BEREZIN TRANSFORM AND TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACES INDUCED BY REGULAR WEIGHTS

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Abstract. Given a regular weight $\omega$ and a positive Borel measure $\mu$ on the unit disc $\mathbb{D}$, the Toeplitz operator associated with $\mu$ is

$$T_{\mu}(f)(z) = \int_{\mathbb{D}} f(\zeta) B_\omega^\mu(\zeta) d\mu(\zeta),$$

where $B_\omega^\mu$ are the reproducing kernels of the weighted Bergman space $A^2_\omega$. We describe bounded and compact Toeplitz operators $T_{\mu}: A^p_\omega \to A^q_\omega$, $1 < q, p < \infty$, in terms of Carleson measures and the Berezin transform

$$\tilde{T}_{\mu}(z) = \frac{\langle T_{\mu}(B_\omega^\mu), B_\omega^\mu \rangle_{A^2_\omega}}{\|B_\omega^\mu\|_{A^2_\omega}^2}.$$

We also characterize Schatten class Toeplitz operators in terms of the Berezin transform and apply this result to study Schatten class composition operators.

1. Introduction and main results

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions in the unit disc $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$. For $0 < p < \infty$ and a nonnegative integrable function $\omega$ on $\mathbb{D}$, the weighted Bergman space $A^p_\omega$ consists of $f \in H(\mathbb{D})$ such that

$$\|f\|_{A^p_\omega}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z) \, dA(z) < \infty,$$

where $dA(z) = \frac{dx \, dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. As usual, $A^p_\omega$ denotes the weighted Bergman space induced by the standard radial weight $(1 - |z|^2)^\alpha$.

A radial weight $\omega$ belongs to the class $\hat{D}$ if $\hat{\omega}(r) = \int_1^r \omega(s) \, ds$ satisfies the doubling condition $\hat{\omega}(r) \leq C \hat{\omega}(1 + r^2)$. Further, a radial weight $\omega \in \mathcal{D}$ is regular, denoted by $\omega \in \mathcal{R}$, if $\omega(r)$ behaves as its integral average over $(r, 1)$, that is,

$$\omega(r) \asymp \frac{\int_r^1 \omega(s) \, ds}{1 - r}, \quad 0 < r < 1.$$

Every standard weight as well as those given in [4.4]–[4.6] are regular. It is easy to see that for each radial weight $\omega$, the norm convergence in $A^2_\omega$ implies the uniform convergence on compact subsets of $\mathbb{D}$, and hence the Hilbert space $A^2_\omega$ is a closed subspace of $L^2_\omega$ and the orthogonal Bergman projection $P_\omega$ from $L^2_\omega$ to $A^2_\omega$ is given by

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta) \overline{B_\omega^\mu(\zeta)} \omega(\zeta) \, dA(\zeta),$$

where $B_\omega^\mu$ are the reproducing kernels of $A^2_\omega$. Recently, those regular weights $\omega$ and $\nu$ for which $P_{\omega}: L^p_\nu \to L^q_\nu$ is bounded were characterized in terms of Bekollé-Bonami type...
conditions [15]. In this paper we consider operators which are natural extensions of the orthogonal projection $P_\omega$. For a positive Borel measure $\mu$ on $\mathbb{D}$, the Toeplitz operator associated with $\mu$ is

$$T_\mu(f)(z) = \int_\mathbb{D} f(\zeta) \overline{B_\omega^\mu(\zeta)} \, d\mu(\zeta).$$

If $d\mu = \Phi \omega dA$ for a non-negative function $\Phi$, then write $T_\mu = T_\Phi$ so that $T_\Phi(f) = P_\omega(f \Phi)$. The operator $T_\Phi$ has been extensively studied since the seventies [31 3 21]. Luecking was probably the one who introduced Toeplitz operators $T_\mu$ with measures as symbols in [17], where he provides, among other things, a description of Schatten class Toeplitz operators $T_\mu : A_\omega^p \to A_\omega^q$ in terms of an $\ell^p$-condition involving a hyperbolic lattice of $\mathbb{D}$. Recently, Zhu [24] gave an alternative characterization in terms of $L^p \left( \frac{dA}{1-|z|^2} \right)$-integrability of the Berezin transform of $T_\mu$ in the widest possible range of the parameters $p$ and $\alpha$. We refer to [23] Chapter 7 for the theory of Toeplitz operators $T_\mu$ acting on $A_\omega^p$ and to [2 11] for descriptions in terms of Carleson measures and the Berezin transform of bounded and compact Toeplitz operators $T_\mu : A_\omega^p \to A_\omega^q$, $1 < p, q < \infty$. The Berezin transform of a bounded linear operator $T : A_\omega^p \to A_\omega^q$ is

$$\tilde{T}(z) = \langle T(b_\omega^p), b_\omega^q \rangle_{A_\omega^q},$$

where $b_\omega^p = \frac{B_\omega^p}{\|B_\omega^p\|_{A_\omega^q}}$ are the normalized reproducing kernels of $A_\omega^q$. Given $0 < p, q < \infty$ and a positive Borel measure $\mu$ on $\mathbb{D}$, we say that $\mu$ is a $q$-Carleson measure for $A_\omega^p$ if the identity operator $Id : A_\omega^p \to L_\mu^q$ is bounded. A description of $q$-Carleson measures for $A_\omega^p$ induced by doubling weights was recently given in [14], see also [18].

One of the main purposes of this study is to characterize, in terms of Carleson measures and the Berezin transform $\tilde{T}_\mu$, those positive Borel measures $\mu$ such that the Toeplitz operator $T_\mu : A_\omega^p \to A_\omega^q$, where $1 < p, q < \infty$ and $\omega \in \mathcal{R}$, is bounded or compact. We also describe Schatten class Toeplitz operators $T_\mu : A_\omega^p \to A_\omega^q$ in terms of their Berezin transforms and show how this result can be used to study Schatten class composition operators induced by symbols of bounded valence.

A simple fact that is repeatedly used in the study of Toeplitz operators on standard Bergman spaces $A_\omega^p$ is the closed formula $(1 - \tau(z))^{-(2+\alpha)}$ of the Bergman reproducing kernel of $A_\omega^p$. This shows that the kernels never vanish, and allows one to easily establish useful pointwise and norm estimates. However, the situation in the case of $A_\omega^p$ with $\omega \in \mathcal{R}$ is more complicated because of the lack of such an explicit expression for $B_\omega^p$. In fact a little perturbation in the weight, that does not change the space itself, might introduce zeros to the kernel functions [20]. This difference causes severe difficulties in the study related to Toeplitz operators on $A_\omega^p$, and forces us to circumvent several obstacles in a different manner. We will shortly indicate the main tools used in the proofs after each result is stated.

We need a bit more of notation to state our first result. For each $1 < p < \infty$ we write $p'$ for its conjugate exponent, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. The Carleson square $S(I)$ on the boundary $T$ of $\mathbb{D}$ is the set $S(I) = \{ re^{it} \in \mathbb{D} : e^{it} \in I, 1 - |I| \leq r < 1 \}$, where $|E|$ denotes the Lebesgue measure of $E \subset T$. We associate to each $a \in \mathbb{D} \setminus \{0\}$ the interval $I_a = \{ e^{i\theta} : |\arg(\alpha e^{-i\theta})| \leq \frac{\pi}{2\|B_\omega^p\|_{A_\omega^q}} \}$, and denote $S(a) = S(I_a)$.

**Theorem 1.** Let $1 < p \leq q < \infty$, $\omega \in \mathcal{R}$ and $\mu$ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:

(i) $T_\mu : A_\omega^p \to A_\omega^q$ is bounded;

(ii) $\frac{\omega(S(\cdot))^{\frac{1}{p} - \frac{1}{q'}}}{\omega(S(\cdot))^{\frac{1}{p} + \frac{1}{q'}}} \in L^\infty$;

(iii) $\mu$ is a $\frac{(p+q')}{pq'}$-Carleson measure for $A_\omega^s$ for some (equivalently for all) $0 < s < \infty$;
Another important fact employed is that, even if the kernels may vanish, among other things, on the duality relation (for compact Toeplitz operators. This result is stated as Theorem 13 and its proof relies, for all the maximal Bergman projection the use of Rademacher functions along with Khinchine’s inequality, the boundedness of range $1 < q < p < \infty$. The equivalence between (ii) and (iv) shows that the Berezin transform $\tilde{T}_\mu$ behaves asymptotically as the average $\mu(S(\cdot))/\omega(S(\cdot))$. By using Fubini’s theorem and the reproducing formula

$$L_z(f) = f(z) = \langle f, B_z^\omega \rangle_{A^2_\omega} = \int_{\mathcal{D}} f(\zeta) \overline{B_z^\omega(\zeta)} \omega(\zeta) dA(\zeta), \quad f \in A^1_\omega, \quad (1.2)$$

we deduce

$$\langle T_\mu(f), g \rangle_{A^2_\omega} = \langle f, g \rangle_{L^2_\mu} \quad (1.3)$$

for each compactly supported positive Borel measure $\mu$ and all $f, g \in A^2_\omega$. This identity shows that Carleson measures and Toeplitz operators are intimately connected, and thus the use of Carleson measures in the proof of Theorem 1 does not come as a surprise. Another key tool in the proof are the $L^p$-estimates of the kernels $B_z^\omega$, obtained in [15 Theorem 1], and a pointwise estimate for $B_z^\omega$ in a sufficiently small Carleson square contained in $S(z)$, given in Lemma 7 below. We also prove a counterpart of Theorem 1 for compact Toeplitz operators. This result is stated as Theorem 13 and its proof relies, among other things, on the duality relation $(A^p_\omega)^* \simeq A^p_\omega$ under the pairing $\langle \cdot, \cdot \rangle_{A^2_\omega}$, valid for all $\omega \in \mathcal{R}$, [15 Corollary 7].

To describe the positive Borel measures such that $T_\mu : A^p_\omega \to A^p_\omega$ is bounded on the range $1 < q < p < \infty$, we write $g(a, z) = |\varphi_a(z)| = \left| \frac{z - a}{1 - \overline{a}z} \right|$, for the pseudohyperbolic distance between $z$ and $a$, and $\Delta(a, r) = \{ z : g(a, z) < r \}$ for the pseudohyperbolic disc of center $a \in \mathbb{D}$ and radius $r \in (0, 1)$.

**Theorem 2.** Let $1 < q < p < \infty$, $0 < r < 1$, $\omega \in \mathcal{R}$ and $\mu$ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:

(i) $T_\mu : A^p_\omega \to A^p_\omega$ is compact;

(ii) $T_\hat{\mu} : A^p_{\hat{\omega}} \to A^p_{\hat{\omega}}$ is bounded;

(iii) $\hat{\mu}(\cdot) = \frac{\mu(\Delta(\cdot, r))}{\omega(\Delta(\cdot, r))} \in L^\frac{p}{p-r}_{\omega}$;

(iv) $\mu$ is a $\left( p + 1 - \left\lfloor \frac{p}{q} \right\rfloor \right)$-Carleson measure for $A^p_\omega$;

(v) $Id : A^p_\omega \to L^{p+1-E}_{\omega}$ is compact;

(vi) $\tilde{T}_\mu \in L^\frac{p}{p-r}_{\omega}$.

Moreover,

$$||T_\mu||_{A^p_\omega \to A^p_\omega} \asymp ||\hat{\mu}||_{L^\frac{p}{p-r}_{\omega}} \asymp ||Id||_{A^p_\omega \to L^{p+1-E}_{\omega}} \asymp ||\tilde{T}_\mu||_{L^\frac{p}{p-r}_{\omega}}.$$ 

Apart from standard techniques, such as a duality relation for Bergman spaces and the use of Rademacher functions along with Khinchine’s inequality, the boundedness of the maximal Bergman projection

$$P^\omega(f)(z) = \int_{\mathbb{D}} |f(\zeta)||B_z^\omega(\zeta)| \omega(\zeta) dA(\zeta)$$

on $L^p_\omega$ for $p \in (1, \infty)$ and $\omega \in \mathcal{R}$ [15 Theorem 5] plays a crucial role in the proof of Theorem 2. Another important fact employed is that, even if the kernels may vanish, by Lemma 8 for each $\omega \in \bar{\mathcal{D}}$ they obey the relation $|B_z^\omega| \asymp B_z^\omega(a)$ on sufficiently small pseudohyperbolic discs centered at $a$. This is used when (iii) is considered, but (iii)
it reduces to the inequality $p \omega$. The hypothesis hence the novelty of Theorem 3 stems from the last part involving the Berezin transform.

Theorem 7.18], see also [24]. Since each standard weight is regular, the cut-off condition of the points $\phi$ motivation are in order. For an analytic self-map $T$ of $D$ of $\involves pseudohyperbolic discs of all sizes, and therefore a suitably chosen covering of $\mathbb{D}$ will be used to deal with this technical obstacle.

As for the statements of our results on Schatten classes, some notation are in order. The polar rectangle associated with an arc $I \subset \mathbb{T}$ is

$$R(I) = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in I, \ 1 - \frac{|I|}{2\pi} \leq |z| < 1 - \frac{|I|}{4\pi} \right\}.$$ Write $z_I = (1 - |I|/2\pi)\xi$, where $\xi \in \mathbb{T}$ is the midpoint of $I$. Let $\mathbb{T}$ denote the family of all dyadic arcs of $\mathbb{T}$. Every arc $I \in \mathbb{T}$ is of the form

$$I_{n,k} = \left\{ e^{i\theta} : \frac{2\pi k}{2^n} \leq \theta < \frac{2\pi (k+1)}{2^n} \right\},$$ where $k = 0, 1, 2, \ldots, 2^n - 1$ and $n = \mathbb{N} \cup \{0\}$. The family $\{R(I) : I \in \mathbb{T}\}$ consists of pairwise disjoint rectangles whose union covers $\mathbb{D}$. For $I_j \in \mathbb{T} \setminus \{I_0,0\}$, we will write $z_j = z_{I_j}$. For convenience, we associate the arc $I_{0,0}$ with the point $1/2$. Given a radial weight $\omega$, we write

$$\omega^*(z) = \int_0^1 \omega(s) \log \frac{s}{|z|} s \, ds, \quad z \in \mathbb{D} \setminus \{0\}.$$ Theorem 3. Let $0 < p < \infty$, $\omega \in \mathbb{D}$ and $\mu$ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:

(i) $T_\mu$ belongs to the Schatten $p$-class $S_p(A^2_\omega)$;

(ii) $\sum_{R_j \in \mathbb{T}} \left( \frac{\mu(R_j)}{\omega^*(z_j)} \right)^p < \infty$;

(iii) $\mu(\Delta(z,r))$ belongs to $L^p \left( \frac{dA}{(1-|z|^2)^2} \right)$ for some (equivalently for all) $0 < r < 1$.

Moreover,

$$|T_\mu|^p \prec \sum_{R_j \in \mathbb{T}} \left( \frac{\mu(R_j)}{\omega^*(z_j)} \right)^p \times \int_D \left( \frac{\mu(\Delta(z,r))}{\omega^*(z)} \right)^p \frac{dA(z)}{(1-|z|^2)^2}.$$ If $\omega \in \mathcal{R}$ such that $\omega^*(\cdot)^p (1-|\cdot|^2)^{2p} \in R$ is also a regular weight, then $T_\mu \in S_p(A^2_\omega)$ if and only if $\tilde{T}_\mu \in L^p_{\omega^*/\omega^*}$ and $|T_\mu|_p \asymp |\tilde{T}_\mu|_{L^p_{\omega^*/\omega^*}}$.

The equivalence of the first three statements were proved in [16, Theorem 1], and hence the novelty of Theorem 3 stems from the last part involving the Berezin transform. The hypothesis $\omega^*(\cdot)^p (1-|\cdot|^2)^{2p} \in R$ is not a restriction for $p \geq 1$, and for $\omega(z) = (1 - |z|^2)^{\alpha}$ it reduces to the inequality $p(2 + \alpha) > 1$. Therefore Theorem 3 is an extension of [23, Theorem 7.18], see also [24]. Since each standard weight is regular, the cut-off condition $\omega^*(\cdot)^p (1-|\cdot|^2)^{2p} \in R$ is in a sense the best possible.

The proof of the last statement of Theorem 3 for $p \geq 1$ follows by standard techniques once the pointwise kernel estimate given in Lemma 3 is available. However, the proof for $0 < p < 1$ is more involved because the reproducing kernels of $A^2_\omega$ with $\omega \in \mathcal{R}$ do not necessarily remain essentially constant in hyperbolically bounded regions, a property which the standard kernels $(1 - \mathfrak{r})^{2+\alpha}$ trivially admit and is used in the proof of [23, Theorem 7.18] concerning the weighted Bergman spaces $A^p_\omega$. This obstacle is circumvented by using subharmonicity and estimates for the $A^p_\omega$-norm of $B^\omega_\omega$ for doubling weights $\omega, \nu \in \mathbb{D}$, obtained in [13, Theorem 1].

Theorem 3 can be applied to study Schatten class composition operators when the inducing symbol $\varphi$ is of finite valence. To state the result, some more notation and motivation are in order. For an analytic self-map $\varphi$ of $\mathbb{D}$, let $\zeta \in \varphi^{-1}(z)$ denote the set of the points $\{\zeta_n\}$ in $\mathbb{D}$, organized by increasing moduli and repeated according to their
multiplicities, such that $\varphi(\zeta_n) = z$ for all $n$. For a radial weight $\varphi$ and $\varphi$ as above, the generalized Nevanlinna counting function is

$$N_{\varphi, \omega^*}(z) = \sum_{\zeta \in \varphi^{-1}(z)} \omega^*(\zeta), \quad z \in \mathbb{D} \setminus \{\varphi(0)\}.$$  

In [10] Theorem 3] it was shown that, for each $\omega \in \hat{\mathbb{D}}$, the composition operator $C_\varphi$ belongs to the Schatten $p$-class $S_p(A^2_\varphi)$ if and only if $N_{\varphi, \omega^*} \in L^p\left(\frac{dA}{(1-|z|^2)^2}\right)$. This condition might be difficult to test in praxis because of the counting function $N_{\varphi, \omega^*}$. Therefore it is natural to look for more workable descriptions. As for this, we observe that by using [15] Theorem 1] one can show that the Berezin transform of $C_\varphi C^*_\varphi$ behaves asymptotically as $\frac{\omega^*(\zeta)}{\omega^*(\varphi(\zeta))}$, and moreover, the condition $\frac{\omega^*(z)}{\omega^*(\varphi(z))} \rightarrow 0, |z| \rightarrow 1^-$, characterizes compact operators $C_\varphi : A^2_\varphi \rightarrow A^2_\varphi$ when $\omega \in \mathcal{R}$ by [16] Theorem 20 and Lemma 23]. Therefore one may ask how close is the condition

$$\int_{\mathbb{D}} \left(\frac{\omega^*(z)}{\omega^*(\varphi(z))}\right)^{\frac{p}{2}} \frac{\omega(z)}{\omega^*(z)} dA(z) < \infty$$  

(1.4)

to describe Schatten class composition operators? The next result shows that this is a description in the case $p > 2$ under the hypothesis of $\varphi$ being of bounded valence.

**Theorem 4.** Let $2 < p < \infty$ and $\omega \in \mathcal{R}$, and let $\varphi$ be a bounded valent analytic self-map of $\mathbb{D}$. Then $C_\varphi \in S_p(A^2_\varphi)$ if and only if (1.4) holds.

Theorem 4 is an extension of [22] Theorem 1.1] to the setting of regular weights. If $\omega(z) = (1 - |z|^2)^{\alpha}$, then the statement in Theorem 4 is not valid for $p(\alpha + 2) \leq 2$ because in this case the condition (1.4) fails for all analytic self-maps $\varphi$. More generally, by using [13] p. 10 (ii)] one can show that if $\omega \in \mathcal{R}$ and $p$ is small enough, then (1.4) fails for each $\varphi$. Moreover, [19] Theorem 3] shows that the statement in Theorem 4 does not remain valid for $\omega \equiv 1$ without the additional hypothesis regarding the valence of $\varphi$.

It is easy to see that each regular weight $\omega$ satisfies $\omega(r) \asymp \omega(t)$ whenever $1 - r \asymp 1 - t$. This asymptotic relation shows that $\omega \in \mathcal{R}$ must be essentially constant in each hyperbolically bounded region, and hence, in particular, $\omega$ may not have zeros. This apparently severe requirement does not cause too much loss of generality in our study. This because in the next section we will show that if $\omega \in \hat{\mathbb{D}}$ satisfies the reverse doubling property $\tilde{\omega}(r) \geq C\tilde{\omega}(1 - \frac{r}{K})$ for some $K > 1$ and $C > 1$, a condition that is satisfied for each $\omega \in \mathcal{R}$, then there exists a differentiable strictly positive weight $W \in \mathcal{R}$ such that $\|\cdot\|_{A^2_\omega}$ and $\|\cdot\|_{A^2_W}$ are comparable. In Section 2 we also discuss the kernel estimates and other auxiliary results. Section 3 is devoted to the study of bounded and compact Toeplitz operators. Schatten class Toeplitz and composition operators are discussed in Sections 4 and 5, respectively.

### 2. Pointwise and norm estimates of Bergman reproducing kernels

We begin with considering the classes of weights appearing in this study and their basic properties. Then we will prove several pointwise and norm estimates for the reproducing kernels, and finally an auxiliary result on weak convergence of normalized kernels is established.

The first auxiliary lemma contains several characterizations of doubling weights and will be repeatedly used throughout the rest of the paper. For a proof, see [12] Lemma 2.1]. All along we will assume $\tilde{\omega}(r) > 0$ for all $0 \leq r < 1$ without mentioning it, for otherwise $A^2_\omega = \mathcal{H}(\mathbb{D})$.

**Lemma A.** Let $\omega$ be a radial weight. Then the following conditions are equivalent:

(i) $\omega \in \hat{\mathbb{D}}$;
(ii) There exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that
\[ \hat{\omega}(r) \leq C \left( \frac{1-r}{1-r} \right)^{\beta} \hat{\omega}(t), \quad 0 \leq r \leq t < 1; \]

(iii) There exist $C = C(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that
\[ \int_0^t \left( \frac{1-t}{1-s} \right)^{\gamma} \omega(s) \, ds \leq C \hat{\omega}(t), \quad 0 \leq t < 1; \]

(iv) The asymptotic equality
\[ \int_1^s s^2 \omega(s) \, ds \asymp \hat{\omega} \left( 1 - \frac{1}{x} \right), \quad x \in [1, \infty), \]
is valid;

(v) $\omega^*(z) \asymp \hat{\omega}(z) (1 - |z|), |z| \to 1^-$;

(vi) There exists $\lambda = \lambda(\omega) > 0$ such that
\[ \int_0^1 \frac{\omega(z)}{|1 - \zeta z|^n} \, dA(z) \asymp \frac{\hat{\omega}(\zeta)}{(1 - |\zeta|)^\lambda}, \quad \zeta \in \mathbb{D}; \]

(vii) There exists $C = C(\omega) > 0$ such that the moments $\omega_n = \int_0^1 r^n \omega(r) \, dr$ satisfy the condition $\omega_n \leq C\omega_{2n}$.

We next briefly discuss radial weights having a kind of reversed doubling property, and then show how this is related to the pointwise condition that defines the class $\mathcal{R}$ of regular weights. More precisely, we show that if $\omega \in \bar{\mathcal{D}}$ satisfies the reverse doubling condition appearing in part (i) of Lemma B below, then one can find a strictly positive $n$ times differentiable weight which belongs to $\mathcal{R}$ and induces the same Bergman space as $\omega$. The next lemma can be found in [17].

**Lemma B.** Let $\omega$ be a radial weight. For each $K > 1$, let $\rho_n = \rho_n(\omega, K)$ be the sequence defined by $\hat{\omega}((\rho_n) = \hat{\omega}(0) K^{-n}$, and for each $\beta \in \mathbb{R}$, write $\omega(\beta)(z) = \omega(z)(1 - |z|)^\beta$. Then the following statements are equivalent:

(i) There exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that $\hat{\omega}(r) \geq C \hat{\omega} \left( 1 - \frac{1-r}{K} \right)$ for all $0 \leq r < 1$;

(ii) There exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that
\[ \hat{\omega}(t) \leq C \left( \frac{1-t}{1-r} \right)^{\beta} \hat{\omega}(r), \quad 0 \leq r \leq t < 1; \]

(iii) For some (equivalently for each) $\beta \in (0, \infty)$, there exists $C = C(\beta, \omega) \in (0, 1)$ such that
\[ \frac{1}{|1-r|^\beta} \int_r^1 \hat{\omega}(t) \beta(1-t)^{\beta-1} \, dt \leq C \hat{\omega}(r), \quad 0 < r < 1. \]

By Lemma B and [13] Lemma 1.1] each $\omega \in \mathcal{R}$ satisfies the reverse doubling condition. The next result shows that if $\omega \in \bar{\mathcal{D}}$ satisfies the reverse doubling condition, then there exists a continuous and locally smooth weight $W$ that induces the same Bergman space as $\omega$.

**Proposition 5.** Let $0 < p < \infty$ and $\omega \in \bar{\mathcal{D}}$, and write $W(r) = W(\omega)(r) = \hat{\omega}(r)/(1-r)$ for all $0 \leq r < 1$. Then $\|f\|_{A^p_W} \asymp \|f\|_{A^p_L}$ for all $f \in H(\mathbb{D})$ if and only if $\omega$ satisfies the reverse doubling condition appearing in part (i) of Lemma B.

**Proof.** Since $\omega$ belongs to $\bar{\mathcal{D}}$ by the hypothesis, so does $W$. Therefore $\|f\|_{A^p_W} \asymp \|f\|_{A^p_L}$ for all $f \in H(\mathbb{D})$ by [13] Theorem 1] if $W(S(a)) \asymp \omega(S(a))$ for all $a \in \mathbb{D} \setminus \{0\}$. Since $\omega$ and $W$ are radial, this is the case if
\[ \hat{W}(r) = \hat{\omega}(r) \int_r^1 \frac{\hat{\omega}(t)}{\hat{\omega}(r)} \frac{1}{1-t} \, dt \asymp \hat{\omega}(r), \quad 0 \leq r < 1. \]
If now \( \omega \in \hat{D} \) satisfies the reverse doubling condition, then Lemma (A)ii) and Lemma (B)ii) applied to the middle term above imply the asymptotic equality we are after.

Conversely, assume that \( \omega \in \hat{D} \) and \( \|f\|_{A^p_\omega} \asymp \|f\|_{A^p_\omega^s} \) for all \( f \in \mathcal{H}(\mathbb{D}) \). Write \( f_a(z) = (1 - az)^{-1} \) for all \( a \in \mathbb{D} \). By Lemma (A)vi) there exists \( \lambda = \lambda(\omega) \geq 0 \) such that
\[
\frac{\hat{\omega}(a)}{(1 - |a|)^\lambda} \asymp \int_{\mathbb{D}} \frac{\omega(z)}{1 - az}^{1/2} dA(z) = \|f_a\|_{A^p_\omega}^p \asymp \|f_a\|_{A^p_{\hat{W}}}^p \asymp \int_0^1 \frac{\hat{\omega}(r)}{(1 - |a|r)^\lambda(1 - r)} dr 
\]
and thus \( \omega \) satisfies the Lemma (B)iii) with \( \beta = 1 \). \( \square \)

Consider now \( \omega \in \hat{D} \) satisfying the reverse doubling condition. Then \( A^p_\omega = A^p_{W_\omega} \) and \( W_\omega \in \mathcal{R} \) by the first part of the proof of Proposition 3. The weight \( W_\omega \) is continuous and strictly positive. Further, the differentiable weight \( W_\omega(r)/(1 - r) \) belongs to \( \mathcal{R} \) and induces the same Bergman space as \( \omega \). Therefore, by repeating the process, for a given \( \omega \in \hat{D} \) satisfying the reverse doubling condition, we can always find a strictly positive \( n \) times differentiable weight that induces the same Bergman space as the original weight \( \omega \). Therefore assuming \( \omega \in \mathcal{R} \) instead of the two doubling conditions is not a severe restriction in our study.

The true advantage of the class \( \mathcal{R} \) is the local smoothness of its weights. It is clear that if \( \omega \in \mathcal{R} \), then for each \( s \in [0, 1) \) there exists a constant \( C = C(s, \omega) > 1 \) such that
\[
C^{-1} \omega(t) \leq \omega(r) \leq C \omega(t), \quad 0 \leq r \leq t \leq r + s(1 - r) < 1.
\]
(2.1)
Therefore, for \( \omega \in \mathcal{R} \) and \( r \in (0, 1) \),
\[
\omega(S(z)) \asymp \hat{\omega}(z)(1 - |z|) \asymp \omega(z)(1 - |z|)^2 \asymp \omega(D(z, r)), \quad z \in \mathbb{D},
\]
(2.2)
where the constants of comparison depend on \( \omega \) and also on \( r \) in the last case. This observation finishes our discussion on basic properties of different classes of weights.

We next turn to kernel estimates. In order to prove our main results, and in particular to deal with the Berezin transform of a Toeplitz operator, we will need asymptotic estimates for the norm of the Bergman reproducing kernel in several spaces of analytic functions in \( \mathbb{D} \). The next result follows by [13, Theorem 1] (see also [13, Lemma 6.2]), Lemma (A) and (2.2).

**Theorem C.** Let \( \omega, \nu \in \hat{D} \), \( 0 < p < \infty \) and \( n \in \mathbb{N} \cup \{0\} \). Then
\[
\|B^p_{\omega}(n)\|_{A^p_\omega} \asymp \int_0^{|z|} \frac{\hat{\nu}(t)}{\hat{\omega}(t)^p(1 - t)^p(1/2)} dt, \quad |z| \to 1^-. \tag{2.3}
\]
In particular, if \( 1 < p < \infty \), \( \omega \in \mathcal{R} \) and \( r \in (0, 1) \), then
\[
\|B^p_{\omega}\|_{A^p_\omega} \asymp \frac{1}{\omega(S(z))^{p-1}} \asymp \frac{1}{\omega(D(z, r))^{p-1}}, \quad z \in \mathbb{D}. \tag{2.4}
\]

As usual, we write \( H^\infty \) for the space of bounded analytic functions in \( \mathbb{D} \), and \( \mathcal{B} \) stands for the Bloch functions, that is, the space of \( f \in \mathcal{H}(\mathbb{D}) \) such that \( \|f\|_B = \sup_{z \in \mathbb{D}} |f'(z)(1 - |z|) + |f(0)| < \infty \).

**Lemma 6.** Let \( \omega \in \hat{D} \). Then
\[
\|B^p_{\omega}\|_B \asymp \frac{1}{\omega(S(z))} \asymp \|B^p_{\omega}\|_{H^\infty}, \quad z \in \mathbb{D}.
\]

**Proof.** Since
\[
B^p_{\omega}(\zeta) = \sum_{n=0}^{\infty} \frac{(\zeta z)^n}{2\omega_n}, \quad (B^p_{\omega})'(\zeta) = \sum_{n=1}^{\infty} \frac{n\zeta^{n-1}z^n}{2\omega_n}, \quad z, \zeta \in \mathbb{D},
\]
the estimate \[ (2.5) \], with \( p = 1, N = 2 \) and \( r = |z|^2 \), together with Lemma \( \square \) yields

\[
\left| (B_z^a)'(z) \right| \leq \sum_{n=1}^{\infty} \frac{n|z|^{2(n-1)} + \int_0^{|z|^2} \frac{1}{\omega(t)(1-t)^2} dt}{\omega(z)(1-|z|^2)^2} \times \frac{1}{\omega(S(z))(1-|z|)}, \quad |z| \to 1^-,
\]

and hence

\[
\frac{1}{\omega(S(z))} \lesssim \|B_z^a\|_{\mathcal{B}}, \quad |z| \to 1^-.
\]

Since \( \|B_z^a\|_{\mathcal{B}} \leq 2 \|B_z^a\|_{H^\infty} \), it remains to establish the desired upper estimate for the \( H^\infty \)-norm. To see this, observe first that

\[
|B_z^a(\zeta)| \leq \sum_{n=0}^{\infty} \frac{|z|^n}{2\omega_n}, \quad z, \zeta \in \mathbb{D}.
\]

Then, by using again the estimate \( \square \), but now with \( p = 1, N = 1 \) and \( r = |z| \), it follows that

\[
\|B_z^a\|_{H^\infty} \leq \sum_{n=0}^{\infty} \frac{|z|^n}{2\omega_n} \times \int_0^{|z|^2} \frac{1}{\omega(t)(1-t)^2} dt \times \frac{1}{\omega(S(z))}, \quad |z| \to 1^-.
\]

This finishes the proof. \( \square \)

We next establish two local pointwise estimates for the Bergman reproducing kernels. To do this, for each \( \delta \in (0, 1] \) and \( a \in \mathbb{D} \setminus \{0\} \), write \( a_\delta = (1 - \delta(1 - |a|))e^{\omega S(a) \cdot a} \). Then \( a_1 = a, |a_\delta| > |a| \) for all \( \delta \in (0, 1) \), and \( \lim_{\delta \to 0^+} a_\delta = a/|a| \).

**Lemma 7.** Let \( \omega \in \widehat{\mathbb{D}} \). Then there exist constants \( c = c(\omega) > 0 \) and \( \delta = \delta(\omega) \in (0, 1] \) such that

\[
|B_{a_\delta}^\omega(z)| \geq \frac{c}{\omega(S(a))}, \quad z \in S(a_\delta), \quad a \in \mathbb{D} \setminus \{0\}.
\]

**Proof.** By Theorem \( \square \) there exists a constant \( C_1 = C_1(\omega) > 0 \) such that \( \|B_{a_\delta}^\omega\|^2_{A_\omega^2} \geq C_1/\omega(S(a)) \) for all \( a \in \mathbb{D} \setminus \{0\} \), and hence

\[
|B_{a_\delta}^\omega(z)| \geq |B_{a_\delta}^\omega(a_\delta)| - |B_{a_\delta}^\omega(a_\delta) - B_{a_\delta}^\omega(z)| = |B_{a_\delta}^\omega(\sqrt{|a_\delta|^2})| - |B_{a_\delta}^\omega(a_\delta) - B_{a_\delta}^\omega(z)|
\]

\[
= \left| \frac{B_{a_\delta}^\omega(\sqrt{|a_\delta|^2}) - B_{a_\delta}^\omega(a_\delta)}{\omega(S(\sqrt{|a_\delta|^2}))} - |B_{a_\delta}^\omega(a_\delta) - B_{a_\delta}^\omega(z)| \right| \geq \frac{C_1}{\omega(S(\sqrt{|a_\delta|^2}))} - |B_{a_\delta}^\omega(a_\delta) - B_{a_\delta}^\omega(z)| \quad (2.7)
\]

Moreover, by \( \square \) and Lemma \( \square \)

\[
|B_{a_\delta}^\omega(a_\delta) - B_{a_\delta}^\omega(z)| \leq \sup_{\zeta \in [a_\delta, z]} |(B_{a_\delta}^\omega)'(\zeta)||z - a_\delta| \leq 2\delta(1 - |a|) \sup_{\zeta \in [a_\delta, z]} |(B_{a_\delta}^\omega)'(\zeta)|
\]

\[
\leq \delta(1 - |a|) \sum_{n=1}^{\infty} \frac{n|a|^n}{\omega_{2n+1}} \leq \frac{\delta C_2}{\omega(S(a))}.
\]

By combining this with \( \square \), and choosing \( \delta = C_1/2C_2 \) we deduce the assertion for \( c = C_1/2 \). \( \square \)

**Lemma 8.** Let \( \omega \in \widehat{\mathbb{D}} \). Then there exists \( r = r(\omega) \in (0, 1) \) such that \( |B_{a_\delta}^\omega(z)| \approx B_{a_\delta}^\omega(a) \) for all \( a \in \mathbb{D} \) and \( z \in \Delta(a, r) \).
Proof. The proof is similar to that of [13] Lemma 6.4. First, use the Cauchy-Schwarz inequality, Theorem C and Lemma A to obtain

\[
|B_\omega^a(z)| \leq \sum_n \frac{|az|^n}{2\omega_{2n+1}} \leq \left( \sum_n \frac{|z|^{2n}}{2\omega_{2n+1}} \right)^{\frac{1}{2}} \left( \sum_n \frac{|a|^{2n}}{2\omega_{2n+1}} \right)^{\frac{1}{2}} = |B_\omega^a(a)|^{\frac{1}{2}} |B_\omega^a(z)|^{\frac{1}{2}}
\]

(2.8)

\[
\leq \frac{|B_\omega^a(a)|^{\frac{1}{2}}}{\sqrt{\omega(z)(1 - |z|)}} \leq \frac{|B_\omega^a(a)|^{\frac{1}{2}}}{\sqrt{\omega(a)(1 - |a|)}} \leq |B_\omega^a(a)|, \quad z \in \Delta(a, r),
\]

for all \( a \in \mathbb{D} \). This gives the claimed upper bound. To obtain the same lower bound, let \( r \in (0, 1) \) and note first that

\[
|B_\omega^a(z)| \geq |B_\omega^a(a)| - \max_{\zeta \in [a, z]} |(B_\omega^a)'(\zeta)||z - a|
\]

\[
\geq |B_\omega^a(a)| - \max_{\zeta \in [a, z]} |(B_\omega^a)'(\zeta)||C(1 - |a|),
\]

where \( C = C(r) > 0 \) is a constant for which \( \sup_{0 < r < r_0} C(r) < \infty \) for each \( r_0 \in (0, 1) \).

Now the Cauchy integral formula and a reasoning similar to that in (2.8) yield

\[
\max_{\zeta \in [a, z]} |(B_\omega^a)'(\zeta)| \lesssim \frac{|B_\omega^a(a)|}{1 - |a|}, \quad a \in \mathbb{D},
\]

and the desired lower bound follows by choosing \( r \) sufficiently small. \( \square \)

The last aim of this section is to show that for each \( \omega \in \mathcal{R} \), the normalized reproducing kernels \( b_{\omega, z}^p = B_\omega^p / \|B_\omega^p\|_{A_\omega^p} \) converge weakly to zero in \( A_\omega^p \), as \( |z| \to 1^- \). To do this, the following growth estimate is used.

Lemma 9. Let \( 0 < p < \infty \) and \( \omega \in \hat{\mathcal{D}} \). Then

\[
|f(z)| = o \left( \frac{1}{(\omega(z)(1 - |z|))^{\frac{1}{p}}} \right), \quad |z| \to 1^-,
\]

for all \( f \in A_\omega^p \).

Proof. Let \( f \in A_\omega^p \) and \( \varepsilon > 0 \). Then there exists \( r \in (0, 1) \) such that

\[
\varepsilon > \int_r^1 M_p(s, f)s\omega(s)\,ds \geq M_p(r, f)\varepsilon\omega(r),
\]

which together with the well-known estimate

\[
M_\infty(r, f) \lesssim \frac{M_p(\frac{1+r}{2}, f)}{(1-r)^{\frac{1}{p}}}, \quad 0 < r < 1,
\]

and the hypothesis \( \omega \in \hat{\mathcal{D}} \) yields the assertion. \( \square \)

The proof of the weak convergence we are after relies on the following known duality relation [15] Corollary 7].

Theorem D. Let \( 1 < p < \infty \) and \( \omega \in \mathcal{R} \). Then \( (A_\omega^p)^* \simeq A_\omega^p \), with equivalence of norms, under the pairing

\[
\langle f, g \rangle_{A_\omega^p} = \int_{\mathbb{D}} f(z)\overline{g(z)}\omega(z)\,dA(z).
\]

(2.9)

With these preparations we can prove the last result of the section.

Lemma 10. Let \( 1 < p < \infty \) and \( \omega \in \mathcal{R} \). Then \( b_{\omega, z}^p \to 0 \) weakly in \( A_\omega^p \), as \( |z| \to 1^- \).
Proof. Let $1 < p < \infty$ and $\omega \in \mathcal{R}$. By Theorem $\text{[D]}$ it suffices to show that
\[
\left| \langle b_{p,z}^\omega, g \rangle_{A^p_z} \right| = \frac{|g(z)|}{\|B_z^\omega\|_{A^p_z}} \to 0, \quad |z| \to 1^-,
\]
for all $g \in A^p_z$. But since $\|B_z^\omega\|_{A^p_z} \gtrless (\hat{\omega}(z)(1 - |z|))^{1-p}$ by Theorem $\text{[C]}$ and $1-p = -p/p'$, the assertion follows by Lemma $\text{[B]}$. \hfill \Box

3. Bounded and Compact Toeplitz Operators

The main objective of this section is to prove Theorems $\text{[I]}$ and $\text{[J]}$ stated in the introduction, and establish a characterization analogous to Theorem $\text{[I]}$ for compact operators $T_\mu : A^p_\omega \to A^q_\omega$, given as Theorem $\text{[K]}$ below. We begin with the following technical result.

Lemma 11. Let $\mu$ be a finite positive Borel measure on $\mathbb{D}$. Then $\text{[L]}$ is satisfied for all $f(z) = \sum_{n=0}^\infty \hat{f}(n)z^n$ and $g(z) = \sum_{n=0}^\infty \hat{g}(n)z^n$ such that $f \in H^\infty$ and $\sum_{n=0}^\infty |\hat{g}(n)| < \infty$.

Proof. Fubini’s theorem and the dominated convergence theorem yield
\[
\langle T_\mu(f), g \rangle = \lim_{s \to 1^-} \int_{|z|<s} \left( \int_{\mathbb{D}} f(\zeta)B_z^\omega(u) d\mu(\zeta) \right) g(u)\omega(u) dA(u) \\
= \lim_{s \to 1^-} \int_{\mathbb{D}} f(\zeta) \left( \int_{|z|<s} g(u)B_z^\omega(u) \omega(u) dA(u) \right) d\mu(\zeta) \\
= \lim_{s \to 1^-} \int_{\mathbb{D}} f(\zeta) \left( \sum_{n=0}^\infty \frac{\hat{g}(n)\zeta^n}{\omega_2n+1} \int_0^s x^{2n+1}\omega(x) dx \right) d\mu(\zeta) = \int_{\mathbb{D}} f(\zeta)\bar{g}(\zeta) d\mu(\zeta),
\]
and the assertion is proved. \hfill \Box

Recall that $b_{p,z}^\omega = B_z^\omega/\|B_z^\omega\|_{A^p_z}$ for all $z \in \mathbb{D}$. If $\mu$ is a finite positive Borel measure on $\mathbb{D}$ and $\omega \in \mathbb{D}$, then by using the definition $\text{[I]}$ of Berezin transform, Lemma $\text{[L]}$ and Theorem $\text{[C]}$ we deduce
\[
\bar{T}_\mu(z) = \langle T_\mu(b_{p,z}^\omega), b_{p,z}^\omega \rangle_{A^p_z} = \frac{\|B_z^\omega\|_{L^p_\omega}^2}{\|B_z^\omega\|_{A^p_z}^2} \asymp \omega(S(z)) \|B_z^\omega\|_{L^p_\omega}, \quad z \in \mathbb{D}. \tag{3.1}
\]
We now embark on the proofs by considering the cases $p \leq q$ and $p > q$ separately.

3.1. Case $1 < p \leq q < \infty$. We first consider bounded Toeplitz operators.

Proof of Theorem $\text{[I]}$. Since $\frac{p+q'}{pq'} \geq 1$ by the hypothesis $q \geq p$, the equivalence (iii) $\Leftrightarrow$ (iv) and the estimate
\[
\|I\|_{A^p_{\omega} \to A^q_{\omega}} \asymp \sup_{I \subset \mathbb{D}} \frac{\mu(S(I))}{\omega(S(I))^{\frac{1}{p} + \frac{1}{q'}}},
\]
follow by $\text{[L]}$ Theorem 1, see also $\text{[K]}$ Theorem 3 and $\text{[L]}$ Theorem 2.1. If $T_\mu : A^p_\omega \to A^q_\omega$ is bounded, then Hölder’s inequality and Theorem $\text{[C]}$ yield
\[
\left| \bar{T}_\mu(z) \right| = \left| \langle T_\mu(b_{p,z}^\omega), b_{p,z}^\omega \rangle_{A^p_z} \right| \leq \|T_\mu\|_{A^p_\omega \to A^q_\omega} \|b_{p,z}^\omega\|_{A^p_z} \leq \|T_\mu\|_{A^p_\omega \to A^q_\omega} \|b_{p,z}^\omega\|_{A^p_z} \|b_{p,z}^\omega\|_{A^p_z} \\
= \|T_\mu\|_{A^p_\omega \to A^q_\omega} \|B_z^\omega\|_{A^p_z} \|B_z^\omega\|_{L^p_\omega} \asymp \|T_\mu\|_{A^p_\omega \to A^q_\omega} \frac{\omega(S(z))}{\omega(S(z))^{\frac{1}{p} + \frac{1}{q'}}}, \quad z \in \mathbb{D},
\]
and hence
\[
\left\| \bar{T}_\mu(\cdot) \right\|_{L^\infty} \preceq \left\| T_\mu \right\|_{A^p_\omega \to A^q_\omega}.
\]
Assume next \( \bar{\bar{T}_\mu} \in L^\infty \), and let \( \delta = \delta(\omega) \) and \( c = c(\omega) \) be those of Lemma 7. Then Theorem 3 and (3.1) give
\[
\frac{\bar{\bar{T}_\mu}(z)}{\omega(S(z))} \geq \int_{S(z)} |B^\omega z_\mu(\zeta)|^2 d\mu(\zeta) \geq c^2 \frac{\mu(S(z))}{\omega(S(z))^2}, \quad z \in \mathbb{D} \setminus \{0\},
\]
and hence \( \mu(S(z)) \lesssim \bar{\bar{T}_\mu}(z) \omega(S(z)) \) for all \( z \in \mathbb{D} \setminus \{0\} \). It follows from Lemma 8 that
\[
\sup_{I} \frac{\mu(S(I))}{\omega(S(I))^{1\frac{1}{p} + \frac{1}{q} - 1}} \lesssim \| \bar{\bar{T}_\mu}(\cdot) \|_{L^\infty},
\]
and hence (ii) \( \Rightarrow \) (iv).

If now \( \mu \) is a \( \frac{\mu(S(z))}{\omega(S(z))} \)-Carleson measure for \( A^\omega_{pq} \), that is, \( \mu \) is a 1-Carleson measure for \( A^\omega_{pq} \) by [14, Theorem 1], then Lemma 11 [18, Theorem 3] and Hölder’s inequality yield
\[
|\langle T_\mu, f \rangle| A^\omega_{pq} \leq \int_D |f(z)g(z)| d\mu(z) \lesssim \|f\|_{A^\omega_{pq}} \|g\|_{A^\omega_{pq}} \left( \int_D |f(z)g(z)| \frac{\mu(S(z))}{\omega(S(z))^{1\frac{1}{p} + \frac{1}{q} - 1}} dA(z) \right)^{\frac{1}{\frac{1}{p} + \frac{1}{q} - 1}},
\]
for all polynomials \( f \) and \( g \). Since polynomials are dense in both \( A^\omega_{pq} \) and \( A^\omega_{pq} \), and \( (A^\omega_{pq})^* \simeq A^\omega_{pq} \) by Theorem 3, it follows that \( T_\mu : A^\omega_{pq} \to A^\omega_{pq} \) is bounded and \( \| T_\mu \|_{A^\omega_{pq} \to A^\omega_{pq}} \lesssim \|f\|_{A^\omega_{pq}} \|g\|_{A^\omega_{pq}} \). This is the right upper bound for \( s = \frac{\mu(S(z))}{\omega(S(z))} \), and the general case follows by an application of [18, Theorem 3].

Now we turn to compact Toeplitz operators.

**Proposition 12.** Let \( 1 < p \leq q < \infty \) and \( \omega \in \mathcal{R} \). If \( T : A^\omega_{pq} \to A^\omega_{pq} \) is a compact linear operator, then
\[
\lim_{|z| \to 1^-} \frac{\bar{\bar{T}_\mu}(z)}{\omega(S(z))^{\frac{1}{p} + \frac{1}{q} - 1}} = 0.
\]

**Proof.** Since \( b^\omega_{pq,z} \to 0 \) weakly in \( A^\omega_{pq} \), as \( |z| \to 1^- \), by Lemma 10 and \( T : A^\omega_{pq} \to A^\omega_{pq} \) is compact, and in particular completely continuous, by the hypothesis, we deduce
\[
\| T(b^\omega_{pq,z}) \|_{A^\omega_{pq}} \to 0, \quad |z| \to 1^-.
\]

By Hölder’s inequality this implies
\[
|\langle T(b^\omega_{pq,z}), b^\omega_{pq,z} \rangle|_{A^\omega_{pq}} \to 0, \quad |z| \to 1^-.
\]

Moreover, by Theorem 3,
\[
\|B^\omega z\|_{A^\omega_{pq}} \|B^\omega z\|_{A^\omega_{pq}} \leq \frac{1}{\omega(S(z))^{1\frac{1}{p} + \frac{1}{q} - 1}} \frac{1}{\omega(S(z))^{1\frac{1}{p} + \frac{1}{q} - 1}} \leq \|B^\omega z\|_{A^\omega_{pq}} \leq \frac{1}{\omega(S(z))^{1\frac{1}{p} + \frac{1}{q} - 1}},
\]
and hence
\[
|\bar{\bar{T}_\mu}(z)| \omega(S(z))^{1\frac{1}{p} + \frac{1}{q} - 1} = |\langle T(b^\omega_{pq,z}), b^\omega_{pq,z} \rangle|_{A^\omega_{pq}} \omega(S(z))^{1\frac{1}{p} + \frac{1}{q} - 1}
\]
\[
\leq |\langle T(b^\omega_{pq,z}), b^\omega_{pq,z} \rangle|_{A^\omega_{pq}} \to 0, \quad |z| \to 1^-,
\]
and the assertion is proved.

The following result is the analogue of Theorem 1 for compact Toeplitz operators.

**Theorem 13.** Let $1 < p \leq q < \infty$, $\omega \in \mathcal{R}$ and $\mu$ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:

(i) $T_\mu : A^p_\omega \to A^q_\omega$ is compact;

(ii) $\lim_{|z| \to 1-} \frac{\overline{T}_\mu(z)}{\omega(S(z))^{\frac{1}{p} - \frac{1}{q}}} = 0$;

(iii) $Id : A^s_\omega \to L^\mu_{\omega^{\mp \frac{p+q}{q}}} |\rightarrow \mu(S(I)) = 0$.

Proof. The equivalence (iii)$\Leftrightarrow$(iv) follows from [18, Theorem 3], see also [13, Theorem 2.1]. If $T_\mu : A^p_\omega \to A^q_\omega$ is compact, then $\lim_{|z| \to 1-} \frac{\overline{T}_\mu(z)}{\omega(S(z))^{\frac{1}{p} - \frac{1}{q}}} = 0$ by Proposition 12. Assume next that (ii) is satisfied, and let $\delta = \delta(\omega) \in (0,1)$ be that of Lemma 4. By the proof of Theorem 1, there exists a constant $C = C(\omega) > 0$ such that $\mu(S(z)) \leq C \overline{T}_\mu(z) \omega(S(z))$ for all $z \in \mathbb{D} \setminus \{0\}$. By applying Lemma A and letting $|z| \to 1^-$, it follows by the assumption (ii) that $\lim_{|z| \to 1-} \frac{\mu(S(I))}{\omega(S(z))^{\frac{1}{p} - \frac{1}{q}}} = 0$, and thus

$Id : A^s_\omega \to L^\mu_{\omega^{\mp \frac{p+q}{q}}}$ is compact by [18, Theorem 3].

Assume now that $Id : A^s_\omega \to L^\mu_{\omega^{\mp \frac{p+q}{q}}}$ is compact for some (equivalently for all) $0 < s < \infty$. Then, by [18, Theorem 3], $Id : A^p_\omega \to L^\mu_{\omega^{\frac{p+q}{q}}}$ and $Id : A^q_\omega \to L^\mu_{\omega^{\frac{p+q}{q}}}$ are compact. Let $\{f_n\}$ be a bounded sequence in $A^q_\omega$. Then the proof of [13, Theorem 2.1] shows that there exists a subsequence $\{f_{n_k}\}$ and $f \in A^q_\omega$ such that $\lim_{k \to \infty} \|f_{n_k} - f\|_{L^\mu_{\omega^{\frac{p+q}{q}}}} = 0$.

Write $\mu_r = \chi_{D(0,r)} \mu$ for $0 < r < 1$. Then Theorem 1 yields

$$\|T_\mu(f_{n_k} - f)\|_{A^q_\omega} \leq \|T_\mu(f_{n_k} - f)\|_{A^q_\omega} + \|(T_\mu - T_{\mu_r})(f_{n_k} - f)\|_{A^q_\omega} \lesssim \|T_\mu(f_{n_k} - f)\|_{A^q_\omega} + \|T_\mu - T_{\mu_r}\|_{A^q_\omega} \leq \|f_{n_k} - f\|_{L^\mu_{\omega^{\frac{p+q}{q}}}} \lesssim \sup_{r \leq \frac{1}{1-r}} \frac{\mu(S(I))}{\omega(S(I))^{\frac{1}{p} + \frac{1}{q}}} \leq \sup_{r \leq \frac{1}{1-r}} \frac{\mu(S(I))}{\omega(S(I))^{\frac{1}{p} + \frac{1}{q}}} \to 0, \quad r \to 1^-,$$

by Theorem 1 and [18, Theorem 3], because $Id : A^q_\omega \to L^\mu_{\omega^{\frac{p+q}{q}}}$ is compact by the hypothesis. Moreover, [13, Theorem 1] and Hölder’s inequality yield

$$\|T_{\mu_r}(f_{n_k} - f), g\|_{A^q_\omega} \leq \int_{\mathbb{D}} \|(f_{n_k} - f)(z)g(z)\| d\mu_r(z) \leq \|f_{n_k} - f\|_{L^\mu_{\omega^{\frac{p+q}{q}}}} \|g\|_{L^\mu_{\omega^{\frac{p+q}{q}}}} \leq \|f_{n_k} - f\|_{L^\mu_{\omega^{\frac{p+q}{q}}}} \|Id\|_{A^q_\omega \to L^\mu_{\omega^{\frac{p+q}{q}}}} \|g\|_{A^q_\omega} \leq \|f_{n_k} - f\|_{L^\mu_{\omega^{\frac{p+q}{q}}}} \|Id\|_{A^q_\omega \to L^\mu_{\omega^{\frac{p+q}{q}}}} \|g\|_{A^q_\omega}.$$
Thus $T_\mu : A^q_p \rightarrow A^q_p$ is compact, and the proof is complete.

3.2. **Case** $1 < q < p < \infty$. We begin with constructing appropriate test functions to be used in the proof of Theorem 2. To do this, some notation is needed. The Euclidean discs are denoted by $D(a, r) = \{ z \in \mathbb{C} : |a - z| < r \}$. A sequence $Z = \{ z_k \}_{k=0}^\infty \subset \mathbb{D}$ is called separated if it is separated in the pseudohyperbolic metric, it is an $\varepsilon$-net for $\varepsilon \in (0, 1)$ if $\mathbb{D} = \bigcup_{k=0}^\infty \Delta(z_k, \varepsilon)$, and finally it is a $\delta$-lattice if it is a 5$\delta$-net and separated with constant $\delta/5$.

**Proposition 14.** Let $1 < p < \infty$, $\omega \in \mathcal{R}$ and $\{ z_j \}_{j=1}^\infty \subset \mathbb{D} \setminus \{ 0 \}$ be a separated sequence. Then $F = \sum_{j=1}^\infty c_j b_{n, z_j} \in A^p_q$ with $\| F \|_{A^p_q} \lesssim \| \{ c_j \}_{j=1}^\infty \|_{\ell^p}$ for all $\{ c_j \}_{j=1}^\infty \in \ell^p$. 

**Proof.** Let $\{ c_j \}_{j=1}^\infty \in \ell^p$, $0 < r < 1$ and $z \in \overline{D(0, \rho)}$ with $0 < \rho < 1$. Then Hölder’s inequality and Theorem C yield

$$
\left| \sum_{j=1}^\infty c_j b_{n, z_j}(z) \right| \leq \| \{ c_j \}_{j=1}^\infty \|_{\ell^p} \left( \sum_{j=1}^\infty |\omega(\Delta(z_j, r))| |b_{n, z_j}(z)|^p \right)^{1/p'} \leq C(\omega, \rho) \| \{ c_j \}_{j=1}^\infty \|_{\ell^p} \omega(\mathbb{D}),
$$

and hence $F \in \mathcal{H}(\mathbb{D})$. Moreover, by Hölder’s inequality, Theorem [C] (2.2), the subharmonicity of $|g|^{p'}$ and (2.1),

$$
|\langle g, F \rangle_{A^2_p}| = \left| \sum_{j=1}^\infty c_j \frac{g(z_j)}{\| b_{n, z_j} \|_{A^2_q}} \right| \lesssim \| \{ c_j \}_{j=1}^\infty \|_{\ell^p} \left( \sum_{j=1}^\infty \omega(\Delta(z_j, r)) |g(z_j)|^{p'} \right)^{1/p'} \lesssim \| \{ c_j \}_{j=1}^\infty \|_{\ell^p} \omega(z) dA(z),
$$

where in the last step the fact that each $z \in \mathbb{D}$ belongs to at most $N$ of the discs $\Delta(z_j, r)$ is also used. Therefore $F$ defines a bounded linear functional on $A^p_q$ with norm bounded by a constant times $\| \{ c_j \}_{j=1}^\infty \|_{\ell^p}$. Since $(A^p_q)^* \simeq A^q_p$ by Theorem [D] this implies $F \in A^q_p$ with $\| F \|_{A^q_p} \lesssim \| \{ c_j \}_{j=1}^\infty \|_{\ell^p}$.

**Proof of Theorem 2** Write $x = x(p, q) = p + 1 - \frac{q}{q}$ for short. Assume first (ii). Take $\{ a_j \}_{j=1}^\infty \subset \mathbb{D} \setminus \{ 0 \}$ a separated sequence. Then Proposition 14 gives

$$
\left\| T_\mu \left( \sum_{j=1}^\infty c_j b_{n, a_j} \right) \right\|_{A^q_p} \lesssim \| T_\mu \|_{A^q_p \rightarrow A^q_p} \| \{ c_j \}_{j=1}^\infty \|_{\ell^p}.
$$

By replacing $c_k$ by $r_k(t)c_k$, where $r_k$ denotes the $k$th Rademacher function, and applying Khinchine’s inequality, we deduce

$$
\| T_\mu \|_{A^q_p \rightarrow A^q_p} \| \{ c_j \}_{j=1}^\infty \|_{\ell^p} \geq \int_{\mathbb{D}} \left( \sum_{j=1}^\infty |c_j|^2 |\mu(b_{n, a_j})| \right)^{q/2} \omega(z) dA(z) \geq \sum_{j=1}^\infty |c_j|^q \int_{\Delta(a_j, s)} |\mu(b_{n, a_j})| |\omega(z) dA(z)|,
$$

where in the last step the fact that each $z \in \mathbb{D}$ belongs to at most $N = N(s)$ of the discs $\Delta(a_j, s)$ is also used. By using the subharmonicity of $|\mu(b_{n, a_j})|^q$ together with (2.1)
and (2.2), and then applying Lemma 8 and Theorem C, we obtain

\[
\int_{\Delta(a_j, s)} |T_\mu(b_\omega^{\mu})(z)|^q \omega(z) \, dA(z) \gtrsim \omega(\Delta(a_j, s)) |T_\mu(b_\omega^{\mu})(a_j)|^q
\]

\[
= \frac{\omega(\Delta(a_j, s))}{\|b_\omega^{\mu}\|_{A_\omega^q}^q} \left( \int_{\Delta(a_j, s)} |b_\omega^{\mu}(\zeta)|^2 \, d\mu(\zeta) \right) ^q
\]

\[
\gtrsim \frac{\omega(\Delta(a_j, s))}{\|b_\omega^{\mu}\|_{A_\omega^q}^q} \left( \int_{\Delta(a_j, s)} |B_\omega^{\mu}(\zeta)|^2 \, d\mu(\zeta) \right) ^q
\]

\[
\gtrsim \left( \frac{\omega(\Delta(a_j, s))}{\omega(\Delta(a_j, s))^{1 + \frac{1}{p - q}}} \right) ^q, \quad 0 < s \leq r(\omega),
\]

where \(r(\omega)\) is that of Lemma 8. This together with (3.2) yields

\[
\sum_{j=1}^{\infty} |c_j|^q \left( \frac{\mu(\Delta(a_j, s))}{\omega(\Delta(a_j, s))^{1 + \frac{1}{p - q}}} \right) ^q \lesssim \|T_\mu\|_{A_\omega^q \to A_\omega^q} \|\{c_j\}_{j=1}^{\infty}\|_{L_p}, \quad 0 < s \leq r(\omega).
\]

(3.3)

Let now \(s \in (r(\omega), 1)\) and \(Z = \{z_j\}_{j=1}^{\infty} \subset \mathbb{D} \setminus \{0\}\) a \(\delta\)-lattice with \(5\delta \leq r(\omega)\). For each \(z_j\) choose \(N = N(s, r(\omega))\) points \(z_{k,j}\) of the \(\delta\)-lattice \(Z\) such that \(\Delta(z_{k,j}, s) \subset \bigcup_{k=1}^{\infty} \Delta(z_{k,j}, r(\omega))\). Then, by (2.1), (2.2) and (3.3),

\[
\sum_{j=1}^{\infty} |c_j|^q \left( \frac{\mu(\Delta(z_j, s))}{\omega(\Delta(z_j, s))^{1 + \frac{1}{p - q}}} \right) ^q \lesssim \sum_{j=1}^{\infty} \sum_{k=1}^{N} |c_j|^q \left( \frac{\mu(\Delta(z_{k,j}, r(\omega)))}{\omega(\Delta(z_{k,j}, r(\omega)))^{1 + \frac{1}{p - q}}} \right) ^q
\]

\[
= \sum_{k=1}^{N} \sum_{j=1}^{\infty} |c_j|^q \left( \frac{\mu(\Delta(z_{k,j}, r(\omega)))}{\omega(\Delta(z_{k,j}, r(\omega)))^{1 + \frac{1}{p - q}}} \right) ^q
\]

\[
\lesssim \|T_\mu\|_{A_\omega^q \to A_\omega^q} \|\{c_j\}_{j=1}^{\infty}\|_{L_p}.
\]

Therefore (3.3) holds for each \(0 < s < 1\) and any \(\delta\)-lattice \(\{z_j\}_{j=1}^{\infty} \subset \mathbb{D} \setminus \{0\}\) with \(5\delta \leq r(\omega)\). The classical duality relation \((\mathbb{H}^p/q)^* \simeq \ell^{p/q}\) now implies

\[
\sum_{j=1}^{\infty} \left( \frac{\mu(\Delta(z_j, s))}{\omega(\Delta(z_j, s))} \right) ^{\frac{ap}{p-q}} \omega(\Delta(z_j, s)) = \sum_{j=1}^{\infty} \left( \frac{\mu(\Delta(z_j, s))}{\omega(\Delta(z_j, s))^{1 + \frac{1}{p - q}}} \right) ^{\frac{ap}{p-q}} \lesssim \|T_\mu\|_{A_\omega^p}^{\frac{ap}{p-q}}.
\]

(3.4)

Let \(0 < r < 1\), and choose \(s = s(r, \delta) \in (0, 1)\) such that \(\Delta(z, r) \subset \Delta(z_j, s)\) for all \(z \in \Delta(z_j, 5\delta)\) and \(j \in \mathbb{N}\). Then (2.1) and (2.2) imply

\[
\|\hat{\mu}_r\|_{L_{\omega}^{\frac{ap}{p-q}}} \leq \sum_{j=1}^{\infty} \int_{\Delta(z_j, 5\delta)} \hat{\mu}_r(z) \left( \frac{\omega(z)}{\omega(z)(1 - |z_j|)^2} \right) ^{\frac{ap}{p-q}} \, dA(z)
\]

\[
\lesssim \sum_{j=1}^{\infty} \left( \frac{\omega(z_j)}{(\omega(z_j)(1 - |z_j|)^2)^{\frac{ap}{p-q}}} \right) ^{\frac{ap}{p-q}} \lesssim \sum_{j=1}^{\infty} \left( \frac{\mu(\Delta(z_j, s))}{\omega(\Delta(z_j, s))^{1 + \frac{1}{p - q}}} \right) ^{\frac{ap}{p-q}}.
\]

Thus (iii) is satisfied and \(\|\hat{\mu}_r\|_{L_{\omega}^{\frac{ap}{p-q}}} \lesssim \|T_\mu\|_{A_\omega^p} \to A_\omega^q\) for each fixed \(0 < r < 1\).
Assume next (iii). By using the subharmonicity of $|f|^r$ together with (2.1) and (2.2), and then Fubini’s theorem and Hölder’s inequality we deduce

$$
\int_D |f(z)|^r \, d\mu(z) \lesssim \int_D \frac{\int_{\Delta(z,r)} |f(\zeta)|^r \omega(\zeta) \, dA(\zeta)}{\omega(\Delta(z,r))} \, d\mu(z)
\approx \int_D \frac{\mu(\Delta(\zeta,r))}{\omega(\Delta(\zeta,r))} |f(\zeta)|^r \omega(\zeta) \, dA(\zeta) \leq \|f\|_{A^p_\infty} \|\tilde{\mu}_r\|_{L^{q/p}_\infty}.
$$

Therefore $\mu$ is a $(p + 1 - \frac{p}{q})$-Carleson measure for $A^p_\infty$, that is, (iv) is satisfied, and $\|\text{Id}\|_{A^p_\infty \rightarrow L^p_\mu} \lesssim \|\tilde{\mu}_r\|_{L^{q/p}_\infty}$. In fact, it follows from [4, Theorem 3.2] and [13, Lemma 1.4] that $\text{Id} : A^p_\infty \rightarrow L^p_\mu$ is bounded if and only if (iii) is satisfied.

The equivalence (iv)$\iff$(v) follows from [18, Theorem 3].

Let us now prove (iv)$\implies$(ii). Since $\mu$ is an $x$-Carleson measure for $A^p_\infty$ by the hypothesis (iv), Lemma 11 Hölder’s inequality and [18, Theorem 3] together with the equality $\frac{p}{q} = \frac{p}{q}$ give

$$\langle T_\mu(f), g \rangle_{A^q_\infty} \leq \int_D |f(z)g(z)| \, d\mu(z) \leq \|f\|_{L^q_\mu} \|g\|_{L^q_\mu'} \leq \|\text{Id}\|_{A^q_\infty \rightarrow L^q_\mu} \|\text{Id}\|_{A^{q'}_\infty \rightarrow L^{q'}_\mu} \|f\|_{A^{q'}_\infty} \|g\|_{A^q_\infty}$$

for polynomials $f$ and $g$. Since polynomials are dense in both $A^q_\infty$ and $A^{q'}_\infty$, and $(A^q_\infty)^* \simeq A^{q'}_\infty$ by Theorem 111 it follows that $T_\mu : A^{q'}_\infty \rightarrow A^q_\infty$ is bounded and $\|T_\mu\|_{A^{q'}_\infty \rightarrow A^q_\infty} \lesssim \|\text{Id}\|_{A^q_\infty \rightarrow L^q_\mu}$.

The implication (ii)$\implies$(i) follows by a general argument. Namely, for $1 < p < \infty$, $A^p_\infty$ is isomorphic to $\ell^p$ by [10, Corollary 2.6] and Lemma C. Moreover, each bounded linear operator $L : \ell^p \rightarrow \ell^q$, $0 < q < p < \infty$, is compact by [6, Theorem I. 2.7, p. 31]. Thus $T_\mu : A^p_\infty \rightarrow A^q_\infty$ is compact.

It remains to prove (iii)$\iff$(vi) and the equivalence of norms $\|\tilde{\mu}_r\|_{L^{q/p}_\infty} \simeq \|\tilde{T}_\mu\|_{L^{q/p}_\infty}$ for each fixed $r \in (0, 1)$. Assume $\tilde{T}_\mu \in L^{q/p}_\infty$, and let first $r \in (0, r(\omega)]$, where $r(\omega)$ is that of Lemma 8. Then Lemma 11 and Theorem C give

$$\tilde{T}_\mu(z) = \int_D |b^*_\omega(\zeta)|^2 \, d\mu(\zeta) \geq \int_{\Delta(z,r)} |b^*_\omega(\zeta)|^2 \, d\mu(\zeta) \approx |b^*_\omega(z)|^2 \mu(\Delta(z,r)) \approx \tilde{\mu}_r(z).$$

Hence $\tilde{\mu}_r \in L^{q/p}_\infty$ and $\|\tilde{\mu}_r\|_{L^{q/p}_\infty} \lesssim \|\tilde{T}_\mu\|_{L^{q/p}_\infty}$. Let now $r \in (r(\omega), 1)$, and let $\{z_j\}$ be a $\delta$-lattice. Further, let $s = s(r, \delta)$ be that of (3.4), and choose $r' = r'(r(\omega))$ such that $\Delta(z, r') \subset \Delta(u, r(\omega))$ for all $z \in \Delta(u, r')$ and $\omega \in \mathbb{D}$. Furthermore, choose $z^n_j \in \Delta(z_j, s)$, $n = 1, \ldots, N$, such that $\Delta(z_j, s) \subset \cup_{n=1}^N \Delta(z^n_j, r')$ for all $j$ and $\text{inf}_n \min_{n \neq m} g(z^n_j, z^m_j) > 0$. 

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Then (3.1), Lemma A, Lemma B, Theorem C and (3.1) yield

\[
\| \hat{\mu}_r \|_{L^{p,q}_{\omega}} \lesssim \sum_{j=1}^{\infty} \frac{\mu(\Delta(z_j, s))}{\omega(\Delta(z_j, s))} \lesssim \sum_{j=1}^{\infty} \sum_{n=1}^{N} \omega(\Delta(z_j, s)) \frac{\mu(\Delta(z_j, r'))}{\omega(\Delta(z_j, s))} \lesssim \sum_{j=1}^{\infty} \sum_{n=1}^{N} \omega(\Delta(z_j, r')) \hat{\mu}_r(\Delta(z_j, r')) \lesssim \sum_{j=1}^{\infty} \sum_{n=1}^{N} \int_{\Delta(z_j, r')} |B_z^r(\nu)|^2 d\mu(\nu) \lesssim \int_{\Delta(z_j, r')} \omega(z) dA(z)
\]

This together with (3.1), Theorem C, and Lemma 6 yield

\[
\|
\int_{\mathbb{D}} h(z) d\mu(z) \lesssim \int_{\mathbb{D}} \left( \frac{1}{(1-|z|)^2} \right) \omega(\Delta(z, r)) dA(z) \lesssim \int_{\mathbb{D}} \left( \frac{1}{\omega(\Delta(z, r))} \right) \omega(z) dA(z)
\]

Then (3.4), Lemma A, Lemma 8, Theorem C and (3.1) yield

\[
\|
\int_{\mathbb{D}} h(z) d\mu(z) \lesssim \int_{\mathbb{D}} \left( \frac{1}{(1-|z|)^2} \right) \omega(\Delta(z, r)) dA(z) \lesssim \int_{\mathbb{D}} \left( \frac{1}{\omega(\Delta(z, r))} \right) \omega(z) dA(z)
\]

This together with (3.1), Theorem C, and Lemma 6 yield

\[
\|
\int_{\mathbb{D}} h(z) d\mu(z) \lesssim \int_{\mathbb{D}} \left( \frac{1}{(1-|z|)^2} \right) \omega(\Delta(z, r)) dA(z) \lesssim \int_{\mathbb{D}} \left( \frac{1}{\omega(\Delta(z, r))} \right) \omega(z) dA(z)
\]

Now assume (iii), and let \( h \) be a positive subharmonic function in \( \mathbb{D} \). Then (2.2), Fubini’s theorem yield

\[
\|
\int_{\mathbb{D}} h(z) d\mu(z) \lesssim \int_{\mathbb{D}} \left( \frac{1}{(1-|z|)^2} \right) \omega(\Delta(z, r)) dA(z) \lesssim \int_{\mathbb{D}} \left( \frac{1}{\omega(\Delta(z, r))} \right) \omega(z) dA(z)
\]

This together with (3.1), Theorem C, and Lemma 6 yield

\[
\|
\int_{\mathbb{D}} h(z) d\mu(z) \lesssim \int_{\mathbb{D}} \left( \frac{1}{(1-|z|)^2} \right) \omega(\Delta(z, r)) dA(z) \lesssim \int_{\mathbb{D}} \left( \frac{1}{\omega(\Delta(z, r))} \right) \omega(z) dA(z)
\]

This finishes the proof.

4. Schatten class Toeplitz operators

The purpose of this section is to prove Theorem 3 or more precisely, the last part of it, and then show that it can not be extended to the whole class \( \mathcal{D} \) of doubling weights. We begin with some necessary notation and definitions, and preliminary results which are well-known in the setting of standard weights [23].

Let \( H \) be a separable Hilbert space. For any non-negative integer \( n \), the \( n \)th singular value of a bounded operator \( T : H \to H \) is defined by

\[
\lambda_n(T) = \inf \{ \|T - R\| : \text{rank}(R) \leq n \},
\]

where \( \| \cdot \| \) denotes the operator norm. It is clear that

\[
\|T\| = \lambda_0(T) \geq \lambda_1(T) \geq \lambda_2(T) \geq \cdots \geq 0.
\]

For \( 0 < p < \infty \), the Schatten \( p \)-class \( \mathcal{S}_p(H) \) consists of those compact operators \( T : H \to H \) whose sequence of singular values \( \{\lambda_n\}_{n=0}^{\infty} \) belongs to the space \( \ell^p \) of \( p \)-summable sequences. For \( 1 \leq p < \infty \), the Schatten \( p \)-class \( \mathcal{S}_p(H) \) is a Banach space with respect to the norm \( \|T\|_p = \|\{\lambda_n\}_{n=0}^{\infty}\|_{\ell^p} \). Therefore all finite rank operators belong to every
Let \( \mathcal{D}(H) \), and the membership of an operator in \( \mathcal{S}_p(H) \) measures in some sense the size of the operator. We refer to [5] and [23, Chapter 1] for more information about \( \mathcal{S}_p(H) \).

The first auxiliary result is well known and its proof is straightforward, so the details are omitted.

**Lemma E.** Let \( H \) be a separable Hilbert space and \( T : H \to H \) a bounded linear operator such that \( \sum_n |\langle T(e_n), e_n\rangle_H| < \infty \) for every orthonormal basis \( \{e_n\} \). Then \( T : H \to H \) is compact.

The next result characterizes positive operators in the trace class \( \mathcal{S}_1(A^2_\omega) \) in terms of their Berezin transforms.

**Theorem 15.** Let \( \omega \in \hat{\mathcal{D}} \) and \( T : A^2_\omega \to A^2_\omega \) a positive operator. Then \( T \in \mathcal{S}_1(A^2_\omega) \) if and only if \( T \in L^1_{\omega/\omega^*} \). Moreover, the trace of \( T \) satisfies

\[
\text{tr}(T) = \int_{\mathcal{D}} \bar{T}(z) \|B^\omega_z\|_{A^2_\omega}^2 \omega(z) \, dA(z) > 0.
\]

Proof. The proof is similar to that of [23, Theorem 6.4], and is included for the convenience of the reader. Fix an orthonormal basis \( \{e_n\}_{n=1}^\infty \) for \( A^2_\omega \). Since \( T \) is positive, [23, Theorem 1.23] and Lemma E show that \( T \in \mathcal{S}_1(A^2_\omega) \) if and only if \( \sum_{n=1}^\infty |\langle T(e_n), e_n\rangle_{A^2_\omega}| < \infty \), and further, \( \text{tr}(T) = \sum_{n=1}^\infty |\langle T(e_n), e_n\rangle_{A^2_\omega}| \). Let \( S = \sqrt{T} \). By the reproducing formula (1.2) and Parseval’s identity, Theorem C and Lemma A we have

\[
\text{tr}(T) = \sum_{n=1}^\infty |\langle T(e_n), e_n\rangle_{A^2_\omega}| = \sum_{n=1}^\infty \|S(e_n)\|^2_{A^2_\omega}
\]

\[
= \sum_{n=1}^\infty \int_{\mathcal{D}} |S(e_n)(z)|^2 \omega(z) \, dA(z) = \int_{\mathcal{D}} \left( \sum_{n=1}^\infty |S(e_n)(z)|^2 \right) \omega(z) \, dA(z)
\]

\[
= \int_{\mathcal{D}} \left( \sum_{n=1}^\infty |\langle e_n, B^\omega_z\rangle_{A^2_\omega}|^2 \right) \omega(z) \, dA(z)
\]

\[
= \int_{\mathcal{D}} \|S(B^\omega_z)\|_{A^2_\omega}^2 \omega(z) \, dA(z)
\]

\[
= \int_{\mathcal{D}} \|T(B^\omega_z)\|_{A^2_\omega}^2 \omega(z) \, dA(z) = \int_{\mathcal{D}} \bar{T}(z) \|B^\omega_z\|_{A^2_\omega}^2 \omega(z) \, dA(z) \times \int_{\mathcal{D}} \bar{T}(z) \frac{\omega(z)}{\omega^*(z)} \, dA(z),
\]

and the assertion is proved. \(\square\)

By combining Theorem 15 with [23, Proposition 1.31] we obtain the following result.

**Lemma 16.** Let \( \omega \in \hat{\mathcal{D}} \) and \( T : A^2_\omega \to A^2_\omega \) a positive operator.

(i) If \( 1 \leq p < \infty \) and \( T \in \mathcal{S}_p(A^2_\omega) \), then \( \bar{T} \in L^p_{\omega/\omega^*} \) with \( \|\bar{T}\|_{L^p_{\omega/\omega^*}} \lesssim \|T\|_p \).

(ii) If \( 0 < p \leq 1 \) and \( \bar{T} \in L^p_{\omega/\omega^*} \), then \( T \in \mathcal{S}_p(A^2_\omega) \) with \( \|T\|_p \lesssim \|\bar{T}\|_{L^p_{\omega/\omega^*}} \).

Recall that

\[
\mathcal{T}_\Phi(f)(z) = P_\omega(f\Phi)(z) = \int_{\mathcal{D}} f(\zeta) \overline{B^\omega_\zeta(\zeta)} \Phi(\zeta) \omega(\zeta) \, dA(\zeta), \quad f \in A^2_\omega,
\]

for each non-negative function \( \Phi \) on \( \mathcal{D} \). We next establish a sufficient condition for \( \mathcal{T}_\Phi \) to belong to \( \mathcal{S}_p(A^2_\omega) \) for \( 1 \leq p < \infty \).

**Proposition 17.** Let \( 1 \leq p < \infty \), \( \omega \in \hat{\mathcal{D}} \) and \( \Phi \in L^p_{\omega/\omega^*} \) positive. Then \( \mathcal{T}_\Phi \in \mathcal{S}_p(A^2_\omega) \) with \( \|\mathcal{T}_\Phi\|_p \lesssim \|\Phi\|_{L^p_{\omega/\omega^*}} \).
Proof. We will follow the proof of [23, Proposition 7.11]. Assume first that Φ has compact support in \( \mathbb{D} \). Then \( \mathcal{T}_\Phi \) is a positive compact operator with canonical decomposition

\[
\mathcal{T}_\Phi(f) = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle_{A^2_\omega} e_n,
\]

where \( \{\lambda_n\} \) is the sequence of eigenvalues of \( \mathcal{T}_\Phi \), and \( \{e_n\} \) is an orthonormal set of \( A^2_\omega \).

Therefore

\[
\lambda_n = \langle \Phi(e_n), e_n \rangle_{A^2_\omega} = \int_\mathbb{D} |e_n(z)|^2 \Phi(z) \omega(z) \, dA(z), \quad n \in \mathbb{N},
\]

by (1.3). Since \( \{\lambda_n\} \) is the sequence of eigenvalues of \( \mathcal{T}_\Phi \), and \( \{e_n\} \) is an orthonormal set of \( A^2_\omega \). Therefore

\[
\lambda_n^p \leq \int_\mathbb{D} |e_n(z)|^2 \Phi(z)^p \omega(z) \, dA(z),
\]

and hence

\[
\sum_{n=1}^{\infty} \lambda_n^p \leq \sum_{n=1}^{\infty} \int_\mathbb{D} |e_n(z)|^2 \Phi(z)^p \omega(z) \, dA(z)
\]

\[
\leq \int_\mathbb{D} B^p(z) \Phi(z)^p \omega(z) \, dA(z) = \int_\mathbb{D} \Phi(z)^p \omega(z) \, dA(z)
\]

by Theorem C. Thus \( \mathcal{T}_\Phi \in \mathcal{S}_p(A^2_\omega) \).

To prove the general case, assume \( \Phi \in L^p(\omega) \). Then Hölder’s inequality and Lemma A yield

\[
\lim_{|a| \to 1^-} \frac{\int_{S(a)} \Phi(z) \omega(z) \, dA(z)}{\omega(S(a))} \leq \left( \int_{S(a)} \Phi(z)^p \omega(z) \, dA(z) \right)^{\frac{1}{p}} \leq \lim_{|a| \to 1^-} \left( \int_{S(a)} \Phi(z)^p \omega(z) \, dA(z) \right)^{\frac{1}{p}} = 0,
\]

and hence \( \mathcal{T}_\Phi : A^2_\omega \to A^2_\omega \) is compact by Theorem [13].

Now write \( \Phi_r = \Phi \chi_{D(0,r)} \), where \( \chi_{D(0,r)} \) is the characteristic function of \( D(0,r) \). Arguing as in \( [1,1] \) it follows that \( \{\mathcal{T}_{\Phi_r}\}_{r \in (0,1)} \) is Cauchy in the Banach space \( (\mathcal{S}_p(A^2_\omega), | \cdot |_p) \).

Hence there exists \( T \in \mathcal{S}_p(A^2_\omega) \) such that \( \lim_{r \to 1^-} |\mathcal{T}_{\Phi_r} - T|_p = 0 \). On the other hand, if \( f \) is a polynomial and \( z \in \mathbb{D} \), then Lemma [11] and Hölder’s inequality yield

\[
|\langle \mathcal{T}_{\Phi_r} - \mathcal{T}_\Phi \rangle(f)(z)| = |\langle (\mathcal{T}_{\Phi_r} - \mathcal{T}_\Phi)(f), B^\omega_z \rangle_{A^2_\omega}|
\]

\[
= \left| \int_{|z| < |\zeta| < 1} f(\zeta) B^\omega_z(\zeta) \Phi(\zeta) \omega(\zeta) \, dA(\zeta) \right|
\]

\[
\leq C|f|_{H^\infty} \int_{|z| < |\zeta| < 1} \Phi(\zeta) \omega(\zeta) \, dA(\zeta)
\]

\[
\leq C|f|_{H^\infty} \left( \int_{|z| < |\zeta| < 1} \Phi(\zeta)^p \omega(\zeta) \, dA(\zeta) \right)^{\frac{1}{p}} \cdot \left( \int_\mathbb{D} \omega^*(\zeta)^{p'-1} \omega(\zeta) \, dA(\zeta) \right)^{\frac{1}{p'}} \to 0, \quad r \to 1^-,
\]

where \( C = C(z) \) is a constant. Thus \( \mathcal{T}_{\Phi_r}(f) \to \mathcal{T}_\Phi(f) \) pointwise for any polynomial \( f \). Since \( \mathcal{T}_{\Phi_r} \) and \( \mathcal{T}_\Phi \) are bounded on \( A^2_\omega \), and polynomials are dense in \( A^2_\omega \), we deduce that \( \mathcal{T}_{\Phi_r}(f) \to \mathcal{T}_\Phi(f) \) pointwise for all \( f \in A^2_\omega \). Therefore \( \mathcal{T}_\Phi = T \in \mathcal{S}_p(A^2_\omega) \).

We will need one more auxiliary result in the proof of Theorem [4].
which in turn implies let shows that Since suffices to prove the last claim which concerns the Berezin transform.

Proof of Theorem 3. □

Proposition 18. Let ω ∈ ℜ, 0 < r < 1 and μ be a finite positive Borel measure on ℂ such that Tμ : Aω → Aω is bounded. Then Tμ : Aω → Aω is bounded, and there exists C = C(ω, r) > 0 such that ⟨Tμ(f), f⟩Aω ≤ C⟨Tμ(f), f⟩Aω for all f ∈ Aω.

Proof. Note first that Tμ : Aω → Aω is bounded by Theorem 1 and [14, Theorem 1], see also [4, Theorem 3.1 and Theorem 4.1]. Let f be a polynomial. Then

\[ |f(\zeta)|^2 \leq \frac{1}{(1 - |\zeta|)^2} \int_{\Delta(\zeta, r)} |f(z)|^2 dA(z) \leq \int_{\Delta(1, \zeta)} \frac{|f(z)|^2}{(1 - |z|)^2} dA(z), \zeta \in \mathbb{D}, \]

and hence Fubini’s theorem, Lemma 11, Lemma A and (2.2) yield

\[ \langle T_\mu(f), f \rangle_{A^2_\omega} = \int_\mathbb{D} |f(\zeta)|^2 d\mu(\zeta) \leq \int_\mathbb{D} \left( \int_{\Delta(\zeta, r)} \frac{|f(z)|^2}{(1 - |z|)^2} dA(z) \right) d\mu(\zeta) \]

\[ = \int_\mathbb{D} \frac{|f(z)|^2}{(1 - |z|)^2} \omega(z) \left( \int_{\Delta(z, r)} d\mu(\zeta) \right) \omega(z) dA(z) \]

\[ \leq \int_\mathbb{D} \frac{|f(z)|^2}{\omega(\Delta(z, r))} \left( \int_{\Delta(z, r)} d\mu(\zeta) \right) \omega(z) dA(z) = \langle T_\mu(f), f \rangle_{A^2_\omega}. \]

Since Tμ : Aω → Aω and Tμ : Aω → Aω are bounded, and polynomials are dense in Aω, it follows that

\[ \langle T_\mu(f), f \rangle_{A^2_\omega} \lesssim \langle T_\mu(f), f \rangle_{A^2_\omega}, \quad f \in A^2_\omega, \]

and the proof is complete. □

Proof of Theorem 3. The conditions (i)–(iii) are equivalent by [16, Theorem 1], so it suffices to prove the last claim which concerns the Berezin transform.

The assertion is valid for p = 1 and ω ∈ ̂D by Theorem 15. For 1 < p < ∞, Lemma 10 shows that Tμ ∈ S_p(A^2_ω) implies ̂Tμ ∈ L^p_{ω/ω⋆}, with \| ̂Tμ\|_{L^p_{ω/ω⋆}} \lesssim \| Tμ\|_p. To see the converse implication, let r ∈ (0, r(ω)), where r(ω) is that of Lemma 8. If ̂Tμ ∈ L^p_{ω/ω⋆}, then ̂μr ∈ L^p_{ω/ω⋆}, with \| ̂μr\|_{L^p_{ω/ω⋆}} \lesssim \| ̂Tμ\|_{L^p_{ω/ω⋆}} by (3.3). Therefore Tμ ∈ S_p(A^2_ω) by Proposition 17 which in turn implies Tμ ∈ S_p(A^2_ω) with \| Tμ\|_p \lesssim \| ̂Tμ\|_{L^p_{ω/ω⋆}} by Proposition 18 and 23 Theorem 1.27.

Let now 0 < p < 1. If ̂Tμ ∈ L^p_{ω/ω⋆}, then Tμ ∈ S_p(A^2_ω) with \| Tμ\|_p \lesssim \| ̂Tμ\|_{L^p_{ω/ω⋆}} by Lemma 10. Conversely, assume that Tμ ∈ S_p(A^2_ω). Then (3.1) yields

\[ ( ̂Tμ(z))^p \asymp (ω^*(z))^p |B^ω_z|^{2p} = (ω^*(z))^p \left( \sum_{R_j \in \mathcal{Y}} \int_{R_j} |B^ω_z(\zeta_j)|^2 d\mu(\zeta) \right)^p \]

\[ \leq (ω^*(z))^p \sum_{R_j \in \mathcal{Y}} \left( \frac{\mu(R_j)}{ω^*(z_j)} \right)^p |B^ω_z(\zeta_j)|^{2p}(ω^*(z_j))^{p}, \quad |z| \geq \frac{1}{2}, \]

where \zeta_j, z such that supζ∈R_j |B^ω_z(ζ)| = |B^ω_z(\zeta_j,z)|. Consequently,

\[ \| ̂Tμ\|_{L^p_{ω/ω⋆}}^p \leq \sum_{R_j \in \mathcal{Y}} \left( \frac{\mu(R_j)}{ω^*(z_j)} \right)^p (ω^*(z_j))^p \int_\mathbb{D} |B^ω_z(\zeta_j)|^{2p}(ω^*(z_j))^{p-1}ω(z) dA(z) \]

\[ \times \sum_{R_j \in \mathcal{Y}} \left( \frac{\mu(R_j)}{ω^*(z_j)} \right)^p (ω^*(z_j))^p \int_\mathbb{D} |B^ω_z(\zeta_j,z)|^{2p}ω^*(z_j)^{p} \left( \frac{1}{1 - |z|^2} \right)^2 dA(z). \]
because $\omega \in \mathcal{R}$. Now, fix $0 < r < 1$ and $\delta = \delta(r) \in (0, 1)$ such that $\Delta(z, r) \subset \Delta(z_j, \delta)$ for all $z \in R_j$. Then, by the subharmonicity of $|B^p_z|^2$ and Fubini’s theorem,

$$
\int_{\mathbb{D}} |B^p_z(z, j)|^2 \frac{\omega^p(z)}{(1 - |z|)^2} dA(z)
\approx \int_{\mathbb{D}} \left( \int_{\Delta(z, r)} |B^p_z(\zeta)|^2 dA(\zeta) \right) \frac{\omega^p(z)}{(1 - |z|)^2} dA(z)
\approx \int_{\mathbb{D}} \left( \int_{\Delta(z, \delta)} |B^p_z(\zeta)|^2 dA(\zeta) \right) \frac{\omega^p(z)}{(1 - |z|)^2} dA(z)
= \frac{1}{(1 - |z|)^2} \int_{\Delta(z, \delta)} \left( \int_{\mathbb{D}} |B^p_z(\zeta)|^2 \frac{\omega^p(z)}{(1 - |z|)^2} dA(\zeta) \right) dA(\zeta).
$$

An application of Theorem C together with Lemma A and the hypothesis that $\omega^*(\cdot)/(1 - |\cdot|)^2$ is a regular weight show that the inner integral above is dominated by a constant times

$$
\int_{0}^{1} \int_{t}^{1} \frac{\omega^p(s)}{(1 - s)^2} ds dt \asymp \int_{0}^{1} \frac{1}{\omega^p(t)(1 - t)^{p+1}} dt \asymp \frac{1}{\omega^p(z)^p(1 - |z|)^p},
$$

and hence

$$
\int_{\mathbb{D}} |B^p_z(z, j)|^2 \frac{\omega^p(z)}{(1 - |z|)^2} dA(z) \lesssim \frac{1}{\omega^p(z)^p(1 - |z_j|)^p} \times \frac{1}{\omega^p(z)^p}, |z_j| \to 1^{-},
$$

by Lemma A. This combined with Theorem 1 and the equivalence (i)$\Leftrightarrow$(iii), proved in Theorem 1, gives the assertion.

In view of Theorems 3 and 15 it is natural to ask whether or not the condition $\tilde{T}_\mu \in L^p_{\omega^p}$ characterizes the Schatten class Toeplitz operators for the whole class $\tilde{D}$ of doubling weights. The next result answers this question in negative.

**Proposition 19.** For each $1 < p < \infty$ there exist $\omega \in \tilde{D}$ and a positive Borel measure $\mu$ on $\mathbb{D}$ such that $\tilde{T}_\mu \in L^p_{\omega^p}$, but $\hat{T}_\mu \notin S_p(A^2_{\omega^p})$.

**Proof.** Let $\omega(z) = \left( (1 - |z|) \left( \log \frac{1 - |z|}{1 - |z_j|} \right)^\alpha \right)^{-1}$, where $\alpha > -1$, and let $d\mu(z) = v(z) dA(z)$, where $v(z) = (1 - |z|)^{-1+\beta} \left( \log \frac{1 - |z|}{1 - |z_j|} \right)^{-\alpha+1-\beta}$ and $0 < \beta < \frac{1}{p}$. Then (3.1), Theorem C and Lemma A yield

$$
\tilde{T}_\mu(z) \asymp \omega^p(z) \int_{\mathbb{D}} |B^p_z(\zeta)|^2 v(\zeta) dA(\zeta) \asymp \frac{\hat{v}(z)}{\omega^p(z)} \asymp (1 - |z|)^{\frac{1}{p}} \left( \log \frac{e}{1 - |z|} \right)^{-\beta}, |z| \geq \frac{1}{2}.
$$

Therefore

$$
\int_{\mathbb{D} \setminus D(0, \frac{1}{2})} \left( \tilde{T}_\mu(z) \right)^p \frac{\omega^p(z)}{\omega^*(z)} dA(z)
\asymp \int_{\mathbb{D} \setminus D(0, \frac{1}{2})} \left( \omega^p(z) \log \frac{1 - |z|}{1 - |z_j|} \right) \left( \log \frac{e}{1 - |z|} \right)^{-\beta} \frac{dA(z)}{(1 - |z|)^2 \log \frac{e}{1 - |z|}}
= \int_{\mathbb{D} \setminus D(0, \frac{1}{2})} \frac{dA(z)}{(1 - |z|)^{\beta p+1}} < \infty,
$$

and thus $\tilde{T}_\mu \in L^p_{\omega^p}$. However, for each $r \in (0, 1), 0$

$$
\tilde{\mu}_r(z) = \frac{\mu(\Delta(z, r))}{\omega^*(z)} \asymp (1 - |z|)^2 v(z) \asymp (1 - |z|)^{\frac{1}{p}} \left( \log \frac{e}{1 - |z|} \right)^{-\beta}, |z| \geq \frac{1}{2}.
$$
and hence
\[
\|\hat{\mu}_r\|^p_{L^p\left(\frac{dA(z)}{(1-|z|)^2}\right)} \geq \int_{\mathbb{D} \setminus D(0, \frac{1}{2})} \left(1 - |z|\right)^{\frac{1}{p}} \left(\log \frac{e}{1 - |z|}\right)^{-\beta}\frac{dA(z)}{(1 - |z|)^2}
\]
\[
= \int_{\mathbb{D} \setminus D(0, \frac{1}{2})} \frac{dA(z)}{(1 - |z|)\left(\log \frac{e}{1 - |z|}\right)^{\beta}} = \infty.
\]

Consequently, \(T_\mu \notin S_p(A^2_\omega)\) by Theorem 3.

The asymptotic relation \(\omega(z)/\omega^*(z) \propto (1 - |z|)^{-2}\), valid for each \(\omega \in \mathcal{R}\) and \(z \in \mathbb{D}\) uniformly bounded away from the origin, has been repeatedly used in this paper. This relation fails for \(\omega \in \hat{\mathcal{D}} \setminus \mathcal{R}\) and, for example, the doubling weight \(\omega(z) = \left[(1 - |z|)\left(\log \frac{e}{1 - |z|}\right)^{\alpha}\right]^{-1}\), where \(\alpha > -1\), satisfies \(\omega(z)(1 - |z|)^2/\omega^*(z) \propto \left(\log \frac{e}{1 - |z|}\right)^{-1}\) \(\to 0\), as \(|z| \to 1^-\). The last result of this section shows that this innocent looking difference is significant concerning the conditions \(T_\mu \in L^1\left(\frac{dA}{(1 - |z|)^2}\right)\) and \(\hat{T}_\mu \in L^1_{\omega^*}\). Therefore one may not replace \(L^1_{\omega^*}\) by \(L^1\left(\frac{dA}{(1 - |z|)^2}\right)\) in the statement of Theorem 19.

**Proposition 20.** There exists \(\omega \in \hat{\mathcal{D}}\) and a positive Borel measure \(\mu\) on \(\mathbb{D}\) such that \(T_\mu \in S_1(A^2_\omega)\) and \(\hat{T}_\mu \notin L^1\left(\frac{dA}{(1 - |z|)^2}\right)\).

**Proof.** Choose \(\omega(z) = \left[(1 - |z|)\left(\log \frac{e}{1 - |z|}\right)^{\alpha}\right]^{-1}\), where \(\alpha > 2\), and \(d\mu(z) = u(z)\,dA(z)\), where \(u(z) = \left(\log \frac{e}{1 - |z|}\right)^{-\beta - \alpha}\) and \(0 < \beta < \min\{1, \alpha - 2\}\). Then, by Lemma A
\[
\|B^\omega\|^2_{L^2_{\omega}} = \sum_{n=0}^{\infty} \frac{|z|^{2n}}{\left((v_n)_{2n+1}\right)^2} u_n \sim \sum_{n=1}^{\infty} \frac{|z|^{2n}}{(n + 1)(\log n)^{\alpha - \beta - 2}} \sim \left(\log \frac{e}{1 - |z|}\right)^{\alpha - \beta - 1},
\]
and hence
\[
\hat{T}_\mu(z) \propto \omega^*(z)\|B^\omega\|^2_{L^2_{\omega}} \propto \left(1 - |z|\right)^{\frac{1}{p}}\left(\log \frac{e}{1 - |z|}\right)^{-\beta}, \quad |z| \geq \frac{1}{2}.
\]
by (5.1). It follows that \(\hat{T}_\mu \notin L^1\left(\frac{dA}{(1 - |z|)^2}\right)\). However,
\[
\int_{\mathbb{D} \setminus D(0, \frac{1}{2})} \frac{\omega(z)}{\omega^*(z)} dA(z) \propto \int_{\mathbb{D}} \frac{dA(z)}{(1 - |z|)\left(\log \frac{e}{1 - |z|}\right)^{\beta + 1}} < \infty,
\]
and hence \(T_\mu \in S_1(A^2_\omega)\) by Theorem 15.

### 5. Schatten class composition operators

The main purpose of this section is to prove Theorem 4. The following result of its own interest plays a role in the proof.

**Proposition 21.** Let \(0 < p < \infty\) and \(\omega \in \hat{\mathcal{D}}\), and let \(\varphi\) be an analytic self-map of \(\mathbb{D}\). Then the condition \(\mathcal{L}_A\) is sufficient if \(0 < p \leq 2\) and necessary if \(2 \leq p < \infty\) for \(C_\varphi\) to belong to \(S_p(A^2_\omega)\).

**Proof.** First observe that
\[
\langle f, C_\varphi(b^\omega_z) \rangle_{A^2_\omega} = \langle C_\varphi(f), b^\omega_z \rangle_{A^2_\omega} = \|B^\omega_z\|_{A^2_\omega}^{-1} \langle C_\varphi(f), B^\omega_z \rangle_{A^2_\omega} = \|B^\omega_z\|_{A^2_\omega}^{-1} f(\varphi(z)), \quad (5.1)
\]
and hence \(C_\varphi(b^\omega_z) = \|B^\omega_z\|_{A^2_\omega}^{-1} B^\omega_z(\varphi(z))\). Consequently,
\[
\|C_\varphi(b^\omega_z)\|^2_{A^2_\omega} = \frac{\|B^\omega_z(\varphi(z))\|^2_{A^2_\omega}}{\|B^\omega_z\|^2_{A^2_\omega}} \sim \frac{\omega(S(z))}{\omega((S(\varphi(z)))))}, \quad z \in \mathbb{D}, \quad (5.2)
\]
by Theorem \textbf{C}. This and Lemma \textbf{A} yield
\[
\int_{\mathbb{D}} \left( \frac{\omega^*(\zeta)}{\omega^*(\varphi(z))} \right)^{\frac{p}{2}} \frac{\omega(z)}{\omega^*(\varphi(z))} dA(z) \asymp \int_{\mathbb{D}} \|C_\omega^*(b^\omega_\varphi)\|_{A^2_\varphi}^p \frac{\omega(z)}{\omega^*(\varphi(z))} dA(z) = \int_{\mathbb{D}} \|\tilde{T}(z)\|_{A^2_\varphi}^p \frac{\omega(z)}{\omega^*(\varphi(z))} dA(z),
\]
where \(T = C_\varphi C_\omega^*\). The assertion follows from \cite{22} Theorem 1.26 and Lemma \textbf{16}.

An alternative way to establish the assertions is to follow the reasoning in \cite{8} p. 1143.

\begin{proof}[Proof of Theorem \textbf{4}] Since \(C_\omega^*\) can be formally computed as
\[
C_\omega^*(f)(z) = \langle C_\omega^* f, B^w_\varphi \rangle_{A^2_\varphi} = \langle f, C_\varphi(B^w_\omega) \rangle_{A^2_\varphi} = \langle f, B^w_\varphi(\varphi) \rangle_{A^2_\varphi},
\]

it follows that
\[
C_\omega^* C_\varphi(f)(z) = \int_{\mathbb{D}} f(\varphi(\zeta)) B^w_\varphi(\varphi(\zeta)) \omega(\zeta) dA(\zeta).
\]

Let \(\mu\) be the pull-back measure defined by \(\mu(E) = \omega(\varphi^{-1}(E))\). Then
\[
C_\omega^* C_\varphi(f)(z) = \int_{\mathbb{D}} f(u) B^w_\varphi(u) d\mu(u) = T_\mu(f)(z),
\]
and hence \(C_\varphi \in S_p(A^2_\varphi)\) if and only if \(T_\mu \in S_p(A^2_\varphi)\) by \cite{23} Theorem 1.26. Therefore, by Theorems \textbf{3} and \textbf{21} it suffices to show that (1.4) implies \(\tilde{T}_\mu \in L^p_{\omega/\omega_*}\). To see this, we use Theorem \textbf{C} to write
\[
\tilde{T}_\mu(z) = \langle \tilde{T}_\mu(b^w_\varphi), b^w_\omega \rangle_{A^2_\varphi} = \int_{\mathbb{D}} \|B^w_\varphi(\varphi(\zeta))\|_{A^2_\varphi}^2 d\mu(\zeta) \asymp \omega(S(z)) \int_{\mathbb{D}} |B^w_\varphi(\varphi(\zeta))|^2 \omega(\zeta) dA(\zeta).
\]

We will now argue as in \cite{22} p. 180. Note first that \cite{13} Theorem 4.2 gives
\[
\tilde{T}_\mu(z) \asymp \omega(S(z)) |B^w_\varphi(\varphi(0))|^2 + \omega(S(z)) \int_{\mathbb{D}} |(B^\omega_\varphi)'(\varphi(\zeta))|^2 |\varphi'(\zeta)|^2 \omega^*(\zeta) dA(\zeta).
\]

Hence it suffices to show that
\[
\Phi(z) = \omega(S(z)) \int_{\mathbb{D}} |(B^\omega_\varphi)'(\varphi(\zeta))|^2 |\varphi'(\zeta)|^2 \omega^*(\zeta) dA(\zeta)
\]

belongs to \(L^p_{\omega/\omega_*}\). To do this we will use Shur’s test with two measures \cite{23} Theorem 3.8. Let
\[
\psi(\zeta) = \frac{\omega^*(\zeta)}{\omega^*(\varphi(\zeta))}, \quad d\nu(\zeta) = \frac{\omega(\varphi(\zeta))}{\omega^*(\varphi(\zeta))} |\varphi'(\zeta)|^2 dA(\zeta)
\]

and
\[
H(z, \zeta) = \frac{|(B^\omega_\varphi)'(\varphi(\zeta))|^2 \omega(S(z)) \omega^*(\varphi(\zeta))^2}{\omega(\varphi(\zeta))},
\]

so that the operator
\[
T(f) = \int_{\mathbb{D}} H(z, \zeta) f(\zeta) d\nu(\zeta)
\]
satisfies \(T(\psi) = \Phi\). Since \(\varphi\) is of bounded valence, we obtain
\[
\int_{\mathbb{D}} H(z, \zeta) d\nu(\zeta) = \omega(S(z)) \int_{\mathbb{D}} |(B^\omega_\varphi)'(\varphi(\zeta))|^2 \omega^*(\varphi(\zeta)) |\varphi'(\zeta)|^2 dA(\zeta)
\]
\[
\asymp \omega(S(z)) \int_{\mathbb{D}} |(B^\omega_\varphi)'(\zeta)|^2 \omega^*(\zeta) dA(\zeta) \asymp 1
\]
Then the condition $0 < p < 2$ is sufficient if

$$
\int_{\mathbb{D}} H(z, \zeta) \frac{\omega(z)}{\omega^*(z)} \frac{\omega^*(\varphi(\zeta))^2}{\omega(\varphi(\zeta))} \left| (B_{\omega}^\varphi)'(\varphi(\zeta)) \right|^2 \omega(z) \, dA(z) = \frac{\omega^*(\varphi(\zeta))^2}{\omega(\varphi(\zeta))} \left| (B_{\omega}^\varphi)'(\varphi(\zeta)) \right|^2 \omega(z) \, dA(z) \lesssim 1,
$$

because $\omega \in \mathcal{R}$. Now $\psi \in L^{p/2}_{\omega^*/\omega^*}$ by the assumption (1.4), and $\nu \lesssim \omega/\omega^*$ by the Schwarz-Pick lemma and the assumption $\omega \in \mathcal{R}$, so $\psi \in L^{p/2}_{\nu/\nu^*}$. Therefore we may apply Schur’s test (with both test functions equal to 1) to deduce that $T$ is a bounded operator from $L^{p/2}_{\nu/\nu^*}$ into $L^{p/2}_{\omega^*/\omega^*}$, and thus, in particular, $T(\psi) = \Phi \in L^{p/2}_{\omega^*/\omega^*}$. Therefore $\tilde{T}_\mu \in L^{p/2}_{\omega^*/\omega^*}$ as desired.

The following result is parallel to Proposition 21. By the Schwarz-Pick lemma, (1.4) implies $\nu \lesssim \omega/\omega^*$ for all $0 < p < \infty$ and $\omega \in \mathcal{R}$, and therefore the case $0 < p < 2$ is of particular interest.

**Proposition 22.** Let $0 < p < \infty$ and $\omega \in \hat{\mathbb{D}}$, and let $\varphi$ be an analytic self-map of $\mathbb{D}$. Then the condition

$$
\int_{\mathbb{D}} \left( \frac{\omega^*(z)}{\omega^*(\varphi(z))} \right)^{\frac{p}{2}} \left| \varphi'(z) \right|^p \frac{(1 - |z|^2)^{p-2}}{(1 - |\varphi(z)|^2)^p} \, dA(z) < \infty \tag{5.3}
$$

is sufficient if $0 < p \leq 2$ and necessary if $2 \leq p < \infty$ for $C_\varphi$ to belong to $\mathcal{S}_p(A_\omega^2)$.

**Proof.** Let first $p \geq 2$. The Schwarz-Pick lemma, a change of variable and a standard inequality yield

$$
\int_{\mathbb{D}} \left( \frac{\omega^*(z)}{\omega^*(\varphi(z))} \right)^{\frac{p}{2}} \left| \varphi'(z) \right|^p \frac{(1 - |z|^2)^{p-2}}{(1 - |\varphi(z)|^2)^p} \, dA(z) \leq \int_{\mathbb{D}} \left( \frac{\omega^*(z)}{\omega^*(\varphi(z))} \right)^{\frac{p}{2}} \frac{|\varphi'(z)|^2}{(1 - |\varphi(z)|^2)^p} \, dA(z)
$$

$$
= \int_{\mathbb{D}} \frac{N_{\varphi,\omega^*(\varphi(z))^2}(\zeta)}{\omega^*(\varphi(z))^2} \frac{dA(\zeta)}{(1 - |\zeta|^2)^2} \leq \int_{\mathbb{D}} \left( \frac{N_{\varphi,\omega^*(\varphi(z))^2}(\zeta)}{\omega^*(\varphi(z))^2} \right)^{\frac{p}{2}} \frac{dA(\zeta)}{(1 - |\zeta|^2)^2},
$$

and hence the assertion follows by [16] Theorem 3. A similar reasoning shows the case $p < 2$. \qed

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