A Study on Arbitrarily Varying Channels with Causal Side Information at the Encoder

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Abstract

In this work, we study two models of arbitrarily varying channels, when causal side information is available at the encoder in a causal manner. First, we study the arbitrarily varying channel (AVC) with input and state constraints, when the encoder has state information in a causal manner. Lower and upper bounds on the random code capacity are developed. A lower bound on the deterministic code capacity is established in the case of a message-averaged input constraint. In the setting where a state constraint is imposed on the jammer, while the user is under no constraints, the random code bounds coincide, and the random code capacity is determined. Furthermore, for this scenario, a generalized non-symmetrizability condition is stated, under which the deterministic code capacity coincides with the random code capacity.

A second model considered in our work is the arbitrarily varying degraded broadcast channel with causal side information at the encoder (without constraints). We establish inner and outer bounds on both the random code capacity region and the deterministic code capacity region. The capacity region is then determined for a class of channels satisfying a condition on the mutual informations between the strategy variables and the channel outputs. As an example, we show that the condition holds for the arbitrarily varying binary symmetric broadcast channel, and we find the corresponding capacity region.
In practice, the statistics of a communication system are not necessarily known in exact, and they may even change over time. The arbitrarily varying channel (AVC) is an appropriate model to describe such a situation, as introduced by Blackwell et al. [8]. Among the motivations for this field of research is the adversarial communication model, where a jammer selects a sequence of channel states in an attempt to disrupt communication.

Considering the AVC without SI, Blackwell et al. determined the random code channel capacity [8], i.e. the capacity achieved by stochastic-encoder stochastic-decoder coding schemes with common randomness. It was also demonstrated in [8] that the random code capacity is not necessarily achievable using deterministic codes. A well-known result by Ahlswede [1] is the dichotomy property presented by the AVC in the absence of state information. Namely, without SI, the deterministic code capacity either equals the random code capacity or else, it is zero.

Subsequently, Ericson [18] and Csiszár and Narayan [15] have established a simple single-letter condition, namely non-symmetrizability, which is both necessary and sufficient for the capacity to be positive in the case of an AVC without state information. The derivation of sufficiency, in [15], is independent of Ahlswede’s work and is based on a subtle decoding rule, analyzed through the method of types.

Csiszár and Narayan also determined the random code capacity [14] and the deterministic code capacity [15] of the AVC, when input and state constraints are imposed on the user and the jammer, respectively. In [15], they show that dichotomy in the notion of [1] does not hold when state constraints are imposed on the jammer. That is, the deterministic code capacity can be lower than the random code capacity, and yet non-zero.

Vast research has been conducted on other AVC models as well. Recently, the arbitrarily varying wiretap channel has been extensively studied, as e.g. in [25, 9, 5, 10, 26, 20]. The multiple user scenario was first studied by Jahn [23], who presented an inner bound on the capacity region of the arbitrarily varying broadcast channel. More recent results on the arbitrarily varying broadcast channel are derived e.g. in [30, 22].

Additional models of interest involve SI available at the encoder. In [4], Ahlswede addressed the AVC with non-causal SI available at the encoder, also referred to as the arbitrarily varying Gel’fand-Pinsker model [19]. The analysis relies on a technique that Ahlswede developed, which is referred to as Ahlswede’s Robustification Technique [3, 4]. This technique was then used in [30], to establish the capacity region of the arbitrarily varying degraded broadcast channel with non-causal SI at the encoder. The AVC with causal SI is addressed in the book by Csiszár and Körner [13], while their approach is independent of Ahlswede’s work. A straightforward application of Ahlswede’s Robustification Technique fails to comply with the causality
In this work, we study two models, analyzed using a modified version of Ahlswede’s Robustification and Elimination Techniques [1, 2, 3, 4]. In particular, we adjust Ahlswede’s Robustification Technique, previously used for the case of non-causal SI, such that it would be applicable in the case of causal SI.

The first model considered in this work is the AVC with input and state constraints when causal SI is available at the encoder. We find lower and upper bounds on the random code capacity. Furthermore we find a lower bound on the deterministic code capacity, for an input constraint that is averaged over the messages. For the case where a state constraint is imposed on the jammer, while the user is under no constraints, the random code bounds coincide, and the random code capacity is determined. In this scenario, a generalized non-symmetrizability condition is stated, under which the deterministic code capacity coincides with the random code capacity.

The second model considered in this work is the arbitrarily varying degraded broadcast channel with causal SI at the encoder (without constraints). Inner and outer bounds on the random code capacity region and the deterministic code capacity region are established. Specifically, Jahn’s inner bound [23] and the dichotomy property are extended to the case where causal SI is available. Furthermore, we find an outer bound, and conditions on the broadcast channel under which the inner and outer bounds coincide and the capacity region is determined. As an example, we show that the condition holds for the arbitrarily varying binary symmetric broadcast channel, and we find the corresponding capacity region.

The manuscript is divided into two main parts. In Chapter 1, we treat the AVC with causal SI in the presence of input and state constraints. In Chapter 2, we treat the arbitrarily varying degraded broadcast channel with causal SI (without constraints).

Chapter 1

Causal Side Information and Constraints

In this chapter, we address the arbitrarily varying channel with causal side information available at the encoder, under input and state constraints.
1.1 Definitions and Previous Results

1.1.1 Notation

We use the following notation conventions throughout. Calligraphic letters $\mathcal{X}, \mathcal{S}, \mathcal{Y}, \ldots$ are used for finite sets. Lowercase letters $x, s, y, \ldots$ stand for constants and values of random variables, and uppercase letters $X, S, Y, \ldots$ stand for random variables. The distribution of a random variable $X$ is specified by a probability mass function (pmf) $P_X(x) = p(x)$ over a finite set $\mathcal{X}$. Let $\mathcal{P}(\mathcal{X})$ denote the set of all pmfs over $\mathcal{X}$.

We use $x^j = (x_1, x_2, \ldots, x_j)$ to denote a constant sequence, with $j \geq 1$. For a pair of integers $i$ and $j$, $1 \leq i \leq j$, we define the discrete interval $[i : j] = \{i, i+1, \ldots, j\}$. A random sequence $X^n$ and its distribution $P_{X^n}(x^n) = p^n(x^n)$ are defined accordingly.

1.1.2 Channel Description

A state-dependent discrete memoryless channel (DMC) $(\mathcal{X} \times \mathcal{S}, W_{Y|X,S}, \mathcal{Y})$ consists of finite input, state and output alphabets $\mathcal{X}, \mathcal{S}, \mathcal{Y}$, respectively, and a collection of conditional pmfs $p(y|x, s)$ over $\mathcal{Y}$. The channel is memoryless without feedback, and therefore $p(y^n|x^n, s^n) = \prod_{i=1}^n W_{Y|X,S}(y_i|x_i, s_i)$. The AVC is a DMC $W_{Y|X,S}$ with a state sequence of unknown distribution, not necessarily independent nor stationary. That is, $S^n \sim q^n(s^n)$ with an unknown joint pmf $q^n(s^n)$ over $\mathcal{S}^n$. In particular, $q^n(s^n)$ can give mass 1 to some state sequence $s^n$. For state-dependent channels with causal SI, the channel input at time $i \in [1 : n]$ may depend on the sequence of past and present states $s^i$. The AVC with causal SI is denoted by $W = \{W_{Y|X,S}\}$.

The compound channel is used as a tool in the analysis. Different models of compound channels are described in the literature. Here, the compound channel is a DMC with a discrete memoryless state, where the state distribution $q(s)$ is not known in exact, but rather belongs to a family of distributions $\mathcal{Q}$, with $\mathcal{Q} \subseteq \mathcal{P}(\mathcal{S})$. That is, the state sequence $S^n$ is independent and identically distributed (i.i.d.) according to $q(s)$, for some pmf $q \in \mathcal{Q}$. We note that this differs from the classical definition of the compound channel, as in [13], where the state is fixed throughout the transmission. The compound channel with causal SI is denoted by $W^\mathcal{Q}$.

1.1.3 Coding

We introduce some preliminary definitions, starting with the definitions of a deterministic code and a random code for the AVC $W$ with causal SI. Note that in general, the term ‘a code’, unless mentioned otherwise, refers to a deterministic code.

Definition 1 (Code). A $(2^{nR}, n)$ code for the AVC $W$ with causal SI consists of the following: a message set $[1 : 2^{nR}]$, where it is assumed throughout that $2^{nR}$ is an integer, a set of $n$ encoding functions $f_i : [1 : 2^{nR}] \times \mathcal{S}^i \rightarrow \mathcal{X}$, for $i \in [1 : n]$, and a decoding function $g : \mathcal{Y}^m \rightarrow [1 : 2^{nR}]$.

At time $i \in [1 : n]$, given a message $m \in [1 : 2^{nR}]$ and a sequence $s^i$, the encoder transmits $x_i = f_i(m, s^i)$. The codeword is then given by

$$x^n = f^n(m, s^n) \triangleq (f_1(m, s_1), f_2(m, s^2), \ldots, f_n(m, s^n)). \quad (1.1)$$
The decoder receives the channel output $y^n$, and finds an estimate of the message $\hat{m} = g(y^n)$. We denote the code by $\mathcal{C} = (f^n(\cdot, \cdot), g(\cdot))$.

We proceed now to coding schemes when using stochastic-encoder stochastic-decoder pairs with common randomness. The codes formed by these pairs are referred to as random codes, a.k.a. correlated codes [4].

**Definition 2 (Random code).** A $(2^{nR}, n)$ random code for the AVC $W$ consists of a collection of $(2^{nR}, n)$ codes $\{\mathcal{C}_\gamma = (f^n_{\gamma}, g_{\gamma})\}_{\gamma \in \Gamma}$, along with a probability distribution $\mu(\gamma)$ over the code collection $\Gamma$. We denote such a code by $\mathcal{C}^\Gamma = (\mu, \Gamma, \{\mathcal{C}_\gamma\}_{\gamma \in \Gamma})$.

Next, we write the definition of Shannon strategy coding with causal SI [27]. Though, we use a different formulation, as e.g. in [16] (see [16, Remark 7.8])

**Definition 3 (Shannon strategy code).** [27] A $(2^{nR}, n)$ Shannon strategy code for the AVC $W$ with causal SI is a $(2^{nR}, n)$ code with an encoder that is composed of an encoding strategy sequence $u^n : [1 : 2^{nR}] \rightarrow U^n$, an encoding function $\xi : U \times S \rightarrow X$, and a decoding function $g : Y^n \rightarrow [1 : 2^{nR}]$. The codeword is then given by

$$x^n = \xi^n(u^n(m), s^n) \triangleq \left[\xi(u_i(m), s_i)\right]_{i=1}^n. \quad (1.2)$$

We denote such a code by $\mathcal{C} = (u^n(\cdot), \xi(\cdot, \cdot), g(\cdot))$.

The definitions above apply to the compound channel $W^Q$ as well.

### 1.1.4 Input and State Constraints

Next, we consider input and state constraints. Let $\phi : \mathcal{X} \rightarrow [0, \infty)$ and $l : \mathcal{S} \rightarrow [0, \infty)$ be some given bounded functions, and define

$$\phi^n(x^n) = \frac{1}{n} \sum_{i=1}^n \phi(x_i), \quad (1.3)$$

and

$$l^n(s^n) = \frac{1}{n} \sum_{i=1}^n l(s_i). \quad (1.4)$$

Let $\Omega > 0$ and $\Lambda > 0$. Below, we specify input constraint $\Omega$ and state constraint $\Lambda$, corresponding to the functions $\phi^n(x^n)$ and $l^n(s^n)$, respectively, for the AVC and the compound channel with causal SI.

We may assume without loss of generality that $0 \leq \Omega \leq \phi_{\text{max}}$ and $0 \leq \Lambda \leq l_{\text{max}}$, where $\phi_{\text{max}} = \max_{x \in \mathcal{X}} \phi(x)$ and $l_{\text{max}} = \max_{s \in \mathcal{S}} l(s)$. It is also assumed that for some $x_0 \in \mathcal{X}$ and $s_0 \in \mathcal{S}$, $\phi(x_0) = l(s_0) = 0$.

**State Constraints**

State constraints are imposed on the compound channel $W^Q$ and the AVC $W$ with causal SI, as specified below. Given some $\Lambda > 0$, define a set of constrained single-letter state distributions,

$$\overline{\mathcal{P}}_\Lambda(\mathcal{S}) \triangleq \{q(s) \in \mathcal{P}(\mathcal{S}) : \mathbb{E}_q l(S) \leq \Lambda\}, \quad (1.5)$$
and a set of constrained \( n \)-fold state distributions,
\[
\mathcal{P}_\Lambda^n(S^n) \triangleq \{ q^n(s^n) \in \mathcal{P}^n(S^n) : q^n(s^n) = 0 \text{ if } l^n(s^n) > \Lambda \}.
\] (1.6)

The set \( \overline{\mathcal{P}}_\Lambda(S) \) represents a state constraint on average, whereas the set \( \mathcal{P}_\Lambda^n(S^n) \) represents a state constraint held almost surely.

We say that a compound channel \( \mathcal{W}^Q \) with causal SI is under a state constraint \( \Lambda \), if the set \( Q \) of state distributions is limited to
\[
Q \subseteq \overline{\mathcal{P}}_\Lambda(S) .
\] (1.7)

As for the AVC \( \mathcal{W} \) with causal SI, it is now assumed that \( l^n(S^n) \leq \Lambda \) w.p. 1, i.e.
\[
q^n(s^n) \in \mathcal{P}^n_\Lambda(S^n) .
\] (1.8)

**Input Constraints**

Consider the AVC \( \mathcal{W} \) with causal SI, under an input constraint as specified below. Attention should be drawn to the fact that, when SI is available and the channel input depends on the state sequence \( S^n \), the input cost depends on the jammer’s strategy \( q^n(s^n) \) as well.

We consider two types of input constraints. We say that the AVC \( \mathcal{W} \) with causal SI is under per message input constraint \( \Omega \), if
\[
\sum_{s^n \in S^n} q^n(s^n)\phi^n(f^n(m, s^n)) \leq \Omega ,
\] for all \( m \in [1 : 2^nR] \) and \( q^n(s^n) \in \mathcal{P}^n_\Lambda(S^n) \).

As for the second type, we say that the AVC \( \mathcal{W} \) with causal SI is under average input constraint \( \Omega \), if
\[
\frac{1}{2^nR} \sum_{m=1}^{2^nR} \sum_{s^n \in S^n} q^n(s^n)\phi^n(f^n(m, s^n)) \leq \Omega ,
\] for all \( q^n(s^n) \in \mathcal{P}^n_\Lambda(S^n) \).

As for the second type, we say that the AVC \( \mathcal{W} \) with causal SI is under average input constraint \( \Omega \), if
\[
\frac{1}{2^nR} \sum_{m=1}^{2^nR} \sum_{s^n \in S^n} q^n(s^n)\phi^n(f^n(m, s^n)) \leq \Omega ,
\] for all \( q^n(s^n) \in \mathcal{P}^n_\Lambda(S^n) \).

Input constraint on the compound channel \( \mathcal{W}^Q \) with causal SI is defined in a similar manner, where (1.9) is taken with respect to i.i.d. state distributions \( q^n(s^n) = \prod_{i=1}^n q(s_i) \), with \( q \in Q \).

**1.1.5 Capacity under Constraints**

We move to the definition of an achievable rate and the capacity of the AVC \( \mathcal{W} \) with causal SI, under input and state constraints. Deterministic codes and random codes over the AVC \( \mathcal{W} \) with causal SI are defined as in Definition 1 and Definition 2, respectively, with the additional constraint (1.9a) or (1.9b) on the codebook.
Define the conditional probability of error of a code $\mathcal{C}$ given a state sequence $s^n \in S^n$ by

$$P_{e|s^n}(\mathcal{C}) \triangleq \frac{1}{2^{nR}} \sum_{m=1}^{2^nR} \sum_{y^n: g(y^n) \neq m} W_{Y^n|X^n, S^n}(y^n | f^n(m, s^n), s^n), \quad (1.10a)$$

where $W_{Y^n|X^n, S^n}(y^n | x^n, s^n) = \prod_{i=1}^{n} W_{Y_i|X_i, s_i}(y_i | x_i, s_i)$. Now, define the average probability of error of $\mathcal{C}$ for some distribution $q^n(s^n) \in \mathcal{P}^n(S^n)$,

$$P_e^{(n)}(q^n, \mathcal{C}) \triangleq \sum_{s^n \in S^n} q^n(s^n) \cdot P_{e|s^n}(\mathcal{C}). \quad (1.10b)$$

**Definition 4** (Achievable rate and capacity under constraints). A code $\mathcal{C} = (f^n, g)$ is a called a $(2^nR, n, \varepsilon)$ code for the AVC $W$, under per message input constraint $\Omega$ and state constraint $\Lambda$, when (1.9a) is satisfied and

$$P_e^{(n)}(q^n, \mathcal{C}) \leq \varepsilon, \quad \text{for all } q^n \in \mathcal{P}_\Lambda^n(S^n). \quad (1.11)$$

We say that a rate $R$ is achievable under per message input constraint $\Omega$ and state constraint $\Lambda$, if for every $\varepsilon > 0$ and sufficiently large $n$, there exists a $(2^nR, n, \varepsilon)$ code for the AVC $W$ under per message input constraint $\Omega$ and state constraint $\Lambda$. The operational capacity is defined as the supremum of all achievable rates, and it is denoted by $C_{\Omega, \Lambda}(W)$. We use the term ‘capacity’ referring to this operational meaning, and in some places we call it the deterministic code capacity in order to emphasize that achievability is measured with respect to deterministic codes.

Analogously to the deterministic case, a $(2^nR, n, \varepsilon)$ random code $\mathcal{C}^{\Gamma} = (\mu, \Gamma, \{\mathcal{C}_\gamma\}_{\gamma \in \Gamma})$ for the AVC $W$, under per message input constraint $\Omega$ and state constraint $\Lambda$, satisfies the requirements

$$\sum_{\gamma \in \Gamma} \mu(\gamma) \left[ \sum_{s^n \in S^n} q^n(s^n) \phi^n(f^n_{\gamma}(m, s^n)) \right] \leq \Omega, \quad \text{for all } m \in [1 : 2^nR], \quad q^n \in \mathcal{P}_\Lambda^n(S^n), \quad (1.12a)$$

and

$$P_e^{(n)}(q^n, \mathcal{C}^{\Gamma}) \triangleq \sum_{\gamma \in \Gamma} \mu(\gamma) \left[ \sum_{s \in S} q^n(s^n) \cdot P_{e|s^n}(\mathcal{C}_\gamma) \right] \leq \varepsilon, \quad \text{for all } q^n \in \mathcal{P}_\Lambda^n(S^n). \quad (1.12b)$$

The capacity achieved by random codes is then denoted by $C_{\Omega, \Lambda}^*(W)$, and it is referred to as the random code capacity.

The definitions above are naturally extended to the compound channel under per message input constraint $\Omega$ and state constraint $\Lambda$, by relaxing the requirements (1.9a), (1.11) and (1.12) to i.i.d. state distributions $q \in Q$. The respective deterministic code capacity and random code capacity, $C_{\Omega, \Lambda}(WQ)$ and $C_{\Omega, \Lambda}^*(WQ)$, are defined accordingly. Furthermore, similar definitions apply to the average input constraint, taking an average over the messages, as in (1.9b). Hence, the deterministic code capacities $\overline{C}_{\Omega, \Lambda}(W), \overline{C}_{\Omega, \Lambda}(WQ)$ and the random code capacities $\overline{C}_{\Omega, \Lambda}^*(W), \overline{C}_{\Omega, \Lambda}^*(WQ)$ are defined accordingly.
1.1.6 In the Absence of Side Information

In this subsection, we briefly review known results for the case where the state is not known to the encoder or the decoder, i.e. SI is not available. For the sake of brevity, we skip the compound channel. Then, consider an AVC without SI, which we denote by $W_0$.

Without Constraints

We begin with the case where there are no constraints, i.e. $\Omega = \phi_{max}$ and $\Lambda = l_{max}$. Then, the subscript ‘$\Omega, \Lambda$’ in the capacity notation is not necessary, and thus omitted.

We cite the random code capacity theorem of the AVC without SI, free of constraints, which was first introduced by Blackwell et al. [8]. Let

$$C^*(W_0) \triangleq \max_{p(x)} \min_{q(s)} I_q(X;Y) = \min_{q(s)} \max_{p(x)} I_q(X;Y).$$

(*1.13*)

Theorem 1. [8] The random code capacity of an AVC $W_0$ without SI, free of constraints, is given by

$$C^*(W_0) = C^*(W_0).$$

(*1.14*)

We note that the expression in (*1.13*) has a game-theoretic minimax interpretation [8, 11, 21, 24]. Now, a well-known result by Ahlswede [1] says that the deterministic code capacity $C(W_0)$ is characterized by the following dichotomy.

Theorem 2 (Ahlswede’s Dichotomy). [1] The capacity of an AVC $W_0$ without SI, free of constraints, either coincides with the random code capacity or else, it is zero. That is, $C(W_0) = C^*(W_0)$ or else, $C(W_0) = 0$.

A necessary and sufficient condition for a positive capacity was established by Ericson [18] and Csiszár and Narayan [15], in terms of the following definition.

Definition 5. A state-dependent DMC $W_{Y|X,S}$ is said to be symmetrizable if for some conditional distribution $J(s|x)$,

$$\sum_{s \in S} W_{Y|X,S}(y|x_1, s) J(s|x_2) = \sum_{s \in S} W_{Y|X,S}(y|x_2, s) J(s|x_1),$$

$$\forall x_1, x_2 \in \mathcal{X}, y \in \mathcal{Y}.$$  (**1.15**)

Equivalently, the channel $\tilde{W}(y|x_1, x_2) = \sum_{s \in S} W_{Y|X,S}(y|x_1, s) J(s|x_2)$ is symmetric, i.e. $\tilde{W}(y|x_1, x_2) = \tilde{W}(y|x_2, x_1)$, for all $x_1, x_2 \in \mathcal{X}$ and $y \in \mathcal{Y}$. We say that such a $J: \mathcal{X} \rightarrow \mathcal{S}$ symmetrizes $W_{Y|X,S}$. We say that the AVC $W_0$ is symmetrizable if the corresponding state-dependent DMC $W_{Y|X,S}$ is symmetrizable.

Theorem 3. [18, 15] An AVC $W_0$ without SI, free of constraints, has a positive capacity $C(W_0) > 0$ if and only if it is not symmetrizable.
Under Constraints

Csiszár and Narayan addressed the AVC $\mathcal{W}_0$ without SI under constraints in [14] and [15]. The focus here is on the case of per message input constraint, although their results apply to the average case as well. Let

$$C_{\Omega, \Lambda}^*(\mathcal{W}_0) \triangleq \min_{q(s) \in \overline{P}_\Lambda(S)} \max_{p(x) : \mathbb{E}\phi(X) \leq \Omega} I_q(X; Y) , \quad (1.16)$$

where $\overline{P}_\Lambda(S)$ is defined in (1.5).

**Theorem 4.** [14] The random code capacity of an AVC $\mathcal{W}_0$ without SI, under per message input constraint $\Omega$ and state constraint $\Lambda$, is given by

$$C_{\Omega, \Lambda}^*(\mathcal{W}_0) = C_{\Omega, \Lambda}^*(\mathcal{W}_0) . \quad (1.17)$$

As for the deterministic code capacity, dichotomy in the classical notion of [1] no longer holds when a state constraint $\Lambda < l_{\max}$ is imposed on the jammer [15]. That is, the capacity of the AVC $\mathcal{W}_0$ can be strictly lower than the random code capacity, and yet non-zero.

For every $p \in \mathcal{P}(X)$ with $\mathbb{E}\phi(X) \leq \Omega$, let

$$\tilde{\Lambda}_0(p) = \min \sum_{x \in X} \sum_{s \in S} p(x)J(s|x)l(s) , \quad (1.18)$$

where the minimization is over all conditional distributions $J(s|x)$ that symmetrize $W_{Y|X,S}$ (see Definition 5). We use the convention that a minimum value over an empty set is $+\infty$. Assume that $\max_{p(x) : \mathbb{E}\phi(X) \leq \Omega} \tilde{\Lambda}_0(p) \neq \Lambda$.

Then, define $C_{\Omega, \Lambda}(\mathcal{W}_0)$ as follows,

$$C_{\Omega, \Lambda}(\mathcal{W}_0) \triangleq 0 , \quad \text{if} \quad \max_{p(x) : \mathbb{E}\phi(X) \leq \Omega} \tilde{\Lambda}_0(p) < \Lambda , \quad (1.19a)$$

and

$$C_{\Omega, \Lambda}(\mathcal{W}_0) \triangleq \max_{p(x) : \mathbb{E}\phi(X) \leq \Omega} \min_{\tilde{\Lambda}_0(p) \geq \Lambda} \max_{q(s) : \mathbb{E}l(s) \leq \Lambda} I_q(X; Y) > 0 ,$$

if $\max_{p(x) : \mathbb{E}\phi(X) \leq \Omega} \tilde{\Lambda}_0(p) > \Lambda . \quad (1.19b)$

**Theorem 5.** [14] The capacity of an AVC $\mathcal{W}_0$ without SI, under per message input constraint $\Omega$ and state constraint $\Lambda$, is given by

$$C_{\Omega, \Lambda}(\mathcal{W}_0) = C_{\Omega, \Lambda}(\mathcal{W}_0) , \quad \text{if} \quad \max_{p(x) : \mathbb{E}\phi(X) \leq \Omega} \tilde{\Lambda}_0(p) \neq \Lambda . \quad (1.20)$$

In particular, if $\mathcal{W}_0$ is non-symmetrizable, $C_{\Omega, \Lambda}(\mathcal{W}_0) = C_{\Omega, \Lambda}^*(\mathcal{W}_0)$. 


1.1.7 In The Presence of Side Information

In this subsection, we briefly review known results for the case where the state is known to the encoder, and no constraints are imposed. The compound channel and the AVC with non-causal SI, free of constraints, were addressed by Ahlswede in [4].

The AVC with causal SI, free of constraints, was addressed in the problem set of the book by Csiszár and Körner [13, Problem 12.18, part (b)]. The corresponding results are stated below. Let

\[ C^*(W) \triangleq \min_{q \in \mathcal{P}(S)} \max_{p(u), \xi(u,s)} I_q(U; Y) , \]

subject to \( X = \xi(U, S) \), where \( U \) is an auxiliary random variable, independent of \( S \), and the maximization is over the pmf \( p(u) \) and the set of all functions \( \xi : U \times S \to X \).

Theorem 6. [13] The random code capacity of the AVC \( W \) with causal SI available at the encoder, free of constraints, is given by

\[ C^*(W) = C^*(W) . \] (1.22)

Theorem 7. [13] The capacity of an AVC \( W \) with causal SI at the encoder, free of constraints, either coincides with the random code capacity or else, it is zero. That is, \( C(W) = C^*(W) \) or else, \( C(W) = 0 \).

This completes our review of previous work, where SI and constraints were considered in separate. Next, we give our results, concerning the combined setting, where SI is available and constraints are imposed.

1.2 Results

1.2.1 The Compound Channel with Causal SI

We present a lower bound on the capacity of the compound channel with causal SI, under per message input constraint \( \Omega \), taking the set of state distributions to be \( \mathcal{Q} = \overline{\mathcal{P}}_\Lambda(S) \). For a given mapping \( \xi : U \times S \to X \), let

\[ \mathcal{P}^*_{\overline{\mathcal{P}}_\Lambda, \xi}(U) \triangleq \left\{ p \in \mathcal{P}(U) : \mathbb{E}_q \phi(\xi(U, S)) \leq \Omega, \text{ for all } q \in \overline{\mathcal{P}}_\Lambda(S) \right\} , \]

where \( (U, S) \sim p(u) \cdot q(s) \). Then, define

\[ R_{low, \Omega, \Lambda}(W) \triangleq \min_{q(s) \in \mathcal{P}^*_{\overline{\mathcal{P}}_\Lambda, \xi}(U)} \max_{\xi \in \mathcal{P}(S)} \max_{p(u) \in \mathcal{P}^*_{\overline{\mathcal{P}}_\Lambda, \xi}(U)} I_q(U; Y) , \]

and

\[ R_{up, \Omega, \Lambda}(W) \triangleq \min_{q(s) \in \mathcal{P}^*_{\overline{\mathcal{P}}_\Lambda, \xi}(U)} \max_{\xi \in \mathcal{P}(S)} \max_{p(u) : \mathbb{E}_q \phi(\xi(U, S)) \leq \Omega} I_q(U; Y) . \]

Observe that \( R_{low, \Omega, \Lambda}(W) \leq R_{up, \Omega, \Lambda}(W) \), since the maximization constraint in (1.24) is taken for all \( q \in \overline{\mathcal{P}}_\Lambda(S) \) (see (1.23)), while the maximization constraint in (1.25) is taken for a particular \( q \in \overline{\mathcal{P}}_\Lambda(S) \).
Lemma 8. Let $W^{P_A(S)}$ be a compound channel with causal SI available at the encoder, under per message input constraint $\Omega$ and state constraint $\Lambda$. The random code capacity and the deterministic code capacity of $W^{P_A(S)}$ are bounded by

\[
\begin{align*}
\mathbb{C}_{\Omega,\Lambda}(W^{P_A(S)}) &\geq R_{\text{low},\Omega,\Lambda}^*(W), \\
\mathbb{C}_{\Omega,\Lambda}^*(W^{P_A(S)}) &\leq R_{\text{up},\Omega,\Lambda}^*(W).
\end{align*}
\]

Furthermore, if $R < R_{\text{low},\Omega,\Lambda}^*(W)$, then for some $a > 0$ and sufficiently large $n$, there exists a $(2^{nR}, n, e^{-an})$ Shannon strategy code over $W^{P_A(S)}$, under per message input constraint $\Omega$.

The proof of Lemma 8 is given in Appendix A.1. It can further be shown that if $\mathbb{C}_{\Omega,\Lambda}(W^{P_A(S)}) > 0$, then $\mathbb{C}_{\Omega,\Lambda}^*(W^{P_A(S)}) = \mathbb{C}_{\Omega,\Lambda}(W^{P_A(S)}) = R_{\text{up},\Omega,\Lambda}^*(W)$. However, this will not be needed here.

1.2.2 The AVC with Causal SI

Random Code Capacity

We give lower and upper bounds on the random code capacity of the AVC $W$ with causal SI under input and state constraints.

We begin with a lemma, which is a restatement of Ahlswede's Robustification Technique (RT) [4] with some modification.

Lemma 9 (Ahlswede’s RT). [4] Let $h : S^n \to [0, 1]$ be a given function. If, for some fixed $\alpha_n \in (0, 1)$, and for all $q^n(s^n) = \prod_{i=1}^n q(s_i)$, with $q \in P_A(S)$,

\[
\sum_{s^n \in S^n} q^n(s^n) h(s^n) \leq \alpha_n,
\]

then,

\[
\frac{1}{n!} \sum_{\pi \in \Pi_n} h(\pi s^n) \leq \beta_n, \quad \text{for all } s^n \in S^n \text{ such that } l^n(s^n) \leq \Lambda,
\]

where $\Pi_n$ is the set of all $n$-tuple permutations $\pi : S^n \to S^n$, and $\beta_n = (n+1)^{|S|} \cdot \alpha_n$.

Originally, Ahlswede’s RT is stated so that (1.27) holds for any $q(s) \in P(S)$, without state constraint (see [4]), but the claim holds also when state constraints are imposed, as here. For completeness, we give the proof of Lemma 9 in Appendix A.2.

Theorem 10. Let $W$ be an AVC with causal SI available at the encoder, under per message input constraint $\Omega$ and state constraint $\Lambda$. Then,

1) the random code capacity of $W$ is bounded by

\[
R_{\text{low},\Omega,\Lambda}^*(W) \leq \mathbb{C}_{\Omega,\Lambda}^*(W) \leq R_{\text{up},\Omega,\Lambda}^*(W),
\]

where $R_{\text{low},\Omega,\Lambda}^*(W)$ and $R_{\text{up},\Omega,\Lambda}^*(W)$ are given by (1.24) and (1.25), respectively.
2) For $\Omega = \phi_{\text{max}}$, i.e. when free of input constraints, the random code capacity of $W$ is given by

$$C_{\Omega,\Lambda}(W) = R^*_{\text{low},\Omega,\Lambda}(W) = R^*_{\text{up},\Omega,\Lambda}(W).$$

(1.30)

Theorem 10 is proved in Appendix A.3. We further note that the result above holds when the input constraint is averaged over the message set as well. The following lemma is the counterpart of a result from [1], stating that a polynomial size of the code collection $\{\mathcal{C}_\gamma\}$ is sufficient. This result is a key observation in Ahlswede’s Elimination Technique (ET), presented in [1], where it is used as a basis for the deterministic code analysis. Here, it will be used to determine a condition under which the deterministic code capacity is identical to the random code capacity of the AVC with causal SI under a state constraint.

Lemma 11. Let $R < C_{\Omega,\Lambda}(W)$. Consider a given $(2^{nR}, n, \varepsilon_n)$ random code $\mathcal{C}^\Gamma = (\mu, \Gamma, \{\mathcal{C}_\gamma\}_{\gamma \in \Gamma})$ for the AVC $W$ with causal SI, under per message input constraint $\Omega$ and state constraint $\Lambda$, where $\lim_{n \to \infty} \varepsilon_n = 0$. Then, for every $\delta > 0$, $0 < \alpha < 1$, and sufficiently large $n$, there exists a $(2^{nR}, n)$ random code $(\mu^*, \Gamma^*, \{\mathcal{C}_\gamma\}_{\gamma \in \Gamma^*})$ such that for all $m \in [1 : 2^{nR}]$ and $q^n \in \mathcal{P}_\Lambda(S^n),$

$$\sum_{\gamma \in \Gamma^*} \mu^*(\gamma) \sum_{s^n \in S^n} q^n(s^n) \phi^n(f^n(\gamma(m, s^n))) \leq \Omega + \alpha,$$

(1.31)

$$P_e^n(q^n, \mathcal{C}^\Gamma) \leq \delta,$$

(1.32)

with the following properties:

1. The size of the code collection is bounded by $|\Gamma^*| \leq n^2$.

2. The code collection is a subset of the original code collection, i.e. $\Gamma^* \subseteq \Gamma$.

3. The distribution $\mu^*$ is uniform, i.e. $\mu^*(\gamma) = \frac{1}{|\Gamma^*|}$ for $\gamma \in \Gamma^*$.

The proof of Lemma 11 is given in Appendix A.4.

**Deterministic Code Capacity**

Here, we consider the AVC with causal SI, under average input constraint $\Omega$ and state constraint $\Lambda$. We establish a lower bound on the capacity for this setting, and we find a condition under which the deterministic code capacity coincides with the random code capacity for the setting where the jammer is under a state constraint while the user is free of constraints. For every encoding mapping $\xi(u, s)$, define an AVC $V_{0}^\xi = \{V_{Y|U,S}^\xi\}$ without SI specified by $V_{Y|U,S}^\xi(y|u, s) = W_{Y|X,S}(y|\xi(u, s), s)$.

Given a function $\xi: U \times S \to X$ and a distribution $p \in \mathcal{P}(U)$, define

$$\tilde{\Lambda}(p, \xi) = \min \sum_{u \in U} \sum_{s \in S} p(u) J(s|u) l(s),$$

(1.33)
where the minimization is over all conditional distributions $J(s|u)$ that symmetrize $V^\xi_{Y|U,S}$ (see Definition 5). Assume that
\[
\max_{p(u)} \tilde{\Lambda}(p, \xi) \neq \Lambda, \text{ for all } \xi : U \times S \to \mathcal{X}. \tag{1.34}
\]
For every $\xi(u, s)$, define the following set. If $V^\xi_{Y|U,S}$ is symmetrizable, define
\[
P_{\Omega, \Lambda, \xi}(U) \triangleq \left\{ p \in \mathcal{P}(U) : \mathbb{E}_q \phi(\xi(U, S)) \leq \Omega, \text{ for all } q \in \mathcal{P}_\Lambda(S), \text{ and } \right. \}
\[
\sum_{u \in U} \sum_{s \in S} p(u)J(s|u)l(s) > \Lambda \right\}, \tag{1.35a}
\]
and if $V^\xi_{Y|U,S}$ is non-symmetrizable,
\[
P_{\Omega, \Lambda, \xi}(U) \triangleq \left\{ p \in \mathcal{P}(U) : \mathbb{E}_q \phi(\xi(U, S)) \leq \Omega, \text{ for all } q \in \mathcal{P}_\Lambda(S) \right\}, \tag{1.35b}
\]
where $(U, S) \sim p(u) \cdot q(s)$.

The intuition behind the definition of $P_{\Omega, \Lambda, \xi}(U)$ above can be explained as follows. For a symmetrizable $V^\xi_{Y|U,S}$, the set defined in (1.35a) consists of distributions $p(u)$ such that every jamming strategy $J(s|u)$, which symmetrizes $V^\xi_{Y|U,S}$, violates the state constraint. That is, $P_{\Omega, \Lambda, \xi}(U)$ consists of distributions for which the jammer is prohibited from using symmetrizing state strategies.

Then, let
\[
R_{\Omega, \Lambda}(\mathcal{W}) \triangleq \min_{q(s) \in \mathcal{P}_\Lambda(S)} \max_{\xi : U \times S \to \mathcal{X}} \max_{p(u) \in P_{\Omega, \Lambda, \xi}(U)} I_q(U; Y). \tag{1.36}
\]
Observe that $R_{\Omega, \Lambda}(\mathcal{W}) \leq R^*_\Omega(\mathcal{W})$ (cf. (1.24) and (1.36)).

Theorem 12. Let $\mathcal{W}$ be an AVC with causal SI, under average input constraint $\Omega$ and state constraint $\Lambda$. Suppose that (1.34) holds. Then,

1) the capacity of $\mathcal{W}$ is lower bounded by
\[
\overline{C}_{\Omega, \Lambda}(\mathcal{W}) \geq R_{\Omega, \Lambda}(\mathcal{W}). \tag{1.37}
\]

2) For $\Omega = \phi_{\max}$, i.e. when free of input constraints, if there exists a function $\xi : U \times S \to \mathcal{X}$, such that $V^\xi_{Y|U,S}$ is non-symmetrizable, the deterministic code capacity is identical to the random code capacity, i.e. $C_{\Omega, \Lambda}(\mathcal{W}) = C^*_\Omega(\mathcal{W}) > 0$, and it is given by
\[
C_{\Omega, \Lambda}(\mathcal{W}) = R^*_\Omega(\mathcal{W}) = R^*_{up, \Omega, \Lambda}(\mathcal{W}). \tag{1.38}
\]

The proof of Theorem 12 is given in Appendix A.5.
1.3 Example

To illustrate our results, we consider the following example of an AVC with causal SI, under a state constraint.

Example 1. Consider an arbitrarily varying noisy-typewriter channel, defined by

$$Y = X + Z \mod 3,$$  \hspace{1cm} (1.39a)

where $X = Z = Y = \{0, 1, 2\}$. The additive noise is defined by $Z = K \cdot S$, with $S \in \{1, 2\}$, and

$$K \sim \text{Bernoulli}(\theta), \ \theta > 0.$$  \hspace{1cm} (1.39b)

Thus, $S$ chooses among two noisy-typewriter DMCs [12]. The channel is under a state constraint $\Lambda$, with

$$l(s) = \begin{cases} 0 & \text{if } s = 1, \\ 1 & \text{if } s = 2. \end{cases}$$  \hspace{1cm} (1.40)

We have the following results. The capacity of the arbitrarily varying noisy-typewriter channel $\mathcal{W}_0$ without SI, under a state constraint $\Lambda$, is given by

$$C_{\phi_{\text{max}},\Lambda}(\mathcal{W}_0) = \begin{cases} \log 3 - h(\theta) - \theta h(\Lambda) & \text{if } 0 \leq \Lambda \leq \frac{1}{2}, \\ \log 3 - h(\theta) - \theta & \text{if } \Lambda \geq \frac{1}{2}, \end{cases}$$  \hspace{1cm} (1.41)

for all $\theta > 0$. The capacity of the arbitrarily varying noisy-typewriter $\mathcal{W}$ with causal

Figure 1.1: The capacity of the arbitrarily varying noisy-typewriter channel as a function of the transition parameter $\theta$. The dashed lines correspond to the capacity $C_{\Omega,\Lambda}(\mathcal{W}_0)$ of the AVC without SI, and the solid lines correspond to the capacity $C_{\Omega,\Lambda}(\mathcal{W})$ of the AVC with causal SI. Each line corresponds to a state constraint $\Lambda = 0, 0.15, 0.25$ and $\Lambda \geq 0.5$, from top to bottom. As the state constraint $\Lambda$ increases, the capacity decreases.
SI, under a state constraint $\Lambda$, is given by

$$C_{\phi_{\text{max}}, \Lambda}(W) = \begin{cases} 
\log 3 - \min (h(\theta) + \theta h(\Lambda), h(\theta) + (1 - \theta)h(\Lambda), h(\theta \ast \Lambda)) & \text{if } 0 \leq \Lambda < \frac{1}{2}, \\
\log 3 - \min (h(\theta) + \theta, h(\theta) + (1 - \theta), 1) & \text{if } \Lambda \geq \frac{1}{2}.
\end{cases}$$

(1.42)

The proof of these results is given in Appendix A.6. Figure 1.1 depicts the capacity of the arbitrarily varying noisy-typewriter channel, as a function of the parameter $\theta$. The dashed lines correspond to the case where there is no SI, and the solid lines correspond to the case where causal SI is available at the encoder. Since $W_{Y|X,S}$ is symmetrizable if and only $\theta = \frac{2}{3}$, the capacity without SI and without constraints is zero only for this value. This is equivalent to a modulo-additive DMC $Y = X + Z \mod 3$ where the noise $Z$ is uniform. On the other hand, with causal SI, the capacity is symmetric around $\theta = \frac{1}{2}$, which resembles the behavior of a BSC, as $K \sim \text{Bernoulli}(\theta)$. Choosing the encoding function $\xi(u, s) = u \cdot s \mod 3$, with $U = \{0, 1, 2\}$, we find that the DMC $V_{Y|U,S}^\xi$ is non-symmetrizable for all $\theta > 0$, thus the capacity of the arbitrarily varying noisy-typewriter with causal SI is positive. Furthermore, the capacity is bounded by $C_{\phi_{\text{max}}, \Lambda=1}(W) \leq C_{\phi_{\text{max}}, \Lambda}(W) \leq C_{\phi_{\text{max}}, \Lambda=0}(W)$, where

$$C_{\phi_{\text{max}}, \Lambda=1}(W) \geq \log 3 - 1 = \log \left(\frac{|X|}{2}\right),$$

(1.43)

$$C_{\phi_{\text{max}}, \Lambda=0}(W) = \log 3 - h(\theta),$$

(1.44)

by (1.42). The lower bound $\log \left(\frac{|X|}{2}\right)$ is the capacity of the standard noisy-typewriter DMC, with $\theta = \frac{1}{2}$. The upper bound (1.44) is the capacity when the state is known to both the encoder and the receiver.

In conclusion of this chapter, we have established lower and upper bounds on the random code capacity, for the single-user AVC with causal SI at the encoder, under input and state constraints. We have then established a lower bound on the deterministic code capacity, for the AVC with causal SI at the encoder, under a state constraint and free of input constraint. For this case, we have also stated a condition under which the deterministic code capacity coincides with the random
code capacity. The next chapter deals with a multiple-user scenario.

Chapter 2

The Arbitrarily Varying Degraded Broadcast Channel

In this chapter, we address the arbitrarily varying degraded broadcast channel with causal SI available at the encoder. It is assumed that there are no constraints.

2.1 Definitions and Previous Results

2.1.1 Channel Description

A state-dependent discrete memoryless broadcast channel \((X \times S, W_{Y_1,Y_2|X,S}: Y_1, Y_2)\) consists of a finite input alphabet \(X\), two finite output alphabets \(Y_1\) and \(Y_2\), a finite state alphabet \(S\), and a collection of conditional pmfs \(p(y_1, y_2|x, s)\) over \(Y_1 \times Y_2\). The channel is memoryless without feedback, and therefore \(p(y_{1,n}, y_{2,n}|x^n, s^n) = \prod_{i=1}^n W_{Y_1,Y_2|X,S}(y_{1,i}, y_{2,i}|x_i, s_i)\). The marginals \(W_{Y_1|X,S}\) and \(W_{Y_2|X,S}\) correspond to user 1 and user 2, respectively. For state-dependent broadcast channels with causal SI, the channel input at time \(i \in [1:n]\) may depend on the sequence of past and present states \(s^i\).

Throughout this chapter, we assume that \(W_{Y_1,Y_2|X,S}\) is a degraded broadcast channel (DBC). Following the definitions by [28], a state-dependent broadcast channel \(W_{Y_1,Y_2|X,S}\) is said to be physically degraded if it can be expressed as

\[
W_{Y_1,Y_2|X,S}(y_1, y_2|x, s) = W_{Y_1|X,S}(y_1|x, s) \cdot p(y_2|y_1), \tag{2.1}
\]

i.e. \((X, S)\xrightarrow{Y_1} Y_2\) form a Markov chain. User 1 is then referred to as the stronger user, whereas user 2 is referred to as the weaker user. More generally, a broadcast channel is said to be stochastically degraded if \(W_{Y_2|X,S}(y_2|x, s) = \sum_{y_1 \in Y_1} W_{Y_1|X,S}(y_1|x, s) \cdot \tilde{p}(y_2|y_1)\) for some conditional distribution \(\tilde{p}(y_2|y_1)\). We note that the definition of degradedness in [23] is equivalent to the definition above when SI is not available, as assumed in [23]. Our results apply to both the physically degraded and the stochastically degraded broadcast channels. Thus, for our purposes, there is no need to distinguish between the two, and we simply say that the broadcast channel is degraded.
The arbitrarily varying degraded broadcast channel (AVDBC) is a discrete memoryless DBC \( W_{Y_1,Y_2|X,S} \) with a state sequence of unknown distribution, not necessarily independent nor stationary. That is, \( S^n \sim q^n(s^n) \) with an unknown joint pmf \( q^n(s^n) \) over \( S^n \). In particular, \( q^n(s^n) \) can give mass 1 to some state sequence \( s^n \). We denote the AVDBC with causal SI by \( \mathcal{B} = \{ W_{Y_1,Y_2|X,S} \} \).

To analyze the AVDBC with causal SI, we consider the compound degraded broadcast channel. Different models of a compound DBC have been considered in the literature, as e.g. in [29] and [6]. Here, we define the compound DBC as a discrete memoryless DBC with a discrete memoryless state, where the state distribution \( q(s) \) is not known in exact, but rather belongs to a family of distributions \( Q \), with \( Q \subseteq \mathcal{P}(S) \). That is, \( S^n \sim \prod_{i=1}^{n} q(s_i) \), with an unknown pmf \( q \in Q \) over \( S \). We denote the compound DBC with causal SI by \( \mathcal{B}^Q \).

2.1.2 Coding

We introduce some preliminary definitions, starting with the definitions of a deterministic code and a random code for the AVDBC \( \mathcal{B} \) with causal SI. Note that in general, the term ‘a code’, unless mentioned otherwise, refers to a deterministic code.

**Definition 6** (A code, an achievable rate pair and capacity region). A \((2^nR_1,2^nR_2,n)\) code for the AVDBC \( \mathcal{B} \) with causal SI consists of the following; two message sets \([1 : 2^nR_1]\) and \([1 : 2^nR_2]\), where it is assumed throughout that \( 2^nR_1 \) and \( 2^nR_2 \) are integers, a set of \( n \) encoding functions \( f_i : [1 : 2^nR_1] \times [1 : 2^nR_2] \times S^i \to \mathcal{X} \), \( i \in [1 : n] \), and two decoding functions, \( g_1 : \mathcal{Y}_1^n \to [1 : 2^nR_1] \) and \( g_2 : \mathcal{Y}_2^n \to [1 : 2^nR_2] \).

At time \( i \in [1 : n] \), given a pair of messages \( m_1 \in [1 : 2^nR_1] \) and \( m_2 \in [1 : 2^nR_2] \) and a sequence \( s^i \), the encoder transmits \( x_i = f_i(m_1, m_2, s^i) \). The codeword is then given by

\[
x^n = f^n(m_1, m_2, s^n) \triangleq (f_1(m_1, m_2, s_1), f_2(m_1, m_2, s^2), \ldots, f_n(m_1, m_2, s^n)) . \tag{2.2}
\]

Decoder 1 receives the channel output \( y^n_1 \), and finds an estimate of the first message \( \hat{m}_1 = g_1(y^n_1) \). Similarly, decoder 2 estimates the second message with \( \hat{m}_2 = g_2(y^n_2) \).

We denote the code by \( \mathcal{C} = (f^n(\cdot, \cdot), g_1(\cdot), g_2(\cdot)) \).

Define the conditional probability of error of \( \mathcal{C} \) given a state sequence \( s^n \in S^n \) by

\[
P^{(n)}_{e|s^n}(\mathcal{C}) = \frac{1}{2^{n(R_1+R_2)}} \sum_{m_1=1}^{2^nR_1} \sum_{m_2=1}^{2^nR_2} \sum_{(y^n_1,y^n_2) \in \mathcal{D}(m_1,m_2)} W_{Y_1^n,Y_2^n|X^n, S^n}(y^n_1,y^n_2|f^n(m_1, m_2, s^n), s^n) ,
\]

where

\[
\mathcal{D}(m_1,m_2) \triangleq \{ (y^n_1,y^n_2) \in \mathcal{Y}_1^n \times \mathcal{Y}_2^n : (g_1(y^n_1), g_2(y^n_2)) = (m_1, m_2) \} . \tag{2.4}
\]

Now, define the average probability of error of \( \mathcal{C} \) for some distribution \( q^n(s^n) \in \mathcal{P}^n(S^n) \),

\[
P^{(n)}_e(q^n, \mathcal{C}) = \sum_{s^n \in S^n} q^n(s^n) \cdot P^{(n)}_{e|s^n}(\mathcal{C}) . \tag{2.5}
\]
We say that \( \mathcal{C} \) is a \((2^{nR_1}, 2^{nR_2}, n, \varepsilon)\) code for the AVDBC \( \mathcal{B} \) if it further satisfies

\[
P_e^{(n)}(q^n, \mathcal{C}) \leq \varepsilon, \quad \text{for all } q^n(s^n) \in \mathcal{P}^n(S^n). \tag{2.6}
\]

We say that a rate pair \((R_1, R_2)\) is achievable if for every \(\varepsilon > 0\) and sufficiently large \(n\), there exists a \((2^{nR_1}, 2^{nR_2}, n, \varepsilon)\) code. The operational capacity region is defined as the closure of the set of achievable rate pairs and it is denoted by \(\mathbb{C}(\mathcal{B})\). We use the term ‘capacity region’ referring to this operational meaning, and in some places we call it the deterministic code capacity region in order to emphasize that achievability is measured with respect to deterministic codes.

We proceed now to define the parallel quantities when using stochastic-encoder stochastic-decoders triplets with common randomness. The codes formed by these triplets are referred to as random codes.

**Definition 7** (Random code). A \((2^{nR_1}, 2^{nR_2}, n)\) random code for the AVDBC \( \mathcal{B} \) consists of a collection of \((2^{nR_1}, 2^{nR_2}, n)\) codes \(\{\mathcal{C}_\gamma = (f^n_\gamma, g_1, g_2, \gamma)\}_{\gamma \in \Gamma}\), along with a probability distribution \(\mu(\gamma)\) over the code collection \(\Gamma\). We denote such a code by \(\mathcal{C}^\Gamma = (\mu, \Gamma, \{\mathcal{C}_\gamma\}_{\gamma \in \Gamma})\).

Analogously to the deterministic case, a \((2^{nR_1}, 2^{nR_2}, n, \varepsilon)\) random code has the additional requirement

\[
P_e^{(n)}(q^n, \mathcal{C}^\Gamma) = \sum_{\gamma \in \Gamma} \mu(\gamma) \sum_{s^n \in S^n} q^n(s^n) p_e^{(n)}(\mathcal{C}_\gamma) \leq \varepsilon, \quad \text{for all } q^n(s^n) \in \mathcal{P}^n(S^n). \tag{2.7}
\]

The capacity region achieved by random codes is denoted by \(\mathbb{C}^*(\mathcal{B})\), and it is referred to as the random code capacity region.

Next, we write the definition of superposition coding [7] using Shannon strategies [27]. See also [28], and the discussion after Theorem 4 therein. Here, we refer to such codes as Shannon strategy codes.

**Definition 8** (Shannon strategy codes). A \((2^{nR_1}, 2^{nR_2}, n)\) Shannon strategy code for the AVDBC \( \mathcal{B} \) with causal SI is a \((2^{nR_1}, 2^{nR_2}, n)\) code with an encoder that is composed of two strategy sequences

\[
u_1^n : [1 : 2^{nR_1}] \times [1 : 2^{nR_2}] \to \mathcal{U}_1^n, \tag{2.8}
\]

\[
u_2^n : [1 : 2^{nR_2}] \to \mathcal{U}_2^n, \tag{2.9}
\]

and an encoding function \(\xi(u_1, u_2, s)\), where \(\xi : \mathcal{U}_1 \times \mathcal{U}_2 \times \mathcal{S} \to \mathcal{X}\), as well as a pair of decoding functions \(g_1 : \mathcal{Y}_1^n \to [1 : 2^{nR_1}]\) and \(g_2 : \mathcal{Y}_2^n \to [1 : 2^{nR_2}]\). The codeword is then given by

\[
x^n = \xi^n(u_1^n(m_1, m_2), u_2^n(m_2), s^n) \triangleq \left[ \xi(u_1^n, m_1, m_2), u_2^n(m_2, s_i) \right]_{i=1}^n. \tag{2.10}
\]

We denote the code by \(\mathcal{C} = (u_1^n, u_2^n, \xi, g_1, g_2)\).
2.1.3 In the Absence of Side Information – Inner Bound

In this subsection, we briefly review known results for the case where the state is not known to the encoder or the decoder, i.e. SI is not available.

Consider a given AVDBC without SI, which we denote by $B_0$. Let

$$R_{0,in}^* \triangleq \bigcup_{p(x,u) \in \mathcal{P}(s)} \bigcap \left\{ (R_1, R_2) : R_2 \leq I_q(U_2; Y_2), R_1 \leq I_q(X; Y_1|U) \right\}$$

(2.11)

In [23, Theorem 2], Jahn introduced an inner bound for the arbitrarily varying general broadcast channel. In our case, where the broadcast channel is assumed to be degraded, Jahn’s inner bound reduces to the following.

Theorem 13 (Jahn’s Inner Bound). [23] Let $B_0$ be an AVDBC without SI. Then, $R_{0,in}^*$ is an achievable rate region using random codes over $B_0$, i.e.

$$C^*(B_0) \supseteq R_{0,in}^*.$$  (2.12)

Now we move to the deterministic code capacity region.

Theorem 14 (Ahlswede’s Dichotomy). [23] The capacity region of an AVDBC $B_0$ without SI either coincides with the random code capacity region or else, its interior is empty. That is, $C(B_0) = C^*(B_0)$ or else, $\text{int}(C(B_0)) = \emptyset$.

By Theorem 13 and Theorem 14, we have that $R_{0,in}^*$ is an achievable rate region, if the interior of the capacity region is non-empty. That is, $C(B_0) \supseteq R_{0,in}^*$, if $\text{int}(C(B_0)) \neq \emptyset$.

Theorem 15. [18, 15, 22] For an AVDBC $B_0$ without SI, the interior of the capacity region is non-empty, i.e. $\text{int}(C(B_0)) \neq \emptyset$, if and only if the marginal $W_{Y_2|X,S}$ is not symmetrizable.

2.2 Results

We present our results on the compound DBC and the AVDBC with causal SI.

2.2.1 The Compound DBC with Causal SI

We now consider the case where the encoder has access to the state sequence in a causal manner, i.e. the encoder has $S^i$.

Inner Bound

First, we provide an achievable rate region for the compound DBC with causal SI. Consider a given compound DBC $B^Q$ with causal SI. Let

$$R_{in}(B^Q) \triangleq \bigcup_{p(u_1,u_2), \xi(u_1,u_2,s) \in \mathcal{Q}} \bigcap \left\{ (R_1, R_2) : R_2 \leq I_q(U_2; Y_2), R_1 \leq I_q(U_1; Y_1|U) \right\}$$

(2.13)
subject to $X = \xi(U_1, U_2, S)$, where $U_1$ and $U_2$ are auxiliary random variables, independent of $S$, and the union is over the pmf $p(u_1, u_2)$ and the set of all functions $\xi: U_1 \times U_2 \times S \to \mathcal{X}$. This can also be expressed as

$$R_{in}(\mathcal{B}^Q) = \bigcup_{p(u_1, u_2), \xi(u_1, u_2, s)} \left\{ (R_1, R_2) : \begin{array}{l} R_2 \leq \inf_{q \in \mathcal{Q}} I_q(U_2; Y_2), \\ R_1 \leq \inf_{q \in \mathcal{Q}} I_q(U_1; Y_1 | U_2) \end{array} \right\}. \quad (2.14)$$

**Lemma 16.** Let $\mathcal{B}^Q$ be a compound DBC with causal SI available at the encoder. Then, $R_{in}(\mathcal{B}^Q)$ is an achievable rate region for $\mathcal{B}^Q$, i.e.

$$C(\mathcal{B}^Q) \supseteq R_{in}(\mathcal{B}^Q). \quad (2.15)$$

Specifically, if $(R_1, R_2) \in R_{in}(\mathcal{B}^Q)$, then for some $a > 0$ and sufficiently large $n$, there exists a $(2^{nR_1}, 2^{nR_2}, n, e^{-an})$ Shannon strategy code over the compound DBC $\mathcal{B}^Q$ with causal SI.

The proof of Lemma 16 is given in Appendix B.1.

**The Capacity Region**

We determine the capacity region of the compound DBC $\mathcal{B}^Q$ with causal SI available at the encoder. In addition, we give a condition, for which the inner bound in Lemma 16 coincides with the capacity region. For every $q \in \mathcal{Q}$, define

$$C(\mathcal{B}^q) \triangleq \bigcup_{p(u_1, u_2), \xi(u_1, u_2, s)} \left\{ (R_1, R_2) : \begin{array}{l} R_2 \leq I_q(U_2; Y_2), \\ R_1 \leq I_q(U_1; Y_1 | U_2) \end{array} \right\}, \quad (2.16)$$

and let

$$R_{out}(\mathcal{B}^Q) \triangleq \bigcap_{q(s) \in \mathcal{Q}} C(\mathcal{B}^q). \quad (2.17)$$

Now, our condition is defined in terms of the following.

**Definition 9.** We say that a function $\xi: U_1 \times U_2 \times S \to \mathcal{X}$ and a set $\mathcal{D} \subseteq \mathcal{P}(U_1 \times U_2)$ achieve both $R_{in}(\mathcal{B}^Q)$ and $R_{out}(\mathcal{B}^Q)$ if

$$R_{in}(\mathcal{B}^Q) = \bigcup_{p(u_1, u_2) \in \mathcal{D}} \bigcap_{q(s) \in \mathcal{Q}} \left\{ (R_1, R_2) : \begin{array}{l} R_2 \leq I_q(U_2; Y_2), \\ R_1 \leq I_q(U_1; Y_1 | U_2) \end{array} \right\}, \quad (2.18a)$$

and

$$R_{out}(\mathcal{B}^Q) = \bigcap_{q(s) \in \mathcal{Q}} \bigcup_{p(u_1, u_2) \in \mathcal{D}} \left\{ (R_1, R_2) : \begin{array}{l} R_2 \leq I_q(U_2; Y_2), \\ R_1 \leq I_q(U_1; Y_1 | U_2) \end{array} \right\}, \quad (2.18b)$$

subject to $X = \xi(U_1, U_2, S)$. That is, the unions in (2.13) and (2.16) can be restricted to the particular function $\xi(u_1, u_2, s)$ and set of strategy distributions $\mathcal{D}$.
Observe that by Definition 9, given a function \( \xi(u_1, u_2, s) \), if a set \( D \) achieves both \( R_{in}(B^Q) \) and \( R_{out}(B^Q) \), then every set \( D' \) with \( D \subseteq D' \subseteq \mathcal{P}(U_1 \times U_2) \) achieves those regions, and in particular, \( D' = \mathcal{P}(U_1 \times U_2) \). Nevertheless, the condition defined below requires a certain property that may hold for \( D \), but not for \( D' \).

**Definition 10.** Given a convex set \( Q \) of state distributions, define the condition \( T^Q \) by the following; for some \( \xi(u_1, u_2, s) \) and \( D \) that achieve both \( R_{in}(B^Q) \) and \( R_{out}(B^Q) \), there exists \( q^* \in Q \) which minimizes both \( I_q(U_2; Y_2) \) and \( I_q(U_1; Y_1|U_2) \), for all \( p(u_1, u_2) \in D \), i.e.

\[
T^Q : \text{For some } q^* \in Q, \quad q^* = \arg \min_{q \in Q} I_q(U_2; Y_2) = \arg \min_{q \in Q} I_q(U_1; Y_1|U_2),
\]

\( \forall p(u_1, u_2) \in D. \) (2.19)

**Theorem 17.** Let \( B^Q \) be a compound DBC with causal SI available at the encoder. Then,

1) the capacity region of \( B^Q \) follows

\[
\mathbb{C}(B^Q) = R_{out}(B^Q), \quad \text{if } \text{int}(\mathbb{C}(B^Q)) \neq \emptyset,
\]

and it is identical to the corresponding random code capacity region, i.e. \( \mathbb{C}^*(B^Q) = \mathbb{C}(B^Q) \) if \( \text{int}(\mathbb{C}(B^Q)) \neq \emptyset \).

2) Suppose that \( Q \subseteq \mathcal{P}(S) \) is a convex set of state distributions. If the condition \( T^Q \) holds, the capacity region of \( B^Q \) is given by

\[
\mathbb{C}(B^Q) = R_{in}(B^Q) = R_{out}(B^Q),
\]

and it is identical to the corresponding random code capacity region, i.e. \( \mathbb{C}^*(B^Q) = \mathbb{C}(B^Q) \).

The proof of Theorem 17 is given in Appendix B.2.

### 2.2.2 The AVDBC with Causal SI

We give inner and outer bounds, on the random code capacity region and the deterministic code capacity region, for the AVDBC \( B \) with causal SI. We also provide conditions, for which the inner bound coincides with the outer bound.

**Random Code Inner and Outer Bounds**

Define

\[
R_{in}^* = \bigcup_{p(u_1, u_2), \xi(u_1, u_2, s)} \bigcap_{q(s)} \left\{ (R_1, R_2) : \begin{array}{l}
R_2 \leq I_q(U_2; Y_2), \\
R_1 \leq I_q(U_1; Y_1|U_2)
\end{array} \right\},
\]

(2.22)
and

\[ R_{\text{out}}^* \triangleq \bigcap_{q(s)} \bigcup_{p(u_1, u_2), \xi(u_1, u_2, s)} \left\{ (R_1, R_2) : \begin{array}{l} R_2 \leq I_q(U_2; Y_2), \\ R_1 \leq I_q(U_1; Y_1|U_2) \end{array} \right\} . \] (2.23)

Now, we define a condition in terms of the following.

**Definition 11.** We say that a function \( \xi : U_1 \times U_2 \times S \to X \) and a set \( \mathcal{D}^* \subseteq \mathcal{P}(U_1 \times U_2) \) achieve both \( R_{\text{in}}^* \) and \( R_{\text{out}}^* \) if

\[ R_{\text{in}}^* = \bigcup_{p(u_1, u_2) \in \mathcal{D}^*} \bigcap_{q(s)} \left\{ (R_1, R_2) : \begin{array}{l} R_2 \leq I_q(U_2; Y_2), \\ R_1 \leq I_q(U_1; Y_1|U_2) \end{array} \right\} , \] (2.24a)

and

\[ R_{\text{out}}^* = \bigcap_{q(s)} \bigcup_{p(u_1, u_2) \in \mathcal{D}^*} \left\{ (R_1, R_2) : \begin{array}{l} R_2 \leq I_q(U_2; Y_2), \\ R_1 \leq I_q(U_1; Y_1|U_2) \end{array} \right\} , \] (2.24b)

subject to \( X = \xi(U_1, U_2, S) \). That is, the unions in (2.22) and (2.23) can be restricted to the particular function \( \xi(u_1, u_2, s) \) and set of strategy distributions \( \mathcal{D}^* \).

**Definition 12.** Define the condition \( \mathcal{T} \) by the following; for some \( \xi(u_1, u_2, s) \) and \( \mathcal{D}^* \) that achieve both \( R_{\text{in}}^* \) and \( R_{\text{out}}^* \), there exists \( q^* \in \mathcal{P}(S) \) which minimizes both \( I_q(U_2; Y_2) \) and \( I_q(U_1; Y_1|U_2) \), for all \( p(u_1, u_2) \in \mathcal{D}^* \), i.e.

\[ \mathcal{T} : \text{For some } q^* \in \mathcal{P}(S), \]

\[ q^* = \arg \min_{q(s)} I_q(U_2; Y_2) = \arg \min_{q(s)} I_q(U_1; Y_1|U_2) \quad \forall p(u_1, u_2) \in \mathcal{D}^* . \]

**Theorem 18.** Let \( \mathcal{B} \) be an AVDBC with causal SI available at the encoder. Then,

1) the random code capacity region of \( \mathcal{B} \) is bounded by

\[ R_{\text{in}}^* \subseteq C^*(\mathcal{B}) \subseteq R_{\text{out}}^* . \] (2.25)

2) If the condition \( \mathcal{T} \) holds, the random code capacity region of \( \mathcal{B} \) is given by

\[ C^*(\mathcal{B}) = R_{\text{in}}^* = R_{\text{out}}^* . \] (2.26)

The proof of Theorem 18 is given in Appendix B.3.

The following lemma is a restatement of a result from [1], stating that a polynomial size of the code collection \( \{ \mathcal{C}_\gamma \} \) is sufficient. This result is a key observation in Ahlswede’s Elimination Technique (ET), presented in [1], and it is significant for the deterministic code analysis.

**Lemma 19.** Consider a given \( (2^{nR_1}, 2^{nR_2}, n, \varepsilon_n) \) random code \( \mathcal{C}^\Gamma = (\mu, \Gamma, \{ \mathcal{C}_\gamma \}_{\gamma \in \Gamma}) \) for the AVDBC \( \mathcal{B} \), where \( \lim_{n \to \infty} \varepsilon_n = 0 \). Then, for every \( 0 < \alpha < 1 \) and sufficiently large \( n \), there exists a \( (2^{nR_1}, 2^{nR_2}, n, \alpha) \) random code \( (\mu^*, \Gamma^*, \{ \mathcal{C}_\gamma^* \}_{\gamma \in \Gamma^*}) \) with the following properties:
1. The size of the code collection is bounded by $|\Gamma^*| \leq n^2$.

2. The code collection is a subset of the original code collection, i.e. $\Gamma^* \subseteq \Gamma$.

3. The distribution $\mu^*$ is uniform, i.e. $\mu^*(\gamma) = \frac{1}{|\Gamma^*|}$, for $\gamma \in \Gamma^*$.

The proof of Lemma 19 follows the same lines as in [1, Section 4] (see also [30]). For completeness, we give the proof in Appendix B.4.

**Deterministic Code Inner and Outer Bounds**

The next theorem characterizes the deterministic code capacity region, which demonstrates a dichotomy property.

**Theorem 20.** The capacity region of an AVDBC $\mathcal{B}$ with causal SI either coincides with the random code capacity region or else, it has an empty interior. That is, $\mathcal{C}(\mathcal{B}) = \mathcal{C}^*(\mathcal{B})$ or else, $\text{int}(\mathcal{C}(\mathcal{B})) = \emptyset$.

The proof of Theorem 20 is given in Appendix B.5. For every function $\xi' : \mathcal{U}_2 \times \mathcal{S} \to \mathcal{X}$, define a DMC $V_{\mathcal{Y}_2|\mathcal{U}_2,\mathcal{S}}^{\xi'}$ specified by $V_{\mathcal{Y}_2|\mathcal{U}_2,\mathcal{S}}^{\xi'}(y_2|u_2, s) = W_{Y_2|X,S}(y_2|\xi'(u_2, s), s)$.

**Corollary 21.** The capacity region of $\mathcal{B}$ is bounded by

\begin{align}
\mathcal{C}(\mathcal{B}) &\supseteq R_{in}^* , \text{ if } \text{int}(\mathcal{C}(\mathcal{B})) \neq \emptyset, \quad (2.27) \\
\mathcal{C}(\mathcal{B}) &\subseteq R_{out}^*. \quad (2.28)
\end{align}

Furthermore, if $V_{\mathcal{Y}_2|\mathcal{U}_2,\mathcal{S}}^{\xi'}$ is non-symmetrizable for some $\xi' : \mathcal{U}_2 \times \mathcal{S} \to \mathcal{X}$, and the condition $\mathcal{F}$ holds, then $\mathcal{C}(\mathcal{B}) = R_{in}^* = R_{out}^*$.

The proof of Corollary 21 is given in Appendix B.6.

To conclude this chapter, we have established inner and outer bounds, on the random code capacity region and the deterministic code capacity region, for the AVDBC $\mathcal{B}$ with causal SI. We also provided conditions, for which the inner bound coincides with the outer bound.

**2.3 Example**

To illustrate the results above, we give the following example.

**Example 2.** [28, Section IV-A] Consider an arbitrarily varying binary symmetric broadcast channel (BSBC),

\[ Y_1 = X + Z_s \mod 2, \]
\[ Y_2 = Y_1 + V \mod 2, \]

where $X, Y_1, Y_2, S, Z_s, V$ are binary, with values in $\{0, 1\}$. The additive noises are distributed according to

\[ Z_s \sim \text{Bernoulli}(\theta_s), \text{ for } s \in \{0, 1\}, \]
\[ V \sim \text{Bernoulli}(\alpha), \]

\[ \theta_s \text{ and } \alpha \text{ are parameters}. \]
with $\theta_0 \leq 1 - \theta_1 \leq \frac{1}{2}$ and $\alpha < \frac{1}{2}$, where $V$ is independent of $(S, Z_S)$. It is readily seen the channel is physically degraded. Define the binary entropy function $h(x) = -x \log x - (1 - x) \log(1 - x)$, for $x \in [0, 1]$, with logarithm to base 2.

We have the following results. The capacity region of the arbitrarily varying BSBC $B_0$ without SI is given by

$$\mathcal{C}(B_0) = \{(0,0)\}.$$  

(2.29)

The capacity region of the arbitrarily varying BSBC $B$ with causal SI is given by

$$\mathcal{C}(B) = \bigcup_{0 \leq \beta \leq 1} \left\{ (R_1, R_2) : \begin{array}{l} R_2 \leq 1 - h(\alpha \beta \theta_1) , \\ R_1 \leq h(\beta \theta_1) - h(\theta_1) \end{array} \right\}.$$  

(2.30)

It will be seen in the achievability proof that the parameter $\beta$ is related to the distribution of $U_1$, and thus the RHS of (2.30) can be thought of as a union over Shannon strategies. The analysis is given in Appendix B.7.

It is shown in Appendix B.7 that the condition $\mathcal{T}$ holds and $\mathcal{C}(B) = R^*_m = R^*_o$. Figure 2.1 provides a graphical interpretation. Consider a DBC $W_{Y_1, Y_2|X,S}$ with random parameters with causal SI, governed by an i.i.d. state sequence, distributed according to $S \sim \text{Bernoulli}(q)$, for a given $0 \leq q \leq 1$, and let $\mathcal{C}(B^q)$ denote the corresponding capacity region. Then, the analysis shows that the condition $\mathcal{T}$ implies that there exists $0 \leq q^* \leq 1$ such that $\mathcal{C}(B) = \mathcal{C}(B^{q^*})$, where $\mathcal{C}(B^{q^*}) \subseteq \mathcal{C}(B^q)$ for every $0 \leq q \leq 1$. Indeed, looking at Figure 2.1, it appears that the regions $\mathcal{C}(B^q)$, for $0 \leq q \leq 1$, form a well ordered set, hence $\mathcal{C}(B) = \mathcal{C}(B^{q^*})$ with $q^* = 1$.

**Appendix A**

**Input and State Constraints: Proofs**

Observe that it suffices to prove the lower bound for the strict input constraint, and the upper bound for the average input constraint. This follows from the fact that the capacity under average input constraint is at least as high as the corresponding capacity under per message input constraint, i.e. $\mathcal{C}_{\Omega, \Lambda}(W^\Omega) \leq \overline{\mathcal{C}}_{\Omega, \Lambda}(W^\Omega)$ and $\mathcal{C}_{\Omega, \Lambda}^*(W^\Omega) \leq \overline{\mathcal{C}}_{\Omega, \Lambda}^*(W^\Omega)$.
Figure 2.1: The capacity region of the AVDBBC in Example 2, the arbitrarily varying binary symmetric broadcast channel. The area under the thick blue line is the capacity region of the AVDBC $B$ with causal SI, with $\theta_1 = 0.005$, $\theta_2 = 0.9$, and $\alpha = 0.2$. The black square at the origin stands for the capacity region of the AVDBC $B_0$ without SI, $C(B_0) = \{(0, 0)\}$. The curves depict $C(B_{\theta})$ for $\theta = 0$, 0.25, 0.5, 0.75, 1, where the capacity region of $B$ is given by $C(B) = R^*_\text{out} = C(B_{\theta})$ for $\theta = 1$ (see (2.23)).

A.1 Proof of Lemma 8

Lower Bound

We construct a code based on Shannon strategies, and decode using joint typicality with respect to a state type, which is "close" to some $q \in \overline{\mathcal{P}}_\Lambda(S)$.

We begin with the following definitions. Basic method of types concepts are defined as in [13, Chapter 2]; including the definition of a type $P_{x^n}$ of a sequence $x^n$; a joint type $P_{x^n, y^n}$ and a conditional type $P_{x^n|y^n}$ of a pair of sequences $(x^n, y^n)$; and a $\delta$-typical set $\mathcal{A}^\delta(P_{X,Y})$ with respect to a distribution $P_{X,Y}(x,y)$. We also define a set of state types $\hat{Q}_n$ by

$$\hat{Q}_n = \{ \hat{P}_{s^n} : s^n \in \mathcal{A}^\delta(q) \text{ for some } q \in \overline{\mathcal{P}}_\Lambda(S) \} , \quad (A.1)$$

where

$$\delta_1 \triangleq \frac{\delta}{2 \cdot |S|} . \quad (A.2)$$

Namely, $\hat{Q}_n$ is the set of types that are $\delta_1$-close to some state distribution $q(s)$ in $\overline{\mathcal{P}}_\Lambda(S)$. A code $\mathcal{C}$ for the compound channel with causal SI is constructed as follows.

**Codebook Generation:** Fix the distribution $P_U(u)$ and the function $\xi(u, s)$ that achieve $R^\star_{\text{low}, \Omega - \varepsilon, \Lambda + \varepsilon}(\mathcal{W})$, where $\varepsilon > 0$ is arbitrarily small. Generate $2^{nR}$ independent sequences $u^n(m), m \in [1 : 2^{nR}]$, at random, each according to $\prod_{i=1}^n P_U(u_i)$. Reveal the codebook to the encoder and the decoder.
Encoding: A message $m \in [1 : 2^{nR}]$ is encoded as follows. If
\[
\sum_{\tilde{s}^n \in \mathcal{S}^n} q^n(\tilde{s}^n)\phi^n(\xi^n(u^n(m), \tilde{s}^n)) \leq \Omega, \quad \text{for all } q \in \overline{\mathcal{P}}_\Lambda(\mathcal{S}),
\] (A.3)
where $q^n(s^n) = \prod_{i=1}^n q(s_i)$, then transmit at time $i \in [1 : n]$, $x_i = \xi(u_i(m), s_i)$. Otherwise, if (A.3) fails to hold for some $q \in \overline{\mathcal{P}}_\Lambda(\mathcal{S})$, transmit $x^n = (a, \ldots, a)$, with an idle symbol $a \in \mathcal{X}$ with $\phi(a) = 0$.

Decoding: As $y^n$ is received, the decoder finds a unique $\hat{m} \in [1 : 2^{nR}]$ such that $(u^n(\hat{m}), y^n) \in \mathcal{A}^\delta(P_U P_{Y|U}^\delta)$, for some $q \in \hat{Q}_n$, where
\[
P_{Y|U}^q(y|u) = \sum_{s \in \mathcal{S}} q(s)W_{Y|X,S}(y|\xi(u, s), s).
\] (A.4)

If there is none, or more than one such $\hat{m} \in [1 : 2^{nR}]$, then the decoder declares an error.

Analysis of Probability of Error: Due to symmetry, we may assume without loss of generality that the user sent the message $m = 1$. Let $q(s) \in \overline{\mathcal{P}}_\Lambda(\mathcal{S})$ denote the actual state distribution chosen by the jammer.

The error event is bounded by the union of the events below. Define
\[
\mathcal{E}_1 = \{U^n(1) \notin \mathcal{A}^\delta(\mathcal{P}_U)\},
\] (A.5)
\[
\mathcal{E}_2 = \{(U^n(1), Y^n) \notin \mathcal{A}^\delta(\mathcal{P}_U P_{Y|U}^\delta) \text{ for all } q' \in \hat{Q}_n\},
\] (A.6)
\[
\mathcal{E}_3 = \{(U^n(m), Y^n) \in \mathcal{A}^\delta(\mathcal{P}_U P_{Y|U}^\delta) \text{ for some } m \neq 1, q' \in \hat{Q}_n\}.
\] (A.7)

Then, the probability of error is bounded by
\[
P_e^{(n)}(q, \mathcal{E}) \leq \Pr(\mathcal{E}_1) + \Pr(\mathcal{E}_2 | \mathcal{E}_1^c) + \Pr(\mathcal{E}_3 | \mathcal{E}_1^c),
\] (A.8)
where the conditioning on $M = 1$ is omitted for convenience of notation. The first term in the RHS of (A.8) tends to zero exponentially as $n \to \infty$, by the law of large numbers and Chernoff’s bound. As for the other terms, observe that given that the event $\mathcal{E}_1$ occurs, i.e. $U^n(1) \in \mathcal{A}^\delta(\mathcal{P}_U)$, we have that for a sufficiently small $\delta > 0$, the requirement
\[
\sum_{s^n \in \mathcal{S}^n} q^n(s^n)\phi^n(\xi^n(U^n(1), s^n)) = \frac{1}{n} \sum_{i=1}^n \sum_{s \in \mathcal{S}} q(s)\phi(\xi(U_i(1), s)) \leq \Omega
\] (A.9)
is held for all $q \in \overline{\mathcal{P}}_\Lambda(\mathcal{S})$. Hence,
\[
X^n = \xi^n(U^n(1), S^n).
\] (A.10)

As for the second term in the RHS of (A.8), we now claim that the event $\mathcal{E}_2$ implies that $(U^n(1), Y^n) \notin \mathcal{A}^\delta(\mathcal{P}_U P_{Y|U}^\delta)$ for all $q'' \in \overline{\mathcal{P}}_\Lambda(\mathcal{S})$. This claim is due to the following. Suppose that $(U^n(1), Y^n) \in \mathcal{A}^\delta(\mathcal{P}_U P_{Y|U}^{q''})$ for some $q'' \in \overline{\mathcal{P}}_\Lambda(\mathcal{S})$. Then, for a sufficiently large $n$, there exists a type $q'(s)$ such that
\[
|q'(s) - q''(s)| \leq \delta_1,
\] (A.11)
for all \( s \in \mathcal{S} \), and by the definition in (A.1), \( q' \in \hat{Q}_n \). Then, (A.11) implies that

\[
|P_{Y|U}^{q'}(y|u) - P_{Y|U}^{q''}(y|u)| \leq |\mathcal{S}| \cdot \delta_1 = \frac{\delta}{2}, \tag{A.12}
\]

for all \( u \in \mathcal{U} \) and \( y \in \mathcal{Y} \) (see (A.2) and (A.4)). Hence, \((U^n(1), Y^n) \in \mathcal{A}^\delta(P_U P_{Y|U}^{q'})\). It follows that if \((U^n(1), Y^n) \notin \mathcal{A}^\delta(P_U P_{Y|U}^{q'})\) for all \( q' \in \hat{Q}_n \), then \((U^n(1), Y^n) \notin \mathcal{A}^{\delta/2}(P_U P_{Y|U}^{q''})\) for all \( q'' \in \overline{\mathcal{P}}_\Lambda(\mathcal{S}) \). Thus,

\[
\Pr(\mathcal{E}_2 | \mathcal{E}_1^c) \leq \Pr\left( (U^n(1), Y^n) \notin \mathcal{A}^{\delta/2}(P_U P_{Y|U}^{q''}) \text{ for all } q'' \in \overline{\mathcal{P}}_\Lambda(\mathcal{S}) | \mathcal{E}_1^c \right) \\
\leq \Pr\left( (U^n(1), Y^n) \notin \mathcal{A}^{\delta/2}(P_U P_{Y|U}^{q''}) | \mathcal{E}_1^c \right). \tag{A.13}
\]

The RHS of (A.13) exponentially tends to zero as \( n \to \infty \) by the law of large numbers and Chernoff’s bound.

We move to the third term in the RHS of (A.8). By the union of events bound and the fact that the number of type classes in \( \mathcal{S}^n \) is bounded by \((n + 1)^{|\mathcal{S}|}\), we have that

\[
\Pr(\mathcal{E}_3 | \mathcal{E}_1^c) \leq (n + 1)^{|\mathcal{S}|} \cdot \sup_{q'' \in \hat{Q}_n} \Pr\left( (U^n(m), Y^n) \in \mathcal{A}^\delta(P_U P_{Y|U}^{q''}) \text{ for some } m \neq 1 | \mathcal{E}_1^c \right) \\
\leq (n + 1)^{|\mathcal{S}|} \cdot 2^{nR} \cdot \sup_{q'' \in \hat{Q}_n} \left[ \sum_{u^n \in \mathcal{U}^n} P_{U^n}(u^n) \cdot \sum_{y^n : (u^n, y^n) \in \mathcal{A}^\delta(P_U P_{Y|U}^{q''})} P_{Y^n}(y^n) \right], \tag{A.14}
\]

where we have defined \( P_Y^q(y) = \sum_{u \in \mathcal{U}, s \in \mathcal{S}} P_U(u) \cdot q(s) \cdot W_{Y|X,S}(y|\xi(u, s), s) \). This follows from (A.10) and the fact that \( U^n(m) \) is independent of \( Y^n \) for every \( m \neq 1 \). Let \( y^n \) satisfy \((u^n, y^n) \in \mathcal{A}^\delta(P_U P_{Y|U}^{q''})\). Then, \( y^n \in \mathcal{A}^{\delta/2}(P_Y^q) \) with \( \delta_2 \triangleq |\mathcal{U}| \cdot \delta \). By Lemmas 2.6 and 2.7 in [13],

\[
P_Y^{q''}(y^n) = 2^{-n(H(P_Y^{q''}) + D(P_Y^{q''}||P_Y^q))} \leq 2^{-nH(P_Y^{q''})} \leq 2^{-n(H_{q''}(Y) - \varepsilon_1(\delta))}, \tag{A.15}
\]

where \( \varepsilon_1(\delta) \to 0 \) as \( \delta \to 0 \). Therefore, by (A.14)–(A.15), along with [13, Lemma 2.13],

\[
\Pr(\mathcal{E}_3 | \mathcal{E}_1^c) \leq (n + 1)^{|\mathcal{S}|} \cdot \sup_{q'' \in \overline{\mathcal{P}}_\Lambda(\mathcal{S})} 2^{-n[I_{q''}(U; Y) - R - \varepsilon_2(\delta)]}, \tag{A.16}
\]

with \( \varepsilon_2(\delta) \to 0 \) as \( \delta \to 0 \). The RHS of (A.16) exponentially tends to zero as \( n \to \infty \), provided that \( R < \min_{q'' \in \overline{\mathcal{P}}_\Lambda(\mathcal{S})} I_{q''}(U; Y) - \varepsilon_2(\delta) \).

\[\square\]

**Upper Bound**

Assume to the contrary that there exists an achievable rate \( R > R^{*}_{\nu, \Omega}(\mathcal{W}) \) using random codes. Thus, for some \( q^*(s) \in \overline{\mathcal{P}}_\Lambda(\mathcal{S}) \), we have that \( R > C_{\nu}(\mathcal{W}^{q^*}) \), where \( C_{\nu}(\mathcal{W}^{q^*}) \triangleq \max_{\xi(u, s), p(u) : \mathbb{E}_q \phi(\xi(U, S)) \leq \Omega} I_{q}(U; Y) \).
The achievability assumption implies that for every $\varepsilon > 0$ and sufficiently large $n$, there exists a $(2^{nR}, n)$ random code $\mathcal{C}'$ for the compound channel $\mathcal{W}_{\mathcal{P}_\Lambda(S)}$ such that $P_{\varepsilon}^{(n)}(q^n, \mathcal{C}') < \varepsilon$ for all i.i.d. state distributions $q \in \mathcal{P}_\Lambda(S)$. If such a code would exist, it could have been used over a random parameter channel with $S^n \sim \prod_{i=1}^n q^*(s_i)$, with causal SI, achieving a rate $R > C_{\Omega}(\mathcal{W}^*)$. This stands in contradiction to Shannon’s fundamental result in [27], hence the assumption is false.

### A.2 Proof of Lemma 9

We state the proof of our modified version of Ahlswede’s RT [1]. The proof follows the lines of [1, Subsection IV-B]. Let $\tilde{s}^n \in S^n$ such that $l^n(\tilde{s}^n) \leq \Lambda$. Denote the type of $\tilde{s}^n \in S^n$ by $\hat{q}$. Observe that $\hat{q} \in \mathcal{P}_\Lambda(S)$.

Given a permutation $\pi \in \Pi_n$,

$$\sum_{s^n \in S^n} q^n(s^n)h(s^n) = \sum_{s^n \in S^n} q^n(\pi s^n)h(\pi s^n) = \sum_{s^n \in S^n} q^n(s^n)h(\pi s^n), \quad (A.17)$$

for every i.i.d. state distribution $q^n(s^n) = \prod_{i=1}^n q(s_i)$, with $q \in \mathcal{P}_\Lambda(S)$, where the first equality holds since $\pi$ is a bijection, and the second equality holds since $q^n$ is i.i.d. Hence, taking $q = \hat{q}$,

$$\sum_{s^n \in S^n} \hat{q}^n(s^n)h(s^n) = \frac{1}{n!} \sum_{\pi \in \Pi_n} \sum_{s^n \in S^n} \hat{q}^n(s^n)h(\pi s^n), \quad (A.18)$$

and by (1.27),

$$\sum_{s^n \in S^n} \hat{q}^n(s^n) \left[ \frac{1}{n!} \sum_{\pi \in \Pi_n} h(\pi s^n) \right] \leq \alpha_n. \quad (A.19)$$

Then,

$$\sum_{s^n : \bar{P}_{\bar{s}^n} = \hat{q}} \hat{q}^n(s^n) \left[ \frac{1}{n!} \sum_{\pi \in \Pi_n} h(\pi s^n) \right] \leq \alpha_n. \quad (A.20)$$

The expression in the square brackets is identical for all sequences $s^n$ of type $\hat{q}$. Thus,

$$\left[ \frac{1}{n!} \sum_{\pi \in \Pi_n} h(\pi \tilde{s}^n) \right] \cdot \sum_{s^n : \bar{P}_{\bar{s}^n} = \hat{q}} \hat{q}^n(s^n) \leq \alpha_n. \quad (A.21)$$

The second sum is the probability of the type class of $\hat{q}$, hence

$$\sum_{s^n : \bar{P}_{\bar{s}^n} = \hat{q}} \hat{q}^n(s^n) \geq \frac{1}{(n+1)|\mathcal{S}|}, \quad (A.22)$$

by [12, Theorem 11.1.4]. The proof follows from (A.21) and (A.22).
A.3 Proof of Theorem 10

Consider the AVC $\mathcal{W}$ per message input constraint $\Omega$ and state constraint $\Lambda$, as specified by (1.8) and (1.9a).

Part 1

Lower Bound

We use Ahlswede’s RT twice, as follows. Let $R < R_{\text{low},\Omega+\Delta}^*(\mathcal{W})$, where $\Delta > 0$ is arbitrarily small. Consider the compound channel with causal SI, under input constraint $\Omega$, with $Q = \overline{\mathcal{P}}_\Lambda(S)$, hence $Q \subseteq \overline{\mathcal{P}}_{\Lambda+\Delta}(S)$. According to Lemma 8, for some $\theta > 0$ and sufficiently large $n$, there exists a $(2^{nR}, n)$ Shannon strategy code $\mathcal{C} = (U^n(m), \xi(u, s), g(y^n))$ for the compound channel $\mathcal{W}^{\overline{\mathcal{P}}_\Lambda(S)}$ with causal SI, such that

$$
\sum_{s^n \in S^n} q^n(s^n) \cdot \mathbb{E} \phi^n(\xi^n(U^n(m), s^n)) \leq \Omega - 2\delta, \text{ for all } m \in [1 : 2^{nR}] . 
$$  \hspace{1cm} (A.23)

and

$$
\mathbb{E} P_{e}^{(n)}(q, \mathcal{C}) = \sum_{s^n \in S^n} q^n(s^n) \cdot \mathbb{E} P_{e|s^n}^{(n)}(\mathcal{C}) \leq e^{-2\theta n} , 
$$  \hspace{1cm} (A.24)

for all i.i.d. state distributions $q^n(s^n) = \prod_{i=1}^n q(s_i)$, with $q \in \overline{\mathcal{P}}_\Lambda(S)$. The expectation in Equations (A.23) and (A.24) is on the ensemble of codebooks, corresponding to the independent i.i.d. random sequences $U^n(m)$, $m \in [1 : 2^{nR}]$, as set in the proof of Lemma 8.

Given such a Shannon strategy code, we have that (1.27) is satisfied with $h_0(s^n) = \mathbb{E} P_{e|s^n}^{(n)}(\mathcal{C})$ and $\alpha_n = e^{-2\theta n}$. Consequently, by Lemma 9, for a sufficiently large $n$,

$$
\frac{1}{n!} \sum_{\pi \in \Pi_n} \mathbb{E} P_{e|\pi s^n}^{(n)}(\mathcal{C}) \leq (n + 1)|S|e^{-2\theta n} \leq e^{-\theta n} , 
$$  \hspace{1cm} (A.25)

for all $s^n \in S^n$ with $l^n(s^n) \leq \Lambda$.

On the other hand, for every Shannon strategy code $\mathcal{C} = (u^n(m), \xi(u, s), g(y^n))$, and for every $\pi \in \Pi_n$,

$$
P_{e|\pi s^n}^{(n)}(\mathcal{C}) \overset{(a)}{=} \frac{1}{2^{nR}} \sum_{m=1}^{2^{nR}} \sum_{y^n: g(y^n) \neq m} W_{Y^n|X^n, S^n}(y^n|\xi^n(u^n(m), \pi s^n), \pi s^n) 
$$

$$
\overset{(b)}{=} \frac{1}{2^{nR}} \sum_{m=1}^{2^{nR}} \sum_{y^n: g(y^n) \neq m} W_{Y^n|X^n, S^n}(\pi y^n|\xi^n(u^n(m), \pi s^n), \pi s^n) 
$$

$$
\overset{(c)}{=} \frac{1}{2^{nR}} \sum_{m=1}^{2^{nR}} \sum_{y^n: g(y^n) \neq m} W_{Y^n|X^n, S^n}(y^n|\pi^{-1}\xi^n(u^n(m), \pi s^n), s^n) , 
$$  \hspace{1cm} (A.26)
where (a) is obtained by plugging $\pi s^n$ and $x^n = \xi^n(\cdot, \cdot)$ in (1.10a); in (b) we simply change the order of summation over $y^n$; and (c) holds because the channel is memoryless. Note that for a Shannon strategy code, $x_i = \xi(u_i, s_i)$, $i \in [1 : n]$, by Definition 3 (see (1.2)). Thus, $\pi^{-1}\xi^n(u^n(m), \pi s^n) = \xi^n(\pi^{-1}u^n(m), s^n)$, and

$$P_{e|\pi s^n}(\mathcal{C}) = \frac{1}{2^{nR}} \sum_{m=1}^{2^{nR}} \sum_{y^n: g(\pi y^n) \neq m} W_{Y^n|X^n, S^n}(y^n|\xi^n(\pi^{-1}u^n(m), s^n), s^n). \quad (A.27)$$

The last expression suggests the use of permutations applied to the encoding strategy and the channel output sequence.

Then, consider the $(2^{nR}, n)$ random code $\mathcal{C}^{\Pi}$, specified by

$$f^n_\pi(m, s^n) = \xi^n(\pi^{-1}U^n(m), s^n), \quad g_\pi(y^n) = g(\pi y^n), \quad \pi \in \Pi_n, \quad (A.28)$$

with a uniform distribution $\mu(\pi) = \frac{1}{|\Pi_n|} = \frac{1}{n!}$. Such permutations can be implemented without knowing $s^n$, hence this coding scheme does not violate the causality requirement.

From (A.27), we see that

$$P_{e|s^n}(\mathcal{C}^{\Pi}) = \sum_{\pi \in \Pi_n} \mu(\pi) \cdot \mathbb{E} P_{e|\pi s^n}(\mathcal{C}), \quad (A.29)$$

for all $s^n \in \mathcal{S}^n$ with $l^n(s^n) \leq \Lambda$. Therefore, together with (A.25), we have that the probability of error of the random code $\mathcal{C}^{\Pi}$ is bounded by

$$P_{e^n}(q^n, \mathcal{C}^{\Pi}) \leq e^{-\theta n}, \quad (A.30)$$

for every $q^n(s^n) \in \mathcal{P}_\Lambda^n(\mathcal{S}^n)$.

It is left for us to verify that the random code $\mathcal{C}^{\Pi}$ obeys the input constraint. To this end, we apply Ahlswede’s RT again. Let $m \in [1 : 2^{nR}]$ and $q(s) \in \overline{\mathcal{P}}_\Lambda(\mathcal{S})$, and let a sequence of i.i.d. random variables $\overline{s}_1, \ldots, \overline{s}_n \sim q(s)$. Define the random variables

$$\Phi_i(m) = \phi(\xi(U_i(m), \overline{s}_i)), \quad \text{for } i \in [1 : n]. \quad (A.31)$$

Then, $\Phi_1(m), \ldots, \Phi_n(m)$ are i.i.d. as well, and by (A.23), $\mathbb{E} \Phi_1(m) \leq \Omega - 2\delta$. Hence, for every $m \in [1 : 2^{nR}]$ and $q \in \overline{\mathcal{P}}_\Lambda(\mathcal{S})$,

$$\Pr\left(\phi^n(\xi^n(U^n(m), \overline{s}^n)) > \Omega - \delta\right) = \Pr\left(\frac{1}{n} \sum_{i=1}^{n} \Phi_i(m) > \Omega - \delta\right) \leq 2^{-n \mathbb{E}(\Omega, \Lambda)},$$

where $\mathbb{E}(\Omega, \Lambda) \triangleq \min_{m \in [1 : 2^{nR}], q \in \overline{\mathcal{P}}_\Lambda(\mathcal{S})} \min_{P_{\Phi}} D(P_{\Phi}||P_{\Phi_1(m)})$, by standard large deviations considerations (see e.g. [12, pp. 362–364]). On the other hand,

$$\Pr\left(\phi^n(\xi^n(U^n(m), \overline{s}^n)) > \Omega - \delta\right) = \sum_{s^n \in \mathcal{S}^n} q^n(s^n) h_m(s^n), \quad (A.32)$$
where \( h_m(s^n) = \Pr (\phi^n(\xi^n(U^n(m), s^n)) > \Omega - \delta) \). Thus, by Lemma 9,
\[
\frac{1}{n!} \sum_{\pi \in \Pi_n} h_m(\pi s^n) \leq (n + 1)^{|S|} \cdot 2^{-n \mathcal{E}(\Omega, \Lambda)} \leq e^{-\theta' n},
\]
(A.33)
for all \( s^n \in S^n \) with \( l^n(s^n) \leq \Lambda \), for some \( \theta' > 0 \) and sufficiently large \( n \).

Then,
\[
\frac{1}{n!} \sum_{\pi \in \Pi_n} \mathbb{E} \phi^n(\xi^n(U^n(m), \pi s^n)) = \frac{1}{n!} \sum_{\pi \in \Pi_n} h_m(\pi s^n) \cdot \mathbb{E} \left( \phi^n(\xi^n(U^n(m), \pi s^n)) \bigg| \phi^n(\xi^n(U^n(m), \pi s^n)) > \Omega - \delta \right) + \frac{1}{n!} \sum_{\pi \in \Pi_n} (1 - h_m(\pi s^n)) \cdot \mathbb{E} \left( \phi^n(\xi^n(U^n(m), \pi s^n)) \bigg| \phi^n(\xi^n(U^n(m), \pi s^n)) \leq \Omega - \delta \right).
\]
(A.34)

To bound the first sum in the RHS of (A.34), we use (A.33) and the fact that \( \phi^n(x^n) \leq \phi_{\max} \), for all \( x^n \in \mathcal{X}^n \). As for the second sum in the RHS of (A.34), observe that the expectation in the last line is bounded by \( (\Omega - \delta) \). Hence,
\[
\frac{1}{n!} \sum_{\pi \in \Pi_n} \mathbb{E} \phi^n(\xi^n(U^n(m), \pi s^n)) \leq \phi_{\max} \cdot e^{-\theta' n} + \Omega - \delta.
\]
(A.35)

It follows that for a sufficiently large \( n \),
\[
\frac{1}{n!} \sum_{\pi \in \Pi_n} \mathbb{E} \phi^n(f^n(\mu, s^n)) = \frac{1}{n!} \sum_{\pi \in \Pi_n} \mathbb{E} \phi^n(\xi^n(U^n(m), \pi s^n)) \leq \Omega,
\]
(A.36)
where the equality is due to (A.28), and the fact that the input constraint is additive (see (1.3)).

Thus, it follows from (A.30) and (A.36) that \( \mathcal{C}_\Pi \) is a \( (2^n R, n, e^{-\theta n}) \) random code for the AVC \( \mathcal{W} \) with causal SI at the encoder, under input constraint \( \Omega \) and state constraint \( \Lambda \).

**Upper Bound**

Assume to the contrary that there exists an achievable rate \( R > R^*_{ap, \Omega + \delta, \Lambda - \delta}(\mathcal{W}) \), using random codes over the AVC \( \mathcal{W} \) with causal SI, under input constraint \( \Omega \) and state constraint \( \Lambda \), where \( \delta > 0 \) is arbitrarily small. That is, for every \( \varepsilon > 0 \) and sufficiently large \( n \), there exists a \( (2^n R, n) \) random code \( \mathcal{C}^\Gamma = (\mu, \Gamma, \{\mathcal{C}_\gamma\}_{\gamma \in \Gamma}) \) for the AVC \( \mathcal{W} \) with causal SI, such that
\[
\sum_{\gamma \in \Gamma} \mu(\gamma) \sum_{s^n \in S^n} q^n(s^n) \phi^n(f^n(\gamma, s^n)) \leq \Omega,
\]
(A.37)
\[
P_e(q^n, \mathcal{C}^\Gamma) \leq \varepsilon,
\]
(A.38)
for all \( m \in [1 : 2^n R] \) and \( q^n(s^n) \in \mathcal{P}_A(S^n) \). In particular, for a kernel, \( P^{(n)}_{e|s^n}(C^T) \leq \varepsilon \), for all \( s^n \in S^n \) such that \( l^n(s^n) \leq \Lambda \).

Consider using the random code \( C^T \) over the compound channel \( \mathcal{W}_{\bar{\mathcal{T}}_{A-\delta}}(S) \) with causal SI under input constraint \( \Omega + \delta \), where \( \delta > 0 \) is arbitrarily small. Let \( \bar{\mathcal{T}}(s) \in \mathcal{P}_{\bar{\mathcal{T}}_{A-\delta}}(S) \) be a given state distribution. Then, define a sequence of i.i.d. random variables \( S_1, \ldots, S_n \sim \bar{\mathcal{T}}(s) \). Letting \( \mathcal{T}^n(s^n) = \prod_{i=1}^{n} \mathcal{T}(s_i) \), the probability of error is bounded by

\[
P_e^{(n)}(q, C^T) \leq \sum_{s^n : l^n(s^n) \leq \Lambda} \bar{q}^n(s^n) P^{(n)}_{e|s^n}(C^T) + \Pr \left( l^n(\bar{S}^n) > \Lambda \right) .
\] (A.39)

Then, the first sum is bounded by (A.38), and the second term vanishes as well by the law of large numbers, since \( \bar{\mathcal{T}}(s) \in \mathcal{P}_{\bar{\mathcal{T}}_{A-\delta}}(S) \).

As for the input constraint, define a random variable \( L \in \Gamma \), with \( L \sim \mu(\ell) \). Then, for every \( m \in [1 : 2^n R] \),

\[
\sum_{\gamma \in \Gamma} \mu(\gamma) \sum_{s^n \in S^n} \bar{q}^n(s^n) \phi^n(f^n_{\gamma}(m, s^n)) = \mathbb{E}_{\bar{\mathcal{T}}} \phi^n(f^n_{L}(m, \bar{S}^n)) \leq e^{-n/4} + \Pr \left( l^n(\bar{S}^n) > \Lambda \right) \leq e^{-n/4} + \phi_{\max} \cdot \varepsilon_n \leq \Omega + \delta ,
\] (A.40)

with \( \varepsilon_n \rightarrow 0 \) as \( n \rightarrow \infty \). The first inequality follows from the law of large numbers, and last inequality is obtained by applying (A.37) to the state distribution \( q^n(s^n) = \Pr(\bar{S}^n = s^n | l^n(\bar{S}^n) \leq \Lambda) \), which is readily seen to satisfy \( q^n \in \mathcal{P}_A(S^n) \).

It follows that the random code \( C^T \) achieves a rate \( R > R_{\mu, \Omega + \delta, \Lambda - \delta}(W) \) over the compound channel \( \mathcal{W}_{\bar{\mathcal{T}}_{A-\delta}}(S) \) under input constraint \( \Omega + \delta \), for an arbitrarily small \( \delta > 0 \), in contradiction to Lemma 8. We deduce that the assumption is false, and \( R > R_{\mu, \Omega, \Lambda}(W) \) cannot be achieved. \( \square \)

**Part 2**

For \( \Omega = \phi_{\max} \), it follows from (1.24) and (1.25) that \( R_{\mu, \Omega}(W) = \min_{q(s) \in \mathcal{P}_A(S)} \max_{p(u), \xi(u, s)} I_q(U; Y) \). Hence, the proof follows from part 1. \( \square \)

**A.4 Ahlswede’s Elimination Technique**

**Proof of Lemma 11.** The proof is an extension of [1, Section 4]. Consider the AVC \( W \) with causal SI, under per message input constraint \( \Omega \) and state constraint \( \Lambda \). Let \( k > 0 \) be an integer, chosen later, and define the random variables

\[
L_1, L_2, \ldots, L_k \text{ i.i.d. } \sim \mu(\ell) .
\] (A.43)

Fix \( m \in [1 : 2^n R] \) and \( s^n \in S^n \), and define the random variables

\[
\Phi_j(m, s) = \phi^n(f^n_{L_j}(m, s^n)) , \quad j \in [1 : k] ,
\] (A.44)
and

\[ \psi_j(s^n) = p(n)_{\varepsilon_j(n)}(\mathcal{C}_{L_j}) \], \quad j \in [1:k], \quad (A.45) \]

which correspond to the code \( \mathcal{C}_{L_j} = (f_{L_j}, g_{L_j}) \) in the code collection \( \{\mathcal{C}_\gamma = (f_{\gamma}, g_{\gamma})\}_{\gamma \in \Gamma} \). Since \( \mathcal{C}_\Gamma \) is a \((2^nR, n, \varepsilon_n)\) random code over the AVC \( W \) with causal SI, under message input constraint \( \Omega \) and state constraint \( \Lambda \), we have that \( \sum_{\gamma} \mu(\gamma) \sum_{s^n} q^n(s^n)p(n)_{\varepsilon_j(n)}(\mathcal{C}_\gamma) \leq \varepsilon_n \), for all \( q^n(s^n) \in \mathcal{P}_\Lambda^n(\mathcal{S}^n) \). In particular, for a kernel, we have that for a given \( m \in [1:2^nR] \) and \( s^n \in \mathcal{S}^n \) with \( l^n(s^n) \leq \Lambda \),

\[ \mathbb{E}\Phi_j(m, s^n) = \sum_{\gamma \in \Gamma} \mu(\gamma)\phi^n(f^n_{\gamma}(m, s^n)) \leq \Omega, \quad (A.46) \]

and

\[ \mathbb{E}\Psi_j(s^n) = \sum_{\gamma \in \Gamma} \mu(\gamma) \cdot p(n)_{\varepsilon_j(n)}(\mathcal{C}_\gamma) \leq \varepsilon_n, \quad (A.47) \]

for all \( j \in [1:k] \). Now take \( n \) to be large enough so that \( \varepsilon_n < \alpha \).

Consider the code \( \mathcal{C}^{\Gamma^*} = (\mu^*, \Gamma^* = [1:k], \{\mathcal{C}_{L_j}\}_{j=1}^k) \) formed by a random collection of codes, with \( \mu^*(j) = \frac{1}{k} \). The event that a “bad code” is chosen is bounded by the union of the following events. Denote the event that the input constraint is violated by

\[ \mathcal{A}_1 = \left\{ \frac{1}{k} \sum_{j=1}^k \Phi_j(m, s^n) > \Omega + \delta, \right\}, \quad (A.48) \]

for some \( (m, s^n) \in [1:2^nR] \times \mathcal{S}^n \) with \( l^n(s^n) \leq \Lambda \) and denote the event that the error requirement is violated by

\[ \mathcal{A}_2 = \left\{ \frac{1}{k} \sum_{j=1}^k \Psi_j(s^n) \geq \alpha, \text{ for some } s^n \in \mathcal{S}^n \text{ with } l^n(s^n) \leq \Lambda \right\}, \quad (A.49) \]

where \( 0 < \alpha < 1 \) and \( \delta > 0 \) are arbitrarily small. Then, by the union of events bound

\[ \Pr(\mathcal{A}_1 \cup \mathcal{A}_2) \leq \Pr(\mathcal{A}_1) + \Pr(\mathcal{A}_2). \quad (A.50) \]

Keeping \( m \) and \( s^n \) fixed, the random variables \( \Phi_j(m, s^n) \) and \( \Psi_j(s^n) \) are each i.i.d., due to \( (A.43) \). Consider the first term in the RHS of \( (A.50) \), \( \Pr(\mathcal{A}_1) \) (see \( (A.48) \)). By standard large deviations considerations, we have that

\[ \Pr\left( \frac{1}{k} \sum_{j=1}^k \Phi_j(m, s^n) > \Omega + \delta \right) \leq 2^{-k(E_j(s^n) - \varepsilon')}, \quad (A.51) \]
with
\[ E_j(s^n) \triangleq \min_{P_{\Phi'}} D(P_{\Phi'}||P_{\Phi_1(m,s^n)}) , \] (A.52)

(see e.g. [12, pp. 362–364]), where \( \varepsilon' > 0 \) is arbitrarily small. Thus, the first term in the RHS of (A.50) is bounded by
\[
\Pr(A_1) \leq \sum_{m \in [1:2^{nR}]} \sum_{s^n \in S^n : p^n(s^n) \leq \Lambda} \Pr \left( \frac{1}{k} \sum_{j=1}^{k} \Phi_j(m, s^n) > \Omega + \delta \right) \] (A.53)
\[
\leq 2^{nR} \cdot |S|^n \cdot 2^{-k(E_j(s^n)-\varepsilon')} . \] (A.54)

Since \( 2^{nR} \cdot |S|^n \) grows only exponentially in \( n \), choosing
\[ k = n^2 \] (A.55)
results in a super exponential decay.

As for the second term in the RHS of (A.50), \( \Pr(A_2) \) (see (A.49)). The technique known as Bernstein’s trick [1] is now applied.
\[
\Pr \left( \sum_{j=1}^{k} \Psi_j(s^n) \geq k\alpha \right) \leq \mathbb{E} \left\{ \exp \left[ \beta \left( \sum_{j=1}^{k} \Psi_j(s^n) - k\alpha \right) \right] \right\} \] (A.56)
\[
= e^{-\beta \alpha} \cdot \mathbb{E} \left\{ \prod_{j=1}^{k} e^{\beta \Psi_j(s^n)} \right\} \] (A.57)
\[
= e^{-\beta \alpha} \cdot \prod_{j=1}^{k} \mathbb{E} \left\{ e^{\beta \Psi_j(s^n)} \right\} \] (A.58)
\[
\leq e^{-\beta \alpha} \cdot \prod_{j=1}^{k} \mathbb{E} \left\{ 1 + e^{\beta \Psi_j(s^n)} \right\} \] (A.59)
\[
\leq e^{-\beta \alpha} \cdot \left( 1 + e^{\beta \varepsilon_n} \right)^k \] (A.60)

where (a) is an application of Chernoff’s inequality; (b) follows from the fact that \( \Psi_j(s^n) \) are independent; (c) holds since \( e^{\beta x} \leq 1 + e^{\beta x} \), for \( \beta > 0 \) and \( 0 \leq x \leq 1 \); (d) follows from (A.47). We take \( n \) to be large enough for \( 1 + e^{\beta \varepsilon_n} \leq e^{\alpha} \) to hold. Thus, choosing \( \beta = 2 \), we have that
\[
\Pr \left( \frac{1}{k} \sum_{j=1}^{k} \Psi_j(s^n) \geq \alpha \right) \leq e^{-\alpha k} = e^{-\alpha n^2} , \] (A.61)

for all \( s^n \in S^n \) with \( p^n(s^n) \leq \Lambda \). Hence, the second term in the RHS of (A.50) is bounded by
\[
\Pr(A_2) \leq \sum_{s^n \in S^n : p^n(s^n) \leq \Lambda} \Pr \left( \frac{1}{k} \sum_{j=1}^{k} \Psi_j(s^n) \geq \alpha \right) \leq |S|^n \cdot e^{-\alpha n^2} . \] (A.62)
By (A.50), (A.54) and (A.62), we have that probability that either the input constraint or the error requirement are violated decays super exponentially with blocklength, namely \( \Pr(A_1 \cup A_2) \sim 2^{-\theta n^2} \), for some \( \theta > 0 \). It follows that there exists a random code \( \mathcal{C}^{\Gamma^*} = (\mu^*, \Gamma^*, \{C_{\gamma^*}^k\}_{j=0}^k) \) for the AVC \( W \), such that for all \( m \in [1 : 2^R] \) and \( q^n \in P^n(\mathcal{S}^n) \),

\[
\sum_{\gamma \in \Gamma^*} \mu^*(\gamma) \sum_{s^n \in \mathcal{S}^n} q^n(s^n) \phi^n(f^n_\gamma(m, s^n)) \leq \Omega + \delta ,
\]

(A.63)

and

\[
P_e^{(n)}(q^n, \mathcal{C}^{\Gamma^*}) = \sum_{s^n \in \mathcal{S}^n} q^n(s^n) P_e^{(n)}(\mathcal{C}^{\Gamma^*}) \leq \alpha ,
\]

(A.64)

as we were set to prove.

\( \square \)

### A.5 Proof of Theorem 12

#### Part 1

Consider the AVC \( W = \{W_{Y|X, S}\} \) with causal SI, under average input constraint \( \Omega \) and state constraint \( \Lambda \). Then, for every encoding mapping \( \xi(u, s) \), consider the AVC \( V_0^\xi = \{V_{Y|U, S}^\xi\} \) without SI, under state constraint \( \Lambda \), and free of input constraint.

Hence, any coding scheme employed over the AVC \( V_0^\xi \) without SI can also be employed over the AVC \( W \) with causal SI, using the encoding function \( \xi(u, s) \), provided that the input constraint \( \Omega \) on \( W \) is satisfied.

Let a type \( P_{U}(u) \) and a function \( \xi(u, s) \) achieve \( R_{\text{low}}; \Omega - 2\varepsilon; \Lambda + \varepsilon \) on \( W \), where \( \varepsilon > 0 \) is arbitrarily small. Hence, \( P_{U}(u) \) is achievable on \( W \), as described below.

**Encoding:** To send a message \( m \in [1 : 2^R] \), do as follows. If

\[
\frac{1}{2^R} \sum_{\tilde{m} \in \mathcal{S}^n} \phi^n(\xi^n(u^n(\tilde{m})), \tilde{s}^n)) \leq \Omega ,
\]

for all \( \tilde{s}^n \in \mathcal{S}^n \) with \( l^n(\tilde{s}^n) \leq \Lambda \) then, at time \( i \in [1 : n] \), transmit \( x_i = \xi(u_i(m), s_i) \). Otherwise, transmit \( x^n = (a, \ldots , a) \), with an idle input symbol \( a \in \mathcal{X} \), with \( \phi(a) = 0 \).

**Decoding:** Use the decoder of the original code \( \mathcal{C}_\xi \), namely \( \hat{m} = g(y^n) \).
Analysis of Probability of Error: Assume without loss of generality that the user sent the message $M = 1$. Denote $M \triangleq 2^{nR}$. For every $s^n \in S$, define a sequence of $M$ random variables given by $Z_m(s^n) = \phi^n(U^n(m), s^n)$, for $m \in [1 : M]$, and consider the event

$$
\mathcal{E}_1 = \left\{ \sum_{m=1}^{2^{nR}} Z_m(s^n) > \Omega, \text{ for some } s^n \in S^n \text{ with } l^n(s^n) \leq \Lambda \right\}.
$$

Then, the probability of error $P_e(n)(q^n, \mathcal{E}_1') = \Pr(g(Y^n) \neq 1)$ is bounded as follows,

$$
P_e(n)(q^n, \mathcal{E}_1') = \Pr(\mathcal{E}_1) \cdot \Pr(g(Y^n) \neq 1 \mid \mathcal{E}_1) + \Pr(\mathcal{E}_1^c) \cdot \Pr(g(Y^n) \neq 1 \mid \mathcal{E}_1^c)
\leq \Pr(\mathcal{E}_1) + \Pr(g(Y^n) \neq 1 \mid \mathcal{E}_1^c),
$$

where the conditioning on $M = 1$ is omitted to simplify notation.

Now, we bound the first term in the RHS of (A.68). Fix $s^n \in S^n$ with $l^n(s^n) \leq \Lambda$. Recall that $U^n(1), \ldots, U^n(M)$ is a sequence of vectors that are independent of each other, where each vector has the same distribution. Therefore, for every given $s^n \in S^n$ with $l^n(s^n) \leq \Lambda$, the sequence $Z_1(s^n), \ldots, Z_M(s^n)$ is i.i.d., hence

$$
\Pr \left( \sum_{m=1}^{M} Z_m(s^n) > \Omega \right) \leq 2^{-M F_0(\Omega)} = 2^{-2^{nR} F_0(\Omega)},
$$

where $F_0(\Omega) \triangleq \min_{s^n \in S^n : l^n(s^n) \leq \Lambda} \min_{\mathcal{E}' \supseteq \Omega} D(P_{Z'} || P_{Z_m(s^n)}) > 0$, by standard large deviations considerations (see e.g. [12, pp. 362–364]). Thus, applying the union bound to (A.67), we have that

$$
\Pr(\mathcal{E}_1) \leq |S^n| \cdot \max_{s^n \in S^n : l^n(s^n) \leq \Lambda} \Pr \left( \sum_{m=1}^{M} Z_m(s^n) > \Omega \right) \leq |S^n| \cdot 2^{-2^{nR} F_0(\Omega)} \quad (A.70)
$$

Hence, $\Pr(\mathcal{E}_1)$ decays to zero double exponentially as $n \to \infty$.

As for the second term in the RHS of (A.68), the probability $\Pr(g(Y^n) \neq 1 \mid \mathcal{E}_1^c)$ vanishes as well, due to the following. Given that the event $\mathcal{E}_1^c$ occurred, we have that $X^n = \xi^n(U^n(1), S^n)$. Then, applying the results by [15], we have that $\mathcal{C}_e$ is a $(2^{nR}, n, \varepsilon_1)$ code over the AVC $V^\xi_0$, where $\varepsilon_1 > 0$ is arbitrarily small. It thus follows that $\Pr(g(Y^n) \neq 1 \mid \mathcal{E}_1^c) \leq \varepsilon_1$. \hfill \square

Part 2

The converse part is a direct consequence of Theorem 10, by which $C_{\Omega, \Lambda}(W) \leq C_{\Omega, \Lambda}^*(W) = R_{\text{up}, \Omega, \Lambda}(W) = R_{\text{up}, \Omega, \Lambda}(W)$, for $\Omega = \phi_{\text{max}}$. In the proof of the direct part, the lemma below is used as a tool.

Lemma 22. [15] If $W_{Y|X,S}(y|x, s)$ is non-symmetrizable, then for every $p \in \mathcal{P}(\mathcal{X})$ with $p(x) > 0$ for all $x \in \mathcal{X}$, we have that $\min_{q \in \mathcal{P}(S)} I_q(X; Y) > 0$. 


Now, assume that there exists a function \( \xi'(u, s) \), such that \( V_{Y|X,S}^\xi(y|u, s) = W_{Y|X,S}(y|u, s) \) is non-symmetrizable. We show that every rate \( R < R_{\text{low}}^*(\Omega, \Lambda) \) can be achieved. The assumption above, along with Lemma 22 and [15, Theorem 2], imply that the capacity without constraints is positive, i.e. \( C_{\phi_{\text{max}}}(W) > 0 \). This, in turn, allows us to use Ahlswede’s ET [1] using the random code constructed in the proof of Theorem 10 to construct a deterministic code (see [15, Section V]).

Let \( R < R_{\text{low}}^*(\Omega, \Lambda) \). By Theorem 10, for some \( \theta > 0 \) and sufficiently large \( n \), there exists a \( (2^{nR}, n, e^{-\theta n}) \) random code for the AVC \( W \) with causal SI, under state constraint \( \Lambda \). Thus, by Lemma 11, for every \( \varepsilon_1 > 0 \) and sufficiently large \( n \), there exists a \( (2^{nR}, n, \varepsilon_1) \) random code \( \mathcal{C}^\gamma = \left( \mu(\gamma) = \gamma_k, \Gamma = [1 : k], \{ \mathcal{C}_{\gamma} = (f^\nu_y, g^\nu_y) \}_{\gamma \in \Gamma} \right) \), for the AVC \( W \) under state constraint \( \Lambda \), with \( k = |\Gamma| \leq n^2 \).

Next, we claim that the code index \( \gamma \in [1 : k] \) can be reliably sent over the AVC \( W \) with causal SI, under state constraint \( \Lambda \). Consider a code for the index \( \gamma \in [1 : k] \), with a blocklength \( \nu \) and rate \( \tilde{R} \). Since \( k \) is polynomial at most, such a code requires a negligible blocklength, i.e. \( \nu = o(n) \). Therefore, the jammer is virtually free of state constraints during this transmission. However, as deduced above, the capacity without state constraints is positive, under the assumptions of part 2 of the theorem, and thus for every \( \varepsilon_2 > 0 \) and sufficiently large \( \nu \), there exists a \( (2^{\nu \tilde{R}}, \nu, \varepsilon_2) \) deterministic code \( \mathcal{C}_i = (f^\nu_y, \tilde{g}^\nu_y) \) to send \( \gamma \in [1 : k] \), where \( \nu = o(n) \) and \( \tilde{R} > 0 \).

Now, consider a code formed by the concatenation of \( \mathcal{C}_i \) as a prefix to a corresponding code in the code collection \( \{ \mathcal{C}_{\gamma} \}_{\gamma \in \Gamma} \). The encoder sends both \( \gamma \) and \( m \), by transmitting \( \tilde{f}^\nu(\gamma, s^\nu) \) and then \( x^n = f^n_m(m, s_{\nu+1}, \ldots, s_{\nu+n}) \). Subsequently, decoding is performed in two stages as well; the index is estimated first, with \( \hat{\gamma} = \tilde{g}(y_1, \ldots, y_\nu) \), and the message is then estimated by \( \hat{m} = g^\nu_\gamma(y_{\nu+1}, \ldots, y_{\nu+n}) \). By the union of events bound, the probability of error is then bounded by \( \varepsilon = \varepsilon_1 + \varepsilon_2 \). That is, the concatenated code is a \( (2^{(\nu+\varepsilon)n \tilde{R}_n}, \nu + n, \varepsilon) \) code over the AVC \( W \) with causal SI, under state constraint \( \Lambda \), where \( \nu = o(n) \), and the rate \( \tilde{R}_n = \frac{n}{\nu + n} \cdot R \) approaches \( R \) as \( n \to \infty \). \( \square \)

### A.6 Analysis of Example 1

We rely on the analysis of Erez and Zamir in [17]. They considered Shannon’s model [27] of a channel with random parameters with causal SI, where the state sequence \( S^n \) is i.i.d. according to a given distribution \( q(s) \). In [17], Erez and Zamir consider a modulo-additive channel,

\[
Y = X + Z_S \mod |\mathcal{X}|,
\]

(A.71)

with \( \mathcal{X} = \mathcal{Z} = \mathcal{Y} = \{0, 1, \ldots, |\mathcal{X}| - 1\} \), such that given \( S = s \), the additive noise is distributed according to \( Z_s \sim p(z|s) \). Let \( \mathcal{U} \) be the index set for the set of all functions \( \xi_u : S \to \mathcal{X} \). It is shown in [17] that the capacity of the modulo-additive
A random parameter channel $W^{y}$ with causal SI is given by

$$C(W^{y}) = \log |\mathcal{X}| - \min_{u \in \mathcal{U}} H(Z_{S} - \xi_{u}(S)). \quad (A.72)$$

For $u \in \mathcal{U}$ that achieves the minimum above, $\xi_{u}(S)$ is interpreted as the minimum error-entropy predictor of $Z_{S}$. The DMC $W_{Y|X,S}$ in Example 1 is a special case of their model.

First, consider the arbitrarily varying noisy-typewriter channel $W_{0}$ without SI, under a state constraint $\Lambda$, when free of input constraints, i.e. $\Omega = \phi_{\max}$. We calculate the random code capacity given by (1.16), due to [14]. Consider a given $0 \leq q \leq 1$, and let

$$S = \begin{cases} 1 & \text{w.p. } 1 - q, \\ 2 & \text{w.p. } q. \end{cases} \quad (A.73)$$

The entropy of the additive noise $Z$ is then given by

$$H_{q}(Z) = h(\theta) + \theta h(q), \quad (A.74)$$

hence,

$$C(W_{0}^{q}) \triangleq \max_{p(x)} I_{q}(X;Y) = \log 3 - h(\theta) - \theta h(q). \quad (A.75)$$

Minimizing over $0 \leq q \leq \Lambda$ yields

$$C_{\Omega,\Lambda}(W_{0}) = \min_{0 \leq q \leq \Lambda} C(W_{0}^{q}) = \begin{cases} \log 3 - h(\theta) - \theta h(\Lambda) & \text{if } 0 \leq \Lambda \leq \frac{1}{2}, \\ \log 3 - h(\theta) - \theta & \text{if } \Lambda \geq \frac{1}{2}. \end{cases} \quad (A.76)$$

and by Theorem 4, due to [14], the random code capacity of the AVC $W_{0}$ without SI, under state constraint $\Lambda$, is given by $C_{\Omega,\Lambda}^{*}(W_{0}) = C_{\Omega,\Lambda}(W_{0})$.

We now claim that $W_{0}$ is non-symmetrizable for all $\theta \neq \frac{2}{3}$, which will imply that the deterministic code capacity is given by $C_{\Omega,\Lambda}(W_{0}) = C_{\Omega,\Lambda}^{*}(W_{0})$, by Theorem 5, due to [15]. Assume to the contrary that $W_{0}$ is symmetrizable and there exists $J(s|x)$ that satisfies (1.15). In particular, denoting $\alpha_{x} = J(2|x)$ for $x \in \{0,1,2\}$, we have that both of the following relations hold for $y \in \{0,1,2\}$,

$$1 - \alpha_{1} \cdot W_{Y|X,S}(y|0,1) + \alpha_{1} \cdot W_{Y|X,S}(y|0,2) = (1 - \alpha_{0}) \cdot W_{Y|X,S}(y|1,1) + \alpha_{0} \cdot W_{Y|X,S}(y|1,2), \quad (A.77a)$$

and

$$1 - \alpha_{2} \cdot W_{Y|X,S}(y|0,1) + \alpha_{2} \cdot W_{Y|X,S}(y|0,2) = (1 - \alpha_{0}) \cdot W_{Y|X,S}(y|2,1) + \alpha_{0} \cdot W_{Y|X,S}(y|2,2). \quad (A.77b)$$

Taking $y = 0$, we have $1 - \theta = \alpha_{0} \cdot \theta = (1 - \alpha_{0}) \cdot \theta$. Since $\theta > 0$, this can only hold for $\alpha_{0} = \frac{1}{2}$ and $\theta = \frac{2}{3}$. Thus, for $\theta \neq \frac{2}{3}$, the AVC $W_{0}$ without SI is non-symmetrizable, and by Theorem 5, $C_{\Omega,\Lambda}(W_{0}) = C_{\Omega,\Lambda}^{*}(W_{0})$. 
For $\theta = \frac{2}{3}$, we have that
\[
C_{\Lambda}(W_0) = \begin{cases} 
\frac{2}{3}(1-h(\Lambda)) & \text{if } 0 \leq \Lambda \leq \frac{1}{2} , \\
0 & \text{if } \Lambda \geq \frac{1}{2} . 
\end{cases} \tag{A.78}
\]
Since the capacity without constraints is zero, Theorem 3 implies that $W_{Y|X,S}$ is symmetrizable for this value of $\theta$. Substituting $y = 0$ and $y = 1$ in (A.77), we find that $W_{Y|X,S}$ can only be symmetrized by $J(s|x)$ such that $\alpha_0 = \alpha_1 = \alpha_2 = \frac{1}{2}$, hence $\sum_{x,s} p(x) J(s|x) l(s) = \frac{1}{2}$ for all $p$. It then follows that $C_{\Lambda}(W_0) = C_{\Lambda}(W_0)$. Therefore, when SI is not available, $C_{\Omega}(W_0) = C_{\Omega}(W_0)$ for all values of $\theta > 0$, and the capacity is thus given by (1.41).

Now, consider the arbitrarily varying noisy-typewriter channel $W$ with causal SI, under state constraint $\Lambda$. We use the formula in (A.72) (by [17]) to find an explicit expression for $C(W)$. There are nine mappings $\xi_u : \mathcal{S} \rightarrow \mathcal{X}$. For $\xi_1(s) = 0, \xi_2(s) = 1$ and $\xi_3(s) = 2$, we have
\[
H(Z - \xi_u(S)) = H(Z) = h(\theta) + \theta h(q), \quad u = 1, 2, 3. \tag{A.79}
\]
For $\xi_4(s) = s, \xi_5(s) = s + 1$ and $\xi_6(s) = s + 2$, we have
\[
H(Z - \xi_u(S)) = H((K - 1) \cdot S) = h(\theta) + (1 - \theta) h(q), \quad u = 4, 5, 6. \tag{A.80}
\]
For $\xi_7(s) = 2s, \xi_8(s) = 2s + 1$ and $\xi_9(s) = 2s + 2$, we have
\[
H(Z - \xi_u(S)) = H((K - 2) \cdot S) = h(\theta \ast q), \quad u = 7, 8, 9, \tag{A.81}
\]
where $\theta \ast q = \theta(1 - q) + (1 - \theta)q$. Therefore,
\[
C(W) = \log 3 - \min (h(\theta) + \theta h(q), h(\theta) + (1 - \theta) h(q), h(\theta \ast q)). \tag{A.82}
\]
Therefore,
\[
R_{\Omega}(W) = \begin{cases} 
\log 3 - \min (h(\theta) + \theta h(\Lambda), h(\theta) + (1 - \theta) h(\Lambda), h(\theta \ast \Lambda)) & \text{if } 0 \leq \Lambda < \frac{1}{2} , \\
\log 3 - \min (h(\theta) + \theta, h(\theta) + (1 - \theta), 1) & \text{if } \Lambda \geq \frac{1}{2} . 
\end{cases} \tag{A.83}
\]
and by part 2 of Theorem 10, the random code capacity of the AVC $W$ with causal SI, under state constraint $\Lambda$, is given by $C_{\Omega}(W) = R^{\ast}_{\Omega}(W) = R^{\ast}_{\Omega}(W)$.

Let us examine the condition in part 2 of Theorem 12. Assume to the contrary that $V_0^\xi$ is symmetrizable. In particular, taking $\xi_{u_1}(s) = s$ and $\xi_{u_2}(s) = 2s$, i.e. $u_1 = 4$ and $u_2 = 7$, we have that for some $\beta_u$, where $\beta_u = J(2|u)$,
\[
(1 - \beta_7)W_{Y|X,S}(y|1,1) + \beta_7 W_{Y|X,S}(y|2,2) = \\
(1 - \beta_4)W_{Y|X,S}(y|2,1) + \beta_4 W_{Y|X,S}(y|1,2). \tag{A.84}
\]
Thus, for \( y = 0 \), we get \( 0 = \theta \), which contradicts our assumption that \( \theta > 0 \), and by part 2 of Theorem 12, \( C_{\Omega, \Lambda} (W) = R_{\text{low}, \Omega, \Lambda} (W) \).

## Appendix B

### AVDBC with Causal SI: Proofs

#### B.1 Proof of Lemma 16

We show that every rate pair \((R_1, R_2) \in R_m (B^Q)\) can be achieved using deterministic codes over the compound DBC \(B^Q\) with causal SI. We construct a code based on superposition coding with Shannon strategies, and decode using joint typicality with respect to a channel state type, which is “close” to some \( q \in Q \).

Define a set \( \hat{Q}_n \) of state types

\[
\hat{Q}_n = \left\{ \hat{P}_{s^n} : s^n \in A^{\delta_1} (q) , \text{ for some } q \in Q \right\} ,
\]

where

\[
\delta_1 \triangleq \frac{\delta}{2 |S|} .
\]

That is, \( \hat{Q}_n \) is the set of types that are \( \delta_1 \)-close to some state distribution \( q(s) \) in \( Q \). Now, a code for the compound DBC with causal SI is constructed as follows

**Codebook Generation:** Fix the distribution \( P_{U_1, U_2} (u_1, u_2) = p(u_1, u_2) \) and the function \( \xi(u_1, u_2, s) \). Generate \( 2^{nR_2} \) independent sequences at random,

\[
u_2^n (m_2) \sim \prod_{i=1}^{n} P_{U_2} (u_{2,i}) , \text{ for } m_2 \in [1 : 2^{nR_2}] .
\]

For every \( m_2 \in [1 : 2^{nR_2}] \), generate \( 2^{nR_1} \) sequences at random,

\[
u_1^n (m_1, m_2) \sim \prod_{i=1}^{n} P_{U_1|U_2} (u_{1,i}|u_{2,i}(m_2)) , \text{ for } m_1 \in [1 : 2^{nR_1}] ,
\]

conditionally independent given \( u_2^n (m_2) \).

**Encoding:** To send a pair of messages \((m_1, m_2) \in [1 : 2^{nR_1}] \times [1 : 2^{nR_2}]\), transmit at time \( i \in [1 : n] \),

\[
x_i = \xi \left( u_{1,i}(m_1, m_2), u_{2,i}(m_2), s_i \right) .
\]
Decoding: Let

\[ P_{U_1,U_2,Y_1,Y_2}(u_1, u_2, y_1, y_2) = \sum_{s \in S} q(s) P_{U_1,U_2}(u_1, u_2) W_{Y_1,Y_2|X,S}(y_1, y_2|\xi(u_1, u_2, s), s) . \]

(B.6)

Observing \( y_2^n \), decoder 2 finds a unique \( \tilde{m}_2 \in [1 : 2^{nR_2}] \) such that

\[ (u_2^n(\tilde{m}_2), y_2^n) \in A^d(P_{U_2}P_{Y_2|U_2}^n) \ , \text{ for some } q \in \hat{Q}_n . \]

(B.7)

If there is none, or more than one such \( \tilde{m}_2 \in [1 : 2^{nR_2}] \), then decoder 2 declares an error.

Observing \( y_1^n \), decoder 1 finds a unique pair of messages \( (\hat{m}_1, \hat{m}_2) \in [1 : 2^{nR_1}] \times [1 : 2^{nR_1}] \) such that

\[ (u_1^n(\hat{m}_2), u_1^n(\hat{m}_1), y_1^n) \in A^d(P_{U_1,U_2}P_{Y_2|U_1,U_2}^n) \ , \text{ for some } q \in \hat{Q}_n . \]

(B.8)

If there is none, or more than such pair \( (\hat{m}_1, \hat{m}_2) \), then decoder 1 declares an error.

Analysis of Probability of Error: By the union of events bound,

\[ P_e^{(n)}(q, \mathcal{E}) \leq \Pr(\tilde{M}_2 \neq 1) + \Pr((\hat{M}_1, \tilde{M}_2) \neq (1, 1)) , \]

(B.9)

where the conditioning on \((M_1, M_2) = (1, 1)\) is omitted for convenience of notation. The error event for decoder 2 is the union of the following events.

\[ \mathcal{E}_{2,1} = \{(U_2^n(1), Y_2^n) \notin A^d(P_{U_2}P_{Y_2|U_2}^n) \text{ for all } q' \in \hat{Q}_n \} , \]

(B.10)

\[ \mathcal{E}_{2,2} = \{(U_2^n(m_2), Y_2^n) \in A^d(P_{U_2}P_{Y_2|U_2}^n) \text{ for some } m_2 \neq 1, q' \in \hat{Q}_n \} . \]

(B.11)

Then, by the union of events bound,

\[ \Pr(\tilde{M}_2 \neq 1) \leq \Pr(\mathcal{E}_{2,1}) + \Pr(\mathcal{E}_{2,2}) . \]

(B.12)

Considering the first term, we claim that the event \( \mathcal{E}_{2,1} \) implies that \((U_2^n(1), Y_2^n) \notin A^d(P_{U_2}P_{Y_2|U_2}^n) \) for all \( q'' \in Q \). Suppose that there exists \( q'' \in Q \) that satisfies \((U_2^n(1), Y_2^n) \in A^{d/2}(P_{U_2}P_{Y_2|U_2}^{q''}) \). Then, for a sufficiently large \( n \), there exists a type \( q'(s) \) such that

\[ |q'(s) - q''(s)| \leq \delta_1 . \]

(B.13)

It can then be inferred that \( q' \in \hat{Q}_n \) (see (B.1)), and

\[ |P_{Y_2|U_2}(y_2|u_2) - P_{Y_2|U_2}^{q''}(y_2|u_2)| \leq |S| \cdot \delta_1 = \frac{\delta}{2} , \]

(B.14)

for all \( u_2 \in U_2 \) and \( y_2 \in Y_2 \) (see (B.2) and (B.6)). Hence, \((U_2^n(1), Y_2^n) \notin A^d(P_{U_2}P_{Y_2|U_2}^{q''}) \). Equivalently, if \((U_2^n(1), Y_2^n) \notin A^{d/2}(P_{U_2}P_{Y_2|U_2}^{q''}) \) for all \( q' \in \hat{Q}_n \), then \((U_2^n(1), Y_2^n) \notin A^{d/2}(P_{U_2}P_{Y_2|U_2}^{q''}) \) for all \( q'' \in Q \). Thus,

\[ \Pr(\mathcal{E}_{2,1}) \leq \Pr\left( (U_2^n(1), Y_2^n) \notin A^{d/2}(P_{U_2}P_{Y_2|U_2}^{q''}) \text{ for all } q'' \in Q \right) \]

\[ \leq \Pr\left( (U_2^n(1), Y_2^n) \notin A^{d/2}(P_{U_2}P_{Y_2|U_2}^{q''}) \right) . \]

(B.15)
The last expression tends to zero exponentially as \( n \to \infty \) by the law of large numbers and Chernoff’s bound.

Moving to the second term in the RHS of (B.12), we use the classic method of types considerations to bound \( \Pr(\mathcal{E}_{2,2}) \). By the union of events bound and the fact that the number of type classes in \( S^n \) is bounded by \((n + 1)^{|S|}\) [13, Lemma 2.2], we have that

\[
\Pr(\mathcal{E}_{2,2}) \leq (n + 1)^{|S|} \cdot \sup_{q' \in \hat{Q}_n} \Pr\left( (U_2^n(m_2), Y_2^n) \in \mathcal{A}^\delta(P_{U_2} P_{Y_2 | U_2}^{q'}) \text{ for some } m_2 \neq 1 \right). \tag{B.16}
\]

For every \( m_2 \neq 1 \),

\[
\Pr \left( (U_2^n(m_2), Y_2^n) \in \mathcal{A}^\delta(P_{U_2} P_{Y_2 | U_2}^{q'}) \right) = \sum_{u_2^n \in U_2^n} P_{U_2}^{u_2^n} \cdot \Pr \left( (u_2^n, Y_2^n) \in \mathcal{A}^\delta(P_{U_2} P_{Y_2 | U_2}^{q'}) \right) = \sum_{u_2^n \in U_2^n} P_{U_2}^{u_2^n} \cdot \sum_{y_2^n : (u_2^n, y_2^n) \in \mathcal{A}^\delta(P_{U_2} P_{Y_2 | U_2}^{q'})} P_{Y_2}^{y_2^n}, \tag{B.17}
\]

where the first equality holds since \( U_2^n(m_2) \) is independent of \( Y_2^n \) for every \( m_2 \neq 1 \). Let \( (u_2^n, y_2^n) \in \mathcal{A}^\delta(P_{U_2} P_{Y_2 | U_2}^{q'}) \). Then, \( y_2^n \in \mathcal{A}^{\delta_2}(P_{Y_2}^{q'}) \) with \( \delta_2 \triangleq |U_2| \cdot \delta \). By Lemmas 2.6 and 2.7 in [13],

\[
P_{Y_2}^{y_2^n} = 2^{-n(H(\hat{P}_{y_2^n}^{q'}) + D(P_{y_2}^{q'} || P_{Y_2}^{y_2^n}))} \leq 2^{-n(H(\hat{P}_{y_2}^{q'}) - \varepsilon_1(\delta))} \leq 2^{-n(H_{q'}(Y_2) - \varepsilon_1(\delta))}, \tag{B.18}
\]

where \( \varepsilon_1(\delta) \to 0 \) as \( \delta \to 0 \). Therefore, by (B.16)–(B.18),

\[
\Pr(\mathcal{E}_{2,2}) \leq (n + 1)^{|S|} \cdot \sup_{q' \in \hat{Q}_n} \left[ 2^{nR_2} \cdot \sum_{u_2^n \in U_2^n} P_{U_2}^{u_2^n} \cdot \left| \{y_2^n : (u_2^n, y_2^n) \in \mathcal{A}^\delta(P_{U_2} P_{Y_2 | U_2}^{q'}) \} \right| \cdot 2^{-n(H_{q'}(Y_2) - \varepsilon_1(\delta))} \right] \leq (n + 1)^{|S|} \cdot \sup_{q' \in \hat{Q}_n} 2^{-n[I_{q'}(U_2; Y_2) - R_2 - \varepsilon_2(\delta)]}, \tag{B.19}
\]

with \( \varepsilon_2(\delta) \to 0 \) as \( \delta \to 0 \), where the last inequality is due to [13, Lemma 2.13]. The RHS of (B.19) tends to zero exponentially as \( n \to \infty \), provided that \( R_2 < \inf_{q' \in \hat{Q}_n} I_{q'}(U_2; Y_2) - \varepsilon_2(\delta) \).

Now, consider the error event of decoder 1. For every \( (m_1, m_2) \in [1 : 2^{nR_1}] \times [1 : 2^{nR_1}] \), define the events

\[
\mathcal{E}_{1,2}(m_2) = \{(U_2^n(m_2), Y_1^n) \in \mathcal{A}^{\delta_1}(P_{U_2} P_{Y_2 | U_2}^{q'}) \text{ for some } q' \in \hat{Q}_n \},
\]

\[
\mathcal{E}_{1,1}(m_1, m_2) = \{(U_2^n(m_2), U_1^n(m_1, m_2), Y_1^n) \in \mathcal{A}^\delta(P_{U_2, U_1} P_{Y_2, Y_1 | U_2, U_1}^{q'}) \text{ for some } q' \in \hat{Q}_n \},
\]
where $\delta_3 \triangleq |U_1|\delta$. Then, the error event is bounded by

$$\left\{(\hat{M}_1, \hat{M}_2) \neq (1, 1)\right\} \subseteq \mathcal{E}_{1,1}(1, 1)^c \cup \bigcup_{m_1 \neq 1} \mathcal{E}_{1,1}(m_1, 1) \cup \bigcup_{m_1 \in [1:2^{R_1}], m_2 \neq 1} \mathcal{E}_{1,1}(m_1, m_2)$$

$$\subseteq \mathcal{E}_{1,1}(1, 1)^c \cup \bigcup_{m_1 \neq 1} \mathcal{E}_{1,1}(m_1, 1) \cup \bigcup_{m_2 \neq 1} \mathcal{E}_{1,2}(m_2), \quad (B.20)$$

where the last line follows from the fact that if the event $\mathcal{E}_{1,1}(m_1, m_2)$ occurs, then $\mathcal{E}_{1,2}(m_2)$ occurs as well. Thus, by the union of events bound,

$$\Pr \left((\hat{M}_1, \hat{M}_2) \neq (1, 1)\right) \leq \Pr (\mathcal{E}_{1,1}(1, 1)^c) + \sum_{m_2 \neq 1} \Pr (\mathcal{E}_{1,2}(m_2)) + \sum_{m_1 \neq 1} \Pr (\mathcal{E}_{1,1}(m_1, 1))$$

$$\leq 2^{-\delta n} + 2^{-n \left(\inf_{q' \in Q} I_{q'}(U_2,Y_1) - R_2 - \varepsilon_3(\delta)\right)} + \sum_{m_1 \neq 1} \Pr (\mathcal{E}_{1,1}(m_1, 1)),$$

$$\quad (B.21)$$

where the last inequality follows from the law of large numbers and type class considerations used before, with $\varepsilon_3(\delta) \to 0$ as $\delta \to 0$. Since the compound DBC is assumed to be degraded, we have that $I_q(U_2,Y_1) \geq I_q(U_2,Y_2)$ for all $q' \in \mathcal{P}(\mathcal{S})$. Thus, taking $R_2 < \inf_{q' \in Q} I_{q'}(U_2,Y_2) - \varepsilon_2(\delta)$ guarantees that the middle term in the RHS of $(B.21)$ tends to zero exponentially as $n \to \infty$. It remains for us to bound the last sum. Using similar type class considerations, we have that for every $q' \in \hat{Q}_n$ and $m_1 \neq 1,$

$$\Pr \left((U_2^n(1), U_1^n(m_1, 1), Y_1^n) \in \mathcal{A}^\delta (P_{U_2, U_1, P_{Y_1|U_2, U_1}^{q'}})\right)$$

$$= \sum_{(u_2^n, u_1^n, y_1^n) \in \mathcal{A}^\delta (P_{U_2, U_1, P_{Y_1|U_2, U_1}^{q'}})} P_{U_2^n}(u_2^n) \cdot P_{U_1^n|U_2^n}(u_1^n|u_2^n) \cdot P_{Y_1^n|U_2^n}(y_1^n|u_2^n)$$

$$\leq 2^{n(H_q(U_2,U_1,Y_1)+\varepsilon_4(\delta))} \cdot 2^{-n(H(U_2) - \varepsilon_4(\delta))} \cdot 2^{-n(H(U_1|U_2) - \varepsilon_4(\delta))} \cdot 2^{-n(H_q(Y_1|U_2) - \varepsilon_4(\delta))}$$

$$= 2^{-n(L_q(U_1,Y_1) - \varepsilon_4(\delta))}, \quad (B.22)$$

where $\varepsilon_4(\delta) \to 0$ as $\delta \to 0$. Therefore, the sum term in the RHS of $(B.21)$ is bounded by

$$\sum_{m_1 \neq 1} \Pr (\mathcal{E}_{1,1}(m_1, 1)) \quad \quad (B.23)$$

$$= \sum_{m_1 \neq 1} \Pr \left((U_2^n(1), U_1^n(m_1, 1), Y_1^n) \in \mathcal{A}^\delta (P_{U_2, U_1, P_{Y_1|U_2, U_1}^{q'}}), \text{ for some } q' \in \hat{Q}_n\right)$$

$$\leq (n + 1)|\mathcal{S}| \cdot 2^{-n \left(\inf_{q' \in Q} I_{q'}(U_1,Y_1|U_2) - R_1 - \varepsilon_5(\delta)\right)}, \quad (B.24)$$

where the last line follows from $(B.22)$, and $\varepsilon_5(\delta) \to 0$ as $\delta \to 0$. The last expression tends to zero exponentially as $n \to \infty$ and $\delta \to 0$ provided that $R_1 < \inf_{q' \in Q} I_{q'}(U_1,Y_1|U_2) - \varepsilon_5(\delta).$
The probability of error, averaged over the class of the codebooks, exponentially decays to zero as $n \to \infty$. Therefore, there must exist a $(2^{nR_1}, 2^{nR_2}, n, \varepsilon)$ deterministic code, for a sufficiently large $n$. \hfill \Box

### B.2 Proof of Theorem 17

#### Part 1

At the first part of the theorem it is assumed that the interior of the capacity region is non-empty, i.e. $\text{int}(\mathcal{C}(\mathcal{B}^Q)) \neq \emptyset$. Denote the marginal compound channels with causal SI, corresponding to user 1 and user 2, by

$$\mathcal{W}_1^Q = \{Q, W_{Y_1|X,s}\}, \quad \text{and} \quad \mathcal{W}_2^Q = \{Q, W_{Y_2|X,s}\} \tag{B.26}$$

respectively. Since the compound DBC is assumed to be degraded, this means that

$$\mathcal{C}(\mathcal{W}_1^Q) \geq \mathcal{C}(\mathcal{W}_2^Q) > 0. \tag{B.27}$$

**Achievability proof.** We show that every rate pair $(R_1, R_2) \in \mathcal{R}_{\text{out}}(\mathcal{B}^Q)$ can be achieved using a code based on Shannon strategies with the addition of a code word suffix. At time $i = n + 1$, having completed the transmission of the messages, the type of the state sequence $s^n$ is known to the encoder. Following the assumption that the interior of the capacity region is non-empty, the type of $s^n$ can be reliably communicated to both receivers as a suffix, while the blocklength is increased by $\nu > 0$ additional channel uses, where $\nu$ is small compared to $n$. The receivers first estimate the type of $s^n$, and then use joint typicality with respect to the estimated type. The details are provided below.

By (B.27), we have that for every $\varepsilon_1 > 0$ and sufficiently large blocklength $\nu$, there exists a $(2^{\nu R_1}, 2^{\nu R_2}, \nu, \varepsilon_1)$ code $\tilde{C} = (\tilde{f}_\nu, \tilde{g}_1, \tilde{g}_2)$ for the transmission of a type $\hat{P}_{s^n}$ at positive rates $\tilde{R}_1 > 0$ and $\tilde{R}_2 > 0$. Since the total number of types is polynomial in $n$ (see [13]), the type $\hat{P}_{s^n}$ can be transmitted at a negligible rate, with a blocklength that grows a lot slower than $n$, i.e.

$$\nu = o(n). \tag{B.28}$$

We now construct a code $\mathcal{C}$ over the compound DBC with causal SI, such that the blocklength is $n + o(n)$, and the rate $R'_n$ approaches $R$ as $n \to \infty$.

**Codebook Generation:** Fix the distribution $P_{U_1,U_2}(u_1, u_2) = p(u_1, u_2)$ and the function $\xi(u_1, u_2, s)$. Generate $2^{nR_2}$ independent sequences $u_2^n(m)$, $m \in [1: 2^{nR_2}]$, at random, each according to $\prod_{i=1}^{n} P_{U_2}(u_{2,i})$. For every $m_2 \in [1: 2^{nR_2}]$, generate $2^{nR_1}$ sequences at random,

$$u_1^n(m_1, m_2) \sim \prod_{i=1}^{n} P_{U_1|U_2}(u_{1,i}|u_{2,i}(m_2)), \text{ for } m_1 \in [1: 2^{nR_1}], \tag{B.29}$$

conditionally independent given $u_2^n(m_2)$. Reveal the codebook of the message pair $(m_1, m_2)$ and the codebook of the type $\hat{P}_{s^n}$ to the encoder and the decoders.
Encoding: To send a message pair \((m_1, m_2) \in [1 : 2^{nR_1}] \times [1 : 2^{nR_2}]\), transmit at time \(i \in [1 : n]\),

\[
x_i = \xi(u_{1,i}(m_1, m_2), u_{2,i}(m_2), s_i) .
\] (B.30)

At time \(i \in [n + 1 : n + \nu]\), knowing the sequence of previous states \(s^n\), transmit

\[
x_i = \tilde{f}_i(\hat{P}_{s^n}, s_{n+1}, \ldots, s_{n+i}) ,
\] (B.31)

where \(\hat{P}_{s^n}\) is the type of the sequence \((s_1, \ldots, s_n)\). That is, the encoded type \(\hat{P}_{s^n}\) is transmitted as a suffix of the codeword. We note that the type of the sequence \((s_{n+1}, \ldots, s_{n+i})\) is not necessarily \(\hat{P}_{s^n}\), and it is irrelevant for that matter since by (B.27), there exists a \((2^{\nu R_1}, 2^{\nu R_2}, \nu, \varepsilon_1)\) code \(\tilde{C} = (\tilde{f}_\nu, \tilde{g}_1, \tilde{g}_2)\) for the transmission of \(\hat{P}_{s^n}\) over the compound DBC with causal SI, with \(\hat{R}_1 > 0\) and \(\hat{R}_2 > 0\).

Decoding: Decoder 2 receives the output sequence \(y_{2}^{n+\nu}\). As a pre-decoding step, the receiver decodes the last \(\nu\) output symbols, and finds an estimate of the type of the state sequence,

\[
\hat{q}_2 = \tilde{g}_2(y_{2,n+1}, \ldots, y_{2,n+\nu}) .
\] (B.32)

Then, given the output sequence \(y_{2}^{n}\), decoder 2 finds a unique \(\hat{m}_2 \in [1 : 2^{nR_2}]\) such that

\[
(u_{2}^{n}(\hat{m}_2), y_{2}^{n}) \in A^\delta(P_{U_2}P_\hat{Y}_2^{\hat{g}_2}[U_2]) .
\] (B.33)

If there is none, or more than one such \(\hat{m}_2 \in [1 : 2^{nR_2}]\), then decoder 2 declares an error.

Similarly, decoder 1 receives \(y_{1}^{n+\nu}\) and begins with decoding the type of the state sequence,

\[
\hat{q}_1 = \tilde{g}_1(y_{1,n+1}, \ldots, y_{1,n+\nu}) .
\] (B.34)

Then, decoder 1 finds a unique pair of messages \((\hat{m}_1, \hat{m}_2) \in [1 : 2^{nR_1}] \times [1 : 2^{nR_2}]\) such that

\[
(u_{2}^{n}(\hat{m}_2), u_{1}^{n}(\hat{m}_1, \hat{m}_2), y_{1}^{n}) \in A^\delta(P_{U_2,U_1}P_\hat{Y}_1^{\hat{g}_1}[U_2,U_1]) .
\] (B.35)

If there is none, or more than one such pair \((\hat{m}_1, \hat{m}_2) \in [1 : 2^{nR_1}] \times [1 : 2^{nR_2}]\), then decoder 1 declares an error.

Analysis of Probability of Error: By symmetry, we may assume without loss of generality that the users sent \((M_1, M_2) = (1, 1)\). Let \(q(s) \in \mathcal{Q}\) denote the actual state distribution chosen by the jammer, and let \(q^n(s^n) = \prod_{i=1}^{n} q(s_i)\). Then, by the union of events bound, the probability of error is bounded by

\[
P_e^{(n)}(q, \mathcal{C}) \leq \Pr(\hat{M}_2 \neq 1) + \Pr((\hat{M}_1, \hat{M}_2) \neq (1, 1)) ,
\] (B.36)

where the conditioning on \((M_1, M_2) = (1, 1)\) is omitted for convenience of notation.
Define the events
\[
\mathcal{E}_{1,0} = \{ \tilde{q}_1 \neq \hat{P}_s \}
\]
\[
\mathcal{E}_{1,1}(m_1, m_2, q') = \{ (U_n^2(m_2), U_1^1(m_1, m_2), Y_1^n) \in \mathcal{A}^\delta(P_{U_2, U_1}) \}
\]
\[
\mathcal{E}_{1,2}(m_2, q') = \{ (U_2^n(m_2), Y_1^n) \in \mathcal{A}^{\delta}(P_{U_2}, P_{Y_1^1 U_2}) \},
\]
and
\[
\mathcal{E}_{2,0} = \{ \tilde{q}_2 \neq \hat{P}_s \}
\]
\[
\mathcal{E}_{2,1}(m_2, q') = \{ (U_2^n(m_2), Y_2^n) \in \mathcal{A}^{\delta}(P_{U_2}, P_{Y_1^2 U_2}) \}
\]
for every \( m_1 \in [1 : 2^{nR_1}] \), \( m_2 \in [1 : 2^{nR_2}] \), and \( q' \in \mathcal{P} \), where \( \delta = |\mathcal{U}| \). The error event of decoder 2 is bounded by
\[
\left\{ \tilde{M}_2 \neq 1 \right\} \subseteq \mathcal{E}_{2,0} \cup \mathcal{E}_{2,1}(1, \tilde{q}_2)^c \cup \bigcup_{m_2 \neq 1} \mathcal{E}_{2,1}(m_2, \tilde{q}_2)
\]
\[
= \mathcal{E}_{2,0} \cup \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1, \tilde{q}_2)^c \right) \cup \left( \bigcup_{m_2 \neq 1} \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(m_2, \tilde{q}_2) \right).
\]
By the union of events bound,
\[
\Pr \left( \tilde{M}_2 \neq 1 \right)
\]
\[
\leq \Pr (\mathcal{E}_{2,0}) + \Pr (\mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1, \tilde{q}_2)^c) + \Pr \left( \bigcup_{m_2 \neq 1} \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(m_2, \tilde{q}_2) \right) \quad \text{ (B.42)}
\]
Since the code \( \tilde{\mathcal{C}} \) for the transmission of the type is a \((2^{\nu R_1}, 2^{\nu R_2}, \nu, \varepsilon_1)\) code, where \( \varepsilon_1 > 0 \) is arbitrarily small, we have that the probability of erroneous decoding of the type is bounded by
\[
\Pr (\mathcal{E}_{1,0} \cup \mathcal{E}_{2,0}) \leq \varepsilon_1 \text{ .} \quad \text{ (B.43)}
\]
Thus, the first term in the RHS of (B.42) is bounded by \( \varepsilon_1 \). Then, we manipulate the last two terms as follows.
\[
\Pr \left( \tilde{M}_2 \neq 1 \right) \leq \sum_{s^n \in \mathcal{A}^{\mathcal{C}_2(q)}} q^n(s^n) \Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1, \tilde{q}_2)^c \mid S^n = s^n \right)
\]
\[
+ \sum_{s^n \in \mathcal{A}^{\mathcal{C}_2(q)}} q^n(s^n) \Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1, \tilde{q}_2)^c \mid S^n = s^n \right)
\]
\[
+ \sum_{s^n \in \mathcal{A}^{\mathcal{C}_2(q)}} q^n(s^n) \Pr \left( \bigcup_{m_2 \neq 1} \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(m_2, \tilde{q}_2) \mid S^n = s^n \right)
\]
\[
+ \sum_{s^n \in \mathcal{A}^{\mathcal{C}_2(q)}} q^n(s^n) \Pr \left( \bigcup_{m_2 \neq 1} \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(m_2, \tilde{q}_2) \mid S^n = s^n \right) + \varepsilon_1 ,
\]
(B.44)
where
\[ \delta_2 \triangleq \frac{1}{2|\mathcal{S}|} \delta. \] (B.45)

Next we show that the first and the third sums in (B.44) tend to zero as \( n \to \infty \).

Consider a given \( s^n \in \mathcal{A}^{\delta_2}(q) \). For notational convenience, denote
\[ q'' = \hat{P}_{s^n}. \] (B.46)

Then, by the definition of the \( \delta \)-typical set, we have that
\[ |q''(s) - q(s)| \leq \delta_2 \text{ for all } s \in \mathcal{S}, \text{ and } q''(s) = 0 \text{ when } q(s) = 0. \]

It follows that
\[
|P_{U_2}(u_2)P_{Y_2|U_2}^{q''}(y|u_2) - P_{U_2}(u_2)P_{Y_2|U_2}^{q}(y|u_2)| \leq \delta_2 \cdot \sum_{s,u_1} P_{U_1|U_2}(u_1|u_2)W_{Y_2|X,S}(y_2|x(u_1,u_2),s) \\
\leq \delta_2 \cdot \sum_{s,u_1} P_{U_1|U_2}(u_1|u_2) = \delta_2 \cdot |\mathcal{S}| = \frac{\delta}{2}. \] (B.47)

for all \( u_2 \in \mathcal{U}_2 \) and \( y_2 \in \mathcal{Y}_2 \), where the last equality follows from (B.45).

Consider the first sum in the RHS of (B.44). Given a state sequence \( s^n \in \mathcal{A}^{\delta_2}(q) \), we have that
\[
\Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1,\hat{q}_2) \mid S^n = s^n \right) \\
= \Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1,P_{s^n}) \mid S^n = s^n \right) \\
= \Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1,q'') \mid S^n = s^n \right) \\
= \Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1,q''), S^n = s^n \right) \cdot \Pr \left( \mathcal{E}_{2,1}(1,q'') \mid S^n = s^n \right), \] (B.48)

where the first equality follows from (B.40), and the second equality follows from (B.46). Then,
\[
\Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1,\hat{q}_2) \mid S^n = s^n \right) \leq \Pr \left( \mathcal{E}_{2,1}(1,q'') \mid S^n = s^n \right) \\
= \Pr \left( \left( U_2^n(1), Y_2^n \right) \notin \mathcal{A}^\delta(P_{U_2}{Y_2|U_2}) \mid S^n = s^n \right). \] (B.49)

Now, suppose that \( (U_2^n(1), Y_2^n) \in \mathcal{A}^{\delta_2}(P_{U_2}{Y_2|U_2}) \), where \( q \) is the actual state distribution. By (B.47), in this case we have that \( (U_2^n(1), Y_2^n) \in \mathcal{A}^\delta(P_{U_2}{Y_2|U_2}). \) Hence, by (B.49), we have that
\[
\Pr \left( \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(1,\hat{q}_2)^c \mid S^n = s^n \right) \\
\leq \Pr \left( \left( U_2^n(1), Y_2^n \right) \notin \mathcal{A}^\delta(P_{U_2}{Y_2|U_2}) \mid S^n = s^n \right). \] (B.50)
The first sum in the RHS of (B.44) is then bounded as follows.

\[
\sum_{s^n \in \mathcal{A}^2(q)} q^n(s^n) \Pr(\mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(m_2, \hat{q}_2)^c \mid S^n = s^n)
\]

\[
\leq \sum_{s^n \in \mathcal{A}^2(q)} q^n(s^n) \Pr\left((U_2^n(1), Y_2^n) \notin \mathcal{A}^{3/2}(P_{U_2}P_{Y_2}^q) \mid S^n = s^n\right)
\]

\[
\leq \sum_{s^n \in \mathcal{A}^2} q^n(s^n) \Pr\left((U_2^n(1), Y_2^n) \notin \mathcal{A}^{3/2}(P_{U_2}P_{Y_2}^q) \mid S^n = s^n\right)
\]

\[
= \Pr\left((U_2^n(1), Y_2^n) \notin \mathcal{A}^{3/2}(P_{U_2}P_{Y_2}^q)\right) \leq \varepsilon_2, \quad \text{(B.51)}
\]

for a sufficiently large \(n\), where the last inequality follows from the law of large numbers.

We bound the third sum in the RHS of (B.44) using similar arguments. If \((U_2^n(m_2), Y_2^n) \in \mathcal{A}^2(P_{U_2}P_{Y_2}^q)\), then \((U_2^n(m_2), Y_2^n) \notin \mathcal{A}^{3/2}(P_{U_2}P_{Y_2}^q)\), due to (B.47).

Thus, for every \(s^n \in \mathcal{A}^2(q)\),

\[
\Pr\left(\bigcup_{m_2 \neq 1} \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(m_2, \hat{q}_2) \mid S^n = s^n\right)
\]

\[
\leq \sum_{m_2 \neq 1} \Pr(\mathcal{E}_{2,1}(m_2, q^n) \mid S^n = s^n)
\]

\[
= \sum_{m_2 \neq 1} \Pr\left((U_2^n(m_2), Y_2^n) \in \mathcal{A}^2(P_{U_2}P_{Y_2}^q) \mid S^n = s^n\right)
\]

\[
\leq \sum_{m_2 \neq 1} \Pr\left((U_2^n(m_2), Y_2^n) \in \mathcal{A}^{3/2}(P_{U_2}P_{Y_2}^q) \mid S^n = s^n\right). \quad \text{(B.52)}
\]

This, in turn, implies that the third sum in the RHS of (B.44) is bounded by

\[
\sum_{s^n \in \mathcal{A}^2(q)} q^n(s^n) \Pr\left(\bigcup_{m_2 \neq 1} \mathcal{E}_{2,0}^c \cap \mathcal{E}_{2,1}(m_2, \hat{q}) \mid S^n = s^n\right)
\]

\[
\leq \sum_{s^n \in \mathcal{A}^2} \sum_{m_2 \neq 1} q^n(s^n) \cdot \Pr\left((U_2^n(m_2), Y_2^n) \in \mathcal{A}^{3/2}(P_{U_2}P_{Y_2}^q) \mid S^n = s^n\right)
\]

\[
= \sum_{m_2 \neq 1} \Pr\left((U_2^n(m_2), Y_2^n) \in \mathcal{A}^{3/2}(P_{U_2}P_{Y_2}^q)\right)
\]

\[
\leq 2^{-n[I_q(U_2; Y_2) - R_2 - \varepsilon_2(\delta)]}, \quad \text{(B.53)}
\]

with \(\varepsilon_2(\delta) \to 0\) as \(\delta \to 0\). The last inequality follows from standard type class considerations. The RHS of (B.53) tends to zero as \(n \to \infty\), provided that

\[
R_2 < I_q(U_2; Y_2) - \varepsilon_2(\delta), \quad \text{(B.54)}
\]

for some \(p(u_1, u_2)\) and \(\xi(u_1, u_2, s)\). Then, it follows from the law of large numbers that the second and fourth sums in the RHS of (B.44) tend to zero as \(n \to \infty\). Thus, by
(B.51) and (B.53), we have that the probability of error of decoder 2, \( \Pr \left( \tilde{M}_2 \neq 1 \right) \), tends to zero as \( n \to \infty \).

Now, consider the error event of decoder 1,

\[
\{ (\tilde{M}_1, \tilde{M}_2) \neq (1, 1) \}
\subseteq \mathcal{E}_{1,0} \cup \mathcal{E}_{1,1}(1, 1, \tilde{q}_1)^c \cup \bigcup_{(m_1, m_2) \neq (1, 1)} \mathcal{E}_{1,1}(m_1, m_2, \tilde{q}_1)
= \mathcal{E}_{1,0} \cup \mathcal{E}_{1,1}(1, 1, \tilde{q}_1)^c \cup \bigcup_{m_1 \neq 1} \mathcal{E}_{1,1}(m_1, 1, \tilde{q}_1) \cup \bigcup_{m_2 \neq 1} \mathcal{E}_{1,1}(m_1, m_2, \tilde{q}_1)
\subseteq \mathcal{E}_{1,0} \cup \mathcal{E}_{1,1}(1, 1, \tilde{q}_1)^c \cup \bigcup_{m_1 \neq 1} \mathcal{E}_{1,1}(m_1, 1, \tilde{q}_1) \cup \bigcup_{m_2 \neq 1} \mathcal{E}_{1,2}(m_2, \tilde{q}_1)
= \mathcal{E}_{1,0} \cup (\mathcal{E}_{c,0}^c \cap \mathcal{E}_{1,1}(1, 1, \tilde{q}_1)^c) \cup \bigcup_{m_1 \neq 1} (\mathcal{E}_{c,0}^c \cap \mathcal{E}_{1,2}(m_2, \tilde{q}_1))
\]  \( \text{(B.55)} \)

where the second inclusion follows from the fact that if the event \( \mathcal{E}_{1,1}(m_1, m_2, \tilde{q}_1) \) occurs, then \( \mathcal{E}_{1,2}(m_2, \tilde{q}_1) \) occurs as well. Thus, by the union of events bound,

\[
\Pr \left( (\tilde{M}_1, \tilde{M}_2) \neq (1, 1) \right)
\leq \Pr (\mathcal{E}_{1,0}) + \Pr (\mathcal{E}_{c,0}^c \cap \mathcal{E}_{1,1}(1, 1, \tilde{q}_1)^c) + \Pr \left( \bigcup_{m_2 \neq 1} \mathcal{E}_{c,0}^c \cap \mathcal{E}_{1,2}(m_2, \tilde{q}_1) \right)
+ \Pr \left( \bigcup_{m_1 \neq 1} \mathcal{E}_{c,0}^c \cap \mathcal{E}_{1,1}(m_1, 1, \tilde{q}_1) \right).
\]  \( \text{(B.56)} \)

By (B.43), the first term is bounded by \( \varepsilon_1 \), and as done above, we write

\[
\Pr \left( (\tilde{M}_1, \tilde{M}_2) \neq (1, 1) \right)
\leq \sum_{s^n \in \mathcal{A}^2(q)} q^n(s^n) \Pr \left( \mathcal{E}_{c,0}^c \cap \mathcal{E}_{1,1}(1, 1, \tilde{P}_{s^n})^c \mid S^n = s^n \right)
+ \sum_{s^n \in \mathcal{A}^2(q)} q^n(s^n) \Pr \left( \bigcup_{m_2 \neq 1} \mathcal{E}_{c,0}^c \cap \mathcal{E}_{1,2}(m_2, \tilde{P}_{s^n}) \mid S^n = s^n \right)
+ \sum_{s^n \in \mathcal{A}^2(q)} q^n(s^n) \Pr \left( \bigcup_{m_1 \neq 1} \mathcal{E}_{c,0}^c \cap \mathcal{E}_{1,1}(m_1, 1, \tilde{P}_{s^n}) \mid S^n = s^n \right)
+ 3 \cdot \Pr \left( S^n \notin \mathcal{A}^2(q) \right) + \varepsilon_1,
\]  \( \text{(B.57)} \)

where \( \delta_2 \) is given by (B.45). By the law of large numbers, the probability \( \Pr \left( S^n \notin \mathcal{A}^2(q) \right) \) tends to zero as \( n \to \infty \). As for the sums, we use similar arguments to those used above.
We have that for a given $s^n \in \mathcal{A}^{\delta_2}(q)$,
\[
|P_{U_1,U_2}(u_1, u_2)P^q_{Y_1|U_1,U_2}(y_1|u_1, u_2) - P_{U_1,U_2}(u_1, u_2)P^q_{Y_1|U_1,U_2}(y_1|u_1, u_2)|
\leq \delta_2 \cdot \sum_{s \in \mathcal{S}} W_{Y_1|X,S}(y_1|u_1, u_2, s) \leq |\mathcal{S}| \cdot \delta_2 = \frac{\delta}{2},
\]
with $q'' = \hat{P}_{s^n}$, where the last equality follows from (B.45).

The first sum in the RHS of (B.57) is bounded by
\[
\sum_{s^n \in \mathcal{A}^{\delta_2}(q)} q^n(s^n) \Pr \left( \mathcal{E}_{1,0}^c \cap \mathcal{E}_{1,1}(1, 1, \hat{P}_{s^n})^c \mid S^n = s^n \right)
\leq \sum_{s^n \in \mathcal{S}^n} q^n(s^n) \Pr \left( (U_2^n(1), U_1^n(1, 1), Y_1^n) \notin \mathcal{A}^{\delta/2}(P_{U_1,U_2}^q P_{Y_1|U_1,U_2}^q) \mid S^n = s^n \right)
= \Pr \left( (U_2^n(1), U_1^n(1, 1), Y_1^n) \notin \mathcal{A}^{\delta/2}(P_{U_1,U_2}^q P_{Y_1|U_1,U_2}^q) \right) \leq \varepsilon_2 .
\]

The last inequality follows from the law of large numbers, with a sufficiently large $n$.

The second sum in the RHS of (B.57) is bounded by
\[
\sum_{s^n \in \mathcal{A}^{\delta_2}(q)} q^n(s^n) \Pr \left( \bigcup_{m_2 \neq 1} \mathcal{E}_{1,0}^c \cap \mathcal{E}_{1,2}(m_2, \hat{P}_{s^n}) \mid S^n = s^n \right) \leq 2^{-n \left( I_q(U_2; Y_1) - R_2 - \varepsilon_3(\delta) \right)} .
\]

This is obtained following exactly the same analysis as for decoder 2. Then, the second sum tends to zero provided that
\[
R_2 \leq I_q(U_2; Y_1) - \varepsilon_3(\delta) .
\]

Since the compound DBC is assumed to be degraded, the requirement $R_2 < I_q(U_2; Y_2)$ suffices.

The third sum in the RHS of (B.57) is bounded by
\[
\sum_{s^n \in \mathcal{A}^{\delta_2}(q)} q^n(s^n) \Pr \left( \bigcup_{m_1 \neq 1} \mathcal{E}_{1,0}^c \cap \mathcal{E}_{1,1}(m_1, 1, \hat{P}_{s^n}) \mid S^n = s^n \right)
\leq \sum_{s^n \in \mathcal{A}^{\delta_2}(q)} q^n(s^n) \Pr \left( \mathcal{E}_{1,1}(m_1, 1, \hat{P}_{s^n}) \mid S^n = s^n \right). 
\]

For every $s^n \in \mathcal{A}^{\delta_2}(q)$, it follows from (B.58) that the event $\mathcal{E}_{1,1}(m_1, 1, \hat{P}_{s^n})$ implies that
\[
(U_2^n(1), U_1^n(m_1, 1), Y_1^n) \in \mathcal{A}^{\delta/2}(P_{U_2,U_1,Y_1}^q) .
\]

Thus, the sum is bounded by
\[
\sum_{s^n \in \mathcal{A}^{\delta_2}(q)} q^n(s^n) \Pr \left( \bigcup_{m_1 \neq 1} \mathcal{E}_{1,0}^c \cap \mathcal{E}_{1,1}(m_1, 1, \hat{P}_{s^n}) \mid S^n = s^n \right)
\leq 2^{-n \left( I_q(U_1; Y_1|U_2) - R_1 - \delta_3 \right)}.
\]
where $\delta_3 \to 0$ as $\delta \to 0$.

We conclude that the RHS of both (B.44) and (B.57) tend to zero as $n \to \infty$. Thus, the overall probability of error, averaged over the class of the codebooks, decays to zero as $n \to \infty$. Therefore, there must exist a $(2^{nR_1}, 2^{nR_2}, n, \varepsilon)$ deterministic code, for a sufficiently large $n$.

Converse proof. Assume to the contrary that there exists an achievable rate pair $(R_1, R_2) \notin \bigcap_{q(s) \in Q} C(B^q)$ using random codes over the compound DBC $B^Q$ with causal SI. Hence, for some state distribution $q^*(s)$ in the closure of $Q$, we have that $(R_1, R_2) \notin C(B^{q^*})$.

The achievability assumption implies that for every $\varepsilon > 0$ and sufficiently large $n$, there exists a $(2^{nR_1}, 2^{nR_2}, n)$ random code $\mathcal{C}^T$ for the compound DBC $B^Q$ with causal SI, with $P_e(n)(q, \mathcal{C}^T) \leq \varepsilon$ for all i.i.d. state distributions $q(s) \in Q$, and in particular, for $q^*(s)$, since $P_e(n)(q, \mathcal{C}^T)$ is continuous in $q$.

Consider the DBC $B^{q^*}$ with causal SI where the state sequence is i.i.d. according $q^*(s)$. If such a random code $\mathcal{C}^T$ would exist, then it could have been used over the DBC $B^{q^*}$, achieving a rate pair $(R_1, R_2) \notin C(B^{q^*})$. This is a contradiction, since the random code capacity region of $B^{q^*}$ is given by $C(B^{q^*})$ [28, Theorem 4]. We deduce that the assumption is false, and $(R_1, R_2) \notin \bigcap_{q(s) \in Q} C(B^q)$ cannot be achieved.

Part 2

We show that when the set of state distributions $Q$ is convex, and the condition $\mathcal{T}^Q$ holds, the capacity region of the compound DBC $B^Q$ with causal SI is given by $C(B^Q) = C^*(B^Q) = R_{in}(B^Q) = R_{out}(B^Q)$ (and this holds regardless of whether the interior of the capacity region is empty or not).

Due to part 1, we have that

$$C^*(B^Q) \subseteq R_{out}(B^Q). \quad (B.67)$$

By Lemma 16,

$$C(B^Q) \supseteq R_{in}(B^Q). \quad (B.68)$$

Thus,

$$R_{in}(B^Q) \subseteq C(B^Q) \subseteq C^*(B^Q) \subseteq R_{out}(B^Q). \quad (B.69)$$

To conclude the proof, we show that the condition $\mathcal{T}^Q$ implies that $R_{in}(B^Q) \supseteq R_{out}(B^Q)$, hence the inner and outer bounds coincide. By Definition 9, if a function $\xi(u_1, u_2, s)$ and a set $\mathcal{D}$ achieve $R_{in}(B^Q)$ and $R_{out}(B^Q)$, then

$$R_{in}(B^Q) = \bigcup_{p(u_1, u_2) \in \mathcal{D}} \left\{ (R_1, R_2) : R_2 \leq \min_{q \in Q} I_q(U_2; Y_2), \quad R_1 \leq \min_{q \in Q} I_q(U_1; Y_1|U_2) \right\}. \quad (B.70a)$$
and
\[
R_{\text{out}}(\mathcal{B}^Q) = \bigcap_{q(s) \in \mathcal{Q}} \bigcup_{p(u_1, u_2) \in \mathcal{D}} \left\{ (R_1, R_2) : \begin{array}{l}
R_2 \leq I_q(U_2; Y_2)
R_1 \leq I_q(U_1; Y_1 | U_2)
\end{array} \right\} . \tag{B.70b}
\]

Hence, when the condition $\mathcal{F}^Q$ holds, we have by Definition 10 that for some $\xi(u_1, u_2, s), \mathcal{D} \subseteq \mathcal{P}(U_1 \times U_2)$, and $q^* \in \mathcal{Q}$,
\[
R_{\text{in}}(\mathcal{B}^Q) = \bigcup_{p(u_1, u_2) \in \mathcal{D}} \left\{ (R_1, R_2) : \begin{array}{l}
R_2 \leq I_{q^*}(U_2; Y_2)
R_1 \leq I_{q^*}(U_1; Y_1 | U_2)
\end{array} \right\} \tag{B.71}
\]
\[
\supseteq R_{\text{out}}(\mathcal{B}^Q) , \tag{B.72}
\]
where the last line follows from (B.70b).

\section*{B.3 Proof of Theorem 18}

\subsection*{Part 1}

First, we explain the general idea. As in Chapter 1, we devise a causal version of Ahlswede’s Robustification Technique (RT) \cite{ahlswede1991, williams1997}. Namely, we use codes for the compound DBC to construct a random code for the AVDBC using randomized permutations. However, in our case, the causal nature of the problem imposes a difficulty, and the application of the RT is not straightforward.

In \cite{ahlswede1991, williams1997}, the state information is non-causal and a random code is defined via permutations of the codeword symbols. This cannot be done here, because the SI is provided to the encoder in a causal manner. We resolve this difficulty using Shannon strategy codes for the compound DBC to construct a random code for the AVDBC, applying permutations to the strategy sequence $(u^n_1, u^n_2)$, which is an integral part of the Shannon strategy code, and is independent of the channel state. The details are given below.

\subsection*{Inner Bound}

We show that the region defined in (2.22) can be achieved by random codes over the AVDBC $\mathcal{B}$ with causal SI, \textit{i.e.} $\mathcal{C}(\mathcal{B}) \supseteq R^*_n$. The proof relies on similar ideas to those in the proof of Theorem 10 in Appendix 10. We start with Ahlswede’s RT, stated below. Let $h : \mathcal{S}^n \to [0, 1]$ be a given function. If, for some fixed $\alpha_n \in (0, 1)$, and for all $q^n(s^n) = \prod_{i=1}^n q(s_i), \text{ with } q \in \mathcal{P}(\mathcal{S}),$
\[
\sum_{s^n \in \mathcal{S}^n} q^n(s^n) h(s^n) \leq \alpha_n , \tag{B.73}
\]
then,
\[
\frac{1}{n!} \sum_{\pi \in \Pi_n} h(\pi s^n) \leq \beta_n , \quad \text{for all } s^n \in \mathcal{S}^n , \tag{B.74}
\]
where $\Pi_n$ is the set of all $n$-tuple permutations $\pi : S^n \to S^n$, and $\beta_n = (n+1)^{|S|} \cdot \alpha_n$.

According to Lemma 16, for every $(R_1, R_2) \in R^*_n$, there exists a $(2^{nR_1}, 2^{nR_2}, n, e^{-2\theta n})$ Shannon strategy code for the compound DBC $B^{P(S)}$ with causal SI, for some $\theta > 0$ and sufficiently large $n$. Given such a Shannon strategy code $\mathcal{C} = (u^n_1(m_1, m_2), u^n_2(m_2), \xi(u_1, u_2, s), g_1(y^n_1), g_2(y^n_2))$, we have that (B.73) is satisfied with $h(s^n) = P_{e|s^n}^{(n)}(\mathcal{C})$ and $\alpha_n = e^{-2\theta n}$. As a result, Ahlswede’s RT tells us that

$$
\frac{1}{n!} \sum_{\pi \in \Pi_n} P_{e|\pi s^n}(\mathcal{C}) \leq (n+1)^{|S|} e^{-2\theta n} \leq e^{-\theta n}, \text{ for all } s^n \in S^n,
$$

(B.75)

for a sufficiently large $n$, such that $(n+1)^{|S|} \leq e^{\theta n}$.

On the other hand, for every $\pi \in \Pi_n$,

$$
P_{e|\pi s^n}(\mathcal{C}) = \frac{1}{2^{n(R_1+R_2)}} \sum_{m_1, m_2} \sum_{(\pi y^n_1, \pi y^n_2) \in D(m_1, m_2)} W_{Y^n_1, Y^n_2 | X^n, S^n}(\pi y^n_1, \pi y^n_2 | \xi^n(u^n_1(m_1, m_2), u^n_2(m_2), \pi s^n), \pi s^n)
$$

\hspace{1cm} (a)

$$
= \frac{1}{2^{n(R_1+R_2)}} \sum_{m_1, m_2} \sum_{(\pi y^n_1, \pi y^n_2) \notin D(m_1, m_2)} W_{Y^n_1, Y^n_2 | X^n, S^n}(\pi y^n_1, \pi y^n_2 | \pi^{-1} \xi^n(u^n_1(m_1, m_2), u^n_2(m_2), \pi s^n), s^n),
$$

\hspace{1cm} (b)

$$
= \frac{1}{2^{n(R_1+R_2)}} \sum_{m_1, m_2} \sum_{(\pi y^n_1, \pi y^n_2) \in D(m_1, m_2)} W_{Y^n_1, Y^n_2 | X^n, S^n}(\pi y^n_1, \pi y^n_2 | \xi^n(\pi^{-1} u^n_1(m_1, m_2), \pi^{-1} u^n_2(m_2), s^n), s^n),
$$

\hspace{1cm} (c)

(B.76)

where (a) is obtained by plugging $\pi s^n$ and $x^n = \xi^n(\cdot, \cdot, \cdot)$ in (2.3) and then changing the order of summation over $(y^n_1, y^n_2)$; (b) holds because the broadcast channel is memoryless; and (c) follows from that fact that for a Shannon strategy code, $x_i = \xi(u_{1,i}, u_{2,i}, s_i), i \in [1 : n]$, by Definition 8. The last expression suggests the use of permutations applied to the encoding strategy sequence and the channel output sequences.

Then, consider the $(2^{nR_1}, 2^{nR_2}, n)$ random code $\mathcal{C}^\Pi$, specified by

$$
f^n_\pi(m_1, m_2, s^n) = \xi^n(\pi^{-1} u^n_1(m_1, m_2), \pi^{-1} u^n_2(m_2), s^n),
$$

(B.77a)

and

$$
g_{\pi}(y^n_1) = g_1(\pi y^n_1), \quad g_{2, \pi}(y^n_2) = g(\pi y^n_2),
$$

(B.77b)

for $\pi \in \Pi_n$, with a uniform distribution $\mu(\pi) = \frac{1}{|\Pi_n|} = \frac{1}{n!}$. Such permutations can be implemented without knowing $s^n$, hence this coding scheme does not violate the causality requirement.

From (B.76), we see that

$$
P_{e|s^n}(\mathcal{C}^\Pi) = \sum_{\pi \in \Pi_n} \mu(\pi) P_{e|\pi s^n}(\mathcal{C}),
$$

(B.78)
for all $s^n \in \mathcal{S}^n$, and therefore, together with (B.75), we have that the probability of error of the random code $\mathcal{C}^\Pi$ is bounded by
\[ P_e^{(n)}(q^n, \mathcal{C}^\Pi) \leq e^{-\theta n}, \tag{B.79} \]
for every $q^n(s^n) \in \mathcal{P}^n(\mathcal{S}^n)$. That is, $\mathcal{C}^\Pi$ is a $(2^{nR_1}, 2^{nR_2}, n, e^{-\theta n})$ random code for the AVDBC $\mathcal{B}$ with causal SI at the encoder. This completes the proof of the inner bound.

**Outer Bound**

We show that the capacity region of the AVDBC $\mathcal{B}$ with causal SI is included within the region defined in (2.23), i.e. $\mathcal{C}^*(\mathcal{B}) \subseteq R_{\text{out}}^*$. The random code capacity region of the AVDBC is included within the random code capacity region of the compound DBC, namely
\[ \mathcal{C}^*(\mathcal{B}) \subseteq \mathcal{C}^*(\mathcal{B}^{\mathcal{P}(\mathcal{S})}) . \tag{B.80} \]
By Theorem 17 we have that $\mathcal{C}(\mathcal{B}^Q) \subseteq R_{\text{out}}(\mathcal{B}^Q)$. Thus, with $Q = \mathcal{P}(\mathcal{S})$,
\[ \mathcal{C}^*(\mathcal{B}^{\mathcal{P}(\mathcal{S})}) \subseteq R_{\text{out}}^* . \tag{B.81} \]
It follows from (B.80) and (B.81) that $\mathcal{C}^*(\mathcal{B}) \subseteq R_{\text{out}}^*$. Since the random code capacity region always includes the deterministic code capacity region, we have that $\mathcal{C}(\mathcal{B}) \subseteq R_{\text{out}}^*$ as well.

**Part 2**

The second equality, $R_{\text{in}}^* = R_{\text{out}}^*$, follows from part 2 of Theorem 17, taking $Q = \mathcal{P}(\mathcal{S})$. By part 1, $R_{\text{in}}^* \subseteq \mathcal{C}^*(\mathcal{B}) \subseteq R_{\text{out}}^*$, hence the proof follows.

**B.4 Proof of Lemma 19**

The proof follows the lines of [1, Section 4]. Let $k > 0$ be an integer, chosen later, and define the random variables
\[ L_1, L_2, \ldots, L_k \text{ i.i.d. } \sim \mu(\ell) . \tag{B.82} \]
Fix $s^n$, and define the random variables
\[ \Omega_j(s^n) = P^{(n)}_{\ell \mid s^n} (\mathcal{C}_{L_j}) , \quad j \in [1 : k] , \tag{B.83} \]
which is the conditional probability of error of the code $\mathcal{C}_{L_j}$ given the state sequence $s^n$.

Since $\mathcal{C}_{\Gamma}$ is a $(2^{nR_1}, 2^{nR_2}, n, \varepsilon_n)$ code, we have that $\sum_\gamma \mu(\gamma) \sum_{s^n} q^n(s^n) P^{(n)}_{\ell \mid s^n} (\mathcal{C}_\gamma) \leq \varepsilon_n$, for all $q^n(s^n)$. In particular, for a kernel, we have that
\[ \mathbb{E} \Omega_j(s^n) = \sum_{\gamma \in \Gamma} \mu(\gamma) \cdot P^{(n)}_{\ell \mid s^n} (\mathcal{C}_\gamma) \leq \varepsilon_n , \tag{B.84} \]
for all $j \in [1 : k]$.

Now take $n$ to be large enough so that $\varepsilon_n < \alpha$. Keeping $s^n$ fixed, we have that the random variables $\Omega_j(s^n)$ are i.i.d., due to (B.82). Next the technique known as Bernstein’s trick [1] is applied.

$$
\Pr\left(\sum_{j=1}^{k} \Omega_j(s^n) \geq k \alpha\right) \leq \mathbb{E}\left\{ \exp\left[ \beta \left( \sum_{j=1}^{k} \Omega_j(s^n) - k \alpha \right) \right] \right\} \quad (B.85)
$$

$$
= e^{-\beta k \alpha} \cdot \mathbb{E}\left\{ \prod_{j=1}^{k} e^{\beta \Omega_j(s^n)} \right\} \quad (B.86)
$$

$$
\leq e^{-\beta k \alpha} \cdot \prod_{j=1}^{k} \mathbb{E}\{ e^{\beta \Omega_j(s^n)} \} \quad (B.87)
$$

$$
\leq e^{-\beta k \alpha} \cdot \prod_{j=1}^{k} \mathbb{E}\{ 1 + e^{\beta \Omega_j(s^n)} \} \quad (B.88)
$$

$$
\leq e^{-\beta k \alpha} \cdot \left( 1 + e^{\beta \varepsilon_n} \right)^k \quad (B.89)
$$

where (a) is an application of Chernoff’s inequality; (b) follows from the fact that $\Omega_j(s^n)$ are independent; (c) holds since $e^{\beta x} \leq 1 + e^{\beta x}$, for $\beta > 0$ and $0 \leq x \leq 1$; (d) follows from (B.84). We take $n$ to be large enough for $1 + e^{\beta \varepsilon_n} \leq e^{\alpha}$ to hold. Thus, choosing $\beta = 2$, we have that

$$
\Pr\left(\frac{1}{k} \sum_{j=1}^{k} \Omega_j(s^n) \geq \alpha\right) \leq e^{-\alpha k}, \quad (B.90)
$$

for all $s^n \in S^n$. Now, by the union of events bound, we have that

$$
\Pr\left(\max_{s^n} \frac{1}{k} \sum_{j=1}^{k} \Omega_j(s^n) \geq \alpha\right) = \Pr\left( \exists s^n : \frac{1}{k} \sum_{j=1}^{k} \Omega_j(s^n) \geq \alpha \right) \quad (B.91)
$$

$$
\leq \sum_{s^n \in S^n} \Pr\left( \frac{1}{k} \sum_{j=1}^{k} \Omega_j(s^n) \geq \alpha \right) \quad (B.92)
$$

$$
\leq |S|^n \cdot e^{-\alpha k}. \quad (B.93)
$$

Since $|S|^n$ grows only exponentially in $n$, choosing $k = n^2$ results in a super exponential decay.

Consider the code $\mathcal{C}^{\Gamma^*} = (\mu^*, \Gamma^* = [1 : k], \{\mathcal{C}_{L_j}\}_{j=1}^{k})$ formed by a random collection of codes, with $\mu^*(j) = \frac{1}{k}$. It follows that the conditional probability of error given $s^n$, which is given by

$$
P_{e|s^n}(\mathcal{C}^{\Gamma^*}) = \frac{1}{k} \sum_{j=1}^{k} P_{e|s^n}(\mathcal{C}_{L_j}), \quad (B.94)
$$
Thus, there exists a random code \( C^{\Gamma^*} = (\mu^{\ast}, \Gamma^{\ast}, \{C_{\gamma^{\ast}}\}_{\gamma^{\ast}}} \) for the AVBC \( B \), such that

\[
P_e^{(n)}(q^n, C^{\Gamma^*}) = \sum_{s^n \in S^n} q^n(s^n) P_e^{(n)}(s^n, C^{\Gamma^*}) \leq \alpha, \quad \text{for all } q^n(s^n) \in \mathcal{P}(S^n). \quad (B.95)
\]

\[\Box\]

### B.5 Proof of Theorem 20

**Achievability proof.** To show achievability, we follow the lines of [1], with the required adjustments. We use the random code constructed in the proof of Theorem 18 to construct a deterministic code.

Let \((R_1, R_2) \in \mathcal{C}^{\ast} (B)\), and consider the case where \(\text{int}(\mathcal{C}(B)) \neq \emptyset\). Namely,

\[
\mathcal{C}(W_1) \supseteq \mathcal{C}(W_2) > 0, \quad (B.96)
\]

where \(W_1 = \{W_{y_1|x,s}\} \) and \(W_2 = \{W_{y_2|x,s}\}\) denote the marginal AVCs with causal SI of the stronger user and the weaker user, respectively. By Lemma 19, for every \(\varepsilon_1 > 0\) and sufficiently large \(n\), there exists a \((2^{R_1}, 2^{R_2}, n, \varepsilon_1)\) random code \(C^\Gamma = (\mu(\gamma) = \frac{1}{k}, \Gamma = [1:k], \{C_{\gamma}\}_{\gamma \in \Gamma})\), where \(C_{\gamma} = (f_n^{1, \gamma}, g_1, g_2, \gamma)\), for \(\gamma \in \Gamma\), and \(k = |\Gamma| \leq n^2\).

Following (B.96), we have that for every \(\varepsilon_2 > 0\) and sufficiently large \(n\), the code index \(\gamma \in [1:k]\) can be sent over \(B\) using a \((2^{R_1}, 2^{R_2}, \nu, \varepsilon_2)\) deterministic code \(C_1 = (f^{1, \gamma}, g_1, g_2)\), where \(\tilde{R}_1 > 0, \tilde{R}_2 > 0\). Since \(k\) is at most polynomial, the encoder can reliably convey \(\gamma\) to the receiver with a negligible blocklength, i.e. \(\nu = o(n)\).

Now, consider a code formed by the concatenation of \(C_1\) as a prefix to a corresponding code in the code collection \(\{C_{\gamma}\}_{\gamma \in \Gamma}\). That is, the encoder sends both the index \(\gamma\) and the message pair \((m_1, m_2)\) to the receivers, such that the index \(\gamma\) is transmitted first by \(\tilde{f}^{1, \gamma}(s^{\nu})\), and then the message pair \((m_1, m_2)\) is transmitted by the codeword \(x^n = f^{1, \gamma}_n(m_1, m_2, s_{\nu+1}, \ldots, s_{n+1})\). Subsequently, decoding is performed in two stages as well; decoder 1 estimates the index at first, with \(\hat{\gamma}_1 = \tilde{g}_1(y_{1,1}, \ldots, y_{1,\nu})\), and the message \(m_1\) is then estimated by \(\hat{m}_1 = g_1(\hat{\gamma}_1, y_{1,\nu+1}, \ldots, y_{1,n+\nu})\). Similarly, decoder 2 estimates the index with \(\hat{\gamma}_2 = \tilde{g}_2(y_{2,1}, \ldots, y_{2,\nu})\), and the message \(m_2\) is then estimated by \(\hat{m}_2 = g_2(\hat{\gamma}_2, y_{2,\nu+1}, \ldots, y_{2,n+\nu})\).

By the union of events bound, the probability of error is then bounded by \(\varepsilon = \varepsilon_1 + \varepsilon_2\), for every joint distribution in \(\mathcal{P}^{\nu+n}(S^{\nu+n})\). That is, the concatenated code is a \((2^{(\nu+n)}\tilde{R}_1, 2^{(\nu+n)}\tilde{R}_2, \nu + n, \varepsilon)\) code over the AVBC \(B\) with causal SI, where \(\nu = o(n)\). Hence, the blocklength is \(n + o(n)\), and the the rates \(\tilde{R}_1, n = \frac{n}{\nu+n} \cdot R_1\) and \(\tilde{R}_2, n = \frac{n}{\nu+n} \cdot R_2\) approach \(R_1\) and \(R_2\), respectively, as \(n \to \infty\). \[\Box\]

**Converse proof.** In general, the deterministic code capacity region is included within the random code capacity region. Namely, \(\mathcal{C}(B) \subseteq \mathcal{C}^{\ast}(B)\). \[\Box\]
B.6  Proof of Corollary 21

First, consider the inner and outer bounds in (2.27) and (2.28). The bounds are obtained as a direct consequence of part 1 of Theorem 18 and Theorem 20. Note that the outer bound (2.28) holds regardless of any condition, since the deterministic code capacity region is always included within the random code capacity region, i.e. \( \mathcal{C}(B) \subseteq \mathcal{C}^*(B) \subseteq R^*_\text{out}. \)

Now, suppose that the marginal \( V_{Y_2U_2S}^{\xi'} \) is non-symmetrizable for some \( \xi' : \mathcal{U}_2 \times \mathcal{S} \rightarrow \mathcal{X} \), and the condition \( \mathcal{T} \) holds. Then, by part 2 of Theorem 12, the capacity of the corresponding single-user AVC is positive, i.e. \( \mathcal{C}(\mathcal{W}_2) > 0 \). Since the AVDBC \( \mathcal{W} \) is assumed to be degraded, we then have that \( \mathcal{C}(\mathcal{W}_1) \geq \mathcal{C}(\mathcal{W}_2) > 0 \), which means that \( \text{int}(\mathcal{C}(B)) \neq \emptyset \). Hence, by Theorem 20, the deterministic code capacity region coincides with the random code capacity region, i.e. \( \mathcal{C}(B) = \mathcal{C}^*(B) \). Then, the proof follows from part 2 of Theorem 18. \( \square \)

B.7  Analysis of Example 2

We begin with the case of an arbitrarily varying BSBC \( B_0 \) without SI. We claim that the single user marginal AVC \( \mathcal{W}_{1,0} \) without SI, corresponding to the stronger user, has zero capacity. Denote \( q \triangleq q(1) = 1 - q(0) \). Then, observe that the additive noise is distributed according to \( Z_{S} \sim \text{Bernoulli}(\varepsilon_q) \), with \( \varepsilon_q \triangleq (1-q) \cdot \theta_0 + q \cdot \theta_1 \), for \( 0 \leq q \leq 1 \). By Theorem 1, \( \mathcal{C}(\mathcal{W}_{1,0}) \leq \mathcal{C}^*(\mathcal{W}_{1,0}) = \min_{0 \leq q \leq 1}[1 - h(\varepsilon_q)] \). Since \( \theta_0 < \frac{1}{2} \leq \theta_1 \), there exists \( 0 \leq q \leq 1 \) such that \( \varepsilon_q = \frac{1}{2} \), thus \( \mathcal{C}(\mathcal{W}_{1,0}) = 0 \). The capacity region of the AVDBC \( B_0 \) without SI is then given by \( \mathcal{C}(B_0) = \{(0,0)\} \).

Now, consider the arbitrarily varying BSBC \( B \) with causal SI. By Theorem 18, the random code capacity region is bounded by \( R^*_\text{in} \subseteq \mathcal{C}^*(B) \subseteq R^*_\text{out} \). We show that the bounds coincide, and are thus tight. Let \( B^\prime \) denote the DBC \( W_{Y_1,Y_2,X,S} \) with causal SI, governed by an i.i.d. state sequence, distributed according to \( S \sim \text{Bernoulli}(q) \). By [28], the corresponding capacity region is given by

\[
\mathcal{C}(B^\prime) = \bigcup_{0 \leq \beta \leq 1} \left\{ (R_1, R_2) : \begin{array}{l}
R_2 \leq 1 - h(\alpha \ast \beta \ast \delta_q), \\
R_1 \leq h(\beta \ast \delta_q) - h(\delta_q)
\end{array} \right\},
\]

where

\[
\delta_q \triangleq (1-q) \cdot \theta_0 + q \cdot (1 - \theta_1),
\]

for \( 0 \leq q \leq 1 \). For every given \( 0 \leq q' \leq 1 \), we have that \( R^*_\text{out} = \bigcap_{0 \leq q \leq 1} \mathcal{C}(B^\prime) \subseteq \mathcal{C}(B^\prime) \). Thus, taking \( q' = 1 \), we have that

\[
R^*_\text{out} \subseteq \bigcup_{0 \leq \beta \leq \frac{1}{2}} \left\{ (R_1, R_2) : \begin{array}{l}
R_2 \leq 1 - h(\alpha \ast \beta \ast \theta_1), \\
R_1 \leq h(\beta \ast \theta_1) - h(\theta_1)
\end{array} \right\},
\]

where we have used the identity \( h(\alpha \ast (1 - \delta)) = h(\alpha \ast \delta) \).
Now, to show that the region above is achievable, we examine the inner bound,

\[
R^*_\text{in} = \bigcup_{p(u_1, u_2), \xi(u_1, u_2, s)} \left\{ (R_1, R_2) : \begin{array}{l}
R_2 \leq \min_{0 \leq q \leq 1} I_q(U_2; Y_2), \\
R_1 \leq \min_{0 \leq q \leq 1} I_q(U_1; Y_1|U_2)
\end{array} \right\}. \tag{B.99}
\]

Consider the following choice of \( p(u_1, u_2) \) and \( \xi(u_1, u_2, s) \). Let \( U_1 \) and \( U_2 \) be independent random variables,

\[
U_1 \sim \text{Bernoulli}(\beta), \text{ and } U_2 \sim \text{Bernoulli}\left(\frac{1}{2}\right), \tag{B.100}
\]

for \( 0 \leq \beta \leq \frac{1}{2} \), and let

\[
\xi(u_1, u_2, s) = u_1 + u_2 + s \mod 2. \tag{B.101}
\]

Then,

\[
\begin{align*}
H_q(Y_1|U_1, U_2) &= H_q(S + Z_S) = h(\delta_q), \\
H_q(Y_1|U_2) &= H_q(U_1 + S + Z_S) = h(\alpha \beta \delta_q), \\
H_q(Y_2|U_2) &= H_q(U_1 + S + Z_S + V) = h(\alpha \beta \delta_q), \\
H_q(Y_2) &= 1,
\end{align*}
\]

where addition is modulo 2, and \( \delta_q \) is given by (B.97b). Thus,

\[
\begin{align*}
I_q(U_2; Y_2) &= 1 - h(\alpha \beta \delta_q), \\
I_q(U_1; Y_1|U_2) &= h(\beta \delta_q) - h(\delta_q),
\end{align*}
\]

hence

\[
R^*_\text{in} \supseteq \bigcup_{0 \leq \beta \leq \frac{1}{2}} \left\{ (R_1, R_2) : \begin{array}{l}
R_2 \leq \min_{0 \leq q \leq 1} 1 - h(\alpha \beta \delta_q), \\
R_1 \leq \min_{0 \leq q \leq 1} h(\beta \delta_q) - h(\delta_q)
\end{array} \right\}. \tag{B.104}
\]

Note that \( \theta_0 \leq \delta_q \leq 1 - \theta_1 \leq \frac{1}{2} \). For \( 0 \leq \delta \leq \frac{1}{2} \), the functions \( g_1(\delta) = 1 - h(\alpha \beta \delta) \) and \( g_2(\delta) = h(\beta \delta) - h(\delta) \) are monotonic decreasing functions of \( \delta \), hence the minima in (B.104) are both achieved with \( q = 1 \). It follows that

\[
C^*(\mathcal{B}) = R^*_\text{in} = R^*_\text{out} = \bigcup_{0 \leq \beta \leq 1} \left\{ (R_1, R_2) : \begin{array}{l}
R_2 \leq 1 - h(\alpha \beta \theta_1), \\
R_1 \leq h(\beta \theta_1) - h(\theta_1)
\end{array} \right\}. \tag{B.105}
\]

It can also be verified that the condition \( \mathcal{T} \) holds (see Definition 12), in agreement with part 2 of Theorem 18. First, we specify a function \( \xi(u_1, u_2, s) \) and a distributions set \( \mathcal{D}^* \) that achieve \( R^*_\text{in} \) and \( R^*_\text{out} \) (see Definition 2.24). Let \( \xi(u_1, u_2, s) \) be as in (B.101), and let \( \mathcal{D}^* \) be the set of distributions \( p(u_1, u_2) \) such that \( U_1 \) and \( U_2 \) are independent random variables, distributed according to (B.100). By the derivation above, the requirement (2.24a) is satisfied. Now, by the derivation in [28, Section IV], we have that

\[
C(\mathcal{B}^\text{in}) = \bigcup_{p(u_1, u_2) \in \mathcal{D}^*} \left\{ (R_1, R_2) : \begin{array}{l}
R_2 \leq I_q(U_2; Y_2), \\
R_1 \leq I_q(U_1; Y_1|U_2)
\end{array} \right\}. \tag{B.106}
\]
Then, the requirement (2.24b) is satisfied as well, hence $\xi(u_1, u_2, s)$ and $D^*$ achieve $R^*_m$ and $R^*_t$. It follows that condition $\mathcal{T}$ holds, as $q^* = 1$ satisfies the desired property with $\xi(u_1, u_2, s)$ and $D^*$ as described above.

We move to the deterministic code capacity region of the arbitrarily varying BSBC $B$ with causal SI. If $\theta_1 = \frac{1}{2}$, the capacity region is given by $C(B) = C^*(B) = \{(0,0)\}$, by (B.105). Otherwise, $\theta_0 < \frac{1}{2} < \theta_1$, and we now show that the condition in Corollary 21 is met. Suppose that $V_{\xi'}^{Y_2|U,S}$ is symmetrizable for all $\xi': U_2 \times S \to X$. That is, for every $\xi'(u_2, s)$, there exists $\lambda_{u_2} = J(1|u_2)$ such that

\begin{equation}
(1 - \lambda_{u_2}) W_{Y_2|X,S}(y_2|\xi'(u_2, 0), 0) + \lambda_{u_2} W_{Y_2|X,S}(y_2|\xi'(u_2, 1), 1) = \frac{1}{2} W_{Y_2|X,S}(y_2|\xi'(u_2, 0), 0) + \frac{1}{2} W_{Y_2|X,S}(y_2|\xi'(u_2, 1), 1)
\end{equation}

(B.107)

for all $u_a, u_b \in U_2, y_2 \in \{0, 1\}$. If this is the case, then for $\xi'(u_2, s) = u_2 + s$ mod 2, taking $u_a = 0, u_b = 1, y_2 = 1$, we have that

\begin{equation}
(1 - \lambda_1) \cdot (\alpha \ast \theta_0) + \lambda_1 \cdot (1 - \alpha \ast \theta_1) = (1 - \lambda_0) \cdot (1 - \alpha \ast \theta_0) + \lambda_0 \cdot (\alpha \ast \theta_1). 
\end{equation}

(B.108)

This is a contradiction. Since $f(\theta) = \alpha \ast \theta$ is a monotonic increasing function of $\theta$, and since $1 - f(\theta) = f(1-\theta)$, we have that the value of the LHS of (B.108) is in $[0, \frac{1}{2}]$, while the value of the RHS of (B.108) is in $(\frac{1}{2}, 1]$. Thus, there exists $\xi': U_2 \times S \to X$ such that $V_{\xi'}^{Y_2|X,S}$ is non-symmetrizable for $\theta_0 < \frac{1}{2} < \theta_1$. As the condition $\mathcal{T}$ holds, we have that $C(B) = R^*_m = R^*_t$, due to Corollary 21. Hence, by (B.105), we have that the capacity region of the arbitrarily varying BSBC $B$ with causal SI is given by (2.30).

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