WOLFF’S PROBLEM OF IDEALS IN THE MULTIPLIER ALGEBRA ON WEIGHTED DIRICHLET SPACE

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Abstract. We establish an analogue of Wolff’s theorem on ideals in $H^\infty(D)$ for the multiplier algebra of weighted Dirichlet space.

In this paper we wish to extend a theorem of Wolff, concerning ideals in $H^\infty(D)$, to the setting of multiplier algebras on weighted Dirichlet spaces. Our techniques will closely follow those used in Banjade-Trent [BT] for the (unweighted) Dirichlet space. The new material requires the boundedness of a certain singular integral operator (Lemma 3) and the boundedness of the Beurling transform (Lemma 4) on some $L^2$ spaces with weights.

In 1962 Carleson [C] proved his famous “Corona theorem” characterizing when a finitely generated ideal in $H^\infty(D)$ is actually all of $H^\infty(D)$. Independently, Rosenblum [R], Tolokonnikov [To], and Uchiyama gave an infinite version of Carleson’s work on $H^\infty(D)$. In an effort to classify ideal membership for finitely-generated ideals in $H^\infty(D)$, Wolff [G] proved the following version:

**Theorem A (Wolff).** If

\[
\left\{ f_j \right\}_{j=1}^n \subset H^\infty(D), \ H \in H^\infty(D) \quad \text{and} \\
\left| H(z) \right| \leq \left( \sum_{j=1}^n |f_j(z)|^2 \right)^{\frac{1}{2}} \quad \text{for all} \ z \in \mathbb{D}, \quad (1)
\]

then

\[ H^3 \in \mathcal{I}(\{f_j\}_{j=1}^n), \]

the ideal generated by $\{f_j\}_{j=1}^n$ in $H^\infty(D)$.

It is known that (1) is not, in general, sufficient for $H$ itself or even for $H^2$ to be in $\mathcal{I}(\{f_j\}_{j=1}^n)$, see Rao [G] and Treil [T].

For the algebra of multipliers on Dirichlet space, the analogue of Wolff’s ideal theorem was established by the authors in [BT].

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the analogue of the corona theorem for the algebra of multipliers on weighted Dirichlet space was established in Kidane-Trent [KT], it seems plausible that Wolff-type ideal results should be extended to the algebra of multipliers on weighted Dirichlet space. This is what we intend to do in this paper.

We use $\mathcal{D}_\alpha$ to denote the weighted Dirichlet space on the unit disk, $\mathbb{D}$. That is,

$$\mathcal{D}_\alpha = \{ f : \mathbb{D} \to \mathbb{C} \mid f \text{ is analytic on } \mathbb{D} \text{ and for } f(z) = \sum_{n=0}^{\infty} a_n z^n, \|f\|_{\mathcal{D}_\alpha}^2 = \sum_{n=0}^{\infty} (n+1)^{\alpha} |a_n|^2 < \infty \}.$$ 

We will use other equivalent norms for smooth functions in $\mathcal{D}_\alpha$ as follows,

$$\|f\|_{\mathcal{D}_\alpha}^2 = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{1-\alpha} dA(z)$$

and

$$\|f\|_{\mathcal{D}_\alpha}^2 = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^{1+\alpha}} d\sigma d\sigma.$$ 

For ease of notation, we will denote $(1-|z|^2)^{1-\alpha} dA(z)$ by $dA_\alpha(z)$. Also, we will consider $\bigoplus_i \mathcal{D}_\alpha$ as an $l^2$-valued weighted Dirichlet space. The norms in this case are exactly as above but we will replace the absolute value by $l^2$-norms. Moreover, we use $\mathcal{H}\mathcal{D}_\alpha$ to denote the harmonic weighted Dirichlet space (restricted to the boundary of $\mathbb{D}$). The functions in $\mathcal{D}_\alpha$ have only vanishing negative Fourier coefficients whereas the functions in $\mathcal{H}\mathcal{D}_\alpha$ may have negative Fourier coefficients which do not vanish. Again, if $f$ is smooth on $\partial \mathbb{D}$, the boundary of the unit disk $\mathbb{D}$, then

$$\|f\|_{\mathcal{H}\mathcal{D}_\alpha}^2 = \int_{-\pi}^{\pi} |f|^2 d\sigma + \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{|f(e^{it}) - f(e^{i\theta})|^2}{|e^{it} - e^{i\theta}|^{1+\alpha}} d\sigma d\sigma.$$ 

We use $\mathcal{M}(\mathcal{D}_\alpha)$ to denote the multiplier algebra of weighted Dirichlet space, defined as: $\mathcal{M}(\mathcal{D}_\alpha) = \{ \phi \in \mathcal{D}_\alpha : \phi f \in \mathcal{D}_\alpha \text{ for all } f \in \mathcal{D}_\alpha \}$, and we will denote the multiplier algebra of harmonic weighted Dirichlet space by $\mathcal{M}(\mathcal{H}\mathcal{D}_\alpha)$, defined similarly (but only on $\partial \mathbb{D}$).
Given \( \{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_{\alpha}) \), we consider \( F(z) = (f_1(z), f_2(z), \ldots) \) for \( z \in \mathbb{D} \). We define the row operator \( M^R_F : \bigoplus_1^{\infty} \mathcal{D}_{\alpha} \to \mathcal{D}_{\alpha} \) by

\[
M^R_F \left( \{h_j\}_{j=1}^{\infty} \right) = \sum_{j=1}^{\infty} f_j h_j \quad \text{for} \quad \{h_j\}_{j=1}^{\infty} \in \bigoplus_1^{\infty} \mathcal{D}_{\alpha}.
\]

Similarly, we define the column operator \( M^C_F : \mathcal{D}_{\alpha} \to \bigoplus_1^{\infty} \mathcal{D}_{\alpha} \) by

\[
M^C_F (h) = \{f_j h\}_{j=1}^{\infty} \quad \text{for} \quad h \in \mathcal{D}_{\alpha}.
\]

We notice that \( \mathcal{D}_{\alpha} \) is a reproducing kernel (r.k.) Hilbert space with r.k.

\[
K_w(z) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^{\alpha}} (zw)^n \quad \text{for} \quad z, w \in \mathbb{D}
\]

and it is well known (see [S]) that

\[
\frac{1}{k_w(z)} = 1 - \sum_{n=1}^{\infty} c_n (zw)^n, \quad c_n > 0, \quad \text{for all} \quad n.
\]

Hence, weighted Dirichlet space has a reproducing kernel with “one positive square” or a “complete Nevanlinna-Pick” kernel. This property will be used to complete the first part of our proof.

We know that \( \mathcal{M}(\mathcal{D}_{\alpha}) \subseteq H^\infty(\mathbb{D}) \), but \( \mathcal{M}(\mathcal{D}_{\alpha}) \neq H^\infty(\mathbb{D}) \) (e.g., \( \sum_{n=1}^{\infty} \frac{2n+1}{n^{2m+1}}, m = \left\lceil \frac{1}{\alpha} \right\rceil + 1, z \in D \), is in \( H^\infty(D) \) but is not in \( \mathcal{D}_{\alpha} \) and so neither in \( \mathcal{M}(\mathcal{D}_{\alpha}) \)). Hence, \( \mathcal{M}(\mathcal{D}_{\alpha}) \not\subseteq H^\infty(\mathbb{D}) \cap \mathcal{D}_{\alpha} \).

Also, it is worthwhile to note that the pointwise hypothesis that \( F(z) F(z)^* \leq 1 \) for all \( z \in \mathbb{D} \) implies that the analytic Toeplitz operators \( T^R_F \) and \( T^C_F \) defined on \( \bigoplus_1^{\infty} H^2(\mathbb{D}) \) and \( H^2(\mathbb{D}) \), in analogy to that of \( M^R_F \) and \( M^C_F \), are bounded and

\[
\|T^R_F\| = \|T^C_F\| = \sup_{z \in \mathbb{D}} \left( \sum_{j=1}^{\infty} |f_j(z)|^2 \right)^{\frac{1}{2}} \leq 1.
\]

But, since \( \mathcal{M}(\mathcal{D}_{\alpha}) \not\subseteq H^\infty(\mathbb{D}) \), the pointwise upperbound hypothesis will not be sufficient to conclude that \( M^R_F \) and \( M^C_F \) are bounded on weighted Dirichlet space. However, \( \|M^R_F\| \leq \sqrt{10} \|M^C_F\| \). Thus, we will replace the natural normalization that \( F(z) F(z)^* \leq 1 \) for all \( z \in \mathbb{D} \) by the stronger condition that \( \|M^C_F\| \leq 1 \).

Then we have the following theorem:
Theorem 1. Let \( H, \{f_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha) \). Assume that

(a) \( \|M_F^*\| \leq 1 \)

and (b) \( |H(z)| \leq \sqrt{\sum_{j=1}^{\infty} |f_j(z)|^2} \) for all \( z \in \mathbb{D} \).

Then there exists \( K(\alpha) < \infty \) and there exists \( \{g_j\}_{j=1}^{\infty} \subset \mathcal{M}(\mathcal{D}_\alpha) \) with

\[ \|M_G^\prime\| \leq K(\alpha) \]

and \( F G^T = H^3 \).

Of course, it should be noted that for only a finite number of multipliers, \( \{f_j\} \), condition (a) of Theorem 1 can always be assumed, so we have the exact analogue of Wolff’s theorem in the finite case.

First, let’s outline the method of our proof. Assume that \( F \in \mathcal{M}_{l^2}(\mathcal{D}_\alpha) \) and \( H \in \mathcal{M}(\mathcal{D}_\alpha) \) satisfy the hypotheses (a) and (b) of Theorem 1. Then we show that there exists a constant \( K(\alpha) < \infty \), so that

\[ M_H^3 M_H^{*3} \leq K(\alpha)^2 M_F^R M_F^{*R}. \]  

(2)

Given (2), a commutant lifting theorem argument as it appears in, for example, Trent [Tr2] completes the proof by providing a \( G \in \mathcal{M}_{l^2}(\mathcal{D}_\alpha) \), so that \( \|M_G^\prime\| \leq K(\alpha) \) and \( F G^T = H^3 \).

But (2) is equivalent to the following: there exists a constant \( K(\alpha) < \infty \) so that, for any \( h \in \mathcal{D}_\alpha \), there exists \( u_h \in \bigoplus_{1}^{\infty} \mathcal{D}_\alpha \) such that

\( i \) \( M_F^R(u_h) = H^3 h \) and

\( ii \) \( \|u_h\|_{\mathcal{D}_\alpha} \leq K(\alpha) \|h\|_{\mathcal{D}_\alpha} \). \( (3) \)

Hence, our goal is to show that (3) follows from (a) and (b). For this we need a series of lemmas.

Lemma 1. Let \( \{c_j\}_{j=1}^{\infty} \in l^2 \) and \( C = (c_1, c_2, ...) \in B(l^2, \mathbb{C}) \). Then there exists \( Q \) such that the entries of \( Q \) are either 0 or \( \pm c_j \) for some \( j \) and \( CC^*I - C^*C = QQ^* \). Also, range of \( Q = \) kernel of \( C \).

We will apply this lemma in our case with \( C = F(z) \) for each \( z \in \mathbb{D} \), when \( F(z) \neq 0 \). A proof of this lemma can be found in Trent [Tr2].

Given condition (b) of Theorem 1 for all \( z \in \mathbb{D} \), \( F \in \mathcal{M}_{l^2}(\mathcal{D}_\alpha) \) and \( H \in \mathcal{M}(\mathcal{D}_\alpha) \) with \( H \) being not identically zero, we lose no generality assuming that \( H(0) \neq 0 \). If \( H(0) = 0 \), but \( H(a) \neq 0 \), let \( \beta(z) = \frac{a-z}{1-\bar{a}z} \).
for $z \in \mathbb{D}$. Then since (b) holds for all $z \in \mathbb{D}$, it holds for $\beta(z)$. So we may replace $H$ and $F$ by $H \circ \beta$ and $F \circ \beta$, respectively. If we prove our theorem for $H \circ \beta$ and $F \circ \beta$, then there exists $G \in \mathcal{M}_1(D_{\alpha})$ so that $(F \circ \beta) G = H \circ \beta$ and hence $F(G \circ \beta^{-1}) = H$ and $G \circ \beta^{-1} \in \mathcal{M}_1(D_{\alpha})$, and we are done. Thus, we may assume that $H(0) \neq 0$ in (b), so $\|F(0)\|_2 \neq 0$. This normalization will let us apply some relevant lemmas from [Tr1].

It suffices to establish (i) and (ii) for any dense set of functions in $D_{\alpha}$, so we will use polynomials. First, we will assume $F$ and $H$ are analytic on $D_{1+\epsilon}(0)$. In this case, we write the most general solution of the pointwise problem on $\overline{D}$ and find an analytic solution with uniform bounds. Then we remove the smoothness hypotheses on $F$ and $H$.

For a polynomial, $h$, we take

$$u_h(z) = F(z)^* (F(z) F(z)^*)^{-1} H^3 h - Q(z) \hat{k}(z),$$

where $\hat{k}(z) \in l^2$ for $z \in \mathbb{D}$.

We have to find $k(z)$ so that $u_h \in \bigoplus_{1}^{\infty} D_{\alpha}$. Thus we want $\partial_z u_h = 0$ in $\mathbb{D}$.

Therefore, we will try

$$u_h = \frac{F^* H^3 h}{F F^*} - Q \left( \frac{Q \star F^* H^3 h}{(F F^*)^2} \right),$$

where $\hat{k}$ is the Cauchy transform of $k$ on $\mathbb{D}$. Note that for $k$ smooth on $\overline{D}$ and $z \in \mathbb{D}$,

$$\hat{k}(z) = -\frac{1}{\pi} \int_D \frac{k(w)}{w - z} \, dA(w) \quad \text{and} \quad \overline{\partial} \hat{k}(z) = k(z) \quad \text{for} \quad z \in \mathbb{D}.$$ 

See [A] for background on the Cauchy transform.

Then it’s clear that $M^R_F(u_h) = H^3 h$ and $u_h$ is analytic. Hence, we will be done in the smooth case if we are able to find $K(\alpha) < \infty$, only depending on $\alpha$ and thus independent of the polynomial, $h$, such that

$$\|u_h\|_{D_{\alpha}} \leq K(\alpha) \|h\|_{D_{\alpha}} \quad (4)$$

**Lemma 2.** Let $w$ be a harmonic function on $\overline{D}$, then

$$\int_D \|Q'w\|_{l^2}^2 \, dA_{\alpha} \leq 8 \|w\|_{H D_{\alpha}}^2.$$ 

Proof of this lemma can be found in [BT].
Lemma 3. Let the operator $T$ be defined on $L^2(\mathbb{D}, dA_\alpha)$ by

$$(T f)(z) = \int_{\mathbb{D}} \frac{f(u)}{(u-z)(1-u\bar{z})} dA_\alpha,$$

for $z \in \mathbb{D}$ and $f \in L^2(\mathbb{D}, dA_\alpha)$. Then

$$||Tf||_{A_\alpha}^2 \leq 4\pi^2 C_\alpha^2 ||f||_{A_\alpha}^2,$$

where $C_\alpha = \frac{8}{\alpha^2}$.

Proof. To show that the singular integral operator, $T$, is bounded on $L^2(\mathbb{D}, dA_\alpha)$, we apply Zygmund’s method of rotations [Z] and apply Schur’s lemma an infinite number of times.

Let $f(z) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} z^j \bar{z}^k$, where $a_{ij} = 0$ except for a finite number of terms. For $z = re^{i\theta}$, we relabel to get

$$f(re^{i\theta}) = \sum_{l=-\infty}^{\infty} f_l(r) e^{i\theta l}, \text{ where } f_l(r) = \sum_{k=0}^{\infty} a_{l+k+k} r^{l+2k}.$$

Then

$$||f||_{A_\alpha}^2 = \sum_{l=-\infty}^{\infty} ||f_l||_{L^2[A_\alpha]}^2,$$

where the measure on $L^2[0, 1]$ is “$(1 - r^2)^{-\alpha} r dr$”.

Now computing as in [BT], we deduce that

$$(T f)(se^{it}) = 2\pi \sum_{l=-\infty}^{\infty} e^{i(l-1)t} (T_l f_l)(s),$$

for

$$(T_l f_l)(s) = \begin{cases} 
-\left( \sum_{n=0}^{l-1} s^{2n} \right) \int_0^1 \chi_{(0,s)}(r) \left( \frac{r}{s} \right)^{1-l} f_l(r) dr \\
+ \frac{1}{1-s^2} \int_0^1 \chi_{(s,1)}(r) \left( rs \right)^{1-l} f_l(r) dr & \text{for } l \leq 0 \\
\frac{1}{1-s^2} \int_0^1 \chi_{(s,1)}(r) \left( \frac{r}{s} \right)^{1-l} f_0(r) r dr & \text{for } l > 0.
\end{cases}$$

By our construction,

$$||Tf||_{A_\alpha}^2 = 4\pi^2 \sum_{l=-\infty}^{\infty} ||T_l f_l||_{L^2[A_\alpha]}^2,$$

where the measure on $L^2[0, 1]$ is “$(1 - r^2)^{1-\alpha} r dr$”. Thus, to prove our lemma it suffices to prove that

$$\sup_l ||T_l f||_{B(L^2[A_\alpha])} \leq C_\alpha < \infty.$$
To illustrate the technique, we show a detailed estimate for $\|T_0\|_{B(L_2^2[0,1])}$. The other cases follow similarly.

Now

$$
\begin{align*}
\int_0^1 \left| T_0 f_0 (se^{it}) \right|^2 (1 - s^2)^{1-\alpha} ds \\
= 2 \int_0^1 \int_0^1 f_0(u) f_0(v) \left( \int_{\max\{u,v\}}^1 \frac{(1 - s^2)^{1-\alpha} ds}{s} \right) udu vdv \\
+ 2 \int_0^1 \int_0^1 f_0(x) f_0(y) \left[ \int_0^{\min\{x,y\}} \frac{s^2(1-s^2)^{1-\alpha}}{(1-s^2)^2} ds \right] xdx ydy.
\end{align*}
$$

\textbf{Claim(I):}

$$
\begin{align*}
\int_0^1 \int_0^1 f_0(u) f_0(v) \left( \int_{\max\{u,v\}}^1 \frac{(1 - s^2)^{1-\alpha} ds}{s} \right) udu vdv \\
\leq \frac{25}{16} \int_0^1 |f_0(u)|^2 (1 - u^2)^{1-\alpha} u du.
\end{align*}
$$

We have

$$
\begin{align*}
\int_0^1 \int_0^1 f_0(u) f_0(v) \left( \int_{\max\{u,v\}}^1 \frac{(1 - s^2)^{1-\alpha} ds}{s} \right) udu vdv \\
\leq \int_0^1 \int_0^1 f_0(u) f_0(v) \left[ \frac{(1 - \max\{u^2, v^2\})^{1-\alpha}}{(1-u^2)^{1-\alpha} (1-v^2)^{1-\alpha}} \ln \left( \frac{1}{\max\{u,v\}} \right) \right] udu vdv.
\end{align*}
$$

We apply Schur’s Test with $p(u) = 1$.

$$
\begin{align*}
\int_0^u \left[ \frac{(1 - v^2)^{1-\alpha}}{(1-u^2)^{1-\alpha} (1-v^2)^{1-\alpha}} \ln \left( \frac{1}{v} \right) \right] (1 - u^2)^{1-\alpha} udu \\
= \frac{1}{2} \ln \left( \frac{1}{v^2} \right) \frac{v^2}{2} \leq \frac{1}{4}.
\end{align*}
$$

Similarly, we get

$$
\int_0^1 \left[ \frac{(1-u^2)^{1-\alpha}}{(1-u^2)^{1-\alpha} (1-v^2)^{1-\alpha}} \ln \left( \frac{1}{u} \right) \right] (1 - u^2)^{1-\alpha} udu \leq 1.
$$
Therefore,

\[
\int_0^1 \left[ \frac{(1 - \max(u^2, v^2))^{1-\alpha}}{(1 - u^2)^{1-\alpha}(1 - v^2)^{1-\alpha}} \ln \left( \frac{1}{\max\{u, v\}} \right) \right] p(u) (1 - u^2)^{1-\alpha} \, du \\
\leq \frac{5}{4} p(v).
\]

**Claim (II):**

\[
\int_0^1 \int_0^1 f_0(x) f_0(y) \left[ \int_0^{\min\{x, y\}} \frac{s^2(1-s^2)^{1-\alpha}}{(1-s^2)^2} \, ds \right] \, dx \, dy \\
\leq \frac{4}{\alpha^2} \int_0^1 |f_0(x)|^2 (1 - x^2)^{1-\alpha} \, dx.
\]

We have

\[
\int_0^1 \int_0^1 f_0(x) f_0(y) \left[ \int_0^{\min\{x, y\}} \frac{s^2(1-s^2)^{1-\alpha}}{(1-s^2)^2} \, ds \right] \, dx \, dy \\
= \int_0^1 \int_0^1 f_0(x) f_0(y) \left[ \frac{1}{2} \int_0^{\min\{x^2, y^2\}} \frac{s}{(1-s)^{1+\alpha}} \, ds \right] \, dx \, dy \\
\leq \int_0^1 \int_0^1 f_0(x) f_0(y) \left[ \frac{1}{2\alpha (1 - \min\{x^2, y^2\})^{\alpha}} \right] \, dx \, dy.
\]

For this term, we take \( p(x) = \frac{1}{(1-x^2)^{\beta}} \), where \( \beta = 1 - \frac{\alpha}{2} \). Then, calculating, we get that

\[
\int_0^y \frac{1}{2\alpha (1 - x^2)^{\alpha+\beta}} \frac{x^2}{(1 - y^2)^{1-\alpha}} \, dx \leq \frac{1}{4\alpha (\beta + \alpha - 1)} \frac{1}{(1 - y^2)^{\beta}}.
\]

Similarly,

\[
\int_y^1 \frac{1}{2\alpha (1 - y^2)^{\alpha}} \frac{y^2}{(1 - x^2)^{1-\alpha}} \, dx \leq \frac{1}{4\alpha (\beta - 1)} \frac{1}{(1 - y^2)^{\beta}}.
\]
Therefore,
\[
\int_0^1 \left[ \frac{1}{2\alpha (1 - \min \{x^2, y^2\})^\alpha (1 - x^2)^{1-\alpha} (1 - y^2)^{1-\alpha}} \right] p(x) (1 - x^2)^{1-\alpha} x dx
\]
\[
= \left( \frac{1}{4\alpha (\beta + \alpha - 1)} + \frac{1}{4\alpha (1 - \beta)} \right) p(y)
\]
\[
= \frac{1}{(4\beta + \alpha - 1) (1 - \beta)} p(y) = \frac{1}{\alpha^2} p(y).
\]
Hence,
\[
\int_0^1 |T_0 f_0(s)|^2 (1 - s^2)^{1-\alpha} s ds \leq C_{\alpha_0}^2 \int_0^1 |f_0(s)|^2 (1 - s^2)^{1-\alpha} s ds,
\]
where \( C_{\alpha_0} = \left[ \frac{5}{2} + \frac{2}{\alpha} \right] \leq \frac{5}{\alpha} \).

Applying Schur’s test for \( l > 1 \) with \( p(x) = \frac{1}{(1 - x^2)^\beta} \), \( \beta = 1 - \frac{\alpha}{2} \), we get the estimate
\[
C_i \leq \frac{5}{\alpha^2},
\]
and for \( l < 0 \) with \( p(x) = \frac{1}{(1 - x^2)^\beta} \), for each of the two terms, respectively, we get the estimate
\[
C_i \leq 6 + \frac{2}{\alpha^2},
\]
independent of \( l \). Thus we conclude that
\[
\sup \|T_i\|_{B(L^2_{A_\alpha}[0,1])} \leq \frac{8}{\alpha^2}.
\]
This finishes the proof of the Lemma.

A classical treatment of the Beurling transform can be found in Zygmund [Z]. For our purposes, we define the Beurling transform formally by
\[
\mathcal{B}(\phi) = \partial_\ast (\hat{\phi}),
\]
where \( \phi \) is in \( C^1(\mathbb{D}) \) and \( \hat{\phi} \) is the Cauchy transform of \( \phi \) on \( \mathbb{D} \).

Lemma 4. Let \( \mathcal{B} \) denote the Beurling transform. Then
\[
\|\mathcal{B}(f)\|_{A_\alpha} \leq \frac{23}{\alpha} \|f\|_{A_\alpha}, \ f \in L^2(\mathbb{D}, dA_\alpha).
\]

Proof. To show that the Beurling transform, \( \mathcal{B} \), is bounded on \( L^2(\mathbb{D}, A_\alpha) \), we again apply Zygmund’s method of rotations [Z] and apply Schur’s lemma.

As in Lemma 3, we take
\[
f(r e^{i\theta}) = \sum_{l=-\infty}^{\infty} f_l(r) e^{il\theta}, \ \text{where} \ f_l(r) = \sum_{k=0}^{\infty} a_{l+k,k} r^{l+2k}.
\]
Then
\[ \|f\|_{A_\alpha}^2 = \sum_{l=-\infty}^{\infty} \|f_l(r)\|_{L_2^2[0,1]}, \]
where the measure on \( L_2^2[0,1] \) is "\((1 - r^2)^{1-\alpha} r dr\)."

Now
\[
\hat{f}(w) = -\frac{2}{2\pi} \int_D \frac{f(z)}{z-w} dA(z)
= 2 \sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} \int_0^{2\pi} \int_0^{|w|} \frac{f_l(r) e^{i(l+n)\theta}}{w^{n+1}} r^{n+1} dr d\sigma(\theta)
- 2 \sum_{l=-\infty}^{\infty} \sum_{n=0}^{\infty} \int_0^{1} \frac{f_l(r) e^{i(l-1-n)\theta} w^n}{r^n} dr d\sigma(\theta). \tag{\star} \]

If we take \( l = 0 \) in (\star), we get that
\[ \hat{f}_0(w) = \frac{2}{w} \int_0^{|w|} f_0(r) r dr. \]

Therefore,
\[ \partial \hat{f}_0(w) = \frac{-2}{w^2} \int_0^{|w|} f_0(r) r dr + \frac{2}{w} f_0(|w|) |w| \frac{\partial(|w|)}{\partial w} \]
\[ = \frac{-2}{w^2} \int_0^{|w|} f_0(r) r dr + \frac{\overline{w}}{w} f_0(|w|), \]
since \( \overline{w} = \frac{\partial|w|^2}{\partial w} = 2|w| \frac{\partial|w|}{\partial w}, \frac{\partial|w|}{\partial w} = \frac{\overline{w}}{|w|}. \)

Thus,
\[
\mathcal{B} f_0(se^{it}) = \partial \hat{f}_0(se^{it}) = e^{-2it} \left[ \frac{-2}{s^2} \int_0^s f_0(r) r dr + f_0(s) \right].
\]

Similarly, a computation shows that
\[
\mathcal{B}(f)(se^{it}) = \sum_{l=-\infty}^{\infty} e^{i(l-2)t} \mathcal{B}_l f_l(s),
\]
for $B_l f_l(s) = \begin{cases} 
-\frac{2}{s^2} \int_0^s f_0(r) r dr + f_0(s) & \text{for } l = 0 \\
-2(l-1)s^{l-2} \int_s^1 \frac{f_1(r)}{r^{l-1}} dr - f_1(s) & \text{for } l \geq 1 \\
-2(1-l)s^{l-2} \int_0^s f_1(r)r^{1-l} dr + f_1(s) & \text{for } l < 0.
\end{cases}$

Thus,

$$||Bf||^2_{A_\alpha} = \sum_{l=-\infty}^{\infty} ||B_l f_l||^2_{L^2_\alpha[0,1]},$$

where the measure on $L^2_\alpha[0,1]$ is "$(1-r^2)^{1-\alpha} r dr$".

**Claim:**

$$\sup_l ||B_l||_{B(L^2_\alpha[0,1])} \leq \frac{23}{\alpha} < \infty.$$

Without loss of generality we may assume that $f_l(s) \geq 0$ for all $l$. For $l < 2$, applying Schur’s test with $p(u) = 1$ or $p(u) = \frac{1}{\sqrt{u}}$, we get that $||B_l||_{B(L^2[0,1])} \leq 7$. The main cases occur for $l \geq 2$. So let $l \geq 2$ be fixed. Then

$$||B_l f_l||_{L^2_\alpha[0,1]} \leq 2 \left( \int_0^1 \left| - (l-1)s^{l-2} \int_s^1 \frac{f_1(r)}{r^{l-1}} dr \right|^2 (1-s^2)^{1-\alpha} s ds \right)^{\frac{1}{2}} + ||f_l||_{L^2_\alpha[0,1]}$$

Now,

$$(l-1)^2 \int_0^1 s^{2(l-2)} \int_0^1 \chi_{(s,1)}(r) \frac{f_1(r)}{r^{l-1}} dr^2 (1-s^2)^{1-\alpha} s ds$$

$$= \int_0^1 \int_0^1 f_1(u)f_1(v) \left[ (l-1)^2 \frac{1}{u^l v^l} \int_0^{\min\{u,v\}} s^{2(l-2)} (1-s^2)^{1-\alpha} s ds \right]$$

$$\left( 1 - u^2 \right)^{1-\alpha} \left( 1 - v^2 \right)^{1-\alpha} u du v dv.$$

Applying Schur’s test with $p(u) = \frac{1}{(1-u^2)^{1-\alpha}}$, then it’s sufficient to show that

$$\int_0^1 \left[ (l-1)^2 \frac{1}{u^l} \int_0^{\min\{u,v\}} s^{2l-3} (1-s^2)^{1-\alpha} ds \right] u du \leq C_l v^l.$$
Since \((1 + s)^{1-\alpha} \leq 2\) and \(\frac{1}{2} \leq \frac{1}{(1+u)^{1-\alpha}} \leq 1\), we will be done if we are able to show

\[
\int_0^1 \left[ (l - 1)^2 \frac{1}{w^l} \int_0^{\min\{u,v\}} s^{2l-3}(1-s)^{1-\alpha} ds \right] udu \leq C_l v'.
\]

So we are trying to prove that

\[
\int_0^v \left[ (l - 1)^2 \frac{1}{w^l} \int_0^u s^{2l-3}(1-s)^{1-\alpha} ds \right] udu \leq C_l v' \quad \text{and}
\]

\[
\int_0^1 \left[ (l - 1)^2 \frac{1}{w^l} \int_0^v s^{2l-3}(1-s)^{1-\alpha} ds \right] udu \leq C_l v'.
\]

Now

\[
\int_0^v \left[ (l - 1)^2 \frac{1}{w^l} \int_0^u s^{2l-3}(1-s)^{1-\alpha} ds \right] udu = \int_0^v \left[ (l - 1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_s^v \frac{du}{u^{l-1}(1-u)^{1-\alpha}} \right] ds.
\]

Let \(t = (1-u)^{\alpha}\) and change variables. Then we get that

\[
\int_0^v \left[ (l - 1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_s^v \frac{du}{u^{l-1}(1-u)^{1-\alpha}} \right] ds
\]

\[
= \int_0^v \left[ \frac{1}{\alpha} (l - 1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_{(1-v)^{\alpha}}^{(1-s)^{\alpha}} \frac{dt}{(1-t^{\alpha})^{(l-2)+1}} \right] ds
\]

\[
= \int_0^v \left[ \frac{1}{\alpha} (l - 1)^2 s^{2l-3}(1-s)^{1-\alpha} \sum_{p=0}^{\infty} \binom{l-2+p}{p} \int_{(1-v)^{\alpha}}^{(1-s)^{\alpha}} t^{p+1} dt \right] ds
\]

\[
\leq \int_0^v \left[ \frac{1}{\alpha} (l - 1)^2 s^{2l-3}(1-s)^{1-\alpha} \sum_{p=0}^{\infty} \frac{(l-2+p)!}{(l-2)!} \frac{((1-s)^{\alpha})^{p+1}}{(p+1)^{\alpha} + 1} \right] ds
\]

\[
\leq \frac{2}{\alpha} \int_0^v \left[ (l - 1)^2 s^{2l-3}(1-s)^{1-\alpha} \sum_{q=1}^{\infty} \frac{(l-3+q)!}{(l-3)!} \frac{(1-s)^q}{(1-s)^{1-\alpha}} \right] ds
\]

\[
= \frac{2}{\alpha} \int_0^v \left[ (l - 1)^2 s^{2l-3} \left( \frac{1}{(1-(1-s))^{l-3+1}} - 1 \right) \right] ds
\]

\[
\leq \frac{2}{\alpha} \int_0^v \left[ (l - 1)^2 s^{2l-3} \left( \frac{1}{s^{l-2}} \right) \right] ds
\]

\[
\leq \frac{2}{\alpha} v'.
\]
Now consider

$$\int_v^1 \left[(l-1)^2 \frac{1}{u'} \int_0^v s^{2l-3}(1-s)^{1-\alpha} ds \right] u'du = \int_0^v \left[(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_v^1 \frac{du}{u'^{1-\alpha}} \right] ds.$$  

Again, change variables with $t = (1-u)$. So

$$\int_0^v \left[(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_v^1 \frac{du}{u'^{1-\alpha}} \right] ds = \int_0^v \left[\frac{1}{\alpha} (l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \int_0^{(1-v)^\alpha} \frac{dt}{(1-t^{1/\alpha})^{l-1}} \right] ds$$

$$= \int_0^v \left[\frac{1}{\alpha} (l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \sum_{p=0}^{\infty} \frac{(l-3+p+1)!}{(l-2)(l-3)! p!} \left[\frac{(1-v)^{p+\alpha}}{p+1}\right] \right] ds$$

$$\leq \frac{2}{\alpha} \int_0^v \left[(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \sum_{q=1}^{\infty} \frac{(l-3+q)!}{(l-3)! q!} \left[\frac{(1-v)^{\alpha}}{(1-v)^{1-\alpha}}\right] \right] ds$$

$$= \frac{2}{\alpha} \int_0^v \left[(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \left(\frac{1}{1-(1-v)^\alpha} \right) \right] ds$$

$$\leq \frac{2}{\alpha} \int_0^v \left[(l-1)^2 s^{2l-3}(1-s)^{1-\alpha} \left(\frac{1-v^{l-2}}{v^{l-2}} \right) \frac{(1-v)^\alpha}{1-v} \right] ds$$

$$= \frac{2 (l-1)}{\alpha} \int_0^v \left[(s^{2l-3} - s^{2l-2}) \left(\frac{1-v^{l-2}}{1-v} \right) \frac{1}{v^{l-2}} \right] ds$$

$$= \frac{2 (l-1)}{\alpha} \left[(\frac{v^{2l-2}}{2l-2} - \frac{v^{2l-1}}{2l-1}) \left(\frac{1-v^{l-2}}{1-v} \right) \frac{1}{v^{l-2}} \right]$$

$$= \frac{2 (l-1)}{\alpha} \left[(\frac{(1-v)}{2l-2} + v \left(\frac{1}{2l-2} - \frac{1}{2l-1}\right)) \left(\frac{1-v^{l-2}}{1-v} \right) \right]$$

$$\leq \frac{1}{\alpha} v' + \frac{2 (l-1) v^{l+1}}{\alpha} \left[\frac{1}{2(l-1)(2l-1)} \left(\frac{1-v^{l-2}}{1-v} \right) \right]$$
\[
\begin{align*}
&= \frac{1}{\alpha} v' + \frac{\alpha}{\alpha} \left[ \frac{1}{2l-1} \left( \frac{1 - v^{l-2}}{1 - v} \right) \right] \\
&\leq \frac{1}{\alpha} v' + \frac{\alpha}{\alpha} \left( \frac{l-2}{2l-1} \right) \\
&\leq \frac{2}{\alpha} v'.
\end{align*}
\]

Therefore,
\[
\int_0^1 \left[ (l-1)^2 \frac{1}{u^l} \frac{1}{v^l} \int_0^\min\{u^2,v^2\} s^{(l-2)} (1 - s)^{1-\alpha} ds \right] p(u)(1 - u^2)^{1-\alpha} u du \\
\leq \frac{4}{\alpha} p(v).
\]

We conclude that
\[
\sup_l \|B_l\|_{B(L_2^\alpha[0,1])} \leq 15 + \frac{8}{\alpha} \leq \frac{23}{\alpha}.
\]

\[\Box\]

**Lemma 5.** If \( Q \) is a multiplier of \( D_\alpha \), then
\[
(1 - |z|^2) |Q'(z)| \leq \|M_Q\|_{B(D_\alpha)} \text{ for all } z \in \mathbb{D}.
\]

**Proof.** Define \( \varphi: D \to D \) as \( \varphi(z) = \frac{Q(z)}{\|M_Q\|_{B(D_\alpha)}} \) for all \( z \in \mathbb{D} \). Now use the Schwarz lemma and the fact that \( \|\varphi\|_{\infty,\mathbb{D}} \leq \|M_\varphi\|_{B(D_\alpha)} \) to complete the proof. \[\Box\]

**Lemma 6.** If \( H \in \mathcal{M}(D_\alpha) \), then \( |H'|^2 dA_\alpha \) is a \( D_\alpha \)-Carleson measure with the constant \( 4\|M_H\|^2_{B(D_\alpha)} \).

**Proof.** To prove the lemma, we need to show that
\[
\int_D |H'|^2 |g|^2 dA_\alpha \leq 4\|M_H\|^2_{B(D_\alpha)} \|g\|^2_{D_\alpha} \text{ for all } g \in D_\alpha.
\]

Let \( g \in D_\alpha \), then
\[\Box\]
\[ \int_D |H'|^2 |g|^2 dA_\alpha = \int_D |(Hg)' - Hg'|^2 dA_\alpha \]
\[ \leq 2 \int_D |(Hg)'|^2 dA_\alpha + 2 \int_D |Hg'|^2 dA_\alpha \]
\[ \leq 2 \int_D |Hg|^2 d\sigma + 2 \int_D |(Hg)'|^2 dA_\alpha + 2 \int_D |Hg'|^2 dA_\alpha \]
\[ \leq 2 \|M_H g\|_{D_\alpha}^2 + 2 \|M_{H}\|_{B(D_\alpha)}^2 \|g\|_{D_\alpha}^2 \]
\[ \leq 4 \|M_H\|_{B(D_\alpha)}^2 \|g\|_{D_\alpha}^2. \]

□

This proves the lemma.

We are now ready to prove Theorem 1.

Proof. First, we will prove the theorem for smooth functions on \( \overline{D} \) and get a uniform bound. Then we will use a compactness argument to remove the smoothness hypothesis.

Assume that (a) and (b) of Theorem 1 hold for \( F \) and \( H \) and that \( F \) and \( H \) are analytic on \( D_{1+\epsilon}(0) \). Our main goal is to show that there exists a constant, \( K(\alpha) < \infty \), independent of \( \epsilon \), so that for any polynomial, \( h \), there exists \( u_h \in \bigoplus \mathcal{D}_\alpha \) such that \( M_{R}(u_h) = H^3 h \) and \( \|u_h\|_{D_\alpha}^2 \leq K(\alpha) \|h\|_{D_\alpha}^2 \).

We take \( u_h = F^* H^3 h - Q \left( \frac{Q^* F^* H^3 h}{(F F^*)^2} \right) \). Then \( u_h \) is analytic and \( M_{R}(u_h) = H^3 h \). We know that
\[ \|u_h\|_{D_\alpha}^2 = \int_{-\pi}^{\pi} \|u_h(e^{it})\|^2 d\sigma(t) + \int_D \|(u_h(z))'\|^2 dA_\alpha(z). \]

Condition (b) implies that
\[ \int_{-\pi}^{\pi} \left| \frac{F^* H^3 h}{FF^*} - Q \left( \frac{Q^* F^* H^3 h}{(F F^*)^2} \right) \right|^2 d\sigma(t) \leq 15 \|h\|_{D_\alpha}^2 \] (see [Tr1]).

Hence, we only need to show that
\[ \int_D \left| \left( \frac{F^* H^3 h}{FF^*} - Q \left( \frac{Q^* F^* H^3 h}{(F F^*)^2} \right) \right)' \right|^2 dA_\alpha(z) \leq K(\alpha)^2 \|h\|_{D_\alpha}^2 \]
for some \( K(\alpha) < \infty \).
Now

\[ \int_{D} \left( \frac{F^* H^3 h}{FF^*} - Q \left( \frac{Q^* F^* H^3 h}{(FF^*)^2} \right) \right)' \|^2 dA_\alpha(z) \]

\[ \leq 5 \int_{D} \left\| \frac{F^* H^3 H' h}{FF^*} \right\|^2 dA_\alpha(z) + 5 \int_{D} \left\| \frac{F^* H^3 h'}{FF^*} \right\|^2 dA_\alpha(z) \]

\[ + 5 \int_{D} \left\| \frac{F^* H^3 h'}{(FF^*)^2} \right\|^2 dA_\alpha(z) + 5 \int_{D} \left\| \frac{Q' \left( \frac{Q^* F^* H^3 h}{(FF^*)^2} \right)}{(FF^*)^2} \right\|^2 dA_\alpha(z) \]

\[ + 5 \int_{D} \left\| Q \left( \frac{Q^* F^* H^3 h}{(FF^*)^2} \right) \right\|^2 dA_\alpha(z). \]

Then

\[(a') = \int_{D} \left\| \frac{F^* 3H^2 H' h}{FF^*} \right\|^2 dA_\alpha(z) = 9 \int_{D} \left\| \frac{F^* H}{\sqrt{FF^*} \sqrt{FF^*}} H H' h \right\|^2 dA_\alpha(z) \]

\[ \leq 9 \int_{D} \left\| H' h \right\|^2 dA_\alpha(z) \]

\[ \leq 36 \left\| M_H \right\|_{B(\mathcal{D}_\alpha)} \left\| h \right\|_{\mathcal{D}_\alpha}^2 \] by Lemma 6.

\[(b') = \int_{D} \left\| \frac{F^* H^3 h'}{FF^*} \right\|^2 dA_\alpha(z) \leq \int_{D} \left\| h' \right\|^2 dA_\alpha(z) \leq \left\| h \right\|_{\mathcal{D}_\alpha}^2. \]

\[(c') = \int_{D} \left\| \frac{F^* H^3 h F^*}{(FF^*)^2} \right\|^2 dA_\alpha(z) = \int_{D} \left\| \frac{F^* F' F^* H^2}{\sqrt{FF^*} \sqrt{FF^*}} H h \right\|^2 dA_\alpha(z) \]

\[ \leq \int_{D} \left\| \frac{F^* F' F^*}{\sqrt{FF^*}} h \right\|^2 dA_\alpha(z) \]

\[ \leq \int_{D} \left\| F^* h \right\|^2 dA_\alpha(z) \leq 4 \left\| h \right\|_{\mathcal{D}_\alpha}^2. \]

We use condition (a) and Lemma 3 to estimate (c').
\[(e') = \int_D \|Q \left( \frac{Q^* F^* H^3 h}{(FF^*)^2} \right) \|^2 dA_\alpha(z)\]
\leq \int_D \| \left( \frac{Q^* F^* H^3 h}{(FF^*)^2} \right) \|^2 dA_\alpha(z) \quad (\text{since } \|Q(z)\|_{B(l^2)} \leq 1)
\leq \left( \frac{23}{\alpha} \right)^2 \int_D \| \frac{Q^* F^* H^3 h}{(FF^*)^2} \|^2 dA_\alpha(z) \quad (\text{by Lemma 4})
\leq 4 \left( \frac{23}{\alpha} \right)^2 \|h\|^2_{D_\alpha}.

So we only need estimate \((d')\). For this, we have
\[
\int_D \|Q' \left( \frac{Q^* F^* H^3 h}{(FF^*)^2} \right) \|^2 dA_\alpha(z) = \int_D \|Q' \hat{w}\|^2 dA_\alpha(z),
\]
where \(\hat{w} = \left( \frac{Q^* F^* H^3 h}{(FF^*)^2} \right)\) is a smooth function on \(\overline{D}\).

Therefore,
\[
\int_D \|Q' \hat{w}\|^2 dA_\alpha(z) \leq 2 \int_D \|Q' \hat{w} - Q' \tilde{w}\|^2 dA_\alpha(z) + 2 \int_D \|Q' \tilde{w}\|^2 dA_\alpha(z),
\]
where \(\tilde{w}(z) = \int_{-\pi}^{\pi} \frac{1 - |e^{i\theta}|^2}{|1 - e^{-i\theta}z|} \hat{w}(e^{i\theta}) d\sigma(t)\) is the harmonic extension of \(\hat{w}\) from \(\partial D\) to \(D\).

Lemma 2 tells us that
\[
\int_D \|Q' \tilde{w}\|^2 dA_\alpha(z) \leq 8 \|\tilde{w}\|^2_{H_\alpha}.
\]
Also, Lemmas 10 and 11 of [KT] imply that there is a \(C_1 < \infty\), independent of \(w\) and \(\alpha\), satisfying
\[
\|\tilde{w}\|^2_{H_\alpha} \leq C_1 \|w\|^2_{A_\alpha}.
\]
But, as we showed above
\[
\|w\|^2_{A_\alpha} = \int_D \| \frac{Q^* F^* H^3 h}{(FF^*)^2} \|^2 dA_\alpha(z) \leq \int_D \|F^* h\|^2 dA_\alpha(z) \leq 4 \|h\|^2_{D_\alpha}.
\]
Thus,
\[
\int_D \|Q' \tilde{w}\|^2 dA_\alpha(z) \leq C_2 \|h\|^2_{D_\alpha},
\]
where $C_2 < \infty$ is independent of $w$ and $\alpha$.

Now we are just left with estimating $(f')$. We have

\[
(f') = \int_D \|Q' \hat{w} - Q \tilde{w}\|^2 dA_\alpha(z)
\]

\[
= \int_D \|Q' \left[ -\frac{1}{\pi} \int_D \frac{w(u)}{u - z} dA(u) - \int_{-\pi}^{\pi} \frac{1 - |z|^2}{1 - e^{-it}z} \tilde{w}(e^{it})d\sigma(t) \right] \|^2 dA_\alpha(z)
\]

\[
= \frac{1}{\pi^2} \int_D \|Q' \int_D w(u) \left[ \frac{1}{u - z} + \frac{\bar{z}}{1 - u\bar{z}} \right] dA(u) \|^2 dA_\alpha(z)
\]

\[
= \frac{1}{\pi^2} \int_D \|Q' \int_D w(u) \left[ \frac{1 - |z|^2}{(u - z)(1 - u\bar{z})} \right] dA(u) \|^2 dA_\alpha(z)
\]

\[
= \frac{1}{\pi^2} \int_D \|Q'(z)(1 - |z|^2) T(w)(z)\|^2 dA_\alpha(z)
\]

\[
\leq \frac{\|M_Q\|^2}{\pi^2} \|T(w)\|^2_{A_\alpha} \quad \text{by Lemma 5}
\]

\[
\leq \frac{256}{\alpha^4} \|M_Q\|^2 \|w\|^2_{A_\alpha} \quad \text{by Lemma 3.}
\]

\[
\leq \frac{1024}{\alpha^4} \|M_Q\|^2 \|h\|^2_{D_\alpha}
\]

By Lemma 9 of [KT], we have $\|M_Q\|_{B(\oplus D_\alpha)} \leq \sqrt{86}$. Combining all these pieces, we see that in the smooth case

\[
\|u_h\|^2_{D_\alpha} \leq K(\alpha)^2 \|h\|^2_{D_\alpha},
\]

where $K(\alpha) = K_1\|M_H\|_{B(\oplus D_\alpha)} + \frac{K_2}{\alpha^4}$, where $K_1 < \infty$ and $K_2 < \infty$ are constants independent of $h, \varepsilon$ and $\alpha$.

By the proof of Theorem 1 in the smooth case, we have

\[
M^R_{F_r} (M^R_{F_r})^* \leq K(\alpha)^2 M^*_H, M^*_H r \quad \text{for } 0 \leq r < 1.
\]

Using a commutant lifting argument, there exists $G_r \in \mathcal{M}(D_\alpha, \oplus D_\alpha)$ so that $M^R_{F_r} M^C_{G_r} = M^R_{G_r}$ and $\|M^R_{G_r}\| \leq K(\alpha)$. Then $M^R_{F_r} \rightarrow M^R_{F_1}$ and $M_{H_r} \rightarrow M_H$ as $r \uparrow 1$ in the $\star-$strong topology.

By compactness, we may choose a net with $G^*_{G_r} \rightarrow G^*$ as $r_\alpha \rightarrow 1^-$.
\[ \mathcal{M}(\mathcal{D}_\alpha, \bigoplus_1^\infty \mathcal{D}_\alpha). \] Also, since \( F^*_r \rightarrow F^* \), we get \( M^*_r H_r = M^*_G C^* M^*_r R \rightarrow M^*_G C^* M^*_r R \) and so \( M^*_G R M^*_C = M^*_H \) with entries of \( G \) in \( \mathcal{M}(\mathcal{D}_\alpha) \) and 
\[ \|M^*_C \| \leq K(\alpha). \]

This ends our proof. \[ \square \]

**References**

[A] M. Andersson, *Topics in Complex Analysis*, Springer-Verlag, 1997.

[AM] J. Agler and J.E. McCarthy, *Pick interpolation and Hilbert spaces*, Amer. Math. Soc. 44 (2002).

[BT] D. Banjade and T. Trent, Wolff’s problem of ideals in the multiplier algebra on Dirichlet space, submitted, [arXiv:1302.5732](https://arxiv.org/abs/1302.5732).

[C] L. Carleson, *Interpolation by bounded analytic functions and the corona problem*, Annals of Math. 76 (1962), 547-559.

[G] J.B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[KT] B. Kidane and T.T. Trent, *The corona theorem for the multiplier algebra on weighted Dirichlet spaces*, Rocky Moun. J. Math. 43 (2013), 1-31.

[R] M. Rosenblum, *A corona theorem for countably many functions*, Int. Equ. Op. Theory 3 (1980), 125–137.

[S] S. Shimorin, *Complete Nevanlinna-Pick property of Dirichlet-type spaces*, J. Func. Anal. 191 (2002), 276-296.

[To] V.A. Tolokonnikov, *Estimate in Carleson’s corona theorem and infinitely generated ideals in the algebras \( H^\infty \)*, Functional Anal., Prilozhen 14 (1980), 85-86, in Russian.

[T] S.R. Treil, *Estimates in the corona theorem and ideal of \( H^\infty \): A problem of T. Wolff*, J. Anal. Math. 87 (2002), 481-495.

[Tr1] T.T. Trent, *An estimate for ideals in \( H^\infty(D) \)*, Integral Equations and Operator Theory 53 (2005), 573-587.

[Tr2] ———, *A corona theorem for the multipliers on Dirichlet space*, Integral Equations and Operator Theory 49 (2004), 123-139.

[Zy] A. Zygmund, *Integrales Singulieres*, Springer-Verlag, 1971.