Hardy spaces associated with Dunkl Transform and Homogeneous type(with a Kernel) *†

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Abstract

This paper mainly contains two pars. In the first part, I will give another characterization of Hardy spaces of the Homogeneous type with a kernel on the real line for $0 < p \leq 1$.

In the second part of this paper, a study for the Hardy spaces associated with the One-Dimensional Dunkl Transform is presented by the tools which is developed in the first part of this paper. The usual analytic function is replaced by the $\lambda$-analytic (pseudo-analytic) function which is based on the $\lambda$-Cauchy-Riemann equations: $D_x u - \partial_y v = 0, \partial_y u + D_x v = 0$ in this paper. $D_x$ is the Dunkl operator: $D_x f(x) = f'(x) + \frac{\lambda}{2} [f(x) - f(-x)]$. Many properties will be developed in the second part of the paper, including maximal functions, atomic decomposition, area integer, interpolation spaces, interpolation of operators, dual spaces and so on.

The theory of the associated Hardy spaces $H^p_{\lambda}(\mathbb{R}^2_+)$ on the half plane $\mathbb{R}^2_+$ for $(p_0 < p \leq 1)$ $p_0 = 2\lambda/2\lambda + 1$ with $\lambda > 0$ extends the results of Muckenhoupt and Stein about Hankel transform to a general case and contains a number of further results.

It is the first time to characterize the Homogeneous type Hardy spaces by a kernel for $0 < p \leq 1$. Other similar papers with a kernel are only the case when $p = 1$, or other papers are the Homogeneous type Hardy spaces without kernel.

It is the first time to study the $\lambda$-analytic functions in $H^p_{\lambda}(\mathbb{R}^2_+)$ and associated it to the Real-Variable $H^p_{\lambda}(\mathbb{R})$ with a pseudo-Poisson kernel for the range of $0 < p \leq 1$.

It is the first time to study the Interpolation of Real-Variable Hardy spaces $H^p_{\lambda}(\mathbb{R})$ with a pseudo-Poisson kernel for the range of $\frac{2\lambda+1}{2\lambda+2} < p \leq 1$.

(It is not difficult to extend the theory of $H^p_{\lambda}(\mathbb{R}^2_+)$ and $H^p_{\lambda}(\mathbb{R})$ with a pseudo-Poisson kernel to other types of Hardy spaces with a kernel on Lie groups (Heisenberg Groups with a kernel). Notice that many similar Hardy spaces have a kernel.)

2000 MS Classification:

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1 Introduction

a. Background. The purpose of this paper is to study the property for Hardy space for the range of $0 < p \leq 1$ associated with the Homogeneous type(with a Homogeneous Kernel) and Dunkl setting in analogy to the ordinary Hardy spaces in the Euclid space. The study of Hardy space originated during 1910’s and 1920’s has over time been transformed into a rich and multifaceted theory. There are two main aspects of this theory: maximal function and atomic decomposition which marks an important step of further development of the real variable theory. Using the grand maximal function, R.R. Coifman first shows that an element in $H^p(\mathbb{R})$ can be decomposed into a sum of a series of basic elements. Then the study on some analytic problem on $H^p(\mathbb{R})$ is summed up to investigate some properties of these basic elements. Thus one of our main objectives is the development of properties of an analogous Hardy space in question.

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Dunkl theory is a generalization of Euclidean Fourier analysis which includes special functions related to root systems. It was developed by C.F. Dunkl in (3)(4)(5)(6)(7)(8). And a systematic study for Hardy spaces associated with the Dunkl transform and Dunkl operator including \( \lambda \)-Poisson integral, \( \lambda \)-Hilbert transform, the \( \lambda \)-translation, the \( \lambda \)-convolution in Zh-K Li and J-Q Liao(12)(13). For more details, please refer to (12), (13). For \( 0 < p < \infty \), we denote by \( L^p_\lambda (\mathbb{R}) \) the set of measurable functions satisfying \( \| f \|_{L^p_\lambda} = \left( \int_{\mathbb{R}} |f(x)|^p |x|^{2\lambda} dx \right)^{1/p} < \infty \), \( c^{-1}_\lambda = 2^{\lambda+1/2} \Gamma(\lambda+1/2) \).

And \( p = \infty \) is the usual \( L^\infty(\mathbb{R}) \) space. For \( \lambda \geq 0 \), The Dunkl operator on the line is:

\[
D_x f(x) = f'(x) + \frac{\lambda}{x} [f(x) - f(-x)]
\]

involving a reflection part. The associated Fourier transform for the Dunkl setting for \( f \in L^1_\lambda(\mathbb{R}) \) is given by:

\[
(\mathcal{F}_\lambda f)(\xi) = c_\lambda \int_{\mathbb{R}} f(x) E_\lambda(-ix\xi) |x|^{2\lambda} dx, \quad \xi \in \mathbb{R}, \quad f \in L^1_\lambda(\mathbb{R})
\]

where \( E_\lambda(-ix\xi) \) is the Dunkl kernel

\[
E_\lambda(iz) = j_{\lambda-1/2}(z) + \frac{i z}{2\lambda+1} j_{\lambda+1/2}(z), \quad z \in \mathbb{C}
\]

where \( j_\alpha(z) \) is the normalized Bessel function

\[
j_\alpha(z) = 2^\alpha \Gamma(\alpha+1) \frac{j_\alpha(z)}{2^\alpha} = \Gamma(\alpha+1) \sum_{n=0}^\infty \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n+\alpha+1)}
\]

Since \( j_{\lambda-1/2}(z) = \cos z \), \( j_{\lambda+1/2}(z) = z^{-1} \sin z \). It follows that \( E_0(iz) = e^{iz} \), and \( \mathcal{F}_0 \) agrees with the usual Fourier transform. We assume \( \lambda > 0 \) in what follows. And the associated \( \lambda \)-translation in Dunkl setting is

\[
\tau_\gamma f(x) = c_\lambda \int_{\mathbb{R}} \mathcal{F}_\lambda f(\xi) E(iz\xi) E(iy\xi) |\xi|^{2\lambda} d\xi, \quad x, y \in \mathbb{R}
\]

The \( \lambda \)-convolution \((f \ast_\lambda g)(x)\) of two appropriate functions \( f \) and \( g \) on \( \mathbb{R} \) associated to the \( \lambda \)-translation \( \tau \) is defined by

\[
(f \ast_\lambda g)(x) = c_\lambda \int_{\mathbb{R}} f(t) \tau_\gamma g(-t) |t|^{2\lambda} dt.
\]

The "Laplace equation" associated with the Dunkl setting is given by:

\[
(\Delta_\lambda u)(x, y) = 0, \quad \text{with} \quad \Delta_\lambda = D^2_x + \frac{\lambda}{y}.
\]

A \( C^2 \) function \( u(x, y) \) satisfying Formula (3) is \( \lambda \)-harmonic. If \( u \) and \( v \) are \( \lambda \)-harmonic functions satisfying \( \lambda \)-Cauchy-Riemann equations:

\[
\begin{cases}
D_x u - \partial_y v = 0, \\
\partial_y u + D_x v = 0
\end{cases}
\]

we say \( F(z) = F(x, y) = u(x, y) + iv(x, y) \) is a \( \lambda \)-analytic function. We define the Hardy space \( H^p_\lambda(\mathbb{R}^2) \) to be the set of \( \lambda \)-analytic functions \( F = u + iv \) on \( \mathbb{R}^2 \) satisfying

\[
\| F \|_{H^p_\lambda(\mathbb{R}^2)} = \sup_{y > 0} \left\{ c_\lambda \int_{\mathbb{R}} |F(x + iy)|^p |x|^{2\lambda} dx \right\}^{1/p}
\]

If \( F(x + iy) \in H^p_\lambda(\mathbb{R}^2) \), we define the boundary value of the real part of \( F(x + iy) \), \( \lim_{y \to 0} Re F(x + iy) \) exists a.e. Thus we can define a space gathering all boundary values of functions in \( H^p_\lambda(\mathbb{R}) \), which is called the real \( H^p_\lambda(\mathbb{R}) \) space. For \( F(x, y) \in H^p_\lambda(\mathbb{R}^2) \), \( H^p_\lambda(\mathbb{R}^2) \), \( H^p_\lambda(\mathbb{R}^2) \)

\[
\left( \frac{\partial}{\partial x} \right)^{\lambda} < p \leq 1 \), we define \( \tilde{H}^p_\lambda(\mathbb{R}) \) as following first:

\[
\tilde{H}^p_\lambda(\mathbb{R}) \triangleq \left\{ \begin{array}{ll}
g(x) : g(x) = \lim_{y \to 0} Re F(t, y) \\
(t, y) \text{ approaches the point } (x, 0) \text{ nontangentially.}
\end{array} \right\}
\]
with the norm:
\[ \|g\|_{H^p_X(\mathbb{R})}^p = \|P^\infty_R g\|_{L^p(\mathbb{R})}^p. \]
The completion of \( \tilde{H}^p_X(\mathbb{R}) \) with the norm \( \| \cdot \|_{H^p_X(\mathbb{R})} \) is denoted as \( H^p_X(\mathbb{R}) \).

b. Summary of Chapter I.

In [18], a type of Homogeneous Hardy Spaces with a Homogeneous Kernel are introduced. Let \( X \) be a topological space, \( \rho \) is a quasi-distance and \( \mu \) is a Borel doubling measure on \( X \). Then Hardy Spaces \( H^p(\mathbb{R}) \) associated to this type \((X, \rho, \mu)\) is investigated in a series of papers. \( H^p(X) \) becomes trivial when \( p \) is near to 1. Let
\[ F(r, x, f) = \int_X K(r, x, y)f(y)d\mu(y)/r, \quad f^\infty(x) = \sup_{r>0}|F(r, x, f)| \]
where \( K(r, x, y) \) is a kind of nonnegative function on \( X \times X \) enjoying several properties. Uchihama showed that for \( 1 - p > 0 \) small enough, the maximal function \( f^\infty(x) \) can be used to characterize the atomic Hardy spaces \( H^p(\mathbb{R}) \).

**Theorem 1.1.** [18] \( \exists p_1, 1 \geq p_1, f \in L^1(\mathbb{R}, \mu) \), such that the following inequality holds:
\[ \|f^\infty\|_{L^p(\mathbb{R}, \mu)} \leq c_1 \|f^\infty\|_{L^p(\mathbb{R}, \mu)} \text{ for } p > p_1 \]
certain \( c_1 \) is a constant depending only on \( \mathbb{R}, 1 \geq \gamma > 0 \).

We will extend the Uchiyama’s result in [18] from \( p_1 \) to \( p \leq p_1 \) (for some \( p_1 \) close to 1) to the range \( \frac{1}{1+p} < p \leq 1 \) under some additional condition \( X = \mathbb{R} \).

In Chapter I, we will introduce first the quasi-distance \( d_\rho(x, y) \) of the topological space \( \mathbb{R} \) endowed with a positive Radon measure \( \mu \) with \( \mu(x, y) = \int_y^x d\mu(t) \) and \( d_\rho(x, y) = |\mu(x, y)| \). The differential and several kinds of spaces of functions associated with the quasi-distance \( d_\rho(x, y) \) will be introduced including Schwartz Class \( S(\mathbb{R}, d_\rho, x) \) in analogy to the Classical Schwartz Class \( S(\mathbb{R}, dx) \). In order to characterize the Homogeneous Hardy space associated with the quasi-distance \( d_\rho(x, y) \), several maximal functions in analogy to the maximal functions in the Classical Hardy space in the Euclid space will be introduced. Let \( f^\infty(x), f^\infty_{S\beta}(x), M^\infty_S f(x), M^\infty_{S\beta} f(x) \) be defined by (2.20), (2.21), (6), (21). We will prove that for \( \beta \geq \gamma > p^{-1} - 1, \phi \in SS\beta \), with \( f \phi(x)dx = 1 \),
\[ \|f^\infty(x)\|_{L^p(\mathbb{R}, \mu)} \approx \|M_{\phi\beta} f(x)\|_{L^p(\mathbb{R}, \mu)} \]
holds when \( f \in L^1(\mathbb{R}, \mu) \).

At last, we will use the \( S(\mathbb{R}, d_\rho, x) \) functions to approximate the kernel function \( K_i(r, x, y) \) to obtain the main results of Chapter I: Theorem(2.39). We could prove that for \( \frac{1}{1+p} < p \leq 1 \), the maximal function \( f^\infty(x) \) can be used to characterize the atomic Hardy spaces \( H^p_\rho(\mathbb{R}) \): for \( i = 1, 2 \)
\[ \|f^\infty(x)\|_{L^p(\mathbb{R}, \mu)} \approx \|f^\infty_i(x)\|_{L^p(\mathbb{R}, \mu)} \]
when \( f \in L^1(\mathbb{R}, \mu) \).

c. Summary of Chapter II.

Chapter II deal mainly with the real methods of Hardy Spaces associated with the Dunkl setting on the real line. By the Theorem(1.1)[18], we could only obtain the results for \( H^p_X(\mathbb{R}) \). But by the results we achieved in Chapter I, we could deal with the real-variable methods for \( H^p_X(\mathbb{R}) \) of the range \( \frac{2+1}{2+1} < p \leq 1 \). Interpolation of spaces, interpolation of operators and dual spaces could also be obtained.

§3.1 comprises the area integral associated with the Dunkl transform. The area integral for the \( \lambda \)-harmonic function in the Dunkl setting was given in Liao’s Doctoral thesis [11].
\[ (S(u)(x))^2 = c_\lambda \int_{\Gamma(0)} (\tau_x(\Delta \lambda u^2))(-t, y)y^{-2\lambda} |t|^{2\lambda} dt dy. \]
The norm of the Hardy spaces is equivalent with the area integral. It serves to characterize \( H^p_X(\mathbb{R}) \) norms and non-tangential convergence.

**Proposition 1.2.** Liao[11] \( u \) is an \( \lambda \)-analytic function on \( \mathbb{R}^2, \) for \( 0 < p < \infty, u_\lambda^* \in L^p_X \) if and only if \( S(u) \in L^p_X, \|S(u)\|_{L^p_X} \approx \|u_\lambda^*\|_{L^p_X} \).

Then some even harder question arise.
Question 1.3. For \( p_0 < p < \infty \), \( F(x,y) = u(x,y) + iv(x,y) \in H^p_\lambda(\mathbb{R}^2_+), \) where \( p_0 = \frac{2\lambda}{2\lambda + 1} \)

\[ \|F\|_{H^p_\lambda(\mathbb{R}^2_+)} \asymp \|\mathcal{S}(u)\|_{L^\infty} . \]

If the question settled, a characterization of the maximal functions in \( H^p_\lambda(\mathbb{R}^2_+) \) could be achieved. Our second result is that we proved the above Question(1.3) in Theorem(3.12) and Proposition(3.13).

In §3.2, the \( \lambda \)-Poisson kernel is introduced. We prove that the \( \lambda \)-Poisson kernel \((\tau_x P_y)(-t)\) is just the kind of kernel in homogeneous space developed in Chapter I. Then together with the tools we developed in the first part of the paper: Theorem(2.39), we could show that \( H^p_\lambda(\mathbb{R}) \) is a kind of homogeneous type Hardy Space for \( \frac{2\lambda}{2\lambda + 1} < p \leq 1 \) in Theorem(3.21). The conclusion is that Hardy space in the Dunkl setting is a kind of homogeneous type space. Then the \( H^p_\lambda(\mathbb{R}) \) norm could be characterized by the maximal functions in the homogeneous type space. Thus the definition of \( H^p_\lambda(\mathbb{R}) \) could be evolved from the properties of \( \lambda \)-analytic functions. \( f \in H^p_\lambda(\mathbb{R}) \cap L^2_\lambda(\mathbb{R}) \cap L^1_\lambda(\mathbb{R}) \) and only if its \( \lambda \)-Poisson nontangential maximal function \( P_\lambda^* f(x) = \sup_{|s-x|<u}(P f)(s,y) \in L^s_\lambda(\mathbb{R}) \cap L^t_\lambda(\mathbb{R}) \cap L^1_\lambda(\mathbb{R}) \). Thus when \( \frac{2\lambda+1}{2\lambda+2} < 1, \) \( \widetilde{H}^p_\lambda(\mathbb{R}) \) could be defined as:

\[ \widetilde{H}^p_\lambda(\mathbb{R}) \triangleq \left\{ f \in L^2_\lambda(\mathbb{R}) \cap L^1_\lambda(\mathbb{R}) : f^*(x) \in L^s_\lambda(\mathbb{R}) \right\} , \]

where \( f^*(x) \) is a kind of maximal functions in homogeneous type space in [18]:

\[ f^*(x) = \sup_{\phi, r} \left\{ \left( \int f(y) \phi(y) d\mu(y) \right) / r : r > 0, \sup \phi \subset B(x, r), \right. \]
\[ L(\phi, s_{\phi}) \leq \frac{1}{2\lambda + 1} \leq r \frac{2\lambda+1}{2\lambda+2}, \|\phi\|_{L^\infty} \leq 1 \} . \]

The completion of \( \widetilde{H}^p_\lambda(\mathbb{R}) \) with the norm \( \| \cdot \|_{\widetilde{H}^p_\lambda(\mathbb{R})} \) is \( H^p_\lambda(\mathbb{R}) \).

§3.3 mainly deals with the atomic decomposition of the \( H^p_\lambda(\mathbb{R}) \) when \( \frac{2\lambda+1}{2\lambda+2} < p \leq 1 \). From Theorem(3.34), we could see that \( H^p_\lambda(\mathbb{R}) \) could be defined as:

\[ H^p_\lambda(\mathbb{R}) = \{ g \text{ is a } \lambda \text{-distribution : } g^*(x) \in L^s_\lambda(\mathbb{R}) \} . \]

By Proposition (3.20) and the maximal function in the homogeneous type space, a function in \( H^p_\lambda(\mathbb{R}) \) could also be decomposed into a sum of serious basic elements: for \( \forall f \in H^p_\lambda(\mathbb{R}) \), we could write \( f(x) = \sum k \lambda_k a_k(x) \) a.e. and in \( H^p_\lambda(\mathbb{R}) \). Then we could obtain

\[ \sum_k |\lambda_k|^p \asymp \|f^*(x)\|_{L^s_\lambda} \asymp \|f\|_{\widetilde{H}^p_\lambda(\mathbb{R})} . \]

Then the theory of atomic decomposition of the \( H^p_\lambda(\mathbb{R}) \): Theorem(3.33)(3.34), can be achieved.

§3.4 and §3.5 introduce two different kinds of atoms: \( p_\lambda \)-atom and \( p_\lambda^* \)-atom for the range \( \frac{2\lambda}{2\lambda+1} < p \leq 1 \). We study the properties of \( p_\lambda \)-atom, and its associated atomic space: \( H^{p_\lambda,\lambda}_\lambda(\mathbb{R}) \) in Proposition(3.46). We could conclude the following conclusions:

\[ H^{p_\lambda,\lambda}_\lambda(\mathbb{R}) = H^p_\lambda(\mathbb{R}) , \text{ for } \frac{2\lambda+1}{2\lambda+2} < p \leq 1 . \]

\[ H^{p_\lambda,\lambda}_\lambda(\mathbb{R}) \subseteq H^p_\lambda(\mathbb{R}) , \text{ for } \frac{2\lambda}{2\lambda+1} < p \leq \frac{2\lambda+1}{2\lambda+2}, \kappa \geq 2 \left(\frac{2\lambda+1}{2\lambda+2} - \frac{1-p}{p} \right) . \]

Also the relation of \( H^p_\lambda(\mathbb{R}^2) \) and \( H^p_\lambda(\mathbb{R}) \) for the range \( \frac{2\lambda}{2\lambda+1} < p \leq 1 \) is characterized in Theorem(3.58). With the theory of atomic decomposition developed, we study some further properties of \( H^p_\lambda(\mathbb{R}) \) for the range \( \frac{2\lambda}{2\lambda+1} < p \leq 1 \). §3.6 mainly deal with the dual spaces of \( H^p_\lambda(\mathbb{R}) \). The conclusion is Theorem(3.58). In §3.7, the interpolation of operators and interpolation of spaces of \( H^p_\lambda(\mathbb{R}) \) will be studied, which is also regarded as a generalization of the Marcinkiewicz interpolation theorem. The results are Theorem(3.62)(3.63) and (3.71). In §3.8, we will study Cesàro operator in \( H^p_\lambda(\mathbb{R}) \). We will show that Cesàro operator can be extended as a bounded operator in \( H^p_\lambda(\mathbb{R}) \) spaces in Theorem(3.79).

c. Summary of the main results. In all, our results are the followings:
1. Theorem (2.39), we extended the Uchiyama’s result in [18] from \( p_1 < p \leq 1 \) (for some \( p_1 \) close to 1) to the range \( \frac{1}{p+1} < p \leq 1 \) under some additional assumptions.

2. We prove the Question (1.3) in Theorem (3.12) and Proposition (3.13).

3. We could show that \( H^p_\lambda(\mathbb{R}) \) is a kind of Homogeneous type Hardy Space for \( \frac{2\lambda+1}{2\lambda+2} < p \leq 1 \) in Theorem (3.21).

4. In Theorem (3.34), \( H^p_\lambda(\mathbb{R})(\frac{2\lambda+1}{2\lambda+2} < p \leq 1) \) can be characterized as:
   \[ H^p_\lambda(\mathbb{R}) = \{ g \text{ is a } \lambda\text{-distribution : } g^\ast(x) \in L^p_\lambda(\mathbb{R}) \} . \]

5. The relation of \( H^p_\lambda(\mathbb{R}^2) \) and \( H^p_\lambda(\mathbb{R}) \) for the range of \( \frac{2\lambda}{2\lambda+1} < p \leq 1 \) is characterized in Theorem (3.59).

6. We study the atomic decomposition of the \( H^p_\lambda(\mathbb{R}) \): Theorem (3.33). And we study the properties of \( p_\lambda \)-atom, and its associated atomic space: \( H^p_{\lambda,\nu}(\mathbb{R}) \) in Proposition (3.46).

7. We study the dual spaces of \( H^p_\lambda(\mathbb{R}) \) in Theorem (3.58), especially
   \[ H^p_\lambda(\mathbb{R})^* = BMO_\lambda, \text{ for } p = 1. \]
   \[ H^p_\lambda(\mathbb{R})^* = L^{p,-1-1}_\lambda(\mathbb{R}), \text{ for } \frac{2\lambda+1}{2\lambda+2} < p \leq 1. \]

8. We study interpolation of operators and interpolation of spaces of \( H^p_\lambda(\mathbb{R}) \) in Theorem (3.62), (3.63) and (3.71).

9. We study the Cesàro operator in \( H^p_\lambda(\mathbb{R}) \) in Theorem (3.79). For \( \frac{2\lambda+1}{2\lambda+2} < p \leq 1 \), \( f \in H^p_\lambda(\mathbb{R}) \)
   \[ \| C_\alpha f \|_{H^p_\lambda(\mathbb{R})} \leq C \| f \|_{H^p_\lambda}. \]

It turns out that many properties developed in the Classical Hardy spaces could be extended to the \( H^p_\lambda(\mathbb{R}) \) spaces when \( \frac{2\lambda+1}{2\lambda+2} < p \leq 1 \).

**d. Notation.** Let \( X \) to be a locally compact Hausdorff space. Let \( B(X) \) be a Borel \( \sigma \)-algebra on \( X \). \( B(X) \subset \mathcal{G} \) is a \( \sigma \)-algebra on \( X \). \( \mu \) is a regular measure on \( X \).

\( \mathcal{G}(\mathbb{R}) \) : the space of \( C^\infty \) functions on \( \mathbb{R} \) with compact support.

\( \mathcal{S}(\mathbb{R}) \) : the space of \( C^\infty \) functions on \( \mathbb{R} \) rapidly decreasing together with their derivatives.

\( L_{\lambda, loc}(\mathbb{R}) \) : the set of locally integrable functions on \( \mathbb{R} \) associated with the measure \( |x|^{2\lambda} dx \).

\( C_\alpha(X) \) : the space of continuous functions on \( X \) with compact support.

\( L^p(X, \varrho, \mu) \) : the set of all \( \varrho \)-measurable functions \( f : X \to \mathbb{R} \) such that \( |f|^p \) is integrable.

Const, c: constant. Different constant may be different in different occurrences.

We use \( A \lesssim B \) to denote the estimate \( |A| \leq CB \) for some absolute universal constant \( C > 0 \), which may vary from line to line. \( A \gtrsim B \) to denote the estimate \( |A| \geq CB \) for some absolute universal constant \( C > 0 \). \( A \approx B \) or \( A \asymp B \) to denote the estimate \( |A| \leq C_1B, |A| \geq C_2B \) for some absolute universal constant \( C_1, C_2 \).

In this paper, let \( I \) to be the Euclid interval : \( I(x_0, \delta_0) = (x_0 - \delta_0, x_0 + \delta_0) = \{ y : |y - x_0| < \delta_0 \} \).

Let \( B \) to be the ball in the homogeneous space in the Dunkl setting: \( B(x_0, r_0) = B_\lambda(x_0, r_0) = \{ y : d_\lambda(y, x_0) < r_0 \} \).

Let \( d_\lambda(x, y) \) denote the distance in the homogeneous space associated with Dunkl setting:

\[
|2\lambda + 1| \int_0^{r_0} t^{p_0 - 1} dt \leq c^{-1}_\lambda 2^{\lambda+1/2} \Gamma(\lambda + 1/2), \Omega \text{ is a domain and } \partial \Omega \text{ is the boundary of the domain}.

\[
d_\mu(x, y) = \text{the distance in the homogeneous space associated with a positive Radon measure } \mu \text{ on the real line}.

Let \( B_\mu \) to be the ball in the homogeneous space: \( B_\mu(x_0, r_0) = \{ y : d_\mu(y, x_0) < r_0 \} \).

If \( E \subseteq \mathbb{R} \), we use \( E^\mu \) to denote the set: \( E^\mu = \{ x \in \mathbb{R} : x \notin E \} \). For two sets \( A \) and \( B \), \( A \setminus B \) means that \( A \setminus B \). Throughout this paper, we assume \( \lambda > 0 \) and \( 0 < \gamma \leq 1 \).

\[
(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} = a(a + 1)(a + 2) \cdots (a + n - 1).
\]
2 Chapter I Homogeneous type Hardy Space on \( \mathbb{R} \) with a Homogeneous Kernel

2.1 Kernel and Maximal function

Let \( X \) be a locally compact Hausdorff space. Let \( B(X) \) be a Borel \( \sigma \)-algebra on \( X \). \( \mathcal{B}(X) \subset \mathcal{G} \) is a \( \sigma \)-algebra on \( X \). \( \mu \) is a regular measure on \( X \). Thus the following conclusions hold in Measure Theory:

**Proposition 2.1.** (i) \( \mu \) is a Radon measure \( \iff \) For any compact set \( K \), \( \mu(K) < \infty \).

(ii) If \( \mu \) is a Radon measure on \( \mathcal{G} \), then \( C_c(X) \) is dense in \( L^p(X, \mu) \).

**Proposition 2.2.** In this paper, \( d_\mu(x, y) \) is a quasi-distance of the topological space \( \mathbb{R} \) endowed with a positive Radon measure \( \mu(x, y) = \int_y^x d\mu(t) \), \( d_\mu(x, y) = |\mu(x, y)| \), satisfying the following condition:

(i) \( d_\mu(x, y) = d_\mu(y, x) \), for any \( x, y \in \mathbb{R} \);

(ii) \( d_\mu(x, y) > 0 \), if \( x \neq y \);

(iii) \( d_\mu(x, z) \leq A(d_\mu(x, y) + d_\mu(y, z)) \), for any \( x, y, z \in \mathbb{R} \);

(iv) \( A^{-1}r \leq \mu(B_\mu(x, r)) \leq r \), for any \( r \in (0, \mu(\mathbb{R})) \);

(v) \( B_\mu(x, r) = \{ y \in X : d_\mu(x, y) < r \} \) form a basis of open neighbourhoods of the point \( x \).

(vi) \( f(u) = \mu(x, u) \) is a continuous bijection on \( \mathbb{R} \) for any fixed \( x \in \mathbb{R} \). (This condition can be replaced by: \( f(u) \) has an inverse function.)

**Definition 2.3** \( (S(\mathbb{R}, dx)) \). Then we begin with the Classic Schwartz Class \( S \): the set of all \( \phi \) on \( \mathbb{R} \) endowed with the Euclidean distance, that are infinitely differentiable and together with all their derivatives, are rapidly decreasing (i.e. remain bounded when multiplied by arbitrary polynomials). On \( S \) one has a denumerable collection of seminorms \( \| \cdot \|_{\alpha, \beta} \) given by

\[
\| \phi \|_{\alpha, \beta} = \sup_{x \in \mathbb{R}} \left| x^\alpha \left( \frac{d}{dx} \right)^\beta \phi(x) \right|.
\]

\( \alpha \) and \( \beta \) are natural numbers. We denote this kind of Schwartz Class as: \( S(\mathbb{R}, dx) \).

**Definition 2.4** \( (S(\mathbb{R}, d_\mu x)) \). If \( \mathbb{R} \) is endowed with a quasi-distance \( d_\mu(x, y) \) as Proposition(2.2), we could define the derivative associated with the quasi-distance \( d_\mu(x, y) \) as:

\[
\frac{d}{d_\mu x} \phi(x) = \lim_{\varepsilon \to 0, d_\mu(x, y) < \varepsilon} \frac{\phi(y) - \phi(x)}{\mu(y, x)}.
\]

Thus the Schwartz Class \( S \) on \( \mathbb{R} \) endowed with a quasi-distance \( d_\mu(x, y) \) could be defined as:

\[
\| \phi \|_{d_\mu(\alpha, \beta)} = \sup_{x \in \mathbb{R}} \left| (d_\mu(x, 0))^\alpha \left( \frac{d}{d_\mu x} \right)^\beta \phi(x) \right| < \infty
\]

for natural numbers \( \alpha \) and \( \beta \). We denote this kind of Schwartz Class as: \( S(\mathbb{R}, d_\mu x) \).

\( \phi(u) \in C(\mathbb{R}, dx) \) means \( \phi(u) \to \phi(u_0) \) as \( u \to u_0 \) in Euclid space, \( \phi(u) \in C(\mathbb{R}, d_\mu x) \) means \( \phi(u) \to \phi(u_0) \) as \( d_\mu(u, u_0) \to 0 \).

**Proposition 2.5.** \( \mu \) is the measure given by Proposition(2.2). \( \forall \psi \in S(\mathbb{R}, d_\mu x) \), and \( supp(\phi) \subset B_\mu(x_0, r_0) \). Then \( \exists \psi(t) \in S(\mathbb{R}, dx) \), \( supp(\psi(t)) \subset [-1, 1] \), satisfying \( \psi \left( \frac{\mu(x_0, u)}{r_0} \right) = \phi(u) \) for \( u \in \{ t : t \in B_\mu(x_0, r_0) \} \) in \( S(\mathbb{R}, d_\mu x) \) space.

**Proof.** Let \( f(u) = \frac{\mu(x_0, u)}{r_0} \) for fixed \( x_0 \) and \( r_0 \). Then \( f(u) \) is a bijection. Thus \( f(x) \) has an inverse function. Let \( g(x) \) be the inverse function of \( f(x) \) satisfying:

\[
g \circ f(u) = u.
\]

Thus \( \forall \phi \in S(\mathbb{R}, d_\mu x) \), we could write \( \phi \) as:

\[
\phi(u) = \phi(g \circ f(u)) = \phi \circ g \left( \frac{\mu(x_0, u)}{r_0} \right).
\]
Thus we could let \( \psi = \phi \circ g \). It is also not difficult to prove that \( \psi \in S(\mathbb{R}, dx) \). Let \( \psi^{(n)}(t) = \frac{d^n}{dt^n} \psi(t) \), then we have:

\[
\frac{d}{d_\mu x} \phi(x) = \lim_\varepsilon \to 0 \frac{\phi(y) - \phi(x)}{\mu(y, x)} = \frac{1}{r_0} \lim_\varepsilon \to 0 \frac{1}{|r_0|} \frac{\mu(x_0, y)}{r_0} - \psi \left( \frac{\mu(x_0, x)}{r_0} \right)
\]

It is also not difficult to have

\[
\left( \frac{d}{d_\mu x} \right)^n \phi(x) = \lim_{\varepsilon \to 0} \left( \frac{d}{d_\mu x} \right)^{n-1} \phi(y) - \left( \frac{d}{d_\mu x} \right)^{n-1} \phi(x)
\]

\[
= - \left( \frac{1}{r_0} \right)^n \psi^{(1)} \left( \frac{\mu(x_0, y)}{r_0} \right).
\]

Notice that \( \mu \) is a bijection on \( \mathbb{R} \), together with the fact \( \phi(x) \in S(\mathbb{R}, d_\mu x) \), thus \( \psi \in S(\mathbb{R}, dx) \). This proves the proposition.

In the same way as Proposition(2.5), we could also have

**Proposition 2.6.** \( \mu \) is the measure given by Proposition(2.2). \( \forall \phi \in C(\mathbb{R}, d_\mu x) \), Then \( \exists \psi(t) \in C(\mathbb{R}, dx) \), satisfying \( \psi \left( \frac{\mu(x_0, u)}{r_0} \right) = \phi(u) \) in \( C(\mathbb{R}, d_\mu x) \) space.

By Proposition (2.5) (2.6), together with the fact that \( S(\mathbb{R}, dx) \) is dense in \( C_0(\mathbb{R}, dx) \), we could know that

**Proposition 2.7.** \( S(\mathbb{R}, d_\mu x) \) is dense in \( C_0(\mathbb{R}, d_\mu x) \).

Then we will introduce kernels \( K_1(r, x, y) \) and \( K_2(r, x, y) \):

**Proposition 2.8.** Let \( K_1(r, x, y) \) be a nonnegative continuous function defined on \( \mathbb{R}^+ \times \mathbb{R} \). Let constant \( A > 0 \) and constant \( 1 \geq \gamma > 0 \) such that

(i) \( K_1(r, x, y) > 1/A \), for \( r > 0, x \in \mathbb{R} \);

(ii) \( 0 \leq K_1(r, x, t) \leq 1 \), for \( r > 0, x, t \in \mathbb{R} \);

(iii) \( \text{For } r > 0, x, t, z \in \mathbb{R} \)

\[
|K_1(r, x, t) - K_1(r, x, z)| \leq \left( \frac{d_\mu(t, z)}{r} \right)^\gamma.
\]

(iv) \( K_1(r, x, y) = 0 \), if \( d_\mu(x, y) > r \).

(v) \( K_1(r, x, y) = K_1(r, y, x) \).

**Proposition 2.9.** Let \( K_2(r, x, y) \) be a nonnegative continuous function defined on \( \mathbb{R}^+ \times \mathbb{R} \). Let constant \( C_i > 0 \), \( i = 1, 2, 3, 4 \) and constant \( 1 \geq \gamma > 0 \) such that

(i) \( K_2(r, x, x) > C_1 \), for \( r > 0, x \in \mathbb{R} \);

(ii) \( 0 \leq K_2(r, x, t) \leq C_2 \left( 1 + \frac{d_\mu(x, t)}{r} \right)^{-\gamma} \), for \( r > 0, x, t \in \mathbb{R} \);

(iii) \( \text{For } r > 0, x, t, z \in \mathbb{R} \), if \( \frac{d_\mu(t, z)}{r} \leq C_3 \min \left( 1 + \frac{d_\mu(x, t)}{r}, 1 + \frac{d_\mu(x, z)}{r} \right) \), then

\[
|K_2(r, x, t) - K_2(r, x, z)| \leq C_4 \left( \frac{d_\mu(t, z)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(x, t)}{r} \right)^{-2\gamma}.\]

(iv) \( K_2(r, x, y) = K_2(r, y, x) \).

**Proposition 2.10.** \( K_i(r, x, y) \) is the kernel as Proposition(2.8)(2.9). Denote \( a_i(r, x) \) as:

\[
a_i(r, x) = \int_\mathbb{R} K_i(r, x, y) d\mu(y)/r.
\]

For \( i = 1, 2 \), \( \exists \) constants \( M_1, M_2 \) with \( 0 < m \leq M \) such that \( a_i(r, x) \) satisfies the following:

\[
m \leq a_i(r, x) \leq M.
\]
**Proof.** We only need to prove the case when \( i = 2 \). By Proposition 2.9(i) and (iii), we could conclude there exist a constant \( \tilde{C} \), when \( d_\mu(x,t) < \tilde{C}r \), \( K_2(r,x,y) > C_1/2 \). Then we could deduce the Proposition.

**Definition 2.11.** For any \( f \in L^1_{\text{loc}}(\mathbb{R}) \), \( 0 < r \leq 1 \), let

\[
F_i(r,x,f) = \int_{\mathbb{R}} K_i(r,x,y) f(y) d\mu(y) / r, \quad f^*_i(x) = \sup_{r > 0} \left| F_i(r,x,f) \right|, \quad f^{\#}_i(x) = \sup_{r > 0, d_\mu(s,x) < r} \left| F_i(r,s,f) \right|
\]

for \( i = 1, 2 \).

\[
f^*_i(x) = \sup_{\phi, r} \left\{ \left| \int_{\mathbb{R}} f(y) \phi(y) d\mu(y) \right| / r : r > 0, \text{supp} \phi \subset B_\mu(x,r), L(\phi, \gamma) \leq r^{-\gamma}, \|\phi\|_{L^\infty} \leq 1 \right\}
\]

and

\[
f^{\#}_i(x) = \sup_{\phi, r} \left\{ \left| \int_{\mathbb{R}} f(y) \phi(y) d\mu(y) \right| / r : r > 0, \text{supp} \phi \subset B_\mu(x,r), L(\phi, \gamma) \leq r^{-\gamma}, \phi \in S(\mathbb{R}, d_\mu x), \|\phi\|_{L^\infty} \leq 1 \right\}.
\]

The Hardy-littlewood maximal function can be defined as:

\[
M_\alpha f(x) = \sup_{r > 0} \frac{1}{r} \int_{B_\mu(x,r)} |f(y)| d\mu(y).
\]

Then \( M_\alpha \) is weak-(1, 1) bounded and \( (p, p) \) bounded for \( p > 1 \).

**Definition 2.12.** \( \phi^{(n)}(x) \), \( H^\alpha(\phi) \), \( |\phi|_\beta \)

For \( \phi \in C(\mathbb{R}, dx) \), \( n \in \mathbb{N} \), \( 1 \geq \alpha \geq 0 \) and \( \beta > 0 \). Let

\[
\{\beta\} = \beta - [\beta]; \quad \lceil \beta \rceil = \max\{n : n \in \mathbb{Z}; n \leq \beta\}.
\]

Denote:

\[
H^\alpha(\phi) = \sup_{x,y \in \mathbb{R}, x \neq y} |\phi(x) - \phi(y)| / |x - y|^\alpha;
\]

and

\[
\phi^{(n)}(x) = \frac{d^n}{dx^n} \phi(x); \quad |\phi|_\beta = H^{\lceil \beta \rceil}(\phi^{(\lceil \beta \rceil)})
\]

Thus we could see that if \( 0 < \beta \leq 1 \)

\[
|\phi|_\beta = H^{\beta}(\phi).
\]

**Proposition 2.13.** For \( \phi \in C(\mathbb{R}, dx) \) \( 1 \geq \alpha \geq 0 \), \( \beta > 0 \) \( H^\alpha(\phi) \leq 1 \), and \( |\phi| \leq 1 \), there exist \( \phi_r(x) \in S(\mathbb{R}, dx) \), satisfying the following property:

(i) \( \lim_{r \to 0} \|\phi_r(x) - \phi(x)\|_{L^\infty} = 0 \),

(ii) \( \|\phi_r(x)\|_{L^\infty} \leq 1 \), \( H^\alpha \phi_r \leq 1 \),

(iii) \( H^\alpha(\phi_r^{(1)}) \leq C \frac{1}{r} \).
Proof.

\[ \rho(x) = \begin{cases} \kappa \exp \left\{ \frac{1}{|x|^\alpha} \right\}, & \text{for } |x| < 1 \\ 0, & \text{for } |x| \geq 1. \end{cases} \]

\( \kappa \) is a constant such that

\[ \int_{\mathbb{R}} \rho(x) \, dx = 1. \]

Then we could see that \( \rho(x) \in S(\mathbb{R}, dx) \). For \( \phi \in C(\mathbb{R}, dx) \), \( \alpha > 0 \), let

\[ \phi_\tau(x) = \int_{\mathbb{R}} \phi(y) \rho \left( \frac{x - y}{\tau} \right) \frac{dy}{\tau} = \int_{\mathbb{R}} \phi(x - y) \rho \left( \frac{y}{\tau} \right) \frac{dy}{\tau}. \]

Thus it is easy to obtain:

\[ H^\alpha \phi_\tau \leq H^\alpha(\phi) \int_{\mathbb{R}} \rho(x) \, dx \leq 1. \]

\[ H^\alpha(\phi_\tau^{(i)}) \leq C \frac{1}{\tau^{\alpha + 1}} (H^\alpha \rho^{(1)}) \sup_{x \in \mathbb{R}} \left| \int_{\mathbb{R}} \phi(y) \chi([-1,1]) \left( \frac{x - y}{\tau} \right) \frac{dy}{\tau} \right| \leq C \frac{1}{\tau^{\alpha + 1}}. \]

This proves the proposition. \( \square \)

From PDE the following conclusion of Hölder spaces holds:

**Proposition 2.14.** \( \beta \geq \beta_1 \geq 0 \). \( n \in \mathbb{Z} \), \( n \leq \beta \). For \( \phi \in S(\mathbb{R}, dx) \), if \( \| \phi(x) \|_\infty \leq 1 \), \( |\phi|_\beta \leq 1 \), then the following holds:

\[ \| \phi^{(n)}(x) \|_\infty \leq C, \quad |\phi|_{\beta_1} \leq C \]

\( C \) is a constant independent on \( \phi \).

**Proposition 2.15.** For \( f \in L_{\text{loc}}, \mu \),

\[ f_\gamma^{(i)}(x) \lesssim f_\gamma^*(x) \quad i = 1, 2. \]

Then if \( f_\gamma^*(x) \in L^p(\mathbb{R}, \mu) \) for \( p > 0 \), we could have

\[ \| f_\gamma^{(i)} \|_{L^p(\mathbb{R}, \mu)} \lesssim \| f_\gamma^* \|_{L^p(\mathbb{R}, \mu)}. \]

**Proof.** When \( i = 1 \), it is easy to see that for fixed \( r \) and \( s \):

\[ \left| K_1(r, s, y) \right| \lesssim 1 \]

\[ L \left( K_1(r, s, y), \gamma \right) \lesssim (r)^{-\gamma} \]

\[ \text{supp} K_1(r, s, y) \subseteq B_\mu(x, 2Ar) \]

Thus we have

\[ f_\gamma^{(i)}(x) \lesssim f_\gamma^*(x) \]

holds. When \( i = 2 \). Choose positive \( \phi(t) \in S(\mathbb{R}, dt) \) satisfying \( \text{supp} \phi(t) \subseteq (-1, 1) \), and \( \phi(t) = 1 \) when \( t \in (-1/2, 1/2) \).

\[ \psi_0, x(t) = \phi \left( \frac{\mu(x, t)}{r} \right), \psi_k, x(t) = \phi \left( \frac{\mu(x, t)}{2^{k+1}r} \right) - \phi \left( \frac{\mu(x, t)}{2^k r} \right), \]

for \( k \geq 1 \).

Thus \( \text{supp} \psi_0, x(t) \subseteq B_\mu(x, r) \) and \( \text{supp} \psi_k, x(t) \subseteq B_\mu(x, 2^{k+1}r) \setminus B_\mu(x, 2^{k-2}r) \) for \( k \geq 1 \). \( \psi_k, x(t) \in S(\mathbb{R}, d_\mu t) \) for \( k \geq 0 \). We have

\[ \sum_{k=0}^\infty \psi_k, x(t) = 1. \]
Then we could conclude:

\[
\begin{align*}
  f_{2\gamma}^N(x) & = \sup_{r>0, d_\mu(s,x) \leq r} \int_{\mathbb{R}} K_2(r, s, y) \sum_{k=0}^{+\infty} \psi_{k,x}(y) f(y) d\mu(y) / r \\
  & \leq \sum_{k=0}^{+\infty} \sup_{r>0, d_\mu(s,x) \leq r} \int_{\mathbb{R}} K_2(r, s, y) \psi_{k,x}(y) f(y) d\mu(y) / r.
\end{align*}
\]

By Proposition(2.9), we could obtain

\[
\begin{align*}
  \left| (1 + 2^k)^{1+\gamma} K_2(r, s, y) \psi_{k,x}(y) \right| & \lesssim 1 \\
  L \left( (1 + 2^k)^{1+\gamma} K_2(r, s, y) \psi_{k,x}(y) \right) & \lesssim (2^k r)^{-\gamma} \\
  \text{supp}(1 + 2^k)^{1+\gamma} K_2(r, s, y) \psi_{k,x}(y) & \subseteq B_\mu(x, A2^{k+2} r).
\end{align*}
\]

Then

\[
\begin{align*}
  f_{2\gamma}^N(x) & = \sup_{r>0, d_\mu(s,x) \leq r} \int_{\mathbb{R}} K_2(r, s, y) f(y) d\mu(y) / r \\
  & \leq \sum_{k=0}^{+\infty} \sup_{r>0, d_\mu(s,x) \leq r} \int_{\mathbb{R}} K_2(r, s, y) \psi_{k,x}(y) f(y) d\mu(y) / r \\
  & \lesssim \sum_{k=0}^{+\infty} (2^k)^{(1 + 2^k)^{-1-\gamma}} f_\gamma^N(x) \\
  & \lesssim f_\gamma^N(x).
\end{align*}
\]

This proves the proposition. \(\square\)

**Proposition 2.16.** For \(f \in L^2(\mathbb{R}, \mu)\), \(i = 1, 2,\)

\[
\lim_{r \to 0} \left( \int_{\mathbb{R}} \left| f(x) - \int_{\mathbb{R}} \frac{f(y)}{d_\mu(r,x)} K_i(r, x, y) d\mu(y) / r \right|^2 d\mu(x) \right)^{1/2} = 0.
\]

**Proof.** Suppose \(N\) large enough, \(N \gg 0\). We denote \(N_0\) and \(N_k\) as following:

\[
N_0 = \min\{x : d_\mu(x, N) < 1, \ x > 0\}, \quad N_k = \min\{x : d_\mu(x, N) < 2^k, \ x > 0\}.
\]

We suppose \(N\) is large enough such that \(N_0\) and \(N_k\) are positive numbers.

We first assume \(f \in C(\mathbb{R}, d_\mu(x)) \bigcap L^2(\mathbb{R}, \mu)\) first. \(\forall \epsilon > 0, \ \exists r_N > 0\) and \(\exists N > 0\), such that the following holds:

\[
\left( \int_{N_0}^{+\infty} |f(x)|^2 d\mu(x) \right)^{1/2} + \left( \int_{-\infty}^{-N_0} |f(x)|^2 d\mu(x) \right)^{1/2} < \epsilon,
\]

and

\[
|f(x) - f(y)| < \epsilon / N
\]

for any \(y \in B(x, r_N), x \in [-N, N] \subset \mathbb{R}\).

Denote the operator \(M_\mu\) as the Hardy-Littlewood operator:

\[
M_\mu f(x) = \sup_{0 < r < 1} \frac{1}{r} \int_{B_\mu(x,r)} |f(y)| d\mu(y).
\]

Denote the operator \(M_\mu^1\) as following:

\[
M_\mu^1 f(x) = \sup_{0 < r < 1} \frac{1}{r} \int_{B_\mu(x,r)} |f(y)| d\mu(y).
\]

Denote \(E_N\) as

\[
E_N = \{x \in \mathbb{R} : x \geq N_0\}.
\]
Thus by the Vitali Lemma, it is easy to obtain:

\[
|\mu \{ x > N : M_{\mu}^{1}f(x) > s \}| \leq \frac{C}{s} \int_{x \in E_{N}} |f(x)|d\mu(x),
\]

for some constant C. Let

\[
f_{1}(x) = \begin{cases} 
  f(x) & \text{for } |f(x)| \geq s \text{ and } x \in E_{N} \\
  0 & \text{for } |f(x)| < s \text{ or } x \in E_{N}^{c}.
\end{cases}
\]

Let \( f_{2}(x) = f(x)\chi_{E_{N}}(x) - f_{1}(x) \), where \( \chi_{E_{N}}(x) \) is the characteristic function of the set \( E_{N} \). Thus

\[
|\mu \{ x > N : M_{\mu}^{1}f_{2}(x) > s \}| = 0.
\]

Then

\[
|\mu \{ x > N : M_{\mu}^{1}f(x) > 2s \}| < |\mu \{ x > N : M_{\mu}^{1}f_{1}(x) > s \}|.
\]

Then we could obtain:

\[
\int_{N}^{+\infty} |M_{\mu}^{1}f(x)|^2d\mu(x) = 2 \int_{0}^{+\infty} 2s \mu \{ x > N : M_{\mu}^{1}f(x) > 2s \} |2ds
\]

\[
\leq \int_{0}^{+\infty} s \mu \{ x > N : M_{\mu}^{1}f_{1}(x) > s \} |ds
\]

\[
\leq \int_{0}^{+\infty} \int_{\{x \in E_{N} : |f(x)| > s\}} sd\mu(x)ds
\]

\[
\leq \int_{\{x \in E_{N} \} \setminus \{0\}} \int_{0}^{+\infty} sdsd\mu(x)
\]

\[
\leq \int_{N_{0}}^{+\infty} |f(x)|^2d\mu(x)
\]

\[
\leq \varepsilon.
\]

In a similar way we could obtain:

\[
\int_{-\infty}^{-N} |M_{\mu}^{1}f(x)|^2d\mu(x) \leq \int_{-\infty}^{N_{0}} |f(x)|^2d\mu(x)
\]

\[
\leq \varepsilon.
\]

When \( i = 1 \), notice that for \( 0 < r < 1 \)

\[
\int_{\mathbb{R}} \frac{f(y)}{a_{1}(r,x)} K_{1}(r,x,y)d\mu(y)/r \leq M_{\mu}^{1}f(x).
\]

Thus we could deduce that for \( 0 < r < 1 \)

\[
\left( \int_{[N, +\infty)} |f(x) - \int_{\mathbb{R}} \frac{f(y)}{a_{1}(r,x)} K_{1}(r,x,y)d\mu(y)/r |^2 d\mu(x) \right)^{1/2}
\]

\[
\leq \left( \int_{[N, +\infty)} |f(x)|^2 d\mu(x) \right)^{1/2} + \left( \int_{[N, +\infty)} |M_{\mu}^{1}f(x)|^2 d\mu(x) \right)^{1/2}
\]

\[
\leq \left( \int_{[N_{0}, +\infty)} |f(x)|^2 d\mu(x) \right)^{1/2}
\]

\[
\leq \varepsilon,
\]

and similarly we could obtain

\[
\left( \int_{(-\infty, N]} |f(x) - \int_{\mathbb{R}} \frac{f(y)}{a_{1}(r,x)} K_{1}(r,x,y)d\mu(y)/r |^2 d\mu(x) \right)^{1/2} \leq \varepsilon.
\]
Notice that \( \text{supp} K_1(r, x, y) \subseteq B_\mu(x, r) \) for the \( y \) variable. Thus when \( r < r_N, x \in [-N, N] \), by the mean value theorem, there exists \( \xi \in B_\mu(x, r_N) \), such that the following holds:

\[
|f(x) - \int_\mathbb{R} \frac{f(y)}{a_1(r, x)} K_1(r, x, y) d\mu(y)/r| = |f(x) - f(\xi)| < \varepsilon/N.
\]

Thus we could obtain that the following inequality holds for \( 0 < r < r_N \)

\[
\left( \int_{[-N, N]} |f(x) - \int_\mathbb{R} \frac{f(y)}{a_1(r, x)} K_1(r, x, y) d\mu(y)/r|^2 d\mu(x) \right)^{1/2} \lesssim \varepsilon \tag{9}
\]

Thus when \( f \in C(\mathbb{R}, d_\mu x) \cap L^2(\mathbb{R}, \mu) \), from Formula (7)(8) and (9), the following holds:

\[
\lim_{r \to 0} \left( \int_\mathbb{R} |f(x) - \int_\mathbb{R} \frac{f(y)}{a_1(r, x)} K_1(r, x, y) d\mu(y)/r|^2 d\mu(x) \right)^{1/2} = 0.
\]

Next we will remove the condition \( f \in C(\mathbb{R}, d_\mu x) \). When \( f \in L^2(\mathbb{R}, \mu) \), noticing that \( C(\mathbb{R}, d_\mu x) \cap L^2(\mathbb{R}, \mu) \) is dense in \( L^2(\mathbb{R}, \mu) \) by Proposition (2.1), we can choose \( \{g_n\} \subset C(\mathbb{R}, d_\mu x) \cap L^2(\mathbb{R}, \mu) \), satisfying

\[
\lim_{n \to +\infty} ||f - g_n||_{L^2(\mathbb{R}, \mu)} = 0.
\]

Thus

\[
\lim_{r \to 0} \left( \int_\mathbb{R} |f(x) - \int_\mathbb{R} \frac{f(y)}{a_1(r, x)} K_1(r, x, y) d\mu(y)/r|^2 d\mu(x) \right)^{1/2} \lesssim \lim_{n \to +\infty} \lim_{r \to 0} \left( \int_\mathbb{R} g_n(x) - \int_\mathbb{R} \frac{g_n(y)}{a_1(r, x)} K_1(r, x, y) d\mu(y)/r|^2 d\mu(x) \right)^{1/2}
\]

\[
+ \lim_{n \to +\infty} \lim_{r \to 0} \left( \int_\mathbb{R} \frac{|f(x) - g_n(x)|}{a_1(r, x)} K_1(r, x, y) d\mu(y)/r|^2 d\mu(x) \right)^{1/2}
\]

\[
\lesssim 0 + \lim_{n \to +\infty} ||f - g_n||_{L^2(\mathbb{R}, \mu)} + \lim_{n \to +\infty} ||M_\mu(f - g_n)||_{L^2(\mathbb{R}, \mu)}
\]

\[
\lesssim 0.
\]

We will prove the Proposition when \( i = 2 \). Denote the operator \( M^{k_0}_\mu \) as following:

\[
M^{k_0}_\mu f(x) = \sup_{0 < r < 1} \frac{1}{2^{k_0} r} \int_{B_\mu(x, 2^{k_0} r)} |f(y)| d\mu(y).
\]

Noticing that for any \( k_0 \in \mathbb{N} \)

\[
\left| \int_\mathbb{R} K_2(r, x, y) f(y) d\mu(y)/r \right| \lesssim \sum_{k=1}^{+\infty} \int_{2^{k-1} r < x(y) \leq 2^k r} (1 + 2^k)^{-1-\gamma} |f(y)| d\mu(y)/r + \int_{d_\mu(x, y) \leq r} |f(y)| d\mu(y)/r
\]

\[
\lesssim \sum_{k=0}^{+\infty} 2^{-k(1+\gamma)} \int_{d_\mu(x, y) \leq 2^k r} |f(y)| d\mu(y)/r.
\]

\[
\lesssim M^{k_0}_\mu f(x) + 2^{-k_0\gamma} M_\mu f(x).
\]

Assume \( f \in C(\mathbb{R}, d_\mu x) \cap L^2(\mathbb{R}, \mu) \). \( k_0 \) is a number such that

\[
2^{-k_0\gamma} ||f||_{L^2(\mathbb{R}, \mu)} < \varepsilon.
\]

Then \( \exists r_N > 0 \) and \( \exists N > 0 \), such that the following holds:

\[
\left( \int_{N, N}^{+\infty} |f(x)|^2 d\mu(x) \right)^{1/2} + \left( \int_{-\infty}^{-N, N} |f(x)|^2 d\mu(x) \right)^{1/2} < \varepsilon,
\]
and
$$|f(x) - f(y)| < \varepsilon/N$$
for any \(y \in B(x, r_N), x \in [-N, N] \subset \mathbb{R}\). Similar to the case when \(i = 1\), we could deduce that:
$$\int_{N}^{+\infty} |M_{\mu}^{k_0} f(x)|^2 d\mu(x) < \int_{N_{k_0}}^{+\infty} |f(x)|^2 d\mu(x) < \varepsilon.$$

Thus we obtain
$$\int_{-N}^{-N_{k_0}} |M_{\mu}^{k_0} f(x)|^2 d\mu(x) \lesssim \int_{-\infty}^{-N_{k_0}} |f(x)|^2 d\mu(x) \lesssim \varepsilon.$$

In a similar way, we could obtain:
$$\left( \int_{[N, +\infty]} \left| f(x) - \int_{\mathbb{R}} \frac{f(y)}{a_2(r, x)} K_2(r, x, y) d\mu(y)/r \right|^2 d\mu(x) \right)^{1/2} \lesssim \varepsilon. \quad (10)$$

When \(x \in [-N, N]\), we have
$$\left( \int_{[-N, N]} \left| f(x) - \int_{\mathbb{R}} \frac{f(y)}{a_2(r, x)} K_2(r, x, y) d\mu(y)/r \right|^2 d\mu(x) \right)^{1/2} \lesssim \varepsilon.$$

Similar to Formula(9),
$$\left( \int_{[-N, N]} \left( \int_{d_\gamma(x, y) \leq r_N} |f(x) - f(y)| K_2(r, x, y) d\mu(y)/r \right)^2 d\mu(x) \right)^{1/2} \lesssim \varepsilon \quad (13)$$
holds. By the mean value theorems for definite integrals, \(\exists \xi_0 \in [-N, N]\), such that the following holds:
$$\left( \int_{[-N, N]} \left( \int_{d_\gamma(x, y) \geq r_N} |f(x) - f(y)| K_2(r, x, y) d\mu(y)/r \right)^2 d\mu(x) \right)^{1/2} \lesssim \varepsilon \quad (14)$$
Notice that:

\[
\int_{d_\mu(x,y) \geq r_N} K_2(r, x, y) d\mu(y) / r = \sum_{k=1}^{+\infty} \int_{2^{k-1} r_N < d_\mu(x,y) \leq 2^k r_N} K_2(r, x, y) d\mu(y) / r \tag{15}
\]

\[
\lesssim \sum_{k=0}^{+\infty} \left( 1 + \frac{2^k r_N}{r} \right)^{-1-\gamma} \frac{2^k r_N}{r} \lesssim \left( \frac{r}{r_N} \right)^\gamma
\]

By Formula (15) and Formula (14), we could obtain:

\[
\left( \int_{[-N,N]} \left( \int_{d_\mu(x,y) \geq r_N} |f(x) - f(y)| K_2(r, x, y) d\mu(y) / r \right)^2 d\mu(x) \right)^{1/2} \lesssim \left( \frac{r}{r_N} \right)^\gamma \left( \int_{[-N,N]} |f(x) - f(\xi_0)|^2 d\mu(x) \right)^{1/2}
\]

Thus we have

\[
\lim_{r \to 0} \left( \int_{[-N,N]} \left( \int_{d_\mu(x,y) \geq r_N} |f(x) - f(y)| K_2(r, x, y) d\mu(y) / r \right)^2 d\mu(x) \right)^{1/2} = 0
\]

Then the above Formula together with Formula (10), (11) and (13), for \( f \in C(\mathbb{R}, d_\mu) \cap L^2(\mathbb{R}, \mu) \) we could obtain:

\[
\lim_{r \to 0} \left( \int_{\mathbb{R}} \left| f(x) - \int_{\mathbb{R}} \frac{f(y)}{a_2(r,x)} K_2(r, x, y) d\mu(y) / r \right|^p d\mu(x) \right)^{1/p} = 0.
\]

When \( f \in L^2(\mathbb{R}, \mu) \), noticing \( C(\mathbb{R}, d_\mu) \cap L^2(\mathbb{R}, \mu) \) is dense in \( L^2(\mathbb{R}, \mu) \), we could also prove the above Formula in the same way when \( i = 1 \). This proves the Proposition. \( \Box \)

This proposition also holds in \( L^p(\mathbb{R}, \mu) \), \( 1 < p \leq \infty \):

**Proposition 2.17.** For \( f \in L^p(\mathbb{R}, \mu) \), \( i = 1, 2 \), \( 1 < p \leq \infty \),

\[
\lim_{r \to 0} \left( \int_{\mathbb{R}} \left| f(x) - \int_{\mathbb{R}} \frac{f(y)}{a_i(r,x)} K_i(r, x, y) d\mu(y) / r \right|^p d\mu(x) \right)^{1/p} = 0
\]

holds.

**Proposition 2.18.** For \( f \in L^1(\mathbb{R}, \mu) \), \( 1 \geq \gamma > 0 \), \( \infty > p > 0 \) we could obtain

\[
f^*_\gamma(x) = f^*_x(x) \quad a.e. x \in \mathbb{R} \text{ in } \mu \text{ measure}.
\]

Further more, if \( \int_{\mathbb{R}} |f^*_x(x)|^p d\mu(x) \leq \infty \) or \( \int_{\mathbb{R}} |f^*_\gamma(x)|^p d\mu(x) \leq \infty \), we could obtain

\[
\int_{\mathbb{R}} |f^*_x(x)|^p d\mu(x) = \int_{\mathbb{R}} |f^*_\gamma(x)|^p d\mu(x) < \infty.
\]

**Proof.** Firstly, we will prove

\[
f^*_\gamma(x) = f^*_x(x) \quad a.e. x \in \mathbb{R} \text{ in } \mu \text{ measure}. \tag{16}
\]

By the definition of \( f^*_\gamma(x) \) and \( f^*_x(x) \) above:

\[
f^*_x(x) = \sup_{\phi, r > 0} \left\{ \int_{\mathbb{R}} |f(y)\phi(y) d\mu(y)| / r : r > 0, \text{supp } \phi \subset B_\mu(x,r), L(\phi, \gamma) \leq r^{-\gamma}, \|\phi\|_{L^\infty} \leq 1 \right\}
\]

and

\[
f^*_\gamma(x) = \sup_{\psi, r > 0} \left\{ \int_{\mathbb{R}} |f(y)\psi(y) d\mu(y)| / r : r > 0, \text{supp } \psi \subset B_\mu(x,r), \psi \in S(\mathbb{R}, d_\mu), L(\psi, \gamma) \leq r^{-\gamma}, \|\psi\|_{L^\infty} \leq 1 \right\}
\]
Then obviously \( f_{\gamma}^{*}(x) \leq f_{\gamma}^{*}(x) \). If \( \phi \) satisfies \( L(\phi, \gamma) \leq r^{-\gamma} \) and \( \text{supp} \phi \subset B_{\mu}(x, r) \), then \( \phi \) is a obviously continuous function with compact support. Thus \( \exists \psi_{n} \in S(\mathbb{R}, d_{\mu}x) \) satisfying \( \lim_{n \to \infty} \| \psi_{n}(t) - \phi(t) \|_{\infty} = 0 \). Denote \( \delta_{n}(x) \) as
\[
\delta_{n}(x) = \left| \int_{B_{\mu}(x, r)} f(y) (\phi(y) - \psi_{n}(y)) d\mu(y) / r \right|.
\]

Then we could conclude:
\[
\delta_{n}(x) \leq M_{\mu} f(x) \| \psi_{n}(y) - \phi(y) \|_{\infty}.
\]

Because \( M_{\mu} \) is weak-(1, 1) bounded, the following holds for any \( \alpha > 0 \):
\[
\lim_{n \to +\infty} \{ x : \delta_{n}(x) > \alpha \} \subset \frac{1}{\alpha} \| f \|_{L^{1}(\mathbb{R}, \mu)} \| \psi_{n}(y) - \phi(y) \|_{\infty} = 0.
\]

Then there exists a sequence \( \{ n_{j} \} \subset \{ n \} \) such that the following holds:
\[
\int_{\mathbb{R}} f(y) \phi(y) d\mu(y) / r = \lim_{n_{j} \to \infty} \int_{\mathbb{R}} f(y) \psi_{n_{j}}(y) d\mu(y) / r, \quad a.e. x \in \mathbb{R} \text{ in } \mu \text{ measure}
\]
for \( f \in L^{1}(\mathbb{R}, \mu) \). Thus we could obtain:
\[
\int_{\mathbb{R}} f(y) \phi(y) d\mu(y) / r \leq f_{\gamma}^{*}(x) \quad a.e. x \in \mathbb{R} \text{ in } \mu \text{ measure}
\]
for any \( \phi \) satisfies \( L(\phi, \gamma) \leq r^{-\gamma} \) and \( \text{supp} \phi \subset B_{\mu}(x, r) \). We could then deduce
\[
\sup_{\phi, r > 0} \left| \int_{\mathbb{R}} f(y) \phi(y) d\mu(y) / r \right| \leq f_{\gamma}^{*}(x) \quad a.e. x \in \mathbb{R} \text{ in } \mu \text{ measure}.
\]

Thus
\[
f_{\gamma}^{*}(x) = f_{\gamma}^{*}(x) \quad a.e. x \in \mathbb{R} \text{ in } \mu \text{ measure}.
\]

Let \( E \) denote a set defined as \( E = \{ x : f_{\gamma}^{*}(x) = f_{\gamma}^{*}(x) \} \). Next we will prove that for any \( x_{0} \in \mathbb{R} \), there is a point \( \varpi_{0} \in E \) such that
\[
f_{\gamma}^{*}(x_{0}) \leq f_{\gamma}^{*}(\varpi_{0}). \tag{17}
\]

Notice that for \( x_{0} \in \mathbb{R} \), there exist \( r_{0} > 0 \) and \( \phi_{0} \) satisfying: \( \text{supp} \phi_{0} \subset B_{\mu}(x_{0}, r_{0}), \phi_{0} \in S(\mathbb{R}, d_{\mu}x), L(\phi_{0}, \gamma) \leq r_{0}^{-\gamma}, \| \phi_{0} \|_{L^{1}} \leq 1 \). Then the following inequality could be concluded:
\[
\left| \frac{1}{r_{0}} \int_{\mathbb{R}} f(y) \phi_{0}(y) d\mu(y) \right| \geq \frac{1}{2} f_{\gamma}^{*}(x_{0}).
\]

\( |\mu(\mathbb{R} \setminus E)| = |\mu(E^c)| = 0 \) implies \( E \) is dense in \( \mathbb{R} \). Then there exists a \( \varpi_{0} \in E \) with \( d_{\mu}(x_{0}, \varpi_{0}) \leq \frac{r_{0}}{4} \). Thus \( \text{supp} \phi_{0} \subset B_{\mu}(\varpi_{0}, 4r_{0}) \) holds. Thus we could obtain
\[
\left| \frac{1}{r_{0}} \int_{\mathbb{R}} f(y) \phi_{0}(y) d\mu(y) \right| \leq Cf_{\gamma}^{*}(\varpi_{0}),
\]

\( C \) is independent on \( f, \gamma \) and \( r_{0} \). Then Formula (17) could be deduced. By Formula (17), we could obtain the following conclusion:
\[
\int_{E} |f_{\gamma}^{*}(x)|^{p} d\mu(x) < \infty \Rightarrow \int_{\mathbb{R}} |f_{\gamma}^{*}(x)|^{p} d\mu(x) = \int_{E} |f_{\gamma}^{*}(x)|^{p} d\mu(x) < \infty. \tag{18}
\]

In the same way, we could conclude that
\[
\int_{\mathbb{R}} |f_{\gamma}^{*}(x)|^{p} d\mu(x) = \int_{E} |f_{\gamma}^{*}(x)|^{p} d\mu(x). \tag{19}
\]
From Formula (16) we could deduce:
\[
\int_E |f^*_\gamma(x)|^p d\mu(x) = \int_E |f^*_\gamma(x)|^p d\mu(x) < \infty
\]
The above Formula together with (18)(19) imply that
\[
\int_{\mathbb{R}} |f^*_\gamma(x)|^p d\mu(x) \leq \int_{\mathbb{R}} |f^*_\gamma(x)|^p d\mu(x) < \infty
\]
holds if \( \int_{\mathbb{R}} |f^*_\gamma(x)|^p d\mu(x) \leq \infty \) or \( \int_{\mathbb{R}} |f^*_\gamma(x)|^p d\mu(x) < \infty \). This proves the proposition. \(\Box\)

**Definition 2.19.** Then we denote \( SS_\beta \) (\( \beta > 0 \)) as
\[
SS_\beta = \{ \phi : \phi \in S(\mathbb{R}, dx), \text{supp} \phi \subset [-1, 1], \|\phi\|_{L^\infty} \leq 1, [\phi]_\beta \leq 1 \}.
\]

Proposition(2.5) and Proposition(2.18) together with Proposition(2.13), we could also define \( f^*_\gamma \) (\( 1 \geq \gamma > 0 \)) and \( f^*_\beta \) (\( \beta > 0 \)) as following:
\[
f^*_\gamma(x) = \sup_{\psi, r > 0} \left\{ \left\| \int_{\mathbb{R}} f(y) \psi \left( \frac{\mu(x, y)}{r} \right) d\mu(y) \right\| / r : r > 0, \psi(t) \in S(\mathbb{R}, dx), \text{supp} \psi(t) \subset [-1, 1], \|\psi\|_{L^\infty} \leq 1, H^\gamma \psi \leq 1 \right\}.
\]

and
\[
f^*_\beta(x) = \sup_{\psi, r > 0} \left\{ \left\| \int_{\mathbb{R}} f(y) \psi \left( \frac{\mu(x, y)}{r} \right) d\mu(y) \right\| / r : r > 0, \psi(t) \in SS_\beta \right\}.
\]

We define \( M_{\phi, \beta} f(x) \) as

**Definition 2.20.**
\[
M_{\phi, \beta} f(x) = \sup_{r > 0} \left\{ \left\| \int_{\mathbb{R}} f(y) \phi \left( \frac{\mu(x, y)}{r} \right) d\mu(y) \right\| / r : r > 0, \phi(t) \in SS_\beta \right\}.
\]

Thus it is easy to see that
\[
f^*_\beta(x) = \sup_{\phi(t) \in SS_\beta} M_{\phi, \beta} f(x).
\]

Let
\[
M_{\phi, \beta}^* f(x) = \sup_{d_\gamma(x, u) < r} M_{\phi, \beta} f(x)
\]

or we could write \( M_{\phi, \beta}^* f(x) \) as
\[
M_{\phi, \beta}^* f(x) = \sup_{d_\gamma(x, u) < r} \left\{ \left\| \int_{\mathbb{R}} f(u) \phi \left( \frac{\mu(x, u) - s}{r} \right) d\mu(u) \right\| / r : r > 0, \phi(t) \in SS_\beta \right\}.
\]

Notice that \( \mu(y, u) = \mu(x, u) - \mu(x, y) \), by letting \( s = \mu(x, y) \) we could also write \( M_{\phi, \beta}^* f(x) \) as following:

**Definition 2.21.**
\[
M_{\phi, \beta}^* f(x) = \sup_{|s| < r} \left\{ \left\| \int_{\mathbb{R}} f(u) \phi \left( \frac{\mu(x, u) - s}{r} \right) d\mu(u) \right\| / r : r > 0, \phi(t) \in SS_\beta \right\}.
\]

\[
M_{\phi, \beta}^* f(x) = \sup_{|s| < r} \left\{ \left\| \int_{\mathbb{R}} f(u) \phi \left( \frac{\mu(x, u) - s}{r} \right) d\mu(u) \right\| / r : r > 0, \phi(t) \in SS_\beta \right\}.
\]

Then we define \( M_{\phi, \beta}^{**} f(x) \) by:

**Definition 2.22.**
\[
M_{\phi, \beta}^{**} f(x) = \sup_{s \in \mathbb{R}, r > 0} \left\{ \left\| \int_{\mathbb{R}} f(u) \phi \left( \frac{\mu(x, u) - s}{r} \right) \left( 1 + \left| \frac{s}{r} \right| \right)^{-N} d\mu(u) \right\| / r : r > 0, \phi(t) \in SS_\beta \right\}.
\]

Then
\[
M_{\phi, \beta} f(x) \lesssim M_{\phi, \beta}^* f(x) \lesssim M_{\phi, \beta}^{**} f(x).
\]
2.2 The characterization of homogeneous Hardy spaces with a kernel

**Definition 2.23** ($H^p_{\mu^\beta}(\mathbb{R})$ and $\tilde{H}^p_{\mu^\beta}(\mathbb{R})$ spaces for $p > 0.$)

$$\tilde{H}^p_{\mu^\beta}(\mathbb{R}) \triangleq \{ g \in L^1(\mathbb{R}, \mu) : g^*_a(x) \in L^p(\mathbb{R}, \mu) \}.$$  

And its norm is

$$\|g\|_{\tilde{H}^p_{\mu^\beta}(\mathbb{R})} = \int_{\mathbb{R}} |g^*_a(x)|^p d\mu(x).$$

Then we define the $H^p_{\mu^\beta}(\mathbb{R})$ to be the completion of $\tilde{H}^p_{\mu^\beta}(\mathbb{R})$ with the $\| \cdot \|_{\tilde{H}^p_{\mu^\beta}(\mathbb{R})}$ norm.

When $1 < p < \infty$, in Classical ways in Harmonic analysis, $H^p_{\mu^\beta}(\mathbb{R}) = L^p(\mathbb{R}, \mu)$, and $\tilde{H}^p_{\mu^\beta}(\mathbb{R})$ is dense in $L^p(\mathbb{R}, \mu)$.

**Proposition 2.24.** For fixed numbers $a \geq b > 0$, $F(x, r)$ is a function defined on $\mathbb{R}^2_+$, its nontangential maximal function $F^*_a(x)$ is defined as

$$F^*_a(x) = \sup_{d_\mu(x,y) < ar} F(y, r).$$

Then we could have

$$\int_{\mathbb{R}} \chi \{ x : F^*_a(x) > \alpha \} d\mu(x) \leq c \frac{a+b}{b} \int_{\mathbb{R}} \chi \{ x : F^*_b(x) > \alpha \} d\mu(x).$$

$c$ is a constant independent on $F$, $a$, $b$, and $\alpha$.

**Proof.** First we could see that $\{ x : F^*_a(x) > \alpha \}$ is an open set. It is also obviously to see that $\{ x : F^*_b(x) > \alpha \} \subseteq \{ x : F^*_a(x) > \alpha \}$,  

when $a \geq b > 0$. If $z \in \{ x : F^*_a(x) > \alpha \}$, then $\exists F(x_0, r_0) > \alpha$ satisfying $d_\mu(z, x_0) < ar_0$. It is obviously that $B_\mu(x_0, b r_0) \subseteq \{ x : F^*_a(x) > \alpha \}$ and $B_\mu(x_0, a r_0) \subseteq \{ x : F^*_a(x) > \alpha \}$. Then we could notice that

$$\frac{|B_\mu(z, (a+b) r_0) \cap \{ x : F^*_a(x) > \alpha \}|_\mu}{|B_\mu(z, (a+b) r_0)|_\mu} \geq \frac{|B_\mu(x_0, b r_0)|_\mu}{|B_\mu(x_0, (a+b) r_0)|_\mu} \geq \frac{b}{a+b}.$$ 

Thus

$$\{ x : F^*_a(x) > \alpha \} \subseteq \left\{ x : M_\mu \chi \{ x : F^*_b(x) > \alpha \} > \frac{b}{a+b} \right\}.$$

$M_\mu$ is the Hardy-Littlewood maximal operator. Because $M_\mu$ is weak-(1, 1). We could obtain:

$$\int_{\mathbb{R}} \chi \{ x : F^*_a(x) > \alpha \} d\mu(x) \leq c \frac{a+b}{b} \int_{\mathbb{R}} \chi \{ x : F^*_b(x) > \alpha \} d\mu(x).$$

This proves the proposition. \[\square\]

When $F^*_b(x) \in L^p(\mathbb{R}, \mu)$, by Proposition(2.24), we could obtain:

$$\int_{\mathbb{R}} |F^*_a(x)|^p d\mu(x) \leq c \left( \frac{a+b}{b} \right) \int_{\mathbb{R}} |F^*_b(x)|^p d\mu(x).$$

(30)

**Proposition 2.25.**

$$\| M^*_{\phi^N} f(x) \|_{L^p(\mathbb{R}, \mu)} \leq c_1 \| M^*_{\phi^N} f(x) \|_{L^p(\mathbb{R}, \mu)} \quad \text{for} \quad p > 0, N > 1/p.$$  

$C$ is independent on $\phi$ and $f$. 
Proof. For $\psi(t) \in SS_\beta$, 
\[
M^*_{\phi \beta N} f(x) = \sup_{s \in \mathbb{R}, r > 0} \left| \int_{\mathbb{R}} f(y) \varphi \left( \frac{\mu(x, y) - s}{r} \right) \left( 1 + \frac{|s|}{r} \right)^{-N} d\mu(y) \right| / r 
\]
\[
\leq \left( \sup_{s \leq r} + \sum_{k=1}^{\infty} \sup_{2^{-k} r < s \leq 2^k r} \right) 2^{-kN} \left| \int_{\mathbb{R}} f(y) \varphi \left( \frac{\mu(x, y) - s}{r} \right) d\mu(y) \right| / r 
\]
\[
\leq \sum_{k=0}^{\infty} 2^{-kN} M^*_{\phi \beta 2^k} f(x) 
\]
Thus together with Formula (30), we could have:
\[
\int_{\mathbb{R}} |M^*_\phi f(x)|^p d\mu(x) \leq C \int_{\mathbb{R}} |M^*_\phi f(x)|^p d\mu(x).
\]
This proves our Proposition.

From Classical Harmonic Analysis, we have the following Proposition:

**Proposition 2.26.** Stein[17] Suppose $\phi, \psi \in S(\mathbb{R}, dx)$, with $\int_{\mathbb{R}} \psi(x) dx = 1$ and $\phi \in SS_\beta$ for some $\beta > 0$. Then there is a sequence $\{\eta^k\}$, $\eta^k \in S(\mathbb{R}, dx)$, so that
\[
\phi = \sum_{k=0}^{\infty} \eta^k * \psi_{2^{-k}}.
\]
$\eta^k$ satisfies
\[
\| \eta^k \|_{a,b} \leq C(2^{-kM}), \quad as \ k \to \infty,
\]
whenever $\| \cdot \|_{a,b}$ is a seminorm, and $M \geq 0$ fixed. $a, b$ are numbers that dependent on $\beta$. $C$ is dependent on $\beta$, not on the particular $\psi$.

From the above Proposition, it is easy to obtain:

**Proposition 2.27.** Stein[17] Suppose $\phi, \psi \in SS_\beta$, with $\int \psi(x) dx = 1$. Then there is a sequence $\{\eta^k\}$, $\eta^k \in S(\mathbb{R}, dx)$, so that
\[
\phi \left( \frac{\mu(y, u)}{r} \right) = \sum_{k=0}^{\infty} \int_{\mathbb{R}} \eta^k \left( \frac{s}{r} \right) \psi \left( \frac{\mu(y, u) - s}{2^{-k} r} \right) ds / 2^{-k r}.
\]
$\eta^k$ satisfies
\[
\| \eta^k \|_{a,b} \leq C(2^{-kM}), \quad as \ k \to \infty.
\]

Now we need to prove that the nontangential maximal operator $M^*_\psi f(x)$ allows control of maximal function $f^{*}_{SS_\beta}(x)$.

**Proposition 2.28.** There exists $\beta > 0$, such that for any $\psi \in SS_\beta$, with $\int \psi(x) dx = 1$ and $p > 0$, the following holds:
\[
\| f^{*}_{SS_\beta} \|_{L^p(\mathbb{R}, \mu)} \leq c \| M^*_\psi f \|_{L^p(\mathbb{R}, \mu)}.
\]
$C$ is dependent on $\beta$.

**Proof.** For any $\phi, \psi \in SS_\beta$, with $\int \psi(x) dx = 1$ by Proposition (2.27), we have
\[
M^*_\phi f(x) = \sup_{r > 0} \left| \int_{\mathbb{R}} f(y) \varphi \left( \frac{\mu(x, y)}{r} \right) d\mu(y) \right| / r \leq \sup_{r > 0} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) \eta^k \left( \frac{s}{r} \right) \psi \left( \frac{\mu(x, y) - s}{2^{-k} r} \right) d\mu(y) ds / 2^{-k r} / r.
\]
Thus by the definition of $M_{\phi;\beta}^*f(x)$ (2.22), we could obtain:

\[
M_{\phi;\beta}f(x) \lesssim \sup_{r > 0} \sum_{k=0}^{\infty} \left| \int_{\mathbb{R}} \int_{\mathbb{R}} f(y) \eta^k \left( \frac{s}{r} \right) \psi \left( \frac{\mu(x, y) - s}{2^{-k}r} \right) d\mu(y) \frac{ds}{2^{-k}r} \right| / r
\]

\[
\lesssim \sup_{r > 0} \sum_{k=0}^{\infty} \int_{\mathbb{R}} \left| f(y) \psi \left( \frac{\mu(x, y) - s}{2^{-k}r} \right) \left( 1 + \frac{|s|}{2^{-k}r} \right)^{-N} d\mu(y) \right| \left| \eta^k \left( \frac{s}{r} \right) \left( 1 + \frac{|s|}{2^{-k}r} \right)^N \right| ds / r
\]

\[
\lesssim M_{\psi;\beta}^*f(x) \sum_{k=0}^{\infty} 2^{-k} \lesssim M_{\psi;\beta}^*f(x),
\]

if $\|\eta^k\|_{a,b} = O(2^{-k(N+1)})$ for a suitable collection of seminorms. Thus

\[
f_{\beta}^*(x) \approx \sup_{\phi \in SS_\beta} M_{\phi;\beta} f(x) \lesssim M_{\psi;\beta}^*f(x),
\]

for all $x \in \mathbb{R}, N > 1/p$ by Proposition(2.25), we could have

\[
\|f_{\beta}^*\|_{L^p(\mathbb{R}, \mu)} \leq c\|M_{\phi;\beta}^*f\|_{L^p(\mathbb{R}, \mu)}.
\]

This proves our proposition.

Proposition 2.29. There exists $\beta > 0$, such that for $p > 0, \phi \in SS_\beta$, with $\int \phi(x)dx = 1$, the following holds:

\[
\|M_{\phi;\beta} f\|_{L^p(\mathbb{R}, \mu)} \leq c\|M_{\phi;\beta}^* f\|_{L^p(\mathbb{R}, \mu)}.
\]

$C$ is dependent on $\beta$.

Proof. Assume first $\|M_{\phi;\beta} f\|_{L^p(\mathbb{R}, \mu)} < \infty$. Let $F = F_\sigma = \{x : f_\sigma(x) \leq \sigma M_{\phi;\beta} f(x)\}$. Thus together with Proposition(2.28), the following holds:

\[
\int_{F_\sigma} |M_{\phi;\beta} f(x)|^p d\mu(x) \leq \sigma^{-p} \int_{F_\sigma} |f(x)|^p d\mu(x) \leq c \sigma^{-p} \int |M_{\phi;\beta}^* f(x)|^p d\mu(x).
\]

Next we will show that for any $q > 0$

\[
M_{\phi;\beta}^* f(x) \leq cM(M_{\phi;\beta} f)^q(x).
\]

Let

\[
f(x, r) = \int_{\mathbb{R}} f(u) \phi \left( \frac{\mu(x, u)}{r} \right) d\mu(u) / r.
\]

Then for any $x \in \mathbb{R}$, there exists $(y, r)$, satisfying $d\mu(x, y) < r$ and $|f(y, r)| \geq M_{\phi;\beta} f(x)/2$. Choose $\delta < 1$ and $x'$ satisfying $d\mu(x', y) < \delta r$. Then there $\exists \xi \in [x', y]$ such that:

\[
|f(x', r) - f(y, r)| \leq \delta r \sup_{x \in B_{\mu}(y, \delta r)} \left| \frac{d}{d\mu x} f(x, r) \right|
\]

\[
\leq \delta \sup_{\xi \in B_{\mu}(y, \delta r)} \left| \int_{\mathbb{R}} f(u) \phi^{(1)} \left( \frac{\mu(x, u) - \mu(x, \xi)}{r} \right) d\mu(u) / r \right|
\]

\[
\leq \delta \sup_{\xi \in B_{\mu}(y, \delta r)} \left| \int_{\mathbb{R}} f(u) \phi^{(1)} \left( \frac{\mu(x, u) - \mu(x, \xi)}{r} \right) d\mu(u) / r \right|
\]

\[
\leq \delta \sup_{|h| \leq 1 + \delta} \left| \int_{\mathbb{R}} f(u) \phi^{(1)} \left( \frac{\mu(x, u) - h}{r} \right) d\mu(u) / r \right|.
\]

Notice that $|h| \leq 1 + \delta < 2$ with $\|H_x^{\beta} \phi^{(1)}(x - h)\|_{\infty} \leq C, \|\phi^{(1)}(x - h)\|_{\infty} \leq C$. By the definition of $f_\beta^*(x)$,

\[
|f(x', r) - f(y, r)| \leq C_0 \delta f_\beta^*(x) \leq C_0 \delta \sigma M_{\phi;\beta}^* f(x) \quad \text{for} \ x \in F.
\]
Taking $\delta$ small enough such that $C_0 \delta \sigma \leq 1/4$, we obtain
\[ f(x', r) \geq \frac{1}{4} M^*_f(x). \]

Thus the following inequality holds:
\[
|M^*_f(x)|^q \leq \frac{1}{B_{\mu}(y, \delta r)} \int_{B_{\mu}(y, \delta r)} 4^q |f(x', r)|^q d\mu(x') \\
\leq \frac{B_{\mu}(x, (1 + \delta)r)}{B_{\mu}(y, \delta r)} \frac{1}{B_{\mu}(x, (1 + \delta)r)} \int_{B_{\mu}(x, (1 + \delta)r)} 4^q |f(x', r)|^q d\mu(x') \\
\leq \frac{1 + \delta}{\delta} \frac{B_{\mu}(x, (1 + \delta)r)}{B_{\mu}(x, \delta r)} \int_{B_{\mu}(x, \delta r)} 4^q |f(x', r)|^q d\mu(x') \\
\leq CM^*_f((M^*_f)^q)(x).
\]

$M_{\mu}$ is the Hardy-Littlewood Maximal Operator. Thus for $p$ satisfying $p > q$ using the maximal theorem for $M_{\mu}$ leads to
\[
\int_F |M^*_\beta f(x)|^p d\mu(x) \leq C \int_F (M^*_\mu((M^*_\beta f)^q)(x))^{p/q} \leq C \int_R |M^*_\beta f(x)|^p d\mu(x). \quad (32)
\]

Combining (31) and (32) together, we could prove the proposition. \hfill \Box

**Proposition 2.30. Stein[17]** Classical Hardy spaces $H^p(\mathbb{R})$ in Euclid space

Let $F = \{\| \cdot \|_{a,b}\}$ be any finite collection of seminorms on $S(\mathbb{R}, dx)$. We denote by $S_F$ the subset of $S(\mathbb{R}, dx)$ controlled by this collection of seminorms:

\[ S_F = \{ \phi \in S(\mathbb{R}, dx) : \| \phi \|_{a,b} \leq 1 \text{ for any } \| \cdot \|_{a,b} \in F \}. \]

Let
\[ M_F f(x) = \sup_{\phi \in S_F} \sup_{t > 0} (f \ast \phi_t)(x). \]

If $f \in H^p(\mathbb{R})$, there exists $F$ such that $M_F f(x) \in L^p(\mathbb{R})$, and
\[ \|f\|_{H^p(\mathbb{R})}^p = \int_\mathbb{R} |M_F f(x)|^p dx. \]

Then every $f \in H^p(\mathbb{R})$ can be written as a sum of $H^p(\mathbb{R})$ atoms:
\[ f = \sum_k \lambda_k a_k \]

in the sense of distribution:
\[ \int_\mathbb{R} f(x) \phi(x) dx = \sum_k \lambda_k a_k(x) \phi(x) dx = \sum_k \lambda_k a_k(x) \phi(x) dx \]

for any $\phi \in S(\mathbb{R}, dx)$. An $H^p(\mathbb{R})$ atom is a function $a(x)$ so that:

(i) $a(x)$ is supported in a ball $B$ in Euclid space;
(ii) $|a(x)| \leq |B|^{-1/p}$ almost everywhere;
(iii) $\int_\mathbb{R} x^n a(x) dx = 0$ for all $n \in \mathbb{Z}$ with $|n| \leq p^{-1} - 1$. Further more
\[ \|f\|_{H^p(\mathbb{R})}^p = \int_\mathbb{R} |M_F f(x)|^p dx \simeq \sum_k \lambda_k^p. \]

For more details please refer to [17].

**Proposition 2.31.** For $\beta \geq \alpha > p^{-1} - 1$,
\[ H^p_{\mu, \beta}(\mathbb{R}) = H^p_{\mu, \alpha}(\mathbb{R}). \]

For any $f \in H^p_{\mu, \beta}(\mathbb{R})$,
\[ C_2 \|f\|_{H^p_{\mu, \beta}(\mathbb{R})} \leq \|f\|_{H^p_{\mu, \alpha}(\mathbb{R})} \leq C_1 \|f\|_{H^p_{\mu, \beta}(\mathbb{R})}, \]

$C_1$ and $C_2$ are independent on $f$. 

Proof. First, with the fact $SS_\beta \subseteq SS_\alpha$ it is easy to see that

$$\tilde{H}_\mu^p(\mathbb{R}) \supseteq H_\mu^p(\mathbb{R}), \quad \|f\|_{H_\mu^p(\mathbb{R})} \leq C\|f\|_{H_\mu^p(\mathbb{R})}$$

for $\beta \geq \alpha > p^{-1} - 1$. Thus we could deduce that $f \in H_\mu^p(\mathbb{R})$, if $f \in \tilde{H}_\mu^p(\mathbb{R})$.

$\mu(x, y)$ is the measure mentioned in Proposition(2.2). Then $P(x) = \mu(x, 0)$ is a bijection on $\mathbb{R}$. Let $P^{-1}(x)$ to be the reverse map of $P(x)$. Let $g(t) = f \circ P^{-1}(t)$. Thus $g(t) \in H^p(\mathbb{R})$, if $f \in \tilde{H}_\mu^p(\mathbb{R})$. And the following equation holds:

$$\|f\|_{H_\mu^p(\mathbb{R})} = \|g\|_{H^p(\mathbb{R})}.$$ 

By Proposition(2.30), $g \in H^p(\mathbb{R})$ can be written as a sum of $H^p(\mathbb{R})$ atoms:

$$g = \sum_k \lambda_k a_k$$

in the sense of distribution. Let $b_k(x) = a_k(P(x))$. Thus functions $\{b_k(x)\}_k$ satisfy:

(i) $b_k(x)$ is supported in a ball $B_k(x_k, r_k)$;
(ii) $|b_k(x)| \leq |B_k(x_k, r_k)|^{-1/p}$ almost everywhere in $\mu$ measure;
(iii) $\int \mu(x, 0)|b_k(x)|d\mu(x) = 0$ for all $n \in \mathbb{Z}$ with $|n| \leq p^{-1} - 1$. Together with proposition(2.5),

$$\int_{\mathbb{R}} f(x)\phi(x)d\mu(x) = \int_{\mathbb{R}} \sum_k \lambda_k b_k(x)\phi(x)d\mu(x) = \sum_k \int_{\mathbb{R}} \lambda_k b_k(x)\phi(x)d\mu(x)$$

for any $\phi(x) \in S(\mathbb{R}, d\mu(x))$, and

$$\|f\|_{H_\mu^p(\mathbb{R})} = \|g\|_{H^p(\mathbb{R})} \leq \sum_k \lambda_k^p,$$

holds. For any $\psi(x) \in SS_\alpha$ satisfying $\int \psi(x)dx = 1$, we have:

$$\int_{B_\mu(x_k, 4r_k)}|b_k(x)|^p d\mu(x) \leq C \int_{B_\mu(x_k, 4r_k)} |M_\mu b_k(x)|^p d\mu(x)$$

$$\leq C \left( \int_{B_\mu(x_k, 4r_k)} |M_\mu b_k(x)|^2 d\mu(x) \right)^{p/2} \left( \int_{B_\mu(x_k, 4r_k)} 1 d\mu(x) \right)^{1-(p/2)} \leq C,$$

C is independent on $\psi$ and $b_k$. For $s \in \mathbb{Z}$, $s \leq \alpha$, by Taylor Expansion, there exist $\xi \in B_\mu(x_k, t)$ such that the following holds:

$$\psi \left( \frac{\mu(t, x)}{r} \right) = \sum_{s=0}^{[\alpha]-1} \frac{1}{s!} \psi^{(s)} \left( \frac{\mu(x_k, x)}{r} \right) \left( \frac{\mu(t, x_k)}{r} \right)^s + \frac{1}{[\alpha]!} \psi^{([\alpha])} \left( \frac{\mu(\xi, x)}{r} \right) \left( \frac{\mu(t, x_k)}{r} \right)^{[\alpha]}.$$

Let

$$P(x, x_k) = \sum_{s=0}^{[\alpha]-1} \frac{1}{s!} \psi^{(s)} \left( \frac{\mu(x_k, x)}{r} \right) \left( \frac{\mu(t, x_k)}{r} \right)^s.$$

Thus we could obtain

$$\left| P(x, x_k) - \psi \left( \frac{\mu(t, x)}{r} \right) \right| \leq \frac{1}{[\alpha]!} \left( \frac{\mu(t, x_k)}{r} \right)^{[\alpha]}.$$

Thus by Proposition(2.14) and the vanishing property of $b_k$ we could have:

$$\int_{B_\mu(x_k, 4r_k)} \left| \int b_k(t)\psi \left( \frac{\mu(t, x)}{r} \right) \frac{d\mu(t)}{r} \right|^p d\mu(x)$$

$$= \int_{B_\mu(x_k, 4r_k)} \left| \int b_k(t) \left( \psi \left( \frac{\mu(t, x)}{r} \right) - P(x, x_k) \right) \frac{d\mu(t)}{r} \right|^p d\mu(x)$$

$$\leq C \int_{B_\mu(x_k, 4r_k)} \left| \int b_k(t) r_k^{\alpha+1-p^{-1}} \right|^p d\mu(x).$$
Notice that $r > |\mu(x, x_k) - r_k|$, $\alpha > p^{-1} - 1$ and $0 < p \leq 1$, thus Formula (35) implies:

$$\int_{B_\mu(x_k, 4r_k)^c} \left| \frac{r_k^{\alpha + 1 - p^{-1}}} {r^{\alpha + 1}} \right|^p d\mu(x) \leq C. \quad (36)$$

Formula (33) and (36) implies:

$$\int_\mathbb{R} |b_k(x)|^p d\mu(x) \leq C$$

$C$ is independent on $\psi$ and $b_k$. Thus

$$\|f\|_{H^p_{\mu_\alpha}(\mathbb{R})} \leq C \sum_k \lambda_k^p \|b_k\|_{H^p_{\mu_\alpha}(\mathbb{R})} \leq C \sum_k \lambda_k^p \leq C\|f\|_{H^p_{\mu_\beta}(\mathbb{R})}.$$

Thus $f \in \tilde{H}^p_{\mu_\alpha}(\mathbb{R})$, if $f \in \tilde{H}^p_{\mu_\beta}(\mathbb{R})$. In all, we could deduce that

$$\tilde{H}^p_{\mu_\alpha}(\mathbb{R}) = \tilde{H}^p_{\mu_\beta}(\mathbb{R}).$$

Together with the fact that $H^p_{\mu_\alpha}(\mathbb{R})$ is the completion of $\tilde{H}^p_{\mu_\alpha}(\mathbb{R})$ with $H^p_{\mu_\alpha}(\mathbb{R})$ norm, and $H^p_{\mu_\beta}(\mathbb{R})$ is the completion of $\tilde{H}^p_{\mu_\beta}(\mathbb{R})$ with $H^p_{\mu_\beta}(\mathbb{R})$ norm, we could deduce that

$$H^p_{\mu_\alpha}(\mathbb{R}) = H^p_{\mu_\beta}(\mathbb{R}).$$

This proves the Proposition. \hfill \square

**Definition 2.32.** Let $\{b_k^{n, p}(x)\}$ be functions as following:
(i) $b_k^{n, p}(x)$ is supported in a ball $B_\mu(x_k, r_k)$;
(ii) $|b_k^{n, p}(x)| \leq |B_\mu(x_k, r_k)|^{-1/p}$ almost everywhere in $\mu$ measure;
(iii) $\int \mu(x, 0) m b_k^{n, p}(x) d\mu(x) = 0$ for all $m \in \mathbb{N}$ with $m \leq n$.

For $n \geq [p^{-1} - 1]$, then $A^{n, p}(\mathbb{R})$ can be defined as

$$A^{n, p}(\mathbb{R}) \triangleq \left\{ f : \int_{\mathbb{R}} f(x) \phi(x) d\mu(x) = \int_{\mathbb{R}} \sum \lambda_k b_k^{n, p}(x) \phi(x) d\mu(x) = \sum \lambda_k b_k^{n, p}(x) \phi(x) d\mu(x) \right\}$$

for any $\phi(x) \in S(\mathbb{R}, d_\mu(x))$. Setting $A^{n, p}(\mathbb{R})$ norm of $f$ by

$$\|f\|_{A^{n, p}(\mathbb{R})} = \inf \left( \sum_k |\lambda_k|^p \right)^{1/p}.$$

Thus by Proposition (2.31) we could conclude that

$$A^{n, p}(\mathbb{R}) = H^p_{\mu_\alpha}(\mathbb{R}) = H^p_{\mu_\beta}(\mathbb{R})$$

for $\beta \geq \alpha > p^{-1} - 1$ and $n \geq [p^{-1} - 1]$. Then we could see that when $n = 0$ the following proposition still holds:

**Proposition 2.33.** For $\beta \geq \gamma > p^{-1} - 1$, $0 < \gamma \leq 1$,

$$A^{0, p}(\mathbb{R}) = H^p_{\mu_\gamma}(\mathbb{R}) = H^p_{\mu_\beta}(\mathbb{R}).$$

**Proposition 2.34.** $K_2(r, x, y)$ is the kernel mentioned in Proposition (2.9), then there exists sequence $\{A^{n, p}_{x, r}(y) : A^{n, p}_{x, r}(y) \in C_c(\mathbb{R}, d_\mu y)\}_n$ satisfying the following:
(i) $A^{n, p}_{x, r}(y) = A^{n, p}_{x, r}(x)$,
(ii) $\text{supp } A^{n, p}_{x, r}(y) \subseteq B_\mu(x, nr)$,
(iii) $\lim_{n \to \infty} \|A^{n, p}_{x, r}(y) - K_2(r, x, y)\|_\infty = 0$, 

$$\text{Hardy spaces associated with the Dunkl Transform and Homogeneous type (with a Kernel)} \quad 22$$
(iv) \(0 \leq A_{x,r}^n(y) \leq C\left(1 + \frac{d_\mu(x,y)}{r}\right)^{-\gamma - 1}\), for \(r > 0, x, y \in \mathbb{R}\),

(v) For \(r > 0, x, t, z \in \mathbb{R}\), if \(\frac{d_\mu(t,z)}{r} \leq C\min\left\{1 + \frac{d_\mu(x,t)}{r}, 1 + \frac{d_\mu(x,z)}{r}\right\}\), then

\[
|A_{x,r}^n(t) - A_{x,r}^n(z)| \leq C\left(\frac{d_\mu(t,z)}{r}\right)^\gamma \left(1 + \frac{d_\mu(x,t)}{r}\right)^{-2\gamma - 1}.
\]

\(C\) is constant independent on \(A_{x,r}^n(y)\) and \(K_2(r, x, y)\).

(vi) For \(r > 0, x, y, z \in \mathbb{R}\),

\[
|A_{x,r}^n(y) - K_2(r, x, y)| \leq C\left(\frac{1}{n}\right)^{\gamma / 2} \left(1 + \frac{d_\mu(x,y)}{r}\right)^{-1 - \gamma / 2}.
\]

(vii) \(A_{x,r}^n(x) > C\), for \(r > 0, x \in \mathbb{R}\).

Proof. Choose a nonnegative function \(\phi(t) \in S(\mathbb{R}) dt\) satisfying \(\phi(t) \leq 1, \|H^\gamma \psi\|_{L^\infty} \leq C\), \(\phi(t) = \phi(-t)\), \(\text{supp}\ \phi(t) \subseteq [-1,1]\), \(\phi(t) = 1\) when \(t \in [-1/2,1/2]\). Let

\[
A_{x,r}^n(y) = K_2(r, x, y) \phi \left(\frac{\mu(x,y)}{nr}\right).
\]

Then we could easily check that sequence \(\{A_{x,r}^n(y)\}\) satisfies (i), (ii), (iii), (iv), (v), (vii).

When \(d_\mu(x,y) < \frac{n}{r^2}\),

\[
|A_{x,r}^n(y) - K_2(r, x, y)| = 0.
\]

When \(d_\mu(x,y) \geq \frac{n}{r^2}\),

\[
|A_{x,r}^n(y) - K_2(r, x, y)| \leq C\left(1 + \frac{d_\mu(x,y)}{r}\right)^{-\gamma - 1} \leq C\left(\frac{1}{n}\right)^{\gamma / 2} \left(1 + \frac{d_\mu(x,y)}{r}\right)^{-1 - \gamma / 2}.
\]

Thus we could conclude that

\[
|A_{x,r}^n(y) - K_2(r, x, y)| \leq C\left(\frac{1}{n}\right)^{\gamma / 2} \left(1 + \frac{d_\mu(x,y)}{r}\right)^{-1 - \gamma / 2}.
\]

(37)

 Proposition 2.35. If \(A_{x,r}(y) \geq 0\) and \(A_{x,r}(y) \in C_c(\mathbb{R}, d_\mu)\), then there exists sequence \(\{a_{x,r}^n(y)\}: a_{x,r}^n(y) \in C_c(\mathbb{R}, d_\mu) \cap S(\mathbb{R}, d_\mu)\) satisfying the following:

(i) \(a_{x,r}^n(y) = a_{x,r}^n(x)\),

(ii) \(\lim_{n \to \infty} \|A_{x,r}(y) - a_{x,r}^n(y)\|_{L^\infty} = 0\),

(iii) \(0 \leq a_{x,r}^n(y) \leq CA_{x,r}(y)\),

(iv) For \(r > 0, x, y, z \in \mathbb{R}\), if \(\frac{d_\mu(t,z)}{r} \leq C\min\left\{1 + \frac{d_\mu(x,t)}{r}, 1 + \frac{d_\mu(x,z)}{r}\right\}\), then

\[
|a_{x,r}^n(y) - a_{x,r}^n(z)| \leq C\left(\frac{d_\mu(t,z)}{r}\right)^\gamma \left(1 + \frac{d_\mu(x,t)}{r}\right)^{-2\gamma - 1}.
\]

\(C\) is constant independent on \(A_{x,r}(y)\) and \(a_{x,r}^n(y)\).

(v) For \(r\) small enough

\[
|a_{x,r}^n(y) - A_{x,r}(y)| \leq C\left(\frac{r}{\gamma}\right)^\gamma \left(1 + \frac{d_\mu(x,y)}{r}\right)^{-2\gamma - 1}.
\]

(vi) \(a_{x,r}^n(x) > C\), for \(r > 0, x \in \mathbb{R}\).

Proof. Let

\[
\rho(x) = \begin{cases} 
\vartheta \exp \left\{\frac{\gamma}{|x| - 1}\right\}, & \text{for } |x| < 1 \\
0, & \text{for } |x| \geq 1.
\end{cases}
\]
θ is a constant such that \( \int \rho(x)dx = 1. \) Let
\[
\frac{a^\tau_{x,r}(y)}{\tau} = \int_{\mathbb{R}} \int_{\mathbb{R}} A_{t_1,r}(t_2) \rho \left( \frac{\mu(x,t_1)}{\tau} \right) \rho \left( \frac{\mu(y,t_2)}{\tau} \right) d\mu(t_1) d\mu(t_2). 
\]
Therefore we obtain (i), (ii), (iii). Next we prove it also satisfies (iv). Notice that
\[
\rho \left( \frac{\mu(y,t_2)}{\tau} \right) = \rho \left( \frac{\mu(z,t_3)}{\tau} \right) \quad \text{and} \quad \frac{d\mu(t_2)}{\tau} = \frac{d\mu(t_3)}{\tau}
\]
hold when
\[
\frac{\mu(y,t_2)}{\tau} = \frac{\mu(z,t_3)}{\tau}.
\]
Thus the following holds:
\[
\left| a^\tau_{x,r}(y) - a^\tau_{x,r}(z) \right| = \int_{\mathbb{R}} \int_{\mathbb{R}} A_{t_1,r}(t_2) \rho \left( \frac{\mu(x,t_1)}{\tau} \right) \rho \left( \frac{\mu(y,t_2)}{\tau} \right) d\mu(t_1) d\mu(t_2)
\]
\[
- \int_{\mathbb{R}} \int_{\mathbb{R}} A_{t_1,r}(t_3) \rho \left( \frac{\mu(x,t_1)}{\tau} \right) \rho \left( \frac{\mu(z,t_3)}{\tau} \right) d\mu(t_1) d\mu(t_3)
\]
\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \left( A_{t_1,r}(t_2) - A_{t_1,r}(t_3) \right) \rho \left( \frac{\mu(x,t_1)}{\tau} \right) \rho \left( \frac{\mu(z,t_3)}{\tau} \right) d\mu(t_1) d\mu(t_3).
\]
Notice that \( \text{supp} \rho(x) \subseteq \{ x : |x| < 1 \}. \) Thus we could have \( d\mu(x,t_1) < \tau, \) \( d\mu(y,t_2) < \tau \) and \( d\mu(z,t_3) < \tau. \) If we choose \( \tau \) small enough, such that \( \frac{d\mu(x,z)}{\tau} \approx \frac{d\mu(t_2,t_3)}{\tau}, \) \( 1 + \frac{d\mu(t_2,t_3)}{\tau} \approx \left( 1 + \frac{d\mu(x,y)}{\tau} \right) \) holds. Then \( \frac{d\mu(t_2,t_3)}{\tau} \leq C \min \{ 1 + \frac{d\mu(t_2,t_1)}{\tau}, 1 + \frac{d\mu(t_3,t_1)}{\tau} \} \)

Thus together with Formula (38), we could conclude
\[
\left| a^\tau_{x,r}(y) - a^\tau_{x,r}(z) \right| \leq C \left( \frac{d\mu(t_2,t_3)}{\tau} \right)^\gamma \left( 1 + \frac{d\mu(t_1,t_2)}{\tau} \right)^{-2\gamma - 1}
\]
\[
\leq C \left( \frac{d\mu(y,z)}{\tau} \right)^\gamma \left( 1 + \frac{d\mu(x,y)}{\tau} \right)^{-2\gamma - 1}.
\]
Thus (iv) holds. We will prove (v) next. Similar to Formula (38), we could obtain:
\[
\left| a^\tau_{x,r}(y) - A_{x,r}(y) \right| = \int_{\mathbb{R}} \int_{\mathbb{R}} A_{t_1,r}(t_2) \rho \left( \frac{\mu(x,t_1)}{\tau} \right) \rho \left( \frac{\mu(y,t_2)}{\tau} \right) d\mu(t_1) d\mu(t_2)
\]
\[
- \int_{\mathbb{R}} \int_{\mathbb{R}} A_{x,r}(y) \rho \left( \frac{\mu(x,t_1)}{\tau} \right) \rho \left( \frac{\mu(z,t_3)}{\tau} \right) d\mu(t_1) d\mu(t_3)
\]
\[
= \int_{\mathbb{R}} \int_{\mathbb{R}} \left( A_{t_1,r}(t_2) - A_{x,r}(y) \right) \rho \left( \frac{\mu(x,t_1)}{\tau} \right) \rho \left( \frac{\mu(z,t_3)}{\tau} \right) d\mu(t_1) d\mu(t_3).
\]
Notice that
\[
\left| A_{t_1,r}(t_2) - A_{x,r}(y) \right| \leq C \left| A_{t_1,r}(t_2) - A_{t_1,r}(y) \right| + C \left| A_{t_1,r}(y) - A_{x,r}(y) \right|
\]
\[
\leq C \left( \frac{d\mu(t_2,y)}{\tau} \right)^\gamma \left( 1 + \frac{d\mu(t_1,y)}{\tau} \right)^{-2\gamma - 1} + C \left( \frac{d\mu(t_1,x)}{\tau} \right)^\gamma \left( 1 + \frac{d\mu(t_1,y)}{\tau} \right)^{-2\gamma - 1}
\]
\[
\leq C \left( \frac{T}{\tau} \right)^\gamma \left( 1 + \frac{d\mu(t_1,y)}{\tau} \right)^{-2\gamma - 1}.
\]
If we choose \( \tau \) small enough, we could conclude:
\[
\left( 1 + \frac{d\mu(t_1,y)}{\tau} \right) \leq \left( 1 + \frac{d\mu(x,y)}{\tau} \right).
\]
Thus we could obtain
\[
\left| A_{t_1,r}(t_2) - A_{x,r}(y) \right| \leq C \left( \frac{T}{\tau} \right)^\gamma \left( 1 + \frac{d\mu(x,y)}{\tau} \right)^{-2\gamma - 1}.
\]
Together with Formula (39), we could conclude
\[ |b_{x,r}(y) - A_{x,r}(y)| \leq C \left( \frac{r}{x} \right)^\gamma \left( 1 + \frac{d_\mu(x,y)}{r} \right)^{-2\gamma - 1}, \tag{40} \]
for \( \tau \) small enough. This proves our proposition.

**Proposition 2.36.** For \( p > \frac{1}{1 + \gamma}, i = 1, 2, f \in L^1(\mathbb{R}, \mu), 0 < \gamma \leq 1 \) and some \( \beta \) satisfying \( \beta > \gamma > p^{-1} - 1 \). Then the following inequality holds:
\[ \|f_{S(x)}\|_{L^p(\mathbb{R}, \mu)} \leq c \|f_{S(x)}\|_{L^p(\mathbb{R}, \mu)}. \]

**Proof.** We only prove the Proposition when \( i = 2 \). Assume first \( r \) and \( x \) fixed. Noticing that \( C_r(\mathbb{R}, dx) \) is dense in \( C_0(\mathbb{R}, dx) \), by Proposition (2.5) (2.6), \( C_r(\mathbb{R}, dx) \) is dense in \( C_0(\mathbb{R}, dx) \).

By the fact that \( K_2(r, x, y) = K_2(r, y, x) \) and \( \int_{\mathbb{R}} K_2(r, x, y) d_\mu(y) / r \geq m > 0 \), together with Proposition (2.34), there exists sequence \( \{ \phi_{x,r}^n(y) : \phi_{x,r}^n(y) \in S(\mathbb{R}, d_\mu(y)) \} \) satisfying the following conditions:
\[
\begin{align*}
\phi_{x,r}^n(y) &= \phi_{y,r}^n(y), \quad \phi_{x,r}^n(y) \in S(\mathbb{R}, d_\mu(y)), \\
\text{supp} \phi_{x,r}^n(y) &\subset B_\mu(x, nr), \quad \int_{\mathbb{R}} \phi_{x,r}^n(y) d_\mu(y) / r \geq m / 2 > 0 \\
L(\phi_{x,r}^n(y), \gamma) &\leq r^{-\gamma}, \phi_{x,r}^n(y) \leq C \left( 1 + \frac{d_\mu(x,y)}{r} \right)^{-\gamma - 1}, \quad \text{for } r > 0 \\
\lim_{n \to \infty} \phi_{x,r}^n(y) &= K_2(r, x, y).
\end{align*}
\]

Thus by Proposition (2.5) (2.6) and (2.35), there exists sequence \( \{ \phi_{x,r}^n(y) : \phi_{x,r}^n(y) \in S(\mathbb{R}, dy) \} \) satisfying the following conditions:
\[
\begin{align*}
\phi_{x,r}^n(y) &= \phi_{y,r}^n(y), \quad \phi_{x,r}^n(y) \in S(\mathbb{R}, dy), \\
\text{supp} \phi_{x,r}^n(y) &\subset B_\mu(x, nr), \quad \int_{\mathbb{R}} \phi_{x,r}^n(y) d_\mu(y) / r \geq m / 2 > 0 \\
L(\phi_{x,r}^n(y), \gamma) &\leq r^{-\gamma}, \phi_{x,r}^n(y) \leq C \left( 1 + \frac{d_\mu(x,y)}{r} \right)^{-\gamma - 1}, \quad \text{for } r > 0 \\
\lim_{n \to \infty} \phi_{x,r}^n(y) &= K_2(r, x, y).
\end{align*}
\]

Thus by Proposition (2.27), for \( \forall \phi(t) \in S_\beta \) satisfying \( \int_{\mathbb{R}} \phi(t) dt = 1 \), the following holds:
\[ \phi \left( \frac{\mu(x,y)}{nr} \right) = \sum_{k=0}^{\infty} \int_{\mathbb{R}} \eta^k \left( \frac{s}{nr} \right) n \phi_y^k \left( \frac{\mu(x,y) - s}{2^{-k}nr} \right) ds \frac{d\mu(y)}{2^{-k}nr} \tag{41} \]

Notice that
\[ \sup_{r > 0} \left| \int_{\mathbb{R}} f(y) \phi \left( \frac{\mu(x,y)}{nr} \right) d_\mu(y) \right| / r = n \sup_{r > 0} \left| \int_{\mathbb{R}} f(y) \phi \left( \frac{\mu(x,y)}{r} \right) d_\mu(y) \right| / r = n M_\phi f(x). \tag{42} \]

Then by Formula (41) and (42) with the fact that \( f \in L^1(\mathbb{R}, \mu) \) we could deduce
\[ nM_\phi f(x) = \sum_{r > 0} \sup_{k=0}^{\infty} \left| \int_{\mathbb{R}} f(y) \eta^k \left( \frac{s}{nr} \right) n \phi_y^k \left( \frac{\mu(x,y) - s}{2^{-k}nr} \right) ds \frac{d\mu(y)}{2^{-k}nr} \right| / r \tag{43} \]

By the fact that
\[ \sum_{k=0}^{\infty} \int_{\mathbb{R}} \eta^k \left( \frac{s}{nr} \right) \left( 1 + \frac{|s|}{2^{-k}nr} \right)^{-N} ds \frac{d\mu(y)}{nr} \leq C \sum_{k=0}^{\infty} 2^{-k}. \]
Together with Formula (43) we could obtain:

\[
M_\phi f(x) \leq C \sup_{r > 0, s \in \mathbb{R}} \int_\mathbb{R} f(y) n \phi_y^n \left( \frac{\mu(x, y) - s}{nr} \right) \left( 1 + \frac{|s|}{nr} \right)^{-N} \frac{d\mu(y)}{nr} \tag{44}
\]

\[
\leq C \left( \sup_{0 \leq s < r} + \sum_{k=1}^{+\infty} \sup_{2^{k-1} \leq s < 2^k} \right) \int_\mathbb{R} f(y) n \phi_y^n \left( \frac{\mu(x, y) - s}{nr} \right) \left( 1 + \frac{|s|}{nr} \right)^{-N} \frac{d\mu(y)}{nr} \\
\leq C \sum_{k=0}^{+\infty} 2^{-(k-1)N} \sup_{0 \leq s < 2^k} \int_\mathbb{R} f(y) \phi_y^n \left( \frac{\mu(x, y) - s}{nr} \right) \frac{d\mu(y)}{r}.
\]

Thus by Formula (44) the following holds:

\[
f_{\beta}^\circ(x) = \sup_{\phi \in \mathcal{S}_\beta} M_\phi f(x) \tag{45}
\]

\[
\leq C \sum_{k=0}^{+\infty} 2^{-(k-1)N} \sup_{0 \leq s < 2^k} \int_\mathbb{R} f(y) \phi_y^n \left( \frac{\mu(x, y) - s}{nr} \right) \frac{d\mu(y)}{r}.
\]

For any positive measure \( \mu, \mu(x, u) \) is a bijection on \( \mathbb{R} \). Then we could let \( s = \mu(x, u) \) satisfying \( d_\mu(x, u) < 2^k r \). Denote

\[
T(x, k, nr) = \sup_{0 \leq s < 2^k r} \int_\mathbb{R} f(y) \phi_y^n \left( \frac{\mu(x, y) - s}{nr} \right) \frac{d\mu(y)}{r} = \sup_{0 \leq d_\mu(x, u) < 2^k} \int_\mathbb{R} f(y) \phi_{u,r}^n(y) \frac{d\mu(y)}{r},
\]

and

\[
(F_n f)(u, r) = \int_\mathbb{R} f(y) \phi_{u,r}^n(y) \frac{d\mu(y)}{r}, \quad (K_2 f)(u, r) = \int_\mathbb{R} f(y) K_2(r, u, y) \frac{d\mu(y)}{r}.
\]

Formula (30) implies

\[
\int_\mathbb{R} |T(x, k, nr)|^p d\mu(x) \leq c (1 + 2^k) \int_\mathbb{R} |T(x, 0, nr)|^p d\mu(x). \tag{46}
\]

For \( N > 1/p \), we could obtain

\[
\int_\mathbb{R} |f_{\beta}^\circ(x)|^p d\mu(x) \leq C \int_\mathbb{R} |T(x, 0, nr)|^p d\mu(x). \tag{47}
\]

\( C \) is dependent on \( \beta \) and \( p \).

By Formula (37) and (40) (let \( \tau = \frac{1}{2m} \)), we could easily obtain

\[
|\phi_{x,r}^n(y) - K_2(r, x, y)| \leq C \left( \frac{1}{n} \right)^{\gamma/2} \left( 1 + \frac{d_\mu(x, u)}{r} \right)^{-1-\gamma/2}. \tag{48}
\]

Thus

\[
|(F_n f)(u, r) - (K_2 f)(u, r)| \leq \int_\mathbb{R} |f(y)| |\phi_{u,r}^n(y) - K_2(r, u, y)| \frac{d\mu(y)}{r} \tag{49}
\]

\[
\leq C \int_\mathbb{R} |f(y)| \left( \frac{1}{n} \right)^{\gamma/2} \left( 1 + \frac{d_\mu(u, y)}{r} \right)^{-1-\gamma/2} \frac{d\mu(y)}{r} \\
\leq C \sum_{k=0}^{+\infty} \left( 2^k \right)^{-1-\gamma/2} 2^k |M\mu f(u)| \left( \frac{1}{n} \right)^{\gamma/2} \\
\leq C |M\mu f(u)| \left( \frac{1}{n} \right)^{\gamma/2},
\]

\( C \) is dependent on \( \gamma \), \( M_\mu \) is the Hardy-Littlewood Maximal Operator. Denote

\[
\delta_n(u) = |(F_n f)(u, r) - (K_2 f)(u, r)|.
\]
Notice that $M_\mu$ is weak-(1, 1) bounded. Then the following holds for any $\alpha > 0$:

$$
\lim_{n \to +\infty} \left| \left\{ x : \delta_n(x) > \alpha \right\} \right| \mu \leq \frac{1}{\alpha} \| f \|_{L^1(\mathbb{R}, \mu)} \left( \frac{1}{n} \right)^{\gamma/2} = 0.
$$

Thus there exists a sequence $\{ n_j \} \subseteq \{ n \}$ such that the following holds:

$$
\lim_{n_j \to +\infty} (F^\mu f)(u, r) = (K_2 f)(u, r), \quad a.e. u \in \mathbb{R} \text{ in $\mu$ measure}
$$

for $f \in L^1(\mathbb{R}, \mu)$. Denote

$$
E = \{ u \in \mathbb{R} : \lim_{n_j \to +\infty} (F^\mu f)(u, r) = (K_2 f)(u, r) \}.
$$

Thus $E$ is dense in $\mathbb{R}$ could be deduced from the fact $|E^C|_\mu = 0$. Notice that for any $x_0 \in \mathbb{R}$, there exists a $(u_0, r_0)$ with $r_0 > 0$, $u_0 \in \mathbb{R}$, $d_\mu(u_0, x_0) < r_0$ such that the following holds:

$$
|(F^\mu f)(u_0, r_0)| \geq \frac{1}{2} |T(x_0, 0, n_j r_0)|.
$$

Because $(F^\mu f)(u, r_0)$ is a continuous function in $u$ variable and $E$ is dense in $\mathbb{R}$. There exists a $\tilde{u}_0 \in E$ with $d_\mu(\tilde{u}_0, x_0) < r_0$ such that

$$
|(F^\mu f)(\tilde{u}_0, r_0)| \geq \frac{1}{4} |T(x_0, 0, n_j r_0)|.
$$

Thus we could deduce that

$$
\sup_{\{ u \in E : d_\mu(u, x_0) < r \}} |(F^\mu f)(u, r)| \sup_{\{ u \in E : d_\mu(u, x_0) < r \}} |(F^\mu f)(u, r)|
$$

Formula (50) together with Fatou lemma, we could conclude:

$$
\lim_{n_j \to +\infty} \int_{\mathbb{R}} |T(x, 0, n_j r)|^p d\mu(x) \leq \lim_{n_j \to +\infty} \sup_{\{ u \in E : d_\mu(u, x_0) < r \}} |(F^\mu f)(u, r)|^p d\mu(x)
$$

$$
\leq C \int_{\mathbb{R}} \lim_{n_j \to +\infty} \sup_{\{ u \in E : d_\mu(u, x_0) < r \}} |(F^\mu f)(u, r)|^p d\mu(x)
$$

$$
\leq C \int_{\mathbb{R}} \sup_{\{ u \in E : d_\mu(u, x_0) < r \}} |(K_2 f)(u, r)|^p d\mu(x)
$$

$$
\leq C \int_{\mathbb{R}} \sup_{\{ u \in E : d_\mu(u, x_0) < r \}} |(K_2 f)(u, r)|^p d\mu(x).
$$

That is

$$
\| f^\mu_{\beta}(x) \|_{L^p(\mathbb{R}, \mu)} \leq C \| f^\mu_{2\beta}(x) \|_{L^p(\mathbb{R}, \mu)}.
$$

This proves our proposition.

\medskip

**Proposition 2.37.** $K_2(x, y, z)$ is a kernel in Proposition (2.9). If $K_2(x, y, z)$ satisfies

$$
|K_2(x, t, u) - K_2(x, z, u)| \leq C \left( \frac{d_\mu(t, z)}{r} \right)^\gamma \left( \frac{d_\mu(x, t)}{r} \right)^{-2\gamma - 1},
$$

for $r > 0, x, t, u \in \mathbb{R}, \frac{d_\mu(t, z)}{r} \leq C_3 \min \{ 1 + \frac{d_\mu(t, x)}{r}, 1 + \frac{d_\mu(x, z)}{r} \}$. Then for any fixed $\alpha$ with $0 < \alpha < \gamma$, the following holds:

$$
0 \leq |K_2(r, a, y) - K_2(r, b, y)| \leq C \left( \frac{d_\mu(a, b)}{r} \right)^\alpha \left( \frac{d_\mu(a, y)}{r} \right)^{-(\gamma - \alpha) - 1}, \text{ for } r > 0, x, t \in \mathbb{R}
$$

and

$$
|\langle (K_2(r, a, y) - K_2(r, b, y)) - (K_2(r, a, z) - K_2(r, b, z)) \rangle|
$$

$$
\leq C \left( \frac{d_\mu(a, b)}{r} \right)^\alpha \left( \frac{d_\mu(y, z)}{r} \right)^{\gamma - \alpha} \left( \frac{d_\mu(a, y)}{r} \right)^{-2(\gamma - \alpha) - 1},
$$

for $d_\mu(a, b) \leq r$, $\frac{d_\mu(\mu, z)}{r} \leq C_3 \min \{ 1 + \frac{d_\mu(a, \mu)}{r}, 1 + \frac{d_\mu(a, \mu)}{r} \}$.\hfill \square
Proof. First, we consider the case when 
\[ d_\mu(a, b) \leq d_\mu(y, z). \]
From the fact that \( d_\mu(a, b) \lesssim r, \frac{d_\mu(y, z)}{r} \leq C_3 \min\{1 + \frac{d_\mu(a, y)}{r}, 1 + \frac{d_\mu(a, z)}{r}\}, \) the following relations could be obtained:
\[
1 + \frac{d_\mu(a, y)}{r} \approx 1 + \frac{d_\mu(b, y)}{r}, \quad 1 + \frac{d_\mu(a, z)}{r} \approx 1 + \frac{d_\mu(b, z)}{r}, \quad \text{and} \quad 1 + \frac{d_\mu(a, z)}{r} \approx 1 + \frac{d_\mu(a, y)}{r}. \tag{51}
\]
Notice that
\[
K_2(r, x, y) = K_2(r, y, x).
\]
Then
\[
|K_2(r, a, y) - K_2(r, b, y)| \leq C \left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1}
\]
\[
\leq C \left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-\gamma - \alpha} \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-\gamma - \alpha - 1}
\]
\[
\leq C \left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-\gamma - \alpha - 1}.
\]
Also we could obtain
\[
|K_2(r, a, y) - K_2(r, b, y)| \leq C \left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1},
\]
and
\[
|K_2(r, a, z) - K_2(r, b, z)| \leq C \left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, z)}{r} \right)^{-2\gamma - 1}.
\]
Together with Formula (51), we could conclude
\[
|(K_2(r, a, y) - K_2(r, b, y)) - (K_2(r, a, z) - K_2(r, b, z))| \leq C \left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1}.
\]
By the fact \( d_\mu(a, b) \leq d_\mu(y, z) \) and \( 1 \lesssim 1 + \frac{d_\mu(a, y)}{r} \), we could obtain:
\[
\left( \frac{d_\mu(a, b)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1} \lesssim \left( \frac{d_\mu(a, b)}{r} \right)^\alpha \left( \frac{d_\mu(y, z)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{\gamma - \alpha - 1} \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2(\gamma - \alpha) - 1}
\]
Then for \( d_\mu(a, b) \leq d_\mu(y, z), \)
\[
|(K_2(r, a, y) - K_2(r, b, y)) - (K_2(r, a, z) - K_2(r, b, z))| \leq C \left( \frac{d_\mu(a, b)}{r} \right)^\alpha \left( \frac{d_\mu(y, z)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{\gamma - \alpha - 1} \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2(\gamma - \alpha) - 1}
\]
holds. In a similar way we will obtain the Formula (53) for the case when \( d_\mu(a, b) \geq d_\mu(y, z) \). Notice that by Formula (51)
\[
|K_2(r, a, y) - K_2(r, a, z)| \leq C \left( \frac{d_\mu(y, z)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1},
\]
and
\[
|K_2(r, b, y) - K_2(r, b, z)| \leq C \left( \frac{d_\mu(y, z)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(b, y)}{r} \right)^{-2\gamma - 1}
\]
\[
\leq C \left( \frac{d_\mu(y, z)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1}.
\]
Then
\[
|(K_2(r, a, y) - K_2(r, b, y)) - (K_2(r, a, z) - K_2(r, b, z))| \leq C \left( \frac{d_\mu(y, z)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1}.
\]
By the fact $d_\mu(a, b) \geq d_\mu(y, z)$ and $1 \lesssim 1 + \frac{d_\mu(a, y)}{r}$, the following holds:

$$
\left( \frac{d_\mu(y, z)}{r} \right)^{\gamma} \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2\gamma - 1} \lesssim \left( \frac{d_\mu(a, b)}{r} \right)^{\alpha} \left( \frac{d_\mu(y, z)}{r} \right)^{\gamma - \alpha} \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2(\gamma - \alpha) - 1}.
$$

Then for $d_\mu(a, b) \geq d_\mu(y, z)$,

$$
|\{K_2(r, a, y) - K_2(r, b, y)\} - (K_2(r, a, z) - K_2(r, b, z))| \leq C \left( \frac{d_\mu(a, b)}{r} \right)^{\alpha} \left( \frac{d_\mu(y, z)}{r} \right)^{\gamma - \alpha} \left( 1 + \frac{d_\mu(a, y)}{r} \right)^{-2(\gamma - \alpha) - 1}.
$$

holds. Formula(52)(53)(54) imply the Proposition.

\[\square\]

**Proposition 2.38.** If $K_2(r, x, y)$ is a kernel satisfying Proposition(2.9). For $p > \frac{1}{1 + \gamma}$, disc $i = 1, 2$, the following holds:

$$
\|f_{2\gamma}^*(x)\|_{L^p(\mathbb{R}, \mu)} \leq C \|f_{2\gamma}^*(x)\|_{L^p(\mathbb{R}, \mu)},
$$

where $C$ is dependent on $p$ and $\gamma$.

**Proof.** We only prove the proposition when $i = 2$. We will prove the proposition follow the above Proposition(2.36). For any fixed $\alpha$ satisfying $0 < \alpha < \gamma$ and $p > \frac{1}{1 + \gamma - \alpha}$. Let

$$
F = \{ x : f_{2\gamma}^*(x) \leq \sigma f_{2\gamma}^*(x) \}.
$$

By Proposition(2.36) and (2.31), we could observe that

$$
\int_{F} |f_{2\gamma}^*(x)|^p d\mu(x) \leq C \frac{1}{\sigma^p} \int_{F} |f_{2\gamma}^*(x)|^p d\mu(x) \leq C \frac{1}{\sigma^p} \int_{\mathbb{R}} |f_{2\gamma}^*(x)|^p d\mu(x) \leq C \frac{1}{\sigma^p} \int \|f_{2\gamma}^*(x)\|^p d\mu(x). (55)
$$

Choose $\sigma^p \geq 2C$, we could obtain

$$
\int_{\mathbb{R}} |f_{2\gamma}^*(x)|^p d\mu(x) \leq \int_{F} |f_{2\gamma}^*(x)|^p d\mu(x). (56)
$$

Denote

$$
Df(x) = \sup_{r > 0} \left| \int_{\mathbb{R}} f(t) K_2(r, x, t) \frac{d\mu(t)}{r} \right|, \quad F(x, r) = \int_{\mathbb{R}} f(t) K_2(r, x, t) \frac{d\mu(t)}{r}.
$$

Next, we will show that for any $q > 0$,

$$
f_{2\gamma}^*(x) \leq C \|M_{\mu}(Df)^q(x)\|^{1/q} \quad \text{for} \ x \in F. (57)
$$

$M_{\mu}$ is the Hardy-Littlewood maximal operator. Fix any $x_0 \in F$, then there exists $(u_0, r_0)$ satisfying $d_\mu(u_0, x_0) < r_0$ such that the following inequality holds:

$$
F(u_0, r_0) > \frac{1}{2} f_{2\gamma}^*(x_0). (58)
$$

Choose $\delta < 1$ small enough and $u$ satisfying $d_\mu(u, u_0) < \delta r_0$, then we could obtain

$$
|F(u, r_0) - F(u_0, r_0)| = \left| \int_{\mathbb{R}} f(y) K_2(r_0, u, y) d\mu(y) \right|/r_0 - \left| \int_{\mathbb{R}} f(y) K_2(r_0, u_0, y) d\mu(y) \right|/r_0 \
\leq \left| \int_{\mathbb{R}} f(y) (K_2(r_0, u, y) - K_2(r_0, u_0, y)) d\mu(y) \right|/r_0.
$$

We could see $(K_2(r_0, u, y) - K_2(r_0, u_0, y))$ as a new kernel. By Proposition(2.37)(2.15), we could obtain:

$$
|F(u, r_0) - F(u_0, r_0)| \leq C\delta^\alpha f_{2\gamma}^*(x_0) \leq C\delta^\alpha \sigma f_{2\gamma}^*(x_0) \quad \text{for} \ x_0 \in F.
$$

Taking $\delta$ small enough such that $C\delta^\alpha \sigma \leq 1/4$, we obtain

$$
F(u, r_0) \geq \frac{1}{4} f_{2\gamma}^*(x_0) \quad \text{for} \ u \in B_\mu(u_0, \delta r_0).
$$
Thus the following inequality holds: for any \( x_0 \in F \),
\[
\left| f_{\gamma,\nu}^\alpha(x_0) \right|^q \leq \frac{1}{|B_\nu(x_0, \delta r_0)|} \int_{B_\nu(x_0, \delta r_0)} 4^q |F(u, r_0)|^q d\mu(u) \\
\leq \frac{B_\nu(x_0, (1 + \delta) r_0)}{B_\nu(x_0, \delta r)} \left\| \frac{1}{B_\nu(x_0, (1 + \delta) r_0)} \right\| \int_{B_\nu(x_0, (1 + \delta) r_0)} 4^q |F(u, r_0)|^q d\mu(u) \\
\leq \frac{1 + \delta}{\delta} \frac{1}{B_\nu(x_0, (1 + \delta) r_0)} \int_{B_\nu(x_0, (1 + \delta) r_0)} 4^q |F(u, r_0)|^q d\mu(u) \\
\leq CM_\nu[(DF)^q](x_0)
\]

C is independent on \( x_0 \). Finally, using the maximal theorem for \( M_\nu \) when \( q < p \) leads to
\[
\int_{F_\nu} \left| f_{2\nu}^\alpha(x) d\mu(x) \right|^p dx \leq C \int_{\mathbb{R}} \{M_\nu[(DF)^q](x)\}^{p/q} d\mu(x) \leq C \int_{\mathbb{R}} |f_\nu^\alpha(x)|^p d\mu(x)
\]
which combined with Formula(55) leads to for any fixed \( \alpha \) satisfying \( 0 < \alpha < \gamma \) and \( p > \frac{1}{1 + \gamma - \alpha} \)
\[
\| f_{\gamma,\nu}^\alpha(x) \|_{L^p(\mathbb{R}, \mu)} \leq C \| f_\nu^\alpha(x) \|_{L^p(\mathbb{R}, \mu)}
\]

C is dependent on \( p \) and \( \alpha \). Next we will remove the number \( \alpha \). Notice that for any \( p > \frac{1}{1 + \gamma} \), let \( p_0 = \frac{1}{2} \left( p + \frac{1}{1 + \gamma} \right) \) satisfying \( p > p_0 > \frac{1}{1 + \gamma} \). Let \( \alpha = 1 + \gamma - \frac{1}{p_0} \), together with the above inequality, then we could obtain the following inequality holds for \( p > \frac{1}{1 + \gamma} \)
\[
\| f_{\gamma,\nu}^\alpha(x) \|_{L^p(\mathbb{R}, \mu)} \leq C \| f_\nu^\alpha(x) \|_{L^p(\mathbb{R}, \mu)}
\]

C is dependent on \( p \) and \( \gamma \). This proves the Proposition.

At last We will prove the following Theorem:

**Theorem 2.39.** For \( \frac{1}{1 + \gamma} < p \leq 1 \), \( 0 < \gamma \leq 1 \), \( f \in L^1(\mathbb{R}, \mu) \), there exists \( \beta > 0 \), such that the following conditions are equivalent:
(i) \( f_{2\beta}^\alpha \in L^p(\mathbb{R}, \mu) \).
(ii) There is a \( f(x) \in SS_\beta \) satisfying \( \int f(x) dx \neq 0 \) so that \( M_{\beta,\beta} f(x) \in L^p(\mathbb{R}, \mu) \).
(iii) \( f_{\gamma,\nu}^\alpha(x) = \sup_{d_\nu(x,y) < r} |F(r, y, f)| \in L^p(\mathbb{R}, \mu) \) for \( i = 1, 2 \).
(iv) \( f_\nu^\alpha(x) = \sup_{r > 0} |F(r, x, f)| \in L^p(\mathbb{R}, \mu) \) for \( i = 1, 2 \).
(v) \( f_\gamma^\alpha \in L^p(\mathbb{R}, \gamma) \).

**Proof.** **Proof of the Theorem(2.39)**
(i) \( \Rightarrow (ii) \) is obvious.
(ii) \( \Rightarrow (i) \) is deduced from Proposition(2.28) and (2.29).
(iii) \( \Rightarrow (v) \) is deduced from Proposition(2.31).
(iv) \( \Rightarrow (i) \) is deduced from Proposition(2.36)
(v) \( \Rightarrow (ii) \) is deduced from Proposition(2.38)
(iii) \( \Rightarrow (iv) \) is obvious.
(v) \( \Rightarrow (iii) \) is deduced from Proposition(2.15) This proves the theorem.

Let \( \tilde{d}(x,y) \) be another distance satisfying \( C_2d_\mu(x,y) \leq \tilde{d}(x,y) \leq C_1d_\mu(x,y) \). \( C_1, C_2 \) are constants independent on \( x, y \in \mathbb{R} \). \( \phi \) is a function satisfying \( L(\phi, \gamma) \leq r^{-\gamma} \) and supp \( \phi \subset B_\mu(x, r) \):
\[
\sup_{x,y \in \mathbb{R}, x \neq y} |\phi(x) - \phi(y)|/d_\mu(x,y)^\alpha \leq r^{-\gamma}.
\]

Obviously we could have:
\[
\sup_{x,y \in \mathbb{R}, x \neq y} |\phi(x) - \phi(y)|/\tilde{d}(x,y)^\alpha \leq r^{-\gamma} \leq C\tilde{r}^{-\gamma}, \tag{59}
\]
and \( \text{supp} \phi \subset B(x, C \tilde{r}) \). \( C \) is independent on \( x, y \in \mathbb{R} \). We denote \( B(x, \tilde{r}) = \{ y : \tilde{d}(x, y) < \tilde{r} \} \), and \( \tilde{L}(\phi, \gamma) \) as following:

\[
\tilde{L}(\phi, \gamma) = \sup_{x, y \in \mathbb{R}, x \neq y} |\phi(x) - \phi(y)|/\tilde{d}(x, y)\gamma. \tag{60}
\]

Let \( f_\gamma^*[x] \) be defined as:

\[
f_\gamma^*[x] = \sup_{\phi, \tilde{r}} \left\{ \int_{\mathbb{R}} f(y)\phi(y)d\mu(y) \middle| \tilde{r} > 0, \text{supp} \phi \subset B(x, \tilde{r}), \tilde{L}(\phi, \gamma) \leq \tilde{r}^{-\gamma}, \| \phi \|_{L^\infty} \leq 1 \right\}
\]

Obviously we could have

\[ C_3 f_\gamma^*[x] \leq f_\gamma^*[x] \leq C_4 f_\gamma^*[x]. \]

\( C_3, C_4 \) are constants independent on \( x \in \mathbb{R} \) and \( f \). Thus we have the following Corollary:

**Corollary 2.40.** Let \( f_\gamma^*[x] \) and \( f_\gamma^*[x] \) to be defined as formula(6) and (60), then

\[
f_\gamma^*[x] \equiv f_\gamma^*[x].
\]

In a similar way we could define \( f_\gamma^*[x] \) as \( f_\gamma^*[x] = \sup_{\tilde{r} > 0, \tilde{d}(s, x) \leq \tilde{r}} |F_\gamma(\tilde{r}, s, f)|. \) Then by Proposition(2.24), we could obtain:

**Corollary 2.41.** For \( 0 < p < \infty, i = 1, 2, \)

\[
\| f_\gamma^*[x] \|_{L^p_x} \geq \| f_\gamma^*[x] \|_{L^p_x}
\]

From above Theorem(2.39) \( H^p_x(\mathbb{R}) \) could be defined as:

**Definition 2.42 ( \( \tilde{H}^p_x(\mathbb{R}) \) and \( H^p_x(\mathbb{R}) \) spaces for \( p > 0 \). ).** Let \( \tilde{H}^p_x(\mathbb{R}) \) denote

\[
\tilde{H}^p_x(\mathbb{R}) \triangleq \{ g \in L^1(\mathbb{R}, \mu) : g_\gamma^*(x) \in L^p(\mathbb{R}, \mu), \text{for } \alpha > p^{-1} - 1 \}.
\]

And its norm is is given by

\[
\| g \|_{\tilde{H}^p_x(\mathbb{R})} = \int_{\mathbb{R}} |g_\gamma^*(x)|^p d\mu(x).
\]

Then \( H^p_x(\mathbb{R}) \) space is the completion of \( \tilde{H}^p_x(\mathbb{R}) \) with its \( \| \cdot \|_{H^p_x(\mathbb{R})} \) norm.

**Remark 2.43.** In fact, Theorem(2.39) still holds if we replace the (iv) \( K_2(r, x, y) = K_2(r, y, x) \) in Proposition(2.9) with:

\[
|K_2(r, t, x) - K_2(r, z, x)| \leq C \left( \frac{d_\mu(t, z)}{r} \right)^\gamma \left( 1 + \frac{d_\mu(x, t)}{r} \right)^{-2\gamma-1}
\]

for \( r > 0, x, t, z \in \mathbb{R}, \frac{d_\mu(t, z)}{r} \leq C_3 \min\{ 1 + \frac{d_\mu(x, t)}{r}, 1 + \frac{d_\mu(x, z)}{r} \} \). \( C \) is a constant in dependent on \( x, t, z, \gamma \).

## 3 Chapter II Hardy spaces on the Dunkl setting

### 3.1 Real Parts of a function in \( H^p_x(\mathbb{R}^2_+) \) and associated maximal functions

**Proposition 3.1.** Let \( \{ f_k \}_k \ (k \in \mathbb{N}) \) be a sequence of \( \lambda \)-analytic functions on the set \( S \). If \( \sum_k |\lambda_k f_k|, \sum_k |\lambda_k \partial_x f_k| \) and \( \sum_k |\lambda_k \partial_y f_k| \ (k \in \mathbb{N}) \) converges uniformly on the set \( S \), then \( \sum_k \lambda_k f_k \ (k \in \mathbb{N}) \) is a \( \lambda \)-analytic function on the set \( S \).

**Proof.** We denote \( D_z \) as

\[
D_z = \frac{1}{2}(D_x + i \partial_y).
\]
A function $F(z) = F(x, y) = u(x, y) + iv(x, y)$ is a $\lambda$-analytic function if and only if $F(z)$ satisfies the following $\lambda$-Cauchy-Riemann equations:

\[
\begin{align*}
D_x u - \partial_y v &= 0, \\
\partial_y u + D_x v &= 0.
\end{align*}
\]

Then the $\lambda$-Cauchy-Riemann equations could be replaced by $D_z F(z) = 0$. Thus a function $F(z) = F(x, y) = u(x, y) + iv(x, y)$ is a $\lambda$-analytic function if and only if

\[
D_z F(z) = 0.
\]

Notice that $\sum_k |\lambda_k f_k|$, $\sum_k |\lambda_k \partial_x f_k|$ $(k \in \mathbb{N})$ and $\sum_k |\lambda_k \partial_y f_k|$ $(k \in \mathbb{N})$ converges uniformly on the set $S$. Thus we could have

\[
\sum_k \lambda_k \partial_y f_k = \partial_y \left( \sum_k \lambda_k f_k \right)
\]
and

\[
\sum_k \lambda_k \partial_x f_k = \partial_x \left( \sum_k \lambda_k f_k \right)
\]
on the set $S$. Thus $\sum_k \lambda_k f_k$ $(k \in \mathbb{N})$ is a $\lambda$-analytic function on the set $S$. □

**Proposition 3.2.** In this paper $\Omega$ is a bounded domain symmetric in $x$: $(x, y) \in \Omega \Rightarrow (-x, y) \in \Omega$. $F(z) = F(x, y) = u(x, y) + iv(x, y)$ is a $\lambda$-analytic function, $u$ and $v$ are real $C^2$ functions satisfying $\lambda$-Cauchy-Riemann equations (62). $u(x, y)$ is an odd or even function in $x$, then

\[
\int_{\partial \Omega} F^2(z)|x|^{2\lambda}dz = 0
\]

**Proof.** $F(z) = F(x, y) = u(x, y) + iv(x, y)$ is a $\lambda$-analytic function, $\tilde{v} = v(-x)$, $u$ and $v$ are real $C^2$ functions satisfying $\lambda$-Cauchy-Riemann equations:

\[
\begin{align*}
D_x u - \partial_y v &= 0, \\
\partial_y u + D_x v &= 0.
\end{align*}
\]

Then $F^2(z) = u^2 - v^2 + 2uvi$ is usually not an analytic function. If $u(x, y)$ is even in $x$ and $v(x, y)$ is odd in $x$, we obtain

\[
\begin{align*}
D_x (u^2 - v^2) - \partial_y (2uv) &= \frac{4\lambda}{x} v^2, \\
\partial_y (u^2 - v^2) + D_x (2uv) &= 0.
\end{align*}
\]

If $u(x, y)$ is odd in $x$ and $v(x, y)$ is even in $x$, the following equations could be achieved:

\[
\begin{align*}
D_x (u^2 - v^2) - \partial_y (2uv) &= -\frac{4\lambda}{x} u^2, \\
\partial_y (u^2 - v^2) + D_x (2uv) &= 0.
\end{align*}
\]

Denote $\Omega^+ = \{(x, y) | (x, y) \in \Omega, x \geq 0\}$, $\Omega^- = \{(x, y) | (x, y) \in \Omega, x \leq 0\}$, and $\tilde{v} = v(-x)$.
From stokes theorem we could obtain

\[
\int_{\partial \Omega^+} F(z)|x|^{2\lambda}dz = \int_{\partial \Omega^+} (u + iv)|x|^{2\lambda}(dx + idy)
\]
\[
= \int_{\Omega^+} \{-u_y + v_x + 2v(\lambda/x) + i(u_x + 2u(\lambda/x) - v_y)|x|^{2\lambda}(dx \wedge dy),
\]
and

\[
\int_{\partial \Omega^-} F(z)|x|^{2\lambda}dz = \int_{\partial \Omega^-} (u + iv)|x|^{2\lambda}(dx + idy)
\]
\[
= \int_{\Omega^-} \{-u_y + v_x + 2v(\lambda/x) + i(u_x + 2u(\lambda/x) - v_y)|x|^{2\lambda}(dx \wedge dy).
\]

As $\Omega$ is a bounded domain symmetric in $x$, then Equation (66) and Equation (65) allow us to write:
\[ \int_{\partial \Omega} F(z) |x|^{2\lambda} dz = \int_{\partial \Omega^+} F(z) |x|^{2\lambda} dz + \int_{\partial \Omega^-} F(z) |x|^{2\lambda} dz \]
\[ = \int_{\Omega} \{-(u_y + D_x v) + i(D_x u - v_y)\} \ |x|^{2\lambda} (dx \wedge dy). \]  

(67)

If \( u(x, y) \) is even in \( x \) and \( v(x, y) \) is odd in \( x \), by (63) and (67), we could have

\[ \int_{\partial \Omega} F^2(z) |x|^{2\lambda} dz = \int_{\Omega} \frac{4\lambda}{x} x^2 |x|^{2\lambda} (dx \wedge dy). \]  

(68)

If \( u(x, y) \) is odd in \( x \) and \( v(x, y) \) is even in \( x \), Formulas (64) and (67) imply that

\[ \int_{\partial \Omega} F^2(z) |x|^{2\lambda} dz = \int_{\Omega} \frac{4\lambda}{x} x^2 |x|^{2\lambda} (dx \wedge dy). \]  

(69)

As \( \Omega \) is a bounded domain symmetric in \( x \), then Formula (68) and (69) imply that

\[ \int_{\partial \Omega} F^2(z) |x|^{2\lambda} dz = 0. \]

\[ \square \]

**Proposition 3.3. (113)**

If \( \frac{2\lambda}{p+1} < p < l \leq +\infty, \delta = \frac{1}{p} - \frac{1}{l}, \) and \( F(x,y) \in H^p_{\lambda}(\mathbb{R}^2_+) \), \( p \leq k < \infty, \) then

(i)

\[ \left( \int_0^{+\infty} y^{\delta(1+2\lambda)-1} \left( \int_{\mathbb{R}} |F(x,y)|^l |x|^{2\lambda} dx \right)^{\frac{1}{l}} dy \right)^{\frac{1}{\delta}} \leq c\|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)}. \]  

(70)

(ii)

\[ \left( \int_{\mathbb{R}} |F(x,y)|^l |x|^{2\lambda} dx \right)^{\frac{1}{l}} \leq cy^{-(1/p-1/l)(1+2\lambda)} \|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)}. \]  

(71)

(iii) If \( 1 \leq p < \infty \) and \( F(x,y) = u(x,y) + iv(x,y) \in H^p_{\lambda}(\mathbb{R}^2_+) \), then \( F(x,y) \) is the \( \lambda \)-Poisson integrals of its boundary values \( F(x) \), and \( F(x) \in L^p_{\lambda}(\mathbb{R}) \).

**Proposition 3.4.**

\( F(x,y) = u(x,y) + iv(x,y) \in H^p_{\lambda}(\mathbb{R}^2_+) \), \( \frac{2\lambda}{p+1} < p \leq 1. \) \( \forall t > 0 \), let \( t \) to be a fixed number. \( F_t(x,y) = u(x,y) + iv(x,y) \), \( u(x,y) \) is an even or odd function in \( x \). \( u_{-t}^+(x) = \sup_{|y-x|<t} u(y) \). Let \( E_t = \{ x \in \mathbb{R} : u_{-t}^+(x) > \sigma \} = \cup_i I_i \). \( \{ I_i \} \) are the open disjoint Euclidean intervals of the open set \( E_t \). Let the tent \( T(I_i) \) defined as:

\[ T(I_i) = \{(x,t) : |x-x_i| \leq r_i - t\}, \]

where \( x_i \) is the center of the interval \( I_i \), \( r_i \) is the radius of the interval \( I_i \):

\[ I_i = (x_i - r_i, x_i + r_i). \]

Let \( \Gamma = \cup_i (\partial T(I_i) \setminus E_t) \cup (\mathbb{R} \setminus E_t) \). Then we could have

\[ \int_{\Gamma} F_t(x,y)^2 |x|^{2\lambda} dz = 0. \]

**Proof.** \( F(x,y) = u(x,y) + iv(x,y) \in H^p_{\lambda}(\mathbb{R}^2_+) \), \( \frac{2\lambda}{p+1} < p \leq 1. \) \( \text{Re}(x,y) \) is an even or odd function in \( x \). If we let \( F_t(x,y) = F(x,y) + t \). And let \( t > 0 \) to be fixed. Then \( F_t \) is a \( \lambda \)-analytic function, and \( F_t(x,y) \) is continuous. Taking \( l = 2, k = 2, \delta = 1/p - 1/2, \) by Formula (70), we could obtain

\[ \left( \int_0^{+\infty} \left( \int_{\mathbb{R}} |F_t(x,y)|^2 |x|^{2\lambda} dx \right) dy \right)^{\frac{1}{2}} \leq c(1/2-1/p)(2\lambda+1)+1/2 \|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)}. \]  

(72)
Let
\[ g_t(x) = \int_{\mathbb{R}} \left( |F_t(x, y)|^2 + |F_t(-x, y)|^2 \right) |x|^{2\lambda} dy. \]

Then
\[ \left( \int_{\mathbb{R}} g_t(x) dx \right)^{\frac{1}{2}} \leq c t^{(1/2-1/p)(2\lambda+1) + 1/2} \|F\|_{H^\delta_N}^{\lambda}. \]

We could know that for \( \forall \varepsilon > 0, \exists N(\varepsilon) \), satisfying
\[ \int_{N(\varepsilon)} g_t(x) dx < \varepsilon/2. \]

Thus \( \exists N \), satisfying \( g_t(N) < \varepsilon/2 \):
\[ \int_{\mathbb{R}} \left( |F_t(N, y)|^2 + |F_t(-N, y)|^2 \right) |x|^{2\lambda} dy < \varepsilon/2. \] (73)

We could deduce from (71) that \( \forall \varepsilon > 0, \exists N_2 \) such that the following holds:
\[ \int_{\mathbb{R}} |F_t(x, N_2)|^2 |x|^{2\lambda} dx < \varepsilon/2. \] (74)

\[ F_t(x, y) = u_t(x, y) + iv_t(x, y), \] \( u_t(x, y) \) is an even or odd function in \( x \), ie \( u_t(x, y) = u_t(-x, y) \).

We define \( u_t(\sigma) = \sup_{|x|<\varepsilon} u_t(y) \). Let \( E_\sigma = \{ x \in \mathbb{R} : u_t(\sigma) > \sigma \} = \bigcup I_i \), \{ \( I_i \) \} are the open disjoint Euclidean intervals of the open set \( E_\sigma \): \( I_i \cap I_j = \emptyset \) when \( i \neq j \). Let \( I_i = (x_i - r_i, x_i + r_i) \), then a Tent is defined as \( T(I_i) = \{(x, t) : |x - x_i| \leq r_i - t \} \). Let \( \Gamma = \bigcup (\partial T(I_i) \setminus E_\sigma) \cup (\mathbb{R} \setminus E_\sigma) \), and \( \Gamma \setminus = (\bigcup (\partial T(I_i) \setminus E_\sigma) \cup (\mathbb{R} \setminus E_\sigma)) \setminus (-N, N) \). \( \text{Re}F(x, y) \) is an even or odd function in \( x \). \( \Omega \) is the domain symmetric in \( x \): \( \Omega = \{(x, y) : -N \leq x \leq N, y_1 \leq y \leq y_1, (x, y_1) \in \Gamma \} \). Then by (61) we have:
\[
\left| \int_{\partial \Omega} F_t(x, y)^2 |x|^{2\lambda} dz \right|
= \left| \int_{I_N} F_t(x, y)^2 |x|^{2\lambda} dz + \int_{N_2} F_t(N, y)^2 |x|^{2\lambda} dy \right.
+ \left. \int_{-N}^{N} F_t(x, N_2)^2 |x|^{2\lambda} dx + \int_{N_2}^{0} F_t(-N, y)^2 |x|^{2\lambda} dy \right|
\lesssim \varepsilon
\]
Together with (73), (74), we could deduce that:
\[ \left| \int_{N} F_t(x, y)^2 |x|^{2\lambda} dz \right| \lesssim \varepsilon. \]

By the arbitrariness of \( \varepsilon > 0 \), we could deduce that:
\[ \int_{\Gamma} F_t(x, y)^2 |x|^{2\lambda} dz = 0. \] (75)

This proves the Proposition.
Proposition 3.5. \( F(x,y) = u(x,y) + iv(x,y) \in H^p_\lambda (\mathbb{R}^d), \frac{2\lambda}{d+1} < p \leq 1, \forall t > 0, \) let \( t \) to be a fixed number, and \( F_t(x,y) = u_t(x,y) + iv_t(x,y) = F(x,y + t). \) Suppose that \( u_t(x,y) \) is an even or odd function in \( x. \) Denote \( u_t(x) = u(x,t) \) and \( v_t(x) = v(x,t). \) Let \( E_\sigma = \{ x \in \mathbb{R} : u_\sigma(x) > \sigma \}. \) Then we could have

\[
| \{ x \in \mathbb{R} : |v_t(x)| \geq \sigma \} | \lambda \leq 3|E_\sigma| + \frac{2}{\sigma^2} \int_0^\sigma s |E_\lambda| ds. \quad (76)
\]

Proof. From Equation (75), we have

\[
\int (u_t(x,y) + iv_t(x,y))^2 |x|^{2\lambda} dz = 0.
\]

Then we take the real part of the above equation to get

\[
Re \int (u_t(x,y) + iv_t(x,y))^2 |x|^{2\lambda} dz = 0.
\]

That is

\[
\int (u_t^2(x,y) - v_t^2(x,y)) |x|^{2\lambda} dx - 2u_t(x,y)v_t(x,y) |x|^{2\lambda} dy = 0.
\]

\[
\Rightarrow 0 = \int (u_t^2(x) - v_t^2(x)) |x|^{2\lambda} dx + \int_{\cup_i (\partial T_i) \setminus E_\sigma} (u_t^2(x,y) - v_t^2(x,y)) |x|^{2\lambda} dx
\]

\[
- \int_{\cup_i (\partial T_i) \setminus E_\sigma} 2u_t(x,y)v_t(x,y) |x|^{2\lambda} dy.
\]

Together with inequality:

\[
\left| \int_{\cup_i (\partial T_i) \setminus E_\sigma} 2u_t(x,y)v_t(x,y) |x|^{2\lambda} dy \right| \leq \int_{\cup_i (\partial T_i) \setminus E_\sigma} (u_t^2(x,y) + v_t^2(x,y)) |x|^{2\lambda} dy.
\]

Then we could obtain:

\[
\int_{\mathbb{R} \setminus E_\sigma} v_t^2(x) |x|^{2\lambda} dx \leq \int_{\mathbb{R} \setminus E_\sigma} u_t^2(x) |x|^{2\lambda} dx + \int_{\cup_i (\partial T_i) \setminus E_\sigma} 2u_t^2(x,y) |x|^{2\lambda} dx.
\]

Thus

\[
\int_{\mathbb{R} \setminus E_\sigma} v_t^2(x) |x|^{2\lambda} dx \leq \int_{\mathbb{R} \setminus E_\sigma} ((u_t)_\sigma^2(x)) |x|^{2\lambda} dx + 2\sigma^2 |E_\sigma| \lambda. \quad (77)
\]

We could also notice that:

\[
\int_{\mathbb{R} \setminus E_\sigma} ((u_t)_\sigma^2(x)) |x|^{2\lambda} dx = 2 \int_0^{+\infty} s \{ x \in \mathbb{R} \setminus E_\sigma : (u_t)_\sigma^2(x) > s \} \lambda ds
\]

\[
= 2 \int_0^\sigma s |E_\lambda| ds. \quad (78)
\]

Then by (77) and (78), we could obtain:

\[
| \{ x \in \mathbb{R} : |v_t(x)| \geq \sigma \} | \lambda \leq |E_\sigma| + | \{ x \in \mathbb{R} \setminus E_\sigma : |v_t(x)| \geq \sigma \} | \lambda
\]

\[
\leq |E_\sigma| + \sigma^{-2} \int_{\mathbb{R} \setminus E_\sigma} v_t^2(x) |x|^{2\lambda} dx
\]

\[
\leq |E_\sigma| + \sigma^{-2} \int_{\mathbb{R} \setminus E_\sigma} ((u_t)_\sigma^2(x)) |x|^{2\lambda} dx + 2|E_\sigma| \lambda
\]

\[
\leq 3|E_\sigma| + \frac{2}{\sigma^2} \int_0^\sigma s |E_\lambda| ds.
\]
Theorem 3.6. \(F(x, y) = u(x, y) + iv(x, y) \in H^p_r(\mathbb{R}^2), \ \frac{2\lambda}{2\lambda + 1} < p \leq 1. \forall t > 0, \) let \( t \) be a fixed number. \( F_t(x, y) = u_t(x, y) + iv_t(x, y) = F(x, y + t), \) \( u_t(x, y) \) is an even or odd function in \( x. \) Then we could have:

\[
\sup_{t > 0} \int_{-\infty}^{+\infty} |v(x, t)|^p |x|^{2\lambda} dx \leq c\|u^+_p\|_{L^p_\mathbb{R}}.
\]

(79)

Proof. By (76), we have

\[
\int_{-\infty}^{+\infty} |v(x, t)|^p |x|^{2\lambda} dx = \int_{-\infty}^{+\infty} v(x) |x|^{2\lambda} dx
\]

\[
= \int_{0}^{+\infty} \sigma^{-1} \{ x \in \mathbb{R} : |v_x(x)| > \sigma \} d\sigma
\]

\[
\leq \int_{0}^{+\infty} 3p \sigma^{p-1} \{ E_\sigma \} d\sigma + \int_{0}^{+\infty} 2p \sigma^{p-3} \int_{0}^{\sigma} E_\sigma d\sigma
\]

\[
= 3 \int_{0}^{+\infty} \{ u^+_p \} (x) |x|^{2\lambda} dx + \int_{0}^{+\infty} \left( 2p \int_{0}^{+\infty} \sigma^{p-3} d\sigma \right) s |E_s| ds
\]

\[
= 3\|u^+_p\|^p_p + \frac{2p}{2 - p} \int_{0}^{+\infty} \sigma^{p-1} |E_\sigma| ds
\]

\[
= \frac{8 - 3p}{2 - p} \|u^+_p\|_{L^p_\mathbb{R}}.
\]

Let \( F(x, y) = u(x, y) + iv(x, y) \) is a \( \lambda \)-analytic function, \( u_0 = (u(x, y) - u(-x, y))/2, \) \( u_e = (u(x, y) + u(-x, y))/2, \) \( v_0 = (v(x, y) - v(-x, y))/2, \) \( v_e = (v(x, y) + v(-x, y))/2, \) \( F_0 = u_0 + iv_e, F_e = u_e + iv_0. \)

Proposition 3.7. \( F_0 = u_o + iv_e, F_e = u_e + iv_0 \) are \( \lambda \)-analytic functions.

Proof. \( F \) is a \( \lambda \)-analytic function, and it satisfies the \( \lambda \)-Cauchy-Riemann equations:

\[
\begin{cases}
D_x u - \partial_y v = 0, \\
\partial_y u + D_x v = 0.
\end{cases}
\]

Then \( F_0 = u_0 + iv_e, F_e = u_e + iv_0 \) both satisfy the \( \lambda \)-Cauchy-Riemann equations:

\[
\begin{cases}
D_x u_0 - \partial_y v_0 = 0, \\
\partial_y u_0 + D_x v_0 = 0.
\end{cases}
\]

\[
\begin{cases}
D_x u_e - \partial_y v_0 = 0, \\
\partial_y u_e + D_x v_0 = 0.
\end{cases}
\]

So \( F_0 = u_0 + iv_e, F_e = u_e + iv_0 \) are \( \lambda \)-analytic functions.

Theorem 3.8. \( F(z) = u(x, y) + iv(x, y) \in H^p_r(\mathbb{R}^2), \) for \( \frac{2\lambda}{2\lambda + 1} < p \leq 1, \) then we could deduce

\[
\|F\|_{H^p_r(\mathbb{R}^2)} \leq c\|u^+_p\|_{L^p_\mathbb{R}}.
\]

Proof. By theorem(3.6) and proposition(3.7), we could have

\[
\|F\|^p_{H^p_r(\mathbb{R}^2)} = \sup_{t > 0} \int_{\mathbb{R}} |u(x, t)|^2 + v(x, t)^2 |x|^{2\lambda} dx
\]

\[
\leq \sup_{t > 0} \int_{\mathbb{R}} (|u(x, t)|^2 + |v_0(x, t)|^2 + |v_e(x, t)|^2) |x|^{2\lambda} dx
\]

\[
\leq c \sup_{t > 0} \int_{\mathbb{R}} (|u(x, t)|^p + |u_0(x)|^p + |u_e(x)|^p + |v_0(x)|^p + |v_e(x)|^p) |x|^{2\lambda} dx
\]

\[
\leq c \int_{\mathbb{R}} |u^+_p(x)|^p |x|^{2\lambda} dx.
\]

(80)
\[ (P.f)(x,y) = c_\lambda \int \int f(t)(\tau_x P_y)(-t)|t|^{2\lambda}dt, \quad \text{for } x \in \mathbb{R}, \ y \in (0,\infty). \] 

(81)

\[ (\tau_x P_y)(-t) \text{ is a } \lambda\text{-Poisson kernel, } P_y(x) = m_\lambda y(y^2 + x^2)^{-\lambda-1}, m_\lambda = 2^{\lambda+1/2}\Gamma(\lambda+1)/\sqrt{\pi}. \] 

Similarly, \( \lambda\text{-Poisson integral of a measure } d\mu \in \mathcal{B}_\lambda(\mathbb{R}) \) can be defined as

\[ (P(d\mu))(x,y) = c_\lambda \int (\tau_x P_y)(-t)|t|^{2\lambda}d\mu(t), \quad \text{for } x \in \mathbb{R}, \ y \in (0,\infty). \] 

(82)

**Proposition 3.9.** ([13]) ([11])

(i) \( \lambda\text{-Poisson kernel } (\tau_x P_y)(-t) \) has the representation

\[ (\tau_x P_y)(-t) = \frac{\lambda \Gamma(\lambda+1/2)}{2^{\lambda - 1/2}\pi} \int_0^\pi \frac{y(1 + \text{sgn}(xt)\cos \theta)}{(y^2 + x^2 + t^2 - 2|xt|\cos \theta)^{\lambda + 1}} \sin^{2\lambda - 1} \theta d\theta. \] 

(83)

For \( f \in L_\lambda^1(\mathbb{R}) \cap L_\lambda^\infty(\mathbb{R}) \),

\[ Pf(x,y) = c_\lambda \int f(t)(\tau_x P_y)(-t)|t|^{2\lambda}dt. \]

We denote \( (\tau_x Q_y)(-t) \) as the conjugate \( \lambda\text{-Poisson kernel}. \) Then we can define conjugate \( \lambda\text{-Poisson integral as } \) for \( f \in L_\lambda^1(\mathbb{R}) \cap L_\lambda^\infty(\mathbb{R}) \), \( (Qf)(x,y) = (f \ast_\lambda Q_y)(x), Q_y(x) = m_\lambda x(y^2 + x^2)^{-\lambda - 1}. \)

\[ (Qf)(x,y) = c_\lambda \int f(t)(\tau_x Q_y)(-t)|t|^{2\lambda}dt, \quad \text{for } x \in \mathbb{R}, \ y \in (0,\infty), \] 

(84)

(ii) We have the representation of conjugate \( \lambda\text{-Poisson kernel } (\tau_x Q_y)(-t) \)

\[ (\tau_x Q_y)(-t) = \frac{\lambda \Gamma(\lambda + 1/2)}{2^{\lambda - 1/2}\pi} \int_0^\pi \frac{(x-t)(1 + \text{sgn}(xt)\cos \theta)}{(y^2 + x^2 + t^2 - 2|xt|\cos \theta)^{\lambda + 1}} \sin^{2\lambda - 1} \theta d\theta. \] 

(85)

(iii) \( \lambda\text{-Hilbert kernel is defined as :} \)

\[ h(x,t) = \frac{\lambda \Gamma(\lambda + 1/2)}{2^{\lambda - 1/2}\pi} (x-t) \int_1^t \frac{(1 + s)(1 - s^2)^{\lambda - 1}}{(x^2 + t^2 - 2xts)^{\lambda + 1}} ds. \]

Then for \( f \in L_\lambda^1(\mathbb{R}) \cap L_\lambda^\infty(\mathbb{R}) \), \( \lambda\text{-Hilbert transform can be defined as} \)

\[ \mathcal{H}_\lambda f(x) = c_\lambda \lim_{\epsilon \to 0^+} \int_{|t-x| > \epsilon} f(t)h(x,t)|t|^{2\lambda}dt. \]

The associated maximal functions are defined as : \( Q_f^+ f(x) = \sup_{|x-t| < \epsilon} (Qf)(s,y), P_f^+ f(x) = \sup_{|x-t| < \epsilon} (Pf)(s,y), F_f^+ (x) = \sup_{|x-t| < \epsilon} F(s,y). \)

**Proposition 3.10.** ([13])

(i) If \( 1 < p < \infty \), \( f(x) \in L_p^\lambda(\mathbb{R}) \) then \( \|Q_f^+ f\|_{L_p^\lambda} \leq C_p^\lambda \|f\|_{L_p^\lambda} \).

(ii) If \( 1 < p < \infty \), \( f(x) \in L_p^\lambda(\mathbb{R}) \) then \( \|P_f^+ f\|_{L_p^\lambda} \leq C_p^\lambda \|f\|_{L_p^\lambda} \).

(iii) If \( \frac{2\lambda}{2\lambda + 1} < p \), and \( F(x) \in H_p^\lambda(\mathbb{R}^d) \), then \( \|F\|_{H_p^\lambda(\mathbb{R}^d)} \approx \|F_f^+ \|_{L_p^\lambda} \).

(iv) If \( 1 \leq p < \infty \), and \( F(x) \in H_p^\lambda(\mathbb{R}^d) \), then \( F(x, y) \) has boundary values. \( F(x, y) \) is the \( \lambda\text{-Poisson integer of its boundary values.} \)

(v) For \( f \in L_\lambda^1(\mathbb{R}) \), \( 1 \leq p < \infty \), its \( \lambda\text{-Poisson integer } Pf(x,y) \) and the conjugate \( \lambda\text{-Poisson integer } Qf(x,y) \) satisfy the generalized Cauchy-Riemann system (62) on \( \mathbb{R}^d \).

(vi) If \( 1 \leq p < \infty \), \( \mathcal{H}_\lambda f(x) = \lim_{y \to 0^+} Qf(x,y) \) exists almost everywhere, and the mapping \( f \to \mathcal{H}_\lambda f \) is strong-(p, p) bounded for \( 1 < p < \infty \) and weakly-(1, 1) bounded.
Proposition 3.11. \((\text{[13]},\text{[11]})\) Let \(p > p_0 = \frac{2\lambda}{2\lambda+1}\) and \(F \in H^p_{\lambda}(\mathbb{R}^2_+)\). Then

(i) For almost every \(x \in \mathbb{R}\), \(\lim F(t,y) = F(x)\) exists as \((t, y)\) approaches the point \((x, 0)\) nontangentially.

(ii) If \(p > \frac{2\lambda}{2\lambda+1}\), then \(\lim_{y \to 0^+} \|F(t,y) - F(x)\|_{L^p_{\lambda}} = 0\), and if \(p \geq 1\), \(\|F\|_{H^p_{\lambda}} = \|F\|_{L^p_{\lambda}}\), and if \(1 > p > \frac{2\lambda}{2\lambda+1}\), \(\|F\|_{H^p_{\lambda}} \geq \|F\|_{L^p_{\lambda}} \geq 2^{1-2/p}\|F\|_{H^p_{\lambda}}\), where \(\|F\|_{L^p_{\lambda}} = (c_{\lambda} \int |F(x)|^p |x|^{2\lambda} dx)^{1/p}\).

(iii) For \(p > \frac{2\lambda}{2\lambda+1}\), \(F \in H^p_{\lambda}(\mathbb{R}^2_+)\) if and only if \(F \in L^p_{\lambda}(\mathbb{R})\), and moreover \(\|F\|_{H^p_{\lambda}} \geq \|F\|_{L^p_{\lambda}} \geq c\|F\|_{H^p_{\lambda}}\).

(iv) Suppose \(p > \frac{2\lambda}{2\lambda+1}\), and \(p_1 > \frac{2\lambda}{2\lambda+1}\), \(F(x,y) \in H^p_{\lambda}(\mathbb{R}^2_+), and if \(F \in L^p_{\lambda}(\mathbb{R})\). Then \(F(x,y) \in H^p_{\lambda}(\mathbb{R}^2_+).\)

(v) \(1 \leq p < \infty\), and \(F = u + iv \in H^p_{\lambda}(\mathbb{R}^2_+)\), then \(F\) is the \(\lambda\)-Poisson integral of its boundary values \(F(x)\), and \(F(x) \in L^p_{\lambda}(\mathbb{R})\).

From Proposition 3.10, Theorem 3.8, we have

Theorem 3.12. \(\frac{2\lambda}{2\lambda+1} < p < \infty\), \(F(x,y) = u(x,y) + iv(x,y) \in H^p_{\lambda}(\mathbb{R}^2_+)\), then

\[\|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)} \asymp \|u\|_{L^p_{\lambda}(\mathbb{R})}\]

Then we could have

Proposition 3.13. \(\frac{2\lambda}{2\lambda+1} < p < \infty\), \(F(x,y) = u(x,y) + iv(x,y) \in H^p_{\lambda}(\mathbb{R}^2_+)\), then

\[\|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)} \asymp \|S(u)\|_{L^p_{\lambda}(\mathbb{R})}\]

Definition 3.14. By Proposition 3.11, Theorem 3.12, we could then define \(\tilde{H}^p_{\lambda}(\mathbb{R})\) \((\frac{2\lambda}{2\lambda+1} < p < \infty)\) as

\[\tilde{H}^p_{\lambda}(\mathbb{R}) \triangleq \left\{ g(x) : g(x) = \lim_{y \to 0^+} Re F(t,y), F \in H^p_{\lambda}(\mathbb{R}^2_+) \cap H^1_{\lambda}(\mathbb{R}^2_+) \cap H^1_{\lambda}(\mathbb{R}^2_+) \right\} \]

(t, y) approaches the point \((x, 0)\) nontangentially \}

with the norm:

\[\|g\|_{\tilde{H}^p_{\lambda}(\mathbb{R})} = \|P^p_{\lambda} g\|_{L^p_{\lambda}(\mathbb{R})}\]

Thus \(\tilde{H}^p_{\lambda}(\mathbb{R})\) is a linear space equipped with the norm: \(\| \cdot \|_{\tilde{H}^p_{\lambda}(\mathbb{R})}\), which is not complete. The completion of \(\tilde{H}^p_{\lambda}(\mathbb{R})\) with the norm \(\| \cdot \|_{\tilde{H}^p_{\lambda}(\mathbb{R})}\) is denoted as \(H^p_{\lambda}(\mathbb{R})\).

From the above Definition 3.14 and Proposition 3.10, we could have the following conclusions:

Proposition 3.15. For \(\frac{2\lambda}{2\lambda+1} < p < \infty\), \(H^p_{\lambda}(\mathbb{R}) \cap H^2_{\lambda}(\mathbb{R}) \cap H^1_{\lambda}(\mathbb{R}) \) is dense in \(H^p_{\lambda}(\mathbb{R})\). For \(1 < p < \infty\), we have \(H^1_{\lambda}(\mathbb{R}) = L^1_{\lambda}(\mathbb{R})\). And we have \(H^1_{\lambda}(\mathbb{R}) \subset L^1_{\lambda}(\mathbb{R})\).

Proposition 3.16. \(\frac{2\lambda}{2\lambda+1} \leq p \leq 1\), \(H^2_{\lambda}(\mathbb{R}^2_+) \cap H^1_{\lambda}(\mathbb{R}^2_+) \cap H^1_{\lambda}(\mathbb{R}^2_+) \) is dense in \(H^1_{\lambda}(\mathbb{R}^2_+)\).

Proof. From([13]), we could know that for \(F(x,y) \in H^p_{\lambda}(\mathbb{R}^2_+), p_0 < p \leq 1, \forall y_0 > 0\)

\[\left( \int_{\mathbb{R}} |F(x,y+y_0)|^2 |x|^{2\lambda} dx \right)^{\frac{1}{2}} \leq c(y_0)^{(1/2-1/p)(1+2\lambda)}\|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)}\]

and

\[\left( \int_{\mathbb{R}} |F(x,y+y_0)|^1 |x|^{2\lambda} dx \right)^{\frac{1}{2}} \leq c(y_0)^{-(1/p-1/1)(1+2\lambda)}\|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)}\]

Thus \(F(x,y+y_0) \in H^2_{\lambda}(\mathbb{R}^2_+) \cap H^1_{\lambda}(\mathbb{R}^2_+)\). By Proposition 3.11(iii), we could see that \(\lim_{y_0 \to 0^+} \|F(\cdot,y+y_0) - F(\cdot,y)\|_{L^p_{\lambda}} = 0\). Then \(H^2_{\lambda}(\mathbb{R}^2_+) \cap H^1_{\lambda}(\mathbb{R}^2_+) \cap H^1_{\lambda}(\mathbb{R}^2_+) \) is dense in \(H^1_{\lambda}(\mathbb{R}^2_+)\). This proves the proposition. \(\Box\)
3.2 Homogeneous type Hardy Space on Dunkl setting

In Definition (3.14), we introduced the real-variable Hardy spaces: $H^p_r(\mathbb{R})$. $H^p_\lambda(\mathbb{R}_2^+)$ spaces of Hardy spaces associated with the pseudo-analytic Hardy spaces $H^p_{\lambda}(\mathbb{R}^2_+)$). We will prove that the $H^p_r(\mathbb{R})$ is a kind of Homogeneous Hardy Space: the $\lambda$-Poisson kernel is a kind of Homogeneous kernel. First we introduce the $\lambda$-Poisson kernel:

\[
(\tau_x P_y)(-t) = \frac{\lambda \Gamma(\lambda + 1/2)}{2^{\lambda - 1/2} \pi} \int_0^\infty \frac{y(1 + \text{sgn}(x) \cos \theta)}{(y^2 + x^2 + t^2 - 2|xt| \cos \theta)^{\lambda + 1}} \sin^{2\lambda - 1} \theta d\theta.
\]

Let $d_\lambda(x, y) = (2\lambda + 1)\int_0^\infty |t|^{2\lambda} dt$, $d_\lambda(x) = (2\lambda + 1)|x|^{2\lambda} dx, d_\lambda(x, y) = (2\lambda + 1)\int_0^\infty |t|^{2\lambda} dt, c_\lambda = 2^{\lambda+1/2}\Gamma(\lambda + 1/2)$. Thus $d_\lambda(x, y) = |\mu_\lambda(x, y)|$. $B(x, r) = B_\lambda(x, r) = \{y : d_\lambda(x, y) < r\}$.

For $\forall f(x) \in L^2_\lambda(\mathbb{R}) \cap L^1(\mathbb{R}) \cap H^p_{\lambda}(\mathbb{R}), \frac{2\lambda + 1}{2\lambda + 3} < p \leq 1$, we introduce a new kernel as following:

\[
K(r, x, t) = \begin{cases} r^{\tau_x P_{|x|-2\lambda}}(-t) & \text{for } r < |x|^{2\lambda+1}, \\ r^{\tau_x P_{\lambda/(2\lambda+1)}}(-t) & \text{for } r \geq |x|^{2\lambda+1}. \end{cases}
\]

(86)

(87)

Then

\[
\sup_{r > 0} \int_{\mathbb{R}} K(r, x, t) f(t) \frac{|t|^{2\lambda} dt}{r} = \sup_{y > 0} (P_y \ast \lambda f)(x).
\]

(88)

From[13] we have the estimation for the $\lambda$-Poisson kernel:

\[
(\tau_x P_y)(-t) = \frac{y|y|^2 + (|x| + |t|)^2}{{|x|}^2 + (x - t)^2} \ln \left( \frac{y^2 + (x - t)^2}{y^2 + (x + t)^2} + 2 \right)
\]

(89)

I will prove the following Proposition (3.17), some of the idea is from Liao’s Ph.D thesis, but his proof is not right.

Proposition 3.17. Let constant $A > 0$; $K(r, x, t)$ is a kernel satisfying the following conditions:

(i) $K(r, x, x) \geq 1$, for $r > 0, x \in \mathbb{R}$;

(ii) $0 \leq K(r, x, t) \leq \left(1 + \frac{d_\lambda(x, t)}{r}\right)^{-\frac{2\lambda + 1}{2\lambda + 3}}$, for $r > 0, x, t \in \mathbb{R}$;

(iii) For $r > 0, x, t, z \in \mathbb{R}$, if $\frac{d_\lambda(t, z)}{r} \leq C \min\{1 + \frac{d_\lambda(x, t)}{r}, 1 + \frac{d_\lambda(x, z)}{r}\}$

\[
|K(r, x, t) - K(r, x, z)| \leq \left(\frac{d_\lambda(t, z)}{r}\right)^{-\frac{2\lambda + 1}{2\lambda + 3}} \left(1 + \frac{d_\lambda(x, t)}{r}\right)^{-\frac{2\lambda + 3}{2\lambda + 1}}.
\]

(iv)

$K(r, x, y) = K(r, y, x)$.

Proof.

$K(r, x, y) = K(r, y, x)$

is deduced from the fact that

\[
(\tau_x P_y)(-t) = (\tau_y P_x)(-x).
\]

Notice that \(s \neq 0\), we have

\[
K(|s|^{2\lambda+1}r, sx, st) = K(r, x, t), \quad d_\lambda(sx, st) = |s|^{-2\lambda-1}d_\lambda(x, t).
\]

Then we need to only prove the theorem for $x = 0$ and $x = 1$. The formulation of the kernel $K(r, x, t)$ could be given by equation (83) and equation (86). First, we will prove $K(r, x, x) \geq A^{-1} > 0$

Case 1 $x = 0$. By equation (87), $y = r^{\frac{1}{2\lambda + 1}}$

\[
K(r, 0, 0) = \frac{r \ast r^{\frac{1}{2\lambda + 1}}}{(r^{\frac{1}{2\lambda + 1}})^{\lambda + 1}} \geq 1.
\]
Case 2 \( x \neq 0 \), we need only to consider the case when \( x = 1 \).
When \( r < 1 \). By equation (86) and equation (87), we have \( y = r < 1 \). Then:

\[
K(r, 1, 1) = \frac{\lambda! (\lambda + 1/2)}{2^{-\lambda-1/2} \pi} \int_0^\pi \frac{r y (1 + \cos \theta)}{(y^2 + 2 - 2 \cos \theta)^{\lambda+1}} \sin^{2\lambda-1} \theta d\theta.
\]

\[
\geq c \int_0^{\pi/4} \frac{r y (1 + \cos \theta)}{(y^2 + 2 - 2 \cos \theta)^{\lambda+1}} \sin^{2\lambda-1} \theta d\theta \\
\geq c.
\]

When \( r \geq 1 \). From equation (87), (89), (86), \( y = r^{\frac{1}{2\lambda+1}} \geq 1 \). Thus we could have:

\[
K(r, 1, 1) \geq \frac{r^{\frac{1}{2\lambda+1}}}{(r^{\frac{2}{2\lambda+1}} + 2)^{\lambda+1}} \\
\geq c.
\]

Second, we will prove that \( 0 \leq K(r, x, t) \leq A \left( 1 + \frac{d_x(x,t)}{r} \right)^{\frac{2(\lambda+1)}{2\lambda+1}} \), for \( r > 0, x, t \in \mathbb{R} \).

Case 1 \( x = 0 \). By equation (87), \( y = r^{\frac{1}{2\lambda+1}} \)

\[
K(r, 0, t) \approx C \left( 1 + \frac{t^2}{r^{2/(2\lambda+1)}} \right)^{-\lambda-1} \leq A \left( 1 + \frac{|t|(2\lambda+1)}{(2\lambda+1)r} \right)^{\frac{2(\lambda+1)}{2\lambda+1}} = A \left( 1 + \frac{d_x(0,t)}{r} \right)^{\frac{2(\lambda+1)}{2\lambda+1}}.
\]

Case 2 \( x \neq 0 \), we need only to consider the case when \( x = 1 \). When \( r \geq 1 \), \( y = r^{\frac{1}{2\lambda+1}} \geq 1 \)

And when \( r < 1 \), \( y = r \). By estimation (89), we could have

\[
K(r, 1, t) \approx \begin{cases} 
\frac{r^{\frac{2\lambda+1}{2\lambda+1}}}{(r^{\frac{2}{2\lambda+1}} + t^2 + 1)^{\lambda+1}} \ln \left( \frac{r^2 + t^2}{r^2 + (t + 1)^2} + 1 \right) & \text{for } t < 0, \\
\frac{r^{\frac{2\lambda+1}{2\lambda+1}}}{(r^{\frac{2}{2\lambda+1}} + t^2 + 1)^{\lambda}} \left( r^{\frac{2\lambda}{2\lambda+1}} + (1-t)^2 \right) & \text{for } t \geq 0.
\end{cases}
\]

\[
\text{when } r \geq 1 \quad K(r, 1, t) \approx \begin{cases} 
\frac{r^2}{(r^2 + t^2 + 1)^{\lambda+1}} & \text{for } t < 0, \\
\frac{r^2}{(r^2 + t^2 + 1)^{\lambda}} \left( r^2 + (1-t)^2 \right) & \text{for } t \geq 0.
\end{cases}
\]

Equation (90) and equation (91) imply that \( K(r, 1, t) \leq C \left( 1 + \frac{d_x(1,t)}{r} \right)^{\frac{2(\lambda+1)}{2\lambda+1}} \) for some constant \( C \). Thus we have established \( 0 \leq K(r, x, t) \leq C \left( 1 + \frac{d_x(1,t)}{r} \right)^{\frac{2(\lambda+1)}{2\lambda+1}} \), for \( r > 0, x, t \in \mathbb{R} \).

At last, if \( \frac{d_x(t,z)}{r} \leq C \min \{1 + \frac{d_x(x,t)}{r}, 1 + \frac{d_x(x,z)}{r}\} \), we will prove

\[
|K(r, x, t) - K(r, x, z)| \leq \left( \frac{d_x(t,z)}{r} \right)^{\frac{1}{2\lambda+1}} \left( 1 + \frac{d_x(x,t)}{r} \right)^{\frac{2(\lambda+1)}{2\lambda+1}}.
\]

for \( r > 0, x, t, z \in \mathbb{R} \). We could see that if

\[
\frac{d_x(t,z)}{r} \leq 1 + \frac{d_x(x,t)}{r},
\]

then

\[
\frac{d_x(x,z)}{r} \leq \left( \frac{d_x(x,t)}{r} + \frac{d_x(t,z)}{r} \right) \leq \left( \frac{d_x(x,t)}{r} + 1 + \frac{d_x(x,t)}{r} \right) \leq 1 + \frac{d_x(x,t)}{r}.
\]

Thus we could have if

\[
\frac{d_x(t,z)}{r} \leq 1 + \frac{d_x(x,t)}{r}.
\]
then
\[ 1 + \frac{d_\lambda(x,z)}{r} \lesssim 1 + \frac{d_\lambda(x,t)}{r}. \]

By the symmetry of \( t \) and \( z \), we could have the estimate:
\[ 1 + \frac{d_\lambda(x,z)}{r} \asymp 1 + \frac{d_\lambda(x,t)}{r}. \] (92)

For \( u \in \mathbb{R} \) satisfying \((u - t)(u - z) \leq 0\), we could obtain
\[
\frac{d_\lambda(u,t)}{r} \lesssim \frac{d_\lambda(t,z)}{r} \lesssim C \min\{1 + \frac{d_\lambda(x,t)}{r}, 1 + \frac{d_\lambda(x,z)}{r}\}.
\]

Thus:
\[ 1 + \frac{d_\lambda(x,u)}{r} \asymp 1 + \frac{d_\lambda(x,t)}{r}, \quad \text{when } (u - t)(u - z) \leq 0. \] (93)

It is enough to prove that if \( \frac{d_\lambda(z)}{r} \leq C \min\{1 + \frac{d_\lambda(x,t)}{r}, 1 + \frac{d_\lambda(x,z)}{r}\} \), we could deduce:
\[
\left( 1 + \frac{d_\lambda(x,t)}{r} \right)^{\frac{2\lambda+2}{2\lambda+1}} |K(r,x,t) - K(r,x,z)| \lesssim \left( \frac{d_\lambda(t,z)}{r} \right)^{\frac{2\lambda+1}{2\lambda+1}}.
\] (94)

Case 1: When \( x = 0 \), we could suppose that \( z > 0 \) first. By the estimation (93), we could obtain the following inequality for \( u(z - u) \leq 0 \):
\[ 1 + \frac{d_\lambda(0,u)}{r} \asymp 1 + \frac{d_\lambda(0,z)}{r} \approx 1 \approx 1 + \frac{u^{2\lambda+1}}{r}. \]

By the mean value theorem for integral, we could have:
\[
\left( 1 + \frac{d_\lambda(0,z)}{r} \right)^{\frac{2\lambda+2}{2\lambda+1}} |K(r,0,t) - K(r,0,z)|
= c_\lambda \left( 1 + \frac{d_\lambda(0,z)}{r} \right)^{\frac{2\lambda+2}{2\lambda+1}} \int_0^r r^{\frac{y}{(y^2+\lambda^2)^{\lambda+1}} - \frac{y}{(y^2+z^2)^{\lambda+1}}} \sin^{2\lambda-1} \theta d\theta
\lesssim \left( 1 + \frac{u^{2\lambda+1}}{r} \right)^{\frac{2\lambda+2}{2\lambda+1}} \frac{yur}{(y^2+u^2)^{\lambda+2}} |t - z|
\lesssim \left( 1 + \frac{u^{2\lambda+1}}{r} \right)^{\frac{2\lambda+2}{2\lambda+1}} \frac{ur^{\frac{2\lambda+2}{2\lambda+1}}}{(r^{2\lambda+1} + u^2)^{\lambda+2}} |t - z|.
\]

\[
\left\{ \begin{array}{ll}
\left( 1 + \frac{u^{2\lambda+1}}{r} \right)^{\frac{2\lambda+2}{2\lambda+1}} \frac{ur^{\frac{2\lambda+2}{2\lambda+1}}}{(r^{2\lambda+1} + u^2)^{\lambda+2}} \leq \frac{1}{r^{\frac{2\lambda+1}{2\lambda+1}}} & \text{for } |u| < r^{\frac{1}{2\lambda+1}}, \\
\left( 1 + \frac{u^{2\lambda+1}}{r} \right)^{\frac{2\lambda+2}{2\lambda+1}} \frac{ur^{\frac{2\lambda+2}{2\lambda+1}}}{(r^{2\lambda+1} + u^2)^{\lambda+2}} \leq \frac{1}{r^{\frac{2\lambda+1}{2\lambda+1}}} & \text{for } |u| \geq r^{\frac{1}{2\lambda+1}}.
\end{array} \right.
\]

Then we have the formula established: for \( \frac{d_\lambda(z)}{r} \leq C \min\{1 + \frac{d_\lambda(x,t)}{r}, 1 + \frac{d_\lambda(x,z)}{r}\} \)
\[ \left( 1 + \frac{d_\lambda(0,t)}{r} \right)^{\frac{2\lambda+2}{2\lambda+1}} |K(r,0,t) - K(r,0,z)| \lesssim \frac{|t - z|}{r^{\frac{2\lambda+1}{2\lambda+1}}} \lesssim \left( \frac{d_\lambda(t,z)}{r} \right)^{\frac{2\lambda+1}{2\lambda+1}}. \]

Case 2: When \( x \neq 0 \). It would be enough to prove Formula (94) when \( x = 1 \). If \( \frac{d_\lambda(|t|,|z|)}{r} \leq C \min\{1 + \frac{d_\lambda(1,|t|)}{r}, 1 + \frac{d_\lambda(1,|z|)}{r}\} \), we will prove:
\[ \left( 1 + \frac{d_\lambda(1,|t|)}{r} \right)^{\frac{2\lambda+2}{2\lambda+1}} |K(r,1,|t|) - K(r,1,|z|)| \lesssim \left( \frac{d_\lambda(|t|,|z|)}{r} \right)^{\frac{2\lambda+1}{2\lambda+1}}. \]
By equation (83), equation (93) and mean value theorem, we could obtain:
\[
\left(1 + \frac{d_{\lambda}(1, |t|)}{r}\right)^{\frac{\lambda + 3}{\lambda + 2}} |K(r, 1, |t|) - K(r, 1, |z|)|
\approx \left(1 + \frac{d_{\lambda}(1, |u|)}{r}\right)^{\frac{\lambda + 3}{\lambda + 2}} \int_0^\pi \left(\frac{r y(1 + \cos \theta) \sin^{\lambda - 1} \theta}{(y^2 + 1 + u^2 - 2|u| \cos \theta)^{\lambda + 1}} - \frac{r y(1 + \cos \theta) \sin^{\lambda - 1} \theta}{(y^2 + 1 + z^2 - 2|z| \cos \theta)^{\lambda + 1}}\right) d\theta
\]
\[
\lesssim \left(1 + \frac{d_{\lambda}(1, |u|)}{r}\right)^{\frac{\lambda + 3}{\lambda + 2}} \int_0^\pi \frac{r y(1 + \cos \theta)(|u| - \cos \theta)}{(y^2 + 1 + u^2 - 2|u| \cos \theta)^{\lambda + 2}} \sin^{\lambda - 1} \theta d\theta d\theta |(|t| - |z|)|
\]
\[
\lesssim \left(1 + \frac{d_{\lambda}(1, |u|)}{r}\right)^{\frac{\lambda + 3}{\lambda + 2}} \int_{-1}^1 \frac{r y(1 + s)(|u| - s)}{(y^2 + 1 + u^2 - 2|u| s)^{\lambda + 2}} (1 - s^2)^{\lambda - 1} (1 + s) ds |(|t| - |z|)|.
\]
We have the following estimation for \(0 \leq s \leq 1\):
\[
\frac{1}{(y^2 + 1 + u^2 - 2|u| s)} < \frac{1}{(y^2 + 1 + u^2 - 2|u|)} \quad \text{and} \quad \frac{|u| - 1}{(y^2 + 1 + u^2 - 2|u| s)} < \frac{|u| - 1}{(y^2 + 1 + u^2 - 2|u|)}.
\]
Then, together with Formula (89), we could obtain
\[
\left| \int_{-1}^1 \frac{r y(1 + s)(|u| - s)}{(y^2 + 1 + u^2 - 2|u| s)^{\lambda + 2}} (1 - s^2)^{\lambda - 1} (1 + s) ds |(|t| - |z|)|
\leq \left( \int_{-1}^1 \frac{r y(1 + s)(|u| - 1)(1 - s^2)^{\lambda - 1} (1 + s)}{(y^2 + 1 + u^2 - 2|u| s)^{\lambda + 2}} ds \right) \left( \int_{-1}^1 \frac{r y(1 - s^2)^{\lambda} (1 + s)}{(y^2 + 1 + u^2 - 2|u| s)^{\lambda + 2}} ds \right) |(|t| - |z|)|
\leq C \left( \frac{|u| - 1}{(y^2 + 1 + u^2 - 2|u|)} \right) \left| (\tau_1 P_y)(-|u|) \right| (|t| - |z|) + C \frac{1}{(y^2 + 1 + u^2)} \left| (\tau_1 P_y)(-|u|) \right| (|t| - |z|)
\leq C \left( |(t| - |z)| \right) \frac{(1 - |u|)^2 + y^2 + (1 + u^2 + y^2)|1 - |u||}{(1 - |u|)^2 + y^2 + (1 + u^2 + y^2)^{\lambda + 1}}.
\]
If \(r < 1\), then \(y = r\).
If \(r < 1\), \(|1 - |u|| \geq 2\), we could deduce \(d_{\lambda}(1, |u|) \approx |u|^{2\lambda + 1}\). Thus the following could be obtained:
\[
\left(1 + \frac{d_{\lambda}(1, |u|)}{r}\right)^{\frac{\lambda + 3}{\lambda + 2}} (|t| - |z|) y r \frac{(1 - |u|)^2 + y^2 + (1 + u^2 + y^2)|1 - |u||}{(1 - |u|)^2 + y^2 + (1 + u^2 + y^2)^{\lambda + 1}}
= \left(1 + \frac{d_{\lambda}(1, |u|)}{r}\right)^{\frac{\lambda + 3}{\lambda + 2}} (|t| - |z|) r^2 \frac{(1 - |u|)^2 + r^2 + (1 + u^2 + r^2)|1 - |u||}{(1 - |u|)^2 + r^2 + (1 + u^2 + r^2)^{\lambda + 1}}
\lesssim \frac{|u|^{2\lambda + 3}}{r \frac{\lambda + 3}{\lambda + 2}} (|t| - |z|) r^2 \frac{(1 - |u|)^2 + r^2 + (1 + u^2 + r^2)|1 - |u||}{(1 - |u|)^2 + r^2 + (1 + u^2 + r^2)^{\lambda + 1}}
\approx \frac{d_{\lambda}(1, |u|)}{r \frac{\lambda + 3}{\lambda + 2}} d_{\lambda}(1, |u|) \frac{1}{r \frac{\lambda + 3}{\lambda + 2}} (|t| - |z|).
\]
For \(r < 1\), \(r/2 \leq |1 - |u|| \leq 2\). We could have \(d_{\lambda}(1, |u|) \approx |1 - |u||\), \(d_{\lambda}(|t|, |z|) \approx r + d_{\lambda}(1, |u|) \lesssim d_{\lambda}(1, |u|), d_{\lambda}(|t|, |z|) \approx |t| - |z|\). Thus
\[
\left(1 + \frac{d_{\lambda}(1, |u|)}{r}\right)^{\frac{\lambda + 3}{\lambda + 2}} (|t| - |z|) y r \frac{(1 - |u|)^2 + y^2 + (1 + u^2 + y^2)|1 - |u||}{(1 - |u|)^2 + y^2 + (1 + u^2 + y^2)^{\lambda + 1}}
= \left(1 + \frac{d_{\lambda}(1, |u|)}{r}\right)^{\frac{\lambda + 3}{\lambda + 2}} (|t| - |z|) r^2 \frac{(1 - |u|)^2 + r^2 + (1 + u^2 + r^2)|1 - |u||}{(1 - |u|)^2 + r^2 + (1 + u^2 + r^2)^{\lambda + 1}}
\lesssim \frac{(1 - |u|)^{2\lambda + 3}}{r \frac{\lambda + 3}{\lambda + 2}} r^2 (|t| - |z|) \frac{1}{|1 - |u||^2} \frac{1}{r \frac{\lambda + 3}{\lambda + 2}} \frac{1}{(1 - |u|)^{\frac{\lambda + 3}{\lambda + 2}}} (|t| - |z|)
\approx \frac{1}{r \frac{\lambda + 3}{\lambda + 2}} \frac{1}{d_{\lambda}(1, |u|)} \frac{1}{r \frac{\lambda + 3}{\lambda + 2}} (|t| - |z|)
\approx \frac{d_{\lambda}(|t|, |z|)}{r \frac{\lambda + 3}{\lambda + 2}}.
\]
For $r < 1$, $|1 - |u|| \leq r/2$. We have $d_\lambda(1, |u|) \approx |1 - |u||, \left| |t| - |z| \right| \approx d_\lambda(|t|, |z|) \lesssim r + d_\lambda(1, |u|) \lesssim r$, then
\[
\left| (1 + \frac{d_\lambda(1, |u|)}{r})^\frac{2 + \lambda}{2 + \lambda} |(t) - |z|| \right| \lesssim \left( \frac{1 - |u|)^2 + y^2 + (1 + u^2 + y^2)|1 - |u||}{1 - |u|^2 + y^2(y^2(1 + u^2 + y^2))^{\lambda + 1}} \right),
\]

Then, for $r \geq 1$, then $y = r^{\frac{1}{\lambda + 1}}$
\[
\left( 1 + \frac{d_\lambda(1, |u|)}{r} \right)^\frac{2 + \lambda}{2 + \lambda} |(|t| - |z|)| \approx \left( \frac{1 - |u|)^2 + y^2 + (1 + u^2 + y^2)|1 - |u||}{1 - |u|^2 + y^2(y^2(1 + u^2 + y^2))^{\lambda + 1}} \right) \lesssim \left( \frac{d_\lambda(|t|, |z|)}{r} \right)^\frac{2 + \lambda}{2 + \lambda}, \text{ for } |1 - |u|| \gtrsim 2r^{\frac{1}{\lambda + 1}}, d_\lambda(1, |u|) \approx u^{2\lambda + 1}
\]

Thus we have proved:
\[
\left( 1 + \frac{d_\lambda(1, |t|)}{r} \right)^\frac{2 + \lambda}{2 + \lambda} |K(r, 1, |t|) - K(r, 1, |z|)| \lesssim \left( \frac{d_\lambda(|t|, |z|)}{r} \right)^\frac{2 + \lambda}{2 + \lambda}
\]

for $\frac{d_\lambda(|t|, |z|)}{r} \leq C \min\{1 + d_\lambda(1, |t|), 1 + d_\lambda(1, |z|)\}$. In the same way, we could prove that:
\[
\left( 1 + \frac{d_\lambda(1, |t|)}{r} \right)^\frac{2 + \lambda}{2 + \lambda} |K(r, 1, |t|) - K(r, 1, |z|)| \lesssim \left( \frac{d_\lambda(|t|, |z|)}{r} \right)^\frac{2 + \lambda}{2 + \lambda}
\]

for $\frac{d_\lambda(|t|, |z|)}{r} \leq C \min\{1 + d_\lambda(1, |t|), 1 + d_\lambda(1, |z|)\}$. Next, we need next to prove that for $\frac{d_\lambda(|t|, |z|)}{r} \leq C \min\{1 + d_\lambda(1, |t|), 1 + d_\lambda(1, |z|)\}$ the following holds:
\[
\left( 1 + \frac{d_\lambda(1, |t|)}{r} \right)^\frac{2 + \lambda}{2 + \lambda} |K(r, 1, |t|) - K(r, 1, |z|)| \lesssim \left( \frac{d_\lambda(|t|, |z|)}{r} \right)^\frac{2 + \lambda}{2 + \lambda}
\]

Notice that if $\frac{d_\lambda(|t|, |z|)}{r} \leq C \min\{1 + d_\lambda(1, |t|), 1 + d_\lambda(1, |z|)\}$, then
\[
1 + \frac{d_\lambda(1, |t|)}{r} \approx 1 + \frac{d_\lambda(1, |z|)}{r}
\]

Then we could have:
\[
\left( 1 + \frac{d_\lambda(1, |t|)}{r} \right)^\frac{2 + \lambda}{2 + \lambda} |K(r, 1, |t|) - K(r, 1, |z|)| \lesssim \left( 1 + \frac{d_\lambda(1, |t|)}{r} \right)^\frac{2 + \lambda}{2 + \lambda} |K(r, 1, |t|) - K(r, 1, 0)| + \left( 1 + \frac{d_\lambda(1, |z|)}{r} \right)^\frac{2 + \lambda}{2 + \lambda} |K(r, 1, |z|) - K(r, 1, 0)|
\]

Together with:
\[
\left( 1 + \frac{d_\lambda(1, |t|)}{r} \right)^\frac{2 + \lambda}{2 + \lambda} |K(r, 1, |t|) - K(r, 1, |z|)| \lesssim \left( \frac{d_\lambda(|t|, |z|)}{r} \right)^\frac{2 + \lambda}{2 + \lambda}.
\]
Then we could prove that for \( \frac{d_3(t, z)}{r} \leq C \min \{ 1 + \frac{d_3(1, t)}{r}, 1 + \frac{d_3(1, z)}{r} \} \)
\[
(1 + \frac{d_\lambda(1, t)}{r})^{ \frac{2\lambda + 3}{2\lambda + 1} } |K(r, 1, t) - K(r, 1, z)| \lesssim \left( \frac{d_\lambda(t, z)}{r} \right)^{ \frac{2\lambda + 3}{2\lambda + 1} }.
\]
This proves the Proposition.

Let \( f^*(x) \) to be defined as:
\[
f^*(x) = \sup_{\phi, r} \left\{ \left| \int f(y) \phi(x) dy \right| / r : r > 0, \text{supp} \phi \subset B_\lambda(x, r), L(\phi, \frac{1}{2\lambda + 1}) \leq r^{-\frac{1}{L}}, \| \phi \|_{L^\infty} \leq 1 \right\}.
\]

**Proposition 3.18.**
\[
\left( \int \sup_{x \in \mathbb{R}} \| f \ast \lambda P_y(x) \|_{L^p(\mathbb{R})} \right)^{1/p} \lesssim \left( \int |f^*(x)|^p |x|^{2\lambda} dx \right)^{1/p}
\]
for \( p > 0, \forall f \in L^\lambda_\lambda(\mathbb{R}) \).

**Proof.** We will prove inequality (95) under extra assumption that \( x \geq 0 \). Let \( \psi(t) = \phi(t^{2\lambda + 1}) \), \( \phi(t) \in \mathcal{S}(\mathbb{R}) \), satisfying \( \phi''(t) \lesssim 1, \phi'(t) \lesssim 1, \text{supp} \phi(t) \subseteq (-1, 1) \), and \( \phi(t) = 1 \) when \( t \in (-1/2, 1/2) \).

Let:
\[
\psi_{0,x}^+(t) = \phi\left( \frac{2\lambda + 1 - x^{2\lambda + 1}}{2\lambda r} \right),
\psi_{k,x}^+(t) = \phi\left( \frac{2\lambda + 1 - x^{2\lambda + 1}}{2k \lambda r} \right) - \phi\left( \frac{2\lambda + 1 - x^{2\lambda + 1}}{2k - 1 \lambda r} \right), \text{for } t \geq x, \ k \geq 1
\]
\[
\psi_{0,x}^-(t) = 0, \psi_{k,x}^-(t) = 0, \text{for } t \leq x, \ k \geq 1
\]
Then we could define \( \psi_{k,x}^-(t) \) as following:
\[
\psi_{k,x}^-(t) = \psi_{k,x}^+(x + d_\lambda(x, t)), \text{for } t \leq x
\]
Let \( \psi_{k,x}(t) \) denote as
\[
\psi_{k,x}(t) = \psi_{k,x}^-(t) + \psi_{k,x}^+(t).
\]
Then \( \sum_{k=0}^{\infty} \psi_{k,x}(t) = 1 \), and \( \text{supp} \psi_{k,x}(t) \subseteq \left( B_\lambda(x, 2^{k+1} r) \setminus B_\lambda(x, 2^{k-2} r) \right) \). when \( k \geq 1 \).

For \( k \neq 0 \), when \( t_2 \geq t_1 \geq t \), by mean value theorem , \( \exists \xi, t_1 \leq \xi \leq t, \exists \xi, \frac{1}{2^{k+1} r} \leq s \leq \frac{1}{2^{k-2} r} \)
\[
\begin{align*}
&\frac{\left| \psi_{k,x}^+(t_1) - \psi_{k,x}^+(t_2) \right|}{d_\lambda(t_1, t_2)^{ \frac{2\lambda + 3}{2\lambda + 1} }} \\
&\leq \left| \phi\left( \frac{2\lambda + 1 - x^{2\lambda + 1}}{2^{k+1} r} \right) - \phi\left( \frac{2\lambda + 1 - x^{2\lambda + 1}}{2^{k-1} r} \right) \right| \frac{\left| t_2^{2\lambda + 1} - t_1^{2\lambda + 1} \right|^{ \frac{2\lambda + 3}{2\lambda + 1}}}{d_\lambda(t_1, t_2)^{ \frac{2\lambda + 3}{2\lambda + 1} }} \\
&\lesssim \left| \left( \xi^{2\lambda + 1} - x^{2\lambda + 1} \right) s \phi''(s) \left( (\xi^{2\lambda + 1} - x^{2\lambda + 1}) \right) - \phi'(s) (\xi^{2\lambda + 1} - x^{2\lambda + 1}) \left| t_2^{2\lambda + 1} - t_1^{2\lambda + 1} \right|^{ \frac{2\lambda + 3}{2\lambda + 1} } - \frac{2\lambda + 3}{2\lambda + 1} \right| \\
&\lesssim \left( \frac{d_\lambda(t_1, t_2)^{ \frac{2\lambda + 3}{2\lambda + 1} }}{2^{k+1} r} - \frac{d_\lambda(t_1, t_2)^{ \frac{2\lambda + 3}{2\lambda + 1} }}{2^{k-2} r} \right) \frac{\left| t_2^{2\lambda + 1} - t_1^{2\lambda + 1} \right|^{ \frac{2\lambda + 3}{2\lambda + 1} } }{d_\lambda(t_1, t_2)^{ \frac{2\lambda + 3}{2\lambda + 1} }} \\
&\lesssim \left( \frac{\left| \left( \xi^{2\lambda + 1} - x^{2\lambda + 1} \right) s \phi''(s) \left( (\xi^{2\lambda + 1} - x^{2\lambda + 1}) \right) - \phi'(s) (\xi^{2\lambda + 1} - x^{2\lambda + 1}) \right| t_2^{2\lambda + 1} - t_1^{2\lambda + 1} }{2^{k+1} r} \right)^{ \frac{2\lambda + 3}{2\lambda + 1} } \\
&\lesssim \left( \frac{1}{2^{k-1} r} \right)^{ \frac{2\lambda + 3}{2\lambda + 1} } \\
&\Rightarrow L \left( \psi_{k,x}^+(t), \frac{1}{2\lambda + 1} \right) \lesssim \left( \frac{1}{2^{k-1} r} \right)^{ \frac{2\lambda + 3}{2\lambda + 1} } .
\end{align*}
\]
By the mean value theorem we have
\[ |\psi_{k,x}^+(t)| = \left| \phi\left(\frac{t^{2\lambda+1} - t^{2\lambda+1}}{2^{k+1}r} \right) - \phi\left(\frac{t^{2\lambda+1} - x^{2\lambda+1}}{2^{k}r} \right) \right| = \left| \phi'(\xi)(\frac{t^{2\lambda+1} - x^{2\lambda+1}}{2^{k+1}r}) \right| \lesssim 1. \]

In all, we obtain the following:
\[
\begin{align*}
\left| \psi_{k,x}^+(t) \right| &= \left| \phi'(\xi)(\frac{t^{2\lambda+1} - x^{2\lambda+1}}{2^{k+1}r}) \right| \lesssim 1 \\
L \left( \psi_{k,x}^+(t), \frac{1}{2^{2\lambda+1}} \right) &\lesssim (2^{-k-1}r^{-1})^{\frac{1}{2\lambda+1}} \\
supp \psi_{k,x}^+(t) &\subseteq \left( B_\lambda(x, 2^{k+1}r) \setminus B_\lambda(x, 2^{k-1}r) \right) \cap \{ t : t > x \}.
\end{align*}
\]

Thus we could deduce from above that:
\[
\begin{align*}
\left| \psi_{k,x}(t) \right| &= \left| \phi'(\xi)(\frac{t^{2\lambda+1} - x^{2\lambda+1}}{2^{k+1}r}) \right| \lesssim 1 \\
L \left( \psi_{k,x}(t), \frac{1}{2^{2\lambda+1}} \right) &\lesssim (2^{-k-1}r^{-1})^{\frac{1}{2\lambda+1}} \\
supp \psi_{k,x}(t) &\subseteq \left( B_\lambda(x, 2^{k+1}r) \setminus B_\lambda(x, 2^{k-1}r) \right) \\
&\quad \cup \left( \{ t : t > x \} \setminus \left( B_\lambda(x, 2^{k+1}r) \setminus B_\lambda(x, 2^{k-1}r) \right) \right).
\end{align*}
\]

For \( k = 0 \), by mean value theorem
\[
\frac{\psi_{0,x}(t_1) - \psi_{0,x}(t_2)}{d_\lambda(t_1, t_2)^{\frac{1}{2\lambda+1}}} = \left| \frac{\phi\left(\frac{t_1^{2\lambda+1} - x^{2\lambda+1}}{2^{k+1}r} \right) - \phi\left(\frac{t_2^{2\lambda+1} - x^{2\lambda+1}}{2^{k+1}r} \right)}{d_\lambda(t_1, t_2)^{\frac{1}{2\lambda+1}}} \right| \lesssim \left( \frac{1}{2\lambda+1} \right)^{\frac{1}{2\lambda+1}}.
\]

We could remove the condition that \( x \geq 0 \). Because when \( x \leq 0 \), let:
\[
\begin{align*}
\psi_{0,x}^+(t) &= \phi\left(\frac{t^{2\lambda+1} - x^{2\lambda+1}}{2^{k+1}r} \right) \\
\psi_{k,x}(t) &= \phi\left(\frac{t^{2\lambda+1} - x^{2\lambda+1}}{2^{k+1}r} \right) - \phi\left(\frac{t^{2\lambda+1} - x^{2\lambda+1}}{2^{k}r} \right), \quad \text{for } t \leq x, \ k \geq 1, \\
\psi_{0,x}^-(t) &= 0, \\
\psi_{k,x}^-(t) &= 0, \quad \text{for } t \geq x, \ k \geq 1.
\end{align*}
\]

Then define \( \psi_{k,x}^-(t) \) as following:
\[
\psi_{k,x}^- (t) = \psi_{k,x}^+ (|x| + d_\lambda(x, t)), \quad \text{for } t \geq x \\
\psi_{k,x}^- (t) = 0, \quad \text{for } t \leq x.
\]

\( K(r, x, t) \) is a kernel defined above:
\[
K(r, x, t) = \left\{ \begin{array}{ll}
r(t_\tau P_x |x|^{2\lambda})(-t) & \text{for } r < |x|^{2\lambda+1} \\
r(t_\tau P_x |x|^{2\lambda+1})(-t) & \text{for } r \geq |x|^{2\lambda+1}.
\end{array} \right.
\]

Then we could have
\[
r t_\tau P_x (-t) = \sum_{k=0}^{+\infty} \psi_{k,x}(t_1) r t_\tau P_x (-t_1) - \psi_{k,x}(t_2) r t_\tau P_x (-t_2) \frac{1}{d_\lambda(t_1, t_2)^{\frac{1}{2\lambda+1}}}.
\]

We need to estimate
\[
\left| \psi_{k,x}(t_1) r t_\tau P_x (-t_1) - \psi_{k,x}(t_2) r t_\tau P_x (-t_2) \right| \frac{1}{d_\lambda(t_1, t_2)^{\frac{1}{2\lambda+1}}}.
\]
When $k \geq 1$, we could see that $d_\lambda(t_1, t_2) \leq 2^{k+1}r$, $2^{k-1}r \leq d_\lambda(x, t_1) \leq 2^{k+1}r$, $2^{k-1}r \leq d_\lambda(x, t_2) \leq 2^{k+1}r$. Thus we could obtain $d_\lambda(t_1, t_2) \leq \min\{d_\lambda(x, t_1) + r, d_\lambda(x, t_2) + r\}$. Thus

$$|K(r, x, t_1) - K(r, x, t_2)| = |rr_x P_y(-t_1) - rr_x P_y(-t_2)| \lesssim \left( \frac{d_\lambda(t_1, t_2)}{r} \right)^{\frac{1+2k}{1+3k}} \left( 1 + \frac{d_\lambda(x, t_1)}{r} \right)^{-\frac{2(k+1)+1}{2+1}} .$$

Then

$$\left| \frac{rr_x P_y(-t_1) - rr_x P_y(-t_2)}{d_\lambda(t_1, t_2)^{\frac{1}{1+3k}}} \right| \lesssim r^{-\frac{1}{1+3k}} (1 + 2^k)^{-\frac{2(k+1)+1}{2+1}} .$$

By Proposition(3.17) we could have

$$rr_x P_y(-t_1) \lesssim \left( 1 + \frac{d(x, t_1)}{r} \right)^{-\frac{2(k+1)+1}{2+1}} \lesssim \left( 1 + 2^k \right)^{-\frac{2(k+1)+1}{2+1}} .$$

Notice that $L \left( (2^{k+1})^{\frac{1}{1+3k}} \psi_{k,x}(t), \frac{1}{2\lambda+1} \right) \lesssim (r^{-1})^{\frac{1}{1+3k}}$, thus we could have

$$\left| \frac{\psi_{k,x}(t_1) - \psi_{k,x}(t_2)}{d_\lambda(t_1, t_2)^{\frac{1}{1+3k}}} \right| \lesssim (2^{-k-1}r^{-1})^{\frac{1}{1+3k}} .$$

From Formula (98), (99) and (100), we could obtain:

$$\frac{\psi_{k,x}(t_1) rr_x P_y(-t_1) - \psi_{k,x}(t_2) rr_x P_y(-t_2)}{d_\lambda(t_1, t_2)^{\frac{1}{1+3k}}} = \left| \frac{\psi_{k,x}(t_1) - \psi_{k,x}(t_2)}{d_\lambda(t_1, t_2)^{\frac{1}{1+3k}}} \right| rr_x P_y(-t_1) + \frac{rr_x P_y(-t_1) - rr_x P_y(-t_2)}{d_\lambda(t_1, t_2)^{\frac{1}{1+3k}}} \psi_{k,x}(t_2) \lesssim (2^{-k-1}r^{-1})^{\frac{1}{1+3k}} + (1 + 2^k)^{\frac{2(k+1)+1}{2+1}} (r^{-1})^{\frac{1}{1+3k}}$$

$$\lesssim \left( 1 + 2^k \right)^{-\frac{2(k+1)+1}{2+1}} (r^{-1})^{\frac{1}{1+3k}} .$$

Thus when $k \geq 1$,

$$\text{supp} \psi_{k,x}(t) rr_x P_y(-t) \subseteq \left( B(x, 2^{k+1}r) \setminus B(x, 2^{k-1}r) \right),$$

$$L \left( (1 + 2^k)^{\frac{2(k+1)+1}{2+1}} \psi_{k,x}(t) rr_x P_y(-t), \frac{1}{2\lambda+1} \right) \lesssim \left( (1 + 2^k) r \right)^{-\frac{1}{1+3k}} .$$

Applying Proposition(3.17), together with $|\psi_{k,x}(t)| \lesssim 1$, we could then have

$$\left( 1 + 2^k \right)^{\frac{2(k+1)+1}{2+1}} \psi_{k,x}(t) rr_x P_y(-t) \lesssim 1$$

That is when $k \geq 1$,

$$\left( 1 + 2^k \right)^{\frac{2(k+1)+1}{2+1}} \psi_{k,x}(t) rr_x P_y(-t) \lesssim 1$$

$$L \left( (1 + 2^k)^{\frac{2(k+1)+1}{2+1}} \psi_{k,x}(t) rr_x P_y(-t), \frac{1}{2\lambda+1} \right) \lesssim \left( (1 + 2^k) r \right)^{-\frac{1}{1+3k}} .$$

Then

$$\sup_{y > 0} |f \ast_{\lambda} P_y(x)| = \sup_{y > 0} \int_{\mathbb{R}} f(t) rr_x P_y(-t) \left| \frac{t^{2\lambda} dt}{r} \right| \lesssim \sum_{k=0}^{+\infty} \sup_{y > 0} \int_{\mathbb{R}} f(t) \psi_{k,x}(t) 2^{k+1} r rr_x P_y(-t) \left| \frac{t^{2\lambda} dt}{2^{k+1} r} \right|$$

$$\lesssim \sum_{k=0}^{+\infty} \left( 1 + 2^k \right)^{-\frac{1}{1+3k}} f^* (x) \lesssim f^*(x).$$
\[ \left( \int_{\mathbb{R}} \sup_{y > 0} |f * \lambda P_y(x)|^p |x|^{2\lambda} \, dx \right)^{1/p} \lesssim \left( \int_{\mathbb{R}} |f^*(x)|^p |x|^{2\lambda} \, dx \right)^{1/p} \quad \forall p > 0, \forall f \in L^1_\lambda(\mathbb{R}). \]

Proposition 3.19. For \( p > \frac{2\lambda+1}{\lambda+1} \), \( \forall f \in L^1_\lambda(\mathbb{R}) \), \( (\lambda > 0) \) we could have
\[ \left( \int_{\mathbb{R}} |f^*(x)|^p |x|^{2\lambda} \, dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}} \sup_{y > 0} |f * \lambda P_y(x)|^p |x|^{2\lambda} \, dx \right)^{1/p}. \]

C is dependent on \( p \) and \( \lambda \).

Proof. The Proposition could be deduced from Theorem (2.39).

Proposition 3.20. Let \( f^*_y(x) = \sup_{|s-x|<y}|f * \lambda P_y(s)| \), for \( p > \frac{2\lambda+1}{\lambda+1} \), \( \forall f \in L^1_\lambda(\mathbb{R}) \)
\[ \left( \int_{\mathbb{R}} |f^*_y(x)|^p |x|^{2\lambda} \, dx \right)^{1/p} \lesssim \left( \int_{\mathbb{R}} f^*(x)^p |x|^{2\lambda} \, dx \right)^{1/p} \]

Proof. The way to prove the inequality \( \left( \int_{\mathbb{R}} |f^*_y(x)|^p |x|^{2\lambda} \, dx \right)^{1/p} \lesssim \left( \int_{\mathbb{R}} |f^*(x)|^p |x|^{2\lambda} \, dx \right)^{1/p} \) is similar to Proposition (3.18). Let \( y = y(s, r) \), and assume \( x \geq 0 \):
\[ y = \begin{cases} \frac{r|s|^{-2\lambda}}{1+2\lambda} & \text{for } r < |s|^{2\lambda+1}, \\ \frac{r|s|^{-2\lambda}}{1+2\lambda+1} & \text{for } r \geq |s|^{2\lambda+1}. \end{cases} \]

First. When \( r \leq |s|^{2\lambda+1} \), by mean value theorem, \( 0 \leq \xi \leq |s|^{-2\lambda-1} \leq 1 \)
\[ |(s \pm r|s|^{-2\lambda})^{2\lambda+1} - s^{2\lambda+1}| \leq |s^{2\lambda+1}(2\lambda + 1)(1 \pm \xi) r|s|^{-2\lambda-1}| \leq 2^{\lambda}(2\lambda + 1) r \]
(102)

When \( r \geq |s|^{2\lambda+1} \), \( |s|^{-\frac{2\lambda+1}{\lambda+1}} \leq 1 \), by mean value theorem, \( 0 \leq \xi \leq 1 \)
\[ |(r^\frac{1}{2\lambda+1} \pm s)^{2\lambda+1} - s^{2\lambda+1}| \leq |(2\lambda + 1)(\xi \pm |s|) r|s|^{-2\lambda-1}| \leq 2^{\lambda}(2\lambda + 1) r. \]
(103)

Thus from formula (102) and (103) we could see that:
\[ \{ x : |x-s| < y \} \subseteq B_\lambda(s, 2^{\lambda}(2\lambda + 2) r), \text{ and } d_\lambda(x, s) < 2^{\lambda}(2\lambda + 2) r. \]

Therefore we could have:
\[ \{ s : |x-s| < y \} \subseteq B_\lambda(x, 2^{\lambda}(2\lambda + 2) r). \]

Together with Formula (101), then \( \exists k_0 = [2\lambda + 1] \in \mathbb{N} \) depending only on \( \lambda \) such that the following conclusions to be valid:
\[ \begin{pmatrix} \left(1 + 2^k\right)^\frac{2\lambda+2}{2k+2} \psi_{k,s}(t) r \tau_s P_y(-t) \lesssim 1 \\ L \left( \left(1 + 2^k\right)^\frac{2\lambda+2}{2k+2} \psi_{k,s}(t) r \tau_s P_y(-t), \frac{1}{2k+1} \right) \lesssim \left(1 + 2^k\right)^{\frac{2\lambda+2}{2k+2}} \\ \sup \psi_{k,s}(t) r \tau_s P_y(-t) \subseteq \left( B_\lambda(x, 2^{k+k_0+1} r) \setminus B_\lambda(x, 2^{k-2} r) \right) \end{pmatrix}. \]

Thus we obtain:
\[ f^*_y(x) = \sup_{|s-x|<y} \left| \int_{\mathbb{R}} f(t) r \tau_s P_y(-t) \frac{|t|^{2\lambda} \, dt}{r} \right| \leq \sum_{k=0}^{+\infty} \sup_{|s-x|<y} \int_{\mathbb{R}} f(t) \psi_{k,s}(t) 2^{k+1} r \tau_s P_y(-t) \frac{|t|^{2\lambda} \, dt}{2^{k+1} r} \right| \lesssim \sum_{k=0}^{+\infty} \left(1 + 2^k\right)^{-\frac{2\lambda+1}{2k+1}} f^*(x) \]
\[ \lesssim f^*(x). \]
We could prove functions defined as following:

**Definition 3.22**

Theorem 3.21. \( \forall (\int \) function \( F(x, y), \) then \( F(x, y) \) is the boundary value of the function \( F(x, y), \) then \( F(x, y) \) is the \( \lambda \)-Poisson integral of the function \( f(x), g(x) = Rf(x). \) Then \( \) and \( \) \( \) lead to the Proposition.

Thus Theorem (2.39) and Proposition (2.24) lead to

\[
\left( \int_{\mathbb{R}} |f^*(x)|^p |x|^{2\lambda} dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}} \sup_{|x-s| < r} |f^*(x)|^p |x|^{2\lambda} dx \right)^{1/p} \leq C \left( \int_{\mathbb{R}} |f^*_\lambda(x)|^p |x|^{2\lambda} dx \right)^{1/p}
\]

(107)

Thus Formula (104) and (107) lead to the Proposition.

Then Theorem (3.21) could be obtained through Proposition (3.19), Proposition (3.18), Proposition (3.13), Proposition (3.20), Proposition (3.16) and Proposition (3.3(iii)):

**Theorem 3.21.** \( \forall p > \frac{2\lambda+1}{2\lambda+2}, \) \( F(x, y) = u(x, y) + iv(x, y) \in H_\lambda^p(\mathbb{R}^2_+) \cap H_{\lambda+1}^p(\mathbb{R}^2_+) \cap H_{\lambda+2}^p(\mathbb{R}^2_+), \) \( f(x) \) is the boundary value of the function \( F(x, y), \) then \( F(x, y) \) is the \( \lambda \)-Poisson integral of the function \( f(x), g(x) = Rf(x). \) Then

\[
\|F\|_{H_\lambda^p(\mathbb{R}^2_+)} \sim \|g^*(x)\|_{L^p_\lambda(\mathbb{R})}
\]

Because \( H^p_\lambda(\mathbb{R}^2_+) \cap H^p_{\lambda+1}(\mathbb{R}^2_+) \cap H^p_{\lambda+2}(\mathbb{R}^2_+) \) is dense in the space \( H^p_\lambda(\mathbb{R}^2_+), \) together with Theorem (3.21), we could have an equivalent definition of \( H^p_\lambda(\mathbb{R}) \) as following:

**Definition 3.22** (\( H^p_\lambda(\mathbb{R}), \bar{H}^p_\lambda(\mathbb{R}), \) and \( MH^p_\lambda(\mathbb{R}) \) space for \( p > \frac{2\lambda+1}{2\lambda+2}, \) and \( \lambda \)-distribution).

\[
\bar{H}^p_\lambda(\mathbb{R}) = \left\{ g \in L^2_\lambda(\mathbb{R}) : g^*(x) \in L^p_\lambda(\mathbb{R}) \right\}
\]

The completion of \( \bar{H}^p_\lambda(\mathbb{R}) \) with the norm \( \| \cdot \|^p_{H^p_\lambda(\mathbb{R})} \) is denoted as \( H^p_\lambda(\mathbb{R}). \) A \( \lambda \)-distribution is a bounded linear functional on \( S(\mathbb{R}, |x|^{2\lambda} dx) \). The space \( MH^p_\lambda(\mathbb{R}) \) is defined as

\[
MH^p_\lambda(\mathbb{R}) = \left\{ g \mid g \text{ is a } \lambda \text{-distribution} : g^*(x) \in L^p_\lambda(\mathbb{R}) \right\}.
\]

Obviously we could have

\[
\bar{H}^p_\lambda(\mathbb{R}) \subseteq H^p_\lambda(\mathbb{R}) \subseteq MH^p_\lambda(\mathbb{R}).
\]

We could prove \( H^p_\lambda(\mathbb{R}) = MH^p_\lambda(\mathbb{R}) \) after the atomic decomposition.

Let \( F(x, y) = G(x, y) + iH(x, y) \) be a \( \lambda \)-analytic function. Let \( F_1(x, y) \) and \( F_2(x, y) \) be \( \lambda \)-analytic functions defined as following:

\[
F_1(x, y) = \frac{G(x, y) + G(-x, y)}{2} + i \frac{H(x, y) - H(-x, y)}{2} = G_1(x, y) + iH_1(x, y),
\]

\[
F_2(x, y) = \frac{G(x, y) - G(-x, y)}{2} + i \frac{H(x, y) + H(-x, y)}{2} = G_2(x, y) + iH_2(x, y).
\]
Proposition 3.23. If \( F(x, y) = G(x, y) + iH(x, y) \in H_\lambda^p(\mathbb{R}_+^2) \), \( \frac{2\lambda}{2\lambda + 1} < p \leq 1 \), then
\[
\|F_1\|_{H_\lambda^p(\mathbb{R}_+^2)}^p + \|F_2\|_{H_\lambda^p(\mathbb{R}_+^2)}^p \geq \|F\|_{H_\lambda^p(\mathbb{R}_+^2)}^p
\]

Proof. Let \( F(x) = F(x, y) - G(x, y) + iH(x, y) \in H_\lambda^p(\mathbb{R}_+^2) \). By Proposition (3.7), \( F_1(x, y) \) and \( F_2(x, y) \) are \( \lambda \)-analytic functions. Then by Theorem (3.8) we could deduce that \( F_1, F_2 \in H_\lambda^p(\mathbb{R}_+^2) \), and their norms satisfy following:
\[
\|(G_1)_p\|_{L_\lambda^p} \approx \|F_1\|_{H_\lambda^p(\mathbb{R}_+^2)}, \|(G_2)_p\|_{L_\lambda^p} \approx \|F_2\|_{H_\lambda^p(\mathbb{R}_+^2)}, \|G_p\|_{L_\lambda^p} \approx \|F\|_{H_\lambda^p(\mathbb{R}_+^2)}.
\]
It is obviously to see that:
\[
\left(\|F_1\|_{H_\lambda^p(\mathbb{R}_+^2)}^p + \|F_2\|_{H_\lambda^p(\mathbb{R}_+^2)}^p\right) \gtrsim \|F\|_{H_\lambda^p(\mathbb{R}_+^2)}^p.
\]

Then we need only to prove:
\[
\|F_1\|_{H_\lambda^p(\mathbb{R}_+^2)}^p + \|F_2\|_{H_\lambda^p(\mathbb{R}_+^2)}^p \lesssim \|F\|_{H_\lambda^p(\mathbb{R}_+^2)}^p
\]

(108)

We could deduce that:
\[
\|(G_1)_p\|_{L_\lambda^p} = \int_{\mathbb{R}} \sup_{|s - x < y|} \left| \frac{G(s, y) - G(-s, y)}{2} \right|^p |x|^{2\lambda} dx \leq \int_{\mathbb{R}} \left( \frac{1}{2p} \sup_{|s - x < y|} |G(s, y)|^p |x|^{2\lambda} dx + \int_{\mathbb{R}} \frac{1}{2p} \sup_{|s + x < y|} |G(s, y)|^p |x|^{2\lambda} dx \right)
\]
\[
= 2^{1-p} \int_{\mathbb{R}} \sup_{|s - x < y|} |G(s, y)|^p |x|^{2\lambda} dx
\]
\[
= 2^{1-p} \|G_p\|_{L_\lambda^p}^p.
\]
In the same way, we could deduced that
\[
\|(G_2)_p\|_{L_\lambda^p} \leq 2^{1-p} \|G_p\|_{L_\lambda^p}^p,
\]
Thus
\[
\|G_p\|_{L_\lambda^p}^p \approx \|(G_1)_p\|_{L_\lambda^p}^p + \|(G_2)_p\|_{L_\lambda^p}^p.
\]
This proves the Proposition.

For any Banach spaces \( A, B, C \) with the same norm \( \| \cdot \| \), let \( \oplus \) to be a symbol denote that
\[
A = B \oplus C
\]
implies that for any \( a \in A \), there exists a unique \( b \in B \) and a unique \( c \in C \) satisfying
\[
a = b + c, \text{ and } \|a\| = \|b\| + \|c\|.
\]

We will define \( MH_\lambda^p(\mathbb{R}) \) as following, from which we could give an atomic decomposition of the \( H_\lambda^p(\mathbb{R}) \)

Definition 3.24. \( MH_\lambda^p(\mathbb{R}), H_\lambda^p(\mathbb{R}) \) and \( \tilde{H}_\lambda^p(\mathbb{R}) \) spaces. \( H_\lambda^p(\mathbb{R}_+^2) \) space. Let \( F_0 = u_o + iv_o, F_e = u_e + iv_e \) and \( F \) be functions as in Proposition (3.7):
\[
MH_\lambda^p(\mathbb{R}) \triangleq \{ g \text{ is odd or even : } g \in MH_\lambda^p(\mathbb{R}) \} \text{ for } p > \frac{2\lambda + 1}{2\lambda + 2}
\]
\[
\tilde{H}_\lambda^p(\mathbb{R}) \triangleq \{ g \text{ is odd or even : } g \in H_\lambda^p(\mathbb{R}) \} \text{ for } p > \frac{2\lambda + 1}{2\lambda + 2}
\]
\[
H_\lambda^p(\mathbb{R}_+^2) \triangleq \{ F_e, F_o : F \in H_\lambda^p(\mathbb{R}_+^2) \} \text{ for } p > \frac{2\lambda}{2\lambda + 1}.
\]
\* denote odd or even: 'o' or 'e'. Thus by Proposition (3.7) and (145), we could obtain

\[ H^p_\lambda(\mathbb{R}^2_+) = H^p_\lambda_0(\mathbb{R}^2_+) \bigoplus H^p_\lambda_0(\mathbb{R}^2_+) \]

Also it is not difficult to prove the following holds:

\[ H^p_\lambda(\mathbb{R}) = H^p_\lambda_x(\mathbb{R}) \bigoplus H^p_\lambda_0(\mathbb{R}), \]

\[ \hat{H}^p_\lambda(\mathbb{R}) = \hat{H}^p_\lambda_x(\mathbb{R}) \bigoplus \hat{H}^p_\lambda_0(\mathbb{R}), \]

and

\[ MH^p_\lambda(\mathbb{R}) = MH^p_\lambda_x(\mathbb{R}) \bigoplus MH^p_\lambda_0(\mathbb{R}). \]

### 3.3 Atomic decomposition of \( MH^p_\lambda(\mathbb{R}) \) for \( \frac{2\lambda+1}{2\lambda+2} < p \leq 1 \)

Let \( B \) be ball in the Homogeneous space, its closure set is denoted by \( \overline{B} \). Let I to be the Euclid interval: \( I(x_0, \delta_0) = (x_0 - \delta_0, x_0 + \delta_0) = \{ y : |y - x_0| < \delta_0 \} \). The original following lemma could be seen in [14], we make a little modification of it.

**Lemma 3.25. [14] Whitney covering Lemma**

Assume that \( O \) is an open set in \( \mathbb{R} \) and \( |O|_\lambda < \infty \). Then there exist two series of \( \{ x_k : x_k \in O \}_k \) and \( \{ r_k : r_k > 0 \}_k \) such that

(i) \( O = \bigcup B(x_k, r_k) \), and \( B(x_k, r_k/4^{2\lambda+1}) \) are mutually disjoint balls.

(ii) \( B(x_k, 36^{2\lambda+1} r_k) \bigcap O^c = \emptyset, \) and \( B(x_k, 108^{2\lambda+1} r_k) \bigcap O^c \neq \emptyset \)

(iii) \( \{ B(x_k, 36^{2\lambda+1} r_k) \}_k \) have the bounded intersection property.

**Definition 3.26 (\( p_\lambda \)-atom ).** A function \( a(x) \) is also a \( p_\lambda \)-atom, if it satisfying the following condition

\[
\begin{align*}
\text{i)} & \quad \| a(x) \|_{L^2_\lambda} \lesssim \frac{1}{|B(x_0, r_0)|^{1/p}} \\
\text{ii)} & \quad \text{supp } a(x) \subseteq B(x_0, r_0), \text{ and } r_0 \frac{1}{x_0} < |x_0/2| \\
\text{iii)} & \quad \int_\mathbb{R} t^k a(t)|t|^{2\lambda} dt = 0 \quad (k = 0, 1, 2, 3 \ldots \kappa) \\
\text{iv)} & \quad 0 < r_0 \frac{1}{x_0} < |x_0/2| \\
\text{v)} & \quad \kappa \geq 2 \left[ (2\lambda + 1) \frac{1-p}{p} \right] \text{ when } \frac{2\lambda}{2\lambda + 1} < p \leq \frac{2\lambda + 1}{2\lambda + 2}, \ \kappa \geq 0 \text{ when } \frac{2\lambda + 1}{2\lambda + 2} < p \leq 1.
\end{align*}
\]

**Proposition 3.27.** Let \( B(x_0, r_0) \) to be the ball in the homogeneous type space: \( B(x_0, r_0) = \{ y : d_\lambda(y, x_0) < r_0 \} \). And let \( I_0 \) be the Euclidean interval: \( I_0 = (x_0 - \delta_2, x_0 + \delta_1) = B(x_0, r_0). \) \( \forall t \in B(x_0, r_0), \) for \( x_0 > 0 \) and \( r_0 \frac{1}{x_0} < |x_0/2| \). The following holds:

\[
\delta_1 < r_0 \frac{1}{x_0} < |x_0/2|, \\
\delta_2 < r_0 \frac{1}{x_0} < |x_0/2|, \quad |x_0| \approx |s| \text{ for } \forall s \in B(x_0, r_0), \\
\delta_1 \approx \delta_2 \approx \frac{r_0}{x_0^\lambda}.
\]

**Proof.** When \( r_0 \frac{1}{x_0} < |x_0/2| \), it is easy to see that:

\[
|x_0| \approx |s| \text{ for } \forall s \in B(x_0, r_0).
\]
To estimate $\delta_1$ and $\delta_2$, we could see that in fact:

$$
\delta_2 = \left| \left( x_0^{2\lambda+1} - r_0 \right)^{1/\lambda} - x_0 \right|
$$

$$
\delta_1 = \left| \left( x_0^{2\lambda+1} + r_0 \right)^{1/\lambda} - x_0 \right|
$$

With the fact that

$$
|y - x|^{2\lambda+1} < |y^{2\lambda+1} - x^{2\lambda+1}|
$$

holds for $x, y > 0$, it is easy to see that $\delta_2 \leq \delta_1 \leq r_0^{1/\lambda}$. Thus we have $B(x_0, r_0) \subseteq I(x_0, \delta_1)$. By Taylor expansion near the origin, for $r_0^{1/\lambda} < |x_0/2|$, we could obtain that

$$
\left| \left( x_0^{2\lambda+1} \pm r_0 \right)^{1/\lambda} - x_0 \right| \approx x_0 \left| \left( 1 \pm \frac{r_0}{x_0^{\lambda+1}} \right)^{1/\lambda} - 1 \right| \approx \frac{r_0}{x_0^{\lambda}}
$$

Therefore we obtain:

$$
\delta_1 \approx \delta_2 \approx \frac{r_0}{x_0^{\lambda}}
$$

This proves the proposition.

**Proposition 3.28.** Let $I(x_0, \delta_0)$ to be the Euclid interval: $(x_0 - \delta_0, x_0 + \delta_0)$. Let $B(x_0, r_0)$ to be the ball in the homogenous space: $B(x_0, r_0) = \{ y : d(x, y) < r_0 \}$ with $r_0^{1/\lambda} < |x_0/2|$. There exists constants $c_1 > 0$ and $c_2 > 0$ independent on $x_0$ and $r_0$, such that the following facts hold:

$$I(x_0, c_2 \frac{r_0}{x_0^{\lambda}}) \subseteq B(x_0, r_0) \subseteq I(x_0, c_1 \frac{r_0}{x_0^{\lambda}})
$$

And the following holds:

$$B(x_0, r_0) \subseteq I(x_0, \frac{r_0}{x_0^{\lambda}})
$$

**Proof.** In fact notice that when $x > 0$ and $y > 0$:

$$|y - x| < |y^{2\lambda+1} - x^{2\lambda+1}|^{1/\lambda}
$$

Then we could obtain $B(x_0, r_0) \subseteq I(x_0, r_0^{1/\lambda})$. By Proposition(3.27), we could obtain that

$$\max_{y, x \in B(x_0, r_0)} |y - x| \approx \frac{r_0}{x_0^{\lambda}}
$$

Therefore there are constants $c_1 > 0$ and $c_2 > 0$ independent on $x_0$ and $r_0$, such that

$$I(x_0, c_2 \frac{r_0}{x_0^{\lambda}}) \subseteq B(x_0, r_0) \subseteq I(x_0, c_1 \frac{r_0}{x_0^{\lambda}})
$$

Hence the proposition holds.

**Proposition 3.29.** \{\hat{B}_{k,j}\}_j is the same as above. The following facts holds:

(i) If $\hat{B}_{k,j} \cap \hat{B}_{k,i} \neq \emptyset$, when $j \neq i$, we have $r_{k,j} \approx r_{k,i}$;

(ii) If $\hat{B}_{k+1,j} \cap \hat{B}_{k,i} \neq \emptyset$, then $r_{k+1,j} < 4^{2\lambda+1}r_{k,i}$, and $\hat{B}_{k+1,j} \subseteq B(x_{k,j}, 18^{2\lambda+1}r_{k,i})$;

(iii) For every $j$, there exists a constant $M$, such that the number of $\hat{B}_{k,i}$ satisfying

$$\hat{B}_{k+1,j} \cap \hat{B}_{k,i} \neq \emptyset$$

is less than $M$. 

Proof. By the Whitney Lemma, \( r_{k,j} \approx \text{dist}(\tilde{B}_{k,j}(O_k)^c) \). Denote \( d_{k,j} = \text{dist}(\tilde{B}_{k,j}, (O_k)^c) \), \( d_{k,i} = \text{dist}(\tilde{B}_{k,i}, (O_k)^c) \). \( s \) is a point satisfying \( s \in \tilde{B}_{k,j} \cap \tilde{B}_{k,i} \). Denote \( d_s = \text{dist}(s, (O_k)^c) \). Thus obviously we have the following conclusions:

\[
d_{k,i} \leq d_s \leq d_{k,i} + r_{k,i}
\]

and

\[
d_{k,j} \leq d_s \leq d_{k,j} + r_{k,j}.
\]

Together with the fact \( r_{k,j} \approx d_{k,j} \) and \( r_{k,i} \approx d_{k,i} \), thus (i) holds.

Suppose \( \tilde{B}_{k+1,j} \cap \tilde{B}_{k,i} \neq \emptyset \). Then

\[
d_\lambda(x_{k+1,j}, x_{k,i}) < 2^{2\lambda+1}(r_{k+1,j} + r_{k,i}).
\]

Notice that \( O_{k+1} \subseteq O_k \), by Lemma(3.25), we could deduce that:

\[
dist(x_{k+1,j}, (O_k)^c) > dist(x_{k+1,j}, (O_{k+1})^c) > 18^{2\lambda+1}r_{k+1,j}.
\]

Thus

\[
18^{2\lambda+1}r_{k+1,j} < dist(x_{k+1,j}, (O_k)^c) < d_\lambda(x_{k+1,j}, x_{k,i}) + dist(x_{k,i}, (O_k)^c) < 2^{2\lambda+1}(r_{k+1,j} + r_{k,i}) + 56^{2\lambda+1}r_{k,i}.
\]

Thus

\[
16^{2\lambda+1}r_{k+1,j} < 64^{2\lambda+1}r_{k,i}.
\]

Thus (ii) holds.

Notice that \( \{B(x_{k,i}, 18^{2\lambda+1}r_{k,i})\}_i \) satisfy the bounded intersection property, thus (iii) is an immediate corollary of (ii). This proves the Proposition.

Let \( \eta(x) \) be a fixed positive smooth even function, satisfying: \( 0 \leq \eta(x) \leq 1 \), \( \eta(x) = \eta(-x) \). \( \eta(x) = 1 \) when \( x \in [-1, 1] \), \( \eta(x) = 0 \) when \( x \in [-2^{\lambda+1}, 2^{\lambda+1}]^c \). There exists a constant \( C \) such that \( |\eta'(x)| \leq C \).

Let \( \zeta_{k,j}(x) = \sum_{x_{k,j} \in (O_k)^c} \eta\left(\frac{x_{k,j} - x}{r_{k,j}}\right) \), then \( \xi_{k,j} \) form a partition of unity for the set \( O_k \), that is to say: \( \chi_{O_k} = \sum_j \xi_{k,j} \) with each \( \xi_{k,j} \) supported in the ball \( \tilde{B}_{k,j} \). By the bounded intersection property of \( \{\tilde{B}_{k,j}\}_j \), there exists constant \( C \) such that \( 1 \leq \sum_j \zeta_{k,j}(x) \leq C \). Then:

\[
\begin{cases}
\zeta_{k,j}(x) = 1, & \text{for } x \in B_{k,j} \\
0 \leq \zeta_{k,j}(x) \leq 1 \text{ and } \zeta_{k,j}'(x) \leq C\frac{x^2}{r_{k,j}}, & \text{for } x \in \tilde{B}_{k,j} \\
\zeta_{k,j}(x) = 0, & \text{for } x \in \left(\tilde{B}_{k,j}\right)^c \end{cases} \quad (109)
\]

Similar to Formula(96), we obtain:

\[
\begin{cases}
|\zeta_{k,j}(x)| \lesssim 1 \\
L(\zeta_{k,j}(x), \frac{1}{2^{\lambda+1}r_{k,j}}) \lesssim (2^{\lambda+1}r_{k,j})^{-\frac{1}{2^{\lambda+1}r_{k,j}}} \\
\text{supp} \zeta_{k,j}(x) \subseteq B(x_{k,j}, 2^{2\lambda+1}r_{k,j}).
\end{cases} \quad (110)
\]

Also

\[
\text{supp} \zeta_{k,j}(x) \subseteq B(x_{k,j}, 2^{2\lambda+1}r_{k,j}).
\]
By the mean value theorem, assume \( y \geq x, \exists l : x \leq l \leq y \):

\[
\left| \frac{\xi_{k,j}(y) - \xi_{k,j}(x)}{d_\lambda(y, x)} \right| = \left| \eta\left(\frac{y^{2\lambda+1} - x^{2\lambda+1}}{r_{k,j}}\right) \left(\sum \eta\left(\frac{y^{2\lambda+1} - z^{2\lambda+1}}{r_{k,z}}\right)^{-1} - \eta\left(\frac{y^{2\lambda+1} - z^{2\lambda+1}}{r_{k,z}}\right) \left(\sum \eta\left(\frac{y^{2\lambda+1} - z^{2\lambda+1}}{r_{k,z}}\right)^{-1}\right)\right) - \eta\left(\frac{x^{2\lambda+1} - x^{2\lambda+1}}{r_{k,x}}\right) \left(\sum \eta\left(\frac{x^{2\lambda+1} - z^{2\lambda+1}}{r_{k,z}}\right)^{-1}\right) \right| \frac{1}{y^{2\lambda+1} - x^{2\lambda+1}}
\]

\[
= \left| \frac{\eta\left(\frac{y^{2\lambda+1} - x^{2\lambda+1}}{r_{k,j}}\right) \sum \eta\left(\frac{y^{2\lambda+1} - z^{2\lambda+1}}{r_{k,z}}\right)^{-1} - \eta\left(\frac{x^{2\lambda+1} - x^{2\lambda+1}}{r_{k,x}}\right) \left(\sum \eta\left(\frac{x^{2\lambda+1} - z^{2\lambda+1}}{r_{k,z}}\right)^{-1}\right)}{y^{2\lambda+1} - x^{2\lambda+1}} \right| (111)
\]

Because \( \forall t \in \tilde{B}_{k,j} \), \( t \approx x_{k,j} \), together with Proposition (3.29) and the fact that there exists a constant \( C \) independent of \( k, j \) such that \( 1 \leq \zeta_{k,j} \leq C \), we could have the following facts from Formula (111):

\[
\left| \frac{\xi_{k,j}(y) - \xi_{k,j}(x)}{d_\lambda(y, x)} \right| \leq C \left( \frac{1}{r_{k,j}} \right) |y^{2\lambda+1} - x^{2\lambda+1}| \frac{1}{y^{2\lambda+1} - x^{2\lambda+1}} \leq (2^{2\lambda+1} r_{k,j})^{-\frac{1}{2\lambda+1}}
\]

\[
\int \xi_{k,j}(x)|x|^{2\lambda} dx \approx |\tilde{B}_{k,j}|_\lambda \approx r_{k,j}
\]

\[
\|\xi_{k,j}(x)\|_\infty \lesssim 1.
\]

That is

\[
\begin{cases}
|\xi_{k,j}(x)| \lesssim 1 \\
L \left( \xi_{k,j}(x), \frac{1}{2\lambda+1} \right) \lesssim (2^{2\lambda+1} r_{k,j})^{-\frac{1}{2\lambda+1}} \\
\text{supp} \xi_{k,j}(x) \subseteq B(x_{k,j}, 2^{2\lambda+1} r_{k,j}) \\
\int \xi_{k,j}(x)|x|^{2\lambda} dx \approx |\tilde{B}_{k,j}|_\lambda \approx r_{k,j}
\end{cases}
\]

(112)

Let \( P^\kappa \) to be the \( \kappa \)-order polynomials. Let \( P^\kappa_{k,j} \) to be the \( \kappa \)-order polynomials with its Hilbert norm: \( f \in P^\kappa \)

\[
\| f \|_{P^\kappa_{k,j}} = \left( \frac{\int |f(x)|^2 \xi_{k,j}(x)|x|^{2\lambda} dx}{\int \xi_{k,j}(x)|x|^{2\lambda} dx} \right) \frac{1}{2}\kappa.
\]

In addition, let \( \{ \pi_{k,j}^l \} \) to be the orthonormal basis associated with the above norm.

**Proposition 3.30.** There exists a constant \( C \) independent of \( j \) and \( k \), such that

\[
\begin{cases}
|\xi_{k,j}(x)\pi_{k,j}^l(x)| \leq C \\
L \left( \xi_{k,j}(x)\pi_{k,j}^l(x), \frac{1}{2\lambda+1} \right) \leq (2^{2\lambda+1} r_{k,j})^{-\frac{1}{2\lambda+1}} \\
\text{supp} \xi_{k,j}(x)\pi_{k,j}^l(x) \subseteq B(x_{k,j}, 2^{2\lambda+1} r_{k,j}) \\
\sup_{t \in \tilde{B}_{k,j}} \left( \frac{r_{k,j}}{2\lambda+1} \right)^m \left( \frac{d}{dt} \right)^m \pi_{k,j}^l(t) \leq C, \quad C \text{ is independent on } j \text{ and } k.
\end{cases}
\]

(113)
Proof. i) \( \text{supp}_{k,j}(x)\pi^i_{k,j}(x) \subseteq B(x_{k,j}, 2^{-1}r_{k,j}) \) is obvious.

ii) Next we will prove \( \left| \xi_{k,j}(x)\pi^i_{k,j}(x) \right| \leq C \). Let \( \widetilde{B}_{k,j} \) defined as above. Let \( I_{k,j}^1 = \int (x_{k,j}, c_1\frac{r_{k,j}}{x_{k,j}}) = \{ y : y \in [x_{k,j} - c_1\frac{r_{k,j}}{x_{k,j}}, x_{k,j} + c_1\frac{r_{k,j}}{x_{k,j}}] \}, \)
\( I_{k,j}^2 = \{ y : y \in [x_{k,j} - c_2\frac{r_{k,j}}{x_{k,j}}, x_{k,j} + c_2\frac{r_{k,j}}{x_{k,j}}] \}. \)

Then by the Proposition(3.28), there are constants \( c_1, c_2 > 0 \) not depending on \( j \) and \( k \) such that the following holds:

\[ I_{k,j}^2 \subseteq \widetilde{B}_{k,j} \subseteq I_{k,j}^1. \]

Together with Proposition(3.27): \( \sup_{t \in B_{k,j}} |t - x_{k,j}| \approx \frac{r_{k,j}}{x_{k,j}} \) and \( |x_{k,j}| \approx |t| \) when \( t \in \widetilde{B}_{k,j} \) and Formula(109), we obtain the following inequality:

\[
1 = \left\| \pi^i_{k,j} \right\|_{L^p_{x_{k,j}}}^2 = \int \left| \pi^i_{k,j}(x)\pi^i_{k,j}(x) \right| x^{2\lambda} dx \\
\approx \frac{1}{r_{k,j}^{2\lambda}} \int_{\widetilde{B}_{k,j}} \left| \pi^i_{k,j}(x)\pi^i_{k,j}(x) \right| x^{2\lambda} dx \\
\geq \frac{1}{r_{k,j}^{2\lambda}} \int_{I_{k,j}^2} \left| \pi^i_{k,j}(x)\pi^i_{k,j}(x) \right| x^{2\lambda} dx \\
\geq \int_{[-1,1]} \left| \pi^i_{k,j}(x)\pi^i_{k,j}(x) + c_2r_{k,j}t \right| x^{2\lambda} \frac{2r_{k,j}}{x_{k,j}^{2\lambda}} dt.
\]

By the fact that any two norms on a finite dimensional space are equivalent, we could obtain for a fixed \( m \in \mathbb{N} \):

\[
\left( \int_{[-1,1]} |p(t)|^2 dt \right)^{1/2} \geq C \sup_{t \in [-1,1]} \left| \left( \frac{d}{dt} \right)^m P(t) \right|, \forall p(t) \in P^m.
\]

If we take \( p(t) = \pi^i_{k,j}(x_{k,j} + c_2\frac{r_{k,j}}{x_{k,j}} t) \), then by the Proposition(3.28), we could obtain:

\[
1 = \left( \int_{[-1,1]} |p(t)|^2 dt \right)^{1/2} \geq C \sup_{t \in [-1,1]} \left| \left( \frac{d}{dt} \right)^m \pi^i_{k,j}(x_{k,j} + c_2\frac{r_{k,j}}{x_{k,j}} t) \right| \tag{114}
\]

\[
\geq C \sup_{t \in I_{k,j}} \left| (\frac{r_{k,j}}{x_{k,j}})^m \pi^i_{k,j}(t) \right| \\
\geq C \sup_{t \in \widetilde{B}_{k,j}} \left| (\frac{r_{k,j}}{x_{k,j}})^m \pi^i_{k,j}(t) \right|.
\]

Let \( m = 0 \) in (114) and together with Formula(112), we could obtain \( \left| \xi_{k,j}(x)\pi^i_{k,j}(x) \right| \leq C. \) Also we could conclude that \( \sup_{t \in \widetilde{B}_{k,j}} \left| \pi^i_{k,j}(t) \right| \leq C. \) Let \( m = 1 \) in (114), inequality \( \sup_{t \in \widetilde{B}_{k,j}} \left| \left( \frac{d}{dt} \right) \pi^i_{k,j}(t) \right| \leq C(\frac{x_{k,j}^{2\lambda}}{r_{k,j}}) \) holds. Thus we have

\[
\left| \sup_{t_1, t_2 \in \widetilde{B}_{k,j}} \frac{\pi^i_{k,j}(t_1) - \pi^i_{k,j}(t_2)}{d_{\lambda}(t_1, t_2)^{2\lambda}} \right| = \left| \sup_{t_1, t_2 \in B_{k,j}} \frac{\pi^i_{k,j}(t_1) - \pi^i_{k,j}(t_2)}{t_1 - t_2} \right| \frac{t_1 - t_2}{d_{\lambda}(t_1, t_2)^{2\lambda}} \leq \frac{x_{k,j}^{2\lambda}}{r_{k,j}} \frac{t_1 - t_2}{d_{\lambda}(t_1, t_2)^{2\lambda}}.
\]

Note that \( \sup_{t \in \widetilde{B}_{k,j}} |t - x_{k,j}| \approx \frac{r_{k,j}}{x_{k,j}} \) and \( |x_{k,j}| \approx |t| \) when \( t \in \widetilde{B}_{k,j} \) by Proposition(3.27). Together
with the mean value theorem, we could obtain from the above inequality that: $\exists \xi \in \tilde{B}_{k,j}$

$$\frac{\left| \sum_{k,j} \frac{\lambda^{2\lambda}}{r_{k,j}} t_1 - t_2 \right|}{d_\lambda(t_1, t_2)\tilde{t}} = \left| \frac{\lambda^{2\lambda}}{r_{k,j}} t_1 - t_2 \right|$$

$$\Rightarrow \frac{\left| \sum_{k,j} \frac{\lambda^{2\lambda}}{r_{k,j}} t_1 - t_2 \right|}{d_\lambda(t_1, t_2)\tilde{t}} \leq \frac{\left| \sum_{k,j} \frac{\lambda^{2\lambda}}{r_{k,j}} t_1 - t_2 \right|}{d_\lambda(t_1, t_2)\tilde{t}}$$

Thus

$$\left| \sup_{t_1, t_2 \in B_{k,j}} \frac{\pi_{k,j}^l(t_1) - \pi_{k,j}^l(t_2)}{d_\lambda(t_1, t_2)\tilde{t}} \right| < \tilde{(r_{k,j})}^{-\frac{\tilde{t}}{\tilde{t}}}.$$}

From the above inequality and Formula (112), the following inequality holds:

$$\left| \frac{\pi_{k,j}^l(t_1)\xi_{k,j}(t_1) - \pi_{k,j}^l(t_2)\xi_{k,j}(t_2)}{d_\lambda(t_1, t_2)\tilde{t}} \right| = \left| \frac{\pi_{k,j}^l(t_1) - \pi_{k,j}^l(t_2)}{d_\lambda(t_1, t_2)\tilde{t}} \xi_{k,j}(t_1) + \frac{\xi_{k,j}(t_1) - \xi_{k,j}(t_2)}{d_\lambda(t_1, t_2)\tilde{t}} \pi_{k,j}^l(t_2) \right| \leq C\tilde{(r_{k,j})}^{-1}.$$}

Hence, the proposition is proved.

**Proposition 3.31.** For $\forall f \in MH^P_{\infty}(\mathbb{R}) \cap L^1(\mathbb{R})$ there exists a unique polynomial $P_{k,j}(x) \in P^\infty$ such that

$$\int \left\{ f(x) - P_{k,j}(x) \right\} Q(x)\xi_{k,j}(x)|x|^{2\lambda}dx = 0, \quad \forall Q(x) \in P^\infty \quad \text{(115)}$$

and unique polynomial $P_{k+1,j}^i \in P^\infty$ such that $\forall Q(x) \in P^\infty$ the following holds:

$$\int \left\{ f(x) - P_{k+1,j}(x) \right\} Q(x)\xi_{k+1,j}(x)|x|^{2\lambda}dx = \int P_{k+1,j}^i(x)Q(x)\xi_{k+1,j}(x)|x|^{2\lambda}dx. \quad \text{(116)}$$

**Proof.** The existence of the $P_{k+1,j}^i$ is obvious. In fact we could let:

$$P_{k,j}(x) = \sum_{n=0}^\infty \frac{\int f(y)\pi_{k,j}^l(y)\xi_{k,j}(y)|y|^{2\lambda}dy}{\int \xi_{k,j}(y)|y|^{2\lambda}dx} \pi_{k,j}^l(x) \quad \text{(117)}$$

$$P_{k+1,j}^i(x) = \sum_{n=0}^\infty \frac{\int \left\{ f(y) - P_{k+1,j}(y) \right\} \pi_{k+1,j}^l(y)\xi_{k,j}(y)|y|^{2\lambda}dy}{\int \xi_{k+1,j}(y)|y|^{2\lambda}dy} \pi_{k+1,j}^l(x) \quad \text{(118)}$$

Formula (118) and (117) are the solutions satisfying (115) and (116) respectively.

The proof of the uniqueness of the solution is also simple. For if $\tilde{P}_{k+1,j}(x)$ is the solution of Formula (115) then we could obtain:

$$\int \left\{ \tilde{P}_{k+1,j}(x) - P_{k+1,j}(x) \right\} Q(x)\xi_{k,j}(x)|x|^{2\lambda}dx = 0, \forall Q(x) \in P^\infty$$

Taking $Q(x) = \tilde{P}_{k+1,j}(x) - P_{k+1,j}(x)$, we obtain

$$\int \left\{ \tilde{P}_{k+1,j}(x) - P_{k+1,j}(x) \right\}^2 \xi_{k,j}(x)|x|^{2\lambda}dx = 0.$$
Noticing that \( \xi_{k,j}(x) \geq 0 \), and using the above relation, we deduce \( \tilde{P}_{k+1,j}(x) = P_{k+1,j}(x) \). By a method similar to the above, we can show that the polynomial \( P^i_{k+1,j}(x) \) satisfying (116) is also unique. Hence the proposition is proved. \( \square \)

For \( P^i_{k+1,j}(x) \) and \( P_{k+1,j}(x) \) the following facts hold:

**Proposition 3.32.** 1) \( \sum_i \sum_j P^i_{k+1,j}(x)\xi_{k+1,j}(x) = 0 \), where the equation holds a.e. and \( L^2_\lambda(\mathbb{R}) \), for \( \forall k \in \mathbb{Z} \).

2) \( \sup_{x \in \tilde{B}_{k+1,j}} |P_{k+1,j}(x)| \leq C2^{k+1} \), \( C \) is independent of \( j, k \).

3) \( \sup_{x \in \mathbb{R}} |P^i_{k+1,j}(x)\xi_{k+1,j}(x)| \leq C2^{k+1} \), \( C \) is independent of \( i, j, k \).

**Proof.** Take \( z \in O_k \) and \( d_\lambda(z, \partial O_k) = d_\lambda(x_k, \partial O_k) \). Then by Proposition (3.30)

\[
\left| \xi_{k,j}(x)\pi^i_{k,j}(x) \right| \lesssim C
\]

\[
L \left( \xi_{k,j}(x)\pi^i_{k,j}(x), \frac{1}{\Sigma^k+1} \right) \lesssim \left( 316^{2\lambda+1+k_{k,j}} \right)^{-1}\tag{119}
\]

From Proposition (3.31), we could write \( P_{k+1,j}(x) \) as:

\[
P_{k,j}(x) = \sum_{n=0}^{\infty} \left( \frac{\int f(y)\pi^i_{k,j}(y)\xi_{k,j}(y)|y|^{2\lambda}dy}{\int \xi_{k,j}(y)|y|^{2\lambda}dy} \right) \pi^i_{k,j}(x).
\]

Then together with formula (119) we obtain

\[
P_{k,j}(x) \lesssim \sum_{n=0}^{\infty} \|\pi^i_{k,j}\|_{\infty} \zeta(z) \leq C2^{k+1} \lesssim 2^{k+1}.
\]

In the same way we obtain

\[
\sup_{x \in \tilde{B}_{k+1,j}} |P^i_{k+1,j}(x)| \leq C2^{k+1}
\]

By the bounded intersection property of \( \{\xi_{k+1,j}\}_j \), we see that for any \( x \in \mathbb{R} \), the number \( j \) satisfying \( \xi_{k+1,j} \neq 0 \) is less than a fixed finite number \( M \). Secondly, for any fixed \( j \), in order to satisfy \( P^i_{k+1,j}(x) \neq 0 \), \( (i, j) \) must satisfy \( B_{k+1,j} \cap \tilde{B}_{k+i} = \emptyset \), thus Proposition (3.29) together with \( \sup_{x \in \tilde{B}_{k+1,j}} |P^i_{k+1,j}(x)| \leq C2^{k+1} \), we obtain

\[
\sum_i \sum_j |P^i_{k+1,j}(x)\xi_{k+1,j}(x)| \leq C2^{k+1}.
\]

Then the following series holds:

\[
\sum_i \sum_j P^i_{k+1,j}(x)\xi_{k+1,j}(x) = \sum_j \left( \sum_i P^i_{k+1,j}(x)\xi_{k+1,j}(x) \right).
\]

Thus in order to show that the first equality of this proposition holds, it suffices to prove that for any \( j \), we have:

\[
\sum_i P^i_{k+1,j}(x) = 0, \ \forall x \in \mathbb{R}
\]

Notice that \( \sum_j \xi_{k,j} = \chi_{O_k} \), \( \text{supp}\xi_{k+1,j} \subseteq O_{k+1} \subseteq O_k \), we have

\[
\int \sum_i P^i_{k+1,j}(x)Q(x)\xi_{k+1,j}(x)|x|^{2\lambda}dx = \int (f(x) - P_{k+1,j}(x))Q(x)\sum_i \xi_{k,i}(x)\xi_{k+1,j}(x)|x|^{2\lambda}dx
\]

\[
= \int (f(x) - P_{k+1,j}(x))Q(x)\xi_{k+1,j}(x)|x|^{2\lambda}dx = 0, \ \forall Q \in P^\infty.
\]
Thus we obtain \( \sum_i P^i_{k+1,j} = 0 \). Also, from the inequality
\[
\sum_i \sum_j \int |P^i_{k+1,j}(x)\xi_{k+1,j}(x)||x|^{2\lambda}dx \leq C2^{k+1}|O_{k+1}|, 
\]
and Lebesgue dominated convergence theorem, we obtain the following equality hold in \( L^1_\lambda(\mathbb{R}) \):
\[
\sum_i \sum_j P^i_{k+1,j}(x)\xi_{k+1,j}(x) = 0 
\]
This proves the proposition.

**Theorem 3.33.** For all \( f \in MH^p_\lambda(\mathbb{R}) \) (\( 0 < p < 1 \)), then we could write \( f(x) = \sum_k \lambda_k a_k(x) \) in \( MH^p_\lambda(\mathbb{R}) \) space. \( a_k(x) \) is a \( p_\lambda \)-atom:
\[
\sum_{k,i} |\lambda_{k,i}|^p \lesssim \|f^*\|_{L^p(\mathbb{R})}^p \approx \|f\|_{L^p(\mathbb{R})}^p 
\]

**Proof.** Let \( \tilde{B}_{k,j} \) and \( O_k \) as above. First, assume \( f \in MH^p_\lambda(\mathbb{R}) \cap L^1_\lambda(\mathbb{R}) \). Set:
\[
b_k(x) = f(x)\chi_{O_k} - \sum_j P_{k,j}(x)\xi_{k,j}(x) = \sum_j \left( f(x) - P_{k,j}(x) \right)\xi_{k,j}(x),
\]

\( b_{k,j}(x) = (f - P_{k,j}(x))\xi_{k,j}(x), \ g_{k,j}(x) = f(x)\chi_{O_k} + \sum_j P_{k,j}(x)\xi_{k,j}(x) \). Then \( f(x) = b_k(x) + g_k(x) \). We have pointed out above that for any \( x \in \mathbb{R} \), the number satisfying \( \xi_{k,j}(x) \neq 0 \) is less than a finite fixed number \( M \). From this and Proposition(3.32), we could easily obtain: \( \|g_k\|_\infty \lesssim 2^k \forall k \in \mathbb{Z} \). Thus the following relation holds uniformly
\[
g_k \to 0, \text{ as } k \to -\infty.
\]
Notice that: \( \text{supp} b_k \subseteq O_k \), and \( |O_k|_\lambda \lesssim \left( \|f^*\|_{L^1_\lambda}/2^k \right)^p \to 0 \), when \( k \to +\infty \),

Thus
\[
b_k \to 0 \ a.e.\mathbb{R}, \text{ as } k \to +\infty.
\]
Denote \( \gamma = \frac{1}{2^{p\lambda+1}} \), and \( SS_\gamma \) as Definition(2.19). Let \( d\mu_\lambda(x) = (2\lambda + 1)|x|^{2\lambda}dx \). By Theorem(2.39), we only need to caculate
\[
M_\phi b_{k,j}(x) = \sup_{t > 0} \left| \int b_{k,j}(y)\phi \left( \frac{\mu_\lambda(x,y)}{t} \right) d\mu_\lambda(y) \right|/t
\]
where \( \phi \in SS_\gamma \) satisfying \( \int \phi(x)dx = 1 \). When \( x \in \tilde{B}_{k,j} \)
\[
M_\phi b_{k,j}(x) \leq Cf^*(x).
\]
(120)
When \( x \in (\tilde{B}_{k,j})^c \), by the vanishing property of \( b_{k,j}(x) \), the following holds:
\[
\int b_{k,j}(y)\phi \left( \frac{\mu_\lambda(x,y)}{t} \right) d\mu_\lambda(y)/t
\]
\[
= \int (f(y) - P_{k,j}(y))\xi_{k,j}(y) \left( \phi \left( \frac{\mu_\lambda(x,y)}{t} \right) - \phi \left( \frac{\mu_\lambda(x,x_{k,j})}{t} \right) \right) d\mu_\lambda(y)/t.
\]
Denote
\[
\psi(x) = \xi_{k,j}(y) \left( \phi \left( \frac{\mu_\lambda(x,y)}{t} \right) - \phi \left( \frac{\mu_\lambda(x,x_{k,j})}{t} \right) \right).
\]
Notice that \( \text{supp} \xi_{k,j}(x) \subseteq B_{x_{k,j}, 2^{k+1}r_{k,j}} \), Thus if \( x \in (\tilde{B}_{k,j})^c \), the following holds for some appropriately small constant \( C \):
\[
d_\lambda(x,y) \approx d_\lambda(x,x_{k,j}) \geq Cr_{k,j}, \ t \geq d_\lambda(x,x_{k,j}) \geq Cr_{k,j}.
\]
Choose \( \tilde{x}_{k,j} \in B(x_{k,j}, 316^{2\lambda+1}r_{k,j}) \cap O_k \). By Formula(112), we could obtain:
Thus together with the bounded intersection property of \( O \), we could obtain the following for Hardy spaces associated with the Dunkl Transform and Homogeneous type (with a Kernel):

Then we could obtain the following for \( x \in \tilde{B}_{k,k,j} \):

Then from Formula (120)(121), we could obtain:

Thus together with the bounded intersection property of \( \{ \tilde{B}_{k,j} \} \), we could conclude

Thus

Let

Using Proposition (3.32), \( \sum_j P_{k+1,j}^j(x)\xi_{k+1,j}(x) = 0 \), \( \sum_j \xi_{k,j}(x) = \chi_{O_k} \), \( \text{supp} \xi_{k+1,j} \subseteq O_{k+1} \subseteq O_k \). Then the above equality can be written as

\[
\begin{align*}
& b_k - b_{k+1} = \sum_i (f - P_{k,i}) \xi_{k,i} - \sum_j (f - P_{k+1,j}) \xi_{k+1,j} \\
& = \sum_i \left\{ (f - P_{k,i}) \xi_{k,i} - \sum_j (f - P_{k+1,j}) \xi_{k,i} \xi_{k+1,j} \right\} \\
& = \sum_i \left\{ (f - P_{k,i}) \xi_{k,i} - \sum_j [(f - P_{k+1,j}) \xi_{k,i} - P_{k+1,j}^i(x)] \xi_{k+1,j} \right\} \\
& = \sum_i h_{k,i}
\end{align*}
\]
where
\[ h_{k,i} = (f - P_{k,i}) \xi_{k,i} - \sum_j \left[ (f - P_{k+1,j}) \xi_{k,i} - P^i_{k+1,j}(x) \right] \xi_{k+1,j}. \]

Obviously, \( \text{supp} \, h_{k,i} \subseteq B(x_{k,i}, 18^{2k+1}r_{k,i}). \) And the supports of \( \{h_{k,i}(x)\}_i \) have the bounded overlapping property. Also we could deduce from Lemma(3.25) that \( |18^{2k+1}r_{k,i}|^{2k+1} < |x_{k,i}|/2. \) We denote \( B^*_k \) as \( B^*_k = B(x_{k,i}, 18^{2k+1}r_{k,i}) = B(x_{k,i}, r_{k,i}) \) where \( r_{k,i} = 18^{2k+1}r_{k,i}. \) In addition, it is easy to see that the following equality holds:
\[ \int h_{k,i}(x)Q(x)|x|^{2\lambda}dx = 0 \quad \forall Q \in \mathcal{P}^\infty. \]

To estimate the size of \( h_{k,i} \), we could write \( h_{k,i}(x) \) as
\[ h_{k,i}(x) = f \xi_{k,i}(x) \chi_{C_{k,i}} - P_{k,i}(x)\xi_{k,i}(x) + \xi_{k,i}(x) \sum_j P_{k+1,j}(x)\xi_{k+1,j}(x) + \sum_j P^i_{k+1,j}(x)\xi_{k+1,j}(x). \]

Then by Proposition(3.32) and bounded overlapping property of \( \{B^*_k\}_i \), we obtain
\[ |h_{k,i}(x)| \lesssim 2^k \]
Thus if we set \( \lambda_{k,i} = 2^k|B^*_k|\lambda \), and \( a_{k,i}(x) = h_{k,i}/\lambda_{k,i} \), then we obtain:
\[ b_k - b_{k+1} = \sum_i \lambda_{k,i}a_{k,i}(x). \quad (125) \]

Noticing Proposition(3.32) and Formula(112), we have \( \sup_{x \in B^*_k} |P_{k,j}(x)\xi_{k,j}| \lesssim 2^k \lesssim f^*(x) \). By Proposition(2.33) \( \left\| a^{\ast}_{k,i} \right\|_{\mathcal{L}^p_{\lambda}(\mathbb{R})} \leq C, \) \( C \) is independent on \( a_{k,i}. \)

Then we could conclude:
\[ \left\| \sum_{i=k}^{+\infty} (b_i - b_{i+1})^\ast(x) \right\|_{\mathcal{L}^p_{\lambda}(\mathbb{R})} \leq C \sum_{i=k}^{+\infty} \left\| (b_i - b_{i+1})^\ast(x) \right\|_{\mathcal{L}^p_{\lambda}(\mathbb{R})} \]
\[ \leq C \sum_{j=k}^{+\infty} \sum_i \lambda_{j,i} \int_{\mathbb{R}} |a^\ast_{j,i}(x)|^p |x|^{2\lambda}dx \]
\[ \leq C \sum_{j=k}^{+\infty} \sum_i 2^{kp}|B^*_k|\lambda \]
\[ \leq C \int_{\{x:2^k < f^*(x)\}} |f^*(x)|^p |x|^{2\lambda}dx. \quad (126) \]

Formula(126) and (124) imply \( f \) could be written as
\[ f = \sum_{k=-\infty}^{+\infty} (b_k - b_{k+1}) \]
in \( MH^p_{\lambda}(\mathbb{R}) \) space. Together with Formula(125), we obtain:
\[ f(x) = \sum_k \sum_i \lambda_{k,i}a_{k,i}(x) \]
holds in \( MH^p_{\lambda}(\mathbb{R}) \), where each \( a_{k,i} \) is a \( p_\lambda \)-atom. At last, it is very easy to verify that:
\[ \sum_{k,i} |\lambda_{k,i}(x)|^p \leq C \sum_{k,i} 2^{kp}|B^*_k|\lambda \leq C \sum_k 2^{kp}|O_k|\lambda \]
\[ \leq C \sum_k \int_{2^k}^{2^{k+1}} \rho \alpha^{p-1} \{|x > 0 : f^*(x) > \alpha\}|_\lambda \, d\alpha \]
\[ \leq C \|f^*\|^p_{L^p(\mathbb{R})} \approx \|f\|^p_{H^p_{\lambda}(\mathbb{R})}. \]
Next we will remove the condition $f \in L^1_\Lambda(\mathbb{R})$. We need to prove that $MH^p_{\Lambda^*}(\mathbb{R}) \cap L^1_\Lambda(\mathbb{R})$ is dense in $MH^p_{\Lambda^*}(\mathbb{R})$ for the range of $\frac{2\lambda+4}{2\lambda+2} < p \leq 1$. Assume that $f$ is a $\lambda$-distribution satisfying $f \in MH^p_{\Lambda^*}(\mathbb{R})$. We can define $b_k$ and $b_{k,j}$ as above. $b_{k,j} = (f - P_{k,j})\xi_{k,j}$, $b_k = \sum_{j} b_{k,j}$. The $b_k$, $b_{k,j}$ are then $\lambda$-distributions with compact support: $\text{supp} b_{k,j} \subseteq B(x_{k,j}, a^* r_{k,j})$ and $\text{supp} b_k \subseteq O_k$. Thus the vanishing property $\int b_{k,j}(x)|x|^2 \, dx = 0$ is valid. Formula(120)(121) still hold from which we can conclude that $\sum_{j} b_{k,j}$ converge unconditionally in $MH^p_{\Lambda^*}(\mathbb{R})$. Thus we could conclude that $g_k = f - b_k = f - \sum_{j} b_{k,j}$ is a well-defined $\lambda$-distributions. Next we will estimate

$$\mathcal{M}_0 g_k(x) = \sup_{t > 0} \left| \int g_k(y) \phi \left( \frac{\mu_{\lambda}(x,y)}{t} \right) \frac{d\mu_{\lambda}(y)}{t} \right|.$$

If we could prove $\mathcal{M}_0 g_k(x) \in L^1_\Lambda(\mathbb{R})$, then we could conclude that $MH^p_{\Lambda^*}(\mathbb{R}) \cap L^1_\Lambda(\mathbb{R})$ is dense in $MH^p_{\Lambda^*}(\mathbb{R})$ for the range of $\frac{2\lambda+4}{2\lambda+2} < p \leq 1$. When $x \in \tilde{B}_{k,j}$, denote $\Lambda = \{j : x \in \tilde{B}_{k,j}\}$, $\Lambda^c = \{j : x \notin \tilde{B}_{k,j}\}$. Then we could write $g_k$ as

$$g_k = \left( f - \sum_{j \in \Lambda} b_{k,j} \right) - \sum_{j \in \Lambda^c} b_{k,j}.$$

Thus $\left( f - \sum_{j \in \Lambda} b_{k,j} \right)$ vanishes on a ball $B(x,r)$. By Proposition(3.29), we could obtain $r_{k,j} \approx r_{k,i}$ if $\tilde{B}_{k,j} \cap \tilde{B}_{k,i} \neq \emptyset$. Thus $r \approx r_{k,j}$. Notice that $\text{supp} \phi(t) \subseteq [-1,1]$, thus if $t \leq Cr_{k,j}$ for some appropriately small constant $C$, then

$$\left| \int \left( f(y) - \sum_{j \in \Lambda} b_{k,j}(y) \right) \phi \left( \frac{\mu_{\lambda}(x,y)}{t} \right) \frac{d\mu_{\lambda}(y)}{t} \right| = 0. \quad (127)$$

On the other hand, if $t \geq Cr_{k,j}$, then we can obtain the following inequality as Proposition(3.32) 2):

$$\left| \int \left( f(y) - \sum_{j \in \Lambda} b_{k,j}(y) \right) \phi \left( \frac{\mu_{\lambda}(x,y)}{t} \right) \frac{d\mu_{\lambda}(y)}{t} \right| \leq c2^k. \quad (128)$$

Similar to Formula(121), we could conclude

$$\left| \int \sum_{j \in \Lambda^c} b_{k,j}(y) \phi \left( \frac{\mu_{\lambda}(x,y)}{t} \right) d\mu_{\lambda}(y) / t \right| \leq \sum_{j \in \Lambda^c} C2^k \left( \frac{r_{k,j}}{d_{\lambda}(x, x_{k,j})} \right)^2. \quad (129)$$

Formula(127)(128)(129) imply that

$$\mathcal{M}_0 g_k(x) \leq c2^k + \sum_j C2^k \left( \frac{r_{k,j}}{r_{k,j} + d_{\lambda}(x, x_{k,j})} \right)^2 \text{ for } x \in O_k. \quad (130)$$

For $x \in O_k^c$, we could easily obtain that

$$\mathcal{M}_0 g_k(x) \leq c f^*(x) + \sum_j C2^k \left( \frac{r_{k,j}}{r_{k,j} + d_{\lambda}(x, x_{k,j})} \right)^2 \text{ for } x \in O_k^c. \quad (131)$$

Also we could obtain that

$$2^k O_k \leq C2^{k(1-p)} \int |f^*(x)|^p d\mu_{\lambda}(x). \quad (132)$$

Formula (130)(131)(132) lead to

$$\int \mathcal{M}_0 g_k(x) d\mu_{\lambda}(x) \leq C2^{k(1-p)} \int |f^*(x)|^p d\mu_{\lambda}(x).$$
Thus \( MH_{\lambda_*}^p(\mathbb{R}) \cap L^1_\lambda(\mathbb{R}) \) is dense in \( MH_{\lambda_*}^p(\mathbb{R}) \) for the range of \( \frac{2\lambda+1}{2\lambda+2} < p \leq 1 \). Thus for any \( f \in MH_{\lambda_*}^p(\mathbb{R}) \), there exist \( f_m \in MH_{\lambda_*}^p(\mathbb{R}) \cap L^1_\lambda(\mathbb{R}) \), such that
\[
\|f - f_m\|_{H_{\lambda_*}^p(\mathbb{R})} \to 0, \quad \text{when } m \to 0.
\]
Without loss of generality, we may assume that \( f_m \) satisfies:
\[
\|f_1\|_{H_{\lambda_*}^p(\mathbb{R})}^p \leq \frac{1}{2} \|f\|_{H_{\lambda_*}^p(\mathbb{R})}^p,
\]
and
\[
\|f_{m+1} - f_m\|_{H_{\lambda_*}^p(\mathbb{R})}^p \leq 2^{-m-1}\|f\|_{H_{\lambda_*}^p(\mathbb{R})}^p, \quad m \in \mathbb{N}.
\]
Set \( t_1 = f_1 \), and \( t_m = f_m - f_{m-1}, (m > 1) \). Then the equality
\[
f = \sum_{m=1}^{+\infty} t_m
\]
holds in \( MH_{\lambda_*}^p(\mathbb{R}) \). Since \( t_m \in MH_{\lambda_*}^p(\mathbb{R}) \cap L^1_\lambda(\mathbb{R}) \), \( t_m \) can be represented as
\[
t_m(x) = \sum_{k,i} \lambda_{k,i}^m a_{k,i}^m(x)
\]
in \( MH_{\lambda_*}^p(\mathbb{R}) \) space, each \( a_{k,i}^m(x) \) is a \( p_\lambda \)-atom. And
\[
\sum_{k,i} |\lambda_{k,i}^m|^p \lesssim \|t_m\|_{H_{\lambda_*}^p(\mathbb{R})}^p.
\]
Now, \( f \) can be written as
\[
f(x) = \sum_{m=1}^{+\infty} \sum_{k,i} \lambda_{k,i}^m a_{k,i}^m(x)
\]
in \( MH_{\lambda_*}^p(\mathbb{R}) \) space. And
\[
\sum_{m=1}^{+\infty} \sum_{k,i} |\lambda_{k,i}^m|^p \lesssim \|f\|_{H_{\lambda_*}^p(\mathbb{R})}^p.
\]
This proves the theorem.

\[\square\]

**Theorem 3.34.** For \( \forall f \in H_{\lambda_*}^p(\mathbb{R})(\frac{2\lambda+1}{2\lambda+2} < p \leq 1) \), we could conclude that
\[H_{\lambda_*}^p(\mathbb{R}) = MH_{\lambda_*}^p(\mathbb{R}).\]
Then we could write \( f(x) \) as \( f(x) = \sum_k \lambda_k a_k(x) \) in the sense of \( H_{\lambda_*}^p(\mathbb{R}) \) space. \( a_k(x) \) is a \( p_\lambda \)-atom.
\[
\sum_k |\lambda_k|^p \asymp \|f\|_{L_{\lambda_*}^p(\mathbb{R})}^p \asymp \|f\|_{H_{\lambda_*}^p(\mathbb{R})}^p.
\]

**Proof.** For \( \frac{2\lambda+1}{2\lambda+2} < p \leq 1 \), by the definition of \( H_{\lambda_*}^p(\mathbb{R}) \), \( MH_{\lambda_*}^p(\mathbb{R}) \) is the completion of \( \hat{H}_{\lambda_*}^p(\mathbb{R}) \) where \( \hat{H}_{\lambda_*}^p(\mathbb{R}) \) is the real boundary value of \( H_{\lambda_*}^p(\mathbb{R}^+_\lambda) \cap H_{\lambda_*}^p(\mathbb{R}^-_\lambda) \). Thus by Theorem(3.33) and Definition(3.24) we could conclude that \( MH_{\lambda_*}^p(\mathbb{R}) \cap L^1_\lambda(\mathbb{R}) \cap L^2_\lambda(\mathbb{R}) \) is dense in \( MH_{\lambda_*}^p(\mathbb{R}) \). From Proposition(3.18), (3.19), we could deduce that \( \hat{H}_{\lambda_*}^p(\mathbb{R}) = MH_{\lambda_*}^p(\mathbb{R}) \cap L^1_\lambda(\mathbb{R}) \cap L^2_\lambda(\mathbb{R}) \). Then we could conclude that
\[H_{\lambda_*}^p(\mathbb{R}) = MH_{\lambda_*}^p(\mathbb{R}).\]
Therefore, \( f \in H_{\lambda_*}^p(\mathbb{R}) \) can be written as
\[
f(x) = \sum_k \lambda_k a_k(x)
\]
in \( H_{\lambda_*}^p(\mathbb{R}) \) space, each \( a_k(x) \) is a \( p_\lambda \)-atom. And
\[
\sum_k |\lambda_k|^p \lesssim \|f\|_{H_{\lambda_*}^p(\mathbb{R})}^p.
\]
By Theorem (3.44), the following inequality holds:

\[
\left( \int_{\mathbb{R}} |f^*(x)|^p |x|^{2\lambda} \, dx \right)^{1/p} \leq \left( \int_{\mathbb{R}} \left| \sum_k \lambda_k a_k^*(x) \right|^p |x|^{2\lambda} \, dx \right)^{1/p} \leq \sum_k \left( \int_{\mathbb{R}} |\lambda_k a_k^*(x)|^p |x|^{2\lambda} \, dx \right)^{1/p} \lesssim \sum_k |\lambda_k|^p.
\]

(134)

Therefore by (133) and (134), we could deduce

\[
\sum_k |\lambda_k|^p \approx \|f^*\|^p_{L^p_s(\mathbb{R})} \approx \|f\|^p_{H^p_s(\mathbb{R})}.
\]

This proves the Theorem.

\section{3.4 Boundness of \(p_\lambda\)-atom}

**Definition 3.35 (\(p_\lambda\)-condition).** If function \(a(x)\) satisfies the following condition, we also say \(a(x)\) satisfying the \(p_\lambda\)-condition:

\[
i \quad \|a(x)\|_{L^p_s(\mathbb{R})} \lesssim \frac{1}{|I(x_0, \delta_0)|^{1/p}_\lambda}.
\]

\[
\begin{align*}
ii & \quad \text{supp } a(x) \subseteq I(x_0, \delta_0), \text{ and } \delta_0 < |x_0/2| \\
iii & \quad \int_{\mathbb{R}} t^k a(t) |t|^{2\lambda} \, dt = 0 \quad (k = 0, 1, 2, 3 \ldots \kappa) \\
iv & \quad 0 < \delta_0 < |x_0/2| \\
v & \quad \kappa \geq 2 \left( 2\lambda + 1 \right) \frac{1-p}{p} \text{ when } \frac{2\lambda}{2\lambda + 1} < p \leq \frac{2\lambda + 1}{2\lambda + 2}. \quad \kappa \geq 0 \text{ when } \frac{2\lambda + 1}{2\lambda + 2} < p \leq 1.
\end{align*}
\]

**Proposition 3.36.** A \(p_\lambda\)-atom \(a(x)\) could also satisfies the \(p_\lambda\)-condition.

**Proof.** Assume \(\text{supp } a(x) \subseteq B(x_0, r_0)\), satisfying \(r_0 \frac{x_0}{2} \approx |x_0/2|\) and \(\|a(x)\|_{L^p_s(\mathbb{R})} \lesssim \frac{1}{|I(x_0, \delta_0)|^{1/p}_\lambda} \).

Then we could let \(\delta_0 = \left( x_0^{2\lambda + 1} + r_0 \right)^{1/2\lambda} - x_0 \), then by Proposition(3.27), \(B(x_0, r_0) \subseteq I(x_0, \delta_0)\) holds. Let \(t = \frac{x_0}{|x_0|} \cdot r_0 \). It is easy to see that \(0 < t < 1/2\) for \(r_0 \frac{x_0}{2} \approx |x_0/2|\). Thus \(\delta_0 = |x_0| |1 + t|^{1/2\lambda} - 1 \approx |x_0| |(1 + 1/2)|^{1/2\lambda} - 1 \approx |x_0|/2\) holds. Together with Proposition(3.27), we could see that

\[
|I(x_0, \delta_0)|_\lambda = \int_{I(x_0, r_0)} |t|^{2\lambda} \, dt \approx |x_0|^{2\lambda} \frac{r_0}{x_0^{2\lambda}} \approx r_0 \approx |B(x_0, \delta_0)|_\lambda.
\]

Thus the proposition holds.

In this section let I denote as following:

\[
I = \left( I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0) \right)^c.
\]

\(a(t)\) satisfies \(p_\lambda\)-condition, and its support set is \(I(x_0, \delta_0)\) which is an Euclid interval with its center \(x_0\) and its radius \(\delta_0\): \(I(x_0, \delta_0) = \{ x : |x - x_0| < \delta_0 \}\). In this section, let \(\langle x, t \rangle_{y,s}, \langle x, t \rangle_s, \langle x, x_0 \rangle_s, \delta_1 \) denote:

\[
\begin{align*}
\langle x, t \rangle_{y,s} &= y^2 + x^2 + t^2 - 2xts \\
\langle x, t \rangle_s &= x^2 + t^2 - 2xts \\
\langle x, x_0 \rangle_s &= x^2 + x_0^2 - 2x_0s \\
\delta_1 &= t^2 - x_0^2 - 2tx_0s + 2x_0^2s = (t - x_0)(x_0 + t - 2x_0)
\end{align*}
\]
Proposition 3.37.

\[
\int \frac{1}{|x| - |x_0|^k} \, dx \lesssim \delta_0^{1-k} \quad (k > 1)
\]

assume \( x_0 > 0 \).

Proof.

\[
\int \frac{1}{|x| - |x_0|^k} \, dx \leq \int_{I(x_0, 4\delta_0)} \frac{1}{|x| - |x_0|^k} \, dx
\]

\[
= \int_{I(0, 4\delta_0)} \frac{1}{|x|} \, dx
\]

\[
\lesssim \delta_0^{1-k}
\]

Proposition 3.38.

\[
\int_{-1}^{1} (1 - bs)^{-\lambda -1} (1 + s)(1 - s^2)^{\lambda -1} \, ds \leq C \frac{1}{1 - |b|}, \quad \forall -1 < b < 1, \quad \lambda > 0
\]

C is depend on \( \lambda \), and independent on \( b \). (In fact \( C \sim 1/\lambda \))

Proof. CASE 1: when \( 0 \leq b < 1 \).

It is obvious to see that when \( 0 \leq b < 1, \lambda > 0 \),

\[
\int_{-1}^{0} (1 - bs)^{-\lambda -1} (1 + s)(1 - s^2)^{\lambda -1} \, ds \lesssim 1.
\]

By the formula of integration by parts and \( 1 - s \leq 1 - bs \) when \( 1 \geq s \geq 0 \) (\( 0 \leq b < 1, \lambda > 0 \)),

we obtain:

\[
\left| \int_{0}^{1} (1 - bs)^{-\lambda -1} (1 + s)(1 - s^2)^{\lambda -1} \, ds \right|
\]

\[
\lesssim \left| \int_{0}^{1} (1 - bs)^{-\lambda -1} (1 - s)^{\lambda -1} \, ds \right|
\]

\[
\lesssim \left| 1 \right| \left| b^{\lambda + 1} - \frac{1}{\lambda} (1 - bs)^{-\lambda -2} (1 - s)^{\lambda} \right|
\]

\[
\lesssim \left| \frac{1}{\lambda} + \frac{\lambda + 1}{\lambda} b \right| \int_{0}^{1} (1 - bs)^{-2} \, ds
\]

\[
\lesssim \frac{1}{1 - b}.
\]

Next we need to prove when \(-1 < b \leq 0 \)

\[
\int_{-1}^{1} (1 - bs)^{-\lambda -1} (1 + s)(1 - s^2)^{\lambda -1} \, ds \lesssim \frac{1}{1 + b}
\]

CASE 2: when \(-1 < b \leq 0 \).

Obviously the following inequality holds:

\[
\int_{0}^{1} (1 - bs)^{-\lambda -1} (1 + s)(1 - s^2)^{\lambda -1} \, ds \lesssim 1.
\]

By the formula of integration by parts and \( 1 + s \leq 1 - bs \) when \(-1 \leq s \leq 0 \) (\( -1 < b \leq 0, \lambda > 0 \))
we obtain:
\[
\begin{align*}
&\left|\int_{-1}^{0} (1 - bs)^{-\lambda} (1 + s)(1 - s^2)^{\lambda-1} ds\right| \\
\leq &\left|\int_{-1}^{0} (1 - bs)^{-\lambda} (1 + s)^{\lambda} ds\right| \\
\leq &\frac{1}{\lambda + 1} (1 - bs)^{-\lambda} (1 + s)^{\lambda + 1} - \int_{-1}^{0} b (1 - bs)^{-\lambda - 2} (1 + s)^{\lambda + 1} ds \\
\leq &\frac{1}{\lambda + 1} - b \int_{-1}^{0} (1 - bs)^{-1} ds \\
&- \ln(1 + b) \\
\leq &\frac{1}{1 + b}.
\end{align*}
\]

By CASE 1 and CASE 2, the inequality:
\[
\int_{-1}^{1} (1 - bs)^{-\lambda - 1} (1 + s)(1 - s^2)^{\lambda - 1} ds \leq C \frac{1}{1 - |b|}, \quad \forall -1 < b < 1, \quad \lambda > 0
\]
holds. Hence the proposition holds.

Thus we could obtain the following Proposition(3.39) from Proposition(3.38):

**Proposition 3.39.**
\[
\int_{-1}^{1} (1 - bs)^{-\lambda - 1} (1 + s)(1 - s^2)^{\lambda - 1} ds \leq C \frac{1}{1 - |b|}, \quad \forall -1 < b < 1, \quad \lambda > 0
\]

*C is depend on \(\lambda\), and independent on \(b\).*

The following Proposition(3.40) could be obtained in a way similar to Proposition(3.38):

**Proposition 3.40.**
\[
\int_{-1}^{1} (1 - bs)^{-\lambda - 1} (1 - s^2)^{\lambda - 1/2} ds \leq C \frac{1}{1 - |b|}, \quad \forall -1 < b < 1, \quad \lambda > 0
\]

*\(C\) is depend on \(\lambda\), and independent on \(b\). (In fact \(C \sim \frac{2}{2\lambda + 1}\)).

**Theorem 3.41** (\(\lambda\)-Hilbert Transform for \(p_{\lambda}\)-condition). For \(\frac{2\lambda}{\lambda + 1} < p \leq 1\), if \(a(t)\) satisfies \(p_{\lambda}\)-condition, with vanishing order \(\kappa \geq 2 \left(2\lambda + 1 - \frac{1 - p}{p}\right)\) then the following holds:
\[
\int_{\mathbb{R}} |\mathcal{H}_\lambda a(x)|^p |x|^{2\lambda} dx \leq C,
\]

*\(C\) is depend on \(\lambda\) and \(p\).*

**Proof.** Assume first that \(x_0 > 0\). Let \(\kappa = 2 \left(2\lambda + 1 - \frac{1 - p}{p}\right)\). Thus \(\kappa\) is an even integer. Let \(n = \kappa/2\). We could write the above integral as:
\[
\int_{\mathbb{R}} |\mathcal{H}_\lambda a(x)|^p |x|^{2\lambda} dx = \int_{I(x_0, 4\delta_0)} |\mathcal{H}_\lambda a(x)|^p |x|^{2\lambda} dx \\
+ \int_{I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)^c} |\mathcal{H}_\lambda a(x)|^p |x|^{2\lambda} dx
\]
\[
= I + II.
\]

We could see the following inequality holds:
\[
4^{2\lambda + 1} \int_{x_0 - \delta_0}^{x_0 + \delta_0} |x|^{2\lambda} dx = \int_{4x_0 - 4\delta_0}^{4x_0 + 4\delta_0} |x|^{2\lambda} dx \geq \int_{4x_0 - \delta_0}^{4x_0 + \delta_0} |x|^{2\lambda} dx.
\]
By Zh-K Li[13][Theorem 5.7], together with Formula (135) we obtain:
\[
I = \int_{I(x_0,4\delta_0) \cup I(-x_0,4\delta_0)} |\mathcal{H}_\lambda a(x)|^p |x|^{2\lambda} \, dx
\]
\[
\leq \left( \int_{I(x_0,4\delta_0) \cup I(-x_0,4\delta_0)} |\mathcal{H}_\lambda a(x)|^2 |x|^{2\lambda} \, dx \right)^{p/2} \left( \int_{I(x_0,4\delta_0) \cup I(-x_0,4\delta_0)} |x|^{2\lambda} \, dx \right)^{1-p/2}
\]
\[
\leq C. \tag{136}
\]

Next we need to prove:
\[
II = \int_{(I(x_0,4\delta_0) \cup I(-x_0,4\delta_0))^c} |\mathcal{H}_\lambda a(x)|^p |x|^{2\lambda} \, dx \leq C. \tag{137}
\]

By Proposition (3.9), when \( x \in (I(x_0,4\delta_0) \cup I(-x_0,4\delta_0))^c \) we could write \( \mathcal{H}_\lambda a(x) \) as:
\[
\mathcal{H}_\lambda a(x) = c_\lambda \int a(t) h(x,t) |t|^{2\lambda} \, dt.
\]

And the \( \lambda \)-Hilbert kernel could be written as:
\[
h(x,t) = \frac{\Lambda(\lambda + 1/2)}{2^{\lambda+1/2} \pi} (x-t) \int_{-1}^{1} (1+s)(1-s^2)^{\lambda-1} \, ds.
\]

Next we need to estimate \( \mathcal{H}_\lambda a(x) \) when \( x \in I = (I(x_0,4\delta_0) \cup I(-x_0,4\delta_0))^c \) (for \( \delta_0 < |x_0/2| \)).

Notice that \( t \in \text{supp} \, a(t) \subseteq I(x_0,\delta_0) \). When \( x \geq 0 \), or \( x < -2x_0 \), the following inequality
\[
|x-x_0| \lesssim (\langle x,x_0 \rangle_s)^{1/2} \tag{138}
\]
holds. It is also obvious to see that the following
\[
|x-x_0s| \lesssim (\langle x,x_0 \rangle_s)^{1/2} \tag{139}
\]
\[
|xs-x_0| \lesssim (\langle x,x_0 \rangle_s)^{1/2} \tag{140}
\]
\[
|x_0 + t - 2xs| \leq |x_0 - x_0s| + |t - xs| \leq ((x,x_0)_s)^{1/2} + ((x,x_0)_s)^{1/2} + |t - x_0| \leq 3 ((x,x_0)_s)^{1/2} \tag{141}
\]
hold. From the above inequality(141), we could obtain the following inequality:
\[
|\delta_1| \leq 3|x - x_0| ((x,x_0)_s)^{1/2}. \tag{142}
\]

For \( \delta_0 < |x_0/2| \), and \( x \in I = (I(x_0,4\delta_0) \cup I(-x_0,4\delta_0))^c \), from Formula (142), we could have:
\[
\frac{|\delta_1|}{\langle x,x_0 \rangle_s} \leq \frac{3|t - x_0|}{\langle x,x_0 \rangle_s^{1/2}} \leq \frac{3|t - x_0|}{|x-x_0|} \leq \frac{3\delta_0}{4\delta_0} = 3/4. \tag{143}
\]

We could see that:
\[
\frac{x-t}{\langle x,t \rangle_s^{\lambda+1}} = \frac{x-x_0}{\langle x,x_0 \rangle_s^{\lambda+1}} + \frac{x_0-t}{\langle x,t \rangle_s^{\lambda+1}} \tag{144}
\]

By the Taylor expansion of formula \( \left( 1 + \frac{\delta_1}{\langle x,x_0 \rangle_s} \right)^{-\lambda-1} \), when \( x \in [-2x_0,0] \cap (I(x_0,4\delta_0) \cup I(-x_0,4\delta_0))^c \), we could obtain:
\[
A = \frac{x-x_0}{\langle x,x_0 \rangle_s^{\lambda+1}} \left( 1 + \frac{\delta_1}{\langle x,x_0 \rangle_s} \right)^{\lambda+1}
\]
\[
= \frac{x-x_0}{\langle x,x_0 \rangle_s^{\lambda+1}} \left[ 1 + \frac{\lambda+1}{1} \left( -\delta_1 \right) + \frac{\lambda+1}{1} \left( \lambda+2 \right) 2! \right. + \cdots + \\
\left. + \frac{(\lambda+1)_n}{(n)!} \left( -\delta_1 \right)^n + \frac{(\lambda+1)_{n+1}}{(n+1)!} \left( -\delta_1 \right)^{n+1} \right], \tag{145}
\]
and
\[
B = \frac{x_0 - t}{\langle x, x_0 \rangle_t^{\lambda+1}} (1 + \frac{\delta_1}{\langle x, x_0 \rangle_t})^{\lambda+1} + \frac{x_0 - t}{\langle x, x_0 \rangle_t^{\lambda+1}} \left[ 1 + \frac{\lambda + 1}{1} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_t} \right)^1 + \frac{(\lambda + 1)(\lambda + 2)}{2!} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_t} \right)^2 + \ldots \right] + \frac{(\lambda + 1)n^{-1}}{(n-1)!} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_t} \right)^{n-1} + \frac{(\lambda + 1)n}{(n)!} \left( \frac{1}{1 + \xi_2} \right)^{\lambda+n+1} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_t} \right)^n. 
\]

By Formula (143), we could have: $\xi_1, \xi_2 \in [-3/4, 3/4]$. Then the inequality holds:
\[
\left( \frac{1}{1 + \xi_1} \right) \leq \left( \frac{1}{1 - 3/4} \right) \leq 4 \text{ and } \left( \frac{1}{1 + \xi_2} \right) \leq \left( \frac{1}{1 - 3/4} \right) \leq 4.
\]

Thus from Proposition (3.9), Formulas (143), (144), (145), (146), (138), (141) and (142), together with the vanishing property of $a(t)$, we obtain: for $x \in [-2x_0, 0]^c \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c$
\[
|\mathcal{H}_\lambda a(x)| = \left| c_\lambda \int a(t) h(x, t) |t|^{2\lambda} dt \right|
\leq \left| \int \int |a(t)| \left| \frac{x - x_0}{\langle x, x_0 \rangle_t^{\lambda+1}} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_t} \right)^{n+1} \right| (1 + s)(1 - s^2)^{\lambda-1}ds |t|^{2\lambda} dt \right|
\leq \left| I(x_0, \delta_0) \right|^{1-(1/p)} \int \int |a(t)| \left| \frac{x - x_0}{\langle x, x_0 \rangle_t^{\lambda+1}} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_t} \right)^{n+1} \right| (1 + s)(1 - s^2)^{\lambda-1}ds |t|^{2\lambda} dt
\leq |I(x_0, \delta_0)|^{1-(1/p)} \int \int |a(t)| \frac{(\delta_0)^{n+1}|x - x_0|^{n/2 + \lambda + 1}|1 + s| |1 - 2x_0s| |s|^{-\lambda-1} ds
\leq C \frac{1}{(x^2 + x_0^2)^\lambda (|x| - |x_0|)^2}.
\]
By the Taylor expansion, we could obtain:

\[
C = \frac{x - x_0 s}{\langle x, x_0 \rangle^{\lambda+1}} \left(1 + \frac{\delta_1}{\langle x, x_0 \rangle} \right)^{\lambda+1} 
\]

\[
= \frac{x - x_0 s}{\langle x, x_0 \rangle^{\lambda+1}} \left[1 + \frac{\lambda + 1}{1} \left(\frac{-\delta_1}{\langle x, x_0 \rangle}\right) + \frac{(\lambda + 1)(\lambda + 2)}{2!} \left(\frac{-\delta_1}{\langle x, x_0 \rangle}\right)^2 + \cdots + \frac{(\lambda + 1)_n}{(n)!} \left(\frac{-\delta_1}{\langle x, x_0 \rangle}\right)^n \right] 
\]

\[
D = \frac{x_0 - t}{\langle x, x_0 \rangle^{\lambda+1}} \left(1 + \frac{\delta_1}{\langle x, x_0 \rangle} \right)^{\lambda+1} 
\]

\[
= \frac{x_0 - t}{\langle x, x_0 \rangle^{\lambda+1}} \left[1 + \frac{\lambda + 1}{1} \left(\frac{-\delta_1}{\langle x, x_0 \rangle}\right) + \frac{(\lambda + 1)(\lambda + 2)}{2!} \left(\frac{-\delta_1}{\langle x, x_0 \rangle}\right)^2 + \cdots + \frac{(\lambda + 1)_{n-1}}{(n-1)!} \left(\frac{-\delta_1}{\langle x, x_0 \rangle}\right)^{n-1} \right] 
\]

\[
E = \frac{x_0(s - 1)}{\langle x, x_0 \rangle^{\lambda+1}} \left(1 + \frac{\delta_1}{\langle x, x_0 \rangle} \right)^{\lambda+1} 
\]

\[
= \frac{x_0(s - 1)}{\langle x, x_0 \rangle^{\lambda+1}} \left[1 + \frac{\lambda + 1}{1} \left(\frac{-\delta_1}{\langle x, x_0 \rangle}\right) + \frac{(\lambda + 1)(\lambda + 2)}{2!} \left(\frac{-\delta_1}{\langle x, x_0 \rangle}\right)^2 + \cdots + \frac{(\lambda + 1)_n}{(n)!} \left(\frac{-\delta_1}{\langle x, x_0 \rangle}\right)^n \right] 
\]

By Formula (143), we could have: \(\xi_3, \xi_4, \xi_5 \in [-3/4, 3/4]\). Thus:

\[
\left(\frac{1}{1 + \xi_i}\right) \leq \left(\frac{1}{1 - 3/4}\right) \leq 4 \text{ for } i = 3, 4, 5. 
\]

Thus Formulas (143), (157), (153), (154), (155), (156), (138), (141) and (142), together with the vanishing property of \(a(t)\), we obtain: for \(x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \bigcup I(-x_0, 4\delta_0)) \subset \mathbb{C}\)

\[
|M_\lambda a(x)| \lesssim \int_{-1}^{1} |a(t)| \left|\frac{x - x_0 s}{\langle x, x_0 \rangle^{\lambda+1}} \left(\frac{-\delta_1}{\langle x, x_0 \rangle}\right)^n + (1 + s)(1 - s^2)^{\lambda-1} ds\right|^{2\lambda} dt
\]

\[
+ \int_{-1}^{1} |a(t)| \left|\frac{x_0 - t}{\langle x, x_0 \rangle^{\lambda+1}} \left(\frac{-\delta_1}{\langle x, x_0 \rangle}\right)^n + (1 + s)(1 - s^2)^{\lambda-1} ds\right|^{2\lambda} dt
\]

\[
+ \int_{-1}^{1} |a(t)| \left|\frac{x_0(s - 1)}{\langle x, x_0 \rangle^{\lambda+1}} \left(\frac{-\delta_1}{\langle x, x_0 \rangle}\right)^n + (1 + s)(1 - s^2)^{\lambda-1} ds\right|^{2\lambda} dt
\]

\[
\lesssim |I(x_0, \delta_0)|^{1/(1/p)} \int_{-1}^{1} \left|\frac{\delta_0}{\langle x, x_0 \rangle^{n/2+\lambda+1}} + (1 + s)(1 - s^2)^{\lambda-1} ds\right|^{2\lambda} dt
\]

\[
+ \int_{-1}^{1} |a(t)| \left|\frac{x_0(s - 1)}{\langle x, x_0 \rangle^{\lambda+1}} \left(\frac{-\delta_1}{\langle x, x_0 \rangle}\right)^n + (1 + s)(1 - s^2)^{\lambda-1} ds\right|^{2\lambda} dt. 
\]

Notice that \(\langle x, x_0 \rangle = x^2 + x_0^2 - 2x_0 s \geq (1 - s^2)x_0^2\) holds for \(\forall x \in \mathbb{R}\). Thus we could obtain the
following inequality:

\[
\int \int_{-1}^1 |a(t)| \left| \frac{x_0(s-1)}{\langle x, x_0 \rangle_s} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_s} \right)^{n+1} \right|^1 (1 + s)(1 - s^2)^{\lambda-1} ds |t|^{2\lambda} dt
\leq \int \int_{-1}^1 |a(t)| \left| \frac{x_0(s-1)}{x_0 (1 - s^2)^{1/2}} \left( \frac{-\delta_0}{\langle x, x_0 \rangle_s} \right)^{n+1} \right|^1 (1 + s)(1 - s^2)^{\lambda-1} ds |t|^{2\lambda} dt
\]

\[
\lesssim |I(x_0, \delta_0)|_{\lambda}^{1-1/(p)} \int_{-1}^1 \left( \frac{\delta_0}{\langle x, x_0 \rangle_s} \right)^{n+1} (1 + s)(1 - s^2)^{\lambda-1/2} ds
\]

By Proposition(3.40), the above inequality(159) implies that:

\[
\lesssim |I(x_0, \delta_0)|_{\lambda}^{1-1/(p)} \int_{-1}^1 \left| \frac{\delta_0}{\langle x, x_0 \rangle_s} \right|^{n+1} (1 + s)(1 - s^2)^{\lambda-1/2} ds
\]

Formula(151) and (160) imply the following inequality holds when \( x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c \):

\[
|\mathcal{H}_a(x)| \leq C |I(x_0, \delta_0)|_{\lambda}^{1-1/(p)} \left( \frac{\delta_0}{\langle x, x_0 \rangle_s} \right)^{n+1} |||x| - |x_0||^{n+2} |||x| + |x_0||^{2\lambda}. \]

From Formula(161) and (152), we obtain that:

\[
|\mathcal{H}_a(x)| \leq C |I(x_0, \delta_0)|_{\lambda}^{1-1/(p)} \left( \frac{\delta_0}{\langle x, x_0 \rangle_s} \right)^{n+1} |||x| - |x_0||^{n+2} |||x| + |x_0||^{2\lambda}
\]

holds for \( x \in (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c \). Because \( \frac{2\lambda}{2\lambda + 1} < p \leq 1 \), we could get \( 0 \leq 2\lambda(1 - p) \leq \frac{2\lambda}{2\lambda + 1} < 1 \). Thus the following holds:

\[
|||x|^{2\lambda(1-p)} \leq |||x| - |x_0||^{2\lambda(1-p)} + |x_0|^{2\lambda(1-p)}. \]

Therefore:

\[
II \leq C |I(x_0, \delta_0)|_{\lambda}^{p-1} \int_{I_0} \left( \frac{\delta_0}{|||x| - |x_0||^{n+2}} \right)^p (|||x| - |x_0||^{2\lambda(1-p)} + |x_0|^{2\lambda(1-p)}) dx 
\]

\[
\approx |I(x_0, \delta_0)|_{\lambda}^{p-1} \int_{I_0} \left( \frac{\delta_0}{|||x| - |x_0||^{n+2}} \right)^p (|||x| - |x_0||^{2\lambda(1-p)}) dx 
\]

\[
+ |I(x_0, \delta_0)|_{\lambda}^{p-1} \int_{I_0} \left( \frac{\delta_0}{|||x| - |x_0||^{n+2}} \right)^p (|x_0|^{2\lambda(1-p)}) dx 
\]

\[
= II_1 + II_2. \quad (163)
\]

\( \kappa \) and \( p \) satisfy the relation: \( \kappa = 2 \left( 2\lambda + 1 \right) \frac{1-p}{p} \). \( n = \kappa/2 \). Therefore we could have:

\[
(n+2)p + 2\lambda(p-1) > 1 \text{ and } (n+2)p > 1.
\]

Then the integral of \( II_1 \) and \( II_2 \) converge:

\[
II_1 = C |I(x_0, \delta_0)|_{\lambda}^{p-1} \int_{I_0} \left( \frac{\delta_0}{|||x| - |x_0||^{n+2}} \right)^p (|||x| - |x_0||^{2\lambda(1-p)}) dx 
\]

\[
\leq C |I(x_0, \delta_0)|_{\lambda}^{p-1} \int_{I_0} \frac{1}{|||x| - |x_0||^{(n+2)p+2\lambda(p-1)}} dx 
\]

\[
\leq C |I(x_0, \delta_0)|_{\lambda}^{p-1} \int_{4\delta_0} \frac{1}{r^{(n+2)p+2\lambda(p-1)}} dr 
\]

\[
\leq C, \quad (164)
\]
We could write the above integral as:

\[ II_2 = C |I(x_0, \delta_0)|^{p-1} \int \left( \frac{(\delta_0)^{n+1}}{|x| - |x_0|} \right)^p |x|^{2\lambda(1-p)} \, dx \]

\[ \leq C x_0^{2(p-1)} (\delta_0)^{p-1} |x_0|^{2\lambda(1-p)} (\delta_0)^{(n+1)p} (\delta_0)^{-(n+2)p+1} \leq C. \]  

(165)

Thus from Formula (136), (163), (164) and (165), the theorem is proved.

**Theorem 3.42.** If \( a(t) \) satisfies \( p_\lambda \)-condition, with vanishing order \( \kappa \geq 2 \left[ (2\lambda + 1) \frac{1 - \frac{p}{\lambda}}{p} \right] \) then the following holds:

\[ \frac{2\lambda}{2\lambda + 4} < p \leq 1 \int_R |(a * \lambda P_y)|^p(x)|x|^{2\lambda} \, dx \leq C, \]

\( C \) is depend on \( \lambda \) and \( p \).

**Proof.** Assume first that \( x_0 > 0 \). Let \( \kappa = 2 \left[ (2\lambda + 1) \frac{1 - \frac{p}{\lambda}}{p} \right] \). Thus \( \kappa \) is an even integer. Let \( n = \kappa/2 \).

We could write the above integral as:

\[ \int_R |(a * \lambda P_y)|^p(x)|x|^{2\lambda} \, dx = \int_{I(x_0, \delta_0)} |(a * \lambda P_y)|^p(x)|x|^{2\lambda} \, dx 
\[ + \int_{I(x_0, \delta_0) \cup I(-x_0, \delta_0)} |(a * \lambda P_y)|^p(x)|x|^{2\lambda} \, dx \]

\[ = III + IV. \]

Using ZhongKai Li [13](Theorem 3.8) and Formula (135), we could get

\[ III = \int_{I(x_0, \delta_0) \cup I(-x_0, \delta_0)} |(a * \lambda P_y)|^p(x)|x|^{2\lambda} \, dx \]

\[ \leq \left( \int_{I(x_0, \delta_0) \cup I(-x_0, \delta_0)} |(a * \lambda P_y)|^2(x)|x|^{2\lambda} \, dx \right)^{p/2} \left( \int_{I(x_0, \delta_0) \cup I(-x_0, \delta_0)} |x|^{2\lambda} \, dx \right)^{1-p/2} \leq C. \]

Next we estimate the following integer:

\[ IV = \int_{I(x_0, \delta_0) \cup I(-x_0, \delta_0)} |(a * \lambda P_y)|^p(x)|x|^{2\lambda} \, dx. \]

By Proposition (3.9), when \( x \in (I(x_0, \delta_0) \cup I(-x_0, \delta_0))^c \) we could write \((a * \lambda P_y)(x)\) as:

\[ (a * \lambda P_y)(x) = c_{\lambda} \int a(t)(\tau_x P_y)(-t)|t|^{2\lambda} \, dt. \]

And the \( \lambda \)-Poisson kernel could be written as:

\[ (\tau_x P_y)(-t) = \frac{\lambda t^{1/2}}{2^{\alpha-1} \pi} \int_0^\pi y(1 + \text{sgn}(x t) \cos \theta) \frac{\sin^{2\lambda-1} \theta \, d\theta} {y^2 + x^2 + t^2 - 2|x|t \cos \theta}. \]

Notice that \( t \in \text{supp} \, a(t) \subseteq I(x_0, \delta_0) \). By the Taylor expansion of formula \( \left( 1 + \frac{4t}{(x, x_0)_{y,s}} \right)^{-\lambda}, \) we could get the following inequality:

\[ \frac{y}{(x, t)_{y,s}^{\lambda + 1}} = \frac{y}{(x, x_0)_{y,s}^{\lambda + 1}} \left( 1 + \frac{\delta_1}{(x, x_0)_{y,s}} \right)^{\lambda + 1} \]

\[ = \frac{y}{(x, x_0)_{y,s}^{\lambda + 1}} \left[ 1 + \frac{\lambda + 1}{1} \left( \frac{-\delta_1}{(x, x_0)_{y,s}} \right) + \frac{(\lambda + 1)(\lambda + 2)}{2!} \left( \frac{-\delta_1}{(x, x_0)_{y,s}} \right)^2 + \cdots \right. \]

\[ + \left. \frac{(\lambda + 1)_n}{n!} \left( \frac{-\delta_1}{(x, x_0)_{y,s}} \right)^n + \frac{(\lambda + 1)_{n+1}}{(n + 1)!} \left( \frac{1}{1 + \zeta} \right)^{\lambda + n + 2} \frac{-\delta_1}{(x, x_0)_{y,s}} \right]^{n+1}. \]

(166)
We could see that:

\[
\left| \frac{\delta_1}{(x,x_0)_{y,s}} \right| \leq \frac{3|t - x_0|}{\|x| - |x_0|} \leq \frac{3|t - x_0|}{\delta_0} = 3/4. \tag{167}
\]

From Formula (167), we could have: \(\xi \in [-3/4, 3/4]\). Thus:

\[
\left( \frac{1}{1 + \xi} \right) \leq \left( \frac{1}{1 - 3/4} \right) \leq 4, \quad \frac{y}{\langle x, x_0 \rangle_{y,s}} \leq \frac{1}{2\langle x, x_0 \rangle_{y,s}^{1/2}}, \text{ and } \frac{1}{(x, x_0)_{y,s}} \leq \frac{1}{\langle x, x_0 \rangle} \tag{168}
\]

From Formula (168), we could have:

\[
\left| \left( \frac{1}{1 + \xi} \right)^{\lambda + n + 2} \frac{y}{\langle x, x_0 \rangle_{y,s}^{\lambda + 1}} \frac{\delta_1^{n + 1}}{(x, x_0)_{y,s}^{n + 1}} \right| \leq C \frac{y}{\langle x, x_0 \rangle_{y,s}^{\lambda + 1}} \frac{|t - x_0|^{n + 1}}{(x, x_0)_{y,s}^{n/2 + \lambda + 1}} \tag{170}
\]

Since we have

\[
|x_0 + t - 2xs| \leq |x_0 - xs| + |t - xs| \leq (\langle x, x_0 \rangle_{s}^{1/2} + \langle x, x_0 \rangle_{s}^{1/2}) + |t - x_0| \leq 3 \langle x, x_0 \rangle_{s}^{1/2},
\]

then the following inequality holds:

\[
|\delta_1| \leq 3|t - x_0| (\langle x, x_0 \rangle_{s}^{1/2}). \tag{170}
\]

Then by Proposition (3.9), Formula(166) we get when \(x \in (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c\) the following holds:

\[
(a \ast \lambda P_y)(x) = c_\lambda \int a(t)(\tau_s P_y)(-t)|t|^{2\lambda} dt
\]

\[
= c_\lambda 2^{\lambda + 1/2 - \pi^{-1}} \lambda \Gamma(\lambda + 1/2) \int a(t) y \int_{-1}^{1} \left( \frac{1 + s}{y^2 + t^2} \right)^{\lambda + 1} ds |t|^{2\lambda} dt
\]

\[
= c_\lambda 2^{\lambda + 1/2 - \pi^{-1}} \lambda \Gamma(\lambda + 1/2) \int a(t) \int_{-1}^{1} \frac{y}{\langle x, x_0 \rangle_{y,s}^{\lambda + 1}} \left[ 1 + \frac{\lambda + 1}{1} \left( \frac{\delta_1}{\langle x, x_0 \rangle_{y,s}} \right)^{1} \right]
\]

\[
+ \frac{(\lambda + 1)(\lambda + 2)}{2!} \left( \frac{\delta_1}{\langle x, x_0 \rangle_{y,s}} \right)^{2} + \cdots + \frac{(\lambda + 1)_n}{(n)!} \left( \frac{\delta_1}{\langle x, x_0 \rangle_{y,s}} \right)^{n} + \frac{(\lambda + 1)_n}{(n + 1)!} \left( \frac{1}{1 + \xi} \right)^{\lambda + n + 2} \left( \frac{\delta_1}{\langle x, x_0 \rangle_{y,s}} \right)^{n + 1}
\]

\[
(1 + s)(1 - s^2)^{\lambda - 1} ds |t|^{2\lambda} dt.
\]

Then the above formula together with the vanishing property of \(a(t)\) we obtain:

\[
|\lambda (a \ast \lambda P_y)(x)| \leq C(\lambda |I(x_0, \delta_0)|^{1/(p)} \left( \frac{\delta_0}{\langle x, x_0 \rangle_{s}^{\lambda/2 + \lambda + 1}} |1 + s| - |x_0|^{n + 2} ||x|| + |x_0||^{2\lambda} \right) \tag{172}
\]

By Formula (151),

\[
|\lambda (a \ast \lambda P_y)(x)| \leq C|I(x_0, \delta_0)|^{1/(p)} \left( \frac{\delta_0}{\langle x, x_0 \rangle_{s}^{\lambda/2 + \lambda + 1}} |1 + s| - |x_0|^{n + 2} ||x|| + |x_0||^{2\lambda} \right) \tag{172}
\]

holds for \(x \in (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0))^c\). Thus the Theorem(3.42) is proved in the same way as Theorem(3.41). This proves the theorem. \(\square\)

**Theorem 3.43.** If \(a(t)\) satisfies \(p_\lambda\)-condition, with vanishing order \(\kappa \geq 2\) \(\left( 2\lambda + 1 \frac{1 - p}{p} \right) \) then

\[
\left( \frac{2\lambda}{2\lambda + 1} < p \leq 1 \right) \int \frac{(|a \ast \lambda Q_y| |p|)(x)|x|^{2\lambda} dx \leq C,
\]

\(C\) is depend on \(\lambda\) and \(p\).
Proof. Assume first that \( x_0 > 0 \). Let \( \kappa = 2 \left[ (2\lambda + 1) \frac{1-p}{p} \right] \). Thus \( \kappa \) is an even integer. Let \( n = \kappa/2 \).

We could write the above integral as:

\[
\int_{\mathbb{R}} |(a * \lambda Q_y)| |x|^{2\lambda} dx = \int_{(I(x_0,4\delta_0) \cup I(-x_0,4\delta_0) \cap I(x_0,4\delta_0))} |(a * \lambda Q_y)| |x|^{2\lambda} dx \\
+ \int_{(I(x_0,4\delta_0) \cup I(-x_0,4\delta_0) \cap I(-x_0,4\delta_0))} |(a * \lambda Q_y)| |x|^{2\lambda} dx \\
= V + VI.
\]

We could have the estimation:

\[
V = \int_{(I(x_0,4\delta_0) \cup I(-x_0,4\delta_0) \cap I(x_0,4\delta_0))} |(a * \lambda Q_y)| |x|^{2\lambda} dx \\
\leq \left( \int_{(I(x_0,4\delta_0) \cup I(-x_0,4\delta_0) \cap I(x_0,4\delta_0))} |(a * \lambda Q_y)|^2 |x|^{2\lambda} dx \right)^{p/2} \left( \int_{(I(x_0,4\delta_0) \cup I(-x_0,4\delta_0))} |x|^{2\lambda} dx \right)^{1-p/2} \\
\leq C \left( \int_{\mathbb{R}} |a(x)|^2 |x|^{2\lambda} dx \right)^{p/2} \left( \left| (I(x_0,4\delta_0) \cap I(-x_0,4\delta_0)) \right|^{1-(p/2)} 2^{1-(p/2)} \right) \leq C.
\]

Next we estimate the following integer:

\[
VI = \int_{(I(x_0,4\delta_0) \cup I(-x_0,4\delta_0) \cap I(x_0,4\delta_0))} |(a * \lambda Q_y)| |x|^{2\lambda} dx.
\]

Notice that \( t \in \text{supp} \mu(t) \subseteq I(x_0,\delta_0) \). By Proposition (3.9), when \( x \in (I(x_0,4\delta_0) \cup I(-x_0,4\delta_0)) \), we could write \( (a * \lambda Q_y)(x) \) as:

\[
(a * \lambda Q_y)(x) = c_{\lambda} \int_{x} a(t)(\tau_x Q_y)(-t)|t|^{2\lambda} dt.
\]

And the conjugate \( \lambda \)-Poisson kernel could be written as:

\[
(\tau_x Q_y)(-t) = \frac{\lambda \Gamma(\lambda + 1/2)}{2^{2-\lambda - 1/2} \pi} \int_{0}^{\pi} (x - t) \left( 1 + \text{sgn}(xt) \cos \theta \right) \left( y^2 + x^2 + t^2 - 2|xt| \cos \theta \right)^{\lambda - 1} \sin^{2\lambda - 1} \theta d\theta.
\]

We could see that:

\[
\frac{x - t}{\langle x, t \rangle_{L^2}} = \frac{x - x_0}{\langle x, t \rangle_{L^2}} + \frac{x_0 - t}{\langle x, x_0 \rangle_{L^2}} = F + G.
\]

By the Taylor expansion of \( \left( 1 + \frac{\delta_1}{\langle x, x_0 \rangle_{L^2}} \right)^{\lambda - 1} \), when \( x \in [-2x_0,0] \cap (I(x_0,4\delta_0) \cup I(-x_0,4\delta_0)) \), we could obtain:\( (\exists \xi) \)

\[
F = \frac{x - x_0}{\langle x, x_0 \rangle_{L^2}} \left( 1 + \frac{\delta_1}{\langle x, x_0 \rangle_{L^2}} \right)^{\lambda + 1} \\
= \frac{x - x_0}{\langle x, x_0 \rangle_{L^2}} \left[ 1 + \frac{\lambda + 1}{2!} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_{L^2}} \right)^2 + \frac{(\lambda + 1)(\lambda + 2)}{3!} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_{L^2}} \right)^3 + \cdots \right] \\
+ \frac{\lambda + 1}{n!} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_{L^2}} \right)^n \left[ \frac{1}{1 + \xi_1} \frac{1 + \xi_1}{(n + 1)!} \right]^{\lambda + n + 2} \left( \frac{-\delta_1}{\langle x, x_0 \rangle_{L^2}} \right)^{n+1},
\]

and
\[ G = \frac{x_0 - t}{\langle x, x_0 \rangle^{\lambda+1}} \left( 1 + \frac{\delta_1}{\langle x, x_0 \rangle} \right)^{\lambda+1} \]

\[ + \frac{x_0 - t}{\langle x, x_0 \rangle^{\lambda+1}} \left[ 1 + \frac{\lambda + 1}{1} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right) + \frac{(\lambda + 1)(\lambda + 2)}{2!} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right)^2 + \ldots \right] \]

\[ + \frac{(\lambda + 1)_{n-1}}{(n - 1)!} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right)^{n-1} + \frac{(\lambda + 1)_n}{(n)!} \frac{1}{1 + \xi_2} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right)^n \frac{\lambda + n + 1}{\lambda + n + 1} \]

We have to estimate the size of formula \( \frac{\delta_1}{\langle x, x_0 \rangle} \). Since \( |\delta_1| \leq 3|t-x_0|\), and \( |x-x_0| \approx (\langle x, x_0 \rangle)^{1/2} \), when \( x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)) \), then:

\[ \left| \frac{\delta_1}{\langle x, x_0 \rangle} \right| \leq \frac{3|t-x_0|}{(\langle x, x_0 \rangle)^{1/2}} \leq \frac{3|t-x_0|}{\|x-x_0\|} \leq \frac{3\delta_0}{4\delta_0} = 3/4, \]

\[ \left( \frac{1}{1 + \xi_1} \right) \leq \left( \frac{1}{1 - 3/4} \right) \leq 4 \quad \text{and} \quad \left( \frac{1}{1 + \xi_2} \right) \leq \left( \frac{1}{1 - 3/4} \right) \leq 4. \]

Let \( d\nu(s) \) denote \((1+s)(1-s^2)^{\lambda-1} ds\). Then by Proposition (3.9) and Formulas (173), (174), (175) for \( x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)) \), we could get

\[ (a \ast \lambda Q_y)(x) = c_\lambda \int a(t)(r_x Q_y)(-t)|t|^{2\lambda}dt \]

\[ \approx \int a(t)(x-t) \int_{-1}^1 (1 + s)(1-s^2)^{\lambda-1}|t|^{2\lambda}dt \]

\[ \approx \int_{-1}^1 a(t) \frac{x-x_0}{\langle x, x_0 \rangle^{\lambda+1}} \left[ 1 + \frac{\lambda + 1}{1} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right) + \frac{(\lambda + 1)_2}{2!} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right)^2 + \ldots \right] \]

\[ + \frac{(\lambda + 1)_n}{(n)!} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right)^n + \frac{(\lambda + 1)_{n+1}}{(n + 1)!} \frac{1}{1 + \xi_1} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right)^{n+1} \times \]

\[ \times \int_{-1}^1 a(t) \frac{x_0-t}{\langle x, x_0 \rangle^{\lambda+1}} \left[ 1 + \frac{\lambda + 1}{1} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right) + \frac{(\lambda + 1)_2}{2!} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right)^2 + \ldots \right] \]

\[ + \frac{(\lambda + 1)_{n-1}}{(n - 1)!} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right)^{n-1} + \frac{(\lambda + 1)_n}{(n)!} \frac{1}{1 + \xi_2} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right)^n \times \int_{-1}^1 a(t) d\nu(s)|t|^{2\lambda}dt. \]

Notice that 0 < \( \langle x, x_0 \rangle \leq \langle x, x_0 \rangle \). Then the above formula together with the vanishing property of \( a(t) \), for \( x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)) \), we obtain:

\[ |(a \ast \lambda Q_y)(x)| \leq \int_{-1}^1 |a(t)| \left| \frac{x-x_0}{\langle x, x_0 \rangle^{\lambda+1}} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right)^{n+1} \right| d\nu(s)|t|^{2\lambda}dt \]

\[ + \int_{-1}^1 |a(t)| \left| \frac{x_0-t}{\langle x, x_0 \rangle^{\lambda+1}} \left( -\frac{\delta_1}{\langle x, x_0 \rangle} \right)^n \right| d\nu(s)|t|^{2\lambda}dt \]

\[ \leq C|I(x_0, \delta_0)| \lambda^{(1/p)} \int_{-1}^1 \left( \frac{(\delta_1)^{n+1}}{(\langle x, x_0 \rangle)^{\lambda+1/p}}(1+s)(1-s^2)^{\lambda-1} ds. \] (176)

We could see that:

\[ \frac{x - t}{\langle x, t \rangle^{\lambda+1}} = \frac{x - x_0 s}{\langle x, x_0 \rangle^{\lambda+1}} + \frac{x_0 - t}{\langle x, t \rangle^{\lambda+1}} + \frac{x_0 (s - 1)}{\langle x, t \rangle^{\lambda+1}} \]

\[ = C_1 + D_1 + E_1. \]

Then we need to estimate \((a \ast \lambda Q_y)(x)\) when \( x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)) \). By the Taylor expansion, we could obtain:
Hardy spaces associated with the Dunkl Transform and Homogeneous type (with a Kernel)

Notice that $0 < \langle x, x \rangle_s \leq \langle x, x \rangle_{y,s}$. Then by Formula (143), we could have: $\xi_3, \xi_4, \xi_5 \in [-3/4, 3/4]$. Thus:

$$
\left( \frac{1}{1 + \xi_i} \right) \leq \left( \frac{1}{1 - 3/4} \right) \leq 4 \text{ for } i = 3, 4, 5.
$$

Notice that $0 < \langle x, x \rangle_s \leq \langle x, x \rangle_{y,s}$. Thus Formulas (177), (178), (179), (180), (143), (181), (138), (141) and (142), together with the vanishing property of a(t), we obtain the following inequality:

$$
|a *_{\lambda} Q_y(x)| \lesssim \int_{-1}^{1} |a(t)| \left| \frac{x - x_0}{\langle x, x \rangle_{y,s}^{\lambda+1}} \left( \frac{\delta_1}{\langle x, x \rangle_s} \right)^n + (1 + s)(1 - s^2)^{\lambda-1} ds \right| |t|^{2\lambda} dt
$$

$$
+ \int_{-1}^{1} |a(t)| \left| \frac{x_0 - t}{\langle x, x \rangle_{y,s}^{\lambda+1}} \left( \frac{\delta_1}{\langle x, x \rangle_s} \right)^n \right| (1 + s)(1 - s^2)^{\lambda-1} ds |t|^{2\lambda} dt
$$

$$
+ \int_{-1}^{1} |a(t)| \left| \frac{x_0(s - 1)}{\langle x, x \rangle_{y,s}^{\lambda+1}} \left( \frac{\delta_1}{\langle x, x \rangle_s} \right)^n \right| (1 + s)(1 - s^2)^{\lambda-1} ds |t|^{2\lambda} dt
$$

$$
\lesssim |I(x_0, \delta_0)|^{-(1/p)} \int_{-1}^{1} \left( \frac{\delta_0}{\langle x, x \rangle_s} \right)^{n+1} (1 + s)(1 - s^2)^{\lambda-1} ds
$$

$$
+ \int_{-1}^{1} |a(t)| \left| \frac{x_0(s - 1)}{\langle x, x \rangle_{y,s}^{\lambda+1}} \left( \frac{\delta_1}{\langle x, x \rangle_s} \right)^n \right| (1 + s)(1 - s^2)^{\lambda-1} ds |t|^{2\lambda} dt.
$$

for $x \in [-2x_0, 0] \cap I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)^c$. Similar to Formula (158), Formula (182) could imply that: for $x \in [-2x_0, 0] \cap I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)^c$

$$
|a *_{\lambda} Q_y(x)| \lesssim |I(x_0, \delta_0)|^{-(1/p)} \left\| \frac{\delta_0^{n+1}}{\|x - x_0\|^{n+2} + \|x\|^{2\lambda}} \right\|.
$$
Formula (151) and (176) imply that: for \( x \in [-2x_0, 0] \cap (I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)) \)
\[
| (a \ast Q_y)(x) | \lesssim |I(x_0, \delta_0)|^{1 - (1/p)} \frac{\delta_0^{n+1}}{|x - x_0|^{n+2}} \frac{|x| + |x_0|^{2\lambda}}{2\lambda}. \tag{184}
\]
Thus similar to Formula (162), we could obtain that
\[
VI = \int_{I(x_0, 4\delta_0) \cup I(-x_0, 4\delta_0)^c} | (a \ast Q_y)|^p(x)|x|^{2\lambda} \leq C.
\]
This proves the theorem. \( \square \)

The above Theorems are base on the \( \lambda \)-Hilbert kernel, \( \lambda \)-Poisson kernel, \( \lambda \)-Conjugate Poisson kernel. By the Taylor Expansion of the kernel, we need \( \kappa \geq 2 \left( 2\lambda + 1 \right) \frac{1 - p}{p} \) vanishing order of the atoms when \( \frac{2\lambda + 1}{2\lambda + 2} < p \leq 1 \). In fact, \( \kappa \geq 0 \) is enough when \( \frac{2\lambda + 1}{2\lambda + 2} < p \leq 1 \).

**Theorem 3.44.** If \( a(t) \) is a \( p_\lambda \)-atom with vanishing order \( \kappa \geq 0 \) \( \left( \frac{2\lambda + 1}{2\lambda + 2} < p \leq 1 \right) \) then \( Pa(x, y) + iQa(x, y) \in H^p_\lambda(\mathbb{R}^2_+) \) and
\[
a(t) \in H^p_\lambda(\mathbb{R}).
\]

**Proof.** From Proposition (2.33)(2.15), the following holds:
\[
\int_{\mathbb{R}} |a^*(x)|^p|x|^{2\lambda} dx \leq C
\]
By Theorem (3.21), we could obtain
\[
\int_{\mathbb{R}} |(a \ast P_y)|^p(x)|x|^{2\lambda} dx \leq C,
\]
\[
\int_{\mathbb{R}} |(a \ast Q_y)|^p(x)|x|^{2\lambda} dx \leq C.
\]
\( C \) is depend on \( \lambda \) and \( p \). Thus By Proposition (3.10)(v), \( Pa(x, y) + iQa(x, y) \) is a \( \lambda \)-analytic function. And \( Pa(x, y) + iQa(x, y) \in H^p_\lambda(\mathbb{R}^2_+) \). This proves the theorem. \( \square \)

Therefore we could define the atomic Hardy space for the Dunkl setting \( H^{p_\lambda}_{\kappa}(\mathbb{R}) \) as following:

**Definition 3.45 (\( H^{p_\lambda}_{\kappa}(\mathbb{R}) \)).** For \( \frac{2\lambda}{2\lambda + 1} < p \leq 1 \), the atomic Hardy space \( H^{p_\lambda}_{\kappa}(\mathbb{R}) \) is defined by
\[
H^{p_\lambda}_{\kappa}(\mathbb{R}) \triangleq \left\{ f \text{ is a } \lambda - \text{distribution} : f = \sum_k \lambda_k a_k(x) \text{ in } H^p_\lambda(\mathbb{R}) \text{ space. } a_k(x) \text{ is a } p_\lambda \text{-atom.} \right\}
\]

Setting \( H^{p_\lambda}_{\kappa}(\mathbb{R}) \) norm of \( f \) by
\[
\| f \|_{H^{p_\lambda}_{\kappa}(\mathbb{R})} = \inf \left( \sum_k |\lambda_k|^p \right)^{1/p}.
\]
Where the infimum is taken over all decompositions of \( f(x) = \sum_k \lambda_k a_k(x) \) above. It is very easy to verify that \( H^{p_\lambda}_{\kappa}(\mathbb{R}) \) is a complete metric space with the distance
\[
\rho(f, g) = \| f - g \|_{H^{p_\lambda}_{\kappa}(\mathbb{R})}^p.
\]
When \( \frac{2\lambda}{2\lambda + 1} < p \leq 1 \), \( f \in H^{p_\lambda}_{\kappa}(\mathbb{R}) \):
\[
f(x) = \sum_k \lambda_k a_k(x),
\]
where \( a_k(x) \) is a \( p \)-atom with \( \sum_k |\lambda_k|^p < +\infty \). By Theorem (3.44), \( P(a_k)(x, y) + iQ(a_k)(x, y) \in H^p_\lambda(\mathbb{R}^2_+) \). From Proposition (3.3),

\[
\sup_{x \in \mathbb{R}} |P(a_k)(x, y) + iQ(a_k)(x, y)| \leq cy^{-(1/p)(1+2\lambda)}.
\]

Then together with Lemma (3.76), we could deduce that \( \sum_k \lambda_k (P(a_k)(x, y) + iQ(a_k)(x, y)) \) is a bounded function in the upper half plane:

\[
\sup_{x \in \mathbb{R}} |\sum_k \lambda_k (P(a_k)(x, y) + iQ(a_k)(x, y))| \leq cy^{-(1/p)(1+2\lambda)} \left( \sum_k |\lambda_k|^p \right)^{1/p} \leq cy^{-(1/p)(1+2\lambda)} \|f\|_{L^p_\lambda(\mathbb{R})}.
\]

Notice that

\[
\left( \int_{\mathbb{R}} |\partial_y (P(a_k)(x, y)))|^p (x)|x|^{2\lambda} \, dx \right)^{1/p} \lesssim \frac{1}{y} \left( \int_{\mathbb{R}} |\partial_y (Q(a_k)(x, y)))|^p (x)|x|^{2\lambda} \, dx \right)^{1/p} \lesssim \frac{1}{y}
\]

and

\[
|\partial_y (P(a_k)(x, y)))| = |D_y (Q(a_k)(x, y)))|, \quad |\partial_y (Q(a_k)(x, y)))| = |D_y (P(a_k)(x, y)))|.
\]

Then by Theorem (3.44) and Proposition (3.1), we could deduce that \( \sum_k P(\lambda_k a_k)(x, y) + i\sum_k Q(\lambda_k a_k)(x, y) \) is a \( \lambda \)-analytic function in \( \mathbb{R}^2_+ \), and \( \sum_k P(\lambda_k a_k)(x, y) + i\sum_k Q(\lambda_k a_k)(x, y) \in H^p_\lambda(\mathbb{R}^2_+) \). Also for \( f(x) \in H^p_{\lambda, \kappa}(\mathbb{R}) \), we could also define \( Pf(x, y), Qf(x, y) \) as following:

\[
Pf(x, y) = \sum_k P(\lambda_k a_k)(x, y), \quad Qf(x, y) = \sum_k Q(\lambda_k a_k)(x, y).
\]

Then by Theorem (3.12) the following inequality holds:

\[
\left\| \sum_k \lambda_k a_k \right\|_{L^p_\lambda(\mathbb{R})} \leq \sup_{y > 0} \left( \int_{\mathbb{R}} |\sum_k \lambda_k a_k(x) \ast \lambda P_y(x) + i \sum_k \lambda_k a_k(x) \ast \lambda Q_y(x)|^p |x|^{2\lambda} \, dx \right)^{1/p} \leq \sum_k \sup_{y > 0} \left( \int_{\mathbb{R}} |\lambda_k a_k \ast \lambda P_y(x)|^p |x|^{2\lambda} \, dx + \int_{\mathbb{R}} |\lambda_k a_k \ast \lambda Q_y(x)|^p |x|^{2\lambda} \, dx \right) \lesssim \sum_k |\lambda_k|^p.
\]

Thus \( \sum_k \lambda_k a_k \in H^p_\lambda(\mathbb{R}) \). Thus by Theorem (3.34) and the definition of the \( H^p_{\lambda, \kappa}(\mathbb{R}) \), the following Proposition holds:

**Proposition 3.46.**

\( H^p_{\lambda, \kappa}(\mathbb{R}) = H^p_\lambda(\mathbb{R}) \), for \( \frac{2\lambda + 1}{2\lambda + 2} < p \leq 1 \), \( \kappa \geq 0 \).

\( H^p_{\lambda, \kappa}(\mathbb{R}) \subseteq H^p_\lambda(\mathbb{R}) \), for \( \frac{2\lambda}{2\lambda + 1} < p \leq \frac{2\lambda + 1}{2\lambda + 2} \), \( \kappa \geq 2 \left( (2\lambda + 1) \frac{1-p}{p} \right) \).

By Theorem (3.41)(3.42)(3.43), and Proposition (2.31) we could obtain:

**Proposition 3.47.** For any \( f \in H^p_{\lambda, \kappa}(\mathbb{R}) \), \( \kappa \geq 0 \) when \( \frac{2\lambda + 1}{2\lambda + 2} < p \leq 1 \). And \( \kappa \geq 2 \left( (2\lambda + 1) \frac{1-p}{p} \right) \) when \( \frac{2\lambda}{2\lambda + 1} \leq p \leq \frac{2\lambda + 1}{2\lambda + 2} \), we could deduce that \( Pf(x, y) + iQf(x, y) \in H^p_\lambda(\mathbb{R}^2_+) \) and the following holds:

\[
\sup_{y > 0} \left( \int_{\mathbb{R}} |f \ast \lambda P_y(x) + if \ast \lambda Q_y(x)|^p |x|^{2\lambda} \, dx \right) \leq C\|f\|_{L^p_\lambda(\mathbb{R})}.
\]

### 3.5 \( p^*_\lambda \)-atom and its relation to Hardy spaces in the Dunkl setting

Let \( B \) be the ball in the homogenous space: \( B(x_0, r_0) = \{ y : d_\lambda(y, x_0) < r_0 \} \). Then we could define a new kind of atom in the Dunkl setting as following:
Definition 3.48 \( (p^*_\lambda\text{-atom}) \). If function \( a(x) \) satisfies the following condition, we also say \( a(x) \) is a \( p^*_\lambda \text{-atom} \):

1. \( \|a(x)\|_{L^\infty} \lesssim \frac{1}{|B(x_0, r_0)|^{1/p}} \)
2. \( \text{supp} \, a(x) \subseteq B(x_0, r_0) \)
3. \( \int_{\mathbb{R}} t^k a(t) |t|^{2\lambda} \, dt = 0 \quad (k = 0, 1, 2, 3 \ldots \kappa) \)
4. \( \kappa \geq 2 \left[ 2\lambda + 1 \right] 1 - \frac{p}{2\lambda + 1} \) when \( \frac{2\lambda}{2\lambda + 1} < p \leq \frac{2\lambda + 1}{2\lambda + 2} \). \( \kappa \geq 0 \) when \( \frac{2\lambda + 1}{2\lambda + 2} < p \leq 1 \).

It is easy to see that if a \( p^*_\lambda \text{-atom} \) satisfies \( 0 < r_0^{-\lambda+\kappa} < |x_0|/2 \), then it will be a \( p_\lambda \text{-atom} \). When \( \lambda = 0 \), then \( p^*_\lambda \text{-atom} \) is just the atom in classical Hardy space.

Let \( B(x_0, r_0) \) be a ball in the Homogeneous Dunkl space satisfying \( 0 < r_0^{-\lambda+\kappa} < |x_0|/2 \). Let \( d\mu_\lambda(x) = (2\lambda + 1)|x|^{2\lambda} \, dx \). Let \( P^\kappa \) to be the \( \kappa \text{-order polynomials} \). Let \( P^\kappa_{B(x_0, r_0)} \) to be the \( \kappa \text{-order polynomials} \) with its Hilbert norm: \( f \in L^2(B(x_0, r_0)) \)

\[
\|f\|_{P^\kappa_{B(x_0, r_0)}} = \left( \frac{\int_{B(x_0, r_0)} |f(x)|^2 |x|^{2\lambda} \, dx}{\int_{B(x_0, r_0)} |x|^{2\lambda} \, dx} \right)^{1/2}.
\]

In addition, let \( \{\pi^i\}_{i=0}^\kappa \subseteq P^\kappa \) to be the orthonormal basis satisfying the following conditions:

\[
\int_{B(x_0, r_0)} \pi^i(t) \pi^j(t) \frac{|t|^{2\lambda} \, dt}{r_0} = \delta_{ij} \}
\]

Thus by Proposition(3.30) we could conclude

\[
\sup_{t \in B(x_0, r_0)} |\pi^i(t)| \leq C,
\]

\[ C \text{ is independent on } \pi^i \text{ and } B(x_0, r_0). \]

And let \( g \in L_{\lambda, \text{loc}}(\mathbb{R}) \). Then there exists a unique \( P^\kappa_{B(x_0, r_0)} g \in P^\kappa \), such that:

\[
\int_{B(x_0, r_0)} \left\{ g(x) - P^\kappa_{B(x_0, r_0)} g(x) \right\} Q(x) |x|^{2\lambda} \, dx = 0, \quad \forall Q(x) \in P^\kappa_{B(x_0, r_0)}.
\]

And

\[
P^\kappa_{B(x_0, r_0)} g \in \chi_{B(x_0, r_0)}(x) \sum_{\alpha \leq \kappa} \pi^\alpha(x) \int_{B(x_0, r_0)} g(t) \pi^\alpha(t) \frac{|t|^{2\lambda} \, dt}{r_0}.
\]

Thus Formula(185)and(187) lead to:

\[
|P^\kappa_{B(x_0, r_0)} g(x)| \leq \chi_{B(x_0, r_0)}(x) \frac{C}{r_0} \int_{B(x_0, r_0)} |g(t)||t|^{2\lambda} \, dt.
\]

Proposition 3.49. If \( a(t) \) is a \( p^*_\lambda \text{-atom} \) for \( \frac{2\lambda + 1}{2\lambda + 2} < p \leq 1 \), then \( a(t) \in H^p_\lambda(\mathbb{R}) \). We can write \( a(t) \) as \( a(t) = \sum \lambda_i a_i(t) \) a.e and \( H^p_\lambda(\mathbb{R}) \). Each \( a_i(t) \) is a \( p_\lambda \text{-atom} \). \( \{\lambda_i\} \) is a sequence of positive numbers with \( \sum |\lambda_i|^p \approx \|a\|^p_{H^p_\lambda(\mathbb{R})} \).

Proof. From Proposition(2.33) we could deduce that \( a(t) \in H^p_\lambda(\mathbb{R}) \). Then from Theorem(3.34), we could deduce Proposition.

Proposition 3.50. Let \( B(x_0, r_0) \) to be a ball in Homogeneous Dunkl space. \( \{\pi^i\}_{i=0}^\kappa \subseteq P^\kappa \) to be the orthonormal basis satisfying the following conditions:

\[
\int_{B(x_0, r_0)} \pi^i(t) \pi^j(t) \frac{|t|^{2\lambda} \, dt}{r_0} = \delta_{ij} \}
\]

\[
1, i = j, \quad 0, i \neq j.
\]
Then we could conclude that

$$
\sup_{t \in B(x_0, r_0)} |\pi^l(t)| \leq C,
$$

(189)

C is independent on $\pi^l$ and $B(x_0, r_0)$.

Proof. Let $d\mu_\lambda(t) = (2\lambda + 1)|t|^{2\lambda}dt$ and $x = f(t) = \mu_\lambda(t, 0) = \int_0^1 (2\lambda + 1)|u|^{2\lambda}du$. Thus $f(t)$ is a bijection on $\mathbb{R}$. Let $f^{-1}(x)$ to be the reverse map of $f(t)$ satisfying the following:

$$
f^{-1} \circ f(t) = t, \quad f \circ f^{-1}(x) = x, \quad \forall x, t \in \mathbb{R}.
$$

Thus

$$
B(x_0, r_0) = \{y : d_\lambda(x_0, y) < r_0\} = \{y : |f(y) - f(x_0)| < r_0\}.
$$

Let $f(x_0) + r_0u = f(y) = (2\lambda + 1) \int_0^y |t|^{2\lambda}dt$, where $-1 < u < 1$. We could deduce that

$$
r_0 du = (2\lambda + 1)|y|^{2\lambda}dy.
$$

Thus we could obtain:

$$
1 = \int_{B(x_0, r_0)} |\pi^l(t)|^2 \frac{|t|^{2\lambda}dt}{r_0} = \int_{B(x_0, r_0)} |\pi^l(f^{-1}(f(y)))|^2 \frac{|y|^{2\lambda}dy}{r_0} = \frac{1}{2\lambda + 1} \int_{[-1, 1]} |\pi^l(f^{-1}(f(x_0) + r_0u))|^2 du.
$$

Let $P^\kappa$ be the $\kappa$-order polynomials. Then we could define $P^\kappa_\lambda$ as following:

$$
P^\kappa_\lambda = \{g \circ f^{-1}(x) : g(t) \in P^\kappa\}.
$$

By the fact that any two norms on a finite dimensional space are equivalent, we could obtain for any $r(x) \in P^\kappa_\lambda$:

$$
\left(\int_{[-1, 1]} |r(x)|^2 dx\right)^{1/2} \geq C \sup_{x \in [-1, 1]} |r(x)|, \quad \forall r(x) \in P^\kappa_\lambda
$$

and C is independent of $r(x)$. Taking $r(u) = \pi^l(f^{-1}(f(x_0) + r_0u))$ we could conclude that:

$$
\left(\int_{[-1, 1]} |\pi^l(f^{-1}(f(x_0) + r_0u))|^2 du\right)^{1/2} \geq C \sup_{u \in [-1, 1]} |\pi^l(f^{-1}(f(x_0) + r_0u))|,
$$

that is

$$
\sup_{t \in B(x_0, r_0)} |\pi^l(t)| \leq C \int_{[-1, 1]} |\pi^l(f^{-1}(f(x_0) + r_0u))|^2 du
$$

$$
\leq C \int_{B(x_0, r_0)} |\pi^l(t)|^2 \frac{|t|^{2\lambda}dt}{r_0}
$$

$$
\leq C.
$$

This proves the proposition. 

Thus we can define $H^{p^\kappa, \kappa}_\lambda(\mathbb{R})$ spaces for $\frac{2\lambda}{2\lambda + 1} < p \leq 1$ as following:

**Definition 3.51** ($H^{p^\kappa, \kappa}_\lambda(\mathbb{R})$). For $\frac{2\lambda}{2\lambda + 1} < p \leq 1$, $\kappa \geq 0$, the atomic Hardy space $H^{p^\kappa, \kappa}_\lambda(\mathbb{R})$ is defined by

$$
H^{p^\kappa, \kappa}_\lambda(\mathbb{R}) \triangleq \left\{ f \text{ is a } \lambda - \text{distribution} : f = \sum_k \lambda_k a_k(x) \text{ in } H^p_\lambda(\mathbb{R}) \text{ space. } a_k(x) \text{ is a } p^\kappa_\lambda\text{-atom.} \right\}
$$

$$
\sum_k |\lambda_k|^p < +\infty.
$$
Setting $H^{p,-\kappa}_\lambda(\mathbb{R})$ norm of $f$ by
\[
\|f\|_{H^{p,-\kappa}_\lambda(\mathbb{R})} = \inf \left( \sum_k |\lambda_k|^p \right)^{1/p}.
\]
Where the infimum is taken over all decompositions of $f = \sum_k \lambda_k a_k(x)$ above. It is very easy to verify that $H^{p,-\kappa}_\lambda(\mathbb{R})$ is a complete metric space with the distance
\[
\rho(f, g) = \|f - g\|_{H^{p,-\kappa}_\lambda(\mathbb{R})}.
\]
Thus the following theorem holds:

**Theorem 3.52.**
\[
H^{p,-\kappa}_\lambda(\mathbb{R}) = H^{p,-\kappa}_\lambda(\mathbb{R}), \quad \text{for } \frac{2\lambda + 1}{2\lambda + 2} < p \leq 1.
\]
And $\forall f \in H^{p,-\kappa}_\lambda(\mathbb{R})$,
\[
\|f\|_{H^{p,-\kappa}_\lambda(\mathbb{R})} \approx \|f\|_{H^{p,-\kappa}_\lambda(\mathbb{R})}.
\]

**Proof.** By Proposition(3.49), we could see that $H^{p,-\kappa}_\lambda(\mathbb{R}) \subseteq H^{p,-\kappa}_\lambda(\mathbb{R})$. It is easy to see that $H^{p,-\kappa}_\lambda(\mathbb{R}) \subseteq H^{p,-\kappa}_\lambda(\mathbb{R})$. By Proposition(3.49), $\forall f \in H^{p,-\kappa}_\lambda(\mathbb{R})$,
\[
\|f\|_{H^{p,-\kappa}_\lambda(\mathbb{R})} \approx \|f\|_{H^{p,-\kappa}_\lambda(\mathbb{R})}.
\]
This proves the theorem. \(\square\)

Then we have the following proposition:

**Proposition 3.53.** $\kappa \geq 0$ when $\frac{2\lambda + 1}{2\lambda + 2} < p \leq 1$. And $\kappa \geq 2\left[\frac{2\lambda + 1}{2\lambda + 2}\right]$ when $\frac{2\lambda + 1}{2\lambda + 2} < p \leq \frac{2\lambda + 1}{2\lambda + 2}$.

For $1 < q < \infty$. $f \in L^q_{B_1}$ means that $|f(x)|^q$ is a integrable function in the ball $B_1 = B_1(x_0, r_0) = \{y : d_\lambda(y, x_0) < r_0\}$, which satisfies $\frac{r_0}{\delta_\lambda} < |x_0|/2$. Let $L^q_{B_1, \kappa}$ to be a subspace of $L^q_{B_1}$:
\[
L^q_{B_1, \kappa} = \{g : P_{B_1}^\kappa g(x) = 0, \forall \alpha \leq \kappa, g(x) \in L^q_{B_1}\}.
\]
Then $L^q_{B_1, \kappa} \subseteq H^q_{\lambda}(\mathbb{R})$.

**Proof.** $\forall g(x) \in L^q_{B_1, \mu}$, let $\delta_\alpha = \left(\frac{2\lambda + 1}{2\lambda + 2} - x_0\right)$, then by Proposition(3.36), $B_1(x_0, r_0) \subseteq I(x_0, \delta_0)$, $\delta_0 < |x_0|/2$ and $\|B_1(x_0, r_0)\| \approx |I(x_0, \delta_0)|$. Then similar to Theorem(3.42) and (3.43), we could prove
\[
\int_{\mathbb{R}} |(g * \lambda P_y)|^q |x|^{2\lambda + 1} dx \lesssim \left(\|g\|_{L^q_{\lambda}(\mathbb{R})}\right)^p \|\lambda P_y\|_{L^q_{\lambda}(\mathbb{R})}^{p-1}\|f(x_0, \delta_0)|^{1-(p/q)}.
\]
and
\[
\int_{\mathbb{R}} |(g * \lambda Q_y)|^q |x|^{2\lambda + 1} dx \lesssim \left(\|g\|_{L^q_{\lambda}(\mathbb{R})}\right)^p \|\lambda P_y\|_{L^q_{\lambda}(\mathbb{R})}^{p-1}\|f(x_0, \delta_0)|^{1-(p/q)}.
\]
Also by Proposition(3.10) $v g * \lambda P_y(x) + i g * \lambda Q_y(x)$ is a $\lambda$-analytic function. Thus $g \in H^q_{\lambda}(\mathbb{R})$ with the following holds:
\[
\|g\|_{H^q_{\lambda}(\mathbb{R})} \lesssim \left(\|g\|_{L^q_{\lambda}(\mathbb{R})}\right)^p |B_1(x_0, r_0)|^{1/p-1/q}.
\]
This proves the proposition. \(\square\)

Thus we could have the following conclusion:

**Proposition 3.54.** $\kappa \geq 0$ when $\frac{2\lambda + 1}{2\lambda + 2} < p \leq 1$. For $1 < q < \infty$. $f \in L^q_B$ means that $|f(x)|^q$ is a integrable function in the ball $B = B(x_0, r_0) = \{y : d_\lambda(y, x_0) < r_0\}$. Let $L^q_{B, \kappa}$ to be a subspace of $L^q_B$:
\[
L^q_{B, \kappa} = \{g : P_{B}^\kappa g(x) = 0, \forall \alpha \leq \kappa, g(x) \in L^q_B\}.
\]
Then $L^q_{B, \kappa} \subseteq H^q_{\lambda}(\mathbb{R})$. 
Let \( d \mu_\lambda(t) = (2\lambda + 1)|t|^{2\lambda}dt \), \( d \mu_\lambda(x, y) = d\lambda(x, y) = \left| \int_y^x (2\lambda + 1)|t|^{2\lambda}dt \right| \). Let \( \alpha = \frac{1}{2\lambda + 1} \). For any \( \psi(x) \in SS_\alpha \) satisfying \( \int \psi(t)dt = 1 \), we have:

\[
\int_{B(x_0,4r_0)} |g^*(x)|^p d\mu_\lambda(x) \leq C \int_{B(x_0,4r_0)} |Mg(x)|^p d\mu_\lambda(x)
\]
\[
\leq C \left( \int_{B(x_0,4r_0)} |Mg(x)|^q d\mu_\lambda(x) \right)^{p/q} \left( \int_{B(x_0,4r_0)} 1 d\mu_\lambda(x) \right)^{1-(p/q)}
\]
\[
\leq C \left( \|g\|_{L^q_\psi(\mathbb{R})} \right)^p \left( |B| \right)^{1-(p/q)} ,
\]

C is independent on \( \psi \) and \( r_0 \). By the vanishing property of \( g \) we could have:

\[
\int_{B(x_0,4r_0)^c} \left| \int g(t) \psi \left( \frac{\mu_\lambda(t, x)}{r} \right) \frac{d\mu_\lambda(t)}{r} \right|^p d\mu_\lambda(x)
\]
\[
= \int_{B(x_0,4r_0)^c} \left| \int g(t) \left( \psi \left( \frac{\mu_\lambda(t, x)}{r} \right) - \psi \left( \frac{\mu_\lambda(x_0, x)}{r} \right) \right) \frac{d\mu_\lambda(t)}{r} \right|^p d\mu_\lambda(x)
\]
\[
\leq C \int_{B(x_0,4r_0)^c} \left( \|g\|_{L^q_\psi(\mathbb{R})} \right)^p \left( r_0 \right)^{p-(p/q)} \left( \frac{r_0}{r_0 + 1} \right)^p d\mu_\lambda(x).
\]

Notice that \( r > |\mu(x, x_k) - r_k|, \alpha > p^{-1} - 1 \) and \( 0 < p \leq 1 \), thus

\[
\int_{B(x_0,4r_0)^c} \left( r_0 \right)^{p-(p/q)} \left( \frac{r_0}{r_0 + 1} \right)^p d\mu_\lambda(x) \leq C r_0^{1-(p/q)}.
\]

Thus we proved for \( g(x) \in L^q_{B,\kappa} \), the following inequality holds:

\[
\|g\|_{H^q_\psi(\mathbb{R})} \lesssim \left( \|g\|_{L^q_\psi(\mathbb{R})} \right) \left( |B| \right)^{1/p-1/q} . \tag{191}
\]

This proves the proposition. \( \square \)

### 3.6 Dual spaces of Hardy spaces for the Dunkl setting

In this section, we shall discuss the dual spaces of the spaces: \( H^p_\lambda(\mathbb{R}) \), for \( \frac{2\lambda + 1}{\lambda + 2} < p \leq 1 \). By Proposition(3.46), they are just the spaces \( H^p_\lambda(\mathbb{R}) \).

**Definition 3.55.** \([BMO_\lambda] \) A locally integrable function \( f \) will be said to belong to \( BMO_\lambda \) is the inequality

\[
\frac{1}{|I|_\lambda} \int_I |f(x) - f_I| |x|^{2\lambda}dx \leq A \tag{192}
\]

holds for all interval in \( \mathbb{R} \). Here \( f_I = \frac{1}{|I|_\lambda} \int_I f(x)|x|^{2\lambda}dx \) denote the mean value of \( f \) over the Euclidean interval \( I \). The above inequality (192) asserts that over any Euclidean interval \( I \), the average oscillation \( f \) is bounded.

The smallest bound \( A \) for which (192) is satisfied is then taken to be the norm of \( f \) in this space, and is denoted by \( \|f\|_{BMO_\lambda} \). Suppose \( g \in L_{\lambda,\text{loc}}(\mathbb{R}) \). As in the proof of previous(115), there exists a unique \( P_I^f g(x) \in P^\kappa \), such that:

\[
\int_I \{g(x) - P_I^f g(x)\} Q(x)|x|^{2\lambda}dx = 0, \quad \forall Q(x) \in P^\kappa.
\]

Then we define the Campanato spaces as following:

**Definition 3.56** (Campanato spaces for the Dunkl setting). Suppose that \( \alpha \geq 0, \kappa \) to be a nonnegative integer. Let

\[
\|g\|_{L^\kappa_\alpha} = \sup_I \left( |I|_\lambda \right)^{-\alpha} \frac{1}{|I|_\lambda} \left( \int_I |g(x) - P_I^f g(x)| |x|^{2\lambda}dx \right).
\]
Proposition 3.57. Let $g \in L_{\lambda, \text{loc}}(\mathbb{R})$, $I$ to be an Euclid interval in $\mathbb{R}$. Then

$$
\sup_{x \in I} |P_I^\lambda g(x)| \lesssim \frac{C}{|I|^\lambda} \int_I |g(x)||x|^{2\lambda} dx.
$$

Therefore, if $g \in L_{\lambda, \text{loc}}(\mathbb{R})$, then

$$
\int_I |P_I^\lambda g(x)||x|^{2\lambda} dx \lesssim C \int_I |g(x)||x|^{2\lambda} dx
$$

where $C$ is independent of $I$ and $g$.

Proof. Obviously, we could direct deduce the second inequality from the first one. Therefore we need to only to show the first one. Let $\{\phi^l_n : l \leq \kappa\}$ denote the Gram-Schmidt orthonormalization of $\{x^{\alpha} : |\alpha| \leq \kappa\}$ on the set $I$ with respect to the weight $1/|I|^\lambda$. That is $\phi^l_n \in P^\kappa$, and

$$
\langle \phi^l_n, \phi^m_n \rangle = \frac{1}{|I|^\lambda} \int_I \phi^l_n(x)\phi^m_n(x)|x|^{2\lambda} dx = \delta_{nm} = \begin{cases} 1, n = m, \\ 0, n \neq m. \end{cases}
$$

By Proposition (3.50), we have $\|\phi^l_n\|_\infty \leq C$. Secondly it is easy to verify that

$$
P_I^\lambda g(x) = \sum_{n \leq \kappa} \langle g, \phi^l_n \rangle \lambda \phi^l_n(x).
$$

Therefore we could deduce the Proposition from the above two relations. Thus the Proposition is proved.

Theorem 3.58. Suppose $\frac{1}{p} + \frac{1}{q} < p \leq 1$, and the nonnegative integer $\kappa \geq 0$. If $g \in \mathcal{L}_{\lambda, \kappa}^{p^{-1} - 1}(\mathbb{R})$, then the linear function defined by

$$
Lf = \int_{\mathbb{R}} f(x)g(x)|x|^{2\lambda} dx
$$

is bounded on some dense subset of $H^{p^*, \kappa}_\lambda(\mathbb{R})$. Conversely, if $L$ is a bounded linear function on $H^{p^*, \kappa}_\lambda(\mathbb{R})$, then there exists a $l(x) \in \mathcal{L}_{\lambda, \kappa}^{p^{-1} - 1}(\mathbb{R})$ such that

$$
Lf = \int_{\mathbb{R}} f(x)l(x)|x|^{2\lambda} dx, \text{  } \forall f \in H^{p^*, \kappa}_\lambda(\mathbb{R}).
$$

The last equality can be comprehended as

$$
Lf = \sum_k \int_{\mathbb{R}} \lambda_k a_k(x)l(x)|x|^{2\lambda} dx, \text{  } \text{if } f = \sum_k \lambda_k a_k(x) \in H^{p^*, \kappa}_\lambda(\mathbb{R}).
$$

Proof. Let us first show $\mathcal{L}_{\lambda, \kappa}^{p^{-1} - 1}(\mathbb{R}) \subseteq \left( H^{p^*, \kappa}_\lambda(\mathbb{R}) \right)^\prime$. Let $g \in \mathcal{L}_{\lambda, \kappa}^{p^{-1} - 1}(\mathbb{R})$. Let $a(x)$ be a $p_\lambda$-atom, and $\text{supp } a(x) \subseteq I$. Then

$$
\left| \int_{\mathbb{R}} a(x)g(x)|x|^{2\lambda} dx \right| = \int_{\mathbb{R}} a(x) \left| g(x) - P_I^\lambda g(x) \right| |x|^{2\lambda} dx \\
\leq \|a(x)\|_{p_\lambda} \left( \int_{I} |g(x) - P_I^\lambda g(x)||x|^{2\lambda} dx \right) \\
\leq \|g\|_{\mathcal{L}_{\lambda, \kappa}^{p^{-1} - 1}}.
$$
Let $g \in L_{\lambda,K}^{p-1,-1}$, and $f = \sum \lambda_k a_k(x) \in H^{p,*}_{K,\alpha}(\mathbb{R})$, where each $a_k$ is a $p_\lambda$-atom. We have
\[
\left| \int_{\mathbb{R}} f(x)g(x) |x|^{2\lambda} dx \right| \leq \sum |\lambda_k| \|g\|_{L^{p-1,-1}_{\lambda,K}} \leq \left( \sum |\lambda_k|^p \right)^{1/p} \|g\|_{L^{p-1,-1}_{\lambda,K}}.
\]

Thus, the former conclusion of the theorem holds. Let us now prove that $\left( H^{p,*}_{K,\alpha}(\mathbb{R}) \right)' \subseteq L^{p-1,-1}_{\lambda,K}(\mathbb{R})$. Let $I = (x_0, r_0)$ be a fixed Euclidean interval in $\mathbb{R}$. Let $L^2_{\lambda,I}$ denote the space of all square integrable functions supported on $I$. This space has the norm $\| \cdot \|_{L^2_{\lambda,I}}$, given by
\[
\|g\|_{L^2_{\lambda,I}} = \left( \int_I |g(x)|^2 |x|^{2\lambda} dx \right)^{1/2}.
\]

Let $L^2_{\lambda,0}$ to be:
\[
L^2_{\lambda,0} = \{ g \in L^2_{\lambda,I} : P_{x}^* g(x) = 0 \}.
\]

Therefore $L^2_{\lambda,0}$ is a closed subspace of $L^2_{\lambda,I}$. By proposition(3.54) Every element $g \in L^2_{\lambda,0}$ is in $H^p_{\lambda}(\mathbb{R})$ and that $\|g\|_{H^p_{\lambda}(\mathbb{R})} \lesssim \left( \|g\|_{L^2_{\lambda,I}(\mathbb{R})} \right) (|I|)^{1/p-1/2}$. Thus if $l(x)$ is a given linear functional on $H^p_{\lambda}(\mathbb{R})$, then $l(x)$ extends to a linear functional on $L^2_{\lambda,0}$ with norm at most $c|I|^{1/p-1/2}$. By Riesz representation theorem for the Hilbert space $L^2_{\lambda,0}$, there exists an element $F^I \in L^2_{\lambda,0}$, so that
\[
l(g) = \int_I F^I(x)g(x) |x|^{2\lambda} dx, \quad \text{if} \quad g \in L^2_{\lambda,0},
\]
with
\[
\left( \int_I |F^I(x)|^2 dx \right)^{1/2} \leq c|I|^{1/p-1/2}.
\]

Hence for each interval $I$, we obtain a function $F^I$. Observe that if $I_1 \subset I_2$ are intervals, then $F^{I_1} - F^{I_2}$ is a polynomial in $P^\nu$ on $I_1$. Thus for $I_1$ and $I_2$ we could have:
\[
(|I_1|)^{1-(1/p)} \left( \int_{I_1} |F^{I_1} - P_{I_2}^* F^{I_1}| |x|^{2\lambda} dx \right) \lesssim (|I_1|)^{1-(1/p)} \left( \int_{I_1} |F^{I_1} - P_{I_2}^* F^{I_1}|^2 |x|^{2\lambda} dx \right)^{1/2} \approx (|I_1|)^{1-(1/p)} \left( \int_{I_1} |F^{I_1}|^2 |x|^{2\lambda} dx \right)^{1/2} \lesssim (|I_1|)^{1-(1/p)} (|I_1|)^{-1/2} (|I_1|)^{1/p-1/2} \lesssim C.
\]

For all $I_1$ satisfying $I_1 \subset I_2$, $F^{I_1} - F^{I_2}$ is polynomials in $P^\nu$ on $I_1$. Take $I_2 = \mathbb{R}$, so $F^\mathbb{R} = F^{I_1}$ in $L^\alpha_{\lambda,K}(\mathbb{R})$ norm. It is easily to see that $F^\mathbb{R} \in L^{p-1,-1}_{\lambda,K}(\mathbb{R})$. This proves the theorem.

**Theorem 3.59.** For $\frac{2\lambda}{2\lambda+1} < p \leq 1$, $F(x,y) \in H^{p}_{\lambda}(\mathbb{R}_+^2)$ if and only if there exists $f \in H^{p}_{\lambda}(\mathbb{R})$ satisfying
\[
\|f\|_{H^{p}_{\lambda}(\mathbb{R}_+^2)} \approx \|F\|_{H^{p}_{\lambda}(\mathbb{R}_+^2)}.
\]

Further more, for $\frac{2\lambda+1}{2\lambda+2} < p \leq 1$, $F(x,y) \in H^{p}_{\lambda}(\mathbb{R}_+^2)$ if and only if there exists $f \in H^{p}_{\lambda}(\mathbb{R})$ satisfying
\[
F(x,y) = Pf(x,y) + iQf(x,y)
\]
in $H^{p}_{\lambda}(\mathbb{R}_+^2)$ space.
Proof. First we will prove that $F(x, y) \in H^p_{\lambda}(\mathbb{R}^2_+) \Rightarrow \exists f \in H^p_{\lambda}(\mathbb{R})$, such that $\|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)} \approx \|f\|_{H^p_{\lambda}(\mathbb{R})}$. For $F(x, y) \in H^p_{\lambda}(\mathbb{R}^2_+)$ (when $\frac{2\lambda}{2\lambda+1} < p \leq 1$), by Proposition(3.16), $H^p_{\lambda}(\mathbb{R}^2_+) \cap H^1_{\lambda}(\mathbb{R}^1_+)$ is dense in $H^p_{\lambda}(\mathbb{R}^2_+)$ when $\frac{2\lambda}{2\lambda+1} < p \leq 1$. Thus there exists a sequence $\{F_n(x, y)\}_n \subseteq H^p_{\lambda}(\mathbb{R}^2_+) \cap H^1_{\lambda}(\mathbb{R}^1_+)$ satisfying

$$\lim_{n \to \infty} \|F - F_n\|_{H^p_{\lambda}(\mathbb{R}^2_+)} = 0.$$ 

Denote the boundary value of $F_n(x, y)$ as $f_n$. By Proposition(3.10)(iv) we have

$$F_n(x, y) = Pf_n(x, y) + iQf_n(x, y).$$

Then by Theorem(3.12)

$$\|F_n\|_{H^p_{\lambda}(\mathbb{R}^2_+)} \approx \|Pf_n\|_{L^p_{\lambda}(\mathbb{R})} = \|f_n\|_{H^p_{\lambda}(\mathbb{R})}.$$ 

Thus $\{f_n\}_n$ is a Cauchy sequence in $H^p_{\lambda}(\mathbb{R})$. Thus there exists a $f \in H^p_{\lambda}(\mathbb{R})$, such that

$$\lim_{n \to \infty} \|f - f_n\|_{H^p_{\lambda}(\mathbb{R})} = 0.$$ 

Thus we obtain

$$\|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)} \approx \|f\|_{H^p_{\lambda}(\mathbb{R})}.$$ 

Next we will prove $f \in H^p_{\lambda}(\mathbb{R}) \Rightarrow \exists \bar{F}(x, y) \in H^p_{\lambda}(\mathbb{R}^2_+)$, such that $\|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)} \approx \|f\|_{H^p_{\lambda}(\mathbb{R})}$. For $f \in H^p_{\lambda}(\mathbb{R})$ (when $\frac{2\lambda}{2\lambda+1} < p \leq 1$), by Proposition(3.15), $H^p_{\lambda}(\mathbb{R}) \cap H^1_{\lambda}(\mathbb{R})$ is dense in $H^p_{\lambda}(\mathbb{R})$ when $\frac{2\lambda}{2\lambda+1} < p \leq 1$. Thus there exists a sequence $\{f_n(x)\}_n \subseteq H^p_{\lambda}(\mathbb{R}) \cap H^1_{\lambda}(\mathbb{R})$ satisfying

$$\lim_{n \to \infty} \|f - f_n\|_{H^p_{\lambda}(\mathbb{R})} = 0.$$ 

Denote $F_n(x, y)$ as

$$F_n(x, y) = Pf_n(x, y) + iQf_n(x, y).$$

Then by Proposition(3.10)(v), we could deduce that $F_n(x, y)$ is a $\lambda$-analytic function, and $F_n(x, y) \in H^p_{\lambda}(\mathbb{R}^2_+)$. Then by Theorem(3.12)

$$\|F_n\|_{H^p_{\lambda}(\mathbb{R}^2_+)} \approx \|f_n\|_{H^p_{\lambda}(\mathbb{R})}.$$ 

Thus $\{F_n\}_n$ is a Cauchy sequence in $H^p_{\lambda}(\mathbb{R}^2_+)$. Thus there exists a $F \in H^p_{\lambda}(\mathbb{R}^2_+)$, such that

$$\lim_{n \to \infty} \|F - F_n\|_{H^p_{\lambda}(\mathbb{R}^2_+)} = 0.$$ 

Thus we obtain

$$\|F\|_{H^p_{\lambda}(\mathbb{R}^2_+)} \approx \|f\|_{H^p_{\lambda}(\mathbb{R})}.$$ 

Further more, when $\frac{2\lambda}{2\lambda+1} < p \leq 1$, by Theorem(3.34), $f \in H^p_{\lambda}(\mathbb{R})$ can be write as a sum of $p_{\lambda}$-atoms: $f(x) = \sum_k \lambda_k a_k(x)$. The $\lambda$-Poisson kernel $(\tau_x P_y)(-t) \in L^{p-1}_{\lambda,n}(\mathbb{R})$ by Proposition(3.17). Thus

$$P\left(\sum_k \lambda_k a_k(x, y)\right) = \sum_k \lambda_k \left(P(a_k)(x, y)\right).$$ 

By Theorem(3.44), $P a_k(x, y) + iQ a_k(x, y) \in H^p_{\lambda}(\mathbb{R}^2_+)$. From Proposition(3.3),

$$\sup_{x \in \mathbb{R}} \left|P(a_k)(x, y) + iQ(a_k)(x, y)\right| \leq cy^{-(1/p)(1+2\lambda)}.$$ 

Then together with Lemma(3.76), we could deduce that $\sum_k \lambda_k \left(P(a_k)(x, y) + iQ(a_k)(x, y)\right)$ is a bounded function in the upper half plane:

$$\sup_{x \in \mathbb{R}} \left|\sum_k \lambda_k \left(P(a_k)(x, y) + iQ(a_k)(x, y)\right)\right| \leq cy^{-(1/p)(1+2\lambda)} \left(\sum_k |\lambda_k|^p\right)^{1/p} \leq cy^{-(1/p)(1+2\lambda)} \|f\|_{H^p_{\lambda}(\mathbb{R})}.$$ 

Notice that
\[
\left( \int_{\mathbb{R}} |\partial_y (P(a_k)(x,y))|^p (x)|x|^{2\lambda} dx \right)^{1/p} \lesssim \frac{1}{y}, \quad \left( \int_{\mathbb{R}} |\partial_y (Q(a_k)(x,y))|^p (x)|x|^{2\lambda} dx \right)^{1/p} \lesssim \frac{1}{y}
\]

and
\[
|\partial_y (P(a_k)(x,y))| = |D_x (Q(a_k)(x,y))|, \quad |\partial_y (Q(a_k)(x,y))| = |D_x (P(a_k)(x,y))|.
\]

Then by Theorem(3.44) and Proposition(3.1), we could deduce that \(\sum_k P(\lambda_k a_k)(x,y) + i \sum_k Q(\lambda_k a_k)(x,y)\) is a \(\lambda\)-analytic function in \(\mathbb{R}^2\), and \(\sum_k P(\lambda_k a_k)(x,y) + i \sum_k Q(\lambda_k a_k)(x,y) \in H^{p}(\mathbb{R}^2)\). Denote \(\sum_k P(\lambda_k a_k)(x,y) + i \sum_k Q(\lambda_k a_k)(x,y) = C(x,y)\). Thus by Theorem(3.12) we have
\[
\lim_{n \to \infty} \|F_n - C\|_{H^p(\mathbb{R}^2)} = \lim_{n \to \infty} \|f_n - f\|_{H^p(\mathbb{R})} = 0.
\]
Thus
\[
F(x,y) = C(x,y)
\]
in \(H^p(\mathbb{R}^2)\) space. Also for \(f \in H^p(\mathbb{R})\), by Theorem(3.34), \(f = \sum_k a_k \in H^p(\mathbb{R})\) spaces. Thus we could define \(Qf(x,y)\) as following:
\[
Qf(x,y) = \sum_k Q(\lambda_k a_k)(x,y).
\]

This prove the theorem. \(\square\)

### 3.7 Interpolation of operators and interpolation of spaces of Hardy spaces for the Dunkl setting

**Definition 3.60 (Weak \((H^p_\lambda, L^p_\lambda)\)).** An operator \(T\) is of weak type if for any \(\alpha \geq 0\), \(f \in H^p_\lambda(\mathbb{R})\) the following holds:
\[
|[x \in \mathbb{R} : |Tf(x)| > \alpha]|_\lambda \leq \left( \frac{c}{\alpha} \|f\|_{H^p_\lambda} \right)^p
\]

\(C\) is a constant independent on \(\alpha\) and \(f\).

**Proposition 3.61.** If \(f \in L^p_\lambda(\mathbb{R})\), \(p > 1\), then for \(\frac{p_2}{p+1} < p_1 \leq p < p_2 \leq \infty\) there exists a decomposition of \(f(x) = g(x) + b(x)\) so that \(g(x) \in L^{p_2}_\lambda(\mathbb{R})\), \(b(x) \in H^{p_1}_\lambda(\mathbb{R})\) and
\[
\|g\|_{L^{p_2}_\lambda} \leq C\alpha^{p_2-p}\|f\|_{L^p_\lambda}^{p_2-p}\]
\[
\||b||_{p_1} \leq C\alpha^{p_1-p}\|f\|_{L^p_\lambda}^{p_1-p}\]
\(C\) is independent on \(f\) and \(\alpha\).

**Proof.** Let us first decompose \(\mathbb{R}\) into a mesh of balls \(B\) in the Homogeneous Dunkl space, whose interiors are disjoint. Clearly if the length of the balls are large enough, we could have
\[
\left( \frac{1}{|B|_\lambda} \int_{B} |f(x)|^p |x|^{2\lambda} dx \right)^{1/p} \leq \alpha.
\]
Further, for each interval \(B\) in the above mesh, we divide it into 2 congruent balls \(\{B'\}\): \(2|B'|_\lambda = |B|_\lambda\). For any \(B' \in \{B'\}\), it must fit either of the following cases:
\[
\left( \frac{1}{|B'|_\lambda} \int_{B'} |f(x)|^p |x|^{2\lambda} dx \right)^{1/p} \leq \alpha, \quad (193)
\]
and
\[
\left( \frac{1}{|B'|_\lambda} \int_{B'} |f(x)|^p |x|^{2\lambda} dx \right)^{1/p} > \alpha. \quad (194)
\]
If \(B'\) satisfies Formula(194), we will not divide the ball \(B'\). Thus
\[
\alpha < \left( \frac{1}{|B'|_\lambda} \int_{B'} |f(x)|^p |x|^{2\lambda} dx \right)^{1/p} \leq \left( \frac{|B|_\lambda}{|B'|_\lambda} \frac{1}{|B|_\lambda} \int_{B} |f(x)|^p |x|^{2\lambda} dx \right)^{1/p} \leq 2^{p-1} \alpha.
\]
If $B'$ satisfies Formula (193), we can incorporate $B'$ by the above method, and the procedure can be carried on continuously. Thus we have

(i) $R = \Omega \cup F$, $\emptyset = \Omega \cap F$.

(ii) $\Omega = \bigcup B_k$, the interior of $\{B_k\}$ are disjoint, satisfying:

$$|\Omega|_\lambda \leq \sum_k |B_k|_\lambda \leq \alpha^{-p} \left( \sum_k \int_{B_k} |f(x)|^p |x|^{2\lambda} dx \right)^{1/p} \leq \alpha^{-p} \|f\|_{L^p_\lambda(R)}^p.$$

(iii) For each $B_k$, the following holds:

$$\alpha < \left( \frac{1}{|B_k|_\lambda} \int_{B_k} |f(x)|^p |x|^{2\lambda} dx \right)^{1/p} \leq 2^{p-1} \alpha.$$

(iv) $|f(x)| \leq \alpha$ a.e. $x \in F$. This is because for $x \in F$

$$|f(x)|^p = \lim_{\text{diam}(B) \to 0, x \in B} \frac{1}{|B|_\lambda} \int_B |f(y)|^p |y|^{2\lambda} dy.$$

For the nonnegative integer $\kappa \geq 0$, let $P^\kappa_{B_k} f(x)$ be a polynomial as Formula (186). Thus by Proposition (3.50)

$$|P^\kappa_{B_k} f(x)| \leq \chi_{B_k}(x) \frac{C}{|B_k|_\lambda} \int_{B_k} |f(y)||y|^{2\lambda} dy$$

C is a constant independent on $f$ and $B_k$. Let

$$g(x) = \begin{cases} f(x), & x \in F, \\ \sum_k P^\kappa_{B_k} f(x), & x \in B_k \subseteq \Omega. \end{cases} \quad (195)$$

Let

$$b(x) = f(x) - g(x) = \sum_k [f(x) - P^\kappa_{B_k} f(x)] \chi_{B_k}(x).$$

Denote

$$b_k(x) = [f(x) - P^\kappa_{B_k} f(x)] \chi_{B_k}(x).$$

We could easily obtain from Proposition (3.54)

$$\|b_k\|_{H^{p_1}_\lambda(R)} \lesssim \left( \|b_k\|_{L^p_\lambda(R)} \right) (|B_k|_\lambda)^{p_1 - p^{-1}}$$

and

$$\|b_k\|_{L^p_\lambda(R)} \lesssim \int_{B_k} |f(t)|^p |t|^{2\lambda} dt.$$ 

Thus

$$\|b\|_{H^{p_1}_\lambda(R)} \lesssim \sum_k \left( \frac{1}{|B_k|_\lambda} \int_{B_k} |f(x)|^p |x|^{2\lambda} dx \right)^{p_1/p} (|B_k|_\lambda) \lesssim \alpha^{p_1} \sum_k (|B_k|_\lambda) \lesssim \alpha^{p_1 - p} \|f\|_{L^p_\lambda(R)}^p.$$
We could then obtain:

\[ \|g\|_{L_p^\alpha(\mathbb{R})} = \int_F |g(t)|^p |t|^{2\lambda} dt + \int_\Omega |g(t)|^p |t|^{2\lambda} dt \]
\[ = \int_F |g(t)|^p |t|^{2\lambda} dt + \sum_k \int_{B_k} |P_{B_k} f(t)|^p |t|^{2\lambda} dt \]
\[ \leq C \alpha^{p-\beta} \int_F |f(t)|^p |t|^{2\lambda} dt + C \sum_k \|P_{B_k} f\|_{L_p^\alpha(\mathbb{R})} |B_k| \lambda \]
\[ \leq C \alpha^{p-\beta} \int_F |f(t)|^p |t|^{2\lambda} dt + C \sum_k \left( \frac{1}{|B_k| \lambda} \int_{B_k} |f(x)| x^{2\lambda} dx \right)^{p_2} |B_k| \lambda \]
\[ \leq C \alpha^{p-\beta} \int_F |f(t)|^p |t|^{2\lambda} dt + C \sum_k \alpha^{p_2} |B_k| \lambda \]
\[ \leq C \alpha^{p-\beta} \int_{\mathbb{R}} |f(t)|^p |t|^{2\lambda} dt. \]

This proves the Proposition.

**Theorem 3.62.** Assume that \( \frac{2\lambda}{2s+2} < p_1 \leq 1 < p_2 < \infty \), and a sublinear operator \( T \) is of weak \( (H_p^\alpha, L_p^\alpha) \) and weak \( (L_p^\alpha, L_p^\alpha) \).

(i) If \( 1 < p \leq p_2 \), then \( T \) is of type \( (L_p^\alpha, L_p^\alpha) \).

(ii) If \( p_1 < p \leq 1 \), \( T \) is of type \( (H_p^\alpha, L_p^\alpha) \).

**Proof.** We will show case (ii) first. When \( \frac{2\lambda}{2s+2} < p \leq 1 \), by Proposition(3.46) and Theorem(3.52) \( \forall f \in H_p^\alpha \), \( f \) can be expressed as a sum of \( p_1^\alpha \)-atoms in \( H_p^\alpha \) spaces:

\[ f(x) = \sum_k \lambda_k a_k(x) \]

Where \( a_k(x) \) is a \( p_1^\alpha \)-atom satisfying the following:

\[ i \quad \|a_k(x)\|_{L_p^\alpha} \lesssim \frac{1}{|B(x_k, r_k)|^{1/p}} \lesssim r_k^{-1/p} \]
\[ ii \quad \text{supp } a_k(x) \subseteq B(x_k, r_k) \]
\[ iii \quad \int a_k(x)|x|^{2\lambda} dx = 0. \]

And the norm of \( f \) in \( H_p^\alpha(\mathbb{R}) \) spaces is equivalent to \( \inf (\sum_k |\lambda_k|^p)^{1/p} \):

\[ \|f\|_{H_p^\alpha(\mathbb{R})} \asymp \left( \sum_k |\lambda_k|^p \right)^{1/p}. \]

When \( a_k(x) \) is a \( p_1^\alpha \)-atom, we will prove the following:

\[ \|T(a_k)\|_{L_p^\alpha(\mathbb{R})} \leq C, \]

where \( C \) is a constant independent on \( a_k(x) \). It is easy to obtain

\[ \|a_k(x)\|_{L_p^\alpha} \lesssim (r_k)^{1/p_2-1/p}. \]

By Proposition(3.54), \( a_k(x) \in H_p^\alpha \). We could obtain:

\[ \|a_k(x)\|_{H_p^\alpha} \lesssim (r_k)^{1/p_1-1/p}. \]
Thus
\[
\|T(a_k)\|_{L_p^p(\mathbb{R})} = p \left( \int_0^M \alpha^{p-1} |\{x \in \mathbb{R} : |T(a)| > \alpha\}|_\lambda d\alpha + \int_M^\infty \alpha^{p-1} |\{x \in \mathbb{R} : |T(a)| > \alpha\}|_\lambda d\alpha \right) \\
= I + II.
\]

Notice that $T$ is of weak $(H_p^p, L_p^0)$, thus
\[
I = p \int_0^M \alpha^{p-1} |\{x \in \mathbb{R} : |T(a)| > \alpha\}|_\lambda d\alpha \\
\leq Cp \int_0^M \alpha^{p-1} \|a_k(x)\|_{H_p^p}^p d\alpha \\
\leq CM^{p-1}\|a_k(x)\|_{L_p^\lambda}^{1-(p_1/p)}.
\]

$T$ is of weak $(L_p^{p_2}, L_p^{p_2})$, thus
\[
II = p \int_M^\infty \alpha^{p-1} |\{x \in \mathbb{R} : |T(a)| > \alpha\}|_\lambda d\alpha \\
\leq Cp \int_M^\infty \alpha^{p-1} \|a_k(x)\|_{L_p^{p_2}}^p d\alpha \\
\leq CM^{p-1}\|a_k(x)\|_{L_p^\lambda}^{1-(p_2/p)}.
\]

Let $M = r_k^{-1/p}$, then
\[
\|T(a_k)\|_{L_p^p(\mathbb{R})} \leq C.
\]

Thus $T$ is of type $(H_p^p, L_p^\lambda)$ for $\frac{2\lambda+1}{\lambda+2} < p_1 < p_1 \leq 1$. Next we will prove case(i) when $1 < p < \infty$. By Proposition(3.61) we could write $f \in L_p^\lambda(\mathbb{R})$ as $f(x) = g(x) + b(x)$ so that $g(x) \in H_p^p(\mathbb{R})$, $b(x) \in H_p^0(\mathbb{R})$ and
\[
\|g\|_{L_p^p} \leq C\alpha^{-p_2}\|f\|_{L_p^p}^p \\
\|b\|_{H_p^0} \leq C\alpha^{-p_1}\|f\|_{L_p^p}^p.
\]

Then
\[
|\{x \in \mathbb{R} : |f(x)| > \alpha\}|_\lambda \leq |\{x \in \mathbb{R} : |g(x)| > \alpha/2\}|_\lambda + |\{x \in \mathbb{R} : |b(x)| > \alpha/2\}|_\lambda \\
\leq C\alpha^{-p_2}\|g\|_{L_p^p}^p + C\alpha^{-p_1}\|b\|_{H_p^0}^p \\
\leq C\alpha^{-p}\|f\|_{L_p^p}^p.
\]

Thus $T$ is weak $(L_p^\lambda, L_p^\lambda)$ for any $p > 1$. By Marcinkiewicz interpolation theorem, $T$ is $(L_p^\lambda, L_p^\lambda)$ for any $p > 1$. This proves the theorem.

**Theorem 3.63.** Assume that $\frac{2\lambda+1}{\lambda+2} < p_1 < p_2 \leq 1$, $\frac{2\lambda+1}{\lambda+2} < \theta_1 < \theta_2 \leq 1$, and a sublinear operator $T$ is $(H_p^p, H_p^{\theta_1})$ and $(H_p^{p_2}, H_p^{\theta_2})$ bounded.

\[
\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2}, \quad \frac{1}{\theta} = \frac{1-t}{\theta_1} + \frac{t}{\theta_2}
\]

for $0 \leq t \leq 1$. Then $T$ is $(H_p^p, H_p^\lambda)$ bounded. In fact, if $p_1 = \theta_1, p_2 = \theta_2, p = \theta$ we could conclude:

\[
\|T\|_{(H_p^p, H_p^\lambda)} = \|T\|_{(H_p^{p_1}, H_p^{\theta_1})_{\theta_1, \theta}} \leq \|T\|_{(H_p^{p_1}, H_p^{\theta_1})_{\theta_1, \theta}} \|T\|_{(H_p^{p_2}, H_p^{\theta_2})_{\theta_1, \theta}}.
\]

**Proof.** When $\frac{2\lambda+1}{\lambda+2} < p_1 \leq p \leq p_2 \leq 1$, by Proposition(3.46) and Theorem(3.52) $\forall f \in H_p^\lambda$, $f$ can be expressed as a sum of $p_\lambda^i$-atoms:

\[
f(x) = \sum_k \lambda_k a_k(x)
\]
holds in $H^p_\lambda$ spaces.

\[ i \quad \|a_k(x)\|_{L^\infty} \lesssim \frac{1}{|B(x_k, r_k)|_{\lambda}^{1/p}} \lesssim r_k^{-1/p} \]

\[ ii \quad \text{supp } a_k(x) \subseteq B(x_k, r_k) \]

\[ iii \quad \int a_k(x)|x|^{2\lambda} \, dx = 0. \]

\[ \|f\|_{H^p_\lambda(\mathbb{R})}^p = \inf \left( \sum_k |\lambda_k|^p \right). \]

By Proposition 3.54, $a_k(x) \in H^p_\lambda$. We could obtain:

\[ \|a_k(x)\|_{H^p_\lambda} \lesssim (r_k)^{1/p_1-1/p} \]

for $i = 1, 2$.

\[ \|T(a_k)\|^\theta_{H^p_\lambda(\mathbb{R})} = \| (T(a_k))^* \|^\theta_{L^\infty(\mathbb{R})} \]

\[ = \theta \left( \int_0^M \alpha^{\theta-1} \left\{ \{ x \in \mathbb{R} : |(T(a))(x)^*| > \alpha \} \right\} \, d\alpha + \int_M^\infty \alpha^{\theta-1} \left\{ \{ x \in \mathbb{R} : |(T(a))(x)^*| > \alpha \} \right\} \, d\alpha \right) \]

\[ = I + II. \]

Notice $T$ is $(H^p_{\lambda_1}, H^\theta_{\lambda_1})$ and $(H^p_{\lambda_2}, H^\theta_{\lambda_2})$ bounded. Thus

\[ I = \theta \int_0^M \alpha^{\theta-1} \left\{ \{ x \in \mathbb{R} : |(T(a))(x)^*| > \alpha \} \right\} \, d\alpha \]

\[ \leq \theta \int_0^M \alpha^{\theta-\theta_1-1} \|T(a_k)\|^\theta_{H^p_{\lambda_1}(\mathbb{R})} \, d\alpha \]

\[ \leq C \int_0^M \alpha^{\theta-\theta_1} \|a_k(x)\|^\theta_{H^p_{\lambda_1}} \, d\alpha \]

\[ \leq CM^{\theta-\theta_1} \alpha^{(1/p_1-1/p)\theta_1}. \]

Similar to $I$, we could obtain:

\[ II \leq CM^{\theta-\theta_2} \alpha^{(1/p_2-1/p)\theta_2}. \]

Let

\[ M = \exp \left\{ \left[ \frac{1}{\theta_2 - \theta_1} \left( \frac{\theta_2}{p_2} - \frac{\theta_1}{p_1} \right) - \frac{1}{p} \right] \ln r_k \right\}, \]

then we could have:

\[ \|T(a_k)\|^\theta_{H^p_\lambda(\mathbb{R})} \leq C \exp \left\{ \left[ \frac{\theta}{p_1} \frac{1}{p_2} - \frac{1}{p_1} + \frac{\theta}{p_2} - \frac{1}{p_2} - \frac{1}{p} \right] \ln r_k \right\}. \]

By

\[ \frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2}, \quad \frac{1}{\theta} = \frac{1-t}{\theta_1} + \frac{t}{\theta_2}, \]

we could obtain:

\[ \frac{1}{p_1} - \frac{1}{p_2} = t, \quad \frac{1}{\theta_1} - \frac{1}{\theta_2} = 1-t. \]

Thus

\[ \frac{\theta}{p_1} \frac{1}{p_2} - \frac{1}{p_1} + \frac{\theta}{p_2} - \frac{1}{p_2} - \frac{1}{p} = 0. \]
Then

\[ \|T(a_k)^\theta\|_{H_\theta^q(\mathbb{R})} \leq C. \]

By Proposition(3.65) and Theorem(3.71) we could conclude:

\[ \|T\|_{(H_\theta^q, H_\theta^r)} = \|T\|_{((H_\theta^q, H_\theta^r), (H_\theta^q, H_\theta^r))} \leq \|T\|_{1-\theta} \|T\|_{\theta} \|H_\theta^q, H_\theta^r\| \]

This proves the Theorem. \(\square\)

Let \(X\) denote a linear topological Hausdorff space. Let \(X_1\) and \(X_2\) to be two quasi-norm linear spaces embedded in \(X\). The direct sum space \(X_1 + X_2 \subset X\) of \(X_1\) and \(X_2\) is defined by

\[ X_1 + X_2 = \{ f : f = f_1 + f_2, f_1 \in X_1, f_2 \in X_2 \}. \]

The \(K\) functional \(K(t, f)\) on \(X_1 + X_2\) is defined by: for \(t > 0\)

\[ K(t, f) = \inf_{f=f_1+f_2} \{ \|f_1\|_1 + t\|f_2\|_2 \}. \]

Then we can proceed to define the interpolation between \(X_1\) and \(X_2\).

**Definition 3.64** (Interpolation spaces of \(X_1\) and \(X_2\)). Suppose \(0 < \theta < 1\) and \(0 < q \leq \infty\). \((X_1, X_2)_{\theta, q}\) is called an interpolation spaces of \(X_1\) and \(X_2\):

\[ (X_1, X_2)_{\theta, q} = \left\{ f \in X_1 + X_2 : \|f\|_{(X_1, X_2)_{\theta, q}} ^{\frac{1}{q}} = \left( \int_0^\infty |t^{-\theta}K(t, f)|^q \frac{dt}{t} \right)^{1/q} \right\}. \]

When \(q = \infty\)

\[ \|f\|_{(X_1, X_2)_{\theta, \infty}} ^{\frac{1}{q}} = \sup_{t>0} t^{-\theta}K(t, f). \]

**Proposition 3.65.** [10] Let \(X, X_1\) and \(X_2\) to be spaces as above. \(T\) is a linear operator on \(X\). \(T\) is \((X_1, X_1)\) and \((X_2, X_2)\) bounded. Let \(0 < \theta < 1\) and \(0 < q \leq \infty\). Then \(T\) is bounded on \((X_1, X_2)_{\theta, q}\).

**Proof.** Let \(f \in (X_1, X_2)_{\theta, q}\). Then \(f = f_1 + f_2\) satisfying \(f_1 \in X_1\) and \(f_2 \in X_2\). Notice that \(T\) is \((X_1, X_1)\) and \((X_2, X_2)\) bounded. Thus we could conclude:

\[ K(t, Tf) \leq \inf_{f=f_1+f_2} \{ \|Tf_1\|_1 + t\|Tf_2\|_2 \} \]

\[ \leq \|T\|_{(X_1, X_1)} \inf_{f=f_1+f_2} \{ \|f_1\|_1 + t\|T\|_{(X_1, X_1)}^{-1} \|T\|_{(X_1, X_2)} \|f_2\|_2 \} \]

\[ \leq \|T\|_{(X_1, X_1)} K(t) \|T\|_{(X_1, X_2)}^{-1} \|T\|_{(X_2, X_2), f}. \]

Then the following holds:

\[ \|Tf\|_{(X_1, X_2)_{\theta, q}} ^{\frac{1}{q}} = \left( \int_0^\infty |t^{-\theta}K(t, Tf)|^q \frac{dt}{t} \right)^{1/q} \]

\[ \leq \|T\|_{(X_1, X_1)} \left( \int_0^\infty |t^{-\theta}K(t)\|T\|_{(X_1, X_2)}^{-1} \|T\|_{(X_2, X_2), f} \|f\|_{(X_1, X_2)_{\theta, q}} ^{\frac{1}{q}} \right) \]

\[ \leq \|T\|_{(X_1, X_1)} \|T\|_{(X_2, X_2)} \|f\|_{(X_1, X_2)_{\theta, q}} ^{\frac{1}{q}}. \]

This proves the Proposition. \(\square\)

Let \(I_0 = [-1, 1]\) be an interval in the Euclid space, \(d\mu_\lambda(x) = (2\lambda + 1)|x|^{2\lambda}dx\) and \(\mathcal{P}^\kappa\) to be the \(\kappa\)-order polynomials as above. We define \(P_{\kappa B(x_0, r_0)}\) as following:

\[ P_{\kappa B(x_0, r_0)} = \{ g(\mu_\lambda(x_0, t)) : g(x) \in P^\kappa \}. \]
with its Hilbert norm: \( f \in P^c_{\lambda B(x_0,r_0)} \)

\[
\|f\|_{P^c_{\lambda B(x_0,r_0)}} = \left( \frac{\int_{B(x_0,r_0)} |f(x)|^2 |x|^{2\lambda} dx}{\int_{B(x_0,r_0)} |x|^{2\lambda} dx} \right)^{\frac{1}{2}}.
\]

In addition, let \( \{\pi^i\}_{i=0}^\infty \) to be the orthonormal basis satisfying the following conditions:

\[
\int_{[-1,1]} \pi^i(t)\pi^j(t) dt = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}
\]

Thus we could obtain:

\[
\int_{B(x_0,r_0)} \frac{2\lambda + 1}{\kappa_0} \pi^i \left( \frac{\mu_\lambda(x_0,t)}{\kappa_0} \right) \pi^j \left( \frac{\mu_\lambda(x_0,t)}{\kappa_0} \right) |t|^{2\lambda} dt = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}
\]

Thus \( \{\pi^i \left( \frac{\mu_\lambda(x_0,t)}{\kappa_0} \right)\}_{i=0}^\infty \) form a orthonormal basis on the space \( P^c_{\lambda B(x_0,r_0)} \). We could obtain the following inequality:

\[
\left( \int_{[-1,1]} |\pi^i(t)|^2 dt \right)^{1/2} \geq C \sup_{t \in [-1,1]} |\pi^i(t)| \geq C \sup_{t \in B(x_0,r_0)} \pi^i \left( \frac{\mu_\lambda(x_0,t)}{\kappa_0} \right).
\]

Thus

\[
\sup_{t \in B(x_0,r_0)} \pi^i \left( \frac{\mu_\lambda(x_0,t)}{\kappa_0} \right) \leq C, \quad (196)
\]

C is independent on \( \pi^i \) and \( B(x_0,r_0) \). And let \( g \in L_{\lambda, \text{loc}}(\mathbb{R}) \). Then there exists a unique \( P^c_{\lambda B(x_0,r_0)} g(x) \in P^c_{\lambda B(x_0,r_0)} \) such that:

\[
\int_{B(x_0,r_0)} \left\{ g(x) - P^c_{\lambda B(x_0,r_0)} g(x) \right\} Q(x)|x|^{2\lambda} dx = 0, \quad \forall Q(x) \in P^c_{\lambda B(x_0,r_0)}.
\]

And

\[
P^c_{\lambda B(x_0,r_0)} g(x) = \chi_{B(x_0,r_0)}(x) \sum_{\alpha \leq \kappa} \pi^\alpha \left( \frac{\mu_\lambda(x_0,t)}{\kappa_0} \right) \int_{B(x_0,r_0)} \frac{2\lambda + 1}{\kappa_0} g(t) \pi^\alpha \left( \frac{\mu_\lambda(x_0,t)}{\kappa_0} \right) |t|^{2\lambda} dt. \quad (197)
\]

Thus Formula (196) and (198) lead to:

\[
|P^c_{\lambda B(x_0,r_0)} g(x)| \leq C \frac{\chi_{B(x_0,r_0)}(x)}{r_0} \int_{B(x_0,r_0)} |g(t)||t|^{2\lambda} dt. \quad (199)
\]

**Proposition 3.66.** Assume \( \frac{2\alpha}{\alpha+1} < p \leq 1, \alpha > 0, \beta = \frac{2\lambda + 1}{\alpha+1} > (p^{-1} - 1) \) and \( f \in L^1_\lambda(\mathbb{R}) \), then \( f \) can be expresses as \( f = g + b \) where \( g \in L^\infty, \|g\|_{\infty} \leq C \alpha, b \in H^p_\lambda(\mathbb{R}) \) and

\[
\|b\|_{H^p_\lambda(\mathbb{R})} \leq C \int_{\{x: f_\alpha^g(x) > \alpha\}} |f_\beta^b(x)|^p |x|^{2\lambda} dx,
\]

where \( C \) is independent of \( f \).
Proof. Similar to Theorem (3.33), for some $1 < a^* < \bar{a} < 3/2$, by letting $O_\alpha = \{x : f_\beta(x) > \alpha\} = \bigcup_i B_i^* = \bigcup_i \tilde{B}_i$, $B_i^* = B(x_i, a^* r_i)$, $\tilde{B}_i = B(x_i, \bar{a} r_i)$ replaced there. $(\tilde{B}_i)$, have the bounded intersection property. We obtain a decomposition of $f = g + \sum_i b_i$ where

$$b_i(x) = \left(f(x) - P^\kappa_{B_i^*} f(x)\right) \xi_i(x),$$

and

$$g(x) = f(x)\chi_{O_\alpha} + \sum_i P^\kappa_{B_i^*} f(x)\xi_i(x), \quad b = \sum_i b_i = f - g,$$

where $P^\kappa_{B_i^*} f(x)$ is defined as Formula (197) ($\forall \kappa \geq 4$, $\text{supp} \xi_i \subseteq B(x_i, a^* r_i)$). Thus similar to Proposition (3.32) we could conclude $P^\kappa_{B_i^*} f(x)\xi_i(x) \leq C \alpha$, then the following holds:

$$g(x) \leq C \alpha.$$

Denote $SS_\beta$ as Definition (2.19). Let $d \mu_\lambda(x) = (2\lambda + 1)|x|^{2\lambda} dx$, $d \mu_\lambda(x, y) = d \lambda(x, y) = \left|\int_y^x (2\lambda + 1)|t|^{2\lambda} dt\right|$, and $\mu_\lambda(x, y) = \int_y^x (2\lambda + 1)|t|^{2\lambda} dt$. By Theorem (2.39), we will estimate

$$b_i^*(x) = \sup_{t > 0} \left|\int b_i(y) \phi \left(\frac{\mu_\lambda(x, y)}{t}\right) d \mu_\lambda(y)\right| / t$$

where $\phi \in S^{\eta}$ satisfying $\int \phi(x) dx \neq 0$. In fact we could choose $\phi(x)$ as

$$\phi(x) = \begin{cases} \exp \left\{ \frac{1}{|x|^2 - 1} \right\}, & \text{for } |x| < 1 \\
0, & \text{for } |x| \geq 1. \end{cases}$$

It is trivial to see that $\phi \in S(\mathbb{R}, dx)$, $H^\beta(\phi) \leq C$, $H^\beta(\phi^{(1)}) \leq C$. $C$ is a constant. Obviously when $x \in B(x_i, \bar{a} r_i)$

$$\sup_{t > 0} \left|\int f(y) \phi \left(\frac{\mu_\lambda(x, y)}{t}\right) \xi_i(y) d \mu_\lambda(y)\right| / t \leq C f^\beta_\beta(x),$$

and with the property $\left|P^\kappa_{B_i^*} f(x)\xi_i(x)\right| \leq C \alpha \leq C f^\beta_\beta(x)$, we could obtain:

$$\sup_{t > 0} \left|\int P^\kappa_{B_i^*} f(y) \phi \left(\frac{\mu_\lambda(x, y)}{t}\right) \xi_i(y) d \mu_\lambda(y)\right| / t \leq C f^\beta_\beta(x).$$

Thus when $x \in B(x_i, \bar{a} r_i)$

$$b_i^*(x) \leq C f^\beta_\beta(x). \quad (200)$$

Then we could conclude

$$b_i^*(x) \leq C f^\beta_\beta(x) \quad \text{for } x \in \mathbb{R}. \quad (201)$$

Next we will estimate the accurate $b_i^*(x)$ when $x \in B(x_i, \bar{a} r_i)^c$. It is easy to see that

$$C r_i \leq C d \mu_\lambda(x, y) \leq t, \quad d \mu_\lambda(x, y) \geq d \mu_\lambda(x, x_i) \quad (202)$$

hold for some appropriately small constant $C$. For any $y_1, y_2 \in B(x_i, \bar{a} r_i)$, suppose $y_1 \leq y_2$ first. There exists a point $y_0$ satisfying $y_1 \leq y_0 \leq y_2$ and $\mu_\lambda(y_1, y_0) = \mu_\lambda(x, y_0)$. Choose $\tilde{x}_i \in B(x_i, 4r_i) \bigcap O_\alpha$. Thus denote $\psi$ as:

$$\psi \left(\frac{\mu_\lambda(x, y)}{t}\right) = \phi \left(\frac{\mu_\lambda(x, y)}{t}\right) - \phi \left(\frac{\mu_\lambda(x, y_0)}{t}\right) - \frac{1}{1!} \phi^{(1)} \left(\frac{\mu_\lambda(x, y_0)}{t}\right) \left(\frac{\mu_\lambda(y_0, y)}{t}\right)$$

$$- \frac{1}{2!} \phi^{(2)} \left(\frac{\mu_\lambda(x, y_0)}{t}\right) \left(\frac{\mu_\lambda(y_0, y)}{t}\right)^2 - \frac{1}{3!} \phi^{(3)} \left(\frac{\mu_\lambda(x, y_0)}{t}\right) \left(\frac{\mu_\lambda(y_0, y)}{t}\right)^3.$$
By Taylor Expansion we could see that there exist $\xi$ between $y$ and $y_0$ such that the following holds:

$$\psi \left( \frac{\mu_\lambda(x,y)}{t} \right) = \frac{1}{4!}\phi^{(4)} \left( \frac{\mu_\lambda(x,y)}{t} \right) \left( \frac{\mu_\lambda(y_0,y)}{t} \right)^4.$$ 

By the vanishing property of $b_i(x)$, the following holds:

$$\int b_i(y)\phi \left( \frac{\mu_\lambda(x,y)}{t} \right) d\mu_\lambda(y)/t = \int b_i(y)\psi \left( \frac{\mu_\lambda(x,y)}{t} \right) d\mu_\lambda(y)/t. \quad (203)$$

Then we could see first

$$\text{supp } \psi \left( \frac{\mu_\lambda(x,y)}{t} \right) \subseteq B(\bar{x}_i, 4d_\lambda(x,x_i)). \quad (204)$$

Formula (202) together with the mean value theorem, we could conclude:

$$\sup_{y \in B(x_i,\bar{a}r_i)} \left| \psi \left( \frac{\mu_\lambda(x,y)}{t} \right) \right| \leq C \left( \frac{r_i}{d_\lambda(x,x_i)} \right)^2. \quad (205)$$

By the fact $\mu_\lambda(y_1,y_0) = \mu_\lambda(y_0,y_2)$, we could also prove that the following holds:

$$\left| \psi \left( \frac{\mu_\lambda(x,y_1)}{t} \right) - \psi \left( \frac{\mu_\lambda(x,y_2)}{t} \right) \right| \leq C \|\phi^{(4)}\| \left| \frac{1}{t^2} \right| \mu_\lambda(y_1,y_2)^{4-\beta} \quad (206)$$

Thus Formula (201)(204)(205)(206) lead to:

$$\left| \int b_i(y)\phi \left( \frac{\mu_\lambda(x,y)}{t} \right) d\mu_\lambda(y)/t \right| \leq C \left( \frac{r_i}{d_\lambda(x,x_i)} \right)^2 f_\beta^*(\bar{x}_i) \leq C\alpha \left( \frac{r_i}{d_\lambda(x,x_i)} \right)^2 \quad (207)$$

when $x \in B(x_i,\bar{a}r_i)^c$. From Formula (200)(207), we have

$$f_\beta^*(x) \leq C f_\beta^*(x) \chi_{B(x_i,\bar{a}r_i)} + C\alpha \left( \frac{r_i}{r_i + d_\lambda(x,x_i)} \right)^2 \quad \text{for } x \in \mathbb{R}.$$ 

From this, it follows that

$$\|b_i\|^p_{H_\Phi^*(\mathbb{R})} \leq C \int_{B(x_i,\bar{a}r_i)} |f_\beta^*(x)|^p |x|^{2\lambda} dx + \int_{B(x_i,\bar{a}r_i)^c} \alpha^p \left( \frac{r_i}{d_\lambda(x,x_i)} \right)^{2p} \mu_\lambda(x).$$

Since $2p > 1$, we have

$$\int_{B(x_i,\bar{a}r_i)^c} \alpha^p \left( \frac{r_i}{d_\lambda(x,x_i)} \right)^{2p} \mu_\lambda(x) = \int_{\bar{a}r_i}^{\infty} \alpha^p \left( \frac{r_i}{r} \right)^{2p} dr \leq C \alpha^p r_i \leq C \alpha^p |B(x_i,r_i)|_\lambda \leq C \int_{B(x_i,r_i)} |f_\beta^*(x)|^p |x|^{2\lambda} dx.$$ 

Thus with the bounded intersection property of the set $\{ \bar{B}_i \}$, we could conclude:

$$\|b\|^p_{H_\Phi^*(\mathbb{R})} \leq C \sum_i \int_{B(x_i,\bar{a}r_i)} |f_\beta^*(x)|^p |x|^{2\lambda} dx \leq C \int_{\{x:f_\beta^*(x) > \alpha\}} |f_\beta^*(x)|^p |x|^{2\lambda} dx.$$ 

This proves the Proposition.
Proposition 3.67. BL[2](5.2.1)
For $0 < p_1 < p_2 \leq \infty$, $0 < \theta < 1$, then:

$$(L^{p_1}(\mathbb{R}), L^{p_2}(\mathbb{R}))_{\theta,p} = L^{p}(\mathbb{R})$$

where $1/p = (1-\theta)/p_1 + \theta/p_2$.

Proposition 3.68. For $0 < p_1 < p_2 \leq \infty$, $0 < \theta < 1$, then:

$$(L^{p_1}_\lambda(\mathbb{R}), L^{p_2}_\lambda(\mathbb{R}))_{\theta,p} = L^{p}_\lambda(\mathbb{R})$$

where $1/p = (1-\theta)/p_1 + \theta/p_2$.

Proof. Let $f(t) = \int_0^1 (2\lambda + 1)|u|^{2\lambda}du$, thus $f(t)$ is a bijection on $\mathbb{R}$. Denote $f^{-1}(t)$ as the reverse map of $f(t)$. For any $0 < q \leq \infty$, $\forall g(t) \in L^q(\mathbb{R})$, $\forall h(t) \in L^q_\lambda(\mathbb{R})$. Denote operator $P_\lambda$ and $P$ as

$$P_\lambda g(t) = g \circ f(t), \quad P h(t) = h \circ f^{-1}(t).$$

Obviously we could have

$$P P_\lambda g = g, \quad P P h = h.$$

and

$$\|P_\lambda g(t)\|_{L^q_\lambda(\mathbb{R})} = \|g(t)\|_{L^q(\mathbb{R})}, \quad \|P h(t)\|_{L^q(\mathbb{R})} = \|h(t)\|_{L^q_\lambda(\mathbb{R})}.$$ 

Thus

$$P_\lambda L^q(\mathbb{R}) = L^q_\lambda(\mathbb{R}) \quad \text{and} \quad PL^q_\lambda(\mathbb{R}) = L^q(\mathbb{R}).$$

Thus by the definition in (3.64), we could see that

$$(PL^{p_1}_\lambda(\mathbb{R}), PL^{p_2}_\lambda(\mathbb{R}))_{\theta,p} = (P L^{p_1}(\mathbb{R}), L^{p_2}(\mathbb{R}))_{\theta,p}.$$ 

Together with Proposition(3.67), we could conclude that

$$(L^{p_1}_\lambda(\mathbb{R}), L^{p_2}_\lambda(\mathbb{R}))_{\theta,p} = (P_\lambda L^{p_1}(\mathbb{R}), P_\lambda L^{p_2}(\mathbb{R}))_{\theta,p} = P_\lambda (L^{p_1}(\mathbb{R}), L^{p_2}(\mathbb{R}))_{\theta,p} = P_\lambda L^p = L^p_\lambda(\mathbb{R}).$$

This proves the Proposition.

□

Proposition 3.69. For $\frac{2\lambda + 1}{2\lambda + 2} < p_1 < \infty$, $0 < \theta < 1$, then:

$$(H^{p_1}_\lambda(\mathbb{R}), L^\infty_\lambda(\mathbb{R}))_{\theta,p} = H^p_\lambda(\mathbb{R})$$

where $1/p = (1-\theta)/p_1$.

Proof. It is obviously that the operator $f \rightarrow f^*(x)$ is of type $(H^q_\lambda(\mathbb{R}), L^q_\lambda(\mathbb{R}))$ bounded for any $q$ with $\frac{2\lambda + 1}{2\lambda + 2} < q \leq \infty$. Therefore by the Proposition(3.68), we could conclude that the above operator maps $(H^{p_1}_\lambda(\mathbb{R}), L^\infty_\lambda(\mathbb{R}))_{\theta,p}$ into $(L^{p_1}_\lambda(\mathbb{R}), L^\infty_\lambda(\mathbb{R}))_{\theta,p} = L^p_\lambda(\mathbb{R})$. And therefore

$$(H^{p_1}_\lambda(\mathbb{R}), L^\infty_\lambda(\mathbb{R}))_{\theta,p} \subseteq H^p_\lambda(\mathbb{R}).$$

In order to prove the inverse inclusion, notice that $H^p_\lambda(\mathbb{R}) \cap L^1_\lambda(\mathbb{R})$ is dense in $H^p_\lambda(\mathbb{R})$. Thus it suffices to show that

$$H^p_\lambda(\mathbb{R}) \cap L^1_\lambda(\mathbb{R}) \subseteq (H^{p_1}_\lambda(\mathbb{R}), L^\infty_\lambda(\mathbb{R}))_{\theta,p}$$

Take a $f \in H^p_\lambda(\mathbb{R}) \cap L^1_\lambda(\mathbb{R})$. Let $\tilde{f}^*$ be the nonincreasing rearrangement of $f^*$:

$$\tilde{f}^*(u) = \inf \{s : \{|x : |f^*(x)| > s|\}_\lambda \leq u, \quad \text{for} \quad u > 0\}.$$

Thus $\tilde{f}^*$ and $f^*$ satisfy:

$$\int_{\mathbb{R}} |f^*(x)|^q |x|^{2\lambda} dx = \int_0^{+\infty} \left|\tilde{f}^*(\alpha)\right|^q d\alpha,$$

and

$$\int_{\{|x : |f^*(x)| > s|\}_\lambda} |f^*(x)|^q |x|^{2\lambda} dx = \int_0^{\left|\{|x : |f^*(x)| > s|\}_\lambda\right|} \left|\tilde{f}^*(\alpha)\right|^q d\alpha.$$
for any $q$ with $\frac{2\lambda+1}{\lambda+2} < q < \infty$ and any $s$ with $s > 0$. For any $t > 0$, taking $\delta = \widetilde{f}^*(t^{p_1})$ in Proposition(3.66), we obtain a decomposition of $f$, $f = g_t + b_t$, satisfying
\[ \|g_t\|_\infty \leq C \delta \]
and
\[ \|b_t\|_{H^s_\lambda(R)} \leq C \int_{\{x: f^*(x) > \delta\}} |f^*(x)|^{p_1} |x|^{2\lambda} dx \leq C \int_0^{\|f^*(\alpha)\|_{p_1}} |f^*(\alpha)|^{p_1} d\alpha. \]
Notice that
\[ \{x: f^*(x) > \delta\} \lambda > \alpha \Longleftrightarrow \widetilde{f}^*(\alpha) > \delta = \widetilde{f}^*(t^{p_1}). \]
Thus
\[ \alpha < t^{p_1}. \]
Then we could conclude
\[ \|b_t\|_{H^s_\lambda(R)} \leq C \int_0^{t^{p_1}} |\widetilde{f}^*(\alpha)|^{p_1} d\alpha. \]
Thus we could have
\[ \int_0^{+\infty} (t^{-\theta} \|g_t\|_\infty)^p \frac{dt}{t} \leq \int_0^{+\infty} \left( t^{-\theta} \delta \right)^p \frac{dt}{t} = C \int_0^{+\infty} \left( \widetilde{f}^*(t) \right)^p dt = C \|f^*\|_{L^p_\lambda(R)} \]
and
\[ \int_0^{+\infty} (t^{-\theta} \|b_t\|_{H^s_\lambda(R)})^p \frac{dt}{t} \leq C \int_0^{+\infty} \left( t^{-\theta p_1} \int_0^{t^{p_1}} |f^*(\alpha)|^{p_1} d\alpha \right)^{p/p_1} \frac{dt}{t} \leq C \int_0^{+\infty} \left( t^{-1} \int_0^{t} |f^*(\alpha)|^{p_1} d\alpha \right)^{p/p_1} dt = C \|f^*\|_{L^p_\lambda(R)}. \]
Noticing that $K(t, f) \leq \|b_t\|_{H^s_\lambda(R)} + t \|g_t\|_\infty$, from above relation, we could conclude that:
\[ \|f\|_{\theta, p} \leq C \|f^*\|_{L^p_\lambda(R)}. \]
This finishes the proof of the Proposition. \qed

**Proposition 3.70.** BL[2][3.11.5]

Let $X_0, X_1$ to be Banach spaces. $0 \leq \theta_0, \theta_1 \leq 1, 0 < \eta < 1$,
\[ \theta = (1 - \eta) \theta_0 + \eta \theta_1. \]

Then for $0 < q < \infty$:
\[ \left( (X_0, X_1)_{\theta_0, q}, (X_0, X_1)_{\theta_1, q} \right)_{\eta, q} = (X_0, X_1)_{\theta, q}. \]

**Theorem 3.71.** For $\frac{2\lambda+1}{\lambda+2} < p_1 < p_2 \leq \infty$, $0 < \theta < 1$, then:
\[ (H^p_\lambda(R), H^{p_2}_\lambda(R))_{\theta, p} = H^p_\lambda(R) \]
where $1/p = (1 - \theta)/p_1 + \theta/p_2$.

**Proof.** We could prove the Theorem by Proposition(3.69) and (3.70). Let $X_1 = L^\infty_\lambda(R)$ and $X_0 = H^p_\lambda(R)$ where $\frac{2\lambda+1}{2\lambda+2} < p_0 < p_1 < p_2 \leq \infty$ satisfying:
\[ (H^p_\lambda(R), L^\infty_\lambda(R))_{\theta_1, p_1} = H^p_\lambda(R) \]
\[ (H^p_\lambda(R), L^\infty_\lambda(R))_{\theta_2, p_2} = H^p_\lambda(R) \]
Thus by Proposition (3.70) we could conclude:

\[ \left( (H^p_\lambda(R), L^\infty_\lambda(R))_{\theta_0, p} \right) = \left( (H^p_\lambda(R), L^\infty_\lambda(R))_{\theta_1, p} \right) \]

for \( 1/p_1 = (1 - \theta_1)/p_0, 1/p_2 = (1 - \theta_2)/p_0, 1/p = (1 - \theta)/p_0 \), and \( 1/p = (1 - \theta)/p_1 + \theta/p_2 \). Thus we could obtain:

\[ \theta_0 = (1 - \theta)\theta_1 + \theta\theta_2. \]

Thus by Proposition (3.70) we could conclude:

\[ \left( (H^p_\lambda(R), L^\infty_\lambda(R))_{\theta_1, p} \right) = (H^p_\lambda(R), L^\infty_\lambda(R))_{\theta_0, p}. \]

Together with Proposition (3.69) we could conclude:

\[ (H^p_\lambda(R), H^p_\lambda(R))_{\theta, p} = H^p_\lambda(R). \]

This proves the theorem.

\[ \square \]

3.8 Cesàro operator in Hardy spaces in the Dunkl setting

In this section, \( a(x) \) is a \( p_\lambda \)-atom (when \( \frac{2\lambda + 1}{2\lambda + 2} < p \leq \infty \)) satisfying:

\[ \begin{align*}
  i & \quad \|a(x)\|_{L^\infty_\lambda} \leq \frac{1}{|B(x_0, r_0)|^{1/p}_\lambda} \\
  ii & \quad \operatorname{supp} a(x) \subseteq B(x_0, r_0), \text{ and } r_0^{1/\lambda} < |x_0/2| \\
  iii & \quad \int_\mathbb{R} a(t)|t|^{2\lambda} dt = 0 \\
  iv & \quad 0 < r_0^{1/\lambda} < |x_0/2|. 
\end{align*} \]

In this section, \( \phi \in SS_{\gamma} (\gamma = \frac{1}{2\lambda + 1}) \), satisfying \( \int \phi(x) dx = 1 \). Let \( \alpha > 0 \). We write \( \phi_\alpha(t) = \alpha(1 - t)^{\alpha - 1} \) for \( 0 < t < 1 \). And let \( d\mu_\lambda(x) = (2\lambda + 1)|x|^{2\lambda} dx, \mu_\lambda(x, y) = \int_y^x (2\lambda + 1)|t|^{2\lambda} dt \) as above. For \( f \in L^p_\lambda(R) (1 \leq p < \infty) \) we define Cesàro operator \( C_\alpha \) as

\[ (C_\alpha f)(x) = \int_0^1 t^{-1} f(t^{-1} x) \phi_\alpha(t) dt. \]

Proposition 3.72. [Minkowski’s inequality] For \( 1 \leq p < \infty, if \)

\[ \int_\mathbb{R} \left( \int_\mathbb{R} |f(x, y)|^p |x|^{2\lambda} dx \right)^{1/p} dy = M < \infty \]

then

\[ \left( \int_\mathbb{R} \left[ \int_\mathbb{R} |f(x, y)|^p |x|^{2\lambda} dx \right]^{1/p} dy \right)^p \leq \int_\mathbb{R} \left[ \int_\mathbb{R} |f(x, y)|^p |x|^{2\lambda} dx \right]^{1/p} dy = M < \infty \]

Proof. Denote \( p, q \) with \( 1 \leq p, q < \infty, 1/p + 1/q = 1 \). Let

\[ F(x) = \int_\mathbb{R} |f(x, y)| dy. \]

Then for any \( \psi \in L^q_\lambda(R) \), we could conclude:

\[ \left| \int_\mathbb{R} F(x)\psi(x)|x|^{2\lambda} dx \right| \leq \int_\mathbb{R} \left[ \int_\mathbb{R} |f(x, y)||\psi(x)| |x|^{2\lambda} dx \right] \]

\[ \leq \int_\mathbb{R} \left[ \int_\mathbb{R} |f(x, y)||\psi(x)| |x|^{2\lambda} dy \right] \]

\[ \leq \int_\mathbb{R} \left[ \int_\mathbb{R} |f(x, y)|^p |x|^{2\lambda} dx \right]^{1/p} \|\psi\|_{L^q_\lambda} dy \]

\[ \leq M\|\psi\|_{L^q_\lambda}. \]
Thus
\[
\left[ \int_{\mathbb{R}} \int_{\mathbb{R}} f(x,y)dy \right]^p \left[ x^2 dx \right]^{1/p} = \sup_{\|\psi\|_{L_1^p}} \left[ \int_{\mathbb{R}} F(x)\psi(x)|x|^{2\lambda} dx \right] \leq M.
\]
This proves the Proposition. \(\square\)

Then by the Minkowski’s inequality in Proposition(3.72), we could know that Cesàro operator \(C_\alpha\) is bounded on \(L_1^p(\mathbb{R})\) (1 \(p < \infty\).

**Proposition 3.73.** Cesàro operator \(C_\alpha\) is bounded on \(L_1^p(\mathbb{R})\) for 1 \(p < \infty\):
\[
\|C_\alpha f\|_{L_1^p} \leq C\|f\|_{L_1^p}
\]
\(C\) is dependent on \(\alpha, p\) and \(\lambda\).

**Proof.** From the Minkowski’s inequality in Proposition(3.72), we could conclude:
\[
\|C_\alpha f\|_{L_1^p} = \left[ \int_0^1 \left( \int_0^1 t^{-1} f\left( t^{-1} x \right) \phi_\alpha(t) dt \right)^p |x|^{2\lambda} dx \right]^{1/p} \leq \left[ \int_0^1 \left( \int_0^1 t^{-1} f\left( t^{-1} x \right)|x|^{2\lambda} dx \right)^p t^{-1} \phi_\alpha(t) dt \right]^{1/p} \leq \left[ \int_0^1 \|f\|_{L_1^p} t^{-1}(2\lambda+1/p) \phi_\alpha(t) dt \right] \leq C\|f\|_{L_1^p}.
\]
\(\square\)

\(a(x)\) is obviously in \(L_1^\lambda(\mathbb{R}) \cap L_\infty^\lambda(\mathbb{R})\). Thus Cesàro operator \(C_\alpha\) can be defined on a \(p_\lambda\)-atom \(a(x)\) as:
\[
(C_\alpha a)(x) = \int_0^1 t^{-1} a\left( t^{-1} x \right) \phi_\alpha(t) dt.
\]
Thus our main purpose in this section is to prove the following inequality:
\[
\|C_\alpha a\|_{H_1^\lambda(\mathbb{R})} \leq C\|a\|_{H_1^\lambda} \leq C
\]
for \(\frac{2\lambda+1}{2\lambda+2} < p \leq 1\). Let \(f(x,r,t)\) be a function defined as following:
\[
f(x,r,t) = \int_\mathbb{R} a(u) \phi\left( \frac{\mu_\lambda(x,ru)}{r} \right) \frac{\lambda^{2\lambda+1}d\mu_\lambda(u)}{r}
\]
for \(x \in \mathbb{R}, r > 0\) and \(1 \geq t > 0\).

**Lemma 3.74.** For \(r > 0\) and \(\frac{2\lambda+1}{2\lambda+2} < p \leq 1\),
\[
\int_\mathbb{R} (C_\alpha a)(y) \phi\left( \frac{\mu_\lambda(x,y)}{r} \right) \frac{d\mu_\lambda(y)}{r} = \int_0^1 t^{-1} f(x,r,t)\phi_\alpha(t) dt.
\]

**Proof.** Notice that the following holds:
\[
\left[ \int_0^1 \left( t^{-1} a\left( t^{-1} y \right) \phi_\alpha(t) \phi\left( \frac{\mu_\lambda(x,y)}{r} \right) \right) \frac{d\mu_\lambda(y)}{r} dt \right] \frac{t^{2\lambda+1}d\mu_\lambda(u)}{r} dt \leq C r_0 * r_0^{-1/p} \frac{1}{r}
\]
Thus we could prove the Lemma by the Fubini Theorem. \(\square\)
Lemma 3.75. For \( x \in \mathbb{R}, r > 0 \) and \( \frac{2\lambda + 1}{2k+2} < p \leq 1 \), \( f(x, r, t) \) is a continuous function in \( t \) variable. For \( 1 \geq t, t_0 > 0 \), we could have:

\[
|f(x, r, t) - f(x, r, t_0)| \leq C r_0 \left( \frac{|x_0|^{2\lambda + 1}}{r} \right)^{1/p} \left( \int_{t_0}^{2\lambda + 1} + 1 \right) |t_0^{2\lambda + 1} - 2^{2\lambda + 1}|.
\]

\( C \) is a constant independent on \( x, r, t, \) and \( a(u) \).

Proof. We could calculate directly:

\[
|f(x, r, t) - f(x, r, t_0)| = \left| \int a(u) \left( \phi \left( \frac{\mu_\lambda(x, t_0u)}{r} \right) t_0^{2\lambda + 1} - \phi \left( \frac{\mu_\lambda(x, tu)}{r} \right) t_0^{2\lambda + 1} \right) d\mu_\lambda(u) \right|
\]

\[
\leq C r_0 \left( \int_0^{2\lambda + 1} \left( \int_0^1 \left( \frac{|x_0|^{2\lambda + 1}}{r} \right)^{1/p} \left( \int_{t_0}^{2\lambda + 1} + 1 \right) |t_0^{2\lambda + 1} - 2^{2\lambda + 1}| \right) dt_0 \right)^{1/p}.
\]

Lemma 3.76. For \( 0 < p \leq 1, i \in \mathbb{N} \), if \( \sum_i |a_i|^p < \infty \), then

\[
\left( \sum_i |a_i| \right)^p \leq \sum_i |a_i|^p.
\]

Proof. Without loss of generality, we may assume that

\[
\sum_i |a_i|^p = 1.
\]

In this case, we have \( |a_i| \leq 1 \) for every \( i \), so

\[
\left( \sum_i |a_i| \right)^p \leq \sum_i |a_i|^p |a_i|^{1-p} \leq \sum_i |a_i|^p = 1.
\]

Then we could obtain:

\[
\left( \sum_i |a_i| \right)^p \leq \sum_i |a_i|^p.
\]

This proves the Lemma.

Lemma 3.77. For \( 0 < p \leq 1, k \in \mathbb{Z}, x \in \mathbb{R} \) and \( r > 0 \) there exists \( \xi_k \in [2^{k-1}, 2^k] \) and \( \xi_k^* \in [1 - 2^k, 1 - 2^{k-1}] \) such that the following holds:

\[
\left| \int_0^1 t^{-1} f(x, r, t) \phi_\alpha(t) dt \right|^p \leq C \sum_{k=-\infty}^{-1} |f(x, r, \xi_k)|^p + C \sum_{k=-\infty}^{-1} 2^{kp_0} |f(x, r, \xi_k^*)|^p
\]

where \( C \) is a constant independent on \( x, r, k \) and \( f \).

Proof. By Lemma(3.76),

\[
\left| \int_0^1 t^{-1} f(x, r, t) \phi_\alpha(t) dt \right|^p \leq \left| \int_0^{1/2} t^{-1} f(x, r, t) \phi_\alpha(t) dt \right|^p + \left| \int_{1/2}^1 t^{-1} f(x, r, t) \phi_\alpha(t) dt \right|^p.
\]

By Lemma(3.76), we could conclude:

\[
\left| \int_0^{1/2} t^{-1} f(x, r, t) \phi_\alpha(t) dt \right|^p = \sum_{k=-\infty}^{-1} \left| \int_{2^{k-1}}^{2^k} t^{-1} f(x, r, t) \phi_\alpha(t) dt \right|^p \leq \sum_{k=-\infty}^{-1} \left| \int_{2^{k-1}}^{2^k} t^{-1} f(x, r, t) \phi_\alpha(t) dt \right|^p.
\]
By Lemma 3.75, \( f(x, r, t) \) is a continuous function in \( t \) variable. Thus by the Mean Value Theorem, we could deduce that there exists \( \xi_k \in [2^{k-1}, 2^k] \), such that the following holds:

\[
\int_{2^{k-1}}^{2^k} t^{-1} |f(x, r, t)| \phi_\alpha(t) dt \leq C |f(x, r, \xi_k)|.
\]

Then

\[
\left| \int_0^{1/2} t^{-1} f(x, r, t) \phi_\alpha(t) dt \right|^p \leq C \sum_{k=-\infty}^{-1} |f(x, r, \xi_k)|^p.
\]

In the same way we could conclude that:

\[
\left| \int_{1/2}^1 t^{-1} f(x, r, t) \phi_\alpha(t) dt \right|^p \leq C \sum_{k=-\infty}^{-1} \sup_{r>0} |f(x, r, \xi_k)|^p.
\]

This proves the Lemma. \( \square \)

**Lemma 3.78.** For \( \frac{2\lambda+1}{2\lambda+2} < p \leq 1 \) and \( 1 \geq t > 0 \)

\[
\sup_{r>0} |f(x, r, t)| \leq Cu^t (t^{-1} x)
\]

where \( C \) is a constant independent on \( p_\lambda \)-atom \( a \), \( t \) and \( x \).

**Proof.** Notice that

\[
\sup_{r>0} |f(x, r, t)| = \sup_{r>0} \left| \int a(u) \phi \left( \frac{\mu_\lambda(x, tu)}{r} \right) \frac{t^{2\lambda+1} d\mu_\lambda(u)}{r} \right|.
\]

Denote

\[
g(u) = \phi \left( \frac{\mu_\lambda(x, tu)}{r} \right).
\]

Then we could prove that

\[
|g(u)| \leq C
\]

\[
L \left( g(u), \frac{1}{2\lambda+1} \right) \leq \left( \frac{t^{2\lambda+1}}{r} \right)^{\frac{1}{2\lambda+1}}
\]

\[
\text{supp}(g) \subseteq B(t^{-1} x, \frac{r}{t^{2\lambda+1}}).
\]

This proves the Lemma. \( \square \)

**Theorem 3.79.** For \( \frac{2\lambda+1}{2\lambda+2} < p \leq 1 \), the Cesàro operator \( C_\alpha \) is bounded on a \( p_\lambda \)-atom in \( H^p_\lambda(\mathbb{R}) \) spaces:

\[
\|C_\alpha a\|_{H^p_\lambda(\mathbb{R})} \leq C \|a\|_{H^p_\lambda} \leq C
\]

and Cesàro operator \( C_\alpha \) can be extended as a bounded operator in \( H^p_\lambda(\mathbb{R}) \) spaces:

\[
\|C_\alpha f\|_{H^p_\lambda(\mathbb{R})} \leq C \|f\|_{H^p_\lambda} \quad \text{for } f \in H^p_\lambda(\mathbb{R}).
\]

**Proof.** By Theorem 2.39 and Lemma 3.74, we could deduce that:

\[
C_\alpha a \bigg|_{H^p_\lambda(\mathbb{R})} = \int_{\mathbb{R}} \sup_{r>0} \left| \int_{\mathbb{R}} (C_\alpha a)(y) \phi \left( \frac{\mu_\lambda(x, y)}{r} \right) \frac{d\mu_\lambda(y)}{r} \right|^p d\mu_\lambda(x)
\]

\[
= \int_{\mathbb{R}} \sup_{r>0} \left| \int_0^1 t^{-1} f(x, r, t) \phi_\alpha(t) dt \right|^p d\mu_\lambda(x).
\]
The above inequality together with Lemma (3.77) imply that:

\[
\|C_\alpha a\|_{H^p_\lambda(\mathbb{R})}^p \leq C \int_{\mathbb{R}} \sup_{r>0} \left( \sum_{k=-\infty}^{-1} |f(x,r,\xi_k)|^p d\mu_\lambda(x) + C \int_{\mathbb{R}} \sup_{r>0} \sum_{k=-\infty}^{-1} 2^{kp\alpha} |f(x,r,\xi_k^r)|^p d\mu_\lambda(x) \right)
\]

\[
\leq C \int_{\mathbb{R}} \sup_{r>0} \sum_{k=-\infty}^{-1} |f(x,r,\xi_k)|^p d\mu_\lambda(x) + C \int_{\mathbb{R}} \sup_{r>0} \sum_{k=-\infty}^{-1} 2^{kp\alpha} |f(x,r,\xi_k^r)|^p d\mu_\lambda(x)
\]

\[
\leq C \int_{\mathbb{R}} \sup_{r>0} \sum_{k=-\infty}^{-1} |f(x,r,\xi_k)|^p d\mu_\lambda(x) + C \sum_{k=-\infty}^{-1} \int_{\mathbb{R}} 2^{kp\alpha} |f(x,r,\xi_k^r)|^p d\mu_\lambda(x).
\]

The last sign of the above inequality is because of the theorem of term-by-term integration for nonnegative measurable sequence of functions. Then by Lemma (3.78), we could obtain:

\[
\|C_\alpha a\|_{H^p_\lambda(\mathbb{R})}^p \leq C \sum_{k=-\infty}^{-1} \left( \int_{\mathbb{R}} |a^r((\xi_k)^r)^{-1}x|^p d\mu_\lambda(x) + C \sum_{k=-\infty}^{-1} \int_{\mathbb{R}} 2^{kp\alpha} |a^r((\xi_k^r)^{-1}x)|^p d\mu_\lambda(x) \right)
\]

\[
\leq C \sum_{k=-\infty}^{-1} \left( 2^{(2\lambda+1)k} + 2^{kp\alpha} \right) \|a\|_{H^p_\lambda}^p.
\]

At last, by Proposition (3.46), Cesàro operator $C_\alpha$ can be extended as a bounded operator in $H^p_\lambda(\mathbb{R})$ spaces:

\[
\|C_\alpha f\|_{H^p_\lambda(\mathbb{R})} \leq C \|f\|_{H^p_\lambda(\mathbb{R})} \quad \text{for} \quad f \in H^p_\lambda(\mathbb{R}).
\]

This proves the theorem. \qed

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