On $d$-divisible graceful $\alpha$-labelings of $C_{4k} \times P_m$

Anita Pasotti

Abstract

In [9] the concept of a $d$-divisible graceful $\alpha$-labeling has been introduced as a generalization of classical $\alpha$-labelings and it has been shown how it is useful to obtain certain cyclic graph decompositions. In the present paper it is proved the existence of $d$-divisible graceful $\alpha$-labelings of $C_{4k} \times P_m$ for any integers $k \geq 1$, $m \geq 2$ for several values of $d$.

Keywords: graceful labeling; $\alpha$-labeling; graph decomposition.

MSC(2010): 05C78.

1 Introduction

We assume familiarity with the basic concepts about graphs.

As usual, we denote by $K_v$ and $K_{m \times n}$ the complete graph on $v$ vertices and the complete $m$-partite graph with parts of size $n$, respectively. Also, let $C_k$, $k \geq 3$, be the cycle on $k$ vertices and let $P_m$, $m \geq 2$, be the path on $m$ vertices. Graphs of the form $C_k \times P_m$ can be viewed as grids on cylinders and they are bipartite if and only if $k$ is even. If $m = 2$, $C_k \times P_2$ is nothing but the prism $T_{2k}$ on $2k$ vertices. For any graph $\Gamma$ we write $V(\Gamma)$ for the set of its vertices and $E(\Gamma)$ for the set of its edges. If $|E(\Gamma)| = e$, we say that $\Gamma$ has size $e$.

Given a subgraph $\Gamma$ of a graph $K$, a $\Gamma$-decomposition of $K$ is a set of graphs, called blocks, isomorphic to $\Gamma$ whose edges partition the edge-set of $K$. Such a decomposition is said to be cyclic when it is invariant under a cyclic permutation of all vertices of $K$. In the case that $K = K_v$ one also speaks of a $\Gamma$-system of order $v$. The problem of establishing the set of values of $v$ for which such a system exists is in general quite difficult. For a survey on graph decompositions see [2].

The concept of a graceful labeling of $\Gamma$, introduced by A. Rosa [10], is quite related to the existence problem of cyclic $\Gamma$-systems. A graceful labeling
of a graph $\Gamma$ of size $e$ is an injective function $f: V(\Gamma) \to \{0, 1, 2, \ldots, e\}$ such that
\[
\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{1, 2, \ldots, e\}.
\]
In the case that $\Gamma$ is bipartite and $f$ has the additional property that its maximum value on one of the two bipartite sets does not reach its minimum on the other one, one says that $f$ is an $\alpha$-labeling. In [10], Rosa proved that if a graph $\Gamma$ of size $e$ admits a graceful labeling then there exists a cyclic $\Gamma$-system of order $2e + 1$ and that if it admits an $\alpha$-labeling then there exists a cyclic $\Gamma$-system of order $2en + 1$ for any positive integer $n$. For a very rich survey on graceful labelings we refer to [6].

Many variations of graceful labelings have been considered. In particular Gnana Jothi [6] defines an odd graceful labeling of a graph $\Gamma$ of size $e$ as an injective function $f: V(\Gamma) \to \{0, 1, 2, \ldots, 2e - 1\}$ such that
\[
\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{1, 3, 5, \ldots, 2e - 1\}.
\]
In a recent paper, see [9], we have introduced the following new definition which is, at the same time, a generalization of the concepts of a graceful labeling (when $d = 1$) and of an odd graceful labeling (when $d = e$).

**Definition 1.1.** Let $\Gamma$ be a graph of size $e = d \cdot m$. A $d$-divisible graceful labeling of $\Gamma$ is an injective function $f: V(\Gamma) \to \{0, 1, 2, \ldots, d(m + 1) - 1\}$ such that
\[
\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{1, 2, 3, \ldots, d(m + 1)\}
\]
\[-\{m + 1, 2(m + 1), \ldots, d(m + 1)\}.
\]
Namely the set $\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\}$ can be divided into $d$ parts $P^0, P^1, \ldots, P^{d-1}$ where $P^i := \{(m + 1)i + 1, (m + 1)i + 2, \ldots, (m + 1)i + m\}$ for any $i = 0, 1, \ldots, d - 1$.

The $\alpha$-labelings can be generalized in a similar way.

**Definition 1.2.** A $d$-divisible graceful $\alpha$-labeling of a bipartite graph $\Gamma$ is a $d$-divisible graceful labeling of $\Gamma$ having the property that its maximum value on one of the two bipartite sets does not reach its minimum value on the other one.

We have to point out that in [9] the above labelings have been called “$d$-graceful ($\alpha$-)labelings”, but the author was unaware that this name is already used in the literature with a different meaning, see [8] and [11].

It is known that there is a close relationship between graceful labelings and difference families, see [1]. In [9] we established relations between $d$-divisible graceful ($\alpha$-)labelings and a generalization of difference families introduced in [3], proving the following theorems.
Theorem 1.3. If there exists a $d$-divisible graceful labeling of a graph $\Gamma$ of size $e$ then there exists a cyclic $\Gamma$-decomposition of $K_{\left(\frac{d}{2}+1\right)\times 2d}$.

Theorem 1.4. If there exists a $d$-divisible graceful $\alpha$-labeling of a graph $\Gamma$ of size $e$ then there exists a cyclic $\Gamma$-decomposition of $K_{\left(\frac{d}{2}+1\right)\times 2dn}$ for any integer $n \geq 1$.

In this paper we determine the existence of $d$-divisible graceful $\alpha$-labelings of $C_{4k} \times P_m$ for several values of $d$. In order to obtain these results, first of all we will find $d$-divisible graceful $\alpha$-labelings of prisms, which correspond to the case $m=2$, and then by induction on $m$ we will be able to construct $d$-divisible graceful $\alpha$-labelings of $C_{4k} \times P_m$ for any $m \geq 2$. For what said above, these results allow us to obtain new infinite classes of cyclic decompositions of the complete multipartite graph in copies of $C_{4k} \times P_m$.

2 $d$-divisible graceful $\alpha$-labelings of prisms

In this section we will investigate the existence of $d$-divisible graceful $\alpha$-labelings of prisms. From now on, given two integers $a$ and $b$, by $[a, b]$ we will denote the set of integers $x$ such that $a \leq x \leq b$.

For convenience, we denote the $2k$ vertices of $T_{2k}$ by $x_1, x_2, \ldots, x_k; y_1, y_2, \ldots, y_k$ where the $x_i$'s are the consecutive vertices of one $k$-cycle and the $y_i$'s are consecutive vertices of the other $k$-cycle and $x_i$ is connected to $y_i$. Clearly $T_{2k}$ has size $e = 3k$ and it is bipartite if and only if $k$ is even. In [4] Frucht and Gallian proved that $T_{2k}$ admits an $\alpha$-labeling if and only if $k$ is even.

Theorem 2.1. The prism $T_{2k}$ admits a $3$-divisible graceful $\alpha$-labeling for every $k \geq 1$.

Proof. We set $O_x = \{x_1, x_3, \ldots, x_{4k-1}\}$, $E_x = \{x_2, x_4, \ldots, x_{4k}\}$, $O_y = \{y_1, y_3, \ldots, y_{4k-1}\}$, $E_y = \{y_2, y_4, \ldots, y_{4k}\}$. Clearly $O_x \cup E_y$ and $O_y \cup E_x$ are the two bipartite sets of $V(T_{2k})$.

Consider the map $f : V(T_{2k}) \rightarrow \{0, 1, \ldots, 12k + 2\}$ defined as follows:

\[
\begin{align*}
  f(x_{2i+1}) &= \begin{cases} 
    6k + 1 & \text{for } i = 0 \\
    8k + 2 - i & \text{for } i \in [1, k] \\
    8k + 1 - i & \text{for } i \in [k + 1, 2k - 1] 
  \end{cases} \\
  f(x_{2i}) &= 4k + i & \text{for } i \in [1, 2k]. \\
  f(y_{2i+1}) &= i & \text{for } i \in [0, 2k - 1] \\
  f(y_{2i}) &= \begin{cases} 
    12k + 3 - i & \text{for } i \in [1, k] \\
    12k + 2 - i & \text{for } i \in [k + 1, 2k] 
  \end{cases}
\end{align*}
\]
We have

\[ f(O_x \cup E_x) = [0, 2k - 1] \cup [4k + 1, 6k] \]
\[ f(O_y \cup E_y) = [6k + 1, 7k] \cup [7k + 2, 8k + 1] \cup [10k + 2, 11k + 1] \cup [11k + 3, 12k + 2]. \]

Hence \( f \) is injective and \( \max f(O_x \cup E_x) < \min f(O_x \cup E_y) \). Now for \( i = 1, \ldots, 4k \) set

\[ \sigma_i = |f(x_{i+1}) - f(x_i)|, \quad \varepsilon_i = |f(y_{i+1}) - f(y_i)|, \quad \rho_i = |f(x_i) - f(y_i)| \quad (1) \]

where the indices are understood modulo \( 4k \). By a direct calculation, one can see that

\[ \sigma_1 = 2k, \]
\[ \{ \sigma_i \mid i = 2, \ldots, 2k + 1 \} = [2k + 1, 4k] \]
\[ \{ \sigma_i \mid i = 2k + 2, \ldots, 4k \} = [1, 2k - 1] \]
\[ \rho_1 = 6k + 1 \]
\[ \{ \rho_i \mid i = 2, \ldots, 2k + 1 \} = [6k + 2, 8k + 1], \]
\[ \{ \rho_i \mid i = 2k + 2, \ldots, 4k \} = [4k + 2, 6k], \]
\[ \{ \varepsilon_i \mid i = 1, \ldots, 2k \} = [10k + 3, 12k + 2], \]
\[ \{ \varepsilon_i \mid i = 2k + 1, \ldots, 4k - 1 \} = [8k + 3, 10k + 1], \]
\[ \varepsilon_{4k} = 10k + 2. \]

Hence \( \{ \sigma_i \mid i = 1, \ldots, 4k \} = [1, 4k] \), \( \{ \rho_i \mid i = 1, \ldots, 4k \} = [4k + 2, 8k + 1] \) and \( \{ \varepsilon_i \mid i = 1, \ldots, 4k \} = [8k + 3, 12k + 2] \). This concludes the proof. \( \square \)

**Theorem 2.2.** The prism \( T_{8k} \) admits a 6-divisible graceful \( \alpha \)-labeling for every \( k \geq 1 \).

**Proof.** Set \( O_x, E_x, O_y, E_y \) as in the proof of previous theorem. Consider the map \( f: V(T_{8k}) \to \{0, 1, \ldots, 12k + 5\} \) defined as follows:

\[
\begin{align*}
  f(x_{2i+1}) &= \begin{cases} 6k + 2 & \text{for } i = 0 \\ 8k + 4 - i & \text{for } i \in [1, k] \\ 8k + 2 - i & \text{for } i \in [k + 1, 2k - 1] \end{cases} \\
  f(x_{2i}) &= 4k + 1 + i \quad \text{for } i \in [1, 2k]. \\
  f(y_{2i+1}) &= i \quad \text{for } i \in [0, 2k - 1] \\
  f(y_{2i}) &= \begin{cases} 12k + 6 - i & \text{for } i \in [1, k] \\ 12k + 4 - i & \text{for } i \in [k + 1, 2k] \end{cases}
\end{align*}
\]
It results

\[ f(O_y \cup E_x) = [0, 2k - 1] \cup [4k + 2, 6k + 1] \]
\[ f(O_x \cup E_y) = [6k + 2, 7k + 1] \cup [7k + 4, 8k + 3] \cup [10k + 4, 11k + 3] \cup [11k + 6, 12k + 5]. \]

Hence \( f \) is injective and \( \max f(O_y \cup E_x) < \min f(O_x \cup E_y) \). Let \( \varepsilon_i, \rho_i, \sigma_i \), for \( i = 1, \ldots, 4k \), be as in (1). It is not hard to see that

\[
\begin{align*}
\{ \sigma_i \mid i = 1, \ldots, 4k \} &= [1, 2k] \cup [2k + 2, 4k + 1] \\
\{ \rho_i \mid i = 1, \ldots, 4k \} &= [4k + 3, 6k + 2] \cup [6k + 4, 8k + 3] \\
\{ \varepsilon_i \mid i = 1, \ldots, 4k \} &= [8k + 5, 10k + 4] \cup [10k + 6, 12k + 5].
\end{align*}
\]

Hence \( f \) is a 6-divisible graceful \( \alpha \)-labeling of \( T_{8k} \). \( \square \)

**Theorem 2.3.** The prism \( T_{8k} \) admits a 12-divisible graceful \( \alpha \)-labeling for every \( k \geq 1 \).

Proof. Also here we set \( O_x, E_x, O_y, E_y \) as in the proof of Theorem [2.1]. We are able to prove the existence of a 12-divisible graceful \( \alpha \)-labeling of \( T_{8k} \) by means of two direct constructions where we distinguish the two cases: \( k \) even and \( k \) odd.

Case 1: \( k \) even.
Consider the map \( f : V(T_{8k}) \to \{0, 1, \ldots, 12k + 11\} \) defined as follows:

\[
f(x_{2i+1}) = \begin{cases} 
6k + 5 & \text{for } i = 0 \\
8k + 8 - i & \text{for } i \in [1, \frac{k}{2}] \\
8k + 7 - i & \text{for } i \in \left[\frac{k}{2} + 1, k\right] \\
8k + 5 - i & \text{for } i \in [k + 1, 2k - 1]\end{cases}\]

\[
f(x_{2i}) = \begin{cases} 
4k + 3 + i & \text{for } i \in [1, \frac{3k}{2}] \\
4k + 4 + i & \text{for } i \in \left[\frac{3k}{2} + 1, 2k\right].\end{cases}\]

\[
f(y_{2i+1}) = \begin{cases} 
i & \text{for } i \in [0, \frac{3k}{2} - 1] \\
i + 1 & \text{for } i \in \left[\frac{3k}{2}, 2k - 1\right]\end{cases}\]

\[
f(y_{2i}) = \begin{cases} 
12k + 12 - i & \text{for } i \in [1, \frac{k}{2}] \\
12k + 11 - i & \text{for } i \in \left[\frac{k}{2} + 1, k\right] \\
12k + 9 - i & \text{for } i \in [k + 1, 2k]\end{cases}\]


It is easy to see that

\[ f(\mathcal{O}_y) = \left[ 0, \frac{3k}{2} - 1 \right] \cup \left[ \frac{3k}{2} + 1, 2k \right] \]

\[ f(\mathcal{E}_x) = \left[ 4k + 4, \frac{11k}{2} + 3 \right] \cup \left[ \frac{11k}{2} + 5, 6k + 4 \right] \]

\[ f(\mathcal{O}_z) = \left[ 6k + 5, 7k + 4 \right] \cup \left[ 7k + 7, \frac{15k}{2} + 6 \right] \cup \left[ \frac{15k}{2} + 8, 8k + 7 \right] \]

\[ f(\mathcal{E}_y) = \left[ 10k + 9, 11k + 8 \right] \cup \left[ 11k + 11, \frac{23k}{2} + 10 \right] \cup \left[ \frac{23k}{2} + 12, 12k + 11 \right] \]

Hence \( f \) is injective and max \( f(\mathcal{O}_y \cup \mathcal{E}_x) = 6k + 4 < 6k + 5 = \min f(\mathcal{O}_z \cup \mathcal{E}_y) \).

Set \( \sigma_i, \varepsilon_i, \rho_i \), for \( i = 1, \ldots, 4k \), as in (1). By a long and tedious calculation, one can see that

\[
\begin{align*}
\{ \sigma_i \mid i = 1, \ldots, 4k \} &= \{ 1, 4k + 3 \} - \{ k + 1, 2k + 2, 3k + 3 \} \\
\{ \rho_i \mid i = 1, \ldots, 4k \} &= \{ 4k + 5, 8k + 7 \} - \{ 5k + 5, 6k + 6, 7k + 7 \} \\
\{ \varepsilon_i \mid i = 1, \ldots, 4k \} &= \{ 8k + 9, 12k + 11 \} - \{ 9k + 9, 10k + 10, 11k + 11 \}.
\end{align*}
\]

This concludes the proof of Case 1.

Case 2: \( k \) odd.

Let now \( f : V(T_{8k}) \rightarrow \{ 0, 1, \ldots, 12k + 11 \} \) defined as follows:

\[
\begin{align*}
f(x_{2i+1}) &= \begin{cases} 
6k + 5 & \text{for } i = 0 \\
8k + 8 - i & \text{for } i \in \left[ 1, k \right] \\
8k + 6 - i & \text{for } i \in \left[ k + 1, \frac{3k - 1}{2} \right] \\
8k + 5 - i & \text{for } i \in \left[ \frac{3k + 1}{2}, 2k - 1 \right]
\end{cases} \\
f(x_{2i}) &= \begin{cases} 
4k + 3 + i & \text{for } i \in \left[ 1, \frac{k + 1}{2} \right] \\
4k + 4 + i & \text{for } i \in \left[ \frac{k + 3}{2}, 2k \right]
\end{cases} \\
f(y_{2i+1}) &= \begin{cases} 
i & \text{for } i \in \left[ 0, \frac{k - 1}{2} \right] \\
i + 1 & \text{for } i \in \left[ \frac{k + 1}{2}, 2k - 1 \right]
\end{cases} \\
f(y_{2i}) &= \begin{cases} 
12k + 12 - i & \text{for } i \in \left[ 1, k \right] \\
12k + 10 - i & \text{for } i \in \left[ k + 1, \frac{3k - 1}{2} \right] \\
12k + 9 - i & \text{for } i \in \left[ \frac{3k + 1}{2}, 2k \right]
\end{cases}
\]

Arguing exactly as in Case 1, one can check that \( f \) is a 12-divisible graceful \( \alpha \)-labeling of \( T_{8k} \). \( \Box \)

**Example 2.4.** The three graphs in Figure [4] show the 3-divisible graceful \( \alpha \)-labeling, the 6-divisible graceful \( \alpha \)-labeling and the 12-divisible graceful \( \alpha \)-labeling of \( T_{24} \) provided by previous theorems.
3 \textit{d}-divisible graceful $\alpha$-labelings of $C_{4k} \times P_m$

In this section using the results of the previous one we will construct $d$-divisible graceful $\alpha$-labelings of $C_{4k} \times P_m$. In particular, since $e = 4k(2m - 1)$ we consider $d = 2m - 1$, $2(2m - 1)$, $4(2m - 1)$. In [7] Jungreis and Reid proved that for any $k, m \geq 2$ not both odd there exists an $\alpha$-labeling of $C_{2k} \times P_m$. For convenience, we denote the vertices of $C_{4k} \times P_m$ as illustrated in Figure 2 and we set $C^i = ((i, 1), (i, 2), \ldots, (i, 4k))$ for any $i = 1, \ldots, m$.

![Figure 2: $C_4 \times P_4$](image)

**Theorem 3.1.** For any integer $k \geq 1$ and $m \geq 2$, $C_{4k} \times P_m$ admits a $(2m - 1)$-divisible graceful $\alpha$-labeling.

Proof. We will prove the result by induction on $m$. If $m = 2$ the thesis follows from Theorem 2.1. Let now $m \geq 2$. Suppose that there exists a

![Figure 1: $T_{24}$](image)
We set
\( f(C^m) = (0, (4k + 1)(2m - 1) - 1, 1, (4k + 1)(2m - 1) - 2, 2, \ldots, \\
(4k + 1)(2m - 1) - k, k, (4k + 1)(2m - 1) - (k + 2), k + 1, \ldots, \\
2k - 1, (4k + 1)(2m - 1) - (2k + 1)). \)

Note that the 3-divisible graceful \( \alpha \)-labeling of \( C_{4k} \times P_2 \) constructed in Theorem 2.1 has this property, in fact \( f(C^2) = (0, 12k + 2, 1, 12k + 2, \ldots, k - 1, 11k + 3, k, 11k + 3, k, 11k + 1, k + 1, \ldots, 2k - 1, 10k + 2). \) So in order to obtain the thesis it is sufficient to construct a \((2m + 1)\)-divisible graceful \( \alpha \)-labeling of \( C_{4k} \times P_{m+1} \) satisfying the same property, namely such that

\[
g(C^{m+1}) = (0, (4k + 1)(2m + 1) - 1, 1, (4k + 1)(2m + 1) - 2, 2, \ldots, \\
(4k + 1)(2m + 1) - k, k, (4k + 1)(2m + 1) - (k + 2), k + 1, \ldots, \\
2k - 1, (4k + 1)(2m + 1) - (2k + 1)).
\]

(2) We set
\[
g((i, j)) = f((i, j)) + (4k + 1) \quad \forall i = 1, \ldots, m, \forall j = 1, \ldots, 4k.
\]

By the hypothesis on \( f(C^m) \) it results
\[
g(C^m) = (4k + 1, (4k + 1)2m - 1, 4k + 2, (4k + 1)2m - 2, 4k + 3, \ldots, \\
(4k + 1)2m - k, 5k + 1, (4k + 1)2m - (k + 2), 5k + 2, \ldots, \\
6k, (4k + 1)2m - (2k + 1)).
\]

So there exists \( j \in [1, 4n] \) such that \( g((m, j)) = (4k + 1)2m - 1. \) We set
\( g(C^{m+1}) \) as in [2] where \( g((m + 1, j)) = 0. \)

Now we will see that \( g : V(C_{4k} \times P_{m+1}) \to \{0, \ldots, (4k + 1)(2m + 1) - 1\} \) defined as above is indeed a \((2m+1)\)-divisible graceful \( \alpha \)-labeling of \( C_{4k} \times P_{m+1}. \) Since \( f(V(C_{4k} \times P_m)) \subseteq [0, (4k + 1)(2m - 1) - 1], \) by the definition of \( g, \) it follows that
\[
g(V(C^1 \cup C^2 \cup \ldots \cup C^m)) \subseteq [4k + 1, (4k + 1)2m - 1].
\]

Also we have
\[
g(V(C^{m+1})) \subseteq [0, 2k - 1] \cup [(4k + 1)(2m + 1) - (2k + 1), (4k + 1)(2m + 1) - 1].
\]
hence \( g \) is an injective function. Since, by hypothesis \( f \) is a \((2m-1)\)-divisible graceful \( \alpha \)-labeling of \( C_{4k} \times P_m, \) we have \( V(C_{4k} \times P_m) = A \cup B \) with \( \max_A f < \min_B f. \) Let \( V(C_{4k} \times P_{m+1}) = C \cup D. \) By the construction, it follows that
\[
g(C) = (f(A) + (4k + 1)) \cup [0, 2k - 1]
\]
\[
g(D) \subseteq (f(B) + (4k + 1)) \cup [(4k + 1)(2m + 1) - (2k + 1), (4k + 1)(2m + 1) - 1]
\]
hence \( \max_C g < \min_D g \). Now we have to consider the differences between adjacent vertices. Since \( f \) is a \((2m-1)\)-divisible graceful \( \alpha \)-labeling of \( C_{4k} \times P_m \), by the construction of \( g \), it results

\[
\bigcup_{i \in [1, m-1]} |f((i, j)) - f((i + 1, j))| \cup \bigcup_{i \in [1, m]} |f((i, j)) - f((i, j + 1))| = \]

\[ [1, (4k + 1)(2m - 1)] - \{\beta(4k + 1) \mid \beta \in [1, 2m - 1]\}\]

where the index \( j \) is taken modulo \( 4k \). Finally it is not hard to check that

\[
\{g((m, j)) - g((m + 1, j)) \mid j \in [1, 4k]\} = \]

\[ [(4k + 1)2m - 4k; (4k + 1)2m - 1]\]

and

\[
\{g((m + 1, j)) - g((m + 1, j + 1)) \mid j \in [1, 4k]\} = \]

\[ [(4k + 1)(2m + 1) - 4k; (4k + 1)(2m + 1) - 1]\]

This concludes the proof. \( \square \)

**Example 3.2.** In Figure 3 we will show the 5-divisible graceful \( \alpha \)-labeling of \( C_4 \times P_3 \), the 7-divisible graceful \( \alpha \)-labeling of \( C_4 \times P_4 \) and the 9-divisible graceful \( \alpha \)-labeling of \( C_4 \times P_5 \) obtained starting from the 3-divisible graceful \( \alpha \)-labeling of \( T_8 = C_4 \times P_2 \) and following the construction illustrated in the proof of Theorem 3.1.

![Figure 3](image_url)
Theorem 3.3. For any integer $k \geq 1$ and $m \geq 2$, $C_{4k} \times P_m$ admits a $2(2m-1)$-divisible graceful $\alpha$-labeling.

Proof. We will prove the result by induction on $m$. If $m = 2$ the thesis follows from Theorem 2.2. Let now $m \geq 2$. Suppose that there exists a $2(2m-1)$-divisible graceful $\alpha$-labeling $f$ of $C_{4k} \times P_m$ with vertices of $C_m$ so labeled:

$$f(C_m) = (0, (4k + 2)(2m - 1) - 1, 1, (4k + 2)(2m - 1) - 2, 2, \ldots, (4k + 2)(2m - 1) - k, k, (4k + 2)(2m - 1) - (k + 3), k + 1, \ldots, 2k - 1, (4k + 2)(2m - 1) - (2k + 2)).$$

We want to show the existence of a $2(2m + 1)$-divisible graceful $\alpha$-labeling $g$ of $C_{4k} \times P_{m+1}$ satisfying the same property, namely such that

$$g(C_{m+1}) = (0, (4k + 2)(2m + 1) - 1, 1, (4k + 2)(2m + 1) - 2, 2, \ldots, (4k + 2)(2m + 1) - k, k, (4k + 2)(2m + 1) - (k + 3), k + 1, \ldots, 2k - 1, (4k + 2)(2m + 1) - (2k + 2)).$$

First of all set

$$g((i, j)) = f((i, j)) + (4k + 2) \quad \forall i = 1, \ldots, m, \forall j = 1, \ldots, 4k.$$  

This implies that there exists $j \in [1, 4m]$ such that $g((m, j)) = (4k + 2)2m - 1$. We set $g(C_{m+1})$ as in (3) where $g((m + 1, j)) = 0$. Arguing exactly as in the previous proof one can prove that $g$ is a $2(2m + 1)$-divisible graceful $\alpha$-labeling $g$ of $C_{4k} \times P_{m+1}$. □

Theorem 3.4. For any integer $k \geq 1$ and $m \geq 2$, $C_{4k} \times P_m$ admits a $4(2m-1)$-divisible graceful $\alpha$-labeling.

Proof. We will prove the result by induction on $m$. If $m = 2$ the thesis follows from Theorem 2.3. Let now $m \geq 2$. We have to distinguish two cases: $k$ even and $k$ odd.

Let $k$ be even. Suppose that there exists a $4(2m - 1)$-divisible graceful $\alpha$-labeling $f$ of $C_{4k} \times P_m$ with vertices of $C_m$ so labeled:

$$f(C_m) = (0, (4k + 4)(2m - 1) - 1, 1, (4k + 4)(2m - 1) - 2, 2, \ldots, (4k + 4)(2m - 1) - \frac{k}{2}, \frac{k}{2}, (4k + 4)(2m - 1) - \left(\frac{k}{2} + 2\right), \frac{k}{2} + 1, \ldots, (4k + 4)(2m - 1) - (k + 1), k, (4k + 4)(2m - 1) - (k + 4), k + 1, \ldots, (4k + 4)(2m - 1) - \left(\frac{3}{2}k + 2\right), \frac{3}{2}k - 1, (4k + 4)(2m - 1) - \left(\frac{3}{2}k + 3\right), \frac{3}{2}k + 1, (4k + 4)(2m - 1) - \left(\frac{3}{2}k + 4\right), \ldots, (4k + 4)(2m - 1) - (2k + 2), 2k, (4k + 4)(2m - 1) - (2k + 3)).$$
We want to show the existence of a $4(2m + 1)$-divisible graceful $\alpha$-labeling $g$ of $C_{4k} \times P_m$ satisfying the same property, namely such that

$$g(C^{m+1}) = (0, (4k + 4)(2m + 1) - 1, 1, (4k + 4)(2m + 1) - 2, 2, \ldots ,$$

$$\ldots, (4k + 4)(2m + 1) - (k + 1), k, (4k + 4)(2m + 1) - (k + 4), k + 1, \ldots ,$$

$$\ldots, (4k + 4)(2m + 1) - 3k + 3,$$

$$\frac{3k + 3}{2}, (4k + 4)(2m + 1) - \frac{3k + 9}{2}, \ldots ,$$

$$\ldots, (4k + 4)(2m + 1) - (2k + 2), 2k, (4k + 4)(2m + 1) - (2k + 3)). \quad (4)$$

First of all set

$$g((i, j)) = f((i, j)) + (4k + 4) \quad \forall i = 1, \ldots , m, \forall j = 1, \ldots , 4k.$$ 

This implies that there exists $j \in [1, 4n]$ such that $g((m, j)) = (4k + 4)2m - 1$. We set $g(C^{m+1})$ as in (4) where $g((m + 1, j)) = 0$.

Let now $k$ be odd. Suppose that there exists a $4(2m - 1)$-divisible graceful $\alpha$-labeling $f$ of $C_{4k} \times P_m$ with vertices of $C^m$ so labeled:

$$f(C^m) = (0, (4k + 4)(2m - 1) - 1, 1, (4k + 4)(2m - 1) - 2, 2, \ldots ,$$

$$\ldots, (4k + 4)(2m - 1) - k, k + 1, (4k + 4)(2m - 1) - (k + 3), k + 2, \ldots ,$$

$$\ldots, (4k + 4)(2m - 1) - \frac{3k + 3}{2}, \frac{3k + 1}{2}, (4k + 4)(2m - 1) - \frac{3k + 7}{2},$$

$$\frac{3k + 3}{2}, (4k + 4)(2m - 1) - \frac{3k + 9}{2}, \ldots ,$$

$$\ldots, (4k + 4)(2m - 1) - (2k + 2), 2k, (4k + 4)(2m - 1) - (2k + 3)).$$

We want to show the existence of a $4(2m + 1)$-divisible graceful $\alpha$-labeling $g$ of $C_{4k} \times P_{m+1}$ satisfying the same property, namely such that

$$g(C^{m+1}) = (0, (4k + 4)(2m + 1) - 1, 1, (4k + 4)(2m + 1) - 2, 2, \ldots ,$$

$$\ldots, (4k + 4)(2m + 1) - k, k + 1, (4k + 4)(2m + 1) - (k + 3), k + 2, \ldots ,$$

$$\ldots, (4k + 4)(2m + 1) - \frac{3k + 3}{2}, \frac{3k + 1}{2}, (4k + 4)(2m + 1) - \frac{3k + 7}{2},$$

$$\frac{3k + 3}{2}, (4k + 4)(2m + 1) - \frac{3k + 9}{2}, \ldots ,$$

$$\ldots, (4k + 4)(2m + 1) - (2k + 2), 2k, (4k + 4)(2m + 1) - (2k + 3)). \quad (5)$$
First of all set

\[ g((i, j)) = f((i, j)) + (4k + 4) \quad \forall i = 1, \ldots, m, \forall j = 1, \ldots, 4k. \]

This implies that there exists \( j \in [1, 4n] \) such that \( g((m, j)) = (4k + 4)2m - 1. \)

We set \( g(C^{m+1}) \) as in (5) where \( g((m + 1, j)) = 0. \)

Arguing exactly in the proof of Theorem 3.1 one can prove that, in both cases, \( g \) is a \( 4(2m + 1) \)-divisible graceful \( \alpha \)-labeling \( g \) of \( C_{4k} \times P_{m+1}. \) \( \square \)

**Example 3.5.** In Figure 4 we have the \( 10 \)-divisible graceful \( \alpha \)-labeling of \( C_{12} \times P_3 \) obtained starting from the \( 6 \)-divisible graceful \( \alpha \)-labeling of \( T_{24} = C_{12} \times P_2 \) shown in Figure 4 and following the construction explained in the proof of Theorem 3.3 and the \( 20 \)-divisible graceful \( \alpha \)-labeling of \( C_{12} \times P_3 \) obtained starting from the \( 12 \)-divisible graceful \( \alpha \)-labeling of \( T_{24} \) shown in Figure 7 and following the construction illustrated in the proof of Theorem 3.4.

![Figure 4: A 10-divisible graceful \( \alpha \)-labeling of \( C_{12} \times P_3 \) and a 20-divisible graceful \( \alpha \)-labeling of \( C_{12} \times P_3 \), respectively.](image)

By virtue of Theorems 1.4, 3.1, 3.3 and 3.4, we have

**Proposition 3.6.** There exists a cyclic \( C_{4k} \times P_m \)-decomposition of \( K_{(4k+1) \times 2(2m-1)n} \) of \( K_{(2k+1) \times 4(2m-1)n} \) and of \( K_{(k+1) \times 8(2m-1)n} \), for any integers \( k, n \geq 1, m > 2. \)
References

[1] R.J.R. Abel and M. Buratti, Difference families, in: CRC Handbook of Combinatorial Designs (C.J. Colbourn and J.H. Dinitz eds.), CRC Press, Boca Raton, FL (2006), 392–409.

[2] D. Bryant and S. El-Zanati, Graph decompositions, in: CRC Handbook of Combinatorial Designs (C.J. Colbourn and J.H. Dinitz eds.), CRC Press, Boca Raton, FL (2006), 477–486.

[3] M. Buratti and A. Pasotti, Graph decompositions with the use of difference matrices, Bull. Inst. Combin. Appl. 47 (2006), 23–32.

[4] R. Frucht and J.A. Gallian, Labeling prisms, Ars Combin., 26 (1988), 69–82.

[5] J.A. Gallian, Dynamic survey of graph labelings, Electron. J. Combin. 17 (2010), DS6, 246pp.

[6] R.B. Gnana Jothi, Topics in Graph Theory, Ph.D. Thesis, Madurai Kamaraj University 1991.

[7] D. Jungreis and M. Reid, Labeling grids, Ars Combin. 34 (1992), 167–182.

[8] M. Maheo and H. Thuiller, On d-graceful graphs, Ars Combin. 13 (1982), 181–192.

[9] A. Pasotti, On d-graceful labelings, to appear on Ars Combin.

[10] A. Rosa, On certain valuations of the vertices of a graph, Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris (1967), 349–355.

[11] P. J. Slater, On k-graceful graphs, Proc. of the 13th S.E. Conf. on Combinatorics, Graph Theory and Computing (1982), 53–57.