Is the multiplicative anomaly relevant?

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Abstract: In a recent work, S. Dowker has shed doubt on a recipe used in computing the partition function for a matrix valued operator. This recipe, advocated by Benson, Bernstein and Dodelson, leads naturally to the so called multiplicative anomaly for the zeta-function regularized functional determinants. In this letter we present arguments in favour of the mentioned prescription, showing that it is the valid one in calculations involving the relativistic charged bosonic ideal gas in the framework of functional analysis.

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In a recent work, Stuart Dowker ¹ has shed doubt on a commonly used manipulation in computing functional determinants related to matrix valued elliptic operators. This recipe is widely used ²–⁵ and can be summarized as follows. Within the one-loop approximation or in the external field approximation, one often has to evaluate Euclidean functional integrals of the kind

\[ Z = \int D\phi_1 D\phi_2 e^{-\int d^4x \phi_i A_{ij}\phi_j} , \]

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where $A$ is a matrix valued differential operator, which we assume to have constant coefficients (this is not a restriction as long as one has to compute the one-loop effective potential). The result of the Gaussian functional integration is

$$Z = (\det A)^{-1/2}.$$  

(2)

The problem is how to compute the functional determinant in Eq. (2). Note that $A$ has matrix elements with discrete (field) and continuous (spacetime) indices. If $A$ is also diagonal in the discrete indices, one has

$$A = \begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}.$$  

(3)

In his work Dowker presents two different ways of computing this determinant, claiming that one of them does not lead to correct results. The first recipe is equivalent to taking the algebraic determinant first and the functional one afterwards, i.e.

$$(\det A)_1 = \det(L_1L_2).$$  

(4)

According to Dowker, it could seem more natural to use the alternative recipe

$$(\det A)_2 = \det L_1 \det L_2.$$  

(5)

This is seen as the implementation of the right way to take the functional determinant of the matrix valued operator, i.e. an ordinary determinant on both the spatial and field indices, at the same moment. The example analyzed is that of two real free scalar fields of different masses, for which the partition function has the form in (1) and the operator is diagonal in the field indices $i, j$ as in (3).

This second recipe seems quite natural indeed, since it is well known that for finite matrices, $\det AB = \det A \det B$. Unfortunately, in the continuum, one needs a regularization and for regularized functional determinants the two recipes may give different answers, because of the existence of the so called multiplicative anomaly, discovered by Wodzicki (see, for example, (8)),

$$\det AB = \det A \det Be^{a(A,B)},$$  

(6)

and, also in very simple cases of physical interest, it is possible to show that $a(A,B)$ is not vanishing (8). Thus the question posed by Dowker is substantial.

In our opinion, both recipes are formally acceptable. In fact if one considers the eigenvalue problem for the matrix valued operator $A$, one arrives at the formal determinant: $\det A = \prod n_1 n_2 \lambda_{n_1} \lambda_{n_2}$, which can as well be rewritten as $\det A = \prod n_2 \lambda_{n_1} \prod n_2 \lambda_{n_2}$. Of course, as rigorous equalities these expressions are restricted to the finite dimensional case (and to a limited class of absolutely convergent situations).

In the following we would like to present arguments in favour of the first recipe, analysing the general validity of both the recipes and also considering as a crucial example the case of a free relativistic charged bosonic field at finite temperature (9). The self-interacting charged scalar field was studied in (3).

We have argued that both recipes actually coincide in the finite dimensional case. One should keep in mind that any extension to the continuous, functional case always starts from a discretization. In particular, the finite dimensional example posed by Dowker at the end of
pretending to prove a discrepancy already at this level does not apply. In fact, let us start considering Dowker statements in general. He supports the inequivalence of the two recipes analysing a generic four by four matrix $A_{\alpha\beta ij}$ diagonal on $i,j$, where the indices $\alpha, \beta = 1, 2$ represent the discretized version of the continuous space-time indices and $i, j = 1, 2$ the fields indices.

Then, the two recipes correspond to $\det_{\alpha,\beta} \det_{i,j} A_{\alpha\beta ij}$ and $\det_{\alpha,\beta,i,j} A_{\alpha\beta ij}$ respectively. For such a generic matrix it is straightforward to see that the two results are different and Dowker’s statement that the first recipe could give inexact results seems correct. The point, though, is that the matrices we encounter in field theory have additional structural requirements. In the continuous limit, the matrix valued differential operator is normally diagonal in the continuous indices, namely

$$A_{i,j}(x,y) \equiv A_{i,j}\delta(x,y).$$

and will therefore have a block-diagonal structure. The determinant of such a matrix is the product of the determinants of the blocks, therefore

$$\det_{\alpha,\beta,i,j} A_{\alpha\beta ij} = \det_{\alpha,\beta} \det_{i,j} A_{\alpha\beta ij},$$

which is exactly the widely used recipe.

A more formal analysis would require the study of the proper discrete matrix, for the functional integral is solely defined as the continuum limit of a discretized lattice version. The derivative there is in fact defined as difference of the values of the field in two neighbouring points of the lattice and the derivative squared has, therefore, terms which are off-diagonal in the spacetime indices, like $\phi(x+1)\phi(x)$. In one dimension the corresponding matrix $A_{\alpha\beta ij}$ would have the structure

$$A = \begin{pmatrix} \Box & \Diamond & 0 & 0 & \cdots \\ 0 & \Box & \Diamond & 0 & \cdots \\ 0 & 0 & \Box & \Diamond & \cdots \\ 0 & 0 & 0 & \Box & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\Box$ and $\Diamond$ represent blocks in the field indices. It is easy to see how, for a matrix with such a structure, the above equality (8) holds again, since the only contributions to the determinant come from the diagonal blocks.

It thus seems reasonable that the first, widely used recipe for calculating the functional determinant is equivalent to the other more rigorous one in the finite limit of field theory. This shows also that the reasons for the presence of the anomaly have to be found in the infinite nature of the functional determinant.

Let us turn our attention to the free charged bosonic field at finite temperature. In order to evaluate functional determinants we will make use of zeta-function regularization. Recall that the one-loop Euclidean partition function, regularized by means of zeta-function techniques, reads

$$\ln Z = -\frac{1}{2} \ln \det \left(L_D \ell^2 \right) = \frac{1}{2} \zeta'(0|L_D) - \frac{1}{2} \zeta(0|L_D) \ln \ell^2,$$

where $\zeta(s|L_D)$ is the zeta function corresponding to $L_D$, a second order elliptic differential operator, $\zeta'(0|L_D)$ its derivative with respect to $s$, and $\ell^2$ is a renormalization scale parameter.
The grand canonical partition function for an ideal charged relativistic boson gas may be written as \( \mu \) is the chemical potential): \[ (10) \]

\[
Z_{\beta,\mu} = \int_{\phi(\tau) = \phi(\tau + \beta)} D\phi e^{-\frac{1}{2} \int_0^\beta d\tau \int d^3x d^3y \phi_i(\tau) A_{ij}(x,y) \phi_j(\tau)},
\]

where the two real fields \( \phi_i, i = 1, 2 \), are chosen to describe the degrees of freedom of the boson gas, and the operator \( A \) has matrix elements given by

\[
A_{ij}(x, y) = \left(L_{ij} + 2\mu \epsilon_{ij} \sqrt{L_\tau}\right) \delta(x, y),
\]

(11)

with \( L_{ij} = \left(L_\tau + L_3 - \mu^2\right) \delta_{ij}, L_3 = -\Delta_3 + m^2 \),

(12)

in which \( \Delta_3 \) is the Laplace operator on \( \mathbb{R}^3 \) (continuous spectrum \( k^2 \)) and \( L_\tau = -\partial^2 \) (discrete spectrum \( \omega_n^2 = \frac{4\pi^2 \omega_n^2}{\beta^2} \)) the Laplace operator on \( S^1 \). In this case, the partition function may be written as \[ (13) \]

\[
Z_{\beta,\mu} = \left(\det\left\{\ell^2 \begin{pmatrix} L_\tau + L_3 - \mu^2 & 2\mu \sqrt{L_\tau} \\ 2\mu \sqrt{L_\tau} & L_\tau + L_3 - \mu^2 \end{pmatrix}\right\}\right)^{-1/2}
\]

The first recipe consists in taking first the algebraic determinant and then the functional determinant. The result is

\[
Z_{\beta,\mu} = \left(\det\left\{\ell^4 \left( L_\tau + L_3 - \mu^2\right)^2 + 4\mu^2 L_\tau \right\}\right)^{-1/2} = \left(\det\left(\ell^4 L_+ L_- \right)\right)^{-1/2},
\]

(14)

where

\[
L_\pm = L_\tau + L_3 + \mu^2 \pm 2\mu \left(L_3\right)^{1/2}.
\]

(15)

In an attempt of using the second recipe, one may observe that every \( 2 \times 2 \) matrix here can always be diagonalized. Then, modulo a trivial functional Jacobian corresponding to the diagonalization, one has

\[
Z_{\beta,\mu} = \left(\det\left\{\ell^2 \begin{pmatrix} L_+ & 0 \\ 0 & L_- \end{pmatrix}\right\}\right)^{-1/2}
\]

(16)

and the answer coming from the second recipe is

\[
Z_{\beta,\mu} = \left(\det(\ell^2 L_+) \det(\ell^2 L_-)\right)^{-1/2}.
\]

(17)

On the other hand —as is usually done in quantum field theory— one can also describe the gas by two complex fields \( \phi \) and \( \phi^* \), defined by

\[
\phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2) \quad \phi^* = \frac{1}{\sqrt{2}} (\phi_1 - i\phi_2).
\]

(18)

The corresponding grand partition function reads:

\[
Z_{\beta,\mu} = \left(\det\left\{\ell^2 \begin{pmatrix} K_+ & 0 \\ 0 & K_- \end{pmatrix}\right\}\right)^{-1/2}
\]

(19)
where now

\[ K_\pm = L_3 + L_\tau - \mu^2 \pm 2i\mu (L_\tau)^{1/2} \]  \hspace{1cm} (20)

Note that we have \( L_+ + L_- = K_+ K_- \), thus the first recipe gives the same answer for the two approaches, namely

\[ Z_{\beta,\mu} = \left( \det \ell^4 (L_+ L_-) \right)^{-1/2} = \left( \det \ell^4 (K_+ K_-) \right)^{-1/2} \]  \hspace{1cm} (21)

The second recipe gives, on its turn,

\[ Z_{\beta,\mu} = \left( \det (\ell^2 L_+) \det (\ell^2 L_-) \right)^{-1/2} = \left( \det (\ell^2 K_+) \det (\ell^2 K_-) \right)^{-1/2} \]  \hspace{1cm} (22)

When the multiplicative anomaly is non vanishing, one of two recipes is in contradiction. In Ref. [13] it has been shown that in odd dimensions, the multiplicative anomaly is vanishing and the two recipes give the same answer. In even dimensions, in particular in \( \mathbb{R}^4 \), the multiplicative anomaly is non vanishing and only the first recipe gives the same partition function for the two approaches, since we have

\[ \ln \left( \ell^2 \det L_+ \right) + \ln \left( \ell^2 \det L_- \right) + a(L_+, L_-) = \ln \left( \ell^2 \det K_+ \right) + \ln \left( \ell^2 \det K_- \right) + a(K_+, K_-) \]  \hspace{1cm} (23)

but, on the contrary,

\[ \ln \left( \ell^2 \det L_+ \right) + \ln \left( \ell^2 \det L_- \right) \neq \ln \left( \ell^2 \det K_+ \right) + \ln \left( \ell^2 \det K_- \right) . \]  \hspace{1cm} (24)

It has also been shown that the statistical sum contribution to the grand partition function is the same and yields the well known expression [10]

\[ S(\beta, \mu) = \sum_i \left[ \ln \left( 1 - e^{-\beta(\sqrt{\lambda_i} + \mu)} \right) + \ln \left( 1 - e^{-\beta(\sqrt{\lambda_i} - \mu)} \right) \right] , \]  \hspace{1cm} (25)

where \( \lambda_i = k^2 + m^2 \). The discrepancy manifests itself in the "vacuum sector", namely in the contribution to the grand partition function linear in \( \beta \) and the presence of the multiplicative anomaly, first recipe, renders the grand partition functions equal and independent by the parametrization of the degrees of freedom of the relativistic ideal gas.

Having performed the above calculation —that seems to leave little chance for discrepancy— one might still ask: how could the apparently more natural second recipe Eq. (5) fail? Well, the answer is, to begin with, that we are dealing with a very elusive mathematical and physical point. In fact, for a direct sum of operators, acting on a corresponding direct sum of independent functional spaces (this is the case in quantum physics when, for instance, a superselection rule is imposed upon the system), we indeed have the factorization property:

\[ \det A = \begin{pmatrix} A_1 & 0 & \cdots \\ 0 & A_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \det A_1 \det A_2 \cdots . \]  \hspace{1cm} (26)

In this situation the product itself of the operators, \( A_1 A_2 \), makes no sense in general, much less its determinant, and the prescription \( \det A_1 \det A_2 \cdots \) is to be used. But things are absolutely different when one is working within a functional space where field mixing and rotations are
allowed, and one just obtains the diagonal expression for the matrix of operators acting in this space as a particular form, after a convenient diagonalization process (this is precisely what happens in our example above). The moral we extract from the outcome of our analysis is that the invariant which is preserved under this process of change of basis is precisely the determinant of the operator matrix, but not the fact that it is equal to the product of the functional determinants of the operators. It is precisely that invariance what lays in the heart of our example: the noncommutative anomaly is the missing term necessary in order to preserve it.

To summarize, we have here carried out what is, in our opinion, a serious consistency check in favour of the first recipe for the calculation of determinants of matrix valued operators and, as a consequence, provided arguments in favour of the relevance of zeta-function regularization and related multiplicative anomaly in quantum field theory. As far as this last issue is concerned, in Ref. [4] we also respond to a criticism appeared recently [14], concerning the use of zeta-function regularization.

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