Automorphism Groups with Cyclic Commutator Subgroup and Hamilton Cycles

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Abstract.

It has been shown that there is a Hamilton cycle in every connected Cayley graph on each group $G$ whose commutator subgroup is cyclic of prime-power order. This paper considers connected, vertex-transitive graphs $X$ of order at least 3 where the automorphism group of $X$ contains a transitive subgroup $G$ whose commutator subgroup is cyclic of prime-power order. We show that of these graphs, only the Petersen graph is not hamiltonian.

Key words: graph, vertex-transitive, Hamilton cycle, commutator subgroup

1 Introduction

Considerable attention has been devoted to the problem of determining whether or not a connected, vertex-transitive graph $X$ has a Hamilton cycle [1], [8], [14]. A graph $X$ is vertex-transitive if some group $G$ of automorphisms of $X$
acts transitively on $V(X)$. If $G$ is abelian, then it is easy to see that $X$ has a Hamilton cycle. Thus it is natural to try to prove the same conclusion when $G$ is “almost abelian.” Recalling that the commutator subgroup of $G$ is the subgroup $G' = \langle x^{-1}y^{-1}xy : x, y \in G \rangle$, and that $G$ is abelian if and only if the commutator subgroup of $G$ is trivial, it is natural to consider the case where the commutator subgroup of $G$ is “small” in some sense. In this vein, K. Keating and D. Witte [10] used a method of D. Marušić [11] to show that there is a Hamilton cycle in every Cayley graph on each group whose commutator subgroup is cyclic of prime-power order. This paper utilizes techniques of B. Alspach, E. Durnberger, and T. Parsons [5], [4], [2] to prove the following result.

**Theorem 1.1.** Let $X$ be a connected vertex-transitive graph of order at least 3. If there is a transitive group $G$ of automorphisms of $X$ such that the commutator subgroup of $G$ is cyclic of prime-power order, then $X$ is the Petersen graph or $X$ is hamiltonian.

Because $K_2$ and the Petersen graph have Hamilton paths, the following corollary is immediate.

**Corollary 1.2.** Let $X$ be a connected vertex-transitive graph. If there is a transitive group $G$ of automorphisms of $X$ such that the commutator subgroup of $G$ is cyclic of prime-power order, then $X$ has a Hamilton path.

## 2 Assumptions and Definitions

**Assumption 2.1.** Throughout this note, $X$ is a connected vertex-transitive graph, $G$ is a group of automorphisms of $X$ that acts transitively on the vertex set $V(X)$, and $G'$ is the commutator subgroup of $G$.

Although the following definitions and results may be stated in more general group-theoretic terms (see [13] or [6]), we state them here in the context of this problem.

**Definition 2.2.** The stabilizer $G_x$ of a vertex $x \in V(X)$ is $\{ g \in G : g(x) = x \}$ and is a subgroup of $G$.

**Lemma 2.3** ([13, 10.1.2, p. 256]). Let $x \in V(X)$ and $g \in G$. Then $G_{gx} = g(G_x)g^{-1}$.

**Corollary 2.4.** If $H$ is a normal subgroup of $G$, then the following are equivalent:

1. $HG_x$ is a normal subgroup of $G$ for some $x \in V(X)$;
(2) $HG_x$ is a normal subgroup of $G$ for every $x \in V(X)$;
(3) $HG_x = HG_y$ for all $x, y \in V(X)$.

**Proof.** Let $g \in G$ and $x \in V(X)$. From the lemma, we know $G_{gx} = g(G_x)g^{-1}$, and since $H$ is normal, we have $H = gHg^{-1}$. So

$$HG_{gx} = (gHg^{-1})(g(G_x)g^{-1}) = g(HG_x)g^{-1}.$$ (1)

(1) $\Rightarrow$ (3). Since $G$ is transitive on $V(X)$, there exists $g \in G$ with $gx = y$. Then, since $HG_x$ is normal, (1) implies $HG_y = HG_x$, as desired.

(3) $\Rightarrow$ (2). Let $g \in G$. From (3), we have $HG_{gx} = HG_x$. Therefore, (1) implies $g(HG_x)g^{-1} = HG_x$, as desired. □

**Corollary 2.5.** For every $x \in V(X)$, the stabilizer $G_x$ does not contain a nontrivial, normal subgroup of $G$.

**Proof.** Let $H$ be a normal subgroup of $G$ that is contained in $G_x$. Lemma 2.3 implies $H \subset G_{gx}$, for all $g \in G$. Since $G$ is acts transitively on $V(X)$, it follows that $H \subset G_y$, for all $y \in V(X)$. Therefore, the identity automorphism of $X$ is the only element of $H$. □

**Definition 2.6.** Let $H$ be a subgroup of $G$, and let $x \in V(X)$. The $H$-orbit of $x$ is $\{hx : h \in H\}$. The $H$-orbits form a partition of $V(X)$, and if $H$ is normal in $G$, then the subgraphs of $X$ induced by distinct $H$-orbits are isomorphic, as $g(Hx) = H(gx)$ in this case.

**Definition 2.7.** Let $H$ be a subgroup of $G$. The quotient graph $X/H$ is that graph whose vertices are the $H$-orbits, and two such vertices $Hx$ and $Hy$ are adjacent in $X/H$ if and only if there is an edge in $X$ joining a vertex of $Hx$ to a vertex of $Hy$. If $H$ is normal in $G$, then the action of $G$ on $V(X)$ factors through to a transitive action of $G/H$ on $V(X/H)$ by automorphisms of $X/H$ and thus $X/H$ is vertex-transitive.

**Lemma 2.8.** If $H$ is a normal subgroup of $G$, then every path in $X/H$ lifts to a path in $X$.

**Proof.** It suffices to show that if $Hx$ is adjacent to $Hy$ in $X/H$, then $x$ is adjacent to some vertex in $Hy$. By definition of $X/H$, we know that some $\tilde{x} \in Hx$ is adjacent to some $\tilde{y} \in Hy$. Next there exists $h \in H$ with $x = h\tilde{x}$, so that $x$ is adjacent to $h\tilde{y} \in Hy$. □

**Definition 2.9.** Let $S$ be a subset of $G$, and assume $S$ is symmetric (that is, $s^{-1} \in S$ for all $s \in S$). The Cayley graph $\text{Cay}(G; S)$ is that graph whose vertices are the elements of $G$, and for vertices $g$ and $h$, there is an edge from $g$ to $h$ if and only if $gs = h$ for some $s \in S$. Since $G$ acts transitively on the
vertices of Cay\((G; S)\) by left multiplication, Cay\((G; S)\) is vertex-transitive. A Cayley graph is connected if and only if \(S\) generates \(G\).

Recall that \(G'\) is a normal subgroup of \(G\) and that the quotient group \(G/G'\) is abelian [13, Thms. 3.4.11 and 3.4.10, p. 59]. Since \(G/G'\) is abelian and transitive on \(V(X/G')\), it follows from the next result that \(X/G'\) is a Cayley graph on the abelian group \(G/(G_xG')\), for any \(x \in V(X)\).

Lemma 2.10 (Sabidussi [12]). If \(G_x\) is trivial for some \(x \in V(X)\), then \(X\) is (isomorphic to) a Cayley graph on \(G\).

3 Preliminaries on the Frattini subgroup

As in Section 2, we assume that Assumption 2.1 holds.

Assumption 3.1. We assume \(G'\) is cyclic of order \(p^k\), where \(p\) is a prime, and that \(X\) has at least three vertices.

Assumption 3.2. We also assume \(X\) is \(G\)-minimal. That is, if \(Y\) is a connected, spanning subgraph of \(X\), such that, for all \(g \in G\), we have \(gY = Y\), then it must be the case that \(Y = X\). (In the case of Cayley graphs, Cay\((G; S)\) is \(G\)-minimal if and only if no proper symmetric subset of \(S\) generates \(G\).) Since a Hamilton cycle in any such subgraph \(Y\) would also be a Hamilton cycle in \(X\), we may assume this without loss of generality.

The main result of this section is Lemma 3.8. A central idea to the proof is that of the Frattini subgroup, defined in [13, §7.3].

Definition 3.3. An element \(g\) of \(G\) is a nongenerator if, for every subset \(S\) of \(G\) such that \(\langle S, g \rangle = G\), we have that \(\langle S \rangle = G\). The Frattini subgroup of \(G\), denoted \(\Phi(G)\), is the set of all nongenerators of \(G\) and is a subgroup of \(G\).

Lemma 3.4. If \(H\) is any subgroup of \(G'\), then \(H\) is normal in \(G\) and \(H^p \subset \Phi(G)\), where \(H^p = \langle h^p : h \in H \rangle\).

Proof. Since \(G'\) is a cyclic normal subgroup of \(G\), we know that every subgroup of \(G'\) is a normal subgroup of \(G\) [9, Thm. 1.3.1(i), p. 9, and Thm. 2.1.2(ii), p. 16]. Therefore \(H\) is normal in \(G\) and hence \(\Phi(H) \subset \Phi(G)\) [13, 7.3.17, p. 162]. Since \(H\) is a cyclic \(p\)-group, it is not difficult to see that \(\Phi(H) = H^p\) [13, 7.3.7, p. 162]. \(\square\)

Lemma 3.5. If \(H\) is a normal subgroup of \(G\) and \(H \subset \Phi(G)\), then \(X/H\) is \(G\)-minimal.
Proof. Let $Y$ be a connected, spanning subgraph of $X/H$ such that for all $g \in G$, we have that $gY = Y$. Let $x \in V(X)$, and let
\[ S = \{ s \in G : sx \text{ is adjacent to } x \text{ in } X \}, \]
and
\[ T = \{ t \in G : Htx \text{ is adjacent to } Hx \text{ in } Y \}. \]
It is straightforward to verify that $G_x S G_x = S$ and $H G_x T G_x = T$. Furthermore, since $Y$ is connected, we see that $T$ generates $G$.

Since $Y$ is a subgraph of $X/H$, it must be the case that $T \subseteq HS$. Hence, since $HT = T$, we have that $T = H(S \cap T)$. Next since $T$ generates $G$ and $H \subseteq \Phi(G)$, we conclude that $S \cap T$ generates $G$. Therefore, letting $Z$ be the spanning subgraph of $X$ whose edge set is
\[ E(Z) = \{ \{ gtx, gx \} : g \in G, \ t \in S \cap T \}, \]
we see that $Z$ is connected. So $Z$ is a connected, spanning subgraph of $X$ such that $gZ = Z$ for all $g \in G$. Since $X$ is $G$-minimal, it follows that $Z = X$ and hence $S \cap T = S$. Therefore $HS = H(S \cap T) = T$, so $X/H = Y$. \qed

Because a $G$-minimal graph has no loops, we have the following corollary.

Corollary 3.6. If $H$ is a normal subgroup of $G$ and $H \subseteq \Phi(G)$, then the subgraph of $X$ induced by each $H$-orbit has no edges.

We now recall (in a weak form) the fundamental work of C. C. Chen and N. F. Quimpo [7].

Theorem 3.7 (Chen-Quimpo [7]). Let $Y$ be a connected Cayley graph on an abelian group of order at least three. Then each edge of $Y$ (except any loop) is contained in some Hamilton cycle of $Y$.

The following helpful result is the main conclusion obtained from our discussion of $G$-minimality and Frattini subgroups. (It also relies on the Chen-Quimpo Theorem.)

Lemma 3.8. If $H$ is a subgroup of $G'$ such that $X/H$ has a Hamilton cycle, then each edge of $X/H$ (except any loop) is contained in some Hamilton cycle of $X/H$.

Proof. If $H = G'$, then we have already seen that $X/G'$ is a Cayley graph on the abelian group $G/(G_x G')$ and hence desired conclusion follows from the Chen-Quimpo Theorem (3.7).

We may now assume $H \neq G'$, which implies $H \subset (G')^p$. So $H \subset \Phi(G)$ by Lemma 3.4; therefore $X/H$ is $G$-minimal by Lemma 3.5. Let $C$ be a Hamilton cycle in $X/H$, and let $Y = \bigcup_{g \in G} gC$. Since $X/H$ is $G$-minimal, we must have $Y = X/H$, and thus every edge of $X/H$ is contained in some Hamilton cycle $gC$. \qed
4 Proof of Theorem 1.1

As before, we assume that Assumptions 2.1, 3.1, and 3.2 still hold. The main conclusions of this section are two propositions which together constitute a proof of Theorem 1.1.

Let us begin by disposing of a trivial case, namely the case when $X/G'$ has only one vertex. Then $G'$ is transitive on $V(X)$. Furthermore, we see from Corollary 2.5 and Lemma 3.4 that $(G')_x = \{e\}$ for each vertex $x$ of $X$. Thus it follows by Lemma 2.10 that $X$ is a Cayley graph on the abelian group $G'$. Then Theorem 3.7 implies that $X$ has a Hamilton cycle if $X$ has order at least 3.

**Lemma 4.1.** Suppose $H$ is a subgroup of $G'$ and that

$$H^p x_1, H^p x_2, \ldots, H^p x_n, H^p x_{n+1}$$

is a path in $X/H^p$ with $H^p x_1 \neq H^p x_{n+1}$. If $Hx_1, Hx_2, \ldots, Hx_n, Hx_{n+1}$ is a Hamilton cycle in $X/H$ (or if we have $n = 2$, $X/H \cong K_2$, and $Hx_1 = Hx_3 \neq Hx_2$), then $X$ has a Hamilton cycle.

**Proof.** By Lemma 2.8, we can lift the path $H^p x_1, H^p x_2, \ldots, H^p x_n, H^p x_{n+1}$ in $X/H^p$ to a path $x_1, x_2, \ldots, x_{n+1}$ in $X$. Since $Hx_1 = Hx_{n+1}$, there exists $\gamma \in H$ such that $\gamma(x_1) = x_{n+1}$. Now, since $x_{n+1} \not\in H^p x_1$, it follows that $\gamma \not\in H^p$, which implies that $\gamma$ generates $H$. Let $P$ be the path $x_1, x_2, \ldots, x_n$. Then the trail $P, \gamma(P), \ldots, \gamma^{|H|-1}(P), x_1$ is a Hamilton cycle in $X$. $\square$

The analysis now breaks into two cases, depending on whether the subgraphs induced by each $G'$-orbit are empty. Since $G'$ is a normal subgroup, all of these subgraphs are isomorphic, and hence either all are empty, or none are.

**Proposition 4.2.** If the subgraph induced by each $G'$-orbit is empty, then $X$ has a Hamilton cycle.

**Proof (cf. [5], [4], [2]).** Let $x_1 \in V(X)$. Since $G/G'$ is abelian, it follows that $G'G_{x_1}$ is a normal subgroup of $G$. Hence, there is a subgroup $H$ of $G'$, such that $HG_{x_1}$ is normal in $G$, but $KG_{x_1}$ is not normal in $G$, for every proper subgroup $K$ of $H$. (It may be the case that $H = G'$ or $H = \{e\}$.) Since $X/H$ is a connected Cayley graph on the group $G/(HG_{x_1})$ (see Lemma 2.10) and the commutator subgroup of $G/HG_{x_1}$ is cyclic, it follows that $X/H$ has a Hamilton cycle or $X/H \cong K_2$ [10].

We may assume that $H \neq \{e\}$, for otherwise $X = X/H$ has a Hamilton cycle, and we are done. Then $H^p \neq H$, and the choice of $H$ implies that $H^p G_{x_1}$ is not normal in $G$. Therefore, since $X$ is connected and vertex-transitive, it follows from Corollary 2.4 that $x_1$ is adjacent to some vertex $u$ such that
$H^pG_{x_1} \neq H^pG_u$. This implies that there exists $\gamma \in G_{x_1}$ such that $\gamma(u) \notin H^p u$. However, since $HG_{x_1} = HG_u$ (see Corollary 2.4), we have that $\gamma(u) \in G_{x_1} u \subseteq HG_u u = Hu$.

Since the subgraph induced by $Hx_1$ is contained in the subgraph induced by $G'x_1$, which has no edges, and $x_1$ is adjacent to $u$, it follows that $u \notin Hx_1$, and thus $\{Hx_1, Hu\}$ is an edge in $X/H$. Therefore, there exists a Hamilton path from $Hx_1$ to $Hu$ in $X/H$ (see Lemma 3.8). This path lifts to a path $x_1, x_2, x_3, \ldots, x_n$ in $X$, where $x_n \in Hu$ (see Lemma 2.8). Since not both of

$H^p u, H^p x_1, H^p x_2, \ldots, H^p x_n$ and $H^p \gamma(u), H^p x_1, H^p x_2, \ldots, H^p x_n$

can be a cycle, Lemma 4.1 implies there is a Hamilton cycle in $X$ as desired. \(\square\)

We now consider the case where the $G'$-orbits do not induce empty graphs. Let us begin with some preliminary observations.

**Lemma 4.3.** If each subgraph induced by each $G'$-orbit is nonempty, then these subgraphs are connected and $p$ is odd.

**Proof.** Suppose that the subgraph induced by $G'x$ is not connected. Since $G'$ is cyclic, this subgraph is circulant, and hence each connected component must be induced by the orbit of some proper subgroup $H$ of $G'$. But $H \subset (G')^p$, and $(G')^p \subset \Phi(G)$ (see Lemma 3.4), and Corollary 3.6 asserts that the subgraph induced by any $H$-orbit has no edges. This contradicts the fact that the connected components of the subgraph induced by $G'x$ do have edges.

We now show that $p$ is odd. Suppose, to the contrary, that $p = 2$. Let $\overline{G} = G/(G')^2$. The commutator subgroup of $\overline{G}$ is $G'/(G')^2$, which has order 2. Because a group of order 2 has no nontrivial automorphisms, this implies that the commutator subgroup of $\overline{G}$ is contained in the center of $\overline{G}$; therefore $\overline{G}$ is nilpotent (of class 2) [9, p. 21]. Since $(G')^2 \subset \Phi(G)$ (see Lemma 3.4), it follows that $G/\Phi(G)$ is nilpotent. Hence $G$ itself is nilpotent [13, 7.4.10, p. 168], so $G' \subset \Phi(G)$ [13, Thm. 7.3.4, p. 160]. Therefore the subgraph induced by each $G'$-orbit is empty (see Corollary 3.6), contradicting our hypothesis. \(\square\)

We can now concisely state several important results of B. Alspach [2], [3]. They have been rephrased in the context of our problem.

**Theorem 4.4 (Alspach).** Assume that the subgraph induced by each $G'$-orbit is nonempty. Then $X$ has a Hamilton cycle if any of the following are true:

1. every vertex of the subgraph induced by a $G'$-orbit has degree at least 3 [3, Thm. 2.4]; or
2. $X/G'$ has only two vertices and $X$ is not the Petersen graph [2, Thm. 2];
(3) the number of vertices of $X/G′$ is odd [3, Thm. 3.7(ii)]; or
(4) there is a Hamilton cycle in $X/G′$ that can be lifted to a cycle in $X$ [3, Thm. 3.9].

Lemma 4.5. Let $x \in V(X)$. If $G_x = G_y$ for all $y \in G′x$, then $X$ has a Hamilton cycle.

Proof. This is essentially the same as the proof of Proposition 4.2; the assumption that the subgraph induced by $G′x$ has no edges was used only to show that $u \not\in Hx_1$, and this follows from the assumption that $G_x = G_y$ for all $y \in G′x$ (and hence for all $y \in Hx$). □

The following lemma shows that we may assume that all the vertices in each $G′$-orbit have different stabilizers. The proof is mainly group-theoretic. The key observation is that the automorphism group of a cycle is a dihedral group. Therefore, if a group of automorphisms acts transitively on the vertices of an odd cycle, then either all vertices have different stabilizers or all vertices have the same stabilizer, depending on whether the group contains a reflection.

Lemma 4.6. Assume that the subgraph induced by each $G′$-orbit is nonempty, and that there are two vertices $x$ and $y$ belonging to the same $G′$-orbit such that $G_x = G_y$. Then $X$ has a Hamilton cycle.

Proof. Let $Y$ be the subgraph of $X$ induced by $G′x$, and let $K = \cap_{v \in G′x} G_v$. (Note that $K$ is a subgroup.) Since every subgroup of $G′$ is normal in $G$ (see Lemma 3.4), it follows that $G′ \cap G_x = \{e\}$ (see Corollary 2.5) and hence $G′ \cap K = \{e\}$. On the other hand, since $G′$ fixes $V(Y)$ setwise, we see from Lemma 2.3 that $G′$ normalizes $K$. Therefore, $[G′, K] \subset G′ \cap K$, so $G′$ must centralize $K$.

By Theorem 4.4(1), if every vertex of $Y$ has degree at least 3, then $X$ has a Hamilton cycle. Thus we may assume that $Y$ is 2-regular. Since $Y$ is connected and has an odd number of vertices (see Lemma 4.3), it follows that $Y$ is a odd cycle. Therefore, we see that $K$ is a subgroup of index at most two in $G_v$, for each $v \in V(Y)$. In fact, from Lemma 4.5, we may assume that the index is exactly two.

Let $A$ be a subgroup of $G_x$ of order two. Since $A$ is not normal in $G$ (see Corollary 2.5), we know that $A$ does not centralize $G′$ (otherwise, it would be the only Sylow 2-subgroup of the normal subgroup $AG′$, and hence $A$ would be normal in $G$). Since $G′$ is a cyclic $p$-group and $p$ is odd, the automorphism group of $G′$ is cyclic [13, 5.7.12, p. 120] and therefore has exactly one element of order 2, namely, inversion. Therefore, the action of $A$ by conjugation inverts $G′$. Since $G′$ has odd order, this means that $e$ is the only element of $G′$ that is centralized by $A$. 8
On the other hand, $A$ must centralize $K$ (since $A \subset G_x$, $G_x$ normalizes $K$, and $K \cap G' = \{e\}$). Thus, we see that $K$ is the centralizer of $AG'$ in $KG'$. Since $AG'$ and $KG'$ are normal, we have that $K$ is a normal subgroup of $G$. Therefore, $K = \{e\}$ (see Corollary 2.5), which implies $G_x = A$ has order 2. Hence, since a group of order 2 has no nontrivial automorphisms, any element of $G$ that normalizes $G_x$ must actually centralize it. In particular, then the conclusion of the preceding paragraph implies that no nontrivial element of $G'$ normalizes $G_x$. This contradicts the fact that $G_x = G_y$ (see Lemma 2.3).

Proposition 4.7. If the subgraph induced by each $G'$-orbit has some edges, then $X$ has a Hamilton cycle or $X$ is the Petersen graph.

Proof (cf. pf. of Prop. 4.2). Let $H$ be the smallest subgroup of $G'$ such that whenever $x$ and $y$ are two adjacent vertices of $X$ not belonging to the same $G'$-orbit, we have $HG_x = HG_y$. (It may be the case that $H = G'$.) Note that, from Theorem 4.4(2), we may assume $X/G'$ has more than two vertices.

Assume for the moment that $H$ is nontrivial. Then $H^p$ is properly contained in $H$, so the minimality of $H$ implies there are two adjacent vertices $x_1$ and $u$, such that $G'x_1 \neq G'\gamma(u)$, and $H^pG_{x_1} \neq H^pG_u$. Thus, there exists $\gamma \in G_{x_1}$ such that $\gamma(u) \notin H^p\gamma$. Since $X/G'$ has more than two vertices, we have that $X/H$ is not the Petersen graph, and from Lemma 3.8 (and induction on the number of vertices in $X$), we know there is a Hamilton path from $Hx_1$ to $Hu$ in $X/H$. This path lifts to a path $x_1, x_2, \ldots, x_n$ in $X$, where $x_n \in Hu$ (see Lemma 2.8). Since not both of

$H^p\gamma(u), H^px_1, H^px_2, \ldots, H^px_n$

and

$H^p, H^px_1, H^px_2, \ldots, H^px_n$

can be a cycle, Lemma 4.1 implies there is a Hamilton cycle in $X$, as desired.

We may now assume $H = \{e\}$. Let $x_1, x_2, \ldots, x_{m+1}$ be a lift in $X$ of a Hamilton cycle in $X/G'$. Because $H = \{e\}$, we must have $G_{x_i} = G_{x_{i+1}}$ for every $i$, so $G_{x_1} = G_{x_{m+1}}$. Therefore, if $x_1 \neq x_{m+1}$, then Lemma 4.6 implies that $X$ has a Hamilton cycle. On the other hand, if $x_1 = x_{m+1}$, then Theorem 4.4(4) yields the same conclusion.

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