Vacuum Persistence and Inversion of Spin Statistics in Strong QED

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The vacuum persistence can be written as the Bose-Einstein distribution in spinor QED and as the Fermi-Dirac distribution in scalar QED exactly in a constant electric field and approximately in time-dependent electric fields. The inverse temperature is determined by the period of charged particle in the Euclidean time and the negative chemical potential by the ratio of the worldline instanton to the inverse temperature. The negativity of chemical potential is due to the vacuum instability under strong electric fields. The inversion of spin statistics in the vacuum persistence is a consequence of the Bogoliubov relations for fermions and bosons.

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I. INTRODUCTION

The vacuum structure of quantum electrodynamics (QED) in external electromagnetic fields exhibits many interesting features such as vacuum polarization and pair production from an electric field due to vacuum instability. The effective action of QED in a constant electromagnetic field has been known by the seminal works by Sauter, Heisenberg, and Euler, and Weisskopf in the 1930s [1] and in a gauge invariant form via the proper-time method by Schwinger [2]. Physics of QED in strong electromagnetic fields has recently attracted much attention partly because X-ray free electron lasers [3] and the intensive light source from extreme light infrastructure (ELI) [4] may produce electric fields near the critical strength ($E_c = 1.3 \times 10^{16}$ V/cm) for electron-position pair production and partly because neutron stars or magnetars are believed to have fields ranging from $10^8$ G to $10^{15}$ G [5], some of which go beyond the critical strength ($B_c = 4.4 \times 10^{13}$ G).

One of interesting aspects of QED effective action is the observation by Müller, Greiner and Rafelski [6] that the vacuum polarization, the real part of the effective action, in a constant electric field reveals an inversion of spin statistics between scalar and spinor QED. They showed that the vacuum polarization of spinor QED can be written as a spectral function times the Bose-Einstein distribution and that of scalar QED as the same spectral function times the Fermi-Dirac distribution as

$$\text{Re}(\mathcal{L}_{\text{eff}}) = -\frac{(2|\sigma| + 1)m^4}{16\pi^2} \int_0^\infty ds \left[ s \ln(s^2 - 1 + i\delta) + \ln\left(\frac{s+1-i\delta}{s-1+i\delta}\right) - 2\right] \frac{1}{e^{\beta s} + (-1)^{2\sigma}},$$

where $\sigma = \pm 1/2$ for spinor QED and $\sigma = 0$ for scalar QED and $\delta$ is an arbitrary small positive number and $\beta = \pi m^2/|qE|$. One may show that the vacuum polarization in a pulsed electric field of Sauter-type $E(t) = E_0 \text{sech}^2(t/\tau)$ [7] also exhibits the spin-statistics inversion [8]. Quite recently it has been shown that the WKB instanton action for a pulsed electric field of Sauter-type configuration in spinor QED more accurately yields the mean number of produced pairs for scalar QED and vice versa [9]. In quantum kinematic approach, the apparent inversion of spin statistics is also shown for the Sauter-type electric field [10] and for an oscillating electric field with the Gaussian envelope [11].

The purpose of this paper is to investigate the inversion of spin statistics in the vacuum persistence, twice the imaginary part of the effective action, both in a constant electric field and in a general electric field. We show that the vacuum persistence of spinor QED takes the Bose-Einstein distribution and that of scalar QED the Fermi-Dirac distribution exactly in a constant electric field and approximately in a general electric field. In the case of a constant
electric field, the temperature is determined by the acceleration of a charged particle in analogy of the Hawking-Unruh effect, as expected from Ref. \[8\]. It is shown that the rest mass energy provides a negative chemical potential to the distributions, which implies the instability of the Dirac sea. For a general electric field, the temperature and the chemical potential for distributions depend not only on the mass and charge but also on the electric profile, for instance, for a Sauter-type electric field, \(E(t) = E_0 \text{sech}^2(t/\tau)\), they are determined by three parameters: the mass \(m\), the field strength \(E_0\) and the duration \(\tau\). This result together with the inversion of spin statistics in the vacuum polarization may shed light on understanding the vacuum structure of QED.

The organization of this paper is as follows. In Sec. II, we review the general relation between the vacuum persistence and the mean number of produced pairs. In Sec. III, we express the vacuum persistence exactly in terms of the Bose-Einstein or Fermi-Dirac distribution in a constant electric field and approximately in time-dependent electric fields, including the Sauter-type electric field. We discuss the temperature and chemical potential from the dynamics of charged particle in electric fields in the Euclidean time. In Sec. IV, we discuss and conclude the inversion of spin statistics.

II. VACUUM PERSISTENCE AND PAIR PRODUCTION RATE

The Heisenberg-Euler effective Lagrangian in a constant electromagnetic field has been obtained using various methods (for references and review, see Ref. \[14\]). One of the methods is the evolution operator method that directly makes use of the Bogoliubov transformation, which transforms the in-vacuum to the out-vacuum \[8\]. It is based on the idea that the scattering amplitude between the in-vacuum and the out-vacuum leads to the effective action \[13\]. For temporal evolution, the out-vacuum operators are expressed in terms of the in-vacuum operators through the Bogoliubov transformation, which enables the effective action to be written as the Bogoliubov coefficients \[16, 17, 18, 19, 20\]. In this section, we shall employ the evolution operator method in Ref. \[8\] to compute the QED effective action in time-dependent electric fields and the mean number of produced pairs.

In the evolution operator method, we use the Bogoliubov transformation between the in-vacuum and the out-vacuum given by

\[
\hat{a}_{\mathbf{k}\sigma}^{\text{out}} = \mu_{\mathbf{k}\sigma} \hat{a}_{\mathbf{k}\sigma}^{\text{in}} + \nu_{\mathbf{k}\sigma}^* \hat{b}_{\mathbf{k}\sigma}^{\text{in}},
\]

where \(\hat{a}\) and \(\hat{b}\) denote the particle and antiparticle operators, respectively. Here, the Bogoliubov coefficients satisfy the relation

\[
|\mu_{\mathbf{k}\sigma}|^2 - (-1)^{2\sigma} |\nu_{\mathbf{k}\sigma}|^2 = 1.
\]

Remarkably the Bogoliubov transformation both in spinor QED and in scalar QED can be written through the evolution operator as

\[
\hat{a}_{\mathbf{k}\sigma}^{\text{out}} = \hat{U}_{\mathbf{k}\sigma}(\hat{a}_{\mathbf{k}\sigma}^{\text{in}}, \hat{b}_{\mathbf{k}\sigma}^{\text{in}}) \hat{a}_{\mathbf{k}\sigma}^{\text{in}} \hat{U}_{\mathbf{k}\sigma}^\dagger(\hat{a}_{\mathbf{k}\sigma}^{\text{in}}, \hat{b}_{\mathbf{k}\sigma}^{\text{in}}),
\]

where the explicit form of \(\hat{U}\) is given in Ref. \[8\]. In particular, Eq. \(\(4\)\) implies that the out-vacuum evolves from the in-vacuum as

\[
|0, \text{out}\rangle = \prod_{\mathbf{k}\sigma} \hat{U}_{\mathbf{k}\sigma} |0, \text{in}\rangle.
\]

Finally, the effective action defined as the scattering amplitude takes the form \[8\]

\[
e^{-i \int dt d^3x \mathcal{L}_{\text{eff}}} = \langle 0, \text{out}|0, \text{in} \rangle = e^{-(-1)^{2\sigma} \sum_{\mathbf{k}\sigma} \mu_{\mathbf{k}\sigma}}.
\]

Thus, the one-loop effective action per unit time and per unit volume is

\[
\mathcal{L}_{\text{eff}} = (-1)^{2\sigma} i \sum_{\mathbf{k}\sigma} \ln(\mu_{\mathbf{k}\sigma}^*),
\]

which leads to the general relation between the vacuum persistence (twice of the imaginary part of the effective action) and the mean number of produced pairs, \(\mathcal{N}_{\mathbf{k}\sigma} = |\nu_{\mathbf{k}\sigma}|^2\):

\[
2(\text{Im} \mathcal{L}_{\text{eff}}) = (-1)^{2\sigma} \sum_{\mathbf{k}\sigma} \ln[1 + (-1)^{2\sigma} \mathcal{N}_{\mathbf{k}\sigma}],
\]

\[8\].
III. INVERSION OF SPIN STATISTICS

In this section we shall study the vacuum persistence exactly in a constant electric field and approximately in time-dependent electric fields, in particular, a pulsed electric field of Sauter-type.

First, we consider the exact case of a constant electric field as in Ref. [6]. In the time-dependent gauge, the Klein-Gordon equation for scalar QED and the Dirac equation for spinor QED have the (spin diagonal) Fourier component [in natural units with \( \hbar = c = 1 \)]

\[
\left[ \partial_t^2 + (k_z - qEt)^2 + 2m \left( \epsilon + \frac{m}{2} \right) + 2i\sigma qE \right] \phi_{k\sigma}(t) = 0, \tag{9}
\]

where \( \sigma = \pm 1/2 \) for spinor QED and \( \sigma = 0 \) for scalar QED and \( \epsilon \) is the excitation energy in the transverse direction in unit of the particle mass:

\[
\epsilon = \frac{k_z^2}{2m}. \tag{10}
\]

The solution to Eq. (9) given by the parabolic cylinder function

\[
\phi_{k\sigma}(t) = D_0(\zeta), \tag{11}
\]

where

\[
\zeta = \sqrt{2qE} e^{i\pi/4} \left( \frac{k_z}{qE} - t \right), \quad p = -\left( \frac{1}{2} + \sigma \right) - i \frac{m(\epsilon + \frac{m}{2})}{qE}, \tag{12}
\]

defines asymptotically the in-vacuum at \( t = -\infty \) and the out-vacuum at \( t = \infty \). Then the Bogoliubov coefficients are

\[
\mu_{k\pm\sigma} = \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{-i(p+1)\pi/2}, \quad \nu_{k\pm\sigma} = e^{-i\pi p}. \tag{13}
\]

Following the renormalization scheme in Ref. [8], the vacuum polarization is then given by

\[
\text{Re}(\mathcal{L}_{\text{eff}}) = (-1)^{2|\sigma| + 1} \frac{(2|\sigma| + 1)(2\pi)^2}{16\pi^2} \text{P} \int_0^{\infty} \frac{ds}{s^2} e^{-m^2s^2} [(qEs)f(s) - g(s)], \tag{14}
\]

where \( \text{P} \) denotes the principal value, and \( f(s) = \csc(qEs), \ g(s) = 1 + (qEs)^2/6 \) for scalar QED and \( f(s) = \cot(qEs), \ g(s) = 1/s - (qEs)^3/3 \) for spinor QED. On the other hand, the vacuum persistence takes the form

\[
2\text{Im}(\mathcal{L}_{\text{eff}}) = -(-1)^{2\sigma} \frac{(2|\sigma| + 1)qE}{2\pi} \int \frac{d^2k_\perp}{(2\pi)^2} \sum_{n=1}^{\infty} \left[ (-1)^{2\sigma} e^{-\frac{2\pi m(\epsilon + \frac{m}{2})}{qE}} \right]^n, \tag{15}
\]

where, in the last factor, the summation over residues from poles along the imaginary axis in the first quadrant has an interpretation of multi-instantons and anti-instantons [21]. Summing over \( n \) and changing the variable \( d^2k_\perp/(2\pi)^2 = mde/2\pi \), we may write the vacuum persistence as

\[
2\text{Im}(\mathcal{L}_{\text{eff}}) = (-1)^{2\sigma} \frac{(2|\sigma| + 1)qEm}{4\pi^2} \int_0^{\infty} dt \ln[1 + (-1)^{2\sigma} N_t], \tag{16}
\]

where \( N_t \) is the mean number of produced pairs:

\[
N_t = e^{-\frac{2\pi m(\epsilon + \frac{m}{2})}{qE}}. \tag{17}
\]

We now wish to give a thermal interpretation of the vacuum persistence [16]. The physical intuition of temperature for a charged particle or antiparticle in a constant electric field is the Hawking-Unruh temperature [22, 23]

\[
T_{\text{HU}} = \frac{a}{2\pi}, \tag{18}
\]

associated with the acceleration \( a \) or the surface gravity \( a \) of a black hole. An accelerated observer measures the thermal state with the Hawking-Unruh temperature from the Minkowski vacuum due to the presence of a horizon. In fact, the acceleration of a charged particle in a constant electric field

\[
a = \frac{qE}{m} \tag{19}
\]
leads to the inverse Hawking-Unruh temperature \[ \beta_0 = \frac{1}{k_B T_{HU}} = \frac{2\pi m}{qE}. \] (20)

Another interpretation of temperature is the acceleration of the reduced mass, \( m/2 \), of a particle and antiparticle pair \[27]. Integrating in, parts, we may write the vacuum persistence as

\[ 2\text{Im}(\mathcal{L}_{\text{eff}}) = \frac{(2|\sigma| + 1)m^2}{2\pi} \int_0^\infty d\epsilon \frac{\epsilon}{e^{\beta_0(\epsilon + \frac{\pi}{2})} + (-1)^{2\sigma}}. \] (21)

A few comments are in order. First, note that Eq. (21) describes the Bose-Einstein distribution for spin-1/2 fermions, whereas it does the Fermi-Dirac distribution for spin-0 scalars. Second, the negative chemical potential \( \eta_0 = -m/2 \) \[24\] in natural units implies that pair production is more favored because the vacuum is unstable against pair production in the presence of strong electric fields. Finally, the connection between the instanton action in Eq. (15) and the Matsubara frequency may be physically feasible since the Bose-Einstein or Fermi-Dirac distribution in Eq. (21) always has an expression in terms of the Matsubara frequency in finite-temperature field theory.

In the second case of a general time-dependent electric field with the gauge potential, \( A_2(t) = -E_0f(t) \), the Klein-Gordon equation and the spin diagonal component of the Dirac equation take the form

\[ [\partial_t^2 + Q_{k\sigma}(t)]\phi_{k\sigma}(t) = 0, \] (22)

where

\[ Q_{k\sigma}(t) = (k_z - qE_0f(t))^2 + m\left(\epsilon + \frac{m}{2}\right) + 2i\sigma qE_0\dot{f}(t). \] (23)

Though the exact solution of Eq. (22) may not be found in general, the WKB method can provide a scheme to find approximately the mean number of produced pairs per unit volume and per momentum, which is given by \[10\]

\[ N(\epsilon, k_z, \sigma) = e^{-S(\epsilon, k_z, \sigma)}, \quad S(\epsilon, k_z, \sigma) = i \int_0^\infty \sqrt{Q_{k\sigma}(t)} dt. \] (24)

Then, the vacuum persistence can be written as

\[ 2\text{Im}(\mathcal{L}_{\text{eff}}) = \frac{m}{2\pi} \sum_{\sigma} \int_0^\infty \frac{dk_z}{2\pi} \int_0^\infty \frac{d\epsilon}{2\pi} \frac{\epsilon e^{i\sigma S(\epsilon, k_z, \sigma)} + (-1)^{2\sigma}}{e^{i\sigma S(\epsilon, k_z, \sigma)} + (-1)^{2\sigma}}. \] (25)

For instance, the Sauter-type electric field \[26\], \( E(t) = E_0\text{sech}^2(t/\tau) \) with \( f(t) = \tau \tanh(t/\tau) \), has the WKB action \[10\]

\[ S(\epsilon, k_z, \sigma) = \pi qE_0\tau^2 \left[ \sqrt{\left(1 + \frac{k_z}{qE_0\tau}\right)^2 + \frac{2m(\epsilon + \frac{m}{2})}{(qE_0\tau)^2}} + \sqrt{\left(1 - \frac{k_z}{qE_0\tau}\right)^2 + \frac{2m(\epsilon + \frac{m}{2})}{(qE_0\tau)^2}} - \lambda \right], \] (26)

where \( \lambda = 2 \) for spinor QED and \( \lambda = \sqrt{1 - 1/(2qE_0\tau^2)} \) for scalar QED. The WKB action up to quadratic order of the momentum is equivalent to the worldline instanton with the prefactor included \[27\]. In the region where the WKB action and the worldline instanton are a good approximation, \( \lambda = 2 \) approximately even for scalar QED and we shall not distinguish spinor QED from scalar QED. Expanding up to quadratic order of momentum and changing the variable \( k_z = (qE_0\tau\sqrt{1 + (m/qE_0\tau)^2})\omega \), we obtain approximately the vacuum persistence per unit volume,

\[ 2\text{Im}(\mathcal{L}_{\text{eff}}) \approx \frac{(2|\sigma| + 1)m^2\tau}{2\pi} \int_{-\infty}^{\infty} d\omega \int_0^\infty d\epsilon \frac{\epsilon}{e^{i\sigma (\epsilon + \frac{m}{qE_0\tau} - \eta)} + (-1)^{2\sigma}}, \] (27)

where the temperature \( \beta \) and the chemical potential \( \eta \) are

\[ \beta = \frac{2\pi m}{qE_0\sqrt{1 + (\frac{m}{qE_0\tau})^2}}, \quad \eta = -m \sqrt{1 + (\frac{m}{qE_0\tau})^2} \] (28)
In the limit of $\tau = \infty$, corresponding to a constant electric field, $\omega = 0$ and the $\omega$-integration becomes unity, which makes Eq. (27) reduce to the vacuum persistence (21) per unit volume and per unit time in the constant electric field, as expected. Further, the temperature and the chemical potential become $\beta_0 = 2\pi m/qE$ and $\eta_0 = -m/2$, respectively. In Ref. 28, the pair-production rate (95) in scalar QED in a spatially localized electric field of Sauter-type, which is the vacuum persistence of this paper, is written as a spectral function times the Fermi-Dirac distribution, in concord with this paper.

Finally, we associate the dynamics of a charged particle or antiparticle in electric fields with the temperature and the chemical potential of distributions. In the Euclidean time, the particle follows a closed trajectory [27]

$$\dot{x}_3^2(u) + \dot{x}_4^2(u) = c^2.$$  

(29)

Here, $u$ is the Euclidean time and $c$ is a constant related with the mass, charge and field configuration. The particle executes a circular motion in the constant electric field, whose period is $\beta_0$ for $c = 1$, whereas it follows a complicated closed path in the Sauter-type electric field, whose period is $\beta$. In both electric fields, the chemical potential is $\eta = -S_0/\beta$, where $S_0$ is the worldline instanton [27]

$$S_0 = \frac{\pi m^2}{qE} \frac{2}{1 + \sqrt{1 + \left(\frac{m}{qE_0}\right)^2}}.$$  

(30)

The worldline instanton $S_0$ is the zero-momentum limit of the WKB action $S(\epsilon = 0, k_z = 0)$ in Eq. (26).

**IV. CONCLUSION**

Müller, Greiner, and Rafelski [6] showed that the vacuum polarization of spinor QED, the real part of the effective action, in a constant electric field can be expressed as a spectral function times the Bose-Einstein distribution and that of scalar QED as the same spectral function times the Fermi-Dirac distribution, thus exhibiting the spin-statistics inversion. In this paper, we showed that the vacuum persistence, twice of the imaginary part of the effective action, also exhibits the inversion of spin statistics between spin or and scalar QED exactly in the constant electric field and approximately in time-dependent electric fields. The inversion of spin statistics is a consequence of the Bogoliubov relations for fermions and bosons, which are equivalent to equal-time commutation relations for fermions and bosons that are closely related with spin statistics.

The temperature and the chemical potential for distributions are determined by the dynamics of charged particle in electric fields. In the constant electric field, the period of a circular motion in the Euclidean time is the inverse temperature and the ratio of the worldline instanton or the zero-momentum of the WKB action to the inverse temperature is the chemical potential. A similar argument holds for the Sauter-type electric field, though the motion is complicated. Another interpretation is the Hawking-Unruh temperature associated with the acceleration of particle in the constant electric field. Though it is not an easy task to find the Hawking-Unruh temperature for general time-dependent electric fields, the dynamical approach based on the worldline instanton or the WKB action does provide the temperature for the distribution in the vacuum persistence.

The inversion of spin statistics of the vacuum polarization has been known for a constant electric field. It would be interesting to see whether such inversion of the vacuum polarization still works for the Sauter-type electric field, whose vacuum persistence exhibits approximately the inversion as shown in this paper.

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