A NOTE ON SMOOTH FORMS ON ANALYTIC SPACES

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Abstract. We give a new proof of the fact that for any proper holomorphic map between reduced analytic spaces there is an associated natural pullback operation on smooth differential forms.

1. Introduction

There is a natural notion of smooth differential forms on any reduced analytic space. The dual objects are the currents. Such forms and currents have turned out to be useful tools in several contexts, e.g., the analytic approach to intersection theory [2, 3] and the $\bar{\partial}$-equation on analytic spaces [1, 8]. It is desirable to be able to take the direct image of a current under a proper holomorphic map $f: X \to Z$ between reduced analytic spaces. By duality this amounts to take pullbacks of smooth forms. In some works, e.g., [2, 3], it is implicitly assumed that this is possible. There is an obvious tentative definition of $f^* \phi$ for a smooth form $\phi$ on $Z$. It is however not completely clear that it gives a well-defined pullback operation, not even if $\phi$ is holomorphic (of positive degree); this case is settled in [5, Corollary 1.0.2]. The problem occurs already if $f$ is the inclusion of a subvariety contained in $Z_{\text{sing}}$. It was proved in [4, III Corollary 2.4.11] that the suggested definition indeed gives a functorial operation on smooth forms on analytic spaces. In this short note we give a new proof of this fact. Our proof is quite elementary except for an application of Hironaka’s theorem.

2. Results

Let $X$ be a reduced analytic space. Recall that, by definition, there is a neighborhood $U$ of any point in $X$ and an embedding $i: U \hookrightarrow D$ in an open set $D \subset \mathbb{C}^N$ such that $U$ can be identified with its image. For notational convenience we will suppress $U$ and say that $i$ is a local embedding of $X$. A smooth $(p, q)$-form $\phi$ on $X_{\text{reg}}$ is smooth on $X$, $\phi \in \mathcal{E}^{p,q}(X)$, if there is a smooth form $\varphi$ in $D$ such that

$$i|_{X_{\text{reg}}}^* \varphi = \phi.$$ 

A different definition is given in [6, Section 3.3], see Remark 2.4 below. If $j: X \to D'$ is another local embedding, then the identity on $X$ induces a biholomorphism $i(X) \sim j(X)$. Thus, again by definition, locally in $D$ and $D'$, there are holomorphic maps $g: D \to D'$ and $h: D' \to D$ such that $i = h \circ j$ and $j = g \circ i$. Since $h^* \varphi$ is smooth in $D'$ and

$$j|_{X_{\text{reg}}}^* h^* \varphi = \phi,$$

it follows that the notion of smooth forms on $X$ is independent of embedding.

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We will write $i^*\varphi$ for the image of $\varphi \in \mathcal{E}(D)$ in $\mathcal{E}(X)$. Let $[i(X)]$ be the Lelong current of integration over $i(X)_{reg}$. The kernel of $i^*$ is closed since

$$i^*\varphi = 0 \iff \varphi \wedge [i(X)] = 0.$$  

Thus, with the quotient topology $\mathcal{E}(X) = \mathcal{E}(D)/\text{Ker } i^*$ is a Fréchet space.

**Theorem 2.1.** Let $f: X \to Z$ be a proper holomorphic map between reduced analytic spaces. There is a well-defined map $f^*: \mathcal{E}(Z) \to \mathcal{E}(X)$ with the following property: If $\phi$ is a smooth form on $Z$, $X \hookrightarrow D_X$ and $Z \hookrightarrow D_Z$ are local embeddings, and $f: D_X \to D_Z$ is a holomorphic extension of $f$, then $f^*\phi$ is the smooth form on $X$ obtained by extending $\phi$ to $D_Z$, pull the extension back under $f$, and pull the result back to $X_{reg}$.

Assume that $g: X \to Y$ and $h: Y \to Z$ are proper holomorphic maps such that $f = h \circ g$. Let $i: X \to D_X$, $j: Y \to D_Y$, and $\iota: Z \to D_Z$ be local embeddings such that $\tilde{g}: D_X \to D_Y$ and $\tilde{h}: D_Y \to D_Z$ are holomorphic extensions of $g$ and $h$, respectively. Notice that $h \circ \tilde{g}$ is a holomorphic extension of $f$. If $\phi \in \mathcal{E}(Z)$ and $\phi = i|_{X_{reg}}^*\varphi$ it follows by Theorem 2.1 that $h^*\phi = j|_{Y_{reg}}^*\tilde{h}^*\varphi$ and $g^*h^*\phi = i|_{X_{reg}}^*\tilde{g}^*h^*\phi = f^*\phi$. Hence,

$$f^*\phi = g^*h^*\phi, \quad \phi \in \mathcal{E}(Z).$$

The space of currents on $X$, $\mathcal{C}(X)$, is the dual of the space of test forms, i.e., compactly supported forms in $\mathcal{E}(X)$, cf. [7] Section 4.2. If $\phi$ is a test form on $Z$ then $f^*\phi$ is a test form on $X$ since $f$ is proper. If $\mu$ is a current on $X$ thus $f_*\mu$ is a current on $Z$ defined by

$$(2.1) \quad f_*\mu \cdot \phi = \mu \cdot f^*\phi.$$

**Corollary 2.2.** Let $f: X \to Z$ be a proper holomorphic map between reduced analytic spaces. Then the induced mapping $f_*: \mathcal{C}(X) \to \mathcal{C}(Z)$ has the property that if $f = h \circ g$, where $g: X \to Y$ and $h: Y \to Z$ are proper holomorphic maps, then

$$f_*\mu = h_*g_*\mu, \quad \mu \in \mathcal{C}(X).$$

**Example 2.3.** Suppose that $i: X \to D$ is an embedding and consider the induced mapping $i_*: \mathcal{C}(X) \to \mathcal{C}(D)$. It follows from (2.1) and the definition of test forms on $X$ that $i_*$ is injective. Thus $\mathcal{C}(X)$ can be identified with its image $i_*\mathcal{C}(X)$. In view of the definition of $\mathcal{C}(X)$ and (2.1) it follows that $i_*\mathcal{C}(X)$ is the set of currents $\mu$ in $D$ such that $\mu \cdot \varphi = 0$ if $i^*\varphi = 0$. Notice in particular that $i_*1 = [i(X)]$.

**Remark 2.4.** Let $i: X \to D$ be a local embedding. In [6] Section 3.3 the space of smooth forms on $X$ is defined as $\mathcal{E}(D)/\mathcal{N}(D)$, where $\mathcal{N}(D)$ is the space of smooth forms $\varphi$ in $D$ such that for any manifold $W$ and any smooth map $g: W \to D$ with $g(W) \subset X$ one has $g^*\varphi = 0$. A priori $\mathcal{N}(D)$ contains $\text{Ker } i^*$. If $\mathcal{N}(D) = \text{Ker } i^*$, then $\mathcal{E}(D)/\mathcal{N}(D)$ coincides with our definition of $\mathcal{E}(X)$ and Theorem 2.1 becomes trivial. It is claimed in [6] Section 3.3 that indeed $\mathcal{N}(D) = \text{Ker } i^*$, but we have not been able to find a proof.

3. **Proofs**

We will prove Theorem 2.1 by showing that the stated property indeed is independent of the choices of extensions. The technical part is contained in Proposition 3.2 cf. [5] Proposition 1.0.1. We begin with the following lemma.
Lemma 3.1. Let $M$ be a reduced analytic space, $N$ a complex manifold, and $p: M \to N$ a proper holomorphic map. If each connected component of $N$ has dimension $d \geq 1$ and $\text{rank}_x p < d$ for all $x \in M_{\text{reg}}$, then $p$ is not surjective on any connected component of $N$.

Proof. We may assume that $N$ is connected. Notice also that if $M$ is smooth then the lemma follows in view of the constant rank theorem. We reduce to this case by induction on $\dim M$. If $\dim M = 0$ the lemma is obvious. Assume that the lemma is true if $\dim M < \delta$.

Let $\dim M = \delta$. Let $M'$ be the union of $M_{\text{sing}}$ and the irreducible components of dimension $< \delta$. Since $p|M'$ satisfies the hypothesis of the statement it follows from the induction hypothesis that $p|M'$ is not surjective. By Remmert’s proper mapping theorem thus $p(M')$ is a proper analytic subset of $N$. If $M = p^{-1}p(M')$ we are done. If not we replace $M$, $N$, and $p$ by $M \setminus p^{-1}(p(M'))$, $N \setminus p(M')$, and $p|M \setminus p^{-1}(p(M'))$, respectively. Since $p|M \setminus p^{-1}(p(M'))$ is proper and $M \setminus p^{-1}(p(M'))$ is smooth the lemma follows for $\dim M = \delta$ by the constant rank theorem.

Proposition 3.2. Let $W \subset V$ be analytic subsets of an open set $D \subset \mathbb{C}^N$ and let $\varphi$ be a smooth form in $D$. If the pullback of $\varphi$ to $V_{\text{reg}}$ vanishes, then the pullback of $\varphi$ to $W_{\text{reg}}$ vanishes.

Proof. We may assume that $W$ is irreducible of dimension $d$. We may also assume that $\varphi$ has positive degree since a smooth function vanishing on $V_{\text{reg}}$ must vanish on $W$ by continuity. The case $d = 0$ is then clear since the pullback of a form of positive degree to discrete points necessarily vanishes. Let $\tilde{\pi}: V' \to V$ be a Hironaka resolution of singularities. Suppose that $W' \subset V'$ is analytic and such that $\tilde{\pi}(W') = W$. Let $\pi = \tilde{\pi}|_{W'}$ and let $\varphi$ be the pullback of $\varphi$ to $W_{\text{reg}}$. Since the pullback of $\varphi$ under $W' \hookrightarrow V' \to V \hookrightarrow D$ is 0, it follows that $\pi^*\varphi = 0$. We will find such $W'$ and $\pi$ such that $\pi^*\varphi = 0$ implies $\varphi = 0$.

To begin with we set $W' = \tilde{\pi}^{-1}(W)$. If $\tilde{\pi}(W_{\text{sing}}') = W$, replace $W'$ by $W_{\text{sing}}'$.

Possibly repeating this we may assume that $\tilde{\pi}(W_{\text{sing}}') \not\subseteq W$. Thus $\tilde{\pi}(W_{\text{sing}}')$ is a proper analytic subset of $W$. Set $\pi = \tilde{\pi}|_{W'}$ and notice that $\pi$ is surjective on $W$.

Let $M = W' \setminus \pi^{-1}(W_{\text{sing}} \cup \pi(W_{\text{sing}}'))$, $\ N = W_{\text{reg}} \setminus \pi(W_{\text{sing}}')$, and $p = \pi|M$. Since $p$ is surjective it follows from Lemma 3.1 that there is $x \in M$ such that $\text{rank}_x p = d$. Since $d$ is the optimal rank of $p$ this holds for $x$ in a non-empty Zariski-open subset of $M$. Let $\tilde{M} = \{x \in M; \text{rank}_x p \leq d - 1\}$ be the complement of this set. Then $\text{rank}_x p|_{\tilde{M}} \leq d - 1$ for all $x \in \tilde{M}_{\text{reg}}$. By Lemma 3.1, $p(\tilde{M}) \not\subseteq N$ and thus a proper analytic subset of $N$.

Now, $N \setminus p(\tilde{M})$ is a dense open subset of $W_{\text{reg}}$ and so it suffices to show that $\varphi = 0$ there. However, $M \setminus p^{-1}(\tilde{M})$ is a (non-empty) open subset of $M$ and in this set $p$ has constant rank $d = \dim W$. Thus, $p$ is locally a simple projection and it follows that if $p^*\varphi = 0$, then $\varphi = 0$.

Proof of Theorem 2.1. Let $\phi \in \mathcal{E}(Z)$ and let $f^*\phi$ be any form on $X_{\text{reg}}'$ obtained by the procedure described in Theorem 2.1. Clearly $f^*\phi$ is smooth on $X$. To see that it is independent of the choices of extensions we may assume that $X$ is irreducible. By Remmert’s proper mapping theorem $f(X)$ is an analytic subset of $Z$. In view of Proposition 2.2 if $\varphi$ is an extension of $\phi$ to $D_Z$, then the pullback $\phi'$ of $\varphi$ to $f(X)_{\text{reg}}$ is independent of the extension. Let $X' := X \setminus f^{-1}(f(X)_{\text{sing}})$ and notice that it is a
non-empty Zariski-open subset of $X$. The restriction of $f$ to $X'_{\text{reg}}$ is a holomorphic map between complex manifolds and it follows that 

$$f^* \phi = f|_{X'_{\text{reg}}}^* \phi'$$

on $X'_{\text{reg}}$. Since $f|_{X'_{\text{reg}}}^* \phi'$ is well-defined and $X'_{\text{reg}}$ is dense in $X$ it follows that $f^* \phi$ is independent of the choices of extensions. □

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