MULTIPLICATIVE MODELS FOR CONFIGURATION SPACES OF ALGEBRAIC VARIETIES

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Abstract. W. Fulton–R. MacPherson [15] found a Sullivan dg-algebra model for the space of $n$-configurations of a smooth compact complex algebraic variety $X$. I. Krž [16] gave a simpler model, $E_n(H)$, depending only on the cohomology ring, $H := H^*X$.

We construct an even simpler and smaller model, $J_n(H)$. We then define another new dg-algebra, $E_n(\hat{H})$, and use $J_n(H)$ to prove that $E_n(\hat{H})$ is a model of the space of $n$-configurations of the non-compact punctured manifold $\hat{X}$, when $X$ is 1-connected. Following an idea of V.G. Drinfel'd [10], we put a simplicial bigraded differential algebra structure on \{\(E_n(\hat{H})\)\}_{n \geq 0}.

1. Introduction and the main results

Let $X$ be a connected space. The topology of ordered configuration spaces

$$F(X,n) := \{(x_1, \ldots, x_n) \in X^n; \ x_i \neq x_j \text{ if } i \neq j\}$$

of $n$ distinct labeled points in $X$ has attracted a considerable attention, over the years.

The cohomology rings $H^*F(\mathbb{R}^2,n)$ have been described by Arnold [1]. In his 1972 thesis, F. Cohen extended Arnold’s computations to all Euclidean spaces; see [7]. For $X$ an $l$-dimensional real oriented manifold, the Leray spectral sequence of the inclusion $F(X,n) \hookrightarrow X^n$ has been described by Cohen-Taylor [8] and further analyzed by Totaro [21]. With field coefficients $\mathbb{K}$, the above Cohen-Taylor spectral sequence converges multiplicatively to $H^*(F(X,n);\mathbb{K})$; it has the property that $E_2 = E_2$, and the differential graded algebra $(E_d, d)$ depends only on $n$ and the cohomology algebra $H^*(X;\mathbb{K})$. See [8], [21]. If $X$ is a smooth projective complex $m$-variety, Totaro [21] showed, over $\mathbb{Q}$, that $E_{2m+1} = E_{\infty}$, and $H^*(F(X,n);\mathbb{Q}) = E_{2m+1}$, as graded algebras.

For a compact oriented real $l$-manifold $X$, it is convenient to view $F(X,n)$ as $X^n \setminus D^nX$, where $D^nX$ denotes the fat diagonal. Using Lefschetz duality, one may thus replace the Betti numbers of $F(X,n)$ by those of the pair $(X^n, D^nX)$, modulo suitable regrading. In this way, Brown and White [6] were able to compute the Betti numbers of $F(X,n)$ in terms of the cohomology algebra of $X$, for $n \leq 3$. For arbitrary $n$, Bendersky and Gitler have constructed in [4] another spectral sequence, converging additively to $H^*(F(X,n);\mathbb{K})$, regraded via Lefschetz duality. They have also proved that $E_2 = E_\infty$ in their spectral...
sequence over $\mathbb{Q}$, when $X$ is rationally formal, in the sense of Sullivan [20] (for instance, when $X$ is compact Kähler manifold, see [9]).

As it turns out, there is an additive isomorphism between the Bendersky-Gitler $E_2$-term and the Cohen-Taylor $E_{l+1}$-term, after regrading (this was proved independently in [17, Theorem 29], and in [13, Theorem 1]). Therefore, the additive part of Totaro’s collapsing result actually holds for $\mathbb{Q}$-formal closed oriented manifolds. Nevertheless, the above spectral sequences for $F(X, 4)$ do not collapse in general, as follows from the example given by Félix and Thomas [13], where $X$ is the sphere tangent bundle of $S^2 \times S^2$.

Our aim in this paper is to go beyond Betti numbers and cohomology algebras. We will describe two new differential graded algebra (DGA) models, for $\mathbb{K}$-homotopy types in characteristic zero (in the sense of Sullivan [20]) of configuration spaces, $F(X, n)$ and $F(\tilde{X}, n)$. Here $X$ is a smooth compact complex algebraic $m$-variety, and $\tilde{X} := X \setminus \text{pt}$ denotes the punctured manifold.

Let $Y$ be a smooth complex algebraic variety. It is known that $Y$ has a convenient compactification. That is, $Y = \tilde{Y} \setminus D$, where $\tilde{Y}$ is smooth compact, and $D \subset \tilde{Y}$ is a divisor with normal crossings. A basic result of Morgan [18, Theorem 9.6 and Corollary 9.7] says then that the $\mathbb{C}$-homotopy type of $Y$ is naturally determined by the $\mathbb{C}$-cohomology algebras of the various intersections of components of $D$, together with the restriction and Gysin maps between them. The same holds true (non-naturally) over $\mathbb{Q}$; see [18, Theorem 10.1].

Consider now $Y = F(X, n)$, where $X$ is a smooth compact complex algebraic variety. In a seminal paper, Fulton and MacPherson [15] constructed a particularly nice compactification $\tilde{Y}$ of this space, in the above sense. Applying Morgan’s theory, they succeeded to describe a DGA model for the $\mathbb{Q}$-homotopy type of $F(X, n)$, depending on $n$, $H^*(X; \mathbb{Q})$, and the Chern classes of $X$. This model was algebraically simplified by Krž [16], whose $\mathbb{Q}$-model for $F(X, n)$ depends only on $n$ and the algebra $H^*(X; \mathbb{Q})$. To describe our models, we begin by introducing a construction which abstracts the key features of the Krž model.

Let $A$ be a unital graded commutative algebra over a field $\mathbb{K}$, $m \geq 1$ a fixed number and $\nabla \in A \otimes A$ a degree $2m$ class which is graded symmetric:

\[(1) \quad T(\nabla) = \nabla,\]

where $T(a \otimes b) := (-1)^{\deg(a) \cdot \deg(b)} (b \otimes a)$ is the graded flip, and which is ‘diagonal’ in the sense that

\[(2) \quad (a \otimes 1) \nabla = (1 \otimes a) \nabla,\]

for any $a \in A$. For example, if $A = H^*(X; \mathbb{K})$, where $X$ is an oriented $2m$-dimensional real manifold, then the diagonal class $\nabla \in H^{2m}(X \times X; \mathbb{K})$ will satisfy (1) and (2) above. Given a subset \( \{i_1 < \ldots < i_k\} \subset \{1, \ldots, n\} \), denote by

\[\iota_{i_1, \ldots, i_k} : A^\otimes k \hookrightarrow A^\otimes n\]

the obvious inclusion that puts the $s$-th factor of $A^\otimes k$ into the $i_s$-th slot of $A^\otimes n$. More generally, for an arbitrary, not necessarily linearly ordered subset \( \{j_1, \ldots, j_k\} \subset \{1, \ldots, n\} \),
there is a unique permutation \( \sigma \in \Sigma_k \) such that \( i_s := j_{\sigma^{-1}(s)}, 1 \leq s \leq k \), is linearly ordered.

We will put
\[
\iota_{i_1, \ldots, i_k} := \iota_{i_1, \ldots, i_k} \circ T_{\sigma},
\]
where \( T_{\sigma} : A^\otimes k \to A^\otimes k \) is the canonical automorphism induced by \( \sigma \). Later in the paper we will also need
\[
\iota_\varphi := \iota_{\varphi(1), \ldots, \varphi(k)} : A^\otimes k \hookrightarrow A^\otimes n,
\]
where \( \varphi : \{1, \ldots, k\} \to \{1, \ldots, n\} \) is an injective map. Set \( \nabla_{ij} := \iota_{ij}(\nabla) \in A^\otimes n \), for \( 1 \leq i \neq j \leq n \).

**Definition 1.** Let \( E_n(A, \nabla) \) be the free graded commutative \( A^\otimes n \)-algebra \( A^\otimes n[G_{ij}] \) with degree \( 2m - 1 \) exterior generators \( G_{ij}, n \geq i > j \geq 1 \), modulo the following relations:
\[
\begin{align*}
G_{ij}G_{ik} &= G_{jk}(G_{ik} - G_{ij}), \quad \text{for } n \geq i > j > k \geq 1, \quad \text{and} \\
\iota_i(x)G_{ij} &= \iota_j(x)G_{ij}, \quad \text{for } x \in A, \ n \geq i > j \geq 1,
\end{align*}
\]
with the differential \( d \) given by \( d(\iota_i(x)) := 0 \) for \( n \geq i \geq 1 \) and \( dG_{ij} := \nabla_{ij} \) for \( n \geq i > j \geq 1 \).

We leave to the reader to verify that symmetry (1) guarantees the compatibility of \( d \) with relations (4), while (2) is necessary for \( d \) to be compatible with (5).

From now on, \( H \) will denote an arbitrary \( 2m \)-dimensional Poincaré duality algebra over a field \( \mathbb{K} \), with a distinguished orientation class \( \omega \in H^{2m} \setminus \{0\} \). Let \( \Delta \in H^\otimes 2 \) be the class of the diagonal,
\[
\Delta := \sum_\alpha (-1)^{\deg(h_\alpha)} h_\alpha \otimes h_\alpha^*,
\]
where \( \{h_\alpha\} \) is a homogeneous \( \mathbb{K} \)-basis of \( H^* \), comprising 1, and \( \{h_\alpha^*\} \) is the Poincaré dual basis with respect to \( \omega \). The couple \( (H, \Delta) \) then clearly fulfills both (1) and (2). The Kržíz-model \( E_n(H) \) is defined by \( E_n(H) := E_n(H, \Delta) \). When \( H = H^*(X; \mathbb{K}) \), where \( X \) is a closed oriented real \( 2m \)-manifold, note also that \( (E_n(H), d) \) coincides with the DGA \( (E_{2m}, d_{2m}) \) coming from the Cohen-Taylor spectral sequence for \( F(X, n) \); see [21].

**Theorem 2** (Fulton-MacPherson [15], Krží [16]). Let \( X \) be a smooth compact complex algebraic variety and \( H = H^*(X; \mathbb{Q}) \). Then the DGA \( (E_n(H), d) \) is a rational model, in the sense of Sullivan, of the configuration space \( F(X, n) \).

Let \( (\Delta^n) := (\Delta_{2,1}, \Delta_{3,1}, \ldots, \Delta_{n,1}) \) denote the ideal generated in \( H^\otimes n \) by the diagonals \( \Delta_{2,1}, \Delta_{3,1}, \ldots, \Delta_{n,1} \). Let \( \pi : H^\otimes n \to H^\otimes n/(\Delta^n) \) be the projection and \( j_i : H \to H^\otimes n/(\Delta^n) \) the composition \( \pi \circ \iota_i, n \geq i \geq 1 \). Our first DGA model, \( J_n(H) \), is basically the Kržíz-model \( E_n(H) \) with \( H^\otimes n \) replaced by \( H^\otimes n/(\Delta^n) \) and \( n - 1 \) generators, \( G_{n1}, G_{n-1,1}, \ldots, G_{21} \), missing:

**Definition 3.** The model \( J_n(H) \) is defined to be the free graded commutative \( H^\otimes n/(\Delta^n) \)-algebra \( H^\otimes n/(\Delta^n)[G_{ij}] \) with degree \( 2m - 1 \) exterior generators \( G_{ij}, n \geq i > j \geq 2 \), modulo the relations:
\[
\begin{align*}
G_{ij}G_{ik} &= G_{jk}(G_{ik} - G_{ij}), \quad \text{for } n \geq i > j > k \geq 2, \quad \text{and} \\
j_i(x)G_{ij} &= j_j(x)G_{ij}, \quad \text{for } x \in H, \ n \geq i > j \geq 2.
\end{align*}
\]
The differential is determined by $d(j_i(x)) := 0$ for $n \geq i \geq 1$ and $dG_{ij} := \pi(\Delta_{ij})$ for $n \geq i > j \geq 2$.

Observe that $J_n(H)$ is not of the form $E_n(A, \nabla)$, but it is very close to it. Though the ‘coefficients’ $H^\otimes n/(\Delta^n)$ of $J_n(H)$ seem more complicated than the coefficients $H^\otimes n$ of the Kříž-model, we will see, in Proposition 11, that the structure of ‘coefficients’ $H^\otimes n/(\Delta^n)$ is actually very simple. The projection $\pi$ clearly generates an epimorphism $\Psi : E_n(H) \rightarrow J_n(H)$ of DGA-algebras with $\Psi(G_{ij}) := G_{ij}$ if $n \geq i > j \geq 2$ and $\Psi(G_{ji}) := 0$, $n \geq j > 2$. Our first main result in this paper, whose proof we postpone to Section 3, reads:

**Theorem 4.** The natural map $\Psi : E_n(H) \rightarrow J_n(H)$ induces a cohomology isomorphism, for an arbitrary even-dimensional Poincaré duality algebra $H$. Therefore, $J_n(H)$ is also a DGA model, in the sense of Sullivan, of the configuration space $F(X, n)$, for $X$ and $H$ as in Theorem 2.

Let $H$ be an arbitrary even-dimensional Poincaré duality algebra, as before. We are going to consider another associated DGA, to be denoted by $E_n(\hat{H})$. To begin with, let $\hat{H}$ be the quotient algebra, $\hat{H} := H/K \cdot \omega$, that is,

$$\hat{H}^i = \begin{cases} H^i, & \text{for } 0 \leq i < 2m, \text{ and} \\ 0, & \text{for } i \geq 2m, \end{cases}$$

with multiplication induced from $H$. Notice that, when $H = H^*(X; \mathbb{K})$, with $X$ a closed oriented real $2m$-manifold, $\hat{H}$ is nothing else but the cohomology algebra of the non-compact punctured manifold $\hat{X} = X \setminus \text{pt}$. Denote by $\hat{\Delta}$ the image of $\Delta$ in $\hat{H}^\otimes 2$. Plainly, conditions (1) and (2) are satisfied by $(\hat{H}, \hat{\Delta})$.

**Definition 5.** The punctured Kříž-model is the differential graded commutative algebra $E_n(\hat{H}) := E_n(\hat{H}, \hat{\Delta})$, with differential denoted by $\hat{d}$.

There is a very intimate relation between the $J$-model from Definition 3 and the punctured Kříž-model, which can be described as follows. Observe first that $H$ is naturally augmented, via an augmentation $\epsilon : H \rightarrow \mathbb{K}$, which makes $\mathbb{K}$ a right $H$-module. Observe also that the map $i_1 : H \rightarrow H^\otimes n$ induces a left differential $(H, 0)$-module structure on $J_n(H)$. The following useful fact will be proved in Section 3:

**Proposition 6.** For any $n \geq 1$, there is an isomorphism of differential graded commutative algebras

$$(E_{n-1}(\hat{H}), \hat{d}) \cong \mathbb{K} \otimes_{(H, 0)} (J_n(H), d).$$

In Section 4 we will derive from Theorem 4 and Proposition 3 the second main result of our paper, which may be viewed as an analog of the fundamental Theorem 2 in a non-compact situation.

**Theorem 7.** Let $X$ be a 1-connected smooth compact complex algebraic variety. Set $H = H^*(X; \mathbb{C})$. Then the differential graded algebra $(E_n(\hat{H}), \hat{d})$ is a $\mathbb{C}$-model, in the sense of Sullivan, of the configuration space $F(X \setminus \text{pt}, n)$.
Our proof entails a careful analysis of the natural fibration
\[ F(X, n - 1) \hookrightarrow F(X, n) \xrightarrow{p} X \ni \text{pt.} \]
Here \( p \) is the projection onto the first coordinate, which is induced by an algebraic map
defined on the Fulton-MacPherson compactification of \( F(X, n) \), see [15]. A key step involves
naturality properties from Morgan’s theory [18], which explains our need to use \( \mathbb{C} \) instead
of \( \mathbb{Q} \) coefficients.

The strength of Theorem 7 is illustrated by Example 18: in the simplest case, corresponding
to \( X = \mathbb{CP}^1 \), we very easily recover the formality of the classifying space \( F(\mathbb{R}^2, n) \) of the
pure braid group, as well as Arnold’s description of its cohomology.

The rich geometry of configuration spaces of manifolds has a natural algebraic analogue,
at the level of \( E \)-models. We will illustrate this principle in \( \S 5 \). In Proposition 35, we endow
the collection \( \{ E_n(\tilde{H}) \}_{n \geq 0} \) with a simplicial structure, in the category of bigraded differential
algebras (DBGA’s); this is based on Drinfel’d’s [10] cosimplicial group structure on Artin
pure braid groups, and on the doubling operations on chord diagrams from the theory of
Vassiliev invariants of links [3]. We also define a coaction map, relating the cohomology of
little cubes to \( \{ E_n(\tilde{H}) \}_{n \geq 0} \), in Proposition 37. It should be pointed out that both above-
mentioned extra structures on DGA models exist
only for punctured models; see Remarks 36 and 38.

2. Hints for applications

The algebra \( E_n(A, \nabla) \) introduced in Definition 1 is not free as a graded commutative
algebra, but it can be presented as a direct sum of free \( A^{\otimes n-k} \)-modules, \( 0 \leq k \leq n - 1 \). To
formulate this statement more precisely, we need the following notation.

For a sequence \( 1 \leq i_1 < \cdots < i_k \leq n \), let \( 1 \leq h_1 < \cdots < h_{n-k} \leq n \) be its ‘complement’,
that is, a sequence such that \( \{ i_1, \ldots, i_k, h_1, \ldots, h_{n-k} \} = \{ 1, \ldots, n \} \). Set
\[ \iota_{i_1, \ldots, i_k}^c := \iota_{h_1, \ldots, h_{n-k}} : A^{\otimes n-k} \to A^{\otimes n}. \]
The following statement is Proposition 2.1 of [5].

**Proposition 8** (Bezrukavnikov [5]). The linear map
\[ \Xi : \bigoplus_{0 \leq k \leq n-1} \bigoplus_{i_k} A^{\otimes n-k} \cdot G_{i_1j_1} G_{i_2j_2} \cdots G_{i_kj_k} \to E_n(A), \]
where \( I_k := \{ 2 \leq i_1 < i_2 < \cdots < i_k \leq n, \ i_1 > j_1 \geq 1, \ldots, i_k > j_k \geq 1 \} \), given by
\[ \Xi(h \cdot G_{i_1j_1} G_{i_2j_2} \cdots G_{i_kj_k}) := \iota_{i_1, \ldots, i_k}^c(h) \cdot G_{i_1j_1} G_{i_2j_2} \cdots G_{i_kj_k}, \]
is an isomorphism of graded vector spaces.

Let now \( H \) be an even-dimensional Poincaré duality algebra over a field \( K \). The direct
sum (5) with the induced differential can be therefore understood as an alternative description
of \( E_n(H) \) or \( E_n(\tilde{H}) \) (depending on whether \( (A, \nabla) \) is \( (H, \Delta) \) or \( (\tilde{H}, \tilde{\Delta}) \)), accessible by
methods of linear algebra.
Example 9. For $n = 3$, the direct sum decomposition of $E_3(H)$ equals:

\[
\begin{align*}
H \otimes^3 & \rightarrow H \otimes^2 \cdot G_{31} \\
& \quad \quad \rightarrow \Delta_{31} \\
H \otimes^2 \cdot G_{31} & \rightarrow H \otimes^2 \cdot G_{21} \\
& \quad \quad \rightarrow \Delta_{21} \\
H \otimes^2 \cdot G_{21} & \rightarrow H \cdot G_{21} G_{31} \\
& \quad \quad \rightarrow -\Delta_{31} \\
H \cdot G_{21} G_{31} & \rightarrow H \cdot G_{21} G_{32} \\
& \quad \quad \rightarrow -\Delta_{32} \\
H \cdot G_{21} G_{32} & \rightarrow H \otimes^2 \cdot G_{32} \\
& \quad \quad \rightarrow \Delta_{32} \\
H \otimes^2 \cdot G_{32} & \rightarrow H \otimes^2 \cdot G_{31} \\
& \quad \quad \rightarrow \Delta_{31} \\
H \otimes^2 \cdot G_{31} & \rightarrow \Delta_{21} \\
& \quad \quad \rightarrow H \otimes^2 \cdot G_{31} \\
& \quad \quad \rightarrow \Delta_{21}
\end{align*}
\]

where the maps are multiplications by indicated elements (modulo Koszul signs).

One can easily adapt Bezrukavnikov’s proof to obtain the following analog of decomposition (6) also for $J_n(H)$.

Proposition 10. The linear map

\[
\Upsilon : \bigoplus_{0 \leq k \leq n-2} H \otimes^{n-k} / (\Delta^{n-k}) \cdot G_{i_1 j_1} G_{i_2 j_2} \cdots G_{i_k j_k} \longrightarrow J_n(H),
\]

where $J_k := \{3 \leq i_1 < i_2 < \cdots < i_k \leq n, \ i_1 > j_1 \geq 2, \ldots, i_k > j_k \geq 2\}$, given by

\[
\Upsilon(h \cdot G_{i_1 j_1} G_{i_2 j_2} \cdots G_{i_k j_k}) := \tilde{f}_{i_1 \ldots i_k}^c(h) \cdot G_{i_1 j_1} G_{i_2 j_2} \cdots G_{i_k j_k}
\]

is an isomorphism of graded vector spaces. In the above display, $\tilde{f}_{i_1 \ldots i_k}^c : H \otimes^{n-k} / (\Delta^{n-k}) \rightarrow H \otimes^n / (\Delta^n)$ is the map induced by $\iota_{i_1 \ldots i_k}^c$.

The left hand side of (7) provides an alternative description of $J_n(H)$ as a direct sum of free $H \otimes^{n-k} / (\Delta^{n-k})$-modules. From this perspective, the following proposition (to be proved in Section 3) is very useful.

Proposition 11. The composition

\[
H \otimes \hat{H} \otimes^{l-1} \hookrightarrow H \otimes^{l} \xrightarrow{\pi} H \otimes^{l} / (\Delta^l)
\]

induces, for any $l \geq 2$, an isomorphism of graded left $H$-modules, where $H$ acts on the first position on both $H \otimes \hat{H} \otimes^{l-1}$ and $H \otimes^{l}$.

Remark 12. Note that all three models, $E_n(H)$, $E_n(\hat{H})$ and $J_n(H)$, are actually bigraded differential algebras (DBGAs). The second (exterior) degree is given by the number of $G_{ij}$-factors, and the differentials are homogeneous, of bidegree $(+1, -1)$. In particular,

\[
H^* F(X, n) = \bigoplus_{p, q \geq 0} H^{p-q} F(X, n),
\]

where $p$ denotes the usual degree and $q$ is the exterior degree (when Theorem 4 applies), and similarly for $\hat{X}$ (when Theorem 7 applies). We can thus define the bigraded Poincaré polynomial

\[
P_{F(X, n)}(s, t) := \sum_{p, q \geq 0} \dim H^{p-q} F(X, n) s^p t^q.
\]
The ordinary Poincaré polynomial is then the specialization \( P_{F(X,n)}(t) := P_{F(X,n)}(1,t) \), and likewise for \( \hat{X} \).

In Examples 13–17, \( X \) is as in Theorem 4, and \( H = H^*(X; \mathbb{Q}) \).

**Example 13.** Let us begin with the simple case of two points (where various methods can be used). Here \( J_2(H) \) is just \( H \otimes \Delta^2 \) with trivial differential, therefore \( F(X,2) \) is a formal space, and

\[
H^*(F(X,2); \mathbb{Q}) = H \otimes \Delta^2.
\]

By Proposition 11, the algebras in the above display are isomorphic, as graded vector spaces, to \( H \otimes oH \), therefore the Poincaré polynomial \( P_{F(X,2)}(s,t) \) of \( F(X,2) \) equals \( P_X(t)(P_X(t) - t^2m) \).

**Example 14.** For three points, the model \( J_3(H) \) reduces to the 2-term complex

\[
H \otimes \Delta^3 / (\Delta_{31}, \Delta_{21}) \xleftarrow{d} \uparrow^{2m-1} H \otimes \Delta^2 / (\Delta_{21}),
\]

where \( \uparrow^{2m-1} \) denotes the suspension iterated \( 2m - 1 \) times and the differential \( d \) is given by

\[
d(h_1 \otimes h_2) = (-1)^{\deg(h_1) + \deg(h_2)}(h_1 \otimes h_2 \otimes 1) \cdot \Delta_{32}, \text{ for } h_1 \otimes h_2 \in H \otimes \Delta^2.
\]

This economical description should be compared with the full Kříž-model for \( F(X,3) \) described in Example 9. Using Proposition 11 we can simplify this complex further to

\[
H \otimes \hat{H} \otimes \Delta^2 \xleftarrow{d} \uparrow^{2m-1} H \otimes \hat{H},
\]

where \( d \) is now given by

\[
d(h \otimes l) = (-1)^{\deg(h) + \deg(l)}[(h \otimes 1 \otimes 1)\Delta_{32} - (h \otimes 1 \otimes l)\Delta_{21} - (-1)^{\deg(h) \deg(l)}(1 \otimes l \otimes h)\Delta_{31}],
\]

for \( h \otimes l \in H \otimes \Delta^2 \). As an exercise, check that \( d(h \otimes l) \) indeed belongs to \( H \otimes \Delta^2 \). Notice also that the complex can be used to compute the full ring structure of the cohomology of \( F(X,3) \), not only its Betti numbers.

**Example 15.** Let \( a, b, c \) and \( d \) be generators of degree 1 and \( V \) a two-dimensional graded vector space concentrated in degree 2 considered as a non-unital algebra with trivial multiplication. Let \( \wedge (\cdot) \) denote the free graded commutative associative algebra functor. Then the rational cohomology algebra of the configuration space \( F(T,3) \) of three points in the two-dimensional torus \( T \) is isomorphic to

\[
\left( \frac{\wedge(a,b,c,d)}{(ab = cd = 0, \ ac = bd)} \oplus V \right) \otimes H^*(T; \mathbb{Q}).
\]

The bigraded Poincaré polynomial \( P_{F(T,3)}(s,t) \) is

\[
(1 + 4t + 3t^2 + 2st^2) \cdot P_T(t) = 1 + 6t + (12 + 2s)t^2 + (10 + 4s)t^3 + (3 + 2s)t^4
\]

and the ordinary Poincaré polynomial \( P_{F(T,3)}(t) \) equals

\[
(1 + 4t + 5t^2) \cdot P_T(t) = 1 + 6t + 14t^2 + 14t^3 + 5t^4.
\]
Example 16. Let $X$ be a Riemann surface of genus $g \geq 2$. Denote by $a_1, \ldots, a_g, b_1, \ldots, b_g$ the standard symplectic basis of $H^1(X; \mathbb{Q})$. Then the kernel of the map $d$ in (8) is the suspension of the ideal generated by $2g^2 + g$ elements,

$$(a_i \otimes a_j + a_j \otimes a_i), \ (b_i \otimes b_j + b_j \otimes b_i), \text{ with } 1 \leq i \leq j \leq g,$$

and $(a_i \otimes b_j + b_j \otimes a_i)$, with $1 \leq i, j \leq g$,

plus one ‘exceptional’ element

$$1 \otimes \omega + 2 \sum_{1}^{g} (a_i \otimes b_j - b_j \otimes a_i) - \omega \otimes 1.$$

The Poincaré polynomial of the kernel of $d$ is $t^3(2g^2 + g + 1 + 2gt)$ and the bigraded Poincaré polynomial of $F(X, 3)$ equals:

$$P_{F(X, 3)}(s, t) = 1 + 6gt + 12g^2t^2 + [8g^3 + (2g^2 + g + 1)s]t^3 + (2g^2 + g + 2gs)t^4.$$ 

Specializing at $s = 1$ gives

$$P_{F(X, 3)}(t) = 1 + 6gt + 12g^2t^2 + (8g^3 + 2g^2 + g + 1)t^3 + (2g^2 + 3g)t^4,$$

which is a formula of [6].

Example 17. One may use the $J$-model to compute the Poincaré polynomial of $F(S^2, n)$, for arbitrary $n$, in the following simple way. Start with $n = 3$, $J_3(H) : H \xrightarrow{\sim} H$ to obtain $H^*(F(S^2, 3)) = \wedge(x_3)$, where $x_3$ is a degree 3 generator, with bigraded Poincaré polynomial $P_{F(S^2, 3)}(s, t) = 1 + st^3$. By induction on $n$, we can find the bigraded Poincaré polynomial

$$P_{F(S^2, n)}(s, t) = (1 + st^3) \prod_{k=2}^{n-2} (1 + kst),$$

thus recovering a classical formula, see [11] Theorem V.7.1 and Corollary V.1.4]. The induction is based on the following easy argument. Filter $J_n(H)$ by $F_0 = J_{n-1}(H)$, $F_1 = J_n(H)$; in the corresponding spectral sequence $E = (E_n, d_n)$ we obtain

$$(E_0, d_0) = (J_{n-1}(H), d) \otimes (\text{Span}_\mathbb{Q}(1, G_{n2}, \ldots, G_{n,n-1}), 0),$$

$$(E_1, d_1) = (H^*(J_{n-1}) \otimes (\text{Span}_\mathbb{Q}(1, G_{n2}, \ldots, G_{n,n-1})), 0),$$

and then $E$ collapses, due to an obvious degree argument.

In Examples 18-21, $X$ is as in Theorem 7 and $H = H^*(X; \mathbb{C})$.

Example 18. Applying Theorem 7 to $X = \mathbb{C}P^1$, for which $\check{H} = \mathbb{C} \cdot 1$, and $\check{d} = 0$, we recover another classical result: the pure braid space $F(\mathbb{R}^2, n)$ is formal, with cohomology ring described by the Arnold relations (4), and bigraded Poincaré polynomial given by

$$P_{F(\mathbb{R}^2, n)}(s, t) = (1 + st)(1 + 2st) \cdots [1 + (n-1)st],$$

for all $n$; see [4], and also [19].
Example 19. For $n = 1$, Theorem 4 reduces to the statement that $\hat{X}$ is a formal space, a result of Avramov [12] (valid for any 1-connected formal closed manifold $X$). For $n = 2$, it follows from isomorphisms (6) that the underlying chain complex of our model $E_2(\hat{H})$ is

$$H^2 \otimes \hat{H} \xleftarrow{\hat{O}} \hat{H} \xrightarrow{d} 2m-1 \hat{H},$$

where $\hat{O}(h) = (-1)^{\deg(h)}(h \otimes 1) \cdot \hat{\Delta}$. We infer that, additively,

$$(9) \quad H^*(\hat{X}, 2; \mathbb{C}) = \hat{O}(\hat{\Delta}) \oplus 2m-1 \text{Ann}(\hat{H}^+) ,$$

where $\text{Ann}(\hat{H}^+) := \{ h \in \hat{H}; hh' = 0, \forall h' \in \hat{H}, \deg(h') > 0 \}$. The full multiplicative structure of $H^*F(\hat{X}, 2)$ can in fact also be described, using decomposition (9).

Example 20. The underlying chain complex of our punctured model $E_3(\hat{H})$ for 3 points takes the shape which we already know from Example 9, namely

$\hat{H} \otimes 3 \xleftarrow{\hat{O}} \hat{H} \otimes 2 \cdot G_{31} \xrightarrow{\hat{O}} \hat{H} \otimes 2 \cdot G_{31} \oplus \hat{H} \otimes 2 \cdot G_{32} \xrightarrow{\hat{O}} \hat{H} \cdot G_{21} G_{31} \xrightarrow{-\hat{O}} -\hat{H} \cdot G_{21} G_{31} \oplus -\hat{H} \cdot G_{21} G_{32}$

or, in a more condensed form,

$$H^3 \otimes \hat{H} \otimes 2 \cdot G_{31} \otimes \hat{H} \otimes 2 \cdot G_{31} \otimes \hat{H} \otimes 2 \cdot G_{32} \xleftarrow{\hat{O}} \hat{H} \cdot G_{21} G_{31} \oplus \hat{H} \cdot G_{21} G_{32} ,$$

where

$$(10) \quad \hat{O}((h_1^3 \otimes 1 \otimes h_3^3) G_{21} + (h_1^2 \otimes h_2^2 \otimes 1)) G_{31} + (h_1^1 \otimes h_2^1 \otimes 1) G_{32}) =$$

$$= (-1)^{\deg(h_1^3) + \deg(h_3^3)} (h_1^3 \otimes 1 \otimes h_3^3) \Delta_{31} + (-1)^{\deg(h_1^2) + \deg(h_2^2)} (h_1^2 \otimes h_2^2 \otimes 1) \Delta_{31} +$$

$$+ (-1)^{\deg(h_1)} (h_1^1 \otimes 1 \otimes h_2^1) \Delta_{31} + (-1)^{\deg(h_1^2) + \deg(h_3^3)} G_{21} +$$

and

$$(11) \quad \hat{O}(h' \cdot G_{21} G_{31} + h'' \cdot G_{21} G_{32}) =$$

$$= - ((-1)^{\deg(h')} (h' \otimes 1 \otimes 1) \Delta_{31} + (-1)^{\deg(h'')} (h'' \otimes 1 \otimes 1) \Delta_{32}) G_{21} +$$

$$+ (-1)^{\deg(h')}(h' \otimes 1 \otimes 1) \Delta_{21} G_{31} + (-1)^{\deg(h'')} (h'' \otimes 1 \otimes 1) \Delta_{21} G_{32} .$$

Let us apply the above three-points punctured model to a 1-connected smooth compact complex algebraic surface $X$. Note that $r := b_2(X) \geq 1$, and that any $b_2 \geq 1$ may be realized, for example by blowing up points in $\mathbb{C}P^2$; see [14, 1.1.1]. Here computations are again easy, since $\hat{H} = \mathbb{C} \cdot 1 \oplus H^2$, and $\Delta = \sum_{i=1}^r x_i \otimes x_i$, where $\{ x_1, \ldots, x_r \}$ denotes a convenient $\mathbb{C}$-basis of $H^2$. 


It is straightforward to see that the 2-cycles of \( E_3(\hat{H}) \), with respect to exterior degree, are given, in (11), by the conditions \( h', h'' \in H^2 \). Likewise, the exterior degree 1-cycles of \( E_3(\hat{H}) \) are given, in (10), by: \( h_1^3, h_1^2, h_2^3 \in H^2 \), if \( r > 1 \). For \( r = 1 \), exceptional 1-cycles of the form

\[
(1 \otimes 1 \otimes h_3^3)G_{21} + (1 \otimes h_2^2 \otimes 1)G_{31} + (h_1^1 \otimes 1 \otimes 1)G_{32},
\]

where \( h_1^1, h_2^2, h_3^3 \in H^2 \) and \( h_1^1 + h_2^2 + h_3^3 = 0 \), must be added. Finally, the bigraded Poincaré polynomial is given by:

\[
P_{F(\hat{X}, 3)}(s, t) = \begin{cases} 
(1 + rt^2)[1 + 2rt^2 + (r^2 - 3)t^4] + st^5[(3r + (3r^2 - 2)t^2] + 2r^2s^2t^8, & \text{for } r > 1, \\
(1 + 3t^2) + st^5(5 + 4t^2) + 2s^2t^8, & \text{for } r = 1.
\end{cases}
\]

**Remark 21.** Even though the natural projection, \( F(\hat{X}, 3) \to \hat{X} \), always has a section (see [11, Lemma II 1.1]), the associated Serre spectral sequence need not collapse. Indeed, for \( X = \mathbb{CP}^2 \), the Poincaré polynomial \( P_X(t) = 1 + t^2 \) does not divide \( P_{F(\hat{X}, 3)}(t) \); see Example 20.

3. Proofs related to the J-model

Let \( H \) be an even-dimensional Poincaré duality algebra over a field \( \mathbb{K} \). In this section we prove Propositions 6 and 11 and Theorem 4. Let us start with Proposition 6, whose proof is the easiest.

**Proof of Proposition 6.** By construction, the differential tensor product \( \mathbb{K} \otimes_{(H, 0)} (J_n(H), d) \) is obtained from the DGA \( (E_n(H), d) \), by first killing \( G_{i1} \) and \( dG_{i1} = \Delta_{i1} \), for \( n \geq i > 1 \), and then also killing \( \iota_1(h) \), for \( h \in H^+ \). It is now easy to check that one obtains in this way a DGA isomorphic to \( (E_{n-1}(\hat{H}), \hat{d}) \).

**Proof of Proposition 11.** Let us show first that the composition

\[
H \otimes \hat{H} \otimes 1 \overset{i}{\to} H \otimes \hat{H} \otimes 1 \overset{\pi}{\to} H \otimes \hat{H} / (\Delta^l)
\]

is an epimorphism. To this end, define the filtration \( \{ F_k := H \otimes k \otimes \hat{H} \otimes l - k \}_{1 \leq k \leq l} \) of \( H \otimes \hat{H} \) and prove, by induction on \( k \), that \( \text{Im}(\pi i) \) contains \( \pi(F_k) \) for each \( 1 \leq k \leq l \). Because \( F_1 = H \otimes \hat{H} / (\Delta^l) \), this would imply the statement.

Because \( F_1 = \text{Im}(i) \), \( \text{Im}(\pi i) \) contains \( \pi(F_1) \). Suppose that we have already proved that \( \text{Im}(\pi i) \supset \pi(F_{k-1}) \), for some \( 2 \leq k < l \), and let \( h \in F_k \setminus F_{k-1} \). We may clearly assume, without loss of generality, that

\[
h = h_1 \otimes \ldots \otimes h_{k-1} \otimes \omega \otimes h_{k+1} \otimes \ldots \otimes h_l.
\]

Note then that

\[
h' := h - \Delta_{k1}(h_1 \otimes \ldots \otimes 1 \otimes \ldots \otimes h_l) \in F_{k-1}.
\]

Since by definition \( \pi(h) = \pi(h') \), the induction gives \( \pi(h) \in \text{Im}(\pi i) \), therefore \( \text{Im}(\pi i) \supset \pi(F_k) \).
Let us prove that the composition (12) is monic. Suppose therefore that \( h \in \text{Ker}(\pi) \), that is
\[
\begin{align*}
  h &= \sum h_1 \otimes \ldots \otimes h_l \in F_1 \cap (\Delta') = \text{Im}(i) \cap (\Delta')
\end{align*}
\]
and prove that then \( h \) actually equals 0. Clearly \( h \) must be of the form
\[
(15) \quad h = \sum_{1 < s \leq l} \Delta_{s1} \cdot (\sum_{1 \leq i \leq p} \varphi_{i}^{(s)} \otimes \ldots \otimes \varphi_{i}^{(s)}) ,
\]
for some \( \Phi^{(s)} := \sum_{1 \leq i \leq p} \varphi_{i}^{(s)} \otimes \ldots \otimes \varphi_{i}^{(s)} \in H^{\otimes l} \). We infer from (13) and (14) that \( \Delta_{s1} \cdot F_k \subset \Delta_{s1} \cdot F_{k-1} + \Delta_{k1} \cdot F_k \), whenever \( 1 < s < l \).

The last possibly nonzero term of (15), say \( \Delta_{q1} \cdot \Phi^{(q)} \), equals, by property (2) of the diagonal,
\[
(16) \quad \Delta_{q1} \cdot (\sum \epsilon \cdot \varphi_{1}^{(q)} \varphi_{q}^{(q)} \otimes \ldots \otimes \varphi_{1}^{(q)} \otimes 1 \otimes \varphi_{q+1}^{(q)} \otimes \ldots \otimes \varphi_{l}^{(q)}) ,
\]
where \( \epsilon \) is an appropriate sign, and it is zero \( \text{mod} \ F_{q-1} \), by assumption. At the same time, the sum from (16) equals, \( \text{mod} \ F_{q-1} \), \( \sum \epsilon \cdot \varphi_{1}^{(q)} \varphi_{q}^{(q)} \otimes \ldots \otimes 1 \otimes \varphi_{q+1}^{(q)} \otimes \ldots \otimes \varphi_{l}^{(q)} \), since \( \Phi^{(q)} \in F_q \).

We conclude that \( \sum \epsilon \cdot \varphi_{1}^{(q)} \varphi_{q}^{(q)} \otimes \ldots \otimes 1 \otimes \varphi_{q+1}^{(q)} \otimes \ldots \otimes \varphi_{l}^{(q)} = 0 \), which implies \( \Delta_{q1} \cdot \Phi^{(q)} = 0 \). The proposition is proved.

**Proof of Theorem 4.** We will work with the Krž–model presented as the direct sum in (6). The main trick is to write \( E_n(H) \) as a bicomplex. Put
\[
E_{pq}^n := \bigoplus H^{\otimes n-(p+q)} \cdot G_{i_1j_1} G_{i_2j_2} \cdots G_{i_{p+q}j_{p+q}} ,
\]
with the sum running over \( i_1, j_1, \ldots, i_{p+q}, j_{p+q} \in I_{p+q} \) as in (3) such that \( \text{card}\{s; \ j_s = 1\} = q \).

Then
\[
E_n(H) = \bigoplus_{0 \leq p+q \leq n-1} E_{pq}^n
\]
and the differential \( d \) clearly splits as \( d = d_1 + d_2 \), where
\[
d_1 : E_{pq}^n \to E_{p,q-1}^n \text{ and } d_2 : E_{pq}^n \to E_{p-1,q}^n.
\]
Let us agree for the rest of this proof to understand by degree the bicomplex degree, that is, elements of \( E_{pq}^n \) have degree \( p+q \).

**Example 22.** The Krž–model \( E_3(H) \) for three points, described explicitly in Example 9, can be organized into a bicomplex whose nontrivial part is:

\[
\begin{array}{ccc}
H \cdot G_{21} G_{31} & \xrightarrow{d_2} & 0 \\
\downarrow d_1 & & \downarrow d_1 \\
H^{\otimes 2} \cdot G_{21} \oplus H^{\otimes 2} \cdot G_{31} & \xrightarrow{d_2} & H \cdot G_{21} G_{32} \\
\downarrow d_1 & & \downarrow d_1 \\
H^{\otimes 3} & \xrightarrow{d_2} & H^{\otimes 2} \cdot G_{32}
\end{array}
\]
Theorem 4 will follow from the following statement.

**Lemma 23.** The columns of the bicomplex \((E^n_{pq}, d)\) are acyclic in positive \(q\)-degrees. The only nontrivial homology of the \(p\)-th column is

\[
H_0(E^n_{ps}, d_1) \simeq \bigoplus_{J_p} H^{\otimes n-p}/(\Delta^{n-p}) \cdot G_{i_1j_1} G_{i_2j_2} \cdots G_{i_pj_p},
\]

where \(J_p\) was introduced in Proposition 10.

Assuming Lemma 23, Theorem 4 immediately follows from the observation that \(J_n(H)\) in presentation (17) is the second term of the obvious spectral sequence related to the bicomplex \((E^n_{pq}, d_1 + d_2)\) and that this sequence is concentrated on the line \(q = 0\). It remains to prove Lemma 23.

Since formula (17) for the 0-th homology is obvious, we need only to prove the acyclicity in positive degrees. This will be done in two steps.

**Step (1)** Let \(T^n_s\) be the extreme left column \(E^n_{0s}\) of \(E^n_{ss}\). We show that all remaining columns \(E^n_{ps}, p \geq 1\), are combinations of complexes \(T^n_s\) with \(s < n\). So it is enough to prove only the acyclicity of \(T^n_s\), for all \(n \geq 1\).

**Step (2)** To prove the acyclicity of \(T^n_s\), we observe that \(T^n_s\) decomposes into the direct sum of two copies of \(T^n_{s-1}\). Using this we reduce the proof of the acyclicity of \(T^n_s\) to the verification that a certain very explicit map is monic.

Let us start with Step (1). For \(n \geq 1\) we denote

\[
(T^n_s, d) := (E^n_{0s}, d_1).
\]

**Claim 24.** For any \(p \geq 1\), the column \((E^n_{ps}, d_1)\) decomposes as

\[
(E^n_{ps}, d_1) = \bigoplus_{J_p} (E^{n-p}_{0s} \cdot G_{i_1j_1} G_{i_2j_2} \cdots G_{i_pj_p}, d_1) \simeq \bigoplus_{J_p} (T^{n-p}_s, d),
\]

where \(J_p\) was defined in Proposition 10.

The claim is obvious, because the differential \(d_1\) by definition does not affect generators \(G_{ij}\) with \(j \geq 2\).

**Example 25.** The complex \((T^n_1, d)\) is just \(H\) with trivial differential. The complex \((T^n_2, d)\) is

\[
H^{\otimes 2} \leftarrow H \cdot G_{21},
\]

with \(d\) given by the multiplication with \(\Delta_{21}\). We see that the right column of the bicomplex \(E^n_{ss}\) described in Example 22 is isomorphic to

\[
(T^n_2, d) \cdot G_{32} \cong \uparrow^{2m-1} (T^n_s, d),
\]

as predicted by Claim 24.
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Let us move to Step (2), that is, prove that the complexes \((T^n_*, d), n \geq 1,\) are acyclic in positive degrees. It follows from the definition that, for \(0 \leq k \leq n - 1,\)

\[ T^n_k = \bigoplus H^{\otimes n-k} \cdot G_{i_1} G_{i_2} \cdots G_{i_k}, \]

where the summation runs over all \(i_1, \ldots, i_k\) with \(2 \leq i_1 < i_2 \cdots < i_k \leq n.\)

Clearly \(T^n_*\) splits into a two-column bicomplex \(T^n_\ast \cong T^n_0 \oplus T^n_1,\) with \(T^n_0\) consisting of summands

\[ H^{\otimes n-q} \cdot G_{i_1} G_{i_2} \cdots G_{i_q}, \]

with \(i_q < n\) and \(T^n_1\) consisting of summands

\[ H^{\otimes n-(q+1)} \cdot G_{i_1} G_{i_2} \cdots G_{i_{q+1}}, \]

of \(T^n_{1+q}\) with \(i_{q+1} = n.\) The differential \(d\) obviously decomposes as \(d = d_1 + d_2,\) where

\[ d_1 : T^n_{pq} \to T^n_{p,q-1} \quad \text{and} \quad d_2 : T^n_{pq} \to T^n_{p-1,q}. \]

The following claim is evident, see also Example 27.

**Claim 26.** One has the following isomorphisms of complexes:

\[ (T^n_0, d_1) \cong (T^{n-1}_*, d) \otimes (H, d = 0) \quad \text{and} \quad (T^n_1, d_1) \cong T^{2m-1}_* \otimes (T^{n-1}_*, d). \]

**Example 27.** The complex \((T^n_4, d)\) splits as:

\[
\begin{array}{ccccc}
H^2 \cdot G_{21} G_{31} & \xrightarrow{d_2} & H \cdot G_{21} G_{31} G_{41} \\
\downarrow d_1 & & \downarrow d_1 \\
H^3 \cdot G_{21} \oplus H^3 \cdot G_{31} & \xrightarrow{d_2} & H^2 \cdot G_{21} G_{41} \oplus H^2 \cdot G_{31} G_{41} \\
\downarrow d_1 & & \downarrow d_1 \\
H^4 & \xrightarrow{d_2} & H^3 \cdot G_{41} \\
\end{array}
\]

We recognize the left column as the left column of the bicomplex in Example 22 tensored with \(H,\) that is, \(T^3_3 \otimes H.\) The right column is \(T^3_\ast \cdot G_{41},\) that is, \(T^{2m-1}_\ast \cdot G_{41}.\) These observations are in perfect harmony with Claim 26.

Assuming inductively that \((T^{n-1}_*, d)\) is acyclic in positive dimensions, the cohomology of \((T^n_\ast, d)\) reduces to the cohomology of a two-term complex

\[ H_0(T^{n-1}_*, d) \otimes H \leftrightarrow T^{2m-1}_0 \otimes H_0(T^{n-1}_*, d) \]

with the differential induced by \(d_2.\) The above map can be easily identified with

\[ H^{\otimes n-1} / (\Delta^{n-1}) \otimes H \leftrightarrow T^{2m-1} H^{\otimes n-1} / (\Delta^{n-1}) \]

where the differential is given by the multiplication with \(\Delta_{n1}.\) So it remains to prove only that the above two-term complex is acyclic at the right term, that is
Claim 28. The map
\[ H^{\otimes n-1}/(\Delta^{n-1}) \to H^{\otimes n-1}/(\Delta^{n-1}) \otimes H \]
sending \([\alpha] \in H^{\otimes n-1}/(\Delta^{n-1})\) to \((\pi \otimes \text{id})((\alpha \otimes 1)\Delta_{n1}) \in H^{\otimes n-1}/(\Delta^{n-1}) \otimes H\) is a monomorphism.

The proof of the claim is simple. Suppose that \(\alpha\) is homogeneous of degree \(s\). The only term in \((\alpha \otimes 1)\Delta_{n1}\) of bidegree \((s, 2m)\) is \(\alpha \otimes \omega\), where \(\omega \in H^{2m}\) is the fundamental class. Therefore \((\pi \otimes \text{id})((\alpha \otimes 1)\Delta_{n1}) = 0\) if and only if \([\alpha] = 0\). This finishes the proof of Lemma 23 and thus also the proof of Theorem 4. □

4. Configuration spaces of punctured manifolds

This section is devoted to the proof of Theorem 7. The bridge between topology and algebra is provided by the natural fibration
\[ F(pX, n-1) \hookrightarrow F(X, n) \to X \ni \text{pt}, \]
where \(p\) is the projection onto the first coordinate. When \(X\) is as in our theorem, (18) above is a fibration of connected spaces, having a 1-connected base with finite Betti numbers. Therefore, the approach initiated by J.C. Moore, to get cochain algebra models for the fiber via differential homological algebra, may be used, in the particularly convenient form of the so-called semifree resolutions introduced by Félix-Halperin-Thomas in [12].

Parts of the theory used below work for arbitrary field coefficients (e.g., the connection between semifree resolutions and fibrations), some other parts (e.g., the theory of Sullivan relative models for fibrations) require characteristic zero field coefficients, while Morgan’s theory of bigraded models for smooth complex varieties from [13, Section 9] needs \(\mathbb{C}\)-coefficients. To simplify matters, we will thus work over \(\mathbb{C}\), throughout this section.

4.1. Semifree resolutions, relative models, and fibrations. We start by recalling some relevant facts from [12]. From now on, all DGA’s \((A, d_A)\) will be tacitly assumed to be homologically connected, that is, \(H^0(A, d_A) = \mathbb{C}\). Given a DGA map, \((B, d_B) \xrightarrow{f} (E, d_E)\), we will assume in §4.1 that \(H^1f\) is monic. The examples we have in mind are
\[ \Omega^*_{dR}(X) \xrightarrow{\Omega^*_{dR}(\cdot)} \Omega^*_{dR}(F(X, n)), \]
coming from the basic fibration [13], via the \(C^\infty\) de Rham DGA functor with complex coefficients, \(\Omega^*_{dR}(\cdot)\). The following definition is taken from [12, §6].

Definition 29. A DGA \((B, d_B)\) is called semifree over \((B, d_B)\) if there exists a graded vector space \(Z^*\) with the following properties:

(i) \(B^* = B^* \otimes Z^*\), as graded vector spaces,
(ii) the graded vector space \(Z^*\) admits a second grading, \(Z^* = \bigoplus_{k \geq 0} Z_k^*\), such that \(Z_0^* = \mathbb{C}\) and \((B \otimes Z_0, d_B) = (B, d_B)\) as DGA’s,
(iii) the product of \(B\) satisfies \(b \otimes z = (b \otimes 1) \cdot (1 \otimes z)\), for \(b \in B\) and \(z \in Z\),
(iv) all subspaces $B \otimes Z_{\leq k}$ are $d_B$-invariant, and, finally
(v) for $k > 0$, the quotient cochain complexes are of the form

$$(B \otimes Z_{\leq k} / B \otimes Z_{<k}, d_B) = (B, d_B) \otimes (Z_k, 0).$$

A DGA map $(B, d_B) \to (E, d_E)$, is said to be a semifree resolution of $f : (B, d_B) \to (E, d_E)$ if $(B, d_B)$ is semifree over $(B, d_B)$, the restriction of $\rho$ to $(B \otimes Z_0, d_B) \equiv (B, d_B)$ equals $f$, and $H^* \rho$ is an isomorphism.

The main source of semifree resolutions is provided by Sullivan relative models of $f$. That is, by DGA’s of the form $(B, d_B) = (B \otimes \wedge W^*, d_B)$, where $W^* = \bigoplus_{k>0} W^k$, and the differential $d_B$ satisfies a certain nilpotence condition, together with a DGA map, $(B, d_B) \to (E, d_E)$, which restricts to $f$ on $B$ and induces a homology isomorphism. If $H^1 f$ is monic, then $f$ has such a relative model, which is a semifree resolution, in the sense of Definition 29. See [12, Proposition 14.3 and Lemma 14.1].

Assume now that $B$ is augmented, by a DGA map $\varepsilon : (B, d_B) \to \mathbb{C}$. In the geometric case from [19], $B^* = O^*_{dR}(X)$ and the augmentation comes from the inclusion, $pt \hookrightarrow X$, via $\Omega^*_{dR}$. Moore’s basic result which relates topology to algebra says that the DGA

$$(\wedge W, d) := \mathbb{C} \otimes_{(B, d_B)} (B, d_B)$$

is a Sullivan model of the fiber, $F(\hat{X}, n-1)$, for any relative model, $(B, d_B)$, of the DGA map $\Omega^*_{dR}(p)$ from [19]. Here, $\mathbb{C}$ is to be considered as a right $(B, d_B)$-module, via $\varepsilon$, and $(B, d_B)$ as a left $(B, d_B)$-module, via the inclusion $(B, d_B) \hookrightarrow (B, d_B)$. See [12, Theorem 15.3 and Theorem 7.10]. Note also that this is the only place where we need the 1-connectivity assumption on $X$.

4.2. Algebraic weak homotopy type and homotopy fiber. Our strategy for proving Theorem 7 is to use the differential tensor product construction from (20) to arrive at the DGA model of $F(\hat{X}, n-1)$ from our theorem. We are thus led to study how this construction depends on its input, that is, on the DGA map $\Omega^*_{dR}(p)$. The next definition comes in naturally.

**Definition 30.** Let $f : B \to E$ and $f' : B' \to E'$ be DGA maps, with $B$ and $B'$ augmented. An elementary weak equivalence, $f \to f'$, consists of a DGA map, $\Phi : E \to E'$, and an augmented DGA map, $\varphi : B \to B'$, such that both $H^* \Phi$ and $H^* \varphi$ are isomorphisms, and such that $\Phi f$ and $f' \varphi$ are Sullivan homotopic DGA maps (notation: $\Phi f \simeq f' \varphi$):
We say that $f$ and $f'$ have the same weak homotopy type (notation: $f \simeq f'$) if there is a chain of DGA maps, $\{f_i : B_i \to E_{i,0}^{1 \leq i \leq l}\}$, as above, together with elementary weak equivalences, $f_i \to f_{i+1}$ or $f_i \leftarrow f_{i+1}$, such that $f_0 = f$ and $f_l = f'$.

**Lemma 31.** Assume $f \simeq f'$, in the sense of the above definition, where $H^1f$ (and consequently also $H^1f'$) are monic. Let $\rho : B = (B \otimes \Lambda W, d_B) \to E$ and $\rho' : B' = (B' \otimes \Lambda W', d_{B'}) \to E'$ be arbitrary relative models, for $f$ and $f'$ respectively. Then the DGA’s $C \otimes (B, d_B)$ and $C \otimes (B', d_{B'})$ have the same Sullivan minimal model.

**Proof.** Plainly, it is enough to treat the case of an elementary weak equivalence, say, $f \to f'$. Suppose first that it is strict, that is, $\Phi f = f' \varphi$ (not only equality up to algebraic homotopy). Consider the pushout construction from [12, § 14(a)]. It provides a DGA map, $\varphi \otimes id : B = (B \otimes \Lambda W, d_B) \to (B' \otimes \Lambda W, d_B') =: \varphi B$, extending $\varphi$ and inducing a homology isomorphism; see [12, Lemma 14.2].

Define a DGA map, $\varphi, \rho : \varphi B \to E'$, which extends $f'$, by setting $\varphi, \rho := \Phi \rho$, on $W$. By construction, $\varphi, \rho$ is a relative model of $f'$ (see [12, § 14(a)]). By [12, Theorem 6.10(ii)], $C \otimes B B$ and $C \otimes B' \varphi B$ have the same minimal model. At the same time ([12, Proposition 14.6]), there is a DGA map, $\Psi : B' \to \varphi B$, extending the identity on $B'$ and such that $\varphi, \rho \circ \Psi \simeq \rho'$. It follows that $\Psi$ induces an isomorphism on homology; hence, by [12, Proposition 6.7(ii)], $C \otimes B B'$ and $C \otimes B' \varphi B$ have the same minimal model. This settles the strict case.

In the general case, the property $\Phi f \simeq f' \varphi$ means the existence of a DGA map, $\mathcal{H} : B \to E' \otimes \Lambda (t, dt)$, such that $e_0 H = \Phi f$ and $e_1 H = f' \varphi$, where $e_i : E' \otimes \Lambda (t, dt) \to E'$ are the DGA maps extending $id_{E'}$, which send $t$ to $i$ and $dt$ to $0$, for $i = 0, 1$.

Pick a relative model of $\mathcal{H}$, $\eta : C = (B \otimes \Lambda T, d_C) \to E' \otimes \Lambda (t, dt)$. Since both $e_0$ and $e_1$ induce homology isomorphisms [12, Lemma 12.5], $e_0 \eta$ is a relative model of $\Phi f$, and $e_1 \eta$ is a relative model of $f' \varphi$. We may now use the strict case to infer that $C \otimes B B$, $C \otimes B' B'$ and $C \otimes B C$ have the same minimal model. \hfill \square

We may rephrase Lemma 31 in the following way. Given a DGA map, $f : B \to E$, with $H^1f$ monic, we may define its homotopy fiber, $F(f)$, to be the minimal model of $C \otimes B B$, where $B$ is any relative model of $f$. If $f \simeq f'$, then $F(f) = F(f')$. In particular, if $\Omega_{\text{IR}}^n(p)$ from [19] has the same weak homotopy type as $f$, then $C \otimes B B$ will be a (not necessarily minimal) model of $F(\hat{X}, n - 1)$, according to our discussion from the end of § 4.4. Recall that $H$ denotes the cohomology algebra $H^*(X; \mathbb{C})$.

**Lemma 32.** Let $\iota_1 : (E_1(H), d) = (H, 0) \to (E_n(H), d)$ be the DGA map induced by the map $\iota_1 : H \to H^\otimes n$ described in Section 1, canonically augmented by $\varepsilon(h) = 0$, for $h \in H^+$, and $\varepsilon(1) = 1$. Then $F(\iota_1)$ is the minimal model of $(E_{n-1}(H), d)$.

**Proof.** According to Theorem 3 and Lemma 31, $F(\iota_1) = F(\Psi \circ \iota_1)$. We claim now that $(J_n(H), d)$ is $(H, 0)$-semifree. Indeed, Propositions 10 and 11 together imply that we may take

$$Z_{k+1} := \bigoplus_{j_k} H^\otimes n-k-1 \cdot G_{i_1j_1} \cdots G_{i_kj_k},$$
for $1 \leq k \leq n - 2$, $Z_1 := (\tilde{H}^\otimes n)\uparrow$ and $Z_0 := \mathbb{C} \cdot 1$. To check the last semifreeness condition from Definition 29, it suffices to recall from Remark 12 that $d$ is homogeneous of degree $-1$, with respect to exterior degree.

Let $\rho : \mathcal{B} \to J_n(H)$ be a relative model of $\Psi \circ \iota_1$. It follows that both $\rho : \mathcal{B} \to J_n(H)$ and $id_{J_n(H)}$ are semifree resolutions of $\Psi \circ \iota_1$. Therefore, $\mathbb{C} \otimes_H \mathcal{B}$ and $\mathbb{C} \otimes_H J_n(H)$ have the same minimal model, by [12, Proposition 6.7(ii)]. To finish the proof of our lemma, it is enough to recall from Proposition 6 that the DGA $\mathbb{C} \otimes_H J_n(H)$ is isomorphic to $E_{n-1}(\tilde{H})$. 

4.3. Replacing forms by $E$-models. Let $X$ be a smooth, compact, complex algebraic variety. Set $H = H^*(X; \mathbb{C})$. We know, from [18] and [15, 16] respectively, that $(H, 0)$ models $\Omega^*_dR(X)$ (the formality property), and $(E_n(H), d)$ models the de Rham DG-algebra $\Omega^*_dR(F(X, n))$. The next proposition links these two things, into the form of a statement about DGA maps. By Lemma 32 and the discussion preceding it, this statement implies Theorem 4.

Proposition 33. The DGA map $\Omega^*_dR(p) : \Omega^*_dR(X) \to \Omega^*_dR(F(X, n))$ has the same weak homotopy type as $\iota_1 : E_1(H) \to E_n(H)$.

The main step in the proof uses Morgan’s results from [18], so we recall his basic constructions. For $Y = \tilde{Y} \setminus D$, a complement of a divisor with normal crossings in a smooth compact complex algebraic variety, Morgan introduced a filtered model, $\mathcal{M}_Y$ (denoted by $E_{C^\infty}(Y)_\mathbb{C}$ in [18, §2]), a DG-algebra of global sections of a sheaf related to $D$. If $Y = \tilde{Y}$, $\mathcal{M}_Y = \Omega^*_dR(Y)$. In general, there is a DGA map,

$$\Omega^*_dR(Y) \xleftarrow{\Phi_Y} \mathcal{M}_Y,$$

inducing an isomorphism in homology; see [18, § 2-3].

Moreover, $\mathcal{M}_Y$ is provided with an increasing filtration $W_*$ induced by the stratification of $D$, and also with the associated increasing filtration, $Dec W_*$, defined in [18, §1]. Morgan’s second construction is the bigraded minimal model, $\mathcal{N}_Y$, of the filtered model $\mathcal{M}_Y$. There is a DGA map,

$$\mathcal{N}_Y = \bigoplus_{n \geq 0} \mathcal{N}^n \xrightarrow{\Phi_Y} \mathcal{M}_Y,$$

inducing an isomorphism in homology. Moreover, each homogeneous component is bigraded: $\mathcal{N}^n = \bigoplus_{r,s \in \mathbb{N}} \mathcal{N}^{r,s}$, $\mathcal{N}^0 = \mathcal{N}^{0,0} = \mathbb{C} \cdot 1$; the differential and the multiplication are homogeneous, of bidegree $(0, 0)$, with respect to the above extra bigrading of $\mathcal{N}_Y$. Finally, the bigrading of $\mathcal{N}_Y$ and the filtration of $\mathcal{M}_Y$ are related by:

$$\Phi(\mathcal{N}^{r,s}) \subset Dec W_{r+s}(\mathcal{M}^n)$$

(see [18, § 6], in particular (6.0) and Theorem 6.6).

Using mixed Hodge diagrams [18, Definition 3.5] and mixed Hodge homotopies [18, § 6.1], Morgan proved the naturality of these two constructions. Given an algebraic map, $f : (\tilde{Y}, Y) \to (\tilde{Z}, Z)$, one can construct a filtered DGA map $\mathcal{M}_f$, a DGA map $\mathcal{N}_f$, which is
homogeneous of bidegree \((0,0)\), with respect to the extra bigradings on minimal models, and also homotopies \(\mathcal{H}\) and \(\mathcal{K}\) which fit into the following diagram:

\[
\begin{array}{c}
\Omega^*_{dR}(Y) \\
\mathcal{H} \\
\Omega^*_{dR}(Z)
\end{array}
\begin{array}{ccc}
\Psi_Y & \mathcal{M}_Y & \Phi_Y \\
\mathcal{M}_f & \mathcal{K} & \mathcal{N}_f \\
\Psi_Z & \mathcal{M}_Z & \Phi_Z \\
\mathbb{C} & \mathbb{C}
\end{array}
\]

(22)

Moreover, \(\mathcal{K} : \mathcal{N}_Z \to \mathcal{M}_Y \otimes \wedge (t, dt)\) has the property that:

\[
\mathcal{K}(\mathcal{N}_Z^{n;r,s}) \subset \text{Dec} W^+(\mathcal{M}_Y \otimes \wedge (t, dt))^n,
\]

where \(W_*(\mathcal{M}_Y \otimes \wedge (t, dt)) := W_*(\mathcal{M}_Y) \otimes \wedge (t, dt)\). See Definitions 3.5 and 3.7, and Propositions 3.6 and 3.9 of [18], for the left square from (22); to construct the right square from (22), use Theorem 6.7 and Corollary 6.8 of [18].

By construction, \(\mathcal{M}_Y\) is a Sullivan model of \(Y\). Let us now consider the decreasing DGA filtration of \(\mathcal{M}_Y\), \(W^* := W_{-*}\), and the associated spectral sequence of DGA’s,

\[
(W^*_{\mathcal{E}_r}(\mathcal{M}_Y), d_r, r \geq 0) \Rightarrow H^*(\mathcal{M}_Y).
\]

A basic result in Morgan’s theory ([18, Theorem 9.6]) says that the DGA \((W^*_{\mathcal{E}_1}(\mathcal{M}_Y), d_1)\) is also a model of \(Y\). We need the following relative version of this result, where \(W^*_{\mathcal{E}_1}(\mathcal{M}_{f})\) denotes the DGA map between \(W^*_{\mathcal{E}_1}(\mathcal{M}_{f})\)–terms induced by the filtered DGA map \(\mathcal{M}_{f}\).

**Proposition 34.** If \(f : (\tilde{Y}, Y) \to (\tilde{Z}, Z)\) is an algebraic map, then the DGA map

\[
\mathcal{M}_{f} : (\mathcal{M}_Z, d) \to (\mathcal{M}_Y, d)
\]

has the same weak homotopy type as the DGA map

\[
W^*_{\mathcal{E}_1}(\mathcal{M}_{f}) : (W^*_{\mathcal{E}_1}(\mathcal{M}_Z), d_1) \to (W^*_{\mathcal{E}_1}(\mathcal{M}_Y), d_1).
\]

**Proof.** Define decreasing DGA filtrations, on both \((\mathcal{N}_Y, d)\) and \((\mathcal{N}_Z, d)\), by \(W^k(\mathcal{N}) := \bigoplus_{r+s \leq n-k} \mathcal{N}^{n;r,s}\). Relation (21) implies that \(\Phi\) is a filtered DGA map: \(\Phi(W^k(\mathcal{N})) \subset W^k(\mathcal{M})\). Obviously, the bihomogeneous DGA map \(\mathcal{N}_{f}\) respects filtrations as well. It is equally easy to check that one has DGA identifications,

\[
(\mathcal{N}_Y, d) \cong (W^*_{\mathcal{E}_1}(\mathcal{N}), d_1),
\]

for both \(Y\) and \(Z\), which give an identification

\[
\mathcal{N}_{f} \cong W^*_{\mathcal{E}_1}(\mathcal{N}_{f}).
\]

We claim that, when applying the \(W^*_{\mathcal{E}_1}(\cdot)\)-functor to the right square from (22), one gets the commutative right square below, in the DGA category.
Note also that the DGA maps \( W_1(\Phi_Y) \) and \( W_1(\Phi_Z) \) induce homology isomorphisms, as follows from Morgan’s proof of [18, Theorem 9.6].

Granting the claim, we may quickly finish the proof of our proposition, as follows. Directly from Definition 30, we deduce from the above diagram elementary weak equivalences,

\[
\Phi_Y : (\mathcal{M}_Y, d) \rightarrow (\mathcal{N}_Y, d) \equiv (W_1(\mathcal{M}_Y), d_1) \rightarrow (W_1(\mathcal{N}_Y), d_1)
\]

\[
\Phi_Z : (\mathcal{M}_Z, d) \rightarrow (\mathcal{N}_Z, d) \equiv (W_1(\mathcal{M}_Z), d_1) \rightarrow (W_1(\mathcal{N}_Z), d_1)
\]

and we are done.

Going back to our commutativity claim, let us start by noting that \( K \) is a filtered DGA map; see (23). We thus get an induced DGA map,

\[
W_0(K) : (W_0(\mathcal{N}_Z), d_0) \rightarrow (W_0(\mathcal{M}_Y \otimes \wedge(t, dt)), d_0) \equiv (W_0(\mathcal{M}_Y), d_0) \otimes \wedge(t, dt).
\]

We thus obtain from the second square of (22) a homotopy commutative DGA square (the left diagram of the following display), and, for the next level of spectral sequences, a strictly commutative one (the right diagram below):

\[
\begin{array}{ccc}
W_0(\mathcal{N}_Y), d_0 & \xrightarrow{W_0(\Phi_Y)} & W_1(\mathcal{M}_Y), d_1 \\
W_0(\mathcal{N}_Z), d_0 & \xrightarrow{W_0(\Phi_Z)} & W_1(\mathcal{M}_Z), d_1 \\
W_0(\mathcal{M}_f), d_0 & \xrightarrow{W_0(K)} & W_1(\mathcal{N}_f), d_0 \\
W_0(\mathcal{M}_f), d_0 & \xrightarrow{W_0(K)} & W_1(\mathcal{N}_f), d_0 \\
W_1(\mathcal{M}_f), d_0 & \xrightarrow{W_1(\Phi_Z)} & W_1(\mathcal{M}_f), d_0
\end{array}
\]

This verifies our claim and thus ends the proof of Proposition 34. \(\square\)

Proof of Proposition 33. We put together a sequence of elementary weak equivalences coming from [18, 15], and [16].

Applying (22) (the left square) and Proposition 34 to the projection onto the first coordinate \( p : (X[n], F(X, n)) \rightarrow (X, X) \) (where \( X[n] \) is the compactification in [15]), we obtain a weak homotopy equivalence \( \Omega d_R(p) \cong W_1(\mathcal{M}_p) \).

The last two elementary equivalences are described in the next diagram:
The DGA $\mathcal{A}^*(X, n)_C$ is the Fulton-MacPherson model from [15, Theorem 8] (with $\mathbb{C}$ coefficients). The DGA isomorphism $\mu_n$ is constructed in [15, §6]. The DGA’s $\mathcal{A}^*(X, 1)_C$ and $\mathcal{W}_E^1(M_X)$ are both equal to $(H, 0)$, and $\mu_1 = id$. The DGA $\mathcal{A}^*(X, n)_C$ is a quotient of a certain free graded $H \otimes^n$-algebra, and the DGA map $\iota_A^1$ between $\mathcal{A}$-models is induced by $\iota_1: H \to H \otimes^n$, like in the case of $E$-models. An easy analysis of the construction of $\mu_n$ shows that the above left DGA square is commutative, and therefore provides an elementary weak equivalence, $\mathcal{W}_E^1(M_p) \to \iota_A^1$.

The DGA map $\kappa_n$, constructed by Krü in [16, §3], induces an isomorphism in homology; see [16, §4]. By construction, $\kappa_n$ is $H \otimes^n$-linear. Consequently, the above right DGA square is commutative, and gives the last needed weak equivalence, $\iota_A^1 \leftarrow \iota_1$.

Thus, Proposition 33 is proved, and the proof of Theorem 7 is complete. \hfill \square

5. A panorama of natural structures on punctured manifolds

In this section, we will define various additional structures on $E$-models, with emphasis on the punctured case. They seem to be both highly natural, and potentially useful for further applications.

Our starting point is like in Section 1: a Poincaré duality algebra, $H$, over a field $\mathbb{K}$, together with an orientation class, $\omega \in H^{2m} \setminus \{0\}$. As noted in Remark 12, the associated models, $E_n(H)$ and $E_n(\hat{H})$, are bigraded differential algebras (DBGA’s).

5.1. Symmetry. The symmetric group $\Sigma_n$ acts (on the left) on both $E_n(H)$ and $E_n(\hat{H})$, by DBGA maps, and the canonical projection, $E_n(H) \to E_n(\hat{H})$, is a $\Sigma_n$-equivariant DBGA map. Geometrically, the above projection corresponds to the natural inclusion, $F(\hat{X}, n) \hookrightarrow F(X, n)$, which is equivariant with respect to the natural $\Sigma_n$-actions on $n$-configurations.

Algebraically, the action of $\sigma \in \Sigma_n$ on the algebra generators is defined as follows. For elements $h_1, \ldots, h_n$ of $H$ or $\hat{H}$, $\sigma \cdot h_1 \otimes \cdots \otimes h_n := \pm h_{\sigma^{-1}1} \otimes \cdots \otimes h_{\sigma^{-1}n}$, where the sign depends on $\sigma$ and the degrees of $\{h_i\}$, according to the standard Koszul sign convention. For both $E_n(H)$ and $E_n(\hat{H})$, $\sigma \cdot G_{ij} := G_{\sigma i, \sigma j}$. Here, it is convenient to replace the definitions from §4 by the following equivalent ones: take as exterior generators all $\{G_{ij}; n \geq i \neq j \geq 1\}$, and add the relations $G_{ij} = G_{ji}$, for all $i \neq j$. As for the projection, $E_n(H) \to E_n(\hat{H})$, it...
is induced by the canonical projection, $H \to \hat{H}$, on $H^\otimes n$, and acts as the identity on the exterior generators, $G_{ij}$.

5.2. Simplicial structure. Set $E_0(\hat{H}) := \mathbb{K}$, and $\mathcal{E}(\hat{H}) := \{E_n(\hat{H})\}_{n \geq 0}$. We will give $\mathcal{E}(\hat{H})$ the structure of a simplicial DBGA. For $H = H^*(S^2)$, this structure is given by Drinfel’d’s [10, p.843] natural cosimplicial group structure of classical (Artin) pure braid groups, $\{\pi_1 F(\mathbb{R}^2, n)\}$, via the cohomology algebra functor; see also Example 18.

Set $D_0 = D_1 := \varepsilon : (\hat{H}, 0) \to E_1(\hat{H}) \to E_0(\hat{H}) = \mathbb{K}$, where $\varepsilon$ denotes the canonical augmentation of the connected graded algebra $\hat{H}$. Set also $S_0 := \eta : E_0(\hat{H}) \to E_1(\hat{H})$, where $\eta$ is the unit of $\hat{H}$.

Fix now $n \geq 1$. The degeneration maps, $S_k : E_n(\hat{H}) \to E_n+1(\hat{H})$, where $0 \leq k \leq n$, correspond to the natural projections, $pr_{k+1} : F(\hat{X}, n + 1) \to F(\hat{X}, n)$, which omit the $(k+1)$-st coordinate. Let us define $S_k$ on the algebra generators. Here and in the sequel, it will be useful to make the following notation. Let $\varphi_k : \{1, \ldots, n\} \to \{1, \ldots, n+1\}$ (where $1 \leq k \leq n+1$) be the order-preserving bijection onto $\{1, \ldots, n+1\} \setminus \{k\}$. Set

$$S_k := t_{\varphi_{k+1}}, \quad \text{on } \hat{H}^\otimes n$$

(where $t_{\varphi_{k+1}}$ is defined like in (3)), and

$$S_k(G_{ij}) := G_{\varphi_{k+1},i;\varphi_{k+1},j}, \quad \text{for } n \geq i > j \geq 1.$$  

The face maps, $D_0$ and $D_{n+1} : E_{n+1}(\hat{H}) \to E_n(\hat{H})$, correspond to the natural inclusions, $L, R : F(\hat{X}, n) \hookrightarrow F(\hat{X}, n+1)$, defined as follows. View $\hat{X}$ from $F(\hat{X}, n)$ as $X \setminus C$, where $C$ is a $2m$-cube. Divide $C$ into two half-cubes, $C := C_1 \cup C_2$, pick a base-point, $p_t \in int(C_1)$, and view $\hat{X}$ from $F(\hat{X}, n+1)$ as $X \setminus C_2$. Then $L(R)$ adds $p_t$ to a given $n$-configuration, on the left (respectively right); see [10]. Algebraically, we define $D_0$ and $D_{n+1}$ as follows. On $\hat{H}^\otimes n+1$:

$$D_k := \varepsilon \otimes id, \quad \text{for } k = 0,$$

$$D_k := id \otimes \varepsilon, \quad \text{for } k = n+1.$$  

On the exterior algebra generators:

$$D_0(G_{ij}) := G_{i-1,j-1}, \quad \text{for } n + 1 \geq i > j > 1,$$

$$D_0(G_{ii}) := 0, \quad \text{for } n + 1 \geq i > j = 1,$$

and

$$D_{n+1}(G_{ij}) := G_{ij}, \quad \text{for } n + 1 > i > j \geq 1,$$

$$D_{n+1}(G_{ii}) := 0, \quad \text{for } n + 1 = i > j \geq 1.$$  

It remains to define the face maps $D_k : E_{n+1}(\hat{H}) \to E_n(\hat{H})$, for $1 \leq k \leq n$. For $X = S^2$, they correspond to the doubling maps, $\delta_k : F(\hat{X}, n) \to F(\hat{X}, n+1)$. They add to a given
n-configuration, \((x_1, \ldots, x_n)\), a new coordinate, \(x'_k\), a copy of \(x_k\), placed to the left of \(x_k\) and close to \(x_k\); see again [10]. Algebraically, on \(\tilde{H}^{n+1}_{\otimes}\):

\[(29)\quad D_k(h_1 \otimes \cdots \otimes h_{n+1}) := h_1 \otimes \cdots \otimes h_{k-1} \otimes (h_k \cdot h_{k+1}) \otimes h_{k+2} \otimes \cdots \otimes h_{n+1},\]

and, for \(n + 1 \geq i \neq j \geq 1:\)

\[(30)\quad\begin{cases} D_k(G_{ij}) := G_{rs}, & \text{if } i = \varphi_k r \text{ and } j = \varphi_k s; \\ D_k(G_{kj}) := G_{ks}, & \text{if } j = \varphi_k s \text{ and } s \neq k; \\ D_k(G_{k,k+1}) := 0, & \end{cases}\]

where \(\varphi_k\) is like in (24).

**Proposition 35.** The maps

\(\{D_k : \tilde{H}^{n+1}_{\otimes} [G_{ij}] \to \tilde{H}^{n}_{\otimes} [G_{ij}]\}_{0 \leq k \leq n+1}\) and \(\{S_k : \tilde{H}^{n}_{\otimes} [G_{ij}] \to \tilde{H}^{n+1}_{\otimes} [G_{ij}]\}_{0 \leq k \leq n}\),

defined as above, induce a simplicial DBGA structure on \(\mathcal{E}(\tilde{H})\).

**Proof.** By construction, all maps are multiplicative and bihomogeneous, on \(\tilde{H}^{n+1}_{\otimes} [G_{ij}]\) and \(\tilde{H}^{n}_{\otimes} [G_{ij}]\) respectively. Starting from definitions (24)-(25), it is straightforward to check that one has induced DBGA maps, \(\{S_k : E_n(\tilde{H}) \to E_{n+1}(\tilde{H})\}_{0 \leq k \leq n}\), as asserted. Similarly, for \(D_0\) and \(D_{n+1}\).

Let us prove this now for \(D_k\), with \(1 \leq k \leq n\). Consider first the quotient algebra, \(\wedge(G_{ij}; n + 1 \geq i > j \geq 1)\), modulo the Arnold relations [4]. After uniform rescaling of degrees (that is, putting all \(G_{ij}\)'s in degree 1, instead of \(2m - 1\), this graded algebra becomes isomorphic to \(H^*F(\mathbb{R}^2, n + 1)\). Moreover, the generator \(G_{ij}\) is dual to \(t^{ij}\), where \(t^{ij} \in H_1F(\mathbb{R}^2, n + 1)\) is the homology class of the standard generator \(\sigma^{ij} \in \pi_1F(\mathbb{R}^2, n + 1)\) from [3, Proposition 3.6]. See also [19]. With these identifications, our formulae (30), defining \(D_k\) on the generators \(G_{ij}\), become dual to those from [3, Definition 2.9], which express the doubling operations on \(H_1F(\mathbb{R}^2, n)\) in terms of the generators \(t^{ij}\). (See [3, §4.4-4.5] for the importance of doubling, viewed in the framework of chord diagrams from the theory of finite type invariants of links.) In this way, \(D_k\) is identified with \(H^*\delta_k\) (where \(\delta_k : F(\mathbb{R}^2, n) \to F(\mathbb{R}^2, n + 1)\) is the geometric doubling), on the exterior generators; in particular, \(D_k\) preserves relations [4]. The preservation of relations [5] follows immediately from definitions (29)-(30), as well as the commutation relations,

\[(31)\quad D_k \delta G_{ij} = \delta D_k G_{ij},\]

in the first two cases from (30). Finally, \(D_k \delta G_{k+1,k} = \sum_\alpha (-1)^{\deg(h_\alpha)} t_k (h_\alpha \cdot h_\alpha^*)\), according to the definitions. This sum equals zero, as asserted, since \(h_\alpha \cdot h_\alpha^* = \omega\), for all \(\alpha\), and \(\omega = 0\) in \(\tilde{H}\).

To finish the proof of our Proposition, it is enough to check the simplicial identities on the algebra generators. On the exterior part, we have identified \(D_k\) with \(H^*\delta_k\), for \(1 \leq k \leq n\). Formulae (27)-(28) readily imply the identifications \(D_0 \equiv H^*L\) and \(D_{n+1} \equiv H^*R\).
respectively. It is equally easy to check that $S_k \equiv H^*(pr_{k+1})$, for $0 \leq k \leq n$, starting from definition (25). Therefore, it is enough to check the cosimplicial identities, for the continuous maps $\{ s^k := pr_{k+1} \}_{0 \leq k \leq n}$, and $\{ \partial^k := \delta_k \}_{1 \leq k \leq n} \cup \{ \partial^0 := L, \partial^{n+1} := R \}$, which is routine.

Similarly, one may check the simplicial identities also on generators coming from $\hat{H}$, by completely straightforward dual calculations.

**Remarks 36.** Formulae (24)–(30) make sense also for $\{ E_n(H) \}_{n}$. While $\{ S_k \}_{k}$ induce DBGA maps, $\{ S_k : E_n(H) \to E_{n+1}(H) \}_{k}$, the others don’t, in general. Indeed, for instance $D_{n+1}dG_{n+1,n} = t_n \omega \neq 0$, while $dD_{n+1}G_{n+1,n} = 0$. This is related to the fact that, for a closed manifold $X$, $pr_{n+1} : F(X, n + 1) \to F(X, n)$ does not need to have a section.

Similarly, $D_n dG_{n+1,n} = e(H) \cdot t_n \omega$, while $dD_n G_{n+1,n} = 0$. Thus, $D_n d \neq dD_n$, when the characteristic of $\mathbb{K}$ does not divide the Euler characteristic $e(H)$. The topological counterpart of the potential failure of type (31) relations, in the closed case, is the fact that the doubling operations, $\{ \delta_k : F(X, n) \to F(X, n + 1) \}_{k}$, have no natural definition, when $X$ does not admit a nowhere zero vector field. 

### 5.3. A 'coaction' map

There is a natural topological action,

$$ F(\hat{X}, n_0) \times F(\mathbb{R}^{2m}, n_1) \times \cdots \times F(\mathbb{R}^{2m}, n_r) \xrightarrow{\alpha_T} F(\hat{X}, n_0 + n_1 + \cdots + n_r), $$

for any partition $T$ of $n := n_0 + n_1 + \cdots + n_r$. Here, $T := \{ T_j \}_{0 \leq j \leq r}$, $\{ 1, \ldots, n \} = T_0 \sqcup T_1 \sqcup \cdots \sqcup T_r$, and $n_j := |T_j|$, for $j = 0, \ldots, r$. For each $j$, denote by $\varphi^j$ the (unique) order-preserving bijection, $\varphi^j : \{ 1, \ldots, n_j \} \xrightarrow{\sim} T_j$. To define the above action map, $\alpha_T$, start by viewing $\hat{X}$ from $F(\hat{X}, n_0)$ as $X \setminus C$, where $C$ is a $2m$-cube. Divide $C$ into two cubes, $C = C' \cup C''$, and view $\hat{X}$ from $F(\hat{X}, n)$ as $X \setminus C''$. Next, subdivide $C'$ into $r$ cubes, $C' = C'_1 \cup \cdots \cup C'_r$. Now, let $x_0$ be an $n_0$-configuration in $\hat{X}$, and let $x_j$ be given $n_j$-configurations in $\mathbb{R}^{2m}$, for $1 \leq j \leq r$. Define $\alpha_T(x_0, x_1, \ldots, x_r)$ to be the $n$-configuration in $\hat{X}$ obtained by putting the coordinates of $x_0$ in $X \setminus C$, on the positions prescribed by $\varphi^0$, and those of $x_j$ (for $1 \leq j \leq r$), suitably rescaled, in $\text{int}(C'_j)$, according to $\varphi^j$-prescriptions.

Note that all spaces $F(\mathbb{C}^m, k)$ are formal, as follows from work by S. Yuzvinsky in [22]. Their cohomology algebras were computed by F.R. Cohen (see [7]); they have exterior generators $\{ G_{ij} : k \geq i > j \geq 1 \}$, in degree $2m - 1$, and defining relations (4).

With these preliminaries, we may now define the algebraic analog of (32), that is, a DBGA map,

$$ q_T : (E_n(\hat{H}), \delta^\hat{H}) \to (E_{n_0}(\hat{H}), \delta^\hat{H}) \bigotimes_{j=1}^{r} H^*F(\mathbb{R}^{2m}, n_j), 0), $$

where the bigrading on $\bigotimes_{j} H^*F(\mathbb{R}^{2m}, n_j)$ comes from putting all $G$-type generators in bidegree $(2m - 1, 1)$, as usual.

Let us define $q_T$ on algebra generators. For $n \geq i \neq j \geq 1$, set

$$ q_T(G_{ij}) := \iota^k(G_{i'j'}), \quad \text{if } i = \varphi^k i' \text{ and } j = \varphi^k j'; $$

$$ q_T(G_{ij}) := 0, \quad \text{otherwise}, $$

where the coordinate $\iota^k$ is from $G_{i'j'}$ to $G_{ij}$, for $i = \varphi^k i'$ and $j = \varphi^k j'$.
where \( \iota^k \) denotes the canonical inclusion into the tensor product of \( E_{n_0}(\hat{H}) \) (respectively \( H^*F(\mathbb{R}^{2m}, n_k) \)), for \( k = 0 \) (respectively \( 1 \leq k \leq r \)). On \( \hat{H}^\otimes n \), which is generated by \( \iota_i(h) \), where \( h \in \hat{H} \) and \( 1 \leq i \leq n \), set:

\[
\left\{
\begin{array}{ll}
q_T\iota_i(h) := \varepsilon(h), & \text{if } i = \varphi^k i' \text{ and } k > 0; \\
q_T\iota_i(h) := \iota'_v(h), & \text{if } i = \varphi^v i',
\end{array}
\right.
\]

where \( \varepsilon \) stands as usual for the canonical augmentation of the connected graded algebra \( \hat{H} \).

**Proposition 37.** The bigraded algebra map,

\[
q_T : \hat{H}^\otimes n [G_{ij}; n \geq i \neq j \geq 1] \to E_{n_0}(\hat{H}) \bigotimes_{j=1}^r (\bigotimes_{i=1}^m H^*F(\mathbb{R}^{2m}, n_j)),
\]

defined by \((34)-(37)\) above, induces a DBGA map, as in \((35)\).

**Proof.** The compatibility of \( q_T \) with the Arnold relations \((41)\) from \( E_n(\hat{H}) \) follows from an easy analysis of the definition given in \((34)\). Likewise, \((5)\)-compatibility is an easy consequence of \((34)-(35)\). Finally, we have to check the commutation of \( q_T \) with differentials, on all generators \( G_{ij} \), \( n \geq i \neq j \geq 1 \). In the second case from \((34)\), as well as in the first case (subcase \( k > 0 \)), this amounts to verifying that \( q_T \Delta_{ij} = 0 \). This in turn follows from the remark that, in all these cases, either \( i \notin T_0 \) or \( j \notin T_0 \). This implies, via \((35)\), that \( q_T \) sends \( \Delta_{ij} \) to zero, since obviously \( (\varepsilon \otimes id)(\Delta) = (\iota'_v \otimes \varepsilon)(\Delta) = (\varepsilon \otimes \varepsilon)(\Delta) = 0 \); see Definition \(5\).

In the remaining case, \((34)-(35)\) readily imply that \( \Delta_{ij} = q_T \Delta_{ij} = \Delta_{ij} \).

**Remark 38.** Again, the existence of the natural topological action \((32)\) is a peculiarity of the punctured case. This phenomenon is also reflected by algebra. For example, take \( T = \{T_0, T_1\} \), with \( T_0 = \{1, \ldots, n\} \) and \( T_1 = \{n+1\} \). Compare \((26)\) and \((28)\) with \((51)\) and \((35)\) to infer that, for this choice of \( T \), \( q_T = D_{n+1} \) from \( \S \).

As noted in Remarks \(36\), the definition of \( q_T \) would make sense also in the closed manifold case, but in that case \( q_T \) would not commute with \( d \).

### 5.4. Connected sum.

We first recall the algebraic analog of the connected sum operation. Let \((H, \omega_H)\) and \((K, \omega_K)\) be two oriented \( K \)-Poincaré duality algebras, of the same formal dimension, \(2m\). Define the underlying graded vector space of their connected sum by \( H \# K := K \cdot 1 \oplus \hat{H}^+ \oplus \hat{K}^+ \oplus K \cdot \omega \), with \( \omega \) in degree \(2m\). Extend the multiplications of \( H \) and \( K \) by setting \( h \cdot k = 0 \), for \( h \in \hat{H}^+ \) and \( k \in \hat{K}^+ \). In this way, \((H \# K, \omega)\) becomes a Poincaré duality algebra, endowed with two (multiplicative) canonical projections, \( \pi_H : H \# K \to \hat{H} \), and \( \pi_K : H \# K \to \hat{K} \).

The connected sum operation for oriented manifolds provides a natural map,

\[
F(\hat{X}, r) \times F(\hat{Y}, s) \to F(X \# Y, r + s),
\]

which sends \((x_1, \ldots, x_r, y_1, \ldots, y_s)\) to \((x_1, \ldots, x_r, y_1, \ldots, y_s)\). Let us define its DBGA analog,

\[
\chi : (E_{r+s}(H \# K), d) \to (E_r(\hat{H}), d) \bigotimes (E_s(\hat{K}), d).
\]
On exterior degree zero algebra generators, set

\[ \chi | (H \# K)^{(r+s)} := \pi^r_H \otimes \pi^s_K. \]  

On \( G \)-type generators, we will define \( \chi \) by

\[
\begin{align*}
\chi(G_{ij}) &:= G^H_{ij}, & \text{for } r \geq i > j \geq 1; \\
\chi(G_{r+i,r+j}) &:= G^K_{ij}, & \text{for } s \geq i > j \geq 1; \\
\chi(G_{ij}) &:= 0, & \text{otherwise}. 
\end{align*}
\]

Proposition 39. Formulae (37)-(38) above define an induced DBGA map, \( \chi \), as in (36).

Proof. By construction, \( \chi : (H \# K)^{(r+s)}[G_{ij}] \to \hat{H}^r [G^H_{ij}] \otimes \hat{K}^s [G^K_{ij}] \) is multiplicative and bihomogeneous. The preservation of relations (4)-(5) is immediate.

To check the commutation relations

\[ d\Delta = d\chi(G_{ij}) = \chi d(G_{ij}), \]

just note that \( \Delta = \omega \otimes 1 + 1 \otimes \omega + \hat{H} + \hat{K}, \) by construction, and then use the definitions. \( \Box \)

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