Strong Unification

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Abstract

We investigate the possibility that unification occurs at strong coupling. We show that, despite the fact the couplings pass through a strong coupling regime, accurate predictions for their low energy values are possible because the couplings of the theory flow to infrared fixed points. We determine the low-energy QCD coupling in a favoured class of strong coupling models and find it is reduced from the weak coupling predictions, lying close to the experimentally measured value. We extend the analysis to the determination of quark and lepton masses and show that (even without Grand Unification) the infra-red fixed point structure may lead to good predictions for the top mass, the bottom to tau mass ratio and tan β. Finally we discuss the implications for the unification scale finding it to be increased from the MSSM value and closer to the heterotic string prediction.

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1 Introduction

The remarkable agreement of the unification predictions for gauge couplings offers the best evidence for a stage of unification of the fundamental forces. Further, the determination of the unification scale close to the (post)diction of string theory may be the first quantitative indication of unification with gravity. However, in detail, the predictions do not quite fit with our expectations, particularly in the case of superstring unification. The evolution of the gauge couplings, with the assumption of the minimal MSSM spectrum, yields a value for the strong coupling, \( \alpha_3^0(M_Z) \approx 0.126 \), rather high when compared with the latest average of experimental determinations \( \alpha_3(M_Z) = 0.118 \pm 0.003 \).

Further, the unification scale, \( M_g^u \), is found to be \( (1 - 3) \times 10^{16} \) GeV, a factor of 20 below the string scale, which is the typical expectation for the weakly coupled heterotic string. The unified coupling at the unification scale is given by \( \alpha_g^u \approx 0.043 \), so the physics around the unification scale \( M_g^u \) lies in the perturbative domain, but it has been argued \(^2\) that this is not acceptable in string theory, as the theory suffers from the “dilaton runaway problem”. In order to stabilise the dilaton one must appeal to non-perturbative effects and the authors of ref \(^2\) and \(^4\) argue that an intermediate value of \( \alpha_g \approx 0.2 \) at \( M_g \) is desired. This, they argue, is large enough to stabilize the dilaton, yet remains perturbative in the sense of quantum field theory \(^1\) to justify the perturbative analysis of the coupling unification.

Ways to eliminate these problems have been widely studied\(^2\). Witten has found \(^5\) that the (10 dimensional) strongly coupled heterotic string theory (M theory) gives a prediction for the unification scale more closely in agreement with the gauge unification value found in the MSSM. However, this does not by itself address the problem of dilaton stability. Stimulated in part by this problem, we have recently explored the case of unification at intermediate values of gauge coupling at the unification scale, for which case perturbation theory still may be used up to the unification scale. We found \(^3\) that the prediction for the strong coupling constant and the unification scale is remarkably insensitive to the addition of massive states\(^3\) which lead to unification at intermediate coupling. In this letter we extend this analysis in two ways. We discuss in detail the prospects for obtaining precision predictions for the case the unified coupling becomes strong. We also consider the implications for Yukawa couplings which are fixed because they flow rapidly to fixed points in the case of unification at strong coupling.

One may easily achieve unification at strong coupling through the addition of a number of additional (massive) multiplets to the MSSM spectrum. It is notable that such additional multiplets occur in the majority of string compactifications, coming in representations vectorlike with respect to the Standard Model gauge group, and hence, likely to acquire mass at the first stage of spontaneous breaking below the compactification scale, which is likely to be much higher than the electroweak breaking scale. Thus, it is reasonable to argue that the MSSM is not the typical case and that we should consider models with additional massive states as standard. However, this seems to destroy the success of the unification predictions which are very sensitive to the addition of such states. In ref \(^6\) we argued that this is not the case for the most promising extensions of the Standard Model have additional states which fill out complete \( SU(5) \) multiplets and these do not change the MSSM unification predictions at one loop order; this is clearly the case for the case of Grand Unification with a Grand Unified group which contains \( SU(5) \) for there may easily be additional Grand Unified multiplets with mass below the unification scale. However, it may also be the case for superstring unification, even though the gauge group below the compactification scale is not Grand Unified.

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\(^1\) String perturbation series are more divergent than field theory series, so small (perturbative) couplings in QFT can generate large (non-perturbative) in string theory.

\(^2\) For an extensive review see \(^4\).

\(^3\) We neglected there the Yukawa effects of the third generation, but the result is still true when they are included, at least for the unification scale \(^4\).
ref. [8] it was argued that level-one string theories with symmetry breaking by Wilson lines provide the most promising superstring compactified models. In these, the coupling constant of the various gauge couplings have the usual \( SU(5) \) unified values, even though the gauge group below compactification may naturally need not be Grand Unified; indeed\(^5\) it may be just the Standard Model gauge group. Furthermore, level-one string theories allow only low lying representations to occur, immediately explaining why quarks and leptons belong only to triplets of \( SU(3) \) and doublets of \( SU(2) \). In such models, with Wilson line breaking, the multiplet structure is predicted to have the generations filling out complete \( SU(5) \) representations\(^2\), just as is observed. Moreover, there is a natural explanation for the Higgs doublets of the MSSM, because there are predicted to be at least two (split) multiplets which do not fill out complete \( SU(5) \) representations. There is only a discrete number of Wilson lines possible and for one of these the split multiplet contains just the Higgs of the MSSM. The only ambiguity in this class of models is that it may contain an additional \( n_5 \) multiplets filling out complete \( 5 + 5 \) representations and \( n_{10} \) complete \( 10 + \bar{10} \) representations.

With this motivation, we now proceed to consider the possibility that gauge unification occurs at strong coupling. We will show that this leads to precise unification predictions and will compute them for the class of models just discussed, in which the additional states leading to strong coupling unification fill out complete \( SU(5) \) representations. We will further show that such models have a very interesting consequence for fermion masses because Yukawa couplings may lie in the domain of attraction of an infrared fixed point of the theory and, due to the strong coupling at unification, they flow very quickly to the fixed point. This leads to detailed predictions for the third generation masses.

2 Strong Unification

Unification at strong coupling was proposed a long time ago\(^1\) as a viable possibility leading to reasonable predictions for the low-energy gauge couplings. Here we reformulate the idea in a way that quantifies the uncertainties in the predictions and refers only to evolution of the couplings once they reach the perturbative domain. At first sight it seems that strong unification does not lend itself to a precise prediction of the gauge couplings, due to the need to determine the evolution in the strong coupling domain. The reason this is not the case is because the ratio of gauge couplings flow to an infrared fixed point. Thus, one has the situation where the boundary conditions for the evolution of couplings in the MSSM are still reliably calculable - at the “intermediate” mass scale, \( M \), of the new vectorlike states, the ratios of the gauge couplings are given by their infra-red fixed point values, corresponding to the theory above this mass scale. For the case the coupling is initially large, the ratio of couplings closely approaches the fixed point, so a determination of the low energy values of the couplings, using these boundary conditions plus two-loop MSSM evolution provides an accurate determination of the couplings.

The two loop renormalisation group equations for the gauge couplings, with no Yukawa interaction, are given by:

\[
\frac{d\alpha_i}{dt} = \tilde{b}_i \alpha_i^2 + \frac{1}{4\pi} \sum_{j=1}^{3} \tilde{b}_{ij} \alpha_i^2 \alpha_j + O(\alpha^4)
\]

with \( i = \{1, 2, 3\} \) and where \( t = \frac{1}{\pi} \ln Q/M_g \); \( M_g \) is the unification scale, and \( \tilde{b}_i \) and \( \tilde{b}_{ij} \) are the appropriate one loop and two loop beta functions respectively. To exhibit the infra-red-fixed-point

\(^4\)For a fuller discussion of the possibilities see [9]

\(^5\)Even though the gauge group is not \( SU(5) \).
(IRFP) structure of these equations we rewrite this equation in the form
\[
\frac{d}{dt} \ln \frac{\alpha_i}{\alpha_k} = \tilde{b}_i \alpha_i - \tilde{b}_k \alpha_k + \frac{1}{4\pi} \sum_{j=1}^{3} \left( \tilde{b}_{ij} \alpha_i - \tilde{b}_{kj} \alpha_k \right) \alpha_j + \mathcal{O}(\alpha^3)
\] (2)

At one-loop order the evolution of this ratio clearly has an IRFP stable fixed point\(^6\) with the fixed point value given by
\[
\left( \frac{\alpha_i}{\alpha_k} \right)^* = \frac{\tilde{b}_k}{\tilde{b}_i}
\] (3)

Provided the gauge couplings are small, the two-loop corrections and above will only give a small correction to this fixed point value. Phenomenologically, this must be the case for, to be viable, the couplings should match at the scale \(M\) the values of the low energy couplings evolved up in energy using the usual MSSM renormalisation group equations. Provided \(M\) is large enough, the values of the couplings are all small (of \(O(1/24)\) for \(M\) near \(10^{16}\) GeV). In the class of models explored here the two loop corrections are further suppressed. This follows because above the intermediate scale the one loop beta functions in the presence of complete \(SU(5)\) multiplets is given by:
\[
\tilde{b}_i = b'_i = \left( \begin{array}{c} \frac{33}{3} + n \\ 1 + n \\ -3 + n \end{array} \right)
\] (4)

where \(n\) represents the linear combination \(^7\) \(n = (n_5 + 3n_{10})/2\). We are particularly interested in the case \(n\) is large for then \(M\) is large and the couplings are driven rapidly to the fixed point ratios. However the two-loop corrections do not grow with \(n\) because, as is clear from \(^{11}\) (cf eqs. (22),(23), (24)), massive states do not contribute two-loop corrections, the usual two-loop contribution to the beta function being cancelled by the massive threshold corrections \(^8\). Thus the two loop effects are suppressed both by the additional power of the coupling at the matching scale, \(M\), and by the factor \(1/n\). We shall investigate the magnitude of these corrections in Section 4.

To summarise we have reformulated the unification of gauge couplings for the case that unification occurs at large coupling via boundary conditions for renormalisation group equations which apply below the scale of new physics beyond the MSSM. The advantage of this is twofold. It requires integration of the renormalisation group equations only in the domain where the couplings are small and perturbation theory applies. It quantifies the uncertainties in the analysis. The latter come from the two loop and higher corrections at the matching scale where these are small; hence the possibility of making accurate predictions for the gauge couplings at low energies even in the case of unification at strong coupling. Note that strong unification \textit{does not} even require the equality of couplings at any scale. In this sense the infra-red structure of the theory substitutes for Grand Unified relations.

2.1 One loop analysis

It is instructive to determine the “strong unification” predictions at one-loop order, to show the general trend, before presenting the results of the full two loop analysis. Below \(M\), the multiplet structure is just that of the MSSM with one-loop beta functions \(b_i\) given by eq(4) with \(n = 0\), \(\tilde{b}_i = b_i = b'_i (n = 0)\). Above \(M\) the beta functions are \(b'_i\). The boundary conditions for the evolution below the scale \(M\) are just
\[
\frac{\alpha_k(M)}{\alpha_i(M)} = \frac{b'_i}{b'_k}
\] (5)

\(^6\)Provided, of course, \(\tilde{b}_i > 0\) which is necessary if all the couplings are to become large at some high scale.

\(^7\)here \(n_5 = N_5 + N_7\) and \(n_{10} = N_{10} + N_{16}\)
From eq(1) we have
\[ \alpha_i^{-1}(M_Z) = \alpha_i^{-1}(M) + \frac{b_i}{2\pi} \ln \left( \frac{M}{M_Z} \right) \] (6)

Using this and the boundary condition gives
\[ \frac{1}{2\pi} \ln \left( \frac{M}{M_Z} \right) = \frac{b'_k \alpha_i^{-1}(M_Z) - b'_i \alpha_k^{-1}(M_Z)}{(b_i - b_k)n} \] (7)

Further, it is straightforward to show that:
\[ (b_1 - b_2)\alpha_3^{-1}(M_Z) = (b_1 - b_3)\alpha_2^{-1}(M_Z) + (b_3 - b_2)\alpha_1^{-1}(M_Z) \] (8)

Using this relation one may obtain \( \alpha_3(M_Z) \) given the experimental measurements of the other two couplings. This relation is identical to that one obtains from the MSSM RGE equations showing that, at one loop order, the predictions are the same in the MSSM and the fixed-point-boundary-condition (FPBC) scheme. Note eq(8) is independent of the value of \( n \) because complete \( SU(5) \) multiplets contribute equally to the one-loop beta functions. As a result, at one loop, the prediction for \( \alpha_3(M_Z) \) in “strong unification” is universal for the class of models with additional complete \( SU(5) \) multiplets. With the measured values of \( \alpha_1(M_Z) \) and \( \alpha_2(M_Z) \) as input, the value for \( \alpha_3(M_Z) \) is 0.1145. Of course, two loop and SUSY threshold corrections should be added to obtain a precision prediction; these will be considered in the next section.

The value of the mass \( M \) of the vectorlike states, additional to the MSSM spectrum, may be obtained from eq(7). Taking \( i = 1 \) and \( k = 2 \) one finds
\[ \frac{1}{2\pi} \ln \left( \frac{M}{M_Z} \right) \approx \frac{29.3n - 136.9}{5.6n} \] (9)

From this, we find that solutions are possible only for \( n \geq 5 \). The value of \( M \) is clearly \( n \) dependent, and varies from \( 10^3 \)GeV for \( n = 5 \) to \( 10^{13} \)GeV for \( n = 20 \) and to \( 10^{16} \) for \( n = 300 \). However, this should not be interpreted as the normal unification scale at which the couplings are equal. The latter point occurs in the strong coupling domain, and thus cannot be precisely determined in perturbation theory. However, one may determine the scale at which the couplings enter the non-perturbative domain. At this point they are evolving rapidly, so it is a reasonable conjecture that they become equal very close to this scale. Remarkably, the scale \( M_{NP} \), at which the couplings become large, turns out to be almost independent of \( n \) and is given by \( M_{NP} \approx 3.10^{16} \)GeV, essentially the same scale as is found in the MSSM for the unification scale!

2.2 Two loop analysis

In this section we present a two loop analysis in which we compute the scale \( M \) at which the couplings are in the fixed point ratio as well as the prediction for \( \alpha_3(M_Z) \). The real unification scale, if any, will not be an output of the scheme.

As usual in making a prediction for the low energy values of the gauge couplings we are faced with the problem of unknown values for the supersymmetric spectrum; this can significantly affect the predictions we make because, at two loop order, one has to take into account the low energy supersymmetric thresholds in one loop. Various scenarios for low supersymmetric energy spectrum have been studied in the minimal supersymmetric standard model (MSSM), and their effect on the value of the unification scale as well as \( \alpha_3^0(M_Z) \) has been extensively discussed [12]. Given this we choose to make our predictions relative to the MSSM prediction calculated with a given SUSY threshold. In the MSSM we have
where the \( \delta^o \) contains the effect of the low energy supersymmetric thresholds and we have ignored the Yukawa couplings effects, which are known to be small. The MSSM variables are labelled with an “o” index to distinguish them from the model based on fixed point scenario.

In the FPBC case we have

\[
\alpha_i^{-1}(M_z) = -\delta^o_i + \alpha_i^{-1}(M) + \frac{b_i}{2\pi} \ln \left[ \frac{M}{M^o_i} \right] + \frac{1}{4\pi} \sum_{j=1}^3 \frac{b_{ij}}{b_j} \ln \left[ \frac{\alpha_j(M)}{\alpha_j^o(M)} \right]
\]

(10)

where, as discussed above, we use the same threshold \( \delta^o \) as in the MSSM. The \( b_i \) and \( b_{ij} \) denote the one loop and two loop beta functions which are just the same as in the MSSM; \( M \) denotes the scale where the couplings \( \alpha_i(M) \) are in the “fixed-point” ratio. Now, from experiment we have well measured values for \( \alpha_1(M_Z) \) and \( \alpha_2(M_Z) \) from the values of electromagnetic coupling and Weinberg angle at \( M_Z \) scale. Therefore, in computing \( \alpha_3(M_Z) \), these values were taken as input from experiment. The FPBC case should comply with this condition, too, and hence \( \alpha_1(M_Z) = \alpha_1^o(M_Z) \) and \( \alpha_2(M_Z) = \alpha_2^o(M_Z) \).

We subtract the eqs. (10), (11) and, using these relations, obtain

\[
0 = \alpha_i^{-1}(M) - \alpha_i^{-1}(M) - \frac{b_i}{2\pi} \ln \left[ \frac{M}{M^o_i} \right] + \frac{1}{4\pi} \sum_{j=1}^3 \frac{b_{ij}}{b_j} \ln \left[ \frac{\alpha_j(M)}{\alpha_j^o(M)} \right] + \frac{1}{4\pi} \frac{b_{3i}}{b_3} \ln \left[ \frac{\alpha_3^o(Mi)}{\alpha_3(M)} \right]
\]

(12)

for the case \( i = \{1, 2\} \) and

\[
\alpha_3^{-1}(M_Z) - \alpha_3^{-1}(M_Z) = \alpha_3^{-1}(M) - \alpha_3^{-1}(M) - \frac{b_3}{2\pi} \ln \left[ \frac{M}{M^o_3} \right] + \frac{1}{4\pi} \sum_{j=1}^3 \frac{b_{3j}}{b_j} \ln \left[ \frac{\alpha_j(M)}{\alpha_j^o(M)} \right] + \frac{1}{4\pi} \frac{b_{33}}{b_3} \ln \left[ \frac{\alpha_3^o(M3)}{\alpha_3(M)} \right]
\]

(13)

for the case \( i = 3 \). At two loop order we may make the approximation

\[
\ln \left[ \frac{\alpha_3^o(M)}{\alpha_3(M)} \right] = \ln \alpha_3^{-1}(M)_{\text{oneloop}} - \ln \alpha_3^{-1}(M)_{\text{oneloop}} = 0
\]

One may readily check that the LHS is indeed numerically very small. This gives

\[
0 = \alpha_i^{-1}(M) - \alpha_i^{-1}(M) - \frac{b_i}{2\pi} \ln \left[ \frac{M}{M^o_i} \right] + \frac{1}{4\pi} \sum_{j=1}^3 \frac{b_{ij}}{b_j} \ln \left[ \frac{\alpha_j(M)}{\alpha_j^o(M)} \right]
\]

(14)

for the case \( i = \{1, 2\} \) and

\[
\alpha_3^{-1}(M_Z) - \alpha_3^{-1}(M_Z) = \alpha_3^{-1}(M) - \alpha_3^{-1}(M) - \frac{b_3}{2\pi} \ln \left[ \frac{M}{M^o_3} \right] + \frac{1}{4\pi} \sum_{j=1}^3 \frac{b_{3j}}{b_j} \ln \left[ \frac{\alpha_j(M)}{\alpha_j^o(M)} \right]
\]

(15)

for \( i = 3 \).

This is a system of three equations with the unknowns: \( \alpha_3(M_Z) \), \( M \) and one of the \( \alpha_i(M) \)’s (say \( \alpha_1(M) \)), as the ratio of any two of them is a known function for any given \( n \) through eq(15). The solution is

\[
\alpha_3^{-1}(M_Z) = \alpha_3^{-1}(M_Z) - \frac{470}{17\pi} \ln \left[ \frac{\alpha_1(M)}{\alpha_1^o(M)} \right] + \frac{17}{14\pi} \ln \left[ \frac{b_1 + n}{b_3 + n} \right] - \frac{15}{2\pi} \ln \left[ \frac{b_1 + n}{b_2 + n} \right]
\]

(16)
Table 1: The value of $\alpha_1$ at the intermediate scale, the strong coupling at $M_Z$, and the intermediate scale obtained using the fixed-point boundary conditions as a function of $n$. Also shown are the bottom to tau mass ratio and the top mass for the case the third generation couplings are in the domain of attraction of the fixed point.

| $n$ | $\alpha_1(M)$ | $\alpha_3(M_Z)$ | $M$ | $\frac{m_\tau}{m_t}(M_Z)$ | $m_t$ | $\tan \beta$ |
|-----|---------------|----------------|-----|--------------------------|------|-------------|
| 6   | 0.020         | 0.1163         | $2.14 \times 10^9$ | 1.62 | 229.26 | 47.15       |
| 8   | 0.023         | 0.1188         | $1.61 \times 10^9$ | 1.62 | 209.87 | 46.99       |
| 10  | 0.025         | 0.1203         | $0.79 \times 10^{10}$ | 1.61 | 204.06 | 46.82       |
| 12  | 0.027         | 0.1213         | $1.04 \times 10^{11}$ | 1.60 | 202.58 | 46.70       |
| 14  | 0.029         | 0.1219         | $6.44 \times 10^{11}$ | 1.59 | 201.51 | 46.64       |
| 16  | 0.030         | 0.1224         | $2.52 \times 10^{12}$ | 1.58 | 200.92 | 46.54       |
| 18  | 0.031         | 0.1228         | $7.23 \times 10^{12}$ | 1.57 | 200.57 | 46.48       |
| 20  | 0.032         | 0.1231         | $1.68 \times 10^{13}$ | 1.57 | 200.34 | 46.43       |
| 22  | 0.033         | 0.1234         | $3.34 \times 10^{13}$ | 1.56 | 200.20 | 46.41       |
| 26  | 0.034         | 0.1238         | $0.96 \times 10^{14}$ | 1.55 | 200.03 | 46.32       |

for the strong coupling and

$$\ln \left[ \frac{M}{M_g^0} \right] = -\frac{2\pi}{n\alpha_g^0} + \frac{(2336 + 341n)}{231n} \ln \left[ \frac{\alpha_1(M)}{\alpha_g^0} \right] + \frac{57 + 7n}{4n} \ln \left[ \frac{b_1 + n}{b_2 + n} \right] - \frac{4(22 + n)}{21n} \ln \left[ \frac{b_1 + n}{b_3 + n} \right]$$ (17)

for the scale $M$, where the value of $\alpha_1(M)$ is given by the root of the nonlinear equation:

$$\alpha_1^{-1}(M) = \frac{b_1 + n}{n\alpha_g^0} - \frac{1168}{231\pi} \left( \frac{b_1 + n}{n} \right) \ln \left[ \frac{\alpha_1(M)}{\alpha_g^0} \right] - \frac{57}{8\pi} \left( \frac{b_1 + n}{n} \right) \ln \left[ \frac{b_2 + n}{b_2 + n} \right] + \frac{44}{21\pi} \left( \frac{b_1 + n}{n} \right) \ln \left[ \frac{b_1 + n}{b_3 + n} \right]$$ (18)

with $b_1 = 33/5, b_2 = 1, b_3 = -3$.

Using these expressions we get the numerical results presented in Table 1 in which we have taken as the reference MSSM prediction $\alpha_3(M_Z) = 0.126$. We see we get a lower value for alpha strong at electroweak scale than in the MSSM and closer to the experimental measurement $\alpha_3(M_Z) = 0.118 \pm 0.003$. Since the couplings are quite small at the high scale $M$, the higher corrections to the boundary conditions are expected to be small. We will estimate these corrections in Section 4.

### 3 Strong Unification and the masses of the third generation

As we have discussed, the addition of massive multiplets to the MSSM is to be expected in viable Grand Unified theories and in many string theories. The effect of such new states is to increase the gauge coupling at unification and can easily make it approach the strong coupling domain. We further remarked that in this domain the fixed point structure of the theory relating the largest Yukawa couplings (and hence third generation masses) to the gauge couplings becomes the dominant effect as the couplings flow rapidly towards the fixed points. In this section we explore these implications in detail.

The renormalisation group equations for the Yukawa couplings in the MSSM are given by

$$\frac{d}{dt} Y_\tau = Y_\tau \left( 3Y_b + 4Y_\tau - \frac{9}{5} \alpha_1 - 3\alpha_2 \right)$$

with $Y_\tau$ being the Yukawa coupling for tau, $Y_b$ for bottom, and $\alpha_1$ and $\alpha_2$ being the gauge couplings.
\[
\frac{d}{dt} Y_b = Y_b \left( Y_t + 6Y_b + Y_r - \frac{7}{15} \alpha_1 - 3 \alpha_2 - \frac{16}{3} \alpha_3 \right) \\
\frac{d}{dt} Y_t = Y_t \left( 6Y_t + Y_b - \frac{16}{15} \alpha_1 - 3 \alpha_2 - \frac{16}{3} \alpha_3 \right)
\]

(19)

where \( Y_j = h_j^2/4\pi \) and \( h_j \) is the Yukawa coupling. If we ignore the smaller gauge couplings \( \alpha_1 \) and \( \alpha_2 \) and we keep only the large top Yukawa coupling \( Y_t \), the last equation has the form

\[
\frac{dY_t}{dt} = Y_t(sY_t - r_3\alpha_3)
\]

(20)

This has a fixed point that relates the top Yukawa coupling to the QCD coupling \( \alpha_3 \) given by [13], [14]:

\[
\left( \frac{Y_t}{\alpha_3} \right)^* = \frac{r_3 + b_3}{s}
\]

(21)

which is infra-red stable if \( r_3 + b_3 > 0 \). In the case of the MSSM, \( b_3 = -3 \), \( r_1 = (13/15, 3, 16/3) \) and \( s = 6 \), then, the fixed point is \( \left( \frac{Y_t}{\alpha_3} \right)^*_{MSSM} = \frac{7}{18} \).

However, as stressed in ref. [13], this fixed point value is not reached for large initial values of the top quark coupling because the range in \( t \) between the Planck scale and the electroweak scale is too small to cause the trajectories to closely approach the fixed point. As demonstrated in [14] a “Quasi-fixed point” governs the value of \( Y_t \) for large initial values of \( Y_t \) and is given by

\[
\left( \frac{Y_t}{\alpha_3} \right)^{QFP} = \frac{\left( \frac{Y_t}{\alpha_3} \right)^*}{1 - \left( \frac{\alpha_3(t)}{\alpha_3(0)} \right)^{B_3}}
\]

(22)

where \( B_3 = \frac{r_3}{\alpha_3} + 1 \). The term \( \left( \frac{\alpha_3(t)}{\alpha_3(0)} \right)^{B_3} \) determines the rate of approach to the fixed point, the smaller this term, the closer the QFP is to the IRSFP. For the MSSM in the case only the top Yukawa coupling is in the domain of attraction of the fixed point, including electroweak corrections, the quasi-fixed point predicts a top quark mass of \( m_t \approx 210 \sin \beta \text{ GeV} \), where \( \tan \beta \) is the ratio of the vacuum expectation values of the Higgs doublets. For the case that the top and the bottom Yukawa couplings are in the domain of attraction of the fixed point, the prediction for the fixed point is \( m_t \approx 190 \text{GeV} \), and the dependence on \( \beta \) disappears in this case, because the top and bottom Yukawas are nearly equal and thus we are in the region with \( \sin \beta \approx 1 \).

As has been shown in [14], the profusion of new fields increase the rate of approach to the fixed point. This follows because the gauge couplings are evolving rapidly so the convergence factor \( \left( \frac{\alpha_3(t)}{\alpha_3(0)} \right)^{B_3} \) is very small. Thus we can expect that the IRSP structure will play an important role for the determination of the couplings in the class of models considered in this paper. In this case one must keep all three gauge couplings, as all are comparable above the scale \( M \). However the analysis is tractable because the ratios of the gauge couplings are given by the infra-red fixed point ratio of eq(3). The renormalisation group equations may be written in the form (see also ref. [16])

\[
\frac{d}{dt} \ln \left( \frac{Y_r}{\alpha_i} \right) = \alpha_i \left( 4 \frac{Y_r}{\alpha_i} + 3 \frac{Y_b}{\alpha_i} - \frac{9 \alpha_1}{5 \alpha_i} - 3 \frac{\alpha_2}{\alpha_i} - (b_i + n) \right)
\]

(23)

\[
\frac{d}{dt} \ln \left( \frac{Y_b}{\alpha_i} \right) = \alpha_i \left( \frac{Y_r}{\alpha_i} + 6 \frac{Y_b}{\alpha_i} + \frac{Y_t}{\alpha_i} - \frac{7 \alpha_1}{15 \alpha_i} - 3 \frac{\alpha_2}{\alpha_i} - \frac{16 \alpha_3}{3 \alpha_i} - (b_i + n) \right)
\]

(24)

\[
\frac{d}{dt} \ln \left( \frac{Y_t}{\alpha_i} \right) = \alpha_i \left( \frac{Y_b}{\alpha_i} + 6 \frac{Y_t}{\alpha_i} - \frac{13 \alpha_1}{15 \alpha_i} - 3 \frac{\alpha_2}{\alpha_i} - \frac{16 \alpha_3}{3 \alpha_i} - (b_i + n) \right)
\]

(25)
n & $m_b(M_Z)$ & $m_t$ \\
6 & 1.41 & 184.58 \\
8 & 1.53 & 185.85 \\
10 & 1.57 & 187.69 \\
12 & 1.59 & 189.20 \\
14 & 1.59 & 190.42 \\
16 & 1.59 & 191.38 \\
18 & 1.59 & 192.17 \\
20 & 1.59 & 192.83 \\
22 & 1.59 & 193.40 \\
26 & 1.58 & 194.27 \\

Table 2: The bottom to tau mass ratio and top mass as a function of $n$ for the case all three generations have equivalent coupling to Higgs states.

where the index $i$ is fixed. Using eq.(3) we get the following fixed points:

$$\left(\frac{Y_\tau}{\alpha_i}\right)^* = b'_i \left( \frac{10}{61} + \frac{143}{305} b'_1 + \frac{30}{61} b'_2 - \frac{40}{61} b'_3 \right)$$

(26)

$$\left(\frac{Y_t}{\alpha_i}\right)^* = b'_i \left( \frac{9}{61} + \frac{136}{915} b'_1 + \frac{27}{61} b'_2 + \frac{136}{183} b'_3 \right)$$

(27)

$$\left(\frac{Y_b}{\alpha_i}\right)^* = b'_i \left( \frac{7}{61} - \frac{23}{915} b'_1 + \frac{21}{61} b'_2 + \frac{160}{183} b'_3 \right)$$

(28)

These fixed point ratios apply at the scale $M$. Below this scale the couplings evolve via the usual MSSM renormalisation group equations, eq(19). Using these fixed point boundary conditions we have integrated these equations numerically, including a SUSY threshold at 300 GeV. This allows us to determine $m_t$, the ratio $m_b/m_\tau$ and $\tan \beta$ at the $M_Z$ scale for the large $\tan \beta$ case. The results are presented in Table 1, with an acceptable value [17] for $m_b/m_\tau$ and a rather high value for $m_t$. For the low $\tan \beta$ case [8] we determine $m_t \sin \beta$, which is approximately 1.08 times the value for $m_t$ given in Table 1. We see that the prediction for $m_t \sin \beta$ is very high. Thus we are driven to the low $\tan \beta$ solution with $\tan \beta$ in the range (1.01-1.3).

The values for the third generation masses are rather sensitive to the number of Yukawa couplings lying in the domain of attraction of the infra-red fixed points. The case just presented corresponds to a near-minimal case where only the third generation Yukawas are large. However, with the addition of numerous vectorlike multiplets, there are many more possible Yukawa couplings which can affect our conclusions. To illustrate this we first consider a model in which each of the quark and lepton families has a coupling to a (heavy) Higgs state, corresponding to the terms in the superpotential $\sum_{i,j=1}^{3} (h_{ij}^u Q_i U_j H^i_2 + h_{ij}^d Q_i U_j H^i_1 + h_{ij}^l L_i e_j H^i_1)$.

Such a model has been proposed as a means of dynamically solving the SUSY flavour problem [13] and for generating structure in the light quark mass matrix [13] via mixing in the Higgs sector so that the two light Higgs doublets of the MSSM are mixtures of $H^1_1$ and of $H^1_2$. In this case the renormalisation group equations are straightforward generalisations

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*This case arises if the light doublet giving mass to the down quarks and leptons is a mixture of two (or more) of the Higgs fields, the dominant component not coupling to the quarks and leptons. Such mixing is discussed in more detail below.
Table 3: The bottom to tau mass ratio and top mass as a function of \( n \) for the couplings of Table 2 plus additional couplings involving heavy doublet quarks and leptons.

| \( n \) | \( \frac{m_b}{m_{\tau}}(M_Z) \) | \( m_t \) |
|-------|----------------|-------|
| 6     | 1.43           | 128.28|
| 8     | 1.59           | 144.44|
| 10    | 1.65           | 154.19|
| 12    | 1.67           | 160.82|
| 14    | 1.68           | 165.68|
| 16    | 1.67           | 169.40|
| 18    | 1.67           | 172.36|
| 20    | 1.67           | 174.78|
| 22    | 1.66           | 176.80|
| 26    | 1.65           | 179.98|

of eq(23). Solving them for the fixed points and using these as boundary conditions for the MSSM renormalisation group equations gives the results shown in Table 2. Finally, we consider a model in which there are also Yukawa couplings involving mixing of the quarks and leptons with the new quark and lepton states belonging to the vectorlike representations. We restrict our example to the case of couplings between additional \( u_L, b_L \) and \( \tau_L \) states to the Higgs and right-handed quarks and leptons via the terms \( \sum_{i,j=1}^{3}(h_{ij}^u Q'_i U_j H^u + h_{ij}^d Q'_i U_j H^d + h_{ij}^l L'_i e_j H^l) \). The results for this case are presented in Table 3. We see from these tables that the result for the ratio \( m_b/m_{\tau} \) is quite stable and remains acceptable for most values of \( n \). The result for \( m_t \) is quite sensitive to the number of Yukawa couplings and to \( n \). We see that even for the large \( \tan \beta \) case when the top mass has a definite prediction such models can lead to a remarkably consistent pattern of third generation masses.

4 Corrections to FPBC formalism

To discuss the corrections to the fixed-point boundary condition formalism, it is most convenient to work with the form Shifman [11] derived for the running couplings

\[
\alpha_1^{-1}(M) = \alpha_g^{-1} + \frac{3}{2\pi} \ln \left[ \frac{M_g}{M_Z L_L} + 2 \frac{\ln M_g}{M_Z c_R} + \frac{1}{3} \frac{\ln M_g}{M_Z q_L} \right] + \frac{8}{3} \ln \frac{M_g}{M_Z u_R} + \frac{2}{3} \ln \frac{M_g}{M_Z d_R} + 2 \ln \frac{M_g}{M_Z H_{u,d}} \right] + \frac{n}{2\pi} \ln \frac{M_g}{\mu_g} \tag{29}
\]

\[
\alpha_2^{-1}(M) = \alpha_g^{-1} - \frac{6}{2\pi} \ln \frac{M_g}{M_Z} \left( \frac{\alpha_g}{\alpha_2(M)} \right)^{1/3} + \frac{1}{2\pi} \ln \frac{2}{3} \sum_{\text{gen}} \left[ \ln \frac{M_g}{M_Z L_L} + \frac{1}{2} \ln \frac{M_g}{M_Z q_L} \right] + \frac{1}{2\pi} \ln \frac{M_g}{M_Z H_{u,d}} + \frac{n}{2\pi} \ln \frac{M_g}{\mu_g} \tag{30}
\]

\[
\alpha_3^{-1}(M) = \alpha_g^{-1} - \frac{9}{2\pi} \ln \frac{M_g}{M_Z} \left( \frac{\alpha_g}{\alpha_3(M)} \right)^{1/3} + \frac{1}{2\pi} \ln \frac{M_g}{M_Z q_L} + \frac{1}{2} \ln \frac{M_g}{M_Z q_L} + \frac{1}{2} \ln \frac{M_g}{M_Z u_R} \tag{31}
\]

9
The advantage of this form for the running of gauge couplings is that (above the supersymmetric
scale) it is exact to all orders. However, the wave function renormalisation coefficients $Z_i$ are only
known perturbatively so one is still confined to a given order in perturbation theory when testing
coupling unification. To two loop order, one must include the values of wave-function renormalisation
coefficients $Z_i$ in one loop only. In this formula the $Z_i$ factors are evaluated at the scale $M$.

The one loop form of these equations gives the FPBC formalism used above. The two loop
corrections came from the terms involving $\ln(\alpha_g/\alpha_2(M))$ and $\ln(\alpha_g/\alpha_3(M))$, the gauge wave function
renormalisation and also from the terms involving the $Z_i$'s, the matter wave function renormalisation.
To good approximation the former do not affect the FPBC predictions for $\alpha_3(M_Z)$. To see this first set
$Z_i = 1$. Now the result follows because the predictions for $\alpha_3(M_Z)$ involves the differences $[\alpha_i^{-1}(M) – \alpha_j^{-1}(M)]$, $i, j = \{1, 2, 3\}$. In these, the $\ln(M_g/M)$ terms coming from the three generations cancel,
leaving just the gauge wave function renormalisation terms proportional to $\ln(M_g/(M(\alpha_g/\alpha_i(M))^{1/3})$ and the Higgs contribution proportional to $\ln(M_g/M)$. The latter has a small coefficient, so we can
effectively absorb the $(\alpha_g/\alpha_i(M))^{1/3}$ term in a redefinition of $M_g$, $M_g \rightarrow M_g/(\alpha_g/\alpha_i(M))^{1/3}$ with $\alpha_i(M) \approx \text{constant}$.

Hence, to a good approximation, we obtain the fixed-point boundary condition formalism provided
we interpret $M_g/ (\alpha_g/\alpha_i(M))^{1/3}$ as an effective scale $M'_g$. We established this for the case we set $Z_i(M) = 1$; thus, one expects corrections to FPBC formalism if this condition is not respected. How large are they? There are two contributions to each $Z_i$, one coming from gauge interactions and one from
Yukawa interactions. For couplings remaining in the perturbative domain, the effects of the gauge
coupling terms alone was analysed in [6] where it was found that the value of $\alpha_i$ increased slightly.
However, as we have stressed, in strong coupling one expects the Yukawa couplings to be large and
their contribution to $Z_i$ factors is opposite to that of gauge interactions, taking the result closer to
the FPBC result. We may check whether this happens in the case unification occurs at intermediate
coupling where the perturbative approach still applies. In this case we may use the one-loop form for
the $Z_i$ factors. Following the argument presented, we have checked in specific cases that, because of
these cancellations, the prediction does lie close to the FPBC predicted value.

Finally it is of interest to consider the expectation for the unification scale. As we have stressed
this is not determined in the case the coupling is really strong at unification. However, for intermediate
values, say of $O(0.3)$, one may use eq(13) to determine $M_g$. Again we find $M_g$ larger than the MSSM value.
Part of this increase follows simply from the fact noted above that part of the two loop
corrections may be absorbed in a change in the unification scale, giving $M_g = M'_g (\frac{\alpha_g}{\alpha_3})^{1/3}$ (For $\alpha_g = 1$
this approximately gives an increase by a factor 3). The remaining effect comes from the two loop
corrections arising from the $Z$ factors. It is interesting that together they take the value of $M_g$ closer
to the weakly coupled heterotic string prediction.

5 Conclusions

Unification at strong coupling is quite likely in extensions of the MSSM which contain additional states
with mass below the unification scale. Since such cases are perhaps the norm, it is important to determine
their implications for gauge coupling unification. We have shown that it is possible to determine
these implications with surprising accuracy given the fact that the gauge coupling evolution involves a
stage of strong coupling. The formulation of the initial boundary conditions at the intermediate scale

$$+rac{1}{2} \ln \frac{M_g}{MZ_W} \] + \frac{n}{2\pi} \ln \frac{M_g}{\mu_g}$$

(31)
in terms of the infra-red fixed point ratios of gauge couplings of the theory above this scale shows that the uncertainties in gauge coupling predictions are of two loop order. These corrections are small since they should be evaluated at the intermediate scale where the couplings are small. The two loop corrections are further suppressed at large $n$, where $n$ specifies the number of additional states, and also by cancellation between two loop effects involving gauge and Yukawa couplings. The net result is that the predicted value of the strong coupling is reduced from the MSSM value coming closer to the experimental value. For unification at intermediate coupling in which perturbation theory may be used above the intermediate scale the unification scale is also raised relative to the MSSM prediction, taking it closer to the heterotic string prediction. The case of strong unification also leads to predictions for quark and lepton masses because the Yukawa couplings are driven towards infra-red fixed points. We have investigated these predictions and found that they may lead to excellent predictions for the third generation masses. It may be hoped that such structure will ultimately shed light on the pattern of light quark and lepton masses too.

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References

[1] Review of Particle Data, Phys. Rev D54 (1996), 83.
[2] M.Dine and N. Seiberg, Phys. Lett. B 162 (1985), 299; T. Banks and M. Dine, Phys. Rev D 50, (1994) 7454; M. Dine and Y. Shirman, Phys. Lett. B377 (1996), 36; M. Dine, preprint hep-th/9508085.
[3] K. S. Babu and J.C. Pati, Phys. Lett. B 384 (1996), 140.
[4] K.R. Dienes, Phys.Rept. 287 (1997) 447.
[5] E. Witten, Nucl. Phys. B471 (1996) 135; P.Horava, Phys. Rev. D 54 (1996), 7561; P.Horava and E. Witten, Nucl. Phys. B475 (1996), 94.
[6] D. Ghilencea, M. Lanzagorta-Saldana, G.G. Ross, Oxford University preprint, OUTP-97-31-P, hep-ph/9707401.
Previous related works on strong coupling: C. Kolda and J. March-Russell, hep-ph/9609480; K. S. Babu and J.C. Pati, Phys. Lett. B 384 (1996), 140; R. Hempfling, Phys. Lett. B351 (1996) 206; B. Brahmachari, U. Sarkar and K. Sridhar, Mod. Phys. Lett. A8 (1993), 3349.
[7] D. Ghilencea and G.G. Ross, work in preparation.
[8] W. Pokorski and G.G. Ross, Oxford University preprint, OUTP-97-34-P, hep-ph/9707402.
[9] K.R. Dienes, A.E. Farragi and J. March-Russell, Nucl. Phys. B467 (1996) 44.
[10] L. Maiani, G. Parisi and R. Petronzio, Nucl. Phys. B136 (1978), 115.
   L. Maiani and R. Petronzio, Phys. Lett. B176 (1986), 120.

[11] M. Shifman, Int. J. Mod. Phys. A11 (1996) 5761 and references therein.

[12] P. Langacker and N. Polonsky, Phys. Rev. D47 (1993), 4028 and references therein.

[13] B. Pendelton and G.G. Ross, Phys. Lett. B98 (1981), 291.

[14] M. Lanzagorta and G.G. Ross, Phys. Lett. B349 (1995), 319.

[15] C.T. Hill, Phys. Rev. D24 (1981) 691;
   C.T. Hill, C.N. Leung and S. Rao, Nucl. Phys. B262 (1985) 517.

[16] M. Bando, J. Sato and K. Yoshioka, Kyoto University preprint, KUNS-1437 HE(TH) 97/04, hep-ph/9703321;
   M. Bando, T. Onogi, J. Sato and T. Takeuchi, CERN preprint CERN-TH/96-363, hep-ph/9612493.

[17] W. de Boer et al., hep-ph/9603356.

[18] M. Lanzagorta and G. G. Ross, Phys. Lett. B364 (1995), 163.

[19] Luis Ibanez and Graham G. Ross, Phys. Lett. B332 (1994) 100;
   Graham G. Ross, Phys. Lett. B364 (1995) 216.