Path-connected Closures of Unitary Orbits

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Abstract. If \( A \) and \( B \) are unital \( \ast \)-algebras and \( \pi : A \to B \) is a unital \( \ast \)-homomorphism, then \( U_B(\pi)^- \) is the set of all \( \ast \)-homomorphisms from \( A \) to \( B \) that are approximately (unitarily) equivalent to \( \pi \). We address the question of when \( U_B(\pi)^- \) is path-connected with respect to the topology of pointwise norm convergence. When \( A \) is singly generated and \( B = B(\ell^2) \), an affirmative answer was given in \( \cite{4} \); we extend this to the case when \( A \) is separable. We also give an affirmative answer when \( A \) is AF and \( B \) is a von Neumann algebra, \( A \) is ASH and \( B \) is a finite von Neumann algebra, or when \( A \) is homogeneous and \( B \) is an arbitrary von Neumann algebra.

1. Introduction

In \( \cite{4} \) D. Hadwin proved that the norm closure of the unitary orbit of an operator in \( B(\ell^2) \) is path connected. In this paper we address the problem of extending this result to representations of separable \( \ast \)-algebras.

Suppose \( A \) and \( B \) are unital \( \ast \)-algebras and \( A \) is separable. We define \( \text{Rep}(A, B) \) as the set of all unital \( \ast \)-homomorphisms from \( A \) to \( B \) with the topology of pointwise norm convergence. Suppose \( \{a_1, a_2, \ldots\} \) is a norm dense subset of the closed unit ball of \( A \). We define a metric \( d = d_{A, B} \) by

\[
d(\pi, \rho) = \sum_{m, n=1}^{\infty} \frac{1}{2^{m+n}} \|\pi(a_n) - \rho(a_n)\|.
\]

Clearly, \( d \) makes \( \text{Rep}(A, B) \) into a complete metric space. When \( B \) is finite-dimensional, \( \text{Rep}(A, B) \) is compact.

Let \( U_B \) denote the group of unitary elements of \( B \). If \( \pi \in \text{Rep}(A, B) \), we define the unitary orbit \( U_B(\pi) \) of \( \pi \) by

\[
U_B(\pi) = \{U^* \pi(\cdot) U : U \in U_B\}.
\]

If \( T \in B \) we define the unitary orbit \( U_B(T) \) of \( T \) by

\[
U_B(T) = \{U^*TU : U \in U_B\}.
\]
It is clear that $\mathcal{U}_B (T)$ corresponds to $\mathcal{U}_B (\pi)$ when $\pi$ is the identity representation of the identity representation of $C^* (T)$.

In this paper we address the problem of when $\mathcal{U}_B (\pi)^-$ is path-connected in $\text{Rep}(A, B)$. In Section 2 we discuss special paths in $\mathcal{U}_B (\pi)^-$. In Section 3 we provide an affirmative answer (Theorem 5) for the case when $A = B$ ($\ell^2$). We reduce the separable case to the singly generated case by tensoring with the algebra $K (\ell^2)$ of compact operators on $\ell^2$. In Section 4 we give an affirmative answer (Theorem 7) when $A$ is AF and $B$ has the property that $\mathcal{U}_{B^p}$ is connected for every projection $p \in B$. We also give an affirmative answer (Theorem 6) when there is an LF $C^*$-algebra $D$ such that $A \subset D \subset A^{##}$, and $B$ is an arbitrary finite von Neumann algebra. In section 5 we give an affirmative answer (Theorem 8) when $A$ is abelian (or homogeneous) and $B$ is an arbitrary von Neumann algebra.

2. Connectedness of $\mathcal{U}_B$ and special paths

An internal path in $\mathcal{U}_B (\pi)^-$ joining $\pi$ to $\rho$ is a continuous map $\gamma : [0, 1] \to \mathcal{U}_B (\pi)^-$ such that $\gamma (0) = \pi$, $\gamma (1) = \rho$ and $\gamma (t) \in \mathcal{U}_B (\pi)$ whenever $0 \leq t \leq 1$. A strong internal path from $\pi$ to $\rho \in \mathcal{U}_B (\pi)^-$ is a continuous map $\gamma : [0, 1] \to \mathcal{U}_B$ such that

$$\lim_{t \to 1^-} \gamma (t)^* \pi \gamma (t) = \rho.$$

In [3, Theorem 3.9] the first author proved that $\mathcal{U}_B (T)^-$ is always path connected when $B = B (\ell^2)$. Actually a slightly stronger result was proved.

**Theorem 1.** [3, Theorem 3.9] Suppose $X \in B (\ell^2)$ and $Y \in \mathcal{U}_{B (\ell^2)} (X)^-$. Then there is a $W$ such that

1. $W$ is unitarily equivalent to $W \oplus W \oplus \cdots$,
2. $X \oplus W$ is unitarily equivalent to $Y \oplus W$,
3. If $C \in B (\ell^2)$ is unitarily equivalent to $X \oplus W$, then
   a. $C \in \mathcal{U}_{B (\ell^2)} (X)^- = \mathcal{U}_{B (\ell^2)} (Y)^-$,
   b. there is a strong internal path in $\mathcal{U}_{B (\ell^2)} (X)^-$ from $X$ to $C$, and
   c. there is a strong internal path in $\mathcal{U}_{B (\ell^2)} (Y)^-$ from $Y$ to $C$.

There is no reason, a priori, that $\mathcal{U}_B (\pi)$ is even connected. It is well-known that if $P$ and $Q$ are projections in a unital C*-algebra $B$ and $\|P - Q\| < 1$, then $P$ and $Q$ are unitarily equivalent [8]. It was proved in [3] that two unital representations $\pi, \rho$ of a finite-dimensional C*-algebras $A$ are unitarily equivalent if and only if $\pi (p)$ is unitarily equivalent to $\rho (p)$ for every minimal projection $p \in A$.

If $\mathcal{U}_B$ is connected, then every $\mathcal{U}_B (\pi)$ must be connected. If $x \in B$ and $\|1 - x\| < 1$, then $(-\infty, -1] \cap \sigma (x) = \emptyset$, so $A (x) = -\log (x) \in B$, $A (x) = A (x)^*$, and $x = e^{iA(x)}$. Since $t \mapsto e^{i(1-t)A(x)}$ is a path in $\mathcal{U}_B$ from $x$ to $1$, we see that $\{x \in \mathcal{U}_B : \|1 - x\| < 1\}$ is contained in the path component $W$ of $1$ in $\mathcal{U}_B$. Since $W = \cup uW$ such that $u \in W$, we see that $W$ is open in $\mathcal{U}_B$. Thus $\mathcal{U}_B$ is connected if and only if it is path-connected. This means that if $\mathcal{U}_B$ is connected, then $\mathcal{U}_B (\pi)$ is path-connected.

**Lemma 1.** If $A$ is finite-dimensional, then for every $B$ and every $\pi \in \text{Rep}(A, B)$, $\mathcal{U}_B (\pi)$ is closed.

**Proof.** It follows from [3, Theorem 2 (4)] that if $\rho \in \mathcal{U}_B (\pi)^-$, then $\rho \in \mathcal{U}_B (\pi)$. \qed
Example 1. B. Blackadar [1] showed that in $\mathcal{B} = \mathbb{M}_2 (C(S^1))$ there are two projections $P, Q$ that are unitarily equivalent, but are not homotopy equivalent. Thus $U_\mathcal{B} (P) = U_\mathcal{B} (P)^*$ is not path-connected. This implies that $U_\mathcal{B}$ is not connected.

We say that a unital C*-algebra $\mathcal{B}$ has property UC if $U_\mathcal{B}$ is connected. The algebra $\mathcal{B}$ has property HUC if, for every projection $P \in \mathcal{B}$, $P \mathcal{B} P$ has property UC. We say that $\mathcal{B}$ is matricially stable if and only if, for every $n \in \mathbb{N}$, $\mathcal{B}$ is isomorphic to $M_n (\mathcal{B})$.

Lemma 2. The following are true:

1. Every von Neumann algebra has property HUC.
2. A direct limit of unital C*-algebras with property HUC has property HUC.
3. Every unital AF algebra has property HUC.
4. If $A$ is a unital C*-algebra and, for every $n \in \mathbb{N}$, $M_n (\mathbb{C})$ has property UC, then $K_1 (A) = 0$.
5. If $B$ is matricially stable, then $B$ has property UC if and only if $K_1 (B) = 0$.

Proof. (1). In a von Neumann algebra $A$ every unitary $U$ can be written $U = e^{iA}$ with $A = A^*$, and the path $g(t) = e^{(1-t)A}$ connects $U$ to 1 in $U_A$. Thus $A$ has property UC. But $PAP$ is a von Neumann algebra for every projection $P \in A$. Thus $A$ has property HUC.

(2). Suppose $\{ A_\lambda : \lambda \in \Lambda \}$ is an increasingly directed family of unital C*-subalgebras of a unital C*-subalgebras $A$ with property UC, and $A = \bigcup_{\lambda \in \Lambda} A_\lambda$. Let $K$ be the connected component of $U_A$ that contains 1. Suppose $U \in U_A$ and $\varepsilon > 0$. Then there is a $\lambda \in \Lambda$ and a unitary $V \in A_\lambda$ such that $\|U - V\| < \varepsilon$. However, if $E_\lambda$ is the connected component of $U_{A_\lambda}$, we have $V \in E_\lambda \subset E$. Since $\varepsilon > 0$, $U \in E^- = E$. Next suppose each $A_\lambda$ has property HUC and $P \in A$ is a projection. Then there is a $\lambda_0 \in \Lambda$ and a projection $Q \in A_{\lambda_0}$ such that $\|P - Q\| < 1$, which implies there is a unitary $W \in A$ such that $P = W^*QW$. Hence

$$PAP = W^*QWAW^*QW = W^* (QAQ).$$

Thus $PAP$ is isomorphic to

$$QAQ = \bigcup_{\lambda \geq \lambda_0} Q A_\lambda Q.$$

Since each $Q A_\lambda Q$ has property UC when $Q \in A_\lambda$, we see that $PAP$ has property UC. Thus $A$ has property HUC.

(3). This follows from (1) and (2).

(4). This follows from the definition of $K_1 (A)$.

(5). This follows from (4).

3. $B (\ell^2)$

In this section we extend Theorem [1] to the case where the single operator is replaced with a representation of a separable C*-algebra. The key idea is a result of C. Olsen and W. Zame [2] that if $A$ is a separable C*-algebra, then $A \otimes K (\ell^2)$ is singly generated. This gives us a general technique for relating the separable case to the singly generated case.

Suppose $A$ is a unital C*-algebra. Let $A^\dagger$ denote the unitization of $A \otimes K (\ell^2)$. If $\pi \in \text{Rep}(A, \mathcal{B})$ we define $\pi^\dagger : A^\dagger \to \mathcal{B}^\dagger$ by

$$\pi^\dagger (\lambda 1 + (a_{ij})) = \lambda 1 + (\pi (a_{ij})).$$
Let $B^\oplus$ be the C*-algebra generated by $B^\dagger$ and $\{\text{diag}(a, a, \ldots) : a \in A\}$.

**Theorem 2.** Suppose $A$ and $B$ are unital C*-algebras and $\pi, \rho \in \text{Rep}(A, B)$. Then

1. The map $\rho \mapsto \rho^\dagger$ from $\text{Rep}(A, B)$ to $\text{Rep}(A^\dagger, B^\dagger)$ is continuous.
2. If $\pi, \rho \in \text{Rep}(A, B)$, then $\rho \in U_{B^\dagger}(\pi)^-$ if and only if $\rho^\dagger \in U_{B^\dagger}(\pi^\dagger)^-$.
3. If $\rho \in U_{B^\dagger}(\pi)^-$ and there is an internal path in $U(\pi)^-$ joining $\pi$ to $\rho$, then there is an internal path in $U_{B^\dagger}(\pi^\dagger)^-$ joining $\pi^\dagger$ to $\rho^\dagger$.
4. If
   - $B^\dagger \subset E$ is a C*-algebra with $e_{11}Ee_{11} = e_{11}B^\dagger e_{11}$,
   - $\rho_1 \in U_{E}(\pi^\dagger)^-$,
   - For every $a \in A$, $\rho_1(\text{diag}(a, 0, 0, \ldots)) = \text{diag}(\rho(a), 0, 0, \ldots)$
   - $U_{B^\dagger}$ is connected,
   - $B^\dagger \subset E$ is a C*-algebra with $e_{11}Ee_{11} = e_{11}B^\dagger e_{11}$, and
   - there is a strong internal path in $U_{E}(\pi^\dagger)^-$ from $\pi^\dagger$ to $\rho_1$,
   then there is a strong internal path in $U_{B^\dagger}(\pi)^-$ from $\pi$ to $\rho$.

**Proof.** (1). This is obvious.

(2). Suppose $\rho \in U_{B^\dagger}(\pi)^-$. Then there is a sequence $\{U_n\}$ in $U_{B^\dagger}$ such that, for every $a \in A$,

$$\lim_{n \to \infty} \|U_n\pi(a)U_n^* - \rho(a)\| = 0.$$ 

For each positive integer $n$, let $W_n = \text{diag}(U_n, \ldots, U_n, 1, 1, 1, \ldots)$ in $B^\dagger$ (with $U_n$ repeated $n$ times). Since

$$\{T \in A^\dagger : \lim_{n \to \infty} \|W_nT - \rho^\dagger(T)\| = 0\}$$

is a unital subalgebra containing the operators $(A_{ij}) \in A^\dagger$ such that,

$$\{(i, j) \in \mathbb{N} \times \mathbb{N} : (i, j) \neq (0, 0)\}$$

is finite, we see that $\rho^\dagger \in U_{A^\dagger}\{\pi^\dagger\}^-.$

Conversely, suppose $\rho^\dagger \in U_{A^\dagger}\{\pi^\dagger\}^-$. Then there is a sequence $\{V_n\}$ in $B^\dagger$ such that, for every $T \in A^\dagger$,

$$\lim_{n \to \infty} \|V_n\pi^\dagger(T)V_n^* - \rho^\dagger(T)\| = 0.$$ 

Since $\pi^\dagger(e_{11}) = \rho^\dagger(e_{11}) = e_{11}$, we see that

$$\lim_{n \to \infty} \|V_ne_{11} - e_{11}V_n\| = \lim_{n \to \infty} \|V_n\pi^\dagger(e_{11})V_n^* - \rho^\dagger(e_{11})\| = 0,$$

and there is a sequence $\{W_n\}$ in $U_{A^\dagger}$ such that

$$\lim_{n \to \infty} \|W_n - e_{11}V_ne_{11}\| = 0.$$

Hence, for every $a \in A$,

$$\lim_{n \to \infty} \|W_n\pi(a)W_n^* - \rho(a)\| = 0.$$

Thus $\rho \in U_{B^\dagger}(\pi)^-.$
(3). Suppose there is an internal path \( \gamma : [0,1] \to \mathcal{U}(\pi)^- \) joining \( \pi \) to \( \rho \). For \( 0 \leq t < 1 \) write \( \gamma(t) = U_t \pi \) with \( U_t \in \mathcal{U}_E \). For each \( 0 \leq t < 1 \) let \( V_t = \text{diag}(U_t, U_{t}, \ldots) \in \mathcal{U}_E \) and let \( \Gamma(t) = V_t \pi^{\dagger} (\Gamma^{\dagger}) V_t^{\dagger} \). Then, for every \( T \in \mathcal{A}^{\dagger} \),
\[
\lim_{t \to 1^-} \| V_t \pi^{\dagger} (T) V_t^{\dagger} - \rho^{\dagger} (T) \| = 0.
\]

(4). Suppose \( \Gamma : [0,1) \to \mathcal{U}_E \) is continuous, and, for every \( T \in \mathcal{A}^{\dagger} \),
\[
\lim_{t \to 1^-} \| \Gamma(t) \pi^{\dagger} (T) \Gamma(t)^* - \rho_1 (T) \| = 0.
\]
Since \( \rho_1(e_{11}) = \pi^{\dagger} (e_{11}) = e_{11} \), we conclude that
\[
\lim_{t \to 1^-} \| \Gamma(t) e_{11} - e_{11} \Gamma(t) \| = \lim_{t \to 1^-} \| \Gamma(t) \pi^{\dagger} (e_{11}) \Gamma(t)^* - \rho_1 (e_{11}) \| = 0.
\]
Since \( \Gamma(t) \) is unitary, there is a \( t_0 \in [0,1) \) such that, whenever \( t_0 \leq t < 1 \), we have \( C_t = e_{11} \Gamma(t) e_{11} \) is invertible in \( \mathcal{A} \) and if
\[
U_t = C_t [C_t^{\dagger} C_t]^{-1/2},
\]
then \( U_t \in \mathcal{U}_A \) and
\[
\lim_{t \to 1^-} \| C_t - U_t \| = 0.
\]
Since \( \mathcal{U}_A \) is connected, there is a continuous map \( t \mapsto U_t \in \mathcal{U}_A \) for \( 0 \leq t \leq t_0 \) so that \( U_0 = 1 \). If, for every \( a \in \mathcal{A} \), we consider \( T_a = \text{diag}(a,0,0,\ldots) \), it is easily seen that
\[
\lim_{t \to 1^-} \| U_t \pi(a) U_t^{\dagger} - \rho(a) \| = 0.
\]

\[\square\]

**Theorem 3.** Suppose \( \mathcal{A} \) is a separable unital \( C^* \)-algebra and \( \pi \in \text{Rep}(\mathcal{A}, B(\ell^2)) \). Then \( \mathcal{U}_{B(\ell^2)} (\pi)^- \) is path-connected.

**Proof.** Suppose \( \rho \in \mathcal{U}_{B(\ell^2)} (\pi)^- \). Then, by Theorem [2], \( \rho^{\dagger} \in \mathcal{U}_{B(\ell^2)} (\pi^{\dagger})^- \).

But \( B(\ell^2)^{\dagger} \subset B(\ell^2 \oplus \ell^2 \oplus \cdots) = \mathcal{E} \). Also, by [7] there is an operator \( T \in \mathcal{A}^{\dagger} \) such that \( \mathcal{A}^{\dagger} = C^* (T) \). Thus \( \rho(T) \in \mathcal{U}_E (\pi(T))^- \). We know from Theorem [1] with \( X = \pi^{\dagger} (T) \) and \( Y = \rho^{\dagger} (T) \), that there is a \( W \in \mathcal{E} \) such \( C \in \mathcal{U}_E (\pi(T))^- \) and a strong internal path from \( \pi(T) \) to \( C \) in \( \mathcal{U}_E (\pi(T))^- \) and a strong internal path in \( \mathcal{U}_E (\rho(T))^- \) from \( \rho(T) \) to \( C \). There is a representation \( \delta \) of \( C^* (T) \) such that \( \delta_0 (T) = W \) and if \( \delta (A) = A \oplus \delta (A) \), we have \( \delta(T) = T \oplus W \). Since \( e_{11} \) and \( \delta (e_{11}) = e_{11} \oplus \delta_0 (e_{11}) \) are projections with infinite rank and infinite corank, there is a unitary operator \( V \) such that \( V^{\dagger} \delta (e_{11}) V = e_{11} \) and \( V^{\dagger} TV \in \mathcal{E} \). Let \( C = V^{\dagger} \delta(T) V \) and \( \rho_1 (\cdot) = V^{\dagger} \delta(\cdot) V \). It follows that there is a \( \sigma \in \text{Rep}(\mathcal{A}, B(\ell^2)) \) such that, for every \( a \in \mathcal{A} \),
\[
\rho_1 (\text{diag}(a,0,0,\cdots)) = (\sigma(a),0,0,\cdots).
\]
Since there is an internal path in \( \mathcal{U}_E (\pi^{\dagger} (T))^- \) from \( \pi^{\dagger} (T) \) to \( \rho_1 (T) \), there is a strong internal path in \( \mathcal{U}_E (\pi^{\dagger})^- \) from \( \pi^{\dagger} \) to \( \rho_1 \). It follows from part (4) of Theorem [2] that there is a strong internal path in \( \mathcal{U}_{B(\ell^2)} (\pi)^- \) from \( \pi \) to \( \sigma \). Similarly, there is a strong internal path in \( \mathcal{U}_{B(\ell^2)} (\rho)^- \) from \( \rho \) to \( \sigma \). Thus there is a path in \( \mathcal{U}_{B(\ell^2)} (\pi)^- = \mathcal{U}_{B(\ell^2)} (\rho)^- \) from \( \pi \) to \( \rho \). \[\square\]
4. AF algebras

Lemma 3. Suppose \( 1 \in \mathcal{A} \subset \mathcal{D} \) are separable unital C*-algebras, \( \mathcal{B} \) is a unital C*-algebra and \( \pi, \rho \in \text{Rep}(\mathcal{D}, \mathcal{B}) \), and suppose \( V, W \in \mathcal{U}_B \) such that

1. for every \( x \in \mathcal{D} \),
   \[
   W^* \rho(x) W = \pi(x),
   \]
2. for every \( x \in \mathcal{A} \),
   \[
   V^* \rho(x) V = \pi(x),
   \]
3. \( \mathcal{U}_{B \cap \rho(A)' \cap B} \) is connected.

Then there is a path \( t \mapsto U_t \) of unitary operators in \( \mathcal{B} \) such that \( U_0 = V \), \( U_1 = W \), and for every \( t \in [0, 1] \) and every \( x \in \mathcal{A} \),

\[
U_t^* \rho(x) U_t = \pi(x).
\]

Proof. We know that, for every \( x \in \mathcal{A} \),

\[
W^* \rho(x) W = V^* \rho(x) V.
\]

Thus \( VW^* = X \in \rho(A)' \cap B \). Thus \( W = U^* V \). Since \( \mathcal{U}_{\rho(A)' \cap B} \) is path connected, there is a path \( t \mapsto X_t \) of unitary elements in \( \rho(A)' \cap B \) such that \( X_0 = 1 \) and \( X_1 = X \). For \( t \in [0, 1] \) let \( U_t = X_t^* V \). Then \( U_t \) is a path in \( \mathcal{U}_B \), \( U_0 = V \) and \( U_1 = X^* V = W \). Moreover, for each \( t \in [0, 1] \) and each \( x \in \mathcal{A} \),

\[
U_t^* \rho(x) U_t = V^* X_t \rho(x) X_t^* V = V^* \rho(x) V = \pi(x).
\]

\( \square \)

Theorem 4. Suppose \( \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A} = [\bigcup_{n \in \mathbb{N}} \mathcal{A}_n]^\circ \) is separable. Suppose \( \pi, \rho \in \text{Rep}(\mathcal{A}, \mathcal{B}) \) such that, for every \( n \in \mathbb{N} \),

1. \( \rho|_{\mathcal{A}_n} \in \mathcal{U}_B \) \( \pi|_{\mathcal{A}_n} \).
2. \( \mathcal{U}_{\rho(\mathcal{A}_n)' \cap B} \) is connected.

Then there is a strong internal path from \( \pi \) to \( \rho \).

Proof. For each \( n \in \mathbb{N} \), choose \( U_n \in \mathcal{U}_B \) such that, for every \( a \in \mathcal{A}_n \),

\[
U_n^* \rho(a) U_n = \pi(a).
\]

It follows from Lemma 3 that we can define a path \( t \mapsto U_t \) from \( [n, n+1] \) so that for \( n \leq t \leq n+1 \) and \( a \in \mathcal{A}_n \), we have

\[
U_t^* \rho(a) U_t = \pi(a).
\]

Thus the map \( t \mapsto U_t \) is continuous, and, for every \( a \in \bigcup_{n \in \mathbb{N}} \mathcal{A}_n \), we have

\[
\lim_{t \to +\infty} \|U_t^* \rho(a) U_t - \pi(a)\| = 0.
\]

Hence, if we define \( \pi_t(\cdot) = U_t^* \rho(\cdot) U_t \) for \( t \in [0, \infty) \) and \( \pi_\infty = \rho \), we have a strong internal path in \( \mathcal{U}_B (\pi)^\circ \) from \( \pi \) to \( \rho \). \( \square \)

Theorem 5. Suppose \( \mathcal{A} \) is a separable unital AF C*-algebra, \( \mathcal{B} \) is a C*-algebra with property HUC, and \( \pi \in \text{Rep}(\mathcal{A}, \mathcal{B}) \). Then \( \mathcal{U}_B (\pi)^\circ \) is path-connected.
Proof. We can assume that \( \ker \pi = 0 \), since \( A/\ker \rho \) is a separable unital AF algebra. Since \( A \) is unital and AF, there is a sequence \( \{ A_n \} \) of unital finite-dimensional C*-subalgebras

\[
1 \in A_1 \subset A_2 \subset \cdots
\]

such that

\[
\left[ \bigcup_{n=1}^{\infty} A_n \right]^\perp = A.
\]

Suppose \( \rho \in U_B(\pi) \). Since each \( A_n \) is finite-dimensional, where approximate equivalence is the same as unitary equivalence, we have \( \rho |_{A_n} \in U_B(\pi |_{A_n}) \) for each \( n \in \mathbb{N} \).

Fix \( n \in \mathbb{N} \) and write \( A_n = M_{s_1}(\mathbb{C}) \oplus \cdots \oplus M_{s_t}(\mathbb{C}) \) and, for \( 1 \leq k \leq t \), let \( \{ e_{ij,k} : 1 \leq i, j \leq s_k \} \) be the system of matrix units for \( M_{s_k}(\mathbb{C}) \). It is easily seen that \( \rho (A_n)' \cap B \) is the set of all

\[
\sum_{k=1}^{t} \sum_{j=1}^{s_k} \rho (e_{j1,k}) \rho (e_{11,k}) x \rho (e_{11,k}) \rho (e_{ij,k})
\]

for \( x \in B \). It follows that \( \rho (A_n)' \cap B \) is isomorphic to

\[
\sum_{1 \leq k \leq t} \rho (e_{11,k}) B \rho (e_{11,k}).
\]

Since \( B \) has property HUC, we see that \( \rho (A_n)' \cap B \) has property UC. The desired conclusion now follows from Theorem 4. \( \square \)

Corollary 1. If \( A \) is a separable unital AF C*-algebra and \( B \) is either an AF C*-algebra or a von Neumann algebra, then, for every \( \rho \in \text{Rep}(A,B) \), \( U_B(\rho) \) is path-connected.

A separable C*-algebra is homogeneous if it is a finite direct sum of algebras of the form \( M_n(C(X)) \), where \( X \) is a compact metric space. A unital C*-algebra is subhomogeneous if it is a unital subalgebra of a homogeneous C*-algebra. Every subhomogeneous von Neumann algebra is homogeneous; in particular, if \( A \) is subhomogeneous, then the second dual \( A^{\#\#} \) of \( A \) is homogeneous. A C*-algebra is approximately subhomogeneous (ASH) if it is a direct limit of subhomogeneous C*-algebras.

A (possibly nonseparable) C*-algebra \( B \) is LF if, for every finite subset \( F \subset B \) and every \( \varepsilon > 0 \) there is a finite-dimensional C*-algebra \( D \) of \( B \) such that, for every \( b \in F \), \( \text{dist}(b,D) < \varepsilon \). Every separable unital C*-subalgebra of a LF C*-algebra is contained in a separable AF subalgebra. See [2] for details.

We are interested in a more general property. We say that a unital C*-algebra \( A \) is strongly LF-embeddable if there is an LF C*-algebra \( D \) such that \( A \subset D \subset A^{\#\#} \). It is easily shown that an ASH algebra is strongly LF-embeddable, i.e., if \( \{ A_\lambda \} \) is an increasingly directed family of subhomogeneous C*-algebras and \( A = (\bigcup \lambda A_\lambda)^{\#\#} \), then \( A \subset (\bigcup \lambda A_\lambda^{\#\#})^{\|\#\|} \subset A^{\#\#} \). The proof of the next theorem relies on results in [5].
Theorem 6. Suppose \( \mathcal{A} \) is a separable strongly \( \text{LF} \) embeddable \( \text{C}^* \)-algebra and \( \mathcal{M} \) is a finite von Neumann algebra. Then, for every \( \pi \in \text{Rep}(\mathcal{A}, \mathcal{M}) \), \( \mathcal{U}_\mathcal{M}(\pi)^- \) is path connected.

Proof. Suppose \( \rho \in \mathcal{U}_\mathcal{M}(\pi)^- \). It follows that there are weak*-weak* continuous unital \(*\)-homomorphisms \( \hat{\pi}, \hat{\rho} : \mathcal{A}^{##} \to \mathcal{M} \) such that \( \hat{\pi}|_\mathcal{A} = \pi \) and \( \hat{\rho}|_\mathcal{A} = \rho \). Since \( \mathcal{A} \) is strongly \( \text{LF} \) embeddable, there is a separable unital AF \( \text{C}^* \)-algebra \( \mathcal{D} \) such that

\[
\mathcal{A} \subset \mathcal{D} \subset \mathcal{A}^{##}.
\]

It follows from [5] Theorem 2 that \( \hat{\rho}|_\mathcal{D} \in \mathcal{U}_\mathcal{M}(\hat{\pi}|_\mathcal{D})^- \). We know from Theorem 5 that \( \mathcal{U}_\mathcal{M}(\hat{\pi}|_\mathcal{D})^- \) is path connected. Thus there is a path in \( \mathcal{U}_\mathcal{M}(\hat{\pi}|_\mathcal{D})^- \) from \( \hat{\pi}|_\mathcal{D} \) to \( \hat{\rho}|_\mathcal{D} \). Restricting to \( \mathcal{A} \), we obtain a path in \( \mathcal{U}_\mathcal{M}(\pi)^- \) from \( \pi \) to \( \rho \).

5. Abelian algebras

Lemma 4. Suppose \( \mathcal{N} \) is a countably generated von Neumann algebra. Then \( \mathcal{N} \) is isomorphic to a direct sum \( \sum_{i \in I} \mathcal{N}_i \) so that each \( \mathcal{N}_i \) acts on a separable Hilbert space. In particular, each \( \mathcal{N}_i \) is \( \sigma \)-finite.

Proof. Suppose \( \mathcal{N} \subset B(H) \) and \( e \in H \) with \( \|e\| = 1 \). Let \( P \) be the orthogonal projection onto \( (\mathcal{N}e)^- \). Thus \( P \in \mathcal{N} \). Let \( P_e \in \mathcal{Z}(\mathcal{N}) \) be the central cover of \( P \). Then the map \( T \mapsto T|_{P_e(H)} \) is a normal isomorphism between \( \mathcal{N}|_{P_e(H)} \) and \( \mathcal{N}|_{P(H)} \). Since \( \mathcal{N} \) is countably generated, \( P(H) \) is separable. Thus \( \mathcal{N}|_{P_e(H)} \) is a direct summand of \( \mathcal{N} \) that is isomorphic to a von Neumann algebra on a separable Hilbert space. The rest of the proof follows from this idea and Zorn’s lemma.

Suppose \( \mathcal{M} \) is a von Neumann algebra and \( T \in \mathcal{M} \). In [3] H. Ding and D. Hadwin defined \( \mathcal{M}\text{-rank}(T) \) to be the Murray von Neumann equivalence class of the orthogonal projection \( \mathfrak{R}(T) \) onto the closure of the range of \( T \). We say \( \mathcal{M}\text{-rank}(S) \leq \mathcal{M}\text{-rank}(T) \) if and only if there is a projection \( P \in \mathcal{M} \) such that \( P \leq \mathfrak{R}(T) \) and \( P \) is Murray von Neumann equivalent to \( \mathfrak{R}(S) \). They proved that if a separable unital \( \text{C}^* \)-algebra is a direct limit of homogeneous algebras, and \( \mathcal{M} \) acts on a separable Hilbert space, then for all \( \pi, \rho \in \text{Rep}(\mathcal{A}, \mathcal{M}) \), \( \rho \in \mathcal{U}_\mathcal{M}(\pi)^- \) if and only if, for every \( x \in \mathcal{A} \),

\[
\mathcal{M}\text{-rank}(\pi(x)) = \mathcal{M}\text{-rank}(\rho(x)).
\]

A key ingredient of the proof of this result was a sequential semicontinuity of \( \mathcal{M}\text{-rank} \) with respect to the \( \ast\)-SOT that was proved when \( \mathcal{M} \) is a von Neumann algebra acting on a separable Hilbert space [3] Theorem 1. We extend this to the general case.

Lemma 5. Suppose \( \mathcal{M} \) is a von Neumann algebra, \( A, B \in \mathcal{M} \) and, for each \( n \in \mathbb{N} \), \( B_n \in \mathcal{M} \) and \( \mathcal{M}\text{-rank}(B_n) \leq \mathcal{M}\text{-rank}(A) \). If \( B_n \to B \) is the \( \ast\)-SOT, then \( \mathcal{M}\text{-rank}(B) \leq \mathcal{M}\text{-rank}(A) \).

Proof. Let \( P_n = \mathfrak{R}(B_n) \), \( Q = \mathfrak{R}(A) \), and, for each \( n \in \mathbb{N} \), choose a partial isometry \( V_n \in \mathcal{M} \) such that \( V_n^*V_n = P_n \) and \( V_nV_n^* \leq Q \). Let \( \mathcal{N} = W^*(\{A, B, B_1, V_1, B_2, V_2, \ldots\}) \).

Clearly, we have, for every \( n \in \mathbb{N} \), that

\[
\mathcal{N}\text{-rank}(B_n) \leq \mathcal{N}\text{-rank}(A).
\]
By Lemma 4, we can write
\[ \mathcal{N} = \bigoplus_{i \in I} \mathcal{N}_i \]
with each \( \mathcal{N}_i \) acting on a separable Hilbert space.

Write
\[ A = \sum_{i \in I} A_i, B = \sum_{i \in I} B_i, B_n = \sum_{i \in I} B_{n,i}, V_n = \sum_{i \in I} V_{n,i}. \]

Since \( \mathcal{R}(A) = \sum_{i \in I} \mathcal{R}(A_i) \) and \( \mathcal{R}(B) = \sum_{i \in I} \mathcal{R}(B_{n,i}) \), for each \( i \in I \), \( \mathcal{N}_i \)-rank\((B_{n,i}) \leq \mathcal{N}_i \)-rank\((A_i) \) and the limit in the \(*\)-SOT of \( B_{n,i} \) is \( B_i \). Thus, by [3] Theorem 1, for each \( i \in I \),
\[ \mathcal{N}_i \)-rank\((B_i) \leq \mathcal{N}_i \)-rank\((A_i) \).

Thus, for each \( i \in I \), there is a partial isometry \( W_i \in \mathcal{N}_i \) such that
\[ W_i^* W_i = \mathcal{R}(B_i) \text{ and } W_i W_i^* \leq \mathcal{R}(A_i). \]

Then \( W = \sum_{i \in I} W_i \) is a partial isometry in \( \mathcal{N} \) such that
\[ W^* W = \mathcal{R}(B) \text{ and } W W^* \leq \mathcal{R}(A). \]

Since we also have \( W \in \mathcal{M} \), we conclude \( \mathcal{M}\text{-rank}(B) \leq \mathcal{M}\text{-rank}(A) \). \hfill \Box

**Corollary 2.** If \( A \) is a unital C*-algebra, \( \mathcal{M} \) is a von Neumann algebra and \( \pi \in \text{Rep}(A, \mathcal{M}) \) and \( \rho \in \mathcal{U}_\mathcal{M}(\pi^-) \), then, for every \( a \in A \),
\[ \mathcal{M}\text{-rank}(\pi(a)) = \mathcal{M}\text{-rank}(\rho(a)). \]

**Proof.** Suppose \( a \in A \). There is a sequence \( \{U_n\} \) in \( \mathcal{U}_\mathcal{M} \) such that
\[ \lim_{n \to \infty} \|U_n^* \pi(a) U_n - \rho(a)\| = \lim_{n \to \infty} \|\pi(a) - U_n \rho(a) U_n^*\| = 0. \]

Also \( \mathcal{M}\text{-rank}(U_n^* \pi(a) U_n) = \mathcal{M}\text{-rank}(\pi(a)) \) and \( \mathcal{M}\text{-rank}(U_n \rho(a) U_n^*) = \mathcal{M}\text{-rank}(\rho(a)) \) for each \( n \in \mathbb{N} \). Thus, by Lemma 5,
\[ \mathcal{M}\text{-rank}(\rho(a)) \leq \mathcal{M}\text{-rank}(\pi(a)) \text{ and } \mathcal{M}\text{-rank}(\pi(a)) \leq \mathcal{M}\text{-rank}(\rho(a)). \] \hfill \Box

Suppose \( A \) is a unital C*-algebra and \( \mathcal{M} \) is a von Neumann algebra and \( \pi : A \to \mathcal{M} \) is a unital \(*\)-homomorphism. Then there is a unique \(*\)-homomorphism \( \hat{\pi} : A^{\#\#} \to \mathcal{M} \) that is weak*-weak* continuous (see [4]).

**Lemma 6.** Suppose \( (X, d) \) is a compact metric space, \( \mathcal{M} \) is a sigma-finite von Neumann algebra, and \( \pi, \rho : C(X) \to \mathcal{M}, \rho \in \mathcal{U}_\mathcal{M}(\pi^-) \). Then there is a sequence \( \mathcal{F}_1, \mathcal{F}_2, \ldots \) of finite disjoint collections of nonempty Borel sets such that
1. \( \sum_{E \in \mathcal{F}_n} \hat{\pi}(\chi_E) = \sum_{E \in \mathcal{F}_n} \hat{\rho}(\chi_E) = 1 \),
2. \( \{ \hat{\pi}(\chi_E) : E \in \mathcal{F}_n \} \subset \text{sp}(\{ \hat{\pi}(\chi_F) : F \in \mathcal{F}_{n+1} \}) \) and \( \{ \hat{\rho}(\chi_E) : E \in \mathcal{F}_n \} \subset \text{sp}(\{ \hat{\rho}(\chi_F) : F \in \mathcal{F}_{n+1} \}) \),
3. For every \( E \in \mathcal{F}_n \), and
\[ \text{diam}(E) < 1/n. \]
4. For every \( E \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n \), \( \hat{\pi}(\chi_E) \) and \( \hat{\rho}(\chi_E) \) are Murray von Neumann equivalent.
PROOF. Let $\text{Bor}(X)$ be the C*-algebra with the supremum norm. We then have

$$C(X) \subset \text{Bor}(X) \subset C(X)^{\#\#}$$

and $\hat{\pi}|_{\text{Bor}(X)}$, $\hat{\rho}|_{\text{Bor}(X)}$ are unital *-homomorphisms.

Let $\Sigma = \{ U \subset X : U$ is open and $\hat{\pi}(\chi_{U^c}) = \hat{\rho}(\chi_{U^c}) = 0 \}$. It is easily shown that if $U, V \in \Sigma$, then $U \setminus V, U \cup V, U \cap V \in \Sigma$. Moreover, if $a \in X$ and $S(a, r) = \{ x \in X : d(a, x) = r \}$ for all $r > 0$, it follows from the fact that $\mathcal{M}$ is $\sigma$-finite that if $E_a = \{ r \in (0, \infty) : \hat{\pi}(\chi_{S(a, r)}) = \hat{\rho}(\chi_{S(a, r)}) = 0 \}$, then $(0, \infty) \setminus E_a$ is countable.

We can assume that $\text{diam}(X) < 1$ and we can let $\mathcal{F}_1 = \{ X \}$.

Suppose $n \in \mathbb{N}$ and $\mathcal{F}_n$ has been defined.

For each $a \in X$, there is an $r_a \in E_a \cap \left(0, \frac{1}{2(n+1)}\right)$. Since $X$ is compact and $\{ \text{ball}(a, r_a) : a \in X \}$ is an open cover with sets in $\Sigma$, there is a finite subcover $\{ U_1, \ldots, U_s \}$. We let $V_1 = U_1$, and $V_k = U_k \setminus \bigcup_{1 \leq j < k} U_j$ for $1 < k \leq s$. Then $\{ V_1, \ldots, V_s \}$ is a disjoint family of open sets in $\Sigma$ with union $V$ such that

$$\hat{\pi}(\chi_V) = \hat{\rho}(\chi_V) = 1.$$ 

We now let

$$\mathcal{F}_{n+1} = \{ V_j \cap W : 1 \leq j \leq s, W \in \mathcal{F}_n, V_j \cap W \neq \emptyset \}.$$ 

If $U \subset X$ is open and nonempty, then there is a continuous $f : X \to [0, 1]$ such that $f(x) = 0$ if and only if $x \in X \setminus U$. Thus the sequence $f^{1/n} \uparrow \chi_U$, which means

$$f^{1/n} \to \chi_U$$

weak* in $C(X)^{\#\#}$. Thus $\pi(f)^{1/n} \uparrow \hat{\pi}(\chi_U)$ and $\rho(f)^{1/n} \uparrow \hat{\rho}(\chi_U)$ in the weak* topology. Thus $\hat{\pi}(\chi_U)$ is the projection onto the closure of the range of $\pi(f)$ and $\hat{\rho}(\chi_U)$ is the projection onto the closure of the range of $\rho(f)$. It follows from Corollary 2 that $\hat{\pi}(\chi_U)$ and $\hat{\rho}(\chi_U)$ are Murray von Neumann equivalent.  

\end{proof}

\begin{theorem}
Suppose $\mathcal{A}$ is a separable unital commutative C*-algebra and $\mathcal{M}$ is a von Neumann algebra. If $\pi \in \text{Rep}(\mathcal{A}, \mathcal{M})$ then $\mathcal{U}_\mathcal{M}(\pi)^{-}$ is path-connected. In fact, for every $\rho \in \mathcal{U}_\mathcal{M}(\pi)^{-}$ there is a strong internal path from $\pi$ to $\rho$.

\begin{proof}
Suppose $\rho \in \mathcal{U}_\mathcal{M}(\pi)^{-}$. Since $\mathcal{A}$ is separable, there is a sequence $\{ U_n \} \in \mathcal{U}_\mathcal{M}$ such that, for every $a \in \mathcal{A}$,

$$\lim_{n \to \infty} \| U_n^{*} \pi(a) U_n - \rho(a) \| = 0.$$ 

Let $\mathcal{N} = W^* (\pi(\mathcal{A}) \cup \rho(\mathcal{A}) \cup \{ U_1, U_2, \ldots \})$. Then $\mathcal{N}$ is a countably generated von Neumann algebra, and $\pi, \rho : \mathcal{A} \to \mathcal{N}$. Hence we can write

$$\mathcal{N} = \bigoplus_{i \in I} \mathcal{N}_i,$$

where each $\mathcal{N}_i$ acts on a separable Hilbert space, and we can write

$$\pi = \bigoplus_{i \in I} \pi_i$$

and

$$\rho = \bigoplus_{i \in I} \rho_i.$$
We also have
\[ \hat{\pi} = \bigoplus_{i \in I} \pi_i \text{ and } \hat{\rho} = \bigoplus_{i \in I} \rho_i. \]

For each \( i \in I \), we can choose a sequence \( F_{n,i} \) of families of nonempty open subsets as in Lemma 6. Since, for each \( i \in I \) and each \( n \in \mathbb{N} \) and each \( E \in F_{n,i} \) we know \( \hat{\pi}_i (\chi_E) \) and \( \hat{\rho}_i (\chi_E) \) are Murray von Neumann equivalent in \( N_i \) and since
\[ \sum_{E \in F_n} \hat{\pi}_i (\chi_E) = \sum_{E \in F_n} \hat{\rho}_i (\chi_E) = 1, \]
there is a unitary \( U_{n,i} \in N_i \) such that
\[ U_{n,i}^* \hat{\pi}_i (\chi_E) U_{n,i} = \hat{\rho}_i (\chi_E) \]
for every \( E \in F_{n,i} \). For each \( n \in \mathbb{N} \), let \( U_n = \bigoplus_{i \in I} U_{n,i} \) for each \( i \in I \), and let \( D_n = \sum_{i \in I} \) sup\{\( \hat{\pi}_i (\chi_E) : E \in F_{n,i} \)\}. Since \( U_n U_{n+1}^\prime \in D_n \), we know from the proof of Lemma 3 that the map \( n \mapsto U_n \) on \( \mathbb{N} \) extends to a continuous map \( t \mapsto U_t = \sum_{i \in I} U_{t,i} \) such that \( U_0 = 1 \), and such that, for every \( n \in \mathbb{N} \), for every \( i \in I \), every \( n \leq t < \infty \), and every \( E \in F_{n,i} \),
\[ U_{t,i}^* \hat{\pi}_i (\chi_E) U_{t,i} = U_{n,i}^* \hat{\pi}_i (\chi_E) U_{n,i} = \hat{\rho}_i (\chi_E). \]
Suppose \( f \in C (X) \) and \( \varepsilon > 0 \). Since \( f \) is uniformly continuous, there is a positive integer \( n_0 \) such that, if \( x, y \in X \) and \( d(x, y) < 1/n_0 \), then \( |f(x) - f(y)| < \varepsilon/2 \).

For each \( i \in I \) and all \( E \in F_{n_0,i} \) we choose \( x_{n_0,i,E} \in E \). Since diam\( (E) < 1/n_0 \), we then have
\[ \|f - f(x_{n_0,i,E})\chi_E\| < \varepsilon/2, \]
so
\[ \left\| \pi_i (f) - \sum_{E \in F_{n_0,i}} f(x_{n_0,i,E}) \hat{\pi}_i (\chi_E) \right\| \leq \varepsilon/2, \]
and
\[ \left\| \rho_i (f) - \sum_{E \in F_{n_0,i}} f(x_{n_0,i,E}) \hat{\rho}_i (\chi_E) \right\| \leq \varepsilon/2. \]
Thus, for \( t \geq n_0 \), we have
\[ \|U_t^* \pi (f) U_t - \rho (f)\| = \sup_{i \in I} \|U_{t,i}^* \pi_i (f) U_{t,i} - \rho_i (f)\| \leq \]
\[ \sup_{i \in I} \left\| U_{t,i}^* \pi_i (f) - \sum_{E \in F_{n_0,i}} f(x_{n_0,i,E}) \hat{\pi}_i (\chi_E) \right\| U_{t,i}, \]
\[ + \sup_{i \in I} \left\| \sum_{E \in F_{n_0,i}} f(x_{n_0,i,E}) \left[ U_{t,i}^* \hat{\pi}_i (\chi_E) U_{t,i} - \hat{\rho}_i (\chi_E) \right] \right\|, \]
\[ + \sup_{i \in I} \left\| \sum_{E \in F_{n_0,i}} f(x_{n_0,i,E}) \hat{\rho}_i (\chi_E) - \rho_i (f) \right\| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon. \]
Thus, the map \( t \mapsto U_t \) is continuous on \([1, \infty)\), and, for every \( f \in C(X)\),
\[
\lim_{t \to \infty} \| U_t \pi(f) U_t^* - \rho(f) \| = 0.
\]

\[ \square \]

**Corollary 3.** Suppose \( \mathcal{A} \) is a separable unital homogeneous \( C^* \)-algebra and \( \mathcal{M} \) is a von Neumann algebra. If \( \pi \in \text{Rep}(\mathcal{A}, \mathcal{M}) \) then \( U_{\mathcal{G}}(\pi)^{-} \) is path-connected. In fact, for every \( \rho \in U_{\mathcal{M}}(\pi)^{-} \) there is a strong internal path from \( \pi \) to \( \rho \).

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