On the Representation Theorem of $G$-Expectations and Paths of $G$–Brownian Motion

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Abstract

We give a very simple and elementary proof of the existence of a weakly compact family of probability measures \( \{P_\theta : \theta \in \Theta \} \) to represent an important sublinear expectation—$G$-expectation $E[\cdot]$. We also give a concrete approximation of a bounded continuous function $X(\omega)$ by an increasing sequence of cylinder functions $L_{ip}(\Omega)$ in order to prove that $C_b(\Omega)$ belongs to the $E[\cdot \cdot]$-completion of the $L_{ip}(\Omega)$.

Keywords: Probability and distribution uncertainty, $G$-normal distribution, $G$-Brownian motion, Continuous paths

1 Introduction

Recently a new stochastic process called $G$-Brownian motion has been introduced in [P3, P4] under a framework of sublinear expectation called $G$-expectation $E$. From the well-known representation theorem of sublinear expectation, a $G$-expectation $E$ can be represented by an upper expectation: $E[\cdot] = \sup_{\lambda \in \Lambda} E_\lambda[\cdot]$, where $\{E_\lambda : \lambda \in \Lambda \}$ is a family of finitely additive linear expectation (see [Huber], [Delb2] and [P5]). In [DHP] Denis, Hu and Peng have introduced a method of optimal stochastic controls (see [DHP], Section 4.1) to construct a weakly compact family of ($\sigma$-additive) probability measures $\{P_\theta : \theta \in \Theta \}$ such that

$$E[X] = \sup_{\lambda \in \Lambda} \int_{\Omega} X(\omega) dP_\lambda.$$

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where $\Omega$ is the space of continuous paths. Since the representation $\sup_{\lambda \in \Lambda} E_\lambda[\cdot]$ is very elementary—only Hahn-Banach theorem is involved—a nature question is: Can we use the family $\{E_\lambda : \lambda \in \Lambda\}$ to find $\{P_\theta : \theta \in \Theta\}$ instead of passing the above mentioned long proof by using sophisticated stochastic control theory?

In this paper we give an affirmative answer to this question. Our method can be regarded as a combination and extension of the original Brownian motion construction approach of Kolmogorov and the Lipschitz cylinder functions $L_{ip}(\Omega)$ (see Section 2 for its definition) introduced in $[P2]$ and $[P3]$. This permits to give a much simpler proof involving only elementary results of probability theory. The proof is short but the importance is obvious since it involves the foundation of the theory of $G$-Brownian motion and the related stochastic calculus.

In this paper, we also give a concrete approximation of a bounded continuous function $X(\omega)$ by an increasing sequence of bounded and Lipschitz functions $L_{ip}(\Omega)$ in order to prove that $C_b(\Omega)$ belongs to the $E[\cdot]$-completion of $L_{ip}(\Omega)$.

This paper is organized as follows: in Section 2, we use Hahn-Banach theorem to prove representation theorem of sublinear expectation. In Section 3, we find a weakly compact family of probability measures to represent $G$-expectation. In Section 4, we prove that every bounded continuous function belongs to the $E[\cdot]$-completion of $L_{ip}(\Omega)$.

2 Basic settings of $G$-Brownian motion and $G$-expectation

We present some preliminaries in the theory of sublinear expectations and the related $G$-Brownian motions. More details of this section can be found in $[P5]$ and $[P2008]$.

Definition 2.1 Let $\Omega$ be a given set and let $\mathcal{H}$ be a linear space of real valued functions defined on $\Omega$ with $c \in \mathcal{H}$ for all constants $c$. $\mathcal{H}$ is considered as the space of our “random variables”. A nonlinear expectation $\hat{E}$ on $\mathcal{H}$ is a functional $\hat{E} : \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) Monotonicity: 
If $X \geq Y$ then $\hat{E}[X] \geq \hat{E}[Y]$.

(b) Constant preserving:
$\hat{E}[c] = c$.

The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a nonlinear expectation space (compare with a probability space $(\Omega, \mathcal{F}, P)$). In this paper we are mainly concerned with sublinear expectation where the expectation $\hat{E}$ satisfies also

(c) Sub-additivity:
$\hat{E}[X] - \hat{E}[Y] \leq \hat{E}[X - Y]$.

(d) Positive homogeneity:
$\hat{E}[\lambda X] = \lambda \hat{E}[X], \ \forall \lambda \geq 0$. 

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If only (c) and (d) are satisfied, \( \hat{E} \) is called a sublinear functional.

The following representation theorem for sublinear expectations is very useful (see [Peng2008] for the proof).

**Lemma 2.2** Let \( \hat{E} \) be a sublinear functional defined on a linear space \( \mathcal{H} \), i.e., (c) and (d) hold for \( \hat{E} \). Then there exists a family \( \mathcal{Q} = \{ E_\theta : \theta \in \Theta \} \) of linear functionals defined on \( \mathcal{H} \) such that

\[
\hat{E}[X] := \sup_{\theta \in \Theta} E_\theta[X], \quad \text{for } X \in \mathcal{H},
\]

and such that, for each \( X \in \mathcal{H} \), there exists a \( \theta \in \Theta \) such that \( \hat{E}[X] := E_\theta[X] \). If we assume moreover that \( \hat{E} \) is a sublinear functional defined on a linear space \( \mathcal{H} \) of functions on \( \Omega \) such that (a) holds (resp. (a), (b) hold) for \( \hat{E} \), then (a) also holds (resp. (a), (b) hold) for \( E_\theta \), \( \theta \in \Theta \).

For a given positive integer \( n \) we will denote by \( (x, y) \) the scalar product of \( x, y \in \mathbb{R}^n \) and by \( |x| = (x, x)^{1/2} \) the Euclidean norm of \( x \). We often consider a nonlinear expectation space \( (\Omega, \mathcal{H}, \hat{E}) \) such that \( X_1, \ldots, X_n \in \mathcal{H} \) implies \( \varphi(X_1, \cdots, X_n) \in \mathcal{H} \) for each \( \varphi \in C_{Lip}(\mathbb{R}^n) \), where \( C_{Lip}(\mathbb{R}^n) \) is the space of real continuous functions defined on \( \mathbb{R}^n \) such that

\[
|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x - y|, \quad \forall x, y \in \mathbb{R}^n,
\]

where \( k \) depends only on \( \varphi \).

We recall some important notions of nonlinear expectations distributions (see [P5,Peng2008]):

**Definition 2.3** Let \( X_1 \) and \( X_2 \) be two \( n \)-dimensional random vectors defined respectively in sublinear expectation spaces \( (\Omega_1, \mathcal{H}_1, \hat{E}_1) \) and \( (\Omega_2, \mathcal{H}_2, \hat{E}_2) \). They are called identically distributed, denoted by \( X_1 \sim X_2 \), if

\[
\hat{E}_1[\varphi(X_1)] = \hat{E}_2[\varphi(X_2)], \quad \forall \varphi \in C_{Lip}(\mathbb{R}^n).
\]

**Definition 2.4** In a sublinear expectation space \( (\Omega, \mathcal{H}, \hat{E}) \) a random vector \( Y = (Y_1, \cdots, Y_n) \), \( Y_i \in \mathcal{H} \) is said to be independent to another random vector \( X = (X_1, \cdots, X_m) \), \( X_i \in \mathcal{H} \) under \( \hat{E}[] \) if for each test function \( \varphi \in C_{Lip}(\mathbb{R}^m \times \mathbb{R}^n) \) we have

\[
\hat{E}[\varphi(X, Y)] = \hat{E}[\hat{E}[\varphi(x, Y)]|x = X].
\]

\( \bar{X} = (\bar{X}_1, \cdots, \bar{X}_m) \) is said to be an independent copy of \( X \) if \( \bar{X} \sim X \) and \( \bar{X} \) is independent to \( X \).
Definition 2.5 \((G\text{-normal distribution})\) A \(d\)-dimensional random vector \(X = (X_1, \ldots, X_d)\) in a sublinear expectation space \((\Omega, \mathcal{F}, \hat{E})\) is called \(G\)-normal distributed if for each \(a, b \geq 0\) we have
\[
a X + b \bar{X}^d = \sqrt{a^2 + b^2} X,
\]
where \(\bar{X}\) is an independent copy of \(X\). Here the letter \(G\) denotes the function
\[
G(A) := \frac{1}{2} \hat{E}[(AX, X)]: S_d \mapsto \mathbb{R}.
\]

Remark 2.6 It is easy to prove that the function \(G\) is a monotonic and sublinear function:
\[
\begin{align*}
G(A + \bar{A}) &\leq G(A) + G(\bar{A}), \\
G(\lambda A) &= \lambda G(A), \quad \forall \lambda \geq 0, \\
G(A) &\geq G(\bar{A}), \text{ if } A \geq \bar{A}.
\end{align*}
\]
From Lemma 2.2 there exists a (bounded) subset \(\Sigma \subset S_d\) such that \(\gamma \geq 0\) for each \(\gamma \in \Sigma\) and
\[
G(A) = \frac{1}{2} \sup_{\gamma \in \Sigma} \operatorname{tr}[A \gamma], \quad A \in S_d.
\]

We often denote \(X \sim \mathcal{N}(0, \Sigma)\). In \([P3, P4, P5, Peng2008]\) it is proved that for each given monotonic and sublinear function \(G\) defined on \(S_d\) there exists a random vector in some sublinear expectation space \((\Omega, \mathcal{F}, \hat{E})\) such that \(X \sim \mathcal{N}(0, \Sigma)\), namely, \(X\) is \(G\)-normal distributed. It is also proved in Peng \([P5, Peng2008]\) that, for each \(a \in \mathbb{R}^d\) and \(p \in [1, \infty)\)
\[
\hat{E}[|\langle a, X \rangle|^p] = \frac{1}{\sqrt{2\pi} \sigma_{aa}^T} \int_{-\infty}^{\infty} |x|^p \exp \left( \frac{-x^2}{2\sigma_{aa}^T} \right) dx,
\]
where \(\sigma_{aa}^T = 2G(aa^T)\).

Definition 2.7 \((P3\text{ and P5})\) Let \(G : S_d \mapsto \mathbb{R}\) be a given monotonic and sublinear function. A process \(\{B_t(\omega)\}_{t \geq 0}\) in a sublinear expectation space \((\Omega, \mathcal{F}, \hat{E})\) is called a \(G\)-Brownian motion if for each \(n \in \mathbb{N}\) and \(0 \leq t_1, \cdots, t_n < \infty\), \(B_{t_1}, \cdots, B_{t_n} \in \mathcal{F}\) and the following properties are satisfied:
(i) \(B_0(\omega) = 0\);
(ii) For each \(t, s \geq 0\), the increment \(B_{t+s} - B_t\) is independent to \((B_{t_1}, B_{t_2}, \cdots, B_{t_n})\), for each \(n \in \mathbb{N}\) and \(0 \leq t_1 \leq \cdots \leq t_n \leq t\);
(iii) \(B_{t+s} - B_{t} \sim \sqrt{s}X\), for \(s, t \geq 0\), where \(X\) is \(G\)-normal distributed.

Let \(\bar{\Omega} = (\mathbb{R}^d)^{[0, \infty)}\) denote the space of all \(\mathbb{R}^d\)-valued functions \((\bar{\omega}_t)_{t \in \mathbb{R}^+}\) and \(\mathcal{B}(\bar{\Omega})\) denote the \(\sigma\)-algebra generated by all finite dimensional cylinder sets. Correspondingly, we denote by \(\Omega = C^d_0(\mathbb{R}^+)\) the space of all \(\mathbb{R}^d\)-valued continuous
functions \((\omega_t)_{t \in \mathbb{R}^+}\), with \(\omega_0 = 0\), equipped with the distance
\[
\rho(\omega^1, \omega^2) := \sum_{i=1}^{\infty} 2^{-i} \left( \max_{t \in [0,i]} |\omega^1_t - \omega^2_t| \right) \land 1.
\]
\(\mathcal{B}(\Omega)\) denotes the \(\sigma\)-algebra generated by all open sets. The corresponding canonical process \(B_t(\omega) = \omega_t\), (resp. \(B_t(\omega) = \omega_t\)) \(t \in [0, \infty)\) for \(\omega \in \Omega\) (resp. \(\omega \in \Omega\)).

The spaces of Lipschitzian cylinder functions on \(\Omega\) and \(\bar{\Omega}\) are denoted respectively by
\[
L_{ip}(\Omega) := \{ \varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_n}) : \forall n \geq 1, t_1, \cdots, t_n \in [0, \infty), \forall \varphi \in C_l.Lip(\mathbb{R}^{d \times n}) \},
\]
\[
L_{ip}(\bar{\Omega}) := \{ \varphi(B_{t_1}, B_{t_2}, \cdots, B_{t_n}) : \forall n \geq 1, t_1, \cdots, t_n \in [0, \infty), \forall \varphi \in C_l.Lip(\mathbb{R}^{d \times n}) \}.
\]

Following \([\text{P3}, \text{P4}]\), we can construct a sublinear expectation \(E\) on \((\Omega, L_{ip}(\Omega))\), called \(G\)-expectation, such that \((B_t(\omega))_{t \geq 0}\) is a \(G\)-Brownian motion. Since the natural correspondence of \(L_{ip}(\Omega)\) and \(L_{ip}(\bar{\Omega})\), we can also construct a sublinear expectation \(\tilde{E}\) on \((\bar{\Omega}, L_{ip}(\bar{\Omega}))\) such that \((\bar{B}_t(\tilde{\omega}))_{t \geq 0}\) is also a \(G\)-Brownian motion. In particular, for each \(0 \leq s < t < \infty\), \(a \in \mathbb{R}^d\) and \(p \in [1, \infty)\),
\[
\tilde{E}[|a, \bar{B}_t - \bar{B}_s|^p] = \frac{1}{\sqrt{2\pi\sigma^2_{aa^T}(t-s)}} \int_{-\infty}^{\infty} |x|^p \exp\left(\frac{-x^2}{2\sigma^2_{aa^T}(t-s)}\right) dx,
\]
where \(\sigma^2_{aa^T} = 2G(aa^T)\).

In \([\text{P3}, \text{P4}, \text{P5}]\) the space \(L_{ip}(\Omega)\) is extended to \(L^p_G(\Omega)\) under the Banach norm \(E[|\cdot|]\) to develop a new type of \(G\)-stochastic calculus, including \(G\)-Itô’s integrals, \(G\)-Itô’s formula and \(G\)-SDE. In \([\text{DHP}]\) a family of weakly compact probability measures has been found to represent \(E\). This representation theorem is essentially important. Indeed, through it we were able to prove in \([\text{DHP}]\) that an element \(Y\) of the abstract Banach space \(L^p_G(\Omega)\) is in fact a quasi-continuous function \(Y = Y(\omega)\) defined on \(\Omega\), with respect to the natural capacity induced by this family. The space \(L^p_G(\Omega)\) is also proved to be identified with the that introduced in \([\text{DenMa}]\). In the next section we give a very simple and elementary proof of this representation theorem.

### 3 \(G\)-Expectation as an upper Expectation

A main objective of this paper is to find a weakly compact family of \((\sigma\text{-additive})\) probability measures on \((\Omega, \mathcal{B}(\Omega))\) to represent \(G\)-expectation \(E\). We need the following Lemmas.

**Lemma 3.1** Let \(0 \leq t_1 < t_2 < \cdots < t_m < \infty\) and \(\{\varphi_n\}_{n=1}^{\infty} \subset C_l.Lip(\mathbb{R}^{d \times m})\) satisfy \(\varphi_n \downarrow 0\). Then \(\tilde{E}[\varphi_n(B_{t_1}, B_{t_2}, \cdots, B_{t_m})] \downarrow 0\).
Proof. We denote by \( X = (\bar{B}_t, \bar{B}_2, \ldots, \bar{B}_{tm}) \). For each \( N > 0 \), it is clear that
\[
\varphi_n(x) \leq k_n^N + \varphi_1(x)1_{|x| > N} \leq k_n^N + \frac{\varphi_1(x)|x|}{N}
\]
for each \( x \in \mathbb{R}^{d \times m} \), where \( k_n^N \triangleq \max_{|x| \leq N} \varphi_n(x) \). Noting that \( \varphi_1(x)|x| \in C_{\text{Lip}}(\mathbb{R}^{d \times m}) \), then we have
\[
\mathbb{E}[\varphi_n(X)] \leq k_n^N + \frac{1}{N}\mathbb{E}[\varphi_1(X)|X|].
\]
It follows from \( \varphi_n \downarrow 0 \) that \( k_n^N \downarrow 0 \). Thus we have \( \lim_{n \to \infty} \mathbb{E}[\varphi_n(X)] \leq \frac{1}{N}\mathbb{E}[\varphi_1(X)|X|] \).
Since \( N \) can be arbitrarily large, we get \( \mathbb{E}[\varphi_n(X)] \downarrow 0 \). \( \square \)

We denote by \( \mathcal{T} := \{ \underline{t} = (t_1, \ldots, t_m) : \forall m \in \mathbb{N}, 0 \leq t_1 < t_2 < \cdots < t_m < \infty \} \).

**Lemma 3.2** Let \( E \) be a finitely additive linear expectation dominated by \( \mathbb{E} \) on \( L_{\text{lip}}(\bar{\Omega}) \). Then there exists a unique probability measure \( Q \) on \( (\bar{\Omega}, \mathcal{B}(\bar{\Omega})) \) such that \( E[X] = E_Q[X] \) for each \( X \in L_{\text{lip}}(\bar{\Omega}) \).

**Proof.** For each fixed \( \underline{t} = (t_1, \ldots, t_m) \in \mathcal{T} \), by Lemma 3.1 for each sequence \( \{ \varphi_n \}_{n=1}^\infty \subset C_{\text{Lip}}(\mathbb{R}^{d \times m}) \) satisfying \( \varphi_n \downarrow 0 \), we have \( E[\varphi_n(\bar{B}_1, \bar{B}_2, \ldots, \bar{B}_{tm})] \downarrow 0 \). By Daniell-Stone’s theorem, there exists a unique probability measure \( Q_{\underline{t}} \) on \( (\mathbb{R}^{d \times m}, \mathcal{B}(\mathbb{R}^{d \times m})) \) such that \( E_Q[\varphi] = E[\varphi(\bar{B}_1, \bar{B}_2, \ldots, \bar{B}_{tm})] \) for each \( \varphi \in C_{\text{Lip}}(\mathbb{R}^{d \times m}) \). Thus we get a family of finite-dimensional distributions \( \{ Q_{\underline{t}} : \underline{t} \in \mathcal{T} \} \). By Daniell-Stone’s theorem, it is easy to check that \( \{ Q_{\underline{t}} : \underline{t} \in \mathcal{T} \} \) is consistent, then by Kolmogorov’s consistent theorem, there exists a probability measure \( Q \) on \( (\bar{\Omega}, \mathcal{B}(\bar{\Omega})) \) such that \( \{ Q_{\underline{t}} : \underline{t} \in \mathcal{T} \} \) is the finite-dimensional distributions of \( Q \). Assume there exists another probability measure \( \bar{Q} \) satisfying the condition, by Daniell-Stone’s theorem, \( Q \) and \( \bar{Q} \) have the same finite-dimensional distributions, then by monotone class theorem, \( Q = \bar{Q} \). The proof is complete. \( \square \)

**Lemma 3.3** There exists a family of probability measures \( \mathcal{R} \) on \( (\bar{\Omega}, \mathcal{B}(\bar{\Omega})) \) such that
\[
\mathbb{E}[X] = \max_{Q \in \mathcal{R}} E_Q[X], \quad \forall X \in L_{\text{lip}}(\bar{\Omega}).
\]

**Proof.** By Lemma 2.2 and Lemma 3.2, it is easy to get the result. \( \square \)

For this \( \mathcal{R} \), we define the associated capacity
\[
\bar{c}(A) := \sup_{Q \in \mathcal{R}} Q(A), \quad A \in \mathcal{B}(\bar{\Omega}).
\]
and upper expectation for each \( \mathcal{B}(\bar{\Omega}) \)-measurable real function \( X \) which makes the following definition meaningful,
\[
\bar{E}[X] := \sup_{Q \in \mathcal{R}} E_Q[X].
\]
Lemma 3.4 For $\bar{B} = \{\bar{B}_t : t \in [0, \infty)\}$, there exists a continuous modification $\tilde{B} = \{\tilde{B}_t : t \in [0, \infty)\}$ of $B$ (i.e. $\tilde{c}(\{\tilde{B}_t \neq \bar{B}_t\}) = 0$, for each $t \geq 0$) such that $\tilde{B}_0 = 0$.

Proof. By Lemma 3.3 we know that $\bar{E} = \tilde{E}$ on $L_{ip}(\Omega)$, from (2) we get
$$\tilde{E}[|\tilde{B}_t - \tilde{B}_s|^4] = \bar{E}[|\bar{B}_t - \bar{B}_s|^4] = d|t - s|^2, \forall s, t \in [0, \infty),$$
where $d$ is a constant depending only on $G$. By generalized Kolmogorov’s criterion for continuous modification with respect to capacity (see Theorem 31 in [DHP]), there exists a continuous modification $\tilde{B}$ of $\bar{B}$. Since $\tilde{c}(\{\tilde{B}_0 \neq 0\}) = 0$, we can set $\tilde{B}_0 = 0$. The proof is complete. □

For each $Q \in \mathcal{P}$, let $Q \circ \tilde{B}^{-1}$ denote the probability measure on $(\Omega, \mathcal{B}(\Omega))$ induced by $\tilde{B}$ with respect to $Q$. We denote by $\mathcal{P}_1 = \{Q \circ \tilde{B}^{-1} : Q \in \mathcal{P}\}$. By Lemma 3.4 we get
$$\tilde{E}[|\tilde{B}_t - \tilde{B}_s|^4] = \bar{E}[|\bar{B}_t - \bar{B}_s|^4] = d|t - s|^2, \forall s, t \in [0, \infty).$$
Applying the well-known result of moment criterion for tightness of Kolmogorov-Chentrov’s type, we conclude that $\mathcal{P}_1$ is tight. We denote by $\mathcal{P} = \overline{\mathcal{P}}_1$ the closure of $\mathcal{P}_1$ under the topology of weak convergence, then $\mathcal{P}$ is weakly compact.

Now, we give the representation of $G$-expectation.

Theorem 3.5 For each continuous monotonic and sublinear function $G : \mathbb{R} \rightarrow \mathbb{R}$, let $E$ be the corresponding $G$-expectation on $(\Omega, \mathcal{B}(\Omega))$. Then there exists a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that
$$E[X] = \max_{P \in \mathcal{P}} E_P[X], \quad \forall X \in L_{ip}(\Omega).$$

Proof. By Lemma 3.3 and Lemma 3.4 we have
$$E[X] = \max_{P \in \mathcal{P}} E_P[X], \quad \forall X \in L_{ip}(\Omega).$$
For each $X \in L_{ip}(\Omega)$, by Lemma 3.1 we get $E[|X - (X \wedge N) \vee (-N)|] \downarrow 0$ as $N \rightarrow \infty$. Noting also that $\mathcal{P} = \overline{\mathcal{P}}_1$, then by the definition of weak convergence, we get the result. □

4 Completion of $L_{ip}(\Omega)$

We denote by $L^0(\Omega)$ the space of all $\mathcal{B}(\Omega)$-measurable real functions and $C_b(\Omega)$ all bounded continuous functions. In section 3, we obtain a weakly compact family
\( \mathcal{P} \) of probability measures on \((\Omega, \mathcal{B}(\Omega))\) to represent \(G\)-expectation \(\mathbb{E} \). For this \( \mathcal{P} \), we define the associated capacity

\[
e^c(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).
\]

and upper expectation for each \( X \in L^0(\Omega) \) which makes the following definition meaningful,

\[
\hat{\mathbb{E}}[X] := \sup_{P \in \mathcal{P}} \mathbb{E}_P[X].
\]

By Theorem 3.5 we know that \( \hat{\mathbb{E}} = \mathbb{E} \) on \( L_{ip}(\Omega) \), thus the \( \mathbb{E}[\cdot \, \mathbb{E}] \)-completion and the \( \hat{\mathbb{E}}[\cdot \, \mathbb{E}] \)-completion of \( L_{ip}(\Omega) \) are the same. We also denote, for \( p > 0 \),

- \( L^p := \{ X \in L^0(\Omega) : \hat{\mathbb{E}}[\|X\|^p] = \sup_{P \in \mathcal{P}} \mathbb{E}_P[\|X\|^p] < \infty \} \);
- \( N^p := \{ X \in L^0(\Omega) : \hat{\mathbb{E}}[\|X\|^p] = 0 \} \);
- \( N := \{ X \in L^0(\Omega) : X = 0, \hat{c}\text{-q.s.} \} \).

It is seen that \( L^p \) and \( N^p \) are linear spaces and \( N^p = N \), for each \( p > 0 \).

We denote by \( \mathbb{L}^p := L^p / N \). As usual, we do not take care about the distinction between classes and their representatives.

Now, we give the following two Propositions which can be found in [DHP].

**Proposition 4.1** For each \( \{X_n\}_{n=1}^\infty \) in \( C_b(\Omega) \) such that \( X_n \downarrow 0 \) on \( \Omega \), we have \( \hat{\mathbb{E}}[X_n] \downarrow 0 \).

**Proposition 4.2** We have

1. For each \( p \geq 1 \), \( \mathbb{L}^p \) is a Banach space under the norm \( \|X\|_p := \left( \hat{\mathbb{E}}[\|X\|^p] \right)^{\frac{1}{p}} \).
2. For each \( p < 1 \), \( \mathbb{L}^p \) is a complete metric space under the distance

\[
d(X, Y) := \hat{\mathbb{E}}[\|X - Y\|^p].
\]

With respect to the distance defined on \( \mathbb{L}^p, p > 0 \), we denote:

- \( \mathbb{L}^p_c \) the completion of \( C_b(\Omega) \).
- \( L^p_G(\Omega) \) the completion of \( L_{ip}(\Omega) \).
Proof. 1. is obvious.

We now prove that \( L^1_G(\Omega) = \mathbb{L}^1_c \). First, we need the following classical approximation Lemma.

**Lemma 4.3** For each \( X \in C_b(\Omega) \) and \( n = 1, 2, \ldots \), we denote
\[
X^{(n)}(\omega) \triangleq \inf_{\omega' \in \Omega} \{ X(\omega') + n \| \omega - \omega' \|_{C^d_0([0,T])} \}, \quad \forall \omega \in \Omega.
\]
Then the sequence \( \{X^{(n)}\}_{n=1}^{\infty} \) satisfies:

1. \(-M \leq X^{(n)} \leq X^{(n+1)} \leq \cdots \leq X, \, M = \sup_{\omega \in \Omega} |X(\omega)|.\)
2. \(|X^{(n)}(\omega_1) - X^{(n)}(\omega_2)| \leq n \| \omega_1 - \omega_2 \|_{C^d_0([0,T])}, \quad \forall \omega_1, \omega_2 \in \Omega.\)
3. \(X^{(n)}(\omega) \uparrow X(\omega), \quad \forall \omega \in \Omega.\)

**Proof.** 1. is obvious.

For 2. We have
\[
X^{(n)}(\omega_1) - X^{(n)}(\omega_2) \\
\leq \sup_{\omega' \in \Omega} \{ [X(\omega') + n \| \omega_1 - \omega' \|_{C^d_0([0,T])}] - [X(\omega') + n \| \omega_2 - \omega' \|_{C^d_0([0,T])}] \} \\
\leq n \| \omega_1 - \omega_2 \|_{C^d_0([0,T])}
\]
and, symmetrically, \(X^{(n)}(\omega_2) - X^{(n)}(\omega_1) \leq n \| \omega_1 - \omega_2 \|_{C^d_0([0,T])}.\) Thus 2 follows.

We now prove 3. For each fixed \( \omega \in \Omega \), let \( \omega_n \in \Omega \) be such that
\[
X(\omega_n) + n \| \omega - \omega_n \|_{C^d_0([0,T])} \leq X^{(n)}(\omega) + \frac{1}{n}.
\]
It is clear that \( n \| \omega - \omega_n \|_{C^d_0([0,T])} \leq 2M + 1, \) or \( \| \omega - \omega_n \|_{C^d_0([0,T])} \leq \frac{2M+1}{n}. \) Since \( X \in C_b(\Omega), \) we get \( X(\omega_n) \to X(\omega) \) as \( n \to \infty. \) We have
\[
X(\omega) \geq X^{(n)}(\omega) \geq X(\omega_n) + n \| \omega - \omega_n \|_{C^d_0([0,T])} - \frac{1}{n},
\]
thus
\[
n \| \omega - \omega_n \|_{C^d_0([0,T])} \leq |X(\omega) - X(\omega_n)| + \frac{1}{n}.
\]
We also have
\[
X(\omega_n) - X(\omega) + n \| \omega - \omega_n \|_{C^d_0([0,T])} \geq X^{(n)}(\omega) - X(\omega) \\
\geq X(\omega_n) - X(\omega) + n \| \omega - \omega_n \|_{C^d_0([0,T])} - \frac{1}{n}.
\]
From the above two relations we obtain
\[ |X^{(n)}(\omega) - X(\omega)| \leq |X(\omega_n) - X(\omega)| + n\|\omega - \omega_n\|_{C^d([0,n])} + \frac{1}{n} \]
\[ \leq 2(|X(\omega_n) - X(\omega)| + \frac{1}{n}) \to 0 \text{ as } n \to \infty. \]
Thus 3 is obtained. □

**Proposition 4.4** For each \( X \in C_b(\Omega) \) and \( \varepsilon > 0 \) there exists a \( Y \in L_{ip}(\Omega) \) such that \( \hat{E}[|Y - X|] \leq \varepsilon. \)

**Proof.** We denote by \( M = \sup_{\omega \in \Omega} |X(\omega)|. \) By Proposition 4.1 and Lemma 4.3, we can find \( \mu > 0, T > 0 \) and \( X \in C_b(\Omega_T) \) such that \( \hat{E}[|X - \bar{X}|] < \varepsilon/3, \)
\[ \sup_{\omega \in \Omega} |X(\omega)| \leq M \]
and
\[ |X(\omega) - \bar{X}(\omega')| \leq \mu \|\omega - \omega'\|_{C^d([0,T])}, \quad \forall \omega, \omega' \in \Omega. \]
Now for each positive integer \( n \), we introduce a mapping \( \omega^{(n)}(\omega) : \Omega \to \Omega \) by
\[ \omega^{(n)}(\omega)(t) = \sum_{k=0}^{n-1} \frac{1}{n} \mathbf{1}_{[t^n_k, t^n_{k+1})}(t) \left[ (t^n_{k+1} - t)\omega(t^n_k) + (t - t^n_k)\omega(t^n_{k+1}) \right] + \mathbf{1}_{[T, \infty)}(t)\omega(t), \]
where \( t^n_k = \frac{kT}{n}, \) \( k = 0, 1, \ldots, n. \) We set \( \bar{X}^{(n)}(\omega) := \bar{X}(\omega^{(n)}(\omega)), \) then
\[ |\bar{X}^{(n)}(\omega) - \bar{X}(\omega^{(n)}(\omega'))| \leq \mu \sup_{t \in [0,T]} |\omega^{(n)}(\omega)(t) - \omega^{(n)}(\omega')(t)| \]
\[ = \mu \sup_{k \in [0, \ldots, n]} |\omega(t^n_k) - \omega'(t^n_k)|. \]
We now choose a compact subset \( K \subset \Omega \) such that \( \hat{E}[1_{K^c}] \leq \varepsilon/6M. \) Since \( \sup_{\omega \in K} \sup_{t \in [0,T]} |\omega(t) - \omega^{(n)}(\omega)(t)| \to 0, \) as \( n \to \infty, \) we then can choose a sufficiently large \( n_0 \) such that
\[ \sup_{\omega \in K} |\bar{X}(\omega) - \bar{X}(\omega^{(n_0)}(\omega))| = \sup_{\omega \in K} |\bar{X}(\omega) - \bar{X}(\omega^{(n_0)}(\omega))| \]
\[ \leq \mu \sup_{\omega \in K} \sup_{t \in [0,T]} |\omega(t) - \omega^{(n_0)}(\omega)(t)| \]
\[ < \varepsilon/3. \]
We set \( Y := \bar{X}^{(n_0)} \), it follows that
\[ \hat{E}[|X - Y|] \leq \hat{E}[|X - \bar{X}|] + \hat{E}[|\bar{X} - \bar{X}^{(n_0)}|] \]
\[ \leq \hat{E}[|X - \bar{X}|] + \hat{E}[1_K|\bar{X} - \bar{X}^{(n_0)}|] + 2M\hat{E}[1_{K^c}] \]
\[ < \varepsilon. \]
The proof is complete. □

By Proposition 4.3, we can easily get $L^1_G(\Omega) = \mathbb{L}_c^1$. Furthermore, we can get $L^p_G(\Omega) = \mathbb{L}_c^p, \forall p > 0$.

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