Anisotropy in Bianchi-type brane cosmologies

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The behavior near the initial state of the anisotropy parameter of the arbitrary type, homogeneous and anisotropic Bianchi models is considered in the framework of the brane world cosmological models. The matter content on the brane is assumed to be an isotropic perfect cosmological fluid obeying a barotropic equation of state. To obtain the value of the anisotropy parameter at an arbitrary moment an evolution equation is derived, describing the dynamics of the anisotropy as a function of the volume scale factor of the Universe. The general solution of this equation can be obtained in an exact analytical form for the Bianchi I and V types and in a closed form for all other homogeneous and anisotropic geometries. The study of the values of the anisotropy in the limit of small times shows that for all Bianchi type space-times filled with a non-zero pressure cosmological fluid, obeying a linear barotropic equation of state, the initial singular state on the brane is isotropic. This result is obtained by assuming that in the limit of small times the asymptotic behavior of the scale factors is of Kasner-type. For brane worlds filled with dust, the initial values of the anisotropy coincide in both brane world and standard four-dimensional general relativistic cosmologies.

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I. INTRODUCTION

The idea [1] that our four-dimensional Universe might be a three-brane, embedded in a higher dimensional space-time, has attracted much attention. According to the brane-world scenario, the physical fields in our four-dimensional space-time, which are assumed to arise as fluctuations of branes in string theories, are confined to the three brane. Only gravity can freely propagate in the bulk space-time, with the gravitational self-couplings not significantly modified. This model originated from the study of a single 3-brane embedded in five dimensions, with the $5D$ metric given by $ds^2 = e^{-f(y)}g_{\mu\nu}dx^\mu dx^\nu + dy^2$, which, due to the appearance of the warp factor, could produce a large hierarchy between the scale of particle physics and gravity. Even if the fifth dimension is uncompactified, standard $4D$ gravity is reproduced on the brane. Hence this model allows the presence of large, or even infinite non-compact extra dimensions. Our brane is identified to a domain wall in a 5-dimensional anti-de Sitter space-time.

The Randall-Sundrum model was inspired by superstring theory. The ten-dimensional $E_8 \times E_8$ heterotic string theory, which contains the standard model of elementary particle, could be a promising candidate for the description of the real Universe. This theory is connected with an eleven-dimensional theory, M-theory, compactified on the orbifold $R^{10} \times S^1/Z_2$. In this model we have two separated ten-dimensional manifolds. For a review of dynamics and geometry of brane Universes see [2].

The static Randall-Sundrum solution has been extended to time-dependent solutions and their cosmological properties have been extensively studied [3]-[37]. In one of the first cosmological applications of this scenario, it was pointed out that a model with a non-compact fifth dimension is potentially viable, while the scenario which might solve the hierarchy problem predicts a contracting Universe, leading to a variety of cosmological problems [12]. By adding cosmological constants to the brane and bulk, the problem of the correct behavior of the Hubble parameter on the brane has been solved by Cline, Grojean and Servant [13]. As a result one also obtains normal expansion during nucleosynthesis, but faster than normal expansion in the very early Universe. The creation of a spherically symmetric brane-world in AdS bulk has been considered, from a quantum cosmological point of view, with the use of the Wheeler-de Witt equation, by Anchordoqui, Nunez and Olsen [14].

The effective gravitational field equations on the brane world, in which all the matter forces except gravity are confined on the 3-brane in a 5-dimensional space-time with $Z_2$-symmetry have been obtained, by using a geometric approach, by Shiromizu, Maeda and Sasaki [15, 16]. The correct signature for gravity is provided by the brane with positive tension. If the bulk space-time is exactly anti-de Sitter, generically the matter on the brane is required to be spatially homogeneous. The contraction of the 5-dimensional Weyl tensor with the normal to the brane $E_{11}$ gives the
leading order corrections to the conventional Einstein equations on the brane. The four-dimensional field equations for the induced metric and scalar field on the world-volume of a 3-brane in the five-dimensional bulk, with Einstein gravity plus a self-interacting scalar field, have been derived by Maeda and Wands \[17\].

Realistic brane-world cosmological models require the consideration of more general matter sources to describe the evolution and dynamics of the very early Universe. The influence of the bulk viscosity of the matter on the brane has been considered, for an isotropic flat Friedmann-Robertson-Walker (FRW) geometry, in \[21\] and, for a Bianchi type I geometry, in \[21\]. The first order rotational perturbations of isotropic FRW cosmological models have been studied in \[22\] and \[23\].

The general solution of the field equations for an anisotropic brane with Bianchi type I and V geometry, with perfect fluid and scalar fields as matter sources, has been obtained in \[27\]. Expanding Bianchi type I and V brane-worlds always isotropize. Anisotropic Bianchi type I brane-worlds with a pure magnetic field and a perfect fluid have also been analyzed \[32\]. Limits on the initial anisotropy induced by the 5-dimensional Kaluza-Klein graviton stresses by using the CMB anisotropies have been obtained by Barrow and Maartens \[33\]. The dynamics of a flat, isotropic brane Universe with two-component matter source: a perfect fluid with a linear barotropic equation of state and a scalar field with a power-law potential has been investigated in \[34\]. Solutions for which the scalar field energy density scales as a power-law of the scale factor (so called scaling solutions) have been obtained and their stability analysis provided.

A family of Bianchi type braneworlds with anisotropy has been constructed in \[35\], by solving the five-dimensional field equations in the bulk. The cosmological dynamics on the brane has been analyzed by also including the Weyl term, and the relation between the anisotropy on the brane and the Weyl curvature in the bulk has been discussed. In these models, it is not possible to achieve geometric anisotropy for a perfect fluid or scalar field – the junction conditions require anisotropic stresses on the brane. But in an anti-de Sitter bulk, the solutions can isotropize and approach a Friedmann type brane. Bianchi I type brane cosmologies with scalar matter self-interacting through combinations of exponential potentials have been studied in \[36\]. Such models correspond in some cases to inflationary Universes. In the brane scenario, as happens in standard four-dimensional general relativity, an increase in the number of fields assists inflation. The asymptotic behavior of Bianchi I brane worlds was considered in \[37\]. As a consequence of the nonlocal anisotropic stresses induced by the bulk, in the limit in which the mean radius goes to infinity the brane does not isotropize and the nonlocal energy does not vanish. The inflation due to the cosmological constant might be prevented by the interaction with the bulk.

The study of anisotropic homogeneous brane world cosmological models \[24\]-\[31\] has shown an important difference between these models and standard four-dimensional general relativity, namely, that brane Universes are born in an isotropic state. For Bianchi type I and V geometries this type of behavior has been found both by exactly solving the gravitational field equations \[27\], or from the qualitative analysis of the model \[26\]. A general analysis of the anisotropy in spatially homogeneous brane world cosmological models has been performed by Coley \[28\], who has shown that the initial singularity is isotropic, and hence the initial conditions problem is naturally solved. Consequently, close to the initial singularity, these models do not exhibit Mixmaster or chaotic-like behavior \[29\]. Based on the results of the study of homogeneous anisotropic cosmological models Coley \[24\] conjectured that the isotropic singularity could be a general feature of brane cosmologies. This conclusion has been contested as a result of a perturbative analysis of the dimensionless shear \[38\]. The existence of a decaying mode in scalar perturbations that grows unbounded in the past seems to suggest that anisotropy also grows unbounded in the limit of small times \[35\]. This result is based on including, through perturbations, generic inhomogeneities in the cosmological model and these are responsible for the unbounded growth of the anisotropy near the singularity. However, in a qualitative numerical study of the asymptotic dynamical evolution of spatially inhomogeneous brane-world cosmological models close to the initial singularity, Coley, He and Lim \[39\] have shown that spatially inhomogeneous $G_2$ brane cosmological models with one spatial degree of freedom always have an initial singularity, which is characterized by the fact that spatial derivatives are dynamically negligible. From the numerical analysis they have also found that there is an initial isotropic singularity in all of these spatially inhomogeneous brane cosmologies, including the physically important cases of radiation and a scalar field source. The numerical studies indicate that the singularity is isotropic for all relevant initial conditions. A similar result has been obtained by using the covariant and gauge-invariant approach for the analysis of the linear perturbations of the isotropic model $\mathcal{F}_b$, which is a past attractor in the phase space of homogeneous Bianchi models on the brane \[40\]. Therefore one can conclude that brane Universes are born with naturally built-in isotropy, contrary to standard four-dimensional general relativistic cosmology \[11\]. The observed large-scale homogeneity and isotropy of the Universe can therefore be explained as a consequence of the initial conditions.

On the other hand most of the studies regarding the behavior of the anisotropy in brane world cosmological models have been done at a qualitative level, and have not provided explicit exact representations for the anisotropy. Also many observationally important questions, like, for example, the maximum value of the anisotropy or the moment in the evolution of the Universe when this maximum occurred have not been answered yet. It is also not clear for what type of equation of state or range of parameters of the cosmological fluid on the brane the initial singularity is isotropic, or what is the effect of an inflationary phase on the initial anisotropy.
It is the purpose of the present paper to consider the evolution and dynamics of the anisotropy of homogeneous anisotropic (Bianchi type) brane world cosmological models in a systematic way. As a first step an evolution equation for the anisotropy parameter (describing differences in the time expansion of the Universe along the three principal axis) is derived. From mathematical point of view it is a separable first order differential equation for Bianchi types I and V, and a Bernoulli type equation for the other Bianchi types. By integrating the evolution equation, one can, generally, obtain the anisotropy parameter as a function of the energy density and the volume scale factor of the Universe. The study of the behavior of the anisotropy parameter near the singular state shows that generally the initial value of this parameter is dependent on the equation of state of the cosmic matter. A high density cosmological fluid obeying a barotropic equation of state starts its evolution on the brane from an isotropic geometry, while the expansion of a pressureless dust, in an anisotropic space-time, is similar to the standard four-dimensional general relativistic one. But for inflationary models the behavior of the anisotropy parameter is similar in both brane world cosmological models and standard four-dimensional general relativity.

The present paper is organized as follows. The basic equations of the brane world cosmological models are presented in Section II. The evolution of the anisotropy of Bianchi type I and V models is considered in Section III. In Section IV the anisotropy of arbitrary Bianchi type models is analyzed. Finally, in Section V, we discuss and conclude our results.

II. GRAVITATIONAL FIELD EQUATIONS IN THE BRANE WORLD MODEL

On the 5-dimensional space-time (the bulk), with the negative vacuum energy \( \Lambda_5 \) and brane energy-momentum as source of the gravitational field, the Einstein field equations are given by

\[
G_{IJ} = k_5^2 T_{IJ}, \quad T_{IJ} = -\Lambda_5 g_{IJ} + \delta(Y) \left[ -\lambda g_{IJ} + T_{IJ}^{\text{matter}} \right],
\]

(1)

In this space-time a brane is a fixed point of the \( Z_2 \) symmetry. In the following capital Latin indices run in the range 0, ..., 4, while Greek indices take the values 0, ..., 3.

Assuming a metric of the form \( ds^2 = (n_I n_J + g_{IJ}) dx^I dx^J \), with \( n_I dx^I = d\chi \) the unit normal to the \( \chi = \text{const.} \) hypersurfaces and \( g_{IJ} \) the induced metric on \( \chi = \text{const.} \) hypersurfaces, the effective four-dimensional gravitational equations on the brane (the Gauss equation), take the form \[12, 16\]:

\[
G_{\mu\nu} = -\Lambda g_{\mu\nu} + k_4^2 T_{\mu\nu} + k_5^2 S_{\mu\nu} - E_{\mu\nu},
\]

(2)

where \( S_{\mu\nu} \) is the local quadratic energy-momentum correction,

\[
S_{\mu\nu} = \frac{1}{12} T T_{\mu\nu} - \frac{1}{4} T^\alpha T_{\nu\alpha} + \frac{1}{24} g_{\mu\nu} \left( 3 T^{\alpha\beta} T_{\alpha\beta} - T^2 \right),
\]

(3)

and \( E_{\mu\nu} \) is the non-local effect from the free bulk gravitational field, the transmitted projection of the bulk Weyl tensor \( C_{IABJ} \), \( E_{IJ} = C_{IABJ} n^A n^B \), with the property \( E_{IJ} \rightarrow E_{\mu\nu} \delta_I^\mu \delta_J^\nu \) as \( \chi \rightarrow 0 \).

The four-dimensional cosmological constant, \( \Lambda \), and the coupling constant, \( k_4 \), are given by \( \Lambda = k_4^2 \left( \Lambda_5 + k_5^2 \lambda^2 / 6 \right) / 2 \) and \( k_4^2 = k_5^2 \lambda / 6 \), respectively, with \( \lambda \) the vacuum energy on the brane.

The Einstein equation in the bulk and the Codazzi equation, also imply that the conservation of the energy momentum tensor on the brane, \( D_\nu T^\mu_\nu = 0 \). Moreover, the contracted Bianchi identities on the brane imply that the projected Weyl tensor should obey the constraint \( D_\nu E^\mu_\nu = k_3^2 D_\nu S^\mu_\nu \).

For any matter fields (scalar field, perfect or dissipative fluids, kinetic gases etc.) the general form of the brane energy-momentum tensor can be covariantly given as \[12\]

\[
T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + \pi_{\mu\nu} + 2 q_{(\mu} u_{\nu)}.
\]

(4)

The decomposition is irreducible for any chosen 4-velocity \( u^\mu \). Here \( \rho \) and \( p \) are the energy density and isotropic pressure, and \( h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \) projects orthogonal to \( u^\mu \). The energy flux obeys \( q_\mu = q_{<\mu>} \), and the anisotropic stress obeys \( \pi_{\mu\nu} = \pi_{<\mu\nu>} \), where angular brackets denote the projected, symmetric and trace-free part:

\[
V_{<\mu>} = h^\nu_{<\mu>} u_\nu, \quad W_{<\mu\nu>} = \left[ h^{(\alpha}_{(\mu} h_{\nu)}^{\beta)} - \frac{1}{3} h^{\alpha\beta} h_{\mu\nu} \right] W_{\alpha\beta}.
\]

(5)

The symmetry properties of \( E_{\mu\nu} \) imply that in general we can decompose it irreducibly with respect to a chosen 4-velocity field \( u^\mu \) as \[18\]

\[
E_{\mu\nu} = -k^4 \left[ \mathcal{U} \left( u_\mu u_\nu + \frac{1}{3} h_{\mu\nu} \right) + \mathcal{P}_{\mu\nu} + 2 \mathcal{Q}_{(\mu} u_{\nu)} \right],
\]

(6)
where \( k = k_5/k_4 \), \( \mathcal{U} \) is a scalar, \( Q_\mu \) a spatial vector and \( P_{\mu \nu} \) a spatial, symmetric and trace-free tensor. For homogeneous models \( Q_\mu = 0 \) and \( P_{\mu \nu} = 0 \). Hence the only non-zero contribution from the 5-dimensional Weyl tensor from the bulk is given by the scalar or "dark radiation" term \( \mathcal{U} \).

In the following we shall consider only homogeneous and anisotropic brane geometries. In particular, those which have space-like surfaces of homogeneity (i.e. a \( G_3 \) acting simply transitively on a \( V_3 \)). The corresponding cosmological models fall into nine classes of equivalence, the so-called Bianchi models. It is useful to classify the nine types into two disjoint groups, depending on the different properties of the isometry groups (the Lie groups), called class A and class B.

At each point of the space-time on the brane we take a local orthonormal tetrad, where the metric can be written as

\[
\, ds^2 = -(\omega^0)^2 + \sum_{i=1}^{3} (\omega^i)^2,
\]

where the differential one-forms \( \omega^\mu \) will be taken as \( \omega^0 = dt \) and \( \omega^i = a_i(t)\Omega_i \), where \( \Omega_i \) are time-independent differential 1-forms and \( a_i(t) = 1, 2, 3 \) are the cosmic scale factors. The time independent 1-forms obey the relations \( d\Omega_i = -\frac{1}{2} c_{ikl} \Omega^k \wedge \Omega^l \), where the \( c_{ikl} \) are the canonical structure constants and \( \wedge \) denotes the exterior product. Although the Ansatz for the metric given by Eq. \( 7 \) is not the most general one that one can chose, it is sufficient to display all the main features of the behavior of Bianchi geometries.

With the help of the scale factors one can define the following variables: \( V = \prod_{i=1}^{3} a_i \) (volume scale factor), \( H_i = \dot{a}_i/a_i, i = 1, 2, 3 \) (directional Hubble parameters), \( H = (1/3) \sum_{i=1}^{3} H_i \) (mean Hubble parameter) and \( \Delta H_i = H_i - H, \quad i = 1, 2, 3 \). By using the definitions of \( H \) and \( V \) we immediately obtain \( H = \dot{V}/3V \), where a dot denotes the derivative with respect to the cosmological time \( t \).

According to the definition of the energy-momentum tensor on the brane, Eq. \( 4 \), in the general case of Bianchi type geometries, the symmetry of the space-time allows different spatial components of \( T_{\mu \nu} \). There are several physical processes that could generate an anisotropic energy momentum tensor, with \( T^1 \neq T^2 \neq T^3 \), like magnetic fields, heat transfer and/or viscous dissipative processes in the cosmological fluid on the brane. The most important of these processes are the bulk viscous type dissipative processes, which are the main sources of entropy generation in the early Universe. However, the effect of the bulk viscosity of the cosmological fluid on the brane can be considered by adding to the usual thermodynamic pressure \( p \) the bulk viscous pressure \( \Pi \) and formally substituting the pressure terms in the energy-momentum tensor by \( p_{eff} = p + \Pi \). Therefore the consideration of the bulk viscosity of the cosmological fluid does not lead to an anisotropic pressure distribution. The viscous dissipative anisotropic stress \( \tau_{\mu \nu} \) of the matter on the brane satisfies the evolution equation \( \tau_{2} h_{\alpha}^\mu h_{\beta}^\nu \tau_{\mu \nu} + \pi_{\alpha \beta} = -2\eta \sigma_{\alpha \beta} - \eta T \left( \tau_{2} v^\nu/2\eta T \right)_{\mu} \pi_{\alpha \beta} \), where \( \eta \) is the shear viscosity coefficient, \( \tau_{2} = 2\eta \beta_{2} \), with \( \beta_{2} \) the thermodynamic coefficient for the tensor dissipative contribution to the entropy density and \( \sigma_{\alpha \beta} \) is the shear tensor. Generally, it is assumed that the dissipative contribution from the shear viscosity in the early Universe can be neglected, \( \eta \approx 0 \). Consequently, in the followings we consider that the anisotropic stresses of the matter on the brane also vanish, \( \pi_{\mu \nu} \approx 0 \). We suppose that in Eq. \( 1 \) the heat transfer is zero, that is, we take \( q_{\mu} = 0 \). All these approximations are standard in the analysis of the physics of the very early Universe. Therefore in the following we assume that the pressure distribution of the cosmological fluid on the brane is isotropic and the fluid pressure satisfies a barotropic equation of state of the form \( p = p(\rho) \).

For any homogeneous model the conservation equations of the energy density of the matter \( \rho \) on the brane and of the dark radiation \( \mathcal{U} \) can be written as

\[
\dot{\rho} + 3(\rho + p) H = 0,
\]

\[
\dot{\mathcal{U}} + 4 H \mathcal{U} = 0,
\]

leading to a general dependence of \( \rho \) and \( \mathcal{U} \) of \( V \) of the form

\[
V = \frac{C_0}{w}, \quad \mathcal{U} = \frac{\mathcal{U}_0}{V^{4/3}},
\]

where

\[
w = \exp \left[ \int \frac{d\rho}{\rho + p(\rho)} \right]
\]

and \( C_0 \geq 0 \) and \( \mathcal{U}_0 \geq 0 \) are constants of integration.
The modified Einstein gravitational field equations on the brane can be written in the form of the standard Einstein four-dimensional field equations,

$$G_{\mu\nu} = k^2 p^{(\text{eff})}_{\mu\nu},$$

where $T^{(\text{eff})}_{\mu\nu} = -\Lambda g_{\mu\nu} / k^2 + T_{\mu\nu} + k^2 S_{\mu\nu} - (1/k^2 E_{\mu\nu})^{[18]}$. Then the effective total energy density, pressure, anisotropic stress and energy flux for a perfect fluid are $\rho^{(\text{eff})} = \Lambda / k^2 + p (1 + \rho / 2\Lambda) + (6/k^2) \mathcal{U}$, $p^{(\text{eff})} = p - \Lambda / k^2 + (\rho/2\Lambda) (\rho + 2p) + (2/k^2) \mathcal{U}$, $\sigma^{(\text{eff})}_{\mu\nu} = (6/k^2) \mathcal{P}_{\mu\nu}$ and $q^{(\text{eff})}_{\mu\nu} = (6/k^2) Q_{\mu\nu}$. Since for homogeneous cosmological models $Q_{\mu} = P_{\mu} = 0$, it follows that in the case of a perfect cosmological fluid there is a close analogy between the gravitational field equations on the brane and standard four-dimensional general relativity, with the role of the standard energy density and pressure played by $\rho^{(\text{eff})}$ and $p^{(\text{eff})}$, respectively. The formal analogy between standard four-dimensional general relativity and brane world cosmology allows the immediate extension of the Collins-Hawking definition of isotropization of a cosmological model to the case of brane Universes.

Hence, we say that a brane world cosmological model approaches isotropy if the following four conditions hold as $t \to \infty$: i) the Universe is expanding indefinitely and $H > 0$ ii) $T^{(\text{eff})00} > 0$ and $T^{(\text{eff})i0} / T^{(\text{eff})00} \to 0$, $i = 1, 2, 3$. $T^{(\text{eff})i0} / T^{(\text{eff})00}$ represents an average velocity of the matter on the brane relative to the surfaces of homogeneity. If this does not tend to zero, the Universe would not approach homogeneous or isotropic i) the anisotropy in the locally measured Hubble constant $\sigma / H$ tends to zero, $\sigma / H \to 0$ and iv) the distortion part of the metric tends to a constant. In condition iii) $\sigma^2 = (1/2) \sigma_{\mu\nu} \sigma^{\mu\nu}$ represents the shear of the normals $n_{\mu}$. For a metric of the form $ds^2 = dt^2 - \exp (2\alpha) \exp (2\beta) [\mathcal{U}^\mu \mathcal{U}^\nu + \Omega^\mu \Omega^\nu]$, where $\Omega^\mu$ are one-forms that are not exact in general, $\alpha$ is a time dependent function and $\beta$ is a symmetric traceless matrix, the shear tensor is defined as $\sigma_{\mu\nu} = [\exp (\beta)]_{\lambda\mu} [\exp (-\beta)]_{\lambda
u} + [\exp (\beta)]_{\lambda\nu} [\exp (-\beta)]_{\lambda\mu}$ [14]. For Bianchi class A and B models the shear is given by $\sigma^2 = \sigma_{\mu\nu} \sigma^{\mu\nu} / 2 = (1/2) \left( \sum_{i=1}^{3} H_i^2 - 3H^2 \right)$.

Spatially homogeneous models can be divided into three classes: those which have less than the escape velocity (i.e., those whose rate of expansion is insufficient to prevent them from recollapsing), those which have just the escape velocity and those whose rate of expansion is sufficient to prevent them from recollapsing), those which have just the escape velocity. Models of the third class do not tend, generally, to isotropy. In fact the only types which can tend toward isotropy at arbitrarily large times are types I, V, V/I0 and V/Ii. For type V/Ii there is no nonzero measure set of these models which tends to isotropy [14]. The Bianchi types that drive flat and open Universes away from isotropy in the Collins-Hawking sense are those of type V/II.

As an indicator of the degree of anisotropy of a cosmological model one can take the mean anisotropy parameter, defined according to [13]

$$A = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{\Delta H_i}{H} \right)^2.$$  

For an isotropic cosmological model $H_1 = H_2 = H_3 = H$ and $A \equiv 0$. The anisotropy parameter is an important indicator of the behavior of anisotropic cosmological models, since in standard four-dimensional general relativity it is finite even for singular states (for example, $A = 2$ for Kasner-type geometries [20]). The time evolution of $A$ is a good indicator of the dynamics of the anisotropy.

For a homogeneous brane world model filled with a perfect fluid satisfying a barotropic equation of state the conditions i), ii) and iv) of the Collins-Hawking definition of isotropization are naturally satisfied. Hence, if we consider only expanding brane worlds, $H > 0$ and condition i) holds. From the choice of the matter content, and due to the symmetries of the energy-momentum tensor, we have $T^{(\text{eff})00} = \rho^{(\text{eff})} > 0$ and $T^{(\text{eff})i0} / T^{(\text{eff})00} \equiv 0$. The choice of the geometry implies that condition iv) is also satisfied. Therefore, the value of the parameter $\sigma / H$ is the main indicator of the isotropic/anisotropic behavior of a brane world cosmological model. As one can see from Eq. (18) the quantity $\sigma / H$ is proportional to the square root of the anisotropy parameter, $\sigma / H \sim \sqrt{\Lambda}$ and so, according to the Collins-Hawking definition, if $A \to 0$, a brane world cosmological model will isotropize in the large time limit $t \to \infty$.

The formal mathematical similarity between standard four-dimensional general relativity and brane world theory can be also used to extend the Hawking-Penrose singularity theorem to brane cosmologies. From the definition of the effective energy density $\rho^{(\text{eff})}$ and of the effective pressure $p^{(\text{eff})}$ it follows that for a linear barotropic fluid with $p = (\gamma - 1)\rho$, $1 \leq \gamma \leq 2$, the effective energy-momentum tensor on the brane satisfies both the strong and weak energy conditions, which can be expressed as $(T^{(\text{eff})\mu\nu} - (1/2) g^{\mu\nu} T^{(\text{eff})}) u_{\mu} u_{\nu} \geq 0$ and $T^{(\text{eff})\mu\nu} u_{\mu} u_{\nu} \geq 0$, respectively, where $u_{\mu}$ is an arbitrary timelike four-vector. The first of these conditions implies that the sum of the local energy density and pressure is non-negative, $\rho^{(\text{eff})} + p^{(\text{eff})} \geq 0$ and $\rho^{(\text{eff})} + 3p^{(\text{eff})} \geq 0$. The second condition requires that the local energy density be non-negative in every observer’s rest frame, $\rho^{(\text{eff})} \geq 0$ and $\rho^{(\text{eff})} + 3p^{(\text{eff})} \geq 0$. Therefore the Bianchi-type brane spacetimes filled with a perfect fluid barotropic fluid are singular, since they satisfy the following conditions: a) $R_{\mu\nu} u^{\mu} u^{\nu} \geq 0$ for all timelike vectors $u^{\mu}$ b) $u_{\mu} R_{\nu\rho\lambda\epsilon} [u^{\rho} u^{\lambda} u^{\epsilon} \neq 0$ for a vector $u^{\mu}$ tangent to some geodesic c) there are no closed time-like curves and d) either i) there is a closed trapped surface or ii) there is a point $p$ for which $u^{\mu}_{\mu} < 0$ for all of the vectors $u^{\mu}$ tangent to the past light cone of $p$. [15].
III. EVOLUTION OF THE ANISOTROPY IN BIANCHI TYPE I AND V MODELS

For a better understanding of the dynamics of the anisotropy in the brane world we consider first in detail the evolution of the mean anisotropy parameter $A$ in Bianchi type I and V geometries. From a formal point of view these two geometries are described by the line element

$$ds^2 = -dt^2 + a_1^2(t)dx^2 + a_2^2(t)e^{-2\alpha x}dy^2 + a_3^2(t)e^{-2\alpha x}dz^2.$$  \hspace{1cm} (14)

The metric for the Bianchi type I geometry formally corresponds to the case $\alpha = 0$, while for the Bianchi type V case we have $\alpha = 1$.

To study the time behavior of the anisotropy parameter $A$ near the initial singular point on the brane we need only the Einstein field equations involving the time derivatives of $H_i, i = 1, 2, 3$, and which are given by

$$\frac{1}{V} \frac{d}{dt} (V H_i) = \Lambda + \frac{2 \alpha^2}{V^{2/3}} + \frac{k_i^2}{2} (\rho - p) - \frac{k_i^4}{12} \rho p + \frac{1}{3} k_i^4 \frac{U_0}{V^{4/3}}, \quad i = 1, 2, 3.$$  \hspace{1cm} (15)

For $\alpha = 0$ we obtain the ($ii$), $i \neq 0$ field equations for Bianchi type I geometry, while $\alpha = 1$ gives the Bianchi type V equations on the brane world.

By summing Eqs. (15) we find

$$\frac{1}{V} \frac{d}{dt} (V H) = \dot{H} + 3H^2 = \Lambda + \frac{2 \alpha^2}{V^{2/3}} + \frac{k_i^2}{2} (\rho - p) - \frac{k_i^4}{12} \rho p + \frac{1}{3} k_i^4 \frac{U_0}{V^{4/3}}.$$  \hspace{1cm} (16)

Subtraction of Eq. (16) from Eqs. (15) gives

$$\Delta H_i = H_i - H = \frac{K_i}{V}, \quad i = 1, 2, 3,$$  \hspace{1cm} (17)

with $K_i, i = 1, 2, 3$ constants of integration satisfying the consistency condition

$$\sum_{i=1}^{3} K_i = 0.$$  \hspace{1cm} (18)

With the use of Eq. (17) the anisotropy parameter defined in Eq. (13) becomes

$$A = \frac{K^2}{V^2 H^2} = \frac{9 K^2}{V^2},$$  \hspace{1cm} (19)

where $K^2 = (1/3) \sum_{i=1}^{3} K_i^2$. Taking the time derivative of Eq. (19) we obtain the following evolution equation for the anisotropy:

$$\frac{dA}{dt} = -2 \frac{\dot{H} + 3H^2}{H} A,$$  \hspace{1cm} (20)

or, equivalently,

$$\frac{1}{A^2} \frac{dA}{dV} = -\frac{2}{3K^2} V \left( \dot{H} + 3H^2 \right) = -\frac{2}{3K^2} V \left[ \Lambda + \frac{2 \alpha^2}{V^{2/3}} + \frac{k_i^2}{2} (\rho - p) - \frac{k_i^4}{12} \rho p + \frac{1}{3} k_i^4 \frac{U_0}{V^{4/3}} \right].$$  \hspace{1cm} (21)

Taking into account that $dp/dV = -(\rho + p)/V$, Eq. (21) can be written as

$$\frac{1}{A^2} \frac{dA}{d\rho} = \frac{2C_0^2}{3K^2} \frac{1}{(\rho + p) w^2} \left[ \Lambda + 2 \alpha^2 C_0^{-2/3} w^{2/3} + \frac{k_i^2}{2} (\rho - p) - \frac{k_i^4}{12} \rho p + \frac{1}{3} k_i^4 U_0 C_0^{-4/3} w^{4/3} \right],$$  \hspace{1cm} (22)

with the general solution given by

$$A(\rho) = -\frac{3K^2}{2C_0^2 \int \left[ \Lambda + 2 \alpha^2 C_0^{-2/3} w^{2/3} + \frac{k_i^2}{2} (\rho - p) - \frac{k_i^4}{12} \rho p + \frac{1}{3} k_i^4 U_0 C_0^{-4/3} w^{4/3} \right] (\rho + p) w^2]^{-1} d\rho - C,$$  \hspace{1cm} (23)

where $C$ is an arbitrary integration constant.
Eq. (23) gives the general representation of the anisotropy parameter as a function of the energy density of the cosmological fluid for the Bianchi type I and V space-times in the brane world scenario. For a cosmological fluid for which the thermodynamic pressure \( p \) obeys a linear barotropic equation of state of the form \( p = (\gamma - 1)\rho \), \( \gamma = \text{const.} \), \( 1 \leq \gamma \leq 2 \), we have \( \rho = \rho_0/V^\gamma \), with \( \rho_0 \geq 0 \) a constant of integration. \( \rho_0 \) can be expressed in terms of \( C_0 \) as \( \rho_0 = C_0^2 \). Hence the anisotropy equation Eq. (23) can be immediately integrated to give the general exact dependence of the anisotropy parameter on the volume scale factor for Bianchi type I and V geometries:

\[
A(V) = \frac{3K^2}{\Lambda V^2 + 3\alpha^2 V^{4/3} + k_5^2 \rho_0 V^{2-\gamma} + \frac{k_5^2}{12} \rho_0^3 V^{2-2\gamma} + k_4 \rho_0 \frac{V^{2/3}}{C + \gamma}},
\]

where the arbitrary integration constant \( C \neq 0 \) is related, via the field equations, to the constant \( K^2 \) by the relation \( C = 2K^2/3 \). The singular state at \( t = 0 \) is characterized by the condition \( V(0) = 0 \). The value of the anisotropy parameter for \( t = 0 \) depends on the equation of state of the cosmological fluid. Hence for \( 1 < \gamma \leq 2 \), from Eq. (24) it follows

\[
\lim_{V\to 0} A(V) = 0, 1 < \gamma \leq 2.
\]

Therefore the singular state of the high density Bianchi type I and V brane cosmological models is isotropic, with \( A(0) = 0 \). For the case of the pressureless dust filled anisotropic brane Universes, \( p = 0 \) and \( \gamma = 1 \). In this case

\[
\lim_{V\to 0} A(V) = \frac{36K^2}{k_5^2 \rho_0^3 + 12C}, \quad \gamma = 1.
\]

The singular state of the dust filled brane Universe is anisotropic, with \( A(0) \neq 0 \). In the case of the standard four-dimensional general relativity (SGR), the behavior of the anisotropy parameter is different from the case of brane cosmological models. SGR is recovered if the limits \( k_5 \to 0 \), \( k \to 0 \) and \( A_5 \to -\infty \) are taken simultaneously. Therefore the anisotropy parameter in standard four-dimensional general relativity \( A^{SGR} \) is given by

\[
A^{SGR}(V) = \frac{3K^2}{(\Lambda V^2 + 3\alpha^2 V^{4/3} + k_5^2 \rho_0 V^{2-\gamma} + C)}. \quad \text{(27)}
\]

Near the singular state,

\[
\lim_{V\to 0} A^{SGR}(V) = \frac{3K^2}{C} > 0, \forall \gamma \in [1, 2]. \quad \text{(28)}
\]

In fact, one can show that for barotropic matter filled standard four-dimensional general relativistic Bianchi type I and V models \( A^{SGR}(0) \leq 2 \). Therefore in the case \( \gamma = 1 \) we obtain the following relation between the initial values of the anisotropy parameters \( A \) and \( A^{SGR} \) of the brane world models and of the SGR, respectively:

\[
A(0) = \frac{A^{SGR}(0)}{1 + \frac{k_5^2 \rho_0^3}{12C}}. \quad \text{(29)}
\]

In Bianchi type I and V geometries there is also a simple proportionality relation between the shear scalar \( \sigma^2 \) and the anisotropy parameter:

\[
\sigma^2 = \frac{1}{2} \sum_{i=1}^{3} H_i^2 - 3H^2 = \frac{3}{2} A H^2. \quad \text{(30)}
\]

IV. BEHAVIOR OF ANISOTROPY IN ARBITRARY TYPE BIANCHI MODELS

For arbitrary type Bianchi geometries the \((ii), i = 1, 2, 3\) components of the gravitational field equations on the brane can be represented in the following general form:

\[
\frac{1}{V} \frac{d}{dt} (V H_i) = F_i (a_1, a_2, a_3) + \Lambda + \frac{k_5}{2} (\rho - p) - \frac{k_5^4}{12} \rho p + \frac{1}{3} \frac{k_4 \rho_0}{V^{2/3}}, \quad i = 1, 2, 3.
\]

\[
\text{(31)}
\]
TABLE I: Values of the constants $c_i, i = 1, 2, 3$, for the Bianchi type A models [44, 46].

| Bianchi type | $c_1$ | $c_2$ | $c_3$ |
|--------------|-------|-------|-------|
| I            | 0     | 0     | 0     |
| II           | 1     | 0     | 0     |
| VI$_0$       | 1     | -1    | 0     |
| VII$_0$      | 1     | 1     | 0     |
| VIII         | 1     | 1     | -1    |
| IX           | 1     | 1     | 1     |

where $F_i(a_1, a_2, a_3), i = 1, 2, 3$ are functions which depend on the Bianchi type. For the class A models [46]

$$F_1(a_1, a_2, a_3) = \frac{(c_2a_2^2 - c_3a_3^2)^2 - (c_1a_1^2)^2}{2V^2},$$

(32)

where the constants $c_i, i = 1, 2, 3$ define the Bianchi type, and are given in the table.

$F_2(a_1, a_2, a_3)$ and $F_3(a_1, a_2, a_3)$ can be obtained by a cyclic permutation of the elements in the numerator of $F_1$.

For Bianchi types V and VI$_h$

$$F_1(a_1, a_2, a_3) = -2a_2^2 + a_0q_0 - 2b^2a_2a_3^2,$$

(33)

$$F_2(a_1, a_2, a_3) = -2a_2^2 + a_0q_0 - 2b^2a_2a_3^2,$$

(34)

$$F_3(a_1, a_2, a_3) = -2a_0^2 - a_0q_0,$$

(35)

with $a_0, q_0$ and $b$ constants.

If we take $q_0 = b = 0$ we obtain Bianchi type V, for $q_0, b \neq 0$ we obtain Bianchi type VI$_h$ ($h \neq 0$), while for $q_0 = -1$ we have Bianchi type III [46].

For Bianchi types IV and VII$_h$ ($h \neq 0$) the functions $F_i$ are slightly more complicated. Thus for Bianchi type IV

$$F_1(a_1, a_2, a_3) = \frac{2}{a_1^2} + \frac{a_3^2}{2a_1^2a_2^2},$$

(36)

$$F_2(a_1, a_2, a_3) = \frac{2}{a_1^2} + \frac{a_3^2}{2a_1^2a_2^2} + \frac{a_3^2}{2a_2^2f^2},$$

(37)

$$F_3(a_1, a_2, a_3) = \frac{2}{a_1^2} - \frac{a_3^2}{2a_1^2a_2^2} + \frac{a_3^2}{2a_2^2f^2},$$

(38)

with $f$ corresponding to the off-diagonal term [47]. For Bianchi type VII$_h$ ($h \neq 0$)

$$F_1(a_1, a_2, a_3) = 2\Delta^2a_1^2a_2^2a_3^2 \left( \frac{\dot{a}_3}{a_3} - \frac{\dot{a}_1}{a_1} - \frac{\dot{a}_2}{a_2} \right),$$

(39)

$$F_2(a_1, a_2, a_3) = -F_1(a_1, a_2, a_3),$$

(40)

$$F_3(a_1, a_2, a_3) \equiv 0,$$

(41)
where $\Delta = \text{constant} \neq 0$. Generally there is one more field equation, the (00) equation, but it will not be used in the proof of the results. By adding Eqs. (31) on obtains

$$
\frac{1}{V} \frac{d}{dt} (V H) = \dot{H} + 3H^2 = F (a_1, a_2, a_3) + \Lambda + \frac{k_2^2}{2} (p - \rho) - \frac{k_4^2}{12} \rho p + \frac{1}{3} k_4^4 U_0 \frac{V}{V^{4/3}},
$$

(42)

where

$$
F (a_1, a_2, a_3) = \frac{1}{3} \sum_{i=1}^{3} F_i (a_1, a_2, a_3).
$$

(43)

Subtraction of Eq. (42) from Eq. (31) and integration of the resulting equation gives

$$
\Delta H_i = H_i - \frac{K_i}{V} + \frac{1}{3V} \int \frac{\Delta F_i [a_1 (V), a_2 (V), a_3 (V)]}{H} dV, i = 1, 2, 3,
$$

(44)

where

$$
\Delta F_i (a_1, a_2, a_3) = F_i (a_1, a_2, a_3) - F (a_1, a_2, a_3), i = 1, 2, 3,
$$

(45)

with the property

$$
\sum_{i=1}^{3} \Delta F_i (a_1, a_2, a_3) = 0,
$$

(46)

and $K_i, i = 1, 2, 3$ are constants of integration satisfying the condition $\sum_{i=1}^{3} K_i = 0$.

Therefore for an arbitrary Bianchi type geometry the anisotropy parameter can be represented in the following exact form:

$$
A = \frac{K^2 + G^2 + L}{V^2 H^2} = 9 \frac{K^2 + G^2 + L}{V^2},
$$

(47)

where

$$
K^2 = \frac{1}{3} \sum_{i=1}^{3} K_i^2,
$$

(48)

$$
G^2 = \frac{1}{27} \sum_{i=1}^{3} \left[ \int \frac{\Delta F_i}{H} dV \right]^2,
$$

(49)

and

$$
L = \frac{2}{9} \sum_{i=1}^{3} K_i \int \frac{\Delta F_i}{H} dV.
$$

(50)

Taking the time derivative of Eq. (47), and changing the time variable to $V$, it follows that in an arbitrary Bianchi type geometry the anisotropy parameter satisfies a Bernoulli type first order differential equation of the form

$$
\frac{dA}{dV} = \left[ \frac{d}{dV} \ln \left( K^2 + G^2 + L \right) \right] A - \frac{2V}{3 \left( K^2 + G^2 + L \right)} \left[ F (a_1, a_2, a_3) + \Lambda + \frac{k_2^2}{2} (p - \rho) - \frac{k_4^2}{12} \rho p + \frac{1}{3} k_4^4 U_0 \frac{V}{V^{4/3}} \right]^2,
$$

(51)

with the general solution given by

$$
A (V) = \frac{3 \left( K^2 + G^2 + L \right)}{\int V \left[ F (a_1, a_2, a_3) + \Lambda + \frac{k_2^2}{2} (p - \rho) - \frac{k_4^2}{12} \rho p + \frac{1}{3} k_4^4 U_0 \frac{V}{V^{4/3}} \right] dV + C},
$$

(52)
where $C$ is an arbitrary constant of integration.

For a brane cosmological fluid obeying a linear barotropic equation of state $p = (\gamma - 1)\rho$, the anisotropy parameter for arbitrary Bianchi type cosmological models is given by

$$A(V) = \frac{3V^{2\gamma}(K^2 + G^2 + L)}{\sum_{i=1}^{3} m_i V^{4m_i}}.$$  

We shall consider in the following the behavior of the anisotropy parameter for arbitrary type Bianchi space-times, filled with a linear barotropic cosmological fluid, obeying an equation of state of the form $p = (\gamma - 1)\rho$, $1 \leq \gamma \leq 2$. For this form of cosmological matter the conditions $\rho(\text{eff}) > 0$, $p(\text{eff}) + \rho(\text{eff}) > 0$ and $\rho(\text{eff}) + 3p(\text{eff}) > 0$ are satisfied on the brane. Therefore the corresponding cosmological models are singular [12]. Since in most Bianchi types one does not know any exact solution, in order to find the behavior of the anisotropy at early times, it is necessary to use some asymptotic solutions obtained, near the singularity, by approximate methods, in the limit of small values of the time parameter. In standard four-dimensional general relativity it is concluded that for all Bianchi types there exists a Kasner-like "vacuum phase" near the singularity, that is, in general, Einstein’s vacuum equations are the first order approximation of the equations with a nonvanishing matter term for $t \to 0$. This idea has been proposed a long time ago by Belinskii, Lifshitz and Khalatnikov [11]. The general argument comes from the consideration of a Kasner type metric $ds^2 = dt^2 - t^{\beta_1}dx^2 - t^{\beta_2}dy^2 - t^{\beta_3}dz^2$, with $p_i, i = 1, 2, 3$ constants satisfying the conditions $\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i^2 = 1$. For a linear barotropic equation of state it follows from the Bianchi identity (8) that energy density of the matter behaves like $\rho \sim t^{-\gamma(p_1+p_2+p_3)}$. Therefore for asymptotes fulfilling the Kasner constraints the matter term may be neglected in comparison with terms coming from the geometric part of the field equations and containing the second time derivatives. A Kasner type behavior near the singularity for a Bianchi type IX brane world geometry has been discussed in [28].

From Eq. (53) one can derive the behavior of the anisotropy parameter near the singular state on the brane for all Bianchi types. We suppose that the brane Universe starts its evolution from a singular state, with $a_i(0) = 0, i = 1, 2, 3$. For $t > 0$ and for an expanding geometry, the scale factors are monotonically increasing functions of time, which for small $t$ can be represented in the Kasner form

$$a_i \sim V^{m_i}, i = 1, 2, 3, \quad (54)$$

with $m_i$ constants satisfying the conditions $0 \leq m_i < 1, i = 1, 2, 3$ and $\sum_{i=1}^{3} m_i = 1$.

The existence of such a representation for the scale factors near the singular point on the brane has been explicitly proven, in the case of Bianchi type I and V geometries, in [27]. The mean Hubble parameter behaves near the singular state like $H \sim V^{-1}$.

With these assumptions it is easy to find the behavior of the integrals involving the functions $F_i$ and $G_i$. For Bianchi class A models, $\int V F[a_1(V), a_2(V), a_3(V)] dV$ can be written, near the singular point, as a sum of terms of the form $V^{4m_i}, i = 1, 2, 3$ and $V^{2m_i+2m_j}, i \neq j$:

$$\int V F[a_1(V), a_2(V), a_3(V)] dV = 3 \sum_{i=1}^{3} \alpha_i V^{4m_i} + 3 \sum_{i \neq j=1}^{3} \beta_{ij} V^{2m_i+2m_j}, \quad (55)$$

where $\alpha_i, i = 1, 2, 3$ and $\beta_{ij}, i, j = 1, 2, 3$ are constants.

Thus, for example, for the integral involving the function $F_1(a_1, a_2, a_3)$ we obtain

$$\int V F_1(a_1, a_2, a_3) dV = \frac{c_1^2}{8m_2} V^{4m_2} - \frac{c_2 c_3}{2(m_2 + m_3)} V^{2(m_2 + m_3)} + \frac{c_3^2}{8m_3} V^{4m_3} - \frac{c_1^2}{8m_1} V^{4m_1}. \quad (56)$$

For the other Bianchi types, the integral is a sum of terms of the form $V^{2-2m_i}, i = 1, 2, 3$ or $V^{2+2m_3-2m_1-2m_2}$ and cyclic permutation of this term. Due to the relation $m_1 + m_2 + m_3 = 1$ the last expression can be transformed to the form $V^{4m_i}, i = 1, 2, 3$. In the limit $V \to 0$, all these terms tend to zero.

The behavior of the function $G^2$ and $L$ can be analyzed in a similar way, and one can easily show that $\lim_{V \to 0} G^2 = \lim_{V \to 0} L = 0$. Therefore for all Bianchi type cosmological models the anisotropy parameter on the brane has the general property $\lim_{V \to 0} A = 0$. In the limit of large times, $V \to \infty$ and all Bianchi models (except type VII) isotropize, with $A \to 0$.

V. DISCUSSIONS AND FINAL REMARKS

The time evolution of the anisotropy parameter for a linear barotropic fluid in the brane world model is very different from the standard four-dimensional general relativistic case. In homogeneous and anisotropic brane world models the
The Universe starts from an isotropic state, with $A = 0$. The anisotropy of the Universe is increasing in time, and reaches a maximum value after a finite time $t_{\text{max}}$. For time intervals so that $t > t_{\text{max}}$, $A$ is a decreasing function of time which generally tends, in the large time limit, to zero. In standard four-dimensional general relativity the Universe starts its evolution from a singular state with maximum anisotropy and reaches, for all Bianchi models except type VII, an isotropic state for $t \to \infty$. For Bianchi type I and V geometries the maximum value of the anisotropy parameter is obtained from the field equations and therefore the initial behavior of $A$ is very different from the standard four-dimensional general relativistic case. In this case the anisotropy parameter has the remarkable property $\lim_{t \to 0} A(t) = \lim_{t \to \infty} A(t) = 0$, in sharp contrast to the SGR case. However, in the case of pressureless dust, the initial values of the anisotropy parameter are identical in both brane world and standard four-dimensional general relativistic cosmological models.

2$\Lambda V + 4\alpha^2 V^{1/3} + (2 - \gamma) k_2^2 \rho_0 V^{1-\gamma} + \frac{k_4^2}{6} \rho_0^2 (1 - \gamma) V^{1-2\gamma} + \frac{2}{3} k^4 U_0 V^{-1/3} = 0$, (57)

which follows from the condition $dA/dV = 0$. In the case of the Bianchi type I brane filled with a stiff cosmological fluid ($\gamma = 2$), and for a negligible small cosmological constant, $\Lambda = 0$, the maximum value of the volume scale factor is given by

$$V_{\text{max}} = \left(\frac{k_3^4 \rho_0^2}{4 k^4 U_0}\right)^{3/8} = \left(\frac{k_3^4 \rho_0^2}{4 U_0}\right)^{3/8}.$$ (58)

The maximum value of the anisotropy parameter is given by

$$A_{\text{max}} = A(V_{\text{max}}) = \frac{3K^2}{k_2^2 \rho_0 + \sqrt{\frac{k_3^4 U_0^{1/4}}{k_4^4}} + C}.$$ (59)

The value of the cosmological time for which the $A$ reaches its maximum value can be found from Eq. (60), and it is given by

$$t_{\text{max}} = \int_0^{V_{\text{max}}} \frac{xdx}{\sqrt{C x^2 + 3 k^4 U_0 x^{5/3} + k_3^4 \rho_0^2/4}}.$$ (60)

The integral in Eq. (60) cannot be expressed in a simple analytical form.

If in the early stages of evolution of a stiff ($\gamma = 2$) cosmological fluid on a Bianchi type I brane the dark radiation term can be neglected, as being negligible small, $U_0 \approx 0$, then the existence of a maximum of $A$ also requires a non-zero cosmological constant, $\Lambda \neq 0$ For $U_0 = 0$ the maximum value of the volume scale factor is given by

$$V_{\text{max}} = \left(\frac{k_3^4 \rho_0^2}{12 \Lambda}\right)^{1/4},$$ (61)

and the time necessary for the brane Universe to reach this state is

$$t_{\text{max}} = \int_0^{V_{\text{max}}} \frac{xdx}{\sqrt{3 \Lambda x^4 + C x^2 + k_3^4 \rho_0^4/4}} = \frac{1}{2 \sqrt{3} \Lambda} \ln \left(1 + \sqrt{\frac{2 k_3^4 \rho_0}{\sqrt{3} \Lambda}}\right).$$ (62)

The maximum value of the anisotropy parameter is, in this case,

$$A_{\text{max}} = A(V_{\text{max}}) = \frac{3K^2}{\rho_0 \left(k_2^2 + \sqrt{\frac{\Lambda}{k_4^4}}\right) + C}.$$ (63)

This type of behavior of $A$, with an initial monotonically increasing evolution from zero up to a maximum value, followed by a decrease to zero, specific to brane world cosmological models, is a direct consequence of the presence, in the gravitational field equations, of the term quadratic in the energy density.

In the present paper we have considered in a systematic manner the time evolution of the anisotropy parameter $A$ in the framework of homogeneous, arbitrary Bianchi type brane world cosmological models, with the matter content consisting of a perfect barotropic cosmological fluid. For Bianchi type I and V exact representations of $A$ can be obtained from the field equations and therefore the initial behavior of $A$ can be explicitly derived from the study of the exact representation near the singularity. In order to obtain the form of $A$ at the initial moment of the cosmological evolution for the other Bianchi types we have used the crucial assumption that near the singularity the metric of the brane world is of Kasner type. For a "normal" matter filled brane world, satisfying a linear barotropic equation of state, the behavior of $A$ is very different from the standard four-dimensional general relativistic case. In this case the anisotropy parameter has the remarkable property $\lim_{t \to 0} A(t) = \lim_{t \to \infty} A(t) = 0$, in sharp contrast to the SGR case. However, in the case of pressureless dust, the initial values of the anisotropy parameter are identical in both brane world and standard four-dimensional general relativistic cosmological models.
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