SIMPLIFIED HYDROSTATIC CARBON BURNING IN WHITE DWARF INTERIORS

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ABSTRACT

We introduce two simplified nuclear networks that can be used in hydrostatic carbon burning reactions occurring in white dwarf interiors. They model the relevant nuclear reactions in carbon–oxygen white dwarfs approaching ignition in Type Ia supernova progenitors, including the effects of the main $e^-$-captures and $\beta$-decays that drive the convective Urca process. They are based on studies of a detailed nuclear network compiled by the authors and are defined by approximate sets of differential equations whose derivations are included in the text. The first network, N1, provides a good first-order estimation of the distribution of ashes and it also provides a simple picture of the main reactions occurring during this phase of evolution. The second network, N2, is a more refined version of N1 and can reproduce the evolution of the main physical properties of the full network to the 5% level. We compare the evolution of the mole fraction of the relevant nuclei, the neutron excess, the photon energy generation, and the neutrino losses between both simplified networks and the detailed reaction network in a fixed temperature and density parcel of gas.

Key words: nuclear reactions, nucleosynthesis, abundances – supernovae: general – white dwarfs

Online-only material: color figures

1. INTRODUCTION

Type Ia supernovae (SNe Ia) are the thermonuclear explosions of white dwarf (WD) stars. They are observable endpoints of stellar evolution, they shape the energy and chemistry evolution of galaxies, and they have been successfully used as distance indicators up to redshifts of $\sim 1.7$ thanks to an empirical decline rate–luminosity relation (Pskovskii 1977; Phillips 1993). This relationship led to the discovery of the acceleration of the universe (Riess et al. 1998; Perlmutter et al. 1999).

The exact nature of SN Ia progenitors is still debated (Hillebrandt & Niemeyer 2000), but most models involve an accreting carbon–oxygen white dwarf (CO WDs) with a mass close to the Chandrasekhar mass (Nomoto et al. 1984). This CO WD would accrete mass from either a slightly evolved main-sequence companion (CO WD + MS), a more evolved red giant star (CO WD + RG), or a helium star (CO WD + He), in the so-called single degenerate scenarios (SD; see, e.g., Hachisu et al. 1996; Li & van den Heuvel 1997; Hachisu et al. 1999a, 1999b; Langer et al. 2000; Han & Podsiadlowski 2004; Meng et al. 2009), or as a result of the merger of two degenerate stars with a combined mass above the Chandrasekhar limit, in the double degenerate scenario (DD; see, e.g., Webbink 1984; Iben & Tutukov 1984). It has been suggested that only CO WDs accreting within a narrow range lead to successful thermonuclear explosions (see, e.g., Nomoto & Kondo 1991; Nomoto et al. 2007), unless additional physical processes are included in the models, e.g., rotation and other three-dimensional effects (see, e.g., Domínguez et al. 2006; Yoon et al. 2007; Pakmor et al. 2010).

The diversity of SN Ia ejecta and the origin of the decline rate–absolute magnitude relation have been understood only recently thanks to new observational techniques and energy transfer codes (see, e.g., Mazzali et al. 2007; Woosley et al. 2007; Kromer & Sim 2009). These efforts have been accompanied by new developments in the physics of the explosion which have shown that even if pure deflagration models can reproduce many SN Ia spectra and light curves (Röpke et al. 2007), in some cases a delayed detonation may be necessary (Khokhlov 1991; Röpke et al. 2007). The physics of both the transition to detonation and ignition of the deflagration wave remains uncertain (Iapichino et al. 2006; Röpke 2007; Iapichino & Lesaffre 2010).

A related problem, rarely addressed in the literature, is how to connect theoretical models with observed systematic differences between SNe Ia occurring in different stellar environments (e.g., Hamuy et al. 1996; Sullivan et al. 2006). These should be related to the presupernova evolution and not to line-of-sight effects or other processes occurring randomly during the explosion (e.g., Kasen et al. 2009; Maeda et al. 2010). Thus, presupernova evolution must play a significant role in the diversity of SN Ia explosions (Lesaffre et al. 2006).

1.1. Presupernova Evolution: From Cooling to Ignition

Here, we will assume that the progenitors of SNe Ia originate in the SD scenario; but note that, even in the standard DD scenario, the core would evolve in a very similar way in the immediate pre-explosion phase (i.e., during the last $\sim 10^3$ yr; see Yoon et al. 2007).

Before an SN Ia progenitor becomes unbound by the explosion it undergoes several distinct phases of evolution (Nomoto et al. 1984; Lesaffre et al. 2006). First, the progenitor WD cools down at almost constant density after its birth, for a period of hundreds or thousands of Myr, depending on the particular formation scenario (the cooling phase). Then, it accretes matter for a period of the order of 1 Myr, making the degenerate star shrink to keep the hydrostatic equilibrium and its core compress adiabatically, the accretion phase. Adiabatic compression at the center and heat diffusion from the hot accreting envelope can make the central temperature increase under degenerate conditions, triggering hydrostatic carbon burning and a new source of energy generation. The energy input from hydrostatic carbon burning will force the star to transport the excess energy at its center convectively, the simmering phase. During this phase, the convective core can grow to engulf most of the star. If the central density is high enough, $e^-$-captures and $\beta$-decays can become...
important. In the presence of a convective core, that process has been called the convective Urca process.

At some point energy deposition will dominate over the energy losses, making the star’s central temperature increase at almost constant density, the thermonuclear flash. Finally, when the temperature is high enough, one or more ignition spots will give rise to nuclear flames that will propagate in the highly convective medium, the thermonuclear runaway, causing a deflagration wave to sweep over the star, sometimes transitioning into a detonation wave. The deflagration and detonation waves will generate most of the kinetic energy and ashes in the ejecta in a few seconds, including radioactive matter which will later power the light curve of the supernova. Since this last phase will occur at very high temperatures, the characteristic burning timescales will be much shorter than the typical weak interaction timescales and the neutron excess of the ejecta will not differ from that of the presupernova star, except for the star’s innermost regions where weak interaction timescales are shorter. Most of the WD’s neutron excess changes will occur before ignition.

1.2. Nuclear Physics and the Convective Urca Process

One of the main obstacles that remains to be solved in order to obtain self-consistent presupernova models up to ignition is the convective Urca process, which was mentioned in the previous section. The following factors make this phase of evolution difficult to solve: (1) the appearance of a fast-growing convective core with a very steep luminosity and composition gradient at its outer edge, the so-called C-flash; (2) a high-density medium with \( e^- \)-capture and \( \beta \)-decay timescales similar to the convective timescales, which have an uncertain effect over the energy budget of the star; (3) a steep density gradient which changes rapidly when the WD approaches the Chandrasekhar mass; (4) a steep composition gradient of the Urca pair \( ^{23}\text{Na} - ^{23}\text{Ne} \) at the threshold density for electron captures, which moves inward as the central density increases; and (5) an increasingly complex set of nuclear reactions as carbon burns and pollutes the WD with its ashes, with a timescale that can be similar to \( e^- \)-capture and \( \beta \)-decay timescales.

It is not clear whether \( e^- \)-captures and \( \beta \)-decays of Urca matter around a threshold density in a convective medium have a cooling or heating effect over time. Many studies over the years have reached different conclusions regarding this point. Paczynski (1972) first suggested that Urca processes have a stabilizing effect over carbon burning, leading to the formation of a neutron star instead of a thermonuclear explosion. Bruenn (1973) realized that \( e^- \)-captures can cause heating by creating holes in the Fermi sea, which dominate over the neutrino losses for the most important Urca pairs. Couch & Arnett (1974) found that significant work must be done to maintain convection when Urca processes occur, with a net cooling effect, but Regev & Shaviv (1975) pointed out that convection cannot develop fast enough to prevent heating. Barkat & Wheeler (1990) summarized the factors controlling the Urca process, but it was later shown that the role of the kinetic energy flux should have been included in their analysis (Mochkovitch 1996; Stein et al. 1999). More recently, Lesaffre et al. (2005) showed that the heating effect of the Urca process depends on the state of mixing of the convective core, that the convective velocities are reduced by Urca processes, and that time-dependent computations with a full nuclear network are needed to understand the effect of Urca processes on the ignition conditions of SNe Ia.

In this study, we have focused on how to accurately treat the increasingly complex nuclear reactions as the WD is polluted with ashes. We do not attempt to answer whether Urca processes cause cooling or heating, but provide a tool for better evaluating the competing heating and cooling mechanisms in a presupernova WD approaching ignition.

In what follows, we will introduce two approximations that use a limited number of nuclei to accurately describe the evolution of the main species that result from the burning of \( ^{12}\text{C} \), the energy deposition rate, the energy losses via neutrinos, and the neutron excess. We will show when these approximations hold, and their potential applications, but first we will introduce a detailed nuclear network which will be used for comparison with the simplified networks.

2. THE FULL NUCLEAR NETWORK

The nuclear reactions within a CO WD approaching explosion are characterized by the burning of \( ^{12}\text{C} \) nuclei at high densities (>10^7 g cm\(^{-3}\)) and high temperatures (>10^9 K) in an environment rich in \( ^{12}\text{C} \) and \( ^{16}\text{O} \) nuclei and relatively devoid of free protons, \( \alpha \)-particles or neutrons. The dominant reactions are \( ^{12}\text{C}(^{12}\text{C}, \alpha )^{20}\text{Ne} \), \( Q = 4.6 \text{ MeV} \), and \( ^{12}\text{C}(^{12}\text{C},^{4}\text{He})^{20}\text{Ne} \), \( Q = 2.2 \text{ MeV} \), both occurring at similar rates (see Figure 1). The nuclear network increases in complexity as the \( ^{12}\text{C} \)-burning pollutes the WD with its ashes, mainly \( ^{20}\text{Ne} \) and \( ^{23}\text{Na} \), but also with protons and \( \alpha \)-particles, which will be processed into additional \( ^{16}\text{O} \) and \( ^{13}\text{C} \) nuclei, as will be shown later.

We have integrated a detailed reaction network at fixed temperature and density compiled by the authors. The system of differential equations defining the network is solved for using the semi-implicit extrapolation method from Bader & Deuflhard (1983). We include species that are part of the reactions known to be most important, or those close to them in a table of nuclides. We include all reaction rates between species of our network that are either in the Reaclib library (Thielemann et al. 1987), or in the weak interaction tables by Oda et al. (1994), as well as improved \( ^{13}\text{N}(e^-, \nu_e)^{13}\text{C} \) rates from Zegers et al. (2008). Screening corrections were also included under the simplification of Gruboske et al. (1973). Recent formalisms that treat carbon burning and screening corrections in alternative

![Comparison of main carbon burning reactions](image_url)

**Figure 1.** Ratio between the two main carbon burning reactions (thermally averaged cross sections) plotted against the temperature in 10^9 K.
ways were not implemented for this work (see, e.g., Gasques et al. 2005; Yakovlev et al. 2006; Spillane et al. 2007).

We have assumed that the initial C/O ratio is always given by the nuclide mass fractions \( X(12C) = 0.3 \) and \( X(16O) = 0.7 \), in order to compare with the work by Chamulak et al. (2008), but reasonable variations of the initial compositions do not affect the validity of our approximations.

Figure 2 shows the main flows in our nuclear network at a temperature of \( 4 \times 10^8 \) K and densities of \( 1 \times 10^9 \) g cm\(^{-3}\) (top) and \( 3 \times 10^9 \) g cm\(^{-3}\) (bottom) when 20% of the original carbon has been burnt.

(A color version of this figure is available in the online journal.)

**Figure 2.** Main flows at a temperature of \( 4 \times 10^8 \) K and densities of \( 1 \times 10^9 \) g cm\(^{-3}\) (top) and \( 3 \times 10^9 \) g cm\(^{-3}\) (bottom) when 20% of the original carbon has been burnt.

To accurately follow the chemistry of the evolving WD progenitor toward explosion, the simultaneous solution of the structure and chemistry of the star is required (Stancliffe 2006). The chemistry equations must be able to reproduce the evolution of the main species and their effect on the star through changes in energy deposition, energy losses, or the electron fraction. This can become computationally demanding if too many species are included and can worsen during the simmering phase of evolution due to known convergence problems (Iben 1978b, 1982), even with approximate theories of convection.

We have found two approximate solutions that achieve the former based on the detailed nuclear network at fixed density and temperature described in this section (see also Chamulak et al. 2008; Piro & Bildsten 2008). The first such approximation will be described in the following section.

### 3. The Simplified Network: First Approximation (N1)

To accurately follow the chemistry of the evolving WD progenitor toward explosion, the simultaneous solution of the structure and chemistry of the star is required (Stancliffe 2006). The chemistry equations must be able to reproduce the evolution of the main species and their effect on the star through changes in energy deposition, energy losses, or the electron fraction. This can become computationally demanding if too many species are included and can worsen during the simmering phase of evolution due to known convergence problems (Iben 1978b, 1982), even with approximate theories of convection.

We have found a first-order approximation to the full nuclear network which will be described in this section. We will refer to this approximation as N1. A more accurate version of this approximation, N2, will be described in Section 5. Both simplified networks use the fact that the dominant carbon burning reactions, \( ^{12}\text{C}(^{12}\text{C}, \alpha)^{15}\text{O} \) and \( ^{12}\text{C}(^{12}\text{C}, \alpha)^{15}\text{N} \), occur at nearly the same rate and the abundances of protons, \( \alpha \)-particles, neutrons, and \( ^{13}\text{N} \) nuclei are approximately at equilibrium most of the time. This can be explained by the complementarity of the relevant reactions (see Figures 3 and 4), which was first noticed by Chamulak et al. (2008) and Piro & Bildsten (2008).

First, when in the carbon burning reaction \( ^{12}\text{C}(^{12}\text{C}, \alpha)^{15}\text{N} \), a proton is released, it will be captured quickly, preferentially in the \( ^{12}\text{C}(p, \gamma)^{13}\text{N} \) reaction. Second, \( ^{13}\text{N} \) will quickly capture an electron in the \( ^{13}\text{N}(e^-, \nu_e)^{13}\text{C} \) reaction, decreasing the pressure and lifting the degeneracy of the gas, and producing a \( ^{13}\text{C} \) nucleus. Third, in the carbon burning reaction \( ^{12}\text{C}(^{12}\text{C}, \alpha)^{20}\text{Ne} \), an \( \alpha \)-particle will be released and quickly captured in the \( ^{13}\text{C}(\alpha, n)^{16}\text{O} \) reaction, thus consuming the \( ^{13}\text{C} \) nucleus just produced, but liberating a neutron. Fourth, the free neutron will be preferentially captured in the \( ^{12}\text{C}(n, \gamma)^{13}\text{C} \) reaction, thus recovering the \( ^{13}\text{C} \) just consumed. Hence, the net effect
Figure 3. Main $^{12}$C-burning reactions and its ashes at low density. Six $^{12}$C nuclei are consumed to produce four nuclei: $^{23}$Na, $^{20}$Ne, $^{16}$O, and $^{13}$C and one electron capture in the reaction $^{13}$N($e^-,\nu_e$)$^{13}$C. The figure is shown as a rotated tree, with parent nuclei connected to children nuclei in the direction indicated by the arrows. Double arrows correspond to electron captures or inverse beta decays.

Figure 4. Same as Figure 3, but at high density. Six $^{12}$C nuclei are consumed to produce four nuclei: $^{23}$Ne, $^{20}$Ne, $^{16}$O, and $^{13}$C and two $e^-$-captures in the reactions $^{13}$N($e^-,\nu_e$)$^{13}$C and $^{23}$Na($e^-,\nu_e$)$^{23}$Ne.

is approximately the burning of six $^{12}$C nuclei and the capture of one electron to be replaced by four nuclei: $^{20}$Ne, $^{23}$Na, $^{16}$O, and $^{13}$C, i.e.,

$$6^{12}\text{C} + e^- \rightarrow ^{23}\text{Na} + ^{20}\text{Ne} + ^{16}\text{O} + ^{13}\text{C}. \quad (1)$$

An additional $e^-$-capture can occur via the $^{23}$Na($e^-,\nu_e$)$^{23}$Ne reaction if the density is above $\approx 1.7 \times 10^9$ g cm$^{-3}$. In this case, two $e^-$-captures can occur and the net effect is

$$6^{12}\text{C} + 2e^- \rightarrow ^{23}\text{Ne} + ^{20}\text{Ne} + ^{16}\text{O} + ^{13}\text{C}, \quad (2)$$

which is schematically shown in Figure 4. If any of the flows above is broken the simplified network will fail and will need a different treatment.

Figure 5. Cross-section ratio for proton captures onto $^{23}$Na. The dominant proton-leak reaction will be $^{23}$Na($p,\alpha$)$^{20}$Ne for all the temperature range considered here.

Now, we will assume that the only relevant reactions are

1. $^{12}$C($^{12}$C, $p$)$^{23}$Na, $Q = 2.24$ MeV,
2. $^{12}$C($^{12}$C, $\alpha$)$^{20}$Ne, $Q = 4.62$ MeV,
3. $^{12}$C($p,\gamma$)$^{13}$N, $Q = 1.94$ MeV,
4. $^{13}$N($e^-,\nu_e$)$^{13}$C, $Q = 2.22$ MeV,
5. $^{13}$C($\alpha, n$)$^{16}$O, $Q = 2.22$ MeV,
6. $^{12}$C($n,\gamma$)$^{13}$C, $Q = 4.95$ MeV,
7. $^{23}$Na($e^-,\nu_e$)$^{23}$Ne, $Q = -4.38$ MeV,
8. $^{23}$Ne($\beta^-$)$^{23}$Na, $Q = 4.38$ MeV.

Note that only mass differences are used to compute the $Q$-values and not the energy of the electrons lost or gained in weak interactions and that the $^{13}$N positron decay reaction must also be included at low densities. The change in electron density, which can be computed assuming charge conservation, can be used with the chemical potential of the electrons to compute the energy changes due to electron gains or losses. The Fermi energy values and not the energy of the electrons lost or gained in weak interactions (7) and (8), we can write the following differential equations:

$$dY^{(12}\text{C})/dt = -Y^{(12}\text{C})\rho N_{A\lambda_1} - Y^{(12}\text{C})\rho N_{A\lambda_2} - Y^{(12}\text{C})Y(p)\rho N_{A\lambda_3} - Y^{(12}\text{C})Y(n)\rho N_{A\lambda_6} + Y^{(13)N}\lambda_3^{inv}, \quad (3)$$

Since ignition can occur at temperatures as high as $8 \times 10^8$ K (see Figure 5 in Lesaffre et al. 2006), it is necessary to consider whether any inverse reaction from the list above can be significant. We have also found that only the inverse of reaction (3), i.e., $^{13}$C($\alpha, n$)$^{16}$O, needs to be considered, since its characteristic timescale can be comparable or smaller than the characteristic timescale of $^{13}$N $e^-$-captures, e.g., we have found that both timescales are similar for a temperature of $8 \times 10^8$ K and a density of $4 \times 10^8$ g cm$^{-3}$. Note that we use reaction rates for the $^{13}$N($e^-,\nu_e$)$^{13}$C reaction from Zegers et al. (2008).

Hence, we consider the following species: $^{12}$C, $^{13}$C, $^{13}$N, $^{16}$O, $^{20}$Ne, $^{23}$Na, $^{23}$Ne, $p$, $\alpha$, and $n$. Defining $\lambda_i \equiv \langle \sigma v \rangle_i$ as the thermally averaged cross section for the strong interactions 1 to 6, or the rate of occurrence per particle per unit time per unit volume of the weak interactions (7) and (8), we can write the following differential equations:

$$dY^{(12}\text{C})/dt = -Y^{(12}\text{C})\rho N_{A\lambda_1} - Y^{(12}\text{C})\rho N_{A\lambda_2} - Y^{(12}\text{C})Y(p)\rho N_{A\lambda_3} - Y^{(12}\text{C})Y(n)\rho N_{A\lambda_6} + Y^{(13)N}\lambda_3^{inv}, \quad (3)$$
The typical timescales for the equilibrium values to be reached

\[ dY(\text{13C})/dt = Y(\text{12C})Y(\text{n})\rho N_A\lambda_5 - Y(\text{13C})Y(\alpha)\rho N_A\lambda_3 + Y(\text{13N})\lambda_4, \]  

\( (4) \)

\[ dY(\text{13N})/dt = Y(\text{12C})Y(\text{p})\rho N_A\lambda_3 - Y(\text{13N})\lambda_4 - Y(\text{13N})\lambda_3^{-1}, \]  

\( (5) \)

\[ dY(\text{16O})/dt = Y(\text{13C})Y(\alpha)\rho N_A\lambda_5, \]  

\( (6) \)

\[ dY(\text{20Ne})/dt = \frac{Y^2(\text{12C})}{2}\rho N_A\lambda_2, \]  

\( (7) \)

\[ dY(\text{23Na})/dt = Y^2(\text{12C})\rho N_A\lambda_1 - Y^2(\text{23Na})\lambda_7 + Y(\text{23Ne})\lambda_8, \]  

\( (8) \)

\[ dY(\text{23Ne})/dt = Y(\text{21Na})\lambda_7 - Y(\text{21Ne})\lambda_8, \]  

\( (9) \)

\[ dY(\text{p})/dt = Y^2(\text{12C})\rho N_A\lambda_1 - Y(\text{12C})Y(\text{p})\rho N_A\lambda_3 + Y(\text{13N})\lambda_3^{-1}, \]  

\( (10) \)

\[ dY(\alpha)/dt = Y^2(\text{12C})\rho N_A\lambda_2 - Y(\text{13C})Y(\alpha)\rho N_A\lambda_5, \]  

\( (11) \)

\[ dY(n)/dt = -Y(\text{12C})Y(n)\rho N_A\lambda_6 + Y(\text{13C})Y(\alpha)\rho N_A\lambda_5, \]  

\( (12) \)

where \( \lambda_5^{-1} \) is the thermally averaged cross section of the inverse reaction \( ^{13}\text{N}(\gamma, \text{p})^{12}\text{C} \).

Numerical experiments with a detailed network show that the mole fractions of protons, \( \alpha \)-particles, neutrons, and \( ^{13}\text{N} \) nuclei will be many orders of magnitude smaller than those of \( ^{12}\text{C}, ^{16}\text{O}, ^{20}\text{Ne} \) and, depending on the density and temperature, \( ^{13}\text{C} \) and \( ^{23}\text{Na} \) or \( ^{23}\text{Ne} \). This means that their evolution is very fast compared to that of the rest of the species, even when very small quantities of \( ^{12}\text{C} \) have been burnt. Hence, we assume that the left-hand side of Equations (5), (10), (11), and (12) is much smaller than each individual term in the right-hand side of their respective equations. Neglecting these time derivatives, we can write the following equilibrium mole fractions for the nuclei \( \text{p}, \alpha, \text{n}, \) and \( ^{13}\text{N} \), which will hereafter be referred to as the trace nuclei:

\[ \bar{Y}(\text{p}) = Y(\text{12C})Y(\gamma)\rho N_A\lambda_5/\lambda_4, \]

\( \bar{Y}(\alpha) = Y(\text{12C})Y(\text{p})\rho N_A\lambda_3/\lambda_4, \]

\( \bar{Y}(\text{n}) = Y(\text{13C})Y(\alpha)\rho N_A\lambda_5/\lambda_4, \]

\( (13) \)

where \( f_{\text{inv}} \), defined as \( f_{\text{inv}} \equiv 1 + \lambda_5^{-1}/\lambda_4 \), indicates the relative strength of the inverse reaction \( ^{13}\text{N}(\gamma, \gamma)^{12}\text{C} \) with respect to \( ^{13}\text{N} \) e\textsuperscript{-} captures. This factor is independent of the composition and we will normally have \( f_{\text{inv}} \approx 1 \), except if we approach ignition at relatively low densities, where \( ^{13}\text{N} \) e\textsuperscript{-} captures cannot compete with the inverse reaction \( ^{13}\text{N}(\gamma, \gamma)^{12}\text{C} \), e.g., in off-center ignition.

If small quantities of \( ^{12}\text{C} \) are burnt, the mole fractions of the trace nuclei will reach the equilibrium values in Equations (13). The typical timescales for the equilibrium values to be reached from either lower or higher abundances can be found by dividing Equations (13) by the positive terms in the right-hand side of Equations (10), (11), (12), and (5), or an arbitrary higher-than-equilibrium mole fraction by the negative terms in the latter equations. Both calculations give the same timescales, assuming \( f_{\text{inv}} = 1 \), namely,

\[ \tau(\text{p}) = \left[ Y(\text{12C})\rho N_A\lambda_5 \right]^{-1}, \]

\[ \tau(\alpha) = \left[ Y(\text{13C})\rho N_A\lambda_3 \right]^{-1}, \]

\[ \tau(\text{n}) = \left[ Y(\text{12C})\rho N_A\lambda_5 \right]^{-1}, \]

\[ \tau(^{13}\text{N}) = \lambda_4^{-1}. \]

\( (14) \)

In contrast, the typical timescale for \( ^{12}\text{C} \)-burning will be

\[ \tau(^{12}\text{C}) \approx \left[ Y(\text{12C})\rho N_A(\lambda_1 + \lambda_2) \right]^{-1}, \]

\( (15) \)

and the typical timescales for \( ^{23}\text{Na} \) electron captures or \( ^{23}\text{Ne} \) \( \beta \)-decays will be \( \lambda_7^{-1} \) and \( \lambda_8^{-1} \), respectively. We compute these timescales at the temperatures and densities relevant for WD interiors (see Table 1) and find that the following relations will be normally satisfied:

\[ \tau(\text{n}) < \tau(\text{p}) < \tau(\alpha) < \tau(^{13}\text{N}) < \tau(^{12}\text{C}). \]

\( (16) \)

Given that the convective timescales found in pre-ignition WDs will normally be bigger than the biggest trace nuclei timescale shown in Table 1, it can be assumed that the trace nuclei will be in equilibrium even within moving convective eddies. The trace nuclei equilibrium mole fractions will be analogous to other equilibrium state variables like the temperature, which is defined by the assumption of local thermodynamic equilibrium inside every point of the star. In fact, in every stellar evolution code, as far as the authors are aware, energy diffusion during convection is computed assuming that local thermodynamic equilibrium is achieved in timescales much shorter than the convective turnover timescale.

Thus, we can assume that the trace nuclei equilibrium mole fractions are given by Equations (13) in every point of the star. Strictly speaking, the reactions needed to reach the equilibrium values will break the assumption of energy conservation under this approximation. However, since the trace nuclei equilibrium mole fractions will be very small compared to the \( ^{12}\text{C} \) mole fraction changes, this effect will be negligible during \( ^{12}\text{C} \)-burning.

It is worth noticing that the trace nuclei equilibrium mole fractions will change with the \( \tau(^{12}\text{C}) \) timescale. For example, when \( ^{13}\text{C} \) nuclei are synthesized by the burning of \( ^{12}\text{C} \), the equilibrium abundance of \( \alpha \)-particles will decrease significantly, as can be seen in Equations (13). Assuming that the mole fraction of \( ^{12}\text{C} \) is constant for this purpose, the \( \alpha \)-particle mole fraction would decrease as \( d\ln Y(\alpha) = -d\ln Y(\text{12C}) \).

With the assumption of trace nuclei equilibrium, we can obtain a simplified set of equations which do not include terms with the trace nuclei typical timescales. Replacing Equations (13) into Equations (3), (4), and (6)--(9), we obtain

\[ dY(\text{12C})/dt = -3Y^2(\text{12C})\rho N_A(\lambda_1 + \lambda_2), \]

\( (17) \)

\[ d\dot{Y}(\text{13C})/dt = Y^2(\text{12C})\rho N_A\lambda_3, \]

\( (18) \)

\[ dY(\text{16O})/dt = Y^2(\text{12C})\rho N_A\lambda_2, \]

\( (19) \)
approximation, which will greatly simplify the computational
accurately follow the chemistry changes in the star under this
become comparable, i.e., assuming
nuclei will be initially faster than that of $^{12}$C, but as significant
which constitutes the system of equations for a simplified
radiative and convective energy transport regions.

\[ dY^{(20)}(20Ne)/dt = \frac{Y^{(12)}(20Ne)}{2} \rho N_A \lambda_2, \quad (20) \]
\[ dY^{(23)}(23Na)/dt = \frac{Y^{(12)}(23Na)}{2} \rho N_A \lambda_1 - Y^{(23)}(23Na) \lambda_7 + Y^{(23)}(23Ne) \lambda_8, \quad (21) \]
\[ dY^{(23)}(23Ne)/dt = Y^{(23)}(23Na) \lambda_7 - Y^{(23)}(23Ne) \lambda_8, \quad (22) \]
which constitutes the system of equations for a simplified
nuclear network.

A quick inspection of these equations shows that only $^{12}$C, $^{23}$Na, and $^{23}$Ne need to be tracked as primary species to
accurately follow the chemistry changes in the star under this approximation, which will greatly simplify the computational
cost of simultaneously solving the structure and chemistry of the star. Moreover, it can be seen that $^{12}$C burns 50% faster than
what one would naïvely obtain using only the two main carbon burning reactions and ignoring the presence of small quantities
of ashes, as first noticed in Piro & Bildsten (2008) and Chamulak et al. (2008).

We can also see that the evolution of the $^{13}$C, $^{16}$O, and $^{20}$Ne
eutrons will be initially faster than that of $^{12}$C, but as significant
amounts of $^{12}$C are burnt their characteristic timescales will
become comparable, i.e., assuming $\lambda_1 \approx \lambda_2$;

\[ \tau^{(12)}(12C) = \frac{2}{3} \left\{ Y^{(12)}(12C) \rho N_A (\lambda_1 + \lambda_2) \right\}^{-1}, \]
\[ \tau^{(13)}(13C) = 2 \frac{Y^{(13)}(13C)}{Y^{(12)}(13C)} \left\{ Y^{(12)}(12C) \rho N_A \lambda_1 \right\}^{-1} \approx 6 \frac{Y^{(13)}(13C)}{Y^{(12)}(13C)} \tau^{(12)}(12C), \]
\[ \tau^{(16)}(16O) = 2 \frac{Y^{(16)}(16O)}{Y^{(12)}(16O)} \left\{ Y^{(12)}(12C) \rho N_A \lambda_2 \right\}^{-1} \approx 6 \frac{Y^{(16)}(16O)}{Y^{(12)}(16O)} \tau^{(12)}(12C), \]
\[ \tau^{(20)}(20Ne) = 2 \frac{Y^{(20)}(20Ne)}{Y^{(12)}(20Ne)} \left\{ Y^{(12)}(12C) \rho N_A \lambda_2 \right\}^{-1} \approx 6 \frac{Y^{(20)}(20Ne)}{Y^{(12)}(20Ne)} \tau^{(12)}(12C). \]

The evolution of $^{23}$Na and $^{23}$Ne will be different and will depend
on the density. Their characteristic timescales will be

\[ \tau^{(23)}(23Na) = \min \left\{ \frac{2 Y^{(23)}(23Na)}{Y^{(12)}(23Na)} \left[ Y^{(12)}(12C) \rho N_A \lambda_1 \right]^{-1}, \right. \]
\[ \left. \frac{Y^{(23)}(23Ne)}{Y^{(23)}(23Na)} \lambda_8^{-1} \right\}, \quad (23) \]
\[ \tau^{(23)}(23Ne) = \min \left\{ \frac{Y^{(23)}(23Ne)}{Y^{(23)}(23Na)} \lambda_7^{-1}, \lambda_8^{-1} \right\}. \quad (24) \]

If the $^{23}$Ne $\beta$-decay timescale were much shorter than the
$^{23}$Na $e^-$-capture timescale ($\lambda_7^{-1} \ll \lambda_8^{-1}$) and the $^{12}$C-burning timescale ($\lambda_8^{-1} \ll \tau^{(12)}(12C)$), which corresponds to the low-density limit and which we call hypothesis H1, $^{23}$Ne nuclei produced by $e^-$-captures from newly synthesized $^{23}$Na would move to an equilibrium value in a timescale $\lambda_8^{-1}$. This value would be obtained neglecting time derivatives in Equation (22):

\[ \bar{Y}^{(23)}(23Ne) = Y^{(23)}(23Na) \lambda_7 \lambda_8^{-1}, \quad (25) \]

and using this value in Equation (21) we obtain

\[ dY^{(23)}(23Na)/dt = \frac{Y^{(12)}(23Na)}{2} \rho N_A \lambda_7. \quad (26) \]

If the $e^-$-capture timescale were much smaller than the $^{23}$Ne $\beta$-decay timescale ($\tau^{(12)}(12C) \ll \lambda_8^{-1}$) and the $^{12}$C-burning timescale ($\lambda_8^{-1} \ll \tau^{(12)}(12C)$), which corresponds to the high-density limit and which we call hypothesis H2, $^{23}$Na would act as trace nuclei and with a timescale $\lambda_8^{-1}$ would reach an equilibrium value, which could be obtained neglecting time derivatives in
Equation (21):
\[
\dot{Y}^{(23)\text{Na}} = \frac{Y^{(12)\text{C}}}{2} \frac{\rho N_A \lambda_1}{\lambda_7} + Y^{(23)\text{Ne}} \frac{\lambda_8}{\lambda_7}. \tag{27}
\]

Assuming this equilibrium value in Equation (22), we obtain
\[
dY^{(23)\text{Ne}}/dt = \frac{Y^{(12)\text{C}}}{2} \rho N_A \lambda_1. \tag{28}
\]

These alternative additional simplifications are useful for understanding the behavior of the simplified network in these extreme cases, but are not valid in the simplification of this section. They are not valid if the \(^{12}\text{C}\)-burning timescale becomes smaller than both the \(^{23}\text{Na}\) \(e^-\)-capture and \(^{23}\text{Ne}\) \(\beta\)-decay timescales, or if the latter timescales are similar to each other. If this is the case, we must solve Equations (21) and (22) exactly under the simplified network.

### 3.1. Dependence on the \(^{12}\text{C}\) Burnt Mole Fraction

We can also derive a set of differential equations that relate the increase in mole fraction of the secondary species with the amount of burnt carbon. First, we divide Equations (18)–(20) by Equation (17) to obtain
\[
\frac{dY^{(13)\text{C}}}{dY^{(12)\text{C}}} = -\frac{1}{3(1 + \lambda_2/\lambda_1)} \approx -0.15, \tag{29}
\]
\[
\frac{dY^{(16)\text{O}}}{dY^{(12)\text{C}}} = -\frac{1}{3(1 + \lambda_1/\lambda_2)} \approx -0.19, \tag{30}
\]
\[
\frac{dY^{(20)\text{Ne}}}{dY^{(12)\text{C}}} = -\frac{1}{3(1 + \lambda_1/\lambda_2)} \approx -0.19. \tag{31}
\]

For \(^{23}\text{Na}\) or \(^{23}\text{Ne}\) the result is not as simple, unless either of the conditions necessary for Equations (26) or (28) are met. In those cases, we would get the following.

1. Under hypothesis H1 (low-density limit),
\[
\frac{dY^{(23)\text{Na}}}{dY^{(12)\text{C}}} \approx -\frac{1}{3(1 + \lambda_2/\lambda_1)} \approx -0.15. \tag{32}
\]

2. Under hypothesis H2 (high-density limit),
\[
\frac{dY^{(23)\text{Ne}}}{dY^{(12)\text{C}}} \approx -\frac{1}{3(1 + \lambda_2/\lambda_1)} \approx -0.15. \tag{33}
\]

### 3.2. Electron Mole Fraction

Similarly, we can write equations for the evolution of the electron mole fraction \(Y_e\). Note that this quantity is closely related to the neutron excess, defined as \(\eta \equiv \sum X_i \eta_i\), with \(\eta_i \equiv (n_i - p_i)/(n_i + p_i)\) and where \(n_i\) and \(p_i\) are the number of neutrons and protons of the respective species. The electron mole fraction and neutron excess are related by the formula \(\eta = 1 - 2Y_e\), which implies \(d\eta/dt = -2dY_e/dt\).

We note that when the \(^{12}\text{C}\)-burning timescale is much smaller than the \(e^-\)-capture timescale \([\tau(12) \ll \lambda_7^{-1}]\) and when \(^{15}\text{Na}\) is in equilibrium, the rate at which \(Y_e\) changes will be the rate at which \(Y^{(13)\text{C}}\) changes due to the reaction \(^{12}\text{C}(12)\text{C}, \rho\)\(^{23}\text{Na}\), i.e.,
\[
\frac{dY_e}{dt} = -\frac{Y^{(12)\text{C}}}{2} \rho N_A \lambda_1 \text{ and } \frac{dY}{dY^{(12)\text{C}}}
\]
\[
= \frac{1}{3(1 + \lambda_2/\lambda_1)} \approx 0.15. \tag{34}
\]

When the \(e^-\)-capture timescale becomes smaller than the \(^{12}\text{C}\)-burning timescale, the \(e^-\)-capture rate will be twice the former result, since \(^{23}\text{Na}\) is produced in the same branch of the network as \(^{13}\text{N}\) (see Figure 4). If both timescales are comparable, the electron mole fraction can be obtained using that \(Y_e = (1 - \eta)/2\) and computing the neutron excess from the exact solution of Equations (17)–(22) and the formula
\[
\eta \approx Y^{(13)\text{C}} + Y^{(23)\text{Na}} + 3Y^{(23)\text{Ne}} + \bar{Y}(n) - \bar{Y}(p) - Y^{(13)\text{N}}. \tag{35}
\]

### 4. LIMITS OF THE SIMPLIFIED NETWORK N1

Now we will examine different ways the assumptions of the simplified network N1 can break down, and with this information build a second more refined simplified network, N2, which includes some of the leak reactions that will be discussed in what follows.

#### 4.1. Proton Leaks

When the \(^{23}\text{Ne}\) \(\beta\)-decay timescale is shorter than the \(^{23}\text{Na}\) \(e^-\)-capture timescale, normally below the threshold density \(\rho_h \approx 1.7\ \text{g cm}^{-3}\), some of the protons that would be captured in the \(^{12}\text{C}(p, \gamma)^{13}\text{N} \text{reaction can leak via the reactions}\(^{23}\text{Na}(p, \alpha)^{20}\text{Ne}\) and \(^{23}\text{Na}(p, \gamma)^{24}\text{Mg}\), which are defined as reactions (9) and (10), as the abundance of \(^{23}\text{Na}\) increases. The ratio of their thermally averaged cross sections is shown in Figure 5.

Conversely, when the \(^{23}\text{Na}\) \(e^-\)-capture timescale is shorter than the \(^{23}\text{Ne}\) \(e^-\)-decay timescale, normally above the threshold density \(\rho_h \approx 1.7\ \text{g cm}^{-3}\), protons can leak via the reaction \(^{23}\text{Ne}(p, n)^{23}\text{Na}\), which is defined as reaction (11), as the abundance of \(^{23}\text{Ne}\) increases. These leak reactions can significantly change the distribution of ashes and energy input as \(^{12}\text{C}\) is burnt, as well as the trace nuclei equilibrium mole fractions, as will be shown later. The abundances of \(^{23}\text{Na}\) and \(^{23}\text{Ne}\) at which the former reaction rates become equal to the \(^{12}\text{C}(p, \gamma)^{13}\text{N} \text{reaction are shown in Figure 6.}

Let us define the cross section for proton captures on either nuclei of the pair \(^{23}\text{Na}–^{23}\text{Ne}\), which we call Urca matter, as \(\lambda_U\), and the cross section for proton captures on \(^{12}\text{C}\) as \(\lambda_C\). The ratio between the rates in both reactions will be
\[
r = \frac{Y^{(23)\text{Na}} \text{ or } (23)\text{Ne}}{Y^{(12)\text{C}}} \lambda_U \lambda_C. \tag{36}
\]

We can use the following approximate relation inferred from the simplified version of the network:
\[
\frac{dY^{(23)\text{Na}} \text{ or } (23)\text{Ne}}{dY^{(12)\text{C}}} \approx -0.15 \tag{37}
\]
and defining \(f\) as the burnt fraction of \(^{12}\text{C}\), we can write
\[
r \approx -0.15\Delta Y^{(12)\text{C}} \lambda_U \lambda_C = 0.15f \lambda_U \lambda_C. \tag{38}
\]
i.e., \( r = 1, f \approx 6.7 \lambda_{12C}/\lambda_{13N} \), or \( \approx 6.7 \times \) the ratio shown in Figure 6. Thus, when 9\% or 4\% of the original 12C is burnt, depending on whether protons leak on 23Na or 23Ne, proton leaks will become significant. From Figure 6, it can be noted that proton leaks will be stronger at temperatures of about 3 \( \times 10^9 \) K. Above temperatures of 5 \( \times 10^9 \) K, the fraction of burnt 12C for proton leaks to be important will change to approximately 27\% and 14\%, when either the 23Ne \( \beta \)-decay timescale or the 23Na electron-capture timescale is shorter, respectively.

Although proton leaks start later when the 23Ne \( \beta \)-decay timescale is shorter, their effect in this case is more difficult to model. Since less 13N will be present due to the bypassing of one of the branches of the network, the amount of \( e^- \)-captures is reduced and less \( \alpha \)-capturing 12C synthesized, opening secondary channels for \( \alpha \)-captures. With more proton leaks, 12C and 16O can become the main targets for \( \alpha \)-captures, depending on the temperature as will be discussed later.

When the 23Na \( e^- \)-capture timescale is shorter, proton leaks will start sooner, which does not significantly change the main results of the simplified network because the end products will be the same after two \( e^- \)-captures and the burning of six 12C nuclei:

\[
6^{12}C + 2e^- \rightarrow 23Na + 20Ne + 16O + 13C. \quad (39)
\]

The main difference will be that now both \( e^- \)-captures will be on 23Na rather than on 13N and 23Na, changing the typical timescale of the network in this leak branch.

The ratios between the relevant thermally averaged cross sections for proton captures at different densities are shown in Figure 7. We summarize the proton-leak reactions in Figures 8 and 9.

### 4.2. Neutron Leaks

A similar analysis can be done to compute the critical mole fraction of secondary nuclei for neutron leaks to be important. In this case, the most important neutron capture reactions are 12C\((n, \gamma)13C\), 20Ne\((n, \gamma)21Ne\), 21Ne\((n, \gamma)22Ne\), and

| Figure 6. Mole fraction ratio between Urca matter and 12C when proton captures onto Urca matter become equal to proton captures onto 13C, plotted against the temperature in units of 10^9 K. Screening corrections are taken into account, using a density of 1 \( \times 10^9 \) g cm\(^{-3}\) for 23Na and 2 \( \times 10^9 \) g cm\(^{-3}\) for 23Ne, below and above the threshold for electron captures. |
| Figure 7. Ratio between thermally averaged cross sections relevant for proton captures at different densities (g cm\(^{-3}\)). Solid lines correspond to \((\lambda_{12C} + \lambda_{13N})/\lambda_{12C}\) and dashed lines, to \(\lambda_{13N}/\lambda_{12C}\) (see Section 4.1). The threshold density for \(e^-\) captures onto 23Na is close to 1.7 \( \times 10^9 \) g cm\(^{-3}\). The density dependence is due to screening corrections. |
| Figure 8. Same as Figure 3, but with proton leaks (dotted arrows). See discussion in the text for neutron and \(\alpha\)-leaks. |

23Na\((n, \gamma)24Na\). The 20Ne and 21Ne neutron capture reactions have very similar cross sections, approximately seven times the 12C neutron capture reaction. The 23Na reaction has a cross section approximately 11 times bigger than the cross section of the 12C neutron capture reaction.

The ratio of the mole fractions of 20Ne and 23Na with respect to 12C can be related to the mole fraction change of 12C using Equations (31) and (32), at densities when the 23Ne \(\beta\)-decay timescale is shorter than the 23Na \(e^-\)-capture timescale:

\[
Y(20Ne) + Y(23Na) \approx -\Delta Y(12C) \left\{ \frac{dY(20Ne)}{dY(12C)} + \frac{dY(23Na)}{dY(12C)} \right\}
\approx -0.34\Delta Y(12C), \quad (40)
\]

or when the 23Na \(e^-\)-capture is shorter than the 23Ne \(\beta\)-decay timescale:

\[
Y(20Ne) + Y(23Na) \approx -0.19\Delta Y(12C). \quad (41)
\]
In the first case, using Equation (40) and using the added cross sections of $^{20}\text{Ne}$, $^{21}\text{Ne}$, and $^{23}\text{Na}$ neutron captures, it can be shown that neutron leaks, i.e., significant neutron captures on species different than $^{12}\text{C}$, occur when the fraction of burnt carbon is approximately 12% of its original amount. In the second case, using Equation (41), it can be shown that neutron leaks will occur when about 38% of the original carbon is burnt.

As first noticed by Chamulak et al. (2007), neutron captures onto $^{56}\text{Fe}$ will be negligible, since the $^{56}\text{Fe}(p, \gamma)^{57}\text{Fe}$ reaction has a cross section approximately 64 times bigger than that of neutron capture onto $^{12}\text{C}$, but the mole fraction of $^{56}\text{Fe}$ is approximately 1250 times smaller than that of $^{12}\text{C}$ at solar metallicity in the $^{12}\text{C}$-rich environment of a WD. Thus, a $^{56}\text{Fe}$ abundance of more than 20 times the solar abundance would be required to compete with the reaction $^{12}\text{C}(n, \gamma)^{13}\text{C}$.

### 4.3. $\alpha$-particle Leaks

Our detailed nuclear network shows that the most important reactions for $\alpha$-captures are $^{13}\text{C}(\alpha, n)^{16}\text{O}$, followed by the much weaker reactions $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$ and $^{16}\text{O}(\alpha, \gamma)^{20}\text{Ne}$. In Figure 10, we show the ratios of the mole fractions of $^{13}\text{C}$ and $^{16}\text{O}$ relative to the mole fraction of $^{13}\text{C}$ necessary for $^{13}\text{C}(\alpha, n)^{16}\text{O}$ to be the dominant $\alpha$-capture reaction. It can be seen that only a minimal amount of $^{13}\text{C}$ is needed for this to be the case: $Y(^{13}\text{C}) \approx 3 \times 10^{-8}$, which is $\approx 10^5$ times lower than the solar metallicity value.

Moreover, using arguments similar to those used for proton and neutron leaks, a fraction of only $4 \times 10^{-8}$ of the original $^{12}\text{C}$ needs to be burnt to reproduce this abundance with zero metallicity, assuming that for every six $^{12}\text{C}$ nuclei one $^{13}\text{C}$ nucleus is produced. Thus, under the temperature range investigated here, one would conclude that these secondary reactions are negligible.

However, if significant neutron leaks occur, $^{13}\text{C}$ can be significantly depleted and these reactions can become important. In fact, after $^{13}\text{C}$ is depleted as a consequence of neutron leaks and the $\alpha$-particle overproduction accompanying the $^{23}\text{Na}(p, \alpha)^{20}\text{Ne}$ proton-leak reaction, $\alpha$-leaks are necessary to reproduce the abundances of $^{16}\text{O}$, $^{20}\text{Ne}$ and to limit the growth of $\alpha$-particles and reproduce the $^{13}\text{C}$ depletion more accurately.

### 5. THE SIMPLIFIED NETWORK: SECOND APPROXIMATION (N2)

We can now include the leak reactions discussed in the previous section and additional ones with the following indices:

- $^{23}\text{Na}(p, \alpha)^{20}\text{Ne}$, $Q = 2.38$ MeV.
- $^{23}\text{Na}(p, \gamma)^{24}\text{Mg}$, $Q = 11.69$ MeV.
- $^{23}\text{Ne}(p, n)^{23}\text{Na}$, $Q = 3.59$ MeV.
- $^{20}\text{Ne}(n, \gamma)^{21}\text{Ne}$, $Q = 6.76$ MeV.
- $^{23}\text{Na}(n, \gamma)^{24}\text{Na}$, $Q = 6.96$ MeV.
- $^{12}\text{C}(\alpha, \gamma)^{16}\text{O}$, $Q = 7.16$ MeV.
- $^{10}\text{O}(\alpha, \gamma)^{20}\text{Ne}$, $Q = 2.84$ MeV.
- $^{21}\text{Ne}(n, \gamma)^{22}\text{Ne}$, $Q = 10.36$ MeV.
- $^{16}\text{O}(\alpha, \gamma)^{24}\text{Mg}$, $Q = 7.55$ MeV.
- $^{21}\text{Ne}(p, \gamma)^{22}\text{Na}$, $Q = 6.74$ MeV.
- $^{20}\text{Ne}(\alpha, \gamma)^{24}\text{Mg}$, $Q = 9.32$ MeV.
- $^{23}\text{Ne}(n, \gamma)^{26}\text{Mg}$, $Q = 5.41$ MeV.
- $^{23}\text{Na}(n, p)^{23}\text{Mg}$, $Q = 1.82$ MeV.
- $^{21}\text{Ne}(n, \alpha)^{26}\text{Mg}$, $Q = 4.84$ MeV.

This list includes proton-leak reactions (9), (10), (11), (17), and (18), neutron-leak reactions (12), (13), and (16), and $\alpha$-leak reactions (14), (15), (19), (20), (21), and (22). Proton leaks are the first to be significant as $^{12}\text{C}$ ashes are produced, but with more ashes produced neutron leaks become relevant too. When this happens, $^{13}\text{C}$ is rapidly depleted and $\alpha$-leaks become important as well.

As can be seen from the reactions in the list, it would be necessary to increase the number of species to account for all proton, neutron and $\alpha$-leaks exactly. To do this, we can either account for $^{24}\text{Mg}$, $^{21}\text{Ne}$, $^{24}\text{Na}$, $^{22}\text{Ne}$, $^{14}\text{N}$, $^{22}\text{Na}$, and $^{26}\text{Mg}$ independently, or we can group some or all of them into an auxiliary variable. Since reactions (16) and (18) depend on the abundance of $^{21}\text{Ne}$ nuclei, we solve for $^{21}\text{Ne}$ independently and group $^{24}\text{Mg}$, $^{24}\text{Na}$, $^{22}\text{Ne}$, $^{14}\text{N}$, $^{22}\text{Na}$, and $^{26}\text{Mg}$ into a dummy leak species.
Thus, while keeping the number of independent variables small and ensuring mass conservation, we can introduce \( Y_L \) as the mole fraction of a dummy nuclei that represents leaks on \(^{24}\text{Mg}, \, ^{24}\text{Na}, \, ^{23}\text{Ne}, \, ^{14}\text{N}, \, ^{22}\text{Na}, \) and \(^{20}\text{Ne} \). We assume that this dummy species has approximately the average mass number of the species it represents, defining it as

\[
Y_L = \max\left\{0, \frac{1 - \sum A_i Y_i}{22}\right\},
\]

where \( A_i \) is the mass number of the respective species and the sum is made over 12C, 13C, 13N, 16O, 20Ne, 21Ne, 23Ne, and 23Na. Since the leak nuclei \(^{24}\text{Mg}, \, ^{24}\text{Na}, \, ^{22}\text{Ne}, \, ^{14}\text{N}, \) and \(^{22}\text{Na} \) can subsequently capture neutrons with cross sections similar to that of \(^{20}\text{Ne} \), for simplicity we will assume that leak nuclei capture neutrons with the \(^{20}\text{Ne} \) cross section.

If we now add new terms associated with the \( p, n, \) and \( \alpha \) leak reactions discussed above into Equations (3)–(12) and assume a stationary solution for the trace nuclei \( p, \alpha, n, \) and \(^{13}\text{N} \), the modified equilibrium mole fractions can be described with the following equations (cf. Equations (13)):

\[
\dot{\tilde{Y}}(p) = \tilde{Y}(p) f_p, \quad \dot{\tilde{Y}}(\alpha) = \tilde{Y}(\alpha) f_\alpha, \\
\dot{\tilde{Y}}(n) = \tilde{Y}(n) f_n, \quad \dot{\tilde{Y}}(^{13}\text{N}) = \tilde{Y}(^{13}\text{N}) f_p,
\]

where the auxiliary variable \( f_p, f_\alpha, \) and \( f_n \) are defined as follows:

\[
f_p \equiv \frac{1 + f_{\text{inv}} Y(\text{23Na}) (\lambda_9 + \lambda_{10}) + Y(\text{23Ne}) (\lambda_{11} + \lambda_{12}) + Y(\text{21Ne}) (\lambda_{13}) + Y(\text{20Ne}) (\lambda_{14})}{Y(\text{12C}) (\lambda_2)},
\]

\[
f_\alpha \equiv K^{-1} \left\{1 + 2 \frac{Y(\text{23Na}) \tilde{Y}(p) \lambda_9}{Y(\text{21C}) \lambda_2}\right\},
\]

\[
f_n \equiv \left\{1 + \frac{Y(\text{23Ne}) \left[\tilde{Y}(p) \lambda_{11} + \tilde{Y}(\alpha) \lambda_{10}\right] + Y(\text{21Ne}) Y(\alpha) \lambda_{22}}{Y(\text{13C}) \tilde{Y}(\alpha) \lambda_5}\right\}^{-1} \times \left\{1 + \frac{\left[Y(\text{20Ne}) + Y(\alpha) \lambda_{12} + Y(\text{21Ne}) \lambda_{16} + Y(\text{23Na}) \lambda_{13}\right]}{Y(\text{12C}) \lambda_6}\right\}^{-1},
\]

where we have neglected the contribution to \( f_p \) from the leak reaction \(^{23}\text{Na}(\alpha, p)^{20}\text{Mg} \), which is generally small. An exact solution that takes into account the additional protons from this or similar reactions can be obtained by multiplying the trace abundances \( \tilde{Y}(p) \) and \( \tilde{Y}(^{13}\text{N}) \) by the factor \( 1 + A \tilde{Y}(p)/(1 - A \tilde{Y}(p)) \), where \( A \equiv 2Y(\text{23Na}) \lambda_{21}/Y(\text{21C}) \lambda_{11} \) and \( B \equiv 2K^{-1} Y(\text{23Ne}) \lambda_{10}/Y(\text{22C}) \lambda_{2}^{-1} \). The factors \( f_\alpha \) and \( f_n \) should be computed by using these modified values.

These formulae can be used to approach the leak regime of the network and can easily be generalized to include more leak reactions involving protons, neutrons or \( \alpha \)-particles. They predict a decrease in the number of free protons and \(^{13}\text{N} \) nuclei at equilibrium and the fact that this decrease starts later when the \(^{23}\text{Ne} \) \( \beta \)-decay timescale is shorter than the \(^{23}\text{Na} \) \( e^- \)-capture timescale, at low densities. Also, they explain the contribution to \( \alpha \)-particles from the proton-leak reaction \(^{23}\text{Ne}(p, \alpha)^{20}\text{Ne} \), the \( \alpha \)-leaks from 12C and 16O captures, the contribution to neutrons from the proton-leak reaction \(^{23}\text{Ne}(p, \alpha)^{20}\text{Ne} \) and the neutron depletion due to \(^{20}\text{Ne}, \, ^{21}\text{Ne}, \) and \(^{23}\text{Na} \) neutron captures under our approximation, as well as a simple estimate of the neutron captures on \(^{24}\text{Mg}, \, ^{24}\text{Na}, \, ^{22}\text{Ne}, \, ^{14}\text{N}, \) and \(^{22}\text{Na} \).

Now, we can rewrite the differential equations describing the evolution of the slowly varying nuclei 12C, 13C, 16O, \(^{20}\text{Ne}, \, ^{23}\text{Na}, \) and \(^{23}\text{Ne} \), using the new trace nuclei equilibrium values and including the leak reactions discussed above, i.e.,

\[
\frac{dY(\text{12C})}{dt} = -Y(\text{12C}) \rho N_A \left\{\tilde{Y}(\text{12C}) (\lambda_1 + \lambda_2) + \tilde{Y}(p) \lambda_3 f_{\text{inv}} + \tilde{Y}(n) \lambda_6 + \tilde{Y}(\alpha) \lambda_{14}\right\},
\]

\[
\frac{dY(\text{13C})}{dt} = -Y(\text{13C}) \rho N_A \left\{Y(\text{12C}) \tilde{Y}(n) \lambda_6 - Y(\text{13C}) \left[\tilde{Y}(\alpha) \lambda_5 + \tilde{Y}(p) \lambda_{17}\right]\right\},
\]

\[
\frac{dY(\text{16O})}{dt} = \rho N_A \left\{Y(\text{13C}) \tilde{Y}(\alpha) \lambda_5 + Y(\text{12C}) \lambda_{14} - Y(\text{16O}) \lambda_{15}\right\},
\]

\[
\frac{dY(\text{20Ne})}{dt} = \rho N_A \left\{\frac{Y^2(\text{12C})}{2} \lambda_2 + Y(\text{23Na}) \tilde{Y}(p) \lambda_9 - Y(\text{20Ne}) \left[\tilde{Y}(n) \lambda_{12} + \tilde{Y}(\alpha) \lambda_{19}\right] + Y(\text{16O}) \tilde{Y}(\alpha) \lambda_{15}\right\},
\]

\[
\frac{dY(\text{23Na})}{dt} = -Y(\text{23Na}) \lambda_7 + Y(\text{23Ne}) \lambda_8 + \rho N_A \left\{\frac{Y^2(\text{12C})}{2} \lambda_1 + Y(\text{23Ne}) \tilde{Y}(p) \lambda_{11} - Y(\text{23Na}) \left[\tilde{Y}(p) \lambda_9 + \lambda_{10}\right] + \tilde{Y}(n) \lambda_{13} + \tilde{Y}(\alpha) \lambda_{21}\right\},
\]

\[
\frac{dY(\text{23Ne})}{dt} = Y(\text{23Na}) \lambda_7 - Y(\text{23Ne}) \lambda_8 - Y(\text{23Ne}) \rho N_A \left[\tilde{Y}(p) \lambda_{11} + \tilde{Y}(\alpha) \lambda_{20}\right],
\]

\[
\frac{dY(\text{21Ne})}{dt} = \rho N_A \left\{Y(\text{20Ne}) \tilde{Y}(n) \lambda_{12} - Y(\text{21Ne}) \tilde{Y}(p) \lambda_{18} + \tilde{Y}(n) \lambda_{16} + \tilde{Y}(\alpha) \lambda_{22}\right\}.
\]

The energy generation rate can be obtained after a straightforward modification of the individual terms above, noting that the combined energy contribution of the inverse reaction \(^{13}\text{N}(\gamma, p)^{12}\text{C} \) and the reaction \(^{13}\text{C}(p, \gamma)^{14}\text{N} \) can be obtained simply by multiplying the \( Q \)-value of reaction (3) to the associated term in the right-hand side of Equation (48).
6. COMPARISON WITH THE DETAILED NETWORK

In what follows we will compare the results of the full nuclear network introduced in Section 2 with those of the simplified nuclear networks N1 and N2 introduced in Sections 3 and 5. We will compare the evolution of the main and trace nuclei mole fractions, the time evolution, and the energy release in the form of photons or neutrinos as a function of the fraction of burnt $^{12}$C nuclei.

In most examples, we have chosen a temperature of $4 \times 10^8$ K and a density of $3 \times 10^9$ g cm$^{-3}$, which are typical of the conditions encountered during the thermonuclear runaway and before ignition in a presupernova CO WD. We chose a temperature and density at which the simplified solution errors would be representative of those encountered at other temperatures and densities after the trace nuclei have reached their equilibrium values. The timescales for the trace nuclei to reach their equilibrium values can be computed from Equation (14).

In Figures 11 and 12, we show the ratio between the $^{12}$C ashes and $^{12}$C mole fraction changes versus the fraction of burnt $^{12}$C in the full and N1 networks. Main ashes and the neutron excess change are shown as continuous (full network) or dot-dashed (N1) lines and trace nuclei are shown as dashed (full network) or dotted (N1) lines. See discussion in the text.

(A color version of this figure is available in the online journal.)

It can be seen that once the slowest evolving trace nuclei reach their equilibrium value and before about 1% of the $^{12}$C has been burnt, both plots show a good agreement between the simplified and full networks for the main $^{12}$C ashes. The exceptions are $^{3}$He-particles, $^{16}$O, and $^{20}$Ne under N1 due to the initial $^{3}$He-leaks from reaction (15) and because we have assumed a pure CO mixture, with $^{13}$C initially absent. Since in this example the $^{23}$Na $\beta$-capture timescale is shorter than the $^{23}$Ne $\beta$-decay timescale, both simplified solutions correctly predict a higher abundance for $^{23}$Ne once both timescales become comparable. Both solutions show a good match for the neutron excess changes too.

However, once the fraction of burnt $^{12}$C is above $\sim$1%, the trace nuclei equilibrium abundances of N1 begin to differ from the exact solution, whereas N2 matches their values even when more than half of the original $^{12}$C has been burnt. The inclusion of the leak reactions in N2 provides a better match for the trace nuclei and, as a consequence, the main $^{12}$C ashes as well, once the trace nuclei equilibrium mole fractions have been reached. Note also that the $^{3}$He-particle, $^{16}$O, and $^{20}$Ne mole fraction changes are well matched in N2 once trace nuclei equilibrium is reached.

In N2, the trace nuclei follow precisely their equilibrium values as $^{12}$C burns, even if these equilibrium values change dramatically. This is because their characteristic timescales are much smaller than the characteristic $^{12}$C-burning
The approximation is lost to about 20% level. Although at the evolution of the species, $^{13}$C, $^{16}$O, and $^{20}$Ne should be exactly flat in this plot according to Equations (29), (30), and (31). Generally speaking, N2 reproduces the full network abundances to the 5% level when the trace nuclei have reached their equilibrium mole fractions. In numerical experiments, N2 loses its accuracy when both $f_p$ and $f_a$ (see Equations (44) and (47)) are lower than 0.5, which is due to the observed $^{13}$C depletion.

Note that in N1 the evolution of the species $^{13}$C, $^{16}$O, and $^{20}$Ne would look flat and approximately 0.19 in this space. The same would be the case for $^{13}$C, although approximately 0.15 (see Equations (29)–(31)). This is only approximately true when the fraction of burnt $^{12}$C is between 2% and 5%. Since we plot absolute values of mole fraction changes, Figure 13 does not show whether the $^{16}$O mole fraction is decreased or increased. In fact, its mole fraction is originally depleted and, only after the fraction of burnt $^{12}$C is about $10^{-4}$, it is increased. The original depletion is caused by $\alpha$-leaks in reaction (15), which increase the mole fraction of $^{20}$Ne.

The $^{21}$Ne mole fraction is larger than the $^{23}$Ne mole fraction only after the burn fraction of $^{12}$C is about $2 \times 10^{-5}$. Interestingly, this makes the neutron excess evolution change from above 0.3 to 0.6 in this transformation. This can be understood noticing that $d\eta/dt = -2dY_e/dt$ and that the value shown in Equation (34) is expected to double at high density.

Note that $^{13}$C is depleted as $^{21}$Ne and leak nuclei increase their abundance. This is because neutron leaks shortcut $^{13}$C neutron captures, as discussed before. When this happens, $f_a$ decreases dramatically (see Equation (46)), and the accuracy of the approximation is lost to about 20% level. Although at the beginning of the integration $f_a$ is also small, N2 matches the full network with great accuracy. To distinguish between the cases when the accuracy of N2 is very good and only a small fraction of $^{12}$C has been burnt from the late loss of accuracy with a big fraction of $^{12}$C burnt, we define the criterion for N2 to be considered an accurate representation of the full network as

$$f_p > 0.5 \text{ and } f_a > 0.5,$$

which should be used in a varying temperature and density integration, such as in real stellar evolution models.

In Figure 14, we show a similar integration as in Figure 13, but at a density of $10^9 \text{ g cm}^{-3}$. We can see that N2 loses its accuracy at a lower fraction of burnt $^{12}$C, which is due to stronger neutron leaks which cause $^{13}$C to be relatively depleted sooner.

(A color version of this figure is available in the online journal.)
6.1. Timing and Energy Generation Comparison

In Figure 15, we show the ratio between the elapsed times of the simplified and full integrations versus the fraction of burnt $^{12}$C. The N1 and N2 subscripts correspond to the solution of the first and second simplified networks, respectively. We can see that the N1 can overestimate the speed at which $^{12}$C burns by more than 20%, whereas the second simplified network reproduces the time evolution to better than 5% error. This is due to the absence of leak reactions in the first approximation, which overestimates the amount of $^{12}$C proton and neutron captures at a given time.

In Figure 16, we show the ratios between the photon and neutrino energy release rates in N1 and N2 and the photon and neutrino energy release rate in the full network. We can see that the photon energy release rates is typically off in N1 by 20% or more, whereas N2 matches them at the 5% level, except after $^{13}$C is depleted. This is because the leak reactions tend to release more energy than the $^{12}$C proton and neutron captures. The neutrino energy release rates are not as accurate because secondary electron captures not included in these networks can have an important contribution, but since photon energy rates are generally much bigger, the error in the net energy release remains at the 5% level.

7. DISCUSSION AND CONCLUSIONS

We have derived two approximate nuclear networks that can be used in the hydrostatic carbon burning regime of CO WDs approaching ignition. The networks have the advantage of being able to accurately track the relevant species during this phase of evolution without having to include the fast-evolving protons, neutrons, $\alpha$-particles, or $^{13}$N nuclei.

Using the same integration method (Bader & Deuflhard 1983), convergence tolerance and initial conditions in one-zone models, we have found that the integration of N1 or N2 subscripts correspond to the solution of the first and second simplified networks, respectively. We can see that the N1 can overestimate the speed at which $^{12}$C burns by more than 20%, whereas the second simplified network reproduces the time evolution to better than 5% error. This is due to the absence of leak reactions in the first approximation, which overestimates the amount of $^{12}$C proton and neutron captures at a given time.

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These accuracies are not valid for nuclei whose mole fractions are not significant, for example, $^{23}$Ne when the $\beta$-decay timescale is much shorter than the $e^-$-capture timescale, since in this case the $^{22}$Ne($n$, γ)$^{23}$Ne reaction will dominate its evolution, a reaction that cannot be included accurately under N2. We have also found that for relatively low densities, below $10^8$ g cm$^{-3}$, the quoted errors can double, but only when significant amounts of $^{12}$C have been burnt, which normally occurs when the density is significantly above this number. Thus, N2 is more accurate for central ignition models, but it could also be used in off-center ignition models.

We have shown how to derive the approximations and have compared them to a detailed network at fixed temperature and density. We have also discussed how the approximations can break down and when they can be used. Since all the details of the derivation are shown, these simplified networks can be further improved straightforwardly. It is also possible to build intermediate approximations between N1 and N2, for example, including $^{21}$Ne in the dummy leak species defined in Equation (42) to remove one independent variable from the solution. We have tested this last approximation in a few cases and the resulting errors appears to be twice the errors in N2.
Although for simplicity we have only shown fixed temperature and density integrations, these approximations can be used in environments with varying temperature and density, such as real stellar evolution models. This is because in the temperature and density range found in pre-ignition WD interiors the trace nuclei reach their equilibrium values much faster than the typical environmental variables vary inside the star, even within strong pre-ignition convective velocity fields.

The networks can account for $^{23}\text{Na}$ or $^{13}\text{N}$ e$^-$-captures for the purposes of following the evolution of the pressure-supporting electrons in presupernova WDs. We have shown that they will be valid even in convective WD interiors. We recommend the use of these networks when the mass fractions of the most abundant species are relevant, namely, $^{12}\text{C}$, $^{16}\text{O}$, $^{13}\text{C}$, $^{20}\text{Ne}$, $^{22}\text{Ne}$, or $^{23}\text{Na}$, or the electron mole fraction, $Y_e$, or the energy generation rates. We do not recommend its use if the abundances of other nuclei not included in this discussion are being studied.

We have introduced an important tool to understand the effect of the convective Urea process on the ignition conditions of SNe Ia. We foresee the application of these simplified networks or their modification in detailed one or multi-dimensional stellar evolution models trying to understand presupernova CO WDs or similar objects (see, e.g., Lesaffre et al. 2006; Zingale et al. 2009; Iapichino & Lesaffre 2010).

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