On APF Test for Poisson Process with Shift and Scale Parameters

A. S. Dabye\(^1\), Yu. A. Kutoyants\(^2\), and E.D. Tanguemp\(^3\)

\(^1,3\)Université Gaston Berger, Saint–Louis, Sénégal
\(^2\)Le Mans University, Le Mans, France
\(^2\)National Research University “MPEI”, Moscow, Russia
\(^2\)Tomsk State University, Tomsk, Russia

Abstract

We propose the goodness of fit test for inhomogeneous Poisson processes with unknown scale and shift parameters. A test statistic of Cramér-von Mises type is proposed and its asymptotic behavior is studied. We show that under null hypothesis the limit distribution of this statistic does not depend on unknown parameters.

Key words: Inhomogeneous Poisson process, parametric basic hypothesis, Cramér-von Mises test, asymptotically parameter free test, scale and shift parameters.

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1 Introduction

The problems of the construction of goodness of fit tests in the case of i.i.d. observations are well studied [15]. Special attention is payed to the case of parametric null hypothesis. Wide class of distributions can be parametrized by the shift and scale parameters, say, \( F \left( \frac{x-\theta_1}{\theta_2} \right) \). In the case of such families several authors showed that the limit distributions of the Kolmogorov-Smirnov and Cramér-von Mises tests statistics do not depend on the unknown
parameters (see [4], [6], [8], [7], [16], [17] and references therein). We call such tests asymptotically parameter free (APF).

For the continuous time stochastic processes the goodness of fit testing is not yet well developed. We can mention here several works for diffusion and Poisson processes [1], [2], [3], [5], [11], [13], [14], [18]. The problem of goodness of fit testing for inhomogeneous Poisson process is interesting because there is a wide literature on the applications of inhomogeneous Poisson process models in different domains (astronomy, biology, image analysis, medicine, optical communication, physics, reliability theory, etc.). Therefore to know if the observed Poisson process corresponds to some parametric family of intensity functions is important.

We consider the problem of goodness of fit testing for inhomogeneous Poisson process which under the null hypothesis has the intensity function with shift and scale parameters. We show that as in the classical case the limit distribution of the Cramer-von Mises type statistics does not depend on these unknown parameters. This allows us to construct the corresponding APF goodness of fit test of fixed asymptotic size.

2 Statement of the problem and auxiliary results

Suppose that we observe \( n \) independents inhomogeneous Poisson processes \( X^n = (X_1, \ldots, X_n) \), where \( X_j = (X_j(t), t \in \mathcal{R}) \) are trajectories of the Poisson processes with the mean function \( \Lambda(t) = \mathbb{E}X_j(t) = \int_{-\infty}^t \lambda(s) ds \).

Here \( \lambda(\cdot) \geq 0 \) is the corresponding intensity function.

Let us remind the construction of GoF test of Cramér-von Mises type in the case of simple null hypothesis. The class of tests \( (\Psi_n)_{n \geq 1} \) of asymptotic size \( \varepsilon \in (0, 1) \) is

\[
\mathcal{H}_\varepsilon = \left\{ \Psi_n : \lim_{n \to \infty} \mathbb{E}_0 \Psi_n = \varepsilon \right\}.
\]

Suppose that the basic hypothesis is simple, say, \( \mathcal{H}_0 : \Lambda(\cdot) = \Lambda_0(\cdot) \), where \( \Lambda_0(\cdot) \) is a known continuous function satisfying \( \Lambda_0(\infty) < \infty \). The alternative is composite (non parametric) \( \mathcal{H}_1 : \Lambda(\cdot) \neq \Lambda_0(\cdot) \). Then we can
introduce the Cramér-von Mises (C-vM) type statistic

\[ \tilde{\Delta}_n = \frac{n}{\Lambda_0(\infty)^2} \int_\mathcal{R} \left[ \hat{\Lambda}_n(t) - \Lambda_0(t) \right]^2 \, d\Lambda_0(t), \]

where \( \hat{\Lambda}_n(t) = \frac{1}{n} \sum_{j=1}^{n} X_j(t) \) is the empirical mean of the Poisson process. It can be verified that under \( \mathcal{H}_0 \) this statistic converges to the following limit:

\[ \tilde{\Delta}_n \Rightarrow \Delta \equiv \int_0^1 W(s)^2 \, ds, \]

where \( W(s) \), \( 0 \leq s \leq 1 \) is a standard Wiener process. Therefore the C-vM type test \( \psi_n(X^n) = \mathbb{1}_{\{\tilde{\Delta}_n > c_\varepsilon\}} \) with the threshold \( c_\varepsilon \) defined by the equation \( \mathbb{P}\{\Delta > c_\varepsilon\} = \varepsilon \) belongs to \( \mathcal{H}_\varepsilon \). This test is *asymptotically distribution free* (ADF) (see, e.g., [3]). Remind that the test is called ADF if the limit distribution of the test statistic under hypothesis does not depend on the mean function \( \Lambda_0(\cdot) \).

Let us consider the case of the parametric null hypothesis. It can be formulated as follows. We have to test the null hypothesis

\[ \mathcal{H}_0 : \Lambda(\cdot) \in \mathcal{L}(\Theta) = \left\{ \Lambda_0(\vartheta, t), \vartheta \in \Theta, t \in \mathcal{R} \right\}, \]

against the alternative \( \mathcal{H}_1 : \Lambda(\cdot) \notin \mathcal{L}(\Theta) \). Here \( \Lambda_0(\vartheta, \cdot) \) is a known mean function of the Poisson process depending on some finite-dimensional unknown parameter \( \vartheta \in \Theta \subset \mathcal{R}^d \). Note that under \( \mathcal{H}_0 \) there exists the *true value* \( \vartheta_0 \in \Theta \) such that the mean of the observed Poisson process \( \Lambda(t) = \Lambda(\vartheta_0, t), t \in \mathcal{R} \).

The C-vM type GoF test can be constructed by a similar way. Introduce the normalized process \( \bar{u}_n(t) \equiv u_n(t, \bar{\vartheta}_n) = \sqrt{n} \left( \hat{\Lambda}_n(t) - \Lambda_0(\bar{\vartheta}_n, t) \right), t \in \mathcal{R} \). Here \( \bar{\vartheta}_n \) is some estimator of the parameter \( \vartheta \), which is (under hypothesis \( \mathcal{H}_0 \)) consistent and asymptotically normal \( \sqrt{n} (\bar{\vartheta}_n - \vartheta_0) \Rightarrow \xi \).

The corresponding C-vM type statistic can be

\[ \Delta_n = \frac{n}{\Lambda_0(\infty, \bar{\vartheta}_n)^2} \int_\mathcal{R} \left( \hat{\Lambda}_n(t) - \Lambda_0(\bar{\vartheta}_n, t) \right)^2 \, d\Lambda_0(\bar{\vartheta}_n, t) \]
Then, under null hypothesis $\mathcal{H}_0$, we can verify the convergence

$$
\bar{u}_n(t) = \sqrt{n} \left( \bar{\Lambda}_n(t) - \Lambda_0(\vartheta_0, t) \right) + \sqrt{n} \left( \Lambda_0(\vartheta_0, t) - \Lambda_0(\bar{\vartheta}_n, t) \right) \\
= W_n(t) - \left\langle \sqrt{n} \left( \bar{\vartheta}_n - \vartheta_0 \right), \frac{\partial \Lambda_0(\vartheta_0, t)}{\partial \vartheta} \right\rangle + o(1)
$$

$$
\Rightarrow W_0(\Lambda_0(\vartheta_0, t)) - \left\langle \xi(\vartheta_0), \dot{\Lambda}_0(\vartheta_0, t) \right\rangle.
$$

Here $\langle , \rangle$ is the scalar product in $\mathbb{R}^d$ and dot means differentiation w.r.t. $\vartheta$.

Let us denote $s = \Lambda_0(\vartheta_0, \infty)^{-1} \Lambda_0(\vartheta_0, t)$ and introduce the vector $G(\vartheta_0, s) = \Lambda_0(\vartheta_0, \infty)^{-1/2} \dot{\Lambda}_0(\vartheta_0, t)$. Then we obtain the convergence

$$
\bar{\Delta}_n \Rightarrow \tilde{\Delta}(\vartheta_0, \Lambda_0) = \int_0^1 \left[ W(s) - \left\langle \xi(\vartheta_0), G(\vartheta_0, s) \right\rangle \right]^2 ds,
$$

where $W(s), 0 \leq s \leq 1$ is standard Wiener process. Here the distribution of the limit random variable $\tilde{\Delta}(\vartheta_0, \Lambda_0)$ depends on the true value $\vartheta_0$ and on the mean function $\Lambda_0(\vartheta_0, \cdot)$.

Therefore if we propose a GoF test based on this statistics, say, $\Phi_n = \mathbb{I}\{\Delta_n > c_\varepsilon\}$, then to find the threshold $c_\varepsilon$ such that $\Phi_n \in \mathcal{K}_\varepsilon$ we have to solve the equation $\mathbb{P}_{\vartheta_0}(\tilde{\Delta}(\vartheta_0, \Lambda_0) > c_\varepsilon) = \varepsilon$. The solution $c_\varepsilon = c_\varepsilon(\vartheta_0, \Lambda_0)$, where $\vartheta_0$ is the unknown true value. There are several possibilities to construct the test belonging $\mathcal{K}_\varepsilon$. One is to calculate the function $c_\varepsilon(\vartheta_0, \Lambda_0)$, verify that this function is continuous w.r.t. $\vartheta$ and then to use the consistent estimator $\hat{\vartheta}_n$ for the threshold $c_\varepsilon(\hat{\vartheta}_n, \Lambda_0)$. Another possibility is to use the linear transformation of the statistic $\bar{u}_n(\cdot)$, which transforms it in the Wiener process (see, e.g., [10] or [11]). In this work we follow the third approach: we show that the limit distribution of the statistic does not depend on $\vartheta_0$.

In particular, the goal of this work is to show that if the unknown parameter is two-dimensional $\vartheta = (\alpha, \beta)$, where $\alpha \in \mathcal{R}$ is the shift and $\beta \in \mathcal{R}_+$ is the scale parameters, then it is possible to construct a test statistic $\hat{\Delta}_n$ whose limit distribution does not depend on $\vartheta_0$. The mean function under null hypothesis is

$$
\Lambda_0(\vartheta, t) = \int_{-\infty}^t \lambda_0 \left( \frac{v - \alpha}{\beta} \right) \, dv, \quad t \in \mathcal{R}.
$$

The proposed test statistic is

$$
\hat{\Delta}_n = \frac{n}{\beta_n^2} \int_{\mathcal{R}} \left[ \hat{\Lambda}_n(t) - \Lambda_0(\hat{\vartheta}_n, t) \right]^2 d\Lambda_0(\hat{\vartheta}_n, t).
$$
Here $\hat{\theta}_n$ is the maximum likelihood estimator (MLE) of the vector parameter $\theta$. We show that $\hat{\Delta}_n \Rightarrow \Delta$, where $\Delta = \Delta (\Lambda_0)$, i.e., the distribution of the random variable $\Delta (\Lambda_0)$ does not depend on $\theta_0$. Remind that the function $\Lambda_0 (t), t \in \mathcal{R}$ is known and therefore the solution $c_{\epsilon} = c_{\epsilon} (\Lambda_0)$ can be calculated before the experiment using, say, numerical simulations.

We are given $n$ independent observations $X^n = (X_1, \ldots, X_n)$ of inhomogeneous Poisson processes $X_j = (X_j (t), t \in \mathcal{R})$ with the mean function $\Lambda (t) = \mathbb{E} X_j (t), t \in \mathcal{R}$. We have to construct a GoF test in the hypothesis testing problem with parametric null hypothesis $\mathcal{H}_0$. More precisely, we suppose that under $\mathcal{H}_0$ the mean function $\Lambda (t)$ is absolutely continuous: $\Lambda' (t) = \lambda_0 (\vartheta_0, t)$. Here $\vartheta_0$ is the true value and the intensity function is $\lambda_0 (\vartheta_0, t) = \lambda_0 \left( \frac{t - \alpha_0 \beta_0}{\beta_0} \right), \vartheta_0 = (\alpha_0, \beta_0) \in \Theta \subset \mathbb{R}^2$. The set $\Theta = (a_1, a_2) \times (b_1, b_2)$ and $b_1 > 0$, where all constants are finite. Therefore if we denote $\Lambda_0 (t) = \int_{-\infty}^t \lambda_0 (v) \, dv, t \in \mathcal{R}$, then the mean function under null hypothesis is

$$\Lambda (t) = \Lambda_0 (\vartheta_0, t) = \beta_0 \Lambda_0 \left( \frac{t - \alpha_0}{\beta_0} \right).$$

It is convenient to use two different functions $\Lambda_0 (\vartheta, t)$ and $\Lambda_0 (t)$ and we hope that such notation will not be misleading.

Therefore, we have the parametric null hypothesis

$$\mathcal{H}_0 : \Lambda (\cdot) \in \mathcal{L} (\Theta),$$

where the parametric family is

$$\mathcal{L} (\Theta) = \left\{ \Lambda (\cdot) : \Lambda (t) = \beta_0 \Lambda_0 \left( \frac{t - \alpha}{\beta} \right), t \in \mathcal{R}, \vartheta = (\alpha, \beta) \in \Theta \right\}. \quad (1)$$

Here $\Lambda_0 (\cdot)$ is a known absolutely continuous function with properties: $\Lambda_0 (-\infty) = 0, \Lambda_0 (\infty) < \infty$.

We consider the class of tests of asymptotic level $\epsilon$:

$$\mathcal{K}_\epsilon = \left\{ \overline{\Psi}_n : \lim_{{n \to \infty}} \mathbb{E}_\vartheta \overline{\Psi}_n = \epsilon, \vartheta \in \Theta \right\}. \quad (2)$$

The test studied in this work is based on the following statistic of C-vM type:

$$\hat{\Delta}_n = \frac{n}{\beta^2_n} \int_\mathcal{R} \left[ \hat{\Lambda}_n (t) - \hat{\beta}_n \Lambda_0 \left( \frac{t - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right]^2 \lambda_0 \left( \frac{t - \hat{\alpha}_n}{\hat{\beta}_n} \right) \, dt. \quad (3)$$
where \( \hat{\vartheta}_n = (\hat{\alpha}_n, \hat{\beta}_n) \) is the MLE. Remind that the log-likelihood ratio for this model of observations is

\[
\ln L (\vartheta, \vartheta_1, X^n) = \sum_{j=1}^{\infty} \int_{\mathbb{R}} \ln \frac{\lambda_0 (\vartheta, t)}{\lambda_0 (\vartheta_1, t_1)} dX_j (t) - n \int_{\mathbb{R}} [\lambda_0 (\vartheta, t) - \lambda_0 (\vartheta_1, t)] dt,
\]

and the MLE \( \hat{\vartheta}_n \) is defined by the equation

\[
L \left( \hat{\vartheta}_n, \vartheta_1, X^n \right) = \sup_{\vartheta \in \Theta} L (\vartheta, \vartheta_1, X^n). \tag{4}
\]

Here \( \vartheta_1 \in \Theta \) is some fixed value.

As we use the asymptotic properties of the MLE \( \hat{\vartheta}_n \), we need some regularity conditions, which we borrow from [12] (see the conditions B1-B5 in the Section 2.1 there).

Note that the derivative (vector)

\[
\dot{\lambda} (\vartheta, t) = \left( \frac{\partial \lambda (\vartheta, t)}{\partial \alpha}, \frac{\partial \lambda (\vartheta, t)}{\partial \beta} \right) = -\lambda' \left( \frac{t - \alpha}{\beta} \right) \left( \frac{1}{\beta}, \frac{t - \alpha}{\beta^2} \right). \tag{5}
\]

Here \( \lambda' (t) = \frac{d\lambda(t)}{dt} \).

Conditions \( \mathcal{R} \)

\( \mathcal{R}_1 \). The intensity function \( \lambda_0 (\cdot) \) is strictly positive and two times continuously differentiable.

\( \mathcal{R}_2 \). For any \( \vartheta_0 \in \Theta \) we have

\[
\lim_{\|\vartheta - \vartheta_0\| \to 0} \int_{\mathbb{R}} \left| \frac{\dot{\lambda}_0 (\vartheta, t)}{\sqrt{\lambda_0 (\vartheta, t)}} - \frac{\dot{\lambda}_0 (\vartheta_0, t)}{\sqrt{\lambda_0 (\vartheta_0, t)}} \right|^2 \lambda_0 (\vartheta_0, t) dt = 0, \tag{6}
\]

\[
\sup_{\vartheta \in \Theta} \int_{\mathbb{R}} \left| \frac{\dot{\lambda}_0 (\vartheta, t)}{\sqrt{\lambda_0 (\vartheta, t)}} \right|^4 \lambda_0 (\vartheta_0, t) dt < \infty. \tag{7}
\]

\( \mathcal{R}_3 \). The function \( \lambda_0 (\cdot) \) satisfies the conditions

\[
\int_{\mathbb{R}} t^2 \lambda_0 (t) dt < \infty, \quad \int_{\mathbb{R}} t^4 |\lambda_0' (t)| dt < \infty. \tag{8}
\]

Of course, we suppose that the expressions under the sign of integrals are integrable in the required sense.
For the consistency of the MLE we need the identifiability condition

\[ I. \text{For any } \nu > 0 \]

\[
\inf_{\|\vartheta - \vartheta_0\| > \nu} \int_{\mathbb{R}} \left[ \sqrt{\lambda_0(\vartheta, t)} - \sqrt{\lambda_0(\vartheta_0, t)} \right]^2 dt > 0.
\]

Note that in the case of shift and scale parameters this condition is fulfilled. Indeed, suppose that for some \( \nu > 0 \) this integral is 0. Then there exists \( \vartheta_1 \neq \vartheta_0 \) \((\|\vartheta_1 - \vartheta_0\| \geq \nu)\) such that \( \lambda(\vartheta_1, t) \equiv \lambda(\vartheta_0, t) \). Recall that the functions are continuous. Therefore \( \lambda \left( \frac{t - \alpha}{\beta} \right) = \lambda \left( \frac{t - \alpha_0}{\beta_0} \right) \), \( t \in \mathbb{R} \) or after the change of variables \( s = \beta_0^{-1}(t - \alpha_0) \) we have

\[ \lambda_0(s) = \lambda_0 \left( \frac{\beta_0}{\beta_1} s - \frac{\alpha_1 - \alpha_0}{\beta_1} \right), \quad s \in \mathbb{R}. \]

Of course, such function \( \lambda_0(\cdot) \notin L_1(\mathbb{R}) \). Hence, the condition of identifiability is fulfilled.

To construct the test statistics we need the following property of the mean function

For all \( \vartheta_0 \in \Theta \)

\[
\sup_{\vartheta \in \Theta} \int_{\mathbb{R}} \left| \dot{\Lambda}_0(\vartheta, t) \right|^2 \lambda(\vartheta_0, t) dt < \infty.
\] (9)

This condition can be expressed in terms of the function \( \lambda_0(\cdot) \) like (6)-(7). Indeed we have

\[
\left| \dot{\Lambda}_0(\vartheta, t) \right|^2 = \lambda_0 \left( \frac{t - \alpha}{\beta} \right)^2 + \left| \Lambda_0 \left( \frac{t - \alpha}{\beta} \right) - \lambda_0 \left( \frac{t - \alpha}{\beta} \right) \right|^2.
\]

As the function \( \lambda_0(\cdot) \) is bounded, it is sufficient to suppose (8) and we obtain (9).

Let us introduce the Fisher information matrix

\[
\mathbb{I}(\vartheta) = \frac{1}{\beta} \begin{pmatrix}
\int_{\mathbb{R}} \frac{\lambda_0'(s)^2}{\lambda_0(s)} \, ds & \int_{\mathbb{R}} \frac{s \lambda_0'(s)^2}{\lambda_0(s)} \, ds \\
\int_{\mathbb{R}} \frac{s \lambda_0'(s)^2}{\lambda_0(s)} \, ds & \int_{\mathbb{R}} s^2 \frac{\lambda_0'(s)^2}{\lambda_0(s)} \, ds
\end{pmatrix} = \frac{1}{\beta} \mathbb{I}_*,
\]

where the matrix \( \mathbb{I}_* \) does not depend on \( \vartheta \). Note that the matrix \( \mathbb{I}_* \) is non degenerate. Indeed, the determinant is

\[
D = \int_{\mathbb{R}} \frac{\lambda_0(s)^2}{\lambda_0(s)} \, ds \int_{\mathbb{R}} \frac{s^2 \lambda_0'(s)^2}{\lambda_0(s)} \, ds - \left( \int_{\mathbb{R}} \frac{s \lambda_0'(s)^2}{\lambda_0(s)} \, ds \right)^2.
\]

7
Remind that by Cauchy-Schwartz inequality
\[
\left( \int_{\mathcal{R}} \frac{s \lambda_0(s)^2}{\lambda_0(s)} \, ds \right)^2 \leq \int_{\mathcal{R}} \frac{\lambda_0(s)^2}{\lambda_0(s)} \, ds \int_{\mathcal{R}} \frac{s^2 \lambda_0(s)^2}{\lambda_0(s)} \, ds.
\]
The equality in Cauchy-Schwartz inequality \((D = 0)\) we obtain if and only if \(|s \lambda_0'(s)| \equiv |\lambda_0'(s)|, \, s \in \mathcal{R}\). Of course such equality is impossible, if \(\lambda'(s) \neq 0\) or \(s \neq \pm 1\). As the function \(\lambda_0(\cdot)\) is positive and differentiable, we have
\[
\int_{\mathcal{R}} \frac{\lambda_0(s)^2}{\lambda_0(s)} \, ds > 0.
\]
We suppose that the intensity function \(\lambda_0(t), \, t \in \mathcal{R}\) is strictly positive because if we have a set of positive Lebesgue measure, where \(\lambda_0(t) = 0\) and the unknown parameters are shift and scale, then the measures induced by the observations will be not equivalent. The properties of the MLE will be different.

Under these conditions, the MLE is uniformly consistent, asymptotically normal \(\sqrt{\frac{n}{\beta_0}} \left( \hat{\vartheta}_n - \vartheta_0 \right) \Rightarrow \zeta \sim \mathcal{N}(0, I^{-1})\) and the polynomial moments converge
\[
\lim_{n \to \infty} \left( \frac{n}{\beta_0} \right)^\frac{p}{2} E \| \hat{\vartheta}_n - \vartheta_0 \|^p = E \| \zeta \|^p.
\]
For the proof see Theorem 2.4 in [12]. Note that the distribution of the vector \(\zeta\) does not depend on \(\vartheta_0\).

3 Main result

Introduce the following random variable:
\[
\Delta_0 = \int_{\mathcal{R}} \left[ W(\Lambda_0(t)) - \langle \zeta, \hat{\Lambda}_0(t) \rangle \right]^2 \, d\Lambda_0(t),
\]
where \(\hat{\Lambda}(\vartheta, t) = \left( -\lambda_0(s), \Lambda_0(s) - s\lambda_0(s) \right)\) and \(W(r), 0 \leq r \leq \Lambda_0(\infty)\) is a Wiener process. The main result of this work is the following theorem.

**Theorem 1** Let the conditions \(\mathcal{R}\) be fulfilled then the test
\[
\Psi_n(X^n) = \mathbb{I}_{\{\Delta_n > c_\varepsilon\}}, \quad P(\Delta_0 > c_\varepsilon) = \varepsilon
\]
belongs to the class \(\mathcal{H}_\varepsilon\).
Proof. We can write
\begin{align*}
\hat{u}_n(t) &= \sqrt{n} \left( \hat{\Lambda}_n(t) - \hat{\beta}_n \Lambda_0 \left( \frac{t - \hat{\alpha}_n}{\hat{\beta}_n} \right) \right) \\
&= \sqrt{n} \left( \hat{\Lambda}_n(t) - \Lambda_0(\vartheta, t) \right) + \sqrt{n} \left( \Lambda_0(\vartheta_0, t) - \Lambda_0(\hat{\vartheta}_n, t) \right) \\
&= W_n(t) - \langle \sqrt{n} (\hat{\vartheta}_n - \vartheta_0), \hat{\Lambda}_0(\vartheta_0, t) \rangle + r_n(t) \equiv u_n(t) + r_n(t) .
\end{align*}
Here the vector \( \hat{\Lambda}_0(\vartheta, t) = \frac{\partial}{\partial \vartheta} \Lambda_0(\vartheta, t) \) and we used the Taylor formula.

We have to show that under the null hypothesis
\begin{align*}
\frac{1}{\beta_n^2} \int_{\mathcal{R}} u_n(t)^2 \lambda_0(\hat{\vartheta}_n, t) \, dt &\longrightarrow \int_{\mathcal{R}} [W(\Lambda_0(s)) + \langle \zeta, V(s) \rangle]^2 \, d\Lambda_0(s) , \quad (12) \\
\int_{\mathcal{R}} r_n(t)^2 \lambda_0(\hat{\vartheta}_n, t) \, dt &\longrightarrow 0 . \quad (13)
\end{align*}
Here \( V(s) = (\lambda_0(s), s\lambda_0(s) - \Lambda_0(s)) \).

The convergences (12), (13) we will prove in several steps.

A. We show that we have the convergence of finite dimensional distributions
\begin{equation}
(\hat{\beta}_n^{-1} \hat{u}_n(t_1), \ldots, \hat{\beta}_n^{-1} \hat{u}_n(t_k)) \Longrightarrow (\hat{u}(s_1), \ldots, \hat{u}(s_k)) , \quad (14)
\end{equation}
where we put \( s_i = \beta_0^{-1} (t_i - \alpha_0) \) and \( \hat{u}(s) = W(\Lambda_0(s)) + \langle \zeta, V(s) \rangle \).

B. We verify the estimate: for \( |t_1| < L, |t_2| < L \) and any \( L > 0 \)
\begin{equation}
E_{\vartheta_0} |\hat{u}_n(t_1) - \hat{u}_n(t_2)|^2 \leq C (1 + L) |t_1 - t_2| , \quad (15)
\end{equation}
where the constant \( C > 0 \) does not depend on \( n \).

C. We show that for any \( \delta > 0 \) there exists \( L > 0 \) such that for all \( n \)
\begin{equation*}
\int_{|t| > L} E_{\vartheta_0} |\hat{u}_n(t)|^2 \lambda_0(\vartheta_0, t) \, dt < \delta .
\end{equation*}

D. We check (13) by direct calculations.
Having A-C by Theorem A.22 in [9] we obtain (12).

To prove A we recall that by the central limit theorem

\[ W_n(t) = \sqrt{\frac{n}{\beta_0}} \left( \hat{\Lambda}_n(t) - \beta_0 \Lambda_0 \left( \frac{t - \alpha_0}{\beta_0} \right) \right) \Rightarrow W \left( \Lambda_0 \left( \frac{t - \alpha_0}{\beta_0} \right) \right), \quad (16) \]

where \( W(r), 0 \leq r \leq \Lambda_0(\infty) \) is a Wiener process. Moreover, the vector \( W_{k,n} \equiv (W_n(t_1), \ldots, W_n(t_k)) \) for any \( k \geq 1 \) and \( t_i \in \mathcal{R} \) is asymptotically normal

\[ W_{k,n} \Rightarrow W_k = \left( W \left( \Lambda_0 \left( \frac{t_1 - \alpha_0}{\beta_0} \right) \right), \ldots, W \left( \Lambda_0 \left( \frac{t_k - \alpha_0}{\beta_0} \right) \right) \right). \]

We know as well that the MLE \( \hat{\vartheta}_n \) is asymptotically normal. The Wiener process \( W(\cdot) \) and the Gaussian vector \( \zeta \) are correlated. To clarify this dependence and to prove the joint asymptotic normality of the MLE and of this vector we recall how the asymptotic normality of the MLE can be proved. We follow below the approach developed by Ibragimov and Khasminskii [9].

Introduce the normalized likelihood ratio \( Z_n(v) = \frac{L(\vartheta_0 + \frac{v}{\sqrt{n}}, X^n)}{L(\vartheta_0, X^n)}, \ v \in \mathbb{V}_n. \)
Here \( \mathbb{V}_n = \left\{ v : \vartheta_0 + \frac{v}{\sqrt{n}} \in \Theta \right\}. \) Under the presented here conditions \( \mathcal{R} \) the random field \( Z_n(v), v \in \mathbb{V}_n \) admits the representation (LAN)

\[ \ln Z_n(v) = \langle v, S_n(\vartheta_0, X^n) \rangle - \frac{1}{2} v^\top I(\vartheta_0) v + m_n, \quad (17) \]

where \( m_n \to 0 \) and the vector

\[ S_n(\vartheta_0, X^n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \int_{\mathcal{R}} \frac{\lambda_0(\vartheta_0, t)}{\lambda_0(\vartheta_0, t)} [dX_j(t) - \lambda_0(\vartheta_0, t) dt] \]

By the central limit theorem

\[ S_n(\vartheta_0, X^n) \Rightarrow S(\vartheta_0) \sim \mathcal{N}(0, I(\vartheta_0)). \quad (18) \]

Let us denote the limit random field

\[ Z(v) = \exp \left\{ \langle v, S(\vartheta_0) \rangle - \frac{1}{2} v^\top I(\vartheta_0) v \right\}, \quad v \in \mathcal{R}^2. \]
Recall that we have the representation

\[ S(\varrho_0) = \sqrt{\beta_0} \int_{\mathbb{R}} \frac{\lambda_0(t - \alpha_0)}{\beta_0} \, dW \left( \Lambda_0 \left( \frac{t - \alpha_0}{\beta_0} \right) \right) \]

with the same Wiener process as in (16). Moreover, for the MLE we have the limit

\[ \sqrt{\frac{n}{\beta_0}} \left( \hat{\varrho}_n - \varrho_0 \right) \Rightarrow \zeta = \mathbb{I}_{\varpi}^{-1} \int_{\mathbb{R}} \frac{\ell(s)}{\lambda_0(s)} \, dW \left( \Lambda_0 \left( s \right) \right), \]

where the vector \( \ell(s) = -\lambda_0'(s) (1, s)^T \) (see (5)). This representation, which we prove below, allows us to say what is the correlation between \( W \left( \Lambda_0 \left( s \right) \right) \) and \( \zeta \):

\[
\mathbb{E} W \left( \Lambda_0(t) \right) \zeta = \mathbb{E} \left[ W \left( \Lambda_0(t) \right) \mathbb{I}_{\varpi}^{-1} \int_{\mathbb{R}} \frac{\ell_0(s)}{\lambda_0(s)} \, dW \left( \Lambda_0 \left( s \right) \right) \right] \\
= \mathbb{I}_{\varpi}^{-1} \int_{-\infty}^{t} \frac{\ell_0(s)}{\lambda_0(s)} \, d\Lambda_0(s) = \mathbb{I}_{\varpi}^{-1} \int_{-\infty}^{t} \ell_0(s) \, ds.
\]

Let us return to the proof of the asymptotic normality of the MLE. The random field \( Z_n(v), v \in \mathbb{V}_n \) we extend on the whole plane \( \mathbb{R}^2 \) continuously decreasing to zero outside of \( \mathbb{V}_n \). Denote \( (\mathcal{C}_0(\mathbb{R}^2), \mathcal{B}) \) the measurable space of the continuous random surfaces tending to zero at infinity with the uniform metrics and Borelian \( \sigma \)-algebra. Introduce the measures \( Q_n \) and \( Q \) induced by the realizations of \( Z_n(\cdot) \) and \( Z(\cdot) \) in the space \( (\mathcal{C}_0(\mathbb{R}^2), \mathcal{B}) \) respectively. Suppose that we already proved the weak convergence

\[ Q_n \Rightarrow Q. \]  

Then we have the convergence of the distributions of the continuous functionals \( \Psi \left( Z_n \right) \) to the distribution of \( \Psi \left( Z \right) \). Consider a convex set \( \mathcal{B} \subset \mathbb{R}^2 \).
We can write

\[
Q_n \left( \sqrt{n} \left( \hat{\vartheta}_n - \vartheta_0 \right) \in \mathbb{B} \right) \\
= Q_n \left( \sup_{\sqrt{n}(\vartheta - \vartheta_0) \in \mathbb{B}} L(\vartheta, X^n) > \sup_{\sqrt{n}(\vartheta - \vartheta_0) \notin \mathbb{B}} L(\vartheta, X^n) \right) \\
= Q_n \left( \sup_{\sqrt{n}(\vartheta - \vartheta_0) \in \mathbb{B}} \frac{L(\vartheta, X^n)}{L(\vartheta_0, X^n)} > \sup_{\sqrt{n}(\vartheta - \vartheta_0) \notin \mathbb{B}} \frac{L(\vartheta, X^n)}{L(\vartheta_0, X^n)} \right) \\
= Q_n \left( \sup_{v \in \mathbb{B}} Z_n(v) > \sup_{v \notin \mathbb{B}} Z_n(v) \right) \rightarrow Q \left( \sup_{v \in \mathbb{B}} Z(v) > \sup_{v \notin \mathbb{B}} Z(v) \right) \\
= Q \left( \mathbb{I}(\vartheta_0)^{-1} S(\vartheta_0) \in \mathbb{B} \right).
\]

Note that \( \psi(z) = \sup_{v \in \mathbb{B}} z(v) - \sup_{v \notin \mathbb{B}} z(v) \) is a continuous functional on the space \((C_0(\mathbb{R}^2), \mathcal{B})\). The random function \( Z(\cdot) \) takes its maximum at the point \( \hat{\vartheta} = \mathbb{I}(\vartheta_0)^{-1} S(\vartheta_0). \) To prove the joint convergence in distribution of the vector \( W_{k,n} \) and \( \hat{v}_n = \sqrt{\frac{n}{\beta_0}} \left( \hat{\vartheta}_n - \vartheta_0 \right) \) we denote \( R_n = (W_{k,n}, Z_n(v), v \in \mathbb{R}^2) \) introduce the product space \( \mathcal{X} = \mathcal{R}^k \times C_0(\mathbb{R}^2) \) with the corresponding Borelian \( \sigma \)-algebra \( \mathcal{B}_\mathcal{X} \). To verify the weak convergence \( R_n \Rightarrow R \), where \( R = (W_k, Z(v), v \in \mathbb{R}^2) \) we

a) prove the convergence of the finite-dimensional distributions

\[
\left( W_{k,n}, Z_n(v_1), \ldots, Z_n(v_m) \right) \Rightarrow \left( W_k, Z(v_1), \ldots, Z_n(v_m) \right)
\]

b) prove the tightness of the corresponding family of measures.

The convergence a) follows from the LAN (17), (18). The prove of b) is a part of the Theorem 1.10.1 in [9]. The conditions \( \mathcal{B} \) are sufficient for the verification of the conditions \( \textbf{B1-B5} \) of the Theorem 1.10.1 in [9]. Therefore we obtain the joint asymptotic normality of the vector \( \left( W_{k,n}, \hat{v}_n \right) \Rightarrow \left( W_k, \zeta \right). \)

Hence we obtain the convergence of the finite-dimensional distributions (14). Let us check \( \textbf{B} \). We have

\[
\hat{u}_n(t_1) - \hat{u}_n(t_2) = W_n(t_1) - W_n(t_2) + \langle \hat{v}_n, \hat{\Lambda}(\vartheta_0, t_1) - \hat{\Lambda}(\vartheta_0, t_2) \rangle.
\]
Hence \((t_1 < t_2)\)
\[
E_{\vartheta_0} \left| W_n(t_1) - W_n(t_2) \right|^2 \\
= E_{\vartheta_0} \left( \frac{1}{\sqrt{n}} \sum_{j=1}^{n} [X_j(t_1) - X_j(t_2) - \Lambda_0(\vartheta_0, t_1) + \Lambda_0(\vartheta_0, t_2)] \right)^2 \\
= \int_{t_1}^{t_2} \lambda_0(\vartheta_0, t) \, dt \leq C |t_2 - t_1|.
\]

For the second term we have
\[
E_{\vartheta_0} \left| \langle \hat{v}_n, \hat{\Lambda}(\vartheta_0, t_1) - \hat{\Lambda}(\vartheta_0, t_2) \rangle \right|^2 \leq E_{\vartheta_0} \left| \hat{v}_n \right|^2 \left| \hat{\Lambda}(\vartheta_0, t_1) - \hat{\Lambda}(\vartheta_0, t_2) \right|^2 \\
\leq C |t_2 - t_1|^2 \leq C |t_2 - t_1|.
\]

The inequality \(C\) follows from the similar estimates.

\[
\int_{t>L} E_{\vartheta_0} \left| W_n(t) \right|^2 \lambda_0(\vartheta_0, t) \, dt = \int_{t>L} \Lambda_0(\vartheta_0, t) \lambda_0(\vartheta_0, t) \, dt \\
< CA_0(\infty) [A_0(\infty) - A_0(L)] \leq \delta
\]

because \(A_0(\infty) - A_0(L) \to 0\) as \(L \to \infty\).

The verification of \(D\) easily follows from the given above estimates.

Now the convergence \(\hat{\Delta}_n \Rightarrow \Delta_0\) is proved as follows. For any \(d > 0\) we take \(L > 0\) such that the estimate \((\ref{15})\) holds. The properties \(A-C\) according to the Theorem A.22 in \([2]\) allow us to obtain the convergence

\[
\int_{-L}^{L} \hat{u}_n(t)^2 \lambda_0(\vartheta_0, t) \, dt \Rightarrow \int_{-L}^{L} [W(\Lambda_0(s)) - \langle \zeta, V(s) \rangle]^2 \lambda_0(s) \, ds.
\]

Therefore we proved that

\[
\int_{\mathbb{R}} \hat{u}_n(t)^2 \lambda_0(\vartheta_0, t) \, dt \Rightarrow \int_{\mathbb{R}} [W(\Lambda_0(s)) - \langle \zeta, V(s) \rangle]^2 \lambda_0(s) \, ds.
\]

Hence the test \(\Psi_n \in \mathcal{K}_\varepsilon\).

There are several related problems which can be solved using such approach. For example, in the case of periodic Poisson process with unknown phase and frequency we have once more the model with shift and scale parameters, but there is an essential difference too. The rate of convergence of the estimate of the frequency is \(n^{3/2}\).

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