A Short Note on Contracting Self-Similar Solutions of the Curve Shortening Flow

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Received: 30 June 2015

Abstract
By the curve shortening flow, the only closed contracting self-similar solutions are circles: we give a very short and intuitive geometric proof of this basic and classical result using an idea of Gage [4].

Keywords and Phrases: Curve shortening flow, self-similar, contracting solutions.

MSC 53C44, 53A04.

1 Introduction
Let \( \gamma \) be a smooth closed curve parametrized by arc length, embedded in the Euclidean plane endowed with its canonical inner product denoted by a single point. A one-parameter smooth family of plane closed curves \((\gamma(\cdot, t))_t\) with initial condition \(\gamma(\cdot, 0) = \gamma\) is said to evolve by the curve shortening flow (CSF for short) if

\[
\frac{\partial \gamma}{\partial t} = \kappa \mathbf{n}
\]  

(1)

where \( \kappa \) is the signed curvature and \( \mathbf{n} \) the inward pointing unit normal. By the works of Gage, Hamilton and Grayson any embedded closed curve evolves to a convex curve (or remains convex if so) and shrinks to a point in finite time[1].

In this note, we are interested by self-similar solutions that is solutions which shapes change homothetically during the evolution. This condition is equivalent to say, after a suitable parametrization, that

\[
\kappa = \varepsilon \gamma \cdot \mathbf{n}
\]  

(2)

with \( \varepsilon = \pm 1 \). If \( \varepsilon = -1 \) (resp. +1), the self-similar family is called contracting (resp. expanding). For instance, for any positive constant \( C \) the concentric circles \((s \mapsto \sqrt{1 - 2t} \cos s, \sin s))\), form a self-similar contracting solution of the CSF shrinking to a point in finite time and as a matter of fact, there is no more example than this.

\[1\]The reader could find a dynamic illustration of this result on the internet page [http://a.carapetis.com/csf/](http://a.carapetis.com/csf/)
one:

by the curve shortening flow, the only closed embedded contracting self-similar solutions are circles.

To the author knowledge, the shortest proof of this was given by Chou-Zhu [3] by evaluating a clever integral. The proof given here is purely geometric and based on an genuine trick used by Gage in [4].

2 A geometric proof

Let $\gamma$ be a closed, simple embedded plane curve, parametrized by arclength $s$, with signed curvature $\kappa$. By reversing the orientation if necessary, we can assume that the curve is counter-clockwise oriented. The length of $\gamma$ is denoted by $L$, the compact domain enclosed by $\gamma$ will be denoted by $\Omega$ with area $A$ and the associated moving Frenet frame by $(t, n)$. Let $\gamma_t = \gamma(t, \cdot)$ be the one parameter smooth family solution of the CSF, with the initial condition $\gamma_0 = \gamma$.

Multiplying (1) by $n$, we obtain

$$\frac{\partial \gamma}{\partial t} \cdot n = \kappa$$

Equations (1) and (3) are equivalent: from (3), one can look at a reparametrization $t \mapsto \varphi(t, s)$ such that $\tilde{\gamma}(t, s) = \gamma(t, \varphi(t, s))$ satisfies (1). A simple calculation leads to an ode on $\varphi$ which existence is therefore guaranteed [3]. From now, we will deal with equation (3).

If a solution $\gamma$ of (3) is self-similar, then there exists a non-vanishing smooth function $t \mapsto \lambda(t)$ such that $\gamma_t(s) = \lambda(t) \gamma(s)$. By (2), this leads to $\lambda'(t) \gamma(s) \cdot n(\gamma(s)) = \kappa(\gamma(s))$, that is $\lambda'(t) \lambda(t) \gamma(s) \cdot n(s) = \kappa(s)$. The function $s \mapsto \gamma(s) \cdot n(s)$ must be non zero at some point (otherwise $\kappa$ would vanish everywhere and $\gamma$ would be a line) so the function $\lambda'\lambda$ is constant equal to a real $\varepsilon$ which can not be zero. By considering the new curve $s \mapsto \sqrt{|\varepsilon|} \gamma\left(s / \sqrt{|\varepsilon|}\right)$ which is still parametrized by arc length, we can assume that $\varepsilon = \pm 1$. In the sequel we will assume that $\gamma$ is contracting, that is $\varepsilon = -1$ which says that we have the fundamental relation:

$$\kappa + \gamma \cdot n = 0$$

An immediate consequence is the value of $A$: indeed, by the divergence theorem and the turning tangent theorem,

$$A = \frac{1}{2} \int_{\gamma} (x dy - y dx) = -\frac{1}{2} \int_0^L \gamma(s) \cdot n(s) ds = \frac{1}{2} \int_0^L k(s) ds = \pi$$

2 The nonembedded closed curves were studied and classified by Abresch and Langer [1]
Therefore, our aim will be to prove that $L = 2\pi$ and we will conclude by using the equality case in the isoperimetric inequality.

The second remark is that the curve is an oval or strictly convex: indeed, by differentiating $\frac{d}{dt} (4)$ and using Frenet formulae, we obtain that $\kappa' = \kappa \gamma \cdot \gamma'$ which implies that $\kappa = Ce^{\gamma^2/2}$ for some non-zero constant $C$. As the rotation index is $+1$, $C$ is positive and so is $\kappa$.

### 2.1 Polar tangential coordinates

As equation $(4)$ is invariant under Euclidean motions, we can assume that the origin $O$ of the Euclidean frame lies within $\Omega$ with axis $\overrightarrow{Ox}$ meeting $\gamma$ orthogonally. We introduce the angle function $\theta$ formed by $-\mathbf{n}$ with the $x$-axis as shown in the figure below:

![Figure 1: Polar tangential coordinates](image-url)
Thus, consider the function defined by \( \gamma(s) = \int_0^s \kappa(u) du \) since \( \theta'(0) = 0 \). As \( \theta' = \kappa > 0 \), \( \theta \) is a strictly increasing function on \( \mathbb{R} \) onto \( \mathbb{R} \). So \( \theta \) can be chosen as a new parameter and we set \( \gamma(\theta) = (\bar{x}(\theta), \bar{y}(\theta)) = \gamma(s) \), \( \bar{t}(\theta) = (-\sin \theta, \cos \theta) \), \( \bar{m}(\theta) = (-\cos \theta, -\sin \theta) \) and we consider the function \( p \) defined by \( p(\theta) = -\bar{r}(\theta) \cdot \bar{m}(\theta) \). As \( \theta(s+L) = \theta(s)+2\pi \), we note that \( \bar{t}, \bar{m} \) and \( p \) are \( 2\pi \)-periodic functions. The curve \( \bar{r} \) is regular but not necessarily parametrized by arc length because \( \overline{r}'(\theta) = \frac{1}{\kappa(\theta)} \gamma'(s) \) and we note \( \kappa \) its curvature. By definition, we have

\[
\bar{x}(\theta) \cos \theta + \bar{y}(\theta) \sin \theta = p(\theta)
\]

which, by differentiation w.r.t. \( \theta \), gives

\[
-\bar{x}(\theta) \sin \theta + \bar{y}(\theta) \cos \theta = p'(\theta)
\]

Thus,

\[
\begin{cases}
\bar{x}(\theta) &= p(\theta) \cos \theta - p'(\theta) \sin \theta \\
\bar{y}(\theta) &= p(\theta) \sin \theta + p'(\theta) \cos \theta
\end{cases}
\]

Differentiating once more, we obtain

\[
\begin{cases}
\bar{x}'(\theta) &= -[p(\theta) + p''(\theta)] \sin \theta \\
\bar{y}'(\theta) &= [p(\theta) + p''(\theta)] \cos \theta
\end{cases}
\]

Since \( \gamma \) is counter-clockwise oriented, we have \( p + p'' > 0 \).

Coordinates \( (\theta, p(\theta))_{0 \leq \theta < 2\pi} \) are called \textit{polar tangential coordinates} and \( p \) is the \textit{Minkowski support function}. By \( (8) \), we remark that the tangent vectors at \( \bar{r}(\theta) \) and \( \bar{r}(\theta+\pi) \) are parallel. We will introduce the \textit{width function} \( w \) defined by

\[
w(\theta) = p(\theta) + p(\theta+\pi)
\]

which is the distance between the parallel tangent lines at \( \bar{r}(\theta) \) and \( \bar{r}(\theta+\pi) \) and we denote by \( \ell(\theta) \) the segment joining \( \bar{r}(\theta) \) and \( \bar{r}(\theta+\pi) \).

With these coordinates, the perimeter has a nice expression:

\[
L = \int_0^{2\pi} \sqrt{(\bar{x}'(\theta))^2 + (\bar{y}'(\theta))^2} \, d\theta = \int_0^{2\pi} [(p + p'') \cos \theta - p' \sin \theta] \, d\theta = \int_0^{2\pi} p(\theta) \, d\theta
\]

(Cauchy formula)

The curvature \( \kappa \) of \( \bar{r} \) is

\[
\kappa = \frac{\bar{x}\bar{y}' - \bar{x}'\bar{y}}{(\bar{x}^2 + \bar{y}^2)^{3/2}} = \frac{1}{p + p''}
\]

and equation \( (8) \) reads \( \kappa = p \). So, finally,

\[
\kappa = p = \frac{1}{p + p''}
\]
2.2 Bonnesen inequality

If \( B \) is the unit ball of the Euclidean plane, it is a classical fact that the area of \( \Omega - tB \) (figure 2) is \( A_{\Omega}(t) = A - Lt + \pi t^2 \) [2].

![Figure 2: The domain \( \Omega - tB \) with positive \( t \)](image)

The roots \( t_1, t_2 \) (with \( t_1 \leq t_2 \)) of \( A_{\Omega}(t) \) are real by the isoperimetric inequality and they have a geometric meaning: indeed, if \( R \) is the circumradius of \( \Omega \), that is the radius of the circumscribed circle, and if \( r \) is the inradius of \( \Omega \), that is the radius of the inscribed circle, Bonnesen [2, 5] proved in the 1920’s a series of inequalities, one of them being the following one:

\[
t_1 \leq r \leq R \leq t_2
\]

Moreover, and this is a key point in the proof, any equality holds if and only if \( \gamma \) is a circle. We also note that \( A_{\Omega}(t) < 0 \) for any \( t \in (t_1, t_2) \).

2.3 End of proof

**Special case:** \( \gamma \) is symmetric w.r.t. the origin \( O \), that is \( \gamma(\theta + \pi) = -\gamma(\theta) \) for all \( \theta \in [0, 2\pi] \), which also means that \( p(\theta + \pi) = p(\theta) \) for all \( \theta \in [0, 2\pi] \). So the width function \( w \) is twice the support function \( p \). As \( 2r \leq w \leq 2R \), we deduce that for all \( \theta \), \( r \leq p(\theta) \leq R \). If \( \gamma \) is not a circle, then one would derive from Bonnesen inequality that \( t_1 < r \leq p(\theta) \leq R < t_2 \). So \( A_{\Omega}(p(\theta)) < 0 \) for all \( \theta \), that is \( \pi p^2(\theta) < Lp(\theta) - \pi \). Multiplying this inequality by \( 1/p = p + p'' > 0 \) and integrating on \([0, 2\pi]\), we would obtain \( \pi L < \pi L \) by Cauchy formula ! By this way, we proved that any symmetric smooth closed curve satisfying \( \gamma \) is a circle. As the area is \( \pi \), the length is \( 2\pi \) of course.
**General case:** using a genuine trick introduced by Gage [4], we assert that

> for any oval enclosing a domain of area $A$, there is a segment $\ell(\theta_0)$ dividing $\Omega$ into two subdomains of equal area $A/2$.

Proof: let $\sigma(\theta)$ be the area of the subdomain of $\Omega$, bounded by $\gamma([\theta, \theta + \pi])$ and the segment $\ell(\theta)$. We observe that $\sigma(\theta) + \sigma(\theta + \pi) = A$. We can assume without lost of generality that $\sigma(0) \leq A/2$. Then we must have $\sigma(\pi) \geq A/2$, and by continuity of $\sigma$ and the intermediate value theorem, there exists $\theta_0$ such that $\sigma(\theta_0) = A/2$ and the segment $\ell(\theta_0)$ proves the claim.

Let $\omega_0$ be the center of $\ell(\theta_0)$. If $\gamma_1$ and $\gamma_2$ are the two arcs of $\gamma$ separated by $\ell(\theta_0)$, we denote by $\gamma_i$ ($i = 1, 2$) the closed curve formed by $\gamma_i$ and its reflection through $\omega_0$. Each $\gamma_i$ is a symmetric closed curve and as $\ell(\theta_0)$ joins points of the curve where the tangent vectors are parallel, each one is strictly convex and smooth (figure 3).

Moreover, each $\gamma_i$ satisfies equation (4) and encloses a domain of area $2 \times A/2 = A$. So we can apply the previous case to these both curves and this gives that length($\gamma_i$) = $2\pi$ for $i = 1, 2$, that is length($\gamma_i$) = $\pi$ which in turn implies that $L = 2\pi$. So $\gamma$ (that is $\gamma$) is a circle. This proves the theorem. 

\[\square\]
References

[1] U. Abresch, J. Langer, The normalized curve shortening flow and homothetic solutions, J. Diff. Geom. 23 (1986), no. 2, 175-196.

[2] Yu. D. Burago, V. A. Zalgaller, Geometric inequalities, Springer-Verlag, Berlin-Heidelberg 1988.

[3] K. S. Chou, X. P. Zhu, The curve shortening problem, Chapman& Hall/CRC, Boca Raton, 2001.

[4] M. E. Gage, An isoperimetric inequality with applications to curve shortening, Duke Mathematical Journal, 50 (3) (1983) 1225-1229.

[5] J. Zhou, F. Chen, The Bonnesen-type inequalities in a plane of constant curvature, J. Korean Math. Soc. 44 (6), (2007) 1363-1372.