Abstract

Using Mackey’s classification of unitary representations of the Poincaré group on massless states of arbitrary helicity we disprove the claim that states with helicity $|h| \geq 1$ cannot couple to a conserved current by constructing such a current.

Keywords: Unitary representation, Poincaré group, Helicity, Massless states, Currents, Supersymmetry, Position operator

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The constitutive feature of a relativistic quantum system is the existence of a unitary representation (of the connected part of the cover) of the Poincaré group: a quantum system is relativistic if its Hilbert space $\mathcal{H}$ carries such a representation. It endows the states with physical properties, e.g. the representation $U_a = e^{iP_a}$ of translations $x \mapsto x + a$ is generated by commuting, hermitean operators $P = (P^0, P^1, P^2, P^3)$, $P \cdot a = P^0 a^0 - P^1 a^1 - P^2 a^2 - P^3 a^3$. Therefore, by definition, $P$ is the four-momentum and its eigenstates are states with definite energy and momentum.

The representation has to be unitary and its generators have to be antihermitean for consistency with the basic postulate of quantum mechanics: if one measures the state $\Psi$ with an ideal device $A$, which for simplicity has discrete, nondegenerate results, then the probability $w$ for the result $a_i$ to occur is

$$w(i, A, \Psi) = | \langle \Lambda_i | \Psi \rangle |^2 .$$

(1)

Here $\Lambda_i$ are the states (they exist by the assumption that the device $A$ is ideal) which yield $a_i$ with certainty, $w(i, A, \Lambda_j) = \delta_{ij}$.

No formulation of quantum theory can do without this statement of what the physical situation is in case that mathematically the system is in a state $\Psi$. We remind the reader of these well known foundations of quantum mechanics only to stress the importance of the antihermiticity of the generators $-iH$ of the representation. If $H$ were not hermitean, then its generated flow in Hilbert space $e^{-iHt} : \Psi(0) \mapsto \Psi(t)$ would not conserve the sum of probabilities, $\sum w(i, A, \Psi) = 1$.

The construction of unitary representations $g \mapsto U_g$ of the Poincaré group by Wigner [1] were shown by Mackey [2] to yield all unitary representations for which for all $\Phi$ and $\Psi$ in $\mathcal{H}$ the maps $g \mapsto \langle \Phi | U_g \Psi \rangle$ are measurable.

By Mackey’s theorems all unitary representations of the Poincaré group are sums or integrals of irreducible representations and each irreducible, unitary representation of the Poincaré group is uniquely specified by its mass shell $\mathcal{M}$, the orbit under the proper Lorentz group $G$ of a momentum $p$, and an irreducible, unitary representation $R$ of its stabilizer $H = \{ h \in G : h p = p \}$.

An orbit $\mathcal{M} = \{ p : p = g p, g \in G \}$ can be viewed as the set $G/H$ of cosets $g H$ on which $G$ acts by left multiplication. The projection of $G$ to the mass shell $G/H$

$$\pi : \begin{cases} G & \rightarrow & G/H \\ g & \mapsto & p = g p \end{cases}$$

(2)

attributes to each Lorentz transformation $g$ the momentum $p = g p$ which is obtained by applying $g$ to $p$. The fiber over a point $p$ consists of all transformations $g$, which map $p$ to $p$. It is the coset $g H$ of the stabilizer $H$ of $p$. Any two fibers are either disjoint or identical and $G$ is the union of all fibers, it is a fiber bundle.

A local section of $G$ is a collection of maps $\sigma_{\alpha}$, enumerated by Greek indices which we exempt from the summation convention. In its domain $\mathcal{U}_\alpha$ the map $\sigma_{\alpha}$ is a right inverse of the projection $\pi$. In other words: $\sigma_{\alpha}$ assigns to each $p \in \mathcal{U}_\alpha$ a transformation $\sigma_{\alpha}(p)$ which maps $p$ to $p$, $\langle \sigma_{\alpha}(p) \rangle \mathcal{U} = p$, $\sigma_{\alpha} : \begin{cases} \mathcal{U}_\alpha \subset G/H & \rightarrow & G \\ p & \mapsto & \sigma_{\alpha}(p) \end{cases}$

(3)

$$\pi \sigma_{\alpha} = \text{id}_{\mathcal{U}_\alpha} .$$

The domains $\mathcal{U}_\alpha$ are required to cover the base manifold $\cup \mathcal{U}_\alpha = G/H$. If this can be done already with one domain, $\mathcal{U} = G/H$, then the section is called a global section and the fiber bundle is trivial, i.e. equivalent to the product of a fiber times the base manifold.

In their common domain two sections differ by their transition function $g_{\alpha \beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \rightarrow H$

$$\sigma_{\beta}(p) = \sigma_{\alpha}(p) g_{\alpha \beta}(p) , \quad h_{\alpha \beta}(p) = h_{\beta \alpha}(p) .$$

(4)
As group multiplication is associative, the transition functions are related by

$$h_{\alpha\beta}h_{\beta\gamma} = h_{\alpha\gamma}$$  \hspace{1cm} (5)

in each intersection $U_\alpha \cap U_\beta \cap U_\gamma$ of three domains.

Let $\mathbb{C}^{2s+1}$ be a space on which a unitary, irreducible representation $R$ of the stabilizer $H$ of some momentum $p$ acts. Consider the space $\mathcal{H}_R$ of functions

$$\Psi : G \to \mathbb{C}^{2s+1}$$  \hspace{1cm} (6)

with a fixed dependence along each fiber (for all $g \in G$ and $h \in H$)

$$\Psi(gh) = R(h^{-1})\Psi(g) .$$  \hspace{1cm} (7)

As $R$ is unitary, the scalar product of the values of two functions $\Phi, \Psi \in \mathcal{H}_R$ at $g$

$$\langle \Phi | \Psi \rangle_g = (\Phi(g), \Psi(g)) = (R(h^{-1})\Phi(g), R(h^{-1})\Psi(g))$$

is constant along each fiber $p = gh$ and therefore a function of $\mathcal{M} = G/H$. This allows to define the scalar product in $\mathcal{H}_R$ as

$$\langle \Phi | \Psi \rangle = \int_{\mathcal{M}} \tilde{d}p \langle \Phi | \Psi \rangle_p .$$  \hspace{1cm} (9)

We choose the integration measure Lorentz invariant and with the conventional normalization factors

$$\tilde{d}p = \frac{d^3p}{(2\pi)^2 \sqrt{m^2 + p^2}} .$$  \hspace{1cm} (10)

Then $\mathcal{H}_R$ becomes a Hilbert space if in addition we require its elements, the wave functions, to be measurable and square integrable with respect to $\tilde{d}p$. $\mathcal{H}_R$ carries a unitary representation of the Poincaré group: translations are represented on the wave functions $\Psi$ multiplicatively

$$(U_g \Psi)(g) = e^{ip\cdot g} \Psi(g) , \hspace{0.5cm} p = g \underline{p} .$$  \hspace{1cm} (11)

Lorentz transformations $\Lambda$ of the wave functions are defined to act by inverse left multiplication of the argument

$$(U_\Lambda \Psi)(g) = \Psi(\Lambda^{-1}g) .$$  \hspace{1cm} (12)

Consistent with (7) they map $\mathcal{H}_R$ unitarily to itself,

$$\langle U_\Lambda \Phi | U_\Lambda \Psi \rangle = \int_{\mathcal{M}} \tilde{d}(Ap) \langle U_\Lambda \Phi | U_\Lambda \Psi \rangle_{Ap} =$$

$$= \int_{\mathcal{M}} \tilde{d}(Ap) \langle \Phi | \Psi \rangle_p = \int_{\mathcal{M}} \tilde{d}p \langle \Phi | \Psi \rangle_p = \langle \Phi | \Psi \rangle .$$  \hspace{1cm} (13)

The map $\Lambda \mapsto U_\Lambda$ is a representation of $G$ in $\mathcal{H}_R$ and is said to be induced by the representation $R$ of the stabilizer $H$. By Mackey’s theorems the representation of the Poincaré group is irreducible if it is induced by an irreducible unitary representation $R$. Vice versa, each unitary, irreducible representation of the Poincaré group is unitarily equivalent to an induced representation. Two of them are equivalent if and only if their mass shells agree and their inducing representations are equivalent.

If the representation $R$ is trivial, $R = 1$, then the wave functions $\Psi \in \mathcal{H}_R$ are constant along each fiber $gH$ and naturally define functions of the mass shell $G/H$.

If $R$ is nontrivial, then by composition with a local section $\sigma_\alpha$ one obtains a one-to-one correspondence of each function $\Psi \in \mathcal{H}_R$ to a vector bundle over $G/H$ consisting of the collection of functions $\Psi_\alpha$,

$$\Psi_\alpha = \Psi \circ \sigma_\alpha : U_\alpha \to \mathbb{C}^{2s+1} ,$$  \hspace{1cm} (14)

which by (7) are related in $U_\alpha \cap U_\beta$ by

$$\Psi_\beta = \Psi(\sigma_\alpha h_{\alpha\beta}) = R(h_{\alpha\beta}^{-1})\Psi_\alpha = R(h_{\beta\alpha})\Psi_\alpha .$$  \hspace{1cm} (15)

A transformation $\Lambda \in G$ maps $\sigma_\alpha(p)$ by left multiplication to $\Lambda \sigma_\alpha(p)$ in the fiber of $(Ap) \in U_\alpha$ for some $\beta$ which by (13) is related to $\sigma_\beta(Ap)$ by an $H$-transformation, the Wigner rotation $W_{\beta\alpha}(\Lambda, p)$,

$$\Lambda \sigma_\alpha(p) = \sigma_\beta(\Lambda p)W_{\beta\alpha}(\Lambda, p) .$$  \hspace{1cm} (16)

Inserted into (12) one obtains the transformation of the wave function $\Psi_\beta$

$$(U_\Lambda \Psi)_\beta(\Lambda p) = (U_\Lambda \Psi)(\sigma_\beta(Ap)) =$$

$$= (U_\Lambda \Psi)(\Lambda \sigma_\alpha(p)W_{\alpha\beta}^{-1}) = R(W_{\beta\alpha})\langle U_\Lambda \Psi(\Lambda \sigma_\alpha(p)) =$$

$$= R(W_{\beta\alpha})\Psi(\sigma_\alpha(p)) = R(W_{\beta\alpha})\Psi_\alpha(p) ,$$

$$= (U_\Lambda \Psi)_\beta(\Lambda p) = R(W_{\beta\alpha}(\Lambda, p))\Psi_\alpha(p) .$$  \hspace{1cm} (17)

The transformation $U_\Lambda$ represents the group $G$ because $R$ represents the stabilizer $H$ and the Wigner rotations satisfy for $p \in U_\alpha$, $\Lambda_1 p \in U_\beta$ and $\Lambda_2 \Lambda_1 p \in U_\gamma$ by (13)

$$W_{\gamma\alpha}(\Lambda_2 \Lambda_1, p) = W_{\gamma\beta}(\Lambda_2, \Lambda_1 p)W_{\beta\alpha}(\Lambda_1, p) .$$  \hspace{1cm} (18)

For $p^0 = \sqrt{m^2 + p^2}$ we spell out the transformations of massive $m > 0$ and massless $m = 0$ states.

The mass shell of a particle with mass $m > 0$ is the orbit of $p = (m, 0, 0, 0),$$

$$\mathcal{M}_m = \{(p^0, \underline{p}) : p^0 = \sqrt{m^2 + \underline{p}^2} , \underline{p} \in \mathbb{R}^3 \} .$$  \hspace{1cm} (19)

The stabilizer of $p$ is the group of rotations, $H = \text{SO}(3)$. Each of its irreducible, unitary representations $R$ is determined by its dimension $2s + 1$, where $s$, the spin of the particle, is nonnegative and half integer or integer.

The manifold $\text{SO}(1, 3)^\dagger$ is the product $\text{SO}(3) \times \mathbb{R}^3$ as each Lorentz transformation $\Lambda = L_p O$ can be uniquely decomposed into a rotation $O$ and a boost $L_p \in \text{SO}(3)$. So there exists a global section $\sigma : \mathcal{M}_m \to \text{SO}(1, 3)^\dagger$ sparing us to write an index $\alpha$ or $\beta$ for local sections.

$$\sigma(p) = L_p , \hspace{0.5cm} L_p \underline{p} = p$$  \hspace{1cm} (20)

Composed with this section, the functions $\Psi \in \mathcal{H}_R$ become wave functions of $\mathcal{M}_m$. 

2
Each Lorentz transformation in the neighbourhood of the unit element is the exponential $\Lambda = e^{\omega}$ of an element of its Lie algebra. As the group multiplication is analytic (and if the local section is analytic) (17) maps analytic states to each other because the parameters $\omega$ of the transformations which do not leave a point in the orbit invariant can be chosen as coordinates in its neighbourhood. On analytic states and only on them the transformation $U_\Lambda$ is given by a series in the transformation parameter $\omega$ times differential and multiplicative antihermitean generators $-iM_{mn}, \quad U_\Lambda = \exp (-\frac{1}{2}\omega_{mn}M_{mn}).$

The Cauchy completion of the analytic states defines an invariant Hilbert space of the irreducible representation. Therefore the Cauchy completion coincides with $H_R.$

Wave functions $\Psi$ which differ only in a set of vanishing measure correspond to the same vector in $H$ because $(\Psi|\Psi) = 0$ if and only if $\Psi = 0$, so the value of the wave function in a point is irrelevant. However, continuous wave functions cannot be changed in a point to some other equivalent continuous function as each equivalence class of functions which coincide nearly everywhere contains at most one and then unique continuous function. States on which the generators of an analytic transformation group act by differential operators are even more restricted to be analytic functions of the orbit and are already determined in each orbit by their behavior in a neighbourhood of one point.

The expansion of (17) gives $(U\Psi)(p) + (\omega p)^i\partial_i(U\Psi)(p)$ to first order and in the latter term we can replace $U\Psi$ by $\Psi$ yielding $\Psi(p) + \frac{1}{2}\omega_{mn}M_{mn}\Psi(p) + (\omega p)^i\partial_i\Psi(p)$.

A rotation $D = e^{\omega}$ coincides with its Wigner rotation $W(D,p), \quad DL_p = L_{Dp}D.$ Let its unitary representation be given by $R(D) = \exp \frac{1}{2}\omega^{ij}\Gamma_{ij}$ with antihermitean matrices, $\Gamma_{ij} = -\Gamma_{ji},$ which represent the Lie algebra of rotations $(i, j, k \in \{1, 2, 3\})$

$$[\Gamma_{ij}, \Gamma_{kl}] = \delta_{ik}\Gamma_{jl} - \delta_{il}\Gamma_{jk} - \delta_{jk}\Gamma_{il} + \delta_{kl}\Gamma_{ij}. \quad (21)$$

So the right side of (17) is $\Psi + \frac{1}{2}\omega^{ij}\Gamma_{ij}\Psi$ up to higher orders and we obtain

$$(-iM_{ij}\Psi)(p) = -(p^i\partial_i \partial_p - p^i\partial_p \partial_i)\Psi(p) + \Gamma_{ij}(U\Psi)(p). \quad (22)$$

As the operators $(J^1, J^2, J^3) = (M_{23}, M_{31}, M_{12})$ generate rotations, they are the components of the angular momentum $J.$ They consist of orbital angular momentum $\tilde{L} = -ip \times \partial_p$ and spin $\tilde{S}$ contributed by the matrices $i\Gamma^i.$ $\tilde{J} = \tilde{L} + \tilde{S}.$ Orbital angular momentum commutes with spin, $[\tilde{L}^i, \tilde{S}^j] = 0.$

The component functions of the differential operator on the right side of (22) are the negative of the infinitesimal motion $\delta p$ of the points $p.$ This motion occurs on the left side of (17). Because of this sign one has to distinguish the variation of a point $p$ from the change of the coordinate functions $h^i : p \mapsto p'$ which change by $\delta h^i = -\delta p^i.$

We choose the basis matrices $l_{mn}$ which generate Lorentz transformations in Minkowski space $\mathbb{R}_{1,3}^+$ with matrix elements $(n = \text{diag}(1, -1, -1, -1))$

$$l_{mn}^r = \delta_m^r \eta_{ns} - \delta_n^r \eta_{ms} \quad (23)$$

consistent with $\omega^r_s = \frac{1}{2}\omega^{mn}l_{mn}^r s$. Their commutators satisfy the Lorentz algebra

$$[l_{kl}, l_{mn}] = -\eta_{km}l_{ln} + \eta_{kn}l_{lm} - \eta_{lm}l_{kn} - \eta_{lk}l_{mn}. \quad (24)$$

So $l_{ij}$ rotates the basis vectors $e_i$ to $e_j (i, j \in \{1, 2, 3\}).$ The infinitesimal boost $l_{0p}$ maps $e_0 \mapsto -e_0$, moves points $p$ by $(l_{0p})^p = -p^0\delta^2 + p^0 = \sqrt{m^2 + p^2},$ and changes wave functions by $p^0\partial_0.$

The calculation of the Wigner rotation $W(l_{0q}, p)$ of a boost $L_0$ in terms of the lengths of $\vec{q}$ and $\vec{p}$ and their included angle is a ‘Herculean task’ [4] or requires ‘tedious manipulations’ [5]. The result is stated in [4] or e.g. [6]. We derive it comparing the products of matrices $l_{s}$ and $w$ which represent in $\text{SL}(2, \mathbb{C})$ [3, chap. 6] boosts $L_p$ with rapidity $a = \text{arctanh}(|\vec{p}|/|\vec{q}|)$ in direction $\vec{n}$ and rotations around an axis $\vec{n}$ by an angle $\delta (a = a/2, \delta = \delta/2)$

$$l_{s} = \text{ch} a - \text{sh} a \vec{n}_a \cdot \vec{\sigma}, \quad w = \text{cos} \delta' - i \sin \delta' \vec{n}_a \cdot \vec{\sigma}. \quad (25)$$

The products of two boosts $bl_{s}$ and of a boost $l_{-c}$ with a rotation $w$ yield

$$(ch b' - sh b' \vec{n}_b \cdot \vec{\sigma})(ch a' - sh a' \vec{n}_a \cdot \vec{\sigma}) = (ch b' ch a' + sh b' sh a' \vec{n}_a \cdot \vec{n}_b)1 - (ch a' sh b' \vec{n}_a + ch b' sh a' \vec{n}_a \cdot \vec{n}_b)s - i sh b' sh a' (\vec{n}_a \times \vec{n}_b) \cdot \vec{\sigma}, (26)$$

$$(ch c' + \vec{n}_c \cdot \vec{\sigma} sh c')(\text{cos} \delta' - i \sin \delta' \vec{n}_a \cdot \vec{\sigma}) = (ch c' \text{cos} \delta' - i \text{sh} c' \text{sin} \delta' \vec{n}_a \cdot \vec{n}_b)1 + sh c' (\text{cos} \delta' \vec{n}_a \cdot \vec{n}_b + \text{sin} \delta' \vec{n}_a \times \vec{n}_b) \cdot \vec{\sigma} - i ch c' \text{sin} \delta' \vec{n}_a \cdot \vec{\sigma}. \quad (26)$$

Both products are equal, $l_{s}l_{s} = l_{-c}w$ (16), if the complex coefficients of 1 and of the $\sigma$-matrices match as the matrices are linearly independent.

The coefficients of 1 agree only if $\vec{n} \cdot \vec{n}_a = 0.$ The axis $\vec{n}$ of the Wigner rotation is orthogonal to the direction $\vec{n}_c$ of the resulting boost and $\vec{n}_c$ lies in the plane spanned by $\vec{n}_a$ and $\vec{n}_b$ because $l_{0}$ and $l_{s}$ are in the subgroup $\text{SO}(1, 2)$ of boosts and rotations in this plane.

With $\vec{n}_a \cdot \vec{n}_a = \text{cos} \varphi$ and $\vec{n}_a \times \vec{n}_b = \vec{n}_c = \vec{n} \text{ sin} \varphi$ the comparison of the coefficients of 1 and of $\vec{n} \cdot \vec{\sigma}$ yields

$$(ch c')(\text{cos} \delta') = (ch b')(ch a') + (sh b')(sh a')(\text{cos} \varphi), \quad (ch c')(\text{sin} \delta') = (sh b')(sh a')(\text{sin} \varphi). \quad (27)$$

\footnote{The summation index $i$ enumerates coordinates of the mass shell, e.g. the spatial components of the momentum. Though the components and the derivatives depend on the coordinates, the differential operator is independent of coordinates. It is the restriction to the mass shell of the vector field $\sum_{m=0}^{3}(\omega p)^m \partial_0$, which is tangent to the Lorentz flow of four-momentum. The derivatives $\partial_0, \quad m = 0, 1, 2, 3,$ cannot be applied separately to the wave functions $\Psi$ as they are no functions of $\mathbb{R}_{1,3}^+$.}
The ratio of both equations determines the looked for angle
\[ \tan \frac{\delta}{2} = \frac{\sin \varphi}{k + \cos \varphi}, \quad k = (\coth \frac{a}{2})(\coth \frac{b}{2}). \] (28)

It has the same sign as \( \varphi \), i.e. \( W(b, a) \) rotates in the direction from \( \vec{n}_a \) to \( \vec{n}_b \).

The derivative of \( \delta \) with respect to \( b \) at \( b = 0 \) is
\[ \frac{d\delta}{db}|_{b=0} = (\sin \varphi) \tanh \frac{a}{2} = (\sin \varphi) \frac{|\vec{p}|}{\rho^0 + m}. \] (29)

As the infinitesimal Wigner rotation of a boost from \( p \) to \( -e_i \) rotates from \( \vec{p} \) to \( -e_i \) in the same sense as from \( e_i \) to \( p \) it is represented by \( \Gamma_{ij}p^i/(\rho^0 + m) \). Combining both infinitesimal changes, we obtain the generators of boosts of wave functions
\[ (-iM_0)\Psi(p) = p^i\partial_{p^i}\Psi(p) + \Gamma_{ij}\frac{p^i}{\rho^0 + m}\Psi(p). \] (30)

The generators \( \{2, 30\} \) are antihermitean with respect to the scalar product \( \langle \rangle \) with the measure \( d\rho \) and represent the Lorentz algebra \( \{24\} \). This is simple to check for \( -iM_{ij}, -iM_{ik} \) and manifest for \( -iM_{ij}, -iM_{ik} \), because \( p^i, \partial_{p^i} \) and \( \Gamma_{ij} \) transform as vectors or products of vectors under the joint rotations of momentum and spin; \( -iM_{ij}, -iM_{ij} = iM_{ij} \) holds as if by miracle,
\[ \left[ p^0\partial_{p^i} + \Gamma_{ik}\frac{p^k}{\rho^0 + m}, p^0\partial_{p^i} + \Gamma_{ij}\frac{p^j}{\rho^0 + m} \right] = \left( p^0\partial_{p^i} + \frac{1}{\rho^0 + m}\left( p^0(p^0 + m)\delta_{ij} - p^ip^j\Gamma_{ij}\right) - \left( p^0\partial_{p^i} + \frac{1}{\rho^0 + m}\left( p^0(p^0 + m)\delta_{ij} - p^ip^j\Gamma_{ij}\right) + \frac{p^0p^j}{(\rho^0 + m)^2}\delta_{ij}\Gamma_{kl} - \delta_{il}\Gamma_{kj} - \delta_{lj}\Gamma_{ik} + \delta_{ik}\Gamma_{lj}\right) = \right. \]
\[ \left. \left. = \left(-p^i\partial_{p^i} - p^i\partial_{p^i} + \Gamma_{ij} \right) \right. \right. \] (31)

For different masses the representations \( U_\alpha \) of translations are inequivalent as the spectra of \( P \) differ. The representations \( U_\Lambda \) of the Lorentz group on states with mass \( m > 0 \), however, are unitarily equivalent by the scale transformation \( U_m \) to the representation on states \( \Psi \) with unit mass \( p^0 = \sqrt{1 + \vec{p}^2} \),
\[ (U_m \Psi)(m p) = m \Psi(p). \] (32)

The mass shell \( M_0 \) of a massless particle, the orbit under the proper Lorentz group of a lightlike momentum such as \( \vec{p} = (1, 0, 0, \ldots, 1) \), is the manifold \( S^{D-2} \times \mathbb{R} \),
\[ M_0 = \{(p^0, \vec{p}) : p^0 = |\vec{p}| > 0, \quad \vec{p} \in S^{D-2} \times \mathbb{R} \}, \] (33)
as \( \vec{p} \neq 0 \) is specified by its direction and its nonvanishing modulus \( |\vec{p}| = c^3 > 0 \). The tip of the lightcone \( p = 0 \) does not belong to the orbit. Already the differential operators \( |\vec{p}| \partial_{\vec{p}} \), which generate the flow of boosts, are not analytic there.

The stabilizer \( H \) of \( \mathbb{R} \) is generated by infinitesimals \( \omega_i \), \( \eta \omega = ((\eta \omega)^T) \) with \( \omega_\alpha^0 + \omega_\alpha^z = 0 \) which consequently are of the form
\[ \omega(a, \hat{\omega}) = \begin{pmatrix} a & a^T \hat{\omega} & -a \end{pmatrix}. \] (34)

Here \( a \) is a \( (D-2) \)-vector and \( \hat{\omega} \) is a \( (D-2) \times (D-2) \) matrix, which generates a rotation \( \omega \in SO(D-2) \). Because \( \omega(a, 0) \) and \( \omega(b, 0) \) commute they generate translations in \( D-2 \) dimensions. They are rotated by \( \omega \). So the stabilizer \( H \) is the Euclidean group \( E(D-2) \) of translations and rotations in \( D-2 \) dimensions.

In \( D = 4 \) the inducing representation \( R \) of an irreducible, massless unitary representation of the Poincaré group is an irreducible, unitary representation of \( E(2) \). Such a representation is characterized by the SO(2) orbit of some point \( q \in \mathbb{R}^2 \) and a unitary representation \( \hat{R} \) of its stabilizer \( \hat{H} \subset SO(2) \). If \( q \neq 0 \), the orbit is a circle and the stabilizer is trivial. \( E(2) \) is represented on wave functions of a circle. Contrary to its denomination ‘continuous spin’ such a representation leads to integer or half integer, though infinitely many, helicities (the Fourier modes of the wave functions). These representations are phenomenologically excluded as infinitely many massless states per given momentum make the specific heat of each cavity infinite.

If \( q = 0 \in \mathbb{R}^2 \), then the orbit is the point \( q = 0 \) and the translations in \( E(2) \) are represented trivially. The stabilizer is \( H = SO(2) \). Each unitary, irreducible representation \( R \) of its cover or each ray representation is one dimensional and represents the rotation \( D_\delta \) by the angle \( \delta \) by multiplication with \( R(D_\delta) = e^{i\hbar \delta h} \), where \( h \) is an arbitrary, real number, which characterizes \( R \).

In \( D = 4 \) there is no global section \( \Psi \) with which to relate wave functions in \( \mathcal{H}_R \) to wave functions of \( M_0 \). Such a section cannot exist as \( S^3 \neq S^2 \times S^1 \) is the Hopf bundle of circles where each circle winds around each other circle.

So we use two local sections \( N_p \) and \( S_p \) which are defined in the north \( U_N \) outside \( A_- = \{ (|p_2|, 0, 0, -|p_1|) \} \) and the south \( U_S \) outside \( A_+ = \{ (|p_3|, 0, 0, |p_2|) \} \)
\[ U_N = \{ p \in M_0 : |p| + p_2 > 0 \}, \quad U_S = \{ p \in M_0 : |p| - p_2 > 0 \}. \] (35)

The section \( N_p = D_p B_p \) boosts \((0, 0, 0, 1)\) in 3-direction to \( B_p \) \((|p|, 0, 0, |p|)\) and then rotates by the smallest angle namely by \( \theta \) around the axis \((-\cos \varphi, \cos \varphi, 0, \varphi)\) along a great circle to \( p = |p| \sqrt{1 - \sin^2 \theta}, \sin \theta \cos \varphi, -\sin \theta \sin \varphi, \cos \theta \). With \( \theta' = \theta/2 \), \( \tan \theta' = \sqrt{(|p| - p_2)/(|p| + p_2)} \) and \( p' = \sqrt{|p|} \)

\[ \text{Only in the 4 cases } D = 1, 2, 4, 8 \text{ does the group manifold } \text{SO}(1, D - 1) \sim \text{SO}(D - 1) \times \mathbb{R}^{D-1} \text{ factorize into } M_0 \times E(D - 2) \sim S^{D-2} \times \text{SO}(D - 2) \times \mathbb{R}^{D-1} \text{.} \]
the preimage of \(N_p\) in \(\text{SL}(2, \mathbb{C})\) is

\[
N_p = \left( \frac{\cos \theta'}{\sin \theta'} e^{i \varphi} \varepsilon - p' \sin \theta' e^{-i \varphi} \right.
\]
\[
= \frac{1}{\sqrt{2}} \left( \frac{\sqrt{|\mathbf{p}| + p_z}}{|\mathbf{p}|} - p_z - i p_y \right) + i \frac{p_z + i p_y}{|\mathbf{p}| + p_z}.
\]

One easily checks that \(N_p\) transforms \(\mathbf{p} = 1 - \sigma^3\) to \(n_p \mathbf{p} (n_p)^t = |\mathbf{p}| - \mathbf{p} \cdot \sigma = \mathbf{p}[3, (6.63)]\) In \(\mathcal{U}_N\) the section \(n_p\) is analytic in \(p\). It is discontinuous on \(\mathcal{A}_-\), as the limit \(\theta \to \pi\) depends on \(\varphi\).

The section

\[
s_p = \left( \frac{\cos \theta' e^{-i \varphi}}{\sin \theta'} - p' \sin \theta' e^{-i \varphi} \right.
\]
\[
= \frac{1}{\sqrt{2}} \left( \frac{p_z + i p_y}{|\mathbf{p}|} - \sqrt{|\mathbf{p}| - p_z} \right)
\]
\[
\frac{p_z - i p_y}{|\mathbf{p}|} + \sqrt{|\mathbf{p}| - p_z}
\]

is defined and analytic in \(\mathcal{U}_S\). The corresponding southern section \(s_p\) rotates \(B \mathbf{p}\) in the 1-3-plane by \(\pi\) to \(|\mathbf{p}|(1, 0, 0, 1)\) and then along a great circle by the smallest angle to \(p\), namely by \(\pi - \theta\) around \(\mathbf{\hat{r}} = (\sin \varphi, -\cos \varphi, 0, 0)\).

In their common domain two sections differ by the multiplication from the right [8] with a momentum dependent matrix in the \(\text{SL}(2, \mathbb{C})\) preimage of \(H = E(2)\)

\[
w = \left( \begin{array}{cc}
e^{-i \delta/2} & 0 \\
0 & e^{i \delta/2}
\end{array} \right).
\]

\(s_p\) differs in \(\mathcal{U}_N \cap \mathcal{U}_S\) from \(n_p\) by a rotation around the 3-axis by \(2\varphi\), the area of the zone (spherical triangle) with vertices \(\mathbf{e}_z, -\mathbf{e}_z\) and through \(\mathbf{e}_\varphi, \mathbf{\hat{r}}/|\mathbf{p}|\).

The angle \(\delta\) of the Wigner rotation \(W(\Lambda, p)\), which accompanies a Lorentz transformation, \(\Lambda N_p = N_{\Lambda p} W(\Lambda, p)\) [11] can be read off from the 22-element of the product of the \(\text{SL}(2, \mathbb{C})\) preimage \(\lambda\) with \(n_p\).

\[
\lambda n_p = n_{\Lambda p} w(\Lambda, p),
\]

as \(\delta/2\) is the phase of \((n_{\Lambda p} w(\Lambda, p) e^{i \delta/2})\) and the Wigner angle \(\delta\) and its infinitesimal value are

\[
\tan \frac{\delta}{2} = \left. \frac{n_z |\mathbf{p}| + n \cdot \mathbf{p}}{((|\mathbf{p}| + p_z)(|\mathbf{p}| - n_{z p} p_z - n_{y p} p_x, p_x)} \right|_{\alpha \to 0},
\]

\[
\frac{d \delta}{d \alpha_{|\alpha=0}} = \left. \frac{n_z |\mathbf{p}| + n \cdot \mathbf{p}}{|\mathbf{p}| + p_z} \right|_{\alpha \to 0}.
\]

As \(R(t^i \hat{\sigma}) = e^{-i \delta}\), the representation \(-i \hbar p_y/((|\mathbf{p}| + p_z)\) of the infinitesimal Wigner rotation accompanies e.g. the infinitesimal rotation in the 3-1-plane.

Analogously one determines the Wigner angle of a boost

\[
\tan \frac{\delta}{2} = \frac{n_z p_y - n_y p_z}{(|\mathbf{p}| + p_z)(|\mathbf{p}| - n_{z p} p_z - n_{y p} p_x, p_x)},
\]

\[
\frac{d \delta}{d \alpha_{|\alpha=0}} = \frac{n_z p_y - n_y p_z}{|\mathbf{p}| + p_z}.
\]

and the generators of boosts, bearing in mind that \(-i M_0\) boosts from \(e_\varphi\) to \(-e_\varphi\),

\[
(-i M_{12} \Psi)_N(p) = \left( -i \hbar \right) \left( -p_2 \partial_{p_2} - p_2 \partial_{p_1} + i \hbar \sqrt{|\mathbf{p}| + p_z} \right) \Psi_N(p),
\]

\[
(-i M_{31} \Psi)_N(p) = \left( -i \hbar \right) \left( -p_z \partial_{p_z} - p_2 \partial_{p_1} + i \hbar \sqrt{|\mathbf{p}| + p_z} \right) \Psi_N(p),
\]

\[
(-i M_{32} \Psi)_N(p) = \left( -i \hbar \right) \left( -p_z \partial_{p_z} - p_2 \partial_{p_1} + i \hbar \sqrt{|\mathbf{p}| + p_z} \right) \Psi_N(p),
\]

\[
(-i M_{10} \Psi)_N(p) = \left( -i \hbar \right) \left( -p_1 \partial_{p_1} - p_2 \partial_{p_2} + i \hbar \sqrt{|\mathbf{p}| + p_z} \right) \Psi_N(p),
\]

\[
(-i M_{30} \Psi)_N(p) = \left( -i \hbar \right) \left( -p_1 \partial_{p_1} - p_2 \partial_{p_2} + i \hbar \sqrt{|\mathbf{p}| + p_z} \right) \Psi_N(p).
\]

In particular the helicity, the angular momentum \(\mathbf{p} \cdot \mathbf{\hat{J}}/|\mathbf{p}|\) in the direction of the momentum \(\mathbf{p}\), is a multiple of \(1\), namely the real number \(h\).

\[
(|p_2 M_{23} + p_y M_{31} + p_z M_{12}) \Psi_N(p) = h |\mathbf{p}| \Psi_N(p).
\]

As \(\mathcal{U}_-\) the differential and multiplicative operators \(-i M_{mn}\) satisfy the Lie algebra [24] of Lorentz transformations for each value of the real number \(h\).

But the functions \((M_{mn}(\Psi))_N\) are not defined on \(\mathcal{A}_-\) and seem to contradict Mackey’s results that analytic states in \(\mathcal{H}_2\) are transformed to each other [12]. One cannot require the wave functions \(\Psi_N\) to vanish on \(\mathcal{A}_-\) because rotations map them to functions which vanish elsewhere. One also cannot turn a blind eye to the singularity [13] taking unwarranted comfort in the misleading argument [3, 4] that one can change a wave function in a set of vanishing measure. The problem is not a set of measure zero but the factor \(p_z/((|\mathbf{p}| + p_z))\) which near to \(|\mathbf{p}|\) grows as \(2(|p_2| p_y/(p_x^2 + p_y^2))\) proportional to \(|p_z|/r\) where \(r = (p_x^2 + p_y^2)^{1/2}\) is the axial distance. For differentiable \(\Psi_N\) the derivative terms of \(M_{12} \Psi\) stay bounded and the integral of \((M_{12} \Psi | M_{12} \Psi)\) [10] diverges as \(4 h^2 |\Psi_N|^2(p_z/|r|)^2\), Cutting out a tube of radius \(\varepsilon\) around on \(\mathcal{A}_-\) and integrating in polar coordinates \(d^3 p = d p_z (r dr d\varphi)\) the integral over \(r \leq \varepsilon\) diverges if \(\varepsilon\) goes to zero, \(h^2 f_0^1 dr/r = h^2 \ln \varepsilon \to \infty\).

If \(\Psi\) does not vanish on \(\mathcal{A}_-\) then for \(|M_{13} \Psi|\) to exist \(\Psi_N\) must not be differentiable there. The derivative term \(-p_z \partial_{p_z}\) has to cancel near \(\mathcal{A}_-\) the multiplicative singularity which is also linear in \(p_z\). This cancellation \(|\partial_{p_z} - 2 i h p_y/(p_x^2 + p_y^2)\) f = 0 and, for \(M_{23} \Psi\) to exist, \(|\partial_{p_z} + 2 i h p_y/(p_x^2 + p_y^2)\) f = 0 determine \(f(p_z, p_y) = e^{-2i h \varphi}\).
If $\Psi_N$ is square integrable and analytic in $\mathcal{U}_N$, then $M_{mn}$ is square integrable and analytic if and only if in $\mathcal{U}_N \cap \mathcal{U}_S$ the function $\Psi_N = e^{-2ih}\Psi_S$ is the product of the gauge transformation $e^{-2ih}\varphi$ times a function $\Psi_S$, which is analytic in $\mathcal{U}_S$,

$$\Psi_S(p) = e^{2ih}\Psi_N(p).$$

(45)

Multiplying (45) with the transition function $e^{2ih}\varphi$ and using (15) one obtains

$$\langle \Psi | \psi \rangle = - (p_x \partial_{p_x} - p_y \partial_{p_y} - i h) \Psi_S(p),$$

$$\langle \Psi | \psi \rangle = - (p_x \partial_{p_x} - p_y \partial_{p_y} + i h \frac{p_y}{|p| - p_z}) \Psi_S(p),$$

$$\langle \Psi | \psi \rangle = (i \bar{p} \partial_{p_x} + i h \frac{p_y}{|p| - p_z}) \Psi_S(p),$$

$$\langle \Psi | \psi \rangle = (i \bar{p} \partial_{p_x} - i h \frac{p_y}{|p| - p_z}) \Psi_S(p),$$

$$\langle \Psi | \psi \rangle = (i \bar{p} \partial_{p_x} - i h \frac{p_y}{|p| - p_z}) \Psi_S(p),$$

(46)

The same generators ensue if, along the lines of the derivation of (13), one reads the Wigner angle $\phi$ from the phase of the 12-element $(\lambda_{sp})_{12} = -[(s \lambda_{sp})_{12}] e^{(i\phi)/2}$.

The transition function $e^{2ih}\varphi$ is defined and analytic in $\mathcal{U}_N \cap \mathcal{U}_S$ only if $2h \in \mathbb{Z}$ is integer. This is why the helicity of a massless particle is integer or half integer. For each other value $-iM_{mn}$ are antihermitean operators in Hilbert space.

For helicity $h \neq 0$ each continuous momentum wave function $\Psi$ has to vanish along some line, a Dirac string in $\mathcal{U}_N$ the phase of $\Psi$ becomes zero, the path cannot be contractible in $\mathcal{U}_N$.

The parameter of SO(3) is given by

$$R(e^{i\delta}) = e^{-i\delta}$$

of rotations around the 3-axis. However, to be induced by an irreducible representation on sections over an orbit does not make the representation of SO(3) irreducible.

To determine its invariant subspaces we recall that each angular momentum multiplet with total angular momentum $j$ contains a state $\Lambda$ which is an eigenvector of $J^3 = M_{12}$ with eigenvalue $j$ and which is annihilated by $J^+ = M_{23} + iM_{31}$

$$(M_{12} - j)\Lambda = 0, \quad (M_{23} + iM_{31})\Lambda = 0.$$

(48)

By (45) these are differential equations for $\Lambda_N$. They become easily solvable if we consider $\Lambda_N$ as a function of the complex stereographic coordinates

$$w = \frac{p_x + i p_y}{|p| + p_z}, \quad \bar{w} = \frac{p_x - i p_y}{|p| + p_z},$$

(49)

which map the northern domain to $\mathbb{C}$. Then the differential equations read

$$(w \partial_w - \bar{w} \partial_{\bar{w}} + w - j)\Lambda_N = 0, \quad (w^2 \partial_w + \partial_{\bar{w}} + h w)\Lambda_N = 0.$$

(50)

Recollecting that $w \partial_w$ measures the homogeneity in $w$, $w \partial_w (w^j) = r(w)^j$, the first equation is solved by $\Lambda_N = w^{j-h} g(|w|^2)$ and the second equation implies $g$ to be homogeneous in $(1 + |w|^2)$ of degree $-j$.

$$(j - h + h + (|w|^2 + 1)g) w^{j-h+1} = 0,$$

$$\Lambda_N(w, \bar{w}) = \frac{w^{j-h}}{(1 + |w|^2)^{j}}.$$  

(51)

The state $\Lambda$ is analytic only if $j - h$ is a nonnegative integer. It is square integrable with respect to the measure

$$d\Omega = d \cos \theta \ d \varphi = d w \ d \bar{w} \ \frac{2i}{(1 + w \bar{w})^2},$$

(52)

if also $j + h$ is nonnegative which is just the restriction to be analytic also in southern stereographic coordinates. They are related to the northern coordinates in their common domain by inversion at the unit circle,

$$w' = \frac{p_x + i p_y}{|p| - p_z} = \frac{1}{w}, \quad \bar{w}' = \frac{p_x - i p_y}{|p| - p_z} = \frac{1}{w},$$

(53)

and $\Lambda_S$ is analytic only if $j + h$ is a nonnegative integer

$$\Lambda_S(w', \bar{w}') = \left( \frac{w}{|w|} \right)^{2h} \Lambda_N = \frac{w^{j+h}}{(1 + |w|^2)^j}.$$  

(54)

So $j \geq |h|$: there is no round photon with $j = 0$.

The representation $R(e^{i\delta}) = e^{-i\delta}$ of SO(2) induces in the space of sections over the sphere no SO(3) multiplet with $j < |h|$ and one multiplet for $j = |h|, |h| + 1, \ldots$

$$n_h(j) = \begin{cases} 0 & \text{if } j < |h| \\ 1 & \text{if } j = |h|, |h| + 1, \ldots \end{cases}.$$  

(55)
The multiplicity $n_h(j)$ exemplifies the Frobenius reciprocity for $G=SO(3)$ and $H=SO(2)$: The representation $h$ of the subgroup $H$ induces each representation $j$ of the group $G$ with the multiplicity $m_j(h)$ with which the restriction of $j$ to the subgroup $H$ contains $h$, $m_j(h) = n_h(j)$.

A complete $SO(3)$ multiplet with spin $s$ at fixed momentum consists of helicities $h = -s, -s+1, \ldots, s$ and induces $SO(3)$ representations $j$ with multiplicity

$$N(j) = \sum_{k=-s}^{s} n_h(j) = \begin{cases} 2j + 1 & \text{if } j \leq s \\ 2s + 1 & \text{if } j \geq s \end{cases} .$$

This fits to the multiplicity $N_{s\otimes l}(j)$ with which total angular momentum $j$ is contained in the product of spin $s$ with orbital angular momentum $l$,

$$N_{s\otimes l}(j) = \begin{cases} 1 & \text{if } j \in \{ s + l, s + l - 1, \ldots, |s - l| \} \\ 0 & \text{else} \end{cases} .$$

As $l$ ranges over $0, 1, 2, \ldots$ one obtains $N(j) = \sum_{l} N_{s\otimes l}(j)$. If each helicity state would induce one multiplet for each $j$, then for integer spin the states of the spin multiplet would induce $2s + 1$ singlets $j = 0$ rather than $N(0) = 1$.

Throughout quantum field theory one uses a continuum basis of momentum eigenstates $\Gamma_p$ to map test functions of $\mathcal{M}_0$ to states $\Psi$ in $\mathcal{H}$,

$$\Psi = \int_{\mathcal{M}} \bar{\Psi}(p) \Gamma_p \ , \ \Psi(p) = (\Gamma_p | \Psi) .$$

Strictly speaking the eigenstates $\Gamma_p$ are distributions with scalar product

$$\langle \Gamma_q | \Gamma_p^T \rangle = (2\pi)^3 2\sqrt{m^2 + p^2} \delta^3(q - p) \mathbf{1} .$$

and only integrals of the continuum basis with test functions are vectors $\Psi$ in $\mathcal{H}$ with the scalar product. This is why we work directly with the wave functions. Moreover the states of massless particles are no functions of $\mathcal{M}_0$ but sections of a nontrivial bundle. To describe them in their appropriate domains with $\Gamma_p, \Gamma_p, \Gamma_p, \Gamma_p, \Gamma_p, \Gamma_p, \Gamma_p, \Gamma_p, \Gamma_p$ in $\mathcal{U}_N \cap \mathcal{U}_S$ and with the restriction on analytic states that their wave functions be analytic in $\mathcal{U}_N$ and $\mathcal{U}_S$ and be related by (53).

The continuum basis transforms contragradiently to the wave functions $\Psi$ by

$$U_\Lambda \Gamma_p = R^T(W(\Lambda, p)) \Gamma_\Lambda p$$

(60) (if $p$ and $\Lambda p$ are covered by the same coordinate patch) such that $U_\Lambda \Psi$ corresponds to the transformed wave function,

$$\int \bar{\Psi}(p) U_\Lambda \Gamma_p \ = \ \int \bar{\Psi}(p) R^T(W(\Lambda, p)) \Gamma_\Lambda p$$

$$= \int \bar{\Psi} (U_\Lambda \Psi)^T (\Lambda p) \Gamma_\Lambda p .$$

The generators of the Poincaré transformation of the continuum basis are the transposed of the generators on the wave function, i.e. their derivative part has the reversed sign from partial integration and the multiplicative part is transposed. This is compatible with the Lorentz algebra as $U_\Lambda$ is linear and therefore commutes with derivatives and matrices which multiply the continuum basis.

Admittedly, the transformation of the continuum basis (60) is more intuitive than (17) and for many purposes it is sufficient to work with one coordinate patch e.g. $\mathcal{U}_N$ and to forget the analyticity requirements (53) for the states on which the generators (43) act. However, employing the continuum basis one is prone to subtle mistakes, e.g. to overlook that helicity states have angular momentum $j$ which is at least $|h|$ (56) or to miss the Dirac string in momentum space on which continuous wave functions have to vanish.

Using a continuum basis Weinberg and Witten conclude that each matrix element $(\Phi | \bar{j} m(0) | \Psi)$ vanishes for states with $|h| > 1/2$ if $\bar{j} m(x) = \bar{e}^{\bar{P} x} \bar{j} m(0) e^{-i P x}$ is a conserved current, $0 = \partial_0 \bar{j} m(x) = i[\bar{P} m, \bar{j} m(x)]$ which transforms as a field $U_D j m(0) U^{-1}_D = \Lambda^{-1} n_j m(0)$.

The conclusion follows from the vanishing of $(\Gamma_{-\bar{P}} | \bar{j} m(0) \Gamma_{\bar{P}})$ in momentum eigenstates. Under rotations $D_\delta$ around $\bar{P}$ they transform by $U_D \Gamma_{-\bar{P}} = e^{-i h \delta} \Gamma_{-\bar{P}}$ and $U_D \Gamma_{\bar{P}} = e^{ih \delta} \Gamma_{\bar{P}}$ as each state has angular momentum $h$ in direction of its momentum, so

$$\langle U_D \Gamma_{-\bar{P}} | U_D \bar{j} m(0) U^{-1}_D \Gamma_{\bar{P}} \rangle$$

$$= e^{-2i h \delta} D^{-1} m \langle \Gamma_{-\bar{P}} | \bar{j} m(0) \Gamma_{\bar{P}} \rangle$$

and the matrix elements constitute an eigenvector of $D$ with eigenvalue $e^{-2i h \delta}$. But a rotation $D_\delta$ of a four vector has only eigenvalues 1 and $e^{i \delta}$. So, if $2|h| > 1$, then

$$\langle \Gamma_{-\bar{P}} | \bar{j} m(0) \Gamma_{\bar{P}} \rangle = 0 .$$

The catch in the argument is that there is no basis of states $\Gamma_p$ for all momenta. The domain $\mathcal{U}_N$ does not cover the south pole. Rotations around some axis $\bar{n}$ are not defined by (17) in $\mathcal{U}_N$ for momenta which lie in the south pole’s orbit under these rotations. In particular, the states $\Gamma_{-\bar{P}}$ and $\Gamma_{\bar{P}}$ are separated by this orbit and do not lie in a common coordinate patch in which $U_D$ acts smoothly.

We show that the conclusion of (53) is wrong and construct a conserved current for arbitrary helicity $h$. For this purpose we enlarge the Hilbert space by an auxiliary multiplet with helicity $h + 1/2$ with states $\Psi_+$ such that these states are two component column vectors $(\Psi_+ , \Psi_h)$. On these states we define in northern coordinates generators $Q_1, Q_2, Q_1 = (Q_1)^\dagger$ and $Q_2 = (Q_2)^\dagger$ to multiply the wave functions $\Psi(p)$ with

$$Q_1(p) = -\frac{p_x - ip_y}{\sqrt{|p| + p_z}} a^\dagger \ , \ Q_2(p) = \sqrt{|p| + p_z} a^\dagger$$

$$Q_1(p) = -\frac{p_x + ip_y}{\sqrt{|p| + p_z}} a \ , \ Q_2(p) = \sqrt{|p| + p_z} a .$$

(64)
where \( a^\dagger \) and \( a \) are the matrices
\[
a^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]
(65)

\( Q_\alpha \) and \( \bar{Q}_\alpha \) commute with \( P^m \) and satisfy the supersymmetry algebra
\[
\{ Q_\alpha, Q_\beta \} = 0 = \{ Q_\alpha, \bar{Q}_\beta \}, \quad \{ Q_\alpha, \bar{Q}_\beta \} = \sigma_r^{m} P_m 1,
\]
(66)

and the Weyl equation
\[
\tilde{a}^0 = \sigma^0, \tilde{a}^3 = -\sigma^3.
\]
(67)

They are analytic in \( U_N \) as \((|p|+p_z)^{\pm 1/2}\) is analytic there. On the negative \( 3\)-axis \( A \) their phase is discontinuous \( (p_x + ip_y)/\sqrt{|p| + p_z} = e^{i\varphi} \sqrt{|p| - p_z} \).

As \((Q_1, Q_2)\) is proportional to the second column of \( n_\hat{p} \), \( Q_\alpha \) satisfy
\[
D_\alpha^{1,\beta} Q_\beta(p) = Q_\alpha(p) e^{i\delta(A,p)/2}
\]
(68)

where \( D(A) \) is the SL(2, \( \mathbb{C} \)) representation of \( \Lambda \) and \( \delta(A,p) \) the Wigner angle. This property, the induced transformation of helicity states
\[
(U_A \Psi)(\Lambda p) = \Psi(p) e^{-i h \delta},
\]
(69)

and that \( Q_\alpha \) is a multiplicative operator allow to simplify
\[
D_\alpha^{1,\beta} (Q_\beta U_A \Psi)(\Lambda p) = D_\alpha^{1,\beta} Q_\beta (U_A \Psi)(\Lambda p) = Q_\alpha(p) e^{-i h \delta/2} \Psi(p) e^{-i h \delta}
\]
(70)

and show (reading \([63]\) for helicity \( h+1/2 \) backwards)
\[
D_\alpha^{1,\beta} (Q_\beta U_A \Psi)(\Lambda p) = (U_A Q_\alpha) \Psi)(\Lambda p).
\]
(71)

So \( Q_\alpha \) are the components of a SL(2, \( \mathbb{C} \)) spinor (\( \tilde{D} \) is the conjugate representation),
\[
U_A Q_\alpha U_A^{-1} = D_\alpha^{1,\beta} Q_\beta, \quad U_A \bar{Q}_\alpha U_A^{-1} = \bar{D}_\alpha^{1,\beta} \bar{Q}_\beta,
\]
(72)

and carry helicity \( 1/2 \).

The supersymmetry generators map analytic sections with helicities \( h+1/2 \) and \( h \) to each other as they provide the relative phase \( e^{i\varphi(r)} \), or a factor \( \sqrt{|p|+p_z} \) which vanishes on \( A_- \), to make \( e^{i(2h+1)\varphi(Q\Psi)} \) and \( e^{i2h\varphi(Q\Psi)} \) analytic in \( U_S \).

In southern coordinates the generators of supersymmetry are related to \([63]\) by
\[
Q_\alpha s(p) = e^{i\varphi} Q_\alpha(p), \quad \bar{Q}_\beta s(p) = e^{-i\varphi} \bar{Q}_\beta(p),
\]
\[
Q_1 s(p) = -\sqrt{|p|-p_z} a^\dagger, \quad Q_2 s(p) = \frac{p_x + ip_y}{\sqrt{|p|-p_z}} a^\dagger,
\]
\[
\bar{Q}_1 s(p) = -\sqrt{|p|-p_z} a, \quad \bar{Q}_2 s(p) = \frac{p_x - ip_y}{\sqrt{|p|-p_z}} a.
\]
(73)

With help of the supersymmetry generators we specify a current \( j^m \) of charge \( q \)
\[
\langle \Psi | j^m(0) \Psi \rangle = \int \tilde{a}^m \tilde{a}^p \Phi^* (p) j^m(p,p') \Psi(p') + \text{H.c.}
\]
(74)

where \( f \) denotes a function of \((p-p')^2 \) with \( f(0) = q \). The current is conserved, \( 0 = \hat{P}_m j^m(p,p') \).

The integral of the density \( j^m(x) = e^{iP^x} j^m(0) e^{-iP^x} \) yields the charge,
\[
\int \tilde{a}^p \tilde{d} a^p d^3x \Phi^* (p) e^{i(P^x-p^x) x} j^0(p,p')(p') \Psi(p')
\]
(75)

because the supersymmetry algebra \( \{ Q_\alpha, Q_\beta \} = P_m \sigma_r^m \) implies \( j^m(p,p') = 2q p^m \).

The multiplet with \( h + 1/2 \) is auxiliary, because only the \( h \)-\( h \)-components of the \( 2 \times 2 \) matrices \( Q_\alpha Q_\beta \) are needed for the current. Explicitly the 4 components of \( j^m(p,p')/f((p-p')^2) \) are
\[
\frac{p_x + ip_y}{\sqrt{|p|+p_z}} \sqrt{|p|+p_z} p_z + \sqrt{|p|+p_z} p_z + \sqrt{|p|+p_z} p_z,
\]
(76)

allowing to check explicitly \( j^m(p,p) = 2q p^m \) and the conservation \( 0 = \hat{P}_m j^m(p,p') \).

Neither the supersymmetry generators nor the current \( j^m \) depend on the value \( h \) of the helicity. There is no obstruction to conserved currents with nonvanishing matrix elements in states with arbitrary helicity.

Massless states do not allow hermitean spatial position operators \( X^i \) which generate translations of the spatial momentum \([11, 12]\). By the Stone von Neumann theorem \([12]\) they act analytically on the subspace of square integrable analytic functions of \( \mathbb{R}^3 \). This space is not mapped to itself by Lorentz transformations. They act analytically on analytic sections over \( S^2 \times \mathbb{R} \). So there is no common dense and invariant subspace of analytic states on which Lorentz transformations and translations of momentum can be expanded and on which the generators can be applied repeatedly.

For \( h \neq 0 \), the partial derivatives of \( \Psi_N \) and \( \Psi_S \) are not in the Hilbert space of square integrable functions as the derivation of \([15]\) shows. Well defined are the covariant
derivatives \( D_N = \partial + A_N \) and \( D_S = e^{2i\hbar \varphi} D_N e^{-2i\hbar \varphi} \),

\[
D_N = \begin{pmatrix} \frac{\partial p_x}{\partial p} & \frac{\partial p_y}{\partial p} & \frac{\partial p_z}{\partial p} \\ \frac{\partial p_x}{\partial p} & \frac{\partial p_y}{\partial p} & \frac{\partial p_z}{\partial p} \\ \frac{\partial p_x}{\partial p} & \frac{\partial p_y}{\partial p} & \frac{\partial p_z}{\partial p} \end{pmatrix}_N - \frac{i\hbar}{|\vec{p}|(|\vec{p}| + p_z)} \begin{pmatrix} p_y & -p_x & 0 \\ -p_x & p_y & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
D_S = \begin{pmatrix} \frac{\partial p_x}{\partial p} & \frac{\partial p_y}{\partial p} & \frac{\partial p_z}{\partial p} \\ \frac{\partial p_x}{\partial p} & \frac{\partial p_y}{\partial p} & \frac{\partial p_z}{\partial p} \\ \frac{\partial p_x}{\partial p} & \frac{\partial p_y}{\partial p} & \frac{\partial p_z}{\partial p} \end{pmatrix}_S + \frac{i\hbar}{|\vec{p}|(|\vec{p}| - p_z)} \begin{pmatrix} p_y & -p_x & 0 \\ -p_x & p_y & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\] (77)

But their commutator yields the field strength of a momentum space monopole of charge \( \hbar \) at \( p = 0 \),

\[
[D_i, D_j] = F_{ij} = \partial_i A_j - \partial_j A_i = i\hbar \varepsilon_{ijk} \frac{p_k}{|\vec{p}|^3}.
\] (78)

The integral over \( F = \frac{1}{2} d^4 p \; d^4 p' F_{ij} \) over each surface \( S \) which encloses the origin

\[
\frac{1}{4\pi} \int_S F = i\hbar
\] (79)

is invariant under each differentiable change of the connection \( A \), the multiplicative part of the covariant derivative, as the Euler derivative of \( F(A) = dA \) with respect to \( A \) vanishes. This is why no such change can make two components of the covariant derivatives commute.

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