Quantum Cramér-Rao bound based on the Bures distance

Yating Ye\textsuperscript{1} and Xiao-Ming Lu\textsuperscript{1,*}

\textsuperscript{1}Department of Physics, Hangzhou Dianzi University, Hangzhou 310018, China

The quantum Fisher information can be different with the Bures metric and becomes discontinuous at the parameter points where the rank of the parametric density operator changes. We show that the invalidation of the QCRB at such parameter points is due to the fact that the symmetric logarithmic derivative operator tends to be unbounded when the value of parameter approaches these singular parameter points. We give an alternative derivation of the QCRB through the Bures distance and show that this form of the QCRB still holds even when the parametric density operator changes its rank.

I. INTRODUCTION

The Fisher information matrix characterizes the least possible mean square error of parameter estimation for a large number of samples \cite{1, 2}. For a finite number of samples, the Fisher information matrix also reveals a lower bound (namely, the Cramér-Rao bound) on estimation errors when the estimator are restricted to be unbiased \cite{3, 4}. These make the Fisher information matrix play a pivotal role in classical parameter estimation theory.

For quantum parameter estimation problems, not only the classical estimators but also the quantum measurements should be taken into consideration for minimizing the estimation errors. Helstrom derived the quantum Cramér-Rao bound (QCRB) by defining the quantum Fisher information (QFI) matrix as an analogue of the classical Fisher information matrix \cite{5–9}. Due to the measurement incompatibility caused by Heisenberg’s uncertainty principle \cite{10}, the quantum estimation problems for multiple parameters are much more intricate than the classical estimation problems \cite{11–19}. For the estimations of a single parameter, the QCRB based on the QFI gives a rather satisfactory approach to revealing the ultimate quantum limit of estimation precision.

However, some recent researches disclosed a defect about the QCRB based on the QFI at the parameter points where the rank of the parametric density operator changes \cite{20–23}. Šafránek demonstrated that the QFI and the Bures metric, which were considered to be equivalent (up to a constant factor) for all cases, can be different at such singular parameter points \cite{20}. Later, Seveso et al. showed that the QCRB based on the QFI does not hold at these limiting cases \cite{21}.

In this work, we revisit the derivation of the QCRB and give an approach to fix the defect mentioned above. We point out that the symmetric logarithmic derivative (SLD) operator is implicitly required to be bounded in the previous derivation of the QCRB. When the rank of density operator changes, the SLD operator becomes unbounded; This is the reason of the invalidation of the QCRB at such singular parameter point. To consolidate the soundness of the QCRB, we give an alternative derivation of the QCRB through the Bures distance. Since the QCRB based on the Bures distance does not require an bounded SLD operator, it still holds at the parameter points where the density operator changes its rank.

This paper is organized as follows. In Sec. II, we give a brief review on the definition and other two expressions of the QFI. In Sec. III, we investigate relation among the discontinuity of the QFI, the invalidation of the QCRB, and the existence of an bounded SLD. In Sec. IV, we gave a new method to derive the QCRB without resorting to the assumption of an bounded SLD operator. In Sec. V, we discuss the validity of the QCRB with a typical example. We summarize our results in Sec. VI.

II. REVIEW ON THE QFI

Let us start by considering the general problem of estimating a single parameter with quantum systems. Assume that the state of the quantum system, denoted by $\rho_0$, depends on an unknown parameter $\theta$. The value of $\theta$ can be estimated by data-processing on the outcome obtained from a measurement performed on the quantum system. A general quantum measurement is mathematically characterized by a positive-operator-valued measure (POVM) $\{E_x \mid E_x \succeq 0, \sum_x E_x = 1\}$ with $x$ denoting the outcome and $1$ being the identity operator. According to Born’s rule in quantum mechanics, the probability of obtaining a outcome $x$ is given by $\text{tr}(E_x \rho_0)$. The data-processing is represented by an estimator $\hat{\theta}$ that maps the measurement outcome $x$ into the estimate for $\theta$. The estimation error is assessed by the mean square error defined as

$$E_\theta := \sum_x \left[ \hat{\theta}(x) - \theta \right]^2 \text{tr}(E_x \rho_0).$$

For all quantum measurements and all unbiased estimators $\hat{\theta}$, which satisfy

$$\sum_x \hat{\theta}(x) \text{tr}(E_x \rho_0) = \theta$$

* luxiaoming@gmail.com
for all possible true values of $\theta$, the estimation error obeys the quantum Cramér bounds (also known as the Helstrom bound):

$$E_\theta \geq \frac{1}{F_\theta},$$

where $F_\theta$ is the QFI whose definition will be given in what follows.

There exist three expressions for the QFI that are often used in the previous works [5, 6, 8, 24]. The first one is the formal definition [5, 6]:

$$F_\theta^{(1)} := \text{tr}(L_\theta^2 \rho_\theta),$$

where $L_\theta$ is the SLD operator defined as the Hermitian operator satisfying

$$\frac{d \rho_\theta}{d \theta} = \frac{1}{2}(L_\theta \rho_\theta + \rho_\theta L_\theta).$$

With the spectral decomposition $\rho_\theta = \sum_j \lambda_j |e_j\rangle \langle e_j|$, the second form of the QFI is given by

$$F_\theta^{(2)} := \sum_{j,k: |\lambda_j + \lambda_k| \neq 0} \left\{ \frac{2}{\lambda_j + \lambda_k} \right\} |\langle e_j| \frac{d \rho_\theta}{d \theta} |e_k\rangle|^2.$$

This expression can be derived by solving the matrix representation of the SLD operator $L_\theta$ with the basis constituted by the eigenvectors of $\rho_\theta$ and then substituting it into the formal definition Eq. (4) of the QFI. The third form of the QFI is given by

$$F_\theta^{(3)} := 8 \lim_{\epsilon \to 0} \frac{1 - \|\sqrt{\rho_\theta} \sqrt{\rho_\theta + \epsilon} \|_{1}}{\epsilon^2},$$

where $\|X\|_p := [\text{tr}(|X|^p)]^{1/p}$ with $|X| := \sqrt{X^* X}$ is the Schatten-p norm of an operator $X$. The quantity $\|\sqrt{\rho_\theta} \sqrt{\rho_\theta + \epsilon} \|_{1}$ is known as Uhlmann’s fidelity between $\rho_\theta$ and $\rho_{\theta + \epsilon}$ [25–27]. This expression is deeply relevant to the Bures distance [28] between two density operators $\rho$ and $\sigma$:

$$d_B(\rho, \sigma) := \sqrt{2(1 - \|\sqrt{\rho} \sqrt{\sigma} \|_{1})}.$$

It can be seen that the third form $F_\theta^{(3)}$ is equivalent to the metric of the Bures distance between two density operators up to an insignificant constant factor 4, i.e.,

$$F_\theta^{(3)} = 4 \lim_{\epsilon \to 0} \frac{d_B(\rho_\theta, \rho_{\theta + \epsilon})^2}{\epsilon^2}.$$ (9)

III. DISCONTINUOUS QFI AND UNBOUNDED SLD

As shown by Šafářnek in Ref. [20], the expressions $F_\theta^{(2)}$ and $F_\theta^{(3)}$, given in Eqs. (6) and (7), can be different at the parameter points where the rank of $\rho_\theta$ changes. The discrepancy occurs when a term excluded from the summation in Eq. (7) for $F_\theta^{(3)}$ has a finite value as the parameter tends to that specific value. At a specific value $\theta'$, it has been shown that

$$\Delta_{\theta'} := F_\theta^{(3)} - F_\theta^{(2)} = \lim_{\theta \to \theta'} \sum_{|\lambda_k| = 0} \left( \frac{d \lambda_k}{d \theta} \right)^2$$

$$= \sum_{|\lambda_k| = 0} 2 \left( \frac{d^2 \lambda_k}{d \theta^2} \right)_{\theta = \theta'},$$

where the last equality is due to L'Hôpital’s rule in calculus [20, 21]. Seveso et al. in Ref. [21] gave an example to show that the QCRB (based on $F_\theta^{(2)}$) may be easily violated at the parameter points where $\Delta_{\theta} > 0$. In what follows, we shall interpret why the QCRB based on $F_\theta^{(2)}$ breaks down at the parameter point where the rank of the density operators changes.

The QCRB was derived with the assumption that there exists an SLD operator, namely, an Hermitian operator $L_\theta$ satisfying Eq. (5). Moreover, it was implicitly assumed that the SLD operator is bounded; Otherwise, the Cauchy-Schwarz inequality invoked in proving the QCRB may not hold. Therefore, the QCRB based on $F_\theta^{(2)}$ always holds, as long as there exists a bounded and Hermitian operator $L_\theta$ satisfying Eq. (5). The expression $F_\theta^{(2)}$ is a consequence of Eq. (4) with solving the bounded SLD operator $L_\theta$ with the basis constituted by the eigenvectors of $\rho_\theta$. It follows from the definition Eq. (5) of the SLD operator and the spectral decomposition $\rho_\theta = \sum_j \lambda_j |e_j\rangle \langle e_j|$ that

$$\frac{\lambda_j + \lambda_k}{2} \langle e_j| L_\theta |e_k\rangle = \langle e_j| \frac{d \rho_\theta}{d \theta} |e_k\rangle.$$

Note that with the spectral decomposition of $\rho_\theta$, the QFI in Eq. (4) can be expressed as

$$F_\theta^{(1)} = \sum_{jk} \lambda_j |\langle e_j| L_\theta |e_k\rangle|^2 = \sum_{jk} \frac{\lambda_j + \lambda_k}{2} |\langle e_j| L_\theta |e_k\rangle|^2,$$

where the second equality is due to the symmetrization with respect to the indices $j$ and $k$. For those indices $j$ and $k$ such that $\lambda_j + \lambda_k = 0$, the corresponding terms in the right hand side of Eq. (12) has no contribution to the summation, if the SLD operator is bounded. In such case, combining Eqs. (11) and (12), we can see that

$$F_\theta^{(1)} = F_\theta^{(2)}.$$ Therefore, the QCRB based on $F_\theta^{(2)}$ holds when there exists an bounded SLD operator.

However, it is not always possible to have a bounded SLD operator [29]. It can be seen from Eq. (11) that the element the SLD operator, $\langle e_j| L_\theta |e_k\rangle$, does not have a finite valued solution, if and only if $\lambda_j + \lambda_k = 0$ and $\langle e_j| \frac{d \rho_\theta}{d \theta} |e_k\rangle \neq 0$ are both satisfied. Since the eigenvalues of $\rho_\theta$ are nonnegative, $\lambda_j + \lambda_k = 0$ implies that both $\lambda_j$ and $\lambda_k$ vanish. In such case, we have $\langle e_j| \frac{d \rho_\theta}{d \theta} |e_k\rangle = (d \lambda_j / d \theta) \delta_{jk}$. So the bounded SLD operator does not exist only when there is an eigenvalue $\lambda_j$ of $\rho_\theta$ such that
\[ \lambda_j = 0 \text{ and } \frac{d\lambda_j}{d\theta} \neq 0, \] which occurs when the parametric density operator changes its rank. When the value of \( \theta \) tends to these points, the SLD elements may diverge in order to fulfill Eq. (11).

As shown in the above argument, the QCRB based on the QFI requires the existence of a bounded SLD operator. When the parametric density operator changes its rank, the QFI is ill-defined and the QCRB can be easily violated. Meanwhile, the expression \( F_{\theta}^{(3)} \), which is derived under the assumption of the existence of an bounded SLD operator, becomes discontinuous.

IV. QCRB BASED ON THE BURES DISTANCE

Since the expression \( F_{\theta}^{(3)} \) is still continuous when the parametric density operator changes its rank, it is natural to ask the question whether the QCRB based on \( F_{\theta}^{(3)} \) holds at such parameter points. To answer this question, we shall give an derivation of the QCRB on mean square error from the perspective of the Bures distance in what follows.

We first give a brief introduction to the notion of amplitudes of density operators [28–31], which will be extensively used later. A bounded operator \( A \) is called an amplitude of a density operator \( \rho \) if it satisfies \( \rho = AA^\dagger \).

The amplitude for a given density operator is not unique, e.g., all \( A = \sqrt{\rho U} \) with \( U \) being an arbitrary unitary operator is an amplitude of \( \rho \). For each amplitude \( A \) of \( \rho \), there corresponds a purification state [28–31]

\[ |A\rangle := \sum_{jk} A_{jk} |j\rangle \otimes |k\rangle \]

in the doubled Hilbert space \( \mathcal{H} \otimes \mathcal{H} \) such that \( \text{tr}_2 |A\rangle\langle A| = \rho \), where \( \{ |j\rangle \} \) is an arbitrary orthonormal basis in the Hilbert space \( \mathcal{H} \) associated with the quantum system, \( A_{jk} \) are the entries of the matrix, and \( \text{tr}_2 \) denotes the partial trace with respect to the second tensor factor of the Hilbert space. We now consider two parameter points \( \theta \) and \( \theta' \).

Let \( A_\theta \) and \( A_{\theta'} \) be the amplitudes of \( \rho_\theta \) and \( \rho_{\theta'} \), respectively, i.e., \( \rho_\theta = A_\theta A_\theta^\dagger \) and \( \rho_{\theta'} = A_{\theta'} A_{\theta'}^\dagger \). The Euclidean distance between the two purification states can be expressed as [31]

\[ \|A_{\theta'} - A_\theta\| = \sqrt{\sum_{jk} \| (A_{\theta'})_{jk} - (A_\theta)_{jk} \|^2} = \sqrt{\text{tr}[(A_{\theta'} - A_\theta)^\dagger (A_{\theta'} - A_\theta)]} = \|A_{\theta'} - A_\theta\|_2, \]

where \( \|X\|_2 = \sqrt{\text{tr}(X^\dagger X)} \) denotes the Schatten-2 norm of an operator \( X \).

We now prove the QCRB based on the Bures distance. The mean square error of estimating the parameter \( \theta \) can be written as

\[ \mathcal{E}_\theta = \sum_x \eta(x, \theta)^2 \left\| \sqrt{E_x A_\theta} \right\|_2^2, \]

where we have defined \( \eta(x, \theta) := \hat{\theta}(x) - \theta \) for brevity. To apply the Cauchy-Schwarz inequality, we insert \( \sum_x E_x = \mathbb{I} \) into the expression of the Euclidean distance between two amplitudes and get

\[ \|A_{\theta'} - A_\theta\|^2 = \sum_x \left\| \sqrt{E_x (A_{\theta'} - A_\theta)} \right\|_2^2. \]

Using Eqs. (15) and (16), it can be shown that

\[ \mathcal{E}_\theta \times \|A_{\theta'} - A_\theta\|^2 \geq \left[ \sum_x |\eta(x, \theta)| \left\| \sqrt{E_x A_\theta} \right\|_2 \sqrt{E_x (A_{\theta'} - A_\theta)} \right]^2 \]

\[ \geq \left\{ \sum_x |\eta(x, \theta)| \left| \text{tr} \left[ A_\theta^\dagger E_x (A_{\theta'} - A_\theta) \right] \right| \right\}^2 =: \alpha. \]

Here, the first inequality is due to a Cauchy-Schwarz inequality \[ \sum_x f(x)^2 \sum_x g(x)^2 \geq \left[ \sum_x f(x) g(x) \right]^2 \] for two real functions \( f(x) = \eta(x, \theta) \left\| \sqrt{E_x A_\theta} \right\|_2 \) and \( g(x) = \left\| \sqrt{E_x (A_{\theta'} - A_\theta)} \right\|_2 \), and the second one is due to a Cauchy-Schwarz inequality \( \|X\|_2 \|Y\|_2 \geq \|X^\dagger Y\| \) for two bounded operators \( X, Y \). Noting that

\[ \left| \text{tr} \left[ A_\theta^\dagger E_x (A_{\theta'} - A_\theta) \right] \right| \geq \Re \text{tr} \left[ A_\theta^\dagger E_x (A_{\theta'} - A_\theta) \right] \]

and \( \sum_x |f_x| \geq |\sum_x f_x| \) for any series of scalars \( f_x \), we get

\[ \alpha \geq \left\{ \sum_x \eta(x, \theta) \Re \text{tr} \left[ A_\theta^\dagger E_x (A_{\theta'} - A_\theta) \right] \right\}^2 = \frac{1}{4} \left\{ \sum_x \eta(x, \theta) \text{tr} \left[ E_x (A_{\theta'} A_\theta^\dagger + A_\theta A_{\theta'}^\dagger - 2A_\theta A_{\theta'}^\dagger) \right] \right\}^2, \]

(19)

For all unbiased estimator \( \hat{\theta} \), \( \sum_x \hat{\theta}(x) \text{tr} \left( E_x A_\theta A_{\theta'}^\dagger \right) = \theta \), which implies that

\[ \sum_x \eta(x, \theta) \text{tr} \left( E_x A_\theta A_{\theta'}^\dagger \right) = \theta' - \theta. \]

Combining Eqs. (19) and (20), we get

\[ \alpha \geq \frac{1}{4} \left\{ \sum_x \eta(x, \theta) \left[ \left\| \sqrt{E_x (A_{\theta'} - A_\theta)} \right\|_2^2 - \left\| \sqrt{E_x A_{\theta'}} \right\|_2^2 \right] \right\}^2 \]

\[ \geq \frac{1}{4} \left[ \theta' - \theta - \sum_x \eta(x, \theta) \left\| \sqrt{E_x (A_{\theta'} - A_\theta)} \right\|_2^2 \right]^2 =: \beta, \]

(21)

Therefore, we get a family of inequalities for the estimation errors:

\[ \mathcal{E}_\theta \geq \|A_{\theta'} - A_\theta\|^2 \beta, \]

(22)
which is our main result in this paper. In what follows, we shall show that the lower bounds in Eq. (22) can be further refined from two perspectives.

On the one hand, the inequality Eq. (22) holds for all amplitudes of \( \rho_0 \) and \( \rho_{0'} \); this supplies us a freedom of choosing specific \( A_0 \) and \( A_{0'} \) to formulate useful lower bounds based on Eq. (22). The Bures-Uhlmann geometry for density operators tell us that [29, 32]

\[
\min_{A_0, A_{0'}} \left\| A_{0'} - A_0 \right\|_2^2 = 2 \left( 1 - \left\| \sqrt{\rho_0} \sqrt{\rho_{0'}} \right\|_1 \right)
\] (23)

and the minimum is attained by the amplitudes satisfying

\[
A_{0',A_{0'}}^\dagger A_{0',A_{0'}} = A_{0',A_{0'}}^\dagger A_0 \geq 0.
\] (24)

Two amplitudes satisfying the above condition are called parallel [31]. For parallel amplitudes \( A_0 \) and \( A_{0'} \) we have

\[
\mathcal{E}_\theta \geq \frac{\beta}{d_B(\rho_0, \rho_{0'})^2}.
\] (25)

Note that the quantity \( \beta \) defined by Eq. (21) still depends on the POVM and estimator.

On the other hand, the inequality Eq. (22) holds for any two parameter points \( \theta \) and \( \theta' \) in the range of the parameter. This supplies us a freedom of choosing the reference parameter point \( \theta' \) to formulate useful lower bounds on the estimation error at \( \theta \) based on Eq. (22). We here consider two infinitesimally neighboring parameter points by defining \( \epsilon := \theta' - \theta \) and taking the limitation \( \epsilon \rightarrow 0 \). Assuming that \( A_0 \) is differentiable, we have \( A_{0'} \approx A_0 + \epsilon \frac{dA_0}{d\theta} \). Therefore, we obtain

\[
\beta \approx \frac{1}{4} \left[ \epsilon - \epsilon^2 \sum_x \eta(x, \theta) \frac{dA_0^\dagger}{d\theta} E_x \frac{dA_0}{d\theta} \right]^2
\] (26)

and thus

\[
\mathcal{E}_\theta \geq \lim_{\epsilon \rightarrow 0} \frac{\beta}{\| A_{0'} - A_0 \|^2} = \lim_{\epsilon \rightarrow 0} \frac{[\epsilon + \mathcal{O}(\epsilon^2)]^2}{4\epsilon^2 \left\| \frac{dA_0}{d\theta} \right\|^2_2} \geq \frac{1}{4} \left\| \frac{dA_0}{d\theta} \right\|^2_2,
\] (27)

where \( \mathcal{O}(\epsilon^2) \) denotes the high order terms.

If we consider the inequality Eq. (22) for two infinitesimally neighboring parameter points with the parallel amplitudes, it follows that

\[
\mathcal{E}_\theta \geq \left[ 4 \lim_{\epsilon \rightarrow 0} \frac{d_B(\rho_0, \rho_{0}+\epsilon)}{\epsilon^2} \right]^{-1} = \frac{1}{\mathcal{F}_\theta^{(3)}}.
\] (28)

Therefore, we obtain the QCRB without resorting to an SLD operator. This bound holds even when \( \mathcal{F}_\theta^{(2)} \) is discontinuous.

V. EXAMPLES

In Ref. [21], Seveso et al. argued that the QCRB \( \mathcal{E}_\theta \geq 1/\mathcal{F}_\theta^{(3)} \) can be violated when the parametric density operator changes its rank through an example, which contradicts our result. In what follows, we will discuss the above-mentioned example and solve the conflict by recognizing an unnoticed error-propagating factor when reparametrizing the statistical model.

Following Ref. [21], we consider an example whose parametric density operator reads

\[
\rho_0 = (1 - \theta) |0\rangle \langle 0 | + \theta |1\rangle \langle 1 |
\] (29)

with \( 0 \leq \theta \leq 1 \). The rank of \( \rho_0 \) changes at the boundary \( \theta = 0 \) and \( \theta = 1 \). The Bures distance is given by

\[
d_B(\rho_0, \rho_{0'})^2 = 2 \left[ 1 - \sqrt{(1 - \theta)(1 - \theta')} - \sqrt{\theta \theta'} \right].
\] (30)

The QFI expressions are given by [21]

\[
\mathcal{F}_\theta^{(3)} = \frac{1}{\theta(1 - \theta)},
\] (31)

which diverges at \( \theta = 0 \) and \( \theta = 1 \), and

\[
\mathcal{F}_\theta^{(2)} = \begin{cases} 1, & \text{if } \theta = 0 \text{ or } 1, \\ \frac{\pi}{|1 - \sigma|}, & 0 < \theta < 1. \end{cases}
\] (32)

Suppose that we have \( n \) copies of the quantum system and perform the projective measurement whose POVM is given by \( E_0 = |0\rangle \langle 0 | \) and \( E_1 = |1\rangle \langle 1 | \) on each copy. Denoted by \( x_j \) the outcome of the quantum measurement on \( j \)-th system, which takes the value \( 0 \) and \( 1 \) with the probabilities \( 1 - \theta \) and \( \theta \), respectively. Define a statistical quantity \( t := \sum_{j=1}^{n} x_j \), which is sufficient for estimating \( \theta \). The probability of \( t \) is given by

\[
\rho_0(t) = \binom{n}{t} \theta^t (1 - \theta)^{n - t}.
\] (33)

Note that

\[
\frac{\partial \ln \rho_0(t)}{\partial \theta} = \frac{t}{\theta} + \frac{n - t}{\theta - 1} - \frac{n\theta - t}{\theta(\theta - 1)}.
\] (34)

implying that the maximum likelihood estimator for this statistical model is given by \( \hat{\theta}(t) = t/n \), whose mean is \( \theta \) and variance is \( \theta(1 - \theta)/n \). Therefore, at the boundary \( \theta = 0 \) and \( \theta = 1 \), the mean-square error of the maximum likelihood estimator vanishes. Since the QFI becomes infinite in such case, the QCRB still holds in such a scenario [21].

Furthermore, Ref. [21] considered a reparameterization of the above example, namely,

\[
\rho_\vartheta = \cos^2 \vartheta |0\rangle \langle 0 | + \sin^2 \vartheta |1\rangle \langle 1 |
\] (35)

with \( 0 \leq \vartheta \leq \pi/2 \). This is a reparameterization of the statistical model Eq. (29) by substituting \( \theta = \sin^2 \vartheta \)
The value of \( \vartheta \) can be inferred by first estimating \( \theta \) in Eq. (29) and then calculating \( \vartheta \) through \( \vartheta = \sin^2 \theta \). Since the maximum likelihood estimator for \( \theta \), as shown in the discussion of the statistical model Eq. (29), vanishes at \( \vartheta = 0 \) and \( \vartheta = \pi/2 \), it was concluded in Ref. [21] that the QCRB \( \mathcal{E}_\vartheta \geq 1/\mathcal{F}_\vartheta^{(3)} \) was violated at \( \vartheta = 0 \) and \( \vartheta = \pi/2 \).

We here point out that when inferring the value of \( \theta \) from the estimates of \( \sin^2 \theta \), a factor from the error propagation has to be taken into consideration in the estimation error. When inferring \( \vartheta \) from the estimates of \( \theta = \sin^2 \theta \), according to the error analysis, see Ref. [33, Chapter 8], we have

\[
\mathcal{E}_\vartheta \bigg| \frac{d\theta}{d\vartheta} \bigg|^{-2} = \frac{\mathcal{E}_\vartheta}{4 \sin^2 \vartheta \cos^2 \vartheta}. \tag{37}
\]

For maximum likelihood estimator of \( \theta \), substituting \( \mathcal{E}_\theta = \theta(1 - \theta)/n \) and \( \theta = \sin^2 \vartheta \) into the above formula, we get

\[
\mathcal{E}_\vartheta = \frac{\theta(1 - \theta)}{4n \sin^2 \theta \cos^2 \theta} = \frac{1}{4n} = \frac{1}{n\mathcal{F}_\vartheta^{(3)}}. \tag{38}
\]

Therefore, the QCRB based on the Bures distance still holds for the statistical model described in Eq. (35).

VI. CONCLUSION

In this work, we have revisited the derivation of the QCRB to investigate its validity at the specific parameter values where the parametric density operator changes its rank. We have ascribed invalidity of the QCRB due to the nonexistence of a bounded SLD. We have alternatively derived the QCRB through the Bures distance. Our approach does not require a bounded SLD operator and thus consolidate the soundness of the QCRB.

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