in \{1, \ldots, t\} \) and any \( v \in V(G_i) \), \( G_i - v \) is a \((1, k_i)\)-polar graph.

Moreover, we claim that for each \( i \in \{1, \ldots, t\} \), the graph \( G_i \) is not \((1, k_i)\)-polar. Suppose for a contradiction that for some \( i \in \{1, \ldots, t\} \), \( G_i \) is a \((1, k_i)\)-polar graph, we will assume \( i = 1 \) without loss of generality. Let \( \{X_1, Y_1\} \) be a \((1, k_1)\)-polar partition of \( G_1 \), and let \( v \in V(G_1) \) such that \( G_1 - v \) is \((1, k_1)\)-polar but it is not \((1, k_1 - 1)\)-polar. Let \( \{A, B\} \) be a \((1, k)\)-polar partition of \( G - v \). For every \( i \in \{1, \ldots, t\} \) define \( A_i \) and \( B_i \) in the following way, \( A_1 = A \cap V(G_1 - v), B_1 = B \cap V(G_1 - v) \), and for each \( j \in \{2, \ldots, t\} \), let \( A_j = A \cap V(G_j) \) and \( B_j = B \cap V(G_j) \). Then \( \{(A \setminus A_1) \cup X_1, (B \setminus B_1) \cup Y_1\} \) is a \((1, k)\)-polar partition of \( G \), a contradiction.

Thus, for each \( i \in \{1, \ldots, t\} \), \( G_i \) is a connected cograph minimal \((1, k_i)\)-polar obstruction that is \((1, k)\)-polar, and in consequence \( \overline{G_i} \) is a disconnected cograph minimal \((k_i, 1)\)-polar obstruction that is a \((k, 1)\)-polar graph. Observe that by Lemmas 2 to 6 this implies that \( \overline{G_i} \) is one of \( 2K_{k_i+1}, K_2 + (\overline{K_2} \oplus K_{k_i}) \) or \( K_1 + (C_4 \oplus K_{k_i-1}) \), and then, \( G_i \) is a disconnected cograph minimal \((k_i, 1)\)-polar obstruction that is \((k_i + 1, 1)\)-polar. Equivalently, we have that \( G_i \) is a connected cograph minimal \((1, k_i)\)-polar obstruction that is \((1, k_i + 1)\)-polar graph.

Finally, by Lemmas 7 and 8 we have that, for \( m = t - 1 + \sum_{i=1}^t k_i \), \( G \) is a cograph minimal \((1, m)\)-polar obstruction that is a \((1, m+1)\)-polar graph, and that \( G \) is not a cograph minimal \((1, \mu)\)-polar obstruction for any integer \( \mu \) with \( 0 \leq \mu < m \). Thus, since we are assuming that \( G \) is a cograph minimal \((1, k)\)-polar obstruction, we have that \( k = m \) and the result follows.

Hence, we are ready to state our main result.

**Theorem 10.** Let \( G \) be a cograph, and let \( s \) be an integer, \( s \geq 2 \). Then \( G \) is a minimal \((s, 1)\)-polar obstruction if and only if it is one of the following:

- One of the four essential obstructions depicted in Figure 1, i.e., \( K_1 + 2K_2, \overline{K_2} + C_4, 2P_3 \) or \( K_1 + (P_3 \oplus \overline{K_2}) \).
- \( 2K_{s+1} \).
• $K_2 + (K_2 \oplus K_s)$.
• $K_1 + (C_4 \oplus K_{s-1})$.
• $(s + 1)K_2$.
• The complement of $G$ is disconnected with components $G_1, \ldots, G_t$, such that $t \leq s$, $G_i$ is the complement of a non-essential disconnected cograph minimal $(s_i, 1)$-polar obstruction and $\sum_{i=1}^{t} s_i = s - t + 1$.

Proof. It is an immediate consequence of all previous lemmas.

To finish this section, we will prove that the four essential obstructions in Figure 1 constitute the set of minimal forbidden induced subgraphs for a cograph to admit an $(s, 1)$-polar partition for some integer $s$, $s \geq 2$.

Lemma 11. Let $s$ be an integer. If $G$ is a cograph minimal $(s, 1)$-polar obstruction that is not essential, then the order of $G$ is at least $s + 1$.

Proof. We will proceed by induction on $s$. The unique cograph minimal $(s, 1)$-polar obstruction for $s = 0$ is $2K_1$, while the unique two cograph minimal $(1, 1)$-polar obstructions are $C_4$ and $2K_2$. This deals with the base case.

Let $s$ be an integer, $s \geq 2$, and suppose that for every integer $N$ such that $N < s$, if $H$ is a non-essential cograph minimal $(N, 1)$-polar obstruction, then $H$ has at least $N + 1$ vertices.

Let $G$ be a non-essential cograph minimal $(s, 1)$-polar obstruction. Observe that if $G$ is disconnected, then by Lemmas 3, 4 and 5, the order of $G$ is strictly greater than $s$. Else, $G$ is a connected cograph and its complement, $\overline{G}$, is a disconnected cograph minimal $(1, s)$-polar obstruction; Lemmas 6 and 9 imply that either $\overline{G}$ is isomorphic to $(s + 1)K_2$, which clearly has strictly more than $s + 1$ vertices, or the components of $\overline{G}$ are $G_1, \ldots, G_t$ for some integer $t \in \{2, \ldots, s\}$, where $G_i$ is a cograph minimal $(1, s_i)$-polar obstruction for $1 \leq i \leq t$ and some nonnegative integer $s_i$, and $\sum_{i=1}^{t} s_i = s - t + 1$. Nevertheless, in the latter case we have by induction hypothesis that for every $i \in \{1, \ldots, t\}$, the order of $G_i$ is at least $s_i + 1$, which implies that

$$|\overline{G}| = |G_1| + \cdots + |G_t| \geq (s_1 + 1) + \cdots + (s_t + 1) = s_1 + \cdots + s_t + t = s - t + 1 + t = s + 1,$$

which ends the proof.

Theorem 12. Let $G$ be a cograph. Then $G$ admits an $(s, 1)$-polar partition for some $s \geq 2$ if and only if it does not contain any of the essential obstructions (Figure 1) as an induced subgraph.
Proof. Let \( G \) be a cograph such that for every integer \( s, s \geq 2 \), \( G \) is not an \((s, 1)\)-polar cograph. Particularly \( G \) is not a \((n, 1)\)-polar cograph, where \( n \) stands for the order of \( G \), and therefore \( G \) contains a cograph minimal \((n, 1)\)-polar obstruction \( H \) as induced subgraph. If \( H \) is not essential, then, by Lemma 11 we have that \( H \) has order at least \( n + 1 \), which is impossible since \( H \) is a subgraph of \( G \). Thus \( G \) contains an essential obstruction as an induced subgraph. The converse implication follows directly from Lemma 2.

\[ \square \]

4 On the number of cograph minimal \((s, 1)\)-polar obstructions

Taking into consideration the number of cograph minimal \((s, 1)\)-polar obstructions for \( s \in \{0, 1, 2\} \), it would seem that the number of this obstructions does not grow too fast. Nonetheless, a quick estimation shows that the growth rate of the families of minimal obstructions is subexponential at best, and we have exponential upper bounds (with an extremely bad overestimation).

Let \( s \) be an integer, \( s \geq 2 \). In view of Lemmas 2 to 4 there are exactly seven disconnected cograph minimal \((s, 1)\)-polar obstructions, namely \( 2K_{s+1}, K_1 + (C_4 \oplus K_{s-1}), K_2 + (K_2 \oplus K_s) \), and the four essential obstructions depicted in Figure 1. Observe that the complements of the first three graphs mentioned above are the unique connected cograph minimal \((1, s)\)-polar obstructions that are \((1, s + 1)\)-polar cographs.

On the other hand, to count the number of connected cograph minimal \((s, 1)\)-polar obstructions is equivalent to count the number of disconnected cograph minimal \((1, s)\)-polar obstructions. Furthermore, by Lemma 3 each disconnected cograph minimal \((1, s)\)-polar obstruction \( G \) with components \( G_0, \ldots , G_k \) satisfies that \( G_i \) is a connected cograph minimal \((1, s_i)\)-polar obstruction that is a \((1, s_i + 1)\)-polar cograph for each \( i \in \{0, \ldots , k\} \), with \( s = s_0 + \cdots + s_k + k \) where each term is a non negative integer. Since there is exactly one connected cograph minimal \((1, s_i)\)-polar obstruction for \( s_i \in \{0, 1\} \), and there are exactly three of them which are connected for \( s_i \geq 2 \) we have the following.

Proposition 13. Let \( s \) be an integer, \( s \geq 2 \). If \( s \) is expressed as a sum of non negative integers, \( s = s_0 + s_1 + \cdots + s_k + k \), and there are exactly \( n \) of the terms \( s_i \) greater than 1, then there are at most \( 3^n \) non isomorphic disconnected cograph minimal \((1, s)\)-polar obstructions \( G \) with connected components \( G_0, \ldots , G_k \) such that \( G_i \) is a cograph minimal \((1, s_i)\)-polar obstruction for each \( i \in \{0, \ldots , k\} \).

Let \( s \) be a non negative integer, and let \( D(s) \) be the number of distinct ways in which \( s \) can be expressed as a sum \( s = s_0 + s_1 + \cdots + s_k + k \), where \( k \geq 1 \) and \( s_i \) is a non negative integer for each \( i \in \{0, 1, \ldots , k\} \), and where we are considering two of this representations of \( s \) as the same when they correspond to a permutation of the terms \( s_i \). Thus, the preceding lemma gives straightforward bounds for the number of disconnected cograph minimal \((1, s)\)-polar cographs in terms of \( D(s) \).

Lemma 14. Let \( s \) be an integer, \( s \geq 2 \). Then the number of disconnected cograph minimal \((1, s)\)-polar obstructions, \( n(s) \), is such that

\[ D(s) \leq n(s) \leq 3^m \cdot D(s) < 3^{s/2} \cdot D(s), \]
where \( m \) is the maximum possible number of terms \( s_i \) greater than one in a decomposition \( s = s_0 + \cdots + s_k + k \) of \( s \) with \( k \geq 1 \).

**Proof.** The left inequality is due to the fact that for each decomposition of \( s \) as a sum of non negative integers \( s = s_0 + \cdots + s_k + k \), there is at least one disconnected cograph minimal \((1, s)\)-polar obstruction. The inequality in the middle is a direct consequence of Lemma 13, while the last inequality follows from the trivial fact that \( m < s/2 \).

It is evident that every non negative integer \( s \) is decomposed in a sum of non negative integers \( s = s_0 + \cdots + s_k + k \) with \( k \geq 1 \), if and only if \( s + 1 \) is decomposed in a sum of positive integers \( s + 1 = s'_0 + \cdots + s'_k \), where \( s'_i = s_i + 1 \) for \( 0 \leq i \leq k \). Thus, the number of distinct ways in which a positive integer \( s \) can be written as a sum of positive integers, denoted \( p(s) \), satisfies the equality \( D(s) = p(s + 1) - 1 \). The parameter \( p(s) \) has been extensively studied, and particularly, Hardy and Ramanujan gave in 1918 the following asymptotic approximation.

**Theorem 15.** Let \( p(n) \) be the number of ways of writing the positive integer \( n \) as a sum of positive integers, where the order of the terms is not considered. Then

\[
p(n) \sim \frac{1}{4n\sqrt{3}} \exp \left( \pi \sqrt{\frac{2n}{3}} \right).
\]

5 Conclusions

Exact lists of cograph minimal \((s, k)\)-polar obstructions are known when \( \max\{s, k\} \leq 2 \), and also when \( \min\{s, k\} = 0 \). The results in the present work seem to indicate that there are too many cograph minimal \((s, k)\)-polar obstructions to expect to find exhaustive lists for arbitrary values of \( s \) and \( k \). Nonetheless, it was a pleasant surprise to find a recursive characterization which is rather simple to obtain all the cograph minimal \((s, 1)\)-polar obstructions. This result makes us wonder whether a similar result may be achieved for any values of \( s \) and \( k \). In particular, we already have some encouraging partial results for the case when \( s = k \), showing that maybe a combination of recursion together with a classification of some families of minimal obstructions may cover the whole family of minimal obstructions.

Also, taking into account the results in [7] and Theorem 12 it seems possible to find the complete list of minimal obstructions to the problem of recognizing \((s, t)\)-polar cographs, for some integer \( t \) and a fixed integer \( s, s \geq 2 \).

References

[1] J.A. Bondy and U.S.R Murty, Graph Theory, Springer, Berlin, 2008.

[2] R.S.F. Bravo, L.T. Nogueira, F. Protti and C. Vianna, Minimal obstructions of \((2, 1)\)-cographs with external restrictions, in: Annals of I ETC - Encontro de Teoria da Computação (CSBC 2016) (2016) Porto Alegre, [http://www.pucrs.br/edipucrs/] ISSN 2175-2761 (In Portuguese).
[3] R. Churchley and J. Huang, On the Polarity and Monopolarity of Graphs, Journal of Graph Theory 76(2) (2014) 138–148.

[4] D. G. Corneil, H. Lerchs and L. Stewart Burlingham, Complement reducible graphs, Discrete Applied Mathematics 3(3) (1981) 163–174.

[5] Z.A. Chernyak and A.A. Chernyak, About recognizing \((\alpha, \beta)\)-classes of polar graphs, Discrete Math. 62 (1986) 133–138.

[6] P. Damaschke, Induced subgraphs and well-quasi-ordering, Journal of Graph Theory 14(4) (1990) 427–435.

[7] T. Ekim, N.V.R. Mahadev and D. de Werra, Polar cographs, Discrete Applied Mathematics 156 (2008) 1652–1660.

[8] T. Ekim, N.V.R. Mahadev and D. de Werra, Corrigendum to “Polar cographs”, Discrete Applied Mathematics 156 (2014) 158.

[9] A. Farrugia, Vertex-partitioning into fixed additive induced-hereditary properties is \(NP\)-hard, Electron. J. Combin. 11 (2004) #R46.

[10] T. Feder, P. Hell, S. Klein and R. Motwani, List Partitions, SIAM J. Discrete Math. 16(3) (2003) 449–478.

[11] T. Feder, P. Hell and W. Hochstättler, Generalized Colourings (Matrix Partitions) of Cographs, in: Graph Theory in Paris, Birkhäuser, 2006, 149–167.

[12] S. Foldes and P.L Hammer, Split graphs, in: Proc. 8th Sout-Eastern Conf. on Combinatorics, Graph Theory and Computing, 1977, 311–315.

[13] P. Hell, Graph partitions with prescribed patterns, European Journal of Combinatorics 35 (2014) 335–353.

[14] P. Hell, C. Hernández-Cruz and C. Linhares-Sales, Minimal Obstructions to 2-polar partitions, Discrete Applied Mathematics, accepted.