A contribution to condition numbers of the multidimensional total least squares problem with linear equality constraint

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Abstract

This paper is devoted to condition numbers of the multidimensional total least squares problem with linear equality constraint (TLSE). Based on the perturbation theory of invariant subspace, the TLSE problem is proved to be equivalent to a multidimensional unconstrained weighted total least squares problem in the limit sense. With a limit technique, Kronecker-product-based formulae for normwise, mixed and componentwise condition numbers of the minimum Frobenius norm TLSE solution are given. Compact upper bounds of these condition numbers are provided to reduce the storage and computation cost. All expressions and upper bounds of these condition numbers unify the ones for the single-dimensional TLSE problem and multidimensional total least squares problem. Some numerical experiments are performed to illustrate our results.

Keywords multidimensional total least squares problem with linear equality constraint; multidimensional total least squares problem; condition number.

AMS subject classifications 65F35, 65F20

1 Introduction

The multidimensional total least square (TLS) model, which arises in many data fitting and estimation problems, finds a “best” fit to the overdetermined system \( Ax \approx B \), where \( A \in \mathbb{R}^{q \times n} (q > n) \) and \( B \in \mathbb{R}^{q \times d} \) are contaminated by some noise. It determines perturbations \( E \) to the coefficient matrix \( A \) and \( F \) to the matrix \( B \) measured by Frobenius norm such that

\[
\min_{E,F} \| [E \ F] \|_F, \quad \text{subject to} \quad (A + E)X = B + F. \tag{1.1}
\]

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After the minimizer $[\hat{E} \ \hat{F}]$ is found such that the corrected system $(A + \hat{E})X = B + \hat{F}$ is consistent, the corresponding solution $X$ is called the TLS solution. The TLS model, was originally proposed in 1901 for data fitting problem [29], but hasn’t caught much attention for a long time. In 1980, Golub and Van Loan [12] introduced this model into the numerical linear algebra area. Since then, it has been attracting more and more attention and now the TLS model is widely applied in a broad class of scientific disciplines such as system identification [18], image processing [27, 28], speech and audio processing [15, 19], etc. An overview of applications, theory, and computational methods of the TLS problem, we refer to [12, 25, 36, 37, 43].

An extension of TLS model is the following multidimensional TLS problem with equality constraint (TLSE):

$$\min_{E,F} \|E \ F\|_F, \quad \text{subject to} \quad (A + E)X = B + F, \ CX = D,$$

where $D \in \mathbb{R}^{n \times d}$ and $C \in \mathbb{R}^{p \times n}$ is of full row rank. When $d = 1$, it reduces to the single-dimensional TLSE problem, which was first presented by Dowling, Degroat, and Linebarger [8] in 1992, where a stable algorithm on the basis of QR and singular value decomposition (SVD) matrix factorizations were proposed. Further investigations on the single-dimensional TLSE were performed in [31], where iteration methods were derived based on the Euler-Lagrange theorem. Recently, Liu et al. [23] investigated uniqueness conditions of the single-dimensional TLSE solution and interpreted the solution as an approximation of the solution to an unconstrained weighted TLS problem (WTLS), with a large weight assigned on the constraint, based on which a QR-based inverse iteration method was presented.

The sensitive analysis and the condition number of a problem are vital in numerical analysis, since the condition number measures the worst-case sensitivity of its solution to small perturbations in the input data. Combined with backward error estimate, an approximate upper bound can be derived for the forward error.

When $C, D$ are zero matrices and $d = 1$, the TLSE problem becomes the standard single dimensional TLS problem, whose first order perturbation analysis and condition numbers have been widely studied [1, 6, 7, 16, 20, 42, 44]. The condition number of the truncated TLS solution of an ill-conditioned TLS problem was studied by Gratton, Tingley-Peloquin, and Ilunga [11], Meng, Diao and Bai [24]. By making use of the perturbation results in [1, 16, 20], and the close relation of the single dimensional TLSE to an unconstrained weighted TLS problems, Liu and Jia [22] derived closed formulae for condition numbers of the single dimensional TLSE problem. Further perturbation results were given in [21], which provides perturbation analysis and tighter bounds for the forward error of the solution, when the perturbation in input data are of different magnitude. The condition numbers and perturbation results in [21, 22] unify those for standard TLS problem [16, 20, 42, 44]. When $C, D$ are nonzero matrices and $d = 1$, under some condition (see (2.12) with $\tilde{\sigma}_{n-p+1} = 0$), the TLSE solution reduces to a solution to the least squares problem with equality constraint (LSE), whose perturbation results were studied in [4, 5, 40], that are also unified by the ones [21] for the TLSE problem.

When $C, D$ are zero matrices and $d > 1$, the TLSE problem becomes the multidimensional TLS problem. In [45], Zheng, Meng and Wei studied the explicit formulae for the condition numbers of the minimum Frobenius norm TLS solution. The condition numbers of multidimensional TLS problem with more than one solution were further studied in [26] by Meng, Zheng and Wei.
To the best of our knowledge, condition numbers of the multidimensional TLSE problem haven’t been addressed in literature. In this paper, we aim to study this issue. With the invariant subspace perturbation theorem, we prove that it is equivalent to a multidimensional weighted TLS problem, with a large weight assigned on the constraint. By making use of the perturbation estimates in [45] for the multidimensional TLS problem, we establish the first order perturbation estimates of the minimum Frobenius norm TLSE solution based on a limit technique, from which Kronecker-product-based normwise, mixed and componentwise condition numbers formulae are derived. In order to reduce the storage and computation cost in these Kronecker-product-based formulae, compact upper bounds of these condition numbers are given. The newly derived results unify those for multidimensional TLS and single-dimensional TLSE problems. Numerical examples are provided to show their tightness.

Throughout the paper, $\| \cdot \|_2$ denotes the Euclidean vector or matrix norm, $I_n$, $0_n$, $0_{m \times n}$ denote the $n \times n$ identity matrix, $n \times n$ zero matrix, and $m \times n$ zero matrix, respectively. If subscripts are ignored, the sizes of identity and zero matrices are suitable with context. For a matrix $M \in \mathbb{R}^{m \times n}$, $M^T$, $M^*$, $\mathcal{R}(M)$, $\sigma(M)(\sigma_{\text{min}}(M))$, $\|M\|_{\text{max}}$ denote the transpose, Moore-Penrose inverse, the column range space, the $i$-th largest (the smallest) singular value, the maximal absolute value of elements of $M$, respectively. vec$(M)$ is an operator, which stacks the columns of $M$ one underneath the other. For matrices $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{s \times t}$, the Kronecker product of $A,B$ is defined by $A \otimes B = [a_{ij} B]$ and its property is listed as follows [14,17]:

$$\begin{align*}
\text{vec}(AXB) &= (B^T \otimes A)\text{vec}(X), \quad (A \otimes B)(C \otimes D) = (AC) \otimes (BD), \\
(A \otimes B)^T &= A^T \otimes B^T, \quad (A \otimes B)^\dagger = A^\dagger \otimes B^\dagger, \quad \|A \otimes B\|_2 = \|A\|_2 \|B\|_2, \\
\text{vec}(A^T) &= \Pi_{(m,n)}\text{vec}(A), \quad \Pi_{(m,n)}(A \otimes B) = (B \otimes A)\Pi_{(t,n)},
\end{align*}$$

where $X \in \mathbb{R}^{n \times s}$, $C \in \mathbb{R}^{n \times k}$, $D \in \mathbb{R}^{t \times r}$ and $\Pi_{(m,n)}$ is an $mn \times mn$ vec-permutation matrix taking the form $\Pi_{(m,n)} = \sum_{i=1}^{m} \sum_{j=1}^{n} E_{ij} \otimes E_{ij}^T$, in which $E_{ij} \in \mathbb{R}^{m \times n}$ has an entry in position $(i, j)$ and all other entries are zero.

2 Preliminaries

In this section we first recall some well known results about multidimensional TLS and single-dimensional TLS problems, after which we give solvability conditions and explicit form for the multidimensional TLS solution.

2.1 The first order perturbation estimate for multidimensional TLS problems

Let $L \in \mathbb{R}^{m \times n}$, $H \in \mathbb{R}^{m \times d}$ ($m \geq n + d$), the multidimensional TLS problem is defined by

$$\min_{E,F} \|[E \quad F]\|_F, \quad \text{s.t.} \quad (L + E)X = H + F. \tag{2.1}$$

Following [12], the TLS problem (2.1) may have no solutions. In order to broad its scope of applications, the generic and nongeneric conditions for TLS solutions were further studied by Van Huffel and Vandewalle [35, 38]. In 1992, Wei [39] redefined the conditions (see Eq. (2.3)) to make the TLS problem (2.1) meaningful in any situation. The condition in (2.3) includes those in [12, 35, 38] as special cases.
SVD is a useful tool to characterize the TLS solution. If the skinny SVD Chapter 2.4 of [L H] is given by
\[
[L \ H] = U \Sigma V^T, \quad \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{n+d}) \in \mathbb{R}^{(n+d) \times (n+d)},
\]
where \(\sigma_i = \sigma_i([L \ H])\) and \(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n+d} \geq 0, U \in \mathbb{R}^{m \times (n+d)} \) and \(V \in \mathbb{R}^{(n+d) \times (n+d)}\) have orthonormal columns. For an integer \(t \in [0, n]\), partition
\[
V = \begin{bmatrix}
V_{11}(t) & V_{12}(t) \\
V_{21}(t) & V_{22}(t)
\end{bmatrix}.
\]
For simplicity, we denote \(V_{ij} = V_{ij}(k)\) for \(i, j = 1, 2\). If
\[
\sigma_i > \sigma_{i+1}, \quad \text{rank}(V_{22}) = d,
\]
holds, a solution to the consistent linear system \(\hat{L}X = \hat{H}\) is defined as a TLS solution to the linear approximation equation \(LX \approx H\), where \(\hat{L} = U_1 \Sigma_1 V_{11}^T\) and \(\hat{H} = U_1 \Sigma_1 V_{21}^T\) with \(U_1, V_1\) being, respectively, the first \(t\) columns of \(U = [U_1 \ U_2]\) and \(V = [V_1 \ V_2]\). The diagonal matrices \(\Sigma_1 = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_t)\) and \(\Sigma_2 = \text{diag}(\sigma_{t+1}, \sigma_{t+2}, \ldots, \sigma_{n+d})\). Among all TLS solutions, the minimum Frobenius norm solution to the compatible system is given by \(X_t = -V_2 \Sigma_2^T\).

In [45], Zheng, Meng and Wei defined the mapping \(\phi: \mathbb{R}^{m(n+d)} \to \mathbb{R}^{nd}\) by \(\phi(c) = \text{vec}(X_t)\) for \(c = \text{vec}([L \ H])\) and provided the first order perturbation analysis of \(\phi(c)\) as
\[
\text{vec}(\Delta X_t) = \phi'(c) \text{vec}(\Delta L \ \Delta H) + \phi'(|\Delta L||\Delta H|) + \phi'(\|\Delta L\|_F^2 + \|\Delta H\|_F^2) = (H_1 + H_2)DZ\text{vec}(\Delta L \ \Delta H) + \phi'(|\Delta L||\Delta H|) + \phi'(\|\Delta L\|_F^2 + \|\Delta H\|_F^2),
\]
where
\[
H_1 = \left( (V_{22} V_{22}^T)^{-1} V_{21} \otimes (V_{12} F_{V_{22}}) \right) , \quad H_2 = \left( V_{22} V_{22}^T \Pi_{(n+d-t,t)} \right),
\]
\[
D = \left( \Sigma_1^2 \otimes I_{n+d-t} - I_t \otimes \Sigma_2 \right)^{-1} \left( L_t \otimes I_{n+d-t} \right), \quad \Sigma_1 \otimes I_{n+d-t},
\]
\[
Z = \begin{bmatrix}
V_1^T \otimes U_1^T \\
\Pi_{(t,n+d-t)}(V_2^T \otimes U_1^T)
\end{bmatrix},
\]
in which \(\Pi_{(n+d-t,t)}\) is a vec-permutation matrix, \(F_{V_{22}} = I - V_{22}^T V_{22}\). From this result, the absolute normwise condition number \(\kappa_{\text{abs}}(X_t, L, H)\) satisfies
\[
\kappa_{\text{abs}}(X_t, L, H) = \|(H_1 + H_2)D\|_2 \leq (1 + \|X_t\|_2^2) \sqrt{\frac{\sigma_i^2 + \sigma_{i+1}^2}{\sigma_i^2 - \sigma_{i+1}^2}},
\]
where the upper bound is proved to be optimal and is attainable for some specific matrices. In particular, for \(t = n\),
\[
\frac{\sqrt{\sigma_n^2 + \sigma_{n+1}^2}}{\|V_{11}\|_2 \|V_{22}\|_2 (\sigma_n^2 - \sigma_{n+1}^2)} \leq \kappa_{\text{abs}}(X_n, L, H) \leq (1 + \|X_t\|_2^2) \sqrt{\frac{\sigma_n^2 + \sigma_{n+1}^2}{\sigma_n^2 - \sigma_{n+1}^2}}.
\]
2.2 Solvability conditions and explicit solution of multi-dimensional TLSE problem

For the multidimensional TLSE problem (1.2), denote \( \tilde{A} = [A \ B], \tilde{C} = [C \ D] \), and assume that the QR factorization of \( \tilde{C}^T \) takes the form:

\[
\tilde{C}^T = \tilde{Q} \begin{bmatrix} \tilde{R}_1 \\ 0 \end{bmatrix}, \quad \tilde{Q} = [\tilde{Q}_1 \ \tilde{Q}_2],
\]

in which \( \tilde{Q}_1 \in \mathbb{R}^{(n+d) \times p} \), \( \tilde{Q}_2 \in \mathbb{R}^{(n+d) \times (n+d-p)} \). Let the skinny SVD of \( \tilde{A}\tilde{Q}_2 \) as

\[
\tilde{A}\tilde{Q}_2 = \tilde{U}\Sigma\tilde{V}^T = [\tilde{U}_1 \ \tilde{U}_2] \begin{bmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & \tilde{\Sigma}_2 \end{bmatrix} [\tilde{V}_1 \ \tilde{V}_2]^T,
\]

where the matrices \( \tilde{U}_1, \tilde{U}_2 \) have, respectively, \( k, (n+d-k-p) \) orthonormal columns. \( \tilde{V}_1 \) is the orthonormal basis of the \( (n+d-p) \times (n+d-p) \) orthogonal matrix \( \tilde{V} \) by taking their first \( k \) columns. The diagonal matrices \( \tilde{\Sigma} = \text{diag}(\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_{n+d-p}), \tilde{\Sigma}_1 = \text{diag}(\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_k), \tilde{\Sigma}_2 = \text{diag}(\tilde{\sigma}_{k+1}, \tilde{\sigma}_{k+2}, \ldots, \tilde{\sigma}_{n+d-p}) \), in which \( 0 \leq k \leq n-p \) is an integer such that

\[
\mathcal{C}(k) : \quad \tilde{\sigma}_1 \geq \tilde{\sigma}_2 \geq \ldots \geq \tilde{\sigma}_k > \tilde{\sigma}_{k+1} \geq \ldots \geq \tilde{\sigma}_{n+d-p}.
\]

For \( d = 1, x_C = C^T D \), and \( r_C = Ax_C - B \), in \( [23] \), Liu et al. proved that if the orthonormal basis of null space of \( \tilde{C} \) is chosen as

\[
\tilde{Q}_2 = \begin{bmatrix} Q_2 & \beta^{-1}x_C \\ 0 & -\beta^{-1} \end{bmatrix}, \quad \beta = (1 + \|x_C\|_2^2)^{1/2},
\]

in which \( Q_2 \) is the orthonormal basis of the null space of \( C \), then under the condition

\[
\sigma_{n-p}(A\tilde{Q}_2) > \sigma_{n-p+1}(A\tilde{Q}_2^T) = \sigma_{n-p+1}(\tilde{A}\tilde{Q}_2) = \tilde{\sigma}_{n-p+1},
\]

the TLSE solution \( x_n \) is unique and can be expressed by

\[
x_n = x_C - \mathcal{K}A^T r_C, \quad \text{for} \quad \mathcal{K} = Q_2(Q_2^T A^T A\tilde{Q}_2 - \tilde{\sigma}_{n-p+1}^2 I_{n-p})^{-1} Q_2^T.
\]

In the following theorem, we give the solvability conditions and explicit form of the solution to the multidimensional case.

**Theorem 2.1** With the notation in (2.7)–(2.9), let \( t = p + k \) and \( \overline{V} = \tilde{Q}_2\tilde{V}_2 \) have the partition

\[
\overline{V} = \tilde{Q}_2\tilde{V}_2 = \begin{bmatrix} \overline{V}_1 & \overline{V}_2 \end{bmatrix} = \begin{bmatrix} \overline{V}_{11} & \overline{V}_{12} \\ \overline{V}_{21} & \overline{V}_{22} \end{bmatrix} = \begin{bmatrix} \overline{V}_1 & \overline{V}_2 \\ \overline{V}_2 & \overline{V}_2 \end{bmatrix}.
\]

If for \( k = n - p \), the condition \( \mathcal{C}(k) \) holds such that \( \overline{V}_{22} \) is nonsingular, then the unique TLSE solution is determined by \( x_n = -\overline{V}_{12}\overline{V}_{22}^{-1}, \) which is also the solution the consistent linear system

\[
\tilde{A}X = \tilde{B}, \quad \text{subject to} \quad CX = D,
\]

(2.14)
where

$$\hat{A} = \hat{U}_1\hat{\Sigma}_1 V_{11}^T, \quad \hat{B} = \hat{U}_1\hat{\Sigma}_2 V_{21}^T.$$  \hfill (2.15)

**Proof.** Let \( \tilde{X} = [X^T - I_d]^T \). Notice that the constraint \( CX = D \) requires \( C\tilde{X} = 0 \), therefore \( \tilde{X} \) lies in the null space of \( \tilde{C} \) spanned by \( \tilde{Q}_2 \). Denote \( \tilde{X} = \tilde{Q}_2 Z \), and write \( A = \tilde{A}\tilde{Q}_1^T + \tilde{A}\tilde{Q}_2^T \), \( E = [E \ F] \). (2.2) becomes

$$\min \| [\hat{E}\tilde{Q}_1 \quad \hat{E}\tilde{Q}_2] \|_F, \quad \text{s.t.} \quad (\hat{A}\tilde{Q}_2 + \hat{E}\tilde{Q}_2)Z = 0,$$

where the restriction only imposed on \( E\tilde{Q}_2 \) means that we can choose optimal \( \hat{E}_s \) such that \( \hat{E}_s\tilde{Q}_1 = 0 \) and \( \hat{A}\tilde{Q}_2 + \hat{E}_s\tilde{Q}_2 \) has a null space with dimension no less than \( d \).

Note that the condition \( \hat{E}_s\tilde{Q}_1 = 0 \) means there exists a matrix \( Y \) such that \( \hat{E}_s^T = \tilde{Q}_2 Y^T \), and (2.16) becomes

$$\min_{\text{rank}(\hat{A}\tilde{Q}_2 + Y) \leq n - p} \| Y \|_F, \quad \text{s.t.} \quad (\hat{A}\tilde{Q}_2 + Y)Z = 0.$$

According to (2.8) and the well-known Eckart-Young theorem [13, Theorem 2.4.8] for the best rank-\((n - p)\) matrix approximation, the optimal \( Y_s \) satisfies \( Y_s = -\hat{U}_2\hat{\Sigma}_2\hat{V}_2^T \), and for this optimal error matrix \( \hat{E}_s = Y_s\tilde{Q}_2 \), the corrected system becomes

$$\hat{A}\tilde{Q}_2 - \hat{U}_2\hat{\Sigma}_2\hat{V}_2^T)Z = 0, \quad \text{or} \quad \hat{U}_1\hat{\Sigma}_1(\tilde{Q}_2\hat{V}_1)^T \tilde{X} = 0.$$

Recall that \( \tilde{X} \in \mathcal{R}(\tilde{Q}_2) \), therefore \( \tilde{X} \) lies in the range of \( \tilde{V}_2 = \tilde{Q}_2\hat{V}_2 \), i.e., there exists an \( d \times d \) matrix \( G \) such that

$$\begin{bmatrix} X \\ -I_d \end{bmatrix} = \begin{bmatrix} \tilde{V}_{12} \\ \tilde{V}_{22} \end{bmatrix} G,$$

from which we obtain \( G = -\tilde{V}_{22}^{-1} \) and the unique solution is given by \( X_n = -\tilde{V}_{12}\tilde{V}_{22}^{-1} \). \hfill \Box

**Remark 2.1** If the condition \( \mathcal{C}(k) \) only holds for \( k < n - p \) such that the right bottom partition \( \tilde{V}_{22} \) in (2.13) is of full row-rank, we define a solution to the linear system (2.14)-(2.15) as a TLSE solution. In this case, \( CX = D \) requires that \( \begin{bmatrix} X \\ -I_d \end{bmatrix} \in \mathcal{R}(\tilde{Q}_2) = \mathcal{R}(\tilde{V}) \), and at the same time we notice that

$$\begin{bmatrix} \hat{A} & \hat{B} \end{bmatrix} \begin{bmatrix} X \\ -I_d \end{bmatrix} = \hat{U}\hat{\Sigma}\hat{V}_1^T \begin{bmatrix} X \\ -I_d \end{bmatrix} = 0,$$

therefore \( \begin{bmatrix} X \\ -I_d \end{bmatrix} \) lies in the range space \( \mathcal{R}(\tilde{V}_2) \) and there exists an matrix \( G \) such that (2.17) holds. From the relation \( \tilde{V}_{22}G = -I_d \), we conclude that \( G = -\tilde{V}_{22}^T + PK \) for \( P = I_{n+d-t} - \tilde{V}_{22}^T\tilde{V}_{22} \) and an arbitrary \( (n + d - t) \times d \) matrix \( K \). Therefore any TLSE solution \( X \) has the form

$$X = -\tilde{V}_{12}\tilde{V}_{22} + \tilde{V}_{12}PK,$$

in which

$$\left( \tilde{V}_{12}\tilde{V}_{22} \right)^T \tilde{V}_{12}P = \tilde{V}_{12}^T \tilde{V}_{12}\tilde{V}_{12}P = \tilde{V}_{22}^T (I - \tilde{V}_{22}^T\tilde{V}_{22}) P = 0,$$

and \( X_i = -\tilde{V}_{12}\tilde{V}_{22}^T \) is the minimum Frobenius norm solution among all TLSE solutions.
3 Close relation of TLSE to an unconstrained weighted TLS problem

When \( d = 1 \), Liu et al. [23] proved that under the genericity condition (2.11), the unique solution of the single dimensional TLSE problem can be interpreted as an approximation of the solution to an unconstrained weighted TLS problem, by assigning a large weight on the constraint.

Similar conclusions can be drawn for the multidimensional case. However, in proving this assertion, we can’t mimic the technique in [23], since some singular values of \( \tilde{A}Q_2 \) characterized by \( \tilde{\Sigma}_2 \) might be multiple in the multidimensional case, and the associated singular vectors are not uniquely determined. We need to generalize Stewart’s result [33] about the asymptotic behavior for the scaled SVD of \( X_\varepsilon = [X_1 \quad \varepsilon X_2] \), based on the following perturbation theorem for invariant subspaces.

**Lemma 3.1** [34] Chp. V, Thm 2.7 Let \( [Z_1 \quad Y_2] \in \mathbb{R}^{n \times n} \) be an orthogonal matrix and \( \mathcal{R}(Z_1) \) is a \( k \)-dimensional simple invariant subspace of \( n \times n \) matrix \( C \) such that

\[
[Z_1 \quad Y_2]^T C[Z_1 \quad Y_2] = \begin{bmatrix} L_1 & H \\ 0 & L_2 \end{bmatrix},
\]

where \( L_1 \) and \( L_2 \) have no common eigenvalues, \( Y_2^T C Z_1 = 0 \) (Here \( \mathcal{R}(Z_1), \mathcal{R}(Y_2) \) are called the right and left invariant subspace of \( C \)). Given a perturbation \( E \), let

\[
[Z_1 \quad Y_2]^T E[Z_1 \quad Y_2] = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}.
\]

Then for perturbations \( \|E\|_2 \) small enough, there is a unique matrix \( P \) such that the columns of

\[
\tilde{Z}_1 = (Z_1 + Y_2 P)(I + P^T P)^{-\frac{1}{2}}, \quad \tilde{Y}_2 = (Y_2 - Z_1 P)(I + P^T P)^{-\frac{1}{2}}
\]

form orthonormal bases for simple right and left invariant subspaces of \( \tilde{C} = C + E \). The representation of \( \tilde{C} \), i.e., \( \tilde{C} \tilde{Z}_1 = \tilde{Z}_1 \tilde{L}_1, \tilde{C} \tilde{Y}_2 = \tilde{Y}_2 \tilde{L}_2 \) with respect to \( \tilde{Z}_1, \tilde{Y}_2 \) are given by

\[
\tilde{L}_1 = (I + P P^T)^{-\frac{1}{2}}[L_1 + E_{11} + (H + E_{12})P](I + P^T P)^{-\frac{1}{2}}, \quad \tilde{L}_2 = (I + P P^T)^{-\frac{1}{2}}[L_2 + E_{22} - P(H + E_{12})](I + P P^T)^{-\frac{1}{2}}.
\]

**Lemma 3.2** Let \( \varepsilon > 0 \) be a small parameter, \( X = [X_1 \quad X_2] \in \mathbb{R}^{m \times n} \) with \( X_1 \in \mathbb{R}^{m \times k} \) being of full column-rank. Denote \( X_\varepsilon = [X_1 \quad \varepsilon X_2], \bar{X}_2 = X_2 - X_1 B \) with \( B = X_1^T X_2 \). Assume that \( X_1 = U_1 S_1 V_1^T, \bar{X}_2 = U_2 \bar{S}_2 V_2^T \) and \( X_\varepsilon = U_\varepsilon S_\varepsilon V_\varepsilon^T \) are the skinny SVDs of \( X_1, \bar{X}_2 \) and \( X_\varepsilon \), respectively, then

\[
S_\varepsilon = \text{diag}(S_1 + \mathcal{O}(\varepsilon^2), \varepsilon \bar{S}_2 + \mathcal{O}(\varepsilon^3)), \quad U_\varepsilon = \begin{bmatrix} U_1 + \mathcal{O}(\varepsilon^2) \\ \bar{U}_2 + \mathcal{O}(\varepsilon^2) \end{bmatrix}, \quad V_\varepsilon = \begin{bmatrix} V_1 + \mathcal{O}(\varepsilon^2) & -\varepsilon B \bar{V}_2 + \mathcal{O}(\varepsilon^3) \\ \varepsilon B^T V_1 + \mathcal{O}(\varepsilon^3) & \bar{V}_2 + \mathcal{O}(\varepsilon^2) \end{bmatrix}.
\]

**Proof.** Let \( G = [X_1 \quad 0_{m \times (n-k)}]^T [X_1 \quad 0_{m \times (n-k)}] \) and

\[
G_\varepsilon = X_\varepsilon^T X_\varepsilon = G + \begin{bmatrix} 0 & \varepsilon X_\varepsilon^T X_2 \\ \varepsilon X_\varepsilon^T X_1 & \varepsilon^2 X_\varepsilon^T X_2 \end{bmatrix} =: G + E
\]
be the perturbed version of $G$. Notice that
\[
[Z_1 \ Y_2] = \begin{pmatrix} k & n-k \\ k & n-k \end{pmatrix} \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}
\]
has orthonormal columns and $Z_1, Y_2$ form orthonormal bases of simple invariant subspaces of $G$ such that the representations of $G$ with respect to $Z_1, Y_2$ are
\[
GZ_1 = Z_1 L_1, \quad GY_2 = Y_2 L_2, \quad \text{for} \quad L_1 = S_1^T S_1, \quad L_2 = 0_{n-k}.
\]

By Lemma 3.1, there exists an $(n-k) \times k$ matrix $P$ such that $\tilde{Z}_1, \tilde{Y}_2$ with structure (3.1) form the orthonormal bases of right and left invariant subspaces of $G_{\epsilon}$, respectively. Substituting the matrices $Z_1, Y_2$ and formula (3.1) into the relation $\tilde{Y}_2^T G_{\epsilon} \tilde{Z}_1 = 0$, one can derive that $(Y_2 - Z_1 P^T) (G + E) (Z_1 + Y_2 P) = 0$. Using (3.3), we obtain
\[
P(S_1^T S_1) = Y_2^T EZ_1 - PZ_1^T EZ_1 + Y_2^T EY_2 P - PZ_1^T EY_2 P,
\]
from which we obtain
\[
P = \epsilon \nabla_2^T (X_1^T X_2)^T V_1 + O(\epsilon^3).
\]

From (3.1)-(3.2), $[\tilde{Z}_1 \ \tilde{Y}_2]$ has the following form
\[
[\tilde{Z}_1 \ \tilde{Y}_2] = \begin{pmatrix} V_1 + O(\epsilon^2) & -\epsilon B V_2 + O(\epsilon^3) \\ \epsilon B^T V_1 + O(\epsilon^3) & V_2 + O(\epsilon^3) \end{pmatrix},
\]
and the representations of $G_{\epsilon}$ with respect to $\tilde{Z}_1, \tilde{Y}_2$ are given by $L_1 = S_1^T S_1 + O(\epsilon^2)$, and
\[
L_2 = (E_{22} - PE_{12}) (1 + O(\epsilon^2)) = (Y_2^T EY_2 - PZ_1^T EY_2) (1 + O(\epsilon^2))
\]
\[
= \epsilon^2 \nabla_2^T [X_2^T X_1 - (X_1^T X_2)^T (X_1^T X_2)] V_2 + O(\epsilon^4)
\]
\[
= \epsilon^2 \nabla_2^T X_1^T X_2 V_2 + O(\epsilon^4) = \epsilon^{5/2} S_1 + O(\epsilon^4).
\]

Note that $G_{\epsilon} = X_2^T X_\epsilon$ is symmetric and has $\tilde{Z}_1$ and $\tilde{Y}_2$ as the bases of its right and left simple invariant subspaces such that $\tilde{Y}_2^T G_{\epsilon} \tilde{Z}_1 = 0$, therefore $H := [\tilde{Z}_1 \ \tilde{Y}_2]$ satisfies
\[
H^T X_\epsilon^T X_{\epsilon} H = \begin{pmatrix} S_1^T S_1 + O(\epsilon^2) & 0 \\ 0 & \epsilon^{25/2} S_2 + O(\epsilon^4) \end{pmatrix},
\]
in which the orthonormal columns of $V_\epsilon := [\tilde{Z}_1 \ \tilde{Y}_2]$ span the right singular subspace of $X_{\epsilon}$, with diagonal entries of $S_1 + O(\epsilon^2), \epsilon S_2 + O(\epsilon^3)$ as its singular values. It’s obvious that
\[
X_{\epsilon} [\tilde{Z}_1 \ \tilde{Y}_2] = [X_1 V_1 + O(\epsilon^2) \quad \epsilon X_2 V_2] + O(\epsilon^3) = [U_1 S_1 + O(\epsilon^2) \quad \epsilon U_2 S_2 + O(\epsilon^3)],
\]
from which we conclude that the left singular matrix $U_{\epsilon}$ of $X_{\epsilon}$ satisfies
\[
U_{\epsilon} = [U_1 + O(\epsilon^2) \quad U_2 + O(\epsilon^2)].
\]
The proof is then complete.
Theorem 3.3 For the multidimensional TLSE problem (1.2), with the notations in (2.7)–(2.13), assume that $V_{22}$ has full row rank, and the minimum Frobenius norm solution $X_i = -V_{12}V_{22}^\dagger$. Denote
\[
L_\varepsilon = W_\varepsilon^{-1}L = \begin{bmatrix} \varepsilon^{-1}C \\ A \end{bmatrix}, \quad H_\varepsilon = W_\varepsilon^{-1}H = \begin{bmatrix} \varepsilon^{-1}D \\ B \end{bmatrix},
\]
(3.4)
where $W_\varepsilon = \text{diag}(\varepsilon I_p, I_q)$ with $\varepsilon$ being a small positive parameter. Consider the multidimensional weighted TLS problem
\[
\min_{\hat{E}, \hat{F}} \| \bar{E} - \hat{F} \|_F \quad \text{subject to} \quad (L_\varepsilon + \hat{E})X_\varepsilon = H_\varepsilon + \hat{F},
\]
(3.5)
then the minimum Frobenius norm solution $X_{i(\varepsilon)}$ tends to $X_i$ as $\varepsilon$ tends to zero.

Proof: To prove the close relation of TLSE solution to the WTLS solution, we need to investigate the right singular vectors of $L_\varepsilon = [L_\varepsilon \quad H_\varepsilon]$ corresponding to small singular values, in which $L_\varepsilon$ has the same left singular vectors as $[\bar{C}^T \quad \varepsilon A^T]$, and their singular values are identical up to multiplication by $\varepsilon^{-1}$.

To apply Lemma 3.2, let $\bar{C}^T = VCSC_U^T$ be the skinny SVD of the full column-rank matrix $\bar{C}^T$, and the SVD of $A\bar{Q}_2^T$ be given by (2.8). It is obvious that
\[
(I_{n+d} - \bar{C}^T\bar{C}^{TT})\bar{A}^T = \bar{Q}_2\bar{Q}_2^T\bar{A}^T = (\bar{Q}_2\bar{V})\bar{\Sigma}^T\bar{U}^T = \bar{\Sigma}^T\bar{U}^T.
\]
By Lemma 3.2, we know that the left and right singular matrices $\bar{V}_\varepsilon$, $\bar{U}_\varepsilon$ of $[\bar{C}^T \quad \varepsilon \bar{A}^T]$ satisfies
\[
\bar{V}_\varepsilon = \begin{bmatrix} V_C + \varepsilon\bar{\sigma}(\varepsilon^2) \\ \bar{V} + \varepsilon\bar{\sigma}(\varepsilon^2) \end{bmatrix}, \quad \bar{U}_\varepsilon = \begin{bmatrix} p \\ n+d-p \end{bmatrix} \begin{bmatrix} \bar{P}_\varepsilon \left( U_C + \varepsilon\bar{\sigma}(\varepsilon^2) \right) \\ p \end{bmatrix}, \quad \bar{Q}_\varepsilon \begin{bmatrix} U + \varepsilon\bar{\sigma}(\varepsilon^2) \end{bmatrix},
\]
(3.6)
where $\bar{V} = \bar{Q}_2\bar{V}_2$, $\bar{P}_\varepsilon = \begin{bmatrix} I_p \\ \varepsilon(A\bar{C}^T)^T \end{bmatrix}$, $\bar{Q}_\varepsilon = \begin{bmatrix} -\varepsilon(\bar{AC}^T)^T \\ I_q \end{bmatrix}$,
(3.7)
and the corresponding singular values are just diagonal entries of $S_C + \varepsilon\bar{\sigma}(\varepsilon^2)$, $\varepsilon\bar{\Sigma} + \varepsilon\bar{\sigma}(\varepsilon^2)$. Therefore the SVD of $L_\varepsilon$ is given by $L_\varepsilon = \bar{U}_\varepsilon S_\varepsilon \bar{V}_\varepsilon^T$ for
\[
\bar{S}_\varepsilon = \text{diag}(\varepsilon^{-1}S_C + \varepsilon\bar{\sigma}(\varepsilon^2), \varepsilon\bar{\Sigma} + \varepsilon\bar{\sigma}(\varepsilon^2)),
\]
(3.8)
and the smallest $n+d-p$ singular values of $L_\varepsilon$ can be approximated by $\bar{\sigma}_i + \varepsilon\bar{\sigma}(\varepsilon^2)$ for $i = 1, \ldots, n+d-p$, and for sufficiently small $\varepsilon$,
\[
\bar{\sigma}_1 + \varepsilon\bar{\sigma}(\varepsilon^2) \geq \cdots \geq \bar{\sigma}_k + \varepsilon\bar{\sigma}(\varepsilon^2) > \bar{\sigma}_{k+1} + \varepsilon\bar{\sigma}(\varepsilon^2) \geq \cdots \geq \bar{\sigma}_{n+d-p} + \varepsilon\bar{\sigma}(\varepsilon^2),
\]
and the bottom right $d \times (n+d-t)$ submatrix in $\bar{V}_\varepsilon$ has full row rank. Therefore the minimum Frobenius norm WTLS solution $X_{i(\varepsilon)}$ to problem (3.5), in the limit, takes the form
\[
\lim_{\varepsilon \to 0^+} X_{i(\varepsilon)} = \lim_{\varepsilon \to 0^+} \left[ -\left( V_{12} + \varepsilon\bar{\sigma}(\varepsilon^2) \right)(V_{22} + \varepsilon\bar{\sigma}(\varepsilon^2)^T \right] = -V_{12}V_{22}^T,
\]
which is exactly $X_i$. The proof of the theorem then follows. \qed
4 Condition numbers of TLSE

Condition numbers measure the sensitivity of the solution to the original data in problems, and they play an important role in numerical analysis.

To evaluate the condition number of the multidimensional TLSE problem, let $m = p + q$, $[\hat{L} \ \hat{H}] = [L \ H] + [\Delta L \ \Delta H]$, where the perturbation $[\Delta L \ \Delta H]$ is sufficiently small. In order to derive the first order perturbation estimate of the TLSE solution, we define the mapping $\phi : \mathbb{R}^{m(n+d)} \rightarrow \mathbb{R}^{nd}$ for the multidimensional TLSE problem (1.2):

$$\phi(c) = \text{vec}(X_t), \quad c = \text{vec}([L \ H]).$$

Define the absolute normwise, relative normwise, mixed and componentwise condition numbers of $X_t$ as follows

$$\kappa_{\text{abs}}(X_t, L, H) = \lim_{\varepsilon \to 0} \sup \left\{ \frac{\|\Delta X_t\|_F}{\|\Delta L\|_F} : \|\Delta H\|_F \leq \varepsilon \|L \ H\|_F \right\},$$

$$\kappa_{\text{rel}}(X_t, L, H) = \lim_{\varepsilon \to 0} \sup \left\{ \frac{\|\Delta X_t\|_F}{\|X_t\|_F} : \|\Delta L\|_F \leq \varepsilon \|L \ H\|_F \right\},$$

$$m(X_t, L, H) = \lim_{\varepsilon \to 0} \sup \left\{ \max_{i,j} \frac{\|\Delta X_{ij}\|}{|X_{ij}|} : |\Delta L| \leq \varepsilon |L|, \quad |\Delta H| \leq \varepsilon |H| \right\},$$

$$c(X_t, L, H) = \lim_{\varepsilon \to 0} \sup \left\{ \frac{\|\Delta X_t\|_{\infty}}{X_t} : |\Delta L| \leq \varepsilon |L|, \quad |\Delta H| \leq \varepsilon |H| \right\},$$

where $| \cdot |$ denotes the componentwise absolute value, $Y \leq Z$ means $y_{ij} \leq z_{ij}$ for all $i, j$, and $\frac{Y}{Z}$ is the entry-wise division defined by $\frac{Y}{Z} := \frac{y_{ij}}{z_{ij}}$ and $\frac{\xi}{0}$ is interpreted as zero if $\xi = 0$ and infinity otherwise.

If $\text{vec}(X_t) = \phi(c)$ is continuous and Fréchet differentiable at the neighbourhood of the point $c$, according to the concept and formulae in [9, 10, 32], the above condition numbers can be formulated as follows:

$$\kappa_{\text{abs}}(X_t, L, H) = \|\phi'(c)\|_2, \quad \kappa_{\text{rel}}(X_t, L, H) = \frac{\|\phi'(c)\|_2}{\|\phi(c)\|_2} \|c\|_2,$$

$$m(X_t, L, H) = \frac{\|\phi'(c)\|_\infty}{\|\phi(c)\|_\infty} \|c\|_\infty, \quad c(X_t, L, H) = \frac{\|\phi'(c)\|_\infty}{\|\phi(c)\|_\infty}.$$

4.1 Normwise condition number

Notice that $\phi'(c)$ is vital for above condition numbers, while a simple and Fréchet differentiable expression of $\phi(c)$ is not easy to derive. To get $\phi'(c)$, as did in [22], we start from the differentiability of the weighted TLS solution $X_{t(e)}$ by defining the mapping for the multidimensional WTLS problem (3.4)-(3.5):

$$\text{vec}(X_{t(e)}) = \phi(c_e), \quad c_e = \text{vec}([L_{e} \ H_{e}]),$$

and then get the first order perturbation estimate $\text{vec}(\Delta X_{t(e)})$ of WTLS solution based on the result in (2.4), from which the first order perturbation estimate of the TLSE solution is derived by taking the limit $\varepsilon \to 0$. The idea of using limit technique to perform perturbation and condition number
analysis of a problem was also used by Wei and De Pierro \cite{9} \cite{11} for equality constrained least squares problem, and by Zheng and Yang \cite{46} for mixed least squares-total least squares problem.

**Lemma 4.1** \cite{45} Let $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ be an $n$-by-$n$ orthogonal matrix with a 2-by-2 partitioning, then

(a) $Q_{11}$ has full column (row) rank if and only if $Q_{22}$ has full row (column) rank;

(b) $\|Q_{11}\|_2 = \|Q_{22}\|_2$, $Q_{11}^T = Q_{11} - Q_{12}Q_{22}Q_{21}$, $Q_{11}^TQ_{21} = -Q_{12}Q_{22}$.

**Theorem 4.2** With the notation in \cite{27} \cite{28}, let the skinny SVD of $\tilde{C}$ be $\tilde{C} = UC\tilde{V}_C^T$ and assume that the condition (2.9) holds with the partition $\tilde{V}_{22}$ in (2.12) of full row rank. Denote

$$P = \begin{bmatrix} I_p \\ 0_{q \times p} \end{bmatrix}, \quad Q = \begin{bmatrix} -(AC^T)^T \\ I_q \end{bmatrix},$$

$$S_1 = \begin{bmatrix} SC & 0 \\ 0 & \Sigma_1 \end{bmatrix}, \quad \tilde{V}_1 = [V_C \quad V_1] = \frac{n}{p} d \begin{bmatrix} \tilde{V}_{11} \\ \tilde{V}_{21} \end{bmatrix}.$$

Then for sufficiently small perturbation $\|([\Delta L \quad \Delta H])_F$, the first order perturbation estimate for the minimum Frobenius norm TLSE solution $X_t = -\tilde{V}_{12}\tilde{V}_{22}$ takes the form

$$\text{vec}(\Delta X_t) = K \text{vec}([\Delta L \quad \Delta H]) + o(\|\Delta L \quad \Delta H\|_F^2),$$

where $K = (H_1 + H_2)\tilde{G}\tilde{Z}$ is exactly the Fréchet derivative $\phi'(c)$ and with $F_{\tilde{V}_{22}} = I - \tilde{V}_{12}\tilde{V}_{22}$.

1. $H_1 = \left(\tilde{V}_{22}\tilde{V}_{22}^T\right)^{-1}\tilde{V}_{21} = \tilde{V}_{12}K_{\tilde{V}_{22}}$, $H_2 = \left(\tilde{V}_{22}^T \tilde{V}_{11}^T\right)\Pi_{(n+d-t,t)}$,

2. $G = (S_2 \otimes I_{n+d-t} - (0_p \otimes I_k) \otimes (\Sigma_2 \otimes \Sigma_2))^{-1} (I_t \otimes \Sigma_2^2 \otimes I_{n+d-t})$,

$$\tilde{Z} = \frac{0_{(n+d) \times p}}{} \left[ \tilde{V}_{11}^T \otimes (Q\tilde{V}_2)^T \right].$$

**Proof.** Assume that the SVD of $\tilde{L}_e = [L_e \quad H_e] = \tilde{U}_e\tilde{S}_e\tilde{V}_e^T$ is given by (3.6)-(3.8), whose factors have partitions as

$$\tilde{U}_e = \begin{bmatrix} \tilde{U}_{1(e)} \\ \tilde{U}_{2(e)} \end{bmatrix} = \begin{bmatrix} P_eU_C \\ Q_e\tilde{U}_1 \end{bmatrix} = \begin{bmatrix} P_eU_C \\ Q_e\tilde{U}_2 \end{bmatrix} + o(e^2),$$

$$\tilde{V}_e = \frac{n}{d} \begin{bmatrix} \tilde{V}_{11(e)} \\ \tilde{V}_{21(e)} \end{bmatrix} = \begin{bmatrix} V_C \quad \tilde{V}_1 \end{bmatrix} = \begin{bmatrix} V_C \quad \tilde{V}_2 \end{bmatrix} + o(e^2),$$

$$\tilde{S}_{1(e)} = \text{diag}(e^{-1}S_C + o(e), \Sigma_1 + o(e^2)), \quad \tilde{S}_{2(e)} = \Sigma_2 + o(e^2),$$

in which $P_e, Q_e$ are given by (3.7).

By applying the result in \cite{24} for the WTLS problem \cite{34} \cite{35}, the first order perturbation estimate of the WTLS solution $X_t(e)$ satisfies

$$\text{vec}(\Delta X_{t(e)}) = \phi'(c_e)\text{vec}([\Delta L_e \quad \Delta H_e]) + o(\|\Delta L_e\|_F^2 + \|\Delta H_e\|_F^2),$$

(4.4)

By applying the result in \cite{24} for the WTLS problem \cite{34} \cite{35}, the first order perturbation estimate of the WTLS solution $X_t(e)$ satisfies

$$\text{vec}(\Delta X_{t(e)}) = \phi'(c_e)\text{vec}([\Delta L_e \quad \Delta H_e]) + o(\|\Delta L_e\|_F^2 + \|\Delta H_e\|_F^2),$$

(4.4)

$$\text{vec}(\Delta X_{t(e)}) = \phi'(c_e)\text{vec}([\Delta L_e \quad \Delta H_e]) + o(\|\Delta L_e\|_F^2 + \|\Delta H_e\|_F^2),$$

(4.4)
where

\[
H_1(\varepsilon) = \left( (\bar{V}_{22}(\varepsilon))^T \bar{V}_{22}(\varepsilon) \right)^{-1} \bar{V}_{21}(\varepsilon) \otimes (\bar{V}_{12}(\varepsilon) F_{\bar{V}_{22}(\varepsilon)}),
\]

\[
H_2(\varepsilon) = \left( \bar{V}_{22}(\varepsilon)^T \otimes \bar{V}_{11}(\varepsilon)^T \right) \Pi_{(n-d-t,t)},
\]

\[
D_\varepsilon = \bar{S}_{1}(\varepsilon) \otimes I_{n+d-t} - I_t \otimes (\bar{S}_{2}(\varepsilon))^T \tilde{S}_{2}(\varepsilon))^{-1} \left[ I_t \otimes \tilde{S}_{2}(\varepsilon) \tilde{S}_{1}(\varepsilon) \otimes I_{n+d-t} \right],
\]

\[
Z_\varepsilon = \begin{bmatrix}
\bar{V}_{1}(\varepsilon)^T \otimes \bar{U}_{2}(\varepsilon)^T \\
\Pi_{(t,n+d-t)}(\bar{V}_{2}(\varepsilon)^T \otimes \bar{U}_{1}(\varepsilon)^T)
\end{bmatrix},
\]

with \( F_{\bar{V}_{22}(\varepsilon)} = I - \bar{V}_{22}(\varepsilon)^T \bar{V}_{22}(\varepsilon) \).

In (4.2), denote \( Y_\varepsilon = D_\varepsilon Z_\varepsilon \text{vec}([\Delta L_\varepsilon \ \Delta H_\varepsilon]) \). Notice that \([\Delta L_\varepsilon \ \Delta H_\varepsilon] = W_\varepsilon^{-1} \begin{bmatrix} \Delta L & \Delta H \end{bmatrix} \) for \( W_\varepsilon = \text{diag}(\varepsilon I_p, I_q) \), therefore

\[
Y_\varepsilon = D_\varepsilon \begin{bmatrix}
\text{vec}\left( \bar{U}_{2}(\varepsilon)^T \Delta L_\varepsilon \Delta H_\varepsilon \bar{V}_{1}(\varepsilon) \right) \\
\text{vec}\left( \bar{V}_{2}(\varepsilon)^T \Delta L_\varepsilon \Delta H_\varepsilon^T \bar{U}_{1}(\varepsilon) \right)
\end{bmatrix} = D_\varepsilon \begin{bmatrix}
\text{vec}\left( \bar{U}_{2}(\varepsilon)^T W_\varepsilon^{-1} \Delta L \Delta H \bar{V}_{1}(\varepsilon) \right) \\
\text{vec}\left( \bar{V}_{2}(\varepsilon)^T W_\varepsilon^{-1} \Delta L \Delta H^T (W_\varepsilon^{-1} \bar{U}_{1}(\varepsilon)) \right)
\end{bmatrix}.
\]

Set \( \tilde{W}_\varepsilon = \text{diag}(\varepsilon I_p, I_q) \) and \( \tilde{S}_{1}(\varepsilon) = \bar{W}_\varepsilon \bar{S}_{1}(\varepsilon) \), then \( \tilde{S}_{1}(\varepsilon) = S_1 + \mathcal{O}(\varepsilon^2) \), and (4.6) becomes

\[
Y_\varepsilon = D_\varepsilon \begin{bmatrix}
\text{vec}\left( \bar{U}_{2}(\varepsilon)^T W_\varepsilon^{-1} \Delta L \Delta H \bar{V}_{1}(\varepsilon) \right) \\
\text{vec}\left( \bar{V}_{2}(\varepsilon)^T W_\varepsilon^{-1} \Delta L \Delta H^T (W_\varepsilon^{-1} \bar{U}_{1}(\varepsilon)) \right)
\end{bmatrix} = G_\varepsilon \tilde{Z}_\varepsilon \text{vec}([\Delta L \ \Delta H]),
\]

where

\[
D_\varepsilon = \left( \bar{S}_{1}(\varepsilon) \otimes I_{n+d-t} - \bar{W}_\varepsilon^2 \otimes (\tilde{S}_{2}(\varepsilon))^T \tilde{S}_{2}(\varepsilon))^{-1} \left[ \bar{W}_\varepsilon^2 \otimes \tilde{S}_{2}(\varepsilon) \tilde{S}_{1}(\varepsilon) \otimes I_{n+d-t} \right],
\]

\[
G_\varepsilon = \left( \bar{S}_{1}(\varepsilon) \otimes I_{n+d-t} - \bar{W}_\varepsilon^2 \otimes (\tilde{S}_{2}(\varepsilon))^T \tilde{S}_{2}(\varepsilon))^{-1} \left[ \bar{W}_\varepsilon^2 \otimes \tilde{S}_{2}(\varepsilon) \tilde{S}_{1}(\varepsilon) \otimes I_{n+d-t} \right],
\]

\[
\tilde{Z}_\varepsilon = \begin{bmatrix}
(\bar{V}_{1}(\varepsilon)^T \bar{W}_\varepsilon^2)^T \otimes (\bar{U}_{2}(\varepsilon)^T \bar{W}_\varepsilon^{-1}) \\
\Pi_{(t,n+d-t)}(\bar{V}_{2}(\varepsilon)^T \otimes \bar{W}_\varepsilon^{-1} \bar{U}_{1}(\varepsilon)^T \bar{W}_\varepsilon^{-1})
\end{bmatrix}.
\]

By the expressions in (4.3), and taking the limit \( \varepsilon \to 0 \) for \( H_{1(\varepsilon)}, H_{2(\varepsilon)}, G_\varepsilon \) and \( \tilde{Z}_\varepsilon \) in (4.5), we obtain the corresponding limit matrices \( H_1, H_2, \tilde{G}, \tilde{Z} \) as (4.2), and \( K = (H_1 + H_2) G \tilde{Z} \) is exactly the Fréchet derivative \( \phi'(c) \).

\[\square\]

**Theorem 4.3** With the notation in Theorem 4.2, the absolute and relative condition numbers of the minimum Frobenius norm TLSE solution \( X_t \) are given by

\[
\kappa^{\text{abs}}(X_t, L, H) = \|(H_1 + H_2) G \tilde{Z}\|_2, \quad \kappa^{\text{rel}}(X_t, L, H) = \|(H_1 + H_2) G \tilde{Z}\|_2 \frac{\|L \ H\|_F}{\|X_t\|_F},
\]

where

\[
\tilde{Z} = \text{diag}\left( \begin{bmatrix} 0_p & 0 \\ 0 & I_k \end{bmatrix} \otimes (\bar{U}_2^T Q)^T, \begin{bmatrix} I_p & 0 \\ -\bar{U}_1^T (\bar{A} \bar{C}^T) U_C & I_k \end{bmatrix} \otimes I_{n+d-t} \right).
\]

In particular, when \( k = n - p \), the term \( H_1 \) diminishes to zero, and \( H_2 = (\bar{V}_{22}^T \otimes \bar{V}_{11}^T)^T \Pi_{(d,n)} \).
Proof. By the condition number formulae, the absolute and relative condition numbers of the solution $X$, are given by

\[
\kappa_{\text{abs}}(X, L, H) = \|\phi'(c)\|_2 = \|K\|_2, \\
\kappa_{\text{rel}}(X, L, H) = \frac{\|\phi'(c)\|_2 \|c\|_2}{\|X\|_F} = \frac{\|K\|_2 \|L\ H\|_F}{\|X\|_F},
\]

in which

\[
\|K\|_2 = \|KK^T\|_2^{1/2} = \|(H_1 + H_2)G\tilde{Z}G^T(H_1 + H_2)\|_2^{1/2},
\]

for

\[
\tilde{Z}Z^T = \begin{bmatrix}
0_p & 0 & 0 \\
0 & I_k & 0 \\
0 & 0 & I_k
\end{bmatrix} \otimes (\tilde{U}_2 Q^T Q \tilde{U}_2) = ZZ^T,
\]

and

\[
M = \begin{bmatrix}
I_p & U_1^T \hat{P}^T U_1 \\
\hat{P}^T U_1 & \hat{P}^T U_1 & \hat{P}^T U_1 & 0
\end{bmatrix}, \\
\tilde{Z} = \text{diag} \left( \begin{bmatrix}
0_p & 0 \\
0 & I_k \\
0 & I_k
\end{bmatrix} \otimes (\tilde{U}_2 Q^T Q \tilde{U}_2) \right), \\
\Pi_{(t,n+d-t)} \left( I_{n+d-t} \otimes M \right) = \Pi_{(t,n+d-t)} \left( I_{n+d-t} \otimes (\tilde{U}_1 (\hat{A} \hat{C}^T) U_C) \right),
\]

Notice that $\Pi_{(n+d-t,t)}$ is an orthogonal matrix, then

\[
\|K\|_2 = \|(H_1 + H_2)G\tilde{Z}\|_2 = \|(H_1 + H_2)G\tilde{Z}\|_2.
\]

When $k = n - p$, $V_{22}$ is nonsingular and $X_n = -V_{12}V_{22}^{-1}$. Moreover, note that the SVDs of $\tilde{A} \hat{Q}_2 \hat{Q}_2^T = \tilde{U} \Sigma \tilde{V}^T$ and $\tilde{C} = U_C S_C V_C^T$ imply $\mathcal{R}(V_C) = \mathcal{R}^T(\tilde{C}) = \mathcal{R}(\tilde{Q}_1)$ and $\mathcal{R}(\tilde{V}) \subseteq \mathcal{R}(\tilde{Q}_2)$, therefore $V_C^T \bar{V} = 0$ and

\[
\hat{V} := [V_C \ V_\bar{V}] = \begin{bmatrix}
\hat{V}_{11} \\
\hat{V}_{12} \\
\hat{V}_{21} \\
\hat{V}_{22}
\end{bmatrix}
\]

is an $(n + d) \times (n + d)$ orthogonal matrix. According to Lemma 4.1(a), $\hat{V}_{11}$ is nonsingular. It follows that $H_1 = 0$, $H_2 = (V_{22}^T \otimes \hat{V}_{11}^{-T}) \Pi_{(d,n)}$. This completes the proof. \hfill \Box

Theorem 4.4 Let

\[
\rho_{\hat{A}C}^{(1)} = 1 + \|C\|_2 + \|\tilde{A} \hat{C}^T \tilde{C}\|_2, \\
\rho_{\hat{A}C}^{(2)} = 1 + \|\hat{C}^T\|_2 + \|\tilde{A} \hat{C}^T\|_2, \\
\eta^\sigma_k = \max \left\{ 1, \sqrt{\frac{\sigma_k^2 + \sigma_{k+1}^2}{\sigma_k^2 - \sigma_{k+1}^2}} \right\},
\]

then for the absolute normwise condition number, we have

\[
\kappa_{\text{abs}}(X, L, H) \leq (1 + \|X\|_2^2) \rho_{\hat{A}C}^{(2)} \eta^\sigma_k.
\]
In particular, when \( k = n - p \), it has the bounds as

\[
\frac{\eta_k^{\sigma}}{\|V_{11}\|_2\|V_{22}\|_2 \rho_{AC}^{(1)}} \leq \kappa^{\text{abs}}(X, L, H) \leq (1 + \|X_n\|_2^2) \rho_{AC}^{(2)} \eta_k^{\sigma}.
\]

**Proof.** Let \( W_0 = \begin{bmatrix} 0_p & 0 \\ 0 & I_k \end{bmatrix} \), it follows that \( \bar{Z} = \Gamma \bar{Z} \) for \( \Gamma = \text{diag}(\Gamma_1, \Gamma_2) \otimes I_{n+d-t} \), and

\[
\Gamma_1 = \text{diag}(W_0, I_{p+k}) \otimes I_{n+d-t}, \quad \Gamma_2 = \text{diag}(S_C, I_k) \otimes I_{n+d-t},
\]

and

\[
\bar{Z} = \text{diag}\left( W_0 \otimes \bar{U}_2^T \bar{Q}^T, \begin{bmatrix} S_C^{-1} & 0 \\ -\bar{U}_1^T (\bar{A}^c)^T U_C & I_k \end{bmatrix} \otimes I_{n+d-t} \right) =: \text{diag}(\bar{Z}_{11}, \bar{Z}_{22}).
\]

Therefore

\[
\kappa^{\text{abs}}(X, L, H) \leq \|H_1 + H_2\|_2 \|\bar{G}\|_2 \|\bar{Z}\|_2, \quad (4.8)
\]

where \( \bar{G} = \Gamma \bar{G} \), and

\[
\bar{G}^T \bar{G} = \left( \begin{bmatrix} S_C^2 & 0 \\ 0 & \bar{\Sigma}_1^T \bar{\Sigma}_1 \end{bmatrix} \otimes I_{n+d-t} + W_0 \otimes (\bar{\Sigma}_1^2 \bar{\Sigma}_2) \right) (S_C^2 \otimes I_{n+d-t} - W_0 \otimes (\bar{\Sigma}_2^T \bar{\Sigma}_2))^{-2},
\]

consists of \((k+1)\) diagonal block \( D^{(i)} \) for \( i = 0, 1, \ldots, k \) satisfying

\[
D^{(0)} = I_{p(n+d-t)}, \quad D^{(i)} = \text{diag}\left( \frac{\sigma_{j}^2 + \sigma_{k+j}^2}{(\sigma_j^2 - \sigma_{k+j}^2)^2} \right), \quad 1 \leq i \leq k, 1 \leq j \leq n + d - t.
\]

Note that \( \frac{\sigma_j^2 + \sigma_{k+j}^2}{(\sigma_j^2 - \sigma_{k+j}^2)^2} \) is an increasing function of \( \eta \) and a decreasing function of \( \sigma \) for \( \sigma > \eta > 0 \), therefore

\[
\|\bar{G}\|_2 = \|\bar{G}^T \|_2^{1/2} = \eta_k^{\sigma}. \quad (4.9)
\]

For the upper bound of \( \|H_1 + H_2\|_2 \), note that \( \bar{V} \) in (4.7) is an orthogonal matrix and \( X_t = -\bar{V}_{12} \bar{V}_{22}^T \), then by applying the CS decomposition (see [13] Theorem 2.6.3) and a similar technique in [45] Theorem 3.6), we have

\[
\|H_1 + H_2\|_2 \leq \|\bar{V}_{22}\|_2^2 = 1 + \|X_t\|_2^2. \quad (4.10)
\]

For the norm of \( \bar{Z} \) and \( \bar{Z}^\dagger \), note that

\[
\|Z_{11}\|_2 \leq \|Q\|_2 = 1 + \|\bar{A}^c Z\|_2, \quad \|Z_{11}\|_2 = (\sigma_{\min}(Q \bar{U}_2))^{-1} = (\sigma_{\min}(1 + \bar{U}_2^T \bar{C}^T \bar{C} \bar{U}_2))^{-1/2} \leq 1
\]

for \( \bar{C} = (\bar{A}^c)^T \). Moreover, with \( (\bar{A}^c)^T U_C \bar{A} = \bar{A} V_C \bar{V}_C^T = \bar{C} \bar{C} \),

\[
\|\bar{Z}_{22}\|_2 = \left\| \begin{bmatrix} S_C^{-1} & 0 \\ -\bar{U}_1^T (\bar{A}^c)^T U_C & I_k \end{bmatrix} \right\|_2 \leq 1 + \|\bar{C}\|_2 + \|\bar{A}^c\|_2 = \rho_{AC}^{(2)},
\]

\[
\|\bar{Z}_{22}^{-1}\|_2 = \left\| \begin{bmatrix} S_C V_C^T & 0 \\ -\bar{U}_1^T A_C V_C^T & I_k \end{bmatrix} \right\|_2 \leq 1 + \|\bar{C}\|_2 + \|\bar{A}^c \bar{C}\|_2 = \rho_{AC}^{(1)}.
\]

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Therefore
\[ \| \hat{Z} \|_2 \leq \rho_{AC}^{(2)}, \quad \| \hat{Z}^T \|_2 \leq \rho_{AC}^{(1)}. \] (4.11)
Substituting (4.9)–(4.11) into (4.8), the upper bound for \( \kappa_{\text{abs}}(X, L, H) \) then follows.

When \( k = n - p \), the upper bound of \( \kappa_{\text{abs}}(X_n, L, H) \) is obvious and for the lower bound, note that
\[
\kappa_{\text{abs}}(X_n, L, H) \geq \sigma_{\text{min}}(H_2) \| \overline{GZ} \|_2 \geq \sigma_{\text{min}}(H_2) \sigma_{\text{min}}(\hat{Z}) \| \| G \|_2 \\
= \frac{\eta_{\text{h}}^\sigma \| H_2 \|_2 \| \hat{Z}^T \|_2}{\| V_{11} \|_2 \| V_{22} \|_2 \rho_{AC}^{(1)} \| \rho_{\text{AC}}^T \|}.
\]
The assertion then follows. \( \Box \)

The upper bounds and lower bounds in Theorem 4.4 reduce to the ones in (2.5)–(2.6) for the TLS problem, when \( C = 0 \) and \( \eta_{\text{h}}^\sigma > 1 \). Moreover, note that \( \| V_{22} \|_2 \geq 1 + \| X \|_2^2 = 1/\sigma_{\text{min}}^2(V_{22}) \), it follows from Theorem 4.4 that the multidimensional TLS problem might be ill-conditioned, when \( \hat{C} \) is ill-conditioned, or the gap between \( \tilde{\sigma}_k \) and \( \tilde{\sigma}_{k+1} \) or \( \sigma_{\text{min}}(V_{22}) \) is small.

When \( d = 1 \), the absolute condition number \( \kappa_{\text{abs}}(X, L, H) \) has a compact form as follows.

**Corollary 4.5** With notations in Theorems 4.2 and 4.4, let \( d = 1, t = k + p \) and \( \tilde{\sigma}_k > \tilde{\sigma}_{k+1} \) such that \( V_{22} \neq 0 \) holds for the approximate system \( Ax \approx b \) subject to \( Cx = d \), then we have
\[
\kappa_{\text{abs}}(X, L, H) = \frac{1}{\| V_{22} \|_2^2} \left( \| (\hat{V}_{21} \otimes (\bar{V}_{12} + x_1 \bar{V}_{22})) + (\hat{V}_{11} + x_1 \hat{V}_{21}) \otimes \bar{V}_{22} \) \| G \|_2 \right) \\
\leq (1 + \| x_1 \|_2^2) \rho_{AC}^{(2)} \eta_{\text{h}}^\sigma,
\]
where \( \tilde{A} = [A \quad b], \tilde{C} = [C \quad d] \). In particular, if \( k = n - p \), then
\[
\frac{\eta_{n-p}^\sigma}{\| V_{11} \|_2 \| V_{22} \|_2 \rho_{AC}^{(1)}} \leq \kappa_{\text{abs}}(X_n, L, H) = \frac{1}{\| V_{22} \|_2^2} \left( \| (\hat{V}_{11} + x_1 \hat{V}_{21}) \otimes \bar{V}_{22} \) \| G \|_2 \right) \\
\leq (1 + \| x_n \|_2^2) \rho_{AC}^{(2)} \eta_{n-p}^\sigma.
\]

**Proof.** Note that when \( d = 1 \), \( V_{22} \in \mathbb{R}^{1 \times (n+1-t)} \) and \( \bar{V}_{22} = \frac{x_1}{\| V_{22} \|_2^2}, \bar{x}_i = -\bar{V}_{12} \bar{V}_{22} = -\bar{V}_{12} \bar{V}_{22}^T/\| V_{22} \|_2^2 \).

\[
H_1 = \left( V_{22} V_{22}^T \right)^{-1} V_{21} = \left( V_{22} V_{22}^T \right)^{-1} V_{21}, \quad H_2 = \left( V_{22} \otimes V_{11}^T \right) \Pi_{(n+1-t),l} = \Pi_{(1,n)} \left( V_{11}^T \otimes \bar{V}_{22} \right) = \frac{1}{\| V_{22} \|_2^2} \left( V_{11}^T \otimes \bar{V}_{22} \right)
\]
where \( \hat{V}_{11} \) is the upper-left \( n \times t \) submatrix of the orthogonal matrix \( \hat{V} \) given in (4.7). By Lemma 4.1(b), we have
\[
\hat{V}_{11}^T = \hat{V}_{11} + x_1 \hat{V}_{21}.
\] (4.12)
The formula for \( \kappa_{\text{abs}} \) then follows, in which \( H_1 = 0 \) when \( k = n - p \). The proof then follows. \( \Box \)

### 4.2 Mixed and componentwise condition numbers

For the mixed and componentwise condition numbers, we have the following results.
**Theorem 4.6** With the notation in Theorem 4.2, then we have mixed and componentwise condition formulae of $X_t$ as follows:

\[
m(X_t, L, H) = \frac{||MN|\text{vec}([|L| \mid |H|])||_\infty}{||X_t||_{\text{max}}},
\]

(4.13)

\[
c(X_t, L, H) = \frac{||MN|\text{vec}([|L| \mid |H|])||_\infty}{\text{vec}(|X_t|)},
\]

(4.14)

where $M = (H_1 + H_2)D^{-1}$, $N = N_1 + N_2$ for

\[
D = S_1^2 \otimes I_{n+d-t} - \begin{bmatrix} 0 & 0 \\ 0 & I_k \end{bmatrix} \otimes (\Sigma_2^T \bar{\Sigma}_2),
\]

\[
N_1 = [0_{(n_d+d) \times p}] \ \bar{V}_1^T \otimes (Q \bar{U}_2 \Sigma_2)^T, \quad N_2 = \Pi_{t,(n_d+d-t)}(\bar{V}_2^T \otimes [PU_{C} \text{vec} Y]) \text{vec}(Q \bar{U}_1 \Sigma_1)^T).
\]

Moreover, they have compact upper bounds as

\[
m^w(X_t, L, H) = \frac{\left| \|\tilde{V}_{11}^T Y^T [\bar{V}_{22}] + |V_{12}^T F_{22} Y^T \tilde{V}_{21}^T (V_{22}^T V_{22})^{-1}| \right|_{\text{max}}}{||X_t||_{\text{max}}},
\]

\[
c^w(X_t, L, H) = \frac{\left| \|\tilde{V}_{11}^T Y^T [\bar{V}_{22}] + |V_{12}^T F_{22} Y^T \tilde{V}_{21}^T (V_{22}^T V_{22})^{-1}| \right|_{\text{max}}}{||X_t||_{\text{max}}},
\]

where $Y = [Q \bar{U}_2 \Sigma_2]^T [|L| \mid |H|][0_{(n_d+d) \times p}] \ |V_1| + |V_2^T||[|L| \mid |H|]^T||[PU_{C} \text{vec} Y]$, (4.15)

from which the $i$-th column of $y_i = Ye_i$ takes the form

\[
y_i = (s_i^2 I_{n+d-t} - \tau_i \Sigma_2^T \bar{\Sigma}_2)^{-1}Ye_i.
\]

(4.16)

Here $s_i$ is the $i$-th diagonal element of $S_1$ and $\tau_i = 1$ for $i > p$ and zero otherwise.

**Proof:** By Theorem 4.2 and the concept of condition numbers, the mixed and componentwise condition numbers of $X_t$ can be formulated

\[
m(X_t, L, H) = \frac{||\tilde{c}'(c) \cdot c||_\infty}{\phi(c)},
\]

\[
= \frac{||(H_1 + H_2)G \bar{Z} \cdot \text{vec}([|L| \mid |H|])||_\infty}{||X_t||_{\text{max}}},
\]

\[
= \frac{M \left( N_1 + (S_1 \otimes I_{n+d-t}) \Pi_{t,(n_d+d-t)}(\bar{V}_2^T \otimes [PU_{C} \text{vec} Y]) \right) \text{vec}([|L| \mid |H|])}{||X_t||_{\text{max}}},
\]

\[
= \frac{M \left( N_1 + (S_1 \otimes I_{n+d-t}) \Pi_{t,(n_d+d-t)}(\bar{V}_2^T \otimes [PU_{C} \text{vec} Y]) \right) \text{vec}([|L| \mid |H|])}{||X_t||_{\text{max}}},
\]

which is bounded by

\[
||MN|\text{vec}([|L| \mid |H|])||_\infty \leq (|H_1| + |H_2|)D^{-1}(|N_1| + |N_2|)\text{vec}([|L| \mid |H|])
\]

\[
\leq (|H_1| + |H_2|)D^{-1}\text{vec}(Y)
\]

\[
= \left( |(\bar{V}_{22}^T \tilde{V}_{11})^T\Pi_{t,(n_d+d-t)}| + |(\bar{V}_{22}^T \tilde{V}_{21})^T \otimes (\bar{V}_{12}^T F_{22})^T \right) \text{vec}(Y)
\]

\[
\leq \text{vec}(\tilde{V}_{11}^T Y^T \bar{V}_{22}^T + |V_{12}^T F_{22} Y^T \tilde{V}_{21}^T (\bar{V}_{22}^T \bar{V}_{22})^{-1}|),
\]

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where \( D \text{vec}(Y) = \text{vec}(Y) \) gives
\[
YS_1^2 - \bar{\Sigma}_2^T \Sigma_2 Y \begin{bmatrix} 0_p & 0 \\ 0 & I_k \end{bmatrix} = \mathbf{0},
\]
from which the upper bound \( m^u(X, L, H) \) follows with the \( i \)-th column of \( Y \) satisfying (4.16).

For the componentwise condition number, it follows that
\[
c(X_t, L, H) = \left\| \frac{\phi'(c)}{\phi(c)} \right\|_\infty = \left\| \frac{|MN|\text{vec}([L \mid H])}{\text{vec}(|X_t|)} \right\|_\infty,
\]
and the upper bound \( c^u(X_t, L, H) \) follows obviously. We then finish the proof of the theorem. \( \square \)

**Corollary 4.7** With notations in Theorem 4.2, if for \( k = n - p, \sigma_k > \sigma_{k+1} \) and \( \bar{\nabla}_{22} \) is nonzero, then the mixed and componentwise condition numbers for the TLSE solution \( X_n = -\bar{\nabla}_{12} \bar{\nabla}_{22} \) satisfies
\[
m(X_n, L, H) = \left\| \frac{|MN|\text{vec}([L \mid H])}{\text{vec}(|X_n|)} \right\|_\infty,
\]
\[
c(X_n, L, H) = \left\| \frac{|MN|\text{vec}([L \mid H])}{\text{vec}(|X_n|)} \right\|_\infty,
\]
where
\[
M = (\bar{\nabla}_{22}^{-T} \otimes \bar{\nabla}_{11}^{-T}) \Pi_{(d,n)} (S_1^2 \otimes I_d - \begin{bmatrix} 0_p & 0 \\ 0 & I_{n-p} \end{bmatrix} \otimes (\bar{\Sigma}_2^T \Sigma_2)^{-1})
\]
\[
N = [0_{(n+d) \times p} \bar{\nabla}_{11}^{-T} \otimes (Q \bar{U}_2 \Sigma_2)^T + \bar{\Pi}_{(d,n)} \left( \bar{\nabla}_{22}^{-T} \otimes [P U_C S_C \quad Q \bar{U}_1 \Sigma_1]^T \right)].
\]
Moreover, they have upper bounds as
\[
m^u(X_n, L, H) = \left\| \frac{\bar{\nabla}_{11}^{-T} Y^T \bar{\nabla}_{22}^{-T}}{\text{vec}(|X_n|)} \right\|_\infty,
\]
\[
c^u(X_n, L, H) = \left\| \frac{\bar{\nabla}_{11}^{-T} Y^T \bar{\nabla}_{22}^{-T}}{\text{vec}(|X_n|)} \right\|_\infty,
\]
where the \( i \)-th column of \( y_i = Ye_i \) satisfies
\[
y_i = (s_i^2 I_d - \tau_i \Sigma_2^T \Sigma_2)^{-1} Ye_i,
\]
in which \( s_i \) is the \( i \)-th diagonal element of \( S_1 \) and \( \tau_i = 1 \) for \( i > p \) and zero otherwise, \( Y \) is given by (4.13) with \( t = n \).

**Theorem 4.8** For the single-dimensional TLSE problem (\( d = 1 \)), assume that for \( k = n - p, \sigma_k > \sigma_{k+1} \) such that \( \bar{\nabla}_{22} \) is nonzero. Then for matrices \( M, N \) given by Corollary 4.7, we have the relation
\[
K = MN = T_1 G(x_n) - T_2,
\]
where \( G(x_n) = [x_n^T \quad -1] \otimes I_{p+q} \), and
\[
T_1 = 2\rho^{-2} \mathcal{K} xu^T - [C_A^\dagger \mathcal{K} A^T], \quad T_2 = \mathcal{K} [I_n \quad 0_{n \times 1}] \otimes u^T,
\]
for \(C_A^\dagger = (I_n - \mathcal{X}A^T A)C_c^\dagger\), \(\rho = \sqrt{1 + \|x\|^2_2}\) and \(u^T = \begin{bmatrix} - r^T (\hat{A}C^\dagger) \\ r^T \end{bmatrix}\) with \(r = Ax - B\).

Proof. Note that when \(k = n - p, d = 1\), in (4.17) and (4.18), \(\pi_{(d,n)} = \pi_{(n,d)} = I_n\), \(\tilde{\Sigma}_2^\dagger \tilde{\Sigma}_2 = \tilde{\sigma}_{\rho+1}^2\) and

\[
D^{-1} := (S^2_1 \otimes I_d - \begin{bmatrix} 0 & 0 \\ 0 & I_{n-p} \end{bmatrix} \otimes (\tilde{\Sigma}_2^\dagger \tilde{\Sigma}_2))^{-1} = \text{diag}(S_1^{-2}, (\tilde{\Sigma}_1^2 - \tilde{\sigma}_{\rho+1}^2 I_{n-p})^{-1}).
\]

Following (4.12), the matrices \(M, N\) in (4.17)–(4.18) take the form

\[
M = \nabla_{22}^T \hat{V}_{11}^T D^{-1} = \nabla_{22}^T [I_n \ x_n] \hat{V}_1 D^{-1} = \nabla_{22}^T [I_n \ x_n] \hat{V}_1 \text{diag}(S_1^{-2}, (\tilde{\Sigma}_1^2 - \tilde{\sigma}_{\rho+1}^2 I_{n-p})^{-1}) = \nabla_{11}^T \otimes (Q \hat{U}_2 \hat{\Sigma}_2^2) + \nabla_{22}^T \otimes \left( [PU_CC \quad Q \hat{U}_1 \hat{\Sigma}_1^2]^T \right),
\]

where \(\hat{V}_1 = [\hat{V}_{11}^T \quad \hat{V}_{21}^T]^T\) and \(\hat{V}_{11}, \hat{V}_{21}\) are defined in Theorem 4.2.

In the following, we first derive an equivalent formula for \((\tilde{\Sigma}_1^2 - \tilde{\sigma}_{\rho+1}^2 I_{n-p})^{-1}\). Let \(\tilde{Q}_2\) be given by (2.10), based on the SVD in (2.8): \(A\tilde{Q}_2 = \tilde{U}_1 \tilde{\Sigma}_1 \tilde{V}_1^T + \tilde{U}_2 \tilde{\Sigma}_2 \tilde{V}_2^T\), partition

\[
\tilde{V} = \begin{bmatrix} \tilde{V}_{11} & \tilde{V}_{12} \\ \tilde{V}_{21} & \tilde{V}_{22} \end{bmatrix}.
\]

Note that \(A\tilde{Q}_2\) is the first \(n - p\) columns of \(\tilde{A}\tilde{Q}_2\), therefore \(A\tilde{Q}_2 = \tilde{U}_1 \tilde{\Sigma}_1 \tilde{V}_{11}^T + \tilde{U}_2 \tilde{\Sigma}_2 \tilde{V}_{12}^T\), from which

\[
Y_0 = (A\tilde{Q}_2)^T (A\tilde{Q}_2) - \tilde{\sigma}_{\rho+1}^2 I_{n-p} = \tilde{V}_{11}^T \left( \text{diag}(\tilde{\Sigma}_1^2, \tilde{\Sigma}_2^2) - \tilde{\sigma}_{\rho+1}^2 I_{n-p} \right) \tilde{V}_{11} \tilde{V}_{12}^T = \tilde{V}_{11} (\tilde{\Sigma}_1^2 - \tilde{\sigma}_{\rho+1}^2 I_{n-p}) \tilde{V}_{11}^T,
\]

in which \(\tilde{V}_{1j}\) satisfies

\[
\nabla_{1j} = \tilde{Q}_2 \tilde{V}_{1j} + \beta^{-1} x_c \tilde{V}_{2j} = \tilde{Q}_2 \tilde{V}_{1j} - x_c \tilde{V}_{2j}, \quad j = 1, 2,
\]

(4.20)

according to the relation \(\nabla = \tilde{Q}_2 \tilde{V}\). Therefore \(\nabla_{22} = -\beta \nabla_{22}\) is nonzero. By Lemma 4.1(a), \(\nabla_{11}\) is nonsingular. From (4.19), we obtain

\[
(\tilde{\Sigma}_1^2 - \tilde{\sigma}_{\rho+1}^2 I_{n-p})^{-1} = \tilde{V}_{11}^T Y_0^{-1} \tilde{V}_{11}.
\]

It should be noted that for any column vector \(z\) and matrices \(M_i\),

\[
M_1 (M_2 \otimes z^T) = (M_1 M_2) \otimes z^T, \quad z^T \otimes M_3 = M_3 (z^T \otimes I),
\]

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therefore with $\hat{V}_1 = [V_C \quad \nabla_1]$.

\[
MN_1 = \begin{bmatrix} I_n & x_n \end{bmatrix} \hat{V}_1 \left[ \begin{array}{c} 0_{p \times (n+1)} \\ \tilde{V}_{11} Y_{10}^{-1} \tilde{V}_{11} V_1^T \end{array} \right] \otimes \left( Q \tilde{U}_2 \tilde{\Sigma}_2 \tilde{V}_{22}^{-1} \right)^T,
\]

\[
MN_2 = \hat{V}_1^T D^{-1} \left( \begin{array}{cc} -x_n^T & 1 \end{array} \right) \otimes \begin{bmatrix} PU_C S_C & Q \tilde{U}_1 \tilde{\Sigma}_1 \end{bmatrix}^T = \left( I_n x_n \right) \hat{V}_1 \left[ \begin{array}{c} 0_{p \times (n+1)} \\ \tilde{V}_{11} Y_{10}^{-1} \tilde{V}_{11} V_1^T \end{array} \right] \otimes \left( Q \tilde{U}_2 \tilde{\Sigma}_2 \tilde{V}_{22}^{-1} \right)^T,
\]

in which

\[
\tilde{U}_2 \tilde{\Sigma}_2 \tilde{V}_{22}^{-1} = A \tilde{Q}_2 \tilde{V}_{22}^{-1} = \tilde{A} \tilde{V}_{22}^{-1} = [A \quad b] \begin{bmatrix} -x_n^p \\ 1 \end{bmatrix} =: -r.
\]

\[
\tilde{U}_1 \tilde{\Sigma}_1 \tilde{V}_{11}^T = \tilde{U}_1 \tilde{\Sigma}_1 \tilde{V}_{11}^T \begin{bmatrix} I_{n-p} \\ 0 \end{bmatrix} = [A \tilde{Q}_2 - \tilde{U}_2 \tilde{\Sigma}_2 \tilde{V}_{22}^{-1} \tilde{V}_{11}^T] \begin{bmatrix} I_{n-p} \\ 0 \end{bmatrix} = A \tilde{Q}_2 - \tilde{U}_2 \tilde{\Sigma}_2 \tilde{V}_{22}^{-1} \tilde{V}_{11}^T = A \tilde{Q}_2 + r \tilde{V}_{22} \tilde{V}_{11}^T,
\]

and

\[
\tilde{V}_{11} \tilde{V}_1^T = \begin{bmatrix} \tilde{V}_{11} \tilde{V}_1^T \\ \tilde{V}_{11} \tilde{V}_2^T \end{bmatrix} = \begin{bmatrix} I_{n-p} - \tilde{V}_{12} \tilde{V}_{12}^T \\ \tilde{V}_{12} \tilde{V}_{12}^T \end{bmatrix} = \begin{bmatrix} I_{n-p} - \tilde{V}_{12} \tilde{V}_{12}^T \\ \tilde{V}_{12} \tilde{V}_{12}^T \end{bmatrix} = \begin{bmatrix} Q_2^T \\ 0 \end{bmatrix} - \tilde{V}_{12} \tilde{V}_{12}^T \tilde{V}_{12}^T = \begin{bmatrix} Q_2^T \\ 0 \end{bmatrix} + \tilde{V}_{12} \tilde{V}_{12}^T |x_n^T - 1|.
\]

Therefore

\[
[I_n \quad x_n] \tilde{V} \tilde{V}_1^T = Q_2.
\]

Moreover, note that the Grevill’s method [2] Chapter 7, Section 5] gives

\[
\tilde{C}^\dagger = \begin{bmatrix} (I_n - \omega^{-1} x_C x_C^T) C^\dagger \\ \omega^{-1} x_C^T C^\dagger \end{bmatrix}, \quad \omega = 1 + ||x_C||^2.
\]

Combining this with the expression for $x_n$ in (2.12) and the residual $r_C = Ax_C - b$, we have

\[
[I_n \quad x_n] \tilde{C}^\dagger = (I_n - \omega^{-1} x_C x_C^T + \omega^{-1} x_C x_C^T) C^\dagger = (I_n - \omega^{-1} x_C x_C^T) C^\dagger,
\]

\[
\tilde{A} \tilde{C}^\dagger = (A - \omega^{-1} x_C x_C^T + \omega^{-1} x_C x_C^T) C^\dagger.
\]

Combining (4.21) - (4.24) with (4.20), we have $u^T = r^T Q^T = \left[ -r^T (A \tilde{C}^\dagger) \quad r^T \right]$, $\mathcal{K} = Q_2 Y_0^{-1} Q_2^T$,
we have
\[
MN_1 = -\left[Q_2Y_0^{-1}Q_2^T\left(|I_{n-p}| 0 - Q_2\tilde{V}_{12}\tilde{V}_{22}^T[x_n^T - 1]\right)\right] \otimes u^T
= -\left[(Q_2Y_0^{-1}Q_2^T)\left(|I_{n-p}| 0 + (\tilde{V}_{12} + x_C\tilde{V}_{22})\tilde{V}_{22}^T[x_n^T - 1]\right)\right] \otimes u^T
= -\left[\mathcal{K}\left(|I_{n-p}| 0 + (-x_n + x_C)\tilde{V}_{22}\tilde{V}_{22}^T[x_n^T - 1]\right)\right] \otimes u^T
= -\left(|\mathcal{K} 0| - \rho^{-2}\mathcal{K}x_n[x_n^T - 1]\right) \otimes u^T
= -\mathcal{K}\left(|I_n| 0 \otimes u^T\right) + (\rho^{-2}\mathcal{K}x_nu^T)([x_n^T - 1] \otimes I_{p+q}),
\]
\[
MN_2 = \left((|I_n - \omega^{-1}\mathcal{K}A^T r_C x_C^T| C^T 0) + Q_2Y_0^{-1}Q_2^T [-\tilde{A}^T I_g]\right)
+ (Q_2Y_0^{-1}Q_2^T)Q_2\tilde{V}_{12}\tilde{V}_{22}u^T\left([-x_n^T 1] \otimes I_{p+q}\right)
= \left((|I_n - \omega^{-1}\mathcal{K}A^T r_C x_C^T| C^T 0) + [-\mathcal{K}A^T(A - \omega^{-1}r_C x_C^T)C^T \mathcal{K}A^T]\right)
+ \mathcal{K}\left(\tilde{V}_{12} + x_C\tilde{V}_{22}\right)\tilde{V}_{22}u^T\left([-x_n^T 1] \otimes I_{p+q}\right)
= \left((|I_n - \mathcal{K}A^T)C^T \mathcal{K}A^T\right) + \mathcal{K}\left(-x_n + x_C\right)\tilde{V}_{22}\tilde{V}_{22}u^T\left([-x_n^T 1] \otimes I_{p+q}\right)
= -\left((|I_n - \mathcal{K}A)C^T \mathcal{K}A^-\rho^{-2}\mathcal{K}x_nu^T\right)([x_n^T - 1] \otimes I_{p+q}),
\]
where \(\tilde{V}_{22}\tilde{V}_{22}^T = \|\tilde{V}_{22}\|^2 = \rho^{-2}\) based on the fact that \([x_n^T - 1] = \rho\tilde{V}_{12}\tilde{V}_{22}^T\) for \(\rho^2 = 1 + \|x_n\|^2\), and \(\mathcal{K}(x_C - x_n) = -\mathcal{K}x_n\) since \(Q_2^T x_C = 0\). Therefore
\[
M(N_1 + N_2) = \left(2\rho^{-2}\mathcal{K}x_nu^T - (|I_n - \mathcal{K}A)C^T \mathcal{K}A^-\right)([x_n^T - 1] \otimes I_{p+q}) - \mathcal{K}(|I_n 0) \otimes u^T),
\]
which is exactly \(\mathcal{K}\). The assertion in the theorem then follows. 

**Remark 4.1** When \(p > 0, k = n - p\) and \(d = 1\), for the single dimensional TLSE problem, Liu and Jia [22] derived the first order perturbation estimate as
\[
\Delta x = K_{L,H} \text{vec}(\Delta L \ \Delta H) + \mathcal{O}((\|\Delta L \ \Delta H\|_F^2),
\]
where \(K_{L,H} = MN = \mathcal{K}\). With this, three types of condition number formulae of the single-dimensional TLSE problem were derived. The result in Corollary 4.8 shows that the newly derived perturbation analysis and condition numbers for the multidimensional case unify those for the single dimensional TLSE problem.

**Remark 4.2** It is observed that the formulae for three types of condition numbers involve the Kronecker product which might lead to large storage and computation cost. For mixed and componentwise condition numbers, we can use their upper bounds as alternations, while for the normwise condition numbers, as did in [45], we can compute
\[
\kappa_{\text{abs}} = \|\tilde{H}\|_2 = \|\tilde{H}^T\tilde{H}\|_2^{1/2}, \quad \text{for} \quad \tilde{H} = (H_1 + H_2)G\tilde{Z} = (H_1 + H_2)D^{-1}\tilde{Z},
\]
by applying the power method to the matrix \(\tilde{H}^T\tilde{H}\), in which \(D\) is defined in Theorem 4.6, and \(\tilde{Z} = [I_t \otimes \tilde{S}_2^T \ S_1 \otimes I_{n+d-1}^T]\tilde{Z}\). In the power scheme, the matrix-vector multiplications associated
Table 5.1: The absolute error of the first order perturbation estimate of vec($\Delta X_t$)

| $\epsilon$ | 10  | 20  | 30  | 40  |
|------------|-----|-----|-----|-----|
| $10^{-4}$  | 1.9e-4 | 6.3e-4 | 5.2e-4 | 3.0e-4 |
| $10^{-5}$  | 5.6e-8 | 2.3e-8 | 3.7e-8 | 2.1e-8 |
| $10^{-6}$  | 2.7e-12 | 3.7e-12 | 1.8e-12 | 1.5e-12 |

with $\tilde{H}$ and $\tilde{H}^T$ can be transformed into Kronecker product-free operations, say for $\tilde{H}f$, where $f = [f_1^T \ f_2^T]^T$ with $f_i = \text{vec}(F_i)$ with $F_1 \in \mathbb{R}^{r \times (p+q)}$, $F_2 \in \mathbb{R}^{(n-d-t) \times t}$, $\tilde{H}f = (H_1 + H_2)D^{-1}\text{vec}\left((Q\tilde{U}_2\tilde{\Sigma}_2)^TF_1 \begin{bmatrix} 0 & 0 & \Sigma_1^T \end{bmatrix} + F_2 \begin{bmatrix} S_C & -U_C^T(\tilde{A}\tilde{C}_1)^T\tilde{U}_1\tilde{\Sigma}_1 \end{bmatrix} \right)$, $g := \tilde{H}f = (H_1 + H_2)D^{-1}\text{vec}\left((\bar{V}_{12} + X_1\bar{V}_{22})^T\bar{V}_{21}^T(\bar{V}_{22}\bar{V}_{22})^{-1} + (\bar{V}_{11} + X_1\bar{V}_{21})^T\bar{V}_{22}^T\right)$, where $t_i = T\epsilon_i$ satisfies $t_i = (s_i^2I_{n+d-t} - \tau_i\tilde{\Sigma}_2^T\tilde{\Sigma}_2)^{-1}((Q\tilde{U}_2\tilde{\Sigma}_2)^TF_1 \begin{bmatrix} 0 & 0 & \Sigma_1^T \end{bmatrix} + F_2 \begin{bmatrix} S_C & -U_C^T(\tilde{A}\tilde{C}_1)^T\tilde{U}_1\tilde{\Sigma}_1 \end{bmatrix})e_i$, in which $s_i, \tau_i$ are the same as those in Theorem 4.6. The Kronecker product-free expression associated with $\tilde{H}^Tg$ can be derived in a similar manner. Here we omit these.

5 Numerical experiments

In this section, we present numerical examples to verify our results. The following numerical tests are performed via MATLAB with machine precision $u = 2.22e-16$ in a laptop with Intel Core (TM) i5-5200U CPU by using double precision.

Example 1 In this example, we generate random multidimensional TLSE problems to verify the rationality of the first order perturbation estimate in Theorem 4.2. The entries in $[C \ D]$ and $[A \ B]$ are generated as random variables uniformly distributed in the interval $(0,1)$, via Matlab command ‘rand(·)’. Set $p = 10, q = 40, n = 40, d = 5$, and let the perturbations to the data be given by $[\Delta C \ \Delta D] = \epsilon \ast \text{rand}(p, n + d)$, $[\Delta A \ \Delta B] = \epsilon \ast \text{rand}(q, n + d)$. Choose $t = 10, 20, 30, 40$ and compute the solutions to the original and perturbed problems via the QR-SVD method. In Table 5.1 we compute the absolute error $\eta_{\Delta X_t} = \|\text{vec}(\Delta X_t) - \text{vec}([\Delta L \ \Delta H])\|_\infty$, with respect to different $\epsilon$. The tabulated results show that $\eta_{\Delta X_t} = \mathcal{O}(\epsilon^2)$, illustrating the rationality of the first order perturbation estimates in Theorem 4.2.
Example 2. In this example, we do some numerical experiments for TLSE from piecewise-polynomial data fitting problem that is modified from [3, Chapter 16] and [21].

Given \(N\) points \((t_i,y_i)\) on the plane, we are seeking to find a piecewise-polynomial function \(f(t)\) fitting the above set of the points, where

\[
f(t) = \begin{cases} f_1(t), & t \leq a, \\ f_2(t), & t > a, \end{cases}
\]

with \(a\) given, and \(f_1(t)\) and \(f_2(t)\) polynomials of degree three or less,

\[
f_1(t) = x_1 + x_2t + x_3t^2 + x_4t^3, \quad f_2(t) = x_5 + x_6t + x_7t^2 + x_8t^3.
\]

The conditions that \(f_1(a) = f_2(a)\) and \(f_1'(a) = f_2'(a)\) are imposed, so that \(f(t)\) is continuous and has a continuous first derivative at \(t = a\). Suppose the \(N\) data are numbered so that \(t_1, \ldots, t_M \leq a\) and \(t_{M+1}, \ldots, t_N > a\). The conditions \(f_1(a) - f_2(a) = 0\) and \(f_1'(a) - f_2'(a) = 0\) leads to the equality constraint \(Cx = d\) for \(x = [x_1, x_2, \ldots, x_8]^T\) and

\[
C = \begin{bmatrix} 1 & a & a^2 & a^3 & -1 & -a & -a^2 & -a^3 \\ 0 & 1 & 2a & 3a^2 & 0 & -1 & -2a & -3a^2 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

The vector \(x\) that minimizes the sum of squares of the prediction errors

\[
\sum_{i=1}^{M} (f_1(t_i) - y_i)^2 + \sum_{i=M+1}^{N} (f_2(t_i) - y_i)^2,
\]

gives \(\min_x \|Ax - b\|_2\), where

\[
A = \begin{bmatrix} 1 & t_1 & t_1^2 & t_1^3 & 0 & 0 & 0 & 0 \\ 1 & t_2 & t_2^2 & t_2^3 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & t_M & t_M^2 & t_M^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & t_{M+1} & t_{M+1}^2 & t_{M+1}^3 \\ 0 & 0 & 0 & 0 & 1 & t_{M+2} & t_{M+2}^2 & t_{M+2}^3 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & t_N & t_N^2 & t_N^3 \end{bmatrix}, \quad b = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \\ y_{M+1} \\ \vdots \\ y_N \end{bmatrix},
\]

and the matrix \(A\) is of 50\% sparsity. If more than one observation vector is allowed, the data fitting problem becomes the multidimensional TLSE problem (1.2).

Take \(M = 200, N = 200\) and sample \(t_i \in [0,1]\) randomly. For a randomly generated piecewise-polynomial function \(f(t)\) with a predetermined \(a\), we compute the corresponding function value \(y_i = f(t_i)\). We add random componentwise perturbations on the data as

\[
\Delta L = 10^{-12} \cdot E_{p+q,8} \odot L, \quad \Delta H = 10^{-12} \cdot E_{p+q,d} \odot H, \quad q = M + N, \quad (5.1)
\]

where \(E_{s,t}\) is the random \(s \times t\) matrix whose entries are uniformly distributed on the interval (0,1), \(\odot\) denotes the entrywise multiplication.

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For simplicity let $\kappa_n, m, c$ denote the relative normwise condition number, mixed and componentwise condition numbers given in Theorem 4.3 and Theorem 4.6, respectively. Set

$$x = \text{vec}(X_t), \quad \epsilon_n = \frac{\|\Delta L \Delta H\|_F}{\|L \Delta H\|_F}, \quad \epsilon_c = \min \{\epsilon : |\Delta L| \leq \epsilon |L|, |\Delta H| \leq \epsilon |H|\},$$

where $t$ is a random integer between $p$ and $n$ such that $\mathbf{V}_{22}$ is of full row rank, and the quantity $\epsilon_n$ is used to evaluate the upper bound of the forward error $\frac{\|\Delta x\|_2}{\|x\|_2}$ via $\epsilon_n \kappa_n$, while $\epsilon_c$ is to derive upper bounds for $\frac{\|\Delta x\|_1}{\|x\|_1}$, $\frac{\|\Delta x\|_\infty}{\|x\|_\infty}$ via mixed and componentwise condition numbers. Moreover we let $\rho = \rho_{\kappa_n}^{(2)} \eta_{\kappa_n}$ be the factor of upper bounds of $\kappa^{\text{abs}}(X_t, L, H)$.

We list numerical results with respect to different $\alpha$, and for each $\alpha$, we generate two different problems and compare the estimated upper bound with actual relative forward errors. It’s observed that for fixed $\alpha$, the problems with a larger $\|X_t\|_2$ and moderate $\rho$ produce larger condition number estimates, which illustrates that the norm $\|X_t\|_2$ is a factor to affect the condition number of TLSE problem. However whether $\|X_t\|_2$ is big or small, the estimated upper bounds of the forward error via $\epsilon_n \kappa_n$, $\epsilon_c m$, $\epsilon_c c$ are about one or two orders of magnitude larger than the corresponding forward error of the solution. Among three upper bounds $\kappa_n, m, c$ of condition numbers, the normwise condition number-based upper bound $\kappa_n$ is acceptable and is about one or two orders of magnitude larger than $\kappa_n$. The upper bounds $m, c$ are more sharp, which are at most one order of magnitude larger than the corresponding exact condition numbers, therefore they are good estimates of their corresponding condition numbers and forward error of the solutions.

| $\alpha$ | $\|X_t\|_2$ | $\rho$ | $\|\Delta x\|_2$ | $\epsilon_n \kappa_n$ | $\epsilon_n \kappa_n^u$ | $\|\Delta x\|_1$ | $\epsilon_c m$ | $\epsilon_c c$ | $\epsilon_c c^u$ |
|----------|-------------|-------|----------------|-----------------|-----------------|----------------|--------------|----------|----------|
| 0.1      | 4.2         | 12.0  | 2.2e-13        | 2.1e-11         | 7.7e-10         | 2.6e-13         | 3.8e-12      | 1.2e-11  | 7.3e-13  |
|          | 1.6e5       | 76.0  | 1.1e-11        | 3.0e-9          | 7.7e-7          | 9.8e-12         | 9.1e-10      | 1.7e-9   | 1.5e-10  |
| 0.3      | 4.2         | 12.0  | 1.2e-13        | 2.1e-11         | 7.5e-10         | 1.7e-13         | 4.0e-12      | 1.4e-11  | 7.2e-12  |
|          | 2.4e5       | 42.0  | 7.4e-12        | 3.5e-9          | 5.2e-7          | 6.4e-12         | 1.2e-9       | 3.0e-9   | 4.3e-11  |
| 0.5      | 5.6         | 12.0  | 1.8e-13        | 4.7e-11         | 8.2e-10         | 2.3e-13         | 1.2e-11      | 3.3e-11  | 2.7e-11  |
|          | 5.3e4       | 68.0  | 5.8e-12        | 1.0e-9          | 3.8e-7          | 6.8e-12         | 7.8e-10      | 1.6e-9   | 1.0e-7   |
| 0.7      | 3.0         | 11.0  | 1.2e-13        | 2.2e-11         | 7.1e-10         | 1.4e-13         | 4.8e-12      | 1.6e-11  | 2.7e-13  |
|          | 1.3e7       | 75.0  | 1.4e-10        | 1.5e-8          | 7.0e-6          | 1.4e-10         | 6.5e-9       | 1.1e-8   | 2.3e-8   |
| 0.9      | 2.3         | 11.0  | 4.0e-14        | 2.2e-11         | 6.9e-10         | 5.8e-14         | 5.7e-12      | 1.9e-11  | 8.0e-14  |
|          | 5.2e8       | 42.0  | 1.0e-9         | 1.0e-7          | 2.7e-5          | 1.3e-9          | 7.6e-8       | 2.1e-7   | 1.7e-9   |

Example 3. This example is modified from [11]. Let $p = d = 5, n = 10, q = 20, k = 3, t = p + k = 8$, and $\mathbf{Q}$ be an arbitrary $(n + d) \times (n + d)$ orthogonal matrix and $\mathbf{Q}_1$ is the submatrix of $\mathbf{Q}$ by taking its first $p$ columns. Let $U_0$ be an arbitrary $p \times p$ orthogonal matrix, $y, z$ be unit column vectors of length $q, n + d$, respectively, set

$$\tilde{C} = [C \, D] = U_0 \text{diag}([1, 0.5, 0.1, 0.1, \kappa_C^{-1}]) \mathbf{Q}_1^T, \quad \tilde{A} = [A \, B] = \tilde{A} \mathbf{Q}_1^T, \quad \tilde{\Sigma} = \text{diag}(10, 8, 1, 1, 1, 1, 1, 1 - \delta/2, 1 - \delta, 1 - 2\delta, 1/6, 1/7, \ldots, 1/10),$$

with $\tilde{A} = (I_q - 2xy^T)[\tilde{\Sigma}] (I_{n+d} - 2ze^T)$,
where $\kappa_C$ is used to control the condition number of $[C \, D]$. Note that $\bar{A}Q_2$ is the last $n + d - p$ columns of $\bar{A}Q$, and by the interlacing theorem of the singular values, the relation $1 = \sigma_j(\bar{A}Q) \geq \sigma_j(\bar{A}Q_2) \geq \sigma_{j+1}(\bar{A}Q)$, for $j = k, k + 1$ and therefore $0 < \delta < 1/12$ can be used to control the gap of the singular values $\bar{\sigma}_k, \bar{\sigma}_{k+1}$ of $\bar{A}Q_2$.

Consider the same perturbation as in (5.1), for different $\kappa_C$ and $\delta$, we compute the forward errors and upper bounds via three types condition numbers in Table 5.3. It's observed that the estimated upper bounds of the forward error via $\varepsilon_n, \varepsilon_c, \varepsilon_c^u$ are about one or two orders of magnitude larger than the corresponding forward error of the solution, even the quantity $\rho$ is very large. For the compact upper bounds $m^u, c^u$ of condition numbers, $m^u, c^u$ are very sharp in most cases, while $\kappa_n^u$ is not robust against the ill-conditioning of $\bar{C}$ and sometimes they are three orders of magnitude larger than $\kappa_n$ and five or six orders of magnitude larger than $\|\Delta x\|_2$.

Table 5.3: Comparisons of forward error and upper bounds for the perturbed TLSE problem

| $\sigma$ | $\|X_t\|^2_2$ | $\rho$ | $\|\Delta x\|_2$ | $\varepsilon_n, \kappa_n$ | $\varepsilon_n, \kappa_n^u$ | $\varepsilon_c, m$ | $\varepsilon_c, m^u$ | $\varepsilon_c, c^u$ | $\varepsilon_c, c^u$ |
|----------|----------------|--------|-----------------|-------------------|-------------------|-----------------|-----------------|-----------------|-----------------|
| $\kappa_C = 10^3$ | | | | | | | | | |
| 0.1 | 2.1 | 2.4e3 | 2.1e-12 | 5.4e-10 | 3.9e-8 | 1.7e-12 | 3.4e-11 | 7.8e-11 | 3.2e-11 | 8.6e-10 | 2.1e-9 |
| 0.01 | 0.71 | 6.7e3 | 4.2e-12 | 2.0e-10 | 6.5e-8 | 4.9e-12 | 1.4e-10 | 1.5e-10 | 1.1e-8 | 1.9e-7 | 3.0e-7 |
| 0.001 | 0.9 | 1.6e5 | 1.1e-10 | 1.2e-8 | 2.4e-6 | 1.4e-10 | 3.1e-9 | 4.2e-9 | 3.6e-10 | 1.1e-8 | 1.3e-8 |
| $\kappa_C = 10^4$ | | | | | | | | | |
| 0.1 | 0.74 | 1.2e5 | 1.1e-10 | 1.5e-8 | 1.4e-6 | 1.2e-10 | 1.1e-9 | 2.3e-9 | 6.5e-10 | 1.4e-8 | 4.5e-8 |
| 0.01 | 2.1 | 9.8e5 | 7.7e-11 | 8.2e-9 | 8.7e-6 | 6.7e-11 | 6.3e-10 | 1.4e-9 | 4.7e-9 | 3.6e-8 | 9.5e-8 |
| 0.001 | 0.51 | 5.5e6 | 1.2e-10 | 2.7e-8 | 7.7e-5 | 1.5e-10 | 3.3e-9 | 5.3e-9 | 3.6e-9 | 9.9e-8 | 2.4e-7 |
| $\kappa_C = 10^5$ | | | | | | | | | |
| 0.1 | 2.4 | 8.9e7 | 2.7e-8 | 9.7e-6 | 1.4e-3 | 3.4e-8 | 6.2e-7 | 1.2e-6 | 6.6e-7 | 2.7e-5 | 1.3e-4 |
| 0.01 | 4.8 | 6.1e8 | 1.2e-7 | 3.5e-5 | 1.2e-2 | 1.1e-7 | 1.2e-6 | 2.9e-6 | 2.2e-6 | 2.8e-5 | 6.7e-5 |
| 0.001 | 2.0 | 4.6e9 | 2.0e-8 | 1.1e-5 | 6.5e-2 | 2.1e-8 | 2.5e-7 | 4.6e-7 | 4.6e-7 | 5.5e-6 | 1.8e-5 |

6 Conclusion

In this paper, we investigate the solution of multidimensional TLSE problem, and prove it is equivalent to the multidimensional weighted TLS solution in the limit sense, with the aid of perturbation theory of invariant subspace. Based on this close relation, the closed formula for the first order perturbation estimate of the minimum Frobenius norm TLS solution $X_t = -\bar{V}_1\bar{V}_2^*$ is derived, from which the expressions for normwise, mixed and componentwise condition numbers of problem TLSE are also presented. Since there expressions involve matrix Kronecker product operations which may make the computation more expensive, we provide compact upper bounds to enhance the computation efficiency. All expressions and upper bounds of these condition numbers generalize those for the single-dimensional TLSE problem [22] and multidimensional TLS problem [26].
Some numerical examples are also given in this paper to demonstrate the effectiveness in estimating the forward errors. Tightness of upper bounds for mixed and componentwise condition numbers are shown in numerical examples, even for ill-conditioned problems, while it is not necessarily true for the upper bounds of the normwise condition number. Therefore in order to derive good estimates of forward errors via normwise condition number, we recommend using power scheme to compute the true value to avoid Kronecker product operations.

References

[1] M. Baboulin, S. Gratton, A contribution to the conditioning of the total least-squares problem, SIAM J Matrix Anal. Appl., 32(3) (2011), pp. 685-699.
[2] A. Ben-Israel, T. N.E. Greville, Generalized inverses, theory and applications, 2nd ed., Springer-Verlag New York, (2003).
[3] S. Boyd, L. Vandenberghe, Introduction to applied linear algebra-vectors, matrices, and least squares, https://web.stanford.edu/boyd/vmls/vmls.pdf, (2017)
[4] A. J. Cox, N. J. Higham, Accuracy and stability of the null space method for solving the equality constrained least squares problem. BIT, 39(1)(1999), pp. 34-50.
[5] H. Diao, Condition numbers for a linear function of the solution of the linear least squares problem with equality constraints. Journal of Computational and Applied Mathematics, 344(2018), pp. 640-65.
[6] H. Diao, Y. Sun, Mixed and componentwise condition numbers for a linear function of the solution of the total least squares problem, Linear Algebra and its Applications 544:1(2018), pp. 1-29.
[7] H. Diao, Y. Wei, P. Xie, Small sample statistical condition estimation for the total least squares problem, Numer. Algorithms, 75(2) (2017), pp. 1-21.
[8] E.M. Dowling, R.D. Degroat, D.A. Linebarger, Total least squares with linear constraints, IEEE International Conference on Acoustics, 5(5), (1992), pp. 341-344.
[9] A.J. Geurts, A contribution to the theory of condition, Numer Math., 39(1) (1982), pp. 85-96.
[10] I. Gohberg, I. Koltracht: Mixed, componentwise, and structured condition numbers. SIAM J. Matrix Anal. Appl. 14 (1993), pp. 688-704
[11] S. Gratton, D. Tittley-Peloquin, J. T. Ilunga, Sensitivity and conditioning of the truncated total least squares solution, SIAM J. Matrix Anal. Appl., 34 (2013), pp. 1257-1276.
[12] G.H. Golub, C.F. Van Loan, An analysis of total least squares problem, SIAM J Matrix Anal Appl., 17(6) (1980), pp. 883-893.
[13] G.H. Golub, C.F. Van Loan, Matrix Computations(4ed.), Johns Hopkins University Press, Baltimore (2013)
[14] A. Graham, Kronecker Products and Matrix Calculus with Application, Wiley, New York, MR0640865 (83g:15001) (1981)

[15] K. Hermus, W. Verhelst, P. Lemmerling, P. Wambacq, S. Van Huffel, Perceptual audio modeling with exponentially damped sinusoids, Signal Processing, 85 (2005), pp. 163-176.

[16] Z. Jia, B. Li, On the condition number of the total least squares problem. Numer. Math. 125(1) (2013), pp. 61-87.

[17] A.N. Langville, W.J. Stewart, The Kronecker product and stochastic automata networks. J. Comput. Appl. Math. 167(2004), pp. 429-447.

[18] P. Lemmerling, B. De Moor, Misfit versus latency, Automatica, 37(2001), pp. 2057-2067.

[19] P. Lemmerling, N. Mastronardi, S. Van Huffel, Efficient implementation of a structured total least squares based speech compression method, Linear Algebra Appl., 366(2003), pp. 295-315.

[20] B. Li, Z. Jia, Some results on condition numbers of the scaled total least squares problem. Linear Algebra Appl. 435(3) (2011), pp. 674-686.

[21] Q. Liu, C. Chen, Q. Zhang, Perturbation analysis for total least squares problems with linear equality constraint, Applied Numerical Mathematics, 161(2021), pp. 69-81.

[22] Q. Liu, Z. Jia, On condition numbers of the total least squares problem with linear equality constraint, arxiv:2008.08233 [math.NA].

[23] Q. Liu, S. Jin, L. Yao, D. Shen, The revisited total least squares problems with linear equality constraint, Applied numerical mathematics, 152(2020), pp. 275-284.

[24] Q. Meng, H. Diao, Z. Bai, Condition numbers for the truncated total least squares problem and their estimations, arXiv:2004.12082[math.NA]

[25] I. Markovsky, S. Van Huffel, Overview of total least squares methods, Signal Processing, 87(2007), pp. 2283-2302.

[26] L. Meng, B. Zheng and Y. Wei, Condition numbers of the multi-dimensional total least squares problems having more than one solution, Numerical Algorithms, 84 (2020) 887-908.

[27] M. Ng, N. Bose, J. Koo, Constrained total least squares for color image reconstruction, Total Least Squares and Errors-in-Variables Modelling III: Analysis, Algorithms and Applications, Kluwer Academic Publishers, (2002), pp. 365-374.

[28] M. Ng, R. Plemmons, F. Pimentel, A new approach to constrained total least squares image restoration, Linear Algebra Appl., 316(2000), pp. 237-258.

[29] K. Pearson, On lines and planes of closest fit to systems of points in space, Phil. Mag., 2(1901), pp. 559-572.
[30] A. R. De Piero, M. Wei, Some new properties of the equality constrained and weighted least squares problem, Linear Algebra and its Applications 320(1-3) (2000), pp. 145-165.

[31] B. Schaffrin, A note on constrained total least squares estimation, Linear Algebra Appl. 417(2006), pp.245-258.

[32] J.R. Rice, A theory of condition. SIAM J. Numer. Anal. 1966; 3:287-310.

[33] G.W. Stewart, On the asymptotic behavior of scaled singular value and QR decompositions, Mathematics of Computation, 43(168)(1984), pp. 483-489.

[34] G. W. Stewart and J.-G. Sun, Matrix Perturbation Theory, Academic Press, Boston, 1990.

[35] S. Van Huffel, On the significance of nongeneric total least squares problems, SIAM J. Matrix Anal. Appl., 13 (1992), pp. 20-35.

[36] S. Van Huffel, P. Lemmerling, eds. Total Least Squares and Errors-in-Variables Modeling: Analysis, Algorithms and Applications, Kluwer, Dordrecht, Boston, London, (2002).

[37] S. Van Huffel, J. Vandewalle, The Total Least Squares Problems: Computational Aspects and Analysis, Vol. 9 of Frontiers in Applied Mathematics, SIAM, Philadelphia, (1991).

[38] S. Van Huffel and J. Vandewalle, Analysis and solution of the nongeneric total least squares problem, SIAM J. Matrix Anal. Appl., 9 (1988), pp. 360-372.

[39] M. Wei, Algebraic relations between the total least squares and least squares problems with more than one solution, Numer. Math., 62 (1992), pp. 123-148.

[40] M. Wei, Perturbation theory for the rank-deficient equality constrained least squares problem. SIAM J. Numer. Anal. 29:5 (1992), pp. 1462-1481.

[41] M. Wei, A. R. De Piero, Upper perturbation bounds of weighted projections, weighted and constrained least squares problems, SIAM J. Matrix Anal. Appl. 21(3) (2000), pp. 931-951.

[42] P. Xie, H. Xiang, Y. Wei, A contribution to perturbation analysis for total least squares problems, Numerical Algorithms, 75(2) (2017), pp. 381-395.

[43] P. Xie, H. Xiang and Y. Wei, Randomized algorithms for total least squares problems, Numer Linear Algebra Appl., 26 (2019) e2219.

[44] L. Zhou, L. Lin, Y. Wei, S. Qiao, Perturbation analysis and condition numbers of scaled total least squares problems. Numer. Algorithms 51(3)(2009), pp. 381-399.

[45] B. Zheng, L. Meng and Y. Wei, Condition numbers of the multidimensional total least squares problem. SIAM J. Matrix Anal. Appl., 38 (2017), pp. 924-948.

[46] B. Zheng , Z. Yang, Perturbation analysis for mixed least squares-total least squares problems. Numer Linear Algebra Appl. 2019;26:e2239. https://doi.org/10.1002/nla.2239