Three-Dimensional Gravity
and String Ghosts

S. Carlip
Department of Physics
University of California
Davis, CA 95616
USA

and

I. I. Kogan\footnote{On leave from ITEP, Moscow, USSR}
Department of Physics
University of British Columbia
Vancouver V6T 2A6
Canada

Abstract

It is known that much of the structure of string theory can be derived from three-dimensional topological field theory and gravity. We show here that, at least for simple topologies, the string diffeomorphism ghosts can also be explained in terms of three-dimensional physics.
1. Introduction

It has been realized for several years that much of string theory can be explained in terms of three-dimensional topological field theory. The key to such a three-dimensional approach was Witten’s observation [1] that the Hilbert space of a Chern-Simons theory can be identified with the space of conformal blocks of an associated two-dimensional conformal field theory. This means that the Chern-Simons path integral on a manifold with boundary \( \Sigma \) has a natural interpretation as the left-moving piece of a conformal field theory amplitude on \( \Sigma \), or, equivalently, as an important part of the integrand of a string theory amplitude.

To obtain a complete amplitude for a conformal field theory, it is necessary to combine left- and right-movers, of course. But this is not hard. As a topological field theory, Chern-Simons theory has a trivial propagator, so the transition amplitude on a three-manifold with the topology \([0, 1] \times \Sigma\) is simply the overlap between the wave functions on the two boundaries. For appropriately chosen wave functions, this overlap integral gives a modular invariant combination of holomorphic and antiholomorphic conformal blocks, that is, a full conformal field theory amplitude. The physical picture is that of a string “thickened” to an annulus; since Chern-Simons theory is topological, it is only the string-like edge excitations of the resulting membrane that contribute to the amplitudes. It is worth noting that additional three-dimensional degrees of freedom can be readily introduced into this picture, since Chern-Simons theory can be viewed as the infinite mass limit of a dynamical topologically massive gauge theory [2, 3, 4].

To understand this construction in more detail, consider a Chern-Simons theory with semisimple gauge group \( G \) and coupling constant \( k \) on a manifold \([0, 1] \times T^2\). The Chern-Simons Hilbert space is spanned by functions \( \Psi_\lambda[A] \) proportional to the level \( k \) Weyl-Kač characters [5, 6], which transform under a unitary representation of the modular group. Physical wave functions, on the other hand, should presumably be exactly diffeomorphism-invariant. But since the transformations of the \( \Psi_\lambda \) are unitary, it is easy to form invariant combinations; one can simply define

\[
\psi^B[A] = \sum_\lambda \Psi_\lambda[B] \Psi_\lambda[A],
\]

(1.1)

where \( B \) is any background field. The inner product derived from Chern-Simons theory then gives [3, 4]

\[
\langle \psi^{B'} | \psi^{B} \rangle = \sum_{\lambda, \lambda'} h^{\lambda \lambda'} \Psi_\lambda[B] \Psi_{\lambda'}[B'],
\]

(1.2)

where \( h \) is the diagonal modular invariant. But (1.2) is simply the partition function for a level \( k \) WZW model with gauge group \( G \) coupled to a background gauge field \( B \) [7]. It is straightforward to generalize this construction to higher genus surfaces, and to include vertex operators, which arise when one incorporates Wilson lines running between the inner and outer boundaries of the three-manifold [1, 2, 3]. A similar approach, introduced in [8], applies to coset models as well, and it seems likely that amplitudes for nondiagonal conformal field theories can be derived from modular invariant wave functions more complicated than (1.1).

There is more to string theory than conformal field theory, of course. It was noted in 1989 that the string theory integration over moduli space has a natural interpretation in terms of three-dimensional gravity [4], and it has recently been shown that the Liouville term can be
derived from the local Lorentz anomaly in Chern-Simons theory \[9\]. Notably missing from
the description so far, however, have been the string theory diffeomorphism ghosts. The
aim of this paper is to present evidence that these ghosts can be understood in terms of an
overlap integral between initial and final gravitational wave functions, in close analogy to the
conformal field theory partition function (1.2).

2. Gravity and Ghosts

Before proceeding further, we should address a possible misconception about ghosts and
three-dimensional gravity. The gravitational action is diffeomorphism-invariant, and one
might hope that string theory ghosts would arise from gauge-fixing the three-dimensional
diffeomorphism group. This is not the case. In canonical form, the gravitational action is \[10\]
\[
S = \int dt \int \Sigma \left( \pi^{ab} \dot{g}_{ab} - N^a \mathcal{H}_a - N \mathcal{H} \right),
\]
where \(g_{ab}\) is the spatial metric, \(\pi^{ab}\) is its canonical conjugate, and
\[
\mathcal{H} = \frac{1}{\sqrt{g}} \left( \pi^{ab} \pi_{ab} - \pi^2 \right) - \sqrt{g} \mathcal{H}^{(2)} R,
\]
\[
\mathcal{H}_a = -2\pi^{ab} \mid_b,
\]
are the Hamiltonian and the two-dimensional diffeomorphism constraints. It is convenient to
choose coordinates in which \(\pi\) depends only on time. Then just as in string theory, gauge-fixing
the two-dimensional diffeomorphisms generates a Faddeev-Popov determinant \(\det P^{\dagger} P\)^{1/2},
where \(P\) is the first-order differential operator that takes vectors to symmetric traceless tensors.
But \(\mathcal{H}_a = (P^{\dagger} \pi)^a\), so the path integral over \(N^a\) generates a delta functional
\(\delta[P^{\dagger} \pi]\) that absorbs this determinant. Similarly, the determinant coming from gauge-fixing the diffeomorphisms
orthogonal to \(\Sigma\) is absorbed by the delta functional generated by the \(N\) integration. This
same phenomenon can be seen in the Chern-Simons formulation of (2+1)-dimensional gravity
\[11\], where the path integral gives a ratio of determinants — the analytic torsion — which
cancel for topologies of the form \([0,1] \times \Sigma\). If three-dimensional gravity is to generate the string
ghost system, the mechanism must be more subtle.

Let us now turn to the gravitational analog of the transition amplitude (1.2). The action
for (2+1)-dimensional gravity on a manifold \([0,1] \times \Sigma\) can be viewed as a Chern-Simons ac-
tion for the gauge group ISO(2,1) \[12\]. In the natural (cotangent bundle) polarization, the
gravitational Hilbert space then consists of functions on a set of SO(2,1) holonomies that char-
acterize the global spacetime geometry \[13\]. These states correspond to Heisenberg picture
wave functions in ordinary quantum mechanics. For our “thickened string” picture, however,
we need Schrödinger picture wave functions, which contain explicit information about the
two-dimensional boundaries on which the string-like excitations live. The problem is essen-
tially the same as that of choosing a time-slicing in ordinary gravity, where it is well known
that wave functions determine the time at which they are defined \[14\].

To understand the choice of time-slicing, it is useful to return to the canonical metric
formulation of (2+1)-dimensional gravity, where the geometric significance of the phase space
variables is clear. The phase space of metrics and momenta can be conveniently divided into
two pieces [13]. The first consists of conformal structures on $\Sigma$ and their conjugate momenta, and has an obvious relevance to string theory. The second consists of the conformal factor and its conjugate, the mean extrinsic curvature $K$ of $\Sigma$. It is natural to fix a time-slicing in terms of this latter set of variables; in particular, Wheeler has argued [16] that York’s choice of $K$ as the time variable [17] is especially attractive from the point of view of the initial value problem and the variational principle. Let us make this choice, and examine the Chern-Simons wave functions for (2+1)-dimensional gravity on a surface of constant mean extrinsic curvature $K$.

The determination of such wave functions is a difficult problem, and for most topologies the solution is not known. If $\Sigma$ has the topology of a torus, however, the complete answer is given in reference [18]. Wave functions depend on the mean curvature $K$ and on two commuting SO(2,1) holonomies $\exp\{\mu J_2\}$ and $\exp\{\lambda J_2\}$, and are given by

$$\psi_{\text{grav}}(\mu, \lambda, K) = \int_F \frac{d^2\tau}{(Im \tau)^2} \left( \frac{\mu - \tau \lambda}{\pi (Im \tau)^{1/2} K} \right) \exp \left\{ -\frac{i|\mu - \tau \lambda|^2}{(Im \tau)K} \right\} \tilde{\chi}(\tau, \bar{\tau}), \quad (2.3)$$

where $\tau$ is the modulus of a torus, the integration is over a fundamental region $F$ for the modular group, and $\tilde{\chi}$ is any automorphic form of weight $1/2$. (In contrast to reference [18], we use the standard notation of string theory, where $\tau$ denotes the torus modulus rather than the mean extrinsic curvature.) The Hamiltonian that generates translations in $K$ is [19]

$$H = \frac{i}{2} K^{-1} \left( \frac{\partial}{\partial \mu} \mu + \frac{\partial}{\partial \lambda} \lambda \right), \quad (2.4)$$

and wave functions (2.3) are eigenstates of $H$ provided that $\tilde{\chi}$ is a Maass form, i.e.,

$$\left(2i(Im \tau) \frac{\partial}{\partial \tau} - \frac{1}{2}\right) \left(2i(Im \tau) \frac{\partial}{\partial \bar{\tau}} + \frac{1}{2}\right) \tilde{\chi}(\tau, \bar{\tau}) = E^2 \tilde{\chi}(\tau, \bar{\tau}). \quad (2.5)$$

In particular, the ground state is obtained by setting

$$\tilde{\chi}^{(0)}(\tau, \bar{\tau}, K) = (Im \tau)^{1/2} \eta^2(\tau) \quad (2.6)$$

in (2.3), where $\eta(\tau)$ is the Dedekind eta function. A straightforward calculation shows that the resulting wave function $\psi^{(0)}_{\text{grav}}(\mu, \lambda, K)$ is independent of $K$.

Note that apart from the $Im \tau$ factor, the automorphic form $\tilde{\chi}^{(0)}$ depends holomorphically on the modulus $\tau$. This may be a general feature for any genus. It is possible to add a gravitational Chern-Simons term to the (2+1)-dimensional action (2.1), thus introducing dynamical topologically massive gravitons. There is then some evidence from perturbation theory that holomorphicity of the wave functions is a necessary condition for a positive total Hamiltonian [20].

The gravitational analog of the inner product (1.2) can now be worked out explicitly. In particular, for the ground state determined by (2.6), we find that

$$\langle \psi^{(0)}_{\text{grav}} | \psi^{(0)}_{\text{grav}} \rangle = \int_F \frac{d^2\tau}{(Im \tau) |\eta(\tau)|^4}. \quad (2.7)$$

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1Since gravity is generally covariant, the “super-Hamiltonian” (2.3) that generates translations in coordinate time vanishes identically. Once a time-slicing is chosen, however, it is meaningful to discuss a Hamiltonian that generates motions between slices.
But this is precisely the diffeomorphism ghost contribution to the string theory torus partition function. Combining this result with (1.2) with $B = 0$, we find a full transition amplitude of

$$\int_{\mathcal{F}} \frac{d^2 \tau}{(Im \tau)} |\eta(\tau)|^4 \sum_{\lambda, \lambda'} h^{\lambda \lambda'} \bar{\Psi}_\lambda [0] \Psi_{\lambda'} [0],$$

which may be recognized as the full string partition function for the torus. Note that the integration over moduli space, with the correct measure, is forced upon us by the integral over $\lambda$ and $\mu$ in the overlap between the initial and final gravitational wave functions. Of course, if the total central charge is nonzero, an additional Liouville contribution must be included in the string partition function; but this term also arises in three dimensions, as a consequence of the local Lorentz anomaly [9].

3. Conclusion

We have now seen that the entire one-loop partition function in string theory can be reconstructed from three-dimensional gravity and topological field theory. All of the string theory ingredients — the conformal field theory partition function, the vertex operators, the Liouville action, the diffeomorphism ghosts, and the integration over moduli space — have straightforward three-dimensional interpretations.

Several important issues remain, however. First, our derivation was based on a particular choice of time-slicing, which, although natural, is by no means unique. Whether the final results depend on this choice is an open question, related to a fundamental problem in quantum gravity, that of understanding how general covariance manifests itself at the quantum level. Second, our derivation involved a particular approach to the quantization of (2+1)-dimensional gravity, based on the Chern-Simons form of the action. If we had started instead with metric variables and used a simple prescription for operator ordering, we would have found wave functions that behaved as automorphic forms of weight $0$ rather than $1/2$ [18, 21]. As yet, we know of no prescription for choosing between these two approaches to quantum gravity. Finally, we have not yet succeeded in generalizing our results to surfaces of genus greater than one, for which the constant mean curvature slicing becomes quite complicated.

These difficulties are real, and may ultimately defeat the attempt to describe string theory in terms of three-dimensional topological field theory. But in view of the success in reproducing the torus amplitude, it seems unlikely that the equivalence of the two- and three-dimensional approaches is merely coincidence.

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