LEBESGUE AND HARDY SPACES FOR SYMMETRIC NORMS I

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Abstract. In this paper, we define and study a class $\mathcal{R}_c$ of norms on $L^\infty(T)$, called continuous rotationally symmetric norms, which properly contains the class $\{\|\cdot\|_p : 1 \leq p < \infty\}$. For $\alpha \in \mathcal{R}_c$ we define $L^\alpha(T)$ and the Hardy space $H^\alpha(T)$, and we extend many of the classical results, including the dominated convergence theorem, convolution theorems, dual spaces, Beurling-type invariant spaces, inner-outer factorizations, characterizing the multipliers and the closed densely-defined operators commuting with multiplication by $z$. We also prove a duality theorem for a version of $L^\alpha$ in the setting of von Neumann algebras.

1. Introduction

Suppose $T$ is the unit circle in the complex plane $\mathbb{C}$ and $m$ is Haar measure (i.e., normalized arc length) on $T$. We let $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, respectively, denote the sets of real numbers, integers, and positive integers. In this paper we focus on norms on $L^\infty(T)$. In particular we focus on norms that are rotationally symmetric (defined below) and we define Lebesgue and Hardy spaces with respect to these norms and extend many classical results concerning $L^p$ and $H^p$ to this setting. Since the family $L^p(T)$ (and $H^p(T)$) spaces are linearly ordered by inclusion, many classical proofs are broken into the cases $1 \leq p \leq 2$ and $2 < p < \infty$. In our setting this dichotomy does not exist, requiring new techniques. In subsequent papers we prove a version of Beurling’s theorem in a vector-valued analogue of $H^p(T)$ (see [13]), we extend results concerning noncommutative $H^p$-spaces (see [14]), and we extend our results to general finite measure spaces with norms symmetric with respect to a group of measure-preserving transformations, we extend our results to analogues of $H^p$-spaces on nice multiply connected domains, and in a final paper we extend our results to $\sigma$-finite measure spaces and corresponding commutative and noncommutative analogues of $H^p$-spaces.

We say that a seminorm $\alpha$ on $L^\infty(T)$ is a gauge norm if

1. $\alpha(1) = 1$,
2. $\alpha(|f|) = \alpha(f)$ for every $f \in L^\infty(T)$.

We say that a seminorm $\alpha$ on $L^\infty(T)$ is rotationally symmetric if it is a gauge norm and

3. $\alpha (f_w) = \alpha (f)$ for every $w \in T$, $f \in L^\infty(T)$,

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where
\[ f_w(z) = f(wz), \]
whenever \( f : \mathbb{T} \to \mathbb{C} \) and \( w \in \mathbb{T} \).

For any measurable \( f : \mathbb{T} \to \mathbb{C} \), we define \( \alpha(f) \) by
\[ \alpha(f) = \sup \{ \alpha(s) : s \text{ is a simple function}, |s| \leq |f| \}. \]

And we define the Banach space \( L^\alpha(\mathbb{T}) \) to be the set of all measurable functions \( f : \mathbb{T} \to \mathbb{C} \) such that \( \alpha(f) < \infty \), and define the (sometimes proper) closed linear subspace \( L^\alpha(\mathbb{T}) \) to be the \( \alpha \)-closure of \( L^\infty(\mathbb{T}) \). For later use we define \( H^\alpha(\mathbb{T}) \) to be the \( \alpha \)-closed linear span of \( \{1, z, z^2, \ldots\} \), which is a closed subspace of \( L^\alpha(\mathbb{T}) \).

We define a seminorm \( \alpha \) to be \textit{continuous} if,
\[ m(E_n) \to 0 \Rightarrow \alpha(\chi_{E_n}) \to 0, \]
and if, in addition, \( L^\alpha(\mathbb{T}) = L^\alpha(\mathbb{T}) \), we say that \( \alpha \) is \textit{strongly continuous}.

Note that, for \( 1 \leq p \leq \infty \), \( \|\cdot\|_p \) is a rotationally symmetric norm that is strongly continuous when \( p < \infty \). Thus the spaces \( L^\alpha(\mathbb{T}) \) and \( H^\alpha(\mathbb{T}) \) are generalizations of the classical Lebesgue spaces \( L^p(\mathbb{T}) \) and the Hardy spaces \( H^p(\mathbb{T}) \). In this paper we extend many of the classical results, often using new techniques, to these more general spaces, including a dominated convergence theorem, convolution theorems, invariant subspace theorems, dual spaces.

More precisely, in Section 2 we describe some of the basic properties and constructions of these seminorms, and show that in many cases (e.g., when they are continuous), \( \|\cdot\|_1 \leq \alpha \leq \|\cdot\|_\infty \), and that \( C(\mathbb{T}) \) is dense in \( L^\alpha(\mathbb{T}) \).

In Section 3 we define the dual seminorm \( \alpha^\prime \) for each continuous rotationally symmetric seminorm \( \alpha \). We show that dual space of \( L^\alpha(\mathbb{T}) \) is \( L^{\alpha^\prime}(\mathbb{T}) \). We also show that \( \alpha'' = \alpha \) holds and we show that \( L^{\alpha^\prime}(\mathbb{T}) \) (and therefore \( H^{\alpha^\prime}(\mathbb{T}) \)) is reflexive whenever \( \alpha \) and \( \alpha^\prime \) are both strongly continuous. Furthermore, we show that a continuous \( \alpha \) is strongly continuous if and only if \( L^\alpha(\mathbb{T}) \) is weakly sequentially complete.

In Section 4 we prove a general continuity theorem and a dominated convergence theorem for \( L^\alpha(\mathbb{T}) \), and we show that each of these theorems characterize \( L^\alpha(\mathbb{T}) \) in \( L^{\alpha^\prime}(\mathbb{T}) \). We also give an example of a continuous norm \( \alpha \) with \( L^{\alpha^\prime}(\mathbb{T}) \neq L^\alpha(\mathbb{T}) \).

In Section 5 we define \( L^\alpha(\mathbb{T}, X) \) for a separable Banach space \( X \), and we discuss properties of the convolution \( f * g \) with \( f \in L^\alpha(\mathbb{T}, X) \) and \( g \in L^1(\mathbb{T}) \) using the fact that the set of continuous functions from \( \mathbb{T} \) to \( X \) is dense in \( L^\alpha(\mathbb{T}, X) \) when \( \alpha \) is continuous. We use convolutions with the Fejer kernel to show that when \( \alpha \) is continuous and \( f \in L^\alpha(\mathbb{T}, X) \), then \( f \) is the \( \alpha \)-limit of the Cesaro sums of the sequence of partial sums of its Fourier series, and we use the Poisson kernel to extend \( f \) to a function from \( \mathbb{D} = \{ z \in \mathbb{C} : |z| \leq 1 \} \) to \( X \).

In Section 6 we study \( H^\alpha(\mathbb{T}) \) when \( \alpha \) is continuous and we show that, as in the classical setting, \( H^\alpha(\mathbb{T}) \) is isometrically isomorphic to \( H^\alpha(\mathbb{D}) \). We characterize \( H^\alpha(\mathbb{T}) \) as the functions in \( L^\alpha(\mathbb{T}) \) whose negative Fourier coefficients vanish and we show \( H^\alpha(\mathbb{T}) = H^1(\mathbb{T}) \cap L^\alpha(\mathbb{T}) \). Similarly, we define \( H^\alpha(\mathbb{T}, X) = H^1(\mathbb{T}, X) \cap L^\alpha(\mathbb{T}) \) to be the functions in \( L^\alpha(\mathbb{T}) \) whose negative Fourier coefficients vanish. We show that, for a separable Banach space \( X \)
\[ L^\alpha(\mathbb{T}, X) = sp^{-\alpha}(\{ zn : n \in \mathbb{Z} \text{ and } x \in X \}) \]
and
\[ H^\alpha(\mathbb{T}, X) = sp^{-\alpha}(\{ zn : n \geq 0 \text{ and } x \in X \}). \]
Only when $\alpha$ is strongly continuous do we get that $f \in H^{\alpha}(\mathbb{D})$ if and only if $f$ is analytic and $\sup \{ \alpha(f(re^{it})) : 0 < r < 1 \} < \infty$.

In Section 7 we prove analogues of Beurling’s invariant subspace theorems for $H^{\alpha}(\mathbb{T})$ when $\alpha$ is continuous. In Section 8, we show that the Riesz-Smirnov inner-outer factorization works in $H^{\alpha}(\mathbb{T})$. In Section 9 we characterize the multipliers of $L^{\alpha}(\mathbb{T})$ and $H^{\alpha}(\mathbb{T})$ and show that they are multiplier pairs in the sense of [5]; we also prove that $L^{\alpha}(\mathbb{T}) = L^{\alpha}(\mathbb{T})$ if and only if $H^{\alpha}(\mathbb{T}) = H^{\alpha}(\mathbb{T})$. In Section 10 we characterize the closed densely defined operators on $H^{\alpha}(\mathbb{T})$ that commute with multiplication by $z$. In the final section we state and prove a corrected version of a theorem in [4] concerning the dual space of a von Neumann algebra version of an $L^{\alpha}$-space.

2. Completing Norms on $L^{\infty}(\mathbb{T})$

We begin by defining two classes of seminorms on $L^{\infty}(\mathbb{T})$. Suppose $f: \mathbb{T} \to \mathbb{C}$ and $w \in \mathbb{T}$. We define $f_w: \mathbb{T} \to \mathbb{C}$ by

$$f_w(z) = f(\overline{wz}).$$

We say that a seminorm $\alpha$ on $L^{\infty}(\mathbb{T})$ is rotationally symmetric if

1. $\alpha(1) = 1$,
2. $\alpha(|f|) = \alpha(f)$ for every $f \in L^{\infty}(\mathbb{T})$,
3. $\alpha(f_w) = \alpha(f)$ for every $w \in \mathbb{T}$, $f \in L^{\infty}(\mathbb{T})$.

To define the smaller class of symmetric gauge norms, we need to define $MP(\mathbb{T})$ to be the set of invertible measure-preserving maps $\phi: \mathbb{T} \to \mathbb{T}$. A seminorm $\beta$ on $L^{\infty}(\mathbb{T})$ is a symmetric gauge seminorm if

1. $\beta(1) = 1$,
2. $\beta(|f|) = \beta(f)$ for every $f \in L^{\infty}(\mathbb{T})$,
3. $\beta(f \circ \phi) = \beta(f)$ for every $f \in L^{\infty}(\mathbb{T})$ and $\phi \in MP(\mathbb{T})$.

We often identify $\mathbb{T}$ with $(0, 1]$ (identifying $e^{2\pi it}$ with $t$) and $m$ with Lebesgue measure on $(0, 1]$. We will make it clear whenever we do this. One place where this is convenient is when we want to talk about increasing or decreasing functions. One particular fact that makes our work more easily understood is the following lemma from [12] Theorem 3.4.1.

**Lemma 2.1.** Suppose $f: (0, 1] \to [0, \infty)$ is measurable. Then there is a unique non-increasing right-continuous function $f^*$ on $(0, 1]$ and an invertible measure-preserving map $\phi: (0, 1] \to (0, 1]$ such that $f^* = f \circ \phi$ a.e. $(m)$.

It follows from the definition that if $\alpha$ is a symmetric gauge norm, then $\alpha(f) = \alpha(|f|) = \alpha(|f^*|)$. Note that if $f = \chi_E$ for $E \subset (0, 1]$ with $m(E) > 0$, then $f^* = \chi_{(0,m(E))}$, which in $\mathbb{T}$ we would represent as $f^* = \chi_{I_m(E)}$, where

$$I_{m(E)} = \{ e^{it} : 0 \leq t \leq 2\pi m(E) \}.$$ 

Therefore if $\beta$ is a symmetric gauge norm, then $\beta(\chi_E) = \beta(\chi_F)$ whenever $m(E) = m(F)$. More generally, $m(E) \leq m(F)$ implies $\chi_{I_{m(E)}} \leq \chi_{I_{m(F)}}$, which implies $\beta(\chi_E) \leq \beta(\chi_F)$, i.e., $\beta(\chi_E)$ is increasing with respect to $m(E)$. Hence, we conclude that $\lim_{m(E) \to 0^+} \beta(\chi_E)$ exists.
We say that a rotationally symmetric seminorm $\alpha$ on $L^\infty(\mathbb{T})$ is continuous if
\[
\lim_{m(E) \to 0^+} \alpha(\chi_E) = 0.
\]
In general, for a rotationally symmetric norm $\alpha$, we do not know $\lim_{m(E) \to 0^+} \alpha(\chi_E)$ exists.

Let $\mathcal{R}$ denote the set of all rotationally symmetric seminorms on $L^\infty(\mathbb{T})$, and let $\mathcal{S}$ denote the set of all symmetric gauge seminorms on $L^\infty(\mathbb{T})$. Clearly, $\mathcal{S} \subset \mathcal{R}$. We let $\mathcal{R}_c$ and $\mathcal{S}_c$ denote, respectively, the continuous seminorms in $\mathcal{R}$ and $\mathcal{S}$. In most of this paper we will be considering elements of $\mathcal{R}_c$.

Although $\alpha \in \mathcal{R}$ is defined only on $L^\infty(\mathbb{T})$, we can define $\alpha(f)$ for all measurable functions $f$ on $\mathbb{T}$ by
\[
\alpha(f) = \sup\{\alpha(s) : s \text{ is a simple function, } |s| \leq |f|\}.
\]
It is clear that $\alpha(f) = \alpha(|f|)$ still holds.

We define $L^\alpha(\mathbb{T})$ to be the completion of $L^\infty(\mathbb{T})$ with respect to $\alpha$, and
\[
L^\alpha(\mathbb{T}) = \{f : \alpha(f) < \infty\}.
\]
If $\alpha \in \mathcal{R}$, we say that $\alpha$ is strongly continuous if and only if $\alpha \in \mathcal{R}_c$ and $L^\alpha(\mathbb{T}) = L^\alpha(\mathbb{T})$.

**Proposition 2.2.** Suppose $\alpha \in \mathcal{R}$ and $f, g : \mathbb{T} \to \mathbb{C}$ are measurable. The following are true.

1. For all measurable functions $f, g$ on $\mathbb{T}$,
   (a) $|f| \leq |g| \implies \alpha(f) \leq \alpha(g)$,
   (b) $\alpha(fg) \leq \alpha(f)\|g\|_\infty$, and
   (c) $\alpha(f) \leq \|f\|_\infty$;
2. $\alpha \in \mathcal{R}_c \iff \limsup_{m(E) \to 0^+} \alpha(\chi_E) = 0$;
3. $\liminf_{m(E) \to 0^+} \alpha(\chi_E) = \inf\{\alpha(\chi_E) : m(E) > 0\}$;
4. If $t > 0$, then
   \[
   \liminf_{m(E) \to 0^+} \alpha(\chi_E) \geq t \iff t\|f\|_\infty \leq \alpha(f) \leq \|f\|_\infty;
   \]
5. If $\alpha \in \mathcal{S}$, then $\alpha \notin \mathcal{S}_c \iff \alpha$ is equivalent to $\|\|_\infty$;
6. If $\alpha \in \mathcal{S} \cup \mathcal{R}_c$, $0 \leq f_1 \leq f_2 \leq \cdots$ and $f_n \to f$ a.e. $m$, then $\alpha(f_n) \to \alpha(f)$;
7. If $\alpha \in \mathcal{R}_c$, then $C(\mathbb{T})^{-\alpha} = L^\alpha(\mathbb{T})$;
8. If $\alpha \in \mathcal{S} \cup \mathcal{R}_c$, then $\|f\|_1 \leq \alpha(f)$;
9. If $\alpha \in \mathcal{S} \cup \mathcal{R}_c$ and $\lambda \in \mathbb{C}$, then
   (a) $\alpha(f + g) \leq \alpha(f) + \alpha(g)$,
   (b) $\alpha(\lambda f) = |\lambda|\alpha(f)$, and
   (c) $\alpha$ is a norm on $L^\alpha(\mathbb{T})$;
10. If $\alpha \in \mathcal{R}_c$, then $(L^\alpha(\mathbb{T}), \alpha)$ is a Banach space and
    
    $\quad L^\infty(\mathbb{T}) \subset L^\alpha(\mathbb{T}) \subset L^\alpha(\mathbb{T}) \subset L^1(\mathbb{T})$.

**Proof.** (1) (a) If $|f| \leq |g|$, then there are two measurable functions $u, v$ with $|u| = |v| = 1$ and $f = g(u + v)/2$, which implies
\[
\alpha(f) \leq \alpha(|ug|) + \alpha(|vg|)/2 = \alpha(|g|) = \alpha(g).
\]
(b) Since $|fg| \leq \|f\|_\infty |g|$ a.e. $m$, it follows from part (a) that
\[
\alpha(fg) = \alpha(|fg|) \leq \alpha(\|f\|_\infty |g|) = \|f\|_\infty \alpha(g).
\]
(c) Since $\alpha(1) = 1$, we know from part (b) that
\[ \alpha(f) = \alpha(f \cdot 1) \leq \alpha(1)\|f\|_\infty = \|f\|_\infty. \]

(2) If $\alpha \in R_c$, then $\lim_{m(E) \to 0^+} \alpha(\chi_E) = 0$, which implies that
\[ \limsup_{m(E) \to 0^+} \alpha(\chi_E) = \lim_{m(E) \to 0^+} \alpha(\chi_E) = 0. \]

On the other hand, if $\limsup_{m(E) \to 0^+} \alpha(\chi_E) = 0$, it follows from
\[ 0 \leq \liminf_{m(E) \to 0^+} \alpha(\chi_E) \leq \limsup_{m(E) \to 0^+} \alpha(\chi_E) = 0 \]
that $\lim_{m(E) \to 0^+} \alpha(\chi_E) = 0$, which means $\alpha \in R_c$.

(3) It is clear that
\[ \liminf_{m(E) \to 0^+} \alpha(\chi_E) = \sup_{r>0} \inf \{ \alpha(\chi_E) : 0 < m(E) < r \}, \]
and $\inf \{ \alpha(\chi_E) : 0 < m(E) < r \}$ is decreasing. Then there is a decreasing sequence
\( \{E_n\} \in \mathbb{T} \) such that when $0 < m(E_n) < r$ for all $n \geq 1$, we have
\[ \alpha(\chi_{E_n}) \to \inf \{ \alpha(\chi_E) : 0 < m(E) < r \}. \]

Suppose $0 < m(E_n) < r$ and $r_1 < r$. Then there exists an $F_n \subset E_n$ such that $0 < m(F_n) < r_1$, which yields $\alpha(\chi_{F_n}) \leq \alpha(\chi_{E_n})$. Hence
\[ \inf \{ \alpha(\chi_E) : 0 < m(E) < r_1 \} \leq \inf \{ \alpha(\chi_{E_n}) : 0 < m(F_n) < r_1 \} \leq \inf \{ \alpha(\chi_{E_n}) : 0 < m(E_n) < r \} \leq \lim_{n \to \infty} \alpha(\chi_{E_n}) = \inf \{ \alpha(\chi_E) : 0 < m(E) < r \}, \]

since $\inf \{ \alpha(\chi_E) : 0 < m(E) < r \}$ is decreasing, which implies that
\[ \lim_{m(E) \to 0^+} \alpha(\chi_E) = \inf \{ \alpha(\chi_E) : m(E) > 0 \}. \]

(4) Suppose $\liminf_{m(E) \to 0^+} \alpha(\chi_E) \geq t$. It follows from part (3) that
\[ \inf \{ \alpha(\chi_E) : m(E) > 0 \} \geq t. \]
Then for any measurable subset $E \subset \mathbb{T}$ with $m(E) > 0$, $\alpha(\chi_E) \geq t$. If $F = \{ x \in \mathbb{T} : |f(x)| \geq \|f\|_\infty \}$, then $m(F) > 0$ and $\alpha(\chi_F) \geq t$. Hence by part (1) we have
\[ \|f\|_\infty \geq \alpha(f) \geq \alpha(\chi_F) \geq \alpha(\|f\|_\infty \chi_F) = \|f\|_\infty \alpha(\chi_F) \geq t \|f\|_\infty. \]

Conversely, if $t \|f\|_\infty \leq \alpha(f) \leq \|f\|_\infty$, then for every measurable set $E \subset \mathbb{T}$, $\alpha(\chi_E) \geq t \|\chi_E\|_\infty = t$, thus $\liminf_{m(E) \to 0^+} \alpha(\chi_E) \geq t$.

(5) It was shown in [3].

(6) The case $\alpha \in S$ was proved in [3]. We can assume $\alpha \in R_c$. Suppose $0 \leq s \leq f$ and $0 \leq t < 1$. Write $s = \sum_{1 \leq k \leq m} a_k \chi_{E_k}$ with $0 < a_k$ for $1 \leq k \leq m$ and $\{E_1, \ldots, E_m\}$ disjoint. If we let $E_{k,n} = \{ \omega \in E_k : t a_k < f_n(\omega) \}$, we see that
\[ E_{k,1} \subset E_{k,2} \subset \cdots \subset E_{k,n} \subset E_k. \]

Since $\alpha$ is continuous,
\[ \alpha(\chi_{E_k} - \chi_{E_{k,n}}) = \alpha(\chi_{E_k \setminus E_{k,n}}) \to 0. \]

Hence
\[ t \alpha(s) = \lim_{n \to \infty} \alpha \left( \sum_{k=1}^m t a_k \chi_{E_{k,n}} \right) \leq \lim_{n \to \infty} \alpha(f_n). \]
Thus for each \( n \) such that \( \alpha \) is actually a norm, \( \| g_n \|_\infty \leq \| f \|_\infty < \infty \) for all \( n \geq 1 \). Since \( \alpha \) is continuous, \( \alpha(\chi_{E_n}) \to 0 \). But

\[
\alpha(g_n - f) \leq 2 \| f \|_\infty \alpha(\chi_{E_n}) \to 0.
\]

Thus \( L^\infty(T) \subset C(T)^{-\alpha} \); hence, \( L^\alpha(T) = L^\infty(T)^{-\alpha} \subset C(T)^{-\alpha} \subset L^\alpha(T) \). Therefore \( L^\alpha(T) = C(T)^{-\alpha} \).

(8) If \( \alpha \in \mathcal{S} \), then it is true \( \mathbb{R} \). We can assume \( \alpha \in \mathcal{R} \).

It is well-known that if \( f \in C(T) \) and \( \omega = e^{2\pi i \theta} \) with \( \theta \) irrational, then

\[
\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{k=1}^{N} |f_{u^k}| - \| f \|_1 \right\|_\infty = 0,
\]

which, by part (1), implies

\[
\| f \|_1 = \alpha(\| f \|_1 \cdot 1) = \lim_{N \to \infty} \alpha \left( \frac{1}{N} \sum_{k=1}^{N} |f_{u^k}| \right) \leq \alpha(f),
\]

since \( \alpha \) is rotationally invariant.

Next suppose \( f \in L^\infty(T) \). If we choose the \( g_n \)'s as in the proof of part (7) (replacing \( \alpha \) with \( \| \cdot \|_1 \)), we get \( \| g_n - f \|_1 \to 0 \). Hence

\[
\| f \|_1 = \lim_{n \to \infty} \| g_n \|_1 \leq \lim_{n \to \infty} \alpha(g_n) = \alpha(f).
\]

Finally, suppose \( f : T \to \mathbb{C} \) is measurable. We can choose a sequence \( \{ s_n \} \) of simple functions such that \( 0 \leq s_1 \leq s_2 \leq \cdots \) and \( s_n(\omega) \to |f(\omega)| \) for every \( \omega \in T \).

It follows from part (6) and the monotone convergence theorem that

\[
\alpha(f) = \lim_{n \to \infty} \alpha(s_n) \leq \lim_{n \to \infty} \| s_n \|_1 = \| f \|_1.
\]

(9) (a) Since \( \alpha(h) = \alpha(|h|) \) for every measurable function \( h \), we may, without loss of generality, assume \( f \) and \( g \) are both nonnegative measurable functions on \( T \). Choose the \( s_n \)'s as in the proof of part (8) to get \( \alpha(f) = \lim_{n \to \infty} \alpha(s_n) \). Similarly, choose another sequence \( \{ r_n \} \) of simple functions such that \( \alpha(g) = \lim_{n \to \infty} \alpha(r_n) \).

It is easy to see \( \{ s_n + r_n \} \) is an increasing sequence of simple functions with \( s_n(w) + r_n(w) \to f(w) + g(w) \) for every \( w \in T \). It follows from part (6) that

\[
\alpha(f + g) = \lim_{n \to \infty} \alpha(s_n + r_n) \leq \lim_{n \to \infty} \alpha(s_n) + \lim_{n \to \infty} \alpha(r_n) = \alpha(f) + \alpha(g).
\]

(b) It follows from the proof in (9a) that

\[
\alpha(\lambda f) = \alpha(|\lambda|f) = \lim_{n \to \infty} \alpha(|\lambda|s_n) = |\lambda| \lim_{n \to \infty} \alpha(s_n) = |\lambda|\alpha(f).
\]

(c) Suppose \( \alpha(f) = 0 \) for some measurable function \( f \) on \( T \). Then, by part (8), \( 0 \leq \| f \|_1 \leq \alpha(f) = 0 \), which implies that \( f = 0 \). Hence in this case, the seminorm \( \alpha \) is actually a norm.
(10) It follows from the definition of \( L^\alpha(T) \) that \( (L^\alpha(T), \alpha) \) is a normed space. To prove the completeness, suppose \( \{f_n\} \) is a sequence in \( L^\alpha(T) \) with \( \sum_{n=1}^{\infty} \alpha(f_n) < \infty \). Then
\[
\infty > \sum_{n=1}^{\infty} \alpha(f_n) \geq \sum_{n=1}^{\infty} \|f_n\|_1 = \sum_{n=1}^{\infty} \int_T |f_n| dm = \int_T \sum_{n=1}^{\infty} |f_n| dm.
\]
If we let \( f = \sum_{n=1}^{\infty} |f_n| \), then \( f \in L^1(T) \). Since \( \{g_N = \sum_{n=1}^N |f_n| : N \geq 1\} \) is an increasing sequence with \( g_N(w) \to f(w) \) a.e. \( (m) \), it follows from part (6) that
\[
\alpha(f) = \lim_{N \to \infty} \alpha(g_N) = \lim_{N \to \infty} \alpha(\sum_{n=1}^{N} |f_n|) = \sum_{n=1}^{\infty} \alpha(|f_n|) = \sum_{n=1}^{\infty} \alpha(f_n) < \infty,
\]
which implies that \( f \in L^\alpha(T) \). Applying part (6) again,
\[
\alpha(f - \sum_{n=1}^{N} |f_n|) = \alpha(\sum_{n=N}^{\infty} |f_n|) = \sum_{n=N}^{\infty} \alpha(|f_n|) = \sum_{n=N}^{\infty} \alpha(f_n) \to 0.
\]
Hence \( \sum_{n=1}^{\infty} f_n \leq \sum_{n=1}^{\infty} |f_n| < \infty \), which tells us that \( (L^\alpha(T), \alpha) \) is a Banach space. Furthermore, it follows from the definition of \( L^\alpha(T) \), \( L^\alpha(T) \) and part (8) that
\[
L^\infty(T) \subset L^\alpha(T) \subset L^\alpha(T) \subset L^1(T).
\]

The following lemma gives ways of constructing examples of rotationally symmetric norms of different types. See Remark 12 for more details. We first construct a sigma-algebra on \( \mathcal{R} \). For each \( f \in L^\infty(T) \), we have a mapping \( \pi_f : \mathcal{R} \to [0, \infty) \) by
\[
\pi_f (\alpha) = \alpha(f).
\]
Note that a net \( \{\alpha_\lambda\} \) in \( \mathcal{R} \) converges pointwise to \( \alpha \) if and only if, for every \( f \in L^\infty(T) \), \( \pi_f (\alpha_\lambda) \to \pi_f (\alpha) \). It follows that the weak topology \( \mathcal{T}(\mathcal{R}) \) on \( \mathcal{R} \) induced by the family \( \{\pi_f : f \in L^\infty(T)\} \) is the topology of pointwise convergence, and the weak topology on \( \mathcal{S} \) induced by \( \{\pi_f|_S : f \in L^\infty(T)\} \) is \( \mathcal{T}(\mathcal{S}) = \{U \cap S : U \in \mathcal{T}(\mathcal{R})\} \) and is the topology of pointwise convergence on \( \mathcal{S} \). We let \( \mathcal{M}(\mathcal{R}) \) denote the smallest \( \sigma \)-algebra on \( \mathcal{R} \) for which each \( \pi_f (f \in L^\infty(T)) \) is measurable. Similarly, \( \mathcal{M}(\mathcal{S}) = \{E \cap S : E \in \mathcal{M}(\mathcal{R})\} \) is the smallest \( \sigma \)-algebra on \( \mathcal{S} \) for which each \( \pi_f \) is measurable.

If \( (\Omega, \mathcal{N}, \lambda) \) is a probability space and \( \rho : \Omega \to \mathcal{R} \), then \( \rho \) is \( \mathcal{M}(\mathcal{R})-\mathcal{N} \) measurable if and only if, for each \( f \in L^\infty(T) \), \( \pi_f \circ \rho : \Omega \to [0, \infty) \) is \( \mathcal{N} \)-measurable. We can uniquely define \( f_\Omega \rho d\lambda \) as a function on \( L^\infty(T) \) by
\[
\left( \int_{\Omega} \rho d\lambda \right)(f) = \int_{\Omega} (\pi_f \circ \rho)(\omega) d\lambda = \int_{\Omega} (\rho(\omega)) (f) d\lambda(\omega).
\]

Lemma 2.3. The following are true.

(1) \( \mathcal{R} \), \( \mathcal{S} \), \( \mathcal{R}_c \), \( \mathcal{S}_c \) are convex;

(2) \( \mathcal{R} \) and \( \mathcal{S} \) are compact in the topology of pointwise convergence, \( \mathcal{S} \) is metrizable, and \( \mathcal{M}(\mathcal{S}) \) is the collection \( \text{Bar}(\mathcal{S}) \) of Borel subsets of \( \mathcal{S} \);

(3) If \( \alpha_1, \alpha_2, \cdots \in \mathcal{R}_c \), and \( t_1, t_2, \cdots > 0 \) with \( \sum_{i=1}^{\infty} t_n = 1 \), then \( \sum_{i=1}^{\infty} t_n \alpha_n \in \mathcal{R}_c \);

(4) Suppose \( \mathcal{E} \) is a nonempty set of seminorms on \( L^\infty(T) \) such that
(a) \( \gamma(|f|) = \gamma(f) \) for every \( f \in L^\infty(T) \) and for every \( \gamma \in \mathcal{E} \) and
(b) \(1 = \sup \{\gamma(1) : \gamma \in \mathcal{E}\}\), and define \(\alpha\) by
\[
\alpha(f) = \sup_{\gamma \in \mathcal{E}} \gamma(f).
\]

Then
(c) If \(\gamma(f) = \gamma(f_w)\) for every \(f \in L^\infty(\mathbb{T})\), for every \(\gamma \in \mathcal{E}\) and every \(\omega \in \mathbb{T}\), then \(\alpha \in \mathcal{R}\);
(d) If \(\gamma(f) = \gamma(f \circ \phi)\) for every \(f \in L^\infty(\mathbb{T})\), for every \(\gamma \in \mathcal{E}\) and every \(\phi \in \mathcal{MP}(\mathbb{T})\), then \(\alpha \in \mathcal{S}\).

(5) If \(h : \mathbb{T} \to \mathbb{C}\) and \(h \geq 0\) and \(0 < \int_{\mathbb{T}} h \, dm \leq 1\), then the maps \(\alpha_h, \beta_h : L^\infty(\mathbb{T}) \to [0, \infty)\) defined by
\[
\alpha_h(f) = \sup_{w \in \mathbb{T}} \int_{\mathbb{T}} |f_w| \, dm
\]
and
\[
\beta_h(f) = \sup_{\phi \in \mathcal{MP}(\mathbb{T})} \int_{\mathbb{T}} |f \circ \phi| \, dm = \int_{\mathbb{T}} |f| \, dm
\]
satisfy, for every \(f \in L^\infty(\mathbb{T})\), \(\alpha_h(f) = \alpha_h(|f|) = \alpha_h(f_w)\) for every \(w \in \mathbb{T}\) and \(\beta_h(f) = \beta_h(|f|) = \beta_h(f \circ \phi)\) for every \(\phi \in \mathcal{MP}(\mathbb{T})\). If \(\int_{\mathbb{T}} h \, dm = 1\), then \(\alpha_h \in \mathcal{R}_c\) and \(\beta_h \in \mathcal{S}_c\).

(6) If \(\alpha \in \mathcal{S}\) and \(t = \lim_{m(E) \to 0^+} \alpha(\chi_E)\), then there is a unique \(\beta \in \mathcal{S}_c\) such that
\[
\alpha = (1 - t) \beta + t \| \cdot \|_\infty.
\]

**Proof.** (1) This is obvious.

(2) Suppose \(\{\alpha_\lambda\}\) is an ultrafilter in \(\mathcal{R}\) (respectively, \(\mathcal{S}\)). Then \(\alpha_\lambda(f) \leq \|f\|_\infty < \infty\) for every \(f \in L^\infty(\mathbb{T})\), which implies that \(\{\alpha_\lambda(f)\}\) is an ultrafilter in the compact set \(\{z \in \mathbb{C} : |z| \leq \|f\|_\infty\}\), so
\[
\alpha(f) = \lim_{\lambda} \alpha_\lambda(f)
\]
exesits for every \(f \in L^\infty(\mathbb{T})\). It is clear that \(\alpha \in \mathcal{R}\) (respectively, \(\mathcal{S}\)). Since every net has a subnet that is an ultrafilter, we see that \(\mathcal{R}\) and \(\mathcal{S}\) are compact. For the proof that \(\mathcal{S}\) is metrizable we identify \((0, 1)\) with \(\mathbb{T}\) (identifying \(t\) with \(e^{2\pi it}\)). Let \(\mathcal{W}\) be the set of all simple functions on \((0, 1)\) of the form \(s = \sum_{k=1}^n r_k \chi_{(a_k-1,a_k)}\) with \(0 = a_0 < a_1 < \cdots < a_n \leq 1\) and \(0 \leq r_1, \ldots, r_n\) and \(a_0, r_1, a_1, \ldots, r_n, a_n\) rational numbers. Clearly \(\mathcal{W} = \{f_1, f_2, \ldots\}\) is countable. We claim that \(\mathcal{T}(\mathcal{S})\) is the weak topology induced by \(\{\pi_{f_n} : n \in \mathbb{N}\}\). Suppose \(\{\alpha_\lambda\}\) is a net in \(\mathcal{S}\) and \(\alpha \in \mathcal{S}\) and \(\lim_n \pi_{f_n} (\alpha_\lambda) = \pi_{f_n} (\alpha)\) for every \(n \in \mathbb{N}\). From the compactness of \(\mathcal{S}\), we can choose a subnet \(\{\alpha_{\lambda_k}\}\) and a \(\beta \in \mathcal{S}\) such that, for every \(f \in L^\infty((0, 1))\), \(\lim_k \pi_f (\alpha_{\lambda_k}) = \pi_f (\beta)\). Since, by definition, \(\pi_g (\gamma) = \gamma (g)\), we see that \(\beta (f) = \alpha (f)\) for every \(f \in \mathcal{W}\). Suppose \(s \in L^\infty((0, 1))\) is a simple function. Then \(|s|\) is a nonnegative simple function, so there is a \(\phi \in \mathcal{MP}((0, 1))\) such that
\[
|s| \circ \phi = \sum_{k=1}^n s_k \chi(b_k-1,b_k)
\]
with \(0 = b_0 < b_1 < \cdots < b_n \leq 1\) and \(s_1, \ldots, s_n \geq 0\). Clearly, there is a sequence \(\{w_n\}\) in \(\mathcal{W}\) such that \(0 \leq w_1 \leq w_2 \leq \cdots\) and \(\lim_{j \to \infty} w_j (t) = (|s| \circ \phi) (t)\) for every \(t \in (0, 1)\), that is, if, for each \(k\), we choose rational numbers \(0 \leq s_{k,1} \leq s_{k,2} \leq \cdots\) converging to \(s_k\) and rational numbers \(b_{k,1} < \cdots < c_{k,2} < c_{k,1} < d_{k,1} < d_{k,2} <\)
\[ \cdots < b_k \text{ with } c_{k,j} \to b_{k-1} \text{ and } d_{k,j} \to b_k, \text{ and we let } w_j = \sum_{k=1}^n s_{k,j} \chi(c_{k,j},d_{k,j}). \]

It follows from part (6) of Lemma 2.2 that

\[ \alpha(s) = \alpha(|s| \circ \phi) = \lim_{j \to \infty} \alpha(w_j) = \lim_{j \to \infty} \beta(w_j) = \beta(s). \]

Since the simple functions are \( \| \cdot \|_\infty \)-dense in \( L^\infty(T) \), we see that \( \alpha = \beta \). Hence \( \alpha \to \alpha \) with respect to \( T(S) \). Thus the weak topology on \( S \) induced by \( \{ \pi_f : f \in W \} \) is \( T(S) \). Since \( W \) is countable, \( S \) is metrizable. Thus \( (S, T(S)) \) is a compact metric space, and \( Bor(S) \) is the smallest \( \sigma \)-algebra for which each \( \pi_f \) \((f \in W) \) measurable, and hence it is the smallest \( \sigma \)-algebra that makes each \( \pi_f \) \((f \in L^\infty(T)) \) measurable.

(3) Let \( \alpha = \sum_{n=1}^\infty t_n \alpha_n \). It easily follows that \( \alpha \) is a rotationally symmetric semi-norm. Suppose \( \sum_{n=1}^\infty t_n = 1 \) and \( \epsilon > 0 \). Then there is an \( N \in \mathbb{N} \) such that \( \sum_{n=N+1}^\infty t_n < \frac{\epsilon}{2} \) for all \( n \geq N \). Thus for any measurable set \( E \subset T \),

\[ \sum_{n=N+1}^\infty t_n \alpha(\chi_E) \leq \sum_{n=N+1}^\infty t_n \cdot 1 < \frac{\epsilon}{2}. \]

Since \( \alpha_1, \alpha_2, \ldots, \alpha_N \) are continuous, there is a \( \delta > 0 \) such that when \( E \subset T \) and \( 0 < m(E) < \delta \), we have \( \alpha_k(\chi_E) < \frac{\epsilon}{2N} \) for all \( 1 \leq k \leq N \). Thus \( n \geq N \) and \( E \subset T \) and \( 0 < m(E) < \delta \)

implies

\[ \alpha(\chi_E) = \sum_{n=1}^N t_n \alpha_n(\chi_E) + \sum_{n=N+1}^\infty t_n \alpha_n(\chi_E) \]

\[ \leq \sum_{n=1}^N t_n \frac{\epsilon}{2N} + \frac{\epsilon}{2} \]

\[ < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]

Hence \( \alpha \) is a continuous rotationally symmetric semi-norm.

(4) Suppose the hypotheses of (4a), (4b), (4c) hold. Then \( \alpha(1) = \sup_{\gamma \in E} \gamma(1) = 1 \), \( \alpha(|f|) = \sup_{\gamma \in E} \gamma(|f|) = \alpha(f) \), and \( \alpha(f_\omega) = \sup_{\gamma \in E} \gamma(f_\omega) = \sup_{\gamma \in E} \gamma(f) = \alpha(f) \).

Since \( E \) is a set of seminorms on \( L^\infty(T) \), \( \sup_{\gamma \in E} \gamma \) is a seminorm, which implies that \( \alpha \) is a seminorm on \( L^\infty(T) \). Hence \( \alpha \in R \). The proof of (4d) is similar.

(5) It is clear that \( \alpha_h(1) = \sup_{\omega \in E} \int_T |1| \omega dm = \int_T \omega dm = 1 \). Since \( h \geq 0 \) and \( 0 < \int_T \omega dm \leq 1 \), it follows that \( \alpha_h(f) \geq 0 \), \( \alpha(\lambda f) = |\lambda| \alpha(f) \) and \( \alpha(f+g) \leq \alpha(f) + \alpha(g) \) for every \( f, g \in L^\infty(T) \) and every \( \lambda \in \mathbb{C} \). Furthermore, suppose \( \epsilon > 0 \). Since \( \int_T \omega dm \leq 1 \), there is a \( \delta > 0 \) such that when \( E \subset T \) and \( m(E) < \delta \), we have \( |\int_E \omega dm| < \epsilon \), thus

\[ |\alpha_h(\chi_E)| = |\int_T h(\chi_E) \omega dm| = |\int_T h \chi_{E^c} \omega dm| = |\int_{E^c} \omega dm| < \epsilon \]

as \( m(E) < \delta \). Therefore \( \lim_{m(E) \to 0^+} \alpha_h(\chi_E) = 0 \), which means \( \alpha_h \in R \). The proof for \( \beta_h \) is similar.

(6) If \( t = 0 \), then \( \alpha \in S_c \) and \( \beta = \alpha \). Suppose \( \lim_{m(E) \to 0} \alpha(\chi_E) = t > 0 \). Then

\[ \|f\|_\infty \geq \alpha(f) \geq \alpha(f\chi_E) \geq t\|f\|_\infty, \]

which means \( 0 < t \leq 1 \) and \( \beta = (\alpha - t \cdot \|\cdot\|_\infty) / (1 - t) \) defines an element of \( S_c \). \( \square \)
3. Dual Norms

Suppose $\alpha$ is a rotationally invariant norm on $L^\infty(\mathbb{T})$. We define the dual norm $\alpha'$ on $L^\infty(\mathbb{T})$ by

$$\alpha'(f) = \sup \{ \left| \int_T fh dm \right| : h \in L^\infty(\mathbb{T}), \alpha(h) \leq 1 \}$$

$$= \sup \{ \int_T |fh| dm : h \in L^\infty(\mathbb{T}), \alpha(h) \leq 1 \}.$$

Lemma 3.1. Suppose $\alpha \in \mathcal{R}$. The following statements are true.

1. $\alpha' \in \mathcal{R}$;
2. $\alpha \in \mathcal{S} \implies \alpha' \in \mathcal{S}$.

Proof. (1) It is clear that $\alpha'$ is a seminorm. Suppose $f \in L^\infty(\mathbb{T})$ and $w \in \mathbb{T}$. It follows from the definition of $\alpha'$ that $\alpha'(1) = 1$, $\alpha'(|f|) = \alpha'(f)$ and $\alpha'(f_w) = \alpha'(f)$. Hence $\alpha' \in \mathcal{R}$.

(2) This assertion was proved in [1].

The following result is probably not new [15, Theorem 5.11]. We will visit these ideas again in Section 10.

Proposition 3.2. Suppose $\alpha \in \mathcal{R}$. Then

1. If $\alpha$ is continuous, then $L^\alpha(\mathbb{T})^2 = \mathcal{L}^\alpha(\mathbb{T})$, i.e., for every $\phi \in L^\alpha(\mathbb{T})^2$, there is a $h \in \mathcal{L}^\alpha(\mathbb{T})$ such that

$$\phi(f) = \int_T fh dm$$

for all $f \in L^\alpha(\mathbb{T})$ and with $\|\phi\| = \alpha'(h)$;
2. If $\alpha$ is continuous, then $\alpha'' = \alpha$;
3. $L^\alpha(\mathbb{T})# = L^\alpha(\mathbb{T})$ and $H^\alpha(\mathbb{T})## = H^\alpha(\mathbb{T})$ if $\alpha$ and $\alpha'$ are both strongly continuous;
4. If $\alpha \in \mathcal{S}$, then $L^\alpha(\mathbb{T})## = L^\alpha(\mathbb{T})$ if and only if $\alpha$ and $\alpha'$ are both strongly continuous.

Proof. (1) Suppose $\alpha \in \mathcal{R}_c$. For any measurable set $E \subset \mathbb{T}$, define

$$\lambda(E) = \phi(\chi_E).$$

Then $\lambda(\emptyset) = \phi(\chi_\emptyset) = \phi(0) = 0$. Also, Since $\phi$ is linear, and since $\chi_{A \cup B} = \chi_A + \chi_B$ if $A$ and $B$ are disjoint, we see $\lambda$ is additive. To prove countable additivity, suppose $E$ is the union of countably many disjoint measurable sets $E_i$, put $A_k = E_1 \cup E_2 \cup \cdots \cup E_k$, and note that

$$m(E - A_k) \to 0 \quad (k \to \infty).$$

The continuity of $\alpha$ implies $\alpha(\chi_{E - A_k}) = \alpha(\chi_E - \chi_{A_k}) \to 0$, and $\phi(\chi_E - \chi_{A_k}) \to 0$. Therefore $\lambda(A_k) \to \lambda(E)$, which is $\lambda(E) = \lambda(\cup_{k=1}^\infty A_k) = \sum_{k=1}^\infty \lambda(A_k)$. So $\lambda$ is a complex measure. It is clear that $\lambda(E) = 0$ if $m(E) = 0$, since then $\|\chi_E\|_\infty = 0$. Thus $\lambda << m$, and the Radon-Nikodym theorem ensures the existence of a function $h \in L^1(\mathbb{T})$ such that, for every measurable $E \subset \mathcal{X}$,

$$\phi(\chi_E) = \lambda(E) = \int_T \chi_E h dm.$$
and such that
\[ \phi(f) = \int_T f h \, dm \]
for every \( f \in L^\infty(T) \) (First consider \( f \) a simple function and use \( \alpha \leq \| \cdot \|_\infty \)).

The uniqueness of \( h \) is clear, for if \( h \) and \( h' \) satisfy (1), then the integral of \( h - h' \) over any measurable set \( E \) of finite measure is 0 (as we see by taking \( \chi_E \) for \( f \)), and the \( \sigma \)-finiteness of \( m \) implies that \( h - h' = 0 \) a.e. (\( m \)). Furthermore, since \( L^\infty(T) \) is dense in \( L^\alpha(T) \), we see
\[ \| \phi \| = \sup \{ |\phi(f)| : f \in L^\infty(T) \}, \alpha(f) \leq 1 \]
\[ = \sup \left\{ \int_T f h \, dm : f \in L^\infty(T) \right\}, \alpha(f) \leq 1 \]
\[ = \alpha'(h). \]

Thus \( h \in L^\alpha(T) \).

(2) Suppose \( f \in L^\infty(T) \) with \( \alpha(f) = 1 \). It follows from
\[ \alpha'(h) = \sup \left\{ \int_T |f h| \, dm : h \in L^\infty(T), \alpha(h) \leq 1 \right\} \]
that
\[ \alpha''(f) = \sup_{h \in L^\infty(T), \alpha(h) \leq 1} \int_T |f h| \, dm \leq \sup_{h \in L^\infty(T), \alpha(h) \leq 1} \alpha'(h) = 1. \]

By the Hahn-Banach theorem, there is a continuous linear functional \( \phi \in L^\alpha(T)^\# \) such that \( \phi(f) = \alpha(f) = 1 \) and \( \| \phi \| = 1 \). Since \( \phi \in L^\alpha(T)^\# \), there is an element \( h \in L^\alpha(T) \) such that \( \phi(|f|) = \int_T |f| h \, dm = 1 \) and \( \alpha'(h) = \| \phi \| = 1 \). Thus
\[ 1 = \int_T |f| h \, dm \leq \sup_{h \in L^\alpha(T), \alpha'(h) \leq 1} \int_T |f| h \, dm = \alpha''(f), \]
and so \( \alpha''(f) = 1 = \alpha(f) \). Next suppose \( f \neq 0 \). Then \( \alpha(\frac{f}{|f|}) = 1 \), and it follows that \( \alpha''(\frac{f}{|f|}) = 1 \), which means \( \alpha''(f) = \alpha(f) \).

(3) This is clear from (1) and (2).

(4) If \( \alpha \in S \) and \( \alpha \) or \( \alpha' \) is not continuous, then one of \( \alpha, \alpha' \) is equivalent to \( \| \cdot \|_\infty \) and the other is equivalent to \( \| \cdot \|_1 \), so \( L^\alpha(T)^{\#\#} \neq L^\alpha(T) \). Also if \( L^\alpha(T) \neq L^\alpha(T) \), then there is a \( 0 \neq \phi \in L^\alpha(T)^\# \) such that \( \phi = 0 \) on \( L^\alpha(T) \) and such a \( \phi \) cannot be written in the form \( \phi(f) = \int_T f h \, dm \), e.g., let \( f = h/|h| \in L^\infty(T) \), which implies \( L^\alpha(T)^{\#\#} \neq L^\alpha(T) \). \( \square \)

**Theorem 3.3.** Suppose \( \alpha \) is a rotationally symmetric norm and \( T : L^\alpha(T) \rightarrow L^1(T) \) is a bounded linear operator such that, for every \( h \in L^\infty(T) \) and every \( g \in L^\alpha(T) \),
\[ T(h g) = h T(g). \]
Then there is an \( f \in L^\alpha'(T) \) such that, for every \( g \in L^\alpha(T) \),
\[ T g = f g. \]
Moreover, \( \| T \| = \alpha'(f) \).

The same conclusion holds if \( L^\alpha(T) \) is replaced with \( L^\alpha(T) \).
Proof. Let \( f = T(1) \). Then \( Tg = fg \) for every \( g \in L^\infty(\mathbb{T}) \). Suppose \( g \in L^\alpha(\mathbb{T}) \).

Define

\[
u(z) = \begin{cases} g(z) & \text{if } |g(z)| \leq 1 \\ 1 & \text{if } |g(z)| > 1 \end{cases}
\]

and

\[
v(z) = \begin{cases} 1 & \text{if } |g(z)| \leq 1 \\ 1/g(z) & \text{if } |g(z)| > 1 \end{cases}.
\]

Then \( u, v \in L^\infty(\mathbb{T}) \), \( v(z) \) is never 0, and \( g = u/v \). Then

\[vT(g) = T(u) = uT(1) = fu\]

implies \( Tg = fg \). Also

\[
\alpha'(f) = \sup_{h \in L^\infty(\mathbb{T}), \alpha(h) \leq 1} \left| \int_T fh \, d\mu \right| \leq ||T|| < \infty.
\]

On the other hand, \( ||Tg||_1 = ||fg||_1 \leq \alpha'(f) \alpha(g) \) implies \( ||T|| \leq \alpha'(f) \). \qed

Corollary 3.4. Suppose \( \alpha \) is a rotationally symmetric norm and \( f : \mathbb{T} \rightarrow \mathbb{C} \) is measurable. Then

\[f \cdot L^\alpha(\mathbb{T}) \subset L^1(\mathbb{T}) \iff f \in \mathcal{L}^{\alpha'}(\mathbb{T}).\]

The preceding theorem yields a Banach space characterization of strongly continuous norms. A Banach space \( X \) is weakly sequentially complete if and only if every weakly Cauchy sequence is weakly convergent.

Theorem 3.5. Suppose \( \alpha \in \mathcal{R}_c \) has dual norm \( \alpha' \). The following are equivalent:

(1) \( L^\alpha(\mathbb{T}) = L^\alpha(\mathbb{T}) \) (\( \alpha \) is strongly continuous);

(2) \( L^\alpha(\mathbb{T}) \) is weakly sequentially complete.

Proof. (1) \( \Rightarrow \) (2) Suppose (1) is true, and suppose \( \{f_n\} \) is a weakly Cauchy sequence in \( L^\alpha(\mathbb{T}) \). Then, by the uniform boundedness theorem, \( s = \sup_{k \geq 1} \alpha (f_k) < \infty \). Also, for every \( h \in \mathcal{L}^{\alpha'}(\mathbb{T}) = L^\alpha(\mathbb{T})^\# \) and every \( u \in L^\infty(\mathbb{T}) \) we have \( \{ \int_T f_n h \, d\mu \} \) is Cauchy, which means

\[
\lim_{n \rightarrow \infty} \int_T f_n h \, d\mu \text{ exists.}
\]

However, \( \{f_n h\} \) is a sequence in \( L^1(\mathbb{T}) \) and \( L^1(\mathbb{T})^\# = L^\infty(\mathbb{T}) \), so it follows that \( \{f_n h\} \) is weakly Cauchy in \( L^1(\mathbb{T}) \). However, \( L^1(\mathbb{T}) \) is weakly sequentially complete \( [15] \), so there is a \( T(h) \in L^1(\mathbb{T}) \), such that, for every \( u \in L^\infty(\mathbb{T}) \), we have

\[
\lim_{n \rightarrow \infty} \int_T f_n h \, d\mu = \int_T T(h) u \, d\mu.
\]

The map \( T : \mathcal{L}^{\alpha'}(\mathbb{T}) \rightarrow L^1(\mathbb{T}) \) is clearly linear. Moreover,

\[
||T(h)||_1 = \sup_{u \in L^\infty(\mathbb{T}), ||u||_\infty \leq 1} \left| \int_T T(h) u \, d\mu \right| = \lim_{n \rightarrow \infty} \left| \int_T f_n h \, d\mu \right| \leq s \alpha'(h),
\]

since \( \left| \int_T f_n h \, d\mu \right| \leq \alpha (f_n) \alpha'(h) \leq s \alpha'(h) ||u||_\infty \). For every \( u, w \in L^\infty(\mathbb{T}) \) and \( h \in \mathcal{L}^{\alpha'}(\mathbb{T}) \), we have

\[
\int_T T(hw) u \, d\mu = \lim_{n \rightarrow \infty} \int_T f_n (hw) u \, d\mu = \int_T T(h) w u \, d\mu.
\]
and we conclude that $T(hw) = T(h)w$. It follows from Theorem 3.3 that there is an $f \in L^{\alpha'}(T) = L^{\alpha}(T) = L^\alpha(T)$ such that $T(h) = fh$ for every $h \in L^{\alpha'}(T)$. From the definition of $T$ we see that $f_n \to f$ weakly.

(2) $\Rightarrow$ (1) Suppose $L^\alpha(T)$ is weakly sequentially complete, and suppose $f \in L^\alpha(T)$. Then we can choose a sequence $\{s_n\}$ of simple functions such that $0 \leq s_1 \leq s_2 \leq \cdots$ and $s_n(z) \to |f(z)|$ for every $z \in T$. If $h \in L^{\alpha'}(T)$ and $h \geq 0$, then, by the monotone convergence theorem,

$$\lim_{n \to \infty} \int_T s_n h dm = \int |f| h dm.$$ 

Hence, since $L^{\alpha'}(T)$ is the linear span of its nonnegative elements, the above limit holds for every $h \in L^{\alpha'}(T)$. Hence $\{s_n\}$ is weakly Cauchy in $L^\alpha(T)$, so there is a $w \in L^\alpha(T)$ such that, for every $h \in L^{\alpha'}(T)$,

$$\lim_{n \to \infty} \int_T s_n h dm = \int wh dm.$$ 

Clearly $|f| = w \in L^\alpha(T)$, which means $f \in L^\alpha(T)$. Thus $L^{\alpha'}(T) = L^\alpha(T)$. \hfill $\Box$

4. Dominated Convergence Theorem on $L^\alpha(T)$

In this section we prove a dominated convergence theorem that generalizes the classical dominated convergence theorem when $\alpha = \|\|_1$. We first prove an extension of the notion of continuity for a norm in $\mathbb{R}$.

**Theorem 4.1. (General Continuity Theorem)** Suppose $\alpha$ is a continuous gauge seminorm and $g \in L^\alpha(T)$. Then $\lim_{m(E) \to 0^+} \alpha(g \chi_E) = 0$.

**Proof.** Suppose $g \in L^\alpha(T)$ and $\epsilon > 0$. Then there is an $f \in L^\infty(T)$ such that $\alpha(g - f) < \frac{\epsilon}{2}$. Since $\alpha$ is continuous, there is $\delta > 0$ such that when $E \subset T$ and $0 < m(E) < \delta$, we obtain $\alpha(\chi_E) < \frac{\epsilon}{2\|f\|_\infty}$. Thus $E \subset T$ and $0 < m(E) < \delta$ implies that

$$\alpha(g \chi_E) = \alpha((g - f)\chi_E + f \chi_E) 
\leq \alpha((g - f)\chi_E) + \alpha(f \chi_E) 
\leq \alpha(g - f)\|\chi_E\|_\infty + \|f\|_\infty \alpha(\chi_E) 
\leq \frac{\epsilon}{2} + \frac{\epsilon}{2\|f\|_\infty} = \epsilon,$$

which gives the result. \hfill $\Box$

**Remark 4.2.** In this remark we identify $T$ with $[0, 1]$. Suppose $0 < t \leq 1$. If we choose a subset $A \subset T$ with $m(A) = t$, and we let $h = \frac{1}{t} \chi_A$, then the norm $\beta_h$ defined in part (5) of Lemma 3.3 is called the Ky Fan norm and is denoted by $\|\|_t$ in $S_c$. Thus, for any measurable $f : [0, 1] \to \mathbb{C}$, we have

$$\|f\|_t = \frac{1}{t} \int_0^t |f(r)|^\star dr,$$

the average over $(0, t)$ of the nonincreasing rearrangement of $|f|$. This fact allows us to focus on nonincreasing nonnegative functions $f$.

Suppose $u : (0, 1] \to [0, 1]$ is any function (maybe not even measurable) with

$$\sup \{u(t) : t \in [0, 1]\} = 1.$$
Then we can define a norm $\beta^u$ by

$$
\beta^u(f) = \sup \{ u(t) \| f \|_t : t \in [0, 1] \} = \sup \{ u(t) \| f \|_t : t \in (0, 1) \}.
$$

It follows from part (4) of Lemma 2.3 that $\beta^u \in S$. It is not hard to show that if $u = \chi(0,1)$, then $\beta^u = \| \cdot \|_\infty$ and if $u = \chi(1)$ or $u(t) = t$, then $\beta^u = \| \cdot \|_1$. Thus we might have $\beta^u \notin S_e$.

We know that $\beta^u \in S_e$ if and only if

$$
\lim_{s \to 0^+} \beta^u(\chi(0,s)) = 0.
$$

However,

$$
u(t) \| \chi(0,s) \|_t = \frac{u(t) \min(s,t)}{t}.
$$

Hence $\beta^u \in S_e$ if and only if

$$
\lim_{s \to 0^+} \sup_{0 < t \leq 1} \frac{u(t) \min(s,t)}{t} = 0.
$$

Since $\sup_{0 < t \leq 1} \frac{u(t) \min(s,t)}{t} \geq \sup_{0 \leq t \leq s} u(t)$, we conclude that $\beta^u \in S_e \Rightarrow \lim_{s \to 0^+} u(s) = 0$.

If $u(t)/t$ is decreasing, then

$$
\sup_{s \leq t \leq 1} \frac{u(t) \min(s,t)}{t} = u(s),
$$

which means that

$$
\beta^u(\chi(0,s)) = \max \left( u(s), \sup_{0 \leq t \leq s} u(t) \right).
$$

In this case we see that $\beta^u \in S_e$ if and only if $\lim_{t \to 0^+} u(t) = 0$.

It is clear that if $f : (0, 1] \to [0, \infty]$ is decreasing, then $\| f \|_t = \frac{1}{t} \int_0^t f(x) \, dx$. So $f \in \mathcal{L}^{\beta^u}(\mathbb{T})$ if and only if

$$
\sup_{0 < t \leq 1} \frac{u(t)}{t} \int_0^t f(x) \, dx < \infty.
$$

It follows from the continuity theorem that, for decreasing $f$, that $f \in \mathcal{L}^{\beta^u}(\mathbb{T})$ if and only if

$$
\lim_{s \to 0^+} \sup_{0 < t \leq s} \frac{u(t)}{t} \int_0^t f(x) \, dx = 0.
$$

So, for example, when $u(t) = \sqrt{t}$, we see that $f(t) = \frac{1}{2\sqrt{t}} \in \mathcal{L}^{\beta^u}(\mathbb{T})$ but not in $\mathcal{L}^{\beta^u}(\mathbb{T})$.

If $\varphi(t) = t/u(t)$ with $\varphi(0) = 0$ is concave and increasing, $\varphi(1) = 1$ and $\lim_{t \to 0^+} u(t) = \lim_{t \to 0^+} \frac{1}{\varphi(t)} = 0$, then $\beta^u$ is called the Marcinkiewicz norm on $\mathcal{L}^{\infty}(\mathbb{T})$ corresponding to $\varphi$, and these norms are continuous but not strongly continuous, i.e., $\mathcal{L}^{\beta^u}(\mathbb{T}) \neq \mathcal{L}^{\beta^u}(\mathbb{T})$. For example, if $u(t) = \sqrt{t}$, $\beta^u$ is such a norm.

We now prove our generalized dominated convergence theorem. The generalization is in two senses. The first is by extending from the $L^p$-norms to continuous rotationally symmetric norms and the second is from greatly extending the notion of dominance. Note that if $f, g : \mathbb{T} \to \mathbb{C}$ are measurable, then $|f| \leq |g|$ if and only if there is a function $u \in \mathcal{L}^{\infty}(\mathbb{T})$ with $\|u\|_\infty \leq 1$ such that $f = ug$. 
Theorem 4.3. (Dominated Convergence Theorem) Suppose \( \alpha \) is a continuous rotationally symmetric norm. Let
\[
G_\alpha = \{ \varphi \in MP (\mathbb{T}) : \forall h \in L^\infty (\mathbb{T}), \; \alpha (h \circ \varphi) = \alpha (h) \}.
\]
Suppose \( g \in L^\alpha (\mathbb{T}) \), let
\[
K = \text{co} (\{ |g \circ \varphi| \; u : \|u\|_\infty \leq 1, \; \varphi \in G_\alpha \}),
\]
and let \( \overline{K}^m \) denote the closure of \( K \) in the topology of convergence in measure.

Suppose \( \{f_n\} \) is a sequence with \( |f_n| \in \overline{K}^m \) (\( n \in \mathbb{N} \)) and such that \( f_n \to f \) in measure.

Then
1. \( f \in L^\alpha (\mathbb{T}) \), and
2. \( \alpha (f_n - f) \to 0 \).

Proof. First note that \( h \in K \) if and only if \( |h| \in K \) since \( h = |h| e^{i\text{Arg}(h)} \) and \( |h| = h e^{-i\text{Arg}(h)} \). Thus \( \{f_n\} \) is in \( \overline{K}^m \) and
\[
K = \text{co} (\{ (g \circ \varphi) \; u : \|u\|_\infty \leq 1, \; \varphi \in G_\alpha \}).
\]
Suppose \( \varepsilon > 0 \). It follows from Theorem 4.1 that there is a \( \delta > 0 \) such that, for every measurable \( E \subseteq \mathbb{T} \), we have \( m (E) < \delta \Rightarrow \alpha (\chi_E) < \varepsilon /3 \). Suppose \( h \in K \). Then there are functions \( u_1, \ldots, u_s \in \text{ball}(L^\infty (\mathbb{T})) \) and \( \varphi_1, \ldots, \varphi_s \in G_\alpha \) and \( 0 \leq t_1, \ldots, t_s \leq 1 \) with \( \sum_{1 \leq k \leq s} t_k = 1 \), such that
\[
h = \sum_{k=1}^{s} t_k u_k (g \circ \varphi_k).
\]

If \( m (E) < \delta \), then \( m (\varphi_k (E)) = m (E) < \delta \) for \( 1 \leq k \leq s \). Hence, we have
\[
\alpha (h \chi_E) \leq \sum_{k=1}^{s} t_k \|u_k\|_\infty \alpha \left( (g \circ \varphi_k) \left( \left[ \chi_E \circ \varphi_k^{-1} \right] \circ \varphi_k \right) \right)
\leq \sum_{k=1}^{s} t_k \alpha \left( (g \chi_{\varphi_k (E)}) \circ \varphi_k \right) = \sum_{k=1}^{s} t_k \alpha \left( g \chi_{\varphi_k (E)} \right)
\leq \sum_{k=1}^{s} t_k \varepsilon /3 = \varepsilon /3.
\]

Case 1. For each \( n \in \mathbb{N} \), \( f_n \in K \). Since \( f_n \to f \) in measure and \( \delta > 0 \), there is an \( N \in \mathbb{N} \) such that \( n, k \geq N \) implies that if \( E_{k,n} = \{ z \in \mathbb{T} : |f_n (z) - f_k (z)| \geq \varepsilon /37 \} \), which implies
\[
\alpha (f_k - f_n) \leq \alpha \left( (f_k - f_n) \chi_{E_{k,n}} \right) + \alpha \left( (f_k - f_n) \chi_{\mathbb{T} \setminus E_{k,n}} \right)
\leq \alpha \left( f_k \chi_{E_{k,n}} \right) + \alpha \left( f_n \chi_{E_{k,n}} \right) + \alpha \left( (\varepsilon /3) \chi_{\mathbb{T} \setminus E_{k,n}} \right)
\leq 3\varepsilon /3 = \varepsilon.
\]
It follows from the fact that \( \varepsilon > 0 \) was arbitrary, then \( \{f_n\} \) is \( \alpha \)-Cauchy, so there is an \( F \in L^\alpha (\mathbb{T}) \) such that \( \alpha (f_n - F) \to 0 \). Since \( \|f_n - F\|_1 \leq \alpha (f_n - F) \), we see that \( f_n \to F \) in measure, which implies \( F = f \).

Case 2. The general case. Since each \( f_n \in \overline{K}^m \), \( f_n \) is a limit in measure of a sequence in \( K \), and it follows from Case 1 that \( f_n \) is an \( \alpha \)-limit of a sequence in \( K \). Hence, for each \( n \in \mathbb{N} \), there is an \( h_n \in K \) such that \( \alpha (f_n - h_n) < 1/n \). Hence
\( f_n - h_n \to 0 \) in measure, which implies \( h_n = f_n - (f_n - h_n) \to f \) in measure, and it follows from Case 1 that \( f \in L^\alpha(\mathbb{T}) \) and \( \alpha(h_n - f) \to 0 \). But 
\[
\alpha(f_n - f) \leq \alpha(f_n - h_n) + \alpha(h_n - f) \to 0,
\]
and the proof is complete. \( \square \)

**Remark 4.4.** We have a few remarks about our dominated convergence theorem.

1. The restriction \( f \in \mathcal{K}^n \) is much more general than \( |f| \leq |g| \). For example, suppose \( \alpha = \| |f| \|_1, E \subseteq \mathbb{T} \) is an arc with \( m(E) = 1/\pi \) and \( g = n\chi_E \). Then if \( \phi_k(z) = z e^{2\pi i/n} \), then \( \sum_{k=1}^n \frac{1}{n} g \circ \phi_k = 1 \). Hence \( 1 \in K \), but any \( f \) with \( |f| \leq |g| \) must be zero off \( E \).

2. Theorem 4.3 and Theorem 4.4 remain true if we replace \((\mathbb{T}, m)\) in Theorem 4.5 and Theorem 4.6 with any finite measure space \((\Omega, \mu)\) and \( \alpha \) with any norm on \( L^\infty(\mu) \) such that
   (a) \( \alpha(1) = 1 \),
   (b) \( \alpha(f) = \alpha(|f|) \) for every \( f \in L^\infty(\mu) \), and
   (c) Whenever \( \{f_n\} \) is a sequence in \( L^\infty(\mu) \) and \( \alpha(f_n) \to 0 \), we have \( f_n \to 0 \) in measure.
   (d) For every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( \mu(E) < \delta \Rightarrow \alpha(\chi_E) < \varepsilon \).

The next corollary follows immediately from the dominated convergence theorem on \( L^\alpha(\mathbb{T}) \).

**Corollary 4.5.** (Monotone Convergence Theorem) Suppose \( \alpha \in \mathcal{R}_c \) and \( f \in L^\alpha(\mathbb{T}) \). If \( 0 \leq f_1 \leq f_2 \leq \cdots \) and \( f_n(x) \to f(x) \) a.e. \((m)\), then \( \alpha(f - f_n) \to 0 \).

It is clear that in the preceding corollary, we have \( \alpha(f_n) \to \alpha(f) \). That this holds for arbitrary measurable functions was proved in part (6) in Proposition 2.2.

**Corollary 4.6.** \((\mathcal{R}_c, \mathcal{T}(\mathcal{R}_c))\) is separable and metrizable.

**Proof.** Let \( \mathcal{F} \) be the linear span of \( \{z^n : n \in \mathbb{Z}\} \) over the field \( \mathbb{Q} + i\mathbb{Q} \) of complex-rational numbers. Then, as in the proof of part (2) of Lemma 2.2 we only need to show that \( \mathcal{T}(\mathcal{R}_c) \) is the weak topology induced by the set \( \{\psi_f : f \in \mathcal{F}\} \). Suppose \( \{\alpha_\lambda\} \) is a net in \( \mathcal{S}_c \) and \( \alpha \in \mathcal{S}_c \) and \( \alpha_\lambda(f) \to \alpha(f) \) for every \( f \in \mathcal{F} \). Since \( \mathcal{S} \) is compact, there is a subnet \( \{\alpha_{\lambda_k}\} \) converging pointwise to \( \beta \in \mathcal{R} \). We only need to show that \( \beta = \alpha \). We know that \( \beta(f) = \alpha(f) \) for every \( f \in \mathcal{F} \), and since the uniform closure of \( \mathcal{F} \) is \( C(\mathbb{T}) \), we conclude \( \beta(f) = \alpha(f) \) for every \( f \in C(\mathbb{T}) \).

Suppose \( K \) is a closed subset of \( \mathbb{T} \). Then there is a function \( h_K : \mathbb{T} \to [0, 1] \) such that \( K = h_K^{-1}(\{1\}) \). Then \( h_K \downarrow \chi_K \) on \( \mathbb{T} \). Suppose \( \{K_1, \ldots, K_m\} \) is a disjoint family of closed subsets of \( \mathbb{T} \) and \( 0 \leq a_1, \ldots, a_m \) and suppose \( s = \sum_{j=1}^m a_j \chi_{K_j} \). Then \( f_n = \sum_{j=1}^m a_j h_{K_j}^n \in C(\mathbb{T}) \) and \( f_n \downarrow s \). It follows that
\[
\beta(s) = \limsup_{n \to \infty} \beta(f_n) = \lim_{n \to \infty} \alpha(f_n) = \alpha(s),
\]
with the last equality following from our dominated convergence theorem. If \( u = \sum_{j=1}^m a_k \chi_{E_k} \), it follows from the regularity of \( m \) that we can choose a sequence \( \{s_n\} \) of simple functions of the form of \( s \) so that
\[
0 \leq s_1 \leq s_2 \leq \cdots
\]
and \( s_n(z) \to u(z) \) a.e. \((m)\). It follows from part (6) of Lemma 2.2 that
\[
\beta(u) = \lim_{n \to \infty} \beta(s_n) \leq \lim_{n \to \infty} \alpha(s_n) = \alpha(s).
\]
Since the simple functions are \( \| \cdot \|_\infty \)-dense in \( L^\infty (\mathbb{T}) \), we conclude that \( \beta (f) \leq \alpha (f) \) for all \( f \in L^\infty (\mathbb{T}) \). It now follows that \( \beta \in \mathcal{S}_\alpha \), and reversing the roles of \( \alpha \) and \( \beta \) in the above arguments, we get \( \alpha \leq \beta \). Hence \( \alpha = \beta \). \( \square \)

**Proposition 4.7.** The following statements are equivalent for a continuous rotationally symmetric norm \( \alpha \):

1. The General Continuity Theorem is true in \( L^\alpha (\mathbb{T}) \);
2. The Dominated Continuity Theorem is true in \( L^\alpha (\mathbb{T}) \);
3. \( \alpha \) is strongly continuous, i.e., \( L^\alpha (\mathbb{T}) = L^\alpha (\mathbb{T}) \).

**Proof.** (1) \( \Rightarrow \) (2) It is clear from Theorem 4.3.
(2) \( \Rightarrow \) (3) Suppose \( f \in L^\alpha (\mathbb{T}) \). Then \( 0 \leq |f| \in L^\alpha (\mathbb{T}) \) and there is a sequence of simple functions \( 0 \leq s_1 \leq s_2 \leq \cdots \) such that \( |s_n| \leq |f| \) and \( s_n (w) \to |f|(w) \) for every \( w \in \mathbb{T} \). It follows from Theorem 4.3 that \( \alpha (s_n - |f|) \to 0 \), and thus \( f \in L^\alpha (\mathbb{T}) \).
(3) \( \Rightarrow \) (1) It is obvious from Theorem 4.3. \( \square \)

5. **Convolution product on \( L^\alpha (\mathbb{T}, X) \)**

Suppose \( X \) is a separable Banach space and \( \alpha \) is a continuous rotationally symmetric norm on \( \mathbb{T} \). Suppose \( f : \mathbb{T} \to X \) is a function. If \( w \in \mathbb{T} \), we define, as in the scalar case, \( f_w : \mathbb{T} \to X \) by \( f_w (z) = f(\overline{w}z) \).
We also define \( |f| : \mathbb{T} \to [0, \infty) \) by
\[
|f| (z) = \| f(z) \|,
\]
i.e., \( |f| = \| \cdot \| \circ f \).

For any rotationally symmetric norm on \( L^\infty (\mathbb{T}) \) we define
\[
\alpha (f) = \alpha (\| \cdot \circ f ) = \alpha (|f|),
\]
and we define
\[
L^\alpha (\mathbb{T}, X) = \{ f |f : \mathbb{T} \to X \text{ is measurable and } |f| \in L^\alpha (\mathbb{T}) \}.
\]
It is easy to show that \( L^\alpha (\mathbb{T}, X) \) is a Banach space with the norm \( \alpha \).

We also define \( C (\mathbb{T}, X) \) to be the set of all continuous functions from \( \mathbb{T} \) to \( X \).

**Lemma 5.1.** If \( \alpha \in \mathcal{R}_c \) and \( X \) is a separable Banach space, then

1. \( L^\alpha (\mathbb{T}, X) \) is the closed linear span of elements of the form \( h(z) = \chi_E (z) x_0 \) with \( E \subset \mathbb{T} \) and \( x_0 \in X \);
2. \( L^\alpha (\mathbb{T}, X) \) is the closed linear span of elements of the form \( h(z) = f(z) x_0 \) with \( f \in C (\mathbb{T}) \) and \( x_0 \in X \);
3. \( C (\mathbb{T}, X)^{\alpha_0} = L^\alpha (\mathbb{T}, X) \);
4. For every \( f \in L^\alpha (\mathbb{T}, X) \), \( \lim_{m(E) \to 0} \alpha (\chi_E f) = 0 \);
5. **Theorem 4.3** is true when \( L^\alpha (\mathbb{T}) \) is replaced with \( L^\alpha (\mathbb{T}, X) \).

**Proof.** (1) Suppose \( \varepsilon > 0 \) and \( f \in L^\alpha (\mathbb{T}, X) \). Suppose \( n \in \mathbb{N} \). Since \( X \) is separable, we can find a disjoint collection \( \{ E_{n1}, E_{n2}, \ldots \} \) of nonempty Borel subsets whose union is \( X \) such that for every \( k \geq 1 \) and every \( x, y \in E_{nk} \), \( \| x - y \| \leq 1/n \). Let \( F_{nk} = f^{-1}(E_{nk}) \) and choose \( x_{nk} \in E_{nk} \). Define \( g_n = \sum_{k=1}^{\infty} x_{nk} \chi_{F_{nk}} : \mathbb{T} \to X \). Then \( g_n \) is measurable and \( \| g_n (z) - f (z) \| \leq 1/n \) for every \( z \in \mathbb{T} \). Hence \( g_n - f \in L^\infty (\mathbb{T}, X) \subset L^\alpha (\mathbb{T}, X) \) and \( g_n = f + (g_n - f) \in L^\alpha (\mathbb{T}, X) \). Since \( \alpha \leq \| \cdot \|_\infty \), it is
clear that \( \alpha (g_n - f) \leq 1/n \to 0 \). Thus the functions of the form \( g = \sum_{k=1}^{\infty} x_k \chi_{F_k} \) are dense in \( L^\alpha (T, X) \). But \( m (\bigcup_{k \in N+1} F_k) \to 0 \) as \( N \to \infty \) implies

\[
\alpha \left( \| \cdot \| \circ \left[ g - \sum_{k=1}^{N} x_k \chi_{F_k} \right] \right) = \alpha \left( \| \cdot \| \circ \left[ g \chi_{\bigcup_{k \in N+1} F_k} \right] \right) \to 0,
\]

which implies \( g \) is the limit in \( L^\alpha (T, X) \) of \( \sum_{k=1}^{N} x_k \chi_{F_k} \), and these are in the linear span of functions of the form \( x_0 \chi_E (z) \).

(2) If \( E \subset T \) is a Borel set, then \( \chi_E \in L^\alpha (T) \), and since \( C (T) \) is dense in \( L^\alpha (T) \), there is a sequence \( \{ f_n \} \) in \( C (T) \) such that \( \alpha (f_n - \chi_E) \to 0 \). If \( h_n (z) = f_n (z) x_0 \), then \( h_n \in C (T, X) \) and

\[
\alpha (h_n - x_0 \chi_E) = \alpha (f_n - \chi_E) \| x_0 \| \to 0.
\]

Hence \( C (T, X)^{-\alpha} \) contains the closed linear span of the functions of the form \( x_0 \chi_E \).

(3) This easily follows from (1) and (2).

(4) If \( f \in L^\alpha (T, X) \), then \( |f| \in L^\alpha (T) \), and \( \alpha (|\chi_E f|) = \alpha (\chi_E |f|) \), the result easily follows from Theorem 4.1.

(5) We first note that \( 0 \leq |f_n| - |f| \leq |f_n - f| \to 0 \) in measure implies \( |f_n| \to |f| \) in measure. Using Theorem 5.2 we get \( |f| \in L^\alpha (T) \), which implies \( f \in L^\alpha (T, X) \). We then replace \( g \) with \( |g| + |f| \) and use the fact that \( |f_n - f| \to 0 \) in measure to apply Theorem 4.3 to get \( \alpha (f_n - f) = \alpha (|f_n - f|) \to 0 \).  

For \( f \in C (T, X) \) we define

\[
\int_T f \, dm = \lim_{n \to \infty} \frac{1}{2^n} \sum_{k=0}^{2^n-1} f (e^{2\pi ik/2^n}) \in X.
\]

The uniform continuity easily implies that the limit converges and it easily follows that

\[
\left\| \int_T f \, dm \right\| \leq \int_T \| f (z) \| \, dm (z) = \| f \|_1 \leq \| f \|_\infty.
\]

Hence \( \mathcal{I} : C (T, X) \to X \) defined by \( \mathcal{I} (f) = \int_T f \, dm \) is a linear mapping and is continuous when we give \( C (T, X) \) either \( \| \cdot \|_1 \) or \( \| \cdot \|_\infty \). Since \( C (T, X) \) is dense in \( L^1 (T, X) \), we see that \( \mathcal{I} \) has a unique continuous norm-one linear extension \( \mathcal{I}' : L^1 (T, X) \to X \), and we will use \( \int_T f \, dm \) to denote \( \mathcal{I}' (f) \).

**Lemma 5.2.** Suppose \( \alpha \) is a continuous rotationally symmetric norm on \( T \) and \( f \in L^\alpha (T, X) \). Then the mapping \( G : T \to L^\alpha (T, X) \) defined by \( G (w) = f_w \) is \( \alpha \)-continuous.

**Proof.** Clearly the set of \( f \in L^\alpha (T, X) \) for which the lemma is true is a closed linear subspace of \( L^\alpha (T, X) \). Since \( X \) is separable, the functions \( f \in L^\alpha (T, X) \) with countable range are dense in \( L^\alpha (T, X) \). Since \( \alpha \) is continuous, the set of simple functions is dense in \( L^\alpha (T, X) \). Suppose \( E \subset T \) is measurable. Then there is an open subset \( U \) of \( T \) such that \( E \subset U \) and \( \alpha (\chi_E - \chi_U) \) is arbitrarily small. However, \( U \) is a countable disjoint union of open arcs, so there is a finite disjoint union \( V \) of open arcs such that \( \alpha (\chi_U - \chi_V) \) is arbitrarily small. Moreover, every open arc is the disjoint union (a.e.) of at most 8 arcs with length at most \( \pi/4 \). It
follows that $L^α(\mathbb{T}, X)$ is the closed linear span of functions $f = x\chi_I$ with $x \in X$ and $I$ an arc with length at most $\pi/4$. It is easy to see from the continuity of $α$ that the lemma is true for such functions $f$.

If $f \in L^α(\mathbb{T}, X)$ and $g \in L^1(\mathbb{T})$, we define the convolution product $f * g : \mathbb{T} \to X$ for almost every $z \in \mathbb{T}$ by

$$(f * g)(z) = \int_{\mathbb{T}} f(\bar{w}z)g(w)dm(w) = \int_{\mathbb{T}} f_w(z)g(w)dm(w).$$

Note that the changes of variable $w \mapsto wz$ or $w \mapsto \bar{w}$ do not change the integral, so

$$(f * g)(z) = \int_{\mathbb{T}} f(w)g(\bar{w}z)dm(w).$$

It is well-known that, if $f \in L^p(\mathbb{T})$ for $1 \leq p < \infty$ and $g \in L^1(\mathbb{T})$, then

$$f * g \in L^p(\mathbb{T}) \text{ and } \|f * g\|_p \leq \|f\|_p\|g\|_1.$$ Our object is to prove the following extension, which is a more general result in $L^α(\mathbb{T}, X)$.

**Theorem 5.3.** Suppose $α$ is a continuous rotationally symmetric norm and $X$ is a separable Banach space. If $f \in L^α(\mathbb{T}, X)$ and $g \in L^1(\mathbb{T})$, then

$$f * g \in L^α(\mathbb{T}, X) \text{ and } α(f * g) \leq α(f)\|g\|_1.$$ 

**Proof.** Suppose $f \in L^α(\mathbb{T}, X)$ and $g \in L^1(\mathbb{T})$ with $g \geq 0$. Then for arbitrary $z \in \mathbb{T}$,

$$(f * g)(z) = \int_{\mathbb{T}} f(\bar{w}z)g(w)dm(w)$$

$$= \int_{\mathbb{T}} f_w(z)\frac{|g(w)|}{\|g\|_1}dm(w)\|g\|_1$$

$$= \int_{\mathbb{T}} f_w(z)d\mu(w)\|g\|_1,$$

where $μ = \frac{|g(w)|}{\|g\|_1}m$ is a probability measure. The convolution can be expressed as

$$f * g = \int_{\mathbb{T}} f_w d\mu(w)\|g\|_1.$$ 

It follows from Lemma 5.2 that

$$f * g \in L^α(\mathbb{T}, X),$$

and

$$α(f * g) = α\left(\int_{\mathbb{T}} f_w d\mu(w)\|g\|_1\right)$$

$$\leq \int_{\mathbb{T}} α(f_w) d\mu(w)\|g\|_1$$

$$= \int_{\mathbb{T}} α(f) d\mu(w)\|g\|_1 = α(f)\|g\|_1.$$

Next suppose $g \in L^1(\mathbb{T})$. Then there are $g_1, g_2, g_3, g_4 \geq 0$ in $L^1(\mathbb{T})$ such that $g$ can be written as

$$g = \text{Re} g^+ - \text{Re} g^- + i(\text{Im} g^+ - \text{Im} g^-) = g_1 - g_2 + ig_3 - ig_4,$$
and hence
\[ f \ast g = \int_T f(w)\|g_1\|_1 - \int_T f(w)\|g_2\|_1 \]
\[ + i \int_T f(w)\|g_3\|_1 - i \int_T f(w)\|g_4\|_1 \]
\[ \in L^\alpha(T, X), \]
Since the definition of convolution product implies \(|f \ast g| \leq |f| \ast |g|\), it follows that
\[ \alpha(f \ast g) = \alpha(|f \ast g|) \leq \alpha(|f| \ast |g|) \leq \alpha(|f|)\|g\|_1 = \alpha(f)\|g\|_1. \]
\[ \square \]

**Definition 5.4.** An approximate identity in \(L^1(T)\) is a net \(\{\phi_\lambda\}\) in \(L^1(T)\) with the properties:

1. \(\phi_\lambda \geq 0\) for all \(\lambda\);
2. \(\int_T \phi_\lambda dm = 1\) for all \(\lambda\);
3. For every subset \(E\) of \(T\) that is the complement of an open neighborhood of \(1\), the net \((\phi_\lambda)\) converges uniformly to \(0\) on \(E\).

**Example 5.5.** (1) The Poisson kernel is defined by
\[ P_r(e^{it}) = \sum_{n=-\infty}^{\infty} r^n e^{int} \]
for \(r \in [0, 1)\). Given a function \(F\) defined on \(T\), we call the function \(f\) defined on \(D\) by
\[ f(re^{it}) = (F \ast P_r)(e^{it}) \]
the Poisson integral of \(F\).

(2) The Fejer kernel is defined by
\[ K_n = \frac{D_0 + D_1 + \cdots + D_n}{n+1}, \]
where \(D_n(z) = \sum_{k=-n}^{n} z^k\) for all \(n \geq 0\). Note that, for any \(F\) defined on \(T\) and any \(k \in \mathbb{Z}\), we have
\[ (F \ast z^k)(z) = \left( (z^k \ast F)(z) = \int_T F(w) \bar{w}^k dm(w) \right) \]
Hence \((F \ast K_n)(z)\) has the form \(\sum_{k=-n}^{n} c_{n,k} z^k\). If we define the \(n^{th}\) Fourier coefficient
\[ \hat{F}(n) = \int_T F(w) \bar{w}^n dm(w), \]
then
\[ c_{n,k} = \frac{(n+1-|k|)}{n+1} \hat{F}(k) \quad \text{for} \quad |k| \leq n, \]
and
\[ \lim_{n \to \infty} c_{n,k} = \hat{F}(k) \]
for \(k \in \mathbb{Z}\).

**Theorem 5.6.** Suppose \(\alpha \in \mathcal{R}_c\) and \(X\) is a separable Banach space. If \(f \in C(T, X)\) and \(\{\phi_\lambda\}\) is an approximate identity, then \(\{f \ast \phi_\lambda\}\) is a net of continuous functions that converges uniformly to \(f\).
Proof. Let \( f_w(z) = f(wz) \). Since \( f \) is continuous and \( \mathbb{T} \) is compact, \( f \) is uniformly continuous on \( \mathbb{T} \), and therefore \( \| f - f_w \|_\infty \to 0 \) as \( w \to 1^- \). Hence for \( \varepsilon > 0 \) there exists a neighborhood \( U \) of 1 in \( \mathbb{T} \) such that \( \| f - f_w \|_\infty < \frac{\varepsilon}{2} \) whenever \( w \in U \). Let \( E = \mathbb{T} \setminus U \), and use properties (1) and (3) in Definition 5.4 to choose \( \lambda_0 \) such that \( \lambda \geq \lambda_0 \) implies \( 0 \leq \phi_\lambda < \frac{\varepsilon}{4\| f \|_\infty} \) on \( E \). Then for arbitrary \( z \in \mathbb{T} \) and \( \lambda \geq \lambda_0 \) we have

\[
\| f(z) - f \ast \phi_\lambda(z) \| \leq \int_{\mathbb{T}} \| f(z) - f(wz) \| \phi_\lambda(w) dm(w)
\leq \int_{U} \| f - f_w \|_\infty \phi_\lambda(w) dm(w) + \int_{\mathbb{R}} \| f \|_\infty \phi_\lambda(w) dm(w)
< \frac{\varepsilon}{2} + 2 \| f \|_\infty \frac{\varepsilon}{4\| f \|_\infty} = \varepsilon.
\]

Properties (2) was used in the last inequality. Therefore \( \{ f \ast \phi_\lambda \} \) converges uniformly to \( f \).

Similarly, if \( z, z_0 \in \mathbb{T} \), then

\[
\| f \ast \phi_\lambda(z) - f \ast \phi_\lambda(z_0) \| \leq \int_{\mathbb{T}} \| f(wz) - f(wz_0) \| \phi_\lambda(w) dm(w) \leq \| f_z - f_{z_0} \|_\infty.
\]

As above, uniform continuity implies \( \| f_z - f_{z_0} \|_\infty \to 0 \) as \( z \to z_0 \), and this implies the continuity of \( f \ast \phi_\lambda \). \( \square \)

**Theorem 5.7.** Suppose \( \alpha \in \mathcal{R}_c \) and \( X \) is a separable Banach space. If \( f \in L^\alpha(\mathbb{T}, X) \) and \( \{ \phi_\lambda \} \) is an approximate identity, then

1. \( f \ast \phi_\lambda \in L^\alpha(\mathbb{T}, X) \), and \( \alpha(f \ast \phi_\lambda) \leq \alpha(f) \);
2. \( \lim_\lambda \alpha(f - f \ast \phi_\lambda) = 0 \).

**Proof.** (1) It is clear that \( \| \phi_\lambda \|_1 = 1 \). If \( f \in L^\alpha(\mathbb{T}, X) \), then by Theorem 5.6 we obtain

\[
f \ast \phi_\lambda \in L^\alpha(\mathbb{T}) \quad \text{and} \quad \alpha(f \ast \phi_\lambda) \leq \alpha(f) \| \phi_\lambda \|_1 = \alpha(f).
\]

(2) Suppose \( f \in L^\alpha(\mathbb{T}, X) \). Since \( C(\mathbb{T}, X) \) is dense in \( L^\alpha(\mathbb{T}, X) \), there is a sequence \( \{ f_n \} \subset C(\mathbb{T}, X) \) such that \( \alpha(f_n - f) \to 0 \). It follows from Theorem 5.6 that \( f_n \ast \phi_\lambda \) converges uniformly to \( f_n \), which, together with Theorem 5.6 and \( \alpha(f_n - f) \to 0 \) implies

\[
\alpha(f \ast \phi_\lambda - f) = \alpha(f \ast \phi_\lambda - f_n \ast \phi_\lambda + f_n \ast \phi_\lambda - f_n + f_n - f)
\leq \alpha((f - f_n) \ast \phi_\lambda) + \alpha(f_n \ast \phi_\lambda - f_n) + \alpha(f_n - f)
\leq \alpha(f - f_n) \| \phi_\lambda \|_1 + \alpha(f_n \ast \phi_\lambda - f_n) + \alpha(f_n - f)
\leq \alpha(f - f_n) + \| f_n \ast \phi_\lambda - f_n \|_\infty + \alpha(f_n - f)
\to 0.
\]

As with Poisson kernel we have the following corollary, which is mostly a special case of Theorem 5.7.

**Corollary 5.8.** Suppose \( \alpha \in \mathcal{R}_c \) and \( X \) is a separable Banach space. If \( f \in L^\alpha(\mathbb{T}, X) \), then

1. \( f \ast P_r \in L^\alpha(\mathbb{T}, X) \) and \( \alpha(f \ast P_r) \leq \alpha(f) \) for \( 0 < r < 1 \);
2. \( \alpha(f \ast P_r) \) is an increasing function of \( r \) on \((0, 1)\);
3. \( \lim_{r \to 1^-} \alpha(f \ast P_r - f) = 0 \).
(4) \( \lim_{r \to 1^{-}} \alpha(f \star P_r) = \alpha(f) \).

Proof. (1) and (3) follows immediately from Theorem 5.7.

(2) If \( 0 \leq r < s < 1 \), then choose \( q \in (0,1) \) such that \( r = sq \), thus \( f \star P_r = f \star P_{sq} = f \star (P_s \star P_q) = (f \star P_s) \star P_q \). By Theorem 5.3, \( \alpha(f \star P_r) = \alpha((f \star P_s) \star P_q) \leq \alpha(f \star P_s) \|P_q\|_1 = \alpha(f \star P_s) \). This implies \( \alpha(f \star P_r) \) is an increasing function of \( r \) on \([0,1)\).

(4) This follows immediately from (3). \( \square \)

6. HARDY CLASSES ON THE CIRCLE AND DISK

We will maintain the distinction throughout this section that \( F \) and \( f \) are functions defined on \( \mathbb{T} \) and \( \mathbb{D} \) respectively that are related by \( f \) being the Poisson integral of \( F \). The Hardy spaces \( H^p(\mathbb{T}) \) were defined as closed subspaces of \( L^p(\mathbb{T}) \) spanned by the set \( \mathcal{P}_+ = \text{span}\{e_n : n \in \mathbb{N}\} \). Closure is with respect to the norm topology of \( L^p(\mathbb{T}) \) for finite \( p \) and the weak* topology for \( p = \infty \). Functions \( F \) in \( H^p(\mathbb{T}) \) can also be characterized by the properties of belonging to \( L^p(\mathbb{T}) \) and having no nonzero Fourier coefficients of negative index.

Suppose \( \alpha \) is a continuous rotationally symmetric norm. We define

\[
H^{\alpha}(\mathbb{T}) = (\text{span}\{e_n : n \in \mathbb{N}\})^{-\alpha} = (\mathcal{P}_+)^{-\alpha},
\]

i.e., \( H^{\alpha}(\mathbb{T}) \) is the closure in the \( \alpha \)-norm of the set of polynomials in \( z \). Based on the convolution theorem on \( L^\alpha(\mathbb{T}) \), we have obtained the corresponding characterization of \( H^{\alpha}(\mathbb{T}) \).

**Theorem 6.1.** Suppose \( \alpha \) is a continuous rotationally symmetric norm. Then

\[
H^{\alpha}(\mathbb{T}) = \{ F \in L^\alpha(\mathbb{T}) : \hat{F}(n) = \int_{\mathbb{T}} F(z) z^{-n} dm(z) = 0, \text{ for all } n < 0 \},
\]

i.e., the functions in \( L^\alpha(\mathbb{T}) \) whose negative Fourier coefficients vanish.

Proof. Let \( M = \{ F \in L^\alpha(\mathbb{T}) : \hat{F}(n) = 0, \text{ for all } n < 0 \} \). It is clear that \( \mathcal{P}_+ \subset M \) and \( M \) is norm closed, then \( \overline{\mathcal{P}_+} = H^{\alpha}(\mathbb{T}) \subset M \).

Conversely, assume \( F \in M \). Then \( F \in L^\alpha(\mathbb{T}) \subset L^1(\mathbb{T}) \) with \( \hat{F}(n) = 0 \) for all \( n < 0 \). Since the partial sums \( S_n(F) = \sum_{k=-n}^n \hat{F}(n)e_n = \sum_{k=0}^n \hat{F}(n)e_n \in \mathcal{P}_+ \) for all \( n \geq 0 \), it follows that the Cesaro means

\[
\sigma_n(F) = \frac{S_0(F) + S_1(F) + \ldots + S_n(F)}{n+1} \in \mathcal{P}_+.
\]

The definition of the Cesaro means and Corollary 5.8 ensure that \( \sigma_n(F) = F \ast K_n \to F \) in \( L^\alpha(\mathbb{T}) \), thus \( F \in \overline{\mathcal{P}_+} = H^{\alpha}(\mathbb{T}) \), which means \( M \subset H^{\alpha}(\mathbb{T}) \), and therefore

\[
H^{\alpha}(\mathbb{T}) = \{ F \in L^\alpha(\mathbb{T}) : \hat{F}(n) = \int_{\mathbb{T}} F(z) z^{-n} dm(z) = 0, \text{ for all } n < 0 \}.
\]

\( \square \)

Recall that

\[
H^1(\mathbb{T}) = \{ F \in L^1(\mathbb{T}) : \hat{F}(n) = \int_{\mathbb{T}} F(z) z^{-n} dm(z) = 0, \text{ for all } n < 0 \},
\]

the following corollary is an immediate consequence.
Corollary 6.2. Suppose $\alpha$ is a continuous rotationally symmetric norm. Then

$$H^\alpha(T) = L^\alpha(T) \cap H^1(T).$$

Based on Corollary 6.2 for each continuous rotationally symmetric norm $\alpha$, we define

$$\mathcal{H}^\alpha(T) = L^\alpha(T) \cap H^1(T),$$

or equivalently,

$$\mathcal{H}^\alpha(T) = \{F \in L^\alpha(T) : \hat{F}(n) = \int_T F(z)z^{-n}dm(z) = 0, \text{ for all } n < 0\}.$$  

If $\alpha$ is a continuous rotationally symmetric norm on $L^\infty(T)$, then it follows from part (10) in Proposition 2.2 that

$$H^\infty(T) \subset H^\alpha(T) \subset H^1(T).$$

We can view $H^1(T) = H^1(D)$, a space of analytic functions on the open unit disk $D$. Since $H^\alpha(T) \subset H^1(T)$, we can view $H^\alpha(T) \subset H^1(D)$ using the Poisson kernel. By Corollary 5.8 it is easy to see that the following result holds.

**Theorem 6.3.** Suppose $\alpha$ is a continuous rotationally symmetric norm, $F \in L^\alpha(T)$, and let $f_r(e^{it}) = f(re^{it}) = (F * P_r)(e^{it})$. Then

1. $f_r \in L^\alpha(T)$;
2. $\alpha(f_r)$ is increasing in $r$;
3. $\alpha(F - f_r) \to 0$ as $r \to 1^-$;
4. $\lim_{r \to 1^-} \alpha(f_r) = \alpha(F)$.

We define

$$H^\alpha(D) = \{f \in H^1(D) : F \in H^\alpha(T)\}.$$ 

Then we can view $H^\alpha(D) = H^\alpha(T)$. Recall that for all $1 \leq p < \infty,$

$$H^p(D) = \{f \in H(D) : \sup_{0 < r < 1} \|f_r\|_p < \infty\},$$

but we can not define $H^\alpha(D)$ in this way. Actually, if $g : D \to \mathbb{C}$ is analytic and

$$\sup_{0 < r < 1} \alpha(g_r) < \infty,$$

then, since $\|\|_1 \leq \alpha$, the radial limit function $G$ is in $H^1(T)$ and when we apply the Poisson kernel to $G$ we get $g$. However, we do not know if $G \in H^\alpha(T)$, because maybe $G \in L^\alpha(T)$ and not in $L^\alpha(T)$. However, when $\alpha$ is strongly continuous, there is no problem.

**Proposition 6.4.** Suppose $\alpha$ is a strongly continuous rotationally symmetric norm on $L^\infty(T)$ and $f : D \to \mathbb{C}$. The following are equivalent:

1. $f \in H^\alpha(D)$;
2. $f \in H(D)$ and $\sup_{0 < r < 1} \alpha(f_r) < \infty$. 
Proof. (1)\(\Rightarrow\) (2) is clear.

(2)\(\Rightarrow\) (1) Since \(f \in H^1(\mathbb{D})\), each \(f_r(e^{it}) = f(re^{it})\) is continuous and if \(0 < r < s\), we have

\[f_r = f_s * P_{r/s},\]

which implies \(\alpha(f_r)\) is monotone in \(r\). Since \(\|\cdot\|_1 \leq \alpha\), the supremum condition implies that \(f \in H^1(\mathbb{D})\), which implies

\[F(e^{it}) = \lim_{r \to 1^-} f_r(e^{it})\]

exists a.e. \((m)\), and \(F \in H^1(\mathbb{T})\), and \(f\) is the Poisson integral of \(F\). Suppose \(\{r_n\}\) is a sequence in \((0, 1)\) with \(r_n \to 1^-\). Then

\[F(e^{it}) = \liminf_{n \to \infty} |f_{r_n}(e^{it})| = \lim_{n \to \infty} \inf_{k \geq n} |f_{r_k}(e^{it})|\]

Since the sequence \(\{\inf_{k \geq n} |f_{r_k}(e^{it})|\}\) is increasing, it follows from part (4) of Proposition 2.2 that

\[\alpha(F) = \lim_{n \to \infty} \alpha(\inf_{k \geq n} |f_{r_n}|) \leq \lim_{n \to \infty} \alpha(f_{r_n}) \leq \sup_{0 < r < 1} \alpha(f_r) < \infty.\]

Since \(\alpha\) is strongly continuous we conclude \(F \in L^\alpha(\mathbb{T}) \cap H^1(\mathbb{T}) = H^\alpha(\mathbb{T})\).

When \(\alpha\) is not strongly continuous but continuous, we need a stricter assumption to get \(f \in H^\alpha(\mathbb{D})\).

**Theorem 6.5.** Suppose \(\alpha\) is a continuous rotationally symmetric norm on \(L^\alpha(\mathbb{T})\). If \(f \in H^1(\mathbb{D})\) and \(F(e^{it}) = \lim_{r \to 1^-} f_r(e^{it})\), then the following are equivalent:

1. \(F \in L^\alpha(\mathbb{T})\);
2. \(f \in H^\alpha(\mathbb{D})\);
3. There is a sequence \(r_n \to 1^-\) such that

   \[\lim_{m, n \to \infty} \alpha(f_{r_n} - f_{r_m}) = 0.\]

Proof. (1) \(\Rightarrow\) (2) Suppose \(f \in H^1(\mathbb{D})\). If \(F \in L^\alpha(\mathbb{T})\), then it follows from Theorem 6.3 that \(f_r = F * P_r \in L^\alpha(\mathbb{T})\), which implies \(f_r \in L^\alpha(\mathbb{T}) \cap H^1(\mathbb{T}) = H^\alpha(\mathbb{T})\), and hence \(f \in H^\alpha(\mathbb{D})\).

(2) \(\Rightarrow\) (3) Suppose \(f \in H^\alpha(\mathbb{D})\). The definition of \(H^\alpha(\mathbb{D})\) implies that \(f\) is the Poisson integral of \(F \in H^\alpha(\mathbb{T})\). By part (3) of Theorem 6.3, \(\alpha(f_r - F) \to 0\), which means \(\{f_r\}\) is \(\alpha\)-Cauchy with respect to \(r\), i.e., if \(\{r_n\}\) is a sequence in \((0, 1)\) with \(r_n \to 1^-\), then \(\lim_{m, n \to \infty} \alpha(f_{r_n} - f_{r_m}) = 0\).

(3) \(\Rightarrow\) (1) Suppose the hypothesis of (3) holds. Then there is a \(F_0 \in L^\alpha(\mathbb{T})\) such that \(\alpha(f_r - F_0) \to 0\). Since \(\|\cdot\|_1 \leq \alpha\), we conclude \(\|f_r - F_0\|_1 \to 0\), and therefore there is a subsequence \(\{f_{r_k}\}\) such that

\[F_0(e^{it}) = \lim_{k \to \infty} f_{r_k}(e^{it}) = \lim_{r_k \to 1^-} f_{r_k}(e^{it}).\]

By the uniqueness of the limit, it follows that \(F = F_0 \in L^\alpha(\mathbb{T})\).

If \(\alpha\) is a continuous rotationally symmetric norm and \(X\) is a separable Banach space, we define

\[H^\alpha(\mathbb{T}, X) = \{F \in L^\alpha(\mathbb{T}, X) : \hat{F}(n) = \int_{\mathbb{T}} F(z)z^{-n}dm(z) = 0, \text{ for all } n < 0\}.\]
We identify $F \in L^1(T, X)$ with a formal Laurent series

$$F \sim \sum_{n=-\infty}^{\infty} \hat{F}(n)z^n,$$

keeping in mind that each $\hat{F}(n)$ is in $X$. We let $S_n(F)(z) = \sum_{k=-n}^{n} \hat{F}(k)z^k$ for $n \geq 0$ and, for each $n \geq 1$ we define

$$\sigma_n(F)(z) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(F)(z) = (F * K_n)(z),$$

where $K_n$ is the Fejer kernel from Example 5.5. The following follows from Theorem 5.7.

**Proposition 6.6.** If $\alpha$ is a continuous rotationally symmetric norm and $X$ is a separable Banach space, then

1. If $F \in L^\alpha(T, X)$, then $\alpha(F - \sigma_n(F)) \to 0$, so $L^\alpha(T, X)$ is the $\alpha$-closed linear span of $\{x \cdot z^n : x \in X, n \in \mathbb{Z}\}$;
2. If $F \in H^\alpha(T, X)$, then $S_n(F) = \sum_{k=-n}^{n} \hat{F}(k)z^k$, so $H^\alpha(T, X)$ is the $\alpha$-closed linear span of $\{x \cdot z^n : x \in X, n \geq 0\}$.

**7. Beurling’s Invariant subspace theorem**

It is well-known that all of the invariant subspaces for the unilateral shift operator, i.e., $M_z$ on $H^2(T)$, have the form $\phi H^2(T)$ for some inner function $\phi$, where an *inner function* is defined to be a member of $H^\infty(T)$ that is unimodular on $T$. The original statement concerning the space $H^2(T)$ of functions on the unit disk $\mathbb{D}$ was proved in 1949 by A. Beurling [2], [11], and was later extended to $H^p(T)$ classes by T. P. Srinivasan [16]. The standard proof for $H^p(T)$ uses the $H^2$-result and considers the only two possible cases $H^p(T) \subset H^2(T)$ or $H^2(T) \subset H^p(T)$. Neither of these relations hold when $H^p(T)$ is replaced with $H^\alpha(T)$, when $\alpha$ is a rotationally symmetric norm on $L^\alpha(T)$. In this section, we aim at a similar result for $H^\alpha(T)$ case by using different techniques.

First, let us review two important topologies. Suppose $\mathcal{H}$ is a Hilbert space.

The *weak operator topology* (WOT) on $\mathcal{B}(\mathcal{H})$ is defined as the weakest topology such that the sets

$$W(T, x, y) := \{A \in \mathcal{B}(\mathcal{H}) : |\langle (T - A)x, y \rangle| < 1\}$$

are open. The sets

$$W(T_i, x_i, y_i; 1 \leq i \leq n) := \bigcap_{i=1}^{n} W(T_i, x_i, y_i)$$

form a base for the WOT topology. A net $\{T_\lambda\}$ converges WOT to an operator $T$ if and only if

$$\lim_{\lambda} T_\lambda x, y = (T x, y) \quad \text{for all } x, y \in \mathcal{H}.$$

Analogously, the *strong operator topology* (SOT) is defined by the open sets

$$S(T, x) := \{A \in \mathcal{B}(\mathcal{H}) : \|\langle (T - A)x \rangle\| < 1\}.$$

A net $\{T_\lambda\}$ converges SOT to $T$ if and only if

$$\lim_{\lambda} T_\lambda x = T x \quad \text{for all } x \in \mathcal{H}.$$
Lemma 7.1. Suppose $X$ is a Banach space and $M$ is a closed linear subspace of $X$. Then $M$ is weakly closed.

Proof. It is clear that $M \subset M'$. Suppose there is $x_0 \in M'$ with $x_0 \notin M$. Then by the Hahn-Banach theorem there is a linear functional $\phi \in X'$ such that $\phi|_M = 0$ and $\phi(x_0) \neq 0$. Since $x_0 \in M'$, there is a net $\{x_\lambda\}$ in $M$ such that $x_\lambda \to x_0$ weakly, which implies that $\phi(x_\lambda) \to \phi(x_0) \neq 0$. But $\phi(x_\lambda) = 0$ for all $\lambda$, which is a contradiction, and therefore $M = \overline{M'}$. This completes the proof. \hfill \Box

Lemma 7.2. Suppose $X$ is a Banach space and $M$ is a closed linear subspace of $X$. Let $A = \{T \in B(X) : TM \subset M\}$. Then $A$ is closed in the weak operator topology.

Proof. Suppose $T \in B(X)$ and $\{T_\lambda\}$ is a net in $A$ with $T_\lambda \to T$ in the weak operator topology. If $x \in M$, then $T_\lambda x \to Tx$ weakly and $T_\lambda x \in M$ for all $\lambda$. It follows from Lemma 7.1 that $M$ is a weakly closed subspace of $X$, which implies $Tx \in M$ for all $x \in M$, and hence $T \in A$, which implies that $A$ is closed in the weak operator topology. \hfill \Box

The following lemma is well-known [8].

Lemma 7.3. Suppose $K_n$ is the Fejer's kernel in Example 5.5. If $f \in L^p(T)$ for $1 \leq p \leq \infty$, then $f * K_n$ converges in norm to $f$ as $r \to 1^-$ when $p$ is finite and in the weak* topology when $p = \infty$.

Lemma 7.4. Suppose $\alpha$ is a continuous rotationally symmetric norm on $L^\alpha(T)$. If $M$ is a closed subspace of $H^\alpha(T)$ invariant under $M$, which means $z M \subset M$, then $H^\infty(T) \cdot M \subset M$.

Proof. It follows from $z M \subset M$ that for any polynomial $P \in P_+$, $P(z)M \subset M$. Suppose $h \in M$ and $\phi \in (H^\alpha(T))^\ell$. Then by the Hahn-Banach theorem, there is a linear functional $\psi \in (L^\alpha(T))^\ell$ such that $\psi|_{H^\alpha(T)} = \phi$ and $\|\psi\| = \|\phi\|$. Since $\alpha$ is continuous, by Proposition 5.5, $(L^\alpha(T))^\ell = L^\alpha(T)$, and therefore there is an $u \in L^\alpha(T)$ such that $\psi(h) = \int_T hudm$, and thus $\|hu\|_1 \leq \alpha(h) \alpha'(u) < \infty$, which implies $hu \in L^1(T)$.

Next suppose $f \in H^\infty(T)$. It is clear that the Cesaro means

$$\sigma_n(f) = \frac{S_0(f) + S_1(f) + \ldots + S_n(f)}{n+1} \in P_+.$$ 

Therefore by Lemma 7.3, $\sigma_n(f) \to f$ in the weak* topology. Since $hu \in L^1(T)$, it follows that

$$\int_T \sigma_n(f)hudm \to \int_T fhudm.$$ 

We also note that $\sigma_n(f)h \in P_+ M \subset M \subset L^\alpha(T)$ and $u \in L^\alpha(T)$, which implies $\sigma_n(f)h \to fh$ weakly, and since $M$ is weakly closed, we see $fh \in M$ for all $f \in H^\infty(T)$, and thus $H^\infty(T)h \subset M$ for all $h \in M$. This completes the proof. \hfill \Box

Proposition 7.5. Suppose $f \in H^\alpha(T)$. Then there are two measurable functions $u, v \in H^\infty(T)$ such that $f = \frac{u}{v}$.

Proof. It is a familiar fact that every function $f$ in $H^1(T)$ can be written as $f = \frac{u}{v}$, where $u, v \in H^\infty(T)$. Since $H^\alpha(T) \subset H^1(T)$, the result follows. \hfill \Box
Proposition 7.6. Suppose \( \alpha \) is a continuous rotationally symmetric norm. Then on the unit ball \( \mathcal{B}(L^\infty(\mathbb{T})) = \{ f \in L^\infty(\mathbb{T}) : \|f\|_\infty \leq 1 \} \), the following statements hold:

1. The \( \alpha \)-topology coincides with the topology of convergence in measure;
2. \( \mathcal{B}(L^\infty(\mathbb{T})) = \{ f \in L^\infty(\mathbb{T}) : \|f\|_\infty \leq 1 \} \) is \( \alpha \)-closed;
3. \( \mathcal{B}(H^\infty(\mathbb{T})) = \{ f \in H^\infty(\mathbb{T}) : \|f\|_\infty \leq 1 \} \) is \( \alpha \)-closed.

Proof. (1) It was shown in [6].

(2) Suppose \( \{ g_n \} \) is a sequence in \( \mathcal{B}(L^\infty(\mathbb{T})) \) with \( \alpha(g_n - g) \to 0 \). Then \( g \in L^\alpha(\mathbb{T}) \) and \( \|g_n - g\|_1 \leq \alpha(g_n - g) \to 0 \), and thus there is a subsequence \( \{ g_{n_k} \} \) with \( g_{n_k} \to g \) a.e. \( (m) \). It follows from \( |g_{n_k}| \leq 1 \) that \( |g| \leq 1 \), and hence \( g \in \mathcal{B}(L^\infty(\mathbb{T})) \). This completes the proof.

(3) Suppose \( \{ g_n \} \) is a sequence in \( \mathcal{B}(H^\infty(\mathbb{T})) \) with \( \alpha(g_n - g) \to 0 \). It is clear that \( H^\infty(\mathbb{T}) \subset L^\infty(\mathbb{T}) \). It follows from part (2) above that \( g \in \mathcal{B}(L^\infty(\mathbb{T})) \). Since \( g_n \in \mathcal{B}(H^\infty(\mathbb{T})) \subset H^\infty(\mathbb{T}) \subset H^\alpha(\mathbb{T}) \) and \( \alpha(g_n - g) \to 0 \), we conclude \( g \in H^\alpha(\mathbb{T}) \), and thus \( g \in \mathcal{B}(H^\infty(\mathbb{T})) \).

The following Lemma is the Krein-Smulian theorem.

Lemma 7.7. Let \( X \) be a Banach space. A convex set in \( X^2 \) is weak* closed if and only if its intersection with \( \{ \phi \in X^2 : \|\phi\| \leq 1 \} \) is weak* closed.

The following theorem is the generalized version of the very important 1949 theorem of Beurling.

Theorem 7.8. Suppose \( \alpha \) is a continuous rotationally symmetric norm and \( M \) is a closed subspace of \( H^\alpha(\mathbb{T}) \). Then \( zM \subset M \) if and only if \( M = \varphi H^\alpha(\mathbb{T}) \) for some inner function \( \varphi \).

Proof. The only if part is obvious. Suppose \( f \in M \subset H^\alpha(\mathbb{T}) \) with \( f \neq 0 \). It follows from Proposition 7.3 that there are two measurable functions \( u, v \in H^\infty(\mathbb{T}) \) such that \( f = \frac{1}{v} \), where \( u \neq 0 \). Then by Lemma 7.4, \( u = f \cdot v \in M \cdot H^\infty \subset M \), which means \( 0 \neq u \in M \cap H^\infty(\mathbb{T}) \). Let

\[
A = \{ u \in H^\infty(\mathbb{T}) : \exists v \in H^\infty(\mathbb{T}), \frac{u}{v} \in M \}.
\]

If \( u \in A \), then there is a \( v \in H^\infty \) such that \( u = \frac{1}{v} \cdot v \in M \cdot H^\infty(\mathbb{T}) \subset M \), and thus \( A \subset H^\infty(\mathbb{T}) \cap M \). On the other hand, suppose \( u \in H^\infty(\mathbb{T}) \cap M \). Since \( 1 \in H^\infty(\mathbb{T}) \) and \( u \in M \), it follows that \( u = \frac{1}{v} \in M \), which implies \( u \in A \). Hence \( A = H^\infty(\mathbb{T}) \cap M \).

Claim: \( A = H^\infty(\mathbb{T}) \cap M \) is weak* closed.

In fact, it is clear that \( A \cap \mathcal{B}(L^\infty(\mathbb{T})) = M \cap \mathcal{B}(L^\infty(\mathbb{T})) \), by Corollary 7.6 we see \( A \cap \mathcal{B}(L^\infty(\mathbb{T})) = \alpha \) closed. Since \( \alpha \) is continuous, it follows from [6] Theorem 4.1 that \( A \cap \mathcal{B}(L^\infty(\mathbb{T})) \) is SOT closed. The fact that \( A \cap \mathcal{B}(L^\infty(\mathbb{T})) \) is convex implies that \( A \cap \mathcal{B}(L^\infty(\mathbb{T})) \) is WOT closed, hence it is weak* closed, since the WOT and the weak*-topology coincide on bounded sets. Therefore it follows from the Krein-Smulian theorem that \( A \) is weak* closed in \( H^\infty(\mathbb{T}) \).

Furthermore, since \( z \cdot M \subset M \) and \( z \cdot H^\infty \subset H^\infty \), we conclude \( A \) is an invariant subspace in \( H^\infty(\mathbb{T}) \) under the unilateral shift operator \( M_z \). Then, by Srinivasan’s theorem in [19], \( M \cap H^\infty(\mathbb{T}) = A = \varphi H^\alpha(\mathbb{T}) \), where \( \varphi \) is inner, so \( \varphi H^\alpha(\mathbb{T}) \subset M \), which implies that

\[
\overline{\varphi H^\alpha(\mathbb{T})} = \overline{\varphi H^\alpha(\mathbb{T})} = \varphi H^\alpha(\mathbb{T}) \subset M = M.
\]
Conversely, suppose \( 0 \neq f \in M \subset H^\alpha (\mathbb{T}) \). Then there are two measurable functions \( u, v \in H^\infty (\mathbb{T}) \) such that \( f = \frac{u}{v} \), where \( v \) is outer. The definition of \( \mathcal{A} \) yields \( 0 \neq u \in \mathcal{A} = \phi H^\infty (\mathbb{T}) \), which implies there is a function \( u_1 \) with \( 0 \neq u_1 \in H^\infty (\mathbb{T}) \) such that \( u = \phi \cdot u_1 \), thus \( f = \phi \cdot \frac{u_1}{v} \), where \( \phi \) is inner, and so \( \frac{u_1}{v} \in L^\alpha (\mathbb{T}) \subset L^1 (\mathbb{T}) \). Since \( v \) is outer, it follows that \( \frac{u_1}{v} \in H^1 (\mathbb{T}) \). Then by Corollary 8.2, \( \frac{u_1}{v} \in H^1 (\mathbb{T}) \cap L^\alpha (\mathbb{T}) = H^\alpha (\mathbb{T}) \), and hence \( f \in \phi H^\alpha (\mathbb{T}) \). This implies \( M \subset \phi H^\alpha (\mathbb{T}) \). □

8. Outer functions in \( H^\alpha (\mathbb{T}) \)

Suppose \( \alpha \) is a continuous rotationally symmetric norm and \( f \in H^\alpha (\mathbb{T}) \). We say that \( f \) is a cyclic vector for \( H^\infty (\mathbb{T}) \) acting on \( H^\alpha (\mathbb{T}) \), i.e., \( (H^\infty (\mathbb{T}) \cdot f)^- = H^\alpha (\mathbb{T}) \). Originally the terms inner and outer were defined for functions in \( H^p (\mathbb{T}) \) by Beurling (see [2]) for \( 0 < p < \infty \). It was shown that a function in \( H^p (\mathbb{T}) \) is outer if and only if it is outer in \( H^1 (\mathbb{T}) \). Since \( H^p (\mathbb{T}) \subset H^1 (\mathbb{T}) \), the term outer was used without reference to \( p \). We prove the same result for \( H^\alpha (\mathbb{T}) \) when \( \alpha \) is a continuous rotationally symmetric norm.

**Theorem 8.1.** Suppose \( \alpha \) is a continuous rotationally symmetric norm and \( f \in H^\alpha (\mathbb{T}) \). Then

\[
(H^\infty (\mathbb{T}) \cdot f)^- = H^\alpha (\mathbb{T}) \iff f \text{ is outer in } H^1 (\mathbb{T})
\]

**Proof.** Let \( M = (H^\infty (\mathbb{T}) \cdot f)^- \). It is clear that \( M \cdot H^\infty (\mathbb{T}) \cdot f \subset H^\infty (\mathbb{T}) \cdot f \), and so \( M \) is a closed invariant subspace of \( H^\alpha (\mathbb{T}) \). It follows from Theorem 7.8 that \( M = (H^\infty (\mathbb{T}) \cdot f)^- = \phi \cdot H^\alpha (\mathbb{T}) \) for some inner function \( \phi \). Since \( f = 1 \cdot f \in (H^\infty (\mathbb{T}) \cdot f)^- = M = \phi \cdot H^\alpha (\mathbb{T}) \), there is a \( g \in H^\alpha (\mathbb{T}) \) such that \( f = \phi g \). If \( f \) is outer, then \( \phi \equiv \text{Constant} \), which implies

\[
M = (H^\infty (\mathbb{T}) \cdot f)^- = \phi \cdot H^\alpha (\mathbb{T}) = H^\alpha (\mathbb{T})
\]

Conversely, suppose \( (H^\infty (\mathbb{T}) \cdot f)^- = H^\alpha (\mathbb{T}) \). Then

\[
H^\infty (\mathbb{T}) \subset H^\alpha (\mathbb{T}) = (H^\infty (\mathbb{T}) \cdot f)^- \subset (H^\infty (\mathbb{T}) \cdot f)^- \| \|
\]

and thus

\[
H^1 (\mathbb{T}) = H^\infty (\mathbb{T}) \| \| \subset (H^\infty (\mathbb{T}) \cdot f)^- \| \| \subset H^1 (\mathbb{T})
\]

which means \( H^1 (\mathbb{T}) = (H^\infty (\mathbb{T}) \cdot f)^- \| \| \). This implies \( f \) is outer. □

The following corollary shows that, given \( f \in H^\alpha (\mathbb{T}) \subset H^1 (\mathbb{T}) \), the two factors in the inner-outer factorization of \( f \) are both in \( H^\alpha (\mathbb{T}) \).

**Corollary 8.2.** (Inner-outer factorization) Suppose \( \alpha \) is a continuous rotationally symmetric norm and \( f \in H^\alpha (\mathbb{T}) \subset H^1 (\mathbb{T}) \), and suppose \( \phi \) is an inner function and \( g \in H^1 (\mathbb{T}) \) is an outer function such that \( f = \phi g \), i.e., \( f = \phi g \) is the Riesz-Smirnov inner-outer factorization of \( f \) in \( H^1 (\mathbb{T}) \). Then \( g \in H^\alpha (\mathbb{T}) \).

**Proof.** Since \( H^\alpha (\mathbb{T}) \subset H^1 (\mathbb{T}) \), the inner-outer factorization in \( H^1 (\mathbb{T}) \) applies of course to functions in \( H^\alpha (\mathbb{T}) \), that is, there exists an inner function \( \phi \in H^\infty (\mathbb{T}) \) and an outer function \( g \in H^1 (\mathbb{T}) \) satisfying \( g(\zeta) = |f(\zeta)| \) for almost every \( \zeta \in \mathbb{T} \) such that \( f = \phi g \). Then it follows from Theorem 8.1 that \( g \) is outer in \( H^\alpha (\mathbb{T}) \), and so we have the desired factorization. □
Theorem 8.3. (Characterization of outer functions) Suppose \( \alpha \) is a continuous rotationally symmetric norm. A nonzero function \( g \in H^\alpha(\mathbb{T}) \) is outer if and only if it has the following property:

for every \( f \in H^\alpha(\mathbb{T}) \), if \( \frac{f}{g} \in L^\alpha(\mathbb{T}) \), then \( \frac{f}{g} \in H^\alpha(\mathbb{T}) \). 

\(^{(\ast)}\)

Proof. Suppose \( g \) is outer in \( H^\alpha(\mathbb{T}) \). Then it follows from (3.1) that \( g \) is outer in \( H^1(\mathbb{T}) \). Since \( \frac{L}{g} \in L^\alpha(\mathbb{T}) \subset L^1(\mathbb{T}) \) and \( f \in H^\alpha(\mathbb{T}) \subset H^1(\mathbb{T}) \), we conclude \( \frac{f}{g} \in H^\alpha(\mathbb{T}) \), and hence \( \frac{f}{g} \in L^\alpha(\mathbb{T}) \cap H^1(\mathbb{T}) = H^\alpha(\mathbb{T}) \).

Conversely, suppose \( g \in H^\alpha(\mathbb{T}) \) satisfy the property \((\ast)\). By the inner-outer factorization, we can write \( g = \phi G \), where \( \phi \) is inner and \( G \in H^\alpha(\mathbb{T}) \) is outer, then \( \phi = \frac{\phi}{g} \) and \( \phi = \frac{\phi}{g} \), being unimodular, is in \( L^\alpha(\mathbb{T}) \). It follows from the property \((\ast)\) that \( \frac{\phi}{\phi} = \frac{\phi}{g} \in H^\alpha(\mathbb{T}) \), thus \( \phi, \frac{\phi}{\phi} \in H^\alpha(\mathbb{T}) \) with \( |\phi| = 1 \), which implies \( \phi \equiv \text{Constant} \), and therefore \( g = \phi G \) is outer. \( \square \)

9. Multipliers of \( H^\alpha(\mathbb{T}) \)

In [5], D. Hadwin and E. Nordgren proved that, if \( \alpha \) is a continuous symmetric gauge norm and if \( Y \) is the set of all measurable complex functions on \( \mathbb{T} \), then \((L^\alpha(\mathbb{T}), Y) \) is a multiplier pair, and if \( f \in L^\alpha(\mathbb{T}) \), then

\[ f \cdot L^\alpha(\mathbb{T}) \subset L^\alpha(\mathbb{T}) \iff f \in L^\infty(\mathbb{T}), \]

i.e., the multipliers of \( L^\alpha(\mathbb{T}) \) are the functions in \( L^\infty(\mathbb{T}) \). In this section we will extend these results to the case in which \( \alpha \) is a continuous rotationally symmetric norm. Another multiplier-type results are Theorem 3.5 and Corollary 3.6. We will also prove similar results for \( H^\alpha(\mathbb{T}) \).

Definition 9.1. \((W, ||\cdot||)\) is a functional Banach space on a nonempty set \( X \) if and only if

1. \( W \) is a vector space of functions from \( X \) to \( \mathbb{C} \);
2. For all \( x \in W \), there is a \( f \in W \) such that \( f(x) \neq 0 \);
3. there is a norm \( ||\cdot|| \) such that \((W, ||\cdot||)\) is a Banach space;
4. For all \( x \in X \), there is a \( r_x \) such that \( |f(x)| \leq r_x \cdot ||f|| \) for all \( f \in W \), i.e., for all \( x \in X \), the map \( E_x : W \to \mathbb{C} \) defined by \( E_x(f) = f(x) \) is a linear bounded functional.

Proposition 9.2. Suppose \( \alpha \) is a continuous rotationally symmetric norm. Then \( H^\alpha(\mathbb{T}) = H^\alpha(\mathbb{D}) \) is a functional Banach space.

Proof. It is clear that \( 1 \in H^\infty(\mathbb{T}) \subset H^\alpha(\mathbb{T}) \) and \((H^\alpha(\mathbb{T}), \alpha)\) is a Banach space.

Furthermore, suppose \( F_n, F \in H^\alpha(\mathbb{T}) \) with \( \alpha(F_n - F) \to 0 \). Since \( \alpha \) is continuous, we see \( ||\cdot||_1 \leq \alpha \), and then \( ||F_n - F||_1 \leq \alpha(F_n - F) \to 0 \). The fact that \((H^1(\mathbb{T}), ||\cdot||_1)\) is a functional Banach space implies \( F_n(z) \to F(z) \) for all \( z \in \mathbb{T} \), and hence \((H^\alpha(\mathbb{T}), \alpha)\) is a functional Banach space. \( \square \)

Lemma 9.3. Suppose \( W \) is a functional Banach space on \( X \), and \( \phi : X \to \mathbb{C} \) and \( \phi W \subset W \). Define \( A : W \to W \) by \( Af = \phi f \). Then \( A \) is bounded.

Proof. Suppose \( f_n \to f \) and \( Af_n = \phi f_n \to g \). Since \( W \) is a functional Banach space, it follows that \( f_n(x) \to f(x) \) for all \( x \in X \). Therefore \((Af_n)(x) = \phi(x)f_n(x) \to g(x)\).

Because \( \phi(x) \in \mathbb{C} \), \( \phi(x)f_n(x) \to \phi(x)f(x) \), and thus \( g(x) = \phi(x)f(x) \) for all \( x \in X \). Therefore \( g = \phi f \), and the closed graph theorem implies that \( A \) is bounded. \( \square \)
If $W$ is a space of (equivalence classes) of functions and $\psi$ is a function such that $\psi W \subset W$, we say that $\psi$ is a multiplier of $W$ and we define the multiplication operator $M_\psi : W \to W$ by

$$M_\psi f = \psi f.$$ 

We now compute the multipliers of $L^\alpha (\mathbb{T})$ and $H^\alpha (\mathbb{T}) = H^\alpha (\mathbb{D})$.

**Theorem 9.4.** (Multipliers on $L^\alpha (\mathbb{T})$ and $H^\alpha (\mathbb{T})$) Suppose $\alpha$ is a continuous rotationally symmetric norm, $\phi : \mathbb{D} \to \mathbb{C}$ is analytic and $\psi : \mathbb{T} \to \mathbb{C}$ is measurable. Then

1. $\psi L^\alpha (\mathbb{T}) \subset L^\alpha (\mathbb{T})$ if and only if $\psi \in L^\infty (\mathbb{T})$. Moreover, $\|\psi\|_\infty = \|M_\psi\|$;
2. $\phi H^\alpha (\mathbb{D}) \subset H^\alpha (\mathbb{D})$ if and only if $\phi \in H^\infty (\mathbb{D})$. Moreover, $\|\phi\|_\infty = \|M_\phi\|$.

**Proof.** (1) Suppose $\alpha (f_n - f) \to 0$ and $\alpha (M_\psi f_n - g) \to 0$. Then $f_n \to f$ in measure and $\psi f_n \to g$ in measure, so we see that $g = \psi f$. Hence, by the closed graph theorem, $M_\psi$ is bounded. Suppose $\varepsilon > 0$ and let $E = \{z \in \mathbb{T} : |\psi(z)| \geq \|M_\psi\| + \varepsilon\}$. Then $|\psi \chi_E| \geq \left(\|M_\psi\| + \varepsilon\right)$, so

$$\|M_\psi\| \alpha (\chi_E) \geq \alpha (M_\psi \chi_E) = \alpha (\psi \chi_E) \geq \left(\|M_\psi\| + \varepsilon\right) \alpha (\chi_E),$$

which implies $\chi_E = 0$, or $m(E) = 0$. Since $\varepsilon > 0$ was arbitrary, we see that $|\psi(z)| \leq \|M_\psi\|$ a.e. ($m$), so $\psi \in L^\infty (\mathbb{T})$ and $\|\psi\|_\infty \leq \|M_\psi\|$. On the other hand, $\alpha (\phi) = \alpha (\psi f) \leq \|\psi\|_\infty \alpha (f)$ implies $\|M_\psi\| \leq \|\psi\|_\infty$.

(2) It follows from Lemma 9.3 that $M_\phi$ is bounded. Suppose $f \in H^\alpha (\mathbb{T})$. Then $\alpha (\phi^\alpha f) = \alpha (\|M_\phi\|^\alpha f) \leq \|M_\phi\|^\alpha \alpha (f)$. Therefore, if $M_\phi = 0$, then $\phi = 0 \in H^\infty (\mathbb{T})$. Otherwise, if $M_\phi \neq 0$, let $\psi = \frac{\phi}{\|M_\phi\|}$. Then

$$\alpha (\psi^\alpha f) = \alpha (\frac{\phi^\alpha}{\|M_\phi\|^\alpha} f) \leq \frac{\|M_\phi\|^\alpha}{\|M_\phi\|^\alpha} \alpha (f) = \alpha (f).$$

It follows from $\phi \in H^\alpha (\mathbb{T})$ that $\phi^\alpha f \in H^\alpha (\mathbb{T})$, thus $\psi^\alpha f \in H^\alpha (\mathbb{T})$. By Corollary 9.2, $H^\alpha (\mathbb{T})$ is a functional Banach space. Therefore for all $x \in \mathbb{T}$, there is an $f \in H^\alpha (\mathbb{T})$ such that $f(x) \neq 0$, and there is a $r_x > 0$ such that

$$|\psi^\alpha (x) f(x)| \leq r_x \alpha (\psi^\alpha f) \leq r_x \alpha (f) < \infty,$$

for all $n \geq 1$. Hence $|\psi(x)| \leq 1$, which means for all $x \in \mathbb{T}$, $|\phi(x)| \leq \|M_\phi\| < \infty$, and therefore $\phi \in H^\infty (\mathbb{T})$ with $\|\phi\|_\infty \leq \|M_\phi\|$. Furthermore, since $\alpha (M_\phi f) = \alpha (\phi f) \leq \|\phi\|_\infty \alpha (f)$, it follows that $\|M_\phi\| \leq \|\phi\|_\infty$. This implies $\|M_\phi\| = \|\phi\|_\infty$. 

Multiplier pairs were created and studied in [5], [6] and [7]. The following result is an easy consequence of our results and results in [5].

**Corollary 9.5.** If $\alpha \in \mathcal{R}_c$, $Y_1$ is the set of all measurable functions topologized by convergence in measure, and $Y_2$ is the set of all analytic functions on $\mathbb{D}$ topologized by uniform convergence on compact subsets, then

1. $(L^\alpha (\mathbb{T}), Y_1)$ is a multiplier pair and $\{M_\psi : \psi \in L^\infty (\mathbb{T})\}$ is a maximal abelian algebra of operators on $L^\alpha (\mathbb{T})$;
2. $(H^\alpha (\mathbb{D}), Y_2)$ is a multiplier pair and $\{M_\phi : \phi \in L^\infty (\mathbb{T})\}$ is a maximal abelian algebra of operators on $H^\alpha (\mathbb{D})$.

We now give a Banach space characterization of the condition $H^\alpha (\mathbb{T}) = \mathcal{H}^\alpha (\mathbb{T})$. We need a characterization of $\alpha (h)$ when $h \in H^1 (\mathbb{T})$. The key ingredient is based on the following result that uses the Herglotz kernel [3].
Lemma 9.6. \( \{ |h| : h \in H^1(\mathbb{T}) \} = \{ \varphi \in L^1(\mathbb{T}) : \varphi \geq 0 \text{ and } \log \varphi \in L^1(\mathbb{T}) \} \). In fact, if \( \varphi \geq 0 \) and \( \varphi, \log \varphi \in L^1(\mathbb{T}) \), then
\[
h(z) = \exp \int_{\mathbb{T}} \frac{w + z}{w - z} \log \varphi(w) \, dm(w)
\]
defines an outer function \( h \) on \( \mathbb{D} \) and \( |h| = \varphi \) on \( \mathbb{T} \).

Lemma 9.7. Suppose \( f \in H^1(\mathbb{T}) \) and \( \alpha \in \mathbb{R} \). Then
\[
\alpha(f) = \sup \{ \|fh\|_1 : h \in H^\infty(\mathbb{T}), \alpha'(h) \leq 1 \}.
\]
Proof. Let \( S = \sup \{ \|fh\|_1 : h \in H^\infty(\mathbb{T}), \alpha'(h) \leq 1 \} \). Suppose \( \varphi \geq 0 \) is a simple function and \( \alpha'(\varphi) \leq 1 \). For each \( \varepsilon > 0 \), \( \varphi_\varepsilon = \frac{\varepsilon + \varphi}{1 + \varepsilon} \geq 0 \) and \( \varphi_\varepsilon \in L^1(\mathbb{T}) \) and since \( \varphi \in L^1(\mathbb{T}) \) and \( \log \left( \frac{1 + \varepsilon}{1 + \varphi} \right) \leq \log (\varphi_\varepsilon) \leq \varphi + \varepsilon \), we see that there is an \( h \in H^1(\mathbb{T}) \) such that \( |h| = \varphi_\varepsilon \). Hence \( h \in H^\infty(\mathbb{T}) \) and \( \alpha'(h) = \alpha'(\varphi_\varepsilon) \leq 1 \). Hence
\[
S \geq \|f\varphi\|_1
\]
for every \( \varepsilon > 0 \). Letting \( \varepsilon \to 0^+ \), we have \( S \geq \|f\varphi\|_1 \). It follows that \( S \geq \alpha(f) \). It is clear that \( S \leq \alpha(f) \). \( \square \)

Theorem 9.8. Suppose \( \alpha \) is a rotationally symmetric norm and \( T : H^\alpha(\mathbb{T}) \to H^1(\mathbb{T}) \) is a bounded linear operator such that, for every \( h \in H^\infty(\mathbb{T}) \) and every \( g \in H^\alpha(\mathbb{T}) \),
\[
T(hg) = hT(g).
\]
Then there is an \( f \in H^{\alpha'}(\mathbb{T}) \) such that, for every \( g \in H^\alpha(\mathbb{T}) \),
\[
Tg = fg.
\]
Moreover, \( \|T\| = \alpha'(f) \). The same conclusion holds when \( H^\alpha(\mathbb{T}) \) is replaced with \( H^\alpha(\mathbb{T}) \).

Proof. Let \( f = T(1) \). Suppose \( h \in H^\alpha(\mathbb{T}) \). Then \( h \in H^1(\mathbb{T}) \), so there are functions \( u, v \in H^\infty(\mathbb{T}) \), with \( v \) outer, such that \( h = u/v \). It follows that
\[
vT(h) = T(u) = uT(1) = uf,
\]
which implies \( T(h) = fh \). The equality \( \|T\| = \alpha'(f) \) follows from Lemma 9.7. \( \square \)

Corollary 9.9. Suppose \( \alpha \) is a rotationally symmetric norm with dual norm \( \alpha' \), and suppose \( f \in H^1(\mathbb{T}) \). Then
\[
f \cdot H^\alpha(\mathbb{T}) \subset H^1(\mathbb{T}) \iff f \in H^{\alpha'}(\mathbb{T}).
\]
Proof. It follows from the closed graph theorem that the map \( T : H^\alpha(\mathbb{T}) \to H^1(\mathbb{T}) \) defined by \( T(g) = fg \) is bounded, and it follows from Theorem 9.8 that \( f \in H^{\alpha'}(\mathbb{T}) \). \( \square \)

We now relate the strong continuity of \( \alpha \in \mathcal{R}_c \) to the condition \( H^\alpha(\mathbb{T}) = H^\alpha(\mathbb{T}) \). The proof of (3)\( \Rightarrow \) (2) below is an adaptation of an argument shown to us by Eric Nordgren.

Theorem 9.10. Suppose \( \alpha \in \mathcal{R}_c \). The following are equivalent:

1. \( L^\alpha(\mathbb{T}) = L^\alpha(\mathbb{T}) \);
2. \( H^\alpha(\mathbb{T}) = H^\alpha(\mathbb{T}) \);
3. \( \alpha \) is weakly continuous.
(3) \( H^\alpha (T) \) is weakly sequentially complete;
(4) \( L^\alpha (T) \) is weakly sequentially complete.

Proof. The statement (1)\(\Rightarrow\)(4) was proved in Theorem 3.5.

To show (2)\(\Rightarrow\)(1), suppose \( \varphi \in L^\alpha (T) \) and \( \varphi \geq 0 \). Then, by Lemma 9.6, there is an \( h \in H^1 (T) \) such that \( |h| = \varphi + 1 \). Hence \( h \in H^\alpha (T) \) and \( h \in H^\alpha (T) \) if and only if \( \varphi = (\varphi + 1) - 1 \in L^\alpha (T) \).

The implication (4)\(\Rightarrow\)(3) follows from the fact that \( H^\alpha (T) \) is a closed subspace of \( L^\alpha (T) \).

We now show (2)\(\Rightarrow\)(3). Suppose (2) holds and suppose \( \{ f_n \} \) is a weakly Cauchy sequence in \( H^\alpha (T) \). Then \( \{ f_n \} \) is a weakly Cauchy sequence in \( L^\alpha (T) \). Following the proof of (1)\(\Rightarrow\)(2) in Theorem 3.5, there is an \( f \in L^\alpha (T) \) such that
\[
\lim_{n \to \infty} \int_T f_n h dm = \int_T f h dm
\]
for every \( h \in L^\alpha (T) \). Thus, for every \( k \geq 0 \), we have
\[
\int_T f z^k dm = 0.
\]
Hence \( f \in H^1 (T) \cap L^\alpha (T) \). Since \( L^\alpha (T) \) is weakly sequentially complete, we can assume that \( \varphi \) is outer. For each positive integer \( n \), let \( E_n = \{ z \in T : |f (z)| > n \} \) and define
\[
\rho_n (z) = \begin{cases} 
1 & \text{if } z \notin E_n \\
1/|f (z)| & \text{if } z \in E_n 
\end{cases}
\]
Since \( f \) is outer and \( f \in H^1 (T) \), \( |f| \) and \( \log |f| \) are in \( L^1 (T) \). Since \( \rho_n \leq 1 \) and \( |\log \rho_n| = \chi_{E_n} \log |\varphi| \), we see that \( \rho_n \) and \( \log \rho_n \) are in \( L^1 (T) \). Hence there is an outer function \( \varphi_n \) such that \( |\varphi_n| = \rho_n \). Since \( |\varphi_n f| = |\rho_n f| \leq n \), we see that \( \varphi_n f \in H^\infty (T) \) for each \( n \in \mathbb{N} \). Also
\[
\|1 - \varphi_n\|^2 = 1 + \|\varphi_n\|^2 - 2 \text{ Re } \varphi_n (0) \leq 2 (1 - \varphi_n (0)).
\]
However, by Lemma 9.6
\[
\varphi_n (0) = \exp \int_T \log \rho_n dm = \exp \int_T \chi_{E_n} \log |f| dm \to 1,
\]
we see that \( \|1 - \varphi_n\| \to 1 \), and by replacing \( \varphi_n \) with a subsequence, if necessary, we can assume that \( \varphi_n (z) \to 1 \) a.e. \( (m) \). Now suppose \( h \in L^\alpha (T) \). Then, \( |f| |h| \in L^1 (T) \), and, for every \( n \in \mathbb{N} \), we have \( |\varphi_n fh| \leq |f| |h| \). It follows from the dominated convergence theorem that
\[
\lim_{n \to \infty} \int_T (\varphi_n f) h dm = \int_T f h dm.
\]
Since \( H^\alpha (T) \subseteq L^\alpha (T) \) and \( L^\alpha (T)^\# = L^\alpha (T) \), we see that \( \{ \varphi_n f \} \) is a weakly Cauchy sequence in \( H^\alpha (T) \). Hence there is an \( F \in H^\alpha (T) \) such that
\[
\lim_{n \to \infty} \int_T \varphi_n h dm = \int_T F h dm.
for every $h \in L^{\alpha'}(\mathbb{T})$. Hence $f = F \in H^{\alpha}(\mathbb{T})$, a contradiction. \hfill \Box

10. Closed densely defined operators in a Multiplier pair

Suppose $X = H^{\alpha}$ (on the unit disk) and $Y = N$ is the set of meromorphic functions in the Nevanlinna class, i.e., functions of the form $f$ with $f, g \in H^{\alpha}$ and $g$ not identically 0. Then $(X, Y)$ is a special multiplier pair. The Smirnov class $N^{\alpha}$ consists of all members of $N$ having a denominator that is an outer function. D. Hadwin, E. Nordgren and Z. Liu [7] have observed that the closed densely defined operators that commute with the unilateral shift on $H^{\alpha}$ are multiplications induced by members of the Smirnov class. In this section, we will give a more general result in $H^{\alpha}$.

Lemma 10.1. If $\phi \in N$ and $\phi \neq 0$, then there exist relatively prime inner functions $u$ and $v$ and outer functions $a$ and $b$ satisfying $|a| + |b| = 1$ a.e. on the unit circle such that

\[ \phi = \frac{vb}{ua} \]

Proof. Recall that an outer function is positive at zero and is uniquely determined by its absolute boundary values, which are necessarily absolutely log integrable. Suppose $\phi$ is a nonzero function in $N$ and the inner-outer factorization is applied to each of the numerator and denominator of $\phi$, so

\[ \phi = \frac{vf_{1}}{uf_{2}} \]

where $u$ and $v$ are relatively prime inner functions and $f_{1}$ and $f_{2}$ are outer functions in $H^{\alpha}$.

Observe that on the unit circle $T$,

\[ \max\{|f_{1}|, |f_{2}|\} \leq |f_{1}| + |f_{2}|, \]

It follows from $f_{1}, f_{2} \in H^{\alpha}$ and $\log(|f_{1}| + |f_{2}|) \leq |f_{1}| + |f_{2}|$ that

\[ -\infty \leq \int_{T} \log |f_{1}| dm \leq \int_{T} \log(|f_{1}| + |f_{2}|) dm \leq \int_{T} (|f_{1}| + |f_{2}|) dm < \infty, \]

and therefore $(|f_{1}| + |f_{2}|)$ is log integrable. Thus there exists an outer function $\psi$ in $H^{\alpha}$ such that $|\psi| = |f_{1}| + |f_{2}|$ a.e. on $T$. Put $a = \frac{f_{1}}{\psi}$ and $b = \frac{f_{2}}{\psi}$ and observe that the definition of $\psi$ implies that $|a| + |b| = 1$ a.e. on $T$ and $\phi = \frac{vb}{u} = \frac{vb}{u\psi}$.

Now we need to show that $a$ and $b$ are outer functions. Since $a = \frac{f_{1}}{\psi}, \psi = |f_{1}| + |f_{2}|$, we see $|a| = \frac{|f_{1}|}{|\psi|} \leq 1$. The fact that $\max\{|f_{1}|, |f_{2}|\} \leq \psi$ shows that

\[ \int_{T} \log |a| dm = \int_{T} (\log |f_{2}| - \log |\psi|) dm \]

\[ \leq \int_{T} \log |f_{2}| dm + \int_{T} |\log |\psi|| dm < \infty, \]

which means $a = \frac{f_{1}}{\psi}$ is outer. Similarly, $b = \frac{f_{2}}{\psi}$ is outer. \hfill \Box

Corollary 10.2. If $\phi \in N$, where $\phi = \frac{uv}{\alpha}$ as in Lemma above, then the graph $\text{Graph}(M_{\phi})$ of $M_{\phi}$ is the closed subset $\{(uag) \oplus (vbg) : g \in H^{\alpha}\}$ of $H^{\alpha} \oplus H^{\alpha}$. 

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Proof. If \( g \in H^\alpha \), then \( uag \in H^\alpha \) and \( M_\phi uag = vbg \in H^\alpha \), thus
\[
\{(uag) \oplus (vbg) : g \in H^\alpha \} \subset \text{Graph}(M_\phi).
\]

For the opposite inclusion suppose both \( f \) and \( \phi f \) belong to \( H^\alpha \). Then
\[
\left| \frac{f}{a} \right| = \left| \frac{a + b}{a} \right| \left| \frac{f}{\phi} \right| = |f| + \|\phi\||f|
\]
on \( \mathbb{T} \), hence \( \frac{f}{a} \in L^\alpha \). Since \( a \) is outer, it follows that \( \frac{f}{a} \in H^\alpha \). Let \( g_1 = \frac{f}{a} \). Then
\[
f = ag_1 \text{ and } u\phi f = u\phi ag_1 = vbg_1.\]

Since \( u \) and \( v \) are relatively prime and \( b \) is outer, the last equation shows that \( u \) is a factor of \( g_1 \), and thus \( g_1 = ug \) for some \( g \in H^\alpha \). We have shown that \( f = uag \) and \( \phi f = vbg \), hence the required inclusion is established. \( \Box \)

**Theorem 10.3.** Suppose \( G \subset H^\alpha \oplus H^\alpha \) is a graph that is invariant under \( M_z \oplus M_z \). Then there is a meromorphic \( \phi \in N \) such that \( G \subset \text{Graph}(M_\phi) \). If the domain of \( G \) is dense in \( H^\alpha \), then \( \phi \) is in the Smirnov class. If, in addition, \( G \) is closed, then \( G \) is in the Smirnov class and \( G = \text{Graph}(M_\phi) \).

**Proof.** The first assertion follows from Corollary 2 in [7]. Next suppose the domain \( D(G) \) is dense in \( H^\alpha \), \( \phi = \frac{u\phi}{w\phi} \) as in Lemma 10.1. \( G \subset \text{Graph}(M_\phi) \) implies that \( D(G) \) is contained in the domain of \( M_\phi \), which is \( D(G) \subset uH^\alpha \subset H^\alpha \). Thus \( uH^\alpha \subset \text{Graph}(M_\phi) \), and so it follows from \( a \) is outer that
\[
H^\alpha = uH^\alpha = L_u(aH^\alpha) = L_u(aH^\alpha) = uH^\alpha,
\]
hence \( u \) is a constant. Therefore \( \phi \in N^+ \).

Suppose \( G \) is closed. Assume \( H^\alpha \oplus H^\alpha \) is given the norm defined by \( \|f \oplus g\| = \alpha(|f| + |g|) \). If we define \( V : H^\alpha \rightarrow H^\alpha \oplus H^\alpha \) by
\[
V(g) = uag \oplus vbg,
\]
then
\[
\|V(g)\| = \|uag \oplus vbg\| = \alpha(|uag| + |vbg|) = \alpha(|a| + |b|)g = \alpha(g).
\]
Thus \( V \) is an isometry from \( H^\alpha \) onto \( \text{Graph}(M_\phi) \). Let \( M \) be the inverse image of \( G \) under \( V \). Then \( M \) is a closed subspace of \( H^\alpha \) and for \( g \in M \), we have
\[
VMzg = V(zg) = uazg \oplus vbzg = (M_z \oplus M_z)Vg \in G,
\]
hence \( M \subset H^\alpha \) is invariant under \( M_z \), it follows from Theorem 7.8 that \( M = wH^\alpha \)
for some inner function \( w \), thus
\[
G = V(M) = \{ uawg \oplus vbwg : g \in H^\alpha \} = (M_w \oplus M_w)\text{Graph}(M_\phi).
\]
It follows that if the domain of \( G \) is dense in \( H^\alpha \), then \( w \) is a constant, hence \( G = \text{Graph}(M_\phi) \). \( \Box \)

As a corollary to the proof we have the following.

**Corollary 10.4.** If \( G \subset H^\alpha \oplus H^\alpha \) is a closed graph that is invariant under \( M_z \oplus M_z \), then there is a meromorphic function \( \phi \) in the Nevanlinna class and an inner function \( w \) such that
\[
G = (M_z \oplus M_z)\text{Graph}(M_\phi).
\]
11. A Corrected Result on von Neumann algebras

Suppose $\mathcal{M}$ is a diffuse type II$_1$ von Neumann algebra acting on a separable Hilbert space. This means that there is a faithful normal tracial state $\tau : \mathcal{M} \to \mathbb{C}$ (i.e., $\tau(ab) = \tau(ba)$, and $\tau(a^*a) = 0 \Rightarrow a = 0$). Suppose $\alpha$ is a symmetric gauge norm on $L^\infty(\mathbb{T})$. In [4] J. Fang, D. Hadwin, E. Nordgren and J. Shen defined the Banach space $L^\alpha(\mathcal{M}, \tau)$ which is the completion of $\mathcal{M}$ with respect to a norm induced by $\alpha$, which they still denote by $\alpha$. They stated a theorem that $L^\alpha(\mathcal{M}, \tau)$ is a reflexive Banach space if and only if $\alpha$ and $\alpha'$ are both continuous. However, as pointed out by Fyodor A. Sukochev in Mathematical Reviews: MR2417813 (2010a:46151), this theorem is not correct. In this section we state and prove the corrected version. We will freely use terminology and notation from [4].

We first describe how $\alpha$ is defined on $\mathcal{M}$. Suppose $\mathcal{A}$ is a masa (i.e., a maximal abelian C*-subalgebra) in $\mathcal{M}$. A theorem of von Neumann says that there is a selfadjoint element $a = a^* \in \mathcal{A}$ such that $\mathcal{A} = W^*(a)$ (the von Neumann algebra generated by $a$). If $Q_s = \chi_{[0,s)}(a)$ denotes the spectral projection of $a$ with respect to the set $[0,s)$, then $\mathcal{A} = W^*(a)$ is generated by the chain $\{Q_s : s \in [0,\infty)\}$ of projections. This chain is contained in a maximal chain $\mathcal{C}$ of projections in $\mathcal{M}$. Since $\tau$ is faithful, $\tau : \mathcal{C} \to [0,1]$ is an injective order-preserving map. Since $\mathcal{M}$ has no minimal projections, $\tau(\mathcal{C})$ must be $[0,1]$. Hence we can write $\mathcal{C} = \{P_t : t \in [0,1]\}$ where $\tau(P_t) = t$ for every $t \in [0,1]$. The map $P_t \mapsto \chi_{\{x \in [0,1] : s(x) \leq t\}}$ extends to an isomorphism from $\mathcal{A} = W^*(C)$ onto $L^\infty(\mathbb{T})$, such that if $b \in \mathcal{A}$ is associated to the function $f \in L^\infty(\mathbb{T})$, then $\tau(b) = \int_\mathbb{T} f dm$. But $L^\infty(\mathbb{T}) = W^*(z)$ where $z(\lambda) = \lambda$. If we let $U \in \mathcal{A}$ be the element associated with $z$, we have that $U$ is a unitary, $\mathcal{A} = W^*(U)$, and such that, for every $h \in L^\infty(\mathbb{T})$,

$$\tau(h(U)) = \int_\mathbb{T} h(z) dm(z).$$

Such a unitary element $U$ in $\mathcal{M}$ is called a Haar unitary and is completely characterized by

$$\tau(U^n) = 0 \text{ for } n \geq 1.$$

Hence, for every selfadjoint element $A \in \mathcal{M}$, $A$ is contained in a masa in $\mathcal{M}$, so there is a Haar unitary $U \in \mathcal{M}$ and a $\varphi \in L^\infty(\mathbb{T})$ such that $A = \varphi(U)$. We define $\alpha(A) = \alpha(\varphi)$. More generally we define $\alpha(T) = \alpha(|T|)$, where $|T| = (T^*T)^{1/2}$.

The difficulty is showing that $\alpha$ is well-defined (i.e., independent of $U$ and $\varphi$) and that $\alpha$ is a norm on $\mathcal{M}$ (see [4]).

**Theorem 11.1.** Suppose $\alpha \in S$ and $\mathcal{M}$ is a diffuse type II$_1$ von Neumann algebra with a faithful tracial state $\tau$ acting on a separable Hilbert space. Then $L^\alpha(\mathcal{M}, \tau)$ is a reflexive Banach space if and only if $\alpha$ and $\alpha'$ are both strongly continuous.

**Proof.** We know that $\mathcal{M}$ contains a Haar unitary $U$ and that $W^*(U)$ is a copy of $L^\infty(\mathbb{T})$ so that $\tau(f(U)) = \int_\mathbb{T} f dm$ and $\alpha(f(U)) = \alpha(f)$ for every $f \in L^\infty(\mathbb{T})$. Hence $L^\alpha(\mathbb{T})$ is isometrically isomorphic to a closed subspace of $L^\alpha(\mathcal{M}, \tau)$. Hence if $L^\alpha(\mathcal{M}, \tau)$ is reflexive, then so is $L^\alpha(\mathbb{T})$, which by part (3) of Theorem 6.2 implies $\alpha$ and $\alpha'$ are both strongly continuous.

Now assume $\alpha$ and $\alpha'$ are both strongly continuous and suppose $\varphi : L^\alpha(\mathcal{M}, \tau) \to \mathbb{C}$ is a continuous linear functional. It was shown in [6] that on the closed unit ball of $\mathcal{M}$, the $\alpha$-topology and the strong operator topology coincide. It follows that if
\{P_n\} is an orthogonal sequence of projections in \(\mathcal{M}\), then
\[
\varphi \left( \sum_{n=1}^\infty P_n \right) = \sum_{n=1}^\infty \varphi (P_n),
\]
which implies \(\varphi : \mathcal{M} \to \mathbb{C}\) is weak*-continuous. Hence there is an \(A \in L^1(\mathcal{M}, \tau)\) such that, for every \(T \in \mathcal{M}\)
\[
\varphi (T) = \tau (AT).
\]
This, and the fact that \(\mathcal{M}\) is dense in \(L^\alpha(\mathcal{M}, \tau)\), implies
\[
\alpha'(A) = \sup \{ \tau (AT) : T \in \mathcal{M}, \ \alpha(T) \leq m \} = \| \varphi \|.
\]
This is the hard part of the proof that \(L^\alpha(\mathcal{M}, \tau)^\# = L^{\alpha'}(\mathcal{M}, \tau)\). Since \(\alpha'' = \alpha\) and \(\alpha'\) is strongly continuous, we see that
\[
L^\alpha(\mathcal{M}, \tau)^{\#\#} = L^{\alpha'}(\mathcal{M}, \tau)^\# = L^\alpha(\mathcal{M}, \tau).
\]

**Remark 11.2.** The proof of the preceding theorem shows that if \(\alpha\) is continuous and \(\alpha'\) is strongly continuous, then \(L^\alpha(\mathcal{M}, \tau)^\# = L^{\alpha'}(\mathcal{M}, \tau)\).

We conclude this section by noting that the analogues of Theorems 3.3 and 3.5 hold in the von Neumann algebra case for symmetric gauge norms. The proofs are easy adaptations and we omit them here. We do need, however, the fact that \(L^1(\mathcal{M}, \tau)^\# = \mathcal{M}\), which, by a theorem of C. Akemann [1], implies that \(L^1(\mathcal{M}, \tau)\) is weakly sequentially complete. A key ingredient is that there is a completion \(\mathcal{Y}\) of \(\mathcal{M}\) in measure (see [9]), which is an algebra containing \(L^1(\mathcal{M}, \tau)\) such that, for every \(h \in \mathcal{Y}\) there exist \(u_1, v_1, u_2, v_2 \in \mathcal{M}\) with \(v_1, v_2\) invertible in \(\mathcal{Y}\) (but maybe not in \(\mathcal{M}\)) such that \(h = v_1^{-1} u_1 = u_2 v_2^{-1}\). Note that since \(\mathcal{M}\) may not be commutative, there is a difference between left \(\mathcal{M}\)-module homomorphisms and right \(\mathcal{M}\)-module homomorphisms, which is reflected in parts (1) and (2) below.

**Theorem 11.3.** Suppose \(\mathcal{M}\) is a II\(_1\) von Neumann algebra with a faithful normal tracial state \(\tau\) and suppose \(\alpha \in \mathcal{S}_c\). Then

1. If \(T : L^{\alpha'}(\mathcal{M}, \tau) \to L^1(\mathcal{M}, \tau)\) is a bounded linear map such that \(T(hg) = hT(g)\) whenever \(h \in \mathcal{M}\) and \(g \in L^{\alpha'}(\mathcal{M}, \tau)\), then there is an \(f \in L^\alpha(\mathcal{M}, \tau)\) such that \(\alpha(f) = \|T\|\) and \(T(h) = hf\) for every \(h \in L^{\alpha'}(\mathcal{M}, \tau)\);
2. If \(T : L^{\alpha'}(\mathcal{M}, \tau) \to L^1(\mathcal{M}, \tau)\) is a bounded linear map such that \(T(gh) = T(g)h\) whenever \(h \in \mathcal{M}\) and \(g \in L^{\alpha'}(\mathcal{M}, \tau)\), then there is an \(f \in L^\alpha(\mathcal{M}, \tau)\) such that \(\alpha(f) = \|T\|\) and \(T(h) = fh\) for every \(h \in L^{\alpha'}(\mathcal{M}, \tau)\);
3. The following are equivalent:
   (a) \(L^\alpha(\mathcal{M}, \tau) = L^\alpha(\mathcal{M}, \tau)\);
   (b) \(\alpha\) is strongly continuous;
   (c) \(L^\alpha(\mathcal{M}, \tau)\) is weakly sequentially complete.

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