COHOMOLOGY OF QUIVER GRASSMANNIANS OF DYNKIN QUIVERS

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Abstract. For a Dynkin quiver, we establish a connection between the cohomology of quiver Grassmannians and the canonical bases of the negative half of the quantized enveloping algebra associated with the quiver. By the categorification of quantized enveloping algebras via KLR algebras, we describe the cohomology of a rigid quiver Grassmannian in term of the corresponding graded dimension of a proper standard module on the corresponding KLR algebra.

1. Introduction

The quiver Grassmannians was introduced to describe the categorification of cluster algebras, for their Euler characteristics play an important role on the categorification of cluster algebras. Hence the geometry of quiver Grassmannians gives rise to a great interest for mathematicians, especially: the geometry of rigid quiver Grassmannians, which enjoy some good properties as the flag varieties such as cellular decomposition (still as a conjecture). Among these properties of quiver Grassmannians, the cohomology (homology) of a quiver Grassmannian is one of the most important topics, for it can directly calculate the Euler characteristic of a quiver Grassmannian. There are some wonderful results related with this question; such as [IEFR].

However, to calculate the cohomology of a quiver Grassmannian explicitly is not clear. There are few works on this topic, which just focus on a specific example of quiver Grassmannian–degeneration of the complete flag variety. For example, in [FR] Fang and Reineke described the support of the family of linear degenerations by the methods of the PBW basis of quantum groups. Meanwhile, in [LS] Lanini and Strickland discovered the cohomology of PBW degeneration of the flag variety surjects onto the cohomology of the original flag variety. In light of their works, we calculate the cohomology of quiver Grassmannians in term of the canonical bases of quantized enveloping algebras.

Let us to illustrate this idea. Let $Q$ be a Dynkin quiver with the set of vertices $I$ and the set of arrows $\Omega$. Denote the source(target) of an arrow $h$ by $s(h)$ (resp: $t(h)$). Fix a field to be complex number field $\mathbb{C}$. Given a dimension vector $\nu + \mu = ((\nu + \mu)_i) \in \mathbb{N}^I$, we define the representation space with dimension vector $\nu + \mu$ as

$$E_{\nu+\mu}(Q) = \bigoplus_{h \in \Omega} \text{Hom}(V_{s(h)}, V_{t(h)})$$

where $\dim V_i = (\nu + \mu)_i$ for all $i \in I$. If both of dimension vectors $\mu$ and $\nu$ also lie in $\mathbb{N}^I$, then we define a smooth variety as
\[ E_{\nu, \mu} := \{(W, y) \mid y \in E_{\nu+\mu}(Q), y_{s}(W_{s(h)}) \subseteq W_{t(h)} \text{ and } \dim W = \mu \} \]

The constant sheaf \( \mathbb{C}_{E_{\nu, \mu}}[\dim E_{\nu, \mu}] \) is a perverse sheaf on \( E_{\nu, \mu} \), which is denoted by \( \mathbb{I}_{E_{\nu, \mu}} \). We construct a map as follows:

\[ q : E_{\nu, \mu} \rightarrow E_{\nu+\mu}(Q) \]

\[ (W, y) \mapsto y \]

The map \( q \) is proper. Given a representation \( M \in E_{\nu+\mu}(Q) \), the quiver Grassmannians is defined by \( q^{-1}(M) \) which is denoted as \( Gr_{\mu}(M) \). Following from the results in [CG], we have that

**Theorem 1.1.** For a representation \( M \) with a dimension vector \( \mu + \nu \), we have that

\[ H^{\bullet}(Gr_{\mu}(M)) \cong H^{\bullet-\dim E_{\nu, \mu}}(i_{M}^{*}q_{*}\mathbb{I}_{E_{\nu, \mu}}), \]

\[ H_{\bullet}(Gr_{\mu}(M)) \cong H^{\dim E_{\nu, \mu}}(q_{*}i_{M}^{!}\mathbb{I}_{E_{\nu, \mu}}) \]

where \( \mathbb{I}_{E_{\nu, \mu}} \) is the constant perverse sheaf over \( E_{\nu, \mu} \) and \( i_{M} : \{M\} \hookrightarrow E_{\nu+\mu}(Q) \)

By the [S] the complex \( q_{*}\mathbb{I}_{E_{\nu, \mu}} \) can be viewed as the multiplication of two canonical bases in the quantum group associated with the quiver. Moreover, by the categorification Theorem we can describe the cohomology of quiver Grassmannians by the irreducible modules on a KLR algebra. We set \( Ch_{q}(H^{\bullet}(Gr_{\mu}(M))) := \sum_{i \in \mathbb{Z}} \dim H^{i}(Gr_{\mu}(M))q^{i} \)

**Theorem 1.2.** Let \( M = M((\nu + \mu)^{0}) \) be a rigid representation for the \( Q \) (all rigid representations are of this form), we have that:

\[ Ch_{q}(H^{\bullet}(Gr_{\mu}(M))) = q^{t_{\nu, \mu}-(\mu, \nu)}\dim_{1_{\nu}, \mu, \nu_{0}}\mathbb{I}((\nu + \mu)^{0}) \]

where \( t_{\nu, \mu} = \dim E_{\nu}(Q) + \dim E_{\mu}(Q) \), \( \mathbb{I}((\nu + \mu)^{0}) \) is a proper standard module corresponding the element \((\nu + \mu)^{0}\) on the corresponding KLR algebra, and \( \dim_{1_{\nu}, \mu, \nu_{0}}\mathbb{I}((\nu + \mu)^{0}) \) is the graded dimension of the module \( 1_{\nu}, \mu, \nu_{0}\mathbb{I}((\nu + \mu)^{0}) \).

As a result, we implicitly generalize the results given by Lanini and Strickland in [LS].

**Corollary 1.3.** Suppose that a rigid representation \( M \) is degenerate to a representation \( N \), it follows

\[ Ch_{q}(H^{\bullet}(Gr_{\mu}(N))) = Ch_{q}(H^{\bullet}(Gr_{\mu}(M))) + f(q) \]

where \( f(q) \in \mathbb{N} \langle q, q^{-1} \rangle \).

2. **The representation of Dynkin quivers**

We first recall some basic facts on Dynkin quivers. Let \( \widehat{Q} = (I, \Omega) \) be a Dynkin quiver, where \( I \) is the set of vertices of the quiver and \( \Omega \) is the set of the arrows of the quiver. We denote the underlying graph of the quiver \( \widehat{Q} \) by \( Q \). For \( h \in \Omega \) we write \( s(h) \) and \( t(h) \) for the source of \( h \) and the target of \( h \), respectively. One can define a representation of the quiver \( \widehat{Q} \) over a field \( k \) in the following way: it is a tuple \((M_{t}, (M_{h}), h \in \Omega)\) where \( M_{t} \) is a vector space over \( k \) and \( M_{h} : M_{s(h)} \rightarrow M_{t(h)} \) is a linear map. We denote the category of representations of the
quiver by $\text{Rep}_k(\overrightarrow{Q})$. Given two representations $M, N \in \text{Rep}_k(\overrightarrow{Q})$, we denote $\text{Hom}_{\overrightarrow{Q}}(M, N)$ by the vector space of $\overrightarrow{Q}$-morphisms between $M$ and $N$ and write $[M, N]$ (resp: $[M, N]^1$) for $\dim(\text{Hom}_{\overrightarrow{Q}}(M, N))$ (resp: $\dim(\text{Ext}_{\overrightarrow{Q}}^1(M, N))$).

For a representation $M \in \text{Rep}_k(\overrightarrow{Q})$, we write $\dim M$ for the tuple $(\dim M_i)_{i \in I}$. Given two representations $M, N$ of the quiver, we define an integral bilinear form $\langle M, N \rangle$ as

$$\langle M, N \rangle = \langle \dim M, \dim N \rangle := \sum_{i \in I} \dim M_i \dim N_i - \sum_{h \in \Omega} \dim M_{s(h)} \dim N_{t(h)}$$

For an acyclic and connected quiver (such as a Dynkin quiver), one can get the following equation:

$$(2.1) \quad \langle M, N \rangle = [M, N] - [M, N]^1$$

Based on the above notations, we define the Cartan matrix $A_Q = (a_{i,j})_{i,j \in I}$ for a quiver $Q$ (which only depends on the underlying graph) by

$$a_{i,j} = \begin{cases} 2 & i = j \\ -(\langle S(i), S(j) \rangle - \langle S(j), S(i) \rangle) & i \neq j \end{cases}$$

where $S(k)$ denotes the simple module corresponding $k \in I$.

It is known that a quiver is a Dynkin quiver if and only if its Cartan matrix $A_Q$ is positive definite. One also defines the Kac-Moody algebra associated with the Cartan matrix $A_Q$, which generated by a vector space $\mathfrak{h}$, elements $\{e_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$ with some conditions. It is known that the Kac-Moody algebras associated with Dynkin quivers are finite dimensional Lie algebras and simple Lie algebras.

The Lie algebra $\mathfrak{g}_Q$ associated with the Dynkin quiver $Q$ could be decomposed into

$$\mathfrak{g}_Q = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha \oplus \mathfrak{b}$$

where $\mathfrak{n}^+$ and $\mathfrak{n}^-$ is generated by $\{e_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$, respectively, and

$$\mathfrak{g}_\alpha = \{u \in \mathfrak{g}_Q : [h, u] = \langle h, \alpha \rangle u, \forall h \in \mathfrak{h} \}$$

We write $R^+ = \{\alpha \in \mathfrak{h}^* : \mathfrak{n}^+ \supset \mathfrak{g}_\alpha \neq 0\}$. The Gabriel’s Theorem is given as follows.

**Theorem 2.1 (Gabriel’s Theorem).** Let $\overrightarrow{Q}$ be a Dynkin quiver, then the map $[V] \mapsto \dim V$ gives a bijection between the set $\text{Ind}(\overrightarrow{Q})$ of isomorphism classes of nonzero indecomposable objects of $\text{Rep}_k(\overrightarrow{Q})$ and the set $R^+$

Note that the number and the dimension of indecomposable representations of a Dynkin quiver do not depend on the orientation $\Omega$ of a Dynkin graph $Q$ and the ground field $k$.

For the Lie algebra $\mathfrak{g}_Q$, its Weyl group $W$ contains an unique longest element $w_0$. Lusztig discovered that an orientation of a fixed Dynkin graph $Q$ could be given by a reduced expression
of $w_0 = s_{i_N} \cdots s_{i_2} s_{i_1} \in W$, then one can construct an ordering on the positive roots $R^+$ as
\[
\begin{align*}
\alpha_1 &= \alpha_{i_1} \\
\alpha_2 &= s_{i_1} (\alpha_{i_2}) \\
&\vdots \\
\alpha_N &= s_{i_1} s_{i_2} \cdots s_{i_{N-1}} (\alpha_{i_N})
\end{align*}
\]
so that $\alpha_k > \alpha_l$ if and only if $k > l$, where the roots $\alpha_{i_j}$ are the simple roots. There is an indecomposable representation $M(\alpha_k)$ of the quiver $\overrightarrow{Q} = (I, \Omega)$ corresponding the root $\alpha_k$ for each $1 \leq k \leq |R^+|$. For these indecomposable representations, it follows:
\[
(2.3) \quad \text{Hom}_{\overrightarrow{Q}}(M(\alpha_a), M(\alpha_b)) = \text{Ext}_{\overrightarrow{Q}}(M(\alpha_b), M(\alpha_a)) = 0 \quad \text{for } a > b
\]

3. LUSZTIG’S GEOMETRIC CONSTRUCTION OF CANONICAL BASES

Lusztig’s geometric construction of canonical bases of negative half of quantum groups is a powerful tool for our paper. The original results concerning geometric construction are given in the cases of the closure field of a finite field. However, the same results are obtained over the complex number field by the works in chapter 6 of [BBD, Chapter 6] via Hall category (see [S, Remark 3.27]). Thus unless specified otherwise, we assume the ground field is the complex number field $\mathbb{C}$ and replace the field $\overline{\mathbb{Q}}_l$ with $\mathbb{C}$. We fix a Dynkin quiver $\overrightarrow{Q}$ and its underlying graph $Q$ in the rest of this section. Due to the results of this section just depend on the underlying graph $Q$, we abbreviate the $\overrightarrow{Q}$ as $Q$ if there is no confusion.

3.1. Perverse sheaves on representation spaces. Given a vector $\nu \in \mathbb{N}^I$, we set the vector space $E_\nu(Q) := \bigoplus_{h \in \Omega} \text{Hom}(V_{s(h)}, V_{t(h)})$ and the linear algebraic group $GL(\nu) := \Pi_{i \in I} GL(V_i)$ where the vector spaces $V_i$ satisfy $\dim V_i = \nu_i$. Lusztig has constructed a family of $GL(\nu)$-equivalent perverse sheaves $\mathcal{P}_\nu$ over $E_\nu(Q)$ to describe the canonical bases of the half-part of the quantum group associated with the quiver $Q$. For the case of Dynkin quivers, by [S, Theorem 2.8] it is known that $\mathcal{P}_\nu = \{IC(\mathcal{O}_\lambda)\}$ where the $\mathcal{O}_\lambda$ runs over the $GL(\nu)$-orbits in the space $E_\nu(Q)$ and the $IC(\mathcal{O}_\lambda)$ is the intersection homology corresponding the orbit $\mathcal{O}_\lambda$. We write $KP(\nu)$ for the set of the $GL(\nu)$-orbits in the space $E_\nu(Q)$

**Example 3.1.** Given a vector $\nu \in \mathbb{N}^I$, the perverse sheaf $\mathcal{C}_{E_\nu(Q)}[\dim E_\nu(Q)]$ on $E_\nu(Q)$ is contained in $\mathcal{P}_\nu$ (see [S, Example 2.5]). Then
\[
\mathcal{C}_{E_\nu(Q)}[\dim E_\nu(Q)]|_{\mathcal{O}_{\text{max}}} = \mathcal{C}_{\mathcal{O}_{\text{max}}}[\dim E_\nu(Q)] = IC(\mathcal{O}_{\text{max}})|_{\mathcal{O}_{\text{max}}}
\]
where the $\mathcal{O}_{\text{max}}$ is a unique open orbit in $E_\nu(Q)$, thus we have $\mathcal{C}_{E_\nu(Q)}[\dim E_\nu(Q)] = IC(\mathcal{O}_{\text{max}})$

From now on, we denote $\mathcal{C}_{E_\nu(Q)}[\dim E_\nu(Q)]$ as $\mathbb{I}_\nu$.

Lusztig also defined the multiplication of the perverse sheaves, for the details one should see [L2] and [L1]. The simple vision is given as follows: Given two vector $\nu, \mu \in \mathbb{N}^I$, we define the variety
$$E_{\nu, \mu} := \{(W, y) \mid y \in E_{\nu+\mu}(Q); y_h(W_{s(h)}) \subseteq W_{t(h)} \text{ and } \dim W = \mu\}$$

Then we construct a map:

$$q : E_{\nu, \mu} \to E_{\nu+\mu}(Q)$$

$$(W, y) \mapsto y$$

The map $q$ is a proper morphism, and then we define the multiplication of $I_\nu$ and $I_\mu$ by

$$I_\nu \ast I_\mu = q! C_{E_{\nu, \mu}}[\dim E_{\nu, \mu} + \dim E_\mu(Q) + \dim E_\mu(Q)]$$

This definition of the multiplication coincides with the original Lusztig’s definition ([S, Section 1.4]).

We write $f_A$ for the Lusztig’s integral form of the negative half of the quantum universal enveloping algebra associated with the quiver $Q$, where $A = \mathbb{Z}[q, q^{-1}]$. Given a number $n \in \mathbb{N}$, we set

$$[n] := \frac{q^n - q^{-n}}{q - q^{-1}} \quad [n]! := \prod_{1 \leq k \leq n} [k]$$

The algebra $f_A$ is a $\mathbb{N}_I$-graded algebra and is generated by elements $\{\theta_i\}_{i \in I}$ subject to the quantum Serre relations

$$\sum_{r+s=1-a_{ij}} (-1)^r \theta_i^{(r)} \theta_j^{(s)} = 0$$

for all $i, j \in I$ and $r \geq 1$, where $\theta_i^{(r)}$ denotes the divided power $\theta_i^r/[r]!$.

The Lusztig’s geometric construction of canonical bases is given as follows.

**Theorem 3.2.** Given a vector $\nu \in \mathbb{N}_I$, let $Q_\nu$ be the semisimple subcategory of derived category $D_{GL(\nu)}(E_\nu(Q))$ generated by $P_\nu$, then there exists an bijection between the Grothendick group of $Q_\nu$ and $f_{A, \nu}$. Moreover, we have that algebraic isomorphism $\bigoplus_{\nu \in \mathbb{N}_I} K(Q_\nu) \cong f_A$ under the operation of the multiplication. The elements in $f_A$ corresponding the elements in $P_\nu$ is called canonical bases.

Note that category $Q_\nu$ doesn’t depend on the ground field $k$. By the facts in [BBD], we know the category defined over $\mathbb{C}$ is equivalent to the category defined over algebraic closed fields $\mathbb{F}_q$, or see [S, Remark 3.27].

From now on, we write $b_\lambda$ for the canonical base in $f_A$ corresponding the perverse sheaf $IC(O_\lambda)$. Specially, we write $b_{0, \nu}$ for the canonical base in $f_A$ corresponding the perverse sheaf $\mathbb{I}_\nu$ in $P_\nu$. 
As a corollary, we have that

$$b_\nu \ast b_\mu = \sum_{\lambda \in KP(\mu + \nu)} \chi_{\nu,\mu}^\lambda b_\lambda$$

(3.5)

where $\chi_{\nu,\mu}^\lambda \in \mathbb{N}[q, q^{-1}]$. There is another way to describe the coefficients $\chi_{\nu,\mu}^\lambda \in \mathbb{N}[q, q^{-1}]$ by Hall algebra of the category $Q = \bigoplus_{\nu \in \mathbb{N}^I} Q_\nu$ as follows (see [S]).

3.2. Hall category. In the rest of this section, we assume the ground field is the algebraic closed field $\mathbb{F}_q$ of a finite field and write $X$ for the space $E_\nu(Q)$. There is the Frobenius element $F$ in the Galois group $Gal(\mathbb{F}_q/\mathbb{F}_q)$ which acts on the variety $X$ and its derived category. Then $X^F = X^0$

A Weil structure on a constructible complex $P$ over $X$ is an isomorphism $j: P \rightarrow F^*P$. It is easy to see that a Weil complex $(P, j)$ gives rise to an action of $F$ on the stalk $P_{x_0}$ of $P$ at each point $x_0 \in X^0(\mathbb{F}_q)$, thus the trace map of $P$ is given by

$$Tr(P) : X^0(\mathbb{F}_q) \rightarrow \mathbb{C}$$

$$x_0 \mapsto \sum_i (-1)^i Tr(F : H^i(P)|_{x_0})$$

where we fix an identification $\overline{Q_I} \cong \mathbb{C}$.

Let $P', P'' \in \mathcal{P}_Q$ and let us write

$$P' \ast P'' = \bigoplus_P M_{P',P''}^P \otimes P$$

(3.6)

where $M_{P',P''}^P = Hom(P' \ast P'', P)$ is the multiplicity complex. Lusztig has showed that $M_{P',P''}^P$ has a Weil structure and the Frobenius eigenvalues over $H^i(M_{P',P''}^P)$ are all equal to $\sqrt{q}$.

We define the algebra $\mathfrak{U}_Q = \bigoplus_{\mu} \mathfrak{U}_\mu$ where $\mathfrak{U}_\mu = \bigoplus_{\nu \in \mathcal{P}_\nu} \mathbb{C}b_\nu$. The multiplication is given by

$$b_{P'} \ast' b_{P''} = \sum_P Tr(M_{P',P''}^P) b_P$$

(3.7)

In [S] there is an isomorphism between $(\mathfrak{f}_A)_{-\sqrt{q}}$ and $\mathfrak{U}_Q$ where $(\mathfrak{f}_A)_{-\sqrt{q}}$ is obtained by replacing the indeterminate of the algebra $\mathfrak{f}_A$ with $-\sqrt{q}$.

In this way, the coefficient $\chi_{\nu,\mu}^\lambda$ could be obtained by

$$\chi_{\nu,\mu}^\lambda(-\sqrt{q}) = Tr(M_{P',P''}^{IC(\nu,\mu)})$$

(3.8)

Note that the above perverse sheaves defined over $\overline{\mathbb{F}_q}$ rather than over $\mathbb{C}$.

4. Quantum shuffle algebras and KLR-algebras

Another way to categorify the half-part of quantum groups is by Khovanov-Lauda-Rouquier algebras (or quiver Hecke algebras).
4.1. Quantum shuffle algebras. Given a finite set $I$, we set $\langle I \rangle$ be the free monoid on $I$, that is, the set of all words $i = [i_1 \cdots i_n]$ for $n \geq 0$ and $i_1, \cdots, i_n \in I$ with multiplication given by concatenation of words.

For a word $i = [i_1 \cdots i_n]$ of length $n$ and a permutation $w \in S_n$, we let
\[
| i | := \alpha_{i_1} + \cdots + \alpha_{i_n}, \quad w(i) := [i_{w^{-1}(1)} \cdots i_{w^{-1}(n)}],
\]
\[
\theta_i := \theta_{i_1} \cdots \theta_{i_n}, \quad \text{deg}(w; i) := - \sum_{1 \leq j \leq k \leq n} \alpha_{i_j} \cdot \alpha_{i_k}.
\]

Setting $\langle I \rangle_\alpha := \{ i : | i | = \alpha \}$, the monomials $\{ \theta_i : i \in \langle I \rangle_\alpha \}$ span $\mathfrak{f}_\alpha$. The quantum shuffle algebra is free $\mathcal{A}$-module $\mathcal{A}(I) = \bigoplus_{\alpha \in \mathbb{Q}^+} \mathcal{A}(I)_\alpha$ on basis $\langle I \rangle$, viewed as an $\mathcal{A}$-algebra via the shuffle product $\circ$ defined on words $i$ and $j$ of length $m$ and $n$, respectively, by
\[
i \circ j := \sum_{\substack{w \in S_{n+m} \\text{s.t.} \ w(i) < w(j) \text{ for all } m+1 \leq w(i) < w(j)}} q^{\text{deg}(w; ij)} w(ij).
\]

There is an injective $\mathcal{A}$-algebra homomorphism
\[
Ch : \mathfrak{f}_\mathcal{A}^* \rightarrow \mathcal{A}(I)
\]
\[
x \mapsto \sum_{i \in \langle I \rangle} (\theta_i, x) i
\]

See [Le]. Generally, we can define the map $Ch : \mathfrak{f}^* \rightarrow \mathbb{Q}(q)\langle I \rangle$ as well. The map is also injective.

Next we fix an arbitrary total order on the set $\Pi = \{ \alpha_1, \cdots, \alpha_r \}$ of simple roots of $\mathfrak{g}_Q$, we set $I = \Pi$, then $\langle I \rangle$ is given the lexicographic order which will be denoted by $<\hspace{1cm}$

**Definition 4.1.** A word $i \in \langle I \rangle$ is called good word, if there exists an element $x \in \mathfrak{f}^*$ such that $i = \text{max}(Ch(x))$. Denote by $\langle I \rangle^+$ the set of all good words.

**Definition 4.2.** A word $i = [i_1 \cdots i_n] \in \langle I \rangle$ is called Lyndon word, if it satisfies the condition
\[
[i_1 \cdots i_n] < [i_j \cdots i_k] \quad \text{for all } 1 \leq j < k \leq n
\]

For Cartan data of finite type (such as Dynkin quivers), the good Lyndon words can be classified by the following fact in [Le].

**Lemma 4.3 ([Le]).** If the Cartan data is of finite type then the map $i \mapsto |i|$ is bijection between the set of good Lyndon words and the set $R^+$ of positive roots.

Given a good word $i \in \langle I \rangle^+$, we can decompose it as $i = j_1 \cdots j_k$ where $j_i$ are good Lyndon words with the condition $j_1 \succ j_2 \cdots \succ j_k$. The above bijection transports the lexicographic order on good Lyndon words to a total order on the set $R^+$ inducing a total order $\succ$ on $\mathbb{Q}^+ := \{ |i| : i \in \langle I \rangle \}$

The induced order $R^+ = \{ |i| : i \in \langle I \rangle \}$ is a convex order, that is,
\[
\beta, \gamma, \beta + \gamma \in R^+, \beta \succ \gamma \Rightarrow \beta \succ \beta + \gamma \succ \gamma.
\]
By [P], there is a bijection between convex orderings of $Q^+$ and reduced expression for the longest element $w_0 \in W$: given a reduced expression $w_0 = s_{i_1} \cdots s_{i_n}$ then corresponding convex ordering on $\Delta_+$ is given by

$$\alpha_{i_1} \prec s_{i_1}(\alpha_{i_2}) \prec \cdots \prec s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})$$

### 4.2. Kostant partition

In [BKM], a Kostant partition of $\alpha \in Q^+$ is a sequence $\lambda = (\lambda_1, \cdots, \lambda_l)$ of positive roots such that $\lambda_1 \succ \cdots \succ \lambda_l$ and $\lambda_1 + \cdots + \lambda_l = \alpha$. Set $\lambda'_k = \lambda_{l+1-k}$. If we construct a convex order on positive root system $R^+$, we define an order $\succ$ on $KP(\alpha) := \{\lambda = (\lambda_1, \cdots, \lambda_l) : \lambda_1 + \cdots + \lambda_l = \alpha\}$ so that $\lambda \succ \kappa$ if and only if both of the following hold:

- $\lambda_1 = \kappa_1, \cdots, \lambda_{k-1} = \kappa_{k-1}$ and $\lambda_k \succ \kappa_k$ for some $k$ such that $\lambda_k$ and $\kappa_k$ are both defined.
- $\lambda'_1 = \kappa'_1, \cdots, \lambda'_{k-1} = \kappa'_{k-1}$ and $\lambda'_k \prec \kappa'_k$ for some $k$ such that $\lambda'_k$ and $\kappa'_k$ are both defined.

For $\lambda = (\lambda_1, \cdots, \lambda_l) \in KP(\alpha)$, we set

$$s_\lambda := \sum_{\beta \in R^+} \frac{1}{2} m_\beta(\lambda)(m_\beta(\lambda) - 1)$$

where $m_\beta$ denotes the multiplicity of $\beta$ in $\lambda$.

If the ordering of the root system $R^+$ is deduced from the good Lyndon words, then the map $i \mapsto |i|$ induces a bijection between the set $KP(\alpha)$ and the set

$$\{j_1, \cdots, j_k\} : j_1 \succ j_2 \cdots \succ j_k$$

such that each $j_i$ is good Lyndon word and $|j_1 \cdots j_k| = \alpha$.

Each word $i$ in $(I)^+_\alpha$ has the unique decomposition $i = j_1 \cdots j_k$ such that all $j_i$ are good Lyndon words with the condition $j_1 \succ j_2 \cdots \succ j_k$. So there is a bijection between $KP(\alpha)$ and $(I)^+_\alpha$

$$\Psi : KP(\alpha) \rightarrow (I)^+_\alpha$$

$$(\lambda_1, \cdots, \lambda_l) \mapsto i(\lambda_1) \cdots i(\lambda_l)$$

Next we define the KLR algebras. See [KL]

**Definition 4.4 ([KL])**. Let $Q$ be a Dynkin quiver and $\alpha \in Q^+$ with $n = ht(\alpha)$. We define the KLR algebra $R_\alpha$ as a unital algebra generated by the elements

$$\{1_i \mid i \in \langle I \rangle_\alpha\} \cup \{x_1, \ldots, x_n\} \cup \{\tau_1, \ldots, \tau_{n-1}\}$$

subject to the following relation:

- $x_kx_l = x_lx_k$;
- the elements $\{1_i \mid i \in \langle I \rangle_\alpha\}$ are mutually orthogonal idempotents whose sum is the identity $1_\alpha \in R_\alpha$;
- $x_k1_i = 1_i x_k$ and $\tau_k1_i = 1_{(k,k+1)i} \tau_k$;
- $(\tau_k x_l - x_{(k,k+1)i} \tau_k)1_i = \begin{cases} 1_i & \text{if } i_k = i_{k+1} \text{ and } l = k+1, \\ -1_i & \text{if } i_k = i_{k+1} \text{ and } l = k, \\ 0 & \text{otherwise}; \end{cases}$
The algebra \( R \) is the image of the identity \( \beta \) an \( \alpha \).

Proposition 4.5 ([KL] or [M]). Let \( \beta, \gamma \in Q^+ \), there is an evident non-unital algebra embedding \( R_\beta \otimes R_\gamma \to R_{\beta+\gamma} \). We denote the image of the identity \( 1_\beta \otimes 1_\gamma \in R_\beta \otimes R_\gamma \) by \( 1_{\beta,\gamma} \in R_{\beta+\gamma} \). Then for an \( R_{\beta+\gamma} \)-module \( U \) and an \( R_\beta \otimes R_\gamma \)-module \( V \), we set

\[
\text{res}^{\beta+\gamma}U := 1_{\beta,\gamma}U; \quad \text{ind}^{\beta+\gamma}V := R_{\beta+\gamma}1_{\beta,\gamma} \otimes_{R_\beta \otimes R_\gamma} V
\]

More generally, given \( \alpha_1, \ldots, \alpha_k \in Q^+ \) such that \( \sum_{1 \leq i \leq k} \alpha_i = \alpha \), there are induction and restriction functors

\[
\text{Ind}_{\alpha_1,\ldots,\alpha_k} : \bigotimes_{1 \leq i \leq k} R_{\alpha_i}\text{-mod} \to R_\alpha\text{-mod}
\]

and

\[
\text{Res}_{\alpha_1,\ldots,\alpha_k} : R_\alpha\text{-mod} \to \bigotimes_{1 \leq i \leq k} R_{\alpha_i}\text{-mod}
\]

The following Mackey-style result is given in [KL] or [M].

Proposition 4.5 ([KL] or [M]). Let \( \alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_k \in Q^+ \) be such that \( \sum_{1 \leq i \leq k} \alpha_i = \sum_{1 \leq i \leq \beta} \beta_i \), then the composite functor \( \text{Res}^{\beta_1,\ldots,\beta_k}_\alpha \circ \text{Ind}_{\alpha_1,\ldots,\alpha_k} \) has a filtration indexed by tuples \( \nu_{ij} \) satisfying \( \alpha_i = \sum_j \nu_{ij} \) and \( \beta_j = \sum_i \nu_{ij} \). The subquotients of this filtration are isomorphic, up to a grading shift, to the composition \( \text{Ind}^\beta_{\nu_{ij}} \circ \text{Res}^\alpha_{\nu_{ij}} \) where \( \text{Res}^\alpha_{\nu_{ij}} : \bigotimes_i R_{\alpha_i} - \text{mod} \to \bigotimes_i (\bigotimes_j R_{\nu_{ij}}) - \text{mod} \) is the tensor product of the \( \text{Res}_{\nu_{ij}} \), \( \tau : \bigotimes_i (\bigotimes_j R_{\nu_{ij}}) - \text{mod} \to \bigotimes_j (\bigotimes_i R_{\nu_{ij}}) - \text{mod} \) is given by permuting the tensor factors and \( \text{Ind}^\beta_{\nu_{ij}} : \bigotimes_j R_{\nu_{ij}} - \text{mod} \to \bigotimes_j R_{\beta j} - \text{mod} \) is the tensor product of the \( \text{Ind}_{\nu_{ij}} \).

4.3. The categorification Theorem. Let \( \text{Rep}(R_\alpha) \) denote the abelian category of finite dimensional \( R_\alpha \)-modules and set

\[
\text{Rep}(H) := \bigoplus_{\alpha \in Q^+} \text{Rep}(R_\alpha)
\]

This is a graded \( k \)-linear monoidal category under the induction \( V \circ U := \text{ind}^{\beta+\gamma}_{\beta,\gamma}(U \boxtimes V) \) for \( U \in \text{Rep}(R_\beta) \) and \( V \in \text{Rep}(R_\gamma) \). Denote by \( |\text{Rep}(R)| = \bigoplus_{\alpha \in Q^+} |\text{Rep}(R_\alpha)| \) its Grothendieck ring which we make into an \( \mathcal{A} \)-algebra so that \( q[V] = [V[1]] \). Moreover, we define \( \text{Dim} V := \)
\sum_{i \in \mathbb{Z}} \dim V_i q^i \text{ and write } \text{hom}(W, V) \text{ for the space of } R\text{-morphisms between } W \text{ and } V \text{ preserving their degrees. Dually, we have the additive category } \text{Proj}(R_{\alpha}) \text{ of finitely generated projective } R_{\alpha}\text{-modules and set }

\text{Proj}(R) := \bigoplus_{\alpha \in Q^+} \text{Proj}(R_{\alpha})

This is also a graded } k\text{-linear monoidal category under the induction, and again the split Grothendieck group } [\text{Proj}(R)] = \bigoplus_{\alpha \in Q^+} [\text{Proj}(R_{\alpha})] \text{ is an } A\text{-algebra. Moreover there is a non-degenerate pairing }

(\cdot, \cdot) : [\text{Proj}(R)] \times [\text{Rep}(H)] \to A

(P, V) := \begin{cases} \text{DimHom}(P, V) & \text{if } P \in \text{Proj}(R_{\alpha}) \text{ and } V \in \text{Rep}(R_{\alpha}) \\ 0 & \text{otherwise} \end{cases}

where

\text{Hom}(P, V) := \sum_{n \in \mathbb{N}} \text{hom}(P[n], V)

For } P_1 \in \text{Proj}(R_{\beta}), P_2 \in \text{Proj}(R_{\gamma}) \text{ and } V \in \text{Rep}(R_{\beta+\gamma}), \text{ we have that }

(\text{res}_{\beta,\gamma} P_1 \otimes P_2, V) = (P_1 \otimes P_2, \text{ind}_{\beta,\gamma} V)

The categorification Theorem is given in [KL] and [VV],et.

**Theorem 4.6** ([KL] and [VV]). \text{There is a unique adjoint pair of } A\text{-algebra isomorphism}

\Phi : \mathfrak{f}_A \sim \to [\text{Proj}(R)], \quad \Phi^* : \mathfrak{f}_A^* \sim \to [\text{Rep}(R)]

\text{such that } \Phi \text{ transports the canonical bases to the self-dual indecomposable projective modules in } \text{Proj}(R) \text{ and } \Phi^* \text{ transports the dual canonical basis to the the self-dual irreducible modules in } \text{Rep}(R).

For an element } \lambda \in KP(\nu), \text{ there exists the corresponding canonical base } b_\lambda \in \mathfrak{f}_\nu. \text{ By the above theorem, we denote the indecomposable projective module corresponding } \lambda \text{ by } P_\lambda \text{ and the irreducible module corresponding } \lambda \text{ by } L(\lambda). \text{ Moreover, in [BKM] or [M] they define a proper standard module } \overline{\Sigma}(\lambda) \text{ as follows: }

\overline{\Sigma}(\lambda) := q^{s_\lambda} L(\lambda_1) \circ \cdots \circ L(\lambda_t)

(4.2)

**Remark 4.7.** In [BKM] we have that } L(\lambda) = \overline{\Sigma}(\lambda) = q^{s_\lambda} L(\lambda_1) \circ \cdots \circ L(\lambda_t), \text{ if } \lambda \text{ is a smallest element of } KP(\nu).

Recalling the equation (3.5), we have the following equation

\begin{align*}
P_{\nu,0} \circ P_{\mu,0} &= \sum_{\lambda \in KP(\mu+\nu)} \chi_{\nu,\mu}^{\lambda} P_{\lambda} \\
\end{align*}

(4.3)
Thus we have

\[ \chi_{\nu,\mu}^\lambda = (P_{\nu,0} \circ P_{\mu,0}, L(\lambda)) \]
\[ = (\text{res}_{\nu,\mu} P_{\nu,0} \boxtimes P_{\mu,0}, L(\lambda)) \]
\[ = (P_{\nu,0} \boxtimes P_{\mu,0}, \text{ind}_{\nu,\mu} L(\lambda)) \]
\[ = \text{Dim}1_{\nu,\mu} P_{\nu,0} \boxtimes P_{\mu,0}, L(\lambda) \]

where \( 1_{\nu,\mu} \) denotes the idempotent of the projective module \( P_{\nu,0} \). In [KL] one defines the map

\[ Ch : \text{[Rep}(R)] \rightarrow A(I) \]
\[ V \mapsto \sum_{i \in \langle I \rangle} (\text{Dim}1_i V)i \]

then \( b_\lambda^* = Ch(L(\lambda)). \)

5. Quiver Grassmannians

In this section, we fix an orientation \( \Omega \) of a Dynkin quiver \( Q \).

5.1. Cohomology of quiver Grassmannians. In the section 2, the reduced expression of the longest element \( w_0 \) adapting the orientation \( \Omega \) gives rise to a convex ordering on \( R^+ \) such that \( \alpha_k > \alpha_l \) for \( k > l \). We set another convex ordering as \( \alpha_k \prec \alpha_l \) for \( k > l \).

By the equation (2.3), we have

\[ \text{Hom}_Q(M(\alpha_a), M(\alpha_b)) = \text{Ext}_Q(M(\alpha_b), M(\alpha_a)) = 0 \quad \text{for } \alpha_a < \alpha_b \]

This ordering leads to a partial order on \( KP(\nu) \) for each \( \nu \in Q^+ \) as the section 4.2. By the definition of \( KP(\nu) \) and the Gabriel’s theorem, we have that the set of the isomorphism classes of representation \( M \) on quiver \( Q \) with \( \dim(M) = \nu \) is equal to the set \( KP(\nu) \), and then there is a bijection between \( KP(\nu) \) and the set of the \( GL(\nu) \)-orbits in \( E_\nu(\widehat{Q}) \), denoted by \( E_\nu(\widehat{Q})/GL(\nu) \), as

\[ KP(\nu) \xrightarrow{\sim} E_\nu(\widehat{Q})/GL(\nu) \]
\[ (\lambda_1, \cdots, \lambda_k) \mapsto \bigoplus_{1 \leq i \leq k} [M(\lambda_i)] \]

where \([M(\lambda_i)]\) denotes the isomorphism class of the indecomposable module \( M(\lambda_i) \) for each root \( \lambda_i \). Hence for an element \( \lambda \in KP(\nu) \) we denote the corresponding \( GL(\nu) \)-orbit as \( O_\lambda \) and the corresponding representation as \( M(\lambda) \).

We recall a partial order on the set \( E_\nu(\widehat{Q})/GL(\nu) \): Given two elements \( \lambda, \kappa \in KP(\nu) \), we define a partial order \( \prec^1 \) so that \( \lambda \prec^1 \kappa \) if and only if we have \( O_\kappa \subset O_\lambda \).
Since $\hat{Q}$ is a Dynkin quiver, the condition $\lambda \prec^1 \kappa$ implies that $[N, M(\lambda)] \leq [N, M(\kappa)]$ and $[M(\lambda), N] \leq [M(\kappa), N]$ for all representations $N$ on the quiver $\hat{Q}$, see [CB]. Denote the Konstant partition associated with the unique open orbit in $E_V(\hat{Q})$ by $\nu^0$. Note that representation $M(\nu^0)$ is a rigid representation.

**Theorem 5.1.** The partition $\nu^0$ is a smallest element of $KP(\nu)$ under the ordering $\prec$.

*Proof.* Suppose for the contradiction that there exists an element $\lambda \prec \nu^0$ in $KP(\nu)$. By the definition of this partial order, we have that $\lambda_k \prec \nu^0$ for some $k$ such that $\lambda_i = \nu^0$ for all $1 \leq i \leq k - 1$. By taking the module $M(\nu^0)$, we get that

$$
[M(\nu^0), M(\nu^0)] = [M(\nu^0), M(\nu^0) \oplus (\oplus_{i \leq k-1} M(\nu^0))] 
= [M(\nu^0), M(\nu^0)] + [M(\nu^0), (\oplus_{i \leq k-1} M(\nu^0))] 
= 1 + [M(\nu^0), (\oplus_{i \leq k-1} M(\nu^0))] 
= 1 + [M(\nu^0), M(\lambda)]
$$

So it contradicts the facts $[N, M(\nu^0)] \leq [N, M(\lambda)]$ for all representations $N$ on the quiver $\hat{Q}$.

From now on, we abbreviate $\hat{Q}$ as $Q$.

Next we recall the concept of quiver Grassmannians. In the section 3, there is a map

$$q : E_{\nu, \mu} \to E_{\nu+\mu}(Q)$$

Given a point $M$ in $E_{\nu+\mu}(Q)$, we call $q^{-1}(M)$ as the quiver Grassmannian with dimension vector $\mu$ for the representation $M$ and denote $q^{-1}(M)$ by $Gr_{\mu}(M)$. As $E_{\nu, \mu}$ is a smooth variety, we denote the perverse sheaf $\mathbb{C}_{E_\nu, \mu}[\dim E_{\nu, \mu}]$ as $\mathbb{I}_{E_\nu, \mu}(Q)$. Denote the imbedding $\{M\} \to E_{\mu+\nu}$ as $i_M$ and $\dim E_\nu(Q) + \dim E_\mu(Q)$ as $t_{\nu, \mu}$

**Lemma 5.2.** For a representation $M$ with a dimension vector $\mu + \nu$, we have that

$$H^i(Gr_{\mu}(M)) \cong H^{i - \dim E_{\nu, \mu}}(i_M^* q^! \mathbb{I}_{E_{\nu, \mu}}),$$

$$H^\bullet(Gr_{\mu}(M)) \cong H^{\dim E_{\nu, \mu} - \bullet}(i_M^* q^! \mathbb{I}_{E_{\nu, \mu}})$$

*Proof.* Follow from the Lemma 8.5.4 in [CG]

**Lemma 5.3.** We have $q^! \mathbb{I}_{E_{\nu, \mu}} = \bigoplus_{\lambda \in KP(\nu+\mu)} V(\lambda) \boxtimes IC(O_\lambda)$ where $V(\lambda)$ are $\mathbb{Z}$-graded vector spaces.

*Proof.* Follow from the BBD Decomposition Theorem [BBD].

As the equations (3.2) and (3.5), if we define the $\text{Dim}(V(\lambda)) := \sum_{i \in \mathbb{Z}} \text{dim} V(\lambda)_i q^i$, then it follows

$$\text{Dim}(V(\lambda)) = q^{\nu, \mu} \text{Dim}(V(\lambda)[t_{\nu, \mu}]) = q^{\nu, \mu} \lambda_{\nu, \mu}^\lambda = q^{\nu, \mu} \text{Dim} 1_{\nu, \mu} \lambda L(\lambda)$$
From now on, we assume that the representation $M$ is a rigid representation, that is: $M = M((\nu + \mu)^0)$ for some dimension vectors $\nu, \mu \in Q^+$ (note that the $Q^+ = N^I$). Set

$$Ch_q(H^*\langle Gr_\mu(M) \rangle) := \sum_{i \in \mathbb{Z}} dimH^i(Gr_\mu(M))q^i$$

**Theorem 5.4.** Let $M = M((\nu + \mu)^0)$ be a rigid representation for the $Q$. It follows:

$$Ch_q(H^*\langle Gr_\mu(M) \rangle) = q^{\nu(\nu + \mu)} Dim1_{\nu,\mu}^0 \Delta((\nu + \mu)^0)$$

**Proof.** By Lemma 5.2 and the fact $supp(\langle IC(\mathcal{O}_\mu) \rangle) \subset \overline{IC(\mathcal{O}_\mu))}$, we have that

$$H^*\langle Gr_\mu(M) \rangle \cong V((\nu + \mu)^0) \otimes H^*_{-dimE_{\nu,\mu}}(Q)(i^*_M IC(\mathcal{O}((\nu + \mu)^0)))$$

For $IC(\mathcal{O}((\nu + \mu)^0)_{\mathcal{O}((\nu + \mu)^0)} = \mathbb{C}_{\mathcal{O}((\nu + \mu)^0)}[dimE_{\nu,\mu}(Q)]$, we have that

$$H^*\langle Gr_\mu(M) \rangle \cong V((\nu + \mu)^0)[dimE_{\nu,\mu}(Q) - dimE_{\nu,\mu}]$$

Note that $dimE_{\nu,\mu} - dimE_{\nu,\mu}(Q) = (\mu, \nu)$. Thus

$$Ch_q(H^*\langle Gr_\mu(M) \rangle) = \sum_{i \in \mathbb{Z}} dimH^i(Gr_\mu(M))q^i$$

$$= q^{-\nu(\nu + \mu)} DimV((\nu + \mu)^0)$$

$$= q^{-\nu(\nu + \mu) + \nu(\nu + \mu)} Dim1_{\nu,\mu}^0 L((\nu + \mu)^0)$$

$$= q^{\nu(\nu + \mu)} Dim1_{\nu,\mu}^0 \Delta((\nu + \mu)^0).$$

for $L((\nu + \mu)^0) = \Delta((\nu + \mu)^0)$ by Lemma 5.1 and Remark 4.7.

**Corollary 5.5.** Suppose that $N >^1 M$ where $M$ is a rigid representation, we have that

$$Ch_q(H^*\langle Gr_\mu(N) \rangle) = Ch_q(H^*\langle Gr_\mu(M) \rangle) + f(q)$$

where $f(q) \in \mathbb{N}[q, q^{-1}]$.

**Proof.** By the Lemma 5.2, we have that

$$q\mathbb{I}_{E_{\nu,\mu}} = \bigoplus_{\lambda \neq (\nu + \mu)^0} V(\lambda) \boxtimes IC(\mathcal{O})_\lambda \bigoplus V((\nu + \mu)^0) \boxtimes \mathbb{I}_{E_{\nu,\mu}}$$

for $\mathbb{I}_{E_{\nu,\mu}}$ is equal to $IC(\mathcal{O}((\nu + \mu)^0)$ (see Example 3.1). Hence:

$$H^*\langle Gr_\mu(N) \rangle = H^*_{-dimE_{\nu,\mu}}(Q)(i^*_N q\mathbb{I}_{E_{\nu,\mu}})$$

$$= H^*_{-dimE_{\nu,\mu}}(Q)(\bigoplus_{\lambda \neq (\nu + \mu)^0} V(\lambda) \boxtimes IC(\mathcal{O})_\lambda \bigoplus V((\nu + \mu)^0) \boxtimes \mathbb{I}_{E_{\nu,\mu}})$$

$$= H^*_{-dimE_{\nu,\mu}}(Q)(\bigoplus_{\lambda \neq (\nu + \mu)^0} V(\lambda) \otimes IC(\mathcal{O})_\lambda) \bigoplus H^*\langle Gr_\mu(M) \rangle$$

for $\mathbb{I}_{E_{\nu,\mu}}$ is the constant perverse sheaf. Thus we have that

$$Ch_q(H^*\langle Gr_\mu(N) \rangle) = Ch_q(H^*\langle Gr_\mu(M) \rangle) + f(q)$$

where $f(q) \in \mathbb{N}[q, q^{-1}]$. 

\[ \square \]
Corollary 5.6. Let us fix an algebraic closed field \( \mathbb{F}_q \). Given a rigid representation \( M = M((\nu + \mu)^0) \), it follows \( |Gr_\mu(M)(\mathbb{F}_q)| = q^{t\nu,\mu - \langle \mu, \nu \rangle} Tr(M^{I_{\nu,\mu}}_{\nu,\mu}) q^2 \). Where \( Tr(M^{I_{\nu,\mu}}_{\nu,\mu}) q^2 \) means it is defined over the field \( \mathbb{F}_q \).

Proof. By [IEFR, Theorem 1 and Corollary 2], the equations (3.7) and (3.8), we have that
\[
|Gr_\mu(M)(\mathbb{F}_q)| = \sum_i \dim H^i(Gr_\mu(M)) q^i = Ch_q(H^*(Gr_\mu(M))) = q^{t\nu,\mu - \langle \mu, \nu \rangle} X_{\nu,\mu} = q^{t\nu,\mu - \langle \mu, \nu \rangle} Tr(M^{I_{\nu,\mu}}_{\nu,\mu}) q^2
\]
\[\square\]

5.2. The ordering arising from an order of set of vertices \( I \). In this section, we assume the ordering of positive roots \( R^+ \) arises from an order of Lyndon words via an order of the set vertices \( I \). Denote the simple roots (the vertices) by \( \{\delta_i\}_{i \in |I|} \).

If there exists an arrow going from \( i \) to \( j \), then by the equations (2.1) and (2.2), it follows
\[
\langle \delta_i, \delta_j \rangle = -\sum_{h \in \Omega, s(h)=i, \tau(h)=j} 1 = [M(\delta_i), M(\delta_j)] - [M(\delta_i), M(\delta_j)] = -[M(\delta_i), M(\delta_j)]
\]
By the equation (5.1), we have that \( \delta_i \prec \delta_j \). From now on, we order the simple roots \( \{\delta_i\}_{i \in |I|} \) so that \( \delta_i \prec \delta_j \) if and only if \( i < j \). Following from [S, Example 2.5]

Lemma 5.7 ([S], Example 2.5). For each dimension vector \( \nu = \sum_i \nu_i \delta_i \), we write \( E_\nu(Q) \) for its representation space, and then the perverse sheaf \( \mathbb{I}_\nu = \mathbb{C}_{E_\nu(Q)}[\dim E_\nu(Q)] \) is
\[
L_{\nu_1 \delta_1, \cdots, \nu_n \delta_n} = \mathbb{I}_\nu
\]
where \( L_{\nu_1 \delta_1, \cdots, \nu_n \delta_n} \) is introduced in [L1].

If we set \( \mathbf{i}_\nu = (\delta_1, \cdots, \delta_1, \delta_2, \cdots, \delta_n, \cdots, \delta_n) \) where the entry \( \delta_i \) has the multiplicity \( \nu_i \) in \( \mathbf{i}_\nu \). By [VV, Remark 1.5], we have an isomorphism of complexes
\[
L_{\mathbf{i}_\nu} = \bigoplus_{w \in S_\nu} \mathbb{I}_\nu[-2l(w)]
\]
This leads to
\[ P_{\nu} = \bigoplus_{w \in S_{\nu}} P_{\nu^0}[-2l(w)] \]  

The Poincaré polynomial of \( S_n \) is
\[ \sum_{w \in S_n} q^{2l(w)} = q^{\frac{1}{2}n(n-1)}[n]! \]  

We set
\[ S_{\nu} := \prod_{1 \leq i \leq n} S_{\nu_i}; \quad s_{\nu} := \prod_{1 \leq i \leq n} \left( \frac{1}{2}(\nu_i - 1)\nu_i \right); \quad [\nu]! := \prod_{1 \leq i \leq n} [\nu_i]! \]  

Thus we have the following lemma:
\[ \sum_{w \in S_{\nu}} q^{2l(w)} = q^{s_{\nu}}[\nu]! \]  

**Lemma 5.8.** From the equation (4.3), we have that
\[ \text{Dim} 1_{i_{\nu} i_{\mu}} L(\lambda) = q^{s_{\nu} + s_{\mu}}[\mu]![\nu]! \chi_{\nu,\mu}^{\lambda} \]  

**Proof.** By the equation (5.3) and the fact \( 1_{i_{\nu} i_{\mu}} L(\lambda) = (P_{\nu} \ast P_{\mu}, L(\lambda)) \), it follows.
\[
\begin{align*}
\text{Dim} 1_{i_{\nu} i_{\mu}} L(\lambda) &= \text{Dim}(P_{\nu} \ast P_{\mu}, L(\lambda)) \\
&= \text{Dim}(P_{\nu} \boxtimes P_{\mu}, \text{res} L(\lambda)) \\
&= \text{Dim}( \bigoplus_{w \in S_{\nu^0}} P_{\nu^0}[-2l(w)] \boxtimes \bigoplus_{v \in S_{\mu}} P_{\mu^0}[-2l(v)], \text{res} L(\lambda)) \\
&= q^{s_{\nu}}[\nu]!q^{s_{\mu}}[\mu]! \text{Dim}(P_{\nu^0} \boxtimes P_{\mu^0}, \text{res} L(\lambda)) \\
&= q^{s_{\nu} + s_{\mu}}[\mu]![\nu]! \chi_{\nu,\mu}^{\lambda}
\end{align*}
\]

**Theorem 5.9.** Let \( M = M((\nu + \mu)^0) \) be a rigid representation with a dimension vector \( \nu + \mu \). Denote \( \nu + \mu \) as \( \kappa = (\kappa_1, \ldots, \kappa_k) \). We have that:
\[
\text{Ch}_q(H^\bullet(Gr_{\mu}(M))) = q^{i_{\nu,\mu} - (\mu,\nu) - s_{\nu} - s_{\mu}} \text{Dim} 1_{i_{\nu} i_{\mu}} \overline{\chi}(\kappa)/[\mu]![\nu]!
\]

**Proof.** By the equation (5.6) and the Theorem 5.4, we have that
\[
\text{Dim} 1_{i_{\nu} i_{\mu}} \overline{\chi}(\kappa) = q^{s_{\nu} + s_{\mu}}[\mu]![\nu]! \chi_{\nu,\mu}^{\kappa}
\]

Given two dimension vectors \( \mu, \nu \in \mathbb{N}^I \), we have that

**Lemma 5.10.** the map \( q : E_{\nu,\mu} \to E_{\nu+\mu}(Q) \) is surjective, then \( i_{\nu} i_{\mu} \leq i((\mu + \nu)^0) \)

**Proof.** If the map \( q : E_{\nu,\mu} \to E_{\nu+\mu}(Q) \) is surjective, then by Theorem 5.4 we have
\[ 1_{i,\nu} \mu_0 \Delta((\nu + \mu)^0) \neq 0. \]

It is equivalent to \( 1_{i,\nu} \mu_0 \Delta((\nu + \mu)^0) \neq 0 \) by Lemma 5.8. By [KR, Corollary 5.4], it follows that the word \( i((\mu + \nu)^0) \) is the largest word in the \( Ch(\Delta((\nu + \mu)^0)) \). Hence \( i_\nu i_\mu \leq i((\mu + \nu)^0) \). \( \square \)

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