Focusing of laser-generated ion beams by a plasma cylinder: similarity theory and the thick lens formula

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It is shown that plasma-based optics can be used to guide and focus highly divergent laser-generated ion beams. A hollow cylinder is considered, which initially contains a hot electron population. Plasma streaming toward the cylinder axis maintains a focusing electrostatic field due to the positive radial pressure gradient. The cylinder works as thick lens, whose parameters are obtained from similarity theory for freely expanding plasma in cylindrical geometry. Because the lens parameters are energy dependent, the lens focuses a selected energy range of ions and works as a monochromator. Because the focusing is due to the quasineutral part of the expanding plasma, the lens parameters depend on the hot electron temperature $T_e$ only, and not their density.

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I. INTRODUCTION

Laser-driven ion sources \cite{1,2,3,4,5,6,7} are considered to be the hot candidates for various important applications in nuclear physics, medicine, biology, material sciences, plasma field tomography \cite{8,9,10}. When multi-terawatt laser pulses are shot on solid state targets, copious amounts of multi-MeV ions - both protons and highly charged heavier ions - are generated \cite{11}. These laser-generated ion beams have picosecond durations and originate from a few micrometer wide virtual source. However, the laser-generated ions are highly divergent and usually are emitted within a cone with some 10-30 degrees opening angle. In addition, they have broad energy spectra. These facts may impede numerous applications for the laser-generated ion beams unless appropriate optics and monochromatizing systems are developed.

Because of their high divergence, one needs very strong fields to collimate the ion beams. Such fields exist only in plasma. However, one cannot exploit the standard technique of self-induced magnetic plasma lensing that is widely used to focus conventionally accelerated ion beams. The reason is that the laser-produced ion beams are charge neutral, i.e. they contain electrons that compensate the ion charge and current.

II. SIMILARITY THEORY OF EXPANDING PLASMA

In the present work we consider ion beam focusing by plasma which already contains a quasistatic electric field. The experimental configuration is the following, Fig. 1 \cite{12}: a laser produced ion beam originates at the point $z = -D$, $x = y = 0$ and propagates toward a hollow metal cylinder of the radius $R$ and length $L$, $L > R$. The axis of the cylinder coincides with the $z$-axis. At the same time, the second laser pulse is shot at the cylinder. This second pulse generates a population of hot electrons, which penetrate through the metal and spread very fast over the inner surface of the cylinder. They exit into vacuum and generate a cloud of space charge at the inner surface. The electric field of this space charge is large enough to ionize the material and to create plasma. As a result at the initial moment we have a cylindrical plasma layer with high electron temperature $T_e$ and low ion temperature $T_i \approx 0$. The plasma begins to expand toward the cylinder axis due to the TNSA (target normal sheath acceleration) mechanism \cite{13}. Normally, the cylinder surface is covered by a thin layer of hydrogen-rich substances. Being the lightest ions, protons are accelerated first and the plasma is usually an electron-proton one regardless of the particular chemical mixture of the cylinder itself.

Plasma dynamics is described by the couple of Vlasov’s equations for electrons and ions and the Maxwell equations
The normalized Vlasov-Maxwell equations \((7)\) reveal that the system dynamics depend on five dimensionless parameters. The first parameter is the ion charge \(Z_i\). The next two parameters are the normalized sound speed \(\alpha_e = c_s/c\) and the normalized Debye length \(\alpha_D = \lambda_D^2/4\pi R^2\), where \(\lambda_D^2 = 4\pi T_e/e^2n_e\). These two parameters define plasma dynamic properties. The remaining two parameters \(d/R\) and \(L/R\) come from the initial system geometry. We are interested in the cylindrical geometry and drop out the parameter \(L/R \rightarrow +\infty\).

Thus, the parameteric dependencies can be written as:

\[
\begin{align*}
\hat{f}_e &= \frac{n_e c^4}{T_e^3} \hat{f}_e \left( \frac{t}{\tau}, \frac{r}{R}, \frac{p}{p_e}, \frac{\alpha_e}{\alpha_D} \right), \\
\hat{f}_i &= \frac{n_e}{(MT_e)^{3/2}} \hat{f}_i \left( \frac{t}{\tau}, \frac{r}{R}, \frac{p}{(MT_e)^{1/2}}, \frac{d}{R}, \frac{Z_i}{\alpha_e}, \frac{\alpha_D}{\alpha_D} \right),
\end{align*}
\]

where \(\hat{f}_e\) and \(\hat{f}_i\) are universal functions. Eqs. \((8)\) already can be used to state exact scaling laws. The requirements \(\alpha_e = \text{const}, \alpha_D = \text{const}\) and \(d/R = \text{const}\) do not fix all the dimensional parameters of the problem, and this allows to scale experimental results.

Yet, the most interesting scalings are obtained in the limit \(\alpha_e \ll 1\) and \(\alpha_D < 1\). Assuming \(\alpha_e \rightarrow 0\) one obtains

\[
\frac{\partial \hat{f}_e}{\partial \hat{r}} \bigg|_{\hat{r}=0} = \frac{\hat{\phi}}{\hat{r}}, \quad \frac{\partial \hat{f}_i}{\partial \hat{r}} = \frac{\hat{f}_i}{\hat{r}},
\]

This means that the electron distribution function can be written as

\[
\hat{f}_e = F_e \left( |\hat{p}_e| - \hat{\phi}, \hat{i}, \frac{Z_i}{\alpha_e}, \frac{d}{R}, \frac{\alpha_D}{\alpha_D} \right),
\]

where \(F_e\) is a universal function. Eq. \((10)\) means that the electron fluid has the same effective temperature at all points.

The formal limit \(\alpha_D \rightarrow 0\) coincides with the quasineutrality condition

\[
Z_i \int \hat{f}_i d^3\hat{p} = \int \hat{f}_e d^3\hat{p}.
\]

Since \(\alpha_D\) is a factor in front of the highest derivative in \((7)\), the quasineutrality condition \((11)\) is violated within the narrow Debye sheath layer of the width \(\propto \lambda_D\). Being very important for problems like ion acceleration this area hardly plays any role in the ion focusing. Because of its narrowness, only the small amount of ion beam on the order of \(\lambda_D/R \propto \sqrt{\alpha_D} \ll 1\) would be influenced by its fields at any particular moment. We neglect this influence.

### III. ION FOCUSING BY HOLLOW PLASMA CYLINDER

In order to describe the focusing, we study properties of the quasineutral part of the expanding plasma cloud.
The quasineutrality $n_e \approx Z_i n_i$ is guaranteed as long as $\alpha_D \to 0$. Here $n_e$ and $n_i$ are the electron and ion densities, $Z_i$ is the ion charge state. At the same time, the plasma density and consequently the electron pressure $P_e = n_e T_e$ vary along the cylinder radius. The pressure gradient is counterbalanced by the radial electric field

$$E = -\frac{1}{en_e} \nabla P_e,$$  \hspace{1cm} (12)

which is developed inside the plasma to satisfy the quasineutrality \[14\]. Because the electron pressure gradient is directed off axis, the developed electric field is directed toward the cylinder axis. It is this field that focuses the injected ions.

Because of the cylindrical symmetry, we neglect any dependencies on the azimuthal angle on the longitudinal coordinate $z$ within the plasma. Then, all distributions depend on the radius $\rho = \sqrt{x^2 + y^2}$ only. To obtain a closed system of equations we take into account the energy conservation law

$$\frac{3}{2} \dot{T} e N_e + \pi M L \int \nu^2 f_i(t, \nu, \rho) d\nu \rho d\rho = \frac{3}{2} T e (0) N_0,$$  \hspace{1cm} (13)

where $N_e$ is the number of hot electrons and $(3/2) T e (0) N_e$ is the laser energy absorbed in the cylinder and stored in the hot electrons. Eq. 13 neglects the energy accumulated in the electromagnetic plasma fields. This assumption is correct provided that the Debye length is much smaller than the cylinder radius $R$, i.e. for $\alpha_D \ll 1$. Eqs. \[14\] and \[15\] show that the electron temperature is equal at all points of the plasma. Thus, the energy conservation law 13 is sufficient to describe the electron dynamics.

The initial ion distribution is

$$f_i(t = 0, p, \rho) = 2 \pi \sigma_i \delta(p) \tilde{F}_0(\rho/R, d/R).$$  \hspace{1cm} (14)

where $\sigma_i$ is the initial surface density of ions participating in the plasma expansion. Because of the quasineutrality condition \[14\], we have $N_e = 2 \pi RL \sigma_i$.

We introduce the dimensionless time-dependent electron temperature $\hat{T}(\hat{t}) = T_e(t)/T_e(0)$ and the ion velocity $\hat{v} = v/c_s$.

The ion Vlasov equation and Eqs. \[12\]–\[15\] rewritten in the dimensionless variables take the form:

$$- \dot{\hat{F}} = \hat{\nabla} \ln n_e,$$  \hspace{1cm} (15)

$$\frac{\partial \hat{F}}{\partial \hat{t}} + \hat{v} \frac{\partial \hat{F}}{\partial \hat{\rho}} + \hat{E} \frac{\partial \hat{F}}{\partial \hat{v}} = 0,$$  \hspace{1cm} (16)

$$\int \hat{v}^2 \hat{f}_i(\hat{t}, \hat{v}, \hat{\rho}) d\hat{v} d\hat{\rho} = 3(1 - \hat{T}),$$  \hspace{1cm} (17)

with the initial condition

$$\hat{f}_i(t = 0, \hat{v}, \hat{\rho}) = F_0(\hat{\rho}, d/R) \delta(\hat{v}).$$  \hspace{1cm} (18)

Eqs. \[14\]–\[18\] contain no dimensional parameters whatsoever. As a consequence, the functions $\hat{T}$, $\hat{f}$ and $\hat{E}$ are universal, i.e., they are not affected by specific values of $d$, $R$, $L$, $\sigma_i$ and $T_e(0)$.

This gives us an opportunity to develop a meaningful similarity theory describing the guidance of laser produced ion beams.

From the normalizations \[4\] we conclude that the electric field $\hat{E}$ developed in the plasma is

$$\hat{E} = \frac{T e(0)}{cR} \hat{E}(t/\tau, \rho/R, d/R),$$  \hspace{1cm} (19)

where $\hat{E}$ is a universal function. It does not depend on the plasma density, but is determined by the hot electron temperature and the cylinder geometry only. This result is valid as long as the Debye length is much smaller than $R$. This means that the uncompensated charge density

$$e \delta n = e(Z_i n_i - n_e) \approx \frac{\nabla E}{2 \pi e R^2} e \delta n(t/\tau, \rho = 0, d/R)$$  \hspace{1cm} (20)

is much smaller than the electron density.

When the laser produced ion beam enters plasma inside the cylinder, it is deflected by the electric field \[19\]. We suppose that the beam has a lower density than the plasma inside the cylinder and thus the beam own fields can be neglected.

To describe the beam ion guiding in plasma we consider ions with the charge state $Z_i$, mass $M_i$ and the initial energy $\hat{E}_0$ being focused by the potential

$$\varphi = -\pi \tau^2 e \delta n_0, \quad \delta n_0 = \delta n(t/\tau, \rho = 0, d/R)$$  \hspace{1cm} (21)

Notice that the charge density $\delta n_0$ depends on time. However, for the most interesting and important case the beam ions pass the cylinder plasma during the time $L/ u_b \ll \tau$, where $u_b = \sqrt{2 \hat{E}_0/M_i}$ and $\tau = R/c_s$ is the plasma evolution time. In this case the dependence of $\delta n$ from time $t$ can be neglected.

Now we are able to estimate influence of the non-neutral Debye sheath with the width $\lambda_D$ on the beam ions motion. This area propagates with the velocity $\propto c_s$ and carries the electric field $E_{nq} \propto \sqrt{n_q T_e}$. The radial momentum of a beam ion is changed by the value

$$\Delta \rho_{nq} \propto e Z_i \sqrt{n_q T_e} \frac{\lambda_D}{c_s} \propto \sqrt{\Delta \hat{E}_{nq} M_b},$$  \hspace{1cm} (22)

where $\Delta \hat{E}_{nq} \propto \frac{M_b}{M_i} T_e$.

The change of a beam ion radial momentum due to the interaction with the quasineutral part of plasma is estimated as

$$\Delta \rho_{q} \propto \frac{T_e}{\tau} \frac{L}{u_b} \propto \sqrt{\Delta \hat{E}_q M_b},$$  \hspace{1cm} (23)
where \( \Delta E_q \propto \left( \frac{L}{v_b \tau} \right)^2 \frac{M_b}{M_e} T_e \).

From Eqs. (22) and (24) one sees that \( \Delta p_i^{\parallel} \gg \Delta p_i^\perp \). Therefore the ions passing through the non-quasineutral edge are strongly deviated than those interacting only with the quasineutral plasma region. However, because the Debye sheath is narrow, the relative number of the strongly declined ions is small and these ions are deviated to different points of space. For these reasons the Debye sheath at the edge of the expanding plasma does not contribute to the ion focusing. It scatters the beam ion instead.

IV. ION LENS FORMULA

To investigate focusing properties of the potential \( \varphi \) we use the well known analogy between the geometrical optics and the classical mechanics. The optical length corresponds to the action \( S \) in the Hamilton-Jacoby equation

\[
\partial_t S + H(\nabla S, r) = 0.
\]

If ions in vacuum are injected at the point \( x = y = 0, z = Z \) then the \( S \) function in vacuum is

\[
S = -E_b t + \sqrt{2M_bE_b((z-Z)^2 + \rho^2)} \approx -E_b t + \sqrt{2M_bE_b(z-Z)} + \frac{\rho^2}{z-Z} \sqrt{\frac{M_bE_b}{2}}.
\]

In our geometry, \( Z = -D \).

If ions are focused at the point \( x = y = 0, z = Z' \) in vacuum, then the action \( S \) is

\[
S = \text{const} - E_b t - \sqrt{2M_bE_b((z-Z')^2 + \rho^2)} \approx \text{const} - E_b t + \sqrt{2M_bE_b(z-Z')} + \frac{\rho^2}{z-Z'} \sqrt{\frac{M_bE_b}{2}}.
\]

The beam ion motion inside the plasma cylinder is described by the Hamiltonian

\[
H = \frac{p^2}{2M_b} + eZ_b \varphi.
\]

The solution of the Hamilton-Jacoby equation inside the plasma can be expanded as

\[
S = \text{const} - E_b t + \sqrt{2E_b M_b z + \frac{1}{2} \beta(z) \rho^2} + \ldots,
\]

where the function \( \beta \) is

\[
\beta(z) = \sqrt{-2\pi Z_b M_b e^2 \delta n_0} \tan \left( z \sqrt{-\frac{\pi Z_b e^2 \delta n_0}{E_b} + C} \right).
\]

Note that when the electrons pull the ions behind themselves (positive pressure gradient) there is an electron excess at the axis \( z \), i.e., \( \delta n_0 < 0 \). The constant \( C \) in (24) is obtained from the continuity conditions of the action \( S \) at the front and rear sides of the plasma cylinder.

Thus, we arrive at the thick lens formula

\[
(Z-g)(Z'+h) = -f^2
\]

where \( g = - (L \cos \epsilon)/(\epsilon \sin \epsilon) \), \( h = g + L \), \( f = L/(\epsilon \sin \epsilon) \) and

\[
\epsilon = \sqrt{Z_b \delta n(t/\tau, 0, d/R) \frac{L}{u_b \tau}}.
\]

In our derivation of Eq. (30) we neglected the change of plasma parameters during the time the beam ions need to pass the cylinder. This means that our analysis is valid if \( \epsilon \ll 1 \).

A parallel beam of ions is obtained if \( Z' = \infty \). This condition is satisfied for \( \epsilon \ll 1 \) if

\[
D = \frac{L}{c^2}.
\]

Thus, the plasma element collimates ions with the energy

\[
E_b \approx \frac{Z_b T_e L D}{N^2 R^2}.
\]

It is worth mentioning that the energy of the collimated ions strongly depends on the dimensionless parameter \( L D/R^2 \) and can be significantly larger than the initial electron temperature.

It is easy to see that for the ion focusing be practical the electron temperature \( T_e \) has to be of the order of several MeVs. Such electron temperatures are routinely produced by multi-terawatt lasers.

Would be the plasma inside the cylinder stationary, then only ions with the selected energy \( E_b \) are collimated. However, the plasma is non-stationary with the characteristic evolution time \( \tau = R/c_s \). The relative change of the plasma parameters during the ion passage time through the cylinder is of the order of \( L/v_b \tau \ll 1 \). This small parameter defines finally the finite energy spectrum width \( \delta E_b \) of the focused ions:

\[
\frac{\delta E_b}{E_b} \propto \frac{L}{v_b \tau} \propto \sqrt{\frac{L}{D}}.
\]

It follows from (34) that the plasma cylinder works as a good monochromator if \( D \gg L \).

To avoid any confusion we emphasize that Eq. (34) describes the quality of a small aperture ion beam only. Of course, different parts of the lens collimate ions of
different energies. For large aperture ion beams the energy spectrum width of the focused ions will be large, $\Delta E_b/\hat{E}_b \propto 1$.

In the preceding part of the paper we consider the focusing by the area near $\rho = \rho_0 < R$. To do so we introduce a new potential

$$\varphi_{\rho_0} = -\pi \varepsilon \delta n_{r_0}, \quad \delta n_{\rho_0} = \frac{E(t, \rho_0/R, d/R)}{2\pi \varepsilon \rho_0 R}. \quad (35)$$

According to Eq. (19) the potential $\varphi_{\rho_0}$ gives the right value of the electric field at $\rho = \rho_0$. Thus the focusing by $\rho = \rho_0$ is obtained from Eq. (36) with $\epsilon \ll 1$ by the substitution $\delta n_0 \rightarrow \delta n_{r_0}$.

Until now we have assumed that the density of the ion beam focused is so small that it does not affect the focusing field of the lens. To find the validity condition for this approach we have to consider the propagating of an ion beam with a given density profile $n_i(\rho)$ through the lens. Using the quasineutrality condition for the system “the lens plasma + the beam” one can easily find the focusing by the area around $\rho = \rho_0$ is not disturbed by the ion beam if

$$|\partial_\rho n_b(\rho_0)| \ll |\partial_\rho n_i(t/\tau, \rho_0)|. \quad (36)$$

Since the plasma lens density gradient can be very large this condition can be much weaker than $n_b \ll n_i$.

V. CONCLUSIONS

In conclusion, we have developed a closed similarity theory of a hollow cylinder as a plasma element for ion beam guiding. Significantly, the beam ions are focused by the quasineutral part of expanding plasma rather than by strong electric fields in the non-quasineutral leading edge of the expanding plasma cloud. The thick lens formula has been obtained with explicit scalings for all of the parameters. We show that the plasma lens collimates only ions with a quite definite energy and may be used for monochromatization of the laser-produced ion beams.

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