Properties and applications of transversal operators

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Abstract: This paper presents some properties and applications of “transversal operators”. Two transversal operators are presented: a “translation” operator $T$ and a “dilation” operator $D$. Such operators are used in common analysis systems including Fourier series analysis, Fourier analysis, Gabor analysis, multiresolution analysis (MRA), and wavelet analysis. Like the unitary Fourier transform operator $\tilde{F}$, the transversal operators $T$ and $D$ are unitary. Demonstrations of the usefulness of these three unitary operators are found in the proofs of results found in some common analytic systems including MRA analysis and wavelet analysis.

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1 Background: operators on linear spaces

1.1 Star-algebras

Definition 1.1 Let $(\mathbb{F}, +, \cdot)$ be a field. Let $X$ be a set and let $+$ be an operator in $X^X$ and $\otimes$ be an operator in $X^{\mathbb{F} \times X}$. The structure $L \triangleq (X, +, \cdot, (\mathbb{F}, +, \cdot))$ is a linear space over the field $(\mathbb{F}, +, \cdot)$ if

1. $\exists \mathbf{0} \in X$ such that $x + \mathbf{0} = x \quad \forall x \in X$ (+ identity)
2. $\exists y \in X$ such that $x + y = 0 \quad \forall x \in X$ (+ inverse)
3. $(x + y) + z = x + (y + z) \quad \forall x, y, z \in X$ (+ is associative)
4. $x + y = y + x \quad \forall x, y \in X$ (+ is commutative)
5. $1 \cdot x = x \quad \forall x \in X$ (identity)
6. $\alpha \cdot (\beta \cdot x) = (\alpha \cdot \beta) \cdot x \quad \forall \alpha, \beta \in S$ and $x \in X$ (- associates with $\cdot$)
7. $\alpha \cdot (x + y) = (\alpha \cdot x) + (\alpha \cdot y) \quad \forall \alpha \in S$ and $x, y \in X$ (- distributes over $+$)
8. $(\alpha + \beta) \cdot x = (\alpha \cdot x) + (\beta \cdot x) \quad \forall \alpha, \beta \in S$ and $x \in X$ (- pseudo-distributes over $+$)

The set $X$ is called the underlying set. The elements of $X$ are called vectors. The elements of $\mathbb{F}$ are called scalars. A linear space is also called a vector space. If $\mathbb{F} \triangleq \mathbb{R}$, then $L$ is a real linear space. If $\mathbb{F} \triangleq \mathbb{C}$, then $L$ is a complex linear space.

All linear spaces are equipped with an operation by which vectors in the spaces can be added together. Linear spaces also have an operation that allows a scalar and a vector to
be “multiplied” together. But linear spaces in general have no operation that allows two
vectors to be multiplied together. A linear space together with such an operator is an algebra.\(^2\)

**Definition 1.2** \(^3\) Let \(A\) be an algebra.
An algebra \(A\) is unital if \(\exists u \in A\) such that \(ux = xu = x\) \(\forall x \in A\)

**Definition 1.3** \(^4\) Let \(A\) be an algebra.
The pair \((A, \ast)\) is a \(*\)-algebra if

1. \((x + y)^* = x^* + y^*\) \(\forall x, y \in A\) \((\text{distributive})\)
2. \((\alpha x)^* = \overline{\alpha}x^*\) \(\forall x \in A, \alpha \in \mathbb{C}\) \((\text{conjugate linear})\)
3. \((xy)^* = y^*x^*\) \(\forall x, y \in A\) \((\text{anti-automorphic})\)
4. \(x^{**} = x\) \(\forall x \in A\) \((\text{involutory})\)

The operator \(\ast\) is called an involution on the algebra \(A\).

**Proposition 1.4** \(^5\) Let \((A, \ast)\) be a unital \(*\)-algebra.
\(x\) is invertible \(\implies\) \(\begin{align*}
1. \quad x^\ast \text{ is invertible } & \quad \forall x \in A \text{ and } \\
2. \quad (x^\ast)^{-1} = (x^{-1})^\ast & \quad \forall x \in A
\end{align*}\)

**Definition 1.5** \(^6\) Let \((A, \|\|)\) be a \(*\)-algebra (Definition 1.3 page 4).
An element \(x \in A\) is hermitian or self-adjoint if \(x^\ast = x\).
An element \(x \in A\) is normal if \(xx^\ast = x^\ast x\).
An element \(x \in A\) is a projection if \(xx = x\) (involutory) and \(x^\ast = x\) (hermitian).

**Theorem 1.6** \(^7\) Let \((A, \|\|)\) be a \(*\)-algebra (Definition 1.3 page 4).
\(x = x^\ast\) and \(y = y^\ast\) \(\implies\) \(\begin{align*}
x + y &= (x + y)^\ast \quad (x + y \text{ is self-adjoint}) \\
x^\ast &= (x^\ast)^\ast \quad (x^\ast \text{ is self-adjoint}) \\
xy &= (xy)^\ast \quad (xy \text{ is hermitian}) \\
xy &= yx \quad \text{commutative}
\end{align*}\)

**Definition 1.7** (Hermitian components) \(^8\) Let \((A, \|\|)\) be a \(*\)-algebra (Definition 1.3 page 4).
The real part of \(x\) is defined as \(\Re x \triangleq \frac{1}{2}(x + x^\ast)\)
The imaginary part of \(x\) is defined as \(\Im x \triangleq \frac{1}{2i}(x - x^\ast)\)

\(^2\) It has been estimated that as of 2005, there are “somewhere between 50,000 and 200,000” algebras.\(^\square\) [62], page v

\(^3\) \[38], page 1

\(^4\) \[109], page 178, \[46], page 241

\(^5\) \[38], page 5

\(^6\) \[109], page 178, \[46], page 242

\(^7\) \[97], page 429

\(^8\) \[97], page 430, \[109], page 179, \[46], page 242
Theorem 1.8 \(^9\) Let \((\mathcal{A}, \ast)\) be a \(*\)-ALGEBRA (Definition 1.3 page 4).
\[
\Re x = (\Re x)^* \quad \forall x \in \mathcal{A} \quad (\Re x \text{ is hermitian})
\]
\[
\Im x = (\Im x)^* \quad \forall x \in \mathcal{A} \quad (\Im x \text{ is hermitian})
\]

Theorem 1.9 (Hermitian representation) \(^{10}\) Let \((\mathcal{A}, \ast)\) be a \(*\)-ALGEBRA (Definition 1.3 page 4).
\[
a = x + iy \iff x = \Re a \quad \text{and} \quad y = \Im a
\]

Definition 1.10 \(^{11}\) Let \(\mathcal{A}\) be an algebra.
The pair \((\mathcal{A}, \|\cdot\|)\) is a \textit{normed algebra} if
\[
\|xy\| \leq \|x\| \|y\| \quad \forall x, y \in \mathcal{A} \quad \text{(multiplicative condition)}
\]
A normed algebra \((\mathcal{A}, \|\cdot\|)\) is a \textit{Banach algebra} if \((\mathcal{A}, \|\cdot\|)\) is also a Banach space.

Proposition 1.11
\((\mathcal{A}, \|\cdot\|)\) is a normed algebra \(\implies\) multiplication is \textit{continuous} in \((\mathcal{A}, \|\cdot\|)\)

Definition 1.12 \(^{12}\) The triple \((\mathcal{A}, \|\cdot\|, \ast)\) is a \textit{\(C^*\) algebra} if
1. \((\mathcal{A}, \|\cdot\|)\) is a Banach algebra and
2. \((\mathcal{A}, \ast)\) is a \(*\)-algebra and
3. \(\|x^*\| = \|x\|^2 \quad \forall x \in \mathcal{A}\).

Theorem 1.13 \(^{13}\) Let \(\mathcal{A}\) be an algebra.
\((\mathcal{A}, \|\cdot\|, \ast)\) is a \textit{\(C^*\) algebra} \(\implies\) \(\|x^*\| = \|x\|\)

1.2 Operators on linear spaces

1.2.1 Operator Algebra

An operator is simply a function that maps from a linear space to another linear space (or to the same linear space).

Definition 1.14 \(^{14}\) A function \(A\) in \(Y^X\) is an \textit{operator} in \(Y^X\) if \(X\) and \(Y\) are both \textit{linear spaces}.

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\(^9\) [97], page 430, [59], page 42
\(^{10}\) [97], page 430, [109], page 179, [47], page 7
\(^{11}\) [109], page 2, [16], page 103, (Theorem IV.9.2)
\(^{12}\) [38], page 1, [46], page 241, [48], [47]
\(^{13}\) [38], page 1, [47], page 4, [46]
\(^{14}\) [63], page 42
Figure 1: Some operator types

Two operators \( A \) and \( B \) in \( Y^X \) are equal if \( Ax = Bx \) for all \( x \in X \). The inverse relation of an operator \( A \) in \( Y^X \) always exists as a relation in \( 2^{XY} \), but may not always be a function (may not always be an operator) in \( Y^X \).

The operator \( I \in X^X \) is the identity operator if \( Ix = I \) for all \( x \in X \).

**Definition 1.15** \(^{15}\) Let \( X^X \) be the set of all operators with from a linear space \( X \) to \( X \). Let \( I \) be an operator in \( X^X \). Let \( \mathcal{O}(X) \) be the identity element in \( X^X \).

\( I \) is the identity operator in \( X^X \) if \( I = \mathcal{O}(X) \).

**1.2.2 Linear operators**

**Definition 1.16** \(^{16}\) Let \( X \triangleq (X, +, \cdot, (F, +, \times)) \) and \( Y \triangleq (Y, +, \cdot, (F, +, \times)) \) be linear spaces.

---

\(^{15}\) [97], page 411

\(^{16}\) [84], page 55, [3], page 224, [66], page 6, [126], page 33
An operator \( L \in \mathcal{L}(X, Y) \) is linear if
1. \( L(x + y) = Lx + Ly \quad \forall x, y \in X \) \hspace{1cm} \text{(additive)}
2. \( L(\alpha x) = \alpha Lx \quad \forall x \in X, \forall \alpha \in \mathbb{F} \) \hspace{1cm} \text{(homogeneous)}

The set of all linear operators from \( X \) to \( Y \) is denoted \( \mathcal{L}(X, Y) \) such that
\[
\mathcal{L}(X, Y) \triangleq \{ L \in \mathcal{L}(X, Y) \mid L \text{ is linear} \}.
\]

**Theorem 1.17** \[17\] Let \( L \) be an operator from a linear space \( X \) to a linear space \( Y \), both over a field \( \mathbb{F} \).

\[
L \text{ is linear} \implies \begin{cases} 1. L(0) = 0 \\ 2. L(-x) = -(Lx) \quad \forall x \in X \\ 3. L(x - y) = Lx - Ly \quad \forall x, y \in X \\ 4. L \left( \sum_{n=1}^{N} \alpha_n x_n \right) = \sum_{n=1}^{N} \alpha_n (Lx_n) \quad x_n \in X, \alpha_n \in \mathbb{F}. \end{cases}
\]

**Theorem 1.18** \[18\] Let \( \mathcal{L}(X, Y) \) be the set of all linear operators from a linear space \( X \) to a linear space \( Y \). Let \( \mathcal{N}(L) \) be the null space of an operator \( L \) in \( \mathcal{L}(X, Y) \) and \( \mathcal{I}(L) \) the image set of \( L \) in \( \mathcal{L}(X, Y) \).

\[
\begin{align*}
\mathcal{L}(X, Y) & \text{ is a linear space (space of linear transforms)} \\
\mathcal{N}(L) & \text{ is a linear subspace of } X \quad \forall L \in \mathcal{L}(X, Y) \\
\mathcal{I}(L) & \text{ is a linear subspace of } Y \quad \forall L \in \mathcal{L}(X, Y)
\end{align*}
\]

**Example 1.19** \[19\] Let \( C([a, b], \mathbb{R}) \) be the set of all continuous functions from the closed real interval \([a, b]\) to \( \mathbb{R} \).
\( C([a, b], \mathbb{R}) \) is a linear space.

**Theorem 1.20** \[20\] Let \( \mathcal{L}(X, Y) \) be the set of linear operators from a linear space \( X \) to a linear space \( Y \). Let \( \mathcal{N}(L) \) be the null space of a linear operator \( L \in \mathcal{L}(X, Y) \).

\[
Lx = Ly \iff x - y \in \mathcal{N}(L) \\
L \text{ is injective} \iff \mathcal{N}(L) = \{0\}
\]

**Theorem 1.21** \[21\] Let \( W, X, Y, \text{ and } Z \) be linear spaces over a field \( \mathbb{F} \).

\[
\begin{align*}
1. L(MN) & = (LM)N \quad \forall L \in \mathcal{L}(X, Y), M \in \mathcal{L}(Y, Z), N \in \mathcal{L}(X, Y) \quad \text{(ASSOCIATIVE)} \\
2. L(M + N) & = (LM) + (LN) \quad \forall L \in \mathcal{L}(X, Y), M \in \mathcal{L}(X, Y), N \in \mathcal{L}(X, Y) \quad \text{(LEFT DISTRIBUTIVE)} \\
3. (L + M)N & = (LN) + (MN) \quad \forall L \in \mathcal{L}(X, Y), M \in \mathcal{L}(X, Y), N \in \mathcal{L}(X, Y) \quad \text{(RIGHT DISTRIBUTIVE)} \\
4. \alpha(LM) & = (\alpha L)M = L(\alpha M) \quad \forall L \in \mathcal{L}(X, Y), M \in \mathcal{L}(X, Y), \alpha \in \mathbb{F} \quad \text{(HOMOGENEOUS)}
\end{align*}
\]

\[17\] \[16\], page 79, \langle Theorem IV.1.1 \rangle, \[53\], page 207, \langle Theorem C.1 \rangle
\[18\] \[97\], pages 98–104, \[16\], pages 80–85, \langle Theorem IV.1.4 and Theorem IV.3.1 \rangle
\[19\] \[32\], page 3
\[20\] \[16\], page 88, \langle Theorem IV.1.4 \rangle
\[21\] \[16\], page 88, \langle Theorem IV.5.1 \rangle
1.3 Operators on Normed linear spaces

1.3.1 Operator norm

**Definition 1.22** Let $\mathcal{L}(X, Y)$ be the space of linear operators over normed linear spaces $X$ and $Y$. The **operator norm** $\|\cdot\|$ is defined as

$$\|A\| = \sup_{x \in X} \{\|Ax\| : \|x\| \leq 1\} \quad \forall A \in \mathcal{L}(X, Y)$$

The pair $(\mathcal{L}(X, Y), \|\cdot\|)$ is the **normed space of linear operators** on $(X, Y)$.

Proposition 1.23 (next) shows that the functional defined in Definition 1.22 (previous) is a norm.

**Proposition 1.23** Let $(\mathcal{L}(X, Y), \|\cdot\|)$ be the normed space of linear operators over the normed linear spaces $X \cong (\mathbb{F}, +, \cdot, (\mathbb{F}, +, \cdot), \|\cdot\|)$ and $Y \cong (\mathbb{F}, +, \cdot, (\mathbb{F}, +, \cdot), \|\cdot\|)$. The functional $\|\cdot\|$ is a norm on $\mathcal{L}(X, Y)$. In particular,

1. $\|A\| \geq 0 \quad \forall A \in \mathcal{L}(X, Y)$ (non-negative) and
2. $\|A\| = 0 \iff A \cong 0 \quad \forall A \in \mathcal{L}(X, Y)$ (non-degenerate) and
3. $\|aA\| = |a| \|A\| \quad \forall a \in \mathbb{F}, A \in \mathcal{L}(X, Y)$ (homogeneous) and
4. $\|A + B\| \leq \|A\| + \|B\| \quad \forall A, B \in \mathcal{L}(X, Y)$ (subadditive).

Moreover, $(\mathcal{L}(X, Y), \|\cdot\|)$ is a **normed linear space**.

**Lemma 1.24** Let $(\mathcal{L}(X, Y), \|\cdot\|)$ be the normed space of linear operators over normed linear spaces $X \cong (\mathbb{F}, +, \cdot, (\mathbb{F}, +, \cdot), \|\cdot\|)$ and $Y \cong (\mathbb{F}, +, \cdot, (\mathbb{F}, +, \cdot), \|\cdot\|)$. $\|L\| = \sup_{x \in \mathcal{L}(X, Y)} \{\|Lx\| : \|x\| = 1\} \quad \forall x \in \mathcal{L}(X, Y)$

**Proposition 1.25** Let $I$ be the identity operator in the normed space of linear operators $(\mathcal{L}(X, X), \|\cdot\|)$. $\|I\| = 1$

**Theorem 1.26** Let $(\mathcal{L}(X, Y), \|\cdot\|)$ be the normed space of linear operators over normed linear spaces $X$ and $Y$.

$$\|Lx\| \leq \|L\| \|x\| \quad \forall L \in \mathcal{L}(X, Y), x \in X$$

$$\|KL\| \leq \|K\| \|L\| \quad \forall K, L \in \mathcal{L}(X, Y)$$
1.3.2 Bounded linear operators

**Definition 1.27** Let $(\mathcal{L}(X, Y), \|\|)$ be a normed space of linear operators. An operator $B$ is **bounded** if $\|B\| < \infty$.

The quantity $\mathcal{B}(X, Y)$ is the set of all **bounded linear operators** on $(X, Y)$ such that $\mathcal{B}(X, Y) \triangleq \{L \in \mathcal{L}(X, Y) | \|L\| < \infty\}$.

**Theorem 1.28** Let $(\mathcal{L}(X, Y), \|\|)$ be the set of linear operators over normed linear spaces $X \triangleq (X, +, \cdot, (F, \oplus, \diamondsuit), \|\|)$ and $Y \triangleq (Y, +, \cdot, (F, \oplus, \diamondsuit), \|\|)$.

The following conditions are all **equivalent**:

1. $L$ is continuous at a **single point** $x_0 \in X \forall L \in \mathcal{L}(X, Y)$ ⇐⇒
2. $L$ is **continuous** (at every point $x \in X$) $\forall L \in \mathcal{L}(X, Y)$ ⇐⇒
3. $\|L\| < \infty$ (L is **bounded**) $\forall L \in \mathcal{L}(X, Y)$ ⇐⇒
4. $\exists M \in \mathbb{R}$ such that $\|Lx\| \leq M \|x\|$ $\forall L \in \mathcal{L}(X, Y), x \in X$

1.3.3 Adjoints on normed linear spaces

**Definition 1.29** Let $\mathcal{B}(X, Y)$ be the space of bounded linear operators on normed linear spaces $X$ and $Y$. Let $X^*$ be the **topological dual space** of $X$.

$B^*$ is the **adjoint** of an operator $B \in \mathcal{B}(X, Y)$ if $f(Bx) = (B^*f)(x) \forall f \in X^*, x \in X$.

**Theorem 1.30** Let $\mathcal{B}(X, Y)$ be the space of bounded linear operators on normed linear spaces $X$ and $Y$.

$(A \oplus B)^* = A^* \oplus B^* \forall A, B \in \mathcal{B}(X, Y)$

$(\lambda A)^* = \lambda A^* \forall \lambda, A \in \mathcal{B}(X, Y)$

$(AB)^* = B^*A^* \forall A, B \in \mathcal{B}(X, Y)$

**Theorem 1.31** Let $\mathcal{B}(X, Y)$ be the space of bounded linear operators on normed linear spaces $X$ and $Y$. Let $B^*$ be the adjoint of an operator $B$.

$\|B\| = \|B^*\| \forall B \in \mathcal{B}(X, Y)$
1.4 Operators on Inner-product spaces

**Theorem 1.32**  Let $A, B \in B(\mathcal{X}, \mathcal{X})$ be bounded linear operators on an inner-product space $\mathcal{X} \triangleq (\mathbb{X}, +, \cdot, (\mathbb{F}, +, \cdot), \langle \cdot \mid \cdot \rangle)$.

\[
\langle Bx \mid x \rangle = 0 \quad \forall x \in \mathcal{X} \iff Bx = 0 \quad \forall x \in \mathcal{X}
\]
\[
\langle Ax \mid x \rangle = \langle Bx \mid x \rangle \quad \forall x \in \mathcal{X} \iff A = B
\]

A fundamental concept of operators on inner-product spaces is the *operator adjoint* (Proposition 1.33 page 10). The adjoint of an operator is a kind of generalization of the conjugate of a complex number in that

- Both are *star-algebras* (Theorem 1.35 page 10).
- Both support decomposition into “real” and “imaginary” parts (Theorem 1.9 page 5).

Structurally, the operator adjoint provides a convenient symmetric relationship between the *range space* and *null space* of an operator (Theorem 1.36 page 11).

**Proposition 1.33**  Let $\mathcal{B}(\mathcal{H}, \mathcal{H})$ be the space of bounded linear operators on a Hilbert space $\mathcal{H}$. An operator $B^*$ is the adjoint of $B \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ if

\[
\langle Bx \mid y \rangle = \langle x \mid B^*y \rangle \quad \forall x, y \in \mathcal{H}.
\]

**Example 1.34**  (Matrix algebra: $A^* = A^H$) In matrix algebra,

- The inner-product operation $\langle x \mid y \rangle$ is represented by $y^H x$.
- The linear operator is represented as a matrix $A$.
- The operation of $A$ on vector $x$ is represented as $Ax$.
- The adjoint of matrix $A$ is the Hermitian matrix $A^H$.

Structures that satisfy the four conditions of the next theorem are known as *-algebras (“star-algebras”, Definition 1.3 page 4). Other structures which are *-algebras include the field of complex numbers $\mathbb{C}$ and any ring of complex square $n \times n$ matrices.

**Theorem 1.35**  (operator star-algebra)  Let $\mathcal{H}$ be a Hilbert space with operators $A, B \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ and with adjoints $A^*, B^* \in \mathcal{B}(\mathcal{H}, \mathcal{H})$. Let $\bar{\alpha}$ be the complex conjugate of some $\alpha \in \mathbb{C}$.
The pair \((\mathcal{H}, \ast)\) is a \(*\)-algebra (star-algebra). In particular,
1. \((A + B)^* = A^* + B^* \quad \forall A, B \in \mathcal{H}\) (DISTRIBUTIVE) and
2. \((\alpha A)^* = \overline{\alpha} A^* \quad \forall A, B \in \mathcal{H}\) (CONJUGATE LINEAR) and
3. \((AB)^* = B^* A^* \quad \forall A, B \in \mathcal{H}\) (ANTIAUTOMORPHIC) and
4. \(A^{**} = A \quad \forall A, B \in \mathcal{H}\) (INVOLUTARY)

Theorem 1.36 \(^{37}\) Let \(\mathcal{Y}^X\) be the set of all operators from a linear space \(X\) to a linear space \(Y\). Let \(\mathcal{N}(L)\) be the NULL SPACE of an operator \(L\) in \(\mathcal{Y}^X\) and \(\mathcal{I}(L)\) the IMAGE SET of \(L\) in \(\mathcal{Y}^X\).

\[
\mathcal{N}(A) = I(A^*)^\perp
\]
\[
\mathcal{N}(A^*) = I(A)^\perp
\]

1.5 Special Classes of Operators

1.5.1 Self-Adjoint Operators

Definition 1.37 \(^{38}\) Let \(B \in \mathcal{B}(\mathcal{H}, \mathcal{H})\) be a bounded operator with adjoint \(B^*\) on a Hilbert space \(\mathcal{H}\).

The operator \(B\) is said to be self-adjoint or hermitian if \(B \cong B^*\).

Example 1.38 (Autocorrelation operator) Let \(x(t)\) be a random process with autocorrelation

\[
R_{xx}(t, u) \triangleq \text{E}[x(t)x^*(u)].
\]

Let an autocorrelation operator \(R\) be defined as \([Rf](t) \triangleq \int_{\mathbb{R}} R_{xx}(t, u)f(u)\, du\).

\[
R = R^*
\]

(The autocorrelation operator \(R\) is self-adjoint)

Theorem 1.39 \(^{39}\) Let \(S : \mathcal{H} \to \mathcal{H}\) be an operator over a Hilbert space \(\mathcal{H}\) with eigenvalues \(\{\lambda_n\}\) and eigenfunctions \(\{\psi_n\}\) such that \(Sw_n = \lambda_n \psi_n\) and let \(\|x\| \triangleq \sqrt{\langle x \mid x \rangle}\).

\[
S = S^* \quad \Rightarrow \quad \begin{cases} 
1. \langle Sx \mid x \rangle \in \mathbb{R} \\
2. \lambda_n \in \mathbb{R} \\
3. \lambda_n \neq \lambda_m \Rightarrow \langle \psi_n \mid \psi_m \rangle = 0
\end{cases}
\]

1. \((\text{the hermitian quadratic form of } S \text{ is real})\)
2. \((\text{eigenvalues of } S \text{ are real})\)
3. \((\text{eigenfunctions associated with distinct eigenvalues are orthogonal})\)

\(^{37}\) \(\text{[112], page 312}\)

\(^{38}\) \(\text{Historical works regarding self-adjoint operators: } \text{[100], page 49, “linearer Operator R selbstadjungiert oder Hermitesch”, [126], page 50, (“self-adjoint transformations”)}\)

\(^{39}\) \(\text{[90], pages 315–316, [79], pages 114–119}\)
1.5.2 Normal Operators

Definition 1.40 Let $\mathcal{B}(X, Y)$ be the space of bounded linear operators on normed linear spaces $X$ and $Y$. Let $N^*$ be the adjoint of an operator $N \in \mathcal{B}(X, Y)$.

$N$ is normal if $N^*N = NN^*$.

Theorem 1.41 Let $\mathcal{B}(H, H)$ be the space of bounded linear operators on a Hilbert space $H$. Let $\mathcal{N}(N)$ be the null space of an operator $N$ in $\mathcal{B}(H, H)$ and $\mathcal{I}(N)$ the image set of $N$ in $\mathcal{B}(H, H)$.

$$N^*N = NN^* \quad \iff \quad \|N^*x\| = \|Nx\| \quad \forall x \in H$$

Theorem 1.42 Let $\mathcal{B}(H, H)$ be the space of bounded linear operators on a Hilbert space $H$. Let $\mathcal{N}(N)$ be the null space of an operator $N$ in $\mathcal{B}(H, H)$ and $\mathcal{I}(N)$ the image set of $N$ in $\mathcal{B}(H, H)$.

$$N^*N = NN^* \quad \Rightarrow \quad \mathcal{N}(N^*) = \mathcal{N}(N)$$

N and $N^*$ have the same null space

1.5.3 Isometric operators

An operator on a pair of normed linear spaces is isometric (next definition) if it is an isometry.

Definition 1.43 Let $(X, +, \cdot, (F, +, \cdot), \|\cdot\|)$ and $(Y, +, \cdot, (F, +, \cdot), \|\cdot\|)$ be normed linear spaces (Definition 3.6 page 33).

An operator $M \in \mathcal{L}(X, Y)$ is isometric if

$$\|Mx\| = \|x\| \quad \forall x \in X.$$

Theorem 1.44 Let $(X, +, \cdot, (F, +, \cdot), \|\cdot\|)$ and $(Y, +, \cdot, (F, +, \cdot), \|\cdot\|)$ be normed linear spaces. Let $M$ be a linear operator in $\mathcal{L}(X, Y)$.

$$\|Mx\| = \|x\| \quad \forall x \in X \quad \iff \quad \|Mx - My\| = \|x - y\| \quad \forall x, y \in X$$

isometric in length

isometric in distance
Isometric operators have already been defined (Definition 1.43 page 12) in the more general normed linear spaces, while Theorem 1.44 (page 12) demonstrated that in a normed linear space \( X, \| Mx \| = \| x \| \iff \| Mx - My \| = \| x - y \| \) for all \( x, y \in X \). Here in the more specialized inner-product spaces, Theorem 1.45 (next) demonstrates two additional equivalent properties.

**Theorem 1.45** \(^4^4\) Let \( B(X, X) \) be the space of bounded linear operators on a normed linear space \( X \equiv (X, +, \cdot, (\mathbb{F}, +, \cdot), \| \cdot \|) \). Let \( N \) be a bounded linear operator in \( L(X, X) \), and \( I \) the identity operator in \( L(X, X) \). Let \( \| x \| \equiv \sqrt{\langle x | x \rangle} \).

The following conditions are all equivalent:

1. \( M^*M = I \)
2. \( \langle Mx | My \rangle = \langle x | y \rangle \; \forall x, y \in X \) (\( M \) is surjective)
3. \( \| Mx - My \| = \| x - y \| \; \forall x, y \in X \) (isometric in distance)
4. \( \| Mx \| = \| x \| \; \forall x \in X \) (isometric in length)

**Theorem 1.46** \(^4^5\) Let \( B(X, Y) \) be the space of bounded linear operators on normed linear spaces \( X \) and \( Y \). Let \( M \) be a bounded linear operator in \( B(X, Y) \), and \( I \) the identity operator in \( B(X, X) \). Let \( \Lambda \) be the set of eigenvalues of \( M \). Let \( \| x \| \equiv \sqrt{\langle x | x \rangle} \).

\[ M^*M = I \implies \left\{ \begin{array}{c} \| M \| = 1 \quad \text{(unit length)} \\ | \lambda | = 1 \quad \forall \lambda \in \Lambda \end{array} \right. \]

\( M \) is isometric

**Example 1.47** (One sided shift operator) \(^4^6\) Let \( X \) be the set of all sequences with range \( \mathbb{W} (0, 1, 2, \ldots) \) and shift operators defined as

1. \( S_r \{ x_0, x_1, x_2, \ldots \} \equiv \{ 0, x_0, x_1, x_2, \ldots \} \) (right shift operator)
2. \( S_l \{ x_0, x_1, x_2, \ldots \} \equiv \{ x_1, x_2, x_3, \ldots \} \) (left shift operator)

1. \( S_r \) is an isometric operator.
2. \( S_r^* = S_l \)

### 1.5.4 Unitary operators

**Definition 1.48** \(^4^7\) Let \( B(X, Y) \) be the space of bounded linear operators on normed linear spaces \( X \) and \( Y \). Let \( U \) be a bounded linear operator in \( B(X, Y) \), and \( I \) the identity operator in \( B(X, X) \).

The operator \( U \) is unitary if \( U^*U = UU^* = I \).

\(^4^4\) \( \equiv \) [97], page 432, (Theorem 7.5.8), \( \equiv \) [84], page 391, (Proposition 5.72), \( \equiv \) [53], pages 226–227, (Theorem C.18).

\(^4^5\) \( \equiv \) [97], page 432

\(^4^6\) \( \equiv \) [97], page 441

\(^4^7\) \( \equiv \) [112], page 312, \( \equiv \) [97], page 431, \( \equiv \) [7], page 209, \( \equiv \) [8], \( \equiv \) [117], \( \equiv \) [125]
Proposition 1.49 Let \( \mathcal{B}(X, Y) \) be the space of bounded linear operators on normed linear spaces \( X \) and \( Y \). Let \( U \) and \( V \) be bounded linear operators in \( \mathcal{B}(X, Y) \).

\[
\begin{align*}
U \text{ is unitary and } V \text{ is unitary} \\
\implies (UV) \text{ is unitary}
\end{align*}
\]

Theorem 1.50 \(^{48}\) Let \( \mathcal{B}(H, H) \) be the space of bounded linear operators on a Hilbert space \( H \). Let \( U \) be a bounded linear operator in \( \mathcal{B}(H, H) \), and \( I(U) \) the image set of \( U \).

The following conditions are equivalent:

\[
\begin{align*}
1. & \quad UU^* = U^*U = I \quad \text{(unitary)} \\
2. & \quad \langle Ux | Uy \rangle = \langle U^*x | U^*y \rangle = \langle x | y \rangle \quad \text{and} \quad I(U) = X \quad \text{(surjective)} \\
3. & \quad \|Ux - Uy\| = \|U^*x - U^*y\| = \|x - y\| \quad \text{and} \quad I(U) = X \quad \text{(isometric in distance)} \\
4. & \quad \|Ux\| = \|x\| \quad \text{and} \quad I(U) = X \quad \text{(isometric in length)}
\end{align*}
\]

Theorem 1.51 \(^{49}\) Let \( \mathcal{B}(H, H) \) be the space of bounded linear operators on a Hilbert space \( H \). Let \( U \) be a bounded linear operator in \( \mathcal{B}(H, H) \), \( \mathcal{N}(U) \) the null space of \( U \), and \( I(U) \) the image set of \( U \).

\[
UU^* = U^*U = I \quad \text{implies} \quad \begin{cases} 
U^{-1} = U^* \\
I(U) = I(U^*) = X \\
\mathcal{N}(U) = \mathcal{N}(U^*) = \{0\} \\
\|U\| = \|U^*\| = 1
\end{cases}
\]

\( \text{(Unit length)} \)

\( ^{48} \) \( ^{\text{[112]}, \text{pages 313–314}, \langle \text{Theorem 12.13} \rangle, \text{[82], page 45}, \langle \text{Proposition 2.6} \rangle, \text{[53], pages 230–231}, \langle \text{Theorem C.20} \rangle} \)

\( ^{49} \) \( ^{\text{[53], page 231}, \langle \text{Theorem C.21} \rangle} \)
2 Background: Harmonic analysis

2.1 Families of functions

This paper is largely set in the space of Lebesgue square-integrable functions \( L^2_{\mathbb{R}} \) (Definition 2.2 page 15). The space \( L^2_{\mathbb{R}} \) is a subspace of the space \( \mathbb{R}^\mathbb{R} \), the set of all functions with domain \( \mathbb{R} \) (the set of real numbers) and range \( \mathbb{R} \). The space \( \mathbb{R}^\mathbb{R} \) is a subspace of the space \( \mathbb{C}^\mathbb{C} \), the set of all functions with domain \( \mathbb{C} \) (the set of complex numbers) and range \( \mathbb{C} \). That is, \( L^2_{\mathbb{R}} \subseteq \mathbb{R}^\mathbb{R} \subseteq \mathbb{C}^\mathbb{C} \). In general, the notation \( Y^X \) represents the set of all functions with domain \( X \) and range \( Y \) (Definition 2.1 page 15). Although this notation may seem curious, note that for finite \( X \) and finite \( Y \), the number of functions (elements) in \( Y^X \) is \(| Y^X | = | Y |^{| X |} \).

Definition 2.1 Let \( X \) and \( Y \) be sets. The space \( Y^X \) represents the set of all functions with domain \( X \) and range \( Y \) such that \( Y^X \triangleq \{ f(x) | f(x) : X \rightarrow Y \} \).

Definition 2.2 Let \( \mathbb{R} \) be the set of real numbers, \( \mathcal{B} \) the set of Borel sets on \( \mathbb{R} \), and \( \mu \) the standard Borel measure on \( \mathcal{B} \). Let \( \mathbb{R}^\mathbb{R} \) be as in Definition 2.1 page 15.

The space of Lebesgue square-integrable functions \( L^2_{(\mathbb{R},\mathcal{B},\mu)} \) (or \( L^2_{\mathbb{R}} \)) is defined as

\[
L^2_{(\mathbb{R},\mathcal{B},\mu)} \triangleq \left\{ f \in \mathbb{R}^\mathbb{R} \bigg| \left( \int_{\mathbb{R}} |f|^2 \right)^{\frac{1}{2}} d\mu < \infty \right\}.
\]

The standard inner product \( \langle \triangle | \nabla \rangle \) on \( L^2_{\mathbb{R}} \) is defined as

\[
\langle f(x) | g(x) \rangle \triangleq \int_{\mathbb{R}} f(x)g^*(x) \, dx.
\]

The standard norm \( \| \cdot \| \) on \( L^2_{\mathbb{R}} \) is defined as \( \| f(x) \| \triangleq \langle f(x) | f(x) \rangle^{\frac{1}{2}} \).

Definition 2.3 Let \( X \) be a set.

The indicator function \( 1 \in \{0,1\}^{2^X} \) is defined as

\[
1_A(x) = \begin{cases} 
1 & \text{for } x \in A \\
0 & \text{for } x \notin A
\end{cases}, \quad \forall x \in X, \, A \in 2^X
\]

The indicator function 1 is also called the characteristic function.
2.2 Trigonometric functions

2.2.1 Definitions

**Lemma 2.4** Let \( \mathcal{C} \) be the space of all continuously differentiable real functions and \( \frac{d}{dx} \in \mathcal{C} \) the differentiation operator. \( \frac{d^2}{dx^2} f + f = 0 \iff \left\{ \begin{align*}
f(x) &= [f](0) \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + \left[ \frac{d}{dx} f \right](0) \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\
&= \left( f(0) + \left[ \frac{d}{dx} f \right](0)x \right) - \left( \frac{f(0)}{2!} x^2 + \frac{\left[ \frac{d}{dx} f \right](0)}{3!} x^3 \right) + \left( \frac{f(0)}{4!} x^4 + \frac{\left[ \frac{d}{dx} f \right](0)}{5!} x^5 \right) + \cdots
\end{align*} \right\} \)

**Definition 2.5** Let \( \mathcal{C} \) be the space of all continuously differentiable real functions and \( \frac{d}{dx} \in \mathcal{C} \) the differentiation operator.

The **cosine** function \( \cos(x) \) is the function \( f \in \mathcal{C} \) that satisfies the following conditions:

\[
\frac{d^2}{dx^2} f + f = 0 \quad f(0) = 1 \quad \left[ \frac{d}{dx} f \right](0) = 0
\]

2nd order homogeneous differential equation \quad 1st initial condition \quad 2nd initial condition

The **sine** function \( \sin(x) \) is the function \( g \in \mathcal{C} \) that satisfies the following conditions:

\[
\frac{d^2}{dx^2} g + g = 0 \quad g(0) = 0 \quad \left[ \frac{d}{dx} g \right](0) = 1
\]

2nd order homogeneous differential equation \quad 1st initial condition \quad 2nd initial condition

**Theorem 2.6** \( \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad \forall x \in \mathbb{R} \)

\( \sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad \forall x \in \mathbb{R} \)

**Proposition 2.7** Let \( \mathcal{C} \) be the space of all continuously differentiable real functions and

\[ f(x) = \cos(x) + \rho \sin(x) + \int_x^1 g(y) \sin(x - y) \, dy. \]

This type of equation is called a **Volterra integral equation of the second type**. References: \[ 37, \] page 371, \[ 92, \] page 157. Volterra equation references: \[ 105, \] page 99, \[ 87, \] [88].
\[ \frac{d}{dx} \in \mathbb{C} \] the differentiation operator. Let \( f'(0) \triangleq \left[ \frac{d}{dx} f \right](0). \]

\[ \frac{d^2}{dx^2} f + f = 0 \iff f(x) = f(0) \cos(x) + f'(0) \sin(x) \quad \forall f \in \mathbb{C}, \forall x \in \mathbb{R} \]

**2nd Order Homogeneous Differential Equation**

**Theorem 2.8** 55 Let \( \frac{d}{dx} \in \mathbb{C} \) be the differentiation operator.

\[ \begin{align*}
\frac{d}{dx} \cos(x) &= -\sin(x) & \forall x \in \mathbb{R} \\
\frac{d}{dx} \sin(x) &= \cos(x) & \forall x \in \mathbb{R}
\end{align*} \]

### 2.2.2 The Complex Exponential

**Definition 2.9** The function \( f \in \mathbb{C} \) is the **exponential function** \( \exp(i x) \triangleq f(x) \) if 

1. \( \frac{d^2}{dx^2} f + f = 0 \) (second order homogeneous differential equation) and
2. \( f(0) = 1 \) (first initial condition) and
3. \( \left[ \frac{d}{dx} f \right](0) = i \) (second initial condition)

**Theorem 2.10** (Euler’s identity) 56

\[ e^{ix} = \cos(x) + i \sin(x) \quad \forall x \in \mathbb{R} \]

**Corollary 2.11**

\[ e^{ix} = \sum_{n \in \mathbb{W}} \frac{(ix)^n}{n!} \quad \forall x \in \mathbb{R} \]

**Corollary 2.12** 57

\[ e^{ix} + 1 = 0 \]

The exponential has two properties that make it extremely special:

- The exponential is an eigenvalue of any LTI operator (Theorem 2.13 page 17).
- The exponential generates a continuous point spectrum for the differential operator.

**Theorem 2.13** 58 Let \( L \) be an operator with kernel \( h(t, \omega) \) and

\[ \hat{h}(s) \triangleq \left\langle h(t, \omega) \mid e^{it} \right\rangle \quad \text{(LAPLACE TRANSFORM).} \]

\[ \begin{align*}
1. & \quad L \text{ is linear and} \\
2. & \quad L \text{ is time-invariant} \end{align*} \implies L e^{st} = \hat{h}^*(s) e^{st} \quad \text{eigenvalue eigenvector} \]

---

55 [110], page 157  
56 [33], [18], page 12  
57 [33], [34], http://www.daviddarling.info/encyclopedia/E/Eulers_formula.html  
58 [95], page 2, …page 2 online: http://www.cmap.polytechnique.fr/~mallat/WTintro.pdf
### 2.2.3 Trigonometric Identities

**Corollary 2.14** (Euler formulas) \(^{59}\)

\[
\cos(x) = \Re\left(e^{ix}\right) = \frac{e^{ix} + e^{-ix}}{2} \quad \forall x \in \mathbb{R}
\]

\[
\sin(x) = \Im\left(e^{ix}\right) = \frac{e^{ix} - e^{-ix}}{2i} \quad \forall x \in \mathbb{R}
\]

**Theorem 2.15** \(^{60}\)

\[e^{(a+b)} = e^a e^b \quad \forall a, b \in \mathbb{C}\]

### 2.3 Fourier Series

The Fourier Series expansion of a periodic function is simply a complex trigonometric polynomial. In the special case that the periodic function is even, then the Fourier Series expansion is a cosine polynomial.

**Definition 2.16** \(^{61}\)

The **Fourier Series operator** \(\hat{F} : L^2_{\mathbb{R}} \rightarrow \ell^2_{\mathbb{R}}\) is defined as

\[
[\hat{F}f](n) \triangleq \frac{1}{\sqrt{\tau}} \int_0^\tau f(x)e^{-i\frac{2\pi}{\tau}nx} \, dx \quad \forall f \in \{f \in L^2_{\mathbb{R}} \mid f \text{ is periodic with period } \tau \}
\]

**Theorem 2.17**

Let \(\hat{F}\) be the Fourier Series operator. The **inverse Fourier Series operator** \(\hat{F}^{-1}\) is given by

\[
[\hat{F}^{-1}(\tilde{x}_n)_{n \in \mathbb{Z}}](x) \triangleq \frac{1}{\sqrt{\tau}} \sum_{n \in \mathbb{Z}} \tilde{x}_n e^{i\frac{2\pi}{\tau}nx} \quad \forall (\tilde{x}_n) \in \ell^2_{\mathbb{R}}
\]

**Theorem 2.18**

The **Fourier Series adjoint** operator \(\hat{F}^*\) is given by

\[
\hat{F}^* = \hat{F}^{-1}
\]

\(^{59}\) [33], [18], page 12

\(^{60}\) [111], page 1

\(^{61}\) [78], page 3
The Fourier Series operator has several nice properties:

- The Fourier Series operator $\hat{F}$ is *unitary* \(^{62}\) (Corollary 2.19 page 19).

- Because $\hat{F}$ is unitary, it automatically has several other nice properties such as being *isometric*, and satisfying *Parseval's equation*, satisfying *Plancherel's formula*, and more (Corollary 2.20 page 19).

**Corollary 2.19** Let $I$ be the identity operator and let $\hat{F}$ be the Fourier Series operator with adjoint $\hat{F}^*$.  

$$\hat{F}\hat{F}^* = \hat{F}^*\hat{F} = I \quad (\hat{F} \text{ is unitary...and thus also normal and isometric})$$

**Proof:** This follows directly from the fact that $\hat{F}^* = \hat{F}^{-1}$ (Theorem 2.18 (page 18)).

**Corollary 2.20** Let $\hat{F}$ be the Fourier series operator, $\hat{F}^*$ be its adjoint, and $\hat{F}^{-1}$ be its inverse.

$$\langle \hat{F}x | \hat{F}y \rangle = \langle \hat{F}^{-1}x | \hat{F}^{-1}y \rangle = \langle x | y \rangle \quad \text{(Parseval's equation)}$$

$$\| \hat{F}x \| = \| \hat{F}^{-1}x \| = \| x \| \quad \text{(Plancherel's formula)}$$

- These results follow directly from the fact that $\hat{F}$ is unitary (Corollary 2.19 page 19) and from the properties of unitary operators (Theorem 1.51 page 14).

**Theorem 2.21** The set

$$\left\{ \frac{1}{\sqrt{\tau}} e^{i \frac{2\pi}{\tau} nx} \right\}_{n \in \mathbb{Z}}$$

is an orthonormal basis for all functions $f(x)$ with support in $[0, \tau]$.

\(^{62}\) *unitary operators*: Definition 1.48 page 13
2.4 Fourier Transform

2.4.1 Properties

Definition 2.22 The Fourier Transform operator $\hat{F}$ is defined as

$$\hat{F}f(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x} \, dx \quad \forall f \in L^2(\mathbb{R},\mathcal{B},\mu).$$

This definition of the Fourier Transform is also called the unitary Fourier Transform.

Remark 2.23 (Fourier transform scaling factor) If the Fourier transform operator $\hat{F}$ and inverse Fourier transform operator $\hat{F}^{-1}$ are defined as

$$\hat{F}f(x) \triangleq A \int_{\mathbb{R}} f(x) e^{-i\omega x} \, dx \quad \text{and} \quad \hat{F}^{-1} \hat{F}f(\omega) \triangleq B \int_{\mathbb{R}} F(\omega) e^{i\omega x} \, d\omega,$$

then $A$ and $B$ can be any constants as long as $AB = \frac{1}{2\pi}$. The Fourier transform is often defined with the scaling factor $A$ set equal to 1 such that $[\hat{F}f(x)](\omega) \triangleq \int_{\mathbb{R}} f(x) e^{-i\omega x} \, dx$. In this case, the inverse Fourier transform operator $\hat{F}^{-1}$ is either defined as

$$[\hat{F}^{-1} f(x)](f) \triangleq \int_{\mathbb{R}} f(x) e^{i2\pi f x} \, dx \quad \text{(using oscillatory frequency free variable $f$)}$$

or

$$[\hat{F}^{-1} f(x)](\omega) \triangleq \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{i\omega x} \, dx \quad \text{(using angular frequency free variable $\omega$)}.$$

In short, the $2\pi$ has to show up somewhere, either in the argument of the exponential ($e^{-i2\pi f t}$) or in front of the integral ($\frac{1}{2\pi} \int \cdots$). One could argue that it is unnecessary to burden the exponential argument with the $2\pi$ factor ($e^{-i2\pi f t}$), and thus could further argue in favor of using the angular frequency variable $\omega$ thus giving the inverse operator definition $[\hat{F}^{-1} f(x)](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\omega x} \, dx$. But this causes a new problem. In this case, the Fourier operator $\hat{F}$ is not unitary (see Theorem 2.25 page 20)—in particular, $\hat{F}\hat{F}^* \neq \mathbf{I}$, where $\hat{F}^*$ is the adjoint of $\hat{F}$; but rather, $\hat{F} \left( \frac{1}{\sqrt{2\pi}} \hat{F}^* \right) = (\frac{1}{\sqrt{2\pi}} \hat{F}^*) \hat{F} = \mathbf{I}$. But if we define the operators $\hat{F}$ and $\hat{F}^{-1}$ to both have the scaling factor $\frac{1}{\sqrt{2\pi}}$, then $\hat{F}$ and $\hat{F}^{-1}$ are inverses and $\hat{F}$ is unitary—that is, $\hat{F}\hat{F}^* = \hat{F}^*\hat{F} = \mathbf{I}$.

Theorem 2.24 (Inverse Fourier transform) Let $\hat{F}$ be the Fourier Transform operator (Definition 2.22 page 20). The inverse $\hat{F}^{-1}$ of $\hat{F}$ is

$$[\hat{F}^{-1} f(x)](\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \hat{F}(\omega) e^{i\omega x} \, d\omega \quad \forall \hat{F} \in L^2(\mathbb{R},\mathcal{B},\mu).$$

Theorem 2.25 Let $\hat{F}$ be the Fourier Transform operator with inverse $\hat{F}^{-1}$ and adjoint $\hat{F}^*$.

$$\hat{F}^* = \hat{F}^{-1}.$$
The Fourier Transform operator has several nice properties:

- \( \hat{F} \) is unitary (Definition 1.48 page 13) (Corollary 2.26—next corollary).
- Because \( \hat{F} \) is unitary, it automatically has several other nice properties (Theorem 2.27 page 21).

**Corollary 2.26** Let \( I \) be the identity operator and let \( \hat{F} \) be the Fourier Transform operator with adjoint \( \hat{F}^* \) and inverse \( \hat{F}^{-1} \).

\[
\hat{F} \hat{F}^* = \hat{F}^* \hat{F} = I, \quad (\hat{F} \text{ is unitary})
\]

**Proof:** This follows directly from the fact that \( \hat{F}^* = \hat{F}^{-1} \) (Theorem 2.25 page 20).  

**Theorem 2.27** Let \( \hat{F} \) be the Fourier transform operator with adjoint \( \hat{F}^* \) and inverse \( \hat{F}^{-1} \). Let \( \|\cdot\| \) be the operator norm with respect to the vector norm \( \|\cdot\| \) with respect to the Hilbert space \( (\mathbb{C}^n, \langle \Delta | \nabla \rangle) \). Let \( \mathcal{R}(A) \) be the range of an operator \( A \).

\[
\begin{align*}
\mathcal{R}(Ff) &= \mathcal{R}(\hat{F}^{-1}g) = L^2_{\mathbb{R}} \\
\|\hat{F}\| &= \|\hat{F}^{-1}\| = 1 & \text{(unitary)} \\
\langle \hat{F}f | \hat{F}g \rangle &= \langle \hat{F}^{-1}f | \hat{F}^{-1}g \rangle = \langle f | g \rangle & \text{(Parseval's equation)} \\
\|\hat{F}f\| &= \|\hat{F}^{-1}f\| = \|f\| & \text{(Plancherel's formula)} \\
\|\hat{F}f - \hat{F}^*g\| &= \|\hat{F}^{-1}f - \hat{F}^{-1}g\| = \|f - g\| & \text{(isometric)}
\end{align*}
\]

**Proof:** These results follow directly from the fact that \( \hat{F} \) is unitary (Corollary 2.26 page 21) and from the properties of unitary operators (Theorem 1.51 page 14).
Theorem 2.28 (Shift relations) Let $\hat{\mathcal{F}}$ be the Fourier transform operator.
\[
\begin{align*}
\hat{\mathcal{F}}[f(x-u)](\omega) &= e^{-i\omega u} \hat{F}(x)(\omega) \\
\hat{\mathcal{F}}(e^{i\alpha x}g(x))(\omega) &= \hat{F}g(x)(\omega - \alpha)
\end{align*}
\]

Theorem 2.29 (Complex conjugate) Let $\hat{\mathcal{F}}$ be the Fourier Transform operator and $\ast$ represent the complex conjugate operation on the set of complex numbers.
\[
\hat{\mathcal{F}}f^*(-x) = [\hat{\mathcal{F}}f(x)]^* \quad \forall f \in L^2_{\mathbb{R},\mathbb{B},\mathcal{M}}
\]

2.4.2 Convolution

Definition 2.30 The convolution operation is defined as
\[
[f \ast g](x) \triangleq f(x) \ast g(x) \triangleq \int_{\mathbb{R}} f(u)g(x-u) \, du \quad \forall f, g \in L^2_{\mathbb{R},\mathbb{B},\mathcal{M}}
\]

Theorem 2.31 (next) demonstrates that multiplication in the “time domain” is equivalent to convolution in the “frequency domain” and vice-versa.

Theorem 2.31 (convolution theorem) Let $\hat{\mathcal{F}}$ be the Fourier Transform operator and $\ast$ the convolution operator.
\[
\begin{align*}
\hat{\mathcal{F}}[f(x) \ast g(x)](\omega) &= \sqrt{2\pi} \hat{F}f(\omega) \hat{F}g(\omega) \quad \forall f, g \in L^2_{\mathbb{R},\mathbb{B},\mathcal{M}} \\
\hat{\mathcal{F}}[f(x)g(x)](\omega) &= \frac{1}{\sqrt{2\pi}} \hat{F}f(\omega) \ast \hat{F}g(\omega) \quad \forall f, g \in L^2_{\mathbb{R},\mathbb{B},\mathcal{M}}^{-1}
\end{align*}
\]

---

\[\text{[9], page 6}\]
PROOF:

\[ \hat{F}[f(x) \star g(x)](\omega) = \hat{F} \left[ \int_{u \in \mathbb{R}} f(u)g(x-u) \, du \right](\omega) \]
by def. of \( \star \) (Definition 2.30 page 22)

\[ = \int_{u \in \mathbb{R}} f(u)[\hat{F}g(x-u)](\omega) \, du \]
\[ = \int_{u \in \mathbb{R}} f(u)e^{-j\omega u} [\hat{F}g(x)](\omega) \, du \]
by Theorem 2.28 page 22

\[ = \frac{1}{\sqrt{2\pi}} \int_{u \in \mathbb{R}} f(u)e^{-j\omega u} \left[ \frac{1}{\sqrt{2\pi}} \int_{v \in \mathbb{R}} [\hat{F}f(\omega)](\omega) \, d\omega \right] \]
\[ \times [\hat{F}g(x)](\omega) \]
by Theorem 2.24 page 20

\[ = \frac{1}{\sqrt{2\pi}} \int_{v \in \mathbb{R}} \left[ \frac{1}{\sqrt{2\pi}} \int_{u \in \mathbb{R}} [\hat{F}f(\omega)](\omega) \, d\omega \right] \left[ \frac{1}{\sqrt{2\pi}} \int_{u \in \mathbb{R}} [\hat{F}g(\omega - v)](\omega - v) \, d\omega \right] \]
by Theorem 2.28 page 22

\[ = \frac{1}{\sqrt{2\pi}} \int_{v \in \mathbb{R}} \left[ \frac{1}{\sqrt{2\pi}} \int_{u \in \mathbb{R}} [\hat{F}f(\omega)](\omega) \, d\omega \right] [\hat{F}g(x)](\omega) \]
by def. of \( \star \) (Definition 2.30 page 22)

2.5 Operations on sequences

2.5.1 Z-transform

Definition 2.32 Let \( X \) be the set of all functions from a set \( Y \) to a set \( X \). Let \( Z \) be the set of integers.

A function \( f \) in \( X \) is a sequence over \( X \) if \( Y = Z \).

A sequence may be denoted in the form \( (x_n)_{n \in Z} \) or simply as \( (x_n) \).

Definition 2.33 Let \( (\mathbb{F}, +, \cdot) \) be a field.

The space of all absolutely square summable sequences \( \ell_2^2 \) over \( \mathbb{F} \) is defined as

\[ \ell_2^2 \triangleq \left\{ (x_n)_{n \in Z} \left| \sum_{n \in Z} |x_n|^2 < \infty \right. \right\} \]

\[ \triangleleft \text{ [19], page 1, [129], page 23, (Definition 2.1), [75], page 31} \]
\[ \triangleleft \text{ [85], page 347, (Example 5.K)} \]
The space \( \ell_2^2 \) is an example of a separable Hilbert space. In fact, \( \ell_2^2 \) is the only separable Hilbert space in the sense that all separable Hilbert spaces are isomorphically equivalent. For example, \( \ell_2^2 \) is isomorphic to \( L_2^2 \), the space of all absolutely square Lebesgue integrable functions.

**Definition 2.34**  The convolution operation \( * \) is defined as
\[
(x_n) * (y_n) \triangleq \left( \sum_{m \in \mathbb{Z}} x_m y_{n-m} \right)_{n \in \mathbb{Z}} \forall (x_n)_{n \in \mathbb{Z}}, (y_n)_{n \in \mathbb{Z}} \in \ell_2^2
\]

**Definition 2.35**  70 The z-transform \( Z \) of \( (x_n)_{n \in \mathbb{Z}} \) is defined as
\[
Z (x_n) (z) \triangleq \sum_{n \in \mathbb{Z}} x_n z^{-n} \forall (x_n)_{n \in \mathbb{Z}} \in \ell_2^2
\]

70 Laurent series: [1], page 49

**Proposition 2.36**  71 Let \( * \) be the convolution operator (Definition 2.34 page 24).
\[
(x_n) * (y_n) = (y_n) * (x_n) \forall (x_n)_{n \in \mathbb{Z}}, (y_n)_{n \in \mathbb{Z}} \in \ell_2^2 \quad (\star \text{ is commutative})
\]

**Theorem 2.37**  72 Let \( * \) be the convolution operator (Definition 2.34 page 24).
\[
Z \left( (x_n) * (y_n) \right) = (Z (x_n)) \cdot (Z (y_n)) \quad \forall (x_n)_{n \in \mathbb{Z}}, (y_n)_{n \in \mathbb{Z}} \in \ell_2^2
\]

2.5.2 Discrete Time Fourier Transform

**Definition 2.38**  The discrete-time Fourier transform \( \hat{F} \) of \( (x_n)_{n \in \mathbb{Z}} \) is defined as
\[
\hat{F} (x_n) (\omega) \triangleq \sum_{n \in \mathbb{Z}} x_n e^{-i\omega n} \forall (x_n)_{n \in \mathbb{Z}} \in \ell_2^2
\]

If we compare the definition of the Discrete Time Fourier Transform (Definition 2.38 page 24) to the definition of the Z-transform (Definition 2.35 page 24), we see that the DTFT is just a special case of the more general Z-Transform, with \( z = e^{i\omega} \). If we imagine \( z \in \mathbb{C} \) as a complex plane, then \( e^{i\omega} \) is a unit circle in this plane. The “frequency” \( \omega \) in the DTFT is the unit circle in the much larger \( z \)-plane as illustrated in Figure 2 (page 25).

71 ⟨ Proposition J.1 ⟩  [53], page 344
72 ⟨ Theorem J.1 ⟩  [53], pages 344–345
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\[ \Re [z] \quad \Im [z] \]
\[ z = e^{i\omega} \]

Figure 2: Unit circle in complex-z plane

Proposition 2.39 73 Let \( \hat{x}(\omega) \triangleq \hat{F}[\{x_n\}](\omega) \) be the discrete-time Fourier transform (Definition 2.38 page 24) of a sequence \( \{x_n\}_{n \in \mathbb{Z}} \) in \( e^2_\mathbb{R} \).

\[ \hat{x}(\omega) = \hat{x}(\omega + 2\pi n) \quad \forall n \in \mathbb{Z} \]

PERIODIC with period \( 2\pi \)

\[ \hat{x}(\omega + 2\pi n) = \sum_{m \in \mathbb{Z}} x_m e^{-(\omega + 2\pi n)m} \]
\[ = \sum_{m \in \mathbb{Z}} x_m e^{-i\omega m} e^{-2\pi mm} \]
\[ = \hat{x}(\omega) \] (2-1)

\[ \hat{x}(1) = \hat{x}(\pi) = 0 \]

Proposition 2.40 74 Let \( \hat{x}(z) \) be the Z-transform (Definition 2.35 page 24) and \( \check{x}(\omega) \) the discrete-time Fourier transform (Definition 2.38 page 24) of \( \{x_n\} \).

\[ \left\{ \sum_{n \in \mathbb{Z}} x_n = c \right\} \quad \Leftrightarrow \quad \left\{ \hat{x}(z) \bigg|_{z=1} = c \right\} \quad \Leftrightarrow \quad \left\{ \check{x}(\omega) \bigg|_{\omega=0} = c \right\} \quad \forall \{x_n\}_{n \in \mathbb{Z}} \in e^2_\mathbb{R} : c \in \mathbb{R} \]

\[ \text{(1) time domain} \quad \text{(2) z domain} \quad \text{(3) frequency domain} \]

73 [53], pages 348–349, (Proposition J.2)
74 [53], pages 349–350, (Proposition J.3)
Proposition 2.41

\[ \sum_{n \in \mathbb{Z}} (-1)^n x_n = c \iff \hat{x}(z) \big|_{z=-1} = c \iff \hat{x}(\omega) \big|_{\omega=\pi} = c \]

(1) in "time"

\( \sum_{n \in \mathbb{Z}} h_{2n} \sum_{n \in \mathbb{Z}} h_{2n+1} = \left( \frac{1}{2} \left( \sum_{n \in \mathbb{Z}} h_n + c \right), \frac{1}{2} \left( \sum_{n \in \mathbb{Z}} h_n - c \right) \right) \)

(4) sum of even, sum of odd

\[ \forall c \in \mathbb{R}, \langle x_n \rangle_{n \in \mathbb{Z}}, \langle y_n \rangle_{n \in \mathbb{Z}} \in l^2(\mathbb{R}) \]

Lemma 2.42

Let \( \hat{f}(\omega) \) be the DTFT (Definition 2.38 page 24) of a sequence \( \langle x_n \rangle_{n \in \mathbb{Z}} \).

REAL-VALUED sequence \( \implies |\hat{x}(\omega)|^2 = |\hat{x}(-\omega)|^2 \quad \forall (x_n)_{n \in \mathbb{Z}} \in l^2(\mathbb{R}) \)

\[ |\hat{x}(\omega)|^2 = |\hat{x}(z)|^2 \big|_{z=e^{i\omega}} = \hat{x}(z)\overline{\hat{x}(z)} \big|_{z=e^{i\omega}} = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} x_n z^{-n} \overline{x_m z^{-m}} \big|_{z=e^{i\omega}} = \sum_{n \in \mathbb{Z}} \left( |x_n|^2 + \sum_{m > n} x_n \overline{x_m} z^{-(m-n)} + \sum_{m < n} x_n \overline{x_m} z^{-(m-n)} \right) \big|_{z=e^{i\omega}} = \sum_{n \in \mathbb{Z}} \left( |x_n|^2 + \sum_{m > n} x_n x_m e^{i\omega(m-n)} + \sum_{m < n} x_n x_m e^{-i\omega(m-n)} \right) = \sum_{n \in \mathbb{Z}} \left( |x_n|^2 + \sum_{m > n} x_n x_m \cos[\omega(m-n)] \right) \]

\[ \text{Proof:} \]

\[ |\hat{x}(\omega)|^2 = |\hat{x}(z)|^2 \big|_{z=e^{i\omega}} = \hat{x}(z)\overline{\hat{x}(z)} \big|_{z=e^{i\omega}} = \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} x_n z^{-n} \overline{x_m z^{-m}} \big|_{z=e^{i\omega}} = \sum_{n \in \mathbb{Z}} \left( |x_n|^2 + \sum_{m > n} x_n \overline{x_m} z^{-(m-n)} + \sum_{m < n} x_n \overline{x_m} z^{-(m-n)} \right) \big|_{z=e^{i\omega}} = \sum_{n \in \mathbb{Z}} \left( |x_n|^2 + \sum_{m > n} x_n x_m e^{i\omega(m-n)} + \sum_{m < n} x_n x_m e^{-i\omega(m-n)} \right) = \sum_{n \in \mathbb{Z}} \left( |x_n|^2 + \sum_{m > n} x_n x_m (e^{i\omega(m-n)} + e^{-i\omega(m-n)}) \right) = \sum_{n \in \mathbb{Z}} \left( |x_n|^2 + \sum_{m > n} x_n x_m 2 \cos[\omega(m-n)] \right) \]

\[ 75 \quad [27], \text{page 123} \]
= \sum_{n \in \mathbb{Z}} |x_n|^2 + 2 \sum_{n \in \mathbb{Z}, m \neq n} x_n x_m \cos[\omega(m - n)]

Since \cos is real and even, then $|\hat{x}(\omega)|^2$ must also be real and even.

**Theorem 2.43** (inverse DTFT) 76 Let $\hat{x}(\omega)$ be the DISCRETE-TIME FOURIER TRANSFORM (Definition 2.38 page 24) of a sequence $(x_n)_{n \in \mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\mathcal{F}^{-1}$ be the inverse of $\mathcal{F}$.

\[
\begin{align*}
\hat{x}(\omega) \triangleq \sum_{n \in \mathbb{Z}} x_n e^{-i\omega n} 
\implies \begin{cases} 
\mathcal{F}(x_n) = \frac{1}{2\pi} \int_{\alpha-\pi}^{\alpha+\pi} \hat{x}(\omega)e^{i\omega n} d\omega & \forall \alpha \in \mathbb{R} \\
(x_n) = \mathcal{F}^{-1}\mathcal{F}(x_n) & \forall (x_n)_{n \in \mathbb{Z}} \in \ell^2_{\mathbb{R}}
\end{cases}
\end{align*}
\]

**Theorem 2.44** (orthonormal quadrature conditions) 77 Let $\hat{x}(\omega)$ be the DISCRETE-TIME FOURIER TRANSFORM (Definition 2.38 page 24) of a sequence $(x_n)_{n \in \mathbb{Z}} \in \ell^2_{\mathbb{R}}$. Let $\delta_n$ be the KRONECKER DELTA FUNCTION at $n$ (Definition 3.12 page 35).

\[
\begin{align*}
\sum_{m \in \mathbb{Z}} x_m y^*_{m-2n} &= 0 \iff \hat{x}(\omega)\hat{y}(\omega) + \hat{x}(\omega + \pi)\hat{y}(\omega + \pi) = 0 & \forall n \in \mathbb{Z}, (x_n, y_n) \in \ell^2_{\mathbb{R}} \\
\sum_{m \in \mathbb{Z}} x^*_m x_n &= \delta_n \iff |\hat{x}(\omega)|^2 + |\hat{x}(\omega + \pi)|^2 = 2 & \forall n \in \mathbb{Z}, (x_n, y_n) \in \ell^2_{\mathbb{R}}
\end{align*}
\]

\[\text{PROOF:} \quad \text{Let } z \triangleq e^{i\omega}.\]

1. Proof that $2 \sum_{n \in \mathbb{Z}} \left[ \sum_{k \in \mathbb{Z}} x_k y^*_{k-2n} \right] e^{-i2\omega n} = \hat{x}(\omega)\hat{y}(\omega) + \hat{x}(\omega + \pi)\hat{y}(\omega + \pi)$:

\[
\begin{align*}
2 \sum_{n \in \mathbb{Z}} \left[ \sum_{k \in \mathbb{Z}} x_k y^*_{k-2n} \right] e^{-i2\omega n} 
&= 2 \sum_{k \in \mathbb{Z}} x_k \sum_{n \in \mathbb{Z}} y^*_{k-2n} e^{i2\omega n} \\
&= 2 \sum_{k \in \mathbb{Z}} x_k \sum_{n \text{ even}} y^*_{k-n} e^{i2\omega n} \\
&= \sum_{k \in \mathbb{Z}} x_k \sum_{n \in \mathbb{Z}} y^*_{k-n} z^{-n} (1 + e^{i2\omega n}) \\
&= \sum_{k \in \mathbb{Z}} x_k \sum_{n \in \mathbb{Z}} y^*_{k-n} z^{-n} + \sum_{k \in \mathbb{Z}} x_k \sum_{n \in \mathbb{Z}} y^*_{k-n} z^{-n} e^{i2\omega n} \\
&= \sum_{k \in \mathbb{Z}} x_k \sum_{m \in \mathbb{Z}} y^*_{m} z^{-(k-m)} + \sum_{k \in \mathbb{Z}} x_k \sum_{m \in \mathbb{Z}} y^*_{m} e^{-i(\omega + \pi)(k-m)} & \text{where } m \triangleq k - n \\
&= \sum_{k \in \mathbb{Z}} x_k z^{-k} \sum_{m \in \mathbb{Z}} y^*_{m} z^{-m} + \sum_{k \in \mathbb{Z}} x_k e^{-i(\omega + \pi)k} \sum_{m \in \mathbb{Z}} y^*_{m} e^{i(\omega + \pi)m}
\end{align*}
\]

\[76 \quad [76], \text{ page 3-95}, \langle (3.6.2) \rangle \]

\[77 \quad [30], \text{ pages 132--137}, \langle (5.1.20),(5.1.39) \rangle \]
\[\sum_{k \in \mathbb{Z}} x_k e^{-i\omega k} + \sum_{k \in \mathbb{Z}} x_k e^{-i(\omega + \pi)k}\]
\[\triangleq \hat{x}(\omega)\hat{y}^* (\omega) + \hat{x}(\omega + \pi)\hat{y}^* (\omega + \pi)\]

(2) Proof that \(\sum_{m \in \mathbb{Z}} x_m y_m^* = 0 \implies \hat{x}(\omega)\hat{y}^* (\omega) + \hat{x}(\omega + \pi)\hat{y}^* (\omega + \pi) = 0\):

\[
0 = 2 \sum_{n \in \mathbb{Z}} \left[ \sum_{k \in \mathbb{Z}} x_k y_{k-2n}^* \right] e^{-i2\omega n} \quad \text{by left hypothesis}
= \hat{x}(\omega)\hat{y}^* (\omega) + \hat{x}(\omega + \pi)\hat{y}^* (\omega + \pi) \quad \text{by item 1}
\]

Thus by the above equation, \(\sum_{n \in \mathbb{Z}} \left[ \sum_{k \in \mathbb{Z}} x_k y_{k-2n}^* \right] e^{-i2\omega n} = 0\). The only way for this to be true is if \(\sum_{k \in \mathbb{Z}} x_k y_{k-2n}^* = 0\).

(3) Proof that \(\sum_{m \in \mathbb{Z}} x_m y_m^* = 0 \iff \hat{x}(\omega)\hat{y}^* (\omega) + \hat{x}(\omega + \pi)\hat{y}^* (\omega + \pi) = 0\):

\[
2 \sum_{n \in \mathbb{Z}} \left[ \sum_{k \in \mathbb{Z}} x_k y_{k-2n}^* \right] e^{-i2\omega n} = \hat{x}(\omega)\hat{y}^* (\omega) + \hat{x}(\omega + \pi)\hat{y}^* (\omega + \pi) \quad \text{by item 1}
= 0 \quad \text{by right hypothesis}
\]

Thus by the above equation, \(\sum_{n \in \mathbb{Z}} \left[ \sum_{k \in \mathbb{Z}} x_k y_{k-2n}^* \right] e^{-i2\omega n} = 0\). The only way for this to be true is if \(\sum_{k \in \mathbb{Z}} x_k y_{k-2n}^* = 0\).

(4) Proof that \(\sum_{m \in \mathbb{Z}} x_m y_m^* = \delta_n \implies |\hat{x}(\omega)|^2 + |\hat{x}(\omega + \pi)|^2 = 2\):

Let \(g_n = x_n\).

\[
2 = 2 \sum_{n \in \mathbb{Z}} \delta_{n \in \mathbb{Z}} e^{-i2\omega n}
= 2 \sum_{n \in \mathbb{Z}} \left[ \sum_{k \in \mathbb{Z}} x_k y_{k-2n}^* \right] e^{-i2\omega n} \quad \text{by left hypothesis}
= \hat{x}(\omega)\hat{y}^* (\omega) + \hat{x}(\omega + \pi)\hat{y}^* (\omega + \pi) \quad \text{by item 1}
\]

Thus by the above equation, \(\sum_{n \in \mathbb{Z}} \left[ \sum_{k \in \mathbb{Z}} x_k y_{k-2n}^* \right] e^{-i2\omega n} = 1\). The only way for this to be true is if \(\sum_{k \in \mathbb{Z}} x_k y_{k-2n}^* = \delta_n\).
2.5.3 Frequency Response

The pole zero locations of a digital filter determine the magnitude and phase frequency response of the digital filter. This can be seen by representing the poles and zeros vectors in the complex z-plane. Each of these vectors has a magnitude $M$ and a direction $\theta$. Also, each factor $(z - z_i)$ and $(z - p_j)$ can be represented as vectors as well (the difference of two vectors). Each of these factors can be represented by a magnitude/phase factor $M_i e^{i\theta_i}$. The overall magnitude and phase of $H(z)$ can then be analyzed.

Take the following filter for example:

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

$$= \frac{(z - z_1)(z - z_2)}{(z - p_1)(z - p_2)}$$

$$= \frac{M_1 e^{i\theta_1} M_2 e^{i\theta_2}}{M_3 e^{i\theta_1} M_4 e^{i\theta_2}}$$

$$= \left( \frac{M_1 M_2}{M_3 M_4} \right) e^{i(\theta_1 - \theta_2)}$$

This is illustrated in Figure 3 (page 30). The unit circle represents frequency in the Fourier domain. The frequency response of a filter is just a rotating vector on this circle. The magnitude response of the filter is just then a vector sum. For example, the magnitude of any $H(z)$ is as follows:

$$|H(z)| = \frac{|(z - z_1)| |(z - z_2)|}{|(z - p_1)| |(z - p_2)|}$$

2.5.4 Filter Banks

Conjugate quadrature filters (next definition) are used in filter banks. If $\hat{x}(z)$ is a low-pass filter, then the conjugate quadrature filter of $\hat{x}(z)$ is a high-pass filter.

Definition 2.45 Let $(x_n)_{n \in \mathbb{Z}}$ and $(y_n)_{n \in \mathbb{Z}}$ be sequences (Definition 2.32 page 23) in $\ell^2_{\mathbb{R}}$ (Definition 2.33 page 23). The sequence $(y_n)$ is a conjugate quadrature filter with shift $N$ with respect to $$(7.2.7), (7.2.8), [123], [122], [98]$$
Figure 3: Vector response of digital filter

\[ y_n = \pm(-1)^n x_{N-n} \]

A conjugate quadrature filter is also called a CQF or a Smith-Barnwell filter. Any triple \((x_n, y_n, N)\) in this form is said to satisfy the conjugate quadrature filter condition or the CQF condition.

**Theorem 2.46**  
Let \( \hat{y}(\omega) \) and \( \hat{x}(\omega) \) be the DTFTs (Definition 2.38 page 24) of the sequences \((y_n)_{n \in \mathbb{Z}}\) and \((x_n)_{n \in \mathbb{Z}}\), respectively, in \( l^2(\mathbb{R}) \) (Definition 2.33 page 23).

\[
\begin{align*}
\left(1\right) \text{CQF in “time”} & \quad \iff \quad \hat{y}(z) = \pm (-1)^N z^{-N} \hat{x}(\frac{-1}{z^*}) \\
& \iff \quad \hat{x}(\omega) = \pm (-1)^N e^{-i\omega N} \hat{y}(\omega + \pi) \\
& \iff \quad x_n = \pm (-1)^N (-1)^n y_{N-n} \\
& \iff \quad \hat{x}(z) = \pm z^{-N} \hat{y}(\frac{-1}{z^*}) \\
& \iff \quad \hat{x}(\omega) = \pm e^{-i\omega N} \hat{y}(\omega + \pi)
\end{align*}
\]

80 [127], page 109, [95], pages 236–238, (7.58), (7.73), [57], pages 256–259, (section 4.5), [130], page 342, (7.2.7), (7.2.8)
3 Background: basis theory

3.1 Linear Combinations in Linear Spaces

**Definition 3.1** Let \( \{ x_n \in X \}_{n=1,2,\ldots,N} \) be a set of vectors in a linear space \( (X, +, \cdot, (F, \iva, \iva)) \).

\[ x = \sum_{n=1}^{N} a_n x_n \]
A vector \( \mathbf{x} \in \mathbf{X} \) is a **linear combination** of the vectors in \( \{ \mathbf{x}_n \} \) if there exists \( \{ \alpha_n \in \mathbb{F} \mid n=1,2,\ldots,N \} \) such that

\[
\mathbf{x} = \sum_{n=1}^{N} \alpha_n \mathbf{x}_n .
\]

**Definition 3.2** \(^{82}\) Let \( \{ \mathbf{x}_n \in \mathbf{X} \mid n=1,2,\ldots,N \} \) be a set of vectors in a linear space \( (\mathbf{X},+,\cdot,(\mathbb{F},+,\times)) \). Let \( A \) be a subset of \( \mathbf{X} \).

The **span** of \( \{ \mathbf{x}_n \} \) is defined as

\[
\text{span}(\mathbf{x}_n) \triangleq \left\{ \sum_{n=1}^{N} \alpha_n \mathbf{x}_n \mid \alpha_n \in \mathbb{F} \right\}.
\]

The set \( \{ \mathbf{x}_n \in \mathbf{X} \} \) **spans** the set \( A \) if \( A \subseteq \text{span}(\mathbf{x}_n) \).

**Definition 3.3** \(^{83}\) Let \( \{ \mathbf{x}_n \in \mathbf{X} \mid n=1,2,\ldots,N \} \) be a set of vectors in a linear space \( \mathbf{L} \triangleq (\mathbf{X},+,\cdot,(\mathbb{F},+,\times)) \).

The set \( \{ \mathbf{x}_n \in \mathbf{X} \mid n=1,2,\ldots,N \} \) is **linearly independent** in \( \mathbf{L} \) if

\[
\sum_{n=1}^{N} \alpha_n \mathbf{x}_n = 0 \quad \implies \quad \alpha_1 = \alpha_2 = \cdots = \alpha_N = 0 .
\]

If the vectors in \( \{ \mathbf{x}_n \} \) are not linearly independent, then they are **linearly dependent**.

An infinite set \( \{ \mathbf{x}_n \in \mathbf{X} \mid n\in\mathbb{N} \} \) is **linearly independent** if every finite subset \( \{ \mathbf{x}_{f(n)} \mid n=1,2,\ldots,N \} \) is **linearly independent**.

**Definition 3.4** \(^{84}\) Let \( \{ \mathbf{x}_n \in \mathbf{X} \mid n=1,2,\ldots,N \} \) be a set of vectors in a linear space \( \mathbf{L} \triangleq (\mathbf{X},+,\cdot,(\mathbb{F},+,\times)) \).

The set \( \{ \mathbf{x}_n \} \) is a **Hamel basis** (also called a **linear basis**) for \( \mathbf{L} \) if

1. \( \{ \mathbf{x}_n \} \) spans \( \mathbf{L} \) and
2. \( \{ \mathbf{x}_n \} \) is linearly independent.

If in addition \( \mathbf{x} = \sum_{n=1}^{N} \alpha_n \mathbf{x}_n \), then \( \sum_{n=1}^{N} \alpha_n \mathbf{x}_n \) is the **expansion** of \( \mathbf{x} \) on \( \{ \mathbf{x}_n \} \), and the elements of \( \{ \alpha_n \} \) are called the **coordinates** of \( \mathbf{x} \) with respect to \( \{ \mathbf{x}_n \} \).

If \( \alpha_N \neq 0 \), then \( N \) is the **dimension** \( \dim \mathbf{L} \) of \( \mathbf{L} \).

---

\(^{82}\) [97], page 86, (3.3.7 Definition), [86], page 44, [118], page 71, (Definition 3.2.5—more general definition)

\(^{83}\) [10], pages 3–4, [25], page 2, [63], page 156, (Definition 5.7)

\(^{84}\) [10], page 4, [84], pages 48–49, (Section 2.4), [139], page 1, [22], page 25, [63], page 125, (Definition 4.1), [60]
3.2 Total sets in Topological Linear Spaces

A linear space supports the concept of the span of a set of vectors (Definition 3.2 page 32). In a topological linear space \( \mathcal{O} \triangleq (X, +, \cdot, (F, +, \times), T) \), a set \( A \) is said to be total in \( \mathcal{O} \) if the span of \( A \) is dense in \( \mathcal{O} \). In this case, \( A \) is said to be a total set or a complete set. However, this use of “complete” in a “complete set” is not equivalent to the use of “complete” in a “complete metric space”. In this text, except for these comments and Definition 3.5, “complete” refers to the metric space definition only.

If a set is both total and linearly independent (Definition 3.3 page 32) in \( \mathcal{O} \), then that set is a Hamel basis (Definition 3.4 page 32).

Definition 3.5 Let \( A \) be the closure of a set \( A \) in a topological linear space \( \mathcal{O} \). Let \( \text{span}(A) \) be the span (Definition 3.2 page 32) of a set \( A \). A set of vectors \( A \) is total (or complete or fundamental) in \( \mathcal{O} \) if \( \text{span}(A) = \mathcal{O} \) (span of \( A \) is dense in \( \mathcal{O} \)).

3.3 Total sets in Banach spaces

Often in a linear space we have the option of appending additional structures that offer useful functionality. One of these structures is the norm (next definition). The norm of a vector can be described as the “length” or the “magnitude” of the vector.

Definition 3.6 Let \( (X, +, \cdot, (F, +, \times)) \) be a linear space and \(|\cdot| \in \mathbb{R}^F \) the absolute value function. A functional \(|\cdot|\|\) in \( \mathbb{R}^X \) is a norm if

1. \(|x|| \geq 0 \quad \forall x \in X \) (strictly positive) and
2. \(|x|| = 0 \iff x = 0 \quad \forall x \in X \) (nondegenerate) and
3. \(|\alpha x|| = |\alpha||x|| \quad \forall x \in X, \alpha \in \mathbb{C} \) (homogeneous) and
4. \(|x + y|| \leq |x|| + |y|| \quad \forall x, y \in X \) (subadditive triangle inequality).

A normed linear space is the tuple \( (X, +, \cdot, (F, +, \times), ||\cdot||) \).

Definition 3.7 Let \( B \triangleq (X, +, \cdot, (F, +, \times), ||\cdot||) \) be a Banach space. \( \triangleq \) represent strong convergence in \( B \). That is,
\( x \triangleq \sum_{n=1}^{\infty} \alpha_n x_n \quad \iff \quad \lim_{N \to \infty} \left\| x - \sum_{n=1}^{N} \alpha_n x_n \right\| = 0 \quad \iff \quad \left\{ \begin{array}{l}
for every \varepsilon > 0, \\
there exists M such that \\
for all N > M, \\
\left\| x - \sum_{n=1}^{N} \alpha_n x_n \right\| < \varepsilon 
\end{array} \right. \)

**Definition 3.8** 89 Let \( B \triangleq (X, +, \cdot, (F, \unicode{x2218}, \unicode{x2217}), \| \cdot \|) \) be a Banach space. Let \( \triangleq \) represent strong convergence (Definition 3.7 page 33) in \( B \).

The countable set \( \{ x_n \in X | n \in \mathbb{N} \} \) is a Schauder basis for \( B \) if for each \( x \in X \)

1. \( \exists (\alpha_n \in F)_{n \in \mathbb{N}} \) such that \( x \triangleq \sum_{n=1}^{\infty} \alpha_n x_n \) (strong convergence in \( B \)) and

2. \( \left\{ \sum_{n=1}^{\infty} \alpha_n x_n \triangleq \sum_{n=1}^{\infty} \beta_n x_n \right\} \implies \{ \langle \alpha_n \rangle = \langle \beta_n \rangle \} \) (coefficient functionals are unique)

In this case, \( \sum_{n=1}^{\infty} \alpha_n x_n \) is the expansion of \( x \) on \( \{ x_n | n \in \mathbb{N} \} \) and the elements of \( \{ \alpha_n \} \) are the coefficient functionals associated with the basis \( \{ x_n \} \).

Coefficient functionals are also called coordinate functionals.

**Definition 3.9** 90 Let \( \{ x_n | n \in \mathbb{N} \} \) and \( \{ y_n | n \in \mathbb{N} \} \) be Schauder bases of a Banach space \( (X, +, \cdot, (F, \unicode{x2218}, \unicode{x2217}), \| \cdot \|) \). \( \{ x_n \} \) is equivalent to \( \{ y_n \} \) if there exists a bounded invertible operator \( R \) in \( X^X \) such that \( Rx_n = y_n \quad \forall n \in \mathbb{Z} \)

**Theorem 3.10** 91 Let \( \{ x_n | n \in \mathbb{N} \} \) and \( \{ y_n | n \in \mathbb{N} \} \) be Schauder bases of a Banach space \( (X, +, \cdot, (F, \unicode{x2218}, \unicode{x2217}), \| \cdot \|) \). \( \{ x_n \} \) is equivalent to \( \{ y_n \} \)

\[ \iff \left\{ \begin{array}{l}
\sum_{n=1}^{\infty} \alpha_n x_n \text{ is convergent} \iff \sum_{n=1}^{\infty} \alpha_n y_n \text{ is convergent}
\end{array} \right. \]
3.4 Total sets in Hilbert spaces

3.4.1 Orthogonal sets in Inner product space

Definition 3.11 Let \( L \triangleq (X, +, \cdot, (\mathbb{F}, \oplus, \otimes)) \) be a linear space. A function \( \langle \triangle | \nabla \rangle \in \mathbb{F}^{X \times X} \) is an **inner product** on \( L \) if

1. \( \langle \alpha x | y \rangle = \alpha \langle x | y \rangle \quad \forall x, y \in X, \forall \alpha \in \mathbb{C} \) (homogeneous) and
2. \( \langle x + y | u \rangle = \langle x | u \rangle + \langle y | u \rangle \quad \forall x, y, u \in X \) (additive) and
3. \( \langle x | y \rangle = \langle y | x \rangle^* \quad \forall x, y \in X \) (conjugate symmetric) and
4. \( \langle x | x \rangle \geq 0 \quad \forall x \in X \) (non-negative) and
5. \( \langle x | x \rangle = 0 \iff x = 0 \quad \forall x \in X \) (non-isotropic)

An inner product is also called a **scalar product**.

The tuple \( (X, +, \cdot, (\mathbb{F}, \oplus, \otimes), \langle \triangle | \nabla \rangle) \) is an **inner product space**.

Definition 3.12

The **Kronecker delta function** \( \delta_n \) is defined as

\[
\delta_n \triangleq \begin{cases} 
1 & \text{for } n = 0 \\
0 & \text{for } n \neq 0 
\end{cases} \quad \forall n \in \mathbb{Z}
\]

Definition 3.13 Let \( (X, +, \cdot, (\mathbb{F}, \oplus, \otimes), \langle \triangle | \nabla \rangle) \) be an inner product space (Definition 3.11 page 35). Two vectors \( x \) and \( y \) in \( X \) are **orthogonal** if

\[
\langle x | y \rangle = \begin{cases} 
0 & \text{for } x \neq y \\
c \in \mathbb{C} \setminus \{0\} & \text{for } x = y
\end{cases}
\]

The notation \( x \perp y \) implies \( x \) and \( y \) are orthogonal. A set \( Y \subseteq 2^X \) is **orthogonal** if \( x \perp y \quad \forall x, y \in Y \). A set \( Y \) is **orthonormal** if it is orthogonal and \( \langle y | y \rangle = 1 \quad \forall y \in Y \). A sequence \( \langle x_n \in X \rangle_{n \in \mathbb{Z}} \) is **orthogonal** if \( \langle x_n | x_m \rangle = c \delta_{nm} \) for some \( c \in \mathbb{R} \setminus \{0\} \). A sequence \( \langle x_n \in X \rangle_{n \in \mathbb{Z}} \) is **orthonormal** if \( \langle x_n | x_m \rangle = \delta_{nm} \).

3.4.2 Orthonormal bases in Hilbert spaces

Definition 3.14 Let \( \{x_n \in X | n = 1, 2, \ldots, N\} \) be a set of vectors in an inner product space (Definition 3.11 page 35) \( \Omega \triangleq (X, +, \cdot, (\mathbb{F}, \oplus, \otimes), \langle \triangle | \nabla \rangle) \).

The set \( \{x_n\} \) is an **orthogonal basis** for \( \Omega \) if \( \{x_n\} \) is orthogonal and is a Schauder basis for \( \Omega \).

The set \( \{x_n\} \) is an **orthonormal basis** for \( \Omega \) if \( \{x_n\} \) is orthonormal and is a Schauder basis for \( \Omega \).
Definition 3.15  Let $H \triangleq (X, +, \cdot, (\mathbb{F}, +, \cdot), \langle \triangledown | \triangledown \rangle)$ be a Hilbert space.

Suppose there exists a set $\{x_n \in X \mid n \in \mathbb{N}\}$ such that $x \triangleq \sum_{n=1}^{\infty} \langle x | x_n \rangle x_n$.

Then the quantities $\langle x | x_n \rangle$ are called the Fourier coefficients of $x$ and the sum $\sum_{n=1}^{\infty} \langle x | x_n \rangle x_n$ is called the Fourier expansion of $x$ or the Fourier series for $x$.

Theorem 3.16  (The Fourier Series Theorem)  Let $\{x_n \in X \mid n \in \mathbb{N}\}$ be a set of vectors in a HILBERT SPACE $H \triangleq (X, +, \cdot, (\mathbb{F}, +, \cdot), \langle \triangledown | \triangledown \rangle)$ and let $\|x\| \triangleq \sqrt{\langle x | x \rangle}$. (A) $\{x_n\}$ is ORTHONORMAL in $H$ $\iff$

\[
\begin{cases}
(1). \quad \langle x_n | x \rangle = H \\ (\{x_n\} is total in H) \\
(2). \quad \langle x | y \rangle = \sum_{n=1}^{\infty} \langle x | x_n \rangle \langle y | x_n \rangle^* \quad \forall x, y \in X \quad (\text{GENERALIZED PARSEVAL'S IDENTITY}) \\
(3). \quad \|x\|^2 \triangleq \sum_{n=1}^{\infty} |\langle x | x_n \rangle|^2 \quad \forall x \in X \quad (\text{PARSEVAL'S IDENTITY}) \\
(4). \quad x = \sum_{n=1}^{\infty} \langle x | x_n \rangle x_n \quad \forall x \in X \quad (\text{FOURIER SERIES EXPANSION})
\end{cases}
\]

Theorem 3.17  Let $H$ be a HILBERT SPACE. $H$ has a SCHAUDER BASIS $\iff$ $H$ is SEPARABLE

Theorem 3.18  Let $H$ be a HILBERT SPACE. $H$ has an ORTHONORMAL BASIS $\iff$ $H$ is SEPARABLE

3.5 Riesz bases in Hilbert space

Definition 3.19  Let $\{x_n \in X \mid n \in \mathbb{N}\}$ be a set of vectors in a separable Hilbert space $H \triangleq (X, +, \cdot, (\mathbb{F}, +, \cdot), \langle \triangledown | \triangledown \rangle)$. $\{x_n\}$ is a Riesz basis for $H$ if $\{x_n\}$ is equivalent (Definition 3.9 page 34) to some orthonormal basis (Definition 3.14 page 35) in $H$.  

\[\begin{align*}
&93 \quad \text{[35], page 27, (Theorem 1.55)}, \quad \text{[139], page 6}, \quad \text{[138], page 6} \\
&94 \quad \text{[10], pages 149–155, (Theorem 9.12)}, \quad \text{[84], pages 360–363, (Theorem 5.48)}, \quad \text{[3], pages 298–299, (Theorem 34.2)}, \quad \text{[25], page 57, (Theorem 34.2)}, \quad \text{[16], pages 52–53, (Theorem II§8.3)}, \quad \text{[63], pages 34–35, (Theorem 1.50)} \\
&95 \quad \text{[11], page 112, (3.4.8)}, \quad \text{[16], page 53, (Theorem II§8.3)} \\
&96 \quad \text{[84], page 357, (Proposition 5.43)} \\
&97 \quad \text{[139], page 27, (Definition 1.8.2)}, \quad \text{[25], page 63, (Definition 3.6.1)}, \quad \text{[63], page 196, (Definition 7.9)}
\]
**Definition 3.20** Let \( \{ x_n \in X \}_{n \in \mathbb{N}} \) be a sequence of vectors in a *separable Hilbert space* \( H \triangleq (X, +, \cdot, (F, +, \triangle), \langle \Delta | \nabla \rangle) \). The sequence \( \{ x_n \} \) is a **Riesz sequence** for \( H \) if

\[
\exists A, B \in \mathbb{R}^+ \text{ such that } A \left( \sum_{n=1}^{\infty} |\alpha_n|^2 \right) \leq \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\|^2 \leq B \left( \sum_{n=1}^{\infty} |\alpha_n|^2 \right) \quad \forall (a_n) \in c_0^2.
\]

**Lemma 3.21** Let \( \{ x_n | n \in \mathbb{N} \} \) be a sequence in a *HILBERT SPACE* \( X \triangleq (X, +, \cdot, (F, +, \triangle), \langle \Delta | \nabla \rangle) \). Let \( \{ y_n | n \in \mathbb{N} \} \) be a sequence in a *HILBERT SPACE* \( Y \triangleq (Y, +, \cdot, (F, +, \triangle), \langle \Delta | \nabla \rangle) \). Let

\[
\begin{cases}
(i). \quad \{ x_n \} \text{ is TOTAL in } X \\
(ii). \quad \text{There exists } A > 0 \text{ such that } A \sum_{n \in C} |a_n|^2 \leq \left\| \sum_{n \in C} a_n x_n \right\|^2 \text{ for finite } C \\
(iii). \quad \text{There exists } B > 0 \text{ such that } \left\| \sum_{n=1}^{\infty} b_n y_n \right\|^2 \leq B \sum_{n=1}^{\infty} |b_n|^2 \quad \forall (b_n) \in c_0^2
\end{cases}
\]

\[
(1). \quad R^* \text{ is a linear bounded operator that maps from } \overline{\text{span}} \{ x_n \} \text{ to } \overline{\text{span}} \{ y_n \} \\
(2). \quad \|R^*\| \leq \frac{B}{A} \text{ and }
(3). \quad \|R\| \leq \frac{B}{A}
\]

**Theorem 3.22** Let \( \{ x_n \in X | n \in \mathbb{N} \} \) be a set of vectors in a *separable Hilbert space* \( H \triangleq (X, +, \cdot, (F, +, \triangle), \langle \Delta | \nabla \rangle) \).

\[
\begin{cases}
\{ x_n \} \text{ is a Riesz basis for } H \\
\exists A, B \in \mathbb{R}^+ \text{ such that } A \sum_{n=1}^{\infty} |\alpha_n|^2 \leq \left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\|^2 \leq B \sum_{n=1}^{\infty} |\alpha_n|^2 \quad \forall (a_n) \in c_0^2
\end{cases}
\]

**Theorem 3.23** Let \( H \triangleq (X, +, \cdot, (F, +, \triangle), \langle \Delta | \nabla \rangle) \) be a separable Hilbert space.
4 MAIN RESULTS: TRANSVERSAL OPERATORS

Daniel J. Greenhoe

{\{x_n \in H \}_{n \in \mathbb{Z}} is a RIESZ BASIS for H } \implies

\[
\begin{align*}
\text{There exists } & \{y_n \in H \}_{n \in \mathbb{Z}} \text{ such that} \\
& \begin{cases}
(1). \quad \{x_n\} \text{ and } \{y_n\} \text{ are BIORTHOGONAL and} \\
(2). \quad \{y_n\} \text{ is also a RIESZ BASIS for } H \text{ and} \\
(3). \quad \exists B > A > 0 \text{ such that}
\end{cases}
\end{align*}
\]

\[
\begin{align*}
& A \sum_{n=1}^{\infty} |a_n|^2 \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\|^2 = \left\| \sum_{n=1}^{\infty} a_n y_n \right\|^2 \leq B \sum_{n=1}^{\infty} |a_n|^2 \\
& \forall \{a_n\} \in \ell^2
\end{align*}
\]

Proposition 3.24 102 Let \{x_n\}_{n \in \mathbb{N}} be a set of vectors in a HILBERT SPACE \( H \triangleq (X, +, \cdot, (F, +, \times), \langle \triangle | \nabla \rangle) \).

\[
\begin{align*}
\{x_n\} \text{ is a RIESZ BASIS for } H \text{ with} \\
& A \sum_{n=1}^{\infty} |a_n|^2 \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\|^2 \leq B \sum_{n=1}^{\infty} |a_n|^2 \\
& \forall \{a_n\} \in \ell^2
\end{align*}
\]

\[
\begin{align*}
\{x_n\} \text{ is a FRAME for } H \text{ with} \\
& 1/B \|x\|^2 \leq \left\| \sum_{n=1}^{\infty} \langle x | x_n \rangle \right\|^2 \leq 1/A \|x\|^2 \\
& \forall x \in H
\end{align*}
\]

4 Main results: transversal operators

4.1 Definitions

The transversal operators \( T \) and \( D \) (next definition) are the main focus of this paper.

Definition 4.1 103

1. \( T \) is the translation operator on \( \mathbb{C}^\mathbb{C} \) defined as
   \[ T_\tau f(x) \triangleq f(x - \tau) \text{ and } T \triangleq T_1 \forall \tau \in \mathbb{C} \]

2. \( D \) is the dilation operator on \( \mathbb{C}^\mathbb{C} \) defined as
   \[ D_\alpha f(x) \triangleq f(\alpha x) \text{ and } D \triangleq \sqrt{2} D_2 \forall \alpha \in \mathbb{C} \]
4.2  Properties

4.2.1  Algebraic properties

Proposition 4.2  Let $T$ be the translation operator (Definition 4.1 page 38).

$$\sum_{n \in \mathbb{Z}} T^n f(x) = \sum_{n \in \mathbb{Z}} T^n f(x+1) \quad \forall f \in \mathbb{R}^\mathbb{R} \quad \left( \sum_{n \in \mathbb{Z}} T^n f(x) \text{ is periodic with period 1} \right)$$

Proof:

$$\sum_{n \in \mathbb{Z}} T^n f(x+1) = \sum_{n \in \mathbb{Z}} f(x-n + 1) \quad \text{by definition of } T \text{ Definition 4.1 page 38}$$

$$= \sum_{m \in \mathbb{Z}} f(x-m) \quad \text{where } m \equiv n-1$$

$$= \sum_{m \in \mathbb{Z}} T^m f(x) \quad \text{by definition of } T \text{ Definition 4.1 page 38}$$

In a linear space, every operator has an inverse. Although the inverse always exists as a relation, it may not exist as a function or as an operator. But in some cases the inverse of an operator is itself an operator. The inverses of the operators $T$ and $D$ both exists as operators, as demonstrated by Proposition 4.3 (next).

Proposition 4.3  Let $T$ and $D$ be as defined in Definition 4.1 page 38.

$T$ has an inverse $T^{-1}$ in $\mathbb{C}^\mathbb{C}$ expressed by the relation

$$T^{-1} f(x) = f(x + 1) \quad \forall f \in \mathbb{C}^\mathbb{C} \quad \text{translation operator inverse}.$$  

$D$ has an inverse $D^{-1}$ in $\mathbb{C}^\mathbb{C}$ expressed by the relation

$$D^{-1} f(x) = \sqrt{2} f\left(\frac{1}{2} x\right) \quad \forall f \in \mathbb{C}^\mathbb{C} \quad \text{dilation operator inverse}.$$  

Proof:

1. Proof that $T^{-1}$ is the inverse of $T$:

$$T^{-1} T f(x) = T^{-1} f(x-1) \quad \text{by Proposition 4.3 page 39}$$

$$= f([x + 1] - 1)$$

$$= f(x)$$

$$= f([x - 1] + 1)$$

$$= T f(x + 1) \quad \text{by definition of } T \text{ (Definition 4.1 page 38)}$$

$$= T T^{-1} f(x)$$

$$\implies T^{-1} T = I = T T^{-1}$$

\[\text{[53], page 3}\]

\[\text{[53], page 3}\]
2. Proof that $D^{-1}$ is the inverse of $D$:

\[
D^{-1}Df(x) = D^{-1}\sqrt{2}f(2x) = \left(\frac{\sqrt{2}}{2}\right)\sqrt{2}f\left(\frac{1}{2}x\right) = f(x) = \sqrt{2}\left[\sqrt{2}f\left(\frac{1}{2}2x\right)\right] = D\left[\frac{\sqrt{2}}{2}f\left(\frac{1}{2}x\right)\right] = DD^{-1}f(x)
\]

$\implies D^{-1}D = I = DD^{-1}$

**Proposition 4.4** Let $T$ and $D$ be as defined in Definition 4.1 page 38. Let $D^0 = T^0 \triangleq I$ be the identity operator.

$D^jT^nf(x) = 2^{j/2}f(2^jx - n)$ \hspace{1em} $\forall j, n \in \mathbb{Z}, f \in \mathbb{C}$

### 4.2.2 Linear space properties

**Definition 4.5** Let $+$ be an addition operator on a tuple $\langle x_n \rangle^N_m$.

The **summation** of $\langle x_n \rangle$ from index $m$ to index $N$ with respect to $+$ is

\[
\sum_{n=m}^{N} x_n \triangleq \begin{cases} 
0 & \text{for } N < m \\
\left(\sum_{n=m}^{N-1} x_n\right) + x_N & \text{for } N \geq m
\end{cases}
\]

An infinite summation $\sum_{n=1}^{\infty} \phi_n$ is meaningless outside some topological space (e.g. metric space, normed space, etc.). The sum $\sum_{n=1}^{\infty} \phi_n$ is an abbreviation for $\lim_{N \to \infty} \sum_{n=1}^{N} \phi_n$ (the limit of partial sums). And the concept of limit is also itself meaningless outside of a topological space.

**Definition 4.6** Let $(X, T)$ be a topological space and $\lim$ be the limit induced by the topology $T$.
\[ \sum_{n=1}^{\infty} x_n \triangleq \sum_{n \in \mathbb{N}} x_n \triangleq \lim_{N \to \infty} \sum_{n=1}^{N} x_n \]

\[ \sum_{n=-\infty}^{\infty} x_n \triangleq \sum_{n \in \mathbb{Z}} x_n \triangleq \lim_{N \to \infty} \left( \sum_{n=0}^{N} x_n \right) + \left( \lim_{N \to -\infty} \sum_{n=-1}^{N} x_n \right) \]

In general the operators \( T \) and \( D \) are noncommutative (\( TD \neq DT \)), as demonstrated by Proposition 4.8 and by the following illustration.

**Proposition 4.7** Let \( T \) and \( D \) be as in Definition 4.1 page 38.

\[ D^{j} T^{n} \left[ f \right] = 2^{-j/2} \left[ D^{j} T^{n} \right] \left[ D^{j} T^{n} \right] \quad \forall j, n \in \mathbb{Z}, f \in \mathbb{C} \]

\[ \text{PROOF:} \]

\[ D^{j} T^{n} \left[ f(x)g(x) \right] = 2^{j/2} f(2^{j} x - n) g(2^{j} x - n) \quad \text{by Proposition 4.4 page 40} \]

\[ = 2^{-j/2} \left[ 2^{j/2} f(2^{j} x - n) \right] \left[ 2^{j/2} g(2^{j} x - n) \right] \]

\[ = 2^{-j/2} \left[ D^{j} T^{n} f(x) \right] \left[ D^{j} T^{n} g(x) \right] \quad \text{by Proposition 4.4 page 40} \]

\[ \text{PROOF:} \]

\[ D^{j} T^{2j/2n} f(x) = 2^{j/2} f(2^{j} x - 2^{j} n) \quad \text{by Proposition 4.7 page 41} \]

\[ = 2^{j/2} f(2^{j} [x - n]) \quad \text{by distributivity property of the field (\( \mathbb{R}, +, \times \))} \]

\[ = T^{n} 2^{j/2} f(2^{j} x) \quad \text{by definition of } T \text{ (Definition 4.1 page 38)} \]

\[ = T^{n} D^{j} f(x) \quad \text{by definition of } D \text{ (Definition 4.1 page 38)} \]

\[ D^{j} T^{n} f(x) = 2^{j/2} f(2^{j} x - n) \quad \text{by Proposition 4.7 page 41} \]

\[ = 2^{j/2} f(2^{j} [x - 2^{-j/2} n]) \quad \text{by distributivity property of the field (\( \mathbb{R}, +, \times \))} \]

\[ = T^{2^{-j/2} n} 2^{j/2} f(2^{j} x) \quad \text{by definition of } T \text{ (Definition 4.1 page 38)} \]

\[ = T^{2^{-j/2} n} D^{j} f(x) \quad \text{by definition of } D \text{ (Definition 4.1 page 38)} \]

**Proposition 4.8** (commutator relation) Let \( T \) and \( D \) be as in Definition 4.1 page 38.

\[ D^{j} T^{n} = T^{2^{-j/2} n} \quad \forall j, n \in \mathbb{Z} \]

\[ T^{j} D^{n} = D^{j} T^{2^{j} n} \quad \forall n, j \in \mathbb{Z} \]

\[ \text{PROOF:} \]

\[ D^{j} T^{2^{j} n} f(x) = 2^{j/2} f(2^{j} x) \quad \text{by Proposition 4.7 page 41} \]

\[ = 2^{j/2} f(2^{j} x) \quad \text{by definition of } T \text{ (Definition 4.1 page 38)} \]

\[ = T^{2^{j} n} 2^{j/2} f(2^{j} x) \quad \text{by definition of } D \text{ (Definition 4.1 page 38)} \]

\[ D^{j} T^{2^{j} n} f(x) = 2^{j/2} f(2^{j} x - 2^{j} n) \quad \text{by Proposition 4.7 page 41} \]

\[ = 2^{j/2} f(2^{j} x - 2^{j} n) \quad \text{by distributivity property of the field (\( \mathbb{R}, +, \times \))} \]

\[ = T^{2^{-j/2} n} 2^{j/2} f(2^{j} x) \quad \text{by definition of } T \text{ (Definition 4.1 page 38)} \]

\[ = T^{2^{-j/2} n} D^{j} f(x) \quad \text{by definition of } D \text{ (Definition 4.1 page 38)} \]
4.2.3 Inner-product space properties

In an inner product space, every operator has an *adjoint* (Proposition 1.33 page 10) and this adjoint is always itself an operator. In the case where the adjoint and inverse of an operator \( U \) coincide, then \( U \) is said to be *unitary* (Definition 1.48 page 13). And in this case, \( U \) has several nice properties (see Proposition 4.14 and Theorem 4.15 page 45). Proposition 4.9 (next) gives the adjoints of \( D \) and \( T \), and Proposition 4.10 (page 43) demonstrates that both \( D \) and \( T \) are unitary. Other examples of unitary operators include the *Fourier Transform operator* \( \tilde{F} \) and the *rotation matrix operator*.

**Proposition 4.9** Let \( T \) be the translation operator (Definition 4.1 page 38) with adjoint \( T^* \) and \( D \) the dilation operator with adjoint \( D^* \).

\[
T^*f(x) = f(x + 1) \quad \forall f \in L^2_\mathbb{R} \quad \text{(translation operator adjoint)}
\]

\[
D^*f(x) = \sqrt{2} f \left( \frac{x}{2} \right) \quad \forall f \in L^2_\mathbb{R} \quad \text{(dilation operator adjoint)}
\]

**Proof:**

1. Proof that \( T^*f(x) = f(x + 1) \):

\[
\langle g(x) | T^*f(x) \rangle = \langle g(u) | T^*f(u) \rangle \quad \text{by change of dummy variable } x \to u
\]

\[
= \langle Tg(u) | f(u) \rangle \quad \text{by definition of adjoint } T^*
\]

\[
= \langle g(u - 1) | f(u) \rangle \quad \text{by definition of } T \text{ (Definition 4.1 page 38)}
\]

\[
= \langle g(x) | f(x + 1) \rangle \quad \text{where } x = u - 1 \implies u = x + 1
\]

\[
\implies T^*f(x) = f(x + 1)
\]

2. Proof that \( D^*f(x) = \sqrt{2} f \left( \frac{x}{2} \right) \):

\[
\langle g(x) | D^*f(x) \rangle = \langle g(u) | D^*f(u) \rangle \quad \text{by change of dummy variable } t \to u
\]

\[
= \langle Dg(u) | f(u) \rangle \quad \text{by definition of } D^*
\]

\[
= \left\langle \sqrt{2} g(2u) | f(u) \right\rangle \quad \text{by definition of } D \text{ (Definition 4.1 page 38)}
\]

\[
= \int_{u \in \mathbb{R}} \sqrt{2} g(2u) f^*(u) \, du \quad \text{by definition of } \langle \triangle | \nabla \rangle
\]

\[
= \int_{x \in \mathbb{R}} g(x) \left[ \sqrt{2} f \left( \frac{x}{2} \right) \right]^* \, dx \quad \text{where } x = 2u
\]

\[
= \left\langle g(x) | \sqrt{2} f \left( \frac{x}{2} \right) \right\rangle \quad \text{by definition of } \langle \triangle | \nabla \rangle
\]

\[
\implies D^*f(x) = \sqrt{2} f \left( \frac{x}{2} \right)
\]
**Proposition 4.10**  
Let $T$ and $D$ be as in Definition 4.1 page 38. Let $T^{-1}$ and $D^{-1}$ be as in Proposition 4.3 page 39.

- $T$ is unitary in $L^2_{\mathbb{R}}$ ($T^{-1} = T^*$ in $L^2_{\mathbb{R}}$).
- $D$ is unitary in $L^2_{\mathbb{R}}$ ($D^{-1} = D^*$ in $L^2_{\mathbb{R}}$).

**Proof:**

$T^{-1} = T^*$ by Proposition 4.3 page 39 and Proposition 4.9 page 42  
$\implies$ $T$ is unitary by the definition of unitary operators (Definition 1.48 page 13)

$D^{-1} = D^*$ by Proposition 4.3 page 39 and Proposition 4.9 page 42  
$\implies$ $D$ is unitary by the definition of unitary operators (Definition 1.48 page 13)

---

### 4.2.4 Normed linear space properties

**Proposition 4.11**  
Let $D$ be the dilation operator (Definition 4.1 page 38).

$$
\begin{cases}
(1). & Df(x) = \sqrt{2}f(2x) \\
(2). & f(x) \text{ is continuous}
\end{cases}
\quad \iff \quad \{ f(x) \text{ is a constant} \} \quad \forall f \in L^2_{\mathbb{R}}
$$

**Proof:**

1. Proof that (1) $\iff$ constant property:

   $$
   Df(x) = \sqrt{2}f(2x)
   \quad \text{by definition of } D \quad \text{(Definition 4.1 page 38)}
   $$
   $$
   = \sqrt{2}f(x)
   \quad \text{by constant hypothesis}
   $$

2. Proof that (2) $\iff$ constant property:

   $$
   \|f(x) - f(x + h)\| = \|f(x) - f(x)\| 
   \quad \text{by constant hypothesis}
   $$
   $$
   = \|0\|
   \quad \text{by nondegenerate property of } \|\cdot\| \quad \text{(Definition 3.6 page 33)}
   $$
   $$
   = 0
   \quad \text{by constant hypothesis}
   $$
   $$
   \leq \varepsilon
   \quad \text{by nondegenerate property of } \|\cdot\| \quad \text{(Definition 3.6 page 33)}
   $$
   $$
   \implies \forall h > 0, \exists \varepsilon \quad \text{such that } \|f(x) - f(x + h)\| < \varepsilon
   $$
   $$
   \iff f(x) \text{ is continuous}
   $$

3. Proof that (1,2) $\iff$ constant property:
(a) Suppose there exists \( x, y \in \mathbb{R} \) such that \( f(x) \neq f(y) \).

(b) Let \( \langle x_n \rangle_{n \in \mathbb{N}} \) be a sequence with limit \( x \) and \( \langle y_n \rangle_{n \in \mathbb{N}} \) a sequence with limit \( y \).

(c) Then

\[
0 < ||f(x) - f(y)|| \quad \text{by assumption in item 3a page 44}
= \lim_{n \to \infty} ||f(x_n) - f(y_n)|| \quad \text{by (2) and definition of \( \langle x_n \rangle \) and \( \langle y_n \rangle \) (item 3b page 44)}
= \lim_{n \to \infty} ||f(2^m x_n) - f(2^\ell y_n)|| \quad \forall m, \ell \in \mathbb{Z}; \text{by (1)}
= 0
\]

(d) But this is a contradiction, so \( f(x) = f(y) \) for all \( x, y \in \mathbb{R} \), and \( f(x) \) is constant.

Note that in Proposition 4.11, it is not possible to remove the continuous constraint outright (next two counterexamples).

**Counterexample 4.12** Let \( f(x) \) be a function in \( \mathbb{R}^R \).

Let \( f(x) \triangleq \begin{cases} 
0 & \text{for } x = 0 \\
1 & \text{otherwise.}
\end{cases} \)

Then \( Df(x) \triangleq \sqrt{2f(2x)} = \sqrt{2f(x)} \), but \( f(x) \) is not constant.

**Counterexample 4.13** Let \( f(x) \) be a function in \( \mathbb{R}^R \). Let \( \mathbb{Q} \) be the set of rational numbers and \( \mathbb{R} \setminus \mathbb{Q} \) the set of irrational numbers.

Let \( f(x) \triangleq \begin{cases} 
1 & \text{for } x \in \mathbb{Q} \\
-1 & \text{for } x \in \mathbb{R} \setminus \mathbb{Q}.
\end{cases} \)

Then \( Df(x) \triangleq \sqrt{2f(2x)} = \sqrt{2f(x)} \), but \( f(x) \) is not constant.

**Proposition 4.14** (Operator norm) Let \( T \) and \( D \) be as in Definition 4.1 page 38. Let \( T^{-1} \) and \( D^{-1} \) be as in Proposition 4.3 page 39. Let \( T^* \) and \( D^* \) be as in Proposition 4.9 page 42. Let \( ||\cdot|| \) and \( \langle \cdot | \cdot \rangle \) be as in Definition 2.2 page 15. Let \( ||\cdot|| \) be the operator norm (Definition 1.22 page 8) induced by \( ||\cdot|| \).

\[
||T|| = ||D|| = ||T^*|| = ||D^*|| = ||T^{-1}|| = ||D^{-1}|| = 1
\]

PROOF: These results follow directly from the fact that \( T \) and \( D \) are unitary (Proposition 4.10 page 43) and from Theorem 1.50 page 14 and Theorem 1.51 page 14.
Theorem 4.15  Let $T$ and $D$ be as in Definition 4.1 page 38. Let $T^{-1}$ and $D^{-1}$ be as in Proposition 4.3 page 39. Let $\|\|$ and $\langle \triangle | \nabla \rangle$ be as in Definition 2.2 page 15.

1. $\|Tf\| = \|Df\| = \|f\| \forall f \in L^2_\mathbb{R}$ (ISOMETRIC IN LENGTH)
2. $\|Tf - Tg\| = \|Df - Dg\| = \|f - g\| \forall f, g \in L^2_\mathbb{R}$ (ISOMETRIC IN DISTANCE)
3. $\|T^{-1}f - T^{-1}g\| = \|D^{-1}f - D^{-1}g\| = \|f - g\| \forall f, g \in L^2_\mathbb{R}$ (ISOMETRIC IN DISTANCE)
4. $\langle Tf | Tg \rangle = \langle Df | Dg \rangle = \langle f | g \rangle \forall f, g \in L^2_\mathbb{R}$ (SUBJECTIVE)
5. $\langle T^{-1}f | T^{-1}g \rangle = \langle D^{-1}f | D^{-1}g \rangle = \langle f | g \rangle \forall f, g \in L^2_\mathbb{R}$ (SUBJECTIVE)

✎ PROOF: These results follow directly from the fact that $T$ and $D$ are unitary (Proposition 4.10 page 43) and from Theorem 1.50 page 14 and Theorem 1.51 page 14.

Proposition 4.16  Let $T$ be as in Definition 4.1 page 38. Let $A^*$ be the adjoint (Proposition 1.33 page 10) of an operator $A$.

$$\left( \sum_{n \in \mathbb{Z}} T^n \right) = \left( \sum_{n \in \mathbb{Z}} T^n \right)^* \quad \text{(The operator } \left[ \sum_{n \in \mathbb{Z}} T^n \right] \text{ is SELF-ADJOINT)}$$

✎ PROOF:

$$\left\langle \left( \sum_{n \in \mathbb{Z}} T^n \right) f(x) \mid g(x) \right\rangle = \left\langle \sum_{n \in \mathbb{Z}} f(x - n) \mid g(x) \right\rangle \quad \text{by definition of } T \text{ (Definition 4.1 page 38)}$$

$$= \sum_{n \in \mathbb{Z}} \left\langle f(x + n) \mid g(x) \right\rangle \quad \text{by commutative property of addition}$$

$$= \sum_{n \in \mathbb{Z}} \left\langle f(u) \mid g(u - n) \right\rangle \quad \text{where } u = x + n$$

$$= \sum_{n \in \mathbb{Z}} \left\langle f(u) \mid g(u - n) \right\rangle \quad \text{by additive property of } \langle \triangle | \nabla \rangle$$

$$= \sum_{n \in \mathbb{Z}} \left\langle f(x) \mid g(x - n) \right\rangle \quad \text{by change of dummy variable: } t \to u$$

$$= \left\langle f(x) \mid \sum_{n \in \mathbb{Z}} T^n g(x) \right\rangle \quad \text{by definition of } T \text{ (Definition 4.1 page 38)}$$

$$\iff \left( \sum_{n \in \mathbb{Z}} T^n \right) = \left( \sum_{n \in \mathbb{Z}} T^n \right)^* \quad \text{by definition of operator adjoint (page 10)}$$

$$\iff \left( \sum_{n \in \mathbb{Z}} T^n \right) \text{ is self-adjoint} \quad \text{by definition of self-adjoint (Definition 1.37 page 11)}$$

Monday 13th October, 2014  ⚡ Properties and applications of transversal operators  ⚡
4.2.5 Fourier transform properties

Proposition 4.17  Let \( T \) and \( D \) be as in Definition 4.1 page 38. Let \( B \) be the \text{TWO-SIDED LAPLACE TRANSFORM} defined as

\[
[BF](s) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-sx} \, dx.
\]

1. \( BT^n = e^{-sn}B \quad \forall n \in \mathbb{Z} \)
2. \( BD^{-j} = D^{-j}B \quad \forall j \in \mathbb{Z} \)
3. \( DB = BD^{-1} \quad \forall n \in \mathbb{Z} \)
4. \( BD^{-1}B^{-1} = B^{-1}D^{-1}B = D \quad \forall n \in \mathbb{Z} \) (\( D^{-1} \) is \text{SIMILAR} to \( D \))
5. \( DBD = D^{-1}BD^{-1} = B \quad \forall n \in \mathbb{Z} \)

\( \Box \)\text{P}ROOF:

\[
BT^n f(x) = BF(x - n) \quad \text{by definition of } T \text{ (Definition 4.1 page 38)}
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x-n)e^{-sx} \, dx \quad \text{by definition of } B
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u)e^{-s(u+n)} \, du \quad \text{where } u \triangleq x - n
\]

\[
= e^{-sn} \left[ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u)e^{-su} \, du \right] \quad \text{by definition of } B
\]

\[
= e^{-sn} BF(x) \quad \text{by definition of } B
\]

\[
BD^{-j} f(x) = B \left[ 2^{j/2} f \left( \frac{2}{j} x \right) \right] \quad \text{by definition of } D \text{ (Definition 4.1 page 38)}
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[ 2^{j/2} f \left( \frac{2}{j} x \right) \right] e^{-sx} \, dx \quad \text{by definition of } B
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[ 2^{j/2} f(u) \right] e^{-s\frac{2}{j}u} \, du \quad \text{let } u \triangleq \frac{2}{j} x \implies x = 2^{-j}u
\]

\[
= \sqrt{2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u)e^{-s\frac{2}{j}u} \, du
\]

\[
= D^{-1} \left[ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u)e^{-su} \, du \right] \quad \text{by Proposition 4.9 page 42 and Proposition 4.10 page 43}
\]

\[
= D^{-j} BF(x) \quad \text{by definition of } B
\]

\[
DB f(x) = D \left[ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-sx} \, dx \right] \quad \text{by definition of } B
\]

\[
= \sqrt{2} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)e^{-2sx} \, dx \quad \text{by definition of } D \text{ (Definition 4.1 page 38)}
\]

\[
= \sqrt{\frac{1}{2}} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f \left( \frac{u}{2} \right) e^{-u\frac{1}{2}} \, du \quad \text{let } u \triangleq 2x \implies x = \frac{1}{2}u
\]
4 MAIN RESULTS: TRANSVERSAL OPERATORS

\[
\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[ e^{-i \frac{u}{2}} f \left( \frac{u}{2} \right) \right] e^{-iu} \, du
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left[ D^{-1} f \right](u) e^{-iu} \, du
\]

\[
= \BD^{-1} f(x)
\]

by Proposition 4.9 page 42 and Proposition 4.10 page 43

\[
= \BD^{-1} f(x)
\]

by definition of \( \BD \)

\[
\BD^{-1} \BD = \BD^{-1} \BD
\]

by previous result

\[
\BD^{-1} \BD = \BD^{-1} \BD
\]

by definition of operator inverse (Definition 1.15 page 6)

\[
\BD^{-1} \BD = \BD^{-1} \BD
\]

by previous result

\[
\BD^{-1} \BD = \BD^{-1} \BD
\]

by definition of operator inverse (Definition 1.15 page 6)

\[
\BD^{-1} \BD = \BD^{-1} \BD
\]

by definition of operator inverse (Definition 1.15 page 6)

\[
\BD^{-1} \BD = \BD^{-1} \BD
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by definition of operator inverse (Definition 1.15 page 6)

\[
\BD^{-1} \BD = \BD^{-1} \BD
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by definition of operator inverse (Definition 1.15 page 6)

\[
\BD^{-1} \BD = \BD^{-1} \BD
\]

by definition of operator inverse (Definition 1.15 page 6)

\[
\BD^{-1} \BD = \BD^{-1} \BD
\]

by definition of operator inverse (Definition 1.15 page 6)

Corollary 4.18  Let \( \BF \) and \( \BD \) be as in Definition 4.1 page 38. Let \( \tilde{f}(\omega) \triangleq \tilde{F}(x) \) be the Fourier Transform (Definition 2.22 page 20) of some function \( f \in \mathcal{L}_R^2 \) (Definition 2.2 page 15).

\[
\begin{align*}
1. \quad \tilde{T}^n f(x) &= e^{-i\omega n} \tilde{F} \\
2. \quad \tilde{F} D^{\frac{j}{2}} &= \tilde{D}^{-\frac{j}{2}} \tilde{F} \\
3. \quad \tilde{D} \tilde{F} &= \tilde{F} D^{-1} \\
4. \quad D &= \tilde{F} D^{-1} \tilde{F} = \tilde{F}^{-1} D^{-1} \tilde{F} \\
5. \quad \tilde{F} &= D \tilde{F} D^{-1} = D^{-1} \tilde{F} D^{-1}
\end{align*}
\]

\% PROOF: These results follow directly from Proposition 4.17 page 46.

\[
\tilde{F} = \BD \big|_{s=i\omega}
\]

Proposition 4.19  Let \( \BF \) and \( \BD \) be as in Definition 4.1 page 38. Let \( \tilde{f}(\omega) \triangleq \tilde{F}(x) \) be the Fourier Transform (Definition 2.22 page 20) of some function \( f \in \mathcal{L}_R^2 \) (Definition 2.2 page 15).

\[
\tilde{F} D^j T^n f(x) = \frac{1}{2^{j/2}} e^{-i \frac{\omega x}{2^{j/2}}} f \left( \frac{\omega}{2^j} \right)
\]

\% PROOF:

\[
\tilde{F} D^j T^n f(x) = D^j \tilde{T}^n f(x) \quad \text{by Corollary 4.18 page 47 (3)}
\]

\[
= D^j e^{-i \omega} \tilde{F} f(x) \quad \text{by Corollary 4.18 page 47 (3)}
\]

\[
= D^j e^{-i \omega} \tilde{f}(\omega) \quad \text{by Corollary 4.18 page 47 (3)}
\]

\[
= 2^{-j/2} e^{-i \frac{\omega x}{2^{j/2}}} f \left( 2^{-j} \omega \right) \quad \text{by Proposition 4.3 page 39}
\]
Proposition 4.20  Let \( T \) be the translation operator (Definition 4.1 page 38). Let \( \hat{f}(\omega) \triangleq \hat{f}(x) \) be the Fourier Transform (Definition 2.22 page 20) of a function \( f \in L^2_\mathbb{R} \). Let \( \hat{\alpha}(\omega) \) be the DTFT (Definition 2.38 page 24) of a sequence \( (a_n)_{n\in\mathbb{Z}} \in \ell^2_\mathbb{R} \) (Definition 2.33 page 23).

\[
\hat{F} \sum_{n\in\mathbb{Z}} a_n T^n \phi(x) = \hat{\alpha}(\omega) \hat{\phi}(\omega) \quad \forall (a_n)_{n\in\mathbb{Z}}, \phi(x) \in L^2_\mathbb{R}.
\]

\[\hat{F} \sum_{n\in\mathbb{Z}} a_n T^n \phi(x) = \sum_{n\in\mathbb{Z}} a_n \hat{F} T^n \phi(x) = \sum_{n\in\mathbb{Z}} a_n e^{-i\omega n} \hat{F} \phi(x) \quad \text{by Corollary 4.18 page 47}
\]

\[= \left[ \sum_{n\in\mathbb{Z}} a_n e^{-i\omega n} \right] \hat{\phi}(\omega) \quad \text{by definition of } \hat{\phi}(\omega)
\]

\[= \hat{\alpha}(\omega) \hat{\phi}(\omega) \quad \text{by definition of DTFT (Definition 2.38 page 24)}
\]

Proof:

\[ \sum_{n\in\mathbb{Z}} T^n f(x) = \sum_{n\in\mathbb{Z}} f(x + n\tau) \quad \text{summation in } \text{“time”}
\]

\[= \sqrt{\frac{2\pi}{\tau}} \hat{F}^{-1} S \hat{F} [f(x)] = \sqrt{\frac{2\pi}{\tau}} \sum_{n\in\mathbb{Z}} \hat{f} \left( \frac{2\pi}{\tau} n \right) e^{i \frac{2\pi}{\tau} nx} \quad \text{operator notation}
\]

\[\text{where } S \in \ell^2_\mathbb{R} \quad \text{is the sampling operator defined as}
\]

\[S[f(x)](n) \triangleq f \left( \frac{2\pi}{\tau} n \right) \quad \forall f \in L^2_{(\mathbb{R},\mathbb{R},\mu)}, \tau \in \mathbb{R}^+
\]

Proof:

(1) Lemma: If \( h(x) \triangleq \sum_{n\in\mathbb{Z}} f(x + n\tau) \) then \( h \equiv \hat{F}^{-1} \hat{F} h \). Proof:

Note that \( h(x) \) is periodic with period \( \tau \). Because \( h \) is periodic, it is in the domain of \( \hat{F} \) and thus \( h \equiv \hat{F}^{-1} \hat{F} h \).
(2) Proof of PSF (this theorem—Theorem 4.21):

\[ \sum_{n \in \mathbb{Z}} f(x + n\tau) = \hat{F}^{-1} \left[ \frac{1}{\sqrt{\tau}} \int_{0}^{\tau} \left( \sum_{n \in \mathbb{Z}} f(x + n\tau) \right) e^{-i\frac{2\pi}{\tau} kx} \, dx \right] \]

by item 1 page 48

\[ = \hat{F}^{-1} \left[ \frac{1}{\sqrt{\tau}} \sum_{n \in \mathbb{Z}} \int_{0}^{\tau} f(x + n\tau) e^{-i\frac{2\pi}{\tau} kx} \, dx \right] \]

by def. of \( \hat{F} \) (Definition 2.16 page 18)

\[ = \hat{F}^{-1} \left[ \frac{1}{\sqrt{\tau}} \sum_{n \in \mathbb{Z}} e^{i\frac{2\pi}{\tau} nu} \int_{u = n\tau}^{u = (n+1)\tau} f(u) e^{-i\frac{2\pi}{\tau} ku} \, du \right] \]

where \( u \triangleq x + n\tau \implies x = u - n\tau \)

\[ = \hat{F}^{-1} \left[ \frac{1}{\sqrt{\tau}} \sum_{n \in \mathbb{Z}} e^{i\frac{2\pi}{\tau} nu} \int_{u = n\tau}^{u = (n+1)\tau} f(u) e^{-i\frac{2\pi}{\tau} ku} \, du \right] \]

by Theorem 2.17 page 18

\[ = \sqrt{\frac{2\pi}{\tau}} \hat{F}^{-1} \left[ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u) e^{-i\left(\frac{2\pi}{\tau}\right)u} \, du \right] \]

by definition of \( \hat{F} \) (page 20)

\[ = \sqrt{\frac{2\pi}{\tau}} \hat{F}^{-1} \left[ \int_{-\infty}^{\infty} f(u) e^{-i\left(\frac{2\pi}{\tau}\right)u} \, du \right] \]

by definition of S

\[ = \sqrt{\frac{2\pi}{\tau}} \sum_{n \in \mathbb{Z}} f\left(\frac{2\pi}{\tau} n\right) e^{\frac{2\pi}{\tau} nx} \]

by Theorem 2.17 page 18

\[\square\]

**Theorem 4.22** (Inverse Poisson Summation Formula—IPSF) Let \( \hat{f}(\omega) \) be the Fourier Transform (Definition 2.22 page 20) of a function \( f(x) \in L^2_{\mathbb{R}}. \)

\[ \sum_{n \in \mathbb{Z}} \mathcal{T}_{\frac{2\pi}{\tau}} f(x) \triangleq \sum_{n \in \mathbb{Z}} f\left(\omega - \frac{2\pi}{\tau} n\right) = \frac{\tau}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} f(n\tau) e^{-i\omega n\tau} \]

\[ \text{summation in "frequency"} \]

\[ \text{summation in "time"} \]

\[\square\]

**Proof:**

[45], page 88
lemma: If \( h(\omega) \triangleq \sum_{n \in \mathbb{Z}} \tilde{f}(\omega + \frac{2\pi}{T} n) \), then \( h \equiv \hat{F}^{-1} \hat{F} h \). Proof:

Note that \( h(\omega) \) is periodic with period \( \frac{2\pi}{T} \):

\[
h(\omega + \frac{2\pi}{T}) = \sum_{n \in \mathbb{Z}} \tilde{f}(\omega + 2\pi \tau + 2\pi \tau n) = \sum_{n \in \mathbb{Z}} \tilde{f}(\omega + (n + 1) 2\pi \tau) = \sum_{n \in \mathbb{Z}} \tilde{f}(\omega + \frac{2\pi}{T} n) \triangleq h(\omega)
\]

Because \( h \) is periodic, it is in the domain of \( \hat{F} \) and is equivalent to \( \hat{F}^{-1} \hat{F} h \).

Proof of IPSF (this theorem—Theorem 4.22):

\[
\sum_{n \in \mathbb{Z}} \tilde{f}(\omega + \frac{2\pi}{T} n) = \hat{F}^{-1} \hat{F} \sum_{n \in \mathbb{Z}} \tilde{f}(\omega + \frac{2\pi}{T} n) = \hat{F}^{-1} \left[ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{f}(\omega) e^{i \omega T k} d\omega \right] = \hat{F}^{-1} \left[ \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{f}(u) e^{-i \omega \tau k} d\omega \right] \text{ where } u \triangleq \omega + \frac{2\pi}{T} n \implies \omega = u - \frac{2\pi}{T} n
\]

\[
= \sqrt{T} \hat{F}^{-1} \left[ \left[ \hat{F}^{-1} \hat{F} \right](\omega) \right] = \sqrt{T} \left[ \hat{F}^{-1} S \hat{F}^{-1} \tilde{f} \right] \text{ by definition of } S
\]

\[
= \sqrt{T} \hat{F}^{-1} \hat{F} \tilde{f}(x) \text{ by definition of } \hat{F} \text{ (Definition 2.22 page 20)}
\]

\[
= \sqrt{T} \hat{F}^{-1} \tilde{f}(-k \tau) \text{ by definition of } S
\]

\[
= \sqrt{T} \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \tilde{f}(-k \tau) e^{i \frac{2\pi}{T} k \omega} \text{ by definition of } \hat{F}^{-1} \text{ (Theorem 2.17 page 18)}
\]

\[
= \frac{T}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} \tilde{f}(-k \tau) e^{i \frac{2\pi}{T} k \omega} \text{ by definition of } \hat{F}^{-1} \text{ (Theorem 2.17 page 18)}
\]
\[
\tau \sum_{m \in \mathbb{Z}} f(m \tau) e^{-i \omega m \tau}
\]
\(\text{let } m \overset{\Delta}{=} -k\)

**Remark 4.23** The left hand side of the Poisson Summation Formula (Theorem 4.21 page 48) is very similar to the Zak Transform \(Z\): \(113\)
\[
(Zf)(t, \omega) \overset{\Delta}{=} \sum_{n \in \mathbb{Z}} f(x + nt) e^{i 2 \pi n \omega}
\]

**Remark 4.24** A generalization of the Poisson Summation Formula (Theorem 4.21 page 48) is the Selberg Trace Formula. \(114\)

### 4.2.6 Basis theory properties

Definition 4.25 and Definition 4.26 define four quantities. In this text, the quantities' notation and terminology are similar to that used in the study of random processes.

**Definition 4.25** \(115\) Let \(\langle \nabla | \triangle \rangle\) be the standard inner product in \(L_\mathbb{R}^2\) (Definition 2.2 page 15).

\[
\begin{align*}
R_{fg}(n) & \overset{\Delta}{=} \langle f(x) | T^n g(x) \rangle, & \text{\(f, g \in L_\mathbb{R}^2\), is the cross-correlation function of } f \text{ and } g. \\
R_{ff}(n) & \overset{\Delta}{=} \langle f(x) | T^n f(x) \rangle, & \text{\(f \in L_\mathbb{R}^2\), is the autocorrelation function of } f.
\end{align*}
\]

**Definition 4.26** \(116\) Let \(R_{fg}(n)\) and \(R_{ff}(n)\) be the sequences defined in Definition 4.25 page 51. Let \(Z (x_n)\) be the \(z\)-transform (Definition 2.35 page 24) of a sequence \(\{x_n\}_{n \in \mathbb{Z}}\).

\[
\begin{align*}
S_{fg}(z) & \overset{\Delta}{=} Z \left[ R_{fg}(n) \right], & \text{\(f, g \in L_\mathbb{R}^2\), is the complex cross-power spectrum of } f \text{ and } g. \\
S_{ff}(z) & \overset{\Delta}{=} Z \left[ R_{ff}(n) \right], & \text{\(f \in L_\mathbb{R}^2\), is the complex auto-power spectrum of } f.
\end{align*}
\]

**Definition 4.27** \(117\) Let \(S_{fg}(z)\) and \(S_{ff}(z)\) be the functions defined in Definition 4.26 page 51.

\[
\begin{align*}
\tilde{S}_{fg}(\omega) & \overset{\Delta}{=} S_{fg} \left( e^{i \omega} \right), & \forall f, g \in L_\mathbb{R}^2, \text{ is the cross-power spectrum of } f \text{ and } g. \\
\tilde{S}_{ff}(\omega) & \overset{\Delta}{=} S_{ff} \left( e^{i \omega} \right), & \forall f \in L_\mathbb{R}^2, \text{ is the auto-power spectrum of } f.
\end{align*}
\]

**Theorem 4.28** \(118\) Let \(\hat{f}(\omega)\) be the Fourier transform (Definition 2.22 page 20) of a function \(f(x) \in L_\mathbb{R}^2\).

\[
\begin{align*}
\tilde{S}_{fg}(\omega) &= 2\pi \sum_{n \in \mathbb{Z}} \hat{f}(\omega + 2\pi n) \tilde{g}^*(\omega + 2\pi n), & \forall f, g \in L_\mathbb{R}^2 \\
\tilde{S}_{ff}(\omega) &= 2\pi \sum_{n \in \mathbb{Z}} |\hat{f}(\omega + 2\pi n)|^2, & \forall f \in L_\mathbb{R}^2
\end{align*}
\]
\[\begin{align*}
\tilde{S}_{\phi\phi}(\omega) & \triangleq \tilde{S}_{\phi}(z) \quad \text{by definition of } \tilde{S}_{\phi} \text{ Definition 4.27 page 51} \\
& = \sum_{n \in \mathbb{Z}} R_{\phi}(n)z^{-n} \quad \text{by definition of } S_{\phi} \text{ Definition 4.26 page 51} \\
& = \sum_{n \in \mathbb{Z}} \langle f(x) | g(x - n) \rangle z^{-n} \quad \text{by Definition 4.27 page 51} \\
& = \sum_{n \in \mathbb{Z}} \langle \tilde{f}(\nu) | \tilde{g}(\nu) e^{i\omega n} \rangle z^{-n} \quad \text{by unitary property of } \tilde{f} \text{ (Theorem 2.27 page 21)} \\
& = \sum_{n \in \mathbb{Z}} \langle \tilde{f}(\nu) | e^{-i\omega n} \tilde{g}(\nu) \rangle z^{-n} \quad \text{by Theorem 2.28 page 22} \\
& = \sum_{n \in \mathbb{Z}} \sqrt{2\pi} \left[ \frac{1}{2\pi} \int_{\mathbb{R}} \tilde{f}(\nu) \tilde{g}^*(\nu) e^{i\omega n} d\nu \right] z^{-n} \quad \text{by Definition 2.2 page 15} \\
& = \sqrt{2\pi} \sum_{n \in \mathbb{Z}} \left[ \tilde{f}^{-1} \left( \sqrt{2\pi} \tilde{f}(\nu) \tilde{g}^*(\nu) \right) \right] e^{-i\omega n} \quad \text{by Theorem 2.24 page 20} \\
& = 2\pi \sum_{n \in \mathbb{Z}} \tilde{f}(\omega + 2\pi n) \tilde{g}^*(\omega + 2\pi n) \quad \text{by IPSF (Theorem 4.22 page 49), } \tau = 1
\end{align*}\]

\[\begin{align*}
\tilde{S}_{\phi\phi}(\omega) = \left. \tilde{S}_{\phi}(z) \right|_{z=e^{i\omega}} \\
& = 2\pi \sum_{n \in \mathbb{Z}} \tilde{f}(\omega + 2\pi n) \tilde{g}^*(\omega + 2\pi n) \quad \text{by previous result} \\
& = 2\pi \sum_{n \in \mathbb{Z}} \tilde{f}(\omega + 2\pi n) \tilde{g}^*(\omega + 2\pi n) \\
& = 2\pi \sum_{n \in \mathbb{Z}} |\tilde{f}(\omega + 2\pi n)|^2
\end{align*}\]

\textbf{Theorem 4.29} \quad \text{Let } \tilde{S}_{\phi\phi} \text{ be the SPECTRAL DENSITY FUNCTION (Definition 4.27 page 51) of a function } \phi(x) \in L^2_{\mathbb{R}}. \text{ Let } 0 < A < B. \text{ Let } \|\cdot\| \text{ be defined as in Definition 2.2 page 15.}

\[
\left\{ A \sum_{n \in \mathbb{N}} |a_n|^2 \leq \|T^n \phi(x)\|^2 \leq B \sum_{n \in \mathbb{N}} |a_n|^2 \quad \forall (a_n) \in \ell^2 \right\} \iff \{ A \leq \tilde{S}_{\phi\phi}(\omega) \leq B \}
\]

\textit{Properties and applications of transversal operators}
(1) lemma:

\[ \left\| \sum_{n \in \mathbb{Z}} a_n T^n \phi(x) \right\|^2 = \left\| \tilde{F} \sum_{n \in \mathbb{Z}} a_n T^n \phi(x) \right\|^2 \]

because \( \tilde{F} \) is unitary (Theorem 2.25 page 20)

\[ = \left\| \tilde{a}(\omega) \tilde{\phi}(\omega) \right\|^2 \]

by Proposition 4.20 page 48

\[ = \int_{\mathbb{R}} |\tilde{a}(\omega)\tilde{\phi}(\omega)|^2 \, d\omega \]

by definition of \( \|\cdot\| \)

\[ = \sum_{n \in \mathbb{Z}} \int_{0}^{2\pi} |\tilde{a}(\omega + 2\pi n)\tilde{\phi}(\omega + 2\pi n)|^2 \, d\omega \]

\[ = \int_{0}^{2\pi} \sum_{n \in \mathbb{Z}} |\tilde{a}(\omega + 2\pi n)|^2 |\tilde{\phi}(\omega + 2\pi n)|^2 \, d\omega \]

by Proposition 2.39 page 25

\[ = \int_{0}^{2\pi} |\tilde{a}(\omega)|^2 \frac{1}{2\pi} 2\pi \sum_{n \in \mathbb{Z}} |\tilde{\phi}(\omega + 2\pi n)|^2 \, d\omega \]

\[ = \frac{1}{2\pi} \int_{0}^{2\pi} |\tilde{a}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega) \, d\omega \]

by def. of \( \tilde{S}_{\phi\phi}(\omega) \) (Theorem 4.28 page 51)

(2) lemma:

\[ \frac{1}{2\pi} \int_{0}^{2\pi} |\tilde{a}(\omega)|^2 \, d\omega = \frac{1}{2\pi} \int_{0}^{2\pi} \left| \sum_{n \in \mathbb{Z}} a_n e^{-i\omega n} \right|^2 \, d\omega \]

by def. of DTFT (Definition 2.38 page 24)

\[ = \frac{1}{2\pi} \int_{0}^{2\pi} \left| \sum_{n \in \mathbb{Z}} a_n e^{-i\omega n} \right|^2 \left| \sum_{m \in \mathbb{Z}} a_m e^{-i\omega m} \right|^* \, d\omega \]

\[ = \frac{1}{2\pi} \int_{0}^{2\pi} \left| \sum_{n \in \mathbb{Z}} a_n e^{-i\omega n} \right|^2 \left| \sum_{m \in \mathbb{Z}} a_m e^{i\omega m} \right| \, d\omega \]

\[ = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} a_n a_m^* \int_{0}^{2\pi} e^{-i\omega (n-m)} \, d\omega \]

\[ = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} a_n a_m^* 2\pi \delta_{nm} \]

\[ = \sum_{n \in \mathbb{Z}} |a_n|^2 \]

by def. of \( \tilde{\delta} \) (Definition 3.12 page 35)

(3) Proof for ( \( \iff \) ) case:

\[ A \sum_{n \in \mathbb{Z}} |a_n|^2 = \frac{A}{2\pi} \int_{0}^{2\pi} |\tilde{a}(\omega)|^2 \, d\omega \]

by lemma 2 page 53

\[ = \frac{1}{2\pi} \int_{0}^{2\pi} |\tilde{a}(\omega)|^2 A \, d\omega \]
\[
\begin{align*}
\leq & \frac{1}{2\pi} \int_{0}^{2\pi} |\tilde{a}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega) \, d\omega \\
= & \left\| \sum_{n \in \mathbb{Z}} a_n T^n \phi(x) \right\|^2 \\
= & \frac{1}{2\pi} \int_{0}^{2\pi} |\tilde{a}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega) \, d\omega \\
= & \frac{1}{2\pi} \int_{0}^{2\pi} |\tilde{a}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega) \, d\omega \\
\leq & \frac{1}{2\pi} \int_{0}^{2\pi} |\tilde{a}(\omega)|^2 B \, d\omega \\
= & \frac{B}{2\pi} \int_{0}^{2\pi} |\tilde{a}(\omega)|^2 \, d\omega \\
= & B \sum_{n \in \mathbb{Z}} |a_n|^2 
\end{align*}
\]

(4) Proof for (\(\implies\)) case:

(a) Let \(Y \triangleq \{ \omega \in [0, 2\pi] \mid |\tilde{S}_{\phi\phi}(\omega) > a \} \)
and \(X \triangleq \{ \omega \in [0, 2\pi] \mid |\tilde{S}_{\phi\phi}(\omega) < a \} \)

(b) Let \(1_A(x)\) be the set indicator (Definition 2.3 page 15) of a set \(A\).

Let \((b_n)_{n \in \mathbb{Z}}\) be the inverse DTFT (Theorem 2.43 page 27) of \(1_Y(\omega)\) such that
\[1_Y(\omega) \triangleq \sum_{n \in \mathbb{N}} b_n e^{-i\omega n} \triangleq \tilde{b}(\omega)\.

Let \((a_n)_{n \in \mathbb{Z}}\) be the inverse DTFT (Theorem 2.43 page 27) of \(1_X(\omega)\) such that
\[1_X(\omega) \triangleq \sum_{n \in \mathbb{N}} a_n e^{-i\omega n} \triangleq \tilde{a}(\omega)\.

(c) Proof that \(a \leq B\):

Let \(\mu(A)\) be the measure of a set \(A\).

\[
\begin{align*}
B \sum_{n \in \mathbb{Z}} |b_n|^2 & \geq \left\| \sum_{n \in \mathbb{Z}} b_n T^n \phi(x) \right\|^2 \\
& = \frac{1}{2\pi} \int_{0}^{2\pi} |\tilde{b}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega) \, d\omega \\
& = \frac{1}{2\pi} \int_{0}^{2\pi} |1_Y(\omega)|^2 \tilde{S}_{\phi\phi}(\omega) \, d\omega \\
& = \frac{1}{2\pi} \int_{Y} |1|^2 \tilde{S}_{\phi\phi}(\omega) \, d\omega \\
\geq & \frac{\alpha}{2\pi} \mu(Y) \\
& = \int_{0}^{2\pi} |1_Y(\omega)|^2 \, d\omega \\
& = \int_{0}^{2\pi} \left| \sum_{n \in \mathbb{N}} b_n e^{-i\omega n} \right|^2 \, d\omega
\end{align*}
\]

by left hypothesis

by lemma 1 page 53

by definition of \(1_Y(\omega)\) (item 4b page 54)

by definition of \(1_Y(\omega)\) (item 4b page 54)

by definition of \(\mu(Y)\) (item 4a page 54)

by definition of \(1_Y(\omega)\) (item 4b page 54)

by definition of \(\{b_n\}\) (item 4b page 54)


\[
\begin{align*}
&= \int_0^{2\pi} |\hat{b}(\omega)|^2 \, d\omega \\
&= \alpha \sum_{n \in \mathbb{Z}} |b_n|^2
\end{align*}
\]

by definition of \(\hat{b}(\omega)\) (item 4b page 54)

\[
\sum_{n \in \mathbb{Z}} |a_n|^2 \leq \left\| \sum_{n \in \mathbb{Z}} a_n T^n \phi(x) \right\|^2
\]

by left hypothesis

\[
= \frac{1}{2\pi} \int_0^{2\pi} |\hat{a}(\omega)|^2 \hat{S}_{\phi\phi}(\omega) \, d\omega
\]

by lemma 1 page 53

\[
= \frac{1}{2\pi} \int_0^{2\pi} |1_X(\omega)|^2 \hat{S}_{\phi\phi}(\omega) \, d\omega
\]

by definition of \(1_X(\omega)\) (Definition 2.3 page 15)

\[
\leq \frac{a}{2\pi} \mu(X)
\]

by definition of \(X\) (item 4a page 54)

\[
\int_0^{2\pi} |1_X(\omega)|^2 \, d\omega
\]

by definition of \(1_X(\omega)\) (Definition 2.3 page 15)

\[
\int_0^{2\pi} \left| \sum_{n \in \mathbb{Z}} a_n e^{-i\omega n} \right|^2 \, d\omega
\]

by definition of \(\langle a_n \rangle\) (lemma 2 page 53)

\[
\int_0^{2\pi} |\hat{a}(\omega)|^2 \, d\omega
\]

by definition of \(\hat{a}(\omega)\) (lemma 2 page 53)

\[
\sum_{n \in \mathbb{Z}} |a_n|^2
\]

by lemma 2 page 53

(d) Proof that \(\hat{S}_{\phi\phi}(\omega) \leq B\):

(i) \(S_{\phi\phi}(\omega) > \alpha\) whenever \(\omega \in Y\) (item 4a page 54).

(ii) But even then, \(\alpha \leq B\) (item 4c page 54).

(iii) So, \(\hat{S}_{\phi\phi}(\omega) \leq B\).

(e) Proof that \(A \leq \alpha\):

Let \(\mu(A)\) be the measure of a set \(A\).

\[
\begin{align*}
\sum_{n \in \mathbb{Z}} |a_n|^2 &\leq \left\| \sum_{n \in \mathbb{Z}} a_n T^n \phi(x) \right\|^2 \\
&\leq \frac{a}{2\pi} \mu(X)
\end{align*}
\]

by definition of \(X\) (item 4a page 54)

\[
\int_0^{2\pi} |1_X(\omega)|^2 \, d\omega
\]

by definition of \(1_X(\omega)\) (Definition 2.3 page 15)

\[
\int_0^{2\pi} \left| \sum_{n \in \mathbb{Z}} a_n e^{-i\omega n} \right|^2 \, d\omega
\]

by definition of \(\langle a_n \rangle\) (lemma 2 page 53)

\[
\int_0^{2\pi} |\hat{a}(\omega)|^2 \, d\omega
\]

by definition of \(\hat{a}(\omega)\) (lemma 2 page 53)

\[
\sum_{n \in \mathbb{Z}} |a_n|^2
\]

by lemma 2 page 53

(f) Proof that \(A \leq \hat{S}_{\phi\phi}(\omega)\):

(i) \(S_{\phi\phi}(\omega) < \alpha\) whenever \(\omega \in X\) (item 4a page 54).

(ii) But even then, \(A \leq \alpha\) (item 4e page 55).

(iii) So, \(A \leq \hat{S}_{\phi\phi}(\omega)\).

In the case that \(f\) and \(g\) are orthonormal, the spectral density relations simplify considerably (next).
**Theorem 4.30**  
Let $\tilde{S}_f$ and $\tilde{S}_g$ be SPECTRAL DENSITY FUNCTIONS (Definition 4.27 page 51).

\[
\langle f(x) | T^n f(x) \rangle = \tilde{\delta}_n \quad \text{(ORTHONORMAL)} \quad \iff \quad \tilde{S}_f(\omega) = 1 \quad \forall f \in L^2_f
\]

\[
\langle f(x) | T^n g(x) \rangle = 0 \quad \iff \quad \tilde{S}_g(\omega) = 0 \quad \forall g \in L^2_g
\]

**Proof:**

1. **Proof that** $\langle f(x) | f(x - n) \rangle = \tilde{\delta}_n \iff \tilde{S}_f(\omega) = 1$: This follows directly from Theorem 4.29 (page 52) with $A = B = 1$ (by Parseval's Identity Theorem 3.16 page 36 since $\{T^n f\}$ is orthonormal).

2. **Alternate proof that** $\langle f(x) | f(x - n) \rangle = \tilde{\delta}_n \implies \tilde{S}_f(\omega) = 1$:

\[
\tilde{S}_f(\omega) = \sum_{n \in \mathbb{Z}} R_f(n)e^{-in\omega} \quad \text{by definition of } \tilde{S}_f \quad \text{(Definition 4.27 page 51)}
\]

\[
= \sum_{n \in \mathbb{Z}} \langle f(x) | f(x - n) \rangle e^{-in\omega} \quad \text{by definition of } R_f \quad \text{(Definition 4.25 page 51)}
\]

\[
= \sum_{n \in \mathbb{Z}} \tilde{\delta}_n e^{-in\omega} \quad \text{by left hypothesis}
\]

\[
= 1 \quad \text{by definition of } \tilde{\delta} \quad \text{(Definition 3.12 page 35)}
\]

3. **Alternate proof that** $\langle f(x) | f(x - n) \rangle = \tilde{\delta}_n \iff \tilde{S}_f(\omega) = 1$:

\[
\langle f(x) | f(x - n) \rangle
\]

\[
= \left\langle \tilde{F}(x) | \tilde{F}(x - n) \right\rangle \quad \text{by unitary prop. of } \tilde{F} \quad \text{(Theorem 2.27 page 21)}
\]

\[
= \left\langle \tilde{f}(\omega) | e^{-i\omega n} \tilde{f}(\omega) \right\rangle \quad \text{by shift property of } \tilde{F} \quad \text{(Theorem 2.28 page 22)}
\]

\[
= \int_\mathbb{R} \tilde{f}(\omega)e^{i\omega n} \tilde{f}(\omega) d\omega \quad \text{by def. of } \langle \triangle | \nabla \rangle \quad \text{(Definition 2.2 page 15)}
\]

\[
= \int_\mathbb{R} |\tilde{f}(\omega)|^2 e^{i\omega n} d\omega
\]

\[
= \sum_{n \in \mathbb{Z}} \int_{2n\pi}^{2(n+1)\pi} |\tilde{f}(\omega)|^2 e^{i\omega n} d\omega
\]

\[
= \sum_{n \in \mathbb{Z}} \int_0^{2\pi} |\tilde{f}(u + 2\pi n)|^2 e^{iu(\omega - 2\pi n)} du \quad \text{where } u \equiv \omega - 2\pi n \implies \omega = u + 2\pi n
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \left[ 2\pi \sum_{n \in \mathbb{Z}} |\tilde{f}(u + 2\pi n)|^2 \right] e^{iu(\omega - 2\pi n)} du
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} \tilde{S}_f(\omega) e^{i\omega n} du \quad \text{by Theorem 4.28 page 51}
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} e^{i\omega n} du \quad \text{by right hypothesis}
\]

\[
= \tilde{\delta}_n \quad \text{by definition of } \tilde{\delta} \quad \text{(Definition 3.12 page 35)}
\]
(4) Proof that \( \langle f(x) \mid g(x - n) \rangle = 0 \implies \tilde{S}_{fg}(\omega) = 0 \):

\[
\tilde{S}_{fg}(\omega) = \sum_{n \in \mathbb{Z}} R_{fg}(n)e^{-i\omega n}
\]

by definition of \( \tilde{S}_{fg} \) (Definition 4.27 page 51)

\[
= \sum_{n \in \mathbb{Z}} \langle f(x) \mid g(x - n) \rangle e^{-i\omega n}
\]

by definition of \( R_{fg} \) (Definition 4.25 page 51)

\[
= \sum_{n \in \mathbb{Z}} 0e^{-i\omega n}
\]

by left hypothesis

\[
= 0
\]

(5) Proof that \( \langle f(x) \mid g(x - n) \rangle = 0 \iff \tilde{S}_{fg}(\omega) = 0 \):

\[
\langle f(x) \mid g(x - n) \rangle = \langle \tilde{F}f(x) \mid \tilde{F}g(x - n) \rangle
\]

by unitary property of \( \tilde{F} \) (Theorem 2.27 page 21)

\[
= \langle \hat{f}(\omega) \mid e^{-i\omega n} \hat{g}(\omega) \rangle
\]

by shift property of \( \tilde{F} \) (Theorem 2.28 page 22)

\[
= \int_{\mathbb{R}} \hat{f}(\omega)e^{i\omega n} \hat{g}^*(\omega) d\omega
\]

by definition of \( \langle \triangle \mid \nabla \rangle \) (Definition 2.2 page 15)

\[
= \int_{\mathbb{R}} \hat{f}(\omega) \hat{g}^*(\omega) e^{i\omega n} d\omega
\]

\[
= \sum_{n \in \mathbb{Z}} \int_{2\pi(n+1)}^{2\pi n} \hat{f}(\omega) \hat{g}^*(\omega) e^{i\omega n} d\omega
\]

where \( u \triangleq \omega - 2\pi n \implies \omega = u + 2\pi n \)

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} \tilde{S}_{fg}(u)e^{iu} du
\]

by Theorem 4.28 page 51

\[
= \frac{1}{2\pi} \int_{0}^{2\pi} 0 \cdot e^{iu} du
\]

by right hypothesis

\[
= 0
\]

Lemma 4.31 121 Let \( \Omega \triangleq (X, +, \cdot, (F, +, \times), T) \) be a topological linear space. Let \( \text{span}A \) be the span of a set \( A \) (Definition 3.2 page 32). Let \( \hat{f}(\omega) \) and \( \hat{g}(\omega) \) be the Fourier transforms (Definition 2.22 page 20) of the functions \( f(x) \) and \( g(x) \), respectively, in \( L^2_{\mathbb{R}} \) (Definition 2.2 page 15). Let \( \tilde{a}(\omega) \) be the DTFT (Definition 2.38 page 24) of a sequence \( \{a_n\}_{n \in \mathbb{Z}} \) in \( \ell^2_{\mathbb{N}} \) (Definition 2.33 page 23).

\[
\begin{align*}
\{ & \{ T^n f \}_{n \in \mathbb{Z}} \text{ is a Schauder basis for } \Omega \text{ and } \\
& \{ T^n g \}_{n \in \mathbb{Z}} \text{ is a Schauder basis for } \Omega \} \implies \\
\exists \{ & a_n \}_{n \in \math{N} \text{ such that } \\
& \hat{f}(\omega) = \tilde{a}(\omega) \hat{g}(\omega) \}
\end{align*}
\]

121 [30], page 140
\begin{proof}
Let $V'_0$ be the space spanned by $\{ T^\sigma \phi | n \in \mathbb{Z} \}$.

$$
\hat{f}(\omega) \triangleq \hat{F}f
= \mathcal{F} \sum_{n \in \mathbb{Z}} a_n T g
= \sum_{n \in \mathbb{Z}} a_n \mathcal{F} T g
= \sum_{n \in \mathbb{Z}} a_n e^{-i\omega n} \mathcal{F} g
= \mathcal{A}(\omega) \mathcal{G}(\omega)
$$

by Definition 2.22 page 20

$$
\mathcal{V}_0 \triangleq \left\{ f(x) \bigg| f(x) = \sum_{n \in \mathbb{Z}} b_n T^\sigma g(x) \right\}
$$

by Riesz basis hypothesis

$$
= \left\{ f(x) \bigg| \mathcal{F}f(x) = \mathcal{F} \sum_{n \in \mathbb{Z}} b_n T^\sigma g(x) \right\}
= \left\{ f(x) \bigg| \hat{f}(\omega) = \mathcal{B}(\omega) \hat{g}(\omega) \right\}
= \left\{ f(x) \bigg| \hat{f}(\omega) = \mathcal{B}(\omega) \mathcal{A}(\omega) \hat{f}(\omega) \right\}
= \left\{ f(x) \bigg| \hat{f}(\omega) = \mathcal{C}(\omega) \hat{f}(\omega) \right\}
$$

where $\mathcal{C}(\omega) \triangleq \mathcal{B}(\omega) \mathcal{A}(\omega)$

$$
= \left\{ f(x) \bigg| f(x) = \sum_{n \in \mathbb{Z}} c_n f(x - n) \right\}
\triangleq V'_0
$$
\end{proof}

\textbf{Theorem 4.32} (Battle-Lemarié orthogonalization) \textsuperscript{122} Let $\hat{f}(\omega)$ be the Fourier Transform (Definition 2.22 page 20) of a function $f \in L^2_\mathbb{R}$.

\begin{enumerate}
\item $\{ T^\sigma g | n \in \mathbb{Z} \}$ is a Riesz basis for $L^2_\mathbb{R}$ and
\item $\hat{f}(\omega) \triangleq \frac{\hat{g}(\omega)}{\sqrt{2\pi \sum_{n \in \mathbb{Z}} |\mathcal{B}(\omega) + 2\pi n|^2}}$
\end{enumerate}

$$
\Rightarrow \left\{ \{ T^\sigma f | n \in \mathbb{Z} \} \text{ is an orthonormal basis for } L^2_\mathbb{R} \right\}
$$

\textsuperscript{122} (137), page 25, (Remark 2.4), (132), page 71, (94), page 72, (95), page 225, (30), page 140, (5.3.3)
4.3 Examples

Example 4.33  (linear functions)\footnote{[65], page 2} Let $\mathbf{T}$ be the \textit{translation operator} (Definition 4.1 page 38). Let $\mathcal{L}(\mathbb{C}, \mathbb{C})$ be the set of all \textit{linear} functions in $L^2_{\mathbb{R}}$.

1. $\{x, \mathbf{T}x\}$ is a \textit{basis} for $\mathcal{L}(\mathbb{C}, \mathbb{C})$ and
2. $f(x) = f(1)x - f(0)Tx \quad \forall f \in \mathcal{L}(\mathbb{C}, \mathbb{C})$
PROOF: By left hypothesis, \( f \) is linear; so let \( f(x) = ax + b \)

\[
f(1)x - f(0)x = f(1)x - f(0)(x - 1)
\]

by Definition 4.1 page 38

\[
= (ax + b)|_{x=1} x - (ax + b)|_{x=0} (x - 1)
\]

by left hypothesis and definition of \( f \)

\[
= (a + b)x - b(x - 1)
\]

\[
= ax + bx - bx + b
\]

\[
= ax + b
\]

\[
= f(x)
\]

by left hypothesis and definition of \( f \)

Example 4.34 (Cardinal Series) Let \( T \) be the translation operator (Definition 4.1 page 38). The Paley-Wiener class of functions \( PW_2^\sigma \) are those functions which are “bandlimited” with respect to their Fourier transform. The cardinal series forms an orthogonal basis for such a space. The Fourier coefficients for a projection of a function \( f \) onto the Cardinal series basis elements is particularly simple—these coefficients are samples of \( f(x) \) taken at regular intervals. In fact, one could represent the coefficients using inner product notation with the Dirac delta distribution \( \delta \) as follows:

\[
\langle f(x) | T^n\delta(x) \rangle \triangleq \int_{\mathbb{R}} f(x)\delta(x - n) \, dx \triangleq f(n)
\]

1. \( \left\{ T^n\frac{\sin(\pi x)}{\pi x} \right\}_{n\in\mathbb{N}} \) is a basis for \( PW_2^\sigma \) and

2. \( f(x) = \sum_{n=1}^{\infty} f(n)T^n\frac{\sin(\pi x)}{\pi x} \)

Cardinal series

\( \forall f \in PW_2^\sigma, \sigma \leq \frac{1}{2} \)

Example 4.35 (Fourier Series)

1. \( \{ D_n e^{ix} \}_{n \in \mathbb{Z}} \) is a basis for \( L(0, 2\pi) \) and

2. \( f(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} \alpha_n D_n e^{ix} \quad \forall x \in (0, 2\pi), f \in L(0, 2\pi) \) where

3. \( \alpha_n \triangleq \frac{1}{\sqrt{2\pi}} \int_{0}^{2\pi} f(x)D_n e^{-ix} \, dx \quad \forall f \in L(0, 2\pi) \)

Example 4.36 (Fourier Transform)

1. \( \{ D_\omega e^{ix} \}_{\omega \in \mathbb{R}} \) is a basis for \( L^2_\mathbb{R} \) and

2. \( f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \tilde{f}(\omega)D_x e^{i\omega x} \, d\omega \quad \forall f \in L^2_\mathbb{R} \) where

3. \( \tilde{f}(\omega) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x)D_\omega e^{-i\omega x} \, dx \quad \forall f \in L^2_\mathbb{R} \)
Example 4.37 (Gabor Transform) 
1. \( \{ (T_r e^{-\pi x^2}) (D_\omega e^{ix}) \mid r, \omega \in \mathbb{R} \} \) is a basis for \( L^2_\mathbb{R} \) and 
2. \( f(x) = \int_{\mathbb{R}} G(\tau, \omega) D_x e^{i\omega} d\omega \quad \forall x \in \mathbb{R}, f \in L^2_\mathbb{R} \) where 
3. \( G(\tau, \omega) \triangleq \int_{\mathbb{R}} f(x) (T_r e^{-\pi x^2}) (D_\omega e^{-ix}) \, dx \quad \forall x \in \mathbb{R}, f \in L^2_\mathbb{R} \)

Example 4.38 (wavelets) Let \( \psi(x) \) be a mother wavelet.
1. \( \{ D_k T^n \psi(x) \mid k, n \in \mathbb{Z} \} \) is a basis for \( L^2_\mathbb{R} \) and 
2. \( \psi(x) = \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \alpha_{k,n} D_k T^n \psi(x) \quad \forall \psi \in L^2_\mathbb{R} \) where 
3. \( \alpha_n \triangleq \int_{\mathbb{R}} \psi(x) D_k T^n \psi^*(x) \, dx \quad \forall \psi \in L^2_\mathbb{R} \)

Definition 4.39
1. \( D_a f(x) \triangleq \sqrt{a} f(ax) \quad \forall f \in \mathbb{R} \) (generalized dilation operator) 
2. \( E_a f(x) \triangleq e^{i2\pi ax} f(x) \quad \forall f \in \mathbb{R} \) (modulation operator)

Example 4.40 (Poisson Summation Formula) Let \( E \) be the modulation operator and \( T \) the translation operator (Definition 4.1 page 38). Let \( \hat{f}(\omega) \) be the Fourier transform of a function \( f(x) \).
\[
\sum_{n \in \mathbb{Z}} E_n \hat{f}(2\pi n) = \frac{1}{\sqrt{2\pi}} \sum_{n \in \mathbb{Z}} T^n f(x) \quad \forall f \in L^2_\mathbb{R}
\]
modulated summation in “frequency” \hspace{1cm} summation in “time”

PROOF: See Theorem 4.21 page 48.

Example 4.41 (B-splines) Let \( M \) be the multiplication operator and \( T \) the translation operator (Definition 4.1 page 38). Let \( N_n(x) \) be the \( n \)th order cardinal B-spline.
\[
N_n(x) = \frac{1}{n} x N_{n-1}(x) - \frac{1}{n} x TN_{n-1}(x) + \frac{n+1}{n} TN_{n-1}(x) \quad \forall n \in \mathbb{N}, x \in \mathbb{R}
\]

Example 4.42 (Fourier Series analysis) Let \( D_\alpha \) be the dilation operator and \( E \) the modulation operator (Definition 4.39 page 61). Let \( \hat{F} \) be the Fourier Series operator.
The inverse Fourier Series operator $\hat{F}^{-1}$ is given by

$$f(n) = \sum_{n \in \mathbb{Z}} \bar{\hat{f}}(n) \frac{1}{\sqrt{n}} D_n e^{-i2\pi nt} = \sum_{n \in \mathbb{Z}} E_n \bar{f}(n) \quad \forall f \in L^2_\mathbb{R}$$

where

$$\bar{f}(\omega) \triangleq \int_0^1 E_{-n} f(x) \, dx$$

\[\text{PROOF:}\] See Theorem 2.17 page 18.

5 Demonstrations/Applications

This section demonstrates some applications of the transversal operators in some well-known analytic systems. In particular, the usefulness of the operators in the proofs of some traditional results is demonstrated.

5.1 Multiresolution analysis

5.1.1 Definition

A multiresolution analysis provides “coarse” approximations of a function in a linear space $L^2_\mathbb{R}$ at multiple “scales” or “resolutions”. Key to this process is a sequence of scaling functions. Most traditional transforms feature a single scaling function $\phi(x)$ set equal to one ($\phi(x) = 1$). This allows for convenient representation of the most basic functions, such as constants.\footnote{126}{The concept of a scaling space was perhaps first introduced by Taizo Iijima in 1959 in Japan, and later as the Gaussian Pyramid by Burt and Adelson in the 1980s in the West. $\Rightarrow$ [94], page 70, $\Rightarrow$ [69], $\Rightarrow$ [20], $\Rightarrow$ [2], $\Rightarrow$ [91], $\Rightarrow$ [4], $\Rightarrow$ [54], $\Rightarrow$ [134], (historical survey)} A multiresolution system, on the other hand, uses a generalized form of the scaling concept:\footnote{127}{\cite{73}, page 8}

1. Instead of the scaling function simply being set equal to unity ($\phi(x) = 1$), a multiresolution analysis (Definition 5.1 page 63) is often constructed in such a way that the scaling function $\phi(x)$ forms a partition of unity such that $\sum_{n \in \mathbb{Z}} T^n \phi(x) = 1$.

2. Instead of there being just one scaling function, there is an entire sequence of scaling functions $\left\{D^{j} \phi(x)\right\}_{j \in \mathbb{Z}}$, each corresponding to a different “resolution”.

\[\Rightarrow\]
**Definition 5.1**  
Let \( \langle V_j \rangle_{j \in \mathbb{Z}} \) be a sequence of subspaces on \( L^2_{\mathbb{R}} \). Let \( A^- \) be the closure of a set \( A \). The sequence \( \langle V_j \rangle_{j \in \mathbb{Z}} \) is a **multiresolution analysis** on \( L^2_{\mathbb{R}} \) if:

1. \( V_j = V^-_j \quad \forall j \in \mathbb{Z} \) (closed) and
2. \( V_j \subset V_{j+1} \quad \forall j \in \mathbb{Z} \) (linearly ordered) and
3. \( \bigcup_{j \in \mathbb{Z}} V_j = L^2_{\mathbb{R}} \) (dense in \( L^2_{\mathbb{R}} \)) and
4. \( f \in V_j \iff Df \in V_{j+1} \quad \forall j \in \mathbb{Z}, f \in L^2_{\mathbb{R}} \) (self-similar) and
5. \( \exists \phi \) such that \( \{ T^n \phi | n \in \mathbb{Z} \} \) is a Riesz basis for \( V_0 \).

A multiresolution analysis is also called an **MRA**. An element \( V_j \) of \( \langle V_j \rangle_{j \in \mathbb{Z}} \) is a **scaling subspace** of the space \( L^2_{\mathbb{R}} \). The pair \( (L^2_{\mathbb{R}}, \langle V_j \rangle) \) is a **multiresolution analysis space**, or MRA space. The function \( \phi \) is the **scaling function** of the MRA space.

The traditional definition of the MRA also includes the following:

6. \( f \in V_j \iff T^n f \in V_j \quad \forall n, j \in \mathbb{Z}, f \in L^2_{\mathbb{R}} \) (translation invariant)
7. \( \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \) (greatest lower bound is 0)

However, Proposition 5.2 (next) and Proposition 5.3 (page 64) demonstrate that these follow from the MRA as defined in Definition 5.1.

**Proposition 5.2**  
Let MRA be defined as in Definition 5.1 page 63. 
\[
\{ \langle V_j \rangle_{j \in \mathbb{Z}} \text{ is an MRA} \} \quad \Longrightarrow \quad \{ f \in V_j \iff T^n f \in V_j \quad \forall n, j \in \mathbb{Z}, f \in L^2_{\mathbb{R}} \}
\]

**Proof:**

\( T^n f \in V_j \)

\[
\iff T^n f \in \text{span}\{ D^m T^n \phi | m \in \mathbb{Z} \} \quad \text{by definition of } \langle \phi \rangle \quad (\text{Definition 5.1 page 63})
\]

\[
\iff \exists (\alpha_n)_{n \in \mathbb{Z}} \text{ such that } T^n f(x) = \sum_{k \in \mathbb{Z}} \alpha_k D^k T^n \phi(x) \quad \text{by definition of } \langle \phi \rangle \quad (\text{Definition 5.1 page 63})
\]

\[
\iff \exists (\alpha_n)_{n \in \mathbb{Z}} \text{ such that } f(x) = T^{-n} \sum_{k \in \mathbb{Z}} \alpha_k D^k T^n \phi(x) \quad \text{by definition of } T \quad (\text{Definition 4.1 page 38})
\]

\[
= \sum_{k \in \mathbb{Z}} \alpha_k T^{-n} D^k T^n \phi(x)
\]

\[
= \sum_{k \in \mathbb{Z}} \alpha_k D^k T^{-2n} \phi(x) \quad \text{by Proposition 4.8 page 41}
\]
\[ \alpha \sum_{\ell \in \mathbb{Z}} \alpha_{\ell+2n} D^\ell \phi(x) \quad \text{where} \quad \ell' \triangleq k - 2n \implies k = \ell' + 2n \]

\[ \beta \sum_{\ell \in \mathbb{Z}} \beta_{\ell} D^\ell \phi(x) \quad \text{where} \quad \beta_{\ell} \triangleq \alpha_{\ell+2n} \]

\[ \iff f \in V_j \quad \text{by definition of \{T^n\phi\} (Definition 5.1 page 63)} \]

Proposition 5.3 Let MRA be defined as in Definition 5.1 page 63.

\[ \left\{ \langle V_j \rangle_{j \in \mathbb{Z}} \text{ is an MRA} \right\} \implies \left\{ \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \right\} \quad \text{(greatest lower bound is 0)} \]

\[ \leq \]

Proof:

1. Let \( P_j \) be the projection operator that generates the scaling subspace \( V_j \) such that

\[ V_j = \{ P_j f \mid f \in L^2_{\mathbb{R}} \} \]

2. lemma: Functions with compact support are dense in \( L^2_{\mathbb{R}} \). Therefore, we only need to prove that the proposition is true for functions with support in \([-R, R]\), for all \( R > 0 \).

3. For some function \( f \in L^2_{\mathbb{R}} \), let \( (f_n)_{n \in \mathbb{Z}} \) be a sequence of functions in \( L^2_{\mathbb{R}} \) with compact support such that

\[ \text{supp} f_n \subseteq [-R, R] \text{ for some } R > 0 \quad \text{and} \quad f(x) = \lim_{n \to \infty} f_n(x). \]

4. lemma: \( \bigcap_{j \in \mathbb{Z}} V_j = \{0\} \iff \lim_{j \to -\infty} \|P_j f\| = 0 \quad \forall f \in L^2_{\mathbb{R}} \). Proof:

\[ \bigcap_{j \in \mathbb{Z}} V_j = \bigcap_{j \in \mathbb{Z}} \{ P_j f \mid f \in L^2_{\mathbb{R}} \} \quad \text{by definition of } V_j \quad \text{(definition 1 page 64)} \]

\[ = \lim_{j \to -\infty} \{ P_j f \mid f \in L^2_{\mathbb{R}} \} \quad \text{by definition of } \bigcap \]

\[ = 0 \iff \lim_{j \to -\infty} \|P_j f\| = 0 \quad \text{by nondegenerate property of } \|\cdot\| \quad \text{(Definition 3.6 page 33)} \]

5. lemma: \( \lim_{j \to -\infty} \|P_j f\| = 0 \quad \forall f \in L^2_{\mathbb{R}} \). Proof:

---

130 \( [137] \), pages 19–28, (Proposition 2.14), \( [64] \), page 45, (Theorem 1.6), \( [106] \), pages 313–314, (Lemma 6.4.28)
Let $\mathbb{1}_A(x)$ be the set indicator function (Definition 2.3 page 15)

\[
\lim_{j \to -\infty} \| P_j f \|^2 \\
= \lim_{j \to -\infty} \| P_j \lim_{n \to \infty} (f_n) \|^2 \\
\leq \lim_{j \to -\infty} B \sum_{n \in \mathbb{Z}} \left\langle P_j \lim_{n \to \infty} (f_n) \left| D'T^n \phi(x) \right| \right\rangle^2 \\
= \lim_{j \to -\infty} B \sum_{n \in \mathbb{Z}} \left\langle \lim_{n \to \infty} (f_n) \left| D'T^n \phi(x) \right| \right\rangle^2 \\
= \lim_{j \to -\infty} B \sum_{n \in \mathbb{Z}} \left\langle 1_{[-R, R](x)} \lim_{n \to \infty} (f_n) \left| D'T^n \phi(x) \right| \right\rangle^2 \\
= \lim_{j \to -\infty} B \sum_{n \in \mathbb{Z}} \left\langle \lim_{n \to \infty} (f_n) \left| 1_{[-R, R]}(x)D'T^n \phi(x) \right| \right\rangle^2 \\
\leq \lim_{j \to -\infty} B \sum_{n \in \mathbb{Z}} \left\| \lim_{n \to \infty} (f_n) \right\|^2 \left\| 1_{[-R, R]}(x)D'T^n \phi(x) \right\|^2 \\
= \lim_{j \to -\infty} B \sum_{n \in \mathbb{Z}} \| f \|^2 \left\| 1_{[-R, R]}(x)D'T^n \phi(x) \right\|^2 \\
= \lim_{j \to -\infty} B \sum_{n \in \mathbb{Z}} \| f \|^2 \left\| D'T^{-j/2}1_{[-R, R]}(x) \left| D'T^n \phi(x) \right| \right\|^2 \\
= \lim_{j \to -\infty} B \sum_{n \in \mathbb{Z}} \| f \|^2 \left\| 2^{j/2}D'T^{-j} \left\{ 1_{[-R, R]}(x) \left| D'T^n \phi(x) \right| \right\} \right\|^2 \\
= \lim_{j \to -\infty} B \sum_{n \in \mathbb{Z}} \| f \|^2 \left\| D'T^{-j} \left\{ 1_{[-R, R]}(x) \left| T^n \phi(x) \right| \right\} \right\|^2 \\
= \lim_{j \to -\infty} B \sum_{n \in \mathbb{Z}} \| f \|^2 \left\| D'T^{-j} \left\{ 1_{[-R, R]}(x) \left| T^n \phi(x) \right| \right\} \right\|^2 \\
= \lim_{j \to -\infty} B \sum_{n \in \mathbb{Z}} \| f \|^2 \left\| 1_{[-R, R]}(2^{-j}x + n) \phi(x) \right\|^2 \\
= \lim_{j \to -\infty} B \sum_{n \in \mathbb{Z}} \| f \|^2 \left\| 1_{[-2^{-j}R, 2^{-j}R] \cup (u \phi(2^{-j}(u - n))) \right\|^2 \\
= B \| f \|^2 \sum_{n \in \mathbb{Z}} \left\| 1_{[-2^{-j}R, 2^{-j}R]}(u) \phi(2^{-j}(u - n)) \right\|^2 \\
= B \| f \|^2 \sum_{n \in \mathbb{Z}} \int_{2^{-j}R}^{2^{-j}R} \left\| \phi(2^{-j}(u - n)) \right\|^2 du \\
\]

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\[ B \|f\|^2 \sum_{n \in \mathbb{Z}} \int_n^n |\phi(0)|^2 \, du = 0 \]

(6) Final step in proof that \( \bigcap V_j = \{0\} \): by lemma 4 page 64 and lemma 5 page 65

The MRA (Definition 5.1 page 63) is more than just an interesting mathematical toy. Under some very “reasonable” conditions (next proposition), as \( j \to \infty \), the scaling subspace \( V_j \) is dense in \( L^2_{\mathbb{R}} \) … meaning that with the MRA we can represent any “reasonable” function to within an arbitrary accuracy.

**Proposition 5.4**

\[
\begin{align*}
(1). & \quad \langle T^n \phi \rangle \text{ is a Riesz sequence} \quad \text{ and } \\
(2). & \quad \phi(0) \text{ is continuous at } 0 \quad \text{ and } \\
(3). & \quad \phi(0) \neq 0
\end{align*}
\]

\[ \implies \left\{ \left( \bigcup_{j \in \mathbb{Z}} V_j \right)^{-} = L^2_{\mathbb{R}} \right\} \] (Dense in \( L^2_{\mathbb{R}} \))

5.1.2 Dilation equation

Several functions in mathematics exhibit a kind of self-similar or recursive property:

\( \Rightarrow \) **If a function** \( f(x) \) **is linear, then** (Example 4.33 page 59)

\[ f(x) = f(1)x - f(0)T_x. \]

\( \Rightarrow \) **If a function** \( f(x) \) **is sufficiently bandlimited, then the Cardinal series** (Example 4.34 page 60) **demonstrates**

\[ f(x) = \sum_{n=1}^{\infty} f(n)T^n \sin\left[ \frac{\pi(x)}{\pi(1)} \right]. \]

\( \Rightarrow \) **B-splines** are another example:

\[ N_n(x) = \frac{1}{n}xN_{n-1}(x) - \frac{1}{n}xT_nN_{n-1}(x) + \frac{n+1}{n}TN_{n-1}(x) \quad \forall n \in \mathbb{Z} \setminus \{1\}, \forall x \in \mathbb{R}. \]

The scaling function \( \phi(x) \) (Definition 5.1 page 63) also exhibits a kind of self-similar property. By Definition 5.1 page 63, the dilation \( Df \) of each vector \( f \) in \( V_0 \) is in \( V_1 \). If \( \{ T^n\phi \}_{n \in \mathbb{Z}} \) is a basis for \( V_0 \), then \( \{ D\phi \}_{n \in \mathbb{Z}} \) is a basis for \( V_1 \), \( \{ D^2T^n\phi \}_{n \in \mathbb{Z}} \) is a basis for \( V_2 \), \ldots ; and in general \( \{ D^jT^n\phi \}_{j \in \mathbb{Z}} \) is a basis for \( V_j \). Also, if \( \phi \) is in \( V_0 \), then it is also in \( V_1 \) (because \( V_0 \subset V_1 \)). And because \( \phi \) is in \( V_1 \) and because \( \{ D^n\phi \}_{n \in \mathbb{Z}} \) is a basis for \( V_1 \), \( \phi \) is a linear combination of the elements in \( \{ D^n\phi \}_{n \in \mathbb{Z}} \). That is, \( \phi \) can be represented as a linear

131 [137], pages 28–31, (Proposition 2.15), [53], pages 35–37, (Proposition 2.3)
combination of translated and dilated versions of itself. The resulting equation is called the **dilation equation** (Definition 5.5, next).

**Definition 5.5**

Let \((L^2_{\mathbb{R}}, (V_j))\) be a multiresolution analysis space with scaling function \(\phi\) (Definition 5.1 page 63). Let \((h_n)_{n \in \mathbb{Z}}\) be a sequence (Definition 2.32 page 23) in \(l^2_{\mathbb{R}}\) (Definition 2.33 page 23). The equation

\[
\phi(x) = \sum_{n \in \mathbb{Z}} h_n DT^n \phi(x) \quad \forall x \in \mathbb{R}
\]

is called the **dilation equation**. It is also called the **refinement equation**, **two-scale difference equation**, and **two-scale relation**.

**Theorem 5.6** (dilation equation) Let an MRA space and scaling function be as defined in Definition 5.1 page 63.

\[
\begin{align*}
\{ (L^2_{\mathbb{R}}, (V_j)) \text{ is an MRA space} & \} \quad \implies \quad \{ \exists (h_n)_{n \in \mathbb{Z}} \text{ such that} \quad \phi(x) = \sum_{n \in \mathbb{Z}} h_n DT^n \phi(x) \quad \forall x \in \mathbb{R} \} \\
\end{align*}
\]

Dilation equation in “time”

**Proof:**

\[
\phi \in V_0 \quad \subseteq V_1 \quad = \bigcap \{ DT^n \phi(x) | n \in \mathbb{Z} \} \\
\implies \exists (h_n)_{n \in \mathbb{Z}} \text{ such that} \quad \phi = \sum_{n \in \mathbb{Z}} h_n DT^n \phi
\]

**Lemma 5.7**

Let \(\phi(x)\) be a function in \(L^2_{\mathbb{R}}\) (Definition 2.2 page 15). Let \(\tilde{\phi}(\omega)\) be the Fourier transform of \(\phi(x)\). Let \(\tilde{h}(\omega)\) be the discrete time Fourier transform of a sequence \((h_n)_{n \in \mathbb{Z}}\).

\[
(A) \quad \phi(x) = \sum_{n \in \mathbb{Z}} h_n DT^n \phi(x) \quad \forall x \in \mathbb{R} \quad \iff \quad \tilde{\phi}(\omega) = \frac{\sqrt{2}}{2} \tilde{h} \left( \frac{\omega}{2} \right) \tilde{\phi} \left( \frac{\omega}{2} \right) \quad \forall \omega \in \mathbb{R} \quad (1)
\]

\[
\iff \quad \tilde{\phi}(\omega) = \tilde{\phi} \left( \frac{\omega}{2N} \right) \prod_{n=1}^{N} \frac{\sqrt{2}}{2} \tilde{h} \left( \frac{\omega}{2^n} \right) \quad \forall n \in \mathbb{N}, \omega \in \mathbb{R} \quad (2)
\]

**Proof:**

The property of translation invariance is of particular significance in the theory of normed linear spaces (a Hilbert space is a complete normed linear space equipped with an inner product).

132 [73], page 7
133 [95], page 228
(1) Proof that \((A) \implies (1)\):
\[
\hat{\phi}(\omega) \triangleq \hat{F}\phi
\]
\[
= \mathbf{F} \sum_{n \in \mathbb{Z}} h_n DT^n \phi(x) \quad \text{by (A)}
\]
\[
= \sum_{n \in \mathbb{Z}} h_n \hat{F} DT^n \phi(x) \quad \text{by Proposition 4.19 page 47}
\]
\[
= \sqrt{2} \left\{ \sum_{n \in \mathbb{Z}} h_n e^{-i \frac{\omega}{2} \frac{n}{N}} \hat{\phi} \left( \frac{\omega}{2} \right) \right\} \quad \text{by definition of DTFT (Definition 2.38 page 24)}
\]
\[
= \sqrt{2} \left[ \sum_{n \in \mathbb{Z}} h_n e^{-i \frac{\omega}{2} \frac{n}{N}} \right] \frac{\hat{\phi}(\omega/2)}{\hat{h}(\omega/2)}
\]
\[
= \sqrt{2} \hat{h} \left( \frac{\omega}{2} \right) \hat{\phi} \left( \frac{\omega}{2} \right) \quad \text{by definition of DTFT (Definition 2.38 page 24)}
\]

(2) Proof that \((A) \iff (1)\):
\[
\phi(x) = \hat{F}^{-1} \hat{\phi}(\omega) \quad \text{by definition of } \hat{\phi}(\omega)
\]
\[
= \hat{F}^{-1} \sqrt{2} \hat{h} \left( \frac{\omega}{2} \right) \hat{\phi} \left( \frac{\omega}{2} \right) \quad \text{by (1)}
\]
\[
= \hat{F}^{-1} \sqrt{2} \sum_{n \in \mathbb{Z}} h_n e^{-i \frac{\omega}{2} \frac{n}{N}} \hat{\phi} \left( \frac{\omega}{2} \right) \quad \text{by definition of DTFT (Definition 2.38 page 24)}
\]
\[
= \sqrt{2} \sum_{n \in \mathbb{Z}} h_n \hat{F}^{-1} e^{-i \frac{\omega}{2} \frac{n}{N}} \hat{\phi} \left( \frac{\omega}{2} \right) \quad \text{by property of linear operators}
\]
\[
= \sqrt{2} \sum_{n \in \mathbb{Z}} h_n \hat{F}^{-1} FDT^n \phi
\]
\[
= \sum_{n \in \mathbb{Z}} h_n DT^n \phi(x)
\]

(3) Proof that \((1) \implies (2)\):
(a) Proof for \(N = 1\) case:
\[
\hat{\phi} \left( \frac{\omega}{2N} \right) \prod_{n=1}^{N} \sqrt{2} \hat{h} \left( \frac{\omega}{2n} \right) \bigg|_{N=1} = \sqrt{2} \hat{h} \left( \frac{\omega}{2} \right) \hat{\phi} \left( \frac{\omega}{2} \right)
\]
\[
= \hat{\phi}(\omega) \quad \text{by (1)}
\]
(b) Proof that \([N \text{ case}] \implies [N+1 \text{ case}]\):
\[
\hat{\phi} \left( \frac{\omega}{2N+1} \right) \prod_{n=1}^{N+1} \sqrt{2} \hat{h} \left( \frac{\omega}{2n} \right) = \left[ \prod_{n=1}^{N} \sqrt{2} \hat{h} \left( \frac{\omega}{2n} \right) \right] \sqrt{2} \hat{h} \left( \frac{\omega}{2N+1} \right) \hat{\phi} \left( \frac{\omega}{2N+1} \right)
\]

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\[ = \tilde{\phi}(\omega/2^N) \prod_{n=1}^{N} \sqrt{2} \hat{h}_n \left( \frac{\omega}{2^n} \right) \]
\[ = \tilde{\phi}(\omega) \]
by \([N \text{ case}] \text{ hypothesis}\)

(4) Proof that (1) \(\iff\) (2):

\[ \tilde{\phi}(\omega) = \tilde{\phi}\left( \frac{\omega}{2^N} \right) \prod_{n=1}^{N} \sqrt{2} \hat{h}_n \left( \frac{\omega}{2^n} \right) \]
by (2)
\[ = \tilde{\phi}\left( \frac{\omega}{2} \right) \sqrt{2} \hat{h}_n \left( \frac{\omega}{2^n} \right) \]
\[ = \sqrt{2} \hat{h}_n \left( \frac{\omega}{2} \right) \tilde{\phi}\left( \frac{\omega}{2} \right) \]

\[ \text{Lemma 5.8} \]
Let \(\phi(x)\) be a function in \(L^2_{\mathbb{R}}\) (Definition 2.2 page 15). Let \(\tilde{\phi}(\omega)\) be the Fourier transform of \(\phi(x)\). Let \(\hat{h}(\omega)\) be the Discrete time Fourier transform of \(\{h_n\}\). Let \(\prod_{n=1}^{\infty} x_n \triangleq \lim_{N \to \infty} \prod_{n=1}^{N} x_n\), with respect to the standard norm in \(L^2_{\mathbb{R}}\).

\[ \left\{ \begin{array}{l} \tilde{\phi}(\omega) = C \prod_{n=1}^{\infty} \sqrt{2} \hat{h}_n \left( \frac{\omega}{2^n} \right) \quad \forall C > 0, \omega \in \mathbb{R} \\ \phi(x) = \sum_{n \in \mathbb{Z}} h_n DT^n \phi(x) \quad \forall x \in \mathbb{R} \quad (1) \end{array} \right. \]
\[ \iff \tilde{\phi}(\omega) = \sqrt{2} \hat{h}_n \left( \frac{\omega}{2} \right) \tilde{\phi}\left( \frac{\omega}{2} \right) \quad \forall \omega \in \mathbb{R} \quad (2) \]
\[ \iff \tilde{\phi}(\omega) = \tilde{\phi}\left( \frac{\omega}{2^N} \right) \prod_{n=1}^{N} \sqrt{2} \hat{h}_n \left( \frac{\omega}{2^n} \right) \quad \forall n \in \mathbb{N}, \omega \in \mathbb{R} \quad (3) \]

\[ \text{PROOF:} \]

(1) Proof that (1) \(\iff\) (2) \(\iff\) (3): by Lemma 5.7 page 67
(2) Proof that (A) $\implies$ (2):
\[
\tilde{\phi}(\omega) = C \prod_{n=1}^{\infty} \sqrt{2} \hat{h}\left(\frac{\omega}{2^n}\right) 
\]
by left hypothesis
\[
= C \sqrt{2} \hat{h}\left(\frac{\omega}{2}\right) \prod_{n=1}^{\infty} \sqrt{2} \hat{h}\left(\frac{\omega}{2^{n+1}}\right) 
\]
\[
= C \sqrt{2} \hat{h}\left(\frac{\omega}{2}\right) \prod_{n=1}^{\infty} \sqrt{2} \hat{h}\left(\frac{\omega/2}{2^n}\right) 
\]
\[
= \sqrt{2} \hat{h}\left(\frac{\omega}{2}\right) \tilde{\phi}\left(\frac{\omega}{2}\right) 
\]
by left hypothesis
\[
\implies \tilde{\phi}(\omega) = \tilde{\phi}\left(\frac{\omega}{2}\right) 
\]
by continuity and definition of $\lim_{N \to \infty} \prod_{n=1}^{N} x_n$

\[
\tilde{\phi}(0) \prod_{n=1}^{\infty} \sqrt{2} \hat{h}\left(\frac{\omega}{2^n}\right) = \lim_{N \to \infty} \tilde{\phi}\left(\frac{\omega}{2^N}\right) \prod_{n=1}^{N} \sqrt{2} \hat{h}\left(\frac{\omega}{2^n}\right) 
\]
by continuity and definition of $\lim_{N \to \infty} \prod_{n=1}^{N} x_n$
\[
= \tilde{\phi}(\omega) 
\]
by (3) and Lemma 5.7 page 67

\[\Box\]

**Proposition 5.9** Let $\phi(x)$ be a function in $L^2_{\mathbb{R}}$ (Definition 2.2 page 15). Let $\hat{\phi}(\omega)$ be the Fourier transform of $\phi(x)$. Let $\hat{h}(\omega)$ be the Discrete time Fourier transform of $\{h_n\}$. Let $\lim_{N \to \infty} \prod_{n=1}^{N} x_n$ with respect to the standard norm in $L^2_{\mathbb{R}}$.

\[
\lim_{N \to \infty} \prod_{n=1}^{N} x_n 
\]

\[
\begin{align*}
\tilde{\phi}(\omega) & \text{ is continuous at } \omega = 0 \\
\Rightarrow \tilde{\phi}(\omega) & = \frac{\sqrt{2}}{2} \hat{h}\left(\frac{\omega}{2}\right) \tilde{\phi}\left(\frac{\omega}{2}\right) \quad \forall \omega \in \mathbb{R} & (2) \\
\Rightarrow \tilde{\phi}(\omega) & = \hat{\phi}\left(\frac{\omega}{2}\right) \prod_{n=1}^{\infty} \sqrt{2} \hat{h}\left(\frac{\omega}{2^n}\right) \quad \forall n \in \mathbb{N}, \omega \in \mathbb{R} & (3) \\
\Rightarrow \tilde{\phi}(\omega) & = \tilde{\phi}(0) \prod_{n=1}^{\infty} \sqrt{2} \hat{h}\left(\frac{\omega}{2^n}\right) \quad \omega \in \mathbb{R} & (4)
\end{align*}
\]

\[\Box\]
Proof that (2) $\iff$ (4): by Lemma 5.8 page 69

Definition 5.10 (next) formally defines the coefficients that appear in Theorem 5.6 (page 67).

**Definition 5.10** Let $(L^2_{\mathbb{R}}, \{V_j\})$ be a multiresolution analysis space with scaling function $\phi$. Let $(h_n)_{n \in \mathbb{Z}}$ be a sequence of coefficients such that $\phi = \sum_{n \in \mathbb{Z}} h_n DT^n \phi$. A **multiresolution system** is the tuple $(L^2_{\mathbb{R}}, \{V_j\}, \phi, (h_n))$. The sequence $(h_n)_{n \in \mathbb{Z}}$ is the **scaling coefficient sequence**. A multiresolution system is also called an **MRA system**. An MRA system is an **orthonormal MRA system** if $\{ DT^n \phi \}_{n \in \mathbb{Z}}$ is orthonormal.

**Definition 5.11** Let $(L^2_{\mathbb{R}}, \{V_j\}, \phi, (h_n))$ be a multiresolution system, and $D$ the dilation operator. The **normalization coefficient at resolution** $n$ is the quantity $\| D^j \phi \|$. 

**Theorem 5.12** Let $(L^2_{\mathbb{R}}, \{V_j\})$ be an MRA system (Definition 5.10 page 71). Let $\text{span} A$ be the **linear span** (Definition 3.2 page 32) of a set $A$.

\[
\text{span}\{ DT^n \phi | n \in \mathbb{Z} \} = V_0 \quad \implies \quad \text{span}\{ D^j DT^n \phi | n \in \mathbb{Z} \} = V_j \quad \forall j \in \mathbb{W} \\
\{ DT^n \phi | n \in \mathbb{Z} \} \text{ is a basis for } V_0 \\
\{ D^j DT^n \phi | n \in \mathbb{Z} \} \text{ is a basis for } V_j
\]

Proof: Proof is by induction:

1. Induction basis (proof for $j = 0$ case):

\[
\text{span}\left\{ D^j DT^n \phi | n \in \mathbb{Z} \right\}|_{j=0} = \text{span}\{ DT^n \phi | n \in \mathbb{Z} \} = V_0 \quad \text{by left hypothesis}
\]

\[\text{[121], page 4}\]
(2) induction step (proof that $j$ case $\implies j+1$ case):

$$\begin{align*}
\text{span} \left\{ D^{j+1}T^n \phi \mid n \in \mathbb{Z} \right\} \\
= \left\{ f \in L^2_{\mathbb{R}} \mid \exists (a_n) \text{ such that } f(x) = \sum_{n \in \mathbb{Z}} a_n D^{j+1}T^n \phi \right\} & \quad \text{by def. of span (Definition 3.2 page 32)} \\
= \left\{ f \in L^2_{\mathbb{R}} \mid \exists (a_n) \text{ such that } f(x) = D \sum_{n \in \mathbb{Z}} a_n D^n T^n \phi \right\} \\
= \left\{ f \in L^2_{\mathbb{R}} \mid \exists (a_n) \text{ such that } D^{-1}f(x) = \sum_{n \in \mathbb{Z}} a_n D^n T^n \phi \right\} \\
= \left\{ [Df] \in L^2_{\mathbb{R}} \mid \exists (a_n) \text{ such that } D^{-1}[Df(x)] = \sum_{n \in \mathbb{Z}} a_n D^n T^n \phi \right\} \\
= D \left\{ f \in L^2_{\mathbb{R}} \mid \exists (a_n) \text{ such that } Df(x) = \sum_{n \in \mathbb{Z}} a_n D^n T^n \phi \right\} & \quad \text{by def. of span (Definition 3.2 page 32)} \\
= D \text{span} \left\{ D^n T^n \phi \mid n \in \mathbb{Z} \right\} & \quad \text{by induction hypothesis} \\
= D \mathcal{V}_j & \quad \text{by self-similar property (Definition 5.1 page 63)} \\
= \mathcal{V}_{j+1}
\end{align*}$$

Example 5.13

In the Haar MRA, the scaling function $\phi(x)$ is the pulse function

$$\phi(x) = \begin{cases} 
  1 & \text{for } x \in [0, 1) \\
  0 & \text{otherwise.}
\end{cases}$$

In the subspace $\mathcal{V}_j (j \in \mathbb{Z})$ the scaling functions are

$$D^j \phi(x) = \begin{cases} 
  (2)^{j/2} & \text{for } x \in [0, (2^{-j})) \\
  0 & \text{otherwise.}
\end{cases}$$

The scaling subspace $\mathcal{V}_0$ is the span $\mathcal{V}_0 = \text{span} \{ T^n \phi \mid n \in \mathbb{Z} \}$. The scaling subspace $\mathcal{V}_j$ is the
### Table: Subspace Transform Approximations

| Subspace | Transform | Approximation |
|----------|-----------|--------------|
| $V_0$    | ![Transform](image1) | ![Approximation](image2) |
| $V_1$    | ![Transform](image3) | ![Approximation](image4) |
| $V_2$    | ![Transform](image5) | ![Approximation](image6) |

Figure 4: Example approximations of $\sin(\pi x)$ in 3 Haar scaling subspaces (see Example 5.13 page 72)

\[
\text{span } V_j \triangleq \text{span } \{ D^j T^n \phi \mid n \in \mathbb{Z} \}. \text{ Note that } \| D^j T^n \phi \| \text{ for each resolution } j \text{ and shift } n \text{ is unity:}
\]

\[
\| D^j T^n \phi \| ^2 = \phi^2 = \int_0^1 |1|^2 \, dx = 1
\]

Let $f(x) = \sin(\pi x)$. Suppose we want to project $f(x)$ onto the subspaces $V_0, V_1, V_2, \ldots$.

The values of the transform coefficients for the subspace $V_j$ are given by
\[
[R_j f(x)](n) = \frac{1}{\|D^T \phi\|^2} \langle f(x) | D^T^n \phi \rangle
\]
\[
= \frac{1}{\|\phi\|^2} \langle f(x) | \phi(2^j x - n) \rangle
\]
\[
= 2^{j/2} \langle f(x) | \phi(2^j x - n) \rangle
\]
\[
= 2^{j/2} \int_{2^{-j} n}^{2^{-j} (n+1)} f(x) \, dx
\]
\[
= 2^{j/2} \int_{2^{-j} n}^{2^{-j} (n+1)} \sin(\pi x) \, dx
\]
\[
= 2^{j/2} \left( -\frac{1}{\pi} \cos(\pi x) \right)_{2^{-j} n}^{2^{-j} (n+1)}
\]
\[
= \frac{2^{j/2}}{\pi} \left[ \cos(2^{-j} n \pi) - \cos(2^{-j} (n+1) \pi) \right]
\]

And the projection \(A_j f(x)\) of the function \(f(x)\) onto the subspace \(V_j\) is
\[
A_j f(x) = \sum_{n \in \mathbb{Z}} \langle f(x) | D^T^n \phi \rangle \, D^T^n \phi
\]
\[
= \frac{2^{j/2}}{\pi} \sum_{n \in \mathbb{Z}} \left[ \cos(2^{-j} n \pi) - \cos(2^{-j} (n+1) \pi) \right] 2^{j/2} \phi(2^j x - n)
\]
\[
= \frac{2^j}{\pi} \sum_{n \in \mathbb{Z}} \left[ \cos(2^{-j} n \pi) - \cos(2^{-j} (n+1) \pi) \right] \phi(2^j x - n)
\]

The transforms into the subspaces \(V_0\), \(V_1\), and \(V_2\), as well as the approximations in those subspaces are as illustrated in Figure 4 (page 73).

5.1.3 Necessary Conditions

\textbf{Theorem 5.14} (admissibility condition) \textit{Let \(\hat{h}(z)\) be the Z-transform (Definition 2.35 page 24) and \(\hat{h}(\omega)\) the discrete-time Fourier transform (Definition 2.38 page 24) of a sequence \(\{h_n\}_{n \in \mathbb{Z}}\). \(\{L_2^2, (V_j), \phi, (h_n)\}\) is an MRA system (Definition 5.10 page 71).}

\[
\iff \left\{ \sum_{n \in \mathbb{Z}} h_n = \sqrt{2} \right\} \iff \left\{ \hat{h}(z) \bigg|_{z=1} = \sqrt{2} \right\} \iff \left\{ \hat{h}(\omega) \big|_{\omega=0} = \sqrt{2} \right\}
\]

\(\textbf{Proof:}\)
(1) Proof that MRA system \(\implies (1):
\[
\sum_{n \in \mathbb{Z}} h_n = \frac{\int_{\mathbb{R}} \phi(x) \, dx}{\int_{\mathbb{R}} \phi(x) \, dx} \sum_{n \in \mathbb{Z}} h_n \\
= \frac{1}{\int_{\mathbb{R}} \phi(x) \, dx} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} h_n \phi(x) \, dx \\
= \frac{1}{\int_{\mathbb{R}} \phi(x) \, dx} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} h_n \sqrt{2} \phi(2y - n) \, dy \\
\text{let } y = \frac{x + n}{2} \implies x = 2y - n \implies dx = 2 \, dy \\
= \frac{2}{\sqrt{2}} \frac{1}{\int_{\mathbb{R}} \phi(x) \, dx} \int_{\mathbb{R}} \sum_{n \in \mathbb{Z}} h_n \mathcal{D} \mathcal{T}^n \phi(y) \, dy \\
\text{by definitions of } \mathcal{T} \text{ and } \mathcal{D} \text{ (Definition 4.1 page 38)} \\
= \sqrt{2} \frac{1}{\int_{\mathbb{R}} \phi(x) \, dx} \int_{\mathbb{R}} \phi(y) \, dy \\
\text{by dilation equation (Theorem 5.6 page 67)} \\
= \sqrt{2}
\]

(2) Alternate proof that MRA system \(\implies (1):
Let \(f(x) \triangleq 1 \quad \forall x \in \mathbb{R}\).
\[
\langle \phi | f \rangle = \left\langle \sum_{n \in \mathbb{Z}} h_n \mathcal{D} \mathcal{T}^n \phi \bigg| f \right\rangle \\
\text{by dilation equation (Theorem 5.6 page 67)} \\
= \sum_{n \in \mathbb{Z}} h_n \langle \mathcal{D} \mathcal{T}^n \phi | f \rangle \\
\text{by linearity of } \langle \triangle | \triangledown \rangle \\
= \sum_{n \in \mathbb{Z}} h_n \langle \phi | (\mathcal{D} \mathcal{T}^n)^* f \rangle \\
\text{by definition of operator adjoint (Theorem 1.35 page 10)} \\
= \sum_{n \in \mathbb{Z}} h_n \langle \phi | (\mathcal{T}^n)^* \mathcal{D}^* f \rangle \\
\text{by property of operator adjoint (Theorem 1.35 page 10)} \\
= \sum_{n \in \mathbb{Z}} h_n \langle \phi | (\mathcal{T}^{-1})^n \mathcal{D}^{-1} f \rangle \\
\text{by unitary property of } \mathcal{T} \text{ and } \mathcal{D} \text{ (Proposition 4.10 page 43)} \\
= \sum_{n \in \mathbb{Z}} h_n \langle \phi | (\mathcal{T}^{-1})^n \sqrt{2} \frac{f}{x} \rangle \\
\text{because } f \text{ is a constant and by Proposition 4.3 page 39} \\
= \sum_{n \in \mathbb{Z}} h_n \langle \phi | \sqrt{2} \frac{f}{x} \rangle \\
\text{by } f(x) = 1 \text{ definition} \\
= \sum_{n \in \mathbb{Z}} h_n \sqrt{2} \langle \phi | f \rangle \\
\text{by property of } \langle \triangle | \triangledown \rangle \\
= \sqrt{2} \langle \phi | f \rangle \sum_{n \in \mathbb{Z}} h_n \\
\implies \sum_{n \in \mathbb{Z}} h_n = \sqrt{2}
\]

(3) Proof that \((1) \iff (2) \iff (3): \text{by Proposition 2.40 page 25.} \)
(4) Proof for \( \Leftrightarrow \) part: by Counterexample 5.15 page 76.

**Counterexample 5.15**  Let \( (L^2_\mathbb{R}, \{V_j\}, \phi, \{h_n\}) \) be an MRA system (Definition 5.10 page 71).

\[
\begin{cases}
\{h_n\} \triangleq \sqrt{2} \delta_{n-1} \triangleq \left\{ \begin{array}{ll}
\sqrt{2} & \text{for } n = 1 \\
0 & \text{otherwise}
\end{array} \right.
\end{cases}
\Rightarrow \{\phi(x) = 0\}
\]

which means

\[
\left\{ \sum_{n \in \mathbb{Z}} h_n = \sqrt{2} \right\} \Rightarrow \{ (L^2_\mathbb{R}, \{V_j\}, \phi, \{h_n\}) \text{ is an MRA system for } L^2_\mathbb{R} \}.
\]

**Proof:**

\[
\phi(x) = \sum_{n \in \mathbb{Z}} h_n D T^n \phi(x) \quad \text{by dilation equation (Theorem 5.6 page 67)}
\]

\[
= \sum_{n \in \mathbb{Z}} h_n \phi(2x - n) \quad \text{by definitions of } D \text{ and } T \text{ (Definition 4.1 page 38)}
\]

\[
= \sum_{n \in \mathbb{Z}} \sqrt{2} \delta_{n-1} \phi(2x - n) \quad \text{by definitions of } \{h_n\}
\]

\[
= \sqrt{2} \phi(2x - 1) \quad \text{by definition of } \phi(x)
\]

\[
\Rightarrow \phi(x) = 0
\]

This implies \( \phi(x) = 0 \), which implies that \( (L^2_\mathbb{R}, \{V_j\}, \phi, \{h_n\}) \) is not an MRA system for \( L^2_\mathbb{R} \) because

\[
\left( \bigcup_{j \in \mathbb{Z}} V_j \right)^{-} = \left( \bigcup_{j \in \mathbb{Z}} \mathfrak{m}(D^j T^n \phi \mid n \in \mathbb{Z}) \right)^{-} \neq L^2_\mathbb{R}
\]

(the least upper bound is not \( L^2_\mathbb{R} \)).

**Theorem 5.16**  (Quadrature condition in “time”)  Let \( (L^2_\mathbb{R}, \{V_j\}, \phi, \{h_n\}) \) be an MRA system (Definition 5.10 page 71).

\[
\sum_{m \in \mathbb{Z}} h_m \sum_{k \in \mathbb{Z}} h_k^* \langle \phi \mid T^{2n-m+k} \phi \rangle = \langle \phi \mid T^n \phi \rangle \quad \forall n \in \mathbb{Z}
\]
\[ \langle \phi \mid T^n \phi \rangle = \left\langle \sum_{m \in \mathbb{Z}} h_m D^m \phi \mid T^n \sum_{k \in \mathbb{Z}} h_k D^k \phi \right\rangle \] by dilation equation (Theorem 5.6 page 67)

\[ = \sum_{m \in \mathbb{Z}} h_m \sum_{k \in \mathbb{Z}} h_k^* \langle \phi \mid (D^m)^* T^n D^k \phi \rangle \] by properties of \( \langle \triangledown \mid \bigtriangledown \rangle \)

\[ = \sum_{m \in \mathbb{Z}} h_m \sum_{k \in \mathbb{Z}} h_k^* \langle \phi \mid (D^m)^* DT^{2m-k} \phi \rangle \] by Proposition 4.8 page 41

\[ = \sum_{m \in \mathbb{Z}} h_m \sum_{k \in \mathbb{Z}} h_k^* \langle \phi \mid T^{m} D T^{2m-k} \phi \rangle \] by Proposition 4.10 page 43

\[ = \sum_{m \in \mathbb{Z}} h_m \sum_{k \in \mathbb{Z}} h_k^* \langle \phi \mid T^{2m-k} \phi \rangle \] by Proposition 1.35 page 10

Theorem 5.24 (next) presents the quadrature necessary conditions of a wavelet system. These relations simplify dramatically in the special case of an orthonormal wavelet system (Theorem 2.44 page 27).

**Theorem 5.17** (Quadrature condition in “frequency”) \(^{136}\)

Let \((L^2_\mathbb{R}, \langle V \rangle, \phi, \langle h \rangle)\) be an MRA system (Definition 5.10 page 71). Let \(\tilde{x}(\omega)\) be the discrete time Fourier transform for a sequence \(\langle x_n \rangle_{n \in \mathbb{Z}}\) in \(L^2_\mathbb{R}\). Let \(\tilde{S}_{\phi\phi}(\omega)\) be the auto-power spectrum (Definition 4.27 page 51) of \(\phi\).

\[ |\tilde{h}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega) + |\tilde{h}(\omega + \pi)|^2 \tilde{S}_{\phi\phi}(\omega + \pi) = 2 \tilde{S}_{\phi\phi}(2\omega) \tag{Note: \(\tilde{S}_{\phi\phi}(\omega) = 1\ for\ orthonormal\ MRA)} \]

**Proof:**

\[ 2 \tilde{S}_{\phi\phi}(2\omega) = 2(2\pi) \sum_{n \in \mathbb{Z}} |\phi(2\omega + 2\pi n)|^2 \] by Theorem 4.28 page 51

\[ = 2(2\pi) \sum_{n \in \mathbb{Z}} \sqrt{\frac{|\tilde{h}(\frac{2\omega + 2\pi n}{2})|^2 |\tilde{\phi}(\frac{2\omega + 2\pi n}{2})|^2}} \] by Lemma 5.7 page 67

\[ = 2 \pi \sum_{n \in \mathbb{Z}} |\tilde{h}(\frac{2\omega + 2\pi n}{2})|^2 |\tilde{\phi}(\frac{2\omega + 2\pi n}{2})|^2 + 2 \pi \sum_{n \in \mathbb{Z}} |\tilde{h}(\frac{2\omega + 2\pi n}{2})|^2 |\tilde{\phi}(\frac{2\omega + 2\pi n}{2})|^2 + 2 \pi \sum_{n \in \mathbb{Z}} |\tilde{h}(\omega + 2\pi n + \pi)|^2 |\tilde{\phi}(\omega + 2\pi n + \pi)|^2 \]

\(^{136}\) [27], page 135, [52], page 110
\[= 2\pi \sum_{n \in \mathbb{Z}} |\hat{h}(\omega)|^2 |\hat{\phi}(\omega + 2\pi n)|^2 + 2\pi \sum_{n \in \mathbb{Z}} |\hat{h}(\omega + \pi)|^2 |\hat{\phi}(\omega + 2\pi n + \pi)|^2 \quad \text{by Proposition 2.39 page 25} \]

\[= |\hat{h}(\omega)|^2 \left(2\pi \sum_{n \in \mathbb{Z}} |\hat{\phi}(\omega + 2\pi n)|^2 \right) + |\hat{h}(\omega + \pi)|^2 \left(2\pi \sum_{n \in \mathbb{Z}} |\hat{\phi}(\omega + 2\pi n + \pi)|^2 \right) \]

\[= |\hat{h}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega) + |\hat{h}(\omega + \pi)|^2 \tilde{S}_{\phi\phi}(\omega + \pi) \quad \text{by Theorem 4.28 page 51} \]

5.1.4 Sufficient conditions

Theorem 5.18 (next) gives a set of sufficient conditions on the scaling function (Definition 5.1 page 63) \(\phi\) to generate an MRA.

**Theorem 5.18** 137 Let an MRA be defined as in Definition 5.1 page 63. Let a Riesz sequence be defined as in Definition 3.20 page 37. Let \(V_j \triangleq \text{span}\{T\phi(x)|_{n\in\mathbb{Z}}\} \).

\[
\begin{align*}
&\text{(1). } \langle T^n\phi \rangle \text{ is a Riesz sequence} \quad \text{and} \quad \\
&\text{(2). } \exists \{h_n\} \text{ such that } \phi(x) = \sum_{n \in \mathbb{Z}} h_n \text{DT}^n\phi(x) \quad \text{and} \quad \\
&\text{(3). } \tilde{\phi}(\omega) \text{ is continuous at } 0 \quad \text{and} \quad \\
&\text{(4). } \tilde{\phi}(0) \neq 0
\end{align*}
\]

\[\implies \{V_j\}_{j\in\mathbb{Z}} \text{ is an MRA} \]

**Proof:** For this to be true, each of the conditions in the definition of an MRA (Definition 5.1 page 63) must be satisfied:

1. Proof that each \(V_j\) is closed: by definition of \(\text{span}\)
2. Proof that \(\langle V_j \rangle\) is linearly ordered:
   \[V_j \subseteq V_{j+1} \iff \text{span}\{D'T^n\phi\} \subseteq \text{span}\{D'(T^n\phi)\} \iff (2)\]
3. Proof that \(\bigcup_{j\in\mathbb{Z}} V_j\) is dense in \(L^2_{\mathbb{R}}\): by Proposition 5.4 page 66
4. Proof of self-similar property:
   \[
   \{f \in V_j \iff Df \in V_{j+1}\} \iff f \in \text{span}\{T^n\phi\} \iff Df \in \text{span}\{DT^n\phi\} \iff (2)
   \]
5. Proof for Riesz basis: by (1) and Proposition 5.4 page 66.

137 \[\{137\}, \text{page 28, (Theorem 2.13), } \{106\}, \text{page 313, (Theorem 6.4.27) }\]
5.2 Wavelet analysis

5.2.1 Definition

The term “wavelet” comes from the French word “ondelette”, meaning “small wave”. And in essence, wavelets are “small waves” (as opposed to the “long waves” of Fourier analysis) that form a basis for the Hilbert space \( L^2_{\mathbb{R}} \).

**Definition 5.19** \(^{139}\) Let \( T \) and \( D \) be as defined in Definition 4.1 page 38. A function \( \psi(x) \) in \( L^2_{\mathbb{R}} \) is a wavelet function for \( L^2_{\mathbb{R}} \) if
\[
\{ D^nT^j\psi \mid j, n \in \mathbb{Z} \}
\]
is a Riesz basis for \( L^2_{\mathbb{R}} \).

In this case, \( \psi \) is also called the mother wavelet of the basis \( \{ D^nT^j\psi \mid j, n \in \mathbb{Z} \} \). The sequence of subspaces \( \{ W_j \} \) is the wavelet analysis induced by \( \psi \), where each subspace \( W_j \) is defined as
\[
W_j \triangleq \text{span}\{ D^nT^j\psi \mid n \in \mathbb{Z} \}.
\]

A wavelet analysis \( \{ W_j \} \) is often constructed from a multiresolution analysis (Definition 5.1 page 63) \( \{ V_j \} \) under the relationship
\[
V_{j+1} = V_j \ast W_j, \quad \text{where} \ast \text{ is subspace addition (Minkowski addition).}
\]
By this relationship alone, \( \{ W_j \} \) is in no way uniquely defined in terms of a multiresolution analysis \( \{ V_j \} \). In general there are many possible complements of a subspace \( V_j \). To uniquely define such a wavelet subspace, one or more additional constraints are required. One of the most common additional constraints is orthogonality, such that \( V_j \) and \( W_j \) are orthogonal to each other.

5.2.2 Dilation equation

Suppose \( \{ T^n\psi \}_{n \in \mathbb{Z}} \) is a basis for \( W_0 \). By Definition 5.19 page 79, the wavelet subspace \( W_0 \) is contained in the scaling subspace \( V_1 \). By Definition 5.1 page 63, the sequence \( \{ DT^n\phi \}_{n \in \mathbb{Z}} \) is a basis for \( V_1 \). Because \( W_0 \) is contained in \( V_1 \), the sequence \( \{ DT^n\phi \}_{n \in \mathbb{Z}} \) is also a basis for \( W_0 \).

**Theorem 5.20** Let \( (L^2_{\mathbb{R}}, \{ V_j \}, \phi, \{ h_n \}) \) be a multiresolution system and \( \{ W_j \}_{j \in \mathbb{Z}} \) a wavelet analysis with respect to \( (L^2_{\mathbb{R}}, \{ V_j \}, \phi, \{ h_n \}) \) and with wavelet function \( \psi \).

\[
\exists \{ g_n \}_{n \in \mathbb{Z}} \text{ such that } \psi = \sum_{n \in \mathbb{Z}} g_n DT^n\phi
\]

\(^{138}\) [127], page ix, \(^{[6]}\), page 191

\(^{139}\) [137], page 17, (Definition 2.1), \(^{[53]}\), page 50, (Definition 2.4), \(^{[53]}\), page 50, (Definition 2.4)
PROOF:

\[ \psi \in W_0 \]
\[ \subseteq V_1 \]
\[ = \text{span}(\mathbf{T}^n \phi(x))_{n \in \mathbb{Z}} \]
\[ \implies \exists (g_n)_{n \in \mathbb{Z}} \text{ such that } \psi = \sum_{n \in \mathbb{Z}} g_n \mathbf{T}^n \phi \]

A wavelet system (next definition) consists of two subspace sequences:

A multiresolution analysis \( (V_j) \) (Definition 5.1 page 63) provides “coarse” approximations of a function in \( L^2_{\mathbb{R}} \) at different “scales” or resolutions.

A wavelet analysis \( (W_j) \) provides the “detail” of the function missing from the approximation provided by a given scaling subspace (Definition 5.19 page 79).

**Definition 5.21** Let \( (L^2_{\mathbb{R}}, (V_j), \phi, (h_n)) \) be a multiresolution system (Definition 5.1 page 63) and \( (W_j)_{j \in \mathbb{Z}} \) a wavelet analysis (Definition 5.19 page 79) with respect to \( (V_j)_{j \in \mathbb{Z}} \). Let \( (g_n)_{n \in \mathbb{Z}} \) be a sequence of coefficients such that \( \psi = \sum_{n \in \mathbb{Z}} g_n \mathbf{T}^n \phi \). A wavelet system is the tuple \( (L^2_{\mathbb{R}}, (V_j), (W_j), \phi, \psi, (h_n), (g_n)) \) and the sequence \( (g_n)_{n \in \mathbb{Z}} \) is the wavelet coefficient sequence.

5.2.3 Necessary conditions

**Theorem 5.22** (quadrature conditions in “time”) Let \( (L^2_{\mathbb{R}}, (V_j), (W_j), \phi, \psi, (h_n), (g_n)) \) be a wavelet system (Definition 5.21 page 80).

1. \[ \sum_{m \in \mathbb{Z}} h_m \sum_{k \in \mathbb{Z}} h_k^* \langle \phi | T^{2n-m+k} \phi \rangle = \langle \phi | T^n \phi \rangle \quad \forall n \in \mathbb{Z} \]
2. \[ \sum_{m \in \mathbb{Z}} g_m \sum_{k \in \mathbb{Z}} g_k^* \langle \phi | T^{2n-m+k} \phi \rangle = \langle \psi | T^n \psi \rangle \quad \forall n \in \mathbb{Z} \]
3. \[ \sum_{m \in \mathbb{Z}} h_m \sum_{k \in \mathbb{Z}} g_k^* \langle \phi | T^{2n-m+k} \phi \rangle = \langle \phi | T^n \psi \rangle \quad \forall n \in \mathbb{Z} \]

**Proof:**

(1) Proof for (1): by Theorem 5.16 page 76.
(2) Proof for (2):
\[ \langle \psi | T^n \psi \rangle = \left\langle \sum_{m \in \mathbb{Z}} g_m D T^m \phi | T^n \sum_{k \in \mathbb{Z}} g_k D T^k \phi \right\rangle \]
by Theorem 5.20 page 79
\[ = \sum_{m \in \mathbb{Z}} g_m \sum_{k \in \mathbb{Z}} g_k^* \langle \phi | T^{m} D T^k \phi \rangle \]
by properties of \( \langle \triangle | \nabla \rangle \)
\[ = \sum_{m \in \mathbb{Z}} g_m \sum_{k \in \mathbb{Z}} g_k^* \langle \phi | (D T^m)\ast T^k D T^k \phi \rangle \]
by definition of operator adjoint (Proposition 1.33 page 10)
\[ = \sum_{m \in \mathbb{Z}} g_m \sum_{k \in \mathbb{Z}} g_k^* \langle \phi | T^{m}D T^{2n} T^k \phi \rangle \]
by operator star-algebra properties (Theorem 1.35 page 10)
\[ = \sum_{m \in \mathbb{Z}} g_m \sum_{k \in \mathbb{Z}} g_k^* \langle \phi | T^{2n-m} D^{-1} D T^{2n} T^k \phi \rangle \]
by Proposition 4.10 page 43
\[ = \sum_{m \in \mathbb{Z}} g_m \sum_{k \in \mathbb{Z}} g_k^* \langle \phi | T^{2n-m+k} \phi \rangle \]

(3) Proof for (3):
\[ \langle \phi | T^n \psi \rangle = \left\langle \sum_{m \in \mathbb{Z}} h_m D T^m \phi | T^n \sum_{k \in \mathbb{Z}} g_k D T^k \phi \right\rangle \]
by Theorem 5.6 page 67 and Theorem 5.20 page 79
\[ = \sum_{m \in \mathbb{Z}} h_m \sum_{k \in \mathbb{Z}} g_k^* \langle \phi | T^{m} D T^k \phi \rangle \]
by properties of \( \langle \triangle | \nabla \rangle \)
\[ = \sum_{m \in \mathbb{Z}} h_m \sum_{k \in \mathbb{Z}} g_k^* \langle \phi | (D T^m)\ast T^k D T^k \phi \rangle \]
by definition of operator adjoint (Proposition 1.33 page 10)
\[ = \sum_{m \in \mathbb{Z}} h_m \sum_{k \in \mathbb{Z}} g_k^* \langle \phi | (D T^m)\ast D T^{2n} T^k \phi \rangle \]
by Proposition 4.8 page 41
\[ = \sum_{m \in \mathbb{Z}} h_m \sum_{k \in \mathbb{Z}} g_k^* \langle \phi | T^{-m} D^{-1} D T^{2n} T^k \phi \rangle \]
by operator star-algebra properties (Theorem 1.35 page 10)
\[ = \sum_{m \in \mathbb{Z}} h_m \sum_{k \in \mathbb{Z}} g_k^* \langle \phi | T^{2n-m} D^{-1} D T^{2n} T^k \phi \rangle \]
by Proposition 4.10 page 43
\[ = \sum_{m \in \mathbb{Z}} h_m \sum_{k \in \mathbb{Z}} g_k^* \langle \phi | T^{2n-m+k} \phi \rangle \]

Proposition 5.23 Let \( (L_2^r, \{ V_j \}, \{ W_j \}, \phi, \psi, \langle h_n \rangle, \langle g_n \rangle) \) be a wavelet system. Let \( \hat{\phi}(\omega) \) and \( \hat{\psi}(\omega) \) be the Fourier transforms of \( \phi(x) \) and \( \psi(x) \), respectively. Let \( \hat{g}(\omega) \) be the discrete time Fourier transform of \( \langle g_n \rangle \).
\[ \hat{\psi}(\omega) = \frac{\sqrt{2}}{\pi} \hat{g}\left(\frac{\alpha}{2}\right) \hat{\phi}\left(\frac{\alpha}{2}\right) \]
PROOF:

\[ \tilde{\psi}(\omega) \triangleq \mathcal{F}\psi \]

\[ = \mathcal{F} \sum_{n \in \mathbb{Z}} g_n D^* \phi \]

by Theorem 5.20 page 79

\[ = \sum_{n \in \mathbb{Z}} g_n \mathcal{F} D^* \phi \]

by Corollary 4.18 page 47

\[ = \sum_{n \in \mathbb{Z}} g_n D^{-1} e^{-i\omega n} \mathcal{F} \phi \]

by Corollary 4.18 page 47

\[ = \sum_{n \in \mathbb{Z}} g_n \sqrt{2} (D^{-1} e^{-i\omega n}) (D^{-1} \mathcal{F} \phi) \]

by Proposition 4.7 page 41

\[ = \sqrt{2} \left( D^{-1} \sum_{n \in \mathbb{Z}} g_n e^{-i\omega n} \right) (D^{-1} \mathcal{F} \phi) \]

\[ = \sqrt{2} \left( D^{-1} \mathcal{F} (g_n) \right) (D^{-1} \mathcal{F} \phi) \]

by definition of \( \mathcal{F} \)

\[ = \sqrt{2} \sqrt{2} \hat{g} \left( \frac{\omega}{2} \right) \hat{\phi} \left( \frac{\omega}{2} \right) \]

by Proposition 4.3 page 39

Theorem 5.24 (next) presents the \textit{quadrature} necessary conditions of a wavelet system. These relations simplify dramatically in the special case of an \textit{orthonormal wavelet system} (Theorem 2.44 page 27).

**Theorem 5.24** (Quadrature conditions in “frequency”) \footnote{\cite{27}, page 135, \cite{52}, page 110} Let \( (L^2(\mathbb{R}^1), (V_j), (W_j), \phi, \psi, (h_n), (g_n)) \) be a wavelet system. Let \( \tilde{s}(\omega) \) be the \textit{discrete time Fourier transform} for a sequence \( (x_n)_{n \in \mathbb{Z}} \) in \( l^2(\mathbb{R}) \). Let \( \tilde{S}_{\phi\phi}(\omega) \) be the \textit{auto-power spectrum} (Definition 4.27 page 51) of \( \phi \), \( \tilde{S}_{\psi\psi}(\omega) \) be the \textit{auto-power spectrum} of \( \psi \), and \( \tilde{S}_{\phi\psi}(\omega) \) be the \textit{cross-power spectrum} of \( \phi \) and \( \psi \).

1. \[ |\tilde{h}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega) + |\tilde{h}(\omega + \pi)|^2 \tilde{S}_{\phi\phi}(\omega + \pi) = 2 \tilde{S}_{\phi\psi}(2\omega) \]
2. \[ |\tilde{g}(\omega)|^2 \tilde{S}_{\phi\phi}(\omega) + |\tilde{g}(\omega + \pi)|^2 \tilde{S}_{\phi\phi}(\omega + \pi) = 2 \tilde{S}_{\psi\psi}(2\omega) \]
3. \[ \tilde{h}(\omega) \tilde{g}^*(\omega) \tilde{S}_{\phi\phi}(\omega) + \tilde{h}(\omega + \pi) \tilde{g}^*(\omega + \pi) \tilde{S}_{\phi\phi}(\omega + \pi) = 2 \tilde{S}_{\phi\psi}(2\omega) \]

**PROOF:**

(1) Proof for (1): by Theorem 5.17 page 77.

\footnote{\cite{27}, page 135, \cite{52}, page 110}
(2) Proof for (2):

\[ 2 \tilde{S}_{\psi\phi}(2\omega) \triangleq 2(2\pi) \sum_{n=\mathbb{Z}} |\tilde{\psi}(2\omega + 2\pi n)|^2 \]

\[ = 2(2\pi) \sum_{n=\mathbb{Z}} |\frac{\sqrt{2}}{2} \check{g} \left( \frac{2\omega + 2\pi n}{2} \right) \phi \left( \frac{2\omega + 2\pi n}{2} \right)|^2 \quad \text{by Lemma 5.7 page 67} \]

\[ = 2\pi \sum_{n=\mathbb{Z}} |\check{g} \left( \frac{2\omega + 2\pi n}{2} \right)|^2 |\phi \left( \frac{2\omega + 2\pi n}{2} \right)|^2 + \]

\[ 2\pi \sum_{n=\mathbb{Z}} |\frac{\sqrt{2}}{2} \check{h} \left( \omega + \pi n \right) \check{g}^* \left( \omega + \pi n \right) \tilde{\phi}^* \left( \omega + \pi n \right)|^2 \]

\[ = 2\pi \sum_{n=\mathbb{Z}} |\check{g} \left( \omega + 2\pi n \right)|^2 |\phi \left( \omega + 2\pi n \right)|^2 + 2\pi \sum_{n=\mathbb{Z}} |\check{g} \left( \omega + 2\pi n + \pi \right)|^2 |\phi \left( \omega + 2\pi n + \pi \right)|^2 \]

\[ = 2\pi \left( 2\pi \sum_{n=\mathbb{Z}} |\tilde{\phi} \left( \omega + 2\pi n \right)|^2 + |\check{g} \left( \omega + \pi \right)|^2 \left( 2\pi \sum_{n=\mathbb{Z}} |\tilde{\phi} \left( \omega + 2\pi n + \pi \right)|^2 \right) \right) \]

\[ = |\check{g} \left( \omega \right)|^2 \tilde{S}_{\phi\phi}(\omega) + |\check{g} \left( \omega + \pi \right)|^2 \tilde{S}_{\phi\phi}(\omega + \pi) \quad \text{by Theorem 4.28 page 51} \]

(3) Proof for (3):

\[ 2\tilde{S}_{\phi\psi}(2\omega) = 2(2\pi) \sum_{n=\mathbb{Z}} \tilde{\phi}(2\omega + 2\pi n)\check{\psi}^*(2\omega + 2\pi n) \]

\[ = 2(2\pi) \sum_{n=\mathbb{Z}} \frac{\sqrt{2}}{2} \check{h} \left( \omega + \pi n \right) \tilde{\phi}^* \left( \omega + \pi n \right) \check{g}^* \left( \omega + \pi n \right) \tilde{\phi}^* \left( \omega + \pi n \right) \quad \text{by Lemma 5.7 page 67} \]

\[ = 2\pi \sum_{n=\mathbb{Z}} \check{h} \left( \omega + \pi n \right) \check{g}^* \left( \omega + \pi n \right) |\tilde{\phi} \left( \omega + \pi n \right)|^2 \]

\[ = 2\pi \sum_{n=\mathbb{Z}} \check{h} \left( \omega + \pi n \right) \check{g}^* \left( \omega + \pi n \right) |\tilde{\phi} \left( \omega + \pi n \right)|^2 \]

\[ + 2\pi \sum_{n=\mathbb{Z}} \check{h} \left( \omega + \pi n \right) \check{g}^* \left( \omega + \pi n \right) |\tilde{\phi} \left( \omega + \pi n \right)|^2 \]

\[ = 2\pi \sum_{n=\mathbb{Z}} \check{h} \left( \omega + 2\pi n + \pi \right) \check{g}^* \left( \omega + 2\pi n + \pi \right) |\tilde{\phi} \left( \omega + 2\pi n + \pi \right)|^2 \]

\[ + 2\pi \sum_{n=\mathbb{Z}} \check{h} \left( \omega + 2\pi n \right) \check{g}^* \left( \omega + 2\pi n \right) |\tilde{\phi} \left( \omega + 2\pi n \right)|^2 \]

\[ = 2\pi \sum_{n=\mathbb{Z}} \check{h} \left( \omega + \pi \right) \check{g}^* \left( \omega + \pi \right) |\tilde{\phi} \left( \omega + 2\pi n + \pi \right)|^2 + 2\pi \sum_{n=\mathbb{Z}} \check{h} \left( \omega \right) \check{g}^* \left( \omega \right) |\tilde{\phi} \left( \omega + 2\pi n \right)|^2 \]

\[ = \check{h} \left( \omega \right) \check{g}^* \left( \omega \right) \left( 2\pi \sum_{n=\mathbb{Z}} |\tilde{\phi} \left( \omega + 2\pi n \right)|^2 \right) \]

\[ + \check{h} \left( \omega + \pi \right) \check{g}^* \left( \omega + \pi \right) \left( 2\pi \sum_{n=\mathbb{Z}} |\tilde{\phi} \left( \omega + 2\pi n + \pi \right)|^2 \right) \]
\[
\begin{align*}
&= \hat{h}(\omega)\hat{g}^*(\omega) \left( 2\pi \sum_{n \in \mathbb{Z}} |\hat{\phi}(\omega + 2\pi n)|^2 \right) + \hat{h}(\omega + \pi)\hat{g}^*(\omega + \pi) \left( 2\pi \sum_{n \in \mathbb{Z}} |\hat{\phi}(\omega + \pi + 2\pi n)|^2 \right) \\
&= \hat{h}(\omega)\hat{g}^*(\omega)\hat{S}_{\hat{\phi}\phi}\hat{\phi}(\omega) + \hat{h}(\omega + \pi)\hat{g}^*(\omega + \pi)\hat{S}_{\hat{\phi}\phi}\hat{\phi}(\omega + \pi) \quad \text{by Theorem 4.28 page 51}
\end{align*}
\]

### 5.2.4 Sufficient condition

In this text, an often used sufficient condition for designing the wavelet coefficient sequence \( (g_n) \) (Definition 5.21 page 80) is the conjugate quadrature filter condition. It expresses the sequence \( (g_n) \) in terms of the scaling coefficient sequence (Definition 5.10 page 71) and a “shift” integer \( N \) as \( g_n = \pm(-1)^n h^*_{N-n} \).

**Theorem 5.25** Let \( (L^2_{\mathbb{R}}, (V_j), (W_j), \phi, \psi, (h_n), (g_n)) \) be a WAVELET SYSTEM (Definition 5.21 page 80). Let \( \hat{g}(\omega) \) be the DTFT (Definition 2.38 page 24) and \( \hat{g}(z) \) the Z-TRANSFORM (Definition 2.35 page 24) of \( (g_n) \).

\[
\begin{align*}
\underbrace{g_n = \pm(-1)^n h^*_{N-n}}_{\text{CONJUGATE QUADRATURE FILTER}} \quad \Leftrightarrow \quad & \hat{g}(\omega) = (\pm(-1)^N e^{-i\omega N}h^*_{0})_{\omega = \pi} \quad (1) \\
\Rightarrow \quad & \sum_{n \in \mathbb{Z}} (-1)^n g_n = \sqrt{2} \quad (2) \\
\Leftrightarrow \quad & \hat{g}(z) \bigg|_{z=-1} = \sqrt{2} \quad (3) \\
\Leftrightarrow \quad & \hat{g}(\omega) \bigg|_{\omega = \pi} = \sqrt{2} \quad (4)
\end{align*}
\]

**Proof:**

(1) Proof that CQF \( \Leftrightarrow \) (1): by Theorem 2.46 page 30

(2) Proof that CQF \( \Leftrightarrow \) (4):

\[
\begin{align*}
\hat{g}(\omega) &= \hat{g}(\omega) \bigg|_{\omega = \pi} \\
&= (\pm(-1)^N e^{-i\omega N}h^*_{0})_{\omega = \pi} \quad \text{by Theorem 2.46 page 30} \\
&= (\pm(-1)^N e^{-i\pi N}h^*_{0}) \\
&= (\pm(-1)^N(-1)^N h^*_{0}) \quad \text{by Proposition 2.39 page 25} \\
&= \sqrt{2} \quad \text{by admissibility condition (Theorem 5.14 page 74)}
\end{align*}
\]

(3) Proof that (2) \( \Leftrightarrow \) (3) \( \Leftrightarrow \) (4): by Proposition 2.41 page 26
5.3 Partition of unity systems

5.3.1 Motivation

A very common property of scaling functions (Definition 5.1 page 63) is the partition of unity property (Definition 5.2 page 86). The partition of unity is a kind of generalization of orthonormality; that is, all orthonormal scaling functions form a partition of unity. But the partition of unity property is not just a consequence of orthonormality, but also a generalization of orthonormality, in that if you remove the orthonormality constraint, the partition of unity is still a reasonable constraint in and of itself.

There are two reasons why the partition of unity property is a reasonable constraint on its own:

- Without a partition of unity, it is difficult to represent a function as simple as a constant.\textsuperscript{141}
- For a multiresolution system \((L^2_{\mathbb{R}}, \{V_j\}, \phi, \{h_n\})\), the partition of unity property is equivalent to \(\sum_{n \in \mathbb{Z}} (-1)^n h_n = 0\) (Theorem 5.2 page 88). As viewed from the perspective of discrete time signal processing, this implies that the scaling coefficients form a “lowpass filter”; lowpass filters provide a kind of “coarse approximation” of a function. And that is what the scaling function is “supposed” to do—to provide a coarse approximation at some resolution or “scale” (Definition 5.1 page 63).

\textsuperscript{141}[73], page 8
5.3.2 Definition and results

**Definition 5.26** A function \( \mathcal{F} \in \mathbb{R} \) forms a partition of unity if
\[
\sum_{n \in \mathbb{Z}} T^n f(x) = 1 \quad \forall x \in \mathbb{R}.
\]

**Theorem 5.27** Let \( (L^2_{\mathbb{R}}, \{V_j\}, \phi, \{h_n\}) \) be a multiresolution system (Definition 5.10 page 71). Let \( \tilde{F} f(\omega) \) be the Fourier transform (Definition 2.22 page 20) of a function \( f \in L^2_{\mathbb{R}} \). Let \( \delta_n \) be the Kronecker delta function.
\[
\sum_{n \in \mathbb{Z}} T^n f = c \quad \iff \quad \left[ \tilde{F} f \right](2\pi n) = \delta_n
\]

**Proof:** Let \( Z_e \) be the set of even integers and \( Z_o \) the set of odd integers.

1. Proof for ( \( \Rightarrow \) ) case:
\[
c = \sum_{m \in \mathbb{Z}} T^m f(x) \quad \text{by left hypothesis}
\]
\[
= \sum_{m \in \mathbb{Z}} f(x - m) \quad \text{by definition of } T \text{ (Definition 4.1 page 38)}
\]
\[
= \sqrt{2\pi} \sum_{m \in \mathbb{Z}} \tilde{f}(2\pi m) e^{i2\pi mx}
\]
\[
= \sqrt{2\pi} \tilde{f}(2\pi n) e^{i2\pi nx} + \sqrt{2\pi} \sum_{m \in \mathbb{Z}} \tilde{f}(2\pi m) e^{i2\pi mx}
\]
\[
\text{real and constant for } n = 0 \quad \text{and complex and non-constant}
\]
\[
\Rightarrow \sqrt{2\pi} \tilde{f}(2\pi n) = c \delta_n \quad \text{because } c \text{ is real and constant for all } n
\]
(2) Proof for ( \( \iff \) ) case:

\[
\sum_{n \in \mathbb{Z}} T^n f(x) = \sum_{n \in \mathbb{Z}} f(x - n) \quad \text{by definition of } T \quad \text{(Definition 4.1 page 38)}
\]

\[
= \sqrt{2\pi} \sum_{n \in \mathbb{Z}} \hat{f}(2\pi n) e^{-i2\pi nx} \quad \text{by PSF (Theorem 4.21 page 48)}
\]

\[
= \sqrt{2\pi} \sum_{n \in \mathbb{Z}} \frac{c}{\sqrt{2\pi}} \hat{\delta}_n e^{-i2\pi nx} \quad \text{by right hypothesis}
\]

\[
= \sqrt{2\pi} \frac{c}{\sqrt{2\pi}} e^{-i2\pi 0x} \quad \text{by definition of } \hat{\delta}_n \quad \text{(Definition 3.12 page 35)}
\]

\[
= c
\]

**Corollary 5.28**

\[
\begin{cases}
\exists g \in L_2^2 \text{ such that } \\
 f(x) = 1_{[-1, 1]}(x) \ast g(x)
\end{cases}
\]

\[
\implies \begin{cases}
 f(x) \text{ generates} \\
 \text{a partition of unity}
\end{cases}
\]

**Example 5.29** All *B-splines* form a partition of unity. All B-splines of order \( n = 1 \) or greater can be generated by convolution with a *pulse* function, similar to that specified in Corollary 5.28 (page 87).

**Example 5.30** Let a function \( f \) be defined in terms of the cosine function (Definition 2.5 page 16) as follows:

\[
f(x) \triangleq \begin{cases}
\cos^2 \left( \frac{\pi}{2} x \right) & \text{for } |x| \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

Then \( f \) forms a *partition of unity*.

Note that \( \hat{f}(\omega) = \frac{1}{2\sqrt{2\pi}} \left[ \frac{2 \sin \omega}{\omega} + \frac{\sin(\omega - \pi)}{(\omega - \pi)} + \frac{\sin(\omega + \pi)}{(\omega + \pi)} \right] \)

and so \( \hat{f}(2\pi n) = \frac{1}{\sqrt{2\pi}} \hat{\delta}_n \);
Example 5.31 (raised cosine) Let a function \( f \) be defined in terms of the cosine function (Definition 2.5 page 16) as follows:

\[
\begin{align*}
\text{Let } f(x) &\equiv \begin{cases} 
1 & \text{for } 0 \leq |x| < 1 - \frac{\beta}{2} \\
\frac{1}{2} \left( 1 + \cos \left[ \frac{\pi}{\beta} \left( |x| - \frac{1 - \beta}{2} \right) \right] \right) & \text{for } \frac{1 - \beta}{2} \leq |x| < \frac{1 + \beta}{2} \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

Then \( f \) forms a partition of unity.

5.3.3 Scaling functions with partition of unity

The \( Z \) transform (Definition 2.35 page 24) of a sequence \( \langle h_n \rangle \) with sum \( \sum_{n \in \mathbb{Z}} (-1)^n h_n = 0 \) has a zero at \( z = -1 \). Somewhat surprisingly, the partition of unity and zero at \( z = -1 \) properties are actually equivalent (next theorem).

Theorem 5.32 Let \( (L^2_{\mathbb{R}}, \langle V_j \rangle, \phi, \langle h_n \rangle) \) be a multiresolution system (Definition 5.10 page 71). Let \( \hat{f}(\omega) \) be the Fourier transform (Definition 2.22 page 20) of a function \( f \in L^2_{\mathbb{R}} \).

\[\hat{f}(-1) = \hat{f}(\pi) = 0\]

\[\sum_{n \in \mathbb{Z}} h_n\]

\[z = e^{i\omega}\]

\[\mathfrak{R} [z]\]

\[\mathfrak{I} [z]\]

\[h(1) = h(\pi) = 0\]

\[h(n)\]

\[\omega\]

\[\mathfrak{I} [z]\]

\[\mathfrak{R} [z]\]

\[\sum_{n \in \mathbb{Z}} h_n\]

\[\omega\]
KRONECKER DELTA FUNCTION.

\[
\sum_{n \in \mathbb{Z}} T^n \phi = c \quad \text{for some } c \in \mathbb{R} \setminus 0 \quad \iff \quad \sum_{n \in \mathbb{Z}} (-1)^n h_n = 0 \quad \iff \quad \sum_{n \in \mathbb{Z}} h_{2n} = \sum_{n \in \mathbb{Z}} h_{2n+1} = \frac{\sqrt{2}}{2}
\]

\(1\) PARTITION OF UNITY
\(2\) ZERO AT \(z = -1\)
\(3\) sum of even = sum of odd = \(\frac{\sqrt{2}}{2}\)

\% PROOF: Let \(Z_e\) be the set of even integers and \(Z_o\) the set of odd integers.

(1) Proof that (1) \(\iff\) (2):

\[
\sum_{n \in \mathbb{Z}} T^n \phi = \sum_{n \in \mathbb{Z}} T^n \left[ \sum_{m \in \mathbb{Z}} h_m DT^m \phi \right] = \sum_{m \in \mathbb{Z}} h_m \sum_{n \in \mathbb{Z}} T^n DT^m \phi = \sum_{m \in \mathbb{Z}} h_m \sum_{n \in \mathbb{Z}} DT^2n T^n \phi = D \sum_{m \in \mathbb{Z}} h_m \sum_{n \in \mathbb{Z}} T^{2n} T^n \phi
\]

by dilation equation (Theorem 5.6 page 67)

\[
= D \sum_{m \in \mathbb{Z}} h_m \left[ \sqrt{\frac{2\pi}{2}} \hat{F}^{-1} S_2 \hat{F}(T^m \phi) \right] = \sqrt{\pi} D \sum_{m \in \mathbb{Z}} h_m \hat{F}^{-1} S_2 e^{-i\omega_0 m} \hat{F} \phi = \sqrt{\pi} D \sum_{m \in \mathbb{Z}} h_m \hat{F}^{-1}(-1)^{km} S_2 \hat{F} \phi = \sqrt{\pi} D \sum_{m \in \mathbb{Z}} h_m \hat{F}^{-1} \left[ \sum_{k \in \mathbb{Z}} (-1)^{km} (S_2 \hat{F} \phi) e^{i \frac{2\pi}{2} kx} \right]
\]

by PSF (Theorem 4.21 page 48) by Corollary 4.18 page 47

by definition of \(S\) (Theorem 4.21 page 48)

\[
= \sqrt{\pi} D \sum_{m \in \mathbb{Z}} h_m \left[ \frac{\sqrt{2}}{2} \sum_{k \in \mathbb{Z}} (-1)^{km} (S_2 \hat{F} \phi) e^{i \frac{2\pi}{2} kx} \right] = \sqrt{\frac{2\pi}{2}} D \sum_{k \in \mathbb{Z}} (S_2 \hat{F} \phi) e^{i \pi kx} \sum_{m \in \mathbb{Z}} (-1)^{km} h_m
\]

\[= \sqrt{\frac{2\pi}{2}} D \sum_{k \in \mathbb{Z}} (S_2 \hat{F} \phi) e^{i \pi kx} \sum_{m \in \mathbb{Z}} (-1)^{km} h_m + \sqrt{\frac{2\pi}{2}} D \sum_{k \in \mathbb{Z}} (S_2 \hat{F} \phi) e^{i \pi kx} \sum_{m \in \mathbb{Z}} (-1)^{km} h_m
\]

\[
= \sqrt{\frac{2\pi}{2}} D \sum_{k \in \mathbb{Z}} (S_2 \hat{F} \phi) e^{i \pi kx} \sum_{m \in \mathbb{Z}} h_m = \sqrt{\frac{\sqrt{2}}{2}}
\]
\[ + \frac{\sqrt{2\pi}}{2} D \sum_{k \in \mathbb{Z}} (S_2 \tilde{F} \phi) e^{i\pi k x} \sum_{m \in \mathbb{Z}} (-1)^m h_m \]

\[ = \sqrt{\pi} D \sum_{k \in \mathbb{Z}} (S_2 \tilde{F} \phi) e^{i\pi k x} \]

by Theorem 5.14 (page 74) and right hyp.

\[ = \sqrt{\pi} D \sum_{k \in \mathbb{Z}} \tilde{\phi}(\frac{2\pi}{2} k) e^{i\pi k x} \]

by definitions of \( \tilde{F} \) and \( S_2 \)

\[ = \sqrt{\pi} D \sum_{k \in \mathbb{Z}} \tilde{\phi}(2\pi k) e^{i2\pi k x} \]

by definition of \( \mathbb{Z}_e \)

\[ = \frac{1}{\sqrt{2}} D \left\{ \sqrt{2\pi} \sum_{k \in \mathbb{Z}} \tilde{\phi}(2\pi k) e^{i2\pi k x} \right\} \]

\[ = \frac{1}{\sqrt{2}} D \sum_{n \in \mathbb{Z}} \phi(x + n) \]

by PSF (Theorem 4.21 page 48)

\[ = \frac{1}{\sqrt{2}} D \sum_n T^n \phi \]

by definition of \( T \) (Definition 4.1 page 38)

The above equation sequence demonstrates that

\[ D \sum_n T^n \phi = \sqrt{2} \sum_n T^n \phi \]

(essentially that \( \sum_n T^n \phi \) is equal to it’s own dilation). This implies that \( \sum_n T^n \phi \) is a constant (Proposition 4.11 page 45).

(2) Proof that (1) \( \implies \) (2):

\[ c = \sum_{n \in \mathbb{Z}} T^n \phi \]

by left hypothesis

\[ = \sqrt{2\pi} \tilde{F}^{-1} S \tilde{F} \phi \]

by PSF (Theorem 4.21 page 48)

\[ = \sqrt{2\pi} \tilde{F}^{-1} S \sqrt{2} \left( D^{-1} \sum_{n \in \mathbb{Z}} h_n e^{-i\omega n} \right) \left( D^{-1} \tilde{F} \phi \right) \]

by Lemma 5.7 page 67

\[ = 2\sqrt{\pi} \tilde{F}^{-1} \left( \frac{S D^{-1} \sum_{n \in \mathbb{Z}} h_n e^{-i\omega n}}{2} \right) \left( S \tilde{F} \phi \right) \]

by Corollary 4.18 page 47

\[ = 2\sqrt{\pi} \tilde{F}^{-1} \left( \frac{S}{\sqrt{2}} \sum_{n \in \mathbb{Z}} h_n e^{-i\frac{\omega n}{2}} \right) \left( S \tilde{F} \phi \right) \]

by Proposition 4.3 page 39

\[ = \sqrt{2\pi} \tilde{F}^{-1} \left( \sum_{n \in \mathbb{Z}} h_n e^{-i\frac{2\pi n}{2}} \right) \left( S \tilde{F} \phi \right) \]

by def. of \( S \) (Theorem 4.21 page 48)
\[
\begin{align*}
&= \sqrt{2\pi} \hat{F}^{-1} \left( \sum_{n \in \mathbb{Z}} h_n (-1)^n \right) (\mathbf{S}^{-1} \mathbf{F} \phi) \\
&= \sqrt{2\pi} \hat{F}^{-1} \left( \sum_{n \in \mathbb{Z}} h_n (-1)^n \right) \left( S \frac{1}{\sqrt{2}} \hat{\phi} \left( \frac{\omega}{2} \right) \right) \quad \text{by def. of } S \text{(Theorem 4.21 page 48)} \\
&= \sqrt{2\pi} \hat{F}^{-1} \left( \sum_{n \in \mathbb{Z}} h_n (-1)^n \right) \left( \frac{1}{\sqrt{2}} \hat{\phi} \left( \frac{2\pi k}{2} \right) \right) \quad \text{by def. of } \hat{F}^{-1} \text{(Theorem 2.17 page 18)} \\
&= \sqrt{\pi} \sum_{k \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} h_n (-1)^n \phi(\pi k) e^{i2\pi k x} \quad \text{by Theorem 5.14 page 74} \\
&= \sqrt{2\pi} \sum_{k \text{ even} \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} h_n (-1)^n \phi(\pi k) e^{i2\pi k x} \\
&\quad + \sqrt{\pi} \sum_{k \text{ odd} \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} h_n (-1)^n \phi(\pi k) e^{i2\pi k x} \\
&= \frac{\sqrt{2\pi}}{\sqrt{2\pi}} \hat{\phi}(0) + \sqrt{\pi} e^{i2\pi x} \sum_{n \in \mathbb{Z}} h_n (-1)^n \sum_{k \in \mathbb{Z}} \phi(\pi [2k + 1]) e^{i4\pi k x} \quad \text{by left hyp. and Theorem 5.27 page 86} \\
\Rightarrow \quad \left( \sum_{n \in \mathbb{Z}} h_n (-1)^n \right) = 0 \quad \text{because the right side must equal } c \\
\end{align*}
\]

(3) Proof that (2) \(\Rightarrow\) (3):

\[
\sum_{n \in \mathbb{Z}_o} h_n = \sum_{n \in \mathbb{Z}_o} h_n = \frac{1}{2} \sum_{n \in \mathbb{Z}} h_n \quad \text{by (2) and Proposition 2.41 page 26} \\
= \frac{\sqrt{2}}{2} \quad \text{by admissibility condition \text{(Theorem 5.14 page 74)}}
\]

(4) Proof that (2) \(\Leftarrow\) (3):

\[
\frac{\sqrt{2}}{2} = \sum_{n \in \mathbb{Z}_e} (-1)^n h_n + \sum_{n \in \mathbb{Z}_o} (-1)^n h_n \quad \text{by (3)}
\]
\[
\sum_{n \in \mathbb{Z}} (-1)^n h_n = 0 \quad \text{by Proposition 2.41 page 26}
\]

Proposition 5.33
\[
\phi(x) \text{ generates a partition of unity} \quad \implies \quad \phi(x) \text{ generates an MRA system.}
\]

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