Dynamical Friction in Stellar Systems: an introduction

Héctor Aceves
Instituto de Astronomía, UNAM. Apartado Postal 877, Ensenada, B.C. 22800, México.

María Colosimo
Facultad de Ciencias, Universidad Nacional del Centro de la Provincia de Buenos Aires. Tandil, Argentina

(Dated: July 26, 2015)

An introductory exposition of Chandrasekhar’s gravitational dynamical friction, appropriate for an undergraduate class in mechanics, is presented. This friction results when a massive particle moving through a “sea” of much lighter star particles experiences a retarding force due to an exchange of energy and momentum. General features of dynamical friction are presented, both in an elementary and in a more elaborate way using hyperbolic two-body interactions. The orbital decay of a massive particle in an homogeneous gravitational system is solved analytically, that leads to an underdamped harmonic oscillator type of motion. A numerical integration of the equation of motion in a more realistic case is done. These results are compared to those of an N-body computer simulation. Several problems and projects are suggested to students for further study.

I. INTRODUCTION

Classical mechanics, perhaps the oldest of the physical sciences, continues to be an area of intensive research, both in its foundations and applications and a source of discussion and examples in teaching. Applications range from the modeling of cellular mechanical processes to solar system dynamics and galactic systems.

In describing nature students learn from their first courses, and particularly in laboratory experiments, that “the forces on a single thing already involve approximation, and if we have a system of discourse about the real world, then that system, at least for the present day, must involve approximations of some kind”; as mentioned by Feynman on introducing the subject of friction.

This phenomenon is usually introduced in textbooks and lectures by considering the slide of a material block on a surface, and a distinction between static and kinetic friction is made. A classical example of the effect of a friction-like force is the motion of a mass attached to a spring inside a viscous medium, where the corresponding differential equation is solved, and its behavior studied. At the end, one invariably needs to state that friction and its origin is a complicated matter, involving complex interactions at the atomic and molecular level among the surfaces in contact.

Several non-typical examples of mechanical friction for introductory courses exist that help both teachers and students alike in lectures on mechanics. All friction related problems are a background for discussing the important connection between the work-energy theorem and dissipative systems.

The purpose of this paper is to bring an example from astronomy closely related to standard mechanical friction, namely: dynamical friction. This process was first introduced in stellar systems by Subrahmanyan Chandrasekhar. In brief, a massive particle experiences a drag force when moving in a “sea” of much lighter star particles by exchanging energy and momentum.

An elementary understanding requires only some basic ideas from mechanics, and hence suitable for presentation in introductory courses.

Dynamical friction is important in astronomical studies of, for example: the fate of galaxy satellites or globular clusters orbiting their host galaxies, the substructure of dark halos surrounding galaxies and the motion of black holes in the centers of galaxies. It has been proposed to explain the formation of binaries in the Kuiper-belt, and the migration of Jupiter-mass planets in other solar systems from the outer parts where they presumably formed to the small orbital distances at which they are observed. It even has been considered in the motion of cosmic strings.

The presentation of this topic to students, in a lower or upper-undergraduate class on mechanics or computational physics will enhance their appreciation of physics in describing nature and expose them to another example of classical mechanics. Furthermore, students will obtain a glimpse of an area of astronomical research important for the understanding of the fate and behavior of stellar systems.

The organization of this paper is as follows. In Section II basic elements of the theory of dynamical friction are presented. Firstly, elementary arguments are used to elucidate them. Secondly, Chandrasekhar’s approximation using two-body hyperbolic Keplerian collisions is considered. In Section III a simple analytical problem for the motion of a massive particle in an ideal homogeneous stellar system is solved; a damped harmonic oscillator is found. In Section IV a more realistic astronomical example that requires the numerical integration of the equation of motion is presented. Comparison with a computer experiment is done afterwards. Final comments as well as some ideas for problems and projects of further study are provided in Section IV. An appendix contains some astronomical units and standard units used in gravitational computer simulations.
where we set $\rho_0 = n_0 m_*$, the background density, and $b_{\text{min}}$ and $b_{\text{max}}$ are a minimum and maximum impact parameter, respectively. Letting $\ln \Lambda$ be the resulting integral, the deceleration of $m$ due to its interaction with an homogenous background of particles stars is

$$\frac{dv_m}{dt} \approx \frac{\pi G^2 \rho_0 m}{v_m^2} \ln \Lambda .$$

The velocity impulse on $m_*$ has a perpendicular $\Delta v_\perp$ and parallel $\Delta v_\parallel$ component; see Figure 1. It is not difficult to see that a mean vector sum of all the $\Delta v_\perp$ contributions vanishes in this case. This is not true however for the mean square of $\Delta v_\perp$, Thus the dynamical friction force is along the line of motion of $m$.

Several key features of dynamical friction are observed from equation (1) in this elementary calculation, that appear also in more elaborate treatments. (1) The deceleration of the massive particle is proportional to its mass $m$, so the frictional force it experiences is directly proportional to $m^2$. (2) The deceleration is inversely proportional to the square of its velocity $v_m$.

B. Chandrasekhar formula

A further step in calculating the effect of dynamical friction is to consider hyperbolic Keplerian two-body encounters. Such analysis was done by Chandrasekhar. The resulting formula is provided in textbooks on stellar dynamics. For completeness such calculations is provided here, following Binney & Tremaine.

Use of well known results from the Kepler problem for two bodies in hyperbolic encounters are used. The two-body problem can be reduced to that of the motion of a particle of reduced mass $\mu = mm_*/(m + m_*)$ about a fixed center of force:

$$\mu \vec{\ddot{r}} = -\frac{\kappa}{r^2} \hat{r} ,$$

where $\kappa = Gm_*$, $r = r_* - r_m$ is the relative vector position of particles $m$ and $m_*$, and $\hat{r}$ its unit vector; see Figure 2. The relative velocity is then $V = v_* - v_m$, and a change in it is

$$\Delta V = \Delta v_* - \Delta v_m .$$

The velocity of the center-of-mass of $m$ and $m_*$ does not change, hence

$$m_* \Delta v_* + m \Delta v_m = 0 .$$

From equations (1) and (7) the change in velocity of $m$ is

$$\Delta v_m = -\left( \frac{m_*}{m + m_*} \right) \Delta V .$$

Once $\Delta V$ is determined, $\Delta v_m$ can be found from equation (6). From the symmetry of the problem, is better
to decompose $\Delta V$ in terms of perpendicular and parallel components:

$$\Delta V = \Delta V_{||} + \Delta V_{\perp},$$  \hspace{1cm} (9)

with

$$|\Delta V_{||}| = V_0 \cos \theta \hspace{1cm} \text{and} \hspace{1cm} |\Delta V_{\perp}| = V_0 \sin \theta,$$  \hspace{1cm} (10)

where $\theta$ is the angle of dispersion and $V_0$ the initial speed at infinity; this being the same after the encounter since only kinetic energy changes are considered; see Figure 2. From geometry, the angle $\alpha$ in Figure 2 is related to the orbit’s eccentricity $e$ by:

$$\cos \alpha = \frac{1}{e} \rightarrow \cot \frac{\theta}{2} = \sqrt{e^2 - 1},$$  \hspace{1cm} (11)

where $\theta + 2\alpha = \pi$. Physically $e$ is given by

$$e = \sqrt{1 + \frac{2EL^2}{\mu V_0^2}},$$  \hspace{1cm} (12)

where $E = \mu V_0^2/2$ in the kinetic energy and $L = \mu b V_0$ the angular momentum magnitude. Since

$$\sin \theta = \frac{2 \tan \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} \hspace{1cm} \text{and} \hspace{1cm} \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}},$$  \hspace{1cm} (13)

after some algebra it is found that

$$|\Delta V_{\perp}| = \frac{2bV_0^3}{G(m + m_*)} \left[ 1 + \frac{b^2 V_0^4}{G^2(m + m_*)^2} \right]^{-1},$$  \hspace{1cm} (14)

$$|\Delta V_{||}| = 2V_0 \left[ 1 + \frac{b^2 V_0^4}{G^2(m + m_*)^2} \right]^{-1}.$$  \hspace{1cm} (15)

Using equation 3 the perpendicular and parallel magnitudes of the components of $\Delta V_m$ follow:

$$|\Delta v_{m,\perp}| = \frac{2bmV_0^3}{G(m + m_*)^2} \left[ 1 + \frac{b^2 V_0^4}{G^2(m + m_*)^2} \right]^{-1},$$  \hspace{1cm} (16)

$$|\Delta v_{m||}| = \frac{2mV_0}{(m + m_*)} \left[ 1 + \frac{b^2 V_0^4}{G^2(m + m_*)^2} \right]^{-1}. $$  \hspace{1cm} (17)

In a homogeneous sea of stellar masses all perpendicular deflections cancel by symmetry. However, the parallel velocity changes are added and the mass $m$ will experience a deceleration.

The calculation of the total drag force due to a set of particles $m_*$ is as follows. Let $f(v_*)$ be the number density of stars. The rate at which particle $m$ encounters stars with impact parameter between $b$ and $b + db$, and velocities between $v_*$ and $v_* + dv_*$, is

$$2\pi bdb \cdot V_0 \cdot f(v_*) d^3v_*,$$  \hspace{1cm} (18)

where $d^3v_*$ is the volume element in velocity space. The total change in velocity of $m$ is found by adding all the contributions of $|\Delta V_m||$ due to particles with impact parameters from 0 to a $b_{\text{max}}$ and then summing over all velocities of stars. At a particular $v_*$ the change is

$$\frac{dv_m}{dt} \bigg|_{v_*} = V_0 \cdot f(v_*) d^3v_* \int_{b_0}^{b_{\text{max}}} |\Delta V_m|| 2\pi bdb. $$  \hspace{1cm} (19)

The required integral is

$$I = \int_{b_0}^{b_{\text{max}}} \left[ 1 + \frac{b^2 V_0^4}{G^2(m + m_*)^2} \right]^{-1} bdb = \int_{b_0}^{b_{\text{max}}} \frac{b}{1 + ab^2} \frac{1}{2a} \int_{s_{\text{min}}}^{s_{\text{max}}} ds,$$

where $a = V_0^2/G^2(m + m_*)^2$ and $s = 1 + ab^2$, with $s_{\text{max}} = 1 + ab_{\text{max}}^2$. Evaluating the integral yields

$$I = \frac{1}{2} G^2(m + m_*)^2 \left[ \frac{V_0^2}{V_0^2} \right] \ln [1 + \Lambda^2],$$

where

$$\Lambda = \frac{b_{\text{max}} V_0^2}{G(m + m_*)} = \frac{b_{\text{max}}}{b_{\text{min}}}.$$  \hspace{1cm} (20)

Putting these results together in equation 19:

$$\frac{dv_m}{dt} \bigg|_{v_*} = 2\pi G^2 \ln(1 + \Lambda^2) m_* (m + m_*)$$

$$\times f(v_*) d^3v_* \frac{v_* - v_m}{|v_* - v_m|^3},$$  \hspace{1cm} (21)

The quantity $\ln \Lambda$ is called the Coulomb logarithm in analogy to an equivalent logarithm found in the theory of strong or close encounters. This may be seen geometrically from Figure 2, were the stronger the deflection the smaller is the parallel component contributing to the slow down of $m$.

The determination of the limits $b_{\text{min}}$ and $b_{\text{max}}$ is not an easy matter and depends on the problem at hand. In this approximation $b_{\text{min}}$ satisfies $V_0^2 = Gm/b_{\text{min}}$, where $V_0$ depends on the relative velocity of $m$ and $m_*$. If the motion of $m$ is relatively slow in comparison to that of
the stars, \( V_0 \) can be approximated for example by the root-mean-square value velocity of stars \( V_{\text{rms}} \). The outer limit \( b_{\text{max}} \) is in principle the radius at which stars no longer can exchange momentum with \( m \). If \( m \) is close to the center of a stellar system \( b_{\text{max}} \) can be taken as a particular scale-radius of the system; for example, where the star density falls to half of its central value.

In typical astronomical applications \( \Lambda \gg 1 \). For example, consider the motion of a massive black hole of mass \( m \approx 10^8 M_\odot \) near the center of a dwarf galaxy. These galaxies have \( V_{\text{rms}} \approx 30 \text{ km s}^{-1} \), characteristic radii \( b_{\text{max}} \approx 3 \text{ kpc} \) and stars of masses \( m_* \approx 1 M_\odot \). Using these values we obtain \( \Lambda \approx 6.3 \times 10^3 \). This allows to use the approximation \( \ln(1 + \Lambda^2) \approx 2 \ln \Lambda \). Note that \( \ln \Lambda \) shows a weak dependence on \( V_0 \) that is usually neglected. Values of \( 2 \lesssim \ln \Lambda \lesssim 20 \) are typically found in astronomical literature.

Now, the integration of equation \((24)\) over the velocity space of stars is required. Writing equation \((24)\) as

\[
\frac{d\mathbf{v}_m}{dt} = G \int_0^\infty \frac{\rho(\mathbf{v}_*) (\mathbf{v}_* - \mathbf{v}_m)}{[\mathbf{v}_* - \mathbf{v}_m]^3} \, d^3 \mathbf{v}_* ,
\]

\[
\rho(\mathbf{v}_*) \equiv 4\pi G (m + m_*) m_* \ln \Lambda f(\mathbf{v}_*),
\]

it is noticed that represents the equivalent problem of finding the gravitational field (acceleration) at the “spatial” point \( \mathbf{v}_m \) generated by the “mass density” \( \rho(\mathbf{v}_*) \). From gravitational potential theory, the acceleration at a particular spatial point \( \mathbf{r} \) is given by

\[
a(\mathbf{r}) = G \int_0^\infty \frac{\rho(\mathbf{r}') (\mathbf{r}' - \mathbf{r}) \, d^3 \mathbf{r}'}{[\mathbf{r}' - \mathbf{r}]^3} = -\frac{Gm}{\mathbf{r}^3} \int_0^r \rho(\mathbf{r}') \, d^3 \mathbf{r}'.
\]

This is the known result that only matter inside a particular radius contributes to the force. In analogy to the gravitational case, the acceleration is given by the total “mass” inside \( v_* < v_m \), is

\[
\frac{d\mathbf{v}_m}{dt} = -c_{\text{df}} \mathbf{v}_m ,
\]

\[
c_{\text{df}} \equiv 16\pi^2 G^2 m_* (m + m_*) \ln \Lambda \int_0^{v_m} f(v_*) v_*^2 \, dv_* .
\]

For an isotropic velocity distribution:

\[
\frac{d\mathbf{v}_m}{dt} = -c_{\text{df}} \mathbf{v}_m ,
\]

\[
c_{\text{df}} \equiv 16\pi^2 G^2 m_* (m + m_*) \ln \Lambda \int_0^{v_m} f(v_*) v_*^2 \, dv_* .
\]

This is called Chandrasekhar dynamical friction formula. It shows that only stars moving slower than \( v_m \) contribute to the drag force on the massive particle.

If stars have a Maxwellian velocity distribution function,

\[
f(v_*) = \frac{n_0}{(2\pi \sigma^2)^{3/2}} \, e^{-v_*^2/(2\sigma^2)} ,
\]

the integral in \((24)\) is done by an elementary method. In dimensionless form it is

\[
I_m = \frac{n_0}{4\pi^3/2} \int_0^\infty \frac{e^{-y^2} y^2 \, dy}{X^3} ,
\]

where \( y^2 = v_*^2/2\sigma^2 \) and \( X \equiv v_m/\sqrt{2\sigma} \). Integrating by parts results in

\[
I_m = \frac{n_0}{4\pi} \left[ \operatorname{Erf}(X) - \frac{X}{\sqrt{\pi}} e^{-X^2} \right] ,
\]

where \( \operatorname{Erf}(x) = (2/\sqrt{\pi}) \int_0^x e^{-y^2} \, dy \) is the error function. If \( \rho_0 = n_0 m_* \), the density of the background of stars, and assume that \( m \gg m_* \), the deceleration of \( m \) inside an homogeneous stellar system with isotropic velocity distribution is:

\[
\frac{d\mathbf{v}_m}{dt} = -\Gamma_{\text{df}} \mathbf{v},
\]

\[
\Gamma_{\text{df}} = \frac{4\pi^2 G^2 m_* (m + m_*) \ln \Lambda}{v_m^3 \sqrt{\pi} e^{-X^2}} \left[ \operatorname{Erf}(X) - \frac{X}{\sqrt{\pi}} e^{-X^2} \right] .
\]

III. AN ANALYTICAL EXAMPLE

A simple application of Chandrasekhar’s formula \((24)\) for an homogeneous spherically symmetric stellar system, although not infinite, is presented. The problem consists in determining the motion of a massive particle \( m \) subject to gravitational and dynamical friction forces. The stellar system has a radius \( R \) and total mass \( M \); see Figure 3. The equation of motion for \( m \) is

\[
\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}_g + \mathbf{F}_{\text{df}} = -m \nabla \varphi(r) + m \mathbf{a}_{\text{df}} ,
\]

where \( \varphi(r) \) is the gravitational potential, and \( \mathbf{a}_{\text{df}} \) is given by equation \((24)\).

Equation \((24)\) is not in general tractable by analytical methods, so some approximations are required. Zhao has found an approximation to the term associated with the velocity distribution in equation \((24)\), namely:

\[
\chi(X) \equiv \frac{1}{X^3} \left[ \operatorname{Erf}(X) - \frac{2X}{\sqrt{\pi}} e^{-X^2} \right] \approx \frac{1}{X^3 + X^3} ,
\]

where
that works to within 10 percent for $0 \leq X < \infty$. When $m$ moves slow in comparison to the velocity dispersion of stars, $v \ll \sigma$, $\chi(0) \approx 3/4$, and when $v = \sigma/2 \chi(1) \approx 3/7$. Note that in the case of a very fast relative motion of $m$ dynamical friction is negligible; a situation analogous to when a block of material slides fast. Using the previous approximation, and considering $\chi = 3/4$, the frictional force $F_{\text{df}}$ in equation (28) becomes:

$$ F_{\text{df}} = ma_{\text{df}} \approx -\frac{3\pi G^2 \ln \Lambda \rho_0 m^2}{(2\sigma)^3} v = -\gamma v, \quad (28) $$

where $\gamma = 3\pi G^2 \ln \Lambda \rho_0 m^2/(2\sigma)^3$.

To determine $F_g$ recall that the potential is related to the density through Poisson equation,

$$ \nabla^2 \varphi(r) = 4\pi G \rho(r), \quad (29) $$

whose solution for a spherically symmetrical system of radius $R$ is

$$ \varphi(r) = -4\pi G \left( \frac{1}{r} \int_0^r \rho(r') r'^2 \, dr' + \int_r^R \rho(r') r \, dr \right). \quad (30) $$

In a constant density $\rho_0$ system the potential is

$$ \varphi(r) = -2\pi G \rho_0 \left( R^2 - \frac{1}{3} r^2 \right), \quad (31) $$

and the gravitational force on $m$ is

$$ F_g = -m \nabla \varphi(r) = -\frac{4}{3} \pi G \rho_0 m r = -k \vec{r}, \quad (32) $$

with $k = 4\pi G \rho_0 m/3$. This is the well known result from introductory mechanics that a particle inside an homogenous gravitational system performs a harmonic motion.

Combining equations (28) and (32) the resulting equation of motion is

$$ m \frac{d^2 \vec{r}}{dt^2} + k \vec{r} + \gamma \vec{v} = 0. \quad (33) $$

This is the same equation, for example, as that of a mass attached to a spring with stiffness constant $k$ inside a medium of viscosity $\gamma$; i.e., a damped harmonic oscillator. The solution of equation (33) in a plane, under arbitrary initial conditions

$$ x(0) = x_0, \quad \dot{x}(0) = u_0; \quad y(0) = y_0, \quad \dot{y}(0) = v_0, $$

where the dot indicates a time derivative, is

$$ x(t) = \frac{e^{-(\beta+\xi)t}}{2R} \left\{ (e^{2\xi t} - 1) u_0 + \left[ (e^{2\xi t} - 1) \beta + (e^{2\xi t} + 1) \xi \right] x_0 \right\}, $$

$$ y(t) = \frac{e^{-(\beta+\xi)t}}{2\xi} \left\{ (e^{2\xi t} - 1) v_0 + \left[ (e^{2\xi t} - 1) \beta + (e^{2\xi t} + 1) \xi \right] y_0 \right\}, \quad (34) $$

where $2\beta = \gamma/m$ and $\xi = \sqrt{\beta^2 - \omega_0^2}$, with $\omega_0^2 = k/m$.

The behavior of $m$ is dictated by the relative values of $\beta$ and $\omega$. The values of $\ln \Lambda$ and $\sigma$ are first to be estimated. Take $b_{\text{max}} \approx R$ and $b_{\text{min}} = GM/V_0^2$. An estimate of $V_0$ may be obtained from the virial theorem that relates the kinetic $T$ and potential energy $W$ of the system by:

$$ 2T = -W. \quad (35) $$

For an homogeneous system of size $R$ the potential energy is

$$ W = -4\pi G \int_0^R \rho_0 M r \, dr = -\frac{3 GM^2}{5 R}, \quad (36) $$

and the kinetic energy is taken as $T = MV_0^2/2$. This leads to

$$ V_0^2 \approx \frac{3}{5} \frac{GM}{R} \approx 3\sigma^2; \quad (37) $$

where the last term provides an estimate of the one-dimensional velocity dispersion under the assumption of isotropy in the velocity distribution of stars. Using equation (37) $\beta$ and $\omega$ are:

$$ \beta = \frac{45}{16} \sqrt{\frac{5}{2}} \frac{G^2 m M \ln \Lambda}{R^3 (GM/R)^{3/2}}, \quad \omega_0 = \sqrt{\frac{GM}{R^2}}. \quad (38) $$

The resulting Coulomb logarithm is $\ln \Lambda = \ln[3M/(5m)]$.

To compare the numerical values of $\beta$ and $\omega_0$ is better to use another system of units than a physical one. Let $G = M = R = 1$, that is a common choice in $N$-body simulations in astronomy; to return to physical units one can use Newton’s law and set $G$ to the appropriate value (see Appendix). In these units, relations (38) become

$$ \beta = \frac{45}{16} \sqrt{\frac{5}{2}} m \ln \left( \frac{3}{5m} \right), \quad \omega_0 = 1. \quad (39) $$

If $m = 1/100$ then $\ln \Lambda \approx 4$ and $\beta \approx 0.2 < \omega_0$. Hence an underdamped harmonic motion for the massive particle results. If $m = 1/10$ then $\ln \Lambda \approx 2$ and $\beta \approx 0.8 \approx \omega_0$, so the motion of $m$ will be strongly damped. Note that an upper limit to $m$ is set when $m = 3/5$, leading to $\ln \Lambda = 0$; i.e., no dynamical friction results. For larger $m$ a negative $\beta$ is obtained. Clearly, the model fails and the behavior of the dynamics is unrealistic.

For cases of interest, where $m \ll M$, it follows that $\beta < \omega_0$ and the resulting motion (34), after some algebra, is

$$ x(t) = \left[ x_0 \cos \omega t + \frac{u_0 + \beta x_0}{\omega} \sin \omega t \right] e^{-\beta t}, $$

$$ y(t) = \left[ y_0 \cos \omega t + \frac{v_0 + \beta y_0}{\omega} \sin \omega t \right] e^{-\beta t}, \quad (40) $$

where $\omega^2 = \omega_0^2 - \beta^2$. Note that a time-scale when the orbit decays $1/e$ is given by $\tau_{\text{df}} = 1/\beta$. 
dynamical friction time scale \( \tau_m \) is at \( \rho \) \( (A \text{ particular case of these are treated next.}) \)

motion of a massive particle inside a non-homogeneous stellar system due to dynamical friction. Initial conditions are \( x_m, y_m = (0.59, 0.44) \). Left panels are for \( m = 0.01 \) and right ones for \( m = 0.02 \). Top panels show the orbit (solid line) in the xy-plane and bottom ones the time evolution of the distance \( r \) from the center. The dynamical friction time scale \( \tau_{df} = 1/\beta \) is indicated by an arrow. Dashed lines are orbits without considering dynamical friction.

Take as example \( m = 0.01 \). If the initial position of \( m \) is at \( (x, y) = (0.59, 0.0) \) and its velocity \( (0.0, 0.44) \) the resulting orbit is that shown in Figure 4 left with solid line. Doubling \( m \) results in the orbit shown in Figure 4 right. The dashed lines in correspond to the orbit of \( m \) without dynamical friction. The effect of increasing the mass of \( m \) on the orbit, and on the decay time \( \tau_{df} \), is clearly appreciated.

This example shows the basic features of, for example, the orbital decay of a satellite galaxy toward the center of its host larger galaxy. It may be applied also to the motion of a massive black hole near the center of a galaxy or star cluster, where to some approximation the gravitational potential can be taken as harmonic. More realistic situations require however the numerical integration of the orbit and/or an N-body computer simulation. A particular case of these are treated next.

**IV. A MORE REALISTIC EXAMPLE**

Chandrasekhar’s formula although derived assuming an infinite homogeneous system may be applied, to some degree, when stellar systems are non-homogeneous. In this case, local values for the density \( \rho(r) \) and the velocity dispersion \( \sigma(r) \) are used. Here the motion of a massive particle \( m \) inside a non-homogeneous stellar system is considered, both using a semi-analytical method and N-body simulation, to illustrate further the application of dynamical friction.

**A. Semi-analytic treatment**

A simple representation of a stellar system, such as a globular cluster or an elliptical galaxy, is provided by the Plummer model. Its potential and stellar density are, respectively,

\[
\varphi(r) = -\frac{GM}{(r^2 + a^2)^{1/2}}, \quad \rho(r) = \frac{3M a^2/4\pi}{(r^2 + a^2)^{3/2}}, \quad (41)
\]

where \( M \) is the total mass, and \( a \) the scale-radius of the system. In a spherical system with isotropic velocity distribution the equation of “hydrostatic” equilibrium is satisfied:

\[
\frac{1}{\rho} \frac{d(\rho \sigma^2)}{dr} = -\frac{d\varphi}{dr} \rightarrow \sigma^2(r) = -\frac{\varphi(r)}{\rho \sigma^2}. \quad (42)
\]

The last result follows from noticing that \( \rho \propto \varphi^5 \), and imposing boundary conditions that both \( \rho \sigma^2 \) and \( \varphi \) go to zero at infinity.

Equations (41) and (42) will be used in equation (28) to compute the orbital motion of a massive particle \( m \). It rests to determine \( b_{\min} \) and \( b_{\max} \). The former is evaluated at local values, \( b_{\min} = Gm/[3\sigma^2(r)] \), and the latter is set fix to \( b_{\max} = a \).

The equation of motion (28) for \( m \) can now be integrated numerically using standard methods or using the one discussed by Feynman for planetary orbits (§9). Here a fourth-order Runge-Kutta algorithm with adaptive step-size was used. The initial conditions for \( m \) are the same as those used in the analytical case.

In Figure 5 the resulting orbit from the numerical integration is shown as a dashed line. Also, the behavior of the \( x \) and \( y \) coordinates, and of the distance \( r \) of \( m \) to the center, as a function of time are shown. The typical decay of the orbit is evident. In the same figure results from an N-body simulation are displayed, that are described next.

**B. N-body simulation**

The use of N-body simulations allows to study more realistically the different dynamical phenomena that occur in stellar systems. Several N-body codes with different degrees of sophistication have been developed for astronomical problems in mind. Some low-N simulations can be run nowadays using a personal computer with publicly available N-body codes.

Barnes’ TREE-CODE in FORTRAN, and some of his public subroutines are used to simulate the motion of \( m \) inside a Plummer model. A numerical realization of
Both approximations overestimate the decay rate of $m$ in comparison to the $N$-body simulation. Taking $b_{\text{max}} = a/5$ leads to a somewhat better agreement, but does not reproduce the $N$-body result. Rather surprisingly, the analytical result does a fair job in reproducing the overall orbital decay in this case.

V. FINAL COMMENTS

The approximations in deriving Chandrasekhar formula limits, obviously, its application to more complex stellar systems than the one considered here. However, it is remarkable that equation (21) leads to reasonably well results when used with values under a local approximation.

In similar vain to the study of the friction between surfaces, dynamical friction is a complex subject. Elaborate calculations based on Brownian motion, linear response theory, resonances, and the fluctuation-dissipation theorem exist. These that are steps forward toward a more complete physical theory for this process.

Instead of listing explicitly some of the shortcomings of Chandrasekhar dynamical friction formula when applied to gravitational systems, the student is encouraged to think on some of them and possible improvements on such formula.

Some ideas that may lead to problems and/or projects for students are:

1. How would the analytical solution considered here be changed if the Plummer model is used? What type of approximations would be required to make? How does $\ln \Lambda$ change?

2. If $\sigma^2$ is a measure of the kinetic energy per unit mass of stars, what is an estimate for its mean increase due to the energy lost by the massive particle during its decay?

3. How would the orbital decay time be changed for different types of initial eccentricities of the massive particle?

4. Consider a star cluster ($m = 10^6 M_\odot$) in circular orbit at a distance of $r = 5$ kpc from the center of our galaxy ($M \approx 6 \times 10^{11} M_\odot$, $R \approx 150$ kpc). Would it be expected to fall to the center within the age of the universe, say $t = 10^{10}$ yr? Typical velocities for stars and dark matter particles at that distance are about 200 km/s, and the scale-radius may be around 5 kpc. What if instead of a star cluster we have a galaxy satellite, such as the Magellanic
Clouds, with $m \approx 10^{10} \, M_{\odot}$ and at a distance of 100 kpc?

5. How do results change if instead of a Plummer model a more pronounced density profile is used, such as the Hernquist model? How does the number of particles $N$ in a simulation affect the decay rate?

6. As the massive particle moves through the stellar system it induces a density wake behind it. Can this be detected in an N-body simulation on a home computer? How about looking for this wake in the phase-space diagram (e.g. a plot of $\dot{x}-x$) of stars near the the massive particle?

7. How good do the local approximation works if instead of a massive particle one has an extended object, small in comparison to its host galaxy?

Textbook problems are designed in general to yield one correct answer, the above ideas for problems are rather vague but this is on purpose. The reason is twofold. On one hand, to promote in students a spirit of research by setting an approximate physical model and to look for the required data and “tools” to solve it; some of them can be found in the references. On the other hand, no single definite answer can be given. A feature proper of the way physics evolves toward describing and understanding nature.

APPENDIX A: ASTRONOMICAL AND N-BODY UNITS

Several quantities in astronomy are so large in comparison to common “terrestrial” values, that special units are used. Table I lists some of these and their equivalences in physical units.

In the mks system of units the Gravitational constant is $G = 6.67 \times 10^{-11} \, m^3 \, kg^{-1} \, s^{-2}$. A natural system of units for gravitational interactions is that where the gravitational constant is set to $G = 1$; in the same way as for quantum systems Planck’s constant is usually set to $\hbar = 1$. On dimensional grounds $[G] = u_m^2 \, u_t / \, u_m$, where $u_m$, $u_t$, and $u_v$ correspond, respectively, to units of mass, length and velocity.

The transformation of $G$ using length units such as kpc or Mpc (10^6 pc) is direct. Choosing $u_l$ and $u_m$ the unit of velocity and of time $u_t$, under an appropriate $G$ value, are

$$u_v = \sqrt{\frac{u_m}{u_l}}, \quad u_t = \sqrt{\frac{u_l^3}{G u_m}}.$$ 

In this way the transformation from N-body units, where $G = M = R = 1$, to physical ones can be made. Table II lists some values for different choices of $u_l$ and $u_m$, and the resulting units of $u_v$ and $u_t$. The entries correspond to using the approximate size and mass of a globular cluster, a disk of a spiral galaxy, and of a cluster of galaxies, respectively, as units $u_l$ and $u_m$.

In the standardized gravitational N-body units, the total energy of a system is $E = -1/4$. This follows from the virial theorem ($2T + W = 0$), where

$$W = -\frac{1}{2} \frac{GM}{R} \rightarrow E = \frac{W}{2} = \frac{GM}{4R}.$$ 

Here $R$ is strictly what is called the virial radius of the system; that does not necessarily coincides with the total extent of the stellar system, but is a very good approximation. The potential energy of a Plummer model is

$$W = \frac{1}{2} \int_0^\infty \rho(r) \varphi(r) 4\pi r^2 \, dr = -\frac{3\pi GM^2}{32} \frac{1}{a}.$$ 

Thus the total energy is $E = -(3\pi GM^2)/(64a)$. In N-body units this leads to a value of the Plummer scale-radius of $a = 3\pi/16$.

| Unit | Equivalence |
|------|-------------|
| Astronomical unit | AU = 1.496 × 10^{11} m |
| Parsec | pc = 2.063 × 10^{3} AU = 3.261 light-years |
| Kiloparsec | kpc = 10^{3} pc |
| Solar mass | $M_{\odot} = 1.989 \times 10^{30} kg$ |
| Year | yr = 3.156 × 10^{7} s |

^a Mean sun-earth distance

| UNITS |
|-------|
| AU = 1.496 × 10^{11} m |
| pc = 2.063 × 10^{3} AU |
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| $M_{\odot} = 1.989 \times 10^{30} kg$ |
| yr = 3.156 × 10^{7} s |

| TABLE II: From N-body units to astronomical |
|---------------------------------------------|
| Stellar system | $u_l$ | $u_m$ | $u_v$ | $u_t$ |
| $u_l$ | $u_m$ | $u_v$ | $u_t$ |
| Solar system | $10^{8}$ | $10^{8}$ | $9.3$ | $5.3$ |
| Galaxy | $10^{11}$ | $207.4$ | $47.2$ |
| Cluster of galaxies | $5 \times 10^{15}$ | $927.4$ | $5271.4$ |
