Propagation of Particles and Strings in One-Dimensional Universe

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We review the basics of the dynamics of closed strings moving along the infinite discretized line $\mathbb{Z}$. The string excitations are described by a field $\varphi_x(\tau)$ where $x \in \mathbb{Z}$ is the position of the string in the embedding space and $\tau$ is a semi-infinite “euclidean time” parameter related to the longitudinal mode of the string. Interactions due to splitting and joining of closed strings are taken into account by a local potential and occur only along the edge $\tau = 0$ of the semi-plane $(x, \tau)$.

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1. Introduction

The standard quantum field theory operates with point-like excitations (particles). From a certain point of view it can be considered as statistical mechanics of interacting random walks. However, there are theories (perhaps the most interesting ones) for which this scheme is too restrictive. Instead of propagation of particles we have to deal with the random motion of extended objects (closed and open strings). For example, it is believed that the multicolour (large $N$) limit of Quantum Chromodynamics \[1\] is equivalent to some kind of string theory. The world surface of of the chromodynamical string then would appear as the result of condensation of high-order planar diagrams.

This was the main motivation for some physicists (including the author) to try to understand the simplest theory of strings whose path integral is given by the sum over all surfaces swept by the string weighed by $\exp[-\text{cosmological constant} \times \text{area}]$. This model can be considered as the two-dimensional generalization of the problem of random walks. The model is correctly defined only if the dimension $D$ of the target space is $\leq 1$; in higher dimensions the appearance of a tachyon with $(\text{mass})^2 \sim 1 - D$ leads to instability of the vacuum and the world sheet of the string degenerates into a ramified one-dimensional object.

The $D = 1$ string has been studied carefully in the last time and in certain sense could be considered solved. Besides the standard quantization due to Polyakov \[2\], there are two gauge-invariant approaches inspired by the discretization of the world sheet through planar graphs. The first approach exploits the possibility to formulate the $D = 1$ bosonic string theory as the quantum mechanics of a large $N$ hermitian matrix \[3\]. This model can be rewritten as an intriguingly simple system of free fermions which upon bosonization turns into a rather unusual field theory with only one interaction vertex \[4\]. The model has been exhaustively solved in references \[5\], \[6\], \[7\], \[8\], \[9\], \[10\] (and references therein). The method of solution appears however rather removed from the traditional image of a string field theory which incorporates factorization, sewing and infinitely many interactions. It is precisely these concepts which appear naturally in the second (loop gas) approach where the embedding space is also discretized \[11\], \[12\], \[13\], \[14\].

In this lecture we will try to explain the logic of the loop gas approach to the string field theory restricting ourselves to the simplest case when the target space of the string is the discretized line $\mathbb{Z}$. \[1\] Our construction will exploit the fact that in the loop gas model

\[1\] The techniques apply equally to the case when the target space is an ADE or ÆDÆ Dynkin diagram \[11\]. In fact, one of the strenghts of the loop gas approach is a unified description of all $D \leq 1$ noncritical strings.
the two components of the dynamics of the world surface - the evolution of the intrinsic geometry (splitting and joining of strings) and the motion in the embedding space - can be completely separated. As a warm-up exercise we will consider first the rather trivial problem of random walks in $\mathbb{Z}$.

2. Random walks in $\mathbb{Z}$

The Euclidean transition amplitude $G(x, x')$ for a quantum relativistic particle is given by the sum over all paths connecting the points $x, x' \in \mathbb{Z}$, each path entering with weight $\exp[- \text{mass} \times \text{length}]$. A path $\Gamma_{xx'}$ in $\mathbb{Z}$ is by definition a map of the proper-time interval $\gamma_\ell = [0, \ell]$ in $\mathbb{Z}$. Each point $\xi \in \gamma_\ell$ has an integer coordinate (height) $x(\xi) \in \mathbb{Z}$. The two-point amplitude then reads

$$G(x, x') = \int_0^\infty d\ell \sum_{\Gamma_{xx'}} e^{-m_0 \ell}$$

where $m_0$ is the bare mass and $\ell$ is the length (= proper time) of the evolution line of the particle.

The sum on the r.h.s. of (1) can be thought of as the partition function of an SOS model defined on the world line of the quantum particle. This sum can be performed as follows. First we notice that the world line embedded in $\mathbb{Z}$ can be divided into segments with constant height (we assume that the height can jump only by $\pm 1$). Then by separating the first segment we write a simple recursion equation for $G(x, x')$:

$$G(x, x') = \frac{1}{m_0} (G(x + 1, x') + G(x - 1, x') + \delta_{x,x'})$$

which is a discretized version of the Klein-Gordon equation.

Considering $x$ as a continuum variable and introducing the parametrization $m_0 = 2\cosh(\pi E)$ we write (2) in the following operator form

$$(e^{\pi E} + e^{-\pi E} - e^{\partial_x} - e^{-\partial_x})G(x, x') = \delta_{x,x'}$$

This equation diagonalizes in the Fourier space (which is periodic with period 2)

$$G(p) = \frac{1}{2\cosh\pi E - 2\cos \pi p}$$

Finally, let us mention that the sum over all closed paths in $\mathbb{Z}$ is formally obtained as the vacuum energy of the following simple field theory

$$Z = \int \prod_{x \in \mathbb{Z}} d\phi_x \exp \left( \sum_x \phi_x (\cosh \partial_x - \cosh(\pi E)) \phi_x \right)$$
and the transition amplitude (1) is given by the two-point correlator

\[ G(x, x') = \langle \phi_x \phi_{x'} \rangle. \]  

(6)

3. Random surfaces in \( \mathbb{Z} \)

It turns out that the above analysis can be generalized to the case of random motion of closed strings immersed in \( \mathbb{Z} \). The closed-string states are determined completely by the intrinsic length \( \ell \) and the occupation \( x \) in the target space \( \mathbb{Z} \). The loop-loop correlator \( G(x, \ell; x', \ell') \) is defined as the sum over all cylindrical surfaces \( S_{xx'} \) immersed in \( \mathbb{Z} \) and having the two loops as boundaries

\[ G(x, \ell; x', \ell') = \int_0^\infty dA \sum_{S_{xx'}} e^{-\Lambda A} \]  

(7)

where the parameter \( \Lambda \) coupled to the area \( A \) of the surface \( S_{xx'} \) is usually called “cosmological constant”. The sum contains an integral over all world-sheet geometries and all possible ways to map given world sheet in \( \mathbb{Z} \). As in the case of the random particle, the world sheet can be divided into domains of constant height \( x \) which are separated by domain walls (fig. 1).

\[ \text{Fig.1. A piece of a surface immersed in } \mathbb{Z}: \text{ domains and domain walls} \]

Our further strategy will be based on the trivial observation that the decomposition of the world sheet of the embedded surface into domains of constant height generates a decomposition of the integration measure in the space of surfaces into a sum over Feynman graphs. In the problem of random surfaces the things are technically more complicated: a domain on the world sheet is characterized not only by the lengths of its boundaries \( \ell_1, \ell_2, \ldots \), but also by its intrinsic geometry (curvature) which can exhibit local fluctuations.
The string interaction constant $\kappa$ is coupled to the total curvature $\chi = 2 - 2h - n$ where $h$ is the number of handles and $n$ is the number of boundaries. The Boltzmann weight of a domain is given by the sum over all possible geometries of a surface having $h$ handles and $n$ boundaries with lengths $\ell_1, \ldots, \ell_n$. This sum can be evaluated using the discretization of the configuration space of surfaces via planar graphs. We will not go into the details since there exists abundant literature on this subject (see, for example, [15] and the references therein). The general $n$-loop amplitude was calculated explicitly in the planar limit $\kappa = 0$ in [16] and [17]. In the simplest case when the area of the domain is zero, this result can be generalized to the case of general topology using the loop equations for the gaussian matrix model [18], [14]

$$W_n(\ell_1, \ell_2, \ldots, \ell_n) = \sqrt{\ell_1 \ldots \ell_n} (\ell_1 + \ldots + \ell_n) e^{-\sqrt{\mu}(\ell_1 + \ldots + \ell_n) + \kappa^2(\ell_1 + \ldots + \ell_n)^3}$$

(Note that this formal expression is divergent for any finite $\kappa$; this is a well known disease of the bosonic string theory.)

It is possible to construct the path integral of the string using these simple amplitudes; then the boundaries between domains of constant height $x$ (i.e., the domain walls) will be distributed densely on the world sheet and the role of cosmological constant will be played by the parameter $\mu$ coupled to the length of the domain walls. The two-loop propagator then has the meaning of the partition function for the SOS model on a cylindrical random surface with fluctuating geometry and constant heights $x$ and $x'$ along its two boundaries.

Similarly to the case of random walks we can construct a “string field theory” for the loop field $\Phi_x(\ell)$ in which the loop-loop correlator is given by the sum over surfaces (8). The partition function of this field theory is given by the following functional integral

$$Z = \int \mathcal{D}\Phi \exp^{-A[\Phi]}$$

$$A = \frac{1}{2} \sum_x \int_0^\infty \frac{d\ell}{\ell} \Phi_x(\ell) [e^{\partial_x} + e^{-\partial_x}]^{-1} \Phi_x(\ell)$$

$$- \sum_{n=1}^\infty \frac{\kappa^{n-2}}{n!} \sum_x \int \frac{d\ell_1}{\ell_1} \ldots \frac{d\ell_n}{\ell_n} W_n(\ell_1, \ell_2, \ldots, \ell_n) \Phi_x(\ell_1) \ldots \Phi_x(\ell_n)$$

The vertices in the interaction potential represent surfaces of various topologies embedded in a single point $x \in \mathbb{Z}$. The kinetic term of the action leads to a “propagator”
which can be thought of as an infinitesimal cylinder connecting two edges with co-ordinates \( x \) and \( x + 1 \). The Feynman graphs constructed in this way represent surfaces embedded in \( \mathbb{Z} \). The weight of a graph with \( h \) handles and \( n \) boundaries depends on the string interaction constant through a factor \( \kappa^{2-2h-n} \).

This diagram technique contains a tadpole vertex \( W_1 \) (disk) and a two-leg vertex \( W_2 \) (cylinder). We will bring it, by a re-definition of the field, to a standard form where the interaction starts with a \( \phi^3 \)-term.

In order to eliminate the tadpole we shift the string field by its expectation value

\[
\Phi_x(\ell) = \langle \Phi_x(\ell) \rangle_{\kappa=0} + \phi_x(\ell) \tag{10}
\]

The classical string field is equal to the sum over all planar surfaces spanning a loop with length \( \ell \). It can be evaluated as the solution of the saddle-point equation for the theory (9) which is equivalent to the loop equation in the planar limit. Its explicit form reads [18]

\[
\langle \Phi_x(\ell) \rangle_{\kappa=0} = \int_{\sqrt{\mu}}^{\infty} dz [(z + \sqrt{z^2 - \mu})^g - (z - \sqrt{z^2 - \mu})^g] e^{-z\ell}
\]

\[
= \mu^{g/2} \frac{2g}{\ell} K_g(\sqrt{\mu\ell}), \quad g = 1
\]

where \( K_g \) is a modified Bessel function

\[
K_g(\sqrt{\mu\ell}) = \int_0^\infty d\tau e^{-\sqrt{\mu\ell}\cosh\tau}\cosh(g\tau)
\]

The space of possible backgrounds is parametrized by the positive constant \( g \); the vacuum in our case is translationary invariant which corresponds to the choice \( g = 1 \). The classical solution with \( g \neq 1 \) describes a string background with non-zero momentum \( p_0 = g - 1 \).

This is the case, for example, when the target is restricted to a chain of \( m - 1 \) points (i.e., the \( A_{m-1} \) Dynkin diagram) and the translational invariance is lost. In this case \( 1 - g = 1/m \). In particular, for \( m = 2 \), the classical field (11) coincides with the tadpole vertex \( W_1 \) given by eq. (9).

The linear change (10) leads to new vertices \( w_n(\ell_1, ..., \ell_n) \) but they have the same structure as the original ones (8). The first ones are [14]

\[
w_2(\ell, \ell')|_{\kappa=0} = \sqrt{\ell e^{-\sqrt{\mu}(\ell+\ell')}} \sqrt{\ell'}
\]

\[
w_3(\ell_1, \ell_2, \ell_3)|_{\kappa=0} = \mu^{-(1/2-g)/2} 3 \prod_{k=1}^{3} \sqrt{\ell_k} e^{-\sqrt{\mu}\ell_k}
\]

\[5\]
\[ w_4(\ell_1, \ell_2, \ell_3, \ell_4)_{\kappa=0} = \mu^{(1/2-g)} \prod_{k=1}^{4} \sqrt{\ell_k} e^{-\sqrt{\mu\ell_k}} \left[ \sum_{k=1}^{4} \ell_k + \frac{1}{2M} (g^2 - \frac{1}{4}) \right] \]  

Note that the two-loop vertex does not change after the shift (10). Indeed, the new vertices make sense of loop amplitudes of a special one-matrix model [14] and it is well known that the two-loop correlator in all matrix models is universal. The planar (\( \kappa = 0 \)) vertices \( w_n \) can be evaluated according to the general formula for the planar loop amplitudes in the one-matrix model [16] [17]

\[ w_n(\ell_1, \ldots, \ell_n) = \left( \frac{d}{dt} \right)^{n-3} \prod_{k=1}^{n} \sqrt{\ell_k} e^{-f\ell_k} |_{t=0} \]  

where the function \( f(M, t) \) is determined by the following “string equation” [14]

\[ t = M^{g-1/2}(M - f) \sum_{k=0}^{\infty} \frac{(1/2 + g)_k (1/2 - g)_k}{k!(k+1)!} \left( \frac{M - f}{2M} \right)^k \]  

so that \( f(M, 0) = M \). We used the standard notation \( (a)_k = a(a+1)...(a+k-1) \). Note that the new vertices coincide with the old ones when \( g = 1/2 \).

The next step is to evaluate the propagator in this string field theory which is given by the sum (7) of cylindrical surfaces immersed in \( \mathbb{Z} \). For this purpose we need to invert the quadratic form in the gaussian part of the action. The latter is given by the sum of the kinetic term in (9) and the two-loop vertex \( w_2 \). This will be the subject the next section.

4. A quantum field theory on the semi-plane

Below we will see that, in specially chosen variables, the string field theory can be driven in to a more usual form of a field theory with local interaction, defined on the semi-plane. The interaction will be concentrated along the edge of the semi-plane. This field theory reminds the effective string field theory of Polchinski [19] in which the interaction increases exponentially along the negative time direction so that the theory is effectively confined to a semi-plane.

The diagonalization of the gaussian (\( \kappa = 0 \)) part of the action is done by means of the formula [20]

\[ \sqrt{\ell} e^{-\sqrt{\mu}(\ell + \ell')} \sqrt{\ell'} = \frac{2}{\pi} \int_0^{\infty} dE E \tan(\pi E) K_iE(\sqrt{\mu}\ell) K_iE(\sqrt{\mu}\ell') \]  

We introduce, following [17], a complete set of delta-function normalized eigenstates [17]

\[ \langle x, \ell | p, E \rangle = \frac{2}{\sqrt{\pi}} \sqrt{\sinh(\pi E)} e^{i\pi px} K_iE(\sqrt{\mu}\ell), \quad E > 0, \quad -1 < p \leq 1 \]
\[ \sum_x \int_0^\infty \frac{d\ell}{\ell} \langle p, E | x, \ell \rangle \langle x, \ell | p', E' \rangle = \delta(E, E')\delta(p, p') \quad (20) \]

Then we write the functional integral in terms of the amplitude \( \varphi(p, E) \) which we normalize as follows

\[ \varphi(p, E) = \frac{1}{\pi} \sqrt{\frac{\pi E \sinh(\pi E)}{\cosh(\pi E)}} \sum_x \int \frac{d\ell}{\ell} \langle p, E | x, \ell \rangle \phi_x(\ell) \quad (21) \]

After that the free part of the action takes the form

\[ \mathcal{A}_{\text{free}} = \frac{1}{2} \int_0^\infty dE \int_{-1}^1 dp \, \varphi(p, E) \mathcal{G}^{-1}(p, E) \varphi(p, E) \quad (22) \]

\[ \mathcal{G}(p, E) = \frac{\pi E \sinh(\pi E) \cos(\pi p)}{2 \cosh(\pi E) [\cosh(\pi E) - \cos(\pi p)]} \quad (23) \]

The general term in the interaction part of the action can be transformed to the \( E \) space using the formula [20]

\[ \int_0^\infty \frac{d\ell}{\ell} \sqrt{\mu e^{\sqrt{\mu} \ell} \ell^k K_i E(\sqrt{\mu} \ell)} = (2\sqrt{\mu})^{-k-1/2}(1/2 + iE)_k(1/2 - iE)_k \frac{\pi^{3/2}}{k!} \cosh(\pi E) \quad (24) \]

It follows from the general form of the interaction (8) that the planar (\( \kappa = 0 \)) \( n \)-point vertex in the \( E \) space will be an odd polynomial of \( E_1, ..., E_n \) of total degree \( 2(n - 3) + 6h \) where \( h \) is the number of handles of the corresponding elementary surface. The lowest vertices (14), (15) in the momentum space are

\[ w_3(E_1, E_2, E_3) = 1 \quad (25) \]

\[ w_4(E_1, E_2, E_3, E_4) = (g^2 - 1/4) + \sum_{k=1}^4 (1/4 + E_k^2) \quad (26) \]

Let us introduce a time variable \( \tau \) dual to the quantum number \( E \) and reformulate the functional integral in terms of the field

\[ \varphi_x(\tau) = \int_{-1}^1 dp \int_0^\infty dE e^{i\pi px} \cos(E\tau) \varphi(E, p), \quad \tau \geq 0 \quad (27) \]

Then the gaussian part of the action takes the form

\[ \mathcal{A}_{\text{free}} = \sum_x \int_0^\infty d\tau \varphi_x(\tau) \frac{\cos(\pi \partial_\tau) - \cosh(\partial_x) \varphi_x(\tau) \cos(\pi \partial_\tau)}{2\pi \partial_\tau \sin(\pi \partial_\tau) \cosh(\pi \partial_x)} \quad (28) \]
The interaction is concentrated along the wall \( \tau = 0 \) in the \((x, \tau)\) space. For example, the two lowest vertices \([23], [26]\) are generated by the potential

\[
\mathcal{A}_{\text{int}} = \sum_x \left( \kappa \frac{[\varphi_x(0)]^3}{3!} + \kappa^2 \left( g^2 \frac{1}{4} \right) \frac{[\varphi_x(0)]^4}{4!} + \kappa^2 \left[ \frac{1}{4} \partial^2 \varphi_x(0) \right] \frac{[\varphi_x(0)]^3}{3!} \right)
\]  

(29)

The field \( \varphi_x(\tau) \) is related to the original field \( \varphi_x(\ell) \) by the relations

\[
\varphi_x(\tau) = \frac{1}{\pi} \frac{\partial \tau \sin(\pi \partial \tau)}{\cos(\partial \tau)} \int_0^\infty d\ell \frac{\ell e^{-\sqrt{\mu \ell} \cosh \tau}}{\cosh^2(\pi \ell) - 2 \cos(\pi \ell) \cos(\partial \tau) \varphi_x(\ell)}
\]  

(30)

\[
\varphi_x(\ell) = \int_0^\infty d\tau e^{-\sqrt{\mu \ell} \cosh \tau} \cos(\partial \tau) \varphi_x(\tau)
\]  

(31)

which follow from the integral representation of the Bessel function \([12]\). The transformation between the eigenstates in the \( \ell \) and \( \tau \) spaces has been first discussed in \([21]\). The field \( \varphi_x(\tau) \) is restricted to the semi-plane \( \tau \geq 0 \), but it can be considered as defined on the whole plane \((x, \tau)\) if the symmetry \( \varphi_x(-\tau) = \varphi_x(\tau) \) is imposed.

Finally let us make the following remark concerning the propagator of our string field theory. It is natural to expect that upon diagonalization it decomposes into a sum of random-walk propagators with different masses. On the other hand, the action \([28]\) does not seem to meet this requirement. The reason is the following. The propagator \([23]\) can be decomposed as a sum of two terms

\[
G(E, p) = G(E, p) - G(E, 1/2)
\]  

(32)

with

\[
G(E, p) = \frac{\partial E \sinh(\pi E)}{2 \cosh(\pi E) - 2 \cos(\pi p)} = \frac{E}{2} \frac{d}{dE} \log(\cosh(\pi E) - \cos(\pi p))
\]  

(33)

The function \([33]\) is, up to a numerical factor, equal to the propagator of a random particle moving in \( \mathbb{Z} \).

This form of the propagator admits simple explanation. The target space of a string theory without embedding consists of a single point; the corresponding momentum space contains a single momentum \( p = 1/2 \). The corresponding string field theory has vanishing propagator and its loop amplitudes are the vertices \([10]\). The full string field theory can be considered as a perturbation of the \( p = 1/2 \) string field theory.

Of course, it is possible to absorb, by re-definition of the vertices, the \( p = 1/2 \) term into the interaction. This is however not advisable since then the \( E \)-integration would produce singularities. These singularities are neatly cancelled for the difference \([32]\).
Note also that the Fourier image of the discrete propagator \( G(E, p) \) is obtained from the two-point correlator of a two-dimensional Euclidean particle by “periodization” with respect to translations \( p \rightarrow p + 2 \)

\[
G(E, p) = \sum_{n=-\infty}^{\infty} \frac{E^2}{E^2 + (p + 2n)^2}
\]  

(34)

4. Loops

Let us evaluate the simplest loop diagram giving the partition function \( Z_{\text{torus}} \) of the noninteracting closed string.

It is convenient first to perform the calculation for fixed momentum \( p \) running along the loop. The corresponding piece of the partition function consists of two terms

\[
Z(p) = Z(1/2) - \int_a^\infty \frac{d\ell}{\ell} \int_0^\infty dE |\langle \ell | E \rangle|^2 \log \left( \frac{\cosh(\pi E) - \cos(\pi p)}{\cosh(\pi E) - \cos(\pi 1/2)} \right)
\]  

(35)

The contribution of the surfaces with momentum \( p = 1/2 \) is equal to the \( \kappa = 0 \) term in the expansion of the vacuum energy of the effective one-matrix model generating the amplitudes (13). A simple calculation (see, for example Appendix C of [22]) yields

\[
Z(1/2) = \frac{g - 1/2}{24} \log(a \sqrt{\mu})
\]  

(36)

The integral over the length \( \ell \) in (35) yields a factor \( \log[1/(a \sqrt{\mu})] \) where \( a \) is a cut-off parameter (this is the diagonal value of the kernel of the regularized identity operator in the \( E \)-space) and the \( E \)-integral equals

\[
\int_0^\infty \frac{dE}{\pi} \log \left( \frac{\cosh(\pi E) - \cos(\pi p)}{\cosh(\pi E)} \right) = \frac{1}{2} (|p| - \frac{1}{2})(|p| - \frac{3}{2}), \quad |p| \leq 1
\]  

(37)

Inserting this (36) and (37) in (35) we find

\[
Z(p) = \frac{1}{24} \left[ g - \frac{1}{2} + 6(1/2 - |p|)^2 \right] \log(a \sqrt{\mu})
\]  

(38)

This result can be applied to discrete target spaces with various geometries since the target space is characterized completely by the spectrum of allowed momenta \( p \). Consider, for example, the string compactified on the circle with \( 2m \) points \( \mathbb{Z}_{2m} \). Summing over all allowed momenta \( p = k/m; \ k = 0, \pm 1, \ldots, \pm (m - 1), m \), we find (for the translationally invariant background \( g = 1 \))

\[
Z_{\text{torus}} = \left( \frac{m^2 + 2}{24m} + 2m \frac{g - 1/2}{24} \right) \log(aM) = \frac{1}{12} \left( m + \frac{1}{m} \right) \log(aM)
\]  

(39)
This is precisely the result obtained the partition function of the string embedded in a continuum compact target space \( \mathbb{Z} \)[23][24].

**Concluding remarks**

The most attractive feature of the loop gas approach is the possibility to construct a simple diagram technique for the loop amplitudes. This is possible because the so-called “special states” corresponding to momentum-independent poles, are inobservable in the string theory with discrete target space. A rather dramatic consequence is the factorization of the interactions which may be traced back to the fact that the string interaction takes place at a single point in the \( x \)-space. Nevertheless, the string theory with target space \( \mathbb{Z} \) seems to be identical to the string theory with continuum embedding space \( \mathbb{R} \), if the right observables are compared. It has been checked [25] that the \( n \)-loop amplitudes in \( x \) space in both theories coincide up to \( n = 4 \). (Actually, since the four loop amplitude in the \( \mathbb{R} \)-string is not yet calculated, the comparison for \( n = 4 \) was restricted to the case of infinitesimal loops.)

Another remark is the the striking similarity of the loop-gas diagram technique with the construction of the interacting string theory by B. Zweibach [26] in which the elementary vertices in the nonpolynomial action are themselves loop amplitudes. A consistent choice of string vertices and propagator give rise to one particular way of decomposing the moduli space of Riemann surfaces. The possibility of choosing different decompositions of moduli space is related to a special symmetry of the string field theory. Such a symmetry might exist in the loop-gas construction as well.

Finally let us mention that the loop gas approach allows to equally construct the open string field theory in the space \( \mathbb{Z} \)[27].
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