Shapes of the Nucleon

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Previously defined spin-dependent quark densities that are matrix elements of specific density operators in proton states of definite spin-polarization generally have an infinite variety of non-spherical shapes. The present application is concerned with both charge and matter densities. We show that the Gross & Agbakpe model nucleon harbors an interesting variety of non-spherical shapes.

I. INTRODUCTION

Recent data [1, 2] showing that the ratio of the proton’s electric and magnetic form factor $G_E/G_M$, falls with increasing momentum transfer $Q^2$ for $1 < Q^2 < 6 \text{ GeV}^2$ and equivalently that $QF_2(Q^2)/F_1(Q^2)$ is approximately constant have created considerable attention. This behavior indicates that the sum of the orbital angular momentum of the quarks in the proton is non-vanishing [3, 4, 5, 6].

It is natural to consider the connection between orbital angular momentum ($l$) and the shape of the nucleon, and one of us showed [7], using the proton model of Ref. [8], that the rest-frame ground-state matrix elements of spin-dependent density operators reveal a host of non-spherical shapes. The use of the spin-dependent density operator is the key feature that allows the detailed connection between orbital, spin and total angular momentum to be revealed in quantum systems. Matrix elements of the non-relativistic spin-density operator have been measured in condensed matter systems [9], revealing the orbital angular momentum content of electron orbitals.

In the model of [3] the relativistic nature of the quarks enters via their lower components of Dirac spinors that are part of the wave function. It is natural to ask if is it necessary that the presence of relativistically moving quarks always causes a non-spherical shape (as determined by matrix elements of spin-dependent density operators). We shall not address this general question here, and consider only the model dependence of the shape. In particular, Gross & Agbakpe (GA) [10] have claimed to construct a manifestly covariant nucleon wave function that contains only $l = 0$ such that “the nucleon would still be spherical” and also describe the measured nucleon electromagnetic form factors. We show that this contention is not correct by evaluating the matrix elements of the spin-dependent density operator using the GA wave function.
II. SPIN-DEPENDENT DENSITY OPERATORS

We begin by explaining how the shapes of a nucleon are exhibited by studying the rest-frame ground-state matrix elements of spin-dependent density operators. The usual charge density operator in non-relativistic quantum mechanics is given by

$$\hat{\rho}(r) = \sum_i \frac{e_i}{e} \delta(r - r_i)$$

(1)

where $e_i/e$ is the charge of the $i$'th particle (in units of the proton charge) and $r_i$ its position operator. Matrix elements of this operator yield the charge density of a system. Suppose the particles also have spin 1/2. Then one can measure the probability that particle is at a given position $r$ and has a spin in an arbitrary, fixed direction specified by a unit vector $n$. The spin projection operator is $(1 + \sigma \cdot n)/2$, so the spin-dependent density operator is

$$\hat{\rho}(r, n) = \sum_i \frac{e_i}{e} \delta(r - r_i) \frac{1}{2}(1 + \sigma \cdot n).$$

(2)

The spin-dependent density allows the presence of the orbital angular momentum to be revealed in the shape of the computed density. Matrix elements of the spin-density operator of Eq. (2) have been measured in condensed matter systems\[9\], revealing the orbital angular momentum content of electron orbitals.

It is worthwhile to consider a simple example of a single charged particle moving in a fixed rotationally invariant potential in a state of quantum numbers: $n, 1, 1/2, m$ we find

$$\rho(r, n) = \langle n, 1, 1/2, m | \hat{\rho}(r, n) | n, 1, 1/2, m \rangle = \frac{R_{n,1/2}^2(r)}{2} \langle m | 1 + 2 \sigma \cdot \hat{n} \cdot \hat{r} - \sigma \cdot n | m \rangle.$$  

(3)

Suppose $\hat{n}$ is either parallel or anti-parallel to the direction of the proton angular momentum defined by the vector $\hat{s}$. The direction of the vector $\hat{s}$ defines an axis (the “z-axis”), and the direction of vectors can be represented in terms of this axis: $\hat{s} \cdot \hat{r} = \cos \theta$. With this notation

$$\rho(r, n = \hat{s}) = R_{n,1/2}^2(r) \cos^2 \theta, \quad \rho(r, n = -\hat{s}) = R_{n,1/2}^2(r) \sin^2 \theta$$

and the non-spherical shape is exhibited. The average of these two cases is a spherical shape as is the average over the direction of $\hat{s}$. The spherical shape claimed in [10] arises from taking an average over the direction of $n$. Averaging over this direction necessarily buries the effects of orbital angular momentum responsible for non-spherical shapes.

The density we have discussed so far is defined in terms of position, but when quantum field theory applies it is convenient to define similar operators that give the probability for a particle to have a given momentum, $K$, and a given direction of spin, $n$. The field-theoretic version of the spin-dependent charge density operator is

$$\hat{\rho}_O(K, n) = \int \frac{d^3r}{(2\pi)^3} e^{iK \cdot r} \bar{\psi}(r) O(\gamma^0 + \gamma \cdot n \gamma_5) \psi(0),$$

(4)

where $O$ is $\hat{Q}/e$, the quark charge operator in units of the proton charge. Here we shall also consider the case in which $O = 1$. Its matrix element along with Eq. (4) gives the spin-dependent matter densities. The quark field operators are evaluated at equal time. There
are no gluon fields in the relativistic constituent quark models used here. The question of color gauge invariance of QCD will be taken up elsewhere. The matrix element of this density operator in a nucleon state of definite total angular momentum defined by the unit vector \( \mathbf{s} \), \(|\Psi_s\rangle\) is

\[
\rho_{O}(\mathbf{K}, \mathbf{n}, \mathbf{s}) \equiv \langle \Psi_s | \hat{\rho}_{O}(\mathbf{K}, \mathbf{n}) | \Psi_s \rangle, \tag{5}
\]

where the subscript \( O = Q, 1 \) specifies the operator used in Eq. (4).

The most general shape of the proton, obtained if parity and rotational invariance are upheld is

\[
\rho_{O}(\mathbf{K}, \mathbf{n}, \mathbf{s}) = A_{O}(\mathbf{K}^2) + B_{O}(\mathbf{K}^2) \mathbf{n} \cdot \mathbf{s} + C_{O}(\mathbf{K}^2) \mathbf{n} \cdot \mathbf{K} \mathbf{s} \cdot \mathbf{K}, \tag{6}
\]

with the last term generating the non-spherical shape. Any wave function that yields a non-zero value of the coefficient \( C_{O}(\mathbf{K}^2) \) represents a system of a non-spherical shape.

In Ref. [7] the spin-dependent charge density of a relativistic three-quark constituent model was evaluated with the result (for the proton):

\[
\rho_{Q}(\mathbf{K}, \mathbf{n}, \mathbf{s}) = \rho(K) \frac{1}{2} (1 + \mathbf{n} \cdot \hat{s} + \gamma(K)(1 - \mathbf{n} \cdot \hat{s} + 2\hat{K} \cdot \mathbf{n}\hat{K} \cdot \hat{s})) \tag{7}
\]

with

\[
\rho(K) \equiv \int d^3k \Phi^2(k, K)(E(K) + m), \quad \gamma(K) \equiv \frac{E(K) - m}{E(K) + m}; \tag{8}
\]

with \( \Phi(k, K) \) the three-quark wave function and \( m \) the constituent quark mass specified in [7]. The result for the neutron is

\[
\rho_{Q}^n(\mathbf{K}, \mathbf{n}, \mathbf{s}) = \rho(K) \frac{1}{18} (1 - \mathbf{n} \cdot \hat{s} + \gamma(K)(1 + \mathbf{n} \cdot \hat{s} - 2\hat{K} \cdot \mathbf{n}\hat{K} \cdot \hat{s})) \tag{9}
\]

The shape for a given value of \( K \) is determined by the ratio \( \gamma(K) \) which reaches a value of 0.6 for \( K = 1 \text{ GeV}/c \). This implies considerable non-sphericity. The probability that a given value of \( K \) is determined by the function \( K^2 \rho(K) \), displayed in Fig. 1 of [7]. The most likely value of \( K \approx 0.25 \text{ GeV}/c \) corresponding to \( \gamma(K) = 0.16 \).

Some special cases of Eq. (7) are interesting. Suppose the quark spin is parallel to the proton spin, \( \mathbf{n} = \hat{s} \), then \( \hat{\rho}_{Q}(K, \mathbf{n} = \hat{s}) = \rho(K)(1 + \gamma(K) \cos^2 \theta) \). For small \( K \) the shape is nearly spherical, but for large \( K \) the \( \cos^2 \theta \) term becomes prominent. On the other hand, the quark spin could be anti-parallel to the proton spin, \( \mathbf{n} = -\hat{s} \). Then we find: \( \hat{\rho}(K, \mathbf{n} = -\hat{s}) = \rho(K)\gamma(K) \sin^2 \theta \), and the shape is that of a torus. We may also take the quark spin perpendicular to the proton spin \( \mathbf{n} \cdot \mathbf{s} = 0 \), so that \( \rho_{Q}(K, \mathbf{n} \cdot \mathbf{s} = 0) = \rho(K)(1 + \gamma(K))/2 + \gamma(K) \sin \theta \cos \theta (\cos \phi m_x + \sin \phi m_y) \), to display the dependence on the azimuthal angle. In each case, the non-spherical nature arises from the term proportional to \( \gamma(K) \) caused by the lower components of the Dirac spinor that is part of the proton wave function. For the neutron one obtains a doughnut shape for \( \mathbf{n} = \hat{s} \), and a peanut for \( \mathbf{n} = -\hat{s} \). Plots of these shapes have appeared in Ref. [7] as well as in the NY Times [11].
For the matter distribution \( \mathcal{O} = 1 \) we find

\[
\rho_1(K, n, s) = \rho(K) \frac{1}{2} (3 + n \cdot \hat{s} + \gamma(K)(3 - n \cdot \hat{s} + 2 \hat{K} \cdot n \hat{K} \cdot \hat{s})).
\] (10)

For a fixed value of \( K \), this density looks more like a sphere than that of Eq. (7). For example, if \( n = -s \) one finds \( \rho_1(K, n) = \rho(K) \left(1 + \gamma \sin^2 \theta\right) \), which is mainly spherical instead of toroid. Similarly if \( n = s \) one finds \( \rho_1(K, n) = \rho(K) (2 + \gamma \cos^2 \theta) \). For the case that \( n \cdot \hat{s} = 0 \), with \( n \) in the \( y \)-direction we find \( \rho_1(K, n) = \rho(K) \left[3/2 + \gamma(3/2 + \cos \theta \sin \theta \cos \phi)\right] \).

### III. MODEL OF GROSS & AGBAKPE

In Ref. [10] explicit effects of quark spinors are avoided by removing a factor of the single-quark free propagator that would appear in a solution of the Bethe-Salpeter equation. The GA nucleon wave function consists of two terms which involve either a scalar or vector diquark. The diquark covariant polarization vector is \( \eta' \). The relativistic impulse approximation is used to evaluate the matrix element of the electromagnetic current operator. In the evaluation of [10] the electromagnetic matrix element, \( J_{\mu}^I \) for a nucleon of isospin \( I \) and four-momentum transfer \( q \) is three times the integrated current of a single off-shell quark,
computed with the spectator diquark system on mass-shell:

\[ \mathcal{J}_i^\mu = \frac{3}{2} \int \frac{m_s^3 d^3 \kappa}{(2\pi)^3 m_s 2E_s(\kappa)} \left\{ j_i^\mu \psi_0(P_+, p)\psi_0(P_-, p) \right. \\
\left. - \frac{1}{9} \gamma_\nu \gamma_5 \gamma_\lambda j_i^\mu \gamma_\lambda \gamma_5 \gamma_\nu \Delta^{\nu\nu'} \psi_1(P_+, p)\psi_1(P_-, p) \right\}, \tag{11} \]

where \( P_\pm \equiv P \pm \frac{1}{2}q \). The CQ current, \( j_i^\mu \), is specified in [11]. In deriving Eq. (11) the polarization vectors of the diquark are summed over, using \( \sum_\eta \eta^{\nu\nu'} \equiv \Delta^{\nu\nu'} = -g^{\nu\nu'} + \frac{\delta^{\nu\nu'} m_s^2}{m_s^2} \). The quantity \( p \) is the on-shell diquark four-momentum \( (p^2 = m_s^2) \), and \( u(P, s) \) is the spinor of the nucleon. The scalar functions \( \psi_{0,1} \) describing the nucleon with scalar \((0)\) and vector \((1)\) diquark systems are chosen to depend only on the variable \( \chi = ((M - m_s)^2 - (P - p)^2)/(2Mm_s) \). For completeness we present the functions \( \psi_{0,1} \):

\[ \psi_0(P, p) = \frac{N_0}{m_s(\beta_1 - 2 + \chi)(\beta_2 - 2 + \chi)}, \tag{12} \]

where \( \beta_1 \) and \( \beta_2 \) are range parameters in units of \( Mm_s \), \( N_0 \) is a normalization constant and

\[ \psi_1(P, p) = \sqrt{\frac{6}{2 + (\chi + 2)^2}} \equiv \mathcal{R}. \tag{13} \]

We are concerned only with rest-frame matrix elements. In this frame \( \chi = 2(E_s(\kappa) - 1) \), where \( E_s(\kappa) = \sqrt{1 + \kappa^2} \) where \( \kappa = \sqrt{1 + p^2/m_s^2} \).

GA claim that their wave function depends only on the magnitude of \( p \), and therefore is spherically symmetric. This is not correct. While the functions \( \psi_{0,1} \) depend only on the magnitude of \( p \), the nucleon wave function contains the polarization vector \( \eta^\mu \) and this yields a non-spherical shape as defined by taking the matrix elements of the operator \( \hat{\rho}_O(K, n) \). Before proceeding we note that the relation between the di-quark momentum \( p \) of GA is simply the negative of the quark momentum \( K \) of [7]: \( p = -K \).

The expectation value of the operator \( \hat{\rho}_O(K, n) \) in the nucleon of total angular momentum \( s \) of the GA model (corresponding to Eq. [3]) is defined as \( \rho_O^{GA}(K, n, s) \). Once again the result is three times the integrated current of a single off-shell quark, calculated with the spectator “diquark” system on mass-shell. A straightforward evaluation yields

\[ \rho_O^{GA}(K, n, s) = \frac{3}{2E(K^2)} \bar{U} \left[ \mathcal{O}_0(\gamma_0 + n \cdot \gamma_5) |\psi_0(K)|^2 \right. \\
\left. + \frac{1}{9} \tau_j \mathcal{O}_1 \tau_j \gamma_{\mu}(\gamma^0 + n \cdot \gamma_5) \gamma_{\nu} \left( \frac{p^\mu p^\nu}{m_s^2} - g^\mu\nu \right) |\psi_1(K)|^2 \left| \right] U, \tag{14} \]

where \( U \) is the Dirac spinor for a nucleon of total angular momentum \( s \) at rest, and \( E(K^2) = \sqrt{K^2 + m_s^2} \). For the charge density: \( \mathcal{O}_0 = 2/3 \) (proton), \(-1/3 \) (neutron), \( \tau_j \mathcal{O}_1 \tau_j = 0 \) (proton), -1
FIG. 2: Matter distributions. The spin direction $\mathbf{s}$ is taken as vertical, $\hat{z}$, with $\mathbf{n} = -\hat{s}$. Right column model of \cite{7}, left column model of \cite{10}.

(neutron). For the matter density: $\mathcal{O}_0 = 1$, $\tau_j \mathcal{O}_1 \tau_j = 3$. Thus we explicitly find

$$\rho_{G\text{Ap}}(\mathbf{K}, \mathbf{n}, \mathbf{s}) = \frac{1}{E(\mathbf{K}^2)} (1 + \mathbf{n} \cdot \mathbf{s}) |\psi_0(\mathbf{K}^2)|^2$$

$$\rho_{G\text{An}}(\mathbf{K}, \mathbf{n}, \mathbf{s}) = -\frac{|\psi_0(\mathbf{K}^2)|^2}{2E(\mathbf{K}^2)} \left[ (1 + \mathbf{n} \cdot \mathbf{s}) + \frac{\mathcal{R}^2}{3} \left( -3 - \frac{2 K^2}{m_s^2} + \mathbf{n} \cdot \mathbf{s} - \frac{2}{m_s^2} \mathbf{s} \cdot \mathbf{Kn} \cdot \mathbf{K} \right) \right]$$

$$\rho_{G1}(\mathbf{K}, \mathbf{n}, \mathbf{s}) = \frac{3|\psi_0(\mathbf{K}^2)|^2}{2E(\mathbf{K}^2)} \left[ (1 + \mathbf{n} \cdot \mathbf{s}) - \frac{\mathcal{R}^2}{3} \left( 3 - \frac{2 K^2}{m_s^2} + \mathbf{n} \cdot \mathbf{s} - \frac{2}{m_s^2} \mathbf{s} \cdot \mathbf{Kn} \cdot \mathbf{K} \right) \right]$$

There are a number of noteworthy features. The non-spherical nature of the distributions arise only from the vector di-quark part of the wave function. This does not contribute to the proton charge and spin-dependent charge densities, leading to a spherical (but polarization dependent) result. The charge distribution of the neutron and the matter distribution of the proton and neutron are each non-spherical with the degree of non-sphericity controlled by the terms $\mathcal{R}$ and $\frac{K^2}{m_s^2}$.

Note that Eq. (15) corresponds to a charge distribution of the proton because the charge operator is of the explicit form: $1/6 + \tau_3/2$ in \cite{11}. That work accounts for the effects of the pion cloud \cite{12,13} in an approximate manner. In general, charge would be carried by both the quarks and the pion. In that case the quark charge operator would take the form $f_+ + f_- \tau_3$, and the vector diquark term would contribute a non-spherical density proportional to $3f_+ - f_-$. In general, quark lines are dressed by loops and the dressing can be applied to the quark charge density as shown in \cite{14}. 
FIG. 3: Matter distributions. The spin direction $\mathbf{s}$ is taken as vertical, $\hat{z}$, with $\mathbf{n} \cdot \hat{s} = 0$. The direction of $\mathbf{n}$ is in the page ($y$ direction). Right column model of [7], left column model of [10]. First row $K = 250$ MeV/c, second row $K = 1$ GeV/c.

IV. COMPARING SHAPES

We turn to the numerical evaluation and display of these shapes, focusing on a comparison of the matter distributions of the Miller [7] and GA models. Specific numerical evaluations of the shapes requires knowing the value of $m_s$, which is not specified in Ref. [10]. Here we note that in the non-relativistic limit $\mathcal{R} \approx 1$ and the quantities $K^2/m_s^2$ and $\gamma$ play the same roles. In that case, we simply set $K^2/m_s^2 = \gamma$. We also consider a case when the quarks are moving relativistically with $K = 1$ GeV/c. Then $\gamma = 0.6$. The relation between $K$ and $\gamma$ is $K = 2m\gamma/(\gamma^2 - 2\gamma + 1)$. It is reasonable to associate the values of $2m$ with the diquark mass $m_s$. In this case, the value $\gamma = 0.6$ corresponds to $K^2/m_s^2 = 3.75$, $\chi = 2.36$, $\mathcal{R} = 0.53$. These numerical values are used to obtain the figures shown below.

The first situation we consider is the one in which the quark spin is parallel to the nucleon spin (total angular momentum). In the model [7] the quark probability is larger at the top than at the side by a factor of $1 + \gamma$. In the model [10] the corresponding factor is 1.24 for quarks of momenta 1 GeV/c. See Fig. 1. In the limit of $K \to \infty$ the ratio would be 2 in both models. Thus the peanut configuration is hiding within the nucleon of [10], albeit with very small probability.

Next we take the quark spin anti-parallel to the nucleon spin: $\mathbf{n} = -\hat{s}$, see Fig. 2. In this case the non-sphericity of the two models is very similar. Both have a toroid shape, with the model [10] having more pronounced effects.

The final situation we consider is that in which the quark spin is perpendicular to that of the nucleon spin, see Fig. 3. We take $\mathbf{n} \cdot \mathbf{s} = 0$ with $\mathbf{n}$ pointing along the right side of the figure. Once again each model nucleon has a significant deformation.
V. SUMMARY

The nucleon is far from round in each of the models considered, and this arises from the relativistic nature of each. The deviation from a spherical shape is associated physically with motion of spin 1/2 quarks moving relativistically within the nucleon, and mathematically with the non-vanishing of the term $C_\alpha(K^2)$ of Eq. [10]. The general nature of $C_\alpha(K^2)$ is a subject for future work.

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