Exact solutions of the associated Camassa-Holm equation

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April 1, 2022

Abstract

Recently the associated Camassa-Holm (ACH) equation, related to the Fuchssteiner-Fokas-Camassa-Holm (FFCH) equation by a hodograph transformation, was introduced by Schiff, who derived Bäcklund transformations by a loop group technique and used these to obtain some simple soliton and rational solutions. We show how the ACH equation is related to Schrödinger operators and the KdV hierarchy, and use this connection to obtain exact solutions (rational and N-soliton solutions). More generally, we show that solutions of ACH on a constant non-zero background can be obtained directly from the tau-functions of known solutions of the KdV hierarchy on a zero background. We also present exact solutions given by a particular case of the third Painlevé transcendent.

1 Introduction

A great deal of interest has been generated by the Fuchssteiner-Fokas-Camassa-Holm (FFCH) equation,

\[ u_T = 2f_X u + f u_X, \quad u = \frac{1}{2} f_{XX} - 2f \] (1.1)

(we are using the slightly non-standard choice of coefficients taken in [21]), which originally appeared in the work of Fuchssteiner and Fokas [3], but was later derived as an equation for shallow water waves by Camassa and Holm [4]. Of particular importance was the discovery [4] that (1.1) admits peaked solitons.
or “peakons”, described in terms of solutions of an associated integrable finite-dimensional dynamical system which has subsequently been related to the Toda lattice \cite{1}. Although the FFCH equation is integrable, and has been shown \cite{10} to be related by a hodograph transformation to the first negative flow of the KdV hierarchy (also known as the AKNS equation \cite{1}),

\[ R(U) U_t = 0, \quad R(U) = \partial_x^2 + 4U + 2U_x \partial_x^{-1}, \]  

(1.2)

it has many non-standard features (for instance, it possesses only the weak Painlevé property \cite{15}) and there is still much to be understood about its solutions. We refer the reader to \cite{21} for a more complete list of references concerning (1.1).

Inspired by \cite{10}, Schiff introduced the associated Camassa-Holm (ACH) equation \cite{21}

\[ p_t = p^2 f_x, \quad f = \frac{p}{4} \left( \log[p] \right)_x - \frac{p^2}{2}, \]  

(1.3)

which (for positive \( u \)) has a one to one correspondence with solutions of the FFCH equation (1.1) given by

\[ p = \sqrt{u}, \quad dx = p \, dX + pf \, dT, \quad dt = dT \]  

(1.4)

(the independent variables \( x, t \) of ACH are denoted \( t_0, t_1 \) in \cite{21}); a solution of ACH where \( p \) has zeros corresponds to a number of solutions of (1.1) where \( u \) has fixed sign. In \cite{21}, a loop group interpretation was given for (1.3), making use of the fact that it is the (zero curvature) compatibility condition for the linear system

\[ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = \begin{pmatrix} 0 & 1/p \\ p/\lambda + 1/p & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \]  

(1.5)

\[ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = \begin{pmatrix} -p_t/2p & \lambda \\ \lambda - 2f & p_t/2p \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \]  

(1.6)

and it was indicated that this ACH equation is part of an integrable hierarchy of zero curvature equations. Automorphisms of the loop group were used to derive two Bäcklund transformations (BTs) for (1.3), and these were applied to the constant background solution \( p = h \) to obtain some interesting solutions, in particular a simple rational solution, one- and two-soliton solutions (all on the same constant background) as well as a superposition formula. By applying the hodograph transformation (1.4) apparently novel solutions of the FFCH equation (1.1) were generated (although it seems that in general the quadratures involved cannot be evaluated explicitly).

In the following we show that, since the hodograph transformation is essentially the same as in \cite{10}, the ACH equation is also related to the inverse KdV
equation (1.2) and has a simple Lax pair of which one part is just a (time-independent) Schrödinger equation. Thus we see that the first BT presented in [21], and the corresponding solutions, can be derived by considering the usual Crum transformation for Schrödinger operators [4] and further show that a solution of ACH on a constant non-zero background may be obtained from the tau-function of a corresponding solution of the KdV hierarchy on a zero background. Rational solutions of ACH are given in terms of the Adler-Moser polynomials [2] for KdV, while the N-soliton on constant background may be obtained immediately from the standard formulae for KdV N-soliton solutions.

In the next section we present some simple properties and solutions of (1.3), as well as summarizing the BTs introduced in [21] and showing how they are related to the Schrödinger equation. Section three contains a demonstration that solutions of (1.3) on a constant background can be obtained from the tau-functions of solutions of the KdV hierarchy vanishing at infinity. The fourth section contains a brief summary of the exact solutions of (1.3) we have obtained, in particular rational and soliton solutions (which are derived as a corollary of the result in section three), as well as a conjectured form of elliptic solutions, and solutions in terms of the third Painlevé transcendent (PIII).

2 Basic properties of the ACH equation

2.1 Simple solutions, Lax pair and tau-function

In order to explore the properties of the ACH equation (1.3), we find it is convenient to rewrite it in two different ways. For analyzing reductions of the equation, it is useful to write it in the form

\[
4[p^{-1}]_t + \left[p^{-1}(pp_{xt} - p_xp_t - 2p^3)\right]_x = 0. \tag{2.1}
\]

The simplest solutions come from the travelling wave reduction (also considered in [21]), and it is easy to see that the general solution of this type is given in terms of the Weierstrass \( \wp \)-function [24],

\[
p(x - ct) = -c\wp(x - ct) + k \tag{2.2}
\]

(\( k \) is constant); the special case \( c = 0 \) yields the solution \( p = \text{const} \). Degenerations of the \( \wp \)-function give the one-pole rational solution and one-soliton solution (both on a constant background \( h \)), respectively

\[
p = h - \frac{h^3}{(x - h^3t)^2}, \quad p = \left(h + \frac{\lambda}{h}\right)\tanh^2\left(\sqrt{\frac{1}{h^2} + \frac{1}{\lambda}}(x + \lambda ht)\right) - \frac{\lambda}{h}. \tag{2.3}
\]

In the form (2.1) it is also straightforward to obtain the scaling similarity reduction of the ACH equation,

\[
p = (2t)^{-\frac{1}{2}}w(z), \quad z = (2t)^{\frac{1}{2}}x,
\]
where \( w(z) \) satisfies the ODE
\[
 w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{1}{z} \left( 2w^2 + \beta \right) - \frac{4}{w};
\] (2.4)

\( ' \) denotes \( d/dz \) and \( \beta \) is an arbitrary constant. This ODE is a special case of the Painlevé transcendent \( \text{PIII} \) [4], and we shall consider it again in section 4. For now we simply observe that for \( \beta = 0 \) it admits the particular solution \( w = (2z)^{\frac{1}{3}} \).

The second way to rewrite the ACH equation is
\[
 U_t = -2p_x, \quad U = -\frac{1}{2} \left( pp_{xx} - \frac{1}{2}p_x^2 + 2 \right). \] (2.5)

This is the best way to write it, as it leads directly to a connection with the KdV hierarchy. In fact, by writing \( p = -\frac{1}{2} \partial_x^{-1} U_t \), it is possible to show directly that \( U \) satisfies the inverse KdV equation (1.2). This suggests that there should be a direct link with Schrödinger operators, and indeed this is the case (I am grateful to Decio Levi and Orlando Ragnisco for pointing this out).

Setting \( \phi = p^{\frac{1}{2}} \psi_1 \) (and using \( \psi_2 = p\psi_{1,x} \)) in (1.5), (1.6) leads to the Lax pair
\[
 (\partial_x^2 + U - 1/\lambda) \phi = 0, \tag{2.6}
\]
\[
 \phi_t = \lambda(p\phi_x - \frac{1}{2}p_x \phi). \tag{2.7}
\]

The two compatibility conditions for this Lax pair (assuming that \( U \) is as yet undefined) are simply the first equation for \( U_t \) in (2.5) and
\[
 (\partial_x^3 + 4U\partial_x + 2U_x)p = 0,
\]
and it is an immediate consequence that \( U \) satisfies (1.2). The latter third order equation for \( p \) can be integrated once to give
\[
 pp_{xx} - \frac{1}{2}p_x^2 + 2U p^2 + g(t) = 0. \tag{2.8}
\]

Equation (2.8) is known as the Ermakov-Pinney equation (see [8] and references). The function \( g(t) \) is arbitrary, but it is clear that the ACH equation (with the definition of \( U \) as in (2.5)) corresponds to the particular choice \( g = 2 \). Thus solutions of the ACH equation correspond to a particular class of solutions of the inverse KdV equation.

We have also applied the WTC generalization of the Painlevé test [23] and found that it is satisfied by the ACH equation. Laurent expansions around a non-characteristic singularity manifold \( \zeta(x, t) = 0 \) have leading order terms \( p = -(\log(\zeta))_{xt} + \ldots \), and this principal balance has resonances \(-1, 2, 4\). Instead of considering such expansions in more detail, we merely note the connection
between the Painlevé property and Hirota’s method \[12\], and thus define the tau-function of ACH by a truncation of this expansion:

\[ p(x,t) = -(\log[\tau])_{xt}. \]

Note that from the ACH equation in the form \( (2.5) \) we can integrate once to find the standard KdV formula for the potential of the Schrödinger operator:

\[ U(x,t) = 2(\log[\tau])_{xx}. \]

The tau-function satisfies a trilinear equation in \( x, t \) (this is a reduction of the trilinear appearing in \[22\]). We observe that for the constant background solutions of ACH constructed in section 4 this \( \tau \) is a tau-function of the KdV hierarchy after rescaling by a factor \( \exp[-x^2/(4h^2) - hxt] \); similarly for the solutions given in terms of the third Painlevé transcendent, \( \tau \) should correspond to the tau-function for PIII introduced by Okamoto \[19\].

2.2 Bäcklund transformations

By the use of loop group techniques, Schiff \[21\] derived two BTs for the ACH equation, which he considered in the original form \( (1.3) \). The first of these BTs is

\[ \tilde{p} = p - s_x, \quad s_x = -(p\lambda)^{-1}s^2 + \lambda p^{-1} + p, \quad s_t = -s^2 + (\log[p])_t s + \lambda(\lambda - 2f), \quad (2.9) \]

where \( \lambda \) is a Bäcklund parameter. A superposition principle was also found for this BT, leading to a formula for the 2-soliton solution of ACH. Schiff’s second BT may be written (after some simplification) as

\[ \tilde{p} = p - (\log[\chi])_{xt}, \quad (pB_x)_x = B(p^{-1} + p\lambda^{-1}), \quad B_t = -\frac{1}{2}(\log[p])_t B + p\lambda B_x, \quad (2.10) \]

where \( \chi \) is determined from the first order equations

\[ \chi_x = p\lambda^{-1}B^2, \quad \chi_t = \lambda(p^2B_x^2 - B^2). \]

We observe that (as also noted in \[21\]) the equations for \( B \) in \( (2.10) \) follow from the linearization of the first Riccati equation in \( (2.9) \) via the substitution \( s = p\lambda(\log[B])_x \); this linearization is just the linear problem \( (1.5) \) when we identify \( B = \psi_1 \). It seems that the second BT actually gives nothing new, as it appears to be equivalent to applying \( (2.9) \) twice with the same Bäcklund parameter each time. Thus we concentrate on the first BT, and observe that it is more convenient to linearize the second Riccati equation in \( (2.9) \) by the substitution \( s = (\log[\phi])_t \), and then it is straightforward to show that the BT \( (2.9) \) is equivalent to

\[ \tilde{p} = p - (\log[\phi])_{xt}, \]
where $\phi$ is a solution of the Lax pair (2.6), (2.7). In fact, it is further possible to show (by a tedious direct calculation) that under this BT the potential of the Schrödinger operator becomes

$$\tilde{U} = U + 2(\log[\phi])_{xx},$$

and also that $\tilde{\phi} = \phi^{-1}$ is a solution of the same Lax pair with $U, p$ replaced by $\tilde{U}, \tilde{p}$. Thus it is apparent that the first BT (2.9) derived by Schiff is equivalent to the well known Crum transformation obtained by factorization of the Schrödinger operator [4], which gives the standard Darboux-Bäcklund transformation for the KdV hierarchy.

Having seen the connection with the Crum transformation, it is clear that it should be possible to express solutions of the ACH equation in terms of known solutions of the KdV hierarchy obtained by this transformation. Before proving a more general statement in the next section, we briefly illustrate this idea by presenting the simplest rational solutions. Applying the first BT (2.9) above starting from the constant background solution $p_0 = h$, and choosing the Bäcklund parameter $\lambda = -h^2$ each time, we obtain solutions of the form

$$p_k = h - (\log[\theta_k])_{xt},$$

where the polynomials $\theta_k$ are given by

$$\theta_0 = 1, \quad \theta_1 = \tau_1, \quad \theta_2 = \tau_1^2 + \tau_2, \quad \theta_3 = \tau_1^6 + 5\tau_2\tau_1^4 + \tau_3 - 5\tau_2^2,$$

where in terms of $x, t$

$$\tau_1 = \tilde{\tau}_1 + x - h^3t, \quad \tau_2 = \tilde{\tau}_2 + 3h^5t, \quad \tau_3 = \tilde{\tau}_3 - 45h^7t$$

(for $\tilde{\tau}_j$ independent of $x, t$). The $\theta_k$ are the Adler-Moser polynomials [2], which are the tau-functions of the vanishing rational solutions of the KdV hierarchy. The $\tau_j$ are the times of the hierarchy, up to a suitable scaling; in the next section we will choose a more canonical normalization for these times.

### 3 Solutions of ACH from KdV

Before stating and proving the main result, we fix our conventions by briefly reviewing well known properties of the KdV hierarchy and its tau-functions (which may be found in many places, [17] for example). The KdV hierarchy is the sequence of evolution equations

$$q_{t_{2j-1}} = 2(P_j[q])_{t_1}$$

(for $j = 1, 2, 3, \ldots$), which arise as the compatibility condition for the Schrödinger equation

$$(\partial_{t_1}^2 + q)\phi = \mu^2\phi$$
with the sequence of linear problems

$$\phi_{t_{2j+1}} = \Pi_j \phi_{t_1} - \frac{1}{2} \Pi_{j,t_1} \phi, \quad \Pi_j[q;\mu] := \sum_{k=0}^j P_{j-k}[q] \mu^{2k}, \quad P_0 = 1 \quad (3.3)$$

(for $j = 0, 1, 2, \ldots$). The $t_{2j-1}$ are the times of the hierarchy (the odd times of the KP hierarchy \([18]\)), and the differential polynomials $P_k[q]$ are the Gelfand-Dikii polynomials \([11]\), which can be defined recursively using a form of the Ermakov-Pinney equation \((2.8)\). Also, in terms of the tau-function $\tau(t_1, t_3, t_5, \ldots)$ of the hierarchy, $q$ and the $P_j$ are given by

$$q = 2(\log[\tau])_{t_1 t_1}, \quad P_j = (\log[\tau])_{t_1 t_{2j-1}}. \quad (3.4)$$

This tau-function satisfies a sequence of bilinear equations \([18]\), but we shall not make use of these here.

To make the connection with the ACH equation we simply observe that for a solution of ACH satisfying $p \to h$ at infinity, it is clear that $U$ as defined in \((2.5)\) satisfies $U \to -1/h^2$. Thus considering the Schrödinger equation \((2.6)\) with such a potential $U$ is equivalent to instead taking a Schrödinger equation \((3.2)\) with potential $q$ vanishing at infinity, when we identify

$$q = U + 1/h^2, \quad \mu = \sqrt{1/h^2 + 1/\lambda}. \quad (3.5)$$

This immediately suggests the following

**Proposition.** Given a tau function $\tau(t_1, t_3, t_5, \ldots)$ for a solution $q(t_1, t_3, t_5, \ldots)$ of the KdV hierarchy satisfying $q \to 0$ as $|t_1| \to \infty$, a corresponding solution of the ACH equation \((2.3)\) on constant background $h$ is given by

$$p = h - (\log[\tau])_{xt}, \quad t_1 = \tilde{t}_1 + x - h^3 t, \quad t_{2j+1} = \tilde{t}_{2j+1} - h^{2j+3}t \quad (j \geq 1) \quad (3.6)$$

(with $\tilde{t}_{2j+1}$ independent of $x, t$). The corresponding Schrödinger potential is given by

$$U = q - 1/h^2 = -1/h^2 + 2(\log[\tau])_{xx}. \quad (3.6)$$

Clearly a more general statement is possible, applying to non-vanishing (non-constant background) solutions also, but for these purposes we are concerned with the explicit appearance of the background parameter $h$. The proof of the proposition is very straightforward, for using \((3.6)\) and \((3.4)\) we see that we can write

$$p = h + \sum_{j=1}^\infty h^{2j+1}(\log[\tau])_{t_1 t_{2j-1}} = \sum_{j=0}^\infty P_j h^{2j+1}.$$
Thus the right hand equation in (2.5) (the Ermakov-Pinney equation) is naturally rewritten as

\[ pp_{t_1,t_1} - \frac{1}{2}p_{t_1}^2 + 2(q - 1/h^2)p^2 + 2 = 0, \]

and by expanding in powers of \( h \) the recursion relations for the Gelfand-Dikii polynomials \( P_j \) are obtained. Hence \( p \) as defined above is automatically a solution of this Ermakov-Pinney equation, and writing everything in terms of the tau-function the ACH equation itself (the left hand equation in (2.3)) is just the tautology \( 2(\log[\tau])_{xxt} = 2(\log[\tau])_{xxt} \).

It is also fairly straightforward to show that, provided \( \mu \) is identified as in (3.5), \( \phi \) satisfying (3.2) and the sequence of linear problems (3.3) provides a solution to the ACH Lax pair (2.6), (2.7) (thus providing an alternative proof of the proposition). In order to show that (2.7) is satisfied, it is necessary to write

\[ \phi_t = -\infty \sum_{j=0}^{\infty} h^{2j+3}\phi_{t_{2j+1}} = -\infty \sum_{j=0}^{\infty} h^{2j+3} \left( \Pi_j \phi_{t_1} - \frac{1}{2} \Pi_j, \phi \right); \]

expanding each \( \Pi_j \) in \( \mu \) and resumming by use of the geometric series

\[ \lambda = \frac{-h^2}{1 - \mu^2 h^2} = -\infty \sum_{k=0}^{\infty} h^{2j+2} \mu^{2j}, \]

and noting that \( t_1 \) derivatives may be replaced by \( x \) derivatives, (2.7) results.

4 Exact solutions

4.1 Rational and soliton solutions

It is now a simple matter to obtain solutions of the ACH equation from known solutions of KdV. For instance, the sequence of rational solutions mentioned at the end of section 2 correspond to tau-functions \( \tau^{(k)} \) which are most easily expressed as Wronskian determinants of odd Schur polynomials,

\[ \tau^{(k)} = [p_{2k-1}, p_{2k-3}, \ldots, p_1] \]

for \( k = 1, 2, 3, \ldots \). The sequence of Schur polynomials may be defined by a generating function, \( \sum_{i=0}^{\infty} p_i \nu^i = \exp \left[ \sum_{j=1}^{\infty} t_j \nu^j \right] \) (see e.g. [18]). The above Wronskians are independent of the even times \( t_{2k} \), and we are using the notation \([\ldots]\) to denote the Wronskian as in [13]. Thus we find the sequence of rational solutions

\[ p^{(k)} = h - (\log[\tau^{(k)}])_{xxt}, \]

where the \( t_j \) are given in terms of \( x, t \) as in (3.4). This agrees with the formulae in section 2, since after rescaling the \( \theta_k \) introduced by Adler and Moser [2] are the same as these \( \tau^{(k)} \).
Similarly it is well known that the tau-functions for soliton solutions of integrable hierarchies can be written as Wronskian determinants \([18, 20]\). For KdV the N-soliton tau-function is built out of \(N\) functions \(\eta_j\) of the form

\[
\eta_j = \exp[\xi(t_1, t_3, \ldots; \mu_j)] + c_j \exp[\xi(t_1, t_3, \ldots; -\mu_j)],
\]

where \(\xi(t_1, t_3, \ldots; \mu) = \sum_{k=1}^{\infty} t_{2k-1} \mu^{2k-1}\). Using the expressions (3.6) for the \(t_{2j-1}\) and defining \(\lambda_j = -h^2(1 - \mu_j^2 h^2)^{-1}\) (which may be written as a geometric series as in the previous section), we find that we may write

\[
\eta_j = \exp\left[\sqrt{\frac{1}{h^2} + \frac{1}{\lambda_j}} (x - h\lambda_j t + x_j)\right] + c_j \exp\left[-\sqrt{\frac{1}{h^2} + \frac{1}{\lambda_j}} (x - h\lambda_j t + x_j)\right],
\]

for \(x_j, c_j\) constants (compare with the 1-soliton formula (2.3)). Then the N-soliton solution of the ACH equation may be written as

\[
p(N) = h - (\log[W(N)])_{xt}, \quad W(N) = [\eta_1, \eta_2, \ldots, \eta_N],
\]

(where all the \(t_1\) derivatives from the KdV Wronskian formula [20] may be replaced by \(x\) derivatives).

### 4.2 Elliptic solutions

Given that the ACH equation admits the simple elliptic solution (2.2), it would be interesting to see whether more general solutions could be obtained from the ansatz

\[
p = -\dot{x}_j \wp(x - x_j) + \text{const},
\]

where the poles \(x_j = x_j(t)\) depend on time, and \(\dot{x}_j = \frac{d}{dt} x_j\). In the light of known results on KdV [16] we would expect that this ansatz should be correct provided that the poles move according to a constrained elliptic Calogero-Moser system.

### 4.3 Solutions in terms of PIII

As we have already seen, the ACH equation has a scaling similarity reduction leading to a particular case (2.4) of PIII, which (with the standard form of PIII as in [4]) corresponds to the choice of parameters \(\alpha = 2, \gamma = 0, \delta = -4\) (and \(\beta\) remaining arbitrary). PIII has a large number of BTs, and some special exact solutions, which are systematically catalogued in [4]. However, for the choice of parameters relevant here, the only BTs that survive are one referred to in [4] as transformation V,

\[
\tilde{w} = \frac{zw'}{w^2} - \frac{\beta + 2}{2w} + \frac{2z}{w^2},
\]

together with its inverse

\[
w = -\frac{z\tilde{w}'}{\tilde{w}^2} - \frac{\beta - 2}{2\tilde{w}} + \frac{2z}{\tilde{w}^2}.
\]
These BTs send a solution to (2.4) for parameter $\beta$ to another solution with parameter $\beta + 4$, $\beta - 4$ respectively.

For this special choice of parameter values there is a hierarchy of special solutions rational in $z^{\frac{1}{3}}$, which can be obtained by applying the BTs to the special seed solution $w_0 = (2z)^{\frac{1}{3}}$ for $\beta = 0$. For example, for $\beta = \pm 4$ (2.4) admits the special solutions

$$w_{\pm 4} = \frac{6z^{\frac{2}{3}} \mp 2^\frac{1}{3}}{3.2^{\frac{1}{3}} z^{\frac{1}{3}}}.$$

Other solutions in this hierarchy can be obtained from table 6 of [4] (on setting the parameters $\mu = 2$, $\kappa = 2^{\frac{1}{3}}$).

## 5 Conclusions

We have shown how the ACH equation introduced by Schiff is related to the KdV hierarchy, and used this connection to construct a variety of exact solutions. We have also found solutions in terms of a particular case of the Painlevé transcendent PIII. By the use of the hodograph transformation (1.4) these solutions of the ACH equation yield solutions of the FFCH equation (1.1), and it would be interesting to study the transformed solutions. We have also found [14] that similar methods apply to the 2+1-dimensional generalization of the FFCH equation introduced in [7].

## 6 Acknowledgements

It is a pleasure to thank Orlando Ragnisco, Decio Levi, Andrew Pickering and Jeremy Schiff for useful discussions. I would also like to thank Sergei Manakov and others present at the Integrable Systems Seminar of Roma 'La Sapienza' for helpful comments. I am very grateful to the Leverhulme Trust for giving me a Study Abroad Studentship in Rome.

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