List types for resource aware languages: an implicit name approach

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Abstract

A novel formalisation of variable control in languages with implicit names based on de Bruijn indices is presented. We design and implement three languages: first, a restricted language with implicit names; then, a restricted calculus with implicit names and explicit substitution, and finally, an extended calculus with implicit names and resource control. We propose a novel concept of list types, which are used to give a simple and manageable definition of linearity. We develop an implementation in Haskell.

Keywords: language design, functional programming, lambda calculus, de Bruijn index, type system, resource awareness, Haskell

1 Introduction

In computation, the control of variable use goes back to Church’s $\lambda I$-calculus and restricted terms [22]. Likewise, in logic, the control of formula use is present in Gentzen’s structural rules [13] which enable a wide class of substructural logics [11]. In programming, the augmented ability to control the use of operations and objects has a wide range of applications which enable, among others, compiling functional languages without garbage collector and avoids memory leaking [38, 37]; inline expansion in compiler optimisations [7]; safe memory management [44]; controlled type discipline as a framework for resource-sensitive compilation [16]; the interpretation of linear formulae as session types that provides a purely logical account of session types [6]. At the core of all these phenomena is the Curry-Howard correspondence of formulae-as-types and proofs-as-terms.

Control: by restriction vs by extension

There are several restricted classes of $\lambda$-terms, where the restrictions are due to the control of variable use. The best known among them are: $\lambda I$-terms, aka relevant terms, where variables occur at least once; $\text{BCK}\lambda$-terms, aka affine terms, where variables occur at most once; $\text{BCI}\lambda$-terms, aka linear terms, where each variable occurs exactly once [22, 18]. E.g. the combinator $K$ is not a $\lambda I$-term and the combinator $S$ is not a $\text{BCK}\lambda$-term. This “control by restriction” approach is widely present in substructural logics [11], substructural type theory [44], linear logic [19, 31], among others. On the other hand, the control of variable use can be achieved by extending the language by operators meant to tightly encode the control. If a variable has to be reused, it will be explicitly duplicated, whereas if the variable is not needed, it will be explicitly erased. These two resource control operators, duplication and erasure, are extensions of the syntax of the $\lambda$-calculus which allow all $\lambda$-terms to become: relevant (only erasure is used), affine (only duplication is used) and linear (both
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erasure and duplication are used). The advantage of this "control by extension" approach is that all \( \lambda \)-terms can be encoded in the extended calculus. Hence, the extended calculi are equivalent, in computational power, to \( \lambda \)-calculus, which is not the case with the restricted calculi. This approach has been developed in different theoretical [3, 23] and applicative [38, 7] settings. From a proof theoretical perspective, such a simply typed extended \( \lambda \)-calculus has a Curry-Howard correspondence with intuitionistic logic with Contraction and Thinning structural rules [42], whereas a restricted \( \lambda \)-calculus corresponds to substructural logic [11, 36] such as relevant or affine logic.

Names: explicit vs implicit The well-known \( \lambda \)-calculus is a calculus with explicit use of variables (names). On the other hand, the calculus with implicit names is de Bruijn notation of \( \lambda \)-calculus that avoids the explicit naming of variables by employing de Bruijn indices [10, 9, 28]. Each variable is replaced by a natural number which is the number of \( \lambda \)'s crossed in order to reach the binder of that variable. For instance in de Bruijn notation, the combinator \( I \equiv \lambda x.x \) is \( \lambda^0 \), the combinator \( K \equiv \lambda x.\lambda y.x \) is \( \lambda\lambda^1 \) and the combinator \( S \equiv \lambda x.\lambda y.\lambda z.x(yz) \) is \( \lambda\lambda\lambda^2(\lambda^0(\lambda^0)) \).

The profound advantage of de Bruijn notation is that \( \alpha \)-conversion, the renaming of bound variables, is not needed, which significantly facilitates implementation and also, in the case of an extended \( \lambda \)-calculus, simplifies the rules.

Foundations In this paper, we study both restricted and extended control of variable use in calculi with implicit names. This means that instead of (explicit) variables we use either de Bruijn indices, or novel \( \mathbb{R} \)-indices. Inspired by de Bruijn indices, \( \mathbb{R} \)-indices provide information about duplication of names. We design and implement three languages. First, a restricted language with implicit names \( \Lambda^L \); then, a restricted language and calculus with implicit names and explicit substitution \( \Lambda^L_\omega \) and \( \lambda^L_\omega \)-calculus, respectively and finally, \( \Lambda^L_\mathbb{R} \), an extended language with implicit names and explicit duplication and erasure. In all introduced calculi, linear terms are defined as specifically typeable terms in systems we call \( \mathcal{L} \)-types. The \( \mathcal{L} \)-type, abbreviated as \( \mathcal{L} \)-type, of a term represents the list of its free indices and is convenient for checking its linearity. This is why we use \( \mathcal{L} \)-typeability as a definition of linearity.

Linearity: indirect vs direct For a reader used to explicit names only, i.e. regular \( \lambda \)-calculus, there exists, at first glance, a simple method to characterise linear terms: it suffices to check that each variable occurs once, which is easy if one creates a new variable for each new abstraction. However with implicit names, aka de Bruijn indices, checking the occurrences of indices with the same number is of no help, because it is well known that a same number may occur several times in a linear term and a term with only occurrences of different numbers, may be non linear. Thus \( \lambda^0(\lambda^0) \) is linear and \( \lambda^0(\lambda^1) \) is not linear. We let the reader imagine examples with thousand \( \lambda \)'s (thousand abstractions) and check whether indices \( 127 \) and \( 828 \) correspond to the same variable or whether those variables are duplicated or not.

Thus for terms with implicit names, checking unique occurrences of numbers (associated with de Bruijn indices) does not work. There is then an indirect method for checking linearity with implicit names which consists of two steps:

1. to translate a \( \lambda \)-term with implicit names into a \( \lambda \)-term with explicit names,
2. to check unique occurrence of variables.

This indirect method is algorithmically costly, since it consists in a translation from implicit names to explicit names. In turn, we give a direct characterisation of linearity with implicit names by means of \( \mathcal{L} \)-types.
Implementation  We worked simultaneously on the development of the $L$-type calculi and on their implementation, because the implementation shapes the development.

The main contributions of this paper are:

- the use of de Bruijn indices (an old concept, in a novel setting) and $®$-indices (a new concept),
- the novel concept of list types, dubbed $L$-types, for characterising linearity in languages with implicit names,
- a formal definition of linearity based on $L$-typeability in three languages, especially in a calculus with explicit substitution and in a calculus with resource control, because in those calculi a formal and direct definition of linearity is hard to state,
- proofs of $L$-type preservation, hence of linearity,
- an implementation in Haskell for framework [29].

The rest of this paper is organised as follows. We first review the background on de Bruijn indices, the $\lambda$-calculus with implicit names, in Section 2. In Section 3, we introduce the notion of list types (abbreviated by $L$-types) and design the language $\Lambda^L$ of restricted terms with implicit names. In Section 4, we extend the notion of $L$-types and design the language $\Lambda^L_\upsilon$ of restricted terms with implicit names and explicit substitution and the corresponding calculus for which we prove type preservation. In Section 5, we modify the notion of $L$-types for the design of the language $\Lambda^L_\upsilon R$ of extended terms with implicit names and resource control and implement it in Haskell. In Section 6, we discuss related work. Section 7 concludes the paper. To facilitate the reader’s comprehension and understanding of the text, the following table highlights the most important introduced notations, along with their informal definitions and references to the sections in which they are formally introduced.

| name    | informal definition                                      | reference to its definition |
|---------|---------------------------------------------------------|-----------------------------|
| $\Lambda$ | The set of terms (with implicit names)                  | Section 2                   |
| $\Lambda^L$ | The set of restricted terms                           | Section 3                   |
| $\Lambda^L_\upsilon$ | The set of restricted terms with explicit substitutions | Section 4                   |
| $\Lambda^L_\upsilon R$ | The set of extended terms with explicit substitutions | Section 4                   |
| $\lambda^L_\upsilon$ | The calculus of $\Lambda^L_\upsilon$               | Section 4                   |
| $\Lambda^L_\upsilon R$ | The set of extended terms with resource control       | Section 5                   |

2 Terms with implicit names $\Lambda$

Our development relies on the paradigm of implicit names in formal calculi. We recall the notion of term with implicit names based on de Bruijn indices [10, 9, 28]. Let us consider the (regular) $\lambda$-terms $K \equiv \lambda x. \lambda y. x$ and $S \equiv \lambda x. \lambda y. \lambda z. x z (y z)$ and the three contractions of the term $SK$

$$(\lambda x. \lambda y. \lambda z. x z (y z)) (\lambda x. \lambda y. x) \rightarrow \lambda y. \lambda z. (\lambda x. \lambda y. x) z (y z) \rightarrow \lambda y. \lambda z. (\lambda y. x) (y z) \rightarrow \lambda y. \lambda z. z.$$
Figure 1: Bourbaki assembly in [4]

Figure 2: Bourbaki assembly of SK

Assume that we want to represent those terms without using variables. Such a variable-free representation is called sometimes Bourbaki assembly, because this variable-free two dimensional representation of terms has been first used by Bourbaki [4] (see Figure 1) and has been called “assembly” [5, 21]. In a contemporary style (eighty five years later and a different context), Figure 1 can be explained as follows. In a system of implicit names and an infix operator ∈, this assembly represents τ((¬(0 ∈ A′)) ∨ (0 ∈ A)), where τ is a binder acting like λ. In a system of explicit names, where τ binds x, this assembly represents τ x.(¬(x ∈ A′)) ∨ (x ∈ A).

Figure 1 resembles Figure 2 (we use here an infix notation for the binary operator “application” and Bourbaki uses prefix notations).

Later and independently, de Bruijn proposed a one dimension variable-free representation using natural numbers\(^2\), called since de Bruijn indices. Each variable is replaced by a natural number which is the number of λ’s crossed in order to reach the binder of that variable. For instance, \(\lambda x.\lambda y.\lambda z.xz(yz)\) is replaced by \(\lambda\lambda\lambda 20(10)\). Indeed, \(x\) is replaced by 2 because one crosses two λ’s to meet its binder, \(y\) is replaced by 1 because one crosses one λ to meet its binder and \(z\) is replaced by 0 because one crosses no λ to meet its binder.

The abstract syntax of terms with de Bruijn notation is the following:

\[
\begin{align*}
t & ::= \ n & | & \lambda t & | & tt
\end{align*}
\]

where \(n\), associated with \(n \in \mathbb{N}\), is an index. The set of all terms with de Bruijn notation will be denoted by \(\Lambda\) and it will be ranged over by \(t, s, \ldots\). We will call them λ-terms or terms without mentioning de Bruijn indices if there is no place for confusion.

Terms with de Bruijn notation are also called terms with implicit names since variables are implicit rather than explicit as in the regular λ-calculus. Using implicit names is convenient because terms with de Bruijn indices represent classes of α-conversion of terms with explicit variables.

We will see that de Bruijn indices also enable simple descriptions of features connected with linearity, duplication and erasure that are otherwise described with cumbersome notations [23, 24, 17]. The formal definition of β-reduction is given in Example 2.

Since \(S \equiv \lambda x.\lambda y.\lambda z.xz(yz)\) is replaced by \(\lambda\lambda\lambda 20(10)\) and \(K \equiv \lambda x.\lambda y.x\) is replaced by \(\lambda\lambda 1\) the above chain of contractions of the term SK is replaced by

\[
(\lambda\lambda\lambda 20(10)) (\lambda\lambda 1) \rightarrow \lambda\lambda((\lambda\lambda 1)0(10)) \rightarrow \lambda\lambda((\lambda\lambda 1)(10)) \rightarrow \lambda\lambda 0.
\]

Figure 3 presents SK and its three contractions. It shows how de Bruijn indices are built from variables (aka explicit names), indicates the links between names and their binders and

\(^2\)This has been popularised by Curien [9]. Notice that de Bruijn and Curien make the indices to start at 1, but the last author proposed in [28] the indices to start at 0, a convention largely adopted since [34].
presents the chain of β-reductions in de Bruijn notation. Notice that in $\lambda\lambda(\lambda1(10))$ the same variable $z$ is associated with two de Bruijn indices, $\lambda$ and $0$ and that the same de Bruijn index $\lambda$ is associated with two variables, $y$ and $z$. In the de Bruijn notation the value of an index associated with a variable depends on the context. Also, notice that $0, 10$ and $20(10)$ are examples of open terms in de Bruijn notation.

![Figure 3: The term SK and its three contractions](image)

The basic reduction considered here is

![Basic Reduction Diagram](image)

Three patterns are of interest in Figure 3:

![Patterns Diagram](image)

The first pattern corresponds to a $\lambda$ that binds no index, the second pattern corresponds to a $\lambda$ that binds exactly one index and the third pattern corresponds to a $\lambda$ that binds two indices. This later pattern is representative, but clearly, there are patterns with more bound indices (see Figure 10, page 28). We propose the control of variable by restricting the language to $\Lambda^c$ in Section 3 and $\Lambda^c_\#$ in Section 4, and by extending the language to $\Lambda^c_\#_v$ in Section 5. In the new language $\Lambda^c_\#_v$, extended with two new operators $\triangledown$ (duplicator) and $\odot$ (erasure), terms are linearised, meaning that only patterns corresponding to a $\lambda$ that binds exactly one index are present in the Bourbaki representation (see Figure 4). This recalls Lamping’s optimal calculus [25], which is described in [20] and in [1] with its connections with linear logic. In $\Lambda_{\#}^c$, we have an atomic substitution, whereas in Lamping’s calculus there is none. Indeed, in Lamping calculus, fans (a kind of duplicators) are propagated. However, the connection should be deepened.
3 Restricted terms $\Lambda^\ell$

In this section, we focus on restricted terms [22] with implicit names [10, 28]. We first define the concept of list types, which we will refer to as $\mathcal{L}$-types. Then we define a type system which assigns $\mathcal{L}$-types to $\lambda$-terms with implicit names and show how this type system singles out linear terms with implicit names. The set of terms typeable with $\mathcal{L}$-types will be denoted by $\Lambda^\ell$.

3.1 $\mathcal{L}$-types for $\Lambda^\ell$

Lists of natural numbers are called $\mathcal{L}$-types for $\Lambda^\ell$.

Definition 1 ($\mathcal{L}$-types). The abstract syntax of $\mathcal{L}$-types is given by

$$\ell ::= [] \mid i :: \ell$$

where $i \in \mathbb{N}$

The empty list is $[]$ and the cons operation, $::$, puts an element in front of a list. We write$^3$ the list made of $1 :: (3 :: (5 :: []))$ as $[1, 3, 5]$. A list is affine if its elements are not repeated. On lists, we define two operations: a binary operation merge, $\triangleleft$, and a unary operation decrement, $\downarrow$.

Definition 2 (Merge). The binary operation $\triangleleft$ which merges two lists is defined as follows:

$$[] \triangleleft \ell = \ell$$

$$(i :: \ell) \triangleleft [] = i :: \ell$$

$$(i_1 :: \ell_1) \triangleleft (i_2 :: \ell_2) =
\begin{cases} 
  i_1 :: (\ell_1 \triangleleft (i_2 :: \ell_2)) & \text{if } i_1 < i_2 \\
  i_2 :: ((i_1 :: \ell_1) \triangleleft \ell_2) & \text{if } i_1 > i_2
\end{cases}$$

Remark 1. Be aware that $\triangleleft$ is not total. For instance if $j$ occurs both in $\ell_1$ and in $\ell_2$ then $\ell_1 \triangleleft \ell_2$ is not defined. Also, note that if two sorted lists are merged, the result is a sorted list.

If all elements of a list are strictly positive, the list is said to be a strictly positive list. We define a unary operation $\downarrow$ on strictly positive lists. The result is either the empty list or the list where all indices of the initial list are decremented.

$^3$Beware ! The reader should not confuse lists as $\mathcal{L}$-types and lists of references.
**Definition 3** (Decrement). The unary operation $\downarrow$ is defined as follows:

$$
\downarrow [] = [] \\
\downarrow ((i + 1) :: \ell) = i :: \downarrow \ell
$$

We assume that the list $(i + 1) :: \ell$ is strictly positive, thus the list $\ell$ is also strictly positive and $\downarrow \ell$ is defined.

The function $\downarrow$ fails if the list contains 0. Said otherwise $\downarrow$ is not total, that is $\downarrow$ is only defined on strictly positive lists.

The type system that defines the set of restricted terms $\Lambda^L$ is given as follows.

**Definition 4** (Terms $\Lambda^L$). A $\lambda^L$-term is a $\lambda$-term that can be typed by the following rules.

\[
\frac{\text{(ind)}}{i : [i]} \quad \frac{\text{(abs)}}{t : 0 :: \ell \quad \lambda t : \downarrow \ell} \quad \frac{\text{(app)}}{t_1 : \ell_1 \quad t_2 : \ell_2 \quad t_1 t_2 : \ell_1 \uplus \ell_2}
\]

The set of all $\lambda^L$-terms is denoted by $\Lambda^L$.

The (abs) rule enforces the constraint that $\lambda x.t$ is well-typed under the condition that $x$ belongs to the set of free variables of $t$, via the condition $t : 0 :: \ell$. If the function $\downarrow$ fails the rule (abs) fails as well. Furthermore, in the (app) rule, the usage of the merge operator $\uplus$ disallows the use of the same free variable in both sub-terms. Likewise, if the operator $\uplus$ fails the rule (app) fails as well. Thus the non determinism of the type system lies in the failures of the functions it uses.

The given type system has no side condition. The same holds for the other two type systems of this paper, namely they have also no side condition for the rules (abs) and (app).

An $L$-type assigned to a term represents the list of natural numbers corresponding to its free implicit names. For instance, $\lambda 052$ has $L$-type $[1, 4]$ since the $L$-type of $052$ is clearly $[0, 2, 5]$ and to obtain the $L$-type of $\lambda 052$ one removes the $0$ which is bound and one decrements the other indices. Moreover, it is a sorted list, as shown by the following Proposition 1.

**Proposition 1** (Sortedness of lists). If $t : \ell$ then $\ell$ is sorted.

**Proof.** $[i]$ is sorted and $\uplus$ and $\downarrow$ preserve sortedness.

**Example 1** (Typing terms).

\[
\begin{array}{c}
1 : [1] & 0 : [0] & 2 : [2] & 0 : [0] & 1 : [1] & 0 : [0] & 0 : [0] \\
10 : [0, 1] & 20 : [0, 2] & 10 : [0, 1] & 10 : [0, 1] & \Lambda 0 : [] & \Lambda 10 : []
\end{array}
\]

Let us highlight the following facts:

1. The term $20 (10)$ is not $L$-typeable since there are two free occurrences of index 0. We cannot merge lists $[0, 2]$ and $[0, 1]$, thus $20 (10)$ does not belong to $\Lambda^L$. 

2. The empty list \([\ ]\) does not start with 0, thus \(\lambda\lambda0\) is not \(L\)-typeable, i.e. it does not belong to \(\Lambda^L\).

We prove further which combinators belong to \(\Lambda^L\) and which do not.

**Proposition 2** (Combinators in \(\Lambda^L\)). For the basic combinators \(S \equiv \lambda\lambda\lambda2(1)\), \(K \equiv \lambda\lambda0\), \(W \equiv \lambda\lambda10\) \(B \equiv \lambda\lambda\lambda2(10)\), \(C \equiv \lambda\lambda\lambda201\), and \(I \equiv \lambda0\) the following holds:

1. The combinators \(S\), \(K\) and \(W\) are not \(\lambda^L\)-terms, i.e. \(S, K, W \not\in \Lambda^L\).
2. The combinators \(B, C\) and \(I\) are \(\lambda^L\)-terms, i.e. \(B, C, I \in \Lambda^L\), moreover \(B : [\ ], C : [\ ]\) and \(I : [\ ]\).

**Proof.**

1. Example 1 proves that \(S \equiv \lambda\lambda\lambda2(1)\not\in \Lambda^L\). Furthermore, \(K \equiv \lambda\lambda1 \not\in \Lambda^L\) and \(W \equiv \lambda\lambda10\not\in \Lambda^L\) since

\[
\begin{array}{c}
\frac{1 : [1]}{\lambda1 : ?} & \frac{1 : [1]}{0 : [0]} & \frac{10 : [0,1]}{0 : [0]} & \frac{100 : ?}{10 : [0,1]}
\end{array}
\]

2. The proof-trees given below prove that \(B \equiv \lambda\lambda\lambda2(10) : [\ ], C \equiv \lambda\lambda\lambda201 : [\ ]\) and \(I \equiv \lambda0 : [\ ]\)

\[
\begin{array}{c}
\frac{1 : [1]}{2 : [2]} & \frac{0 : [0]}{2 : [2]} & \frac{0 : [0]}{1 : [1]}
\end{array}
\]

\[
\begin{array}{c}
\frac{2(10) : [0,1,2]}{\lambda2(1) : [0,1]} & \frac{201 : [0,1,2]}{\lambda201 : [0,1]} & 0 : [0]
\end{array}
\]

\[
\begin{array}{c}
\frac{\lambda2(1) : [0]}{\lambda\lambda2(1) : [\ ]} & \frac{\lambda201 : [0]}{\lambda\lambda201 : [\ ]}
\end{array}
\]

\[
\begin{array}{c}
\frac{\lambda0 : [\ ]}{\lambda0 : [\ ]}
\end{array}
\]

\[
\frac{\lambda0 : [\ ]}{\lambda0 : [\ ]}
\]

**Proposition 3.** If \(t\) is a BCI-term, i.e. a term generated by the combinators \(B, C\) and \(I\), then \(t : [\ ]\).

**Proof.** By Proposition 2 (2) and the fact that application is closed for terms typeable with empty lists, i.e. if \(t : [\ ]\) and \(s : [\ ]\), then \(ts : [\ ]\).
Remark We do not treat reduction in $\Lambda^L$, or more precisely, reduction using implicit substitution. We will treat fully reduction in the framework of explicit substitution in $\Lambda^L$ (Section 4) and in the extended language $\Lambda^L_\oplus$ (Section 5). Consequently the reader will find no $\beta$-reduction and no statement of a theorem of type preservation in this section. For a discussion the reader is invited to look at Section 4.2.

4 Restricted terms with explicit substitution $\Lambda^L_\upsilon$

In this section, we focus on terms with implicit names and explicit substitution. We start from the $\lambda_\upsilon$-calculus, a simple calculus with explicit substitution introduced by Lescanne in [27]. First, we modify the syntax and define restricted terms, dubbed $\lambda_\upsilon$-terms, by typeability with $L$-types. We then prove $L$-type preservation under reduction. The design of the language is inspired by [28].

The set of plain $\lambda_\upsilon$-terms, denoted by $\Lambda_\upsilon$, is given by the following syntax:

$$
\begin{align*}
t & ::= n \mid \lambda t \mid tt \mid t[s] \\
s & ::= t/ \mid \uparrow(s) \mid \uparrow
\end{align*}
$$

A term $t$ can be a natural number $n$ (i.e. a de Bruijn index), an abstraction, an application or a substituted term, where a substitution can be one of the following three: a slash $t/$, a lift $\uparrow(s)$ or a shift $\uparrow$.

The rewriting rules of the $\lambda_\upsilon$-calculus are given in Figure 5.

$$
\begin{align*}
(\lambda t_1) t_2 & \xrightarrow{\lambda_\upsilon} t_1[t_2/] \quad \text{(B)} \\
(t_1 t_2)[s] & \xrightarrow{\lambda_\upsilon} (t_1[s]) (t_2[s]) \quad \text{(App)} \\
(\lambda t)[s] & \xrightarrow{\lambda_\upsilon} \lambda (l[\uparrow(s)]) \quad \text{(Lambda)} \\
0[t/] & \xrightarrow{\lambda_\upsilon} t \quad \text{(FVar)} \\
n + 1[t/] & \xrightarrow{\lambda_\upsilon} n \quad \text{(RVar)} \\
0[\uparrow (s)] & \xrightarrow{\lambda_\upsilon} 0 \quad \text{(FVarLift)} \\
n + 1[\uparrow (s)] & \xrightarrow{\lambda_\upsilon} n[s][\uparrow] \quad \text{(RVarLift)} \\
n[\uparrow] & \xrightarrow{\lambda_\upsilon} n + 1 \quad \text{(VarShift)}
\end{align*}
$$

Figure 5: The rewriting system for $\lambda_\upsilon$-calculus

Example 2 ($\beta$-reduction). Let us write $t \downarrow_\upsilon$ the normalisation of $t$ using the set of rules

$$
u = \{(\text{App}), (\text{Lambda}), (\text{FVar}), (\text{RVar}), (\text{FVarLift}), (\text{RVarLift}), (\text{VarShift})\}.
$$

Said otherwise, one eliminates in $t$ all the occurrences of substitutions. Given a $\beta$-redex $(\lambda t_1) t_2$, its $\beta$-reduction is the term $(t_1[t_2/]) \downarrow_\upsilon$. This means that after the substitution $[t_2/]$ has been introduced, it is eliminated.

In what follows $\xrightarrow{\lambda_\upsilon}$ is the transitive closure of the rewriting relation $\xrightarrow{\lambda_\upsilon}$. In order to characterise linearity by a type system, we consider two kinds of objects:

- $[\uparrow^i]$ is called an updater and abbreviated as $[i]$, $i = 0, 1, ...$, whereas
• \([\tau^i(t/)]\) is called simply a substitution and abbreviated as \(\{t, i\}\), \(i = 0, 1, \ldots\).

According to the introduced abbreviations, we propose an alternative syntax that will be used in the definition of terms \(\Lambda_n^\Sigma\):

\[
t := n | \lambda t | tt | t[i] | t\{t, i\}
\]

Furthermore, we propose an alternative rewriting system for \(\lambda\nu\)-calculus, given in Figure 6, which is in accordance with the new syntax introduced above.

\[
\begin{array}{ccc}
(\lambda t_1) t_2 & \xrightarrow{\lambda t_1} & t_1\{t_2, 0\} \quad (B_{in}) \\
(t_1 t_2)[i] & \xrightarrow{\lambda t_1} & t_1[i] t_2[i] \quad (App_1) \\
(t_1 t_2)[t_3, i] & \xrightarrow{\lambda t_1} & t_1\{t_3, i\} t_2\{t_3, i\} \quad (App_1) \\
(\lambda t)[i] & \xrightarrow{\lambda t_1} & \lambda(t[i + 1]) \quad (Lambda_1) \\
(\lambda t_1)\{t_2, i\} & \xrightarrow{\lambda t_1} & \lambda(t_1\{t_2, i + 1\}) \quad (Lambda_1) \\
0(t, 0) & \xrightarrow{\lambda t_1} & t \quad (FVar_1) \\
n + 1\{t, 0\} & \xrightarrow{\lambda t_1} & n \quad (RVar_1) \\
0(t, i + 1) & \xrightarrow{\lambda t_1} & 0 \quad (FVarLift_1) \\
n + 1\{t, i + 1\} & \xrightarrow{\lambda t_1} & n\{t, i\}[0] \quad (RVarLift_1) \\
0[i + 1] & \xrightarrow{\lambda t_1} & 0 \quad (FVarLift_1) \\
n + 1[i + 1] & \xrightarrow{\lambda t_1} & n[i][0] \quad (RVarLift_1) \\
n[i][0] & \xrightarrow{\lambda t_1} & n + 1 \quad (VarShift_1)
\end{array}
\]

Figure 6: An alternative rewriting system for \(\lambda\nu\)-calculus

**Proposition 4.** The rewriting system given by the rules in Figure 6 is computationally equivalent to the rewriting system of \(\lambda\nu\) given in Figure 5.

**Proof.** Let us show, on four rules, as a paradigm for the others, the computational equivalence of the systems of Figure 5 and Figure 6.

- **Consider rule** \((B_{in})\): In Figure 5 and Figure 6, \((B)\) and \((B_{in})\) have the same left-hand side. In Figure 5 the right-hand side is \(t_1[t_2/]\) that is \(t_1[\tau^0(t_2/)]\) which writes \(t_1\{t_2, 0\}\) in the syntax of Figure 6.

- **Consider rule** \((\text{Lambda}_1)\): Written in the syntax of Figure 5, left-hand side \((\lambda t)[i]\) is \((\lambda t)[\tau^i(t/)]\) and right-hand side \(t[i + 1]\) is \(\lambda(t[\tau^{i+1}(t/)]\). Therefore right-hand side of rule \((\text{Lambda}_1)\) of Figure 6 is obtained from left-hand side of rule \((\text{Lambda}_1)\) by application of rule (\(\text{Lambda}\)) of Figure 5.

- **Consider rule** \((\text{Lambda}_{11})\): Written in the syntax of Figure 5, left-hand side \((\lambda t_1)\{t_2, i\}\) is \((\lambda t_1)[\tau^i(t_2/)]\) and right-hand side \(\lambda(t_1\{t_2, i + 1\})\) is \(\lambda(t_1[\tau^{i+1}(t_2/)]\). Therefore right-hand side of rule \((\text{Lambda}_{11})\) of Figure 6 is obtained from left-hand side of rule \((\text{Lambda}_{11})\) by application of rule (\(\text{Lambda}\)) of Figure 5.
Consider rule \((\text{RVarLift}\{\})\): Written in the syntax of Figure 5, left-hand side \(n + 1\{t, i+1\}\) is \(n + 1[\↑(t/i)]\) and right-hand side \(n\{t, i\}[0]\) is \(n[\↑(t/i)][↑]\). Therefore right-hand side of rule \((\text{RVarLift}\{\})\) of Figure 6 is obtained from left-hand side of rule \((\text{RVarLift}\{\})\) by application of rule \((\text{RVarLift}\{\})\) of Figure 5.

\[\]

4.1 \(\mathcal{L}\)-types for \(\Lambda_{\nu}^\mathcal{L}\)

Just like in the case of \(\Lambda_{\nu}^\mathcal{C}\), \(\mathcal{L}\)-types for \(\Lambda_{\nu}^\mathcal{L}\) provide information on free indices of a \(\lambda_{\nu}^\mathcal{L}\)-term. In a declaration \(t : \ell\), the type \(\ell\) represents a sorted list of free indices of \(t\). In order to define the language \(\Lambda_{\nu}^\mathcal{L}\) we need to extend the notion of \(\mathcal{L}\)-types with a new operation and the notion of filters on \(\mathcal{L}\)-types.

The operation \textit{increment}, denoted by \(\↑\), increments all the elements of a list and should not be confused with \(\downarrow\), defined in Definition 3, which decrements the indices of a list.

\textbf{Definition 6 (Increment).} The unary operation \(\↑\) is defined as follows:

\[
\begin{align*}
\↑[] &= [] \\
\↑(i :: \ell) &= (i + 1) :: \↑\ell
\end{align*}
\]

By \(\↑\) we denote \(i\) applications of the operation \(\↑\), i.e. \(\↑^0 \ell = \ell\) and \(\↑^{i+1} = \↑\↑^i \ell\). Since \textit{shift} is not defined in the same context as \textit{increment}, there is no confusion despite both use the same symbol.

\textbf{Filters on lists}

In order to ease list manipulation, we introduce filters on lists. Given a predicate \(p\) on naturals and a list \(\ell\), then \((p \mid \ell)\) is the list filtered by the predicate.

\[
\begin{align*}
(p \mid []) &= [] \\
(p \mid i :: \ell) &= \text{if } p(i) \text{ then } i :: (p \mid \ell) \text{ else } (p \mid \ell)
\end{align*}
\]

We will consider three basic predicates

\[
\begin{align*}
< i &= \text{def} \lambda k . k < i \\
> i &= \text{def} \lambda k . k > i \\
\geq i &= \text{def} \lambda k . k \geq i
\end{align*}
\]

We can modify a predicate \(p\) into \(p_{[i \leftarrow e]}\) which is \(p\) in which each free occurrence of \(i\) is replaced by \(e\). Assume that predicates are made of

- constants,
- free variables,
- basic predicates \(< i, > i, \text{ and } \geq i,\)
- logical connectors,
- functions on the naturals, like \(\lambda k . k + 1\)

\]
we define \( p_i \) by induction as follows (we denote an expression by \( expr \) and we assume \( k \) is not the same variable as \( i \)):

\[
\begin{align*}
& (\langle expr \rangle | i) = def \lambda k. k < expr | i \\quad \text{and} \quad k \neq i \\
& (\rangle expr \rangle | i) = def \lambda k. k > expr | i \\
& (\geq expr | i) = def \lambda k. k \geq expr | i \\
& (p \lor q) | i = def p | i \lor q | i \\
& (p \land q) | i = def p | i \land q | i
\end{align*}
\]

The substitution in expressions over the naturals is done as usual, as the substitution in universal algebra.

**Example 3** (Predicates).

\[
\begin{align*}
& (\langle 3 | [0, 2, 3, 4] \rangle = [0, 2] \\
& (\geq 3 | [0, 2, 3, 4]) = [3, 4] \\
& (\rangle 3 | [0, 2, 3, 4]) = [4] \\
& \uparrow (\geq 3 | [0, 2, 3, 4]) = \uparrow [3, 4] = [4, 5] \\
& \downarrow (\geq 3 | [0, 2, 3, 4]) = \downarrow [3, 4] = [2, 3] \\
& (\langle i + 1 \rangle | [i, j] \rangle = \langle (j + 1) + 1 \rangle = \langle j + 2 \rangle \\
& (\geq (i + 1) | [i, j] \rangle = \geq 1
\end{align*}
\]

Now, we can prove the following auxiliary lemma, containing list related properties needed in the proof of type preservation.

**Lemma 1.** Let \( \ell, \ell_1, \ell_2 \) and \( \ell_3 \) be sorted lists. The following equations hold, if all lists that appear in the equations are defined.

\[
\begin{align*}
a) & \quad \ell_1 \uparrow \ell_2 = \ell_2 \uparrow \ell_1; \\
b) & \quad \ell_1 \uparrow (\ell_2 \uparrow \ell_3) = (\ell_1 \uparrow \ell_2) \uparrow \ell_3; \\
c) & \quad (p \parallel \ell_1) \uparrow (p \parallel \ell_2) = (p \parallel \ell_1 \parallel \ell_2); \\
d) & \quad \uparrow \ell_1 \uparrow \ell_2 = \uparrow (\ell_1 \parallel \ell_2); \\
e) & \quad \downarrow \ell_1 \downarrow \ell_2 = \downarrow (\ell_1 \parallel \ell_2); \\
f) & \quad \uparrow (p \parallel \ell) = (p \parallel \ell) \uparrow \ell, \\
g) & \quad \downarrow (p \parallel \ell) = (p \parallel \ell) \downarrow \ell. \\
\end{align*}
\]

**Proof.**

- **a)** Cases \( \parallel \ell \) and \( \parallel \ell \) are by definition. Consider the case \( \langle n_1 \parallel \ell_1 \rangle \parallel (n_2 \parallel \ell_2) \) with \( n_1 < n_2 \):

\[
\begin{align*}
\langle n_1 \parallel \ell_1 \rangle \parallel (n_2 \parallel \ell_2) & = n_1 \parallel (\ell_1 \parallel (n_2 \parallel \ell_2)) \quad \text{(by definition)} \\
& = n_1 \parallel ((n_2 \parallel \ell_2) \parallel \ell_1) \quad \text{(by induction)} \\
& = (n_2 \parallel \ell_2) \parallel (n_1 \parallel \ell_1) \quad \text{(by definition)}.
\end{align*}
\]

Case \( n_2 < n_1 \) is symmetric.
b) Cases where at least one of $\ell_1$, $\ell_2$ or $\ell_3$ is $[]$ are easy. For the general case, consider $n_1 < n_2 < n_3$. The other cases are on the same pattern.

\[
(n_1 :: \ell_1) \downarrow ((n_2 :: \ell_2) \downarrow (n_3 :: \ell_3))
= (n_1 :: \ell_1) \downarrow (n_2 :: (\ell_2 \downarrow (n_3 :: \ell_3)))
= n_1 :: (\ell_1 \downarrow (n_2 :: (\ell_2 \downarrow (n_3 :: \ell_3))))
= n_1 :: ((\ell_1 \downarrow (n_2 :: \ell_2)) \downarrow (n_3 :: \ell_3))
= (n_1 :: (\ell_1 \downarrow (n_2 :: \ell_2))) \downarrow (n_3 :: \ell_3)
= ((n_1 :: \ell_1) \downarrow (n_2 :: \ell_2)) \downarrow (n_3 :: \ell_3)
\]

c) By case and induction

\[
(p | []) \downarrow (p | \ell) = [] \downarrow (p | \ell) = (p | \ell) = (p | []) \uparrow \ell
\]

\[
(p | i :: \ell) \downarrow (p | []) = (p | i :: \ell) \downarrow []
= (p | i :: \ell) = (p | (i :: \ell) \uparrow [])
\]

General case and sub-case $i_1 < i_2$ and $\neg p(i_1)$:

\[
(p | i_1 :: \ell_1) \downarrow (p | i_2 :: \ell_2) = (p | \ell_1) \downarrow (p | i_2 :: \ell_2)
= (p | \ell_1 \downarrow (i_2 :: \ell_2)) = (p | i_1 :: (\ell_1 \downarrow (i_2 :: \ell_2)))
= (p | (i_1 :: \ell_1) \downarrow (i_2 :: \ell_2))
\]

by induction. Sub-case $i_1 < i_2$ and $p(i_1)$:

\[
(p | i_1 :: \ell_1) \downarrow (p | i_2 :: \ell_2) = (i_1 :: (p | \ell_1)) \downarrow (p | i_2 :: \ell_2)
= i_1 :: ((p | \ell_1) \downarrow (p | i_2 :: \ell_2))
= i_1 :: (p | \ell_1 \downarrow (i_2 :: \ell_2)) = (p | i_1 :: (\ell_1 \downarrow (i_2 :: \ell_2)))
= (p | (i_1 :: \ell_1) \downarrow (i_2 :: \ell_2)).
\]

The sub-cases $i_2 < i_1$ ($\neg p(i_2)$ and $p(i_2)$) are similar.

d) Also by case and induction

\[
(\uparrow []) \downarrow (\uparrow \ell) = [] \downarrow (\uparrow \ell) = \uparrow \ell = \uparrow ([]) \downarrow \ell
\]

\[
(\uparrow (i :: \ell)) \downarrow (\uparrow []) = (\uparrow (i :: \ell)) \downarrow [] = \uparrow (i :: \ell) = \uparrow ((i :: \ell) \downarrow [])
\]

case $i_1 < i_2$ (hence $(i_1 + 1) < (i_2 + 1)$):

\[
(\uparrow (i_1 :: \ell_1)) \downarrow (\uparrow (i_2 :: \ell_2)) = ((i_1 + 1) :: \uparrow \ell_1) \downarrow ((i_2 + 1) :: \uparrow \ell_2)
= (i_1 + 1) :: ((\uparrow \ell_1) \downarrow ((i_2 + 1) :: \uparrow \ell_2))
= (i_1 + 1) :: ((\uparrow \ell_1) \downarrow ((i_2 :: \ell_2)))
= (i_1 + 1) :: (\uparrow \ell_1 \downarrow (i_2 :: \ell_2))
= \uparrow (i_1 :: (\ell_1 \downarrow (i_2 :: \ell_2))) = \uparrow ((i_1 :: \ell_1) \downarrow (i_2 :: \ell_2))
\]

Case $i_2 < i_1$ is similar.
e) This proof is similar to the proof of d).

f) Also by case and induction

\[ \uparrow (p \mid i \ell) = \uparrow \ell = \uparrow \ell = (p_{[\ell \to \ell + 1]} \mid \uparrow \ell) = (p_{[\ell \to \ell + 1]} \mid \uparrow \ell) \]

Sub-case \( \neg p(i) \), hence \( \neg p_{[\ell \to \ell + 1]}(i + 1) \)

\[ \uparrow (p \mid i \ell) = \uparrow (p \mid \ell) \]

by definition of the filter

\[ = (p_{[\ell \to \ell + 1]} \mid \uparrow \ell) \]

by induction

\[ = (p_{[\ell \to \ell + 1]} \mid (i + 1) \ell) \]

by definition of the filter

\[ = (p_{[\ell \to \ell + 1]} \mid \uparrow (i \ell)) \]

by definition of \( \uparrow \)

Sub-case \( \neg p(i) \), hence \( p_{[\ell \to \ell + 1]}(i + 1) \)

\[ \uparrow (p \mid i \ell) = \uparrow (i \ell, (p \mid \ell)) \]

by definition of the filter

\[ = (i + 1) \ell, \uparrow (p \mid \ell) \]

by definition of \( \uparrow \)

\[ = (i + 1) \ell, (p_{[\ell \to \ell + 1]} \mid \uparrow \ell) \]

by induction

\[ = (p_{[\ell \to \ell + 1]} \mid \uparrow (i \ell)) \]

by definition of the filter

g) Works like f).

\[ \square \]

**Definition 7** (Terms \( \Lambda_\ell^C \)). A \( \lambda^C_\ell \)-term is a plain \( \lambda^\ell \)-term that can be \( \ell \)-typed by the rules of *Figure 7*. The set of all \( \lambda^C_\ell \)-terms is denoted by \( \Lambda_\ell^C \).

![Figure 7: Typing rules for \( \Lambda_\ell^C \)](image)

About rule \( \text{sub}_q \) we may notice that we assume \( t_2 : \ell_2 \) although this assumption is not used in the consequence.\(^4\) This guarantees the well-formedness of \( t_2 \).

Like for \( \Lambda_\ell^C \), we observe that in the typing tree of a \( \ell \)-typed closed term, we meet only sorted lists with unique occurrence of free indices. This suggests to define linearity in \( \Lambda^\ell \) by \( \ell \)-typeability.

**Definition 8** (Linearity of \( \lambda^\ell \)-terms). A \( \lambda^\ell \)-term is said to be linear if \( t : [] \).

\(^4\)A similar situation occurs with intersection types for explicit substitution [26], with the rule (drop).
4.2 Reduction of $\Lambda^L_v$

Let us recall that the $\Lambda^L_v$-calculus is equipped with the rewrite system given in Figure 6. In the following theorem, we prove that $L$-types are preserved by reduction.

**Theorem 1** ($L$-type preservation). If $t : \ell$ and $t \xrightarrow{\lambda v_i}$ $t'$, then $t' : \ell$.

**Proof.** Assume that $t$ matches the left-hand side of one of the rules given in Figure 6.

\[ B_{in} : (\lambda t_1) t_2 \xrightarrow{\lambda v_i} t_1 \{ t_2, 0 \} \]

The left-hand side and the right-hand side of the rule can be typed as follows:

\[
\begin{array}{c}
\lambda t_2 : \downarrow \ell_1 & \quad t_2 : \ell_2 \\
(\lambda t_1) t_2 : \downarrow \ell_1 \uparrow \ell_2
\end{array}
\]

\[
\begin{array}{c}
t_1 : 0 :: \ell_1 & \quad t_1 : 0 :: \ell_1 \\
t_2 : \ell_2 & \quad t_2 : \ell_2
\end{array}
\]

\[
t_1 \{ t_2, 0 \} : (\langle 0 \mid \ell_1 \rangle) \uparrow (\langle 0 \mid \ell_1 \rangle) \uparrow^0 \ell_2
\]

Successfully typing the left-hand side means $\downarrow \ell_1 \cap \ell_2 = \emptyset$. If this is the case, then $((\langle 0 \mid \ell_1 \rangle) \uparrow (\langle 0 \mid \ell_1 \rangle)) \uparrow^0 \ell_2 = \emptyset$ holds, so the right-hand side can be successfully typed.

The equality $\downarrow \ell_1 \uparrow \ell_2 = (\langle 0 \mid \ell_1 \rangle) \uparrow (\langle 0 \mid \ell_1 \rangle) \uparrow^0 \ell_2$ comes from

- $(\langle 0 \mid \ell_1 \rangle) = \emptyset$
- $(\langle 0 \mid \ell_1 \rangle) = \ell_1$
- $\uparrow^0 \ell_2 = \ell_2$

\[ \text{App} \| : (t_1 t_2)[i] \xrightarrow{\lambda v_1} t_1[i] t_2[i] \]

For the left-hand side of the rule we get

\[
\begin{array}{c}
t_1 : \ell_1 & \quad t_2 : \ell_2 \\
t_1 t_2 : \ell_1 \uparrow \ell_2
\end{array}
\]

\[
(t_1 t_2)[i] : (\langle i \mid \ell_1 \rangle \downarrow (\geq i \mid \ell_1)) \downarrow (\geq i \mid \ell_1 \downarrow \ell_2)
\]

For the right-hand side we get

\[
\begin{array}{c}
t_1 : \ell_1 & \quad t_2 : \ell_2 \\
t_1[i] : (\langle i \mid \ell_1 \rangle \downarrow (\geq i \mid \ell_1)) \downarrow (\geq i \mid \ell_1 \downarrow \ell_2)
\end{array}
\]

\[
(t_2[i]) : (\langle i \mid \ell_2 \rangle \downarrow (\geq i \mid \ell_2)) \downarrow (\geq i \mid \ell_2)
\]

Typing the left-hand side, means $\ell_1 \cap \ell_2 = \emptyset$. As a consequence, $((\langle i \mid \ell_1 \rangle \downarrow (\geq i \mid \ell_1)) \cap (\langle i \mid \ell_2 \rangle \downarrow (\geq i \mid \ell_2))) = \emptyset$ holds, so the right-hand side can be successfully typed. From Lemma 1, we conclude that

\[
\langle i \mid \ell_1 \downarrow \ell_2 \rangle \uparrow (\geq i \mid \ell_1 \downarrow \ell_2) = ((\langle i \mid \ell_1 \rangle \downarrow (\langle i \mid \ell_2 \rangle \downarrow (\geq i \mid \ell_1 \downarrow \ell_2))) \downarrow (\langle i \mid \ell_2 \rangle \downarrow (\geq i \mid \ell_2)))
\]

\[
= ((\langle i \mid \ell_1 \rangle \downarrow (\geq i \mid \ell_1))) \downarrow (\langle i \mid \ell_2 \rangle \downarrow (\geq i \mid \ell_2))
\]
App: \( (t_1 t_2)\{t_3, i\} \xrightarrow{\lambda t} t_1\{t_3, i\} t_2\{t_3, i\} \)

For the left-hand side, with \( i \in \ell_1 \) we get

\[
\begin{array}{c}
t_1 : \ell_1 \\
t_2 : \ell_2 \\
t_3 : \ell_3 \\
\hline
(\ell_1 \ell_2)\{t_3, i\} : (\langle i | \ell_1 \uparrow \ell_2 \rangle \uparrow \downarrow (i | \ell_1 \uparrow \ell_2) \uparrow \uparrow i \ell_3 \quad i \in \ell_1
\end{array}
\]

If the right-hand side is successfully typed, then \( \ell_1 \cap \ell_2 = [] \), and since \( i \in \ell_1 \), then \( i \notin \ell_2 \). For the right-hand side, we get

\[
\begin{array}{c}
t_1 : \ell_1 \\
t_3 : \ell_3 \\
t_2 : \ell_2 \\
t_3 : \ell_3 \\
\hline
(\ell_1 \ell_2)\{t_3, i\} : (\langle i | \ell_1 \uparrow \ell_2 \rangle \uparrow \downarrow (i | \ell_1 \uparrow \ell_2) \uparrow \uparrow i \ell_3 \quad i \notin \ell_2
\end{array}
\]

Like the previous cases, it is straightforward to show that whenever the left-hand side is typeable, the right-hand side is typeable as well.

Here also, from Lemma 1, we get

\[
(\langle i \uparrow | \ell_1 \uparrow \ell_2 \rangle \uparrow \downarrow (i \uparrow | \ell_1 \uparrow \ell_2) \uparrow \uparrow i \ell_3 = (\langle i \uparrow | \ell_1 \uparrow \ell_2 \rangle \uparrow \downarrow (i \uparrow | \ell_1) \uparrow \uparrow i \ell_3 \uparrow \uparrow (i \uparrow | \ell_2) \uparrow \downarrow (i \uparrow | \ell_2)).
\]

The case \( i \notin \ell_1, i \in \ell_2 \) is similar. Let us look now at case \( i \notin \ell_1 \uparrow \ell_2 \). For the left-hand side we have

\[
\begin{array}{c}
t_1 : \ell_1 \\
t_2 : \ell_2 \\
t_3 : \ell_3 \\
\hline
(\ell_1 \ell_2)\{t_3, i\} : (\langle i \uparrow | \ell_1 \uparrow \ell_2 \rangle \uparrow \downarrow (i \uparrow | \ell_1 \uparrow \ell_2)
\end{array}
\]

For the right-hand side we have

\[
\begin{array}{c}
t_1 : \ell_1 \\
t_3 : \ell_3 \\
t_2 : \ell_2 \\
t_3 : \ell_3 \\
\hline
(\ell_1 \ell_2)\{t_3, i\} : (\langle i \uparrow | \ell_1 \uparrow \ell_2 \rangle \uparrow \downarrow (i \uparrow | \ell_1 \uparrow \ell_2)
\end{array}
\]

Whenever \( \ell_1 \cap \ell_2 = [] \) holds and we can type the left-hand side of the rule, \( (\langle i \uparrow | \ell_1 \rangle \uparrow \downarrow (i \uparrow | \ell_1)) \cap (\langle i \uparrow | \ell_2 \rangle \uparrow \downarrow (i \uparrow | \ell_2)) = [] \) holds and the right-hand side of the rule can be typed.

From Lemma 1 we conclude that we obtain equal types for both left-hand side and right-hand side of the rule.

**Lambda**: \( (\lambda t)[i] \xrightarrow{\lambda t} \lambda(t[i + 1]) \)

Left-hand and right-hand sides of the rule can be typed as follows:

\[
\begin{array}{c}
t : 0 :: \ell \\
\hline
\lambda t : \downarrow \ell
\end{array}
\]

\[
\begin{array}{c}
t[i + 1] : 0 :: (i + 1 | \ell) \\
\hline
\lambda(t[i + 1]) : \downarrow ((i + 1 | \ell) \uparrow (i + 1 | \ell))
\end{array}
\]
The equality \((< i \downarrow \ell) \uparrow ((i + 1) \downarrow \ell) = \downarrow ((< i + 1 \downarrow \ell) \uparrow ((i + 1) \downarrow \ell))\) is a consequence of Lemma 1 d) e) and g).

**Lambda**

\[
\lambda t_1\{t_2, i\} \xrightarrow{\lambda v_i} \lambda (t_1\{t_2, i + 1\})
\]

First, we consider case \(i + 1 \not\in \ell_1\) (with the same calculation as in the case **Lambda**):

\[
\frac{t_1 : 0 :: \ell_1\quad t_2 : \ell_2}{(\lambda t_1\{t_2, i\}) : (\ell_1 \downarrow ((i + 1) \downarrow \ell_1)) \quad \ell_2 \downarrow ((i + 1) \downarrow \ell_1)}
\]

From Lemma 1 we can conclude that the type of the term on the left-hand side of the rule and the type of the term on the right-hand side of the rule are equal.

Next, let us look at the case \(i + 1 \in \ell_1\)

\[
\frac{t_1 \in \ell_1\quad t_2 : \ell_2}{(\lambda t_1\{t_2, i\}) : (\ell_1 \downarrow ((i + 1) \downarrow \ell_1)) \quad \ell_2 \downarrow ((i + 1) \downarrow \ell_1)}
\]

The equality \((< 0 | [0]) \uparrow 0 \ell = \ell\) comes from the fact that \((< 0 | [0]) \uparrow 0 \ell = [\ell].

**RVar**

\[
\frac{n + 1\{t, 0\} \xrightarrow{\lambda v_i} n}{n : [n]}\quad n + 1 : [n + 1] \quad t : \ell\quad n + 1\{t, 0\} : (\ell_1 \downarrow ((i + 1) \downarrow \ell_1)) \quad (\ell_2 \downarrow ((i + 1) \downarrow \ell_1)) \quad n : [n]}
\]

The equality of the types comes from the fact that \((< 0 | [0]) = [\ell]\) and \((\ell_1 \downarrow ((i + 1) \downarrow \ell_1)) = [\ell].\]
\[
F_{\text{VarLift}}(i) : \quad 0\{t, i + 1\} \xrightarrow{\lambda \text{eq}} 0
\]

\[
0 : [0] \quad t : \ell
\]

\[
0\{t, i + 1\} : (< i + 1 \mid [0]) \uparrow \downarrow (> i + 1 \mid [0]) \quad i + 1 \in [0]
\]

\[
0 : [0]
\]

The equality of the types comes from \( (< i + 1 \mid [0]) = [0] \) and \( \downarrow (> i + 1 \mid [0]) = [] \).

\[
R_{\text{VarLift}}(i) : \quad n + 1\{t, i + 1\} \xrightarrow{\lambda \text{eq}} n\{t, i\}[0]
\]

We will consider three cases, depending on the numbers \( i \) and \( n \). First, we consider the case where \( i < n \)

\[
n + 1 : [n + 1] \quad t : \ell
\]

\[
n + 1\{t, i + 1\} : (< i + 1 \mid [n + 1]) \uparrow \downarrow (> i + 1 \mid [n + 1]) \quad i + 1 \in [n + 1]
\]

\[
n : [n] \quad t : \ell
\]

\[
n\{t, i\} : \downarrow [n] \quad i \in [n]
\]

Since \( i < n \), we have \( i + 1 < n + 1 \), and it holds that \( (< i + 1 \mid [n + 1]) = [] \) and \( \downarrow (> i + 1 \mid [n + 1]) = \downarrow [n + 1] = [n] \), so the types are equal.

Next, we consider the case where \( i = n \).

\[
n + 1 : [n + 1] \quad t : \ell
\]

\[
n + 1\{t, i + 1\} : ((< i + 1 \mid [n + 1]) \uparrow \downarrow (> i + 1 \mid [n + 1])) \uparrow i + 1 \in [n + 1]
\]

\[
n : [n] \quad t : \ell
\]

\[
n\{t, i\} : \uparrow i \ell \quad i \in [n]
\]

From \( i = n \), we obtain \( i + 1 = n + 1 \), and it follows that \( (< i + 1 \mid [n + 1]) \uparrow \downarrow (> i + 1 \mid [n + 1]) = [] \), hence the types are equal.

Finally, we consider the case \( i > n \).

\[
n + 1 : [n + 1] \quad t : \ell
\]

\[
n + 1\{t, i + 1\} : (< i + 1 \mid [n + 1]) \uparrow \downarrow (> i + 1 \mid [n + 1]) \quad i + 1 \in [n + 1]
\]

\[
n : [n] \quad t : \ell
\]

\[
n\{t, i\} : \uparrow i \ell \quad i \in [n]
\]

Since \( i > n \), we have \( i + 1 > n + 1 \), and it follows that \( \downarrow (> i + 1 \mid [n + 1]) = [] \). Hence, the types are equal.

We see that in all three cases we have typed both the term on the left-hand side and the term on the right-hand side of the rule with the same type.
\[ \text{RVarLift} : \quad n + 1[i + 1] \xrightarrow{\lambda\nu_1} n[i][0] \]

Left-hand side and right-hand side of the rule can be typed as follows:

\[
\begin{align*}
n + 1: [n + 1] \\
(n + 1)[i + 1]: (& i + 1 \mid [n + 1]) \uparrow (\geq i + 1 \mid [n + 1])
\end{align*}
\]

\[
\begin{align*}
n : [n] \\
(n[i]): (< i \mid [n]) \uparrow (\geq i \mid [n])
\end{align*}
\]

\[
\begin{align*}
(n[i][0]): ((< i \mid [n]) \uparrow (\geq i \mid [n]))
\end{align*}
\]

From Lemma 1 we get

\[
(< i + 1 \mid [n + 1]) \uparrow (\geq i + 1 \mid [n + 1]) = (\geq i \mid [n]) \uparrow (\geq i \mid [n])
\]

\[ \text{VarShift} : \quad n[0] \xrightarrow{\lambda\nu_1} n + 1 \]

Left-hand side and right-hand side of the rule can be typed as follows:

\[
\begin{align*}
n : [n] \\
(n[0]): (< 0 \mid [n]) \uparrow (\geq 0 \mid [n])
\end{align*}
\]

\[
\begin{align*}
n + 1: [n + 1]
\end{align*}
\]

Since we have that

- \((< 0 \mid [n]) = []\),
- \((\geq 0 \mid [n]) = [n]\), and
- \((\uparrow [n]) = [n + 1]\),

it follows that \(((< 0 \mid [n]) \uparrow (\geq 0 \mid [n])) = [n + 1]\). 

\[ \square \]

Let us point out that this constructive proof of \(L\)-type preservation enables a constructive evaluator for terms in \(\Lambda^L\). Currently this implementation is in Haskell. Indeed Theorem 1 and Definition 8 entail the correctness of \(\lambda^L\)-calculus. If we remember that linearity is defined as \(L\)-typedness when the type is []\, we obtain the following corollary.

**Corollary 1** (Preservation of linearity). *If a \(\lambda\nu\)-term \(t\) is linear and \(t \xrightarrow{\lambda\nu_1} t'\) then \(t'\) is linear.*

An open question is whether this proof of preservation of linearity for a \(\lambda\)-calculus with implicit names (de Bruijn indices) yields easily a proof of preservation of linearity for a calculus with explicit names (nominal logic [35]). We refer to the discussion of Berghofer and Urban [2] to let the reader figure out the work that remains to be done.

## 5 Extended terms with resource control \(\Lambda^R\)

In the previous sections resource control was achieved by restricting the sets \(\Lambda\) and \(\Lambda\nu\) to \(\Lambda^L\) and \(\Lambda^L\), respectively. In this section, we take a dual approach and extend \(\Lambda\) with explicit operators performing erasure and duplication on terms in order to obtain full resource control. The goal is to design a language capable to linearise all \(\lambda\)-terms. We adapt \(L\)-types and use them to define terms \(\Lambda^L\) and to characterise linear terms in \(\Lambda^R\).
The abstract syntax of $\lambda_{\mathcal{R}}$-terms, plain terms with resources and implicit names, is generated by the following grammar:

$$t, s ::= (n, \alpha) \mid \lambda t \mid t s \mid (n, \alpha) \odot t \mid (n, \alpha) \triangledown t$$

where $(n, \alpha)$ is an $\mathcal{R}$-index, $\odot$ denotes the erasure of index in a term, and $\triangledown$ denotes the duplication of index in a term. The set of all $\lambda_{\mathcal{R}}$-terms is denoted by $\Lambda_{\mathcal{R}}$.

**$\mathcal{R}$-indices**

An $\mathcal{R}$-index is the pair $(n, \alpha)$, where $n$ is a natural number and $\alpha$ is a string of booleans. For convenience, we will use the following abbreviations: $0 \equiv false$ and $1 \equiv true$. Therefore $\alpha$ will be a string of 0's and 1's. Whether 0 and 1 refer to natural numbers or to booleans will be easily distinguished; so we consider that using those notations will introduce no confusion. In $(n, \alpha)$, $n$ corresponds to an index in $\Lambda$ and $\alpha$ represents duplications of the index. The empty string of booleans, corresponding to absence of duplications, is denoted by $\varepsilon$. For instance, if $(n, \varepsilon)$ is duplicated, it is represented by $(n, 0)$ and $(n, 1)$; if it is triplicated, it can be represented by $(n, 0), (n, 10)$ and $(n, 11)$ (or by $(n, 00), (n, 01)$ and $(n, 1)$).

In the following example and in Subsection 5.1 we introduce informally notions corresponding to $\lambda_{\mathcal{R}}$-terms, which will be formally defined in Subsection 5.2.

**Example 4.**

- The term $\lambda x.y$ is represented in $\Lambda_{\mathcal{R}}$ by the term $\lambda(0, \varepsilon) \odot (1, \varepsilon)$.
- The term $\lambda x.(x(\lambda y.xy))$ is represented in $\Lambda_{\mathcal{R}}$ by the term $\lambda((0, \varepsilon) \triangledown ((0, 0) (\lambda(1, 1) (0, \varepsilon))))$.
- The linear term $\lambda x.\lambda y.x y$ is represented in $\Lambda_{\mathcal{R}}$ by the term $\lambda(1, \varepsilon) (0, \varepsilon)$, that has neither $\triangledown$ nor $\odot$, since it is linear and needs no resource control.
- The open term with multiple occurrences of a free variable $x z (y z)$ is represented in $\Lambda_{\mathcal{R}}$ by the term $(0, \varepsilon) \triangledown ((2, \varepsilon) (0, 0) ((1, \varepsilon) (0, 1)))$ where $(2, \varepsilon)$ represents the free variable $x$, $(1, \varepsilon)$ represents the free variable $y$, and $(0, 0)$ and $(0, 1)$ correspond to the two occurrences of the variable $z$.
- Term $\lambda x.x x x$ is discussed in Example 8.

Several more examples of $\lambda_{\mathcal{R}}$-terms will be elaborated in the following subsection.

### 5.1 A bestiary of $\lambda_{\mathcal{R}}$-terms

In this section, we examine basic and well known terms.

**The term I**

$$I = \lambda(0, \varepsilon).$$
This corresponds to the term $\lambda x.x$ in the $\lambda$-calculus with explicit names. $(\mathbf{0}, \varepsilon)$ means that there is no $\lambda$ between the $(\mathbf{0})$-index $(\mathbf{0}, \varepsilon)$ and its binder and that there is no duplication.

The term $K$

$$K = \lambda \lambda (\mathbf{0}, \varepsilon) \circ (\mathbf{1}, \varepsilon).$$

In $\lambda$-calculus, $K$ is written $\lambda x.\lambda y.x$. In $\Lambda$, $K$ is written $\lambda \lambda \mathbf{1}$. The index $\mathbf{0}$ does not occur in $\mathbf{1}$, but since we want $\Lambda_{(\mathbf{0})}$-terms to be linear, we make it to occur anyway, thus we write $(\mathbf{0}, \varepsilon) \circ (\mathbf{1}, \varepsilon)$. Notice that $\varepsilon$ is the second component of all the $(\mathbf{0})$-indices since there is no duplication. Recall that the term $\lambda \lambda (\mathbf{0}, \varepsilon) \circ (\mathbf{1}, \varepsilon)$ correspond to the term $\lambda \lambda \mathbf{1}$. The translation from $\Lambda_{(\mathbf{0})}$ to $\Lambda$ will be introduced in Section 5.4. This term also corresponds to the term

$$\lambda x.\lambda y. y \circ x$$

using the notations of [17] and to the term

$$\lambda x.\lambda y. W_{\lambda y}(x).$$

using the notations of [23].

The term $S$

$$S = \lambda \lambda \lambda (\mathbf{0}, \varepsilon) \triangledown (\mathbf{0}, \varepsilon) (\mathbf{0}, 0) ((\mathbf{0}, \varepsilon) (\mathbf{0}, 1))$$

In $\lambda$-calculus, $S$ is written $\lambda x.\lambda y.\lambda z.xz(yz)$ and in $\Lambda$, $S$ is written $\lambda \lambda \lambda (\mathbf{2} \ 0 \ (\mathbf{1} \ 0))$. We notice the double occurrence of $z$ in $\lambda$-calculus and of $\mathbf{0}$ in $\Lambda$. Therefore a duplication is necessary. From the $(\mathbf{0})$-index $(\mathbf{0}, \varepsilon)$ it creates two indices $(\mathbf{0}, 0)$ and $(\mathbf{0}, 1)$. Where the second component $0$ is the string of length $1$ made of $0$ alone and the second component $1$ is the string of length $1$ made of $1$ alone. This term can be written using the notations of [17] as the term

$$\lambda x.\lambda y.\lambda z. (z \langle z_0 \ x \ z_0 \ y \ z_1 \rangle)$$

or using the notations of [23] as the term

$$\lambda x.\lambda y.\lambda z. (C_{\varepsilon}^z (x \ z_0 \ y \ z_1)).$$

The term $5$

$$5 = \lambda \lambda ((\mathbf{1}, \varepsilon) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0))$$

$5$ represents the Church numeral $5$. Recall that in $\lambda$-calculus, $5$ is written $\lambda f.\lambda x.(f(f(f(f(f(x))))))$ and in $\Lambda$, $\lambda \lambda (1(1(1(1(1))))))$. Since $1$ is repeated five times, we need four duplications. If we compute $5$ other ways, we can get other forms. For instance, as the result of $3 + 2$:

$$\lambda \lambda ((\mathbf{1}, \varepsilon) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0))$$

or as the result of $2 + 3$:

$$\lambda \lambda ((\mathbf{1}, \varepsilon) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0))$$

or as the result of $3 + 1 + 1$:

$$\lambda \lambda ((\mathbf{1}, \varepsilon) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0) \ \triangledown (\mathbf{1}, 0))$$

21
The four above forms correspond to the same term in Λ, namely \( \lambda\lambda(1(1(1(1(1))))). \) The translation readback from \( \Lambda_{\mathbb{R}} \) to Λ will be defined in Section 5.4.

**The terms \( ff \) and \( tt \)**

The \( \mathbb{R} \)-term \( ff \) (i.e. the boolean false) is \( \lambda(\underline{0}, \varepsilon) \odot \lambda(\underline{0}, \varepsilon) \) and the \( \mathbb{R} \)-term \( tt \) (i.e. the boolean true, that is also the combinator K) is \( \lambda(\lambda(\underline{0}, \varepsilon) \odot (\underline{1}, \varepsilon)) \).

**The Curry fixpoint combinator**

The Curry fixpoint combinator \( Y \) is:

\[
Y = \lambda L_0, \varepsilon \, \varepsilon ((\lambda (L_1, 0) (L_1, 0)) (\lambda (L_1, 1) (L_1, 0)))
\]

and in notations of [17]:

\[
\lambda x. (x^{(\alpha_0)} \lambda y. (x_0 (y^{(y_0)} y_0 y_1))) (\lambda y. (x_0 (y^{(y_0)} y_0 y_1))
\]

or using the notations of [23]:

\[
\lambda x. (C_x^{(\alpha_0)} (\lambda y. (x_0 (C_y^{y_0} y_0 y_1))) (\lambda y. (x_0 (C_y^{y_0} y_0 y_1))))
\]

### 5.2 \( \mathcal{L} \)-types for \( \Lambda_{\mathbb{R}} \)

In this setting, lists of \( \mathbb{R} \)-indices are called \( \mathcal{L} \)-types for \( \Lambda_{\mathbb{R}} \).

**Definition 9 (\( \mathcal{L} \)-types for \( \Lambda_{\mathbb{R}} \)).** The abstract syntax of \( \mathcal{L} \)-types for \( \Lambda_{\mathbb{R}} \) is given by

\[
\ell ::= [] | (\underline{n}, \alpha) :: \ell
\]

where \((\underline{n}, \alpha)\) is an \( \mathbb{R} \)-index.

Operations \( \dagger \) and \( \downarrow \) are defined in Section 3 for lists on \( \mathbb{N} \). Here we apply \( \dagger \) to lists of \( \mathbb{R} \)-indices. For that, we have to define an order on the set of all \( \mathbb{R} \)-indices. We define first an order on strings of booleans.

**Definition 10 (Order on strings of booleans).** An order \( <_L \) on strings of booleans is defined as

\[
0 :: \ell <_L 1 :: \ell \quad \varepsilon <_L b :: \ell \quad \ell_1 <_L \ell_2 \quad b :: \ell_1 <_L b :: \ell_2
\]

In other words, \( <_L \) is the lexicographic extension on lists of the order \( 0 < 1 \).

**Definition 11 (Order on \( \mathbb{R} \)-indices).** An order \( <^\mathbb{R} \) on \( \mathbb{R} \)-indices is defined as

\[
\underline{n_1} < \underline{n_2} \quad (\underline{n_1}, \alpha_1) <^\mathbb{R} (\underline{n_2}, \alpha_2) \quad \alpha_1 <_L \alpha_2 \quad (\underline{n}, \alpha_1) <^\mathbb{R} (\underline{n}, \alpha_2)
\]

In other words, \( <^\mathbb{R} \) is the lexicographic product \( < \times <_L \) of the orders \( < \), on the naturals and \( <_L \) on strings of booleans. By \( \leq^\mathbb{R} \) we denote the relation \( <^\mathbb{R} \lor = \) and the relation \( \leq^\mathbb{R} \) is total.
Definition 12 (Merge). A binary operation which merges two lists of \( \mathbb{R} \)-indices is defined as follows:

\[
\begin{align*}
\emptyset \triangleright \ell & = \ell \\
((n, \alpha) :: \ell) \triangleright \emptyset & = (n, \alpha) :: \ell \\
((n_1, \alpha_1) :: \ell_1) \triangleright ((n_2, \alpha_2) :: \ell_2) & = \begin{cases} 
(n_1, \alpha_1) \circ (n_2, \alpha_2) & \text{if } (n_1, \alpha_1) <^\mathbb{R} (n_2, \alpha_2) \\
(n_2, \alpha_2) \circ (n_1, \alpha_1) & \text{if } (n_2, \alpha_2) <^\mathbb{R} (n_1, \alpha_1) 
\end{cases} \\
& \quad \text{then } ((n_1, \alpha_1) :: \ell_1) \triangleright ((n_2, \alpha_2) :: \ell_2) \\
\end{align*}
\]

Remark 2. The function \( \triangleright \) is not total.

If a list \( \ell \) is an empty list or it contains only indices with strictly positive first component, we write \( \ell \in \text{List}^+ \).

Definition 13 (Decrement). Given a list \( \ell \), assume that we have a proof that \( \ell \in \text{List}^+ \), we can define operation \( \downarrow \) on this list:

\[
\begin{align*}
\downarrow \emptyset & = \emptyset \\
\downarrow ((n + 1, \alpha) :: \ell) & = (n, \alpha) :: \downarrow \ell 
\end{align*}
\]

All properties proved in Lemma 1 hold also for the lists of \( \mathbb{R} \)-indices. We omit the proof, due to the lack of space and the fact that it is analogous to the proof of Lemma 1.

By means of \( \mathcal{L} \)-typeability, we single out meaningful (well-formed) plain terms with resources and implicit names.

Definition 14 (Terms \( \Lambda^{\mathcal{L}}_{\mathbb{R}} \)). A \( \lambda^{\mathcal{L}}_{\mathbb{R}} \)-term is a plain \( \lambda_{\mathbb{R}} \)-term that can be \( \mathcal{L} \)-typed by the rules of Figure 8.

The set of all \( \lambda^{\mathcal{L}}_{\mathbb{R}} \)-terms is denoted by \( \Lambda^{\mathcal{L}}_{\mathbb{R}} \).

The following example illustrates the Definition 14 by \( \mathcal{L} \)-typing the \( \lambda_{\mathbb{R}} \)-term \( \text{SK} \).

Example 5.
(2, ε) : (2, ε) (0, 0) : (0, 0) (1, ε) : (1, ε) (0, 1) : (0, 1)  
(2, ε) (0, 0) : (0, 0, (2, ε)) (1, ε) (0, 1) : (0, 1, (1, ε))  
(0, ε) ∨ ((2, ε) (0, 0) ((1, ε) (0, 1))) : (0, ε) (1, ε) (2, ε)  
λ((0, ε) ∨ ((2, ε) (0, 0) ((1, ε) (0, 1)))) : (0, ε)  
λλ((0, ε) ∨ ((2, ε) (0, 0) ((1, ε) (0, 1)))) : []  
(λλλ((0, ε) ∨ ((2, ε) (0, 0) ((1, ε) (0, 1)))) (λλ((0, ε) ∨ (1, ε)) : []  
λλ((0, ε) ∨ (1, ε)) : []  
λ((0, ε) ∨ (1, ε)) : []  
λ((0, ε) ∨ (1, ε)) : []  
λλ((0, ε) ∨ (1, ε)) : []

Notice that we abstract with λ (see Definition 14) only ⌦-index of the form (0, ε). Further, the definition of ⌦ ensures that in an L-typed term an ⌦-index can occur at most once (Definition 12). The other binder, namely duplication, binds two ⌦-indices of the form (n, α0) and (n, α1) and produces a new ⌦-index (n, α). Closed terms are terms in which each ⌦-index is bound.

**Proposition 5** (Closedness). If \( t : [] \) then \( t \) is closed.

**Proof.** If \( t : \ell \), then \( \ell \) is the set of free ⌦-indices in the term. Therefore, if \( \ell \) is empty then \( t \) has no free ⌦-index and \( t \) is closed. \( \square \)

There are actually two rules which eliminate ⌦-indices, namely abs and dup. But when dup eliminates two indices (n, α0) and (n, α1), it introduces (n, α). Therefore if a term is closed, all the ⌦-indices are checked for linearity when abstracted by λ. This justifies the following definition of linearity.

**Definition 15** (Linearity of \( \text{λ}_{\text{L}} \)-terms). A \( \text{λ}_{\text{L}} \)-term \( t \) is said to be linear if \( t : [] \).

Similarly as in Section 3.1 and 4.1, the notion of \( \text{L} \)-typeability has enabled the characterisation of linearity of \( \text{λ}_{\text{L}} \)-terms.

### 5.3 Reduction in \( \text{L}_{\text{R}} \)

We define rewriting rules for normal forms w.r.t. \( \odot \) and \( \triangledown \) and we prove type preservation. Consequently, linearity is preserved. Those rules are inspired by [17]. Basically, we propagate \( \triangledown \) in the term and pull \( \odot \) out of the term.

First, we define replacement of an ⌦-index in a term. By \( t [\langle n, \alpha \rangle \leftarrow \langle m, \beta \rangle] \) we denote a term obtained from term \( t \) by replacing recursively the ⌦-index \( \langle n, \alpha \rangle \) by \( \langle m, \beta \rangle \).

**Definition 16** (Replacement). Let \( \text{cond}(\alpha, \delta, n, k) \) the condition

\[ n \neq k \land \forall \gamma \in \{0, 1\}^* \delta \neq \alpha \gamma. \]

Notice that this can be written also

\[ n \neq k \land \lnot (\alpha \text{ prefix } \delta). \]
Replacement $t \lfloor (n, \alpha) \leftrightarrow (m, \beta) \rfloor$ is defined as:

\[
\begin{align*}
(n, \alpha \gamma) \lfloor (n, \alpha) \leftrightarrow (m, \beta) \rfloor &= (m, \beta \gamma) \\
(k, \delta) \lfloor (n, \alpha) \leftrightarrow (m, \beta) \rfloor &= (k, \delta) & \text{if } \text{cond}(\alpha, \delta, n, k) \\
(t_1, t_2) \lfloor (n, \alpha) \leftrightarrow (m, \beta) \rfloor &= t_1 \lfloor (n, \alpha) \leftrightarrow (m, \beta) \rfloor \lfloor (m, \alpha) \leftrightarrow (m, \beta) \rfloor \\
\lambda t \lfloor (n, \alpha) \leftrightarrow (m, \beta) \rfloor &= \lambda(t \lfloor (n+1, \alpha) \leftrightarrow (m+1, \beta) \rfloor) \\
((k, \delta) \ast t) \lfloor (n, \alpha) \leftrightarrow (m, \beta) \rfloor &= (k, \delta) \lfloor (n, \alpha) \leftrightarrow (m, \beta) \rfloor \ast t \lfloor (n, \alpha) \leftrightarrow (m, \beta) \rfloor, & \ast \in \{\circ, \nabla\}
\end{align*}
\]

The rewriting system for $\lambda_{R}^{\text{L}}$-terms is given by the rules in Figure 9.

\[
\begin{align*}
\lambda(n + 1, \alpha) \circ t & \rightarrow (n, \alpha) \circ \lambda t \\
(n, \alpha) \nabla (\lambda t) & \rightarrow \lambda((n + 1, \alpha) \nabla t) \\
((n, \alpha) \circ (t_1, t_2)) & \rightarrow ((n, \alpha) \circ t_1) \circ t_2 \\
(t_1, (n, \alpha) \circ t_2) & \rightarrow (n, \alpha) \circ (t_1, t_2) \\
((n, \alpha) \nabla (t_1, t_2)) & \rightarrow ((n, \alpha) \nabla t_1) \circ t_2, & \text{if } (\n, \alpha, 0) \in t_1 \land (\n, \alpha, 1) \in t_1 \\
(n, \alpha) \nabla (t_1, t_2) & \rightarrow t_1 ((n, \alpha) \nabla t_2), & \text{if } (\n, \alpha, 0) \in t_2 \land (\n, \alpha, 1) \in t_2 \\
(n, \alpha) \circ (m, \beta) \circ t & \rightarrow (m, \beta) \circ (n, \alpha) \circ t, & \text{if } n < m \\
(n, \alpha) \nabla (m, \alpha) \circ t & \rightarrow t ((n, \alpha) \nabla (m, \alpha)) \\
(n, \alpha) \nabla (m, \alpha) \circ t & \rightarrow t \lfloor (n, \alpha, 0) \leftrightarrow (m, \alpha, 0) \rfloor \\
(n, \alpha) \nabla (m, \alpha) \circ t & \rightarrow t \lfloor (n, \alpha, 1) \leftrightarrow (m, \alpha, 1) \rfloor \\
(n, \alpha) \nabla (m, \beta) \circ t & \rightarrow (m, \beta) \circ (n, \alpha) \nabla t, & \text{if } n \neq m \lor (\beta \neq \alpha, 0 \land \beta \neq \alpha, 1) \\
(n, \alpha) \nabla ((m, \alpha) \nabla t) & \rightarrow (n, \alpha) \nabla ((m, \alpha) \nabla t) \lfloor (n, \alpha, 0) \leftrightarrow (m, \alpha, 0) \rfloor \lfloor (n, \alpha, 0) \leftrightarrow (m, \alpha, 1) \rfloor \lfloor (n, \alpha, 1) \leftrightarrow (m, \alpha, 1) \rfloor)
\end{align*}
\]

Figure 9: The rewriting system for $\lambda_{R}^{\text{L}}$-terms

**Example 6** (Reducing SKK in $\lambda_{R}^{\text{L}}$). As an example, we propose to reduce the term $\text{SKK}$, from Subsection 5.1:

\[
\lambda \lambda \lambda 0, 1, \varepsilon \nabla ((2, \varepsilon)(0, 1)(0, 0)((1, \varepsilon)(0, 1))) \lambda \lambda (0, 1, \varepsilon) \circ (0, 1, \varepsilon) \lambda (0, 1, \varepsilon) \circ (0, 1, \varepsilon)
\]

This will be an application of the Haskell implementation. Let us call skk the Haskell term that implements $\text{SKK}$ and $\beta$ the function in Haskell that contracts one step in $\lambda_{R}^{\text{L}}$, in other words, one step of rule $(\lambda \circ \circ)$, followed by a normalisation by the other rules. We perform four applications of $\beta$. After each application of $\beta$ we apply a readback, to see the corresponding $\lambda$-term. readback is defined in Definition 18, but the reader may figure out what it means.

\[\text{25}\]
Theorem 2 (Ł-type preservation). If \( t : \ell \) and \( t \to t' \), then \( t' : \ell \).

Proof. Assume that \( t \) matches the left-hand side of one of the rules in Figure 9. We consider the following two cases.

\[ (\lambda - \circ) : \quad \lambda(n + 1, \alpha) \circ t \to (n, \alpha) \circ \lambda t \]

Rule \( \lambda - \circ \) preserves type. Indeed
Both the term on the left-hand side and the term on the right-hand side of the rule are typed with the same type.

\[(\triangledown - \lambda) : \triangledown t \rightarrow \lambda \triangledown t\]

Rule \((\triangledown - \triangledown)\) preserves type. Indeed

\[
\begin{array}{c}
t : [(\ell) \triangledown (\ell) : \triangledown\ell] \Downarrow [(\ell) \triangledown (\ell) : \triangledown\ell] \\
\lambda t : [(\ell) \triangledown (\ell) : \triangledown\ell] \Downarrow [(\ell) \triangledown (\ell) : \triangledown\ell] \\
\end{array}
\]

Both the term on the left-hand side and the term on the right-hand side of the rule are typed with the same type.

Proving that other rules preserve \(L\)-type is straightforward.

\[\square\]

Example 7 \((\triangledown - \triangledown)\).

\[
\begin{align*}
\lambda((\ell) \triangledown (\ell, 1) \triangledown ((\ell, 0) \triangledown (\ell, 10) \triangledown (\ell, 11))) & \rightarrow \\
\lambda((\ell) \triangledown (\ell, 0) \triangledown ((\ell, 00) \triangledown (\ell, 01) \triangledown (\ell, 1))) & \\
\end{align*}
\]

See Figure 10.

Corollary 2 (Preservation of linearity). If a \(\lambda\)-term \(t\) is linear and \(t \rightarrow t'\) then \(t'\) is also linear.

5.4 Correspondence with \(\Lambda\)

In order to establish a correspondence between the introduced language \(\Lambda\) and the well-known system \(\Lambda\), we define two translations: \textbf{read} : \(\Lambda \rightarrow \Lambda\) and \textbf{readback} : \(\Lambda \rightarrow \Lambda\).

Definition 17 \textbf{read}. \textbf{read} : \(\Lambda \rightarrow \Lambda\)

- \textbf{read} \(\ell = (\ell, \varepsilon)\)
- \textbf{read} \(\lambda t = \text{let } u = \text{read } t \text{ in if } (\ell, \varepsilon) \in u \text{ then } \lambda u \text{ else } \lambda (\ell, \varepsilon) \odot u\)
- \textbf{read}(t_1, t_2) = \text{rename 0 } t_1 \text{ rename 1 } t_2\)

where

- \textbf{rename} 0 \(\ell\) replaces every \(\odot\)-index of the form \((\ell, \alpha)\) in the list \(\ell\) of \(\odot\)-indices by the corresponding \(\odot\)-index of the form \((\ell, \alpha0)\) and similarly \textbf{rename} 1 \(\ell\) replaces all \(\odot\)-index of the form \((\ell, \alpha)\) in the list by the corresponding \(\odot\)-index of the form \((\ell, \alpha 1)\).
− \( t_1^* \cap t_2^* \) is a short notation for the list of \( \otimes \)-indices that occur both in \( \text{read}(t_1) \) and in \( \text{read}(t_2) \).

− \( \bigwedge_{(\ell, \gamma) \in \ell} (k, \gamma) \) \( t \) is the iteration of the duplicator \( \bigwedge \). In other words, if \( \ell \) is the list \( \{(k_1, \gamma_1), (k_2, \gamma_2), \ldots, (k_n, \gamma_n)\} \), then \( \bigwedge_{(\ell, \gamma) \in \ell} (k, \gamma) \) \( t = (k_1, \gamma_1) \bigwedge (k_2, \gamma_2) \bigwedge \ldots (k_n, \gamma_n) \bigwedge t \).

The translation \( \text{read} \) is the formalisation of the translations presented in Example 4 and corresponds to the Haskell function \( \text{readLR} \) (l. 151 in \text{Lambda_R_dB.hs}).

**Definition 18** (\( \text{readback} \)). \( \text{readback} : \Lambda_{\otimes} \rightarrow \Lambda \)

- \( \text{readback} (\eta, \alpha) = \eta \)
- \( \text{readback} (\lambda t) = \lambda (\text{readback} t) \)
- \( \text{readback} (t_1 t_2) = (\text{readback} t_1) (\text{readback} t_2) \)
- \( \text{readback} ((\eta, \alpha) \odot t) = \text{readback} t \)
- \( \text{readback} ((\eta, \alpha) \bigwedge t) = \text{readback} t \).

**Proposition 6** (Correctness of \( \text{read} \)). \( \lambda t. \text{readback} \ (\text{read} t) : \Lambda \rightarrow \Lambda \) is the identity on \( \Lambda \). In other words,

\( \text{readback} \ (\text{read} t) = t. \)

The function \( \lambda t. \text{read} \ (\text{readback} t) : \Lambda_{\otimes} \rightarrow \Lambda_{\otimes} \) is an interesting function which associates with a term \( t \) another term with a somewhat standard disposition of \( \odot \) and \( \bigwedge \), which we call standardisation of the term.

![Figure 10: \( \lambda((0 \otimes 0) \otimes 0) \) and antecedents by \( \text{readback} \) as terms with two duplications](image)

Evidently, the same non-linear \( \lambda \)-term may correspond to several \( \lambda_{\otimes} \)-terms. For instance, this is the case for the term \( \lambda((0 \otimes 0) \otimes 0) \) (a \( \lambda_{\otimes} \) instance of \( \lambda x.xxx \)) illustrated by the following example and pictured in Figure 10.
Example 8. Consider the term \( \lambda(\underline{0}, \varepsilon)(\underline{0}, 1) \uplus (\underline{0}, \underline{0})(\underline{0}, 10)(\underline{0}, 11) \). This is actually term \( \lambda x.xxx \) in explicit name and no duplication.

\[
\text{readback}(\lambda(\underline{0}, \varepsilon)(\underline{0}, 1) \uplus (\underline{0}, \underline{0})(\underline{0}, 10)(\underline{0}, 11)) = \lambda(\underline{0}, \underline{0}) 0
\]

but

\[
\text{read}(\lambda(\underline{0}, \underline{0}) 0) = \lambda(\underline{0}, \varepsilon)(\underline{0}, 0) \uplus ((\underline{0}, 00)(\underline{0}, 01))(\underline{0}, 1)
\]

Hence

\[
\text{read} \circ \text{readback}(\lambda(\underline{0}, \varepsilon)(\underline{0}, 1) \uplus (\underline{0}, \underline{0})(\underline{0}, 10)(\underline{0}, 11)) = \lambda(\underline{0}, \varepsilon)(\underline{0}, 0) \uplus ((\underline{0}, 00)(\underline{0}, 01))(\underline{0}, 1)
\]

The reader may notice that, in both terms, the first duplication is \( \underline{0}, \varepsilon \). But the reader may also notice that the second duplication is \( \underline{0}, 1 \) in the first term and \( \underline{0}, 0 \) in the second term. So they are not the same. Choosing \( \underline{0}, 0 \) over \( \underline{0}, 1 \) is somewhat canonical. This corresponds to choosing the leftmost diagram in Figure 10. The fourth diagram corresponds to

\[
\lambda(\underline{0}, \varepsilon)(\underline{0}, 0) \uplus (\underline{0}, 1)(\underline{0}, 00)(\underline{0}, 11)(\underline{0}, 1)
\]

and the fifth diagram corresponds to

\[
\lambda(\underline{0}, \varepsilon)(\underline{0}, 0) \uplus (\underline{0}, 1)(\underline{0}, 10)(\underline{0}, 0)
\]

We let the reader write the \( \Lambda_{\mathcal{R}} \) term corresponding to the third diagram of Figure 10. There are 12 ways to write the term \( \lambda(\underline{0}, \underline{0}) 0 \) in \( \Lambda_{\mathcal{R}} \) and to draw corresponding diagrams. The reader may devise the omitted cases.

5.5 Implementation of \( \Lambda_{\mathcal{R}} \) in Haskell

We implemented the whole \( \lambda_{\mathcal{R}} \) in Haskell,\(^5\) where the data type for \( \Lambda_{\mathcal{R}} \) is as follows:

```haskell
data RTerm = App RTerm RTerm
  | Abs RTerm
  | Ind Int [Bool]
  | Era Int [Bool] RTerm
  | Dup Int [Bool] RTerm
```

We give here a flavor of the implementation. We have defined functions \text{read} and \text{readback}. As presented in the previous section \text{readback} is relatively easy to define, by just forgetting duplications and erasures. Function \text{read}, denoted by \text{readLR}, is defined in Haskell as follows:

\(^5\)For a better presentation see this.
L-types for resource aware languages

-- Given a list of indices and a term,
-- dupTheIndices applies all the duplications of that list to that term
dupTheIndices :: [(Int,[Bool])] -> RTerm -> RTerm
dupTheIndices [] t = t
dupTheIndices ((i,alpha):l) t = Dup i alpha (dupTheIndices l t)

-- 'consR' is a function used in 'readLR'
-- given a boolean and an index, put the boolean (0 or 1)
-- in front of all the alpha parts associated with the index
consR :: Bool -> Int -> RTerm -> RTerm
consR b i (App t1 t2) = App (consR b i t1) (consR b i t2)
consR b i (Abs t) = Abs (consR b (i+1) t)
consR b i (Ind j beta) = if i==j
    then Ind j (b:beta)
    else Ind j beta
consR b i (Era j beta t) = if i==j
    then Era j (b:beta) (consR b i t)
    else Era j beta (consR b i t)
consR b i (Dup j beta t) = if i==j
    then Dup j (b:beta) (consR b i t)
    else Dup j beta (consR b i t)

indOf is a function that extracts the indices of a term; ? is an infix operator which returns a boolean, i ? t returns True if and only if i occurs in t.

readLR :: Term -> RTerm
readLR (Ap t1 t2) = let rt1 = readLR t1
                        rt2 = readLR t2
                        indToIndR i = (i,[])
                        commonInd = sort (indOf t1 'intersect' indOf t2)
                        pt1 = foldl (.) id (map (consR False) commonInd) rt1
                        pt2 = foldl (.) id (map (consR True) commonInd) rt2
                        in dupTheIndices (map indToIndR commonInd) (App pt1 pt2)
readLR (Ab t) = if 0 ? t then Abs (readLR t) else Abs (Era 0 [] (readLR t))
readLR (In i) = Ind i []

We also present the Haskell code for test of linearity and closedness:

-- (iL t) returns the list of free (R)-de Bruijn indices of t
-- if all the binders of the term binds one and only one (R)-index.
remove :: Eq a => a -> [a] -> Maybe [a]
remove _ [] = Nothing
remove x (y:1) = if x == y then Just 1
                        else case (remove x 1) of
                                Nothing -> Nothing
                                Just 1' -> Just (y:1')
\texttt{iL :: RTerm -> Maybe [(Int,[Bool])]}\\
\texttt{iL (Ind n alpha) = Just [(n,alpha)]}\\
\texttt{iL (Abs t) =}\\
case \texttt{iL t of}\\
\hspace{1em}\texttt{Nothing \to Nothing}\\
\hspace{1em}\texttt{Just u \to case remove (0,[]) u of}\\
\hspace{2em}\texttt{Nothing \to Nothing}\\
\hspace{2em}\texttt{Just u' \to case find (((==) 0).fst) u' of}\\
\hspace{3em}\texttt{Just _ \to Nothing}\\
\hspace{3em}\texttt{Nothing \to Just $ map ((i,a)->(i-1,a)) u}$\\
\texttt{iL (App t1 t2) =}\\
case \texttt{iL t1 of}\\
\hspace{1em}\texttt{Nothing \to Nothing}\\
\hspace{1em}\texttt{Just u1 \to case iL t2 of}\\
\hspace{2em}\texttt{Nothing \to Nothing}\\
\hspace{2em}\texttt{Just u2 \to if null (u1 ‘intersect’ u2) then Just(u1 ++ u2) else Nothing}$\\
\hspace{2em}\texttt{iL (Era n alpha t) = case iL t of}\\
\hspace{3em}\texttt{Nothing \to Nothing}\\
\hspace{3em}\texttt{Just u -> Just ((n,alpha):u)}\\
\hspace{2em}\texttt{iL (Dup n alpha t) =}\\
case \texttt{iL t of}\\
\hspace{1em}\texttt{Nothing \to Nothing}\\
\hspace{1em}\texttt{Just u \to if (n,alpha++[False]) ‘elem’ u \&\&}\\
\hspace{2em}\texttt{(n,alpha++[True]) ‘elem’ u}\\
\hspace{2em}\texttt{then Just ((n,alpha):(delete (n,alpha++[False]) (delete (n,alpha++[True]) u)))}\\
\hspace{2em}\texttt{else Nothing}$\\
\texttt{-- is linear in the sense that all the binders bound one and only one index.}\\
\texttt{isLinearAndClosed t = case iL t of}\\
\hspace{1em}\texttt{Nothing \to False}\\
\hspace{1em}\texttt{Just u \to u == []}$\\
The $\beta$-reduction of $\lambda_\mathbb{R}$-terms is in \texttt{GitHub}.

6 Discussion and Related work

Compared to languages with explicit names, like $\lambda\text{lxr}$ [23] or the language of [17], $\lambda_\mathbb{R}$ is a simpler calculus, because, we can tell exactly how the $\mathbb{R}$-indices are duplicated, since we have a tight control on the way those indices are built. As consequences, there are fewer basic rules...
and a simple implementation is possible. For instance, if we consider a rough quantitatively
aspect, the calculus of [23] has 19 rules and 6 congruences, the system of [17] has 18 rules
(9 basic rules and 8 rules for substitution) and 4 congruences, whereas our system $\lambda_{\mathcal{G}}$ has 12
rules and no congruences.

Linear logic [18] is, among others, an approach to linearity of $\lambda$-calculus and certainly the
most popular and there is a vast literature on the subject. By 1998, people at CMU [33] collected
463 entries. Let us cite a few papers that address the implementation of the linear $\lambda$-calculus
by a calculus of explicit substitution related to linear logic [14, 15, 8]. More specifically, in
those calculi, the type system is this of linear logic, with connectors like $\multimap$ (linear implication),
$\otimes$ (tensor) and $!$ (exponential modality).

The two approaches share a common focus on linear $\lambda$-terms, addressing similar objects.
However, they diverge in their treatment of types: the authors of [14, 15, 8] use a different
set of types, namely the specific set of types of linear logic, whereas, in this paper, a given
$\lambda$-term can receive a pair of types, namely a first type usual in $\lambda$-calculus (simple type, system
$F$, calculus of construction, etc.) to characterise its computing behaviour and a second type
($L$-type) telling its linearity. A different approach to tracking resource usage in type systems
is presented by McBride [32], where resource-annotated contexts distinguish between data that
is observed and data that is consumed. Unlike our systems, which assign $L$-types to control
linearity, McBride uses annotations to track usage constraints, ensuring that certain terms can
be referenced without explicit duplication or erasure. This perspective aligns with our treatment
of $L$-types, which regulate resource usage in a different type-theoretic setting.

The $L$-types of our systems address a notion of correctness which is somewhat orthogonal to
this of standard types (such as simple types or higher order types). A term is well $L$-typed if it is
linear and we prove, thanks to $L$-type preservation, that linearity is preserved by reduction. The
two notions of types are orthogonal in the sense that standard types say something about the
result (the term is an integer or a boolean, for instance) whereas $L$-types say something about
the internal features of the terms (the term is linear). Since we do not characterise the “result”
of a computation, but only the structure of the term, there is no notion of “progress” associated
with $L$-types, there is only a notion of “preservation” (terms stay linear along their reduction,
i.e. $L$-type is preserved). However, it is possible to introduce other standard type systems, such
as simple types, intersection types or system $F$, to further characterise computational properties
of $L$-typed terms. As the continuation of the presented research, we intent to explore such a
hierarchy of type systems for the language $\Lambda_{\mathcal{G}}$.

We have developed the concept of $L$-types with the aim to characterise linear $\lambda$-terms.
Linear $\lambda$-calculus with implicit names, namely de Bruijn indices, is also in the focus of research
presented in [40, 39]. More precisely, in [40] the authors have introduced the calculus of explicit
substitutions with the affine and linear types. They have extended simple type system with
affine and linear functional types, in order to control the consumption of resources. However,
these types are classic types in the sense explained above.

The language $\Lambda_{\mathcal{G}}$ has connection with the differential $\lambda$-calculus of Erhard and Regnier [12]
where the duplicator $\nabla$ is a non-commutative differential operator (similar to their $D$) and the
black-hole $\odot$ corresponds to an empty iteration of $\nabla$ (like $D^0$). Therefore $\lambda_{\mathcal{G}}$ can be considered
as a non-commutative differential $\lambda$-calculus, where iterations are no more done on natural
numbers, but on lists of $\text{Bool}$. These observations merit to be deepened.

Paul Tarau and Valeria de Pavia address a similar problem ([41] Section 4.3) in their attempt
to generate closed linear $\lambda$-terms. Anyway in functional programming, programs of interest are
those with no free (undeclared) variable.
The implementation in Agda [30], currently under development, covers some topics, e.g. explicit substitution, which are not present in [43].

7 Conclusion

This paper introduces a novel approach to dealing with resource-related properties of formal languages, specifically focusing on the concept of linearity. This concept, although informally clear and intuitive, can be challenging to formally define and successfully implement. Our approach relies on using implicit names (such as de Bruijn indices or novel ®-indices) and L-types (which represent the list of free indices in a term). This approach enables simple formal definitions of linearity through L-typeability.

To illustrate our approach, we introduced three new languages with implicit names. The first language is the most straightforward, as it comprises traditional λ-terms with unique occurrences of each variable (BCIλ-terms). The second language addresses an abstract implementation of β-reduction through explicit substitution. The third language is concerned with resource control, featuring explicit duplication and explicit erasure.

For each of those languages, we introduce a specific list type system, which is used to give a simple and manageable definition of linearity. Those types represent lists of free implicit names: de Bruijn indices for the first two calculi, and new ®-indices for the calculus with resource control.

To summarise, we have addressed the concepts of linearity, implicit names and explicit resource control (duplication and erasure) by introducing and implementing in Haskell [29] list types, ®-indices and three new languages: ΛL, ΛLυ and ΛL®.

As for ongoing and further work, we plan to develop a direct method for characterising affiness and relevance in the implicit names framework. Moreover, the Agda implementation, which is currently under development [30], will be furthered.

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