THE WEYL CURVATURE TENSOR 
OF HYPERFACES 
UNDER THE MEAN CURVATURE FLOW 

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ABSTRACT. In this paper, we study the evolution of the Weyl curvature tensor $W$ of hypersurfaces in $\mathbb{R}^{n+1}$ under the mean curvature flow. We find a bound for the Weyl curvature tensor of hypersurfaces during the evolution in terms of time. As a consequence, we suppose that the initial hypersurface is conformally flat, i.e., $W = 0$ at $t = 0$ and then we find an upper estimate for $W$ during the evolution in terms of time.

1. Introduction

Let $\varphi_0 : M \rightarrow \mathbb{R}^{n+1}$ be a smooth immersion from an $n$-dimensional manifold $M$ to $\mathbb{R}^{n+1}$. The mean curvature flow of $\varphi_0$ is defined as a family of smooth immersions, $\varphi_t : M \rightarrow \mathbb{R}^{n+1}$ for $t \in [0, T)$ such that setting $\varphi(p, t) = \varphi_p(t)$, the map $\varphi : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$ is a smooth solution of the following PDE

\[
\begin{cases}
\frac{\partial}{\partial t} \varphi(p, t) &= H(p, t)\nu(p, t), \\
\varphi(p, 0) &= \varphi_0(p),
\end{cases}
\]

where $H(p, t)$ and $\nu(p, t)$ are the mean curvature and the unit normal of the hypersurfaces $M_t = \varphi(\cdot, t)(M)$, respectively. Notice that the field $H(p, t)\nu(p, t)$ is independent of the sign of the normal vector field $\nu(p, t)$.

The mean curvature flow was first introduced in 1956 by Mullins in [8] and independently by Brakke in 1978 from the viewpoint of geometric measure theory [1]. Since then, this flow has been widely studied. It has been studied in the...
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smooth setting by Huisken in 1984. He showed in [5] that if the initial hypersurface is convex and closed then the flow has an unique solution in $[0, T)$, where $T$ is the maximal time where the flow exists, and as $t \to T$, the hypersurfaces converge to a point. He also studied the singularity of mean curvature flow when the initial hypersurface of the flow is non-convex [6]. One of the interesting subjects in mean curvature flow is studying how geometric quantities and their properties change when the hypersurface evolves under the flow. For example, if the initial hypersurface is compact, then it remains compact during the evolution too, or if the initial hypersurface is compact and its mean curvature $H$ is nonnegative, then $H$ will be positive under the mean curvature flow [5,6].

The Weyl curvature tensor has been studied in other flows like the Ricci flow. For a Riemannian manifold $(M^n, g)$ the Ricci flow is defined by

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$

(2)

where $R_{ij}$ is the Ricci curvature of $M$. For example, it has been shown in [7] that if the Weyl curvature tensor remains identically zero then either the Ricci tensor is proportional to the metric or it has an eigenvalue of multiplicity $(n - 1)$ and another of multiplicity 1. The evolution equation of Weyl curvature tensor of hypersurfaces under the mean curvature flow is more complicated than (2) because the evolution of metric tensor of the hypersurface under the mean curvature flow has one more term, it is

$$\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij} = -2R_{ij} - 2h^k_i h_{kj}.$$  

(3)

In this paper, we present an estimate of the evolution of the Weyl curvature tensor of hypersurfaces in a Euclidean space under the mean curvature flow and its application under the assumption of conformal flatness of the initial hypersurface. The Weyl curvature tensor $W$ is used in general relativity. Closely related is the Ricci curvature tensor whose behaviour with respect to the Ricci flow played the central role in the famous solution of the geometrization conjecture in dimension 3 by Perelman, Hamilton and others.

The main results are Theorem 4.1 and Corollary 4.2. Theorem 4.1. says that if the initial hypersurface has the Weyl curvature tensor bounded by some constant $M$, then the evolving hypersurfaces also have the Weyl curvature tensor bounded by a specific formula involving $M$ and some other constants. Corollary 4.2 is then the special case when $M = 0$. The proof of Theorem 4.1 is based on the so-called “Maximum principle” recalled here as Theorem 2.3. Roughly, it says that if the variation of a function $u$ (such as $|W|$) can be appropriately estimated in terms of the Laplacian, the gradient and a locally Lipschitz function $F$, then function $u$ is estimated by the solution of an initial value problem defined by $F$. Hence, the method of the proof is to estimate the variation of the Weyl curvature tensor achieved in Corollary 3.8 and to obtain the estimate for the
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Weyl tensor itself by the above principle. The estimate of the variation takes several pages due to the fact that all the components of the tensor have to be estimated. Many notions in classical differential geometry are used such as the Riemannian metric, Ricci tensor, the second fundamental form, ∗-product and various technical manipulations of tensors.

Since for all surfaces and three dimensional hypersurface W is zero, it will be assumed \( n \geq 4 \).

2. Preliminaries and the geometry of hypersurfaces

In this section some definitions and formulas for hypersurfaces in \( \mathbb{R}^{n+1} \) are introduced (see \([5]\)).

Throughout the paper, we consider \( n \)-dimensional complete hypersurfaces immersed in \( \mathbb{R}^{n+1} \), that is, for the pair \((M, \varphi)\) \( M \) is an \( n \)-dimensional smooth manifold with empty boundary and \( \varphi : M \rightarrow \mathbb{R}^{n+1} \) is a smooth immersion. Also, the Euclidean metric in \( \mathbb{R}^{n+1} \) is denoted by \( \langle , \rangle \).

The metric \( g \) on \( M \) is obtained by pulling back the Euclidean metric by \( \varphi \), i.e.,

\[
g_{ij} = g \left( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = \left\langle \frac{\partial \varphi}{\partial x_i}, \frac{\partial \varphi}{\partial x_j} \right\rangle,
\]

for \( 1 \leq i, j \leq n \). The inverse metric and the area element of \( M \) are denoted by

\[
g^{ij} = (g_{ij})^{-1} \quad \text{and} \quad \sqrt{g} = \sqrt{\det g_{ij}},
\]

respectively.

The metric \( g \) of \( M \) extended to tensors is given by

\[
g(T, S) = g_{i_1 s_1} \ldots g_{i_k s_k} g^{j_1 r_1} \ldots g^{j_l r_l} T_{j_1 \ldots j_l} S_{r_1 \ldots r_l}^{i_1 \ldots i_k},
\]

where \( T \) and \( S \) are two \((k, l)\)-tensors, and the norm of a tensor is given by

\[
|T| = \sqrt{g(T, T)}.
\]

The induced covariant derivative on \((M, g)\) of a vector field is

\[
\nabla_j X^i = \frac{\partial X^i}{\partial x_j} + \Gamma^i_{jk} X^k,
\]

where \( \Gamma^i_{jk} \) are the Christoffel symbols that means

\[
\Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{ji} \right) \quad i, j, k = 1, 2, \ldots, n.
\]
The gradient of a function \( f \) and the divergence of vector field \( X \) at a point \( p \in M \) are defined, respectively, as following
\[
g(\nabla f(p), w) = df_p(w),
\]
for all
\[
w \in T_p(M) \quad \text{and} \quad \text{div}X = tr\nabla X = \nabla_i X^i = \frac{\partial}{\partial x_i}X^i + \Gamma^i_{ik}X^k.
\]
The Laplacian \( \Delta T \) of a tensor \( T \) is defined as
\[
\Delta T = g^{ij}\nabla_i \nabla_j T.
\]
The second fundamental form \( A = h_{ij} \) of \( M \) is the symmetric 2-form defined as
\[
h_{ij} = \left\langle \nu, \frac{\partial^2 \varphi}{\partial x_i \partial x_j} \right\rangle = \left\langle \frac{\partial \nu}{\partial x_i}, \frac{\partial \varphi}{\partial x_j} \right\rangle,
\]
where \( \nu \) is the unit normal vector at every point of \( M \). The mean curvature \( H \) is the trace of \( A \), that is, \( H = g^{ij}h_{ij} \). We also have
\[
|A|^2 = g^{ij}g^{kl}h_{ik}h_{jl}.
\]
The Gauss-Weingarten relations are
\[
\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \Gamma^k_{ij} \frac{\partial \varphi}{\partial x^k} + h_{ij} \nu, \quad \frac{\partial}{\partial x_j} \nu = -h_{ji}g^{ls} \frac{\partial \varphi}{\partial x_s}.
\]
The Riemann curvature tensor, Ricci tensor and the scalar curvature for hypersurfaces can be expressed by means of the second fundamental form as follows
\[
R_{ijkl} = h_{ik}h_{jl} - h_{il}h_{jk}, \quad \text{(4)}
\]
\[
Ric_{ij} = g^{kl}R_{ijkl} = Hh_{ij} - h_{il}g^{lk}h_{kj}, \quad \text{(5)}
\]
\[
r = g^{ij}Ric_{ij} = H^2 - |A|^2. \quad \text{(6)}
\]
The Codazzi equations
\[
\nabla_i h_{jk} = \nabla_j h_{ik} = \nabla_k h_{ij}
\]
show the symmetry properties of the covariant derivative of \( A \).

In [3,4] Hamilton used the \( T^* \) as a tensor which is formed by a sum of terms that each of them obtained by contracting some indices of the pair \( T \) and \( S \) with the metric \( g_{ij} \). One of the important and useful property of \( *- \)product is
\[
|T^* S| \leq C|T||S|,
\]
where the constant \( C \) depends only on the algebraic “structure” of \( T \) \( S \).

Let \((M, g)\) be an \( n \)-dimensional Riemannian manifold, a conformal change of metric \( g \) is defined as \( \tilde{g} = \sigma g \), where \( \sigma \) is a positive function on \( M \). A Riemannian manifold \((M, g)\) is called conformally flat if it is a coefficient of the Euclidean metric.
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The Weyl curvature tensor on \((M, g)\) is a \((0, 4)\)-tensor locally defined by

\[
W_{ijkl} = R_{ijkl} - \frac{1}{n-2} (R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}) + \frac{r}{(n-1)(n-2)} (g_{ik}g_{jl} - g_{il}g_{jk}).
\] (7)

For manifolds with \(\dim M \leq 3\) the Weyl curvature tensor is zero. The following theorem is the theorem of Weyl [9].

**Theorem 2.1.** A necessary and sufficient condition for a Riemannian manifold \(M\) to be conformally flat is that \(W = 0\).

At the end of this section we state the Maximum principle (see [2, Lemma 2.12]).

**Theorem 2.2** (Maximum Principle). Let \(u : M \times [0, T) \rightarrow \mathbb{R}\) be a smooth function on a closed manifold satisfying \(\frac{\partial u}{\partial t} \leq \Delta_{g(t)} u + \langle X, \nabla u \rangle + F(u)\) and \(u(x, 0) \leq c\) for all \(x \in M\), where \(g(t)\) is a 1-parameter family of metrics and \(F\) is a locally Lipschitz. Let \(\phi(t)\) be the solution to the initial value problem

\[
\begin{aligned}
\frac{d\phi}{dt} &= F(\phi), \\
\phi(0) &= 0.
\end{aligned}
\]

Then \(u(x, t) \leq \phi(t)\) for all \(x \in M\) and \(t \in [0, T)\) such that \(\phi(t)\) exists.

It will be assumed that the second fundamental forms therefore \(|A|^2\) and \(|\nabla A|^2\) for initial hypersurface are bounded [5].

3. Evolution equations

In this section, the evolution equations for metric, normal vector, second fundamental form and the mean curvature are mentioned. For the proof of these equations see [5]. Moreover, the evolution equations for the Riemannian curvature tensor, Ricci tensor, scalar curvature and the Weyl curvature tensor are computed.

**Lemma 3.1.** The following evolution equations hold for \(M_t = \varphi(., t)(M)\)

\[
\frac{\partial}{\partial t} g_{ij} = -2H h_{ij},
\] (8)

\[
\frac{\partial}{\partial t} g^{ij} = 2H h^{ij},
\] (9)

\[
\frac{\partial}{\partial t} \nu = -\nabla H,
\] (10)

\[
\frac{\partial}{\partial t} \Gamma^i_{jk} = \nabla H \ast A + H \ast \nabla A = \nabla A \ast A.
\] (11)
Proposition 3.2. The second fundamental form satisfies the evolution equation
\[ \frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2H h_{il} g^{ls} h_{sj} + |A|^2 h_{ij}. \] (12)

And it follows that
\[ \frac{\partial}{\partial t} h_{ij}^2 = \Delta h_{ij}^2 + |A|^2 h_{ij}, \] (13)
\[ \frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2|\nabla A|^2 + 2|A|^4, \] (14)
\[ \frac{\partial}{\partial t} H = \Delta H + |A|^2 H. \] (15)

Now we want to compute the evolution of the Riemannian curvature tensor. By using (12) at first we compute the evolutions of its terms
\[ \frac{\partial}{\partial t} (h_{ik} h_{jl}) = h_{jl} \frac{\partial h_{ik}}{\partial t} + h_{ik} \frac{\partial h_{jl}}{\partial t} \]
\[ = h_{jl}(\Delta h_{ik} - 2H h_{ir} g^{rs} h_{sk} + |A|^2 h_{ik}) \]
\[ + h_{ik}(\Delta h_{jl} - 2H h_{jr} g^{rs} h_{sl} + |A|^2 h_{jl}) \]
\[ = h_{jl} \Delta h_{ik} + h_{ik} \Delta h_{jl} - 2H h_{ik} h_{jr} g^{rs} h_{sk} \]
\[ - 2H h_{ik} h_{jr} g^{rs} h_{sl} + 2|A|^2 h_{ik} h_{jl}. \] (16)

Now the Laplacian of the product of two tensors yields
\[ \frac{\partial}{\partial t} (h_{ik} h_{jl}) = \Delta (h_{ik} h_{jl}) - 2\langle \nabla h_{ik}, \nabla h_{jl} \rangle - \]
\[ 2H h_{jl} h_{ir} g^{rs} h_{sk} - 2H h_{ik} h_{jr} g^{rs} h_{sl} + 2|A|^2 h_{ik} h_{jl}. \] (16)

Similarly, for the other components of the Riemannian curvature tensor we can write
\[ \frac{\partial}{\partial t} (h_{il} h_{jk}) = \Delta (h_{il} h_{jk}) - 2\langle \nabla h_{il}, \nabla h_{jk} \rangle - \]
\[ 2H h_{jk} h_{ir} g^{rs} h_{sl} - 2H h_{il} h_{jr} g^{rs} h_{sk} + 2|A|^2 h_{il} h_{jk}. \] (17)

Hence, we can state the following lemma.

Lemma 3.3. The evolution equation for the Riemannian tensor \( R \) satisfies
\[ \frac{\partial R_{ijkl}}{\partial t} = \Delta R_{ijkl} + 2\langle \nabla h_{il}, \nabla h_{jk} \rangle - 2\langle \nabla h_{ik}, \nabla h_{jl} \rangle + \]
\[ 2H h_{k}^{r} R_{ijlr} - 2H h_{l}^{r} R_{ijkr} + 2|A|^2 R_{ijkl}. \] (18)
Proof. By applying the equations (11), (12), (16) and (17) we have
\[
\frac{\partial R_{ijkl}}{\partial t} = \frac{\partial (h_{ik}h_{jl})}{\partial t} - \frac{\partial (h_{il}h_{jk})}{\partial t}
\]
\[
= \Delta (h_{ik}h_{jl}) - \Delta (h_{il}h_{jk}) - 2\langle \nabla h_{ik}, \nabla h_{jl} \rangle
\]
\[+ 2\langle \nabla h_{il}, \nabla h_{jk} \rangle - 2Hh_{iji}g^{rs}h_{sk} + 2Hh_{ijk}h_{ir}g^{rs}h_{sl}
\]
\[- 2Hh_{ik}h_{jr}g^{rs}h_{sl} + 2Hh_{ii}h_{jr}g^{rs}h_{sk} + 2|A|^2(h_{ik}h_{jl} - h_{il}h_{jk})
\]
\[= \Delta R_{ijkl} - 2\langle \nabla h_{ik}, \nabla h_{jl} \rangle + 2\langle \nabla h_{il}, \nabla h_{jk} \rangle - 2Hh_{ski}g^{rs}(h_{ir}h_{jl} - h_{il}h_{jr})
\]
\[- 2Hh_{skj}g^{rs}(h_{ik}h_{jr} - h_{ir}h_{jk}) + 2|A|^2(h_{ik}h_{jl} - h_{il}h_{jk}).
\]
We obtain (18) by replacing all the statements inside the parentheses by the Riemannian curvature tensor with proper indices in (11). 

**Proposition 3.4.** The evolution equations for the Ricci tensor and the scalar curvature are
\[
\frac{\partial R_{ij}}{\partial t} = \Delta R_{ij} + 2\langle \nabla h_{il}, h^i_j \rangle - 2\langle \nabla h_{ij}, \nabla H \rangle
\]
\[+ 2|A|^2 R_{ij} - 2Hh^i_{i}R_{js},
\]
\[
\frac{\partial r}{\partial t} = \Delta r + 2|\nabla A|^2 - 2|\nabla H|^2 + 2|A|^2 r.
\]

Proof. From (5) we have
\[
\frac{\partial R_{ij}}{\partial t} = \frac{\partial H}{\partial t}h_{ij} + H \frac{\partial h_{ij}}{\partial t} - \frac{\partial h_{il}}{\partial t}h^l_j - h_{ij} \frac{\partial h^l_i}{\partial t}
\]
and by applying the formulas in Proposition 3.2 we have
\[
\frac{\partial R_{ij}}{\partial t} = (\Delta H + |A|^2 H)h_{ij} + H(\Delta h_{ij} - 2Hh_{ir}g^{rs}h_{sj} + |A|^2 h_{ij})
\]
\[+ h_{ij}H - 2Hh_{ir}g^{rs}h_{sl} + |A|^2 h_{il}h^l_j - h_{il}(\Delta h^l_j + |A|^2 h^l_j)
\]
\[= h_{ij}H - 2Hh_{ir}g^{rs}h_{sj} + 2|A|^2 Hh_{ij} - 2Hg^{rs}h_{ir}(H_{js} - h_{sl}h^l_j)
\]
\[- h_{ij}H - 2Hh_{ir}g^{rs}h_{sl} + 2|A|^2 h_{il}h^l_j - 2|A|^2 h_{il}h^l_j
\]
\[= \Delta (h_{ij}H) - 2\langle \nabla h_{ij}, \nabla H \rangle + 2|A|^2 R_{ij} - 2Hg^{rs}h_{ir}R_{sj}
\]
\[- \Delta (h_{il}h^l_j) + 2\langle \nabla h_{il}, h^l_j \rangle.
\]
Which concludes the first formula. For the second one using (6) we have
\[
\frac{\partial r}{\partial t} = \frac{\partial}{\partial t}(H^2 - |A|^2).
\]
Now, the equation (14) yields
\[ \frac{\partial r}{\partial t} = 2H \frac{\partial H}{\partial t} - \Delta |A|^2 + 2|\nabla A|^2 - 2|A|^4 \]
and (15) implies
\[ \frac{\partial r}{\partial t} = 2H(\Delta H + |A|^2H) \]
\[ \quad - \Delta |A|^2 + 2|\nabla A|^2 - 2|A|^4 \]
\[ = \Delta H^2 - 2|\nabla H|^2 + 2|A|^2H^2 \]
\[ \quad - \Delta |A|^2 + 2|\nabla A|^2 - 2|A|^4 \]
\[ = \Delta H^2 - \Delta |A|^2 \]
\[ + 2|A|^2(H^2 - |A|^2) + 2|\nabla A|^2 - 2|\nabla H|^2 \]
\[ = \Delta r + 2|\nabla A|^2 - 2|\nabla H|^2 + 2|A|^2r. \]

Now we can compute the evolution of the Weyl curvature tensor but at first the evolution of its terms in relation (7) are calculated. We have
\[ \frac{\partial}{\partial t}(R_{ik}g_{jl}) = \frac{\partial R_{ik}}{\partial t}g_{jl} + R_{ik} \frac{\partial g_{jl}}{\partial t}, \]
and from (19) and (8) we get
\[ \frac{\partial}{\partial t}(R_{ik}g_{jl}) = (\Delta(R_{ik}) + 2\langle \nabla h^s_{ir}, \nabla h^r_{k} \rangle - 2\langle \nabla h_{ik}, \nabla H \rangle \]
\[ + 2|A|^2R_{ik} - 2Hh^s_iR_{ks} \)g_{jl} \]
\[ + R_{ik}(-2Hh_{jl}), \]

since \( \nabla_k g_{ij} = 0 \), then \( \Delta g_{ij} = 0 \). Therefore,
\[ \Delta(R_{ik}g_{jl}) = g_{jl}\Delta R_{ik}, \]
hence we have
\[ \frac{\partial}{\partial t}(R_{ik}g_{jl}) = \Delta(R_{ik}g_{jl}) + 2g_{jl}\langle \nabla h^s_{ir}, \nabla h^r_{k} \rangle - 2g_{jl}\langle \nabla h_{ik}, \nabla H \rangle \]
\[ + 2|A|^2R_{ik}g_{jl} - 2Hh^s_iR_{js} \]
\[ - 2Hh_{jl}R_{ik}. \]
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So, we imply

\[
\frac{\partial}{\partial t}(R_{ikgjl} + R_{jilgik} - R_{ilgjk} - R_{jkgil}) \\
= \Delta(R_{ikgjl} + R_{jilgik} - R_{ilgjk} - R_{jkgil}) \\
+ 2g_{jl}\langle \nabla h_{ir}, \nabla h_{ik} \rangle + 2g_{ik}\langle \nabla h_{jr}, \nabla h_{il} \rangle - 2g_{jk}\langle \nabla h_{ir}, \nabla h_{il} \rangle \\
- 2g_{il}\langle \nabla h_{jr}, \nabla h_{ik} \rangle - 2g_{jl}\langle \nabla h_{ik}, \nabla H \rangle - 2g_{ik}\langle \nabla h_{jl}, \nabla H \rangle \\
+ 2g_{jk}\langle \nabla h_{il}, \nabla H \rangle + 2g_{il}\langle \nabla h_{jk}, \nabla H \rangle + 2|A|^2 R_{ikgjl} \\
+ 2|A|^2 R_{jilgik} - 2|A|^2 R_{ilgjk} - 2|A|^2 R_{jkgil} - 2Hh_{i}^{s}g_{jl}R_{ks} \\
- 2Hh_{j}^{s}g_{ik}R_{ls} + 2Hh_{i}^{s}g_{jk}R_{ls} + 2Hh_{j}^{s}g_{il}R_{ks} \\
- 2Hh_{jl}R_{ik} - 2H_{i}^{s}g_{ik}R_{jl} + 2Hh_{jk}R_{il} + 2Hh_{il}R_{jk}
\]

Also, by using (8) we can easily obtain

\[
\frac{\partial}{\partial t}(g_{ikgjl} - g_{ilgjk}) = -2Hg_{jl}h_{ik} - 2Hg_{ik}h_{jl} \\
+ 2Hg_{il}h_{jk} + 2Hg_{jl}h_{ik}.
\]

Therefore by using (7) and these last two relations, we can compute the evolution of the Weyl curvature tensor

\[
\frac{\partial W_{ijkl}}{\partial t} = \frac{\partial R_{ijkl}}{\partial t} - \frac{1}{n-2} \frac{\partial}{\partial t}(R_{ikgjl} + R_{jilgik} - R_{ilgjk} - R_{jkgil}) \\
+ \frac{1}{(n-1)(n-2)} \frac{\partial r}{\partial t}(g_{ikgjl} - g_{ilgjk}) \\
+ \frac{r}{(n-1)(n-2)} \frac{\partial}{\partial t}(g_{ikgjl} - g_{ilgjk}).
\]
Now by taking into account these two relations, \([14]\) and the evolution equations for the Ricci curvature and the scalar curvature, we have

\[
\frac{\partial W_{ijkl}}{\partial t} = \Delta R_{ijkl} + 2\langle \nabla h_{il}, \nabla h_{jk} \rangle - 2\langle \nabla h_{ik}, \nabla h_{jl} \rangle + 2H h^r_k R_{ijlr} - 2H h^r_l R_{ijkr} + 2|A|^2 R_{ijkl}
\]

\[- \frac{1}{n-2} \Delta (R_{ik} g_{jl} + R_{jl} g_{ik} - R_{il} g_{jk} - R_{jk} g_{il})
\]

\[- \frac{1}{n-2} [2g_{jl} \langle \nabla h_{ir}, \nabla h^r_k \rangle + 2g_{ik} \langle \nabla h_{jr}, \nabla h^r_l \rangle - 2g_{jk} \langle \nabla h_{ir}, \nabla h^r_l \rangle - 2g_{il} \langle \nabla h_{jr}, \nabla h^r_k \rangle - 2g_{jl} \langle \nabla h_{ik}, \nabla H \rangle - 2g_{ik} \langle \nabla h_{jl}, \nabla H \rangle + 2g_{jk} \langle \nabla h_{il}, \nabla H \rangle + 2g_{il} \langle \nabla h_{jk}, \nabla H \rangle]
\]

\[+ \frac{2|A|^2}{n-1} (R_{ik} g_{jl} + R_{jl} g_{ik} - R_{il} g_{jk} - R_{jk} g_{il})
\]

\[- \frac{2H h^s_k}{n-1} (R_{ks} g_{jl} + R_{jl} g_{ks} - R_{ls} g_{jk} - R_{jk} g_{ls})
\]

\[+ \frac{2H h^s_l}{n-1} (R_{ks} g_{il} + R_{il} g_{ks} - R_{ls} g_{ik} - R_{ik} g_{ls})
\]

\[+ \frac{1}{(n-1)(n-2)} \Delta [r (g_{ik} g_{jl} - g_{il} g_{jk})]
\]

\[+ \frac{2|\nabla A|^2}{n-1} - \frac{2|\nabla H|^2}{n-1} (g_{ik} g_{jl} - g_{il} g_{jk})
\]

\[+ \frac{2|A|^2 r}{(n-1)(n-2)} (g_{ik} g_{jl} - g_{il} g_{jk})
\]

\[+ \frac{r}{(n-1)(n-2)} (-2H g_{jl} h_{ik} - 2H g_{ik} h_{jl} + 2H g_{il} h_{jk} + 2H g_{jk} h_{il})
\]

in which, these two relations hold

\[h^r_k R_{ijlr} = h^i_r R_{kljr}, \quad h^r_i R_{ijkr} = h^i_r R_{klir}, \quad (21)\]
We can arrange this as

\[
\frac{\partial W_{ijkl}}{\partial t} = \Delta W_{ijkl} + 2|A|^2 W_{ijkl} + 2 \langle \nabla h_{il}, \nabla h_{jk} \rangle - 2 \langle \nabla h_{ik}, \nabla h_{jl} \rangle - \frac{1}{n-2} \left[ 2g_{jl} \langle \nabla h_{ir}, \nabla h_{ik}^* \rangle + 2g_{ik} \langle \nabla h_{jr}, \nabla h_{il}^* \rangle - 2g_{jk} \langle \nabla h_{ir}, \nabla h_{il}^* \rangle - 2g_{il} \langle \nabla h_{jr}, \nabla h_{ik}^* \rangle - 2g_{jl} \langle \nabla h_{ik}, \nabla H \rangle - 2g_{ik} \langle \nabla h_{jl}, \nabla H \rangle + 2g_{jk} \langle \nabla h_{il}, \nabla H \rangle + 2g_{il} \langle \nabla h_{jk}, \nabla H \rangle \right] + 2H h^s_i R_{kljs} - 2H h^s_i R_{klis} + \frac{2H h^s_i}{n-2} (R_{ks} g_{il} + R_{il} g_{ks} - R_{ls} g_{ik} - R_{ik} g_{ls}) - \frac{2H h^s_i}{n-2} (R_{ks} g_{jl} + R_{jl} g_{ks} - R_{ls} g_{jk} - R_{jk} g_{ls}) + \frac{2|\nabla A|^2 - 2|\nabla H|^2}{(n-1)(n-2)} (g_{ik} g_{jl} - g_{il} g_{jk}) + \frac{r}{(n-1)(n-2)} (-2H g_{ij} h_{ik} - 2H g_{ik} h_{jl} + 2H g_{il} h_{jk} + 2H g_{jk} h_{il}).
\]

We can arrange this as

\[
\frac{\partial W_{ijkl}}{\partial t} = \Delta W_{ijkl} + 2|A|^2 W_{ijkl} + 2 \langle \nabla h_{il}, \nabla h_{jk} \rangle - 2 \langle \nabla h_{ik}, \nabla h_{jl} \rangle - \frac{1}{n-2} \left[ 2g_{jl} \langle \nabla h_{ir}, \nabla h_{ik}^* \rangle + 2g_{ik} \langle \nabla h_{jr}, \nabla h_{il}^* \rangle - 2g_{jk} \langle \nabla h_{ir}, \nabla h_{il}^* \rangle - 2g_{il} \langle \nabla h_{jr}, \nabla h_{ik}^* \rangle - 2g_{jl} \langle \nabla h_{ik}, \nabla H \rangle - 2g_{ik} \langle \nabla h_{jl}, \nabla H \rangle + 2g_{jk} \langle \nabla h_{il}, \nabla H \rangle + 2g_{il} \langle \nabla h_{jk}, \nabla H \rangle \right] - 2H h^s_i R_{kljs} - \frac{1}{n-2} (R_{ls} g_{jk} + R_{jk} g_{ls} - R_{ks} g_{jl} - R_{jl} g_{ks}) + \frac{r}{(n-1)(n-2)} (g_{jl} g_{sk} - g_{jk} g_{sl}) + 2H h^s_i R_{klis} - \frac{1}{n-2} (R_{ks} g_{il} + R_{il} g_{ks} - R_{ls} g_{ik} - R_{ik} g_{ls}) + \frac{r}{(n-1)(n-2)} (g_{ik} g_{sl} - g_{il} g_{sk}) + \frac{2|\nabla A|^2 - |\nabla H|^2}{(n-1)(n-2)} (g_{ik} g_{jl} - g_{il} g_{jk}).
\]

The phrases inside the second and the third brackets can be replaced by the Weyl curvature tensor with respect to the proper indices. Hence, we have
\[
\frac{\partial W_{ijkl}}{\partial t} = \Delta W_{ijkl} + 2|A|^2W_{ijkl} + 2\langle \nabla h_{il}, \nabla h_{jk} \rangle - 2\langle \nabla h_{ik}, \nabla h_{jl} \rangle - \frac{1}{n-2} \left[ 2g_{ji}\langle \nabla h_{ir}, \nabla h_{rk}^* \rangle + 2g_{ik}\langle \nabla h_{jr}, \nabla h_{ik}^* \rangle - 2g_{jk}\langle \nabla h_{ir}, \nabla h_{l}^* \rangle - 2g_{il}\langle \nabla h_{jr}, \nabla h_{k}^* \rangle - 2g_{jl}\langle \nabla h_{ik}, \nabla H \rangle - 2g_{ik}\langle \nabla h_{jl}, \nabla H \rangle \right] \\
+ 2Hh_j^s w_{klis} - 2Hh_i^s w_{kljs} \\
+ \frac{2(|\nabla A|^2 - |\nabla H|^2)}{(n-1)(n-2)} (g_{ik}g_{jl} - g_{il}g_{jk}).
\]

Therefore, we can state the following theorem.

**Theorem 3.5.** The evolution equation for the Weyl curvature tensor is

\[
\frac{\partial W_{ijkl}}{\partial t} = \Delta W_{ijkl} + 2|A|^2W_{ijkl} + 2Hh_j^s W_{klis} - 2Hh_i^s W_{kljs} \\
+ 2\langle \nabla h_{il}, \nabla h_{jk} \rangle - 2\langle \nabla h_{ik}, \nabla h_{jl} \rangle - \frac{1}{n-2} \left[ 2g_{ji}\langle \nabla h_{ir}, \nabla h_{rk}^* \rangle + 2g_{ik}\langle \nabla h_{jr}, \nabla h_{ik}^* \rangle - 2g_{jk}\langle \nabla h_{ir}, \nabla h_{l}^* \rangle - 2g_{il}\langle \nabla h_{jr}, \nabla h_{k}^* \rangle - 2g_{jl}\langle \nabla h_{ik}, \nabla H \rangle - 2g_{ik}\langle \nabla h_{jl}, \nabla H \rangle \right] \\
+ \frac{2(|\nabla A|^2 - |\nabla H|^2)}{(n-1)(n-2)} (g_{ik}g_{jl} - g_{il}g_{jk}).
\]

Now we can use the *-product to simplify this evolution equation, because we want to use the Maximum principle. Therefore, we have the following theorem.

**Theorem 3.6.** The evolution equation for the Weyl curvature tensor can be written as

\[
\frac{\partial}{\partial t} W = \Delta W + A * A * W + A * A + \nabla A * \nabla A.
\] (22)

Also by applying (22), we state following theorem.

**Theorem 3.7.** The variation of \(|W|^2\) satisfies

\[
\frac{\partial |W|^2}{\partial t} \leq \Delta |W|^2 + a_1|A|^2|W|^2 + a_2|A|^2|W| + a_3|\nabla A|^2|W|, \tag{23}
\]

where \(a_1\), \(a_2\) and \(a_3\) are real numbers depending on the initial hypersurface.
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Proof.
\[
\frac{\partial}{\partial t} |W|^2 = \frac{\partial}{\partial t} \langle W, W \rangle = 2 \langle \frac{\partial W}{\partial t}, W \rangle = 2 \langle \Delta W, W \rangle + \langle A * A * W, W \rangle + \langle A * A, W \rangle + \langle \nabla A * \nabla A, W \rangle = \Delta |W|^2 - 2|\nabla W|^2 + A * A * W + \nabla A * \nabla A * W \\
\leq \Delta |W|^2 + a_1 |A|^2 |W|^2 + a_2 |A|^2 |W| + a_3 |\nabla A|^2 |W|.
\]

Since it was supposed that $|A|$ and $|\nabla A|$ are bounded, we conclude the following corollary.

**Corollary 3.8.** The variation of the norm of the Weyl curvature tensor holds
\[
\frac{\partial}{\partial t} |W|^2 \leq \Delta |W|^2 + a|W|^2 + b,
\]
where $a$ and $b$ are real numbers depending on the initial hypersurface.

Proof. Since $|A|$ and $|\nabla A|$ are bounded, then the inequality in the previous theorem can be written in the form
\[
\frac{\partial}{\partial t} |W|^2 \leq \Delta |W|^2 + b_1 |W|^2 + b_2 |W|,
\]
on the other hand, for any two real numbers $x$ and $y$ we have $x^2 + xy \leq 2x^2 + \frac{y^2}{2}$, then
\[
\frac{\partial}{\partial t} |W|^2 \leq \Delta |W|^2 + b_1 (|W|^2 + \frac{b_2}{b_1} |W|) \leq \Delta |W|^2 + 2b_1 |W|^2 + \frac{b_2^2}{2b_1} = \Delta |W|^2 + a|W|^2 + b,
\]
where we set $a = 2b_1$ and $b = \frac{b_2}{2b_1}$. □

4. Weyl curvature tensor estimate

In this section, it is supposed that the Weyl curvature tensor is bounded at time $t = 0$, then an upper bound for it during the evolution is estimated. Finally, it is assumed that the initial hypersurface is the conformally flat. Then an estimate for the Weyl curvature tensor is concluded.

**Theorem 4.1.** Let us have $|W(p,0)| \leq M$. Then
\[
|W(p,t)|^2 \leq \left( M + \frac{b}{a} \right) e^{at} - \frac{b}{a} \text{ for all } t \in [0,T),
\]
where $a$ and $b$ are the real numbers depending on the initial hypersurface $M$. 

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Proof. By taking into account the equation (24) and the Maximum principle, the corresponding solution to the ODE is

$$\frac{d\phi}{dt} = a\phi + b; \quad \phi(0) = M.$$  

It implies

$$\frac{d\phi}{a\phi + b} = dt,$$

where the solution of this differential equation is

$$\phi(t) = e^{at} \left( M + \frac{b}{a} \right) - \frac{b}{a}.$$

By considering the solution and the Maximum principle, the proof will be completed. \(\square\)

**Corollary 4.2.** If the initial hypersurface is conformally flat, i.e., \(|W(0, p)| = 0\) for every \(p \in M\), then under the mean curvature flow we have

$$|W(p, t)|^2 \leq \frac{b}{a} e^{at} - \frac{b}{a} < \frac{b}{a} e^{at}.$$  

Proof. It is sufficient if in the previous theorem we put \(M = 0\). \(\square\)

Acknowledgement. We are very thankful to the referee who helped to improve our paper.

REFERENCES

[1] BRAKKE K.A.: *The motion of surface by by its mean curvature*. Mathematical Notes Vol. 20, Princeton University Press, Princeton, NJ, 1978.

[2] CHOW, B.—LU, P.—NI, L.: *Hamilton’s Ricci Flow*. In: Graduate Studies in Mathematics, Vol. 77, AMS, Providence RI, Science Press Beijing, New York, 2006.

[3] HAMILTON, R.S.: *Three-manifolds with positive Ricci curvature*. J. Differ. Geom. 17 (1982), 255–306.

[4] HAMILTON, R.S.: *Four-manifolds with positive curvature operator*, J. Differ. Geom. 24 (1986), 153–179.

[5] HUISKEN, G.: *Flow by mean curvature of convex surfaces into sphere*, J. Differ. Geom. 20 (1984), 237–266.

[6] HUISKEN, G.: *Asymptotic behavior for singularities of the mean curvature flow*, J. Differ. Geom. 31 (1990), 285–299.

[7] CATINO, G.—MANTEGAZZA, C.: *The evolution of the Weyl tensor under the Ricci flow*, Ann. Inst. Fourier (Grenoble) 61 (2012), no. 4, 1407–1435.

[8] MULLINS, W.W.: *Two-dimensional motion of idealized grain boundaries*, J. Appl. Phys. 27 (1956), 900–904.

[9] YANO, K.—KON, M.: *Structures on manifolds*. Series in Pure Mathematics Vol. 3. World Scientific Publishing Co., Singapore, 1984.

Received August 18, 2020

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