Remarks on Lin-Nakamura-Wang’s paper

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Abstract

Theorem 1.2 in their paper arXiv:1904.00999v1 [math.AP] 30 Mar 2019 “Reconstruction of unknown cavity by single measurement” is not valid.

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1 A counter example

In [6] they state1 if $\overline{D} \not\subset \overline{G}$, then $I(G) = \infty$. However, in this note we give a simple example that $\overline{D} \not\subset \overline{G}$, however $I(G) = 0$.

Let $\Omega = \{x \in \mathbb{R}^2 | |x| < R\}$ with $R > 1$ and $D = \{x \in \mathbb{R}^2 | |x| < 1\}$. Let $u$ solve

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega \setminus \overline{D}, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial D, \\
u(R \cos \theta, R \sin \theta) = (R + \frac{1}{R}) \cos \theta, & \theta \in [0, 2\pi].
\end{cases}
\] (1.0)

Note that the solution has the explicit form

\[u(r \cos \theta, r \sin \theta) = \left(r + \frac{1}{r}\right) \cos \theta.\]

The key point of this note is the following trivial fact: $u$ has an extension to the domain $\tilde{\Omega} = \{x \in \mathbb{R}^2 | 0 < |x| < R\} = \Omega \setminus \{0\}$ as a solution of the Laplace equation.

Let $0 < \delta < 1$ and choose $G = \{x \in \mathbb{R}^2 | |x| < 1 - \delta\}$. We have $\overline{G} \subset D$ and thus $\overline{D} \not\subset \overline{G}$.

Given $\epsilon > 0$ let $g \in H^{1/2}(\partial \Omega)$ be an arbitrary function such that the solution $z_g$ of

\[
\begin{cases}
\Delta z_g = 0 & \text{in } \Omega, \\
z_g = g & \text{on } \partial \Omega
\end{cases}
\]

satisfies

\[\|z_g\|_{H^1(G)} < \epsilon.\] (1.1)

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1Please refer to their paper [6] for the symbols used in this note without explanation.
By Lemma 2.1 in [6] we have
\[
\int_{\partial \Omega} \partial_\nu w \cdot g \, ds = - \int_{\partial D} u \cdot \partial_\nu z_g \, ds,
\]
where \( w = u - v \) and \( v \) solves
\[
\begin{aligned}
\Delta v &= 0 \quad \text{in } \Omega, \\
v &= u \quad \text{on } \partial \Omega.
\end{aligned}
\]
Let \( \tilde{u} \) denote the harmonic extension of \( u \) into \( \tilde{\Omega} \), that is
\[
\tilde{u}(r \cos \theta, r \sin \theta) = \left( r + \frac{1}{r} \right) \cos \theta.
\]
Let \( C = \{ x \in \mathbb{R}^2 \mid |x| = 1 - \delta' \} \) with \( \delta < \delta' < 1 \). We have \( C \subset G \).

Write
\[
- \int_{\partial D} u \cdot \partial_\nu z_g \, ds = \int_{\partial D} (\partial_\nu u \cdot z_g - u \cdot \partial_\nu z_g) \, ds = \int_{\partial D} (\partial_\nu \tilde{u} \cdot z_g - \tilde{u} \cdot \partial_\nu z_g) \, ds.
\]
Since \( \tilde{u} \) and \( z_g \) are harmonic in \( 1 - \delta' < |x| < 1 \), one has the expression
\[
\int_{\partial D} (\partial_\nu \tilde{u} \cdot z_g - \tilde{u} \cdot \partial_\nu z_g) \, ds = \int_C (\partial_\nu \tilde{u} \cdot z_g - \tilde{u} \cdot \partial_\nu z_g) \, ds.
\]
Thus (1.2) becomes
\[
\int_{\partial \Omega} \partial_\nu w \cdot g \, ds = \int_C (\partial_\nu \tilde{u} \cdot z_g - \tilde{u} \cdot \partial_\nu z_g) \, ds.
\]
It is easy to see that this right-hand side has the bound \( O(\|z_g\|_{H^1(G)}) \). Thus the condition (1.1) yields
\[
\left| \int_{\partial \Omega} \partial_\nu w \cdot g \, ds \right| \leq C \epsilon,
\]
where \( C \) is independent of \( g \). Hence \( I_\epsilon(G) \leq C \epsilon \) and \( I(G) = \lim_{\epsilon \downarrow 0} I_\epsilon(G) = 0 \).

2 Looking at the example in Section 1 a little more

Let \( u \) be the solution of (1.0) and \( \tilde{u} \) its harmonic extension to \( \tilde{\Omega} \). In this section \( G \) denotes an arbitrary open subset of \( \Omega \) such that \( \overline{G} \subset \Omega \) and \( \Omega \setminus \overline{G} \) is connected. In this section we prove

**Proposition 2.1.**

(a) If \((0, 0) \in G\), then \( I(G) = 0 \).

(b) If \((0, 0) \notin \overline{G}\), then, for all \( \epsilon \), \( I_\epsilon(G) = \infty \).

**Proof.** First we prove (a). In this case one can find a circle \( S \) centered at \((0, 0)\) such that \( S \subset G \). At this time, the following equation is obtained as in the previous section:
\[
\int_{\partial \Omega} \partial_\nu w \cdot g \, ds = \int_S (\partial_\nu \tilde{u} \cdot z_g = \tilde{u} \cdot \partial_\nu z_g) \, ds.
\]
Note that \( z_g \) is the same as before. Thus this together with (1.2) yield \( I_\epsilon(G) \leq C \epsilon \) with a positive constant \( C \) independent of \( g \). And hence \( I(G) = \lim_{\epsilon \downarrow 0} I_\epsilon(G) = 0 \).
Next we prove (b). For this we claim the identity:

$$\int_{\partial\Omega} \partial_{
u} w \cdot g \, ds = -2\pi \nabla z_g(0,0) \cdot e_1,$$

(2.1)

where $e_1 = (1, 0)^T$.

First of all admit equation (2.1) and move on. Consider the case $(0,0) \notin \overline{G}$. One can find an open disc $B$ centered at $(0,0)$ and radius $t_0$ such that $\overline{B} \subset \Omega \setminus \overline{G}$. Let $B_t = \{x \in \mathbb{R}^2 \mid |x| < t\}$ with $0 < t < t_0$. Since the function

$$E_t(x) = \log |x - te_1|$$

is harmonic in a neighbourhood of $\overline{G} \cup \overline{B_{t/2}}$, the Runge approximation property yields: there exists a sequence $\{g_j\}$ such that

$$\lim_{j \to \infty} \|z_{g_j} - E_t\|_{H^1(G \cup B_{t/2})} = 0.$$  (2.2)

Then an interior regularity estimate yields $z_{g_j}$ together with its all derivatives converges to $E_t$ and the corresponding derivatives compact uniformly in $B_{t/2}$. Thus (2.1) yields

$$\lim_{j \to \infty} \int_{\partial\Omega} \partial_{
u} w \cdot g_j \, ds = \frac{2\pi}{t}.$$  (2.3)

Note also that we have

$$\lim_{j \to \infty} \|z_{g_j}\|_{H^1(G)} = \|E_t\|_{H^1(G)}.$$  (2.4)

Given $\epsilon > 0$ define

$$\tilde{g}_j = \frac{\epsilon}{2\|E_t\|_{H^1(G)}} g_j.$$  (2.5)

Since the map $g \mapsto z_g$ is linear, we have

$$\|z_{\tilde{g}_j}\|_{H^1(G)} = \frac{\epsilon}{2\|E_t\|_{H^1(G)}} \|z_{g_j}\|_{H^1(G)} < \epsilon$$

for all $j \gg 1$.

And (2.3) gives

$$\lim_{j \to \infty} \int_{\partial\Omega} \partial_{
u} w \cdot \tilde{g}_j \, ds = \frac{2\pi}{t} \cdot \frac{\epsilon}{2\|E_t\|_{H^1(G)}}.$$  (2.6)

Since $\overline{B} \cap \overline{G} = \emptyset$, Lebesgue's dominated convergence theorem gives $\lim_{t \downarrow 0} \|E_t\|_{H^1(G)} = \|E_0\|_{H^2(G)} < \infty$. Thus the right-hand side on (2.6) blows up as $t \downarrow 0$. This yields $I_\epsilon(G) = \infty$.

□

Remarks.

(i) The case $(0,0) \in \partial G$ seems delicate (at the present time).

(ii) This type of sequence satisfying (2.2) has been used in the probe method [2] which aims at reconstructing unknown discontinuities such as cavities, inclusions and cracks. However, the probe method employs the Dirichlet-to-Neumann map, i.e., infinitely many pairs of the Cauchy data of the governing equation. Instead in the proof of (b) a single pair of Cauchy data is fixed and sequences $z_{g_j}$ produced by infinitely many $g_j$ are used as test functions.

(iii) The choices of $\{g_j\}$ in two cases (a) and (b) are different. Since we do not know the position of $(0,0)$ in advance, we have the question: what is the good choice of $\{g_j\}$ common to two cases. This is also a problem about the no response test.
2.1 Proof of (2.1)

Same as before, we have, for all circles $S_\eta$ centered at (0, 0) with radius $\eta \in ]0, 1[

$$\int_{\partial \Omega} \partial_\nu w \cdot g \, ds = \int_{S_\eta} (\partial_\nu \tilde{u} \cdot z_g - \tilde{u} \cdot \partial_\nu z_g) \, ds.$$ 

We compute the limit of this right-hand side as $\eta \downarrow 0$.

First we have

$$\int_{S_\eta} \partial_\nu \tilde{u} \cdot z_g \, ds = \left(1 - \frac{1}{\eta^2}\right) \eta \int_0^{2\pi} \cos \theta \cdot z_g(\eta \cos \theta, \eta \sin \theta) \, d\theta$$

$$= -\left(1 - \frac{1}{\eta^2}\right) \eta \int_0^{2\pi} \sin \theta \cdot \frac{d}{d\theta} \{z_g(\eta \cos \theta, \eta \sin \theta)\} \, d\theta$$

$$= -\left(1 - \frac{1}{\eta^2}\right) \eta^2 \int_0^{2\pi} \sin \theta \cdot \nabla z_g(\eta \cos \theta, \eta \sin \theta) \cdot (\cos \theta, \sin \theta)^T \, d\theta$$

$$\rightarrow \int_0^{2\pi} \sin \theta \cdot \nabla z_g(0, 0) \cdot (\cos \theta, \sin \theta)^T \, d\theta$$

$$= -\pi \nabla z_g(0, 0) \cdot e_1.$$ 

Second we have

$$\int_{S_\eta} \tilde{u} \cdot \partial_\nu z_g \, ds$$

$$= (\eta^2 + 1) \int_0^{2\pi} \cos \theta \cdot \nabla z_g(\eta \cos \theta, \eta \sin \theta) \cdot (\cos \theta, \sin \theta)^T \, d\theta$$

$$\rightarrow \pi \nabla z_g(0, 0) \cdot e_1.$$ 

This completes the proof.

3 One cannot apply Fatou’s lemma

The key point of their argument on page 5 is the definiteness of the signature of $\partial_{\nu_x} F_\alpha(x, y)$ for $x \in N_{y_0} \cap \partial D$ and $y \rightarrow y_0$ along the axis of the cylinder $N_{y_0}$. Here we give an example of $D$ that does not ensure this property.

Let $D$ be a bounded domain and in $x_3 < 0$. We assume that $y_0 = (0, 0, 0) \in \partial D$ and $N_{y_0} \cap \partial D$ is flat and included in the plane $x_3 = 0$. Thus $\nu_x = \nu_{y_0} = e_3$.

Let $E(x) = \frac{1}{|x|}$. We have

$$\partial_3 E(x) = -\frac{x_3}{|x|^3}$$

and

$$\partial_3^2 E(x) = \frac{1}{|x|^5} (3x_3^2 - |x|^2).$$
Since \( a = \nu_{y_0} = e_3 \), we have, for all \( x \in N_{y_0} \cap \partial D \) and \( y = (0, 0, y_3) \) with \( 0 < y_3 << 1 \)

\[
\partial_{\nu_x} F_a(x, y) = -\partial^2_{x} E(x - y)
\]

and thus

\[
\partial_{\nu_x} F_a(x, y) = -\frac{1}{|x - y|^3}(2y_3^2 - x_1^2 - x_2^2).
\]

Therefore we have

(i) if \( x_1^2 + x_2^2 < 2y_3^2 \), then \( \partial_{\nu_x} F_a(x, y) < 0 \);

(ii) if \( x_1^2 + x_2^2 > 2y_3^2 \), then \( \partial_{\nu_x} F_a(x, y) > 0 \).

Thus as \( y_3 \downarrow 0 \) the sign of the function \( \partial_{\nu_x} F_a(x, y) \) of \( x \in N_{y_0} \cap \partial D \) can not have a definite sign.

This implies, one can not apply Fatou’s lemma as done (3.4) in this simplest case.

4 Another reason of invalidness of (3.5) on page 5: A heuristic explanation

Even general case one can not obtain (3.5). Its heuristic explanation is the following.

Since \( a = \nu_{y_0} \), if \( x \in N_{y_0} \cap \partial D \) we expect

\[
\partial_{\nu_x} F_a(x, y) \sim -\partial^2_{\nu_{y_0}} E(x - y).
\]

However, \( E \) satisfies the Laplace equation we have

\[
\partial^2_{\nu_{y_0}} E(x - y) = -(\partial^2_{x_1} + \partial^2_{x_2}) E(x - y),
\]

where \( x_1 \) and \( x_2 \) are tangential directions at \( y_0 \). Thus we can expect

\[
\partial_{\nu_x} F_a(x, y) \sim (\partial^2_{x_1} + \partial^2_{x_2}) E(x - y).
\]

Then the integral

\[
\int_{N_{y_0} \cap \partial D} u(x) \cdot \partial_{\nu_x} F_a(x, y) ds(x)
\]

may become

\[
\sim \int_{N_{y_0} \cap \partial D} u(x) \cdot (\partial^2_{x_1} + \partial^2_{x_2}) E(x - y) ds(x).
\]

Then applying integration by parts to this right-hand, one can reduce the singularity of integrand twice and gets an integral and additional terms which are bounded as \( y \to y_0 \).

5 Some comments on references

In [3] (1999!) using a single set of the Cauchy data, we have already given the reconstruction formula of the convex hull of unknown polygonal cavity \( D \) and done its numerical testing in [5].

The method developed in this paper is called the enclosure method and based on the asymptotic behaviour of the integral with respect to a large parameter \( \tau \)

\[
\int_{\Omega} \partial_{\nu} w g ds,
\]

where \( g = e^{\tau x \cdot (\omega + i\omega^\perp)} \) with two unit vectors \( \omega \) and \( \omega^\perp \) perpendicular each other. Note that in this case \( z_g(x) = e^{\tau x \cdot (\omega + i\omega^\perp)} \).
Besides, in the case when $\Omega$ is an ellipse, even though the homogeneous background is unknown, the enclosure method works and yields a reconstruction formula of the convex hull of the union of the polygonal cavity and the focal points of $\Omega$ by using a single flux corresponding to a band-limited surface potential [4].

These informations are missed in [6].

6 Extendability

The point is the extendability of the potential $u$ from $\Omega \setminus \overline{D}$ across $\partial D$ into $D$, for example, if $\partial D$ is a real analytic surface, then by applying the Cauchy-Kovalevskaia theorem one has such an extension locally. In this case, we can prove that, by doing the procedure above locally around $y_0 \in \partial D \setminus \mathcal{G}$ on page 5 in [6], (3.5) in [6] is not valid. The enclosure method in [3] catches a corner where one can not have an extension of the potential (due to Friedman-Isakov’s extension argument [1] under the condition $\text{diam } D < \text{dist } (D, \partial \Omega)$).

So at least we have to find an argument that employs explicitly the impossibility of applying the Cauchy-Kovalevskaia theorem on $\partial D$.

7 Conclusion

The problem is not simple and still unsolved! I guess the complete version of the no response test with a single measurement tells us the limit of the extension of the solution (continuation as a solution of the governing equation). Proposition 2.1 is an evidence of this belief.

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