The Quantum Formalism and the GRW Formalism

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Abstract

The Ghirardi–Rimini–Weber (GRW) theory of spontaneous wave function collapse is known to provide a quantum theory without observers, in fact two different ones by using either the matter density ontology (GRWm) or the flash ontology (GRWf). Both theories are known to make predictions different from those of quantum mechanics, but the difference is so small that no decisive experiment can as yet be performed. While some testable deviations from quantum mechanics have long been known, we provide here something that has until now been missing: a formalism that succinctly summarizes the empirical predictions of GRWm and GRWf. We call it the GRW formalism. Its structure is similar to that of the quantum formalism but involves different operators. In other words, we establish the validity of a general algorithm for directly computing the testable predictions of GRWm and GRWf. We further show that some well-defined quantities cannot be measured in a GRWm or GRWf world.

PACS: 03.65.Ta. Key words: quantum theory without observers; Ghirardi–Rimini–Weber (GRW) theory of spontaneous wave function collapse; empirical predictions; quantum measurement theory; predicted deviations from quantum mechanics; primitive ontology; limits on knowledge; positive-operator-valued measure (POVM); completely positive superoperator.

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1 Introduction

This paper is about the derivation of statistical predictions for macroscopic behavior from a specific microscopic physical model. That is common in statistical physics. A bit unusual, though, is that the microscopic model we study was developed for explaining quantum mechanics. Indeed, in order to obtain a quantum theory without observers, and thus to solve the measurement problem and other paradoxes of quantum mechanics, it has been suggested that one should incorporate spontaneous collapses of the wave function into the laws of nature by replacing the Schrödinger evolution with a stochastic and nonlinear evolution law. The simplest and best known proposal for such a law is due to Ghirardi, Rimini, and Weber (GRW) [28, 9] (see [6] for a review of collapse theories). This is the framework we are concerned with in this paper. Our goal is to obtain the axioms of quantum mechanics as theorems in the GRW theory.

To complete the GRW theory, one needs to specify a choice of primitive ontology (PO) and a law determining how the wave function governs the PO (see [3] for a discussion). Two possibilities for the PO and its law have been proposed: the matter density ontology and the flash ontology, leading to two different theories we shall denote GRWm and GRWf, respectively, in the following. We recall their definitions in Section 2. It is known that GRWm and GRWf are empirically equivalent, i.e., that they make exactly and always the same empirical predictions [3]; we describe the reasons in Section 2.1. The first purpose of this paper is to derive what these predictions actually are. By “empirical predictions” we mean those predictions that can be tested in experiment; we will see that there are also predictions that cannot be so tested. The totality of all empirical predictions of a theory we also call the empirical content of the theory.
While GRWm and GRWf are designed to imitate quantum mechanics, they have been known since their inception to deviate from quantum mechanics, and a number of particular predictions differing from those of quantum mechanics have been identified \cite{28, 39, 36, 31, 1} (for an overview of attempts to test GRW theories against quantum mechanics, see \cite{6}). Nonetheless, in practice the GRW theories tend to agree extremely well with quantum mechanics: for small systems, collapses are too rare to be noticed, while the breakdown of macroscopic superpositions is hard to test because of decoherence (for explicit figures about how closely GRW theories agree with quantum mechanics, see \cite{5}). Thus, the theorems we prove are not precisely the axioms of quantum mechanics, but very close.

Is there a general scheme of predictions, or an algorithm for directly calculating the predictions, of the GRW theories, in particular where they differ from quantum mechanics? In this paper, we answer this question in the positive and provide a formalism, which we call the \textit{GRW formalism}, summarizing the empirical predictions of the GRWm and GRWf theories. (Indeed, GRWm and GRWf give rise to the same formalism; they have to, because they are empirically equivalent.) The GRW formalism is analogous to the quantum formalism of orthodox quantum theory that describes the results of quantum experiments in terms of operators as observables, spectral measures, and the like. The main difference between the two formalisms lies in the relevant operators.

The second main innovation of this paper, besides the formulation of the GRW formalism, concerns the nature of the argument used in deriving it: the argument is based on the primitive ontology of the theory.

A third contribution of this paper is to identify questions that possess a unique answer in a GRW world but cannot be answered by the inhabitants of that world by means of any experiment. The following question is presumably of this type: How many collapses occurred in a certain system during the time interval \([t_1, t_2]\)?

### 1.1 A First Look at the GRW Formalism

The GRW formalism can be formulated in a way similar to the formalism of quantum mechanics using operators in Hilbert space. We will give the complete formulation in Section 6. Put succinctly, the difference between the quantum and the GRW formalism is different evolution, different operators.

“Different evolution” means that the unitary Schrödinger evolution is replaced by a master equation for the density matrix \(\rho_t\) (a Lindblad equation, or quantum dynamical semigroup):

\[
\frac{d\rho_t}{dt} = -\frac{i}{\hbar}[H, \rho_t] + \lambda \sum_{k=1}^N \int d^3x \Lambda_k(x)^{1/2} \rho_t \Lambda_k(x)^{1/2} - N\lambda \rho_t . \tag{1}
\]

For readers who are not familiar with this type of equation, we note that the term \(-\frac{i}{\hbar}[H, \rho_t]\) represents the unitary evolution, with \(H\) the Hamiltonian, while the further
terms, the deviation from the unitary evolution, have the effect that the evolution (1) transforms “pure states into mixed states,” i.e., transform density matrices that are 1-dimensional projections into ones that are not. Equation (1) holds for the density matrix $\rho_t$ corresponding to the probability distribution of the random GRW wave function $\Psi_t$, arising from a fixed initial wave function $\Psi_{t_0}$. Concerning the notation, $\lambda > 0$ is a constant, and the positive operators $\Lambda_k(x)$ are the collapse rate operators (see Section 2 for the definition).

“Different operators” means that “observables” are associated with different operators than in quantum mechanics. This requires some explanation. A precise statement (which forms a crucial part of the GRW formalism) is that with every experiment $\mathcal{E}$, there is associated a positive-operator-valued measure (POVM) $E(\cdot)$ such that the probability distribution of the random outcome $Z$ of $\mathcal{E}$, when performed on a system with density matrix $\rho$, is given by

$$P(Z \in B) = \text{tr}(\rho E(B))$$  \hspace{1cm} (2)

for all sets $B$. This statement, the main theorem about POVMs, is valid in quantum mechanics as well as in GRW theories, but the POVM $E^{\text{GRW}}(\cdot)$ associated with $\mathcal{E}$ in a GRWm or GRWf world is different from the POVM $E^{\text{Qm}}(\cdot)$ associated with $\mathcal{E}$ in quantum mechanics. We prove this statement in Section 4. However, we do not compute any specific operators for specific experiments, but derive only an abstract and general characterization of $E^{\text{GRW}}(\cdot)$.

When talking about every experiment, we mean that any possible future advances of technology are included. The assumptions that define our concept of “experiment” are: it involves a system (the object on which the experiment is performed) and an apparatus; it is possible to consider the same experiment for different states of the system, whereas changing the apparatus counts as considering a different experiment; at the time at which the experiment begins, the system and the apparatus are not entangled.

Some colleagues that we have discussed this topic with have found it difficult to imagine how GRW could lead to different operators. When speaking of different operators, we were asked, does that mean that the momentum operator is no longer $-i\hbar \nabla$? No, it does not mean that. It means that, given any experiment in a quantum world, one can consider the same experiment in a GRWm or GRWf world, and the statistics of the outcome of that experiment are different from those in quantum mechanics—given by a different operator, or different POVM. Which operator should be called the “momentum operator” remains a matter of convention, and indeed there are reasons to call $-i\hbar \nabla$ the “momentum operator” also in the GRW theories. Similarly, we would say that the “position observable” is the same in the GRW theories as in quantum mechanics, even

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1Here $P(Z \in B)$ denotes the probability of the event $Z \in B$; sets are always assumed to be measurable. The notion of “POVM” is defined in Section 3.1.

2Some “observables” of the quantum formalism—the momentum, angular momentum, and energy operators—are the generators of symmetries of the theory, such as translation, rotation, and time translation invariance. By virtue of Noether’s theorem, then, they commute with the Hamiltonian. Since GRWm and GRWf, too, are translation, rotation, and time translation invariant (if the interaction potential is), the same self-adjoint operators occur here in the role of generators of symmetries (and
though concrete experimental designs for “measuring position” may lead to different outcome statistics than in quantum mechanics.

We were also asked, when speaking of different operators, whether we refer to the Heisenberg picture? No, we do not. The question means this: If the time evolution is not unitary then the Heisenberg picture (or whatever replaces it for a master equation such as \(\text{(1)}\)) should attribute to all observables different operators than standard quantum mechanics. But the “different operators” arise even in the Schrödinger picture: If the observation of the system (i.e., the period of its interaction with the apparatus) begins at time \(s\) and ends at \(t\), then one is supposed, according to the GRW formalism, to evolve the system’s density matrix until time \(s\) using \(\text{(1)}\) in the Schrödinger picture, and insert into the formula \(\text{(2)}\) the resulting \(\rho_s\), corresponding to what one feeds into the apparatus.3

Maybe the reason why many physicists find it difficult to understand that the GRW formalism involves different operators arises from regarding the operators of quantum mechanics as something that came into the theory by means of a second postulate besides the Schrödinger equation, the measurement postulate. From such a picture one might expect that the measurement postulate should remain unchanged, and, hence, also the operators, even when the Schrödinger equation is modified. The GRW perspective, however, forces us to proceed differently since it contains no measurement postulate, and predictions must be derived instead from postulates about the primitive ontology. This makes it evident that the measurement postulate and the Schrödinger equation actually never were independent, and that the operators depend on the evolution law, for example because the experiment’s outcome depends on the evolution law of the apparatus. The GRW perspective also forces us to make precise what it means to say that a certain observable has operator \(A\). We take it to mean that \(A\) encodes the outcome statistics, in the sense that the relevant experiment has outcome statistics given by \(\text{(2)}\) with \(E(\cdot)\) the spectral projection-valued measure (PVM) of \(A\).

The master equation \(\text{(1)},\) or very similar equations, also arise in the theory of decoherence. As a closely related fact, the GRW formalism would in principle also hold in a hypothetical quantum world in which decoherence is inevitable and affects every system in the same way, corresponding to \(\text{(1)}\). (In practice, of course, decoherence, due to interaction with the environment, cannot correspond to \(\text{(1)}\) in exactly the same way for every system because different systems have different environments and interact with their environments in different ways.) Let us underline the difference between deriving the GRW formalism from the quantum formalism together with the right dose of decoherence corresponding to \(\text{(1)}\), and deriving it from GRWm or GRWf: A deriv-

3But some connection with the Heisenberg picture exists indeed: keep in mind that the main theorem about POVMs concerns any experiment \(\mathcal{E}\); for example, \(\mathcal{E}\) could consist of waiting for a while \(\Delta t\) and then “measuring position.” Then, the quantum operator associated with \(\mathcal{E}\) is the Heisenberg-evolved position operator, \(\hat{Q}_\mathcal{E} = e^{iH\Delta t}\hat{Q}e^{-iH\Delta t}\), and the reader will be able to imagine that in GRWm or GRWf there is a different operator (in fact, a POVM) associated with \(\mathcal{E}\).
tion starting from quantum mechanics would assume statements about the outcomes of experiments (the measurement postulate) to deduce other statements about the outcomes of experiments. When starting from GRWm or GRWf, in contrast, we assume statements about the primitive ontology, and derive that, e.g., pointers point in certain directions.

It is an interesting side remark that Bohmian mechanics \cite{bohm} can be so modified as to become empirically equivalent to GRWm and GRWf. This modified version is described in \cite{grw} under the name “MBM.” Its empirical content is also summarized by the GRW formalism. As a consequence, the empirical content of the GRW theories can as well be obtained with a particle ontology, and is not limited to the flash and matter density ontologies.

1.2 Role of the Primitive Ontology

What is the connection between empirical predictions and primitive ontology (PO)? Since the PO is described by the variables $\xi$ giving the distribution of matter in space and time, a statement like “the experiment $E$ has the outcome $z$” should mean that the PO of the apparatus indicates the value $z$. For example, if the apparatus displays the outcome by a pointer pointing to a particular position on a scale, what it means for the outcome to be $z$ is that the matter of the pointer is, according to the PO, in the configuration corresponding to $z$. Thus, the outcome $Z$ is a function of the PO,

$$Z = \zeta(\xi).$$  \hspace{1cm} (3)

Precursors of our treatment of the connection between predictions and PO can be found in \cite{prl, jpa, jcp, ptp, prl, prl, prl}, in some of which this connection was implicit, or hinted at, or briefly mentioned. In Bohmian mechanics \cite{bohm}, a similar connection between PO and the empirical predictions was explicitly made in \cite{bohm}; however, researchers working on Bohmian mechanics have essentially always been aware of this connection—much in contrast to those working on collapse theories, who tended to focus on the wave function and forget about the PO.

The fact that GRWm and GRWf have the same formalism, despite their difference in PO, may suggest that the PO is not so relevant after all. That is true for practical applications which require working out the figures of the predictions, but not for the theoretical analysis of GRW theories, for their logical structure, or for their definition, as the considerations in this paper exemplify.

1.3 Status of the Derivation

It may seem as if the GRW formalism were a rather trivial consequence of the master equation \cite{grw}. So it is perhaps useful to make a list of what is nontrivial about our derivation of the GRW formalism:

- It is not a priori clear that a GRW formalism should exist.
The existence of a GRW formalism had not been noticed for 20 years.

Since the predictions of GRWm and GRWf deviate from those of quantum mechanics, it is not obvious that they can be summarized by any small number of simple rules.

The derivation of the GRW formalism has a status similar to that of the quantum formalism from Bohmian mechanics (see, e.g., [25]), a result implying in particular that there is no possibility of experimentally testing Bohmian mechanics against standard quantum mechanics. If that claim is non-obvious (after all, some authors have claimed the contrary), then so should be the GRW formalism.

The non-linearity of the GRW evolution of the wave function $\Psi_t$ might have suggested against the existence of a GRW formalism using linear operators. On the other hand, the master equation (1) is linear in $\rho_t$, a crucial fact for deriving the GRW formalism. Still, this fact alone does not imply the GRW formalism.

Our assertion about the GRW formalism concerns the PO. In detail, it states that the matter density function $m(x,t)$ of GRWm and the set $F$ of flashes in GRWf are such that macroscopic apparatuses display certain results with certain probabilities.

Our derivation of the GRW formalism is based on an analysis of the behavior of the PO. Such an analysis was not done in [7, 8].

Our derivation applies to the matter density ontology and to the flash ontology. We do not make claims for any other ontology.

The defining laws of GRWm and GRWf, unlike the ordinary axioms of quantum mechanics, do not refer to observations, but to the wave function and the PO. Thus, the empirical predictions are not immediate from the defining laws of the theory but require a derivation.

To the extent that it is not obvious how the PO variables $m(x,t)$ and $F$ behave, it is not obvious how macroscopic apparatuses (built out of the elements of the PO) behave.

It has often been noted that there are situations in which $m(x,t)$ and $F$ behave in an unexpected, surprising, or counter-intuitive way. (See, e.g., [6, p. 347], [3, footn. 5].)

\[\text{4For example, we do not know of a way of deriving the GRW formalism from GRWm other than exploiting the empirical equivalence to GRWf (or MBM [4]), even though (1) is valid in GRWm.}\]

\[\text{5However, there are reasons why every reasonable ontology suitable for the stochastic GRW wave function evolution law should lead to the same empirical predictions. Similarly, the empirical contents of CSLm, the Continuous Spontaneous Localization theory [35, 27, 6] with the matter density ontology, or with any other reasonable ontology, can presumably be summarized by a formalism very similar to the GRW formalism.}\]
Every physicist knows rules for what can be concluded about measurement results if the wave function is such-and-such. These rules, however, cannot be used in the derivation of the GRW formalism, partly because the GRW theories are not quantum mechanics, and partly because it is the aim of the derivation (and of this paper) to deduce, and not to presuppose, rules for the results of experiments.

- Our derivation makes no use of the rules of standard quantum mechanics for predicting results of experiments given the wave function.
- Our derivation makes no use of any customs of standard quantum mechanics for how to interpret or use wave functions.
- In particular, operators as observables emerge from an analysis of the GRW theories, they are not postulated; in fact, they are not even mentioned in the definition of the GRW theories.
- Certain wave functions may easily suggest certain macro-states, but this does not mean that the configuration of the PO looks like this macro-state. Our derivation makes no use of such suggestive assumptions.

As a consequence of our analysis, there are severe limits on the epistemic access to microscopic details of the PO variables \( m(x, t) \) or \( F \). In other words, there are limits to the extent to which one can measure \( m(x, t) \) or \( F \). This fact can be regarded as an instance of surprising behavior of the PO (as mentioned above), and underlines that it is not obvious which functions of the PO are observable.

The issue we mentioned in the last item of the list deserves more comment. It turns out to be impossible to measure, with any reasonable microscopic accuracy, the matter density \( m(x, t) \) in GRWm (or, presumably, the set \( F \) of flashes in GRWf), unless information about the wave function of the system is available. Limitations on the observers' access to \( m(x, t) \) were described before in [13]; we describe here several similar limitations. As a particular example, one might wish to measure the number of collapses that occur in a certain system (e.g., a tiny drop of water) during a chosen time interval, in analogy for example to the measurement of the number of radioactive decay events in a sample of radioactive matter. Heuristic considerations suggest, perhaps surprisingly, that it is impossible to measure the number of collapses, with any accuracy and reliability better than what one could estimate without any measurement at all. In other words, the precise number of collapses is empirically undecidable, and thus GRWm and GRWf entail sharp limits on knowledge. In a GRWm or GRWf world, certain facts are kept secret from its inhabitants. Note that this situation does not arise from anything like a conspiratorial character of the theory, but simply as a consequence of the defining equations; after all, we do not make postulates about what can or cannot be measured but analyze the theory. Similar limits on knowledge are known for Bohmian mechanics, where for example it turns out to be impossible to measure the (instantaneous) velocity of a particle [25, 26], unless information about the wave function is available; as another example, it turns out to be impossible to distinguish empirically between certain different versions of Bohmian mechanics (see [30] for a discussion).
A question we do not address here is how to do scattering theory for GRW theories. But we briefly state the problem. Normal quantum scattering theory (see, e.g., [22]) involves limits $t \to \infty$, which would be inappropriate in GRW theories because one consequence of GRW theories is long run “universal warming,” since every collapse tends to increase energy, as it makes the wave function narrower in the position representation and therefore wider in the momentum representation. In the limit $t \to \infty$, scattered wave packets in a GRW world would therefore always end up with infinite energy, and uniformly distributed over all spatial directions. From a practical point of view, the time scale of free flight in real scattering experiments ($\sim 10^{-2}$ s) is much smaller than the time scale of universal warming ($\sim 10^{15}$ years [28, p. 481]), usually even much smaller than the time scale of collapse ($\sim 10^8$ years), but much larger than the time scale of the interaction process. Thus, a simple and quite appropriate method of predicting the scattering cross section in a GRW world is to take the limit $t \to \infty$ for the unitary evolution, which is the dominating part of the evolution of the wave function $\Psi_t$ over the relevant time scale. But this is to ignore the difference between the predictions of GRW theories and quantum mechanics for scattering theory, and the question remains how to compute GRW corrections to the quantum formulas for scattering cross sections.

Another question we do not address here is: Which condition characterizes the regime in which the GRW theories empirically agree with quantum mechanics?

Finally, although the GRW formalism is valid for both GRWm and GRWf, the status of the derivation is very different for the two theories. While we derive the GRW formalism as precise theorems from GRWf, we do not know of a similar derivation from GRWm. In fact, the only way we know of to derive it for GRWm is by exploiting the empirical equivalence with GRWf, and the argument for the empirical equivalence is not as mathematical in character as the derivation of the GRW formalism from GRWf.

2 The GRWm and GRWf Theories

GRWm was essentially proposed by Ghirardi et al. [13] and Goldstein [29], and taken up in [6, 2, 34, 16, 42, 3, 7, 8]. GRWf was proposed by Bell in [9] and taken up in [12, 32, 29, 40, 2, 34, 16, 41, 42, 3, 45]. For a detailed discussion of these two choices of PO see [3]. Both GRWm and GRWf are non-relativistic theories. The relativistic GRWf theory proposed in [40] has a more complex mathematical structure than GRWf and is not covered by the considerations in this paper. A third type of PO was proposed for collapse theories on lattices by Dowker et al. [19, 20, 21], which will not be considered here.

2.1 The GRW Jump Process in Hilbert Space

In both GRWm and GRWf the evolution of the wave function follows, instead of the Schrödinger equation, a stochastic jump process in Hilbert space, called the GRW process. We shall summarize this process as follows.
Consider a quantum system of (what would normally be called) \(N\) “particles,” described by a wave function \(\Psi = \Psi(x_1, \ldots, x_N), x_i \in \mathbb{R}^3, i = 1, \ldots, N\). For any point \(x\) in \(\mathbb{R}^3\), define on the Hilbert space of the system the collapse rate operator

\[
\Lambda_i(x) = \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-\frac{(Q_i - x)^2}{2\sigma^2}},
\]

where \(Q_i\) is the position operator of “particle” \(i\). Here \(\sigma\) is a new constant of nature of order \(10^{-7}\) m.

Let \(\Psi_{t_0}\) be the initial wave function, i.e., the normalized wave function at some time \(t_0\) arbitrarily chosen as initial time. Then \(\Psi\) evolves in the following way:

1. It evolves unitarily, according to Schrödinger’s equation, until a random time \(T_1 = t_0 + \Delta T_1\), so that

\[
\Psi_{T_1} = U_{\Delta T_1} \Psi_{t_0},
\]

where \(U_i\) is the unitary operator \(U_i = e^{-\frac{i}{\hbar}Ht}\) corresponding to the standard Hamiltonian \(H\) governing the system, e.g., given, for \(N\) spinless particles, by

\[
H = -\sum_{k=1}^{N} \frac{\hbar^2}{2m_k} \nabla_k^2 + V,
\]

where \(m_k, k = 1, \ldots, N\), are the masses of the particles, and \(V\) is the potential energy function of the system. \(\Delta T_1\) is a random time distributed according to the exponential distribution with rate \(N\lambda\) (where the quantity \(\lambda\) is another constant of nature of the theory\(^6\) of order \(10^{-15}\) s\(^{-1}\)).

2. At time \(T_1\) it undergoes an instantaneous collapse with random center \(X_1\) and random label \(I_1\) according to

\[
\Psi_{T_1} \mapsto \Psi_{T_1+} = \frac{\Lambda_{I_1}(X_1)^{1/2} \Psi_{T_1}}{\|\Lambda_{I_1}(X_1)^{1/2} \Psi_{T_1}\|}.
\]

\(I_1\) is chosen at random in the set \(\{1, \ldots, N\}\) with uniform distribution. The center of the collapse \(X_1\) is chosen randomly with probability distribution

\[
P(X_1 \in dx_1 | \Psi_{T_1}, I_1 = i_1) = \langle \Psi_{T_1} | \Lambda_{i_1}(x_1) | \Psi_{T_1} \rangle dx_1 = \|\Lambda_{i_1}(x_1)^{1/2} \Psi_{T_1}\|^2 dx_1.
\]

3. Then the algorithm is iterated: \(\Psi_{T_1+}\) evolves unitarily until a random time \(T_2 = T_1 + \Delta T_2\), where \(\Delta T_2\) is a random time (independent of \(\Delta T_1\)) distributed according to the exponential distribution with rate \(N\lambda\), and so on.

\(^6\)Pearle and Squires \cite{PeS76} have argued that \(\lambda\) should be chosen differently for every “particle,” with \(\lambda_i\) proportional to the mass \(m_i\).
Thus, if, between time $t_0$ and any time $t > t_0$, $n$ collapses have occurred at the times $t_0 < T_1 < T_2 < \ldots < T_n < t$, with centers $X_1, \ldots, X_n$ and labels $I_1, \ldots, I_n$, the wave function at time $t$ will be

$$
\Psi_t = \frac{L_{[t_0,t]}(F_n) \Psi_{t_0}}{\|L_{[t_0,t]}(F_n) \Psi_{t_0}\|}
$$

(9)

where $F_n = \{(X_1, T_1, I_1), \ldots, (X_n, T_n, I_n)\}$, and

$$
L_{[t_0,t]}(F_n) = \lambda^{n/2} e^{-N\lambda(t-t_0)/2} \times
$$

$$
\times U_{t-T_n} \Lambda_{I_{n}}(X_n)^{1/2} U_{T_{n-1}-T_{n-1}} \Lambda_{I_{n-1}}(X_{n-1})^{1/2} U_{T_{n-2}-T_{n-2}} \cdots \Lambda_{I_1}(X_1)^{1/2} U_{T_{1}-t_0}.
$$

(10)

(The scalar factor in the first line will be convenient for future use.) Since $T_i$, $X_i$, $I_i$ and $n$ are random, $\Psi_t$ is also random. We will also call $\Psi_t$ the collapsed wave function, particularly when in need to contrast it with the "uncollapsed" wave function $U_{t-t_0} \Psi_{t_0}$.

It should be observed that—unless $t_0$ is the initial time of the universe—also $\Psi_{t_0}$ should be regarded as random, being determined by the collapses that occurred at times earlier than $t_0$. However, given $\Psi_{t_0}$, the statistics of the future evolution of the wave function is completely determined; for example, the joint distribution of the first $n$ collapses after $t_0$, with particle labels $I_1, \ldots, I_n \in \{1, \ldots, N\}$, is

$$
P(X_1 \in dx_1, T_1 \in dt_1, I_1 = i_1, \ldots, X_n \in dx_n, T_n \in dt_n, I_n = i_n | \Psi_{t_0}) = \frac{\|L(f_n) \Psi_{t_0}\|^2 dx_1 dt_1 \cdots dx_n dt_n}{\|L(f_n) \Psi_{t_0}\|^2 dx_1 dt_1 \cdots dx_n dt_n},
$$

(11)

with $f_n = \{(x_1, t_1, i_1), \ldots, (x_n, t_n, i_n)\}$ and

$$
L(f_n) = \lambda^{n/2} e^{-N\lambda(t-t_0)/2} \times
$$

$$
\times \Lambda_{I_{n}}(x_n)^{1/2} U_{t_n-t_{n-1}} \Lambda_{I_{n-1}}(x_{n-1})^{1/2} U_{t_{n-1}-t_{n-2}} \cdots \Lambda_{I_1}(x_1)^{1/2} U_{t_1-t_0}.
$$

(12)

(The symbol $1_C$ is 1 if the condition $C$ is satisfied and 0 otherwise.) The expression (12) is the same as $L_{[t_0,t_0]}(f_n)$ defined in (10) (and is explicitly set to zero if the $t_k$ are not ordered increasingly).

We have described the law for the evolution of the wave function. We now turn to the primitive ontology (PO). In the subsections below we present two versions of the GRW theory, based on different choices of the PO, namely the matter density ontology (in Section 2.2) and the flash ontology (in Section 2.3).

### 2.2 GRWm

In GRWm, the PO is given by a field: We have a variable $m(x, t)$ for every point $x \in \mathbb{R}^3$ in space and every time $t \geq t_0$, defined by

$$
m(x, t) = \sum_{i=1}^{N} m_i \int_{\mathbb{R}^3} dx_1 \cdots dx_N \delta(x_i - x) \left| \Psi_t(x_1, \ldots, x_N) \right|^2.
$$

(13)
In words, one starts with the $|\Psi|^2$-distribution in configuration space $\mathbb{R}^{3N}$, then obtains the marginal distribution of the $i$-th degree of freedom $x_i \in \mathbb{R}^3$ by integrating out all other variables $x_j$, $j \neq i$, multiplies by the mass associated with $x_i$, and sums over $i$. Alternatively, (13) can be rewritten as

$$m(x, t) = \langle \Psi_t | \bar{\Lambda}(x) | \Psi_t \rangle$$

(14)

with $\bar{\Lambda}(x) = \sum_i m_i \delta(\hat{Q}_i - x)$.

The field $m(\cdot, t)$ is supposed to be understood as the density of matter in space at time $t$. GRWm is a theory about the behavior of matter with density $m(\cdot, t)$ in three-dimensional space.

2.3 GRWf

According to GRWf, the PO is given by “events” in space-time called flashes, mathematically described by points in space-time. In GRWf, histories of matter are not made of world lines but of world points. The flashes form the set

$$F = \{(X_1, T_1), \ldots, (X_k, T_k), \ldots\}$$

(with $T_1 < T_2 < \ldots$), or, when we consider labeled flashes,

$$F = \{(X_1, T_1, I_1), \ldots, (X_k, T_k, I_k), \ldots\}$$

with $I_k \in \mathcal{L} = \{1, \ldots, N\}$, the set of labels. The GRWf law of the flashes asserts that there is a flash at the center $(X, T)$ of every collapse, with the appropriate label. Accordingly, equation (11) gives the joint distribution of the first $n$ flashes, after some initial time $t_0$.

Note that if the number $N$ of the degrees of freedom in the wave function is large, as in the case of a macroscopic object, the number of flashes is also large (if $\lambda = 10^{-15}$ s$^{-1}$ and $N = 10^{23}$, we obtain a rate of $10^8$ flashes per second). Therefore, for a reasonable choice of the parameters of the GRWf theory, a cubic centimeter of solid matter contains more than $10^8$ flashes per second. Such large collections of flashes can form macroscopic shapes, such as tables and chairs. That is how we find an image of our world in GRWf.

We should add that the mathematical scheme of GRWf that we have introduced here is not the most general one possible. The flash rate operators $\Lambda(x)$ do not have to be of the form (14) but could be other positive operators [41], they could depend on time, $\Lambda(x) = \Lambda_t(x)$, and they could even be allowed to depend on the previous flashes [45]. (The latter case occurs in the relativistic GRWf theory presented in [40].) The considerations in this paper are still valid if the $\Lambda(x)$ are other positive operators than in (14) and if they depend on time, but we do not consider the case in which they depend on the previous flashes. For the sake of concreteness readers can simply take $\Lambda(x)$ to be the multiplication operators (4).
2.4 Empirical Equivalence

As already remarked, it is known that GRWf and GRWm are empirically equivalent, i.e., they make always and exactly the same predictions [3]. In other words, there is no conceivable experiment (including future advances in technology) that could distinguish between GRWf and GRWm. This follows from the following even stronger statement: When applying the flash ontology and the matter density ontology to the same wave function $\Psi$ obtained from the GRW process, the two PO histories are macro-history equivalent, i.e., all macroscopic facts come out the same way.

Let us elaborate on this statement. What we mean is to consider a realization of the GRW jump process in Hilbert space as described in Section 2.1 (that is, $\Psi_t$ for every $t$), and then both the GRWm world and the GRWf world associated with this $\Psi$, defined by $m(x,t)$ as in (13) for every $t$, respectively by putting a flash at the center of every collapse of $\Psi$. What we mean by macro-history equivalence is that the macroscopic world history is the same in both worlds, including, e.g., the weather in a particular place at a particular time, lottery numbers, and more generally the exact sequence of outcomes of any experiment. This is more than empirical equivalence, as the latter requires not that all random events come out the same way in two worlds, but only that the outcome statistics are the same, or, put differently, that one cannot conclude from the two macroscopic histories which one is governed by which of two theories. Clearly, macro-history equivalence implies empirical equivalence.

For GRWf and GRWm, macro-history equivalence holds with overwhelming probability. That is, although there do exist wave functions $\Psi$ for which the macroscopic facts in the GRWf world are different from those in the GRWm world, such wave functions are extremely improbable to come out of the GRW process.

Here is the argument. It suffices to consider a macroscopic amount of matter, which we call the “pointer” (though it could also be, e.g., the shape of ink on paper), that can either be in position 1 or position 2 at time $t$, and a wave function of the form $\Psi_t = c_1 \Phi_1 + c_2 \Phi_2$, where $\Phi_i$ is concentrated on configurations in which the pointer is in position $i$; we assume $\|\Phi_i\| = 1$ and $|c_1|^2 + |c_2|^2 = 1$. If, in GRWm, the matter of the pointer is in position 1, then this means that $m(1,t) \gg m(2,t)$; thus, $|c_1|^2 \gg |c_2|^2$; thus, it is much more likely that flashes occur at position 1 than at position 2; thus, with probability near 1, in GRWf the matter is also in position 1.

To appreciate how close to 1 this probability is, recall that, as a consequence of the GRW process for $\Psi_t$, it is overwhelmingly likely that either $|c_1|^2$ or $|c_2|^2$ will become exorbitantly small within a fraction of a second (see 8 for concrete numbers in a realistic scenario). Note also that, in the unlikely event that many flashes occur in position 2 between $t$ and $t + \Delta t$ and thus create a discrepancy between the pointer position in GRWf and that in GRWm, the associated collapses would shrink the size of $c_1$ to a considerable extent; so much indeed, if the number of flashes in position 2 is sufficient, that $|c_1(t + \Delta t)|^2$ is close to zero and $|c_2(t + \Delta t)|^2$ close to 1; as a consequence, $m(1,t + \Delta t) \ll m(2,t+\Delta t)$. That is, even in the unlikely event of a discrepancy, it cannot persist longer than the time it takes the collapses centered at position 2 to make $|c_1(t + \Delta t)|^2$ small.
2.5 Systems

Since we have not specified, in the definition of the GRW theories, which kinds of systems the defining equations, such as (9) through (12), apply to, they a priori apply only to the universe as a whole. For any system, being a subsystem of the universe, equations of the same kind may or may not apply, but there is no need, and indeed no room, for postulates about this because the equations for the universe will determine what is true about any subsystem. Hence, the wave function $\Psi$ we were talking of is the wave function of the universe. However, in our analysis of the empirical predictions of GRWm and GRWf, we will have to consider systems: the system corresponding to those instruments which comprise the apparatus for the experiment and, most importantly, the system upon which the experiment is performed. For this, it will be helpful to formalize the notion of system, as well as that of the wave function of a system.

To begin to approach such a notion, note that usually a system corresponds to some of the “configuration variables” in the wave function,

$$\Psi = \Psi(q) = \Psi(q_{\text{sys}}, q_{\text{env}})$$

where $q = (x_1, \ldots, x_N)$ is the configuration variable of the universe, $q_{\text{sys}}$ that of the system, and $q_{\text{env}}$ that of its environment (the rest of the world); defining a system amounts to splitting the universe in two parts, the system and its environment. For example, $q_{\text{sys}}$ may correspond to a certain collection of “particle variables”, say

$$q_{\text{sys}} = (x_1, \ldots, x_M) \quad \text{and} \quad q_{\text{env}} = (x_{M+1}, \ldots, x_N).$$

Since for the GRW theories, the configuration variables do not play a fundamental role, our mathematical definition of “system” is formulated in different terms, namely in terms of the Hilbert space and of the primitive ontology.

For our purposes, a system is defined by two ingredients:

- A splitting of Hilbert space according to

$$\mathcal{H} = \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{env}}.$$ 

For example, such a splitting is provided by (15) according to $\mathcal{H}_{\text{sys}} = L^2(q_{\text{sys}})$, $\mathcal{H}_{\text{env}} = L^2(q_{\text{env}})$, and $\mathcal{H} = L^2(q)$.

- A splitting of the PO; this means, in GRWf, a splitting of the flashes according to

$$F = F_{\text{sys}} \cup F_{\text{env}}, \quad F_{\text{sys}} \cap F_{\text{env}} = \emptyset,$$

or, in GRWm, a splitting of the matter density according to

$$m(x, t) = m_{\text{sys}}(x, t) + m_{\text{env}}(x, t).$$

In each of the two theories, we assume that the splitting is defined either through a subset $\mathcal{L}_{\text{sys}} \subseteq \mathcal{L}$ of the set of labels (corresponding to different types of
flashes/collapses), or through a region \( R_{sys} \subseteq \mathbb{R}^3 \) in space, or a combination of both: In GRWf, a flash belongs to \( F_{sys} \) if and only if it occurs in \( R_{sys} \) and its label belongs to \( \mathcal{L}_{sys} \); \( F_{env} := F \setminus F_{sys} \). In GRWm, \( m_{sys} \) is the contribution to \( m(x,t) \) from labels in \( \mathcal{L}_{sys} \) at locations in \( R_{sys} \):

\[
m_{sys}(x,t) = \sum_{i \in \mathcal{L}_{sys}} m_i \int_{\mathbb{R}^3} dx_1 \cdots dx_N \delta(x_i - x) |\Psi_t(x_1, \ldots, x_N)|^2,
\]

and \( m_{env} = m - m_{sys} \). The set \( F_{sys} \subseteq F \) of the system’s flashes may happen to be empty, but even in that case the definition of the system in terms of \( \mathcal{L}_{sys} \) and \( R_{sys} \) will be useful. In the example of (16), \( \mathcal{L}_{sys} = \{1, \ldots, M\} \), while \( R_{sys} \) does not play a role.

The example provided by (15) suggests that everything that could be considered a system in orthodox quantum mechanics also defines a system in the sense of our definition.

We say that the system has wave function \( \psi_{sys} \) if the wave function of the universe factorizes according to

\[
\Psi = \psi_{sys} \otimes \psi_{env}
\]

with \( \psi_{sys} \in \mathcal{H}_{sys} \) and \( \psi_{env} \in \mathcal{H}_{env} \). Since it follows that not every system has a wave function at every time, it will also be useful to say that the system has reduced density matrix \( \rho_{sys} \) if

\[
\rho_{sys} = \text{tr}_{env} |\Psi\rangle\langle\Psi|
\]

with \( \text{tr}_{env} \) the partial trace over \( \mathcal{H}_{env} \).

We call a system a GRW system if it has an autonomous GRW dynamics, i.e., if it behaves as if it were alone in the universe. We postpone the exact definition of what that means to Section 7.1.2; there we will also show that a system is a GRW system if and only if it does not interact with its environment.

### 3 Mathematical Tools

#### 3.1 POVM

Recall that, while many quantum experiments are associated with self-adjoint operators, this is not the most general case, which corresponds to positive-operator-valued measures (POVMs, also known as “generalized observables”; see [17] and Section 4 of [25] for an introduction). Leaving out some technical details of the mathematical definition, we recall that a POVM on the set \( \Omega \) acting on \( \mathcal{H} \) is a mapping

\[
E : \mathcal{A} \to \mathcal{L}(\mathcal{H})
\]

from a \( \sigma \)-algebra \( \mathcal{A} \) over \( \Omega \) (the family of all subsets of \( \Omega \) regarded as “measurable”) to the space of bounded operators on the Hilbert space \( \mathcal{H} \), with the properties that
(i) $E(B)$ is a positive self-adjoint operator for every $B \in \mathcal{A}$, (ii) $E(\Omega) = I$, the identity operator, and (iii) $E(\cdot)$ is $\sigma$-additive, i.e., for pairwise disjoint $B_1, B_2, \ldots \in \mathcal{A}$

$$E\left(\bigcup_{k=1}^{\infty} B_k\right) = \sum_{k=1}^{\infty} E(B_k). \quad (24)$$

(All subsets and functions we consider will be assumed to be measurable with respect to the relevant $\sigma$-algebras. A positive operator $S$ with $S \leq I$ is also called an \textit{effect} in the literature \cite{33}, and a POVM also an \textit{effect-valued measure}.) By virtue of the spectral theorem, the self-adjoint operators correspond to special POVMs, the projection-valued measures (PVMs) on the real axis. In many cases relevant to us, $\Omega$ will be a finite or countable set; in that case, the POVM is determined by the operators associated with singleton sets, $E_\omega = E(\{\omega\})$, according to

$$E(B) = \sum_{\omega \in B} E_\omega, \quad (25)$$

and any collection of positive operators $(E_\omega)_{\omega \in \Omega}$ such that

$$\sum_{\omega \in \Omega} E_\omega = I \quad (26)$$

defines a POVM. We will thus often identify the POVM with the collection $(E_\omega)_{\omega \in \Omega}$.

The following two very simple observations about POVMs will be used in the course of this paper:

\textbf{Function Property.} If the distribution of the random variable $X$ depends on a system’s wave function $\psi$ via a POVM $D(\cdot)$, $\mathbb{P}(X \in A) = \langle \psi | D(A) | \psi \rangle$, and if the random variable $Y$ is a function of $X$, $Y = f(X)$, then the distribution of $Y$ is also given by a POVM:

$$\mathbb{P}(Y \in B) = \langle \psi | E(B) | \psi \rangle \quad \text{with} \quad E(B) = D\left(f^{-1}(B)\right). \quad (27)$$

\textbf{Reduction Property.} If $D(\cdot)$ is a POVM on $\Omega$ acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$, and if $\phi \in \mathcal{H}_2$ has $\|\phi\| = 1$, then the partial scalar product

$$E(B) = \langle \phi | D(B) | \phi \rangle \quad (28)$$

defines a POVM $E(\cdot)$ on $\Omega$ acting on $\mathcal{H}_1$.

\section{3.2 The Distribution of the Flashes}

In GRWf, the joint distribution of all flashes, as a functional of the initial wave function $\Psi_{t_0}$, is given by a POVM $G(\cdot)$, called the \textit{history POVM}. Let us elaborate on this statement.
Reformulating (11), the joint distribution of the first \( n \) flashes is given by a POVM \( G_n(\cdot) \) on \( \Omega_n = (\mathbb{R}^3 \times \mathbb{R} \times \mathcal{L})^n \) (where \( \mathbb{R}^3 \) represents space, \( \mathbb{R} \) time, and \( \mathcal{L} \) is the set of labels),

\[
\mathbb{P}(F_n \in df_n) = \langle \Psi_t | G_n(df_n) | \Psi_t \rangle
\tag{29}
\]

with \( df_n = dx_1 dt_1 \cdots dx_n dt_n \) a “volume element” around \( f_n = \{(x_1, t_1), \ldots, (x_n, t_n, i_n)\} \) in \( \Omega_n \) and

\[
G_n(B) = \int_B df_n L^*(f_n) L(f_n)
\tag{30}
\]

with \( L(f_n) \) defined by (12) (and \( L^* \) denoting the adjoint of \( L \)). It is easy to convince oneself that \( G_n(\cdot) \) is a POVM; see [45] for a rigorous proof.

It is no surprise now that also the joint distribution of all flashes is given by a POVM \( G(\cdot) \); see [44] for a rigorous proof. The space on which \( G(\cdot) \) lives is \( \Omega_{[s,t]} \), where \( \Omega_{[s,t]} \) means the space of all histories of flashes in the time interval \([s,t]\) (possibly \( t = \infty \)), i.e., the set of all discrete (finite or countable) subsets of \( \mathbb{R}^3 \times [s,t] \times \mathcal{L} \).

Finally, consider \( F_{[s,t]} \), the set of all flashes during the time interval \([s,t]\). Since it trivially is a function of \( F \), the set of all flashes, by the function property (27) its distribution is given by a POVM \( G_{[s,t]}(\cdot) \) on \( \Omega_{[s,t]} \). We can specify it explicitly for \( s = t_0 \) and \( t < \infty \):

\[
G_{[t_0,t]}(B) = \int_B df L_{[t_0,t]}^*(f) L_{[t_0,t]}(f)
\tag{31}
\]

with \( df \) the Lebesgue measure, which exists on the space \( \Omega_{[t_0,t]}^\infty \) of finite flash patterns because it exists on each sector

\[
\Omega_{[t_0,t]}^n = \left\{ f \subset \mathbb{R}^3 \times [t_0, t] \times \mathcal{L} : \# f = n \right\}.
\tag{32}
\]

Note that, since \( F_{[t_0,t]} \) is almost surely finite for \( t < \infty \), \( G_{[t_0,t]} \) is concentrated on \( \Omega_{[t_0,t]}^\infty = \bigcup_{n=0}^{\infty} \Omega_{[t_0,t]}^n \).

### 3.3 The Conditional Probability Formula

Set, for the ease of notation, \( t_0 = 0 \). A simple and important consequence of the distribution law (11) of the flashes is the conditional probability formula, which asserts that, for \( 0 < s < t \leq \infty \),

\[
\mathbb{P}_{\Psi_0}(F_{[s,t]} \in B \mid F_{[0,s]}) = \mathbb{P}_{\Psi_s}(F_{[s,t]} \in B).
\tag{33}
\]

Here, \( \mathbb{P}_{\Psi_0} \) means the distribution obtained starting from the wave function \( \Psi_0 \), and \( \mathbb{P}_{\Psi_s} \) the one obtained starting from \( \Psi_s \) at time \( s \). Note that the dependence on \( F_{[0,s]} \) of the right hand side is through \( \Psi_s \), which is a function of \( F_{[0,s]} \). In words, the conditional probability formula asserts that the conditional distribution of the flashes after time \( s \), given the flashes before \( s \), coincides with the distribution obtained from starting the universe at time \( s \) with wave function \( \Psi_s \).

This formula is the ultimate reason why it is natural in GRWf to regard the collapsed (GRW) wave function \( \Psi_s \) as the wave function at time \( s \): because the distribution of the
future flashes after \( s \) (given that the past was what it was) agrees with the distribution arising from \( \Psi_s \) as the initial wave function at time \( s \).

The conditional probability formula follows from the Markov property of the stochastic GRW process \( \Psi_t \), defined by

\[
P_{\Psi_0}(\Psi_t \in B | \Psi_{s'} = \psi_s' \forall s' \in [0, s]) = P_{\Psi_0}(\Psi_t \in B | \Psi_s = \psi_s)
\]

for every \( t > s \). The Markov property means that the process is memoryless. To see how the conditional probability formula follows, note first that the history of the wave function between 0 and \( s \) is determined by (and, conversely, determines) the flashes between 0 and \( s \), so that conditioning on \( \Psi_{s'} = \psi_s' \forall s' \in [0, s] \) amounts to the same thing as conditioning on \( F_{[0, s)} = f_{[0, s)} \). Furthermore, the Markov property implies (by varying \( s \)) that \((34)\) still holds when the event \( \Psi_t \in B \) is replaced with the event \( (\Psi_{t_1}, \ldots, \Psi_{t_k}) \in B' \) for \( s < t_1 < \ldots < t_k \), or in fact any event about the future after \( s \). Again, the future history of the wave function is in one-to-one correspondence with the future flashes, so that

\[
P_{\Psi_0}(F_{[s, t)} \in B | F_{[0, s)}) = P_{\Psi_0}(F_{[s, t)} \in B | \Psi_s),
\]

and the right hand side coincides with the right hand side of \( (33) \). Alternatively, an algebraic-analytic derivation of the conditional probability formula can be found in Appendix A.

## 4 How Operators Emerge

We will formulate and derive the GRW formalism in Section 6. At this stage, we can already understand, in a particularly easy way, how operators emerge from GRWf, and that is why we present this aspect first.

We give a simple derivation for the main theorem about POVMs in GRWf, i.e., for the fact that in GRWf, as in quantum mechanics, there is a POVM \( E(\cdot) \) for every experiment, so that the probability distribution of the outcome of the experiment, when performed on a system with wave function \( \psi \), is given by \( \langle \psi | E(\cdot) | \psi \rangle \). To appreciate the substance of this derivation it is relevant to realize that the definition of GRWf did not mention operators as observables. Thus, operators as observables were not put in, they come out by themselves.

Many physicists find such a situation hard to imagine, and that is why this point deserves a separate section. Many physicists are used to thinking that the central role of operators in quantum theory, particularly in view of their non-commutativity, constitutes a crucial departure from classical physics, and, even more, from any kind of theory describing an objective reality, or any kind of theory that can be understood as clearly as a classical theory. According to this widespread view, the non-commutativity of operators entails that reality itself is paradoxical and will forever remain incomprehensible to us mortals. This view is often connected to the key word “complementarity.” Now the shocking result is that the same non-commuting operators appear in GRWf, a
theory describing an objective reality which can indeed be understood as clearly (and as easily) as a classical theory!

This is not so shockingly new since the same can be said of Bohmian mechanics (see, e.g., \[11, 25\]), and since it has been clear for 20 years that GRW theories make almost the same predictions as quantum mechanics \[28, 9\]. Nonetheless, it is worthwhile to get a good grasp of how exactly this can be so, how non-commuting operators can emerge from a theory describing non-paradoxical reality.

Here is the derivation. Recall from Section 3.2 that the joint distribution of all flashes after time \(t\) is given by a POVM \(G_{(t, \infty)}(\cdot)\) on the appropriate space \(\Omega_{(t, \infty)}\) of flash histories and the wave function of the universe \(\Psi_t\) at time \(t\). Let \(t\) be the time at which the experiment begins. Consider splitting the universe into a system (the object of the experiment), the apparatus of the experiment, and the rest of the world. It so happens that for the argument that follows, the division between apparatus and the rest of the world is irrelevant, so we put the two together and call them the environment (of the system). The division between the system and its environment corresponds to a splitting of the Hilbert space into \(H = H_{\text{sys}} \otimes H_{\text{env}}\); the splitting \(F = F_{\text{sys}} \cup F_{\text{env}}\) of the flashes is not needed in this section. We assume independence between the system and the environment immediately before \(t\), so that \[\Psi_t = \psi \otimes \phi.\] (36)

Here \(\phi\) is fixed, being part of the characterization of the experiment, while \(\psi\), the initial state of the system upon which the experiment is performed, is allowed to vary in the system Hilbert space \(\mathcal{H}_{\text{sys}}\). The outcome \(Z\) of the experiment is a function of the pattern \(F\) of flashes,

\[Z = \zeta(F)\] (37)

with \(\zeta : \Omega_{(t, \infty)} \to \mathcal{Z}\), where \(\mathcal{Z}\) is the value space of the experiment. That is so because the flashes define where the pointers point, and what the shape of the ink on a sheet of paper is. (It would even be realistic to assume that \(Z\) depends only on the flashes of the apparatus, but this restriction is not needed for the further argument.) Therefore, the distribution of the random outcome \(Z\) is given by

\[
\mathbb{P}(Z \in B) = \mathbb{P}(F \in \zeta^{-1}(B)) = \langle \Psi_t | G \circ \zeta^{-1}(B) | \Psi_t \rangle = \langle \psi | E(B) | \psi \rangle \quad \forall B \subseteq \mathcal{Z},
\] (38)

Readers may worry that the factorization condition (36) never holds because of the symmetrization postulate: As soon as both the system and the apparatus contain electrons, the wave function has to be anti-symmetric in the electron variables \(x_i\), so that a splitting as in (16), grouping some variables \(x_i\) together as “system variables” and others as “environment variables,” would conflict with (36). The answer is, (36) can hold nevertheless, as follows: For identical particles, the indices of the variables \(x_1, \ldots, x_N\) are mere mathematical labels, and the splitting into system and environment should not be based on these unphysical labels but instead on regions of space. Indeed, if \(R_{\text{sys}} \subseteq \mathbb{R}^3\) is a region of space such that both \(R_{\text{sys}}\) and \(\mathbb{R}^3 \setminus R_{\text{sys}}\) have positive volume then \(\mathcal{H}(\mathbb{R}^3) = \mathcal{H}(R_{\text{sys}}) \otimes \mathcal{H}(\mathbb{R}^3 \setminus R_{\text{sys}})\), where \(\mathcal{H}(S)\) is the fermionic (or bosonic) Fock space over \(L^2(S)\). Since a fermionic wave function can be represented by a vector \(\Psi \in \mathcal{H}(\mathbb{R}^3)\), it can indeed factorize in the splitting based on \(R_{\text{sys}}\).
where the first scalar product is taken in the Hilbert space of the universe and the second in the Hilbert space of the system (i.e., the object of the experiment), and $E(\cdot)$ is the POVM given by

$$E(B) = \langle \phi | G \circ \zeta^{-1}(B) | \phi \rangle \quad \forall B \subseteq \mathcal{Z},$$

(39)

where the scalar product is a partial scalar product in the Hilbert space of the environment. Thus, for every experiment in GRWf the distribution of outcomes is given by a POVM $E(\cdot)$ on $\mathcal{Z}$, which is what we wanted to show.

At this point, we would like to go through the derivation again, carefully keeping track of the ingredients in the argument:

- The distribution of flashes in GRWf is given by a POVM $G(\cdot)$. In more detail:
  - $G(\cdot)$ is a POVM on the total Hilbert space $\mathcal{H} = \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{env}}$, where $\mathcal{H}_{\text{sys}}$ is the Hilbert space of the system and $\mathcal{H}_{\text{env}}$ that of its environment, including all apparatus.
  - What we really want is, of course, the conditional distribution of the flashes, given what happened up to the time $t$ when the experiment begins. By the conditional probability formula (33), this distribution is $\langle \Psi_t | G_{[t,\infty)}(\cdot) | \Psi_t \rangle$ with $\Psi_t$ the (collapsed) wave function at time $t$.

- The outcome $Z$ of an experiment in a GRWf world must be a function of the flashes (usually, just of the flashes belonging to some apparatus), $Z = \zeta(F)$.

- By the function property (27) of POVMs, the distribution of the outcome is also given by a POVM on $\mathcal{H}$.

- Consider a particular setting of the experiment, as encoded in $\phi \in \mathcal{H}_{\text{env}}$; ask for the dependence of the distribution of the outcome $Z$ on the wave function $\psi \in \mathcal{H}_{\text{sys}}$ of the object. In particular, assume factorization, $\Psi_t = \psi \otimes \phi$.

- By the reduction property (28) of POVMs, the distribution of $Z$ as a function of $\psi$ is given by a POVM $E(\cdot)$ on $\mathcal{H}_{\text{sys}}$.

We close this section with a few remarks.

- The POVMs corresponding to different experiments may well, and typically will, not commute. Even the single POVM $E(\cdot)$ may be non-commuting, in the sense that $E(B_1)$ does not commute with $E(B_2)$ for suitable sets $B_1, B_2 \subseteq \mathcal{Z}$. The simple derivation above, just a few lines long, shows how non-commuting operators can emerge from a picture of reality (a random set of flashes) that is completely coherent, clear, easy-to-understand, complementarity-free, and paradox-free. Why do different experiments correspond to different POVMs? Because they correspond
to different choices of the interaction Hamiltonian between the system and the apparatus, as well as different choices of $\phi$.

- Since we know that the predictions of GRWf and GRWm are very close to those of quantum mechanics for all presently feasible experiments, for these experiments the POVM $E(\cdot) = E^{\text{GRW}}(\cdot)$ should be very close to $E^{\text{Qu}}(\cdot)$, the POVM predicted by quantum mechanics. For a principled consideration see Section 6.4.

- We called the result of our reasoning the “main theorem about POVMs” in GRWf. Let us be explicit about the mathematical theorem that is involved here. It was formulated before as Theorem 8 in [45] and asserts the following:

Let $H = H_{\text{sys}} \otimes H_{\text{env}}$ be a separable Hilbert space, $G(\cdot)$ a POVM on a measurable space $(\Omega, \mathcal{A}_\Omega)$ acting on $H_{\text{env}}$ with $\|\phi\| = 1$, and $\zeta : (\Omega, \mathcal{A}_\Omega) \to (\mathcal{Z}, \mathcal{A}_Z)$ a measurable function. For every $\psi \in H_{\text{sys}}$ with $\|\psi\| = 1$, let $\Psi_t = \psi \otimes \phi$, let $F$ be a random variable in $\Omega$ with distribution $\langle \Psi_t | G(\cdot) | \Psi_t \rangle$, and let $Z = \zeta(F)$. Then there is a POVM $E(\cdot)$ on $(\mathcal{Z}, \mathcal{A}_Z)$ acting on $H_{\text{sys}}$ so that the distribution of $Z$ is $\langle \psi | E(\cdot) | \psi \rangle$.

The proof of this theorem is a straightforward application of the function property (27) and the reduction property (28) of POVMs. What is important for us here is to appreciate the relevance of this theorem as the appropriate mathematical formalization in GRWf of the physical statement of the main theorem about POVMs, viz. of the statement that with every experiment $\mathcal{E}$, there is associated a POVM $E(\cdot)$ such that the probability distribution of the random outcome $Z$ of $\mathcal{E}$, when performed on a system with wave function $\psi$, is given by $P(Z \in B) = \langle \psi | E(B) | \psi \rangle$.

- If the wave function $\phi$ of the environment were not fixed but random, we would still end up with a POVM, as long as $\phi$ is independent of $\psi$ (at least conditionally on all information available to us about the experimental setup): we would have to replace (39) by

$$E(B) = \int \mu(d\phi) \langle \phi | G \circ \zeta^{-1}(B) | \phi \rangle,$$

with $\mu$ the distribution of $\phi$.

- The reader may find it confusing that part of the characterization of the experiment was the specification of $\phi$, the wave function of the system’s environment: After all, it will be practically impossible to repeat the experiment with the same $\phi$, as $\phi$ comprises everything outside the system; for example, when we try to repeat the experiment at a later time, the moons of Jupiter will have moved, and the state of the lab will have changed as it will contain records of the previous experiment. So for practical purposes it is important that $E(\cdot)$ as given by (39) does not depend on all details of $\phi$, but only on a few features of $\phi$ that we can control—and thus repeat. Mathematically, however, (39) provides the correct POVM, and (38) the

\[8\]

[8] From the point of view of the entire universe, from which the Hamiltonian may be regarded as fixed once and for all, the relevant choice would lie only in that of $\phi$. 

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correct distribution, no matter whether we are able to evaluate or control this expression.

• Note that the derivation did not assume any pre-determined time at which the experiment is over. It allows that the time at which the outcome $Z$ can be read off depends on $Z$ itself, a situation that occurs, e.g., in a time-of-arrival measurement, with $Z$ the time when a detector clicks.

• What if factorization $\Psi_t = \psi \otimes \phi$ is not exactly satisfied, but only approximately? Then the probability distribution of the outcome $Z$ is still approximately given by $\langle \psi | E(B) | \psi \rangle$.

Then for any $B \subseteq \mathcal{Z}$,

$$\left| \mathbb{P}(Z \in B) - \langle \psi | E(B) | \psi \rangle \right| < 3 \| \Delta \Psi \| .$$

This estimate conveys that the relevant measure for quantifying the size of the deviation from perfect factorization is the $L^2$ norm of the deviation $\Delta \Psi$.

• We do not know of a similar derivation of the main theorem about POVMs from GRWm, mainly because the probability distribution of the random function $m(\cdot, t)$ is not given by a POVM. Nevertheless a derivation from GRWm has been given in [7], however one that is rather different in character: It requires great effort and yields a limited result, as it assumes a special, idealized type of experiment and, since it allows for small errors in the outcome statistics, does not show that the outcome statistics is exactly given by a POVM.

5 The Quantum Formalism

Before we formulate the GRW formalism, we formulate for comparison the standard quantum formalism in the way relevant to us. We begin with the simplified version that one learns in beginner’s courses and that suffices for many applications.

The Simplified Quantum Formalism.

• A system isolated from its environment has at every time $t$ a density matrix $\rho_t$ which evolves according to the unitary Schrödinger evolution,

$$\frac{d\rho_t}{dt} = -\frac{i}{\hbar} [H_{\text{sys}}, \rho_t].$$

---

9To see this, write $\mathbb{P}(Z \in B)$ as $\langle \Psi_t | G \circ \zeta^{-1}(B) | \Psi_t \rangle$; insert (41); use $0 \leq G \circ \zeta^{-1}(B) \leq I$ to bound the term quadratic in $\Delta \Psi$ by $\| \Delta \Psi \|^2$; use the Cauchy–Schwarz inequality and $|c| < 1$ to bound the cross terms by $2\| \Delta \Psi \| \| \Delta \Psi \|$; use that $1 - |c|^2 = \| \Delta \Psi \|^2$; in total, by the triangle inequality, obtain the bound $2\| \Delta \Psi \|/(1 + \| \Delta \Psi \|) < 3\| \Delta \Psi \|$ provided $\| \Delta \Psi \| < 1/2$.  

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• With suitable experiments $E$ there is associated a self-adjoint operator $A$ on $\mathcal{H}_{sys}$ (called the “observable”) with pure point spectrum; let its spectral decomposition be

$$A = \sum_z zP_z,$$  \hspace{1cm} (44)

with $P_z$ the projection to the eigenspace with eigenvalue $z$. When the experiment $E$ is performed on a system with density matrix $\rho$, the outcome $Z$ is random with probability distribution

$$\mathbb{P}(Z = z) = \text{tr}(P_z \rho).$$  \hspace{1cm} (45)

• In case $Z = z$, the density matrix immediately after the experiment is

$$\rho' = \frac{P_z \rho P_z}{\text{tr}(P_z \rho)}.$$  \hspace{1cm} (46)

The last rule contains the standard kind of collapse of the wave function, induced by “the observer.”

We will need a more general formulation since the above formalism applies only to a narrow class of experiments, usually called “ideal measurements.” And for this we will need some more mathematical notions.

5.1 Mathematical Tool: Completely Positive Superoperators

We recall that the trace class $\text{TRCL}(\mathcal{H})$ is (roughly speaking) the space of all operators with finite trace. It contains in particular the density matrices.

By a superoperator $\mathcal{C}$ we mean a $\mathbb{C}$-linear mapping $\mathcal{C} : \text{TRCL}(\mathcal{H}_1) \rightarrow \text{TRCL}(\mathcal{H}_2)$. A superoperator $\mathcal{C}$ is called completely positive if for every integer $k \geq 1$ and every positive operator $\rho \in \mathbb{C}^{k \times k} \otimes \text{TRCL}(\mathcal{H}_1)$, $(I_k \otimes \mathcal{C})(\rho)$ is positive, where $I_k$ denotes the identity operator on $\mathbb{C}^{k \times k}$ [15, 33]. (Completely positive superoperators are also often called completely positive maps. If for every density matrix $\rho$, $\text{tr}\mathcal{C}(\rho) \leq 1$ (as will be the case for all superoperators that we consider in this paper) then $\mathcal{C}$ is also called a quantum operation [33].)

Completely positive superoperators arise as a description of how a density matrix changes under the collapse caused by an experiment: If $\rho$ is the density matrix before the collapse, then $\mathcal{C}(\rho)/\text{tr}\mathcal{C}(\rho)$ is the density matrix afterwards. The simplest example of a completely positive superoperator is

$$\mathcal{C}(\rho) = P \rho P,$$  \hspace{1cm} (47)

where $P$ is a projection. Note that for a density matrix $\rho$, $\mathcal{C}(\rho)$ is not, in general, a density matrix because completely positive superoperators do not, in general, preserve the trace.

In order to prove the complete positivity of a given superoperator, the following facts are useful: If $\rho_2$ is a density matrix on $\mathcal{H}_2$ then the mapping $\mathcal{C} : \text{TRCL}(\mathcal{H}_1) \rightarrow$
\( TRCL(\mathcal{H}_1 \otimes \mathcal{H}_2) \) given by \( C(\rho) = \rho \otimes \rho_2 \) is completely positive. Conversely, the partial trace \( \rho \mapsto \text{tr}_2 \rho \) is a completely positive superoperator \( TRCL(\mathcal{H}_1 \otimes \mathcal{H}_2) \rightarrow TRCL(\mathcal{H}_1) \). For any bounded operator \( R : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \), \( \rho \mapsto R\rho R^* \) is a completely positive superoperator \( TRCL(\mathcal{H}_1) \rightarrow TRCL(\mathcal{H}_2) \), where \( R^* : \mathcal{H}_2 \rightarrow \mathcal{H}_1 \) is the adjoint of \( R \). The composition of completely positive superoperators is completely positive. Positive multiples of a completely positive superoperator are completely positive. Finally, when a family of completely positive superoperators is summed or integrated over, the result is completely positive. Indeed, these rules suffice for all cases we will encounter in this paper.

For example, the master equation (1) of the GRW evolution has the property that the solution \( \rho_t \) as a function of the initial datum \( \rho_0 \) is given by a completely positive superoperator \( S_{[0,t]} \), \( \rho_t = S_{[0,t]} \rho_0 \) (indeed, \( S_{[0,t]} \) is trace-preserving). Here is a simple path to that conclusion: Since composition preserves complete positivity, it suffices to convince oneself that \( \rho_{t+dt} \) as a function of \( \rho_t \) is given by a completely positive superoperator; using the rules in the previous paragraph (in a non-rigorous way), this is the case for both the unitary contribution \( \exp(-iH dt/\hbar)\rho_t \exp(iH dt/\hbar) \) and the non-unitary terms in (1).

A canonical form of completely positive superoperators is provided by the theorem of Choi and Kraus [15, 33] (also sometimes connected with the name of Stinespring), which asserts that for every bounded completely positive superoperator \( C : TRCL(\mathcal{H}_1) \rightarrow TRCL(\mathcal{H}_2) \) there exist bounded operators \( R_i : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) so that
\[
C(\rho) = \sum_{i \in \mathcal{I}} R_i \rho R_i^*,
\]
(48)
where \( \mathcal{I} \) is a finite or countable index set.

5.2 The Formalism

We are now prepared for formulating the quantum formalism in greater generality. Without an essential loss of generality, we only consider experiments with discrete value space \( \mathcal{Z} \), i.e., experiments for which the set \( \mathcal{Z} \) of possible outcomes is finite or countable. The reason why this is essentially no restriction is that every experiment in practice has limited accuracy, and thus indeed only a finite number of possible outcomes. Nevertheless it is sometimes convenient to consider a continuous variable \( z \), and indeed, as far as the main theorem about POVMs, or (49), is concerned, we can allow \( \mathcal{Z} \) to be any measurable space (i.e., set with a \( \sigma \)-algebra), including the possibility of a continuous variable \( z \). However, when trying to formulate the collapse rule (51) for a continuous variable \( z \), difficulties arise that lie outside the scope of this paper.

The Quantum Formalism.

- A system isolated from its environment has at every time \( t \) a density matrix \( \rho_t \) which evolves according to the unitary Schrödinger evolution (13).
• With every experiment $\mathcal{E}$ with discrete value space $\mathcal{Z}$, beginning at time $s$ and ending at time $t$, there is associated a POVM $(E^{\text{Qu}}_z)_{z \in \mathcal{Z}}$ on $\mathcal{Z}$ acting on $\mathcal{H}_{\text{sys}}$. When the experiment $\mathcal{E}$ is performed on a system with density matrix $\rho_s$, the outcome $Z$ is random with probability distribution

$$P(Z = z) = \text{tr}(\rho_s E^{\text{Qu}}_z).$$

(49)

• With $\mathcal{E}$ is further associated a family $(C^{\text{Qu}}_z)_{z \in \mathcal{Z}}$ of completely positive superoperators acting on $\text{TRCL}(\mathcal{H}_{\text{sys}})$ with the consistency property that, for all trace class operators $\rho$,

$$\text{tr}(\rho E^{\text{Qu}}_z) = \text{tr}C^{\text{Qu}}_z(\rho).$$

(50)

In case $Z = z$, the density matrix of the system at time $t$ (immediately after the experiment) is

$$\rho_t = \rho' = \frac{C^{\text{Qu}}_z(\rho_s)}{\text{tr}C^{\text{Qu}}_z(\rho_s)}.$$

(51)

Since readers may not be familiar with this formulation of the quantum formalism, we elucidate it a bit in the following subsections. We begin with a remark.

The assumption that the experiment is over at a fixed time $t$ is not in all practical cases satisfied, for example when the experiment measures the time at which a detector clicks. To keep this discussion simple, we postpone the discussion of experiments whose duration is random (i.e., decided upon by the experiment itself) to Section 8.

5.3 First Example

To begin with, the simplified quantum formalism is contained in the full quantum formalism in the following way: Let $\mathcal{Z}$ be the spectrum of the self-adjoint operator $A$ (a finite or countable set since we assume pure point spectrum), $E^{\text{Qu}}(\cdot)$ the spectral PVM of $A$,

$$E^{\text{Qu}}_z = P_z,$$

(52)

and

$$C^{\text{Qu}}_z(\rho) = P_z \rho P_z,$$

(53)

for every operator $\rho$ in the trace class. Then, the consistency property (50) is satisfied since

$$\text{tr}(\rho E^{\text{Qu}}_z) = \text{tr}(\rho P_z) = \text{tr}(P_z \rho P_z) = \text{tr}C^{\text{Qu}}_z(\rho).$$

Eqs. (49) and (51) reduce to (45) and (46).

In general, the set $\mathcal{Z}$ need not be a subset of $\mathbb{R}$. For example, an element of $\mathcal{Z}$—an outcome of the experiment—could be a list of numbers ($\mathcal{Z} \subseteq \mathbb{R}^n$), or simply a name like “up” or “down.”
5.4 Consistency Between Superoperators and POVM

Using the Choi–Kraus theorem

\[
\mathcal{C}_z(\rho) = \sum_{i \in \mathcal{I}_z} R_{z,i} \rho R_{z,i}^* \tag{54}
\]

(where we dropped the superscript “Qu” for ease of notation), we can show that the POVM \(E(\cdot)\) associated with \(E\) is completely determined by the \((\mathcal{C}_z)_{z \in \mathcal{Z}}\) according to

\[
E_z = \sum_{i \in \mathcal{I}_z} R_{z,i}^* R_{z,i} \tag{55}
\]

To see this, note that the consistency property (50) implies, with (54), that

\[
\text{tr}(\rho E_z) = \text{tr} \mathcal{C}_z(\rho) = \text{tr} \sum_{i \in \mathcal{I}_z} R_{z,i} \rho R_{z,i}^* = \text{tr} \sum_{i \in \mathcal{I}_z} R_{z,i}^* R_{z,i} \rho .
\]

Specializing to \(\rho = |\psi\rangle \langle \psi|\), we obtain

\[
\langle \psi| E_z |\psi\rangle = \left\langle \psi \left| \sum_{i \in \mathcal{I}_z} R_{z,i}^* R_{z,i} \right| \psi \right\rangle
\]

for every \(\psi\), and therefore (55). Moreover, it follows from (50) by summing over all \(z \in \mathcal{Z}\) that \(\sum_{z \in \mathcal{Z}} \mathcal{C}_z\) is trace-preserving.

Conversely, suppose the \((\mathcal{C}_z)_{z \in \mathcal{Z}}\) are given and that the superoperator \(\sum_{z \in \mathcal{Z}} \mathcal{C}_z\) is trace-preserving. Then (55) defines a POVM \(E(\cdot)\) satisfying (50): \(R_{z,i}^* R_{z,i}\) is a positive operator, and \(E(\mathcal{Z}) = I\) because, for every vector \(\psi\) in Hilbert space,

\[
\langle \psi| E(\mathcal{Z}) |\psi\rangle = \text{tr} \left( |\psi\rangle \langle \psi| \sum_{z \in \mathcal{Z}} \sum_{i} R_{z,i}^* R_{z,i} \right) = \sum_{z \in \mathcal{Z}} \sum_{i} \text{tr} \left( R_{z,i} |\psi\rangle \langle \psi| R_{z,i}^* \right) = \sum_{z \in \mathcal{Z}} \mathcal{C}_z \left( |\psi\rangle \langle \psi| \right) = \text{tr} \left( |\psi\rangle \langle \psi| \right) = ||\psi||^2 .
\]

To see that (50) holds, note that

\[
\text{tr}(\rho E_z) = \sum_{i \in \mathcal{I}_z} \text{tr}(\rho R_{z,i}^* R_{z,i}) = \sum_{i \in \mathcal{I}_z} \text{tr}(R_{z,i} \rho R_{z,i}^*) = \text{tr} \mathcal{C}_z(\rho) .
\]

5.5 Another Example

Here is an example illustrating how the POVM \(E(\cdot)\) and the superoperators \(\mathcal{C}_z\) arise, and how to do calculations with them. Suppose we carry out two experiments \(E_1\) and \(E_2\) in a row on the same system with a break of \(t\) time units in between, and regard the entire procedure as one experiment \(E\) whose outcome \(Z\) is given by the pair \((Z_1, Z_2)\) of outcomes of \(E_1\) and \(E_2\). Suppose we know the POVMs \(E_{1,z_1}\) and \(E_{2,z_2}\) (for the ease of notation, we drop the superscript “Qu”) as well as the superoperators \(\mathcal{C}_{1,z_1}\) and \(\mathcal{C}_{2,z_2}\),
and want to determine the POVM $E_z = E_{(z_1,z_2)}$ and the superoperators $\mathcal{C}_z = \mathcal{C}_{(z_1,z_2)}$ corresponding to $\mathcal{E}$. For example, $\mathcal{E}_1$ and $\mathcal{E}_2$ could be ideal measurements as described in the simplified quantum formalism. We will see that in that case $\mathcal{E}$ is (in general) not itself an ideal measurement, and $E(\cdot)$ a proper POVM (i.e., not a PVM).

The value space of $\mathcal{E}$ is $\mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2$. The joint distribution of $Z_1$ and $Z_2$, if the system starts with density matrix $\rho$, is

$$P(Z_1 = z_1, Z_2 = z_2) = P(Z_1 = z_1) P(Z_2 = z_2 | Z_1 = z_1) =$$

$$= \text{tr}(\rho E_{1,z_1}) \text{tr}(e^{-iHt/\hbar} \mathcal{C}_{1,z_1}^{*} \rho e^{iHt/\hbar} E_{2,z_2}) =$$

[using the consistency property (50)]

$$= \text{tr}(e^{-iHt/\hbar} \mathcal{C}_{1,z_1}^{*} \rho e^{iHt/\hbar} E_{2,z_2}) =$$

[using the Choi–Kraus theorem for $\mathcal{C}_{1,z_1}$]

$$= \text{tr}(e^{-iHt/\hbar} \sum_i R_{1,z_1,i}^{*} \rho R_{1,z_1,i} e^{iHt/\hbar} E_{2,z_2}) =$$

$$= \text{tr}(\rho \sum_i R_{1,z_1,i}^{*} e^{iHt/\hbar} E_{2,z_2} e^{-iHt/\hbar} R_{1,z_1,i}) =$$

$$= \text{tr}(\rho E_{(z_1,z_2)})$$

with

$$E_{(z_1,z_2)} = \sum_i R_{1,z_1,i}^{*} e^{iHt/\hbar} E_{2,z_2} e^{-iHt/\hbar} R_{1,z_1,i}.$$ (57)

Note that this expression defines a POVM, since each summand is a positive operator and $E(\mathcal{Z}_1 \times \mathcal{Z}_2) = I$:

$$\sum_{z_1} \sum_{z_2} E_{(z_1,z_2)} = \sum_{z_1} \sum_i R_{1,z_1,i}^{*} e^{iHt/\hbar} \sum_{z_2} E_{2,z_2} e^{-iHt/\hbar} R_{1,z_1,i} =$$

$$= \sum_{z_1} \sum_i R_{1,z_1,i}^{*} R_{1,z_1,i} = \sum_{z_1} E_{1,z_1} = I.$$

In case $\mathcal{E}_1$ and $\mathcal{E}_2$ are ideal measurements, the formula (57) reduces to

$$E_{(z_1,z_2)} = P_{1,z_1} e^{iHt/\hbar} P_{2,z_2} e^{-iHt/\hbar} P_{1,z_1}.$$ (58)

If $P_{1,z_1}$ commutes with $e^{iHt/\hbar} P_{2,z_2} e^{-iHt/\hbar}$ (equivalently, if the self-adjoint operators $A_1$ and $e^{iHt/\hbar} A_2 e^{-iHt/\hbar}$ commute) then $E(z_1, z_2)$ is itself a projection, and $E(\cdot)$ is a PVM, but in general it is not.
The final density matrix after $E_2$, given that the outcomes were $Z_1 = z_1$ and $Z_2 = z_2$, is

$$
\rho' = \rho_2 = \frac{\mathcal{E}_{2,z_2}(e^{-iHt/\hbar}\rho_1e^{iHt/\hbar})}{\text{tr} \mathcal{E}_{2,z_2}(e^{-iHt/\hbar}\rho_1e^{iHt/\hbar})} = \frac{\mathcal{E}_{2,z_2}(e^{-iHt/\hbar}\mathcal{E}_{1,z_1}(\rho)e^{iHt/\hbar})}{\text{tr} \mathcal{E}_{2,z_2}(e^{-iHt/\hbar}\mathcal{E}_{1,z_1}(\rho)e^{iHt/\hbar})}.
$$

That is, the superoperators corresponding to $E$ are given by the composition law

$$
\mathcal{E}_{(z_1,z_2)}(\rho) = \mathcal{E}_{2,z_2}(e^{-iHt/\hbar}\mathcal{E}_{1,z_1}(\rho)e^{iHt/\hbar}),
$$

which is completely positive as a composition of three completely positive superoperators: $\mathcal{E}_{1,z_1}$, the unitary evolution, and $\mathcal{E}_{2,z_2}$. If $E_1$ and $E_2$ are ideal measurements, so that $\mathcal{E}_{1,z_1}$ and $\mathcal{E}_{2,z_2}$ are of the form \([53]\), then

$$
\mathcal{E}_{(z_1,z_2)}(\rho) = P_{2,z_2}e^{-iHt/\hbar}P_{1,z_1}\rho P_{1,z_1}e^{iHt/\hbar}P_{2,z_2},
$$

which is not itself of the form \([53]\), unless $t = 0$ and $P_{1,z_1}$ commutes with $P_{2,z_2}$. It thus exemplifies how $E$ can be different from \([53]\).

### 5.6 The Law of Operators

How does one know which is the POVM $(E^\text{Qu}_z)_{z \in \mathcal{Z}}$ and which the family $(\mathcal{C}^\text{Qu}_z)_{z \in \mathcal{Z}}$ of superoperators associated with $E$? In practice, this is part of the working knowledge, and it is sometimes obtained by trial and error, or by symmetry arguments, or other methods of guessing. It is also often suggested by “quantization rules”, but we prefer here a rule that is generally valid (and does not appeal to classical physics).

**The Quantum Law of Operators.**

- Suppose we are given the density matrix $\rho_{\text{app}}$ for the ready state of the apparatus, its Hamiltonian $H_{\text{app}}$, and the interaction Hamiltonian $H_I$. Let

  $$
  U_t = e^{-i(H_{\text{sys}}+H_{\text{app}}+H_I)t}
  $$

  be the unitary Schrödinger evolution operator for the composite (system + apparatus). Let the experiment $E$ start at time $s$ and be finished at time $t$, so that the result can be read off at time $t$ from the apparatus.\(^{10}\) Let $P_{\text{app}}^z$ be the projection to the subspace of apparatus states in which the pointer is pointing to the value $z$. Then

  $$
  E^\text{Qu}_z = \text{tr}_{\text{app}} \left( [I_{\text{sys}} \otimes \rho_{\text{app}}] U_{t-s}^* [I_{\text{sys}} \otimes P_{\text{app}}^z] U_{t-s} \right)
  $$

  and

  $$
  \mathcal{C}^\text{Qu}_z(\rho) = \text{tr}_{\text{app}} \left( [I_{\text{sys}} \otimes P_{\text{app}}^z] U_{t-s} [\rho \otimes \rho_{\text{app}}] U_{t-s}^* [I_{\text{sys}} \otimes P_{\text{app}}] \right),
  $$

  where $\text{tr}_{\text{app}}$ denotes the partial trace over the Hilbert space of the apparatus. We check the consistency property \([50]\) in Appendix \([4]\).

\(^{10}\)This assumption is to be understood in an operational sense: It is assumed that we humans can read off the result when looking at the apparatus. This is different from assuming that the result can be read off from the wave function of (the system and) the apparatus, which is notoriously not the case, a fact known as the measurement problem of quantum theory.
In other words, the superoperator $C_{Qu}$ is obtained by solving the Schrödinger equation for the apparatus together with the system, then collapsing the joint density matrix as if applying the collapse rule to a “quantum measurement” of the pointer position, and then computing the reduced density matrix of the system.

To obtain that $E_{Qu}(\cdot)$ is a POVM, we need that $\sum_{z \in \mathcal{Z}} C_{Qu}^z$ is trace-preserving. Indeed,

$$
\text{tr} \left( \sum_{z \in \mathcal{Z}} C_{Qu}^z(\rho) \right) = \sum_{z \in \mathcal{Z}} \text{tr} \left( U_{t-s}[\rho \otimes \rho_{\text{app}}]U^*_{t-s}[I_{\text{sys}} \otimes P_{\text{app}}^z] \right) = \text{tr} \left( U_{t-s}[\rho \otimes \rho_{\text{app}}]U^*_{t-s}[I_{\text{sys}} \otimes \sum_{z \in \mathcal{Z}} P_{\text{app}}^z] \right) = \text{tr} \left( U_{t-s}[\rho \otimes \rho_{\text{app}}]U^*_{t-s} \right) = \text{tr} \rho,
$$

provided

$$
\sum_{z \in \mathcal{Z}} P_{\text{app}}^z = I_{\text{app}}. \tag{65}
$$

(This equation amounts to the statement that the experiment always has some outcome. This is normally not true, as, e.g., the apparatus might get destroyed by some accident with small but nonzero probability. However, we may deal with this trivial problem by assuming that the set $\mathcal{Z}$ of all possible outcomes contains one element representing the possibility that the experiment was not properly carried out.)

6 The GRW Formalism

6.1 The Formalism

The GRW formalism is very similar to the quantum formalism. There are only three differences: (i) the unitary Schrödinger evolution (43) between the experiments is replaced with the master equation (1); (ii) the POVM $E_{GRW}(\cdot)$ associated with an experiment $E$ as its “observable” may be different from $E_{Qu}(\cdot)$, and (iii) the superoperators $C_{GRW}^z$ (encoding the “observer-induced collapse”) may be different from $C_{Qu}^z$. Thus, it reads as follows.

The GRW Formalism.

- A system isolated from its environment has at every time $t$ a density matrix $\rho_t$ which evolves according to the master equation (1).

- With every experiment $E$ with discrete value space $\mathcal{Z}$, beginning at time $s$ and ending at time $t$, there is associated a POVM $E_{GRW}(\cdot)$ on $\mathcal{Z}$ acting on $\mathcal{H}_{sys}$. When the experiment $E$ is performed on a system with density matrix $\rho_s$, the outcome $Z$ is random with probability distribution

$$
\mathbb{P}(Z = z) = \text{tr}(\rho_s E_{GRW}^z). \tag{66}
$$
With $\mathcal{E}$ is further associated a family $\left(\mathcal{C}_{Gr}^{GR}\right)_{z \in \mathcal{Z}}$ of completely positive superoperators acting on $TRCL(\mathcal{H}_{sys})$ with the consistency property that for all trace-class operators $\rho$,

$$\text{tr}(\rho E_{Gr}^{GR}) = \text{tr} \mathcal{C}_{Gr}^{GR}(\rho).$$

(67)

In case $\mathcal{Z} = \mathbb{Z}$, the density matrix of the system at time $t$ immediately after the experiment $\mathcal{E}$ is

$$\rho_t = \rho' = \frac{\mathcal{C}_{Gr}^{GR}(\rho_s)}{\text{tr} \mathcal{C}_{Gr}^{GR}(\rho_s)}.$$

(68)

For the same reasons as for the quantum formalism, we assume a discrete value space $\mathcal{Z}$. In theories (like GRWm and GRWf) with a clear PO, on the other hand, one might consider experiments using an “analog” rather than “digital” display, for example ones in which the outcome is displayed as the center-of-mass position of a pointer. However, even in this case it is reasonable to regard the outcome as discrete, since it is hard to regard microscopic details of the pointer’s PO as a means to display information about the outcome.

Corresponding to the simplified quantum formalism, one can also formulate a simplified GRW formalism, which coincides with the simplified quantum formalism: For suitable (but not all) experiments $\mathcal{E}$ it so happens that $E_{Gr}^{GR}(\cdot)$ is a PVM (i.e., that $E_{Gr}^{GR}(B)$ is a projection for all subsets $B \subseteq \mathcal{Z}$), that $\mathcal{Z}$ is a subset of $\mathbb{R}$, and that $\mathcal{C}_{Gr}^{GR}(\rho) = P_z \rho P_z$ for suitable projections $P_z$. In this case, all the data encoding information about $\mathcal{E}$ needed for computing outcomes (i.e., $\mathcal{Z}$, $E_{Gr}^{GR}(\cdot)$, and $\left(\mathcal{C}_{Gr}^{GR}\right)_{z \in \mathcal{Z}}$) can be encoded into a single self-adjoint operator, $A = \sum_{z \in \mathcal{Z}} z P_z$.

The GRW Law of Operators.

• Suppose we are given the density matrix $\rho_{app}$ for the ready state of the apparatus, its Hamiltonian $H_{app}$, and the interaction Hamiltonian $H_1$, so that $H = H_{sys} + H_{app} + H_1$. Let the experiment $\mathcal{E}$ start at time $s$ and be finished at time $t$, and let $\zeta : \Omega_{(s,t)} \rightarrow \mathcal{Z}$ be the function that reads off the outcome of $\mathcal{E}$ from the flashes between $s$ and $t$. Then $E_{Gr}^{GR}(\cdot)$ is given by (39), or, equivalently,

$$E_{Gr}^{GR} = \text{tr}_{app} \int_{\zeta^{-1}(z)} df \left[ I_{sys} \otimes \rho_{app} \right] L^*_s(f) L_s(t)(f),$$

(69)

and

$$\mathcal{C}_{Gr}^{GR}(\rho) = \text{tr}_{app} \int_{\zeta^{-1}(z)} df L_{s,t}(f) [\rho \otimes \rho_{app}] L^*_s(f).$$

(70)

We check the consistency property (67) in Appendix B.

Before we begin the derivation of the GRW formalism, we have to elucidate a bit more what exactly it asserts.
6.2 Isolated System

The “system” is mathematically represented, as described in Section 2.5, by a splitting $\mathcal{H} = \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{env}}$ of Hilbert space, as well as a splitting $F = F_{\text{sys}} \cup F_{\text{env}}$ of the flashes, grounded in either a set $\mathcal{L}_{\text{sys}}$ of labels or a region $R_{\text{sys}} \subseteq \mathbb{R}^3$ (or both) selecting $F_{\text{sys}}$. When we say that a system is isolated or does not interact with its environment, we mean two things: First, the Hamiltonian does not contain an interaction term, that is,
\[ H = H_{\text{sys}} \otimes I_{\text{env}} + I_{\text{sys}} \otimes H_{\text{env}}. \] (71)

Second, the collapse operators associated with flashes of the system act only on $\mathcal{H}_{\text{sys}}$ but not on $\mathcal{H}_{\text{env}}$, and vice versa:
\[ \Lambda_i(x) = \begin{cases} \Lambda^\text{sys}_i(x) \otimes I_{\text{env}} & \text{if } i \in \mathcal{L}_{\text{sys}} \text{ and } x \in R_{\text{sys}} \\ I_{\text{sys}} \otimes \Lambda^\text{env}_i(x) & \text{otherwise.} \end{cases} \] (72)

This second condition, apart from expressing that the splitting $\mathcal{H} = \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{env}}$ is compatible with the splitting $F = F_{\text{sys}} \cup F_{\text{env}}$, is necessary because otherwise the system could, despite the absence of an interaction Hamiltonian, interact through collapses with the environment; e.g., an initial product wave function could become entangled.

6.3 Density Matrix

Density matrices can arise in two ways: either as representing a statistical mixture (or ensemble) of wave functions, or as the reduced density matrix of a system entangled with another system (which we will call system $b$ in the following, while system $a$ is the system of interest). Both types of density matrices are allowed in the GRW formalism: the system under consideration may be entangled with system $b$ (but not to the apparatus of the experiment), and the wave function (of the two systems together) may be random. It is part of the statement of the GRW formalism that, in this case, (i) the density matrix $\rho_t$ of the system still evolves according to the master equation (1) as long as it remains isolated (from system $b$, from the apparatus, and from everything else); (ii) the statistics of the outcome $Z$ depends only on the density matrix of the system; (iii) in case $Z = z$ the system’s reduced density matrix after the experiment is given by (68).

6.4 Smallness of Deviations

In this subsection, we characterize the “quantum regime” of the GRW theories, i.e., the regime in which the GRW formalism agrees with the quantum formalism. We do so in a sketchy way by comparing the laws of operators in the quantum and the GRW formalism, (63) and (69), which we repeat here for convenience:
\begin{align*}
E^\text{Qu}_Z &= \text{tr}_{\text{app}} \left( [I_{\text{sys}} \otimes \rho_{\text{app}}] U_{t-s}^* \left[ I_{\text{sys}} \otimes P_{z_{\text{app}}} \right] U_{t-s} \right), \\
E^\text{GRW}_Z &= \text{tr}_{\text{app}} \int_{\zeta^{-1}(z)} \text{df} \left[ I_{\text{sys}} \otimes \rho_{\text{app}} \right] L^*_z \left( f \right) L_z \left( f \right). \hspace{1cm} (73, 74)
\end{align*}
The goal is to describe conditions under which one can see in the above two formulas that
\[ E^{GRW}_z \approx E^{Qu}_z. \] (75)

A first condition is that during the experiment, collapses are likely to occur only in the apparatus, not in the system:
\[ \mathbb{P}(F_{\text{sys}}^{[s,t]} = \emptyset) \approx 1. \] (76)

To ensure this condition, the collapse rate per particle \( \lambda \) was chosen so small that for a microscopic system (e.g., \( N_{\text{sys}} = 10^4 \)) the average time between collapses \( 1/N_{\text{sys}} \lambda \approx 10^4 \) years is still much larger than the duration of any practical experiment.

The next step concerns the calibration function \( \zeta \) that specifies how to read off the result \( Z \) from the flashes of the apparatus. In the quantum law of operators (73) it is implicit that the result is read off from the apparatus at the end of the experiment, i.e., at time \( t \). To make a comparison possible, we should assume as well that \( \zeta \) in (74) depends only on the flashes near \( t \). So let \( 0 < \varepsilon \ll t - s \), let us divide the flashes during \([s,t)\] into the “early flashes” \( F_{\text{early}} = F_{[s,t-\varepsilon)} \) and the “late flashes” \( F_{\text{late}} = F_{[t-\varepsilon,t)} \), and let us assume that \( \zeta \) depends only on the late flashes:
\[ \zeta(F_{[s,t)}) = \zeta(F_{[t-\varepsilon,t)}). \] (77)

While \( \varepsilon \) must be big enough to allow a macroscopic number of late flashes, we assume it is so small that the unitary time evolution can be neglected:
\[ U_{\varepsilon} \approx I. \] (78)

We draw some conclusions from the assumptions so far. By the definition (10) of \( L_{[s,t)} \),
\[ L_{[s,t)}(f) = L_{[t-\varepsilon,t)}(f_{\text{late}}) L_{[s,t-\varepsilon)}(f_{\text{early}}). \] (79)

By (78),
\[ L_{[t-\varepsilon,t)}(f_{\text{late}}) = \lambda^#f_{\text{late}}/2 e^{-N\lambda\varepsilon/2} \prod_{(x,t',i) \in f_{\text{late}}} \Lambda_i(x)^{1/2}, \] (80)
where the order of factors in the product is arbitrary since the \( \Lambda \) are all (in the position representation) multiplication operators and thus commute. Let us set
\[ \Lambda(z) := \int_{\zeta^{-1}(z)} df_{\text{late}} L_{[t-\varepsilon,t)}(f_{\text{late}}) L_{[t-\varepsilon,t)}(f_{\text{late}}) \] (81)

\[ = \int_{\zeta^{-1}(z)} df_{\text{late}} \lambda^#f_{\text{late}} e^{-N\lambda\varepsilon} \prod_{(x,t',i) \in f_{\text{late}}} \Lambda_i(x). \] (82)

Given the meaning of \( P_{\text{app}}^{\text{app}} \) and \( \Lambda_i(x) \), given that \( P_{\text{app}}^{\text{app}} \) also is (at least approximately) a multiplication operator, and taking into account (76), it is plausible that
\[ \Lambda(z) \approx I_{\text{sys}} \otimes P_{\text{app}}^{\text{app}}. \] (83)
We thus obtain that
\[ E_z^{\text{GRW}} \approx \text{tr}_{\text{app}} \int df_{\text{early}} \left[ I_{\text{sys}} \otimes \rho_{\text{app}} \right] L_{[s,t-\epsilon]}^* (f_{\text{early}}) \left[ I_{\text{sys}} \otimes P_{z}^{\text{app}} \right] L_{[s,t-\epsilon]} (f_{\text{early}}) \] (84)
\[ \approx \text{tr}_{\text{app}} \int df \left[ I_{\text{sys}} \otimes \rho_{\text{app}} \right] L_{[s,t]}^* (f) \left[ I_{\text{sys}} \otimes P_{z}^{\text{app}} \right] L_{[s,t]} (f). \] (85)

In the last line, we have replaced \( \varepsilon \) by 0 and renamed \( f_{\text{early}} \) into \( f \).11

Comparison with (73) shows that the difference to \( E_{z}^{\text{Qu}} \) lies in replacing \( U_{t-s} \) by \( L_{[s,t]} (f) \) and integrating over \( f \). By the definition (10) of \( L_{[s,t]} (f) \), it contains unitary operators \( U_{k_{k+1}-t_k} \) which, were there no collapse operators in between, would combine to exactly \( U_{t-s} \). Thus, (85) agrees with \( E_{z}^{\text{Qu}} \) under the second condition that the early collapses of the apparatus do not change the statistics of the outcome \( Z \). Put differently, the second condition says that the result of the GRW collapse process for the wave function \( \Psi \) of system and apparatus is more or less the same as if \( \Psi \) evolved unitarily until \( t \) and then GRW collapses took place at \( t \). In most cases, this condition will presumably be true because of decoherence, i.e., because wave packets that are disjoint in the configuration space of the apparatus and associated with different outcomes \( z \) will remain disjoint under the unitary evolution, at least until time \( t \). A more detailed and careful analysis of the second condition is non-trivial and has been carried out in [8] for a concrete example model of an apparatus in the framework of QMUPL [18], a spontaneous collapse model very similar to the GRW process.

7 Derivation of the GRW Formalism

After some preparatory considerations in Section 7.1, we prove the GRW formalism from GRWf in Section 7.2, followed by an example in Section 7.3.

7.1 Density Matrix

We need to collect some facts about density matrices in GRWf.

7.1.1 Statistical Density Matrix

Set, for ease of notation, \( t_0 = 0 \). Since the wave function \( \Psi_t \) is random, with its distribution there is associated the density matrix
\[ \rho_t = \mathbb{E} |\Psi_t \rangle \langle \Psi_t| = \int_{\mathcal{H}} \mathbb{P}_\Psi (\Psi_t \in d\Phi) |\Phi \rangle \langle \Phi|, \] (86)

---

11By the way, the expression (85) corresponds to the following naive way of computing predictions of GRW theories without properly taking into account the role of the primitive ontology: Starting from \( \Psi_s = \psi \otimes \phi \), let \( \Psi \) evolve according to the GRW process until time \( t \), and if \( [I_{\text{sys}} \otimes P_{z}^{\text{app}}] \Psi_t \approx \Psi_t \) (i.e., if \( \Psi_t \) approximately lies in the subspace to which \( P_{z}^{\text{app}} \) projects) then say that the outcome is \( z \). This is likely the case for some \( z \) because macroscopic superpositions are unlikely to survive for long.
where $S(\mathcal{H}) = \{ \Psi \in \mathcal{H} : \|\Psi\| = 1 \}$ is the unit sphere in Hilbert space $\mathcal{H}$. In other words, $\rho_t$ is the density matrix of a large ensemble of systems, each of which started with the same initial wave function $\Psi_0$ but experienced collapses independently of the other systems.

We note that the density matrix $\rho_t$ obeys the master equation (1). But the validity of (1) is even wider: Suppose that even the initial wave function $\Psi_0$ is random, with distribution given by any probability measure $\mu_0$ on $S(\mathcal{H})$. Then, for $t > 0$, $\Psi_t$ is doubly random, because of the random initial wave function and of the stochastic GRW evolution, with distribution

$$\mu_t(\cdot) = \int \mu_0(d\Psi_0) \mathbb{P}_{\Psi_0}(\Psi_t \in \cdot),$$

Again, the corresponding density matrix

$$\rho_t = \mathbb{E}_{\mu_0}|\Psi_t\rangle\langle \Psi_t| = \int \mu_t(d\Psi) |\Psi\rangle\langle \Psi|$$

obeys (1). To see this, note that it satisfies

$$\rho_t = \int \mu_0(d\Psi_0) \int \mathbb{P}_{\Psi_0}(\Psi_t \in d\Phi) |\Phi\rangle\langle \Phi|,$$

where the inner integral obeys (1), so that $\rho_t$ is a mixture of solutions of (1) and therefore is itself a solution of (1).

Alternatively, $\rho_t$ can directly be expressed in terms of $\rho_0$ according to

$$\rho_t = \mathcal{S}_{[0,t]} \rho_0 = \int_{\Omega_{[0,t]}} df L_{[0,t]}(f) \rho_0 L_{[0,t]}^*(f).$$

From this the master equation (1) can be obtained by differentiation with respect to $t$. As a by-product, it can be read off from (90) that the mapping $\mathcal{S}_{[0,t]} : \rho_0 \mapsto \rho_t$ obtained by evolving the density matrix $\rho$ according to the master equation (1) is a completely positive superoperator.

The following proposition is a consequence of the fact that the distribution of flashes is given by a POVM: If the initial wave function $\Psi_0$ is random with distribution $\mu_0$, then the distribution of the flashes depends only on the density matrix $\rho_0$ associated with $\mu_0$,

$$\mathbb{P}(F \in \cdot) = \int \mu_0(d\Psi_0) \mathbb{P}_{\Psi_0}(F \in \cdot) = \int \mu_0(d\Psi_0) \langle \Psi_0|G(\cdot)|\Psi_0\rangle = \text{tr}(\rho_0 G(\cdot))$$

with

$$\rho_0 = \int_{S(\mathcal{H})} \mu_0(d\Psi_0) |\Psi_0\rangle\langle \Psi_0|. $$

In other words, if two probability distributions $\tilde{\mu}_0$ and $\mu_0$ have the same density matrix, $\tilde{\rho}_0 = \rho_0$, then they lead to the same distribution of the PO. For comparison, this is not
true in Bohmian mechanics or GRWm: there, \( \hat{\mu}_0 \) and \( \mu_0 \) may lead to different trajectories \([10, 23]\) respectively to different probability distributions of the \( m \) function \([3]\).

Since \( \hat{\mu}_0 \) and \( \mu_0 \) lead to the same distribution of flashes, we may write \( P_{\rho_0} \) for that distribution. This also means that we can simply talk of the flash process for a given initial density matrix, as opposed to the flash process for a given initial wave function. As time proceeds, the density matrix determining the distribution of the flashes evolves according to the master equation in the sense that

\[
P_{\rho_0}(F_{[t,\infty)} \in B) = P_{\rho_t}(F \in B),
\]

(93)

where the right hand side refers to the distribution of the flashes when starting with \( \rho_t \) at time \( t \). This fact follows from the conditional probability formula by averaging over \( F_{[0,t)} \).

### 7.1.2 The Marginal Probability Formula

The marginal probability formula expresses that a system which does not interact with its environment is itself governed by GRWf, even if the system is entangled with the environment. (Note that this is not true, e.g., in Bohmian mechanics, where the trajectories of the system’s particles depend on the configuration of the environment, even in the absence of interaction. As we will see, it is not true in GRWm either.)

The marginal probability formula says that for an isolated system,

\[
P_{\Psi_0}(F_{\text{sys}} \in B) = P_{\rho_{\text{sys}}}(B).
\]

(94)

Here, \( P_{\Psi_0} \) is the distribution of the flashes in a universe starting with wave function \( \Psi_0 \) at time \( t_0 = 0 \), and \( P_{\Psi_0}(F_{\text{sys}} \in \cdot) \) is the marginal distribution of the system’s flashes; \( \rho_{\text{sys}} = \text{tr}_{\text{env}} |\Psi_0\rangle\langle \Psi_0| \) is the reduced density matrix of the system; finally, \( P_{\rho_{\text{sys}}} \) is the distribution of flashes in a universe containing nothing but the system and starting with density matrix \( \rho_{\text{sys}} \) at time 0 in the sense of equation (91):

\[
P_{\rho_{\text{sys}}}(\cdot) = \text{tr}\left( \rho_{\text{sys}} G_{\text{sys}}(\cdot) \right).
\]

(95)

We provide a proof of the marginal probability formula in Appendix C.

The marginal probability formula was first derived by Bell [9] for the purpose of proving a no-signalling theorem for GRWf. To see the connection, suppose the system is Alice’s lab, which does not interact with Bob’s lab (e.g., because they are, when considering the relevant time intervals, spacelike separated); then the distribution of the flashes in Alice’s lab, and thus in particular the distribution of the outcome of any experiment, does not depend on external fields at work in Bob’s lab, nor on the common wave function \( \Psi_0 \) except through the reduced density matrix \( \rho_{\text{sys}} \).

The marginal probability formula should not be confused with the following simple consequence of the function property \([27]\): Since \( F_{\text{sys}} \) is a function of \( F \), its distribution is given by a POVM \( E(\cdot) \), namely

\[
P_{\Psi_0}(F_{\text{sys}} \in B) = \langle \Psi_0 | E(B) | \Psi_0 \rangle.
\]

(96)
The marginal probability formula goes further in two respects: First, its right hand side depends only on the reduced density matrix \( \rho_{\text{sys}} \), and not on the entire wave function \( \Psi_0 \); second, the POVM \( G_{\text{sys}}(\cdot) \) is not just some POVM but exactly the one that would govern the flashes if the universe contained nothing but the system.

A related fact is the *independence property*: If a system does not interact with its environment and is initially disentangled from its environment, then the flashes of the system and those of the environment are stochastically independent, i.e., their joint distribution is a product:

\[
\mathbb{P}_{\text{sys} \otimes \text{env}}(F_{\text{sys}} \in B_{\text{sys}}, F_{\text{env}} \in B_{\text{env}}) = \mathbb{P}_{\text{sys}}(F_{\text{sys}} \in B_{\text{sys}}) \mathbb{P}_{\text{env}}(F_{\text{env}} \in B_{\text{env}}). \tag{97}
\]

Moreover, in that case the wave function \( \Psi_t \) remains a product at later times.

In GRWm there is a formula that is in a way analogous to the marginal probability formula of GRWf, as it connects \( \rho_{\text{sys}} \) to the PO of the system, namely to \( m_{\text{sys}} \) as introduced in (21). However, it is much weaker as it connects \( \rho_{\text{sys}} \) not to the entire future history of the PO, for \( t \geq 0 \), but just to the PO at \( t = 0 \). This formula reads

\[
m_{\text{sys}}(x, t = 0) = \sum_{i \in \mathcal{L}_{\text{sys}}} m_i \int dq_{\text{sys}} \delta(q_{\text{sys},i} - x) \langle q_{\text{sys}} | \rho_{\text{sys}} | q_{\text{sys}} \rangle \tag{98}
\]

assuming, for simplicity, that the system is defined in terms of a label set \( \mathcal{L}_{\text{sys}} \), not of a region \( R_{\text{sys}} \). As before, \( \rho_{\text{sys}} = \text{tr}_{\text{env}} |\Psi_0 \rangle \langle \Psi_0 | \). The formula implies that a different wave function \( \tilde{\Psi}_0 \neq \Psi_0 \) with \( \text{tr}_{\text{env}} |\tilde{\Psi}_0 \rangle \langle \tilde{\Psi}_0 | = \text{tr}_{\text{env}} |\Psi_0 \rangle \langle \Psi_0 | \) would lead to the same \( m_{\text{sys}} \). An analogous statement holds in Bohmian mechanics: the marginal distribution of \( Q_{\text{sys}} \) at \( t = 0 \) depends only on \( \rho_{\text{sys}} \). Note that in GRWm, \( m_{\text{sys}} \) cannot be obtained from a statistical density matrix.

Returning to GRWf, we call a system a *GRW system* if the distribution of the flashes of the system (after time 0) is given by \( \rho_{\text{sys}} \) (at time 0), i.e., if (94) holds. The marginal probability formula thus asserts that *every isolated system is a GRW system*—a system whose PO behaves as if the system were alone in the universe. Conversely, if a system is not isolated then it cannot be expected to be a GRW system since the interaction with the environment should affect the pattern of flashes.

Now that we have the concept of GRW system, one conclusion we can draw is that the reasoning of Section 4 applies not just to the universe as a whole but also when the object and the apparatus together form a GRW system. In this case, the POVM \( E_{\text{GRW}}^{\text{sys}}(\cdot) \) depends only on the apparatus.

A variant of the marginal probability formula asserts the following: If a system is isolated during \([0, t)\) then

\[
\mathbb{P}_{\Psi_0}(F_{[0,t]}^{\text{sys}} \in B) = \mathbb{P}_{\rho_{\text{sys}}}(F_{[0,t]} \in B). \quad \tag{99}
\]

Here, a system can stop being isolated because the Hamiltonian or the collapse rate operators are time-dependent, \( H = H_t \) and \( \Lambda_i(x) = \Lambda_{i,t}(x) \).

The fact (99) follows from the first version (94) of the marginal probability formula: Consider a hypothetical universe whose time-dependent Hamiltonian \( H_t \) and collapse
operators $\Lambda_{i,t}(x)$ are whatever we choose. Then, for a fixed initial wave function $\Psi_0$, the distribution of flashes during $[0, t)$ will depend on our choices of $H_s$ for all $s \in [0, t)$, but not for $s \geq t$. So we need not specify $H_t$ before time $t$. For example, if the system is initially isolated, we can turn on the interaction with its environment at time $t$, and the distribution of the flashes up to time $t$ is the same as it would have been if the system were isolated forever, and thus given by (94).

7.1.3 The Marginal Master Equation

The marginal master equation

$$(\rho_t)_\text{sys} = (\rho_{\text{sys}})_0$$

expresses the related fact that also the reduced density matrix of the system, when isolated from but entangled with its environment, evolves according to the master equation (1). This is a general fact about the master equation, which can also be expressed by saying that when the system and the environment do not interact, the following diagram commutes:

Here, $(\mathbb{1})_\text{sys}$ means the master equation (1) applied to the system. In words, the marginal of the master equation is again a version of the master equation: the version that would hold if the universe contained nothing but the system. In another notation, $\mathcal{H}_\text{sys} \circ \text{tr}_{\text{env}} = \text{tr}_{\text{env}} \circ \mathcal{H}_{[0, t)}$. The marginal master equation allows us to write $\rho_t^{\text{sys}}$ instead of either $(\rho_{\text{sys}})_t$ or $(\rho_t)_\text{sys}$.

We provide an analytic-algebraic proof of the marginal master equation in Appendix D. Alternatively, here are two derivations from the marginal probability formula: First, since the distribution of the flashes of the system depends only on $(\rho_{\text{sys}})_t=0$, and for an isolated system the collapses associated with the flashes of the environment (as well as the Hamiltonian evolution) act trivially on $\mathcal{H}_\text{sys}$, $(\rho_t)_\text{sys}$ depends only on $(\rho_{\text{sys}})_t=0$, and since for an empty environment $(\rho_t)_\text{sys}$ would trivially equal $(\rho_{\text{sys}})_t$, (100) must be generally true. Second, the significance of the density matrix associated with the system at time $t$ lies in governing the distribution of the flashes after $t$. Thus if, as the marginal probability formula tells us, the distribution of the system’s flashes after $t$ is the same as if the system were alone in the universe and started with $(\rho_{\text{sys}})_0$, namely $\text{tr}((\rho_{\text{sys}})_t G^{\text{sys}}_{[t, \infty)}(\cdot))$, then the system’s density matrix at time $t$ must be $(\rho_{\text{sys}})_t$. On the other hand, by the marginal probability formula applied to time $t$, the distribution of the system’s flashes after $t$ is $\text{tr}((\rho_t)_\text{sys} G^{\text{sys}}_{[t, \infty)}(\cdot))$, so the density matrix at time $t$ must be $(\rho_t)_\text{sys}$. (Mathematically, this argument assumes that the family of operators $\{G^{\text{sys}}_{[t, \infty)}(B) : \text{any } B\}$, is sufficiently rich.)
7.1.4 State and Primitive Ontology

From our discussion of density matrices in GRWf we can conclude that the appropriate notion of state of a GRW system is that of density matrix. Let us explain.

While in classical mechanics the notion of state at time \( t \) used to mean “phase point”, i.e., “a mathematical datum that determines the PO after \( t \),” the meaning has shifted, with the advent of quantum mechanics, to a statistical notion which, in our framework, can be characterized as “a mathematical datum that determines the probability distribution of the PO after \( t \).” For example, in classical mechanics a state in the latter sense would correspond to a probability distribution over states in the former sense. On the basis of the latter sense, what is the appropriate notion of “state” in the various versions of quantum mechanics?

In Bohmian mechanics, the wave function \( \psi_t \) (or, in fact, the 1-dimensional subspace spanned by it) is such a datum, as it determines the distribution \( \mathbb{P} \) of the configuration \( Q_t \) and, given \( Q_t \), the further motion of the configuration, \( (Q_{t'})_{t' \geq t} \); thus, \( \psi_t \) determines the distribution \( \mathbb{P}_{\psi_t} \) of \( (Q_{t'})_{t' \geq t} \). If \( \psi_t \) is random, then its distribution \( \mu \) is a “state,” as the distribution of \( (Q_{t'})_{t' \geq t} \) then is

\[
\mathbb{P}_\mu = \int \mu(d\psi) \mathbb{P}_\psi.
\]

However, the statistical density matrix \( \rho = \int \mu(d\psi) |\psi\rangle \langle \psi| \) associated with \( \mu \) is not a “state” since, as mentioned already, some distribution \( \tilde{\mu} \neq \mu \) may have the same statistical density matrix \( \tilde{\rho} = \rho \) but lead to a different distribution \( \mathbb{P}_{\tilde{\mu}} \neq \mathbb{P}_\mu \) of the trajectories \([10, 23]\). (One can postulate a Bohm-type law of motion involving a density matrix instead of a wave function, but then yet other trajectories would come out.)

Now consider a system. If it is disentangled from its environment, i.e., \( \Psi = \psi_{\text{sys}} \otimes \psi_{\text{env}} \) with \( \Psi \) the wave function of the universe, and if the system is isolated from the environment (so it stays disentangled) then \( \psi_{\text{sys}} \) is a state for the system. The same is still true if the system possesses an effective wave function \([24]\), i.e., if the wave function \( \Psi \) of the universe is of the form \( \Psi = \psi_{\text{sys}} \otimes \psi_{\text{env}} + \Psi_\perp \) and \( Q_{t'} \) never enters the support of \( \Psi_\perp \).

In general, however, the system’s wave function, namely its conditional wave function \([24]\) \( \psi_{\text{sys}}(q_{\text{sys}}) = \Psi(q_{\text{sys}}, Q_{\text{env}}) \), is not a state because the trajectory of \( Q_{\text{sys}} \) depends on more than just \( \psi_{\text{sys}} \) at time \( t \). For the same reason, the reduced density matrix \( \rho_{\text{sys}} = \text{tr}_{\text{env}} |\Psi\rangle \langle \Psi| \) is not a state.

In GRWm, the situation is the same as in Bohmian mechanics: The wave function is a state but the (statistical) density matrix is not \([3]\); a system that is disentangled and isolated possesses a wave function which is a state; the reduced density matrix of a system that is entangled with its environment is not a state (in spite of \([18]\), as the probability distribution of \( m_{\text{sys}} \) at later times depends on more than \( \rho_{\text{sys}} \) at time \( t \).)

\[\text{To see this, consider for example } \Psi(t = 0) = 2^{-1/2}(|u⟩|1⟩ + |d⟩|2⟩), \text{ where } |u⟩, |d⟩ \text{ are orthonormal vectors in } H_{\text{sys}} \text{ and } |1⟩, |2⟩ \text{ in } H_{\text{env}}, \text{ and suppose that } \Psi \text{ quickly collapses to either } |u⟩|1⟩ \text{ or } |d⟩|2⟩; \text{ contrast this with } \Psi(t = 0) = 2^{-1/2}(|l⟩|1⟩ + |r⟩|2⟩), \text{ where } |l⟩ = 2^{-1/2}(|u⟩ + |d⟩) \text{ and } |r⟩ = 2^{-1/2}(|u⟩ - |d⟩), \text{ and suppose that } \Psi \text{ quickly collapses to either } |l⟩|1⟩ \text{ or } |r⟩|2⟩. \text{ Then } \rho_{\text{sys}} = \frac{1}{2} |u⟩⟨u| + \frac{1}{2} |d⟩⟨d| =\]

39
In GRWf, however, the reduced or statistical density matrix is indeed a state for any isolated system. This is just a way of re-formulating the marginal probability formula.

In orthodox quantum mechanics, it is more or less the results of experiments that are regarded as the PO, and a state is what determines the distribution of the results of experiments. Thus, as in GRWf, the reduced or statistical density matrix $\rho$ is a state for any isolated system, since the distribution of the result of an experiment, acting only on the system and associated with POVM $E^{Qu}(\cdot)$ acting on $\mathcal{H}_{sys}$, is given by $\text{tr}(\rho E^{Qu}(\cdot))$.

### 7.2 Derivation of the GRW Formalism

The first rule of the GRW formalism asserts in particular that with an isolated system at every time $t$ there is associated a density matrix $\rho_t$. Indeed, there is associated a density matrix $\rho_t$ no matter if the system is isolated or not. Let us formulate how that density matrix is defined:

The system, called system $a$ in the following, may be entangled with another system called system $b$. The wave function $\Psi$ of $a$ and $b$ together at time $t$ may be random with distribution $\mu_t$ on $\mathfrak{S}(\mathcal{H}_a \otimes \mathcal{H}_b)$; $\mu_t$ may be determined by a preparation procedure, by previous flashes, or both. Then

$$\rho_t = \rho_t^a = \int \mu_t(d\Psi) \text{tr}_b |\Psi\rangle\langle\Psi|.$$  \tag{103} $\text{(103)}$

Now the marginal master equation implies that, as long as system $a$ is isolated, $\rho_t$ evolves according to the master equation $[\mathbb{I}]$. This yields already the first rule of the GRW formalism.

A derivation of the second rule—asserting that the outcome statistics is of the form $\text{tr}(\rho E(\cdot))$—was given in Section 4 except for the possible entanglement between system $a$ and system $b$. So let us derive the second rule in this more general situation. Suppose that

1. the experiment $\mathcal{E}$ involves a splitting of the world into four parts: system $a$ (the “object” of $\mathcal{E}$), system $b$, the apparatus of $\mathcal{E}$, and the rest of the world;

2. $\mathcal{E}$ begins at time $s$ and ends at time $t$\textsuperscript{13}

3. system $a$, system $b$, and the apparatus together form a GRW system (i.e., are isolated) during the time interval $[s, t]$ and possess a wave function;

4. at time $s$, the apparatus is not entangled with system $a$ and $b$,

$$\Psi_s = \psi_{a b} \otimes \phi,$$  \tag{104} $\text{(104)}$

\textsuperscript{13}This assumption will be relaxed in Section 8 where we allow that the experiment’s run-time is not fixed before the experiment.
where $\psi_{ab}$ is the (possibly random) wave function of systems $a$ and $b$ together at time $s$, $\phi$ is the (possibly random) wave function of the apparatus at time $s$;

5. $\psi_{ab}$ and $\phi$ are independent random variables;

6. during $[s, t]$ the apparatus interacts only with system $a$, while system $a$ and the apparatus do not interact with system $b$;

7. the outcome $Z$ is a function $\zeta$ of the flashes of the apparatus during $[s, t]$; this assumption can be weakened by allowing that $Z$ is read off from the flashes of both system $a$ and the apparatus,

$$Z = \zeta(F^a_s \cup F^{app}_s),$$

while we need to exclude a direct dependence of $Z$ on the flashes of system $b$.

By the conditional probability formula (33), we can regard $\Psi_s$ as the (random) initial wave function. By the marginal probability formula (applied to $a \cup app$ instead of $a$) and assumption 6 above, the joint distribution of the flashes of system $a$ and the apparatus (given $\Psi_s$) is given by

$$P(F^a_s \cup F^{app}_s \in A | \Psi_s) = \text{tr}(\rho^a_{Psi} G_{a \cup app}^{a \cup app}(A))$$

with

$$\rho^a_{Psi} = \text{tr}_b|\Psi_s\rangle\langle\Psi_s| = (\text{tr}_b|\psi_{ab}\rangle\langle\psi_{ab}|) \otimes |\phi\rangle\langle\phi|$$

and $G_{a \cup app}^{a \cup app}$ the history POVM for system $a$ and the apparatus during $[s, t]$. Let $E$ denote averaging over the random wave function $\Psi_s$; using that $\phi$ is stochastically independent of $\psi_{ab}$, we obtain that

$$\rho_{a \cup app} := E\rho_{Psi}^a = (E \text{tr}_b|\psi_{ab}\rangle\langle\psi_{ab}|) \otimes (E|\phi\rangle\langle\phi|) = \rho_a \otimes \rho_{app}$$

with $\rho_a$ and $\rho_{app}$ the density matrices—as defined in (103)—of system $a$ and the apparatus, respectively. As a consequence,

$$P(Z = z) = E\text{P}(F^a_s \cup F^{app}_s \in \zeta^{-1}(z) | \Psi_s)$$

$$= E\text{tr}^{\Psi_s}(\rho_{a \cup app} G_{a \cup app}^{a \cup app}(\zeta^{-1}(z)))$$

$$= \text{tr}(\rho_{a \cup app} G_{a \cup app}^{a \cup app}(\zeta^{-1}(z)))$$

$$= \text{tr}(\rho_a E^{GRW}_z)$$

with $E^{GRW}_z$ given by (69)—the GRW law of operators. The operators $E^{GRW}_z$ form a POVM because of the function property (27) and the reduction property (28). This completes the derivation of the second rule.

Now we turn to the third rule—the collapse rule. According to the definition (103) of the density matrix of system $a$, the density matrix $\rho_t = \rho_t^a$ after the experiment is
obtained by the partial trace over the environment (here, \(b\) and the apparatus) from the density matrix associated with the probability distribution representing the preparation procedure. In this case, the preparation procedure consists of the preparation before the experiment \(\mathcal{E}\), the experiment \(\mathcal{E}\) itself, and conditioning on the outcome \(Z = z\). Thus,

\[
\rho' = \rho_t' = \rho_a' = E\left(\text{tr}_{b,\text{app}}|\Psi_t\rangle\langle\Psi_t|Z = z\right) \tag{113}
\]

with \(E\) the average over both the random wave function \(\Psi_s\) of \(a \cup b \cup \text{app}\) before \(\mathcal{E}\) and the flashes during \([s, t]\), conditional on the outcome \(z\) of the experiment. This expression can be rewritten as

\[
\frac{1}{\mathcal{N}} \text{tr}_{\text{app}} \int_{\zeta^{-1}(z)} df L_{[s, t]}(f) [\rho_s \otimes \rho_s^{\text{app}}] L_{[s, t]}^*(f) \tag{114}
\]

which agrees with (70), the GRW law of superoperators. This completes our derivation of the GRW formalism.

### 7.3 An Example: Consecutive Experiments

The above derivation is simple but powerful. To illustrate how much so, we now consider an example case, extract a prediction from the GRW formalism, and derive the same prediction directly from the distribution of the flashes. The example is in fact a problem of general interest, namely to compute the joint probability distribution of the outcomes of subsequent experiments on the same system ("system \(a\"), say \(\mathcal{E}_2\) after \(\mathcal{E}_1\). Suppose \(\mathcal{E}_1\) is carried out during the time interval \([s_1, t_1]\), \(\mathcal{E}_2\) during \([s_2, t_2]\) with \(0 \leq s_1 < t_1 \leq s_2 < t_2 < \infty\), and system \(a\) is isolated during \([t_1, s_2]\).

To compute such probabilities is exactly the purpose of the third rule, which leads to the equation

\[
\mathbb{P}(Z_2 = z_2|Z_1 = z_1) = \text{tr}\left(\mathcal{J}^a_{[t_1, s_2]} \left[ \frac{\mathcal{G}_{1,z_1}(\rho_{s_1})}{\text{tr} \mathcal{G}_{1,z_1}(\rho_{s_1})} \right] E_2(z_2) \right), \tag{115}
\]

where \(\mathcal{J}^a_{[t_1, s_2]}\) is the superoperator evolving a density matrix according to the master equation (1) for system \(a\) from time \(t_1\) to time \(s_2\).

To derive (115) directly from the distribution of the flashes amounts to computing the conditional distribution of the positions of the two pointers, each of which consists of flashes. Thus, the conditional probability of \(Z_2 = z_2\), given that \(Z_1 = z_1\), is of the form

\[
\mathbb{P}\left(\zeta_2(F_{[s_2, t_2]}) = z_2 \mid \zeta_1(F_{[s_1, t_1]}) = z_1\right). \tag{116}
\]

We will compute this quantity and show that it equals the right hand side of (113). But before, let us step back and make the setting explicit. We make the same assumptions 1–7 as in Section 7.2 for both experiments, \(\mathcal{E}_1\) and \(\mathcal{E}_2\). At time \(t_1\), system \(a\) will be
entangled with both system \( b = b_1 \) and the apparatus of \( E_1 \). For the discussion of \( E_2 \), the apparatus of \( E_1 \) will count as part of the system \( b_2 \); i.e., \( b_2 = b_1 \cup \text{app}_1 \). We explicitly allow that \( E_2 \) may depend on \( Z_1 \). For example, experimenters may let the decision which “observable” to “measure” depend on \( Z_1 \); or the time \( s_2 \) at which \( E_2 \) starts may depend on \( Z_1 \) (say, because the experimenters check the apparatus more carefully if \( Z_1 \) was unexpected); or system \( a \) may consist of several smaller systems (say, 200 electrons), and it may depend on \( Z_1 \) (say, the outcomes of 200 Stern–Gerlach experiments) which of these subsystems \( E_2 \) will act on (say, only those whose outcome was “up”). In particular, assumption 5 includes that, at time \( s_2 \), the wave function of apparatus 2 is conditionally independent of the wave function of \( a \cup b_1 \cup \text{app}_1 \), given that \( Z_1 = z_1 \).

The preparation procedure of system \( a \) for \( E_2 \) consists of: the preparation procedure for \( E_1 \); the time evolution until \( s_2 \); and conditioning on \( Z_1 = z_1 \). As a consequence, by the definition (103) of the density matrix of system \( a \) at time \( s_2 \), is the conditional density matrix given \( Z_1 = z_1 \), and it will be useful to first collect some facts about such density matrices.

### 7.3.1 Conditional Density Matrix

A frequently relevant special case of the statistical density matrix associated with a probability distribution over wave functions corresponds to conditioning on an event concerning the flashes between \( t_0 \) and \( t \).

If a GRW world at time \( t_0 = 0 \) has random wave function \( \Psi_0 \) with probability distribution \( \mu_0 \) on \( S(H) \) then at time \( t > 0 \) the conditional density matrix given \( B \subseteq \Omega_{[0,t)} \) is defined to be

\[
\rho_{t|B} = \mathbb{E}_{\mu_0}\left( |\Psi_t\rangle\langle\Psi_t| \big| F_{[0,t)} \in B \right),
\]

provided \( \mathbb{P}_{\mu_0}(F_{[0,t)} \in B) > 0 \). As a consequence,

\[
\mathbb{P}_{\rho_{t|B}}(C) = \mathbb{E}_{\mu_0} \left( \mathbb{P}_{\Psi_t}(F_{[t,\infty)} \in C) \big| F_{[0,t)} \in B \right).
\]

It is a straightforward observation that the conditional density matrix for fixed \( B \) evolves with \( t \) according to the master equation (1): for \( 0 < s < t < \infty \),

\[
\rho_{t|B} = \mathcal{S}_{[s,t)} \rho_{s|B},
\]

where \( \mathcal{S}_{[s,t)} \) is again the superoperator that evolves a density matrix from time \( s \) to time \( t \) according to the master equation (1). This fact is a simple consequence of the definition (117) in the case that the event \( B \) concerns only \( F_{[0,s)} \), rather than all of \( F_{[0,t)} \).

---

14 Although the kind of entanglement that first comes to mind when talking about entanglement between system and apparatus (viz., a superposition of macroscopically different states corresponding to different outcomes) is essentially absent in a collapse theory (as that was the motivation for introducing the spontaneous collapses), a small degree of such entanglement may survive the collapses and, more importantly, entanglement between the system and microscopic degrees of freedom of the apparatus may persist despite the collapses.
Another basic fact (that is easy to prove) is that the conditional density matrix $\rho_t|B$ can be expressed in terms of the initial density matrix $\rho_0$ in the following way analogous to (90):

$$\rho_0 = \mathbb{E}_{\mu_0}(|\Psi_0\rangle\langle\Psi_0|) = \int_{S(\mathcal{H})} \mu_0(d\Phi)|\Phi\rangle\langle\Phi|$$

in the following way analogous to (90):

$$\rho_t|B = \frac{1}{\mathcal{N}} \int_B df \; L_{[0,t]}(f) \rho_0 L^*_0(f)$$

with $\mathcal{N}$ the normalizing constant,

$$\mathcal{N} = \text{tr} \int_B df \; L_{[0,t]}(f) \rho_0^t L^*_0(f)$$

A further useful fact about conditional density matrices is the following alternative variant of the conditional probability formula, which expresses the conditional distribution of the future flashes after time $s$, given that $F_{[0,s]} \in B$, in terms of the conditional density matrix $\rho_s|B$: for $0 < s < t \leq \infty$,

$$\mathbb{P}_{\mu_0}(F_{[s,t]} \in C|F_{[0,s]} \in B) = \mathbb{P}_{\rho_s|B}(C),$$

where $\mathbb{P}_{\mu_0}$ means the distribution of flashes when starting at time 0 with a random wave function $\Psi_0$ with distribution $\mu_0$, and the right hand side means the distribution of flashes when starting with the density matrix $\rho_s|B$ at time $s$. The above formula (123) follows from the conditional probability formula by taking the appropriate average. Indeed, since for any random variable $X$ and any event $C$,

$$\mathbb{P}(C|X) = g(X) \text{ implies } \mathbb{P}(C|X \in B) = \mathbb{E}(g(X)|X \in B),$$

the conditional probability formula (123) implies

$$\mathbb{P}_{\Psi_0}(F_{[s,t]} \in C|F_{[0,s]} \in B) = \mathbb{E}_{\Psi_0}\left(\mathbb{P}_{\Psi_s}(F_{[s,t]} \in C)|F_{[0,s]} \in B\right),$$

and the right hand side, after averaging over $\Psi_0$, equals $\mathbb{P}_{\rho_s|B}(C)$ by virtue of (118).

It is a generalization, but also a special case, of the above alternative conditional probability formula (123) that later flashes, those after $t < \infty$, are governed by the time-evolved conditional density matrix $\rho_t|B = \mathcal{F}_{[s,t]} \rho_s|B$, i.e.,

$$\mathbb{P}_{\mu_0}(F_{[t,\infty]} \in C|F_{[0,s]} \in B) = \mathbb{P}_{\rho_t|B}(C).$$
7.3.2 Conditional Distribution of Pointer Positions

Let us now turn to evaluating (116). By the alternative conditional probability formula (123), the conditional distribution of the flashes of \( a \cup b_1 \cup \text{app}_1 \cup \text{app}_2 \) after time \( t_1 \), given \( Z_1 = z_1 \), is the same as if \( a \cup b_1 \cup \text{app}_1 \cup \text{app}_2 \) started at time \( t_1 \) with density matrix

\[
\rho_{a \cup b_1 \cup \text{app}_1 \cup \text{app}_2}^{t_1|Z_1 = z_1}.
\]

Since from \( t_1 \) onwards, the system \( a \cup \text{app}_2 \) does not interact any more with \( b_1 \cup \text{app}_1 \), the marginal probability formula tells us that the distribution of flashes of \( a \cup \text{app}_2 \) is the same as if \( a \cup \text{app}_2 \) were alone in the world and started at time \( t_1 \) with density matrix

\[
\rho_{a \cup \text{app}_2}^{t_1} = \text{tr}_{b_1 \cup \text{app}_1} \rho_{a \cup b_1 \cup \text{app}_1 \cup \text{app}_2}^{t_1|Z_1 = z_1}.
\]  

(127)

By (93), the distribution of the flashes in \( a \cup \text{app}_2 \) after \( s_2 \) is the same as if \( a \cup \text{app}_2 \) were alone in the world and started at time \( s_2 \) with density matrix

\[
\rho_{a \cup \text{app}_2}^{s_2} = \mathcal{S}_{[t_1, s_2]} \rho_{a \cup \text{app}_2}^{t_1}.
\]  

(128)

By the marginal master equation, this equals

\[
\rho_{a \cup \text{app}_2}^{s_2} = \text{tr}_{b_1 \cup \text{app}_1} \rho_{a \cup b_1 \cup \text{app}_1 \cup \text{app}_2}^{s_2|Z_1 = z_1}.
\]  

(129)

As a consequence of the assumption that the wave function of apparatus 2 at time \( s_2 \) is conditionally independent of the wave function of \( a \cup b_1 \cup \text{app}_1 \), given \( Z_1 = z_1 \), we have that

\[
\rho_{a \cup b_1 \cup \text{app}_1 \cup \text{app}_2}^{s_2|Z_1 = z_1} = \rho_{a \cup \text{app}_2}^{s_2|Z_1 = z_1} \otimes \rho_{\text{app}_1}^{s_2|Z_1 = z_1}.
\]  

(130)

Hence,

\[
\rho_{a \cup \text{app}_2}^{s_2} = \left( \text{tr}_{b_1 \cup \text{app}_1} \rho_{a \cup b_1 \cup \text{app}_1}^{s_2|Z_1 = z_1} \right) \otimes \rho_{\text{app}_2}^{s_2|Z_1 = z_1}.
\]  

(131)

Thus, the flashes of \( a \cup \text{app}_2 \) after \( s_2 \) are distributed as if \( a \) started at time \( s_2 \) with density matrix

\[
\rho_{a}^{s_2} = \mathcal{S}_{[t_1, s_2]} \text{tr}_{b_1 \cup \text{app}_1} \rho_{a \cup b_1 \cup \text{app}_1}^{s_2|Z_1 = z_1}.
\]  

(132)

Thus, the conditional probability of \( Z_2 = z_2 \) given \( Z_1 = z_1 \) is given by

\[
\mathbb{P}(Z_2 = z_2|Z_1 = z_1) = \text{tr} \left( \mathcal{S}_{[t_1, s_2]} \left[ \text{tr}_{b_1 \cup \text{app}_1} \rho_{a \cup b_1 \cup \text{app}_1}^{s_2|Z_1 = z_1} \right] E_2(z_2) \right).
\]  

(133)

Comparison with (115) shows that the joint distribution of \( Z_1 \) and \( Z_2 \) comes out right if

\[
\mathcal{C}_{1, z_1}(\rho_{s_1}) = \text{tr}_{\text{app}_1} \rho_{a \cup \text{app}_1}^{s_1|Z_1 = z_1} = \frac{1}{N} \text{tr}_{\text{app}_1} \int df L_{[s_1, t_1]}(f) \rho_{s_1}^{a \cup \text{app}_1} L^*_{[s_1, t_1]}(f) = \frac{1}{N} \text{tr}_{\text{app}_1} \int df L_{[s_1, t_1]}(f) \rho_{s_1}^{a \cup \text{app}_1} L^*_{[s_1, t_1]}(f)
\]

which is equivalent to the equation (120), the GRW law of superoperators.
8 Random Run-Time

We now discuss the case in which duration of the experiment $E$ is not fixed. Rather, we assume that the time at which $E$ is finished is itself a random quantity $T$, generated by $E$ itself. The starting time $s$, in contrast, is assumed to be fixed. We assume that $T$ can take values from a finite or countable set $\mathcal{T} \subseteq [s, \infty)$ (just as we assumed that the value space $\mathcal{Z}$ is discrete).

8.1 GRW Formalism for Random Run-Time

The assumption $Z = \zeta(F)$ now gets complemented by the assumption $T = \tau(F)$, i.e., the finishing time $T$ can be read off from the flashes. On top of that, we assume that the random variable $T = \tau(F)$ is a stopping time in the sense of the theory of stochastic processes, i.e., that $\tau$ is such that

\[ \tau(F) \leq t \] depends only on $F_{[s,t]}$, \hspace{1cm} (134)

i.e., on the flashes up to time $t$. In other words, we require that it is possible to read off from the flashes up to time $t$ whether the experiment is over yet. In the terminology of the theory of stochastic processes, the space $\Omega_{[s,\infty)}$ of all possible flash patterns is naturally equipped with a filtration $(\mathcal{A}_{[s,t)})_{t \in (s,\infty)}$, where $\mathcal{A}_{[s,t)}$ is the $\sigma$-algebra of all events that depend only on the flashes before $t$ (i.e., it is the collection of those $B \subseteq \Omega_{[s,\infty)}$ such that for any two $f, f' \in \Omega_{[s,\infty)}$ with $f_{[s,t)} = f'_{[s,t)}$, either both $f, f' \in B$ or both $f, f' \notin B$); our assumption that $T$ is a stopping time means that for each $t$, the event $\{T \leq t\}$, regarded as the set $\{f : \tau(f) \leq t\}$, belongs to $\mathcal{A}_t$.

Moreover, we assume that

\[ \text{if } \tau(F) = t \text{ then } \zeta(F) \text{ depends only on } F_{[s,t)}. \] \hspace{1cm} (135)

That is, when the experiment is over it must be possible to read off the result from the flashes so far. In the terminology of the theory of stochastic processes, this assumption is that $Z$ is adapted to the $\sigma$-algebra $\mathcal{A}_{[s,T)}$ of all events that depend only on the flashes before $T$ (i.e., it is the collection of those $B \subseteq \Omega_{[s,\infty)}$ such that for any two $f, f' \in \Omega_{[s,\infty)}$ with $\tau(f) = \tau(f') =: t$ and $f_{[s,t)} = f'_{[s,t)}$, either both $f, f' \in B$ or both $f, f' \notin B$).

In this setting, the formalism reads as follows.

The GRW Formalism for Random Run-Time.

- A system isolated from its environment has at every time $t$ a density matrix $\rho_t$ which evolves according to the master equation $\rho_t$.

- With every experiment $E$ starting at time $s$ with a discrete set $\mathcal{Z}$ of possible outcomes and a discrete set $\mathcal{T} \subseteq [s, \infty)$ of possible finishing times, there is associated a POVM $E^{GRW}(\cdot)$ on $\mathcal{Z} \times \mathcal{T}$ acting on $\mathcal{H}_{\text{sys}}$. When the experiment $E$ is performed on a system with density matrix $\rho_s$, the outcome $Z$ and the time $T$ at which $E$ is finished are random with joint probability distribution

\[ \mathbb{P}(Z = z, T = t) = \text{tr}(\rho_s E^{GRW}_{z,t}) \] \hspace{1cm} (136)
With $\mathcal{E}$ is further associated a family $(C^{\text{GRW}}_{z,t})_{z \in \mathcal{Z}, t \in \mathcal{T}}$ of completely positive superoperators acting on $\text{TRCL}(\mathcal{H}_{\text{sys}})$ with the consistency property that for all trace-class operators $\rho$,

$$\text{tr}(\rho E^{\text{GRW}}_{z,t}) = \text{tr} C^{\text{GRW}}_{z,t} (\rho).$$  \hfill (137)

In case $Z = z$ and $T = t$, the density matrix of the system at time $t$ immediately after the experiment $\mathcal{E}$ is

$$\rho' = \rho_t = \frac{C^{\text{GRW}}_{z,t}(\rho_s)}{\text{tr} C^{\text{GRW}}_{z,t}(\rho_s)}.$$  \hfill (138)

**The GRW Law of Operators for Random Run-Time.**

- Suppose we are given the density matrix $\rho_{\text{app}}$ for the ready state of the apparatus, its Hamiltonian $H_{\text{app}}$, and the interaction Hamiltonian $H_I$, so that $H = H_{\text{sys}} + H_{\text{app}} + H_I$. Let the experiment $\mathcal{E}$ start at time $s$, let $\zeta : \Omega_{[s,\infty)} \to \mathcal{Z}$ be the function that reads off the outcome of $\mathcal{E}$ from the flashes, and let $\tau : \Omega_{[s,\infty)} \to \mathcal{T}$ be the function that reads off the finishing time of $\mathcal{E}$ from the flashes. Then

$$E^{\text{GRW}}_{z,t} = \text{tr}_{(s,t)} \left( [I_{\text{sys}} \otimes \rho_{\text{app}}] G(\zeta^{-1}(z) \cap \tau^{-1}(t)) \right)$$

$$= \text{tr}_{(s,t)} \left( \int_{\zeta^{-1}(z) \cap \tau^{-1}(t)} df [I_{\text{sys}} \otimes \rho_{\text{app}}] L^*_{(s,t)}(f) L_{(s,t)}(f) \right);$$  \hfill (140)

and

$$C^{\text{GRW}}_{z,t}(\rho) = \text{tr}_{(s,t)} \left( \int_{\zeta^{-1}(z) \cap \tau^{-1}(t)} df L_{(s,t)}(f) [\rho \otimes \rho_{\text{app}}] L^*_{(s,t)}(f) \right).$$  \hfill (141)

Concerning (140), note that by assumptions (134) and (135), the set $\zeta^{-1}(z) \cap \tau^{-1}(t) \subseteq \Omega_{[s,\infty)}$ is of the form $A \times \Omega_{[t,\infty)}$ for suitable $A \subseteq \Omega_{[s,t)}$. We wrote $\zeta^{-1}(z) \cap \tau^{-1}(t)$ for $A$ in the domain of the integral in (140) and (141); that is, the domain of the integral is to be regarded as a subset of $\Omega_{[s,t)}$, so that the integration variable $f$ is a history of flashes in the time interval $[s, t)$ and thus can be inserted into $L_{(s,t)}$. To see that (140) is the same as (139), note that

$$G(\zeta^{-1}(z) \cap \tau^{-1}(t)) = G(A \times \Omega_{[t,\infty)}) = G_{(s,t)}(A) = \int_A df L^*_{(s,t)}(f) L_{(s,t)}(f)$$

using (31). We check the consistency condition (137) in Appendix B.
8.2 Derivation

The biggest difference to the derivation of the GRW formalism for fixed run-time is that we now have to consider a system that is isolated from its environment only during the random time interval \([s,T]\).

In particular, we need a version of the marginal probability formula for stopping times: Consider a system “sys” (such as \( \text{sys} \cup \text{app} \)), let \( T \) be a stopping time adapted to \( \text{sys} \) (i.e., a function \( \tau \) of \( F_{\text{sys}}^{[0,\infty)} \) such that the event \( T = \tau(F_{\text{sys}}) \leq t \) depends only on \( F_{[0,t)}^{\text{sys}} \)), and let \( A_{[0,T)}^{\text{sys}} \) be the \( \sigma \)-algebra of events depending only on the flashes of \( \text{sys} \) up to time \( T \). If the system is isolated during \([0,T)\) then

\[
P_{\Psi_0}(F_{\text{sys}} \in B) = P_{\rho_{\text{sys}}}(B) \quad \forall B \in A_{[0,T)}^{\text{sys}}.
\]

(142)

Put differently, this means for every \( t > 0 \) and every \( B \subseteq \Omega_{[0,t)}^{\text{sys}} \),

\[
P_{\Psi_0}(F_{[0,t)}^{\text{sys}} \in B \mid \tau(F_{\text{sys}}) = t) = P_{\rho_{\text{sys}}}(F_{[0,t)} \in B \mid F = t).
\]

(143)

This fact follows from the marginal probability formula (94) in much the same way as the version (99) for a system that is isolated during the deterministic interval \([0,t)\). Consider a hypothetical universe whose time-dependent Hamiltonian \( H_t \) and collapse operators \( \Lambda_{i,t}(x) \) are whatever we choose, and a fixed initial wave function \( \Psi_0 \), so that the distribution of flashes during \([0,t)\) will not depend on our choices of \( H_s \) for \( s \geq t \). So we need not specify before \( t \) whether we will turn on the interaction at \( t \) or not, and we could make this decision depend on the flashes up to time \( t \). Since this choice does not affect the distribution of the flashes before \( t \), this distribution is the same as it would have been if the system were isolated forever, and thus given by (94).

Together with the marginal probability formula for stopping times we also obtain a marginal master equation for stopping times: If a system is isolated during \([0,T)\) then its reduced density matrix will evolve during \([0,T)\) according to the master equation for the system.

With these tools, the derivation of the GRW formalism for random run-time follows the same lines as the derivation of the GRW formalism for fixed run-time.

8.3 Quantum Formalism for Random Run-Time

The quantum version is exactly the same, except with (i) the master equation (1) replaced by the Schrödinger equation, (ii) the POVM \( E^{\text{GRW}}(\cdot) \) replaced by a different one \( E^{\text{Qu}}(\cdot) \), and (iii) the superoperators \( C^{\text{GRW}}_{z,t} \) replaced by different ones \( C^{\text{Qu}}_{z,t} \). Here is how those operators are determined.

The Quantum Law of Operators for Random Run-Time.

- Given the density matrix \( \rho_{\text{app}} \) for the ready state of the apparatus, its Hamiltonian \( H_{\text{app}} \), and the interaction Hamiltonian \( H_I \), so that \( H = H_{\text{sys}} + H_{\text{app}} + H_I \), and \( U_t = \exp(-\frac{i}{\hbar} H t) \) is the unitary Schrödinger evolution operator for system + apparatus.
Let $P_{app}^{z,t}$ be the projection to the subspace of apparatus states in which $E$ is over, the pointer for the outcome is pointing to $z$, and the pointer for the time when $E$ was over is pointing to $t$. Then

$$E_{z,t}^{Qu} = \text{tr}_{app}\left([I_{\text{sys}} \otimes \rho_{app}]U_{t-s}^* [I_{\text{sys}} \otimes P_{app}^{z,t}]U_{t-s}\right)$$

(144)

and

$$C_{z,t}^{Qu}(\rho) = \text{tr}_{app}\left([I_{\text{sys}} \otimes P_{app}^{z,t}]U_{t-s}[\rho \otimes \rho_{app}]U_{t-s}^* [I_{\text{sys}} \otimes P_{app}^{z,t}]\right).$$

(145)

We check the consistency condition in Appendix B.

How do we know that (144) and (145) are the right formulas? Given their complexity, and given that the consideration of random run-times is somewhat intricate, we need a derivation. But derivation from what? One possibility is to derive them from Bohmian mechanics, where the quantum formalism is thought of as a consequence of more fundamental postulates. Another possibility is to base a derivation on the following principle: The predictions should not depend on where to put the “Heisenberg cut” between (quantum) system and (classical) apparatus. In particular, the predictions should be invariant under moving the Heisenberg cut outward, i.e., they should not change when we treat the apparatus as a quantum system and assume a “super-apparatus” for “measuring” where the pointer of the apparatus points. With the aid of this principle one could, for example, derive the quantum formalism from postulates that exclusively concern position measurements (applied to the pointer). Similarly, one could derive the formalism for random run-time from the formalism for fixed run-time, by assuming that all outcomes $Z$ and times $T$ get recorded, while all apparatus is included in the Schrödinger evolution until a very late time, at which all experiments are expected to be finished, when we use the super-apparatus to read out the records. The details of such a derivation shall not be worked out here.

9 Genuine Measurements

Genuine measurements are experiments for determining values of the variables of the theory, as opposed to quantum measurements, which do not actually measure anything in the ordinary sense of the word, i.e., do not measure any pre-existing property. Genuine measurements in GRWm, for example, would be experiments determining $m(x,t)$, or the wave function, or some functional thereof. In this section we discuss the possibilities and limits of genuine measurements in GRWm and GRWf. We plan to provide a more thorough discussion in a future work [5].

9.1 Limits on Knowledge

We show that it is impossible to measure, with microscopic accuracy,
(i) the matter density \(m(x,t)\) in GRWm

(ii) the wave function \(\psi_t\) of a system in either GRWm or GRWf.

Furthermore, we conjecture that it is also impossible to measure

(iii) the space-time pattern of flashes \(F\) in GRWf or of the collapse centers in GRWm

(iv) the number \(C_{[s,t]}\) of collapses in a system during \([s,t]\) in either GRWm or GRWf

In other words, the exact values of these variables are \textit{empirically undecidable}. In contrast, it is possible to measure, with certain accuracy and reliability, the macroscopic equivalence class of either \(m(\cdot, t)\) in GRWm, or of \(F\) in GRWf, or of \(\psi_t\) in both GRWm and GRWf. Further genuine measurements are possible when information about the wave function is provided, as we will explain in Section 9.2.

Let us compare this situation to that of Bohmian mechanics. Also Bohmian mechanics entails limits on knowledge: for example there is no experiment in a Bohmian world that will reveal the velocity of a given particle (unless information about its wave function is given) \textsuperscript{25, 26}. On the other hand, there is no limitation in Bohmian mechanics to measuring the position of a particle, except that doing so will alter the particle’s wave function, and thus its future trajectory. Here we encounter a basic difference between Bohmian mechanics and GRWm: the configuration of the PO can be measured in Bohmian mechanics but not in GRWm. (In Bohmian mechanics, the configuration of the PO corresponds to the positions of all particles, while in GRWm it corresponds to the \(m(x,t)\) function for all \(x \in \mathbb{R}^3\).) In GRWf, for comparison, there is nothing like a configuration of the PO \textit{at time} \(t\), of which we could ask whether it can be measured. There is only a space-time \textit{history} of the PO, which we may wish to measure. Bohmian mechanics is an example of a world in which the history of a system cannot be measured without disturbing its course, and indeed disturbing it all the more drastically the more accurately we try to measure it. This suggests that also in GRWf, measuring the pattern of flashes may entail disturbing it—and thus finding a pattern of flashes that is different from what would have occurred naturally (i.e., without intervention). This kind of measurement is not what was intended when wishing to measure the history of flashes.

About (iv), we conjecture further that, unless information about the system’s wave function is given, no experiment can reveal any information at all about the random number \(C_{[s,t]}\). This means the following. Without any experiment we can say that \(C_{[s,t]}\) has a Poisson distribution with expectation \(N\lambda(t - s)\), i.e.,

\[
\mathbb{P}(C_{[s,t]} = n) = \frac{1}{n!} (N\lambda(t - s))^n e^{-N\lambda(t-s)},
\]

where \(N\lambda\) is the collapse rate of the system (see after (6)). The conjecture is that no experiment on the system can produce an outcome \(Z\) such that the conditional distribution \(\mathbb{P}(C_{[s,t]} = n|Z)\) would be narrower than (146), or indeed in any way different from (146).

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9.2 If Information About the Initial Wave Function Is Given

Further genuine measurements are possible when information about the wave function is provided. What does that mean? For example, while there presumably exists no experiment that could measure the number $C_{s,t}$ of collapses between $s$ and $t$ on any given system with any wave function, there do exist experiments that work for one particular wave function $\psi$ and can, for a system with initial wave function $\psi_s = \psi$, disclose at least partial information about this number. We describe a concrete example experiment which, though it does not reliably determine $C_{s,t}$, can estimate $C_{s,t}$ better than one could without performing any experiment.

Suppose $\psi$ is the wave function of a single electron and a superposition of two wave packets,

$$\psi = \frac{1}{\sqrt{2}}|\text{here}\rangle + \frac{1}{\sqrt{2}}|\text{there}\rangle,$$  \hspace{1cm} (147)

as may result from a double-slit setup. Suppose, for simplicity, that the Hamiltonian of the system vanishes, so that the time evolution is trivial, and that the time span $t - s$ is of the order $1/N\lambda$, so that $\mathbb{P}(C_{s,t} = 0)$ is neither close to 1 nor close to 0. We ask whether $C_{s,t}$ is zero or nonzero, i.e., whether a collapse has occurred. Without any experiment, one can only say that the probability that a collapse has occurred is

$$p = 1 - e^{-N\lambda(t-s)}.$$ \hspace{1cm} (148)

The following experiment provides further information. The task is equivalent to determining whether $\psi_t = \frac{1}{\sqrt{2}}|\text{here}\rangle + \frac{1}{\sqrt{2}}|\text{there}\rangle$ or $\psi_t = |\text{here}\rangle$ or $\psi_t = |\text{there}\rangle$, since any collapse would effectively reduce (147) to either $|\text{here}\rangle$ or $|\text{there}\rangle$. To this end, carry out a “quantum measurement of the observable” $O$ given by the projection to the 1-dimensional subspace spanned by (147). If the result $Z$ was zero, then it can be concluded that a collapse has occurred, or $C_{s,t} > 0$. If the result $Z$ was 1, nothing can be concluded with certainty (since also $|\text{here}\rangle$ and $|\text{there}\rangle$ lead to a probability of $1/2$ for the outcome to be 1). However, in this case the (Bayesian) conditional probability that a collapse has occurred is less than $p$ (and thus $Z$ is informative about $C_{s,t}$):

$$\mathbb{P}(C_{s,t} > 0 | Z = 1) = \frac{\mathbb{P}(Z = 1 | C_{s,t} > 0) \mathbb{P}(C_{s,t} > 0)}{\mathbb{P}(Z = 1 | C_{s,t} > 0) \mathbb{P}(C_{s,t} > 0) + \mathbb{P}(Z = 1 | C_{s,t} = 0) \mathbb{P}(C_{s,t} = 0)} = \frac{\frac{1}{2}p}{\frac{1}{2}p + 1 \cdot (1 - p)} = \frac{p}{2 - p} < p.$$  

Thus, in every case the experiment can retrodict $C_{s,t}$ with greater reliability than it could have been predicted a priori.

This leads us to the question whether it is possible, for a known initial wave function, to determine reliably whether a collapse has occurred or not. We conjecture that the answer is no, and that indeed no other experiment can retrodict $C_{s,t}$ for the initial wave function (147) with greater reliability than the quantum measurement of $O$. 

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We close this subsection with another example of how information about $\psi$ can help with genuine measurements. Trivially, if we want to measure a system's wave function $\psi$ then every given information about $\psi$ is of immediate advantage. Concretely, if we know that $\psi$ is one of the mutually orthogonal vectors $\phi_1, \ldots, \phi_n$, then a “quantum measurement of the observable” $A = \sum_k k |\phi_k\rangle \langle \phi_k|$ will reveal which one it is.

9.3 The Quadratic Functional Argument

We now return to the problem of measuring quantities without knowing the initial wave function. There is a simple argument, the quadratic functional argument, that will prove the impossibility claims (i) and (ii). This argument was first used, to our knowledge, in [25] in the context of Bohmian mechanics, and goes as follows. If the same experiment is supposed to measure a quantity $Z$ for every initial wave function $\psi$ of the system, then the probability distribution of $Z$ must be a quadratic functional of $\psi$, i.e., $P(Z = z) = \langle \psi | E(z) | \psi \rangle$ for some POVM $E(\cdot)$, since, by the GRW formalism, for every experiment the distribution of its results is a quadratic functional of $\psi$. This allows us to conclude that a quantity whose distribution is not quadratic in $\psi$ cannot be measured.

We list some such quantities:

- The wave function $\psi$ itself, since its distribution as a functional of $\psi$ is a $\delta$ distribution at $\psi$. (More explicitly, $Z = \psi$ would have the distribution $P(Z \in B) = 1_B(\psi)$, which is 1 if $\psi \in B$ and 0 otherwise, for any subset $B \subset \mathbb{Z} = S(\mathcal{H})$ of the unit sphere in Hilbert space. Unlike $\psi \mapsto \langle \psi | E(B) | \psi \rangle$, the step function $1_B$ is not a quadratic function.)

- Also the distribution of the wave function $\psi_t$ at a later time $t$, arising from the initial $\psi = \psi_s$ through the GRW evolution, is not quadratic in $\psi$.

- The distribution of $m(x,t)$ is not quadratic in $\psi_s$; in fact, for $t = s$ it is a $\delta$ distribution. More generally, any quantity that is deterministic in $\psi_s$, i.e., given by a functional of $\psi_s$, has a $\delta$ distribution not quadratic in $\psi_s$.

We have thus proved statements (i) and (ii). Since the distribution of $F$ is in fact a quadratic functional of $\psi$, the quadratic functional argument does not yield statement (iii). If we could measure wave functions, we would be able to detect collapses by measuring the wave function before and after; but we cannot.

Let us now turn to the heuristic behind the conjecture that flashes cannot be measured. Here is a very simple, non-rigorous argument suggesting this. Suppose we had an apparatus capable of detecting flashes in a system. Think of the wave function of system and apparatus together as a function on configuration space $\mathbb{R}^{3N}$. There is a region in configuration space containing the configurations in which the apparatus display reads “no flash detected so far,” and another region, disjoint from the first, containing the configurations in which the display reads “one flash detected so far.”
Recall that a flash in the system leads to a change in the wave function of the form
\[ \psi \to \psi' = \Lambda_i(x)\psi / \| \Lambda_i(x)\psi \| , \]
where \( \Lambda_i(x) \) is a multiplication operator. But for such a change it is impossible to push the wave function from the first region to the second.

Here is a somewhat similar argument concerning the wish to measure the location \( X \) of a flash. For simplicity, let us assume the system consists of a single “particle,” and let us further assume we are given the following information about a system’s wave function: It is a superposition of finitely many disjoint packets \( \psi\ell \), each so narrow that the width is much smaller than \( \sigma \), and any two so very disjoint that the distance is much greater than \( \sigma \). Then a collapse will essentially remove all but one of these packets. Now a collapse acting on the system can indeed force the apparatus into a particular state, for example if the wave function of system and apparatus together before the collapse was
\[ \sum \psi\ell \otimes \phi\ell , \tag{149} \]
where \( \phi\ell \) may be a state in which the apparatus displays the location of \( \psi\ell \) as the location of the flash. The state \( (149) \) may arise from the initial state \( (\sum \psi\ell) \otimes \phi_0 \) by means of the interaction between the system and the apparatus. However, in case no flash occurs, the reduced density matrix of the system arising from the state \( (149) \) would be \( \sum \ell |\psi\ell\rangle \langle \psi\ell| \), which leads to a different distribution of flashes than the pure state \( (\sum \psi\ell) (\sum \psi\ell) \). This means that the presence of the apparatus has altered the distribution of the future flashes. Moreover, the state \( (149) \) represents essentially a quantum position measurement, and will collapse most probably because of flashes associated with the apparatus, thus forcing the first system flash to occur at the location the was the outcome of the position measurement.

## 10 Conclusions

We have formulated a GRW formalism that is analogous to, but not the same as, the quantum formalism and summarizes the empirical content of both GRWm and GRWf. We have given a derivation of the GRW formalism based on the primitive ontology (PO). We have further shown that several quantities that are real in the GRWm or GRWf worlds cannot be measured by the inhabitants of these worlds. These were the main contributions of this paper. Derivations of the empirical predictions of GRW theories have been given before in \[28, 9, 6, 7, 8\], but with two gaps: First, these derivations did not pay attention to the role of the PO. Second, these derivations focused on how to obtain the quantum probabilities from GRW theories, and thus ignored the (usually tiny) differences between the empirical predictions of GRW theories and those of quantum mechanics. On the other hand, earlier derivations of empirical deviations from quantum mechanics \[28, 39, 36, 6, 31, 1\] focused on particular experiments but did not provide a general formalism.

It has played an important role for our analysis that the GRW theories are given by explicit equations. Other collapse theories, for example that of Penrose \[37, 38\], are formulated in a more vague way that still permits to arrive at concrete testable
predictions deviating from quantum mechanics but does not permit any general theorems about arbitrary experiments. The concreteness of the GRW theories also has (what may seem like) disadvantages, as it gives the theory a flavor of arbitrariness, and that of a being “merely a toy model,” as opposed to a serious theory. For example, arbitrariness may be seen in the existence of the two parameters λ and σ (whose values remain unknown until experiments confirm deviations from quantum mechanics), or in the choice of the Gaussian (could it not be another function instead of a Gaussian?), or in the assumption that collapses are instantaneous, or in other aspects. But in the end of the day it is the concreteness of the GRW theories, or their explicit character, that paves the way for their successful analysis. In this paper in particular, theorems are established about the GRW theories, and this would not have been possible if the GRW theories had not been defined by unambiguous mathematics. Since we are dealing with concrete equations, we can derive precisely what predictions these equations entail—with rather unexpected results, such as the emergence of a simple operator formalism.

It has also played an important role to be explicit about the PO, i.e., to say clearly what the PO is and to specify an equation governing the PO, namely (13) respectively (13). To provide such an equation is somewhat unusual; instead it is often silently assumed that when ψt is the wave function of a live cat then there is a live cat. Our derivation of the GRW formalism relied on this equation (13), which makes the structure of the argument explicit and simple.

Another question arises once the GRW formalism is formulated: Should we not, given that the GRW formalism summarizes the empirical contents of GRWm/GRWf, keep only the GRW formalism as a physical theory and abandon GRWm and GRWf? No. From a positivistic point of view it would seem so because in this view only empirical predictions are regarded as scientific, meaningful statements. But in our view, this position is exaggerated. The goal of science is not only to summarize empirical observations but also to explore explanations of the observations. It is entirely reasonable to ask for a theory that speaks about reality and not about observations, i.e., for a quantum theory without observers.

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A Proof of the Conditional Probability Formula

Note first that, for \( t < \infty \),

\[
L_{[0,t]}(f) = L_{[s,t]}(f_{[s,t]}) L_{[0,s]}(f_{[0,s]})
\]

(150)

and

\[
\int_{\Omega_{[s,t]}} df L^*_{[s,t]}(f) L_{[s,t]}(f) = G_{[s,t]}(\Omega_{[s,t]}) = I,
\]

(151)
the identity operator. As a consequence,
\[
\mathbb{P}_{\psi_0}\left(F_{[s,t]} \in B \mid F_{[0,s]} = f_{[0,s]}\right) =
\]
\[
= \left\langle \psi_0 \left| \int_B df_{[s,t]} L^*_{[0,t]}(f_{[0,s]} \cup f_{[s,t]}) L_{[0,t]}(f_{[0,s]} \cup f_{[s,t]}) \right| \psi_0 \right\rangle
\]
\[
= \left\langle \psi_0 \left| \int_{\Omega(s,t)} df_{[s,t]} L^*_{[0,t]}(f_{[0,s]} \cup f_{[s,t]}) L_{[0,t]}(f_{[0,s]} \cup f_{[s,t]}) \right| \psi_0 \right\rangle
\]
\[
= \left\langle \psi_0 \left| L^*_{[0,s]}(f_{[0,s]}) \left( \int_B df_{[s,t]} L^*_{[s,t]}(f_{[s,t]}) L_{[s,t]}(f_{[s,t]}) \right) L_{[0,s]}(f_{[0,s]}) \right| \psi_0 \right\rangle
\]
\[
= \left\langle \psi_s \left| \int_B df_{[s,t]} L^*_{[s,t]}(f_{[s,t]}) L_{[s,t]}(f_{[s,t]}) \right| \psi_s \right\rangle = \mathbb{P}_{\psi_s}(F_{[s,t]} \in B) .
\]
This proves the conditional probability formula for \( t < \infty \). The one for \( t = \infty \) follows from the one for finite \( t \) because in the \( \sigma \)-algebra of \( \Omega_{[s,\infty)} \), the family \( A_{\text{finite}} \) of events depending only on a finite amount of time form a \( \cap \)-stable generator, and thus the two measures \( \mathbb{P}_{\psi_0}(F_{[t,\infty]} \in \cdot \mid F_{[0,s]} = f_{[0,s]} \) and \( \mathbb{P}_{\psi_s}(F_{[s,\infty]} \in \cdot) \) coincide since they coincide on \( A_{\text{finite}} \).

\section{Check of Consistency Conditions}

We provide the proofs of the equations expressing the consistency property between the POVM \( E(\cdot) \) and the superoperator \( \mathcal{E}_z \) as defined in the various versions of the law of operators. We often use the following mathematical fact: If \( \mathcal{H}_{a \cup b} = \mathcal{H}_a \otimes \mathcal{H}_b \), \( S_a \) is an operator on \( \mathcal{H}_a \), and \( T_{a \cup b} \) is an operator on \( \mathcal{H}_{a \cup b} \) then
\[
S_a \text{ tr}_b T_{a \cup b} = \text{ tr}_b ([S_a \otimes I_b] T_{a \cup b}) ,
\]
where \( \text{ tr}_b \) means the partial trace.

To check the consistency property \((50)\) between \((63)\) and \((64)\), note that
\[
\text{ tr}(\rho E^\text{Qu}_z) = \text{ tr}\left( [\rho \otimes I_{\text{env}}] [I_{\text{sys}} \otimes \rho_{\text{app}}] U^*_{t-s} [I_{\text{sys}} \otimes P^\text{app}_z] U_{t-s} \right) =
\]
\[
= \text{ tr}\left( [\rho \otimes \rho_{\text{app}}] U^*_{t-s} [I_{\text{sys}} \otimes P^\text{app}_z] U_{t-s} \right) =
\]
\[
= \text{ tr}\left( U_{t-s} \rho \otimes \rho_{\text{app}} U^*_{t-s} [I_{\text{sys}} \otimes P^\text{app}_z] \right) =
\]
\[
= \text{ tr}\left( [I_{\text{sys}} \otimes P^\text{app}_z] U_{t-s} \rho \otimes \rho_{\text{app}} U^*_{t-s} [I_{\text{sys}} \otimes P^\text{app}_z] \right) = \text{ tr}\mathcal{E}^\text{Qu}_z(\rho) ,
\]
where \( \text{ tr} \) always means the trace, sometimes on \( \mathcal{H}_{\text{sys}} \) and sometimes on \( \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{env}} \).
To check the consistency property (67) between (69) and (70), note that
\[
\text{tr}(\rho E^\text{GRW}_z) = \text{tr}\left([\rho \otimes I_{\text{app}}] \int d\tau [I_{\text{sys}} \otimes \rho_{\text{app}}] L^*_{(s,t)}(f) L_{(s,t)}(f)\right) = \\
= \text{tr} \int d\tau L_{(s,t)}(f) [\rho \otimes \rho_{\text{app}}] L^*_{(s,t)}(f) = \text{tr} \mathcal{E}_{z}^\text{GRW}(\rho).
\]

To check the consistency condition (137) between (140) and (141), note that
\[
\text{tr}(\rho E^\text{GRW}_{z,t}) = \text{tr}\left([\rho \otimes I_{\text{app}}] \int d\tau [I_{\text{sys}} \otimes \rho_{\text{app}}] L^*_{(s,t)}(f) L_{(s,t)}(f)\right) = \\
= \text{tr} \int d\tau L_{(s,t)}(f) [\rho \otimes \rho_{\text{app}}] L^*_{(s,t)}(f) = \text{tr} \mathcal{E}_{z,t}^\text{GRW}(\rho).
\]

To check the consistency condition between (144) and (145), note that
\[
\text{tr}(\rho E^\text{Qu}_{z,t}) = \text{tr}\left([\rho \otimes I_{\text{app}}] [I_{\text{sys}} \otimes \rho_{\text{app}}] U^*_{t-s} [I_{\text{sys}} \otimes P_{z,t}^\text{app}] U_{t-s}\right) = \\
= \text{tr}\left([\rho \otimes \rho_{\text{app}}] U^*_{t-s} [I_{\text{sys}} \otimes P_{z,t}^\text{app}] U_{t-s}\right) = \\
= \text{tr}\left([I_{\text{sys}} \otimes P_{z,t}^\text{app}] U_{t-s} [\rho \otimes \rho_{\text{app}}] U^*_{t-s} [I_{\text{sys}} \otimes P_{z,t}^\text{app}]\right) = \text{tr} \mathcal{E}_{z,t}^\text{Qu}(\rho).
\]

\[\text{C Proof of the Marginal Probability Formula}\]

Let us suppose for simplicity that the set $F_{\text{sys}}$ of the flashes of the system corresponds, not to a region $R_{\text{sys}}$ in space, but to a subset $\mathcal{L}_{\text{sys}}$ of labels. (We come back to the other case below.) Thus, the assumption (72) now reads
\[
\Lambda_i(x) = \begin{cases} 
\Lambda^\text{sys}_i(x) \otimes I_{\text{env}} & \text{if } i \in \mathcal{L}_{\text{sys}} \\
I_{\text{sys}} \otimes \Lambda^\text{env}_i(x) & \text{if } i \notin \mathcal{L}_{\text{sys}}.
\end{cases}
\tag{153}
\]

Since the system is not interacting with its environment, as expressed by (71), we have that $U_t = e^{-iH_{\text{sys}}/\hbar} \otimes e^{-iH_{\text{env}}/\hbar} = U^\text{sys}_t \otimes U^\text{env}_t$. Now it follows that
\[
L(f) = L_{\text{sys}}(f_{\text{sys}}) \otimes L_{\text{env}}(f_{\text{env}}),
\tag{154}
\]
where $f_{\text{sys}}$ (respectively $f_{\text{env}}$) is the set of flashes belonging to the system (respectively the environment) and, as the notation suggests,
\[
L_{\text{sys}}\left(\{(x_1, t_1, i_1), \ldots, (x_n, t_n, i_n)\}\right) = \lambda^{n/2} \mathcal{L}_{t_0 < t_1 < \cdots < t_n} \lambda^{\sum_{i=1}^{n} \Lambda^\text{sys}_{t_i}(x_i) \Lambda^\text{sys}_{t_{i-1}}(x_{i-1})} \times \\
\times \Lambda^\text{sys}_{t_n}(x_n)^{1/2} U^\text{sys}_{t_n - t_{n-1}} \cdots \Lambda^\text{sys}_{t_1}(x_1)^{1/2} U^\text{sys}_{t_1 - t_0}.\tag{155}
\]
with \( N_{\text{sys}} = \# \mathcal{L}_{\text{sys}} \), and similarly for \( L_{\text{env}} \). As a consequence,
\[
G(B_{\text{sys}} \times B_{\text{env}}) = G_{\text{sys}}(B_{\text{sys}}) \otimes G_{\text{env}}(B_{\text{env}}),
\]
where \( G_{\text{sys}} \) (respectively \( G_{\text{env}} \)) is the POVM that would govern the system (respectively the environment) if it were alone in the universe. (In particular, the marginal of \( G_{\text{sys}} \) for the first \( n \) flashes is given by
\[
G_{\text{sys},n}(B) = \int_B df L_{\text{sys}}(f)^* L_{\text{sys}}(f) \quad \forall B \subseteq \Omega_n,
\]
in parallel to (30)). From (156) we obtain the marginal probability formula:
\[
\mathbb{P}_{\Psi_0}(F_{\text{sys}} \in B_{\text{sys}}) = \langle \Psi_0 | G(B_{\text{sys}} \times \Omega_{\text{env}}) | \Psi_0 \rangle = \\
= \langle \Psi_0 | G_{\text{sys}}(B_{\text{sys}}) \otimes I_{\text{env}} | \Psi_0 \rangle = \text{tr}\left( \rho_{\text{sys}} G_{\text{sys}}(B_{\text{sys}}) \right)
\]
with \( \rho_{\text{sys}} = \text{tr}_{\text{env}} | \Psi_0 \rangle \langle \Psi_0 | \).

The case in which the system is defined in terms of a region \( R_{\text{sys}} \) is best treated in the framework described in [41, 45] that generalizes the GRW process to arbitrary positive operators \( \Lambda_i(x) \). In that framework, eq.s (10) and (12) get modified, and it is then easy to arrive at (154) starting from (72).

**D Proof of the Marginal Master Equation**

We now provide a proof of the fact, described around (101), that for two non-interacting but entangled systems \( a \) and \( b \), also the reduced density matrix of system \( a \) evolves according to the appropriate version of the master equation (1). This follows from two ingredients: first,
\[
L(f) = L_a(f_a) \otimes L_b(f_b),
\]
which is shown as eq. (154) in the proof of the marginal probability formula in Appendix C and second,
\[
\rho_{t}^{a \cup b} = \int_{\Omega_{[0,t]}} df L_{[0,t]}(f) \rho_{0}^{a \cup b} L_{[0,t]}^*(f) ,
\]
which is the representation (90) of the density matrix corresponding to the random GRW wave function, applied to system \( a \) and system \( b \) together. Now it follows, using \( f = f_a \cup f_b \), that
\[
\rho_{t}^{a} = \text{tr}_b \rho_{t}^{a \cup b} = \text{tr}_b \int df_a \int df_b [L_{[0,t]}^a(f_a) \otimes L_{[0,t]}^b(f_b)] \rho_{0}^{a \cup b} [L_{[0,t]}^a(f_a)^* \otimes L_{[0,t]}^b(f_b)^*] = \\
= \int df_a L_{[0,t]}^a(f_a) \text{tr}_b \rho_{0}^{a \cup b} I_a \otimes \int df_b L_{[0,t]}^b(f_b) L_{[0,t]}^b(f_b)^* L_{[0,t]}^a(f_a)^* = \\
= I_a
\]
\[ \int df_a L^a_{[0,t]}(f_a) \left( \text{tr}_b \rho^0_{[0,t]} \right) L^a_{[0,t]}(f_a)^* = \int df_a L^a_{[0,t]}(f_a) \rho^a_{0,t} L^a_{[0,t]}(f_a)^*, \]

which means that the reduced density matrix \( \rho^a_t \) satisfies the appropriate version of the master equation (I).

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