Taylor–Couette flow of a fractional second grade fluid in an annulus due to a time-dependent couple

M. Imran, M. Kamran, M. Athar, A.A. Zafar
Abdus Salam School of Mathematical Sciences, GC University
68-B, New Muslim Town, Lahore, Pakistan
drminranchaudhry@gmail.com

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Abstract. Exact solutions for the velocity field and the associated shear stress, corresponding to the flow of a fractional second grade fluid between two infinite coaxial cylinders, are determined by means of Laplace and finite Hankel transforms. The motion is produced by the inner cylinder which is rotating about its axis due to a time-dependent torque per unit length $2\pi R_1 f(t)$. The solutions that have been obtained satisfy all imposed initial and boundary conditions. For $\beta \to 1$, respectively $\beta \to 1$ and $\alpha_1 \to 0$, the corresponding solutions for ordinary second grade fluids and Newtonian fluids, performing the same motion, are obtained as limiting cases.

Keywords: fractional second grade fluid, velocity field, shear stress, exact solutions.

1 Introduction

The governing equation that describes the flow of a Newtonian fluid is the Navier–Stokes equation. However, some materials such as clay coatings, drilling muds, suspensions, certain oils and greases, polymer melts, elastomers, many emulsions have been treated as non-Newtonian fluids and they cannot be described by the Navier–Stokes equation. For this reason, many non-Newtonian models or constitutive equations have been proposed and most of them are empirical or semi-empirical. One of the most popular models for non-Newtonian fluids is the model that is called the second grade fluid or fluid of second grade. It is reasonable to use the second grade fluid model to do numerical calculations. This is particularly so due to the fact that the calculations will generally be simpler. The constitutive equation of a second grade fluid is a linear relation between the stress and the square of the first Rivlin–Ericksen tensor and the second Rivlin–Ericksen tensor [1]. This constitutive equation has three coefficients. There are some restrictions on these coefficients due to the Clausius–Duhem inequality and also due to the assumption that Helmholtz free energy is minimum in equilibrium. A comprehensive discussion on the restrictions for these coefficients has been given by Dunn and Fosdick [2] and Dunn and Rajagopal [3]. The restrictions on the two coefficients have not been confirmed by experiments and the sign of the moduli is the subject of much controversy.
During the last years, the fractional calculus has achieved a great success in the description of the complex dynamics. In particular, it has been found to be quite flexible in describing the viscoelastic behavior \[4, 5\]. A very good fit of the experimental data was achieved when the Maxwell model was used with its first-order derivatives replaced by fractional-order derivatives \[6\]. Especially, the rheological constitutive equations with fractional derivatives play an important role in the description of the behavior of polymer solutions and melts. In other cases, it has been shown that the constitutive equations employing fractional derivatives are linked to molecular theories \[7\]. The list of their applications is quite long, including fractal media, fractional wave diffusion, fractional Hamiltonian dynamics as well as many other topics in physics \[8\]. In the last time, a lot of papers regarding these fluids have been published but we remember here only a part of those concerning generalized second grade fluids \[9–19\].

Here, the velocity field and the adequate shear stress, corresponding to the flow of a second grade fluid with fractional model in an annular region between two infinite coaxial cylinders, are determined by means of Laplace and the finite Hankel transforms. The motion is produced by the inner cylinder which is moving about its axis due to a time-dependent torque. The solutions that have been obtained satisfy all imposed initial and boundary conditions. For \(\beta \to 1\), respectively \(\beta \to 1\) and \(\alpha_1 \to 0\), the corresponding solutions for ordinary second grade and Newtonian fluids, performing the same motion, are obtained as limiting cases.

2 Governing equations

The flows to be here considered have the velocity field of the form \[20–22\]

\[
v = v(r, t) = w(r, t) e_\theta,
\]

where \(e_\theta\) is the unit vector along the \(\theta\)-direction of the cylindrical coordinate system \(r, \theta\) and \(z\). For such flows the constraint of incompressibility is automatically satisfied. The non-trivial shear stress \(\tau(r, t) = S_{r\theta}(r, t)\) corresponding to such a motion of a second grade fluid is given by \[23\]

\[
\tau(r, t) = \left(\mu + \alpha_1 \frac{\partial}{\partial t}\right) \left[\frac{\partial w(r, t)}{\partial r} - \frac{w(r, t)}{r}\right],
\]

where \(\mu\) is the viscosity and \(\alpha_1\) is a material modulus. In the absence of a pressure gradient in the flow direction and neglecting body forces, the balance of the linear momentum leads to the relevant equation \[24, 25\]

\[
\rho \frac{\partial w(r, t)}{\partial t} = \left(\frac{\partial}{\partial r} + \frac{2}{r}\right) \tau(r, t).
\]

Eliminating \(\tau(r, t)\) between Eqs. (2) and (3), we get the governing equation

\[
\frac{\partial w(r, t)}{\partial t} = \left(\nu + \alpha \frac{\partial}{\partial t}\right) \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2}\right) w(r, t),
\]
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where \( \nu = \mu/\rho \) is the kinematic viscosity of the fluid, \( \rho \) is its constant density and \( \alpha = \alpha_1/\rho \).

Generally, the governing equations for a fractional second grade fluid (FSGF) are derived from those of the ordinary fluids by replacing the inner time derivatives of an integer order with the so called Riemann–Liouville operator \([26]\)

\[
D_\beta^\mu f(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t f(\tau) \left( \frac{1}{(t-\tau)^\beta} \right) \, d\tau, \quad 0 \leq \beta < 1,
\]

where \( \Gamma(\cdot) \) is the Gamma function.

Consequently, the governing equations corresponding to the motion (1) of a FSGF are (cf. \([21, \text{Eqs. (2) and (4)}]\) with \( \lambda = 0 \))

\[
\frac{\partial w(r,t)}{\partial t} = \left( \nu + \alpha D_\beta^\mu \right) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) w(r,t); \quad (5)
\]

\[
\tau(r,t) = \left( \mu + \alpha_1 D_\beta^\mu \right) \left( \frac{\partial}{\partial r} - \frac{1}{r} \right) w(r,t), \quad (6)
\]

where the new material constant \( \alpha_1 \) (for simplicity, we are keeping the same notation) goes to the initial \( \alpha_1 \) for \( \beta \to 1 \).

In this paper, we are interested into the motion of a FSGF whose governing equations are given by Eqs. (5) and (6). The fractional partial differential equations (6), with adequate initial and boundary conditions, can be solved in principle by several methods, the integral transforms technique representing a systematic, efficient and powerful tool. The Laplace transform will be used to eliminate the time variable and the finite Hankel transform to remove the spatial variable. However, in order to avoid the lengthy calculations of residues and contour integrals, the discrete inverse Laplace transform will be used.

### 3 Rotational flow between two infinite cylinders

Consider an incompressible FSGF at rest in the annular region between two infinitely co-axial cylinders. At time \( t = 0^+ \) let the inner cylinder of radius \( R_1 \) be set in rotation about its axis by a time dependent torque per unit length \( 2\pi R_1 ft^2 \), while the outer cylinder of radius \( R_2 \) is held stationary. Owing to the shear, the fluid between cylinders is gradually moved, its velocity being of the form (1). The governing equations are given by Eqs. (5) and (6) and the appropriate initial and boundary conditions are (see \([20, \text{Eqs. (5.2), (5.3)}]\))

\[
w(r,0) = 0; \quad r \in [R_1, R_2], \quad (7)
\]

\[
\tau(R_1, t) = \left( \mu + \alpha_1 D_\beta^\mu \right) \left( \frac{\partial}{\partial r} - \frac{1}{r} \right) w(r, t) \bigg|_{r=R_1} = ft^2; \quad (8)
\]

\[
w(R_2, t) = 0; \quad t \geq 0,
\]

where \( f \) is a constant.
3.1 Calculation of the velocity field

Applying the Laplace transform to the equations (5) and (8) and using the initial condition (7), we get

\[ q \mathcal{F}(r, q) = (\nu + \alpha q^\beta) \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \mathcal{F}(r, q); \quad r \in (R_1, R_2), \]

(9)

\[ \mathcal{F}(R_1, q) = (\mu + \alpha_1 q^\beta) \left( \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \mathcal{F}(r, q) \bigg|_{r=R_1} = \frac{2f}{q^2}; \quad \mathcal{F}(R_2, q) = 0, \]

(10)

where \( \mathcal{F}(r, q) \) and \( \mathcal{F}(R_1, q) \) are the Laplace transforms of the functions \( w(r, t) \) and \( \tau(R_1, t) \), respectively. We denote by [21, Eq. (32)]

\[ \mathcal{F}(r, q) = \frac{2f}{q^2}. \]

(11)

the finite Hankel transform of the function \( w(r, q) \), where

\[ B(r, r_n) = J_1(rr_n)Y_2(R_1r_n) - J_2(R_1r_n)Y_1(rr_n), \]

(12)

\( r_n \) being the positive roots of the equation \( B(R_2, r) = 0 \) and \( J_p(\cdot) \), \( Y_p(\cdot) \) are the Bessel functions of the first and second kind of order \( p \).

The inverse Hankel transform of \( \mathcal{F}(r_n, q) \) is given by [21, Eq. (35)]

\[ \mathcal{F}(r, q) = \frac{\pi^2}{2} \sum_{n=1}^{\infty} \frac{r_n^2 J_1^2(R_2r_n)}{J_2^2(R_1r_n) - J_2^2(R_2r_n)} \mathcal{F}(r_n, q). \]

(13)

By means of Eq. (10) and of the identity

\[ J_1(z)Y_2(z) - J_2(z)Y_1(z) = -\frac{2}{\pi z}, \]

we can easily prove that

\[ \int_{R_1}^{R_2} r \left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right) \mathcal{F}(r, q) B(r, r_n) \, dr \]

\[ = -r_n^2 \mathcal{F}(r_n, q) + \frac{2}{\pi r_n} \left( \frac{\partial}{\partial r} - \frac{1}{r} \right) \mathcal{F}(r, q) \bigg|_{r=R_1}. \]

(14)

Multiplying Eq. (9) by \( r B(r, r_n) \) and integrating the result with respect to \( r \) from \( R_1 \) to \( R_2 \) and using Eqs. (10) and (14), we find that

\[ \mathcal{F}(r_n, q) = \frac{4f}{\pi r_n} \frac{1}{q^2} \rho q + \frac{1}{\mu r_n^2} + \alpha_1 q^\beta r_n^2 + \frac{1}{\mu r_n^2}. \]

(15)
Writing $\varpi_H(r_n, q)$ under the equivalent forms,

$$\varpi_H(r_n, q) = \frac{4f}{\mu r_n^2} \left( \frac{1}{q^3} - \frac{1 + \alpha r_n^2 q^{\beta - 1}}{q^2 \left( q + (\nu + \alpha q^\beta) r_n^2 \right)} \right) = \frac{4f}{\mu r_n^2} \left[ \frac{1}{q^3} - \frac{q^{-\beta - 2} + \alpha r_n^2 q^{-3}}{(q^{1-\beta} + \alpha r_n^2) + \nu r_n^2 q^{-\beta}} \right], \quad (16)$$

and applying the Hankel transform to Eq. (16) and using the identities

$$\left( q^{1-\beta} + \alpha r_n^2 \right) + \nu r_n^2 q^{-\beta} = \sum_{k=0}^{\infty} \left( -\nu r_n^2 \right)^k q^{-\beta k} \quad (17)$$

$$\pi \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r, r_n)}{r_n [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} = \frac{1}{2} \left( \frac{R_1}{R_2} \right)^2 \left( r - \frac{R_2}{r} \right), \quad (18)$$

we find that

$$\varpi(r, q) = \frac{f}{\mu} \left( \frac{R_2}{R_2} \right)^2 \left( r - \frac{R_2}{r} \right) \left( r - \frac{R_2}{r} \right) \frac{1}{q^3} - \frac{2\pi f}{\mu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r, r_n)}{r_n [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \sum_{k=0}^{\infty} \frac{(-\nu r_n^2)^k}{(q^{1-\beta} + \alpha r_n^2)^{k+1}} \left[ q^{-\beta k-\beta - 2} + \alpha r_n^2 q^{-\beta k-3} \right]. \quad (19)$$

Now applying the inverse Laplace transform to Eq. (19), we find for the velocity field the suitable expression [18]

$$w(r, t) = \frac{f}{2\mu} \left( \frac{R_1}{R_2} \right)^2 \left( r - \frac{R_2}{r} \right) \left( r - \frac{R_2}{r} \right) t^2 - \frac{2\pi f}{\mu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2 r_n) B(r, r_n)}{r_n [J_2^2(R_1 r_n) - J_1^2(R_2 r_n)]} \times \sum_{k=0}^{\infty} (-\nu r_n^2)^k \left[ G_{1-\beta, -\beta k-\beta - 2, k+1}(-\alpha r_n^2, t) \right. \left. + \alpha r_n^2 G_{1-\beta, -\beta k-\beta - 3, k+1}(-\alpha r_n^2, t) \right], \quad (20)$$

where the generalized function $G_{a,b,c}(d, t)$ is defined by [27, Eqs. (97) and (101)]

$$G_{a,b,c}(d, t) = L^{-1}\left\{ \frac{q^d}{(q^a - d)^c} \right\} = \sum_{j=0}^{\infty} \frac{d^j \Gamma(c + j)}{\Gamma(c) \Gamma(j + 1) \Gamma((c + j)a - b)}; \quad (21)$$

Re(ac - b) > 0, $\left. \frac{d}{q^d} \right| < 1$.

### 3.2 Calculation of the shear stress

Applying the Laplace transform to Eq. (6), we find that

$$\tau(r, q) = (\mu + \alpha_1 q^\beta) \left( \frac{\partial}{\partial r} - \frac{1}{r} \right) \varpi(r, q). \quad (22)$$

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In order to get a suitable form for $\tau(r, t)$, we rewrite Eq. (15) under the equivalent form

$$
\bar{\omega}(r_n, q) = \frac{4f}{\pi \nu r_n^3} \beta^3(\mu + \alpha q^3) - \frac{4f}{\pi \nu r_n^3} q^2(\mu + \alpha q^3)(q + \alpha q^2 r_n^2 + \nu r_n^2).
$$

Applying the inverse Hankel transform to Eq. (23) and using Eq. (13) and the identity (18), we find that

$$
\bar{\omega}(r, q) = \frac{\left(\frac{R_1}{R_2}\right)^2}{\beta^2} \left(\frac{r - R_2^2}{R_2^2}\right) \frac{1}{q^2(\mu + \alpha q^3)} - 2\pi f \sum_{n=1}^{\infty} \frac{J_n^2(R_2 r_n) B_1(r, r_n)}{r_n[J_n^2(R_1 r_n) - J_n^2(R_2 r_n)]} \times \frac{1}{q^2(\mu + \alpha q^3)(q + \alpha q^2 r_n^2 + \nu r_n^2)}.
$$

Introducing Eq. (24) into Eq. (22), it results that

$$
\tau(r, q) = \left(\frac{R_1}{r}\right)^2 \frac{2 f}{\beta^2} + 2\pi f \sum_{n=1}^{\infty} \frac{J_n^2(R_2 r_n) B_1(r, r_n)}{J_n^2(R_1 r_n) - J_n^2(R_2 r_n)} \frac{1}{q^2(\mu + \alpha q^2 r_n^2 + \nu r_n^2)},
$$

or equivalently (see also Eq. (17))

$$
\tau(r, q) = \left(\frac{R_1}{r}\right)^2 \frac{2 f}{\beta^2} + 2\pi f \sum_{n=1}^{\infty} \frac{J_n^2(R_2 r_n) B_1(r, r_n)}{J_n^2(R_1 r_n) - J_n^2(R_2 r_n)} \times \sum_{k=0}^{\infty} (-\nu r_n^2)^k q^{-\beta k - \beta - 2} \left(q^{-\beta - \alpha r_n^2} + 1\right)^{k+1},
$$

where $B_1(r, r_n) = J_0(r r_n) Y_2(R_1 r_n) - J_2(R_1 r_n) Y_0(r r_n)$.

Now taking the inverse Laplace transform of both sides of Eq. (26), we get for the shear stress $\tau(r, t)$ the expression

$$
\tau(r, t) = \left(\frac{R_1}{r}\right)^2 \frac{2 f}{\beta^2} + 2\pi f \sum_{n=1}^{\infty} \frac{J_n^2(R_2 r_n) B_1(r, r_n)}{J_n^2(R_1 r_n) - J_n^2(R_2 r_n)} \times \sum_{k=0}^{\infty} (-\nu r_n^2)^k G_{1-\beta, -\beta k - \beta - 2, k+1}(-\alpha r_n^2, t).
$$

4 The special case $\beta \to 1$ (second grade fluid)

Making $\beta \to 1$ into Eqs. (20) and (27), we obtain the similar solutions

$$
\tau_{\beta \to 1}(r, t) = \left(\frac{R_1}{R_2}\right)^2 \frac{2 f}{2\mu} \left(\frac{r - R_2^2}{r}\right) \frac{1}{\mu^2} \sum_{n=1}^{\infty} \frac{J_n^2(R_2 r_n) B_1(r, r_n)}{r_n[J_n^2(R_1 r_n) - J_n^2(R_2 r_n)]} \times (1 + \alpha r_n^2) \sum_{k=0}^{\infty} (-\nu r_n^2)^k G_{0, -3, k+1}(-\alpha r_n^2, t),
$$

$$
\tau_{\beta \to 1}(r, t) = \left(\frac{R_1}{R_2}\right)^2 \frac{2 f}{2\mu} \left(\frac{r - R_2^2}{r}\right) \frac{1}{\mu^2} \sum_{n=1}^{\infty} \frac{J_n^2(R_2 r_n) B_1(r, r_n)}{r_n[J_n^2(R_1 r_n) - J_n^2(R_2 r_n)]} \times (1 + \alpha r_n^2) \sum_{k=0}^{\infty} (-\nu r_n^2)^k G_{0, -3, k+1}(-\alpha r_n^2, t),
$$

$$
\tau_{\beta \to 1}(r, t) = \left(\frac{R_1}{R_2}\right)^2 \frac{2 f}{2\mu} \left(\frac{r - R_2^2}{r}\right) \frac{1}{\mu^2} \sum_{n=1}^{\infty} \frac{J_n^2(R_2 r_n) B_1(r, r_n)}{r_n[J_n^2(R_1 r_n) - J_n^2(R_2 r_n)]} \times (1 + \alpha r_n^2) \sum_{k=0}^{\infty} (-\nu r_n^2)^k G_{0, -3, k+1}(-\alpha r_n^2, t),
$$

$$
\tau_{\beta \to 1}(r, t) = \left(\frac{R_1}{R_2}\right)^2 \frac{2 f}{2\mu} \left(\frac{r - R_2^2}{r}\right) \frac{1}{\mu^2} \sum_{n=1}^{\infty} \frac{J_n^2(R_2 r_n) B_1(r, r_n)}{r_n[J_n^2(R_1 r_n) - J_n^2(R_2 r_n)]} \times (1 + \alpha r_n^2) \sum_{k=0}^{\infty} (-\nu r_n^2)^k G_{0, -3, k+1}(-\alpha r_n^2, t),
$$

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and

\[ \tau_{SG}(r, t) = \left( \frac{R_1}{r} \right)^2 f t^2 + \frac{2\pi f}{\nu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2r_n)B_1(r, r_n)}{J_2^2(R_1r_n) - J_1^2(R_2r_n)} \times \sum_{k=0}^{\infty} \left( -\nu r_n^2 \right)^k \mathcal{G}_{0,-k-3,k+1}(-\alpha r_n^2, t), \]  

(29)

corresponding to a second grade fluid.

Now, in view of the identity

\[ \sum_{k=0}^{\infty} \left( -\nu r_n^2 \right)^k \mathcal{G}_{0,-(k+3),k+1}(-\alpha r_n^2, t) \]

\[ = 1 + \frac{\alpha r_n^2}{\nu r_n^2} \left[ \frac{\exp \left( -\frac{\nu r_n^2 t}{1 + \alpha r_n^2} \right)}{1 + \alpha r_n^2} \right], \]

Eqs. (28) and (29) can be written under the simplest forms

\[ w_{SG}(r, t) = \frac{f}{2\mu} \left( \frac{R_1}{R_2} \right)^2 \left( r - \frac{R_2}{r} \right) \left( t^2 - \frac{2\alpha_1 t}{\mu} \right) - \frac{2\pi f}{\nu \mu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2r_n)B(r, r_n)}{r_n^2 \left[ J_2^2(R_1r_n) - J_1^2(R_2r_n) \right]} \times \left\{ t - \frac{1 + \alpha r_n^2}{\nu r_n^2} \left( 1 - \exp \left( -\frac{\nu r_n^2 t}{1 + \alpha r_n^2} \right) \right) \right\}, \]

(30)

and

\[ \tau_{SG}(r, t) = \left( \frac{R_1}{r} \right)^2 f t^2 + \frac{2\pi f}{\nu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2r_n)B_1(r, r_n)}{r_n^2 \left[ J_2^2(R_1r_n) - J_1^2(R_2r_n) \right]} \times \left\{ t - \frac{1 + \alpha r_n^2}{\nu r_n^2} \left( 1 - \exp \left( -\frac{\nu r_n^2 t}{1 + \alpha r_n^2} \right) \right) \right\}. \]

(31)

5  Newtonian case

Making \( \alpha_1 \) and then \( \alpha \rightarrow 0 \) into Eqs. (30) and (31), the velocity field

\[ w_N(r, t) = \frac{f}{2\mu} \left( \frac{R_1}{R_2} \right)^2 \left( r - \frac{R_2}{r} \right) t^2 - \frac{2\pi f}{\nu \mu} \sum_{n=1}^{\infty} \frac{J_1^2(R_2r_n)B(r, r_n)}{r_n^2 \left[ J_2^2(R_1r_n) - J_1^2(R_2r_n) \right]} \times \left\{ t - \frac{1}{\nu r_n^2} \left( 1 - e^{-\nu r_n^2 t} \right) \right\}, \]

(32)

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and the associated shear stress

\[
\tau_N(r, t) = \left(\frac{R_1}{r}\right)^2 ft^2 + \frac{2\pi f}{\nu} \sum_{n=1}^{\infty} \frac{J_n^2(R_2r_n)B_1(r, r_n)}{r_n^2 J_n^2(R_1r_n) - J_n^2(R_2r_n)} \times \left\{ t - \frac{1}{\nu r_n^2} \left( 1 - e^{-\nu r_n^2 t} \right) \right\},
\]

(33)
corresponding to a Newtonian fluid are obtained.

### 6 Conclusions

The purpose of this note is to provide exact analytic solutions for the velocity field \(w(r, t)\) and the shear stress \(\tau(r, t)\) corresponding to the unsteady rotational flow of a fractional second grade fluid between two infinite coaxial cylinders, the inner cylinder being set in rotation about its axis by a time-dependent shear. The solutions that have been obtained, presented under series form in terms of usual Bessel \((J_1(\cdot)\) and \(J_2(\cdot)\)) and generalized \(G_{a,b,c}(\cdot, t)\) functions, satisfy all imposed initial and boundary conditions. They can be easily specialized to give the similar solutions for ordinary second grade and Newtonian fluids.

The large time solutions corresponding to second grade fluids (see Eqs. (30) and (31))

\[
w_{LSG}(r, t) = \frac{f}{2\mu} \left( \frac{R_1}{R_2} \right)^2 \left( r - \frac{R_2}{r} \right) \left( t^2 - \frac{2\alpha_1}{\mu} \right) - \frac{2\pi f}{\nu} \sum_{n=1}^{\infty} \frac{J_n^2(R_2r_n)B(r, r_n)}{r_n^2 J_n^2(R_1r_n) - J_n^2(R_2r_n)} \left\{ t - \frac{1 + \alpha r_n^2}{\nu r_n^2} \right\},
\]

(34)

\[
\tau_{LSG}(r, t) = \left( \frac{R_1}{r} \right)^2 ft^2 + \frac{2\pi f}{\nu} \sum_{n=1}^{\infty} \frac{J_n^2(R_2r_n)B_1(r, r_n)}{r_n^2 J_n^2(R_1r_n) - J_n^2(R_2r_n)} \times \left\{ t - \frac{1 + \alpha r_n^2}{\nu r_n^2} \right\},
\]

(35)

are different of those corresponding to Newtonian fluids.

In order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity \(w(r, t)\) and the shear stress \(\tau(r, t)\) given by Eqs. (20) and (27), have been drawn against \(r\) for different values of the time \(t\) and of the material parameters. Figs 1(a) and 1(b). show the influence of time on the fluid motion. From these figures it is clearly seen that the velocity as well as the shear stress in absolute value are increasing functions of \(t\). In Figs. 2(a) and 2(b), it is shown the influence of the kinematic viscosity \(\nu\) on the fluid motion. It is clearly seen from these figures that the velocity and shear stress (in absolute value) are increasing functions of \(\nu\). The influence of the material parameter \(\alpha\) on the fluid motion is shown by Figs 3. It shows that the velocity is an increasing function, while the shear stress (in absolute value) is a decreasing function with respect
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Fig. 3. Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (20) and (27) for $t = 17 \text{ s}$, $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $\nu = 0.0015$, $\mu = 30$, $\beta = 0.4$ and different values of $\alpha$.

Fig. 4. Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ given by Eqs. (20) and (27) for $t = 14 \text{ s}$, $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $\nu = 0.0015$, $\mu = 40$, $\alpha = 0.07$ and different values of $\beta$.

Fig. 5. Profiles of the velocity $w(r,t)$ and shear stress $\tau(r,t)$ corresponding to the Newtonian, ordinary second grade, fractional second grade fluids, for $t = 5 \text{ s}$, $R_1 = 0.3$, $R_2 = 0.5$, $f = -2$, $\nu = 0.002$, $\mu = 30$, $\alpha = 0.04$ and $\beta = 0.5$. 
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References

1. A.C. Eringen, *Nonlinear Theory of Continuum Media*, McGraw-Hill, New York, 1962.

2. J.E. Dunn, R.L. Fosdick, Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second grade, *Arch. Ration. Mech. Anal.*, 56, pp. 191–252, 1974.

3. J.E. Dunn, K.R. Rajagopal, Fluids of differential type-critical review and thermodynamic analysis, *Int. J. Eng. Sci.*, 33, pp. 689–729, 1995.

4. D.Y. Song, T.Q. Jiang, Study on the constitutive equation with fractional derivative for the viscoelastic fluids - modified Jeffreys model and its application, *Rheol. Acta*, 37, pp. 512–517, 1998.

5. R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific Press, Singapore, 2000.

6. N. Makris, D. F. Darsusf, M. C. Constantinou, Dynamic analysis of generalized viscoelastic fluids, *J. Eng. Mech.*, 119, pp. 1663–1679, 1993.

7. C. Friedrich, Relaxation and retardation functions of the Maxwell model with fractional derivatives, *Rheol. Acta*, 30, pp. 151–158, 1991.

8. A. Heibig, L.I. Palade, On the rest state stability of an objective fractional derivative viscoelastic fluid model, *J. Math. Phys.*, 49, pp. 043101-1–22, 2008.

9. W.C. Tan, F. Xian, L. Wei, An exact solution of unsteady Couette flow of generalized second grade fluid, *Chin. Sci. Bul.*, 47, pp. 1226–1228, 2002.

10. W.C. Tan, M.Y. Xu, Unsteady flows of a generalized second grade fluid with the fractional derivative model between two parallel plates, *Acta Mech. Sin.*, 20, pp. 471–476, 2004.

11. W.C. Tan, T. Masuoka, Stokes first problem for a second grade fluid in a porous half-space with heated boundary, *Int. J. Non-Linear Mech.*, 40, pp. 515–522, 2005.

12. M. Khan, S. Nadeem, T. Hayat, A.M. Siddiqui, Unsteady motions of generalized second grade fluid, *Math. Comput. Modelling*, 41, pp. 629–637, 2005.

13. F. Shen, W. C. Tan, Y. Zhao, T. Masuoka, The Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivative model, *Nonlinear Anal., Real World Appl.*, 7, pp. 1072–1080, 2006.

14. M. Khan, S. Wang, Flow of a generalized second grade fluid between two side walls perpendicular to a plate with fractional derivative model, *Nonlinear Anal., Real World Appl.*, 10, pp. 203–208, 2009.

*Nonlinear Anal. Model. Control, Vol. 16, No. 1, 47–58, 2011*
15. M. Nazar, Corina Fetecau, A.U. Awan, A note on the unsteady flow of a generalized second grade fluid through a circular cylinder subject to a time-dependent shear stress, *Nonlinear Anal., Real World Appl.*, 11, pp. 2207–2214, 2010.

16. A. Mahmood, C. Fetecau, N.A. Khan, M. Jamil, Some exact solutions of the oscillatory motion of a generalized second grade fluid in an annular region of two cylinders, *Acta Mech. Sin.*, 26, pp. 541–550, 2010.

17. M. Athar, M. Kamran, M. Imran, On the unsteady rotational flow of a fractional second grade fluid through a circular cylinder, *Meccanica*, 2010, DOI 10.1007/s11012-010-9373-1.

18. C. Fetecau, A.U. Awan, M. Athar, A note on “Taylor–Couette flow of a generalized second grade fluid due to a constant couple”, *Nonlinear Anal. Model. Control*, 15(2), pp. 155–158, 2010.

19. M. Kamran, M. Imran, M. Athar, Exact solutions for the unsteady rotational flow of a generalized second grade fluid through a circular cylinder, *Nonlinear Anal. Model. Control*, 15(4), pp. 437–444, 2010.

20. R. Bandelli, K.R. Rajagopal, Start-up flows of second grade fluids in domains with one finite dimension, *Int. J. Non-Linear Mech.*, 30, pp. 817–839, 1995.

21. D. Tong, Y. Liu, Exact solutions for the unsteady rotational flow of non-Newtonian fluid in an annular pipe, *Int. J. Eng. Sci.*, 43, pp. 281–289, 2005.

22. C. Fetecau, A.U. Awan, Corina Fetecau, Taylor–Couette flow of an Oldroud-B fluid in a circular cylinder subject to a time-dependent rotation, *Bull. Math. Soc. Sci. Math. Roumanie*, 52(100)(2), pp. 117–128, 2009.

23. M. Athar, M. Kamran, C. Fetecau, Taylor–Couette flow of a generalized second grade fluid due to a constant couple, *Nonlinear Anal. Model. Control*, 15(1), pp. 3–13, 2010.

24. C. Fetecau, Corina Fetecau, Starting solutions for motion of a second grade fluid due to longitudinal and torsional oscillations of a circular cylinder, *Int. J. Eng. Sci.*, 44, pp. 788–796, 2006.

25. T. Hayat, M. Khan, M. Ayub, Some analytical solutions for second grade fluid flows for cylindrical geometries, *Math. Comput. Modelling*, 43, pp. 16–29, 2006.

26. I. Podlubny, *Fractional Differential Equations*, Academic press, San Diego 1999.

27. C.F. Lorenzo, T.T. Hartley, Generalized functions for the fractional calculus, NASA/TP-1999-209424/REV1, 1999.