Q-deformed $SU(1, 1)$ and $SU(2)$ Squeezed and Intelligent States and Quantum Entanglement

Mohamed Taha Rouabah$^1$, Mohamed Farouk Ghiti$^1$, Nouredinne Mebarki$^1$

$^1$Laboratoire de Physique Mathématique et Physique Subatomique, Mentouri University Constantine 1, Route Ain El Bey, 25017 Constantine, Algeria

The intelligent states (IS) associated with the $su_q(1, 1)$ and $su_q(2)$ $q$-deformed Lie algebra are investigated. The eigenvalue problem is also discussed.

I. INTRODUCTION

Intelligent states are quantum states which minimize uncertainty relations for non-commuting quantum observables [1–5]. In the last years there exists a great interest in various properties, applications and generalizations of intelligent states [5–11]. One of the reasons for this interest is the close relationship between intelligent states and squeezing. A generalization of squeezed states for an arbitrary dynamical symmetry group leads to the intelligent states [6–9]. In particular, the concept of squeezing can be naturally extended to the intelligent states associated with the $SU(2)$ and $SU(1, 1)$ Lie groups. An important possible application of squeezing properties of the $SU(2)$ and $SU(1, 1)$ intelligent states is the reduction of the quantum noise in spectroscopy [10] and interferometry [7–12] and hence improve measurement precision. On the other hand, quantum groups are a generalization of symmetry groups which have been used successfully in physics. A general feature of spaces carrying a quantum group structure is that they are noncommutative and inherit a well-defined mathematical structure from quantum group symmetries. In this paper we consider a $q$-deformation of $su(1, 1)$ and $su(2)$ Lie algebras and their IS using the Dyson realization [13].

II. $SU_Q(1, 1)$ INTELLIGENT STATES

The $su(1, 1)$ Lie algebra is spanned by the three generators $K_1, K_2$ and $K_3$ which satisfy the following commutation relations:

$$[K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_+, K_-] - 2K_0. \quad (1)$$

The Casimir operator for any irreducible representation is $K^2 = k(k - 1)I$. Thus a representation of $su(1, 1)$ is determined by the parameter $k$ called the Bergman index. The corresponding Hilbert space is spanned by the complete orthonormal basis $|n, k\rangle$. Since $SU(1, 1)$ is a non compact group, all irreducible representation are of infinite dimensions. Here we shall only deal with the representation known as the positive discrete series in which:

$$K_0 |n, k\rangle = (n + k) |n, k\rangle, \quad K_+ |n, k\rangle = \sqrt{(n + 1)(n + 2k)} |n + 1, k\rangle, \quad K_- |n, k\rangle = \sqrt{n(n + 2k - 1)} |n - 1, k\rangle. \quad (2)$$

On the other hand, the uncertainty relation limits the precise knowledge of conjugate physical quantities of a system. The state which minimize the uncertainty relation can describe the quantum system as precisely as possible. First for a given two self-adjoint operators $A$ and $B$, one can obtain, using the Cauchy-Schwartz inequality, the uncertainty relation:

$$\langle (\Delta A)^2 \rangle \langle (\Delta B)^2 \rangle = \frac{1}{4} \langle [A, B] \rangle^2. \quad (3)$$

where the variance and expectation value are given by $\Delta A = \langle A \rangle^2 - \langle A \rangle^2$ and $\langle \psi | A | \psi \rangle$ respectively. A state is called intelligent if it satisfy the strict inequality in (3). It is well known [2] that such state, or IS, must satisfy the eigenvalue equation

$$(A + i\lambda B) |\psi\rangle = \eta |\psi\rangle, \quad (4)$$
where λ is a positive real parameter and η a complex number. The $SU(1, 1)$ intelligent states (IS) can be derived by considering the special case of Eq. (4), where $| \psi \rangle$ are solutions of the eigenvalues problem

$$(K_1 - i \lambda K_2) | \psi \rangle = \eta | \psi \rangle.$$  \hspace{1cm} (5)

It is convenient to rewrite equation (5) in term of $K_\pm$ as

$$(\alpha K_+ + \beta K_-) | \psi \rangle = 2\eta | \psi \rangle,$$  \hspace{1cm} (6)

where $\alpha = 1 + \lambda$ and $\beta = 1 - \lambda$.

Let us expand the state $\psi$ on the basis $| n, k \rangle$

$$| \psi \rangle = \sum_{n=0}^{\infty} c_n(k) | n, k \rangle,$$  \hspace{1cm} (7)

and apply (6) to obtain the recurrence relation among the coefficients $c_n$ as fellow [14]

$$\alpha \sqrt{(n + 1)(n + 2k)} c_{n+1} + \beta \sqrt{n(n + 2k - 1)} c_{n-1} = 2\eta c_n.$$  \hspace{1cm} (8)

The $q$-deformed algebra $su_q(1, 1)$ is given as [15]

$$[Q_0, Q_\pm] = Q_\pm, \quad [Q_+, Q_-] = -2 [Q_0]_q,$$  \hspace{1cm} (9)

where the $q$-deformation is defined as

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$  \hspace{1cm} (10)

One can obtain the explicit form of the $q$-deformed generators following [16] and [17] in Dyson realization as

$$Q_0 = K_0 = N + k,$$

$$Q_- = K_- \sqrt{[N]q \over N},$$

$$Q_+ = K_+ \sqrt{[N]q [N + 2k - 1]q \over N + 2k - 1},$$  \hspace{1cm} (11)

where $N$ is the number operator and $k$ is assumed to be a positive integer or half odd integer. We require the conjugate relation $Q_-| n, k \rangle = Q_+ | n, k \rangle$ to be independent of the realizations. These generators act on the ket as

$$Q_- | n, k \rangle = \sqrt{[n]q (n + 2k - 1)} | n - 1, k \rangle,$$

$$Q_+ | n, k \rangle = \sqrt{[n + 1]q (n + 2k)} {n + 2k \over n + 2k} | n + 1, k \rangle.$$  \hspace{1cm} (12)

The $q$-deformed $SU(2)$ IS are solution of the following eigenvalue equation

$$(\alpha Q_+ + \beta Q_-) | \psi \rangle_q = 2\eta | \psi \rangle_q.$$  \hspace{1cm} (13)

Substituting Eq.(7) in Eq.(13), and applying (12) we obtain the recurrence relation among the coefficients $c_n$ as follows:

$$\alpha \sqrt{[n + 1]q (n + 2k)} c_{n+1} + \beta \sqrt{[n]q (n + 2k - 1)} {n + 2k - 1 \over n + 2k - 1} c_{n-1} = 2\eta c_n.$$  \hspace{1cm} (14)

Assuming $c_n = (\beta q / \alpha q) d_n$ where $\alpha_q = \alpha \sqrt{[n + 1]q \over n + 1}$ and $\beta_q = \beta \sqrt{[n]q \over n + 2k - 1}$, we obtain:

$$\frac{1}{2} \sqrt{(n + 1)(n + 2k)} d_{n+1} + \frac{1}{2} \sqrt{n(n + 2k - 1)} d_{n-1} - z d_n = 0,$$  \hspace{1cm} (15)
where \( z^{(q)} = \frac{\eta}{\sqrt{\alpha_q \beta_q}} \).

Comparing Eq.(15) with the Pollaczek polynomials \( P_n \) [18–21] the solution to the eigenvalue equation (13) is directly the Pollaczek polynomials, namely,

\[
d_n = P_n(z^{(q)},k) = i^n \left( \frac{\Gamma[n + 2k]}{n!\Gamma[2k]} \right)^{n/2} F_1(\eta, n, k + i z^{(q)}; 2k; 2). \tag{16}
\]

We thus obtain the final result for the IS in the form

\[
| \psi \rangle_q = c_0 \sum_{n=0}^{\infty} i^n \left( \frac{\Gamma[n + 2k]}{n!(1 + \lambda)\Gamma[2k]} \right)^{n/2} F_1(\eta, n, k + i z^{(q)}; 2k; 2) | n, k \rangle,
\]

where the normalization factor \( c_0 \) has the form

\[
| c_0 |^2 = \sum_{n=0}^{\infty} i^n \left( \frac{\Gamma[n + 2k]}{n!(1 + \lambda)\Gamma[2k]} \right)^{n/2} F_1(\eta, n, k + i z^{(q)}; 2k; 2)^2. \tag{18}
\]

III. \( SU_q(2) \) INTELLIGENT STATES

\( su(2) \) generators satisfy the algebra

\[
[J_0, J_\pm] = J_\pm, \quad [J_+, J_-] = 2J_0,
\]

and the Casimir operator is given as \( C = J_0(J_0 + 1) - J_-J_+ \). We introduce the eigenstates of the angular momentum as \( | j, n-j \rangle \) where

\[
\begin{align*}
J_0 \ | j, n-j \rangle &= (n-j) \ | j, n-j \rangle, \\
J_+ \ | j, n-j \rangle &= \sqrt{j(j+1) - (n-j)(n-j+1)} \ | j, n-j+1 \rangle, \\
J_- \ | j, n-j \rangle &= \sqrt{j(j+1) - (n-j)(n-j-1)} \ | j, n-j-1 \rangle,
\end{align*}
\]

where \( n = 0, 1, 2, \ldots, 2j \). The Hilbert space is finite dimensional with dimension \( 2j + 1 \). IS corresponding to \( SU(2) \) generators satisfy the following eigenvalue equation [3]

\[
(J_1 - i \lambda J_2) \ | \psi \rangle = \eta \ | \psi \rangle,
\]

using the raising and lowering angular momentum operators we may write Eq.(21) in the following form

\[
(\alpha J_- + \beta J_+) \ | \psi \rangle = 2\eta \ | \psi \rangle. \tag{22}
\]

We may also expand \( | \psi \rangle \) in terms of the angular momentum eigenstates \( | j, n-1 \rangle \) as

\[
| \psi \rangle = \sum_{n=0}^{\infty} c_n(j) \ | j, n-j \rangle. \tag{23}
\]

Substituting Eq.23 in Eq.22, we obtain the recurrence relation for the coefficients \( c_n \).

The \( su_q(2) \) IS can be studied in close analogy with the previous section and, therefore, we will describe briefly \( SU_q(2) \) algebra only. The \( q \)-deformed \( su(2) \) algebra is given as [2]

\[
[Q_0, Q_\pm] = Q_\pm, \quad [Q_+, Q_-] = 2 [Q_0]_q. \tag{24}
\]

One can obtain the explicit form of the \( q \)-deformed generators following [3] and [4] in Dyson realization as

\[
\begin{align*}
Q_0 &= J_0 = j - N, \\
Q_- &= J_- \sqrt{\frac{[N]_q}{N}}, \\
Q_+ &= J_+ \sqrt{\frac{[N]_q [J - N + 1]_q}{N(J - N + 1)}},
\end{align*}
\]

where \( [N]_q = \frac{\Gamma_q(N+1)}{\Gamma_q(N+1)} \).
where N is the number operator. We require the conjugate relation \( Q_{-}^\dagger = Q_{+} \) and \( Q_{0}^\dagger = Q_{0} \) to be independent of the realizations. These generators act on the ket as

\[
Q_{-} \mid j, n - j \rangle = \sqrt{n} q(2j - n + 1) \mid j, n - j - 1 \rangle
\]

\[
Q_{+} \mid j, n - j \rangle = \sqrt{n + 1} q(2j - n + 1) \mid j, n - j + 1 \rangle.
\]  

(26)

The q-deformed SU(2) IS are solution of the following eigenvalue equation

\[
(\alpha Q_{-} + \beta Q_{+}) \mid \psi \rangle_q = 2\eta \mid \psi \rangle_q.
\]  

(27)

Let us now consider the eigenvalue problem (27) we apply Eq. (26) to obtain the recurrence relation among the coefficients \( c_n \) as follows: Substituting Eq. (23) in (27) and applying Eq. (26) we obtain the following recurrence relation among the coefficients \( c_n \):

\[
\alpha \sqrt{n + 1} q(2j - n)c_{n+1} + \beta \sqrt{n} q(2j - n + 1) \frac{2j - n + 1}{2j - n + 1} c_{n-1} = 2\eta c_n.
\]  

(28)

Assuming \( c_n = (\beta' q/\alpha') \frac{c}{q} d_n \) where \( \alpha' = \alpha \sqrt{n + 1} q \) and \( \beta' = \beta \sqrt{n} q \), we obtain:

\[
\frac{1}{2} \sqrt{(n + 1)(n + 2j)} d_{n+1} + \frac{1}{2} \sqrt{n(n + 2j + 1)} d_{n-1} - z' d_n = 0,
\]  

(29)

where \( z'(q) = \frac{\eta}{\sqrt{\alpha' q/\beta'} q} \).

Comparing Eq. (29) with the Pollaczek polynomials \( P_n(\theta, b) [5–7] \), the solution to the eigenvalue equation (29) is directly the Pollaczek polynomials, namely

\[
d_n = P_n(z'(q), j) = i^n \left( \frac{\Gamma[n + 2j]}{n! \Gamma[2j]} \right)^{n/2} 2F_1 (-n, j + iz'(q); 2j; 2).
\]  

(30)

Thus, we obtain the final result for the IS in the form

\[
\mid \psi \rangle_q = c_0 \sum_{n=0}^{2j} i^n \left( \frac{(1 - \lambda) \Gamma[n + 2j]}{n!(1 + \lambda) \Gamma[2j]} \right)^{n/2} 2F_1 (-n, j + iz'(q); 2j; 2) \mid j, n - j + 1 \rangle,
\]  

(31)

where the normalization factor \( c_0 \) has the form

\[
| c_0 |^2 = \sum_{n=0}^{2j} i^n \left( \frac{(1 - \lambda) \Gamma[n + 2j]}{n!(1 + \lambda) \Gamma[2j]} \right)^{n/2} | 2F_1 (-n, j + iz'(q); 2j; 2) |^2.
\]  

(32)

IV. CONCLUSION

A remarkable property in a q-deformed bipartite composite system is the existence of a natural entangled structure for a non-classical value of the q-deformation parameter (\( q \neq 1 \)). It appears to be useful for extending the horizon of studies on entangled non orthogonal states so as to incorporate systems with quantum algebraic symmetries [15]. Composite systems with quantum symmetries, such as anyons for instance, are natural candidates.

V. ACKNOWLEDGMENT

We are very grateful to the Algerian Ministry of education and research, DGRSDT and ANDRU for the financial support.

[1] R. Jackiw, Journal of Mathematical Physics 9, 339 (1968), URL http://scitation.aip.org/content/aip/journal/jmp/9/3/10.1063/1
2. C. Aragone, G. Guerri, S. Salamo, and J. L. Tani, Journal of Physics A: Mathematical, Nuclear and General 7, L149 (1974), URL http://stacks.iop.org/0301-0015/7/i=15/a=001.
3. C. Aragone, E. Chalbaud, and S. Salamo, Journal of Mathematical Physics 17, 1963 (1976), URL http://scitation.aip.org/content/aip/journal/jmp/17/11/10.1063/1.522835.
4. G. V. Berghe and H. D. Meyer, Journal of Physics A: Mathematical and General 11, 1569 (1978), URL http://stacks.iop.org/0305-4470/11/i=8/a=017.
5. K. Wódkiewicz and J. H. Eberly, J. Opt. Soc. Am. B 2, 458 (1985), URL http://josab.osa.org/abstract.cfm?URI=josab-2-3-458.
6. M. M. Nieto and D. R. Truax, Phys. Rev. Lett. 71, 2843 (1993), URL http://link.aps.org/doi/10.1103/PhysRevLett.71.2843.
7. M. Hillery and L. Moidinow, Phys. Rev. A 48, 1548 (1993), URL http://link.aps.org/doi/10.1103/PhysRevA.48.1548.
8. D. Yu and M. Hillery, Quantum Optics: Journal of the European Optical Society Part B 6, 37 (1994), URL http://stacks.iop.org/0954-8998/6/i=1/a=005.
9. D. A. Trifonov, Journal of Mathematical Physics 35, 2297 (1994), URL http://scitation.aip.org/content/aip/journal/jmp/35/5/10.1063/1.530553.
10. G. S. Agarwal and R. R. Puri, Phys. Rev. A 49, 4968 (1994), URL http://link.aps.org/doi/10.1103/PhysRevA.49.4968.
11. C. C. Gerry and R. Grobe, Phys. Rev. A 51, 4123 (1995), URL http://link.aps.org/doi/10.1103/PhysRevA.51.4123.
12. C. Brif and Y. Ben-Aryeh, Journal of Physics A: Mathematical and General 27, 8185 (1994), URL http://stacks.iop.org/0305-4470/27/i=24/a=025.
13. F. J. Dyson, Phys. Rev. 102, 1217 (1956), URL http://link.aps.org/doi/10.1103/PhysRev.102.1217.
14. G. M. A. Al-Kader and A.-S. F. Obada, Physica Scripta 78, 035401 (2008), URL http://stacks.iop.org/1402-4896/78/i=3/a=035401.
15. J. Jimbo, Yang-Baxter Equation in Integrable Systems (World Scientific, Singapore, 1989).
16. T. Curtright and C. Zachos, Physics Letters B 243, 237 (1990), ISSN 0370-2693, URL http://www.sciencedirect.com/science/article/pii/037026939090845W.
17. P. Oh and C. Rim, Reports on Mathematical Physics 40, 285 (1997), ISSN 0034-4877, URL http://www.sciencedirect.com/science/article/pii/S0034487797929265.
18. A. Erdélyi, The Hypergeometric Function in Higher Transcendental Functions, vol. 1 (McGraw-Hill, New York, 1953).
19. A. Erdélyi, Orthogonal Polynomials in Higher Transcendental Functions, vol. 2 (McGraw-Hill, New York, 1953).
20. B. Nagel, Spectra and Generalized Eigenfunctions of the One- and Two-Mode Squeezing Operators in Quantum Optics, Springer Netherlands, Dordrecht, 1995), pp. 211-220, ISBN 978-94-015-8543-9, URL https://doi.org/10.1007/978-94-015-8543-9_19.
21. M. Hillery, Phys. Rev. A 36, 3796 (1987), URL http://link.aps.org/doi/10.1103/PhysRevA.36.3796.