Non-existence of certain Galois representations with a uniform tame inertia weight: A resume

By

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Abstract

In this paper, we announce some results on the non-existence of certain semistable Galois representations. We apply them to a conjecture of Rasmussen and Tamagawa.

§ 1. Main results

Our main concern in this paper is the non-existence of certain semistable Galois representations of a number field. Let $\ell$ be a prime number and $K$ a number field of degree $d$ and discriminant $d_K$. Choose an algebraic closure $\bar{K}$ of $K$. Fix non-negative integers $n,r$ and $w$, and a prime number $\ell_0 \neq \ell$. Put $\bullet := (n, \ell_0, r, w)$. Let $\text{Rep}_{\mathbb{Q}_\ell}(G_K)^\bullet$ be the set of isomorphism classes of $n$-dimensional $\ell$-adic representations $V$ of the absolute Galois group $G_K = \text{Gal}(\bar{K}/K)$ of $K$ which satisfy the following four conditions:

(A) For any place $\lambda$ of $K$ above $\ell$, the restriction of $V$ to the decomposition group of (an extension to $\bar{K}$ of) $\lambda$ is semistable and has Hodge-Tate weights in $[0, r]$.

(B) For some place $\lambda_0$ of $K$ above $\ell_0$, the representation $V$ is unramified at $\lambda_0$ and the characteristic polynomial $\det(T - \text{Fr}_{\lambda_0}|V)$ has rational integer coefficients. Furthermore, the roots of the above characteristic polynomial have complex absolute value $q_{\lambda_0}^{w/2}$ for every embedding $\mathbb{Q}_\ell$ into $\mathbb{C}$. Here $\text{Fr}_{\lambda_0}$ and $q_{\lambda_0}$ are the arithmetic Frobenius and the order of the residue field of $\lambda_0$, respectively.

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(C) For any finite place $v$ of $K$ not above $\ell$, the action of the inertia group at $v$ on $\bar{V}$ is unipotent. Here $\bar{V}$ is a residual representation of $V$.

(D) The representation $\bar{V}$ has a filtration of $G_K$-modules

$$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{n-1} \subset \bar{V}_n = \bar{V}$$

such that $\bar{V}_k$ has dimension $k$ for each $0 \leq k \leq n$.

Furthermore, we denote by $\text{Rep}_{\mathbb{Q}_{\ell}}(G_K)^{\text{cycl}}$ the subset of $\text{Rep}_{\mathbb{Q}_{\ell}}(G_K)$ whose elements $V$ satisfy the additional property (E) below:

(E) For each $1 \leq k \leq n$, the $G_K$-action on the quotient $\bar{V}_k/\bar{V}_{k-1}$ is given by a power of the mod $\ell$ cyclotomic character $\chi_{\ell}$.

**Example 1.1.** Let $X$ be a proper smooth scheme over $K$ which has semistable reduction everywhere and has good reduction at some place of $K$ above $\ell_0$. Let $n$ be the $w$-th Betti number of $X(\mathbb{C})$ and $w \leq r$. Then the dual of $H_{\text{et}}^w(X_K, \mathbb{Q}_{\ell})$ satisfies the conditions (A), (B) and (C).

Our main results are the following:

**Theorem 1.2** ([O], Theorem 3.10). Suppose that $w$ is odd or $w > 2r$. Then there exists an explicit constant $C$ depending only on $d_K, n, \ell_0, r$ and $w$ such that $\text{Rep}_{\mathbb{Q}_{\ell}}(G_K)^{\text{cycl}}$ is empty for any prime number $\ell > C$.

**Theorem 1.3** ([O], Theorem 3.11). Suppose that $w$ is odd or $w > 2r$. Then there exists an explicit constant $C'$ depending only on $K, n, \ell_0, r$ and $w$ such that $\text{Rep}_{\mathbb{Q}_{\ell}}(G_K)^{\text{cycl}}$ is empty for any prime number $\ell > C'$ which does not split in $K$.

The key of the proofs of the above Theorems is a relation between tame inertia weights and Frobenius weights. This relation is obtained by a result of Caruso [Ca] which gives an upper bound of tame inertia weights of semistable Galois representations.

**§ 2. Rasmussen-Tamagawa Conjecture**

We describe an application, which is a special case of the Rasmussen-Tamagawa conjecture ([RT]) related with the finiteness of the set of isomorphism classes of abelian varieties with constrained prime power torsion. Our work is motivated by this conjecture. We denote by $K_{\ell}$ the maximal pro-$\ell$ extension of $K(\mu_{\ell})$ which is unramified away from $\ell$.

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1A residual representation $\bar{V}$ is not uniquely defined (it depends on the choice of a $G_K$-stable lattice), but the validity of conditions (C), (D) or (E) does not depend on the choice of $\bar{V}$. 
Definition 2.1. Let $g \geq 0$ be an integer. We denote by $\mathcal{A}(K, g, \ell)$ the set of $K$-isomorphism classes of abelian varieties $A$ over $K$, of dimension $g$, which satisfy the following equivalent conditions:

1. $K(A[\ell^\infty]) \subset \bar{K}_\ell$;
2. The abelian variety $A$ has good reduction outside $\ell$ and $A[\ell]$ admits a filtration
   $$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{2g-1} \subset \bar{V}_{2g} = A[\ell]$$
   such that $\bar{V}_k$ has dimension $k$ for each $0 \leq k \leq 2g$. Furthermore, for each $1 \leq k \leq 2g$, the $G_K$-action on the space $\bar{V}_k/\overline{V}_{k-1}$ is given by a power of the mod $\ell$ cyclotomic character $\chi_\ell$.

The equivalence of (1) and (2) follows from the criterion of Néron-Ogg-Shafarevich and Lemma 3 of [RT]². The set $\mathcal{A}(K, g, \ell)$ is a finite set because of the Shafarevich conjecture proved by Falting. Rasmussen and Tamagawa conjectured that this set is in fact empty for any $\ell$ large enough:

Conjecture 2.2 ([RT], Conjecture 1). The set $\mathcal{A}(K, g, \ell)$ is empty for any prime $\ell$ large enough.

It is known that this conjecture holds under the following conditions:

(i) $K = \mathbb{Q}$ and $g = 1$ ([RT], Theorem 2);
(ii) $K$ is a quadratic number field other than the imaginary quadratic fields of class number one and $g = 1$ ([RT], Theorem 4).

We consider the semistable reduction case of Conjecture 2.2.

Definition 2.3. (1) We denote by $\mathcal{A}(K, g, \ell)_{\text{st}}$ the set of $K$-isomorphism classes of abelian varieties in $\mathcal{A}(K, g, \ell)$ with everywhere semistable reduction.

(2) We denote by $\mathcal{A}(K, g, \ell_0, \ell)_{\text{st}}$ the set of $K$-isomorphism classes of abelian varieties $A$ over $K$ with everywhere semistable reduction, of dimension $g$, which satisfy the following condition: The abelian variety $A$ has good reduction at some place of $K$ above $\ell_0$ and $A[\ell]$ admits a filtration of $G_K$-modules
   $$\{0\} = \bar{V}_0 \subset \bar{V}_1 \subset \cdots \subset \bar{V}_{2g-1} \subset \bar{V}_{2g} = A[\ell]$$
   such that $\bar{V}_k$ has dimension $k$ for each $0 \leq k \leq 2g$.

We have $\mathcal{A}(K, g, \ell)_{\text{st}} \subset \mathcal{A}(K, g, \ell_0, \ell)_{\text{st}}$. We can show the following easily as corollaries of Theorems 1 and 2³:

²Lemma 3 of [RT] is stated in the setting $K = \mathbb{Q}$. However, an easy argument allows us to extend this setting to any number field $K$.
³Rasmussen and Tamagawa have shown the emptiness of the set $\mathcal{A}(K, g, \ell)_{\text{st}}$ for $\ell$ large enough by using the result of [Ra] instead of [Ca] (unpublished).
Corollary 2.4 ([O], Corollary 4.5). There exists an explicit constant $D$ depending only on $d_K$ and $g$ such that the set $\mathcal{A}(K, g, \ell)_{st}$ is empty for any prime number $\ell > D$.

Corollary 2.5 ([O], Corollary 4.6). There exists an explicit constant $D'$ depending only on $K, g$ and $\ell_0$ such that the set $\mathcal{A}(K, g, \ell_0, \ell)_{st}$ is empty for any prime number $\ell > D'$ which does not split in $K$.

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