Jump transformations and an embedding of $\mathcal{O}_\infty$ into $\mathcal{O}_2$

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Abstract

A measurable map $T$ on a measure space induces a representation $\Pi_T$ of a Cuntz algebra $\mathcal{O}_N$ when $T$ satisfies a certain condition. For such two maps $\tau$ and $\sigma$ and representations $\Pi_\tau$ and $\Pi_\sigma$ associated with them, we show that $\Pi_\tau$ is the restriction of $\Pi_\sigma$ when $\tau$ is a jump transformation of $\sigma$. Especially, the Gauss map $\tau_1$ and the Farey map $\sigma_1$ induce representations $\Pi_{\tau_1}$ of $\mathcal{O}_\infty$ and that $\Pi_{\sigma_1}$ of $\mathcal{O}_2$, respectively, and $\Pi_{\tau_1} = \Pi_{\sigma_1}|_{\mathcal{O}_\infty}$ with respect to a certain embedding of $\mathcal{O}_\infty$ into $\mathcal{O}_2$.

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1 Introduction

The purpose of this paper is to show a new relation between dynamical systems and operator algebras. The former means jump transformations in metric number theory and the later does an embedding of Cuntz algebras. In this section, we show our motivation and main theorem.
1.1 Motivation

We explain our motivation in this subsection. Mathematical details will be shown in § 1.2, § 1.3 and § 3.

As is well known, metric number theory has originated as Gauss’ problem about the asymptotic behavior of iterations of the regular continued fraction transformation. As a modern style, the theory is formulated as a measure theoretical dynamical system [5, 11]. For a measure space \((X, \mu)\), let \(T\) be a measurable map from \(X\) to \(X\) which may not be invertible. We call the triplet \((X, \mu, T)\) a (measure theoretical) dynamical system. For example, the Gauss map (or the continued fraction transformation) \(\tau_1\) on the closed interval \([0, 1]\) defined by

\[
\tau_1(x) \equiv \begin{cases} 
1/x - \left\lfloor \frac{1}{x} \right\rfloor & (x \neq 0), \\
0 & (x = 0),
\end{cases}
\]

(1.1)
gives an important dynamical system \(([0, 1], \lambda, \tau_1)\) with respect to the Lebesgue measure \(\lambda\) where \(\lfloor \cdot \rfloor\) denotes Gauss’ symbol (or the floor function). The second and the third authors have studied a generalization of the continued fraction transformation in metric number theory [2].

On the other hand, the first author has studied representations of Cuntz algebras and Perron-Frobenius operators [6]. A dynamical system \((X, \mu, T)\) induces a representation \((L^2(X, \mu), \Pi_T)\) of a Cuntz algebra when \((X, \mu, T)\) satisfies a certain condition. Properties of \((L^2(X, \mu), \Pi_T)\) are characterized by \((X, \mu, T)\). As an application, such construction is used to construct type III factor representations of Cuntz-Krieger algebras ([7], § 1.3).

These two different studies have the following common problem in dynamical system.

**Problem 1.1** For a given map \(T\) on a set \(X\), find a \(T\)-invariant (or \(T\)-preserving) measure \(\mu\) of \(X\), that is,

\[
\mu(T^{-1}(E)) = \mu(E)
\]

(1.2)

for every \(\mu\)-measurable subset \(E\) of \(X\).

For example, a solution of Gauss’ problem is given by an invariant measure of the Gauss map [5, 11], and there exists a relation between an invariant measure and the representation of Cuntz-Krieger algebra arising from dynamical system [6].
The new idea in this paper is to show a relation between (the construction of) jump transformation and an embedding of \( \mathcal{O}_\infty \) into \( \mathcal{O}_2 \) where \( \mathcal{O}_\infty \) and \( \mathcal{O}_2 \) denote Cuntz algebras which will be explained in \( \S \) 1.2.1. Let \( \tau \) and \( \sigma \) be two maps, and let \( \Pi_\tau \) and \( \Pi_\sigma \) denote representations associated with them, respectively. We show that \( \Pi_\tau \) is the restriction of \( \Pi_\sigma \) when \( \tau \) is a jump transformation of \( \sigma \) (Theorem 1.4). Especially, the Gauss map \( \tau_1 \) and the Farey map \( \sigma_1 \) induce representations \( \Pi_{\tau_1} \) of \( \mathcal{O}_\infty \) and that \( \Pi_{\sigma_1} \) of \( \mathcal{O}_2 \), respectively, and \( \Pi_{\tau_1} = \Pi_{\sigma_1} |_{\mathcal{O}_\infty} \) with respect to a certain embedding of \( \mathcal{O}_\infty \) into \( \mathcal{O}_2 \) (\( \S \) 3.2).

1.2 Representations of Cuntz algebras arising from dynamical systems

In this subsection, we explain representations of Cuntz algebras arising from dynamical systems according to [6].

1.2.1 Cuntz algebras

For \( N = 2, 3, \ldots, +\infty \), let \( \mathcal{O}_N \) denote the Cuntz algebra \([3]\), that is, a \( \mathbb{C}^* \)-algebra which is universally generated by \( s_1, \ldots, s_N \) satisfying

\[
\begin{align*}
    s_i^* s_j &= \delta_{ij} I \quad \text{for } i, j = 1, \ldots, N, \quad (1.3) \\
    \sum_{i=1}^{N} s_i s_i^* &= I \quad \text{if } N < +\infty, \quad (1.4) \\
    \sum_{i=1}^{k} s_i s_i^* &\leq I \quad \text{for } k = 1, 2, \ldots \text{ (if } N = +\infty) \quad (1.5)
\end{align*}
\]

where \( I \) denotes the unit of \( \mathcal{O}_N \).

Since \( \mathcal{O}_N \) is simple, that is, there is no nontrivial closed two-sided ideal, any unital homomorphism from \( \mathcal{O}_N \) to a \( \mathbb{C}^* \)-algebra is injective. If \( t_1, \ldots, t_N \) are elements of a unital \( \mathbb{C}^* \)-algebra \( \mathfrak{A} \) such that \( t_1, \ldots, t_N \) satisfy the relations of canonical generators of \( \mathcal{O}_N \), then the correspondence \( s_i \mapsto t_i \) for \( i = 1, \ldots, N \) is uniquely extended to a \( * \)-embedding of \( \mathcal{O}_N \) into \( \mathfrak{A} \) from the uniqueness of \( \mathcal{O}_N \). Therefore we call such a correspondence among generators by an embedding of \( \mathcal{O}_N \) into \( \mathfrak{A} \).

Assume that \( s_1, \ldots, s_N \) are realized as operators on a Hilbert space \( \mathcal{H} \). According to (1.3) and (1.4), \( \mathcal{H} \) is decomposed into orthogonal subspaces as \( s_1 \mathcal{H} \oplus \cdots \oplus s_N \mathcal{H} \). Since \( s_i \) is an isometry, \( s_i \mathcal{H} \) has the same dimension as \( \mathcal{H} \). From this, we see that there is no finite dimensional representation
of $O_N$ which preserves the unit. The following illustration is helpful in understanding $s_1, \ldots, s_N$:

![Diagram]

1.2.2 Representations of Cuntz algebras

We introduce branching function systems. Let $(X, \mu)$ be a measure space and $\Omega_N \equiv \{1, \ldots, N\}$ for $2 \leq N < \infty$ and $\Omega_\infty \equiv \mathbb{N} = \{1, 2, 3, \ldots\}$. Let $T$ be a measurable map on $(X, \mu)$. Define the new measure $\mu \circ T$ on $X$ by $(\mu \circ T)(E) \equiv \mu(T(E))$ for $E \subset X$. For $2 \leq N \leq \infty$, a family $f = \{f_i : i \in \Omega_N\}$ of maps on $X$ is a branching function system if

(i) $f_i$ is a measurable map from $X$ to $X$ for each $i$,

(ii) if $R_i \equiv f_i(X)$, then $\mu(X \setminus \bigcup_{i \in \Omega_N} R_i) = 0$ and $\mu(R_i \cap R_j) = 0$ when $i \neq j$, and

(iii) there exists the Radon-Nikodým derivative $\Phi_{f_i}$ of $\mu \circ f_i$ with respect to $\mu$ and $\Phi_{f_i} > 0$ almost everywhere in $X$ for each $i$.

A map $F$ on $X$ is called the coding map of a branching function system $f = \{f_i : i \in \Omega_N\}$ if $F \circ f_i = \text{id}_X$ for each $i$.

Let $L_2(X, \mu)$ denote the Hilbert space of all square integrable complex valued-functions on $X$. For a branching function system $f = \{f_i : i \in \Omega_N\}$ with the coding map $F$, define the family $\{S(f_i) : i \in \Omega_N\}$ of operators on $L_2(X, \mu)$ by

$$\{S(f_i)\phi\}(x) \equiv \chi_{R_i}(x) \cdot (\Phi_{F}(x))^{1/2} \cdot \phi(F(x)) \quad (\phi \in L_2(X, \mu), x \in X) \quad (1.6)$$

where $\chi_{R_i}$ denotes the characteristic function of $R_i$. From this, adjoint operators $\{S(f_i)^* : i \in \Omega_N\}$ are given as follows:

$$\{S(f_i)^*\phi\}(x) = (\Phi_{f_i}(x))^{1/2} \cdot \phi(f_i(x)) \quad (\phi \in L_2(X, \mu), x \in X). \quad (1.7)$$

Then $S(f_i)$ is an isometry and

$$S(f_i)S(f_j) = S(f_i \circ f_j) \quad (i, j \in \Omega_N) \quad (1.8)$$
where \((f_i \circ f_j)(x) \equiv f_i(f_j(x))\). Let \(\{s_i : i \in \Omega_N\}\) denote canonical generators of \(O_N\). For a branching function system \(f = \{f_i : i \in \Omega_N\}\),

\[
\pi_f(s_i) \equiv S(f_i) \quad (i \in \Omega_N),
\]

defines a representation \((L^2(X, \mu), \pi_f)\) of the Cuntz algebra \(O_N\) because \(\{S(f_i) : i \in \Omega_N\}\) satisfy relations of canonical generators of \(O_N\). Especially, branching function systems \(f = \{f_1, f_2\}\) and \(g = \{g_i : i \in \mathbb{N}\}\) define a representation of \(O_2\) and that of \(O_{\infty}\), respectively. Examples will be shown in §3.

**Remark 1.2** For a dynamical system \((X, \mu, T)\), a branching function system with the coding map \(T\) is not unique even if it exists. Therefore the representation \(\pi\) of \(O_N\) arising from \((X, \mu, T)\) depends on the choice of a branching function system with the coding map \(T\).

### 1.3 Main theorem

In this subsection, we show our main theorem. For this purpose, we introduce jump transformation according to [11].

**Definition 1.3** Let \((X, \mu, T)\) be a dynamical system. Assume that \(A\) is a measurable subset of \(X\) such that

\[
\mu\left(X \setminus \bigcup_{n \geq 1} T^{-n}(A)\right) = 0. \tag{1.10}
\]

(i) For the pair \((T, A)\), the map \(e\) from \(X\) to \(\mathbb{Z}_{\geq 0} \equiv \{0, 1, 2, 3, \ldots\}\) is called the first entry time of \((T, A)\) if

\[
e(x) \equiv \min\{k \in \mathbb{Z}_{\geq 0} : T^k(x) \in A\} \quad (x \in X). \tag{1.11}
\]

(ii) For the pair \((T, A)\), define the map \(J_{T,A}\) from \(X\) to \(X\) by

\[
J_{T,A}(x) \equiv T^{e(x)+1}(x) \quad (x \in X). \tag{1.12}
\]

We call \(J_{T,A}\) the jump transformation of \((T, A)\).

We briefly explain the meaning of jump transformation as follows. The assumption (1.10) means that it surely takes finite time such that \(x\) enters the subset \(A\) with respect to the discrete time evolution

\[
x \mapsto T(x) \quad \tag{1.13}
\]
for almost all \( x \in X \). From this explanation, we can also understand the meaning of the first entry time. For \( x \in X \), assume \( n \equiv e(x) < \infty \). Then the transformation \( J_{T,A} \) maps \( x \) as follows:

\[
A \psi \\
\xrightarrow{\psi} \\
\xrightarrow{T(x)} \\
\xrightarrow{T^2(x)} \\
\xrightarrow{\cdots} \\
\xrightarrow{T^n(x)} \\
\xrightarrow{T^{n+1}(x)} \\
J_{T,A}
\]

Let \( \{t_1, t_2\} \) and \( \{s_n : n \in \mathbb{N}\} \) denote canonical generators of Cuntz algebras \( \mathcal{O}_2 \) and \( \mathcal{O}_\infty \), respectively. Assume that \( \mathcal{O}_\infty \) is embedded into \( \mathcal{O}_2 \) by

\[
s_n = t_2^{n-1} t_1 \quad (n \geq 1)
\]

where we write \( t_0^2 \equiv I \) for convenience. This embedding induces the restriction of representations of \( \mathcal{O}_2 \) on \( \mathcal{O}_\infty \):

\[
\text{Rep}\mathcal{O}_2 \ni \pi \mapsto \pi|_{\mathcal{O}_\infty} \in \text{Rep}\mathcal{O}_\infty.
\]

**Theorem 1.4** Let \( (X, \mu, \sigma) \) be dynamical system such that \( \sigma \) is the coding map of some branching function system \( f = \{f_1, f_2\} \). Assume that \( A_1 \equiv f_1(X) \) satisfies (1.10) with respect to \( (X, \mu, \sigma) \). Let \( \pi_f \) denote the representation of \( \mathcal{O}_2 \) associated with \( f \) in (1.9). Let \( J_{\sigma,A_1} \) be as in (1.12) for \((\sigma,A_1)\). Then the following holds:

(i) There exists a branching function system \( g = \{g_n : n \in \mathbb{N}\} \) on \( (X, \mu) \) with the coding map \( J_{\sigma,A_1} \) such that the representation \( \pi_g \) of \( \mathcal{O}_\infty \) coincides with the restriction \( \pi_f|_{\mathcal{O}_\infty} \) of \( \pi_f \) on \( \mathcal{O}_\infty \) with respect to the embedding in (1.14), that is,

\[
\pi_f|_{\mathcal{O}_\infty} = \pi_g.
\]

(ii) If \( \phi \) is the density of an invariant measure with respect to \( \sigma \), that is,

\[
\nu(E) \equiv \int_E \phi(x) \, d\mu(x) \quad (E \subset X)
\]

defines a \( \sigma \)-invariant measure \( \nu \) of \( X \), and \( \phi \) belongs to \( L_1(X, \mu) \), then the function \( \psi \) on \( X \) defined by

\[
\psi(x) = \left\{ \left( \pi_f(t_1)^* \sqrt{\phi}(x) \right) \right\}^2 \quad (x \in X),
\]

is the density of an invariant measure of \( J_{\sigma,A_1} \) where \( \sqrt{\phi}(x) \equiv \sqrt{\phi(x)} \).
We summarize Theorem 1.4(i) as follows:

| Map | Cuntz algebra |
|-----|---------------|
| \( \sigma \) | representation of \( \mathcal{O}_2 = C^* \{ \{ t_1, t_2 \} \} \) |
| \( \tau \) | representation of \( \mathcal{O}_\infty = C^* \{ \{ s_n : n \in \mathbb{N} \} \} \) |
| \( \tau = J_{\sigma, A} \) | \( s_n = t_2^{n-1} t_1 \) |

**Remark 1.5**

(i) We stress that jump transformation (1.12) and the embedding (1.14) are independently introduced in different studies. Theorem 1.4 is a casual discovery. We see that the index \( e(x) + 1 \) of \( T^{e(x)+1}(x) \) and the index \( n - 1 \) of the element \( t_2 \) of the equation \( s_n = t_2^{n-1} t_1 \) fit like a glove in the proof of Theorem 1.4. This is an interesting accidental coincidence.

(ii) From Theorem 1.4, we see that a jump transformation gives a geometric realization of the embedding of \( \mathcal{O}_\infty \) into \( \mathcal{O}_2 \) in (1.14). On the other hand, the embedding in (1.14) appears in the construction of a unitary isomorphism between the Bose-Fock space and the Fermi-Fock space [8]. From this, we wonder the existence of a behind mathematical structure among dynamical system, operator algebra and quantum field theory.

(iii) Apparently, an embedding of \( \mathcal{O}_\infty \) into \( \mathcal{O}_2 \) is not unique. For example, the following is one of other embeddings of \( \mathcal{O}_\infty \) into \( \mathcal{O}_2 \):

\[
s_n = t_2^{n-1} (t_1 t_2 t_1^* + t_1^2 t_2^*) \quad (n \geq 1).
\]

Therefore the choice of the embedding (1.14) is also an interesting accident.

In § 2 we prove Theorem 1.4. In § 3 we show examples of Theorem 1.4.

## 2 Proof of Theorem 1.4

We prove Theorem 1.4 in this section.

**Theorem 2.1** ([11], § 19.2.1) Let \( (X, \mu, T) \) be a dynamical system and assume that \( \mu \) is \( T \)-invariant and a measurable subset \( A \) of \( X \) satisfies (1.10). Define the new measure \( \nu \) by

\[
\nu(E) \equiv \mu(T^{-1}(E) \cap A) \quad (E \subset X).
\]

Then \( \nu \) is invariant with respect to the jump transformation of \((T, A)\).
Lemma 2.2 Let \((X, \mu, T)\) be a dynamical system, and let \(\nu_1\) and \(\nu_2\) be other measures on \(X\). Assume that \(\nu_1, \nu_2\) and \(\mu \circ T\) are absolutely continuous with respect to \(\mu\). We write \(\psi_i \equiv d\nu_i/d\mu\) for \(i = 1, 2\) and \(\Phi_T \equiv d(\mu \circ T)/d\mu\). Then the following are equivalent:

(i) \(\nu_1 = \nu_2 \circ T\).

(ii) \(\psi_1 = \Phi_T \cdot (\psi_2 \circ T)\) almost everywhere in \(X\).

Proof. By using the chain rule of Radon-Nikodým derivatives, the statement holds.

From Lemma 2.2 the following holds.

Corollary 2.3 For \(a, b \in \mathbb{R}\), \(a < b\), let \([a, b]\) and \((a, b)\) denote the closed interval and the open interval. Let \(T\) be a piecewise differentiable map from \((a, b)\) to \((a, b)\) with the differential \(T'\) and let \(\nu_1\) and \(\nu_2\) be measures on \([a, b]\). Assume that \(\nu_1, \nu_2\) and \(\lambda \circ T\) are absolutely continuous with respect to the Lebesgue measure \(\lambda\) on \([a, b]\). Define \(\psi_i \equiv d\nu_i/d\lambda\) for \(i = 1, 2\). Then the following are equivalent:

(i) \(\nu_1 = \nu_2 \circ T\).

(ii) \(\psi_1 = |T'| \cdot (\psi_2 \circ T)\) almost everywhere in \([a, b]\) where \(|T'|(x) \equiv |T'(x)|\) for \(x \in (a, b)\).

Proof of Theorem 1.4 (i) Define the branching function system \(g = \{g_n : n \in \mathbb{N}\}\) on \(X\) by

\[
g_n \equiv f_2^{n-1} \circ f_1 \quad (n \geq 1). \tag{2.2}
\]

Since \(f\) is a branching function system, we can verify that \(g\) is a branching function system on \(X\). From (1.14) and (1.8), (1.15) holds. If \(G\) denotes the coding map of \(g\), then

\[
G(x) = g_n^{-1}(x) = (f_1^{-1} \circ f_2^{-n+1})(x) \quad \text{when } x \in g_n(X), n \in \mathbb{N}. \tag{2.3}
\]

Remark \(g_n(X) = f_2^{n-1}(A_1)\) for each \(n \in \mathbb{N}\). On the other hand, when \(x \in g_n(X), e(x) = n - 1\). From this, \(J_{\sigma, A_1}\) satisfies that

\[
J_{\sigma, A_1}(x) = \sigma^n(x) = (f_1^{-1} \circ f_2^{-n+1})(x). \tag{2.4}
\]
From (1.12), \( G = J_{\sigma,A_1} \) holds almost everywhere in \( X \). Hence the statement holds.

(ii) Define measures \( \nu_{\infty} \) and \( \nu_2 \) of \( X \) by

\[
\nu_{\infty}(E) \equiv \int_E \psi(x) \, d\mu(x), \quad \nu_2(E) \equiv \int_E \phi(x) \, d\mu(x)
\]

for each measurable subset \( E \) of \((X,\mu)\). From (1.17) and the definition of \( \pi_f \),

\[
\psi(x) = \left\{ S(f_1)^* \sqrt{\phi} \right\}(x)^2 = \Phi_f(x) \cdot \phi(f_1(x)).
\]

From Lemma 2.2,

\[
\nu_{\infty}(E) = (\nu_2 \circ f_1)(E) = \nu_2(\sigma^{-1}(E) \cap A_1).
\]

From Theorem 2.1 the statement holds.

3 Examples

We show examples of Theorem 1.4 in this section. For convenience, we show formulae of representations as follows. Assume that \( X = [a,b] \) and \( T \) is a piecewise differentiable map on \( X \) which is the coding map of a branching function system \( \{ f_i : i \in \Omega_N \} \) on \( X \) for \( 2 \leq N \leq \infty \). Then (1.6) is given as follows:

\[
\{ \pi_f(s_i) \phi \}(x) = \chi_{R_i}(x) \sqrt{|T'(x)|} \phi(T(x)),
\]

\[
\{ \pi_f(s_i^*) \phi \}(x) = \sqrt{|f_i'(x)|} \phi(f_i(x)) \]

for \( i \in \Omega_N \).

3.1 Tent map

As an introductory example of jump transformation, we show a tent map. Define the map \( \Lambda \) by \( \Lambda(x) \equiv 1 - |2x - 1| \) on \( x \in [0,1] \). The map \( \Lambda \) is called a tent map [4]. Let \( A_1 \equiv [0,1/2] \), and let \( f_2 \equiv (\Lambda|_{[0,1/2]})^{-1} \) and \( f_1 \equiv (\Lambda|_{[1/2,1]})^{-1} \). Then

\[
f_1(x) = 1 - \frac{1}{2} x, \quad f_2(x) = \frac{1}{2} x.
\]
Define $g_n \equiv f_2^{n-1} \circ f_1$ for $n \geq 1$. Then

$$g_n(x) = \frac{2 - x}{2^n}.$$  \hspace{1cm} (3.4)

The coding map $G$ of $\{g_n : n \in \mathbb{N}\}$ is given by

$$G(x) = 2 - 2^n x \quad \text{when } 1/2^n < x \leq 1/2^{n-1}. \hspace{1cm} (3.5)$$

Hence the jump transformation is given as follows:

$$J_{\Lambda, A_1}(x) = G(x) = 2 - 2^{\lceil \log_2 x^{-1} \rceil + 1} \cdot x \quad (x \in (0, 1]). \hspace{1cm} (3.6)$$

Both of these invariant probability measures are the Lebesgue measure. For $f = \{f_1, f_2\}$ in (3.3) and $g = \{g_n : n \in \mathbb{N}\}$ in (3.4), the representation $(L_2[0,1], \pi_f)$ of $\mathcal{O}_2$ and that $(L_2[0,1], \pi_g)$ of $\mathcal{O}_\infty$ are given as follows:

$$\begin{cases}
\{\pi_f(t_1)\phi\}(x) = \chi_{[1/2,1]}(x)\sqrt{2}\phi(\Lambda(x)), \\
\{\pi_f(t_2)\phi\}(x) = \chi_{[0,1/2]}(x)\sqrt{2}\phi(\Lambda(x)),
\end{cases} \hspace{1cm} (3.7)$$

$$\{\pi_g(s_n)\phi\}(x) = \chi_{[2^{-n},2^{-n+1}]}(x)2^{n/2}\phi(G(x)) \quad (n \geq 1). \hspace{1cm} (3.8)$$

From (1.8), we can verify that

$$\pi_f(s_n) = \pi_f(t_2)^{n-1}\pi_f(t_1) = S(f_2)^{n-1}S(f_1) = S(f_2^{n-1} \circ f_1) = S(g_n) = \pi_g(s_n)$$

for each $n$. Hence $\pi_f|_{\mathcal{O}_\infty} = \pi_g$. 

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3.2 Gauss map and Farey map

It is well known that the Gauss map $\tau_1$ in (1.1) is the jump transformation of the Farey map $\sigma_1$ [12, 9, 11], which is defined on $[0, 1]$ by
\[
\sigma_1(x) \equiv \frac{2}{1 + |1 - 2x|} - 1.
\] (3.9)

Define two measures $\gamma$ and $\theta$ on $[0, 1]$ by
\[
d\gamma(x) \equiv \frac{1}{\log 2} \frac{dx}{x + 1}, \quad \text{and} \quad d\theta(x) \equiv \frac{dx}{x}.
\] (3.10, 3.11)

The measure $\gamma$ is called Gauss’ measure [5, p.16]. Then $\gamma$ and $\theta$ are invariant measures of $\tau_1$ and $\sigma_1$, respectively [12], where $dx$ denotes the Lebesgue measure on $[0, 1]$. Remark that the former is finite but the later is not.

For $k \geq 1$, define $g_k \equiv (\tau_1)^{-1}$ on $\frac{1}{k+1} < x \leq \frac{1}{k}$, and $f_1 \equiv (\sigma_1|_{[1/2, 1]})^{-1}$ and $f_2 \equiv (\sigma_1|_{[0, 1/2]})^{-1}$. Then $\{g_k : k \in \mathbb{N}\}$ and $\{f_1, f_2\}$ are branching function systems on $[0, 1]$ with the coding map $\tau_1$ and $\sigma_1$, respectively. Furthermore,
\[
g_k(x) = \frac{1}{x + k} \quad (k \geq 1),
\] (3.12)
\[
f_1(x) = \frac{1}{x + 1}, \quad f_2(x) = \frac{x}{x + 1}.
\] (3.13)
for $x \in [0, 1]$. We see that $g_k = f_2^{k-1} \circ f_1$ for $k \geq 1$. From these, we obtain representations $\pi_{\tau_1}$ and $\pi_{\sigma_1}$ of $O_\infty$ and $O_2$ on $L_2[0, 1]$, respectively which are written as follows:

$$\{\pi_{\tau_1}(s_n)\phi\}(x) = \frac{\chi_{[1/(n+1),1/n]}(x)}{x} \phi(\tau_1(x)) \quad (n \in \mathbb{N}),$$

$$\begin{cases} 
\{\pi_{\sigma_1}(t_1)\phi\}(x) = \frac{\chi_{[1/2,1]}(x)}{x} \phi(\sigma_1(x)), \\
\{\pi_{\sigma_1}(t_2)\phi\}(x) = \frac{\chi_{[0,1/2]}(x)}{1-x} \phi(\sigma_1(x))
\end{cases}$$

for $x \in [0, 1]$ and $\phi \in L_2[0, 1]$.

**Remark 3.1** Remark that the invariant measure $\theta$ of the Farey map in (3.11) is not finite. Therefore Theorem 1.4(ii) cannot apply to this case. However, if one forgets that the operator $\pi_{\sigma_1}(t_1)^*$ is defined as an operator from $L_2[0, 1]$ to $L_2[0, 1]$, then we see that Theorem 1.4(ii) can apply to also this case as follows: By definition,

$$\{\pi_{\sigma_1}(t_1)^* \sqrt{\rho_0}\}(x) = \frac{1}{x+1} \phi \left( \frac{1}{x+1} \right) \quad (\phi \in L_2[0, 1], x \in [0, 1]).$$

Although the density $\rho_0(x) \equiv \frac{1}{x}$ of $\theta$ does not belong to $L_1[0, 1]$,

$$\left\{ \{\pi_{\sigma_1}(t_1)^* \sqrt{\rho_0}\}(x) \right\}^2 = \left\{ \frac{1}{x+1} \sqrt{\frac{1}{x+1}} \right\}^2 = \frac{1}{x+1} \quad (x \in [0, 1]).$$

Therefore the $\left\{ \{\pi_{\sigma_1}(t_1)^* \sqrt{\rho_0}\}(x) \right\}^2$ is the density of Gauss’ measure $\gamma$ in (3.10) up to scalar multiple. Hence Theorem 1.4(ii) holds for this case.

We summarize relations between maps and Cuntz algebras in this example as follows:

| map                        | Cuntz algebra                  |
|----------------------------|--------------------------------|
| Gauss map                  | representation of $O_\infty$    |
| Farey map                  | representation of $O_2$         |
| jump transformation        | embedding of $O_\infty$ into $O_2$ |
3.3 A generalization of continued fraction transformation

In this subsection, we show an example of jump transformation associated with a generalization of continued fraction transformation. Define the map \( \tau_2 \) on \([0, 1]\) by

\[
\tau_2(x) = \begin{cases} 
\frac{1}{2^{k-1}x} - 1 & (2^{-k} < x \leq 2^{-k+1}, k \in \mathbb{N}), \\
0 & (x = 0). 
\end{cases} 
\] (3.18)

The map \( \tau_2 \) was introduced by Chan [1]. The invariant probability measure \( \mu_2 \) of \( \tau_2 \) is given as follows ([1], (2.20)):

\[
d\mu_2(x) = \frac{1}{\log \frac{4}{3}} \frac{dx}{(x + 1)(x + 2)} \quad (x \in [0, 1]). 
\] (3.19)

For \( k \in \mathbb{N} \), define \( g_k \equiv \left( \tau_2|_{X_k} \right)^{-1} \) on \( X_k \equiv (2^{-k}, 2^{-k+1}] \). Then

\[
g_k(x) = \frac{1}{2^{k-1}(x + 1)} \quad (x \in [0, 1)). 
\] (3.20)

Define the map \( \sigma_2 \) from \([0, 1]\) to \([0, 1]\) by

\[
\sigma_2(x) = \begin{cases} 
2x & (0 \leq x \leq \frac{1}{2}), \\
\frac{1}{x} - 1 & (\frac{1}{2} \leq x \leq 1). 
\end{cases} 
\] (3.21)

Then \( \tau_2 \) is the jump transformation of \( (\sigma_2, A_1) \) for \( A_1 \equiv [0, 1/2] \), and Gauss’ measure \( \gamma \) in (3.10) is \( \sigma_2 \)-invariant.
Define maps $f_1$ and $f_2$ on $[0,1]$ by

$$f_1(x) \equiv \frac{1}{x+1}, \quad f_2(x) \equiv \frac{1}{2} x.$$ \hfill (3.22)

Then $\{f_1, f_2\}$ is a branching function system on $[0,1]$ with the coding map $\sigma_2$. We see that

$$g_k(x) = f_2^{k-1} \circ f_1 \quad (k \geq 1).$$ \hfill (3.23)

For $f = \{f_1, f_2\}$ in (3.22), the representation $\pi_f$ of $\mathcal{O}_2$ on $L_2[0,1]$ is given by

$$\begin{cases}
\{\pi_f(t_1)\phi\}(x) \equiv \chi_{[1/2,1]}(x)(1+x)\phi\left(\frac{1}{2} - 1\right), \\
\{\pi_f(t_2)\phi\}(x) \equiv \chi_{[0,1/2]}(x)\sqrt{2}\phi(2x).
\end{cases}$$ \hfill (3.24)

for $\phi \in L_2[0,1]$ and $x \in [0,1]$. From this,

$$\begin{cases}
\{\pi_f(t_1^*\phi)\}(x) = \frac{1}{1+x}\phi\left(\frac{1}{x+1}\right), \\
\{\pi_f(t_2^*\phi)\}(x) = \frac{1}{\sqrt{2}}\phi\left(\frac{1}{2} x\right).
\end{cases}$$ \hfill (3.25)

Define $\phi_0(x) \equiv \frac{1}{\log 2} \frac{1}{x+1}$. From Theorem 1.4(ii),

$$\psi(x) = \left\{\pi(t_1^*\sqrt{\phi_0})\phi_0(x)\right\}^2 = \frac{1}{(1+x)^2} \frac{1}{1+\frac{1}{1+x}} = \frac{1}{(1+x)(2+x)}.$$ \hfill (3.19)

In this way, the density $\psi$ of $\tau_2$-invariant measure $\mu_2$ in (3.19) is given up to scalar multiple.

For further generalizations of $\tau_1$ and $\tau_2$, see [2]. About operator theory associated with continued fractions, see [5, 10, 11].

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