PROBABILISTIC SOLUTION OF THE AMERICAN OPTIONS

ALİ SüLEYMAN ÜSTÜNEL

Abstract. The existence and uniqueness of probabilistic solutions of variational inequalities for the general American options are proved under the hypothesis of hypoellipticity of the infinitesimal generator of the underlying diffusion process which represents the risky assets of the stock market with which the option is created. The main tool is an extension of the Itô formula which is valid for the tempered distributions on \( \mathbb{R}^d \) and for nondegenerate Itô processes in the sense of the Malliavin calculus.

1. Introduction

The difficulty to justify the validity of the probabilistic solutions of the American options is well-known. This is in fact due to the lack of regularity of the classical solutions of the variational inequalities (cf.\[2\]) which are satisfied by the value function which characterizes the Snell envelope (cf. \[10\] for a recent survey about this subject). In particular the value function is not twice differentiable hence the Itô formula is not applicable to apply the usual probabilistic techniques. In the case of Black and Scholes model, there are some results using extensions of the Itô formula for the Brownian motion, which, however, are of limited utility for more general cases.

In this note we give hopefully more general results in the sense that the option is constructed by the assets which obey to a general, finite dimensional stochastic differential equation with deterministic coefficients, i.e., a diffusion process. The basic hypothesis used is the nondegeneracy of this diffusion in the sense of the Malliavin calculus (cf. \[11\]): recall that an \( \mathbb{R}^d \)-valued random variable \( F = (F_1, \ldots, F_d) \), defined on a Wiener space is called nondegenerate (cf. \[11, 17, 18\]) if it is infinitely Sobolev differentiable with respect to the Wiener measure and if the determinant of the inverse of the matrix \( ((\nabla F_i, \nabla F_j)_H : i, j \leq d) \), where \( \nabla \) denotes the Sobolev derivative on the Wiener space, is in all the \( L^p \)-spaces w.r. to the Wiener measure. In this case, the mapping \( f \to f \circ F \), defined from the smooth functions on \( \mathbb{R}^d \) to the space of smooth functions on the Wiener space extends continuously to a linear mapping, denoted as \( T \to T(F), T \in S'(\mathbb{R}^d) \), from the tempered distributions \( S'(\mathbb{R}^d) \) to the space of Meyer distributions on the Wiener space (cf. \[11, 17, 18\]). Similarly, if \( F \) is replaced with an Itô process whose components satisfy similar regularity properties, we obtain an Itô formula for \( T(F_t) - T(F_s), 0 < s \leq t \), where the stochastic integral should be treated as a distribution-valued Gaussian divergence and the absolutely continuous term is a Bochner integral concentrated in some negatively indexed Sobolev space. Moreover, if this latter term is a positive distribution, then the resulting integral is a Radon measure on the Wiener space due to a well-known result about the positive Meyer distributions on the Wiener space (cf. \[15, 16, 17, 18\]).

Having summarized the technical tools that we use, let us explain now the main results of the paper: for the uniqueness result we treat two different situations; namely the first one where the coefficients are time dependent and the variational inequality is interpreted as an evolutionary variational inequality in \( S'(\mathbb{R}^d) \). The second one concerns the case where the coefficients are time-independent and we interpret it as an inequality in the space \( D'(0, T) \otimes S'(\mathbb{R}^d) \) with a boundary condition, which is of course more general than the first one. In both cases the operators are supposed only to be hypoelliptic; a hypothesis which is far more general than the ellipticity hypothesis used in \[2\]. The homogeneity in time permits us more generality since, in this case the time-component regularization by the mollifiers of the solution candidates preserve their property of being negative distributions, hence measures. The existence is studied in the last section using the similar techniques and we
obtain as a by product some regularity results about the solution of the variational inequality. In particular, we realize there that even if the density of the underlying diffusion has zeros, there is still a solution on the open set which corresponds to the region of \([0, T] \times \mathbb{R}^d\) where the density is strictly positive.

2. Preliminaries and notation

Let \(W\) be the classical Wiener space \(C([0, T], \mathbb{R}^n)\) with the Wiener measure \(\mu\). The corresponding Cameron-Martin space is denoted by \(H\). Recall that the injection \(H \hookrightarrow W\) is compact and its adjoint is the natural injection \(W^* \hookrightarrow H^* \subset L^2(\mu)\).

Since the translations of \(\mu\) with the elements of \(H\) induce measures equivalent to \(\mu\), the Gâteaux derivative in \(H\) direction of the random variables is a closable operator on \(L^p(\mu)\)-spaces and this closure will be denoted by \(\nabla\) cf., for example \([4, 17, 18]\). The corresponding Sobolev spaces (the equivalence classes) of the real random variables will be denoted as \(\mathbb{D}_{p, k}(\Phi)\), where \(k \in \mathbb{N}\) is the order of differentiability and \(p > 1\) is the order of integrability. If the random variables are with values in some separable Hilbert space, say \(\Phi\), then we shall define similarly the corresponding Sobolev spaces and they are denoted as \(\mathbb{D}_{p, k}(\Phi), p > 1, k \in \mathbb{N}\). Since \(\nabla : \mathbb{D}_{p, k} \to \mathbb{D}_{p, k-1}(H)\) is a continuous and linear operator its adjoint is a well-defined operator which we represent by \(\delta\). \(\delta\) coincides with the Itô integral of the Lebesgue density of the adapted elements of \(\mathbb{D}_{p, k}(H)\) (cf.\([17, 18]\)).

For any \(t \geq 0\) and measurable \(f : W \to \mathbb{R}_+\), we note by

\[
P_t f(x) = \int_W f\left(e^{-t}x + \sqrt{1-e^{-2t}}y\right)\mu(dy),
\]

it is well-known that \((P_t, t \in \mathbb{R}_+)\) is a hypercontractive semigroup on \(L^p(\mu), p > 1\), which is called the Ornstein-Uhlenbeck semigroup (cf.\([4, 17, 18]\)). Its infinitesimal generator is denoted by \(-\mathcal{L}\) and we call \(\mathcal{L}\) the Ornstein-Uhlenbeck operator (sometimes called the number operator by the physicists).

The norms defined by

\[
\|\phi\|_{p,k} = \|(I + \mathcal{L})^{k/2}\phi\|_{L^p(\mu)}
\]

are equivalent to the norms defined by the iterates of the Sobolev derivative \(\nabla\). This observation permits us to identify the duals of the space \(\mathbb{D}_{p, k}(\Phi); p > 1, k \in \mathbb{N}\) by \(\mathbb{D}_{q, -k}(\Phi')\), with \(q^{-1} = 1 - p^{-1}\), where the latter space is defined by replacing \(k\) in \((2.1)\) by \(-k\), this gives us the distribution spaces on the Wiener space \(W\) (in fact we can take as \(k\) any real number). An easy calculation shows that, formally, \(\delta \circ \nabla = \mathcal{L}\), and this permits us to extend the divergence and the derivative operators to the distributions as linear, continuous operators. In fact \(\delta : \mathbb{D}_{q, k}(H \otimes \Phi) \to \mathbb{D}_{q, k-1}(\Phi)\) and \(\nabla : \mathbb{D}_{q, k}(\Phi) \to \mathbb{D}_{q, k-1}(H \otimes \Phi)\) continuously, for any \(q > 1\) and \(k \in \mathbb{R}\), where \(H \otimes \Phi\) denotes the completed Hilbert-Schmidt tensor product (cf., for instance \([11, 17, 18]\)). We shall denote by \(\mathbb{D}(\Phi)\) and \(\mathbb{D}'(\Phi)\) respectively the sets

\[
\mathbb{D}(\Phi) = \bigcap_{p > 1, k \in \mathbb{N}} \mathbb{D}_{p, k}(\Phi),
\]

and

\[
\mathbb{D}'(\Phi) = \bigcup_{p > 1, k \in \mathbb{N}} \mathbb{D}_{p, -k}(\Phi),
\]

where the former is equipped with the projective and the latter is equipped with the inductive limit topologies. A map \(F \in \mathbb{D}'(\mathbb{R}^d)\) is called nondegenerate if \(\det \gamma \in \cap_p L^p(\mu)\), where \(\gamma\) is the inverse of the matrix \((\nabla F_i, \nabla F_j)_H, i, j \leq d\) and \((\cdot, \cdot)_H\) denotes the scalar product in \(H\). For such a map, it is well-known that \([11, 17, 18]\) the map \(f \to f \circ F\) from \(S(\mathbb{R}^d) \to \mathbb{D}\) has a linear, continuous extension to \(S'(\mathbb{R}^d) \to \mathbb{D}'\), where \(S(\mathbb{R}^d)\) and \(S'(\mathbb{R}^d)\) denote the space of rapidly decreasing functions and tempered distributions on \(\mathbb{R}^d\), respectively. In fact, due to the “polynomially increasing” character
of the tempered distributions, the range of this extension is much smaller than $\mathbb{D}'$, in fact it is included in
\[
\mathbb{D}' = \bigcap_{p > 1} \bigcup_{k \in \mathbb{N}} \mathbb{D}_{p-k}.
\]
This notion has been extended in [15] and used to give an extension of the Itô formula as follows:

**Theorem 1.** Assume that $(X_t, t \in [0, T])$ is an $\mathbb{R}^d$-valued non-degenerate Itô process with the decomposition
\[
dX_t = b_t dt + \sigma_t dW_t
\]
where $b \in \mathbb{D}^a(L^2([0, T]) \otimes \mathbb{R}^d)$ and $\sigma \in \mathbb{D}^a(L^2([0, T]) \otimes \mathbb{R}^d \otimes \mathbb{R}^n)$, where the upper index $a$ means adapted to the Brownian filtration. Assume further that $p > 0$ and $k$ are integers such that $\mathbb{D}_{p-k}(H)$ is of finite total variation from $[0, t]$ to $\mathbb{S}((\mathbb{R}^d)')$. Then, for any $T \in \mathbb{S}(\mathbb{R}^d)$ and $0 < s < t \leq 1$, we have
\[
T(X_t) - T(X_s) = \int_s^t A_u T(X_u) du + \int_s^t (\partial T(X_u), \sigma_u dW_u),
\]
where $A_u = \frac{1}{2} \sigma_i \sigma_j \partial_i \partial_j + b_i(u) \partial_i$, the first integral is a Bochner integral in $\mathbb{D}'$ and the second one is the extended divergence operator explained above.

**Remark 1.** The conditions under which the hypothesis (2.2) holds are extremely well-studied in the literature, cf. [7, 8].

**Remark 2.** The divergence operator acts as an isomorphism between the spaces $\mathbb{D}^a_{p,k}(H)$ and $\mathbb{D}_{p,k}$ for any $p > 1$, $k \in \mathbb{R}$, cf. [10].

**Remark 3.** We can extend the above result easily to the case where $t \to T_t$ is a continuous map of finite total variation from $[0, T]$ to $\mathbb{S}(\mathbb{R}^d)$ in the sense that, the mapping $t \to (T_t, g)$ is of finite total variation on $[0, T]$ for any $g \in \mathbb{S}(\mathbb{R}^d)$. In fact, the kernel theorem of A. Grothendieck implies that $T_t$ can be represented as
\[
T_t = \sum_{i=1}^\infty \lambda_i \alpha_i(t) F_i,
\]
where $(\lambda_i) \in \ell^1$, $(\alpha_i)$ is bounded in the total variation norm and $(F_i)$ is bounded in $\mathbb{S}'(\mathbb{R}^d)$. Using this decomposition, it is straightforward to show that
\[
T(t, X_t) - T(s, X_s) = \int_s^t A_u T(u, X_u) du + \int_s^t T(u, X_u) + \int_s^t (\partial T(u, X_u), \sigma_u dW_u).
\]
where the second integral is defined as
\[
\int_s^t T(u, X_u) = \sum_{i=1}^\infty \lambda_i \int_s^t F_i(X_u) d\alpha_i(u)
\]
and the right hand side is independent of any particular representation of $T_t$. Integrals are concentrated in $\mathbb{D}'$.

We can prove easily the following result using the technique described in [15, 18]:

**Theorem 2.** Assume that $(l_t, t \in [0, 1])$ is an Itô process
\[
dl_t = m_t dt + \sum_i z_i^l dW^i_t,
\]
with $m, z^i \in \mathbb{D}^p(L^2[0, T])$, then we have
\[
 l_t T(t, X_t) - l_s T(s, X_s) = \int_s^t l_u A_u T(u, X_u) du + \int_s^t l_u T(du, X_u)
 + \int_s^t l_u (\partial T(u, X_u), \sigma_u dW_u) + \int_s^t T(u, X_u) m_u du
 + \int_s^t T(u, X_u) \sum_i z^i u dW^i_u + \int_s^t (\partial T(u, X_u), \sigma_u z_u) du
\]
where $z = (z^1, \ldots, z^n)$.

An important feature of the distributions on the Wiener space is the notion of positivity: we say that $S \in \mathbb{D}'$ is positive if for any positive $\varphi \in \mathbb{D}$, we have $S(\varphi) = \langle S, \varphi \rangle \geq 0$. An important result about the positive distributions is the following (cf. [1, 11, 14, 17, 18]):

**Theorem 3.** Assume that $S$ is a positive distribution in $\mathbb{D}'$, then there exists a positive Radon measure $\nu_S$ on $W$ such that
\[
\langle S, \varphi \rangle = \int_W \varphi d\nu_S,
\]
for any $\varphi \in \mathbb{D} \cap C_b(W)$. In particular, if a sequence $(S_n)$ of positive distributions converge to $S$ weakly in $\mathbb{D}'$, then $(\nu_{S_n})$ converges to $\nu_S$ in the weak topology of measures.

**Remark 4.** In fact we can write, for any $\varphi \in \mathbb{D}$
\[
\langle S, \varphi \rangle = \int_W \tilde{\varphi} d\nu_S,
\]
where $\tilde{\varphi}$ denotes a redefinition of $\varphi$ which is constructed using the capacities associated to the scale of Sobolev spaces $(\mathbb{D}_{p,k}, p > 1, k \in \mathbb{N})$, cf. [3].

3. **Uniqueness of the solution of parabolic variational inequality**

Assume that $(X_t^x, 0 \leq s \leq t \leq T)$ is a diffusion process governed by an $\mathbb{R}^n$-valued Wiener process $(W_t, t \in [0, T])$. We assume that the diffusion has smooth, bounded drift and diffusion coefficients $b(t, x), \sigma(t, x)$ defined on $[0, T] \times \mathbb{R}^d$, with values in $\mathbb{R}^d$ and $\mathbb{R}^d \otimes \mathbb{R}^n$ respectively and we denote by $A_t$ its infinitesimal generator. We shall assume that $X_t^x$ is nondegenerate for any $0 \leq s < t \leq T$. $\partial / \partial t + A_t$ is hypoelliptic and
\[
\int_{s+\varepsilon}^t (\det \gamma_\varepsilon^s)^p dv \in L^1(\mu)
\]
for any $0 < s < t \leq T$ and $\varepsilon > 0$, where $\gamma_\varepsilon$ is the inverse of the matrix $(\langle \nabla X_u^{s,i}, \nabla X_u^{s,j} \rangle_H : i, j \leq d)$.

Suppose that $f \in C_b(\mathbb{R}^d)$ and we shall study the following partial differential inequality whose solution will be denoted by $u(t, x)$:

**Theorem 4.** Assume that $u \in C_b([0, T] \times \mathbb{R}^d)$ such that $t \rightarrow \langle u(t, \cdot), g \rangle$ is of finite total variation on $[0, T]$ for any $g \in \mathcal{S}(\mathbb{R}^d)$ and that it satisfies the following properties:

\[
\frac{\partial u}{\partial t} + A_t u - ru \leq 0, \quad u \geq f \text{ in } [0, T] \times \mathbb{R}^d
\]
\[
\frac{\partial u}{\partial t} + A_t u - ru (f - u) = 0,
\]
\[
U(T, x) = f(x),
\]
where all the derivatives are taken in the sense of distributions, in particular the derivative w.r. to $t$ is taken using the $C^\infty$-functions of compact support in $[0, T]$. Then
\[
u(t, x) = \sup_{\tau \in \mathcal{Z}_{T,T}} E f(X_\tau^x) \exp - \int_T^T r(s, X^x_s(x)) ds,
\]
where \( Z_{t,T} \) denotes the set of all the stopping times with values in \([t,T]\) and \( r \) is a smooth function on \([0,T] \times \mathbb{R}^d\).

**Proof:** We shall prove the case \( t = 0 \). Let us denote by \( l \) the process defined as \( l_t = \exp -\int_0^t r(s,X_s)\,ds \). From Theorem 2 we have, for any \( \varepsilon > 0 \),

\[
(3.6) \quad l_t(u(t,X_t)) = -\varepsilon u(\varepsilon,X_t) - \int_0^t l_s(A_su(s,X_s)\,ds - (ru)(s,X_s)\,ds + u(ds,X_s)) = M^\varepsilon_t
\]

where \( M^\varepsilon_t \) is a \( \mathbb{D}' \)-valued martingale difference, i.e., denoting by \( E[\cdot|\mathcal{F}_s] \) the extension of the conditional expectation operator to \( \mathbb{D}' \) we have \( E[M^\varepsilon_t|\mathcal{F}_s] = M^\varepsilon_s \) for any \( \varepsilon \leq s \leq t \). Note also that \( K_tu = \frac{\partial u}{\partial x} + A_t u - ru \leq 0 \) hence its composition with \( X_t \) is a negative measure and this implies that the integral at the l.h.s. of (3.6) is a negative distribution on the Wiener space. Consequently we have

\[
(3.7) \quad M^\varepsilon_t \geq l_t(u(t,X_t)) - l_s u(\varepsilon,X_s)
\]

in \( \mathbb{D}' \). For \( \alpha > 0 \), let \( P^\alpha \) be the Ornstein-Uhlenbeck semigroup and define \( M^\alpha_\varepsilon \) as

\[
M^\alpha_\varepsilon = P^\alpha M^\varepsilon_t.
\]

Then \( (M^\alpha_\varepsilon)^{\varepsilon \geq \varepsilon} \) is a continuous martingale (in the ordinary sense). From the inequality (3.7), we have, for any \( \tau \in Z_{\varepsilon,T} \),

\[
M^\alpha_\varepsilon \geq P^\alpha(l_t(u(t,X_t)) - l_s u(\varepsilon,X_s))|_{t=\tau}.
\]

Taking the expectation of both sides, we get

\[
E[l_\varepsilon u(\varepsilon,X_\varepsilon)] \geq E[P^\alpha(l_t(u(t,X_t))|_{t=\tau})]
\]

for any \( \alpha > 0 \), hence we also have

\[
E[l_\varepsilon u(\varepsilon,X_\varepsilon)] \geq E[l_\tau u(\tau,X_\tau)]
\]

for any \( \varepsilon > 0 \) which is arbitrary and finally we obtain

\[
u(0,x) \geq E[l_\tau u(\tau,X_\tau)]
\]

for any \( \tau \in Z_{0,T} \).

To show the reverse inequality let \( D = \{(s,x) : u(s,x) \neq f(x)\} \) and define

\[
\tau_x = \inf \{s : (s,X^0_s) \in D^r\}.
\]

Since \( K \) is hypoelliptic, and since \( K \) is continuous on \( D \), \( u \) is smooth in \( D \). If \( \mu\{\tau_x = 0\} = 1 \), from the continuity of \( u \) we have

\[
u(0,x) = f(x) = E[l_\tau u(\tau_x,X^0_{\tau_x})],
\]

hence the supremum is attained in this case. If \( \mu\{\tau_x \neq 0\} > 0 \), then from the \( 0-1 \)-law \( \mu\{\tau_x \neq 0\} = 1 \) and \( \tau_x \) is predictable. Let \( (\tau_n,n \geq 1) \) a sequence of stopping times announcing \( \tau_x \). From the classical Itô formula, we have

\[
l_{\tau_n} u(\tau_n, X_{\tau_n}) - u(0, x) = \int_0^{\tau_n} l_s (\sigma^* \partial u)(s, X_s) \cdot dW_s.
\]

By the hypothesis the l.h.s. is uniformly integrable with respect to \( n \in \mathbb{N} \), consequently we obtain

\[
u(0,x) = \lim_n E[l_{\tau_n} u(\tau_n, X_{\tau_n})] = E[l_\tau u(\tau,X_\tau)]
\]

hence \( \tau_x \) realizes the supremum.

In the homogeneous case the finite variation property of the solution follows directly from the quasi-variational inequality:

---

1Such an extension is licit since the conditional expectation operator commutes with the Ornstein-Uhlenbeck semigroup.
Let now \( (P_\nu F \) from the hypothesis the distribution \( \langle 0 \) and \( \langle 0 \) on \( M \) where \( \langle 0 \) for any \( \langle 0 \) as in the preceding theorem, we have from Theorem 2

\[
\text{The relations (3.8) and (3.9) are to be understood in the weak sense. This means that}
\]

\[
\text{Let } \eta_\delta \text{ be a mollifier on } \mathbb{R} \text{ and let } \eta_\epsilon \text{ be a family of positive smooth functions on } (0, T), \text{ equal to unity on the interval } [\epsilon, T - \epsilon] \text{, converging to the indicator function of } [0, T] \text{ pointwise. Define } u^{\delta, \epsilon} \text{ as}
\]

\[
\nu = \frac{\partial u}{\partial t} + Au - ru
\]

is a negative measure on \((0, T) \times \mathbb{R}^d\). A simple calculation gives

\[
\frac{\partial u^{\delta, \epsilon}}{\partial t} + Au^{\delta, \epsilon} - ru^{\delta, \epsilon} = \rho_\delta \ast (\eta_\epsilon' u) + \rho_\delta \ast (\eta_\epsilon u) + \rho_\delta \ast (\eta_\epsilon ru) - ru^{\delta, \epsilon}.
\]

As in the preceding theorem, we have from Theorem 2

\[
l_t u^{\delta, \epsilon}(t, X_t) - l_a u^{\delta, \epsilon}(a, X_a) - \int_a^t l_s K_s u^{\delta, \epsilon}(s, X_s)ds = M_t^{\delta, \epsilon, a},
\]

where \( M_t^{\delta, \epsilon, a} \) is a \( \mathbb{D}' \)-martingale difference. Since \( \nu \) is a negative measure, we get the following inequality in \( \mathbb{D}' \):

\[
M_t^{\delta, \epsilon, a} \geq l_t u^{\delta, \epsilon}(t, X_t) - l_a u^{\delta, \epsilon}(a, X_a) - \int_a^t [\rho_\delta \ast (\eta_\epsilon' u) + \rho_\delta \ast (\eta_\epsilon u) - ru^{\delta, \epsilon}](s, X_s)ds
\]

Let now \((P_\alpha, \alpha \geq 0)\) be the Ornstein-Uhlenbeck semigroup. Then \((P_\alpha M_t^{\delta, \epsilon, a}, a \leq t \leq T)\) is a real valued martingale difference, consequently, we have

\[
0 = E[(P_\alpha M_t^{\delta, \epsilon, a})_{t=\tau}] \\
\geq E \left[ P_\alpha \left( l_t u^{\delta, \epsilon}(t, X_t) - l_a u^{\delta, \epsilon}(a, X_a) - \int_a^t [\rho_\delta \ast (\eta_\epsilon' u) + \rho_\delta \ast (\eta_\epsilon u) - ru^{\delta, \epsilon}](s, X_s)ds \right)_{t=\tau} \right],
\]
for any stopping time $\tau$ with values in $[\varepsilon, T - \varepsilon]$. By letting $\alpha \to 0$, we get by continuity
\[ 0 \geq E \left[ l_{\tau} u^{k, \varepsilon}(\tau, X_{\tau}) - l_{\alpha} u^{k, \varepsilon}(a, X_{a}) - \int_{a}^{\tau} \left[ \rho_\delta * (\eta_\varepsilon u) + \rho_\delta * (\eta_\varepsilon ru) - ru^{\delta, \varepsilon}(s, X_{s}) ds \right] \right]. \]

Let us choose $a > 0$ and let then $\varepsilon, \delta \to 0$. Note that $\eta_\varepsilon' \to \delta_0 - \delta_T$ (i.e., the Dirac measures at 0 and at $T$), by the choice of $a$ and by the weak convergence of measures and by the dominated convergence theorem, we obtain
\[ \lim_{\varepsilon, \delta \to 0} E \int_{a}^{\tau} (\rho_\delta * (\eta_\varepsilon u)) (s, X_{s}) ds = 0. \]

Again from the dominated convergence theorem we have
\[ \lim_{\varepsilon, \delta \to 0} E \int_{a}^{\tau} \rho_\delta * (\eta_\varepsilon ru) - ru^{\delta, \varepsilon}(s, X_{s}) ds = 0. \]

Consequently
\[ E[l_{\tau} u(a, X_{a})] \geq E[l_{\tau} u(\tau, X_{\tau})] \geq E[l_{\tau} f(X_{\tau})], \]
for any stopping time $\tau$ with values in $[a, T - a]$, since $a > 0$ is arbitrary, the same inequality holds also for any stopping time with values in $[0, T]$; hence
\[ u(0, x) \geq E[l_{\tau} f(X_{\tau})] \]
for any stopping time $\tau \in \mathcal{Z}_{0,T}$ and we obtain the first inequality:
\[ u(0, x) \geq \sup_{\tau \in \mathcal{Z}_{0,T}} E[l_{\tau} f(X_{\tau})]. \]

The proof of the reverse inequality is exactly the same that of Theorem 3 due to the hypoellipticity hypothesis. \[ \square \]

4. Existence of the solutions

In this section, under the hypothesis of the preceding section, we shall prove that the function defined by the Snell envelope (cf. [3]) of the American option satisfies the variational inequality (3.8) and the equality (3.9). We start with a lemma:

**Lemma 1.** Assume that $Z = (Z_t, t \in [0, T])$ is a uniformly integrable, real-valued martingale on the Wiener space. Let $Z^\kappa = (Z^\kappa_t, t \in [0, T])$ be defined as $Z^\kappa_t = P_\kappa Z_t$, where $P_\kappa$ is the Ornstein-Uhlenbeck semigroup at the instant $\kappa > 0$. Then $(Z^\kappa_t, t \in [0, T])$ is a uniformly integrable martingale with
\[ E[(Z^\kappa, Z^\kappa)^{1/2}] \leq c E[(Z, Z)^{1/2}], \]
where $c$ is a constant independent of $Z$ and $\kappa$. In particular, if $Z$ has the representation
\[ Z_T = \int_{0}^{T} (m_s, dW_s), \]
with $m \in L^1(\mu, H)$ optional, then
\[ P_\kappa Z_T = \int_{0}^{T} e^{-\kappa}(P_\kappa m_s, dW_s). \]
Theorem 6. Assume that \((X_t^z)\) is a hypoelliptic diffusion such that, for any \(\varepsilon > 0\),
\[
\int_0^T (\det \gamma^c) p dv \in L^1(\mu)
\]
for any \(p > 1\). Let \(p(s, t; x, y)\), \(s < t\), \(x, y \in \mathbb{R}^d\) be the density of the law of \(X_t^z(x)\) and denote by \(S_{0, z}\) the open set
\[
S_{0, z} = \{(s, y) \in (0, T) \times \mathbb{R}^d : s > 0, p(0, s; z, y) > 0\}.
\]
Then, for any \(z \in \mathbb{R}^d\), \(u\) is a solution of the variational inequality (3.8, 3.9, 3.10) in \(D'(S_{0, z})\). If \(S_{0, z} = (0, T) \times \mathbb{R}^d\) for any \(z \in \mathbb{R}^d\), then \(u\) is a solution of the variational inequality (3.8, 3.9, 3.10) in \(D'(0, T) \otimes D'(\mathbb{R}^d)\).

Proof: From the optimal stopping results, we know that \(u\) is a bounded, continuous function and \(t \to u(t, x)\) is monotone, decreasing (cf. [3]). Moreover
\[
u(t, X_t^z) l_t - u(0, x) = M_t + B_t
\]
is a supermartingale where \(X_t^z = X_t^0(x)\) and we denoted by \(M\) its martingale part and by \(B\) its continuous, decreasing process part. In particular \(d\mu \times d\mu\) defines a negative measure \(\gamma\) on \([0, T] \times C([0, T], \mathbb{R}^d)\). We can write \(u(ds, x)\) as the sum \(u_{ac}(s, x)ds + u_{sing}(ds, x)\) where \(u_{ac}\) is defined as the absolutely continuous part of \(u\) and \(u_{sing}\) is the singular part. We have, from the extended Itô formula,
\[
u(t, X_t^z) l_t - u(\varepsilon, X_\varepsilon) = \int_\varepsilon^t \left( A_s u - ru + u_{ac} \right)(s, X_s) ds + u_{sing}(ds, X_s) + \int_\varepsilon^t (\sigma \partial u)(s, X_s) dW_s + \int_\varepsilon^t (\partial u)(s, X_s) dW_s
\]
hence regularizing both parts by the Ornstein-Uhlenbeck semigroup, from Lemma 4 we get
\[
B_t^\varepsilon = \int_\varepsilon^t (A_s u - ru + u_{ac})(s, X_s) ds + u_{sing}(ds, X_s)
\]
\[
M_t^\varepsilon = \int_\varepsilon^t ((\sigma \partial u)(s, X_s) dW_s + (\partial u)(s, X_s) dW_s).
\]
Consequently, for any \(\alpha \in D(0, T)\) and \(\phi \in \mathbb{D}\)
\[
E\left[\phi \int_0^T \alpha(s) dB_s\right] = \int \alpha \otimes \phi d\gamma
\]
\[
= \int_{(0, T)} \alpha(s) ((A_s u - ru + u_{ac})(s, X_s) ds + u_{sing}(ds, X_s), \phi)
\]
and this quantity is negative for any $\alpha \in D_+(0, T)$ and $\phi \in D_+$. Let now $0 \leq g \in \mathcal{S}(\mathbb{R}^d)$ and assume that $(t_i, i \leq m)$ is a partition of $[0, T]$. Define $\xi_m$ as

$$\xi_m(t, w) = \sum_i 1_{[t_i, t_{i+1}]}(t) g(X_{t_i}).$$

Then it is immediate from the hypothesis about the diffusion process $(X_t)$ that $(\xi_m, m \geq 1)$ converges to $(g(X_s)1_{[0,T]}(s), s \in [0, T])$ in $\mathcal{D}(L^p([0, T]))$ for any $p \geq 1$ and $(\xi_m(s, \cdot)$ converges to $g(X_s)$ in $\mathcal{D}$ for any fixed $s \in [0, T]$ as the partition pace tends to zero. Let us represent $u(ds, \cdot)$, using the kernel theorem (cf. [3, 13]), as

$$u(ds, \cdot) = \sum_{k=1}^{\infty} \lambda_k T_k \otimes \alpha_k,$$

where $(\lambda_k) \in l^1$, $(T_k) \subset \mathcal{S}'(\mathbb{R}^d)$ is bounded and $(\alpha_k)$ is a sequence of measures on $[0, T]$, bounded in total variation norm. It follows then

$$u(ds, X_s) = \sum_{k=1}^{\infty} \lambda_k T_k(X_s) \alpha_k(ds)$$

and this some is convergent in $V([0, T]) \otimes \mathcal{D}_{n-k}$ for some $k \in \mathbb{N}$ and $p > 1$, in the projective topology, where $V([0, T])$ denotes the Banach space of measures on $[0, T]$ under the total variation norm. Since

$$\sup_{s \in [0, T]} \|\xi_m(s, X_s)\|_{p,l} \leq \sup_{s \in [0, T]} \|g \circ X_s\|_{p,l},$$

uniformly in $m \in \mathbb{N}$, for any $p, l$ and since

$$\|\xi_m(s, \cdot) - \xi_n(s, \cdot)\|_{p,l} \to 0$$

as $m, n \to \infty$ for any $p, l$ and $s \in [0, T]$, we obtain

$$\lim_{m \to \infty} \int_{(0,T)} \delta(s) \langle \xi_m(s, \cdot) - g(X_s), u(ds, X_s) \rangle = 0$$

for any $\delta \in \mathcal{D}(0, T)$ from the dominated convergence theorem. The above relation implies in particular that we have

$$\int_{(0,T)} \alpha(s) \langle (A_s u - ru + u_{ac})(s, \cdot), p_{0,s}g \rangle ds + \int_{(0,T)} \alpha(s) \langle u_{sing}(ds, \cdot), g p_{0,s} \rangle \leq 0,$$

with smooth, positive $\alpha$ and $g$, where the brackets in the integral correspond to the duality between $\mathcal{D}(\mathbb{R}^d)$ and $\mathcal{D}'(\mathbb{R}^d)$. For the functions of support in $(0, T)$, we can replace the term

$$\int_{(0,T)} \alpha(s) \langle u(ds, X_s), g(X_s) \rangle$$

by

$$\int_{(0,T)} \alpha(s) \langle \frac{\partial}{\partial s} u(s, X_s), g(X_s) \rangle$$

where $\partial/\partial s$ denotes the derivative in $\mathcal{D}'(0, T)$. Since $\alpha$ and $g$ are arbitrary, we obtain the inequality [3, $\S$] in $\mathcal{D}'(S_{0,x})$. If $S_{0,x} = (0, T) \times \mathbb{R}^d$, then we have the inequality in the sense of distributions on $(0, T) \times \mathbb{R}^d$.

To complete the proof, let $D$ be the set defined as

$$D = \{ (s, x) \in (0, T) \times \mathbb{R}^d : u(s, x) = f(x) \}.$$

Then we have

$$\int_0^T 1_{D^c}(s, X_s) dB_s = 0.$$

PROBABILISTIC SOLUTION OF THE AMERICAN OPTIONS 9
almost surely (cf. [3]). Let $C = -B$, then for any smooth function $\eta \in \mathcal{D}(0, T) \otimes \mathcal{S}(\mathbb{R}^d)$ such that $\eta \leq 1$, we have

$$
0 = E \int_0^T 1_{D^c}(s, X_s) dC_s \\
\geq E \int_0^T \eta(s, X_s) dC_s \\
= -\int_{(0, T)} \langle (A_s u - ru)(s, X_s) ds + u(ds, X_s), \eta(s, X_s) \rangle \\
\geq 0,
$$

where, the second equality follows from the estimates above. Hence

$$A_s u - ru + \frac{\partial}{\partial s} u = 0$$

as a distribution on the set $S_{0,x} \cap D^c$, by the hypoellipticity, the equality is everywhere on this set. If $S_{0,x} = (0, T) \times \mathbb{R}^d$, then we obtain the relation (3.9).

**Remark 6.** From the general theory, we can express the martingale part of $(l_t u(t, X_t), t \in [0, T])$ as

$$
\int_0^T (H_s, dW_s)
$$

where $H$ is an adapted process which is locally integrable. On the other hand we have

$$M^2_t = \int_t^T (\sigma(s, X_s) \partial u(s, X_s), dW_s)$$

where the r.h.s. is to be interpreted in a negatively indexed Sobolev space on the Wiener space. Using Lemma [7] we obtain the identity

$$H_s = \sigma(s, X_s) \partial u(s, X_s)$$

ds x d\mu-a.s., in particular we have

$$E \left[ \left( \int_0^T |\sigma(s, X_s) \partial u(s, X_s)|^2 ds \right)^{1/2} \right] < \infty.$$

**References**

[1] H. Airault and P. Malliavin: “Intégration géométrique sur l’espace de Wiener”. Bull. Sci. Math., 112, no.1, 3-52, 1988.

[2] A. Bensoussan and J.L. Lions: Applications des inéquations variationnelles en contrôle stochastique. Dunod, 1978.

[3] N. El Karoui: Les aspects probabilistes du contrôle stochastique. Lecture Notes in Mathematics, 876, p. 72-238. Springer, 1981.

[4] A. Grothendieck: Produits tensoriels topologiques et espaces nucléaires. Mem. Amer. Math. Soc., no. 16. 1955.

[5] N. Ikeda and S. Watanabe: Stochastic Differential Equations and Diffusion Processes. North Holland, Amsterdam (Kodansha Ltd., Tokyo), 1981.

[6] D. Lamberton: “Optimal stopping and American options”. To appear in Daiwa Lecture Series, Kyoto, 2008.

[7] P. Malliavin: Stochastic Analysis. Springer, 1997.

[8] P.-A. Meyer: “Un cours sur les intégrales stochastiques”. Séminaire de Probabilités X, p. 246-354. Lect. Notes in Math., Springer, 1976.

[9] H. H. Schaefer: Topological Vector Space. Graduate Texts in Math., Springer Verlag, 1970.
[14] H. Sugita: “Positive generalized Wiener functions and potential theory over abstract Wiener spaces”. Osaka J. Math. 25, no. 3, p. 665–696, 1988.
[15] A. S. Üstünel: “Extension of the Itô Calculus via the Malliavin Calculus”. Stochastics, Vol. 23, p. 353–375, 1988.
[16] A. S. Üstünel: “Representation of the distributions on Wiener space and stochastic calculus of variations”. Journal of Functional Analysis, Vol. 70, p. 126–139, 1987.
[17] A. S. Üstünel: Introduction to Analysis on Wiener Space. Lecture Notes in Math. Vol. 1610. Springer, 1995.
[18] A. S. Üstünel: Analysis on Wiener Space and Applications. Electronic text at the site http://www.finance-research.net/
[19] A. S. Üstünel and M. Zakai: Transformation of Measure on Wiener Space. Springer Verlag, 1999.

A.S. Üstünel, Telecom-Paristech (formerly ENST), Dept. Infres, 46, rue Barrault, 75013 Paris, France email: ustunel@telecom-paristech.fr