Few-body nuclear reactions

A Deltuva
Centro de Física Nuclear, University of Lisbon, P-1649-003 Lisbon, Portugal
E-mail: deltua@cii.fc.ul.pt

Abstract. Three- and four-body scattering is described in the framework of exact Faddeev-type integral equations. They are solved numerically using momentum-space partial-wave basis. The Coulomb interaction between charged particles is included using a novel implementation of the screening and renormalization method. The technique is applied to three- and four-nucleon scattering. Furthermore, the method has been extended successfully to elastic, transfer, and breakup reactions in three-body-like nuclear systems. Examples are deuteron scattering on stable nuclei ranging from $^4$He to $^{58}$Ni and proton scattering on weakly bound two-body system such as $^{11}$Be, $^{15}$C, and $^{17}$O. These calculations allow to evaluate the accuracy of traditional approximate nuclear reaction approaches like the continuum-discretized coupled-channels (CDCC) method but also to test novel dynamical models such as nonlocal optical potentials.

1. Introduction
The application of exact Faddeev three-body theory to the description of nuclear reactions that are dominated by three-body degrees of freedom has been precluded in the past by the difficulty in dealing with the long-range Coulomb interaction between charged particles. Instead, approximate theoretical methods like the distorted-wave Born approximation (DWBA) or the continuum-discretized coupled-channels (CDCC) method [1] were used to analyze data resulting from the three-body-like nuclear reactions. However, recently also the exact Faddeev-type description of few-body nuclear reactions has become possible due to the novel implementation [2, 3] of the screening and renormalization method [4, 5, 6, 7] for including the Coulomb interaction between charged particles. This latter framework, though being technically and computationally most complicated and expensive, has an advantage that, once numerically well-converged results are obtained, all discrepancies with the experimental data can be attributed solely to the shortcomings of the used interaction models or to the inadequacy of the few-body model. The present paper recalls some recent achievements in the Faddeev-type description of few-body nuclear reactions.

Section 2 gives three-body scattering equations. Section 3 describes the inclusion of the Coulomb interaction. Section 4 presents selected results for three-body-like nuclear reactions. Four-nucleon scattering equations and example results are given in section 5. Section 6 contains summary.

2. Three-body scattering equations
An exact treatment of the quantum three-body scattering problem is provided by both Faddeev [8] and Alt, Grassberger, and Sandhas (AGS) equations [9] that are equivalent to the Schrödinger
The AGS equations are applicable only to short-range potentials \( v \). Inclusion of the Coulomb interaction employed numerical techniques are described in great detail in Refs. [10, 11].

A system of integral equations with two continuous variables, the values of Jacobi momenta. The \( \psi_{\gamma} \) is the potential for the pair \( \gamma \); we use the odd-man-out notation. The channel states \( |\psi_{\gamma}\rangle \) for \( \gamma = 1, 2, 3 \) are the eigenstates of the corresponding channel Hamiltonian \( H_\gamma = H_0 + v_\gamma \) with the energy eigenvalue \( E \); thus, \( |\psi_{\gamma}\rangle \) is a product of the bound state wave function for pair \( \gamma \) and a plane wave with fixed on-shell momentum corresponding to the relative motion of particle \( \gamma \) and pair \( \gamma \) in the initial or final state. The channel states \( |\psi_0\rangle \) are the eigenstates of \( H_0 \) with the same energy eigenvalue \( E \) and describe the free motion of three particles. Observables of elastic scattering are calculated from the matrix elements with \( \beta = \alpha \), those of breakup are given by \( \beta = 0 \) while \( 0 \neq \beta \neq \alpha \) correspond to transfer reactions.

We solve the AGS equations using momentum-space partial-wave basis where they become a system of integral equations with two continuous variables, the values of Jacobi momenta. The employed numerical techniques are described in great detail in Refs. [10, 11].

### 3. Inclusion of the Coulomb interaction

The AGS equations are applicable only to short-range potentials \( v_\gamma \). The Coulomb interaction \( w_{\gamma C} \), due to its long range, does not satisfy the mathematical properties required for the formulation of the standard scattering theory. However, since in nature the Coulomb potential is always screened, one could expect that the physical observables become insensitive to the screening provided it takes place at sufficiently large distances \( R \) and, therefore, the \( R \to \infty \) limit should correspond to the proper Coulomb. This was proved by Taylor [4] in the context of the two-particle system: though the on-shell screened Coulomb transition matrix diverges in the \( R \to \infty \) limit, after renormalization by (an equally) diverging phase factor it converges as a distribution to the well known proper Coulomb amplitude and therefore yields identical results for the physical observables. A similar renormalization relates screened and proper Coulomb wave functions [12]. Therefore, to include the long-range Coulomb force we follow the idea of screening and renormalization that was extended in Refs. [2, 6, 13, 14, 15] to three-particle scattering where only two particles are charged. The screened Coulomb potential

\[
 w_{\gamma R}(r) = w_{\gamma C}(r) \ e^{-\langle r/R \rangle^\alpha}
\]  

(3)

is added to the nuclear potential \( v_{\gamma N} \), the standard scattering theory is applicable, and the AGS equations (1) are solved with the resulting potential \( v_\gamma = v_{\gamma N} + w_{\gamma R} \) yielding the AGS transition operators \( U^{(R)}_{\beta \alpha} \) that depend on the screening radius \( R \). On-shell matrix elements \( \langle \psi_\beta | U^{(R)}_{\beta \alpha} | \psi_\alpha \rangle \) do not have a \( R \to \infty \) limit. However, as demonstrated in Refs. [2, 6, 13], the three-particle amplitudes can be decomposed into long-range and Coulomb-distorted short-range parts, where the quantities diverging in that limit are of two-body nature, i.e., the on-shell transition matrix \( T_{\alpha R}^{m} \) derived from the screened Coulomb potential between spectator \( \alpha \) and the center of mass (c.m.) of the bound pair \( \alpha \), the corresponding wave function, and the screened Coulomb wave

\[
 T_{\gamma R} = v_\gamma + v_\gamma G_0 T_{\gamma R}
\]  

(2)
function for the relative motion of two charged particles in the final state. Those quantities, renormalized according to Ref. [4], in the $R \to \infty$ limit converge to the two-body Coulomb scattering amplitude $\langle \psi_\beta | U_{\beta \alpha}^{C} | \psi_\alpha \rangle$ (in general, as a distribution) and to the corresponding Coulomb wave functions, respectively, thereby yielding the three-particle scattering amplitudes in the proper Coulomb limit

$$\langle \psi_\beta | U_{\beta \alpha}^{(C)} | \psi_\alpha \rangle = \delta_{\beta \alpha} \langle \psi_\alpha | T_{\alpha \alpha}^{C,m} | \psi_\alpha \rangle + \lim_{R \to \infty} \{ Z_{\beta R}^{-1/2} \langle \psi_\beta | U_{\beta \alpha}^{(R)} - \delta_{\beta \alpha} T_{\alpha \alpha}^{C,m} | \psi_\alpha \rangle Z_{\alpha R}^{-1/2} \}, \quad (4)$$

$$\langle \psi_0 | U_{0 \alpha}^{(C)} | \psi_\alpha \rangle = \lim_{R \to \infty} \{ z_{R}^{-1/2} \langle \psi_0 | U_{0 \alpha}^{(R)} | \psi_\alpha \rangle Z_{\alpha R}^{-1/2} \}. \quad (5)$$

The renormalization factors $Z_{\alpha R}$ and $z_R$ are diverging phase factors given in Refs. [2, 4, 5, 13]. The $R \to \infty$ limit in Eqs. (4) and (5) has to be calculated numerically, but due to the short-range nature of the corresponding operators it is reached with sufficient accuracy at finite $R$. Thus, for including the Coulomb interaction in three-body nuclear reactions with only two charged particles one only needs to solve the standard AGS equations (1) with short-range potentials. However, the screened Coulomb interaction, due to its longer range, compared to the nuclear interaction, brings additional difficulties: quasisingular nature of the potential and slow convergence of the partial-wave expansion. The right choice of the screening, i.e., the parameter $n$ in Eq. (3), is essential in resolving those difficulties. For the fast convergence with $R$ we have to ensure that the screened Coulomb potential $w_{\alpha R}(r)$ approximates well the proper Coulomb potential $w_{\alpha C}(r)$ for distances $r$ smaller than the screening radius $R$ and simultaneously vanishes smoothly but rapidly for $r > R$, providing a comparatively fast convergence of the partial-wave expansion. In Refs. [2, 13] we found the optimal values $3 \leq n \leq 6$.

![Figure 1](https://example.com/figure1.png)

**Figure 1.** Convergence of the $\alpha + d \to \alpha + p + n$ breakup cross section with the screening radius $R$ in selected kinematical configurations at 15 MeV $\alpha$ lab energy. Results obtained with screening radius $R = 10$ fm (dotted curves), 15 fm (dash-dotted curves), and 20 fm (solid curves) are compared. Results without Coulomb (dashed curves) are given as reference for the size of the Coulomb effect. The experimental data are from Ref. [16].

The most important criterion for the reliability of the screening and renormalization method is the convergence of the observables with the screening radius $R$ used to calculate the Coulomb-distorted short-range part of the amplitudes. In Fig. 1 it is shown for $\alpha + d \to \alpha + p + n$ breakup reaction in two kinematical configurations; more examples can be found in Refs. [2, 13, 17]. In most cases the convergence is impressively fast and only becomes slower for the observables at very low energies. Furthermore, as demonstrated in Ref. [18], our results for proton-deuteron elastic scattering agree well over a wide energy range with those of Ref. [19] obtained from
the variational solution of the three-nucleon Schrödinger equation in configuration space with
the inclusion of an unscreened Coulomb potential and imposing the proper Coulomb boundary
conditions explicitly.

4. Results for three-body nuclear reactions
The method of screening and renormalization in the framework of momentum-space AGS
equations has been successfully applied in the three-nucleon system \[2, 13, 20\], i.e., for
proton-deuteron elastic scattering, breakup, and radiative capture, and for photo- and
electrodisintegration of \(^3\)He. Furthermore, it was extended to three-body nuclear reactions
\[17, 21\] like deuteron (d) scattering on a stable nucleus \(A\) or proton (p) scattering on a weakly
bound two-body system (An) consisting of the core \(A\) and the neutron \(n\); all elastic, transfer,
charge-exchange, and breakup reactions, allowed by the chosen Hamiltonian, were calculated on
the same footing. Those reactions have been calculated previously using approximate methods
like DWBA or CDCC. The exact Faddeev-type calculations allow to evaluate the accuracy of
those approaches. Recent comparison \[22\] of Faddeev/AGS and CDCC results revealed that
CDCC is indeed a reliable method to calculate \(d + A\) elastic and breakup cross sections but
may lack accuracy for transfer reactions such as \(p + ^{11}\)Be \(\rightarrow d + ^{10}\)Be and for breakup of one-
nucleon halo nuclei \(p + ^{11}\)Be \(\rightarrow p + n + ^{10}\)Be. Furthermore, the Faddeev/AGS momentum-space
framework allows for an inclusion of a novel dynamic input like energy-dependent \[23\] or nonlocal
\[24, 25\] optical potentials (OP). Especially important nonlocality effects were found in deuteron
stripping and pickup reactions (d, p) and (p, d) involving both stable \[26\] and exotic nuclei \[27\].
Examples are given in Figs. 2 — 4.

![Figure 2](image)

**Figure 2.** Differential cross section divided by Rutherford cross section
and deuteron vector analyzing power for deuteron elastic scattering from \(^{16}\)O and \(^{40}\)Ca
nuclei at \(E_d = 56\) MeV. Predictions of nonlocal OP (solid curves) and
equivalent local OP (dashed-dotted curves) are compared with the experimental data from Ref. \[28\].

5. Four-nucleon scattering
Exact description of the four-nucleon scattering is given by the Faddeev-Yakubovsky equations
\[32\] for the wave-function components or by the equivalent AGS equations \[33\]. We use isospin
formalism and solve the symmetrized AGS equations \[34\] for the transition operators \(U_{\alpha\sigma}\). We
consider only reactions with initial (final) two-cluster states, i.e., \(1+3\) and \(2+2\), that correspond
1.0
10.0
1.0
10.0
0
60
120
180
0
60
120
180
Figure 3. Differential cross section for $d + ^{16}O \rightarrow p + ^{17}O(5/2^+)$ (left side) and excited state $1/2^+$ (right side) at $E_d = 25.4$ and 36.0 MeV. Curves as in Fig. 2. The experimental data are from Ref. [29].

Figure 4. Differential cross section for $d + ^{10}Be \rightarrow p + ^{11}Be$ transfer to the $^{11}Be$ ground state $1/2^+$ at $E_d = 12$ and 25 MeV as function of the c.m. scattering angle. Curves as in Fig. 2. The experimental data are from Refs. [30, 31].

to $\alpha(\beta) = 1$ and 2, respectively:

$$
U_{11} = -(G_0 T G_0)^{-1} P_{34} - P_{34} U_1 G_0 T G_0 U_{11} + U_2 G_0 T G_0 U_{11}, \\
U_{21} = (G_0 T G_0)^{-1} (1 - P_{34}) + (1 - P_{34}) U_1 G_0 T G_0 U_{11}, \\
U_{12} = (G_0 T G_0)^{-1} P_{34} U_1 G_0 T G_0 U_{12} + U_2 G_0 T G_0 U_{12}, \\
U_{22} = (1 - P_{34}) U_1 G_0 T G_0 U_{22}.
$$

Here $G_0$ is the free resolvent, $P_{ij}$ is the permutation operator of particles $i$ and $j$, and $T = v + vG_0 T$ is the two-nucleon transition matrix, $v$ being the potential. The subsystem $1 + 3$ and $2 + 2$ AGS transition operators are calculated as

$$
U_\alpha = P_\alpha G_0^{-1} + P_\alpha T G_0 U_\alpha, \\
P_1 = P_{12} P_{23} + P_{13} P_{23}, \\
P_2 = P_{13} P_{24}.
$$
The on-shell elements of the four-nucleon AGS transition operators \( \langle \Psi_\beta | U_{\beta\alpha} | \Psi_\alpha \rangle \) with initial (final) state Faddeev amplitudes

\[ |\Psi_\alpha\rangle = G_0 TP_\alpha |\Psi_\alpha\rangle \quad (13) \]

are closely related to the scattering amplitudes [34].

In order to include the Coulomb interaction between protons we use the screening and renormalization approach. Long- and Coulomb-distorted short-range parts in the scattering amplitudes are separated [35, 36]. The former is of two-body nature and its \( R \to \infty \) limit is known analytically. The Coulomb-distorted short-range part is calculated by solving symmetrized AGS equations numerically where the screened Coulomb potential is added to the nuclear proton-proton potential; the \( R \to \infty \) limit is reached with sufficient accuracy at finite screening radii \( R \) as demonstrated in Ref. [35].

In Figs. 5 through 7 we show examples for elastic, charge-exchange, and transfer reactions in the four-nucleon system. The two-nucleon interactions we use are charge-dependent Bonn (CD Bonn) [37], AV18 [38], inside nonlocal outside Yukawa (INOY04) potential by Doleschall [39], and the one derived from chiral perturbation theory at next-to-next-to-next-to-leading order (N3LO) [40]. Although here we do not include a three-nucleon force, its presence is simulated by using the potential INOY04 that fits both \( ^3\text{He} \) and \( ^3\text{H} \) experimental binding energies (7.72 MeV and 8.48 MeV, respectively). The results for the two-baryon potential CD Bonn + \( \Delta \) [41] allowing for a virtual excitation of a nucleon to a \( \Delta \)-isobar and thereby yielding consistent effective three- and four-nucleon forces are qualitatively similar and can be found in Ref. [42].

Most of the experimental data are quite well described [35, 36, 43] at least by some of the used two-nucleon force models, but there exist also several discrepancies, e.g., for the neutron-\( ^3\text{H} \) total cross section [44] and for the proton analyzing power in the proton-\( ^3\text{He} \) elastic scattering [35, 45, 46] and in the \( p + ^3\text{H} \to n + ^3\text{He} \) charge-exchange reaction [36].

![Graph](image)

**Figure 5.** Differential cross section and proton analyzing power of elastic proton-\(^3\text{H}\) scattering at 4.15 MeV proton lab energy. Results obtained with realistic two-nucleon potentials CD Bonn (solid curves), AV18 (dashed curves), INOY04 (dashed-dotted curves), and N3LO (dotted curves) are compared with the experimental data from Ref. [47].

### 6. Summary

Exact Faddeev/AGS equations in the momentum-space framework were solved for the description of few-body nuclear reactions. The Coulomb interaction was included using the method of screening and renormalization. Fully converged results were obtained for three- and four-nucleon scattering and for three-body-like nuclear reactions.
Figure 6. Differential cross section and proton analyzing power $A_y$ of $p + ^3H \rightarrow n + ^3He$ charge-exchange reaction at 2.48 and 6 MeV proton lab energy. Curves as in Fig. 5. The cross section data are from Refs. [48] (circles) and [49] (squares) at 2.48 MeV, and from Ref. [50] at 6 MeV. $A_y$ data are from Ref. [51] at 2.48 MeV and from Ref. [52] at 6 MeV.

Figure 7. Differential cross section and deuteron analyzing power $iT_{11}$ of $d + d \rightarrow p + ^3H$ and $d + d \rightarrow n + ^3He$ reactions at 3 MeV deuteron lab energy. Curves as in Fig. 5. The cross section data are from Refs [53] (squares) and [54] (circles). Analyzing power data are from Ref [54] for $d + d \rightarrow p + ^3H$ and from Ref. [55] for $d + d \rightarrow n + ^3He$. 
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