MBL and topological order in disordered interacting Ising-Majorana chains

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We numerically explore \( \mathbb{Z}_2 \)-symmetric disordered interacting Ising-Majorana chains at high energy. A very rich phase diagram emerges with two distinct many-body localization (MBL) regimes separated by a broad thermal phase. We find that localization-protected topological edge states survive interactions, and are always associated to many-body spectral pairing. In contrast, weak interactions destabilize the infinite randomness critical point towards thermalization, and we show evidences that its close vicinity also gets delocalized by weak interactions, as long as the typical localization length is larger than the avalanche threshold \( \xi > (\ln 2)^{-1} \).

**Main results**— Building on state-of-the-art numerical simulations, this Letter explores the \( \mathbb{Z}_2 \)-symmetric disordered and interacting Ising-Majorana (IM) chain model

\[
\mathcal{H}_{IM} = \sum_i J_i \sigma_i^x \sigma_{i+1}^x + h_i \sigma_i^z + g \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^z \sigma_{i+2}^z \right),
\]

for which we provide the infinite-temperature phase diagram in Fig. 1. The Kramers-Wannier duality [27] of the IM model leads to symmetric phase boundaries, with a broad intervening ergodic phase [23] emerging between two different MBL regimes: a featureless paramagnet (PM) and a spin-glass order (SG) order. Linked to the (non-interacting) topological ordered phase of the TFI model, the quantum SG order exhibits cat-states for all energies, with a global double degeneracy of the many-body spectrum between the two parity sectors (even or odd under the action of \( \mathbb{P} = \prod_i \sigma_i^z \)). This is an example of MBL-protected topological order with localized Majorana edge states [28], as previously discussed by Huse et al. [22]. Here we remarkably identify a unique phase having both MBL SG and a paired spectrum, and hence a single transition towards ergodicity which occurs when the typical localization length exceeds the “avalanche” threshold \( \xi^* \equiv (\ln 2)^{-1} \) [29]. This has strong consequences for the weakly interacting limit: not only the IRC is unstable towards thermalization [24], but its close vicinity may also get delocalized by weak interactions, as long as \( \xi > \xi^* \). Two possible scenarios will be discussed in the light of our extensive exact diagonalization (ED) calculations.

**Introduction**— Many-body localization (MBL) in quantum interacting systems has attracted a lot of attention over the past two decades [1–10]. While the first (analytical) studies addressed the fate of the non-interacting Anderson insulator against weak interactions [3, 4], most of the numerical studies then focused on strongly interacting one-dimensional (1D) models, such as the random-field Heisenberg chain [6, 8], for which there is a global consensus for an ergodicity-breaking transition \([8, 11–14]\). In addition to convincing experimental observations [15–18], the existence of MBL has also been proven by Imbrie [19] for interacting random Ising chains governed by \( \mathcal{H} = \mathcal{H}_{TFI} + V_{\text{xy}}, \)

\[
\mathcal{H}_{TFI} = \sum_i J_i \sigma_i^x \sigma_{i+1}^x + h_i \sigma_i^z
\]

is the \( \mathbb{Z}_2 \)-symmetric non-interacting transverse-field Ising (TFI) chain model, further perturbed by interactions of the form \( V_{\text{xy}} = \sum_i \Gamma_i^x \sigma_i^x \), that explicitly breaks the \( \mathbb{Z}_2 \) symmetry. Pekker et al. [20] instead considered a \( \mathbb{Z}_2 \)-symmetric perturbation \( V_{\text{xx}} = \sum_i \delta_i \sigma_i^x \sigma_{i+1}^x \), as did Kjäll et al. [21] with \( V_{\text{xx}} = \sum_i \delta_i \sigma_i^x \sigma_{i+2}^x \), where in both cases the parity operator \( \mathbb{P} = \prod_i \sigma_i^z \) commutes with the interacting Hamiltonian.

**FIG. 1.** Infinite-temperature phase diagram of the random interacting Ising-Majorana chain model Eq. (2), obtained using shift-invert exact diagonalization. In the plane \( \delta - q (\delta = \ln J - \ln t, q = \text{the interaction}) \), a broad ergodic phase intervenes between two MBL regimes: a featureless paramagnet (MBL PM) and a spin-glass order (MBL SG) with a global double degeneracy of the many-body spectrum associated to topological edge states. The most probable scenario (thick colored lines) has a single transition for MBL order and spectral pairing. A 2nd transition is not excluded, with a tiny intermediate (green region) MBL SG regime + unpaired spectrum.
Interacting Ising-Majorana chain model— Before presenting our numerical results, we first discuss some key properties of the IM Hamiltonian, Eq. (2). As previously mentioned, this model preserves the $\mathbb{Z}_2$ parity symmetry, $[\mathcal{H}_{\text{IM}}, \mathcal{P}] = 0$, such that one can independently diagonalize $\mathcal{H}_{\text{IM}}$ in each (even and odd) parity sectors. As also exploited in Refs. [23, 24], there is a duality transformation which ensures a symmetric phase boundary under $\delta = \ln (J/h) \rightarrow -\delta$ [27], as seen in Fig. 1.

If one rewrites Pauli operators as Majorana fermions:
$$\gamma_{2j-1} = \sigma^y_j \prod_{k<j} \sigma^z_k, \quad \text{and} \quad \gamma_{2j} = \sigma^y_j \prod_{k<j} \sigma^z_k,$$
the original IM spin model reads
$$\mathcal{H}_{\text{IM}} = \sum_k -3\tau_k \gamma_k \gamma_{k+1} - 9\tau_k \gamma_{k+1} \gamma_{k+2} \gamma_{k+3},$$
with hopping terms $t_{2j-1} = h_j$, $t_{2j} = J_j$, and interaction strength $g$. Ground-state properties have been obtained in some limiting cases, such as the random non-interacting chain by Fisher [25], and for the interacting problem in the absence [30] or presence of disorder [33–34]. Interestingly, the Majorana edge states localized at the boundaries of a non-interacting open chain [35, 36] in the topological regime ($\delta > 0$) appear robust against disorder or weak interactions [33]. On the other hand, IRC at $\delta = 0$ was found [34] unstable towards localization and topological ordering for $g > 0$ (repulsive interactions), while attraction is also relevant but could apparently drive IRC to a different (Ising) criticality [34].

Anyhow, in our case at high energy the interaction sign is expected to be irrelevant, as confirmed for the XXZ chain [37], such that we work with $g > 0$, assuming similar results for attractive couplings. We then take box distributions for both random couplings $J_i$ and fields $h$, $P_{J/h} = \text{Box}[0, W_{J/h}]$ being uniform between 0 and $W_{J/h}$, with $W_J = W_h^{-1} = W$ so that the control parameter is $\delta = \ln J - \ln h = 2 \ln W$. In the rest of the paper, we present and discuss in details the results of numerical simulations that led us to the phase diagram shown in Fig. 1. Using the shift-invert ED technique [8, 38], we have studied open IM chains Eq. (2) up to $L = 16$ sites [39], and computed $\sim 100$ mid-spectrum (infinite-temperature) eigenpairs for each parity sector ($\rho = \pm 1$), for $\sim 500 - 1000$ independent random samples.

Emergent ergodicity between distinct MBL phases— We first illustrate in Fig. 2 the sequence of three phases which appear for finite interaction: MBL PM — Ergodic — MBL SG, upon varying $\delta$ (corresponding to an horizontal scan in the phase diagram at $g = 0.5$). Fig. 2 shows two classic estimates for these phases and associated transitions: the average half-chain von-Neumann entropy $S_{\text{VN}}(L/2)$ [40], probing area vs. volume-law entanglement, and the average (parity-resolved) gap-ratio $\tau$ [41, 42] which diagnoses a global change in the spectral statistics: Poisson vs. GOE, occurring in each parity sector. A broad thermal (ergodic) phase emerges for $|\delta| \lesssim 4$ in between two regimes showing MBL physics with uncorrelated Poisson spectral statistics and area-law entanglement. Note however that in contrast with the gap ratio, the entanglement does not respect dual symmetry $\delta \rightarrow -\delta$. Indeed, $S_{\text{VN}}$ vanishes at large negative $\delta$, while $S_{\text{VN}} \rightarrow \ln 2$ at large positive $\delta$. This signals a quantum-order with a “cat-state” structure for the eigenstates, of the form (in the $\sigma^x$ basis)
$$|\psi\rangle \propto |\uparrow\downarrow\cdots\uparrow\downarrow\rangle_x + |\downarrow\uparrow\cdots\downarrow\uparrow\rangle_x,$$
where $p = \pm 1$ is the parity sector. In addition, we expect such even and odd eigenstates to be pair-wise degenerate across the full many-body spectrum, as we will elaborate on further below.

Spin correlations— Spontaneous breaking of $\mathbb{Z}_2$ symmetry associated to magnetic order with $\langle \sigma^z \rangle \neq 0$ only occurs in the thermodynamic limit. One can nevertheless probe it on finite IM chains using the squared spin correlator $C_{ij} = \langle \sigma^z_i \sigma^z_j \rangle^2$ (the square integrates out sign fluctuations at high-energy), or the more frequently used spin-glass structure factor [21] $\chi^S_L = \frac{1}{i} \sum_{i,j} C_{ij}$. The later provides a reasonably good qualitative estimate for detecting SG order since $\chi^S_L$ diverges with $L$, while it goes to a constant outside the quantum-ordered regime. However, there is no reason to expect a finite-size crossing of $\chi^S_L$ precisely at the critical point [21, 23, 43, 44] because the long distance decay of the critical correlations is not known. Moreover, the integrated form of $\chi^S_L$ enhances non-universal short-range effects, and it is safer to directly focus on two-point correlators.

![Figure 2](image_url)
Typical pairwise correlators $C_{\text{gap}}(r) = e^{\ln x_{i} x_{i+r}}$ [45] are shown in Fig. 3 for three representative situations. (a) Deep in the MBL SG regime, $C_{\text{gap}}(r)$ rapidly saturates to a roughly size-independent finite value, thus testifying for magnetic order. (b) In the middle of the thermal region we nicely verify the $2^{-L}$ behavior, expected from eigenstate thermalization hypothesis (ETH) [46, 47]. (c) The MBL PM phase displays short-range correlations, exponentially decaying with additional oscillations which can be understood in the limit $J_i \ll 1$ where the 2nd neighboring terms $g\sigma_i^z\sigma_{i+2}^z$ dominates over nearest-neighbor, thus strongly reducing odd-distant correlations.

**Quantum (topological) order and spectral pairing**—First observed by Kitaev [35] for clean chains, the ordered phase of free fermions ($\delta > 0$) has non-trivial topological properties, with zero-energy modes (ZM) localized at the boundaries of an open chain: the so-called “unpaired” Majorana edge states. For random TFI chains, ZM operators commuting with $H_{\text{TFI}}$ when $\delta > 0$ can also be explicitly defined [36, 48], but not in the presence of interactions. Nevertheless, building on the non-interacting limit, one expects end-to-end correlations $\langle \sigma_i^z\sigma_{-i}^z \rangle$ to be a good proxy for Majorana edge modes [48].

In addition, ZM imply a double (even-odd) degeneracy of the entire many-body spectrum [22, 36] (see Fig. 1), with an exponentially small level splitting $\Delta_{\text{parity}} \sim e^{-L/\xi}$, where $\xi$ is the edge mode localization length. From a global spectroscopic point of view, this parity degeneracy competes with the many-body level spacing $\Delta_{\text{mb}} \sim e^{-sL}$ ($s$ is the entropy density) such that the spectral pairing is resolved only if $\Delta_{\text{parity}} \ll \Delta_{\text{mb}}$, i.e. for $\xi < 1/s$. In practice, the detection of the topological pairing is achieved using the (parity-mixed) gap ratio $r_i = \min(\Delta_i, \Delta_{i+1})/\max(\Delta_i, \Delta_{i+1})$, where the individual gaps $\Delta_i$ are computed within the many-body spectrum when both parity sectors are mixed. In the MBL topological regime, we expect $r_i \sim \Delta_{\text{parity}}/\Delta_{\text{mb}} \rightarrow 0$ if $\xi < 1/s$, while Poisson statistics should arise when $\xi > 1/s$ (as well as for a non-topological MBL phase), where $\mathcal{P} = \ln 4 - 1$ takes the same value as the parity-resolved ratio $\tau$. Interestingly, the ergodic regime also manifests in the mixed spectral statistics, where two GOE blocks yield $\mathcal{P}_{\text{GOE}}^2 \approx 0.4234$, as shown recently [49]. Table 1 summarizes these spectral features.

Ref. [22] conjectured that two types of MBL orders may emerge, with and without spectral pairing. This issue is addressed in Fig. 4 where we show $T = \infty$ ($s = \ln 2$) results for a vertical scan in the phase diagram, collected at $\delta = 2.5$. Several estimates for the MBL SG — ergodic transition are shown: (a-c) parity-resolved and parity-mixed gap ratios; (d) the von-Neuman entropy density; (e-g) end-to-end correlators. As previously observed (Fig. 2), here also the area to volume-law entanglement transition coincides with the Poisson — GOE change in the parity-resolved spectral statistics, both observed for $g_c = 0.23\pm 0.03$. In addition, boundary correlations (capturing SG order and topological edge states) also display a clear ordering transition in the same region, as established by the best power-law fit $\left[\sigma_i^z\sigma_{-i}^z \right] \sim L^{-\omega_k}$ obtained at $g_c = 0.24$ with $\omega_k \approx 0.17$ (contrasting with free-fermions where $\omega_k \approx 1$ [50]). This presumably rules out the possibility of an intermediate MBL PM regime [21, 22], a result also strongly supported by the parity-mixed gap ratio $\mathcal{P}$ shown in Fig. 4 (a-c). Indeed, one sees a unique MBL regime associated to spectral pairing for $g \leq g_c$, followed by an ergodic phase where $\tau$ takes its GOE value $\approx 0.5307$ together with its parity-mixed counterpart $\mathcal{P}$ which saturates to its GOE2 value $\approx 0.4234$, as expected for two GOE blocks [49].

The infinite-temperature pairing transition signals that the typical localization length $\xi$ (controlling the parity gap decay) has reached $\xi^* = (\ln 2)^{-1}$, a value which strikingly coincides with the so-called avalanche threshold [29]. At this stage, it is very instructive to make a small detour to the other side of the phase diagram ($\delta < 0$) where there is no topological MBL order. Instead, both MBL PM and ergodic regimes have exponential decaying end-to-end correlations $\left[\sigma_i^z\sigma_{-i}^z \right] \sim e^{-L/\xi_0}$ controlled by the localization length $\xi_0$, as we confirm in Fig. 4 (g). The $g$-dependence of $\xi_0$, at $\delta = -2.5$, Fig. 4 (h), remarkably establishes that the MBL PM — ergodic transition (at $g \sim 0.2$) is again characterized by $\xi_0 = \xi^* = (\ln 2)^{-1}$, further meeting the avalanche criterion [29].

| $\mathcal{P}_{\text{mixed}}$ | Ergodic | MBL | MBL + paired spectrum |
|---------------------------|--------|-----|-----------------------|
| $\ln 4 - 1$ | $e^{(\xi - \xi^*)}$ | $0$ |
| $\mathcal{P}_{\text{resolved}}$ | $0.5307$ (GOE) | $0$ |

**TABLE 1.** Parity-mixed $\mathcal{P}$ and parity-resolved $\tau$ values for the gap ratios across the different regimes.

![FIG. 3. ED results for open IM chains at $g = 0.5$, showing the decay of typical correlators $C_{\text{gap}}(r)$ for three representative cases. (a) In the MBL SG regime ($\delta = 6.58$) spin-glass order is clear, with a notable boundary enhancement. (b) Ergodic ($\delta = 0$): ETH behavior is evident after a few sites: $C_{\text{gap}}(r) \gtrsim 5 \rightarrow 2^{-L}$ (grey line). (c) MBL PM ($\delta = -6.58$): we observe fast (even-odd oscillating and size-independent) exponential decay $C_0 \exp[-r/\xi]$, with fitting parameters ($C_0, \xi_{\text{even}} = (0.41, 0.35)$ and ($C_0, \xi_{\text{odd}} = (3.2 \times 10^{-5}, 0.408$) for the available range (fits are shown by grey lines).
Avalanche, infinite randomness and weak interactions—The main idea behind the avalanche instability [29] is that a small (rare) ergodic region embedded inside an otherwise localized system can nucleate a growing ergodic surrounding (ultimately thermalizing the whole system), only if the typical localization length $\xi > \xi^* = (\ln 2)^{-1}$, a critical threshold fixed by the many-body spacing in the middle of the spectrum. So far, our numerics have adhered this avalanche condition, for both sides of the phase diagram, leading to the remarkable consequence that MBL SG is always accompanied by spectral pairing.

Building further on this, the condition $\xi > \xi^*$ is also satisfied without interaction (where the typical localization length is $1/\delta$ [25]) when $|\delta| < \ln 2$. We therefore expect ergodic instability upon infinitesimal interaction for such a region $-[\ln 2, \ln 2]$ surrounding IRC, thus extending the already discussed $\delta = 0$ case [24]. This is checked numerically in Fig. 5 at $\delta = 0, 0.5, 1$, where both $S_N(L/2)/L$ and $\bar{\tau}$ experience strong drifts which only stop for a non-zero interaction $g_c \approx 0.05$ for $\delta = 1$. Our results for $\delta = 0$ in Fig. 5 (a,d), which confirm Ref. [24], appear very similar to $\delta = 0.5$, see Fig. 5 (b,e), thus supporting such a scenario.

Conclusions—The $\mathbb{Z}_2$-symmetric disordered and interacting Ising-Majorana chain model Eq. (2) provides a very rich infinite-temperature phase diagram, as shown in Fig. 1. First, we confirm the emergence of two distinct MBL regimes, separated by a broad ergodic phase, as already observed in Refs. [23, 24]. Our key new result is that localization-protected topological order which survives weak interactions is always associated to many-body spectral pairing. Indeed, the avalanche criterion $\xi = (\ln 2)^{-1}$ for ergodic instability is the very same condition required to resolve spectral pairing on top of the natural many-body spacing. A crucial consequence of this finding is that weak interactions destabilize not only the infinite randomness critical point towards thermalization, but also its close vicinity, as long as the typical localization length remains larger than the avalanche threshold.

Nevertheless, it is important to take a critical step back on these conclusions, simply because our results come from numerical simulations which are far from the thermodynamic limit. This prompts us to remain careful, in particular we cannot entirely exclude the existence of an intermediate phase (thin green region in Fig. 1) where MBL SG could occur without spectral pairing, while our finite-size numerics
appear more consistent with a single spectral-paired MBL SG regime. We finally note that finite-size effects associated to the avalanche instability [51, 52] may point towards an even wider opening of the ergodic phase at weak interaction.

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