The Bifurcation analysis of Prey-Predator Model in The Presence of Stage Structured with Harvesting and Toxicity

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Abstract—For a mathematical model the local bifurcation like pitchfork, transcritical and saddle node occurrence condition is defined in this paper. With the existing of toxicity and harvesting in predator and prey it consist of stage-structured. Near the positive equilibrium point of mathematical model on the Hopf bifurcation with particular emphasis it established. Near the equilibrium point $E_0$ the transcritical bifurcation occurs it is described with analysis. And it shown that at equilibrium points $E_1$ and $E_2$ happened the occurrence of saddle-node bifurcation. At each point the pitch fork bifurcation occurrence is not happened. For the occurrence of local bifurcation illustration there used some numerical simulation.

Keywords: Ecological model, Equilibrium point, Local bifurcation, Hopf bifurcation.

1. Introduction

The prey and predator model is an important topic at present, as it is used to solve many problems in the ecology, nature and other sciences. The prey system includes several interactions, including interactions, competition [10], co-existence, food chain [7] and age stage[9]. The ecological models of age stage are more logical than models that do not contain phase structure. In addition, there are several factors that affect the system, for example, harvesting, disease, toxicity, shelter and others. Sometimes, differences in any parameter in the system can lead to complex behavior that leads to system instability, Causing a bifurcation that is main the qualitative change in the behavior of a dynamic system as a result of changing one of its coefficients.

For defining the ordinary nonlinear differential equations the oscillatory solutions of a system and the stable state the bifurcation theory is consider as mathematical tool. More complex features like exotic attractors, the emergence and disappearance of equilibrium and periodic orbits are the examples. To understand the nonlinear dynamic systems results of the bifurcation theory and model used as fundamental. To identify the complex model controllers it also help.

The bifurcations separated into two chief classes: nearby bifurcations and worldwide bifurcations. Nearby bifurcations, which can be broke down altogether through changes in the neighborhood security properties of equilibria, occasional circle or other invariant sets as parameters cross through basic edges, for example, saddle hub, transcritical, pitchfork, period-multiplying (flip), Hopf and Neimark (auxiliary Hopf) bifurcation. Worldwide bifurcations happen when bigger invariant sets, for example, intermittent circles, slam into equilibria. This causes changes in the topology of the directions in stage space which can’t be limited to a little neighborhood.

Moreover, a Hopf bifurcation implies the appearance or the vanishing of an occasional circle over a neighborhood change in the properties of the dependability around a fixedpoint. More accurately, it is a neighborhood bifurcation where a fixed purpose of a dynamical framework loses steadiness, as a couple of complex conjugate eigenvalues (of the linearization around the fixed point) cross the unpredictable

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plane fanciful hub [4]. Under sensible general recommendations about the dynamical framework, a little range breaking points cycle branches from the fixed point. A Hopf bifurcation is otherwise called a Poincare–Andronov–Hopf bifurcation, named after Henri Poincaré, Eberhard Hopf, and Aleksander Andronov.

In the past few years, this theory has evolved considerably through the use of new ideas and methods and their introduction into the theory of dynamic systems. For an ecological system having of a stage structured prey a predator in [6] Majeed research the occurrence of Hopf bifurcation and local bifurcation. In prey population with a refuge-stage structure in [8] Majeed and Naji described near each of the equilibrium points of a prey-predator model the existance of local bifurcation. In [3] Naji, Majeed and Kadhim introduced with refuge near each of the equilibrium points of a stage structured prey food web model the Hopf bifurcation and local bifurcation. With two functional responses and refuge the Hopf bifurcation near the positive point of a stage structured prey food chain model and near each of the equilibrium points the local bifurcation represented by Ali and Majeed in [5]. With the non-refugees prey it show the connection between the two predators.

Finally, in this paper, a set of basic results and methods in local bifurcation theory around all equilibrium points and a Hopf bifurcation theory around the positive equilibrium point for a system include age stage with harvesting and toxicity which depends on a single parameter $\mu$ is presented and discussed.

2. Model formulation

In this area, the model comprises of two species prey and predator, every specie isolated into two classes: one is youthful and other is experienced, which are meant to their population size at time $T$ by $X(T)$, $Y(T)$, $Z(T)$ and $W(T)$ for juvenile prey, develop prey, youthful predator and develop predator separately. Presently, it is referenced in [2] so as to define the elements of such framework, the accompanying presumptions are viewed:

- with grown up rates $\eta_1$ and $\eta_2$ respectively the predator and immature prey grown up to be mature.
- The immature prey depends completely in its feeding on mature prey that growth logistically with an intrinsic growth rate $r$ and carrying capacity $K > 0$ in absence of mature predator. Also the immature predator depends completely in its feeding on mature predator that consumes the immature and mature prey with the classical Lotka-Volterra functional response with consumption rates $\theta_1$ and $\theta_2$, respectively, therefore the predator species growth due to attack by mature predator on immature and mature prey with conversion rates $0 < e_1 < 1$ and $0 < e_2 < 1$.
- It shows mature predator, mature prey, immature predator and immature prey’s toxicity coefficients and catchability coefficients through $\phi_i$ and $\delta_i, i=1,2,3,4$.

According above assumptions, the model is formulated as follows:
\[
\frac{dX}{dt} = rY \left(1 - \frac{Y}{K}\right) - \eta_1 X - \delta_1 X^2 - \varphi_1 X - \theta_1 XW
\]
\[
\frac{dY}{dt} = \eta_1 X - \delta_2 Y^2 - \varphi_2 Y - \theta_2 YW
\]
\[
\frac{dZ}{dt} = n_1 e_1 \theta_1 XW + n_2 e_2 \theta_2 YW - \eta_2 Z - \delta_3 Z - \varphi_3 Z - \gamma_1 Z
\]
\[
\frac{dW}{dt} = \eta_2 Z + (1 - n_1) e_1 \theta_1 XW + (1 - n_2) e_2 \theta_2 YW - \delta_4 W - \varphi_4 W - \gamma_2 W
\]

By using given parameters and variables it can reduce the number of parameters:
\[
t = rT, \quad x = \frac{X}{K}, \quad y = \frac{Y}{k}, \quad z = \frac{Z}{k}, \quad w = \frac{W}{k}, \quad \alpha_i = \frac{\eta_i}{r}, \quad u_j = \frac{\delta_j}{r}, \quad h_j = \frac{\varphi_j}{r}, \quad d_i = \frac{\gamma_i}{r}
\]
\[
\beta_i = r \frac{\theta_i k}{r}, \quad \beta_{i+2} = \frac{n_i e_i \theta_i k}{r}, \quad \beta_{i+4} = \frac{(1 - n_i) e_i \theta_i k}{r}, \quad i = 1, 2 \quad \text{and} \quad j = 1, 2, 3, 4.
\]

Then dimensional system (1) becomes:
\[
\frac{dx}{dt} = x \left(\frac{y(1 - y)}{x} - (\alpha_1 + h_1) - u_1 x - \beta_1 w\right) = xf_1(x, y, z, w)
\]
\[
\frac{dy}{dt} = y \left(\frac{\alpha_1 x}{y} - u_2 y - h_2 - \beta_2 w\right) = yf_2(x, y, z, w)
\]
\[
\frac{dz}{dt} = z \left(\frac{\beta_3 xw}{z} + \beta_4 yw}{z} - (\alpha_2 + u_3 + h_3 + d_1)\right) = zf_3(x, y, z, w)
\]
\[
\frac{dw}{dt} = w \left(\frac{\alpha_2 z}{w} + \beta_5 x + \beta_6 y - (u_4 + h_4 + d_2)\right) = wf_4(x, y, z, w)
\]

**Theorem 1[2]:** It is uniformly bounded complete solutions of system (2).

3. **Equilibrium points of system[2] stability analysis and existence**

The system (2) has maximum 3 points of nonnegative equilibrium that are as follows:

- The equilibrium point is locally asymptotically stable using the given constraint. And it is denoted by \(E_0 = (0, 0, 0, 0)\) that is always there.

\[
h_2 > 1.
\]

- The equilibrium point \(E_1 = (\bar{x}, \bar{y}, 0, 0)\), exists uniquely in \(\text{Int.} \, R^2_+\), if the condition hold:

\[
\alpha_1 + h_1 < \frac{\alpha_1}{h_2}.
\]

Under the given situation the \(E_1\) is consider as locally asymptotically stable:

\[
\frac{1}{2} (u_4 + h_4 + d_2) > (\beta_5 \bar{x} + \beta_6 \bar{y}).
\]

\[
\left(\alpha_2 + u_3 + h_3 + d_1\right) \left((u_4 + h_4 + d_2) - (\beta_5 \bar{x} + \beta_6 \bar{y})\right) > \alpha_2 (\beta_3 \bar{x} + \beta_4 \bar{y}).
\]

However, it is a saddle (unstable) point otherwise.
Finally the positive equilibrium point $E_2 = (\dot{x}, \dot{y}, \dot{z}, \dot{w})$, exists in Int. $R_+^4$, if the condition holds:

$$\dot{x} > \frac{(u_2 \dot{y} + h_2)}{\alpha_1}.$$  \(8\)

At $E_2$ the system (2)’s Jacobian matrix can be defined as:

$$J_2 = J_2(E_2) = \begin{bmatrix} *_{ai} *_{aj} \end{bmatrix}_{4 \times 4},$$  \(9\)

where $a_{11} = -(\alpha_1 + h_1) - 2u_1 \dot{x} - \beta_4 \dot{w} < 0$, $a_{12} = 1 - 2\dot{y}$, $a_{13} = 0$,

$$a_{14} = -\beta_1 \dot{x} < 0, \quad a_{21} = a_1 > 0, \quad a_{22} = -2u_2 \dot{y} - h_2 - \beta_2 \dot{w} < 0,$$

$$a_{23} = 0, \quad a_{24} = -\beta_2 \dot{y} < 0, \quad a_{31} = \beta_3 \dot{w} > 0, \quad a_{32} = \beta_4 \dot{w} > 0,$$

$$a_{33} = -(\alpha_2 + u_3 + h_3 + d_1) < 0, \quad a_{34} = \beta_3 \dot{x} + \beta_4 \dot{y} > 0, \quad a_{41} = \beta_5 \dot{w} > 0,$$

$$a_{42} = \beta_6 \dot{w} > 0, \quad a_{43} = a_2 > 0, \quad a_{44} = \beta_5 \dot{x} + \beta_6 \dot{y} - (u_4 + h_4 + d_2).$$

Then the characteristic equation of $J_2$ is given by:

$$\lambda^4 + A_1 \lambda^3 + A_2 \lambda^2 + A_3 \lambda + A_4 = 0,$$  \(10\)

where:

$$A_1 = -(\rho_0 + \rho_1) > 0,$$

$$A_2 = \rho_2 + \rho_0 \rho_1 + \rho_3 - \rho_4,$$

$$A_3 = -\rho_0 \rho_2 - \rho_1 \rho_3 + \rho_5 \rho_6 - \rho_7 - \rho_8 + \rho_9 \rho_{10} - (\rho_{11} + \rho_{12}),$$

$$A_4 = \rho_2 \rho_3 - \rho_6 \rho_{13} + \rho_{33} \rho_7 (\rho_7 + \rho_8) - \rho_9 \rho_{14} + \rho_{11} \rho_{12} - \rho_{43} (\rho_{15} + \rho_{16}),$$

with:

$$\rho_0 = \alpha_{11} + \alpha_{22} < 0, \quad \rho_1 = \alpha_{33} + \alpha_{44}, \quad \rho_2 = \alpha_{33} a_{44} - \alpha_{34} a_{43},$$

$$\rho_3 = \alpha_{11} a_{22} - \alpha_{12} a_{21}, \quad \rho_4 = \alpha_{24} a_{42} + \alpha_{14} a_{41} < 0, \quad \rho_5 = \alpha_{11} + \alpha_{33} < 0,$$

$$\rho_6 = \alpha_{34} a_{42} < 0, \quad \rho_7 = \alpha_{24} a_{24} a_{41}, \quad \rho_8 = \alpha_{24} a_{42} < 0, \quad \rho_9 = \alpha_{44} a_{41} < 0,$$

$$\rho_{10} = \alpha_{22} + \alpha_{33} < 0, \quad \rho_{11} = \alpha_{24} a_{32} a_{43} < 0, \quad \rho_{12} = \alpha_{14} a_{31} a_{43} < 0,$$

$$\rho_{13} = \alpha_{11} a_{33} > 0, \quad \rho_{14} = \alpha_{22} a_{33} > 0, \quad \rho_{15} = \alpha_{12} a_{24} a_{31}, \quad \rho_{16} = \alpha_{14} a_{21} a_{32}.$$

With negative real parts it has the roots when it use Routh-Hawirtz criterion equation (10) if and only if $A_i > 0$, $i = 1,3,4$ and $\Delta = (A_1 A_2 - A_3) A_3 - A_2^2 A_4 > 0$.

Clearly, $A_i > 0$ if the following conditions hold:

$$y > \frac{1}{\lambda},$$  \(11\)

$$\left(\beta_5 \dot{x} + \beta_6 \dot{y}\right) > (u_4 + h_4 + d_2) - \frac{a_2 (\beta_3 \dot{x} + \beta_4 \dot{y})}{(\alpha_2 + u_4 + h_4 + d_2)},$$  \(12\)

$$\frac{\beta_5}{\beta_6} < \frac{-\alpha_1 \beta_1 \dot{x}}{(1 - 2\dot{y})} \frac{\beta_2 \dot{y}}{\beta_4},$$  \(13\)

Straightforward computation shows that:

$$\Delta = P_1 - P_2,$$  \(14\)

where $P_1 = (\rho_1 \rho_2 A_3)^2 + A_3 [\rho_1 \rho_4 - \rho_0 (\rho_2 + \rho_4)] - \rho_2^2 \rho_2 \rho_3 - \rho_0^2 \rho_1 A_3 - \rho_2^2 \rho_0 A_3 - \rho_1 \rho_2 A_3$,

$$P_2 = A_3 [(\rho_1 \rho_2)^2 A_3 + \rho_1 (\rho_4 - (\rho_0^2 + \rho_2 + \rho_1 \rho_0)) - \rho_0 \rho_3 (\rho_3 - \rho_4)] - \rho_2^2 \rho_2 \rho_3.$$


Hence, $\Delta$ will be positive if in addition of the situation (11-14) the given condition hold:

$$P_1 > P_2 \quad (15)$$

4. Analysis of Local bifurcation

For local bifurcation Sotomayor’s theorem application is suitable in the given theorems.

The system (2)’s Jacobian matrix is:

$$J = \begin{bmatrix} a_{ij} \end{bmatrix}_{4 \times 4}, \quad (16)$$

where,

$$
\begin{align*}
    a_{11} &= -(\alpha_1 + h_1) - 2u_1 x - \beta_1 w, \quad a_{12} = 1 - 2 y, \quad a_{13} = 0, \\
    a_{14} &= -\beta_1 x, \quad a_{21} = \alpha_1, \quad a_{22} = -2u_2 y - h_2 - \beta_2 w, \quad a_{23} = 0, \\
    a_{24} &= -\beta_2 y, \quad a_{31} = \beta_3 w, \quad a_{32} = \beta_4 w, \quad a_{33} = -(\alpha_2 + u_3 + h_3 + d_1), \\
    a_{34} &= \beta_3 x + \beta_4 y, \quad a_{41} = \beta_5 w, \quad a_{42} = \beta_6 w, \quad a_{43} = \alpha_2, \\
    a_{44} &= \beta_5 x + \beta_6 y - (u_4 + h_4 + d_2).
\end{align*}
$$

For any nonzero vector it can verify that $V = (v_1, v_2, v_3, v_4)^T$ we have:

$$D^2 f(\mu)(X, \mu)(V, V) = \begin{bmatrix} b_{ij} \end{bmatrix}_{4 \times 1}, \quad (17)$$

where:

$$
\begin{align*}
    b_{11} &= 2 \left[ u_1 v_1^2 - v_2^2 - \beta_1 v_1 v_4 \right], \\
    b_{21} &= -2 \left[ u_2 v_2 + \beta_2 v_2 v_4 \right], \\
    b_{31} &= 2 \left[ \beta_3 v_1 v_4 + \beta_4 v_2 v_4 \right], \\
    b_{41} &= 2 \left[ \beta_5 v_1 v_4 + \beta_6 v_2 v_4 \right],
\end{align*}
$$

and

$$D^3 f(\mu)(X, \mu)(V, V, V) = [0, 0, 0, 0]^T. \quad (18)$$

**Theorem 2:** Suppose that the following conditions are satisfied:

$$h_2 < 1 \quad (19)$$

$$u_1 h_2^2 v_1^{[0]} \neq \alpha_1 \left( u_2 + \alpha_2 v_1^{[0]} \right) \quad (20)$$

Then system (2) at the equilibrium point $E_0 = (0, 0, 0, 0)$ with the parameter value,

$$\alpha_1' = \frac{h_1 h_2}{1 - h_2},$$

**Proof:** The Jacobian matrix $J(E_0)$ of system (2) is:

$$J_0 = J(E_0) = \begin{bmatrix} c_{ij} \end{bmatrix}_{4 \times 4}, \quad (21)$$

where:

$$
\begin{align*}
    c_{11} &= -(\alpha_1 + h_1), \quad c_{12} = 1, \quad c_{13} = 0, \quad c_{14} = 0, \\
    c_{21} &= \alpha_1, \quad c_{22} = -h_2, \quad c_{23} = 0, \quad c_{24} = 0, \\
    c_{31} &= 0, \quad c_{32} = 0, \quad c_{33} = -(\alpha_2 + u_3 + h_3 + d_1), \quad c_{34} = 0, \\
    c_{41} &= 0, \quad c_{42} = 0, \quad c_{43} = \alpha_2, \quad c_{44} = -(u_4 + h_4 + d_2).
\end{align*}
$$
at the equilibrium point $E_0$ has zero eigenvalue (say $\lambda_{0x} = 0$) at $\alpha_1 = \alpha_1^*$, and the Jacobian matrix $\mathbf{f}_0$ with $\alpha_1 = \alpha_1^*$ becomes:

$$\mathbf{f}_0' = \mathbf{f}(\alpha_1 = \alpha_1^*) = [\mathbf{c}]_{4 \times 4'},$$

where $c_{ij} = c_{ij}$ for all $i, j = 1, 2, 3, 4$ except $c_{11} = -(\alpha_1^* + h_1)$ and $c_{21}^* = \alpha_1^*$.

Now, let $\Psi^{[0]} = \left( v_1^{[0]}, v_2^{[0]}, v_3^{[0]}, v_4^{[0]} \right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{0x} = 0$.

Thus, $(\mathbf{f}_0' - \lambda_{0x} \mathbf{I})\Psi^{[0]} = 0$, which gives:

$$v_2^{[0]} = \frac{\alpha_1^*}{h_2} v_1^{[0]}, \quad v_3^{[0]} = 0, \quad v_4^{[0]} = 0 \quad \text{and} \quad v_1^{[0]} \text{ any nonzero real number.}$$

Let $\Psi^{[0]} = \left( \psi_1^{[0]}, \psi_2^{[0]}, \psi_3^{[0]}, \psi_4^{[0]} \right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{0x} = 0$ of the matrix $\mathbf{f}_0'$.

Then we have: $(\mathbf{f}_0' - \lambda_{0x} \mathbf{I})\Psi^{[0]} = 0$.

By solving this equation for $\Psi^{[0]}$, we obtain:

$$\Psi^{[0]} = \left( \psi_1^{[0]}, \frac{1}{h_2} \psi_1^{[0]}, 0, 0 \right)^T,$$

where $\psi_1^{[0]}$ any nonzero real number.

Now, consider:

$$\frac{\partial \mathbf{f}}{\partial \alpha_1} = \mathbf{f}_x(X, \alpha_1) = \left( \frac{\partial f_1}{\partial \alpha_1}, \frac{\partial f_2}{\partial \alpha_1}, \frac{\partial f_3}{\partial \alpha_1}, \frac{\partial f_4}{\partial \alpha_1} \right)^T = (-x, x, 0, 0)^T.$$  

Now, since

$$D \mathbf{f}_{\alpha_1}(X, \alpha_1) = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $D \mathbf{f}_{\alpha_1}(X, \alpha_1)$ represents the derivative of $D \mathbf{f}_{\alpha_1}(X, \alpha_1)$ with respect to $X = (x, y, z, w)^T$. Furthermore, it is observed that:

$$D \mathbf{f}_{\alpha_1}(X, \alpha_1^*) \Psi^{[0]} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{1}{h_2} v_1^{[0]} = \begin{bmatrix} -v_1^{[0]} \\ v_1^{[0]} \\ 0 \\ 0 \end{bmatrix},$$

$$(\Psi^{[0]})^T [D \mathbf{f}_{\alpha_1}(X, \alpha_1^*) \Psi^{[0]}] = \left( \frac{1}{h_2} - 1 \right) v_1^{[0]} \psi_1^{[0]}.$$  

So, according to the condition (19), we obtain:

$$(\Psi^{[0]})^T [D \mathbf{f}_{\alpha_1}(X, \alpha_1^*) \Psi^{[0]}] \neq 0.$$  

Now, by substituting $\Psi^{[0]}$ in (17), we get:
\[
D^2f(E_0, \alpha_1^*)(V^{[0]}, V^{[0]}) = \begin{bmatrix}
2(u_1(v_1^{[0]}) - (\frac{\alpha_1^*}{h_2}v_1^{[0]})^2) \\
-2(\frac{\alpha_1^*}{h_2}v_1^{[0]}) \\
0 \\
0
\end{bmatrix}.
\]

Hence, it is obtained that: \((\psi^{[0]})^T D^2f(E_0, \alpha_1^*)(V^{[0]}, V^{[0]}) = 2Q^*V_1^{[0]}\psi_1^{[0]},\)

where: 
\[
Q^* = \left[\frac{u_1h_2v_1^{[0]} - \alpha_1^*(u_1v_1^{[0]} + u_2)}{h_2^2}\right].
\]

So, if the condition (20) is satisfied, we obtain that:

\[
(\psi^{[0]})^T D^2f(E_0, \alpha_1^*)(V^{[0]}, V^{[0]}) \neq 0.
\]

**Theorem 3:** Suppose that the following conditions are satisfied:

\[
\bar{y} < \frac{1}{2} \tag{22}
\]

\[
\alpha_1 > \frac{2u_2\bar{y}(\alpha_1 + h_1 + 2u_1\bar{x})}{(1 - 2\bar{y})} \tag{23}
\]

\[
u_1 L_1^2 L_4 + L_3 [L_5(\beta_3 L_1 + \beta_4) + L_6(\beta_5 L_1 + \beta_6)] \\
\neq \left(\frac{L_4(1 + \beta_4 L_1 L_3) + \frac{u_2}{v_2^{[0]} + \beta_2 L_3}}{\alpha_1^*} + \beta_6\right), \tag{24}
\]

where:

\[
L_1 = \left(\frac{\bar{c}_{14} L_3 - \bar{c}_{12}}{\bar{c}_{11}}\right), \quad L_2 = -\frac{\bar{c}_{34}}{\bar{c}_{33}} L_3, \quad L_3 = -\frac{\bar{c}_{13} L_4^{[0]} + \beta_4}{\bar{c}_{11} \bar{c}_{22} - \bar{c}_{12}^2},
\]

\[
\bar{L}_4 = -\frac{\bar{c}_{21}}{\bar{c}_{11}}, \quad \bar{L}_5 = -\frac{\bar{c}_{34}}{\bar{c}_{33}} \bar{L}_6, \quad \bar{L}_6 = \frac{\bar{c}_{11} \bar{c}_{21} - \bar{c}_{12} \bar{c}_{22}}{\bar{c}_{34} \bar{c}_{43} - \bar{c}_{33} \bar{c}_{44}}.
\]

Then system (2) at the equilibrium point \(E_1 = (\bar{x}, \bar{y}, 0, 0)\) with the parameter value:

\[
\bar{h}_2 = \frac{\alpha_1^*(1 - 2\bar{y}) - 2u_2\bar{y}(\alpha_1^* + h_1 + 2u_1\bar{x})}{(\alpha_1^* + h_1 + 2u_1\bar{x})}
\]

possesses saddle-node bifurcation at \(E_1 = (\bar{x}, \bar{y}, 0, 0)\).

**Proof:** The Jacobian matrix \(J(E_1)\) of system (2) is:

\[
J_1 = J(E_1) = [d_{ij}]_{4 \times 4}, \tag{25}
\]

where:

\[
d_{11} = -(\alpha_1 + h_1) - 2u_1\bar{x}, \quad d_{12} = 1 - 2\bar{y}, \quad d_{13} = 0, \quad d_{14} = -\beta_1 \bar{x},
\]

\[
d_{12} = 1 - 2\bar{y}, \quad d_{13} = 0, \quad d_{14} = -\beta_1 \bar{x},
\]

\[
d_{11} = -(\alpha_1 + h_1) - 2u_1\bar{x}, \quad d_{12} = 1 - 2\bar{y}, \quad d_{13} = 0, \quad d_{14} = -\beta_1 \bar{x},
\]

\[
d_{11} = -(\alpha_1 + h_1) - 2u_1\bar{x}, \quad d_{12} = 1 - 2\bar{y}, \quad d_{13} = 0, \quad d_{14} = -\beta_1 \bar{x}.
\]
With \( h_2 = \tilde{h}_2 \) at the equilibrium point that is defined by \( E_1 \). It has Jacobian matrix \( J_1 \) and zero eigenvalue (say \( \lambda_{1y} = 0 \)) at \( h_2 = \tilde{h}_2 \):

\[
J_1 = J(h_2 = \tilde{h}_2) = [c_{ij}]_{4 \times 4},
\]

where, \( \tilde{c}_{ij} = d_{ij} \) for all \( i, j = 1, 2, 3, 4 \) except \( \tilde{c}_{ij} = -2u_2y - \hat{h}_2 \).

Now, let \( V^{[1]} = (v_1^{[1]}, v_2^{[1]}, v_3^{[1]}, v_4^{[1]})^T \) corresponding to the eigenvalue \( \lambda_{1y} = 0 \). It will be the eigenvector.

Thus \( (J_1 - \lambda_{1y}I)V^{[1]} = 0 \), which gives:

\[
V^{[1]} = (L_1v_2^{[1]}, L_2v_2^{[1]}, L_3v_2^{[1]}),
\]

where, \( v_2^{[1]} \) any non-zero real number, with \( L_1, L_2, L_3 \) which are mentioned in the state of the theorem.

Let \( \psi^{[1]} = (\psi_1^{[1]}, \psi_2^{[1]}, \psi_3^{[1]}, \psi_4^{[1]})^T \) be the eigenvector associated with the eigenvalue \( \lambda_{1y} = 0 \) of the matrix \( J_1^T \).

Then we have:

\[
(J_1^T - \lambda_{1y}I)\psi^{[1]} = 0. \quad \text{for } \psi^{[1]} \text{ by getting the solution of this equation, it obtain}
\]

\[
\psi^{[1]} = (L_4\psi_2^{[1]}, L_2\psi_2^{[1]}, L_5\psi_2^{[1]}, L_6\psi_2^{[1]}),
\]

where, \( \psi_2^{[1]} \) any non-zero real number, with \( L_4, L_5, L_6 \) which are mentioned in the state of the theorem.

Now, consider:

\[
\frac{\partial f}{\partial h_2} = f_h(X, h_2) = \left( \frac{\partial f_1}{\partial h_2}, \frac{\partial f_2}{\partial h_2}, \frac{\partial f_3}{\partial h_2}, \frac{\partial f_4}{\partial h_2} \right)^T = (0, -y, 0, 0)^T.
\]

So, \( f_h(E_1, \tilde{h}_2) = (0, -\tilde{y}, 0, 0)^T \), and hence \( (\psi^{[1]}_1)^T f_h(E_1, \tilde{h}_2) \neq 0 \). It satisfied saddle-node bifurcation according to the theorem of Sotomayor:

\[
\text{Now, since } \quad Df_h(E_1, \tilde{h}_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

where, \( Df_h(X, h_2) \) represents the derivative of \( Df_h(X, h_2) \) with respect to \( X = (x, y, z, w)^T \).

Furthermore, it is observed that:
\[
Df_{\tilde{h}_2}(E_1, \tilde{h}_2)V^{[1]} = \begin{bmatrix}
0 & 0 & 0 & [L_1 v_2^{[1]}] \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & [L_2 v_2^{[1]}] \\
0 & 0 & 0 & [L_3 v_2^{[1]}]
\end{bmatrix} = \begin{bmatrix}
0 \\
-\psi_2^{[1]} \\
0 \\
0
\end{bmatrix},
\]

\((\psi^{[1]})^T [Df_{\tilde{h}_2}(E_1, \tilde{h}_2)V^{[1]}] = -v_2^{[1]} \psi_2^{[1]} \neq 0.\)

Now, by substituting \(V^{[1]}\) in (17), we get:

\[
D^2f_{\tilde{h}_2}(E_1, \tilde{h}_2)(V^{[1]}_1, V^{[1]}_2) = \begin{bmatrix}
2 \left[ u_1 L_1^2 - 1 - \beta_1 L_1 L_3 \right] (v_2^{[1]})^2 \\
-2 \left[ u_2 + \beta_2 L_3 v_2^{[1]} \right] v_2^{[1]} \\
2 \left[ \beta_3 L_1 L_3 + \beta_4 L_3 \right] (v_2^{[1]})^2 \\
2 \left[ \beta_5 L_1 L_3 + \beta_6 L_3 \right] (v_2^{[1]})^2
\end{bmatrix}.
\]

Hence, it is obtained that:

\((\psi^{[1]})^T D^2f_{\tilde{h}_2}(E_1, \tilde{h}_2)(V^{[1]}_1, V^{[1]}_2) = 2 \tilde{Q} (v_2^{[1]})^2 \psi_2^{[1]},\)

where, \(\tilde{Q} = u_1 L_1^2 L_4 + L_3 [L_5 (\beta_3 L_1 + \beta_4) + L_6 (\beta_5 L_1 + \beta_6)] - \left( L_4 (1 + \beta_1 L_1 L_3) + \frac{u_2}{v_2^{[1]}} + \beta_2 L_3 \right),\)

with \(\tilde{L}_i; \ i = 1,3,4,5,6\) which are mentioned in the state of the theorem.

So, if the condition (24), is satisfied, we obtain that:

\((\psi^{[1]})^T D^2f_{\tilde{h}_2}(E_1, \tilde{h}_2)(V^{[1]}_1, V^{[1]}_2) \neq 0.\)

**Theorem 4:** Suppose that the following conditions are satisfied:

\(\gamma < \frac{1}{2},\) \hspace{0.5cm} (26)

\(d_{11}d_{22} < d_{12}d_{21}\) \hspace{0.5cm} (27)

\(\beta_5 \dot{x} + \beta_6 \dot{y} > (u_4 + h_4)\) \hspace{0.5cm} (28)

\(\dot{L}_1 \left( u_1 \dot{L}_4 + \beta_3 \dot{L}_6 + \beta_5 \right) + \dot{L}_3 \left( \beta_4 \dot{L}_6 + \beta_6 \right) \neq \left( \dot{L}_2 \left( \beta_2 \dot{L}_1 + \beta_4 \dot{L}_6 \right) + \dot{L}_2 \dot{L}_5 \left( \frac{u_2}{v_2^{[2]}} + \beta_2 \right) \right),\) \hspace{0.5cm} (29)

where:

\(\dot{L}_1 = -\left( \frac{d_{12}}{d_{11}} \right) \dot{L}_2 + \frac{d_{14}}{d_{11}} \dot{L}_3 \), \hspace{0.5cm} \(\dot{L}_2 = \left( \frac{d_{14}d_{21} - d_{11}d_{24}}{d_{11}d_{22} - d_{12}d_{21}} \right), \hspace{0.5cm} \dot{L}_3 = -\left( \frac{d_{34} + d_{31} \dot{L}_1 + d_{32} \dot{L}_2}{d_{33}} \right),\)
\[ \dot{L}_4 = -\left( \frac{d_{21} \dot{L}_5 + (d_{31} \dot{L}_6 + d_{41})}{d_{11}} \right), \dot{L}_5 = \left( \frac{d_{11} d_{42} - d_{12} (d_{31} \dot{L}_6 + d_{41}) - d_{11} d_{32} \dot{L}_6}{d_{12} d_{21} - d_{11} d_{22}} \right) \]

\[ \dot{L}_6 = -\frac{d_{43}}{d_{33}} \]

Then system (2.2) at the equilibrium point \[ E_2 = (\dot{x}, \dot{y}, \dot{z}, \dot{w}) \] with the parameter value:

\[ \dot{d}_2 = \frac{\dot{Q}_2}{d_{33}(d_{11} d_{22} - d_{12} d_{21}) \rho_3} \]

where:

\[ \dot{Q}_2 = d_{33} \left( \rho_7 + \rho_8 + \left( \beta_5 \dot{x} + \beta_6 \dot{y} - (u_4 + h_4) \right) \rho_3 \right) - \rho_6 \rho_{13} - \rho_9 \rho_{14} + d_{11} \rho_{11} + d_{22} \rho_{12} 
- d_{43} (\rho_{15} + \rho_{16}) \]

**Proof:** If and only if \[ A_4 = 0 \] it have the zero eigen value (say \( \lambda_{2w} = 0 \)) given by eq. (10), has the characteristic equation. And a non-hyperbolic equilibrium point is considered \( E_2 \). With parameter \( d_2 = d_2 \) at the equilibrium point \( E_2 \), the system (2) jacobian matrix become:

\[ J_2 = J_2 \left( d_2 = d_2 \right) = [a_{ij}]_{4 \times 4}, \]

where, \( a_{ij} = \dot{a}_{ij} \) for all \( i, j = 1, 2, 3, 4 \) except \( a_{44} = \beta_5 \dot{x} + \beta_6 \dot{y} - (u_4 + h_4 + \dot{d}_2) \).

Note that, \( \dot{d}_2 > 0 \) provided that conditions (26), (27) and (28) hold.

Now, Let \( \Psi^{[2]} = (\psi_1^{[2]}, \psi_2^{[2]}, \psi_3^{[2]}, \psi_4^{[2]})^T \) with the eigenvalue \( \lambda_{2w} = 0 \) of \( J_2 \) it will be the eigenvector

Thus \( (J_2 - \lambda_{2w} I) \psi^{[2]} = 0 \), which gives:

\[ \psi^{[2]} = \left( L_4 \psi_1^{[2]}, L_5 \psi_2^{[2]}, L_6 \psi_3^{[2]}, L_7 \psi_4^{[2]} \right)^T, \]

where, \( \psi_4^{[2]} \) any nonzero number, with, \( L_4, L_5 \) and \( L_6 \) which are mentioned in the state of the theorem.

Let \( \Psi^{[2]} = \left( \psi_1^{[2]}, \psi_2^{[2]}, \psi_3^{[2]}, \psi_4^{[2]} \right)^T \) with the eigenvalue \( \lambda_{2w} = 0 \) of \( J_2 \) it will be the eigenvector

Then we have:

\[ (J_2 - \lambda_{2w} I) \psi^{[2]} = 0. \]

For \( \Psi^{[2]} \) getting the solution

we obtain:

\[ \psi^{[2]} = \left( L_4 \psi_1^{[2]}, L_5 \psi_2^{[2]}, L_6 \psi_3^{[2]}, L_7 \psi_4^{[2]} \right)^T, \]

where, \( \psi_4^{[2]} \) any nonzero number, with, \( L_4, L_5 \) and \( L_6 \) which are mentioned in the state of the theorem.

Now, \( \frac{\partial f}{\partial d_2} = f_d(X, d_2) = \left( \frac{\partial f_1}{\partial d_2}, \frac{\partial f_2}{\partial d_2}, \frac{\partial f_3}{\partial d_2}, \frac{\partial f_4}{\partial d_2} \right)^T = (0, 0, 0, -w)^T. \)

So, \( f_d \left( E_2, d_2 \right) = (0, 0, 0, -w)^T \), and hence \( (\psi^{[2]})^T f_d \left( E_2, d_2 \right) = -w \psi_4^{[2]} \neq 0. \)

Moreover, by substituting \( \psi^{[2]} \) in (17), we get:
\[
D^2 f_d \left( E_2, d_2 \right) \left( V^{[2]}, V^{[2]} \right) = \begin{bmatrix}
2 \left( u_1 \left( \dot{L}_1 \right)^2 - L_2 - \beta_1 \dot{L}_1 \right) \left( v_4^{[2]} \right)^2 \\
-2 \left( u_2 \dot{L}_2 - \beta_2 \dot{L}_2 v_4^{[4]} \right) v_4^{[2]} \\
2 \left( \beta_3 \dot{L}_1 + \beta_4 \dot{L}_2 \right) \left( v_4^{[2]} \right)^2 \\
2 \left( \beta_5 \dot{L}_1 + \beta_6 \dot{L}_2 \right) \left( v_4^{[2]} \right)^2
\end{bmatrix}
\]

Hence, it is obtained that: \((\Psi^{[2]})^T D^2 f \left( E_2, d_2 \right) \left( V^{[2]}, V^{[2]} \right) = 2Q_4 v_4^{[2]} \psi_4^{[2]}\),

where:
\[
Q_i = L_1 \left( u_1 \dot{L}_1 L_4 + \beta_3 \dot{L}_6 + \beta_5 \right) + L_2 \left( \beta_4 \dot{L}_6 + \beta_6 \right) - \left( \dot{L}_4 \left( \dot{L}_2^2 + \beta_1 L_1 \right) + L_2 \dot{L}_5 \left( \frac{u_2}{v_4^{[2]}} + \beta_2 \right) \right),
\]

with \(\dot{L}_i; i = 1, 2, 4, 5, 6\) which are mentioned in the state of the theorem.

So, if the condition (29), is satisfied, we obtain that: \((\Psi^{[2]})^T D^2 f \left( E_2, d_2 \right) \left( V^{[2]}, V^{[2]} \right) \neq 0\)

It has a saddle-node bifurcation based on theorem of Sotomayor of system(2) at \(E_2 = (\dot{x}, \dot{y}, \dot{z}, \dot{w})\), but not experience a transcritical bifurcation at \(E_2\), with the parameter \(d_2\) passes through the bifurcation value \(d_2 = \dot{a}'\).

5. Analysis of Hopf bifurcation

near the positive equilibrium point of the system(2) the happening of Hopf bifurcation is shown in the theorem given below:

Theorem 5: Suppose that the locally conditions \((11 - 15)\) with the following conditions are satisfied:

\[
K_1 > K_2(30) \\
K_3 > K_4(31) \\
K_5 > \max \left\{ \left( \alpha_1 + 2 u_1 \dot{x} + \beta_1 \dot{w} \right) (G_2 + K_3 + K_4), K_6 \right\} (32) \\
K_7 < K_8(33)
\]

\[
\frac{4A_1A_2 - A_1^2}{4} < A_3 < \min \left\{ \frac{A_1A_2}{2}, \frac{c_{33} \rho_6 - \rho_{11} - \rho_{22} \rho_2}{(\rho_1 + \rho_{22})} A_1 \right\} (34)
\]

where
\[
K_4 = a_{22} \rho_6 - \rho_1 \rho_2 - a_{22} \rho_1^2 - a_{22}^2 \rho_1 \\
K_5 = a_{33} \rho_6 - \rho_6 - \rho_{11} \\
K_3 = \left( \rho_1 + a_{22} \right) \left( \rho_2 + \rho_6 \right) + \rho_{11} + \rho_{12} - \rho_9 \rho_{10} + \rho_7 \left( \rho_2 + 2 a_{22}^2 \rho_2 - a_{12} a_{21} - \rho_6 \right) + \rho_4 (\rho_6 - \rho_2) + a_{22} \rho_1 \left( \rho_1^2 - \rho_4 + a_{22}^2 + \rho_2 - 2 a_{12}^2 \rho_2 \right) + \left( a_{12} a_{21} - a_{22}^2 \right) \rho_6
\]
\[ K_4 = \dot{a}_{33}(p_7 + p_8) + \dot{a}_{22}(p_{11} + 2p_6) - (p_6 + p_9)p_{14} - \dot{a}_{33}\rho_6(2p_1 - 1) + 2\left(\rho_1 + \dot{a}_{22}\right)p_{11} - \left(\dot{a}_{22}\rho_1 - p_6 + p_2\right)p_6 - \dot{a}_{43}(p_{15} + \rho_1) \]

\[ K_5 = \rho_6\left(\dot{a}_{33}(\rho_1^2 - 2p_6) + \rho_2\rho_1 + \dot{a}_{22}(\rho_1 + 3\rho_1 - 2)\right) + \dot{a}_{22}^2 K_2 - 2\left(a_{22}\rho_1 + \dot{a}_{33}(p_7 + p_8) - \dot{a}_{43}(p_{15} + \rho_1) - \rho_9p_{14}\right)\left(\rho_1 + \dot{a}_{22}\right) - \left(2\dot{a}_{22} + \rho_1\right)p_{11} + \dot{a}_{12}\dot{a}_{21}\left(\dot{a}_{22}\rho_6 + p_1\left(2\rho_2 - \dot{a}_{22}^2\right) - \rho_4\right) + \left(2\rho_6 - \dot{a}_{22}\rho_1 - p_2\right)\left(\rho_7 + p_6\right) + p_{11} + p_{12} - p_9p_{10} \]

\[ K_6 = \dot{a}_{22}\left(-p_7^2\rho_2 - p_6 + p_4\left(2p_2 + \dot{a}_{22}\rho_1 + p_7^2\right) - \dot{a}_{22}\rho_1\rho_2 + p_6\rho_1\right) + \rho_4\left(p_4 - p_2 - a_{33}\rho_1\right) - \dot{a}_{22}\rho_6 + \dot{a}_{12}\dot{a}_{21}\left(\dot{a}_{22}(p_7 + p_8) + \rho_1\right) - \rho_4\left(\rho_7 + p_8\right) + p_{11} + p_{12} - p_9p_{10} \]

\[ K_7 = \rho_{11}\rho_2\left(\dot{a}_{22}\left(p_2 - p_4 + \dot{a}_{22}\rho_4\right) - \dot{a}_{33}\rho_6\right) + \rho_4\left(a_{33}\rho_6 - \dot{a}_{22}^2\rho_2\right) + \dot{a}_{12}\dot{a}_{21}\left(p_6\rho_{14} - \dot{a}_{22}\rho_1\left(\rho_7^2 - p_4 + \dot{a}_{22}\rho_1\right) + \dot{a}_{12}\dot{a}_{21}\right) + \dot{a}_{22}\left(\rho_7^2 - p_9\rho_{10}\right) + \dot{a}_{22}\rho_1 - p_4\right) + \rho_1\left(p_2 - p_4\right)\left(\rho_7 + p_8\right) + p_{11} + p_{12} - p_9p_{10} \]

\[ K_8 = -\left(2\dot{a}_{33}\rho_6 - \dot{a}_{22}\rho_2 + \dot{a}_{12}\dot{a}_{21}\rho_1 - \dot{a}_{12}\dot{a}_{21}\dot{a}_{22}\right)\left(\rho_7 + p_8\right) + \rho_{11} + p_{12} - p_9p_{10} \]

Then at the parameter value \( h_1 \), near the point \( E_2 \) it has Hopf bifurcation.

**Proof:** system (2) equation of characteristic assume at \( E_2 \) which is given by eq. (10), then by using the Hopf bifurcation theorem, for \( n=4 \), we need to find a parameter say \( \left( h_1 \right) \) to verify the necessary and sufficient conditions for the Hopf bifurcation to satisfy that: \( A_i \left( h_1 \right) > 0 ; i = 1,3,4 \), \( \Delta i \left( h_1 \right) > 0 \), \( A_i^3 \left( h_1 \right) - 4 \Delta _i \left( h_1 \right) > 0 \) and \( \Delta_2 \left( h_1 \right) = 0 \), where \( A_i ; i = 1,3,4 \) represents the coefficients of the characteristic eq. (10).

Straightforward computation gives that:
\( A_i \left( h_1 \right) > 0 ; i = 1,3,4 \) and \( \Delta_i \left( h_1 \right) > 0 \) under the locally conditions (11-16), while \( A_i^3 \left( h_1 \right) - 4 \Delta _i \left( h_1 \right) > 0 \), the condition (34) contains.

It is analyzed that \( \Delta_2 = 0 \) gives that: \( A_3(A_4A_2 - A_3) - A_2^2A_4 = 0 \)

Straightforward computation we get:
\[ G_4h_1^3 + G_2h_1^2 + G_3h_1 + G_4 = 0 \]
where:
\[ G_i = K_1 - K_2 \]
\[ G_2 = 3 \left( \alpha_1 + 2u_1 \dot{x} + \beta_1 \dot{w} \right) G_1 + (K_3 - K_4) \]
\[ G_3 = \left( \alpha_1 + 2u_1 \dot{x} + \beta_1 \dot{w} \right) \left( G_2 + (K_3 - K_4) \right) - (K_5 - K_6) \]
\[ G_4 = \left( \alpha_1 + 2u_1 \dot{x} + \beta_1 \dot{w} \right) \left( G_3 - 2 \left( \alpha_1 + 2u_1 \dot{x} + \beta_1 \dot{w} \right) G_1 - (K_3 - K_4) \right) + (K_7 - K_8) \]

where \( K_i; \ i = 1,2,3,4,5,6,7,8 \) which are mentioned in the state of the theorem.

Clearly, \( G_i > 0, \ i = 1,2, \) and \( G_j < 0, \ j = 3,4 \) provided that in addition to the conditions (11 - 15), the conditions (30 - 33) holds. Note that, by using Descartes rule of sign eq. (35), has a unique positive root \( h_1 \).

Now, at \( h_1 \) the characteristic equation given by eq. (10) can be written as:
\[ \left( \dot{x}^2 + \frac{A_3}{A_1} \right) \left( \ddot{x} + A_1 \dot{x} + \frac{\Delta_1}{A_1} \right) = 0 \]
that has 4 roots:
\[ \lambda_{1,2} = \pm i \sqrt{\frac{A_3}{A_1}} \text{ and } \lambda_{3,4} = \frac{1}{2} \left( -A_1 \pm \sqrt{A_1^2 - 4 \frac{\Delta_1}{A_1}} \right). \]

\[ \lambda_1 = \varepsilon_1 + i\varepsilon_2, \lambda_2 = \varepsilon_1 - i\varepsilon_2, \lambda_{3,4} = \frac{1}{2} \left( -A_1 \pm \sqrt{A_1^2 - 4 \frac{\Delta_1}{A_1}} \right). \]

Clearly, \( Re(\lambda_N(h_1)) \big|_{h_1=h_1^*} = \varepsilon_1(h_1^*) = 0, N = 1,2 \) At \( h_1 = h_1^* \) for Hopf bifurcation the 1st sufficient and necessary condition is fulfilling. It must prove the condition of transversality according to verify:
\[ \hat{\Theta}(h_1^*) \hat{\Psi}(h_1^*) + \hat{\Gamma}(h_1^*) \hat{\Phi}(h_1^*) \neq 0, \]
where \( \hat{\Theta}, \hat{\Psi}, \hat{\Gamma} \) and \( \hat{\Phi} \) are given in lemma (1) in [1]. Note that for \( h_1 = h_1^* \) we have \( \varepsilon_1(h_1^*) = 0 \) and \( \varepsilon_2(h_1^*) = \sqrt{\frac{A_3}{A_1}} \), thus gives the following simplifications:
\[ \hat{\Psi}(h_1^*) = -2 \frac{A_3}{A_1} \hat{\chi}(h_1^*), \]
\[ \hat{\Phi}(h_1^*) = 2 \frac{\varepsilon_2}{A_1} \left( A_1 A_2 - 2 A_3 \right), \]
\[ \hat{\Theta}(h_1^*) = A_4' \hat{\chi}(h_1^*) - \frac{A_3}{A_1} A_2' \hat{\chi}(h_1^*), \]
\[ \hat{\Gamma}(h_1^*) = \varepsilon_2 \left( A_5' \hat{\chi}(h_1^*) - \frac{A_3}{A_1} A_1' \hat{\chi}(h_1^*) \right), \]

where:
\[ A_1' = \frac{dA_1}{dh_1} \bigg|_{h_1=h_1^*} = -1, \quad A_2' = \frac{dA_2}{dh_1} \bigg|_{h_1=h_1^*} = -(\rho_1 + c_{22}), \]
\[ A_3' = \frac{dA_3}{dh_1} \bigg|_{h_1=h_1^*} = \rho_2 + c_{22} \rho_1, \quad A_4' = \frac{dA_4}{dh_1} \bigg|_{h_1=h_1^*} = -c_{23} \rho_2 + c_{33} \rho_6 - \rho_{11}. \]

Then, we get that:
\[ \hat{\Theta}(h_1^*) \hat{\Psi}(h_1^*) + \hat{\Gamma}(h_1^*) \hat{\Phi}(h_1^*) = K_9 + K_{10} \neq 0, \] where:
\begin{align*}
K_9 &= 2 A_3 \left( c_{22} \rho_2 + \rho_{11} - c_{33} \rho_6 - \frac{A_2}{A_1} (\rho_1 + c_{22}) \right), \\
K_{10} &= 2 \frac{A_3}{A_1^2} \left[ \rho_2 + c_{22} \rho_1 + \frac{A_3}{A_1} (A_1 A_2 - 2 A_3) \right].
\end{align*}

Now, according to condition (35), we have:
\[ \dot{\Theta}(h) \dot{\Psi}(h) + \dot{\Gamma}(h) \dot{\Phi}(h) \neq 0. \]

6. System Numerical Analysis

This segment described, the dynamical conduct of framework (2) is read numerically for one lot of parameters and various arrangements of starting focuses. The destinations of this examination are explore the impact of fluctuating the estimation of every parameter on the dynamical conduct of framework (2) and affirm our acquired expository outcomes. It is seen that, for the accompanying arrangement of theoretical parameters that fulfills solidness states of the positive harmony point.

\[ \alpha_i = 0.5, 0.2, \ u_j = h_j = d_i = 0.1, \ \beta_{j+2} = 0.3, \ \beta_i = 0.6, \ i = 1, 2 \text{ and } j = 1, 2, 3, 4 \quad (36) \]

Figure 1: (a) with \(\alpha_1 = 0.005\) for a given data set on equation (36) the time series (b) with \(\alpha_1 = 0.02\) for a given data set on equation (36) the time series (c) with \(\alpha_1 = 0.1\) for a given data set on equation (36) the time series.

Figure 2: (a) with \(u_2 = 1\) for a given data set on equation (36) the time series (b) with \(u_2 = 1.5\) for a given data set on equation (36) the time series (c) with \(u_2 = 1.5\) for a given data set on equation (36) the time series.
7. Conclusions and discussion

In this paper, we proposed and broke down a biological model that portrayed the dynamical conduct of the stage-organized in prey just as predator with nearness gathering and poison. The model included four non-straight self-governing differential conditions that depict the elements of four diverse populace, to be specific first youthful prey $X(T)$, develop prey $Y(T)$, juvenile predator $Z(T)$ and develop predator $W(T)$. The states of the event the neighborhood bifurcation, for example, saddle hub, transcritical and pitchfork of framework (2) has been established, also specific accentuation on the Hopf bifurcation close to the positive harmony purpose of numerical model. After the investigation and examination, it is seen that the transcritical bifurcation happens close to the balance point $E_0$ just as the event of seat hub bifurcation at harmony points $E_1$ and $E_2$. It merits referencing, there are no plausibility event of the pitch fork bifurcation at each point. At long last, some numerical recreation has been utilized to delineation the event of nearby bifurcation of this model.

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