A High-Performance Sparse Tensor Algebra Compiler in Multi-Level IR

Ruiqin Tian  
Pacific Northwest National Laboratory  
ruiqin.tian@pnnl.gov

Bin Ren  
William & Mary  
bren@cs.wm.edu

Luizheng Guo  
Pacific Northwest National Laboratory  
luen@pnnl.gov

Jiajia Li  
Pacific Northwest National Laboratory  
jiajia.li@pnnl.gov

Gokcen Kestor  
Pacific Northwest National Laboratory, UC Merced  
gokcen.kestor@pnnl.gov

ABSTRACT

Tensor algebra is widely used in many applications, such as scientific computing, machine learning, and data analytics. The tensors represented real-world data are usually large and sparse. There are tens of storage formats designed for sparse matrices and/or tensors and the performance of sparse tensor operations depends on a particular architecture and/or selected sparse format, which makes it challenging to implement and optimize every tensor operation of interest and transfer the code from one architecture to another. We propose a tensor algebra domain-specific language (DSL) and compiler infrastructure to automatically generate kernels for mixed sparse-dense tensor algebra operations, named COMET. The proposed DSL provides high-level programming abstractions that resemble the familiar Einstein notation to represent tensor algebra operations. The compiler performs code optimizations and transformations for efficient code generation while covering a wide range of tensor storage formats. COMET compiler also leverages data reordering to improve spatial or temporal locality for better performance. Our results show that the performance of automatically generated kernels outperforms the state-of-the-art sparse tensor algebra compiler, with up to 20.92x, 6.39x, and 13.9x performance improvement, for parallel SpMV, SpMM, and TTM over TACO, respectively.

1 INTRODUCTION

Tensor algebra is at the core of numerous applications in scientific computing, machine learning, and data analytics. Tensors are a generalization of matrices to any number of dimensions, which are often large and sparse. Sparse tensors are used to represent a multifactor or multirelational dataset, and has found numerous applications in data analysis and mining [36, 54, 64] for health care [3, 46], natural language processing [15, 51], machine learning [43, 59], and social network analytics [77], among many others.

Developing optimized kernels for sparse tensor algebra methods is complicated. First, sparse tensors are often stored in a compressed form (indexed data structures) and computational kernels need to efficiently loop over the nonzero elements of the tensor inputs. Second, iterating over nonzero elements highly depends on the particular storage format employed, hence many algorithms exist to implement the same operation, each targeting a specific format. Finally, applications may use multiple formats concurrently throughout the computation and mix different formats in the same operation to achieve high performance. When tensors with different storage formats are used in the same operation, there are two options: converting one (or both) tensor(s), which is time-consuming especially if the tensor is only used once, or developing an algorithm that can efficiently iterate over both formats simultaneously, which lacks generality and requires different implementations for each combination of tensor formats [10, 44].

The current solutions implement ad hoc high-performance approaches for particular computer architecture and/or format. Most of these algorithms tackle specific problems and domains and conveniently store sparse tensors in a format that exploits the characteristics of the problem. This approach has originated tens of different formats [10, 18, 40, 50, 62] to represent sparse tensors. Some of them are storage-efficient for specific inputs [10, 18, 37, 67] or evenly nonzero distributions across rows/columns [18, 49]; some are better affiliated to specific tensor computations, e.g., sparse matrix-vector multiplication [72, 73] versus sparse tensor-matrix multiplication [10, 61]; others are particularly designed for different computer architectures, such as CPUs [41, 72] versus GPUs [47, 48]. On the other hand, it is infeasible to manually write optimized code for each tensor algebra expressions considering the all possible combinatorial combinations of tensor operations and formats.

To solve the above challenges, we present a sparse tensor algebra compiler, named COMET, that is agnostic to storage formats: as opposed to a library of sparse tensor methods, where the methods are statically defined, a compiler can automatically and dynamically generate efficient tensor algebra kernel specifically optimized mixed dense/sparse tensor expressions. COMET Domain-Specific Language (DSL) is a highly-productive language that provides high-level programming abstractions that resemble the familiar Einstein notations [20] to represent tensor operations. COMET is based on the Multi-Level Intermediate Representation (MLIR) [38] framework recently introduced by Google to building reusable and extensible compiler infrastructures. The key benefit of building on top of MLIR is its built-in performance portability. In the COMET multi-level Intermediate Representation (IR), domain-specific, application-dependent optimizations are performed at higher levels of the IR stack where operations resemble programming languages’ abstractions and can be optimized based on the operations semantics. Generic, architecture-specific optimizations are, instead, performed at lower-levels, where simpler operations are mapped to the memory hierarchy and to processor’s registers.
To enable modular code generation with respect to formats and combination of formats, we employ four storage format attributes—dense, compressed unique, compressed non-unique, and singleton—which are assigned to each tensor dimension [33]. By properly combining those attributes in each dimension, it is possible to express common sparse tensor compressed formats, such as COO, CSR, DCSR, ELLPACK, CSF and Mode-generic. COMET code generation algorithm analyzes the dimension attributes and produces code to efficiently iterate over the nonzero elements of the input tensors. Since the number of storage format attributes is far lower than all possible combinations of storage formats, the code generation algorithm is greatly simplified and yet can support most of the commonly used sparse tensor storage formats and arbitrary combinations of those. This approach lets users not only mix and match storage format desired for their applications but also can enable custom formats without modifying the underlying compiler infrastructure. Once the loop form of a computation has been generated at the IR, COMET either lowers the code for sequential or parallel execution. In the former case, COMET produces a high-quality LLVM IR (which we show in this work has better loop unrolling and vectorization than an equivalent LLVM IR produced by clang); in the latter case, instead, COMET lowers code to the async dialect for asynchronous task execution based on LLVM co-routines Compared to hand-tuned libraries [12, 40, 41, 50, 62, 73] and source-to-source compilers [31–33], our approach is more portable, flexible, and adaptable, as emerging architectures and storage formats can be added without re-engineering the computational algorithms. Finally, COMET employs the state-of-the-art data reordering algorithm [42] to increase spatial and temporal locality on a modern processor.

We evaluated COMET with 2833 sparse matrices and six tensors from the SuiteSparse Suite Matrix Collection [19], FROSTT Tensor Collection [60] and BIGtensor [26]. Our results show that COMET can generate efficient code for multi-threaded CPU architectures from high-level descriptions of the algorithms. Compared to state-of-the-art high-productivity tensor algebra languages and compiler, COMET provides on average 2.29x, up to 6.26x, performance improvements over TACO compiler for sequential Sparse-Matrix Dense-Matrix (SpMM). We also show that asynchronous task execution outperforms OpenMP parallelization, especially for small input matrices, where runtime overhead is predominant. Our results show up to 6.39x and 13.9x speedup over TACO for SpMM and TTM, respectively. Finally, data reordering achieves up to 3.89x and 7.41x performance improvements for parallel SpMV and SpMM kernels, respectively, over the original COMET.

To the best of our knowledge, COMET is the first MLIR-based compiler that integrates generic code generation for arbitrary input formats, data reordering, and automatic parallelization within the same framework. COMET can improve end-user application performance while supporting efficient code generation for a wider range of formats specialized for different application and data characteristics. This paper makes the following contributions:

- We introduce the COMET DSL, an intuitive yet powerful and flexible language to implement dense and sparse tensor algebra algorithms;
- We propose an MLIR-based compiler that automatically generates efficient sequential and parallel code for a tensor expression with dense and mixed operands while supporting the important sparse tensor storage formats.
- We integrate the state-of-the-art data reordering algorithm to enhance data locality.
- We provide an exhaustive experimental evaluation and show that COMET generally outperforms state-of-the-art tensor compiler for both sequential and parallel execution.

2 BACKGROUND AND MOTIVATION

There exist various compressed and uncompressed formats to store sparse matrices and tensors in the literature, including COOrdinate (COO), Compressed Sparse Row (CSR), Double Compressed Sparse Row (DCSR), ELLPACK, Compressed Sparse Fiber (CSF), and Mode-Generic [12, 17, 22, 29, 48]. The specific format chosen to represent data in an application generally depends on the expected characteristics of the data itself and how these impact other desired properties, such as performance of a computational kernel or memory footprint (which is particularly important in the case of very large, multi-dimensional tensors).

Each format is important for different reasons. COO [5, 58] is commonly used to store sparse matrices and tensors, such as the Matrix Market exchange format [2] and the FROSTT sparse tensor format [60]. While COO is the most natural format, it is not necessarily the most performant format. CSR [70] is for sparse matrices, which compresses row indices as pointers to row beginning positions to avoid duplicated storage and increase performance for memory bandwidth-bound computation such as Sparse-Matrix Dense-Vector (SpMV). DCSR [16] further compresses zero rows by adding an extra pointer to nonzero rows based on the CSR format. With an extra level of compression on rows, DCSR is more efficient than CSR for highly sparse (hypersparse) data. The ELLPACK [29] format is efficient for matrices that contain a bounded number of nonzeros per row, such as matrices that represent well-formed meshes. CSF [62] generalizes the DCSR or CSR matrix format to high-order tensors that compress every dimension. Mode-General format [10] is a generic representation of semi-sparse tensors with one or more dense dimensions stored as dense blocks with the coordinates of the blocks stored in COO.

An application might need any or even several of these formats based on its needs, which makes it important to support computation with various tensor storage formats and their combinatorial combinations. The main challenge is that the computational kernel needs to effectively iterate over each sparse input tensor stored in different storage formats. This problem is especially more complicated for expressions that involve multiple operands.

Because of the large number of storage formats and possible combinations, most state-of-the-art sparse tensor libraries support only a few sparse formats (and generally only binary operations) or convert tensors to an internal storage format, thereby potentially losing the performance, memory footprint, or other advantages that a specific format may offer. A compiler, on the other hand, can automatically generate the efficient code for specific input formats and their combinations, increasing flexibility, adaptivity to new formats, and portability to various hardware platforms. To
def main() {
# Index Label Definition
IndexLabel \{a\} = [?];
IndexLabel \{b\} = [?];
IndexLabel \{c\} = [?];

# Tensor Definition
Tensor<\text{double}> A[\{a, b\}, \text{CSR}];
Tensor<\text{double}> B[\{b, c\}, \text{Dense}];
Tensor<\text{double}> C[\{a, c\}, \text{Dense}];

# Tensor Readfile Operation
A[\{a, b\}] = \text{space_read}(\text{filename});

# Tensor Fill Operation
B[\{b, c\}] = 1.0;
C[\{a, c\}] = 0.0;

# Tensor Contraction
C[\{a, c\}] = A[\{a, b\}] * B[\{b, c\}];
}

Listing 1: An example SPACe program for Sparse Matrix-times-Dense-Matrix operation.

achieve this goal, two important requirements need to be satisfied: 1) a unified way to represent important sparse storage formats (Section 4) and 2) an efficient algorithm to generate specific code for a given expression and its particular input formats (Section 6).

3 COMET OVERVIEW

COMET consists of a DSL for tensor algebra computations, a progressive lowering process to map high-level operations to low-level architectural resources, a series of optimizations performed in the lowering process, and various IR dialects to represent key concepts, operations, and types at each level of the multi-level IR. This section reviews the key characteristics of our compiler framework. COMET is based on the MLIR framework [38], a compiler infrastructure to build reusable and extensible compilers and IRs. MLIR supports the compilation of high-level abstractions and domain-specific constructs and provides a disciplined, extensible compiler pipeline with gradual and partial lowering. Users can build domain-specific compilers and customized IRs (called dialect), as well as combining existing IRs, opting into optimizations and analysis.

Our previous work focuses on dense high-dimensional tensor contractions. The compiler reformulates tensor contractions as a sequence of transpose and matrix-matrix multiplication operations, then generates efficient code by several code optimizations (e.g., loop tiling, micro kernel). The detailed description of previous work and its performance results for important tensor expressions from the Northwest Chemistry framework (NWChem) [66] can be found in [1]. This work, instead, focuses on sparse tensor algebra.

Figure 1 shows an example COMET program for an SpMM operation. The IndexLabel operation defines an index label. It can assign the size of the index with a scalar number. If the size is unknown in static time, then use a question mark (?) (Lines 3-5). The Tensor operation defines new tensors (Lines 8-10); the SpMM operation is defined at Line 20. In particular, the matrix A is stored in the CSR format while the matrix B and the result matrix C are dense. Note that there is no specific operation for SpMM at language level, nor the programmer needs to explicitly state the format of each input tensor while contracting the two tensors. COMET atomically derives the specific operation from the format of input tensors and the index labels. COMET internally annotates each tensor with storage format attributes, devises the storage formats used in the contraction, and properly passes this information down to the IR stack when lowering the code. COMET can generate the appropriate code according to the input tensor storage formats (Section 6.2).

The code generation in COMET follows a progressive lowering approach where optimizations are applied at different levels. Figure 1 shows the compilation pipeline of COMET, where our contributions are annotated by the dashed box. Users express their computation in a high-level tensor algebra DSL (Section 5). First, the COMET DSL is lowered to a Sparse Tensor Algebra (TA) IR, the first dialect in the COMET IR stack. The language operators, types, and structures are first mapped to an abstract syntax tree and then to the TA dialect. The TA dialect contains domain-specific concepts, such as multi-dimensional tensors, contractions, and tensor expressions. Our compiler framework applies high-level optimizations and transformation leveraging semantics information carried from the DSL. For example, COMET tracks the input tensors’ definitions and annotates each tensor with storage format attributes on each dimension, based on the index label definitions.

Next, our compiler lowers the Tensor Algebra (TA) IR code to lower levels of the IR stack, which follows different paths depending on the operation and input formats. Dense tensor algebra operations are lowered first to linear algebra dialect, then to Structured Control Flow (SCF) dialect, and finally to standard dialect. Sparse linear algebra operations are lowered to SCF dialect which is a loop represented in the MLIR framework. At this point, COMET employs generic optimizations during the lowering steps but also considers additional information about the final target architecture. For CPU execution, the code is lowered to the Low-Level Virtual Machine (LLVM) dialect for sequential execution and asyn dialect to models asynchronous execution at a higher-level and then to proper LLVM IR for final assembly and linking.
Figure 2: Example matrix and tensor represented in different formats. Each format is a combination of the storage format attributes.

4 TENSOR STORAGE FORMAT

As reported in Section 2, to support multiple sparse storage formats a compiler needs a uniform way to represent each tensor in memory. This internal storage formats need to preserve the characteristics of the original format, e.g., data compression or performance for specific sparse patterns, while allowing a unified algorithm to generate efficient code for each computational expression. COMET defines a set of storage format attributes for each dimension to represent various sparse tensor formats. Code generation is then based on each dimension's storage format attributes rather than the whole format, which greatly reduces the number of formats and combinations that a compiler needs to support. Importantly, COMET does not convert the original data layout into a different storage format. Instead, the storage format attributes are used to compose meta-data information that describes the original format, i.e., the data layout of the original format is preserved in memory and retains the original characteristics (compression, locality, etc.).

Representing every tensor dimension separately has been shown to be an effective way to generalize tensor storage formats and support efficient code generation [34]. Representing each dimension independently makes it easier to manage, adapt, and convert formats and to generate computational kernels uniformly. COMET defines the following four storage format attributes borrowed from [34, 40, 62]:

**Dense (D).** This dimension is in the “dense” format, i.e., all coordinates in this dimension will be accessed during the computations. For this format, we only use one scalar number stored in the pos array to represent the size of this dimension, such as the row dimension in Figure 2a(3).

**Compressed Unique (CU).** This dimension is in a “compressed unique” format, i.e., the coordinates of nonzero elements in this dimension are compressed, and only the unique (no duplication) ones are stored in the array crd. It uses another array pos to store the start position of each unique coordinate, such as the row dimension Figure 2a(4), where the elements 1 and 2 are in the same row, but only one row coordinate is stored in row_crd array.

**Compressed Nonunique (CN).** This dimension is in a “compressed non-unique” format, i.e., all the coordinates of nonzero elements will be recorded in crd array, and every coordinate in the crd array will be accessed one by one. CN then stores the start and the length of the crd array to the pos array, such as the row dimension Figure 2a(2), where all the row coordinates of the nonzeros are stored in row_crd array, row_pos only stores the start and the length of the row_crd array.

**Singleton (S).** The dimension is in a “singleton” format, i.e., all the nonzero coordinates are recorded to the array crd without any other information, such as the column dimension Figure 2a(2), only the column coordinates of the nonzeros are stored in row_crd array.

Internally, each tensor dimension is described by two arrays, a position (pos) and a coordinate (crd) array. D only uses the pos array to store the size of the dimension; the compressed storage format attributes CU and CN use both pos and crd arrays to store the nonzero coordinates and their positions; S only uses the crd array to store the nonzero coordinates in the dimension.

Furthermore, Figure 2 shows two examples that store a sparse matrix and a sparse tensor, respectively, in three formats (COO, CSR, and DCSR) with the representation of varied storage format attributes combinations. By properly combining the tensor storage format attributes, COMET can represent the important sparse storage formats, including COO, CSR, DCSR, BCSR, CSB, ELLPACK, CSF and Mode-generic, in a uniform way, while retaining each format’s characteristics.

5 COMET LANGUAGE DEFINITION

COMET provides a high-level Tensor Algebra DSL that increases portability and productivity by allowing scientists to reason about their algorithms implementation in their familiar notation and
syntax. Specifically, COMET DSL allows scientists 1) to express concepts and operations in a form that closely resembles their familiar notations and 2) to convey domain-specific information to the compiler for better program optimization. For example, our language represents Einstein mathematical notation and provides users with an interface to express tensor algebra semantics. The same COMET program can be lowered to different architectures, and the lowering steps can follow different optimizations and lowering algorithms, allowing COMET to produce high-quality code for target architectures without excessive burden on the programmer (see Section 6). This work extends the COMET tensor algebra language to support sparse tensor algebra operations and syntax, the storage formats described in the previous sections.

Furthermore, we extend COMET to support dynamic data types. As discussed above, Figure 1 shows an example of a COMET program. In the COMET language, a tensor object refers to a multi-dimensional array of arithmetic values that can be accessed by indices. Range-based index labels construct (Index<label>) represent the range of indices expressed through a scalar, a range, or a range with increment. Index labels can be used both for constructing a tensor or for representing a tensor operation. Different from the original COMET compiler [1], Index<label>s can now be defined as static or dynamic. Static Index<label>s explicitly state the size of the dimension (Line 5) while dynamic Index<label>s (Lines 3 and 4) only indicate that there exists a dimension, but the size will be determined later on during the execution of the program. Dynamic and static index labels differ in that dynamic index labels indicate an unknown size through a question mark (?) operator while static index labels explicitly state the size of the dimension through a scalar value.

A tensor is constructed by defined static or dynamic index labels and by declaring the sparsity of each dimension, according to the internal storage format described in the previous section. In Figure 1 tensor A is stored in CSR format, while tensors B and C are stored in dense format. Note that COMET provides convenient notation to represent the most common tensor storage format, avoiding the need to specify the storage format for each dimension, as described in the comments at Lines 8-10. Internally, however, COMET reasons in terms of sparsity on each dimension when generating code.

In the example COMET program in Figure 1, the tensor A, B, and C are initialized with a tensor file by space_read(), the constant value 1.0, and the constant value 0.0, respectively. The function space_read() first reads a tensor from the file in COO format and then converts it to our internal storage format (see Section 4) to represent CSR. We implement space_read() as a runtime function, and it can be called in the COMET program directly.

The last line in the program performs the SpMM operation. However, users need not explicitly state that the operation is an SpMM but can simply use the common tensor contraction * operator. COMET will infer that the operator refers to an SpMM operation from the storage format of the input tensors, in this case, a sparse matrix and a dense matrix, and will generate the proper code to iterate over the specific storage format through rules generated from the definition of storage format attributes. Also, note that COMET employs index labels to determine the type of operation to perform. For example, the * operator refers to a tensor contraction

if the contraction indices are adjacent or to element-wise operation otherwise. In Figure 1, the index label b is used as contraction indices between A and B (adjacent or internal indices), thus the operator * refers to a tensor contraction. Therefore, COMET can not only support tensor contraction but are generally applicable to many other operations as well. Conclusively, the COMET TA language simplifies writing tensor algebra program by supporting common programming paradigms and enables users to express high-level concepts in their familiar notations.

6 COMPILATION PIPELINE

We introduce sparse tensor algebra dialect in MLIR to support mix dense/sparse tensor algebra computation with a wide range of storage formats. We use format attributes to represent each dimension sparsity format in a uniform way in the proposed TA IR. COMET compiler generates efficient code based on the represented format attribute per dimension. This section describes the compiler framework, which consists of two main parts: 1) a sparse MLIR TA dialect to represent tensor storage formats and operations, and 2) code generation algorithms to generate efficient serial and parallel code starting from the proposed TA DSL.

6.1 Sparse Tensor Algebra Dialect

COMET supports a uniform tensor storage format based on the attributes described in Section 4 and the tensor algebra operations supported in our DSL. Figure 3 shows the generated tensor algebra IR for the SpMM program in Listing 1. The rest of this section details the various operation in the sparse TA dialect.

Static/Dynamic Index Labels. The sparse tensor algebra dialect supports two types of index label, static and dynamic. If the dimension size of the index is known in compile-time, COMET uses ta.index_label_static to represent the index label. It has three operands, which represent the start, end, and step value on this index. ta.index_label_dynamic is used to represent the index label when the dimension size is unknown in compile time. ta.index_label_dynamic has two operands, the start, and step
value on this index. The end value on this index will be known in runtime.

**Sparse Tensor Declaration.** In sparse tensor algebra dialect, the tensor is declared with *ta. tensor_decl* operation. The operands of *ta. tensor_decl* are the index labels of the tensor. It can contain an arbitrary number of operands, which means it can declare arbitrary dimensional tensor. *ta. tensor_decl* operation also contains storage format attributes of the tensor in each dimension for sparse tensors.

**Sparse Tensor Operations.** The sparse TA dialect also defines the tensor algebra operations supported by COMET. For example, the tensor contraction *ta. tc* operation for an SpMM computation (shown in line 20 of Figure 3) takes two input tensors and computes the result of the contraction. The first and second operands (%A and %B) are input tensors, and the third operand (%C) is the output tensor. "ta.sptensor<tensor<?x132>, tensor<?x132>, tensor<?x132>, tensor<?x132>, tensor<?xf64>>" is the data type for %A, while "tensor<?x32xf64>" is the data type for %B, and "tensor<?x32xf64>" is the data type for %C. "()" represents the return type which is void.

We introduce formats attribute to extend the original *ta. tc* to provide the storage format information of each input tensor. In line 20 of Figure 3, the first tensor is in CSR format, while the second and third are all Dense tensors. The code in the figure shows that each input tensors is associated with its storage format information. We also introduce indexing_maps to *ta. tc* to represent the indices of each tensor. The *indexing_maps* helps propagate indices information along with the lowering stack. The tensor expression and the storage format information will be further propagated down to the lower level of the IR to provide the format attribute in each dimension when generating the computational code.

**Sparse Tensor Data Type.** As described in Section 4, a tensor *T* consists of *k* dimensions *d_i* for 0 ≤ *i* ≤ *k* − 1, where every dimension *d_i* is associated with a uniform storage attribute *a <i>* ∈ {D, CU, CN, S}. COMET associates two arrays crd and pos to each dimension to describe the storage format (meta-data). In the TA dialect, we define a sparse tensor as a *struct* data structure, which contains the nonzero indices in each dimension and their values.

Figure 4 shows how a 2D sparse matrix is represented in our TA dialect. In Figure 4, *ta. sptensor_construct* is the function to construct the sparse tensor struct, which is implemented as an operation in the TA dialect. The *sptensor_construct* operation takes the pos and crd arrays in each dimension (%A1pos, %A1crd, %A2pos, %A2crd) and the nonzero values (%Aval) as input, and returns a *ta. sptensor* type data structure that represents a sparse tensor in the TA dialect. The tensor types within *ta. sptensor* represent the pos and crd arrays corresponding to each dimension of the tensor itself (see Section 4). In the *ta. sptensor* structure, the type of %A1pos, %A1crd, %A2pos, %A2crd are tensor<?x132>, the type of %Aval is tensor<?xf64>.

### 6.2 Sparse Code Generation Algorithm

COMET lowers the code from high-level COMET DSL language to low-level machine code in multiple lowering steps.

**DSL Lowering.** The first step in our compilation pipeline consists of lowering the high-level COMET DSL into the sparse TA dialect. Figure 3 shows the TA dialect corresponding to the COMET code presented in Figure 1. In Figure 3, "ta." represents the tensor algebra dialect. The *indexLabel* operation in COMET DSL will be lowered either into a *ta. indexLabel_static* operation or a *ta. indexLabel_dynamic* operation (e.g., Lines 9-11 in Figure 3) based on whether the size of the dimension represented by the index label is known or unknown at compile time. The *ta. indexLabel* operation has three parameters (%A, %a, and %b), which are the start, the end, and the iteration step values in the dimension represented by the index label. The *indexLabel* at Lines 3-4 of Figure 1 has an unknown size, so it will be lowered into the *ta. indexLabel_dynamic* operation, which only contains the start value of the dimension. The dimension size will be inferred during the runtime.

**Progressive Lowering.** Next, the sparse TA dialect is further translated to lower MLIR dialects. We describe this lowering process in two parts, early lowering and late lowering.

First, in the early lowering step, COMET lowers all the operations in the sparse TA dialect, except the *ta. tc* operation. In particular, the *ta. tensor_decl* operation, which declares a tensor, is lowered into *alloca* and *tensor_load* operations, which are standard dialect operations in *std* dialect for dense. For sparse, *ta. tensor_decl* operations are lowered into more, a composition of *alloca* and *tensor_load* operations for pos and crd arrays to store the coordinates of nonzeros in each dimension, and *val* array to store nonzero values. These coordinates of nonzeros are later used by *ta. sptensor_construct* operation (Figure 4) to construct a sparse tensor. To fill the pos, crd, and val arrays, the *ta. generic_call* operation is invoked to call the *space_read()* function. The *ta. generic_call* operation is then lowered to the *call* operations in the MLIR *std* dialect. The *ta. fill* operation initializes dense tensors with identical values. The *ta. fill* operation will be lowered into the *fill* operation in the MLIR *linalg* dialect. The *ta. return* operation returns the function, and is lowered into *return* operation in the MLIR *std* dialect. The *ta. indexLabel* operations is lowered into the *ta. indexLabel_static* operation when the index label is identified from the input file.

Second, in the late lowering step, *ta. tc* operations are lowered into the MLIR *scf* (structure control flow) dialect operations. Figure 6 describes the lowering algorithm to *ta. tc* with an example mix sparse dense tensor contraction operation, where a sparse tensor *A* times a dense tensor *B*, and the output can be either sparse or dense. The algorithm takes *ta. tc* as input, and automatically generates the computational kernel code of a combination of *scf* and *std* dialects. *ta. tc* is the sparse tensor algebra dialect of the tensor contraction operation presented at Line 20 in Figure 1.
shown at Line 20 in Figure 3 t a. tc operation is lowered based on the code generation algorithm in Figure 6.

Figure 6 shows COMET’s code generation algorithm that consists of three key steps. This algorithm is general, applicable to varied tensor algebra operations, and can generate arbitrary index permutations. Moreover, in contrast to TACO, COMET can generate sparse output. Take tensor expression $C_{ik} = A_{ij} * B_{jk}$ as an example, and assume the format of $A$ is [D, CU], $B$ is [D, D] and $C$ is [D, D], respectively. The basic idea of this code generation is as follows:

**Step-I** (Line 1 to Line 3) collects both index information as well as the format attribute of each index. The above sample tensor expression has three indices (all-Indices = (i, j, k)). The order of these indices matters, and is decided by tensor access orders. The format attribute of each index is decided by the usage of this index. If this index appears in dense input tensors only, its format attribute is D; otherwise, the format attribute is decided by the corresponding dimension of the sparse tensor. For the above sample tensor expression, the format attribute of index $i$ is D and $j$ is CU (both decided by sparse input tensor $A$), and $k$ is D (decided by dense input tensor $B$), respectively. After collecting this information, this algorithm defines three index variables (oldx$A$, oldx$B$ and oldxC) to access the value array of tensor $A$, $B$, and $C$, respectively (Line 3).

**Step-II** (Line 4 to Line 19) iterates each index to generate loop structure code (as the algorithm line starting with “emit” shows). It leverages the aforementioned definition of each storage format attribute to find nonzero coordinates in each dimension via pos and crd arrays (e.g., d_pos and d_crd in the algorithm). Table 1 shows the sample loop code in C language for each format attribute. Besides generating loop structure code for each index, this step also updates three index variables (oldx$T$, $T$ ∈ (A, B, C)) that will be used for inner-most computation. If the format attribute of an index (e.g., $d$) is D, i.e. $d$ only appears in dense tensors, then oldx$T = oldxT * d_SIZE + arg$, where $T$ denotes all dense tensors that contain index $d$. arg is the coordinate on index $d$ (i.e., the argument of the generated loop for index $d$) and $d_SIZE$ is index $d$’s dimension size. If the format attribute of index $d$ is sparse (e.g., CU), this step handles sparse tensors and dense tensors separately. For sparse tensors $T$ that contain index $d$, oldx$T = oldxT + d_crd[arg]$, where $d_crd$ is the crd array of index $d$ and $d_crd[arg]$ is the coordinate.

**Step-III** (Line 20) generates inner-most computation code to load values from $A[oldxA]$ and $B[oldxB]$, compute their product, and update $C[oldxC]$, after step-II generates oldx$T$ for tensor $T$ ($T$ ∈ (A, B, C)).

### Table 1: Generated code to access nonzeros coordinates

| Attr | Corresponding code |
|------|--------------------|
| D    | for $i$ from pos[0] to pos[3]: |
| CU   | for $j$ from pos[4] to pos[7]: |
| CN   | for $k$ from pos[8] to pos[11]: |
| S    | idx = crd[0]: |

$$%A1SIZE_{i32} = load \%A1pos[%c0]: memref<7x32>;$$

```c
1. \%
2. A1SIZE = index_cast %A1SIZE_{i32} : i32 to index
3. scf for %i = %c0 to %A1SIZE_{i32} step 1:
4. \%next_i = addi %i, 1, 1:
5. \%A2pos_start_i = load %A2pos[%c1]: memref<8x32>:
6. \%A2pos_start = index_cast %A2pos_start_i : i32 to index
7. \%A2pos_end_i = index_cast %A2pos_end_i : i32 to index
8. scf for %arg1 = %A2pos_start to %A2pos_end step %c1:
9. \%i32 = load %A2dpos[%arg1]: memref<7x32>;
10. \%j = index_cast %Nj_i32 : i32 to index
11. scf for %k = %c0 to %c32 step %c1:
12. \%Cvalue_old = load %C[%i, %k]: memref<32x8>
13. \%Cvalue = add %Cvalue, %product : memref<32x8>
14. store \%Cvalue, %C[%i, %k]: memref<32x8>
19. }}
```

Figure 5: Lowered scf dialect code example for SpMM in the CSR format. The right side numbers represent line numbers in Algorithm 6

### 6.3 Parallel Code Generation

For sequential execution COMET lowers the scf dialect to the 11vm IR dialect and then to proper LLVM IR for assembly and linking. For parallel execution, instead, the scf dialect is lowered to the async dialect (See Figure 1). In details, we developed a pass to lower scf. For loops to scf. parallel loops and the latter to the async dialect. The async dialect encapsulates the semantics of an asynchronous task-based parallel runtime in which computational tasks are spawn and asynchronously executed by parallel worker threads. Currently, MLIR supports a task continuation stealing approach (like Cilk [14]) in which the control is returned to the parent task after spawning. The dialect provides semantics primitives to synchronize the execution of tasks. COMET lowers those asynchronous task execution primitives to LLVM co-routines in LLVM IR, which is then passed to the assembler and linker to create a
binary. As Figure 7d shows, the MLIR asynchronous runtime introduces relatively low overhead during execution, which improves performance, especially for small computations.

7 DATA REORDERING

The distribution of the nonzero entries in sparse matrices/tensors can significantly affect the performance of sparse matrix/tensor algebra computations. Reordering [27, 42] is the de facto technique to optimize the memory access pattern caused by uneven data distribution. Different from existing compiler frameworks [30, 32] which apply reordering to iterations, we apply reordering to matrices and tensors to optimize their memory access patterns.

We borrow from the reordering algorithm presented in [42] (LexiOrder), extended it to support sparse matrices, and implemented it in the COMET runtime (tensor_reorder()). The LexiOrder algorithm is built on top of the doubly lexical ordering algorithm [45, 52] with some optimization techniques to advance its overall efficiency and availability on some concern cases. The basic idea of the LexiOrder algorithm is to sort a specific dimension (either rows or columns for matrices) in an iteration using the doubly lexical ordering algorithm and sort all dimensions in turn across iterations. The algorithm’s objective is to cluster all nonzero entries around the diagonal to increase spatial and temporal locality.

8 EVALUATION

In this section we evaluate COMET against state-of-the-art high-level compiler frameworks and DSL for dense and sparse tensor algebra. Specifically, we compare our results against TACO [34], a tensor algebra compiler that performs automatic source-to-source transformation from TACO DSL to sequential C++, Parallel OpenMP, and data-parallel CUDA. For brevity, we evaluated the performance of selected benchmarks with a single storage format – matrices (CSR) and tensors (CSF), though our compiler can operate on other formats as well. All results reported are the average of 25 runs.

8.1 Experimentation Setup

We performed our experiments on a compute node equipped with two Intel Xeon Gold 6126 sockets running at 2.60GHz. Each CPU socket consists of 12 processing core (for a total of 24 cores). The system features 192 GB of DRAM memory. We compiled COMET, TACO, and all the benchmarks with -O3 and c1ang 12.8 and use the most recent MLIR version at the time of writing this manuscript.

We use as input datasets 2833 matrices and six tensors of different sizes and shapes chosen from the SuiteSparse Matrix Collection [19], the FROSTTT Tensor Collection [60], and BiGtensor [26]. The SuiteSparse Matrix Collection is a growing dataset of sparse matrices in real-world applications. The dataset is widely used in the numerical linear algebra community for performance evaluation. The FROSTTT Tensor Collection is a composition of open-source sparse tensor datasets from various data sources that are difficult to collect. The BiGtensors dataset is a tensor database that contains large-scale tensors for large-scale tensor analysis. Our input datasets represent the most important HPC domains in scientific computing, including chemistry, structural engineering, various linear solvers, computer graphics and vision, and molecular dynamics. We provide the description of the six tensors in Table 2.

| Name       | Size          | Nonzeros | Domain                      |
|------------|---------------|----------|-----------------------------|
| NELL-1     | 2,902,330 x   | x        | 145599552                  |
|            | 2,145,368 x   |          | Natural Language Processing|
|            | 25,495,389    |          |                             |
| NELL-2     | 12,092 x 9184 | x        | 76879419                   |
|            | 28,818        |          | Natural Language Processing|
| delicious  | 3d            | x        | 140,126,181                |
|            | 17,262,471 x  |          | Tags from Delicious website|
|           | 2,480,308     |          |                             |
| flickr-3d  | 319,686 x     | x        | 112,890,310                |
|            | 28,153,045 x  |          | Tags from Flickr website   |
|            | 1,607,193     |          |                             |
| vast-2015-mc1-3d | 165,427 x 11,374 x 2 |          | 20,021,854                 |
| freelancer-music [26] | 23,344,794 x 223,344,784 x 166 |          |                             |

Table 2: Description of sparse tensors

8.2 Sparse Tensor Operations

We define the sparse tensor operations considered in COMET below.

**SpMV.** The Sparse Matrix-times-Vector (SpMV or SpMSpV), \( y = X \times v \), is the multiplication of a sparse matrix \( X \in \mathbb{R}^{I \times J} \) with a dense vector \( v \in \mathbb{R}^{J} \). \( y_i = \sum_{j=1}^{J} x_{ij} v_j \).

**SpMM.** The Sparse Matrix-times-Matrix (SpMM or SpGEMM), \( Y = X \times U \), is the multiplication of a sparse matrix \( X \in \mathbb{R}^{I \times J} \) with a dense matrix \( U \in \mathbb{R}^{J \times N} \). \( y_{ij} = \sum_{k=1}^{N} x_{ik} u_{kj} \).

**SpTTV.** The Sparse Tensor-Times-Vector (SpTTV) [7] in mode \( n \), \( Y = X \times_n v \), is the multiplication of a sparse tensor \( X \in \mathbb{R}^{I_1 \times I_2 \times I_3} \) with a dense vector \( v \in \mathbb{R}^{I_n} \). Given \( n = 1 \), \( y_{i_1,i_2,i_3} = \sum_{i_n=1}^{I_n} x_{i_1,i_2,i_3,i_n} v_{i_n} \). This results in a two-dimensional \( I_2 \times I_3 \) tensor which has one less dimension.

**SpTTM.** The Sparse Tensor-Times-Matrix (SpTTM) [7, 35] in mode \( n \), denoted by \( Y = X \times_n U \), is the multiplication of a sparse tensor \( X \in \mathbb{R}^{I_1 \times I_2 \times I_3} \) with a dense matrix \( U \in \mathbb{R}^{N \times K} \). Mode-1 TTM results in a \( R \times I_2 \times I_3 \) tensor, and its operation is defined as \( y_{r,i_2,i_3} = \sum_{i_1=1}^{I_1} x_{i_1,i_2,i_3} u_{i_1 r} \). Also, note that \( R \) is typically much smaller than \( I_n \) in low-rank decompositions, typically \( R < 100 \).

SpMV and SpMM widely appear in applications from scientific computing, such as direct or iterative solvers [39, 71], to data-intensive domains [76], graph analytics [41]. SpTTV and SpTTM are computational kernels of popular tensor decompositions, such as the Tucker decomposition [35, 59, 75], tensor power method [4, 69], for a variety of applications, including (social network, electrical grid) data analytics, numerical simulation, machine learning.

8.3 Performance Evaluation

**SpMV and SpMM.** We measured the performance of COMET and TACO while running SpMV and SpMM with each of the 2833 matrices for sequential and parallel execution. We present the experimental results in Figure 7, where COMET and TACO are represented in red and blue dots respectively. In the plot, the x-axis represents a matrix (2,833 matrices, ordered by increasing number of nonzeros) and the y-axis execution time (lower is better). As we can see from the plots, COMET achieves better performance than TACO on sequential SpMV (Figure 7c and parallel SpMV (Figure 7b), and comparable performance on sequential SpMV and parallel SpMM (Figures 7a and 7d, respectively). For sequential execution, COMET
outperforms TACO by up to 6.26x for SpMM (average 2.29x) and by up to 2.14x for SpMV (average 0.94x). A comparison of COMET and TACO generated LLVM IR codes shows that COMET results in more optimized code with better SIMD (or vectorization) utilization and loop unrolling. For both SpMV and SpMM, take SpMM as an example. The utilization of many SIMD instructions in TACO is only half of that in COMET (e.g., TACO only uses 2 lanes while COMET uses 4 lanes). COMET unrolls multiple loops by 8 while TACO unrolls them by 2. Although the generated LLVM IR for both SpMV and SpMM show similar differences, the effect of better vectorization and loop unrolling are more evident for larger computation (SpMM). These results highlight one of the major goals of MLIR and MLIR-based compilers: by leveraging higher-level semantics information and progressive lowering steps, it is possible to produce a more aggressive and higher-quality LLVM IR that, eventually, results in higher performance and resource utilization.

For parallel SpMV, COMET achieves an average of 20.92x speedup over TACO. Especially for small matrices, COMET outperforms TACO by a significant margin, however, after further inspection, we realized that this performance difference is due to the overhead introduced by the underlying parallel runtime. COMET uses an asynchronous task-based programming model based on LLVM co-routines while TACO leverages OpenMP. For small computation, LLVM co-routines introduce less overhead than OpenMP threading (which is beneficial for larger parallel regions). As we can see from Figure 7d, when there is enough computation for each OpenMP thread, the runtime overhead is amortized and both COMET and TACO perform similarly.

**Reordering.** By reordering data in memory, COMET attempts to increase spatial and temporal locality to achieve higher performance. The plots in Figure 8 show COMET performance when reordering data compared to original case (no reordering). Figure 8 shows that, indeed, in many cases there is significant advantage of reordering data, with up to 3.41x (average 1.04x), 3.89x (average 1.03x), 7.12x (average 1.12x), and 7.14x (average 1.13x) for SpMV sequential, SpMV parallel, SpMM sequential, and SpMM parallel, respectively. However, we also note that there might be significant performance degradation, especially for parallel execution. We further analyzed the reasons for this disparity and identified load imbalance as the primary source of performance degradation. Our
reordering algorithm attempts to cluster nonzeros on the top-left corner of sparse matrices. In an ideal case, after reordering the nonzeros are distributed around the matrix diagonal.

Figure 9 shows a case in which reordering results in high performance improvements. In this case, the nonzero elements originally around the first column are distributed around the diagonal. Figure 9, instead, shows a case in which reordering reduces performance. In this case, the nonzeros are clustered around the top-left corner, thus threads that operate on the top rows have more work to perform compared to threads that operate on the bottom rows, which results in load imbalance and performance degradation.

TTV and TTM. We also compare COMET with TACO on TTV and TTM with six sparse tensors on CPU and multi-threads and with reordering optimization on and off. Figure 10 illustrates the experimental results. TACO does not generate parallel code if the output tensor is stored in sparse format, even if instructed to do so, thus the results in the Figure for parallel execution are with respect to sequential execution of the TACO benchmarks. For sequential TTV, COMET performs comparably to TACO. With reordering, COMET’s performance is degraded on five of six sparse tensors. For parallel TTV, COMET performs significantly better than TACO with up to 12.5× and on average 8× speedup. With reordering, COMET’s performance is degraded on five of six sparse tensors except for delicious-3d. As for the case of SpMV and SpMM, we observed similar load imbalance issues. For sequential TTM, COMET performs better than TACO with up to 3.3× and on average 2.53× speedup. With reordering, COMET achieves better performance on three out of six sparse tensors. For parallel TTM, COMET performs significantly better than TACO with up to 13.9× and on average 8.13× speedup. With reordering, COMET’s performance is degraded on five of six sparse tensors except for vast-2015-mc1-3d.

Our results show that reordering tensors have a significant (positive or negative) impact on performance, more than for matrices. One possible reason is that the LexiOrder algorithm reorders all dimensions of data simultaneously, which means the data locality is the best when accessing all the dimensions in conjunction, as in conjunction. The sparse tensor operation MTTKRP [42] follows this behavior to gain a good performance speedup. However, this does not mean that the indices in every dimension get good locality when accessing the vector or matrix in TTV or TTM, potentially leading to low performance. We will investigate alternative reordering algorithms and adaptive methods in future work.

9 RELATED WORK

Compiler for Tensor Algebra. Compiler techniques have been used to drive irregular computation in tensor algebra [8, 25, 28, 34, 65]. TCE [25] is a compiler optimization framework that focuses on dense tensor contraction operations in quantum chemistry. TTC [65] is a compiler framework that carries out a composition of high-performance tensor transpose strategies for GPUs. TACO [34] is a compiler that generates code for given tensor algebra expressions and used as a higher-level domain-specific language for tensor algebra. Kim et al. [28] use similar compiler techniques for high-performance tensor contractions but focus on its application on Graphics Processing Unit (GPU). Different from existing works, we develop a high-performance sparse tensor algebra compiler using MLIR, which supports both serial and parallel code generation and enables better portability and adaptability.

Domain-specific Libraries for Tensor Algebra. There have been a collection of tensor algebra libraries developed [21, 23, 24, 53, 55–57, 63, 68]. FLAME [24] is a library aiming for the derivation and implementation of tensor algebra operations on CPUs. Later, serial linear algebra libraries are extended to run on distributed parallel systems [21, 53, 56, 57]. On the other hand, these libraries are extended to support sparse tensor algebra operations using different sparse tensor formats [23, 55, 63]. Tensor algebra libraries favor scientific computing and are widely utilized in scientific application development. By contrast, COMET transparently implements tensor algebra algorithms per se and can compile most types of sparse tensor formats and automatically generate efficient code.

Tensor Algebra Optimization. Plenty of work [6, 9, 12, 13, 37, 40, 62, 74] leverage reordering to optimize tensor algebra with respect to distinct tensor formats for different tensor operations and heterogeneous architectures. Kjolstad et al. [31, 33] reorder loops of tensor algebra computations to improve the data locality. Smith et al. [62] use reordering to enable high-performance tensor factorization operations. Yang et al. [74] identify an efficient memory access pattern for high-performance SpMM operations through merge-based load balancing and row-major coalesced memory access. Other works, such as [11, 18, 40, 47], to name a few, design high-performance algorithms considering computer architecture characteristics using techniques like register blocking, cache blocking, and reordering. COMET.
10 CONCLUSION
In this work, we present a high-performance sparse tensor algebra compiler, called COMET, and a high-productive DSL to support next-generation tensor operations. Our DSL enables high-level programming abstractions that resemble the familiar Einstein notation to express tensor algebra operations. COMET is based on the MLIR framework, which allows us to build portable, adaptable, and extensible compilers. COMET provides an effective and efficient code generator which supports most tensor storage formats through an internal storage format based on four dimension attributes and a novel code generation algorithm. Furthermore, we incorporate a data reordering algorithm to increase the data locality. The evaluation results reveal that COMET outperforms competing for baseline sparse tensor algebra compiler TACO with up to 20.92x, 6.39x, and 13.9x performance improvement for SpMV, SpMM, and TTM computation results reveal that COMET outperforms competing for baseline sparse tensor algebra compiler TACO with up to 20.92x, 6.39x, and 13.9x performance improvement for SpMV, SpMM, and TTM computation results reveal that COMET outperforms competing for baseline sparse tensor algebra compiler TACO with up to 20.92x, 6.39x, and 13.9x performance improvement for SpMV, SpMM, and TTM.

11 ACKNOWLEDGEMENT
This research is supported by PNNL Laboratory Directed Research and Development Program (LDRD), Data-Model Convergence Initiative, project DuoMO: A Compiler Infrastructure for Data-Model Convergence.

REFERENCES
[1] [n.d.]. (n.d.).
[2] 2013. National Institute of Standards and Technology. http://math.nist.gov/MatrixMarket/formats.html.
[3] Evrim Acar, Canan Aykut-Bingo, Haluk Bingo, Rasimnoo Bro, and Bülent Yener. 2007. Multiway analysis of epilepsy tensors. Bioinformatics 23, 13 (2007), 110–118.
[4] Animashree Anandkumar, Rong Ge, and Majid Janzamin. 2017. Analyzing tensor power method dynamics in complete regime. The Journal of Machine Learning Research 18, 1 (2017), 752–791.
[5] Hartwig Anzt, Terry Cojocan, Chen Yen-Chen, Jack Dongarra, Goran Flegar, Pratik Nayak, Stanimire Tomov, Yuhuang Tan, and Weichung Wang. 2020. Load-balancing sparse matrix vector product kernels on GPUs. ACM Transactions on Parallel Computing (TOPC) 7, 1 (2020), 1–26.
[6] Alexander A Auer, Gerald Baumgartner, David E Bernholdt, Alina Bibireata, Venkatesh Chopppla, Daniel Cocoriva, Xiaoyang Gao, Robert Harrison, Srimat Krishnamoorthy, Sandhya Krishna, et al. 2006. Automatic code generation for many-body electronic structure methods: the tensor contraction engine. Molecular Physics (2006).
[7] Brett W. Bader and Tamara G. Kolda. 2007. Efficient MATLAB computations with sparse and factored tensors. SIAM Journal on Scientific Computing 30, 1 (Dec. 2007), 205–231. https://doi.org/10.1137/060667689
[8] Riyad Baghdadi, Jessica Ray, Malek Ben Raddhane, Emanuele Del Sozzo, and Saman Amarasinghe. 2019. Tiramisu: A polyhedral compiler for expressing fast and portable code. In 2019 IEEE/ACM International Symposium on Code Generation and Optimization (CGO). IEEE, 193–205.
[9] Muthu Baskaran, Benoit Meister, and Richard Lethin. 2014. Low-overhead load-balanced scheduling for sparse tensor computations. In 2014 IEEE/ACM International Symposium on Code Generation and Optimization (CGO). IEEE, 193–205.
[10] Muthu Baskaran, Benoit Meister, Nicolas Vasilache, and Richard Lethin. 2012. Efficient and scalable computations with sparse tensors. In 2012 IEEE Conference on High Performance Extreme Computing. IEEE, 1–6.
[11] Nathan Bell and Michael Garland. 2008. Efficient sparse matrix-vector multiplication on throughput-oriented processors. In Proceedings of the conference on high performance computing networking, storage and analysis.
[12] Aart J. C. Bik and Harry A. G. Wijshoff. 1993. Compilation Techniques for Sparse Matrix Computations (ICS '93): Association for Computing Machinery, New York, NY, USA, 416–424.
[13] Robert Blumenthal, Christopher F. Joerg, Bradley C Kuizmaa, Charles E Leiserson, Keith H Randall, and Yuli Zhou. 1995. Cilk: An efficient multithreaded runtime system. ACM SigPlan Notices 30, 8 (1995), 207–216.
[14] Guillaume Bouchard, Jason Naradowsky, Sebastian Riedel, Tim Rocktaschel, and Andreas Vlachos. 2015. Matrix and tensor factorization methods for natural language processing. In Proceedings of the 53rd Annual Meeting of the Association for Computational Linguistics and the 7th International Joint Conference on Natural Language Processing: Tutorial Abstracts. 16–18.
[15] Aydin Buluç and John R Gilbert. 2008. On the representation and multiplication of hypersparse matrices. In 2008 IEEE International Symposium on Parallel and Distributed Processing. IEEE, 1–11.
[16] Yuedan Chen, Guoqing Xiao, M Tamer Ozu, Chubu Liu, Albert Y Zomaya, and Tao Li. 2020. aeSpTV: An adaptive and efficient framework for sparse tensor-vector product kernel on a high-performance computing platform. IEEE Transactions on Parallel and Distributed Systems 31, 10 (2020), 2329–2345.
[17] Lee W Choi, Amik Singh, and Richard W Vuduc. 2010. Model-driven autotuning of sparse matrix-vector multiply on GPUs. ACM sigplan notices 45, 5 (2010), 115–126.
[18] Timothy A Davis and Yifan Hu. 2011. The University of Florida sparse matrix collection. ACM Transactions on Mathematical Software (TOMS) (2011).
[19] Albert Einstein. 1923. Die grundlage der allgemeinen relativitätstheorie. In Das Relativitätsprinzip. Springer.
[20] Evgeniy Efizanovsky, Michael Worrnit, Tomasz Kusi, Arie Landau, Dmitry Zuev, Kirill Khistyaev, Prashant Manohor, Ilya Kaliman, Andreas Drevu, and Anna I Krylov. 2013. New implementation of high-level correlated methods using a general block tensor library for high-performance electronic structure calculations. Journal of computational chemistry (2013).
[21] Xiaowen Feng, Hai Jin, Ran Zheng, Kan Hu, Jingxian Zeng, and Zhuyuan Shao. 2011. Optimization of sparse matrix-vector multiplication with variant CSR on GPUs. In 2011 IEEE 17th International Conference on Parallel and Distributed Systems (IEE), 165–172.
[22] Matthew Fishman, Steven R White, and E Miles Stoudbin. 2020. The Tensornum software library for tensor network calculations. arXiv preprint arXiv:2007.14822 (2020).
[23] John A Gunnels, Fred G Gustavson, Greg M Henry, and Robert A Van De Geijn. 2001. FLAME: Formal linear algebra methods environment. ACM Transactions on Mathematical Software (TOMS) (2001).
[24] So Hirata. 2003. Tensor contraction engine: Abstraction and automated parallel implementation of configuration-interaction, coupled-cluster, and many-body perturbation theories. The Journal of Physical Chemistry A (2003).
[25] Inah Jeon, Evangelos E Papalexakis, U Kang, and Christer Faloutsos. 2015. HaTens2: Billion-scale Tensor Decompositions. In IEEE International Conference on Data Engineering (ICDE).
[26] Peng Jiang, Changwan Hong, and Gagan Agrawal. 2020. A novel data transformation and execution strategy for accelerating sparse matrix multiplication on GPUs. In Proceedings of the 26th ACM SIGPLAN Symposium on Principles and Practice of Parallel Programming. 376–388.
[27] Jinsung Kim, Aravind Sukumaran-Rajam, Vineeth Thumma, Siriram Krishnamoorthy, Ajay Panyala, Louis-Noël Pouchet, Atanans Rountev, and Ponnuswamy Sadayappan. 2019. A code generator for high-performance tensor contractions on gpus. In 2019 IEEE/ACM International Symposium on Code Generation and Optimization (CGO). IEEE.
[28] David R Kincaid, Thomas C Oppe, and David M Young. 1989. ITPACK 2D user’s guide. Technical Report. Texas Univ., Austin, TX (USA). Center for Numerical Analysis.
[29] Vladimir Kiriansky, Yumming Zhang, and Saman Amarasinghe. 2016. Optimizing indirect memory references with milk. In Proceedings of the 2016 International Conference on Parallel Architectures and Compilation. 299–312.
[30] Fredrik Kjolstad, Peter Ahrens, Shoaib Kamil, and Saman Amarasinghe. 2019. Tensor algebra compiler with workspaces. In 2019 IEEE/ACM International Symposium on Code Generation and Optimization (CGO). IEEE, 180–192.
[31] Fredrik Kjolstad, Peter Ahrens, Shoaib Kamil, and Saman Amarasinghe. 2017. Iaco: A Tool to Generate Tensor Algebra Kernels. In 2017 32nd IEEE/ACM International Conference on Automatic Software Engineering (ASE). 943–948. https://doi.org/10.1109/ASE.2017.8115709
[32] Fredrik Kjolstad, Shoaib Kamil, Stephen Chou, David Lugato, Shoaib Kamil, and Saman Amarasinghe. 2017. The Tensor Algebra Compiler. Proc. ACM Program. Lang. 1, OOPSLA, Article 77 (Oct. 2017), 29 pages. https://doi.org/10.1145/3133901
[33] Fredrik Kjolstad, Shoaib Kamil, Stephen Chou, David Lugato, and Saman Amarasinghe. 2017. The Tensor algebra compiler. In Proceedings of the ACM on Programming Languages 1, OOPSLA (2017), 1–29.
[34] Tamara G Kolda and Brett W Bader. 2009. Tensor decompositions and applications. SIAM review 51, 3 (2009), 455–506.
[35] Tamara G Kolda and John Mersman. 2008. Scalable tensor decompositions for multi-aspect data mining. In 2008 Eighth IEEE international conference on data mining. IEEE, 363–372.
[36] Kornillon Kourtis, Vasilis Karakasis, Georgios Goumas, and Nektarios Koziris. 2011. CSS: an extended compression format for spmv on shared memory systems.
[38] Christian Lattner, Jacques Pienaar, Mehdi Amini, Uday邦dhwala, River Riddle, Albert Cohen, Tatsiana Shetipsman, Andy Davie, Nicolai Vasilache, and Oleksandr Zinin.ko. 2028. MLIR: A Compiler Infrastructure for the End of Moore’s Law. arXiv preprint arXiv:2002.11054 (2020).

[39] Benjamin C Lee, Richard W Vuduc, James W Demmel, and Katherine A Yelick. 2004. Performance models for evaluation and automatic tuning of symmetric sparse matrix-vector multiply. In International Conference on Parallel Processing, 2004. ICPP 2004. IEEE, 169–176.

[40] Jiajia Li, Jimeng Sun, and Richard Vuduc. 2018. HiCOO: hierarchical storage of sparse tensors. In SC18: International Conference for High Performance Computing, Networking, Storage and Analysis. IEEE, 238–252.

[41] Jiajia Li, Guiming Tan, Mingyu Chen, and Ninghui Sun. 2013. SMAT: an efficient and effective sparse tensor reordering. In Proceedings of the ACM International Conference on Supercomputing. 227–237.

[42] Xupeng Li, Bin Cui, Yiru Chen, Wentao Wu, and Ce Zhang. 2017. Mlog: Towards declarative in-database machine learning. Proceedings of the VLDB Endowment 10, 12 (2017), 1933–1956.

[43] Benjamin C Lee, Richard W Vuduc, James W Demmel, and Katherine A Yelick. 2004. Performance models for evaluation and automatic tuning of symmetric sparse matrix-vector multiply. In Proceedings of the 34th ACM SIGPLAN conference on Programming language design and implementation. 117–126.

[44] Xupeng Li, Bora Uçar, Umit V Catalyurek, Jimeng Sun, Kevin Barker, and Richard Vuduc. 2019. Efficient and effective sparse tensor reordering. In Proceedings of the ACM International Conference on Supercomputing. 227–237.

[45] Shaden Smith and George Karypis. 2017. Accelerating the Tucker decomposition with compressed sparse tensors. In European Conference on Parallel Processing. Springer, 653–664.

[46] Shaden Smith, Niranjay Ravindran, Nicholas D Sidiropoulos, and George Karypis. 2015. SPLATT: Efficient and parallel sparse tensor-matrix multiplication. In 2015 IEEE International Parallel and Distributed Processing Symposium. IEEE, 61–70.

[47] Edgar Solomonik, Devin Matthews, Jeff R Hammond, John F Stanton, and James Demmel. 2014. A massively parallel tensor contraction framework for coupled-cluster computations. J. Parallel and Distrib. Comput. (2014).

[48] Qingguan Song, Hancheng Ge, James Caverlee, and Xia Hu. 2019. Tensor completion algorithms in big data analytics. ACM Transactions on Knowledge Discovery from Data (TKDD) 13, 1 (2019), 1–48.

[49] Paul Springer, Aravind Sankaran, and Paolo Bientinesi. 2016. TTC: A tensor tensorization compiler for multiple architectures. In Proceedings of the 3rd ACM SIGPLAN International Workshop on Libraries, Languages, and Compilers for Array Programming.

[50] Hubertus JF Van Dam, Danyou Wang, Jarek Nieplocha, Edwardo Araujo, Therun, Tatsuya Yokota and Andrzej Cichocki. 2014. Fast and guaranteed tensor decomposition via sketching. arXiv preprint arXiv:1506.04448 (2015).

[51] James B White and Ponnuswamy Sadayappan. 1997. On improving the performance of sparse matrix-vector multiplication. In Proceedings Fourth International Conference on High-Performance Computing. IEEE, 66–71.

[52] Richard W Vuduc and Hyun-Jin Moon. 2005. Fast sparse matrix-vector multiplication by exploiting variable block structure. In International Conference on High Performance Computing and Communications. Springer, 807–816.

[53] Endong Wang, Qing Zhang, Bo Shen, Guangyang Zhang, Xiaowei Lu, Qing Wu, and Yajuan Wang. 2014. Intel math kernel library. In High-Performance Computing on the Intel® Xeon Phi™. Springer.

[54] Qingguan Song, Hsiao-Yu Tung, Alexander Smola, and Animashree Anandkumar. 2015. Fast and guaranteed tensor decomposition via sketching. arXiv preprint arXiv:1506.04448 (2015).

[55] Samuel Williams, Leonid Oliker, Richard Vuduc, John Shalf, Katherine Yelick, and James Demmel. 2007. Optimization of sparse matrix-vector multiplication on emerging multicore platforms. In SC’07: Proceedings of the 2007 ACM/IEEE Conference on Supercomputing. IEEE, 1–12.

[56] Tatsuya Yokota and Andrzej Cichocki. 2014. Multilinear tensor rank estimation via sparse Tucker decomposition. In 2014 Joint 7th International Conference on Soft Computing and Intelligent Systems (SCIS) and 15th International Symposium on Advanced Intelligent Systems (ISIS). IEEE, 478–483.

[57] Shengen Yan, Chao Li, Yunquan Zhang, and Huynh Z. 2014. yasPMV: yet another SpMV framework on GPUs. Acm Sigplan Notices 49, 8 (2014), 107–118.

[58] Carl Yang, Aydun Buluç, and John D Owens. 2018. Design principles for sparse matrix multiplication on the GPU. In European Conference on Parallel Processing. Springer.

[59] Robert Paige and Robert E Tarjan. 1987. Three partition refinement algorithms. ACM Transactions on Mathematical Software (TOMS) 13, 1 (1987), 793–989.

[60] Jack Pauley, Bryan Marker, Robert A van de Geijn, and Katherine Yelick. 2014. Exploiting symmetry in tensors for high performance: Multiplication with compressed sparse tensors. In Proceedings of the 15th ACM SIGKDD international conference on Knowledge discovery and data mining. 727–736.

[61] Shaden Smith, Tatsuya Yokota, and Andrzej Cichocki. 2014. Multilinear tensor rank estimation via sparse Tucker decomposition. In 2014 Joint 7th International Conference on Soft Computing and Intelligent Systems (SCIS) and 15th International Symposium on Advanced Intelligent Systems (ISIS). IEEE, 478–483.

[62] Samuel Williams, Leonid Oliker, Richard Vuduc, John Shalf, Katherine Yelick, and James Demmel. 2007. Optimization of sparse matrix-vector multiplication on emerging multicore platforms. In SC’07: Proceedings of the 2007 ACM/IEEE Conference on Supercomputing. IEEE, 1–12.

[63] Tatsuya Yokota and Andrzej Cichocki. 2014. Multilinear tensor rank estimation via sparse Tucker decomposition. In 2014 Joint 7th International Conference on Soft Computing and Intelligent Systems (SCIS) and 15th International Symposium on Advanced Intelligent Systems (ISIS). IEEE, 478–483.

[64] Qingguan Song, Hsiao-Yu Tung, Alexander Smola, and Animashree Anandkumar. 2015. Fast and guaranteed tensor decomposition via sketching. arXiv preprint arXiv:1506.04448 (2015).

[65] James B White and Ponnuswamy Sadayappan. 1997. On improving the performance of sparse matrix-vector multiplication. In Proceedings Fourth International Conference on High-Performance Computing. IEEE, 66–71.