A SPLITTING RESULT FOR THE ALGEBRAIC $K$-THEORY OF PROJECTIVE TORIC SCHEMES

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ABSTRACT. Suppose $X$ is a projective toric scheme defined over a ring $R$ and equipped with an ample line bundle $\mathcal{L}$. We prove that its $K$-theory has a direct summand of the form $K(R)^{k+1}$ where $k \geq 0$ is minimal such that $\mathcal{L}^{\otimes (-k-1)}$ is not acyclic. Using a combinatorial description of quasi-coherent sheaves we interpret and prove this result for a ring $R$ which is either commutative, or else left noetherian.

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Let $R$ be a commutative ring with unit. An $n$-dimensional polytope $P \subseteq \mathbb{R}^n$ with integral vertices determines a projective toric $R$-scheme $X_P$ together with a family of line bundles $\mathcal{O}_{X_P}(k)$ (a family of quasi-coherent sheaves of $\mathcal{O}_{X_P}$-modules which are locally free of rank 1). To a given chain complex of quasi-coherent sheaves $\mathcal{F}^\bullet$ we associate an $R$-module chain complex $\check{\Gamma}(\mathcal{F}^\bullet)$ given by first forming a Čech complex in each chain degree of $\mathcal{F}^\bullet$, and then taking the total complex of the resulting twofold chain complex.

Set $n_P = 0$ if $P$ has integral points in its interior; otherwise, let $n_P \geq 1$ be such that the dilated polytope $(n_P + 1)P$ has lattice points in its interior, but $n_P P$ does not. This number can be characterised in different ways: $n_P \geq 0$ is minimal among non-negative integers $k$ such that $\mathcal{O}_{X_P}(-k - 1)$ is not acyclic; also, $n_P$ is the number of distinct integral roots of the Ehrhart polynomial of $P$.

We will show that there is a homotopy equivalence of $K$-theory spaces $K(X_P) \simeq K(R) \times \ldots \times K(R) \times K(X_P, [n_P])$ where $K$-theory is defined using perfect complexes of sheaves and modules, and the last factor on the right denotes the $K$-theory of those perfect complexes $\mathcal{F}^\bullet$ of quasi-coherent sheaves on $X_P$ for which $\check{\Gamma}(\mathcal{F}^\bullet(k))$ is acyclic for $0 \leq k \leq n_P$. In fact, we will prove slightly more: by exploiting a strictly combinatorial viewpoint of sheaves on toric varieties we can prove the corresponding result for a unital ring $R$ which is commutative, or else left noetherian.

A corresponding result has been proved by the author in a “non-linear” context, replacing modules by topological spaces [Hit09]. It must also be pointed out that the splitting result is, in general, far from optimal: a lot of $K$-theoretical information can be left over in the factor $K(X_P, [n_P])$. For example, if $n_P = 0$ (which can be guaranteed by first replacing $P$ by its dilate $(n + 1)P$) the splitting results merely gives a version of reduced $K$-theory. But in the other extreme, if $P$ is an $n$-dimensional simplex with volume $1/n!$ then $X_P = \mathbb{P}^n$ is $n$-dimensional projective space; one can show that $n_P = n$ and $K(\mathbb{P}^n, [n]) \simeq *$ in this case so that we recover the known splitting results for projective spaces, generalised to suitable non-commutative ground rings (Theorem 3.3.1). There are, of course, intermediate cases; suffice it to mention that 0/1-polytopes do not have interior lattice points and hence lead to non-trivial splittings with $n_P \geq 1$. Interesting examples of 0/1-polytopes arise as Stanley’s order and chain polytope associated to finite posets [Sta86].

The paper is divided into three parts. In §1 we introduce the combinatorial framework for sheaves on projective toric varieties, and give a first formulation of the main result, Theorem 1.5.1. We also prove that for a commutative ring $R$ we recover the usual notions of algebraic geometry. In
§2 we develop some algebraic geometry from the combinatorial viewpoint, allowing for a non-commutative ground ring $R$: we define twisting sheaves and study the Čech complex of various of complexes of sheaves. Of major importance is the finiteness Theorem 2.6.1 which asserts that the Čech complex of a perfect complex is a perfect complex of $R$-modules. In the left noetherian case this is quite straightforward, while the non-noetherian commutative case requires noetherian approximation (descent). Finally, §3 contains a detailed formulation of the main theorem, and its proof.

We assume some familiarity with basic homological algebra as presented by Weibel [Wei95], Waldhausen $K$-theory [Wal85] and its formulation in an algebro-geometric setting by Thomason and Trobaugh [TT90]. We mention a few conventions used in this paper. Chain complexes are topologically indexed: differentials lower the degree by 1. If needed, modules are considered as chain complexes concentrated in chain degree 0. The term “module” without any qualification refers to a left module.

1. The $K$-theory of projective toric varieties

1.1. Complexes of sheaves on projective toric varieties. We start by a combinatorial description of quasi-coherent sheaves on toric schemes defined by a polytope.

Let $P \subseteq \mathbb{R}^n$ be an $n$-dimensional polytope with integral vertices. Each non-empty face $F$ of $P$ gives rise to a cone (the set of finite linear combinations with non-negative real coefficients)

$$T^P_F = T_F = \text{cone} \{ p - f \mid p \in P, f \in F \},$$

the tangent cone of $F$, and hence to an additive monoid $S^P_F = S_F = T_F \cap \mathbb{Z}^n$.

Lemma 1.1.1. Let $F \subseteq G$ be a pair of non-empty faces of $P$. Then there exists a vector $v_{F,G} \in \mathbb{Z}^n \cap T_F$ such that

$$T_G = T_F + \mathbb{R}v_{F,G} \quad \text{and} \quad S_G = S_F + \mathbb{Z}v_{F,G}.$$

In the former case the symbol “+” denotes Minkowski sum, the set of sums of elements of the indicated cones; in the latter, the symbol “+” denotes the submonoid of $\mathbb{Z}^n$ generated by sums of elements of the indicated monoids.

Proof. This is Proposition 1.2.2 of [Ful93], applied to the dual cones $\sigma = T_F^\vee$ and $\tau = T_G^\vee$. □

Now let $R$ denote an arbitrary ring with unit. We then have monoid $R$-algebras $A^P_F = A_F = R[S_F]$ for every non-empty face $F$ of $P$; an element $v \in S_F$ gives rise to the element $x = 1 \cdot v \in A_F$. Note that $A_F$ is not commutative unless $R$ is, but that the element $x$ lies in the centre of $A_F$. Moreover, monoid generators of $S_F$ give rise to $R$-algebra generators of $A_F$, which is thus a finitely generated algebra by Gordan’s Lemma ([Ful93, 1.2.1], applied to the dual cone $\sigma = T_F^\vee$). The previous Lemma implies immediately:
Lemma 1.1.2. For $F \subseteq G$ the algebra $A_G$ is obtained from the algebra $A_F$ by localising by a single element $x_{F,G}$ in the centre of $A_F$. Consequently, for every $p \in A_G$ there is $N \geq 0$ such that $x_{F,G}^N \cdot p \in A_F$. □

Replacing $P$ by an integral dilate $kP = \{ kp \mid p \in P \}$, $k \geq 1$, and replacing the face $F$ by its dilate $kF$ does not change these cones, monoids and algebras. That is, $T^P_F = T^k_{kF}$, and similarly for $S_F$ and $A_F$.

Definition 1.1.3 (Presheaves). Let $F(P)_0$ denote the set of non-empty faces of $P$, partially ordered by inclusion. We write Ch$(R)$ for the category of (possibly unbounded) chain complexes of $R$-modules.

1. A presheaf on $P$ is a functor
   \[ Y: F(P)_0 \rightarrow \text{Ch}(R), F \mapsto Y_F \]
equipped with extra data which turns the object $Y_F$ into a chain complex of (left) $A_F$-modules, such that for each pair of non-empty faces $F \subseteq G$ of $P$ the structure map $Y_F \rightarrow Y_G$ is a map of $A_F$-module chain complexes.

2. A map of presheaves is a natural transformation of functors such that its $F$-component is an $A_F$-linear map, for each $F \in F(P)_0$. The category of presheaves is denoted by Pre$(P)$.

3. A map $f: Y \rightarrow Z$ of presheaves is called a quasi-isomorphism, or a weak equivalence, if all its components $f^F$ are quasi-isomorphisms of chain complexes of modules.

Definition 1.1.4 (Quasi-coherent sheaves). (1) An object $Y \in \text{Pre}(P)$ is called a quasi-coherent sheaf, or sheaf for short, if for each pair of non-empty faces $F \subseteq G$ of $P$, the adjoint structure map
   \[ A_G \otimes_{A_F} Y_F \rightarrow Y_G \]
is an isomorphism of $A_G$-module chain complexes.

(2) The full subcategory of Pre$(P)$ consisting of sheaves is denoted qCoh$(P)$.

Definition 1.1.5 (Homotopy sheaves). (1) A presheaf $Y \in \text{Pre}(P)$ is called a homotopy sheaf if for each pair of non-empty faces $F \subseteq G$ of $P$ the adjoint structure map
   \[ A_G \otimes_{A_F} Y_F \rightarrow Y_G \]
is a quasi-isomorphism of chain complexes.

(2) The full subcategory of Pre$(P)$ consisting of homotopy sheaves is denoted hCoh$(P)$.

Every sheaf is a homotopy sheaf, and qCoh$(P)$ is a full subcategory of hCoh$(P)$. Moreover, the notion of a homotopy sheaf is homotopy invariant in the following sense: If $f: Y \rightarrow Z$ is a weak equivalence of presheaves, then $Y$ is a homotopy sheaf if and only if $Z$ is a homotopy sheaf. This is true since $A_G$ is a localisation of $A_F$ by 1.1.2 for every pair of non-empty faces $F \subseteq G$ of $P$, so that the functor $A_G \otimes_{A_F}$ - is exact.
A chain complex of modules over some ring is called \textit{strict perfect} if it is bounded and consists of finitely generated projective modules. It is called \textit{perfect} if it is quasi-isomorphic to a strict perfect complex. In fact, a complex $C$ is perfect if and only if there exists a strict perfect complex $B$ and a quasi-isomorphism $B \xrightarrow{\cong} C$.

\textbf{Definition 1.1.6} (Perfect complexes and vector bundles). \begin{enumerate}
\item The homotopy sheaf $Y \in \text{hCoh}(P)$ is called a \textit{perfect complex} if for each $F \in F(P)_0$ the chain complex $Y^F$ is a perfect complex of $A_F$-modules.
\item The full subcategory of $\text{hCoh}(P)$ consisting of perfect complexes is denoted by $\text{Perf}(P)$.
\item A homotopy sheaf $Y \in \text{hCoh}(P)$ is called a \textit{homotopy vector bundle} if for each $F \in F(P)_0$ the chain complex $Y^F$ is a strict perfect complex of $A_F$-modules.
\item The full subcategory of $\text{hCoh}(P)$ consisting of homotopy vector bundles is denoted $\text{hVect}(P)$.
\end{enumerate}

The notion of a perfect complex is homotopy invariant in the manner described above for homotopy sheaves.

In algebraic geometry, a perfect complex can be replaced up to quasi-isomorphism by a bounded chain complex of vector bundles provided the scheme under consideration has an ample line bundle (as is the case for a projective scheme) or, more generally, has an ample family of line bundles. In the homotopy world the replacement is possible without reference to such additional structure.

\textbf{Lemma 1.1.7.} For $Y \in \text{Perf}(P)$ there is a homotopy vector bundle $Z \in \text{hVect}(P)$ together with a quasi-isomorphism $\zeta: Z \xrightarrow{\cong} Y$.

\textbf{Proof.} By definition of perfect complexes there is, for each $F \in F(P)_0$, a bounded chain complex $V^F$ of finitely generated $A_F$-modules together with a quasi-isomorphism $\nu^F: V^F \xrightarrow{\cong} Y^F$.

Let $\mathcal{P}(d)$ denote the sub-poset of $F(P)_0$ of faces of dimension at most $d$. For $F$ a vertex of $P$ define $Z^F = V^F$ and $\xi^F = \nu^F$; this defines a $\mathcal{P}(0)$-diagram $Z$ and a quasi-isomorphism $Z \xrightarrow{\nu^F} Y|_{\mathcal{P}(0)}$.

Suppose we have constructed a quasi-isomorphism of $\mathcal{P}(d - 1)$-diagrams $Z \xrightarrow{\nu^F} Y|_{\mathcal{P}(d-1)}$ such that each component of $Z$ is a strict perfect complex over the appropriate algebra. We show how to extend this data to $\mathcal{P}(d)$.

Fix a $d$-dimensional face $F$ of $P$, and let

$$L^F Z = \colim_{G \subseteq F, \dim G < d} A_F \otimes_{A_G} Z^G.$$ 

This is a strict perfect complex of $A_F$-modules. We define $L^F Y$ by a similar colimit with $Z^G$ replaced by $Y^G$. These come equipped with canonical maps $L^F Z \xrightarrow{\nu^F} L^F Y \xrightarrow{\nu^F} Y^F$ induced by the maps $\zeta^G$ defined before, and the structure maps of $Y$. Up to homotopy, the composition factors
over the quasi-isomorphism $\nu^F$ (this follows from [Wei95, 10.4.7] together
with the fact that $\nu^F$ induces an isomorphism of hom-sets $\text{hom}(L^F Z, V^F) \cong \text{hom}(L^F Z, Y^F)$ in the derived category of $A_F$). Let $Z^F$ be the mapping cylinder of $L^F Z \longrightarrow V^F$; a homotopy then determines a map $\zeta^F : Z^F \longrightarrow Y^F$
such that the two compositions

\[ L^F Z \longrightarrow Z^F \longrightarrow Y^F \quad \text{and} \quad L^F Z \longrightarrow L^F Y \longrightarrow Y^F \]

agree. Since $Z^F \cong V^F$ the map $\zeta^F$ is a quasi-isomorphism. For $G \subset F$ define

a structure map $Z^G \longrightarrow Z^F$ as the composition $Z^G \longrightarrow L^F \longrightarrow Z^F$, considered as maps of $A_G$-modules.

Performing this construction for each $d$-dimensional face of $P$ yields a $P(d)$-diagram $Z$ together with a quasi-isomorphism $\zeta : Z \longrightarrow Y|_{P(d)}$. At the $n$th step we arrive at the assertion of the Lemma.

1.2. **Algebraic $K$-theory.** The category $\text{Perf}(P)$ carries the structure of a “complicial biWaldhausen category” in the sense of Thomason and Trobaugh [TT90, 1.2.11]; as ambient abelian category we choose the category $\text{Pre}(P)$. The weak equivalences are as defined in [1.1.3]. The cofibrations are the degreewise split injections $Y \longrightarrow Z$ in $\text{Perf}(P)$ with cokernel in $\text{Perf}(P)$.

**Definition 1.2.1.** The *algebraic $K$-theory of $P$* is defined to be the $K$-theory space of the complicial biWaldhausen category $\text{Perf}(P)$. In symbols,

\[ K(P) = \Omega |wS\text{Perf}(P)| \]

where the symbol “$w$” denotes the subcategory of weak equivalences as usual.

1.3. **Justification of terminology.** Let now $R$ be a commutative ring with unit. The polytope $P$ determines a projective $R$-scheme $X_P$, obtained from the affine schemes $U_F = \text{Spec } A_F$ by gluing $U_F$ and $U_G$ along their common open subscheme $U_{F \lor G}$. A chain complex $F$ of quasi-coherent $O_{X_P}$-modules gives rise, by evaluation on open sets $U_F$, to a sheaf

\[ Y_F : F \mapsto \Gamma(U_F, F) \]  

(1.3.0.1)

as defined in [1.1.3]. The categories of chain complexes of quasi-coherent $O_{X_P}$-modules and of perfect complexes of quasi-coherent $O_{X_P}$-modules in the sense of [TT90, 2.2.10] are equivalent, via this construction, to the categories $\text{qCoh}(P)$ and $\text{Perf}(P) \cap \text{qCoh}(P)$, respectively.

**Lemma 1.3.1.** Every homotopy sheaf $Y \in \text{hCoh}(P)$ can be functorially replaced by a chain complex of quasi-coherent $O_{X_P}$-modules $F$ in such a way that $\Gamma(U_F, F)$ and $Y^F$ are quasi-isomorphic. More precisely, there exists a homotopy sheaf $\tilde{Y} \in \text{hCoh}(P)$, and there exist maps $Y_F \longrightarrow \tilde{Y} \longrightarrow Y$, cf. (1.3.0.1), which restrict to quasi-isomorphisms of chain complexes on $F$-components for every $F \in F(P)_0$. Moreover, this data can be chosen to depend on $Y$ in a functorial manner.
Proof. This is the content of [Hüt10, 4.4.1]. In short, the homotopy sheaf $\tilde{Y}$ is a fibrant replacement of $Y$ with respect to a suitable model structure on $\text{Pre}(P)$ (the replacement can be chosen functorially in $Y$), and $F$ is the limit of the $F(P)_0$-diagram of quasi-coherent $\mathcal{O}_{X_P}$-modules $F \mapsto j^*_F(\tilde{Y}_F)$. Here $j^F$ is the inclusion $U_F \subset X_P$, $j^*_F$ is push-forward along $j$, and $\tilde{Y}_F$ is the the chain complex of quasi-coherent $\mathcal{O}_{U_F}$-modules associated to the complex of modules $Y_F$. □

For $Y \in \text{Perf}(P)$ the chain complex $F$ of Lemma 1.3.1 is a perfect complex in the sense of [TT90, 2.2.10]. Conversely, every perfect complex $F$ of quasi-coherent $\mathcal{O}_{X_P}$-modules gives rise to an object $Y_F: F \mapsto \Gamma(U_F, F)$ of $\text{Perf}(P)$, by [TT90, 2.4.3] applied to the affine schemes $U_F$.

Definition 1.3.2. The algebraic $K$-theory $K(X_P)$ of $X_P$ is the algebraic $K$-theory space of the complicial bi-Waldhausen category of perfect complexes of quasi-coherent $\mathcal{O}_{X_P}$-modules, equipped with weak equivalences the quasi-isomorphisms, and cofibrations the degreewise split monomorphisms with cokernel a perfect complex.

This is the “right” definition by [TT90, 3.6] since the scheme $X_P$ is quasi-compact (a finite union of affine schemes) and semi-separated (with open affine subschemes $U_F$ as semi-separating cover).

Proposition 1.3.3. The algebraic $K$-theory space $K(X_P)$ of $X_P$ is homotopy equivalent to $K(P)$.

Proof. The functor sending a perfect complex $F$ to its associated object $Y_F$ of $\text{Perf}(P)$,

$$Y_F: F \mapsto \Gamma(U_F, F)$$

is exact and induces, in view of Lemma 1.3.1 an equivalence of derived categories. By [TT90, 1.9.8] this implies that $K(X_P) \simeq K(P)$ as claimed. □

1.4. Lattice points and Ehrhart polynomials. We need to introduce a classical result on counting lattice points in polytopes. Let $P \subseteq \mathbb{R}^n$ be an $n$-dimensional polytope with integral vertices. For a non-negative integer $k$, let $N_P(k)$ denote the number of integral points in the dilation $kP$ of $P$. Since $0P = \{0\} \subset \mathbb{R}^n$ we have $N_P(0) = 1$.

Theorem 1.4.1 (Ehrhart Theorem). There is a unique polynomial $E_P(x)$ with rational coefficients, called the EHRHART polynomial of $P$, such that $E_P(k) = N_P(k)$ for all non-negative integers $k$. The polynomial $E_P(x)$ has degree $n$, constant term 1 and leading coefficient $\text{vol}(P)$ (with volume normalised so that $\text{vol}([0,1]^n) = 1$). Moreover, if $j$ is a negative integer then $(-1)^n E_P(j)$ is the number of integral points in the interior of the dilation $jP$.

The reader can find a proof of this remarkable theorem in [MS05, 12.16] or [Bar08, §18]. — From the geometric meaning of the EHRHART polynomial we deduce:
Corollary 1.4.2. All integral zeros of \( E_P(x) \) are negative, and the set of integral zeros of \( E_P(x) \) is of the form \( \{-n_P, -n_P + 1, \ldots, -1\} \) for some integer \( n_P \in \mathbb{Z} \). The number \( n_P \) is minimal among integers \( k \geq 0 \) such that \( -(k+1)P \) has integral points in its interior.

1.5. The main result. We can now formulate a preliminary version of the main result of this paper.

Theorem 1.5.1. Let \( P \subseteq \mathbb{R}^n \) be an \( n \)-dimensional polytope with integral vertices, and let \( n_P \) denote the number of distinct integral roots of its Ehrhart polynomial. Let \( R \) be a ring with unit. Suppose that \( R \) is commutative, or else left noetherian. Then there is a homotopy equivalence of \( K \)-theory spaces

\[
K(P) \simeq K(R)^{n+1} \times K(P, [n_P])
\]

where \( K(R) \) denotes the \( K \)-theory space of the ring \( R \), defined using perfect complexes of \( R \)-modules, and \( K(P, [n_P]) \) denotes the \( K \)-theory space of a certain subcategory of the category \( \text{Perf}(P) \).

In view of Proposition 1.3.3, this implies a splitting result for the algebraic \( K \)-theory of projective toric \( R \)-schemes provided \( R \) is a commutative ring with unit.

The proof of Theorem 1.5.1 will be given in §3.2 and will contain explicit descriptions of the homotopy equivalence and of \( K(P, [n_P]) \).

2. Čech cohomology

2.1. Double complexes and the spectral sequence argument. We work over an arbitrary unital ring \( R \). Let \( D_{*,*} \) be a double complex of \( R \)-modules; that is, we are given \( R \)-modules \( D_{p,q} \) for \( p, q \in \mathbb{Z} \), “horizontal” and “vertical” differentials

\[
\partial^h : D_{p,q} \rightarrow D_{p-1,q} \quad \text{and} \quad \partial^v : D_{p,q} \rightarrow D_{p,q-1}
\]

with \( \partial^h \circ \partial^h = 0 \) and \( \partial^v \circ \partial^v = 0 \), such that

\[
\partial^h \circ \partial^v = -\partial^v \circ \partial^h.
\]

Its total complex \( \text{Tot} D_{*,*} \) is a chain complex with \( \text{Tot}(D_{*,*})_n = \bigoplus_{p+q=n} D_{p,q} \) and differential \( \partial = \partial^h + \partial^v \), cf. [Wei95] 1.2.6.

Let \( D'_{*,*} \) be another double complex of \( R \)-modules. A map of double complexes \( f : D_{*,*} \rightarrow D'_{*,*} \) is a collection of \( R \)-linear maps \( D_{p,q} \rightarrow D'_{p,q} \) which commute with vertical and horizontal differentials.

Proposition 2.1.1 (The spectral sequence argument). Let \( D_{*,*} \) be a double complex of \( R \)-modules. Suppose that \( D_{*,*} \) is concentrated in the first \( n+1 \) columns so that \( D_{p,*} = 0 \) if \( p < 0 \) or \( p > n \), or that \( D_{*,*} \) is concentrated in the first \( n+1 \) rows so that \( D_{*,q} = 0 \) if \( q < 0 \) or \( q > n \).
(1) There is a convergent spectral sequence
\[ E^1_{p,q} = H^h_q(D_{s,t}) \Rightarrow H^{p+q}_\text{Tot} D_{s,t} \]  
(2.1.1.1)

with \( E^2_{p,q} = H^h_p H^v_q(D_{s,t}) \). Here \( H^h \) denotes taking homology modules with respect to the horizontal differential \( \partial^h \), and \( H^v \) denotes taking homology modules with respect to the vertical differential \( \partial^v \).

(2) Suppose that \( D'_{s,t} \) is another double complex of R-modules concentrated in the first \( n+1 \) columns or rows. Suppose that the map of double complexes \( f: D_{s,t} \longrightarrow D'_{s,t} \) induces an isomorphism on horizontal homology modules. Then \( f \) induces a quasi-isomorphism \( \text{Tot}(f): \text{Tot} D_{s,t} \longrightarrow \text{Tot} D'_{s,t} \).

Proof. The spectral sequence in (1) arises in the standard way from a filtration of \( \text{Tot} D_{s,t} \) by the rows of \( D_{s,t} \), see [Wei95, §5.6] for details of the construction, and [Wei95, 5.2.5] for convergence. The result of (2) now follows from convergence of the bounded spectral sequences for \( D_{s,t} \) and \( D'_{s,t} \), together with the fact that by hypothesis \( f \) induces an isomorphism of spectral sequences on the \( E^1 \)-term. \( \square \)

2.2. Čech cohomology. Assume now that we have oriented the faces of \( P \) so that we have incidence numbers \([F:G]\) in \( \{-1, 0, 1\} \) at our disposal.

Definition 2.2.1. (1) Given a diagram \( A: F(P)_0 \longrightarrow R\text{-Mod} \) we define its Čech complex to be the bounded chain complex \( \check{\Gamma}(A) = \check{\Gamma}(A) \) given by
\[ \check{\Gamma}(A)_s := \bigoplus_{\dim F=n-s} A^F, \]
the sum extending over all non-empty faces of \( P \), with differentials given by \( A^G \nleftrightarrow_{[F:G]} A^F \) for the pair \( F \subseteq G \) of non-empty faces of \( P \).

(2) Let \( Y: F(P)_0 \longrightarrow \text{Ch}(R) \) be a diagram of chain complexes of \( R \)-modules. We define the Čech complex \( \check{\Gamma}(Y) = \check{\Gamma}(Y) \) of \( Y \) to be the total complex of the double chain complex of \( R \)-modules
\[ D_{s,t}(Y) = D_{s,t} = \bigoplus_{\dim F=n-s} Y^F_t \]  
(2.2.1.1)

with horizontal differentials given by \( Y^G_t \nleftrightarrow_{[F:G]} Y^F_t \) and vertical differential given by the differential in \( Y \) multiplied by the sign \((-1)^s\), cf. [Wei95, 1.2.5].

(3) Any presheaf \( Y \in \text{Pre}(P) \) can be considered as a diagram of chain complexes of \( R \)-modules, and we define its Čech complex \( \check{\Gamma}(Y) = \check{\Gamma}(Y) \) as in (2).

Remark 2.2.2. (1) The homology modules of \( \check{\Gamma}(A) \) in \( \check{\Gamma}(A) \) are isomorphic to higher derived inverse limits of the diagram \( A \); more precisely, \( \lim^k(A) \cong H_{n-k}\check{\Gamma}(A) \). See [H{"u}t04, 2.19] for a proof.
(2) If $A: F(P)_0 \rightarrow R$-Mod is a constant diagram with value $A^F = M$ for all $F$, then $H_n \Gamma(A) = M$ and $H_k \Gamma(A) = 0$ for $k \neq n$. This follows easily from (1), or from the observation that $\Gamma(A)$ is dual to the chain complex computing cellular homology of the polytope $P$ with coefficients in $M$, up to re-indexing.

(3) If $R$ is a commutative ring and $Y \in \text{qCoh}(P)$ is concentrated in chain degree 0, then $Y$ determines a quasi-coherent sheaf $F$ on the scheme $X_P$. By [Hüt04, 2.18] we have isomorphisms $H_{n-k} \Gamma(Y) \cong H^k(X_P, F)$, the $k$th cohomology module of $X_P$ with coefficients in $F$.

(4) For the diagram $Y$ in 2.2.1 (2) $D^* \star$ is concentrated in the first $n+1$ columns so that $D^* p, \star = 0$ if $p < 0$ or $p > n$. We have $D^* t = \Gamma(Y_t)$ which is a chain complex computing $\lim_{\rightarrow} Y_t$. The double chain complex $D^* \star$ thus gives rise to a convergent (homological) spectral sequence

$$E^1_{s,t}(Y) = \lim_{\rightarrow} \dim_{F} H_q(Y^F) \Rightarrow H_{s+t} \Gamma(Y),$$

(2.2.2.1)
cf. Proposition 2.1.1 (1).

Remark 2.2.3. There is another standard spectral sequence which we will have occasion to use. Let $Y \in \text{Pre}(P)$, or more generally, let $Y$ be a diagram $F(P)_0 \rightarrow \text{Ch}(R)$. Filtration by columns yields a convergent $E^1$-spectral sequence

$$E^1_{p,q}(Y) = \bigoplus_{\dim F = n-p} H_q(Y^F) \Rightarrow H_{p+q} \Gamma(Y),$$

cf. [Wei95, 5.6.1]; by Remark 2.2.2 (1), $E^2_{p,q}(Y) = \lim_{\rightarrow} H_{n-p} H_q(Y)$.

Proposition 2.2.4. Formation of Čech complexes is homotopy invariant. More precisely, let $f: Y \rightarrow Z$ be a map of $F(P)_0$-diagrams of $R$-module chain complexes. Suppose that for each $F \in F(P)_0$ the $F$-component of $f$ is a quasi-isomorphism. Then $f$ induces a quasi-isomorphism $\Gamma(Y) \rightarrow \Gamma(Z)$.

Proof. Consider the spectral sequence 2.2.3 for $Y$ and $Z$. By hypothesis, the map $f$ induces an isomorphism of $E^1$-spectral sequences $E^1_{*,*}(Y) \cong E^1_{*,*}(Z)$, hence induces an isomorphism of their abutments. But this is just a reformulation of the claim. □

2.3. Line bundles determined by $P$. The polytope $P$ determines a family of objects of $\text{qCoh}(P)$ as follows. For $k \in \mathbb{Z}$ we define

$$\mathcal{O}(k): F \mapsto \mathcal{O}(k)^F = R[kF + T_F \cap Z^n],$$

considered as a diagram of chain complexes concentrated in degree 0. Here $T_F$ is the tangent cone of $P$ at $F$, and $kF + T_F = \{kf + v \mid f \in F, v \in T_F \}$ is the Minkowski sum of the dilation $kF$ of $F$ and the cone $T_F$. The symbol $R[S]$ means the free $R$-module with basis $S$. 

It is not difficult to see that $\mathcal{O}(k)$ is an object of $\text{qCoh}(P)$; the $A_F$-module structure of $\mathcal{O}(k)^F$ is induced by the translation action of the monoid $T_F \cap \mathbb{Z}^n$ on the set of integral points in $kF + T_F$. In fact, $\mathcal{O}(k)^F$ is a free $A_F$-module of rank 1.

**Proposition 2.3.1.** For $k \in \mathbb{Z}$ and $j \in \mathbb{N}$ there is an isomorphism

$$H^j_{\mathbb{I}}(\mathcal{O}(k)) \cong \begin{cases} R[kP \cap \mathbb{Z}^n] & \text{if } j = n \text{ and } k \geq 0, \\ R[(\text{int } kP) \cap \mathbb{Z}^n] & \text{if } j = 0 \text{ and } k < 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** For $R = \mathbb{C}$ this is the standard calculation of cohomology of the line bundles $\mathcal{O}(k)$ on the toric variety $X_P$. Details are contained, for example, in [Hüt09, 2.5.3]; the calculation given there remains valid for arbitrary rings with unit. □

It follows from Theorem 1.4.1 that the number $n_P$ defined in Corollary 1.4.2 can be characterised as being minimal among those $k \geq 0$ for which $\mathcal{O}(-k - 1)$ is not acyclic (i.e., $\mathbb{I}\mathcal{O}(-k - 1)$ is not acyclic).

**2.4. Twisting sheaves.**

**Definition 2.4.1.** Let $Y \in \text{Pre}(P)$ and $k \in \mathbb{Z}$. We define the $k$th twist of $Y$, denoted $Y(k)$, as the objectwise tensor product of $\mathcal{O}(k)$ and $Y$. Explicitly,

$$Y(k) : F \mapsto Y(k)^F = \mathcal{O}(k)^F \otimes_{A_F} Y^F,$$

with structure maps induced by those of $\mathcal{O}(k)$ and $Y$.

By definition $Y(k)^F$ is isomorphic to the tensor product of $Y^F$ with a free $A_F$-module of rank 1, hence $Y(k)^F$ is non-canonically isomorphic to $Y^F$. The following properties are easily verified:

**Lemma 2.4.2.**

1. For $Y \in \text{Pre}(P)$ and $k, \ell \in \mathbb{Z}$ there is an isomorphism $Y(k)(\ell) \cong Y(k + \ell)$. Moreover $Y(0) \cong Y$, this last isomorphism being natural in $Y$.
2. If $Y \in \text{qCoh}(P)$ then $Y(k) \in \text{qCoh}(P)$ for every $k \in \mathbb{Z}$.
3. If $Y \in \text{hCoh}(P)$ then $Y(k) \in \text{hCoh}(P)$ for every $k \in \mathbb{Z}$.
4. If $Y \in \text{Perf}(P)$ then $Y(k) \in \text{Perf}(P)$ for every $k \in \mathbb{Z}$. □

**2.5. Quasi-coherent functors.**

**Definition 2.5.1.** A quasi-coherent functor $Y$ is an object $Y \in \text{qCoh}(P)$ which is concentrated in chain degree 0.

**Lemma 2.5.2.** Let $Y$ be a quasi-coherent functor, let $F \in F(P)_0$, and let $s_F \in Y^F$. Then there exists $k \in \mathbb{Z}$ and a map $f : \mathcal{O}(k) \longrightarrow Y$ such that $s_F$ is in the image of the $F$-component of $f$. 
Proof. The proof is a translation of the corresponding algebro-geometric fact [Har77] II.5.14 (b)] into combinatorial language. To begin with, we may assume that each face of $P$ has a lattice point in its relative interior. Indeed, we can replace $P$ by its dilate $(n+1)P$; note that this does not change the poset $F(P)_0$, nor does it change the cones, monoids and algebras constructed from $P$. Twisting translates easily: the sheaf $O(1)$ computed with respect to $(n+1)P$ is precisely the sheaf $O(n+1)$ computed with respect to $P$.

All algebras and modules constructed from $P$ are $R$-submodules of the free $R$-module $R[\mathbb{Z}^n]$. An element $v_F \in \mathbb{Z}^n$ gives rise to an element $x_F = 1 \cdot v_F \in R[\mathbb{Z}^n]$. We think of the $x$-symbols as multiplicative, that is, we write $x_F/x_G$ for the module element associated to the vector $v_F - v_G$.

Now choose, for each $F \in F(P)_0$, a lattice point $v_F$ in the relative interior of $F$. It is not difficult to see that $O(N)^G = R[NG + T_G] \cap \mathbb{Z}^n$, considered as a free $R$-module, has basis given by $Nv_G + S_G \subseteq \mathbb{Z}^n$ so that

$$O(N)^G = x_G^N A_G , \; N \in \mathbb{Z}.$$ 

Moreover, if $p \in O(N)^G$ then $x_F^M p \in O(M+N)^G$ for all $M \geq 0$. In fact, $x_F^M p$ is the image of $x_F^M \otimes p$ under the isomorphism $O(M) \otimes O(N) \cong O(M+N)$ restricted to $G$-components.

As notational convention, given an element $x \in \mathbb{Z}^G$ for a coherent functor $Z$ we call the image $x|_H$ of $x$ under the structure map $Z^G \rightarrow Z^H$ the restriction of $x$ to $H$.

Let us now start with the actual proof. By definition of tangent cones $v_F - v_G \in T_G$ for each $G \in F(P)_0$ so that $x_F/x_G \in A_G$. One can show the following:

**2.5.2.1.** For each $G \in F(P)_0$ we have

$$T_{F \vee G} = T_G + R(v_F - v_G) \quad \text{and} \quad S_{F \vee G} = S_G + Z(v_F - v_G)$$

where $F \vee G$ is the join of $F$ and $G$, that is, the smallest face of $P$ containing $F \cup G$. Consequently, the algebra $A_{F \vee G}$ is obtained from $A_G$ by localising by the single element $x_F/x_G$ in the centre $Z(A_G)$ of $A_G$.

This implies that for a large enough positive integer $N$ the element $s_G = (x_F/x_G)^N \cdot s_F|_{F \vee G}$ is in $Y^{F \vee G} \cong A_{F \vee G} \otimes A_G Y^G$, where $s_F$ is the element given in the formulation of the Lemma. We may pick an integer $N$ which works for all $G \in F(P)_0$ simultaneously. Then for all $G$, $x_G^N \otimes s_G$ is an element of $Y(N)^G = (O(N) \otimes Y)^G$ which restricts to $x_F^N \otimes s_F|_{F \vee G} \in (O(N) \otimes Y)^{F \vee G}$.

(Note here that $(x_F^N \otimes s_F)|_{F \vee G} = x_F^N \otimes (s_F|_{F \vee G})$.)

Now let $G \subseteq H \in F(P)_0$ be an arbitrary pair of non-empty faces of $P$. We do not know whether the elements $x_G^N \otimes s_G|_H$ and $x_H^N \otimes s_H$ agree, but we know that after restricting further to $F \vee H$ both agree with $x_F^N \otimes s_F|_{F \vee H}$. Consequently, using 2.5.2.1 again, for all large integers $M$ we have equality

$$\left(\frac{x_F}{x_H}\right)^M \cdot x_G^N \otimes s_G|_H = \left(\frac{x_F}{x_H}\right)^M \cdot x_H^N \otimes s_H \in Y(N)^H.$$
Now multiplication with $x^N_H$ yields an isomorphism $Y(N)^H \cong Y(M + N)^H$ so that the above equality becomes

$$(x^M_F x^N_G) \otimes s_G|_H = (x^M_F x^N_H) \otimes s_H \in Y(M + N)^H,$$

with both elements restricting to $x^{M+N}_F \otimes s_F|_{F \vee H}$ on $F \vee H$.

To sum up, we have shown that the family of elements $(x^M_F x^N_G) \otimes s_G, G \in F(P)_0,$ determines an element $e$ in $\lim \leftarrow Y(M + N)$, and hence an $R$-module homomorphism $R \longrightarrow \lim \leftarrow Y(M + N)$ which sends $1 \in R$ to $e$. But then, by forcing equivariance, there is a map $O(0) \longrightarrow Y(M + N)$ which sends $1 \in A_G$ to $(x^M_F x^N_G) \otimes s_G$. Twisting by $-(M + N)$ yields a map $O(-M - N) \longrightarrow Y$ such that $s_G$ is in the image of the $G$-component. This applies in particular to $G = F$ which is the case of the Lemma.

**Corollary 2.5.3.** Let $Y$ be a quasi-coherent functor such that for all $F \in F(P)_0$ the $A_F$-module $Y^F$ is finitely generated. Then there are finitely many numbers $n_i \in \mathbb{Z}$ and a map

$$\bigoplus_i O(n_i) \longrightarrow Y$$

which is surjective on each component.

**Proof.** For $F \in F(P)_0$ choose generators $s^F_1, \cdots, s^F_{\ell(F)}$. By Lemma 2.5.2 there are maps $f^F_i : O(n^F_i) \longrightarrow Y$ such that $s^F_i$ is in the image of $f^F_i$. The required map is given by the sum

$$\bigoplus_{F \in F(P)_0} \bigoplus_{i=1}^{\ell(F)} O(n^F_i) \longrightarrow Y.$$ 

□

**Lemma 2.5.4.** Let $R$ be a left noetherian ring. Let $Y$ be a quasi-coherent functor such that for all $F \in F(P)_0$ the $A_F$-module $Y^F$ is finitely generated. Then $H_k \hat{\Gamma}(Y)$ is trivial for $k < 0$ and $k > n$, and is a finitely generated $R$-module for all $k \in \mathbb{Z}$.

**Proof.** This follows the pattern of [Har77, III.5.2]. Triviality of $H_k \hat{\Gamma}(Y)$ for $k < 0$ and $k > n$ is immediate as $\hat{\Gamma}(Y)$ is concentrated in degrees 0 to $n$, by construction.

The Lemma is true for a finite sum of quasi-coherent functors of the form $O(k)$, by the calculation in Proposition 2.3.1. By Corollary 2.5.3 we can find a surjection $Z \longrightarrow Y$ with $Z$ a finite sum of $O(k)$'s. Let $K$ denote the kernel; this is a quasi-coherent functor as well. By construction we obtain a short exact sequence of chain complexes

$$0 \longrightarrow \hat{\Gamma}K \longrightarrow \hat{\Gamma}Z \longrightarrow \hat{\Gamma}Y \longrightarrow 0.$$ 

Now use increasing induction on $k$ on the corresponding exact sequence snippet

$$H_{k+1}\hat{\Gamma}Z \longrightarrow H_{k+1}\hat{\Gamma}Y \longrightarrow H_k\hat{\Gamma}K,$$
starting with \( k = -1 \); by choice of \( Z \) the module on the left is finitely generated, and by what has been established by induction the module on the right is finitely generated as well. Since \( R \) is left noetherian it follows that the middle module is finitely generated. □

**Proposition 2.5.5.** Suppose \( R \) is left noetherian. Let \( Y \in \hCoh(P) \) be such that \( Y^F_k \) is finitely generated as an \( A^F \)-module for all \( F \in F(P)_0 \) and all \( k \in \mathbb{Z} \). Then \( \check{\Gamma}(Y) \) is a (possibly unbounded) chain complex with finitely generated homology modules.

**Proof.** Since \( Y \) is a homotopy sheaf \( H_q(Y): F \mapsto H_q(Y^F) \) is a quasi-coherent functor (this uses Lemma 1.1.2 and the fact that taking homology is compatible with localisation). Since \( R \) is left noetherian, and since all the modules \( Y^F_k \) are finitely generated, the modules \( H_q(Y^F) \) are finitely generated as well. It follows from Lemma 2.5.4, applied to \( H_q(Y) \), and from Remark 2.2.2 (1) that all the entries of the \( E^2 \)-term of the spectral sequence 2.2.3 are finitely generated \( R \)-modules. Since this spectral sequence is concentrated in columns 0 to \( n \), its abutment \( H_{p+q}\check{\Gamma}Y \) consists of finitely generated \( R \)-modules as well (making use the fact that \( R \) is left noetherian once again). □

### 2.6. Finiteness of the Čech complex

We are now going to prove the following fundamental finiteness result:

**Theorem 2.6.1.** Let \( R \) be a unital ring. Suppose that \( R \) is commutative, or else left noetherian. Let \( Y \in \Perf(P) \). Then \( \check{\Gamma}(Y) \) is a perfect complex of \( R \)-modules.

**Proof.** Suppose first that \( R \) is left noetherian. Since \( Y \) is a perfect complex, there exists a homotopy vector bundle \( V \in \hVect(P) \) together with a quasi-isomorphism \( V \rightarrow Y \) by Lemma 1.1.7. Then the induced map \( \check{\Gamma}V \rightarrow \check{\Gamma}Y \) is a quasi-isomorphism by Proposition 2.2.4. Since \( V \) is bounded so is \( \check{\Gamma}V \).

Since \( V^F \) consists of projective \( A^F \)-modules and since \( A^F \) is free as an \( R \)-module, \( \check{\Gamma}V \) consists of projective \( R \)-modules, and has finitely generated homology modules by Proposition 2.5.5. But this means that \( \check{\Gamma}V \) is chain homotopy equivalent to a strict perfect complex of \( R \)-modules [Ros94, 1.7.13].

Now suppose that \( R \) is commutative, but not noetherian. Then \( Y \) can be replaced, up to quasi-isomorphism, by a bounded complex \( V \) in \( \hVect(P) \cap qCoh(P) \); this is true since the toric scheme \( X_P \) is projective over \( \Spec R \) and thus has an ample line bundle, so we can appeal to [TT90, 2.3.1 (d)]. Then \( \check{\Gamma}(Y) \simeq \check{\Gamma}(V) \), and it is enough to prove the Theorem for \( V \) only.

The complex \( V \) descends to a noetherian subring \( R_0 \). More precisely, write \( qCoh(P)_0 \) for the category \( qCoh(P) \) defined over a subring \( R_0 \) instead of \( R \), and similarly for \( hVect(P)_0 \). Then by noetherian approximation [TT90 Appendix C], there is a noetherian subring \( R_0 \) of \( R \), and a bounded complex \( V_0 \in hVect(P)_0 \cap qCoh(P)_0 \) such that \( V = R \otimes_{R_0} V_0 \). By the first part of the proof there is a strict perfect complex \( B_0 \) of \( R_0 \)-modules which is
chain homotopy equivalent to \( \hat{\Gamma} V_0 \). But then \( \hat{\Gamma} V \cong R \otimes_{R_0} \hat{\Gamma} V_0 \) is homotopy equivalent to \( R \otimes_{R_0} B_0 \), and the latter is strict perfect. \( \square \)

2.7. Canonical sheaves and suspension of chain complexes.

**Definition 2.7.1.**

1. For \( k \in \mathbb{Z} \) and \( C \in \text{Ch}(R) \) we define \( \mathcal{O}(k) \otimes C \) to be the sheaf given by
   \[
   (\mathcal{O}(k) \otimes C)^F = \mathcal{O}(k)^F \otimes_R C \quad \text{for } F \in F(\mathcal{P})_0
   \]
   with structure maps induced by those of \( \mathcal{O}(k) \). We call \( \mathcal{O}(k) \otimes C \) the \( k \)-th canonical sheaf associated to \( C \).

2. Let \( C \) be an \( R \)-module chain complex. We denote by \( \text{con}(C) \) the constant \( F(\mathcal{P})_0 \)-diagram with value \( C \) and identity structure maps.

3. The \( n \)-th suspension \( C[[n]] \) of the chain complex \( C \in \text{Ch}(R) \) is defined by \( C[[n]]_k = C_{k-n} \), and multiplying the differentials with the sign \( (-1)^n \).

**Lemma 2.7.2.** Let \( C \in \text{Ch}(R) \). If \( k < 0 \) is an integer such that \( E_P(k) = 0 \) (i.e., such that \( kP \) has no lattice points in its interior), then \( \hat{\Gamma}(\mathcal{O}(k) \otimes C) \) is acyclic.

**Proof.** Let \( D_{*,*} = D_{*,*}(\mathcal{O}(k) \otimes C) \) be the double chain complex associated to \( \mathcal{O}(k) \otimes C \), cf. [2.2.1.1]. The \( A_F \)-module \( \mathcal{O}(k)^F \cong A_F \) is a free \( R \)-module for each \( F \in F(\mathcal{P})_0 \), and the \( \check{\text{Cech}} \) complex \( \hat{\Gamma}\mathcal{O}(k) \) has the property that the image of each chain module under the differential is a free \( R \)-module (since all maps are given by inclusion of bases). Hence by the KÜNNETH formula [Wei95 3.6.1] the homology of the horizontal chain complex \( D_{*,p} \) fits into a short exact sequence
   \[
   H_k(\hat{\Gamma}\mathcal{O}(k)) \otimes_R C_p \longrightarrow H_k(\hat{\Gamma}\mathcal{O}(k) \otimes_R C_p) \longrightarrow \text{Tor}_1^R(H_{n-1} \hat{\Gamma}\mathcal{O}(k), C_p) \,.
   \]
   By hypothesis \( E_P(k) = 0 \) so that \( \hat{\Gamma}\mathcal{O}(k) \) is acyclic by Proposition [2.3.1]. So first and third term of the short exact sequence are trivial, hence so is the middle term.

   But this means that the spectral sequence [2.2.2.1] associated to \( D_{*,*} \) has trivial \( E^1 \)-term, hence its abutment \( H_* \hat{\Gamma}(\mathcal{O}(k) \otimes C) \) is trivial too. This proves the Lemma. \( \square \)

**Lemma 2.7.3.** For every chain complex \( C \in \text{Ch}(R) \) we have a canonical quasi-isomorphism \( C[[n]] \cong \hat{\Gamma}\text{con}(C) \).

**Proof.** Let \( A_{*,*} \) denote the complex \( C \) considered as a double chain complex concentrated in column \( n \). That is, we’re looking at the double chain complex with \( A_{n,k} = C_k \), \( A_{j,k} = 0 \) for \( j \neq n \), and vertical differential the differential of \( C \) multiplied with the sign \( (-1)^n \). Then the total complex of \( A_{*,*} \) is precisely \( C[[n]] \).
The double chain complex $A_{s,t}$ maps into the double chain complex $D_{s,t} = D_{s,t}(\text{con}(C))$ associated to $\text{con}(C)$ by the diagonal map $\Delta$ given by

$$C_t \longrightarrow \bigoplus_{\dim F = n} C_t = D_{n,t}, \ t \in \mathbb{Z}.$$ 

This defines indeed a map of double chain complexes by the properties of incidence numbers; more precisely, the horizontal chain complex $D_{s,t}$ is the tensor product of $C_t$ with the dual of the chain complex computing the integral cellular homology of $P$, and the map of horizontal chain complexes $A_{s,t} \longrightarrow D_{s,t}$ is the tensor product of $C_t$ with the dual of the augmentation map. In particular, the map is a quasi-isomorphism with respect to “horizontal” homology. By the spectral sequence argument 2.1.1 (2), $\Delta$ induces a quasi-isomorphism

$$\Gamma \cong \Gamma \left( \text{con}(C) \right).$$

Lemma 2.7.4. For every chain complex $C \in \text{Ch}(R)$ we have a canonical map $\text{con}(C) \longrightarrow \mathcal{O}(0) \otimes C$, induced by the inclusions of $F$-components

$$\text{con}(C)^F = C \cong R[[0]] \otimes_R C \longrightarrow R[\mathbb{Z}^n \cap C_F] \otimes_R C = (\mathcal{O}(0) \otimes C)^F,$$

which in turn induces a quasi-isomorphism $\Gamma \cong \Gamma \left( \mathcal{O}(0) \otimes C \right)$.

Proof. First recall that $\mathcal{O}(0)^F = A_F = R[S_F]$ for each $F \in F(P)_0$; this means that we can consider $\mathcal{O}(0)^F$ as a $\mathbb{Z}^n$-graded $R$-module with homogeneous components 0 or $R$. The double chain complex $D_{s,t}(\mathcal{O}(0) \otimes C)$ associated to $\mathcal{O}(0) \otimes C$ is a double chain complex of $\mathbb{Z}^n$-graded $R$-modules with horizontal and vertical differentials respecting the grading. We can thus concentrate on homogeneous components one at a time. As explained in [Hüt09] 2.5.3, the component of degree 0 $\in \mathbb{Z}^n$ of the horizontal chain complex $D_{s,t}(\mathcal{O}(0) \otimes C)$ is the tensor product of $C_t$ with the dual of the chain complex calculating the integral cellular homology of $P$ so that its horizontal homology is concentrated in column $n$ and has value $C_t$. For all homogeneous degrees different from 0 the chain complex is acyclic (loc. cit.).

On the other hand, we have a canonical isomorphism of diagrams

$$\text{con}(C) \cong \text{con}(R) \otimes C$$

where $\text{con}(R)$ is the constant diagram with value $R$, considered as a diagram of chain complexes concentrated in chain degree 0. We can think of $R$ as a $\mathbb{Z}^n$-graded module concentrated in degree 0 $\in \mathbb{Z}^n$. Thus the associated double chain complex $D_{s,t}(\text{con}(R) \otimes C)$ consists of $\mathbb{Z}^n$-graded modules with differentials preserving the grading. In homogeneous degree 0, the horizontal chain complexes $D_{s,t}(\text{con}(R) \otimes C)$ and $D_{s,t}(\mathcal{O}(0) \otimes C)$ agree, in non-zero degrees the horizontal chain complex $D_{s,t}(\text{con}(R) \otimes C)$ is the zero-complex.

We have an obvious map of double complexes

$$\omega: D_{s,t} \left( \text{con}(R) \otimes C \right) \longrightarrow D_{s,t} \left( \mathcal{O}(0) \otimes C \right),$$

□
the inclusion of degree 0 components, which by the previous two paragraphs induces an isomorphism on horizontal homology modules. By the spectral sequence argument 2.1.1 (2) the composite
\[
\tilde{\Gamma}\con(C) \cong \tilde{\Gamma}(\con(R) \otimes C) = \text{Tot} D_{*,*}(\con(R) \otimes C)
\]
is thus a quasi-isomorphism. □

2.8. A model for suspension. For any presheaf \( Y \in \text{Pre}(P) \) we let \( Y[n] \) denote the \( n \)th suspension of \( Y \), that is, the diagram given by \( F \mapsto Y[n]^F = Y^F[n] \), cf. Definition 2.7.1 (3), with structure maps induced by those of \( Y \).

Definition 2.8.1. Let \( F \) be a face of \( P \) (possibly empty), and let \( Y \in \text{Pre}(P) \). We define a new presheaf \( F_*Y \) by the rule
\[
F_*Y : G \mapsto Y^{G\cap F}
\]
where \( G \cap F \) is the join of \( G \) and \( F \), that is, the smallest face of \( P \) containing \( G \cup F \), and \( (F_*Y)^G = Y^{F\cap G} \) is considered as an \( A_G \)-module chain complex by restriction of scalars.

Remark 2.8.2. If \( R \) is a commutative ring and \( Y \in \text{qCoh}(P) \) is a sheaf, let \( F \) denote the chain complex of quasi-coherent sheaves on \( X_P \) determined by \( Y \). Then \( F_*Y \) corresponds to \( j^* \left( F \big|_{U_F} \right) = j^* (\tilde{Y}_F) \) where \( j : U_F \to X_P \) is the inclusion.

The construction of \( F_*Y \) is natural in \( Y \): For \( F \subseteq F' \) a pair of faces of \( P \), the structure maps of \( Y \) induce a map of presheaves \( F_*Y \to F'_*Y \). Moreover, every entry of \( F_*Y \) is an \( A_F \)-module, by restriction of scalars \( (A_0 = R \text{ here}) \) so that \( \tilde{\Gamma} F_*Y \) is a chain complex of \( A_F \)-modules. Hence the following definition is meaningful:

Definition 2.8.3. For \( Y \in \text{Pre}(P) \) we define a new presheaf \( \sigma Y \) by
\[
F \mapsto (\sigma Y)^F = \tilde{\Gamma}(F_*Y) .
\]

Lemma 2.8.4. Let \( Y \in \text{Pre}(P) \). There is a natural map of presheaves
\[
\alpha : \gamma[n] \to \sigma Y
\]
which is a quasi-isomorphism on each component. In particular, if \( Y \) is a homotopy sheaf then so is \( \sigma Y \), and if \( Y \) is a perfect complex so is \( \sigma Y \).

Proof. Let \( \gamma Y \) denote the presheaf \( (\gamma Y)^F = \tilde{\Gamma}\con(Y^F) \). For any face \( F \in F(P)_0 \) the structure maps of \( Y \) induce a map of diagrams \( \con(Y^F) \to F_*Y \) and thus a map \( \tilde{\Gamma}\con(Y^F) \to \tilde{\Gamma} F_*Y \). This construction is natural in \( F \) so we obtain maps of presheaves
\[
\gamma[n] \to \gamma Y \to \sigma Y ,
\]
the first one consisting of the canonical quasi-isomorphisms of Lemma \[2.7.3\]. The composition of these two maps is the \(\alpha\) of the Lemma, and we are left to prove that the map

\[
(\gamma Y)^F = \hat{\Gamma} \text{con}(Y^F) \longrightarrow \hat{\Gamma} F_* Y = (\sigma Y)^F
\]
is a quasi-isomorphism for each \(F \in F(P)_0\); we will use the (by now familiar) spectral sequence comparison argument.

Write \(\text{st } F = \{ G \in F(P)_0 \mid G \supseteq F \}\), a sub-poset of \(F(P)_0\). Given a diagram \(A: F(P)_0 \longrightarrow \text{R-Mod}\) we can consider its restriction \(A|_{\text{st } F}\) to the poset \(\text{st } F\). Conversely, a diagram \(B: \text{st } F \longrightarrow \text{R-Mod}\) can be extended to a diagram

\[
F_* B: F(P)_0 \longrightarrow \text{R-Mod} \quad , \quad G \mapsto B^{G \cap F}.
\]

In fact, extension and restriction form an adjoint pair, with restriction being the left adjoint. Both functors are exact.

Now let \(B: \text{st } F \longrightarrow \text{R-Mod}\) be given, and let \(B \longrightarrow I^*\) be an injective resolution of \(B\). Then \(F_* B \longrightarrow F_* I^*\) is an injective resolution of \(F_* B\). Consequently, we have

\[
\lim_{\leftarrow} q B = H^q \lim_{\text{st } F} I^* \\
= H^q \text{hom} ((\text{con } R)|_{\text{st } F}, I^*) \\
= H^q \text{hom}(\text{con } R, F_* I^*) \\
= H^q \lim_{\leftarrow} F_* I^* \\
= \lim_{\leftarrow} q F_* B .
\]

On the other hand, \(\text{st } F\) has minimal element \(F\) so that \(\lim B = B^F\), and \(\lim\) is exact; that is,

\[
\lim_{\leftarrow} q B = \begin{cases} B^F & \text{if } q = 0 \\ 0 & \text{else.} \end{cases}
\]

These calculations apply in particular to \(B = A|_{\text{st } F}\) for \(A\) an \(F(P)_0\)-diagram of \(R\)-modules. Since \((F_* (A|_{\text{st } F}))^F = A^F\) this means that

\[
\lim_{\leftarrow} q F_* (A|_{\text{st } F}) = \begin{cases} A^F & \text{if } q = 0 \\ 0 & \text{else.} \end{cases}
\]

The calculations also imply that the obvious map \(\text{con } (B^F)|_{\text{st } F} \longrightarrow B\), for arbitrary \(B\) as before and its adjoint \(B^F \longrightarrow F_* B\) induce isomorphisms

\[
\lim_{\leftarrow} q (\text{con } B^F)|_{\text{st } F} \cong \lim_{\leftarrow} q B \quad \text{and} \quad \lim_{\leftarrow} q \text{con } B^F \cong \lim_{\leftarrow} q F_* B
\]
as both are the identity on \(F\)-components.

Let us return to the map

\[
(\gamma Y)^F = \hat{\Gamma} \text{con}(Y^F) \longrightarrow \hat{\Gamma} F_* Y = (\sigma Y)^F .
\]

It is induced by a map of double chain complexes

\[
D_{*,*}(\text{con } Y^F) \longrightarrow D_{*,*}(F_* Y)
\]
which, when restricted to $t$th horizontal chain complexes, is a map of chain complexes computing the higher derived limits $\lim_{n \to \infty}^{\leftarrow}$ of the diagrams $A = \text{con} Y^F_t$ and $B = Y_{[s]} F$, respectively, by Remark 2.7.2 [1] by the calculation above, applied to $A$ and $B = Y_{[s]} F$, this yields an isomorphism on homology modules for all values of $\ast$. In other words, the $E^1$-terms of the two spectral sequences are isomorphic, hence their abutments are as well, so the map in question is a quasi-isomorphism as claimed.

2.9. Relating $O(0) \otimes \bar{\Gamma}$ to $n$th suspension. We proceed to construct a map between the functors $O(0) \otimes \bar{\Gamma}$ and $n$th suspension which will be used later in $K$-theoretical computations.

2.9.1. Let $Y \in \text{Pre} (P)$, and fix $F \in F(P)_0$. The structure maps of $Y$ induce a canonical map $Y^F \xrightarrow{\rho^F} \lim F^* Y$ and thus, by forcing equivariance, a map

\[
\rho^F : O(0) \otimes Y^F \xrightarrow{\rho^F} F^* Y.
\]

(2.9.1.1)

Lemma 2.9.2. For $Y \in h\text{Coh} (P)$ the map $\bar{\Gamma}(\rho^F) : \bar{\Gamma}(O(0) \otimes Y^F) \longrightarrow \bar{\Gamma} F^* Y$ is a quasi-isomorphism.

Proof. The map $\bar{\Gamma}(\rho^F)$ fits into a commutative square diagram

\[
\begin{array}{ccc}
\bar{\Gamma}(O(0) \otimes Y^F) & \xrightarrow{\bar{\Gamma}(\rho^F)} & \bar{\Gamma} F^* Y \\
\downarrow & & \downarrow \\
\bar{\Gamma} \text{con } (Y^F) & \xleftarrow{\ast} & Y^F[n]
\end{array}
\]

Left, bottom and right map are quasi-isomorphisms by Lemmas 2.7.3 and 2.8.4 respectively, hence the top map is a quasi-isomorphism as well.

2.9.3. Let $Y \in \text{Pre} (P)$. The structure maps of $Y$ induce, for each $F \in F(P)_0$, a map of presheaves $Y \longrightarrow F^* Y$ and hence a map of chain complexes $\bar{\Gamma} Y \longrightarrow \bar{\Gamma} F^* Y = (\sigma Y)^F$. Since this is natural in $F$ we obtain a map

\[
\bar{\Gamma} Y \longrightarrow \lim_{F \in F(P)_0} \bar{\Gamma} F^* Y = \lim_{F \in F(P)_0} (\sigma Y)^F
\]

which, by forcing equivariance, defines a map

\[
\beta : O(0) \otimes \bar{\Gamma} Y \longrightarrow \sigma Y.
\]

(2.9.3.1)

Lemma 2.9.4. Let $Y \in h\text{Coh} (P)$. Then $\bar{\Gamma}(\beta) : \bar{\Gamma}(O(0) \otimes \bar{\Gamma} Y) \longrightarrow \bar{\Gamma}(\sigma Y)$ is a quasi-isomorphism of chain complexes of $R$-modules.

Before we delve into the proof, we fix conventions regarding triple chain complexes. Suppose we have a threefold $R$-module chain complex $A_{*,*,*}$: a $\mathbb{Z}^3$-indexed collection of $R$-modules $A_{*,*,*}$ together with pairwise commuting
differentials

\[ \partial_x: A_{x,y,z} \longrightarrow A_{x-1,y,z} \]
\[ \partial_y: A_{x,y,z} \longrightarrow A_{x,y-1,z} \]
\[ \partial_z: A_{x,y,z} \longrightarrow A_{x,y,z-1} \]

which square to trivial maps. That is, we are looking at an object of the category \( \text{Ch}(\text{Ch}(\text{Ch}(R))) \). Then we can define new differentials by

\[ \partial_x = \tilde{\partial}_x, \quad \partial_y = (-1)^x \tilde{\partial}_y, \quad \partial_z = (-1)^{x+y} \tilde{\partial}_z \]

which are easily checked to anti-commute: \( \partial_i \partial_j = (\delta_{i,j} - 1) \partial_j \partial_i \) (where \( \delta_{i,j} \) is the usual Kronecker delta symbol). We say that the graded module \( A_{*,*,*} \) together with the maps \( \partial_i, i = x, y, z \) is a triple chain complex (in analogy to the established usage of the term double complex in the literature). We define the total complex \( \text{Tot}^{x,y,z}(A_{*,*,*}) \) by setting

\[ \text{Tot}^{x,y,z}(A_{*,*,*})_n = \bigoplus_{x+y+z=n} A_{x,y,z} \]
equipped with the differential defined by \( \partial = \partial_x + \partial_y + \partial_z \). This is an \( R \)-module chain complex, the relation \( \partial^2 = 0 \) is easily verified.

The relevant observation here is that one can do the totalisation in two steps. Define \( \text{Tot}^{y,z}(A_{*,*,*}) \) by

\[ \text{Tot}^{y,z}(A_{*,*,*})_{p,q} = \bigoplus_{y+z=q} A_{p,y,z} ; \]

this is a double chain complex when equipped with “horizontal” differential \( \partial^h = \partial_x \) and “vertical” differential \( \partial^v = \partial_y + \partial_z \). It is a matter of tracing definitions to see that we have an equality of chain complexes

\[ \text{Tot}(\text{Tot}^{y,z}(A_{*,*,*})) = \text{Tot}^{x,y,z}(A_{*,*,*}) . \]

Similarly, we can define a double chain complex \( \text{Tot}^{x,z}(A_{*,*,*}) \) by

\[ \text{Tot}^{x,z}(A_{*,*,*})_{p,q} = \bigoplus_{x+z=p} A_{x,q,z} ; \]
equipped with “horizontal” differential \( \partial^h = \partial_x + \partial_z \) and “vertical” differential \( \partial^v = \partial_y \). We then have an equality of chain complexes

\[ \text{Tot}(\text{Tot}^{x,z}(A_{*,*,*})) = \text{Tot}^{x,y,z}(A_{*,*,*}) . \]

Proof of Lemma 2.9.4. The map \( \tilde{\Gamma}(\beta) \) is a map of \( R \)-module chain complexes which can, in fact, be described by a map of triple chain complexes. In more detail, define

\[ A_{x,y,z} = \bigoplus_{\dim G = n-x} \bigoplus_{\dim F = n-y} \mathcal{O}(0)^G \otimes_R Y_z^F , \]
equipped with differentials $\partial_z$ given by the differential in $Y^F$, and differentials $\partial_x$ and $\partial_y$ determined by incidence numbers $[G_1 : G_2]$ and $[F_1 : F_2]$, respectively. Similarly, define

$$B_{x,y,z} = \bigoplus_{\dim G = n-x} \bigoplus_{\dim F = n-y} Y_z^{G \lor F},$$

equipped with differentials as above. Then both $A_{x,y,z}$ and $B_{x,y,z}$ are threefold chain complexes and thus determine, by modification of the differentials, triple chain complexes as explained above. The structure maps of $Y$ induce a map of triple chain complexes $\gamma: A_{x,y,z} \longrightarrow B_{x,y,z}$. A tedious but straightforward tracing of signs and direct sums involved shows:

(i) $D_{s,x}(O(0) \otimes \Gamma Y) = \text{Tot}^{y,z}(A_{s,s,s}),$

(ii) $D_{s,s}(\sigma Y) = \text{Tot}^{y,z}(B_{s,s,s}),$

(iii) $D_{s,s}(\beta) = \text{Tot}^{y,z}(\gamma).$

Since $\text{Tot} \circ \text{Tot}^{y,z} = \text{Tot}^{x,y,z}$ this implies that

$$\text{Tot}^{x,y,z}(\gamma) = \Gamma(\beta). \quad (2.9.4.1)$$

Let us now consider the double chain complex map $\text{Tot}^{x,z}(\gamma)$. We claim that $\text{Tot}^{x,z}(\gamma)$ induces a quasi-isomorphism on horizontal chain complexes. By the usual spectral sequence argument $2.1.1$ this implies that $\text{Tot}^{x,y,z}(\gamma) = \text{Tot}(\text{Tot}^{x,z}(\gamma))$ is a quasi-isomorphism of $R$-module chain complexes, hence so is $\Gamma(\beta)$ in view of $(2.9.4.1)$.

Fix an index $q \in \mathbb{Z}$. The $q$th row of the source of $\text{Tot}^{x,z}(\gamma)$ has $p$th entry

$$\bigoplus_{\dim F = n-q} \bigoplus_{\dim G = n-x} O(0)^G \otimes_R Y^F_z$$

and is thus of the form

$$\bigoplus_{\dim F = n-q} \Gamma(O(0) \otimes Y^F),$$

up to the (constant!) sign $(-1)^q$ in the differential of $Y^F$. In particular this is non-trivial only for $0 \leq q \leq n$.

The $q$th row of the target of $\text{Tot}^{x,z}(\gamma)$ has $p$th entry

$$\bigoplus_{\dim F = n-q} \bigoplus_{\dim G = n-x} Y_z^{G \lor F} = \bigoplus_{\dim F = n-q} \bigoplus_{\dim G = n-x} (F_y)^G_z$$

and is thus of the form

$$\bigoplus_{\dim F = n-q} \Gamma F_y,$$

up to the (constant!) sign $(-1)^q$ in the differential of $F_y$. In particular this is non-trivial only for $0 \leq q \leq n$.

Thus in row $q$ the map $\text{Tot}^{x,z}(\gamma)$ is, up to sign $(-1)^q$ in the differentials of source and target, the direct sum of the maps $\rho^F$ defined in $(2.9.1.1)$ with $\dim F = n-q$. By Lemma $2.9.2$ this means that $\text{Tot}^{x,z}(\gamma)$ is a quasi-isomorphism of horizontal chain complexes as claimed. \hfill $\square$
3. Splitting the $K$-theory

Let $R$ be a ring with unit. For this entire section we assume that $R$ is commutative, or else left noetherian.

3.1. Reduced $K$-theory. Recall that an $R$-module chain complex $C$ is called perfect if it is quasi-isomorphic to a strict perfect complex, that is, a bounded complex $B$ of finitely generated projective $R$-modules; if this is the case, there will always be a quasi-isomorphism $B \to C$. Write $\text{Perf}(R)$ for the category of perfect chain complexes of $R$-modules. We write $K(R)$ for the $K$-theory space of the complicial biWaldhausen category $\text{Perf}(R)$ equipped with quasi-isomorphisms as weak equivalences, and the degreewise split monomorphisms with cokernel in $\text{Perf}(R)$ as cofibrations.

For an $n$-dimensional polytope $P \subset \mathbb{R}^n$ we have defined the category $\text{Perf}(P)$ of perfect complexes in 1.1.6; recall that an object of $\text{Perf}(P)$ is a diagram indexed by the face lattice of $P$ with values in perfect chain complexes of modules over different rings, subject to a gluing condition. The $K$-theory space of $\text{Perf}(P)$ is denoted by $\tilde{K}(P)$, cf. §1.2.

Let $\text{Perf}(P)^{[0]}$ denote the full subcategory of those $Y \in \text{Perf}(P)$ such that $\tilde{\Gamma}(Y)$ is acyclic, cf. Definition 2.2.1 (3). This is a complicial biWaldhausen category with the usual conventions. Its associated $K$-theory space is called the reduced $K$-theory of $P$ and denoted $\tilde{\tilde{K}}(P)$.

We call a map $f : Y \to Z$ in $\text{Perf}(P)$ an $h_{[0]}$-equivalence if $\tilde{\Gamma}(f)$ is a quasi-isomorphism; with respect to these maps as weak equivalences, $\text{Perf}(P)$ is (yet another) complicial biWaldhausen category. Note that every quasi-isomorphism in $\text{Perf}(P)$ is an $h_{[0]}$-equivalence as the functor $\tilde{\Gamma}$ preserves quasi-isomorphisms by Proposition 2.2.3.

We will need the functor

$$\psi_0 : \text{Perf}(R) \to \text{Perf}(P), \quad C \mapsto \mathcal{O}(0) \otimes C.$$ 

It is easy to see that $\psi_0$ takes values in perfect complexes. Indeed, for $C \in \text{Perf}(C)$ there is a perfect complex of $R$-modules $D$ which is quasi-isomorphic to $C$. Since $A_F$ is a free $R$-module, for each $F \in F(P)_0$, taking tensor product with $A_F$ over $R$ is exact. Consequently, $\psi_0(Y)^F$ is quasi-isomorphic to $A_F \otimes_R D$ which is a strict perfect complex of $A_F$-modules.

**Proposition 3.1.1.** There is a fibration sequence of $K$-theory spaces

$$\tilde{K}(P) \to K(P) \xrightarrow{\tilde{\Gamma}} K(R)$$

which has a section up to homotopy and up to sign induced by the functor $\psi_0$. Hence there is a splitting up to homotopy

$$\tilde{K}(P) \times K(R) \simeq K(P).$$

**Proof.** By the Fibration Theorem [Wal85, 1.6.4] the sequences of exact functors of biWaldhausen categories

$$(\text{Perf}(P)^{[0]}, w) \xrightarrow{\subseteq} (\text{Perf}(P), w) \xrightarrow{(\text{Perf}(P), h_{[0]})} (3.1.1.1)$$

where \( w \) stands for quasi-isomorphisms as weak equivalences, induces a fibration sequence of \( K \)-theory spaces.

We have exact functors

\[
\text{Perf}(R) \xrightarrow{\psi_0} (\text{Perf}(P), h_{[0]}) \quad \text{and} \quad (\text{Perf}(P), h_{[0]}) \xrightarrow{\Gamma} \text{Perf}(R),
\]

the latter being well defined by Theorem 2.6.1. By Lemmas 2.7.3 and 2.7.4 we have a natural weak equivalence of functors from the \( n \)-th suspension \( C \mapsto C[n] \) to the composition \( \Gamma \circ \psi_0 \). Since suspension induces a self homotopy equivalence on the \( K \)-theory space \( K(R) \), the functor \( \Gamma \) is surjective on homotopy groups.

By Lemmas 2.8.4 and 2.9.4 there is a chain of natural transformations of functors represented by

\[
Y[n] \xrightarrow{\sigma Y} \mathcal{O}(0) \otimes \Gamma(Y) = \psi_0 \circ \Gamma(Y)
\]

which is in fact a chain of \( h_{[0]} \)-equivalences of functors. Thus \( \Gamma \) is injective on homotopy groups.

In total, we have shown that the functor \( \Gamma \) induces a homotopy equivalence from the \( K \)-theory of the base of the fibration sequence (3.1.1.1) to \( K(R) \). The resulting fibration sequence

\[
\tilde{K}(P) \longrightarrow K(P) \xrightarrow{\Gamma} K(R)
\]

has section up to homotopy and up to sign induced by \( \psi_0 \) (as the composition \( \Gamma \circ \psi_0 \) is weakly equivalent to \( n \)-th suspension, just as argued above) which yields the desired splitting.

\[\square\]

3.2. **Further splitting.** If \( \Gamma(\mathcal{O}(-1)) \) happens to be acyclic, we can split off a further copy of \( K(R) \) from the reduced \( K \)-theory \( \tilde{K}(P) \); acyclicity of \( \Gamma \mathcal{O}(-j) \) for \( j > 1 \) allows to iterate the procedure. The argument is virtually the same as in §3.1 but the functors involve additional twisting. We record the details.

For a given integer \( k \geq 0 \), let \( \text{Perf}(P)^{[k]} \) denote the full subcategory of those \( Y \in \text{Perf}(P) \) such that \( \Gamma Y(j) \) is acyclic for \( 0 \leq j \leq k \). This is a complicial biWaldhausen category with the usual conventions. Its associated \( K \)-theory space is denoted \( K(P,[k]) \); in particular, \( K(P,[0]) = \tilde{K}(P) \).

We call a map \( f : Y \longrightarrow Z \) in \( \text{Perf}(P) \) an \( h_{[k]} \)-equivalence if \( \Gamma f(j) \) is a quasi-isomorphism for \( 0 \leq j \leq k \); with respect to these maps as weak equivalences, \( \text{Perf}(P) \) is (yet another) complicial biWaldhausen category. Note that every quasi-isomorphism in \( \text{Perf}(P) \) is an \( h_{[k]} \)-equivalence as both twisting and the functor \( \Gamma \) preserve quasi-isomorphisms.

We will need the functors

\[
\psi_k : \text{Perf}(R) \longrightarrow \text{Perf}(P), \ C \mapsto \mathcal{O}(k) \otimes C
\]

\((k \in \mathbb{Z} \) here). One can show that this functor takes indeed values in perfect complexes; the argument is as for \( \psi_0 \). As a matter of notation, let us also
introduce the $k$th twist functor

$$\theta_k : \text{Perf} (P) \rightarrow \text{Perf} (P), Y \mapsto Y(k).$$

Recall that the polytope $P$ determines a polynomial $E_P(x)$ with rational coefficients such that $|E_P(-j)|$ is the number of integral points in the interior of $-jP$ for integers $j \geq 1$, cf. Theorem 1.4.1. It follows that if $E_P(-j) = 0$ for some $j > 1$, then $E_P(-\ell) = 0$ for $0 < \ell \leq j$. Let $n_P$ be the number of distinct integral roots of $E_P(x)$; then $n_P \in [0,n]$, and if $n_P \neq 0$ then $n_P$ is maximal among the negatives of integer roots of $E_P(x)$.

**Proposition 3.2.1.** For $1 \leq \ell \leq n_P$ there is a fibration sequence of $K$-theory spaces

$$K(P, [\ell]) \rightarrow K(P, [\ell - 1]) \xrightarrow{\Gamma \circ \theta_\ell} K(R)$$

which has a section up to homotopy and up to sign induced by the functor $\psi_{-\ell}$. Consequently, we have a homotopy equivalence

$$K(P, [\ell]) \times K(R) \simeq K(P, [\ell - 1]).$$

**Proof.** The sequence of biWaldhausen categories

$$(\text{Perf} (P)[\ell], w) \subseteq (\text{Perf} (P)[\ell-1], w) \rightarrow (\text{Perf} (P)[\ell-1], h_{[\ell]})$$

induces a fibration sequence of $K$-theory spaces, by the Fibration Theorem [Wal85, 1.6.4]. We have to prove that the $K$-theory of its base is homotopy equivalent, via $\tilde{\Gamma} \circ \theta_\ell$, to $K(R)$, where we have written $\theta_\ell$ for the $\ell$th twist functor $Y \mapsto Y(\ell)$.

First note that $\psi_{-\ell}$ restricts to an exact functor

$$\psi_{-\ell} : \text{Perf} (R) \rightarrow (\text{Perf} (P)[\ell-1], h_{[\ell]}).$$

We have to check that $\tilde{\Gamma} \circ \psi_j$, when applied to a complex of the form $\psi_{-\ell}(C)$, produces an acyclic chain complex for $0 \leq j < \ell$. But for $j$ in this range we have $-n_P \leq -\ell + j < 0$ so that $E_P(-\ell + j) = 0$. It follows that

$$\tilde{\Gamma} \circ \theta_j \circ \psi_{-\ell}(C) \cong \tilde{\Gamma}C(-\ell + j)$$

is acyclic by Lemma 2.7.2.

Now the composition $(\tilde{\Gamma} \circ \theta_\ell) \circ \psi_{-\ell} \cong \tilde{\Gamma} \psi_0$ is weakly equivalent to the $n$th suspension endo-functor $C \mapsto C[n]$ of $\text{Ch} (R)$, by Lemmas 2.7.3 and 2.7.4, hence $\tilde{\Gamma} \circ \theta_\ell$ induces a surjection on homotopy groups of $K$-theory spaces.

According to Lemmas 2.8.4 and 2.9.4 the composition $\psi_0 \circ \tilde{\Gamma}$ is connected to the $n$th suspension functor by a chain of $h_{[0]}$-equivalences depicted

$$Y[n] \xrightarrow{\sim} \sigma Y \xrightarrow{\psi_0 \circ \tilde{\Gamma}(Y)}$$

(where the first map is actually a weak equivalence). Replacing $Y$ by its $\ell$th twist $Y(\ell)$ yields a chain of $h_{[0]}$-equivalences

$$\theta_\ell Y[n] \xrightarrow{\sim} \sigma Y(\ell) \xrightarrow{\psi_0 \circ \tilde{\Gamma} \circ \theta_\ell(Y)}.$$  (3.2.1.1)
Twisting by \(-\ell\) again leaves us with a chain of natural maps

\[ Y[n] \xrightarrow{\theta_{-\ell}} \theta_{-\ell}(\sigma Y(\ell)) \xrightarrow{\nu} \psi_{-\ell} \circ \theta_{\ell}(Y). \]

We claim that the maps \(\mu\) and \(\nu\) are \(h[\ell]\)-equivalences. This is clear for \(\mu\) since \(\mu\) is in fact a weak equivalence. As for \(\nu\), given an integer \(j\) with \(0 \leq j \leq \ell\) we have to prove that application of \(\hat{\Gamma} \circ \theta_j\) to the map \(\nu\) produces a quasi-isomorphism of \(R\)-module chain complexes. For \(0 \leq j < \ell\) this is true since both source and target of the resulting map of chain complexes are acyclic. Indeed, \(\hat{\Gamma} \circ \theta_j(\theta_{-\ell}(\sigma Y(\ell))) \simeq \hat{\Gamma}Y(j)[n] \simeq 0\) since \(Y\) is an object of \(\text{Perf}(P)^{[\ell-1]}\), while \(\hat{\Gamma} \circ \theta_j \circ \psi_{-\ell} \circ \hat{\Gamma} \circ \theta_{\ell}(Y) = \hat{\Gamma} \circ \psi_{j-\ell} \circ \tilde{\Gamma} \circ \theta_{\ell}(Y) \simeq 0\) by Lemma 2.7.2 applied to the chain complex \(C = \tilde{\Gamma} \circ \theta_{\ell}(Y)\), since \(E_P(j-\ell) = 0\).

For \(j = \ell\) note that \(\theta_{\ell}(\nu)\) is the map \(\sigma Y(\ell) \xrightarrow{\psi_0} \hat{\Gamma}Y(\ell)\) which is an \(h[0]\)-equivalence according to Lemma 2.9.4 (applied to \(Y(\ell)\) instead of \(Y\)).

We have thus verified the claim. But this means that \(n\)th suspension and \(\psi_{-\ell} \circ \tilde{\Gamma} \circ \theta_{\ell}\) induce homotopic maps on the \(K\)-theory space of \((\text{Perf}(P)^{[\ell-1]}, h[\ell])\) so that \(\hat{\Gamma} \circ \theta_{\ell}\) induces a map of \(K\)-theory spaces which is injective on homotopy groups.

This proves that \(K(\text{Perf}(P)^{[\ell-1]}, h[\ell]) \simeq K(R)\) via the functor \(\hat{\Gamma} \circ \theta_{\ell}\). The resulting fibration sequence

\[ \tilde{K}(P, [\ell]) \longrightarrow K(P, [\ell - 1]) \xrightarrow{\Gamma \circ \theta_{\ell}} K(R) \]

has a section up to homotopy and up to sign induced by \(\psi_{-\ell}\) as the composition \(\hat{\Gamma} \circ \psi_{\ell} \circ \psi_{-\ell}\) is weakly equivalent to \(n\)th suspension, just as argued above.

We are now in a position to return to the main theorem of the paper:

**Theorem 1.5.1.** Let \(P \subseteq \mathbb{R}^n\) be an \(n\)-dimensional lattice polytope, and let \(nP\) be the number of distinct integral roots of its Ehrhart polynomial \(E_P(x)\). Let \(R\) be a ring with unit; suppose that \(R\) is commutative, or else left Noetherian. Then there is a homotopy equivalence of \(K\)-theory spaces

\[ K(P) \simeq K(R)^{1+n_P} \times K(P, [nP]) \]

where \(K(R)\) denotes the \(K\)-theory of the ring \(R\), and \(K(P, [nP])\) denotes the \(K\)-theory of those perfect complexes \(X \in \text{Perf}(P)\) which satisfy \(\hat{\Gamma}(Y(j)) \simeq 0\) for \(0 \leq j \leq n_P\). If \(nP = 0\) this expresses the tautological splitting \(K(P) \simeq K(R) \times K(P)\) where \(K(P) = K(P, [0])\).

**Proof.** This follows by concatenating the homotopy equivalences from Propositions 3.1.1 and 3.2.1 for \(\ell = 1, 2, \cdots, n_P\). \(\square\)

### 3.3. Algebraic \(K\)-theory of projective space.

**Theorem 3.3.1.** Let \(\Delta^n\) be an \(n\)-dimensional simplex with volume \(1/n!\). Then \(n_{\Delta^n} = n\) and \(K(\Delta^n, [n]) \simeq \ast\) so there is a homotopy equivalence

\[ K(\Delta^n) \simeq K(R)^{n+1}. \]
Let us remark first that an \( n \)-dimensional simplex with volume \( 1/n! \) can be transformed, by integral translation and a linear map in \( GL_n(\mathbb{Z}) \), into the standard simplex with vertices \( 0, e_1, \cdots, e_n \in \mathbb{R}^n \). Up to isomorphism, the algebraic data associated to \( \Delta^n \) does not change so that we may assume \( \Delta^n \) to be a standard simplex to begin with. Its \textit{Ehrhart} polynomial is \( E_{\Delta^n}(x) = (x+1)(x+2) \cdots (x+n)/n! \) which has precisely \( n_{\Delta^n} = n \) integral roots.

In case of a commutative ring \( R \) we have \( X_{\Delta^n} = \mathbb{P}^n_R \), projective \( n \)-space over \( R \), and the splitting of Theorem 3.3.1 corresponds to the known splitting of \( K \)-theory of projective \( n \)-space which in turn is a special case of Quillen’s “projective bundle” theorem in \( K \)-theory applied to the trivial vector bundle of rank \( n+1 \) over the affine scheme \( \text{Spec } R \) [Qui73, §8.2] [TT90, 4.1].

\textbf{Proof of 3.3.1.} It is enough to prove the following assertion:

\textbf{3.3.1.1.} Let \( Y \in \mathit{hCoh}(\Delta^n) \) be such that \( \Gamma(Y(\ell)) \) is acyclic for all \( \ell \) with \( 0 \leq \ell \leq n \). Then the chain complexes \( Y^F \) are acyclic for all \( F \in F(\Delta^n)_0 \).

For then the map \( Y \longrightarrow 0 \) in \( \mathit{hCoh}(\Delta^n) \) gives a weak equivalence of endo-functors of \( \text{Perf}(P)^{[n]} \) from the identity to the zero functor. Consequently, the identity map of \( K(\Delta^n, [n]) \) is null homotopic so that \( K(\Delta^n, [n]) \simeq * \). The theorem now follows from the splitting result 1.5.1 and the fact that \( n_{\Delta^n} = n \).

The above assertion roughly states that the sheaves \( \mathcal{O}(k), 0 \leq k \leq n \), generate the derived category of \( \mathit{hCoh}(P) \). This point of view has been pursued, in a model category context, in [Hut10, 3.3.5]. We sketch the argument for the reader’s convenience.

Suppose \( Y \in \mathit{hCoh}(\Delta^n) \) has the property that all the structure maps \( Y^F \longrightarrow Y^G \) are quasi-isomorphisms, for all pairs \( F \subseteq G \) of non-empty faces of \( \Delta^n \). For fixed \( F \in F(\Delta^n)_0 \) the structure maps then induce a chain of quasi-isomorphisms of diagrams

\[
Y \longrightarrow \mathit{con} Y_{\Delta^n} \longrightarrow \mathit{con} Y^F.
\]

By Proposition 2.2.3 and Lemma 2.7.3 we obtain quasi-isomorphisms of chain complexes of \( R \)-modules

\[
\Gamma Y \longrightarrow \Gamma(\mathit{con} Y_{\Delta^n}) \longrightarrow \Gamma(\mathit{con} Y^F) \longrightarrow Y^F[n].
\]

So if in addition \( \Gamma(Y) = \Gamma(Y(0)) \) is acyclic we know that \( Y^F[n] \) and hence \( Y^F \) is acyclic as well. — It is thus sufficient to prove the following claim:

\textbf{3.3.1.2.} Let \( Y \in \mathit{hCoh}(\Delta^n) \) be such that \( \Gamma(Y(\ell)) \) is acyclic for all \( \ell \) with \( 0 \leq \ell \leq n \). Then the structure maps \( Y^F \longrightarrow Y^G \) are quasi-isomorphisms for all pairs \( F \subseteq G \) of non-empty faces of \( \Delta^n \).

It is in fact enough to consider structure maps of the form \( Y^F \longrightarrow Y^{v \vee F} \) for \( F \in F(\Delta^n)_0 \) and a vertex \( v \) of \( P \).
As remarked above, $\Delta^n$ is isomorphic to a standard $n$-simplex with vertices $0, e_1, \cdots, e_n \in \mathbb{R}^n$; the isomorphism can be chosen to map any vertex of $\Delta^n$ to 0. In view of this symmetry it is enough to prove the following:

**3.3.1.3.** Suppose that $\Delta^n$ is a standard $n$-simplex, and suppose that $Y \in h\text{Coh}(\Delta^n)$ is such that $\check{\Gamma}(Y(\ell))$ is acyclic for all $\ell$ with $0 \leq \ell \leq n$. Then for every face $F$ of $\Delta^n$ not containing 0 the structure map $Y^F \longrightarrow Y^{0\cap F}$ is a quasi-isomorphism.

This assertion is proved by induction on $n$, the case $n = 0$ being trivial as $\text{Pre}(\Delta^0) = h\text{Coh}(\Delta^0) = \text{Ch}(R\text{-Mod})$.

So let $n > 0$. For every face $F$ of $\Delta^n$ there is an obvious inclusion of sets $kF + T_F \subseteq (k + 1)F + T_F$, $0 \leq k < n$, which is an equality if and only if $0 \in F$. Hence we have corresponding maps $\mathcal{O}(k) \longrightarrow \mathcal{O}(k + 1)$ and $Y(k) \longrightarrow Y(k + 1)$ which are identities if $0 \in F$. We obtain short exact sequences in $\text{Pre}(\Delta^n)$

$$0 \longrightarrow Y(k) \longrightarrow Y(k + 1) \longrightarrow Z(k + 1) \longrightarrow 0$$

(3.3.1.4)

where $Z(k + 1)$ is, a priori, simply a name for the cokernel. However, since taking cokernels commutes with tensor products we see that $Z(k + 1)$ is indeed the $k$th twist of $Z(1) = \text{coker} (\mathcal{O}(0) \longrightarrow \mathcal{O}(1))$. As the functor $\check{\Gamma}$ preserves short exact sequences we conclude that $\check{\Gamma}Z(k + 1)$ is acyclic for $0 \leq k < n$ since $\check{\Gamma}Y(k)$ and $\check{\Gamma}Y(k + 1)$ are acyclic in this range by hypothesis.

We now have to analyse the diagram $Z(k + 1)$ in more detail. If $0 \in F$ the map $Y(k)^F \longrightarrow Y(k + 1)^F$ is the identity, as remarked above, so $Z(k + 1)^F = 0$. — Suppose now that $F$ is a face of $\Delta^n$ with $0 \notin F$. There is in fact a maximal face $\Delta^{n-1}$ of $\Delta^n$ not containing 0, and $F$ is a face of $\Delta^{n-1}$. We will argue that $Z(k + 1)$, when restricted to $F(\Delta^{n-1})$, is naturally an object of $h\text{Coh}(\Delta^{n-1})$ with $\check{\Gamma}_{\Delta^{n-1}}Z(k + 1)$ being acyclic for $0 \leq k < n$. By induction this implies:

**3.3.1.5.** The chain complex $Z(1)^F$ is acyclic for all $F \in F(\Delta^{n-1})$ and hence for all $F \in F(\Delta^{n-1})_0$.

First let $\mathbb{R}^{n-1}$ denote the affine hull of $\Delta^{n-1}$, turned into a vector space by distinguishing a lattice point as origin. It comes equipped with its own integer lattice $\mathbb{Z}^{n-1} = \mathbb{Z}^n \cap \mathbb{R}^{n-1}$. Let $F \in F(\Delta^{n-1})_0$. Then

$$(((k + 1)F + T_F) \cap \mathbb{Z}^n) \setminus (((k + 1)F + T_F) \cap \mathbb{Z}^n) = ((k + 1)F + T_F^{\mathbb{Z}^{n-1}}) \cap \mathbb{Z}^{n-1} \setminus \mathbb{Z}^n,$$

the barrier cones on the left being computed in $\mathbb{R}^n$, the barrier cone on the right being computed in $\mathbb{R}^{n-1}$. This translates into an isomorphism

$$\text{coker} (\mathcal{O}_{\Delta^n}(k)^F \longrightarrow \mathcal{O}_{\Delta^n}(k + 1)^F) \cong \mathcal{O}_{\Delta^{n-1}}(k + 1)^F.$$  \hspace{1cm} (3.3.1.6)

By considering $k = -1$ we obtain from this an algebra isomorphism

$$A_F^\mathbb{Z} / \mathcal{O}_{\Delta^n}(-1)^F = \mathcal{O}_{\Delta^n}(0)^F / \mathcal{O}_{\Delta^n}(-1)^F \cong A_F^\mathbb{Z}^{n-1}$$

(3.3.1.7)

and thus an algebra epimorphism $A_F^{\mathbb{Z}} \longrightarrow A_F^{\mathbb{Z}^{n-1}}$. The isomorphism displayed in (3.3.1.7) is used to equip $Z(k + 1)^F$ with a natural $A_F^{\mathbb{Z}^{n-1}}$-module
structure while the isomorphism (3.3.1.6) is used to verify that twisting with respect to $\Delta^n$ and $\Delta^{n-1}$, respectively, is compatible. A straightforward 5-lemma argument shows that $Z(k)$, when considered as an object of $\text{Pre}(\Delta^{n-1})$, is indeed a homotopy sheaf. Finally, from the definition of Čech complexes it follows that the chain complexes $\check{\Gamma}(\Delta^n(Z(k + 1))$ and $\check{\Gamma}((\Delta^{n-1})_0)$ agree up to re-indexing by 1. In total, this means that $Z(k + 1) \in \text{hCoh}(\Delta^{n-1})$ satisfies the induction hypotheses as claimed.

We have thus verified 3.3.1.5.

From the short exact sequence (3.3.1.4), restricted to $F$-components, we infer that the map $\sigma_F : Y(0)^F \rightarrow Y(1)^F$ is a quasi-isomorphism of chain complexes, hence so is the map from $Y(0)^F$ to the colimit of the infinite sequence

$$Y(0)^F \xrightarrow{\cong} Y(1)^F \xrightarrow{\sigma_F} Y(0)^F \xrightarrow{\cong} Y(1)^F \xrightarrow{\sigma_F} Y(0)^F \xrightarrow{\cong} \cdots.$$  

Here every second map is a fixed isomorphism between $Y(0)^F$ and $Y(1)^F$. It is not difficult to see that the colimit of this sequence is isomorphic to $A_{0/V} \otimes_{A_F} Y^F$ (this follows from the fact that the cone $T_{0/V}$ is obtained from the cone $T_F$ by forming Minkowski sum with a single ray spanned by the negative of a vector in $T_F \cap \mathbb{Z}^n$). Now the composite

$$Y^F \rightarrow A_{0/V} \otimes_{A_F} Y^F \rightarrow Y^{0/V}$$

is a structure map of $Y$, and both constituent maps are quasi-isomorphisms: the first by what we have just shown, the second by the stipulation that $Y$ be a homotopy sheaf. In total, we have verified that the structure map $Y^F \rightarrow Y^{0/V}$ is a quasi-isomorphism. We are done. \hfill $\Box$

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