Sparse Signal Detection in Heteroscedastic Gaussian Sequence Models: Sharp Minimax Rates

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Abstract

Given a heterogeneous Gaussian sequence model with unknown mean \( \theta \in \mathbb{R}^d \) and known covariance matrix \( \Sigma = \text{diag}(\sigma_1^2, \ldots, \sigma_d^2) \), we study the signal detection problem against sparse alternatives, for known sparsity \( s \). Namely, we characterize how large \( \epsilon^* > 0 \) should be, in order to distinguish with high probability the null hypothesis \( \theta = 0 \) from the alternative composed of \( s \)-sparse vectors in \( \mathbb{R}^d \), separated from 0 in \( L^t \) norm \( (t \in [1, \infty]) \) by at least \( \epsilon^* \). We find non-asymptotic minimax upper and lower bounds over the minimax separation radius \( \epsilon^* \) and prove that they are always matching. We also derive the corresponding minimax tests achieving these bounds. Our results reveal new phase transitions regarding the behavior of \( \epsilon^* \) with respect to the level of sparsity, to the \( L^t \) metric, and to the heteroscedasticity profile of \( \Sigma \). In the case of the Euclidean (i.e. \( L^2 \)) separation, we bridge the remaining gaps in the literature.

1 Introduction

Global testing against structured alternatives is a canonical problem in modern statistics. Under the minimax hypothesis testing framework formulated by [Bur79, Ing82, Ing87, IIS03], the central object of interest is the minimax separation radius—intuitively, this corresponds to the smallest separation between the null and alternative hypotheses so that consistent detection is possible. From a mathematical perspective, it is particularly interesting to characterize the effect of the problem parameters on the separation radius.

Motivated by questions arising from genomics, communications, social sciences etc., diverse global testing problems have been rigorously investigated, and their associated separation radii have been characterized—see e.g. [Bur79, Ing82, Ing87, IIS03, DJ04, HJ10, ACCP11, ITV10, TCJJ11] and references therein.

The gaussian sequence model furnishes arguably the most canonical setup to explore the fundamental thresholds for global testing. Starting with the original works of [Bur79, Ing82, Ing87, IIS03], the global testing problem has been carefully studied in this context under diverse structured alternatives, and under many different notions of separation. In the modern age of big-data, sparse alternatives are particularly important. The detection thresholds for gaussian sequence model under sparse alternatives has been derived in [IIS03, DJ04, HJ10, ACCP11, ITV10]. However, most of the existing literature focuses on the homoscedastic sequence model, where the error variances are all equal. In this work, we go beyond the homoscedastic case, and derive the separation radius for heteroscedastic Gaussian sequence models under sparse alternatives.

Hereafter, we write \( [k] = \{1, \ldots, k\} \) for any positive integer \( k \). For any positive integer \( d \), any vector \( u \in \mathbb{R}^d \), and any \( t > 0 \), we write \( \|u\|_t = \left( \sum_{j=1}^d u_j^t \right)^{1/t} \), \( \|u\|_{\infty} = \max_{i=1}^d |u_i| \) and \( \|u\|_0 = \text{Card} \{j \in [d] : u_j \neq 0\} \).
1.1 Model

Fix $d$ real numbers $\sigma_1 \geq \cdots \geq \sigma_d > 0$, which we assume to be known throughout the paper. For some unknown $\theta \in \mathbb{R}^d$, suppose we observe $X = (X_1, \ldots, X_d)$ where $\forall j \in [d]: X_j = \theta_j + \sigma_j \xi_j$. Here, $\xi_j \sim \mathcal{N}(0, \sigma_j^2)$ are independent and distributed as normal random variables with mean $\theta_j \in \mathbb{R}$ and variance $\sigma_j^2$. We shall often denote this observation scheme as $X \sim \mathcal{N}(\theta, \Sigma)$ where $\theta \in \mathbb{R}^d$ and $\Sigma = \text{diag}(\sigma_1^2, \ldots, \sigma_d^2)$. For some $t \in [1, \infty]$, some known sparsity $s \in [d]$, and some $\epsilon > 0$, we define $\Theta(\epsilon, s, t) = \{ \theta \in \mathbb{R}^d : \|\theta\|_t \geq \epsilon$ and $\|\theta\|_0 \leq s \}$ and consider the following testing problem

$$H_0 : \theta = 0 \quad \text{against} \quad H_1 : \theta \in \Theta(\epsilon, s, t).$$

Here, the parameter $\epsilon > 0$ induces a separation between the two hypotheses, and our goal is to characterize how large $\epsilon$ should be for the testing problem (1) to be feasible in a sense defined below. The $L^2$ separation (i.e. the $t = 2$ case) is by far the most studied case in the literature [Bar02, ITV10, LLM12, CCT17, KG21, LGS21]. In this paper, we go beyond the $L^2$ case and study general separations in $L^t$ norm for any $t \in [1, \infty]$.

A test $\psi$ is defined as a measurable function of the data $X$ taking its values in $\{0, 1\}$: $\psi : \mathbb{R}^d \rightarrow \{0, 1\}$. In the minimax paradigm, one measures the quality of any test by its risk. We assume that the matrix $\Sigma$ is fixed and known, and for any $\theta \in \mathbb{R}^d$, we let $P_\theta$ denote the probability measure associated with the law of $X \sim \mathcal{N}(\theta, \Sigma)$. Then the risk of test $\psi$ is defined as the sum of its type I and type II error probabilities:

$$R(\psi, \epsilon, s, t, \Sigma) := P_0 (\psi = 1) + \sup \left\{ P_\theta (\psi = 0) \mid \theta \in \Theta(\epsilon, s, t) \right\}. \quad (2)$$

Note that in equation (2), the notation $P_\theta$ denotes the probability distribution of $\mathcal{N}(\theta, \Sigma)$ and implicitly depends on $\Sigma$. The minimax risk represents the infimal risk among all possible tests, and can be understood as the risk of the best test if it exists:

$$R^*(\epsilon, s, t, \Sigma) := \inf_\psi R(\psi, \epsilon, s, t, \Sigma). \quad (3)$$

In the above definition, the infimum is taken over all tests $\psi$. Note that if $R^*(\epsilon, s, t, \Sigma) = 1$, then random guessing is optimal (the test $\tilde{\psi}$ that accepts $H_0$ with probability 1/2 independently of the data $X$ achieves a risk equal to 1). Therefore, we say that the testing problem (1) is feasible if, for some tolerance $\eta \in (0, 1)$ fixed in advance, we have $R^*(\epsilon, s, t, \Sigma) \leq \eta < 1$. The sparsity $s$, the metric-inducing norm $\| \cdot \|_t$ and the covariance matrix $\Sigma$ being fixed, the difficulty of this testing problem is entirely characterized by the separation parameter $\epsilon > 0$. Noting that $R^*(\epsilon, s, t, \Sigma)$ decreases with respect to $\epsilon$, our goal is therefore to determine the smallest value of $\epsilon$ ensuring feasibility of Problem (1). This value is referred to as the minimax separation radius, defined as

$$\epsilon^*(s, t, \Sigma) = \inf \left\{ \epsilon > 0 \mid R^*(\epsilon, s, t, \Sigma) \leq \eta \right\}. \quad (4)$$

Note that we drop the dependency of $\epsilon^*(s, t, \Sigma)$ on $\eta$ as this parameter is assumed to be a fixed constant throughout the paper. Our goal is to characterize the minimax separation radius $\epsilon^*(s, t, \Sigma)$ up to multiplicative constants depending only on $\eta$ and $t$.

1.2 Motivation and background

Consider the following case of practical interest: we aim to test for genetic effects on a particular disease outcome. To do so, a procedure could be to test each gene individually, by computing a test statistic that
Heteroscedasticity arises naturally in many settings. We include two motivating examples, which can be reduced to the model considered in the present paper.

(i) Inverse problems—The heteroscedastic sequence model can arise naturally in statistical inverse problems, as described in [LLM12]. Fix a Hilbert space $H$ with inner product $(\cdot, \cdot)$, and suppose that $T$ denotes a known linear operator on $H$. Assume moreover that, for an unknown element $f \in H$, our observations consist of the following sequence model
\[
Y(g) = (Tf, g) + \sigma \varepsilon(g), \quad \forall g \in H,
\]
where $\varepsilon(g)$ is a centered gaussian with variance $(g, g)$. If the operator $T$ is compact, there exist two orthonormal bases of $H$, denoted as $(\psi_j)_j$ and $(\phi_j)_j$, the following singular value decomposition holds
\[
T\phi_j = \lambda_j \psi_j \quad \text{and} \quad T^*\psi_j = \lambda_j \phi_j, \quad \text{for any } j \in \mathbb{N},
\]
where $T^*$ is the adjoint of $T$. In particular, the observations $(Y(\psi_j))_{j \in \mathbb{N}}$ are independent and distributed as $Y(\psi_j) \sim \mathcal{N}(\lambda_j \theta_j, \sigma^2)$, where $\theta_j = (f, \phi_j)$ for any $j \in \mathbb{N}$. This is clearly equivalent to the heteroscedastic model introduced above, with $\sigma_j^2 = \sigma^2 / \lambda_j^2$. We note that under an appropriate choice of basis $\phi_j$, the representation of the function $f$ is often sparse—we refer the interested reader to [Joh02, LLM12] for an in-depth discussion of this phenomenon.

(ii) Linear regression with orthogonal design—Suppose that we observe $(X_1, y_1), \ldots, (X_n, y_n) \in \mathbb{R}^d \times \mathbb{R}$, $n \geq 1$, following the linear model
\[
\forall i \in [n] : \quad y_i = X_i^\top \theta + \xi_i, \quad \text{where} \quad \xi_i \overset{iid}{\sim} \mathcal{N}(0, \sigma)
\]
Assume that the columns of the design $X := [X_1, \ldots, X_n]$ are orthogonal (not necessarily orthonormal), and for simplicity, assume that $n = d$. Then $X$ can be decomposed as $X = U \Sigma^{1/2}$, where $U^\top U = I_n$ and $\Sigma$ is diagonal. Setting $y = (y_1, \ldots, y_n)^\top$, $\xi = (\xi_1, \ldots, \xi_n)^\top$ and $z = (XX^\top)^{-1}X y$, we exactly obtain that $z = \theta + w$, where $w = (XX^\top)^{-1}X \xi \sim \mathcal{N}(0, \sigma^2 \Sigma^{-1})$ in the present case. Moreover, the regressor $\theta$ is often assumed to be sparse in many applications in genetics, communications, and compressed sensing. We refer the interested reader to [ACC11, FTV10, CCC+19, CCC+22] for a discussion of the relevant motivations. Therefore, observing $z \sim \mathcal{N}(\theta, \sigma^2 \Sigma^{-1})$ for sparse $\theta$ and diagonal $\Sigma$ allows us to exactly recover the setting of Problem (i). Our results consequently open up interesting developments in this setting of practical interest, which is left for future inquiries.

The aim of this paper is two-fold:

- To go beyond isotropic noise and study the interplays between sparsity and heteroscedasticity.
- To go beyond the Euclidean separation ($t = 2$), by providing a complete overview of the results for the $L^t$ separation, $t \in [1, \infty]$.

We explain here why these two goals are natural and challenging.

**Heteroscedasticity:** Heteroscedasticity arises naturally in many settings. We include two motivating examples, which can be reduced to the model considered in the present paper.
Exploring heteroscedasticity turns out to be significantly more complex than testing with isotropic noise, even for $t = 2$, as the transition phenomena depend on intricate interactions between the sparsity level and the heteroscedasticity profile of $\Sigma$. Unfortunately, even for $t = 2$, this dependence makes it impossible to express $\epsilon^*(s, t, \Sigma)$ in closed form in general, and our expressions are fundamentally implicit in nature.

$L^t$ separations: The geometry induced by the $\| \cdot \|_t$ metric changes which signals are easier to detect. Since we aim at detecting signals $\theta$ for which $\| \theta \|_t$ is large, the $L^1$ norms for large $t$ are more adapted to detect imbalanced signals, such as, for instance, those with one dominating coordinate. The extreme case of the $L^\infty$ norm can be used if we aim at detecting if at least one gene has an influence at all. In contrast, the $L^1$ norm enables us to better detect signals with numerous tiny coordinates, but large total mass.

We motivate in more detail the case of the $L^1$, $L^2$ and $L^\infty$ norms, which stand out as important particular cases of our results and could have implications in related settings. Another goal of the present paper is also to thoroughly study the interpolation between those three cases of interest.

1. The $L^1$ distance importantly represents the total variation metric over discrete and continuous distributions. It turns out that our results in $L^1$ separation are not direct analogs of the case $t = 2$, and instead exhibit much more intricate phase transitions. Considering that multinomial data are fundamentally heteroscedastic, our results could find applications of independent interest in the important field of distribution learning and testing, where sparse alternatives have recently received much attention [BM21, DK22, Kip22, Kip21, KD21]. We provide a more detailed discussion about our results’ implications for multinomial testing in Section 6.2.

2. The $L^2$ or Euclidean separation has a natural geometric interpretation, and its smoothness properties make it an ideal choice for studying signal detection. As such, it constitutes a canonical case to compare results with existing literature. In Euclidean separation, signal detection under general covariance matrix is well understood [HS03, LLM12]. Signal detection under sparsity and isotropic noise is also well understood [CCT17]. However, the interaction of the two constraints raises important challenges and is just starting to be explored (see Subsection 1.3).

3. It turns out that the case of the $L^\infty$ has occurred naturally and extensively in the literature of hypothesis testing under sparse alternatives [CGM21, CLX13, CLX14, CLL18, CZZZ17, LQL21, XCC18, XY20, YLXL22, ZC17]. Indeed, for homoscedastic variance profiles it is easy to see that testing in $L^\infty$-separation is naturally equivalent to testing with a sparsity of $s = 1$. Our results supplement this philosophy for heteroscedastic variance profiles and complete the picture of hypothesis testing against sparse alternatives in Gaussian sequence models under any $L_r$-norm.

Going beyond the $L^2$ case presents numerous technical challenges. For instance, when $t \in [1, 2)$, the functional $\| \cdot \|_t$ is not twice continuously differentiable, which raises complications for constructing chi-square type test statistics, which are based on degree 2 polynomials in the coordinates $X_j$’s. We further discuss the technical challenges raised by the $L^1$ separation in the Sections below.

1.3 Prior results and contributions

Prior results: As remarked above, global null testing for the gaussian sequence model has been extensively studied, for various notions of separation and alternatives. In the Gaussian mean model, minimax testing under sparse alternatives was pioneered by [Ing97, DJ04] in the asymptotic where $s = d^\beta$, for $\beta \in (0, 1)$.
Nonasymptotic rates were first given in [Bar02], matching up to an extra logarithmic gap in the upper bound, which was later closed by [CCT17]. However, all of the above papers only restrict to isotropic noise. It turns out that non-isotropic gaussian noise still represents a challenge in sparse signal detection. Important attempts to go beyond isotropic noise include [KG21], dealing with correlated noise structures, and more closely, [LLMT22] considering the same heteroscedastic sequence model (Problem 1) studied in this paper, but only with Euclidean separation ($t = 2$). Leveraging techniques developed in [Bar02], they obtained non-asymptotic upper and lower bounds on the minimax separation radius, which did not match for specific profiles of the covariance matrix $\Sigma$.

**Contributions:** In this paper, we provide a complete understanding of sparse signal detection with diagonal noise covariance matrix and general $L^t$ separation, $t \in [1, \infty]$. We derive upper and lower bounds for the minimax separation radius that are always matching, and explicitly construct the corresponding minimax tests. All of our results are non-asymptotic. We uncover new interplays between sparsity, the $L^t$ metric, and to the heteroscedasticity profile of $\Sigma$, and thoroughly study the corresponding phase transitions. To the best of our knowledge, the matching upper and lower bounds for diagonal covariance had not been established in the literature, even for the $L^2$ separation.

**Organization.** The rest of the paper is structured as follows. In Section 2, we present the case where $t \geq 2$ before moving to the case $t \in [1, 2]$ in Section 3 and $t = \infty$ in Section 4. We give some examples in Section 5, and conclude with a discussion of our results and some directions for future inquiry in Section 6.

**Notation** We denote by $\mathbb{N}^*$ the set of positive integers. For any $k \in \mathbb{N}^*$, we denote by $I_k$ the identity matrix of size $k$. Let $\eta > 0$. For any two real-valued functions $f$ and $g$, we write $f \lesssim g$ (resp. $f \gtrsim g$) when there exists a constant $c(\eta, t) > 0$ (resp. $C(\eta, t) > 0$) depending only on $\eta$ and $t$, such that $c(\eta, t) \cdot g \leq f$ (resp. $f \leq C(\eta, t) \cdot g$). We write $f \asymp g$ if $g \lesssim f$ and $f \lesssim g$. We respectively denote by $x \vee y$ and $x \wedge y$ the maximum and minimum of the two real values $x$ and $y$, and we set $x_+ = x \vee 0$. Note that the constants denoted by $C$ or $c$, depending on $\eta$ and $t$, are allowed to take different values on each appearance. We also denote by $\text{TV}(P, Q)$ the total variation between any two probability measures $P, Q$ defined over the same measurable space $(\mathcal{X}, \mathcal{U})$. For any $d \in \mathbb{N}^*$, and for any property $P(j)$ over index $j \in [d]$, we set $\max\{j \in [d] : P(j)\} = 0$ if for any $j \in [d]$, $P(j)$ is false.

## 2 Minimax rates in $L^t$ separation for $t \in [2, \infty)$

In this Section, we study Problem 1 with $t \in [2, \infty)$, and highlight that all of these problems can fundamentally be approached in a similar way. The $L^2$ separation stands out as an important special case of this Section’s results, as it is extensively studied in the literature. Here we bridge the remaining gaps in Euclidean separation, which were left as an open question in [LLMT22]. To present the main result, we first need to introduce a few definitions.

Let $\beta \in \mathbb{R}$ be the unique solution to the equation

$$\frac{\sum_{j=1}^{d} \sigma_j^t \exp \left(-\beta / \sigma_j^2\right)}{\sqrt{\sum_{j=1}^{d} \sigma_j^{2t} \exp \left(-\beta / \sigma_j^{2t}\right)}} = \frac{s}{2},$$

and let $\lambda = \sqrt{\beta_+}$. \hfill (5)

We first observe that the values $\beta$ and $\lambda$ in (5) are well-defined: Lemma 10 guarantees that equation (5) admits a unique solution, by ensuring that the left-hand side is continuous, strictly decreasing, and tends to
appears in the expression of the minimax separation radius \( \epsilon \) problem under study, namely: the sparsity \( s \), the heteroscedasticity profile of \( \Sigma \), and the metric-inducing norm \( \| \cdot \| \). The information captured by \( \lambda \) is the essential key to our problem, as this value fundamentally appears in the expression of the minimax separation radius \( \epsilon^*(s, t, \Sigma) \), in our lower bounds, and in the construction of our minimax optimal tests. Unfortunately, the value of \( \lambda \) cannot be expressed explicitly as a function of the \( \sigma_j \)'s in general, although it is possible to solve equation (10) for some specific profiles of \( \Sigma \) (see Section 5). Theorem 1 below states the expression of the minimax separation radius.

**Theorem 1.** Assume that \( t \geq 2 \). Let \( \lambda \) be defined as in equation (5) and let \( \nu = \left[ \sum_{j=1}^{d} \sigma_j^2 e^{-\lambda^2/\sigma_j^2} \right]^{1/2} \). Then the following hold.

i. [Lower Bound] There exists a small constant \( c \) depending only on \( \eta \), such that

\[
\epsilon^*(s, t, \Sigma)^t \geq c \left( \lambda^t s + \nu^t \right).
\]

ii. [Upper Bound] There exists a large enough constant \( C' \) depending only on \( \eta \) such that the test \( \hat{\psi} \) defined in (17) satisfies

\[
\begin{align*}
\mathbb{P}_{\theta} (\hat{\psi} = 1) &\leq \eta/2 \quad \text{if} \quad \theta = 0, \\
\mathbb{P}_{\theta} (\hat{\psi} = 0) &\leq \eta/2 \quad \text{if} \quad \|\theta\|_0 \leq s \text{ and } \|\theta\|_t^t \geq C' \left( \lambda^t s + \nu^t \right).
\end{align*}
\]

Therefore, \( \epsilon^*(s, t, \Sigma)^t \leq C' \left( \lambda^t s + \nu^t \right) \).

Theorem 1 immediately establishes that \( \epsilon^*(s, t, \Sigma)^t \approx \lambda^t s + \nu^t \). Another expression of \( \epsilon^*(s, t, \Sigma)^t \) is given in the following corollary, which will be useful for further connections and discussions in later sections.

**Corollary 1.** Under the assumptions of Theorem 1 we have that

\[
\epsilon^*(s, t, \Sigma)^t \approx \lambda^t s + \nu^t, \quad \text{where} \quad \nu^t = \left[ \sum_{j=1}^{d} \sigma_j^2 \exp \left( -\lambda^2 / \sigma_j^2 \right) \right]^{1/2},
\]

\[
\approx \lambda^t s + \sum_{j \leq \lambda^t s} \sigma_j^{2t}, \quad \text{where} \quad j_\ast = \max \left\{ j \in [d] \mid \sigma_j \geq \lambda \right\}.
\]

Theorem 1 is proved in Section 3. Proofs of the lower and upper bound are provided in subsections B.1 and B.2 respectively. Further, the simplification in (7) of Corollary 1 is proved in Lemma 11.

Equation (7) will be interesting to compare with the opposite case \( t \in [1, 2] \) covered in Theorem 2, Section 3.

Finally, a few remarks are in order regarding the implications of Theorem 1. We organize them along three subsections regarding the special case of \( L^2 \)-separation, proof ideas behind the lower bounds, and motivations behind the upper bounds in the theorem.

### 2.1 \( L^2 \)-Separation

The \( L^2 \) separation has attracted the strongest attention in signal detection under gaussian sequence models. In the next corollary, we therefore collect our result in Euclidean separation, and specifically discuss its connections with and differences from existing literature.

**Corollary 2.** Let \( \lambda_2 = \lambda^2 \) where \( \lambda \) is defined as in equation (5) for \( t = 2 \). Then

\[
\epsilon^*(s, 2, \Sigma)^2 \approx \lambda_2 s + \nu_2 \quad \text{where} \quad \nu_2 = \left( \sum_{j=1}^{d} \sigma_j^4 \exp \left( -\lambda_2 / \sigma_j^2 \right) \right)^{1/2}.
\]
A few remarks are in order regarding Corollary 2.

(a) Corollary 2 is best understood when compared with classical results in the isotropic case for \( t = 2 \). In particular, assuming that \( \sigma_1 = \cdots = \sigma_d := \sigma \) and using \( I_d \) to denote the identity matrix of size \( d \), [CCT17] derives

\[
\epsilon^*(s, 2, \sigma^2 I_d) \simeq \begin{cases} 
\sigma d^{1/4} & \text{if } s \geq \sqrt{d}, \\
\sigma \sqrt{s \log(1 + d/s^2)} & \text{otherwise}.
\end{cases}
\]

Further, the elbow at \( s = \sqrt{d} \) in the expression above can be replaced by \( s = c \sqrt{d} \) for any absolute constant \( c \) without affecting the rate, up to multiplicative constants depending only on \( c \). Indeed, this result can be recovered from Corollary 2 in the present paper. To see this note that in the homoscedastic model \( \beta \) solves \( \sqrt{d} \exp(-\beta/2 \sigma^2) = s/2 \), i.e. \( \beta = 2 \sigma^2 \log \left( \frac{2 \sqrt{d}}{s} \right) \). Therefore two cases emerge.

- If \( s > 2 \sqrt{d}/e \), then \( \sigma^2 > \lambda_2 \) and \( j_* = d \). Therefore, equation (9) yields that \( \epsilon^*(s, 2, \sigma^2 I_d) = \sigma d^{1/4} \).
- Otherwise, if \( s < 2 \sqrt{d}/e \), then \( \sigma^2 \leq \lambda_2 \) and \( j_* = 0 \) so that the minimax separation radius scales as \( \epsilon^*(s, 2, \sigma^2 I_d) \simeq \sqrt{\lambda_2 s} \simeq \sigma \sqrt{s \log(1 + d/s^2)} \).

(b) Noticeably, the isotropic case involves an extreme phase transition: we either have \( j_* = 0 \) or \( j_* = d \). In this case, all of the coordinates exclusively belong to the dense set of coordinates \( \{ j \leq j_* \} \) or to the sparse set of coordinates \( \{ j > j_* \} \). In our heteroscedastic model, however, the phase transition is more subtle. When \( s = d \), all of the coordinates contribute to the dense regime \( \left( \sum_{j=1}^d \sigma_j^4 \right)^{1/4} \). When we let \( s \) decrease from \( d \) to 1, the cut-off \( j_* \) progressively shifts from \( d \) to 0.

(c) This progressive shift between the dense and sparse regimes is reflected in our expression of \( \epsilon^*(s, 2, \Sigma) \), which involves two contributions. We recall that in the fully dense case \( s = d \), the minimax separation radius \( \epsilon^*(d, 2, \Sigma)^2 \) is known to be \( \left( \sum_{j=1}^d \sigma_j^4 \right)^{1/2} \) (see e.g. [LLM12] Propositions 1 and 2). In comparison, the dense contribution \( \left( \sum_{j=1}^{j_*} 1 \right)^{1/4} \) in Corollary 2 therefore represents the separation obtained by only testing the first \( j_* \) coordinates with a sparsity \( s' = j_* \). The second contribution in this rate corresponds to the term \( \lambda_2 s \). In the isotropic case, the term \( \lambda_2 s \) is responsible for the rate \( \sigma \sqrt{s \log(d/s^2)} \) when \( s \ll \sqrt{d} \).

(d) When \( \lambda_2 = 0 \), which, by Lemma [10] is equivalent to the condition \( s/2 \geq \sqrt{\sum_{j=1}^d \sigma_j^2} \), it holds that \( \epsilon^*(s, 2, \Sigma) \simeq \epsilon^*(d, 2, \Sigma) \). In other words, sparsity does not help. In the homoscedastic case, this phenomenon arises when \( s \geq \sqrt{d} \). In the heteroscedastic case, the elbow at \( s = \sqrt{d} \) is replaced by an elbow at \( s = \text{Tr}(\Sigma)/\text{Tr}(\Sigma)^{1/2} \). This quantity is commonly referred to as the stable rank of \( \Sigma \), and represents a suitable notion of intrinsic dimension.

(e) Note that for general \( t \geq 2 \), still by Lemma [10] we have \( \lambda = 0 \) whenever \( s \geq \text{Tr}((\Sigma^2)^{t/2})/\sqrt{\text{Tr}(\Sigma^2)} \). Sparsity never helps in this case. We also note that the intrinsic dimension depends on \( t \).
2.2 Lower bounds for $t \geq 2$

In our lower bound construction, we use Le Cam’s two points method by defining a prior distribution over the parameter space $\Theta = \mathbb{R}^d$, which we detail here. We distinguish between two cases.

If $s < c(\eta)$ for a sufficiently large constant $c(\eta)$ depending only on $\eta$, we use the immediate relation $\epsilon^*(s, t, \Sigma) \geq \epsilon^*(1, t, \Sigma)$ and further show that $\epsilon^*(1, t, \Sigma) \geq c(\lambda + \nu)$, where $c > 0$ is a sufficiently small constant. To do so, we propose a combination of two 1-sparse priors, each being separated from the null hypothesis in $L^t$ norm by $c\lambda$ and $c\nu$ respectively. Moreover, we prove that no test can distinguish them from the null hypothesis with high probability. This is made precise in Lemma 3 from Appendix A.1, where we give the expression of the corresponding priors.

Conversely, if $s \geq c(\eta)$, our prior is as follows. We define a random vector $\theta \in \mathbb{R}^d$ whose coordinates $\theta_j$ are mutually independent and satisfy

$$\forall j \in [d] : \theta_j = b_j \omega_j \gamma_j,$$

where $b_j \sim \text{Ber}(\pi_j)$, $\omega_j \sim \text{Rad}(\frac{1}{2})$ are mutually independent. Here,

$$\pi_j = \frac{\sigma_j \exp(-\lambda^2/\sigma_j^2)}{\sqrt{\sum_{j=1}^d \sigma_j^{2n} \exp(-\lambda^2/\sigma_j^2)}}$$

$$\gamma_j = \sigma_j \arg \sinh\left[ c \cdot \exp\left(\lambda^2/\sigma_j^2\right) \right] \approx \lambda + \sigma_j,$$

for some small enough $c$ depending only on $\eta$. In other words, each coordinate $\theta_j$ takes the value 0 with probability $1 - \pi_j$, and $\pm \gamma_j$ with probability $\pi_j \in [0, 1]$. This vector $\theta$ therefore has a random sparsity $\sum_{j=1}^d b_j$, whose expectation is $\sum_{j=1}^d \pi_j = s/2$ by the definition of $\lambda$ from (5).

We now describe the intuitions behind the latter prior construction. The parameters $\pi_j$ and $\gamma_j$ in (11) and (12) are found by solving the following optimization problem

$$\max_{\gamma, \pi} \sum_{j=1}^d \gamma_j \pi_j \quad \text{s.t.} \quad \sum_{j=1}^d \pi_j = s/2, \quad \pi_j \in [0, 1] \quad \forall j \in [d],$$

$$\sum_{j=1}^d \pi_j^2 \sinh^2\left(\frac{\lambda_j^2}{2\sigma_j^2}\right) \leq c',$$

for some sufficiently small constant $c'$ depending on $\eta$. The interpretation of Problem (13) is as follows: Among all priors of the form (10), Problem (13) finds the parameters $(\pi_j)_j$ and $(\gamma_j)_j$ maximizing $\mathbb{E}\|\theta\|_t^* = \sum_{j=1}^d \gamma_j^2 \pi_j$ under the constraints that $\|\theta\|_0 \leq s$ with high probability, and that no test can distinguish this prior from the null hypothesis with sufficient probability.

More precisely, the condition $\sum_{j=1}^d \pi_j = s/2$ guarantees that our prior’s sparsity is at most $s$ with high probability, and the condition $\sum_{j=1}^d \pi_j^2 \sinh^2\left(\gamma_j^2/2\sigma_j^2\right) \leq c'$ ensures that

$$\text{TV}^2(P_{\text{prior}}, P_0) \leq \chi^2(P_{\text{prior}}, P_0) \leq c'$$

(see equation (63)), which we refer to as the indistinguishability condition. In the last equation, we denoted by $P_{\text{prior}}$ the probability distribution induced by the prior (10).

This variational problem elucidates how the phase transitions arise in the behavior of $\epsilon^*(s, t, \Sigma)$. We recall that $j_*$ is the index where the transition $\sigma_j \leq \lambda$ occurs (see equation (7)). Now from (12), we can deduce
that $j_*$ is also the index where the transition $\gamma_j \leq c \sigma_j$ occurs. Therefore, over $\{1, \ldots, j_*\}$, the following linearization holds: $\sinh(\gamma_j^2/2\sigma_j^2) \approx \gamma_j^2/\sigma_j^2$. Conversely, over $\{j_*+1, \ldots, d\}$, the following relation holds: $\sinh^2(\gamma_j^2/2\sigma_j^2) \approx \exp(\gamma_j^2/\sigma_j^2^2)$. These two parts therefore exhibit fundamentally different behaviors in the analysis of the lower bound. For $j \leq j_*$, the perturbation $\gamma_j \propto \sigma_j$ coincides with the optimal perturbation that one would set in absence of sparsity (it suffices to evaluate our results at $s = d$, which implies $\lambda = 0$, so that $\gamma_j \propto \sigma_j$ for any $j \in [d]$ by (12)). For $j > j_*$, the perturbation $\gamma_j \propto \lambda$ is more surprising, as it does not depend on $\sigma_j$.

Moreover, we note that $\pi_j \propto \sigma_j^2 \exp(-\lambda^2/\sigma_j^2)$, which implies $\pi_1 \geq \cdots \geq \pi_d$, with a very fast decay of $\pi_j$ with $\sigma_j \ll \lambda^2$. In other words, our sparse prior preferably selects and perturbs the coordinates with largest standard deviations $\sigma_j$’s. This makes intuitive sense: If $\sigma_j = 0$, then under $H_0$, one should observe $X_j = 0$ a.s., so that the optimal value of $\pi_j$ should be 0. This stands in contrast with the lower bound proposed in the paper [LLM12], which also considers Problem (1) for $t = 2$. In the latter paper, the prior is defined by perturbing some coordinates $\{\theta_j : j \in J\}$, for a set $J$ selected uniformly at random among subsets of $[d]$ of cardinality $s$, leading to sub-optimality in specific regimes.

Finally, for $t \geq 2$, the constraint $\pi_j \in [0,1]$ is never saturated. To anticipate on Subsection 3.2 we will see that this constraint can be saturated for $t < 2$, giving rise to a third regime in this case, which we will refer to as the fully dense regime.

Finally, the reason why we need to define two different priors when $r \geq c(\eta)$ or $s < c(\eta)$ is as follows. In the homoscedastic case [Bar02, LLM12, CCT17, KG21], the optimal sparse prior can be defined by selecting the support of $\theta$ uniformly at random among all subsets of $\{1, \ldots, d\}$ with cardinality $s$. In our case, we would like to select exactly $s$ coordinates without replacement in $[d]$, each coordinate being selected with probability proportional to $\pi_j$. This leads to technical difficulties, which we circumvent using the prior (10), which has independent coordinates but random sparsity, equal to $\sum_{j=1}^d b_j$. In expectation, our prior has sparsity $\sum_{j=1}^d \pi_j \leq s/2$, with a standard deviation of $\left(\sum_{j=1}^d \pi_j(1-\pi_j)\right)^{1/2} \leq \sqrt{s}$. By Chebyshev’s inequality our prior’s sparsity is therefore at most $s$ with probability $1-\eta/10$, provided that $s$ is greater than a constant depending only on $\eta$ (see equation (12)). In the opposite case $s < c(\eta)$, such a prior would not be at most $s$-sparse with high probability. This is why we adopt another strategy, which is to set a 1-sparse prior, selecting only one coordinate $\{1, \ldots, d\}$, each coordinate $j$ being selected with probability proportional to $\pi_j$.

### 2.3 Upper bounds for $t \geq 2$

In this subsection, we describe the test achieving the upper bound in Theorem 1. Let $\lambda$ and $\nu$ be defined as in equation (5) and (6) respectively, and let

$$\tau = C_\nu \lambda^t + \nu^t/s \quad \text{where } C_\nu = (4t)^t.$$  

$$\alpha_j = \mathbb{E}\left[|Z_j|^t \mid |Z_j|^t > \tau\right] \quad \forall j > j_*, \quad \text{where } Z_j \sim \mathcal{N}(0, \sigma_j^2).$$  

Subsequently we define our test statistics as follows:

$$T_{\text{dense}} = \sum_{j \leq j_*} |X_j|^t - \mathbb{E}_{H_0} |X_j|^t,$$

$$T_{\text{sparse}} = \sum_{j > j_*} (|X_j|^t - \alpha_j)1 \{|X_j|^t > \tau\}.$$


For some large enough constant $C$ depending only on $\eta$, we finally define the test functions

\[
\begin{align*}
\psi_{\text{dense}} &= 1\{T_{\text{dense}} \geq C \left( \sum_{j \leq j_*} \sigma_j^2 \right)^{1/2} \}, \\
\psi_{\text{sparse}} &= 1\{T_{\text{sparse}} > C\rho \}, \\
\psi &= \psi_{\text{dense}} \lor \psi_{\text{sparse}}
\end{align*}
\]

where $\rho = \lambda^t s + \nu^t$.

We can now compare the upper bound from Theorem 11 with the literature in the isotropic case \cite{CCT17} for $t = 2$. Recalling the notation from Corollary 2 when $\sigma_1 = \cdots = \sigma_d =: \sigma$ the test statistic used in \cite{CCT17} is defined as follows

\[
T = \begin{cases}
\sum_{j=1}^d (X_j^2 - \sigma^2) & \text{if } s > \sqrt{d}, \\
\sum_{j=1}^d (X_j^2 - \sigma^2) 1_{|X_j| > \sigma \sqrt{\log(1+d/s^2)}} & \text{otherwise},
\end{cases}
\]

(19)

(20)

where $\alpha_s = \mathbb{E} \left[ X^2 \mid X^2 > 2 \log(1 + \frac{d}{s^2}) \right]$. We can compare this with our results from Theorem 1. As noted above, when $s > 2\sqrt{d/e}$, our result implies that $j_* = d$ and our dense test statistic coincides with (19).

On the other hand, when $s \leq 2\sqrt{d/e}$, we have $j_* = 0$ and our sparse test coincides with (20). Contrary to \cite{CCT17}, our phase transition occurs at $s = 2\sqrt{d/e}$ instead of $s = \sqrt{d}$, which only affects the multiplicative constants in the rates. Noticeably, equations (19) and (20) show that the isotropic case requires only one test at a time. To understand this, it suffices to note that in the isotropic case, $j_*$ is always equal to 0 or to $d$, meaning that either the dense region $\{1, \ldots, j_*\}$ or the sparse region $\{j_*+1, \ldots, d\}$ is empty. In contrast, under heteroscedastic noise, two fundamentally different contributions coexist in the rate, requiring the use of two separate tests to handle both regimes.

We can also analyze the truncation parameters of the sparse test statistics. If $s \leq \sqrt{d}$, the isotropic sparse test (20) requires the truncation $X_j^2 > 2\sigma^2 \log(1 + \frac{d}{s^2}) > 2\lambda_2$. In comparison, our sparse test $T_{\text{sparse}}$ requires the truncation $X_j^2 > \tau_2 = C\lambda_2 + \nu_2/s$, which is larger than $2\lambda_2$. The reason is that setting $\tau_2 = 2\lambda_2$ would cause $\text{Var}_n[T_{\text{sparse}}]$ to be too large when the dense regime dominates, that is, when $\nu_2 \gg \lambda_2 s$. However, when the sparse regime dominates, i.e. when $\nu_2 \lesssim \lambda_2 s$, we recover $\tau_2 \sim \lambda_2$. In the isotropic case \cite{CCT17}, truncating at $2\lambda_2$ is sufficient, since $\nu_2/s$ never dominates over $2\lambda_2$ when $s \leq \sqrt{d}$ (see Remark 1).

Finally, we compare our results with \cite{LLM12} where the authors also consider the heteroscedastic Gaussian sequence model under $L^2$ separation. Specifically \cite{LLM12} proposes to combine the test $1\left\{ \sum_{j=1}^d X_j^2 > t_{d,1-\alpha/2}(\sigma) \right\}$ with $1\{ \max_j X_j^2/\sigma_j^2 > q_{d,1-\alpha/2} \}$. For completeness, we give the expressions of $t_{d,1-\alpha}$ and $q_{d,1-\alpha}$. Here $\alpha$ denotes the type-I error probability; moreover, for any $\delta > 0$, the quantity $t_{d,1-\delta}$ denotes the quantile of order $1 - \delta$ of $\sum_{j=1}^d \xi_j^2$, with $\xi_j \sim \mathcal{N}(0, \sigma_j^2)$, and $q_{d,1-\delta}$ denotes the quantile of order $1 - \delta$ of $\max_j \xi_j^2$. They obtain an upper bound of the order of

\[
\epsilon^*(s, 2, \Sigma)^2 \asymp \left( \sum_{j=1}^d \sigma_j^4 \right)^{1/2} \land \sum_{j \neq j_0} \sigma_j^2 \log n.
\]

In comparison, we only use the chi-square test $T_{\text{dense}}$ over the first $j_*$ coordinates, which allows us to reduce the term $\left( \sum_{j=1}^d \sigma_j^4 \right)^{1/4}$ to $\left( \sum_{j \leq j_*} \sigma_j^4 \right)^{1/4}$. Moreover, we use the test statistic $T_{\text{sparse}}$, which allows us to take into account the number of coordinates exceeding a certain value (namely $\tau$), rather than deciding in favor of $H_1$ if one value exceeds a suitable threshold.
3 Minimax rates in $L^t$ separation for $t \in [1, 2]$

In this Section, we examine Problem (11) with $t \in [1, 2]$. We emphasize that the case $t < 2$ is significantly more challenging than $t \geq 2$. Namely, it leads to more intricate phase transitions due to the fact that $\| \cdot \|_t$ is less smooth when $t < 2$ than when $t \geq 2$. Once again we begin with a few definitions and some intermediate lemmas which will help us present the main result of the section. First we let

$$a = \frac{4t}{4-t}, \quad b = \frac{4-2t}{4-t}.$$  \hfill (21)

We define for any $x \geq 0$,

$$j_*(x) := \max \{ j \in [d] : \sigma_j \geq x \}.$$  \hfill (22)

Thereafter, we use the convention that $\frac{1}{0} = +\infty$; moreover, for some small enough constant $c_\nu = c_\nu(\eta)$ depending only on $\eta$, and for any $x \geq 0$, we define $\overline{\nu}(x)$ as the solution to the equation

$$\sum_{j \leq j_*(x)} \frac{\sigma_j^t}{\overline{\nu}(x)} \wedge \frac{\sigma_j^t}{x^{1-2t}\overline{\nu}(x)} + \sum_{j > j_*(x)} \frac{\sigma_j^t}{\overline{\nu}(x)} \exp \left( -\frac{x^2}{\sigma_j^t} + 1 \right) = c_\nu.$$  \hfill (23)

We note that the equation above always admits a unique solution. To see this, we can note that, $j_*(x)$ being fixed, the left-hand side is a continuous function of $\overline{\nu}(x)$ that decreases from $+\infty$ to 0. We now also set

$$f(x) = \sum_{j \leq j_*(x)} \frac{1}{1 + \frac{\sigma_j^4}{x^{1-2t}\overline{\nu}(x)}} + \sum_{j > j_*(x)} \frac{\sigma_j^t}{\overline{\nu}(x)} \exp \left( -\frac{x^2}{\sigma_j^t} + 1 \right).$$  \hfill (24)

$$\lambda := \inf f^{-1}\left(\left\{ s/2 \right\} \right).$$  \hfill (25)

$$\nu = \overline{\nu}(\lambda).$$  \hfill (26)

$$i_* = i_*(\lambda) = \max \left\{ j \leq j_* : \sigma_j^4 \geq \lambda^{4-t}\nu^t \right\}.$$  \hfill (27)

$$j_* = j_*(\lambda).$$  \hfill (28)

Lemma 12 in Appendix C ensures that the quantities $\lambda$ and $\nu$ given in (24) and (26) are well-defined. With this we are ready to present the main result regarding the minimax separation radius $\epsilon^*(s,t,\Sigma)$ for $t \in [1, 2]$.

**Theorem 2.** Let $t \in [1, 2]$ and $\lambda$ and $\nu = \nu(\lambda)$ be defined as in (25) and (26). Then the following hold

i. [Lower Bound] There exists a small constant $c$ depending only on $\eta$, such that

$$\epsilon^*(s,t,\Sigma)^t \geq c(\lambda^t s + \nu^t).$$

ii. [Upper Bound] There exists a large enough constant $C'$ depending only on $\eta$ such that the test $\psi^*$ defined in (45) satisfies

$$\begin{align*}
\mathbb{P}_\theta (\psi^* = 1) &\leq \eta/2 \quad \text{if} \quad \theta = 0, \\
\mathbb{P}_\theta (\psi^* = 0) &\leq \eta/2 \quad \text{if} \quad \|\theta\|_0 \leq s \text{ and } \|\theta\|_t^t \geq C'(\lambda^t s + \nu^t).
\end{align*}$$

Therefore, $\epsilon^*(s,t,\Sigma)^t \leq C'(\lambda^t s + \nu^t).$

Theorem 1 immediately establishes $\epsilon^*(s,t,\Sigma)^t \asymp (\lambda^t s + \nu^t)$ when $t \in [1, 2]$. It turns out that the separation behavior admits another representation which will be useful for interpretations and our later discussions. For easy reference we collect them in the following corollary.
Corollary 3. Under the assumptions of Theorem 2 we have

\[ \epsilon^*(s, t, \Sigma)^t \approx \lambda^s s + \nu^t. \]

Moreover, writing \( \sigma_{\leq i_*} = (\sigma_1, \ldots, \sigma_{i_*}) \) and recalling (21), this expression can be rewritten as

\[ \epsilon^*(s, t, \Sigma)^t \approx \lambda^s s + \|\sigma_{\leq i_*}\|^t_\alpha. \]  \hspace{1cm} (29)

The proof of Theorem 2 is detailed in Appendix C with proofs of the lower and upper bound provided in subsections C.1 and C.2 respectively. Further, the simplification claimed in equation (29) is proved in Lemma 13, item 2. Now we once again provide a detailed discussion regarding the implications of Theorem 2.

### 3.1 Regimes of Minimax Separation

This rate is a combination of two terms. Similarly to the case \( t \geq 2 \), the first term \( \lambda^s s \) cannot be expressed explicitly as a function of the \( \sigma_j \)'s in general. However, it is possible to solve the equations (23), (24) and (25) for some specific profiles of \( \Sigma \) (see Section 5). The second term \( \|\sigma_{\leq i_*}\|^t_\alpha \) is best understood when compared with the rate in the absence of sparsity. When \( s = d \), our result directly implies that the minimax separation radius in \( L^t \) norm scales as \( \epsilon^*(d, t, \Sigma)^t \approx \|\sigma\|^t_\alpha \). On the dense part \( (\sigma_1, \ldots, \sigma_{i_*}) \), sparsity is therefore irrelevant. We call this the “fully dense” regime, as in the lower bound, this contribution is obtained by setting a non-sparse prior (see Subsection 3.2). This term showcases an interesting interpolation between the norms. When \( t \in [1, 2] \), the minimax separation radius is expressed in terms of the \( L^a \) norm where \( a = a(t) = \frac{4L}{1-t^2} \). This duality has also been highlighted for testing discrete distributions without sparsity in \( \ell^t \) norm, \( t \in [1, 2] \) in the paper [CC22] (see Subsection 6.2 for further details).

Our next lemma helps us present more insights to the results by providing a more interpretable expression for \( \nu^t \).

**Lemma 1.** There exist two large constants \( C_1, C_2 > 1 \) depending only on \( c_\nu \) (hence independent of \( \nu \)), which can be made arbitrarily large provided that \( c_\nu \) is small enough, such that \( \nu^t \in \left[ C_1 \bar{\nu}^t, C_2 \bar{\nu}^t \right] \), where

\[ \bar{\nu}^t := \left[ \sum_{j \leq i_*} \sigma_j^t \right]^{1/a} + \frac{1}{\chi^{2-t}} \left( \sum_{j = i_* + 1}^{j_*} \sigma_j^t \right)^{1/2} + \sum_{j > j_*} \sigma_j^t \exp \left( -\lambda^t/\sigma_j^t \right). \]

Note that the expression of \( \nu \) from Lemma 1 involves three contributions. Surprisingly, the last two terms \( \nu_{\text{inter}}^t \) and \( \nu_{\text{sparse}}^t \) never dominate in the rate (see Lemma 13). One could therefore believe that only two regimes coexist in the minimax separation radius. However, this is not the case. In fact, the minimax separation radius contains three regimes: a fully dense regime over \( (\sigma_1, \ldots, \sigma_{i_*}) \), an intermediate regime over \( (\sigma_{i_*+1}, \ldots, \sigma_{j_*}) \) and a sparse regime over \( (\sigma_{j_*+1}, \ldots, \sigma_d) \), the intermediate and sparse regimes being hidden in the term \( \lambda^t s \). As will be discussed in Subsections 3.2 and 3.3, the three regimes involve very different phenomena, the intermediate one, however, sharing similarities with the other two. This is reflected by our upper bound, which requires three tests. When \( t = 2 \), we get \( a = 4 \), and the fully dense and intermediate parts merge into one single regime.

Finally, we note that although Theorem 2 can be made to encompass the case where \( t = 2 \), the method presented in Section 2 is more intuitive in that special case.
3.2 Lower bounds for \( t \in [1, 2] \)

**Proposition 1.** Let \( \lambda \) be defined as in \((25)\) and let \( \nu = \nu(\lambda) \) be the solution to equation \((24)\) for \( x = \lambda \). Then there exists a small constant \( c \) depending only on \( \eta \), such that

\[
\epsilon^x t \geq c(\nu^t + \lambda^t s).
\]

To prove the lower bound, we once again use Le Cam’s two points method. Again, we distinguish between two cases.

If \( s < c(\eta) \) for a sufficiently large constant \( c(\eta) \) depending only on \( \eta \), then we bound from below \( \epsilon^x(s, t, \Sigma) \) by \( \epsilon^x(1, t, \Sigma) \) and propose a combination of 1-sparse priors separated from the null hypothesis by \( c\lambda \) and \( c\nu \) respectively, and that cannot be distinguished from the null hypothesis with high probability (see Lemma 3 for details).

Conversely, if \( s \geq c(\eta) \), we define the following prior distribution over the parameter space \( \Theta = \mathbb{R}^d \). We define a random vector \( \theta \in \mathbb{R}^d \) whose coordinates \( \theta_j \) satisfy \( \forall j \in [d] : \theta_j = b_j \omega_j \gamma_j \), where \( b_j \sim \text{Ber}(\pi_j) \), \( \omega_j \sim \text{Rad}(\frac{1}{\nu}) \) are mutually independent, and where the parameters \( \pi_j, \gamma_j \) are given in Table 1 below.

| \( \pi_j \) | \( j \leq i_* \) | \( i_* < j \leq j_* \) | \( j > j_* \) |
|---|---|---|---|
| \( \gamma_j \) | \( c\sigma_j^2 / \nu^t \) | \( \sigma_j^2 \) | \( \sigma_j^2 \exp \left( \frac{-\lambda^t}{\sigma_j^2} \right) \) |
| \( \gamma_j^t \) | \( c\lambda^t \) | \( c\lambda^t \) |

Table 1: Values of the prior parameters for \( t \in [1, 2] \).

In Table 1, \( c \) denotes some small enough constant depending only on \( \eta \). The parameters in the above table are found by solving the variational problem \((13)\) for \( t \leq 2 \), which is significantly more involved than in the case \( t \geq 2 \), as the constraint \( \pi_j \leq 1 \) can be saturated. This gives rise to a new phase transition occurring at \( i_* \), on top of the phase transition at \( j_* \). As a reminder, \( j_* \) is the index after which the linearization \( \sinh^2 \left( \frac{\gamma_j^t}{2\sigma_j^2} \right) < \frac{\gamma_j^t}{\sigma_j^2} \) no longer holds and has to be replaced by \( \sinh^2 \left( \frac{\gamma_j^t}{2\sigma_j^2} \right) \approx \exp \left( \frac{\lambda^t}{\sigma_j^2} \right) \).

The indices \( j \leq i_* \) form the fully dense regime, which characterizes the largest values of the \( \sigma_j \)’s. In this regime, the Bernoulli parameters \( \pi_j \) are all equal to 1 (in other words, the optimal prior has no sparsity at all) and the optimal perturbations \( \gamma_j^t = \sigma_j^2 / \nu^t \) are proportional to the values that would be optimal in absence of sparsity (up to the rescaling factor \( 1 / \nu^t \)). In this regime, sparsity is irrelevant. As soon as \( \pi_j \) no longer saturates the constraint “\( \pi_j = 1 \)”, a phase transition occurs in the behavior of \( \gamma_j \). An interesting phenomenon arises concerning the decay of \( \gamma_j \). The perturbation \( \gamma_j^t \) first decreases from \( j = 1 \) to \( j = i_* \) until it reaches \( c\lambda^t \). After \( i_* \) it remains equal to \( c\lambda^t \) independently of \( \sigma_j \). Over \( \{i_* + 1, \ldots, d\} \), \( \gamma_j \) therefore does not exhibit any phase transition, contrary to \( \pi_j \). This is very surprising, given that the two parts \( j \leq j_* \) and \( j > j_* \) exhibit fundamentally different behaviors in the analysis of the lower bound, and it was unexpected to observe that the intermediate zone \( j \in \{i_* + 1, \ldots, j_*\} \) and the sparse zone \( j > j_* \) share the same magnitude of the \( \gamma_j \)’s.
3.3 Upper bounds for $t \in [1, 2]$

In this subsection, we describe the tests achieving the rate in Theorem 2. Let $\lambda$ and $\nu$ be defined as in (25) and (26) respectively. We let

$$\tau = C_t \lambda t + \nu t / s,$$

where $C_t = (4t)^t$.

and now define the test statistics as follows:

$$T_{fdense} = \sum_{j \leq i^*} \frac{1}{\sigma_j^2} \left( X_j^2 - \sigma_j^2 \right),$$

$$T_{inter} = \sum_{j = i^* + 1}^j X_j^2 - \sigma_j^2,$$

$$T_{sparse} = \sum_{j > i^*} \left( |X_j|^t - \alpha_j \right) \mathbb{1}\{|X_j|^t > \tau\}.$$

For some large enough constant $C$ depending only on $\eta$, we finally define the test functions as follows

$$\psi_{fdense} = 1\left\{ T_{fdense} \geq C \left( \sum_{j \leq i^*} \sigma_j^2 \right)^{1/2} \right\},$$

$$\psi_{inter} = 1\left\{ T_{inter} \geq C \frac{\lambda t}{\lambda t} \sum_{j = i^* + 1}^j \sigma_j^4 \right\},$$

$$\psi_{sparse} = 1\left\{ T_{sparse} > C \rho \right\},$$

$$\psi^* = \psi_{fdense} \lor \psi_{inter} \lor \psi_{sparse},$$

We prove the following Proposition which complements the lower bound from Proposition 1 by a matching upper bound.

**Proposition 2.** Recall the definition of $\rho$ in (34). There exists a large enough constant $C'$ depending only on $\eta$ such that

$$\begin{cases} P_{\theta} (\psi^* = 1) \leq \eta / 2 & \text{if } \theta = 0, \\ P_{\theta} (\psi^* = 0) \leq \eta / 2 & \text{if } \|\theta\|_0 \leq s \text{ and } \|\theta\|_t \geq C' \rho. \end{cases}$$

Proposition 2 is proved in Appendix C.2. As already evoked, the test $\psi^*$ is a combination of three tests, each one being tailored for the three different regimes. The first two test statistics $T_{fdense}$ and $T_{inter}$ share the similarity of being non-truncated chi-square test statistics. Their reweightings are however different. The last two test statistics $T_{inter}$ and $T_{sparse}$ are both non-reweighted test statistics.

4 Minimax rates in $L^\infty$ separation

In this section, we consider the case of the $L^\infty$ separation. We highlight an interesting connection with the results from Sections 2 and 3. Namely, for any $t \geq 1$, testing in $L^\infty$ separation is equivalent to testing in any $L^t$ separation with $s = 1$. This makes intuitive sense: For $t = \infty$, the problem reduces to detecting signals $\theta$ whose largest coordinate $|\theta_j|$ exceeds a certain threshold. Among such vectors, the most difficult ones to detect are clearly 1-sparse, and their $L^t$ norm consequently coincides with their $L^\infty$ norm. The next lemma makes this claim precise.
Lemma 2. The following relations hold for any $s \in [d]$ and any $t \geq 1$:
$$
\epsilon^*(s, \infty, \Sigma) = \epsilon^*(1, \infty, \Sigma) = \epsilon^*(1, t, \Sigma).
$$

This Lemma connects all results of the paper for $s = 1$, by showing that $\epsilon^*(1, t, \Sigma)$ is independent of $t \in [1, \infty]$, which is not immediate from Theorems 1 and 2. To obtain the expression of $\epsilon^*(s, \infty, \Sigma)$, we therefore evaluate any $\epsilon^*(1, t', \Sigma)$ and choose $t' = 2$ for simplicity. The following theorem gives the behavior of the minimum separation radius in $L^\infty$ separation:

Theorem 3. Define $\lambda$ and $\nu$ respectively as in (5) and (6) by taking $t' = 2$ and $s' = 1$ in these equations.

i. There exist two constants $C, c > 0$ depending only on $\eta$ such that for any $s \in [d]$, we have $c(\lambda + \nu) \leq \epsilon^*(s, \infty, \Sigma) \leq C(\lambda + \nu)$.

ii. Moreover, there exists a large enough constant $C'$ depending only on $\eta$ such that the test $\psi^*$ defined in (36) satisfies
$$
\begin{cases}
P_\theta (\psi^* = 1) \leq \eta/2 & \text{if } \theta = 0, \\
P_\theta (\psi^* = 0) \leq \eta/2 & \text{if } \|\theta\|_\infty \geq C'(\lambda + \nu).
\end{cases}
$$

Theorem 3 is a corollary of Lemma 2 and Theorem 1. The proof of Theorem 3 is done in Appendix D.1 and relies on the following test. For some large enough constant $C > 0$ depending only on $\eta$, we define the test rejecting if at least one coordinate $|X_j|$ exceeds the threshold $C\rho$, where $\rho = \lambda + \nu$:
$$
\psi^* = 1\left\{ \exists j \in [d] : |X_j| > C\rho \right\}.
$$
(36)

As in the cases $t \in [1, 2]$ and $t \geq 2$, the term $\lambda + \nu$ cannot be expressed as an explicit function of the $\sigma_j$'s in general. However, in the isotropic case, we recover a well-known result. Indeed, if $\sigma_1 = \cdots = \sigma_d =: \sigma$, then $\lambda + \nu \asymp \sigma\sqrt{\log(d)}$. This result makes intuitive sense. In the isotropic case, the largest value $X_j$ where $X_j \sim N(0, 1)$ is of the order of $\sigma\sqrt{\log(d)}$ with high probability. The test (36) achieving this rate is equivalent to checking that no value observed exceeds this threshold. Note also that our bound does not depend on $s$. This is also clear: To detect in $L^\infty$ separation, the worst possible perturbations are 1-sparse.

5 Examples

To provide more insight into our results we now discuss some examples for specific heterogeneity profiles $\Sigma$. The proofs of the results presented in this Section can be found in Appendix E.

5.1 Isotropic case

Assume that $\sigma_1 = \cdots = \sigma_d = \sigma$. Up to multiplicative constants, the minimax separation radius $\epsilon^*(s, t, \sigma^2 I_d)$ can be expressed as in Table 2 below:

In the isotropic case for $t \geq 2$, sparsity does not help for testing when $s \geq C\sqrt{d}$. However, this is no longer the case for $t < 2$: sparsity always improves the rates as soon as $s \ll d$. The case $t \geq 2$ has been investigated in [Gut19] in the isotropic case and without sparsity.

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If \( s \geq C \sqrt{d} \), then
\[
\epsilon^*(s,t,\Sigma) \approx \begin{cases} 
\frac{\alpha_j^{1/2} t}{d^{\alpha/2} s} & \text{if } s \geq C \sqrt{d}.
\end{cases}
\]
otherwise.

5.3 Exponentially decreasing variances

Let \( \alpha \in (0,1] \) and assume that \( \forall j \in [d] : \sigma_j = \alpha^j \). Let \( j_1 = \min\{ j \in [d] : \alpha^j < \alpha/4 \} \) if this minimum is taken over a non-empty set, and set \( j_1 = d + 1 \) otherwise. Then the minimax separation radius satisfies
\[
\forall t \geq 2 : \epsilon^*(s,t,\Sigma) \approx \epsilon^*(s,t,\alpha^2 I_{j_1}) \approx \begin{cases} 
\alpha_j^{1/2} t & \text{if } s \geq C \sqrt{j_1},
\alpha^j s \log^{t/2} \left( \frac{2 \sqrt{j_1}}{s} \right) & \text{otherwise}.
\end{cases}
\]

Note that this result encompasses the isotropic case from Subsection 5.1. For non-pathological decays of the values \( \sigma_j \)'s, that is, for \( \alpha \leq 1 - \delta \) where \( \delta > 0 \) is some fixed constant, the index \( j_1 \) will typically be a constant depending on \( \delta \). Namely: \( j_1 \leq \log_{\alpha-1}(4) \leq \log(1-\delta)-1(4) \). Therefore, as soon as \( s \) is greater than a constant (depending on \( \delta \)), the minimax separation radius will further simplify as \( \epsilon^*(s,t,\Sigma) \approx \delta \alpha = \sigma_1 \), regardless of the sparsity. This makes intuitive sense: When the \( \sigma_j \)'s decay exponentially fast, the intrinsic dimension of the data, given by \( d_{\text{intrinsic}} = \text{Tr}(\Sigma^{1/2})/\sqrt{\text{Tr}(\Sigma)} \), is of the order of a constant. Therefore, sparsity should not be relevant if \( s \geq \sqrt{d_{\text{intrinsic}}} = \text{Cste} \) (see the discussion in Subsection 5.1).

However, the rate can be more subtle if \( \alpha \) approaches 1, and equation (37) reveals that whenever \( \sigma_j = \alpha^j \) for \( j = 1 \ldots, d \), the testing problem is essentially as difficult in the isotropic case with covariance matrix \( \alpha^2 I_{j_1} \). Here, the dimension \( j_1 \) can understood as the size of the set \( \{1, \ldots, j_1\} \) on which the values \( \sigma_j \) can be considered as constant: \( \sigma_j \in [\sigma_1, \sigma_1], \forall j \in [j_1] \).

6 Discussion

In this Section, we discuss further connections and implications of our results in comparison to the literature along with possible future directions.
6.1 Minimax estimation of $\|\theta\|_t$ for $t \geq 1$

A natural question is to compare this paper’s results about testing with the corresponding task of estimating $\|\theta\|_t$ for $t \geq 1$. Estimation of non-smooth functionals have been considered in \cite{CL11, JHVI15, WY19, WY16, CV19, FS17, BI21}. Some techniques used in the present paper can be linked with techniques developed in \cite{CCT17}, and more closely, in \cite{CCT20}. The paper \cite{CCT20} considered the problem of estimating $\|\theta\|_t$ over $\{\theta \in \mathbb{R}^d : \|\theta\|_0 \leq s\}$ given an observation $X \sim \mathcal{N}(\theta, \sigma^2 I_d)$. The difficulty of this estimation problem is characterized by the minimax estimation risk defined as

$$R_{s,d}(\sigma, t) := \inf_{\hat{T}} \sup_{\theta \in \mathbb{R}^d} \mathbb{E}\left[\|\hat{T} - \|\theta\|_t\|^2\right],$$

where the infimum is taken over all estimators $\hat{T} : \mathbb{R}^d \to \mathbb{R}$. In Table 3 below, we collect the results of \cite{CCT20} for $t \geq 1$, and compare them with the results of the present paper. To preserve homogeneity, we give here the expression of the square root of the minimax estimation risk of $\|\theta\|_t$, namely $R_{s,d}^{1/2}(\sigma, t)$, as well as our expression for the minimax separation radius in $L^s$ norm $\epsilon^*(s, t, \sigma^2 I_d)$. We denote by $2\mathbb{N}^*$ the set of positive even integers.

| $t \notin 2\mathbb{N}^*$ | $s \leq \sqrt{d}$ | $s > \sqrt{d}$ (Lower bound) | $s > \sqrt{d}$ (Upper bound) |
|-----------------|----------------|-------------------|-------------------|
| $R_{s,d}^{1/2}(\sigma, t)$ | $\sigma s^t \log^2 \left(1 + \frac{d}{s^2}\right)$ | $\sigma s^t \log^2 \left(1 + \frac{s^2}{d}\right)$ | $\sigma s^t 1 + \frac{s^2}{d}$ |
| $\epsilon^*(s, t, \sigma^2 I_d), t < 2$ | $\epsilon^*(s, t, \sigma^2 I_d), t \geq 2$ | $\epsilon^*(s, t, \sigma^2 I_d)$ | $\epsilon^*(s, t, \sigma^2 I_d)$ |

| $t \in 2\mathbb{N}^*$ | $s \leq \sqrt{d}$ | $s > \sqrt{d}$ |
|-----------------|----------------|----------------|
| $R_{s,d}^{1/2}(\sigma, t)$ | $\sigma s^t \sqrt{\log \left(1 + \frac{d}{s^2}\right)}$ | $\sigma d^t$ |
| $\epsilon^*(s, t, \sigma^2 I_d)$ | $\epsilon^*(s, t, \sigma^2 I_d)$ | $\epsilon^*(s, t, \sigma^2 I_d)$ |

**Table 3:** Comparison of the minimax estimation and testing problems across $t \in [1, \infty)$.

Our analysis of the isotropic case can be found in Subsection 5.1 (in Subsection 5.1, the lower bounds for $s \leq C\sqrt{d}$ involve the term $\log \left(2\sqrt{d}/s\right)$, but we use the fact that $\log(1+d/s^2) \approx \log \left(2\sqrt{d}/s\right)$ when $s \leq \sqrt{d}$). Interestingly, for $s \leq \sqrt{d}$ and for any $t \geq 1$, $R_{s,d}^{1/2}(\sigma, t)$ and $\epsilon^*(s, t, \sigma^2 I_d)$ are always of the same order. This is reflected in the similarity between our test statistic $T_{sparse}$ and the estimator used in \cite{CCT20} in the sparse zone $s \leq \sqrt{d}$. We recall that in this regime, for some constant $C_t$:

$$T_{sparse} = \sum_{j=1}^d (|X_j|^t - \alpha_j) 1_{|X_j|^t > \tau}, \quad \text{where } \tau = C_t \lambda^{t} + \nu^{t}/s \approx \sigma^t \log^{t/2} \left(1 + \frac{d}{s^2}\right).$$

In comparison, the estimator of $\|\theta\|_t$ used in \cite{CCT20} is as follows:

$$\hat{T} = \left(\hat{N}_t\right)^{1/t} \quad \text{where } \hat{N}_t = \sum_{j=1}^d \left(|X_j|^t - \alpha_j\right) 1 \left\{ |X_j|^t > 2\sigma^t \log \left(1 + d/s^2\right) \right\}.$$
In the estimation problem, the constant 2 in the indicator function is important, in order to balance the bias and variance of the estimator. In the testing problem, the constant 2 can be replaced by any sufficiently large constant, only at the price of a larger constant in the upper bound.

However, when \( s > \sqrt{d} \), the square root estimation rate \( R_{s,d}(\sigma,t) \) is always at least as large as the rate of testing \( \epsilon^*(s,t,\sigma^2 I_d) \). The only case where the two quantities coincide for \( s > \sqrt{d} \) is when \( t \) is an even integer. In this case, the functional \( \theta \mapsto \|\theta\|_t^t \) is sufficiently smooth to ensure that there exists unbiased estimators with much faster rates than for other \( L^t \) norms, which are not smooth.

6.2 Multinomial testing

The case of the \( L_1 \) separation is an interesting special case of our results and could be of independent interest. Indeed, in the context of discrete distributions, the \( L_1 \) distance is proportional to the total variation distance, and is therefore commonly used for multinomial testing [VV17, BB20, LWCS22, GP22, BW19, Can20, Can22]. In this Subsection, we set \( P = \{ p = (p_1, \ldots, p_d) \in \mathbb{R}_+^d \mid \sum_{j=1}^d p_j = 1 \} \), and denote by \( M(n,p) \) the multinomial distribution with parameters \( n \in \mathbb{N}^* \) and \( p \in P \). We also denote by \( \text{Unif}(d) = (\frac{1}{d}, \ldots, \frac{1}{d}) \) the uniform distribution over \( \{1, \ldots, d\} \). We also fix a histogram \( N \sim M(n,p) \). Multinomial testing against sparse alternatives has been considered in [BM21]. Namely, the authors considered the following global testing problem:

\[
p = \text{Unif}(d) \quad \text{against} \quad H_1 : \quad \left\{ \begin{array}{l}
p \in P \\
\|p - \text{Unif}(d)\|_1 \geq \epsilon \quad \text{and} \quad \|p - \text{Unif}(d)\|_0 \leq s.
\end{array} \right.
\]  

In the asymptotic \( s = d^{1-\alpha} \) for \( \alpha \in (0,1) \), they proved that the minimax separation radius \( \epsilon^* = \epsilon^*(s,n,d) \) for Problem (38) scales as

\[
\epsilon^* \asymp \frac{s}{d} \wedge \begin{cases} 
\frac{\sqrt{s}}{\sqrt{\epsilon d^{1/4}}} & \text{if } \alpha \leq \frac{1}{2} \\
\frac{\sqrt{\log d}}{nd} & \text{if } \alpha > \frac{1}{2}.
\end{cases}
\]

The term \( s/d \) represents the impossibility regime: Any distribution \( p \in P \) such that \( \|p - \text{Unif}(d)\|_0 \leq s \) necessarily satisfies \( \|p - \text{Unif}(d)\|_1 \leq 2s/d \). The second term interestingly bears similarity with our results in \( L_1 \) separation. Indeed, for \( t = 1 \) and \( \Sigma = \sigma^2 I_d \), Theorem 2 yields (see Subsection 5.1):

\[
\epsilon^*(s,1,\sigma^2 I_d) \asymp \begin{cases} 
\sigma d^{1/4} \sqrt{s} & \text{if } s \geq C\sqrt{d}, \\
s \sqrt{\log (2\sqrt{d}/s)} & \text{otherwise}.
\end{cases}
\]

The second term in the rate (39) is exactly analogous to (40) when \( \sigma^2 = 1/nd \), which is proportional to the variance of \( N_j/n \) where \( N_j \) denotes the \( j \)-th coordinate of the histogram \( N \). This comparison therefore proves that the testing problem (38) is either impossible or analogous to a Gaussian testing problem in \( L_1 \) separation, and that the correlation between the coordinates of \( X \) do not affect the minimax rates. Further interplays between correlation and sparsity in signal detection have been thoroughly discussed in [KG21], in the case of an isotropic covariance matrix and with Euclidean separation. The results of the present paper could therefore find natural applications to the local analog of Problem (38), which is left for future work.
In the absence of sparsity, the paper [VV17] considered the following local testing problem in multinomials:

\[ p = p_0 \quad \text{against} \quad H_1 : \ p \in \{ q \in \mathcal{P} : \| q - p_0 \|_1 \geq \epsilon \}, \quad (41) \]

where \( p_0 \) is a fixed and known distribution in the class \( \mathcal{P} \). The result is as follows: Assume without loss of generality that \( p_0(1) \geq \cdots \geq p_0(d) \), and, for any \( \delta > 0 \), define \( p_{0,-\delta} = (0, p_2, \ldots, p_j, 0, \ldots, 0) \) where \( j = \max \{ j : \sum_{i \geq j} p_0(i) > \delta \} \). Then the minimax separation radius for Problem (43) is defined as the solution to the equation:

\[ Ce = \sqrt{\frac{\| p_{0,-\delta} \|_{L2/3}}{n}} + \frac{1}{n}, \quad (42) \]

for some absolute constant \( C > 0 \). See [BW19] and [CC22] Appendix D for the equivalence between (42) and the formulation of the results in [VV17]. See also [BCG19] for a further discussion about the relation between the \( \ell^1 \) and the \( \ell^{2/3} \) norms. The \( 2/3 \)-norm exhibits some similarities with the Gaussian testing problem (11) for \( t = 1 \) and \( s = d \). Indeed, in light of our Theorem 2, we get \( \epsilon^*(d, 1, \Sigma) = \| \sigma \|_{4/3} \). Fixing \( J = \max \{ j : \sum_{i \geq j} p_0(i) > \epsilon \} \), the term \( \sqrt{\| p_{0,-\epsilon} \|_{L2/3}/n} \) is exactly analogous to \( \| \tilde{\sigma} \|_{4/3} \) where \( \forall i \in \{ 2, \ldots, J \} : \tilde{\sigma}_j = \sqrt{p_0(j)/n} \) which is proportional to the standard deviation of \( N_j/n \). We can take this analogy further by comparing with the results in [CC22], which considered the problem

\[ p = p_0 \quad \text{against} \quad H_1 : \ p \in \{ q \in \mathcal{P} : \| q - p_0 \|_t \geq \epsilon \}, \quad (43) \]

for \( t \in [1, 2] \). The authors proved that, for a suitably defined index \( I \in \{ 1, \ldots, d \} \), the minimax separation radius for Problem (43) scales as

\[ \epsilon^* = \sqrt{\frac{\| p_{0,-\epsilon} \| r}{n}} + \frac{\| p_{0,I} \|_{(2-t)/t}^{(2-t)/t}}{n^{2(t-1)/t}} + \frac{1}{n}, \]

where \( r = \frac{2t}{t-1} \) and \( p_{0,-\epsilon} = (p_2, \ldots, p_I) \) and \( p_{0,I} = (0, \ldots, 0, p_{I+1}, \ldots, p_d) \). The term \( \sqrt{\| p_{0,-\epsilon} \| r/n} \) can therefore be written as \( \| \tilde{\sigma}' \|_a \) where \( \forall j \in \{ 2, \ldots, I \} : \tilde{\sigma}'_j = \sqrt{p_0(j)/n} \) which is once again proportional to the standard deviation of \( N_j/n \). The paper [Wag15] also highlighted this duality in the global case rather than in the local one. The paper [CC21] considered an analogous version of Problem (43), for Hölder-continuous densities.

### 6.3 Toward general covariance matrix \( \Sigma \)?

The natural and important case of a general covariance matrix \( \Sigma \), not necessarily diagonal, stands out as a highly non-trivial extension of our results, and goes far beyond the scope of this paper. Solving this problem would allow for important developments in sparse linear regression. Indeed, the linear regression problem in the low dimensional regime \( d \leq n \) can be linked with the gaussian sequence model as follows. Suppose that we observe \( (X, y) \) satisfying

\[ y = X^T \theta + \xi \quad \text{where} \quad \begin{cases} y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n \\ \xi = (\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n \\ X = [X_1, \ldots, X_n] \in \mathbb{R}^{d \times n} \text{ has rank } d. \end{cases} \]

Writing \( z = (XX^\top)^{-1}Xy \) and \( w = (XX^\top)^{-1}X\xi \), we can rewrite our model as \( z = \theta + w \sim \mathcal{N}(\theta, \Sigma) \), where \( \Sigma = X^\top(XX^\top)^{-2}X \). Despite its importance for many practical applications, the optimal detection rate
for sparse signals θ in this setting remains unknown. This is due to the major technical challenges arising when combining sparsity with non-isotropic noise. One of them is the heterogeneity of the eigenvalues of Σ, addressed in this paper.

Another important challenge is that the sparsity basis (i.e. the canonical basis of \( \mathbb{R}^d \)) might not be aligned with the basis of eigenvectors of Σ. An important step toward that direction is [KG21], which thoroughly solves the case \( \Sigma = (1 - \gamma)I_d + \gamma 11^\top \) where \( 1 = (1, \ldots, 1)^\top \in \mathbb{R}^d \), for any \( \gamma \in [0, 1] \), and in Euclidean separation. This paper develops important techniques to tackle the non-alignment of the two bases, and elegantly decomposes the problem under study into separate isotropic detection problems. In this regard, it does directly address heteroscedasticity. We believe that the present paper captures fundamentally different phenomena from those uncovered in [KG21], and hope that these two papers will constitute a basis to further explore sparse signal detection with arbitrary covariance matrix.

7 Conclusion and future work

In this paper, we solved the problem of sparse signal detection in the heteroscedastic Gaussian sequence model with a diagonal covariance matrix Σ, for any \( L^1 \) separation, \( t \geq 1 \). The present paper is a step toward addressing the much more ambitious case of general Σ, and it will be interesting to see how the present results can be combined with correlations in the noise. The \( L^1 \) separation distance also opens up interesting future directions for sparse testing in multinomials, which is left for future work. Another avenue of research for future work is to translate the present results concerning the heteroscedastic Gaussian sequence model to the setting of testing sparse linear regression with non-isotropic design, generalizing [ITV10].

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Lemma 3. (Case of constant sparsity) Assume that $s \leq C$ where $C > 0$ is a constant depending only on $\eta$. Then it holds that $e^{s}(s, t, \Sigma) \geq c(\lambda + \nu)$ for some sufficiently small constant $c > 0$, depending only on $\eta$, where $\lambda$ and $\nu$ are defined as in [25], (6) if $t \in [2, \infty)$ and as in [25], (26) if $t \in [1, 2]$.

Note that for $t = 2$, two definitions are possible for the quantities $\lambda$ and $\nu$, but the conclusion of the lemma still holds in both cases.

Proof of Lemma 3 We first set some notation and distinguish between two cases.

1. For $t \in [2, \infty)$, let $\beta$, $\nu$, $j_*$ be defined as in [25], (6), (7) and let $\pi_j = \frac{2}{s} \lambda e^{-\beta/\sigma_j^2} \sqrt{\frac{\sum_{j=1}^d \sigma_i^{2}}{\sum_{j=1}^d \sigma_{j}^{2}}} \exp \left(-\beta/\sigma_j^2\right)$ for any $j \in [d]$. 

2. For $t \in [1, 2]$, define $\lambda$, $\nu$, $j_*$ as in [25], (26), (28), and for all $j \in [d]$, define $\pi_j = \frac{2}{s} \lambda e^{-\beta/\sigma_j^2} + \frac{2\sigma_i^4}{s\nu^2\lambda^2} \Pi_{i,j < j_*} + \frac{2\sigma_i^4}{s\nu^2\lambda^2} \Pi_{i,j \leq j_*} \Pi_{j_* < j}. $

For $t = 2$, choose either one of the definitions above. In both cases, note that $\sum_{j=1}^d \pi_j = 1$. This is a consequence of equation [25] if $t \geq 2$ and of [26] for $t \in [1, 2]$.

Now, let $p \sim \text{Multi}(\pi_1, \ldots, \pi_d, 1)$ and define the random vector $\theta$ such that $\forall j \in [d] : \theta_j = c\lambda 1_{j=p}$ where $c > 0$ is a small enough constant depending only on $\eta$. We denote by $\Pi$ the prior distribution over $\theta$, defined such that $\mathbb{P}_\Pi(\forall j \in [d] : \theta_j = c\lambda 1_{j=p}) = \pi_j$. Let $\mathbb{P}_\text{prior} = \mathbb{E}_{\theta \sim \Pi} [\mathcal{N}(\theta, \Sigma)]$ denote the corresponding mixture of normal distributions $\mathcal{N}(\theta, \Sigma)$ where $\theta \sim \Pi$.

First, we have $||\theta||_t = c\lambda$ and $||\theta||_0 = 1 \leq s$ a.s., so that $\theta \in \Theta(c\lambda, s, t)$ a.s. We can now compute the $\chi^2$ divergence between our prior and the null distribution. To do so, note that conditionally on $p = j$, we have

$$
\forall x = (x_1, \ldots, x_d) \in \mathbb{R}^d : \mathbb{P}_\text{prior}(x \mid p = j) = \frac{e^{-(x_j - c\lambda)^2/2\sigma_j^2}}{\sqrt{2\pi\sigma_j^2}} \prod_{k \neq j} e^{-x_k^2/2\sigma_k^2}.
$$

Therefore,

$$
1 + \chi^2(\mathbb{P}_\text{prior} \parallel \mathbb{P}_0) = \int_{\mathbb{R}^d} \left(\frac{\mathbb{P}_\text{prior}}{\mathbb{P}_0}(x)\right)^2 \mathbb{P}_0(x) dx = \int_{\mathbb{R}^d} \sum_{j=1}^d \pi_j \mathbb{P}_0(x) \exp \left\{ -\frac{(x_j - c\lambda)^2 - x_j^2}{2\sigma_j^2} \right\} \frac{1}{\sqrt{2\pi\sigma_j^2}} \prod_{k \neq j} e^{-x_k^2/2\sigma_k^2} dx.
$$

Then,

$$
1 + \chi^2(\mathbb{P}_\text{prior} \parallel \mathbb{P}_0) = \sum_{i \neq j} \pi_i \pi_j \int_{\mathbb{R}^2} \exp \left\{ -\frac{(x_i - c\lambda)^2 - x_i^2}{2\sigma_i^2} - \frac{(x_j - c\lambda)^2 - x_j^2}{2\sigma_j^2} \right\} \left(4\pi^2\sigma_i^2\sigma_j^2\right)^{-1/2} \exp \left\{ -\frac{x_i^2}{2\sigma_i^2} - \frac{x_j^2}{2\sigma_j^2} \right\} dx_i dx_j.
$$
\[ + \sum_{j=1}^{d} \pi_j^2 \int_{\mathbb{R}} \exp \left\{ -\frac{(x_i - c\lambda)^2 - x_i^2}{\sigma_i^2} \right\} \left( 2\pi \sigma_j^2 \right)^{-1/2} \exp \left\{ -\frac{x_j^2}{2\sigma_j^2} \right\} dx_j \]

\[ = \sum_{i \neq j} \pi_i \pi_j + \sum_{j=1}^{d} \pi_j^2 \int_{\mathbb{R}} \left( 2\pi \sigma_j^2 \right)^{-1/2} e^{-x_j^2/\sigma_j^2} \exp \left\{ -\frac{(x_j - 2c\lambda)^2}{2\sigma_j^2} \right\} dx_j \]

\[ = \sum_{i \neq j} \pi_i \pi_j + \sum_{j=1}^{d} \pi_j^2 \exp \left\{ e^{2\lambda^2/\sigma_j^2} \right\} = 1 + \sum_{j=1}^{d} \pi_j^2 \left( \exp \left\{ e^{2\lambda^2/\sigma_j^2} \right\} - 1 \right) \text{ recalling that } \sum_{j=1}^{d} \pi_j = 1 \]

\[ = 1 + \sum_{j \leq j^*} \pi_j^2 \left( e^{\sigma_j^2} - 1 \right) + 4 \sum_{j > j^*} \pi_j^2 \frac{e^{-\lambda^2/\sigma_j^2}}{\nu^j} \exp \left\{ -\frac{\lambda^2}{\sigma_j^2} \right\} \left( \exp \left\{ e^{2\lambda^2/\sigma_j^2} \right\} - 1 \right) \]

\[ \leq 1 + 2e^2 + \sup_{\alpha > 0} \left[ e^{-(1-c^2)\alpha} - e^{-\alpha} \right] , \]

for \( c \) small enough. Now it remains to prove that the family of functions

\[ \left\{ f_c : \alpha \in \mathbb{R}_+ \rightarrow e^{-(1-c^2)\alpha} - e^{-\alpha} \mid c > 0 \right\} \]

uniformly converges to 0 when \( c \downarrow 0 \).

Let \( \delta > 0 \) and let \( A > 0 \) be such that \( e^{-A^2/2} \leq \delta \). We note that the family of functions \( f_c \) is continuous over the compact set \([0, A]\), converges pointwise to 0 when \( c \downarrow 0 \), which is a continuous function, and that \( (f_c)_{c>0} \) decreases when \( c \downarrow 0 \). Therefore, by Dini’s theorem, \( (f_c)_c \) uniformly converges to 0 as \( c \downarrow 0 \) over \([0, A]\). Now, let \( c > 0 \) such that \( \forall \alpha \in [0, A] : |f_c(\alpha)| \leq \delta \), then by definition of \( A \), we also have \( \forall \alpha \geq A : |f_c(\alpha)| \leq \delta \). Combining the two guarantees proves the desired uniform convergence over \( \mathbb{R}_+ \).

For \( c \) small enough, we therefore have \( \text{TV}(\mathbb{P}_{\text{prior}}, \mathbb{P}_0) \leq \sqrt{\chi^2(\mathbb{P}_{\text{prior}} \parallel \mathbb{P}_0)} \leq c' \) (see e.g. [tsy08]) where \( c' \) can be arbitrarily small. Therefore,

\[ R^*(\epsilon, s, t, \Sigma) = \inf_{\psi} \int_{\mathbb{R}} R(\psi, \epsilon, s, t, \Sigma) = \inf_{\psi} \left\{ \mathbb{P}_0 (\psi = 1) + \sup_{\theta} \mathbb{P}_\theta (\psi = 0) \mid \theta \in \Theta(\epsilon, s, t) \right\} \]

\[ \geq \inf_{\psi} \left\{ \mathbb{P}_0 (\psi = 1) + \mathbb{E}_{\theta \sim \Pi} [\mathbb{P}_\theta (\psi = 0)] \right\} = \inf_{\psi} \left\{ 1 - \mathbb{P}_0 (\psi = 0) + \mathbb{P}_{\text{prior}} [\psi = 0] \right\} \]

\[ = 1 - \sup_{A \in \mathcal{B}(\mathbb{R}^d)} \left\{ \mathbb{P}_{\text{prior}} (A) - \mathbb{P}_0 (A) \right\} = 1 - \text{TV}(\mathbb{P}_0, \mathbb{P}_{\text{prior}}) \geq 1 - c' > \eta, \]

for \( c' \) smaller than a quantity depending only on \( \eta \). At the last line, we denoted by \( \mathcal{B}(\mathbb{R}^d) \) the family of Borel subsets of \( \mathbb{R}^d \). Since for any point \( \theta \) of the prior, we have \( ||\theta||_t = c\lambda \), then by definition of \( \epsilon^*(s, t, \Sigma) \), we have proved that \( \epsilon^*(s, t, \Sigma) \geq c\lambda \).

We now prove that \( c' \) \( (s, t, \Sigma) \geq c\lambda \). Note that if \( \lambda + \nu \propto \lambda \), then the result is clear. Otherwise, we will show that the case \( \nu \gg \lambda \) can only arise if \( \nu \leq C\sigma_1 \), and we will show that \( \epsilon^*(s, t, \Sigma) \geq c\sigma_1 \).

If \( t \in [1, 2] \), then recalling the notation of Lemma [13], we have \( \lambda + \nu \propto \lambda + \nu_{d\text{dense}} \) since \( s \leq C \). If \( i_s = 0 \), then \( \lambda + \nu \propto \lambda \), and we have proved \( \epsilon^*(s, t, \Sigma) \geq c\lambda \propto \lambda + \nu \). Otherwise, we note that

\[ 1 \geq \sum_{j \leq i_s} \pi_j = \frac{2i_s}{s}, \text{ so that } i_s \leq s \leq C. \]

Therefore, \( \nu_{d\text{dense}} = \|\sigma_{\leq i_s}\|_a \leq \sigma_1 i_s^{1/a} \leq C\sigma_1 \). Moreover, if \( t \geq 2 \), then since \( s \leq C \), we have

\[ \nu^2 = \sum_{j=1}^{d} \sigma_j^2 e^{-\lambda^2/\sigma_j^2} = \sum_{j=1}^{d} \frac{8\pi}{2} \sigma_j^4 \nu^j \leq C\sigma_1^4 \nu^4, \text{ so that } \nu \leq C\sigma_1. \]
We now set a different prior over the parameter space. Namely, we define a random vector \( \theta' \) such that \( \forall j \in \{2, \ldots, d\} : \theta'_j = 0 \) and \( \theta'_1 = b \cdot c \sigma_1 \) for a sufficiently small constant \( c > 0 \) depending only on \( \eta \) and where \( b \sim \text{Ber}(1/2) \). We let \( P'_\text{prior} \) denote the probability distribution corresponding to this mixture. Denoting by KL the KL-divergence between probability distributions (see for instance [Tsy08]), we can compute

\[
\text{KL}(P'_\text{prior} \mid P_0) \leq \frac{1}{2} \text{KL}(P_{\theta^{(i)}} \mid P_{\theta^{(0)}}) \leq \text{KL}(N(0, \sigma_1^2) \mid N(c \sigma_1, \sigma_1^2)) \leq c^2,
\]

so that \( \text{TV}(P'_\text{prior}, P_0) \leq \sqrt{\text{KL}(P'_\text{prior} \mid P_0)} \leq c \). By the same argument as in [44], we can conclude that \( \epsilon^*(s, t, \Sigma) \geq c \sigma_1 \geq c \nu \) so that \( \epsilon^*(s, t, \Sigma) \geq c(\lambda + \nu) \). The proof is complete.

\[\Box\]

A.2 Generalities for upper bounds

**Lemma 4.** (Analysis of \( T_{\text{sparse}} \)) Let \( \lambda, \nu, j, \tau, \rho \) be defined as in [5], [6], [14], [18] for \( t \geq 2 \) and as in [25], [26], [28], [30], [34] for \( t \in [1, 2] \). Let \( T_{\text{sparse}} \) be defined as in [10] for \( t \geq 2 \) and [31] for \( t \in [1, 2] \). Then, there exists a constant \( C_0 \) depending only on \( t \), such that when \( \|\theta\|_0 \leq s \), we have

| \( \mathbb{E}^2 T_{\text{sparse}} \) | Under \( H_0 \) | When \( \|\theta_{>j}\|_l^t \geq 4 \bar{C} \rho \) |
|---|---|---|
| \( \forall T_{\text{sparse}} \) | \( = 0 \) | \( \geq \bar{C} \rho / 2^{t+2} \) |
| \( \forall T_{\text{sparse}} \) | \( \leq C_0 \rho \) | \( \leq c \mathbb{E}^2_\phi [T_{\text{sparse}}] \).

In the last cell, \( c \) is a constant depending only on \( \eta \) and \( t \), that can be made arbitrarily small provided that \( \bar{C} = \bar{C}(\eta, t) \) is large enough, and \( \bar{C} \) can be chosen independently of \( C_0 \).

**Proof.**

1. Under \( H_0 \), \( T_{\text{sparse}} \) is centered by definition. As for the variance, by independence, we get:

\[
\mathbb{V}[T_{\text{sparse}}] = \sum_{j > j_\ast} \mathbb{E} \left[ (|X_j|^t - \alpha_j)^2 \mathbb{I}\{X_j|^t > \tau\} \right]
\]

\[
= \sum_{j > j_\ast} \mathbb{E} \left[ |X_j|^{2t} \mathbb{I}\{X_j|^t > \tau\} \right] - \alpha_j^2 \mathbb{P}[|X_j|^t > \tau]
\]

\[
\leq \sum_{j > j_\ast} \mathbb{E} \left[ |X_j|^{2t} \mathbb{I}\{X_j|^t > \tau\} \right] \leq C \rho^2 \quad \text{by Lemma 6}
\]

2. Let \( \|\theta_{>j}\|_l^t \geq 4 \bar{C} \rho \) and let

\[ I = \left\{ j > j_\ast : |\theta_j|^t \geq \bar{C} \tau \right\}. \]

We write \( \theta_l = (\theta_j)_{j \in I} \) and \( \theta_{>j} = (\theta_j)_{j > j_\ast} \). Then we have \( \|\theta_l\|_l^t \geq \frac{1}{2} \|\theta_{>j}\|_l^t, \|\theta_l\|_l^t \geq \bar{C} \rho \). Indeed,

\[
\sum_{j \notin I} |\theta_j|^t \leq s \bar{C} \tau \leq 2 \bar{C} \rho \leq 2^{t+2} \|\theta_{>j}\|_l^t, \quad \text{so that} \quad \|\theta_l\|_l^t = \|\theta_{>j}\|_l^t - \sum_{j \notin I} |\theta_j|^t \geq \frac{1}{2} \|\theta_{>j}\|_l^t.
\]

Throughout the proof, we shall use the fact that for any \( j \in I \) and any \( \alpha \geq 1 \), we have

\[
\mathbb{E}[|\xi_j|^\alpha] \leq C(\alpha) \sigma_j^\alpha \leq C(\alpha) \lambda^\alpha \leq C(\alpha) \tau^\alpha / \lambda \leq \frac{C(\alpha)}{C(\alpha) \tau^\alpha / \lambda} |\theta_j|^\alpha \quad \text{by definition of } I.
\]

(45)

For the first inequality, see for example [Win12]. Now, we can now bound from below \( \mathbb{E} T \) as follows:
• Fix any $j \in I$. Noting that $| \cdot |^t$ is convex, we have for any $u \in \mathbb{R}: |u|^t + |1-u|^t \geq 2^{1-t}$, so that for any $a, b \in \mathbb{R}$, we get $|a+b|^t \geq 2^{1-t}|a|^t + |b|^t$ (set $u = -b/a$ when $a \neq 0$). We therefore get

$$E \left[ (|X_j|^t - \alpha_j) \mathbb{1} \{ |X_j|^t > \tau \} \right] \geq E \left[ (2^{1-t}|\theta_j|^t - |\xi_j|^t - \alpha_j) \mathbb{1} \{ |X_j|^t > \tau \} \right]$$

$$\geq E \left[ 2^{1-t}|\theta_j|^t \mathbb{1} \{ |X_j|^t > \tau \} \right] - E[|\xi_j|^t - \alpha_j]$$

$$\geq 2^{-t}|\theta_j|^t - C|\sigma_j|^t - \alpha_j \quad \text{by Lemma 8 for } C \text{ large enough}$$

$$\geq 2^{-t}|\theta_j|^t - (C + C_{10})\tau,$$

(46)

where $C_{10}$ is the constant from Lemma 8.

• Fix any $j > j_s$ such that $j \in S \setminus I$. We have

$$E \left[ (|X_j|^t - \alpha_j) \mathbb{1} \{ |X_j|^t > \tau \} \right] \geq -\alpha_j \geq -C_{10}\tau.$$  

(47)

• Fix any $j > j_s$ such that $j \not\in S$. Then by definition of $\alpha_j$, 

$$E \left[ (|X_j|^t - \alpha_j) \mathbb{1} \{ |X_j|^t > \tau \} \right] = 0.$$  

(48)

Combining equations (46), (47) and (48), we can conclude that

$$ET_{\text{simple}} \geq 2^{-t}||\theta||^t_t - (1 + 2C_{10})s\tau \geq 2^{-t-1}||\theta||^t_t \quad \text{choosing } C \text{ large enough, by definition of } I,$$

so that $ET_{\text{simple}} \geq 2^{-t-2}||\theta_{j > j_s}||^t_t$, which proves the claim.

We now move to the variance term. Again, there are three cases.

1. If $j \in S \setminus I$, then, using $E|X_j|^{2t} \leq C|\theta_j|^{2t} + C\sigma_j^{2t}$ and Lemma 9, we get

$$\forall \left[ (|X_j|^t - \alpha_j) \mathbb{1} \{ |X_j|^t \geq \tau \} \right] \leq E \left[ (|X_j|^t - \alpha_j) \mathbb{1} \{ |X_j|^t > \tau \} \right] \leq 2E|X_j|^{2t} + 2\alpha_j^2$$

$$\leq C\theta_j^{2t} + (C + C_{10})\tau^2 \leq C \left( 1 + \frac{C_{10}}{C} \right) \tau^2.$$  

(49)

2. If $j \notin S$: we are back to the analysis of the variance under $H_0$, which allows us to directly conclude that

$$\sum_{j \notin S} \forall \left[ (|X_j|^t - \alpha_j) \mathbb{1} \{ |X_j|^t > \tau \} \right] \leq \sum_{j \notin S} E \left[ X_j^{2t} \mathbb{1} \{ |X_j|^t > \tau \} \right] \leq C\rho^2 \quad \text{by Lemma 8}$$

(50)

3. If $j \in I$, we prove that there exists a small constant $c > 0$ such that $\forall \left[ (|X_j|^t - \alpha_j) \zeta_j \right] \leq c|\theta_j|^{2t}$. We define the random variable $\zeta_j = \mathbb{1} \{ |X_j|^t \geq \tau \}$ and $q_j = E \zeta_j$. Note that $E \left[ |X_j|^t \right] \leq 2^{t-1}E \left[ |\theta_j|^t + |\xi_j|^t \right] \leq C \left( |\theta_j|^t + \sigma_j^2 \right)$. Therefore, we have

$$\forall \left[ (|X_j|^t - \alpha_j) \zeta_j \right] = E \left[ (|X_j|^t - \alpha_j)^2 \zeta_j \right] - E^2 \left[ (|X_j|^t - \alpha_j) \zeta_j \right]$$

$$= \left\{ E \left[ X_j^{2t} \zeta_j \right] - 2\alpha_j E \left[ |X_j|^t \zeta_j \right] + \alpha_j^2 q_j \right\} - \left\{ E^2 \left[ |X_j|^t \zeta_j \right] - 2\alpha_j q_j E \left[ |X_j|^t \zeta_j \right] + \alpha_j^2 q_j^2 \right\}$$

$$\leq \left\{ E \left[ X_j^{2t} \zeta_j \right] - E^2 \left[ |X_j|^t \zeta_j \right] \right\} + \alpha_j^2 q_j + 2\alpha_j q_j E \left[ |X_j|^t \zeta_j \right]$$

$$\leq \left\{ E \left[ X_j^{2t} \zeta_j \right] - E^2 \left[ |X_j|^t \zeta_j \right] \right\} + C_{10} \tau^2 + 2C_{10} E \left[ |X_j|^t \right]$$
To control the term \( \{ E \left[ |X_j|^{2t} \zeta_j \right] - E^2 \left[ |X_j|^t \zeta_j \right] \} \), there are two cases.

- If \( t \leq 2 \), then we have

\[
E \left[ |X_j|^{2t} \zeta_j \right] \leq E \left[ X_j^t \right]^{t/2} = \left[ \theta_j^t + 6 \theta_j^t \sigma_j^2 + 3 \sigma_j^4 \right]^{t/2}
\leq \left[ \theta_j^t + 6 \theta_j^t \frac{\sigma_j^2}{C^2 t} + 3 \frac{\theta_j^t}{C^4 t} \right]^{t/2}
\leq \theta_j^{2t} \left( 1 + \frac{9}{C^2 t} \right)^{t/2}
\]

(51)

Now, we define a large constant \( C^* \), and in what follows, we first take \( C^* \) sufficiently large before taking \( \tilde{C} \in (C^*, +\infty) \) sufficiently large for fixed \( C^* \). Defining \( z_j = 1 \{ |\xi_j| \leq C^* \sigma_j \} \), we note that

\[
\forall j \in I : \quad E \left[ |\xi_j| z_j \zeta_j \right] \leq C^* \sigma_j E \left[ z_j \zeta_j \right] \leq \frac{C^*}{\tilde{C}} |\theta_j| E \left[ z_j \zeta_j \right] \leq |\theta_j| E \left[ z_j \zeta_j \right] \text{ if } \tilde{C} \geq C^*.
\]

(53)

Therefore, we get

\[
E \left[ |X_j|^t \zeta_j \right] \geq E^t \left[ |X_j| \zeta_j \right] \geq E^t \left[ |\theta_j| - |\xi_j| \right]^{t} \zeta_j \geq E \left[ (|\theta_j| - |\xi_j|) z_j \zeta_j \right]^{t}
\leq \left( \frac{C^*}{\tilde{C}} \right)^t \left[ \left| \theta_j \right| \left( 1 - \frac{C^*}{\tilde{C}} \right)^t \left( 1 - \frac{C^*}{\tilde{C}} \right)^t \right]
\leq \left| \theta_j \right|^t \left( 1 - \frac{C^*}{\tilde{C}} \right)^t \left( 1 - \frac{C^*}{\tilde{C}} \right)^t \left( 1 - \frac{C^*}{\tilde{C}} \right)^t \text{ if } \tilde{C} \text{ is large enough for fixed } C^*.
\]

(54)

Note that the constant \( \left( 1 - \frac{C^*}{\tilde{C}} \right)^t \left( 1 - \frac{C^*}{\tilde{C}} \right)^t \left( 1 - \frac{C^*}{\tilde{C}} \right)^t \) can be made arbitrarily close to 1 provided that we choose successively \( C^* \) and \( \tilde{C} \) large enough. Now, combining equations (51), (52) and (54), we get

\[
\forall \left( |X_j|^t - \alpha_j \right) \zeta_j \leq \theta_j^{2t} \left( 1 + \frac{9}{C^2 t} \right)^{t/2} |\theta_j|^{2t} \left( 1 - \frac{C^*}{\tilde{C}} \right)^{2t} \left( 1 - \frac{C^*}{\tilde{C}} \right)^{2t} + \frac{C^*}{\tilde{C}} |\theta_j|^{2t}
\leq \left| \theta_j \right|^{2t} c^t,
\]

(56)

where the constant \( c \) can be made arbitrarily small by successively choosing \( C^* \) and \( \tilde{C} \) large enough.

- If \( t \geq 2 \), we use the following Taylor expansion: For any \( \alpha \geq 2 \), for any \( \theta_j, \xi_j \in \mathbb{R} \), there exists \( \xi_j^* \in (0, \xi_j) \) such that

\[
|\theta_j + \xi_j|^\alpha = |\theta_j|^\alpha + \zeta_j^{(\alpha)}
\]
where \( z_j^{(\alpha)} : = \alpha |\theta_j^{\alpha - 1} \xi_j| \text{sign}(\theta_j) + \frac{\alpha(\alpha - 1)}{2} \xi_j^2 |\theta_j + \xi_j|^{\alpha - 2} \). \hfill (57)

For \( j \in I \), we can bound \( \mathbb{E} |z_j^{(\alpha)}| \) from above as follows

\[
\mathbb{E} |z_j^{(\alpha)}| \leq \alpha C \bar{C}^{-1/t} |\theta_j|^{\alpha} + \frac{\alpha(\alpha - 1)}{2} C \left\{ |\theta_j|^\alpha \sqrt{\alpha^{1/t}} + |\theta_j|^\alpha \right\} \leq C |\theta_j|^{\alpha} / C^{1/t},
\]

for some constant \( C \) depending on \( \alpha \). We now have

\[
\mathbb{E} \left[ |X_j|^{2t} \zeta_j \right] \leq \mathbb{E} \left[ |\theta_j|^{2t} + |z_j^{(2t)}| \right] \leq |\theta_j|^{2t} \left( 1 + \frac{C}{C^{1/t}} \right),
\]

\[
\mathbb{E} \left[ |X_j|^{t} \zeta_j \right] \geq \left( |\theta_j|^t - |z_j^{(t)}| \right) \zeta_j \geq |\theta_j|^t q_j - \mathbb{E} \left[ |z_j^{(t)}| \right] \geq |\theta_j|^t \left( q_j - \frac{C}{C^{1/t}} \right) \quad \text{by (58),}
\]

Combining the above two relations, we obtain

\[
\mathbb{E} \left[ |X_j|^{2t} \zeta_j \right] - \mathbb{E}^2 \left[ |X_j|^{t} \zeta_j \right] \leq |\theta_j|^{2t} \left( 1 + \frac{C}{C^{1/t}} \right) - |\theta_j|^{2t} \left( q_j - \frac{C}{C^{1/t}} \right)^2 \leq c |\theta_j|^{2t}, \quad \text{by (59)}
\]

where the constant \( c \) can be made arbitrarily small by choosing \( C \) large enough.

Now, combining equations (49), (50) and (56) for \( t \leq 2 \) or (58) for \( t \geq 2 \), we get

\[
\forall T \leq \sum_{j \geq j_*} C \tau^2 + C \rho^2 + \sum_{j \in I} c |\theta_j|^{2t} \leq C \tau^2 + C \rho^2 + c \| \theta_i \|^2_{2t}
\]

\[
\leq C \rho^2 + c \| \theta_{>j_*} \|^2_{t} \quad \text{since } \| \cdot \|_{2t} \leq \| \cdot \|_{t}
\]

\[
\leq \frac{C}{4C} \| \theta_{>j_*} \|^t_{t} + c \| \theta_{>j_*} \|^2_{t} \quad \text{by assumption}
\]

\[
\leq 2c \| \theta_{>j_*} \|^2_{t}, \quad \text{provided that } C \text{ is large enough}
\]

\[
\leq 8c \mathbb{E}^2(T_{\text{sparse}}).
\]

Lemma 5. Let \( Y \sim \mathcal{N}(0, 1) \) and let \( \alpha \geq 1 \) and \( x > 0 \) such that \( x^2 \geq 2(\alpha - 1) \). Then we have

\[
\mathbb{E} \left[ |Y|^{\alpha} \mathbbm{1}_{|Y| \geq x} \right] \leq 4x^{\alpha-1}e^{-x^2/2}.
\]

Proof of Proposition. \hfill \square

By integration by parts, we have

\[
\int_{x}^{\infty} y^\alpha e^{-y^2/2}dy = x^{\alpha-1}e^{-x^2/2} + (\alpha-1) \int_{x}^{\infty} y^\alpha e^{-y^2/2}dy
\]

\[
\leq x^{\alpha-1}e^{-x^2/2} + \frac{1}{2} \int_{x}^{\infty} y^\alpha e^{-y^2/2}dy \quad \text{using that } y^2 \geq \alpha - 1 \text{ over } [x, +\infty),
\]

so that

\[
\mathbb{E} \left[ |Y|^{\alpha} \mathbbm{1}_{|Y| \geq x} \right] = 2 \int_{x}^{\infty} y^\alpha e^{-y^2/2}dy \leq 4x^{\alpha-1}e^{-x^2/2}.
\]

Lemma 6. Let \( j_* \), \( \tau \) and \( \rho \) be respectively defined as in (7), (14) and (18) for \( t \geq 2 \) and as in (28), (30) and (54) for \( t \in [1, 2] \). Assume that for any \( j > j_* \), we have \( X_j \sim \mathcal{N}(0, \sigma_j^2) \). Then there exists a constant \( C > 0 \) depending only on \( t \), such that

\[
\sum_{j > j_*} \mathbb{E} \left[ |X_j|^{2t} \mathbbm{1}_{\{ |X_j| > \tau \}} \right] \leq C \rho^2.
\]
Proof of Lemma 6. We apply Lemma 3 with \( \alpha = 2t \) and \( x = C_i^{1/t} \lambda / \sigma_j \geq C_i^{1/t} \), which satisfies \( x^2 \geq 2(\alpha - 1) \).

\[
\sum_{j > j_*} \mathbb{E} \left[ |X_j|^{2t} 1 \{ |X_j|^{t} > \tau \} \right] = \sum_{j > j_*} \sigma_j^{2t} \mathbb{E} \left[ |Y|^{2t} 1 \{ |Y|^{t} > \tau / \sigma_j \} \right]
\]

where \( Y \sim \mathcal{N}(0, 1) \)

\[
\leq \sum_{j > j_*} \sigma_j^{2t} \cdot 4v_j^{2t-1}e^{-v_j^2/2} = 4 \sum_{j > j_*} \tau^{2-1/t} \sigma_j e^{-v_j^2/2} \quad \text{by Lemma 5, where } v_j = \frac{\tau^{1/t}}{\sigma_j}
\]

\[
\leq C \rho^2 \quad \text{by Lemma 4}
\]

\[
\square
\]

Lemma 7. Let \( j_*, \tau \) and \( \rho \) be respectively defined as in (7), (14) and (18) for \( t \geq 2 \) and as in (28), (30) and (54) for \( t \in [1, 2] \). Then there exists a constant \( C > 0 \) depending only on \( t \), such that

\[
\sum_{j > j_*} \tau^{2-1/t} \sigma_j e^{-v_j^2/2} \leq C \rho^2.
\]

Proof. For any \( j > j_* \), we let \( v_j = \tau^{1/t}/\sigma_j \). Recalling that \( C_t = (4t)^t \geq 2^t \), we have

\[
v_j^2 = \frac{(C_t \lambda^t + \nu^t/s)^{2/t}}{\sigma_j^2} \geq \frac{1}{2} \left( \frac{C_t \lambda^2}{\sigma_j^2} + \frac{\nu^2}{s^{2/t} \sigma_j^2} \right) \geq \frac{2\lambda^2}{\sigma_j^2} + \frac{\nu^2}{2s^{2/t} \sigma_j^2}.
\]

We have

\[
\sum_{j > j_*} \tau^{2-1/t} \sigma_j e^{-v_j^2/2} \leq \sum_{j > j_*} \tau^{2-1/t} \sigma_j \exp \left( -\frac{\lambda^2}{\sigma_j^2} + \frac{\nu^2}{4s^{2/t} \sigma_j^2} \right)
\]

\[
\leq \tau^{2-1/t} \sum_{j > j_*} \sigma_j \exp \left( -\frac{\lambda^2}{\sigma_j^2} \right) \left( \frac{\sigma_j^{s^{1/t}}}{\nu^t} \right)^{t-1} \quad \text{using that } e^{-x^2/4} \leq Cx^{1-t} \text{ for some } C = C(t)
\]

\[
\leq C \tau^{2-1/t} \frac{s^{1-1/t}}{\nu^{t-1}} \sum_{j > j_*} \sigma_j \exp \left( -\frac{\lambda^2}{\sigma_j^2} \right) \leq C \tau^{2-1/t} \frac{s^{1-1/t}}{\nu^{t-1}} s
\]

\[
= C(\tau s)^2 \frac{\nu}{(\tau s)^{1/t}} \leq C \tau^{2} s^2 \leq C \rho^2.
\]

\[
\square
\]

Lemma 8. Assume that, for some \( j \in \{1, \ldots, d\} \) and that for some large enough constant \( C \geq 1 \), and some arbitrary real number \( \tau > 0 \) satisfying \( \sigma_j^2 \leq \tau \), we have \( |\theta_j| \geq C \tau \) and let \( X_j \sim \mathcal{N}(\theta_j, \sigma_j^2) \). Then

\[
\mathbb{P} \left( |X_j|^{t} \leq \tau \right) \leq e^{-(C-1)^2/2}.
\]

Proof of Lemma 8. Assume without loss of generality that \( \theta_j \geq 0 \). We have

\[
\mathbb{P} \left( |X_j|^{t} \leq \tau \right) = \mathbb{P} \left( \mathcal{N} \left( \frac{\theta_j}{\sigma_j}, 1 \right)^{t} \leq \frac{\tau}{\sigma_j} \right) \leq \mathbb{P} \left( \mathcal{N} \left( \frac{\theta_j}{\sigma_j}, 1 \right)^{t} \leq \frac{\tau^{1/t}}{\sigma_j} \right)
\]

\[
= \mathbb{P} \left( \mathcal{N}(0,1) \leq \frac{\tau^{1/t} - \theta_j}{\sigma_j} \right) \leq \exp \left( -\frac{(\tau^{1/t} - \theta_j)^2}{2\sigma_j^2} \right) \leq e^{-(C-1)^2/2},
\]

using the relations \( \theta_j \geq C \tau \geq C \sigma_j \).

\[
\square
\]
Lemma 9. Let $0 < \sigma_j^2 \leq \tau$ be any two positive numbers and let $X_j \sim \mathcal{N}(0, \sigma_j^2)$. Then for some absolute constant $C_0 > 0$

$$\alpha_j := \mathbb{E}[|X_j|^\tau | |X_j|^\tau > \tau] \leq C_0 \tau.$$  

Proof of Lemma 9. Letting $Z_j \sim \mathcal{N}(0, 1)$, we have by Lemma 4 and Lemma 4 from [CCT17]:

$$\alpha_j = \mathbb{E}[|X_j|^\tau | |X_j|^\tau > \tau] = \frac{\mathbb{E}[|X_j|^\tau \{ |X_j|^\tau > \tau \}]}{\mathbb{P}( |X_j|^\tau > \tau )} = \sigma_j^2 \frac{\mathbb{E}[|Z_j|^\tau \{ |Z_j|^\tau > \tau/\sigma_j^2 \}]}{\mathbb{P}( |Z_j|^\tau > \tau/\sigma_j^2 )} \leq C\sigma_j^2 \left( \frac{\tau^{1/\tau}}{\sigma_j^2} \right)^{t-1} \frac{\exp\left(-\tau^{2/\tau} \sigma_j^2 \right)}{\sigma_j^2 \exp\left(-\tau^{2/\tau} \sigma_j^2 \right)} =: C \tau.$$

\[\square\]

B Proof of Theorem 1

We need the following lemma before we begin the proof of the theorem.

Lemma 10. The following function is continuous and strictly decreasing over $\mathbb{R}$:

$$\phi : \left\{ \begin{array}{l}
\mathbb{R} \rightarrow \mathbb{R} \\
\beta \mapsto \frac{\sum_{j=1}^d \sigma_j^2 \exp\left(-\beta/\sigma_j^2 \right)}{\sqrt{\sum_{j=1}^d \sigma_j^2 \exp\left(-\beta/\sigma_j^2 \right)}}
\end{array} \right..$$

Proof of Lemma 10. The function $\phi$ is clearly differentiable hence continuous, and

$$\phi'(\beta) = \frac{\left( \sum_{j=1}^d \sigma_j^2 \exp\left(-\beta/\sigma_j^2 \right) \right)^2 - 2 \left( \sum_{j=1}^d \sigma_j^2 \exp\left(-\beta/\sigma_j^2 \right) \right) \left( \sum_{j=1}^d \sigma_j^4 \exp\left(-\beta/\sigma_j^2 \right) \right)}{2 \left( \sum_{j=1}^d \sigma_j^4 \exp\left(-\beta/\sigma_j^2 \right) \right)^{3/2}}$$

$$= \frac{1}{2} \left( \sum_{j=1}^d \sigma_j^4 \exp\left(-\beta/\sigma_j^2 \right) \right)^{-3/2} \left( \sum_{j=1}^d \exp\left(-\beta/\sigma_j^2 \right) \right)^{-2} \times \left( \sum_{j=1}^d \sigma_j^2 \exp\left(-\beta/\sigma_j^2 \right) \right)^2 \left( \sum_{j=1}^d \sigma_j^4 \exp\left(-\beta/\sigma_j^2 \right) \right)^{-2} - 2 \left( \sum_{j=1}^d \sigma_j^4 \exp\left(-\beta/\sigma_j^2 \right) \right)^{-2} \left( \sum_{j=1}^d \exp\left(-\beta/\sigma_j^2 \right) \right)^{-2} \left( \sum_{j=1}^d \sigma_j^2 \exp\left(-\beta/\sigma_j^2 \right) \right)^2 \left( \sum_{j=1}^d \sigma_j^4 \exp\left(-\beta/\sigma_j^2 \right) \right)^{-2} \left[ \mathbb{E}_{Y \sim \mu} Y^2 - 2 \mathbb{E}_{Y \sim \mu} Y^4 \right] < 0.$$

At the last line, we have defined the probability measure $\mu = \sum_{j=1}^d w_j \delta_x$, where $w_j = \frac{\exp\left(-\beta/\sigma_j^2 \right)}{\sum_{j=1}^d \exp\left(-\beta/\sigma_j^2 \right)}$ and where $\delta_x$ denotes the Dirac measure at point $x \in \mathbb{R}$. We also used Jensen’s inequality combined with the fact that, when $Y \sim \mu$, we have $\mathbb{E}Y^4 > 0$ as by assumption, $\sigma_j > 0, \forall j \in [d]$.  

\[\square\]
Lemma 11. Let $t \geq 2$ and let $\lambda, \nu$ and $j_*$ be defined as in [3], [10] and [7]. We let

$$s_{\text{sparse}} = \frac{1}{\nu} \sum_{j > j_*} \sigma_j^t \exp \left( -\lambda^2 / \sigma_j^2 \right),$$

$$\nu_{\text{sparse}}^t = \sqrt{\nu} \sum_{j > j_*} \sigma_j^t \exp \left( -\lambda^2 / \sigma_j^2 \right),$$

$$\nu_{\text{dense}}^t = \sqrt{\sum_{j \leq j_*} \sigma_j^t}.$$

Up to absolute constants, we have $\nu \asymp \nu_{\text{dense}} + \nu_{\text{sparse}}$ and $\nu_{\text{sparse}}^t \leq \nu^t \lambda^t s_{\text{sparse}}$.

In particular, it holds that $\nu^t + \lambda^t s \asymp \nu_{\text{dense}}^t + \lambda^t s$.

Remark 1. In the isotropic case $\sigma_j = \sigma$, $\forall j \in [d]$ and for $t = 2$, we recall that $\lambda_2$ plays the role of $\lambda^2$. Consider the case where $s \leq 2 \sqrt{d/e}$. Then, we have $j_* = d$ so that $\nu_{\text{dense}} = 0$. Therefore, it holds that $\nu^t \leq \nu_{\text{sparse}}^t \leq \lambda_2 s_{\text{sparse}} \leq \lambda_2 s$, hence the term $\nu/s$ never dominates over $\lambda_2 s$.

Proof of Lemma 11. By definition of $j_*$, the following relations are true up to absolute constants:

$$\nu = \sqrt{\sum_{j = 1}^{d} \sigma_j^{2t} \exp \left( -\lambda^2 / \sigma_j^2 \right)} \asymp \sqrt{\sum_{j \leq j_*} \sigma_j^{2t} \exp \left( -\lambda^2 / \sigma_j^2 \right) + \sum_{j > j_*} \sigma_j^{2t} \exp \left( -\lambda^2 / \sigma_j^2 \right)}$$

$$\asymp \sqrt{\sum_{j > j_*} \sigma_j^{2t}} + \sqrt{\sum_{j > j_*} \sigma_j^{2t} \exp \left( -\lambda^2 / \sigma_j^2 \right)}$$

$$= \nu_{\text{dense}}^t + \nu_{\text{sparse}}^t.$$

Moreover, we have

$$\nu_{\text{sparse}}^t = \sum_{j > j_*} \sigma_j^{2t} \exp \left( -\lambda^2 / \sigma_j^2 \right) \leq \lambda^t \sum_{j > j_*} \sigma_j^t \exp \left( -\lambda^2 / \sigma_j^2 \right) = \nu^t \lambda^t s_{\text{sparse}}.$$

We now prove that $\nu^t + \lambda^t s \asymp \nu_{\text{dense}}^t + \lambda^t s$. If $\nu_{\text{sparse}} \leq \nu_{\text{dense}}$, this relation is clear. Otherwise, $\nu^t \leq 2\nu_{\text{sparse}}^t$ so that by [61], we have $\nu_{\text{sparse}}^t \leq 2\lambda^t s_{\text{sparse}}$. Therefore, $\nu^t \leq 2\nu_{\text{sparse}}^t \leq 4\lambda^t s_{\text{sparse}} \leq 4\lambda^t s$, so that $\nu^t + \lambda^t s \asymp \lambda^t s \asymp \nu_{\text{dense}}^t + \lambda^t s$.

B.1 Proof of Lower Bounds for $t \geq 2$

Proof of Theorem 11. First, if $s \leq C$ for some constant $C$ depending only on $\eta$, then by Lemma 3, we have $c(s, t, \Sigma)^t \geq c(\lambda + \nu)^t \asymp \lambda^t s + \nu^t$. From now on, we will assume that $s$ is larger than a sufficiently large constant $c(\eta)$ depending only on $\eta$.

We denote by $\Pi$ the prior distribution over $\theta$, defined such that $\forall b \in \{0, 1\}^d, \forall \omega \in \{\pm 1\}^d$:

$$P \Pi (\theta = (b_j \omega_j)_{j = 1}^d) = \frac{1}{2^d} \Pi_{j = 1}^d \pi_j^{b_j} (1 - \pi_j)^{1 - b_j}.$$

Let $P_{\text{prior}} = E_{\theta \sim \Pi} \left[ N(\theta, \Sigma) \right]$ denote the corresponding mixture of normal distributions $N(\theta, \Sigma)$ where $\theta \sim \Pi$. Note that if $\beta \geq 0$, then $\sum_{j = 1}^d \pi_j = s/2$, otherwise, if $\beta < 0$, then by monotonicity (see Lemma 10), we have $\sum_{j = 1}^d \pi_j \leq s/2$. Our prior has a random sparsity, equal to $\sum_{j = 1}^d b_j$. Therefore, with high probability, its sparsity is at most $s$. To justify this, note that

$$E \left[ \sum_{j = 1}^d b_j \right] = \sum \pi_j \leq s/2, \quad \text{and} \quad \forall \left[ \sum_{j = 1}^d b_j \right] = \sum \pi_j (1 - \pi_j) \leq s/2.$$
Provided that \( s \geq 20/\eta \), we have by Chebyshev’s inequality,

\[
P \left( \sum_{j=1}^{d} b_j > s \right) \leq \frac{\sqrt{\sum_{j=1}^{d} b_j}}{(s/2)^2} \leq \frac{2}{s} \leq \frac{\eta}{10}.
\]  

(62)

We use Lemma 23 in \([LGS21]\) to compute \( \chi^2 (P_{\text{prior}} \| P_0) \). Let \( \theta, \theta' \) be two independent random variables with distribution \( II \). Then

\[
1 + \chi^2 (P_{\text{prior}} \| P_0) = \mathbb{E}_{\theta, \theta'} \left[ \exp \left( \theta^T \Sigma^{-1} \theta' \right) \right] = \prod_{j=1}^{d} \mathbb{E}_{\theta_j, \theta'_j} \left[ \exp \left( \theta_j \theta'_j \sigma_j^{-2} \right) \right]
\]

\[
= \prod_{j=1}^{d} \left( 1 - \pi_j^2 \right) + \pi_j^2 \left( \frac{1}{2} \exp \left( \frac{\gamma_j^2}{\sigma_j^2} \right) + \frac{1}{2} \exp \left( \frac{\gamma_j^2}{\sigma_j^2} \right) \right)
\]

\[
= \prod_{j=1}^{d} \left[ 1 + 2\pi_j^2 \sinh^2 \left( \frac{\gamma_j^2}{2\sigma_j^2} \right) \right] \leq \exp \left[ \sum_{j=1}^{d} \pi_j^2 \cdot 2 \sinh^2 \left( \frac{\gamma_j^2}{2\sigma_j^2} \right) \right]
\]

\[
= \exp \left[ \sum_{j=1}^{d} \pi_j^2 \left( \sqrt{1 + c^2 \exp \left( 2\lambda^2/\sigma_j^2 \right)} - 1 \right) \right]
\]

\[
\leq \exp \left\{ c \sum_{j=1}^{d} \pi_j^2 \exp \left( \lambda^2/\sigma_j^2 \right) \right\}
\]

(63)

(64)

There are two cases.

1. First case: \( \beta \leq 0 \) i.e. \( \lambda = 0 \), then the relation (64) simplifies as:

\[
1 + \chi^2 (P_{\text{prior}} \| P_0) \leq \exp \left[ c \sum_{j=1}^{d} \pi_j^2 \right] = \exp(c),
\]

ensuring indistinguishability if \( c \) is small enough. Moreover, using that \( \arg \sinh u \geq \log(1 + u) \), we conclude that the prior’s squared \( L^2 \) norm concentrates on

\[
\sum_{j=1}^{d} \pi_j^2 \gamma_j^4 \geq \frac{\sum_{j=1}^{d} \sigma_j^{2t}}{\sqrt{\sum_{j=1}^{d} \sigma_j^{2n}}} \log (1 + c) = \log(1 + c) \sqrt{\sum_{j=1}^{d} \sigma_j^{2t}},
\]

which exactly corresponds to the dense case. Note that, since \( \lambda = 0 \), this quantity also coincides with \( \log(1 + c) \sqrt{\sum_{j=1}^{d} \sigma_j^{2t}} e^{-\lambda^2/\sigma_j^2} + \frac{1}{2} \lambda^2 s \).

2. Second case: \( \lambda > 0 \). Then the relation (64) simplifies as:

\[
1 + \chi^2 (P_{\text{prior}} \| P_0) \leq \exp \left[ c \sum_{j=1}^{d} \pi_j^2 \exp \left( \lambda^2/\sigma_j^2 \right) \right] = \exp \left[ c \cdot \frac{\sum_{j=1}^{d} \sigma_j^{2t} \exp \left( -\lambda^2/\sigma_j^2 \right)}{\sum_{j=1}^{d} \sigma_j^{2n} \exp \left( -\lambda^2/\sigma_j^2 \right)} \right] = \exp(c).
\]

Moreover, we have that \( \arg \sinh u \geq \log(1 + u) \). Note also that, for \( x \geq 1 \) and \( c < 1 \), by concavity of \( x \mapsto x^c \), the function \( x^c \) is always below its tangent in \( x = 1 \) so that \( x^c \leq 1 + c(x-1) \leq 1 + cx \). Therefore,
we have the relation $\log(1 + cx) \geq c \log x$. Moreover, for $x \geq 1$ we also have $\log(1 + cx) \geq \log(1 + c)$, so that $\log(1 + cx) \geq c \log x \lor \log(1 + c)$. We now apply this for $x = \exp(\lambda^2/\sigma_j^2)$, which yields that the prior’s squared $L^2$ norm concentrates on

$$
\sum_{j=1}^d \pi_j \gamma_j^2 \geq \sum_{j=1}^d \sigma_j^{2t} \exp \left( -\lambda^2/\sigma_j^2 \right) \log^{t/2} \left( 1 + c \exp \left( \lambda^2/\sigma_j^2 \right) \right)
$$

$$
\geq \log^{t/2} (1 + c) \sqrt{\sum_{j=1}^d \sigma_j^{2t} \exp \left( -\lambda^2/\sigma_j^2 \right) \lor c^t \lambda^t \frac{\sum_{j=1}^d \sigma_j^2 \exp \left( -\lambda^2/\sigma_j^2 \right)}{\sum_{j=1}^d \sigma_j^{2t} \exp \left( -\lambda^2/\sigma_j^2 \right)}}
$$

$$
= \log^{t/2} (1 + c) \sqrt{\sum_{j=1}^d \sigma_j^{2t} \exp \left( -\lambda^2/\sigma_j^2 \right) \lor c^t \lambda^t s/2}.
$$

\[\square\]

### B.2 Upper bounds for $t \geq 2$

**Proof of Theorem 27.** We recall the definitions of the tests $T_{\text{dense}}$ and $T_{\text{sparse}}$ from equations (16).

We write $\theta_{\leq j} = (\theta_1, \ldots, \theta_j)$, and $\theta_{> j} = (\theta_{j+1}, \ldots, \theta_d)$, where we recall that $j_* = \max \{ j \in [d] : \sigma_j \geq \lambda \}$.

We show that there exist large enough constants $C_0, C_1$ and $\bar{C}$ such that, when $\|\theta\|_0 \leq s$, we have:

|                | Under $H_0$ | If $\|\theta_{\leq j_*}\|_t^i \geq \bar{C}j$ | Under $H_0$ | If $\|\theta_{> j_*}\|_t^i \geq \bar{C}j$ |
|----------------|-------------|----------------------------------------|-------------|----------------------------------------|
| $E^2 T_{\text{dense}}$ | $0$         | $\geq C_1 \|\theta_{\leq j_*}\|_t^2$       | $0$         | $\geq \bar{C}j/2^{t+2}$                |
| $\forall T_{\text{dense}}$ | $\leq C_0 \sum_{j \leq j_*} \sigma_j^{2t}$ | $\leq \bar{C}^{-2/t} E_{\theta}^2 [T_{\text{dense}}]$   | $\forall T_{\text{sparse}}$ | $\leq C_0 \rho$ | $\leq c E_{\theta}^2 [T_{\text{sparse}}]$ |

where in the last cell, $c$ is a constant depending only on $\eta$ and $t$, that can be made arbitrarily small provided that $\bar{C}$ is large enough, and $\bar{C}$ can be chosen independently of $C_0$. Proposition 2 is proved by combining the above relations with Chebyshev’s inequality.

**Analysis of $T_{\text{dense}}$.** Under $H_0$, $T_{\text{dense}}$ is centered by definition. Its variance under $H_0$ can be bounded from above as follows

$$
\forall T_{\text{dense}} = \sum_{j \leq j_*} \forall |X_j|^t \leq \sum_{j \leq j_*} E|X_j|^{2t} \leq C \sum_{j \leq j_*} \sigma_j^{2t},
$$

for some constant $C$ depending on $t$ (see for example [Win12]).

Now, if $\|\theta_{\leq j_*}\|_t^i \geq \bar{C}j$ and $\|\theta\|_0 \leq s$, then, writing $X_j = \theta_j + \xi_j$, we will prove that $\sum_{j \leq j_*} E_{\theta}|X_j|^t - E|\xi_j|^t \geq 1/4^t \|\theta\|_t^i$. We have $E_{\theta}|X_j|^t \geq |\theta_j|^t P(\theta_j \xi_j \geq 0) \geq 1/2 |\theta_j|^t$. Therefore, if $|\theta_j|^t \geq 4 E|\xi_j|^t$, then

$$
E_{\theta}|X_j|^t - E|\xi_j|^t \geq 1/4 |\theta_j|^t.
$$

Otherwise, we can assume that $|\theta_j|^t < 4 E|\xi_j|^t$. We use a Taylor expansion that is analogous to (57), except that we swap the roles of $\xi_j$ and $\theta_j$. For fixed $\xi_j$, we define the function $\phi(\theta_j) = |\xi_j + \theta_j|\rho$, which is twice
continuously differentiable. For any \( \theta_j \in \mathbb{R} \), there exists \( \theta_j' \in [0, \theta_j] \) such that \( \phi(\theta_j) = \phi(0) + \theta_j\phi'(0) + \frac{1}{2}\theta_j^2\phi''(\theta_j') \), or equivalently

\[
|\theta_j + \xi_j|^\alpha = |\xi_j|^\alpha + \alpha|\theta_j^{-1}\xi_j| \text{sign}(\theta_j\xi_j) + \frac{\alpha(\alpha - 1)}{2}|\theta_j|^2|\xi_j + \theta_j'|^{\alpha - 2},
\]

for any \( \alpha \geq 2 \). Taking the expectation, for \( \alpha = t \), gives

\[
E|\theta_j + \xi_j|^{t} = E|\xi_j|^t + \frac{t(t - 1)}{2}E|\xi_j + \theta_j'|^{\alpha - 2}
\geq E|\xi_j|^t + \frac{t(t - 1)}{2}\theta_j^2 E\left[|\xi_j|^{t - 2}\mathbb{1}\{\theta_j\xi_j > 0\}\right]
= E|\xi_j|^t + C\theta_j^2 s_j^{-2}
\geq E|\xi_j|^t + C\theta_j^2 C''(t - 4)|\theta_j|^{t - 2}
\]

recalling that \( |\theta_j|^t < 4E|\xi_j|^t =: C\sigma_j^t \)

\[
= E|\xi_j|^t + C''|\theta_j|^t.
\]

In both cases, we have found a constant \( C'' \) depending only on \( t \) such that

\[
E|\theta_j + \xi_j|^t - E|\xi_j|^t \geq C''E|\theta_j|^t,
\]

which yields

\[
E\theta |\theta_j|^t \geq C''|\theta_j|^t.
\]

We now turn to the variance term. We have the following classical inequalities (see [IIS03]):

\[
(|x| + |y|)^\alpha \leq 2^{\alpha - 1}(|x|^\alpha + |y|^\alpha),
\]

\[
|\alpha x + y|^\alpha - |x|^\alpha \leq \alpha 2^{\alpha - 1}|x| \left(|x|^\alpha + |y|^\alpha \right),
\]

true for any \( x, y \in \mathbb{R} \) and \( \alpha \geq 1 \). Therefore,

\[
\forall|\theta_j + \xi_j|^t = \min_{x \in \mathbb{R}} E\left[\left(|\theta_j + \xi_j|^t - x\right)^2\right] \leq E\left[\left(|\theta_j + \xi_j|^t - |\theta_j|^t\right)^2\right]
\leq t^22^{t - 2}E\left[|\xi_j|^2 \left(|\xi_j|^{t - 1} + |\theta_j|^{t - 1}\right)^2\right]
\leq t^22^{t - 1}E\left[|\xi_j|^2 \left(|\xi_j|^{2t - 2} + |\theta_j|^{2t - 2}\right)\right] \leq C \left( \sigma_j^{2t} + \sigma_j^2 |\theta_j|^{2t - 2}\right).
\]

Now, setting \( u = t \) and \( v = \frac{t}{t - 1} \), we have \( \frac{1}{u} + \frac{1}{v} = 1 \), so that by Hölder’s inequality

\[
\sum_{j \leq j_*} \sigma_j^2 |\theta_j|^{2t - 2} \leq \left( \sum_{j \leq j_*} \sigma_j^2 \right)^{1/u} \left( \sum_{j \leq j_*} |\theta_j|^{2t} \right)^{1/v} = \|\sigma_{\leq j_*}\|_2^2 \|\theta_{\leq j_*}\|_{2t}^{2t - 2}
\leq e^{1/t} \tilde{C}^{-2/t} \|\theta_{\leq j_*}\|_{2t}^{2t} \leq c E^2 [T_{\text{dense}}],
\]

where in the last line we used the fact that, since \( \forall j \leq j_* : \sigma_j \geq \lambda \), we have \( \sqrt{t}\|\sigma_{\leq j_*}\|_{2t}^t \leq \nu^t \leq \rho \leq \frac{1}{3\|\theta_{\leq j_*}\|_{2t}^t} \).

**Analysis of** \( T_{\text{sparce}} \): See Lemma [4]

\[\square\]

### C Proof of Theorem 2

We first present a couple of useful lemmas before presenting the detailed proofs of the lower and upper bounds corresponding to Theorem 2.
Lemma 12. The functions $\nu(x)$ and $f(x)$ are continuous with respect to $x$. Moreover, $\lim_{x \to +\infty} f(x) = d$ and

$$\lim_{x \to +\infty} f(x) = 0.$$

Proof of Lemma 12. In this proof, we will use the notation from Section 3. Fix $x_0 \geq 0$ and recall (22) and (28). Note that $j_*(x)$ is always left-continuous: $j_*(x) = \lim_{y \to x^-} j_*(y)$. Therefore, from (23), $\nu(x)$ is a left-continuous function. Now, we show that $x \mapsto j_*(x)$ and $x \mapsto \nu(x)$ are right-continuous functions. Fix $x_0 \in \mathbb{R}$; we always have $\sigma_{j_*} \geq x_0$ by definition. If $\sigma_{j_*} > x_0$, then $x \mapsto j_*(x)$ is clearly continuous on a neighborhood of $x_0$ and so is $x \mapsto \nu(x)$. Otherwise, we have $\sigma_{j_*} = x_0$ and we show that $x \mapsto j_*(x)$ and $x \mapsto \nu(x)$ are still right-continuous in $x_0$. Define

$$J(x_0) = \{j : \sigma_j = x_0 = \sigma_{j_*}\}.$$  

By (23), we have that $\nu'(x_0) \geq C \frac{\sigma_{j_*}}{\nu(x_0)^{a-1}} \wedge \frac{\sigma_j^4}{x_0 2^t \nu(x_0)} = C \frac{x_0^a}{\nu(x_0)^{a-1}} \wedge \frac{x_0 2^t}{\nu(x_0)}$, so that $\nu(x_0) \geq x_0$ and consequently, $\forall j \in J(x_0) : \frac{\sigma_j^a}{\nu(x_0)^{a-1}} \wedge \frac{\sigma_j^4}{x_0 2^t \nu(x_0)} = \frac{x_0^a}{\nu(x_0)}$. We can now write

$$\nu'(x_0) = \sum_{j \leq j_* \quad j \notin J(x_0)} \frac{\sigma_j^a}{\nu(x_0)^{a-1}} \wedge \frac{\sigma_j^4}{x_0 2^t \nu(x_0)} + \sum_{j > j_*} |J(x_0)| \frac{x_0 2^t}{\nu(x_0)} + \sum_{j > j_* \quad \sigma_j = x_0} \frac{\sigma_j^2}{\nu(x_0)} \exp\left(-\frac{x_0^2}{\sigma_j^2} + 1\right).$$

(67)

Noting that for a sufficiently small $\delta > 0$, we have for any $x \in (x_0, x_0 + \delta)$

$$\nu'(x) = \sum_{j \leq j_* \quad j \notin J(x_0)} \frac{\sigma_j^a}{\nu(x_0)^{a-1}(x)} \wedge \frac{\sigma_j^4}{x_0 2^t \nu(x_0)} + \sum_{j > j_* \quad \sigma_j = x_0} \frac{\sigma_j^2}{\nu(x_0)} \exp\left(-\frac{x_0^2}{\sigma_j^2} + 1\right).$$

Now, the two sets of summation indices are fixed when $x \in (x_0, x_0 + \delta)$, so that the right-hand side is clearly continuous with respect to $x$ over $(x_0, x_0 + \delta)$. Therefore, writing $\nu(x_0^+) = \lim_{x \to x_0^+} \nu(x)$, we get

$$\nu'(x_0^+) = \sum_{j \leq j_* \quad j \notin J(x_0)} \frac{\sigma_j^a}{\nu(x_0^+)} \wedge \frac{\sigma_j^4}{x_0 2^t \nu(x_0^+)} + \sum_{j > j_* \quad \sigma_j = x_0} \frac{\sigma_j^2}{\nu(x_0^+)} \exp\left(-\frac{x_0^2}{\sigma_j^2} + 1\right).$$

Comparing with (67), we note that $\nu(x_0^+)$ and $\nu(x_0)$ solve the same equation, hence $\nu(x_0^+) = \nu(x_0)$ by uniqueness of the solution of (23). This proves that $\nu$ is right-continuous in $x_0$, concluding the proof of the continuity of $\nu$. The continuity of $f$ can be proved by exactly following the same steps.

When $x \to 0^+$, we have $j_*(x) \to d$ and $\nu(x) \to e^{-1/a} \|\sigma\|_a$. Therefore, $f(x) \to \sum_{j=1}^d 1 = d$.

When $x \to \infty$, we have $j_*(x) = 0$ and $\nu(x) = \left(\sum_{j=1}^d c_j \sigma_j^2 \exp\left(-\frac{x_0^2}{\sigma_j^2}\right)\right)^{1/2t}$ for $x$ large enough. Therefore, still for $x$ large enough, we have $f(x) = \sqrt{\sum_{j=1}^d \sigma_j^2 \exp\left(-\frac{x_0^2}{\sigma_j^2}\right)} \to 0$. \[ \square \]
Lemma 13. Recall the notation of Section 3 Writing $\sigma_{\leq i^*} = (\sigma_1, \ldots, \sigma_{i^*})$ and $\sigma_{\text{int}} = (\sigma_{i^*+1}, \ldots, \sigma_j)$, we let

\[
\begin{align*}
  s_{f_{\text{dense}}} &= i_*, \\
  s_{\text{inter}} &= \sum_{j = i^*+1}^{j^*} \frac{\sigma_j^4}{\nu a - t}, \\
  s_{\text{sparse}} &= \sum_{j > j^*} \frac{e\sigma_j^4}{\nu} \exp \left( -\lambda^2/\sigma_j^2 \right), \\
  \nu_{f_{\text{dense}}} &= \|\sigma_{\leq i^*}\|_a, \\
  \nu_{\text{inter}} &= \lambda^{t-2} \|\sigma_{\text{int}}\|^2_4, \\
  \nu_{\text{sparse}} &= \sqrt{\sum_{j > j^*} e\sigma_j^4} \exp \left( -\lambda^2/\sigma_j^2 \right).
\end{align*}
\]

Note that by equations (24) and (25), we have $\frac{\nu}{2} = s_{f_{\text{dense}}} + s_{\text{inter}} + s_{\text{sparse}}$, and by Lemma 7, we have $\nu t \approx \nu_{f_{\text{dense}}} + \nu_{\text{inter}} + \nu_{\text{sparse}}$. Then the following relations hold:

1. $\lambda^t \left(s_{f_{\text{dense}}} + s_{\text{inter}}\right) \leq 2\nu_t$. In particular, we have $\nu^t + \lambda^t s_{\text{sparse}} \approx \nu^t + \lambda^t s$.

2. $\nu^t_{\text{inter}} = \nu^t \lambda^t s_{\text{inter}}$ and $\nu^t_{\text{sparse}} \leq \nu^t \lambda^t \sigma^2$. In particular: $\nu^t + \lambda^t s \approx \nu^t_{f_{\text{dense}}} + \lambda^t s$.

Proof of Lemma 13. 1. We have by Lemma 1

\[
\begin{align*}
  \lambda^t s_{f_{\text{dense}}} &= \lambda^t i_* = \frac{i_* \lambda^t}{\nu a - t} \leq \frac{1}{\nu a - t} \sum_{j = i^*}^j \sigma_j^2 \leq \nu^t \quad \text{by definition of } i_* \text{ from (27)},
\end{align*}
\]

\[
\begin{align*}
  \lambda^t s_{\text{inter}} &= \sum_{j = i^*+1}^{j^*} \frac{\sigma_j^4}{\lambda^2 - 2t \nu^t} \leq \nu^t.
\end{align*}
\]

2. We have by definition of $s_{\text{int}}$ and $\nu_{\text{int}}$:

\[
\begin{align*}
  \lambda^t s_{\text{inter}} &= \sum_{j = i^*+1}^{j^*} \frac{\sigma_j^4}{\lambda^2 - 2t \nu^t} = \frac{\nu^t}{\nu a - t}.
\end{align*}
\]

Moreover, we have by definition of $s_{\text{sparse}}$ and $\nu_{\text{sparse}}$:

\[
\begin{align*}
  \lambda^t s_{\text{sparse}} &= \sum_{j > j^*} \frac{e\sigma_j^4}{\nu} \exp \left( -\lambda^2/\sigma_j^2 \right) \geq \sum_{j > j^*} \frac{e\sigma_j^4}{\nu} \exp \left( -\lambda^2/\sigma_j^2 \right) = \frac{\nu^t_{\text{sparse}}}{\nu^t}.
\end{align*}
\]

Finally, we prove that $\nu^t + \lambda^t s \approx \nu^t_{f_{\text{dense}}} + \lambda^t s$. If $\max(\nu_{f_{\text{dense}}}, \nu_{\text{int}}, \nu_{\text{sparse}}) = \nu_{f_{\text{dense}}}$, then the result is clear. Otherwise, assume first that $\nu_{f_{\text{dense}}} \leq \nu_{\text{int}} \leq \nu_{\text{sparse}}$. Then we have $\nu^t \leq 3\nu^t_{\text{sparse}}$, which, by equation (68), yields $\nu^t_{\text{sparse}} \leq 3\lambda^t s_{\text{sparse}}$. In particular, $\nu^t \leq 9\lambda^t s$, so that $\lambda^t s + \nu^t_{f_{\text{dense}}} \approx \lambda^t s + \nu^t \approx \lambda^t s$. Proceeding similarly if $\nu_{f_{\text{dense}}} \leq \nu_{\text{sparse}} \leq \nu_{\text{int}}$ concludes the proof.

\[\square\]

C.1 Lower bounds

Proof of Theorem 2. First, if $s \leq C$ for some constant $C$ depending only on $\eta$, then by Lemma 3 we have $\epsilon^*(s, t, \Sigma)^t \geq c(\lambda + \nu)^t \approx \lambda^t s + \nu^t$. From now on, we will assume that $s$ is larger than a sufficiently large constant $c(\eta)$ depending only on $\eta$.

We recall the definition of the prior from Subsection 5.2. We can bound from above the $\chi^2$ divergence between this prior and $\mathbb{P}_0$ as in (63):

\[
1 + \chi^2 (\mathbb{P}_{\text{prior}} \mid \mathbb{P}_0) \leq \exp \left[ \sum_{j=1}^d \pi_j \cdot 2 \sinh^2 \left( \frac{\gamma^2}{2\sigma_j^2} \right) \right].
\]
Recall the notation from Section 3. Lemma 1 ensures that \( \nu \geq \sigma_j \) for any \( j \leq i_* \), so that \( \gamma_j^2 = c \frac{\sigma_j^2}{\nu} \leq c \sigma_j^2 \).

Moreover, by definition of \( j_* \), we also have \( \gamma_j = c \lambda \leq \sigma_j \) for any \( j \in \{i_* + 1, \ldots, j_*\} \). Therefore, on the dense part \( \{1, \ldots, j_*\} \), we can use the relation \( \sinh(x) \leq 2x \) which holds for any \( x \leq 1 \). We get:

\[
2 \sum_{j \leq j_*} \pi_j^2 \sinh^2 \left( \frac{\gamma_j^2}{2 \sigma_j^2} \right) \leq \sum_{j \leq j_*} \pi_j^2 \cdot \frac{\gamma_j^4}{\sigma_j^4} = c^4 \sum_{j \leq i_*} \frac{\sigma_j^4}{\nu^4} + \frac{c^4}{\lambda^4 - 2t \nu^2} \sum_{j = i_* + 1}^{j_*} \sigma_j^4 \leq 2c^4 \quad \text{by Lemma 1} \quad (69)
\]

Now, for any \( j > j_* \), we use \( \sinh(x) \leq e^x/2 \) to get

\[
2 \sum_{j > j_*} \pi_j^2 \sinh^2 \left( \frac{\gamma_j^2}{2 \sigma_j^2} \right) \leq 2 \sum_{j > j_*} \frac{\sigma_j^2}{\nu^2} \exp \left( -2\lambda^2 / \sigma_j^2 \right) \cdot \frac{\exp \left( \lambda^2 / \sigma_j^2 \right)}{4} = \sum_{j > j_*} \frac{\sigma_j^2}{\nu^2} \exp \left( \lambda^2 / \sigma_j^2 \right).
\]

By Lemma 1 the latter quantity can be made arbitrarily small provided that \( c_\nu \) is sufficiently small. Combining this fact with (69), we conclude that \( \chi^2 \left( \mathbb{P}_{\theta \mid \theta_0} \right) \) can be made arbitrarily small by choosing \( c \) and \( c_\nu \) small enough, ensuring the indistinguishability condition.

By definition of \( \lambda \) from (25), we have \( f(\lambda) = s/2 \) so that \( \sum_{j=1}^d \pi_j = \frac{s}{2} \). Therefore, with high probability, the prior’s sparsity is at most \( s \), provided that \( s \) is greater than a sufficiently large constant depending only on \( \eta \). Now, letting \( s_{\text{sparse}} = \sum_{j > j_*} \pi_j \), the prior’s \( L^t \) norm raised to the power \( t \) concentrates on

\[
\frac{\gamma_j \pi_j}{\nu} = \sum_{j \leq i_*} \frac{\sigma_j^4}{\nu^4 - \nu^2} + c \sum_{j = i_*}^{j_*} \frac{\sigma_j^4}{\lambda^4 - 2t \nu^2} + c \lambda^t s_{\text{sparse}}
\]

\[
\geq \sum_{j \leq i_*} \frac{\sigma_j^4}{\nu^4 - \nu^2} + c \sum_{j = i_*}^{j_*} \frac{\sigma_j^4}{\lambda^4 - 2t \nu^2} + c \nu^t \sum_{j > j_*} \frac{\lambda^t}{\sigma_j} \sum_{j > j_*} \exp \left( -\frac{\lambda^2}{\sigma_j^2} \right) \quad \text{since} \quad \lambda \geq \sigma_j \quad \text{for} \quad j > j_*
\]

\[
\geq c \left( \nu^t + \frac{\lambda^t}{2} s_{\text{sparse}} \right), \quad \text{by Lemma 1}
\]

Now, by Lemma 13 we have \( \nu^t + \lambda^t s_{\text{sparse}} \preceq \nu^t + \lambda^t s \). This concludes the proof.

\( \square \)

### C.2 Upper bounds for \( t \in [1, 2] \)

**Proof of Theorem 2**. Theorem 2 is proved by combining Lemmas 14 and Chebyshev’s inequality. \( \square \)

**Lemma 14.** We write \( \theta_{\leq i_*} = (\theta_1, \ldots, \theta_{i_*}) \), and \( \theta_{\text{int}} = (\theta_{i_*+1}, \ldots, \theta_{j_*}) \). There exist two large enough constants \( C_0 \) and \( \tilde{C} \) such that, when \( \|\theta\|_0 \leq s \) the following relations hold:

|                  | Under \( H_0 \) | When \( \|\theta_{\leq i_*}\|_1^t \geq \tilde{C} \rho \) | Under \( H_0 \) | When \( \|\theta_{\text{int}}\|_t \geq \tilde{C} \rho \) |
|------------------|-----------------|-----------------|-----------------|-----------------|
| \( \mathbb{E}^2 T_{f_{\text{dense}}} \) | = 0             | \( \geq \tilde{C}^4 \rho \sum_{j \leq i_*} \sigma_j^4 \) | = 0             | \( \geq 4 \tilde{C}^2 \sum_{j = i_*+1}^{j_*} \sigma_j^4 \) |
| \( \forall T_{f_{\text{dense}}} \)   | = 2 \( \sum_{j \leq i_*} \sigma_j^4 \) | \( \leq c \mathbb{E}^2 \left[ T_{f_{\text{dense}}} \right] \) | = 2 \( \sum_{j = i_*+1}^{j_*} \sigma_j^4 \) | \( \leq c \mathbb{E}^2 \left[ T_{\text{inter}} \right] \) |

In the above table, the constant \( c \) can be made arbitrarily small provided that \( C \) is large enough.

**Proof of Lemma 14.**
1. Analysis of $T_{\text{dense}}$: Under $H_0$, the relations $\mathbb{E}T_{\text{dense}} = 0$ and $\forall T_{\text{dense}} = 2 \sum_{j \geq i^*} \sigma_j^2$ are clear. Under the alternative, assume that $\|\theta_{i^*}\|_t^4 \geq \tilde{C}_\rho \geq C \left[\sum_{j \geq i^*} \sigma_j^2\right]^{t/a}$ by Lemma I. By the Hölder inequality, we have

$$\mathbb{E}T_{\text{dense}} = \sum_{j \leq i^*} \frac{\theta_j^2}{\sigma_j^2} \geq \frac{\|\theta_{i^*}\|_t^4}{(\sum_{j \leq i^*} \sigma_j^2)^{(t-2)/2}} \geq \tilde{C}_\rho^{2/t} \left(\sum_{j \leq i^*} \sigma_j^2\right)^{1/2}.$$ 

As for the variance, we have

$$\forall T_{\text{dense}} = 4 \sum_{j \leq i^*} \frac{\theta_j^2 \sigma_j^2}{\sigma_j^2} + 2 \sum_{j \leq i^*} \sigma_j^2 = 4 \sum_{j \leq i^*} \frac{\theta_j^2}{\sigma_j^2} \sigma_j^2 + 2 \sum_{j \leq i^*} \sigma_j^2 \leq 4 \sqrt{\sum_{j \leq i^*} \frac{\theta_j^2}{\sigma_j^2} \sum_{j \leq i^*} \sigma_j^2 + 2 \sum_{j \leq i^*} \sigma_j^2} \quad \text{by the Cauchy-Schwarz inequality}$$

$$\leq 4 \sum_{j \leq i^*} \frac{\theta_j^2}{\sigma_j^2} \frac{1}{C^2/t} \mathbb{E} \left[T_{\text{dense}}\right] + \frac{2}{C^4} \mathbb{E}^2 \left[T_{\text{dense}}\right] \leq c \mathbb{E}^2 \left[T_{\text{dense}}\right].$$

Note that we did not need to use the fact that $\|\theta\|_0 \leq s$ in the analysis of $T_{\text{dense}}$.

2. Analysis of $T_{\text{inter}}$: Under $H_0$, the relations $\mathbb{E}T_{\text{inter}} = 0$ and $\forall T_{\text{inter}} = 2 \sum_{j > i^*} \sigma_j^4$ are clear. Now, assume that $\|\theta\|_0 \leq s$ and $\|\theta_{i^*}\|_t^4 \geq \tilde{C}_\rho \geq \tilde{C}_s \lambda^t s + \frac{\tilde{C}_s}{\lambda^{2-t}} \left[\sum_{j > i^*} \sigma_j^4\right]^{1/2}$. Note that

$$\frac{1}{C^d} \|\theta_{i^*}\|_t^4 \geq \lambda^t s + \frac{1}{\lambda^{2-t}} \left[\sum_{j > i^*} \sigma_j^4\right]^{1/2} \geq \inf_{\lambda > 0} \lambda^t s + \frac{1}{(\lambda^t)^{2-t}} \left[\sum_{j > i^*} \sigma_j^4\right]^{1/2} = 2s^{1-t/2} \left[\sum_{j > i^*} \sigma_j^4\right]^{t/4}. \quad (70)$$

Now, by Hölder’s inequality, we can bound from below the expectation term as follows

$$\mathbb{E}T_{\text{inter}} = \sum_{j > i^*} \theta_j^2 = \|\theta_{i^*}\|_t^2 \geq \|\theta_{i^*}\|_t^2 \geq 4C^2/t \left[\sum_{j > i^*} \sigma_j^4\right]^{1/2}. \quad (71)$$

As for the variance, we have $\forall \left[T_{\text{inter}}\right] = 2 \sum_{j > i^*} \sigma_j^4 + 4 \sum_{j > i^*} \sigma_j^4 \theta_j^2$, and by the Cauchy-Schwarz inequality:

$$\sum_{j > i^*} \sigma_j^4 \theta_j^2 \leq \sqrt{\sum_{j > i^*} \sigma_j^4} \sum_{j > i^*} \theta_j^4 \leq \frac{1}{4C^2/t} \mathbb{E} \left[T_{\text{inter}}\right] \|\theta_{i^*}\|_t^2 \leq \frac{1}{4C^2/t} \mathbb{E}^2 \left[T_{\text{inter}}\right]. \quad \text{by (71)}.$$ 

Therefore, sill by (71):

$$\forall \left[T_{\text{inter}}\right] \leq \frac{1}{C^4/t} \mathbb{E}^2 \left[T_{\text{inter}}\right] + 4 \cdot \frac{1}{4C^2/t} \mathbb{E}^2 \left[T_{\text{inter}}\right] \leq c \mathbb{E}^2 \left[T_{\text{inter}}\right].$$
C.3 Proof of Lemma 1

Proof of Lemma 1. By definition of \( i_* \) from (24) and \( \nu \) from (26), we have

\[
\nu = \frac{A}{\nu^2} + \frac{B}{\nu^2},
\]

where \( A = \sum_{j \leq i_*} \sigma_j^2 \) and \( B = \sum_{j > i_*} \sigma_j^2 \exp \left( -\frac{\lambda^2}{\sigma_j^2} + 1 \right) \).

Therefore, we have \( \nu \geq (\nu^{-1} A)^{1/a} \lor (\nu^{-1} B)^{1/2t} \), hence:

\[
\nu^t \geq \frac{\nu^{-t/a}}{2} A^{1/a} + \frac{\nu^{-1/2}}{2} B^{1/2}.
\]

Setting \( C_1 = \frac{\nu^{-t/a}}{2} \lor \frac{\nu^{-1/2}}{4} \) yields the lower bound part of the claim. For the upper bound part, note that equation (72) yields

\[
\frac{\nu}{2} \leq \frac{A}{\nu^2} \lor \frac{B}{\nu^2},
\]

therefore, we have \( \nu \leq (2A/\nu)^{1/a} \lor (2B/\nu)^{1/2t} \) so that

\[
\nu^t \leq (2A/\nu)^{1/a} \lor (2B/\nu)^{1/2t} \leq (2A/\nu)^{t/a} + (2B/\nu)^{1/2}.
\]

Taking \( C_2 = (2\nu)^{-t/a} \lor (2\nu)^{-1/2} \) concludes the proof.

\[\square\]

D Proof of Theorem 3

Proof of Lemma 2. In this proof, we let \( \epsilon^*_s(s) = \epsilon^*(1, t, \Sigma) \) for any \( s \in [d] \) and \( t \in [1, \infty] \), and we also define \( \epsilon_i^* = \epsilon_i^*(1) \).

1. We first prove that \( \epsilon^*_i(s) = \epsilon^*_i \). The inequality \( \epsilon^*_i(s) \geq \epsilon^*_i \) is clear. To prove the converse bound, we define \( \Pi \) as a prior over the parameter space

\[
\Theta(\epsilon_i^*(s), s, \infty) = \left\{ \theta \in \mathbb{R}^d : \|\theta\|_\infty \geq \epsilon_i^*(s) \text{ and } \|\theta\|_0 \leq s \right\}
\]

and denote by \( P_\Pi = \mathbb{P}_{\theta \sim \Pi} \left[ \mathcal{N}(\theta, \Sigma) \right] \) the corresponding mixture induced by \( \Pi \). Assume moreover that TV(\( P_\Pi, \mathbb{P}_0 \)) \( \leq 1 - \eta \). For any \( \theta \in \mathbb{R}^d \), we let \( \arg\max_{j \in [d]} \|\theta_j\|_0 \) denote the smallest index \( i \) such that \( \|\theta_i\|_0 = \|\theta\|_\infty \). We also let \( \phi(\theta) = \left( \theta_{j=i} \right)_{j \in [d]} \) be the vector obtained by zeroing out all coordinates of \( \theta \) except for the first extremal one. Note that \( \phi(\theta) \) is always 1-sparse and that \( \|\phi(\theta)\|_\infty = \|\theta\|_\infty \) by construction. We consider the new prior \( \phi\Pi \) by

\[
P_{\phi\Pi} = \mathbb{E}_{\theta \sim \Pi} \left[ \mathcal{N}(\phi(\theta), \Sigma) \right].
\]

Therefore, \( \phi\Pi \) is a prior over \( \Theta(\epsilon^*(s), 1, \infty) \) and clearly, TV(\( P_{\phi\Pi}, \mathbb{P}_0 \)) \( \leq \) TV(\( P_\Pi, \mathbb{P}_0 \)) \( \leq 1 - \eta \), which proves by (41) that \( \epsilon^*_i(s) \leq \epsilon^*_i \).

2. Now, we prove that \( \epsilon^*_i = \epsilon_i^* \). We let \( \psi_{\infty} \) be the test defined in (36) and \( \psi_t \) be the test defined in (35) for \( t \in [1, 2] \), or in (17) for \( t \in [2, \infty] \). Let \( \theta \in \mathbb{R}^d \) such that \( \|\theta\|_0 = 1 \) and \( \|\theta\|_\infty > \epsilon_i^* \). Then we also have \( \|\theta\|_t > \epsilon_i^* \), hence

\[
\mathbb{P}_0(\psi_t = 1) + \mathbb{P}_\theta(\psi_t = 0) \leq \eta
\]

Therefore, \( \epsilon^*_i = \epsilon_i^* \).
\[ R(\psi_t, \epsilon_t^*, 1, \infty, \Sigma) = \mathbb{P}_0(\psi_t = 1) + \sup \left\{ \mathbb{P}_0(\psi_t = 0) \mid \theta \in \Theta(\epsilon_t^*, 1, \infty) \right\} \leq \eta \]
\[ \Rightarrow R^*(\epsilon_t^*, 1, \infty, \Sigma) \leq \eta \implies \epsilon_t^* \geq \epsilon_\infty. \]

The proof of the converse bound \( \epsilon_t^* \leq \epsilon_\infty \) is analogous.

\[ \square \]

D.1 Upper bound for \( t = \infty \)

We recall the expression of our test \( \psi^* = 1 \{ \exists j \in [d]: |X_j| > C(\lambda + \nu) \} \), where \( \lambda \) and \( \nu \) are respectively defined as in (3) and (4) by taking \( t' = 2 \) and \( s' = 1 \) in these equations. We first show that for any small constant \( c > 0 \), provided that \( s = 1 \), we have

\[ C \geq \frac{8}{\sqrt{2\pi c}}, \]

for any small constant \( c > 0 \), provided that \( C \) is large enough. Note that in Lemma 4, since \( s = 1 \), we have \( \tau = \rho \), where the notation \( \rho \) represents \( \rho^2 \) in the present proof.

\[ \begin{align*}
\mathbb{P}_0(\psi^* = 0) &= \mathbb{P}_0(\forall j \in [d]: |X_j| \leq C\rho) = \prod_{j=1}^{d} \left( 1 - \mathbb{P}(|X_j| > C\rho) \right) \\
&\geq \prod_{j=1}^{d} \left( 1 - \frac{4}{\sqrt{2\pi C\rho}} \exp \left( -\frac{C^2\rho^2}{\sigma_j^2} \right) \right) \quad \text{by Lemma 4 from [CCT17]} \\
&\geq \exp \left\{ -\sum_{j=1}^{d} \frac{40}{\sqrt{2\pi C\rho}} \exp \left( -\frac{C^2\rho^2}{\sigma_j^2} \right) \right\} \quad \text{using that} \quad 1 - x \geq e^{-10x} \quad \text{for} \quad x \in \left[ 0, \frac{1}{2} \right] \\
&\geq \exp \{-c\} \geq 1 - 2c, \quad \text{by Lemma 4}
\end{align*} \]

Under \( H_0 \), taking \( C \geq \frac{8}{\sqrt{2\pi c}} \), we have

\[ \mathbb{P}_0(\psi^* = 0) = \mathbb{P}_0(\forall j \in [d]: |X_j| \leq C\rho) = \prod_{j=1}^{d} \left( 1 - \mathbb{P}(|X_j| > C\rho) \right) \]

\[ \geq \prod_{j=1}^{d} \left( 1 - \frac{4}{\sqrt{2\pi C\rho}} \exp \left( -\frac{C^2\rho^2}{\sigma_j^2} \right) \right) \quad \text{by Lemma 4 from [CCT17]} \\
\geq \exp \left\{ -\sum_{j=1}^{d} \frac{40}{\sqrt{2\pi C\rho}} \exp \left( -\frac{C^2\rho^2}{\sigma_j^2} \right) \right\} \quad \text{using that} \quad 1 - x \geq e^{-10x} \quad \text{for} \quad x \in \left[ 0, \frac{1}{2} \right] \\
\geq \exp \{-c\} \geq 1 - 2c, \quad \text{by Lemma 4}
\]

for any small constant \( c > 0 \), provided that \( C \) is large enough. Note that in Lemma 4, since \( s = 1 \), we have \( \tau = \rho \), where the notation \( \rho \) represents \( \rho^2 \) in the present proof.

\[ \begin{align*}
\mathbb{P}_0(\psi^* = 0) &= \mathbb{P}_0(\forall j \in [d]: |X_j| \leq C\rho) = \prod_{j=1}^{d} \left( 1 - \mathbb{P}(|X_j| > C\rho) \right) \\
&\geq \prod_{j=1}^{d} \left( 1 - \frac{4}{\sqrt{2\pi C\rho}} \exp \left( -\frac{C^2\rho^2}{\sigma_j^2} \right) \right) \quad \text{by Lemma 4 from [CCT17]} \\
&\geq \exp \left\{ -\sum_{j=1}^{d} \frac{40}{\sqrt{2\pi C\rho}} \exp \left( -\frac{C^2\rho^2}{\sigma_j^2} \right) \right\} \quad \text{using that} \quad 1 - x \geq e^{-10x} \quad \text{for} \quad x \in \left[ 0, \frac{1}{2} \right] \\
&\geq \exp \{-c\} \geq 1 - 2c, \quad \text{by Lemma 4}
\end{align*} \]

for any small constant \( c > 0 \), provided that, for fixed \( C \), the constant \( C' \) is large enough. At the last line, we used the fact that \( \sigma_j \geq \epsilon \rho \) for any \( j \in [d] \).

E Proof of examples

E.1 Isotropic case

In this Subsection, assume that \( \sigma_1 = \cdots = \sigma_d = \sigma \). Assume that \( t \geq 2 \). We have by equation (5):

\[ s/2 = \frac{d \alpha^4 \exp(-\beta/\sigma^2)}{\sqrt{d} \sigma^2 \exp(-\beta/\sigma^2)} = \sqrt{d} \exp\left(-\beta/2\sigma^2\right), \quad \text{so that} \quad \beta = 2\sigma^2 \log\left(\frac{2\sqrt{d}}{s}\right). \]
If $s \geq 2\sqrt{d}$, then $\lambda = 0$ so that $\nu^t = \sqrt{d}\sigma^t$ and $\epsilon^*(s, t, \sigma^2 I_d) \approx \sigma d^{1/2t}$. Otherwise, by the definition of $\nu$ in (6):

$$\nu^t = \sqrt{d}\sigma^t \exp\left(-\beta/\sigma^2\right) = \sqrt{d}\sigma^t \frac{s}{2\sqrt{d}} = \sigma^t s/2,$$

so that $\epsilon^*(s, t, \sigma^2 I_d)^t \approx \nu^t + \lambda^t s \approx \sigma^t s \log^{t/2} \left(\frac{s}{\sqrt{s}}\right)$.

Now, assume that $t \leq 2$. If $s = d$, then from (25) we have $\lambda = 0$ and $\nu = \|\sigma\|_a$. Otherwise, we have $i_s = 0$. If $s \geq \sqrt{d}$, we have $\lambda = \left(\sum_{j=1}^d \sigma_j^4\right)^{1/4}/\sqrt{s} = \sigma d^{1/4}/\sqrt{s}$ and $j_s = d$. Therefore, $\epsilon^*(s, t, \sigma^2 I_d)^t \approx \nu^t + \lambda^t s \approx \sigma^t d^{1/4}s^{1-t/2}$. If $s < \sqrt{d}$, then $j_s = 0$ and the analysis follows the same lines as in the case $t \geq 2$:

$$s/2 = \sum_{j=1}^d \frac{\sigma_j}{\nu_j^t} \exp\left(-\lambda^2/\sigma^2\right) = \sum_{j=1}^d \frac{\sqrt{d}}{\nu_j^t} \exp\left(-\lambda^2/\sigma^2\right)$$

and

$$\nu^t = \sqrt{d}\sigma^t \exp\left(-\frac{\lambda^2}{\sigma^2}\right).$$

i.e. $s/2 = \sqrt{d} \exp\left(-\frac{\lambda^2}{2\sigma^2}\right)$ hence $\lambda^2 = 2\sigma^2 \log \left(\frac{2\sqrt{d}}{s}\right)$.

Therefore, we have $\epsilon^*(s, t, \sigma^2 I_d)^t \approx \sigma^t s \log^{t/2} \left(\frac{2\sqrt{d}}{s}\right)$.

### E.2 Polynomially increasing variances

Assume that $\alpha \geq 1$ and $t \geq 2$. We have

$$s/2 = \sum_{j=1}^d j \epsilon_j^a \exp\left(-\lambda^2/\sigma^2\right) \geq \sqrt{\sum_{j=1}^d d^2 j \epsilon_{2\alpha} \exp\left(-\lambda^2/\sigma^2\right)} \geq \sum_{j=1}^d \sigma_j \exp\left(-\lambda^2/\sigma^2\right) \approx \sqrt{d} \exp\left(-c\lambda^2/d^{2\alpha}\right).$$

Similarly:

$$s/2 = \sum_{j=1}^d j \epsilon_j^a \exp\left(-\lambda^2/\sigma^2\right) \leq \sqrt{\sum_{j=1}^d d^2 j \epsilon_{2\alpha} \exp\left(-\lambda^2/\sigma^2\right)} \leq \frac{2}{\sqrt{2\alpha}} \sum_{j=1}^d \sigma_j \exp\left(-\lambda^2/\sigma^2\right) \approx \sqrt{d} \exp\left(-c'\lambda^2/d^{2\alpha}\right).$$

Therefore, we have $\lambda \approx_\alpha d^\alpha \sqrt{\log(d/s^2)}$ if $s \leq C\sqrt{d}$ and $\lambda = 0$ otherwise. Therefore, if $s \geq C\sqrt{d}$, then we have $\nu^t = \sum_{j=1}^d j^{2\alpha} \approx d^{2\alpha+1}$, so that $\epsilon^*(s, t, \Sigma) \approx \nu \approx d^\alpha d^{1/2t} = \sigma_{\max} d^{1/2t}$, where $\sigma_{\max} = \sigma_d = \max_j \sigma_j$. Otherwise,

$$\nu^t = \sum_{j=1}^d j^{2\alpha} \exp\left(-\lambda^2/j^{2\alpha}\right) \leq 2 \sum_{j=1}^d j^{2\alpha} \exp\left(-\lambda^2/j^{2\alpha}\right) \approx \alpha d^{2\alpha+1} \left(\frac{s^2}{d}\right) \leq d^{2\alpha} s^2,$$

therefore, $\epsilon^*(s, t, \Sigma) \approx \lambda s^{1/t} \approx d^\alpha s^{1/t} \sqrt{\log(d/s^2)}$.

### E.3 Exponentially decreasing variances

Assume first that $t \geq 2$. If $\alpha t^d \geq \alpha^t/4$, then we are back to the isotropic case and $\epsilon^*(s, t, \Sigma) \approx \epsilon^*(s, t, \alpha I_d)$. Otherwise, we have $\alpha^t < \alpha/4$. Let $j_0 = \min \left\{ j : \alpha^j \leq \alpha/2 \right\}$ and $j_1 = \min \left\{ j : \alpha^j \leq \alpha/4 \right\}$. Then,

$$\sum_{j > j_1} \sigma_j^t \exp\left(-\beta/\sigma^2\right) \leq \alpha^{(j_1+1)} \frac{1 - \alpha^{t-d-j_1}}{1 - \alpha^t} \exp\left(-16\beta/\alpha^2\right) \leq \frac{\alpha^t/4}{1 - \alpha^t} \exp\left(-16\beta/\alpha^2\right).$$

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Moreover,
\[ \sum_{j < j_0} \sigma_j^t \exp\left(\frac{-\beta}{\sigma_j^2}\right) \geq \sum_{j < j_0} \sigma_j^t \exp\left(-4\beta / \alpha^2\right) = \frac{\alpha^t - \alpha^{j_0} t}{1 - \alpha^t} \exp\left(-4\beta / \alpha^2\right) \geq \frac{\alpha^t / 2}{1 - \alpha^t} \exp\left(-4\beta / \alpha^2\right). \]

Therefore, we always have \( \sum_{j = 1}^{d} \sigma_j^t \exp\left(-\beta / \sigma_j^2\right) \approx \sum_{j \leq j_1} \sigma_j^t \exp\left(-\beta / \sigma_j^2\right). \) Proceeding similarly, we can also get \( \sum_{j = 1}^{d} \sigma_j^{2t} \exp\left(-\beta / \sigma_j^2\right) \approx \sum_{j \leq j_1} \sigma_j^{2t} \exp\left(-\beta / \sigma_j^2\right). \) Now, for \( j \leq j_1, \) we have \( \sigma_j^t \in [\alpha^t / 4^t, \alpha^t], \) so that:

\[ s / 2 \approx \frac{\sum_{j \leq j_1} \sigma_j^t \exp\left(-\beta / \sigma_j^2\right)}{\sqrt{\sum_{j \leq j_1} \sigma_j^{2t} \exp\left(-\beta / \sigma_j^2\right)}} \approx \sqrt{j_1} \exp\left(-\beta / C' \alpha^2\right), \quad \text{hence} \quad \beta \approx C' \alpha^2 \log\left(C \sqrt{j_1} / s\right), \]

for some constants \( C, C' \) depending only on \( t. \) Moreover, \( \nu^{2t} \approx \sum_{j \leq j_1} \sigma_j^{2t} \exp\left(-\beta / \sigma_j^2\right). \) We exactly recover the analysis of the isotropic case from Subsection E.1. In other words, it holds that \( \varepsilon^*(s, t, \Sigma) \approx \varepsilon^*(s, t, \alpha^2 I_{j_1}), \) where by definition of \( j_1, \) we have \( j_1 \approx \log^{-1}(1/\alpha). \)