Observation of distinct phase transitions in a nonlinear optical Ising machine

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Optical Ising machines promise to solve complex optimization problems with an optical hardware acceleration advantage. Here we study the ground state properties of a nonlinear optical Ising machine realized by spatial light modulator, Fourier optics, and second-harmonic generation in a nonlinear crystal. By tuning the ratio of the light intensities at the fundamental and second-harmonic frequencies, we experimentally observe two distinct ferromagnetic-to-paramagnetic phase transitions: a second-order phase transition where the magnetization changes to zero continuously and a first-order phase transition where the magnetization drops to zero abruptly as the effective temperature increases. Our experimental results are corroborated by a numerical simulation based on the Monte Carlo Metropolis-Hastings algorithm, and the physical mechanism for the distinct phase transitions can be understood with a mean-field theory. Our results showcase the flexibility of the nonlinear optical Ising machine, which may find potential applications in solving combinatorial optimization problems.
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ombinatorial optimization is ubiquitous and fundamental in many areas of science, engineering, finance, and social networks. Many optimization problems, such as the traveling salesman problem, the graph coloring problem, the Boolean satisfiability problem, spin glass dynamics, protein folding, etc., belong to the non-deterministic polynomial time (NP) hard or NP-complete class which can be formulated as finding the ground states of Ising spin models. Because of the computational complexity, it is usually challenging to find the exact solutions of general Ising models with traditional electronic computers. In the past years, many physical systems including the superconducting circuits, stochastic nanomagnets, trapped ions, complementary metal-oxide semiconductor devices, injection-locked laser networks, polariton condensates, etc., have been applied to realize an Ising simulator and to solve the optimization problems with heuristic search algorithms.

Among these physical implementations, optical Ising machines are particularly attractive because of their capability of parallelism, low energy consumption, and operation at the speed of light. In addition to the promising approach based on a network of degenerate optical parametric oscillators and Fourier optics are also being actively pursued in the optical community. By encoding the spins in the SLM-modulated binary phase of an incident beam and measuring the light intensity at the focal plane, a fully connected large-scale optical Ising machine with configurable two-body spin-spin interactions can be realized. Furthermore, by including a second-harmonic (SH) light generation through nonlinear crystal and measuring the superposition of the pump light and SH light intensities, we recently have realized a more general Ising model with both two-body and four-body spin interactions. Our nonlinear optical Ising machine realizes a special class of the so-called k-local Hamiltonian which is useful for polynomial unconstrained binary optimization or high order binary optimization problems.

A natural question that arises is whether such a nonlinear optical Ising machine can be used to solve optimization problems more efficiently. As the first step to address this important question, we experimentally and theoretically investigate the ground state magnetic phases of the nonlinear optical Ising machine for different four-body spin interaction coefficient and effective temperature. The ground state of the Ising model at different temperatures provides important information for the energy landscape. Understanding the ground state phase diagram can assist the search for the true ground state of the Ising model in the simulated annealing. Our main finding is that we can identify two distinct types of phase transitions with the order parameter—the magnetization—changes either continuously or abruptly to zero as the increase of temperature, corresponding to a second-order and a first-order phase transition respectively. We point out that similar Ising models with nearest-neighbor two-body and local four-body spin interactions have been theoretically explored in the 1970s. To the best of our knowledge, our results represent the first experimental observation of two types of phase transitions in a configurable optical Ising model with fully connected two-body and four-body spin interactions.

Results

Experimental setup and Ising Hamiltonian. The schematic of the experimental setup for the nonlinear optical Ising machine is shown in Fig. 1a. We use a pump laser of wavelength $\lambda \sim 1550.9\text{nm}$ incident on a SLM. A lens with focal length $F = 200 \text{mm}$ is used to focus the light into a temperature-stabilized periodically poled lithium niobate (PPLN) crystal. It generates a SH light at $\lambda_h \sim 775.5 \text{nm}$. After passing through another lens, the pump and SH lights are separated by a dichroic mirror (DM), then coupled into the single mode fibers and detected by two separate power meters. The measurements are sent to the recurrent feedback, which updates the spins on the SLM according to the Monte Carlo Metropolis-Hastings algorithm to find the ground state of the given Ising problem.

The Hamiltonian of our Ising spin model is defined as the superposition of the pump and SH light intensities (see Methods),

$$H = -\sum_{ij} J_{ij} \sigma_i \sigma_j - \gamma \sum_{i,j,s,r} K_{ijrs} \sigma_i \sigma_j \sigma_s \sigma_r,$$

where we have multiplied $-1$ for pump light intensity (so that a ferromagnetic phase is favored at $T = 0$) and $\gamma$ is a tunable parameter. The two terms correspond to a two-body and a four-body spin interaction, with the following explicit expressions for the spin interactions

$$J_{ij} = 2\pi (w_i^f)^2 \frac{a^4}{\lambda^2} \xi_i \xi_j \xi_i \xi_j,$$

$$K_{ijrs} = \frac{1}{2} \pi a^2 (w_i^f)^2 \frac{a^8}{\lambda^4} \xi_i \xi_j \xi_s \xi_r,$$

where $w_i^f = \exp[-\frac{\pi(w_i^f)^2}{\lambda^2}(x_i^2 + y_i^2)]$ and $w_i^f = \exp[-\frac{\pi(w_i^f)^2}{\lambda^2}(x_i + y_i)^2]$. It is worth pointing out that in the pump non-diffraction and non-depletion regime, the two-body and four-body spin interactions are not full-rank matrices. They can be decomposed as $J_{ij} = P_i P_j$ with $P_i = \sqrt{2\pi w_i^f a^2 \xi_i^2} / \lambda$ and $K_{ijrs} = Q_i Q_j Q_s Q_r$ with $Q_i = \sqrt{\pi a^4 a^4 \xi_i^2} / \sqrt{2\pi \lambda^2}$. Thus, the two-body spin interaction $J_{ij}$ is a matrix of rank one and the reduced four-body interaction $Q_{ijrs}$ is a matrix of rank $N$ as the fiber mixes the light from different pixels (see Fig. 1b, c). Such low-rank two-body and four-body spin interactions may impose some limitations to the SLM-based optical Ising machine for solving complex optimization problems. This can be circumvented, for example, by adding a scattering medium in front of the detector, as proposed in the recent work. Alternatively, the relative locations of the Fourier lenses can be changed to induce non-negligible diffraction inside the crystal, or the pump power can be increased to enter the pump-depletion regime.

Experimental results and numerical simulations. Our main experimental observations are summarized in Fig. 2a–c. For a fixed effective temperature $T$, the Ising energy decreases to its lowest value after some Monte-Carlo iterations and the corresponding magnetization increases from zero (as we start with an initial random spin configuration) to a finite value that depends on the four-body coefficient. As shown in Fig. 2c, the magnetization changes to zero continuously as a function of temperature for $y = 0, 100$ while it drops to zero abruptly for $y = 500$, indicating two different types of phase transitions. These experimental observations can be reproduced with a full Monte Carlo numerical simulation. Using the same Metropolis-Hasting algorithm, we randomly choose one spin each time to flip in order to reduce the total energy. The spin flip is accepted with the Boltzmann probability, similar to that adopted in the experiment.
This allows the system to evolve out of energy local minima and have more chance to reach the ground state. As shown in Fig. 2d, the numerical results are qualitatively the same as the experimental results. One of the mechanisms for the discrepancy is related to the optical loss, which is not captured in our theoretical model. We further have performed another measurement with the same pump power and better SH coupling into the fiber. Figure 3 shows the experimental results for the measured magnetization as a function of temperature for different $\gamma$ values. The pump power for the new measurement is about 1.3 times larger and thus the SH power is about 1.7 times larger compared to Fig. 2c. According to the Hamiltonian Eq. (1), the critical $\gamma$ separating the second-order and the first-order phase transitions will be about 1.3 times smaller. This allows the system to evolve out of energy local minima and have more chance to reach the ground state. As shown in Fig. 2d, the numerical results are qualitatively the same as the experimental results. One of the mechanisms for the discrepancy is related to the optical loss, which is not captured in our theoretical model. We further have performed another measurement with the same pump power and better SH coupling into the fiber. Figure 3 shows the experimental results for the measured magnetization as a function of temperature for different $\gamma$ values. The pump power for the new measurement is about 1.3 times larger and thus the SH power is about 1.7 times larger compared to Fig. 2c. According to the Hamiltonian Eq. (1), the critical $\gamma$ separating the second-order and the first-order phase transitions will be about 1.3 times smaller. This allows the system to evolve out of energy local minima and have more chance to reach the ground state. As shown in Fig. 2d, the numerical results are qualitatively the same as the experimental results. One of the mechanisms for the discrepancy is related to the optical loss, which is not captured in our theoretical model. We further have performed another measurement with the same pump power and better SH coupling into the fiber. Figure 3 shows the experimental results for the measured magnetization as a function of temperature for different $\gamma$ values. The pump power for the new measurement is about 1.3 times larger and thus the SH power is about 1.7 times larger compared to Fig. 2c. According to the Hamiltonian Eq. (1), the critical $\gamma$ separating the second-order and the first-order phase transitions will be about 1.3 times smaller. This allows the system to evolve out of energy local minima and have more chance to reach the ground state. As shown in Fig. 2d, the numerical results are qualitatively the same as the experimental results. One of the mechanisms for the discrepancy is related to the optical loss, which is not captured in our theoretical model. We further have performed another measurement with the same pump power and better SH coupling into the fiber. Figure 3 shows the experimental results for the measured magnetization as a function of temperature for different $\gamma$ values. The pump power for the new measurement is about 1.3 times larger and thus the SH power is about 1.7 times larger compared to Fig. 2c. According to the Hamiltonian Eq. (1), the critical $\gamma$ separating the second-order and the first-order phase transitions will be about 1.3 times smaller. This allows the system to evolve out of energy local minima and have more chance to reach the ground state. As shown in Fig. 2d, the numerical results are qualitatively the same as the experimental results. One of the mechanisms for the discrepancy is related to the optical loss, which is not captured in our theoretical model. We further have performed another measurement with the same pump power and better SH coupling into the fiber. Figure 3 shows the experimental results for the measured magnetization as a function of temperature for different $\gamma$ values. The pump power for the new measurement is about 1.3 times larger and thus the SH power is about 1.7 times larger compared to Fig. 2c. According to the Hamiltonian Eq. (1), the critical $\gamma$ separating the second-order and the first-order phase transitions will be about 1.3 times smaller.
According to Landau misalignment, etc. In Eq. (8), the magnetization drops to zero continuously at $\gamma = 0$, 100, 500. The same results obtained from full Monte-Carlo numerical simulation. Error bars in each measurement are less than the size of the dots. Errors in the experiment could be arise due to the slight laser power fluctuations, temperature variation, and optical misalignment, etc. In (c, d), the magnetization drops to zero continuously at $\gamma = 1$ (blue), 100 (orange) and abruptly at $\gamma = 500$ (green) with the thick lines the guides to the eyes.

Nonuniform spin interaction induced spatially structured magnetic phase. For $\gamma < 0$, our experimental observation (Fig. 5a) and Monte-Carlo numerical simulation (Fig. 5b) indicate that there is a spatially structured magnetic phase appearing: the inner part of the SLM exhibits a ferromagnetic phase with all the spins pointing to the same direction. The latter matrix has larger elements if $r_i + r_j > 0$. To reduce magnetization, the spins located at the opposite side of the SLM favor opposite directions and the spins next to each other favor the same direction in order to reduce the four-body interaction energy. On the other hand, at the inner part of the SLM where the light intensity is stronger, the two-body interaction dominates and thus a ferromagnetic phase with all the spins point to the same direction is preferred. Consequently, a domain wall that separates the two regions of opposite spins is formed in the SLM (see the inset of Fig. 5a, b).

Conclusion

In summary, we have performed systematic investigations of the magnetic phases of the nonlinear optical Ising machine. The exhibited rich phase diagram is a direct consequence of the competition between two-body and four-body spin interactions at different effective temperature. The great flexibility of the nonlinear optical Ising machine may be useful for solving optimization problems where the four-body spin interaction coefficient is non-zero.
interactions through sum frequency generation, four-wave mixing, and high harmonic generations. It can be used for the study of q-state Potts model\textsuperscript{56,57} and the development of self-learning Monte-Carlo algorithm\textsuperscript{58}, etc.

### Methods

#### Design of nonlinear optical Ising machine

We use a mode-locked laser of wavelength $\lambda = 1550.9$ nm with an average power of 70mW as the pump light. The full width half maximum of the pump beam is $w_p = 3.8$ mm which is incident on the SLM (Santec SLM-100, 1440 × 1050 pixels, pixel length $a = 10 \mu m$). The region of interest on the SLM is defined as a square lattice of $N = 20 \times 20$ giant spins with each spin consisting of 20 × 20 pixels of the same phase which is modulated to be 0 or π to generate a random initial spin configuration. The unmodulated portion of the pump light is deflected by an optimized blazed grating. A lens with focal length $F = 200$ mm is used to focus the modulated beam into a temperature-stabilized periodically poled lithium niobate (PPLN) crystal with a poling period of $\Lambda = 19.36 \mu m$ (5mol.% MgO-doped PPLN, length 1cm from HC Photonics). It generates a SH light at $\lambda_2 = 775.5$ nm\textsuperscript{59}. After passing through another lens of the same focal length, the pump and SH lights are separated by a dichroic mirror (DM) and then coupled into the single mode fibers (SMF-28) using fiber collimators with aspheric lenses Thorlabs C220TMD-C and A375TM-B, respectively, and detected by the power meters (Thorlabs PM-100D with sensors S132C and S130C). The measurements are sent to the computer MATLAB interface, which completes the feedback loop by updating the SLM\textsuperscript{56}. The spin flipping during each iteration is accepted or rejected according to a Boltzmann probability function $P =$ exp($-\Delta U/T$), where $\Delta U = E_{\text{new}} - E_{\text{old}}$ is the change in energy of a target Hamiltonian and $T$ is the effective temperature (see Fig. 1a).

#### Theoretical Model

Since the phase of the light incident on the SLM is modulated to be either 0 or $\pi$, we can model the electric field of the incident light with the following discretized form

$$E_p(x) = \sum_{i} \xi_i \text{rect}\left(\frac{x - x_i}{a}\right),$$

where $x = (x, y)$ denotes the spatial coordinate on the SLM plane, $x_i$ is the position of the $i^{th}$ pixel, $a$ is the pixel length, $x_i = \pm 1$ is the phase of the pump light, and $\xi_i = E_0 e^{-(5r)^2/2\sigma^2}$ corresponds to the amplitude of the Gaussian beam, $\text{rect}(x) = \text{rect}(x)\text{rect}(y) = 1$ for $|x| < 0.5$ and $|y| < 0.5$ is the rectangular function. The wave reflected by the SLM passes through a lens of focal length $F$. On the focal plane, the
wave is transformed to the Fourier domain according to
\[ \hat{E}_0(x') = \frac{1}{\sqrt{A}} \int E_0(x)e^{i2\pi x' - i2\pi x} \, dx, \]
where \( x' = (x', y') \) denotes the spatial coordinate on the focal plane. A straightforward calculation gives
\[ \hat{E}_0(x') = \frac{d}{\sqrt{A}} \sum_{n} e^{i\frac{\pi}{\beta x_n}} \sin\left(\frac{\pi x_n}{\beta A}\right), \]
where \( \sin(x) = \sin(\pi x')/\sin(\pi y') \) denotes the two-dimensional sinc function. The PPN nonlinear crystal placed at the focal point of the lens generates SH light of frequency \( \omega_0 \). In the pump non-diffraction and non-depletion regime, the pump light in the nonlinear crystal does not change while the SH light can be obtained as \( \hat{E}_0(x') = A_N^0 \hat{E}_0(x') \) where \( A_N = i\omega_0^2/L(2\epsilon_0^3) \). \( L \) is the length of the nonlinear crystal, \( \epsilon_0^2 \) is the second-order susceptibility, \( c \) is the speed of light, \( n_0 = 2n_2/\beta L \) is the wave number of the SH light, and \( n_0 \) is the index of refraction. After passing through two lenses of focal length \( f \) and \( f', \) the pump light and SH light are subsequently coupled into the fibers and then the intensities are measured. The light intensity coupled into the fiber can be defined as \( P = \frac{1}{2}(\hat{E}_0(u)\hat{E}_0^*(u))^{\frac{1}{2}} \), where \( u = (u, v) \) and \( \hat{E}_0(u) = \sqrt{K} \exp(-\frac{\pi}{\epsilon u^2}) \) are the spatial coordinate and lowest optical mode (characterized by the width \( \epsilon \) which is slightly different for the pump and SH lights) of the fiber, respectively. The Hamiltonian of our ising spin model is defined as the superposition of the pump and SH light intensities (see Eq. (1)).

### Data availability
The data that support the plots and other findings of this paper are available from the corresponding author upon a reasonable request.

### Code availability
The code is available from the corresponding author upon a reasonable request.

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Author contributions

Y.H. and C.Q. supervised the project. S.K. and T.B. built the experimental setup. S.K. carried out the experiments and analyzed the data. Z.L. and C.Q. performed the theoretical calculations. All authors contributed to the scientific discussion of this paper.

Competing interests

The authors declare no competing interests.

Additional information

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