Links between quantum chaos and counting problems

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Abstract

We show that Hurwitz numbers may be generated by certain correlation functions which appear in quantum chaos.

Key words: Hurwitz number, Klein surface, products of random matrices

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First, in short we present two different topics: Hurwitz numbers which appear in counting of branched covers of Riemann and Klein surfaces, and the study of spectral correlation functions of products of random matrices which belong to independent (complex) Ginibre ensembles.

There are a lot of studies on extracting information about Hurwitz numbers, on the one hand side, from integrable systems, as it was done in [51], [52], [21] and further developed in [43], [44], [6], [7], [23], [48], [27], [66], [15], [18], [49] (see also reviews [29] and [33]) and from matrix integrals [39], [22], [34] on the other hand. (Actually the point that there is a special family of tau functions which were introduced in [35] and [55] and studied in [56], [25], [53], [24], [59], [58], [26] where links with matrix models were written down which describes perturbation series in coupling constants of a number of matrix models, and these very tau functions, called hypergeometric ones, count also special types of Hurwitz numbers. This article is based on [48], [59] and [54] and it was the content of my talk in Bialowiez meeting “XXXVI Workshop in Geometric Method in Physics, 3-8 June 2017”. In the last paper we put known results in quantum chaos [1], [2], [3], The results of our the work should be compared to ones obtained in [31], [5] and [12].

1 Counting of branched covers

Let us consider a connected compact surface without boundary Ω and a branched covering \( f : \Sigma \to \Omega \) by a connected or non-connected surface \( \Sigma \). We will consider a covering \( f \) of the degree \( d \). It means that the preimage \( f^{-1}(z) \) consists of \( d \) points \( z \in \Omega \) except some finite number of points. This points are called critical values of \( f \).

Consider the preimage \( f^{-1}(z) = \{p_1, \ldots, p_k\} \) of \( z \in \Omega \). Denote by \( \delta_i \) the degree of \( f \) at \( p_i \). It means that in the neighborhood of \( p_i \) the function \( f \) is homeomorphic to \( x \mapsto x^{\delta_i} \). The set \( \Delta = (\delta_1, \ldots, \delta_\ell) \) is the partition of \( d \), that is called topological type of \( z \).

For a partition \( \Delta \) of a number \( d = |\Delta| \) denote by \( \ell(\Delta) \) the number of the non-vanishing parts (\( |\Delta| \) and \( \ell(\Delta) \) are called the weight and the length of \( \Delta \), respectively). We denote a partition and its Young diagram by the same letter. Denote by \( (\delta_1, \ldots, \delta_\ell) \) the Young diagram with rows of length \( \delta_1, \ldots, \delta_\ell \) and corresponding partition of \( d = \sum \delta_i \).

Fix now points \( z_1, \ldots, z_f \) and partitions \( \Delta^{(1)}, \ldots, \Delta^{(\ell)} \) of \( d \). Denote by

\[
\mathcal{C}_{\Delta(z_1, \ldots, z_f)}(d; \Delta^{(1)}, \ldots, \Delta^{(\ell)})
\]

the set of all branched covering \( f : \Sigma \to \Omega \) with critical points \( z_1, \ldots, z_f \) of topological types \( \Delta^{(1)}, \ldots, \Delta^{(\ell)} \).

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Coverings \( f_1 : \Sigma_1 \to \Omega \) and \( f_2 : \Sigma_2 \to \Omega \) are called isomorphic if there exists an homeomorphism \( \varphi : \Sigma_1 \to \Sigma_2 \) such that \( f_1 = f_2 \varphi \). Denote by \( \text{Aut}(f) \) the group of automorphisms of the covering \( f \). Isomorphic coverings have isomorphic groups of automorphisms of degree \( |\text{Aut}(f)| \).

Consider now the set \( C_{\Omega(z_1, \ldots, z_p)}(d; (\Delta^{(1)}, \ldots, \Delta^{(p)}) \) of isomorphic classes in \( \hat{C}_{\Omega(z_1, \ldots, z_p)}(d; (\Delta^{(1)}, \ldots, \Delta^{(p)}) \) . This is a finite set. The sum

\[
H_E^E(d; (\Delta^{(1)}, \ldots, \Delta^{(p)})) = \sum_{f \in C_{\Omega(z_1, \ldots, z_p)}(d; (\Delta^{(1)}, \ldots, \Delta^{(p)})) \frac{1}{|\text{Aut}(f)|}
\]

don’t depend on the location of the points \( z_1, \ldots, z_p \) and is called Hurwitz number. Here \( E \) denotes the number of the branch points, and \( \Delta \) is the Euler characteristic of the base surface.

In case it will not produce a confusion we admit 'trivial' profiles \( (1^d) \) among \( \Delta^1, \ldots, \Delta^r \) in \( \Pi \) keeping the notation \( H_E^E \) though the number of critical points now is less than \( r \).

In case we count only connected covers \( \Sigma \) we get the connected Hurwitz numbers \( H_{\text{con}}^E(d; (\Delta^{(1)}, \ldots, \Delta^{(p)})) \).

The Hurwitz numbers arise in different fields of mathematics: from algebraic geometry to integrable systems. They are well studied for orientable \( \Omega \). In this case the Hurwitz number coincides with the weighted number of holomorphic branched coverings of a Riemann surface \( \Omega \) by other Riemann surfaces, having critical points \( z_1, \ldots, z_p \in \Omega \) of the topological types \( \Delta^{(1)}, \ldots, \Delta^{(p)} \) respectively. The well known isomorphism between Riemann surfaces and complex algebraic curves gives the interpretation of the Hurwitz numbers as the numbers of morphisms of complex algebraic curves.

Similarly, the Hurwitz number for a non-orientable surface \( \Omega \) coincides with the weighted number of the dianalytic branched coverings of the Klein surface without boundary by another Klein surface and coincides with the weighted number of morphisms of real algebraic curves without real points [11, 45, 46]. An extension of the theory to all Klein surfaces and all real algebraic curves leads to Hurwitz numbers for surfaces with boundaries may be found in [9, 17].

Riemann-Hurwitz formula related the Euler characteristic of the base surface \( E \) and the Euler characteristic of the \( d \)-fold cover \( \Omega' \) as follows:

\[
e' = dE + \sum_{i=1}^{p} (\ell(\Delta^{(i)}) - d) = 0
\]

where the sum ranges over all branch points \( z_i, i = 1, 2, \ldots \) with ramification profiles given by partitions \( \Delta^1, i = 1, 2, \ldots \) respectively, and \( \ell(\Delta^{(i)}) \) denotes the length of the partition \( \Delta^{(i)} \) which is equal to the number of the preimages \( f^{-1}(z_i) \) of the point \( z_i \).

**Example 1.** Let \( f : \Sigma \to \mathbb{C}P^1 \) be a covering without critical points. Then, each \( d \)-fold cover is a union of \( d \) Riemann spheres: \( \mathbb{C}P^1 \sqcup \cdots \sqcup \mathbb{C}P^1 \), then \( \deg f = d \) and \( H_{\text{con}}^E(d) = \frac{1}{d} \).

**Example 2.** Let \( f : \Sigma \to \mathbb{C}P^1 \) be a \( d \)-fold covering with two critical points with the profiles \( \Delta^{(1)} = \Delta^{(2)} = (d) \). (One may think of \( f = x^d \).) Then \( H_{\text{con}}^E(d; (d), (d)) = \frac{1}{d^2} \). Let us note that \( \Sigma \) is connected in this case (therefore \( H_{\text{con}}^E(d; (d), (d)) = H_{\text{con}}^E(d; (d), (d)) \)) and its Euler characteristic \( e' = 2 \).

**Example 3.** The generating function for the Hurwitz numbers \( H_{\text{con}}^E(d; (d), (d)) \) from the previous example may be written as

\[
F(h^{-1}p^{(1)}, h^{-1}p^{(2)}):= h^{-2}\sum_{d>0} H_{\text{con}}^2(d; (d), (d)) p_d^{(1)}p_d^{(2)} = h^{-2}\sum_{d>0} \frac{1}{d} p_d^{(1)}p_d^{(2)}
\]

Here \( p^{(i)} = (p^{(i)}_1, p^{(i)}_2, \ldots), i = 1, 2 \) are two sets of formal parameters. The powers of the auxiliary parameter \( \frac{1}{h} \) count the Euler characteristic of the cover \( e' \) which is \( 2 \) in our example. Then thanks to the known general statement about the link between generating functions of ”connected” and ”disconnected” Hurwitz numbers (see for instance [39]) one can write down the generating function for the Hurwitz numbers for covers with two critical points, \( H_{\text{con}}^E(d; (\Delta^{(1)}, \Delta^{(2)}) \), as follows:

\[
\tau(h^{-1}p^{(1)}, h^{-1}p^{(2)}) = e^{F(h^{-1}p^{(1)}, h^{-1}p^{(2)})}
\]

\[
e^{-2}\sum_{d>0} \frac{1}{d} p_d^{(1)}p_d^{(2)} = \sum_{d>0} \sum_{\Delta^{(1)}, \Delta^{(2)}} H_{\text{con}}^E(d; (\Delta^{(1)}, \Delta^{(2)}) h^{e'-1}p_{\Delta^{(1)}}^{(1)}p_{\Delta^{(2)}}^{(2)}
\]
where \( p_{\Delta(i)} := p_{d1}^{(i)} p_{d2}^{(i)} p_{d3}^{(i)} \cdots, \ i = 1, 2 \) and where \( e' = \ell(\Delta^{(1)}) + \ell(\Delta^{(2)}) \) in agreement with (2) where we put \( f = 2 \). From (3) it follows that the profiles of both critical points coincide, otherwise the Hurwitz number vanishes. Let us denote this profile \( \Delta \), and \( |\Delta| = d \) and from the last equality we get

\[
H^{2,2}(d; \Delta, \Delta) = \frac{1}{z_\Delta}
\]

Here

\[
z_\Delta = \prod_{i=1}^{\infty} i^{m_i} \cdot \prod_{i=1}^{\infty} \frac{m_i!}{m_i!}
\]

where \( m_i \) denotes the number of parts equal to \( i \) of the partition \( \Delta \) (then the partition \( \Delta \) is often denoted by \( (1^{m_1} \cdot 2^{m_2} \cdots) \)).

**Example 4.** Let \( f : \Sigma \to \mathbb{R}P^2 \) be a covering without critical points. Then, if \( \Sigma \) is connected, then \( \Sigma = \mathbb{R}P^2 \), \( \deg f = 1 \) or \( \Sigma = S^2 \), \( \deg f = 2 \). Next, if \( d = 3 \), then \( \Sigma = \mathbb{R}P^2 \bigsqcup \mathbb{R}P^2 \bigsqcup \mathbb{R}P^2 \) or \( \Sigma = \mathbb{R}P^2 \bigsqcup \sum S^2 \). Thus \( H^{1,0}(3) = \frac{1}{3^2} + \frac{1}{3^2} = \frac{2}{3} \).

**Example 5.** Let \( f : \Sigma \to \mathbb{R}P^2 \) be a covering with a single critical point with profile \( \Delta \), and \( \Sigma \) is connected. Note that due to (2) the Euler characteristic of \( \Sigma \) is \( e' = \ell(\Delta) \). (One may think of \( f = z^d \) defined in the unit disc where we identify \( z \) and \( -z \) if \( |z| = 1 \). In case we cover the Riemann sphere by the Riemann sphere \( z \to z^m \) we get two critical points with the same profiles. However we cover by the Riemann sphere, then we have the composition of the mapping \( z \to z^m \) on the Riemann sphere and the factorization by antipodal involution \( z \to -\frac{1}{z} \). Thus we have the ramification profile \( (m, m) \) at the single critical point 0 of \( \mathbb{R}P^2 \). The automorphism group is the dihedral group of the order \( 2m \) which consists of rotations on \( \frac{2\pi}{m} \) and antipodal involution \( z \to -\frac{1}{z} \). Thus we get that

\[
H_{\text{con}}^{1,1}(2m; (m, m)) = \frac{1}{2m}
\]

From (2) we see that \( 1 = \ell(\Delta) \) in this case. Now let us cover \( \mathbb{R}P^2 \) by \( \mathbb{R}P^2 \) via \( z \to z^d \). From (2) we see that \( \ell(\Delta) = 1 \). For even \( d \) we have the critical point 0, in addition each point of the unit circle \( |z| = 1 \) is critical (a folding), while from the beginning we restrict our consideration only on isolated critical points. For odd \( d = 2m - 1 \) there is the single critical point 0, the automorphism group consists of rotations on the angle \( \frac{2\pi}{2m-1} \). Thus in this case

\[
H_{\text{con}}^{1,1}(2m - 1; (2m - 1)) = \frac{1}{2m - 1}
\]

**Example 6.** The generating series of the connected Hurwitz numbers with a single critical point from the previous Example is

\[
F(h^{-1}p) = \frac{1}{h^2} \sum_{m>0} p_m^2 H_{\text{con}}^{1,1}(2m; (m, m)) + \frac{1}{h} \sum_{m>0} p_{2m-1} H_{\text{con}}^{1,1}(2m - 1; (2m - 1))
\]

where \( H_{\text{con}}^{1,1} \) describes \( d \)-fold covering either by the Riemann sphere \( (d = 2m) \) or by the projective plane \( (d = 2m - 1) \). We get the generating function for Hurwitz numbers with a single critical point

\[
\tau(h^{-1}p) = e^{F(h^{-1}p)} = e^{\sum_{m>0} \frac{1}{m^2} p_m^2 + \sum_{m=0} p_m} = \sum_{d>0} \sum_{|\Delta|=d} h^{-\ell(\Delta)} p_{\Delta} H_{\text{con}}^{1,1}(d; \Delta)
\]

where \( a = 0 \) and if \( \Delta = (1^d) \), and where \( a = 1 \) and otherwise. Then \( H^{1,1}(d; \Delta) \) is the Hurwitz number describing \( d \)-fold covering of \( \mathbb{R}P^2 \) with a single branch point of type \( \Delta = (d_1, \ldots, d_l) \), \( |\Delta| = d \) by a (not necessarily connected) Klein surface of Euler characteristic \( e' = \ell(\Delta) \). For instance, for \( d = 3 \), \( e' = 1 \) we get \( H^{1,1}(3; \Delta) = \frac{1}{4} \delta_{\Delta,(3)} \). For unbranched coverings (that is for \( a = 0 \), \( e' = d \)) we get formula (4).
where \( \Delta = (\delta_1, \ldots, \delta_\ell) \) is a partition and each \( p_{\delta_i} \) is defined by (8).

In our notation one can write

\[
\tau_1^{2KP}(X,Y) = \tau_1^{2KP}(p(X),p(Y)) = \sum_{\Delta} \frac{1}{z_\Delta} P_\Delta(X)P_\Delta(Y)
\]  

### Combinatorial approach.

The study of the homomorphisms between the fundamental group of the base Riemann surface of genus \( g \) (the Euler characteristic is respectively \( e = 2 - 2g \)) with \( f \) marked points and the symmetric group in the context of the counting of the non-equivalent \( d \)-fold covering with given
of profiles \(\Delta^i, i = 1, \ldots, F\) results to the following equation (for instance, for the details, see Appendix A written by Zagier for the Russian edition of [20] or works [10], [20])

\[
\prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1} X_1 \cdots X_f = 1
\]  

(14)

where \(a_j, b_j, X_i \in S_d\) and where each \(X_i\) belongs to the cycle class \(C_{\Delta^i}\). Then the Hurwitz number \(H^{2g,F}(d, \Delta^1, \ldots, \Delta^F)\) is equal to the number of solutions of equation (14) divided by the order of symmetric group \(S_d\) (to exclude the equivalent solutions obtained by the conjugation of all factors in (14) by elements of the group. In the geometrical approach each conjugation means the re-enumeration of \(d\) sheets of the cover).

For instance Example 3 considered above counts non-equivalent solutions of the equation \(X_1 X_2 = 1\) with given cycle classes \(C_{\Delta^1}\) and \(C_{\Delta^2}\). Solutions of this equation consist of all elements of class \(C_{\Delta^1}\) and inverse elements, so \(\Delta^2 = \Delta^1 =: \Delta\). The number of elements of any class \(C_\Delta\) (the cardinality of \(|C_\Delta|\)) divided by \(|\Delta|!\) as we got in the Example 3.

For Klein surfaces (see [11], [20]) instead of (14) we get

\[
\prod_{j=1}^{g} R_j^2 X_1 \cdots X_f = 1
\]  

(15)

where \(R_j, X_i \in S_d\) and where each \(X_i\) belongs to the cycle class \(C_{\Delta^i}\). In (15), \(g\) is the so-called genus of non-orientable surface which is related to its Euler characteristic \(e\) as \(e = 1 - g\). For the projective plane \((e = 1)\) we have \(g = 0\), for the Klein bottle \((e = 1)\) \(g = 1\).

Consider unbranched covers of the torus (equation (14)), projective plane and Klein bottle (15). In this we put each \(X_i = 1\) in (14) and (15). Here we present three pictures, for the torus \((e = 1)\) (14) and Klein bottle \((e = 0)\) which may be obtained by the identification of square’s edges. We get \(aba^{-1} b^{-1} = 1\) for torus, \(abab = 1\) for the projective plane and \(abab^{-1} = 1\) for the Klein bottle.

3 pictures.

Consider unbranched coverings \((F = 0)\). For the real projective plane we have \(g = 1\) in (15) only one \(R_1 = ab\). If we treat the projective plane as the unit disk with identified opposite points of the border \(|z| = 1\), then \(R\) is related to the path from \(z\) to \(-z\). For the Klein bottle \((g = 2\) in (15)) there are \(R_1 = ab\) and \(R_2 = b^{-1}\).

To avoid confusions in what follows we will use the notion of genus and the notations \(g\) only for Riemann surfaces, while the notion of the Euler characteristic \(e\) we shall use both for orientable and non-orientable surfaces.

2 Random matrices. Complex Ginibre ensemble.

On this subject there is an extensive literature, for instance see [1], [2], [3], [61], [62].

We will consider integrals over complex matrices \(Z_1, \ldots, Z_n\) where the measure is defined as

\[
d\Omega(Z_1, \ldots, Z_n) = \prod_{\alpha=1}^{n} \mu(Z_{\alpha}) = c \prod_{\alpha=1}^{n} \prod_{i,j=1}^{N} \delta R(Z_{\alpha})_{ij} \delta \Omega(Z_{\alpha})_{ij} e^{-[(Z_{\alpha})_{ij}]^2}
\]  

(16)

where the integration range is \(\mathbb{C}^{N^2} \times \cdots \times \mathbb{C}^{N^2}\) and where \(c\) is the normalization constant defined via \(\int d\Omega(Z_1, \ldots, Z_n) = 1\).

We treat this measure as the probability measure. The related ensemble is called the ensemble of \(n\) independent complex Ginibre ensembles. The expectation of a quantity \(f\) which depends on entries of the matrices \(Z_1, \ldots, Z_n\) is defined by

\[
\mathbb{E}(f) = \int f(Z_1, \ldots, Z_n) d\Omega(Z_1, \ldots, Z_n).
\]
Let us introduce the following products
\[
X := (Z_1 C_1) \cdots (Z_n C_n) \quad \text{(17)}
\]
\[
Y_i := Z_{i,n}^{d} Z_{i,n-1}^{d} \cdots Z_i^d, \quad 0 < t < n \quad \text{(18)}
\]
where $Z'_\alpha$ is the Hermitian conjugate of $Z_\alpha$. We are interested in correlation functions of spectral invariants of matrices $X$ and $Y_i$.

We denote by $x_1, \ldots, x_N$ and by $y_1, \ldots, y_N$ the eigenvalues of the matrices $X$ and $Y_i$, respectively. Given partitions $\lambda = (\lambda_1, \ldots, \lambda_l)$, $\mu = (\mu_1, \ldots, \mu_k)$, $l, k \leq N$. Let us introduce the following spectral invariants
\[
P_\lambda(X) = p_{\lambda_1}(X) \cdots p_{\lambda_l}(X), \quad P_\mu(Y_i) = p_{\mu_1}(Y_i) \cdots p_{\mu_k}(Y_i) \quad \text{(19)}
\]
where each $p_m(X)$ is defined via (3).

For a given partition $\lambda$, such that $d := |\lambda| \leq N$, let us consider the spectral invariant $P_\lambda$ of the matrix $XY_i$ (see (12)). We have

**Theorem 1.** $X$ and $Y_i$ are defined by (17)-(18). Denote $e = 2 - 2g$.

(A) Let $n > t = 2g \geq 0$. Then
\[
E(P_\lambda(XY_{2g})) = z_\lambda \sum_{\Delta^1, \ldots, \Delta^{n-2g+1}} H^{2-2g,n+2-2g}(d; \lambda, \Delta^1, \ldots, \Delta^{n-2g+1})P_{\Delta_{n-2g+1}}(C') P_\Delta(C_{2g+1}) \prod_{i=1}^{n-2g} P_{2} \left(C_{2g+1+i} \right) \quad \text{(20)}
\]
where
\[
C' = C_1 \cdots C_{2g-1}, \quad C'' = C_2 C_4 \cdots C_{2g} \quad \text{(21)}
\]

(B) Let $n > t = 2g + 1 \geq 1$. Then
\[
E(P_\lambda(XY_{2g+1})) = z_\lambda \sum_{\Delta^1, \ldots, \Delta^{n-2g+1}} H^{2-2g,n+2-2g}(d; \lambda, \Delta^1, \ldots, \Delta^{n-2g+1})P_{\Delta_{n-2g}}(C') P_{\Delta_{n-2g+1}}(C''') \prod_{i=1}^{n-2g-1} P_{\Delta}(C_{2g+1+i}) \quad \text{(22)}
\]
where
\[
C' = C_1 C_3 \cdots C_{2g+1}, \quad C''' = C_2 C_4 \cdots C_{2g} \quad \text{(23)}
\]

**Corollary 1.** Let $|\lambda| = d \leq N$ as before, and let each $C_i = I_N$ ($N \times N$ unity matrix). Then
\[
\frac{1}{z_\lambda} E(P_\lambda(XY_{2g})) = \frac{1}{z_\lambda} E(P_\lambda(XY_{2g+1})) = N^{nd-\ell(\lambda)} \sum_{e'} N^{e'} S^{e'}_E(\lambda) \quad \text{(24)}
\]
where $E = 2 - 2g$ and where
\[
S^{e'}_E(\lambda) := \sum_{\Delta^1, \ldots, \Delta^{n+e-1}} H^{E,n+E}(d; \lambda, \Delta^1, \ldots, \Delta^{n+e-1}), \quad L = -\ell(\lambda) + nd + e' \quad \text{(25)}
\]
is the sum of Hurwitz numbers counting all $d$-fold coverings with the following properties:
(i) the Euler characteristic of the base surface is $E$
(ii) the Euler characteristic of the cover is $e'$
(iii) there are at most $F = n + E$ critical points

The item (ii) in the Corollary follows from the equality $P_\Delta(I_N) = N^{\ell(\Delta)}$ (see (12) and (5)) and from the Riemann-Hurwitz relation which relates Euler characteristics of a base and a cover via branch points profile's lengths (see (2)):
\[
\sum_{i=1}^{n+e-1} \ell(\Delta^i) = -\ell(\lambda) + (F - E)d + e'
\]
In our case $F = E = n$. 

6
Theorem 2. $X$ and $Y_t$ are defined by \[17\]–\[18\].
(A) If $|\lambda| \neq |\mu|$ then $E(\mathbf{P}_\lambda(X)P_{\mu}(Y_1)) = 0$.
(B) Let $|\lambda| = |\mu| = d$ and $n - 1 > t = 2g + 1 \geq 1$.
Then
\[E(\mathbf{P}_\lambda(X)P_{\mu}(Y_{2g+1})) =
\]
\[
z_{\lambda}z_{\mu} \sum_{\Delta^1, \ldots, \Delta^{n-2g}} H^{2-2g,n+2-2g}(d; \lambda, \mu, \Delta^1, \ldots, \Delta^{n-2g}) P_{\Delta^{n-2g-1}}(C') P_{\Delta^{n-2g}}(C_n C'') \prod_{i=1}^{n-2g-2} P_{\Delta^i}(C_{2g+1+i})
\]
where $C'$ and $C''$ are given by \[27\].
(C) Let $|\lambda| = |\mu| = n > t = 2g \geq 0$.
Then
\[E(\mathbf{P}_\lambda(X)P_{\mu}(Y_{2g})) =
\]
\[
z_{\lambda}z_{\mu} \sum_{\Delta^1, \ldots, \Delta^{n-2g}} H^{2-2g,n+2-2g}(d; \lambda, \mu, \Delta^1, \ldots, \Delta^{n-2g}) P_{\Delta^{n-2g-1}}(C') P_{\Delta^{n-2g}}(C_n C'') \prod_{i=1}^{n-2g} P_{\Delta^i}(C_{2g+i})
\]
where $C'$ and $C''$ are given by \[27\].

Corollary 2. Let $|\lambda| = d \leq N$ as before, and let each $C_i = I_N$. Then
\[
\frac{1}{z_{\lambda}z_{\mu}} E(\mathbf{P}_\lambda(X)P_{\lambda}(Y_{2g})) = \frac{1}{z_{\lambda}E(\mathbf{P}_\lambda(X)P_{\lambda}(Y_{2g+1})) = \frac{1}{z_{\lambda}E(\mathbf{P}_\lambda(Y_{2g}))) = \frac{1}{z_{\lambda}}E(\mathbf{P}_\lambda(XY_{2g})) = \frac{1}{z_{\lambda}}E(\mathbf{P}_\lambda(Y_{2g+1}))
\]

Theorem 3. $X$ and $Y_t$ are defined by \[17\]–\[18\].
(A) Let $n - 1 > t = 2g + 1 \geq 0$.
Then
\[E(\mathbf{P}_\lambda(X)_{1BKP}(Y_{2g+1})) =
\]
\[
z_{\lambda} \sum_{\Delta^1, \ldots, \Delta^{n-2g}} H^{1-2g,n+1-2g}(d; \lambda, \Delta^1, \ldots, \Delta^{n-2g}) P_{\Delta^{n-2g-1}}(C') P_{\Delta^{n-2g}}(C_n C'') \prod_{i=1}^{n-2g-2} P_{\Delta^i}(C_{2g+1+i})
\]
where $C'$ and $C''$ are given by \[27\].
(B) Let $n > t = 2g \geq 0$.
Then
\[E(\mathbf{P}_\lambda(X)_{1BKP}(Y_{2g})) =
\]
\[
z_{\lambda} \sum_{\Delta^1, \ldots, \Delta^{n-2g}} H^{1-2g,n+1-2g}(d; \lambda, \Delta^1, \ldots, \Delta^{n-2g}) P_{\Delta^{n-2g-1}}(C') P_{\Delta^{n-2g}}(C_n C'') \prod_{i=1}^{n-2g} P_{\Delta^i}(C_{2g+i})
\]
where $C'$ and $C''$ are given by \[27\].

The sketch of proof.
The character Frobenius-type formula by Mednykh-Pozdnyakova-Jones \[17\], \[20\]
\[H^{\xi,k}(d; \Delta^1, \ldots, \Delta^k) = \sum_{|\lambda| = d} \frac{\dim \lambda}{d!} \varphi_\lambda(\Delta^1) \cdots \varphi_\lambda(\Delta^k)
\]
where $\dim \lambda$ is the dimension of the irreducible representation of $S_d$, and
\[\varphi_\lambda(\Delta^{(i)}) := |C_{\Delta^{(i)}}| \frac{\chi_\lambda(\Delta^{(i)})}{\dim \lambda}, \quad \dim \lambda := \chi_\lambda((1^d))
\]
$\chi_\lambda(\Delta)$ is the character of the symmetric group $S_d$ evaluated at a cycle type $\Delta$, and $\chi_\lambda$ ranges over the irreducible complex characters of $S_d$ (they are labeled by partitions $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ of a given weight.
\( d = |\lambda| \). It is supposed that \( d = |\lambda| = |\Delta^1| = \cdots = |\Delta^k| \). \( |C_\Delta| \) is the cardinality of the cycle class \( C_\Delta \) in \( S_d \).

Then we use the characteristic map relation [38]:

\[ s_\lambda(p) = \frac{\text{dim}_\lambda}{d!} \left( p_d^d + \sum_{\Delta} \varphi_\lambda(\Delta) p_\Delta \right) \tag{33} \]

where \( p_\Delta = p_{\Delta^1} \cdots p_{\Delta^k} \) and where \( \Delta = (\Delta^1, \ldots, \Delta^k) \) is a partition whose weight coincides with the weight of \( \lambda : |\lambda| = |\Delta| \). Here

\[ \text{dim}_\lambda = d! s_\lambda(p_\infty), \quad p_\infty = (1, 0, 0, \ldots) \tag{34} \]

is the dimension of the irreducible representation of the symmetric group \( S_d \). We imply that \( \varphi_\lambda(\Delta) = 0 \) if \( |\Delta| \neq |\lambda| \).

Then we know how to evaluate the integral with the Schur function via Lemma used in [59] and [49] (for instance see [38] for the derivation).

**Lemma 1.** Let \( A \) and \( B \) be normal matrices (i.e. matrices diagonalizable by unitary transformations). Then Below \( p_\infty = (1, 0, 0, \ldots) \).

\[
\int_{\mathbb{C}^{n^2}} s_\lambda(AZBZ^+) e^{-\text{tr}ZZ^+} \prod_{i,j=1}^n d^2Z = \frac{s_\lambda(A)s_\lambda(B)}{s_\lambda(p_\infty)} \tag{35}
\]

and

\[
\int_{\mathbb{C}^{n^2}} s_\mu(AB) s_\lambda(Z^+B) e^{-\text{tr}ZZ^+} \prod_{i,j=1}^n d^2Z = \frac{s_\lambda(AB)}{s_\lambda(p_\infty)} \delta_{\mu,\lambda}. \tag{36}
\]

To prove Theorem 1 we use that we can equate the integral over \( E(\tau^{2KP}(XY)) \) using this Lemma and (6) and then compare it to the same integral where now we use (9). To prove Theorem 2 in the similar way we equate \( E(\tau^{2KP}(X)\tau^{2KP}(Y)) \). To prove Theorem 3 we similarly \( E(\tau^{2KP}(X)\tau^{2KP}(Y)) \) in the same way taking into account also (10).

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Let us recall that the characters of the unitary group $U(N)$ are labeled by partitions and coincide with the so-called Schur functions $\phi$. A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a set of nonnegative integers $\lambda_i$ which are called parts of $\lambda$ and which are ordered as $\lambda_i \geq \lambda_{i+1}$. The number of non-vanishing parts of $\lambda$ is called the length of the partition $\lambda$, and will be denoted by $\ell(\lambda)$. The number $|\lambda| = \sum_i \lambda_i$ is called the weight of $\lambda$. The set of all partitions will be denoted by $\mathbb{P}$. 

A Partitions and Schur functions

Let us recall that the characters of the unitary group $U(N)$ are labeled by partitions and coincide with the so-called Schur functions $\phi$. A partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a set of nonnegative integers $\lambda_i$ which are called parts of $\lambda$ and which are ordered as $\lambda_i \geq \lambda_{i+1}$. The number of non-vanishing parts of $\lambda$ is called the length of the partition $\lambda$, and will be denoted by $\ell(\lambda)$. The number $|\lambda| = \sum_i \lambda_i$ is called the weight of $\lambda$. The set of all partitions will be denoted by $\mathbb{P}$. 

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The Schur function labelled by $\lambda$ may be defined as the following function in variables $x = (x_1, \ldots, x_N)$:

$$s_{\lambda}(x) = \frac{\det \left[ x_j^{\lambda_i + N} \right]_{i,j}}{\det \left[ x_j^{\lambda_i} \right]_{i,j}}$$

(37)

in case $\ell(\lambda) \leq N$ and vanishes otherwise. One can see that $s_{\lambda}(x)$ is a symmetric homogeneous polynomial of degree $|\lambda|$ in the variables $x_1, \ldots, x_N$, and $\deg x_i = 1$, $i = 1, \ldots, N$.

**Remark 1.** In case the set $x$ is the set of eigenvalues of a matrix $X$, we also write $s_{\lambda}(X)$ instead of $s_{\lambda}(x)$.

There is a different definition of the Schur function as quasi-homogeneous non-symmetric polynomial of degree $|\lambda|$ in other variables, the so-called power sums, $p = (p_1, p_2, \ldots)$, where $\deg p_m = m$.

For this purpose let us introduce

$$s_{\{h\}}(p) = \det [s_{\{h_{i+j-N}\}}(p)]_{i,j},$$

where $\{h\}$ is any set of $N$ integers, and where the Schur functions $s_{\{h\}}$ are defined by $e^{\sum_{m>0} \frac{1}{m} p_m z^m} = \sum_{m>0} s_{\{h\}}(p) z^m$. If we put $h_i = \lambda_i - i + N$, where $N$ is not less than the length of the partition $\lambda$, then

$$s_{\lambda}(p) = s_{\{h\}}(p).$$

(38)

The Schur functions defined by (37) and by (38) are equal, $s_{\lambda}(p) = s_{\lambda}(x)$, provided the variables $p$ and $x$ are related by the power sums relation

$$p_m = \sum_i x_i^m$$

(39)

In case the argument of $s_{\lambda}$ is written as a non-capital fat letter the definition (38), and we imply the definition (37) in case the argument is not fat and non-capital letter, and in case the argument is capital letter which denotes a matrix, then it implies the definition (37) with $x = (x_1, \ldots, x_N)$ being the eigenvalues.

It may be easily checked that

$$s_{\lambda}(p) = (-1)^{|\lambda|} \lambda^{\ast \nu}(\lambda^{\ast})$$

(40)

where $\lambda^{\ast \nu}$ is the partition conjugated to $\lambda$ (in [38] it is denoted by $\lambda^\ast$). The Young diagram of the conjugated partition is obtained by the transposition of the Young diagram of $\lambda$ with respect to its main diagonal. One gets $\lambda_1 = \ell(\lambda^{\ast \nu})$.

**B Matrix integrals as generating functions of Hurwitz numbers from [48], [49]**

In case the base surface is $\mathbb{CP}^1$ the set of examples of matrix integrals generating Hurwitz numbers were studied in works [14], [39], [7], [12], [34], [36], [66]. One can show that the perturbation series in coupling constants of these integrals (Feynman graphs) may be related to TL (KP and two-component KP) hypergeometric tau functions. It actually means that these series generate Hurwitz numbers with at most two arbitrary profiles (An arbitrary profile corresponds to a certain term in the perturbation series in the coupling constants which are higher times. The TL and 2-KP hierarchies there are two independent sets of higher times which yeilds two critical points for Hurwitz numbers).

Here, very briefly, we will write down few generating series for the $\mathbb{RP}^2$ Hurwitz numbers. These series may be not tau functions themselves but may be presented as integrals of tau functions of matrix argument. (The matrix argument, which we denote by a capital letter, say $X$, means that the power sum variables $p$ are specified as $p_i = \text{tr} X^i$, $i > 0$. Then instead of $s_{\lambda}(p)$, $\tau(p)$ we write $s_{\lambda}(X)$ and $\tau(X)$).

If a matrix integral in examples below is a BKP tau function then it generates Hurwitz numbers with a single arbitrary profile and all other are subjects of restrictions identical to those in $\mathbb{CP}^1$ case mentioned above. In all examples $V(x, p) := \sum_{m>0} \frac{1}{m} x^m p_m$. We also recall the notation $p_{\infty} = (1,0,0,\ldots)$. We also recall that numbers $H^{\mu,\nu}(d, \ldots)$ are Hurwitz numbers only in case $d \leq N$, $N$ is the size of matrices.
For more details of the \( \mathbb{R}P^2 \) case see [45]. New development in [45] with respect to the consideration in [50] is the usage of products of matrices. Here we shall consider a few examples. All examples include the simplest BKP tau function, of matrix argument, \( X \), defined by (compare to (??))

\[
\tau^B_1(X) := \sum_\lambda s_\lambda(X) = e^{\frac{i}{2} \sum_{m>0} \frac{1}{m} (\text{tr} X^m)^2 + \sum_{m>0, odd} \frac{1}{m} \text{tr} X^m} = \frac{\det \frac{1}{1+X}}{\det \frac{1}{1} (I_N \otimes I_N - X \otimes X)}
\]

(41)

as the part of the integration measure. Other integrands are the simplest KP tau functions \( \tau^\text{Kp}_1(X, \mathbf{p}) := e^{\text{tr} V(X, \mathbf{p})} \) where the parameters \( \mathbf{p} \) may be called coupling constants. The perturbation series in coupling constants are expressed as sums of products of the Schur functions over partitions and are similar to the series we considered in the previous sections.

**Example 1.** The projective analog of Okounkov’s generating series for double Hurwitz series as a model of normal matrices. From the equality

\[
(2\pi i^{-1})^\frac{1}{2} e^{\frac{i}{2} \sum_{m>0} \frac{1}{m} (\text{tr} X^m)^2 + \sum_{m>0, odd} \frac{1}{m} \text{tr} X^m} = \int_{\mathbb{R}} e^{xi_n \zeta_0 + (\xi_1 - \xi_2)^2} \zeta_1 dx_1,
\]

in a similar way as was done in [55] using \( \varphi_\lambda(\Gamma) = \sum_{(i,j) \in \lambda} (j - i) \), one can derive

\[
e^{n|\lambda| \zeta_0 \varphi_\lambda(\Gamma)} \delta_{\lambda, \mu} = K \int \lambda s_\lambda(M) s_\mu(M^\dagger) \det (MM^\dagger)^{n\zeta_0} e^{-\frac{1}{2} \zeta_1 (\text{tr} \log(MM^\dagger))^2} dM
\]

where \( K \) is unimportant multiplier, where \( M \) is a normal matrix with eigenvalues \( z_1, \ldots, z_N \) and \( \log |z_i| = x_i \), and where \( dM = dU \prod_{i<j} |z_i - z_j|^2 \prod_{i=1}^N d^2 z_i \). Then the \( \mathbb{R}P^2 \) analogue of Okounkov’s generating series may be presented as the following integral (51) may be written

\[
\sum_{\lambda, \mu \leq N} e^{n|\lambda| \zeta_0 + \zeta_1 \varphi_\lambda(\Gamma)} s_\lambda(\mathbf{p}) = K \int e^{V(M, \mathbf{p})} \varphi_\lambda(\Gamma) \delta_{\lambda, \mu} \tau^B_1(M^\dagger) dM
\]

(42)

Recall that in the work [51] there were studied Hurwitz numbers with an arbitrary number of simple branch points and two arbitrary profiles. In our analog, describing the coverings of the projective plane, an arbitrary profile only one, because, unlike the Toda lattice, the hierarchy of BKP has only one set of (continuous) higher times.

A similar representation of the Okounkov \( \mathbb{C}P^1 \) was earlier presented in [8].

Below we use the following notations

- \( dU \) is the normalized Haar measure on \( U(N) \): \( \int_{U(N)} dU = 1 \)
- \( Z \) is a complex matrix

\[
d\Omega(Z, Z^\dagger) = \pi^{-n^2} e^{-\text{tr}(ZZ^\dagger)} \prod_{i,j=1}^N d\Re Z_{ij} d\Im Z_{ij}
\]

- Let \( M \) be a Hermitian matrix the measure is defined

\[
dM = \prod_{i \leq j} d\Re M_{ij} \prod_{i<j} d\Im M
\]

It is known [33]

\[
\int s_\lambda(Z) s_\mu(Z^\dagger) d\Omega(Z, Z^\dagger) = (N)_\lambda \delta_{\lambda, \mu}
\]

(43)

where \( (N)_\lambda := \prod_{(i,j) \in \lambda} (N + j - i) \) is the Pochhammer symbol related to \( \lambda \). A similar relation was used in [53, 26, 59, 4, 88], for models of Hermitian, complex and normal matrices.

By \( I_N \) we shall denote the \( N \times N \) unit matrix. We recall that

\[
s_\lambda(I_N) = (N)_\lambda s_\lambda(p_\infty), \quad s_\lambda(p_\infty) = \frac{\dim \lambda}{d!}, \quad d = |\lambda|
\]
Example 2. Three branch points. The generating function for $\mathbb{R}P^2$ Hurwitz numbers with three ramification points, having three arbitrary profiles:

$$
\sum_{\lambda, \ell(\lambda) \leq N} s_\lambda(p^{(1)}) s_\lambda(\Lambda) s_\lambda(p^{(2)})
\frac{(s_\lambda(p^{(1)})^2)}{s_\lambda(p^{(2)})} = \int \tau^B_1(Z_1 \Lambda Z_2) \prod_{i=1,2} e^{V(trZ_i^t, p^{(i)})} d\Omega(Z_i, Z_i^t)
$$

(44)

If $p^{(2)} = p(q, t)$ with any given parameters $q, t$, and $\Lambda = I_N$ then (44) is the hypergeometric BKP tau function.

Example 3. 'Projective' Hermitian two-matrix model. The following integral

$$
\int \tau^B_1(cM_2)e^{trV(M_1, p) + tr(M_1 M_2)} dM_1 dM_2 = \sum_{\lambda} \frac{c|\lambda|}{N|\lambda|} s_\lambda(p)
$$

where $M_1, M_2$ are Hermitian matrices is an example of the hypergeometric BKP tau function.

Example 4. Unitary matrices. Generating series for projective Hurwitz numbers with arbitrary profiles in $n$ branch points and restricted profiles in other points:

$$
\sum_{d \geq 0} c^d (d!)^{1-m} \sum_{\lambda, \ell(\lambda) \leq N} \frac{(\dim \lambda)}{(\dim)}^{2-m} \frac{s_\lambda(I_N)}{s_\lambda(p^{(i)})} = \int \tau^B_1(U_2) e^{trV(U_1, p) + tr(cU_1 U_2)} dU_1 dU_2 = \sum_{\ell(\lambda) \leq N} c|\lambda| s_\lambda(p)
$$

(45)

Here $p^{(i)}$ are parameters. This series generate certain linear combination of Hurwitz numbers for base surfaces with Euler characteristic $2 - m$, $m \geq 0$. In case $n = 1$ this BKP tau function may be viewed as an analogue of the generating function of the so-called non-connected Bousquet-Melou-Schaeffer numbers (see Example 2.16 in [33]). In case $n = m = 1$ we obtain the following BKP tau function

$$
\int \tau^B_1(U_2) e^{trV(U_1, p)} dU_1 dU_2 = c|\lambda| s_\lambda(p)
$$

Example 5. Integrals over complex matrices. A paradigmatic example of Belyi curves generating function [69, 14] is as follows:

$$
\sum_{i=1}^N \sum_{D_{i+1}^{(i)}} c^d H_{i+1, i}^{(i)}(d; \Delta^{(1)}, \ldots, \Delta^{(n)}) \prod_{i=1}^n p^{(i)}_{\Delta^{(i)}} = \sum_{\lambda} c^{(i)} \frac{(d!)^{m-2} s_\lambda(p^{(i)})}{(\dim\lambda)^{m-2} s_\lambda(p^{(i)})}
$$

(46)

$$
\int \tau^B_1(Z_1 \Lambda Z_2) \prod_{i=1,2} e^{V(trZ_i^t, p^{(i)})} d\Omega(Z_i, Z_i^t)
$$

(47)

where $E = 2 - m$ is the Euler characteristic of the base surface.

The series in the following example generates the projective Hurwitz numbers themselves where get rid of the factor $(N)_\lambda$ in the sum over partitions we use mixed integration over $U(N)$ and over complex matrices:

$$
\sum_{D_{i+1}^{(i)}} c^d H_{1, i}^{(i)}(d; \Delta^{(1)}, \ldots, \Delta^{(n)}) \prod_{i=1}^n p^{(i)}_{\Delta^{(i)}} = \sum_{\lambda, \ell(\lambda) \leq N} c^{(i)} \frac{(d!)^{m-2} s_\lambda(p^{(i)})}{(\dim\lambda)^{m-2} s_\lambda(p^{(i)})}
$$

(48)

$$
\int \tau^B_1(cU_1 Z_1^{(i)} \ldots Z_{n-i}^{(i)} p^{(n)}) \tau^B_1(U) dU \prod_{i=1}^{n-1} \tau^B_1(Z_i, p^{(i)}) d\Omega(Z_i, Z_i^t)
$$

(49)

Here $Z_i, i = 1, \ldots, n - 1$ are complex $N \times N$ matrices and $U \in U(N)$. As in the previous examples one can specify all sets $p^{(i)} = p(q, t, i)$, $i = 1, \ldots, n$ except a single one which in this case has the meaning of the BKP higher times.