A Rigorous Path Integral for Supersymmetric Quantum Mechanics and the Heat Kernel

Dana S. Fine\textsuperscript{1} and Stephen F. Sawin\textsuperscript{2}

Abstract

In a rigorous construction of the path integral for supersymmetric quantum mechanics on a Riemann manifold, based on Bär and Pfäffle’s use of piecewise geodesic paths, the kernel of the time evolution operator is the heat kernel for the Laplacian on forms. The path integral is approximated by the integral of a form on the space of piecewise geodesic paths which is the pullback by a natural section of Mathai and Quillen’s Thom form of a bundle over this space. In the case of closed paths, the bundle is the tangent space to the space of geodesic paths, and the integral of this form passes in the limit to the supertrace of the heat kernel.

\textsuperscript{1}University of Massachusetts Dartmouth, dfine@umassd.edu

\textsuperscript{2}Fairfield University, sawin@cs.fairfield.edu
Introduction

In [B-P] Bär and Pfaffle construct a path integral representation of the heat kernel for a general Laplacian on a Riemann manifold. They express the path integral as an integral over piecewise geodesic paths in the limit as \( n \), the number of pieces, approaches infinity. In this note, we begin with the Lagrangian for \( N = 1 \) supersymmetric quantum mechanics (SUSYQM), restrict the action to piecewise geodesic paths, and identify the resulting expression as a form on a finite-dimensional manifold. This form derives directly from Mathai and Quillen’s universal Thom form. We interpret the integral of the top part of this form over the finite-dimensional space as defining an approximation to the path integral representing the kernel of the SUSYQM time evolution operator.

Applying Bär and Pfaffle’s arguments to evaluate the appropriate large-\( n \) limit shows the partition functions for piecewise geodesic paths with fixed endpoints converge to the heat kernel for the Laplacian on forms. Precisely, we prove a corollary to Bär and Pfaffle’s Theorems 2.8 and 6.1:

**Corollary 3.5.1** For any sequence of partitions \( t_1, t_2, \ldots, t_n \) such that \( \max_i(t_i) \to 0 \) and \( \sum_i t_i \to t \) and for any form \( \alpha \) on \( M \)

\[
\lim K(t_1)K(t_2) \cdots K(t_n)\alpha = e^{-t\Delta/2}\alpha
\]

where \( \Delta \) is the Laplace-Beltrami operator on forms. Moreover, for some such sequence of partitions

\[
\lim K(t_1) * K(t_2) * \cdots * K(t_n) \to K_\Delta(x, y; t)
\]

uniformly, where \( K_\Delta \) is the heat kernel of \( \Delta \) (the kernel of \( e^{-t\Delta/2} \)).

Here the kernel \( K(t) \) of the operator \( \mathfrak{R}(t) \) is the pullback (by a certain natural section) of Mathai and Quillen’s Thom form on the bundle \( TM \times M \to M \times M \) restricted to an open subset. In fact, the indicated \( n \)-fold \(*\)-product expresses an integration of the analogous Mathai-Quillen Thom form on a bundle over \( M^{n+1} \) restricted to an open subset and pulled back by a section. The base space of this bundle fibers further to become a bundle over \( M \times M \), on which the \( n \)-fold \(*\)-product becomes an integration over the fibers.

The import of this corollary is that the finite-dimensional partition functions which directly approximate the kernel of the time evolution operator \( e^{-t\Delta/2} \) converge to the heat kernel. Further, for closed paths based at a given point, this yields a rigorous path integral expression for the supertrace of the heat kernel. This path integral is the large-\( n \) limit of the Mathai-Quillen Euler form integrated over the finite-dimensional manifold.

Getzler [G] uses stochastic integrals due to Stroock [S], and asymptotics of the heat operator for the Laplacian on spinors due to Patodi [P], to calculate the supertrace of this heat operator as a rigorous path integral. Rogers [R1] uses stochastic analysis techniques to express the heat operator on forms in terms of a supersymmetric generalization of Wiener integrals and thereby obtains a
path integral expression for the supertrace of the heat operator. The novelty of our approach is in constructing a rigorous path integral that directly links the heat operator to the SUSYQM time evolution operator and the Mathai-Quillen construction.

These results confirm Alvarez-Gaumé’s [A] and Witten’s [W] now-standard arguments, which express the supertrace of the heat operator heuristically as a path integral. Our approach to rigorizing these arguments is sufficiently direct to see the relation, as derived formally by Blau [B], between SUSYQM and Mathai & Quillen’s universal Thom form [M-Q].

Acknowledgements: It is our pleasure to thank Christian Bär and Steve Rosenberg for helpful comments on the draft of this paper.

1 Preliminaries and Notation

We review the key facts needed from Riemannian geometry and fix notation, most of which follows Berline, Getzler and Vergne [B-G-V].

1.1 Notation for Riemannian geometry

Let $M$ be a compact oriented $2m$-dimensional Riemann manifold. In a coordinate patch let $\partial_\mu$ be the corresponding basis of tangent fields, $\psi^\mu$ be the dual basis of one-forms, and $\iota_\mu$ be the odd derivation on forms defined by $\iota_\mu \psi^\nu = \delta^\nu_\mu$.

The metric $g_{\mu\nu} = (\partial_\mu, \partial_\nu)$ determines Christoffel symbols

$$\Gamma^\gamma_{\mu\nu} = \frac{1}{2} g^{\gamma\rho}(\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} - \partial_\nu g_{\rho\mu}) = \frac{1}{2} g^{\gamma\rho}(g_{\rho\nu,\mu} + g_{\rho\mu,\nu} - g_{\mu\nu,\rho}), \quad (1.1)$$

(indices after the comma denote differentiation in that coordinate) in terms of which the Levi-Civita connection is

$$\nabla_\mu (Y^\nu \partial_\nu) = (\partial Y^\nu / \partial x^\mu) \partial_\nu + \Gamma^\nu_{\mu\rho} Y^\rho \partial_\nu.$$  

The operator $\nabla$ extends to a one-form with values in differential operators on forms by

$$\nabla_\mu = \iota_\mu d - \Gamma^\eta_{\mu\nu} \psi^\nu \iota_\eta.$$  

The curvature $R$ of the Levi-Civita connection is a smooth two-form with values in linear transformations on the fiber. Acting on the coordinate basis, it is

$$R(\partial_\pi, \partial_\eta) \cdot \partial_\nu = R_{\pi\eta}^{\nu} \partial_\nu,$$

where

$$R_{\mu\nu\gamma} = \Gamma^\delta_{\nu\gamma,\mu} - \Gamma^\delta_{\mu\gamma,\nu} + \Gamma^\delta_{\mu\chi} \Gamma^\chi_{\nu\gamma} - \Gamma^\delta_{\nu\chi} \Gamma^\chi_{\mu\gamma}. \quad (1.2)$$

The element of the dual basis is more commonly denoted $dx^\mu$. We use $\psi^\mu$ in anticipation of the interpretation in terms of supersymmetric variables in 1.4 below.
We will freely raise and lower all four indices on $R$ with the metric, keeping track of the order by spacing. With this convention, the symmetries of $R$ are

$$R_{\mu\nu\pi\eta} = R_{\pi\eta\mu\nu} = -R_{\nu\mu\pi\eta} \quad R_{\mu\nu\pi\eta} + R_{\pi\mu\nu\eta} + R_{\eta\nu\pi\mu} = 0.$$  

The Ricci tensor is

$$\text{Ricci}_{\sigma\tau} = R_{\sigma\mu\nu}^\mu. \quad (1.3)$$

### 1.2 Laplace-Beltrami and heat kernels

The Laplace-Beltrami operator $\Delta$ on the space $\Omega(M)$ of forms is

$$\Delta = -g^{\mu\nu}(\nabla_\mu \nabla_\nu - \Gamma^\sigma_{\mu\nu} \nabla_\sigma) - \text{Ricci}_\pi^\eta t_\pi - \frac{1}{2}R_{\mu\eta\pi\nu}^\pi \psi_\mu^\eta \psi_\pi^\nu t_\pi. \quad (1.4)$$

The evolution operator $e^{-t\Delta/2}$ is a semigroup of operators on $\Omega(M)$ depending on a parameter $t \in [0, \infty)$ such that for $\alpha \in \Omega(M)$ $\alpha_t = e^{-t\Delta/2}\alpha$ is a solution to the heat equation

$$(\Delta/2 + \partial_t)\alpha_t = 0$$

with $\alpha_0 = \alpha$ as initial conditions.

The *heat kernel*, a smooth map $K_\Delta$ from $(0, \infty)$ to sections of $\Omega(M \times M)$, provides an integral representation of the time evolution operator. Explicitly, for $\alpha \in \Omega(M)$

$$(e^{-t\Delta/2}\alpha)(x) = K_\Delta \star \alpha = \int_{y \in M} K_\Delta(x, y; t)\alpha(y)$$

where on the right-hand side we wedge the forms over $y$ together, take the top form piece, and integrate over the second factor of $M$. In general operators on $\Omega(M)$ are represented by forms in $\Omega(M \times M)$, with operator composition

$$K_1 \star K_2(x, z) = \int_{y \in M} K_1(x, y)K_2(y, z). \quad (1.5)$$

### 1.3 Riemann Normal Coordinates

Orthonormal coordinates on $T_y M$ extend via $\exp_y$ to coordinates on a patch of $M$ called Riemann normal coordinates. In Riemann normal coordinates lines from the origin are geodesics with length consistent with the coordinates, and the following hold, where $\vec{x} = x^\mu \partial_\mu$ is the tangent vector at $y$ corresponding to $x$ and $|\vec{x}|$ is its length

$$g^{\mu\nu}(x) = \delta^{\mu\nu} - \frac{1}{3}R_{\sigma\tau}^\mu(0)x^\sigma x^\tau + O(|\vec{x}|^3), \quad (1.6)$$

$$\Gamma^\delta_{\mu\gamma}(x) = -\frac{1}{3} \left[ R_{\mu\nu\gamma}(0) + R_{\gamma\nu\mu}(0) \right] x^\nu + O(|\vec{x}|^2). \quad (1.7)$$
Finally, any vector $v \in T_xM$ defines two vectors in $T_yM$: the first is $\vec{v} = (d\exp_y)^{-1}v$; the second is the parallel translate $v^{||}$ of $v$ along the geodesic from $x$ to $y$. These are related by

$$v^{||} = \vec{v} + \frac{1}{6} R(\vec{x}, \vec{v}) \cdot \vec{x} + O(|\vec{x}|^3)|v|.$$  

(1.8)

Whenever we work in Riemann normal coordinates we implicitly restrict attention to a patch within the injectivity radius of the center, small enough that there is a unique geodesic from the center to each point.

### 1.4 Supersymmetric variables

If $V$ is a vector space, we represent elements of $\Lambda(V^*)$ by formulas involving an anticommuting element $\psi$ of $V$. For example, given a basis $e_1, \ldots, e_n$ of $V$, an antisymmetric matrix $\omega_{\mu\nu}$ determines an element $\omega(\psi)$ of $\Lambda^2(V^*)$ via $\omega(\psi) = \frac{1}{2} \omega_{\mu\nu} \psi^\mu \psi^\nu$, with $\psi = \psi^\mu e_\mu$. In the latter expansion of $\psi$, each $\psi^\mu$ is an anticommuting numerical variable. On the other hand, each $\psi^\mu$ in the expansion of an element of $\Lambda(V^*)$ is a map sending the real element $v \in V$ to a real number; namely, its component $v^\mu$ in the given basis. Thus $\psi^1, \ldots, \psi^n$ also represents the basis of $V^*$ dual to $e_1, \ldots, e_n$. In this interpretation $\psi = \psi^\mu e_\mu$ is then an expression for the identity map $dx^\mu e_\mu$ on $V$ composed with the natural map from $V$ to the exterior algebra $\Lambda(V)$. In calculations it is usually easier to work with $\psi$ as denoting an anticommuting tangent vector; to interpret the resulting expressions, it is helpful to remember it means this identity map.

By the same token we can consider $\rho$ as an anticommuting variable in $V^*$ which will be used in formulas representing elements of $\Lambda(V)$. In this context $\rho_\mu$ replaces $e_\mu$ in the usual expressions. Equivalently, $\rho$ represents the identity map on $V^*$.

Most often $\rho$ will be used inside a Berezin integral. This integration is defined for $f$ an anticommuting polynomial, in terms of a volume form on $V$, by $\oint f(\rho) \, d\rho$ is the volume of the dim($V$) degree piece of $f$. For example if $V$ is $2m$ dimensional, with a basis chosen so $\psi^1 \cdots \psi^{2m}$ is the volume form, and $g(\psi)$ is an anticommuting polynomial in $\psi$, then the Berezin integral over $\rho$ is

$$\oint e^{i(\rho, \psi)} g(\psi) \, d\rho = \oint \frac{(-1)^m}{(2m)!} \rho_{\mu_1}^1 \cdots \rho_{\mu_{2m}}^{2m} g_{\nu_1 \cdots \nu_l} \psi_{\nu_1} \cdots \psi_{\nu_l} \, d\rho.$$  

The right-hand side denotes the 0-degree part of $g$ times the volume form on $V$.

In this paper $\psi$ and $\rho$ will be anticommuting elements of the tangent and cotangent spaces, respectively, at a point in $M$, so that the formulas involving
them will describe forms on $M$. Two examples serve to illustrate Berezin integration in this context and to provide formulas we will require in Section 3. With $\psi_x$, $\rho^y$ and $\psi_y$ denoting anticommuting tangent and cotangent vectors at points $x$ and $y$ in $M$, $\psi^y_x$ representing the parallel transport of $\psi_x$ from $x$ to $y$ along some path connecting them, and $\alpha \in \Lambda(T^*_y M)$,

$$\oint \oint e^{\langle \rho^y,\psi^y_x \rangle - \psi_y} \alpha d\rho^y d\psi_y = \alpha^y. \quad (1.9)$$

Here $\alpha^y$ is $\alpha$ parallel transported along the given path. Thus we have an operator that can implement parallel transport. (Of course, parallel transport could be replaced by any linear map.) The key to this calculation is that the coefficient of the top form in $\rho$ is proportional to $[\psi^y_x]^1 - \psi^y_x 1$. The top-form piece of the product of this with $\alpha$ will include terms like $\psi^y_x \psi^y_x \psi^y_x \psi^y_x \alpha_{14} \alpha_{14} (\psi^y_x)^1 (\psi^y_x)^4$ leading, after integration with respect to $\psi_y$, to $\alpha^y$ on the right-hand side. Likewise, for $\mu \in \{1, \ldots, 2m\}$,

$$\oint \oint \rho^y \alpha^y \psi^y_x \alpha d\rho^y d\psi_y = i (\lambda_{\mu} \alpha) = (1.10)$$

In this notation, $f(x, \psi_x)$, for $f$ smoothly varying in $x$ and an antisymmetric multinomial in $\psi_x$, corresponds to a smooth differential form $f$ on $M$. Moreover, $\int \oint f(x, \psi_x) d\psi_x dx$ is the integral $\int f$ of the top-form part of $f$ over $M$.

# 2 Discrete Approximation to the SUSYQM Lagrangian

In this section we define a sequence of finite-dimensional subspaces of the space of paths in $M$ on which we interpret the $N=1$ supersymmetric quantum mechanical Lagrangian as a form. This form describes a kernel which is an operator product of a number of copies of a simpler kernel described by a form $\mathcal{K}^{qm}$ on a $4m$-dimensional space. We will ultimately apply Bär & Pfaiffle’s arguments to show that, as the dimension of the subspaces increases, the product of kernels converges uniformly to the kernel of the Laplace-Beltrami heat operator.

## 2.1 Short geodesics

A short geodesic is a geodesic of length less than the injectivity radius of $M$. The space of short geodesics is isomorphic to $M^{(2)}$, the subspace of $M^2$ consisting of pairs of points within the injectivity radius of each other. (We take our paths as oriented but not parameterized; later we will choose parameterizations). Let Path$_n$ denote the space of $n$-segment piecewise short geodesic paths in $M$, and let Path$_n(x, y)$ denote the subspace of those going from $y$ to $x$. Path$_n$ is
isomorphic to $M^{(n+1)}$, the subspace of $M^{n+1}$ in which each successive point is within the injectivity radius of the previous.

If $\sigma_t$ is a short geodesic in Path 1 the isomorphism with $M^{(2)}$ sends $\sigma$ to $(x, y)$, where $x = \sigma_1$ and $y = \sigma_0$. Note the unconventional choice of a path going from $y$ to $x$. This is necessitated by the standard conventions of kernels and operators.

### 2.2 Tangents to short geodesics

If $\sigma_t$ is a geodesic, represent a tangent vector to it in the space of geodesics by a tangent field $\psi_t \in T_{\sigma_t}M$ along $\sigma$.

Let $\psi_1 \in T_{\sigma_0}M$ be the parallel translate of $\psi_t$ from $\sigma_t$ to $\sigma_0$ along $\sigma$ according to the Levi-Civita connection. Suppose $\sigma$ from $t = 0$ to $t = 1$ is a short geodesic, and take $\psi_t$ to be tangent to a one-parameter family of short geodesics. Since each geodesic in this family is determined by its endpoints, $\psi_t$ should be determined by $\psi_0$ and $\psi_1$.

**Lemma 2.2.1** If $\sigma_t$ is a geodesic path mapping $[0, 1]$ to $M$, $\psi_t$ is a tangent field along $\sigma$, $d = d(\sigma_0, \sigma_1)$ is the distance between the endpoints of $\sigma$, and $|\psi| = \max(|\psi_0|, |\psi_1|)$, then

$$
\psi_t = t\psi_1 + (1 - t)\psi_0 + \frac{t^3 - t}{6}R(\dot{\sigma}_0, \psi_1) \cdot \dot{\sigma}_0
- \frac{t^3 - 3t^2 + 2t}{6}R(\dot{\sigma}_0, \psi_0) \cdot \dot{\sigma}_0 + O(d^3)|\psi|
$$

(2.1)

where $R$ is computed at $\sigma_0$, and $\dot{\sigma}_t = \partial_t \sigma_t$.

**Proof:** Since the result is linear in $\psi_1$ and $\psi_0$, we prove it when $\psi_0 = 0$. The case $\psi_1 = 0$ and thus the general case follow from reversing the parameterization.

In Riemann normal coordinates centered at $\sigma_0$, $\psi_1 = (d\exp_{\sigma_0})\tilde{\psi}_1$ for some $\tilde{\psi}_1 \in T_{\sigma_0}M$.

Extend $\tilde{\psi}_1$ to a path of tangent vectors as

$$
\tilde{\psi}_t = t\tilde{\psi}_1.
$$

Note that because lines through the origin are geodesics in Riemann normal coordinates, this path of tangent vectors describes a tangent vector to the space of geodesics.

Applying Eq. (1.8) to $\tilde{\psi}_t$ and $\tilde{\psi}_1$ gives

$$
\psi_t = \tilde{\psi}_t + \frac{t^2}{6}R(\dot{\sigma}_0, \tilde{\psi}_1) \cdot \dot{\sigma}_0 + O(d^3)|\psi|
$$

$$
\psi_1 = \tilde{\psi}_1 + \frac{1}{6}R(\dot{\sigma}_0, \tilde{\psi}_1) \cdot \dot{\sigma}_0 + O(d^3)|\psi|
$$

\footnote{In what follows, the components of this vector field could be either real or anticommuting numbers. Since our application of the lemma below will be to the anticommuting case, we use $\psi$ to denote a generic vector.}
\[ \psi_1^\parallel = t \psi_1^\parallel + \frac{t^3 - t}{6} R(\dot{\sigma}_0, \psi_1^\parallel) \cdot \dot{\sigma}_0 + O(d^3 |\psi|). \]

Reversing the parameterization and assuming \( \psi_1 = 0 \) introduces the terms
\[ (1 - t) \psi_0 + \frac{(1 - t)^3 - (1 - t)}{6} R(\dot{\sigma}_1, \psi_0^\parallel) \cdot \dot{\sigma}_1, \]
where the parallel transport is from \( \sigma_0 \) to \( \sigma_1 \). After parallel transporting back to \( \sigma_0 \) in the second term, these become the additional terms the lemma requires. Note this substitution is permitted to the given order, since \( \dot{\sigma} \) is parallel along \( \sigma \), and the difference between applying the curvature and metric at \( \sigma_1 \) and applying them at \( \sigma_0 \) after parallel transport is of order \( d^3 |\psi| \).

**Remark**
The scale of the parameterization is of course arbitrary in the above lemma. \( \psi \) is determined by its value at any two points of \( \sigma \), and Eq. (2.1) continues to describe this dependence with the parameter \( t \) adjusted appropriately.

### 2.3 The SUSYQM Lagrangian

The action for \( N = 1 \) supersymmetric quantum mechanics on the manifold \( M \) is
\[ S(\sigma, \psi, \rho, t) = \int_0^t \left( \frac{\dot{\sigma}_r^2}{2} + i \langle \rho^r, (\nabla_{\dot{\sigma}} \psi)_r \rangle - \frac{1}{4} \langle \rho^r, R(\psi_r, \psi_r) \cdot \rho^r \rangle \right) dr. \]

where \( \sigma \) is an element of the space of paths in \( M \), \( \psi_r \) is an anticommuting element of the tangent to the space of paths, and \( \rho^r \) is an anticommuting variable modeled on the dual to the tangent space of the space of paths. In a pairing \( \int_0^t \langle \rho^r, \psi_r \rangle dr \), the end result is (at least formally) a one-form on the space of paths with values in linear functions in \( \rho \). The partition function for SUSYQM on \( M \) is
\[ Z = \int \mathcal{D} \psi e^{S(\sigma, \psi, \rho, t)}. \]

The (formal) Berezin integration in \( \rho \) produces a form on the space of paths. The “top form piece” of this form is integrated over the space of paths to give the partition function. Taking the paths to have fixed endpoints, the partition function is a path integral representation for the kernel of the time evolution operator or the Feynman propagator.

Given a family of paths \( \sigma \), we may think of \( \dot{\sigma} \) and \( \psi \) as vector fields on \( M \), which must necessarily commute, since the paths locally define coordinate curves which are integral curves for \( \psi \) and \( \dot{\sigma} \). Thus, in the action we may replace \( \nabla_{\dot{\sigma}} \psi \) with \( \nabla_\psi \dot{\sigma} \). We thereby recognize the Lagrangian as (formally) exactly the Mathai-Quillen Thom form on the tangent bundle to the space of paths, pulled back by the section \( \dot{\sigma} \). The connection is the Levi-Civita connection determined by the metric \( \int_0^t (X_t, Y_t) dt \). This observation and its formal consequences are due to Blau [13].

It is of course the integral over the infinite-dimensional space of paths that makes the links between the heat kernel, the partition function, and a Mathai-Quillen integral purely formal. However, if we interpret the path integral by
restricting it to a sequence of finite-dimensional subspaces that in a reasonable
sense approximate the whole space of paths, the arguments are correct on the
finite-dimensional approximating spaces.

We approximate the space of continuous paths $\sigma : [0, t] \to M$ with $\sigma(0) = y$ and $\sigma(t) = x$ by $\text{Path}_n(x, y)$. We choose positive numbers $t_1, \ldots, t_n$ such that $t = \sum_{i=1}^{n} t_i$ and parameterize each path in $\text{Path}_n(x, y)$ so that the first segment is the image of $[0, t_1]$ parameterized proportionally to arclength (so the segment is a parameterized geodesic), the second segment is the image of $[t_1, t_1 + t_2]$ parameterized proportionally to arclength, and so forth. Let $\text{Path}_n(x, y; t_1, \ldots, t_n)$ denote the space of paths in $\text{Path}_n$ parameterized in this way so that the parameter length of the $i$th geodesic segment is $t_i$. In the computation of the approximation to the partition function, $\psi$ will become an anticommuting vector tangent to $\text{Path}_n$. This tangent space has dimension $2m(n + 1)$, since it consists of vectors $\psi_1, \ldots, \psi_{n+1}$ with $\psi_i \in T_{x_i}M$ and with the $x_i$ denoting the $(n + 1)$ endpoints of the geodesic segments.

The situation with $\rho$ is a bit more complicated: The quantum mechanical state space consists of sums of anticommuting polynomials in $\psi_i^\mu$ with coefficients depending on $x_i$; these correspond to forms on $M^{n+1}$. The path integral will give a kernel of the time evolution operator which will act on the form on $\sigma_0$ representing the initial state. Thus we should think of the form at $\sigma_0$ as already being determined, so that the space in which $\psi$ lives is the space of all tangent vectors extending a given tangent vector at $\sigma_0$. The variable $\rho$ should thus not be a dual tangent vector at each of $n + 1$ terminal points of the geodesic pieces, as we might naively expect, because it should have no component dual to the tangent space at $\sigma_0$. Thus $\rho$ will consist of dual vectors $\rho^1, \rho^2, \ldots, \rho^n$, with each $\rho^i$ an anticommuting element in $T_{y_i}^* M$, where $x_i$ is the final point of the $i$th segment. The pairing of $\rho$ and $\psi$ is given by

$$
\int_0^t \langle \rho^r, \psi_r \rangle \, dr = \sum_{i=1}^{n} t_i \langle \rho^i, \psi_i \rangle .
$$

Note that in local coordinates $x_i^\mu$ in a neighborhood of $x_i$, the pointwise pairing on the right-hand side is $\langle \rho^i, \psi_i \rangle = \rho_i^\mu \psi_i^\mu$.

Thus the natural restriction of the path integral to the space of piecewise geodesic paths is

$$
K_{qm}^n(x, y; t_1, \ldots, t_n)) = \prod_{i=1}^{n} (2\pi t_i)^{-m} \int_{\text{Path}_n(x, y; t_1, \ldots, t_n)} \int \cdots \int
$$

$$
\exp \left[ \sum_{i=1}^{n} -\frac{|\dot{\sigma}_i|^2}{2} t_i + i t_i \langle \rho^i, (\nabla_x \psi)_i \rangle \\
- \frac{t_i}{4} \langle \rho^i, R(\psi_i, \psi_i) \cdot \rho^i \rangle \right] \, d\rho^1 \cdots d\rho^n
$$

(2.2)

where $\dot{\sigma}_i$ denotes tangent the final point $x_i$ of the $i$th geodesic segment.
The normalization factor out front is chosen to make the trivial case $M = \mathbb{R}^{2m}$ with the Euclidean metric work out right.

The time evolution operator associated to the kernel above is a composition of operators, each corresponding to one geodesic piece and each having $K^m_1$ as its kernel. In other words (suppressing the spatial variables)

$$K^m_1(t_1, t_2, \ldots, t_n) = K^m_1(t_1) \ast K^m_1(t_2) \ast \cdots \ast K^m_1(t_n).$$

### 2.4 Expressing $K^m_1$ as a form on $M^{(2)}$

In this section we explicitly evaluate the form $K^m_1$, in terms of geometric invariants. It is natural to rescale the parameterization length to 1, and adjust the meaning of $\dot{\sigma}$ accordingly, to obtain the following form on the same path parameterized from 0 to 1

$$K^m_1(x, y; t) = (2\pi t)^{-m} \int \exp \left[ -\frac{||\dot{\sigma}||^2}{2t} \right. $$

$$+ \left. i \left\langle \rho^x, (\nabla \dot{\sigma})_x \right\rangle - \frac{t}{4} \left( \rho^x \cdot R(\psi_x, \psi_x) \right) \right] d\rho^x.$$

This is a form on Path$_1 \cong M^{(2)}$, and can be expressed as such. First $\dot{\sigma}$ in Riemann normal coordinates centered at $x$ is $-\vec{y}$. From Lemma 2.2.1, by taking the derivative with respect to $t$ at $t = 1$ in Eq. 2.1 and parallel transporting everything to $x = \sigma_1$, we get

$$(\nabla \dot{\sigma})_x = \psi_x - \psi_y || + \frac{1}{3} R(\vec{y}, \psi_x) \cdot \vec{y} + \frac{1}{6} R(\vec{y}, \psi_y ||) \cdot \vec{y} + O(||\vec{y}||^3) \psi.$$

So

$$K^m_1(x, y; t) = (2\pi t)^{-m} \int \exp \left[ -\frac{||\vec{y}||^2}{2t} - \frac{t}{4} \left( \rho^x \cdot R(\psi_x, \psi_x) \right) \right. $$

$$\left. + i \left\langle \rho^x, \psi_x - \psi_y || + \frac{1}{3} R(\vec{y}, \psi_x) \cdot \vec{y} + \frac{1}{6} R(\vec{y}, \psi_y ||) \cdot \vec{y} \right\rangle + O(||\vec{y}||^3) \right] d\rho^x.$$

### 2.5 Shifting $\psi_y$ to $\psi_x$

Suppose $\eta$ and $\pi$ are indices for an orthonormal basis of $T_y M$. Suppose $f(\rho)$ is an anticommuting polynomial in the $\rho_1, \ldots, \rho_{2m}$ excluding $\rho_\eta$, and $g(\psi)$ is an anticommuting polynomial in the $\psi_1, \ldots, \psi_{2m}$ excluding $\psi_\pi$. Then

$$\oint i \rho_\pi \left( \psi_x - \psi_y || \right)^\eta f(\rho) g(\psi) \exp \left[ i \left\langle \rho, \psi_x - \psi_y || \right\rangle \right] d\rho =$$

$$\oint f(\rho) g(\psi) d\eta \exp \left[ i \left\langle \rho, \psi_x - \psi_y || \right\rangle \right] d\rho.$$

In particular, within an integral against $\exp \left[ i \left\langle \rho, \psi_x - \psi_y || \right\rangle \right]$, we have

$$\frac{i}{6} \left\langle \rho, R(\vec{y}, \psi_y ||) \cdot \vec{y} \right\rangle = \frac{i}{6} \left\langle \rho, R(\vec{y}, \psi_x) \cdot \vec{y} \right\rangle - \frac{1}{6} (\vec{y}, \text{Ricci} \cdot \vec{y}).$$
So defining
\[ H(x, y; t) = (2\pi t)^{-m} \exp \left[ -\frac{1}{2t} d(x, y)^2 \right] \] (2.3)
for \( x, y \in M \) within the injectivity radius of each other and \( t > 0 \) (and zero otherwise) this gives
\[
K^{qm}(x, y; t) = H(x, y; t) \oint \exp \left[ i \left( \rho^x, \psi_x - \psi_y + \frac{1}{2} R(\bar{y}, \psi_x) \cdot \bar{y} \right) - \frac{1}{6} (\bar{y}, \text{Ricci} \cdot \bar{y}) - \frac{t}{4} (\rho^x, R(\psi_x, \psi_x) \cdot \rho^x) + O(|\bar{y}|^3) \right] d\rho^x. \] (2.4)

2.6 Mathai-Quillen on paths and loops

The vector bundle \( TM \times M \to M \times M = M^2 \) restricts to a bundle over the open submanifold \( M^{(2)} \cong \text{Path}_1(M) \). A natural section of this bundle assigns to each \( (x, y) \in M^{(2)} \) the tangent vector \( \bar{y} \) at \( x \) (or equivalently, in terms of \( \text{Path}_1(M) \), the vector \( -\dot{\sigma}_1 \)). The Levi-Civita connection on \( M \) extends to a connection on this bundle, in terms of which one can easily verify that \( K^{qm}(t) \) gives the pullback of the Mathai-Quillen Thom form on this bundle via the section. We note that for finite positive \( t \) this gives a closed form on \( M^{(2)} \), but not a compactly supported closed form. Likewise the form on \( \text{Path}_n(M) \cong M^{(n+1)} \) whose integral gives \( K^{qm} \) is the pullback by the corresponding section of the Mathai-Quillen form on the bundle \( T(M \times \cdots \times M) \times M \to M^{n+1} \) restricted to the subset \( M^{(n+1)} \) (after absorbing the \( t_i \)'s into the metric on the various factors of \( M \)).

Instead of paths we can consider piecewise geodesic loops. Here it is natural to consider the kernel Eq. (2.2) with not only the points \( x_0 \) and \( x_n \) identified but also \( \psi_0 \) and \( \psi_n \) identified. That is, we identify \( x \) and \( y \) and wedge the form over \( x \) with the form over \( y \) (the form over \( x \) coming first). The integral of the resulting form on \( M \) is the supertrace of the kernel on the left-hand side of Eq. (2.2). Proving that as \( n \) goes to infinity the latter kernel converges to the heat kernel will show this integral converges to the supertrace of the heat kernel. The ability to connect the supertrace of the heat kernel to the integral of the pullback of the Mathai-Quillen form for a tangent bundle, through an intervening limit, is strong circumstantial evidence that this is a productive way of interpreting the supersymmetric path integral.

3 Strong Convergence of the Time Evolution Operator

Bär and Pfäffle \([B-P]\) offer a rigorous expression for various heat kernels as a kind of path integral. Specifically they use a form of Chernoff’s theorem to prove the following result:
Theorem 3.0.1 (Bär, Pfaffle) Suppose $K(x, y; t) \in E_x \otimes E_y$ is a smooth one-parameter family of kernels (with positive real parameter $t$) representing the family of operators $\mathcal{K}(t)$ on a Euclidean vector bundle $E$ that satisfy the following three assumptions:

1. $||\mathcal{K}(t)|| = 1 + O(t)$ for small $t$, where the norm is as an operator on the space of smooth functions with the supremum norm.

2. On each $\alpha \in \Gamma(M, E)$
   \[
   \lim_{t \to 0} \frac{(\mathcal{K}(t)\alpha - \alpha)}{t} \to -\Delta \alpha,
   \]
   in the supremum norm where $\Delta$ is a generalized Laplacian on $E$.

3. For each $y$
   \[
   \lim_{t \to 0} K(x, y; t) = \delta(x, y)
   \]
   as a distribution.

If $t_1, t_2, \ldots, t_n$ is called a partition, then for any sequence of partitions in which $\max_i t_i \to 0$ and $\sum_i t_i \to t$ and for any form $\alpha$ on $M$

\[
\lim \mathcal{K}(t_1)\mathcal{K}(t_2)\cdots\mathcal{K}(t_n)\alpha = e^{-t\Delta/2}\alpha.
\]

Moreover, for some such sequence of partitions

\[
\lim \mathcal{K}(t_1) \ast \mathcal{K}(t_2) \ast \cdots \ast \mathcal{K}(t_n) \to K_\Delta(x, y; t)
\]

uniformly, where $K_\Delta$ is the heat kernel of $\Delta$, i.e. the kernel of $e^{-t\Delta/2}$, and we suppress the spatial variables in $K$.

Remark Bär and Pfaffle work with $\Delta$ rather than $\Delta/2$, which of course amounts to nothing more than a rescaling of $t$ by a factor of 2. However, in the usual scaling of the physics literature, the time evolution operator corresponds to $e^{-t\Delta/2}$, so we follow this convention.

3.1 Applying the theorem to $K_{\text{qm}}$

Bär and Pfaffle apply this theorem to operators constructed from heat kernel asymptotics to give their path integral formulation. It is possible to relate Eq. (2.4) to the kernel in their Theorem 6.1 (note that their paths are parameterized in the opposite direction, and thus signs on all integrals are reversed), thus showing that supersymmetric quantum mechanics path integral restricted to piecewise short geodesic paths approaches the heat kernel for the Laplace Beltrami operator on forms as the number of pieces goes to infinity (for certain sequences of parameterization lengths). Instead we will check directly that the SUSYQM Lagrangian satisfies the assumptions of Theorem 3.0.1, thus achieving the same result. The check is a simple calculation that involves no sophisticated
understanding of heat kernel asymptotics and seems closer in spirit to path integral arguments.

Write $K^\text{qm}(t)$ for $K^\text{qm}(x,y;t)$ when the spatial variables are to be understood, and $\mathcal{R}^\text{qm}(t)$ for the operator represented by this kernel.

**Proof of Assumption 1:** The operator norm of $K^\text{qm}(t)$ is $1 + O(t)$. By compactness we can check this pointwise at each $x$, and because $K^\text{qm}(t)$ is zero outside the injectivity radius we can do the calculation inside a coordinate patch in Riemann normal coordinates. It suffices to let $\mathcal{R}^\text{qm}$ act on a function times a covariantly constant form, and the result follows from the fact that $H(x,y;t)$ has operator norm 1. □

**Proof of Assumptions 2 and 3:** If $\alpha$ is a form on $M$, we must show

$$\lim_{t \to 0} \left( \mathcal{R}^\text{qm}(t) \alpha - \alpha \right)/t = -\frac{\Delta}{2} \alpha$$

where $\Delta$ is the Laplace-Beltrami operator on forms Eq. (1.4). Again, we may check at a specific point $x$, and we may assume $\alpha$ is zero outside the geodesic neighborhood of $x$. We may also assume $\alpha$ is simply a function times a covariantly constant form, so that $\alpha^||| = f(y)\alpha(x,\psi_x)$, where the parallel transport from $y$ to $x$ is along the minimal geodesic.

Working in Riemann normal coordinates centered at $x$ so that

$$\det^{1/2}(g)(\bar{y}) = 1 + \frac{1}{6} \text{Ricci}_{\sigma\tau} y^\sigma y^\tau + O(|\bar{y}|^3),$$

(3.1)

and writing $H(\bar{y};t)$ for the expression of $H(x,y;t)$ in these coordinates, gives

$$\mathcal{R}^\text{qm}(t)\alpha = \int H(\bar{y};t) \oint \int \exp \left[ i \left( \rho^\tau, \psi_x - \psi_y^\tau \right) + \frac{1}{2} R(\bar{y}, \psi_x) \cdot \bar{y} \right]$$

$$- \frac{1}{6} (\bar{y}, \text{Ricci} \cdot \bar{y}) - \int \left( \rho^\tau, R(\psi_x, \psi_x) \cdot \rho^\tau \right) + O(|\bar{y}|^3) \right]$$

$$\cdot \alpha(\bar{y}, \psi_y) \, d\rho^\tau d\psi_y d\bar{y}$$

$$= \int H(\bar{y};t) \oint \int \exp \left[ i \left( \rho^\tau, \psi_x^\tau - \psi_y^\tau \right) + \frac{1}{2} R(\bar{y}, \psi_x^\tau) \cdot \bar{y} \right]$$

$$- \frac{1}{6} (\bar{y}, \text{Ricci} \cdot \bar{y}) - \int \left( \rho^\tau, R(\psi_x^\tau, \psi_x^\tau) \cdot \rho^\tau \right) + O(|\bar{y}|^3) \right]$$

$$\cdot \alpha(\bar{y}, \psi_y) \, d\rho^\tau d\psi_y d\bar{y}$$

$$= \int \left[ 1 - \frac{1}{6} \text{Ricci}_{\sigma\tau} y^\sigma y^\tau + \frac{i}{2} \rho^\tau R_\pi \eta \sigma y^\pi (\psi_x^\eta) \rho^\eta + O(|\bar{y}|^3) \right]$$

$$\cdot \left( 1 - \frac{1}{4} R_{\pi\eta}^{\nu\pi} R^{\nu\sigma} (\psi_x^\eta) \rho^\pi \rho^\eta \right) \exp \left[ i \left( \rho^\eta, \psi_x^\eta \right) - \psi_y^\eta \right]$$

$$\cdot \alpha(\bar{y}, \psi_y) \, d\rho^\tau d\psi_y d\bar{y}$$

$$= \int H(\bar{y};t) f(y) \left[ 1 - \frac{1}{6} \text{Ricci}_{\sigma\tau} y^\sigma y^\tau + \frac{i}{2} R_\pi \eta \sigma y^\pi \psi_x^\eta \tau + O(|\bar{y}|^3) \right]$$

$$\cdot \det^{1/2}(g) d\bar{y} \left( 1 + \frac{1}{4} R_{\mu\eta}^{\nu\pi} \psi_x^\mu \psi_x^\eta \tau + O(t^{3/2}) \right) \alpha(0, \psi_x)$$

13
where we have applied Eqs. (1.9) and (1.10). Now, if \( f \) is a smooth function on \( \mathbb{R}^{2m} \), then

\[
\int H(\tilde{y}; t)f(\tilde{y})\,dy^1 \cdots dy^{2m} = \int (2\pi t)^{-m}\exp \left[ -\frac{1}{2t}|\tilde{g}|^2 \right] f(\tilde{y})\,dy^1 \cdots dy^{2m} = f(0) - t(\Delta f)(0) + O(t^2)
\]

(3.2)

where \( \Delta f = -\delta^{\mu\nu}\partial_{\mu}\partial_{\nu} f \). Since, according to Eq. (3.1), \( [1 - \frac{1}{t}\text{Ricci}_{\sigma\tau}y^\sigma y^\tau] \det^{1/2}(g) = 1 + O(|\tilde{g}|^2) \), Eq. (3.2) implies the term linear in \( t \) coming from the integral over \( \mathbb{R}^{2m} \) is just \(-\frac{1}{2}\Delta [\int f(\tilde{y}) (1 + \frac{1}{2}R_{\sigma\tau\nu\rho}y^\sigma y^\tau \tau) \,d\tilde{y}] \big|_{\tilde{y}=\tilde{0}} \). That is,

\[
R_{\alpha}^{\gamma}(t)\alpha = f(0) \left( 1 + \frac{t}{2}\text{Ricci}_{\sigma\tau}^{\gamma} \tau + \frac{t}{4}R_{\mu\eta}^{\gamma\nu\pi} \psi_0^{\eta} \psi_0^{\nu} t_{\nu} \tau \right) \alpha(0, \psi_x) + \frac{t}{2}(\delta^{\mu\nu}\partial_{\mu}\partial_{\nu} f)(0)\alpha(0, \psi_x).
\]

Thus the required \( t \)-derivative is

\[
\lim_{t\to 0} (\mathcal{R}_{\alpha}^{\gamma}(t)\alpha - \alpha) / t = \frac{1}{2}(\delta^{\mu\nu}\partial_{\mu}\partial_{\nu} f)(0)\alpha(0, \psi_x)
\]

\[
+ \left( \frac{1}{2}\text{Ricci}_{\sigma\tau}^{\gamma} \tau + \frac{1}{4}R_{\mu\eta}^{\gamma\nu\pi} \psi_0^{\eta} \psi_0^{\nu} t_{\nu} \tau \right) f(0)\alpha(0, \psi_x).
\]

On the other hand \( \nabla_\mu \alpha = 0 \) since it is covariantly constant, so in Riemann normal coordinates, with the derivatives acting at 0, the right-hand side of Assumption 2 is

\[
-\frac{\Delta}{2} \alpha = -\frac{1}{2}\Delta_0 f(x)\alpha(0, \psi_x) = \frac{1}{2}(\delta^{\mu\nu}\partial_{\mu}\partial_{\nu} f)(0)\alpha(0, \psi_x) + \frac{1}{2} \left( \text{Ricci}_{\sigma\tau}^{\gamma} \tau + \frac{1}{4}R_{\mu\eta}^{\gamma\nu\pi} \psi_0^{\eta} \psi_0^{\nu} t_{\nu} \tau \right) f(0)\alpha(0, \psi_x).
\]

Assumption 3 is an analogous but simpler calculation where we consider \( \int K(x, y; t)\alpha(x)\,dx \) for a smooth \( \alpha \) and require it to converge to \( \alpha(y) \) as \( t \) goes to zero. \( \square \)

**Corollary 3.1.1** For any sequence of partitions \( t_1, t_2, \ldots, t_n \) such that \( \max_i(t_i) \to 0 \) and \( \sum_i t_i \to t \) and for any form \( \alpha \) on \( M \)

\[
\lim \mathcal{R}(t_1)\mathcal{R}(t_2)\cdots\mathcal{R}(t_n)\alpha = e^{-t\Delta/2}\alpha
\]

where \( \Delta \) is the Laplace-Beltrami operator on forms. Moreover, for some such sequence of partitions

\[
\lim K(t_1) * K(t_2) * \cdots * K(t_n) \to K_{\Delta}(x, y; t)
\]

uniformly, where \( K_{\Delta} \) is the heat kernel of \( \Delta \) (the kernel of \( e^{-t\Delta/2} \)).

**Remark** Thus the approximation \( K_{\text{qm}}(x, y; t_1, \ldots, t_n) \) to the kernel of the time evolution operator for supersymmetric quantum mechanics converges to the heat kernel for \( \Delta \) in the large partition limit.
References

[A] L. Alvarez-Gaumé: Supersymmetry and the Atiyah-Singer index theorem. Commun. Math. Phys. 90, 161 (1983)

[B-G-V] Nicole Berline, Ezra Getzler, and Michèele Vergne. Heat Kernels and Dirac Operators. Berlin: Springer, 2004

[B] Matthias Blau: The Mathai-Quillen formalism and topological field theory. J. Geom. Phys. 11 (1-4), 95–127 (1993) Infinite-dimensional geometry in physics (Karpacz, 1992).

[B-P] Christian Bär and Frank Pfäffle: Path integrals on manifolds by finite dimensional approximations. AP/07032731v1.

[M-Q] Varghese Mathai and Daniel Quillen: Supercalibrations, Thom classes, and equivariant characteristic classes. Topology 25, 85–110 (1986)

[G] Ezra Getzler: A short proof of the local Atiyah-Singer index theorem. Topology 25, 111–117 (1986)

[R1] Alice Rogers: Stochastic calculus in superspace. II. Differential forms, supermanifolds and the Atiyah-Singer index theorem. J. Phys. A 25 (22), 6043–6062 (1992)

[R2] Alice Rogers: Supersymmetry and Brownian motion on supermanifolds. Infin. Dimens. Anal. Quantum Probab. Relat. Top. 6 (suppl.), 83–102 (2003)

[P] V. K. Patodi: Curvature and the eigenforms of the Laplace operator. J. Diff. Geom 5, 233–249 (1971)

[S] D. W. Stroock: On certain systems of parabolic equations. Comm. Pure Appl. Math. 23, 447-457 (1970)

[W] E. Witten: Supersymmetry and Morse theory. J. Diff. Geom. 17, 661–692 (1982)