Another refinement of the right-hand side of the Hermite–Hadamard inequality for simplices

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Abstract. We establish a new refinement of the right-hand side of the Hermite–Hadamard inequality for convex functions of several variables defined on simplices.

Mathematics Subject Classification. Primary 26D150.

Keywords. Convex function, Simplex, Hertmite–Hadamard inequality.

The classical Hermite–Hadamard inequality states that if \( f: I \to \mathbb{R} \) is a convex function then for all \( a < b \in I \) the inequality

\[
 f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}
\]

is valid. This powerful tool has found numerous applications and has been generalized in many directions (see e.g. [1,2]). One of those directions is its multivariate version:

**Theorem 1.** ([1]) Let \( f: U \to \mathbb{R} \) be a convex function defined on a convex set \( U \subset \mathbb{R}^n \) and \( \Delta \subset U \) be an \( n \)-dimensional simplex with vertices \( x_0, x_1, \ldots, x_n \). Then

\[
 f(b_\Delta) \leq \frac{1}{\text{Vol} \Delta} \int_\Delta f(x)dx \leq \frac{f(x_0) + \cdots + f(x_n)}{n+1},
\]

(1)

where \( b_\Delta = \frac{x_0 + \cdots + x_n}{n+1} \) is the barycenter of \( \Delta \) and the integration is with respect to the \( n \)-dimensional Lebesgue measure.

The aim of this note is to prove a refinement of the right-hand side of (1) stated in Theorem 2.

Let us start with a set of definitions.

A function \( f: I \to \mathbb{R} \) defined on an interval \( I \) is called convex if for any \( x, y \in I \) and \( t \in (0,1) \) the inequality
f(tx + (1 - t)y) ≤ tf(x) + (1 - t)f(y)

holds.

If $U$ is a convex subset of $\mathbb{R}^n$, then a function $f : U \to \mathbb{R}$ is convex if its restriction to every line segment in $U$ is convex.

For $n + 1$ points $x_0, \ldots, x_n \in \mathbb{R}^n$ in general positions the set $\Delta = \text{conv}\{x_0, \ldots, x_n\}$ is called an $n$-dimensional simplex. If $K$ is a nonempty subset of the set $N = \{0, \ldots, n\}$ of cardinality $k$, the set $\Delta_K = \text{conv}\{x_i : i \in K\}$ is called a face (or a $k - 1$-face) of $\Delta$. The point $b_K = \frac{1}{k} \sum_{i \in K} x_i$ is called a barycenter of $\Delta_K$. The barycenter of $\Delta$ will be denoted by $b$. By $\text{card} \, K$ we shall denote the cardinality of the set $K$.

For each $k - 1$-face $\Delta_K$ we calculate the average value of $f$ over $\Delta_K$ using the formula

$$\text{Avg}(f, \Delta_K) = \frac{1}{\text{Vol}(\Delta_K)} \int_{\Delta_K} f(x) \, dx,$$

where the integration is with respect to the $k - 1$-dimensional Lebesgue measure (in case $k = 1$ this is the counting measure).

For $k = 1, 2, \ldots, n + 1$ we define

$$A(k) = \frac{1}{\binom{n + 1}{k}} \sum_{K \subset N \atop \text{card} \, K = k} \text{Avg}(f, \Delta_K).$$

Note that the right-hand side of the inequality (1) can be rewritten as $A(n + 1) \leq A(1)$. It turns out, that

**Theorem 2.** The following chain of inequalities holds:

$$A(n + 1) \leq A(n) \leq \cdots \leq A(2) \leq A(1).$$

In the proof we shall use the following

**Lemma 1.** ([3, Theorem 4.1]) If $K_i = N \setminus \{i\}$ and $b$ is the barycenter of $\Delta$, then

$$\text{Avg}(f, \Delta) \leq \frac{1}{n + 1} f(b) + \frac{n}{n + 1} \frac{1}{n + 1} \sum_{i=0}^{n} \text{Avg}(f, \Delta_{K_i}).$$

**Proof of Theorem 2.** We shall prove first the inequality $A(n + 1) \leq A(n)$. Let us use the notation from Lemma 1. For $i = 0, 1, \ldots, n$ we have

$$b_{K_i} = \frac{1}{n} \sum_{j=0}^{n} x_j = \frac{1}{n} \left( \sum_{j=0}^{n} x_j - x_i \right) = \frac{1}{n} ((n + 1)b - x_i). \quad (2)$$

Summing (2) we obtain

$$b = \frac{1}{n + 1} \sum_{j=0}^{n} b_{K_j}. \quad (3)$$
Now using Lemma 1 and the convexity of $f$ applied to (3) we get

\[
\text{Avg}(f, \Delta) \leq \frac{1}{n+1} f(b) + \frac{n}{n+1} \frac{1}{n+1} \sum_{i=0}^{n} \text{Avg}(f, \Delta_{K_i})
\]

\[
\leq \frac{1}{n+1} \frac{1}{n+1} \sum_{i=0}^{n} f(b_{K_i}) + \frac{n}{n+1} \frac{1}{n+1} \sum_{i=0}^{n} \text{Avg}(f, \Delta_{K_i}),
\]

thus, by the left-hand side of (1)

\[
\leq \frac{1}{n+1} \frac{1}{n+1} \sum_{i=0}^{n} \text{Avg}(f, \Delta_{K_i}) + \frac{n}{n+1} \frac{1}{n+1} \sum_{i=0}^{n} \text{Avg}(f, \Delta_{K_i})
\]

\[
= \frac{1}{n+1} \sum_{i=0}^{n} \text{Avg}(f, \Delta_{K_i}).
\]

This shows the inequality $A(n+1) \leq A(n)$.

Let $K \subset N$ be a set of cardinality $k > 1$. Applying the above reasoning to $\Delta_{K}$ we obtain

\[
\text{Avg}(f, \Delta_{K}) \leq \frac{1}{k} \sum_{K' \subset K, \text{card } K' = k-1} \text{Avg}(f, \Delta_{K'}).
\]

Summing the above for all $k$-element subsets we get

\[
\sum_{K \subset N, \text{card } K = k} \text{Avg}(f, \Delta_{K}) \leq \frac{1}{k} \sum_{K \subset N, \text{card } K = k} \sum_{K' \subset K, \text{card } K' = k-1} \text{Avg}(f, \Delta_{K'})
\]

\[
= \frac{n-k+2}{k} \sum_{K' \subset K, \text{card } K' = k-1} \text{Avg}(f, \Delta_{K'}).
\]

The equality follows from the fact that every $k - 2$-face belongs to $n - k + 2$ distinct $k - 1$-faces so every term $\text{Avg}(f, \Delta_{K'})$ appears in the sum exactly $n - k + 2$ times. Dividing both sides by $\binom{n+1}{k}$ we get $A(k) \leq A(k-1)$, which completes the proof. \qed

Just for completeness note that a similar refinement of the left-hand side of (1) can be found in [4, Corollary 2.6]. It reads as follows:

**Theorem 3.** For a nonempty subset $K$ of $N$ define the simplex $\Sigma_K$ as follows: let $A_K$ be the affine span of $\Delta_K$ and $A'_K$ be the affine space of the same dimension, parallel to $A_K$ and passing through the barycenter of $\Delta$. Then $\Sigma_K = \Delta \cap A'_K$. 

For $k = 1, 2, \ldots, n + 1$ let
\[ B(k) = \frac{1}{\binom{n+1}{k}} \sum_{K \subset N, \text{card } K = k} \text{Avg}(f, \Sigma_K). \]

Then
\[ f(b) = B(1) \leq B(2) \leq \cdots \leq B(n + 1) = \text{Avg}(f, \Delta). \]

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Received: January 25, 2018
Revised: June 23, 2018