On the Cubicity of AT-free graphs and Circular-arc graphs

L. Sunil Chandran, Mathew C. Francis *, and Naveen Sivadasan **

Abstract. A unit cube in k dimensions (k-cube) is defined as the Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where $R_i$ (for $1 \leq i \leq k$) is a closed interval of the form $[a_i, a_i + 1]$ on the real line. A graph $G$ on $n$ nodes is said to be representable as the intersection of $k$-cubes (cube representation in $k$ dimensions) if each vertex of $G$ can be mapped to a $k$-cube such that two vertices are adjacent in $G$ if and only if their corresponding $k$-cubes have a non-empty intersection. The cubicity of $G$ denoted as $\text{cub}(G)$ is the minimum $k$ for which $G$ can be represented as the intersection of $k$-cubes.

An interesting aspect about cubicity is that many problems known to be NP-complete for general graphs have polynomial time deterministic algorithms or have good approximation ratios in graphs of low cubicity. In most of these algorithms, computing a low dimensional cube representation of the given graph is usually the first step.

We give an $O(bw \cdot n)$ algorithm to compute the cube representation of a general graph $G$ in $bw + 1$ dimensions given a bandwidth ordering of the vertices of $G$, where $bw$ is the bandwidth of $G$. As a consequence, we get $O(\Delta)$ upper bounds on the cubicity of many well-known graph classes such as AT-free graphs, circular-arc graphs and co-comparability graphs which have $O(\Delta)$ bandwidth. Thus we have:

1. $\text{cub}(G) \leq 3\Delta - 1$, if $G$ is an AT-free graph.
2. $\text{cub}(G) \leq 2\Delta + 1$, if $G$ is a circular-arc graph.
3. $\text{cub}(G) \leq 2\Delta$, if $G$ is a co-comparability graph.

Also for these graph classes, there are constant factor approximation algorithms for bandwidth computation that generate orderings of vertices with $O(\Delta)$ width. We can thus generate the cube representation of such graphs in $O(\Delta)$ dimensions in polynomial time.

Keywords : Cubicity, bandwidth, intersection graphs, AT-free graphs, circular-arc graphs, co-comparability graphs.

1 Introduction

Let $\mathcal{F} = \{S_x \subseteq U : x \in V\}$ be a family of subsets of a universe $U$, where $V$ is an index set. The intersection graph $\Omega(\mathcal{F})$ of $\mathcal{F}$ has $V$ as vertex set, and two distinct
vertices $x$ and $y$ are adjacent if and only if $S_x \cap S_y \neq \emptyset$. Representations of graphs as the intersection graphs of various geometrical objects is a well studied topic in graph theory. Probably the most well studied class of intersection graphs are the interval graphs, where each $S_x$ is a closed interval on the real line. A restricted form of interval graphs, that allow only intervals of unit length, are indifference graphs.

A well known concept in this area of graph theory is the cubicity, which was introduced by F. S. Roberts in 1969 [11]. This concept generalizes the concept of indifference graphs. A unit cube in $k$ dimensions ($k$-cube) is a Cartesian product $R_1 \times R_2 \times \cdots \times R_k$ where $R_i$ (for $1 \leq i \leq k$) is a closed interval of the form $[a_i, a_i + 1]$ on the real line. Two $k$-cubes, $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ are said to have a non-empty intersection if and only if the intervals $x_i$ and $y_i$ have a non-empty intersection for $1 \leq i \leq k$. For a graph $G$, its cubicity is the minimum dimension $k$, such that $G$ is representable as the intersection graph of $k$-cubes. We denote the cubicity of a graph $G$ by $\text{cub}(G)$. The graphs of cubicity at most 1 are exactly the class of indifference graphs.

If we require that each vertex correspond to a $k$-dimensional axis-parallel box $R_1 \times R_2 \times \cdots \times R_k$ where $R_i$ (for $1 \leq i \leq k$) is a closed interval of the form $[a_i, b_i]$ on the real line, then the minimum dimension required to represent $G$ is called its boxicity denoted as $\text{box}(G)$. Clearly $\text{box}(G) \leq \text{cub}(G)$ for any graph $G$ because cubicity is a stricter notion than boxicity.

It has been shown that deciding whether the cubicity of a given graph is at least 3 is NP-hard [15].

In many algorithmic problems related to graphs, the availability of certain convenient representations turn out to be extremely useful. Probably, the most well-known and important examples are the tree decompositions and path decompositions. Many NP-hard problems are known to be polynomial time solvable given a tree(path) decomposition of the input graph that has bounded width. Similarly, the representation of graphs as intersections of “disks” or “spheres” lies at the core of solving problems related to frequency assignments in radio networks, computing molecular conformations etc. For the maximum independent set problem which is hard to approximate within a factor of $n^{(1/2)-\varepsilon}$ for general graphs, a PTAS is known for disk graphs given the disk representation [4, 1] and an FPTAS is known for unit disk graphs [14]. In a similar way, the availability of cube or box representation in low dimension make some well known NP hard problems like the max-clique problem, polynomial time solvable since there are only $O((2n)^k)$ maximal cliques if the boxicity or cubicity is at most $k$. Though the complexity of finding the maximum independent set is hard to approximate within a factor $n^{(1/2)-\varepsilon}$ for general graphs, it is approximable to a log $n$ factor for boxicity 2 graphs (the problem is NP-hard even for boxicity 2 graphs) given a box or cube representation [2, 3].

It is easy to see that the problem of representing graphs using $k$-cubes can be equivalently formulated as the following geometric embedding problem. Given an undirected unweighted graph $G = (V, E)$ and a threshold $t$, find an embedding $f : V \rightarrow \mathbb{R}^k$ of the vertices of $G$ into a $k$-dimensional space (for the minimum
possible $k$) such that for any two vertices $u$ and $v$ of $G$, $||f(u) - f(v)||_{\infty} \leq t$ if and only if $u$ and $v$ are adjacent. The norm $|| \cdot ||_{\infty}$ is the $L_{\infty}$ norm. Clearly, a $k$-cube representation of $G$ yields the required embedding of $G$ in the $k$-dimensional space. The minimum dimension required to embed $G$ as above under the $L_{\infty}$ norm is called the sphericity of $G$. Refer [9] for applications where such an embedding under $L_{\infty}$ norm is argued to be more appropriate than embedding under $L_2$ norm. The connection between cubicity and sphericity of graphs were studied in [6,8].

As far as we know, the only known upper bound for the cubicity of general graphs (existential or constructive) is by Roberts [11], who showed that $\text{cub}(G) \leq 2n/3$ for any graph $G$ on $n$ vertices. The cube representation of special class of graphs like hypercubes and complete multipartite graphs were investigated in [11,8,10].

**Linear Ordering and Bandwidth.** Given an undirected graph $G = (V, E)$ on $n$ vertices, a linear ordering of $G$ is a bijection $f : V \rightarrow \{1, \ldots, n\}$. The width of the linear ordering $f$ is defined as $\max_{(u,v) \in E} |f(u) - f(v)|$. The bandwidth minimization problem is to compute $f$ with minimum possible width. The bandwidth of $G$ denoted as $\text{bw}(G)$ is the minimum possible width achieved by any linear ordering of $G$. A bandwidth ordering of $G$ is a linear ordering of $G$ with width $\text{bw}(G)$. Our algorithm to compute the cube representation of a graph $G$ takes as input a linear ordering of $G$. The smaller the width of this ordering, the lesser the number of dimensions of the cube representation of $G$ computed by our algorithm. It is NP-hard to approximate the bandwidth of $G$ within a ratio better than $k$ for every $k \in \mathbb{N}$ [13]. Feige [5] gives a $O(\log^3(n) \sqrt{\log n \log \log n})$ approximation algorithm to compute the bandwidth (and also the corresponding linear ordering) of general graphs. For bandwidth computation, several algorithms with good heuristics are known that perform very well in practice [12].

### 1.1 Our results

We summarize below the results of this paper.

1. For any graph $G$, $\text{cub}(G) \leq \text{bw}(G) + 1$
2. For an AT-free graph $G$ with maximum degree $\Delta$, $\text{cub}(G) \leq 3\Delta - 1$
3. For a circular-arc graph $G$ with maximum degree $\Delta$, $\text{cub}(G) \leq 2\Delta + 1$
4. For a co-comparability graph $G$ with maximum degree $\Delta$, $\text{cub}(G) \leq 2\Delta$

### 1.2 Definitions and Notations

All the graphs that we consider will be simple, finite and undirected. For a graph $G$, we denote the vertex set of $G$ by $V(G)$ and the edge set of $G$ by $E(G)$. For a vertex $u \in V$, let $d(u)$ denote its degree (the number of outer neighbors of $u$). The maximum degree of $G$ is denoted by $\Delta(G)$ or simply $\Delta$ when the graph under consideration is clear. For a vertex $u \in V(G)$, we denote the set of neighbors of $u$ by $N_G(u)$. By definition, $N_G(u) = \{v \in V(G) \mid (u,v) \in E(G)\}$. Again,
Definition 1 (Unit interval representation). Given an indifference graph \( I(V, E) \), the unit interval representation is a mapping \( f: V \rightarrow \mathbb{R} \) such that for any two vertices \( u, v \), \(|f(u) - f(v)| \leq 1\) if and only if \((u, v) \in E\).

Note that this is equivalent to mapping each vertex of \( I \) to the unit interval \([f(u), f(u) + 1]\) so that two vertices are adjacent in \( I \) if and only if the unit intervals mapped to them overlap. Now, consider the mapping \( g: V \rightarrow \mathbb{R} \) given by \( g(u) = xf(u) \) where \( x \in \mathbb{R} \). It can be easily seen that for any two vertices \( u, v \), \(|g(u) - g(v)| \leq x\) if and only if \((u, v)\) is an edge in \( I \). \( g \) thus corresponds to an interval representation of \( I \) using intervals of length \( x \). We call such a mapping \( g \) a unit interval representation of \( I \) with interval length \( x \).

Definition 2 (Indifference graph representation). The indifference graphs \( I_1, \ldots, I_k \) constitute an indifference graph representation of a graph \( G \) if \( G = I_1 \cap \cdots \cap I_k \).

Theorem 1 (Roberts[11]). A graph \( G \) has \( \text{cub}(G) \leq k \) if and only if it has an indifference graph representation with \( k \) indifference graphs.

2 Cubicity and bandwidth

2.1 The construction

We show that given a linear ordering of the vertices of \( G \) with width \( b \), we can construct an indifference graph representation of \( G \) using \( b + 1 \) indifference graphs.

Theorem 2. If \( G \) is any graph with bandwidth \( b \), then \( \text{cub}(G) \leq b + 1 \).

Proof. Let \( n \) denote \(|V(G)|\) and let \( A = u_1, u_2, \ldots, u_n \) be a linear ordering of the vertices of \( G \) with width \( b \), i.e., if \((u_i, u_j) \in E(G)\), then \(|i - j| \leq b\).

We construct \( b + 1 \) indifference graphs \( I_0, I_1, \ldots, I_b \), such that \( G = I_0 \cap I_1 \cap \cdots \cap I_b \). Let \( f_i \) denote the unit interval representation of \( I_i \).

Construction of \( I_0 \):

Since \( I_0 \) has to be a supergraph of \( G \), we have to make sure that every edge in \( E(G) \) has to be present in \( E(I_0) \). \( b \) being the bandwidth of the linear ordering.
of vertices taken, a vertex \( u_j \) is not adjacent in \( G \) to any vertex \( u_k \) when \( |j - k| > b \). Now, we define \( f_0 \) in such a way that \( E(I_0) = \{(u_j, u_k) \mid |j - k| < b\} \cup \{(u_j, u_k) \mid |j - k| = b \text{ and } (u_j, u_k) \in E(G)\} \). The definition of \( f_0 \) can be explained as the following procedure. We first assign the interval \([j, j + b] \) to vertex \( u_j \), for all \( j \). This makes sure that \( u_j \) is not adjacent to any vertex \( u_k \), if \( k > j + b \). Now, each vertex is adjacent in \( I_0 \) to exactly the \( b \) vertices preceding and following it in \( A \). Now, for each vertex \( u_j \) where \( j > b \), we shift \( f_0(u_j) \), the unit interval for \( u_j \), slightly to the right (by \( \epsilon \)) if \( u_j \) is not adjacent to \( u_{j-b} \) in \( G \) so that \( f_0(u_j) \) becomes disjoint from \( f_0(u_{j-b}) \). Along with \( f_0(u_j) \), all the intervals that start after \( f_0(u_j) \) are also shifted right by \( \epsilon \). This procedure is done for vertices \( u_{b+1}, \ldots, u_n \) in that order. Our choice of a small value for \( \epsilon \) ensures that \( I_0 \) is still a supergraph of \( G \).

\( f_0 \) is a unit interval representation for \( I_0 \) with interval length \( b \) defined as follows. Let \( \epsilon = 1/n^2 \).

\[
\begin{align*}
    f_0(u_j) &= j, \text{ for } j \leq b \\
    f_0(u_j) &= f(u_{j-b}) + b, \text{ for } j > b \text{ and } (u_{j-b}, u_j) \in E(G) \\
    f_0(u_j) &= f(u_{j-b}) + b + \epsilon, \text{ for } j > b \text{ and } (u_{j-b}, u_j) \notin E(G)
\end{align*}
\]

**Construction of \( I_i \), for \( 1 \leq i \leq b \):**

We split the sequence of vertices \( A \) into blocks \( B_0^i, \ldots, B_{p-1}^i \) of vertices of size \( b \) starting from the vertex \( i \) where the last block \( B_{p-1}^i \) may have less than \( b \) vertices. Formally, \( B_t^i = \{u_{i+b}, \ldots, u_{i+b(t+1)-1}\} \), for \( 1 \leq t < p - 1 \), and \( B_{p-1}^i = \{u_{i+b(p-1)}, \ldots, u_n\} \). Let \( s_t^i \) denote the vertex \( u_{i+b} \), or the first vertex (in the ordering \( A \)) in block \( B_t^i \). We now define \( f_i \), the unit interval representation for \( I_i \) with interval length 2, as follows:

\[
    f_i(u_j) = 2, \text{ if } j < i
\]

Let \( u \) be a vertex in \( V(G) - \{u_j \mid j < i\} \) and let \( u \in B_t^i \).

\[
\begin{align*}
    f_i(u) &= t, \text{ if } u = s_t^i \\
    &= t + 2, \text{ if } (u, s_t^i) \in E(G) \\
    &= t + 3, \text{ if } (u, s_t^i) \notin E(G)
\end{align*}
\]

**Claim.** \( I_0 \) is an indifference supergraph of \( G \).

**Proof.** First we observe that for any vertex \( u_j \), \( j \leq f_0(u_j) \leq j + 1/n \). This is because \( f_0(u_j) \leq f_0(u_{j-b}) + b + \epsilon \) where \( \epsilon = 1/n^2 \). Now, consider an edge \((u_j, u_k)\) of \( G \) where \( j < k \). Since the width of the input linear ordering \( A \) is \( b \), we have \( k - j \leq b \). Now we consider the following two cases. If \( k - j \leq b - 1 \) then \( f(u_k) - f(u_j) \leq k + 1/n - j \leq b - 1 + 1/n < b \). Since each interval in \( I_0 \) has length \( b \), it follows that \((u_j, u_k) \in E(I_0)\). If \( k - j = b \) then from the definition of \( f_0 \), it follows that \( f_0(u_k) = f_0(u_{k-b}) + b = f_0(u_j) + b \). Thus \( f_0(u_k) - f_0(u_j) \leq b \) implying that \((u_j, u_k) \in E(I_0)\).
Claim. $I_i$ for $1 \leq i \leq b$ is an indifference supergraph of $G$.

Proof. Consider the indifference graph $I_i$. Let $(u_j, u_k)$ be any edge in $E(G)$. We assume without loss of generality that $j < k$. If $j < i$, then $k < i + b$ and therefore, $u_k \in B_i^t$. In this case, $f_i(u_j) = 2$ and $0 \leq f_i(u_k) \leq 3$ and so we have $|f_i(u_j) - f_i(u_k)| \leq 2$. Now, consider the case when $j \geq i$. Let $u_j \in B_i^t$. Since $|j - k| \leq b$, we have $u_k \in B_i^t \cup B_i^{t+1}$. From the definition of $f_i$, it is clear that $|f_i(u_j) - f_i(u_k)| \leq 2$ if $u_j \neq s_i^t$. Now, if $u_j = s_i^t$, then either $u_k \in B_i^t$, in which case $f_i(u_k) = t + 2$, or $u_k = s_{i+1}^t$, in which case $f_i(u_k) = t + 1$. But in both cases, $|f_i(u_j) - f_i(u_k)| \leq 2$. Therefore, we have $f_i(u_j) \cap f_i(u_k) \neq \emptyset$ which implies that $(u_j, u_k) \in E(I_i)$.

It remains to show that $G = I_0 \cap \cdots \cap I_b$. To do this, it suffices to show that for any $(u_j, u_k) \notin E(G)$, there exists an $I_i$, $0 \leq i \leq b$ such that $(u_j, u_k) \notin E(I_i)$. Let $j < k$. Case $k - j \geq b$. In this case, we claim that $(u_j, u_k) \notin E(I_0)$. This is because of the following. If $k - j = b$ then clearly $f_0(u_k) - f_0(u_j) = b + e$ and thus $(u_j, u_k) \notin E(I_0)$. Now, if $k - j > b$ then $f_0(u_k) \geq f_0(u_k-b) + b > f_0(u_j) + b$ observing that $f_0(u_1) < f_0(u_2) < \cdots < f_0(u_n)$. Thus $(u_j, u_k) \notin E(I_0)$. Now the remaining case is $k - j < b$. Consider the graph $I_t$ where $t \equiv j \mod b$. Let $t$ be such that $u_j \in B_i^t$. Therefore, $bt + l \leq j < b(t+1) + l$. This implies that $bt + l = bt + j \mod b = j$ and so we have $s_i^t = u_j$. Therefore, $u_k \in B_i^t$ since $k - j + b = b(t+1) + l$. Now, from the definition of $f_i$, we have $f_i(u_k) = t$ and $f_i(u_j) = t + 3$. Thus, $|f_i(u_j) - f_i(u_k)| > 2$ and hence $(u_j, u_k) \notin E(I_i)$ as required.

Thus $I_0, \ldots, I_b$ is a valid indifference graph representation of $G$ using $b + 1$ indifference graphs which establishes that cub$(G) \leq b + 1$. ■

2.2 The algorithm

Our algorithm to compute the cube representation of $G$ in $b+1$ dimensions given a linear ordering of the vertices of $G$ with width $b$ constructs the indifference supergraphs of $G$, namely, $I_0, \ldots, I_b$ using the constructive procedure used in the proof of Theorem 2. It is easy to verify that this algorithm runs in $O(b \cdot n)$ time where $b$ is the width of the input linear arrangement and $n$ is the number of vertices in $G$.

3 Applying our results

Theorem 2 can be used to derive upper bounds for the cubicity of several special classes of graphs such as circular arc graphs, co-comparability graphs and AT-free graphs.

Corollary 1. If $G$ is a circular-arc graph, cub$(G) \leq 2\Delta + 1$, where $\Delta$ is the maximum degree of $G$. 


Corollary 2. If \( G \) is a co-comparability graph, then \( \text{cub}(G) \leq 2\Delta \), where \( \Delta \) is the maximum degree of \( G \).

Proof. Let \( V \) denote \( V(G) \) and let \( |V| = n \). Since \( \overline{G} \) is a comparability graph, there exists a partial order \( \prec \) in \( \overline{G} \) on the node set \( V \) such that \( (u,v) \in E(\overline{G}) \) if and only if \( u \prec v \) or \( v \prec u \). This partial order gives a direction to the edges in \( E(\overline{G}) \). We can run a topological sort on this partial order to produce a linear
Proof. Let $G$ be an AT-free graph, say, $f : V \to \{1, \ldots, n\}$. The topological sort ensures that if $u \prec v$, then $f(u) < f(v)$. Now, let $(u, v) \in E(G)$ and let $w$ be a vertex such that $f(u) < f(w) < f(v)$. We will show that $w$ is adjacent to either $u$ or $v$ in $G$. Suppose not. Then $(u, w), (w, v) \in E(G)$ and therefore $u \prec w$ and $w \prec v$. Now, by transitivity of $\prec$, this implies that $u \prec v$, which means that $(u, v) \in E(G)$ — a contradiction. Therefore, any vertex $w$ such that $f(u) < f(w) < f(v)$ in the ordering $f$ is adjacent to either $u$ or $v$. Since the maximum degree of $G$ is $\Delta$, there can be at most $2\Delta - 2$ vertices between with $f(\cdot)$ value between $f(u)$ and $f(v)$. Thus, the width of the ordering given by $f$ is at most $2\Delta - 1$ and by Theorem 2, we have our bound on cubicity.

A caterpillar is a tree such that a path (called the spine) is obtained by removing all its leaves. In the proof of Theorem 3.16 of [7], Kloks et al. show that every connected AT-free graph $G$ has a spanning caterpillar subgraph $T$, such that adjacent nodes in $G$ are at a distance at most four in $T$. Moreover, for any edge $(u, v) \in E(G)$ such that $u$ and $v$ are at distance exactly four in $T$, both $u$ and $v$ are leaves of $T$. Let $p_1, \ldots, p_k$ be the nodes along the spine of $G$.

**Corollary 3.** If $G$ is an AT-free graph, $\cub(G) \leq 3\Delta - 1$, where $\Delta$ is the maximum degree of $G$.

**Proof.** Let $L_i$ denote the set of leaves of $T$ adjacent to $p_i$. Clearly, $|L_i| \leq \Delta$ and $L_i \cap L_j = \emptyset$ for $i \neq j$. For any set $S$ of vertices, let $\langle S \rangle$ denote an arbitrary ordering of the vertices in set $S$. Let $< \langle \cdot \rangle$ denote ordering with just one vertex $u$ in it. If $\alpha = u_1, \ldots, u_s$ and $\beta = v_1, \ldots, v_t$ are two orderings of vertices in $G$, then let $\alpha \circ \beta$ denote the ordering $u_1, \ldots, u_s, v_1, \ldots, v_t$. Let $\mathcal{A} = < L_1 > \circ < p_1 > \circ < L_2 > \circ < p_2 > \circ \cdots \circ < L_k > \circ < p_k >$ be a linear ordering of the vertices of $G$. One can use the property of $T$ stated before the theorem to easily show that $\mathcal{A}$ is a linear ordering of the vertices of $G$ with width at most $3\Delta - 2$. The corollary will then follow from Theorem 2.

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