Characterization of Termination for Linear Loop Programs

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Abstract
We present necessary and sufficient conditions for the termination of linear homogeneous programs. We also develop a complete method to check termination for this class of programs. Our complete characterization of termination for such programs is based on linear algebraic methods. We reduce the verification of the termination problem to checking the orthogonality of a well determined vector space and a certain vector, both related to loops in the program. Moreover, we provide theoretical results and symbolic computational methods guaranteeing the soundness, completeness and numerical stability of the approach. Finally, we show that it is enough to interpret variable values over a specific countable number field, or even over its ring of integers, when one wants to check termination over the reals.

1 Introduction

Static program analysis [1, 2, 3] is used to check that a software is free of defects, such as buffer overflows or segmentation faults, which are safety
properties, or termination, which is a liveness property. Verification of temporal properties of infinite state systems [4] is another example. Proving termination of \textbf{while} loop programs is necessary for the verification of liveness properties that any well behaved and engineered system, or any safety critical embedded system, must guarantee. We could list here many verification approaches that are only practical depending on the facility with which termination can be automatically determined. More recent work on automated termination analysis of imperative loop programs has focused on partial decision procedures based on the discovery and synthesis of ranking functions. Such functions map the loop variable to a well-defined domain where their value decreases at each iteration of the loop [5, 6]. Several interesting approaches, based on the generation of \textit{linear} ranking functions, have been proposed [7, 8] for loop programs where the guards and the instructions can be expressed in a logic supporting linear arithmetic. For the generation of such functions there are effective heuristics [9, 6] and, in some cases, there are also complete methods [10]. On the other hand, it is easy to generate a simple linear terminating loop program that does not have a linear ranking function. In these cases, complete synthesis methods [10] fail to provide a conclusion about the termination or nontermination of such programs.

In this work we are motivated by the termination problem for linear \textbf{while} loop programs. In this class of loop programs, the loop condition is a conjunction of linear inequalities and the assignments to each of the variables in the loop instruction block are of an affine or linear form. In matrix notation, \textit{linear loop programs} can be represented as

\[ \text{while } (Bx > b), \{ x := Ax + c \}, \]

for \( x \) and \( c \) in \( \mathbb{R}^n \), \( b \) in \( \mathbb{R}^m \), and \( A \) and \( B \) real matrices of size \( n \times n \) and \( m \times n \), respectively. The termination analysis for this class of linear programs can be reduced to the termination problem of homogeneous programs with one loop condition, \textit{i.e.} when \( m = 1 \), \( b \) is zero and \( c \) is a zero vector [11, 12]. The really difficult step being the reduction to \( m = 1 \), while the reduction to \( b \) and \( c \) being zero is immediate. We focus on the termination of this type of program with one loop condition, and obtain results as sharp and complete as one could hope. At this point, it is worth mentioning some recent work on \textit{asymptotically non-terminating initial variable values} generation techniques [13]. Amongst many other results, we obtain methods that can be adapted here in order to extend our termination analysis for general linear programs, \textit{i.e.} when \( m \) is arbitrary.
Despite tremendous progress over the years [11, 14, 15, 16, 17, 18, 19, 20],
the problem of finding a practical, sound and complete method for determining termination or nontermination remains very challenging for this class of programs, and for all initial variable values. We also note that some earlier works [21, 22] have inspired the methods developed here.

We summarize our contributions as follows:

**Preliminary result:** First we prove a sufficient condition for the termination of homogeneous linear programs. This result is also stated in [12], but some shortcomings in that proof sketch require further elaboration. We closed those gaps in a solid mathematical way, with some obstacles not being so easy to overcome. We return to this point in more detail at Remark 3.1. Our new proof of this sufficient condition requires non-trivial topological and algebraic arguments. On the other hand, this sufficient condition is not a necessary condition for termination of linear homogeneous programs. Before we list our main contributions, it is important to note that the works [12, 11] produce some decidability results for this type of programs. However, for programs with one loop condition, our characterization of termination is much simpler, very explicit, and straightforwardly leads to much faster algorithm for checking termination. See also Section 7 for a more detailed comparison.

**Main contributions:**

(i) We give a necessary and sufficient condition (NSC, for short) for the termination of linear homogeneous programs with one loop condition. In fact, this NSC exhibits a complete characterization of termination for such programs, and gives decidability results for all initial variable values.

(ii) Moreover, departing from this NSC, we show the scalability of our approach by demonstrating that one can directly extract a sound and complete computational method to determine termination of such programs. We reduce the termination analysis to the problem of checking if a specific vector, related to the loop encoding condition, belongs to a specific vector space related to the eigenvalues of the matrix encoding assignments to the loop variables. The analysis of our associated algorithms shows that our method has a much better computational time complexity. We show that the method, based on three computational steps running in polynomial time complexity, is of a lower complexity...
than basic routines that form the mathematical foundations of previous methods [12, 11].

(iii) We provide theoretical results guaranteeing the soundness and completeness of the termination analysis while restricting variable interpretations over a specific countable sub-ring of $\mathbb{R}^n$. In other words, we show that it is enough to interpret variable values over a specific countable field — a number field, or even its ring of integers, — when one wants to check the termination over the reals. By so doing, we circumvent difficulties such as rounding errors. Those results enable our symbolic computational methods to rely on closed-form algebraic expression and numbers.

The rest of this article is organized as follows. Section 2 is a preliminary section where we introduce our computational model for programs, the notations for the rest of the paper, and some key notions of linear algebra used to develop our computational methods. Section 3 develops our theoretical results and a very useful necessary and sufficient condition, in Subsection 3.2, which allows us to propose the complete computational method illustrated in Section 4 and fully described in Section 5. In the important Section 6, we show that it is enough to interpret the variable values over a countable field in order to determine program termination over the reals. We provide a discussion of related works in Section 7. Finally, Section 8 concludes the paper.

2 Linear Algebra and Linear Loop Programs

We recall classical facts from linear algebra. Let $E$ be a real vector space and $A \in \text{End}_\mathbb{R}(E)$, the space of $\mathbb{R}$-linear maps from $E$ to itself. Let $E^*$ be the set of linear functionals in $E$. We denote by $\mathcal{M}(p, q, \mathbb{R})$ the space of $p \times q$ matrices, and if $p = q$ we simply write $\mathcal{M}(p, \mathbb{R})$. We will denote by $\mathbb{K}$ the $\mathbb{R}$ or $\mathbb{C}$ fields. If $A \in \mathcal{M}(m, n, \mathbb{K})$, with entry $a_{i,j}$ in position $(i, j)$, we will sometimes denote it by $(a_{i,j})$. If $B$ is a basis for $E$, we denote by $A_B = \text{Mat}_B(A)$ the matrix of $A$ in the basis $B$, and we have $A_B \in \mathcal{M}(n, \mathbb{R})$. Let $I_n$ be the identity matrix in $\mathcal{M}(n, \mathbb{R})$, and $\text{id}_E$ the identity of $E$. The transpose of the matrix $A = (a_{i,j})$ is the matrix $A^T = (b_{i,j})$ where $b_{i,j} = a_{j,i}$. The kernel of $A$, also called its nullspace and denoted by $\text{Ker}(A)$, is the set $\{v \in \mathbb{K}^n \mid A \cdot v = 0_{\mathbb{K}^m}\}$. Let $A$ be a square matrix in $\mathcal{M}(n, \mathbb{K})$. A nonzero
vector $x \in \mathbb{K}$ is an eigenvector of $A$ associated with an eigenvalue $\lambda \in \mathbb{K}$ if 
\[ A \cdot x = \lambda x, \ \text{i.e.,} \ (A - \lambda I_n) \cdot x = 0. \]
The nullspace of $(A - \lambda I_n)$ is called the eigenspace of $A$ associated with eigenvalue $\lambda$. A non-zero vector $x$ is said to be a generalized eigenvector of $A$ corresponding to $\lambda$ if 
\[ (A - \lambda I_{kn}) \cdot x = 0. \]
The nullspace of $(A - \lambda I_{kn})$ is called the eigenspace of $A$ associated with eigenvalue $\lambda$. A non-zero vector $x$ is said to be a generalized eigenvector of $A$ corresponding to $\lambda$ if 
\[ (A - \lambda I_{kn})^k \cdot x = 0 \]
for some positive integer $k$. The spaces $\text{Ker}((A - \lambda I_{kn})^k)$ form an increasing larger sequence of subspaces of $E$, which is stationary for $k \geq e$, for some $e \leq n$. We call the subspace $\text{Ker}((A - \lambda I_{kn})^e) = \text{Ker}((A - \lambda I_{kn})^n)$ the generalized eigenspace of $A$ associated with $\lambda$, and its nonzero elements are exactly the generalized eigenvectors. We denote by $\langle \cdot, \cdot \rangle$ the canonical scalar product on $\mathbb{R}^n$. As it is standard in static program analysis, a primed symbol $x'$ refers to the next state value of $x$ after a transition is taken. Next, we present transition systems as representations of imperative programs and automata as their computational models.

**Definition 2.1.** In a transition system $(x, L, T, l_0, \Theta)$, $x = (x_1, \ldots, x_n)$ is a set of variables, $L$ is a set of locations and $l_0 \in L$ is the initial location. A state is given by an interpretation of the variables in $x$. A transition $\tau \in T$ is given by a tuple $\langle l_{\text{pre}}, l_{\text{post}}, q_\tau, \rho_\tau \rangle$, where $l_{\text{pre}}$ and $l_{\text{post}}$ designate the pre- and post-locations of $\tau$, respectively, and the transition relation $\rho_\tau$ is a first-order assertion over $x \cup x'$. The transition guard $q_\tau$ is a conjunction of inequalities over $x$. $\Theta$ is the initial condition, given as a first-order assertion over $x$. The transition system is said to be linear when $\rho_\tau$ is an affine form.

A loop program, defined next, is a special kind of transition system. We also establish some matrix notations to represent loop programs, where the effects of sequential linear assignments are described as simultaneous updates. Departing from sequential instructions, we use syntactic and common propagation procedures to obtain the equivalent simultaneous systems expressed in matrix notations (see Definition 2.2).

**Definition 2.2.** Let $P = \langle x, \{l\}, T, l, \Theta \rangle$ be a transition system with $x = (x_1, \ldots, x_n)$ and $T = \{\langle l, l, q_\tau, \rho_\tau \rangle\}$. Then $P$ is a linear loop program if:

- The transition guard is a conjunction of linear inequalities. We represent the loop condition in matrix form as $Fx > b$ where $F \in \mathcal{M}(m, n, \mathbb{R})$ and $b \in \mathbb{R}^m$. Which means that each coordinate of the vector $Fx$ is greater than the corresponding coordinate of vector $b$.

- The transition relation is an affine or linear form. We represent the linear assignments in matrix form as $x := Ax + c$, where $A \in \mathcal{M}(n, \mathbb{R})$ and $c \in \mathbb{R}^n$. 

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The most general linear loop program \( P = P(A, F, b, c) \) is thus written
\[
\text{while } (Fx > b) \{ x := Ax + c \}.
\]

In this work, one needs first to focus mainly on the following class of linear loop programs.

**Definition 2.3.** We denote by \( P^H \) the set of programs where all linear assignments consist of homogeneous expressions, and where the linear loop condition consists of at most one inequality.

If \( P \) is in \( P^H \), then \( P \) will be interpreted in matrix terms as
\[
\text{while } ((f, x) > 0) \{ x := Ax \},
\]
where \( f \) is a \((n \times 1)\)-vector corresponding to the loop condition, and \( A \in \mathcal{M}(n, \mathbb{R}) \) is related to the list of assignments in the loop. In this case, we say that \( P \) has a homogeneous form and it will be identified as \( P(A, f) \).

Consider a program \( P(A, f) \), where \( A \in \mathcal{M}(n, \mathbb{R}) \), \( f \in \mathcal{M}(1, n, \mathbb{R}) \). Alternatively, we may consider \( A \in \text{End}_\mathbb{R}(E) \), \( f \in E^* \) and write
\[
P(A, f) : \text{while } (f(x) > 0) \{ x := Ax \}.
\]
Fixing a basis \( B \) of \( E \) we can write \( A = \text{Mat}_B(A) \), \( f = \text{Mat}_B(f) \), \( x = \text{Mat}_B(x) \), and so on. We now define termination for such programs.

**Definition 2.4.** Program \( P(A, f) \) terminates on input \( x \in E \) if and only if there exists \( k \geq 0 \) such that \( f(A^k(x)) \) is not positive. Alternatively, for \( A \in \mathcal{M}_n(\mathbb{R}) \), and \( f \in \mathcal{M}_{1,n}(\mathbb{R}) \), we say that \( P(A, f) \) terminates on input \( x \in \mathbb{R}^n \), if and only if there exists \( k \geq 0 \), such that \( \langle A^k x, f \rangle \) is not positive. Thus, a program \( P(A, f) \) is non-terminating if and only if there exists an input \( x \in E \) such that \( f(A^k(x)) > 0 \) for all \( k \geq 0 \). In matrix terms, \( P(A, f) \) is non-terminating on input \( x \in \mathbb{R}^n \) if and only if \( \langle A^k x, f \rangle > 0 \) for all \( k \geq 0 \).

### 3 Linear Program Termination

First we prove a *sufficient* condition for the termination of homogeneous linear programs, already stated in [12]. We note that the proof of sufficiency in [12] does not go through, and needed to be amended, which was not a trivial task. Then we present the main result, which provides the first *necessary and sufficient* condition for the termination problem for the class of linear homogeneous programs.
3.1 Sufficiency and Homogeneous Linear Programs

We prove a sufficient condition for the termination of programs $P(A, f) \in P_H^H$, written

$$\text{while } (f^T x > 0) \{ x := Ax \}.$$  

**Theorem 3.1.** Let $n$ be a positive integer, and let $P(A, f) \in P_H^H$. If $P(A, f)$ is non-terminating, then $A$ has a positive eigenvalue. □

In the following discussion, we provide the complete proof of Theorem 3.1.

Before we complete the proof, which is a mix of topological and algebraic arguments, we need first to state the following lemmas and propositions.

We first recall some basic facts about generalized eigenspaces. Let $E$ be an $\mathbb{R}$-vector space of finite dimension, and let $A \in \text{End}_\mathbb{R}(E)$. Let $E'$ be a subspace of $E$. We say that $E'$ is $A$-stable if $A(E') \subseteq E'$. If $\lambda \in \mathbb{R}$, we denote by $E_\lambda(A)$ the subspace \{ $x \in E | \exists k \geq 0, (A - \lambda \text{id}_E)^k(x) = 0$ \}. This space is non zero if and only if the input vector $x$ is an eigenvector of $A$. In this case, it is called the generalized eigenspace corresponding to $\lambda$. If $\chi_A$ is the characteristic polynomial of $A$, if $d_\lambda$ is the multiplicity of the monomial $(X - \lambda)$ in $\chi_A(X)$, which may be 0 if $\lambda$ is not an eigenvalue, then $E_\lambda(A) = \text{Ker}(A - \lambda \text{id}_E)^{d_\lambda}$. It is obvious that $E_\lambda(A)$ is $A$-stable. We denote by $\text{Spec}(A)$ the set of real eigenvalues of $A$. The following property of generalized eigenspaces was stated in the preliminaries.

**Proposition 3.1.** Let $E$ be an $\mathbb{R}$-vector space of finite dimension, and let $A$ belong to $\text{End}_\mathbb{R}(E)$. Then $E_\lambda(A) = \text{Ker}(A - \lambda \text{id}_E)^{d_\lambda}$, for some $d_\lambda \leq n$. In particular, $E_\lambda(A) = \text{Ker}(A - \lambda \text{id}_E)^n$. □

Proof. We just said that one can choose $d_\lambda$ to be such that $(X - \lambda)^{d_\lambda} \mid \chi_A$. Hence, $d_\lambda \leq d^\circ(\chi_A) = n$ (with $d^\circ$ beeing the standart notation for polynomial degree.). □

We will also need the following lemma.

**Lemma 3.1.** Let $E^*$ be the space $\text{Hom}_\mathbb{R}(E, \mathbb{R})$, where $E$ is a finite dimensional vector space, and $f_0, \ldots, f_m$ be linear forms in $E^*$. Then this family spans $E^*$ if and only if $\cap_{i=0}^m \text{Ker}(f_i) = \{0\}$. □

Proof of Lemma 3.1. In the following we use the notation $\text{Vect}(v_1, \ldots, v_u)$ to describe the vector space spaned by the elements $v_1, \ldots, v_u$. Suppose that $f_0, \ldots, f_m$ spans $E^*$. If $x$ belongs to $\cap_{i=0}^m \text{Ker}(f_i)$, then $x$ belongs to the
kernel of any element of $E^*$. But then, if $B = (e_1, \ldots, e_n)$ is a basis of $E$, and $B^* = (e_1^*, \ldots, e_n^*)$ is its dual basis, we have $x = x_1 e_1 + \cdots + x_n e_n$, and $e_i^*(x) = x_i = 0$. Hence, $x = 0$. Conversely, if $\bigcap_{i=0}^{m} \ker(f_i) = \{0\}$, let $g_1, \ldots, g_r$ be a maximal linearly independent family in $f_0, \ldots, f_m$. Hence, $Vect(g_1, \ldots, g_r) = Vect(f_0, \ldots, f_m)$. We thus have $r \leq n$ because $\dim(E^*) = \dim(E) = n$ and $\bigcap_{i=1}^{r} \ker(g_i) = \{0\}$. If $r$ was strictly smaller than $n$, then $\bigcap_{i=1}^{r} \ker(g_i)$ would be an intersection of $r$ subspaces of co-dimension 1. Hence, it would be of co-dimension at most $r$, i.e., $\bigcap_{i=1}^{r} \ker(g_i)$ would be of dimension at least $n - r > 0$, which is a contradiction. Thus $r = n$, and $(g_1, \ldots, g_r)$ is a basis of $E^*$. It follows that $Vect(f_0, \ldots, f_m) = E^*$.

Before proving Lemma 3.3, we recall and prove the following standard lemma.

**Lemma 3.2.** Let $A$ be an endomorphism of a real vector space $E$, and let $\lambda$ be an eigenvalue of $A$. There is a supplementary space $E'$ of $E_{\lambda}(A)$, i.e., $E = E_{\lambda}(A) \oplus E'$, and two polynomials $C$ and $D$ in $\mathbb{R}[X]$, such that $C(A)$ is the projection on $E_{\lambda}(A)$ with respect to $E'$, and $D(A)$ is the projection on $E'$ with respect to $E_{\lambda}(A)$. In particular $E'$ is also $A$-stable, and for any $A$-stable subspace $L$ of $E$, we have $L = L \cap E_{\lambda}(A) \oplus L \cap E'$.

*Proof.* Let $\chi_A = (X - \lambda)^d Q$, with $Q(\lambda) \neq 0$. By the kernel decomposition lemma, we have

$$E = \ker(A - \lambda I_d)^d \oplus \ker(Q(\lambda)).$$

We set $E' = \ker(Q(\lambda))$. It is thus $A$-stable. Moreover, by Bezout’s identity, there are $P$ and $P'$ in $\mathbb{R}[X]$, such that

$$P(u) \circ (A - \lambda I_d)^d + P'(u) \circ Q(A) = I_d.$$

We set $C = P(X - \lambda)^d$, and $D = P'(A) \circ Q(u)$. Finally, if $L$ is $A$-stable, we always have

$$L \cap E_{\lambda}(A) \oplus L \cap E' \subset L.$$

Now write an element $l$ of $L$ as $l_1 + l_2$, with $l_1 \in E_{\lambda}(A)$, and $l_2 \in E'$. We get $B(A)(l) = l_1$. But $L$ being $A$-stable, it is also $D(A)$-stable as well. Hence, $l_1 \in L$. Similarly we have $l_2 \in L$, thus

$$L = L \cap E_{\lambda}(A) \oplus L \cap E',$$

completing the proof.
We will use the following result about quotient vector spaces.

**Lemma 3.3.** Let $E$ be an $\mathbb{R}$-vector space, let $A \in \text{End}_{\mathbb{R}}(E)$, and suppose that $L$ is a $A$-stable subspace of $E$. Let $\overline{A} : E/L \to E/L$ be the element of $\text{End}_{\mathbb{R}}(E/L)$ defined by $\overline{A}(x + L) = \overline{A}(x) + L$. Then $\text{Spec}(\overline{A}) \subset \text{Spec}(A)$. More generally, for any $\lambda \in \text{Spec}(\overline{A})$, the generalized eigenspace $E_\lambda(A)$ maps surjectively to $E_\lambda(A)$ in $E/L$.

**Proof of Lemma 3.3.** Let $B_1$ be a basis for $L$, and $B_2$ be a basis for any supplementary space. Call $B_2$ the image of the elements of $B_2$ in $E = E/L$. Then $B_2$ is a basis of $E$. With $B = B_1 \cup B_2$, $\text{Mat}_B(A)$ is of the form

$$
\begin{pmatrix}
X & Y \\
0 & Z
\end{pmatrix}
$$

Then $X = \text{Mat}_{B_1}(A|_L)$, $Z = \text{Mat}_{B_2}(\overline{A})$, and the second statement follows from this second fact.

Now if $\overline{x}$ belongs to $E_\lambda(\overline{A})$, then $(\overline{A} - \lambda I_d)^a \overline{x} = \overline{0}$ for some $a \geq 0$. This means that $(A - \lambda I_d)^a x \in L$.

We write $x = x_\lambda + x' \in E_\lambda(A) \oplus E'$, for $E'$ as in Lemma 3.2. Then $(A - \lambda I_d)^a x = (A - \lambda I_d)^a x_\lambda + (A - \lambda I_d)^a x'$, with $(A - \lambda I_d)^a x_\lambda \in E_\lambda(A)$, and $(A - \lambda I_d)^a x' \in E'$. Let $d$ be the multiplicity of $\lambda$ as a root of $\chi_A$. For $k$ large enough such that $kd \geq a$, we have $(A - \lambda I_d)^{kd} x_\lambda = 0$ and $(A - \lambda I_d)^{kd} x = (A - \lambda I_d)^{kd} x'$. Taking $P \in \mathbb{R}[X]$ as in the proof of Lemma 3.2 we have that $P(A) \circ (A - \lambda I_d)^d$ is the identity when restricted to $E'$. In particular, this implies that $x' = P(A)^k (A - \lambda I_d)^{kd} x$, and thus $x' \in L$. Finally, we obtain $\overline{x} = \overline{x}_\lambda$, and this concludes the proof, as $x_\lambda \in E_\lambda(A)$.

We say that a subset of $\mathbb{R}^n$ is a convex cone if it is convex, and it is also stable under multiplication by elements of $\mathbb{R}_{>0}$. It is obvious that an intersection of convex cones is still a convex cone, and so one can speak of the convex cone spanned by a subset of $\mathbb{R}^n$.

**Proposition 3.2.** Let $C$ be a convex cone of $\mathbb{R}^n$. Assume that $C$ is non reducible to zero, and is contained in the closed cone

$$
\Delta = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \forall i, x_i \geq 0 \}.
$$

If $A$ is an invertible endomorphism of $\mathbb{R}^n$, with $A(C) \subset C$, then $A$ has a positive eigenvalue.
Proof. Consider $C' = C - \{0\}$. Then $C'$ is also a convex cone. It is obviously still stable under multiplication by elements of $\mathbb{R}_{>0}$. Moreover, if $x$ and $y$ belong to $C'$, then the vector $tx + (1 - t)y$ belongs to $C$ by convexity, for $t \in [0, 1]$. But it cannot be equal to zero, as both $x$ and $y$ have non-negative coefficients, this would imply that $x$ or $y$ is null, which is a contradiction.

Now let $H_1$ be the affine hyperplane $H_1 = \{x \in \mathbb{R}^n, x_1 + \ldots + x_n = 1\}$, and let $f$ be the linear form on $\mathbb{R}^n$ defined by $f : x \mapsto x_1 + \cdots + x_n$, so that $H = f^{-1}(\{1\})$. This linear form is positive on $\Delta$, and so we can define the projection $p : \Delta - \{0\} \to H$ given by

$$x \mapsto \frac{1}{f(x)} x.$$ 

It is obviously continuous. We call $C_1$ the set $p(C')$. We claim that $C_1 = C' \cap H_1$ and, in particular, it is convex. Indeed, $C_1 \subset H_1$ by definition, and $C_1 \subset C'$ because $C'$ is stable under $\mathbb{R}_{>0}$. Conversely, the restriction of $p$ to $C' \cap H_1$ is the identity, and so $C_1$ contains $C' \cap H_1 = p(C' \cap H_1)$. It is also clearly stable under the continuous map

$$s = p \circ A : \Delta - \{0\} \to H_1,$$

as $A(C') \subset C'$. In particular, its closure $\overline{C_1}$ is stable under $s$ as well. It is convex and compact, as a closed subset of the compact set

$$\{x \in \mathbb{R}^n, \forall i, x_i \geq 0, x_1 + \cdots + x_n = 1\}.$$

According to Brouwer’s fixed point theorem, this implies that $s$ has a fixed $x$ point in $\overline{C_1} \subset \Delta - \{0\}$. But we then have $A(x) = f(x)x$. As $f(x) > 0$ for any $x$ in $\Delta - \{0\}$. This proves the lemma.

Finally we will prove the following statement equivalent to Theorem 3.1.

We just rewrite the statement of Theorem 3.1 in terms of morphisms, which are more convenient to work with.

**Theorem 3.2.** Let $E$ be an $\mathbb{R}$-vector space of dimension $n$, let $A$ be an endomorphism of $E$, and let $f$ be a nonzero linear form on $E$. If there exists a vector $x \in E$ such that $f(A^k(x)) > 0$ for all $k \geq 0$, then $A$ has a positive eigenvalue.
Proof. We prove the result by induction on \( n \). When \( n = 1 \), we can identify \( E \) with \( \mathbb{R} \). Then \( A \) is of the form \( x \mapsto t_A x \), for some nonzero \( t_A \), and \( \{ f > 0 \} \) is either \( \mathbb{R}_{>0} \), or \( \mathbb{R}_{<0} \). Hence, \( x \) belongs to \( \mathbb{R}_{>0} \), or to \( \mathbb{R}_{<0} \), and \( t_A^k x \) belongs to the same half-space for every \( k \geq 0 \). Hence, \( t_A > 0 \).

Now if \( A \) is non invertible, we can replace \( E \) by the image of \( A \), \( \text{Im}(A) \), and \( x \) by \( A(x) \), so that the hypothesis are still verified by \( A \)'s restriction to \( \text{Im}(A) \). But since \( \text{Im}(A) \) is a subspace of \( E \) of strictly smaller dimension, we get the result using the induction hypothesis. We are thus left with the case when \( A \) is invertible. Let \( m \) be the maximal non negative integer such that \( (f, f \circ A^1, \ldots, f \circ A^m) \) is a linearly independent family of \( E^* \). It is easy to see that \( L = \cap_{k \geq 0} \ker(f \circ A^k) \) is equal to \( \cap_{k=0}^m \ker(f \circ A^k) \). Hence, it is \( A \)-stable. The space \( L \) is a proper subspace of \( E \) because it is contained in \( \ker(f) \). Taking the quotient space \( E = E/L \), the linear map \( A \) induces \( \bar{A} : E \rightarrow \bar{E} \), and \( f \) induces a linear form \( \bar{f} \) on \( E \). By letting \( \bar{x} \) be the image of \( x \) in \( E \), the quadruplet \((\bar{E}, \bar{A}, \bar{f}, \bar{x})\) still satisfies the hypothesis of the theorem. If \( L \) is not zero, using the induction we conclude that the linear map \( \bar{A} \) has a positive eigenvalue \( \lambda > 0 \). But \( \lambda \) is necessarily an eigenvalue of \( A \) by Lemma \[3.3\] and we are done in this case. Finally, assume that \( L = \{0\} \). Then \((e^*_1 = f, e^*_2 = f \circ A, \ldots, e^*_n = f \circ A^m)\) is a basis of \( E^* \). In particular \( m = n - 1 \), according to Lemma \[3.1\]. Take \((e_1, \ldots, e_n)\) as its dual basis in \( E \), and identify \( E \) with \( \mathbb{R}^n \), given this basis. Then \( A^k(x) \) belongs to the space \( \{v \mid \forall i, v_i > 0\} \subset \Delta \) for all \( k \geq 0 \). Hence, the convex cone \( C \) is spanned by this family as well. It is clearly \( A \)-stable, and is not reduced to zero as it contains \( x \). We conclude by applying Proposition \[3.2\].

This also concludes the proof of Theorem \[3.1\] as Theorem \[3.2\] is an equivalent statement written in terms of the morphisms \( A = \text{Mat}_B(A) \) and \( f = \text{Mat}_B(f) \). Theorem \[3.1\] says that the linear program terminates when \( A \) has no positive eigenvalue. But one cannot conclude on the termination problem using Theorem \[3.1\] when \( A \) has at least one positive eigenvalue. As we already mentioned, Theorem \[3.1\] is stated in \[12\]. But the proof given therein contains certain flaws that we now expose.

Remark 3.1. The argument of \[12\] applies the Brouwer’s fixed point theorem to a subspace of the projective space \( P(\mathbb{R}^n) \), and not \( \mathbb{R}^{n-1} \) as stated in \[12\]. However, this is not an Euclidian space, and so convexity is not well defined in it. Hence, one cannot apply Brouwer’s fixed point theorem to such a set. Moreover, using notation as in the proof of Theorem 1 in \[12\], the closure \( NT^i \) of the set \( NT \) can contain zero. For example as soon as all, real or
while $(3x - y > 0)$ {
    $x := 3x - 2y$;
    $y := 4/3x - 5/3y$;
} /*...*/

while $(z > 0)$ {
    $x := x + y$;
    $z := -z$;
} /*...*/

Figure 1: Examples of homogeneous linear programs

Theorem 3.1 provides a sufficient condition for the termination of linear programs. In other words, Theorem 3.1 says that the linear program terminates when there is no positive eigenvalues. But one can not conclude on the termination problem using Theorem 3.1 if there exists at least one positive eigenvalue. Intuitively, we could say that Theorem 3.1 provides us with a decidability result for the termination problem considering the subclass of linear program where the associated assignment matrix $A$ has no positive eigenvalues. In the following example, we illustrate situations where Theorem 3.1 applies and when it does not.

Example 3.1. Consider the homogeneous linear program 1a denoted by $P(A, v)$, and depicted in Figure 1. The associated matrix $A = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$ correspond to the simultaneous updates representing the sequential loop assignments, and the vector $v$ encoding the loop condition, is $v = (3, -1)$. The eigenvalues of $A$ are the complex numbers: $1 + 2i$ and $1 - 2i$. As $S$ does not have any positive eigenvalues, we can use Theorem 3.1 and conclude that program $P(A, v)$ terminates on all possible inputs.

Example 3.2. Now consider the homogeneous linear program 1b depicted in Figure 1 denoted by $P(A_1, v_1)$. The associated matrix $A_1$ representing the simultaneous updates is given by $A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Its eigenvalues...
are 1 and −1. As \( A \) has a positive eigenvalues, one can not determine the termination of \( P(A_1, v_1) \) using Theorem 3.1. In the following sections we will see how to handle this case in an automated and efficient.

In the next subsection we generalize Theorem 3.1, obtaining stronger results.

### 3.2 Necessity and Sufficiency for Termination of Linear Programs

Theorem 3.3 provides a necessary and sufficient condition for the termination of programs \( P(A, f) \in P_\text{H} \)

\[
\text{while } (f^T x > 0) \{ x := Ax \}.
\]

**Theorem 3.3.** Let \( A \in M_n(\mathbb{R}) \) and let \( f \neq 0 \) be in \( \mathbb{R}^n \). Then program \( P(A, f) \)

\[
\text{while } (f^T x > 0) \{ x := Ax \}
\]

terminates if and only if for every positive eigenvalue \( \lambda \) of \( A \), the generalized eigenspace \( E_\lambda(A) \) is orthogonal to \( f \), i.e., \( f^T E_\lambda(A) = \langle f, E_\lambda(A) \rangle = 0 \).

In order to prove Theorem 3.3 we first restate it in equivalent linear algebraic terms.

**Theorem 3.4.** Let \( E \) be an \( \mathbb{R} \)-vector space of finite dimension \( n \), let \( A \) be an endomorphism of \( E \), and let \( f \) be a nonzero linear form on \( E \). Then there exists a vector \( x \in E \) with \( f(A^k(x)) > 0 \) for all \( k \geq 0 \) if and only if there is \( \lambda > 0 \) in \( \text{Spec}(A) \) such that \( E_\lambda(A) \not\subset \text{Ker}(f) \).

**Proof.** First suppose that there is a \( \lambda > 0 \) in \( \text{Spec}(A) \) with \( E_\lambda(A) \not\subset \text{Ker}(f) \). Then there is some \( r \geq 1 \) such that \( \text{Ker}(A - \lambda \text{Id}_E)^{r-1} \subset \text{Ker}(f) \). But we also have \( \text{Ker}(A - \lambda \text{Id}_E)^r \not\subset \text{Ker}(f) \). Let \( x \) be an element of \( \text{Ker}(A - \lambda \text{Id}_E)^r - \text{Ker}(f) \) such that \( f(x) > 0 \). This is always possible because \( \text{Ker}(A - \lambda \text{Id}_E)^r - \text{Ker}(f) \) is stable under \( y \mapsto -y \). Because \( x \in \text{Ker}(A - \lambda \text{Id}_E)^r \), it is clear that \( \lambda x \in \text{Ker}(A - \lambda \text{Id}_E)^{r-1} \). Let \( L \) be \( \text{Ker}(A - \lambda \text{Id}_E)^{r-1} \), and let \( \overline{E} = E/L \). As \( L \) is \( A \)-stable, \( \overline{A} \) is well defined, and \( \overline{A}(\overline{x}) = \lambda \overline{x} \) because \( A(x) - \lambda x \in L \). Moreover, \( L \subset \text{Ker}(f) \). Hence, \( \overline{f} \) is well defined and \( \overline{f}(\overline{A}^k(\overline{x})) = f(A^k(x)) \) for every \( k \geq 0 \). As \( \overline{A}^k(\overline{x}) = \lambda^k \overline{x} \), we deduce that \( f(A^k(x)) = \lambda^k f(x) > 0 \) for all \( k \geq 0 \).
Conversely, suppose that there exists a vector \( x \in E \), such that \( f(A^k(x)) > 0 \) for all \( k \geq 0 \). We prove by induction on \( n \) that \( A \) has an eigenvalue \( \lambda > 0 \) such that \( E_\lambda(A) \) is not contained in \( \text{Ker}(f) \). If \( n = 1 \), then \( A : t \mapsto \lambda t \) for \( \lambda \in \mathbb{R} \), and so, \( \lambda^k(f(x)) > 0 \) for all \( k \geq 0 \). This implies \( \lambda > 0 \), and so \( E_\lambda(A) = E \) is not be contained in \( \text{Ker}(f) \). If \( n > 1 \), according to Theorem 3.2 we know that \( A \) admits a positive eigenvalue \( \mu \). If \( E_\mu(A) \) is not a subset of \( \text{Ker}(f) \) we are done. If \( L = E_\mu(A) \subset \text{Ker}(f) \), we consider \( \overline{E} = E/L \). This vector space is of dimension less than \( n \) and so \( f(\overline{A}^k(x)) = f(A^k(x)) > 0 \) for all \( k \geq 0 \). By the induction hypothesis, there is some \( \lambda > 0 \) in \( \text{Spec}(\overline{A}) \) such that \( E_\lambda(\overline{A}) \not\subset \text{Ker}(\overline{f}) \). But \( \lambda \) belongs to \( \text{Spec}(A) \) according to Lemma 3.3 and \( E_\lambda(A) \) maps surjectively on \( E_\lambda(\overline{A}) \) according to this same Lemma. In particular, we have \( \overline{f}(E_\lambda(\overline{A})) = f(E_\lambda(A)) \), but the left hand side is not reduced to zero in this equality. Hence, \( f(E_\lambda(A)) \neq \{0\} \), i.e., \( E_\lambda(A) \not\subset \text{Ker}(f) \), concluding the proof.

This argument proves Theorem 3.3 as it is a direct corollary of Theorem 3.4 with \( A = \text{Mat}_B(A) \) and \( f = \text{Mat}_B(f) \). Theorem 3.3 gives a necessary and sufficient condition that we can use as the foundation to build a complete procedure for checking termination. In order to determine termination, we have to check, for each positive eigenvalue, if the vector \( f \), encoding the loop condition, is orthogonal to the associated generalized eigenspace. In other words we want to verify if \( f \) is orthogonal to the nullspace \( \text{Ker}((A - \lambda I_n)^n) \).

**Example 3.3.** Consider the program \( P(A_1, v_1) \) depicted in Figure 4 that we denoted as \( P(A_1, v_1) \). The matrix \( A_1 \) is given in Example 3.1. The vector encoding the loop condition is \( v_1 = e_3 = (0, 0, 1) \). We recall that \( A_1 \) has eigenvalues 1 and \(-1\). The generalized eigenspace \( E_1(A_1) \) is equal to \( \text{Vect}(e_1, e_2) \), where \( e_1 \) and \( e_2 \) are the first two vectors of the canonical basis of \( \mathbb{R}^3 \). Hence \( E_1(A_1) \) is orthogonal to \( v_1 \). According to Theorem 3.3, program \( P(A, w) \) terminates.

**Example 3.4.** Now we change the loop condition of program 4 depicted in Figure 4 to \( (y > 0) \). Then, we obtain the program \( P(A_1, v_2) \) with the new considered loop condition encoded as \( v_2 = e_2 = (0, 1, 0) \). The eigenvalues of \( A_1 \) are (still) 1 and \(-1\), and the generalized eigenspace \( E_1(A_1) = \text{Vect}(e_1, e_2) \). Hence \( E_1(A) \) is not orthogonal to \( v_2 \), because it contains \( v_2 \). Theorem 3.3 tells us the program \( P(A_1, v_2) \) does not terminate.

In both of these examples, we are able to determine the termination or nontermination of the corresponding program using Theorem 3.3. On the
other hand, Theorem 3.1 does not allow us to conclude anything about the termination of these programs, since the assignment matrix $A'$ exhibit at least one positive eigenvalue. In order to avoid the computation of basis for generalized eigenspaces, we first introduce the space \( \text{Row}_\text{Space}(M) \), and use the next lemma. If $M \in M(m, n, \mathbb{R})$, then \( \text{Row}_\text{Space}(M) \) denotes the vector subspace of \( \mathbb{R}^n \) spanned by the row vectors of $M$.

**Lemma 3.4.** Let $M$ be a matrix in \( M(m, n, \mathbb{K}) \). Then every vector in the nullspace of $M$ is orthogonal to every vector in \( \text{Row}_\text{Space}(M) \).

**Proof.** Let $w \in \text{Ker}(M)$, and let $v$ be in the column space of $M^\top$. We denote by $\{c_1, ..., c_m\}$ the set of column vectors of $M^\top$. Then, there exists a vector $k \in \mathbb{R}^m$ such that $v = M^\top \cdot k$, since $v$ is a linear combination of the column vectors of $M^\top$. Now we have

\[
\langle w, v \rangle = w^\top \cdot v = w^\top \cdot M^\top \cdot k = (M \cdot w)^\top \cdot k = 0,
\]

because $w \in \text{Ker}(M)$ and $M \cdot w = 0$.

From Lemma 3.4, a basis of \( \text{Row}_\text{Space}(M) \) is a basis of the orthogonals of \( \text{Ker}(M) \). Thus, for the square matrix $A$, a vector $v$ is orthogonal to \( \text{Ker}((A - \lambda I_n)^n) \), i.e., $\langle E_\lambda(A), v \rangle = 0$, if and only if $v \in \text{Row}_\text{Space}((A - \lambda I_d)^n)$. We directly deduce the following corollary.

**Corollary 3.1.** Let $A \in M_n(\mathbb{R})$ and $v \neq 0 \in \mathbb{R}^n$. The program $P(A, v)$ terminates if and only if for every positive eigenvalue $\lambda$ of $A$ $v$ is in the vector space \( \text{Row}_\text{Space}((A - \lambda I_d)^n) \).

**Proof.** By Lemma 3.4 the basis of \( \text{Row}_\text{Space}((A - \lambda I_d)^n) \) is a basis of the orthogonals of \( \text{Ker}((A - \lambda I_d)^n) \). We then apply Theorem 3.3.

### 4 Running Example

In practice, we can use Corollary 3.1 to support three fast computational steps, as illustrated in the following example.

**Example 4.1.** (Running example) Consider a program $P(A, v)$ where

\[
A = \begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & 1 \\
0 & 0 & 0 & 2
\end{pmatrix}, \quad \text{and} \quad v = \begin{pmatrix}
-1 \\
-1 \\
1 \\
1
\end{pmatrix}.
\]

**Step 1:** We compute the list $e_\lambda$ of positive eigenvalues for $A$. The result is:
Hence, we have three positive eigenvalues, namely, \( \lambda_1 = 2 \), \( \lambda_2 = 2 - \sqrt{2} \), \( \lambda_3 = 2 + \sqrt{2} \), with multiplicities 2, 1 and 1, respectively.

**Step 2:** We compute the matrix \( E_\lambda = (A - \lambda I_n)^n \) for \( \lambda = 2 + \sqrt{2} \). The result is:

\[
(A - (e[i])*Id_m)^d
\begin{bmatrix}
18 & 16*sqrt(2) & 14 & -4*sqrt(2) \\
16*sqrt(2) & 32 & 16*sqrt(2) & -14 \\
14 & 16*sqrt(2) & 18 & -12*sqrt(2) \\
0 & 0 & 0 & 4
\end{bmatrix}
\]

**Step 3:** We check if \( v \in \text{Row}_\text{Space}(E_\lambda) \):
Here we use a standard procedure from linear algebra to check if a given vector belongs to a vector-space spanned by a given set of vectors. We compute the unique reduced row echelon form of matrix \( E_\lambda^T \). For that we run a Gaussian elimination on the rows using the Gauss-Jordan elimination algorithm. The generated matrix, below on the left, provides us with a linearly independent basis for \( \text{Row}_\text{Space}(E_\lambda) \). We remove the rows containing only zero entries, and we augment the computed basis with the vector \( v^T \) by appending it as the last row. We obtain the matrix below on the right.

\[
\text{(E[i]).echelon_form()}
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & sqrt(2) & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\text{block_matrix([[Er[i]], [V.T]])}
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & sqrt(2) & 0 \\
0 & 0 & 0 & 1 \\
-1 & -1 & 1 & 1
\end{bmatrix}
\]

Finally, we generate its reduced row echelon form obtaining matrix \( R_\lambda S_\lambda \):

\[
\text{block_matrix([[Er[i]], [V.T]].echelon_form()}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
------
0 & 0 & 0 & 1
\end{bmatrix}
\]
From the Gauss-Jordan elimination properties, it is well-known that \( v \) belongs to the space \( \text{Row}_\lambda(E) \) if and only if \( R_{\text{S}_\lambda}(n, n+1) = 0 \). Here we have \( R_{\text{S}_\lambda}(n, n+1) = 1 \), which means that \( v \) is not in \( \text{Row}_\lambda(E) \). Thus, by Corollary 3.1, we conclude that program \( P(A, v) \) is nonterminating.

As we show in Example 4.1, we avoid the computation of generalized eigenspaces in practice. Instead, use the exact algorithm associated to Corollary 3.1.

5 A Complete Procedure to Check Termination

We use the necessary and sufficient conditions provided by Theorem 3.3 and its related practical Corollary 3.1 to build a sound and complete procedure to check the termination of linear programs. Moreover, the method so obtained is based on few computational steps associated with fast numerical algorithms.

The pseudo code depicted in Algorithm 1 illustrates the strategy. It takes as input the number of variables, the chosen field where the variables are interpreted, the assignment matrix \( A \) and the vector \( w \) encoding the loop condition. We first compute the list of positive eigenvalues (lines 1 and 2 in Algorithm 1). If this list is empty we can then state that the loop is terminating (lines 3 and 4). Otherwise, we continue the analysis using the nonempty list of positive eigenvalues. For each positive eigenvalues \( e'[i] \) we first need to compute the matrix \( E_i = (A - e'[i]I_n)^n \) (line 6). Using Corollary 3.1, we know that the loop is terminating if and only if \( w \) is in the \( \text{Row}_\cdot \) of \( (A - e'[i]I_n)^n \) for every positive eigenvalue \( e'[i] \). In other words, for each positive eigenvalue, we have to check if \( w \) is in the vector space spanned by the basis of the \( \text{Row}_\cdot \) of the associated matrix \( E_i \). In order to do so, one first needs to consider the linearly independent vectors \( \{r_1, ..., r_n\} \) that form a basis of the \( \text{Row}_\cdot \). This basis is obtained from the list of the non-zero row vectors of the computed \emph{reduced row echelon form} of \( E_i \) (lines 7 and 8). The efficient way to check if \( w \) is in the vector space spanned by the basis \( \{r_1, ..., r_n\} \) comprises the following computational steps: (i) We build the augmented matrix \( E_A \) formed by the row vectors \( r_1, ..., r_n \) and \( w^\top \) (line 9); (ii) We compute the \emph{reduced row echelon form} of matrix \( E_A \) (line 8). For that we apply \emph{Gaussian elimination} on the rows. This reduced, canonical form is
unique and is computed exactly by the Gauss-Jordan elimination method; (iii) We know that the added vector \( w \) is in the vector space spanned by \( r_1, \ldots, r_n \) if and only if the bottom right entry of the reduced row echelon matrix \( E_R \) is null. Thus if \( E_R(n, n+1) \neq 0 \), we conclude that there exists a positive eigenvalue \( e'[i] \) such that \( w \) is not in \( \text{Row}_\text{Space}(A - e'[i]I_n)^n \), which is equivalent to saying that the loop is nonterminating (lines 11 and line 12). Otherwise if he have exhausted the list of positive eigenvalues and always found that \( w \) is in the \( \text{Row}_\text{Space} \) of the associated matrix, we conclude that the loop is terminating (line 13).

### Algorithm 1: Termination_linear_Loop \((n, \mathbb{K}, A, w)\)

/*Checking the termination for linear homogeneous programs.*/

**Data:** \( n \) the number of program variables, \( \mathbb{K} \) the field, \( P(A, w) \in P_{\mathbb{N}} \)

where \( A \in M(n, \mathbb{K}) \) and \( w \in M(n, 1, \mathbb{K}) \)

**Result:** Determine the Termination/Nontermination

begin

1. \( \{e[1], \ldots, e[r]\} \leftarrow \text{eigenvalues}(A); \)
2. \( \{e'[1], \ldots, e'[s]\} \leftarrow \text{strictly positives}(\{e[1], \ldots, e[r]\}); \)
3. if \( \{e'[1], \ldots, e'[s]\} = \emptyset \) then there is no positive eigenvalues.
   return TERMINANT;
4. for \( i = 1 \) to \( s \) do
5. \( E \leftarrow (A - e'[i]I_n)^n; \)
6. \( E_{rrf} \leftarrow \text{echelon_form}(E); \)
7. \( E'_{rrf} \leftarrow \text{remove_zero_row}(E_{rrf}); \)
8. \( E_A \leftarrow \text{augment_row}(E'_{rrf}, w^\top); \)
9. \( E_R \leftarrow \text{echelon_form}(E_A); \)
10. if \( E_R(n, n+1) \neq 0 \) then
   return NONTERMINANT;
11. return TERMINANT;

The function \text{echelon_form} computes the reduced row echelon form by Gauss-Jordan elimination, and its time complexity is of order \( O(n^3) \). We interpret the variables in a specified field, i.e. an extension of \( \mathbb{Q} \), chosen according to the discussion in Section 6. By using efficient mathematical packages, e.g. Maple, Mathematica, Sage, Lapack or Eispack, one can ob-
tain the eigenvalues as closed-form algebraic expressions, \textit{i.e.} the solution of an algebraic equation in terms of its coefficients, relying only on addition, subtraction, multiplication, division, and the extraction of roots. Also, it is well known that with \( n < 5 \), the eigenvalues computed by the function \texttt{eigenvalues} are already exhibited as such algebraic numbers. Moreover, the algorithm for eigenvalue computation has a time complexity that is of order \( O(n^3) \), and so the overall time complexity of the algorithm \texttt{Termination_linear_Loop} remains of the same order of time complexity.

In Table 1 we list some experimental results. The column \texttt{Set-i} refers to a set of loops generated randomly. The column \texttt{#Loops} gives the number of loops treated. We use the countable subsets described in Section 6. The column \texttt{Dim} refers to the dimension of the initial systems, \textit{i.e.}, the number of variables. The column \texttt{#T} shows the number of programs found to be terminating, and the column \texttt{#NT} gives the number of loop programs found to be non-terminating. Finally, column \texttt{CPU/s[T]} refers to cpu time results while checking all the terminating loop programs, and column \texttt{CPU/s[N]} gives the cpu time taken to check nontermination. The column \texttt{CPU/s[total]} gives cpu time results, in seconds, for deciding about termination for the given set of 500 loops. We have implemented our prototype using \texttt{Sage} \cite{23} with interfaces written in \texttt{Python}. By so doing, we were able to access several useful mathematical packages. As expected, we can see that more nonterminating programs were found, as they are easier to write. Note also that it takes much more time to prove termination than to prove nontermination.

6 Variables Over Countable Sets

In this section, we show that to check the termination of a linear program \( P(A,v) \) with one loop condition over \( \mathbb{R}^n \), we can restrict the analysis to the case where the variable belongs to a countable subset of \( \mathbb{R}^n \), depending on \( A \). First, we study an example, which is already interesting in itself, and which will prove that we cannot restrict the interpretation of the variable over the field \( \mathbb{Q} \) of rational numbers if we want to prove the termination for all real inputs. We start with two elements of \( \mathbb{Q}(\sqrt{2}) - \mathbb{Q} \), which are conjugate under the Galois group \( \text{Gal}_{\mathbb{Q}}(\mathbb{Q}(\sqrt{2})) \), of opposite signs, and the negative one of absolute value strictly greater than the positive one. For instance, take \( \lambda^- = -1 - \sqrt{2} \), and \( \lambda^+ = -1 + \sqrt{2} \). They are the roots of the polynomial
\(P(X) = (X - \lambda^-)(X - \lambda^+) = X^2 + 2X - 1\). Now let \(A = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}\) be the associated companion matrix, so that its characteristic polynomial is \(P\), and its eigenvalues are \(\lambda^-\) and \(\lambda^+\). Its generalized eigenspaces are easy to compute. We find \(E_{\lambda^-}(A) = \mathbb{R}.e^-\) and \(E_{\lambda^+}(A) = \mathbb{R}.e^+\) with \(e^- = \begin{pmatrix} 1 \\ \lambda^- \end{pmatrix}\) and \(e^+ = \begin{pmatrix} 1 \\ \lambda^+ \end{pmatrix}\). Now let \(v = (1, 0)^T\). We have \(<v, e^+> = 1\) and so, according to Theorem 3.3, the program \(P_1 = P(A, v)\), associated to \(A\) and \(v\), does not terminate. We can actually find the points of \(\mathbb{R}^2\) for which the program is not terminating.

**Proposition 6.1.** Let \(A, v\) and \(P_1\) be as above. Then program \(P_1\) does not terminate for an initial condition \(x \in \mathbb{R}^2\) if and only if \(x \in E_{\lambda^+}(A)\) and \(<x, v> > 0\), i.e. \(x \in \mathbb{R}_{>0}.e^+\).

**Proof.** If \(x = t.e^+\), with \(t > 0\), then \(A^k(x) = t\lambda^{+k}.x\), and \(<v, A^k(x) >= t\lambda^{+k} > 0\) for all \(k \geq 0\). Hence, the program does not terminate with such an \(x\) as initial condition. Conversely, suppose that \(x\) satisfies \(<v, A^k(x) > 0\) for all \(k \geq 0\). Decompose \(x\) on the basis \((e^-, e^+). Then x = s.e^- + t.e^+, and A^k(x) = s\lambda^{-k}.e^- + t\lambda^{+k}.e^+, so that <v, A^k(x) >= s\lambda^{-k} + t\lambda^{+k}. Suppose that \(s\) is not zero. As \(|\lambda^-| > |\lambda^+|\), for \(k\) large enough, the scalar \(<v, A^k(x) >\) will be of the same sign as \(s\lambda^{-k}\), which alternates positive and negative. Since this is absurd, \(s = 0\). Now as \(<v, A^k(x) >= t\lambda^{+k}\), this implies that \(t > 0\), and so the proposition holds.

Proposition 6.1 leads us to the following corollary.

**Corollary 6.1.** With \(A\) and \(v\) as above, program \(P_1\) is terminating on \(\mathbb{Q}^2\), but not on \(\mathbb{R}^2\).

**Proof.** We already saw that \(P_1\) does not terminate on \(\mathbb{R}^2\). Now let \(x\) be an element of \(\mathbb{Q}^2\). If \(P_1\) was not terminating with \(x\) as an initial value, this would imply that \(x\) is in \(\mathbb{R}_{>0}.e^+\), according to Lemma 6.1. However, no element of \(\mathbb{Q}^2\) belongs to \(\mathbb{R}_{>0}.e^+\) because the quotient of the coordinates of \(e^+\) is irrational. This implies that \(P_1\) terminates on \(\mathbb{Q}^2\).

This proves that even if \(A\) and \(v\) are rational, one cannot guarantee the termination over the reals if the interpretation of the variables are restricted.
to rationals. It is clear that one cannot hope to produce any valid conjecture of this type if $A$ and $v$ have wild coefficients, like transcendentals, for example. However, when $A$ and $v$ have algebraic coefficients, using Corollary 3.1 one can find a simple remedy. It is indeed enough to replace $\mathbb{Q}$ by a finite extension of the field $\mathbb{Q}$. Such an extension $K$ is called a number field, and is known to be countable. Indeed, it is a $\mathbb{Q}$-vector space of finite dimension, i.e., $K = \mathbb{Q}.k_1 \oplus \cdots \oplus \mathbb{Q}.k_l$ for some $l \geq 1$, and elements $k_i$ in $K$. It is, moreover, known that $K$ is the fraction field of its ring of integers $O_K$, which is a free $\mathbb{Z}$-module of finite type. In fact $O_K = \mathbb{Z}.o_1 \oplus \cdots \oplus \mathbb{Z}.o_l$ for the same $l \geq 1$, and where the elements $o_i$ can be chosen equal to the $k_i$, for well chosen $k_i$'s. We say that a number field is real if it is a subfield of $\mathbb{R}$. Notice that in the mathematical literature a totally real number field is a number field with only real embeddings in $\mathbb{C}$. Here what we call real is thus weaker than totally real.

**Theorem 6.1.** Let $A \in M_n(\mathbb{R})$, $v \neq 0 \in \mathbb{R}^n$, and suppose that their coefficients are actually in $\mathbb{Q}$ or, more generally, in a real number field $K$. Then there is a well-determined real finite extension $L$ of $\mathbb{Q}$, or of $K$ in the general case, which is contained in $\mathbb{R}$ and such that the program $P(A,v)$, associated to $A$ and $v$, terminates if and only if it terminates on the countable set $L^n$. We can choose $L$ to be the extension $\mathbb{Q}(\lambda_1, \ldots, \lambda_l)$ of $\mathbb{Q}$, or $K(\lambda_1, \ldots, \lambda_l)$ in general, spanned by the positive eigenvalues $(\lambda_1, \ldots, \lambda_l)$ of $A$. It is actually enough to check the termination of the program on $O^n_L$.

**Proof.** We deal with the general case. The reader not familiar with field extensions can just replace $K$ by $\mathbb{Q}$. It is obvious that if the program terminates, it terminates on $L^n$ for any subset $L$ of $\mathbb{R}$. Now let $\lambda_1, \ldots, \lambda_r$ be the positive eigenvalues of $A$. They are all roots of the minimal (or characteristic) polynomial $Q$ of $A$, which is in $K[X]$. Hence they are all algebraic on $K$, and so also on $\mathbb{Q}$ as $K/\mathbb{Q}$ is finite. Let $L = K(\lambda_1, \ldots, \lambda_r) \subset \mathbb{R}$. Suppose that the program $P_i$ does not terminate. Then there is some $i \in \{1, \ldots, r\}$, such that $<E_{\lambda_i}, v> \neq 0$ according to Corollary 3.3. Let $r$ be the positive integer such that $\text{Ker}((A - \lambda_i I_n)^r) \not\subset v^\perp$, but $\text{Ker}((A - \lambda_i I_n)^{r-1}) \subset v^\perp$. As in the proof of Theorem 3.4 for any $x$ in $\text{Ker}((A - \lambda_i I_n)^r) - \text{Ker}((A - \lambda_i I_n)^{r-1})$, such that $<v, x> > 0$, the program does not terminate. We fix such an $x$. Since both spaces $\text{Ker}((A - \lambda_i I_n)^r)$ and $\text{Ker}((A - \lambda_i I_n)^{r-1})$ are defined by linear equations with coefficients in $L$, there is a basis of $\text{Ker}((A - \lambda_i I_n)^r)$ with coefficients in $L^n$ containing a basis of $\text{Ker}((A - \lambda_i I_n)^{r-1})$ with coefficients in $L^n$. It is easy to see that this implies that $L^n \cap [\text{Ker}((A - \lambda_i I_n)^r) - \text{Ker}((A - \lambda_i I_n)^{r-1})] \neq \emptyset$. Therefore, $P_i$ does not terminate on $L^n$ as well.
Ker((A − λI_n)^r−1]) is dense in Ker((A − λI_n)^r) − Ker((A − λI_n)^r−1), because L contains Q which is dense in \(\mathbb{R}\). Hence, there is a sequence \(x_k\) in \(L^n \cap [Ker((A − λI_n)^r) − Ker((A − λI_n)^r−1)]\) which approaches \(x\). In particular, \(\langle v, x_k \rangle > 0\) for \(k\) large enough. Thus the program does not terminate on \(x_k\) when \(k\) is such that \(\langle v, x_k \rangle > 0\). This shows that \(P_1\) does not terminate on \(L^n\). The fact that \(P_1\) does not terminate on \(O_L\) is a trivial consequence of the fact that any element of \(L\) is the quotient of two elements of \(O_L\). In particular, if \(P_1\) does not terminate on \(x \in L^n\), take \(a > 0\) in \(O_L\) such that \(ax \in O_L^2\). Then the program does not terminate on \(ax\).

We now show how Theorem 6.1 applies on our previous example.

**Example 6.1.** For the program associated to matrix \(A = \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}\) and vector \(v = (1, 0)\top\), we get \(L = \mathbb{Q}(\lambda^+) = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} : a \in \mathbb{Q}, b \in \mathbb{Q}\}\). Its ring of integers is \(O_L = \mathbb{Z}(\lambda^+) = \mathbb{Z}(\sqrt{2}) = \{a + b\sqrt{2} : a \in \mathbb{Z}, b \in \mathbb{Z}\}\). Theorem 6.1 asserts that, as the program \(P(A, v)\) is non terminating, it is already non terminating on \(O_L^2\). Indeed, take \(x^+\) as an initial value, then \(x^+ = \begin{pmatrix} 1 \\ -1 + \sqrt{2} \end{pmatrix}\) is in \(O_L^2\), and we saw that \(P(A, v)\) does not terminate on \(x^+\).

**7 Discussion**

The important papers [11, 12], treating homogeneous linear programs, can be seen, at first, as closely related to our results. The sufficient condition fully proved and established as our preliminary results in section 3.1 was first stated in [12]. On the other hand, the sufficient conditions proposed in [11, 12] are not necessary conditions for the termination of homogeneous linear programs and, thus, it is not obvious that one can obtain from those results a direct encoding leading deterministically to a practical algorithm. The treatment in [12] can be divided in two parts. First, the interesting sketch of the proof for the sufficient condition leaves space for elaboration. We completed it in a solid mathematical way. We found obstacles that were not obvious how to circumvent. Like applying Brouwer’s fixed point theorem to appropriate spaces, and having 0 in the closure of the orbit of a variable under the action of the transition matrix. The second part provides a lengthy procedure to check for termination. It comprises 3 reductions, a case
analysis, and long and costly symbolic computations (whose complexities are \( \max(O(n^6), O(n^m^3)) \) where \( n \) is the number of variables and \( m \) the number of conditions \[24\].) Also ideas presented in \[11\] are based on the approach proposed in \[12\], while considering termination analysis over the integers. Similar points could be raised concerning the work in \[11\], that is, a complex procedure is proposed, one that appears lengthy and costly. In fact, it is not clear to us if those approaches give rise to simple and fast algorithms. Instead, we have a more direct and clear statement which naturally provides a simple algorithm to check termination, as illustrated by our examples, and with much better complexity. Moreover, we show that it is enough to interpret the variable values over a countable number field, or over its ring of integers, in order to determine program termination over the reals.

In a recent work about asymptotically nonterminating values (ANT) generation \[13\], we also provided new and efficient techniques to extend our results to general affine loop programs, \( i.e. \), programs with several loop conditions. We defer this discussion to another companion article, where more practical details will be presented, together with some experiments.

The generated ANT set can be used directly as preconditions for termination or it can be intersected efficiently with another given preconditions, provided by other static analysis methods for instance.

Our main results, Theorem 3.3 and its Corollary 3.1, with a direct encoding as in Algorithm 1, together with the results in Section 6, guaranteeing the symbolic computation while circumventing rounding errors, are evidences of the novelty of our approach.

8 Conclusions

We presented the first necessary and sufficient condition for the termination of linear homogeneous loop programs. This condition leads to a sound and complete procedure for checking termination for this class of programs. The analysis of the associated algorithms shows that the new method operates in fewer computational steps than all known routines that support the mathematical foundations of previous methods. Section 6, and especially the example therein, introduces the important notion of the locus of initial variables values for which a linear program terminates. In that example, it allowed us to decide if the program terminates on all rational initial variables values. Actually, these methods can be vastly generalized in order to treat
the termination problem for linear programs on rational initial values. How-
however, we suspect that this development it will involve some Galois theo-
ry, as well as our results on asymptotically non terminating variable values [13],
and so we prefer to pursue this investigation in the near future.

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Table 1: Experimental results on randomly generated linear loop programs

| RandSet | #Loops | Dim | #T | #NT | CPU/s\[T\] | CPU/s\[N\] | CPU/s\[total\] |
|---------|--------|-----|----|-----|------------|------------|--------------|
| Set-1   | 500    | 3   | 152| 348 | 10.02      | 8.79       | 18.24        |
| Set-2   | 500    | 3   | 195| 305 | 8.97       | 9.11       | 18.08        |
| Set-3   | 500    | 3   | 233| 267 | 15.07      | 12.78      | 27.85        |
| Set-4   | 500    | 3   | 223| 277 | 12.49      | 10.42      | 22.91        |
| Set-5   | 500    | 3   | 246| 254 | 12.52      | 11.59      | 24.11        |
| Set-6   | 500    | 3   | 222| 278 | 13.30      | 10.35      | 23.66        |
| Set-7   | 500    | 4   | 122| 378 | 27.8       | 16.51      | 44.31        |
| Set-8   | 500    | 4   | 184| 316 | 42.67      | 21.90      | 64.57        |
| Set-9   | 500    | 4   | 145| 355 | 31.91      | 18.05      | 49.97        |
| Set-10  | 500    | 4   | 171| 329 | 41.16      | 22.37      | 63.54        |
| Set-11  | 500    | 4   | 185| 315 | 43.03      | 24.22      | 67.25        |
| Set-12  | 500    | 4   | 176| 324 | 40.36      | 19.95      | 50.32        |
| Set-13  | 500    | 5   | 183| 317 | 126.24     | 66.95      | 193.20       |
| Set-14  | 500    | 5   | 227| 273 | 155.80     | 81.29      | 237.10       |
| Set-15  | 500    | 5   | 178| 322 | 103.90     | 43.47      | 146.57       |
| Set-16  | 500    | 5   | 161| 339 | 169.92     | 54.00      | 223.92       |
| Set-17  | 500    | 5   | 174| 326 | 171.92     | 66.75      | 238.68       |
| Set-18  | 500    | 5   | 158| 342 | 174.91     | 70.32      | 245.24       |
| Set-19  | 500    | 6   | 141| 359 | 236.0      | 70.19      | 306.20       |
| Set-20  | 500    | 6   | 173| 327 | 387.80     | 105.69     | 493.50       |
| Set-21  | 500    | 6   | 192| 308 | 342.70     | 101.89     | 444.59       |
| Set-22  | 500    | 6   | 188| 312 | 352.40     | 165.41     | 517.81       |
| Set-23  | 500    | 6   | 227| 273 | 402.71     | 174.56     | 577.28       |
| Set-24  | 500    | 6   | 184| 316 | 385.00     | 190.94     | 575.94       |
| Set-25  | 500    | 7   | 171| 329 | 851.18     | 194.21     | 1044.39      |
| Set-26  | 500    | 7   | 139| 361 | 699.03     | 174.65     | 873.68       |
| Set-27  | 500    | 7   | 166| 334 | 876.62     | 238.94     | 1115.56      |