Abstract

The initial – boundary value problem for a general balance law in a bounded domain is proved to be well posed. Indeed, we show the existence of an entropy solution, its uniqueness and its Lipschitz continuity as a function of time, of the initial datum and of the boundary datum. The proof follows the general lines in [4], striving to provide a rigorous treatment and detailed references.

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1 Introduction

This paper is devoted to the well posedness of a general scalar balance law in an $n$-dimensional bounded domain, that is of

$$
\begin{aligned}
\partial_t u + \text{Div} f(t, x, u) &= F(t, x, u) \quad (t, x) \in I \times \Omega \\
u(0, x) &= u_o(x) \quad x \in \Omega \\
u(t, \xi) &= u_b(t, \xi) \quad (t, \xi) \in I \times \partial \Omega.
\end{aligned}
$$

(1.1)

A key reference in this context is the classical paper by Bardos, Leroux and Nédélec [4]. There, the “correct” definition of solution to (1.1) is selected, in the spirit of the definition given by Kružkov in the case $\Omega = \mathbb{R}^n$, see [15, Definition 1]. A proof of the existence, uniqueness and continuous dependence of the solution from the initial data is described in [4] in the case $u_b = 0$.

For its relevance, since its publication, the well posedness of (1.1) proved in [4] was refined or explained in various text books, mostly in particular cases. For instance, the case $f = f(u)$, $F = 0$ and $u_b = 0$ is detailed in [7, Section 6.9], while non homogeneous boundary conditions are considered in [21, Section 15.1], always in the case $f = f(u)$, $F = 0$. A different type of boundary condition is considered, for instance, in [2].

Below, we aim at a presentation which covers the general case (1.1), which is self contained and with precise references to the elliptic or parabolic results required. Where possible, we also seek to underline which regularity is necessary at which step. As a result, we also obtain further estimates on the solution to (1.1).

As in [4] and [15], existence of solution is obtained through the vanishing viscosity technique. The usual term $\varepsilon \Delta u$ is added on the right hand side of the equation in (1.1), turning it into the parabolic problem

$$
\begin{aligned}
\partial_t u_\varepsilon + \text{Div} f(t, x, u_\varepsilon) &= F(t, x, u_\varepsilon) + \varepsilon \Delta u_\varepsilon \quad (t, x) \in I \times \Omega \\
u_\varepsilon(t, x) &= u_o(x) \quad x \in \Omega \\
u_\varepsilon(t, \xi) &= u_b(t, \xi) \quad (t, \xi) \in I \times \partial \Omega,
\end{aligned}
$$

(1.2)
which is first considered under stricter conditions (regularity and compatibility of the data). Classical results from the parabolic literature \cite{9,16,17} can then be applied to ensure the existence of a solution $u_\varepsilon$ to \eqref{1.2} in the case $u_\varepsilon = 0$. To pass to the limit $\varepsilon \to 0$, suitable bounds on $u_\varepsilon$ are necessary. First, the $L^\infty$ bound \eqref{3.3} is fundamental. In this connection, we note that the similar bound \cite{14} Formula \eqref{9} lacks a term that should be present also in the case $u_\varepsilon = 0$ considered therein, refer to Section \ref{3} for more details. Then, a tricky $BV$ bound allows to prove that the family of solutions to \eqref{1.2} is relatively compact in $L^1$, so that the limit of any convergent subsequence of the $u_\varepsilon$ solves \eqref{1.1}.

The next step is the problem with $u_\varepsilon \neq 0$, a situation hardly considered in the literature. To rigorously extend the existence of solutions to the non homogeneous case, a time and space dependent translation in the $u$ space of the solution to \eqref{1.1} is necessary. This leads on one side to the need of solving an elliptic problem and, on the other side, to prove that this translation does indeed give a solution to \eqref{1.1}. An \textit{ad hoc} adaptation of the doubling of variables technique from \cite{15} makes this latter proof possible. However, to get the necessary estimates on the translated balance law \eqref{6.18}, strict regularity requirements on the elliptic problem are necessary, see Lemma \ref{6.1}. All this leads to keep, in the present work, strict regularity assumptions on $u_\varepsilon$ and the condition that $u_\varepsilon(0,\xi) = 0$ for all $\xi \in \partial \Omega$.

At this stage, the existence of solutions to \eqref{1.1} is proved, under rather strict conditions on initial and boundary data. A further use of the doubling of variables technique allows to prove the Lipschitz continuous dependence of the solution form the initial and boundary data. Remarkably, this technique allows to obtain a proof that essentially relies only on the definition of solution, in a generality wider than that available for the existence of solutions. Finally, we thus obtain at once also the uniqueness of solutions to \eqref{1.1} and to relax the necessary condition on the initial datum.

The bounds on the total variation of the solution have a key role throughout this work. First, they are obtained in the case $u_\varepsilon = 0$, similarly to what is done in \cite{4}, see \eqref{4.1}–\eqref{4.2}. This bound depends on the total variation of the initial datum and on various norms of the flow $f$ and of the source $F$. The translation that allows to pass to the non homogeneous problem leads to consider a translated balance law, where the translated flow and source depend on an extension of the boundary data, see \eqref{6.18}. Therefore, the bound on the total variation of the solution to the translated problem depends on high norms of the boundary datum, see \eqref{4.5}, which in the end imposes to keep the condition $u_\varepsilon(0,\xi) = 0$ for all $\xi \in \partial \Omega$.

The paper is organized as follows. The next section is devoted to the main result, which is obtained through estimates on the parabolic approximation \eqref{1.2} to \eqref{1.1}, presented in Section \ref{3}. Then, Section \ref{4} accounts for the hyperbolic results. All proofs are deferred to sections \ref{5}, \ref{6}, and \ref{7}. A final appendix gathers useful information on the trace operator.

## 2 Notations, Definitions and Main Result

Throughout, $\mathbb{R}^+ = [0, +\infty[$, $B(x,r)$ denotes the open ball centered at $x$ with radius $r > 0$. The closed real interval $I = [0, T]$ is fixed, $T$ being completely arbitrary. For the divergence of a vector field, possibly composed with another function, we use the notation

$$\text{Div } f \left( t, x, u(t,x) \right) = \text{div } f \left( t, x, u(t,x) \right) + \partial_u f \left( t, x, u(t,x) \right) \cdot \text{grad } u(t,x).$$

The Lebesgue $n$ dimensional measure of $\Omega$ is denoted $\mathcal{L}^n(\Omega)$, while the Hausdorff $n-1$ dimensional measure of $\partial \Omega$ is $\mathcal{H}^{n-1}(\partial \Omega)$.

We use below the following standard assumptions, where $\ell \in \mathbb{N}$ and $\alpha \in [0,1]$:

1. $(\Omega_{\ell,\alpha})$ $\Omega$ is a bounded open subset of $\mathbb{R}^n$ with piecewise $C^{\ell,\alpha}$ boundary $\partial \Omega$ and exterior unit normal vector $\nu$.
2. $(f)$ $f \in C^2(\Sigma; \mathbb{R}^n)$, $\partial_u f \in L^\infty(\Sigma; \mathbb{R}^n)$, $\partial_u \text{div } f \in L^\infty(\Sigma; \mathbb{R})$.
3. $(F)$ $F \in C^2(\Sigma; \mathbb{R})$, $\partial_u F \in L^\infty(\Sigma; \mathbb{R})$.  

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(C) \( u_o \in (\text{BV} \cap L^\infty)(\Omega; \mathbb{R}) \) and \( u_b \in (\text{BV} \cap L^\infty)(I \times \partial \Omega; \mathbb{R}) \).

Above and in the sequel, we denote
\[
\Sigma = I \times \Omega \times \mathbb{R}.
\]

Above, we followed the choice in [21, Chapter 10] of a boundary data with bounded total variation. Refer to [19, Section 2.6] and [20] for a generalization to \( L^\infty \) boundary data. For a definition of functions of bounded variation on a manifold, refer for instance to [14, Definition 3.1].

Our starting point is the definition of solution, which originates in the work of Vol’pert [22], see also [3, 7, 15, 21]

**Definition 2.1** ([3, p.1028]). Let \( \Omega \) satisfy \((\Omega_{2,0})\). Fix \( u_o \) and \( u_b \) satisfying (C). A solution to \((1.1)\) on \( I \) is a map \( u \in (L^\infty \cap \text{BV})(I \times \Omega; \mathbb{R}) \) such that for any test function \( \varphi \in C^0_c([-\infty, T[ \times \mathbb{R}^n; \mathbb{R}^n] \) and for any \( k \in \mathbb{R} \),
\[
\int_I \int_{\Omega} \left\{ |u(t,x) - k| \partial_t \varphi(t,x) + \text{sgn}(u(t,x) - k) \left( f(t,x,u) - f(t,x,k) \right) \cdot \text{grad} \varphi(t,x) \\
+ \text{sgn}(u(t,x) - k) \left( F(t,x,u) - \text{div} f(t,x,k) \right) \varphi(t,x) \right\} \, dx \, dt \\
- \int_I \int_{\partial \Omega} \text{sgn}(u_b(t,\xi) - k) \left( f(t,\xi,(\text{tr} u)(t,\xi)) - f(t,\xi,k) \right) \cdot \nu(\xi) \varphi(t,\xi) \, d\xi \, dt \geq 0
\]
Above, \( \text{tr} u(t,\xi) \) denotes the trace of the map \( x \to u(t,x) \) on \( \partial \Omega \) evaluated at \( \xi \). More information and references on the trace operator are collected in the Appendix.

Now, we recall consequences of Definition 2.1 specifying the sense in which the initial datum is attained.

**Proposition 2.2.** Let \((\Omega_{2,0}), (f), (F)\) and (C) hold. Let \( u \in (\text{BV} \cap L^\infty)(I \times \Omega; \mathbb{R}) \) be a solution to \((1.1)\) in the sense of Definition 2.1. Then, there exists a set \( \mathcal{E} \subset I \) of Lebesgue measure 0 such that
\[
\lim_{t \to 0^+, t \in \mathcal{E}} \int_{\Omega} |u(t,x) - u_o(x)| \, dx = 0.
\]
The proof is deferred to Section 7.

A further similar consequence of the above definition of solution and of the properties of the trace operator is the following Proposition. It gives information on the way in which the values of the boundary data are attained by the solution.

**Proposition 2.3.** Let \((\Omega_{2,0}), (f), (F)\) and (C) hold. Let \( u \in (\text{BV} \cap L^\infty)(I \times \Omega; \mathbb{R}) \) be a solution to \((1.1)\) in the sense of Definition 2.1. Then, for all \( k \in \mathbb{R} \) and for almost every \((t,\xi) \in I \times \partial \Omega \),
\[
\left[ \text{sgn} \left( \text{tr} u(t,\xi) - k \right) - \text{sgn} \left( u_b(t,\xi) - k \right) \right] \left[ f(t,\xi,(\text{tr} u)(t,\xi)) - f(t,\xi,k) \right] \cdot \nu(\xi) \geq 0.
\]
Moreover, for almost every \((t,\xi) \in I \times \partial \Omega \)
\[
\min_{k \in I(t,\xi)} \left[ \text{sgn} \left( \text{tr} u(t,\xi) - u_b(t,\xi) \right) \right] \left[ f(t,\xi,\text{tr} u(t,\xi)) - f(t,\xi,k) \right] \cdot \nu(\xi) = 0,
\]
where \( I(t,\xi) = \left\{ k \in \mathbb{R} : \left( u_b(t,\xi) - k \right) \left( k - \text{tr} u(t,\xi) \right) \geq 0 \right\} \).

The proof is deferred to Section 7. In other words, \((2.3)\) states that \( \text{tr} u \) and \( u_b \) may differ whenever the jump between them gives rise to waves exiting \( \Omega \).

Recall now the classical concept of entropy – entropy flux pair, in the general case \((1.1)\).

**Definition 2.4.** An entropy – entropy flux pair for equation \( \partial_t u + \text{Div} f(t,x,u) = F(t,x,u) \) is a pair of functions \((E,F)\) such that:
1. \( \mathcal{E} \in C^2(\mathbb{R}; \mathbb{R}) \) and \( \mathcal{F} \in C^2(I \times \Omega \times \mathbb{R}; \mathbb{R}^n) \);

2. \( \mathcal{E} \) is convex;

3. for all \( (t, x, u) \in \Sigma \), \( \mathcal{E}'(u) \partial_u f(t, x, u) = \partial_u \mathcal{F}(t, x, u) \).

In the case of the general balance law (1.1), the differential form of the entropy inequality is

\[
\partial_t \mathcal{E}(u(t, x)) + \text{Div} \mathcal{F}(t, x, u(t, x)) \leq \mathcal{E}'(u(t, x)) \left( \mathcal{F}(t, x, u(t, x)) - \text{div} f(t, x, u(t, x)) \right) + \text{div} \mathcal{F}(t, x, u(t, x)).
\]

Particular cases of this expression are considered, for instance, in [5, 6, 7, 10, 13, 21].

**Definition 2.5.** Let \( \Omega \) satisfy (\( \Omega_{2,0} \)). Fix \( u_o \) and \( u_b \) satisfying (C). An entropy solution to (1.1) is a map \( u \in (L^\infty \cap BV)(I \times \Omega; \mathbb{R}) \) such that for any entropy – entropy flux pair \((\mathcal{E}, \mathcal{F})\) and for any \( \varphi \in C^1_c([-\infty, T[ \times \mathbb{R}^n; \mathbb{R}^+), \) the following inequality holds:

\[
\int_I \int_{\Omega} \left\{ \mathcal{E}(u(t, x)) \partial_t \varphi(t, x) + \mathcal{F}(t, x, u(t, x)) \cdot \text{grad} \varphi(t, x) \\
+ \left[ \mathcal{E}'(u(t, x)) \left( \mathcal{F}(t, x, u(t, x)) - \text{div} f(t, x, u(t, x)) \right) + \text{div} \mathcal{F}(t, x, u(t, x)) \right] \varphi(t, x) \right\} dx \, dt \\
+ \int_I \int_{\partial \Omega} \left[ \mathcal{F}(t, \xi, u_b(t, \xi)) \\
- \mathcal{E}'(u_b(t, \xi)) f(t, \xi, u(t, \xi)) + f(t, \xi, \text{tr} u(t, \xi)) \right] \nu(\xi) \varphi(t, \xi) \, d\xi \, dt \geq 0.
\]

Formally, Definition 2.1 is a “particular” case of Definition 2.5 obtained choosing as entropy – entropy flux pair the maps

\[
\mathcal{E}(u) = |u - k| \quad \text{and} \quad \mathcal{F}(t, x, u) = \text{sgn}(u - k) \left( f(t, x, u) - f(t, x, k) \right),
\]

for \( k \in \mathbb{R} \). However, the two definitions actually coincide.

**Proposition 2.6.** Definitions 2.1 and 2.5 are equivalent for bounded solutions.

This Proposition is well known and its proof is briefly sketched in Section 7.

We are now ready to state the main result of this paper.

**Theorem 2.7.** Let \( T > 0 \), \( \alpha \in [0, 1] \), and assume that (\( \Omega_{3,n} \)), (f) and (F) hold. Fix an initial datum \( u_0 \in (L^\infty \cap BV)(\Omega; \mathbb{R}) \) and a boundary datum \( u_b \in C^{3,\alpha}(I \times \partial \Omega; \mathbb{R}) \) with \( u_b(0, \xi) = 0 \) for all \( \xi \in \partial \Omega \). Then, problem (1.1) admits a unique solution \( u \in C^{0,1}(I; L^1(\Omega; \mathbb{R})) \). Moreover, the following estimates hold:

\[
\|u(t)\|_{L^\infty(\Omega; \mathbb{R})} \leq \left( \|u_0\|_{L^\infty(\Omega; \mathbb{R})} + \|u_b\|_{L^\infty([0,t] \times \partial \Omega; \mathbb{R})} \right) e^{c_1 t} + \frac{c_2 + \|\partial_t u_b\|_{L^\infty([0,t] \times \partial \Omega; \mathbb{R})}}{c_1} (e^{c_1 t} - 1) \tag{2.5}
\]

\[
\text{TV}(u(t)) \leq C(\Omega, f, F, t) \left( \|u_0\|_{C^{3,\alpha}([0,t] \times \partial \Omega; \mathbb{R})} + \|u_b\|_{C^{3,\alpha}([0,t] \times \partial \Omega; \mathbb{R})} \right) \\
\times \left( 1 + t + \|u_b\|_{L^\infty(\Omega; \mathbb{R})} + \text{TV}(u_0) \right) \times \exp \left( C(\Omega, f, F, t)(1 + \|u_b\|_{C^{3,\alpha}([0,t] \times \partial \Omega; \mathbb{R})}) t \right) \tag{2.6}
\]

\[
\|u(t) - u(s)\|_{L^1(\Omega; \mathbb{R})} \leq \left( \sup_{\tau \in [s,t]} \text{TV}(u(\tau)) \right) |t - s| \tag{2.7}
\]

for \( t, s \in I \), where \( c_1, c_2 \) and \( C(\Omega, f, F, t) \) are independent of the initial and boundary data, see (5.1) and (6.43).
The proof consists of the lemmas and propositions in the sections below, together with the final bootstrap procedure presented in Section 7. The Lipschitz continuous dependence of the solution from the initial and boundary data is stated and proved in Theorem 4.3.

**Remark 2.8.** The above estimate (2.5) shows that the solution \( u \) is in \( L^\infty(I \times \Omega; \mathbb{R}) \). By (2.6), we also have \( u(t) \in BV(\Omega; \mathbb{R}) \) for every \( t \in I \). The Lipschitz continuity in time ensured by (2.7) then implies that \( u \in (L^\infty \cap BV)(I \times \Omega; \mathbb{R}) \), as required in Definition 2.1. This can be proved using exactly the arguments in [5, Section 2.5, Proof of Theorem 2.6].

### 3 The Parabolic Problem (1.2)

All proofs of the statements in this Section are deferred to Section 5. Note that the results in this section are obtained without requiring that \( u_0 = 0 \).

The next Lemma provides the existence of classical solutions to the parabolic problem (1.2).

**Lemma 3.1.** Fix \( \alpha \in [0, 1] \). Let conditions \((\Omega_{2, \alpha})\), \((f)\) and \((F)\) hold. Assume moreover that there exists a function \( \bar{u} \in C^{2, \delta}(I \times \Omega; \mathbb{R}) \), with \( \delta \in [\alpha, 1] \), such that

\[
\partial_t \bar{u}(0, \xi) + \text{Div } f(0, \xi, \bar{u}(0, \xi)) = F(0, \xi, \bar{u}(0, \xi)) + \varepsilon \Delta \bar{u}(0, \xi)
\]

for all \( \xi \in \partial \Omega \). (3.1)

Then, setting

\[
u_a(x) = \bar{u}(0, x) \quad \text{for all } x \in \Omega \quad \text{and} \quad \nu_b(t, \xi) = \bar{u}(t, \xi) \quad \text{for all } (t, \xi) \in I \times \partial \Omega,
\]

there exists a unique solution \( u_\varepsilon \) to (1.2) of class \( C^{2, \gamma}(I \times \Omega; \mathbb{R}) \), for a suitable \( \gamma \in ]\delta, 1[ \).

We now provide an \( L^\infty \)–estimate for the solution \( u_\varepsilon \) to (1.2). It is important to note that we obtain a bound that holds uniformly in \( \varepsilon \), see (3.3).

**Lemma 3.2.** Fix \( \alpha \in [0, 1] \). Let conditions \((\Omega_{2, \alpha})\), \((f)\) and \((F)\) hold. Assume moreover there exists a function \( \bar{u} \in C^{2, \delta}(I \times \Omega; \mathbb{R}) \), for \( \delta \in [\alpha, 1] \), such that (3.1) holds. Let \( u_\varepsilon \) be a solution to (1.2) with \( u_0 \) and \( u_\varepsilon \) as in (3.2). Then, for all \( t \in I \),

\[
\|u_\varepsilon\|_{L^\infty([0, t] \times \Omega; \mathbb{R})} \leq \left( \|u_0\|_{L^\infty(\Omega; \mathbb{R})} + \|u_\varepsilon\|_{L^\infty([0, t] \times \partial \Omega; \mathbb{R})} \right) e^{c_1 t} + \frac{c_2}{c_1} \left( e^{c_1 t} - 1 \right),
\]

where \( c_1, c_2 \) are constants depending on the \( L^\infty \) norms of \( \text{div } f, \partial_u \text{div } f, F \) and \( \partial_u F \), as defined in (5.1).

We remark that, also in the case \( u_0 = 0 \), due to the presence of the second addend in the right hand side, the above estimate (3.3) significantly differs from the \( L^\infty \) bound [3, Formula (9)], which can not be true. Indeed, the estimate [3, Formula (9)] implies that the solution to (1.2) with \( u_0 = 0 \) and \( u_\varepsilon = 0 \) is \( u = 0 \), which is false as, for instance, the case where \( f(t, x, u) = -x \) and \( F = 0 \) clearly shows.

Consider now problem (1.2) with homogeneous boundary condition, i.e., \( u_0(t, \xi) = 0 \) for \( (t, \xi) \in I \times \partial \Omega \). In the next Lemma we partly follow [3, Theorem 1], [7, Chapter 6, § 6.9] and [10, Chapter 4]. Introduce the notation

\[
U(t) = [-M(t), M(t)] \quad \text{with} \quad M(t) = \left( \|u_0\|_{L^\infty(\Omega; \mathbb{R})} + \|u_\varepsilon\|_{L^\infty([0, t] \times \partial \Omega; \mathbb{R})} \right) e^{c_1 t} + \frac{c_2}{c_1} \left( e^{c_1 t} - 1 \right),
\]

as in (3.3) and (5.1).

**Lemma 3.3.** Fix \( \delta \in [0, 1] \). Let conditions \((\Omega_{2, \delta})\), \((f)\) and \((F)\) hold. Assume moreover that \( u_0 \in C^{2, \delta}(\Omega; \mathbb{R}) \) is such that \( u_\varepsilon(\xi) = 0 \) for all \( \xi \in \partial \Omega \) and \( u_\varepsilon(t, \xi) = 0 \) for \( (t, \xi) \in I \times \Omega \). Let \( u_\varepsilon \in C^2(I \times \Omega; \mathbb{R}) \) be a solution to (1.2). Then,

\[
\text{TV} \left( u_\varepsilon(t) \right) \leq L_\varepsilon(t)
\]

(3.5)
for \( t, s \in I \), where

\[
\mathcal{L} \varepsilon(t) = (A_1 + A_2 t + A_3 \| \text{grad} u_0 \|_{L^1(\Omega; \mathbb{R}^n)} + \varepsilon \| \Delta u_0 \|_{L^1(\Omega; \mathbb{R}^n)}) e^{A_4 t}.
\]  

(3.7)

Above, \( A_1, A_2, A_3, A_4 \) are constants depending on \( n, \Omega \) and on norms of \( Df \) and \( F \), see (5.29).

In particular, they are independent of \( \varepsilon \) and of \( u_0 \).

\section{The Hyperbolic Problem \((1.1)\)}

In the particular case of homogeneous boundary condition, we study the convergence of the sequence \((u_\varepsilon)\) as \( \varepsilon \) tends to 0. We also prove that the limit function is a solution to problem \((1.1)\), with homogeneous boundary condition.

\textbf{Proposition 4.1.} Fix \( \delta \in [0,1] \). Let conditions \((\Omega_2, \delta)\), \((f)\) and \((F)\) hold. Assume moreover that \( u_0 \in C^{2,\delta}(\Omega; \mathbb{R}) \) is such that \( u_0(\xi) = 0 \) for \( \xi \in \partial \Omega \), and \( u_0(t, \xi) = 0 \) for \( (t, \xi) \in I \times \Omega \).

Then, the family of solutions \( u_\varepsilon \) to (4.1) is relatively compact in \( L^1 \). Any cluster point \( u_\infty \in L^1(I \times \Omega; \mathbb{R}) \) of this family is a solution to (1.1), with \( u_\infty = 0 \), in the sense of Definition 2.1. Moreover, the following estimates hold:

\[
\| u_\infty(t) \|_{L^\infty(\Omega; \mathbb{R})} \leq \| u_0 \|_{L^\infty(\Omega; \mathbb{R})} e^{c_1 t} + \frac{c_2}{c_1} (e^{c_1 t} - 1),
\]

(4.1)

\[
\text{TV} (u_\infty(t)) \leq \mathcal{L}(t),
\]

\[
\| u_\infty(t) - u_\infty(s) \|_{L^1(\Omega; \mathbb{R})} \leq \mathcal{L}(\max \{ t, s \}) | t - s |,
\]

(4.2)

for \( t, s \in I \), where

\[
\mathcal{L}(t) = (A_1 + A_2 t + A_3 \| \text{grad} u_0 \|_{L^1(\Omega; \mathbb{R}^n)}) e^{A_4 t}.
\]

(4.3)

Above, \( c_1, c_2, A_1, A_2, A_3, A_4 \) are constants depending on \( n, \Omega \) and on norms of \( Df \) and \( F \), see (5.1) and (5.29), all independent of the initial datum.

Note that \( u_\infty \in (L^\infty \cap BV)(I \times \Omega; \mathbb{R}) \), see Remark 2.8.

\textbf{Theorem 4.2.} Fix \( \alpha \in (0,1) \). Let conditions \((\Omega_3, \alpha)\), \((f)\) and \((F)\) hold. Assume moreover that \( u_0 \in C^{3,\alpha}(I \times \partial \Omega; \mathbb{R}) \) and \( u_0 \in C^{2,\delta}(\Omega; \mathbb{R}) \), with \( \delta \in ]0,1[ \), are such that

\[
u_0(\xi) = 0 = u_0(0, \xi) \quad \text{for all } \xi \in \partial \Omega.
\]

(4.4)

Then, there exists a unique solution \( u \in C^{0,1}(I; L^1(\Omega; \mathbb{R})) \) to (1.1) in the sense of Definition 2.1. Moreover, the following bounds hold:

\[
\| u(t) \|_{L^\infty(\Omega; \mathbb{R})} \leq \left( \| u_0 \|_{L^\infty(\Omega; \mathbb{R})} + \| u_0 \|_{L^\infty([0,t] \times \partial \Omega; \mathbb{R})} \right) e^{c_1 t} + \frac{c_2 + \| \partial_t u_0 \|_{L^\infty([0,t] \times \partial \Omega; \mathbb{R})}}{c_1} (e^{c_1 t} - 1),
\]

(4.5)

\[
\text{TV} (u(t)) \leq c(\Omega, f, F, t) \left( \| u_0 \|_{C^{3,\alpha}([0,t] \times \partial \Omega; \mathbb{R})} + \| u_0 \|_{C^{3,\alpha}([0,t] \times \partial \Omega; \mathbb{R})} \right) \times (1 + t + \text{TV} (u_0))
\]

\[
\times \exp \left( c(\Omega, f, F, t)(1 + \| u_0 \|_{C^{3,\alpha}([0,t] \times \partial \Omega; \mathbb{R})}) \right)
\]

(4.6)

\[
\| u(t) - u(s) \|_{L^1(\Omega; \mathbb{R})} \leq \left( \sup_{\tau \in [s,t]} \text{TV} (u(\tau)) \right) | t - s |,
\]

for \( t, s \in I \), where \( c_1, c_2 \) and \( c(\Omega, f, F, t) \) are independent of the initial and boundary data, see (5.1) and (6.43).
Above, Remark 2.8 applies and guarantees that $u \in (L^\infty \cap BV)(I \times \Omega; \mathbb{R})$.

Following [4] Theorem 2, we extend [21] Theorem 15.1.5 to the case of balance laws with time and space dependent flow and source.

**Theorem 4.3.** Let $(\Omega_{2,0})$, (f) and (F) hold. Set

$$L_f = \|\partial_u f\|_{L^\infty(\Sigma; \mathbb{R}^n)} \quad \text{and} \quad L_F = \|\partial_u F\|_{L^\infty(\Sigma; \mathbb{R})}.$$ 

Assume that the initial data $u_o, v_o$ and the boundary data $u_b, v_b$ satisfy (C). If $u$ and $v$ are the corresponding solutions to (1.1) in the sense of Definition 2.7 then, for all $t \in I$, the following estimate holds

$$\int_\Omega |u(t,x) - v(t,x)| \, dx \leq e^{L_f t} \int_\Omega |u_o(x) - v_o(x)| \, dx + L_f \int_0^t e^{L_f (t-\tau)} \int_{\partial\Omega} |u_b(\tau, \xi) - v_b(\tau, \xi)| \, d\xi \, d\tau.$$

**5 Proofs Related to the Parabolic Problem**

**Proof of Lemma 3.1.** To improve the readability, we write $u$ instead of $u_\varepsilon$. We apply [9] Chapter 7, § 4, Theorem 9. To this aim, in the notation of [9] § 4, we verify the required assumptions with reference to $Lu = f(t,x,u,\text{grad } u)$, where

$$Lu = \varepsilon \Delta u - \partial_t u$$

$$f(t,x,u,w) = \text{div } f(t,x,u) + \partial_u f(t,x,u) \cdot w - F(t,x,u).$$

The boundary and initial data $\psi$ in [9] corresponds here to the function $\bar{u}$. The required $C^{2,\alpha}$ regularity of $S = I \times \partial \Omega$ is ensured by the hypothesis. The parabolicity condition [4] Chapter 7, § 2, p.191, (A)] holds with $H_\varepsilon = \varepsilon$. The condition [4] Chapter 7, § 4, p.204, (B')] on the coefficients of $L$ is immediately satisfied: the only non-zero coefficient is the constant $\varepsilon$. By hypothesis, the function $\bar{u}$ is in $C^{2,\delta}$, for $\alpha < \delta < 1$. The Hölder continuity of $f$ follows from (f) and (F).

Concerning [9] Chapter 7, § 2, p.203, Formula (4.10)], it reads:

$$u f(t,x,u,0) = u \left(\text{div } f(t,x,u) - F(t,x,u)\right)$$

$$\leq |u| \left(\|\text{div } f(\cdot,\cdot,0)\|_{L^\infty(I \times \Omega; \mathbb{R})} + \|F(\cdot,\cdot,0)\|_{L^\infty(I \times \Omega; \mathbb{R})}\right)$$

$$+ u^2 \left(\|\partial_u \text{div } f\|_{L^\infty(I \times \Omega \times \mathbb{R}; \mathbb{R})} + \|\partial_u F\|_{L^\infty(I \times \Omega \times \mathbb{R}; \mathbb{R})}\right)$$

$$\leq A_1 u^2 + A_2$$

for suitable positive $A_1, A_2$, by (f) and (F).

Passing to [9] Chapter 7, § 2, p.205, Formula (4.17)]

$$|f(t,x,u,w)| \leq \|\text{div } f(t,x,u)\| + \|\partial_u f(t,x,u)\| w + |F(t,x,u)|$$

$$\leq \left(\|\text{div } f(\cdot,\cdot,0)\|_{L^\infty(I \times \Omega; \mathbb{R})} + \|F(\cdot,\cdot,0)\|_{L^\infty(I \times \Omega; \mathbb{R})}\right)$$

$$+ \left(\|\partial_u \text{div } f\|_{L^\infty(I \times \Omega \times \mathbb{R}; \mathbb{R})} + \|\partial_u F\|_{L^\infty(I \times \Omega \times \mathbb{R}; \mathbb{R})}\right) |w|$$

$$+ \|\partial_u f\|_{L^\infty(I \times \Omega \times \mathbb{R}; \mathbb{R}^n)} \|w\|$$

for a non decreasing function $A$ and a positive scalar $\mu$, by (f) and (F). Lastly, the compatibility condition $Lu(0,x) = f(0,x,\bar{u},\text{grad } \bar{u})$ on $\partial \Omega$ holds by (3.1).

We can thus apply [9] Chapter 7, § 4, Theorem 9, obtaining the existence of a solution $u_\varepsilon$ to (1.2) in the class $C^{2,\gamma}(I \times \Omega; \mathbb{R})$ for $0 < \gamma < 1$. Moreover, [9] Chapter 7, § 4, Theorem 6] ensures the uniqueness of the solution. The verification that the necessary assumptions are satisfied is here immediate. □
Proof of Lemma 3.2 For the sake of readability, we write \( u \) instead of \( u_\varepsilon \). We use \[16\] Chapter 1, § 2, Theorem 2.9, which refers to
\[
\partial_t u - \sum_{i,j=1}^n a_{ij}(t, x, u, \text{grad } u) \partial_{ij}^2 u + a(t, x, u, \text{grad } u) = 0
\]
where, in the present case,
\[
a_{ij}(t, x, u, p) = \varepsilon \delta_{ij} \quad \text{for } i, j = 1, \ldots, n, \]
\[
a(t, x, u, p) = \text{div } f(t, x, u) + \partial_u f(t, x, u) \cdot p - F(t, x, u).
\]
Condition \((\Omega_2, \alpha)\) ensures the necessary regularity of the domain. By \((f)\) and \((F)\), the regularity requirements on \(a_{ij}\) and \(a\) are met. Moreover,
\[16\] Formula (2.29):
\[
\sum_{i,j=1}^n a_{ij}(t, x, u, 0) \xi_i \xi_j = \varepsilon \|\xi\|^2 \geq 0
\]
\[16\] Formula (2.32):
\[
u a(t, x, u, 0) = u \text{div } f(t, x, u) - u F(t, x, u) \geq -\Phi(|u|) |u|
\]
where \(b_2 = 0\) in \[16\] Chapter 1, § 2, Formula (2.32),
\[
\Phi(|u|) = c_1 |u| + c_2 \quad \text{and} \quad c_1 = 1 + \|\partial_u \text{div } f\|_{L^\infty(\Omega \times \Omega \times \mathbb{R}; \mathbb{R})} + \|\partial_u F\|_{L^\infty(\Omega \times \Omega \times \mathbb{R}; \mathbb{R})},
\]
\[
c_2 = \left\|\text{div } f(\cdot, \cdot, 0)\right\|_{L^\infty(\Omega \times \mathbb{R}; \mathbb{R})} + \left\|F(\cdot, \cdot, 0)\right\|_{L^\infty(\Omega \times \mathbb{R}; \mathbb{R})}.
\]
Note that \(\Phi\) is nondecreasing, positive and condition \[16\] Chapter 1, § 2, Formula (2.32) holds. Hence, \[16\] Chapter 1, § 2, Theorem 2.9 applies and the solution \( u \) to \[1.2\] satisfies \[16\] Formula (2.34) with \( \varphi(\xi) = (c_2/c_1) (\xi^{c_1} - 1) \), so that
\[
\|u\|_{L^\infty([0, t] \times \Omega, \mathbb{R})} \leq \left(\|u_0\|_{L^\infty(\Omega, \mathbb{R})} + \|u_b\|_{L^\infty([0, t] \times \partial \Omega, \mathbb{R})}\right) e^{c_1 t} + \frac{c_2}{c_1} \left(\xi^{c_1 t} - 1\right).
\]
completing the proof. \qed

Proof of Lemma 3.3 First, define \( w_\varepsilon \in C^{2, \delta}(\Omega; \mathbb{R}) \) as solution to the elliptic problem
\[
\begin{align*}
\Delta w_\varepsilon &= -\Delta u_\varepsilon + \frac{1}{\varepsilon} \text{div } f(0, x, 0) + \frac{1}{\varepsilon} \partial_u f(0, x, 0) \cdot \text{grad } u_\varepsilon(x) - \frac{1}{\varepsilon} F(0, x, 0) \quad x \in \Omega, \\
w_\varepsilon(\xi) &= 0 \quad \xi \in \partial \Omega.
\end{align*}
\]
The elliptic problem above admits a unique solution \( w_\varepsilon \in C^{2, \delta}(\Omega; \mathbb{R}) \) thanks to \[18\] Chapter 3, § 1, Theorem 1.3. Indeed, with reference to the equation \( Lw_\varepsilon(x) = f(x) \) where
\[
Lw_\varepsilon = \sum_{i,j=1}^n a_{ij}(x) \partial_{ij}^2 w_\varepsilon + \sum_{i=1}^n a_i(x) \partial_i w_\varepsilon + a(x) w_\varepsilon = \Delta w_\varepsilon
\]
\[
f(x) = -\Delta u_\varepsilon + \frac{1}{\varepsilon} \text{div } f(0, x, 0) + \frac{1}{\varepsilon} \partial_u f(0, x, 0) \cdot \text{grad } u_\varepsilon(x) - \frac{1}{\varepsilon} F(0, x, 0),
\]
the hypotheses of \[18\] Chapter 3, § 1, Theorem 1.3 are all satisfied: the coefficients of \( L \) belong to \( C^\delta(\Omega; \mathbb{R}) \) and satisfy the ellipticity condition; we have \( a(x) = 0 \); the boundary \( \partial \Omega \) is of class \( C^{2, \delta} \) by hypothesis; the function \( f \) is in \( C^\delta(\Omega; \mathbb{R}) \) thanks to the hypothesis on \( u_\varepsilon \), to \((f)\) and \((F)\); the homogeneous boundary condition implies that, in the notation of \[18\] Chapter 3, § 1, \( \varphi = 0 \), which is clearly in \( C^{2, \delta}(\partial \Omega; \mathbb{R}) \).

Define now \( u_\varepsilon(t, x) = u_\varepsilon(x) + w_\varepsilon(x) \) for every \((t, x) \in I \times \Omega\): this function \( u_\varepsilon \) belongs to \( C^{2, \delta}(I \times \Omega; \mathbb{R}) \) and it satisfies \((3.1)\) and \((3.2)\), with \( w_\varepsilon = 0 \). Since \( \partial \Omega \) is of class \( C^{2, \delta} \), it is also of class \( C^{2, \alpha} \) for any \( \alpha \in [0, \delta) \). Hence, Lemma 3.1 yields that there exists a unique solution \( u_\varepsilon \in C^{2, \gamma}(I \times \Omega; \mathbb{R}) \), for a \( \gamma \in [0, 1] \), to \((1.2)\) with \( u_b = 0 \).
Following \cite{1, 7, 10}, for \( \eta > 0 \) introduce the functions
\[
\sigma_\eta(z) = \begin{cases} 
z - \eta/2 & z > \eta \\
z^2/(2\eta) & z \in [-\eta, \eta] \\
-\varepsilon + 3\eta/2 & z < -\eta, 
\end{cases} \\
\sigma'_\eta(z) = \begin{cases} 
1 & z > \eta \\
\varepsilon/\eta & z \in [-\eta, \eta] \\
-1 & z < -\eta.
\end{cases}
\] (5.2)

Note that \( u_z \) is of class \( C^2 \), hence, by \cite{1.2}, \( \Delta u_z \) is of class \( C^1 \) and we can differentiate with respect to \( t \) the equation in \cite{1.2}:
\[
\partial_t^2 u_z(t, x) + \text{Div} \left( \partial_t f (t, x, u_z(t, x)) + \partial_u f (t, x, u_z(t, x)) \partial_t u_z(t, x) \right) = \\
= \partial_t F (t, x, u_z(t, x)) + \partial_u F (t, x, u_z(t, x)) \partial_t u_z(t, x) + \partial_t \Delta u_z(t, x).
\] (5.3)

Multiply by \( \sigma'_\eta (\partial_t u_z(t, x)) \) and integrate over \( \Omega \) each term above to obtain
\[
\int_{\Omega} \partial_{t}^2 u_z(t, x) \sigma'_\eta (\partial_t u_z(t, x)) \, dx = \frac{d}{dt} \int_{\Omega} \partial_{t} u_z(t, x) \sigma'_\eta (v) \, dv \, dx \eta = 0 \int_{\Omega} \partial_t u_z(t, x) \, dx.
\] (5.4)

Concerning the second term on the first line of (5.3), we have
\[
\int_{\Omega} \text{Div} \partial_t f (t, x, u_z(t, x)) \sigma'_\eta (\partial_t u_z(t, x)) \, dx
\]
\[
= \int_{\Omega} \left[ \text{div} \partial_t f (t, x, u_z(t, x)) \sigma'_\eta (\partial_t u_z(t, x)) + \partial_u \partial_t f (t, x, u_z(t, x)) \text{grad } u_z(t, x) \sigma'_\eta (\partial_t u_z(t, x)) \right] \, dx
\]
\[
\geq - \mathcal{L}^n(\Omega) \| \text{div} \partial_t f \|_{L^\infty([0, t] \times \Omega \times t(t); \mathbb{R})} - \| \text{grad } u_z(t) \|_{L^1(\Omega; \mathbb{R}^n)} \| \partial_u \partial_t f \|_{L^\infty([0, t] \times \Omega \times t(t); \mathbb{R}^n)} \] (5.5)

and
\[
\int_{\Omega} \text{Div} \left( \partial_u f (t, x, u_z(t, x)) \right) \partial_t u_z(t, x) \sigma'_\eta (\partial_t u_z(t, x)) \, dx
\]
\[
= \int_{\Omega} \sigma'_\eta (\partial_t u_z(t, \xi)) \partial_u f (t, \xi, u_z(t, \xi)) \cdot \nu(\xi) \partial_t u_z(t, \xi) \, d\xi
\]
\[
- \int_{\Omega} \partial_t u_z(t, x) \partial_u f (t, x, u_z(t, x)) \cdot \text{grad } \partial_t u_z(t, x) \sigma''_\eta (\partial_t u_z(t, x)) \, dx
\]
\[
\geq - \| \partial_u f \|_{L^\infty([0, t] \times \Omega \times t(t); \mathbb{R}^n)} \int_{\Omega} \| \partial_t u_z(t, \xi) \| \, d\xi
\]
\[
- \| \partial_u f \|_{L^\infty([0, t] \times \Omega \times t(t); \mathbb{R}^n)} \int_{\Omega} \| \partial_t u_z(t, x) \| \, dx
\]
\[
\geq - \| \partial_u f \|_{L^\infty([0, t] \times \Omega \times t(t); \mathbb{R}^n)} \int_{\Omega} \| \partial_t u_z(t, \xi) \| \, d\xi
\]
\[
- \| \partial_u f \|_{L^\infty([0, t] \times \Omega \times t(t); \mathbb{R}^n)} \int_{\Omega} \| \partial_t u_z(t, x) \| \, dx
\]
\[
\eta \to 0 \int_{\Omega} \| \partial_t u_z(t, \xi) \| \, d\xi
\]
\[
u \to 0.
\]

where, in the last limit, we used \cite{3} Lemma 2.

To estimate the first two terms on the second line of (5.3), we compute:
\[
\int_{\Omega} \left( \partial_t F (t, x, u_z(t, x)) + \partial_u F (t, x, u_z(t, x)) \partial_t u_z(t, x) \right) \sigma'_\eta (\partial_t u_z(t, x)) \, dx
\]
\[
\leq \mathcal{L}^n(\Omega) \| \partial_t F \|_{L^\infty([0, t] \times \Omega \times t(t); \mathbb{R})} + \| \partial_u F \|_{L^\infty([0, t] \times \Omega \times t(t); \mathbb{R})} \| \partial_t u_z(t) \|_{L^1(\Omega; \mathbb{R}^n)}.
\] (5.6)
To bound the last term on the second line of (5.3), we proceed as follows:

\[ \varepsilon \int_{\Omega} \partial_t \Delta u_\varepsilon(t, x) \sigma_\eta' \left( \partial_t u_\varepsilon(t, x) \right) \, dx \]

\[ = \varepsilon \int_{\partial \Omega} \partial_t \text{grad} \, u_\varepsilon(t, \xi) \cdot \nu(\xi) \sigma_\eta' \left( \partial_t u_\varepsilon(t, \xi) \right) \, d\xi - \int_{\Omega} \left\| \partial_t \text{grad} \, u_\varepsilon(t, x) \right\|^2 \sigma_\eta'' \left( \partial_t u_\varepsilon(t, x) \right) \, dx \]

\[ \leq \varepsilon \int_{\partial \Omega} \partial_t \text{grad} \, u_\varepsilon(t, \xi) \cdot \nu(\xi) \sigma_\eta' \left( \partial_t u_\varepsilon(t, \xi) \right) \, d\xi \]

\[ = \varepsilon \int_{\partial \Omega} \text{grad} \, \sigma_\eta \left( \partial_t u_\varepsilon(t, \xi) \right) \cdot \nu(\xi) \, d\xi \]

\[ = \varepsilon \int_{\partial \Omega} \text{grad} \, \sigma_\eta \left( \partial_t u_\varepsilon(t, \xi) \right) \cdot \nu(\xi) \, d\xi \]

Integrate (5.3) in time over \([0, t]\), using (5.4), (5.5) and (5.6) to obtain

\[ \left\| \partial_t u_\varepsilon(t) \right\|_{L^1(\Omega; R^n)} \leq \left\| \partial_t u_\varepsilon(0) \right\|_{L^1(\Omega; R^n)} \]

\[ + \mathcal{L}^n(\Omega) \left( \text{div} \, \partial_t f \right)_{L^\infty([0, t] \times \Omega \times U(t); R^n)} \]

\[ + \int_0^t \left\| \partial_t \text{grad} \, \sigma_\eta \left( \partial_t u_\varepsilon(t, \xi) \right) \right\|_{L^1(\Omega; R^n)} \, d\tau \]

\[ \leq \left\| \partial_t u_\varepsilon(0) \right\|_{L^1(\Omega; R^n)} + \mathcal{L}^n(\Omega) \left( \text{div} \, f \right)_{L^\infty([0, t] \times \Omega \times U(t); R^n)} + \left\| \partial_t \text{grad} \, \sigma_\eta \left( \partial_t u_\varepsilon(t, \xi) \right) \right\|_{L^1(\Omega; R^n)} \]

\[ \leq \left\| \partial_t u_\varepsilon(0) \right\|_{L^1(\Omega; R^n)} + \mathcal{L}^n(\Omega) \left( \text{div} \, f \right)_{L^\infty([0, t] \times \Omega \times U(t); R^n)} + \left\| \partial_t \text{grad} \, \sigma_\eta \left( \partial_t u_\varepsilon(t, \xi) \right) \right\|_{L^1(\Omega; R^n)} \right\|_{L^1(\Omega; R^n)} \]

\[ \leq \left( \left\| \partial_t \text{grad} \, \sigma_\eta \left( \partial_t u_\varepsilon(t, \xi) \right) \right\|_{L^1(\Omega; R^n)} + \left\| \text{grad} \, u_\varepsilon(\tau) \right\|_{L^1(\Omega; R^n)} \right) \, d\tau . \]

Using the parabolic equation (1.2), we can estimate the first term in the right hand side above as follows:

\[ \left\| \partial_t u_\varepsilon(0) \right\|_{L^1(\Omega; R^n)} \leq \left\| \partial_t f \right\|_{L^\infty([0, t] \times \Omega \times U(t); R^n)} \left\| \text{grad} \, u_\varepsilon \right\|_{L^1(\Omega; R^n)} \]

\[ + \mathcal{L}^n(\Omega) \left( \left\| \text{div} \, f \right\|_{L^\infty([0, t] \times \Omega \times U(t); R^n)} + \left\| \partial_t \text{grad} \, \sigma_\eta \left( \partial_t u_\varepsilon(t, \xi) \right) \right\|_{L^1(\Omega; R^n)} \right) \]

\[ \leq \left( \left\| \partial_t \text{grad} \, \sigma_\eta \left( \partial_t u_\varepsilon(t, \xi) \right) \right\|_{L^1(\Omega; R^n)} + \left\| \text{grad} \, u_\varepsilon(\tau) \right\|_{L^1(\Omega; R^n)} \right) \, d\tau \]

As noted above, \( \Delta u_\varepsilon \) is of class \( C^1 \) and, for \( j = 1, \ldots, n \), we can differentiate the equation in (1.2) with respect to \( x_j \) to obtain

\[ \partial_t \partial_j u_\varepsilon(t, x) + \text{Div} \frac{d}{dx_j} f(t, x, u_\varepsilon(t, x)) = \frac{d}{dx_j} F(t, x, u_\varepsilon(t, x)) \]

\[ + \Delta \partial_j u_\varepsilon(t, x). \]

Multiply by \( \sigma_\eta' \left( \partial_j u_\varepsilon(t, x) \right) \) and integrate each term in (5.9) over \( \Omega \):

\[ \int_{\Omega} \partial_t \partial_j u_\varepsilon(t, x) \sigma_\eta' \left( \partial_j u_\varepsilon(t, x) \right) \, dx = \int_{\Omega} \frac{d}{dt} \sigma_\eta \left( \partial_j u_\varepsilon(t, x) \right) \, dx = \frac{d}{dt} \int_{\Omega} \sigma_\eta \left( \partial_j u_\varepsilon(t, x) \right) \, dx . \]

To estimate the second term in the left hand side of (5.9), we follow [7] Chapter 6, Proof of Lemma 6.9.5, use the equality \( \text{grad} \, \sigma_\eta \left( \partial_j u_\varepsilon(t, x) \right) = \sigma_\eta' \left( \partial_j u_\varepsilon(t, x) \right) \) and the Divergence Theorem:

\[ \int_{\Omega} \text{Div} \frac{d}{dx_j} f(t, x, u_\varepsilon(t, x)) \sigma_\eta' \left( \partial_j u_\varepsilon(t, x) \right) \, dx \]
we can now elaborate (5.14) as follows:

\[ \int_{\Omega} \text{Div} \partial_j f(t, x, u_z(t, x)) \sigma'_{\eta} (\partial_j u_z(t, x)) \, dx \]
\[ + \int_{\Omega} \text{Div} \left( \partial_u f(t, x, u_z(t, x)) \partial_j u_z(t, x) \right) \sigma'_{\eta} (\partial_j u_z(t, x)) \, dx \]
\[ = \int_{\Omega} \text{div} \partial_j f(t, x, u_z(t, x)) \sigma'_{\eta} (\partial_j u_z(t, x)) \, dx \]
\[ + \int_{\Omega} \partial_u \partial_j f(t, x, u_z(t, x)) \partial_j u_z(t, x) \sigma'_{\eta} (\partial_j u_z(t, x)) \, dx \]
\[ + \int_{\Omega} \text{Div} \left( \partial_u f(t, x, u_z(t, x)) \right) \partial_j u_z(t, x) \sigma'_{\eta} (\partial_j u_z(t, x)) \, dx \]
\[ + \int_{\Omega} \partial_u f(t, x, u_z(t, x)) \cdot \text{grad} \sigma_{\eta} (\partial_j u_z(t, x)) \, dx \]
\[ \geq - \mathcal{L}^u(\Omega) \| \text{grad} div f \|_{L^\infty([0, T] \times U(t) ; \mathbb{R}^n)} \]
\[ - \| \text{grad} \partial_u f \|_{L^\infty([0, T] \times U(t) ; \mathbb{R}^n ; \mathbb{R}^n)} \int_{\Omega} \left| \partial_j u_z(t, x) \sigma'_{\eta} (\partial_j u_z(t, x)) \right| \, dx \]
\[ + \int_{\Omega} \text{Div} \left( \partial_u f(t, x, u_z(t, x)) \right) \partial_j u_z(t, x) \sigma'_{\eta} (\partial_j u_z(t, x)) \, dx \]
\[ - \int_{\Omega} \text{Div} \left( \partial_u f(t, x, u_z(t, x)) \right) \sigma_{\eta} (\partial_j u_z(t, x)) \, dx \]
\[ + \int_{\partial \Omega} \sigma_{\eta} (\partial_j u_z(t, \xi)) \partial_u f(t, \xi, u_z(t, \xi)) \cdot \nu(\xi) \, d\xi . \]

For later use, note that, for \( \xi \in \partial \Omega, \) [1.2] is the equality

\[ \partial_u f(t, \xi, u_z(t, \xi)) \cdot \nu(\xi) \partial_{u_z}(t, \xi) = \varepsilon \Delta u_z(t, \xi) + F(t, \xi, u_z(t, \xi)) - \text{div} f(t, \xi, u_z(t, \xi)) . \]

Hence, thanks also to the fact that

\[ \partial_u u_z(\nu_j = \partial_j u_z, \] (5.15)

we can now elaborate (5.14) as follows:

\[ \int_{\partial \Omega} \sigma_{\eta} (\partial_j u_z(t, \xi)) \partial_u f(t, \xi, u_z(t, \xi)) \cdot \nu(\xi) \, d\xi \]
\[ = \int_{\partial \Omega} \frac{\sigma_{\eta}(\partial_j u_z(t, \xi))}{\partial_j u_z(t, \xi)} \left( \varepsilon \Delta u_z(t, \xi) + F(t, \xi, u_z(t, \xi)) - \text{div} f(t, \xi, u_z(t, \xi)) \right) \nu(\xi) \, d\xi . \] (5.16)

Here we used the fact that \( \sigma_{\eta}(z) = o(z) \) for \( z \to 0 \), so that the map \( z \to \sigma_{\eta}(z)/z \) is well defined also at \( z = 0 \). Pass now to the first term in the right hand side of (5.9):

\[ \int_{\Omega} \frac{d}{dx_j} F(t, x, u_z) \sigma'_{\eta} (\partial_j u_z(t, x)) \, dx \]
\[ = \int_{\Omega} \left( \partial_j F(t, x, u_z(t, x)) + \partial_u F(t, x, u_z(t, x)) \right) \partial_j u_z(t, x) \sigma'_{\eta} (\partial_j u_z(t, x)) \, dx \]
\[ = \int_{\Omega} \partial_j F(t, x, u_z(t, x)) \sigma'_{\eta} (\partial_j u_z(t, x)) \, dx \]
while the last term on the right hand side of (5.9) gives

\[ \varepsilon \int_\Omega \Delta \partial_j u_\varepsilon(t, x) \sigma'_\eta \left( \partial_j u_\varepsilon(t, x) \right) \, dx \]

\[ = \varepsilon \int_\Omega \text{div} \, \text{grad} \, \partial_j u_\varepsilon(t, x) \sigma'_\eta \left( \partial_j u_\varepsilon(t, x) \right) \, dx \]

\[ = \varepsilon \int_\Omega \sigma'_\eta \left( \partial_j u_\varepsilon(t, \xi) \right) \text{grad} \, \partial_j u_\varepsilon(t, \xi) \cdot \nu(\xi) \, d\xi - \varepsilon \int_\Omega \text{grad} \, \partial_j u_\varepsilon(t, x) \cdot \text{grad} \, \sigma'_\eta(t, x) \, dx \]

\[ = \varepsilon \int_\Omega \sigma'_\eta \left( \partial_j u_\varepsilon(t, \xi) \right) \text{grad} \, \partial_j u_\varepsilon(t, \xi) \cdot \nu(\xi) \, d\xi - \varepsilon \int_\Omega \sigma''_\eta(t, x) \left\| \text{grad} \, \partial_j u_\varepsilon(t, x) \right\| \, dx \]

\[ \leq \varepsilon \int_\Omega \sigma'_\eta \left( \partial_j u_\varepsilon(t, \xi) \right) \text{grad} \, \partial_j u_\varepsilon(t, \xi) \cdot \nu(\xi) \, d\xi . \]

Integrate (5.9) in time over \([0, t]\), using (5.10), (5.11), (5.13), (5.16), (5.17) and (5.18) to obtain

\[ \int_\Omega \sigma_\eta \left( \partial_j u_\varepsilon(t, x) \right) \, dx \]

\[ \leq \int_\Omega \sigma_\eta \left( \partial_j u_\varepsilon(0, x) \right) \, dx \]

\[ + \mathcal{L}^n(\Omega) \, t \left\| \text{grad} \, \text{div} \, f \right\|_{L^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \]

\[ + \int_0^t \left\| \text{grad} \, \partial_j u_\varepsilon \right\|_{L^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \left\| \partial_j u_\varepsilon(\tau) \right\|_{L^1(\Omega, \mathbb{R})} \, d\tau \]

\[ - \int_0^t \int_\Omega \text{Div} \left[ \partial_j u_\varepsilon(\tau, x) \sigma'_\eta \left( \partial_j u_\varepsilon(\tau, x) \right) - \sigma_\eta \left( \partial_j u_\varepsilon(\tau, x) \right) \right] \, dx \, d\tau \]

\[ - \int_0^t \int_\Omega \sigma'_\eta \left( \partial_j u_\varepsilon(\tau, \xi) \right) \text{grad} \, \partial_j u_\varepsilon(\tau, \xi) \cdot \nu(\xi) \, d\xi \, d\tau \]

\[ + \mathcal{L}^n(\Omega) \, t \left\| \text{grad} \, \text{div} \, F \right\|_{L^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \]

\[ + \int_0^t \left\| \partial_j u_\varepsilon \right\|_{L^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \left\| \partial_j u_\varepsilon(\tau) \right\|_{L^1(\Omega, \mathbb{R})} \, d\tau \]

\[ + \int_0^t \int_\Omega \varepsilon \sigma'_\eta \left( \partial_j u_\varepsilon(\tau, \xi) \right) \text{grad} \, \partial_j u_\varepsilon(\tau, \xi) \cdot \nu(\xi) \, d\xi \, d\tau \]

\[ \leq \int_\Omega \sigma_\eta \left( \partial_j u_\varepsilon(0, x) \right) \, dx \]

\[ + \mathcal{L}^n(\Omega) \, t \left( \left\| \text{grad} \, \text{div} \, f \right\|_{L^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} + \left\| \partial_j u_\varepsilon \right\|_{L^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \right) \]

\[ + \left( \left\| \text{grad} \, \partial_j u_\varepsilon \right\|_{L^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} + \left\| \partial_j u_\varepsilon \right\|_{L^\infty([0,t] \times \Omega \times \mathcal{U}(t); \mathbb{R}^n)} \right) \int_0^t \left\| \partial_j u_\varepsilon(\tau) \right\|_{L^1(\Omega, \mathbb{R})} \, d\tau \]

\[ - \int_0^t \int_\Omega \text{Div} \left[ \partial_j f(\tau, x, u_\varepsilon(\tau, x)) \right] \left[ \partial_j u_\varepsilon(\tau, x) \sigma'_\eta \left( \partial_j u_\varepsilon(\tau, x) \right) - \sigma_\eta \left( \partial_j u_\varepsilon(\tau, x) \right) \right] \, dx \, d\tau \]

\[ + \varepsilon \int_0^t \int_\Omega \left( \sigma'_\eta \left( \partial_j u_\varepsilon(\tau, \xi) \right) \text{grad} \, \partial_j u_\varepsilon(\tau, \xi) \cdot \nu(\xi) - \sigma_\eta \left( \partial_j u_\varepsilon(\tau, \xi) \right) \Delta u_\varepsilon(\tau, \xi) \nu(\xi) \right) \, d\xi \, d\tau \]

\[ - \int_0^t \int_\Omega \sigma'_\eta \left( \partial_j u_\varepsilon(\tau, \xi) \right) \text{grad} \, \partial_j u_\varepsilon(\tau, \xi) \cdot \nu(\xi) \, d\xi \, d\tau . \]
To compute the limit $\eta \to 0$, consider first the latter three terms above separately:

$$\lim_{\eta \to 0} [5.20] = 0 \quad (5.23)$$

Concerning (5.21), following [7] Proof of Lemma 6.9.5], it is useful to recall the following relations, based on (5.15):

$$\partial_{\nu}(\partial_{\nu}u_{\varepsilon}) = \partial_{\nu}^{2}u_{\varepsilon} \nu_{j} + O(1) \partial_{\nu}u_{\varepsilon} \quad \text{and} \quad \Delta u_{\varepsilon} = \partial_{\nu}^{2}u_{\varepsilon} + O(1) \partial_{\nu}u_{\varepsilon}. \quad (5.24)$$

Here and in what follows, by $O(1)$ we denote a constant dependent only on the geometry of $\Omega$. In particular, $O(1)$ is independent of the flow $f$, of the source $F$ and of the initial datum $u_{0}$. Then, using (5.24) and the boundedness of $\sigma_{\eta}(z)/z$ and of $\sigma_{\eta}^{\prime}$,

$$\sigma_{\eta}^{\prime}(\partial_{\nu}u_{\varepsilon}(\tau, \xi)) \frac{\partial_{\nu}(\partial_{\nu}u_{\varepsilon}(t, \xi))}{\partial_{\nu}u_{\varepsilon}(\tau, \xi)} = \left( \sigma_{\eta}^{\prime}(\partial_{\nu}u_{\varepsilon}(\tau, \xi)) - \frac{\sigma_{\eta}(\partial_{\nu}u_{\varepsilon}(\tau, \xi))}{\partial_{\nu}u_{\varepsilon}(\tau, \xi)} \right) \Delta u_{\varepsilon}(\tau, \xi) \nu_{j}(\xi) + O(1) \partial_{\nu}u_{\varepsilon}(\tau, \xi),$$

whence, by [8] Lemma A.3, see also [10] Chapter 4,

$$\lim_{\eta \to 0} [5.21] = O(1) \varepsilon \int_{0}^{t} \int_{\partial \Omega} \partial_{\nu}u_{\varepsilon}(\tau, \xi) \, d\xi \, d\tau$$

$$\leq O(1) \varepsilon \int_{0}^{t} \int_{\partial \Omega} \|\text{grad } u_{\varepsilon}(\tau, \xi)\| \, d\xi \, d\tau$$

$$\leq O(1) \varepsilon \int_{0}^{t} \int_{\Omega} |\Delta u_{\varepsilon}(\tau, x)| \, dx \, d\tau$$

$$\leq O(1) \int_{0}^{t} \int_{\Omega} |\partial_{\nu}u_{\varepsilon}(\tau, x)| \, dx \, d\tau$$

$$\quad + O(1) \|\partial_{\nu}f\|_{L^{\infty}([0, t] \times \Omega \times U(t) ; \mathbb{R}^{n})} \int_{0}^{t} \int_{\Omega} \|\text{grad } u_{\varepsilon}(\tau, x)\| \, dx \, d\tau$$

$$\quad + O(1) L^{n}(\Omega) \varepsilon \int \left( \|\text{div } f\|_{L^{\infty}([0, t] \times \Omega \times U(t) ; \mathbb{R}^{n})} + \|F\|_{L^{\infty}([0, t] \times \Omega \times U(t) ; \mathbb{R})} \right). \quad (5.25)$$

Passing to (5.22), we have the following estimate that holds uniformly in $\eta$:

$$[5.22] \leq H^{n-1}(\partial \Omega) t \left( \|\text{div } f(\cdot, \cdot, 0)\|_{L^{\infty}([0, t] \times \partial \Omega ; \mathbb{R}^{n})} + \|F(\cdot, \cdot, 0)\|_{L^{\infty}([0, t] \times \partial \Omega ; \mathbb{R})} \right). \quad (5.26)$$

Insert now (5.23), (5.25) and (5.26) in (5.19), (5.22) to obtain

$$\int_{\Omega} |\partial_{\nu}u_{\varepsilon}(t, x)| \, dx$$

$$\leq \int_{\Omega} |\partial_{\nu}u_{\varepsilon}(0, x)| \, dx$$

$$\quad + L^{n}(\Omega) t \left( \|\text{grad } f\|_{L^{\infty}([0, t] \times \Omega \times U(t) ; \mathbb{R}^{n})} + \|\text{grad } F\|_{L^{\infty}([0, t] \times \Omega \times U(t) ; \mathbb{R}^{n})} \right)$$

$$\quad + O(1) L^{n}(\Omega) t \left( \|\text{div } f\|_{L^{\infty}([0, t] \times \Omega \times U(t) ; \mathbb{R}^{n})} + \|F\|_{L^{\infty}([0, t] \times \Omega \times U(t) ; \mathbb{R})} \right)$$

$$\quad + H^{n-1}(\partial \Omega) t \left( \|\text{div } f(\cdot, \cdot, 0)\|_{L^{\infty}([0, t] \times \partial \Omega ; \mathbb{R}^{n})} + \|F(\cdot, \cdot, 0)\|_{L^{\infty}([0, t] \times \partial \Omega ; \mathbb{R})} \right)$$

$$\quad + \left( \|\text{grad } \partial_{\nu}f\|_{L^{\infty}([0, t] \times \Omega \times U(t) ; \mathbb{R}^{n} \times n)} + \|\partial_{\nu}F\|_{L^{\infty}([0, t] \times \Omega \times U(t) ; \mathbb{R})} \right) \int_{0}^{t} \|\partial_{\nu}u_{\varepsilon}(\tau)\|_{L^{1}([\Omega \times \mathbb{R})} \, d\tau$$

$$\quad + O(1) \int_{0}^{t} \|\partial_{\nu}u_{\varepsilon}(\tau)\|_{L^{1}([\Omega \times \mathbb{R})} \, d\tau$$

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Summing the inequalities (5.7), (5.8) and (5.27)–(5.28) we obtain the estimate
\[ \sum_{j=1}^{n} \text{ grad } u_j(t) \|_{L^1(\Omega;\mathbb{R}^n)} \leq \sqrt{n} \| \text{ grad } u_j(0) \|_{L^1(\Omega;\mathbb{R}^n)} + O(1) \int_0^t \| \text{ grad } u_j(\tau) \|_{L^1(\Omega;\mathbb{R}^n)} d\tau \]
Summing over \( j = 1, \ldots, n \) and using the notation \( O(1) \), we get
\[ \| \text{ grad } u_z(t) \|_{L^1(\Omega;\mathbb{R}^n)} \leq \sqrt{n} \| \text{ grad } u_z(0) \|_{L^1(\Omega;\mathbb{R}^n)} + O(1) t \int_0^t \| \text{ grad } u_z(\tau) \|_{L^1(\Omega;\mathbb{R}^n)} d\tau \]
× \[ \| \text{ grad } \partial_t u_z(\tau) \|_{L^1(\Omega;\mathbb{R}^n)} + \| \text{ grad } u_z(t) \|_{L^1(\Omega;\mathbb{R}^n)} \ \text{ for} \ (5.27) \]
Summing the inequalities (5.7), (5.8) and (5.27)–(5.28) we obtain the estimate
\[ \| \partial_t u_z(t) \|_{L^1(\Omega;\mathbb{R}^n)} + \| \text{ grad } u_z(t) \|_{L^1(\Omega;\mathbb{R}^n)} \leq \ A_1 + A_2 t + A_3 \| \text{ grad } u_o \|_{L^1(\Omega;\mathbb{R}^n)} + \epsilon \| \Delta u_o \|_{L^1(\Omega;\mathbb{R}^n)} \]
\[ + A_4 \int_0^t \left[ \| \partial_t u_z(\tau) \|_{L^1(\Omega;\mathbb{R}^n)} + \| \text{ grad } u_z(\tau) \|_{L^1(\Omega;\mathbb{R}^n)} \right] d\tau, \]
where
\[ A_1 = O(1) \left[ \| \text{ div } f \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} + \| F \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} \right] \]
\[ A_2 = O(1) \left[ \| \text{ grad } \text{ div } f \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} + \| \text{ grad } F \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} \right] \]
\[ + \| \text{ div } f \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} + \| F \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} \]
\[ + \| \text{ grad } \partial_t f \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} + \| \partial_t F \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} \]
\[ + \| \text{ div } f(\cdot,0) \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} + \| F(\cdot,0) \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} \]
\[ A_4 = O(1) \left[ \| \partial_u f \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} \right] \]
\[ + \| \text{ grad } \partial_u f \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} + \| \partial_u F \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} \]
\[ + \| \text{ grad } u_o \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} + \| \Delta u_o \|_{L^\infty([0,t] \times \Omega \times \mathbb{R}^n)} \]
Note that the \( A_4 \) are increasing with \( t \). Hence, an application of Gronwall Lemma yields
\[ \| \partial_t u_z(t) \|_{L^1(\Omega;\mathbb{R}^n)} + \| \text{ grad } u_z(t) \|_{L^1(\Omega;\mathbb{R}^n)} \]
\[ \leq \left( A_1 + A_2 t + A_3 \| \text{ grad } u_o \|_{L^1(\Omega;\mathbb{R}^n)} + \epsilon \| \Delta u_o \|_{L^1(\Omega;\mathbb{R}^n)} \right) e^{A_4 t} \]
From the inequality above, (3.5) follows easily, introducing the notation (3.7). Noting that
\[ \| u_z(t) - u_z(s) \|_{L^1(\Omega;\mathbb{R}^n)} \leq \int_s^t \| \partial_t u_z(\tau) \|_{L^1(\Omega;\mathbb{R}^n)} d\tau, \]
we obtain (3.6), concluding the proof.
6 Proofs Related to the Hyperbolic Problem

Proof of Proposition 4.1. The family $u_\varepsilon$ of solutions to (1.2) as constructed in Lemma 3.3 is uniformly bounded in $L^1(I \times \Omega; \mathbb{R})$ by (3.3). It is also totally bounded in $L^1(I \times \Omega; \mathbb{R})$ thanks to (3.5).

To prove that cluster point of the $u_\varepsilon$ is a solution to (1.1) in the sense of Definition 2.1, we introduce $k \in \mathbb{R}$ and a test function $\varphi \in C^2(\mathbb{R} \times \mathbb{R}; \mathbb{R}^+ \setminus \{0\})$. We multiply equation (1.2) by $\sigma'_\eta(u_\varepsilon(t,x) - k) \varphi(t,x)$, with $\eta > 0$ and $\sigma'_\eta$ as in (5.2). Then, we integrate over $I \times \Omega$:

$$
\int_I \int_\Omega \left( \partial_t u_\varepsilon(t,x) + \text{Div} \left( f(t,x,u_\varepsilon(t,x)) \right) - F \left( t,x,u_\varepsilon(t,x) \right) \right) \sigma'_\eta(u_\varepsilon(t,x) - k) \varphi(t,x) \, dx \, dt = \int_I \int_\Omega \varepsilon \Delta u_\varepsilon(t,x) \sigma'_\eta(u_\varepsilon(t,x) - k) \varphi(t,x) \, dx \, dt.
$$

(6.1)

Consider each term in (6.1) separately. Integrate by part the first term:

$$
\int_I \int_\Omega \partial_t u_\varepsilon(t,x) \sigma'_\eta(u_\varepsilon(t,x) - k) \varphi(t,x) \, dx \, dt = \int_I \int_\Omega \frac{d}{dt} \sigma'\eta(u_\varepsilon(t,x) - k) \varphi(t,x) \, dx \, dt
$$

(6.2)

Concerning the second term in the left hand side of (6.1), first integrate by part, then add and subtract $\int_I \int_\Omega f(t,x,k) \cdot \text{grad} \left( \sigma'_\eta(u_\varepsilon(t,x) - k) \varphi(t,x) \right) \, dx \, dt$. After some rearrangements,

$$
\int_I \int_\Omega \text{Div} \left( f(t,x,u_\varepsilon(t,x)) \right) \sigma'_\eta(u_\varepsilon(t,x) - k) \varphi(t,x) \, dx \, dt
$$

(6.3)

We do not modify the third term in the left hand side of (6.1). Passing to the right hand side of (6.1), we have:

$$
\int_I \int_\Omega \varepsilon \Delta u_\varepsilon(t,x) \sigma'_\eta(u_\varepsilon(t,x) - k) \varphi(t,x) \, dx \, dt
$$
Using (6.2), (6.3) and (6.4), equation (6.1) becomes
\[\begin{align*}
&= \varepsilon \int_I \int_{\partial \Omega} \text{grad } u_r(t, \xi) \sigma'_\eta(-k) \varphi(t, \xi) \cdot \nu(\xi) \, d\xi \, dt \\
&\quad - \varepsilon \int_I \int_{\Omega} \text{grad } u_r(t, x) \cdot \text{grad } \left( \sigma'_\eta(u_r(t, x) - k) \varphi(t, x) \right) \, dx \, dt \\
&= \varepsilon \int_I \int_{\partial \Omega} \sigma'_\eta(-k) \varphi(t, \xi) \text{grad } u_r(t, \xi) \cdot \nu(\xi) \, d\xi \, dt \\
&\quad - \varepsilon \int_I \int_{\Omega} \sigma'_\eta(u_r(t, x) - k) \text{grad } u_r(t, x) \cdot \text{grad } \varphi(t, x) \, dx \, dt \\
&\quad - \varepsilon \int_I \int_{\Omega} \|\text{grad } u_r(t, x)\|^2 \sigma''_\eta(u_r(t, x) - k) \varphi(t, x) \, dx \, dt.
\end{align*}\]

Using (6.2), (6.3) and (6.4), equation (6.1) becomes
\[\begin{align*}
&\int_I \int_{\Omega} \sigma'_\eta(u_r(t, x) - k) \partial_t \varphi(t, x) \, dx \, dt \\
&\quad + \int_I \int_{\Omega} \sigma'_\eta(u_r(t, x) - k) \left( f(t, x, u_r(t, x)) - f(t, x, k) \right) \cdot \text{grad } \varphi(t, x) \, dx \, dt \\
&\quad + \int_I \int_{\Omega} \sigma'_\eta(u_r(t, x) - k) \left( f(t, x, u_r(t, x)) - f(t, x, k) \right) \cdot \text{grad } u_r(t, x) \varphi(t, x) \, dx \, dt \\
&\quad + \int_I \int_{\Omega} \sigma'_\eta(u_r(t, x) - k) \left( F(t, x, u_r(t, x)) - \text{div } f(t, x, k) \right) \varphi(t, x) \, dx \, dt \\
&\quad + \int_{\Omega} \sigma'_\eta(u_0(x) - k) \varphi(0, x) \, dx \\
&\quad - \int_I \int_{\Omega} \sigma'_\eta(-k) \left( f(t, \xi, 0) - f(t, \xi, k) \right) \varphi(t, \xi) \cdot \nu(\xi) \, d\xi \, dt \\
&= \varepsilon \int_I \int_{\Omega} \sigma'_\eta(u_r(t, x) - k) \text{grad } u_r(t, x) \cdot \text{grad } \varphi(t, x) \, dx \, dt \\
&\quad + \varepsilon \int_I \int_{\Omega} \|\text{grad } u_r(t, x)\|^2 \sigma''_\eta(u_r(t, x) - k) \varphi(t, x) \, dx \, dt \\
&\quad - \varepsilon \int_I \int_{\Omega} \sigma'_\eta(-k) \varphi(t, \xi) \text{grad } u_r(t, \xi) \cdot \nu(\xi) \, d\xi \, dt.
\end{align*}\]

Choose now any sequence \(\varepsilon_m\), with \(m \in \mathbb{N}\), and call \(u_\infty\) the \(L^1\) limit of a convergent subsequence. For the sake of readability, we write \(u_r\) instead of \(u_{\varepsilon_m}\). The left hand side of (6.5)–(6.6) converges to the same expression with \(u_r\) replaced by \(u_\infty\). The first term in the right hand side can be treated as follows:
\[\varepsilon_m \int_I \int_{\Omega} \sigma'_\eta \left( u_r(t, x) - k \right) \text{grad } u_r(t, x) \cdot \text{grad } \varphi(t, x) \, dx \, dt \]
\[\geq - \varepsilon_m \int_I \int_{\Omega} \sigma'_\eta \left( u_r(t, x) - k \right) \text{grad } u_r(t, x) \cdot \text{grad } \varphi(t, x) \, dx \, dt \]
\[\geq - \varepsilon_m \|\text{grad } \varphi\|_{L^\infty(I \times \Omega, \mathbb{R}^n)} \|\text{grad } u_r\|_{L^1(I \times \Omega, \mathbb{R}^n)} \]
\[m \to +\infty 0,\]

since \(\varepsilon_m\) is a multiplicative coefficient in the estimate (3.5) of \(\|\text{grad } u_r\|_{L^1(I \times \Omega, \mathbb{R}^n)}\), see (3.7).

The second term in the right hand side of (6.5)–(6.6) is non negative.
To compute the limit as \(m \to +\infty\) of the third term in the right hand side of (6.5)–(6.6),
introduce a function $\Phi_h \in C^2_c(\mathbb{R}^n; [0, 1])$ with the following properties:

$$\Phi_h(\xi) = 1 \text{ for all } \xi \in \partial \Omega,$$
$$\Phi_h(x) = 0 \text{ for all } x \in \Omega \text{ such that } B(x, h) \subset \Omega,$$
$$\|\nabla \Phi_h\|_{L^\infty(\Omega; \mathbb{R}^n)} \leq 1/h.$$  

(6.7)

Then, using equation [1.2] and integration by parts, except for the constant $c'_n(-k)$, the considered term becomes

$$\int_I \int_{\partial \Omega} \varphi(t, \xi) \grad u_c(t, \xi) \cdot \nu(\xi) \, d\xi \, dt$$
$$= \epsilon_m \int_I \int_{\partial \Omega} \varphi(t, \xi) \Phi_h(\xi) \grad u_c(t, \xi) \cdot \nu(\xi) \, d\xi \, dt$$
$$= \epsilon_m \int_I \int_{\Omega} \left( \Delta u_c(t, x) \varphi(t, x) \Phi_h(x) + \grad u_c(t, x) \cdot \grad \left( \varphi(t, x) \Phi_h(x) \right) \right) \, dx \, dt$$
$$= \int_I \int_{\Omega} \left( \partial_t u_c + \text{Div } f(t, x, u_c(t, x)) - F(t, x, u_c(t, x)) \right) \varphi(t, x) \Phi_h(x) \, dx \, dt$$
$$+ \epsilon_m \int_I \int_{\Omega} \grad u_c(t, x) \cdot \grad \left( \varphi(t, x) \Phi_h(x) \right) \, dx \, dt$$
$$= \int_I \int_{\Omega} u_0(x) \varphi(0, x) \Phi_h(x) \, dx - \int_I \int_{\Omega} u_c(t, x) \partial_t \varphi(t, x) \Phi_h(x) \, dx \, dt$$
$$+ \int_I \int_{\partial \Omega} f(t, x, u_c(t, x)) \varphi(t, x) \Phi_h(x) \cdot \nu(\xi) \, d\xi \, dt$$
$$- \int_I \int_{\Omega} f(t, x, u_c(t, x)) \cdot \grad \left( \varphi(t, x) \Phi_h(x) \right) \, dx \, dt$$
$$- \int_I \int_{\Omega} F(t, x, u_c(t, x)) \varphi(t, x) \Phi_h(x) \, dx \, dt$$
$$+ \epsilon_m \int_I \int_{\Omega} \grad u_c(t, x) \cdot \grad \left( \varphi(t, x) \Phi_h(x) \right) \, dx \, dt$$
$$= \int_{\Omega} u_0(x) \varphi(0, x) \Phi_h(x) \, dx$$
$$- \int_I \int_{\Omega} \left( u_c(t, x) \partial_t \varphi(t, x) + f(t, x, u_c(t, x)) \right) \grad \varphi(t, x)$$
$$+ F(t, x, u_c(t, x)) \varphi(t, x) - \epsilon_m \grad u_c(t, x) \cdot \grad \varphi(t, x) \right) \Phi_h(x) \, dx \, dt$$
$$- \int_I \int_{\Omega} \left( f(t, x, u_c(t, x)) - \epsilon_m \grad u_c(t, x) \right) \cdot \grad \Phi_h(x) \varphi(t, x) \, dx \, dt$$
$$+ \int_I \int_{\partial \Omega} f(t, x, 0) \varphi(t, x) \Phi_h(x) \cdot \nu(\xi) \, d\xi \, dt.$$  

(6.8)

Let $m \to +\infty$:

$$\lim_{m \to +\infty} \left[ (6.8) \right].$$
\[
\text{Therefore, in the limit } m \rightarrow \infty, \text{ we obtain that the equality (6.5)–(6.6) implies the inequality}
\[
\int_I \int_{\partial \Omega} \sigma_\eta (u_\infty (t, x) - k) \, d\xi \, dt \\
+ \int_I \int_{\Omega} \sigma'_\eta (u_\infty (t, x) - k) \left( f (t, x, u_\infty (t, x)) - f (t, x, k) \right) \cdot \text{grad } \varphi (t, x) \, dx \, dt \\
+ \int_I \int_{\Omega} \sigma'_\eta (u_\infty (t, x) - k) \left( F (t, x, u_\infty (t, x)) - \text{div } f (t, x, k) \right) \varphi (t, x) \, dx \, dt \\
+ \int_I \int_{\Omega} \sigma_\eta (u_\infty (t, x) - k) \varphi (0, x) \, dx \\
- \int_I \int_{\partial \Omega} \sigma'_\eta (-k) \left( f (t, \xi, 0) - f (t, \xi, k) \right) \varphi (t, \xi) \cdot \nu (t, \xi) \, d\xi \, dt \\
\geq \int_I \int_{\partial \Omega} \sigma'_\eta (-k) \left( f (t, \xi, 0) - f (t, \xi, \text{tr } u_\infty (t, \xi)) \right) \varphi (t, \xi) \cdot \nu (t, \xi) \, d\xi \, dt .
\]

Let now \( \eta \rightarrow 0 \). Thanks to \cite{4} Lemma 2], to the choice \text{(5.2)} of \( \sigma_\eta \) and of its derivative, we get
\[
\int_I \int_{\Omega} \left| u_\infty (t, x) - k \right| \partial_x \varphi (t, x) \, dx \, dt \\
+ \int_I \int_{\Omega} \text{sgn } (u_\infty (t, x) - k) \left( f (t, x, u_\infty (t, x)) - f (t, x, k) \right) \cdot \text{grad } \varphi (t, x) \, dx \, dt \\
+ \int_I \int_{\Omega} \text{sgn } (u_\infty (t, x) - k) \left( F (t, x, u_\infty (t, x)) - \text{div } f (t, x, k) \right) \varphi (t, x) \, dx \, dt \\
+ \int_{\Omega} \left| u_\infty (x) - k \right| \varphi (0, x) \, dx \\
- \int_{\partial \Omega} \text{sgn } (-k) \left( f (t, \xi, \text{tr } u_\infty (t, \xi)) - f (t, \xi, k) \right) \varphi (t, \xi) \cdot \nu (t, \xi) \, d\xi \, dt \\
\geq 0,
\]

that is \text{(2.1)} in the case \( u_0 = 0 \). Hence \( u_\infty \) is a solution to \text{(1.1)} in the sense of Definition \text{2.1}.

As a consequence of \text{(3.3)} in Lemma \text{3.2}, \( u_\infty \) satisfies the \( L^\infty \) estimate
\[
\| u_\infty \|_{L^\infty ([0, t] \times \Omega ; \mathbb{R})} \leq \| u_0 \|_{L^\infty (\Omega ; \mathbb{R})} e^{c_1 t} + \frac{c_2}{c_1} \left( e^{c_1 t} - 1 \right),
\]
where \( c_1, c_2 \) are defined in (5.1). Thanks to the lower semicontinuity in \( L^1 \) of the total variation, see \([9, \text{Remark 3.5}]\), the bound (3.5) in Lemma 3.3 gives

\[
\text{TV} (u_\infty(t)) \leq \liminf_{\varepsilon \to 0} \text{TV} (u_\varepsilon(t)) \leq \liminf_{\varepsilon \to 0} \mathcal{L}_\varepsilon(t) = \mathcal{L}(t)
\]

with \( \mathcal{L}_\varepsilon(t) \) and \( \mathcal{L}(t) \) as defined in (3.7) and (4.2).

From (3.6) in Lemma 3.3 we have for \( t, s \in I \)

\[
\| u_\infty(t) - u_\infty(s) \|_{L^1(\Omega; \mathbb{R})} = \lim_{\varepsilon \to 0} \| u_\varepsilon(t) - u_\varepsilon(s) \|_{L^1(\Omega; \mathbb{R})} \\
\leq \lim_{\varepsilon \to 0} \mathcal{L}_\varepsilon(\max\{t, s\}) |t - s| \\
= \mathcal{L}(\max\{t, s\}) |t - s| ,
\]

concluding the proof. \( \square \)

The following Lemma will be of use in the proof of Theorem 4.2.

**Lemma 6.1.** Let \( k \in \mathbb{N} \) with \( k \geq 2 \), \( \Omega \) satisfy \( (\Omega_{k,\alpha}) \) and fix \( \psi \in C^{k,\alpha}(I \times \partial \Omega; \mathbb{R}) \). Then, the elliptic problem

\[
\begin{cases}
\Delta z(t, x) = 0 & (t, x) \in I \times \Omega \\
z(t, \xi) = \psi(t, \xi) & (t, \xi) \in I \times \partial \Omega
\end{cases}
\]  

(6.11)

admits a unique solution \( z \in C^{k,\alpha}(I \times \bar{\Omega}); \mathbb{R}) \). Moreover,

\[
\| z \|_{L^\infty([0, t] \times \Omega; \mathbb{R})} \leq \| \psi \|_{L^\infty([0, t] \times \partial \Omega; \mathbb{R})} , \\
\| \text{grad} z \|_{L^\infty([0, t] \times \Omega; \mathbb{R}^n)} \leq \| \psi \|_{C^{0,\alpha}([0, t] \times \partial \Omega; \mathbb{R})} , \\
(\mathbf{k} \geq 3) \| \partial^k z \|_{L^\infty([0, t] \times \Omega; \mathbb{R})} \leq \| \partial^k \psi \|_{L^\infty([0, t] \times \partial \Omega; \mathbb{R})} , \\
(\mathbf{k} \geq 3) \| Dz \|_{W^{1,\infty}([0, t] \times \Omega; \mathbb{R})} \leq O(1) \| \psi \|_{C^{3,\alpha}(\partial \Omega; \mathbb{R})} .
\]  

(6.12) (6.13) (6.14) (6.15)

**Proof.** We verify that the assumptions of \([9, \text{Chapter 3, \S 8, Theorem 20}]\), in the case \( p = k + 2 \), hold. With reference to the notation of \([9, \text{Chapter 3, \S 8, Theorem 20}]\), for any \( t \in I \) and for \( i = 0, \ldots, k \), consider the problem

\[
\begin{cases}
L z_i(t) = f \text{ in } \Omega \\
z_i(t) = \partial^i \psi(t) \text{ in } \partial \Omega
\end{cases}
\]  

(6.16)

where \( L = \Delta \) is an elliptic operator, \( f = 0 \) is of class \( C^{k-2,\alpha} \), \( \partial \Omega \) is of class \( C^{k,\alpha} \), \( \partial^i \psi(t) \) is of class \( C^{k,\alpha} \) in \( x \).

Therefore, for any \( t \in I \), (6.11) admits a solution \( z_i(t) \in C^{k,\alpha}(\bar{\Omega}; \mathbb{R}) \).

Thanks to the form of \( L \) in (6.16) and to the continuity of \( \partial^i \psi \), \([9, \text{Chapter 2, \S 7, Theorem 20}]\) can be applied, ensuring the uniqueness of the solution to (6.11).

Concerning the regularity in \( t \), remark that \( \partial^i \psi \) is of class \( C^{k-i,\alpha} \) in \( t \). Hence, for \( i = 0, \ldots, k \), by the Maximum Principle \([9, \text{Chapter 2, \S 7, Theorem 19}]\) for any \( x \in \Omega, t \in I \), \( h \) sufficiently small such that \( t + h \in I \), considering separately the cases \( i < k \) and \( i = k \),

\[
i < k : \left| z_i(t + h, x) - z_i(t, x) \right| \leq \sup_{\xi \in \partial \Omega} \left| \frac{\partial^i \psi(t + h, \xi) - \partial^i \psi(t, \xi)}{h} - \partial^{i+1} \psi(t, h) \right| \\
\leq \sup_{\xi \in \partial \Omega} \left| \partial^{i+1} \psi(t + h, \xi) - \partial^{i+1} \psi(t, \xi) \right| \\
\leq \left\{ \begin{array}{ll}
C h^\alpha & i + 1 < k, \\
C h^\alpha & i + 1 = k, \end{array} \right.
\]

\[
i = k : \left| z_k(t + h, x) - z_k(t, x) \right| \leq \sup_{\xi \in \partial \Omega} \left| \partial^k \psi(t + h, \xi) - \partial^k \psi(t, \xi) \right| \leq C h^\alpha .
\]
where $C$ is the Hölder constant of $\partial_t^k \psi$. Hence, $z = z_0$ is of class $C^{k,a}$ in both $t$ and $x$.

Concerning the bounds on $z$ and on its derivatives, note that (6.12) immediately follow from the Maximum Principle [9, Formula (7.5)]. The same result applies also to $\partial_t z = \Delta \partial_t z$, yielding (6.14), whenever $k \geq 3$. The Boundary Schauder Estimate [9, Chapter 3, p.86] provides the bound for grad $z$, grad$^2 z$, grad $\partial_t z$ and $\partial_{tt} z$, proving (6.13) and (6.15). □

We recall the following result from [15], to be used in the proof below.

Lemma 6.2 ([15 Lemma 2]). Fix positive $r$ and choose $\rho \in [0, \min\{r, T\}]$. Let $w \in L^\infty(I \times B(0, r); \mathbb{R})$. For $h \in [0, \rho]$, define

$$A_h = \left\{ (t, X, s, Y) \in (I \times \mathbb{R}^N)^2 : |t - s| \leq h, \quad \|X - Y\| \leq h, \quad \|X + Y\|/2 \in [0, r - \rho] \right\}.$$ 

Then, \( \lim_{h \to 0^+} \frac{1}{h^2 + N} \int_{A_h} |w(t, X) - w(s, Y)| \, dt \, ds \, dY = 0. \)

Proof of Theorem 4.2 Define $z$ as the solution to (6.11) with $\psi(t, \xi) = u_\nu(t, \xi)$. Lemma 6.1 applies, ensuring the existence and uniqueness of a solution $z$ of class $C^{3,\alpha}$. Note that $z(0, x) = \bar{0}$ for all $x \in \Omega$. For all $\bar{k} \in \mathbb{R}$ and for all $\hat{\varphi} \in C^2_0([-\infty, T] \times \mathbb{R}^n; \mathbb{R}^+)$ the following equality holds

$$\begin{align*}
\int \int_{\Omega} & \left( z(s, y) - \bar{k} \right) \partial_s \hat{\varphi}(s, y) + \text{sgn} \left( z(s, y) - \bar{k} \right) \partial_s z(s, y) \hat{\varphi}(s, y) \right) dy \, ds \\
&+ \int |\hat{\varphi}(0, y) dy = 0.
\end{align*}
$$ (6.17)

We now apply Proposition 4.1 to the problem

$$\begin{align*}
\partial_t v + \text{Div} g(t, x, v) &= G(t, x, v) \\
v(0, x) &= u_\alpha(x) \\
v(t, \xi) &= 0 \quad (t, \xi) \in I \times \partial \Omega \\
\text{where } g(t, x, v) &= f(t, x, v + z(t, x)), \\
G(t, x, v) &= F(t, x, v + z(t, x) - \partial_t z(t, x)).
\end{align*}
$$ (6.18)

To this aim, we verify the necessary assumptions. Clearly, (\Omega_{2,\delta}) holds. By assumption, $u_\alpha \in C^{2,\delta}(\Omega; \mathbb{R})$ and $u_\alpha(\xi) = 0$ for all $\xi \in \Omega$. By construction, the boundary data along $I \times \partial \Omega$ is zero. To verify that also (F) and (F) hold for $g$ and $G$, simply use the assumptions on $f$, $F$ and apply Lemma 6.1. Call $v$ the solution to (6.18) as constructed in Proposition 4.1. By Definition 2.1, for all $\bar{k} \in \mathbb{R}$ and for all $\hat{\varphi} \in C^2_0([-\infty, T] \times \mathbb{R}^n; \mathbb{R}^+),

$$0 \leq \int \int_{\Omega} \left[ v(t, x) - \bar{k} \right] \partial_t \hat{\varphi}(t, x) \\
+ \text{sgn} \left[ v(t, x) - \bar{k} \right] \left[ g(t, x, v(t, x)) - g(t, x, \bar{k}) \right] \cdot \text{grad} \hat{\varphi}(t, x) \\
+ \text{sgn} \left[ v(t, x) - \bar{k} \right] \left[ G(t, x, v(t, x)) - \text{div} g(t, x, \bar{k}) \right] \hat{\varphi}(t, x) \right) dx \, dt \\
+ \int_{\partial \Omega} u_\alpha(x) - \bar{k} \hat{\varphi}(0, x) \, dx \\
- \int_{\partial \Omega} \text{sgn}(-\bar{k}) \left[ g(t, \xi, v(t, \xi)) - g(t, \xi, \bar{k}) \right] \cdot v(\xi) \hat{\varphi}(t, \xi) \, d\xi \, dt \\
= \int \int_{\Omega} \left[ v(t, x) - \bar{k} \right] \partial_t \hat{\varphi}(t, x) \\
+ \text{sgn} \left[ v(t, x) - \bar{k} \right] \left[ f(t, x, v(t, x) + z(t, x)) - f(t, x, \bar{k} + z(t, x)) \right] \cdot \text{grad} \hat{\varphi}(t, x) \\
+ \text{sgn} \left[ v(t, x) - \bar{k} \right] \left[ F(t, x, v(t, x) + z(t, x)) - \partial_t z(t, x) - \text{div} f(t, x, \bar{k} + z(t, x)) \right]
$$ (6.19)
\[
\times \hat{\psi}(t, x) \, dx \, dt \\
\int_\Omega |u_\omega(x) - \hat{k}| \hat{\psi}(0, x) \, dx \\
\int_0^\infty \int_\Omega \sgn(\hat{k}) \left[ f(t, x, \text{tr} v(t, x) + z(t, x)) - f(t, x, \hat{k} + z(t, x)) \right] \\
\times \cdot v(\xi) \hat{\psi}(t, \xi) \, d\xi \, dt .
\] (6.20)

We now verify that the map
\[
u(t, x) = v(t, x) + z(t, x)
\]
is a solution to (1.1) in the sense of Definition 2.1. To this aim, we suitably modify the doubling of variables technique by Kružkov, see [15]. Let \( \tilde{v} \) be a solution to (1.1) in the sense of Definition 2.1. To this aim, we suitably modify the doubling of variables technique by Kružkov, see [15]. Let \( k = k - v(t, x) \) in (6.17) and \( \hat{k} = k - z(s, y) \) in (6.19)–(6.20) for \( k \in \mathbb{R} \). Integrate (6.17) with respect to \( t \) and \( x \) over \( I \times \Omega \), integrate (6.19)–(6.20) in \( s \) and \( y \) over \( I \times \Omega \). Add the resulting expressions, with as test function the map \( \psi_h = \psi_h(t, x, s, y) \) defined by
\[
\psi_h(t, x, s, y) = \varphi \left( \frac{t + s}{2} \right) Y_h(t - s) \prod_{i=1}^n Y_h(x_i - y_i),
\] (6.21)
with \( \varphi \in C_c^\infty(\mathbb{R}; \mathbb{R}^+) \) and \( Y_h \) defined as follows. Let \( Y \in C_c^\infty(\mathbb{R}; \mathbb{R}^+) \) be such that
\[
Y(z) = \varphi \left( \frac{t + s}{2} \right) Y_h(t - s) \prod_{i=1}^n Y_h(x_i - y_i),
\] (6.21)
with \( \varphi \in C_c^\infty(-\infty, T] \times \mathbb{R}^n; \mathbb{R}^+ \) and \( Y_h \) defined as follows. Let \( Y \in C_c^\infty(\mathbb{R}; \mathbb{R}^+) \) be such that
\[
Y(h(z)) = \varphi \left( \frac{t + s}{2} \right) Y_h(t - s) \prod_{i=1}^n Y_h(x_i - y_i),
\] (6.21)
with \( \varphi \in C_c^\infty(-\infty, T] \times \mathbb{R}^n; \mathbb{R}^+ \) and \( Y_h \) defined as follows. Let \( Y \in C_c^\infty(\mathbb{R}; \mathbb{R}^+) \) be such that
\[
\int_\mathcal{H} Y_h(z) \, dz = 1 .
\] (6.22)

and define \( Y_h(z) = \frac{1}{h} Y \left( \frac{z}{h} \right) \). Obviously, \( Y_h \in C_c^\infty(\mathbb{R}; \mathbb{R}^+) \), \( Y_h(-z) = Y_h(z) \), \( Y_h(z) = 0 \) for \( |z| \geq h \), \( \int_\mathcal{H} Y_h(z) \, dz = 1 \) and \( Y_h \to \delta_0 \) as \( h \to 0 \), where \( \delta_0 \) is the Dirac delta in 0.

We temporarily require also that
\[
h \in [0, h_*] \quad \text{and} \quad \varphi(t, x) = 0 \quad \text{for all} \quad x \quad \text{such that} \quad B(x, h_*) \cap (\mathbb{R}^n \setminus \Omega) \neq \emptyset \] (6.23)
f for a fixed positive \( h_* \). We therefore obtain:
\[
0 \leq \int_I \int_{\mathcal{H}} \int_\Omega \left[ |v(t, x) + z(s, y) - k| \left( \partial_t \psi_h(t, x, s, y) + \partial_z \psi_h(t, x, s, y) \right) \right. \\
\left. + \sgn \left[ v(t, x) + z(s, y) - k \right] \left( \partial_t \psi_h(t, x, s, y) + \partial_z \psi_h(t, x, s, y) \right) \right] \\
\times \left[ f \left( t, x, u(t, x) \right) - f \left( t, x, z(t, x) - z(s, y) + k \right) \right] \cdot \text{grad}_x \psi_h(t, x, s, y) \\
+ \sgn \left[ v(t, x) + z(s, y) - k \right] \left( \partial_t z(s, y) - \partial_z t(z, x) \right) \psi_h(t, x, s, y) \\
\left. - \sgn \left[ v(t, x) + z(s, y) - k \right] \text{div} \left( t, x, z(t, x) - z(s, y) + k \right) \psi_h(t, x, s, y) \right) \\
+ \sgn \left[ v(t, x) + z(s, y) - k \right] F \left( t, x, u(t, x) \right) \psi_h(t, x, s, y) \right) \, dx \, dt \, dy \, ds \\
+ \int_I \int_{\mathcal{H}} \int_\Omega \left| u_\omega(x) + z(s, y) - k \right| \psi_h(0, x, s, y) \, dx \, dy \, ds \\
+ \int_I \int_{\mathcal{H}} \int_\Omega \left| v(t, x) - k \right| \psi_h(t, x, 0, y) \, dx \, dy \, dt .
\] (6.29)

To compute the limit as \( h \to 0 \), consider the terms above separately. First, proceeding as in [15, Formule (3.5)–(3.7)], thanks to (6.21), we have
\[
\lim_{h \to 0} (6.24) = \int_I \int_\Omega \left| u(t, x) - k \right| \partial_t \psi(t, x) \, dx \, dt .
\] (6.32)
To deal with (6.25–6.26) we simplify the notation by introducing the map

\[ \Upsilon(t, x, s, y) = \text{sgn} \left[ v(t, x) + z(s, y) - k \right] \left[ f(t, x, u(t, x)) - f(t, x, z(t, x) - z(s, y) + k) \right], \]

so that

\[ \Upsilon(t, x, s, y) \cdot \text{grad}_x \psi_h(t, x, s, y) \]

\[ = \Upsilon(t, x, t, x) \cdot \text{grad} \varphi(t, x) \, Y_h(t - s) \prod_{j=1}^n Y_h(x_j - y_j) \]

\[ + \Upsilon(t, x, t, x) \cdot \left( \text{grad} \varphi \left( \frac{t + s}{2}, x \right) - \text{grad} \varphi(t, x) \right) Y_h(t - s) \prod_{j=1}^n Y_h(x_j - y_j) \]

\[ + \left( \Upsilon(t, x, s, y) - \Upsilon(t, x, t, x) \right) \cdot \text{grad} \varphi \left( \frac{t + s}{2}, x \right) Y_h(t - s) \prod_{j=1}^n Y_h(x_j - y_j) \]

\[ + \sum_{i=1}^n \Upsilon_i(t, x, t, x) \varphi \left( \frac{t + s}{2}, x \right) Y_h(t - s) Y_h'(x_i - y_i) \prod_{j \neq i}^n Y_h(x_j - y_j) \]

\[ + \sum_{i=1}^n \left[ \Upsilon_i(t, x, s, y) - \Upsilon_i(t, x, t, x) \right] \varphi \left( \frac{t + s}{2}, x \right) Y_h(t - s) Y_h'(x_i - y_i) \prod_{j \neq i}^n Y_h(x_j - y_j). \]

Then,

\[ \int_I \int_{\Omega} \int_I \int_{\Omega} \left[ (6.34) \right] \, dy \, ds \, dx \, dt = \int_I \int_{\Omega} \Upsilon(t, x, t, x) \cdot \text{grad} \varphi(t, x) \, dx \, dt. \]

To deal with (6.35), recall that \( |Y_h| \leq (Y(0)/h) \chi_{[-h, h]} \) and apply Lemma 6.2 with

\[ N = 2n + 1, \quad X = (x, t, x), \quad Y = (x, t, y), \quad w(s, Y) = \frac{Y(0)^{n+1}}{h^{n+1}} \Upsilon(t, x, t, x) \cdot \text{grad} \varphi \left( \frac{t + s}{2}, x \right), \]

so that

\[ \lim_{h \to 0} \int_I \int_{\Omega} \int_I \int_{\Omega} \left[ (6.35) \right] \, dy \, ds \, dx \, dt = 0. \]

Similarly, to deal with (6.36), apply Lemma 6.2 with

\[ N = 2n + 1, \quad X = (x, t, x), \quad Y = (x, t, y), \quad w(s, Y) = \frac{Y(0)^{n+1} \| \text{grad} \varphi \|_{L^\infty(I \times \mathbb{R}^n; \mathbb{R}^n)} \Upsilon(t, x, s, y),} {h^{n+1}} \]

so that

\[ \lim_{h \to 0} \int_I \int_{\Omega} \int_I \int_{\Omega} \left[ (6.36) \right] \, dy \, ds \, dx \, dt = 0. \]

The term (6.37) vanishes, since

\[ \int_I \int_{\Omega} \int_I \int_{\Omega} \left[ (6.37) \right] \, dy \, ds \, dx \, dt = \int_{I - h}^{I+h} \cdots \int_{x_i - h}^{x_i + h} Y_h'(x_i - y_i) \, dy_i \cdots \, dx \, dt = 0. \]

Finally, to estimate (6.38), recall that \( |Y_h'| \leq \left( \|Y'\|_{L^\infty(\mathbb{R}^n; \mathbb{R}^2)} / h^2 \right) \chi_{[-h, h]} \) and use Lemma 6.2 with

\[ N = 2n + 1, \quad X = (x, t, x), \quad Y = (x, t, y), \quad w(s, Y) = \frac{Y(0)^n \| Y' \|_{L^\infty(\mathbb{R}^n; \mathbb{R}^n)} \| \varphi \|_{L^\infty(I \times \mathbb{R}^n; \mathbb{R}^n)} \Upsilon(t, x, s, y),} {h^{n+2}} \]

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so that, thanks to $2n + 1 \geq n + 2$,

$$\lim_{h \to 0} \iint_{\Omega} \iint_{\Omega} \left| [6.38] \right| dy \, ds \, dx \, dt = 0.$$ 

Hence

$$\lim_{h \to 0} \left[ 0.25 \times 0.26 \right] = \lim_{h \to 0} \iint_{\Omega} \iint_{\Omega} \iint_{\Omega} Y(t, x, s, y) \cdot \nabla \psi_h(t, x, s, y) dy \, ds \, dx \, dt$$

$$= \iint_{\Omega} \nabla \varphi(t, x) \, dx \, dt$$

$$= \iint_{\Omega} \nabla \left( f(t, x, u(t, x)) - f(t, x, k) \right) \cdot \nabla \varphi(t, x) \, dx \, dt. \quad (6.39)$$

Note that setting

$$N = n, \quad X = x, \quad Y = y$$

and

$$w(s, y) = \frac{Y(0)^{n+1}}{h^{n+1}} \| \varphi \|_{L^\infty(\mathbb{R}^n; \mathbb{R})} \partial_z z(s, y)$$

in Lemma 6.2, we obtain

$$\lim_{h \to 0} [0.27] = \lim_{h \to 0} \iint_{\Omega} \iint_{\Omega} \iint_{\Omega} \left| \partial_z z(t, x) - \partial_z z(s, y) \right| \psi_h(t, x, s, y) dy \, ds \, dx \, dt = 0.$$ 

Omitting now the integrals in (6.28), we have

$$[6.28] = - \mathrm{sgn} \left[ v(t, x) + z(s, y) - k \right] \nabla f(t, x, z(t, x) - z(s, y) + k) \psi_h(t, x, s, y)$$

$$= - \mathrm{sgn} \left[ u(t, x) - k \right] \nabla f(t, x, k) \varphi(t, x) Y_h(t - s) \prod Y_h(x_i - y_i)$$

$$- \mathrm{sgn} \left[ u(t, x) - k \right] \nabla f(t, x, k) \left[ \varphi \left( \frac{t + s}{2}, x \right) - \varphi(t, x) \right] Y_h(t - s) \prod Y_h(x_i - y_i)$$

$$- \mathrm{sgn} \left[ u(t, x) - k \right] \left( \nabla f(t, x, z(t, x) - z(s, y) + k) = - \nabla f(t, x, k) \right)$$

$$\times \varphi \left( \frac{t + s}{2}, x \right) Y_h(t - s) \prod Y_h(x_i - y_i)$$

$$- \left( \mathrm{sgn} \left[ u(t, x) + z(s, y) - z(t, x) - k \right] - \mathrm{sgn} \left[ u(t, x) - k \right] \right)$$

$$\times \nabla f(t, x, z(t, x) - z(s, y) + k) \varphi \left( \frac{t + s}{2}, x \right) Y_h(t - s) \prod Y_h(x_i - y_i).$$

A repeated application of Lemma 6.2 together with standard estimates, yields

$$\lim_{h \to 0} [6.28] = - \iint_{\Omega} \mathrm{sgn} \left[ u(t, x) - k \right] \nabla f(t, x, k) \varphi(t, x) \, dx \, dt. \quad (6.40)$$

The term (6.29) is treated similarly, since

$$[6.29] = \mathrm{sgn} \left[ v(t, x) + z(s, y) - k \right] F(t, x, u(t, x)) \psi_h(t, x, s, y)$$

$$= \mathrm{sgn} \left[ u(t, x) - k \right] F(t, x, u(t, x)) \varphi(t, x) Y_h(t - s) \prod Y_h(x_i - y_i)$$

$$+ \mathrm{sgn} \left[ u(t, x) - k \right] F(t, x, u(t, x)) \left[ \varphi \left( \frac{t + s}{2}, x \right) - \varphi(t, x) \right] Y_h(t - s) \prod Y_h(x_i - y_i)$$

$$+ \left( \mathrm{sgn} \left[ u(t, x) + z(s, y) - z(t, x) - k \right] - \mathrm{sgn} \left[ u(t, x) - k \right] \right) F(t, x, u(t, x))$$

$$= \mathrm{sgn} \left[ u(t, x) - k \right] F(t, x, u(t, x)) \varphi(t, x) Y_h(t - s) \prod Y_h(x_i - y_i)$$

$$+ \left( \mathrm{sgn} \left[ u(t, x) + z(s, y) - z(t, x) - k \right] - \mathrm{sgn} \left[ u(t, x) - k \right] \right) F(t, x, u(t, x))$$
\[ \times \varphi \left( \frac{t + s}{2}, x \right) Y_h(t - s) \prod Y_h(x_i - y_i), \]

so that further applications of Lemma 6.2 lead to

\[ \lim_{h \to 0} \left[ (6.29) \right] = \int_I \int_{\Omega} \int \operatorname{sgn} [u(t, x) - k] \ F(t, x, u(t, x)) \ \varphi(t, x) \, dx \, dt. \quad (6.41) \]

To deal with \( (6.30) \) and \( (6.31) \), introduce the function

\[ \Upsilon(x, s, y) = |u_m(x) + z(s, y) - k| + |v(s, x) - k| \]

and, exploiting the symmetry \( \Upsilon(x, s, y) = \Upsilon(-x) \), we obtain

\[ (6.30) + (6.31) \]

\[ \int_I \int_{\Omega} \int \Upsilon(x, 0, x) \varphi(0, x) \ Y_h(s) \prod Y_h(x_i - y_i) \, dy \, dx \, ds \]

\[ + \int_I \int_{\Omega} \int \Upsilon(x, 0, x) \left( \varphi \left( \frac{s}{2}, x \right) - \varphi(0, x) \right) Y_h(s) \prod Y_h(x_i - y_i) \, dy \, dx \, ds \]

\[ + \int_I \int_{\Omega} \int \Upsilon(x, s, y) \varphi \left( \frac{s}{2}, x \right) Y_h(s) \prod Y_h(x_i - y_i) \, dy \, dx \, ds \]

\[ \leq \frac{1}{2} \int_I \int_{\Omega} \int \Upsilon(x, 0, x) \varphi(0, x) \, dx \]

\[ + \frac{\|\partial_t \varphi\|_{L^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R})}}{2} \int_I \int_{\Omega} \left| \Upsilon(x, 0, x) - \Upsilon(x, 0, x) \right| Y_h(s) \prod Y_h(x_i - y_i) \, dy \, dx \, ds. \]

Both the two latter terms vanish in the limit \( h \to 0 \). Indeed, by Lemma 6.2 for a.e. \( s \in [0, h] \), we have that \( \int_I \int_{\Omega} \left| \Upsilon(x, s, y) - \Upsilon(x, 0, x) \right| \, dy \, dx \to 0 \). Hence,

\[ \lim_{h \to 0} \left[ (6.30) + (6.31) \right] = \int_I \int_{\Omega} |u_m(x) - k| \ \varphi(0, x) \, dx . \quad (6.42) \]

We can now summarize the computations: thanks to \( (6.30), (6.31), (6.40), (6.41) \) and \( (6.42) \), in the limit \( h \to 0 \) \( (6.24)-(6.31) \) becomes

\[ \int_I \int_{\Omega} |u(t, x) - k| \partial_t \varphi(t, x) \, dx \, dt \]

\[ + \int_I \int_{\Omega} \operatorname{sgn} (u(t, x) - k) \left( f(t, x, u(t, x)) - f(t, x, k) \right) \cdot \operatorname{grad} \varphi(t, x) \, dx \, dt \]

\[ + \int_I \int_{\Omega} \operatorname{sgn} (u(t, x) - k) \left( F(t, x, u(t, x)) - \operatorname{div} f(t, x, k) \right) \varphi(t, x) \, dx \, dt \]

\[ + \int_I \int_{\Omega} |u_m(x) - k| \varphi(0, x) \, dx \geq 0, \]

which holds under the choice \( (6.23) \) of \( \varphi \). To pass to an arbitrary test function as in Definition 2.1, substitute \( \varphi(t, x) \) with \( (1 - \Phi_h(x)) \varphi(t, x) \), where \( \varphi \in C^2_c([-\infty, T] \times \mathbb{R}^n; \mathbb{R}^+ \) and \( \Phi_h \) is as in \( (6.7) \),

\[ \int_I \int_{\Omega} |u(t, x) - k| \partial_t \varphi(t, x) (1 - \Phi_h(x)) \, dx \, dt \]
In the limit $h \to 0$, the first 4 lines above converge to the first 3 lines in the left hand side in (2.1) of Definition 2.1 by the Dominated Convergence Theorem. Concerning the latter term, use Lemma A.6 and Lemma A.4, which can be applied since the function $(w_1, w_2) \to \sgn(w_1 - w_2) (f(t, x, w_1) - f(t, x, w_2))$ is Lipschitz continuous, see [15, Lemma 3]. We therefore obtain that

\[
- \lim_{h \to 0} \iint_{\Omega} \sgn \left( u(t, x) - k \right) \left( f \left( t, x, u(t, x) \right) - f \left( t, x, k \right) \right) \cdot \nabla \varphi(t, x) \, dx \, dt = - \iint_{\Omega} \sgn \left( \text{tr} \ u(t, \xi) - k \right) \left( f \left( t, \xi, \text{tr} \ u(t, \xi) \right) - f \left( t, \xi, k \right) \right) \cdot \nu(\xi) \, dx \, dt
\]

where to get to the last line, we used the following fact:

\[
- \iint_{\Omega} \left( \sgn \left( \text{tr} \ u(t, \xi) - k \right) - \sgn \left( u_0(t, \xi) - k \right) \right) \times \left( f \left( t, \xi, \text{tr} \ u(t, \xi) \right) - f \left( t, \xi, k \right) \right) \cdot \nu(\xi) \, dx \, dt
\]

\[
= - \iint_{\Omega} \left( \sgn \left( \text{tr} \ v(t, \xi) + z(t, \xi) - k \right) - \sgn \left( z(t, \xi) - k \right) \right) \times \left( f \left( t, \xi, \text{tr} \ v(t, \xi) + z(t, \xi) \right) - f \left( t, \xi, k \right) \right) \cdot \nu(\xi) \, dx \, dt
\]

\[
= - \iint_{\Omega} \left( \sgn \left( \text{tr} \ v(t, \xi) - (k - z(t, \xi)) \right) - \sgn \left( - (k - z(t, \xi)) \right) \right) \times \left( g \left( t, \xi, \text{tr} \ v(t, \xi) \right) - g \left( t, \xi, k - z(t, \xi) \right) \right) \cdot \nu(\xi) \, dx \, dt
\]

\[
\leq 0,
\]

since $\varphi \geq 0$ and by (2.2) in Proposition 2.3 applied to $v$ as solution to (6.18)

\[
\left( \sgn \left( \text{tr} \ v(t, \xi) + k \right) - \sgn \left( -k \right) \right) \left( g \left( t, \xi, \text{tr} \ v(t, \xi) \right) - g \left( t, \xi, k \right) \right) \cdot \nu(\xi) \geq 0
\]

for all $k \in \mathbb{R}$ and for a.e. $(t, \xi) \in I \times \partial \Omega$. This completes the first part of the proof: the existence of a solution to (1.1) in the sense of Definition 2.1.

Consider now the $L^\infty$ estimate. Recall (6.18), (5.1) and Proposition 4.1 so that

\[
\|u(t)\|_{L^\infty (\Omega; \mathbb{R})} \leq \|v(t)\|_{L^\infty (\Omega; \mathbb{R})} + \|z(t)\|_{L^\infty (\Omega; \mathbb{R})}
\]

\[
\leq \|u_0\|_{L^\infty (\Omega; \mathbb{R})} e^{c_1 t} + \frac{c_2 + \|\partial_t z\|_{L^\infty (0, t; \times \Omega; \mathbb{R})}}{c_1} (e^{c_1 t} - 1)
\]

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Recall now that $TV \left( z \right)$ on elliptic problems to bound $TV \left( z \right)$ with $A$ being an upper bound for $\|Df\|_{L^\infty(\Omega;\mathbb{R})}$.

To obtain the $TV$ bound, we use Proposition 4.1 to estimate $TV \left( z \right) \leq O \left( \frac{c_2 + \|\partial_\Omega u_0\|_{L^\infty([0,t] \times \partial \Omega, \mathbb{R})}}{c_1} \right) e^{c_1 t} - 1$,

which proves the $L^\infty$ estimate (4.4).

To obtain the $TV$ bound, we use Proposition 4.1 to estimate $TV \left( v \right)$ and standard estimates on elliptic problems to bound $TV \left( z \right)$. To this aim, we call $A_i(g)$, for $i = 1, \ldots, 4$, the quantities defined in (5.29), but with norms of $g$ and $G$ over $[0, t] \times \Omega \times V(t)$, where $V(t) = [-M_v(t), M_v(t)]$, with $M_v(t)$ being an upper bound for $\|v\|_{L^\infty([0,t] \times \Omega;\mathbb{R})}$ as in (5.4). Clearly, $V(t) \subseteq U(t) = [-M_v(t), M_v(t)]$. By (5.29), and Lemma 6.1, we have:

$A_1(g) \leq O(1) \left[ \|\text{div} f\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} + \|f\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} \right.
\left. + \|\partial_{\Omega} f\|_{L^\infty([0,t] \times \Omega \times \Omega;\mathbb{R})} \right]
\leq O(1) \left[ \|Df\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} + \|f\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} \right.
\left. + \left( 1 + \|Df\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} \right) \|u_0\|_{C^{2,\alpha}([0,t] \times \partial\Omega;\mathbb{R})} \right] =: A_1$

$A_2(g) \leq O(1) \left[ \|Df\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} + \|f\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} \right.
\left. + \left[ 1 + \|Df\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} + \|\partial_{\Omega} f\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} \right] \|Dz\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} \right]
\leq O(1) \left[ \|Df\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} + \|f\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} \right.
\left. + \left[ 1 + \|Df\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} + \|\partial_{\Omega} f\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} \right] \|u_0\|_{C^{2,\alpha}([0,t] \times \partial\Omega;\mathbb{R})} \right] =: A_2$

$A_3(g) \leq O(1) + \|\partial_{\Omega} f\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} \right]$

$A_4(g) \leq O(1) \left[ 1 + \|Df\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} + \|\partial_{\Omega} f\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} \right.
\left. + \|\partial_{\Omega} f\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} \right]
\leq O(1) \left[ 1 + \|Df\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R}))} + \|\partial_{\Omega} f\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R)))} \right.
\left. + \|\partial_{\Omega} f\|_{L^\infty([0,t] \times \Omega \times U(t); \mathbb{R)))} \right] =: A_4$

which proves the bound

$TV \left( v(t) \right) \leq \left( A_1 + A_2 + A_3 \right) TV \left( u_0 \right) e^{A_4 t}$.

Recall now that $TV \left( u \right) \leq TV \left( v \right) + TV \left( z \right)$ and, by Lemma 6.1, $TV \left( z \right) \leq L^\infty(\Omega) \|u_0\|_{C^{2,\alpha}([0,t] \times \partial\Omega;\mathbb{R})}$.

The proof is completed.
Proof of Theorem 4.3 Assume preliminarily that \( u_a \in C^2(\Omega; \mathbb{R}) \) and \( u_b \in C^2(I \times \partial \Omega; \mathbb{R}) \).

Let \( \varphi \in C^2([-\infty, T[ \times \mathbb{R}^n; \mathbb{R}^+ \) be a test function as in Definition 2.1 with
\[
\varphi(0, x) = 0 \quad \text{for all} \quad x \in \mathbb{R}^n, \\
\varphi(t, \xi) = 0 \quad \text{for all} \quad (t, \xi) \in I \times \partial \Omega.
\] (6.44)

Define
\[
\psi_h(t, x, s, y) = \varphi \left( \frac{t + s}{2}, \frac{x + y}{2} \right) Y_h(t - s) \prod_{i=1}^n Y_h(x_i - y_i)
\] (6.45)

where \( Y_h \) is defined in (6.22). We now use the doubling of variables method, see [15]. In inequality (2.1), set \( k = v(s, y) \) and use as test function the map \( \psi_h = \psi_h(t, x, s, y) \) for a fixed point \( (s, y) \) and integrate over \( I \times \Omega \) with respect to \( (s, y) \):
\[
\int_I \int_{\Omega} \int_I \int_{\Omega} \left\{ \left| u(t, x) - v(s, y) \right| \partial_t \psi_h(t, x, s, y) \right. \\
+ \text{sgn} \left( u(t, x) - v(s, y) \right) \left[ f \left( t, x, u(t, x) \right) - f \left( t, x, v(s, y) \right) \right] \cdot \text{grad}_x \psi_h(t, x, s, y) \\
+ \text{sgn} \left( u(t, x) - v(s, y) \right) \left[ F \left( t, x, u(t, x) \right) - \text{div} f \left( t, x, v(s, y) \right) \right] \psi_h(t, x, s, y) \right\} dx \, dt \, dy \, ds \\
+ \left. \int_I \int_{\Omega} \int_I \int_{\Omega} \left| u_a(x) - v(s, y) \right| dx \, dy \, ds \, \geq 0.
\]

In the same way, starting from the inequality (2.1) for the function \( v = v(s, y) \), set \( k = u(t, x) \), consider the same test function \( \psi_h = \psi_h(t, x, s, y) \) and integrate over \( I \times \Omega \) with respect to \( (t, x) \):
\[
\int_I \int_{\Omega} \int_I \int_{\Omega} \left\{ \left| v(s, y) - u(t, x) \right| \partial_t \psi_h(t, x, s, y) \right. \\
+ \text{sgn} \left( v(s, y) - u(t, x) \right) \left[ f \left( s, y, v(s, y) \right) - f \left( s, y, u(t, x) \right) \right] \cdot \text{grad}_y \psi_h(t, x, s, y) \\
+ \text{sgn} \left( v(s, y) - u(t, x) \right) \left[ F \left( s, y, v(s, y) \right) - \text{div} f \left( s, y, u(t, x) \right) \right] \psi_h(t, x, s, y) \right\} dy \, ds \, dx \, dt \\
+ \left. \int_I \int_{\Omega} \int_I \int_{\Omega} \left| v_a(y) - u(t, x) \right| dy \, dx \, dt \, \geq 0.
\]

Summing the last two inequalities above, we obtain:
\[
0 \leq \int_I \int_{\Omega} \int_I \int_{\Omega} \left\{ \left| u(t, x) - v(s, y) \right| \left( \partial_t \psi_h(t, x, s, y) + \partial_s \psi_h(t, x, s, y) \right) \right. \\
+ \text{sgn} \left( u(t, x) - v(s, y) \right) \left[ f \left( t, x, u(t, x) \right) - f \left( t, x, v(s, y) \right) \right] \cdot \text{grad}_x \psi_h(t, x, s, y) \\
+ \text{sgn} \left( v(s, y) - u(t, x) \right) \left[ f \left( s, y, v(s, y) \right) - f \left( s, y, u(t, x) \right) \right] \cdot \text{grad}_y \psi_h(t, x, s, y) \\
+ \text{sgn} \left( u(t, x) - v(s, y) \right) \left[ F \left( t, x, u(t, x) \right) - F \left( s, y, v(s, y) \right) \right] \\
+ \text{div} f \left( s, y, u(t, x) \right) - \text{div} f \left( t, x, v(s, y) \right) \right\} dx \, dt \, dy \, ds \\
+ \int_I \int_{\Omega} \int_I \int_{\Omega} \left| u_a(x) - v(s, y) \right| dx \, dy \, ds \\
+ \int_I \int_{\Omega} \int_I \int_{\Omega} \left| v_a(y) - u(t, x) \right| dy \, dx \, dt \cdot
\] (6.47)

We follow the proof of [15] Theorem 1. As \( h \to 0 \), the first integral in the 5 lines \([6.46] \cdots [6.47]\), can be treated exactly as in [15], leading to the following analog of [15] Formula (3.12):
\[
\lim_{h \to 0^+} [6.46] \cdots [6.47]
\]
To compute the second integral \((6.48)\), observe preliminarily that

\[
\text{Hence:}
\]

\[
\|u_0(x) - v(s, y)\| \leq \|u_0(x) - v(s, y)\| - \|u_0(x) - v(s, x)\| \\
+ \|u_0(x) - v(s, x)\| - \|u_0(x) - v(0+, x)\| \\
+ \|u_0(x) - v(0+, x)\| \\
\leq \|v(s, y) - v(s, x)\| + \|v(s, x) - v(0+, x)\| + \|u_0(x) - v(0+, x)\|
\]

Hence:

\[
\begin{align*}
\int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |v(s, y) - v(s, x)| \, dx \, dy \, ds \\
+ \int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |v(s, x) - v(0+, x)| \, dx \, dy \, ds \\
+ \int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |u_0(x) - v(0+, x)| \, dx \, dy \, ds
\end{align*}
\]

(6.50)

To compute the second integral \([6.48]\), observe preliminarily that

\[
\begin{align*}
\lim_{h \to 0^+} |u_0(x) - v(s, y)| & \leq |u_0(x) - v(s, y)| - |u_0(x) - v(s, x)| \\
& \quad + |u_0(x) - v(s, x)| - |u_0(x) - v(0+, x)| \\
& \quad + |u_0(x) - v(0+, x)| \\
& \leq |v(s, y) - v(s, x)| + |v(s, x) - v(0+, x)| + |u_0(x) - v(0+, x)|.
\end{align*}
\]

Hence:

\[
\begin{align*}
\int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |v(s, y) - v(s, x)| \, dx \, dy \, ds \\
+ \int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |v(s, x) - v(0+, x)| \, dx \, dy \, ds \\
+ \int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |u_0(x) - v(0+, x)| \, dx \, dy \, ds.
\end{align*}
\]

(6.51)

(6.52)

(6.53)

Compute the limit as \(h \to 0^+\) of the three lines separately. First, apply Lemma 6.2 in the case of a function \(w\) depending only on the space variable to obtain

\[
\lim_{h \to 0^+} \int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |v(s, y) - v(s, x)| \, dx \, dy \, ds = 0.
\]

Second, by Lemma A.2

\[
\lim_{h \to 0^+} \int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |v(s, x) - v(0+, x)| \, dx \, dy \, ds = 0.
\]

Third, by the choice of the function \(Y_h\) and \((6.44)\)

\[
\lim_{h \to 0^+} \int_{\Omega} \phi(0, x) |u_0(x) - v(0+, x)| \, dx = 0,
\]

proving that \(\lim_{h \to 0^+} [6.49] = 0\). The term \((6.50)\) is treated exactly in the same way. Hence,

\[
\lim_{h \to 0^+} \int_{\Omega} \int_{\Omega} \psi_h(0, x, s, y) |v(s, x) - v(0+, x)| \, dx \, dy \, ds = 0,
\]

so that

\[
0 \leq \int_{\Omega} \int_{\Omega} \left\{|u(t, x) - v(t, x)| \, \partial_t \phi(t, x) \\
+ \text{sgn} \left(u(t, x) - v(t, x)\right) \left[f \left(t, x, u(t, x)\right) - f \left(t, x, v(t, x)\right)\right] \text{grad} \phi(t, x) \\
+ \text{sgn} \left(u(t, x) - v(t, x)\right) \left[F \left(t, x, u(t, x)\right) - F \left(t, x, v(t, x)\right)\right] \phi(t, x)\right\} \, dx \, dt.
\]

(6.54)

For \(h > 0\), recall the function \(\Phi_h \in C^2_c(\mathbb{R}^n; [0, 1])\) defined in \((6.7)\). Let \(\Psi \in C^2_c([0, T]; \mathbb{R}^n)\) with \(\Psi(0) = 0\). Note that for any \(h > 0\) sufficiently small, the map

\[
\phi_h(t, x) = \Psi(t) \left(1 - \Phi_h(x)\right) \quad \text{for} \quad (t, x) \in \mathbb{R}^n_{\geq 0}, T \times \mathbb{R}^n
\]
satisfies (6.44). Introduce this test function \( \varphi_h \) in (6.54) and pass to the limit \( h \to 0 \) to obtain:

\[
0 \leq \int_{\Omega} \left\{ |u(t, x) - v(t, x)| \right\} \Psi(t) + \operatorname{sgn} (u(t, x) - v(t, x)) \left[ F(t, x, u(t, x)) - F(t, x, v(t, x)) \right] \Psi(t) \, dx \, dt \tag{6.55}
\]

\[
- \int_{\partial \Omega} \operatorname{sgn} (\operatorname{tr} u(t, \xi) - \operatorname{tr} v(t, \xi)) \left[ f(t, \xi, \operatorname{tr} u(t, \xi)) - f(t, \xi, \operatorname{tr} v(t, \xi)) \right] \cdot \nu(\xi) \Psi(t) \, d\xi \, dt,
\]

where we used Lemma A.6 and Lemma A.4 which can be applied since the function \( (u, v) \to \operatorname{sgn}(u - v) \left( f(t, x, u) - f(t, x, v) \right) \) is Lipschitz continuous, see [15] Lemma 3.

To ease readability, we now omit the dependence on \((t, \xi)\) of \( f, \operatorname{tr} u, \operatorname{tr} v, u_b, v_b, \nu \). Apply (2.2) to \( u \) choosing \( k = \operatorname{tr} v \) and to \( v \) choosing \( k = \operatorname{tr} u \):

\[
- \operatorname{sgn} (\operatorname{tr} u - \operatorname{tr} v) \left[ f(\operatorname{tr} u) - f(\operatorname{tr} v) \right] \cdot \nu \leq - \operatorname{sgn} (u_b - \operatorname{tr} v) \left[ f(\operatorname{tr} u) - f(\operatorname{tr} v) \right] \cdot \nu, \\
- \operatorname{sgn} (\operatorname{tr} u - \operatorname{tr} v) \left[ f(\operatorname{tr} u) - f(\operatorname{tr} v) \right] \cdot \nu \leq - \operatorname{sgn} (v_b - \operatorname{tr} u) \left[ f(\operatorname{tr} v) - f(\operatorname{tr} u) \right] \cdot \nu.
\]

Hence,

\[
- \operatorname{sgn} (u_b - \operatorname{tr} v) \left[ f(\operatorname{tr} u) - f(\operatorname{tr} v) \right] \cdot \nu \\
\leq \frac{1}{2} \left[ \operatorname{sgn} (v_b - \operatorname{tr} u) - \operatorname{sgn} (u_b - \operatorname{tr} v) \right] \left[ f(\operatorname{tr} u) - f(\operatorname{tr} v) \right] \cdot \nu. \tag{6.56}
\]

The second line in (6.56) attains the following values:

| \( u_b - \operatorname{tr} v > 0 \) | \( u_b - \operatorname{tr} v = 0 \) | \( u_b - \operatorname{tr} v < 0 \) |
|---|---|---|
| \( v_b - \operatorname{tr} u > 0 \) | 0 | \( \frac{1}{2} (f(\operatorname{tr} u) - f(\operatorname{tr} v)) \cdot \nu \) |
| \( v_b - \operatorname{tr} u = 0 \) | \( \frac{1}{2} (f(\operatorname{tr} v) - f(\operatorname{tr} u)) \cdot \nu \) | 0 |
| \( v_b - \operatorname{tr} u < 0 \) | \( f(\operatorname{tr} v) - f(\operatorname{tr} u) \cdot \nu \) | \( \frac{1}{2} (f(\operatorname{tr} v) - f(\operatorname{tr} u)) \cdot \nu \) |

Clearly, we can reduce our study to two cases highlighted in the table above. Applying (2.3) to \( u \) with \( k = u_b \) and to \( v \) with \( k = v_b \) leads to

\[
\operatorname{sgn} (\operatorname{tr} u - u_b) \left[ f(\operatorname{tr} u) - f(u_b) \right] \cdot \nu \geq 0, \tag{6.57}
\]

\[
\operatorname{sgn} (v_b - \operatorname{tr} v) \left[ f(\operatorname{tr} v) - f(v_b) \right] \cdot \nu \geq 0. \tag{6.58}
\]

Let \( J = J(t, \xi) = \left\{ k \in \mathbb{R}: (u_b(t, \xi) - k) (k - v_b(t, \xi)) \geq 0 \right\} \). Focus on each case separately.

**Case I:** \( u_b - \operatorname{tr} v \leq 0 \) and \( v_b - \operatorname{tr} u \geq 0 \).

1. If \( u_b \leq v_b \leq \operatorname{tr} v \leq u_b \) or \( u_b \leq \operatorname{tr} u \leq v_b \leq v_b \), then

\[
\left( f(\operatorname{tr} u) - f(\operatorname{tr} v) \right) \cdot \nu \leq \sup_{s,r \in J} \left\| f(s) - f(r) \right\|,
\]

where \( \| \cdot \| \) represents the Euclidean norm in \( \mathbb{R}^n \).

2. If \( \operatorname{tr} v \leq v_b \leq u_b \leq \operatorname{tr} u \leq v_b \leq \operatorname{tr} v \), then

by (6.57) \( \Rightarrow f(\operatorname{tr} u) \cdot \nu \leq f(u_b) \cdot \nu \), 

by (6.58) \( \Rightarrow f(\operatorname{tr} v) \cdot \nu \geq f(v_b) \cdot \nu \).

Hence we have

\[
\left( f(\operatorname{tr} u) - f(\operatorname{tr} v) \right) \cdot \nu \leq \left( f(u_b) - f(v_b) \right) \cdot \nu \leq \sup_{s,r \in J} \left\| f(s) - f(r) \right\|.
\]
Case II: \( \mathfrak{u} - \mathfrak{v} \geq 0 \) and \( \mathfrak{u} - \mathfrak{v} \leq 0 \).

1. If \( \mathfrak{v} \leq \mathfrak{v} \leq \mathfrak{u} \leq \mathfrak{v} \leq \mathfrak{v} \), then
   \[ (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|. \]

2. If \( \mathfrak{v} \leq \mathfrak{v} \leq \mathfrak{u} \leq \mathfrak{v} \leq \mathfrak{v} \), then
   \[ (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|. \]

   Then we have
   \[
   (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \leq (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|.
   
   Hence, \[ (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \leq (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|.
   
3. If \( \mathfrak{u} \leq \mathfrak{v} \leq \mathfrak{u} \leq \mathfrak{v} \), by \[ (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \geq (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \] and using the fact that \( \mathfrak{v} \leq \mathfrak{v} \leq \mathfrak{u} \), we get
   \[
   (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \leq (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|.
   
4. If \( \mathfrak{v} \leq \mathfrak{v} \leq \mathfrak{u} \leq \mathfrak{v} \), by \[ (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \leq (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \] and using the fact that \( \mathfrak{u} \leq \mathfrak{u} \leq \mathfrak{v} \), we obtain
   \[
   (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \leq (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|.
   
Hence, \[ (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \leq (f(\mathfrak{v}) - f(\mathfrak{u})) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|.

By \[ (6.57) \] we get
\[
\int f(\mathfrak{v}(t)) \cdot \nu \leq \sup_{s,r \in J} \|f(s) - f(r)\|.
\]

Since \( \Psi \) assumes only positive values, we can estimate \[ (6.55) \] by
\[
0 \leq \int_I \int_{\mathcal{V}} \left\{ |u(t,x) - v(t,x)| \Psi(t) + \|\partial_u F\|_{L^\infty(\Sigma;\mathbb{R}^n)} |u(t,x) - v(t,x)| \Psi(t) \right\} dx dt
+ \|\partial_u F\|_{L^\infty(\Sigma;\mathbb{R}^n)} \int_I \int_{\mathcal{V}} |u(t,x) - v(t,x)| \Psi(t) d\xi dt.
\]
Introduce $\tau, t$ such that $0 < \tau < t < T$. Note that the map $s \to \Psi_h(s)$ defined by
\[
\Psi_h(s) = \alpha_h(s - \tau - h) - \alpha_h(s - t - h),
\]
where $\alpha_h(z) = \int_{-\infty}^{z} Y_h(\zeta) \, d\zeta$
and $Y_h$ as in (6.22),
satisfies (6.44). Hence, we substitute $\Psi_h$ for $\Psi$ in (6.59). Observe that $\Psi_h \to \chi_{[\tau, t]}$ and $\Psi'_h \to \delta_{\tau} - \delta_t$
as $h$ tends to 0. At the limit we obtain
\[
0 \leq \int_{\Omega} |u(\tau, x) - v(\tau, x)| \, dx - \int_{\Omega} |u(t, x) - v(t, x)| \, dx
+ \|\partial_t F\|_{L^\infty(\Sigma; \mathbb{R})} \int_{\tau}^{t} \int_{\Omega} |u(s, x) - v(s, x)| \, dx \, ds
+ \|\partial_t f\|_{L^\infty(\Sigma; \mathbb{R}^n)} \int_{\tau}^{t} \int_{\partial\Omega} |u_b(s, \xi) - v_b(s, \xi)| \, d\xi \, ds.
\]
A Gronwall type argument yields
\[
\int_{\Omega} |u(t, x) - v(t, x)| \, dx
\leq e^{\|\partial_t F\|_{L^\infty(\Sigma; \mathbb{R})}(t-\tau)} \int_{\Omega} |u(\tau, x) - v(\tau, x)| \, dx
+ \|\partial_t f\|_{L^\infty(\Sigma; \mathbb{R}^n)} \int_{\tau}^{t} e^{(t-\tau-s)} \|\partial_t F\|_{L^\infty(\Sigma; \mathbb{R})} \int_{\partial\Omega} |u_b(s, \xi) - v_b(s, \xi)| \, d\xi \, ds.
\]
In the limit $\tau \to 0$ for a.e. $\tau$, an application of Proposition 2.2 completes the proof when $u_o \in C^2(\Omega; \mathbb{R})$ and $u_b \in C^2(I \times \partial\Omega; \mathbb{R})$. The general case now follows by a straightforward regularization argument. \hfill \framebox[0.07\textwidth]{□}

7 Proofs Related to Section 2

Proof of Proposition 2.2. Let $M = \max\{\|u\|_{L^\infty(I \times \Omega; \mathbb{R})}, \|u_o\|_{L^\infty(\Omega; \mathbb{R})}, \|u_b\|_{L^\infty(I \times \partial\Omega; \mathbb{R})}\}$. We first prove that choosing $k \in ]-\infty, -M[ \cap M, +\infty[$, the terms containing $k$ in the left hand side in (2.1) vanish. Indeed, assuming $k < -M$, observe that
\[
|u(t, x) - k| = u(t, x) - k \quad \text{sgn} \, (u(t, x) - k) = 1
|u_o(x) - k| = u_o(x) - k \quad \text{sgn} \, (u_o(t, \xi) - k) = 1.
\]
Therefore, the terms containing $k$ in the left hand side in (2.1) are:
\[
\int_{I} \int_{\Omega} (-k \partial_t \varphi(t, x) - f(t, x, k) \cdot \nabla \varphi(t, x) - \text{div} \, f(t, x, k) \, \varphi(t, x)) \, dx \, dt
- \int_{\Omega} k \varphi(0, x) \, dx + \int_{I} \int_{\partial\Omega} f(t, \xi, k) \cdot \nu(\xi) \varphi(t, \xi) \, d\xi \, dt
= \int_{I} \int_{\Omega} - \text{div} \, f(t, x, k) \, \varphi(t, x) \, dx \, dt + \int_{I} \int_{\partial\Omega} f(t, \xi, k) \cdot \nu(\xi) \varphi(t, \xi) \, d\xi \, dt
= 0.
\]
The inequality (2.1) now reads

\[
0 \leq \int_I \int_\Omega \left\{ u(t,x) \partial_t \varphi(t,x) + f(t,x,u) \cdot \nabla \varphi(t,x) + F(t,x,u) \varphi(t,x) \right\} \, dx \, dt \\
+ \int_\Omega \varphi(0,x) u_o(x) \, dx - \int_I \int_{\partial \Omega} f(t,\xi,(\text{tr} \, u)(t,\xi)) \cdot \nu(\xi) \varphi(t,\xi) \, d\xi \, dt
\]

\[
= \int_I \int_\Omega \left\{ u(t,x) \partial_t \varphi(t,x) + f(t,x,u) \cdot \nabla \varphi(t,x) + F(t,x,u) \varphi(t,x) \right\} \, dx \, dt \\
+ \int_\Omega \varphi(0,x) u_o(x) \, dx - \int_I \int_{\partial \Omega} \text{tr} \, \left\{ f(t,\xi,u(t,\xi)) \right\} \cdot \nu(\xi) \varphi(t,\xi) \, d\xi \, dt
\]

\[
= \int_I \int_\Omega \left\{ u(t,x) \partial_t \varphi(t,x) - \varphi(t,x) \text{div} f(t,x,u) + F(t,x,u) \varphi(t,x) \right\} \, dx \, dt \\
+ \int_\Omega \varphi(0,x) u_o(x) \, dx,
\]

where we apply Lemma A.4 and the Divergence Theorem. Choose \( \varphi_k \in C^2_c([0,\infty[^2 ; \mathbb{R}^n ; \mathbb{R}^+]) \) as

\[
\varphi_k(t,x) = \vartheta_k(t) \psi(x),
\]

where \( \vartheta_k \in C^2_c([0,T];[0,1]) \) is such that

\[
\vartheta_k(0) = 1, \\
\vartheta_k(t) = 0 \text{ for all } t \geq 1/|k|, \\
\sup_k \frac{1}{|k|} \| \vartheta_k' \|_{C^0} < +\infty,
\]

while \( \psi \in C^2_c(\mathbb{R}^n ; \mathbb{R}^+) \). Hence,

\[
0 \leq \int_I \int_\Omega \left\{ u(t,x) \vartheta_k'(t) \psi(x) + (F(t,x,u) - \text{div} f(t,x,u)) \vartheta_k(t) \psi(x) \right\} \, dx \, dt \\
+ \int_\Omega \psi(x) u_o(x) \, dx.
\]

Pass now to the limit for \( k \to -\infty \). Observe that, by the Dominated Convergence Theorem,

\[
\lim_{k \to -\infty} \int_I \int_\Omega \left( F(t,x,u) - \text{div} f(t,x,u) \right) \vartheta_k(t) \psi(x) \, dx \, dt = 0.
\]

Thanks to Lemma A.5 and to the Dominated Convergence Theorem we also have

\[
\lim_{k \to -\infty} \int_I \int_\Omega u(t,x) \vartheta_k'(t) \psi(x) \, dx \, dt = - \int_\Omega u(0+,x) \psi(x) \, dx.
\]

Then, in the case \( k < -M \), (2.1) reduces to

\[
0 \leq \int_\Omega \psi(x) u_o(x) \, dx - \int_\Omega u(0+,x) \psi(x) \, dx.
\]

If \( k > M \) the signs in (7.1) are opposite and analogous computations show that (2.1) reduces to

\[
0 \geq \int_\Omega \psi(x) u_o(x) \, dx - \int_\Omega u(0+,x) \psi(x) \, dx.
\]

Hence,

\[
\int_\Omega \psi(x) \left( u_o(x) - u(0+,x) \right) \, dx = 0 \text{ for all } \psi \in C^2_c(\mathbb{R}^n ; \mathbb{R}^+).
\]
We then obtain that

\[ 0 = \int_{\Omega} \psi(x) \left( u_\alpha(x) - u(0+, x) \right) dx \]

\[ = \int_{\Omega} \psi(x) \left( u_\alpha(x) - \lim_{t \to 0^+} \frac{1}{t} \int_0^t u(\tau, x) d\tau \right) dx \]

\[ = \lim_{t \to 0^+} \frac{1}{t} \int_{\Omega} \int_0^t \psi(x) \left( u_\alpha(x) - u(\tau, x) \right) d\tau dx \]

\[ = \lim_{t \to 0^+} \frac{1}{t} \int_{\Omega} \int_0^t \psi(x) \left( u_\alpha(x) - u(\tau, x) \right) dx d\tau. \]

Therefore, there exists a set \( \mathcal{E} \subset I \) with measure 0 such that

\[ \lim_{t \to 0^+, t \in \mathcal{E}} \int_{\Omega} \psi(x) \left( u_\alpha(x) - u(t, x) \right) dx = 0 \]

and, by the arbitrariness of \( \psi \), the proof is completed. \( \square \)

**Proof of Proposition 2.3** Let \( \Psi \in C_1^2([0, T[ \times \mathbb{R}^n; \mathbb{R}^+) \) and \( \Phi_h \) as in (6.7). Write (2.1) with \( \varphi(t, x) = \Psi(t, x) \Phi_h(x) \) and take the limit as \( h \to 0 \). For all \( k \in \mathbb{R} \):

\[ \lim_{h \to 0} \int_{I \setminus \mathcal{E}} \left| u(t, x) - k \right| \partial_t \Psi(t, x) \Phi_h(x) dx dt = 0; \]

\[ \lim_{h \to 0} \int_{I \setminus \mathcal{E}} \operatorname{sgn}(u(t, x) - k) \left( f(t, x, u) - f(t, x, k) \right) \cdot \nabla \Psi(t, x) \Phi_h(x) dx dt = 0; \]

\[ \lim_{h \to 0} \int_{I \setminus \mathcal{E}} \operatorname{sgn}(u(t, x) - k) \left( f(t, x, u) - f(t, x, k) \right) \cdot \nabla \Phi_h(x) \Psi(t, x) dx dt = 0; \]

\[ \int_{\partial \Omega} \operatorname{sgn} \left( \partial_t \psi(t, \xi) - k \right) \left( f(t, \xi, \partial_t \psi(t, \xi)) - f(t, \xi, k) \right) \cdot \nu(\xi) \Psi(t, \xi) d\xi dt = 0; \]

\[ \lim_{h \to 0} \int_{\partial \Omega} \operatorname{sgn}(u_0(t, \xi) - k) \left[ f(t, \xi, \partial_t \psi(t, \xi)) - f(t, \xi, k) \right] \cdot \nu(\xi) \Psi(t, \xi) d\xi dt = 0; \]

where we used the Dominated Convergence Theorem, Lemma A.6 and Lemma A.4. The latter Lemma can be used since the function \((u, k) \to \operatorname{sgn}(u - k) \left( f(t, x, u) - f(t, x, k) \right)\) is Lipschitz continuous by [15]. Lemma 3. Therefore,

\[ \int_{\partial \Omega} \left[ \operatorname{sgn} \left( \partial_t \psi(t, \xi) - k \right) - \operatorname{sgn} \left( u_0(t, \xi) - k \right) \right] \left[ f(t, \xi, \partial_t \psi(t, \xi)) - f(t, \xi, k) \right] \cdot \nu(\xi) \Psi(t, \xi) d\xi dt \geq 0. \]

Hence,

\[ \left[ \operatorname{sgn} \left( \partial_t \psi(t, \xi) - k \right) - \operatorname{sgn} \left( u_0(t, \xi) - k \right) \right] \left[ f(t, \xi, \partial_t \psi(t, \xi)) - f(t, \xi, k) \right] \cdot \nu(\xi) \geq 0 \quad (7.2) \]

almost everywhere on \([0, T[ \times \partial \Omega\) for all \( k \in \mathbb{R} \). Inequality (7.2) is reduced to (2.3) by taking \( k \) in the interval \( I(t, \xi) \). \( \square \)
implies that $C$. We have also the uniform bounds $u$ sequence $\tilde{E}$ convergent to $E$ as in (6.7). Let $\Psi_m = 1 - \Phi_{1/m}$, with $\Phi_{1/m}$ as in (5.7). Let

$$u_m(x) = \Psi_m(x) \tilde{u}_m(x) \quad \text{for all } x \in \Omega.$$  

By construction, $\lim_{m \to +\infty} \|u_m - u_0\|_{L^1(\Omega; \mathbb{R})} = 0$, so that $u_m$ is a Cauchy sequence in $L^1(\Omega; \mathbb{R})$. We have also the uniform bounds

$$\|u_m\|_{L^1(\Omega; \mathbb{R})} \leq \|u_0\|_{L^1(\Omega; \mathbb{R})} ;$$

$$TV(u_m) \leq \|\text{grad } \Psi_m\|_{L^1(\Omega; \mathbb{R}^n)} \|\tilde{u}_m\|_{L^1(\Omega; \mathbb{R}^n)} + \|\Psi_m\|_{L^\infty(\Omega; \mathbb{R})} \|\text{grad } \tilde{u}_m\|_{L^1(\Omega; \mathbb{R}^n)} + TV(u_0).$$

Since for any $m \in \mathbb{N} \setminus \{0\}$ we have that $u_m(\xi) = u_0(0, \xi) = 0$, Theorem 4.2 applies to (1.1) with initial datum $u_m$ and boundary datum $u_0$, yielding the existence of a solution $u_m$ to (1.1) in the sense of Definition 2.1 which satisfies the estimates (4.4), (4.5) and (4.6). Theorem 4.3 then implies that

$$\int_{\Omega} u_m(t, x) - u_m'(t, x) \, dx \leq e^{L_F t} \int_{\Omega} u_0^m(x) - u_0^{m'}(x) \, dx,$$

proving that the sequence $u_m(t)$ satisfies the Cauchy condition in $L^1(\Omega; \mathbb{R})$ uniformly in $t \in I$.

Call $u = \lim_{m \to \infty} u_m$. We now verify that $u$ solves (1.1). By Proposition 2.6 each $u_m$ satisfies (2.4) for any $C^2$ entropy $\mathcal{E}_t(u) = \sqrt{1 + (u - k)^2}$ and for any $\varphi \in C^0_c(\mathbb{R}; \mathbb{R}^n; \mathbb{R}^+)$:

$$\int_I \int_{\Omega} \left\{ \mathcal{E}_t(u_m(t, x)) \partial_t \varphi(t, x) + \mathcal{F}_t(t, x, u_m(t, x)) \cdot \text{grad } \varphi(t, x) + \left[ \mathcal{E}_t'(u_m(t, x)) \left( F(t, x, u_m(t, x)) - \text{div } f(t, x, u_m(t, x)) \right) + \text{div } \mathcal{F}_t(t, x, u_m(t, x)) \right] \varphi(t, x) \right\} \, dx \, dt$$

$$+ \int_I \int_{\partial \Omega} \left[ \mathcal{F}_t(t, x, u_m(t, \xi)) - \mathcal{E}_t'(u_m(t, \xi)) \left( f(t, x, u_m(t, \xi)) - f(t, \xi, \text{tr } u_m(t, \xi)) \right) \right] \varphi(0, x) \, dx \, dt \geq 0.$$
In the limit $m \to +\infty$, since $u_m$ converges in $L^1$ to $u$, $\text{tr} u_m$ converges to $\text{tr} u$ by Lemma A.2 which can be applied thanks to the estimate (4.5). Hence, we have

$$
\int_I \int_\Omega \left\{ \mathcal{E}_i(u(t,x)) \partial_t \varphi(t,x) + \mathcal{F}_i(t,x,u(t,x)) \cdot \text{grad} \varphi(t,x) + \mathcal{E}'_i(u(t,x)) \left( F(t,x,u(t,x)) - \text{div} f(t,x,u(t,x)) \right) + \text{div} \mathcal{F}_i(t,x,u(t,x)) \right\} \varphi(t,x) \, dx \, dt 
- \int_I \int_\partial A \left[ \mathcal{F}_1(t,x,u_b(t,\xi)) - \mathcal{E}'_1(u_b(t,\xi)) \left( f(t,\xi,u_b(t,\xi)) - f(t,\xi,\text{tr} u(t,\xi)) \right) \right] \nu(\xi) \varphi(t,\xi) \, d\xi \, dt \geq 0.
$$

In the limit $l \to +\infty$, we have the convergences $\mathcal{E}_i \to \mathcal{E}$ and $\mathcal{F}_i \to \mathcal{F}$, with $\mathcal{E}(u) = |u - k|$ and $\mathcal{F}(t,x,u) = \text{sgn}(u - k) \left( f(t,x,u) - f(t,x,k) \right)$, so that

$$
\int_I \int_\Omega \left\{ |u(t,x) - k| \partial_t \varphi(t,x) + \text{sgn}(u(t,x) - k) \left( f(t,x,u) - f(t,x,k) \right) \cdot \text{grad} \varphi(t,x) + \text{sgn}(u(t,x) - k) \left( F(t,x,u) - \text{div} f(t,x,k) \right) \varphi(t,x) \right\} \, dx \, dt 
- \int_I \int_\partial A \text{sgn}(u_b(t,\xi) - k) \left( f(t,\xi,\text{tr} u(t,\xi)) - f(t,\xi,k) \right) \cdot \nu(\xi) \varphi(t,\xi) \, d\xi \, dt \geq 0.
$$

Finally, observe that

$$
\int_\Omega |u(0,x) - k| \varphi(0,x) \, dx \leq \int_\Omega |u_0(x) - k| \varphi(0,x) \, dx 
+ \int_\Omega \left| u_0(x) - u^m_0(x) \right| \varphi(0,x) \, dx \underset{m \to +\infty}{\overset{\text{by } 7.3}{\to}} 0 
+ \int_\Omega \left| u^m_0(x) - u_m(0,x) \right| \varphi(0,x) \, dx \underset{m \to +\infty}{\overset{\text{by Proposition } 2.2}{\to}} 0 
+ \int_\Omega \left| u_m(0,x) - u(0,x) \right| \varphi(0,x) \, dx \underset{m \to +\infty}{\overset{\text{since } u_m \to u \text{ in } L^1}{\to}} 0,
$$

concluding the existence proof.

The bounds directly follow from (4.4), (4.5) and (4.6), thanks to the properties (7.4) and (7.5) of the sequence $u^m_0$.

## A Appendix: The Trace Operator

A relevant role is played by the trace operator which we recall here from [8, Paragraph 5.3].

**Definition A.1.** Let $A \subset \mathbb{R}^n$ be bounded with Lipschitz boundary. The trace operator is the map $\text{tr}_A : BV(A;\mathbb{R}) \to L^1(\partial A;\mathbb{R})$ such that for all $\varphi \in C^1(\mathbb{R}^n;\mathbb{R}^n)$ and for all $w \in BV(A;\mathbb{R})$,

$$
\int_{\partial A} \left( \left( \text{tr}_A w(\xi) \right) \varphi(\xi) \cdot \nu(\xi) \, d\xi \right) = \int_A w(x) \, \text{div} \varphi(x) \, dx + \int_A \varphi(x) \, d(\nabla w(x)).
$$

Below, when no misunderstanding arises, we omit the dependence of the trace operator from the set. First of all, we recall without proof the following two lemmas.

**Lemma A.2** ([8, Paragraph 5.3, Theorem 1 and Theorem 2]). Let $A \subset \mathbb{R}^n$ be bounded with Lipschitz boundary. Fix $w \in BV(A;\mathbb{R})$. Then, the trace operator is a bounded linear operator and for $H^{n-1}$-a.e. $\xi \in \partial A$,

$$
\lim_{r \to 0+} \frac{1}{\mathcal{L}^n(B(\xi,r) \cap A)} \int_{B(\xi,r) \cap A} |w(x) - \left( \text{tr} w \right)(\xi)| \, dx = 0.
$$
Lemma A.3 ([R] Paragraph 5.3, Remark to Theorem 2). Let $A \subset \mathbb{R}^n$ be bounded with Lipschitz boundary. Fix $w \in \text{BV}(A; \mathbb{R}) \cap C^0(A; \mathbb{R})$. Then, $(\text{tr} w)(\xi) = w(\xi)$ for $\mathcal{H}^{n-1}$-a.e. $\xi \in \partial A$.

Recall also the following property.

Lemma A.4. Let $A \subset \mathbb{R}^n$ be bounded with Lipschitz boundary. Fix $w \in \text{BV}(A; \mathbb{R})$ and $h \in C^{0,1}(\mathbb{R}; \mathbb{R})$. Then, $\text{tr}(h \circ w) = h \circ (\text{tr} w)$ for $\mathcal{H}^{n-1}$-a.e. $\xi \in \partial A$.

**Proof.** For any $\xi \in \partial A$ and for $r > 0$ sufficiently small, compute:

\[
\frac{1}{\mathcal{L}^n(B(\xi, r) \cap A)} \int_{B(\xi, r) \cap A} \left| (h \circ w)(x) - \left(\text{tr}(h \circ w)(\xi)\right) \right| \, dx
\]

\[
\frac{1}{\mathcal{L}^n(B(\xi, r) \cap A)} \int_{B(\xi, r) \cap A} \left| (h \circ w)(x) - h \left(\text{tr}(w)(\xi)\right) \right| \, dx
\]

and both addends in the right hand side above vanish as $r \to 0^+$ by Lemma A.2.

The next two Lemmas relate the values attained by the trace of $u$ with limits at the boundary of integrals of $u$.

Lemma A.5. Let $T > 0$ and $u \in \text{BV}([0, T]; \mathbb{R})$. Choose a sequence $\varphi_k \in C_c^1([0, T]; [0, 1])$ such that $\varphi_k(0) = 1$, $\varphi_k(t) = 0$ for all $t \geq 1/k$ and $\sup_k \frac{1}{k} \left\| \varphi'_k \right\|_{L^\infty([0, T]; \mathbb{R}^n)} < +\infty$. Then,

\[
\lim_{k \to +\infty} \int_0^T u(t) \varphi'_k(t) \, dt = -u(0+).
\]

Above, we used the standard notation $u(0+) = \text{tr}_{[0, T]} u(0)$.

**Proof.** Denote $c = \sup_k \frac{1}{k} \left\| \varphi'_k \right\|_{L^\infty([0, T]; \mathbb{R}^n)}$. Compute:

\[
\left| \int_0^T u(t) \varphi'_k(t) \, dt + u(0+) \right| = \left| \int_0^{1/k} u(t) \varphi'_k(t) \, dt + u(0+) \right|
\]

\[
\leq \left| \int_0^{1/k} \left( u(t) - u(0+) \right) \left| \varphi'_k(t) \right| \, dt \right| + \left| u(0+) \int_0^{1/k} \varphi'_k(t) \, dt + u(0+) \right|
\]

\[
\leq \frac{c}{1/k} \left| \int_0^{1/k} \left( u(t) - u(0+) \right) \, dt \right|
\]

which vanishes as $k \to +\infty$ by Lemma A.2.

Lemma A.6. Let $\Omega$ satisfy (\Omega_{2,0}) and $u \in \text{BV}(\Omega; \mathbb{R})$. Choose a sequence $\chi_k \in C_c^1(\mathbb{R}^n; [0, 1])$ such that $\chi_k(\xi) = 1$ for all $\xi \in \partial \Omega$, $\chi_k(x) = 0$ for all $x \in \Omega$ with $B(x, 1/k) \subseteq \Omega$ and moreover $\sup_k \frac{1}{k} \left\| \nabla \chi_k \right\|_{L^\infty(\Omega; \mathbb{R}^n)} < +\infty$. Then, $\lim_{k \to +\infty} \int_\Omega u(x) \text{grad} \chi_k(x) \, dx = \int_{\partial \Omega} \text{tr}_\Omega u(\xi) \nu(\xi) \, d\xi$.

**Proof.** Let $e_i$ be the $i$-th vector of the standard basis in $\mathbb{R}^n$. Set $\Omega_k = \left\{ x \in \Omega : d(x, \partial \Omega) \leq 1/k \right\}$. Then:

\[
\left( \int_\Omega u(x) \text{grad} \chi_k(x) \, dx - \int_{\partial \Omega} \text{tr}_\Omega u(\xi) \nu(\xi) \, d\xi \right) \cdot e_i
\]

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\[\int_{\Omega} u(x) \partial_i \chi_k(x) \, dx - \int_{\partial\Omega} \text{tr} u(\xi) \chi_k(\xi) e_i \cdot \nu(\xi) \, d\xi \]

\[= \int_{\Omega} u(x) \partial_i \chi_k(x) \, dx - \int_{\partial\Omega} \text{tr} u(\xi) \chi_k(\xi) e_i \cdot \nu(\xi) \, d\xi \]

\[= - \int_{\partial\Omega} \chi_k(x) \, d(\nabla u)_i(x)\]

where Definition 3.1 was used to obtain the last expression. By the Dominated Convergence Theorem, \(\lim_{k \to +\infty} \int_{\Omega} \chi_k(x) \, d(\nabla u)_i(x) = 0\), completing the proof. \(\square\)

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**References**

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 2000.

[2] B. Andreianov and K. Sbihi. Well-posedness of general boundary-value problems for scalar conservation laws. *Transactions AMS*, page in print, 2015.

[3] C. Bardos, D. Brézis, and H. Brezis. Perturbations singulières et prolongements maximaux d’opérateurs positifs. *Arch. Rational Mech. Anal.*, 53:69–100, 1973/74.

[4] C. Bardos, A. Y. le Roux, and J.-C. Nédélec. First order quasilinear equations with boundary conditions. *Comm. Partial Differential Equations*, 4(9):1017–1034, 1979.

[5] A. Bressan. *Hyperbolic systems of conservation laws*, volume 20 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[6] M. Bulíček, P. Gwiazda, and A. Świerczewska-Gwiazda. Multi-dimensional scalar conservation laws with fluxes discontinuous in the unknown and the spatial variable. *Math. Models Methods Appl. Sci.*, 23(3):407–439, 2013.

[7] C. M. Dafermos. *Hyperbolic conservation laws in continuum physics*, volume 325 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 2010.

[8] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992.

[9] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[10] G. Gagneux and M. Madaune-Tort. *Analyse mathématique de modèles non linéaires de l’ingénierie pétrolière*, volume 22 of *Mathématiques & Applications (Berlin) [Mathematics & Applications]*. Springer-Verlag, Berlin, 1996. With a preface by Charles-Michel Marle.

[11] E. Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[12] H. Hanche-Olsen and H. Holden. The Kolmogorov-Riesz compactness theorem. *Expo. Math.*, 28(4):385–394, 2010.

[13] K. H. Karlsen and N. H. Risebro. On the uniqueness and stability of entropy solutions of nonlinear degenerate parabolic equations with rough coefficients. *Discrete Contin. Dyn. Syst.*, 9(5):1081–1104, 2003.

[14] D. Kröner, T. Müller, and L. M. Strehlau. Traces for functions of bounded variation on manifolds with applications to conservation laws on manifolds with boundary. *ArXiv e-prints*, Mar. 2014.

[15] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.
[16] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Urал’ceva. Linear and quasilinear equations of parabolic type. Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I., 1968.

[17] O. A. Ladyženskaja and N. N. Urал’ceva. A boundary-value problem for linear and quasi-linear parabolic equations. Dokl. Akad. Nauk SSSR, 139:544–547, 1961. English transl., Soviet Math. Dokl. 2 (1961), 969–972.

[18] O. A. Ladyženskaja and N. N. Urал’ceva. Linear and quasilinear elliptic equations. Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis. Academic Press, New York-London, 1968.

[19] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. Weak and measure-valued solutions to evolutionary PDEs, volume 13 of Applied Mathematics and Mathematical Computation. Chapman & Hall, London, 1996.

[20] F. Otto. Initial-boundary value problem for a scalar conservation law. C. R. Acad. Sci. Paris Sér. I Math., 322(8):729–734, 1996.

[21] D. Serre. Systems of conservation laws. 2. Cambridge University Press, Cambridge, 2000. Geometric structures, oscillations, and initial-boundary value problems, Translated from the 1996 French original by I. N. Sneddon.

[22] A. I. Vol’pert. Spaces BV and quasilinear equations. Mat. Sb. (N.S.), 73 (115):255–302, 1967.