The QCD Coupling Parameter Derived from the Uncertainty Principle and a Model for Quark Vacuum Fluctuations

David Batchelor

Radiation Effects & Analysis Group
National Aeronautics and Space Administration’s Goddard Space Flight Center
Mail Code 561.4
Greenbelt, MD 20771
(Dated: July 30, 2010)

Abstract

The magnitude of the strong interaction is characterized by $\alpha_s$, the coupling parameter in Quantum Chromodynamics (QCD), a parameter with an unexplained value in the Standard Model. In this paper, a candidate explanation for $\alpha_s$ is derived from (1) the lifetime of quark-antiquark pairs in vacuum fluctuations given by the Uncertainty Principle, (2) the variation of $\alpha_s$ as a function of energy in QCD, and (3) classical relativistic dynamics of the quarks and antiquarks. A semiclassical model for heavy quark-antiquark vacuum fluctuations is described herein, based on (2) and (3). The model in this paper predicts the measured value of $\alpha_s(M_{Z^0})$ to be 0.121, which is in agreement with recent measurements within statistical uncertainties.

PACS numbers: PACS: 03.65.-w, 03.65.Sq, 12.38.Aw, 12.39.-x, 12.39.Pn, 14.65.Dw, 14.65.Fy, 14.65.Ha

David.A.Batchelor@nasa.gov
I. INTRODUCTION

As a result of the Uncertainty Principle, vacuum fluctuations occur consisting of a particle and its antiparticle created by the vacuum and annihilated in a short lifetime $\Delta t$. A fluctuation consisting of a quark and its antiquark interacting via a gluon can occur in perturbation theory in QCD ([1], [2], [3]), the quantum field theory that successfully describes quark interactions with great accuracy.

For educational purposes, the author developed a classical model for the dynamics of such a quark-antiquark pair – a model that was intended to contrast the quantum mechanical prediction of the pair lifetime with the classical prediction. A surprising result was found: the pair lifetime in the classical model agrees to a good approximation with the lifetime in quantum mechanics.

The model is described completely in this paper. The model becomes semiclassical in the usual sense in a natural way. Because the pair lifetime from the model agrees so precisely with the quantum mechanical lifetime, particularly in the case of bottom quarks, the expressions for the lifetime in the two theories enable us to solve for the QCD strength parameter $\alpha_s$ solely from these theoretical considerations.

In the Standard Model, QCD is a physical asymptotically free field theory for any appropriate value of the input parameter $\Lambda$, which is set via experimental measurement. Within the QCD theory alone it would be impossible to establish $\alpha_s$ without recourse to a measurement. But the Uncertainty Principle and semiclassical QCD are not one and the same, and because we find in this work that the Uncertainty Principle and semiclassical QCD come to agreement on the vacuum fluctuation lifetime, we can logically derive what the physical measurement must be. It is shown below that this enables us to ground $\alpha_s$ on the value of $\bar{\hbar}$.

II. THE MODEL BASIS – ENERGY CONSERVATION

If a quark and its antiquark are positioned at rest in their center of mass reference frame and are released, then in general their mutual attraction will draw them to a collision at the origin where they will annihilate. Photons or other particles would result, given sufficient energy. However, the quark and antiquark experience a potential energy $U(R)$ as a
FIG. 1: Feynman diagram of a virtual quark-antiquark pair (VQAP). Time increases from bottom to top of figure. The curling segment represents gluon interaction.

function of their separation distance that could reduce the mass-energy of the system, if the separation $R$ between particles is small enough. The energy conservation relation

$$\varepsilon = 2\gamma m_q c^2 + U(R) = 0$$

is possible to satisfy, where the first term is the total relativistic energy of the particles, kinetic plus rest mass. ($R = 2r$ with $r$ the radius of either particle from the center of mass.) In classical physics, the collision would not yield any energy and so no photons or particles could be emitted.

If the time-reversed ballistic trajectory occurred, with the vacuum spontaneously creating a quark-antiquark pair obeying Eq. (1), then the particles only could move apart in one-dimensional motion to reach turning points separated by $R_{\text{max}}$. Continuing this trajectory so that the particles fall from the turning points back to the origin, they would disappear back into the vacuum, like a virtual quark-antiquark pair (VQAP; see Fig. 1 for the Feynman diagram). This is similar to virtual electron-positron pairs, as discussed by Greiner (p. 3 of ref. [5]) and Sakurai (p. 139 of ref. [6]).

Quantum theory implies that this two-particle system of quarks would obey the time-energy uncertainty relationship for the energy fluctuation $\Delta \varepsilon$ (p. 139 of ref. [6])

$$\Delta \varepsilon \Delta t = \frac{1}{2} \hbar = 5.273 \times 10^{-28} \text{erg s.}$$

The energy fluctuation is $\Delta \varepsilon = 2m_q c^2$, since the mass-energy of each quark contributes $m_q c^2$. The quantum-mechanical lifetime of the fluctuation is $\Delta t$. Since $\hbar$ is the quantum of action, Eq. (2) establishes an action integral $A$ that characterizes a VQAP that has $\Delta \varepsilon = 2m_q c^2$.

The first purpose of this paper is to present classical computations of $\Delta t$ and the action integral for the trajectory described above, which turn out to give results that satisfy Eq. (2)
remarkably well, provided that the QCD interaction between the particles is well-described by the potential energy function.

The second purpose of the paper follows from the fact that QCD cannot specify the value of $\alpha_s$ at arbitrary energy or 4-momentum scale $Q$ without an established measurement of $\alpha_s$ at some particular energy $\mu$ \([7]\); but once the renormalized coupling $\alpha_s(\mu^2)$ is measured, then QCD precisely gives the variation of $\alpha_s$ as a function of energy (the “running” coupling). The present paper offers a theory based on the action integral that establishes the value of $\alpha_s$ at the energy scale of twice the bottom quark mass-energy to good approximation. This enables one to use the QCD running coupling $\alpha_s(Q^2)$ to determine the coupling strength in general in the usual way to good approximation.

### III. QCD POTENTIAL ENERGY FUNCTION

As discussed in detail by Lucha et al. (especially pp. 161-162 of ref. \([4]\)), a potential energy function serves to describe the bound states of heavy quarks (charm, bottom, and top). For light quarks the QCD interaction is not satisfactorily described by a potential energy function and will not be attempted here. Here we apply the standard potential energy treatment to the heavy charm, bottom and top quarks.

We use the standard Cornell potential \([8], [9]\)

$$V(R) = -\frac{4}{3} \frac{\alpha_s \hbar c}{R} + aR,$$  \(3\)

where $\alpha_s$ is the dimensionless QCD strong coupling strength and $a \approx 0.25$ GeV$^2$. The second term, $aR$, is only significant for $R > 10^{-13}$ cm. We will not need to consider the $aR$ term, since the first term with the Coulomb-like dependence turns out to strongly dominate the potential because $R_{\text{max}} \ll 10^{-13}$ cm for VQAPs.

The model VQAP is a form of bound state. The standard way to account for the variation of $\alpha_s(Q^2)$ in a quark bound state is to let $\alpha_s$ depend on the quark masses and use $Q^2 = (m_1 + m_2)^2$, with the $m_i$ the quark masses (see p. 129 of ref. \([10]\)). To model a VQAP we may then compute $\alpha_s$ to leading order (Eq. (6) of Ref. \([7]\)). So the QCD coupling is then given by

$$\alpha_s(Q^2) = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}, \quad \text{(4)}$$
where \( \beta_0 \) is defined by
\[
\beta_0 = \frac{33 - 2n_f}{12\pi},
\]
(5)
n\(_f\) is the number of quark flavors with masses much less than \( m_1 + m_2 \), and \( \Lambda \) is the QCD scale energy,
\[
\Lambda^2 = \frac{\mu^2}{e^{1/(\beta_0\alpha_s(\mu^2))}}.
\]
(6)
\( \mu \) is approximately 0.093 GeV, assuming that \( \mu \equiv M_{Z^0} = 91.2 \) GeV (the mass-energy of the \( Z_0 \) particle), \( \alpha_s(M_{Z^0}) = 0.119 \pm 0.002 \). This is a typical value of \( \Lambda \) for a one-loop approximation ([7] p. R31).

For the running coupling near the charm quark mass \( n_f = 4 \), near the bottom quark mass \( n_f = 5 \), and near the top quark mass \( n_f = 6 \). We now can use \( m_1 = m_2 = m_q \) for each quark mass \( m_q \) in the expression for \( Q^2 \) and compute \( \alpha_s \) from Eq. (4) for a running coupling. Then with a charm quark current mass \( m_q = m_c \) of 1.27 GeV/\( c^2 \) we have \( \alpha_s(2m_c c^2) = 0.228 \); with a bottom quark current mass \( m_q = m_b \) of 4.2 GeV/\( c^2 \) we have \( \alpha_s(2m_b c^2) = 0.167 \); with a top quark current mass \( m_q = m_t \) of 173.1 GeV/\( c^2 \) we have \( \alpha_s(2m_t c^2) = 0.109 \) [11].

**IV. CLASSICAL BALLISTIC TRAJECTORY AND LIFETIME OF VQAP**

Let us calculate the trajectory lifetime \( t_{vq} \) for a VQAP. If one solves the energy equation ([11]) for the dynamics using \( V(R) \), the nonrelativistic potential energy, then the particle velocities nevertheless exhibit relativistic motion, approaching \( c \) asymptotically at the origin \( r = 0 \). Thus it is necessary to correct the potential energy for relativistic effects. Jackson ([12], p. 553) demonstrates how this is done by using the relativity factor \( \gamma \) defined in Eq. (10) below. For this trajectory of linear motion, the transformation \( R \rightarrow \gamma R \) in the expression for \( V(R) \) performs the appropriate modification of the potential energy function (since we are considering the center-of-mass reference frame). The potential energy function in Eq. (3) becomes
\[
U(R) = -\frac{4}{3\gamma R} \alpha_s \hbar c.
\]
(7)
With this \( U(R) \) we can solve Eq. (11) for \( R_{max} \) at the turning point (where \( \gamma = 1 \):
\[
2 m_q c^2 \equiv -U(R_{max}) = \frac{4}{3} \alpha_s \frac{\hbar}{R_{max}} c
\]
(8)
\[
R_{max} = \frac{2}{3} \frac{\alpha_s \hbar}{m_q c}.
\]
(9)
For the charm quark current mass \( m_c \) of 1.27 GeV/\( c^2 \equiv 2.26 \times 10^{-21} \) g, we find the charm VQAP has an \( R_{\text{max}} = 2.35 \times 10^{-15} \) cm.

Checking the terms in Eq. (3) for \( V(R_{\text{max}}) \) shows that the first term is about -100 times the second term. This confirms that the quarks are so deep in the potential well that the \( aR \) term of the Cornell potential can be neglected in solving the problem.

Let us define the time from appearance of a quark at the origin \( r = 0 \) to the time that the quark stops at the turning point \( r = \frac{1}{2} R_{\text{max}} \) as \( \frac{1}{2} t_{\text{vq}} \). The \( t_{\text{vq}} \) is the classical equivalent of the quantum-mechanical \( \Delta t \) that we seek. We note that

\[
\gamma^2 \equiv \frac{1}{1 - \beta^2} \quad \beta = \frac{1}{c} \frac{dr}{dt},
\]

and solve for \( dt \), which we shall integrate. We rewrite the energy equation (11) with \( \zeta \equiv R/R_{\text{max}} \) as

\[
\gamma^2 = \zeta^{-1} \Rightarrow dt = \frac{dr}{c \sqrt{1 - \zeta}}.
\]

The time for the particle to fall from \( r = R_{\text{max}}/2 \) back to \( r = 0 \) is also \( \frac{1}{2} t_{\text{vq}} \), so we have

\[
t_{\text{vq}} = 2 \int_0^{R_{\text{max}}/2} \frac{dr}{c \sqrt{1 - \zeta}} = \frac{R_{\text{max}}}{c} \int_0^1 \frac{d\zeta}{\sqrt{1 - \zeta}} = \frac{R_{\text{max}}}{c} \frac{\sqrt{\pi} \Gamma(1)}{\Gamma(\frac{3}{2})} = 2 \frac{R_{\text{max}}}{c} = \frac{4 \alpha_s \hbar}{3 m_q c^2}
\]

The integral is given in ref. (13), p. 974. The value of \( t_{\text{vq}} \) is the total time for either quark to travel from \( r = 0 \) to its turning point and back to \( r = 0 \).

For the charm quark, with \( m_q = m_c \approx 2.26 \times 10^{-24} \) g, we find the trajectory lifetime of the VQAP to be \( t_{\text{vq}} \approx 1.57 \times 10^{-25} \) s. In comparison, the standard lifetime of the charm VQAP, given by the Uncertainty Principle expressed in Eq. (2), is \( \Delta t = \hbar/(4m_c c^2) \approx 1.29 \times 10^{-25} \) s. So \( t_{\text{vq}} \) from the classical computation is approximately 22% larger than \( \Delta t \). This is remarkably close agreement of the classical lifetime with the quantum mechanical lifetime.

For the bottom quark, with \( m_q = m_b \approx 7.48 \times 10^{-24} \) g, we find the trajectory lifetime of the VQAP to be \( t_{\text{vq}} \approx 3.47 \times 10^{-26} \) s. In comparison, the standard lifetime of the bottom VQAP, given by the Uncertainty Principle expressed in Eq. (2), is \( \Delta t = \hbar/(4m_b c^2) \approx 3.90 \times 10^{-26} \) s. So \( t_{\text{vq}} \) from the classical computation is approximately 11% smaller than \( \Delta t \). This also is remarkably close agreement of the classical lifetime with the quantum mechanical lifetime.

For the top quark, with \( m_q = m_t \approx 3.08 \times 10^{-22} \) g, we find the trajectory lifetime of the VQAP to be \( t_{\text{vq}} \approx 5.51 \times 10^{-28} \) s. In comparison, the standard lifetime of the top VQAP, given by the Uncertainty Principle expressed in Eq. (2), is \( \Delta t = \hbar/(4m_t c^2) \approx 9.48 \times 10^{-28} \) s.
s. So $t_{vq}$ from the classical computation is approximately 42% smaller than $\Delta t$. This also is remarkably close to the quantum mechanical lifetime.

V. ACTION INTEGRAL FOR THE TRAJECTORY

A key step in quantizing a classical model to make it a semiclassical model of a quantum system is computation of the action integral. In the present model, that is done as follows. The expression for the integral of action associated with a potential function $U$ acting on a particle, in the relativistic case, is given by Lanczos ([14], p. 321):

$$A = -\int_{t_1}^{t_2} U \frac{ds}{c}$$

(13)

where $ds = c\,dt/\gamma$. Considering the integrated action of the potential energy field in a VQAP, we compute the field action integrated over $t_{vq}$:

$$A = -2 \int_{0}^{t_{vq}/2} \left( -\frac{4\alpha_s\hbar c}{3\gamma R} \right) \frac{dt}{\gamma} = \frac{8\alpha_s\hbar}{3} \int_{0}^{R_{max}/2} dr \frac{\gamma^2 R \sqrt{1 - \zeta}}{\gamma}$$

(14)

$$= \frac{4\alpha_s\hbar}{3} \int_{0}^{1} \frac{d\zeta}{\zeta^{-1}\sqrt{1 - \zeta}} = \frac{4\alpha_s\hbar}{3} \int_{0}^{1} \frac{d\zeta}{\sqrt{1 - \zeta}} = \frac{8\alpha_s\hbar}{3}$$

For the charm VQAP, $\alpha_s(2m_c c^2) = 0.228$ and therefore $A = 0.61\hbar$. This action integral is only 22% larger than the exact VQAP quantum fluctuation action in Eq. (2), $\frac{1}{2}\hbar$.

In the case of the bottom quark, the action integral for the model of the VQAP is found by substituting $\alpha_s(2m_b c^2) = 0.167$ into Eq. (15), and we find $A = 0.45\hbar$. This is 10% lower than the quantum mechanical action for a VQAP.

In the case of the top quark, the action integral for the model of the VQAP is found by substituting $\alpha_s(2m_t c^2) = 0.109$ into Eq. (15), and we find $A = 0.29\hbar$. This is 42% lower than the quantum mechanical action for a VQAP.

VI. SPIN-SPIN INTERACTION EFFECTS

As noted by Lichtenberg (p. 133 of ref. [10]) the spin-spin interaction between quarks is the source of electromagnetic mass splittings among hadron isospin multiplets. Here it will be shown that spin-spin interaction between the quark and antiquark in a VQAP is significant for the charm and top quark cases. Spin-spin interaction would modify the
dynamics of the semiclassical model and lead to a different action integral and trajectory lifetime, so its influence needs to be quantified.

The creation of a quark-antiquark pair from the vacuum should conserve angular momentum, so we consider the case in which the particles have spin parallel to the motion axis but pointing in opposite directions. The potential energy function in Eq. (7) becomes (p. 185)

\[ U(R) = -\frac{4\alpha_s}{3\gamma R^3} \frac{\hbar c}{(\gamma R)^3} - \frac{2\mu_q^2}{(\gamma R)^3} \]

where

\[ \mu_q = \frac{g Q_q}{2 m_q c^2} \]

is the magnetic moment of a quark with electric charge \( Q_q \) and \( g \) is the Landé factor, which will be taken as equal to 2 for present purposes.

The energy equation Eq. (1) in this case becomes

\[ \varepsilon = 2\gamma m_q c^2 - \frac{4\alpha_s}{3\gamma R^3} \frac{\hbar c}{(\gamma R)^3} - \frac{2\mu_q^2}{(\gamma R)^3} = 0 \]

Again we can derive the separation of the particles at the turning point \( R_{\text{max}}^* \) by letting \( \gamma \rightarrow 1 \). It is convenient to use \( \zeta = R/R_{\text{max}} \) as a dimensionless variable again, and with that substitution, the energy equation becomes

\[ \zeta_{\text{max}}^3 - \zeta_{\text{max}}^2 - \Delta = 0 \quad \text{with} \quad \Delta = \frac{27 Q_q^2 \alpha_s^3 \hbar c}{32 m_q^2 c^6} . \]

The parameter \( \Delta \) is a measure of the magnitude of the spin-spin interaction relative to the QCD interaction; \( \Delta \rightarrow 0 \) recovers the previous case in Eq. (1).

Solutions of this new energy equation (18) are deferred for a later paper, but the values of \( \Delta \) for each of the heavy quarks are illuminating. The electric charges of the charm and top quarks are both \( \frac{2}{3} e \), but the charge of the bottom quark is only \( -\frac{1}{3} e \). Consequently the parameter \( \Delta \) is 0.23 for the charm quark and 2.1 for the top quark. For the bottom quark, \( \Delta = 0.15 \), which indicates that the importance of spin-spin effects in the dynamics of the model is minimal for the bottom quark.

The smallest value of \( \Delta \) is associated with the quark that exhibits the best agreement between \( \Delta t \) and \( t_{\nu q} \), the bottom quark. This suggests that the bottom quark case is best for using the model to characterize the QCD interaction without perturbations from the spin-spin effects.
In the above cases, the model’s representation of the bottom VQAP inherently is approximately quantized – a remarkable agreement between a quantum-mechanical characteristic of a dynamical system and the classical description of it. In comparison, semiclassical models for mesons, which achieve excellent agreement with measurements of meson masses \[15, 16\] need to be formulated with additional quantization conditions that introduce the factor \(\hbar\). We have not imposed any quantization conditions upon the trajectory in this dynamical model. The bottom quark model herein achieves approximate quantization at \(\alpha_s(2m_b c^2)\) based upon only the measured value of \(\alpha_s(M_{Z^0})\), the QCD theoretical energy dependence of \(\alpha_s(Q^2)\), and relativistic dynamical theory (Eqs. (1) and (7)).

Preliminary work to investigate the discrepancy between \(t_{vq}\) and \(\Delta t\) in the cases of the charm quark and top quark has been performed as the author will show elsewhere \[17\]: for the charm quark and top quark, electromagnetic spin-spin interactions become important and influence \(t_{vq}\) and \(A\) in such a way as to bring into closer agreement the classical and quantum results.

VII. CONCLUSIONS

The salient logic in this paper’s result is the following. The measurement of \(\alpha_s(M_{Z^0})\) and the one-loop \(\Lambda\) obtained from the so-called ‘modified minimal subtraction scheme’ of renormalization theory \[7\] predict \(\alpha_s(2m_b c^2)\). From this we may use the classical ballistic trajectory lifetime of the VQAP to compute the \(t_{vq}\) and action \(A\), obtaining \(A \approx \frac{1}{2}\hbar\) in approximate agreement with quantum mechanics. From this we may go further.

Since \(\hbar\) is a more universal and fundamental parameter than \(\alpha_s\), \(\hbar\) intuitively would seem to be the governing parameter in the action equation Eq. (15).

If \(\alpha_s(2m_b c^2)\) equalled 3/16 then \(A\) would exactly equal \(\frac{1}{2}\hbar\). Setting Eq. (4) equal to 3/16, \(Q^2\) equal to \((2m_b c^2)^2\), and solving for \(\Lambda\) yields \(\Lambda = 0.106\) GeV instead of the standard 0.093 GeV. With this value of \(\Lambda\), Eq. (4) gives \(\alpha_s(91.2\) GeV \(\equiv M_{Z^0}) = 0.121\). This is only 2% different from the measured value upon which the accuracy of QCD depends, and is within the statistical uncertainty in \(\alpha_s(M_{Z^0})\) quoted in Ref. [7].

We now have a mathematical link between the measured \(\alpha_s(M_{Z^0})\) and the action integral for QVAPs from the Uncertainty Principle, \(\frac{1}{2}\hbar\). This logical sequence is equally valid in reverse. We may take as starting point the action \(A\) and infer that the action integral \(\frac{1}{2}\hbar\) is
what governs the value of $\alpha_s(M_{Z^0})$. This reverse argument from $A = \frac{1}{2}\hbar$ through Eqs. \((15), (12),\) and \((1)\) to $\alpha_s(M_{Z^0})$ is the derivation mentioned in this paper’s title.

The Uncertainty Principle is more general than QCD, and the reason that this derivation works is that the fluctuation lifetime $\Delta t$ is a different condition than the semiclassical model lifetime which is derived from phenomenological QCD. Because these two theoretical lifetimes agree in the special bottom quark case, we can now consider the QCD input parameter $\alpha_s(M_{Z^0})$ to be derivable from our theories.

This good agreement between the classical trajectory lifetime and the quantum uncertainty lifetime at the key mass-energy of the bottom quark is surprising, but it may have a simple physical explanation: if the vacuum creates these particles in motion at $v \approx c$, then their de Broglie wavelengths $\lambda = h/p$ should be small, and wave packets that are small relative to $R_{max}$ would represent the particles well. Ballistic dynamics of point masses then would serve as a good approximation for the particle motions and the agreement of the ballistic and quantum mechanical timescales would be accounted for. The length scale of any VAP is usually characterized in standard literature by assuming that $v \approx c$ \([6]\).

The semiclassical model described herein may be used to produce predictions for experiments involving VQAPs that are interacting with other particles, instead of the unobserved VQAPs that were modelled in this paper.

Acknowledgments

The author is grateful to NASA’s Goddard Space Flight Center for support during this research.

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