Quantization of linearly polarized cosmological models with two Killing vector fields

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Abstract. We discuss the quantization of two-Killing symmetry reductions of general relativity with cosmological interpretation. In particular we will focus on the obtention of unitary evolution operators in closed form for the class of time-dependent quadratic Hamiltonians that appear in this context. This will be done by extending to field theoretical models the techniques developed for the exact quantization of the harmonic oscillator with time-dependent frequency. The abstract operators that we will derive are useful both to discuss different field representations and the problem of unitary implementation of dynamics. Finally, we consider in some detail the applications of the formalism developed to the Gowdy and Schmidt cosmological models.

1. Introduction

Despite the remarkable successes of string theories and loop quantum gravity, we still do not have a satisfactory field theory describing gravity and quantum mechanics in an unified way. Symmetry reductions of general relativity play an important role in this context since they provide simple models in which quantum gravity effects can be studied. In particular, the Bianchi models and two-Killing vector reductions of general relativity have received a lot of attention owing to their applications in astrophysics and cosmology.

Two-Killing symmetry reductions are widely considered appealing testing grounds for quantum gravity due to the fact that (in contrast to the Bianchi models) they still have local degrees of freedom as well as (restricted) diffeomorphism invariance, two of the features of the gravitational theory that make its quantization rather complicated. These models differ from each other in the spacetime topology and the isometry group. Here we will focus on the Gowdy T³ and Schmidt models, in which the action of the toroidal symmetry group U(1) × U(1) is defined so that the spatial sections are topologically T³ or ℝ × T², respectively, providing inhomogeneous cosmological frameworks with initial singularities.

A. Corichi, J. Cortez, H. Quevedo, and, independently, C. G. Torre were the first to point out the apparent impossibility of implementing the dynamics of these models in a unitary way, and hence the non-existence of a Schrödinger picture of time evolution [1, 2]. However, A. Corichi,
2. Gowdy and Schmidt models: Hamiltonian formalism

Let \((^{(4)}M \simeq \mathbb{R} \times^{(3)}\Sigma, ^{(4)}g_{ab})\) be a globally hyperbolic spacetime and suppose an effective, smooth, proper, and with no degenerare orbits action of the isometry group \(U(1) \times U(1)\) on the connected and oriented 3-manifold \(^{(3)}\Sigma\). The spatial topology is then fixed to be of the form \(^{(3)}\Sigma \simeq \mathbb{X} \times S^1 \times S^1\), where \(\mathbb{X}\) denotes the circle \(S^1\) (this way we obtain the so called Gowdy \(T^3\) model) or the real line \(\mathbb{R}\) (Schmidt model).

We will focus here on the linearly polarized case, where the isometry group is generated by a pair of mutually orthogonal, commuting, spacelike, and globally defined hypersurface-orthogonal Killing vector fields \((\xi^a, \sigma^a)\). Under this assumption it can be shown [5] that the 3+1-dimensional Einstein equation is equivalent to the Einstein-Klein-Gordon equations corresponding to 2+1-dimensional gravity \(^{(3)}M, g_{ab}\) coupled to an axi-symmetric zero rest mass scalar field. Here \(^{(3)}M := (^{(4)}M/U(1) \simeq \mathbb{R} \times \mathbb{X} \times S^1\) denotes the space of orbits defined by one of the Killing vector fields, say \(\xi^a\), and the 3-metric \(g_{ab}\) is obtained from the induced metric \(^{(3)}g_{ab}\) on \(^{(3)}M\) through the conformal rescaling \(g_{ab} := \lambda^{(3)}g_{ab}\), where \(\lambda^{(4)}g_{ab}\xi^a\xi^b > 0\). In this case, the remaining Killing vector field \(\sigma^a\) satisfies the vanishing of the Lie derivatives \(L_\sigma g_{ab} = 0\), \(L_\sigma \lambda = 0\), and the massless scalar field is given by \(\phi/\sqrt{16\pi G_3} =: \phi/\sqrt{8\pi G_3}\), where \(G_3\) is the Newton constant per unit length in the direction of the \(\xi\)-symmetry orbits.

We obtain a Hamiltonian formulation of the theory by introducing a spacelike foliation of the 2-manifolds –topologically \(\mathbb{R} \times \mathbb{X}\)– that are everywhere orthogonal to the closed orbits of \(\sigma^a\). This is accomplished by defining spacelike level hypersurfaces of a suitable scalar function \(t\), and a dynamical vector field \(t^a = Nn^a + N^a\dot{x}^a\) such that \(t^a\nabla_a t = 1\). Here \(n^a\) and \(\dot{x}^a\) are the unit, future-pointing, timelike, normal vector field to the foliation and the unit vector field tangent to each slice, respectively. \(N\) and \(N^a\) are the usual lapse and shift functions. Then we proceed to write the 2+1-Einstein-Hilbert action in its canonical form. Just as expected the lapse and shift functions act as Lagrange multipliers enforcing two first class constraints \(C = 0\), \(C_a = 0\). Therefore, the canonical variables are restricted to lie in a constraint surface. We also find that the Hamiltonian of the system can be expressed as a sum of two constraint functionals \(H[N, N^a] = C[N] + C[N^a]\), where \(C[N] := \int_\Sigma NCdx\), \(C[N^a] := \int_\Sigma N^a C_adx\).

Thus, the Hamiltonian is identically zero on the constraint surface regardless of \(\mathbb{X}\), so we must proceed to deparametrize the theory by selecting one of the Hamiltonian vector fields associated to the constraints to generate the time evolution, and gauge fixing the others. This is done by introducing a suitable phase space variable to play the role of time. The fact that symmetry forces the area density of the group orbits \(\tau := g_{ab}\sigma^a\sigma^b\) to have a timelike gradient [6] suggests its identification with the time variable \(t\) by imposing suitable gauge fixing conditions.

The main difference between the Gowdy and Schmidt models, as far as the deparametrization is concerned, is that in the Gowdy model a global degree of freedom characterized by a canonical pair \((Q, P)\) of spatial constants arises naturally (in addition to the usual local degrees of freedom). It is always possible to describe \(Q\) and \(P\) as constants of motion through a suitable choice of the gauge fixing conditions and a set of canonical transformations [7].

After deparametrization, all the true local degrees of freedom reside in the scalar field \(\phi\)
and its canonically conjugate momentum $p_\phi$, as expected in a 2+1-dimensional theory of gravity coupled to matter, and the 3-metric becomes (from now on, take $P \equiv 0$ in the non-compact case)

$$g_{ab} = e^\gamma \left( -\nabla_a t \nabla_b t + \nabla_a x \nabla_b x \right) + e^2 P t^2 \nabla_a \sigma \nabla_b \sigma ,$$

where $(t, x, \sigma) \in \mathbb{R}^+ \times \mathbb{R} \times S^1$. Here the new scalar field $\gamma(x, t)$ satisfies the equation $e^P \gamma' = p_\phi \dot{\phi}'$, where the prime denotes $\partial_t$. Notice that this metric displays a curvature singularity at $t = 0$. It must be said that the shift function cannot be totally fixed in the compact case, so the metric must include it. However, we can use a suitable redefinition of the angular variable $x$ in order to obtain the expression (1) (see [8]). The reduced action takes the form (we choose units such that $8G_3 = 1$)

$$S_R = \int_{t_1}^{t_2} \left( P \dot{Q} + \int_{\mathcal{X}} p_\phi \dot{\phi} \, dx - H_R(t) \right) \, dt ,$$

where the dot denotes $\partial_t$, and

$$H_R(t) = \int_{\mathcal{X}} \left( \frac{p_\phi^2}{2} + t \dot{\phi}^2 \right) \, dx$$

is the time-dependent Hamiltonian of the system. The Hamilton equations for the local degrees of freedom derived from it are

$$\dot{\phi} = \frac{1}{2t} p_\phi , \quad \dot{p}_\phi = 2t \phi'' \Rightarrow -\dot{\phi} + \phi'' - \frac{1}{t} \phi = 0 .$$

This second order differential equation corresponds to the Klein-Gordon equation for a massless scalar field (independent of the $\sigma$ coordinate) on a fixed, 3-dimensional, time-dependent background spacetime $(\mathcal{M}, g_{ab})$, where $\mathcal{M} \simeq \mathbb{R}^+ \times \mathbb{R} \times S^1$, and the metric is given by

$$\dot{g}_{ab} = -\nabla_a t \nabla_b t + \nabla_a x \nabla_b x + t^2 \nabla_a \sigma \nabla_b \sigma ,$$

that shows, again, the singular behavior of the system at $t = 0$.

Therefore, the reduced canonical phase space $\Gamma_R$ of the Gowdy model can be decomposed as the direct sum $\Gamma_R = \Gamma_0 \oplus \Gamma$, where $\Gamma_0$ is coordinatized by the canonical pair $(Q, P)$, and $\Gamma$ by the canonical pair $(\phi, p_\phi)$. $\Gamma_R$ is simply $\Gamma$ in the Schmidt case. In these models there is a quadratic conserved momentum

$$\Lambda = \int_{\mathcal{X}} p_\phi \phi' \, dx$$

as a result of the invariance of the equation (4) under constant translations $x \mapsto x + x_0$, $\forall x_0 \in \mathbb{X}$. In the compact case it is forced to be zero, and hence $\Gamma_R$ does not have a linear structure. For this reason, it is preferable to postpone the reduction by the global constraint (6) to the quantization program, where is imposed as an operator that annihilates the physical quantum states.

We proceed now to quantize the classical theory by promoting the dynamical variables that describe the global and local degrees of freedom $(Q, P, \phi, p_\phi)$ to quantum operators $(\hat{Q}, \hat{P}, \hat{\phi}, \hat{p}_\phi)$. Let $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}$ be the (kinematical) Hilbert space of the system, ignoring the existence of the global constraint (6). $\mathcal{H}_0$ is the Hilbert space associated to the canonical pair $(\hat{Q}, \hat{P})$; it can be identified as $L^2(\mathbb{R})$, and hence a possible representation is $Q\psi = q\psi$, $P\psi = -i\partial_q \psi$, with $\psi = \psi(q) \in L^2(\mathbb{R})$ (notice that we take $\hbar = 1$ in this paper). $\mathcal{H}$ denotes the Hilbert space related to the local degrees of freedom; notice that evolution is non trivial only in this field sector. We will discuss the problem of defining a Fock space representation for $\mathcal{H}$ in section 4.

The physical Hilbert space of the Gowdy model is then the subspace of $\mathcal{H}$ corresponding to the kernel of the quantum operator associated to the constraint (6), whereas it is simply equivalent to $\mathcal{H}$ in the Schmidt case.
3. Evolution operators

Let us consider now the problem of finding the formal quantum evolution operator of a system with a classical time-dependent Hamiltonian of the general form

\[ H(t) = \int_{\mathbb{X}} \left( \frac{1}{4} \pi^2(x) + \Lambda(t) \pi(x) \varphi(x) + \omega^2(t) \varphi^2(x) \right) dx, \tag{7} \]

where \((\varphi(x), \pi(x))\) is a canonical pair of fields and momenta defined on \(\mathbb{X}\). Notice that the Hamiltonians of the Gowdy and Schmidt models are particular cases of (7) because the function of time that appears in the \(p_\varphi\)-term of (3) can be eliminated by a simple redefinition of the time variable. By doing this, and introducing a cross term in (3) to make its expression more general, we obtain a Hamiltonian that belongs to the general class (7).

First of all, we can diagonalize this Hamiltonian by writing

\[
\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{X}} \cos \left( kx - \frac{\pi}{4} \right) \varphi(k) d\mu(k),
\]

\[
\pi(x) = \sqrt{\frac{2}{\pi}} \int_{\mathbb{X}} \cos \left( kx - \frac{\pi}{4} \right) \pi(k) d\mu(k),
\]

where \((\varphi(k), \pi(k))\) is also a canonical pair for all \(k \in \mathbb{X}\), with \(\mathbb{X}\) denoting the real line for \(\mathbb{X} = \mathbb{R}\) and the integers when \(\mathbb{X} = \mathbb{S}^1\). Here the measure \(d\mu(k)\) simply refers to the fact that the previous integrals extend to the real line or become a sum over the integers, respectively. Then, in order to quantize the system we promote the field and its momentum to formal algebraic objects \(\varphi(k) \rightarrow \hat{\varphi}(k), \pi(k) \rightarrow \hat{\pi}(k)\) satisfying the usual canonical commutation relations \([\hat{\varphi}(k), \hat{\pi}(q)] = i\delta(k, q)\), where \(\delta(k, q)\) is a Dirac or a Kronecker delta. In terms of them, we obtain the following quantum Hamiltonian

\[ \hat{H}(t) = \frac{1}{2} \int_{\mathbb{X}} \left( \hat{\varphi}^2(k) + \Lambda(t) [\hat{\pi}(k) \hat{\varphi}(k) + \hat{\varphi}(k) \hat{\pi}(k)] + k^2 \omega^2(t) \hat{\varphi}^2(k) \right) d\mu(k). \tag{9} \]

We can use the techniques previously applied by other authors to compute the explicit solution for the evolution operator for an one-dimensional quantum harmonic oscillator with a time-dependent frequency (see for example \[9\]) to obtain the formal quantum evolution operator \(\hat{U}(t, t_0)\) associated to (9). In this way, \(\hat{U}(t, t_0)\) can be written down in closed form as the product

\[ \hat{U}(t, t_0) = \hat{D}(t, t_0)\hat{S}(t, t_0)\hat{R}(t, t_0), \tag{10} \]

with

\[
\hat{D}(t, t_0) := \exp \left( -\frac{i}{2} \int_{\mathbb{X}} \left[ \frac{\hat{\rho}(k, t_0)}{\hat{\rho}(k, t)} - \frac{\hat{\rho}(k, t)}{\hat{\rho}(k, t_0)} \right] \hat{\varphi}^2(k) d\mu(k) \right),
\]

\[
\hat{S}(t, t_0) := \exp \left( \frac{i}{2} \int_{\mathbb{X}} \left[ \log \frac{\hat{\rho}(k, t_0)}{\hat{\rho}(k, t)} \right] \left\{ \hat{\pi}(k) \left[ \hat{\pi}(k) - \left( \frac{\hat{\rho}(k, t_0)}{\hat{\rho}(k, t)} - \Lambda(t_0) \right) \hat{\varphi}(k) \right] \hat{\varphi}(k) d\mu(k) \right\},
\]

\[
\hat{R}(t, t_0) := \exp \left( -\frac{i}{2} \int_{\mathbb{X}} \left[ \int_{t_0}^{t} d\tau \right] \left\{ \frac{\hat{\varphi}^2(k)}{\hat{\rho}^2(k, t_0)} \right\} + \rho^2(k, t_0) \left[ \hat{\pi}(k) - \left( \frac{\hat{\rho}(k, t_0)}{\hat{\rho}(k, t_0)} - \Lambda(t_0) \right) \hat{\varphi}(k) \right]^2 \right) d\mu(k). \tag{13} \]
Here $\rho(k,t)$ is any solution to the Ermakov-Pinney equation $\ddot{\rho}(k,t) + \Omega^2(k,t)\rho(k,t) = \rho^{-3}(k,t)$, where we have defined $\Omega^2(k,t) := k^2\omega^2(t) - \Lambda^2(t)$. It can be shown that the function $\rho(k,t)$ is positive definite and $\hat{U}(t,t_0)$ is independent of the choice of this solution [4].

4. Fock space representations

Let us suppose that we take a Fock space $\mathcal{F}$ and write the fields and momenta in terms of creation and annihilation operators

$$\begin{align*}
\hat{\varphi}(k) &= f(k)\hat{a}_k + \bar{f}(k)\hat{a}_k^\dagger, \\
\hat{\pi}(k) &= g(k)\hat{a}_k + \bar{g}(k)\hat{a}_k^\dagger,
\end{align*}
$$

where the functions $f, g : \mathbb{R} \rightarrow \mathbb{C}$ satisfy the condition $f(k)\bar{g}(k) - \bar{f}(k)g(k) = i$ in virtue of the canonical commutation relation $[\hat{a}_k, \hat{a}_k^\dagger] = \delta(k,q)$, but are otherwise arbitrary at this stage.

In the Gowy and Schmidt models, the Hilbert space $\mathcal{H}$ is usually assumed to have the structure of a symmetric Fock space $\mathcal{F}_s(\mathcal{H}_P)$ over a one-particle Hilbert space $\mathcal{H}_P$. In a rigorous mathematical sense, the space $\mathcal{H}_P$ is constructed by defining a complex form $J : \mathcal{S} \rightarrow \mathcal{S}$ (compatible with the natural symplectic structure $\Omega$) on the space $\mathcal{S}$ of real solutions to the equation (4), i.e. a linear symplectic transformation such that $J^2 = -1\delta_{\mathcal{S}}$. Thus, $\mu_J(\phi_1, \phi_2) = \Omega(J\phi_1, \phi_2)/2$ defines a symmetric bilinear form on $\mathcal{S}$. Choosing the map $J$ such that $\mu_J$ is positive definite, $\langle \phi_1, \phi_2 \rangle_J := \mu_J(\phi_1, \phi_2) - i\Omega(\phi_1, \phi_2)/2$ is a Hermitian inner product on the complex vector space $\mathcal{S}_J$, where multiplication by complex scalars $z \in \mathbb{C}$ is defined with respect to the norm $\| \cdot \|_J := \sqrt{\langle \cdot, \cdot \rangle_J}$. Notice however that in general there is not a preferred choice of the complex form $J$, and hence we can define different one-particle Hilbert spaces $\mathcal{H}_P$ yielding non unitarily equivalent quantum theories. It is easy to show that every choice of a complex form corresponds in this context to a pair of functions $(f, g)$.

We want to find $f$ and $g$ functions such that the operators $\hat{D}(t,t_0), \hat{S}(t,t_0)$, and $\hat{R}(t,t_0)$ –with normal ordered exponents to prevent the appearance of infinite phases– are unitary. Let us consider a general quantum operator of the type

$$\exp(i\hat{F}) = \exp \left( i \int_{\mathbb{R}} [\chi_1(k)\hat{a}_k\hat{a}_k + \bar{\chi}_1(k)\hat{a}_k^\dagger\hat{a}_k^\dagger + 2\chi_2(k)\hat{a}_k^\dagger\hat{a}_k] \, d\mu(k) \right),
$$

where $\chi_{1,2,3}$ are real functions of $k$. Notice that $\hat{D}(t,t_0), \hat{S}(t,t_0)$, and $\hat{R}(t,t_0)$ are particular cases of it with a parametric dependence on $t$ and $t_0$. It is possible to know that if the exponent (multiplied by $i$) defines a self-adjoint operator by studying the auxiliary dynamics in a fictitious time parameter $s$ defined by the exponent taken as a classical Hamiltonian $F = \int_{\mathbb{R}} [\chi_1(k)a_k\bar{a}_k + \bar{\chi}_1(k)\bar{a}_k\hat{a}_k + 2\chi_2(k)\hat{a}_k\bar{a}_k] \, d\mu(k)$. The time evolution (in the Heisenberg picture) of the modes $a_k$, $\bar{a}_k$ –defined by the classical field and momentum $\varphi(k) = f(k)a_k + \bar{f}(k)\bar{a}_k, \pi(k) = g(k)a_k + \bar{g}(k)\bar{a}_k$– is given by the linear equations $\dot{a}_k/s = \{a_k, F\}, \dot{\bar{a}}_k/s = \{\bar{a}_k, F\}$, whose solutions have a linear dependence on the initial conditions $a_k(0), \bar{a}_k(0)$.

$$\begin{align*}
a_k(s) &= a_k(s)a_k(0) + \beta_k(s)\bar{a}_k(0), \\
\bar{a}_k(s) &= \overline{(a_k(s))}.
\end{align*}
$$

Using the theory of unitary implementation of canonical transformations (see [2, 10]), we obtain:

(i) The transformation (16) is unitarily implementable for each $s$ if and only if the modulo of the Bogoliubov coefficient $\beta_k(s)$ is square-summable with respect to the measure $d\mu(k)$

$$\int_{\mathbb{R}} |\beta_k(s)|^2 \, d\mu(k) < \infty.
$$

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(ii) Furthermore, the transformation (16) is implementable as a continuous, unitary, one-parameter group if verifies the strong continuity condition in the auxiliary parameter $s$

$$\lim_{s \to s_0} \int_{\mathbb{R}} |a_k(s) - a_k(s_0)|^2 \, d\mu(k) = 0, \quad \forall s_0 \in \mathbb{R}. \quad (18)$$

Then, the transformation (16) can be implemented as a continuous unitary group $\{\hat{U}(s)\}_s$ such that $\hat{U}^\dagger(s)\hat{a}_k\hat{U}(s) = \alpha_k(s)\hat{a}_k + \beta_k(s)\hat{a}_k^\dagger$, and its infinitesimal generator $\hat{F}$ is a self-adjoint operator, as a consequence of Stone’s theorem.

5. Gowdy and Schmidt models: Unitary evolution

Let us consider again the problem of quantizing the Gowdy and Schmidt models. It was previously shown that the reduced classical Hamiltonian of these models is given by (3). After redefining the time parameter as $t = e^{2T}$, using the cosine transform (8), and promoting the fields and momenta to formal operators we get the quantum Hamiltonian

$$\hat{H}_R(T) = \frac{1}{2} \int_{\mathbb{R}} \left( \phi^2(k) + k^2 e^{2T} \hat{\phi}^2(k) \right) \, d\mu(k),$$

that belongs to the general class of Hamiltonians (9) whose evolution operators we have constructed before (take $\Lambda(T) = 0$ and $\omega^2(T) = e^{2T}$). In this case it is impossible to define a Fock representation –i.e. a pair $(f, g)$– leading to a unitary evolution for the field $\phi$ that naturally encodes the local gravitational degrees of freedom of the theory. It is possible to understood the difficulties to get a unitary evolution operator in this context as the impossibility of fulfilling all the unitarity conditions (17,18) for some of the operators used to build $\hat{U}(t, t_0)$. In particular, we find that there are problems with the factor $\hat{S}(t, t_0)$. However, the way this operator fails to be unitary suggests the introduction at the Lagrangian level of the new scalar field

$$\xi(x,t) := \sqrt{t} \phi(x,t). \quad (19)$$

This kind of transformation has been previously used in the Gowdy model to study its WKB regime [11], and the unitary implementation of its dynamics [3]. Using the cosine transform and promoting the fields and momenta to formal operators, we obtain the quantum Hamiltonian

$$\hat{H}_R(t) = \frac{1}{2} \int_{\mathbb{R}} \left( \phi^2(k) + \frac{1}{2t} [\hat{\phi}(k)\hat{\xi}(k) + \hat{\xi}(k)\hat{\phi}(k)] + k^2 \hat{\xi}^2(k) \right) \, d\mu(k), \quad (20)$$

that again belongs to the general class (9), with $\Lambda(t) = 1/(2t)$ and $\omega^2(t) = 1$. In this case it is possible to define a unitary evolution for the field $\xi$ with the Fock representation

$$f(k) := 1/\sqrt{2(1 + |k|)}, \quad g(k) := -i\sqrt{(1 + |k|)/2}. \quad (21)$$

Indeed, it is straightforward to check that the operators $\hat{D}(t, t_0)$, $\hat{S}(t, t_0)$, and $\hat{R}(t, t_0)$ verify the unitary conditions (17,18) for this choice of $f$ and $g$ functions. In this way, the quantum evolution operator $\hat{U}(t, t_0) : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H} = \mathcal{H}_0 \otimes \check{\mathcal{H}}$, is given by $\hat{U}(t, t_0) = \mathbb{I} \otimes \hat{U}(t, t_0)$, with $\hat{U}(t, t_0)$ taking the form (10).

6. Conclusions and Comments

In this paper we have clarified and confirmed the results obtained in [3], i.e. the existence of a perfectly well defined and unitary evolution for the new scalar field $\xi$, but making an important contribution: we have been able to compute the evolution operator explicitly in closed form. The formalism introduced in section 3 allows us to study representations for the field and momentum operators different to the usual Fock ones, and the techniques developed here are also useful to study the generalization of the coherent and squeezed states for the harmonic quantum oscillator with time-dependent frequency to the Gowdy and Schmidt models.
Acknowledgments

The authors want to thank J. Cortez, I. Garay, L. Garay, J. M. Martín García, G. Mena Marugán, and M. Varadarajan for their insightful comments, and D. Gómez Vergel acknowledges the financial support provided by CSIC through the Introduction to Research assistantship and I3P programs.

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