From calmness to Hoffman constants for linear inequality systems*

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Abstract

In this paper we deal with (finite) linear inequality systems parameterized by their right-hand side. We show that the sharp Hoffman constant at a given nominal parameter may be expressed as the maximum of the calmness moduli of the feasible set mapping at finitely many feasible points (extreme points when the nominal feasible set contains no lines). Recalling then a point-based expression for these calmness moduli, we are able to give a practically computable expression for the sharp Hoffman constant. A comparative analysis with other expressions in the literature is carried out.

Key words. Hoffman constants, calmness, linear inequality systems, feasible set mapping.

Mathematics Subject Classification: 90C31, 49J53, 15A39, 90C05

1 Introduction

Concerning linear inequality systems parameterized by their right-hand side, the celebrated Hoffman lemma [8] (see Section 2 for details) is a result of global nature as far as it works for any parameter making the system consistent and any point of the Euclidean space. We can also find in the literature related semi-local results as far as they work around a nominal (given) parameter and any point in the Euclidean space, leading to the concept of Hoffman constant at a this parameter (see e.g. Azé and Corvellec [1] and

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Zălinescu [20]). In this paper we relate this semi-local Hoffman constant with the local concept of calmness modulus, which involves parameters and points, both around nominal ones.

We consider the linear inequality system

$$\sigma(b) := \{a_t^t x \leq b_t, \ t \in T = \{1, ..., m\}\},$$  

where $x \in \mathbb{R}^n$ is the decision variable, regarded as a column-vector, the prime stands for transposition, $a_t \in \mathbb{R}^n$ is fixed for each $t$ and $b = (b_t)_{t \in T} \in \mathbb{R}^T \equiv \mathbb{R}^m$ is the parameter to be perturbed. Alternatively, system (1) may be written as

$$Ax \leq b,$$  

where the $t$-th row of matrix $A$ is $a_t^t$, $t \in \{1, ..., m\}$. The associated feasible set mapping, $F : \mathbb{R}^T \Rightarrow \mathbb{R}^n$, is given by

$$F(b) := \{x \in \mathbb{R}^n : a_t^t x \leq b_t \text{ for all } t \in T\}.$$  

The classical (sharp) Hoffman constant of $F$ concerns the whole graph of $F$ and, accordingly, only depends on matrix $A$ (see e.g. Peña et al. [15] and Klatte and Thiere [11]). More specifically, with $\mathbb{R}^n$ being endowed with an arbitrary norm $\| \cdot \|$ and appealing to the notation of [15], the pioneer Hoffman lemma [8] establishes the existence of a sharpest constant $H(A)$ such that

$$d(x, F(b)) \leq H(A) \max_{t \in T} [a_t^t x - b_t]_+ \text{ for all } x \in \mathbb{R}^n \text{ and all } b \in \text{dom } F,$$  

where $[\alpha]_+ := \max\{\alpha, 0\}$ denotes the positive part of $\alpha \in \mathbb{R}$ and $\text{dom } F := \{b \in \mathbb{R}^T : F(b) \neq \emptyset\}$ is the domain of $F$. Note that

$$\max_{t \in T} [a_t^t x - b_t]_+ = d(b, F^{-1}(x)),$$  

for all $x \in \mathbb{R}^n$ and all $b \in \mathbb{R}^T$, provided that the parameter space, $\mathbb{R}^T$, is endowed with the supremum norm $\|b\|_\infty := \max_{t=1,...,m} |b_t|$, and where we appeal to the notions of point-to-set distance and inverse multifunction defined few lines below in a more general setting. In this way (3) may be written in the variational form

$$d(x, F(b)) \leq H(A) d(b, F^{-1}(x)),$$  

for all $x \in \mathbb{R}^n$ and all $b \in \text{dom } F$. (4)
By fixing $b = \overline{b} \in \text{dom} \mathcal{F}$ in (4), a semi-local inequality appears (providing some right-hand side sensitivity analysis), giving rise to the so-called Hoffman modulus of $\mathcal{F}$ at $\overline{b}$, represented by $\text{Hof} \mathcal{F}(\overline{b})$; see Section 2 for the formal definition. Different authors have provided explicit formulae/estimations for this modulus and, for comparative purposes, the current paper recalls in Theorem 3 a specific expression for $\text{Hof} \mathcal{F}(\overline{b})$ traced out from [1]. After fixing $b = \overline{b} \in \text{dom} \mathcal{F}$, when focused on a neighborhood of a nominal point $\overline{x} \in \mathcal{F}(\overline{b})$, the well-studied calmness modulus of $\mathcal{F}$ at $(\overline{b}, \overline{x})$, denoted by $\text{clm} \mathcal{F}(\overline{b}, \overline{x})$, provides a local variational measure of the expansion of feasible solutions around $\overline{x}$ under parameter perturbations. A point-based (only involving the nominal data $(\overline{b}, \overline{x})$) formula, recalled in Theorem 1 for completeness, is provided in [3, Theorem 4] (see also [4, Section 2]).

Now we describe the main contributions of this paper. Keeping the previous notation in mind, the original motivation of the present work came from relating the semi-local variational measure $\text{Hof} \mathcal{F}(\overline{b})$ with the local one $\text{clm} \mathcal{F}(\overline{b}, x)$ for an appropriate choice of points $x \in \mathcal{F}(\overline{b})$. This relationship is established in Theorem 5. One important feature of this result is the fact that it only requires to look at finitely many feasible points, which makes our expression for $\text{Hof} \mathcal{F}(\overline{b})$, as the maximum of finitely many calmness moduli, practically implementable. In contrast, infinitely many points $x$ are involved to determine $\text{Hof} \mathcal{F}(\overline{b})$ by the formula given in [1] (see Example 4) and we cannot confine to feasible points $x \in \mathcal{F}(\overline{b})$. Moreover, as a consequence of Theorem 5, $H(A)$ can be also expressed in terms of the calmness moduli of $\mathcal{F}$ at certain pairs $(b, x) \in \text{gph} \mathcal{F}$ (the graph of $\mathcal{F}$); see Remark 6 for details.

In addition, the current paper analyzes two intermediate related upper Lipschitz properties, intended to complete the picture of local/semi-local/global approaches to the expansion rate of the feasible set. For the sake of generality, the referred properties are introduced for generic multifunctions between metric spaces and the differences among them are clarified.

The structure of the paper is as follows: Section 2 introduces the necessary notation and gathers the preliminary results needed later on. Section 3 analyzes the relationships among different upper-Lipschitz type properties for generic multifunctions and their moduli, as well as provides illustrative counter-examples. Section 4 is focused on linear inequality systems, establishes Theorem 5 and answers some questions which can naturally arise. The paper finishes with a short section of conclusions.
2 Preliminaries

To start with, we recall some definitions for a multifunction $\mathcal{M} : Y \rightrightarrows X$ between metric spaces with both distances being denoted by $d$. In this general context, elements $y \in Y$ may be interpreted as parameters and points $x \in X$ as solutions associated with $y$. Mapping $\mathcal{M}$ is said to be calm at $(\bar{y}, \bar{x}) \in \text{gph} \mathcal{M}$ if there exist neighborhoods $V$ of $\bar{y}$ and $U$ of $\bar{x}$ along with a constant $\kappa \geq 0$ such that

$$d(x, \mathcal{M}(\bar{y})) \leq \kappa d(y, \bar{y})$$

for all $y \in V$ and all $x \in \mathcal{M}(y) \cap U$, \hspace{1cm} (5)

where the point-to-set distance $d(x, \Omega)$ is defined as $\inf \{d(x, \omega) : \omega \in \Omega\}$ for $x \in X$ and $\Omega \subset X$, with $\inf \emptyset := +\infty$, so that $d(x, \emptyset) = +\infty$. Since this paper is concerned with nonnegative constants, throughout the paper we use the convention $\sup \emptyset := 0$.

It is well-known (cf. [5, Theorem 3H.3 and Exercise 3H.4]) that the calmness of $\mathcal{M}$ at $(\bar{y}, \bar{x})$ is equivalent to the metric subregularity of $\mathcal{M}^{-1}$, given by $y \in \mathcal{M}^{-1}(x) \iff x \in \mathcal{M}(y)$, at $(\bar{x}, \bar{y})$, i.e., the existence of a (possibly smaller) neighborhood $U$ of $\bar{x}$ and a constant $\kappa \geq 0$ such that

$$d(x, \mathcal{M}(\bar{y})) \leq \kappa d(y, \mathcal{M}^{-1}(x))$$

for all $x \in U$. \hspace{1cm} (6)

For more details about these and other Lipschitz-type properties, the reader is addressed to the monographs [5, 10, 14, 17].

It is also known that the infimum of $\kappa$ satisfying (5) for some associated $U$ and $V$ equals the infimum of $\kappa$ satisfying (6) for some $U$, and it is called the calmness modulus of $\mathcal{M}$ at $(\bar{y}, \bar{x})$, denoted by $\text{clm} \mathcal{M} (\bar{y}, \bar{x})$. We have

$$\text{clm} \mathcal{M} (\bar{y}, \bar{x}) = \limsup_{(y,x) \to (\bar{y},\bar{x})} \frac{d(x, \mathcal{M}(\bar{y}))}{d(y, \bar{y})} = \limsup_{x \to \bar{x}} \frac{d(x, \mathcal{M}(\bar{y}))}{d(\bar{y}, \mathcal{M}^{-1}(x))},$$

under the convention $\frac{0}{0} := 0$, where $\limsup$ is understood as the supremum (maximum, indeed) of all possible sequential upper limits (i.e., with $(y, x)$ being replaced with elements of sequences $\{(y_r, x_r)\}_{r \in \mathbb{N}}$ converging to $(\bar{y}, \bar{x})$ as $r \to \infty$).

As a generalization of the well-known (sharp) Hoffman constant $H(A)$ referred to in [3] associated with $\mathcal{M}$, we define the global Hoffman constant of $\mathcal{M}$ as

$$H(\mathcal{M}) := \sup_{y \in \text{dom} \mathcal{M}} \frac{d(x, \mathcal{M}(y))}{d(y, \mathcal{M}^{-1}(x))}.$$
Observe that both $\text{clm} \, \mathcal{M} (\mathbf{y}, \mathbf{x})$ and $H(\mathcal{M})$ take their values in $[0, +\infty]$. When confined to our feasible set mapping $\mathcal{F}$, the following two theorems provide exact formulae for $\text{clm} \, \mathcal{F} (\mathbf{b}, \mathbf{x})$, with $(\mathbf{b}, \mathbf{x}) \in \text{gph} \, \mathcal{F}$, and for $H(\mathcal{F})$ ($= H(\mathcal{A})$ in our setting). First, we introduce the necessary notation: Given $S \subset \mathbb{R}^k$, $k \in \mathbb{N}$, we denote by $\text{conv} \, S$ and $\text{cone} \, S$ the convex hull and the conical convex hull of $S$, respectively. It is assumed that $\text{cone} \, S$ always contains the zero-vector $0_k$, in particular $\text{cone}(\emptyset) = \{0_k\}$. From the topological side, $\text{int} \, S$, $\text{cl} \, S$, and $\text{bd} \, S$ stand, respectively, for the (topological) interior, closure, and boundary of $S$. If $\|\cdot\|$ is any norm in $\mathbb{R}^k$, its corresponding dual norm is denoted by $\|\cdot\|^*$, i.e., $\|u\|^* = \max_{\|x\| \leq 1} |u'x|$. In any metric space $(Z, d)$, the closed ball centered at $z \in Z$ with radius $r > 0$ is denoted by $B(z, r)$, whereas $B(S, r) := \{z \in Z : d(z, S) \leq r\}$, for $S \subset Z$, denotes the $r$-enlargement of $S$.

For any $(b, x) \in \text{gph} \, \mathcal{F}$, $T_b(x)$ represents the set of active indices of system $\sigma(b)$ at $x$, defined as

$$T_b(x) := \{t \in T : a'_t x = b_t\},$$

and $\mathcal{D}_b(x)$ denotes the family of all subsets $D \subset T_b(x)$ such that system

$$\begin{cases} a'_t d = 1, & t \in D, \\ a'_t d < 1, & t \in T_b(x) \setminus D \end{cases}$$

is consistent (in the variable $d \in \mathbb{R}^n$); i.e., $\{a_t, t \in D\}$ is contained in some hyperplane which leaves $\{0_n\} \cup \{a_t, t \in T_b(x) \setminus D\}$ on one of its two associated open half-spaces.

Now we gather the announced results on the calmness modulus and Hoffman constant of $\mathcal{F} : \mathbb{R}^T \rightrightarrows \mathbb{R}^n$, recalling that the parameter space $\mathbb{R}^T$ is endowed with the supremum norm $\|\cdot\|_\infty$, the space of variables $\mathbb{R}^n$ is equipped with an arbitrary norm $\|\cdot\|$, and $d^*$ represents the distance associated with $\|\cdot\|^*$.

**Theorem 1** \cite{3, Theorem 4} Given $(\mathbf{b}, \mathbf{x}) \in \text{gph} \, \mathcal{F}$, we have

$$\text{clm} \, \mathcal{F}(\mathbf{b}, \mathbf{x}) = \max_{D \in \mathcal{D}_b(x)} d^* (0_n, \text{conv} \{a_t, t \in D\})^{-1}.$$

**Remark 1** The expression above may be rewritten in terms of the concept of end of a nonempty convex set $C \subset \mathbb{R}^n$ introduced in \cite{[9]} (see also \cite{[13]}), defined as

$$\text{end} \, C := \{u \in \text{cl} \, C : \exists \mu > 1 \text{ such that } \mu u \in \text{cl} \, C\}.$$

In these terms, we have $\text{clm} \, \mathcal{F}(\mathbf{b}, \mathbf{x}) = d^* (0_n, \text{end} \, \text{conv} \{a_t, t \in T_b(x)\})^{-1}.$
The following result comes straightforwardly from \cite{15} Formula (3)] for our choice of norms. For the sake of completeness we give a short sketch of the proof. A version of this result when \( \mathbb{R}^n \) is endowed with the Euclidean norm can be found in \cite{11} Theorem 2.7.

**Theorem 2** We have

\[
H(A) = \max_{J \in 2^T} d_+(0_n, \text{conv} \{a_t \mid t \in J\})^{-1}.
\]

**Proof.** According to \cite{15} Formula (3)] and the subsequent comments therein, we only have to prove that condition \( 0_n \notin \text{conv} \{a_t \mid t \in J\} \) is equivalent to the consistency of system \( \{a'_t x < 0, t \in J\} \), and this follows, for instance, from equivalence \( (iv) \Leftrightarrow (v) \) in \cite{7} Theorem 6.1.

The next result comes straightforwardly from \cite{15} Formula (4)] with the trivial observation that instead of all linearly independent \( \{a_t \mid t \in J\} \), with \( J \in 2^T \), we can confine ourselves to those which are maximal with respect to the inclusion order. Indeed, the result also follows from Theorem 2 since the sufficiency of considering those \( \{a_t \mid t \in J\} \) which are linearly independent comes from \cite{11} Lemma 3.1. Here \( A_J \) stands for the matrix whose rows are \( a'_t \), with \( t \in J \).

**Corollary 1** We have

\[
H(A) = \max_{\text{rank } A_J = \text{rank } A \atop \{a_t \mid t \in J\} \text{ lin. indep.}} d_+(0_n, \text{conv} \{a_t \mid t \in J\})^{-1}.
\]

As pointed out in the introduction, this paper is focused on Hoffman stability (in a sense defined later on) of \( F \) at \( b \). There are different notions to be explored, depending on where parameter \( b \in \text{dom } F \) and point \( x \in \mathbb{R}^n \) are allowed to vary. We will see in Theorem 3 that most of these properties lead to the same constant for multifunctions having a closed and convex graph, as our \( F \), and that this constant, which depends on \( b \), may be strictly smaller than \( H(A) \).

The rest of preliminary results, mainly Theorem 4, are postponed to the beginning of Section 4, after a general discussion on semi-local properties of multifunctions has been done in Section 3.

Finally, let us comment that there are many other authors who have contributed to the study of Hoffman constants and their relationship with other concepts (as Lipschitz constants). Additional references can be obtained from the reference list of \cite{1, 11, 15, 20}. At this moment we mention Belousov and Andronov \cite{2}, Li \cite{12} and Robinson \cite{16}.
3 From calmness to Hoffman constants for a generic multifunction

The purpose of this section is to introduce two new properties that live between calmness and Hoffman ones, as well as to study their relationship with the already known Lipschitz upper semicontinuity property, the latter going back to the classical work of Robinson \[16\]. Throughout this section $\mathcal{M} : Y \rightrightarrows X$ is a generic multifunction between metric spaces $Y$ and $X$ (with both distances denoted by $d$). Following the terminology by Uderzo \[18, Definition 2.1(iii)\], $\mathcal{M}$ is said to be Lipschitz upper semicontinuous at $\overline{y} \in \text{dom} \mathcal{M}$ if there exists a neighborhood $V$ of $\overline{y}$ along with a constant $\kappa \geq 0$ such that

$$d(x, \mathcal{M}(y)) \leq \kappa d(y, \overline{y}) \text{ for all } y \in V \text{ and all } x \in \mathcal{M}(y).$$

(9)

Equivalently, (9) may be written as $e(\mathcal{M}(y), \mathcal{M}(\overline{y})) \leq \kappa d(y, \overline{y})$ for all $y \in V$, where $e(A, B) := \sup_{x \in A} d(x, B)$ is the Hausdorff excess of $A$ over $B$, with $A, B \subset X$. The associated Lipschitz upper semicontinuity modulus, denoted by Lipusc$\mathcal{M}(\overline{y})$, is defined as the infimum of constants $\kappa$ satisfying (9) for some associated $V$.

Next we introduce some related properties. To this aim, we use the following notation: Given $\overline{y} \in \text{dom} \mathcal{M}$ and $\varepsilon > 0$, the mapping $\mathcal{M}_{\varepsilon} : Y \rightrightarrows X$ is given by

$$\mathcal{M}_{\varepsilon}(y) := \mathcal{M}(y) \cap B(\mathcal{M}(\overline{y}), \varepsilon) \text{ for } y \in Y.$$

(For simplicity in the notation we obviate the dependence of $\mathcal{M}_{\varepsilon}$ on $\overline{y}$.)

**Definition 1** Given $\overline{y} \in \text{dom} \mathcal{M}$, we say that:

(i) $\mathcal{M}$ is uniformly calm at $\overline{y}$ if there exist a neighborhood $V$ of $\overline{y}$ along with $\varepsilon > 0$ and $\kappa \geq 0$ such that

$$d(x, \mathcal{M}(\overline{y})) \leq \kappa d(y, \overline{y}) \text{ for all } y \in V \text{ and all } x \in \mathcal{M}_{\varepsilon}(y),$$

(10)

or, equivalently, if $\mathcal{M}_{\varepsilon}$ is Lipschitz upper semicontinuous at $\overline{y}$ for some $\varepsilon > 0$.

(ii) $\mathcal{M}$ is Hoffman stable at $\overline{y}$ if there exists $\kappa \geq 0$ such that

$$d(x, \mathcal{M}(\overline{y})) \leq \kappa d(y, \overline{y}) \text{ for all } y \in \text{dom} \mathcal{M} \text{ and all } x \in \mathcal{M}(y),$$

or, equivalently, if

$$d(x, \mathcal{M}(\overline{y})) \leq \kappa d(\overline{y}, \mathcal{M}^{-1}(x)) \text{ for all } x \in X.$$
The corresponding moduli naturally appear. Specifically, we call Hoffman modulus of $\mathcal{M}$ at $\overline{y}$, $\text{Hof} \mathcal{M}(\overline{y})$, to the infimum of constants $\kappa$ satisfying (11), while $\text{ucm} \mathcal{M}(\overline{y})$ represents the modulus of uniform calmness of $\mathcal{M}$ at $\overline{y}$ which is defined as the infimum of constants $\kappa$ satisfying (10) for some associated $V$ and $\varepsilon > 0$. It is straightforward to check that

$$\text{ucm} \mathcal{M}(\overline{y}) = \inf_{\varepsilon > 0} \text{lipusc} \mathcal{M}_x(\overline{y}).$$

(12)

Roughly speaking, the uniform calmness of $\mathcal{M}$ at $\overline{y}$ entails the calmness of $\mathcal{M}$ at any $(\overline{y}, x)$ for all $x \in \mathcal{M}(\overline{y})$ with the same calmness constant $\kappa$, the same neighborhood $V$ of $\overline{y}$, and a common radius $\varepsilon$ for all neighborhoods of points $x \in \mathcal{M}(\overline{y})$, say $U_x := B(x, \varepsilon)$. Example 2 below shows that the calmness of $\mathcal{M}$ at $(\overline{y}, x)$ for all $x \in \mathcal{M}(\overline{y})$ does not ensure the uniform calmness of $\mathcal{M}$ at $\overline{y}$.

As it occurs with the calmness property, the uniform calmness turns out to be equivalent to a certain metric regularity type property, showing that neighborhood $V$ in Definition 1(i) is negligible. The key fact is that points $x \in \mathcal{M}(\overline{y})$ which are required to satisfy (10) are those which are sufficiently close to $\mathcal{M}(\overline{y})$.

Remark 2 The statement of Proposition 1 does not hold for $\kappa = 0$. To see this, take $\mathcal{M} : \mathbb{R} \to \mathbb{R}$ (single-valued) given by $\mathcal{M}(y) := \max \{0, y - 1\}$. Clearly (i) holds for $V = ]-1, 1[$ and $\kappa = 0$, whereas (ii) works for $\varepsilon > 0$ if and only if $\kappa \geq \varepsilon / (1 + \varepsilon)$.
Corollary 2 Let \( \overline{y} \in \text{dom} \, \mathcal{M} \). We have:

(i) \( \mathcal{M} \) is uniformly calm at \( \overline{y} \) if and only if there exist \( \varepsilon > 0 \) and \( \kappa \geq 0 \) such that

\[
d(x, \mathcal{M}(\overline{y})) \leq \kappa d(\overline{y}, \mathcal{M}^{-1}(x)) \quad \text{for all } x \in B(\mathcal{M}(\overline{y}), \varepsilon).
\] (13)

(ii) The modulus of uniform calmness can be expressed as follows

\[
ucl \, \mathcal{M}(\overline{y}) = \inf \{ \kappa \geq 0 : \exists \varepsilon > 0 \text{ such that } (13) \text{ holds} \}.
\]

Proof. Both (i) and (ii) come from the fact that uniform calmness at \( \overline{y} \) with associated elements \( V, \varepsilon > 0 \) and \( \kappa \geq 0 \) in (10) entails the same property with \( V, \varepsilon > 0 \) and \( \kappa > k \). Hence the conclusions follow straightforwardly from Proposition 1.

It is clear that

\[
\text{Hof} \, \mathcal{M}(\overline{y}) = \sup_{(y,x) \in \text{gph} \, \mathcal{M}} d(x, \mathcal{M}(\overline{y})) = \sup_{x \in X} d(x, \mathcal{M}(\overline{y})).
\] (14)

Next we provide characterizations of Lipusc \( \mathcal{M}(\overline{y}) \) and uclm \( \mathcal{M}(\overline{y}) \) in terms of certain upper limits, which allow for a better understanding of these concepts and a clear relationship among all moduli introduced in the paper.

Proposition 2 Let \( \mathcal{M} : Y \rightrightarrows X \) be a multifunction between metric spaces and let \( \overline{y} \in \text{dom} \, \mathcal{M} \), then

(i) Lipusc \( \mathcal{M}(\overline{y}) = \limsup_{y \to \overline{y}} \left( \sup_{x \in \mathcal{M}(y)} d(x, \mathcal{M}(\overline{y})) / d(y, \overline{y}) \right) \);

(ii) uclm \( \mathcal{M}(\overline{y}) = \limsup_{d(x, \mathcal{M}(\overline{y})) \to 0} d(x, \mathcal{M}(\overline{y})) / d(\overline{y}, \mathcal{M}^{-1}(x)) \).

Proof. (i) For the sake of simplicity, let us denote by \( s \) the right-hand side of (i) and

\[
K := \{ \kappa \geq 0 : \exists V \text{ neighborhood of } \overline{y} \text{ verifying } (9) \}.
\] (15)

We start by establishing inequality `\( \leq \)`.

Since Lipusc \( \mathcal{M}(\overline{y}) = \inf K \), we can write Lipusc \( \mathcal{M}(\overline{y}) = \lim_{r \to \infty} \kappa_r \) for some \( \{ \kappa_r \} \subset K \). For each \( r \) take a neighborhood \( V_r \) associated with \( \kappa_r \) according to (15) and define

\[
\kappa_r := \sup_{y \in V_r \cap B(\overline{y}, 1/r)} \left( \sup_{x \in \mathcal{M}(y)} d(x, \mathcal{M}(\overline{y})) / d(y, \overline{y}) \right) \leq \kappa_r.
\]
By definition $\overline{\kappa}_r \in K$, having $V_r \cap B(y, 1/r)$ as an associated neighborhood, so that we have $\text{Lipusc } M(y) = \lim_{r \to \infty} \overline{\kappa}_r$.

Finally, for each $r$, consider any $y_r \in V_r \cap B(y, 1/r)$ such that $\overline{\kappa}_r - \frac{1}{r} \leq \sup_{x \in M(y)} \frac{d(x, M(y))}{d(y_r, y)} \leq \overline{\kappa}_r$. Obviously, $\{y_r\}_{r \in \mathbb{N}}$ converges to $y$, and then

$$\text{Lipusc } M(y) = \lim_{r \to \infty} \sup_{x \in M(y)} \frac{d(x, M(y))}{d(y_r, y)} \leq s.$$ 

In order to prove ‘$\geq$’ in $(i)$, we may assume the nontrivial case $s > 0$ and write

$$s = \lim_{r \to \infty} \sup_{x \in M(y_r)} \frac{d(x, M(y))}{d(y_r, y)},$$

for some $\{y_r\}_{r \in \mathbb{N}}$ converging to $y$. It is clear that we may replace $\{y_r\}_{r \in \mathbb{N}}$ with a suitable subsequence (denoted as the whole sequence for simplicity) such that $y_r \in V_r$, and then

$$s \leq \lim_{r \to \infty} \kappa_r = \text{Lipusc } M(y).$$

$(ii)$ The procedure is analogous to the previous one by considering

$$\tilde{K} = \{\kappa \geq 0 : \exists \varepsilon > 0 \text{ such that } (13) \text{ holds}\}.$$ 

As a direct consequence of the expressions $(7)$ and $(14)$ for $\text{clm } M(y, x)$ and $\text{Hof } M(y)$, respectively, together with $(12)$ and the previous proposition, we conclude the following corollary.

**Corollary 3** Let $y \in \text{dom } M$. We have

$$\sup_{x \in M(y)} \text{clm } M(y, x) \leq \text{ulcm } M(y) \leq \text{Lipusc } M(y) \leq \text{Hof } M(y). \quad (16)$$

**Remark 3** The previous corollary yields $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$, where:

$(i)$ $M$ is Hoffman stable at $y$;

$(ii)$ $M$ is Lipschitz upper semicontinuous at $y$;

$(iii)$ $M$ is uniformly calm at $y$;

$(iv)$ $M$ is calm at every $(y, x) \in \text{gph } M$.

The next three examples show that all converse implications in the previous remark may fail for a suitable multifunction.
Example 1 Let $\mathcal{M} : \mathbb{R} \rightarrow \mathbb{R}$ be given by $\mathcal{M}(y) = \{h_r(y), r \in \mathbb{N}\}$, where

$$h_r(y) = \begin{cases} 
  r + y & \text{if } y \leq \frac{1}{r}, \\
  r + \frac{1}{r} + r(y - \frac{1}{r}) & \text{if } y > \frac{1}{r}.
\end{cases}$$

For $\overline{y} = 0$, it is easy to check that $\operatorname{clm} \mathcal{M}(\overline{y}, x) = 1$ for all $x \in \mathcal{M}(\overline{y})$. Hence, $\sup_{x \in \mathcal{M}(\overline{y})} \operatorname{clm} \mathcal{M}(\overline{y}, x) = 1$. Nevertheless, it is impossible to find $\varepsilon > 0$ that meets the conditions for uniform calmness; i.e., $\operatorname{uclm} \mathcal{M}(\overline{y}) = +\infty$. More specifically, take $\varepsilon_r := r^{-1} + r^{-1/2}$ for all $r \in \mathbb{N}, r \geq 8$ (to ensure $\varepsilon_r < 1/2$), and consider $y_r := r^{-1} + r^{-3/2}$ and $x_r := h_r(y) = r + r^{-1} + r^{-1/2} \in \mathcal{M}_{\varepsilon_r}(y_r)$. Then

$$d(x_r, \mathcal{M}(0)) = \frac{r^{-1} + r^{-1/2}}{r^{-1} + r^{-3/2}} \rightarrow +\infty \text{ as } r \rightarrow +\infty.$$

Example 2 Consider $\mathcal{M} : \mathbb{R} \rightarrow \mathbb{R}$ (single-valued) given by $\mathcal{M}(y) = 0$ if $y \leq 0$ and $\mathcal{M}(y) = 1$ if $y > 0$. It is clear that $\mathcal{M}$ is uniformly calm at $y = 0$ (take $\varepsilon = 1/2$) but not Lipschitz upper semicontinuous by just considering $y_r = 1/r$ for $r \in \mathbb{N}$.

Example 3 Let $\mathcal{M} : \mathbb{R} \rightarrow \mathbb{R}$ be given by

$$\mathcal{M}(y) = [0, 1] \text{ if } y < 0, \quad \mathcal{M}(y) = [0, +\infty[ \text{ if } y \geq 0.$$

It is clear that $\mathcal{M}$ is Lipschitz upper semicontinuous, with zero modulus, at any $y \in \mathbb{R}$. Nevertheless, it is not Hoffman stable at any $\overline{y} < 0$.

The next theorem establishes that all inequalities in (16) become equalities under the convexity of $\operatorname{gph} \mathcal{M}$ together with the closedness of $\mathcal{M}(\overline{y})$, provided that $Y$ and $X$ are normed spaces. As an obvious consequence, all properties in Remark 3 become equivalent in such a case. Firstly, we include two lemmas.

Lemma 1 Let $X$ be a normed space and $\emptyset \neq C \subset X$ be a closed set. Take any $x \in X$ such that there exists a best approximation, $\overline{x}$, of $x$ in $C$. Then $\overline{x}$ is a best approximation of $x_\lambda := (1 - \lambda) \overline{x} + \lambda x$ in $C$ for all $\lambda \in [0, 1]$.

Proof. Reasoning by contradiction, suppose that for some $\lambda \in [0, 1]$ there exists $\hat{x} \in C$ such that $\|\hat{x} - x_\lambda\| < \|\overline{x} - x_\lambda\|$. Then

$$\|\hat{x} - x\| \leq \|\hat{x} - x_\lambda\| + \|x_\lambda - x\| < \|\overline{x} - x_\lambda\| + \|x_\lambda - x\| = \lambda \|\overline{x} - x\| + (1 - \lambda) \|\overline{x} - x\| = \|\overline{x} - x\|,$$

which contradicts the fact that $\overline{x}$ is a best approximation of $x$ in $C$. ■
Lemma 2 \( \text{Let } M : Y \Rightarrow X \text{ be a multifunction between normed spaces } Y \text{ and } X, \text{ and assume that } \text{gph } M \text{ is a nonempty convex set. Let } \overline{y} \in \text{dom } M \text{ and suppose that } M (\overline{y}) \text{ is closed. Consider any } (y, x) \in \text{gph } M \text{ and let } \bar{x} \text{ be a best approximation of } x \text{ in } M (\overline{y}), \text{ then}
\)
\[
\frac{d (x, M (\overline{y}))}{d (y, \overline{y})} \leq \text{clm } M (\overline{y}, \bar{x}).
\]

Proof. \( \text{By the convexity assumption, for each } \lambda \in [0, 1), \)
\[
(x_{\lambda}, y_{\lambda}) := (1 - \lambda) (\bar{x}, \overline{y}) + \lambda (x, y) \in \text{gph } M.
\]
\( \text{According to lemma 1, } \bar{x} \text{ is also a best approximation of } x_{\lambda} \text{ in } M (y_{\lambda}), \text{ for each } \lambda \in [0, 1]. \text{ Therefore,}
\)
\[
\frac{d (x_{\lambda}, M (y_{\lambda}))}{d (y_{\lambda}, y_{\lambda})} = \frac{d (x_{\lambda}, \overline{y})}{d (y_{\lambda}, \overline{y})} = \frac{d (x_{\lambda}, \overline{y})}{d (y_{\lambda}, \overline{y})}, \text{ for all } \lambda \in [0, 1].
\]
\( \text{Since, letting } \lambda \to 0, \text{ we have } (x_{\lambda}, y_{\lambda}) \to (\bar{x}, \overline{y}), \text{ by the definition of the calmness modulus (recall (7)) we conclude}
\)
\[
\text{clm } M (\overline{y}, \bar{x}) \geq \limsup_{\lambda \to 0} \frac{d (x_{\lambda}, M (y_{\lambda}))}{d (y_{\lambda}, y_{\lambda})} = \frac{d (x, M (\overline{y}))}{d (y, \overline{y})}.
\]

Theorem 3 \( \text{Let } M : Y \Rightarrow X, \text{ with } Y \text{ and } X \text{ being normed spaces, and assume that gph } M \text{ is a nonempty convex set. Let } \overline{y} \in \text{dom } M \text{ with } M (\overline{y}) \text{ closed. Then one has}
\)
\[
\sup_{x \in M (\overline{y})} \text{clm } (\overline{y}, x) = \text{ulcm } (\overline{y}) = \text{Lipusc } (\overline{y}) = \text{Hof } M (\overline{y}).
\]

Proof. \( \text{We only have to prove Hof } M (\overline{y}) \leq \sup_{x \in M (\overline{y})} \text{clm } (\overline{y}, x), \text{ according to (16)}.
\)
\( \text{Take any } (\bar{y}, \bar{x}) \in \text{gph } M \text{ and let } \bar{x} \text{ be a best approximation of } \bar{x} \text{ in } M (\overline{y}). \text{ Lemma 2 ensures that}
\)
\[
\frac{d (\bar{x}, M (\overline{y}))}{d (\bar{y}, \overline{y})} \leq \text{clm } (\overline{y}, \bar{x}) \leq \sup_{x \in M (\overline{y})} \text{clm } (\overline{y}, x).
\]
\( \text{Then, recalling (11), we conclude}
\)
\[
\text{Hof } M (\overline{y}) = \sup_{(\bar{y}, \bar{x}) \in \text{gph } M} \frac{d (\bar{x}, M (\overline{y}))}{d (\bar{y}, \overline{y})} \leq \sup_{x \in M (\overline{y})} \text{clm } (\overline{y}, x).
\]
We finish this section by observing that the sharp Hoffman constant for
the whole graph can be larger than the Hoffman modulus for a specific \( \overline{y} \). Just consider \( \mathcal{M} : \mathbb{R} \rightarrow \mathbb{R} \) be given by
\[
\mathcal{M}(y) = [-\infty, y] \text{ if } y < 0, \quad \mathcal{M}(y) = [-\infty, 0] \text{ if } y \geq 0.
\]
Then clearly Hof \( \mathcal{M} (\overline{y}) = 1 \) if \( \overline{y} < 0 \) and Hof \( \mathcal{M} (\overline{y}) = 0 \) if \( \overline{y} \geq 0 \); so that \( H(\mathcal{M}) = 1 \).

4 Calmness and Hoffman moduli for linear inequality systems

The main preliminary result to be discussed in the present section is Theorem 4 below, which can be traced out from Azé and Corvellec [1], originally formulated in terms of constants playing the role of \( \kappa^{-1} \) multiplying in the left-hand side of inequalities of the type (11). As commented in Section 1, one important feature of this result is the fact that infinitely many points \( x \) are considered to determine Hof \( F(\overline{b}) \) and, as the subsequent Example 4 (also from [1]) shows, we cannot confine to feasible points \( x \in F(\overline{b}) \). In contrast, our Theorem 5 provides a point-based expression for Hof \( F(\overline{b}) \) which only requires to look at finitely many feasible points. In order to describe the referred preliminaries we need some extra notation.

Recalling that \( T = \{1, \ldots, m\} \), let us consider the supremum function at the nominal value of the parameter and the associated ‘index mapping’ given by
\[
f(x) := \sup_{t \in T} (a'_t x - \overline{b}_t), \quad J(x) = \{t \in T : a'_t x - \overline{b}_t = f(x)\}, \text{ for } x \in \mathbb{R}^n.
\]
Observe that \( F(\overline{b}) = [f \leq 0] := \{x \in \mathbb{R}^n : f(x) \leq 0\} \). Although in the present paper the results are not written in terms of subdifferentials, let us also observe that, from the well-known Valadier’s formula,
\[
\partial f(x) = \text{conv} \{a_t : t \in J(x)\},
\]
where \( \partial f(x) \) stands for the usual subdifferential of convex analysis (see e.g. [17]). In particular, if \( \overline{b} \in \text{dom} F \) and \( f(x) > 0 \), then \( 0_n \notin \text{conv} \{a_t : t \in J(x)\} \), since such an \( x \) is not a global minimizer of the convex function \( f \).
**Theorem 4** [1] Lemma 3.1, Theorem 3.3 and Proposition 3.5] For \( \overline{b} \in \text{dom} \mathcal{F} \) we have

\[
(\text{Hof} \mathcal{F}(\overline{b}))^{-1} = \min \{d_\ast(0_n, \text{conv}\{a_t, t \in J(x)\}) : f(x) > 0\}
\]

\[
= \min \left\{d_\ast(0_n, \text{conv}\{a_t, t \in J\}) : J \subset J(x), f(x) > 0, \{a_t, t \in J\} \text{ lin. indep.} \right\},
\]

and both minima can be confined to those \( x \in \mathbb{R}^n \) such that \( 0 < f(x) \leq \varepsilon \) for any given \( \varepsilon > 0 \).

**Remark 4** In the expressions for \( (\text{Hof} \mathcal{F}(\overline{b}))^{-1} \) of the previous theorem, although the minimum is taken on a finite subset, since \( J(x) \subset T \) for all \( x \), infinitely many values of \( x \) are involved.

The following example shows that the expressions providing \( (\text{Hof} \mathcal{F}(\overline{b}))^{-1} \) in Theorem 4 cannot be extended to feasible points, i.e. condition \( f(x) > 0 \) is essential.

**Example 4** [1] Example 3.1] Let \( n = 2, m = 3, a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, a_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, a_3 = \begin{pmatrix} \delta \\ \delta \end{pmatrix} \), with \( 0 < \delta < 1/2 \), and let \( \overline{b} = 0_3 \). The admissible \( J \)'s in Theorem 4 are \{1\}, \{2\}, and \{1, 2\}, yielding \( \text{Hof} \mathcal{F}(\overline{b}) = \sqrt{2} \). If we could admit \( x = 0_2 \), then \{3\}, \{1, 3\} and \{2, 3\} would be also admissible \( J \)'s, yielding a nonsharp constant \( \left(\delta \sqrt{2}\right)^{-1} \). In this example we also see that \( \text{Hof} \mathcal{F}(\overline{b}) \) does depend on \( \overline{b} \). For instance, \( \text{Hof} \mathcal{F}((0, 0, -\delta)') = \left(\delta \sqrt{2}\right)^{-1} \).

The remaining part of this section is focused on the computation of \( \text{Hof} \mathcal{F}(\overline{b}) \), for \( \overline{b} \in \text{dom} \mathcal{F} \), through the calmness moduli, \( \text{clm} \mathcal{F}(\overline{b}, x) \), at a certain finite choice of elements \( (\overline{b}, x) \in \text{gph} \mathcal{F} \), where we can apply Theorem 1. In this way, taking Theorem 3 into account, we compute semi-local stability measures (which study the behavior of the whole sets \( \mathcal{F}(\overline{b}) \) with respect a nominal \( \overline{b} \)) in terms of other measures of local nature. The following lemma constitutes a key tool in the proof of the subsequent theorem. Here we recall the notation from Section 2.

**Lemma 3** Let \( x^1, x^2 \in \mathcal{F}(\overline{b}) \) such that \( T_{\overline{b}}(x^1) \subset T_{\overline{b}}(x^2) \). Then,

(i) \( D_{\overline{b}}(x^1) \subset D_{\overline{b}}(x^2) \);

(ii) \( \text{clm} \mathcal{F}(x^1, \overline{b}) \leq \text{clm} \mathcal{F}(x^2, \overline{b}) \).
Proof. (i) Given $D \in \mathcal{D}_b(x^1)$, let us see that $D \in \mathcal{D}_b(x^2)$. First, consider $d := x^1 - x^2$ and observe that,

$$\begin{cases}
  a_i' d = 0, & t \in T_b(x^1), \\
  a_i' d = a_i' x^1 - a_i' x^2 < b_t - b_t = 0, & t \in T_b(x^2) \setminus T_b(x^1).
\end{cases}$$

Now, recalling (8), the fact that $D \in \mathcal{D}_b(x^1)$ ensures the existence of $d \in \mathbb{R}^n$ such that

$$\begin{cases}
  a_i' d = 1, & t \in D, \\
  a_i' d < 1, & t \in T_b(x^1) \setminus D.
\end{cases}$$

For every $\alpha > 0$, we consider a new vector $d_\alpha := d + \alpha d$; observe that

$$\begin{cases}
  a_i' d_\alpha = a_i' d + \alpha a_i' d = 1, & t \in D, \\
  a_i' d_\alpha = a_i' (d + \alpha d) < 1, & t \in T_b(x^1) \setminus D,
\end{cases}$$

for any $\alpha > 0$. Since $a_i' d < 0$ for $t \in T_b(x^2) \setminus T_b(x^1)$, we can choose $\alpha$ large enough (any $\alpha > \max_{t \in T_b(x^2) \setminus T_b(x^1)} \frac{a_i' d - 1}{a_i' d}$ will do it) to make $a_i' (d + \alpha d) < 1$ for all $t \in T_b(x^2) \setminus T_b(x^1)$. This proves (i).

Statement (ii) follows straightforwardly from Theorem 1.

The following theorem appeals to the set

$$\mathcal{E}(\overline{b}) := \text{extr} (\mathcal{F}(\overline{b}) \cap \text{span} \{a_t : t \in T\}),$$

where $\text{span}S$ denotes the linear hull of $S \subset \mathbb{R}^n$ and, provided that $S$ is convex, extr$S$ stands for the set of extreme points of $S$. Recall that $x \in \text{extr}S$ if it cannot be expressed as a convex combination of two points of $S \setminus \{x\}$. Observe that, as a consequence of well-know results about polyhedral sets, $\mathcal{E}(\overline{b})$ is always a nonempty and finite set; moreover,

$$\mathcal{E}(\overline{b}) = \text{extr} \mathcal{F}(\overline{b}) \iff \text{extr} \mathcal{F}(\overline{b}) \neq \emptyset;$$

in fact, extr$\mathcal{F}(\overline{b}) \neq \emptyset$ if and only if $\mathcal{F}(\overline{b})$ does not contain any line, which is equivalent to the fact that span $\{a_t : t \in T\} = \mathbb{R}^n$. This construction is inspired by the one of [12, p. 142], and used in [6] to compute the calmness modulus of the optimal value function of linear optimization problems.

Theorem 5 Let $\overline{b} \in \text{dom} \mathcal{F}$. Then

$$\text{Hof} \mathcal{F}(\overline{b}) = \max_{x \in \mathcal{E}(\overline{b})} \text{clm} \mathcal{F}(\overline{b}, x).$$
Proof. The inequality \( \text{Hof} \ F(b) \geq \max_{x \in \mathcal{E}(b)} \text{clm} \ F(b, x) \) for all \( b \in \text{dom} \ F \) holds directly from the definitions.

Let us see that \( \text{Hof} \ F(b) \leq \max_{x \in \mathcal{E}(b)} \text{clm} \ F(b, x) \) for all \( b \in \text{dom} \ F \).

Firstly, Theorem 3 establishes
\[
\text{Hof} \ F(b) = \sup_{x \in F(b)} \text{clm} \ F(b, x).
\]

Our next goal is to select a finite subset of points to play the role of \( F(b) \) in the previous equality. To this purpose, observe that from Lemma 3 (ii), together with the finiteness of \( T \), and then of the family \( T_b = \{ T_b(x) : x \in F(b) \} \), we can write
\[
\sup_{x \in F(b)} \text{clm} \ F(b, x) = \sup_{x \in M(b)} \text{clm} \ F(b, x),
\]
where
\[
M(b) := \{ x \in F(b) : T(x) \text{ is maximal w.r.t. the inclusion order in } T \}.
\]

Now, let us see that \( M(b) \) may be replaced with \( \mathcal{E}(b) \) in (17). Specifically, let us prove that for every \( x \in M(b) \) there exists \( y \in \mathcal{E}(b) \) with \( T(x) = T(y) \). Write \( x = y + z \), where \( y \in \text{span} \{ a_t : t \in T \} \) and \( z \in \{ a_t : t \in T \}^\perp \) (the orthogonal subspace to \( \{ a_t : t \in T \} \)). Since \( a'_tx = a'_ty \) for all \( t \in T \), \( y \in F(b) \) and \( T(x) = T(y) \).

Let us prove that \( y \in \mathcal{E}(b) \) arguing by contradiction. Since \( y \in P := F(b) \cap \text{span} \{ a_t : t \in T \} \), let us assume that \( y \) is not an extreme point of polyhedron \( P \). Write
\[
P = \{ w \in \mathbb{R}^n : a'_tw \leq b_t, \ t \in T; \ u'_jw = 0, \ j \in J \},
\]
where \( \{ u_j : j \in J \} \) is a basis of \( \{ a_t : t \in T \}^\perp \), with \( J = \emptyset \) if \( \text{span} \{ a_t : t \in T \} = \mathbb{R}^n \). Under the current assumption, the rank of \( \{ a_t, \ t \in T(y) \}; \ u_j, \ j \in J \) is (strictly) less than \( n \). Then, there exists
\[
0_n \neq u \in \{ a_t : t \in T(y) ; \ u_j, \ j \in J \}^\perp.
\]

Observe that there must exist \( t_0 \in T \setminus T(y) \) such that \( a'_{t_0}u \neq 0 \). Assume without loss of generality that \( a'_{t_0}u > 0 \) and take
\[
\nu = \min_{a'_tu > 0, \ t \in T \setminus T(y)} \frac{b_t - a'_ty}{a'_tu} = \frac{b_{t_1} - a'_{t_1}y}{a'_{t_1}u},
\]
where
for some \( t_1 \in T \setminus T_\mathcal{B} (y) \) with \( a'_t u > 0 \). Then, one easily checks

\[
a'_t (y + \alpha u) = b_t, \ t \in T_\mathcal{B} (y) \cup \{ t_1 \}, \quad a'_t (y + \alpha u) \leq b_t, \ t \in T \setminus (T_\mathcal{B} (y) \cup \{ t_1 \}),
\]

which entails \( y + \alpha u \in F (\overline{b}) \) and \( T_\mathcal{B} (y) \subseteq T_\mathcal{B} (y + \alpha u) \), leading to a contradiction with the maximality of \( T_\mathcal{B} (y) \).

Finally,

\[
\text{Hof} \ F (\overline{b}) = \sup_{x \in M (\overline{b})} \text{clm} \ F (\overline{b}, x) = \max_{x \in E (\overline{b})} \text{clm} \ F (\overline{b}, x) .
\]

\[\blacksquare\]

**Remark 5** Observe that the equivalence relation

\[
x^1 \sim x^2 \text{ iff } T_\mathcal{B} (x^1) = T_\mathcal{B} (x^2), \ \text{with } x^1, x^2 \in F (\overline{b}),
\]

induces a finite partition of \( F (\overline{b}) \), say \( F (\overline{b}) = \cup_{i \in R} F_i \), \( R \) being a finite set of indices. In this way, we may define \( T_i := T_\mathcal{B} (x) \) and \( \kappa_i := \text{clm} \ F (\overline{b}, x) \), whenever \( x \in F_i \) (observe that \( T_\mathcal{B} (\cdot) \) and \( \text{clm} \ F (\overline{b}, \cdot) \) are constant on \( F_i \)). Hence,

\[
\text{Hof} \ F (\overline{b}) = \sup_{x \in F (\overline{b})} \text{clm} \ F (\overline{b}, x) = \max_{i \in R} \kappa_i = \max_{i \in M} \kappa_i ,
\]

where \( M \subset R \) is formed by those \( i \in R \) such that \( T_i \) is maximal in the family \( \{ T_i : i \in R \} \) with respect to inclusion order (recall again Lemma 3(ii)).

Then, looking at the proof of the previous theorem we can choose a representative element of each equivalence class, say \( x_i \in F_i , i \in M \). Thus, by decomposing \( x_i = y_i + z_i \), with \( y_i \in \text{span} \{ a_t : t \in T \} \) and \( z_i \in \{ a_t : t \in T \}^\perp \), we have that \( y_i \in E (\overline{b}) \). So, each element of \( E (\overline{b}) \) can be seen as a canonical representative of its equivalence class.

**Remark 6** As a consequence of Theorem 5 we can write the sharp Hoffman constant \( H (A) \) as follows:

\[
H (A) = \max_{b \in \text{dom} \ F} \text{Hof} \ F (b) = \max_{b \in \text{dom} \ F} \max_{x \in E (b)} \text{clm} \ F (b, x) .
\]

Indeed, if the maximum in Corollary 1 is attained at \( J \in 2^T \) such that \( \text{rank} \ A_J = \text{rank} \ A \) and \( \{ a_t : t \in J \} \) is linearly independent, we have

\[
H (A) = \text{Hof} \ F (b^J) = \text{clm} \ F (b^J, 0_n) ,
\]

17
where \( b^J \) is defined as \( b^J_t = 0 \) if \( t \in J \) and \( b^J_t = 1 \) otherwise. Observe that 
\[ 0_n \in F(b^J), \ T_{b^J}(0_n) = J \] and \( 0_n \in E(b^J) \). Furthermore, the associated system \( a'_t d = 1 \) for \( t \in J, \ d \in \mathbb{R}^n \) is consistent, and hence \( J \) is the only maximal element of \( D_{b^J}(0_n) \).

When moving from the global \( H(A) \) to the semi-local \( Hoff \ F(\bar{b}) \), it is clear from Example [3] that not all \( J \)’s are admissible within those with \( \text{rank} \ A_j = \text{rank} \ A \) and \( \{a_t, \ t \in J\} \) being linearly independent.

We finish this section with two illustrative examples of Theorem [5] and the role played by the so-called strongly redundant constraints in relation to \( H(A) \). Recall that the inequality indexed by \( t_0 \in T \) (or \( t_0 \) itself) is said to be strongly redundant for \( \sigma(\bar{b}) = \{a'_t x \leq \bar{b}_t, \ t \in T\} \) if \( \sup \{a'_t x : a'_t x \leq \bar{b}_t, \ t \in T \setminus \{t_0\}\} < \bar{b}_t \) (see [2], Section 4.3). Because of the well-known continuity properties of the optimal value in linear programming, it is clear that the strongly redundant constraints of \( \sigma(\bar{b}) \) have no effect on \( Hoff \ F(\bar{b}) \).

**Example 5** Given the system \( \{x_1 \leq b_1, \ x_2 \leq b_2, \ -x_1 + 4x_2 \leq b_3\} \) we present three general situations depending on \( b \).

**Case 1:** If \(-b_1 + 4b_2 < b_3\), then \( E(b) = \{b_1, (4b_2 - b_3)\} \) with \( T(b_1) = \{1, 2\} \) and \( T(4b_2 - b_3) = \{2, 3\} \). Their associated calmness moduli are \( \sqrt{2} \) and 1 respectively, hence \( Hoff F(b) = \sqrt{2} \).

**Case 2:** If \(-b_1 + 4b_2 = b_3\), then \( E(b) = \{b_1\} \) with \( T(b_1) = \{1, 2, 3\} \). In this set up its associated calmness modulus coincide with the Hoffman modulus, which is \( \sqrt{5}/2 \).

**Case 3:** If \(-b_1 + 4b_2 > b_3\), the second constraint becomes strongly redundant. Then \( E(b) = \{b_1\} \) with \( T(b_1) = \{1, 3\} \). Again both calmness and Hoffman moduli reach the value \( \sqrt{5}/2 \).

All different scenarios are covered here, thus \( H(A) = \max_{b \in \mathbb{R}^3} Hoff \ F(b) = \sqrt{2} \). This quantity is obtained in case 1, where the system has no strongly redundant inequalities.

The following example is closely related to Example [3].

**Example 6** Given the system \( \{x_1 \leq b_1, \ x_2 \leq b_2, \ x_1 + x_2 \leq b_3\} \) we present again three general situations depending on \( b \).

**Case 1:** If \( b_1 + b_2 < b_3 \), the third constraint is strongly redundant. Then \( E(b) = \{b_1\} \) with \( T(b_1) = \{1, 2\} \). Hence, \( clm \ F(b, \{b_1\}) = Hoff F(b) = \sqrt{2} \).

**Case 2:** If \( b_1 + b_2 = b_3 \), \( E(b) = \{b_1\} \) and \( T(b_1) = \{1, 2, 3\} \). With only one
extreme point, $\text{clm} \, \mathcal{F}(b, \binom{b_1}{b_2}) = \text{Hof} \, \mathcal{F}(b) = 1$.

**Case 3:** If $b_1 + b_2 > b_3$, then $\mathcal{E}(b) = \left\{ \binom{b_1}{b_3}, \binom{b_3 - b_2}{b_2} \right\}$ with $T(\binom{b_1}{b_3}) = \{1, 3\}$ and $T(\binom{b_3 - b_2}{b_2}) = \{2, 3\}$. Both calmness moduli are the same, so $\text{clm} \, \mathcal{F}(b, x) = \text{Hof} \, \mathcal{F}(b) = 1$.

Clearly $H(A) = \sqrt{2}$, and it cannot be obtained without a strongly redundant inequality.

## 5 Conclusions

We have analyzed different properties oriented to quantify the semi-local upper-Lipschitz behavior of set-valued mappings between metric spaces, where by ‘semi-local’ we mean the study of the whole image set with respect to parameter perturbations (a similar use of this term can be found, for instance, in [19, Definition 2.1]). Both Hoffman stability (11) and uniform calmness (10) constitute intermediate steps between calmness and global Hoffman properties. All these semi-local properties are shown to be equivalent (and with the same rate/modulus) for closed-convex-graph mappings (Theorem 3). This is the case of the feasible set mapping of linear inequality systems under right-hand side perturbations.

For this feasible set mapping we succeed in giving a practically computable expression (Theorem 5), as far as it reduces the computation of Hoffman modulus to computing finitely many calmness moduli. When the nominal feasible set has no lines, we just have to consider its extreme points. On the other hand, a point-based formula for computing calmness moduli can be traced out from [3, Section 4]. We include some comments oriented to integrate our results in the existing literature.

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