CATEGORIES OF SYMPLECTIC TORIC MANIFOLDS AS PICARD STACK TORSORS

EUGENE LERMAN

Abstract. We outline a proof that the stack of symplectic toric G-manifolds over a fixed orbit space W is a torsor for the stack of symplectic toric G-principal bundles over W.

Contents

1. Introduction 1
   Acknowledgments 2
2. Definitions, notation, conventions 2
3. A multiplication on the stack of symplectic toric principal G-bundles. 3
4. An action of the stack of symplectic toric principal bundles on the symplectic toric manifolds 6
References 9

1. Introduction

In an influential paper [D] Delzant classified compact symplectic toric manifolds. Recently, using some ideas from [L], Karshon and I [KL] extended the classification to non-compact symplectic toric manifolds: If \((M, \omega)\) is a symplectic manifold with a completely integrable action of a torus \(G\) and an associated moment map \(\mu : M \to g^*\), then \(W = M/G\) is naturally a manifold with corners [DH]. Furthermore the orbital moment map \(\bar{\mu} : W \to g^*\) induced by \(\mu\) is locally an embedding that maps the corners of \(W\) to unimodular cones in \(g^*\). That is, \(\bar{\mu}\) is a unimodular local embedding.

It was easy to classify the isomorphism classes of symplectic toric manifolds \((M', \omega', \mu' : M \to g^*)\) with orbit space \(W\) and orbital moment map \(\bar{\mu}' = \bar{\mu}\). These classes are in bijective correspondence with degree 2-cohomology classes of \(W\) with coefficients in \(Z_G \times \mathbb{R}\), where \(Z_G\) denotes the integral lattice of the torus \(G\).

However, showing that given a unimodular local embedding \(\psi : W \to g^*\) there exists a symplectic toric manifold \((M, \omega, \mu)\) with \(M/G = W\) and \(\bar{\mu} = \psi\) turned out to be hard. We dealt with this problem by defining the category \(\text{STB}_\psi(W)\) of symplectic toric \(G\)-principal bundles (with corners) over \(W\) and constructing a functor

\[
\mathbf{c} : \text{STB}_\psi(W) \to \text{STM}_\psi(W)
\]

(\(\mathbf{c}\) for “collapse” or “cut”), where \(\text{STM}_\psi(W)\) denotes the category of symplectic toric manifolds with orbit space \(W\) and orbital moment map \(\psi\). Since \(\text{STB}_\psi(W)\) contains the pullback of the symplectic toric principal \(G\)-bundle \(T^*G \to g^*\) by \(\psi : W \to g^*\), this proved the \(\text{STM}_\psi(W)\) was non-empty as well. In this paper I clarify the relation between the two categories, the categories of toric principal bundles with corners and of toric manifolds. The idea is to use the language of stacks. Observe that

Key words and phrases. symplectic toric manifold, Picard stack.

This research is partially supported by the National Science Foundation.

2

arXiv:0908.2783v2 [math.SG] 26 Aug 2009.
• \( \text{STB}_\psi(W) \) and \( \text{STM}_\psi(W) \) are groupoids.
• For every open set \( U \subset W \) we have groupoids \( \text{STB}_\psi(U) \) and \( \text{STM}_\psi(U) \); for an inclusion of two open sets \( V \hookrightarrow U \) we have obvious restriction functors \( |_V : \text{STB}_\psi(U) \to \text{STB}_\psi(V) \) and \( |_V : \text{STM}_\psi(U) \to \text{STM}_\psi(V) \). Thus we have two (strict!) presheaves \( \text{STB}_\psi \) and \( \text{STM}_\psi \) of groupoids on the category \( \mathcal{O}(W) \) of open subsets of \( W \): \( U \hookrightarrow \text{STB}_\psi(U) \) and \( U \hookrightarrow \text{STM}_\psi(U) \).
• The two presheaves \( \text{STB}_\psi \) and \( \text{STM}_\psi \) are actually stacks.

The main result of the paper is the following theorem.

1.1. Theorem. Let \( G \) be a torus and \( \psi : W \to g^* \) a unimodular local embedding. Then

• The stack \( \text{STB}_\psi \) of symplectic toric principal \( G \)-bundles over \( W \) is a Picard stack.
• There is an action of \( \text{STB}_\psi \) on the stack \( \text{STM}_\psi \) of symplectic toric manifolds over \( W \) making \( \text{STM}_\psi \) into a \( \text{STB}_\psi \)-torsor.
• In particular each choice of a global object of \( \text{STM}_\psi \) defines an isomorphism of stacks \( \text{STB}_\psi \) and \( \text{STM}_\psi \).

Acknowledgments. I thank Yael Karshon and Anton Malkin for our fruitful collaborations without which this paper won’t be possible.

2. Definitions, Notation, Conventions

Notation and conventions. Given a category \( A \) we write \( A \in A \) to indicate that \( A \) is an object of \( A \). Given a functor \( F : A \to B \) we will often describe it by only indicating what it does on objects.

A torus is a compact connected abelian Lie group. We denote the Lie algebra of a torus \( G \) by \( g \), the dual of the Lie algebra, \( \text{Hom}(g, \mathbb{R}) \), by \( g^* \) and the integral lattice, \( \ker(\exp : g \to G) \), by \( \mathbb{Z}_G \). When a torus \( G \) acts on a manifold \( M \), we denote the action of an element \( g \in G \) by \( m \mapsto g \cdot m \) and the vector field induced by a Lie algebra element \( \xi \in g \) by \( \xi_M \); by definition, \( \xi_M(m) = \frac{d}{dt}|_{t=0} \exp(t\xi) \cdot m \). We write the canonical pairing between \( g^* \) and \( g \) as \( \langle \cdot, \cdot \rangle \). Our convention for a moment map \( \mu : M \to g^* \) for a Hamiltonian action of a torus \( G \) on a symplectic manifold \( (M, \omega) \) is that it satisfies

\[
d(\mu, \xi) = -\omega(\xi_M, \cdot).
\]

The moment map \( \mu \) \( G \)-invariant; we call the induced map \( \bar{\mu} : M/G \to g^* \) the orbital moment map. We say that the triple \( (M, \omega, \mu : M \to g^* ) \) is a symplectic toric \( G \)-manifold if the action of \( G \) on \( M \) is effective and if

\[
\dim M = 2 \dim G.
\]

Given a principal \( G \)-bundle \( P \) we write the action of \( G \) on \( P \) as a left action. There is no problem with that since \( G \) is abelian.

We denote the positive orthant \( \{ x \in \mathbb{R}^k \mid x_i \geq 0, 1 \leq i \leq k \} \) by \( \mathbb{R}_+^k \).

2.1. Definition. Let \( g^* \) denote the dual of the Lie algebra of the torus \( G \). A unimodular cone in \( g^* \) is a subset \( C \) of \( g^* \) of the form

\[
C = \{ \eta \in g^* \mid \langle \eta - \epsilon, v_i \rangle \geq 0 \text{ for all } 1 \leq i \leq k \},
\]

where \( \epsilon \) is a point in \( g^* \) and \( \{ v_1, \ldots, v_k \} \) is a basis of the integral lattice of a subtorus \( K \) of \( G \).

2.3. The set \( C = g^* \) is a unimodular cone with \( k = 0 \), with \( \{ v_1, \ldots, v_k \} = \emptyset \), and with \( K \) the trivial subgroup \( \{ 1 \} \).

2.4. Definition. Let \( W \) be a manifold with corners and \( g^* \) the dual of the Lie algebra of a torus \( G \). A smooth map \( \psi : W \to g^* \) is a unimodular local embedding if for each point \( x \) in \( W \) there exists a unimodular cone \( C \subset g^* \) and open sets \( T \subset W \) and \( U \subset g^* \) such that \( \psi(T) = C \cap U \) and such that \( \psi|_T : T \to C \cap U \) is a diffeomorphism of manifolds with corners.
The definition is justified by

2.5. **Proposition.** Let \((M, \omega, \mu : M \to \mathfrak{g}^*)\) be a symplectic toric \(G\)-manifold. Then the orbit space \(M/G\) is a manifold with corners, and the orbital moment map \(\tilde{\mu} : M/G \to \mathfrak{g}^*\) is a unimodular local embedding.

**Proof.** See [KL]. \(\square\)

We now introduce the category of symplectic toric manifolds over a unimodular local embedding:

2.6. **Definition** (The category \(\text{STM}_\psi(W)\) of symplectic toric \(G\)-manifolds over \(\psi : W \to \mathfrak{g}^*\)). Let \(\psi : W \to \mathfrak{g}^*\) be a unimodular local embedding of a manifold with corners \(W\) into the dual of the Lie algebra of a torus \(G\).

An object of the category \(\text{STM}_\psi(W)\) is a pair \(((M, \omega, \mu : M \to \mathfrak{g}^*), \varpi : M \to W)\), where \((M, \omega, \mu : M \to \mathfrak{g}^*)\) is a symplectic toric \(G\) manifold and \(\varpi : M \to W\) is a quotient map for the action of \(G\) on \(M\) with \(\mu = \psi \circ \pi\).

A morphism \(\varphi\) from \(((M, \omega, \mu : M \to \mathfrak{g}^*), \varpi : M \to W)\) to \(((M', \omega', \mu' : M' \to \mathfrak{g}^*), \varpi' : M' \to W)\) is a \(G\)-equivariant symplectomorphism \(\varphi : M \to M'\) such that \(\varpi' \circ \varphi = \varpi\).

2.7. **Remark.** We may informally write \(\varpi : M \to W\) or even \(M\) for an object of \(\text{STM}_\psi(W)\) and \(\varphi : M \to M'\) for a morphism between two objects.

2.8. **Definition** (The category \(\text{STB}_\psi(W)\) of symplectic toric \(G\)-principal bundles over \(\psi : W \to \mathfrak{g}^*\)). An object of \(\text{STB}_\psi(W)\) is a principal \(G\)-bundle \(P \to W\) equipped with a symplectic form \(\sigma\) and a moment map \(\mu : P \to \mathfrak{g}^*\) such that \(\mu = \psi \circ \pi\). The morphisms in this category are \(G\)-equivariant symplectomorphisms that commute with the maps to \(W\).

2.9. **Remark.** The standard lifted action of a torus \(G\) on its cotangent bundle \(T^*G\) makes \(T^*G\) into a symplectic toric \(G\)-bundle over the identity map \(\text{id} : \mathfrak{g}^* \to \mathfrak{g}^*\). If \(\psi : W \to \mathfrak{g}^*\) is a unimodular local embedding, the pullback of \(T^*G \to \mathfrak{g}^*\) by \(\psi\) gives a symplectic toric \(G\) bundle over \(\psi : W \to \mathfrak{g}^*\). Thus, for any unimodular local embedding \(\psi : W \to \mathfrak{g}^*\), the category \(\text{STB}(\psi : W \to \mathfrak{g}^*)\) is non-empty.

2.10. **Remark.** Suppose \(\pi : (P, \sigma) \to W\) is a symplectic toric \(G\)-bundle. Then it is easy to check that for any connection 1-form \(A\) on \(P \to W\) the 2-form

\[
\sigma - d(\psi \circ \pi, A)
\]

is basic. (Recall that \(\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}\) denotes the canonical pairing.) Hence any symplectic 2-form \(\sigma\) on the symplectic toric \(G\)-principal bundle is of the form

\[
(2.11) \quad \sigma = d(\psi \circ \pi, A) + \pi^* \beta
\]

for some connection 1-form \(A\) on \(P\) and a closed 2-form \(\beta\) on \(W\). Conversely, since \(\psi\) is locally an embedding, the 2-form \(d(\psi \circ \pi, A)\) is non-degenerate for any connection 1-form \(A\). Hence all symplectic \(G\)-invariant forms on \(P\) so that \(\psi \circ \pi\) is the corresponding moment map has to be of the form \(2.11\).

3. **A multiplication on the stack of symplectic toric principal \(G\)-bundles.**

Fix a torus \(G\) and a unimodular local embedding \(\psi : W \to \mathfrak{g}^*\) of a manifold with corners \(W\) into the dual of the Lie algebra of \(G\). The following observation must be well known to experts. If \(\pi : P \to W, \pi' : P' \to W\) are two principal \(G\)-bundles over a manifold with corners \(W\), then their fiber product \(P \times_W P'\) is a \(G \times G\)-principal bundle over \(W\). Dividing out by the action of \(G\) given by

\[
g \cdot (p, p') = (g \cdot p, g^{-1} \cdot p')
\]
produces a principal $G$-bundle
\[ P \otimes P' := (P \times_W P')/G \]
over $W$: the induced $G$-action on $P \otimes P'$ is given by
\[ a \cdot [p,p'] = [a \cdot p, a \cdot p'], \]
where $[p,p']$ denotes the orbit of $(p,p') \in P \times W P'$ in $P \otimes P'$. Naturally if $P$ and $P'$ are principal bundles over an open subset $U \subset W$ rather than the whole of $W$, then so is $P \otimes P'$. One can show that the product $\otimes$ turns the gerbe $BG$ over the site $\text{Op}(W)$ of open subsets of $W$ into a Picard stack.

3.1. **Proposition.** For any open subset $U$ of $W$ and any two symplectic toric principal $G$-bundles $\pi : (P, \sigma) \to U$, $\pi' : (P', \sigma') \to U$ over $U$ the tensor product $P \otimes P' \to U$ is naturally a symplectic toric principal $G$-bundle.

**Proof.** We claim that the restriction of the 2-form $\sigma + \sigma' \in \Omega^2(P \times P')$ to $P \times_W P'$ is basic with respect to the action of $G$ given, as above, by $g \cdot (p,p') = (g \cdot p, g^{-1} \cdot p')$ and descends to a non-degenerate form on $P \otimes P'$. Since this is a local claim, we may assume that $\psi|_U$ is an embedding or, better yet, that $U \subset g^\times$. Then the moment map $\mu$ for the action of $G$ on $(P \times P', \sigma + \sigma')$ is
\[ \mu = \pi - \pi'. \]
Hence
\[ \mu^{-1}(0) = \{(p,p') \in P \times P' | \pi(p) = \pi'(p')\} = P \times_U P' \]
Since the closed 2-form $(\sigma + \sigma')|_{\mu^{-1}(0)}$ is $G$-basic and is degenerate precisely in the directions of $G$-orbits, it descends to a symplectic form $\sigma \otimes \sigma'$ on $P \otimes P'$. It is easy to check that the remaining $G = (G \times G)/G$-action on $P \otimes P'$ makes $(P \otimes P', \sigma \otimes \sigma')$ into a symplectic toric $G$-bundle over $U$. \hfill \Box

3.2. **Notation.** We write
\[ (P, \sigma) \otimes (P', \sigma') := (P \otimes P', \sigma \otimes \sigma'). \]

3.3. It is easy to extend $\otimes$ to arrows: if $f : (P_1, \sigma_1) \to (P'_1, \sigma'_1)$ and $g : (P_2, \sigma_2) \to (P'_2, \sigma'_2)$ are two arrows in $\text{STB}_\psi(U)$, then $(f \otimes g) : (P_1 \times P_2, \sigma_1 + \sigma_2) \to (P'_1 \times P'_2, \sigma'_1 + \sigma'_2)$ descends to a map $f \otimes g : (P_1 \otimes P_2, \sigma_1 \otimes \sigma_2) \to (P'_1 \otimes P'_2, \sigma'_1 \otimes \sigma'_2)$ of symplectic toric $G$-principal bundles over $U$ for any $U \in \text{Op}(W)$.

3.4. It is tedious to check that the multiplication $\otimes$ defined above gives rise to a structure of the Picard stack $\text{STB}_\psi$ of symplectic toric principal $G$-bundles over $\text{Op}(W)$.

For example, the canonical isomorphism $(P \times P', \sigma + \sigma') \to (P' \times P, \sigma' + \sigma)$ descends to a natural isomorphism
\[ \tau_{P,P'} : (P, \sigma) \otimes (P', \sigma') \to (P', \sigma') \otimes (P, \sigma). \]
Similarly, the canonical isomorphism $(P_1 \times (P_2 \times P_3), \sigma_1 + (\sigma_2 + \sigma_3)) \to ((P_1 \times P_2) \times P_3, (\sigma_1 + \sigma_2) + \sigma_3)$ together with a version of reduction in stages produces a natural isomorphism
\[ \theta_{P_1,P_2,P_3} : (P_1, \sigma_1) \otimes ((P_2, \sigma_2) \otimes (P_3, \sigma_3)) \to ((P_1, \sigma_1) \otimes (P_2, \sigma_2)) \otimes (P_3, \sigma_3) \]
and so on. The only possibly non-trivial claim is that for any open set $U \in \text{Op}(W)$ and any symplectic toric principal $G$-bundle $(P, \sigma) \in \text{STB}_\psi(U)$, the functor $(-) \otimes (P, \sigma) : \text{STB}_\psi(U) \to \text{STB}_\psi(U)$ of multiplication by $(P, \sigma)$ is an equivalence of categories. The proof of this claim is the same, mutatis mutandis, as the proof of Theorem 4.6 below. We omit it.

3.5. **Proposition.** Let $\pi : (P, \sigma) \to U$ be an object of $\text{STB}_\psi(U)$. A Lagrangian section $s : U \to P$, if it exists, defines a natural isomorphism $s : \text{id}_U \Rightarrow (-) \otimes (P, \sigma)$ between the identity functor $\text{id}_U$ on $\text{STB}_\psi(U)$ and the functor $(-) \otimes (P, \sigma)$ of multiplication by $(P, \sigma)$. 

4
Lemma. The proof for symplectic toric principal bundles is the same.

Proof. Let \( f : (Q_1, \tau_1) \to (Q_2, \tau_2) \) be an arrow in \( \text{STB}_\psi(U) \). Consider the map \( s_{Q_i} : (Q_i, \tau_i) \to (Q_i, \tau_i) \otimes (P, \sigma), i = 1, 2, \) is given by

\[
s_{Q_i}(q) = [q, s(\pi_i(q))]
\]

where, as before, \([q, p] \in (Q_i \times_U P)/G\) the orbit of \((q, p) \in Q_i \times_U P\) and \(\pi_i : Q_i \to U\) denotes the projection. It is clearly a map of principal \(G\)-bundles. Moreover, since \(s\) is Lagrangian, \(s_{Q_i}\) is symplectic. Recall that \((f \otimes id_P)([q, p]) = [f(q), p]\) by definition. Hence

\[
s_{Q_2} \circ f(q) = [f(q), s(\pi_2(f(q))].
\]

Now, \(\pi_2(f(q)) = \pi_1(q)\). Therefore

\[
(f \otimes id_P) \circ s_{Q_1}(q) = [f(q), s(\pi_1(q))] = s_{Q_2} \circ f(q).
\]

\(\square\)

3.6. Remark. Note that if \(f : Q_1 \to Q_2\) is a map of principal \(G\)-bundles (not necessarily symplectic!), then \(f \otimes id_P : Q_1 \otimes P \to Q_2 \otimes P\) is a map of principal bundles, and we still have

\[ (f \otimes id_P) \circ s_{Q_1} = s_{Q_2} \circ f. \]

3.8. Note the global neutral object of \((\text{STB}_\psi, \otimes, \tau, \theta)\) is the pullback \(W \times_{\mathfrak{g}^*} T^*G\) of the canonical symplectic toric principal \(G\)-bundle \(T^*G \to \mathfrak{g}^*\) by the map \(\psi : W \to \mathfrak{g}^*\).

We end the section with two technical observations that we will need later on.

3.9. Lemma. If \(V, U\) are two open sets in \(W\) and \(V \subset U\) is dense in \(U\), then the restriction functors \(|_V : \text{STM}_\psi(U) \to \text{STM}_\psi(V)\) and \(|_V : \text{STB}_\psi(U) \to \text{STB}_\psi(V)\) are faithful.

Proof. If \(V \subset U\) is dense, then for any \(M \in \text{STM}_\psi(U)\) the restriction \(M|_V\) is dense in \(M\). Therefore, for any two arrows \(f, g : M \to M'\) in \(\text{STM}_\psi(U)\)

\[
f|_M|_V = g|_M|_V \implies f = g.
\]

The proof for symplectic toric principal bundles is the same. \(\square\)

3.10. Lemma. Suppose an open subset \(U\) of \(W\) is contractible. Then

(3.10.i) Any object of \(\text{STB}_\psi(U)\) are isomorphic. Consequently the stack \(\text{STB}_\psi\) is a gerbe.

(3.10.ii) Any object of \(\text{STB}_\psi(U)\) has a Lagrangian section.

Proof. Since \(U\) is contractible all principal \(G\)-bundles over \(U\) are trivial, hence isomorphic. Therefore, given two objects \(\pi : (P, \sigma) \to U, \pi' : (P', \sigma') \to U\) of \(\text{STB}_\psi(U)\) we may assume \(P = P'\). By remark 2.10 \(\sigma = d(\psi \circ \pi, A) + \pi^*\beta\) and \(\sigma' = d(\psi \circ \pi, A) + \pi'^*\beta'\) for a connection 1-form \(A\) on \(P\) and closed 2-forms \(\beta, \beta'\) on \(U\). By Poincare lemma \(\beta - \beta' = d\gamma\) for some 1-form \(\gamma\) on \(U\). Hence

\[
\sigma' = \sigma + \pi^*(d\gamma).
\]

By [KL] Lemma 5.10 there exists a gauge transformation \(\phi : P \to P\) with \(\phi^*\sigma' = \sigma\). This proves 3.10.i.

By 3.10.i we may assume \((P, \sigma) = (U \times G, d(\psi \circ \pi, g^{-1}dg))\) where \(g^{-1}dg\) denotes the Maurer-Cartan form on the torus \(G\). Then \(s : U \to U \times G\) given by \(s(x) = (x, 1)\) is a desired Lagrangian section. \(\square\)
4. An action of the stack of symplectic toric principal bundles on the symplectic toric manifolds

Next we construct an action of $\text{STB}_\psi$ on $\text{STM}_\psi$. We first observe that

4.1. **Lemma.** For any symplectic toric principal $G$-bundle $(\pi : (P, \sigma) \to U) \in \text{STB}_\psi(U)$ and any symplectic toric $G$-manifold $(\varpi : (M, \omega) \to U) \in \text{STM}_\psi(U)$ over $U$, the fiber product

$$P \times_U M = \{(p, m) \mid \pi(p) = \varpi(m)\}$$

is a manifold with a free action of $G$ given by $g \cdot (p, m) = (g \cdot p, g^{-1} \cdot m)$.

Moreover, the form $\sigma + \omega|_{P \times_U M}$ descends to a $G$-invariant symplectic form $\sigma \ast \omega$ on the quotient

$$P \ast M := (P \times_U M)/G$$

making $(P \ast M, \sigma \ast \omega) \to U$ into a symplectic $G$-toric manifold over $U$.

**Proof.** This is similar to the proof of 3.1. The difference is that it is not completely obvious that the fiber product $P \times_U M$ is a manifold (without corners), since $P$ is a manifold with corners.

It is no loss of generality to assume that $G = \mathbb{R}^n/\mathbb{Z}^n$, $U \subset \mathfrak{g}^* \cong \mathbb{R}^n$ is of the form $U = \mathbb{R}_+^k \times \mathbb{R}^\ell$ for some $k, \ell$ with $k + \ell = n$, $P = U \times G$ with $\pi : U \times G \to U$ given by $\pi(p, q) = p$ and $M = C^k \times \mathbb{R}^\ell \times \mathbb{R}^\ell/\mathbb{Z}^\ell$ with $\varpi : M \to U$ given by $\varpi(z, \eta, \theta) = (|z|^2, \ldots, |z_k|^2, \eta_1, \ldots, \eta_\ell)$. Then

$$P \times_U M = \{(p, q, z, \eta, \theta) \in U \times G \times C^k \times \mathbb{R}^\ell \times \mathbb{R}^\ell/\mathbb{Z}^\ell \mid (p_1, \ldots, p_n) = (|z_1|^2, \ldots, |z_k|^2, \eta_1, \ldots, \eta_\ell)\}$$

is the image of the map $\nu : G \times M \to P \times M = U \times G \times M$ given by

$$\nu(g, z, \eta, \theta) = ((|z_1|^2, \ldots, |z_k|^2, \eta_1, \ldots, \eta_\ell), g(z, \eta, \theta)).$$

Since for any coordinate function $f : U \times G \times M \to \mathbb{R}$ the composite $f \circ \nu$ is smooth, $\nu$ is smooth. And, in fact, $P \times_U M$ is the product of the graph of a smooth map

$$\lambda : C^k \to \mathbb{R}_+^k, \quad \lambda(z) = (|z_1|^2, \ldots, |z_k|^2)$$

with $G \times \mathbb{R}^\ell \times (\mathbb{R}^\ell/\mathbb{Z}^\ell)$. Hence $\nu$ is an embedding of a manifold without corners into a manifold with corners. This proves that $P \times_U M$ is a manifold without corners.

The rest of the argument is the same as in Proposition 3.1. \(\square\)

4.2. It is easy to see that if $\varpi : (M, \omega) \to W$ is a symplectic toric manifold over $\psi : W \to \mathfrak{g}^*$ and $\pi : (P, \sigma) \to U$ is a symplectic principal $G$ bundle over an open subset $U \hookrightarrow W$ of $W$, then the fiber product $P \times_W M$ is a manifold and $P \ast M := (P \times_W M)/G$ is naturally a symplectic toric manifold over $U$. Indeed it is $(P, \sigma) \ast ((M, \omega)|_U)$. We will denote it, by a slight abuse of notation, by $(P, \sigma) \ast (M, \omega) = (P \ast M, \sigma \ast \omega)$.

4.3. The map $\ast$ extends to arrows. The pairs of arrows $f : (P, \sigma) \to (P', \sigma')$ and $g : (M, \omega) \to (M', \omega')$ in $	ext{STB}_\psi(U)$ and $	ext{STM}_\psi(U)$ respectively define a $G \times G$ equivariant symplectomorphism $(f, g) : (P \times M, \sigma + \omega) \to (P' \times M', \sigma' + \omega')$ which maps $P \times_U M$ to $P' \times_U M'$. Hence it induces an arrow $f \ast g : (P, \sigma) \ast (M, \omega) \to (P', \sigma') \ast (M', \omega')$ in $	ext{STB}_\psi(U)$.

We leave it to the reader to check that

$$\ast : \text{STB}_\psi(U) \times \text{STM}_\psi(U) \to \text{STM}_\psi(U)$$

is a functor. It is also not hard to check that $\ast$ commutes with restrictions: for any $V, U$ open subsets of $W$ with $V \subset U$ the diagram

$$\begin{array}{ccc}
\text{STB}_\psi(U) \times \text{STM}_\psi(U) & \xrightarrow{\ast} & \text{STM}_\psi(U) \\
\downarrow V & & \downarrow V \\
\text{STB}_\psi(V) \times \text{STM}_\psi(V) & \xrightarrow{\ast} & \text{STM}_\psi(V)
\end{array}$$

(4.4)
commutes. Thus
\[ * : \text{STB}_\psi \times \text{STM}_\psi \to \text{STM}_\psi \]
is a strict map of presheaves of groupoids.

We will not check all the details necessary to show that \( * \) defines an action of the Picard stack \( \text{STB}_\psi \) on the stack \( \text{STM}_\psi \). Our goal is to prove

4.5. **Theorem.** The action \( * \) of the Picard stack \( \text{STB}_\psi \) of symplectic toric principal \( G \)-bundles over a unimodular local embedding \( \psi : W \to g^* \) on the stack \( \text{STM}_\psi \) of symplectic toric manifold over \( \psi \) makes \( \text{STM}_\psi \) into an \( \text{STB}_\psi \)-torsor. That is, the functor
\[ a : \text{STB}_\psi \times \text{STM}_\psi \to \text{STM}_\psi \times \text{STM}_\psi, \quad a((P, \sigma), (M, \omega)) := ((M, \omega), (P, \sigma) * (M, \omega)) \]
is an isomorphism of stacks.

**Theorem 4.5** is an easy consequence of a seemingly weaker result, which is of an independent interest:

4.6. **Theorem.** For any symplectic toric manifold \( \varphi : (M, \omega) \to W \) over a unimodular local embedding \( \psi : W \to g^* \) the functor
\[ F_M : \text{STB}_\psi \to \text{STM}_\psi, \quad F_M((P, \sigma)) := (P, \sigma) * (M, \omega) \]
is an isomorphism of stacks.

**Proof of Theorem 4.5 assuming 4.6.** To show that \( a \) is essentially surjective, given \( (M, \omega), (M', \omega') \in \text{STM}_\psi(U) \) we need to find \( (P, \sigma) \in \text{STB}_\psi(U) \) so that
\[ (P, \sigma) * (M, \omega) \]
is isomorphic to \( (M', \omega') \).

By **4.6**, the functor \( F_M(U) : \text{STB}_\psi(U) \to \text{STM}(U) \), \( (P, \sigma) \mapsto (P, \sigma) * (M, \omega) \) is essentially surjective. Hence for any \( (M', \omega') \in \text{STM}_\psi(U) \) there is \( (P, \sigma) \in \text{STB}_\psi(U) \) so that **4.7** holds.

Next suppose \( (f, h), (g, k) : ((P, \sigma), (M, \omega)) \to ((P', \sigma'), (M', \omega')) \) are two arrows in \( \text{STB}_\psi(U) \times \text{STM}_\psi(U) \) with
\[ a(f, h) = a(g, k). \]

Then
\[ h = k \quad \text{and} \quad f * h = g * k. \]

Hence
\[ g * \text{id}_M = (g * k) \circ (\text{id}_P * h^{-1}) = (f * h) \circ (\text{id}_P * h^{-1}) = f * \text{id}_M. \]

By **4.6** again, \( g = f \). Hence \( a \) is faithful.

Finally we argue that \( a \) is full. Suppose we have an arrow \( (h, f) : ((M, \omega), (P * M, \sigma * \omega)) \to ((M', \omega'), (P' * M', \sigma' * \omega')) \) in \( \text{STM}_\psi(U) \times \text{STM}_\psi(U) \). We want to find an arrow \( \tilde{f} : (P, \sigma) \to (P', \sigma') \) in \( \text{STB}_\psi(U) \) with
\[ (h, f) = a(\tilde{f}, h) \equiv (h, \tilde{f} * h). \]

Consider \( (\text{id}_P * h^{-1}) \circ f : P * M \to P' * M \). By **4.6** there is an arrow \( \tilde{f} : (P, \sigma) \to (P', \sigma') \) with
\[ \tilde{f} * \text{id}_M = (\text{id}_P * h^{-1}) \circ f. \]

Then
\[ f = (\text{id}_P * h^{-1})^{-1} \circ (\tilde{f} * \text{id}_M) = (\text{id}_P * h) \circ (\tilde{f} * \text{id}_M) = \tilde{f} * h. \]

Thus \( a \) is full. \( \square \)

As a preparation for our proof of **4.6**, we prove a number of lemmas. But first,

4.8. **Notation.** Given a manifold with corners \( W \) we denote its interior by \( \overset{\circ}{W} \). For any open subset \( U \) of \( W \) we set
\[ \overset{\circ}{U} = U \cap \overset{\circ}{W}. \]
4.9. **Lemma.** Suppose $U \in \text{Op}(W)$ is such that $\tilde{U}$ is contractible. Then the functor 

$$F_M(U) : \text{STB}_\psi(U) \to \text{STM}_\psi(U), (f : (P, \sigma) \to (P', \sigma')) \mapsto f \ast \text{id}_M$$

is faithful.

**Proof.** Since $\tilde{U}$ is contractible, the symplectic toric principal $G$-bundle $M|_{\tilde{U}} \to \tilde{U}$ has a Lagrangian section by 3.10. By 3.5, $F_M(\tilde{U}) = F_M(U)|_{\tilde{U}} : \text{STB}_\psi(\tilde{U}) \to \text{STB}_\psi(\tilde{U}) = \text{STM}_\psi(\tilde{U})$ is isomorphic to the identity functor, hence is faithful. Since the diagram

$$\begin{array}{ccc}
\text{STB}_\psi(U) & \xrightarrow{F_M(U)} & \text{STM}_\psi(U) \\
\downarrow_{\tilde{U}} & & \downarrow_{\tilde{U}} \\
\text{STB}_\psi(\tilde{U}) & \xrightarrow{F_M(U)|_{\tilde{U}}} & \text{STM}_\psi(\tilde{U})
\end{array}$$

commutes and since the functors $|_{\tilde{U}}$ are faithful by 3.9, the functor $F_M(U)$ is faithful as well. \qed

4.10. **Lemma.** Let $U \in \text{Op}(W)$ be contractible. Suppose further that $\tilde{U}$ is contractible as well. Then the functor

$$F_M(U) : \text{STB}_\psi(U) \to \text{STM}_\psi(U)$$

is full.

**Proof.** Step 1. Let $f : P \ast M \to P \ast M$ be an arrow in $\text{STM}_\psi(U)$ (to streamline our notation we are no longer explicitly keeping track of the symplectic forms). We will argue that there is an arrow $\tilde{f} : P \to P'$ in $\text{STB}_\psi(U)$ with $\tilde{f} \ast \text{id}_M = f$. By [152] Theorem 3.1 there is a smooth map $h : U \to G$ with

$$f(x) = h(\varpi(x)) \cdot x$$

for all $x \in P \ast M$ (here, again, $\varpi : P \ast M \to U$ denotes the orbit map). Define a gauge transformation $\tilde{f} : P \to P$ of the principal $G$ bundle $\pi : P \to U$ by

$$\tilde{f}(p) := h(\pi(p)) \cdot p.$$

Then for any point $[p, m] \in P \ast M$ we have

$$(\tilde{f} \ast \text{id}_M)[p, m] = [h(\pi(p)) \cdot p, m] = h(\pi(p)) \cdot [p, m] = h(\varpi([p, m])) \cdot [p, m] = f([p, m]).$$

It remains to check that $\tilde{f}$ actually preserves the symplectic form on $P$. Since $P|_{\tilde{U}}$ is dense in $P$, it’s enough to check that $\tilde{f}|_{\tilde{U}} := \tilde{f}|_{P|_{\tilde{U}}}$ is symplectic. Since $\tilde{U}$ is contractible by assumption, the bundle $M|_{\tilde{U}} \to \tilde{U}$ has a Lagrangian section. By 3.5 and 3.6 we have a symplectic $G$-equivariant diffeomorphism $\alpha : P|_{\tilde{U}} \to (P \ast M)|_{\tilde{U}} = P|_{\tilde{U}} \ast M|_{\tilde{U}}$ so that the diagram

$$\begin{array}{ccc}
P|_{\tilde{U}} & \xrightarrow{\alpha} & P \ast M |_{\tilde{U}} \\
| & \downarrow f & | \\
| \tilde{f} |_{\tilde{U}} & \xrightarrow{\alpha} & P \ast M |_{\tilde{U}}
\end{array}$$

commutes (all maps are maps of principal $G$-bundles). But we also know that $\alpha$ and $f$ are symplectic. Hence $\tilde{f}|_{\tilde{U}}$ is symplectic. Therefore $\tilde{f}$ is symplectic.

**Step 2.** We now argue that $F_M(U) : \text{Hom}(P, P') \to \text{Hom}(P \ast M, P' \ast M)$ is onto for any pair of symplectic toric principal $G$-bundles $P, P'$ over $U$. Since $U$ is contractible, $P$ and $P'$ are isomorphic by 3.9. Let $\psi : P \to P'$ denote this isomorphism. Then for any $\phi \in \text{Hom}(P \ast M, P' \ast M)$, the map $(\psi \ast \text{id}_M) \circ \phi$ is in $\text{Hom}(P \ast M, P \ast M)$. By Step 1,

$$(\psi \ast \text{id}_M) \circ \phi = \tau \ast \text{id}_M$$
for some $\tau \in \text{Hom}(P, P)$. Hence

$$\phi = (\psi^{-1} \circ \tau) \ast \text{id}_M,$$

and we are done. \hfill \Box

4.11. Notation. Given an open cover $\{U_i\}$ we write $U_{ij}$ for the double intersection $U_i \cap U_j$ and $U_{ijk}$ for $U_i \cap U_j \cap U_k$.

4.12. Lemma. Suppose $\mathcal{A}$ and $\mathcal{B}$ are two stacks over the site $\text{Op}(W)$ of open subsets of a manifold with corners $W$ thought of as strict presheaves of groupoids and $F : \mathcal{A} \to \mathcal{B}$ a map of strict presheaves, so that for any pair $U'' \hookrightarrow U$ of open sets the diagram

$$
\begin{array}{ccc}
\mathcal{A}(U) & \xrightarrow{F(U)} & \mathcal{B}(U) \\
| & | & |
\mathcal{A}(U') & \xrightarrow{F(U')} & \mathcal{B}(U')
\end{array}
$$

commutes. If for any $U \in \text{Op}(W)$ there is a cover $\{U_i\}$ so that $F(U_i) : \mathcal{A}(U_i) \to \mathcal{B}(U_i)$, $F(U_{ij}) : \mathcal{A}(U_{ij}) \to \mathcal{B}(U_{ij})$ and $F(U_{ijk}) : \mathcal{A}(U_{ijk}) \to \mathcal{B}(U_{ijk})$ are all equivalences of categories for all $i, j, k$ then $F(U)$ is also an equivalence of categories. Hence $F : \mathcal{A} \to \mathcal{B}$ is an isomorphism of stacks.

Proof. Since $\mathcal{A}$ is a stack, for any $U \in \text{Op}(U)$ and any cover $\{U_i\}$ of $U$ the category $\mathcal{A}(U)$ is equivalent to the descent category $\text{Desc}((\{U_i\}, \mathcal{A})$ defined by the cover $\{U_i\}$. The conditions on $F$ guarantee that the induced functor

$$F : \text{Desc}((\{U_i\}, \mathcal{A}) \to \text{Desc}((\{U_i\}, \mathcal{B})$$

between the descent categories is an equivalence. It follows that $F(U)$ is an equivalence of categories. \hfill \Box

Proof of 4.6. We need to show that for any open subset $U$ of $W$, the functor

$$F_M(U) : \text{STB}_\psi(U) \to \text{STM}_\psi(U)$$

is an equivalence of categories.

If $U$ is contractible the cohomology $H^2(U, \mathbb{Z}_G \times \mathbb{R})$ is 0. Hence, by [KL, Theorem 1.8], all objects of $\text{STM}_\psi(U)$ are isomorphic. Therefore $F_M(U)$ is essentially surjective. Furthermore if $U$ is contractible, then $F_M(U)$ is faithful by 4.9. By 4.10 $F_M(U)$ is full.

We have seen (4.3) that $F_M$ strictly commutes with restrictions. Now choose an open cover $\{U_i\}$ of $W$ so that

- all sets $U_i$, $U_{ij} := U_i \cap U_j$ and $U_{ijk} := U_i \cap U_j \cap U_k$ are contractible and
- their interiors $\bar{U}_i$, $\bar{U}_{ij}$ and $\bar{U}_{ijk}$ are contractible as well.

This can be achieved, for example, by choosing a triangulation [Goresky, Johnson] of the manifold with corners $W$ compatible with its stratification into manifolds and using open stars of the triangulation as the elements of the cover. By 4.12 the functor $F_M$ is an isomorphism of stacks. \hfill \Box

References

[D] T. Delzant, Hamiltoniens périodiques et image convexe de l’application moment, Bull. Soc. Math. France 116 (1988), 315-339.

[DH] A. Douady and L. Hérault, Arrondissement des variétés à coins, Appendix to: A. Borel and J.-P. Serre, Corners and arithmetic groups, Comment. Math. Helv. 48 (1973), 436 491.

[Goresky] R.M. Goresky, Triangulation of stratified objects. Proc. Amer. Math. Soc. 72 (1978), no. 1, 193-200.

[HS] A. Haefliger and E. Salem, Actions of tori on orbifolds, Ann. Global Anal. Geom. 9 (1991), 37–59.

[Johnson] F.E.A. Johnson, On the triangulation of stratified sets and singular varieties. Trans. Amer. Math. Soc. 275 (1983), no. 1, 333–343.
[KL] Y. Karshon and E. Lerman, Non-compact symplectic toric manifolds. \url{arXiv:0907.2891v1 [math.SG]}.

[L] E. Lerman, Contact toric manifolds. \textit{J. Symplectic Geom.} Volume 1, Number 4 (2002), 785-828.

[Math] J.N. Mather, Stability of $C^\infty$ mappings. II. Infinitesimal stability implies stability. \textit{Ann. of Math. (2)} 89 (1969) 254–291.

Department of Mathematics, The University of Illinois at Urbana-Champaign, 1409 W. Green Street, Urbana, IL 61801, U.S.A.

\textit{E-mail address:} leman@math.uiuc.edu