SEPARATING PATH AND IDENTITY TYPES IN PRESHEAF MODELS OF UNIVALENT TYPE THEORY

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Abstract. We give a collection of results regarding path types, identity types and univalent universes in certain models of type theory based on presheaves.

The main result is that path types cannot be used directly as identity types in any Orton-Pitts style model of univalent type theory with propositional truncation in presheaf assemblies over the first and second Kleene algebras.

We also give a Brouwerian counterexample showing that there is no constructive proof that there is an Orton-Pitts model of type theory in presheaves when the universe is based on a standard construction due to Hofmann and Streicher, and path types are identity types. A similar proof shows that path types are not identity types in internal presheaves in realizability toposes as long as a certain universe can be extended to a univalent one.

We show that one of our key lemmas has a purely syntactic variant in intensional type theory and use it to make some minor but curious observations on the behaviour of cofibrations in syntactic categories.

1. Introduction

In the cubical set model of homotopy type theory, and more generally in members of the classes of models considered by Gambino and Sattler, by Van den Berg and Frumin, and by Orton and Pitts, the most basic notion of identity is that of path type. In that construction one uses exponentiation with an interval object $I$. Given an object $X$, we call $X^I$ the path type on $X$ and think of it as the collection of paths between two elements of $X$. This can then be used to produce path objects for any fibration $X \to Y$ via the mapping path factorisation.

In order to interpret identity types in type theory as path types it is necessary to show that the reflexivity (or “constant paths”) map $r^X : X \to X^I$ is a trivial cofibration. In many natural examples however, it is difficult to show that this map is a trivial cofibration, or even just a cofibration. For example, in [4], Bezem, Coquand and Huber gave a definition of path type, but did not show how to interpret identity types that strictly satisfy the $J$-computation rule. In [22] the author gave both an explanation for why this was difficult constructively as well as a solution. The explanation was that the definition of fibration used in the BCH model leads to an awfs where the trivial cofibrations always have pointwise decidable image. However, when $X$ is the nerve of a complete metric space, the map $r^X$ is essentially the inclusion of constant paths into the set of all paths in the usual topological sense. One then gives a Brouwerian counterexample to show that there is no constructive proof that such inclusions are pointwise decidable. The solution was to use a second, more elaborate construction to obtain identity types that do satisfy the $J$-computation rule.

Of course, one way to prevent this Brouwerian counterexample from causing problems is to use a different definition of fibration for interpreting types. From an abstract point of view this can be achieved by taking any monomorphism to be a cofibration, by definition. The reflexivity map $r^X : X \to X^I$ is in general a (split) monomorphism and so a cofibration. From here one can show that it is in
fact a trivial cofibration and thereby use it to implement identity types. From a syntactic point of view it can be achieved by adding so called regularity or normality conditions to the definition of Kan operations, which state that composition along a degenerate open box is the identity. Indeed, at one point in the development of cubical type theory, Coquand and collaborators did use such a regularity condition. Although this was never published, it was in many ways very successful and did lead to a version of type theory where path types are identity types, with some higher inductive types and is believed to be consistent and have good computational properties. The only problem with this approach, as discovered by Dan Licata, is that it is completely unclear how to construct a universe satisfying univalence. See [9] for some informal discussion of this issue online.

The aim of this paper is to give a wide class of counterexamples that apply not just to one model, but to a range of different categories with varying definitions of cofibration and thereby also varying definitions of fibration. We will look at a class of structures based on models of type theory in presheaves and in particular presheaf assemblies (for instance cubical assemblies as defined by Uemura in [25]). We develop two basic techniques.

(1) Using the assumption that path types are already identity types to show that certain maps have to be cofibrations.

(2) Using a univalent universe to show that certain cofibrations are pointwise stable under double negation.

We will then combine these to derive statements that are non constructive and in realizability models outright false.

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2. General Set Up

We work over a setting based on a definition due to Van den Berg and Frumin in [27], in turn based on a definition due to Gambino and Sattler in [7]. We assume that the reader is already familiar with the notions used there such as left/right lifting problems, wfs’s and pushout products. The Van den Berg-Frumin definition can also be seen as a purely homotopical reformulation of the Orton-Pitts axioms in [17]. See e.g. [23, Section 7.5.2] for a discussion of the precise relation between these two approaches. We weaken the Van den Berg-Frumin definition in a few ways. Most importantly, we weaken the requirement that the underlying category is a topos to locally cartesian closed category with finite colimits. This is necessary to even include presheaf assemblies as an example. We drop the requirement that cofibrations are classified by a single universal cofibration, which is the object $Cof$ in the Orton-Pitts formulation, and $\Sigma$ in the the Van den Berg-Frumin formulation.

We also have no strictness condition (Orton and Pitts’ axiom $ax_9$), although there is an important point here. We will derive some statements very much related to strictness from the assumption that a univalent universe exists.

Essentially we consider locally cartesian closed categories with a good notion of cofibration and interval object, where cofibrations are closed under pullback and finite union, and generating trivial cofibrations are given by the pushout product of a cofibration with an endpoint inclusion. Formally, we state this as follows.

Let $\mathcal{C}$ be a locally cartesian closed category with all finite colimits. We assume we are also given a class of maps whose elements we call cofibrations and an interval object $\delta_0, \delta_1 : 1 \to I$. We use these to define the following classes of maps.
Definition 2.1. (1) We say a map is a **trivial fibration** if it has the right lifting property against every cofibration.

(2) We say a map is a **fibration**, if it has the right lifting property against $\delta_0 \times m$ and $\delta_1 \times m$ for every cofibration $m$.

(3) We say an object $X$ is **fibrant** if the unique map $X \to 1$ is a fibration.

(4) We say an object $X$ is **cofibrant** if the unique map $0 \to X$ is a cofibration.

(5) We say a map is a **trivial cofibration** if it has the left lifting property against every fibration.

We assume throughout that all of the following conditions are satisfied.

(1) We assume that cofibrations are closed under pullback.

(2) We assume that cofibrations are closed under binary unions.

(3) We are given connections on $I$ as defined in [27].

(4) We assume $\delta_0$ and $\delta_1$ are disjoint as subobjects of $\mathbb{I}$.

(5) We assume that $\delta_0$ and $\delta_1$ are cofibrations.

(6) We assume that every map factors as a cofibration followed by a trivial fibration.

(7) We assume that for every map $X$, the map $0 \to X$ is a cofibration. That is, every object is cofibrant.

There are two main ways to satisfy the requirement that every map factors as a cofibration followed by a trivial fibration. One way is to assume that cofibrations form a dominance on $C$, which is the case if the remaining Orton-Pitts axioms are assumed (this is the approach taken by Van den Berg and Frumin). The other way is to assume cofibrations are the left class in a cofibrantly generated wfs, generated using a version of the small object argument. This could be an approach using external transfinite colimits, such as Garner’s small object argument [8], but could also be an internal version such as the one developed by the author in [24].

We require that path objects are, by definition, constructed using exponentiation with the interval, as defined below.

Definition 2.2. Given any object $X$, we define the **path object on $X$**, to be the object $X^I$ (which we will also denote $P^X$) together with the maps $r^X$, $p_0^X$ and $p_1^X$, where $r^X : X \to X^I$ is the constant map, and $p_0^X, p_1^X : X^I \to X^1 \cong X$ are given by composition with $\delta_0$ and $\delta_1$ respectively.

The link between identity types in type theory and very good path objects is one of the key ideas in homotopy type theory. See for example the well known results of Gambino and Garner in [6] or Awodey and Warren in [2]. In order for path types to be used as identity types along these lines it is necessary for $r^X$ to be a trivial cofibration. We focus on the condition that $r^X$ is just a cofibration, which of course follows from the assumption that $r^X$ is a trivial cofibration.

Definition 2.3. We say **path types are identity types** if for every fibrant object $X$, the map $r^X$ is a cofibration.

Recall that $P$ can be extended to a fibred functor over cod as follows. Given a map $f : X \to Y$, we define $P(f)$ to be given by the pullback below, where the bottom map $Y \to P(Y)$ corresponds to the projection $Y \times 1 \to Y$ under

\[ 1 \text{In fact with a little work one can show the converse also holds, but we don’t need that here, since we are not constructing identity types but giving conditions that imply path objects are not identity types.} \]
the adjunction.

\[
P(f) \xrightarrow{\delta} P(X) \xleftarrow{\delta} P(Y)
\]

When \( f \) is clear from the context, we will also write \( P(f) \) as \( P_Y(X) \). One can further define maps \( r^f : X \rightarrow Y \), \( p_0^f : P_Y(X) \rightarrow X \) and \( p_1^f : P_Y(X) \rightarrow X \) to produce a factorisation of the diagonal map \( \Delta : X \rightarrow X \times_Y X \) in the slice category \( \mathcal{C}/Y \).

Note that for \( Y = 1 \), \( P_Y(X) \) is just \( P_X \).

For some of our results, including the main theorem, we will need a notion of propositional truncation. For this, we use the definitions below.

**Definition 2.4.** We say \( f : X \rightarrow Y \) is an hproposition if it is a fibration and the map \( P_Y(X) \rightarrow X \times_Y X \) is a trivial fibration.

We state below what it means for \( \mathcal{C} \) to have propositional truncation, although technically we will never require this to hold for \( \mathcal{C} \) itself. Instead we will assume that small maps are closed under propositional truncation, in a sense that we will define later (definition 4.3).

**Definition 2.5.** We say \( \mathcal{C} \) has propositional truncation if every fibration \( f : X \rightarrow Y \) factors as follows, where \( g \) is an hproposition and \( i \) has the left lifting property against every hproposition.

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{i} & & \downarrow{g} \\
\|X\| & & \|X\| \\
\end{array}
\]

We note that the axioms suffice to check a few basic propositions.

**Proposition 2.6.** Every trivial cofibration is a cofibration.

*Proof.* It suffices to show that every generating trivial cofibration is a cofibration. That is, for every cofibration \( m : A \rightarrow B \), and for \( i = 0, 1 \), \( \delta_i \times m \) is a cofibration. However, the pushout product \( \delta_i \times m \) can be viewed as a union of cofibrations into \( B \times I \), and we assumed that cofibrations are closed under finite union. \( \square \)

**Proposition 2.7.** Every trivial fibration has a section.

*Proof.* This easily follows from the assumption that every object is cofibrant. \( \square \)

Recall from [27, Section 3.1] that I can be used to define a notion of homotopy and so also homotopy equivalence, as well as the stronger notion of strong homotopy equivalence.

**Proposition 2.8.** A fibration \( f : X \rightarrow Y \) between fibrant objects \( X \) and \( Y \) is a trivial fibration if and only if it is a strong homotopy equivalence.

*Proof.* See [27, Proposition 4.1] \( \square \)

**Proposition 2.9.** Every map between fibrant objects factors as a homotopy equivalence followed by a fibration.

*Proof.* See [27, Proposition 4.3]. \( \square \)

**Proposition 2.10.** Dependent products preserve fibrations.

*Proof.* See [27, Proposition 4.5]. \( \square \)
Proposition 2.11. Every object is a subobject of a fibrant object.

Proof. Given an object $X$, we factorise the map $X \to 1$ as a cofibration followed by a trivial fibration, to get $X \to X' \to 1$, where the map $X \to X'$ is a cofibration and in particular a monomorphism and the map $X' \to 1$ is a trivial fibration, and so in particular a fibration. □

Proposition 2.12. Hpropositions are closed under arbitrary pullbacks.

Proof. This follows from the fact that path types, fibrations and trivial fibrations are closed under pullbacks. □

Proposition 2.13. If $f : X \to Y$ is an hproposition and has a section then it is a trivial fibration.

Proof. Let $s : Y \to X$ be a section of $f$ and let $t : X \times_Y X \to P_Y X$ be a section of the map $P_Y X \to X \times_Y X$. We exhibit $f$ as a retract of the trivial fibration $p_f : P_Y(X) \to X$, which shows it is also a trivial fibration.

\[
\begin{array}{c}
X \xrightarrow{(\text{pr}_1, s \circ f)} X \times_Y X \xrightarrow{t} P_Y X \xrightarrow{p_f} X \\
\downarrow f \quad \downarrow s \quad \downarrow t \quad \downarrow p_f \quad \downarrow f \\
Y \xrightarrow{\text{pr}_1} X \xrightarrow{f} Y
\end{array}
\]

□

Lemma 2.14. Suppose we are given two monomorphisms $m_0 : A_0 \to B$ and $m_1 : A_1 \to B$, with at least one of $m_0$ and $m_1$ a cofibration, and an hproposition $f : X \to B$ together with two maps $t_0$ and $t_1$ in the following commutative diagram.

\[
\begin{array}{c}
X \xrightarrow{t_0} A_0 \xrightarrow{m_0} B \xrightarrow{m_1} A_1 \\
\downarrow f \\
X \xrightarrow{f} B
\end{array}
\]

Write the union of $m_0$ and $m_1$ as $m : A_0 \cup A_1 \to B$.

Then there is a map $t : A_0 \cup A_1 \to X$ making the following diagram commute.

\[
\begin{array}{c}
X \xrightarrow{t} A_0 \cup A_1 \xrightarrow{m} B
\end{array}
\]

Proof. Without loss of generality say that $m_0$ is a cofibration.

First, observe that the result would be trivial if we knew that $t_0$ and $t_1$ agreed on $A_0 \cap A_1$. We therefore aim to produce a new map $t'_1$ ensuring that $t_0$ and $t'_1$ agree on $A_0 \cap A_1$.

If we pullback $f$ along $m_1$, then the resulting map $m_1^*(f) : m_1^*(X) \to A_1$ is also an hproposition, and using $t_1$ we can show it has a section. We deduce by proposition 2.13 that it is a trivial fibration. Furthermore, observe that the inclusion $t_1 : A_0 \cap A_1 \to A_1$ is a pullback of $m_0$, and so a cofibration. We will define a lifting problem of $t_1$ against $m_1^*(f)$. Let $t_0 : A_0 \cap A_1 \to m_1^*(X)$ be the pullback of $t_0$ along $m_1$, so that if $\pi_0 : m_1^*(X) \to X$ is one of the projections in the pullback then...
π₀ ∘ t₀ = t₀ ∘ t₀. Then let j be a diagonal filler in the following diagram.

We then define t₁′ to be π₀ ∘ j. One can then verify that f ∘ t₁′ = m₁ and t₁′ ∘ t₁ = t₀ ∘ t₀, and we can now easily define the required t using the universal property of the union.

**Proposition 2.15.** Let X be an object of C. The following are equivalent.

1. The map r^X : X → X^X is an isomorphism.
2. The map r^X : X → X^X is a regular epimorphism.
3. The statement “every function 1 → X is constant” holds in the internal language of C.

*Proof.* Note that r^X is a split monomorphism in any case (with retraction X^δ₀). It follows that 1 and 2 are equivalent.

Showing 2 and 3 are equivalent is straightforward. □

**Definition 2.16.** We say an object X is discrete if one of the equivalent conditions in proposition 2.15 holds.

**Proposition 2.17.** Let X be an inhabited object of C. The following are equivalent.

1. The constant function map 2 → 2^X is an isomorphism.
2. The constant function map 2 → 2^X is an epimorphism.
3. The statement “all functions from X to 2 are constant” holds in the internal language.
4. (When C has a subobject classifier) the following statement holds in the internal language “if U and V are disjoint subobjects of X such that X = U ∪ V, then either X = U or X = V.”

**Definition 2.18.** We say an inhabited object X is connected if one of the equivalent conditions in proposition 2.17 holds.

**Proposition 2.19.**

1. If X has decidable equality and the interval is connected, then X is discrete.
2. If X is discrete and Y is any object, then X^Y is discrete.
3. If X is discrete and m : Z → X is a subobject then Z is discrete.
4. If C is a category of presheaves over a category C, the interval is representable and C has finite products then every constant presheaf is discrete.
5. Every map between discrete objects is a fibration.

*Proof.* 4 was already observed by Uemura in [25, Proposition 4.7]. The rest are straightforward. □

Finally we recall the following notions of ¬¬-stability, density and separation.

**Definition 2.20.** Let m : X → Y be a monomorphism in a category C. We say m is ¬¬-stable if the following statement holds in the internal language of C.

∀ y ∈ Y, ¬¬(∃ x ∈ X, m(x) = y) → (∃ x ∈ X, m(x) = y)

We say m is ¬¬-dense if the following statement holds in the internal language.

∀ y ∈ Y, ¬¬(∃ x ∈ X, m(x) = y)
We say an object $X$ is $\sim\sim$-separated if the diagonal map $X \to X \times X$ is $\sim\sim$-stable, or equivalently, if the following statement holds in the internal language.

$$\forall x, y \in X, \sim\sim(x = y) \to x = y$$

Suppose $C$ is a category of internal presheaves over an internal category $E$. We say a monomorphism $m$ in $C$ is $\sim\sim$-stable if the underlying map in $E/\text{Ob}(C)$ is $\sim\sim$-stable, or equivalently if the following statement holds in the internal language of $E$: for every object $e$ of $C$, $m_e$ is $\sim\sim$-stable. We similarly define pointwise $\sim\sim$-dense and pointwise $\sim\sim$-separated.

3. Cofibrations when Path Types are Identity Types

In this section we will use the assumption that path types are identity types to show that certain maps are cofibrations.

**Lemma 3.1.** Suppose that $\mathbb{I}$ is connected. Suppose that $C$ possesses a natural number object $\mathbb{N}$. Suppose that path types are identity types. Then the map $1 \to 2^\mathbb{N}$ given by $\lambda x.\lambda n.0$ is a cofibration.

**Proof.** First, note that $\mathbb{I}$ is not necessarily fibrant, but we can embed it in a fibrant object $\hat{\mathbb{I}}$ using proposition 2.11. Write $i$ for the inclusion $\mathbb{I} \to \hat{\mathbb{I}}$.

Since $\mathbb{I}$ is connected, $\mathbb{N}$ is fibrant. Since dependent products preserve fibrations, $\hat{\mathbb{N}}$ is also fibrant. Hence by the assumption that path types are identity types, $r^{\hat{\mathbb{N}}}$ is a cofibration. We will show that the map $1 \to 2^\mathbb{N}$ is a cofibration by exhibiting it as a pullback of $r^{\hat{\mathbb{N}}}$.

We define the map $e: 2^\mathbb{N} \to (\hat{\mathbb{N}})^\mathbb{I}$ as below.

$$e(\alpha)(i)(n) = \begin{cases} i(0) & \alpha(n) = 0 \\ i(i) & \alpha(n) = 1 \end{cases}$$

To check that $\lambda x.\lambda n.0$ is a pullback of $r^{\hat{\mathbb{N}}}$ along $e$, it suffices to show, in the internal logic of $C$, that for all $\alpha \in 2^\mathbb{N}$, $\alpha = \lambda n.0$ if and only if $e(\alpha)$ lies in the image of $r^{\hat{\mathbb{N}}}$, which is the case if and only if $e(\alpha)$ is constant as a function $\mathbb{I} \to \hat{\mathbb{N}}$.

It is easy to check that $e(\lambda n.0)$ is constant. Hence we just show the converse, that if $e(\alpha)$ is constant then for all $n \in \mathbb{N}$, $\alpha(n) = 0$.

So suppose that $\alpha \in 2^\mathbb{N}$ and $e(\alpha)$ is constant. Let $n$ be an element of $\mathbb{N}$. We know that $\alpha(n) = 0$, or $\alpha(n) = 1$, so to show that $\alpha(n) = 0$ it suffices to show that $\alpha(n) \neq 1$. Suppose that $\alpha(n) = 1$. Since $e(\alpha)$ is constant, for all $i, i' \in \mathbb{I}$ we have $e(\alpha)(i)(n) = e(\alpha)(i')(n)$, and so $i = i'(\lambda n.0)$. In particular, applying this to the endpoints 0 and 1, we have $i(0) = i(1)$ and so $0 = 1$ since $i$ is monic. But then we get a contradiction by the disjointness of the endpoints. We have now shown $\alpha(n) = 0$. This applies for arbitrary $n$, and so $\alpha = \lambda n.0$. $\square$

We next show that if we have exact quotients then we can in fact show that all monomorphisms are cofibrations. This doesn’t apply in presheaf assemblies, but does work for the usual definition of presheaves and in fact for any II-pretopos.

**Theorem 3.2.** Suppose that $C$ is a II-pretopos (i.e. $C$ has exact quotients) and path types are identity types. Then any monomorphism is a cofibration.

**Proof.** Let $m: A \to B$ be any monomorphism. We need to show it is a cofibration.

Working in the internal logic of $C$, we define an equivalence relation on $I \times B$. Given $(i, b)$ and $(i', b')$ in $I \times B$, we set $(i, b) \sim (i', b')$ if $b = b'$ and either $i = i'$ or $m^{-1}((b))$ is inhabited. It is straightforward to verify that this is an equivalence relation.

By proposition 2.11 there exists a fibrant object $X$ such that $I \times B/\sim$ is a subobject of $X$. Say $\iota: I \times B/\sim \to X$ is the subobject inclusion.
We define \( e : B \to X^1 \) to be the map that sends each \( b \in B \) to \( \lambda i. i ([i, b]) \).

Now, still reasoning internally in \( C \), we show that for each \( b \in B \), \( m^{-1} \{ b \} \) is inhabited if and only if \( e(b) \) lies in the image of \( r^X \).

Suppose first that \( m^{-1} \{ b \} \) is inhabited. Then for any \( i, i' \in I \), we have \( [i, b] = [(i', b)] \). Hence \( e(b) = \lambda i. i ([i, b]) \) is a constant function, and so lies in the image of \( r^X \).

Conversely, suppose that \( e(b) = \lambda i. i ([i, b]) \) lies in the image of \( r^X \). Then it is a constant function, and in particular we have \( e([0, b]) = e([1, b]) \), where 0 and 1 are the images of \( \delta_0 \) and \( \delta_1 \) respectively. Since \( e \) is monic, we deduce \( [0, b] = [1, b] \).

Since quotients are exact, we now have \( (0, b) \sim (1, b) \), and so either \( 0 = 1 \), or \( m^{-1} \{ b \} \) is inhabited. But we assumed the endpoints are disjoint and so we do have that \( m^{-1} \{ b \} \) is inhabited as required.

Since \( r^X \) is monic, we can now deduce that \( e \) fits into a pullback diagram as below.

\[
\begin{array}{ccc}
A & \xrightarrow{m} & X \\
\downarrow & & \downarrow \phi^X \\
B & \xrightarrow{e} & X^1
\end{array}
\]

This witnesses \( m \) as a pullback of the cofibration \( r^X \), and so is itself a cofibration, as required. \( \square \)

4. Cofibrations and Univalent Universes

We specialise to the case where \( C \) is a category of internal presheaves. That is, we fix a locally cartesian closed category \( E \) with all finite colimits and an internal category \( C \) in \( E \). We then take \( C \) to be the category of internal presheaves over \( C \). We will follow the convention that \( C \) is non trivial, in the sense that \( C \) has at least one object.

**Definition 4.1.** Suppose we are given a map \( \xi : \hat{U} \to U \). We say a map \( f : X \to Y \) is \( U \)-small if it is a pullback of \( \xi \) along a map \( g : Y \to U \). We will refer to such a \( g \) as a classifying map for \( f \).

**Definition 4.2.** We say \( \xi : \hat{U} \to U \) is a universe if the following hold.

1. Every isomorphism is \( U \)-small.
2. \( U \)-Small maps are closed under composition.
3. \( U \)-Small maps are closed under dependent product.
4. \( U \)-Small maps are closed under pairwise coproduct.
5. \( U \)-Small maps are closed under mapping path spaces.

We say a universe \( \xi : \hat{U} \to U \) is a homotopical universe if in addition to the above, we have the following.

1. \( \xi \) is a fibration (or equivalently every \( U \)-small map is a fibration).
2. \( U \) is fibrant.

**Definition 4.3.** We say a homotopical universe \( U \) is closed under propositional truncation if every \( U \)-small fibration \( f \) factors as a map with the left lifting property against all hpropositions followed by a \( U \)-small hproposition.

We will often view \( \xi : \hat{U} \to U \) as a family of types \( \xi (x) \) indexed by the elements \( x \) of \( U \).

Note that using path objects and dependent products, we can translate one of the definitions of equivalence from type theory (as appear for instance in [26, Chapter 4]) into the formulation we are using here. We write \( \text{Equiv}(X, Y) \) for the object of equivalences from \( X \) to \( Y \). Observe that for any of the usual definitions
it is straightforward to show that every isomorphism is an equivalence. We use \( \text{Equiv}(X,Y) \) to define univalence as follows.

**Definition 4.4.** We say a homotopical universe \( \text{El}: \tilde{U} \to U \) is univalent if the first projection \( \Sigma_{X,Y} \text{Equiv}(X,Y) \to U \) is a trivial fibration.

**Remark 4.5.** In any case \( \Sigma_{X,Y} \text{Equiv}(X,Y) \to U \) is a fibration by our other conditions. Hence by proposition 2.8 it is a trivial fibration if and only if the statement that it is contractible holds in the model. This is equivalent to the univalence axiom holding in the model.

**Definition 4.6.** We fix a map \( u: 1 \to U \) such that the following is a pullback, and refer to it as the unit type.

\[
\begin{array}{ccc}
1 & \to & \tilde{U} \\
\downarrow & & \downarrow \\
1 & \to & U \\
\end{array}
\]

When \( C \) is a category of (possibly internal) presheaves over a category \( C \) we use the following notation. Since \( u \) is a global section of the presheaf \( U \), we can think of it as a choice of elements \( u(c) \) for each object \( c \) in the category \( C \).

We clearly have the following proposition.

**Proposition 4.7.** Suppose that \( f: X \to Y \) is a pullback of \( \text{El}: \tilde{U} \to U \) along a map of the form \( u \circ ! \), where \( !: Y \to 1 \) is the unique map \( Y \to 1 \). Then \( f \) is an isomorphism.

We will also need the following observations.

**Proposition 4.8.** Let \( U \) be a homotopical universe. Suppose that \( m: A \to B \) is a cofibration, \( A \) and \( B \) are both \( U \)-small objects and \( B \) is discrete. Then \( m \) is a small fibration and furthermore an hproposition.

**Proof.** We first note that since \( B \) is discrete, the subobject \( A \) must be too. Since \( m \) is a map between discrete objects it is a fibration by proposition 2.19. However, we still need to show that it is a small fibration. We first replace \( A \) with the mapping path space, which we view as a small fibration \( A' \to B \). Explicitly, we can define \( A' \) as a type internally in type theory with the following definition (where the equality is implemented using path types).

\[ A'(b) := \Sigma_{a:A} m(a) = b \]

This gives a well defined small fibration since \( B \) and \( A \) are small types, and universes are closed under path types. We now note that since \( B \) is discrete, we in fact have an isomorphism \( A' \cong A \), and so \( m \) is a small fibration as required. \( \square \)

**Lemma 4.9.** Let \( U \) be a homotopical universe closed under propositional truncation. Suppose that \( m: A_1 \to B \) are cofibrations, \( A_1 \) and \( B \) are \( U \)-small objects and \( B \) is discrete (but \( A_0 \cup A_1 \) is not necessarily small). By proposition 4.8 a \( U \)-small propositional truncation \( \| A_0 + A_1 \| \) exists. Then we have maps \( A_0 \cup A_1 \to \| A_0 + A_1 \| \) and \( \| A_0 + A_1 \| \to A_0 \cup A_1 \) forming commutative triangles in the following diagram.

\[
\begin{array}{ccc}
A_0 \cup A_1 & \to & \| A_0 + A_1 \| \\
\downarrow & & \downarrow \\
B & \to & \| A_0 + A_1 \| \\
\end{array}
\]
Proof. We first construct the map $A_0 \sqcup A_1 \to \|A_0 + A_1\|$.

We clearly have maps $t_i$ for $i = 0, 1$ in the following commutative diagram.

Since $\|A_0 + A_1\|$ is an hproposition and each $m_i$ is a cofibration we can apply lemma 2.14 to get the required map $A_0 \sqcup A_1 \to \|A_0 + A_1\|$.

We next construct the map $\|A_0 + A_1\| \to A_0 \sqcup A_1$. Since $m: A_0 \sqcup A_1 \to B$ is a map between discrete objects, it is a fibration, albeit not necessarily small. Since it is a monomorphism and a fibration, it is an hproposition. Hence we can obtain the required map as a filler in the following lifting problem.

We will now see the first key lemma of this section. We will later give a more general lemma, but this one is simpler and therefore easier to understand, and is already useful in presheaf assemblies where all objects are pointwise \(\neg\neg\)-separated.

Lemma 4.10. Let $C$ be a category of internal presheaves in a locally cartesian closed category with finite colimits and disjoint coproducts. Suppose we are given a univalent universe $El: \tilde{U} \to U$ and two maps $m_0: A_0 \to B$ and $m_1: A_1 \to B$ satisfying the following conditions.

1. $U$ is pointwise \(\neg\neg\)-separated.
2. $U$ is closed under propositional truncation.
3. Both $m_0$ and $m_1$ are cofibrations.
4. $B$ is discrete.
5. $A_0$, $A_1$ and $B$ are $U$-small (but note that $A_0 \sqcup A_1$ does not need to be $U$-small).

Write $m$ for the union $A_0 \sqcup A_1 \to B$.

Then $m$ is pointwise \(\neg\neg\)-stable.

Proof. We first note that since $B$ is discrete, each $m_i$ is a small fibration by proposition 4.8.

We avoided assuming that $A_0 \sqcup A_1$ is small. We note however, that the “homotopy union” of $A_0$ and $A_1$ is necessarily small, since $U$ is closed under coproducts and propositional truncation. Explicitly, we define another small fibration $f: C \to B$ using the definition below.

Let $\gamma: B \to U$ be a classifying map for $f$.

Since cofibrations are closed under unions, $m$ is a cofibration. Since $U$ is univalent the map $\pi_X: \Sigma_X \Sigma_Y \to \Sigma_Y \Sigma_X$ is a trivial fibration. We will aim to define a lifting problem of $m$ against $\pi_X$, as illustrated below.

$$C(b) := \|A_0(b) + A_1(b)\|$$
We take the bottom map \( B \to U \) to be \( \gamma \), which we recall was a classifying map for the small fibration \( f : C \to B \).

The next step is to construct the top map \( \alpha \) of the lifting problem, which needs to map from \( A_0 \cup A_1 \) to \( \Sigma_{X,Y} \Sigma_{Y,U} \text{Equiv}(X,Y) \). Note that this amounts to constructing maps \( \xi, \zeta : A_0 \cup A_1 \to U \) together with an equivalence \( e \) over \( A_0 \cup A_1 \) between \( \xi^*(U) \) and \( \zeta^*(U) \). First note that in order for the lifting problem to be a commutative square, we are forced to take \( \xi \) to be \( \gamma \circ m \).

The key to the proof is that we define \( \zeta \) to be \( \text{wo!}_{A_0 \cup A_1} \). Informally, the \( Y \) component of the map from \( A_0 \cup A_1 \) to \( \Sigma_{X,Y} \Sigma_{Y,U} \text{Equiv}(X,Y) \) is constantly equal to the unit type.

It still remains to construct the equivalence \( e \). Since the definition of equivalence does not require defining small types we no longer need to work “inside \( U \).” Therefore, as we stated above, it suffices to construct an equivalence in \( \mathcal{C} \) over \( A_0 \cup A_1 \) between \( m^*(f) \) and the identity on \( A_0 \cup A_1 \). Note that it suffices to show that the map \( m^*(f) : m^*(C) \to A_0 \cup A_1 \) is a strong homotopy equivalence. Hence by proposition 2.8 it suffices to show it is a trivial fibration. Recall that we constructed \( f : C \to B \) by interpreting the type \( \| A_0(b) + A_1(b) \| \), which is an hproposition. Hence the pullback \( m^*(f) \) is also an hproposition. Therefore to show it is a trivial fibration, it suffices by proposition 2.13 to show it has a section, which easily follows from lemma 4.9.

So we do have a well defined map \( \alpha : A_0 \cup A_1 \to \Sigma_{X,Y} \Sigma_{Y,U} \text{Equiv}(X,Y) \) such that \( \pi_X \circ \alpha = \gamma \) and \( \pi_Y \circ \alpha = \text{wo!}_{A_0 \cup A_1} \). Let \( j \) be a diagonal filler as in (1).

We now use all of this to show \( m \) is locally \( \sim \sim \)-stable. We recall that we are working in a category of presheaves over a category \( \mathcal{E} \) and switch to the internal logic of \( \mathcal{E} \). Let \( c \) be an object of \( \mathcal{C} \) and let \( b \in B(c) \). Suppose that \( m^{-1}_c(b) \) is not empty. We need to show that it is inhabited.

Note that if \( m^{-1}_c(b) \) was inhabited, then the upper triangle in the lifting diagram would imply that \( \pi_Y(j_c(b)) = u(c) \). We can therefore deduce that \( \pi_Y(j_c(b)) \) is not equal to \( u(c) \). However, we can now apply the fact that \( U \) is pointwise \( \sim \sim \)-separated to show that in fact \( \pi_Y(j_c(b)) \) is equal to \( u(c) \). Furthermore, for all \( \sigma : c' \to c \) in \( \mathcal{C} \), we have that \( m^{-1}_c(B(\sigma)(b)) \) is not empty, and so we similarly can show that \( \pi_Y(j_{c'}(B(\sigma)(b))) \) is equal to \( u_{c'}(B(\sigma)(b)) \). Therefore, if \( b : yc \to B \) is the map corresponding to \( b \) under Yoneda, then the composition \( \pi_Y \circ j \circ b \) factors through the unit type \( u : 1 \to U \). Hence the pullback of \( Y \) along \( b \) is an isomorphism. Furthermore, we can pullback the equivalence to obtain an equivalence between \( b^*(Y) \) and \( b^*\|A_0 + A_1\| \). We deduce that \( b^*\|A_0 + A_1\| \) has a section. This gives us an element of \( \|A_0 + A_1\| \) in the fibre of \( b \).

Finally, applying the map \( \|A_0 + A_1\| \to A_0 \cup A_1 \) from lemma 4.9 gives us an element of \( A_0 \cup A_1 \) in the fibre of \( b \) as required.

We will now aim towards another, more general result, which allows us to replace the assumption that \( U \) is \( \sim \sim \)-separated with a much weaker (but more complicated) requirement. We will further assume that the universe contains “contractibility representations” in a sense that we will define below.

**Definition 4.11.** Let \( V \) be a universe in \( \mathcal{E} \). A **weakly \( \sim \sim \)-stable unit** is a unit \( u : 1 \to V \) such that the following holds in the internal logic of \( \mathcal{E} \).

1. \( \text{El}(u) \) has exactly one element.
2. For all \( x \in V \), if \( \sim \sim(x = u) \), then \( \text{El}(x) \) has at most one element.

A key idea is that although the definition of weakly \( \sim \sim \)-stable unit still sounds a little strong when working in intuitionistic logic, it does hold in constructive set theory, using the axiom of extensionality. An earlier version of this idea is mentioned by Orton and Pitts in [17, Remark 8.7].
Lemma 4.12. Work over CZF + Inacc, take $E$ to be the category of sets, and $V$ to be an inaccessible set. Then $V$ has a weakly $\neg\neg$-stable unit.

Proof. We take $z$ to be the small set $\{\emptyset\}$ (which is the usual implementation of the terminal object in Set anyway). Suppose that $x$ is a (small) set and that the double negation of $x = z$ holds. Now let $y$ be any element of $x$. Suppose that $y$ contains an element $w$. Then $y \neq \emptyset$, and so $y \notin z$. Hence $x \neq z$ by extensionality, contradicting the double negation of $x = z$. But we have now shown that every element $y$ of $x$ is empty, and so $x$ is a subset of $\{\emptyset\}$. We can now deduce that $x$ has at most one element, as required. \qed

We note furthermore that there is another example of a weakly $\neg\neg$-stable unit in the effective topos, or more generally any realizability topos. In [21 Section 3], Streicher observed that one can construct universes in realizability toposes using ideas developed by Awodey, Butz, Simpson and Streicher in [1], as follows.

Assuming the existence of an inaccessible ordinal $\kappa$ one can obtain a set sized version of McCarty’s model of IZF from [10] by truncating the definition of $V(A)$ at level $\kappa$. Streicher then makes this into an object $\mathbf{Mc}(A)$ of the realizability topos using the same definition of equality as used in set theory to obtain a universe.

Lemma 4.13. For any pca $A$, the universe $\mathbf{Mc}(A)$ defined above possesses a weakly $\neg\neg$-stable unit.

Proof. Since $V_\kappa(A)$ is a model of set theory, we can carry out exactly the same argument as in lemma 4.12 internally in the model. Since the definition of equality in the topos is the same as in the set theoretic model, it follows that we do get a weakly $\neg\neg$-stable unit in the topos. \qed

As before, we will exploit the fact that we are working in a category of presheaves.

Definition 4.14. Let $U$ be a universe in $C$. We say a pointwise weakly $\neg\neg$-stable unit is a map $u : 1 \to U$ with the following property. In the internal logic of $E$ we have that for every $c \in C$ and every $x \in U(c)$, if $\neg\neg x = u(c)$ then $\text{El}(c, x)$ has at most one element.

Definition 4.15. If $E$ has a universe $V$, recall that we can define the Hofmann-Streicher universe $V_C$ in $C$ as follows. Given an object $c \in C$, we take $(V_C)_c$ to be the collection of “small presheaves” on the category $\int_y \text{yc}$, as defined by Hofmann and Streicher in [10]. The action of morphisms is defined via composition. Defining this internally in $E$ takes a little care, but this has been done by Uemura in [25 Section 4.1].

Definition 4.16. Let $V$ be a universe in $E$. A homotopical Hofmann-Streicher universe on $V$ is a homotopical universe $U$ together with a map $i : U \to V_C$ with the following property. Let $\chi : Y \to V_C$ be any map. Then $\chi$ factors through $i$ if and only if the pullback of $\text{El}$ along $\chi$ is a fibration.

Remark 4.17. In [25], Uemura used techniques developed byLicata, Orton, Pitts and Spitters in [14] to extend a Hofmann-Streicher universe in cubical assemblies to a homotopical Hofmann-Streicher universe.

Lemma 4.18. Suppose that $V$ has a weakly $\neg\neg$-stable unit and $U$ is a homotopical Hofmann-Streicher universe on $V$. Then $U$ has a pointwise weakly $\neg\neg$-stable unit.

Proof. Suppose that $u : 1 \to V$ is a weakly $\neg\neg$-stable unit in $V$. First note that $V_C$ has a pointwise weakly $\neg\neg$-stable unit $u' : 1 \to V_C$ defined as follows. In the internal logic of $E$ we need to define a map $1 \to V_C(c)$ for each object $c$ of $C$. An element of $V_C(c)$ consists first of a map $\Sigma_{d \in C} \text{hom}(d, c) \to V$. We define each such map
to be constantly equal to $u$. We take the action on morphisms to be the identity everywhere.

Next, note that the identity on 1 is an isomorphism and so a fibration, and so $u'$ does factor through $i: U \to V_C$ to give a map $u'': 1 \to U$.

We finally need to check that $u''$ is pointwise weakly $\neg\neg$-stable. That is, we need to show, in the internal logic of $\mathbb{E}$, that for all $c \in C$ and every $x \in U(c)$, if $\neg\neg x = u''(c)$ then $\text{El}(x)$ has at most one element. First, note that if $\neg\neg x = u''(c)$, then also $\neg\neg i(x) = u'(c)$. It follows that for every $d \in C$, and every $f: d \to c$, we have $\neg\neg i(x)(d, f) = u$. In particular we have $\neg\neg i(x)(c, 1_c) = u$. Hence $\text{El}(i(x)(c, 1_c))$ has at most one element. But this is precisely the definition of $\text{El}(i(x))(c)$. □

**Definition 4.19.** Let $f: X \to Y$ be a small hproposition with respect to some universe $U$. We say a contractibility representation for $f$ is a small fibration $g: Z \to Y$ that has a section $z_0: Y \to Z$, and such that there is a homotopy equivalence $e$ in the following diagram, where $\text{IsContr}_{z_0}(Z)$ is the result of interpreting $\Pi_{z: Z \ z = z_0}$ using path types and dependent products in the usual way.

$$
\begin{array}{ccc}
X & \xrightarrow{e} & \text{IsContr}_{z_0}(Z) \\
\downarrow{f} & & \downarrow{=} \\
Y & & \\
\end{array}
$$

**Remark 4.20.** One can construct contractibility representations working internally in homotopy type theory under reasonable conditions about the existence of higher inductive types. For example, given an hproposition $X$, one can show using univalence that if the suspension $\text{Susp}(X)$ exists then it is a contractibility representation of $X$. Alternatively one can also use set quotients together with univalence.

**Lemma 4.21.** Suppose that $C$ is a category of presheaves over an internal category $\mathcal{C}$ in $\mathbb{E}$ and that all of the following hold.

1. $\mathcal{C}$ has a univalent universe $U$.
2. $U$ has a pointwise weakly $\neg\neg$-stable unit.
3. $U$ is closed under propositional truncation.
4. $U$ has contractibility representations for all small hpropositions.

Let $A_0$, $A_1$ and $B$ be small and discrete objects of $\mathcal{C}$. If $m_i: A_i \to B$ are cofibrations for $i = 0, 1$ then the union $m: A_0 \cup A_1 \to B$ is pointwise $\neg\neg$-stable.

**Proof.** We start by following the same proof as for lemma 4.10. We recall that this allows us to define the small hproposition $f: C \to B$ defined as below.

$$C(b) := ||A_0(b) + A_1(b)||$$

Next, let $g: D \to B$ be a contractibility representation for $f$, and let $\beta: B \to U$ be a classifying map for $g$.

As before, we next define a map $\alpha: A \to \Sigma_{X, U} \Sigma_{Y, U} \text{Equiv}(X, Y)$ that we will use along with $\beta$ to define a lifting problem.

Given $a \in A$, we need to define small types $X(a)$ and $Y(a)$, together with an equivalence $e$ between them. As before, we are forced to take $X(a)$ to be $\beta(m(a))$ in order for the square to commute. We take $Y(a)$ to be the pointwise weakly $\neg\neg$-stable unit.

We now need to construct an equivalence between $m^*(D)$ and the unit type over $A$. First, following the proof of lemma 4.10 we note that we can use lemma 4.9 to construct a section of $m^*(C)$. It follows that we can construct a section of $\text{IsContr}(m^*(D))$, and so we obtain the required equivalence from the observation that any two contractible types are equivalent.
Now since $m$ is a cofibration by assumption, we have a diagonal filler $j$ in the diagram below.

\[
\begin{array}{ccc}
A & \xrightarrow{\Sigma_X U \Sigma_Y U} & \text{Equiv}(X,Y) \\
\downarrow m & & \downarrow \\
B & \xrightarrow{\beta} & U
\end{array}
\]

We take $y: B \rightarrow U$ to be the composition of $j$ with the projection to the $Y$ component. We will write $Y$ for the pullback of $\text{El}$ along $y$.

We now recall that we are working in a category of internal presheaves and switch to the internal logic of $E$. Let $c$ be an object of $C$ and let $b \in B(c)$. We will deduce that $m^{-1}({\{b\}})$ is inhabited from its double negation. As before, note that if $m^{-1}({\{b\}})$ is inhabited, then the upper triangle of (2) implies that $y_c(b)$ is equal to $u$ for all $c \in C$. Hence, if $m^{-1}({\{b\}})$ is not inhabited, then $y_c(b)$ is not not equal to $u(c)$. Since $u(c)$ is weakly $\neg\neg$-stable, we deduce that if $m^{-1}({\{b\}})$ is not not inhabited then $Y(c; b)$ has at most one element for every $c \in C$. Hence, by the same argument as in lemma 4.10 the pullback of $Y$ along the map $\beta: y_c \rightarrow B$ is an hproposition. Since the pullback of $D$ along $\beta$ is equivalent to the pullback of $Y$ over $y_c$, it is also an hproposition. We can now use the definition of contractibility representation to show that $\beta^*(C)$ has a section.

Finally, as in the proof of lemma 4.10 using lemma 4.9 and the discreteness of $B$ we can deduce that $m^{-1}({\{b\}})$ is inhabited.

\[\square\]

5. The Counterexamples

We now give the counterexamples. We first show that it is impossible to take identity types to be path types in certain models of univalent type theory in presheaf assemblies. We assume that the reader is familiar with standard definitions and results in realizability. See e.g. [28] for a good introduction. Recall that the lesser limited principle of omniscience is defined as follows.

**Definition 5.1.** The lesser limited principle of omniscience (LLPO) states that if $\alpha: N \rightarrow 2$ is a binary sequence such that $\alpha(n) = 1$ for at most one $n$, then either $\alpha(2n) = 0$ for all $n$, or $\alpha(2n + 1) = 0$ for all $n$.

**Lemma 5.2.** Suppose the following.

1. $E$ is a locally cartesian closed category with finite colimits and disjoint coproducts.
2. $C$ is a category of internal presheaves in $E$.
3. $C$ possesses a class of cofibrations and an interval satisfying our general conditions.
4. The interval object $1$ in $C$ is connected.
5. $C$ has a univalent universe $U$, satisfying the following
   a. $N$ is $U$-small
   b. $U$ is closed under propositional truncation.
   c. $U$ is pointwise $\neg\neg$-separated.
6. Path types are identity types.

Then LLPO holds in the internal logic of $E$.

**Proof.** We aim to apply lemmas 3.1 and 4.10.

We define $B$ to consist of those $\alpha(n)$ in $2^N$ such that $\alpha(n) = 1$ at most once. Note that $\Delta$ preserves exponentials, limits, colimits and the natural number object. Hence $\Delta B$ is also a subobject of $2^N$ in $C$ and so discrete.
We define two subobjects $A_0$ and $A_1$ of $B$ as below.

$$A_0 := \{ \alpha \in B \mid \forall n \in \mathbb{N}, \alpha(2n) = 0 \}$$

$$A_1 := \{ \alpha \in B \mid \forall n \in \mathbb{N}, \alpha(2n + 1) = 0 \}$$

Observe that $\Delta(A_0)$ and $\Delta(A_1)$ can both be written as pullbacks of the constant map $1 \to 2^\mathbb{N}$, which is a cofibration by lemma 3.1, and so the inclusions are both cofibrations.

We need to check that $\Delta(B)$, $\Delta(A_0)$ and $\Delta(A_1)$ are small. Since small maps are closed under dependent products, composition and path types, we can implement the types below as small fibrant objects, where equality is interpreted using path types.

$$B' := \Sigma_{\alpha : 2^{\mathbb{N}}} \Pi_{n,m : \mathbb{N}} \alpha(n) = 1 \times \alpha(m) = 1 \to n = m$$

$$A'_0 := \Sigma_{\alpha : 2^{\mathbb{N}}} \Pi_{n : \mathbb{N}} \alpha(2n) = 0$$

$$A'_1 := \Sigma_{\alpha : 2^{\mathbb{N}}} \Pi_{n : \mathbb{N}} \alpha(2n + 1) = 0$$

Since all objects involved are discrete, the same definition applies whether we interpret these types “extensionally” (i.e. using the internal language of $C$ in the usual way) or “intensionally” (i.e. using the homotopical structure, and in particular path types for the equalities). Furthermore, as we observed earlier $\Delta$ preserves limits, exponentials and the natural number object, and so we see that $\Delta(B)$, $\Delta(A_0)$ and $\Delta(A_1)$ are respectively isomorphic to $B'$, $A'_0$ and $A'_1$, and therefore small.

We define $A$ to be the union of $A_0$ and $A_1$, with $m$ the inclusion $A \hookrightarrow B$. Since $\Delta$ preserves unions and cofibrations are closed under unions, $\Delta(m)$ is a cofibration.

We can therefore apply lemma 5.2 to show that $m$ is $\sim\sim$-stable. However one can check that $A$ is $\sim\sim$-dense in $B$, and that LLPO precisely states that every element of $B$ belongs to $A$. It follows that LLPO holds in $E$. □

**Theorem 5.3.** Let $C$ be a category of presheaf assemblies over either of the pca’s $K_1$ or $K_2$. Assume the axiom of excluded middle in the meta theory. Then it is impossible to satisfy all of the following conditions.

1. $C$ possesses a class of cofibrations and an interval satisfying our general conditions.
2. There is a univalent universe containing $\mathbb{N}$ and closed under propositional truncation.
3. The interval object $1$ in $C$ is connected.
4. Path types are identity types.

**Proof.** It is well known that in categories of assemblies every object is $\sim\sim$-separated, as long as we assume excluded middle in the meta theory (see e.g. [23, Section 3.1]). Hence any object in a category of internal presheaves over assemblies is pointwise $\sim\sim$-separated. In particular this applies to any universe. It is also well known that LLPO fails in assemblies over $K_1$ and over $K_2$. For example, Richman proved in [19, Theorem 5] that in the presence of countable choice a weak form of LLPO is not consistent with Church’s thesis. However, both Church’s thesis and countable choice hold in the effective topos. A similar argument applies in function realizability (i.e. realizability over $K_2$); see e.g. the proof of [18, Corollary 7.23].

We apply lemma 5.2. □

We now turn to examples based on exact categories, specifically ordinary presheaves and internal presheaves in realizability toposes.

**Theorem 5.4.** Let $\text{Inacc}$ be the axiom that every set is an element of an inaccessible set (where we define inaccessible to include closure under subsets). We work over $\text{IZF + Inacc}$. 

Suppose that the following hold.

1. We are given a small category $C$ with finite products.
2. The category of presheaves $\text{Set}^{C^{op}}$ has a class of cofibrations and an interval satisfying our general conditions.
3. The interval is connected.
4. $C$ possesses a univalent Hofmann-Streicher universe on an inaccessible set.
5. Path types are identity types.

Then we deduce the law of excluded middle.

Proof. By lemma 4.12, any inaccessible set $V$ has a weakly $\neg\neg$-stable unit. We deduce by lemma 4.18 that any homotopical universe on $V$ has a pointwise weakly $\neg\neg$-stable unit. Hence by lemma 4.21 we see that if $m$ is a monomorphism in $\text{Set}$ and $\Delta(m)$ is a cofibration then $m$ is $\neg\neg$-stable. However, by the assumption that path types are identity types and theorem 3.2, all monomorphisms are cofibrations. Applying this to $\Delta(m)$ where $m$ is any monomorphism in $\text{Set}$, we deduce that every monomorphism in $\text{Set}$ is $\neg\neg$-stable. Excluded middle follows. $\square$

We now consider internal categories in realizability toposes. We first note that realizability toposes are never boolean (except for the trivial case).

Lemma 5.5. Suppose that $A$ contains two distinct elements $x \neq y$ (we say $A$ is non trivial). Then $\text{RT}(A)$ is not a boolean topos.

Proof. We need to show that $\top : 1 \to 2$ is not a subobject classifier, so it suffices to find a monomorphism that is not a pullback of $1 \to 2$. We take this monomorphism to be the canonical map $2 \to \nabla 2$.

Since $A$ is non trivial we have $0 \neq 1$, by a similar proof to [28, Proposition 1.3.1, part iii]. Therefore all maps from $\nabla 2$ to $2$ are constant, and clearly the map $2 \to \nabla 2$ is not the pullback along either of the constant maps. $\square$

Theorem 5.6. There is no category $C$ satisfying all of the following conditions.

1. $C$ is a category of internal presheaves over an internal category $C$ in a realizability topos $\text{RT}(A)$ where $A$ is non trivial.
2. $C$ has finite products.
3. $C$ has a class of cofibrations and interval object satisfying our general conditions.
4. The interval object is connected.
5. $C$ has a univalent homotopical Hofmann-Streicher universe on a McCarty universe $\text{Mc}(A)$.
6. Path types are identity types.

Proof. By lemma 4.13, any McCarty universe $\text{Mc}(A)$ has a weakly $\neg\neg$-stable unit. Hence any homotopical Hofmann-Streicher universe on $\text{Mc}(A)$ has a pointwise weakly $\neg\neg$-stable unit by lemma 4.18. We deduce by lemma 4.21 that for every monomorphism $m$ in $\text{RT}(A)$, if $\Delta(m)$ is a cofibration then $m$ is $\neg\neg$-stable. However, by the assumption that path types are identity types and theorem 3.2, all monomorphisms are cofibrations. Hence all monomorphisms in $\text{RT}(A)$ are $\neg\neg$-stable, including the subobject classifier $1 \to \Omega$, and so we deduce the law of excluded middle, contradicting lemma 5.5. $\square$

We observe that all of the results for internal presheaves apply in particular to the degenerate case where the internal category is trivial. In this case, for instance, pointwise $\neg\neg$-separated is the same as $\neg\neg\neg$-separated, whereas in general it is usually weaker. Also note that in this case the Hofmann-Streicher universe on $V$ is $V$ itself. We will apply this to two realizability toposes in particular: the effective topos and
the Kleene-Vesley topos. These were studied from a homotopical point of view respectively by Van den Berg and Frumin in [27] and the author in [24, Section 8.2]. In both cases our argument depends only on the choice of interval object and is independent of the choice of cofibrations, as long as they satisfy our general conditions. In particular for these results we don’t need to assume path types are identity types.

Recall (from [27]) that we can define an interval object in the effective topos on \( \nabla 2 \), the uniform object with 2 elements.

**Theorem 5.7.** Suppose we are given a class of cofibrations in the effective topos, that together with the interval object \( \nabla 2 \) satisfies our general conditions. Then there is no univalent homotopical universe on a McCarty universe, and no \( \neg \neg \)-separated univalent universe closed under propositional truncation.

**Proof.** Note that any map \( 1 \to 2^N \) can be viewed as a pullback of an endpoint inclusion \( 1 \to \nabla 2 \). We can therefore use lemma [4.21] together with the same argument as in theorem 5.3. \( \square \)

We recall that countably based \( T_0 \)-spaces embed into the function realizability topos, as shown by Bauer [3]. The subcategory \( KV \) consists of maps that are both computable and continuous and hence one can view the usual topological interval as an interval object in \( KV \). It is straightforward to find a connection structure for the interval using the usual topological definitions. It is currently unclear what the best choice of cofibration for \( KV \) is, but the theorem below applies in any case, as long as our general conditions are satisfied. We note that there is at least one non trivial example given by taking all monomorphisms to be cofibrations.

**Theorem 5.8.** Suppose we are given a class of cofibrations in the Kleene-Vesley topos, that together with the topological interval object \([0,1]\) satisfies our general conditions. Then there is no univalent universe on a McCarty universe, and no \( \neg \neg \)-separated univalent universe closed under propositional truncation.

**Proof.** We first show that the map \( 1 \to 2^N \) defined to be constantly \( \lambda n.0 \) can be viewed as a pullback of the endpoint inclusion \( \delta_0 \): \( 1 \to [0,1] \). We define a continuous map \( h : 2^N \to [0,1] \) by taking \( h(\alpha) \) to be \( \Sigma_{n=0}^{\infty} 2^{-\alpha(n)} \). This is evidently computable, and so does define a map in \( KV \), and it is straightforward to check that \( \lambda n.0 \) is the pullback of the \( \delta_0 \) along \( h \).

We can therefore use lemma [4.21] together with the same argument as in theorem 5.3, and the observation that \( LLPO \) fails in \( KV \). \( \square \)

Both of the above results are specific to McCarty universes and \( \neg \neg \)-separated universes. This of course leaves open the possibility of constructing univalent universes in a completely different way. One such possibility is to find a constructive version of the definition by Shulman in [20, Section 3].

### 6. Cofibrations in Homotopy Type Theory

Although technically the results so far are specific to Orton-Pitts models of type theory, they illustrate ideas that might turn out to be more widely applicable. In particular the use of \( \neg \neg \)-separation in presheaf assemblies matches up well with definitional equality in type theory. In assemblies the \( \neg \neg \)-stable propositions are those that are “free of computational information.” That is, we don’t need to be told a particular realizer to show they are realized; we can guess a realizer uniformly and if they are true then the realizer works. Meanwhile equality in the underlying category is used to implement definitional equality in type theory. This is something that should be “free of computational information,” in the sense that we don’t need
a proof term to exist to show two terms are definitionally equal. We are (or perhaps should be) able to work out whether two terms are are equal or not just by looking at the terms themselves without being given any extra computational information.

We will use this idea to show that there is a purely type theoretic version of the construction used in lemma 4.10. We will use this to give interesting proofs of some minor results regarding the syntactic category of homotopy type theory. This also suggests that in future work it may be possible to obtain a syntactic version of the main theorem (as will be discussed further in the conclusion).

We work over homotopy type theory as defined in [26]. Recall that we can use the syntax of type theory to define a category that is referred to as the syntactic category or classifying category. As shown by Lumsdaine in [15], building on the results of Gambino and Garner [6], one can define a notion of cofibration in the syntactic category of type theory, and in the presence of suitable higher inductive types one in fact gets a model structure. We define display maps to be maps of the form $\Gamma. X \to \Gamma$. We then define trivial cofibrations to be maps with the left lifting property against display maps and cofibrations to be maps with the left lifting property against those display maps $\Gamma. X \to \Gamma$ where there is a term witnessing that $X$ is contractible.

Following the notation in [26] we write $U$ for the universe of small types and $*$ for the unique element of the unit type $1 : U$. We will follow the convention of using the half adjoint definition of equivalence. Hence a term witnessing the equivalence is a tuple containing maps in both directions as components (in addition to the terms witnessing that the maps are mutually inverse and the half adjoint coherence term).

In order to state the results that follow, we define the notion of canonical map in the syntactic category type theory.

**Definition 6.1.** We say a map $\sigma : \Delta \to \Gamma$ in the syntactic category is canonical if for every map $\tau : \Xi \to \Gamma$ we can effectively decide whether or not there exists a map $\mu$ in the diagram below.

\[
\begin{array}{c}
\Xi \\
\downarrow^\mu \\
\Delta \\
\downarrow^\sigma \\
\Gamma
\end{array}
\]

Moreover, if $\mu$ exists then it is unique and we can find it effectively.

**Proposition 6.2.** If $\sigma : \Delta \to \Gamma$ is canonical then it is a monomorphism.

**Proof.** This follows from the uniqueness condition in the definition of canonical. $\square$

We now give our syntactic version of lemma 4.10.

**Lemma 6.3.** Let $m : \Gamma. A \to \Gamma. B$ be a cofibration over $\Gamma$. Then there is a raw term $r$ whose only free variables either belong to $\Gamma$ or are equal to a fresh variable $y$ such that for any context $\Delta$, any $\sigma : \Delta \to \Gamma$ and any terms $a$ and $b$ with $\Delta \vdash a : A[\sigma]$, $\Delta \vdash b : B[\sigma]$ and $\Delta \vdash m[\sigma][x/a] \equiv b$, we have the following.

1. $\Delta \vdash r[\sigma][y/b] : \|\Sigma x. A[\sigma] m = b\|
2. $\Delta \vdash r[\sigma][y/b] \equiv ([a, \text{refl}_{m[x/a]}])$

**Proof.** Write $C$ for the type $\|\Sigma x. A m = y\|$. Then we have the valid judgement $y : B \vdash C(y) : U$. Note that if we reindex along $m$, we get the type $x : A \vdash C[y/m] : U$. In any case $C$ is an hproposition, and in context $x : A$ we can clearly construct an inhabitant $|[x, \text{refl}_{m(x)}]|$ of $C[y/m(x)]$. Hence in context $x : A$, $C[y/m]$ is contractible and so we can construct a term witnessing that $C[y/m]$ is equivalent

\[
\Delta \vdash r[\sigma][y/b] : \|\Sigma x. A[\sigma] m = b\|
\]

Moreover, if $\mu$ exists then it is unique and we can find it effectively.

**Proposition 6.2.** If $\sigma : \Delta \to \Gamma$ is canonical then it is a monomorphism.

**Proof.** This follows from the uniqueness condition in the definition of canonical. $\square$

We now give our syntactic version of lemma 4.10.
to the unit type \(1 : U\). We take the component witnessing the map \(1 \to C[y/m]\) to be \(\lambda z.(x, \text{refl}_{m(x)})\). We omit writing out the other components of the equivalence.

Since \(m\) is a cofibration by assumption, we can deduce by univalence that there is a type \(D\) with \(\Gamma, y : B \vdash D : U\) together with a term witnessing that \(D\) is equivalent to \(C\) in context \(\Gamma, y : B\). Explicitly, we use a diagonal filler in the lifting problem below.

\[
\begin{array}{ccc}
\Gamma, A & \xrightarrow{(\lambda x.1, \lambda z.((x, \text{refl}_{m(x)})))} & \Sigma_{C, U} \Sigma_{D, U} \text{Equiv}(D, C) \\
\Gamma, B & \xrightarrow{e} & U
\end{array}
\]

We write \(e\) for the component of the equivalence that witnesses the map \(D \to C\) and omit writing the other components of the equivalence. We will take \(r\) to be \(e\ast\).

The upper triangle law for the diagonal filler tells us that when we reindex along \(m\) we get the following definitional equalities.

\[
\Gamma, x : A \vdash D[y/m] \equiv 1 : U
\]
\[
\Gamma, x : A \vdash e[y/m] \equiv \lambda z.((x, \text{refl}_{m(x)})) : 1 \to C[y/m]
\]

Substituting in \(\sigma\) and \(a\) we deduce the following.

\[
\Delta \vdash D[\sigma][y/m][\sigma][x/a] \equiv 1
\]
\[
\Delta \vdash e[\sigma][y/m][\sigma][x/a] \equiv \lambda z.((a, \text{refl}_{m(x/a)}))
\]

Using \(\Delta \vdash m[\sigma][x/a] \equiv b\) we deduce the following.

\[
\Delta \vdash D[\sigma][y/m][\sigma][x/a] \equiv D[\sigma][y/b]
\]
\[
\Delta \vdash e[\sigma][y/m][\sigma][x/a] \equiv e[\sigma][y/b]
\]

We then combine \((5)\) with \((7)\) and \((6)\) with \((8)\) to get the following.

\[
\Delta \vdash D[\sigma][y/b] \equiv 1
\]
\[
\Delta \vdash e[\sigma][y/b] \equiv \lambda z.((a, \text{refl}_{m(x/a)}))
\]

From \((9)\) we derive \(\Delta \vdash \ast : D[\sigma][y/b]\), and so we can derive the following judgement.

\[
\Delta \vdash e[\sigma][y/b][\ast] : C[\sigma][y/b]
\]

But this is the same as \((3)\).

From \((10)\) we can derive the following.

\[
\Delta \vdash e[\sigma][y/b][\ast] \equiv ((a, \text{refl}_{m(x/a)})) : C[y/b]
\]

But this is the same as \((4)\). \(\square\)

**Lemma 6.4.** Suppose that we are given types \(A\) and \(B\) in a context \(\Gamma\) and a term \(\Gamma, x : A \vdash m : B\). Suppose that the truncation map \((\Gamma, y : B, \Sigma_{\Sigma Am = y}) \to (\Gamma, y : B, \|\Sigma_{\Sigma Am = y}\|)\) is canonical and that we have decidable type checking. Then the map \((1\Gamma, m) : \Gamma.A \to \Gamma.B\) is canonical.

**Proof.** Suppose we are given a map \(\tau : \Xi \to \Gamma.B\). Note that we can split up \(\tau\) as \((\sigma, b)\) where \(\sigma : \Xi \to \Gamma\) and \(\Xi \vdash b : B[\sigma]\).

Let \(r\) be a raw term as in the statement of lemma 6.3. We first use decidable type checking to decide whether the following judgement is valid.

\[
\Xi \vdash r[\sigma][y/b] : \|\Sigma_{\Sigma Am}[\sigma] = b\|
\]

If it is not valid we say there is no such term \(a\) satisfying the condition. If it is valid, we continue.
Now using the assumption that the truncation map is canonical, we can effectively decide whether or not there exists a term \( c \) satisfying the following.

\[
\Xi \vdash c : \Sigma_{x : A} m[\sigma] = b
\]
\[
\Xi \vdash r[\sigma][y/b] \equiv |c|
\]

If the check returns false we say there is no such term \( a \) satisfying the condition. If it is valid, then we can effectively find such a term \( c \), and we continue.

We next use type checking to decide if the following judgement is valid.

\[
\Xi \vdash c \equiv (\pi_0 a, \text{refl}_{m[\sigma|x/\pi_0 a]})
\]

If so, then we have found a suitable term taking \( a := \pi_0 c \), otherwise we say there is no such term.

We now need to show that the term \( a \) is unique and that if any of the three checks above returns false then there really is no such term \( a \). It suffices for both to show that if \( a \) is any such term then all the checks above return true, and that for the resulting term \( c \) we have \( \Xi \vdash a \equiv \pi_0 c \). So, let \( a \) be any term such that \( \Xi \vdash a : A \) and \( \Xi \vdash m[\sigma|x/a] \equiv b \).

Lemma 6.3 tells us that we have the judgements below.

\[
\Xi \vdash r[\sigma][y/b] : \|\Sigma_{x : A} m[\sigma] = b\|
\]
\[
\Xi \vdash r[\sigma][y/b] \equiv (a, \text{refl}_{m[\sigma|x/a]})
\]

Then the first judgement tells us that the first type check must have returned true. Next, the two judgements together with canonicity for truncation tell us that the second test must have returned true, and that for the resulting term \( c \) we have \( \Xi \vdash a \equiv (a, \text{refl}_{m[\sigma|x/a]}) \).

We can now deduce that \( \Xi \vdash \pi_0 c \equiv a \), and that the final type check must have also returned true, as required.

We will now use the lemma to prove a couple of minor results about canonical maps. In each case, the result itself isn’t so interesting so much as that we can prove them without using strong normalisation, or something similar.

The first result is analogous to the kind of construction that was very useful when we were working semantically. Unfortunately, it is currently unclear if there are any new non trivial examples of applications when working syntactically. However, we do have the minor observation that coproduct inclusions are monic, in the sense that if \( \Gamma \vdash a, a' : A \) and \( \Gamma \vdash \text{inl}(a) \equiv \text{inl}(a') : A + B \) then \( \Gamma \vdash a \equiv a' : A \).

**Theorem 6.5.** Suppose that we have decidable type checking. Suppose that \( A \) and \( B \) are types in context \( \Gamma \), \( m \) is a term \( \Gamma, x : A \vdash m : B \), that we are given a term witnessing that \( m \) is an embedding and that \((1_\Gamma, m) : \Gamma.A \rightarrow \Gamma.B\) is a cofibration. Then \((1_\Gamma, m)\) is canonical.

*Proof.* Since \( m \) is an embedding, by definition the type \( \Sigma_{x : A} m = y \) is an hproposition in context \( \Gamma, y : B \). It easily follows that the truncation map \( \Gamma.B.\Sigma_{x : A} m = y \rightarrow \Gamma.B.\|\Sigma_{x : A} m = y\| \) has a retraction, and hence is canonical. We can now apply lemma 6.4.

**Theorem 6.6.** Suppose that for all closed types \( A \) and \( B \) and maps between singleton contexts \( m : (A) \rightarrow (B) \), the truncation map \( (y : B.\Sigma_{x : A} m = y) \rightarrow (y : B.\|\Sigma_{x : A} m = y\|) \) is canonical, and that we have decidable type checking.

Then for any closed types \( A \) and \( B \), any cofibration \( m : (A) \rightarrow (B) \) is canonical.

*Proof.* This is a special case of lemma 6.4 where we take \( \Gamma \) to be empty.
Similar results can be obtained from a well known theorem by Nicolai Krans [13, Section 8.4]. By a similar (but easier) argument to theorem 6.4, one can use Kraus’ result to show that if A is an inhabited transitive type and decidable type checking holds, then the truncation map \( \Gamma.A \to \Gamma.\|A\| \) is canonical. It is straightforward to check that truncation maps are cofibrations, but it is also an instance of a general principle by Lumsdaine [15], stating that point constructors of higher inductive types are always cofibrations.

In fact, the proof Kraus used applies not just to truncation maps, but to any cofibration, as long as the domain satisfies the requirement of having terms witnessing it is transitive and inhabited. Hence a cofibration \( m: \Gamma.A \to \Gamma.B \) is canonical whenever \( A \) is transitive and inhabited, and in fact it follows that a cofibration is monic whenever \( A \) is transitive (but not necessarily inhabited). For example, when \( A \) is transitive and \( R \) is a binary relation on \( A \), the set quotient map \( A \to A/R \) is always monic (but obviously not always “homotopy monic”).

7. Conclusion

7.1. Towards a Proof that Path Types are not Identity Types. The results here and in particular section 6 suggest that similar results might hold in general in type theory with univalence. Roughly speaking, I expect that in any type theory with a notion of path type that behaves similar to an exponential, it is impossible to simultaneously satisfy all three of the following requirements.

1. Path types are definitionally isomorphic to identity types.
2. Univalence and all the higher inductive types defined in [26] are derivable.
3. The type theory has good computational properties such as strong normalisation, decidable type checking and canonicity.

Unfortunately, even formulating this statement precisely is a difficult task. For example, to even give a good definition of what a type theory is in general remains an area of active research. For this reason this doesn’t deserve to be called a “conjecture;” sometimes the term “hypothesis” is used for such statements.

To be clear, even if the hypothesis is correct, it allows for consistent type theories where any two of the three conditions are satisfied.

For example, cubical type theory as appears in [5] would be an example of a type theory satisfying 2 and 3. In [11], Huber showed that cubical type theory does satisfy canonicity and suggests that the technique could be extended to also show the other good computational properties hold.

Earlier versions of cubical type theory that feature the regularity condition are likely examples of theories satisfying 1 and 3.

One approach to obtaining a theory satisfying conditions 1 and 2 is to build on work by Isaev in [12]. This contained a definition of a type theory \( \text{coe}_1 + \sigma + \text{Path + wUA} \), with a built in notion of path type and coercion where coercion satisfies a computation rule denoted \( \sigma \), akin to regularity, that allows one to implement identity types as path types. It also satisfies a weak (but computationally meaningful) version of univalence, denoted wUA. However, no claim is made regarding decidability of type checking, canonicity or strong normalisation.

7.2. Towards a Proof that Path Types are Identity Types. Although the aim of this paper was towards finding counterexamples, we note that the hypothesis in the previous section remains just a hypothesis and so could easily be false.

In particular, for each of the three examples satisfying only two of the conditions, there is the possibility that the hypothesis can be falsified by showing that in fact the third remaining condition does hold. Namely, one could show that the hypothesis is false through any of the following.
(1) Showing that cubical type theory can be extended with extra computational rules that allow us to use path types as identity types, while retaining its good computational properties.

(2) Showing that in fact it is possible to construct a univalent universe in cubical type theory with a regularity axiom.

(3) Showing that Isaev’s $\text{coe}_1 + \sigma + \text{Path} + \text{wUA}$ does have decidable type checking, strong normalisation and canonicity, and moreover this remains true if it is extended with a universe satisfying full univalence and with higher inductive types.

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