On the star partition dimension of comb product of cycle and complete graph

Ridho Alfarisi¹, Darmaji¹ and Dafik²
¹Department of Mathematics, Sepuluh Nopember Institute of Technology, Surabaya, Indonesia
²CGANT-University of Jember, Indonesia
E-mail: alfarisi38@gmail.com, darmaji@matematika.its.ac.id, d.dafik@unej.ac.id

Abstract. Let G = (V, E) be a connected graphs with vertex set V(G), edge set E(G) and S ⊆ V(G). For an ordered partition Π = {S₁, S₂, S₃, . . . , Sₖ} of V(G), the representation of a vertex v ∈ V(G) with respect to Π is the k-vectors r(v)[Π] = (d(v, S₁), d(v, S₂), . . . , d(v, Sₖ)), where d(v, Sᵈ) represents the distance between the vertex v and the set Sᵈ, defined by d(v, Sᵈ) = min{d(v, x)|x ∈ Sᵈ}. The partition Π of V(G) is a resolving partition if the k-vectors r(v)[Π], v ∈ V(G) are distinct. The minimum resolving partition Π is a partition dimension of G, denoted by pd(G). The resolving partition Π = {S₁, S₂, S₃, . . . , Sₖ} is called a star resolving partition for G if it is a resolving partition and each subgraph induced by Sᵈ, 1 ≤ i ≤ k, is a star. The minimum k for which there exists a star resolving partition of V(G) is the star partition dimension of G, denoted by spd(G). Finding a star partition dimension of G is classified to be a NP-Hard problem. Furthermore, the comb product between G and H, denoted by G ⊿ H, is a graph obtained by taking one copy of G and |V(G)| copies of H and grafting the i-th copy of H at the vertex o to the i-th vertex of G. By definition of comb product, we can say that V(G ⊿ H) = {(a, u)|a ∈ V(G), u ∈ V(H)} and (a, u)(b, v) ∈ E(G ⊿ H) whenever a = b and uv ∈ E(H), or ab ∈ E(G) and u = v = o. In this paper, we will study the star partition dimension of comb product of cycle and complete graph, namely Cₙ ⊿ Kₘ and Kₘ ⊿ Cₙ for n ≥ 3 and m ≥ 3.

Keywords: Star resolving partition, star partition dimension, comb product, cycle, complete graph.

1. Introduction

All graphs in this paper are nontrivial, finite, simple, undirected and connected, for detail definition of graph see [1, 2]. The concept of metric dimension was independently introduced by Slater [3] and Harrary and Melter [4]. In his paper, Slater called this concept as a locating set. Chartrand et.al [5] introduced the concept of partition dimension of G. They introduced the same concept of resolving partition with a partition dimension of graphs. For S ⊆ V(G) with vertex v ∈ V(G), the distance between v and S is d(v, S) = min{d(v, x)}. Given k—partition Π = {S₁, S₂, S₃, . . . , Sₖ} of V(G). The representation v to Π is defined by ordered r(v)[Π] = (d(v, S₁), d(v, S₂), . . . , d(v, Sₖ)) for every v ∈ V(G). If Π of V is a resolving partition then the different vertices of G have distinct representations, i.e., for every pair of vertices u, v ∈ V, r(u)[Π] ≠ r(v)[Π]. The minimum of k such that the resolving partition Π is partition dimension of G is called the partition dimension number of G, denoted by pd(G).
Furthermore, Chartrand et al. [6] determined the bounds of the metric dimensions for any connected graphs and determined the metric dimensions of some well known families of graphs such as tree, path, and complete graph. Furthermore, Saenpholphat et al. in [7] studied a natural extension of partition dimension, they studied a particular case of resolving partition, i.e., connected resolving partition. II = \{S_1, S_2, ..., S_k\} is a connected resolving partition if it is a resolving partition and each subgraph induced by \(S_i, 1 \leq i \leq k\), is connected in \(G\). The minimum \(k\) for which there is a connected resolving partition of \(V(G)\) is said to be connected partition dimension number of \(G\), denoted by \(\text{cpd}(G)\).

In this paper we consider a particular case of resolving partition, namely star resolving partition. We say \(II = \{S_1, S_2, ..., S_k\}\) is said to be a star resolving partition if it is a resolving partition and each subgraph induced by \(S_i, 1 \leq i \leq k\), is star in \(G\). The minimum \(k\) for which there is a star resolving partition of \(V(G)\) is the star partition dimension number of \(G\), denoted by \(\text{spd}(G)\).

There are many results related to the star partition dimension study. Saenpholphat et al. in [7] studied the connected partition dimension of graphs, Steiner in [8] on the \(k\)-path partition of graphs. Ghemeci [9] studied on star partition dimension of Trees. Furthermore, Ghemeci and Tomescu [10] determine on star partition dimension of the generalized of gear graph, Amalia [11] studied star partition dimension of generalized windmill graph.

**Lemma 1.1** [6] Let \(II = \{S_1, S_2, ..., S_k\}\) be a resolving partition of \(G\) and \(u, v \in V(G)\). If \(d(u, w) = d(v, w)\) for all \(w \in V(G) - \{u, v\}\), then \(u\) and \(v\) belong to distinct elements of \(II\).

**2. Main Results**

Let \(G\) and \(H\) be two connected graphs. Let \(o\) be a vertex of \(H\). The comb product between \(G\) and \(H\), denoted by \(G \triangleright H\), is a graph obtained by taking one copy of \(G\) and \(|V(G)|\) copies of \(H\) and grafting the \(i\)-th copy of \(H\) at the vertex \(o\) to the \(i\)-th vertex of \(G\). By the definition of comb product, we can say that \(V(G \triangleright H) = \{(a, u) | a \in V(G), u \in V(H)\}\) and \((a, u)(b, v) \in E(G \triangleright H)\) whenever \(a = b\) and \(uv \in E(H)\), or \(ab \in E(G)\) and \(u = v = o\). In this section, we will show the results related to star partition dimension of comb product of cycle and complete graph namely \(C_m \triangleright K_n\) and \(K_n \triangleright C_m\) for \(n \geq 3\) and \(m \geq 3\).

There are two theorems that we have found in this research. We will prove it in separate way by introducing an observation in each theorem. The first, we will deal with the star partition dimension of \(C_m \triangleright K_n\).

**Observation 2.1** Let \(C_m\) and \(K_n\), respectively be a cycle and complete graph. The order and the size of \(C_m \triangleright K_n\) are \(|V(C_m \triangleright K_n)| = mn\) and \(|E(C_m \triangleright K_n)| = \frac{n^2m - mn + 2m}{2}\), respectively.

**Proof:** The comb product between \(C_m\) and \(K_n\), denoted by \(C_m \triangleright K_n\) is a graph obtained by taking one copy of \(C_m\) and \(|V(C_m)|\) copies of \(K_n\) and grafting the \(i\)-th copy of \(K_n\) at the vertex \(o\) to the \(i\)-th vertex of \(C_m\). A graph \(C_m \triangleright K_n\) has vertex set \(V(C_m \triangleright K_n) = \{y_{j,i} | 1 \leq j \leq m, 1 \leq i \leq n\}\) and edge set \(E(C_m \triangleright K_n) = \{y_{j,i}y_{j+1,i} | 1 \leq j \leq m - 1\} \cup \{y_{m,i}y_{1,i+1}\} \cup \{y_{j,i}y_{j,i+k} | 1 \leq j \leq m, 1 \leq i \leq n, 1 \leq k \leq n - i\}\). Thus, \(|V(C_m \triangleright K_n)| = mn\) and \(|E(C_m \triangleright K_n)| = \frac{n^2m - mn + 2m}{2}\). \(\square\)

Figure 1 (a) shows an example of \(C_m \triangleright K_n\).

**Theorem 2.1** Let \(C_m\) and \(K_n\) be two connected graphs of order \(m \geq 3\) and \(n \geq 3\), respectively. Then star partition dimension of comb product of cycle and complete graph is \(\text{spd}(C_m \triangleright K_n) = m(n - 1)\).

**Proof:** We will determine the star partition dimension of \(C_m \triangleright K_n\). The first step, we will determine the upper bound of star partition dimension obtained by construction star resolving partition. Suppose that \(\Pi_s = \{S_1, S_2, S_3, ..., S_{m(n-1)}\}\) where \(S_{(n-1)(j-1)+i} = \{y_{j,i} | 1 \leq j \leq m, 2 \leq i \leq n\}\), and \(S_{(n-1)j-n+2} = \{y_{j,i} | 1 \leq j \leq m\}\). It can be seen that vertices in
resolving partition that consists of upper bound of star partition dimension is trivial graph, it induces a star graph and every vertex $v$ of $(\text{spd}_C \sigma)$ there are two cases as follow:

The proof.

Thus, it concludes the lower bound of star partition dimension $\text{C}_m \triangleright K_n$ is $\text{spd}(\text{C}_m \triangleright K_n) \geq (m - 1)$. Now, we consider the subgraph is induced of vertices in one partition class $\Pi_S$. Without loss of generality, suppose that $\Pi_S = \{S_1, S_2, S_3, \ldots, S_{m(n-1)-1}\}$ then there are two cases as follow:

a) If any vertices $u, v \in V(\text{C}_m \triangleright K_n)$ with $u, v$ pass subgraph $K_n$ then, based on Lemma 1.1, it will be obtained a same vertex representation to $\Pi_S$. Given a partition class $S_{nm-1} = \{y_{m,n-1}, y_{m,n}\}$, the vertices $y_{m,n-1}$ and $y_{m,n}$ have the same distance to all vertices in $V(\text{C}_m \triangleright K_n) - \{y_{m,n-1}, y_{m,n}\}$ or it can be written $d(y_{m,n-1}, w) = d(y_{m,n}, w)$ where $w \in V(\text{C}_m \triangleright K_n) - \{y_{m,n-1}, y_{m,n}\}$. It implies that $r(y_{m,n-1}\Pi_S) = r(y_{m,n}\Pi_S)$. Thus, the partition class $\Pi_S$ with $|\Pi_S| = m(n-1) - 1$ is not star resolving partition even though the partition class $S_{nm-1}$ induces star graph.

b) Suppose we take a partition class of $S_{m(n-1)(m-2)+1} = \{y_{m-1,2}\}$, $S_{m(n-1)(m-1)+1} = \{y_{m,2}\}$, $S_{m(n-1)(m-1)-n+2} = \{y_{m-1,1}\}$ and $S_{(n-1)m-n+2} = \{y_{m,1}\}$. If we combine it in one partition class of $S_k$ then we can see that the partition class $S_k$ does not induces a star graph. Thus the $\Pi_S$ is not star resolving partition even though the partition class $S_k$ has distinct vertex representation to $\Pi_S$.

Thus, we obtain that $\Pi_S$ with cardinality $|\Pi_S| = m(n-1) - 1$ is not star resolving partition. It concludes the lower bound of star partition dimension $\text{C}_m \triangleright K_n$ is $\text{spd}(\text{C}_m \triangleright K_n) \geq (m - 1)$.

Therefore the star partition dimension $\text{spd}(\text{C}_m \triangleright K_n) = (m - 1)$ for $n \equiv 0(\text{mod} 3)$. It concludes the proof.

Now, we are dealing with the star partition dimension of $K_n \triangleright C_m$.

**Figure 1.** (a) A graph of $C_m \triangleright K_n$, (b) Construction of star resolving partition $\Pi_S$.
Theorem 2.2 Let $K_n \triangleright C_m$ be a connected graph. The order and the size of $K_n \triangleright C_m$ are $|V(K_n \triangleright C_m)| = mn$ and $|E(K_n \triangleright C_m)| = \frac{n^2 + 2mn - n}{2}$, respectively.

**Proof:** The comb product between $K_n$ and $C_m$, denoted by $K_n \triangleright C_m$ is a graph obtained by taking one copy of $K_n$ and $|V(K_n)|$ copies of $C_m$ and grafting the $i-$th copy of $C_m$ at the vertex $o$ to the $i-$th vertex of $K_n$. A graph $K_n \triangleright C_m$ has the vertex set $V(K_n \triangleright C_m) = \{y_{i,j} | 1 \leq j \leq m, 1 \leq i \leq n\}$ and the edge set $E(K_n \triangleright C_m) = \{y_{i,1}y_{i+1,1} | 1 \leq i \leq n, 1 \leq k \leq n - i\} \cup \{y_{i,1}y_{i,m} | 1 \leq i \leq n\} \cup \{y_{i,j}y_{i,j+1} | 1 \leq i \leq n, 1 \leq j \leq m - 1\}$. Thus, $|V(K_n \triangleright C_m)| = mn$ and $|E(K_n \triangleright C_m)| = \frac{n^2 + 2mn - n}{2}$.

Figure 2 (a) shows an example of comb product of $K_n \triangleright C_m$.

**Theorem 2.2** Let $K_n$ and $C_m$ be a connected graphs with order $n \geq 3$ and $m \geq 3$, respectively. Then star partition dimension of comb product of complete graph and cycle is

$$\text{spd}(K_n \triangleright C_m) = \begin{cases} n\left(\frac{m}{3}\right) + 1, & \text{if } 3 \leq m \leq 4 \\ n\left(\frac{m}{3}\right), & \text{if } m \equiv 0(\text{mod } 3), m \equiv 2(\text{mod } 3) \\ n\left(\frac{m}{3}\right) + 1, & \text{if } m \equiv 1(\text{mod } 3) \end{cases}$$

**Proof:** Now, we will determine the star partition dimension of $K_n \triangleright C_m$. We will divide the proof in three cases.

**Case 1:** For $m \equiv 0(\text{mod } 3)$

The upper bound of star partition dimension obtained by constructing a star resolving partition. Suppose that $\Pi_S = \{S_1, S_2, S_3, ..., S_{n\left(\frac{m}{3}\right)}\}$ obtained from the following

$$S_{n\left(\frac{m}{3}\right)}(i-1)+k = \{y_{i,j} | 1 \leq k \leq \frac{m}{3}, 3(k-1) + 1 \leq j \leq 3k, 1 \leq i \leq n\}$$

It can be seen that vertices in $S_{n\left(\frac{m}{3}\right)}(i-1)+k$ induces a star $K_{1,2}$. Thus, it can be shown the representation every vertex $v \in V(K_n \triangleright C_m)$ is distinct to $\Pi_S$. Thus, $\Pi_S = \{S_1, S_2, S_3, ..., S_{n\left(\frac{m}{3}\right)}\}$ is a star resolving partition that consists of $n\left(\frac{m}{3}\right)$ partition class. The cardinality of $\Pi_S$ is $|\Pi_S| = n\left(\frac{m}{3}\right)$. However, $\Pi_S$ is not necessarily to have minimum cardinality. Thus, the upper bound of star partition dimension $K_n \triangleright C_m$ is $\text{spd}(K_n \triangleright C_m) \leq n\left(\frac{m}{3}\right)$.
Furthermore, the lower bound of the star partition dimension $K_n \triangleright C_m$ can be obtained by Lemma 1.1. Now, we consider the subgraph which is induced by vertices in one partition class of a star graph. If $\Pi_S$ has cardinality $|\Pi_S| = n(\frac{m}{3}) - 1$, then there are at least one partition class that does not induces a star graph. Notice that the vertices in partition class $\Pi_S$ are vertices of $V(C_m)$. Without loss of generality, suppose that $\Pi_S = \{S_1, S_2, S_3, ..., S_{n(\frac{m}{3})-1}\}$, then there are partition class that does not induces a star graph namely $S_{n(\frac{m}{3})-1} = \{y_{m,j}\ | \ 3(k-1) + 1 \leq j \leq 3k\}$. Thus, we obtain that $\Pi_S$ of the cardinality $|\Pi_S| = n(\frac{m}{3}) - 1$ is not star resolving partition. It implies that the lower bound of the star partition dimension $K_n \triangleright C_m$ is $\text{spd}(K_n \triangleright C_m) \geq n(\frac{m}{3})$. Once we have the same upper bound and lower bound of star partition dimension $K_n \triangleright C_m$ then star partition dimension $\text{spd}(K_n \triangleright C_m) = n(\frac{m}{3})$ for $m \equiv 0(\text{mod } 3)$ and $m \geq 5$.

For $m = 3$. Suppose lower bound of the star partition dimension is $\text{spd}(K_n \triangleright C_m) \geq n(\frac{m}{3})$. Based on Lemma 1.1, there are some vertex representations in $V(K_n \triangleright C_m)$, namely $r(y_{i,2})\Pi_S = r(y_{i,3})$ for $1 \leq i \leq n$ cause the vertices $y_{i,2}$ and $y_{i,3}$ have to be on different partition class. Thus that lower bound of the star partition dimension:

$$\text{spd}(K_n \triangleright C_m) \geq \frac{(m}{3} + 1) + \frac{(m}{3} + 1) + ... + \frac{(m}{3} + 1) = n\left(\frac{m}{3} + 1\right)$$

Thus, the lower bound of the star partition dimension $P_n \triangleright C_m$ is $\text{spd}(K_n \triangleright C_m) \geq n(\frac{m}{3} + 1)$.

Now, we will determine the upper bound by constructing the star resolving partition namely $\Pi_S = \{S_1, S_2, S_3, ..., S_{n(\frac{m}{3} + 1)}\}$ by $S_i = \{y_{i,1}, y_{i,2}\ | \ 1 \leq i \leq n\}$ and $S_{n+i} = \{y_{i,3}\ | \ 1 \leq i \leq n\}$. It can be seen that vertices in $S_i$ induce a star graph $K_{n,1}$ and vertices in $S_{n+i}$ are singleton partition class that obtain trivial graph of star graph. Thus, it can be shown that the representation of every vertex $v \in V(K_n \triangleright C_m)$ is distinct to $\Pi_S$. The results of observation shows that the coordinate representation of all vertices in $K_n \triangleright C_m$ for $m = 3$ is unique. Thus, $\Pi_S = \{S_1, S_2, S_3, ..., S_{n(\frac{m}{3} + 1)}\}$ is star partition dimension which consists of $n(\frac{m}{3} + 1)$ partition class. Now the cardinality of $\Pi_S$ is $|\Pi_S| = n\left(\frac{m}{3} + 1\right)$. However, $\Pi_S$ is not necessarily to have minimum cardinality. Thus, the upper bound of star partition dimension $K_n \triangleright C_m$ is $\text{spd}(K_n \triangleright C_m) \leq n(\frac{m}{3} + 1)$. It concludes that the partition dimension $K_n \triangleright C_m$ is $\text{spd}(K_n \triangleright C_m) = n\left(\frac{m}{3} + 1\right)$ for $m = 3$.

Case 2: For $m \equiv 1(\text{mod } 3)$. The upper bound of star partition dimension is obtained by construction star resolving partition. Suppose $\Pi_S = \{S_1, S_2, S_3, ..., S_{n(\frac{m-1}{3}+1)}\}$ where $S_{\frac{m-1}{3}(i-1)+k} = \{y_{i,j}\ | \ 1 \leq k \leq \frac{m-1}{3}, 3(k-1) + 3 \leq j \leq 3k + 2, 1 \leq i \leq n\}$ and $S_{n(\frac{m-1}{3})+1} = \{y_{i,1}\ | \ 1 \leq i \leq n\}$. It is clearly to see that for all $v \in V(K_n \triangleright C_m)$, the graph $K_n \triangleright C_m$ has distinct vertex representation to $\Pi_S$. Thus, $\Pi_S = \{S_1, S_2, S_3, ..., S_{n(\frac{m-1}{3}+1)}\}$ is a star resolving partition consists of $n\left(\frac{m-1}{3}\right) + 1$ partition class. Now, the cardinality of $\Pi_S$ is $|\Pi_S| = n\left(\frac{m-1}{3} + 1\right)$. However, the partition class $\Pi_S$ is not necessarily to have minimum cardinality. Thus, the upper bound of star partition dimension $K_n \triangleright C_m$ is $\text{spd}(K_n \triangleright C_m) \leq n\left(\frac{m-1}{3} + 1\right)$. Furthermore, the lower bound of star partition dimension $K_n \triangleright C_m$ is obtained by Lemma 1.1. Now, we consider the subgraph which is induced by vertices in one partition class of a star graph. Thus, if $\Pi_S$ has cardinality $|\Pi_S| = n\left(\frac{m-1}{3}\right)$, then there are at least one partition class that does not induces a star graph $K_{n,1}$. Notice that vertices in partition class $\Pi_S$ are vertices of $V(K_n \triangleright C_m)$, without loss of generality, suppose $\Pi_S = \{S_1, S_2, S_3, ..., S_{n(\frac{m-1}{3})}\}$, then there are partition class that does not induces a star graph namely $S_{n(\frac{m-1}{3})+1} = \{y_{m,j}\ | \ m - 1 \leq j \leq m\} \cup \{y_{i,j}\ | \ 1 \leq i \leq n\}$. Thus, the cardinality $\Pi_S$ is $|\Pi_S| = n\left(\frac{m-1}{3}\right)$ and $|\Pi_S|$ is not star resolving partition. Therefore, the lower bound of star partition dimension $K_n \triangleright C_m$ is $\text{spd}(K_n \triangleright C_m) \geq$
n\left(\frac{m-1}{3}\right) + 1. It concludes that the star partition dimension \(spd(K_n \triangleright C_m) = n\left(\frac{m-1}{3}\right) + 1\) for \(m \equiv 1 \pmod{3}\) and \(m \geq 5\).

For \(m = 4\). Suppose the lower bound of the star dimension partition is \(spd(K_n \triangleright C_m) \geq n\left(\frac{m-1}{3}\right) + 1\). Based on Lemma 1.1, there are vertex representation in \(V(K_n \triangleright C_m)\) namely \(r(y_i,2) = r(y_i,4)\) for \(1 \leq i \leq n\) which cause vertices \(y_i,2\) and \(y_i,4\) have to be on different partition class such that the lower bound of the star partition dimension:

\[
spd(K_n \triangleright C_m) \geq \left(\frac{m-1}{3} + 1\right) + \left(\frac{m-1}{3} + 1\right) + \ldots + \left(\frac{m-1}{3} + 1\right) = n\left(\frac{m-1}{3} + 1\right) \text{ times}
\]

Thus, the lower bound of the star partition dimension \(K_n \triangleright C_m\) is \(spd(K_n \triangleright C_m) \geq n\left(\frac{m-1}{3} + 1\right)\).

Now, we will determine upper bound by constructing a star resolving partition namely \(\Pi_S = \{S_1, S_2, S_3, \ldots, S_n(\frac{m-1}{3}+1)\}\) by \(S_i = \{y_i,1, y_i,2, y_i,3\} | 1 \leq i \leq n\) and \(S_n+i = \{y_i,4\} | 1 \leq i \leq n\).

It can be seen that vertices in \(S_i\) induces a star graph \(K_{1,2}\) and vertices in \(S_n+i\) is singleton partition class that obtain trivial graph of star graph. Thus, it can be shown the representation of all \(v \in V(K_n \triangleright C_m)\) are distinct to \(\Pi_S\). The results of observation shows the representation of all vertices in \(K_n \triangleright C_m\) for \(m = 4\) is unique. Thus, \(\Pi_S = \{S_1, S_2, S_3, \ldots, S_n(\frac{m-1}{3}+1)\}\) is star partition dimension consists of \(n\left(\frac{m}{4} + 1\right)\) partition class. Now, the cardinality of \(\Pi_S\) is \(|\Pi_S| = n\left(\frac{m-1}{3} + 1\right)\). However, the partition class \(\Pi_S\) is not necessarily to have a minimum cardinality. Therefore, the upper bound of star partition dimension \(K_n \triangleright C_m\) is \(spd(P_n \triangleright C_m) \leq n\left(\frac{m-1}{3} + 1\right)\). Once we have got the upper bound and lower bound of star partition dimension \(n\left(\frac{m-1}{3} + 1\right) \leq spd(K_n \triangleright C_m) \leq n\left(\frac{m-1}{3} + 1\right)\), then the partition dimension \(K_n \triangleright C_m\) is \(spd(K_n \triangleright C_m) = n\left(\frac{m-1}{3} + 1\right)\) for \(m = 4\).

**Case 3:** For \(m \equiv 2 \pmod{3}\). The upper bound of star partition dimension obtained by constructing star resolving partition. Suppose \(\Pi_S = \{S_1, S_2, S_3, \ldots, S_n(\frac{m-2}{3}+n)\}\) where \(S_n(\frac{m-2}{3}+i+k) = \{y_i,j\} | 1 \leq k \leq \frac{m-2}{3}, 3(k-1) + 3 \leq j \leq 3k + 2, 1 \leq i \leq n\) and \(S_n(\frac{m-2}{3}+i) = \{y_i,1 \leq i \leq n, 1 \leq j \leq 2\}\). It can be seen that vertices in \(S_n(\frac{m-2}{3}+i+k)\) and \(S_n(\frac{m-2}{3}+i)\) is a star resolving partition that consists of \(n\left(\frac{m-2}{3}\right) + n\) partition class. Therefore, the cardinality of \(\Pi_S\) is \(|\Pi_S| = n\left(\frac{m-2}{3}\right) + n = n\left(\frac{m-1}{3}\right)\). However, the partition class \(\Pi_S\) is not necessarily to have minimum cardinality. Thus, the upper bound of star partition dimension \(K_n \triangleright C_m\) is \(spd(K_n \triangleright C_m) \leq n\left(\frac{m-1}{3}\right)\).

Furthermore, the lower bound of star partition dimension \(K_n \triangleright C_m\) is obtained by Lemma 1.1. Now, we will consider the subgraph is induced of vertices in one partition class of a star graph. If \(\Pi_S\) has cardinality \(|\Pi_S| = n\left(\frac{m+1}{3}\right) - 1\), then there are at least one partition class that does not induce a star graph. Notice that all vertices in partition class \(\Pi_S\) are vertices of \(V(K_n \triangleright C_m)\).

Without loss of generality, suppose that \(\Pi_S = \{S_1, S_2, S_3, \ldots, S_n(\frac{m+1}{3})-1\}\) then there is partition class that does not induce a star graph namely \(S_n(\frac{m+1}{3})-1 = \{y_i,1, y_i,3 \mid 1 \leq i \leq n\}\). Thus, we have obtained that \(\Pi_S\) with cardinality \(|\Pi_S| = n\left(\frac{m+1}{3}\right) - 1\) which is not star resolving partition. Thus, the lower bound of star partition dimension \(K_n \triangleright C_m\) is \(spd(K_n \triangleright C_m) \geq n\left(\frac{m+1}{3}\right)\).

Since we have got the upper bound and lower bound of star partition dimension \(K_n \triangleright C_m\) is \(n\left(\frac{m+1}{3}\right) \leq spd(K_n \triangleright C_m) \leq n\left(\frac{m+1}{3}\right)\), then star partition dimension \(spd(K_n \triangleright C_m) = n\left(\frac{m+1}{3}\right)\) for \(n \equiv 2 \pmod{3}\).

Based on three cases above, it can be seen that \(spd(K_n \triangleright C_m) = n\left(\frac{m}{3}\right)\) for \(m = 3\) and \(spd(K_n \triangleright C_m) = n\left(\frac{m-1}{3} + 1\right)\) for \(m = 4\). These can be rewritten into \(spd(K_n \triangleright C_m) = n\left(\frac{m}{3}\right)\) for \(m = 3\).
for $3 \leq m \leq 4$. Furthermore, $\text{spd}(K_n \triangleright C_m) = n\left(\frac{m}{3}\right)$ for $m \equiv 0(\text{mod } 3)$ and $m \geq 5$ and $\text{spd}(K_n \triangleright C_m) = n\left(\frac{m+1}{3}\right)$ for $m \equiv 2(\text{mod } 3)$ and $m \geq 5$, thus for value $m$, it can be combined into $\text{spd}(K_n \triangleright C_m) = n\left\lceil \frac{m}{3} \right\rceil$. While $\text{spd}(K_n \triangleright C_m) = n\left(\frac{m+1}{3}\right) + 1$ for $m \equiv 1(\text{mod } 3)$, $m \geq 5$, it can be combined into $\text{spd}(K_n \triangleright C_m) = n\left\lceil \frac{m}{3} \right\rceil + 1$. It concludes the proof.

3. Conclusion

In this paper we have given an asymptotically tight result on the star partition dimension of comb product of cycle and complete graph. The result shows that the star partition dimension of the comb product of cycle and complete graph as follows:

$$\text{spd}(C_m \triangleright K_n) = m(n - 1)$$

$$\text{spd}(K_n \triangleright C_m) = \begin{cases} 
  n\left(\left\lfloor \frac{m}{3} \right\rfloor \right) + 1, & \text{if } 3 \leq m \leq 4 \\
  n\left\lfloor \frac{m}{3} \right\rfloor, & \text{if } m \equiv 0(\text{mod } 3), m \equiv 2(\text{mod } 3) \\
  n\left\lfloor \frac{m}{3} \right\rfloor + 1, & \text{if } m \equiv 1(\text{mod } 3)
\end{cases}$$

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