Limit theorems for projections of random walk on a hypersphere

Max Skipper

August 25, 2009

Abstract

We show that almost any one-dimensional projection of a suitably scaled random walk on a hypercube, inscribed in a hypersphere, converges weakly to an Ornstein-Uhlenbeck process as the dimension of the sphere tends to infinity. We also observe that the same result holds when the random walk is replaced with spherical Brownian motion. This latter result can be viewed as a “functional” generalisation of Poincaré’s observation for projections of uniform measure on high dimensional spheres; the former result is an analogous generalisation of the Bernoulli-Laplace central limit theorem. Given the relation of these two classic results to the central limit theorem for convex bodies, the modest results provided here would appear to motivate a functional generalisation.

Keywords: random walk, functional central limit theorem, convex bodies

1 Introduction

Let $S^{d-1}_r$ be the spherical surface, centered at the origin, of radius $\sqrt{r}$ and let $X$ be uniformly distributed on $S^{d-1}_d$. A classic observation dating back to Maxwell, Poincaré and Borel is that the distributions of the first $K$ coordinates of $X$ converge to independent standard normals as $d \to \infty$; see Diaconis and Freedman [2] for an historical account. Recently there has been much work done on a generalisation of this result which seeks to replace $S^{d-1}_d$ with an arbitrary convex body $\mathcal{K} \in \mathbb{R}^d$ and the coordinates of $X$ with arbitrary linear projections $\langle \theta, X \rangle$, $\theta \in S^{d-1}_1$. If $X$ is now uniformly distributed on $\mathcal{K}$, what is now known as the central limit theorem for convex bodies asserts that, under suitable conditions on $\mathcal{K}$, the law of $\langle \theta, X \rangle$ is approximately Gaussian for most $\theta \in S^{d-1}_1$. We refer to Klartag [8] for precise statements of the theorem complete with quantitative definitions of the words “approximately” and “most”, as well as an overview of previous work; see Milman [9] for a more recent account including improved estimates for special cases.

Relative to the central limit theorem for convex bodies, the results presented in this note take the classical Maxwell-Poincaré-Borel observation in another direction — replacing an observation of projections of uniform measure on $S^{d-1}_d$ with
an observation of projections of two particular “uniform” processes on $S_{d-1}^d$. The two processes considered are spherical Brownian motion (SBM) on $S_{d-1}^d$ and a nearest neighbour random walk on the hypercube $\{-1,1\}^d$ (inscribed in $S_{d-1}^d$); they are referred to as “uniform” only because their invariant measures are uniform on their support. Assuming $\theta^{(d)} \in S_{d-1}^d$ and letting $X^{(d)}$ denote either of the above mentioned processes started at $x^{(d)}$, our main result states that if $|\theta^{(d)}|_\infty \to 0$ and $\langle \theta^{(d)}, x^{(d)} \rangle \to u$, then $\langle \theta^{(d)}, X^{(d)} \rangle$ converges weakly to an Ornstein-Uhlenbeck (OU) process $U$ started at $u$. (The condition on the $\infty$-norm of $\theta^{(d)}$ is unnecessary in the SBM case.) Just as the Maxwell-Poincaré-Borel observation represents a special case of a “naïve” (non-quantitative) central limit theorem for convex bodies, this modest result represents a special case of a naïve functional central limit theorem for convex bodies — developments and applications of which we hope to report in a subsequent paper.

Indeed, the original impetus for this work was derived from the practical problem of how to extract macroscopic dynamics from a randomly evolving system where an explicit microscopic description is given. Typically, the microscopic behaviour is modelled by a large system of coupled stochastic differential or difference equations driven by continuous or discrete Markov processes. In contrast, the dynamics of interest are those of a smaller number of functionals of the microscopic variables which are, in general, non-Markov. Since usually solutions must be obtained numerically, the main objective is to find a self-contained approximate description of the sought-after dynamics without needing to fully resolve the dynamics of the larger system; see Givon et al [3] for an informative survey.

A particularly relevant example is the Ehrenfest model of heat exchange between two isolated bodies, first published in 1907 in an effort to reconcile the irreversibility and recurrence in Boltzmann’s kinetic theory of gases. (See Takács [14] for an historical account of early work and Kac [7] for a discussion of Zermelo’s irreversibility/recurrence paradox.) The original model involves $d$ balls — representing energized gas molecules — distributed among two urns — the isolated bodies. The microscopic dynamics are such that at each time increment a ball is drawn out at random and placed in the opposite urn from whence it came. The macroscopic variable of interest is the number of balls in the first urn.

As is well-known, one may describe the allocation of the balls in the Ehrenfest model by the vector $X(n) \in \{0,1\}^d$ where $X_i(n) = 1$ if the $i$th ball is in the first urn after $n$ transitions and $X_i(n) = 0$ otherwise. Moreover, the Ehrenfest dynamics imply that $X = \{X(n)\}_{n \in \mathbb{N}_0}$ is a random walk on the hypercube $\{0,1\}^d$. What is special about this example is that because the $d$ microscopic variables (balls) are exchangeable, the macroscopic Ehrenfest process $\sum_{i=1}^d X_i$ is also Markov and hence an exact self-contained description is readily obtained. Nevertheless, it wasn’t until 40 years after the publication of the model that Kac [7] managed to derive the transition probabilities. As part of his work Kac found the transition probabilities of a suitably normalised Ehrenfest process and provided a sketch of how they converge to those of an OU-process as $d \to \infty$.

As far as we know, the most general extension of the Ehrenfest model that has some overlap with the work here is that given by Schach [11]. Schach’s model
consists of $d$ balls distributed among $K$ urns where at each transition a ball is moved from urn $j$ to urn $k$ with probability proportional to the number of balls in urn $j$ and a given number $p_{jk}$. Again, the microscopic variables are (the locations of) the balls and the macroscopic variables are the numbers of balls in each urn. Again, the macroscopic variables are Markov. Schach shows that as $d \to \infty$ the suitably normalised $K$-variate macroscopic process converges weakly to a $K$-variate OU-process. He also includes an account of earlier work and discusses applications of his results.

The Ehrenfest models are examples of models in which the (normalised) macroscopic process retains the Markov property and may be reasonably approximated by a diffusion process. Since the publication of Schach’s work, a powerful theory has been developed which gives conditions for weak convergence in such circumstances; see Stroock and Varadhan [13], Chapter 11. Like Schach’s results, Theorem 2.1 below — concerning the weak convergence of projections of SBM — also follows as a consequence of this general theory. (Despite this, we are not aware that the result has been made known explicitly.) On the other hand, Theorem 2.2 — concerning random walk on the hypercube — can not be deduced from the same theory. This is quite simply because an arbitrary projection of the random walk on the hypercube is non-Markov for finite $d$. Hence, it is the proof of Theorem 2.2 that occupies the better part of the sequel.

2 Set-up and main results

Unless otherwise stated, we continue to adopt the notational convention that vectors appear in bold typeface and the value $x_j$ is assumed to be the $j$th component of a vector $x$. In addition, a $\mathcal{D}$ above a binary relation indicates that the relation holds in the sense of probability law, while $:\equiv$ indicates a notational definition. Also, $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ denotes the set of non-negative integers and for any $m \in \mathbb{N}$, $[m] \equiv \{1, \ldots, m\}$. Any convergence statements made in the sequel are intended to be understood with respect to the limit $d \to \infty$.

Define the OU-process $U = \{U_t\}_{t \geq 0}$ as the diffusion process with drift and diffusion coefficients given by $b(u) \equiv -u$ and $a(u) \equiv 2$. That is, the infinitesimal generator of $U$ is given by

$$
\mathcal{L} \equiv \frac{\partial^2}{\partial u^2} - u \frac{\partial}{\partial u}.
$$

(2.1)

Here and subsequently, we shall assume $U_0 = u$ is deterministic so that $U$ is a Gaussian process.

Let $X^{(d)} = \{X_t^{(d)}\}_{t \geq 0}$ denote a random walk on $\mathcal{S}_d^{d-1}$ with $X_t^{(d)}$ representing the location of the walker at time $t$. For each $d \in \mathbb{N}$ we choose a ‘direction’ $\theta^{(d)} \in \mathcal{S}_1^{d-1}$ and define the ‘projected process’ $Y^{(d)} = \{Y_t^{(d)}\}_{t \geq 0}$ by

$$
Y_t^{(d)} \equiv \langle \theta^{(d)}, X_t^{(d)} \rangle,
$$

where $\langle \cdot, \cdot \rangle$ denotes the conventional inner product. Note that $Y^{(d)}$ depends on $\theta^{(d)}$ but that this is not explicitly highlighted in the notation. We will assume that $X^{(d)}$
starts from a given initial position \( X_0^{(d)} = x^{(d)} \) so that \( Y_0^{(d)} = y^{(d)} := \langle \theta^{(d)}, x^{(d)} \rangle \) is deterministic.

Though we make the assumption that \( X^{(d)} \) starts at the deterministic point \( X_0^{(d)} = x^{(d)} \), it will become evident that analogous results hold when \( X^{(d)} \) is a stationary random walk, i.e. \( X_0^{(d)} \) is distributed uniformly on the state-space. Commuting the role of randomness in \( X^{(d)} \) and \( \theta^{(d)} \) implies a randomized central limit theorem for the case when \( X_0^{(d)} \) is again fixed but \( \theta^{(d)} \) is chosen uniformly from the state-space (and then normalised). Note also that while our theorems will be stated only for 1-dimensional projections of \( X^{(d)} \), the results are readily extended, via the Cramér-Wold device, to \( K \)-dimensional projections of \( X^{(d)} \), \( K < \infty \).

2.1 Continuous case: spherical Brownian motion

Here we take \( X^{(d)} \) to be SBM on \( S_{d-1} \). Using the definition stated in Itô and McKean \[5\], SBM on \( S_{d-1} \) is the unique diffusion process with infinitesimal generator:

\[
\Delta_d = \sum_{i \in [d]} \frac{\partial^2}{\partial x_i^2} - \frac{1}{d} \sum_{i,j \in [d]} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - \frac{d-1}{d} \sum_{i \in [d]} x_i \frac{\partial}{\partial x_i}.
\] (2.2)

Alternative yet equivalent characterisations of SBM are given in Stroock \[12\] and Rogers and Williams \[10\].

**Theorem 2.1.** Let \( X^{(d)} \) be SBM on \( S_{d-1} \). If \( y^{(d)} \to u \), then

\( Y^{(d)} \xrightarrow{D} U \).

**Proof.** By the symmetry of SBM it follows that \( Y^{(d)} \) is Markov. What’s more, its infinitesimal generator is determined from that of \( X^{(d)} \) simply by studying the action of \( \Delta_d \) on functions dependent only on \( y = \theta^{(d)} \cdot x \). From the expression given in (2.2), it follows immediately that the infinitesimal generator of \( Y^{(d)} \) is given by

\[
L_d = -\frac{d-1}{d} y \frac{\partial}{\partial y} + \left(1 - \frac{y^2}{d}\right) \frac{\partial^2}{\partial y^2}.
\] (2.3)

We read off the drift and diffusion coefficients as \( b_d(y) := -(d-1)y/d \) and \( a_d(y) := 2(1 - y^2/d) \) respectively. Now, since (i): \( U \) is the unique process started at \( U_0 = u \) with infinitesimal generator \( L \); (ii): \( a_d \) and \( b_d \) are continuous and bounded uniformly in \( d \) on compact subsets of \( \mathbb{R} \); (iii): \( a_d \) (resp. \( b_d \)) converges pointwise to \( a \) (resp. \( b \)) on compact subsets of \( \mathbb{R} \); the result follows by Theorem 11.1.4, page 264, of Stroock and Varadhan \[13\].

\[\square\]

2.2 Discrete case: random walk on a hypercube

From here on we take \( X^{(d)} \) to be a simple, ‘lazy’, nearest neighbour random walk (LNNRW) on the vertices of the hypercube \( B^d \), where \( B := \{-1, 1\} \). Two vertices in
are nearest neighbours if they differ in exactly one coordinate. We assume that at regular clock pulses, separated by time intervals of length \( \delta := \frac{2p}{d} \), a LNNRW on \( B^d \) is ‘lazy’ (remains stationary) with probability \( 1 - p \), or moves to any given one of its \( d \) nearest neighbour vertices with equal probability \( \frac{p}{d} \). We assume \( p \in (0, 1) \) may depend on \( d \) (e.g. \( p = \frac{d}{d+1} \)).

**Theorem 2.2.** Let \( X^{(d)} \) be LNNRW on \( B^d \). If \( y^{(d)} \to u \) and \( |\theta^{(d)}|_\infty \to 0 \), then

\[
Y^{(d)} \xrightarrow{D} U. \tag{2.4}
\]

**Proof.** We apply the general program of Billingsley [1]. The convergence of the finite dimensional distributions of \( Y^{(d)} \) to those of \( U \) is given by Lemma 3.2; tightness of the sequence \( \{Y^{(d)}\}_{d \in \mathbb{N}} \) is established by Lemma 3.4. \( \square \)

**Remark 2.1.** The extra condition appearing in Theorem 2.2 that was absent from Theorem 2.1 is due to the lack of complete spherical symmetry of the LNNRW. As an example of why some condition on \( \theta^{(d)} \) is necessary, consider the choice \( \theta^{(d)} = (1, 0, \ldots, 0) \in S^{d-1} \) for each \( d = 1, 2, \ldots \). In this case it is clear that \( Y^{(d)} \) is a two-valued process and thus in no way can approach a diffusion limit as \( d \to \infty \).

**Remark 2.2.** It is possible to generalise the LNNRW model along the lines of Schach’s multivariate urn model without changing the conclusions of Theorem 2.2; see Remark 3.1.

### 3 Results for random walk on the cube

We begin with a concrete characterisation of LNNRW on \( B^d \). Since \( X_1^{(d)} \) gives the location of the random walker at the (continuous) real time \( t \), we will also adopt the alternative notation \( X^{(d)}(n) \equiv X_t^{(d)}, \ n\delta \leq t < (n + 1)\delta \), so that \( X^{(d)}(n) \) represents the location of the walker after \( n \) clock pulses. Here and subsequently let \( X_1, X_2, \ldots \) be a sequence of i.i.d. copies of \( X^{(1)} \) (LNNRW on the 1-cube) and let \( M^{(d)}(n) \sim \text{Mult}(n; \frac{1}{d}, \ldots, \frac{1}{d}) \) denote a multinomial random vector with parameters \( (n; \frac{1}{d}, \ldots, \frac{1}{d}) \). Since at any clock pulse the two choices of where to walk and whether to walk are interchangeable, a moment’s reflection will confirm that

\[
X^{(d)}(n) \xrightarrow{D} (X_1(M_1^{(d)}(n)), \ldots, X_d(M_d^{(d)}(n))). \tag{3.1}
\]

Now, let \( Z^{(d)} \) be a discrete-time random process on \( B^d \) with i.i.d. coordinate processes, each of which is equal in distribution to any of the identically distributed, but dependent, coordinate processes of \( X^{(d)} \). That is, for each \( n \in \mathbb{N}_0 \),

\[
Z^{(d)}(n) := (X_1(B_1^{(d)}(n)), \ldots, X_d(B_d^{(d)}(n))), \tag{3.2}
\]

where \( B^{(d)}(n) \) is a vector of independent \( \text{Bi}(n; \frac{1}{d}) \) binomial random variables each with parameters \( (n; \frac{1}{d}) \). The proximity of the moments of the finite dimensional distributions of \( X^{(d)} \) to those of \( Z^{(d)} \) will be the result that’s useful in the sequel.
Now we introduce some notation that helps us to precisely state and prove the result we require.

Due to the symmetry of $\mathcal{B}^d$ and the arbitrariness of $\theta^{(d)}$, we may, without loss of generality, restrict our attention to only a single choice of initial position. Thus, we henceforth assume that $\mathbf{X}^{(d)}_0$ starts at $\mathbf{x}^{(d)} = (1, \ldots, 1) \in \mathcal{B}^d$.

Pre-empting the treatment of the finite dimensional distributions of $Y^{(d)}$, let $0 = t_0 < t_1 < \cdots < t_K < \infty$ be a sequence of $[0, \infty)$-valued times and let $n_0, n_1, \ldots, n_K$ be the corresponding sequence of $\mathbb{N}_0$-valued ‘$\delta$-counts’ such that $n_k$ is the integer part of $t_k/\delta$; note that $n_k$ depends on $d$. From here on we shall often refrain from indicating the dependence on $d$ explicitly with the superscript $(d)$. We shall also utilize the shorthand $V(n_k) \equiv V_k$ for any vector-valued process $V \in \mathbb{R}^d$.

For any multi-index $k \in [K]$, let $M_k' \sim \text{Mult}(n_k - n_{k-1}; \frac{1}{d}, \ldots, \frac{1}{d})$ and $B_k'$ be a vector of i.i.d. $\text{Bi}(n_k - n_{k-1}; \frac{1}{d})$ random variables, such that $M_k = \sum_{j=1}^k M_j'$ and $B_k = \sum_{j=1}^k B_j'$, and introduce

$$X_k' \overset{\mathcal{D}}{=} (X_1(M_1'), \ldots, X_d(M_{kd}')), \quad Z_k' \overset{\mathcal{D}}{=} (X_1(B_{k1}'), \ldots, X_d(B_{kd}')).$$  

In view of the Markov property we may deduce that for any $i$:

$$X_{ki} \overset{\mathcal{D}}{=} \prod_{j=1}^k X_{ji}', \quad Z_{ki} \overset{\mathcal{D}}{=} \prod_{j=1}^k Z_{ji},$$

assuming that the $X_k'$ (resp. $Z_k'$) are mutually independent.

Now, fix a multi-index $i = (i_1, \ldots, i_L) \in [d]^L$ and constants $l_1, \ldots, l_K$ such that $l_1 + \cdots + l_K = L$. For each $k \in [K]$, let $L_k := l_1 + \cdots + l_K, L_k' := L_{k-1} + 1$ and $J_k(i)$ be the set containing precisely those $j \in [d]$ that occur with odd multiplicity in the multi-index $(i_{1j}, \ldots, i_{Lj})$ of length $l_k + \cdots + l_K$. We will also need $\eta_k(i) := |J_k(i)|$.

**Lemma 3.1.** For any multi-index $i = (i_1, \ldots, i_L) \in [d]^L$,

$$\mathbb{E}\left\{ \prod_{k=1}^K \prod_{l=L_k}^{L_k'} X_{kli} \right\} = \prod_{k=1}^K (1 - \eta_k(i)\delta)^{n_k-n_{k-1}}, \quad (3.3)$$

$$\mathbb{E}\left\{ \prod_{k=1}^K \prod_{l=L_k}^{L_k'} Z_{kli} \right\} = \prod_{k=1}^K (1 - \delta)^{\eta_k(i)(n_k-n_{k-1})}. \quad (3.4)$$

**Proof.** We prove only (3.3), the proof of (3.4) is analogous. Fix $i$ and set $J_k \equiv J_k(i)$.

$$\mathbb{E}\left\{ \prod_{k=1}^K \prod_{l=L_k}^{L_k'} X_{kli} \right\} = \mathbb{E}\left\{ \prod_{k=1}^K \prod_{l=L_k}^{L_k'} \prod_{j=1}^{k} X_{ji}' \right\} = \prod_{k=1}^K \mathbb{E}\left\{ \prod_{j=1}^{k} X_{ji}' \right\} = \prod_{k=1}^K \mathbb{E}\left\{ \prod_{l=L_k}^{L_k'} X_{kli}' \right\}.$$

Now, using the fact that $(X_{ki}')^r = X_{ki}'$ if $r$ is odd and $(X_{ki}')^r = 1$ otherwise, we see that

$$\mathbb{E}\left\{ \prod_{l=L_k'}^{L_k} X_{kli}' \right\} = \mathbb{E}\left\{ \prod_{j \in J_k} \mathbb{E}\{ X^{(1)}(M_{kj}) | M_{kj}' \} \right\} = \mathbb{E}\left\{ \prod_{j \in J_k} \lambda^{M_{kj}'} \right\}, \quad (3.5)$$
where $\lambda := 1 - 2p$ is the non-unit eigenvalue of the transition probability matrix of $X^{(1)}$. The result now follows by noting that $\mathbb{E}\left\{ \prod_{j \in J_k} \lambda^{M_{kj}} \right\}$ is the probability generating function, evaluated at $\lambda$, of $\sum_{j \in J_k} M_{kj}' \sim \text{Bi}\left(n_k - n_{k-1}, \frac{\eta_k(i)}{d}\right)$. □

**Remark 3.1.** All quantitative information that is used in the proof of Theorem 2.2 can be traced back to Lemma 3.1, which itself hinges on certain independence assumptions and the valuation of $\mathbb{E}\left\{ \prod_{j \in J_k} \lambda^{M_{kj}} \right\}$ in (3.5). Thus, generalisations of the LNNRW model that leave Theorem 2.2 unchanged become apparent. For example, with reference to (3.1), if we exchange $M^{(d)}(n)$ for $N^{(d)}(n) \sim \text{Mult}(n; \phi^{(d)})$ and the i.i.d. $\mathcal{B}$-valued LNNRW’s $X_1, X_2, \ldots$ for the independent $\mathcal{B}$-valued recurrent Markov chains $V_1^{(d)}, V_2^{(d)}, \ldots$, the respective transition probability matrices of which have non-unit eigenvalues $\lambda_1^{(d)}, \lambda_2^{(d)}, \ldots$, then provided there exists a $\delta$ dependent on $d$ such that $\phi_j^{(d)} (1 - \lambda_j^{(d)}) \delta^{-1} \to 1$ for each $j$, the conclusion of Theorem 2.2 remains valid.

### 3.1 Convergence of finite dimensional distributions

**Lemma 3.2.** If $|\theta^{(d)}|_\infty \to 0$ and $y^{(d)} \to u$, then

$$(Y_{t_1}, \ldots, Y_{t_k}) \overset{D}{\rightarrow} (U_{t_1}, \ldots, U_{t_k}).$$

**Proof.** By the Cramér-Wold device (Billingsley [1], Theorem 7.7) it is enough to show that for any $\phi \in \mathbb{R}^K$,

$$\Psi_d := \sum_{k=1}^K \phi_k Y_{t_k}^{(d)} = \sum_{k=1}^K \phi_k \langle \theta^{(d)}, X_k^{(d)} \rangle \overset{D}{\rightarrow} \Gamma := \sum_{k=1}^K \phi_k U_{t_k}.$$  

Since the (Gaussian) law of $\Gamma$ is uniquely determined by its sequence of moments, this can be achieved through a method of moments argument (Gut [4], p. 237) by showing that,

$$\mathbb{E}\left\{ \Psi_d^L \right\} \to \mathbb{E}\left\{ \Gamma^L \right\}, \quad L = 1, 2, \ldots.$$  

(3.6)

Now, let $\Upsilon_d := \sum_{k=1}^K \phi_k \langle \theta^{(d)}, Z_{k}^{(d)} \rangle = \sum_{j \in \mathcal{Y}} \theta_j^{(d)} \xi_j^{(d)}$ be the sum of $d$ independent random variables, with $\xi_j^{(d)} := \sum_{k=1}^K \phi_k z_{kj}^{(d)}$, and set $\sigma_d^2 := \text{Var}\{\Upsilon_d\}$. By taking appropriate linear combinations of formula (3.4) with $K = 1, 2$ and $L = 1, 2$, it is straightforward to show that the first two moments of $\Upsilon_d$ converge to those of $\Gamma$. Moreover, the fact that $|\theta^{(d)}|_2 = 1$ and the condition that $|\theta^{(d)}|_\infty \to 0$ is enough to ensure there exists an $r > 2$ such that

$$\sum_{j=1}^d \sigma_d^{-r} \mathbb{E}|\theta_j^{(d)} \xi_j^{(d)}| \mathbb{E}|\theta_j^{(d)} \xi_j^{(d)}|^r = \sigma_d^{-r} \mathbb{E}|\xi_1^{(d)}|^r - \mathbb{E}(|\xi_1^{(d)}|)^r \sum_{j=1}^d |\theta_j^{(d)}|^r \to 0,$$

so that the Lyapounov condition (Gut [4], p. 339) is satisfied. Hence we may conclude from Lyapounov’s central limit theorem that $\Upsilon_d \to \Gamma$. What’s more, we may use the Marcinkiewicz-Zygmund inequalities (Gut [4], p. 146) to verify that
\[ \mathbb{E}\{\Upsilon_d^L\} \] is bounded for each \( L \in \mathbb{N} \), implying \( \Upsilon_d^L \) is uniformly integrable for each \( L \in \mathbb{N} \) and thus \( \mathbb{E}\{\Upsilon_d^L\} \to \mathbb{E}\{\Gamma^L\} \) for each \( L \in \mathbb{N} \) (Billingsley [1], Theorem 5.4). Finally, since \( \mathbb{E}\{\Psi_d^L - \Upsilon_d^L\} \to 0 \) for each \( L \in \mathbb{N} \) (Lemma 3.3 below), we conclude that \( \mathbb{E}\{\Psi_d^L\} \to \mathbb{E}\{\Gamma^L\} \), as required. \( \square \)

Before coming to the proof of Lemma 3.3 cited above, we need to cover an intermediary result. Recall that a set \( \pi \) of non-empty subsets of a finite set \( \Sigma \) is a partition of \( \Sigma \) if the elements of \( \pi \) are mutually disjoint and \( \Sigma = \cup_{\sigma \in \pi} \sigma \). For two partitions \( \pi = \{\pi_1, \ldots, \pi_m\} \) and \( \nu = \{\nu_1, \ldots, \nu_l\} \) of the same finite set, we will write \( \nu \prec \pi \) if \( l < m \) and each \( \nu_j \) is a union of \( \pi_j \)'s. We write \( \nu \preceq \pi \) if either \( \nu \prec \pi \) or \( \nu = \pi \).

To every multi-index \( i = (i_1, \ldots, i_L) \in [d]^L \), or equivalently, mapping \( i : [L] \to [d] : l \mapsto i_l \), there corresponds a partition of \( [L] \):

\[ \nu_i := \{i^{-1}(k) \subset [L] : k \in [d] \} \setminus \{\emptyset\}, \]

where \( \emptyset \) denotes the empty set and \( i^{-1}(k) := \{l \in [L] : i_l = k\} \) is the pre-image of \( k \) under the mapping \( i \). For any partition \( \pi = \{\pi_1, \ldots, \pi_m\} \) of \( [L] \), define

\[ I_\pi := \{i \in [d]^L : \nu_i = \pi\}, \]

so that for each \( i \in I_\pi \) there exists \( m \) distinct numbers \( b_1, \ldots, b_m \) such that for each \( k \in [m] \), \( i_l = b_k \) for all \( l \in \pi_k \). In addition, define \( I_{\prec \pi} = \cup_{\nu \prec \pi} I_\nu \) and \( I_{\preceq \pi} = I_\pi \cup I_{\prec \pi} \) so that, in particular, \( I_{\preceq \pi} \) contains precisely those \( i \in [d]^L \) where there exist not-necessarily distinct numbers \( b_1, \ldots, b_m \in [d] \) such that for each \( k \in [m] \), \( i_l = b_k \) for all \( l \in \pi_k \).

**Proposition 3.1.** Given an array \( (a_{jl}) : j \in [d], l \in [L] \), set \( A_s := \sum_{j \in [d]} \prod_{l \in s} a_{jl} \) for any \( s \subset [L] \). Let \( (c_{\pi,\nu})_{\nu \preceq \pi} \) be the triangular array of constants, indexed by partitions of \([L]\), that satisfies the recursion:

\[ c_{\pi,\nu} = -\sum_{\nu \preceq \mu \preceq \pi} c_{\mu,\nu} \text{ for } \nu \prec \pi, \text{ and } c_{\pi,\pi} = 1. \]

Then,

\[ \sum_{i \in I_\pi} \prod_{l \in [L]} a_{i_l} = \sum_{\nu \preceq \pi} c_{\pi,\nu} \prod_{s \in \nu} A_s. \]  \( (3.7) \)

**Proof.** The proof is via induction on \( |\pi| \). As the first step: when \( |\pi| = 1 \) we must have \( \pi = \{[L]\} \) and \( I_\pi = \{i \in [d]^L : \exists k \in [d] \text{ s.t. } i = (k, \ldots, k)\} \) so that

\[ \sum_{i \in I_\pi} \prod_{l \in [L]} a_{i_l} = \sum_{k \in [d]} \prod_{l \in [L]} a_{kl} = A_{[L]}. \]

Assuming the induction hypothesis \((3.7)\) to be true for any partition \( \pi \) of \([L]\) such that \( |\pi| \leq m \), we now proceed with the induction step. If \( \pi = \{\pi_1, \ldots, \pi_{m+1}\} \)
is a partition of \([L]\), then

\[
\sum \prod_{i \in \ell, l \in [L]} a_{i\ell} = \sum \prod_{i \in I_{\leq l}, l \in [L]} a_{i\ell} - \sum \prod_{i \in I_{< l}, l \in [L]} a_{i\ell}
\]

\[
= \sum_{i \in [d]^{m+1}} \prod_{k=1}^{m+1} \prod_{l \in I_k} a_{i\ell} - \sum \sum_{\nu < \pi} c_{\nu,\mu} \prod A_s = \prod_{k=1}^{m+1} \sum_{j \in [d]} \prod_{l \in I_k} a_{j\ell} - \sum_{\nu < \pi} c_{\nu,\mu} \prod A_s
\]

\[
= \prod_{k=1}^{m+1} \sum_{\nu < \pi} (\sum_{\mu \leq \nu} c_{\nu,\mu}) \prod A_s = \sum_{\mu \leq \pi} \sum_{s} c'_{\pi,\mu} \prod A_s,
\]

where \(c'_{\pi,\pi} = 1\) and \(c'_{\pi,\mu} = -\sum_{\mu \leq \nu < \pi} c_{\nu,\mu}\). Hence \(c'_{\pi,\nu} = c_{\pi,\nu}\) as required. \(\square\)

**Remark 3.2.** Note that the number of terms in the sum on the left hand side of (3.7) depends only on \(d\) whereas the number of terms in the sum on the right hand side of the same equation depends only on \(L\).

**Lemma 3.3.** For each \(L \in \mathbb{N}\), \(\mathbb{E}\{\Psi_d^L - \Upsilon_d^L\} \rightarrow 0\).

**Proof.** Let \(t \in \mathbb{N}_0^K\) be such that \(l_1 + \cdots + l_K = L\). Define the shorthand

\[
E_d(t) := \mathbb{E}\{\langle \theta, X_1 \rangle^{l_1} \cdots \langle \theta, X_K \rangle^{l_K}\} - \mathbb{E}\{\langle \theta, Z_1 \rangle^{l_1} \cdots \langle \theta, Z_K \rangle^{l_K}\}
\]

and introduce \(\Pi_L := \{I_\pi : \pi\) is a partition of \([L]\}\) as a partition of \([d]^{L}\) into disjoint subsets of multi-indices over which \(\eta(i)\) remains constant. Agree to allow \(\eta(\pi) \equiv \eta(i)\) whenever \(i \in I_\pi\). Now, after using a multinomial expansion we get

\[
|\mathbb{E}\{\Psi_d^L - \Upsilon_d^L\}| \leq (|\phi_1| + \cdots + |\phi_K|)^L \max_t |E_d(t)|, \tag{3.8}
\]

so that it suffices to show that \(E_d(t) \rightarrow 0\) for each admissible \(t\).

Using Lemma 3.1, we have

\[
E_d(t) = \sum_{i \in [d]^{L}} \left[ \mathbb{E}\{\prod_{k=1}^{K} \theta_{i_{L_k}} X_{k_{i_k}}\} - \mathbb{E}\{\prod_{k=1}^{K} \theta_{i_{L_k}} Z_{k_{i_k}}\} \right]
\]

\[
= \sum_{\pi \in \Pi_L} f_d(\pi) \left(\sum_{i \in I_\pi} \prod_{l=1}^{L_i} \theta_{i_l}\right), \tag{3.9}
\]

where for all \(\pi \in \Pi_L\),

\[
f_d(\pi) := \prod_{k=1}^{K} (1 - \eta_k(\pi) \delta)^{n_k - n_{k-1}} - \prod_{k=1}^{K} (1 - \delta)^{\eta_k(\pi)(\eta_k - n_{k-1})} \rightarrow 0.
\]

9
By assumption: \(\sum_{j \in |d|} \theta_j \rightarrow u\), \(\sum_{j \in |d|} \theta_j^2 = 1\) and for all \(r > 2\), \(\sum_{j \in |d|} \theta_j^r \leq \max_{j \in |d|} |\theta_j|^r \sum_{j \in |d|} |\theta_j|^2 \rightarrow 0\); hence, replacing \(a_j\) with \(\theta_j\) for all \(l\), we conclude via Proposition 3.1 that \(\sum_{i \in \mathcal{I}_r} \prod_{l=1}^L \theta_{i_l}\) remains bounded as \(d \rightarrow \infty\); see Remark 3.2. Noting that \(|\Pi_l|\) is independent of \(d\), it follows that the sum (3.9) consists of a fixed number of terms each of which tends to zero as \(d \rightarrow \infty\), thus completing the proof.

3.2 Tightness

Lemma 3.4. If \(y^{(d)} \rightarrow u\), then \(\{Y^{(d)}\}_{d=1}^\infty\) is tight.

Proof. We appeal to the tightness criterion of Theorem 4.1, page 355, Jacod and Shiryaev \[6\]. Let \(\varepsilon > 0\) and let \(C\) denote a generic constant independent of \(d\) and \(t_3\). Since there is zero probability of a jump discontinuity at time zero, it suffices to show that

\[
\mathbb{P}\left( |Y_{t_3}^{(d)} - Y_{t_1}^{(d)}| \geq \varepsilon, |Y_{t_1}^{(d)} - Y_{t_2}^{(d)}| \geq \varepsilon \right) \leq \frac{C}{\varepsilon^3 (t_3 - t_1)^2}; \quad d = 2, 3, \ldots \quad (3.10)
\]

To establish this result we follow the same method as used on page 459 of Schach \[11\]. In what follows, we assume \(d \geq 2\) and drop the \((d)\) notation once again.

Lemma 3.5.

\[
\mathbb{P}\left( |Y_{t_2} - Y_{t_1}| \geq \varepsilon \mid Y_{t_1} \right) \leq \frac{4}{\varepsilon^2 (n_2 - n_1)\delta} \left(1 + Y_{t_1}^2\right). \quad (3.11)
\]

Proof. Define \(\nu_k(x) = 1 - k x \delta\) and let

\[
f(x) = 1 - \nu_2^{n_2-n_1}(x) + Y_{t_1}^2 \left\{ \nu_2^{n_2-n_1}(x) - 2\nu_1^{n_2-n_1}(x) + 1 \right\}.
\]

Then by the Mean Value Theorem,

\[
f(1) - f(0) = \mathbb{E}\left\{ (Y_{t_2} - Y_{t_1})^2 \mid Y_{t_1}\right\} \leq 2(n_2 - n_1)\delta \left[1 + 2Y_{t_1}^2\right]. \quad (3.12)
\]

The result now follows by Chebyshev’s inequality.

Lemma 3.6. If \(y^{(d)} \leq C\), then for any \(L \in \mathbb{N}\),

\[
|\mathbb{E}\{Y_{t_3}^L\}| \leq C.
\]

Proof. First, if \(d \leq L\) we may use the trivial bound \(|\mathbb{E}\{Y_{t_3}^L\}| \leq L^\frac{L}{2} \leq C\). If, on the other hand, \(d > L\), then for any \(i \in [d]^L\), \(0 \leq \eta_3(i)\delta \leq 2\). Thus, following analogous arguments to those used in the proof of Lemma 3.3, we see that

\[
|\mathbb{E}\{Y_{t_3}^L\}| = \left| \sum_{i \in [d]^L} \left( \prod_{l=1}^L \theta_{i_l} \right) (1 - \eta_3(i)\delta)^{(n_3-n_2)} \right| \leq \sum_{\pi \in \Pi_L} \left| \sum_{i \in \mathcal{I}_r} \prod_{l=1}^L \theta_{i_l} \right| \leq C.
\]
Lemma 3.7.

\[ \mathbb{P} \{ |Y_{t_3} - Y_{t_2}| \geq \varepsilon, |Y_{t_2} - Y_{t_1}| \geq \varepsilon \} \leq \frac{C}{\varepsilon^3} (n_3 - n_2)(n_2 - n_1)^{\frac{1}{2}} \delta^{\frac{3}{2}}. \tag{3.13} \]

**Proof.** Utilising the Markov property of \( X \), the Cauchy-Schwarz inequality and Lemmas 3.5 and 3.6, we obtain:

\[ \mathbb{P} \{ |Y_{t_3} - Y_{t_2}| \geq \varepsilon, |Y_{t_2} - Y_{t_1}| \geq \varepsilon \} = \mathbb{E} [ \mathbb{P} \{ |Y_{t_3} - Y_{t_2}| \geq \varepsilon, |Y_{t_2} - Y_{t_1}| \geq \varepsilon \} | X_{t_2} ] \]

\[ \leq \frac{C(n_3 - n_2)\delta}{\varepsilon^2} \mathbb{E} [ \mathbb{P} \{ |Y_{t_2} - Y_{t_1}| \geq \varepsilon \} ] \cdot (1 + Y_{t_2}^2)^{\frac{1}{2}} \]

\[ \leq \frac{C(n_3 - n_2)\delta}{\varepsilon^2} [ \mathbb{E} [ \mathbb{P} \{ |Y_{t_2} - Y_{t_1}| \geq \varepsilon \} ] ]^{\frac{1}{2}} \]

\[ \leq \frac{C(n_3 - n_2)(n_2 - n_1)^{\frac{1}{2}} \delta^{\frac{3}{2}}}{\varepsilon^3} \mathbb{E} [ 1 + Y_{t_1}^2 ]^{\frac{1}{2}} \]

\[ \leq \frac{C(n_3 - n_2)(n_2 - n_1)^{\frac{1}{2}} \delta^{\frac{3}{2}}}{\varepsilon^3}. \]

We now verify that (3.13) implies the tightness condition (3.10). Suppose first that \( n_3 > n_2 > n_1 \geq 0 \). Then clearly \( n_3 - n_2 \geq 1, n_2 - n_1 \geq 1 \) and \( n_3 - n_1 \geq 2 \) from which it follows immediately that

\[ (n_3 - n_2)(n_2 - n_1)^{\frac{1}{2}} \leq (n_3 - n_1 - 1)^{\frac{3}{2}} \leq (t_3 - t_1)^{\frac{3}{2}} \delta^{-\frac{3}{2}}. \]

Moreover, if \( n_3 = n_2 \) and/or \( n_2 = n_1 \), the above inequality is trivially satisfied and thus it in fact holds for \( n_3 \geq n_2 \geq n_1 \geq 0 \).

\[ \square \]

**Acknowledgements**

The author would like to thank Terry Lyons for suggesting the problem and Stephen Buckley, Svante Janson and Gesine Reinert for helpful suggestions that led to improvements of the proofs.
References

[1] P. Billingsley. *Convergence of probability measures*. John Wiley & Sons Inc., New York, 1968.

[2] P. Diaconis and D. Freedman. A dozen de Finetti-style results in search of a theory. *Ann. Inst. H. Poincaré Probab. Statist.*, 23(2, suppl.):397–423, 1987.

[3] D. Givon, R. Kupferman, and A. Stuart. Extracting macroscopic dynamics: model problems and algorithms. *Nonlinearity*, 17(6):R55–R127, 2004.

[4] A. Gut. *Probability: a graduate course*. Springer Texts in Statistics. Springer, New York, 2005.

[5] Ki. Itô and H. P. McKean, Jr. *Diffusion processes and their sample paths*. Springer-Verlag, Berlin, 1974. Second printing, corrected, Die Grundlehren der mathematischen Wissenschaften, Band 125.

[6] J. Jacod and A. N. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2003.

[7] M. Kac. Random walk and the theory of Brownian motion. *Amer. Math. Monthly*, 54:369–391, 1947.

[8] B. Klartag. A central limit theorem for convex sets. *Invent. Math.*, 168(1):91–131, 2007.

[9] Emanuel Milman. On Gaussian marginals of uniformly convex bodies. *J. Theoret. Probab.*, 22(1):256–278, 2009.

[10] L. C. G. Rogers and D. Williams. *Diffusions, Markov processes, and martingales*. Vol. 2. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2000. Itô calculus, Reprint of the second (1994) edition.

[11] S. Schach. Weak convergence results for a class of multivariate Markov processes. *Ann. Math. Statist.*, 42:451–465, 1971.

[12] D. W. Stroock. On the growth of stochastic integrals. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 18:340–344, 1971.

[13] D. W. Stroock and S. R. S. Varadhan. *Multidimensional diffusion processes*, volume 233 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1979.

[14] L. Takács. On an urn problem of Paul and Tatiana Ehrenfest. *Math. Proc. Cambridge Philos. Soc.*, 86(1):127–130, 1979.