On Multi-step MLE-process for Markov Sequences

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Abstract

We consider the problem of the construction of the estimator-process of the unknown finite-dimensional parameter in the case of the observations of nonlinear autoregressive process. The estimation is done in two or three steps. First we estimate the unknown parameter by a learning relatively short part of observations and then we use the one-step MLE idea to construct an-estimator process which is asymptotically equivalent to the MLE. To have the learning interval shorter we introduce the two-step procedure which leads to the asymptotically efficient estimator-process too. The presented results are illustrated with the help of two numerical examples.

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1 Introduction

This work is devoted to the problem of finite-dimensional parameter estimation in the case of observations of Markov sequence in the asymptotics of
large samples. The observations are \( X^n = (X_0, X_1, X_2, \ldots, X_n) \). For simplicity of exposition we take as a model of observations a nonlinear time series satisfying the relation

\[
X_j = S(\vartheta, X_{j-1}) + \varepsilon_j, \quad j = 1, 2, \ldots
\] (1)

and the initial value \( X_0 \) is given too. The random variables \((\varepsilon_j)_{j \geq 1}\) are i.i.d. with some known smooth density function \( g(x) \). The function \( S(\vartheta, x) \) is supposed to be known and smooth with respect to \( \vartheta \). It can be verifies that under the supposed regularity conditions the family of measures corresponding to these model of observations is locally asymptotically normal (LAN). Our goal is to construct a sequence (we say process) of estimators \( \vartheta^{*}_n = (\vartheta_{k,n}, k = N + 1, \ldots, n) \), where \( N \ll n \). By the first \( N + 1 \) observations \( X^N = (X_0, X_1, \ldots, X_N) \) we estimate the parameter \( \vartheta \) and the obtained preliminary estimator \( \bar{\vartheta}_N \) we use in the construction of the estimator process \( \vartheta^{*}_n \). This construction is based on the modification of the well-known one-step maximum likelihood estimator (MLE) procedure introduced by Le Cam in 1956 [10] for LAN families of distributions. In the proofs we follow the similar work [8] devoted to parameter estimation in the case of ergodic diffusion process. Such estimator-processes appeared in the works devoted to the problem of approximation of the solution of backward stochastic differential equations (see review in [7]). As the initial estimator is constructed by a relatively small number of observations \( N \sim n^\delta \) with \( \delta < 1 \) the rate of convergence of the preliminary estimator is “bad” \( \sqrt{N} \sim n^{\delta/2} \)

\[
\sqrt{N} (\bar{\vartheta}_N - \vartheta) \Rightarrow \mathcal{N}(0, \mathbb{D}(\vartheta))
\]

and we have to improve this rate up to the optimal \( \sqrt{n} \) and to improve the limit variance up to the optimal.

Therefore this work is devoted to adaptive estimation for LAN family [3]. The structure of our estimator-processes is in some sense close to that of the Fisher-scoring algorithm, but the proposed realization is different because we have to improve the rate of convergence. The idea to use a preliminary estimator with a “bad” rate of convergence in the one-step MLE framework to obtain asymptotically efficient one was used by Skorohod and Khasminskii [15] and the idea to improve the rate of convergence of preliminary estimator using multi-step Newton-Raphson procedure was realized by Kamatani and Uchida [6]. In the work [15] it was considered the problem of parameter estimation for partially observed diffusion processes and in [6] it was considered the problem of parameter estimation by the discrete time observations of the diffusion process in the asymptotics of high frequency observations, i.e.,
they supposed that the step of discretization tends to zero. We consider the multi-step procedure of one-step MLE type for Markov sequences. Another particularity of the presented work is the following. We propose a sequence of estimators, which can be easily calculated and the same time it has the same asymptotic properties as the asymptotically efficient MLE. This means that these estimators are asymptotically normal and that its limit variance is the inverse Fisher information matrix.

The properties of the parameter estimators for nonlinear time series and Markov sequences, of course, are well-known. Let us mention here the works by Roussas [13], Ogata and Inagaki [12], Varakin and Veretennikov [17]). More about statistical problems for time series can be found in the monographs by Veretennikov [18], Taniguchi and Kakizawa [16], Fan and Yao [4], and the references therein.

Note that we take the time series (1) just for simplicity of expositions. The proposed results can be generalized on the more general Markov sequences defined by their transition density if we suppose that this density satisfies to the corresponding regularity conditions.

The process \((X_j)_{j \geq 0}\) has a transition density

\[
\pi(\vartheta, x, x') = g(x' - S(\vartheta, x)).
\]

It depends on the parameter \(\vartheta\) and defines the probability of reaching the state \(x'\) after sojourning in the state \(x\). The parameter \(\vartheta\) takes its values in some open, convex, bounded set \(\Theta \subset \mathbb{R}^d\).

The construction of the one-step MLE-process in this work is done in two steps. On the first step we estimate the unknown parameter by the observations \(X^N = (X_0, X_1, \ldots, X_N)\) on the learning interval \(j \in [0, N]\). As preliminary estimator we can take the MLE, Bayes estimator (BE), estimator of the method of moments (EMM) or any other estimator, which is consistent and asymptotically normal.

Let us recall some of them. The MLE is defined as follows. Introduce the likelihood function

\[
V(\vartheta, X^n) = \pi(\vartheta, X_0) \prod_{j=1}^n \pi(\vartheta, X_{j-1}, X_j), \quad \vartheta \in \Theta.
\]

We suppose that the observations are strictly stationary and therefore the density of the initial value is the density of the invariant measure \(\pi(\vartheta, x)\).
The maximum likelihood estimator we introduce as usual by the equation
\[ V(\hat{\vartheta}_n, X^n) = \sup_{\vartheta \in \Theta} V(\vartheta, X^n), \quad (3) \]
If this equation has many solutions then we can take any of them as the MLE.

It is known that under the regularity conditions the MLE is consistent and asymptotically normal:
\[ \sqrt{n}(\hat{\vartheta}_n - \vartheta) \Rightarrow N(0, I(\vartheta))^{-1}. \quad (4) \]
Here \( I(\vartheta) \) is the Fisher information matrix
\[ I(\vartheta) = E_{\vartheta} \left[ \dot{\ell}(\vartheta, X_0, X_1) \dot{\ell}(\vartheta, X_0, X_1)^T \right], \]
where \( \ell(\vartheta, x, x') = \ln \pi(\vartheta, x, x'). \) The dot means the derivation w.r.t. \( \vartheta \) and \( T \) means the transpose of a matrix.

As \( \pi(\vartheta, x, x') = g(x' - S(\vartheta, x)) \) we can write
\[ \begin{align*}
I(\vartheta) &= E_{\vartheta} \left[ \dot{\ell}(X_j - S(\vartheta, X_{j-1})) \dot{\ell}(X_j - S(\vartheta, X_{j-1}))^T \right] \\
&= E_{\vartheta} \left[ \frac{g'(X_j - S(\vartheta, X_{j-1}))^2 S(\vartheta, X_{j-1})}{g(X_j - S(\vartheta, X_{j-1}))^2} \right] \\
&= E_{\vartheta} \left[ \frac{g'(\epsilon_j)}{g(\epsilon_j)} \right]^2 E_{\vartheta} \left[ \dot{S}(\vartheta, \xi) \dot{S}(\vartheta, \xi)^T \right] \\
&= I_g E_{\vartheta} \left[ \dot{S}(\vartheta, \xi) \dot{S}(\vartheta, \xi)^T \right], \quad (5)
\end{align*} \]
where we used the equality \( X_j - S(\vartheta, X_{j-1}) = \epsilon_j \) and denoted
\[ I_g = \int \frac{g'(x)^2}{g(x)} \, dx. \]
Moreover the MLE is asymptotically efficient. There are several definitions of the asymptotically efficient estimators. One of them is the following: an estimator \( \hat{\vartheta}_n^* \) is called asymptotically efficient if it satisfies the relation: for all \( \vartheta_0 \in \Theta \)
\[ \lim_{\delta \to 0} \lim_{n \to \infty} \sup_{|\vartheta - \vartheta_0| < \delta} E_{\vartheta} W(\sqrt{n}(\hat{\vartheta}_n^* - \vartheta)) = EW(\zeta I(\vartheta_0)^{-1/2}). \quad (6) \]
Here $W(u), u \in \mathbb{R}^d$ is a loss function satisfying the usual conditions. Note that it can be bounded, polynomial $W(u) = |u|^p, u \in \mathbb{R}^d$ with $p > 0$ or other (see, e.g., [3]) and $\zeta$ is a Gaussian vector $\zeta \sim \mathcal{N}(0, \mathbb{I})$, $\mathbb{I}$ is a unit $d \times d$ matrix. Remind that for all estimators $\bar{\vartheta}_n$ the following Hajek-Le Cam’s type lower bound

$$\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{|\bar{\vartheta}_n - \vartheta| < \delta} \mathbb{E}_\vartheta W (\sqrt{n (\bar{\vartheta}_n - \vartheta)}) \geq \mathbb{E} W \left( \zeta \mathbb{I} (\vartheta_0)^{-1/2} \right)$$

(7)

holds (see, e.g. [3]). That is why (6) indeed defines the asymptotically efficient estimator.

Note that these properties of the MLE were established in several works. We mention here [12] and [17] (in the one-dimensional case $d = 1$).

As preliminary estimator we can use as well the BE. Recall its definition and properties. Suppose that the unknown parameter $\vartheta \in \Theta$ is a random vector with the prior density $p(\vartheta), \vartheta \in \Theta$. The function $p(\cdot)$ is continuous, bounded and positive. The BE for the quadratic loss function has the following representation:

$$\bar{\vartheta}_n = \frac{\int_\Theta \vartheta p(\vartheta) V(\vartheta, X^n) \, d\vartheta}{\int_\Theta p(\vartheta) V(\vartheta, X^n) \, d\vartheta}$$

This estimator under regularity conditions is consistent, asymptotically normal

$$\sqrt{n}(\hat{\vartheta}_n - \vartheta_0) \Longrightarrow N(0, \mathbb{I}(\vartheta_0)^{-1})$$

(8)

and asymptotically efficient for the polynomial loss functions. For the proof see [11].

Recall also the properties of the estimator of the method of moments. Suppose that the vector-function $q(x) \in \mathbb{R}^d$ is such that the system of equations

$$m(\vartheta) = t, \quad \vartheta \in \Theta$$

where

$$m(\vartheta) = \mathbb{E}_\vartheta q(\xi)$$

has a unique solution $\vartheta = \vartheta(t)$. Introduce the function $h(t)$ inverse to the function $m(\vartheta)$, i.e., $\vartheta = m^{-1}(t) = h(t)$. Then the EMM is defined as follows

$$\bar{\vartheta}_n = h \left( \frac{1}{n} \sum_{j=1}^n q(X_j) \right).$$
It is known that under regularity conditions this estimator is consistent and asymptotically normal
\[
\sqrt{n} \left( \bar{\theta}_n - \theta \right) \implies \mathcal{N} \left( 0, \mathbb{C}(\theta) \right),
\]
where \( \mathbb{C}(\theta) \) is the matrix defined, for example, in [11]. Moreover the moments of the EMM converge too (see [11] for the conditions and proof). We use such estimator as preliminary one in the numerical simulation Example 2 below.

In this work the construction of the multi-step MLE is based on the score-function. Let us recall the definition and some properties of it. Introduce the log-likelihood ratio function
\[
L(\theta, X^n) = \ln \pi(\theta, X_0) + \sum_{j=1}^{n} \ln \pi(\theta, X_{j-1}, X_j).
\] (9)
The normalized score-function is (for simplicity of exposition we omit the term with initial value)
\[
\Delta_n(\varphi_0, X^n) = \frac{1}{\sqrt{n}} \frac{\partial L(\varphi, X^n)}{\partial \varphi} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{g'(X_j - S(\varphi, X_{j-1}))}{g(X_j - S(\varphi, X_{j-1}))} \dot{S}(\varphi, X_{j-1}).
\]
If we denote the true value \( \varphi = \varphi_0 \), then we have
\[
\Delta_n(\varphi_0, X^n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \frac{g'(\varepsilon_j)}{g(\varepsilon_j)} \dot{S}(\varphi_0, X_{j-1}).
\]
Note that (\( i < j \))
\[
E_{\varphi} \left( \frac{g'(\varepsilon_i)}{g(\varepsilon_i)} \frac{g'(\varepsilon_j)}{g(\varepsilon_j)} \dot{S}(\varphi, X_{i-1}) \dot{S}(\varphi, X_{j-1})^T \right) = E_{\varphi} \left( \frac{g'(\varepsilon_i)}{g(\varepsilon_i)} \dot{S}(\varphi, X_{i-1}) E_{\varphi} \left( \frac{g'(\varepsilon_j)}{g(\varepsilon_j)} \dot{S}(\varphi, X_{j-1})^T | \mathcal{F}_{j-1} \right) \right) = 0
\]
because
\[
E_{\varphi} \left( \frac{g'(\varepsilon_j)}{g(\varepsilon_j)} \dot{S}(\varphi, X_{j-1})^T | \mathcal{F}_{j-1} \right) = E \left( \frac{g'(\varepsilon_j)}{g(\varepsilon_j)} \right) E_{\varphi} \left( \dot{S}(\varphi, X_{j-1})^T | \mathcal{F}_{j-1} \right)
\]
and
\[
E \left( \frac{g'(\varepsilon_j)}{g(\varepsilon_j)} \right) = \int_{-\infty}^{\infty} g'(x) \, dx = 0.
\]
Therefore by the central limit theorem
\[
\Delta_n(\varphi_0, X^n) \implies \mathcal{N} \left( 0, I(\varphi_0) \right),
\]
where \( I(\varphi_0) \) is the Fisher information matrix defined in [5].
2 Main result

Suppose that we have a Markov sequence $X^n = (X_j)_{j=0,...,n}$ with the transition density $\pi(\cdot)$ depending on some unknown finite-dimensional parameter $\vartheta \in \Theta$. The set $\Theta \subset \mathbb{R}^d$ is open, bounded. Our goal is to construct on-line recurrent estimator of this parameter. Therefore we need for each $j$ to have an estimator $\vartheta_{j,n}^*$ with good properties, i.e., this estimator can be easily calculated and the same time it has to be asymptotically optimal in some sense. We call such sequence of estimators $\vartheta_{j,n}^*, j = 1, \ldots, n$ estimator-process. We propose a construction of such estimator in two steps. We slightly change the statement of the problem. Introduce the learning part $X^N = (X_0, X_1, \ldots, X_N)$ of observations $X^n = (X_0, X_1, \ldots, X_n)$, where $N = \lfloor n \delta \rfloor$ ($N$ is the integer part of $n\delta$) and the parameter $\delta < 1$ will be chosen later.

We say that a family of random variables $\{\eta_n(\vartheta), n = 1, 2, \ldots\}$ is tight uniformly on compacts $K \subset \Theta$ if for any $\varepsilon > 0$ and any compact $K$ there exists a constant $C > 0$ such that

$$\sup_{\vartheta \in K} \mathbb{P}(|\eta_n(\vartheta)| > C) \leq \varepsilon.$$ 

Throughout the paper we suppose that the following conditions are fulfilled.

Conditions $\mathcal{R}$.

1. The time series $(X_j)_{j \geq 0}$ is strictly stationary and has a unique invariant distribution with the density function $\pi(\vartheta, x)$.

2. The preliminary estimator $\tilde{\vartheta}_n$ is such that $\sqrt{n}(\tilde{\vartheta}_n - \vartheta)$ is tight uniformly on compacts $K \subset \Theta$.

3. The function $S(\vartheta, x) \in C^3$, the density $g(\cdot) > 0$ and $g(\cdot) \in C^3$. The derivatives $\partial^i \ell(\vartheta, x, x')/\partial \vartheta^i, i = 1, 2, 3$ of the function $\ell(\vartheta, x, x') = \ln \pi(\vartheta, x, x')$ are uniformly on $\vartheta$ majorated by quadratically integrable functions, i.e.,

$$\sup_{\vartheta \in \Theta} \left\| \frac{\partial^i \ell(\vartheta, x, x')}{\partial \vartheta^i} \right\| \leq R_i(x, x'), \quad i = 1, 2, 3,$$

where $\mathbb{E}_\vartheta |R_i(X_{j-1}, X_j)|^2 < C$ and the constant $C > 0$ does not depend on $\vartheta$.

4. We have
• the law of large numbers
\[ \frac{1}{n} \sum_{j=1}^{n} \ell(\vartheta, X_{j-1}, X_j) \vartheta \leftarrow \mathbb{I}(\vartheta), \quad (10) \]

• the central limit theorem
\[ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \ell(\vartheta, X_{j-1}, X_j) \rightarrow \mathcal{N}(0, \mathbb{I}(\vartheta)), \quad (11) \]

• the family of random variables
\[ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \left[ \ell(\vartheta, X_{j-1}, X_j) + \mathbb{I}(\vartheta) \right] \quad (12) \]
is tight uniformly on compacts \( \mathbb{K} \subset \Theta \).

5. The information matrix \( \mathbb{I}(\vartheta) \) is Lipschitz
\[ |\mathbb{I}(\vartheta_1) - \mathbb{I}(\vartheta_2)| \leq L |\vartheta_1 - \vartheta_2| \quad (13) \]
and is uniformly in \( \vartheta \in \Theta \) non-degenerate and bounded
\[ 0 < \inf_{\vartheta \in \Theta} \inf_{|\lambda|=1} \lambda^T \mathbb{I}(\vartheta) \lambda, \quad \sup_{\vartheta \in \Theta} \sup_{|\lambda|=1} \lambda^T \mathbb{I}(\vartheta) \lambda < \infty. \quad (14) \]

Here \( \lambda \in \mathbb{R}^d \).

Note that as preliminary estimator \( \hat{\vartheta}_N \) we can take the MLE, the BE or the EMM. All of them have the required properties (under additional regularity conditions, which we do not mention here). The details can be found in \([12], [17], [11]\) or any other work describing their properties. The conditions for \((10)-(12)\) can be found, for example, in \([4], [16], [18]\). The condition \((13)\) can be verified if we have the corresponding smoothness of the density of invariant distribution \( \pi(\vartheta, x) \) (see, e.g. \([2]\)).

We construct the one-step MLE-process \( \vartheta_{k,n}^*, k = N + 1, \ldots, n \) as follows. Introduce the variable \( s \in [\tau_{\delta}, 1] \), where \( \tau_{\delta} = n^{-1+\delta} \rightarrow 0 \) and put \( k = [sn] \), where \( [a] \) means the integer part of \( a \). Let us write \( \vartheta_{k,n}^* = \vartheta_{s,n}^* \) and consider the estimator-process \( \vartheta_n^* = \{\vartheta_{s,n}^*, s \in [\tau_{\delta}, 1]\} \). Our goal is to construct an estimator process \( \vartheta_n^* \) asymptotically optimal for all \( s \in [\tau_{\delta}, 1] \). Recall that the MLE \( \vartheta_{s,n} \) constructed by the first \( k = [sn] \) observations is asymptotically efficient and for example,
\[ \sqrt{sn} \left( \vartheta_{s,n} - \vartheta \right) \rightarrow \mathcal{N}(0, \mathbb{I}(\vartheta)^{-1}), \quad s \in [\delta, 1]. \]
Note that to solve the equation
\[
\sup_{\vartheta \in \Theta} V(\vartheta, X^{[sn]}) = V(\hat{\vartheta}_{s,n}, X^{[sn]})
\]
for all \(s \in [\tau_\delta, 1]\) is computationally rather difficult problem, except some particular examples. Therefore it is better to seek another estimators, which have the same limit covariance matrix as the MLE (which is asymptotically efficient) for all \(s \in (\tau_\delta, 1]\) and which can be calculated in more simple way.

We consider two different situations depending on the length of the learning interval \([0, N]\). If \(N = [n^\delta]\) (here \([a]\) is integer part of \(a\)) with \(\frac{1}{2} < \delta < 1\) then we construct the one-step MLE-process and if we take the preliminary interval shorter, i.e., \(N = [n^\delta]\) with \(\frac{1}{4} < \delta \leq \frac{1}{2}\), then we introduce an intermediate estimator and only after that we can construct the two-step MLE-process. Therefore we consider below these two situations separately.

2.1 Case \(N = [n^\delta], \frac{1}{2} < \delta < 1\)

We proceed as follows. Let us fix \(s \in [\tau_\delta, 1]\) and slightly modify the vector score-function
\[
\Delta_k(\vartheta, X_N^k) = \frac{1}{\sqrt{k}} \sum_{j=N+1}^{k} \ell(\vartheta, X_{j-1}, X_j),
\]
where \(k = [sn] \to \infty\). Introduce the one-step MLE-process
\[
\vartheta_{s,n}^* = \hat{\vartheta}_N + \frac{1}{\sqrt{k}} \Pi(\hat{\vartheta}_N)^{-1} \Delta_k(\hat{\vartheta}_N, X_N^k), \quad \tau_\delta \leq s \leq 1
\]

Here and below for simplicity of notation this writing means that \(N\) is the integer part of \(n^\delta\).

**Theorem 1** Suppose that the conditions \(\mathcal{R}\) are fulfilled, then for all \(s \in (0, 1]\)
\[
\sqrt{k}(\vartheta_{s,n}^* - \vartheta) \Rightarrow \mathcal{N}(0, I(\vartheta)^{-1})
\]  
(15)
and this estimator-process is asymptotically efficient for the bounded loss functions in (6).
Proof. Note that for any \( s > 0 \) (\( s \leq 1 \)) we have \( s > \tau_\delta \) for \( n > s^{1-s} \). We can write
\[
\sqrt{k}(\vartheta^*_n - \vartheta) = \sqrt{k}(\bar{\vartheta}_N - \vartheta) + \mathbb{I}(\bar{\vartheta}_N)^{-1} \Delta_k(\bar{\vartheta}_N, X^k_N) \\
= \sqrt{k}(\bar{\vartheta}_N - \vartheta) + \mathbb{I}(\bar{\vartheta}_N)^{-1} \Delta_k(\vartheta, X^k_N) \\
+ \mathbb{I}(\bar{\vartheta}_N)^{-1} \left[ \Delta_k(\bar{\vartheta}, X^k_N) - \Delta_k(\vartheta, X^k_N) \right].
\]

We have
\[
\Delta_k(\bar{\vartheta}, X^k_N) - \Delta_k(\vartheta, X^k_N) = \int_0^1 \langle (\bar{\vartheta}_N - \vartheta), \dot{\Delta}_k(\vartheta + v (\bar{\vartheta}_N - \vartheta), X^k_N) \rangle \, dv.
\]
Hence (below \( \vartheta_v = \vartheta + v (\bar{\vartheta}_N - \vartheta) \))
\[
\sqrt{k}(\bar{\vartheta}_N - \vartheta) + \mathbb{I}(\bar{\vartheta}_N)^{-1} \left[ \Delta_k(\bar{\vartheta}, X^k_N) - \Delta_k(\vartheta, X^k_N) \right] \\
= \sqrt{k}(\bar{\vartheta}_N - \vartheta) \mathbb{I}(\bar{\vartheta}_N)^{-1} \left[ \mathbb{I}(\bar{\vartheta}_N) + \frac{1}{\sqrt{k}} \int_0^1 \dot{\Delta}_k(\vartheta_v, X^k_N) \, dv \right].
\]
Further
\[
\mathbb{I}(\bar{\vartheta}_N) + \frac{1}{\sqrt{k}} \int_0^1 \dot{\Delta}_k(\vartheta_v, X^k_N) \, dv = \mathbb{I}(\vartheta) + \frac{1}{\sqrt{k}} \dot{\Delta}_k(\vartheta, X^k_0) - \frac{1}{\sqrt{k}} \dot{\Delta}_k(\vartheta, X^k_{N-1}) \\
+ \frac{1}{\sqrt{k}} \int_0^1 \left[ \dot{\Delta}_k(\vartheta_v, X^k_N) - \dot{\Delta}_k(\vartheta, X^k_N) \right] \, dv \\
= \frac{1}{k} \sum_{j=1}^k \left[ \ell(\vartheta, X_{j-1}, X_j) + \mathbb{I}(\vartheta) \right] + O \left( \frac{N}{k} \right) + O \left( n^{-\frac{3}{2}} \right),
\]
because
\[
\frac{1}{\sqrt{k}} \dot{\Delta}_k(\vartheta, X^k_{N-1}) = \frac{1}{k} \sum_{j=1}^{N-1} \ell(\vartheta, X_{j-1}, X_j) = O \left( \frac{N}{k} \right) = O \left( n^{-1+\delta} \right),
\]
\[
|\mathbb{I}(\bar{\vartheta}_N) - \mathbb{I}(\vartheta)| \leq L |\bar{\vartheta}_N - \vartheta| = O \left( n^{-\frac{3}{2}} \right)
\]
and
\[
\frac{1}{\sqrt{k}} \int_0^1 \left[ \dot{\Delta}_k(\vartheta_v, X^k_N) - \dot{\Delta}_k(\vartheta, X^k_N) \right] \, dv = O \left( \bar{\vartheta}_N - \vartheta \right) = O \left( n^{-\frac{3}{2}} \right).
\]
Here and in the sequel \( O \left( n^{-\varepsilon} \right) \) means that \( n^\varepsilon O \left( n^{-\varepsilon} \right) \) is bounded in probability uniformly on compacts \( \mathbb{K} \), i.e., for any \( \varepsilon > 0 \) there exists \( C_1 > 0 \) such that
\[
\sup_{\vartheta \in \mathbb{K}} \mathbb{P}_\vartheta \left( n^\varepsilon \left| O \left( n^{-\varepsilon} \right) \right| > C_1 \right) \leq \varepsilon.
\]
For example,
\[
\sup_{\vartheta \in \mathbb{K}} \mathbb{P}_\vartheta \left( \frac{1}{k} \sum_{j=1}^{N-1} |\ell(\vartheta, X_{j-1}, X_j)| > C_1 \right) \leq \frac{1}{kC_1} \sum_{j=1}^{N-1} \sup_{\vartheta \in \mathbb{K}} |\ell(\vartheta, X_{j-1}, X_j)| \\
\leq \frac{1}{kC_1} \sum_{j=1}^{N-1} \sup_{\vartheta \in \mathbb{K}} |R_2(X_{j-1}, X_j)| \leq CN/C_k.
\]

Recall that \( \mathbb{E}_\vartheta \ell(\vartheta, X_{j-1}, X_j) = -I(\vartheta) \). Hence by the central limit theorem (12) we have
\[
\frac{1}{\sqrt{k}} \sum_{j=1}^{k} \left[ \ell(\vartheta, X_{j-1}, X_j) + I(\vartheta) \right] \implies \mathcal{N}(0, \mathbb{D}(\vartheta))
\]
with some \( \mathbb{D}(\vartheta) \).

Therefore
\[
\sqrt{k}(\vartheta_{s,n}^* - \vartheta) = \mathbb{I}(\hat{\vartheta}_N)^{-1} \Delta_k(\vartheta, X_N^k) + n^{\frac{\delta}{2}}(\mathbb{I}_N - \vartheta) \left[ n^{-\frac{1}{2}} O \left( n^{-\frac{1}{2}} \right) + n^{\frac{1}{2}} O \left( n^{-1+\delta} \right) + n^{\frac{1}{2}} O \left( n^{-\frac{3}{2}} \right) \right] \\
= \mathbb{I}(\vartheta)^{-1} \Delta_k(\vartheta, X_N^k) + o(1) \implies \mathcal{N}(0, \mathbb{I}(\vartheta)^{-1})
\]
where we used once more the central limit theorem (11).

Therefore the one-step MLE-process \( \vartheta_{s,n}^* = (\vartheta_{s,n}^*, \tau_\delta < s \leq 1) \) for all \( s \in (\tau_\delta, 1] \) is uniformly in \( \vartheta \in \mathbb{K} \) asymptotically normal (15). Hence for the bounded loss functions \( W(\cdot) \) we obtain the convergence
\[
\lim_{n \to \infty} \sup_{|\vartheta - \vartheta_0| < \delta} \mathbb{E}_\vartheta W \left( \sqrt{n}(\vartheta_{n}^* - \vartheta) \right) = \sup_{|\vartheta - \vartheta_0| < \delta} \mathbb{E} W \left( \zeta \mathbb{I}(\vartheta)^{-1/2} \right).
\]
Now (6) follows from the continuity of the Fisher information.

### 2.2 Case \( N = n^{\delta} \), \( \frac{1}{4} < \delta \leq \frac{1}{2} \)

The choice of the learning period of observations \( N = \lceil n^{\delta} \rceil \) with \( \delta \in (1/2, 1) \) allows us to construct an estimator process for the values \( s \in (\tau_\delta, 1] \) only. It can be interesting to see if it is possible to take more short learning interval and therefore to have the estimator-process for the larger time interval. Our goal is to show that the learning period can be \( N = \lceil n^{\delta} \rceil \) with \( \delta \in (1/4, 1/2] \).
Below we follow the construction which was already realized in [8] in the case of ergodic diffusion process.

Suppose that \( N = \lfloor n^\delta \rfloor \) with \( \delta \in (1/4, 1/2] \). The asymptotically efficient estimator we construct in three steps. By the first \( N \) observations as before we obtain the preliminary estimator \( \bar{\vartheta}_N \) which is asymptotically normal with the rate \( \sqrt{N} \), i.e.,

\[
 n^{2\delta} (\bar{\vartheta}_N - \vartheta) \implies \mathcal{N}(0, \mathbb{B}(\vartheta)).
\]

This can be the same estimator as in the preceding case. It can be, for example, the EMM, BE or MLE.

The two-step MLE-process \( \vartheta^{**}_n = (\vartheta^{**}_{s,n}, k = N + 1, \ldots, n) \) we construct as follows. Fix some \( s \in (\tau_3, 1] \), \( \tau_3 = n^{-1+\delta} \) and introduce the second preliminary estimator-process (as before \( k = \lfloor sn \rfloor \))

\[
 \bar{\vartheta}_{k,2} = \bar{\vartheta}_N + \frac{1}{\sqrt{k}} \mathbb{I} \left( \bar{\vartheta}_N \right)^{-1} \Delta_k(\bar{\vartheta}_N, X^k), \quad k = N + 1, \ldots, n, \quad (16)
\]

where

\[
 \Delta_k(\vartheta, X^k) = \frac{1}{\sqrt{k}} \sum_{j=1}^k \ell(\vartheta, X_{j-1}, X_j).
\]

Then we show that the random sequence \( n^{1/4+\varepsilon} (\bar{\vartheta}_{k,2} - \vartheta) \) with some \( \varepsilon > 0 \) is bounded in probability (tight).

Finally, using this estimator-process and the one-step procedure of Theorem 1 we obtain the asymptotically efficient estimator

\[
 \vartheta^{**}_{s,n} = \bar{\vartheta}_{k,2} + \frac{1}{\sqrt{k}} \mathbb{I} \left( \bar{\vartheta}_{k,2} \right)^{-1} \Delta_k(\bar{\vartheta}_{k,2}, X^k). \quad (17)
\]

In the next theorem we realize this program.

**Theorem 2** Suppose that the conditions of regularity are fulfilled, then the estimator \( \vartheta^{**}_{s,n} \) defined by (16) and (17) for all \( s \in (0, 1] \) is asymptotically normal

\[
 \sqrt{k}(\vartheta^{**}_{s,n} - \vartheta) \implies \mathcal{N}(0, \mathbb{I}(\vartheta)^{-1})
\]

and asymptotically efficient for the bounded loss functions.

**Proof.** The only thing to proof is the tightness of the sequence of random vectors \( n^{1/4+\varepsilon} (\bar{\vartheta}_{k,2} - \vartheta) \) because if it is tight, then the proof of Theorem 2 follows from the Theorem 1. Let us fix some \( \varepsilon \in (0, \frac{1}{4}) \).
For the estimator-process $\bar{\vartheta}_{k,2}$ defined by \[10\] we can write
\[
n^\frac{1}{4} + \varepsilon \left( \bar{\vartheta}_{k,2} - \vartheta \right) = n^\frac{1}{4} + \varepsilon \left( \bar{\vartheta}_N - \vartheta \right) + \frac{n^\frac{1}{4} + \varepsilon}{\sqrt{k}} \mathbb{I} \left( \bar{\vartheta}_N \right)^{-1} \Delta_k(\bar{\vartheta}_N, X^k)
\]
\[
= n^\frac{1}{4} + \varepsilon \left( \bar{\vartheta}_N - \vartheta \right) + \frac{n^\frac{1}{4} + \varepsilon}{\sqrt{k}} \mathbb{I} \left( \bar{\vartheta}_N \right)^{-1} \Delta_k(\bar{\vartheta}, X^k)
\]
\[
+ \frac{n^\frac{1}{4} + \varepsilon}{\sqrt{k}} \mathbb{I} \left( \bar{\vartheta}_N \right)^{-1} \left( \bar{\vartheta}_N - \vartheta \right) \dot{\Delta}_k(\bar{\vartheta}_N, X^k).
\]

Note that $\Delta_k(\vartheta, X^k)$ is asymptotically normal and therefore
\[
n^\frac{1}{4} + \varepsilon \frac{1}{\sqrt{k}} \mathbb{I} \left( \bar{\vartheta}_N \right)^{-1} \Delta_k(\bar{\vartheta}, X^k) \rightarrow 0,
\]
because $n^\frac{1}{4} + \varepsilon k^{-\frac{1}{2}} \rightarrow 0$. Further
\[
n^\frac{1}{4} + \varepsilon \left( \bar{\vartheta}_N - \vartheta \right) + \frac{n^\frac{1}{4} + \varepsilon}{\sqrt{k}} \mathbb{I} \left( \bar{\vartheta}_N \right)^{-1} \left( \bar{\vartheta}_N - \vartheta \right) \dot{\Delta}_k(\bar{\vartheta}_N, X^k)
\]
\[
= n^\frac{1}{4} + \varepsilon \left( \bar{\vartheta}_N - \vartheta \right) R_n,
\]
where
\[
R_n = n^\frac{1}{4} + \varepsilon - \frac{1}{2} \left[ \mathbb{I} + \mathbb{I} \left( \bar{\vartheta}_N \right)^{-1} \frac{1}{k} \sum_{j=1}^{k} \ell \left( \bar{\vartheta}_N, X_{j-1}, X_j \right) \right].
\]

We have by the law of large numbers
\[
\frac{1}{k} \sum_{j=1}^{k} \ell(\vartheta, X_{j-1}, X_j) \rightarrow -\mathbb{I}(\vartheta).
\]

From the regularity conditions it follows that
\[
\left| \mathbb{I} \left( \bar{\vartheta}_N \right)^{-1} - \mathbb{I}(\vartheta)^{-1} \right| \leq C |\bar{\vartheta}_N - \vartheta|,
\]
\[
k^{-1/2} \left| \dot{\Delta}_k(\bar{\vartheta}, X^k) - \dot{\Delta}_k(\vartheta, X^k) \right| \leq \frac{1}{k} \sum_{j=1}^{k} |R_3(\bar{\vartheta}, X_{j-1}, X_j)| |\bar{\vartheta}_N - \vartheta|.
\]

Therefore we verified the tightness of the sequence $n^\frac{1}{4} + \varepsilon \left( \bar{\vartheta}_{k,2} - \vartheta \right)$. Now the proof of the Theorem 2 follows from the proof of the Theorem 1.
3 Examples

We consider below two examples. The first one is new and the second example was already discussed in the previous work in the context of the study of the Bayesian estimators and the estimators of the method of moments [11]. In the first example we construct the preliminary MLE and the one-step MLE-process. In the second example we construct the preliminary EMM, the second preliminary estimator-process and then the two-step MLE-process.

3.1 Example 1.

Let us consider the problem of the construction of the one-step MLE-process in the case of observations $X^n = (X_0, X_1, \ldots, X_n)$ of the time series

$$X_j = \frac{(X_{j-1})^2}{1 + \vartheta |X_{j-1}|} + \varepsilon_j, \quad \vartheta \in (2, 5),$$

where $(\varepsilon_j)_{j \geq 1} \sim N(0, 1)$.

Note that this time series has invariant distribution. The density of it we estimate with the help of gaussian kernel-type estimator $K(\cdot)$:

$$\hat{\pi}_n(x) = \frac{1}{nh_n} \sum_{j=1}^{n} K\left(\frac{X_j - x}{h_n}\right), \quad K(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

where the width $h_n = n^{-1/5}$.

On the Figure we present the estimator of the invariant density in the case $n = 10^5$ and $\vartheta = 2, 5$.

First we define the MLE constructed on the learning sequence $X^N = (X_0, X_1, \ldots, X_N)$. For the conditional density function $\pi(\vartheta, X_{j-1}, X_j)$ of the Markov sequence [18], we have the representation

$$\pi(\vartheta, X_{j-1}, X_j) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left[\frac{(X_j - \frac{(X_{j-1})^2}{1 + \vartheta |X_{j-1}|})^2}{\sqrt{2\pi}}\right]},$$

Hence the log-likelihood ratio function is

$$L_N(\vartheta, X^N) = \ln \pi_0 (X_0) + \sum_{j=1}^{N} \left( -\frac{1}{2} \ln 2\pi - \frac{1}{2} \left[ X_j - \frac{(X_{j-1})^2}{1 + \vartheta |X_{j-1}|} \right]^2 \right).$$

$$= \ln \pi_0 (X_0) + \sum_{j=1}^{N} \ell(\vartheta, X_{j-1}, X_j), \quad \vartheta \in (2, 5).$$
Figure 1: Estimator of invariant density $\pi(\vartheta, x)$ for $\vartheta = 2.5$ and $n = 10^5$

To find the MLE $\hat{\vartheta}_N$ we have to solve the maximum likelihood equation

$$\frac{\partial L}{\partial \vartheta} = \sum_{j=1}^{N} \ell(\vartheta, X_{j-1}, X_j) = 0, \quad \vartheta \in (2, 5),$$

which has the following form

$$\sum_{j=1}^{N} \frac{|X_{j-1}|^3}{(1 + \vartheta |X_{j-1}|)^2} \left( -X_j + \frac{(X_{j-1})^2}{1 + \vartheta |X_{j-1}|} \right) = 0, \quad \vartheta \in (2, 5).$$

Now we construct the one-step MLE-process $\vartheta_{n}^* = (\vartheta_{k,n}, N + 1 \leq k \leq n)$ based on this preliminary estimator $\hat{\vartheta}_N$ as follows. The normalized score-function is

$$\Delta_k(\vartheta, X^k) = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} \frac{|X_{j-1}|^3}{(1 + \vartheta |X_{j-1}|)^2} \left( -X_j + \frac{(X_{j-1})^2}{1 + \vartheta |X_{j-1}|} \right),$$

where $N + 1 \leq k \leq n$. Finally the one-step MLE-process has the following representation

$$\vartheta_{k,n}^* = \hat{\vartheta}_N + \frac{1}{\tilde{I}_k(\hat{\vartheta}_N)k} \sum_{j=1}^{k} \frac{|X_{j-1}|^3}{(1 + \vartheta_N |X_{j-1}|)^2} \left( -X_j + \frac{(X_{j-1})^2}{1 + \vartheta_N |X_{j-1}|} \right),$$
where $N+1 \leq k \leq n$ and $I_k(\hat{\vartheta}_N) = 0.001$ is the Fisher information calculated as follows

$$I_k(\hat{\vartheta}_N) = -\frac{1}{k} \sum_{j=1}^{k} \ell(\hat{\vartheta}_N, X_{j-1}, X_j).$$

More detailed analysis shows that with such definition of the empirical Fisher information the main result of this work Theorem 1 is valid. Therefore the estimator-process $\vartheta^*_n$ is asymptotically normal with the same limit variance as that of the MLE.

The realization of the simulated one-step MLE-process for $n = 10^5$ is shown on the Figure 2. We can see that the initial estimator $\hat{\vartheta}_N$ is far from the true value and that the trajectory of one-step MLE-process approaches to the true value.

### 3.2 Example 2.

Let us consider another example, where preliminary estimator is EMM. Our goal is to illustrate the convergence of the one- and two-step MLE-processes, when the initial estimator is EMM ("bad" rate and "bad" limit variance).

Introduce the time series

$$X_j = X_{j-1} + 3 \frac{\vartheta - X_{j-1}}{1 + (X_{j-1} - \vartheta)^2} + \varepsilon_j, \quad j = 1, \ldots, n, \quad (20)$$

where $(\varepsilon_j)_{j \geq 1}$ are i.i.d. standard Gaussian random variables and $X_0$ is given. The unknown parameter $\vartheta \in \Theta = (-1, 1)$. This example was already discussed in the work [11] to illustrate the properties of the BE and EMM.

This process has ergodic properties and its invariant density can be estimated as in the Example 1 with the help of the kernel-type estimator. The result of such estimation can be found in [11].

We construct two estimator-processes: one-step and two-step. Our goal is to construct the estimator-processes $\vartheta^*_n$ and $\vartheta^{**}_n$, which are asymptotically equivalent to the MLE and therefore are asymptotically efficient. The same time their calculation is much more simple than that of the MLE.

We start with the one-step MLE-process. As described before we construct this estimator in two steps. First we need to calculate a consistent preliminary estimator $\tilde{\vartheta}_N$ by the initial observations $X_1, \ldots, X_N$, where $N = n^\delta$ with
Figure 2: One-step MLE-process for $n = 10^5$ observations and $\vartheta = 2.5$

$\delta \in (\frac{1}{2}, 1)$. Note that the unknown parameter for this model of observations is the shift parameter and that the invariant density function is symmetric with respect to $\vartheta$. Hence we can take the EMM

$$\bar{\vartheta}_N = \frac{1}{N} \sum_{j=1}^{N} X_j \rightarrow \vartheta, \quad N = n^{3/4}.$$  

Of course, the limit variance of the EMM $\bar{\vartheta}_N$ is greater than that of the BE, but this estimator is much more easier to calculate.
The score-function process is

\[ \Delta_k(\vartheta, X^k) = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} \ell(\vartheta, X_{j-1}, X_j), \quad N + 1 \leq k \leq n. \]

where

\[ \ell(\vartheta, x, x') = 3 \left( x' - x - 3 \frac{\vartheta - x}{1 + (\vartheta - x)^2} \right) \frac{1 - (\vartheta - x)^2}{(1 + (\vartheta - x)^2)^2}. \]
Therefore we can calculate the one-step MLE-process as follows

\[
\vartheta^*_k,n = \bar{\vartheta}_N + \frac{3}{I_k k} \sum_{j=1}^{k} \left( X_j - X_{j-1} - 3 \frac{\bar{\vartheta}_N - X_{j-1}}{1 + (\bar{\vartheta}_N - X_{j-1})^2} \right) \frac{1 - (\bar{\vartheta}_N - X_{j-1})^2}{(1 + (\bar{\vartheta}_N - X_{j-1})^2)^2}.
\]

Here \(I_k\) is the empirical Fisher information. Its calculation in this example can be found in \([11]\). Note that \(I(\vartheta) = I\) as usual with the shift parameter. Remind that by the Theorem \([1]\) this estimator is asymptotically normal.

The simulated one-step MLE-processes are shown on the Figure 3 and 4 for \(n = 10^3\) and \(n = 10^4\) respectively.

Figure 4: One-step MLE-process for \(n = 10^4\) and \(\vartheta = 0.5\).

On the Figure 3 the preliminary EMM \(\bar{\vartheta}_N = 0.45\) that is close to the true value of parameter \(\vartheta = 0.5\). We obtain this estimator based on the
learning interval of $N = 178$ observations. And we can observe the sequence of estimator $\vartheta_n^\ast = (\vartheta_{k,n}^\ast, k = N + 1; \ldots, n)$ that is asymptotically efficient.

On the Figure 4 the preliminary EMM $\tilde{\vartheta}_N = 0.56$ that is close to the true value $\vartheta = 0.5$. We obtain this estimator by the first $N = 10^3$ observations. We can see that the estimator-process $\vartheta_n^\ast = (\vartheta_{k,n}^\ast, k = N + 1; \ldots, n)$ tends to the true value.

Let us illustrate the two-step MLE-process. Now we take $N = n^{3/8}$.

![Figure 5: Second preliminary and two-step MLE-processes. $n = 10^3$, $\vartheta = 0.5$](image)

We consider two cases: one with $n = 10^3$ observations and the second with $n = 10^4$ observations.

On the Figure 5 the preliminary EMM $\tilde{\vartheta}_N = 0.4$ that is far from the true
Figure 6: Second preliminary and two-step MLE-processes. $n = 10^4$, $\vartheta = 0.5$ value $\vartheta = 0.5$. We obtain this estimator based on the learning interval of $N = 1000^{3/8} \approx 13$ observations. Then we obtain the second preliminary estimator-process $\bar{\vartheta}_{k,2}, k = N + 1, \ldots, n$ (continuous line) and see that it tends to the true value. The two-step MLE-process $\vartheta_n^{**}$ (dashed line) is closer to the true value and as well tends to the true value.

On the Figure the preliminary EMM $\bar{\vartheta}_N = 0.54$ that is close to the true value $\vartheta = 0.5$. We obtain this estimator based on the learning interval of $N = 10000^{3/8} = 32$ observations. Then we obtain the second preliminary estimator-process $\bar{\vartheta}_{k,2}, k = N + 1, \ldots, n$ (continuous line) and see that it tends to the true value. The two-step MLE-process $\vartheta_n^{**}$ (dashed line) is closer to the true value and as well tends to the true value.
4 Discussion

Two-step MLE-process allows us to estimate the parameter $\theta$ for the values $k$ satisfying the condition $n^{1/4} < k \leq n$. If we need a shorter learning interval, say, $[1, n\delta]$ with $\delta \in \left(\frac{1}{8}, \frac{1}{4}\right]$, then we have to study the three-step MLE-process, i.e., we use a preliminary estimator $\bar{\theta}_N$ and two estimator-processes like (16).

Note that the proposed one-step MLE-process can be written in the recurrent form. Indeed, the estimator $\vartheta_{k+1,n}^*$ we can write as follows:

\[
\vartheta_{k+1,n}^* = \bar{\theta}_N + \frac{1}{\sqrt{k + 1}} \{ \bar{\theta}_N \}^{-1} \Delta_{k+1} \left( \bar{\theta}_N, X_k^{k+1} \right)
\]

\[
= \bar{\theta}_N + \frac{1}{k + 1} \{ \bar{\theta}_N \}^{-1} \left[ \sum_{j=1}^k \hat{\ell} \left( \bar{\theta}_N, X_{j-1}, X_j \right) + \hat{\ell} \left( \bar{\theta}_N, X_k, X_{k+1} \right) \right]
\]

\[
= \frac{k}{k + 1} \left[ \bar{\theta}_N + \frac{1}{k} \{ \bar{\theta}_N \}^{-1} \sum_{j=1}^k \hat{\ell} \left( \bar{\theta}_N, X_{j-1}, X_j \right) \right] + \frac{1}{k + 1} \bar{\theta}_N
\]

\[
+ \frac{1}{k + 1} \{ \bar{\theta}_N \}^{-1} \hat{\ell} \left( \bar{\theta}_N, X_k, X_{k+1} \right)
\]

\[
= \frac{k}{k + 1} \vartheta_{k,n}^* + \frac{1}{k + 1} \bar{\theta}_N + \frac{1}{k + 1} \{ \bar{\theta}_N \}^{-1} \hat{\ell} \left( \bar{\theta}_N, X_k, X_{k+1} \right)
\]

The obtained presentation

\[
\vartheta_{k+1,n}^* = \frac{k}{k + 1} \vartheta_{k,n}^* + \frac{1}{k + 1} \bar{\theta}_N + \frac{1}{k + 1} \{ \bar{\theta}_N \}^{-1} \hat{\ell} \left( \bar{\theta}_N, X_k, X_{k+1} \right)
\]

allows us to calculate $\vartheta_{k+1,n}^*$ using the values $\bar{\theta}_N, \vartheta_{k,n}^*$ and observations $X_k, X_{k+1}$ only.

The similar structure can be obtained for the two-step MLE-process too.

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