REDUCTION AND INTEGRABILITY

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Abstract. We discuss the relationship between the integrability of a dynamical system invariant under a Lie group action and its reduced integrability, i.e. integrability of the corresponding reduced system.

1. Introduction

The aim of this note is to address the following question (Q):

Given a manifold $M$ (eventually with singularities), a Lie group $G$ which acts on $M$ (in such a way that the quotient $M/G$ is a manifold with singularities), and a vector field $X$ on $M$ which is invariant under the action of $G$. Denote by $X/G$ the projection of $X$ on $M/G$. What is the relationship between the integrability of $(M, X)$ and the integrability of $(M/G, X/G)$ (a.k.a. the reduced integrability of $(M, X, G)$)?

The above question is very natural, since dynamical systems often admit natural symmetry groups, and by integrability of a problem in classical mechanics one often means its reduced integrability. It seems to me, however, that Question (Q) has not been formally addressed anywhere in the literature, and that’s why this note.

We will consider two different cases: Hamiltonian and non-Hamiltonian. For simplicity, we will assume that $G$ is a compact group. We will show that, when the action of $G$ is Hamiltonian, Hamiltonian integrability is the same as reduced Hamiltonian integrability. In the non-Hamiltonian case, integrability still implies reduced integrability, though the inverse needs not be true. The proof of these facts is elementary: we simply play with the dimensions of various spaces and their intersections, quotients, etc.

2. Hamiltonian integrability

Let $(M, \Pi)$ be a real Poisson manifold, with $\Pi$ being the Poisson structure. Let $H$ be a (smooth or analytic) function on $M$, and $X_H$ be the corresponding Hamiltonian vector field. Assume that we have found a set $\mathcal{F}$ of first integrals of $X_H$, i.e. each $F \in \mathcal{F}$ is a function on $M$ which is preserved by $X_H$ (equivalently, $\{F, H\} = 0$, where $\{,\}$ denotes the Poisson bracket as usual). Denote by $\text{ddim} \mathcal{F}$ the functional dimension of $\mathcal{F}$, i.e. the maximal number of functionally independent functions in $\mathcal{F}$. To avoid pathologies, we will always assume that the functional dimension of the restriction of $\mathcal{F}$ to any open subset of $M$ is equal to $\text{ddim} \mathcal{F}$. (This is automatic in the analytic case).

We will associate to $\mathcal{F}$ the space $\mathcal{X} = \mathcal{X}_\mathcal{F}$ of Hamiltonian vector fields $X_F$ such that $X_F(G) = 0$ for all $G \in \mathcal{F}$ and $F$ is functionally dependent of $\mathcal{F}$ (i.e. the...
functional dimension of the union of $\mathcal{F}$ with the function $F$ is the same as the functional dimension of $\mathcal{F}$). Clearly, $X_H$ belongs to $\mathcal{X}$, and the vector fields in $\mathcal{X}$ commute pairwise. Denote by $\text{ddim} \mathcal{X}$ the functional dimension of $\mathcal{X}$, i.e. the maximal number of vector fields in $\mathcal{X}$ whose exterior (wedge) product does not vanish.

With the above notations, we have the following definition, due essentially to Nekhoroshev [8] and Mischenko and Fomenko [7]:

**Definition 2.1.** A Hamiltonian vector field $X_H$ on an $m$-dimensional Poisson manifold $(M, \Pi)$ is called *Hamiltonianly integrable* with the aid of a set of first integrals $\mathcal{F}$, if $m = \text{ddim} \mathcal{F} + \text{ddim} X_\mathcal{F}$. It is called *properly Hamiltonianly integrable* if $\mathcal{F}$ satisfies the following additional properness condition:

There are $q$ functions $F_1, ..., F_q$ in $\mathcal{F}$, where $q = \text{ddim} \mathcal{F}$, which are functionally independent, and whose joint “moment map” $(F_1, ..., F_q) : M \to \mathbb{R}^q$ (here we consider the real case) is a proper map from $M$ to its image, and there are $p$ vector fields $X_1, ..., X_p$ in $\mathcal{X}$, where $p = \text{ddim} \mathcal{X}$, such that the image of their singular set \{ $x \in M, X_1 \wedge X_2 \wedge ... \wedge X_p(x) = 0$ \} under the map $(F_1, ..., F_q) : M \to \mathbb{R}^q$ is nowhere dense in $\mathbb{R}^q$.

**Remarks**

1. The above notion of integrability is often called *generalized Liouville integrability*, or also *non-commutative integrability* by Mischenko-Fomenko, due to the fact that the functions in $\mathcal{F}$ do not Poisson-commute in general, and in many cases one may choose $\mathcal{F}$ to be a finite-dimensional non-commutative Lie algebra of functions (under the Poisson bracket). When the functions in $\mathcal{F}$ Poisson-commute, we get back to the classical integrability à la Liouville.

2. We always have $m \leq \text{ddim} \mathcal{F} + \text{ddim} \mathcal{X}$ (even for non-integrable systems), because the vector fields in $\mathcal{X}$ are tangent to the common level sets of the functions in $\mathcal{F}$. Thus the integrability condition $m = \text{ddim} \mathcal{F} + \text{ddim} \mathcal{X}$ is a maximality, or fullness, condition on $\mathcal{F}$.

3. Casimir functions of $(M, \Pi)$, i.e. functions whose Hamiltonian vector fields vanish, must be functionally dependent of $\mathcal{F}$ in the integrable case - otherwise we could add them to $\mathcal{F}$ to increase the functional dimension of $\mathcal{F}$, which contradicts the above remark about $m \leq \text{ddim} \mathcal{F} + \text{ddim} \mathcal{X}$.

4. It follows directly from the fact that if $X_F \in \mathcal{X}$ then $F$ is functionally dependent of $\mathcal{F}$ that we have $\text{ddim} \mathcal{X} \leq \text{ddim} \mathcal{F}$. We didn’t mention the rank of the Poisson structure $\Pi$ in the above definition, but of course $\text{ddim} \mathcal{X} \leq 1/2 \text{rank} \Pi$.

When $m = \text{ddim} \mathcal{F} + \text{ddim} \mathcal{X}$ and $\text{ddim} \mathcal{X} < 1/2 \text{rank} \Pi$, some authors also say that the system is *super-integrable* (because in this case one finds more first integrals than necessary for integrability).

5. We prefer to use the term *Hamiltonian integrability*, to contrast it with the *non-Hamiltonian integrability* discussed in the next section.

6. Under the additional properness condition, one get a natural generalization of the classical Liouville theorem [8, 7]: the manifold $M$ is foliated by invariant isotropic tori on which the flow of $X_H$ is quasi-periodic (thus the behavior of $X_H$ is very regular, justifying the word “integrable”), and there also exist local generalized action-angle coordinates. The existence of action-angle coordinates for Liouville-integrable systems is often referred to as Arnold-Liouville theorem, though it was probably first proved by Mineur [6].
Assume that there is a Hamiltonian action of a Lie group $G$ on $(M, \Pi)$, given by an equivariant moment map $\pi : M \rightarrow \mathfrak{g}^*$, where $\mathfrak{g}$ denotes the Lie algebra of $G$, such that the following conditions are satisfied:

1) The action of $G$ on $M$ is proper, so that the quotient space $M/G$ is a singular manifold whose ring of functions may be identified with the ring of $G$-invariant functions on $M$.

2) Recall that the image $\pi(M)$ of $M$ under the moment map $\pi : M \rightarrow \mathfrak{g}^*$ is saturated by symplectic leaves (i.e. coadjoint orbits) of $\mathfrak{g}^*$. Denote by $s$ the minimal codimension in $\mathfrak{g}^*$ of a coadjoint orbit which lies in $\pi(M)$. Then we assume that there exist $s$ functions $f_1, ..., f_s$ on $\mathfrak{g}^*$, which are invariant on the coadjoint orbits which lie in $\pi(M)$, and such that for almost every point $x \in M$ we have $df_1 \wedge ... \wedge df_s(\pi(x)) \neq 0$.

For example, when $G$ is compact and $M$ is connected, then the above conditions are satisfied automatically.

If $\pi(M)$ contains a generic point of $\mathfrak{g}^*$, then $s = \text{ind} \, \mathfrak{g}$, where $\text{ind} \, \mathfrak{g}$ denotes the index of $\mathfrak{g}$, i.e. the corank of the corresponding linear Poisson structure on $\mathfrak{g}^*$, and if there are (ind $\mathfrak{g}$) global functionally independent Casimir functions on $\mathfrak{g}^*$ they may play the role of required functions $f_1, ..., f_s$ in the above assumption. If $\pi(M)$ lies in the singular part of $\mathfrak{g}^*$ then $s$ may be greater than the index of $\mathfrak{g}$.

Since $\Pi$ is preserved by $G$, it can be projected to a Poisson structure, denoted by $\Pi/G$, on $M/G$. In the case when $M$ is a symplectic manifold, the symplectic leaves of $(M/G, \Pi/G)$ are known as Marsden-Weinstein reductions.

Let $H$ be a function on $M$ which is invariant by $G$. Then $H$ may be viewed as the pull-back of a function $h$ on $M/G$ via the projection $p : M \rightarrow M/G, H = p^*(h)$. Denote by $X_H$ (resp., $X_h$) the Hamiltonian vector field of $H$ (resp., $h$) on $(M, \Pi)$ (resp., $(M/G, \Pi/G)$). Of course, $X_H$ is $G$-invariant, and its projection to $M/G$ is $X_h$.

With the above notations and assumptions, we have:

**Theorem 2.2.** If the system $(M/G, X_h)$ is Hamiltonianly integrable, then the system $(M, X_H)$ also is Hamiltonianly integrable. If $G$ is compact and $(M/G, X_h)$ is properly Hamiltonianly integrable, then $(M, X_H)$ also is properly Hamiltonianly integrable.

**Proof.** Denote by $\mathcal{F}'$ a set of first integrals of $X_h$ on $M/G$ which provides the integrability of $X_h$, and by $\mathcal{X}' = X_{\mathcal{F}'}$ the corresponding space of commuting Hamiltonian vector fields on $M/G$. We have $\dim M/G = p' + q'$ where $p' = \text{ddim} \, \mathcal{X}'$ and $q' = \text{ddim} \, \mathcal{F}'$.

Recall that, by our assumptions, there exist $s$ functions $f_1, ..., f_s$ on $\mathfrak{g}^*$, which are functionally independent almost everywhere in $\pi(M)$, and which are invariant on the coadjoint orbits which lie in $\pi(M)$. Here $s$ is the minimal codimension in $\mathfrak{g}^*$ of the coadjoint orbits which lie in $\pi(M)$. We can complete $(f_1, ..., f_s)$ to a set of $d$ functions $f_1, ..., f_s, f_{s+1}, ..., f_d$ on $\mathfrak{g}^*$, where $d = \dim G = \dim \mathfrak{g}$ denotes the dimension of $\mathfrak{g}$, which are functionally independent almost everywhere in $\pi(M)$.

Denote by $\mathcal{F}$ the pull-back of $\mathcal{F}'$ under the projection $p : M \rightarrow M/G$, and by $F_1, ..., F_d$ the pull-back of $f_1, ..., f_d$ under the moment map $\pi : M \rightarrow \mathfrak{g}^*$. Note that, since $H$ is $G$-invariant, the functions $F_i$ are first integrals of $X_H$. And of course, $\mathcal{F}$ is also a set of first integrals of $X_H$. Denote by $\mathcal{F}$ the union of $\mathcal{F}$ with $(F_{s+1}, ..., F_d)$. (It is not necessary to include $F_1, ..., F_s$ in this union, because these functions are $G$-invariant and project to Casimir functions on $M/G$, which implies that they are
functionally dependent of \( \mathcal{F} \)). We will show that \( X_H \) is Hamiltonianly integrable with the aid of \( \mathcal{F} \).

Notice that, by assumptions, the coadjoint orbits of \( \mathfrak{g}^* \) which lie in \( \pi(M) \) are of generic dimension \( d-s \), and the functions \( f_{s+1}, \ldots, f_d \) may be viewed as a coordinate system on a symplectic leaf of \( \pi(M) \) at a generic point. In particular, we have

\[
< df_{s+1} \wedge \ldots \wedge df_d, X_{f_{s+1}} \wedge \ldots X_{f_d} > \neq 0,
\]

which implies, by equivariance :

\[
< dF_{s+1} \wedge \ldots \wedge dF_d, X_{F_{s+1}} \wedge \ldots X_{F_d} > \neq 0.
\]

Since the vector fields \( X_{F_{s+1}}, \ldots, X_{F_d} \) are tangent to the orbits of \( G \) on \( M \), and the functions in \( \mathcal{F} \) are invariant on the orbits of \( G \), it implies that the set \( (F_{s+1}, \ldots, F_d) \) is “totally” functionally independent of \( \mathcal{F} \). In particular, we have:

\[
\text{ddim } \mathcal{F} = \text{ddim } \mathcal{F}' + \text{ddim } (F_{s+1}, \ldots, F_d) = q' + d - s,
\]

where \( q' = \text{ddim } \mathcal{F}' \). On the other hand, we have

\[
\dim M = \dim M/G + (d - k) = p' + q' + d - k,
\]

where \( p' = \text{ddim } \mathcal{X}_{\mathcal{F}'} \), and \( k \) is the dimension of a minimal isotropic group of the action of \( G \) on \( M \). Thus, in order to show the integrability condition

\[
\dim M = \text{ddim } \mathcal{F} + \text{ddim } \mathcal{X}_{\mathcal{F}},
\]

it remains to show that

\[
\text{ddim } \mathcal{X}_{\mathcal{F}} = \text{ddim } \mathcal{X}_{\mathcal{F}'} + (s - k).
\]

Consider the vector fields \( Y_1 = X_{F_1}, \ldots, Y_d = X_{F_d} \) on \( M \). They span the tangent space to the orbit of \( G \) on \( M \) at a generic point. The dimension of such a generic tangent space is \( d - k \). It implies that, among the first \( s \) vector fields, there are at least \( s - k \) vector fields which are linearly independent at a generic points: we may assume that \( Y_1 \wedge \ldots \wedge Y_{s-k} \neq 0 \).

Let \( X_{h_1}, \ldots, X_{h_{p'}} \) be \( p' \) linearly independent (at a generic point) vector fields which belong to \( \mathcal{X}_{\mathcal{F}'} \), where \( p' = \text{ddim } \mathcal{X}_{\mathcal{F}'} \). Then we have

\[
X_{\mathcal{F}^*(h_1)}, \ldots, X_{\mathcal{F}^*(h_{p'})}, Y_1, \ldots, Y_{s-k} \in \mathcal{X}_{\mathcal{F}},
\]

and these \( p' + s - k \) vector fields are linearly independent at a generic point. (Recall that, at each point \( x \in M \), the vectors \( Y_1(x), \ldots, Y_{s-k}(x) \) are tangent to the orbit of \( G \) which contains \( x \), while the linear space spanned by \( X_{\mathcal{F}^*(h_1)}, \ldots, X_{\mathcal{F}^*(h_{p'})} \) contains no tangent direction to this orbit).

Thus we have \( \dim \mathcal{X}_{\mathcal{F}} \geq p' + s - k \), which means that \( \dim \mathcal{X}_{\mathcal{F}} = p' + s - k \) (because, as discussed earlier, we always have \( \dim \mathcal{F} + \dim \mathcal{X}_{\mathcal{F}} \leq \dim M \)). We have proved that if \( (M/G, X_h) \) is Hamiltonianly integrable then \( (M, X_H) \) also is.

Now assume that \( G \) is compact and \( (M/G, X_h) \) is properly Hamiltonianly integrable: there are \( q' \) functionally independent functions \( g_1, \ldots, g_{q'} \in \mathcal{F}' \) such that \( (g_1, \ldots, g_{q'}) : M/G \to \mathbb{R}^{q'} \) is a proper map from \( M/G \) to its image, and \( p' \) Hamiltonian vector fields \( X_{h_1}, \ldots, X_{h_{p'}} \) in \( \mathcal{X} \) such that on a generic common level set
of \((g_1, \ldots, g_{q'})\) we have that \(X_{h_1} \wedge \ldots \wedge X_{h_{q'}}\) does not vanish anywhere. Then it is straightforward that
\[
p^*(g_1), \ldots, p^*(g_{q'}), F_{s+1}, \ldots, F_d \in F
\]
and the map
\[
(p^*(g_1), \ldots, p^*(g_{q'}), F_{s+1}, \ldots, F_d) : M \to \mathbb{R}^{d'+d-s}
\]
is a proper map from \(M\) to its image. More importantly, on a generic level set of this map we have that the \((q' + s - k)\)-vector \(X_{p^*(h_1)} \wedge \ldots \wedge X_{p^*(h_{q'})} \wedge Y_1 \wedge \ldots \wedge Y_{s-k}\) does not vanish anywhere. To prove this last fact, notice that \(X_{p^*(h_1)} \wedge \ldots \wedge X_{p^*(h_{q'})} \wedge Y_1 \wedge \ldots \wedge Y_{s-k}(x) \neq 0\) for a point \(x \in M\) if and only if \(X_{p^*(h_1)} \wedge \ldots \wedge X_{p^*(h_{q'})}(x) \neq 0\) and \(Y_1 \wedge \ldots \wedge Y_{s-k}(x) \neq 0\) (one of these two multi-vectors is transversal to the \(G\)-orbit of \(x\) while the other one “lies on it”), and that these inequalities are \(G \times \mathbb{R}^{d'}\)-invariant properties, where the action of \(\mathbb{R}^{d'}\) is generated by \(X_{p^*(h_1)}, \ldots, X_{p^*(h_{q'})}\).

Remark. Recall from Equation (2.2) above that we have
\[
d \dim X_F = \dim X_{F'} = s - k,
\]
where \(k\) is the dimension of a generic isotropic group of the \(G\)-action on \(M\), and \(s\) is the (minimal) corank in \(\mathfrak{g}^*\) of a coadjoint orbit which lies in \(\pi(M)\). On the other hand, the difference between the rank of the Poisson structure on \(M\) and the reduced Poisson structure on \(M/G\) can be calculated as follows:
\[
(2.3) \quad \text{rank} \ \Pi - \text{rank} \ \Pi/G = (d - k) + (s - k)
\]
Here \((d - k)\) is the difference between \(\dim M\) and \(\dim M/G\), and \((s - k)\) is the difference between the corank of \(\Pi/G\) in \(M/G\) and the corank of \(\Pi\) in \(M\). It follows that
\[
(2.4) \quad \text{rank} \ \Pi - 2 \dim X_F = \text{rank} \ \Pi/G - 2 \dim X_{F'} + (d - s)
\]
In particular, if \(d - s > 0\) (typical situation when \(G\) is non-Abelian), then we always have
\(\text{rank} \ \Pi - 2 \dim X_F > 0\) (because we always have \(\text{rank} \ \Pi/G - 2 \dim X_{F'} \geq 0\) due to integrability), i.e. the original system is always super-integrable with the aid of \(F\). When \(G\) is Abelian (implying \(d = s\)), and the reduced system is Liouville-integrable with the aid of \(F'\) (i.e. \(\text{rank} \ \Pi/G = 2 \dim X_{F'}\)), then the original system is also Liouville-integrable with the aid of \(F\).

Remark. Following Mischenko-Fomenko \([\text{[Mischenko-Fomenko]}\), we will say that a hamiltonian system \((M, \Pi, X_H)\) is non-commutatively integrable in the restricted sense with the aid of \(F\), if \(F\) is a finite-dimensional Lie algebra under the Poisson bracket and \((M, \Pi, X_H)\) is Hamiltonianly integrable with the aid of \(F\). In other words, we have an equivariant moment maps \((M, \Pi) \to \mathfrak{f}^*\), where \(\mathfrak{f}\) is some finite-dimensional Lie algebra, and if we denote by \(f_1, \ldots, f_n\) the components of this moment map, then they are first integrals of \(X_H\), and \(X_H\) is Hamiltonianly integrable with the aid of this set of first integrals. Theorem 2.2 remains true, and its proof remains the same if not easier, if we replace Hamiltonian integrability by non-commutative integrability in the restricted sense. Indeed, if \(M \to \mathfrak{g}^*\) is the equivariant moment map of the symmetry group \(G\), and if \(M/G \to \mathfrak{h}^*\) is an equivariant moment map which provides non-commutative integrability in the restricted sense on \(M/G\), then the map \(M \to \mathfrak{h}^*\) (which is the composition \(M \to M/G \to \mathfrak{h}^*\)) is an equivariant moment map which commutes with \(M \to \mathfrak{g}^*\), and the direct sum of this two maps, \(M \to \mathfrak{f}^*\) where \(\mathfrak{f} = \mathfrak{g} \oplus \mathfrak{h}\), will provide non-commutative integrability in the restricted sense on \(M\).
The above remarks show that the notion of Hamiltonian integrability (or non-commutative integrability if you prefer), rather than integrability à la Liouville, is the most natural one when dealing with systems admitting (non-Abelian) symmetry groups.

Examples. 1) The simplest example which shows an evident relationship between reduction and integrability is the classical Euler top: it can be written as a Hamiltonian system on $T^*SO(3)$, invariant under a natural Hamiltonian action of $SO(3)$, is integrable with the aid of a set of four first integrals, and with 2-dimensional isotropic invariant tori. 2) The geodesic flow of a bi-invariant metric on a compact Lie group is also properly Hamiltonianly integrable: in fact, the corresponding reduced system is trivial (identically zero).

It is known that most Hamiltonianly integrable systems are also integrable à la Liouville, i.e. there exists a set of first integrals which commute pairwise and which make the system integrable - see e.g. [5] for a very long discussion on this subject. A question of similar kind, which is directly related to the inverse of Theorem 2.2, is the following:

If $X_H$ is Hamiltonianly integrable with the aid of $\mathcal{F}_1$, and if $\mathcal{F}_2$ is another set of first integrals of $X_H$ which contains $\mathcal{F}_1$, then is it true that $X_H$ is also Hamiltonianly integrable with the aid of $\mathcal{F}_2$? In particular, let $\mathcal{F}_H$ denotes the set of all first integrals of $X_H$. If $X_H$ is Hamiltonianly integrable, then is it true that it is integrable with the aid of $\mathcal{F}_H$?

A related question is the following:

Suppose that $\mathcal{F} = (F_1, ..., F_q)$ is a set of independent first integrals of $X_H$ such that regular common level sets of these functions $(F_1, ..., F_q)$ are isotropic submanifolds of $(M, \Pi)$. Is it true that $X_H$ is integrable with the aid of $\mathcal{F}$?

Remark that the inverse to the later question is always true. And if we can say YES to the later question, then we can also say YES to the former one, because adding first integrals has the affect of minimizing invariant submanifolds, and a submanifold of an isotropic submanifold is again an isotropic submanifold. It is easy to see that, at least in the smooth proper case, the answer to the above two questions is YES: smooth proper Hamiltonian integrability of a Hamiltonian system is equivalent to the singular foliation of the Poisson manifold by invariant isotropic tori (This fact is similar to the classical Liouville theorem).

For the following theorem, we use the same notations and preliminary assumptions as in Theorem 2.2:

**Theorem 2.3.** If $G$ is compact, and if the Hamiltonian system $(M, X_H)$ is Hamiltonianly integrable with the aid of $\mathcal{F}_H$ (the set of all first integrals), then the reduced Hamiltonian system $(M/G, X_h)$ is also Hamiltonianly integrable. The same thing holds in the smooth proper case.

**Proof.** By assumptions, we have $\dim M = p + q$, where $q = \dim \mathcal{F}_H$ and $p = \dim X_{\mathcal{F}_H}$, and we can find $p$ first integrals $H_1, ..., H_p$ of $H$ such that $X_{H_1}, ..., X_{H_p}$ are linearly independent (at a generic point) and belong to $X_{\mathcal{F}_H}$. In particular, we have $X_{H_i}(F) = 0$ for any $F \in \mathcal{F}$ and $1 \leq i \leq p$. 
An important observation is that the functions $H_1, ..., H_p$ are $G$-invariant. Indeed, if we denote by $F_1, ..., F_p$ the components of the equivariant moment map $\pi : M \to \mathfrak{g}^*$ (via an identification of $\mathfrak{g}^*$ with $\mathbb{R}^d$), then since $H$ is $G$-invariant we have $\{H, F_j\} = 0$, i.e. $F_j \in \mathcal{F}_H$, which implies that $\{F_j, H_i\} = 0 \ \forall 1 \leq i \leq d, \ 1 \leq j \leq p$, which means that $H_i$ are $G$-invariant.

Denote by $h_i$ the projection of $H_i$ on $M/G$ (recall that the projection of $H$ on $M/G$ is denoted by $h$). Then the Hamiltonian vector fields $X_{h_i}$ belong to $\mathcal{X}_R$, i.e. $h_i$ are first integrals of $H$, implying $\{h_i, p^*(f)\} = 0$, or $\{h_i, f\} = 0$, where $p$ denotes the projection $M \to M/G$.

To prove the integrability of $X_{h_i}$, it is sufficient to show that
\begin{equation}
dim M/G \leq \dim \mathcal{F}_h + \dim (X_{h_1}, ..., X_{h_p})
\end{equation}
But we denote by $r$ the generic dimension of the intersection of a common level set of $p$ independent first integrals of $X_H$ with an orbit of $G$ in $M$, then one can check that
\[p - \dim (X_{h_1}, ..., X_{h_p}) = \dim \mathcal{X}_{R_H} - \dim (X_{h_1}, ..., X_{h_p}) = r\]
and
\[q - \dim \mathcal{F}_h = \dim \mathcal{F}_H - \dim \mathcal{F}_h \leq (d - k) - r\]
where $(d - k)$ is the dimension of a generic orbit of $G$ in $M$. To prove the last inequality, notice that functions in $\mathcal{F}_h$ can be obtained from functions in $\mathcal{F}_H$ by averaging with respect to the $G$-action. Also, $G$ acts on the (separated) space of common level sets of the functions in $\mathcal{F}_H$, and isotropic groups of this $G$-action are of (generic) codimension $(d - k) - r$.

The above two formulas, together with $p + q = \dim M = \dim M/G + (d - k)$, implies Inequality (2.3) (it is in fact an equality).

We will leave the proper case to the reader as an exercise. ♦

3. Non-Hamiltonian integrability

The interest in non-Hamiltonian integrability comes partly from the fact that there are many non-Hamiltonian (e.g. non-holonomic) systems whose behaviors are very similar to that of integrable Hamiltonian systems, see e.g. [1, 2]. In particular, in the proper case, the manifold is foliated by invariant tori on each of which the system is quasi-periodic. Another common point between (integrable) Hamiltonian and non-Hamiltonian systems is that their local normal form theories are very similar and are related to local torus actions, see e.g. [1, 2]. The notion of non-Hamiltonian integrability was probably first introduced by Bogoyavlenskij [2], who calls it broad integrability, in his study of tensor invariants of dynamical systems. Let us give here a definition of non-Hamiltonian integrability, which is similar to the ones found in [1, 2, 3, 4, 10]:

**Definition 3.1.** A vector field on a (eventually singular) manifold $M$ is called non-Hamiltonianly integrable with the aid of $(\mathcal{F}, \mathcal{X})$, where $\mathcal{F}$ is a set of functions on $M$ and $\mathcal{X}$ is a set of vector fields on $M$, if the following conditions are satisfied:

a) Functions in $\mathcal{F}$ are first integrals of $\mathcal{X}$: $X(F) = 0 \ \forall F \in \mathcal{F}$.

b) Vector fields in $\mathcal{X}$ commute pairwise and commute with $\mathcal{X}$: $[Y, Z] = [Y, X] = 0 \ \forall Y, Z \in \mathcal{X}$.

c) Functions in $\mathcal{F}$ are common first integrals of vector fields in $\mathcal{X}$: $Y(F) = 0 \ \forall Y \in \mathcal{X}, F \in \mathcal{F}$.

d) $\dim M = \dim \mathcal{F} + \dim \mathcal{X}$. 


If, moreover, there exist \( p \) vector fields \( Y_1, \ldots, Y_p \in \mathcal{X} \) and \( q \) functionally independent functions \( F_1, \ldots, F_q \in \mathcal{F} \), where \( p = \text{ddim} \mathcal{X} \) and \( q = \text{ddim} \mathcal{F} \), such that the map 

\[
(F_1, \ldots, F_q) : M \to \mathbb{R}^q
\]

is a proper map from \( M \) to its image, and for almost any level set of this map the vector fields \( Y_1, \ldots, Y_p \) are linearly independent everywhere on the level set, then we say that \( X \) is properly non-Hamiltonianly integrable with the aid of \((\mathcal{F}, \mathcal{X})\).

**Remarks.**

1. It is straightforward that, in the proper case, the manifold is a (singular) foliation by invariant tori (common level sets of some first integrals) on each of which the vector field \( X \) is quasi-periodic. (This is similar to the classical Liouville theorem).

2. If a Hamiltonian system is (properly) Hamiltonianly integrable, then it is also (properly) non-Hamiltonianly integrable, though the inverse is not true: it may happen that the invariant tori are not isotropic, see e.g. [2, 4] for a detailed discussion about this question.

One of the main differences between the non-Hamiltonian case and the Hamiltonian case is that reduced non-Hamiltonian integrability does not imply integrability. In fact, in the Hamiltonian case, we can lift Hamiltonian vector fields from \( M/G \) to \( M \) via the lifting of corresponding functions. In the non-Hamiltonian case, no such canonical lifting exists, therefore commuting vector fields on \( M/G \) do not provide commuting vector fields on \( M \). For example, consider a vector field of the type

\[
X = a_1 \partial/\partial x_1 + a_2 \partial/\partial x_2 + b(x_1, x_2) \partial/\partial x_3
\]

on the standard torus \( \mathbb{T}^3 \) with periodic coordinates \((x_1, x_2, x_3)\), where \( a_1 \) and \( a_2 \) are two incommensurable real numbers \((a_1/a_2 \notin \mathbb{Q})\), and \( b(x_1, x_2) \) is a smooth function of two variables. Then clearly \( X \) is invariant under the \( S^1 \)-action generated by \( \partial/\partial x_3 \), and the reduced system is integrable. On the other hand, for \( X \) to be integrable, we must be able to find a function \( c(x_1, x_2) \) such that

\[
[X, \partial/\partial x_1 + c(x_1, x_2) \partial/\partial x_3] = 0.
\]

This last equation does not always have a solution (it is a small divisor problem, and depends on \( a_1/a_2 \) and the behavior of the coefficients of \( b(x_1, x_2) \) in its Fourier expansion), i.e. there are choices of \( a_1, a_2, b(x_1, x_2) \) for which the vector field \( X \) is not integrable.

However, non-Hamiltonian integrability still implies reduced integrability. Before formulating a precise result, let us mention a question similar to the one already mentioned in the previous section:

For a vector field \( X \) on a manifold \( M \), denote by \( \mathcal{F}_X \) the set of all first integrals of \( X \), and by \( \mathcal{X}_X \) the set of vector fields which preserve each function in \( \mathcal{F} \) and commute with \( X \). Suppose that \( X \) is non-Hamiltonianly integrable. Is it then non-Hamiltonianly integrable with the aid of \((\mathcal{F}_X, \mathcal{X}_X)\)? In other words, is it true that vector fields in \( \mathcal{F} \) commute pairwise and \( \text{ddim} \mathcal{X}_X + \text{ddim} \mathcal{F}_X = \text{dim} M \)?

It is easy to see that the answer to the above question is YES in the proper non-Hamiltonianly integrable case, under the additional assumption that the orbits of \( X \) are dense (i.e. its frequencies are incommensurable) on almost every invariant torus (i.e. common level of a given set of first integrals \( \mathcal{F} \)). In this case \( \mathcal{X}_X \) consists of the vector fields which are quasi-periodic on each invariant torus. Another case where the answer is also YES arises in the study of local normal forms of analytic integrable vector fields, see e.g. [10].
Theorem 3.2. Let $X$ be a smooth properly non-Hamiltonianly integrable system on a manifold $M$ with the aid of $(F_X, \mathcal{X}_X)$, and $G$ be a compact Lie group acting on $M$ which preserves $X$. Then the reduced system on $M/G$ is also properly non-Hamiltonianly integrable.

Proof. Let $\mathcal{X}_G^X$ denote the set of vector fields which belong to $\mathcal{X}_X$ and which are invariant under the action of $G$. Note that the elements of $\mathcal{X}_G^X$ can be obtained from the elements of $\mathcal{X}_X$ by averaging with respect to the $G$-action.

A key ingredient of the proof is the fact $\text{ddim } \mathcal{X}_G^X = \text{ddim } \mathcal{X}_X$. (To see this fact, notice that near each regular invariant torus of the system there is an effective torus action (of the same dimension) which preserves the system, and this torus action must necessarily commute with the action of $G$. The generators of this torus action are linearly independent vector fields which belong to $\mathcal{X}_G^X$ - in fact, they are defined locally near the union of $G$-orbits which by an invariant torus, but then we can extend them to global vector fields which lie in $\mathcal{X}_G^X$).

Therefore, we can project the pairwise commuting vector fields in $\mathcal{X}_G^X$ from $M$ to $M/G$ to get pairwise commuting vector fields on $M/G$. To get the first integrals for the reduced system, we can also take the first integrals of $X$ on $M$ and average them with respect to the $G$-action to make them $G$-invariant. The rest of the proof of Theorem 3.2 is similar to that of Theorem 2.3. ♦

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