ROBUST PRECONDITIONING FOR A MIXED FORMULATION OF PHASE-FIELD FRACTURE PROBLEMS

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Abstract. In this work, we consider fracture propagation in nearly incompressible and (fully) incompressible materials using a phase-field formulation. We use a mixed form of the elasticity equation to overcome volume locking effects and develop a robust, nonlinear and linear solver scheme and preconditioner for the resulting system. The coupled variational inequality system, which is solved monolithically, consists of three unknowns: displacements, pressure, and phase-field. Nonlinearities due to coupling, constitutive laws, and crack irreversibility are solved using a combined Newton algorithm for the nonlinearities in the partial differential equation and employing a primal-dual active set strategy for the crack irreversibility constraint. The linear system in each Newton step is solved iteratively with a flexible generalized minimal residual method (GMRES). The key contribution of this work is the development of a problem-specific preconditioner that leverages the saddle-point structure of the displacement and pressure variable. Four numerical examples in pure solids and pressure-driven fractures are conducted on uniformly and locally refined meshes to investigate the robustness of the solver concerning the Poisson ratio as well as the discretization and regularization parameters.

1. Introduction

Phase-field fracture modeling \cite{37,43} emerged from a variational formulation introduced in \cite{20,10} is an attractive model approach to simulate crack propagation in solids. To date, displacement-based formulations have been used in the large majority of investigations \cite{8,43,57,55,51,13,2,12,40,39} in which classical ideas from (non-fractured) solids were employed by introducing a Lagrange multiplier for the pressure variable. However, considering (nearly) incompressible solids, these models are subject to locking effects, i.e., the values of the displacement field are underestimated. For this reason, mixed phase-field formulations have been recently developed in \cite{40,39} in which classical ideas from (non-fractured) solids were employed by introducing a Lagrange multiplier for the pressure variable. With help of a mixed form we get a stable problem formulation up to the incompressible limit \cite{5}. However only sparse direct solvers, e.g., \cite{17}, were utilized in these previous studies for solving the arising linear equation systems.

The main purpose of the current work is to propose (for the first time) a preconditioned iterative linear solver for solving mixed formulations of phase-field fracture problems. Therein, we deal with three unknowns, namely displacements, pressure, and phase-field, $U := (u, p, \varphi)$. For classical, displacement-based $(u, \varphi)$ formulations iterative linear and multigrid methods are known. We also note that we consider fractures in pure solids as well as pressurized cracks (Sneddons’s test, see e.g., \cite{49,9,48}). The reader should not
confound the pressure $p$ introduced due to the mixed formulation with the pressure $\rho$, which is imposed inside the crack region in pressurized (i.e., pressure-driven) configurations.

The first study with a clear focus on linear solvers is [19]. Therein, a nonlinear Gauss-Seidel scheme was proposed together with a Schur complement based preconditioner for the linear systems. A parallel GMRES (generalized minimal residual) solver with diagonal preconditioner with algebraic multigrid preconditioning was developed in [26, 27]. Earlier versions were used in [25, 38], however without studying the parallel performance and scalability. A GMRES solver with a matrix-free geometric multigrid preconditioner was later suggested in [32] with a subsequent parallel version in [31]. An overall summary of these developments can be found in the PhD thesis of Jodlbauer [30].

A Galerkin finite element discretization yields a nonlinear system of the form $MU = F$ with a $(3 \times 3)$ block matrix $M \in \mathbb{R}^{n \times n}, U \in \mathbb{R}^{n}$, and $F \in \mathbb{R}^{n}$. For inf-sup stability a Taylor-Hood element $Q_{2}^{3}/Q_{1}$ is used for the $(u, p)$ system. Here, $Q_{2}^{3}$ denotes a continuous finite element space with bi-quadratic finite elements (we restrict the discussion to quadrilateral finite elements here). We note that computational comparisons to stabilized low-order equal-order finite elements were undertaken in [93]. However, it was found that this approach cannot be recommended for the mixed phase-field fracture formulation combined with high Poisson ratios. The stabilizing terms contain $\nabla p$ with mesh-dependent coefficients leading to large gradients in the crack region.

The discretized system is nonlinear, for which we employ Newton’s method as a nonlinear solver. Inside, the linear system is non-symmetric and therefore, we use a GMRES method. The key contribution is the development of a block triangular preconditioner. Individual blocks are approximated with inner solves using the conjugate gradient method (CG) and algebraic multigrid (AMG) from the ML package [52, 21]. The development of a block triangular preconditioner is summarized in Section 5.

The main challenge in developing the preconditioner is the interaction of various model, discretization, and material parameters to obtain a robust approach. These are the spatial discretization parameter $h$ and the Poisson ratio $\nu$ (related to the Lamé coefficient $\lambda$) up to the incompressible limit $\nu = 0.5$, and the regularization parameter $\kappa$ and the crack bandwidth $\epsilon$. We note that the basis of this work was developed later suggested in [32] with a subsequent parallel version in [31]. An overall summary of these developments can be found in the PhD thesis of the second author [39] and some preliminary results were published in [24].

The outline of this paper is as follows: In Section 2 the notation and governing equations are introduced. Next, Section 3 is the main part in which we first summarize the discretization and nonlinear solver. Then, the iterative solver and a Schur-type preconditioner are derived. Afterward, in Section 4 four numerical experiments are conducted to substantiate the performance of our algorithmic developments. Our work is summarized in Section 5.

2. Notation and Governing Equations

Let $\Omega$ be an open and smooth two-dimensional domain and $T := (0, T_{\text{end}})$ is a time (i.e., loading) interval with the partition $0 := t_{0} < t_{1} < \ldots < t_{N} := T_{\text{end}}$. The lower-dimensional crack is approximated by a phase-field indicator function $\varphi : (\Omega \times T) \to [0, 1]$ with $\varphi = 0$ in the crack and $\varphi = 1$ in the unbroken area. The bandwidth of the zone between broken and unbroken is named $\epsilon$. Further, a displacement function is defined as $u : (\Omega \times T) \to \mathbb{R}^{2}$. In the following, the scalar-valued $L^{2}$-product is denoted by $(x, y) := \int_{\Omega} x \cdot y \, d\Omega$, whereas the vector-valued $L^{2}$-product is described by $(X, Y) := \int_{\Omega} X : Y \, d\Omega$, with the Frobenius product $X : Y$ of two vectors $X$ and $Y$. We define the usual Sobolev spaces $V := H_{0}^{1}(\Omega)^{2}, W := H^{1}(\Omega)$ and a convex subset $K := \{ \varphi \in W \mid 0 \leq \varphi \leq \varphi_{n-1} \leq 1 \text{ a.e. in } \Omega \} \subset W$ and $\mathcal{U} := L_{2}(\Omega)$. Further, the degradation function is defined as $g(\varphi) := (1-\kappa)\varphi^{2} + \kappa$, where $\kappa$ is a sufficiently small regularization parameter. The stress tensor is defined as $\sigma(u) := 2\mu E_{\text{lin}}(u) + \lambda \text{tr}(E_{\text{lin}}(u))I$ with a linearized strain tensor $E_{\text{lin}}(u) := \frac{1}{2}(\nabla u + \nabla u^{T})$, material dependent Lamé coefficients $\lambda$ and $\mu$, and the two-dimensional identity matrix $I$. The critical energy release rate is denoted as $G_c$. Based on this notation, the pressurized phase-field fracture model in its classical form can be formulated as follows [54]:

**Problem 1** (Pressurized phase-field fracture).

Let a (constant) pressure $\rho \in L^{\infty}(\Omega)$ and the initial value $\varphi(0) := \varphi^{0}$ be given. Given the previous timestep data $\varphi_{n-1} := \varphi(t_{n-1}) \in K$. Find $u := u^{n} \in V$ and $\varphi := \varphi^{n} \in K$ for loading steps $n = 1, 2, \ldots, N$ with
\{u, \varphi\} \in V \times K\ such\ that
\[
(g(\tilde{\varphi})\sigma(u), E_{\text{lin}}(w)) + (\tilde{\varphi}^2 \rho, \nabla \cdot w) = 0 \quad \forall w \in V,
\]
\[
(1 - \kappa)(\varphi \sigma(u) : E_{\text{lin}}(u), \psi - \varphi) + 2(\varphi \rho \nabla \cdot u, \psi - \varphi)
+ G_c \left(-\frac{1}{\epsilon}(1 - \varphi, \psi - \varphi) + \epsilon(\nabla \varphi, \nabla(\psi - \varphi))\right) \geq 0 \quad \forall \psi \in K.
\]

In the elasticity part, a linear-in-time extrapolation with \(\tilde{\varphi} := \tilde{\varphi}(\varphi^{n-1}, \varphi^{n-2})\) is used in the phase-field variable \(\varphi\) to obtain a convex functional \([25]\). Therein, for \(\varphi^{n-2}\) at \(n = 1\), we set \(\varphi^{-1} := \varphi^0\).

Based on Problem 1 and following \([10]\), we introduce a pressure \(p := \lambda \text{tr}(E_{\text{lin}}(u))\), which is a Lagrange multiplier. As mentioned in the introduction, \(p\) and the crack pressure \(\rho\) (see for instance \([54]\) and therein itself denoted as \(p\)) should not be mixed up.

**Problem 2** (Pressurized phase-field fracture in mixed form).
Let \(\rho \in L^\infty(\Omega)\) be given and the initial value \(\varphi(0) := \varphi^0\) be given. Given the previous time step data \(\varphi^{n-1} \in K\). Find \(u := u^n \in V, p := p^n \in U\) and \(\varphi := \varphi^n \in K\) for loading steps \(n = 1, 2, \ldots, N\) with \(U := \{u, p, \varphi\} \in V \times U \times K\) such that
\[
(g(\tilde{\varphi})\sigma(u, p), E_{\text{lin}}(w)) + (\tilde{\varphi}^2 \rho, \nabla \cdot w) = 0 \quad \forall w \in V,
\]
\[
g(\tilde{\varphi})(\nabla \cdot u, q) - \left(-\frac{1}{\lambda} p, q\right) = 0 \quad \forall q \in U,
\]
\[
(1 - \kappa)(\varphi \sigma(u, p) : E_{\text{lin}}(u), \psi - \varphi) + 2(\varphi \rho \nabla \cdot u, \psi - \varphi)
+ G_c \left(-\frac{1}{\epsilon}(1 - \varphi, \psi - \varphi) + \epsilon(\nabla \varphi, \nabla(\psi - \varphi))\right) \geq 0 \quad \forall \psi \in K,
\]
where the stress tensor is defined as \(\sigma(u, p) := 2\mu E_{\text{lin}}(u) + pI\).

**Remark.** It is clear that by setting \(\rho = 0\), we obtain a phase-field formulation for fracture in pure solids. With this, we can investigate our preconditioner for both situations, namely fracture in solids and pressurized cracks.

### 3. Discretization and Numerical solution

For the spatial discretization of Problem 2 we employ a Galerkin finite element method in each incremental step, where the domain \(\Omega\) is partitioned into quadrilaterals \([14]\) with the discrete spaces \(V_h, U_h, \) and the convex set \(K_h \subset \mathcal{W}_h\). To fulfill a discrete inf-sup condition, stable Taylor-Hood elements with continuous bi-quadratic shape functions \((Q_2)^3\) for the displacement field \(u\) and bilinear shape functions \((Q_1)^2\) for the pressure variable \(p\) and the phase-field variable \(\varphi\) are used as in \([10]\).

**3.1. Nonlinear solver.** The nonlinear solution algorithm is based on a combined method. First, nonlinearities arising from the PDE (partial differential equation) are treated with a standard line-search assisted Newton scheme. The crack irreversibility is handled with a primal-dual active set method. The combination of both techniques yields one single nonlinear Newton iteration; see \([25]\) for further details.

**Problem 3** (Discretized pressurized phase-field fracture in mixed form).
Define \(u^n_h := u_h(t_n), p^n_h := p_h(t_n)\) and \(\varphi^n_h := \varphi_h(t_n)\) at the loading step \(t_n\). Let \(\tilde{\varphi}_h := \tilde{\varphi}_h(\varphi_{h-1}^n, \varphi_{h-2}^n)\) be the discrete linear-in-time extrapolation. Find \(U^n_h := (u^n_h, p^n_h, \varphi^n_h) \in V_h \times U_h \times K_h\) for all loading steps \(n = 1, 2, \ldots, N\) such that
\[
A(U^n_h)(\Psi_h - \Phi^n_h) = A_1(U^n_h)(w_h) + A_2(U^n_h)(q_h) + A_3(U^n_h)(\psi_h - \varphi^n_h) \geq 0
\]
with \( \Phi^n_h = (0, \varphi^n_h) \) and for all \( \Psi_h := (w_h, q_h, \psi_h) \in V_h \times U_h \times K_h \), and where

\[
\begin{align*}
A_1(U^n_h)(w_h) &= g(\tilde{\varphi}_h) \left( \sigma(u^n_h, p^n_h), E_{lin}(w_h) \right) + (\tilde{\varphi}_h \rho \nabla \cdot w_h), \\
A_2(U^n_h)(q_h) &= g(\tilde{\varphi}_h) \left( \nabla \cdot u^n_h, q_h \right) - \frac{1}{\lambda} (p^n_h, q_h), \\
A_3(U^n_h)(\psi_h - \varphi^n_h) &= (1 - \kappa) \left( \varphi^n_h \sigma(u^n_h, p^n_h) : E_{lin}(u^n_h), \psi_h - \varphi^n_h \right) \\
&\quad + 2(\varphi^n_h \rho \nabla \cdot u^n_h, \psi_h - \varphi^n_h) \\
&\quad + G_c \left( \frac{1}{\epsilon} (1 - \varphi^n_h), \psi_h - \varphi^n_h \right) + \epsilon \left( \nabla \varphi^n_h, \nabla (\psi_h - \varphi^n_h) \right),
\end{align*}
\]

where \( \sigma(u^n_h, p^n_h) := 2\mu E_{lin}(u^n_h) + p^n_h I \).

In order to treat the inequality constraint in \( K_h \), we employ a primal-dual active set method as explained in [25] and use the function space \( W_h \) for approximating \( \varphi \). Then, at each loading step \( n \), we have the following Newton iteration indexed by \( k \). We set as initial guess \( U^{n,0}_h := U^{n-1}_h \) and iterate for \( k = 1, 2, 3, \ldots \):

\[
\nabla A(U^n_h)(\delta U^{n,k}_h, \Psi) = -A(U^n_h)(\Psi) \quad \forall \Psi \in V_h \times U_h \times W_h.
\]

The directional derivative \( \nabla A(U^{n,k}_h)(\delta U^{n,k}_h, \Psi) \) in direction \( \delta U^{n,k}_h \) for \( U^{n,k}_h \in V_h \times U_h \times W_h \) is given by

\[
\nabla A(U^{n,k}_h)(\delta U^{n,k}_h, \Psi) = g(\tilde{\varphi}^{n,k}_h) \left( \sigma(\delta u^{n,k}_h, \delta p^{n,k}_h), E_{lin}(w_h) \right)
\]

\[
+ g(\tilde{\varphi}^{n,k}_h) \left( \nabla \cdot \delta u^{n,k}_h, q_h \right) - \frac{1}{\lambda} (\delta p^{n,k}_h, q_h)
\]

\[
+ (1 - \kappa) \left( \varphi^{n,k}_h 2\mu E_{lin}(\delta u^{n,k}_h) : E_{lin}(u^{n,k}_h) + E_{lin}(u^{n,k}_h) : E_{lin}(\delta u^{n,k}_h), \psi_h \right)
\]

\[
+ 2(\varphi^{n,k}_h \rho \nabla \cdot \delta u^{n,k}_h, \psi_h) + (1 - \kappa) (\varphi^{n,k}_h \rho \delta p^{n,k}_h I : E_{lin}(u^{n,k}_h), \psi_h)
\]

\[
+ (1 - \kappa) \left( \delta \varphi^{n,k}_h \sigma(u^{n,k}_h, p^{n,k}_h) : E_{lin}(u^{n,k}_h), \psi_h \right) + 2(\delta \varphi^{n,k}_h \rho \nabla \cdot u^{n,k}_h, \psi_h)
\]

\[
+ G_c \left( \frac{1}{\epsilon} \delta \varphi^{n,k}_h, \psi_h \right) + G_c \epsilon \left( \nabla \delta \varphi^{n,k}_h, \nabla \psi_h \right).
\]

### 3.2. Linear solution and Schur-type preconditioning

For the arising linear systems \( M \delta \vec{u} = F \) inside Newton’s method, a GMRES method is used, which is right-preconditioned [47] with a Schur-type preconditioner \( P^{-1} \). As usual, the goal when developing \( P^{-1} \) is to have the eigenvalues of \( (MP^{-1}) \) be independent of discretization, regularization parameters and coefficients of the problem.

#### 3.2.1. Preconditioning the \((3 \times 3)\) linear system

The system matrix \( M_{\text{mixed}} \) of the mixed phase-field fracture from the modified mixed problem formulation has the following block structure [24]:

\[
M_{\text{mixed}} = \\
\begin{pmatrix}
M^{uu} & M^{up} & M^{u\varphi} \\
M^{pu} & M^{pp} & M^{p\varphi} \\
M^{\varphi u} & M^{\varphi p} & M^{\varphi\varphi}
\end{pmatrix}
\begin{pmatrix}
\begin{pmatrix}
g(\tilde{\varphi})A_u & g(\tilde{\varphi})B^T & 0 \\
0 & S & 0 \\
0 & 0 & L
\end{pmatrix}
\end{pmatrix},
\]

where block \( A_u \) is the mass matrix of the displacements, \( B \) and \( B^T \) are symmetric off-diagonal blocks coupling \( u \) and \( p \), and \( M_p \) is the mass matrix of the pressure variable. The blocks \( E, F \) and \( L \) from Equation (1) consist of the entries from the phase-field equation, where \( L \) is Laplacian-like. For the entry-wise definition of the blocks, we refer to [39, page 169].

A typical block factorization of the system matrix yields the preconditioner (details can be found in [39, Chapter 6])

\[
P_{\text{mixed}} = \\
\begin{pmatrix}
g(\tilde{\varphi})A_u & g(\tilde{\varphi})B^T & 0 \\
0 & S & 0 \\
0 & 0 & L
\end{pmatrix},
\]

for \( M_{\text{mixed}} \), where \( S \) is the Schur complement block defined as

\[
S = -\frac{1}{\lambda} M_p - g(\tilde{\varphi})B^T \cdot [g(\tilde{\varphi})A_u]^{-1} \cdot g(\tilde{\varphi})B.
\]
It is not feasible to construct $S^{-1}$ or even $S$ exactly, as this would result in a dense matrix. This means that exact evaluation of $P^{-1}_{\text{mixed}}$ is also not a feasible option, but it helps us to design an appropriate preconditioner by approximating the action of $P^{-1}_{\text{mixed}}$ with an operator $\hat{P}_{\text{mixed}}^{-1}$ defined below. Note that all eigenvalues of $M_{\text{mixed}}\hat{P}_{\text{mixed}}^{-1}$ are equal to one and GMRES would converge in at most two iterations \[44\] [6].

Without considering the last row and column of $M_{\text{mixed}}$ and $P^{-1}_{\text{mixed}}$ (the phase-field), this is a typical saddle-point problem with a penalty term, where block triangular preconditioners are a common choice \[6\], first considered by Bramble and Pasciak in 1988 \[11\], and frequently used for Stokes-type problems \[15\] and the Oseen equations \[33\], and the Oseen equations \[33\], where mesh-independent convergence can be observed.

To be able to efficiently apply $P^{-1}_{\text{mixed}}$, we require approximations of the inverses of the Laplacian-like matrices $L$, of $g(\tilde{\varphi})A_u$, and of the Schur complement matrix $S$. With spectrally equivalent approximations, this would result in an optimal preconditioner \[6\] yielding an eigenvalue distribution independent of mesh size $h$ and other problem parameters and therefore constant GMRES iterations numbers independent of mesh size and problem parameters. Since multigrid methods allow for mesh-independent convergence \[23\], algebraic or geometric multigrid methods are the method of choice.

The approximation of the inverse of $S$ turns out to be more challenging. It is well-known, e.g., \[53\], for inf-sup stable discretizations of the linear elasticity problem, the Schur-complement is spectrally equivalent to the mass matrix. In our case, for $\lambda \to \infty$, $g(\tilde{\varphi})$ acts like a varying viscosity. It is common to scale the mass matrix with the inverse of the viscosity for Stokes interface problems \[45\] or variable viscosity Stokes problems, e.g., \[22\] \[11\], which yields

$$\hat{S}^{-1} := -\left(\frac{1}{\lambda} + \frac{g(\tilde{\varphi})}{2\mu} M_p\right)^{-1},$$

as an approximation of the inverse of $S$ in our situation. Under sufficient regularity, $\kappa > 0$ and if the coefficient can be assumed to be constant, $\hat{S}^{-1}$ is spectrally equivalent to $S^{-1}$ \[45\]. For the incompressible limit $\nu = 0.5$, the Schur complement approximation becomes

$$\hat{S}^{-1} = -\left(\frac{g(\tilde{\varphi})}{2\mu} M_p\right)^{-1}.$$

**Remark** (Differences to Stokes-type problems). Commonly, this Schur complement approach is used for Stokes-type problems and incompressible fluid dynamics, see, e.g. \[18\]. Even if the elasticity part of the considered phase-field fracture problem has a similar saddle-point structure, aside from the phase-field function, material and regularization parameters complicate the situation: $\lambda \to \infty$ leads to a purely $\kappa$-dependent block $\hat{S}^{-1}$, and $\kappa \to 0$ increases the condition number of the block $(g(\tilde{\varphi})A_u)^{-1}$ in the crack, where $\varphi = 0$. While the approximation of $\hat{S}^{-1}$ is spectrally equivalent with respect to the mesh size, it is not robust with respect to large viscosity variations, or in our case minimum and maximum value of $g(\tilde{\varphi})$ throughout the domain. For the Stokes interface problem with a viscosity jump with single interface, the scaled mass matrix is spectrally equivalent independent of the magnitude of the jump \[45\], which is the case in our situation. This will be visible in Section 4. We hypothesize that a better Schur complement could be a weighted BFRT preconditioner presented in \[40\], but a thorough investigation is future work.

### 3.2.2. Preconditioning algorithm.

As discussed above, the evaluation of the preconditioner

$$\hat{P}_{\text{mixed}}^{-1} = \begin{pmatrix}
(g(\tilde{\varphi})A_u)^{-1} & -A_u^{-1}B^T \hat{S}^{-1} & 0 \\
0 & \hat{S}^{-1} & 0 \\
0 & 0 & L^{-1}
\end{pmatrix},$$

requires efficient approximations to the exact inverses of $A_u$, $\hat{S}$, and $L$. Iterative solvers like GMRES of course only require the result of a matrix-vector product with the preconditioner $\hat{P}_{\text{mixed}}^{-1}$, see \[47\] and inside our basis software deal.II \[21\], see \[36\].

First, we approximate $L^{-1}$ by a single $V$-cycle of algebraic multigrid (AMG). Second, for $(g(\tilde{\varphi})A_u)^{-1}$ we use an inner Conjugate Gradient (CG) solve, which, in turn, is preconditioned by one $V$-cycle of algebraic multigrid. Finally, the action of $\hat{S}^{-1}$ is either done using a single $V$-cycle of AMG or, in Figures 4 and 8, using CG preconditioned by AMG.
With this, the matrix-vector product \( \hat{P}^{-1}_{\text{mixed}} \vec{x} \) with \( \vec{x} = (x_u, x_p, x_\varphi)^T \) given as

\[
\hat{P}^{-1}_{\text{mixed}} \vec{x} = \left( \begin{array}{c}
(g(\varphi)A_u)^{-1}x_u - (g(\varphi)A_u)^{-1}g(\varphi)BT\hat{S}^{-1}x_p \\
\hat{S}^{-1}x_p \\
L^{-1}x_\varphi
\end{array} \right),
\]

is built up step by step. In deal.II \cite{dealII}, the preconditioners given to solver classes need a `vmult()` member function \cite{dealII}. Then, our final algorithm is designed as follows:

**Algorithm 1.** Evaluation of \( \hat{P}^{-1}_{\text{mixed}} \vec{x} \):

1. Approximate \( \hat{S}^{-1} \) via AMG and compute \( q := \hat{S}^{-1}x_p \);
2. Compute \( r := x_u - g(\varphi)BTq \);
3. Approximate \( (g(\varphi)A_u)^{-1} \) via CG preconditioned with AMG and compute \( s := (g(\varphi)A_u)^{-1}r \);
4. Approximate \( L^{-1} \) via AMG and compute \( t := L^{-1}x_\varphi \);
5. Return the result \( (s, q, t)^T \).

### 4. Numerical tests

In this section, we consider four different numerical experiments to substantiate our algorithmic developments and to investigate the performance of the nonlinear solver, linear solver and preconditioner.

#### 4.1. Test cases and presentation of our results

To facilitate the readability of the tables from the next sections, we give an overview, how to read them. For the four tests, we conduct numerical studies with different emphases: we investigate robustness in \( h, \kappa, \lambda, \epsilon, \) we use different models (‘primal’ from Problem 1 versus ‘mixed’ from Problem 2) and different finite element discretizations. In the top row of each table, we summarize the key aspect of the current numerical study: the name of the example, the observed task, the modeling, and – if required – further test-specific settings. The white rows in the tables correspond to results based on the primal phase-field fracture model (solved with pfm-cracks \cite{pfm-cracks}) or to reference values. The colored rows belong to computations based on the mixed model and \( Q2_{\text{ref}}Q1_{\text{ref}} \) for \( \nu = 0.2 \) (yellow), \( \nu = 0.4999 \) (blue) and \( \nu = 0.5 \) (red). A more saturated shading denotes a finer mesh size.

The four test configurations with attributes are given in the following:

- **Section 4.3** a hanging block with an initial slit for \( \nu = 0.2, 0.4999 \) and 0.5, uniform mesh refinement, mixed \( (Q2_{\text{ref}}Q1_{\text{ref}}) \) versus primal \( (Q2_{\text{ref}}Q1_{\text{ref}}) \), \( \epsilon \) fixed and \( \epsilon = 2h, \kappa = 10^{-2} \);
- **Section 4.4** Sneddon’s test \cite{Sneddon} \cite{Sneddon} for \( \nu = 0.2, 0.4999 \) and 0.5, uniform mesh refinement, mixed \( (Q2_{\text{ref}}Q1_{\text{ref}}) \), \( \epsilon \) fixed and \( \epsilon = 2h, \kappa = 10^{-2}, 10^{-8} \);
- **Section 4.5** Sneddon’s test layered \cite{Sneddon} \cite{Sneddon} for \( \nu = 0.2, 0.4999 \) and 0.5 in the inner domain, adaptive mesh refinement (geometric), mixed \( (Q2_{\text{ref}}Q1_{\text{ref}}) \), \( \epsilon = h, \kappa = 10^{-2}, 10^{-8} \);
- **Section 4.6** single-edge notched tension test for \( \nu = 0.3, 0.45, \) and 0.49, adaptive mesh refinement (predictor-corrector scheme), mixed \( (Q2_{\text{ref}}Q1_{\text{ref}}) \), \( \epsilon = 4h, \kappa = 10^{-8} \);

With the help of numerical studies, we investigate the robustness of the new Schur-type preconditioner via evaluating the required number of linear iterations for different mesh sizes, Poisson ratios, \( \kappa \), and different finite element discretizations. Besides, we discuss challenges and point out difficulties.

#### 4.2. Implementation details

The software developed for this paper is a major extension built upon pfm-cracks \cite{pfm-cracks} \cite{pfm-cracks}, which is an open-source code available at \url{https://github.com/tjhei/cracks}. This project is built on the finite element library deal.II \cite{dealII}, which offers scalable parallel algorithms for finite element computations. The deal.II library in turn uses functionality from other libraries such as Trilinos \cite{Trilinos} \cite{Trilinos} for linear algebra, including the Trilinos ML AMG preconditioner \cite{Trilinos} \cite{Trilinos}. The GMRES stopping criterion is a relative tolerance of \( 10^{-5} \). CG uses a relative tolerance of \( 10^{-6} \) for the inner solves with a maximum of 200 iterations. The Newton iteration stops when an absolute tolerance of \( 10^{-7} \) is reached. We use four CPUs on a single machine with four Intel E7 v3 CPUs for all computations.

#### 4.3. Hanging block with initial slit

As a first test configuration, we consider a hanging block test with an initial geometrical slit of length 2.0 mm with an interpolated initial condition \( \varphi = 0 \) in the crack; see Figure 1. The force acting on the hanging block is reduced to \( f = -8.0 \cdot 10^{-7} \) N/mm \(^2\). In Figure 1 on the right, the solution of the phase-field function is given on the deformed block for \( \nu = 0.2 \) on a uniform refined
mesh with 41,924 degrees of freedom (DoFs). We evaluate the displacement in the $y$-direction in a certain point (0, 1.99) on the lower opening crack lip.

Table 1. A hanging block with an initial slit for $\nu = 0.2$ and $\nu = 0.4999$, uniform mesh refinement, mixed ($Q_2^cQ_1^cQ_1^c$) versus primal ($Q_2^cQ_1^c$), $\epsilon$ fixed, $\kappa = 10^{-2}$. Problem size in #DoFs, average number of GMRES iterations ($\varnothing$ln) per Newton/active set (AS) step, the average number of inner CG iterations ($\varnothing$CG) per linear iteration, number of Newton/AS (#AS), and goal functional displacement in a certain point ($u_y(0, 1.99)$). Applied force $f = -8 \cdot 10^{-7}$ N/mm$^2$.

| model       | FR | $\nu$  | $h$  | $\epsilon$ | #DoFs | $\varnothing$ln | $\varnothing$CG | #AS | $u_y(0, 1.99)$ |
|-------------|----|--------|------|-------------|-------|-----------------|-----------------|-----|---------------|
| mixed       | 25 | 0.2    | 0.353| 0.707       | 2,804 | 4               | 24              | 3   | -0.3871       |
| mixed       | 25 | 0.2    | 0.176| 0.707       | 10,724| 4               | 25              | 3   | -0.5199       |
| mixed       | 25 | 0.2    | 0.088| 0.707       | 41,924| 10              | 32              | 32  | -0.4919       |
| mixed       | 25 | 0.2    | 0.044| 0.707       | 165,764| 4               | 36              | 31  | -0.0825       |
| mixed       | 25 | 0.2    | 0.022| 0.707       | 659,204| 8               | 50              | 53  | -0.0824       |
| mixed       | 25 | 0.2    | 0.011| 0.707       | 2,629,124| 8            | 79              | 38  | -0.0815       |
| mixed       | 25 | 0.2    | 0.353| 0.707       | 2,804 | 1               | 3               | 3   | -0.3368       |
| mixed       | 25 | 0.2    | 0.176| 0.707       | 10,724| 1               | 3               | 3   | -0.4156       |
| mixed       | 25 | 0.2    | 0.088| 0.707       | 41,924| 5               | 5               | 5   | -0.4434       |
| mixed       | 25 | 0.2    | 0.044| 0.707       | 149,059| 5               | 5               | 5   | -0.0818       |
| mixed       | 25 | 0.2    | 0.022| 0.707       | 593,027| 7               | 7               | 35  | -0.0820       |
| mixed       | 25 | 0.2    | 0.011| 0.707       | 2,665,689| 8            | 8               | 35  | -0.0810       |
| primal      | 25 | 0.4999| 0.353| 0.707       | 2,804 | 10              | 24              | 3   | -0.2181       |
| primal      | 25 | 0.4999| 0.176| 0.707       | 10,724| 9               | 25              | 3   | -0.2869       |
| primal      | 25 | 0.4999| 0.088| 0.707       | 41,924| 6               | 32              | 29  | -0.1295       |
| primal      | 25 | 0.4999| 0.044| 0.707       | 165,764| 7               | 38              | 36  | -0.0576       |
| primal      | 25 | 0.4999| 0.022| 0.707       | 659,204| 10              | 52              | 38  | -0.0585       |
| primal      | 25 | 0.4999| 0.011| 0.707       | 2,629,124| 11           | 80              | 41  | -0.0578       |
| primal      | 25 | 0.4999| 0.353| 0.707       | 2,804 | 1               | 3               | 3   | -0.2077       |
| primal      | 25 | 0.4999| 0.176| 0.707       | 10,724| 1               | 3               | 3   | -0.2788       |
| primal      | 25 | 0.4999| 0.088| 0.707       | 41,924| 1               | 3               | 3   | -0.4084       |
| primal      | 25 | 0.4999| 0.044| 0.707       | 149,059| 9             | 3               | 31  | -0.0567       |
| primal      | 25 | 0.4999| 0.022| 0.707       | 593,027| 7             | 3               | 31  | -0.0564       |
| primal      | 25 | 0.4999| 0.011| 0.707       | 2,665,689| 8            | 8               | 34  | -0.5076       |

Tables 1 and 2 show the iteration numbers of numerical tests for the hanging block with a slit for three Poisson ratios $\nu$ and $h$ refinement. For the incompressible limit $\nu = 0.5$, Table 2 presents the results for $\epsilon$ fixed, and further in the pink rows, results for $\epsilon = 2h$ are listed. The nearly constant number of GMRES iterations confirms the robustness in $\epsilon$ for $\nu = 0.5$, tested for the hanging block with a slit on five levels of uniform refined meshes; see the last five rows in Table 2.
Table 2. A hanging block with an initial slit for \( \nu = 0.5 \), uniform mesh refinement, \( Q_2^pQ_1^pQ_1^p \) elements, \( \epsilon \) fixed and \( \epsilon = 2h, \kappa = 10^{-2} \). Problem size in \#DoFs, average number of GMRES iterations (\( \#CG \)) per Newton/active set (AS) step, the average number of inner CG iterations (\( \#CG \)) per linear iteration, number of GMRES/AS (\( \#AS \)), and goal functional displacement in a certain point (\( u_\theta(0,1.99) \)). Applied force \( f = -8 \cdot 10^{-7} \text{N/mm}^2 \).

| model     | \( \nu \)  | \( h \)  | \( \epsilon \) | \#DoFs | \#CG | \#AS | \( u_\theta(0,1.99) \) |
|-----------|------------|------------|----------------|--------|------|------|----------------------|
| mixed \( Q_2^pQ_1^pQ_1^p \) | 0.5 | 0.353 | 0.707 | 2804 | 9 | 23 | 3 | -0.0578 |
| mixed \( Q_2^pQ_1^pQ_1^p \) | 0.5 | 0.176 | 0.707 | 10724 | 9 | 24 | 3 | -0.2835 |
| mixed \( Q_2^pQ_1^pQ_1^p \) | 0.5 | 0.088 | 0.707 | 41924 | 7 | 32 | 33 | -0.0955 |
| mixed \( Q_2^pQ_1^pQ_1^p \) | 0.5 | 0.044 | 0.707 | 165764 | 6 | 37 | 38 | -0.0584 |
| mixed \( Q_2^pQ_1^pQ_1^p \) | 0.5 | 0.022 | 0.707 | 658436 | 9 | 53 | 36 | -0.0583 |
| mixed \( Q_2^pQ_1^pQ_1^p \) | 0.5 | 0.011 | 0.707 | 2429124 | 11 | 80 | 39 | -0.0579 |
| mixed \( Q_2^pQ_1^pQ_1^p \) | 0.5 | 0.353 | 0.707 | 2804 | 9 | 23 | 3 | -0.2166 |
| mixed \( Q_2^pQ_1^pQ_1^p \) | 0.5 | 0.176 | 0.353 | 10724 | 7 | 25 | 4 | -0.1033 |
| mixed \( Q_2^pQ_1^pQ_1^p \) | 0.5 | 0.088 | 0.176 | 41924 | 6 | 30 | 14 | -0.0701 |
| mixed \( Q_2^pQ_1^pQ_1^p \) | 0.5 | 0.044 | 0.088 | 165764 | 5 | 36 | 109 | -0.0572 |
| mixed \( Q_2^pQ_1^pQ_1^p \) | 0.5 | 0.022 | 0.044 | 658436 | 7 | 40 | 805 | -0.0516 |

**Remark** (High iteration numbers in the primal-dual active set method). In Table 2 in the pink rows, many active set/Newton iterations are required for \( \epsilon \to 0 \). Here, not the Poisson ratio is responsible, but the refinement in \( h \) and \( \epsilon \). For finer meshes with small \( \epsilon \), the active set algorithm oscillates between a certain non-equal number of active nodes from the constraint. This effect leads to high total Newton iterations, even if the Newton algorithm converges fast; see also [25, Figure 14].

The number of CG iterations does not depend significantly on the size of \( \kappa \) for this test setup. Further, the required CG iterations seem to be independent of \( \lambda \) but sensitive to the mesh size. Aside from the robustness in \( h \) and \( \lambda \), we confirm the robustness in \( \kappa \) for the hanging block test with a slit. Details on that can be found in [39] page 105).

4.4. **Sneddon’s pressure-driven cavity.** As a second example, we consider a benchmark test [18], which is motivated by the book of Sneddon [50] and Sneddon and Lowengrub [49]. We restrict ourselves to a 1d fracture \( C \) on a 2d domain \( \Omega = (-10,10)^2 \) as depicted on the left in Figure 2. In this domain, an initial crack with length \( 2l_0 = 2.0 \) and thickness \( h \) of two cells is prescribed with the help of the phase-field function \( \varphi \), i.e., \( \varphi = 0 \) in the crack and \( \varphi = 1 \) elsewhere. As boundary conditions, the displacements \( u \) are set to zero on \( \partial \Omega \). We use homogeneous Neumann conditions for the phase-field variable, i.e., \( \epsilon \partial_n \varphi = 0 \) on \( \partial \Omega \). The driving force is given by a constant pressure \( p = 10^{-3} \text{Pa} \) in the interior of the crack. An overview of the parameter setting is given in Figure 2 on the right.

![Figure 2](image_url)
Two quantities of interest are discussed: the crack opening displacement (COD) and the total crack volume (TCV). The analytical solution (from [49]) can be computed via

$$\text{COD}_{\text{ref}} = \frac{2 \pi p_0^2}{E'} \left(1 - \frac{x^2}{t_0^2}\right)^{1/2},$$

where $E' := \frac{E}{1 - \nu^2}$, $E$ is the Young modulus and $\nu$ is the Poisson ratio. The TCV can be computed numerically with

$$\text{TCV} = \int_{\Omega} u(x, y) \cdot \nabla \varphi(x, y) d(x, y).$$

The analytical solution (from [49]) is given by

$$\text{TCV}_{\text{ref}} = \frac{2 \pi p_0^2}{E'}. $$

In Table 3, for $\kappa = 10^{-8}$, the average number of CG iterations increases with a decreasing mesh size. We observe an increase in the CG iteration numbers in particular for the incompressible limit $\nu = 0.5$ and finer meshes, where we finally do not get convergence in the solver for smaller $h$. Already for $\nu = 0.4999$ and a problem size of less than 300 000 DoFs, the average number of CG iterations is above 100.

**Remark (Difficulties considering small $\kappa$).** In Table 3, compared to Table 3, we can evaluate the impact of the setting of $\kappa$. We compute Sneddon’s test for different mesh sizes $h$, fixed bandwidth $\epsilon$, for three Poisson ratios $\nu = 0.2, 0.4999, \text{ and } \nu = 0.5$, and for a small and large regularization parameter $\kappa = 10^{-2} \text{ and } \kappa = 10^{-8}$ to evaluate its impact on the behavior of the CG solver. These solver dependencies on $\kappa$ have a natural correspondence in error estimates. For a decoupled linearized system, such estimates are shown in [54], Section 5.5). A numerical error analysis for this test on a good choice of $\kappa$ can be found in [54].

Further, we observe an increased number of CG iterations for high Poisson ratios. The number of GMRES iterations in particular for the incompressible limit $\nu = 0.5$ increases with a decreasing mesh size. We can evaluate the impact of $\kappa$ on the behavior of the CG solver. These solver dependencies on $\kappa$ have a natural correspondence in error estimates. For a decoupled linearized system, such estimates are shown in [54], Section 5.5). A numerical error analysis for this test on a good choice of $\kappa$ can be found in [54].

**Table 3.** Sneddon’s pressure-driven cavity in 2d. Average number of GMRES iterations ($\#\text{lin}$) per Newton step ($\#\text{AS}$), the average number of CG iterations ($\#\text{CG}$) per linear iteration. Based on the newly developed mixed model with $Q_2^2Q_1^3Q_1^1$ elements for different problem sizes and setting of the length scale parameter $\epsilon$ for three Poisson ratios. Quantities of interest: COD$_{\text{max}}$ and TCV and $\kappa = 10^{-8}$. Uniform refined meshes.

| FE  | $\nu$  | $h$  | $\epsilon$ | $\#\text{DoFs}$ | $\#\text{lin}$ | $\#\text{AS}$ | $\text{COD}_{\text{max}}$ | TCV |
|-----|-------|------|------------|-----------------|----------------|----------------|--------------------------|-----|
| $Q_2^2Q_1^3Q_1^1$ | 0.2 | 0.707 | 1.414 | 16 484 | 3 | 26 | 4 | 0.00282 | 0.0240 |
| $Q_2^2Q_1^3Q_1^1$ | 0.2 | 0.353 | 1.414 | 64 964 | 6 | 28 | 6 | 0.00270 | 0.0189 |
| $Q_2^2Q_1^3Q_1^1$ | 0.2 | 0.176 | 1.414 | 257 924 | 9 | 55 | 4 | 0.00266 | 0.0184 |
| $Q_2^2Q_1^3Q_1^1$ | 0.2 | 0.088 | 1.414 | 1027 844 | 12 | 31 | 5 | 0.00252 | 0.0150 |
| ref. [49] | 0.2 | | | | | | | | 0.0019200 | 0.00663 |
| $Q_2^2Q_1^3Q_1^1$ | 0.4999 | 0.707 | 1.414 | 16 484 | 3 | 31 | 6 | 3.0383e-05 | 0.000257 |
| $Q_2^2Q_1^3Q_1^1$ | 0.4999 | 0.353 | 1.414 | 64 964 | 7 | 46 | 8 | 3.6024e-05 | 0.000254 |
| $Q_2^2Q_1^3Q_1^1$ | 0.4999 | 0.176 | 1.414 | 257 924 | 10 | 107 | 39 | 3.9899e-05 | 0.000252 |
| $Q_2^2Q_1^3Q_1^1$ | 0.4999 | 0.088 | 1.414 | 1027 844 | 14 | 57 | 52 | 4.2265e-05 | 0.000250 |
| ref. [49] | 0.4999 | | | | | | | | 0.0015500 | 0.004713 |
| $Q_2^2Q_1^3Q_1^1$ | 0.5 | 0.707 | 1.414 | 16 484 | 3 | 31 | 3 | 2.9937e-20 | 7.1504e-20 |
| $Q_2^2Q_1^3Q_1^1$ | 0.5 | 0.353 | 1.414 | 64 964 | 6 | 25 | 2 | 3.128e-19 | 2.3835e-19 |
| $Q_2^2Q_1^3Q_1^1$ | 0.5 | 0.176 | 1.414 | 257 924 | 5 | 59 | 7 | 3.9309e-19 | 7.8081e-19 |
| $Q_2^2Q_1^3Q_1^1$ | 0.5 | 0.088 | 1.414 | 1027 844 | 11 | 37 | 3 | 2.4585e-14 | 1.2562e-14 |
| $Q_2^2Q_1^3Q_1^1$ | 0.5 | 0.707 | 1.414 | 16 484 | 11 | 37 | 3 | 2.4585e-14 | 1.2562e-14 |
| $Q_2^2Q_1^3Q_1^1$ | 0.5 | 0.353 | 0.707 | 64 964 | 6 | 32 | 3 | 2.3632e-18 | 1.0069e-17 |
| $Q_2^2Q_1^3Q_1^1$ | 0.5 | 0.176 | 0.353 | 257 924 | 10 | 30 | 3 | 6.5953e-18 | 1.4749e-16 |
| $Q_2^2Q_1^3Q_1^1$ | 0.5 | 0.088 | 0.176 | 1027 844 | 14 | 38 | 3 | 1.2397e-18 | 2.6778e-18 |
| ref. [49] | 0.5 | | | | | | | | 0.0015500 | 0.0047124 |

This observation is confirmed by the numerical results from Table 4, where a CG solver preconditioned with AMG is used to approximate $S^{-1}$. The numerical results in Table 4 are based on the same tests as in
Table 4. Sneddon’s pressure-driven cavity in 2d. Average number of GMRES iterations (#lin) per Newton step (#AS), the average number of CG iterations (\#CG) per linear iteration, CG plus AMG is used for \((g(\tilde{\varphi})A_u)^{-1}\) and \(S^{-1}\). Based on the newly developed mixed model with \(Q_2^cQ_1^cQ_1^c\) elements for different problem sizes and setting of the length scale parameter \(\epsilon\) for two Poisson ratios. Uniform refined meshes.

| FE | \(\nu\) | \(h\) | \(c\) | \#DoFs | \#lin | \#CG \((g(\tilde{\varphi})A_u)^{-1}\) | \#CG \(S^{-1}\) | #AS |
|----|----|----|----|--------|--------|-----------------|-----------------|----|
| \(Q_2^cQ_1^c\) | 0.4999 | 0.707 | 1.414 | 16484 | 3 | 26 | 4 | 3 |
| \(Q_2^cQ_1^c\) | 0.4999 | 0.353 | 1.414 | 64964 | 8 | 56 | 6 | 8 |
| \(Q_2^cQ_1^c\) | 0.4999 | 0.176 | 1.414 | 257924 | 6 | 106 | 6 | 38 |
| \(Q_2^cQ_1^c\) | 0.4999 | 0.088 | 1.414 | 1027844 | 6 | 42 | 6 | 69 |
| \(Q_2^cQ_1^c\) | 0.5 | 0.707 | 1.414 | 16484 | 10 | 36 | 1 | 3 |
| \(Q_2^cQ_1^c\) | 0.5 | 0.353 | 1.414 | 64964 | 6 | 26 | 6 | 8 |
| \(Q_2^cQ_1^c\) | 0.5 | 0.176 | 1.414 | 257924 | 6 | 63 | 6 | 37 |
| \(Q_2^cQ_1^c\) | 0.5 | 0.088 | 1.414 | 1027844 | 7 | 41 | 6 | 101 |

Table 3 but for \(\nu = 0.4999\) and \(\nu = 0.5\). The number of linear iterations is moderate, and at most six CG iterations are needed for \(S^{-1}\).

As expected in Tables 3, 4 and 5, considering the quantities of interest \(\text{COD}_{\text{max}}\) and TCV, they get vanishingly small for high Poisson ratios. This is what we expected for incompressible solids: a closed domain does not change its volume; the opening of the initial crack in the interior of the domain is avoided. For \(\nu = 0.2\), the quantities of interest are acceptable compared to the reference values. Also for \(\nu = 0.2\), since all computations are conducted with uniformly refined meshes, moderate problem sizes, and fixed \(\epsilon\), we cannot expect excellent results in the quantities of interest.

Table 5. Sneddon’s pressure-driven cavity. Average number of GMRES iterations (#lin) per Newton step (#AS), average number of CG iterations (#CG) per linear iteration. Based on the newly developed mixed model with \(Q_2^cQ_1^cQ_1^c\) elements for different problem sizes and setting of the length scale parameter \(\epsilon\) for three Poisson ratios. Quantities of interest: \(\text{COD}_{\text{max}}\) and TCV and \(\kappa = 10^{-2}\). Uniform refined meshes.

| FE | \(\nu\) | \(h\) | \(c\) | \#DoFs | \#lin | \#CG | \#AS | \(\text{COD}_{\text{max}}\) | TCV |
|----|----|----|----|--------|--------|------|------|-----------------|------|
| \(Q_2^cQ_1^c\) | 0.2 | 0.707 | 1.414 | 16484 | 2 | 16 | 4 | 0.002248 | 0.0224 |
| \(Q_2^cQ_1^c\) | 0.2 | 0.353 | 1.414 | 64964 | 8 | 18 | 4 | 0.00227 | 0.0173 |
| \(Q_2^cQ_1^c\) | 0.2 | 0.176 | 1.414 | 257924 | 9 | 15 | 5 | 0.00206 | 0.0145 |
| \(Q_2^cQ_1^c\) | 0.2 | 0.088 | 1.414 | 1027844 | 15 | 28 | 5 | 0.00190 | 0.0129 |
| \(Q_2^cQ_1^c\) ref. [49] | 0.2 | | | | 0.0019200 | 0.0060 |
| \(Q_2^cQ_1^c\) | 0.4999 | 0.707 | 1.414 | 16484 | 13 | 16 | 3 | 3.0853e-05 | 0.000269 |
| \(Q_2^cQ_1^c\) | 0.4999 | 0.353 | 1.414 | 64964 | 8 | 18 | 14 | 3.1739e-05 | 0.000242 |
| \(Q_2^cQ_1^c\) | 0.4999 | 0.176 | 1.414 | 257924 | 6 | 18 | 93 | 3.3667e-05 | 0.000224 |
| \(Q_2^cQ_1^c\) | 0.4999 | 0.088 | 1.414 | 1027844 | 7 | 29 | 65 | 3.4596e-05 | 0.000216 |
| \(Q_2^cQ_1^c\) ref. [49] | 0.4999 | | | | 0.0015601 | 0.004713 |
| \(Q_2^cQ_1^c\) | 0.5 | 0.707 | 1.414 | 16484 | 9 | 14 | 3 | 1.7339e-19 | 5.8895e-19 |
| \(Q_2^cQ_1^c\) | 0.5 | 0.353 | 1.414 | 64964 | 9 | 18 | 14 | 2.3734e-19 | 5.6268e-18 |
| \(Q_2^cQ_1^c\) | 0.5 | 0.176 | 1.414 | 257924 | 11 | 15 | 14 | 5.6547e-20 | 6.0926e-18 |
| \(Q_2^cQ_1^c\) | 0.5 | 0.088 | 1.414 | 1027844 | 5 | 26 | 39 | 7.7351e-19 | 2.2733e-17 |
| \(Q_2^cQ_1^c\) | 0.5 | 0.707 | 1.414 | 16484 | 9 | 14 | 3 | 1.5881e-19 | 5.8895e-19 |
| \(Q_2^cQ_1^c\) | 0.5 | 0.353 | 0.707 | 64964 | 6 | 17 | 3 | 1.9290e-19 | 1.8057e-18 |
| \(Q_2^cQ_1^c\) | 0.5 | 0.176 | 0.353 | 257924 | 6 | 18 | 3 | 4.1847e-19 | 2.1156e-18 |
| \(Q_2^cQ_1^c\) | 0.5 | 0.088 | 0.176 | 1027844 | 10 | 26 | 3 | 2.4801e-18 | 9.5514e-18 |
| \(Q_2^cQ_1^c\) ref. [49] | 0.5 | | | | 0.0015600 | 0.0047124 |

In Table 5 the same computations are conducted as in Table 3 and Table 4 for \(\kappa = 10^{-2}\) to discuss the statement of Remark 4.4. The COD values are close to the reference values. Here, a large regularization parameter \(\kappa = 10^{-2}\) stabilizes the block \((g(\tilde{\varphi})A_u)^{-1}\). Further, the linear iterations are stable, and also the inner CG iterations are relatively constant. In the last four rows of Table 5 similar to Table 2 results of four tests with \(\epsilon = 2h\) are listed to check the robustness in \(\epsilon\) for \(\nu = 0.5\), which can be confirmed for Sneddon’s benchmark test.
4.5. **Sneddon’s pressure-driven cavity, layered.** As a fourth test case, the pressure-driven cavity from [48] is modified similarly to [5]. We consider a two-dimensional domain $\Omega = (-20, 20)^2$. In contrast to the previous Sneddon test, a compressible layer of size 10 is added around the incompressible domain to allow deforming of the solid on a finite domain. So the Poisson ratio changes over the domain for the layered Sneddon test. We expect to get better results concerning COD and TCV on a finite domain compared to the reference values on an infinite domain. A sketch of the geometry is given in Figure 3 on the left. The setting of the material and numerical parameters is the same as in the previous section.

![Figure 3](image)

**Figure 3.** Left: Geometry of the two-dimensional Sneddon’s test with a compressible layer of size 10. Further, the inside of the initial crack is assumed to be compressible [5]. Right: Close zoom-in to the geometrically refined mesh around the crack, used in Table 7.

In Figure 3 on the right, a zoom-in snapshot of the inner domain is given to show the geometric refinement for the tests in Tables 6, 7, and 8. Aside from the adaptively refined mesh, we set $\epsilon = h$, depending on the current mesh size. The total numbers of degrees of freedom (#DoFs) on $\Omega$ are listed in the numerical results in Tables 6 to 8.

In Table 6, the results for the Sneddon test in 2d with a compressible layer around a possibly incompressible domain are given for three Poisson ratios and adaptively refined meshes, with $\epsilon = h$, and $\kappa = 10^{-2}$. We choose $\kappa = 10^{-2}$ to avoid the effects of $\kappa$ on the inner CG iterations. For large $\kappa$, the computed quantities of interest $\text{COD}_{\text{max}}$ and TCV do not converge to the correct physics ($\kappa \approx 0$), however they still converge, but to values corresponding to large $\kappa$ material’s physics. In Table 6, the numbers of GMRES iterations are moderate for $\nu = 0.2$. For higher Poisson ratios, we observe high linear iteration numbers. The incompressibility and the mesh adaptivity seem to significantly impact the linear solver. We observe the same effects for $\kappa = 10^{-8}$ in Table 7.

In Table 7, the numerical results of the same tests are given as in Table 6 for $\kappa = 10^{-8}$. Analogously to Table 6, Table 8 contains the numerical results for the Sneddon test layered for high Poisson ratios and small $\kappa$. In contrast to Table 7, we approximate $\hat{S}^{-1}$ with a CG solver which is preconditioned with AMG.

The results of $\text{COD}_{\text{max}}$ and TCV in Tables 7 and 8 look promising for all three Poisson ratios. For $\nu = 0.5$, the solver does not converge with sufficiently small $\kappa$ and $h \to 0$. An explanation is given in Remark 4.4 (Section 4.4). In Table 8 for high Poisson ratios, the modified approximation of $\hat{S}^{-1}$ changes the behavior of the linear solver. With a relative tolerance of $10^{-6}$ for the preconditioned CG solver for $(g(\tilde{\varphi})A_u)^{-1}$ and $\hat{S}^{-1}$, we observe that more GMRES iterations are required. The number of linear iterations is relatively high, but nearly constant for $\nu = 0.4999$ and $\nu = 0.5$. The number of linear iterations increases for higher Poisson ratios with adaptive refined meshes and $\epsilon = h$. The results of $\text{COD}_{\text{max}}$ and TCV match the manufactured reference values.

In Figure 4, the solutions of $u_x$, $u_y$, $p$, and $\varphi$ are presented as zoom-in snapshots for $\nu = 0.5$ with a compressible layer, based on Table 8. Especially the pressure field (upper left snapshot) is expected to have zero values in the interior of the crack and the maximal values in the crack tip on the left and the right of the pre-defined initial crack. Further, in Figure 4, the mesh on the finest refinement level is given on the bottom left. On the bottom right, the crack zone is shown, on which the computed solutions are presented above to get an impression of the mesh size around the fracture.
Table 6. Sneddon’s pressure-driven cavity layered. Average number of GMRES iterations (∇lin) per Newton step (#AS). Computations based on the newly developed mixed model with $Q_2^h Q_1^2$ elements for different problem size, $\epsilon = h$ for three Poisson ratios. Quantities of interest: $\text{COD}_{\text{max}}$ and TCV and $\kappa = 10^{-2}$. Geometrically refined mesh in the area around the crack zone as depicted in Figure 3.

| FE | $\nu$ | $h$ | #DoFs | $\#\text{lin}$ | #AS | $\text{COD}_{\text{max}}$ | TCV  |
|----|------|-----|-------|---------------|-----|-----------------|------|
| $Q_2^h Q_1^2$ | 0.2 | 0.353 | 257 924 | 18 | 3 | 0.00214077 | 0.0097207 |
| $Q_2^h Q_1^2$ | 0.2 | 0.176 | 263 604 | 25 | 4 | 0.00188194 | 0.0069353 |
| $Q_2^h Q_1^2$ | 0.2 | 0.088 | 282 484 | 20 | 3 | 0.00163459 | 0.0055415 |
| $Q_2^h Q_1^2$ | 0.2 | 0.044 | 350 804 | 17 | 3 | 0.00136002 | 0.0044906 |
| $Q_2^h Q_1^2$ | 0.2 | 0.022 | 610 164 | 19 | 4 | 0.00104379 | 0.0034504 |
| $Q_2^h Q_1^2$ | 0.2 | 0.011 | 1 620 244 | 24 | 6 | 0.00071731 | 0.0024168 |
| ref. [49] | 0.2 | 0.0055 | 6 506 324 | 28 | 6 | 0.00439863 | 0.0015294 |

| FE | $\nu$ | $h$ | #DoFs | $\#\text{lin}$ | #AS | $\text{COD}_{\text{max}}$ | TCV  |
|----|------|-----|-------|---------------|-----|-----------------|------|
| $Q_2^h Q_1^2$ | 0.4999 | 0.353 | 257 924 | 40 | 2 | 0.00205349 | 0.0108334 |
| $Q_2^h Q_1^2$ | 0.4999 | 0.176 | 263 604 | 52 | 2 | 0.00168136 | 0.0069892 |
| $Q_2^h Q_1^2$ | 0.4999 | 0.088 | 282 484 | 58 | 3 | 0.00143863 | 0.0052347 |
| $Q_2^h Q_1^2$ | 0.4999 | 0.044 | 350 804 | 58 | 3 | 0.00122931 | 0.0041947 |
| $Q_2^h Q_1^2$ | 0.4999 | 0.022 | 610 164 | 62 | 4 | 0.00099910 | 0.0033284 |
| $Q_2^h Q_1^2$ | 0.4999 | 0.011 | 1 620 244 | 150 | 4 | 0.00073784 | 0.0022468 |
| ref. [49] | 0.4999 | 0.0055 | 6 506 324 | 318 | 7 | 0.00048045 | 0.0016595 |

Table 7. Sneddon’s pressure-driven cavity layered. Average number of GMRES iterations (∇lin) per Newton step (#AS). Computations with $Q_2^h Q_1^2$ elements for different problem size, $\epsilon = h$ for three Poisson ratios. Quantities of interest: $\text{COD}_{\text{max}}$ and TCV and $\kappa = 10^{-8}$. Geometrically refined mesh as depicted on the right in Figure 3.

| FE($\nu$, p, w) | $\nu$ | $h$ | #DoFs | $\#\text{lin}$ | #AS | $\text{COD}_{\text{max}}$ | TCV  |
|----------------|------|-----|-------|---------------|-----|-----------------|------|
| $Q_2^h Q_1^2$ | 0.2 | 0.353 | 257 924 | 10 | 3 | 0.00242526 | 0.0107193 |
| $Q_2^h Q_1^2$ | 0.2 | 0.176 | 263 604 | 20 | 3 | 0.00221789 | 0.0080140 |
| $Q_2^h Q_1^2$ | 0.2 | 0.088 | 282 484 | 18 | 3 | 0.00208683 | 0.0090646 |
| $Q_2^h Q_1^2$ | 0.2 | 0.044 | 350 804 | 29 | 6 | 0.00200814 | 0.0064862 |
| $Q_2^h Q_1^2$ | 0.2 | 0.022 | 610 164 | 26 | 4 | 0.00196329 | 0.0062530 |
| $Q_2^h Q_1^2$ | 0.2 | 0.011 | 1 620 244 | 32 | 3 | 0.00193890 | 0.0061344 |
| ref. [49] | 0.2 | 0.0055 | 6 506 324 | 40 | 3 | 0.00192609 | 0.0060733 |

| FE($\nu$, p, w) | $\nu$ | $h$ | #DoFs | $\#\text{lin}$ | #AS | $\text{COD}_{\text{max}}$ | TCV  |
|----------------|------|-----|-------|---------------|-----|-----------------|------|
| $Q_2^h Q_1^2$ | 0.4999 | 0.353 | 257 924 | 40 | 2 | 0.00223391 | 0.0116929 |
| $Q_2^h Q_1^2$ | 0.4999 | 0.176 | 263 604 | 73 | 5 | 0.00187338 | 0.0077187 |
| $Q_2^h Q_1^2$ | 0.4999 | 0.088 | 282 484 | 229 | 4 | 0.00168693 | 0.0060788 |
| $Q_2^h Q_1^2$ | 0.4999 | 0.044 | 350 804 | 511 | 4 | 0.00159278 | 0.0053537 |
| $Q_2^h Q_1^2$ | 0.4999 | 0.022 | 610 164 | 601 | 6 | 0.00154436 | 0.0050158 |
| $Q_2^h Q_1^2$ | 0.4999 | 0.011 | 1 620 244 | 655 | 5 | 0.00151941 | 0.0048527 |
| ref. [49] | 0.4999 | 0.0055 | 6 506 324 | 641 | 5 | 0.00150668 | 0.0047248 |

| FE($\nu$, p, w) | $\nu$ | $h$ | #DoFs | $\#\text{lin}$ | #AS | $\text{COD}_{\text{max}}$ | TCV  |
|----------------|------|-----|-------|---------------|-----|-----------------|------|
| $Q_2^h Q_1^2$ | 0.5 | 0.353 | 257 924 | 40 | 2 | 0.00223391 | 0.0116929 |
| $Q_2^h Q_1^2$ | 0.5 | 0.176 | 263 604 | 73 | 5 | 0.00187338 | 0.0077187 |
| $Q_2^h Q_1^2$ | 0.5 | 0.088 | 282 484 | 227 | 4 | 0.00168693 | 0.0060782 |
| ref. [49] | 0.5 | 0.0055 | 6 506 324 | 641 | 5 | 0.00150668 | 0.0047248 |
Table 8. Sneddon’s pressure-driven cavity layered with $Q_2^c Q_1^c Q_1^c$ elements and $\epsilon = h$. Average number of GMRES iterations ($\dot{\text{lin}}$) per Newton step ($\dot{\text{AS}}$). CG plus AMG is used for $(g(\hat{\varphi}) A_h)^{-1}$ and $\hat{S}^{-1}$; the average number of CG iterations for $(g(\hat{\varphi}) A_h)^{-1}$ is 38 for $\nu = 0.4999$ and 36 for $\nu = 0.5$. The average number of CG iterations for $\hat{S}^{-1}$ is 8 for $\nu = 0.4999$ and 7 for $\nu = 0.5$. Computations for different problem size, $\epsilon = h$ for three Poisson ratios. Quantities of interest: COD$_{\text{max}}$ and TCV and $\kappa = 10^{-8}$. Geometrically refined mesh as depicted on the right in Figure 2.

| FE$(u, p, \varphi)$ | $\nu$ | $h$ | #DoFs | $\dot{\text{lin}}$ | #AS | COD$_{\text{max}}$ | TCV |
|---------------------|-------|-----|-------|-----------------|-----|----------------|------|
| $Q_2^c Q_1^c Q_1^c$ | 0.4999 | 0.353 | 257,924 | 40 | 2 | 0.00223914 | 0.0116630 |
| $Q_2^c Q_1^c Q_1^c$ | 0.4999 | 0.176 | 263,604 | 70 | 4 | 0.00187365 | 0.0077192 |
| $Q_2^c Q_1^c Q_1^c$ | 0.4999 | 0.088 | 282,484 | 165 | 4 | 0.00168093 | 0.0060788 |
| $Q_2^c Q_1^c Q_1^c$ | 0.4999 | 0.044 | 350,804 | 153 | 4 | 0.00159278 | 0.0053537 |
| $Q_2^c Q_1^c Q_1^c$ | 0.4999 | 0.022 | 610,164 | 145 | 5 | 0.00154446 | 0.0050158 |
| $Q_2^c Q_1^c Q_1^c$ | 0.4999 | 0.011 | 1,620,244 | 139 | 5 | 0.00151941 | 0.0048527 |
| $Q_2^c Q_1^c Q_1^c$ | 0.4999 | 0.0055 | 5,606,324 | 148 | 5 | 0.00150668 | 0.0047724 |
| ref. [25] | 0.4999 | | | | | 0.00150019 | 0.0047130 |
| $Q_2^c Q_1^c Q_1^c$ | 0.5 | 0.353 | 257,924 | 40 | 2 | 0.00223891 | 0.0116629 |
| $Q_2^c Q_1^c Q_1^c$ | 0.5 | 0.176 | 263,604 | 70 | 4 | 0.00187338 | 0.0077187 |
| $Q_2^c Q_1^c Q_1^c$ | 0.5 | 0.088 | 282,484 | 165 | 4 | 0.00168093 | 0.0060788 |
| $Q_2^c Q_1^c Q_1^c$ | 0.5 | 0.044 | 350,804 | 153 | 4 | 0.00159254 | 0.0053528 |
| $Q_2^c Q_1^c Q_1^c$ | 0.5 | 0.022 | 610,164 | 145 | 5 | 0.00154414 | 0.0050151 |
| $Q_2^c Q_1^c Q_1^c$ | 0.5 | 0.011 | 1,620,244 | 139 | 5 | 0.00151920 | 0.0048521 |
| $Q_2^c Q_1^c Q_1^c$ | 0.5 | 0.0055 | 5,606,324 | 148 | 5 | 0.00150648 | 0.0047718 |
| ref. [25] | 0.5 | | | | | 0.00150000 | 0.0047124 |

4.6. Single edge notched pure tension test. As the last example, we use the single-edge notched tension test from Miehe et al. [42] testing with three Poisson ratios. We use the predictor-corrector scheme from Heister et al. [25] for two steps of adaptive mesh refinement on four times uniformly refined mesh with a phase-field threshold of 0.5. The parameter setting is the same as in [42] but we use the mixed problem formulation and discretization from Section 2 and vary the Poisson ratio; see Table 9.

Table 9. Parameter setting for three tests with different Poisson’s ratios for the single-edge notched tension test with $\kappa = 10^{-8}$, and $\epsilon = 4 \cdot h$. The maximal number of DoFs is given in the last column for the test cases. For all tests, four uniform ($h = 0.011$) and two adaptive refinement steps are conducted with a phase-field threshold of 0.5 for predictor-corrector.

| $\nu$ | $\mu$ | $\lambda$ | #DoFs |
|-------|-------|-------|-------|
| 0.3   | $80,77 \cdot 10^3$ | 121.15 $\cdot 10^3$ | 19,584 |
| 0.45  | $80,77 \cdot 10^3$ | 726.93 $\cdot 10^3$ | 19,704 |
| 0.49  | $80,77 \cdot 10^3$ | 3957.73 $\cdot 10^3$ | 19,498 |

We consider the bulk and crack energy as two further numerical quantities of interest. The bulk energy $E_B$ can be computed via

$$ E_B(u, \varphi) = \int_{\Omega} (g(\hat{\varphi}) \psi(E_{\text{lin}}(u))) \, d(x, y), $$

where the strain energy functional is defined as

$$ \psi(E_{\text{lin}}(u)) := \mu \text{tr} (E_{\text{lin}}(u)^2) + \frac{1}{2} \lambda \text{tr} (E_{\text{lin}}(u))^2. $$

Here, no manufactured reference values are provided and we only present values computed numerically. Further, we compute the crack energy $E_C$ via

$$ E_C(u, \varphi) = \frac{G_C}{2} \int_{\Omega} \left( \frac{(|\varphi - 1|^2}{\epsilon} + |\nabla \varphi|^2 \right) \, d(x, y). $$

Again, no manufactured reference values are provided. At least for $\nu = 0.3$, we can compare our results for $E_B$ and $E_C$ with reference values from the literature, e.g., [1] [30]. In Figures 5 and 6 on the left side, the bulk and the crack energy are plotted versus the incremental step number. On the right of Figures 5 to 6
the average number of linear iterations and the number of Newton/AS steps are plotted. The number of linear iterations behaves differently for \( \nu = 0.3 \) from the results for higher Poisson ratios. While for \( \nu = 0.3 \), in Figure 5 on the right, the linear iterations decrease if the crack starts propagating, in Figure 6 the linear iterations increase up to an average of more than 70 iterations at the end of the crack simulations.

In Figure 7 snapshots of the pressure field and phase-field are given for \( \nu = 0.49 \), where – to the author’s knowledge – no reference values are available in the literature. The crack paths look similar as for \( \nu = 0.3 \), but a slight asymmetry is visible in the crack path. We decided to present the crack path during the simulation to depict the pressure field with the maximal value in front of the crack tip while the pressure values in the crack are zero. The computed bulk and crack energies in Figure 5 fit well to results in the literature, e.g., [25]. The bulk energy increases until the critical energy release rate is reached, and the crack energy increases when the crack propagates while the bulk energy releases. Also, in Figure 6 the bulk and crack energy curves fit the observed crack pattern in Figure 7. For \( \nu = 0.49 \) with snapshots in the last column in Figure 7, no comparable results in the literature are available. The crack pattern differs from the snapshots for smaller Poisson ratios. We observe that the crack has an orientation to the upper left corner, and a second crack develops from the singularity in the corner, where non-homogeneous Dirichlet boundary conditions and Neumann boundary conditions meet. In the first column in Figure 7 the pressure
and phase-field solution is given for $\nu = 0.3$ after total failure. The crack propagates from the center of the geometry to the left boundary, as we expect it. Further, one can see a pure zero pressure field after total failure.

5. Conclusions

In this work, a preconditioner for a mixed formulation phase-field fracture model that is robust in $h, \epsilon,$ and $\lambda$ was developed and tested on four numerical examples for different Poisson ratios up to the incompressible limit, namely $\nu \to 0.5$ yielding $\lambda \to \infty$. For the first test case, a hanging block with a slit, we confirmed the robustness and efficiency of the physics-based preconditioner, discretized with $Q^2_c Q^2_c Q^2_c$ finite elements. To the best of the authors’ knowledge, in the last test case the well-known single edge notched tension test was considered for higher Poisson ratios for the first time. For $\nu = 0.49$ a non-symmetric crack behavior and crack initiation from the upper left corner singularity was observed. In Sneddon’s test case and $\kappa = 10^{-8}$, an impact of $\kappa$ on the condition of the $\kappa$-dependent block entries of the system matrix could be explicitly seen.

It is well-known that from a phase-field perspective the regularization parameter $\epsilon$ is challenging, in particular its choice in relation to $h$. However, we found in this paper that from a preconditioner perspective the first regularization parameter $\kappa$ (in the bulk term of the displacement equation) causes difficulties instead. Basically, we deal with an elliptic (Laplacian) term where diffusion ranges from $\kappa \approx 10^{-8}$ in the crack region to 1, a difference of 8 orders of magnitude. We expect that a carefully designed geometric multigrid preconditioner or a weighted BFBT preconditioner might handle this situation better. We emphasize that
h, ϵ and λ is a significant contribution, which has not yet been studied so far in the published literature. A second future extension would be thermodynamically consistent constitutive materials laws, namely incorporating stress splitting in σ(u, p).

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