**H∞ performance for load frequency control systems with random delays**

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**ABSTRACT**

This paper investigates the problem of \( H_\infty \) performance analysis for PI-type load frequency control (LFC) of power systems with random delays. By taking the probability distribution characteristic of communication delays into account in the LFC design, the power systems with a PI controller are modelled as stochastic time-delay systems. Furthermore, a delay-product-type augmented Lyapunov-Krasovskii functional (LKF) is constructed, and a new extended reciprocally convex matrix inequality combining Wirtinger-based integral inequality with convex combination approach is utilized to reduce the conservatism of main results. As a result, less conservative \( H_\infty \) performance criteria are derived, which guarantee the asymptotically stable in the mean-square of the considered systems. Numerical examples are also provided to illustrate the superiority of our proposed methods.

**1. Introduction**

Load frequency control (LFC) has been widely utilized in large interconnected power systems with multiple control areas, which is one of the major measures to maintain the balance between the load and generation in a specified control area (Sharma et al., 2019; Wen et al., 2016; Yan & Xu, 2019). It should be noted that, dedicated communication channels have been made use of to transmit control signals between remote terminal units (RTUs) and a control centre in traditional centralized LFC schemes, the problems due to communication delays have been ignored by most previous research work (Fu et al., 2020; Jiang et al., 2012; Xiong et al., 2018; C. K. Zhang et al., 2013). However, with the emergence of numerous private networks and the employment of open communication networks, some challenging problems have appeared on account of limited network bandwidth, such as communication delays and data losses, which exerts potential threats on the stable operation of power systems (Peng et al., 2018; Sargolzaei et al., 2016; Singh et al., 2016). As a matter of fact, for a given communication channel based on transmission control protocol/internet protocol, communication delay is often random, and varies in an interval (Peng & Zhang, 2016). Generally speaking, random delays are characterized by means of Bernoulli-distributed stochastic variable, and this type of random interval delays could occur with a high probability in one subinterval and the opposite probability of occurring in another subinterval (Jia et al., 2019). Consequently, the delay intervals and corresponding occurrence probability should be fully considered, which is significant to obtain less conservative results. However, the probability distribution characteristic of communication delays was rarely considered in most existing studies. Therefore, it is of great significance to study the influence of random delays on LFC systems.

To LFC systems with random delays and load disturbance, the \( H_\infty \) performance level and the upper bounds of time delay are two major factors to judge the conservatism of the derived criteria. In order to further reduce the conservatism, sustained efforts have been made mainly on two aspects, one is to construct an appropriate LKF, the other is to estimate the derivatives of the LKF more accurately, such as delay-partitioning approach (Ko et al., 2018), augmented LKF (W. I. Lee et al., 2018; Zeng et al., 2019), LKF with triple-integral and quadruple-integral terms (Qian, Li, Chen, et al., 2020), LKF with delay-product terms (Li et al., 2019; Qian, Xing, et al., 2020; C. F. Shen et al., 2020; C. K. Zhang, He, Jiang, Wang, et al., 2017), Jensen’s inequality (Qian, Li, Zhao, et al., 2020), Wirtinger-based integral inequality (Qian et al., 2019), free-matrix-based integral inequality (Zeng et al., 2015), auxiliary-function-based inequality (P. G. Park et al., 2015), Bessel-Legendre inequality (W. I. Lee et al., 2018; Seuret & Gouaisbaut, 2018) and reciprocally convex combination techniques in different forms (P. G. Park et al., 2011; C. K.
Zhang, He, Jiang, Wang, et al., 2017; C. K. Zhang, He, Jiang, Wu, et al., 2017; R. M. Zhang et al., 2019). Furthermore, various $H_\infty$ performance criteria for LFC systems have been put forward and researches on this problem are still going on. For instance, in Wen et al. (2016), by integrating the communication delays and event triggered control in the formulated model, and utilizing free-weighting matrix approach, the $H_\infty$ performance criteria of LFC systems were derived. In Peng et al. (2018), an adaptive time-delay LFC model was developed, and reciprocally convex combination technique was applied in the derivation of main results, which can obtain improved $H_\infty$ performance criteria and reduce the number of decision variables. By introducing the single and double integral items in LKF construction, and employing Jensen’s inequality along with reciprocally convex combination approach, the delay-distribution-dependent $H_\infty$ performance and stability criteria were presented in Peng and Zhang (2016). In Cheng et al. (2020), by considering transmission delays and denial-of-service attacks in the LFC design, and employing piecewise LKFs together with novel analysis methods, sufficient conditions were developed with $H_\infty$ performance. In H. Zhang et al. (2020), a new model based on the area control error and time-varying delays was established, then an suitable LKF and extended Wirtinger’s inequality was used, which had better $H_\infty$ performance by the number of packets sent and average sampling period. By building an accurate model with a degree of packet losses and introducing an appropriate LKF, then exploiting Wirtinger-based inequality to estimate the integral terms, the desired $H_\infty$ performance index of multi-area LFC systems was attained in Peng et al. (2017). It should be noted that, there is still plenty of room in how to coordinate LKF construction with estimating techniques efficiently, which helps to get $H_\infty$ performance criteria with less conservative.

In order to estimate the infinitesimal operators of constructed LKF more accurately, the integral terms in single and double forms are separated precisely by using delay-partitioning method. Then the single integral terms are estimated by an extended reciprocally convex matrix inequality together with Wirtinger-based integral inequality, and the double integral items are estimated by Jensen’s inequality, by which the constructed LKF and the estimating methods fit together effectively to reduce the conservatism of the main results.

**Notation:** Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n \times m}$ denote the $n$-dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively. $P > 0 (< 0)$ means that $P$ is a positive (negative) definite matrix. $E[\cdot]$ is the mathematical expectation of $\cdot$. $A^T$ and $A^{-1}$ represent the transpose and the inverse of $A$. $I_{n \times n}$ and $0_{n \times n}$ stand for the $n \times n$ identity matrix and $n \times n$ zero matrix, respectively. $\ast$ in the matrix denotes the symmetric term. $\text{diag} (\cdots)$ denotes a block diagonal matrix, $\text{col} (x_1, x_2, \ldots, x_n) = [x_1^T \, x_2^T \, \cdots \, x_n^T]^T$ and $\text{Sym}(X) = X + X^T$.

### 2. Problem formulation and preliminaries

The schematic diagram of one-area delayed LFC systems with proportional-integral (PI) controller is presented in Figure 1, and its state-space equation is indicated as:

$$
\begin{align*}
\dot{x}(t) &= \tilde{A}x(t) + \tilde{B}u(t) + \tilde{B}_W\omega(t) \\
\dot{y}(t) &= \tilde{C}x(t)
\end{align*}
$$

(1)

where

$$
\tilde{x}(t) = \begin{bmatrix} \Delta f \\ \Delta P_m \\ \Delta P_d \end{bmatrix}, \quad \omega(t) = \Delta P_d,
$$

$$
\tilde{A} = \begin{bmatrix}
-D & \frac{1}{M} & 0 \\
0 & -\frac{1}{T_{ch}} & \frac{1}{T_{ch}} \\
-\frac{1}{RT_g} & 0 & -\frac{1}{T_g}
\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},
$$

$$
\tilde{B}_W = \begin{bmatrix} -\frac{1}{M} \\ 0 \\ 0 \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} \beta^T \\ 0 \\ 0 \end{bmatrix}
$$

and $\Delta f$, $\Delta P_m$, $\Delta P_d$, $\Delta P_g$ are the deviations of frequency, the turbine/generator mechanical output, generator valve position and the disturbance of load, respectively. $M$, $D$, $\tilde{R}$, $T_{ch}$, $T_g$ denote the moment of inertia of the generator, the generator damping constant, speed droop, time constant of the turbine and time constant of the governor, respectively.
As we all know, there is no net tie-line power exchange in single-area power systems. It can be seen from Figure 1 that, as the output of the system (1), the area control error (ACE) is defined as:

$$\tilde{y}(t) = ACE = \beta \Delta f$$  \hspace{1cm} (2)

where $\beta > 0$ is frequency bias factor. Moreover, ACE is also acted as the input of the designed controller, so the following PI-based controller can be designed:

$$u(t) = -K_P\text{ACE} - K_I \int ACE$$ \hspace{1cm} (3)

where $K_P$ and $K_I$ are proportional and integral gains, and $\int ACE$ is the integration of ACE.

As depicted in Figure 1, the communication delay from ACE to the PI-based controller (3) is defined by an exponential block $e^{-sh(t)}$. Denote

$$y(t) \overset{\Delta}{=} \begin{bmatrix} \tilde{y}(t) & \int \tilde{y}(t) \end{bmatrix}^T, \quad K \overset{\Delta}{=} [K_P \ K_I]$$ \hspace{1cm} (4)

the PI-based controller (3) can be further written as:

$$u(t) = -Ky(t - h(t))$$ \hspace{1cm} (5)

where $h(t)$ is a time-varying delay satisfying:

$$0 \leq h(t) \leq h, \quad h(t) \leq \mu$$

where $h$ and $\mu < 1$ are constants.

In this paper, the information about the probability distribution of time-varying delay $h(t)$ is employed.

**Assumption 2.1:** To describe the probability distribution of the time-varying delay $h(t)$, define two sets and functions by

$$\Omega_1 = \left\{ t : h(t) \in [0,h_0) \right\} \quad \text{and} \quad \Omega_2 = \left\{ t : h(t) \in [h_0,h] \right\}$$

$$h_1(t) = \begin{cases} h(t) & \text{for } t \in \Omega_1 \\ \bar{h}_1 & \text{for } t \in \Omega_2 \end{cases},$$

$$h_2(t) = \begin{cases} \bar{h}_2 & \text{for } t \in \Omega_1 \\ h(t) & \text{for } t \in \Omega_2 \end{cases}$$

$h_1(t) \leq \mu_1 < 1$ and $h_2(t) \leq \mu_2 < 1$

where $h_0 \in [0,h]$, $\bar{h}_1 \in [0,h_0)$, $\bar{h}_2 \in [h_0,h]$. Obviously, $\Omega_1 \cup \Omega_2 = \mathbb{R}^+$ and $\Omega_1 \cap \Omega_2 = \emptyset$. It is easy to know that $t \in \Omega_1$ means the event $h(t) \in [0,h_0)$ occurs and $t \in \Omega_2$ means the event $h(t) \in [h_0,h]$ occurs. Therefore, a stochastic variable $\alpha(t)$ can be defined as

$$\alpha(t) = \begin{cases} 1 & \text{for } t \in \Omega_1, \\ 0 & \text{for } t \in \Omega_2. \end{cases}$$

**Assumption 2.2:** The stochastic variable $\alpha(t)$ is a Bernoulli distributed sequence with

$$\begin{cases} \text{Prob} \{ \alpha(t) = 1 \} = \mathbb{E} [\alpha(t)] = \alpha_0 \\ \text{Prob} \{ \alpha(t) = 0 \} = 1 - \mathbb{E} [\alpha(t)] = 1 - \alpha_0 \end{cases}$$

where $0 \leq \alpha_0 \leq 1$ is a constant.

**Remark 2.1:** As is well known, the real power system is a system with high nonlinearity and time-varying characteristics. In practice, modern power systems usually require a wide area open communication network to transmit information concerned. The usage of these networks causes inevitable unreliable factors, such as time delays, packet losses, latent faults, and etc. Similarly, these nonlinear disturbances may occur randomly as a result of some environment reasons. Therefore, the stochastic variable $\alpha(t)$ is introduced in this paper to describe such randomly occurring phenomenon, which has universality and application prospect.

According to the above analysis, the following delay-distribution-dependent PI controller can be taken to replace the general form shown in (5):

$$u(t) = -\alpha(t)Ky(t - h_1(t)) - (1 - \alpha(t))Ky(t - h_2(t)) \hspace{1cm} (6)$$

Defining $x(t) = [\Delta f \ \Delta P_m \ \Delta P_v \ \int ACE]^T$ and substituting (6) into (1), it can be obtained:

$$\begin{cases} \dot{x}(t) = Ax(t) - \alpha(t)BKCx(t - h_1(t)) - (1 - \alpha(t))BKCx(t - h_2(t)) + B_\omega \omega(t) \\ BKCx(t - h_2(t)) + B_\omega \omega(t) \\ y(t) = Cx(t) \\ x(t) = \phi(t), \quad t \in [\bar{h},0] \end{cases} \hspace{1cm} (7)$$
where
\[
A = \begin{bmatrix}
\frac{1}{M} & \frac{1}{M} & 0 & 0 \\
0 & -\frac{1}{T_{ch}} & \frac{1}{T_{ch}} & 0 \\
\frac{1}{RT_g} & \beta & 0 & 0 \\
\beta & 0 & 0 & 0
\end{bmatrix}, \\
B = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix},
\]
\[
B_ω = \begin{bmatrix}
\frac{1}{M} \\
0 \\
0 \\
0
\end{bmatrix}, \\
C = \begin{bmatrix}
β & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\]
and \(x(t) \in \mathbb{R}^n\) is the state vector. The initial condition \(ϕ(t)\) denotes a vector-valued continuous function of \(t \in [-h, 0]\).

The main objective of this paper is to derive the less conservatism \(H_∞\) performance criteria, and guarantee system (7) is asymptotically mean-square stable. In order to obtain main results, the following definition and lemmas are required.

**Definition 2.1 (B. Shen et al., 2011):** Given a scalar \(γ > 0\), under the zero initial condition, the system (7) is said to be asymptotically mean-square stable with \(H_∞\) performance level \(γ\), if the following inequality holds
\[
\mathbb{E} \left\{ \int_0^∞ y^T(t) y(t) \, dt \right\} < γ^2 \int_0^∞ ω^T(t) ω(t) \, dt.
\]

**Lemma 2.1 (Qian et al., 2019):** For a positive definite matrix \(R > 0\), the following inequality holds for all continuously differential function \(ω(s)\) in \([α, β]\) \(→ \mathbb{R}^n\):
\[
\int_α^β ω^T(s) R ω(s) \, ds ≥ \frac{1}{β - α} \left( \int_α^β ω(s) \, ds \right)^T \\
× R \left( \int_α^β ω(s) \, ds \right) + \frac{3}{β - α} σ^T R σ
\]
where \(σ = \int_α^β ω(s) \, ds - \frac{β}{β - α} \int_α^β ω(s) \, ds \, dθ\).

**Lemma 2.2 (C. K. Zhang, He, Jiang, Wu, et al., 2017):** For a real scalar \(α \in (0, 1)\), symmetric matrices \(χ_1 > 0\), \(χ_2 > 0\), and any matrices \(S_1, S_2\), the following matrix inequality holds:
\[
\begin{bmatrix}
\frac{1}{α}χ_1 & 0 \\
0 & \frac{1}{1-α}χ_2
\end{bmatrix} \geq \begin{bmatrix}
χ_1 + (1-α) \bar{Σ}_1 & (1-α) S_1 + α S_2 \ \\
S_2 & α \bar{Σ}_2
\end{bmatrix}
\]
where \(\bar{Σ}_1 = χ_1 - S_2 χ_2^{-1} S_2^T, \bar{Σ}_2 = χ_2 - S_1^T χ_1^{-1} S_1\).

### 3. Main results

In this section, by constructing a novel delay-product augmented LKF and employing appropriate analytical methods, some improved \(H_∞\) performance and stability criteria for the considered systems are given. Some notations are shown to simplify the representation of the following parts:

\[
ψ_1(t) = \int_{t-h_1(t)}^{t} x(s) h_1(t) \, ds, \\
ψ_2(t) = \int_{t-h_0}^{t} x(s) h_0 \, ds, \\
ψ_3(t) = \int_{t-h_2(t)}^{t} x(s) h_2(t) \, ds, \\
ψ_4(t) = \int_{t-h_0}^{t} x(s) h_0 \, ds, \\
ψ_5(t) = \int_{t-h_1(t)}^{t} x(s) h_1(t) \, ds \, dθ, \\
ψ_6(t) = \int_{t-h_0}^{t} \int_{t-h_1(t)}^{t} x(s) h_0 h_1(t) \, ds \, dθ, \\
ψ_7(t) = \int_{t-h_2(t)}^{t} \int_{t-h_0}^{t} x(s) h_2(t) h_0 \, ds \, dθ, \\
ψ_8(t) = \int_{t-h_0}^{t} \int_{t-h_2(t)}^{t} x(s) h_0 h_2(t) \, ds \, dθ,
\]

\(ξ(t) = \text{col} (x(t), x(t-h_1(t)), x(t-h_0), x(t-h_2(t)), x(t-h), h_1(t), h_2(t), h_0, h_0, h_2(t), h_1(t), h_0)\).

**Theorem 3.1:** For some given positive scalars \(α_0, h_0, h, μ_1, μ_2, γ\) and a matrix \(K\), system (7) is asymptotically mean-square stable with \(H_∞\) performance \(γ\), if there exist symmetric positive definite matrices \(W_m, M_m \in \mathbb{R}^{2n \times 2n}\) \((m = 1, 2)\), symmetric matrices \(P \in \mathbb{R}^{9n \times 9n}, L_i \in \mathbb{R}^{2n \times 2n}\) \((i = 1, 2, 3, 4)\) and \(Q_j \in \mathbb{R}^{2n \times 2n}\) \((j = 1, 2, \ldots, 8)\), and any matrices \(T_k, S_k\) \((k = 1, 2, 3, 4)\), \(N_1\) and \(N_2\) with appropriate dimensions, such that the following LMIs hold:

\[
\begin{bmatrix}
W_1 & \text{col}_8 \left( W_{11}, W_{12}, W_{13}, W_{14}, W_{15}, W_{16}, W_{17}, W_{18} \right) \\
\text{col}_8 \left( W_{21}, W_{22}, W_{23}, W_{24}, W_{25}, W_{26}, W_{27}, W_{28} \right) & N_1
\end{bmatrix} < 0
\]
\[
\begin{align*}
\sum_{i=1}^{8} \Xi_1(h_1(t), h_2(t) = h) & = \Psi + \mathbb{N} \begin{bmatrix}
\mathcal{F}_1^T S_2 & \mathcal{F}_1^T S_3 \\
\ast & \tilde{\theta}_2 & 0 \\
\ast & \ast & -\tilde{\theta}_3
\end{bmatrix} < 0 \\
\sum_{i=1}^{8} \Xi_2(h_1(t) = h_0, h_2(t) = h_0) & = \Psi + \mathbb{N} \begin{bmatrix}
\mathcal{F}_3^T S_1 & \mathcal{F}_3^T S_4 \\
\ast & -\tilde{\theta}_1 & 0 \\
\ast & \ast & -\tilde{\theta}_4
\end{bmatrix} < 0 \\
\sum_{i=1}^{8} \Xi_3(h_1(t) = h_0, h_2(t) = h) & = \Psi + \mathbb{N} \begin{bmatrix}
\mathcal{F}_3^T S_1 & \mathcal{F}_3^T S_3 \\
\ast & -\tilde{\theta}_1 & 0 \\
\ast & \ast & -\tilde{\theta}_3
\end{bmatrix} < 0
\end{align*}
\]

(9) (10) (11)

\[
\psi_{[h_1(t), h_2(t)]} > 0, \quad \forall h_1(t) \in \{0, h_0\}, h_2(t) \in \{h_0, h\}
\]

(12)

\[
\begin{align*}
Q_1(t) > 0, & \quad Q_3(t) > 0, \quad Q_5(t) > 0, \quad Q_7(t) > 0 \\
\tilde{\theta}_1 > 0, & \quad \tilde{\theta}_2 > 0, \quad \tilde{\theta}_3 > 0, \quad \tilde{\theta}_4 > 0
\end{align*}
\]

(13) (14)

where

\[
\begin{align*}
\Xi_1[h_1(t), h_2(t)] & = \text{Sym}[\Pi_1^T \rho \Pi_2] \\
\Xi_2[h_1(t), h_2(t)] & = 2\Pi_3^T L_1 \Pi_4 + 2\Pi_5^T L_2 \Pi_6 + 2\Pi_7^T L_3 \Pi_8 \\
& + 2\Pi_8^T L_4 \Pi_10 + \mu_1 \Pi_9^T L_1 \Pi_13 - \mu_1 \Pi_5^T L_2 \Pi_5 \\
& + \mu_2 \Pi_8^T L_3 \Pi_7 - \mu_2 \Pi_7^T L_4 \Pi_9 \\
\Xi_3[h_1(t), h_2(t)] & = \Pi_{11}^T Q_1(t) \Pi_{11} - \Pi_{12}^T Q_7(t) \Pi_{12} \\
& - \Pi_{13}^T (Q_3(t) - Q_6(t)) \Pi_{13} \\
& - (1 - \mu_1) \Pi_{14}^T (Q_5(t) - Q_6(t)) \Pi_{14} \\
& - (1 - \mu_2) \Pi_{15}^T (Q_5(t) - Q_7(t)) \Pi_{15} \\
\Xi_4[h_1(t), h_2(t)] & = \Pi_{11}^T \left\{ h_0^2 W_1 + (h - h_0)^2 W_2 + \frac{h_0^2}{2} M_1 \\
& + \frac{(h - h_0)^2}{2} M_2 \right\} \Pi_{11} \\
\Xi_5[h_1(t), h_2(t)] & = \text{Sym}[h_0 \mathcal{F}_1^T \tilde{\theta}_1 \mathcal{F}_2] + h_0 h_1(t) \mathcal{F}_2^T \tilde{\theta}_2 \mathcal{F}_2 \\
& + \text{Sym}[h_0 \mathcal{F}_3^T \tilde{\theta}_2 \mathcal{F}_4] \\
& + h_0 (h_0 - h_1(t)) \mathcal{F}_4^T \tilde{\theta}_4 \mathcal{F}_4 \\
& + \text{Sym}[(h - h_0) \mathcal{F}_3^T \tilde{\theta}_3 \mathcal{F}_6] \\
& + h_0 (h_0 - h_1(t)) \mathcal{F}_4^T \tilde{\theta}_4 \mathcal{F}_6 \\
& + \text{Sym}[(h - h_0) \mathcal{F}_3^T \tilde{\theta}_3 \mathcal{F}_6] \\
& + (h - h_0)(h - h_2(t)) \mathcal{F}_8^T \tilde{\theta}_4 \mathcal{F}_8 \\
& + (h - h_0)(h - h_2(t)) \mathcal{F}_8^T \tilde{\theta}_4 \mathcal{F}_8 \\
\Xi_6[h_1(t), h_2(t)] & = -\frac{2h_0 - h_1(t)}{h_0} \mathcal{F}_1^T \tilde{\theta}_1 \mathcal{F}_1 \\
& - \frac{h_0 + h_1(t)}{h_0} \mathcal{F}_3^T \tilde{\theta}_2 \mathcal{F}_3 \\
& - \frac{h_0 - h_1(t)}{h_0} \text{Sym}[\mathcal{F}_1^T S_1 \mathcal{F}_3] \\
& - \frac{h_1(t)}{h_0} \text{Sym}[\mathcal{F}_1^T S_2 \mathcal{F}_3] \\
\Xi_7[h_1(t), h_2(t)] & = -\frac{2h - h_0 - h_2(t)}{h - h_0} \mathcal{F}_5^T \tilde{\theta}_3 \mathcal{F}_5 \\
& - \frac{h - 2h_0 + h_2(t)}{h - h_0} \mathcal{F}_7^T \tilde{\theta}_4 \mathcal{F}_7 \\
& - \frac{h - h_2(t)}{h - h_0} \text{Sym}[\mathcal{F}_5^T S_3 \mathcal{F}_7] \\
& - \frac{h_2(t) - h_0}{h - h_0} \text{Sym}[\mathcal{F}_5^T S_4 \mathcal{F}_7] \\
\Xi_8[h_1(t), h_2(t)] & = -2\Pi_{14}^T M_1 \Pi_{16} - 2\Pi_{17}^T M_1 \Pi_{17} \\
& - 2\Pi_{18}^T M_2 \Pi_{18} - 2\Pi_{19}^T M_2 \Pi_{19} \\
\psi & = \text{Sym}(\gamma \Theta) \\
\gamma & = e_1^T V_1 + e_6^T V_2, \quad \Theta = e_1^T C^T C e_1 - \gamma^2 e_1^T e_1 \\
\Pi_1 & = [e_1^T h_1(t) e_1^T] (h_0 - h_1(t)) e_1^T \\
& (h_2(t) - h_0) e_1^T (h_2(t) - h_0) e_1^T \\
& (h_0 - h_1(t)) e_1^T (h_0 - h_1(t)) e_1^T \\
& (h_2(t) - h_0) e_1^T (h_2(t) - h_0) e_1^T \\
\Pi_2 & = [e_6^T e_1^T (1 - \mu_1) e_2^T e_3^T e_1^T h_1(t) e_1^T] (1 - \mu_2) e_2^T e_4^T (1 - \mu_2) e_4^T e_5^T \\
& (h_1(t) e_1^T (1 - \mu_1) h_1(t) e_1^T) (1 - \mu_1) (h_0 - h_1(t)) e_2^T - (h_0 - h_1(t)) e_2^T \\
& (h_2(t) - h_0) e_3^T (1 - \mu_2) (h_2(t) - h_0) e_3^T \\
& (1 - \mu_2) (h_2(t) - h_0) e_4^T - (h_2(t) - h_0) e_4^T e_4^T \\
\Pi_3 & = [e_1^T e_1^T]^T, \quad \Pi_5 = [e_1^T e_1^T]^T, \quad \Pi_7 = [e_1^T e_1^T]^T, \quad \Pi_9 = [e_1^T e_1^T]^T \\
\Pi_4 & = [h_1(t) e_6^T e_1^T (1 - \mu_1) e_2^T e_2^T e_1^T]^T, \quad \Pi_6 = [h_0 - h_1(t) e_6^T] (1 - \mu_1) e_2^T e_2^T e_2^T \\
& + (1 - \mu_1) (h_0 - h_1(t)) e_2^T e_2^T (1 - \mu_1) e_1^T e_1^T \\
& + (h_2(t) - h_0) e_6^T (h_2(t) - h_0) e_6^T e_6^T \\
& (1 - \mu_2) e_6^T e_6^T e_6^T \\
\Pi_8 & = [(h_2(t) - h_0) e_6^T e_6^T (1 - \mu_2) e_6^T (1 - \mu_2) e_6^T e_6^T]^T \\
\Pi_{10} & = [(h - h_2(t)) e_6^T e_6^T e_6^T]^T
\end{align*}
\]
\[ (1 - \mu_2) e_t^T - e_t^T \mu_2 e_t^T + \mu_2 e_t^T \]

\[
\begin{align*}
\Pi_{11} &= [e_1^T e_6^T, \quad \Pi_{12} = [e_5^T e_{10}^T, \\
\Pi_{13} &= [e_3^T e_8^T, \quad \Pi_{14} = [e_2^T e_7]^T, \\
\Pi_{15} &= [e_4^T e_9^T, \quad \Pi_{16} = [e_{15}^T e_1^T - e_1^T]^T, \\
\Pi_{17} &= [e_{16}^T e_2^T - e_{12}^T]^T, \\
\Pi_{18} &= [e_{17}^T e_3^T - e_{13}^T]^T, \\
\Pi_{19} &= [e_{18}^T e_4^T - e_{14}^T]^T.
\end{align*}
\]

\[
\begin{align*}
\mathcal{F}_1 &= [0 e_1^T - e_2^T - 2 e_{15}^T - e_1^T - e_2^T + 2 e_{11}^T]^T, \\
\mathcal{F}_3 &= [0 e_2^T - e_3^T - 2 e_{16}^T - e_2^T - e_3^T + 2 e_{12}^T]^T, \\
\mathcal{F}_5 &= [0 e_3^T - e_4^T - 2 e_{17}^T - e_3^T - e_4^T + 2 e_{13}^T]^T, \\
\mathcal{F}_7 &= [0 e_4^T - e_5^T - 2 e_{18}^T - e_4^T - e_5^T + 2 e_{14}^T]^T, \\
\mathcal{F}_9 &= [0 e_{10}^T 0 e_{11}^T 0]^T, \\
\mathcal{F}_6 &= [0 e_{12}^T 0 e_{13}^T 0]^T, \\
\mathcal{F}_8 &= [0 e_{14}^T 0 e_{14}^T 0]^T.
\end{align*}
\]

\[
\begin{align*}
\vartheta_1 &= \text{diag}(\varrho_1, 3 \varrho_1), \quad \vartheta_2 = \text{diag}(\varrho_2, 3 \varrho_2), \\
\vartheta_3 &= \text{diag}(\varrho_3, 3 \varrho_3), \quad \vartheta_4 = \text{diag}(\varrho_4, 3 \varrho_4)
\end{align*}
\]

\[
\begin{align*}
\vartheta_1 &= \frac{\mu_1}{h_0} Q_2 + W_1 + \frac{h_0 - h_1(t)}{h_0} M_1, \\
\vartheta_2 &= \frac{\mu_1}{h_0} Q_4 + W_1, \\
\vartheta_3 &= \frac{\mu_2}{h_0} Q_6 + W_2 + \frac{h - h_2(t)}{h - h_0} M_2, \\
\vartheta_4 &= \frac{\mu_2}{h_0} Q_8 + W_2.
\end{align*}
\]

\[
\begin{align*}
\psi_{\eta_1(t), \eta_2(t)}(x) &= \begin{bmatrix}
 P + L_A & \sqrt{h_1(t)} A_1^T T_1^T \\
 & \vdots \\
 & \sqrt{h_2(t) - h_0} A_3^T T_3^T \\
 & \sqrt{h_2(t) - h_0} A_4^T T_4^T \\
 & \sqrt{h_2(t) - h_0} A_5^T T_5^T \\
\end{bmatrix}
\end{align*}
\]

\[
L_A = A_1^T (h_1(t) L_1 + (h_0 - h_1(t)) L_2 + (h_2(t) - h_0) L_3 + (h_1(t) - h_2(t)) L_4) A_1
\]

\[
+ \text{Sym} \{ A_1^T L_1 A_2 + A_1^T L_2 A_3 + A_1^T L_3 A_4 + A_1^T L_4 A_5 \}
\]

\[
\begin{align*}
&+ \frac{2 h_0 - h_1(t)}{h_0^2} A_1^T L_1 A_2 \\
&+ \frac{h_0 + h_1(t)}{h_0^2} A_1^T L_2 A_3 \\
&+ \frac{2 h - h_0 - h_2(t)}{(h - h_0)^2} A_1^T L_3 A_4 \\
&+ \frac{h - 2 h_0 + h_2(t)}{(h - h_0)^2} A_1^T L_4 A_5 \\
&+ \text{Sym} \left\{ \frac{h_0 - h_1(t)}{h_0^2} A_1^T T_1 A_3 + \frac{h_1(t) - h_0}{h_0} A_1^T T_2 A_3 + \frac{h - h_2(t)}{(h - h_0)^2} A_1^T T_3 A_5 \\
&+ \frac{h_2(t) - h_0}{(h - h_0)^2} A_1^T T_4 A_5 \right\}
\end{align*}
\]

\[
A_1 = \text{col}(e_1, 0), \quad A_2 = \text{col}(0, e_2), \\
A_3 = \text{col}(0, e_3), \quad A_4 = \text{col}(0, e_4), \\
A_5 = \text{col}(0, e_5)
\]

\[
\begin{align*}
e_i &= [0_{n \times (i - 1)})]_{n \times n} 0_{n \times (19 - i)}]_T, \\
i &= 1, 2, \ldots, 9.
\end{align*}
\]

**Proof:** Define the Lyapunov-Krasovskii functional candidate as follows:

\[
V(x_t) = \sum_{i=1}^{5} V_i(x_t)
\]

where

\[
\begin{align*}
V_1(x_t) &= \eta_1^T(t) P \eta_1(t) \\
V_2(x_t) &= \eta_2^T(t) L_1 \eta_2(t) + \eta_3^T(t) L_2 \eta_2(t) \eta_3(t) \\
&+ \eta_4^T(t) L_3 \eta_4(t) + \eta_5^T(t) L_4 \eta_5(t) \\
V_3(x_t) &= \int_{t-h_1(t)}^{t} \eta_6^T(s) Q_1(s) \eta_6(s) ds \\
&+ \int_{t-h_1(t)}^{t-h_2(t)} \eta_6^T(s) Q_3(s) \eta_6(s) ds \\
&+ \int_{t-h_2(t)}^{t-h_3(t)} \eta_6^T(s) Q_5(s) \eta_6(s) ds
\end{align*}
\]
\[
+ \int_{t-h}^{t} \eta_{1}^{T}(s) \dot{Q}(t) \eta_{2}(s) \, ds \tag{18}
\]

\[
V_{4}(x_{t}) = h_{0} \int_{t-h}^{t} \int_{0}^{\theta} \eta_{2}^{T}(s) W_{1} \eta_{2}(s) \, ds \, d\theta
\]

\[
+ (h-h_{0}) \int_{t-h}^{t} \int_{0}^{\theta} \eta_{1}^{T}(s) W_{2} \eta_{2}(s) \, ds \, d\theta
\]

\[
V_{5}(x_{t}) = \int_{t-h}^{t} \int_{0}^{t} \int_{0}^{\theta} \eta_{1}^{T}(s) M_{1} \eta_{1}(s) \, ds \, du \, d\theta
\]

\[
+ \int_{t-h}^{t} \int_{0}^{t} \int_{0}^{\theta} \eta_{2}^{T}(s) M_{2} \eta_{2}(s) \, ds \, du \, d\theta
\]

\[
\tag{19}
\]

\[
\eta_{1}(t) = \text{col}[x(t), h_{1}(t)\varphi_{1}(t), (h_{0} - h_{1}(t))\varphi_{2}(t),
\]

\[
(h_{2}(t) - h_{0})\varphi_{3}(t), (h - h_{2}(t))\varphi_{4}(t),
\]

\[
h_{1}(t)\varphi_{5}(t), (h_{0} - h_{1}(t))\varphi_{6}(t), (h_{2}(t) - h_{0})\varphi_{7}(t),
\]

\[
(h - h_{2}(t))\varphi_{8}(t)
\]

\[
\eta_{2}(t) = \text{col}[x(t), \varphi_{1}(t)], \quad \eta_{3}(t) = \text{col}[x(t), \varphi_{2}(t)],
\]

\[
\eta_{4}(t) = \text{col}[x(t), \varphi_{3}(t)]
\]

\[
\eta_{5}(t) = \text{col}[x(t), \varphi_{4}(t)], \quad \eta_{6}(s) = \text{col}[x(s), \dot{x}(s)]
\]

\[
L_{1}(t) = h_{1}(t)L_{1}, \quad L_{2}(t) = (h_{0} - h_{1}(t))L_{2},
\]

\[
L_{3}(t) = (h_{2}(t) - h_{0})L_{3}, \quad L_{4}(t) = (h - h_{2}(t))L_{4}
\]

\[
Q_{1}(t) = Q_{1} - h_{1}(t)Q_{2}, \quad Q_{3}(t) = Q_{3} + (h_{0} - h_{1}(t))Q_{4}
\]

\[
Q_{5}(t) = Q_{5} - (h_{2}(t) - h_{0})Q_{6}, \quad Q_{7}(t) = Q_{7} + (h - h_{2}(t))Q_{8}
\]

\[
\tag{20}
\]

Remark 3.1: As we all know, in order to improve the $H_{\infty}$ performance level, choosing an appropriate LKF is crucial. In this paper, the single integral terms \( \int_{t-h}^{t} x(s) \, ds \), \( \int_{t-h_{0}}^{t} x(s) \, ds \), \( \int_{t-h_{0}}^{t} x(s) \, ds \) and the double integral terms \( \int_{t-h}^{t} \int_{0}^{\theta} x(s) \, ds \, d\theta \), \( \int_{t-h}^{t} \int_{0}^{\theta} x(s) \, ds \, d\theta \), \( \int_{t-h}^{t} \int_{0}^{\theta} x(s) \, ds \, d\theta \), \( \int_{t-h}^{t} \int_{0}^{\theta} x(s) \, ds \, d\theta \), \( \int_{t-h}^{t} \int_{0}^{\theta} x(s) \, ds \, d\theta \) are augmented in $V_{3}(x_{t})$, which establishes more relations among some new cross items. Moreover, the augmented delay-product nonlinear integral items are introduced in $V_{2}(x_{t})$ and delay-product-type functional method is extended to single integral terms in $V_{3}(x_{t})$. The matrices $P$, $L_{i}(t) \ (i = 1, 2, 3, 4)$ in the constructed LKF are just symmetrical, not positive definite, and $Q_{i}(t) \ (j = 1, 3, 5, 7)$ are delay-dependent. Different from the existing constant variable matrices $L_i$ and $Q_j$, delay-dependent matrices can fully capture more information of time delay. Furthermore, $x(t)$ and $\dot{x}(t)$ are augmented in the single, double and triple integral terms of $V_{m}(x_{t}) \ (m = 3, 4, 5)$, so that the relationships between LKF and state information is deepened, all of which play a vital role in obtaining new $H_{\infty}$ performance conditions with less conservatism.

First, in order to ensure the positive definiteness of $V(x_{t})$, $V_{1}(x_{t}) + V_{2}(x_{t})$ can be written together and expressed as below:

\[
V_{1}(x_{t}) + V_{2}(x_{t})
\]

\[
= \eta_{1}^{T}(t) \begin{pmatrix} P + h_{1}(t) & \begin{pmatrix} \dot{e}_{1} \\ \dot{e}_{2} \end{pmatrix} \end{pmatrix} L_{1} \begin{pmatrix} \dot{e}_{1} \\ \dot{e}_{2} \end{pmatrix} \]

\[
+ (h_{0} - h_{1}(t)) \begin{pmatrix} \dot{e}_{1} \\ \dot{e}_{3} \end{pmatrix} L_{2} \begin{pmatrix} \dot{e}_{1} \\ \dot{e}_{3} \end{pmatrix}
\]

\[
+ (h_{2}(t) - h_{0}) \begin{pmatrix} \dot{e}_{1} \\ \dot{e}_{4} \end{pmatrix} L_{3} \begin{pmatrix} \dot{e}_{1} \\ \dot{e}_{4} \end{pmatrix}
\]

\[
+ (h - h_{2}(t)) \begin{pmatrix} \dot{e}_{1} \\ \dot{e}_{5} \end{pmatrix} L_{4} \begin{pmatrix} \dot{e}_{1} \\ \dot{e}_{5} \end{pmatrix}
\]

\[
\eta_{1}(t)
\]

\[
\tag{21}
\]

If $\Psi(h_{1}(t), h_{2}(t)) > 0$ holds, according to Schur complement lemma, conditions $L_{i} > 0$, $L_{i} > 0$, $L_{3} > 0$, $L_{4} > 0$ and $P + L_{A} - \frac{h_{1}(t)}{h_{0}} L_{A}^{T} L_{A}^{-1} L_{T_{1}} A_{3} - \frac{h_{0} - h_{1}(t)}{h_{0}} A_{2}^{T} T_{2} L_{A}^{-1} T_{2}^{T} A_{2} - \frac{h_{2}(t) - h_{0}}{h_{0}} A_{2}^{T} T_{3} L_{A}^{-1} T_{3} A_{3} - \frac{h_{0} - h_{2}(t)}{h_{0}} A_{2}^{T} T_{4} L_{A}^{-1} T_{4} A_{4} > 0$ can be obtained. Therefore, by utilizing Lemma 2.2, for any matrices $T_{i} \ (i = 1, 2, 3, 4)$,

\[
\begin{align*}
&\frac{A_{2}^{T} L_{1} A_{2}}{h_{1}(t)} + \frac{A_{2}^{T} L_{2} A_{3}}{h_{0} - h_{1}(t)} + \frac{A_{2}^{T} L_{3} A_{4}}{h_{2}(t) - h_{0}} + \frac{A_{2}^{T} L_{4} A_{5}}{h - h_{2}(t)}
\end{align*}
\]

\[
\begin{align*}
&\frac{A_{2}^{T} L_{1} A_{2}}{h_{1}(t)} + \frac{A_{2}^{T} L_{2} A_{3}}{h_{0} - h_{1}(t)} + \frac{A_{2}^{T} L_{3} A_{4}}{h_{2}(t) - h_{0}} + \frac{A_{2}^{T} L_{4} A_{5}}{h - h_{2}(t)}
\end{align*}
\]

\[
\begin{align*}
&\geq \frac{2h_{0} - h_{1}(t)}{h_{0}^{2}} A_{2}^{T} L_{1} A_{2} + \frac{h_{0} + h_{1}(t)}{h_{0}^{2}} A_{2}^{T} L_{2} A_{3}
\end{align*}
\]
Hence, we have the following inequalities:

\[
\begin{align*}
&\frac{2h - h_0 - h_2 (t)}{(h - h_0)^2} A_T^2 T_3 A_4 \\
&+ \frac{h - 2h_0 + h_2 (t)}{(h - h_0)^2} A_T^2 L_4 A_5 \\
&+ \text{Sym} \left\{ \frac{h_0 - h_1 (t)}{h_0^2} A_T^2 T_1 A_3 + \frac{h_1 (t)}{h_0} A_T^2 T_2 A_3 \\
&+ \frac{h - h_2 (t)}{(h - h_0)^2} A_T^2 T_3 A_5 + \frac{h_2 (t) - h_0}{(h - h_0)^2} A_T^2 T_4 A_4 \right\} \\
&- \frac{h_1 (t)}{h_0} A_T^2 T_1 T_1^{-1} T_1 A_3 - \frac{h_0 - h_1 (t)}{h_0} A_T^2 T_2 L_2^{-1} T_2 A_2 \\
&- \frac{h_2 (t) - h_0}{(h - h_0)^2} A_T^2 T_3 T_3^{-1} T_3 A_5 \\
&- \frac{h - h_2 (t)}{(h - h_0)^2} A_T^2 T_4 L_4^{-1} T_4 A_4 
\end{align*}
\]

(22)

Based on the above analysis, by using convex combination approach, \( V_1 (x_t) + V_2 (x_t) > \varepsilon \| x(t) \|^2 \) can be ensured for a sufficiently small \( \varepsilon > 0 \) if \( \Psi_{[h_1 (t), h_2 (t)]} > 0 \) holds. In consequence, the positive definiteness of \( V(x_t) \) can be guaranteed by \( W_m, M_m (m = 1, 2) > 0 \) and conditions (12), (13).

**Remark 3.2:** It can be clearly discovered that, the delay-product nonintegral terms can be selected differently depend on actual situations. In LKF construction, delay-product nonintegral terms such as \( \eta^T_2 (t) L_1 (t) \eta_2 (t) \) and \( \eta^T_4 (t) L_2 (t) \eta_3 (t) \) are introduced in \( V_2 (x_t) \), which fully utilizes the information of time delay in the coefficients before symmetric matrices \( L(i = 1, 2, 3, 4) \). Moreover, by considering \( V_1 (x_t) \) and \( V_2 (x_t) \) together, we can obtain that

\[
V_1 (x_t) + V_2 (x_t) \geq \eta^T_1 (t) \Psi_{[h_1 (t), h_2 (t)]} \eta_1 (t)
\]

From the above inequality, we can see that conditions \( P > 0 \) and \( L_i (i = 1, 2, 3, 4) > 0 \) are relaxed as \( \Psi_{[h_1 (t), h_2 (t)]} > 0 \). In other words, the delay-product-type functional method can make the constructed LKF have a more general form since the restrictions of some conditions are defined loosely. It is worth to mention that the delay-product-type functional approach has not been applied to deal with the problem of random delays for LFC systems before.

Defining the infinitesimal operator \( \mathcal{L} \) of \( V(x_t) \) as follows

\[
\mathcal{L} V (x_t) = \lim_{\Delta \to 0^+} \frac{1}{\Delta} \left[ \mathbb{E} \left( V (x_{t+\Delta}) | x_t \right) - V (x_t) \right]
\]

it can be obtained

\[
\begin{align*}
\mathcal{L} V_1 (x_t) &= 2 \eta^T_1 (t) P \eta_1 (t) \\
&= \xi^T (t) \left( \Pi_1^T \Pi_2 + \Pi_2^T \Pi_1 \right) \xi (t) \\
&= \xi^T (t) \Xi_{\Pi_1 \Pi_2} \xi (t) \tag{25}
\end{align*}
\]

\[
\begin{align*}
\mathcal{L} V_2 (x_t) &= 2 \eta^T_4 (t) L_1 (t) \eta_2 (t) + 2 \eta^T_4 (t) L_2 (t) \eta_3 (t) \\
&+ 2 \eta^T_4 (t) L_3 (t) \eta_4 (t) + 2 \eta^T_5 (t) L_4 (t) \eta_5 (t) \\
&+ \eta^T_4 (t) [\mu_1 L_1] \eta_2 (t) + \eta^T_5 (t) [-\mu_1 L_2] \eta_3 (t) \\
&+ \eta^T_4 (t) [\mu_2 L_3] \eta_4 (t) + \eta^T_5 (t) [-\mu_2 L_4] \eta_5 (t) \\
&= \xi^T (t) \Xi_{\Pi_1 \Pi_2} \xi (t) \tag{26}
\end{align*}
\]

\[
\begin{align*}
\mathcal{L} V_3 (x_t) &= \eta^T_3 (t) \Xi_{\Pi_1 \Pi_2} \xi (t) \\
&- \mu_1 \int_{t-h_1 (t)}^{t} \eta_6^T (s) Q_2 \eta_6 (s) \, ds
\end{align*}
\]
\[ - \mu_1 \int_{t-h_1(t)}^{t-h_2(t)} \eta_6^T(s)Q_4 \eta_6(s) \, ds \\
- \mu_2 \int_{t-h_2(t)}^{t-h_0} \eta_6^T(s)Q_6 \eta_6(s) \, ds \\
- \mu_2 \int_{t-h_2(t)}^{t-h_0} \eta_6^T(s)Q_8 \eta_6(s) \, ds \] 

(27) \]

where

\[ \mathcal{L}V_4(x_t) = \eta_6^T(t) \left( \frac{h_0^2}{2}W_1 + (h - h_0)^2W_2 \right) \eta_6(t) \]

\[ - h_0 \int_{t-h_0}^{t} \eta_6^T(s)W_1 \eta_6(s) \, ds \\
- (h - h_0) \int_{t-h_1(t)}^{t-h_0} \eta_6^T(s)W_2 \eta_6(s) \, ds \] 

(28) \]

\[ \mathcal{L}V_5(x_t) = \eta_6^T(t) \left( \frac{h_0^2}{2}M_1 + \frac{(h - h_0)^2}{2}M_2 \right) \eta_6(t) \]

\[ - \int_{t-h_0}^{t} \int_{\theta}^{t} \eta_6^T(s)M_1 \eta_6(s) \, ds \, d\theta \\
- \int_{t-h_1(t)}^{t-h_0} \int_{\theta}^{t} \eta_6^T(s)M_2 \eta_6(s) \, ds \, d\theta \] 

(29) \]

The nonintegral terms in \( \mathcal{L}V_4(x_t) \) and \( \mathcal{L}V_5(x_t) \) are defined in \( \mathcal{Z}_{4h_1(t),h_2(t)} \). By taking single integral terms in (27), (28) and (29) into consideration together, we have

\[ \mathcal{J}_1 = -h_0 \int_{t-h_1(t)}^{t} \eta_6^T(s) \] 

\[ \times \frac{\mu_1}{h_0} \left( Q_2 + W_1 + \frac{h_0 - h_1(t)}{h_0}M_1 \right) \eta_6(s) \, ds \]

\[ \mathcal{J}_2 = -h_0 \int_{t-h_0}^{t} \eta_6^T(s) \left( \frac{\mu_1}{h_0}Q_4 + W_1 \right) \eta_6(s) \, ds \]

\[ \mathcal{J}_3 = -(h - h_0) \int_{t-h_2(t)}^{t-h_0} \eta_6^T(s) \] 

\[ \times \left( \frac{\mu_2}{h-h_0}Q_6 + W_2 + \frac{h - h_2(t)}{h-h_0}M_2 \right) \eta_6(s) \, ds \]

\[ \mathcal{J}_4 = -(h - h_0) \int_{t-h_1(t)}^{t-h_2(t)} \eta_6^T(s) \] 

\[ \times \left( \frac{\mu_2}{h-h_0}Q_8 + W_2 \right) \eta_6(s) \, ds \] 

(30) \]

By choosing the integral inequality introduced in Lemma 2.1 to bound \( \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3 \) and \( \mathcal{J}_4 \), the following inequalities can be attained

\[ \mathcal{J}_1 \leq - \frac{h_0}{h_1(t)} \left[ \begin{array}{c} \int_{t-h_1(t)}^{t} \eta_6(s) \, ds \\
- h_0 \int_{t-h_0}^{t} \eta_6(s) \, ds \\
- 2 \frac{h_0}{h_1(t)} \int_{t-h_1(t)}^{t} \eta_6(s) \, ds \, d\theta \\
- 2 \frac{h_0}{h_1(t)} \int_{t-h_1(t)}^{t} \eta_6(s) \, ds \, d\theta \\
\end{array} \right]^T \]

\[ \times \tilde{\vartheta}_1 \left[ \begin{array}{c} \int_{t-h_1(t)}^{t} \eta_6(s) \, ds \\
- h_0 \int_{t-h_0}^{t} \eta_6(s) \, ds \\
- 2 \frac{h_0}{h_1(t)} \int_{t-h_1(t)}^{t} \eta_6(s) \, ds \, d\theta \\
- 2 \frac{h_0}{h_1(t)} \int_{t-h_1(t)}^{t} \eta_6(s) \, ds \, d\theta \\
\end{array} \right] \]

\[ = - \frac{h_0}{h_1(t)} \bar{\xi}^T(t) \left( \mathcal{F}_1 + h_1(t) \mathcal{F}_2 \right) \tilde{\vartheta}_1 \]

\[ \times \left( \mathcal{F}_1 + h_1(t) \mathcal{F}_2 \right) \bar{\xi}(t) \]
\[
\begin{align*}
    \mathcal{J}_2 &\leq - \frac{h_0}{h_0 - h_1(t)} \xi^T(t) \left[ \int_{t-h_0}^{t-h_1(t)} \eta_6(s) \, ds \right] T \\
    &\quad \times \hat{\vartheta}_2 \left[ \int_{t-h_0}^{t-h_1(t)} \eta_6(s) \, ds \right]
\end{align*}
\]

\[
\begin{align*}
    \mathcal{J}_3 &\leq - \frac{h-h_0}{h_2(t) - h_0} \xi^T(t) \left[ \int_{t-h_2(t)}^{t-h_0} \eta_6(s) \, ds \right] T \\
    &\quad \times \hat{\vartheta}_3 \left[ \int_{t-h_2(t)}^{t-h_0} \eta_6(s) \, ds \right]
\end{align*}
\]

Thus from (31) to (34), it follows that

\[
\mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3 + \mathcal{J}_4
\]

\[
\leq - \xi^T(t) \left( \mathbb{S}_6[h_1(t), h_2(t)] + \frac{h_0}{h_1(t)} \mathcal{F}_1^T \hat{\vartheta}_1 \mathcal{F}_1 + \frac{h_0}{h_2(t) - h_0} \mathcal{F}_5^T \hat{\vartheta}_2 \mathcal{F}_5 \right) \xi(t)
\]

Besides, according to Lemma 2.2, there exists constant matrix \( S_j (j = 1, 2, 3, 4) \) with appropriate dimensions such that

\[
\xi^T(t) \left( \mathcal{F}_1^T \hat{\vartheta}_1 \mathcal{F}_1 - \frac{h_0}{h_0 - h_1(t)} \mathcal{F}_3^T \hat{\vartheta}_2 \mathcal{F}_3 \right) \xi(t)
\]

\[
= - \xi^T(t) \left[ \mathcal{F}_1 \mathcal{F}_3 \right]^T \left[ \begin{array}{cc}
    \frac{h_0}{h_1(t)} & 0 \\
    0 & \frac{h_0}{h_0 - h_1(t)}
\end{array} \right] \mathcal{F}_1 \mathcal{F}_3 \xi(t)
\]

\[
\leq \xi^T(t) \left( \mathbb{S}_6[h_1(t), h_2(t)] + \Phi_1 \right) \xi(t)
\]

(36)
where
\[ \Phi_1 = \frac{h_0 - h_1(t)}{h_0} \mathcal{F}^T S_2 \tilde{S}_2^{-1} S_2^T F_1 + \frac{h_1(t)}{h_0} \mathcal{F}^T S_0 \tilde{S}_1^{-1} S_1 F_3 \]
\[ \Phi_2 = \frac{h - h_2(t)}{h - h_0} \mathcal{F}^T S_0 \tilde{S}_1^{-1} S_1 F_5 \]

Then, applying Jensen’s inequality to estimate the double integral items \(Z_5, Z_6, Z_8\) and \(Z_9\) in (29) yields the following
\[ Z_5 + Z_6 + Z_8 + Z_9 \]
\[ \leq - \frac{2}{h_1^2(t)} \left( \int_{t_1}^t \int_{t_1}^t \eta_6(s) \text{ d} s \text{ d} \theta \right)^T \]
\[ \times M_1 \left( \int_{t_1}^t \int_{t_1}^t \eta_6(s) \text{ d} s \text{ d} \theta \right) \]
\[ - \frac{2}{(h_0 - h_1(t))^2} \left( \int_{t_1}^t \int_{t_1}^t \eta_6(s) \text{ d} s \text{ d} \theta \right)^T \]
\[ \times M_1 \left( \int_{t_1}^t \int_{t_1}^t \eta_6(s) \text{ d} s \text{ d} \theta \right) \]
\[ - \frac{2}{(h - h_2(t))^2} \left( \int_{t_1}^t \int_{t_1}^t \eta_6(s) \text{ d} s \text{ d} \theta \right)^T \]
\[ \times M_2 \left( \int_{t_1}^t \int_{t_1}^t \eta_6(s) \text{ d} s \text{ d} \theta \right) \]
\[ \leq - \frac{2}{h_1^2(t)} \left( \int_{t_1}^t \int_{t_1}^t \eta_6(s) \text{ d} s \text{ d} \theta \right)^T \]
\[ \times M_1 \left( \int_{t_1}^t \int_{t_1}^t \eta_6(s) \text{ d} s \text{ d} \theta \right) \]
\[ \leq - \frac{2}{h_1^2(t)} \left( \int_{t_1}^t \int_{t_1}^t \eta_6(s) \text{ d} s \text{ d} \theta \right)^T \]
\[ \times M_1 \left( \int_{t_1}^t \int_{t_1}^t \eta_6(s) \text{ d} s \text{ d} \theta \right) \]
\[ = \xi^T(t) \Xi \dot{\xi}(t) \] (38)

Remark 3.3: Integral delay-product in \(V_5(x_t)\) is useful to reduce the conservatism of the derived conditions, and the key point is how to deal with the integrals in the infinitesimal operators of delay-product LKF skillfully. In order to combine all integral terms into a similar form and estimate them together, we separate
\[ \int_{t_1}^t \int_{t_1}^t \eta_6(s) W_1 \eta_6(s) \text{ d} s \text{ d} \theta \]
\[ \text{and} \]
\[ \int_{t_1}^t \int_{t_1}^t \eta_6(s) W_2 \eta_6(s) \text{ d} s \text{ d} \theta \]
\[ \text{in} \quad \mathcal{L} V_4(x_t) \quad \text{into} \quad \int_{t_1}^t \int_{t_1}^t \eta_6(s) W_1 \eta_6(s) \text{ d} s \text{ d} \theta, \quad -h_0 \int_{t_1}^t \int_{t_1}^t \eta_6(s) W_1 \eta_6(s) \text{ d} s \text{ d} \theta, \quad -h_0 \int_{t_1}^t \int_{t_1}^t \eta_6(s) W_2 \eta_6(s) \text{ d} s \text{ d} \theta, \quad \text{and} \]
\[ \int_{t_1}^t \int_{t_1}^t \eta_6(s) W_2 \eta_6(s) \text{ d} s \text{ d} \theta \]

Also, the derivatives of triple integrals \(\int_{t_1}^t \int_{t_1}^t \eta_6(s) W_1 \eta_6(s) \text{ d} s \text{ d} \theta\) and \(\int_{t_1}^t \int_{t_1}^t \eta_6(s) W_2 \eta_6(s) \text{ d} s \text{ d} \theta\) in \(\mathcal{L} V_5(x_t)\) are divided into \(-h_0 \int_{t_1}^t \eta_6(s) W_1 \eta_6(s) \text{ d} s \text{ d} \theta, \quad -h_0 \int_{t_1}^t \eta_6(s) W_2 \eta_6(s) \text{ d} s \text{ d} \theta, \quad -h_0 \int_{t_1}^t \eta_6(s) W_2 \eta_6(s) \text{ d} s \text{ d} \theta, \quad \text{and} \]
\[ \int_{t_1}^t \int_{t_1}^t \eta_6(s) W_2 \eta_6(s) \text{ d} s \text{ d} \theta \]

For any matrices \(N_1\) and \(N_2\) with appropriate dimensions, from the system (7), the following zero equality holds
\[ 0 = 2 \left[ x^T(t) N_1 + x^T(t) N_2 \right] \left[ A x(t) - a_0 B K C x(t - h_1(t)) \right. \]
\[ \left. - (1 - a_0) B K C x(t - h_2(t)) + B_0 \omega(t) - \chi^T(t) \right] \]
\[ = \xi^T(t) \text{Sym \{ \Upsilon(\theta) \}} \xi(t) \] (39)

Combining the equalities and inequalities from (25) to (39) and taking the expectation, we can derive that
\[ \mathbb{E} \{ \mathcal{L} V(x_t) \} \leq \mathbb{E} \{ \xi^T(t) (\Xi + \Phi_1 + \Phi_2) \xi(t) \} \] (40)

where \(\Xi = \sum_{i=1}^6 \Xi h_i(t) J_{b_i}(t) + \Psi\).

To analyze the \(H_\infty\) performance, we introduce the following performance index
\[ J(t) = \mathbb{E} \left\{ \int_0^t \left( y^T(s) y(s) - \gamma^2 \omega^T(s) \omega(s) \right) \text{ d}s \right\} \] (41)

By considering the zero initial condition, it can be obtained
\[ J(t) \leq \mathbb{E} \{ \xi^T(t) (\Xi + \mathbb{N} + \Phi_1 + \Phi_2) \xi(t) \} \] (42)

Then by considering the inequality (42) and employing Schur complement lemma, when inequalities (8)-(11) hold, \(J(t) < 0\) can be obtained. Letting \(t \to \infty\), the condition in Definition 2.1 is guaranteed. Therefore, the closed-loop system (7) is asymptotically mean-square stable with \(H_\infty\) performance \(\gamma\). This completes the proof.
When \( \alpha(t) = 1 \), that is, there is only one delay interval with \( h(t) = h_1(t) \), \( \hat{h}(t) \leq \mu \), system (7) decreases to

\[
\begin{aligned}
\dot{x}(t) &= Ax(t) + A_{d}x(t - h(t)) + B_{w} \omega(t) \\
y(t) &= Cx(t) \\
x(t) &= \phi(t), \quad t \in [-h, 0]
\end{aligned}
\] (43)

By making use of the similar methods in the derivation of Theorem 3.1, we have Corollary 3.1.

**Corollary 3.1:** For some given positive scalars \( h, \mu, \gamma \), system (43) is asymptotically stable with \( H_{\infty} \) performance \( \gamma \), if there exist symmetric positive definite matrices \( W, M \in \mathbb{R}^{2n \times 2n} \), symmetric matrices \( \bar{P} \in \mathbb{R}^{5n \times 5n} \), \( \bar{L}_i \in \mathbb{R}^{2n \times 2n} \) \((i = 1,2)\) and \( \bar{Q}_j \in \mathbb{R}^{2n \times 2n} \) \((j = 1,2,3,4)\), and any matrices \( T_k, S_k \) \((k = 1,2)\), \( N_1 \) and \( N_2 \) with appropriate dimensions, such that the following LMIs hold:

\[
\begin{align*}
\sum_{i=1}^{7} \bar{E}_{[h(t)]=0} &+ \bar{\psi} + \bar{N} \bar{F}_T \bar{S}_2 < 0 \\
\sum_{i=1}^{7} \bar{E}_{[h(t)]=h} &+ \bar{\psi} + \bar{N} \bar{F}_T \bar{S}_1 < 0 \\
\bar{\Psi}_{[h(t)]} > 0, \quad \forall h(t) \in (0, h) \\
\bar{Q}_1(t) > 0, \quad \bar{Q}_3(t) > 0, \quad \bar{\psi}_1 > 0, \quad \bar{\psi}_2 > 0
\end{align*}
\] (44)

\[
\bar{\Psi}_{[h(t)]} = \begin{bmatrix} \bar{P} + \bar{L}_A \frac{\sqrt{h(t)}}{h} \bar{A}_3 \bar{T}_1 \bar{P} + \bar{W} & \sqrt{h(t)} \bar{A}_3 \bar{T}_1 \bar{A}_2 \bar{T}_2 \\ \star & \bar{L}_1 \end{bmatrix} < 0
\]

where

\[
\begin{align*}
\bar{E}_{1[h(t)]=0} &= \text{Sym}\{\bar{P}\bar{T}_1 \bar{P} \bar{T}_2\} \\
\bar{E}_{2[h(t)]=0} &= 2\bar{P}_{1} \bar{T}_1 \bar{P}_1 + 2\bar{P}_{2} \bar{T}_2 \bar{P}_6 + \mu \bar{P}_{1} \bar{T}_1 \bar{P}_3 \\
&\quad - \mu \bar{P}_{2} \bar{T}_2 \bar{P}_5 \\
\bar{E}_{3[h(t)]=0} &= \bar{P}_{1} \bar{T}_1 \bar{Q}_1(t) \bar{P}_7 - \bar{P}_{1} \bar{T}_1 \bar{Q}_1(t) \bar{P}_8 \\
&\quad - (1 - \mu) \bar{P}_{1} \bar{T}_1 \bar{Q}_1(t) \bar{Q}_9 \\
\bar{E}_{4[h(t)]=0} &= \bar{P}_{1} \bar{T}_1 \left( h^2 \bar{W} + h^2 \bar{M} \right) \bar{P}_7 \\
\bar{E}_{5[h(t)]=0} &= \text{Sym}\{h \bar{F}_T \bar{T}_1 \bar{P}_7 \bar{F}_2 + hh(t) \bar{F}_T \bar{T}_2 \bar{P}_7 \bar{F}_2 \} \\
&\quad + \text{Sym}\{h \bar{F}_T \bar{T}_1 \bar{P}_7 \bar{F}_2 \bar{T}_2 \bar{F}_2 \bar{T}_2 \} \\
\bar{E}_{6[h(t)]=0} &= \frac{-2h - h(t)}{h} \bar{F}_T \bar{T}_2 \bar{P}_7 \bar{F}_2 \bar{F}_2 \\
&\quad + \frac{-h - h(t)}{h} \text{Sym}\{h \bar{F}_T \bar{T}_1 \bar{P}_7 \bar{F}_3 \bar{F}_3 \} \\
\bar{E}_{7[h(t)]=0} &= \frac{-2h - h(t)}{h} \bar{F}_T \bar{T}_1 \bar{P}_7 \bar{F}_2 \bar{F}_2 \\
&\quad + \frac{-h - h(t)}{h} \text{Sym}\{h \bar{F}_T \bar{T}_1 \bar{P}_7 \bar{F}_3 \bar{F}_3 \}
\end{align*}
\]

When \( \omega(t) = 0 \) in system (43), the corresponding stability criterion can be deduced from Corollary 3.1 as follows.

**Corollary 3.2:** For given positive scalars \( h \) and \( \mu \), system (43) is asymptotically stable if there exist symmetric positive definite matrices \( W, M \in \mathbb{R}^{2n \times 2n} \), symmetric matrices \( \bar{P} \in \mathbb{R}^{5n \times 5n} \), \( \bar{L}_i \in \mathbb{R}^{2n \times 2n} \) \((i = 1,2)\) and \( \bar{Q}_j \in \mathbb{R}^{2n \times 2n} \) \((j = 1,2,3,4)\), and any matrices \( T_k, S_k \) \((k = 1,2)\), \( N_1 \) and \( N_2 \) with appropriate dimensions, such that the following LMIs hold:

\[
\begin{align*}
\bar{E}_{1[h(t)]=0} &= \text{Sym}\{\bar{P}\bar{T}_1 \bar{P} \bar{T}_2\} \\
\bar{E}_{2[h(t)]=0} &= 2\bar{P}_{1} \bar{T}_1 \bar{P}_1 + 2\bar{P}_{2} \bar{T}_2 \bar{P}_6 + \mu \bar{P}_{1} \bar{T}_1 \bar{P}_3 \\
&\quad - \mu \bar{P}_{2} \bar{T}_2 \bar{P}_5 \\
\bar{E}_{3[h(t)]=0} &= \bar{P}_{1} \bar{T}_1 \bar{Q}_1(t) \bar{P}_7 - \bar{P}_{1} \bar{T}_1 \bar{Q}_1(t) \bar{P}_8 \\
&\quad - (1 - \mu) \bar{P}_{1} \bar{T}_1 \bar{Q}_1(t) \bar{Q}_9 \\
\bar{E}_{4[h(t)]=0} &= \bar{P}_{1} \bar{T}_1 \left( h^2 \bar{W} + h^2 \bar{M} \right) \bar{P}_7 \\
\bar{E}_{5[h(t)]=0} &= \text{Sym}\{h \bar{F}_T \bar{T}_1 \bar{P}_7 \bar{F}_2 + hh(t) \bar{F}_T \bar{T}_2 \bar{P}_7 \bar{F}_2 \} \\
&\quad + \text{Sym}\{h \bar{F}_T \bar{T}_1 \bar{P}_7 \bar{F}_2 \bar{T}_2 \bar{F}_2 \bar{T}_2 \} \\
\bar{E}_{6[h(t)]=0} &= \frac{-2h - h(t)}{h} \bar{F}_T \bar{T}_2 \bar{P}_7 \bar{F}_2 \bar{F}_2 \\
&\quad + \frac{-h - h(t)}{h} \text{Sym}\{h \bar{F}_T \bar{T}_1 \bar{P}_7 \bar{F}_3 \bar{F}_3 \} \\
\bar{E}_{7[h(t)]=0} &= \frac{-2h - h(t)}{h} \bar{F}_T \bar{T}_1 \bar{P}_7 \bar{F}_2 \bar{F}_2 \\
&\quad + \frac{-h - h(t)}{h} \text{Sym}\{h \bar{F}_T \bar{T}_1 \bar{P}_7 \bar{F}_3 \bar{F}_3 \}
\end{align*}
\]
Corollary 3.2 \[ (j = 1, 2, 3, 4), \text{ and } \text{any matrices } T_k, S_k \ (k = 1, 2), N_1 \text{ and } N_2 \text{ with appropriate dimensions, such that the LMIs (44)-(47) hold.} \]

4. Numerical simulations and analysis

In this segment, a second-order example and one-area LFC system are provided to illustrate the effectiveness of main results. Moreover, the time delay in one-area LFC systems is considered as a random delay with the probability distribution characteristic, which shows a significant improvement in the stable operating regions and the disturbance attenuation ability of power systems.

Example 4.1: Consider the following parameters in the system (43) with \( \omega(t) = 0 \):

\[
A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_d = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}
\]

Table 1. Admissible upper bounds \( h \) for given \( \mu \).

\[
\begin{array}{cccc}
\mu & 0.1 & 0.2 & 0.5 & 0.8 \\
Seuret and Gouaisbaut (2018) (N = 1) & 4.80 & 3.99 & 2.79 & 2.42 \\
T.H. Lee and Park (2017) & 4.8257 & 4.1274 & 3.1131 & 2.6763 \\
W.I. Lee et al. (2018) & 4.908 & 4.199 & 3.166 & 2.867 \\
Zeng et al. (2019) & 4.921 & 4.218 & 3.221 & 2.792 \\
Li et al. (2019) & 4.996 & 4.308 & 3.251 & 2.867 \\
Chen and Chen (2019) & 4.943 & 4.270 & 3.322 & 2.899 \\
M.J. Park et al. (2018) & 4.9445 & 4.2747 & 3.3058 & 2.8505 \\
Seuret and Gouaisbaut (2018) (N = 4) & 5.01 & 4.29 & 3.19 & 2.70 \\
Corollary 3.2 & 5.237 & 4.602 & 3.475 & 3.361 \\
\end{array}
\]

Table 2. Maximum delay upper bounds \( h \propto (K_P, K_I) \) with \( \mu = 0.9 \).

| \( h \) | \( K_P \) | \( K_I \) |
|---|---|---|
| 0.05 | 38.735 | 24.475 |
| 0.10 | 33.785 | 20.144 |
| 0.20 | 20.180 | 11.686 |
| 0.40 | 3.568 | 3.499 |
| 0.60 | 2.685 | 2.650 |
| 1.00 | 1.898 | 1.782 |

The purpose of this example is to compare the admissible upper bounds \( h \) by various approaches, which can check the conservatism of the stability conditions.

Table 1 lists the admissible upper bounds \( h \) obtained by different methods for various \( \mu \). When \( \mu = 0.8 \), by applying the methods in W. I. Lee et al. (2018) and Chen and Chen (2019), the admissible upper bounds are \( h = 2.735 \) and \( h = 2.899 \), and the result achieved by Corollary 3.2 is \( h = 3.361 \). Hence, it can be seen obviously that the admissible upper bounds of Corollary 3.2 are larger than those in above works, which verifies the progressiveness of our applied methods.

Example 4.2: For one-area closed-loop LFC system, the following parameters are considered:

\[
T_{ch} = 0.3, \quad T_g = 0.1, \quad \hat{R} = 0.05, \quad D = 1.0, \quad \beta = 21, \quad M = 10
\]

A. Result comparison and analysis

For various controller gains \( K_P \) and \( K_I \), Table 2 shows the maximum delay upper bounds \( h \) of system (43) with \( \omega(t) = 0 \) based on Corollary 3.2. It can be discovered that PI controller gains have a significant impact on affecting delay margins. When \( K_P \) is fixed, the maximum delay upper bound \( h \) decreases with the increase of \( K_I \). However, the relationship between delay upper bound \( h \) and \( K_P \) is more complicated. When \( K_I \) is fixed, in most situations \( h \) decreases first and then increases with the increase of \( K_P \). Therefore, all of these regulations can be regarded as auxiliary conditions for designing PI controllers, which have a positive effect on obtaining larger stable operating regions for power systems.

Table 3 gives more comparative results of the maximum delay upper bounds \( h \) with Jiang et al. (2012) and Peng and Zhang (2016) based on Corollary 3.2. We can see clearly that the results obtained by our methods are obviously larger than that acquired by other methods, which
means that the methods applied in this work have distinct advantages in calculating the delay margins of real networks.

For the given conditions of $\mu = 0.5$ and $\gamma = 1$, Table 4 provides the maximum delay upper bounds $h$ under various $K_P$ and $K_I$ based on Corollary 3.1. It should be pointed out that delay upper bound $h$ becomes smaller with the increase of $K_P$ and $K_I$, which reveals the stable operating regions of power systems is closely related to PI-based controller gains.

Table 5 presents the maximum delay upper bounds $h$ with $\gamma = 1$ under controller gains $K_P = 0.4$, $K_I = 0.4$. The maximum delay upper bound achieved by Corollary 3.1 is $h = 0.594$. Comparing with the obtained results by Theorem 3.1, it is easily found that the larger maximum delay upper bounds $h$ can be obtained by taking the probability distribution characteristic of time delay into consideration, which confirms the accuracy of our results.

For the prescribed conditions of $\mu = 0.5$ and $h = 2$, Table 6 lists the allowable minimum $\gamma_{\min}$ by different $K_P$ and $K_I$ based on Corollary 3.1. It is worth to mention that allowable minimum $\gamma_{\min}$ becomes larger with the increase of $K_P$ and $K_I$, which reflects the disturbance attenuation ability of power systems is also closely contact with PI-based controller gains.

Table 7 shows the allowable minimum $\gamma_{\min}$ with $h = 2$ under controller gains $K_P = 0.2$, $K_I = 0.6$. The allowable minimum $\gamma_{\min}$ attained by Corollary 3.1 is $\gamma_{\min} = 4.799$. Comparing with the calculated results by Theorem 3.1, we can easily figure out that the smaller $H_\infty$ performance index $\gamma_{\min}$ can be obtained by considering the non-uniform distribution delay characteristic in the analysis of power systems, which verifies the effectiveness of our results.

For $K_P = 0.15$, $K_I = 0.1$, $\mu_1 = \mu_2 = 0.5$ and the same delay upper bound $h = 2$, the allowable minimum $H_\infty$ performance index $\gamma_{\min}$ based on different methods are listed in Table 8. Through the comparative results with Jiang et al. (2012) and Peng and Zhang (2016), it can be seen apparently that our results are much smaller than those obtained by other methods, which show the less conservative of our methods.

### Table 4. Maximum delay upper bounds $h$ with $\gamma = 1$.

| $h$   | $K_P$ | $K_I$ |
|-------|-------|-------|
| 0     | 0.05  | 0.1   |
| 0.05  | 0.1   | 0.15  |
| 0.1   | 0.2   | 0.2   |
| 0.2   | 0.4   | 0.4   |
| 0.4   | 0.6   | 0.6   |
| 0.6   | 1.0   | 1.0   |

### Table 5. Maximum delay upper bounds $h$ with $K_P = 0.4$, $K_I = 0.4$.

| Methods                     | $h$   |
|-----------------------------|-------|
| Corollary 3.1               | 0.594 |
| Theorem 3.1 with $\alpha_0 = 0.4$, $h_0 = 1.0$ | 1.158 |
| Theorem 3.1 with $\alpha_0 = 0.4$, $h_0 = 1.5$ | 1.649 |
| Theorem 3.1 with $\alpha_0 = 0.6$, $h_0 = 1.5$ | 1.642 |
| Theorem 3.1 with $\alpha_0 = 0.6$, $h_0 = 2.0$ | 2.138 |
| Theorem 3.1 with $\alpha_0 = 0.8$, $h_0 = 2.0$ | 2.127 |

### Table 6. Allowable minimum $\gamma_{\min}$ with $h = 2$.

| $\gamma_{\min}$ | $K_P$ | $K_I$ |
|------------------|-------|-------|
| $0$              | 2.075 | 2.165 |
| 0.05             | 2.176 | 2.266 |
| 0.1              | 2.280 | 2.340 |
| 0.2              | 2.570 | 2.653 |
| 0.4              | 3.331 | 3.406 |
| 0.6              | 5.043 | 5.232 |
| 0.8              | 10.695| 11.400|

### Table 7. Allowable minimum $\gamma_{\min}$ with $K_P = 0.2$, $K_I = 0.6$.

| Methods                     | $\gamma_{\min}$ |
|-----------------------------|------------------|
| Corollary 3.1               | 4.799 |
| Theorem 3.1 with $\alpha_0 = 0.2$, $h_0 = 1.0$ | 1.572 |
| Theorem 3.1 with $\alpha_0 = 0.2$, $h_0 = 1.5$ | 1.795 |
| Theorem 3.1 with $\alpha_0 = 0.4$, $h_0 = 1.5$ | 1.896 |
| Theorem 3.1 with $\alpha_0 = 0.4$, $h_0 = 1.8$ | 1.247 |
| Theorem 3.1 with $\alpha_0 = 0.6$, $h_0 = 1.8$ | 1.378 |

### Figure 2. State response trajectory with $K_P = 0.4$, $K_I = 0.4$ by Corollary 3.1.
probability distribution characteristic. In the simulation, the load fluctuation is chosen as $\omega(t) = 0.1pu$. Based on the different conditions listed in Tables 5 and 7, Figures 2 and 4 present the frequency response trajectory of the system (43) without considering probability distribution characteristic, then Figures 3 and 5 give the frequency response trajectories of the system (7) with considering probability distribution characteristic. It can be seen clearly from the simulation results in Figure 2–5 that all the state variables converge to their equilibrium points, which confirm the veracity of our theoretical results.

5. Conclusion

In this paper, $H_{\infty}$ performance for PI-type LFC of power systems with random delays have been investigated. By introducing new vectors and delay-dependent matrices, a delay-product-type augmented LKF has been constructed, and a novel extended reciprocally convex matrix inequality combining with Wirtinger-based integral inequality have been employed to tackle with the integral terms effectively, which can utilize more information of time delay and improve the estimation accuracy. According to applied optimal analysis methods, less conservative delay-dependent $H_{\infty}$ performance and stability criteria have been developed. Finally, two numerical examples have been carried out to illustrate the effectiveness of our theoretical results and the improvement of the proposed methods. In the future, we will consider the influence of other stochastic factors in power systems, construct more reasonable LKF and propose new integral inequalities to further cut down the conservatism of main results.

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