Peak–peak correlations in the cosmic background radiation from cosmic strings

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ABSTRACT
We examine the two-point correlation function of local maxima in temperature fluctuations at the last scattering surface when this stochastic field is modified by the additional fluctuations produced by straight cosmic strings via the Kaiser–Stebbins effect. We demonstrate that one can detect the imprint of cosmic strings with tension $G\mu \gtrsim 1.2 \times 10^{-8}$ on noiseless 1 arcmin resolution cosmic microwave background (CMB) maps at 95 per cent confidence interval. Including the effects of foregrounds and anticipated systematic errors increases the lower bound to $G\mu \gtrsim 9.0 \times 10^{-8}$ at 2σ confidence level. Smearing by beams of the order of 4 arcmin degrades the bound further to $G\mu \gtrsim 1.6 \times 10^{-7}$. Our results indicate that two-point statistics are more powerful than one-point statistics (e.g. number counts) for identifying the non-Gaussianity in the CMB due to straight cosmic strings.

Key words: cosmic background radiation – cosmology: theory – early Universe – large-scale structure of Universe.

1 INTRODUCTION
The origins of the seeds of present large-scale structures in the Universe are still debated. It is believed that they are mainly primordial and produced some time after big bang. In this framework, there are two approaches: (1) the freezing-in of quantum fluctuations of a scalar field during the so-called inflationary epoch (Guth 1981; Liddle & Lyth 1993; Steinhardt 1995; Liddle 1999); and/or (2) topological defects as sources (Kibble 1976, 1980). Indeed topological defects can be formed during phase transitions between different vacuum states in an expanding universe, and cosmic strings (CSs) are predicted by quantum field theory in cosmology (Kibble 1976; Zeldovich 1980; Vilenkin 1981; Vachaspati & Vilenkin 1984; Vilenkin 1985; Shellard 1987; Hindmarsh et al. 1995; Allen et al. 1996; Khlopov 1999; Vilenkin & Shellard 2000; Sakellariadou 2007; Bevis et al. 2008; Depies 2009; Bevis et al. 2010). Both the inflationary and topological defects scenarios predict the same features for the cosmic microwave background (CMB) power spectrum on large scales. However, at the intermediate and small scales, due to differences in the super-horizon-scale behaviour of perturbations in these theories, the predictions are completely different.

The inflationary Λ cold dark matter (ΛCDM) paradigm is consistent with today’s high-precision observations of the CMB. Nevertheless, from both the theoretical and the observational points of view there are many motivations for other sources of anisotropies, e.g. in hybrid inflation models, the braneworld paradigm and superstring theory, production of topological defects is crucial and inevitable (Copeland et al. 1994; Sakellariadou 1997; Majumdar & Davis 2002; Sarangi & Tye 2002; Pogosian et al. 2003; Copeland, Myers & Polchinski 2004; Dvali & Vilenkin 2004; Tye 2008). For more recent observational results, see Bevis et al. (2008), Dvorkin, Wyman & Hu (2011), Ringeval & Bouchet (2012) and Kuroyanagi et al. (2013).

A CS network which consists of infinite strings, loops and junctions of strings can generate gravitational waves as the universe evolves. Astrophysical evidence of CS depends on (1) the inter-commuting probability and (2) the dimensionless string tension $G\mu/c^2 \equiv \Lambda^2/M_{\text{Plank}}^2$, where $G$ is Newton’s constant, $\mu$ is the mass per unit length of the CS and $\Lambda$ is the energy scale of the string-creation epoch (Vilenkin & Shellard 2000; Bevis et al. 2008, 2010). We set $c = 1$ throughout this paper. Determining bounds on the value of $\mu$ directly means limiting the basis of fundamental theories for CS production. In addition, observing CSs not only is a kind of observational evidence for such theories, but also provides an opportunity to rule out or confirm theoretical models of particle physics.

To infer a reliable prediction and find the observational signatures of CSs, it is essential to understand the statistical properties of a typical CS network. Since calculating the CS components is still a challenging topic, subsequently, all previous studies especially from the simulation point of view have relied on...
one or more simplifications (for more details, see Bevis et al. 2007, 2008). In order to examine the evolution of a CS network, the most important key is the scaling solution, which is consistent with the numerical simulations (Vilenkin 1981; Albrecht & Turok 1985; Kibble 1985; Bennett 1986a,b; Bennett & Bouchet 1988, 1989, 1990; Albrecht & Turok 1989; Allen & Shellard 1990; Shellard & Allen 1990; Austin, Copeland & Kibble 1993; Martins & Shellard 1996, 2006; Vincent, Antunes & Hindmarsh 1998; Vanchurin, Olum & Vilenkin 2005, 2006; Polchinski & Rocha 2006, 2007; Olum & Vanchurin 2007; Ringeval, Sakellariadou & Bouchet 2007; Dubath, Polchinski & Rocha 2008; Fraisse et al. 2008; Blanco-Pillado, Olum & Shlaer 2011; Kurayyanagi et al. 2013). Several different groups have independently established various codes as well as theoretical methods to explore the evolution of a CS network and their imprints on different observations such as CMB (Vilenkin 1981; Albrecht & Turok 1985; Kibble 1985; Bennett 1986a,b; Bennett & Bouchet 1988, 1989, 1990; Albrecht & Turok 1989; Allen & Shellard 1990; Shellard & Allen 1990; Austin et al. 1993; Martins & Shellard 1996, 2006; Vincent et al. 1998; Yamaguchi, Yokoyama & Kawasaki 2000; Moore & Shellard 2001; Vanchurin et al. 2005; Polchinski & Rocha 2006; Vanchurin et al. 2006; Olum & Vanchurin 2007; Polchinski & Rocha 2007; Ringeval et al. 2007; Dubath et al. 2008; Fraisse et al. 2008; Blanco-Pillado et al. 2011). Initial conditions and equations of motion have major roles in the simulation of a CS network (Vachaspati & Vilenkin 1984; Blanco-Pillado et al. 2011). During the evolution of CSs, the intercommute phenomenon produces closed loops and makes kinks. These loops lose their energy through gravitational radiations or particle production, depending on their size. The probability of this intercommutation in principle is not unity. Allen & Shellard used a high-resolution total-variation-non-increasing (Sod 1985) algorithm to simulate the behaviour of CSs in an expanding universe during the radiation-dominated era (Allen & Shellard 1990). The same simulations but in matter-dominated epoch have been done in Yamaguchi et al. (2000) and Moore & Shellard (2001). They found that long strings have scaling behaviour and exhibit significant small-scale-structure kinks and short-wavelength propagating modes. In addition, loops produce peaks at scales very smaller than the horizon. On the other hand, the Abelian–Higgs model and one-dimensional effective theory—the Nambu–Goto action—are suitable to simulate a CS network (Moore & Shellard 1998; Vincent et al. 1998; Fraisse et al. 2008). The scaling behaviour, the correlation of a CS network and the evaluation of dominant decay mechanisms in different epochs have been investigated in Vilenkin (1981), Kibble (1985), Bennett (1986a,b), Bennett & Bouchet (1988, 1989), Albrecht & Turok (1985, 1989), Allen & Shellard (1990), Austin et al. (1993), Martins & Shellard (1996), Vincent et al. (1998), Vanchurin et al. (2005), Vanchurin et al. (2006), Martins & Shellard (2006), Polchinski & Rocha (2006, 2007), Ringeval et al. (2007), Olum & Vanchurin (2007), Dubath et al. (2008), Fraisse et al. (2008) and Blanco-Pillado et al. (2011). Historically, after many extensive numerical and theoretical studies concerning the CS network, the cosmological as well as astrophysical effects of CSs have attracted much attention. One of the most pioneering studies was done by Bouchet, Bennet & Stebbins (1988). Since at the time of importance for CMB computations the difference in scale is enormous, idealizations for CSs such as zero-width limit and unconnected segmentation of a CS network are hence justified (Kasuya & Kawasaki 2000; Bevis et al. 2007).

There are some important ways to investigate the effect of a CS network on the temperature fluctuations at the last scattering surface: (i) Nambu–Goto simulation of connected CSs; (ii) model based on a stochastic ensemble of unconnected segments (Allen et al. 1996, 1997; Albrecht, Battye & Robinson 1997; Contaldi, Hindmarsh & Magueijo 1999; Landriau & Shellard 2004); and (iii) Abelian–Higgs model on a lattice and evolution of CS networks are evolved in terms of their corresponding fields (Kasuya & Kawasaki 2000; Bevis et al. 2007). Following previous approaches, the full Boltzmann equations are solved to compute the CMB fluctuations from a CS network and all relevant physics to first order are taken into account (Coulson et al. 1994; Allen et al. 1996; Landriau & Shellard 2003). The fourth way, the so-called statistical approach, has been adopted by some groups (Perivolaropoulos 1993a,b; Moessner, Perivolaropoulos & Brandenberger 1994; Jeong & Smoot 2005; Amsel, Berger & Brandenberger 2008; Stewart & Brandenberger 2009; Danos & Brandenberger 2010a; Movahed & Khosravi 2011). In the latter approach, analytical and numerical models have been proposed to investigate the effect of CSs on the CMB by means of the nature of CSs. The main part of this approach is based on counting random multiple impulses, inflicted on photon trajectories by CS networks between the time of recombination and the present era. It is well known that the direct implication based on the explicit recognition of discontinuity in the fluctuations of the CMB is a unique signature of straight CSs, namely the Kaiser–Stebbins effect (Kaiser & Stebbins 1984; Allen et al. 1997; Pen, Seljak & Turok 1997). It has been demonstrated that to infer reliable results not only one should use robust methods, but also a combination of powerful methods is necessary (Starck, Aghanim & Forni 2004). Usually, in the first three approaches, the contribution of various components on the CMB map is modelled in the spherical or Fourier spaces, and then by integrating on the power spectrum the effect will be emerged. In contrast, in the latter method, the effect of topological defects is modelled in the real space (Perivolaropoulos 1993a,b; Moessner et al. 1994; Jeong & Smoot 2005; Amsel et al. 2008; Stewart & Brandenberger 2009; Danos & Brandenberger 2010a). As explained in more details in Stewart & Brandenberger (2009) and Movahed & Khosravi (2011), since the statistical isotropy is valid as a major statistical property [Hajian & Souradeep 2003, 2006; Movahed et al. 2011; Ade et al. (Planck Collaboration) 2013b], one can compute the two-point correlation function (TPCF) at those scales, which have not been affected by the boundary condition. The superposition of fluctuations produced by CSs can generate new and extra peaks in temperature fluctuations, consequently finding such statistically meaningful footprints in the map in comparison with a pure Gaussian signature, including instrumental noise, may potentially help us to get a deep insight into the CS detections and put constraints on the free parameters of CS theoretical models. Finally, because of the phase coefficients in the Fourier analysis, it seems that many trivial imprints of CSs diminish or at least are mixed with other observational phenomena; therefore, this is another motivation to investigate the imprint of CSs in the real space. To this end, we used the method introduced by Perivolaropoulos (1993a,b), Moessner et al. (1994), Jeong & Smoot (2005), Amsel et al. (2008), Stewart & Brandenberger (2009) and Danos & Brandenberger (2010a) in this paper. To make the simulation more reliable, it is interesting to do the following tasks. According to Martins & Shellard (2006), the number of CSs in the radiation and matter eras changes. So, it may be proper to improve our code to take this effect into account. The intercommutation of straight strings to produce loops has been considered in Blanco-Pillado et al. (2011); however, in our current approach, one can ignore this contribution (Stewart & Brandenberger 2009; Movahed & Khosravi 2011). The junction of CSs is other interesting point that can be used to improve our results (Tye, Wasserman & Wyman 2005; Bevis & Saffin 2008;
Urrestilla & Vilenkin 2008; Yamauchi et al. 2010a). Investigation of E-mode and B-mode polarizations is another area of interest (Pogosian, Wasserman & Wyman 2006; Bevis et al. 2007).

There are many constraints on the upper as well as lower values of CS parameters from theoretical and observational perspectives. Pulsar timing and photometry based on gravitational microlensing and gravitational waves require $10^{-15} < G_\mu < 10^{-8}$ (Blinnikov & Khlopov 1982; Gasilov, Maslyankin & Khlopov 1985; Oknyanskij 1999; Damour & Vilenkin 2005; Jenet et al. 2006; Battye & Moss 2010; Pshirkov & Tunits 2010a,b). The COSMOS survey requires $G_\mu < 3 \times 10^{-7}$ (Christiansen et al. 2010). The 21-cm signature of CSs has been investigated in Brandenberger et al. (2010). In another paper by Hernandez & Brandenberger, it has been demonstrated that the CS signal has the overall thermal noise of an individual pixel in the Square Kilometre Array for string tensions $G_\mu > 2.5 \times 10^{-8}$ (Hernandez & Brandenberger 2012). The LIGO and VIRGO collaborations have determined $7 \times 10^{-9} < G_\mu < 1.5 \times 10^{-7}$ (LIGO & VIRGO Collaboration 2009). In a recent paper, the stochastic gravitational waves from the European Pulsar Timing Array place the constraint $G_\mu < 5.3 \times 10^{-7}$ (Sanidas, Battye & Stappers 2012). Based on a probable global earthquake there is another lower limit on $G_\mu$ which is down to 10 orders of magnitude smaller than the cosmological events (Motohashi & Suyama 2013).

Another strong constraint on $G_\mu$ comes from the investigation of temperature fluctuations at the last scattering surface (Pogosian & Vachaspati 1999; Simatoss & Perivolaropoulos 2000; Landreau & Shellard 2003; Fraisse et al. 2008). The accumulation of anisotropies induced by CSs on the fluctuations at the last scattering surface can be divided into two categories: (1) anisotropies related to pre-recombination processes and created by the Kaiser–Stebbins effect (Kaiser & Stebbins 1984) and (2) the decay of string loops, which results in a stochastic background of gravitational waves (Kaiser & Stebbins 1984; Fraisse et al. 2008). CMB analyses bound the CS tension to be $G_\mu < 6.4 \times 10^{-7}$ (Kaiser & Stebbins 1984; Perivolaropoulos 1993a; Bevis, Hindmarsh & Kunz 2004; Fraisse 2005; Wyman, Pogosian & Wasserman 2005, 2006; Bevis et al. 2007; Fraisse et al. 2008; Battye & Moss 2010). The effects of CSs on the skewness of the one-point probability distribution of CMB temperatures (Yamauchi et al. 2010a), the TT power spectrum (Yamauchi et al. 2010b) and the B-mode polarization (Ma et al. 2010) have also been considered.

Various detection methods have been explored, including wavelet domain Bayesian denoising: $G_\mu > 6.3 \times 10^{-10}$ (Hammond, Wiaux & Vanderheynst 2009), the Canny algorithm: $G_\mu > 5.5 \times 10^{-8}$ (Stewart & Brandenberger 2009; Danos & Brandenberger 2010a,b) and level-crossing analysis: $G_\mu > 4 \times 10^{-9}$ and $G_\mu > 5.8 \times 10^{-9}$ without and in the presence of instrumental noise, respectively (Movahed & Kosravi 2011).

Recent observational constraints via WMAP and the South Pole Telescope yield $G_\mu < 1.7 \times 10^{-7}$ at 95 per cent confidence interval (Dvorkin et al. 2011). In the mentioned paper, the aspect of the polarization power spectrum to put robust constraints on the properties of CSs have been discussed. Other computations expressed $G_\mu < 0.7 \times 10^{-6}$ with $f_{10} = 0.11$ (the fractional contribution of CSs on the temperature power spectrum at $\ell = 10$) (Bevis et al. 2008). The constraints on CSs from future CMB polarization come from Foreman, Moss & Scott (2011). The non-Gaussianity imposed by CSs has been considered by Starck et al. (2004), Ringeval (2010), Hobson et al. (1999) and Barreiro & Hobson (2001). The more complementary and recent considerations concerning non-Gaussianity due to CSs can be found in Hindmarsh, Ringeval & Suyama (2009, 2010). In Hindmarsh, Ringeval & Suyama (2009, 2010), the bispectrum and trispectrum by CSs generated by Nambu–Goto CS simulations in the Friedmann–Lemaître–Robertson–Walker universe and those given by analytic calculations have been computed. Planck 2013 results put the following conservative upper bounds, due to post-recombination string contributions, on the tension of CSs: based on the Nambu–Goto string model $G_\mu < 1.5 \times 10^{-7}$ for $f_{10} = 0.015$ and $G_\mu < 1.3 \times 10^{-7}$ for $f_{10} < 0.01$, and using the Abelian–Higgs model $G_\mu < 3.2 \times 10^{-7}$ for $f_{10} < 0.028$ [Ade et al. (Planck Collaboration) 2013a].

In the current study, we concentrate on the discontinuities and fluctuations in the CMB map caused by CSs, arising from the Kaiser–Stebbins effect (Kaiser & Stebbins 1984). This phenomenon can produce observational consequences on the anisotropies in the CMB with high degree of reliability on the high-resolution map.

In what follows we study the TPCF of local maxima or minima in the observed temperature maps (Bond & Efstathiou 1987; Heavens & Sheth 1999; Heavens & Gupta 2001) to see if this is a useful probe of the extra roughness in the temperature distribution induced by strings. Previous work has shown that although the bispectrum of all pixels in a map at 5.5 arcmin resolution is not sensitive to a CS component, the TPCF of local maxima is, especially on scales of the order of 10–20 arcmin (Heavens & Gupta 2001). The main goal of the present work is to examine the capability of the clustering approach to detect CS components and to quantify the limits of $G_\mu$ which such a measurement can place based on the Kaiser–Stebbins phenomenon.

Section 2 describes how we generate mock maps of the primordial Gaussian CMB, and how we incorporate the effects of straight CSs. In essence this is a straightforward combination of the algorithms in Heavens & Sheth (1999) with Movahed & Kosravi (2011). We then identify peaks in these maps and study if the one- and two-point statistics of these peaks agree with the Gaussian prediction, quantifying the statistical significance of the differences. The final section summarizes why we conclude that two-point statistics are much more efficient in identifying the presence of CSs compared to one-point statistics.

2 SIMULATION AND ANALYSIS OF MOCK CMB MAPS

This section describes how we simulate maps of the last scattering surface (Perivolaropoulos 1993a,b; Moessner et al. 1994; Stewart & Brandenberger 2009; Danos & Brandenberger 2010a,b; Movahed & Kosravi 2011). At first, our code creates pure Gaussian fluctuations corresponding to the standard inflationary model with $\Lambda$CDM components in a flat universe following Bond & Efstathiou (1987). However, our program can be easily modified to other cosmological models for this purpose. Secondly, anisotropies produced by straight CSs by the Kaiser–Stebbins effect, from the last scattering surface up to the present, are simulated following Perivolaropoulos (1993a), Stewart & Brandenberger (2009), Danos & Brandenberger (2010a,b) and Movahed & Kosravi (2011). Our method of simulation differs from that used to produce the CS maps analysed by Heavens & Gupta (2001), where the CS contribution to the energy momentum tensor was modelled using Fourier methods (Pogosian & Vachaspati 1999).

For reasons discussed in detail in Stewart & Brandenberger (2009), Danos & Brandenberger (2010a,b) and Movahed & Kosravi (2011), to simulate the Kaiser–Stebbins lensing due to moving CSs we work in real space, where straight strings produce random jumps on the background radiation field. The scaling behaviour of straight strings means that the number of strings crossing...
a given Hubble volume is fixed to $M_{\text{min}} = 10$ (Bennett & Bouchet 1988). The CSs possess relativistic velocities; consequently, after $2t_H$ ($t_H$ is the Hubble time) an entirely new network of CSs provides new kicks to the CMB photons. The two signals are superposed, then smeared by our model of the instrumental beam, after which we add instrumental noise. Finally, we identify peaks in the simulated maps and measure the peak–peak correlation, showing results after averaging over a large ensemble of realizations.

### 2.1 Mock Gaussian CMB map

In what follows, the size and resolution of the simulated map are $\Theta$ and $R$, respectively. Thus, a $\Theta = 10^\circ$ map at $R = 1$ arcmin requires $600 \times 600$ pixels. The rms instrumental noise, $\sigma_{\text{noise}}$, and the full width at half-maximum (FWHM) of the detector are used to take into account additional effects on the simulated maps. We set $\sigma_{\text{noise}} = 10 \mu K$ which is appropriate for the South Pole Telescope [Ruhl et al. (SPT Collaboration) 2004; Keisler et al. 2011]. We also use FWHM = 4 arcmin to illustrate our results.

For making Gaussian maps, all that is required is the initial power spectrum $C_i$. For this, we use the CAMB software (Lewis, Challinor & Lasenby 2000) with parameters appropriate for a $\Lambda$CDM model, that is, consistent with the WMAP7, supernova type Ia and Sloan Digital Sky Survey data sets. We use this $C_i$ to generate a 2D Gaussian random field following Bond & Efstathiou (1987). Since we are interested in relatively small angular scales, we work in the flat sky approximation, following Heavens & Sheth (1999).

To add the effects of strings, we follow Stewart & Brandenberger (2009), Danos & Brandenberger (2010a,b) and Movahed & Khosravi (2011). This means that we ignore the contribution of CS loops, since their size is smaller than our map resolution (1 arcmin). In contrast, the characteristic length scale of straight CSs is the horizon scale.

### 2.2 Combination of simulated components

Before starting to combine different components, for the sake of convenience, let us take note of the notations used. Throughout this paper, we use $G$ for the Gaussian component, $S$ for the string part, $B$ stands for the beam effect and $N$ is for the noise part. In addition, for the combined map, Gaussian+string+beam+noise, we use $GSBN$.

When combining the Gaussian ($G$) and string ($S$) components, we are careful to ensure that, at $\ell < 200$ (Danos & Brandenberger 2010a; Movahed & Khosravi 2011), the total power is the same as that observed. In practice, this means that in each pixel $(i, j)$ we set the fluctuation, $F \equiv (T(i, j) - \langle T \rangle) / \langle T \rangle$, to be

$$F_G(i, j) + F_S(i, j) + F_B(i, j) + F_N(i, j), \quad (1)$$

where $\omega < 1$ is chosen so that the amplitude of the power spectrum

$$C_G^{(G+S)} = \omega^2 C_G^{(G)} + C_G^{(S)} \quad (2)$$

is close to that observed at $\ell < 200$. Since $C_G^{(S)}$ depends on $\ell$, determination of the appropriate value of $\omega$ is done by a likelihood analysis.

The beam smearing is modelled by

$$C_B^{(G+S)} = C_G^{(G+S)} e^{-\Gamma^2[(\ell+1)/\ell]} \quad (3)$$

with $\Gamma = \text{FWHM} / \sqrt{8 \ln 2}$ (Bond & Efstathiou 1987; Heavens & Sheth 1999).

### 2.3 Peak counts in mock maps

We have checked that the number density of peaks we identify in the Gaussian maps agrees with that expected from theory. When the peak height is expressed in units of the rms temperature, $\delta = \Delta T / T$, this prediction depends only on the shape of the power spectrum $C_\ell$ (Bond & Efstathiou 1987). Since CSs modify $C_\ell$ at high $\ell$, it is interesting to see if the peak counts predicted by $C_\ell^{(G+S)}$ provide a good description of the peak abundances in the $G + S$ maps, even though the maps themselves are not Gaussian. If not, then peak counts alone allow one to distinguish between a purely Gaussian model and one with an additional component. To compute theoretically the number density of extrema of a typical 2D Gaussian stochastic field in the flat sky approximation, we should construct the multivariate distribution function of the underlying field. The components of the mentioned distribution function are $W = \{F, F_G, F_S, F_B, F_N\}$. Here $F_{\alpha \beta} \equiv \delta_{\alpha \beta} F_{\alpha \beta}$. Consequently, according to the notation introduced

![Figure 1. Contribution of various components to the total angular power spectrum of temperature fluctuations. The solid line shows the power spectrum for a WMAP7 ACDM model (from CAMB). The symbols show a simulated Gaussian map with resolution $R = 1$ arcmin and size $10^\circ$. The long-dashed line shows the contribution from CSs having $G_\mu = 8 \times 10^{-3}$; the dot-dashed line shows a power law $(\ell + 1)C_\ell \sim \ell^{-0.90^{+0.05}_{-0.05}}$. The double-dot-dashed curve shows the window associated with a beam having FWHM = 20 arcmin. Clearly, the CS component is most easily detected at $\ell \sim 6000$.](image)
by Rice (1954) and Peacock & Heavens (1985), the multivariate distribution function of six variables for a Gaussian process reads

\[ p(W) = \frac{1}{(2\pi)^3 |\text{det} \mathbf{M}|} e^{-\frac{1}{2}(W^T \mathbf{M}^{-1} W)} \]  

(5)

where \( \mathbf{M} \) is the covariance matrix of the underlying variables, namely \( \mathbf{M} = \langle WW^T \rangle \). Therefore, the number density of extrema for a purely Gaussian CMB map in the range of \( \vartheta, \vartheta + d\vartheta \) is \( n(\vartheta) \, d\vartheta \) and is given by

\[ n(\vartheta) = \int p(W) |\text{det} \mathbf{F}_\vartheta| \, dW. \]  

(6)

The above expression can be integrated analytically in the Gaussian case and becomes (Bardeen et al. 1986; Bond & Efstathiou 1987):

\[ n(\vartheta) = \frac{1}{(2\pi)^{3/2} \gamma^2} e^{-\frac{\vartheta^2}{2}} G(\Psi, \Psi \vartheta), \]  

(7)

where

\[ G(\Psi, \Psi \vartheta) \equiv (\Psi^2 \vartheta^2 - \Psi^2) \left\{ 1 - \frac{1}{2} \text{erfc} \left[ \frac{\Psi \vartheta}{\sqrt{2(1 - \Psi^2)}} \right] \right\} \]

\[ + \Psi \vartheta (1 - \Psi^2) e^{-\frac{\vartheta^2}{2}} \frac{e^{\frac{-\vartheta^2}{2}}}{\sqrt{2\pi(1 - \Psi^2)}} \left\{ 1 - \frac{1}{2} \text{erfc} \left[ \frac{\Psi \vartheta}{\sqrt{2(1 - \Psi^2)(3 - 2\Psi^2)}} \right] \right\} \]  

(8)

in which \( \text{erfc}(\cdot) \) stands for the complementary error function. According to the notation explained in Bond & Efstathiou (1987), the so-called spectral parameters \( \Psi \) and \( \gamma \) in equations (7) and (8) are defined as \( \Psi \equiv \frac{\sigma_1^2}{\sigma_0^2} \) and \( \gamma \equiv \frac{\sqrt{2} \sigma_1^2}{\sigma_0^2} \), where

\[ \sigma_0^2 \equiv \langle F(r)^2 \rangle = \frac{1}{(2\pi)^2} \int S(|k|) \, dk, \]

\[ \sigma_1^2 \equiv \left\langle \left( \frac{\partial}{\partial x} F(r) \right)^2 \right\rangle = \frac{1}{(2\pi)^2} \int k^2 S(|k|) \, dk, \]  

(9)

and \( S(|k|) \) is the spectral density. The number density of peaks in a 2D Gaussian map as a function of \( \vartheta \) has been plotted in the upper panel of Fig. 3 for various values of spectral parameters. We also

**Figure 3.** The left-hand upper panel indicates the normalized number density of peaks in a typical 2D Gaussian field, while the right-hand upper panel shows the normalized cumulative peaks for the same map as a function of \( \vartheta \) for various values of \( \Psi \). The lower panels correspond to the same just for our simulated Gaussian CMB map (size = 10° and resolution \( R = 1 \) arcmin) by taking ensemble averages with \( \Psi = 0.320 \pm 0.020 \) and \( \gamma = (1.050 \pm 0.040) \times 10^{-3} \) at 1σ confidence interval.
Figure 4. Abundance of density of peaks as a function of the peak threshold level for pure Gaussian maps and upon adding a CS component. Upper left-hand panel: $G\mu = 2 \times 10^{-8}$. Upper right-hand panel: $G\mu = 1 \times 10^{-7}$. Middle left-hand panel: $G\mu = 8 \times 10^{-7}$. Results for a Gaussian map (Gaussian–GS) having the same power spectrum as the $G+S$ map has also been indicated in this panel. Middle right-hand panel: for $G\mu = 8 \times 10^{-7}$ with beam effect. The lower panels show the difference between the theoretical Gaussian field prediction for the number density of the peak and what measured in the simulated maps.

compute the number density of our simulated pure Gaussian CMB map and is illustrated in the lower panel of Fig. 3.

Fig. 4 shows that, for $G\mu \leq 10^{-7}$ (top panels), the peak counts in $G$ and $G+S$ are almost indistinguishable. For larger values of $G\mu$, the peak counts are noticeably different from one another (middle left-hand panel), with the distribution being shifted to smaller mean values when CSs are present. However, the measurements are each well described by Gaussian peaks theory (solid line) (Bond & Efstathiou 1987) with their respective power spectra ($C_l^{G}$ for the circles and $C_l^{G+S}$ for the triangles), even though the $G+S$ maps themselves are not Gaussian. Thus, given only the observed $C_l$ and the peak counts, it will not be possible to determine if there is a CS component in the maps. Beam smoothing makes the counts indistinguishable up to even larger values, $G\mu \leq 8 \times 10^{-7}$ (middle right-hand panel of Fig. 4). The lower panel of Fig. 4 shows $\Delta n(\theta) \equiv n_{\text{com}}(\theta) - n_{\text{obs}}(\theta)$, where ‘com.’ refers to the numerical result and ‘the.’ corresponds to the theoretical prediction.

Recently, Pogosyan, Pichon & Gay (2011) derived expressions for the number density of extrema in weakly non-Gaussian 2D fields. They showed that various non-Gaussian models could be distinguished by means of $n(\theta)$. Our analysis demonstrates that, at least for the non-Gaussianity due to straight CSs, this does not work in our case. In addition, Rossi, Chingangbom & Park (2011) used excursion sets such as regions above or below a temperature threshold and their clustering to examine the contribution of primordial non-Gaussianity on the CMB signal. They also observed to the optimum value of threshold, namely $\theta = 2.00 \sigma_0$, for discrimination between the Gaussian and local non-Gaussian cases. From the lower panel of Fig. 4, it is evident that in examining CSs, the sensitivity of the number density of peaks for a wide range of thresholds is flat, so there is no priority in selecting the value of the threshold for further computation.

2.4 Two-point statistics

Although we have demonstrated that peak counts in our $G+S$ maps are consistent with those in a Gaussian field having the same $C_l$, direct inspection of the maps themselves (top panel of Fig. 5) shows that they have quite different morphologies. The CS component seems to add small-scale random noise on top of the original Gaussian CMB signal. We turn therefore to the use of two-point peak statistics for distinguishing between the two maps.

To this end, we measure the TPCF of peaks in our Gaussian maps, our $G+S$ maps and our Gaussian–GS maps. For each value of $G\mu$, map size, resolution scale and beam size, we have generated ensembles of $\sim 100$ maps. The lower panels of Figs 5 and 6 show results from averaging over 100 realizations of maps with a $\theta = 10^\circ$ map at $R = 1$ arcmin and $\theta > 1 \sigma_0$. It must be pointed out that the plots do not show the entire range of scales we simulated, but only those that we believe to be accurate, free of cosmic variance and the boundary effect. There are obvious differences between the TPCF in the $G$ and $G+S$ maps, with the latter having substantially more signal on small scales. Although the beam erases some of this (Figs 5 and 6), a residual effect remains. This signal is rather different from that measured in a Gaussian field which has the same $C_l$ (what we called Gaussian–GS previously). So, we conclude that this is indeed a promising method for identifying the CS component in the maps.

The lower panel of Fig. 5 shows explicitly that, although the peak counts were unable to distinguish between GSB and Gaussian–GS maps (Fig. 4), the TPCF on scales $\geq 12$ arcmin can. Fig. 6 shows that the ability to discriminate depends on $G\mu$ and the beam size FWHM.

It has been demonstrated that long strings exhibit significant small-scale kinks and short-wavelength propagating modes (Albrecht & Turok 1989; Bennett & Bouchet 1989, 1990; Allen & Shellard 1990; Shellard & Allen 1990; Vilenkin & Shellard 2000; Yamaguchi et al. 2000; Moore & Shellard 2001). In addition, we are interested in the value of $G\mu$ in the range of less than almost $2 \times 10^{-7}$. If $f_{\text{iso}} \sim 0.1$, then strings should dominate for $\ell > 3000$ and remain above the thermal Sunyaev–Zel’dovich effect and for $\ell < 1000$ with an acceptable $G\mu$ the effect of CSs and the texture is ignorable (Kaiser & Stebbins 1984; Allen et al. 1997; Pen et al. 1997; Fraisse 2005). In addition, according to Fig. 1 and the paper by Fraisse et al. (2008), the power spectrum of CSs behaves as $\ell^{-0.9}$ (Fig. 1); consequently, the contribution of CSs for large modes, namely $200 < \ell < 1000$, is less than $1–2$ orders of magnitude; in addition, since in this work we are working on a local sky map instead of a full sky map, namely $\ell_{\text{min}} > 51$ for map size $10^\circ$ and $\ell_{\text{max}} > 102$ for map size $5^\circ$, we do not expect that CSs have an effective role in the large modes. To make the footprint of CSs in the peak–peak correlation function of the CMB more obvious, we took almost a large value of $G\mu = 8 \times 10^{-7}$ in the lower panel of Fig. 5. We also checked the consistency between the number
CMB peak correlations from cosmic strings

2.5 Quantitative limits

To quantify this, we first compute Student’s t-test based on

\[ t(\theta) = \frac{\bar{\xi}(\theta) - \bar{\xi}(\theta)}{\bar{\sigma}_0(\theta)} \]  

where \( \bar{\xi}(\theta) \) is the TPCF and \( \sigma(\theta) \) is the mean standard deviation of each term in the numerator. The symbols \( \odot \) and \( \otimes \) correspond to the \( G + S \) and \( G \) measurements and to \( (G + S)B \) and \( GB \) with beam effect, respectively. For each \( \theta \), the corresponding p-values, \( p(\theta) \), are calculated. Degrees of freedom based on the t-distribution function are \( 2N_{\text{sim}} - 2 \), where \( N_{\text{sim}} \) is the number of simulated maps.

\[ \chi^2 = -2 \sum \ln p(\theta) \]  

The final p-value related to \( \chi^2 \) is calculated based on the chi-square distribution function with \( 2(\theta_{\text{max}} - \theta_{\text{min}})/\Delta \theta - 2 \) degrees of freedom. Fig. 7 shows this p-value as a function of \( G_\mu \) for various maps with \( \Theta = 10^\circ \). We have drawn lines at \( p = 0.0027 \) and 0.0455, since these correspond to 3\( \sigma \) and 2\( \sigma \) significance levels. This shows that the TPCF can detect CSs at 95 per cent confidence level, provided \( G_\mu \gtrsim 1.2 \times 10^{-8} \) in maps without instrumental noise. If noise is present, with rms \( \sigma_{\text{noise}} = 10 \mu K \), then this limit increases to \( G_\mu \gtrsim 9.0 \times 10^{-8} \). Including beam smearing further degrades our limits: the minimum detectable CS becomes \( G_\mu \gtrsim 1.6 \times 10^{-7} \) at 2\( \sigma \) confidence interval.

Table 1 summarizes our results.

To minimize the effect of cosmic variance more than the previous task, we also construct new quantities (Rossi et al. 2011):

\[ \bar{\xi}(\theta) = \frac{\xi(\theta) - \xi(\theta)}{\bar{\xi}(\theta)} \]  

and again compute

\[ \bar{t}(\theta) = \frac{\bar{\xi}(\theta)}{\bar{\sigma}_0(\theta)} \]  

Figure 6. TPCF of peaks above \( \theta = 1\sigma_0 \) for different values of \( G_\mu \). Here FWHM is 4 arcmin. \( \Delta \xi(\theta) \) corresponds to the difference between the TPCFs of various cases indicated in the plots.

We then define \( \chi^2 = -2 \sum \ln p(\theta) \). The final p-value related to \( \chi^2 \) is calculated based on the chi-square distribution function with \( 2(\theta_{\text{max}} - \theta_{\text{min}})/\Delta \theta - 2 \) degrees of freedom. Fig. 7 shows this p-value as a function of \( G_\mu \) for various maps with \( \Theta = 10^\circ \). We have drawn lines at \( p = 0.0027 \) and 0.0455, since these correspond to 3\( \sigma \) and 2\( \sigma \) significance levels. This shows that the TPCF can detect CSs at 95 per cent confidence level, provided \( G_\mu \gtrsim 1.2 \times 10^{-8} \) in maps without instrumental noise. If noise is present, with rms \( \sigma_{\text{noise}} = 10 \mu K \), then this limit increases to \( G_\mu \gtrsim 9.0 \times 10^{-8} \). Including beam smearing further degrades our limits: the minimum detectable CS becomes \( G_\mu \gtrsim 1.6 \times 10^{-7} \) at 2\( \sigma \) confidence interval. Table 1 summarizes our results.

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and again compute

\[ \bar{t}(\theta) = \frac{\bar{\xi}(\theta)}{\bar{\sigma}_0(\theta)} \]  

Figure 5. Upper panels: comparison of a \( G + S \) map (left-hand panel) with a pure Gaussian map which has the same total power spectrum (right-hand panel); the blue dots show peaks above \( \theta = 0.5\sigma_0 \). In both cases, \( G_\mu = 8 \times 10^{-7} \), the resolution \( R = 1 \) arcmin and the map size shown is \( 5^\circ \times 5^\circ \). The morphology of these two maps is quite different. Lower panels: TPCF of \( \theta = 1.0 \sigma_0 \) peaks in these maps and differences between them.
CS signal is particularly strong on arcmin and smaller scales. Some of this signal is removed if the beam size of the experiment is larger than this scale. For a 4 arcmin beam, the limit is $G\mu \gtrsim 1.2 \times 10^{-7}$ at 2\(\sigma\) confidence interval. Broader beams further degrade the limit on $G\mu$.

We have argued that two-point statistics of peaks (the pair correlation function) are better than one-point statistics (peak number counts) for distinguishing between models. Our results suggest that the $n$-point correlation functions of peaks can be used for similar purposes. This is interesting in view of the previous work showing that the three-point statistics of all pixels are not very informative.

A final remark is that it could be interesting to use more realistic models (Landriau & Shellard 2003; Fraisse et al. 2008; Hindmarsh et al. 2009, 2010; Landriau & Shellard 2010) to simulate a map taking all contributions of CSs into account and apply our method to examine the effect of CSs in our future works.

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