Laguerre polynomials method in the valon model

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Abstract
We used the Laguerre polynomials method for determination of the proton structure function in the valon model. We have examined the applicability of the valon model with respect to a very elegant method, where the structure of the proton is determined by expanding valon distributions and valon structure functions on Laguerre polynomials. We compared our results with the experimental data, the Gluck–Jimenez-Delgado–Reya parameterization and the Donnachie–Landshoff model. Having checked, this method gives a good description of the proton structure function in the valon model.

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1. Introduction
Structure functions in lepton–nucleon deep-inelastic scattering (DIS) are the established observables probing quantum chromodynamics (QCD), the theory of the strong interaction, and in particular the structure of the nucleon. The structure functions provide unique information about the deep structure of the hadrons and most importantly, they form the backbone of our knowledge of the parton densities. In QCD, structure functions are defined as the convolution of the universal parton momentum distributions inside the proton and coefficient functions, which contain information about the boson–parton interaction. The parton distributions in the proton have been studied extensively in a wide range of both $x$ and $Q^2$, as they are accurate but inconvenient to describe analytically. Here we elaborate on the valon model, which can be very useful in the study of hadronic structure, in particular when the experimental data are scarce. A valon has its own cloud of partons which can be calculated in pQCD. This structure is universal and independent of the hosting hadron [1, 2].

The valon model [1–4] is a phenomenological model which is proven to be very useful in its application to many areas of hadron physics. A valon is defined to be a dressed valence quark in QCD with a cloud of gluons and sea quarks and antiquarks. Its structure can be resolved at high enough $Q^2$ probes. In the scattering process, the virtual emission and absorption of gluon in a valon becomes bremsstrahlung and pair creation, which can be calculated in QCD. At a sufficiently low $Q^2$, the internal structure of a valon cannot be resolved, and hence it behaves as a structureless valence quark. At such a low value of $Q^2$, the nucleon is considered as a bound state of three valons, UUD for proton. The binding agent is assumed to be very soft gluons or pions. Let us denote the distribution of a valon in a nucleon by $G_N^v(y)$ for each valon $v$. It satisfies the normalization condition $\int_0^1 G_N^v(y) \, dy = 1$ and the momentum sum rule $\sum_v \int_0^1 G_N^v(y) \, dy = 1$, where the sum runs over all valons in nucleon $N$. The organization of the paper is as follows. In section 2, we will give a brief description of the valon model. In section 3, we consider the Laguerre polynomials method on which this work is based on; then we will calculate the nucleon structure in terms of the valon distributions and the valon structure functions on the Laguerre polynomials. Finally, in section 4, the numerical calculations will be outlined; then we discuss some qualitative implications of the Laguerre polynomials method on the structure of the nucleon in the valon model.

2. Valon distribution and valon structure function
In the preceding section, we discussed the valon structure of a nucleon. In this subsection, we will give the distributions in a valon. The nucleon structure function is related to the valon structure function by the convolution theorem as follows [1–5]:

$$F_N^N(x, Q^2) = \sum_v \int_0^1 G_N^v(y) F_v^2 \left( \frac{x}{y}, Q^2 \right) \frac{dy}{y},$$

(1)
where the summation is over the three valons and \( G^s \) (y) is the probability of finding a valon that carries the momentum fraction y of the hadron, and \( F_2^u(x, Q^2) \) is the structure function of a valon. The structure function of a \( U \)-type valon can be written as

\[
F_2^U(z, Q^2) = \frac{4}{9} z(G_\bar{p} + G_\bar{q} + \frac{1}{3} z(G_p + G_\bar{p} + G_\bar{q} + G_q),
\]

where \( G \) are the valence and sea quark distribution functions, and defined the probability function for \( v \)-valon to have a momentum fraction \( z \) in the nucleon. A similar expression can be written for the \( D \)-type valon. The structure of a valon can be written in terms of favored distribution \((G_i)\) and unfavored distribution \((G_{af})\), as for the \(D\) and \(U\) valon structure functions we have \([1–5]\)

\[
F_2^U(z, Q^2) = \frac{4}{9} [G_i(z, Q^2) + 2G_{af}(z, Q^2)],
\]

(3)

\[
F_2^D(z, Q^2) = \frac{1}{9}[G_i(z, Q^2) + 11G_{af}(z, Q^2)],
\]

(4)

\[
F_2^U(z, Q^2) = \frac{2}{9}[G_i(z, Q^2) + G_{NS}(z, Q^2)],
\]

(5)

\[
F_2^D(z, Q^2) = \frac{1}{9}[2G_i(z, Q^2) + G_{NS}(z, Q^2)],
\]

(6)

where \( G_i \) and \( G_{NS} \) are defined by singlet and nonsinglet components, respectively. The relation between favored and unfavored distributions with singlet and nonsinglet distributions has the following forms:

\[
G_i = \frac{1}{2f}[G^S + (2f - 1)G^{NS}],
\]

(7)

\[
G_{af} = \frac{1}{2f}[G^S - G^{NS}],
\]

(8)

where \( f \) is the number of active flavors \((f = 3 \text{ or } 4)\). In the momentum representation, we have

\[
M_2(n, Q^2) = \int_0^1 x^{-2} F_2(x, Q^2) \text{d}x
\]

(9)

and

\[
M_\alpha(n, Q^2) = \int_0^1 x^{-1} G_\alpha(x, Q^2) \text{d}x,
\]

(10)

where \( \alpha = S, N, S \). In order to estimate the structure function moments, by inserting equations (9) and (10) into equation (1), we obtain

\[
M^\alpha(n, Q^2) = \sum_v M^\alpha_v(n) M^\alpha_v(n, Q^2).
\]

(11)

We assume a general parameterization form for the \(U\) and \(V\) valons as follows:

\[
G^{U,V}(y_1, y_2, y_3) = \alpha(y_1 y_2 y_3)^\rho y_1^\delta y_2^\beta (y_1 + y_2 + y_3 - 1),
\]

(12)

where \( \alpha \) and \( \beta \) are the two free parameters that can be evaluated from the experimental data, and \( \alpha \) is a normalization coefficient. Here \( y_i \) is the momentum fraction of the \( i \)-th valon. The \( U \)- and \( D \)-type valon distributions can be obtained by integration over the specified variable as

\[
G^U(y) = \int dy_2 dy_3 G^{U,V}(y, y_2, y_3) = B(a + 1, a + 2, 2)^{-1} y^a (1 - y)^{a+1},
\]

(13)

\[
G^D(y) = \int dy_1 dy_2 G^{U,V}(y_1, y_2, y) = B(b + 1, 2a + 2)^{-1} y^b (1 - y)^{2a+1},
\]

(14)

where \( B(m, n) \) is the Euler beta-function. The normalization factor has been fixed by requiring

\[
\int_0^1 G^U(y) \text{d}y = \int_0^1 G^D(y) \text{d}y = 1.
\]

(15)

Consequently, moments of these valon distributions are calculated \([1–6]\) according to the Mellin transformation from equation (10) for a nucleon:

\[
U_n = \frac{B(a + n, a + b + 2)}{B(a + 1, a + b + 2)},
\]

(16)

\[
D_n = \frac{B(b + n, 2a + 2)}{B(b + 1, 2a + 2)},
\]

(17)

with \( a = 0.65 \) and \( b = 0.35 \). Therefore, the valon distributions can be obtained as

\[
G^U = 7.98 y^{0.65} (1 - y)^2,
\]

(18)

\[
G^D = 6.01 y^{0.35} (1 - y)^{2.3}.
\]

(19)

Now, we can go to the \( N \)-moment space for defining the moments of these quark distribution functions (valence, sea and gluons) \([5, 7, 8]\) as:

\[
M_{ns}(n, s) = 2U(n) M^{NS}(n, s),
\]

(20)

\[
M_{ds}(n, s) = D(n) M^{NS}(n, s),
\]

(21)

where \( M^S \) and \( M^{NS} \) are the moments of the singlet and nonsinglet valon structure functions and \( M_{gs}(n, s, \rho) \) is a quark-to-gluon evolution function and defined into \( d_{gs}, d_{gs(-)}, d_{gs} \) and \( \rho \) where they are anomalous dimensions \([1, 2, 5, 7, 8]\) as follows:

\[
M^{NS}(n, s) = e^{-d_{gs}s},
\]

(24)

\[
M^S(n, s) = \frac{1}{2} (1 + \rho) e^{-d_{gs}s} + \frac{1}{2} (1 - \rho) e^{-d_{gs}s},
\]

\[
M_{gs}(n, s) = (d_{gs(-)} - d_{gs})^{-1} d_{gs} e^{d_{gs}s} - e^{-d_{gs}s}.
\]

(25)
Here, $s$ is defined by

$$s = \ln \left[ \frac{\ln(\frac{Q^2}{x^2})}{\ln(\frac{Q^2}{x^2})} \right],$$  \hspace{1cm} (26)$$

where $Q_0^2$ and $\Lambda$ are our initial scales. In order to determine the parton distributions, we have used a fit to a set of the experimental data [9, 10] for a single value of $s$ or $Q^2$. [Then we fit the moments by a beta function, as they can be written by [3, 4, 6]]

$$xq_i(x) = a_i (1 - x)^b x^c,$$  \hspace{1cm} (27)

$$xq_i(x) = a_i^b (1 - x)^c (1 + d_i x + e_i x^{0.5}),$$  \hspace{1cm} (28)

where the subscript $i$ stands for sea or gluon, and $xq_i$ the sea and gluon distribution functions. The free parameters in equations (27) and (28) are further considered to be functions of $s$, as given in the appendix. Therefore, we obtained the parton distributions for any valon that can be used in the valon structure function.

3. Laguerre polynomials to valon model

So far, the structure of a nucleon in the valon distributions has been determined. Now, we will use an elegant and fast numerical method to determine the proton structure function in the valon model. Therefore, we concentrate on the Laguerre polynomials in our determinations. In the Laguerre polynomials method [11, 12], the Laguerre polynomials are defined as

$$(n + 1)L_n(x) = (2n + 1 - x)L_n(x) - n L_{n-1}(x)$$  \hspace{1cm} (29)$$

and orthogonality condition is defined as

$$\int_0^{\infty} e^{-\hat{x}} L_n(\hat{x}) L_m(\hat{x}) d\hat{x} = \delta_{n,m}. \hspace{1cm} (30)$$

The general integrable function $f(e^{-\hat{x}})$ is transformed into the sum

$$f(e^{-\hat{x}}) = \sum_{n=0}^{N} f(n) L_n(\hat{x}), \hspace{1cm} (31)$$

where

$$f(n) = \int_0^{\infty} e^{-\hat{x}} L_n(\hat{x}) f(e^{-\hat{x}}) d\hat{x}. \hspace{1cm} (32)$$

In what follows, we calculate the proton structure function in the valon model using the Laguerre polynomials method. We used the variable transformations, $x = e^{-\hat{x}}$, $y = e^{-\hat{y}}$ to get the valonic structure function form to the Laguerre polynomials form. Then, we combined and expanded each term of this equation on Laguerre polynomials according to equations (31) and (32) and used these properties as

$$\int_0^{\infty} d\hat{y} L_n(\hat{x} - \hat{y}) L_m(\hat{y}) = L_{n+m}(\hat{x}) - L_{n+m+1}(\hat{x}).$$  \hspace{1cm} (33)$$

We obtained an equation that determines $F_2^p(x, Q^2)$ in terms of the Laguerre polynomials, namely

$$F_2^p(n, Q^2) = \sum_{m=0}^{n} \tilde{G}_z(m)[F_2^p(n - m, Q^2) - F_2^p(n - m - 1, Q^2)], \hspace{1cm} (34)$$

where

$$F_2^p(n, Q^2) = \int_0^{\infty} d\hat{x} e^{-\hat{x}} F_2^p(e^{-\hat{x}}, Q^2) L_n(\hat{x}), \hspace{1cm} (35)$$

and

$$\tilde{G}_z(m) = \int_0^{\infty} d\hat{y} e^{-\hat{y}} G_z(e^{-\hat{y}}) L_m(\hat{y}), \hspace{1cm} (36)$$

as $F_2^p(x)$ is defined according to equations (3) and (4), which are accompanied with respect to equations (27) and (28) and their coefficients as given in the appendix, and $G_z(y)$ is defined according to equations (18) and (19), respectively. Therefore, we find the solution of the proton structure function in the valon model defined by solving this recursion relation as

$$F_2^p(x, Q^2) = \sum_{n=0}^{N} F_2^p(n, Q^2) L_n \left( \frac{1}{x} \right), \hspace{1cm} (37)$$

where $F_2^p(n, Q^2)$ is the proton structure function with respect to the Laguerre model and is defined by equations (34)–(36) as the coefficients in these equations is obtained with respect to the valon model. This result is completely general and gives the expression for the proton structure function with respect to the Laguerre polynomials model. Here we can expand the integrable functions till a finite order $N = 30$, as we can make these series converge in the numerical determinations.

4. Results and Discussion

We computed the predictions for the proton structure function with all details in the kinematic range where it has been measured by the H1 Collaboration [9, 10] and compared our results with the Donnachie–Landshoff (DL) model [13–15] based on hard pomeron exchange, and with the Gluck–Jimenez–Delgado–Reya (GJR) parameterization [16]. Our numerical predictions are presented as functions of $x$ for $Q^2 = 22.5 \text{GeV}^2$. The results are presented in figure 1, where they are compared with the H1 data and with the results obtained with the help of other standard parameterizations. The curves represent the proton structure functions based on a fit to all data. We compared our results with predictions of $F_2^p$ in perturbative QCD, where the input densities are given by GJR parameterizations [16]. Also, we compared our results with the two pomeron fits as seen in figure 1. The agreement between the Laguerre polynomials method for the proton structure function in the valon model and data at low and high $x$ is remarkably good, because at low $x$ the gluon distributions dominate. Therefore, the good agreement indicates that the Laguerre polynomials method in the valon model for the proton structure function has a good asymptotic behavior and it is compatible with both the data and the other standard models at $x$ values. Hence, this model has the advantage that we get a very elegant solution for the proton.
shows the shape of the distribution
Hwa R C 1980
28 22 27 66 66 67 hep-ph/9909328v1
Arash F 2003 arXiv:
hep-ph/0307247v1 hep-ph/9904264v1
Arash F 2003
557
Figure 1. Comparison of the proton structure function by using the Laguerre polynomials in valon model, with the experimental data [9, 10], DL model [13–15] and GJR parameterization [16] at \( Q^2 = 22.5 \text{ GeV}^2 \).

Figure 2. Comparison of the parton distributions in proton with GJR parameterization [16] at \( Q^2 = 22.5 \text{ GeV}^2 \).

structure function. Figure 2 shows the shape of the distribution functions in equations (27) and (28) for the valence and the sea quarks at \( Q^2 = 22.5 \text{ GeV}^2 \).

In summary, we have used the Laguerre polynomials method to describe the proton structure function in the valon model. The proton structure can be determined in terms of the valon distributions and the valon structure functions with respect to the Laguerre polynomials. To confirm the method and results, the calculated values are compared with the H1 data on the proton structure function. It is shown that there is good agreement with experimental H1 data for \( F_2 \) if one takes into account the total errors and is consistent with a higher order QCD calculations of \( F_2 \), which essentially show an increase as \( x \) decreases. We observed that the calculation results are consistent with the two pomeron models, thus implying that the Regge theory and perturbative evolution may be made compatible at small-\( x \). Also, this model gives a good description of the parton distributions at low and high \( x \) values.

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Appendix
Here we will give the functional form of parameters of equations (27) and (28) in terms of \( s \) as follow: Coefficients for \( u \) valence in \( U \) valon are
\[
\begin{align*}
a_u &= 16.860 - 8.382x + 1.352x^2, \\
b_u &= 1.775 + 0.693x + 0.014x^2 \\
c_u &= 1.794 - 0.699x + 0.104x^2.
\end{align*}
\]

Coefficients for \( d \) valence in \( D \) valon are
\[
\begin{align*}
a_d &= 13.119 - 8.66x + 1.65x^2, \\
b_d &= 7.129 - 3.5x + 0.901x^2 \\
c_d &= 1.971 - 0.9502x + 0.163x^2.
\end{align*}
\]

Coefficients for sea quarks in each valon are
\[
\begin{align*}
a_{sea} &= -0.206 + 0.190s - 0.022s^2, \\
b_{sea} &= -0.884 + 0.484s - 0.104s^2, \\
c_{sea} &= 12.089 - 4.532s + 0.939s^2, \\
d_{sea} &= -2.564 + 5.937s - 1.531s^2,
\end{align*}
\]

and
\[
\begin{align*}
e_{sea} &= -8.623 + 4.982s - 1.123s^2.
\end{align*}
\]

Coefficients for gluons in each valon are
\[
\begin{align*}
a_{gluon} &= 13.8745 - 22.3304s + 12.7885s^2 - 2.4801s^3, \\
b_{gluon} &= 4.6810 - 8.4594s + 4.7656s^2 - 0.9209s^3, \\
c_{gluon} &= -24.5652 + 50.4661s - 30.147s^2 + 6.0738s^3, \\
d_{gluon} &= -0.8839 + 0.0403s - 0.0174s^2.
\end{align*}
\]

References
[1] Hwa R C 1980 Phys. Rev. D 22 1593
Hwa R C 1995 Phys. Rev. D 51 85
[2] Hwa R and Yang C B 2002 Phys. Rev. C 66 025204
Hwa R and Yang C B 2002 Phys. Rev. C 66 025205
[3] Arash F 2003 arXiv:hep-ph/0307247v1
Arash F and Khorramian A N 2003 Phys. Rev. C 67 045201
Arash F and Khorramian A N 1999 arXiv:hep-ph/9904264v1
Arash F and Khorramian A N 1999 arXiv:hep-ph/9909328v1
[4] Arash F 2003 Phys. Lett. B 557 38
Arash F 2004 Phys. Rev. D 9 054024
[5] Hwa R C and Zahir S 1981 Phys. Rev. D 23 2539
  Hwa R C and Lam C S 1982 Phys. Rev. D 26 2338
[6] Mirgalili A et al 2010 J. Phys. G: Nucl. Part. Phys. 37 105003
[7] Degrand T A 1979 Nucl. Phys. B 151 485
[8] Hinchliffe I and Llewellyn Smith C H 1977 Nucl. Phys. B 128 93
[9] Adams A et al (E665 Collaboration) 1995 Phys. Rev. Lett. 75 1466
  Ahmed T et al (H1 Collaboration) 1995 Nucl. Phys. B 439 471
[10] Adloff C and H1 Collaboration 2001 Eur. Phys. J. C 21 33
[11] Schoeffel L 1999 Nucl. Instrum. Methods A 423 439
  Coriano C and Savkli C 1999 Comput. Phys. Commun. 118 236
[12] Rezaei B and Boroun G R 2011 Nucl. Phys. A 857 42
[13] Donnachie A and Landshoff P V 1992 Phys. Lett. B 296 257
[14] Donnachie A and Landshoff P V 1998 Phys. Lett. B 437 408
[15] Donnachie A and Landshoff P V 2002 Phys. Lett. B 550 160
  Landshoff P V 2002 arXiv:hep-ph/0203084
[16] Gluck M, Jimenez-Delgado P and Reya E 2008 Eur. Phys. J. C53 355