Fractional Leibniz integral rules for Riemann-Liouville and Caputo fractional derivatives and their applications

Ismail T. Huseynov, Arzu Ahmadova, Nazim I. Mahmudov

Department of Mathematics, Faculty of Arts and Sciences, Eastern Mediterranean University, Mersin 10, Gazimagusa, TRNC, Turkey

Abstract
In recent years, the theory for Leibniz integral rule in the fractional sense has not been able to get substantial development. As an urgent problem to be solved, we study a Leibniz integral rule for Riemann-Liouville and Caputo type differentiation operators with general fractional-order of \(0 < \alpha < n\), \(n \in \mathbb{N}\). A rule of fractional differentiation under integral sign with general order is necessary and applicable tool for verification by substitution for candidate solutions of inhomogeneous multi-term fractional differential equations. We derive explicit analytical solutions of generalized Bagley-Torvik equations in terms of recently defined bivariate Mittag-Leffler type functions that based on fractional Green’s function method and verified solutions by substitution in accordance by applying the fractional Leibniz integral rule. Furthermore, we study an oscillator equation as a special case of differential equations with multi-orders via the Leibniz integral rule.

Keywords: Caputo fractional derivative, Riemann-Liouville fractional derivative, Leibniz integral rule, Bagley-Torvik equation, oscillator equation, bivariate Mittag-Leffler function

1. Introduction
Fractional calculus is a generalization of the classical differential calculus which has attracted growing attention due to the applications for many problems in science and engineering such as reaction-diffusion systems [1], viscoelasticity [2], electrical circuits [3, 4], control theory [5], stochastic analysis [6] and time-delay systems [7].

One of the most frequently encountered tools in the theory of fractional calculus is furnished by the Riemann-Liouville \(RL_t^\alpha f(t)\) and Caputo \(C_t^\alpha f(t)\) fractional differentiation operators. A fractional analogue of Leibniz rule for differentiation is crucial and useful properties of these operators. Podlubny in [8], Baleanu and Trujillo in [9] give a proof the Leibniz rule for Riemann-Liouville and Caputo type derivatives; these results are stated respectively below.

Assume that \(0 < \alpha < 1\) and \(f, g : [t_0, T] \subset \mathbb{R} \rightarrow \mathbb{R}\) with all their derivatives are continuous. Then

\[
RL_t^\alpha \{f(t)g(t)\} = \sum_{k=0}^{\infty} \binom{\alpha}{k} f^{(k)}(t) RL_t^{\alpha-k} g(t), \quad t \in (t_0, T), \tag{1.1}
\]

\[
C_t^\alpha \{f(t)g(t)\} = \sum_{k=0}^{\infty} \binom{\alpha}{k} f^{(k)}(t) RL_t^{\alpha-k} g(t) - \frac{(t - t_0)^{-\alpha}}{\Gamma(1 - \alpha)} f(t_0)g(t_0), \quad t \in (t_0, T), \tag{1.2}
\]

where binomial coefficients satisfy the identity:

\[
\binom{\alpha}{0} = 1, \quad \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\ldots(\alpha-k+1)}{k!}.
\]

*Corresponding author
Email addresses: ismail.huseynov@emu.edu.tr (Ismail T. Huseynov), arzu.ahmadova@emu.edu.tr (Arzu Ahmadova), nazim.mahmudov@emu.edu.tr (Nazim I. Mahmudov)

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Another essential property of Riemann-Liouville fractional differentiation operator is obtained by Podlubny in \cite{8} which is called fractional Leibniz integral rule stated as below:

\[
R_L^0 D_t^\alpha \int_0^t K(t, \tau) d\tau = \lim_{\tau \to t^-} R_L^0 D_t^{\alpha-1} K(t, \tau) + \int_0^t R_L^0 D_\tau^\alpha K(t, \tau) d\tau, \quad \alpha \in (0, 1), \tag{1.3}
\]

where lower terminal \( t_0 = 0 \).

The following important particular case for convolution operator whenever we have \( K(t - \tau) f(\tau) \) instead of \( K(t, \tau) \), the relationship \( 1.3 \) takes the form:

\[
R_L^0 D_t^\alpha \int_0^t K(t - \tau) f(\tau) d\tau = \lim_{\tau \to t^-} f(t - \tau)^{R_L^0 D_t^{\alpha-1}} K(\tau) + \int_0^t R_L^0 D_\tau^\alpha K(\tau) f(t - \tau) d\tau, \quad \alpha \in (0, 1). \tag{1.4}
\]

It is important to note that the above tools are necessary for checking by substitution method for fractional differential equations with variable and constant coefficients.

Fractional differential equations (FDEs) are differential equations involving derivatives of arbitrary (fractional) order. FDEs provide one of the most accurate tools to describe hereditary properties of natural phenomenon. Using fractional derivatives instead of integer-order derivatives allows us for the modeling of a wider variety of behaviours. However, sometimes, FDEs involving one fractional order of differentiation are not sufficient to demonstrate physical processes. Therefore, recently, several authors have studied more general types of fractional-order models, such as multi-term equations \cite{10, 11, 12, 13, 14} and multi-dimensional systems \cite{4, 6, 7, 15, 16, 17, 18}.

Multi-term differential equations with fractional-order have been studied and solved using various mathematical methods, of which we mention a few as follows. Luchko and several collaborators \cite{10, 11} have used the method of operational calculus to solve multi-term FDEs with constant coefficients with regard to various types of fractional derivatives. Bazhlekova \cite{12} has considered multi-term fractional relaxation equations with Caputo fractional derivatives by using a Laplace transform technique, and studied the fundamental and impulse-response solutions of the initial value problem (IVP). Kaczorek and Idczak \cite{17} have considered existence and uniqueness results and a Cauchy formula for the analytical solution of the time-varying linear system with Caputo fractional derivative. Pak et al. \cite{13} has recently investigated multi-term FDEs with variable coefficients using a new method to construct analytical solutions.

As one of the important special cases of multi-term FDEs, Bagley-Torvik equations have been discussed in terms of analytical \cite{8, 13, 20, 21} and numerical methods \cite{22, 23}. Bagley-Torvik equations with \( \frac{1}{2} \)-order or \( \frac{3}{4} \)-order derivative describe the motion of real physical systems in a Newtonian fluid \cite{19}. In 1984, Bagley and Torvik \cite{19} have considered the following Cauchy problem under the homogeneous initial conditions:

\[
m p''(r) + \frac{2S \sqrt{\mu \rho}}{m} \left( C D_{0+}^\alpha y \right)(r) + ky(r) = g(r), \quad r > 0, \tag{1.5}
\]

\[
g(0) = y'(0) = 0,
\]

where \( \left( C D_{0+}^\alpha y \right)(\cdot) \) is Caputo fractional differential operator of order \( \alpha = \frac{1}{2} \) or \( \alpha = \frac{3}{4} \), \( S \)- an area of the rigid plate, \( \mu \)-viscosity, \( \rho \)-fluid density, \( m \)-mass, \( k \)-spring of stiffness and \( g(\cdot) \)-an external force. An analytical solution of \( 1.5 \) has introduced by Podlubny \cite{8} in the form:

\[
g(r) = \int_0^r G(r - \tau) g(\tau) d\tau, \quad r > 0, \tag{1.6}
\]

with

\[
G(r) = \frac{1}{m} \sum_{l=0}^{\infty} \frac{(-1)^l l!}{l!} \left( \frac{k}{m} \right)^l r^{2l+1} \mathcal{E}(l) \frac{2^l}{2^l} \left( \frac{-2S \sqrt{\mu \rho}}{m} \right)^{2l+1} + \frac{\left( \frac{-2S \sqrt{\mu \rho}}{m} \right)^{\frac{3}{2}}}{\mathcal{E}(l)} + \frac{\left( \frac{-2S \sqrt{\mu \rho}}{m} \right)^{\frac{5}{2}}}{\mathcal{E}(l)} + \cdots,
\]

where

\[
\mathcal{E}(l) = \sum_{j=0}^{\infty} \frac{(2l+1)!!}{2^l l!} \left( \frac{-2S \sqrt{\mu \rho}}{m} \right)^{2l+1},
\]

and

\[
\frac{2^l}{2^l} = \left\{ \begin{array}{ll} 1 & \text{if } l \text{ is even}, \\ 0 & \text{if } l \text{ is odd}. \end{array} \right.
\]
where $\mathcal{E}^{(l)}_{\alpha,\beta} (\cdot)$ is the $l$th-derivative of two-parameter Mittag-Leffler function. In \cite{20}, Mahmudov et al. have studied explicit analytical solutions for several families of generalized multidimensional Bagley-Torvik equations with permutable matrices. In \cite{21}, Wang et al. have modified the following Bagley-Torvik equation

$$y''(r) + \mu \left( C D^2_{0^+} y \right)(r) + g(r) = 0, \quad \mu, r > 0, \quad (1.7)$$

where $\alpha = \frac{1}{2}$ or $\alpha = \frac{3}{2}$, to the sequential FDEs and introduced a general solution of \cite{17} by using the technique related to characteristic roots. The numerical point of view Diethelm and Ford in \cite{22} have used linear multi-steps, Srivastava et al. in \cite{23} have applied wavelet approach to obtain approximate solutions of the Bagley-Torvik equations.

Therefore, the plan of this paper is systematized as below. Section 2 is a mathematical preliminary section where we recall main definitions and results from fractional calculus, special functions and necessary lemmas from fractional differential equations. Section 3 is devoted to formulating the Leibniz integral rule for higher order derivatives of Lebesgue integration which depends on parameter in classical sense. In Section 4, we have introduced fractional differentiation under the integral sign in Riemann-Liouville and Caputo sense. Moreover, we have considered the derivative of convolution operator which has more importance for differential equations with classical or fractional order. In Section 5, we have acquired explicit analytical solutions of Bagley-Torvik equations with Riemann-Liouville and Caputo type fractional derivatives in terms of recently defined bivariate Mittag-Leffler type functions in accordance with fractional Green’s function method and tested the candidate solutions by using our newly defined tools which are natural generalization of well-known Leibniz integral rule. At the end, in Section 6 we give the conclusions and future directions.

2. Mathematical preliminaries

We embark on this section by briefly introducing the essential structure of fractional calculus, special functions and fractional differential operators (for the more salient details on the matter, see the textbooks \cite{8, 23, 26, 28, 42, 44}). We begin by defining some notations, Riemann-Liouville and Caputo fractional differentiation operators which are fundamental for fractional calculus and fractional differential equations.

Let $\mathbb{R}^n$ be Euclidean space and $J$ be some interval of the real line, i.e. $J \subset \mathbb{R}$. We suppose that $J = [t_0, T]$ for some $t \in J$ and denote $\hat{J} = (t_0, T)$. Assume that $f : \hat{J} \to \mathbb{R}$ is an absolutely continuous function.

**Definition 2.1** (\cite{28, 42, 44}). The Riemann-Liouville derivative operator of fractional order $n - 1 < \alpha \leq n$ for $n \in \mathbb{N}$ is defined by

$$\left( ^{RL}_{t_0} D^\alpha_t g \right)(t) = \frac{d^n}{dt^n} \left( \int_{t_0}^{t} \frac{g(s)}{(t-s)^{\alpha-n+1}} ds \right), \quad t \in \hat{J}, \quad (2.1)$$

where $\int_{t_0}^{t} \frac{g(s)}{(t-s)^{\alpha-n+1}} ds$ is the Riemann-Liouville integral operator of order $\alpha > 0$ which is defined by

$$\left( t_0 I^\alpha_t g \right)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} g(s) ds, \quad t \in \hat{J}. \quad (2.2)$$

Furthermore, the following equality holds true:

$$\left( ^{RL}_{t_0} D^\alpha_t \left( t_0 I^\alpha_t g \right) \right)(t) = g(t), \quad \alpha > 0, \quad t \in \hat{J}. \quad (2.3)$$

**Definition 2.2** (\cite{8, 23, 27}). The Caputo derivative operator of fractional order $n - 1 < \alpha \leq n$ for $n \in \mathbb{N}$ is defined by

$$\left( ^{C}_{t_0} D^\alpha_t g \right)(t) = \int_{t_0}^{t} \frac{d^n}{ds^n} \frac{g(s)}{(t-s)^{\alpha-n+1}} ds, \quad t \in \hat{J}. \quad (2.4)$$

Moreover, the next relation holds true:

$$\left( ^{C}_{t_0} D^\alpha_t \left( t_0 I^\alpha_t g \right) \right)(t) = g(t), \quad \alpha > 0, \quad t \in \hat{J}. \quad (2.5)$$
The relationship between Riemann-Liouville and Caputo fractional derivatives are as follows:

\[(\mathcal{C}_{t_0}^\alpha D_t^\alpha g)(t) = (\mathcal{R}_0^L D_t^\alpha g)(t) - \sum_{k=0}^{n-1} \frac{(t-t_0)^{k-\alpha} f^{(k)}(t_0)}{\Gamma(k-\alpha+1)}, \quad n-1 < \alpha \leq n, \quad n \in \mathbb{N}. \tag{2.6}\]

The following results are useful in solving fractional differential equations.

**Definition 2.3** ([17, 48]). A function \( g \) is said to be *exponentially bounded* on \([0, \infty)\) if it satisfies an inequality of the form:

\[|g(t)| \leq Me^{\sigma t}, \quad t \geq T,\]

for some real constants \( M > 0, \ T > 0 \) and \( \sigma \in \mathbb{R} \).

**Definition 2.4** ([17, 48]). If \( g : [0, \infty) \to \mathbb{R} \) is exponentially bounded for \( t \geq 0 \), then the Laplace integral transform \( \mathcal{L} \{g(t)\}(s) \) defined by

\[G(s) = \mathcal{L} \{g(t)\}(s) = \int_0^\infty e^{-st} g(t) dt,\]

exists for \( s \in \mathbb{C} \) and is an analytic function of \( s \) for \( \Re(s) > 0 \) and Laplace inversion formula is defined as

\[\mathcal{L}^{-1} \{G(s)\}(t) := \frac{1}{2\pi i} \int_L e^{st} G(s) ds,\]

where \( g(t) = \mathcal{L}^{-1} \{G(s)\}(t), \ t \geq 0 \) and \( L \) is a closed contour which enclosing the poles (singularities) of \( g \).

**Definition 2.5** ([3]). The Laplace integral transform of Riemann-Liouville fractional derivative of order \( \alpha \in (n-1, n], \ n \in \mathbb{N} \) is given by:

\[\mathcal{L} \{ (\mathcal{R}_0^L D_t^\alpha y) \}(s) = s^\alpha Y(s) - \sum_{k=1}^{n} s^{k-1} (\mathcal{R}_0^L D_t^\alpha y)^{(k)}(0), \tag{2.7}\]

where \( Y(s) \) represents the Laplace transform of the function \( y(t) \).

**Remark 2.1** ([3]). In the special cases, the Laplace integral transform of the Riemann-Liouville fractional differentiation is:

- If \( \alpha \in (0, 1] \), then
  \[\mathcal{L} \{ (\mathcal{R}_0^L D_t^\alpha y) \}(s) = s^\alpha Y(s) - (\mathcal{R}_0^L D_t^{\alpha-1} y)(0).\]

- If \( \alpha \in (1, 2] \), then
  \[\mathcal{L} \{ (\mathcal{R}_0^L D_t^\alpha y) \}(s) = s^\alpha Y(s) - (\mathcal{R}_0^L D_t^{\alpha-1} y)(0) - s (\mathcal{R}_0^L D_t^{\alpha-2} y)(0).\]

**Definition 2.6.** The Laplace integral transform of Caputo fractional derivative of order \( \alpha \in (n-1, n], \ n \in \mathbb{N} \) is given by:

\[\mathcal{L} \{ (\mathcal{C}_0^\alpha D_t^\alpha y) \}(s) = s^\alpha Y(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} y^{(k)}(0), \tag{2.8}\]

where \( Y(s) \) represents the Laplace transform of the function \( y(t) \).

**Remark 2.2.** In the special cases, the Laplace integral transform of the Caputo fractional differentiation is:
• If $\alpha \in (0,1]$, then
  $$\mathcal{L}\left\{\left(\frac{C}{0} D^\alpha y\right) (t)\right\}(s) = s^\alpha Y(s) - s^{\alpha-1} y_0, \quad \text{where} \quad y_0 = y(0).$$

• If $\alpha \in (1,2]$, then
  $$\mathcal{L}\left\{\left(\frac{C}{0} D^\alpha y\right) (t)\right\}(s) = s^\alpha Y(s) - s^{\alpha-1} y_0 - s^{\alpha-2} y'_0, \quad \text{where} \quad y_0 = y(0) \quad \text{and} \quad y'_0 = y'(0).$$

**Definition 2.7** ([47, 48]). Let $f$ and $g$ be both piece-wise continuous functions on $[0, \infty)$. Then the integral in
$$f * g := (f * g)(t) = \int_0^t f(t-s)g(s)ds,$$

is called the convolution operator of two functions $f$ and $g$ which is well-defined and finite for any $t \geq 0$ and it has the commutativity property:
$$f * g = g * f.$$

**Theorem 2.1** ([47, 48]). Suppose that $f$ and $g$ are piece-wise continuous and exponentially bounded functions on $[0, \infty)$. Then the Laplace transform of convolution operator of two functions $f$ and $g$, given on $[0, \infty)$, has the following property:
$$\mathcal{L}\{ (f * g) (t) \} (s) = \mathcal{L}\{ f (t) \} (s) \mathcal{L}\{ g (t) \} (s), \quad s \in \mathbb{C}.$$

The Mittag-Leffler function is a generalization of the exponential function, first proposed in 1903 [29] as a single-parameter function of one variable, defined using a convergent infinite series. Extensions to two, three and multi-parameters are well known and thoroughly studied in textbooks such as [26, 31] which are involving single power series in one variable [32, 33, 34]. Extensions to two, three, or more variables, involving correspondingly double, triple, or multiple power series, have been studied more recently [18, 35, 36, 37].

**Definition 2.8** ([29]). The classical Mittag-Leffler function is defined by
$$E_\alpha(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(i\alpha + 1)}, \quad \alpha > 0, t \in \mathbb{R}.$$

**Remark 2.3** ([25]). The Mittag-Leffler functions are often used in a form where the variable inside the brackets is not $t$ but a fractional power $t^\alpha$, or even a constant multiple $\lambda t^\alpha$, as follows:
$$E_\alpha(\lambda t^\alpha) = \sum_{i=0}^{\infty} \frac{\lambda^i t^{i\alpha}}{\Gamma(i\alpha + 1)}, \quad \alpha > 0, t, \lambda \in \mathbb{R}.$$

The two-parameter Mittag-Leffler function [26] is given by
$$E_{\alpha,\beta}(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(i\alpha + \beta)}, \quad \alpha > 0, \beta \in \mathbb{R}, t \in \mathbb{R}.$$

The $l$-th derivative of two-parameter Mittag-Leffler function [26] is defined by
$$\frac{d^l}{dt^l} E_{\alpha,\beta}(t) = E_{\alpha,\beta}^{(l)}(t) = \sum_{i=0}^{\infty} \frac{(i+l)!}{i!} \frac{t^i}{\Gamma(i\alpha + l\alpha + \beta)}, \quad l \in \mathbb{N}, \alpha > 0, \beta \in \mathbb{R}, t \in \mathbb{R}.$$

The three-parameter Mittag-Leffler function [30] is determined by
$$E_{\alpha,\beta}^{\gamma}(t) = \sum_{i=0}^{\infty} \frac{(\gamma)_i}{\Gamma(i\alpha + \beta)} \frac{t^i}{i!}, \quad \alpha > 0, \beta, \gamma \in \mathbb{R}, t \in \mathbb{R},$$

where $(\gamma)_i$ is the Pochhammer symbol denoting $\frac{\Gamma(\gamma + i)}{\Gamma(\gamma)}$. These series are convergent, locally uniformly in $\tau$, provided the $\alpha > 0$ condition is satisfied. Note that
$$E_{\alpha,\beta}^{1}(t) = E_{\alpha,\beta}(t), \quad E_{\alpha,1}(t) = E_{\alpha}(t), \quad E_{1}(t) = \exp(t).$$
The next lemma includes Laplace integral transform of three-parameter Mittag-Leffler function which will be used throughout the proof of Lemma 2.2.

**Lemma 2.1.** For \( \alpha > \beta > 0, \lambda \in \mathbb{R}, l \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) and \( \Re(s) > 0 \), we have:

\[
\mathcal{L}^{-1}\left\{\frac{1}{(s^\alpha - \lambda s^\beta)^{l+1}}\right\}(\tau) = t^{(l+1)\alpha-1}\sum_{k=0}^{\infty} \binom{l+k}{k} \frac{\lambda^k k^{\alpha-\beta}}{\Gamma(k(\alpha-\beta)+(l+1)\alpha)}
\]

Taking inverse Laplace transform of the above function, we get the desired result:

\[
\mathcal{L}^{-1}\left\{\frac{1}{(s^\alpha - \lambda s^\beta)^{l+1}}\right\}(t) = \sum_{k=0}^{\infty} \lambda^k \binom{l+k}{k} \mathcal{L}^{-1}\left\{\frac{1}{s^{k(\alpha-\beta)+(l+1)\alpha}}\right\}(t)
\]

which is the required result. We have required an extra condition on \( s \) for convergence of the binomial type series in the Laplace domain, namely that

\[ s^{\alpha-\beta} > |\lambda| \]

However, this condition can be removed at the end, by analytic continuation of both sides of the identity, to give the desired result for all \( s \in \mathbb{C} \) satisfying \( \Re(s) > 0 \). The proof is complete.

**Definition 2.9 (33).** We consider the bivariate Mittag-Leffler function defined by

\[
E_{\alpha,\beta,\gamma}^\delta(u, v) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\delta)_{l+k}}{\Gamma(l\alpha + k\beta + \gamma)} \frac{u^l v^k}{l! k!}, \quad \alpha, \beta > 0, \gamma, \delta \in \mathbb{R}, u, v \in \mathbb{R}.
\]  

(2.9)

If we write \( u = \lambda t^\alpha \) and \( v = \mu t^\beta \) for a single variable \( t \), and multiply by a power function \( t^{\gamma-1} \), we derive the following univariate version:

\[
t^{\gamma-1}E_{\alpha,\beta,\gamma}^\delta(\lambda t^\alpha, \mu t^\beta) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\delta)_{l+k}}{\Gamma(l\alpha + k\beta + \gamma)} \frac{\lambda^l \mu^k}{l! k!} t^{\alpha+k\beta+\gamma-1}.
\]  

(2.10)

Note that when \( \delta = 1 \),

\[
E_{\alpha,\beta,\gamma}^1(\lambda t^\alpha, \mu t^\beta) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(1)_{l+k}}{\Gamma(l\alpha + k\beta + \gamma)} \frac{\lambda^l \mu^k}{l! k!} t^{\alpha+k\beta+\gamma-1}
\]

\[
= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(l+k)!}{l! k!} \frac{\lambda^l \mu^k}{\Gamma(l\alpha + k\beta + \gamma)} t^{\alpha+k\beta+\gamma-1}
\]

\[
= \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \binom{l+k}{k} \frac{\lambda^l \mu^k}{\Gamma(l\alpha + k\beta + \gamma)} t^{\alpha+k\beta+\gamma-1}.
\]
Proof. We have the following formula for Caputo derivatives of power functions \(8, 27\):

\[
\mathcal{L}^{-1}\left\{ \frac{s^{\gamma}}{s^{\alpha} - \mu s^{\beta} - \lambda} \right\}(t) = t^{\alpha - \gamma - 1} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(l + k) \lambda^l k^l (\alpha - \beta + \gamma)}{(l + l + k) \Gamma(l + k + (\alpha - \beta) + \alpha - \gamma)}
\]

\[
= t^{\alpha - \gamma - 1} E_{\alpha, \alpha - \beta, \alpha - \gamma}(\lambda t^{\alpha}, \mu t^{\alpha - \beta}).
\]

Lemma 2.2. For \(\alpha > \beta, \alpha > \gamma, \lambda, \mu \in \mathbb{R}\) and \(\Re(s) > 0\), the following result holds true:

\[
\mathcal{L}^{-1}\left\{ \frac{s^{\gamma}}{s^{\alpha} - \mu s^{\beta} - \lambda} \right\}(t) = t^{\alpha - \gamma - 1} E_{\alpha, \alpha - \beta, \alpha - \gamma}(\lambda t^{\alpha}, \mu t^{\alpha - \beta}).
\]

Proof. \(\frac{s^{\gamma}}{s^{\alpha} - \mu s^{\beta} - \lambda}\) can be written via a series expansion as follows:

\[
\frac{s^{\gamma}}{s^{\alpha} - \mu s^{\beta} - \lambda} = \frac{s^{\gamma}}{s^{\alpha} - \mu s^{\beta} - 1} \frac{1}{s^{\alpha - \beta} - s^{\alpha - \gamma}} = \sum_{l=0}^{\infty} \frac{\lambda l^{\gamma}}{s^{(l+1)\alpha}(1 - \frac{\mu}{s^{\alpha - \beta}})^{l+1}}
\]

Then applying Lemma 2.1 to the last expression, we acquire that

\[
\frac{s^{\gamma}}{s^{\alpha} - \mu s^{\beta} - \lambda} = \sum_{l=0}^{\infty} \lambda l^{\gamma} s^{(l+1)\alpha} \sum_{k=0}^{\infty} \frac{(l + k) \mu^k}{s^{\alpha - \beta}} \frac{0}{l^{\alpha - \gamma}}
\]

Taking inverse Laplace transform of the aforementioned function, we attain:

\[
\mathcal{L}^{-1}\left\{ \frac{s^{\gamma}}{s^{\alpha} - \mu s^{\beta} - \lambda} \right\}(t) = \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(l + k) \mu^k}{s^{\alpha \gamma}} \frac{1}{l^{\alpha - \gamma}} E_{\alpha, \alpha - \beta, \alpha - \gamma}(\lambda t^{\alpha}, \mu t^{\alpha - \beta}),
\]

which is the desired result. We have required extra conditions on \(s\) for convergence of the binomial type series in the Laplace domain, namely that

\[
s^{\alpha - \beta} > |\mu|,
\]

\[
|s^{\alpha} - \mu s^{\beta}| > |\lambda|.
\]

However, these conditions can be removed at the end, by analytic continuation of both sides of the identity, to give the desired result for all \(s \in \mathbb{C}\) satisfying \(\Re(s) > 0\). The proof is complete.

Lemma 2.3. For any parameters \(\alpha, \beta, \gamma, \lambda, \mu \in \mathbb{R}\) satisfying \(\alpha, \beta > 0\) and \(\gamma - 1 > |\alpha|\), we have

\[
\mathcal{C}_0 D_t^\gamma \left[ t^{\alpha - \gamma - 1} E_{\alpha, \alpha - \beta, \alpha - \gamma}(\lambda t^{\alpha}, \mu t^{\alpha - \beta}) \right] = t^{\alpha - \gamma - 1} E_{\alpha, \alpha - \beta, \alpha - \gamma}(\lambda t^{\alpha}, \mu t^{\alpha - \beta}), \quad t > 0.
\]

Proof. We have the following formula for Caputo derivatives of power functions \(8, 27\):

\[
\mathcal{C}_0 D_t^\eta \left( \frac{t^{\eta}}{\Gamma(\eta + 1)} \right) = \begin{cases} 
\frac{t^{\eta}}{\Gamma(\eta + 1)}, & \eta > |\nu|, \\
0, & \eta = 0, 1, 2, \ldots, |\nu|, \\
\text{undefined}, & \text{otherwise}.
\end{cases}
\]

(2.12)
Therefore, the given condition \( \gamma - 1 > |\alpha| \), from (2.12) we can attain

\[
\frac{C_0 D_t}{C_0 D_t} \left[ t^{\gamma-1} E_{\alpha,\beta,\gamma}(\lambda t^\alpha, \mu t^\beta) \right] = \frac{C_0 D_t}{C_0 D_t} \left[ \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(l+k)^l}{k!} \Gamma(\lambda t^\alpha + k + \gamma - 1) \right] = t^{\gamma-\alpha} E_{\alpha,\beta,\gamma-\alpha}(\lambda t^\alpha, \mu t^\beta), \quad t > 0.
\]

The proof is complete. \( \square \)

**Lemma 2.4.** For any parameters \( \alpha, \beta, \gamma, \lambda, \mu \in \mathbb{R} \) satisfying \( \alpha, \beta, \gamma > 0 \), we have

\[
\frac{RL D_t}{RL D_t} \left[ t^{\gamma-1} E_{\alpha,\beta,\gamma}(\lambda t^\alpha, \mu t^\beta) \right] = t^{\gamma-\alpha} E_{\alpha,\beta,\gamma-\alpha}(\lambda t^\alpha, \mu t^\beta), \quad t > 0.
\] (2.13)

**Proof.** We have the following formula for Riemann-Liouville derivatives of power functions [8, 27]:

\[
\frac{RL D_t}{RL D_t} \left( \frac{t^\eta}{\Gamma(\eta + 1)} \right) = \frac{t^{\eta-\nu}}{\Gamma(\eta - \nu + 1)}, \quad \nu, \eta \in \mathbb{R}, \quad \eta > -1.
\] (2.14)

Therefore, given the condition \( \gamma > 0 \), in accordance with (2.13) we will get the same result with (2.11). The proof is complete. \( \square \)

**Definition 2.10.** Let \( \lambda_i, \mu_j \in \mathbb{R} \), \( \alpha_i, \beta_j \in \mathbb{R} \), \( i = 1, 2, \ldots, p \), \( j = 1, 2, \ldots, q \). Generalized Wright function or Fox-Wright function \( p \Psi_q(\cdot) : \mathbb{R} \rightarrow \mathbb{R} \) is defined by

\[
p \Psi_q(t) = \prod_{k=0}^{p} \Gamma(\lambda_i + \alpha_i, \beta_j, \mu_j) \frac{t^k}{k!}.
\] (2.15)

The Fox-Wright function was established by Fox [45] and Wright [46]. If the following condition holds

\[
\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1,
\]

then the series in (2.15) is convergent for arbitrary \( t \in \mathbb{R} \).

3. **Leibniz integral rule**

In this section, we formulate Leibniz integral rule for higher order derivatives on Lebesgue integration. It is known that according to the suitable conditions, we can differentiate under the integral sign for Lebesgue integrals [38]. We begin with the first derivative of a Lebesgue integral on \( X \subseteq \mathbb{R} \).

**Theorem 3.1** ([38]). Assume that \( X, Y \subseteq \mathbb{R} \) are intervals. Suppose also that the function \( f : X \times Y \rightarrow \mathbb{R} \) satisfies the following assumptions:

(a) For every fixed \( y \in Y \), the function \( f(\cdot, y) \) is measurable on \( X \);

(b) The partial derivative \( \frac{\partial}{\partial y} f(x, y) \) exists for every interior point \( (x, y) \in X \times Y \);

(c) There exists a non-negative integrable function \( g \) such that \( \left| \frac{\partial}{\partial y} f(x, y) \right| \leq g(x) \) exists for every interior point \( (x, y) \in X \times Y \);

(d) There exists \( y_0 \in Y \) such that \( f(x, y_0) \) is integrable on \( X \).
Then for every $y \in Y$, the Lebesgue integral
\[
\int_X f(x, y) \, dx
\]
exists. Furthermore, the function $F : Y \to \mathbb{R}$, defined by
\[
F(y) = \int_X f(x, y) \, dx
\]
for every $y \in Y$, is differentiable at every interior point of $Y$, and the derivative of $F(y)$ satisfies
\[
F'(y) = \int_X \frac{\partial}{\partial y} f(x, y) \, dx.
\]

The well-known rule for the differentiation of an integral depending on a parameter with the upper limit also depends on the same parameter, namely:

**Corollary 3.1** \(\text{[39]}\). If $X = (y_0, y)$ and assumptions of Theorem 3.1 are fulfilled, then the following relation holds true for all $y \in Y$:
\[
\frac{d}{dy} \int_{y_0}^{y} f(x, y) \, dx = \int_{y_0}^{y} \frac{\partial}{\partial y} f(x, y) \, dx + \lim_{x \to y-0} f(x, y), \quad y \in X. \tag{3.1}
\]

So, the formula of differentiation under the integral sign for $K(t, s)$ with respect to $t$ is
\[
\frac{d}{dt} \int_{t_0}^{t} K(t, s) \, ds = \frac{d}{dt} \left( \int_{t_0}^{t} K(t, s) \, ds \right) + \lim_{s \to t-0} K(t, s), \quad t \in \mathbb{J}. \tag{3.2}
\]

Using the formula (3.2), we define the second-order derivative of the integral depending on $t$:
\[
\frac{d^2}{dt^2} \int_{t_0}^{t} K(t, s) \, ds = \frac{d}{dt} \left( \frac{d}{dt} \int_{t_0}^{t} K(t, s) \, ds \right) = \frac{d}{dt} \left( \lim_{s \to t-0} K(t, s) + \int_{t_0}^{t} \frac{\partial}{\partial t} K(t, s) \, ds \right)
\]
\[
= \frac{d}{dt} \lim_{s \to t-0} K(t, s) + \frac{d}{dt} \int_{t_0}^{t} \frac{\partial}{\partial t} K(t, s) \, ds
\]
\[
= \frac{d}{dt} \lim_{s \to t-0} K(t, s) + \lim_{s \to t-0} \frac{\partial}{\partial t} K(t, s) + \int_{t_0}^{t} \frac{\partial^2}{\partial t^2} K(t, s) \, ds
\]
\[
= \sum_{l=1}^{\infty} \frac{d^{l-1}}{dt^{l-1}} \lim_{s \to t-0} \frac{\partial^{2-l}}{\partial t^{2-l}} K(t, s) + \int_{t_0}^{t} \frac{\partial^2}{\partial t^2} K(t, s) \, ds, \quad t \in \mathbb{J}.
\]

Then, the third-order differentiation of the integral will be:
\[
\frac{d^3}{dt^3} \int_{t_0}^{t} K(t, s) \, ds = \frac{d}{dt} \left( \frac{d^2}{dt^2} \int_{t_0}^{t} K(t, s) \, ds \right)
\]
\[
= \frac{d}{dt} \left( \frac{d}{dt} \lim_{s \to t-0} K(t, s) + \lim_{s \to t-0} \frac{\partial}{\partial t} K(t, s) + \int_{t_0}^{t} \frac{\partial^2}{\partial t^2} K(t, s) \, ds \right)
\]

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Theorem 3.2. Let the function \( K: \mathbb{J} \times \mathbb{J} \to \mathbb{R} \) be such that the following assumptions are fulfilled:

(a) For every fixed \( t \in \mathbb{J} \), the function \( \frac{\partial^n}{\partial t^n} K(t, s) \) is measurable on \( \mathbb{J} \) and integrable on \( \mathbb{J} \) with respect to \( s \) for some \( t^* \in \mathbb{J} \);

(b) The partial derivative \( \frac{\partial^n}{\partial t^n} K(t, s) \) exists for every interior point \( (t, s) \in \mathbb{J} \times \mathbb{J} \);

(c) There exists a non-negative integrable function \( g(s) \) such that \( |\frac{\partial^n}{\partial t^n} K(t, s)| \leq g(s) \) for every interior point \( (t, s) \in \mathbb{J} \times \mathbb{J} \);

(d) The derivative \( \frac{\partial^{n-1}}{\partial t^{n-1}} \lim_{s \to t^n} K(t, s), l = 1, 2, \ldots, n \) exists for every interior point \( (t, s) \in \mathbb{J} \times \mathbb{J} \).

Then, the following relation holds true for the \( n \)th derivative under Lebesgue integration for \( n \in \mathbb{N} \):

\[
\frac{d^n}{dt^n} \int_{t_0}^{t} K(t, s)ds = \sum_{l=1}^{n} \frac{d^{n-1}}{dt^{n-1}} \lim_{s \to t^n} K(t, s) + \int_{t_0}^{t} \frac{\partial^n}{\partial t^n} K(t, s)ds, \quad t \in \mathbb{J}.
\] (3.3)

Proof. Using mathematical induction principle, we prove above theorem. It is obvious that the equation (3.3) is true for \( n = 1 \) \( \mathbb{J} \):

\[
\frac{d}{dt} \int_{t_0}^{t} K(t, s)ds = \lim_{s \to t_0} K(t, s) + \int_{t_0}^{t} \frac{\partial}{\partial t} K(t, s)ds.
\]

We assume that (3.3) holds true for \( n = k \):

\[
\frac{d^k}{dt^k} \int_{t_0}^{t} K(t, s)ds = \sum_{l=1}^{k} \frac{d^{k-1}}{dt^{k-1}} \lim_{s \to t^k} K(t, s) + \int_{t_0}^{t} \frac{\partial^k}{\partial t^k} K(t, s)ds, \quad t \in \mathbb{J}.
\]
We prove that (3.3) is true for \( n = k + 1 \):

\[
\frac{d^{k+1}}{dt^{k+1}} \int_{t_0}^t K(t, s)ds = \frac{d}{dt} \left( \frac{d^k}{dt^k} \int_{t_0}^t K(t, s)ds \right)
= \frac{d}{dt} \left( \sum_{l=1}^k \frac{d^{l-1}}{dt^{l-1}} \lim_{s \to t-0} \frac{\partial^{k-l}}{\partial t^{k-l}} K(t, s) + \int_{t_0}^t \frac{d^k}{dt^k} K(t, s)ds \right)
= \frac{d}{dt} \left( \lim_{s \to t-0} \frac{\partial^{k-1}}{\partial t^{k-1}} K(t, s) + \frac{d}{dt} \lim_{s \to t-0} \frac{\partial^{k-2}}{\partial t^{k-2}} K(t, s) + \cdots + \frac{d^{k-1}}{dt^{k-1}} \lim_{s \to t-0} K(t, s) \right)
+ \int_{t_0}^t \frac{d^k}{dt^k} K(t, s)ds
= \frac{d}{dt} \lim_{s \to t-0} \frac{\partial^{k-1}}{\partial t^{k-1}} K(t, s) + \frac{d^2}{dt^2} \lim_{s \to t-0} \frac{\partial^{k-2}}{\partial t^{k-2}} K(t, s) + \cdots + \frac{d^k}{dt^k} \lim_{s \to t-0} K(t, s)
+ \lim_{s \to t-0} \frac{\partial}{\partial t^k} K(t, s) + \int_{t_0}^t \frac{\partial^{k+1}}{\partial t^{k+1}} K(t, s)ds
= \sum_{l=1}^{k+1} \frac{d^{l-1}}{dt^{l-1}} \lim_{s \to t-0} \frac{\partial^{k-l+1}}{\partial t^{k-l+1}} K(t, s) + \lim_{s \to t-0} \frac{\partial^k}{\partial t^k} K(t, s) + \int_{t_0}^t \frac{\partial^{k+1}}{\partial t^{k+1}} K(t, s)ds
= \sum_{l=1}^{k+1} \frac{d^{l-1}}{dt^{l-1}} \lim_{s \to t-0} \frac{\partial^{k-l+1}}{\partial t^{k-l+1}} K(t, s) + \int_{t_0}^t \frac{\partial^{k+1}}{\partial t^{k+1}} K(t, s)ds, \quad t \geq t_0.
\]

Therefore, the formula (3.3) holds true for all \( n \in \mathbb{N} \) and \( t \geq t_0 \). \( \square \)

The following important particular case must be defined for convolution operator of the functions \( f \) and \( g \).

**Corollary 3.2.** If \( K(t, s) = f(t-s)g(s) \) and \( t_0 = 0 \), and assumptions of Theorem 3.2 are satisfied, then the following relation is true for any \( n \in \mathbb{N} \):

\[
\frac{d^n}{dt^n} \int_0^t f(t-s)g(s)ds = \sum_{l=1}^n \frac{d^{l-1}}{dt^{l-1}} \lim_{s \to t-0} \frac{\partial^{n-l}}{\partial t^{n-l}} f(t-s) \frac{d^{l-1}}{dt^{l-1}} \lim_{s \to t-0} g(s)
+ \int_0^t \frac{\partial^n}{\partial t^n} f(t-s)g(s)ds, \quad t > 0.
\]

**Proof.** If we write \( f(t-s)g(s) \) instead of \( K(t, s) \) in (3.3), then we obtain

\[
\frac{d^n}{dt^n} \int_0^t f(t-s)g(s)ds = \sum_{l=1}^n \frac{d^{l-1}}{dt^{l-1}} \left( \lim_{s \to t-0} \left( \frac{\partial^{n-l}}{\partial t^{n-l}} f(t-s)g(s) \right) \right)
+ \int_0^t \frac{\partial^n}{\partial t^n} f(t-s)g(s)ds
= \sum_{l=1}^n \frac{d^{l-1}}{dt^{l-1}} \left( \lim_{s \to t-0} \frac{\partial^{n-l}}{\partial t^{n-l}} f(t-s) \lim_{s \to t-0} g(s) \right)
\]
Thus, the proof is complete. \(\square\)

4. Fractional Leibniz integral rules

Now, we are starting to prove fractional Leibniz integral rule for Riemann-Liouville fractional derivative of order \(\alpha \in (n-1,n], n \in \mathbb{N}\). For this, firstly, let us consider partial Riemann-Liouville fractional differentiation operator of order \(n-1 < \alpha < n\), \(n \in \mathbb{N}\) with respect to \(t\) of a function \(K(t,s)\) of two variables \((t,s) \in \mathbb{J} \times \mathbb{J}\), \(K: \mathbb{J} \times \mathbb{J} \to \mathbb{R}\), defined by \([28, 42, 44]\):

\[
\mathcal{K}_0^t J_t^t \mathcal{K}(t,s) = \frac{\partial}{\partial t} \mathcal{K}_0^t J_t^t \mathcal{K}(t,s) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K(s,\tau)ds, \quad t \in \mathbb{J};
\]

where \(\mathcal{K}_0^t J_t^t \mathcal{K}\) is the partial Riemann-Liouville integral operator of order \(\alpha > 0\) which is given by:

\[
\mathcal{K}_0^t J_t^s \mathcal{K}(t,s) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} K(s,\tau)ds, \quad \text{for} \quad t \in \mathbb{J}.
\]

The following important result in the theory of fractional calculus was first proposed by Podlubny \([8]\) for \(\alpha \in (0,1]\) in Riemann-Liouville sense as follows:

\[
\mathcal{R}^n_{\mathcal{T}_0} \mathcal{K}(t,s) = \lim_{\alpha \to 0} \mathcal{R}^n_{\mathcal{T}_0} \mathcal{K}(t,s) + \int_{\mathcal{T}_0}^{t} \mathcal{R}^n_{\mathcal{T}_0} \mathcal{K}(t,s)ds, \quad t > \mathcal{T}_0.
\]

Now, we are going to state and prove the following theorem for more general case where \(\alpha \in (n-1,n], n \in \mathbb{N}\) which is more useful tool for the testing particular solution of inhomogeneous linear multi-order differential equations with variable coefficients. Note that Matychyn has proposed \([11]\) Leibniz integral rule for Riemann-Liouville derivative of order \(0 < \alpha \leq 1\) on Lebesgue integration.

**Theorem 4.1.** Let the function \(K: \mathbb{J} \times \mathbb{J} \to \mathbb{R}\) be such that the following assumptions are fulfilled:

(a) For every fixed \(t \in \mathbb{J}\), the function \(\hat{K}(t,s) = \mathcal{R}^n_{\mathcal{T}_0} \mathcal{K}(t,s)\) is measurable on \(\mathbb{J}\) and integrable on \(\mathbb{J}\) with respect to some \(t^* \in \mathbb{J}\);

(b) The partial derivative \(\mathcal{R}^n_{s} \mathcal{K}(t,s)\) exists for every interior point \((t,s) \in \mathbb{J} \times \mathbb{J}\);

(c) There exists a non-negative integrable function \(g\) such that \(\left| \mathcal{R}^n_{s} \mathcal{K}(t,s) \right| \leq g(s)\) for every interior point \((t,s) \in \mathbb{J} \times \mathbb{J}\);

(d) The derivative \(\mathcal{R}^n_{s} \mathcal{K}(t,s)\) exists for every interior point \((t,s) \in \mathbb{J} \times \mathbb{J}\);

Then, the following relation holds true for fractional derivative in Riemann-Liouville sense under Lebesgue integration:

\[
\mathcal{R}^n_{\mathcal{T}_0} \mathcal{K}(t,s) = \sum_{l=1}^{n} \mathcal{R}^n_{\mathcal{T}_0} \mathcal{K}(t,s) + \int_{\mathcal{T}_0}^{t} \mathcal{R}^n_{\mathcal{T}_0} \mathcal{K}(t,s)ds, \quad t \in \mathbb{J}.
\]
Proof. Using the Definition 2.1 and Fubini’s theorem 38, we have

\[ \mathcal{R}L_{t_0}^{\alpha} \int_{t_0}^{t} K(t, s) ds = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} d\tau \int_{t_0}^{\tau} K(\tau, s) ds \]

\[ = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} K(\tau, s) ds d\tau \]

\[ = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{s}^{t} (t-\tau)^{n-\alpha-1} K(\tau, s) d\tau ds \]

\[ = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{s}^{t} (t-\tau)^{n-\alpha-1} K(\tau, s) d\tau \]

\[ = \frac{d^n}{dt^n} \int_{s}^{t} RL_{s}^{n-\alpha} K(t, s) ds. \]

Using the formula 3.3 for the last part of above expression, we get a desired result:

\[ \mathcal{R}L_{t_0}^{\alpha} \int_{t_0}^{t} K(t, s) ds = \sum_{l=1}^{n} \lim_{s \to t^{-}} \frac{d^{l-1}}{dt^{l-1}} \partial_{\alpha-1}^{n-l} s I_{s}^{n-\alpha} K(t, s) + \int_{t_0}^{t} \partial_{\alpha}^{n} s I_{s}^{n-\alpha} K(t, s) ds \]

\[ = \sum_{l=1}^{n} \lim_{s \to t^{-}} RL_{s}^{l} D_{s}^{n-l} K(t, s) + \int_{t_0}^{t} RL_{s}^{l} D_{s}^{n} K(t, s) ds, \quad t \in \hat{J}. \]

Corollary 4.1. If we have \( K(t, s) = f(t-s)g(s), t_0 = 0 \) and assumptions of Theorem 4.1 are fulfilled, then following equality holds true for convolution operator in Riemann-Liouville sense for any \( n \in \mathbb{N} \):

\[ \mathcal{R}L_{0}^{\alpha} \int_{0}^{t} f(t-s)g(s) ds = \sum_{l=1}^{n} \lim_{s \to t^{-}} RL_{s}^{l} D_{s}^{n-l} f(t-s) \frac{d^{l-1}}{dt^{l-1}} \lim_{s \to t^{-}} g(s) \]

\[ + \int_{0}^{t} RL_{s}^{l} D_{s}^{n} f(t-s)g(s) ds, \quad t > 0. \]  

(4.3)

Proof. If we write \( f(t-s)g(s) \) instead of \( K(t, s) \) in (4.2), then we obtain

\[ \mathcal{R}L_{0}^{\alpha} \int_{0}^{t} f(t-s)g(s) ds = \sum_{l=1}^{n} \frac{d^{l-1}}{dt^{l-1}} \left( \lim_{s \to t^{-}} RL_{s}^{l} D_{s}^{n-l} f(t-s)g(s) \right) \]

\[ + \int_{0}^{t} RL_{s}^{l} D_{s}^{n} f(t-s)g(s) ds \]

\[ = \sum_{l=1}^{n} \lim_{s \to t^{-}} RL_{s}^{l} D_{s}^{n-l} f(t-s) \lim_{s \to t^{-}} g(s) \]

\[ = \sum_{l=1}^{n} \lim_{s \to t^{-}} RL_{s}^{l} D_{s}^{n} f(t-s) \lim_{s \to t^{-}} g(s) \]

\[ = \sum_{l=1}^{n} \lim_{s \to t^{-}} RL_{s}^{l} D_{s}^{n-l} f(t-s)g(s). \]
Thus, the proof is complete.

Then, we are going to introduce fractional differentiation under the integral sign in Caputo sense which will be useful for checking the candidate solutions of fractional differential equations with multi-orders. For this, firstly, let us consider partial Caputo fractional differentiation operator of order \( \alpha \) with respect to \( t \), firstly, let us consider partial Caputo fractional differentiation operator of order \( \alpha \) with respect to \( t \), we have

\[
\frac{D_t^\alpha}{D_t^\alpha} f(t-s)g(s) = \frac{1}{\Gamma(n-\alpha+1)} \int_0^t (t-s)^{n-\alpha-1} g(s)ds, \quad t > 0.
\]

More generally, the fractional Leibniz integral rule for fractional derivative of order \( \alpha \) with respect to \( t \) of a function \( K(t,s) \) of two variables \( (t,s) \in \mathbb{J} \times \mathbb{J} \), \( K : \mathbb{J} \times \mathbb{J} \to \mathbb{R} \), defined by \( \mathbb{R} \):

\[
C_{t_0}^t D_t^\alpha K(t,s) = \frac{1}{\Gamma(n-\alpha+1)} \int_0^t (t-s)^{n-\alpha-1} \frac{\partial^n}{\partial s^n} K(s,\tau)ds, \quad t \in \mathbb{J}, \tag{4.4}
\]

Matyshyn and Onyshchenko \( [40] \) showed that the fractional Leibniz integral rule for Caputo fractional derivative coincide with Riemann–Liouville one when \( \alpha \in (0,1] \):

\[
\frac{C_{t_0}^t}{D_t^\alpha} \int_{t_0}^t K(t,s)ds = \frac{1}{\Gamma(n-\alpha+1)} \int_0^t (t-s)^{n-\alpha-1} \frac{\partial^n}{\partial s^n} K(s,\tau)ds, \quad t \in \mathbb{J}.
\]

More generally, the fractional Leibniz integral rule for fractional derivative of order \( \alpha \in (n-1,n] \), \( n \geq 2 \) in Caputo sense is stated and proved in the following theorem.

**Theorem 4.2.** Let the function \( K : \mathbb{J} \times \mathbb{J} \to \mathbb{R} \) be such that the following assumptions are fulfilled.

(a) For every fixed \( t \in \mathbb{J} \), the function \( K(t,s) = C_{t_0}^t D_t^\alpha K(t,s) \) is measurable and integrable on \( \mathbb{J} \) with respect to some \( t^* \in \mathbb{J} \);

(b) The partial derivative \( C_{t_0}^t D_t^\alpha K(t,s) \) exists for every interior point \( (t,s) \in \mathbb{J} \times \mathbb{J} \);

(c) There exists a non-negative integrable function \( g \) such that \( |C_{t_0}^t D_t^\alpha K(t,s)| \leq g(s) \) for every interior point \( (t,s) \in \mathbb{J} \times \mathbb{J} \);

(d) The integral \( \int_{t_0}^t I_t^{n-\alpha} \left\{ \frac{d^{l-1}}{dt^{l-1}} \lim_{s \to t-0} \frac{\partial^{n-l}}{\partial \tau^{n-l}} K(t,\tau) \right\} , \quad l = 1, \ldots, n, \quad n \in \mathbb{N} \) exists for every interior point \( (t,s) \in \mathbb{J} \times \mathbb{J} \).

Then, the following relation holds true for fractional derivative in Caputo sense under Lebesgue integration:

\[
\frac{C_{t_0}^t}{D_t^\alpha} \int_{t_0}^t K(t,s)ds = \frac{1}{\Gamma(n-\alpha+1)} \int_0^t (t-s)^{n-\alpha-1} \frac{\partial^n}{\partial s^n} K(t,s)ds, \quad t \in \mathbb{J}. \tag{4.5}
\]

**Proof.** Using the Definition \( \mathbb{R} \), Fubini’s theorem \( \mathbb{R} \), and the formula \( \mathbb{R} \), we have

\[
\frac{C_{t_0}^t}{D_t^\alpha} \int_{t_0}^t K(t,s)ds = \frac{1}{\Gamma(n-\alpha+1)} \int_0^t (t-s)^{n-\alpha-1} \frac{\partial^n}{\partial s^n} K(t,s)ds
\]

\[
= \frac{1}{\Gamma(n-\alpha+1)} \int_0^t (t-s)^{n-\alpha-1} \sum_{l=1}^n \frac{d^{l-1}}{dt^{l-1}} \lim_{\tau \to s-0} \frac{\partial^{n-l}}{\partial \tau^{n-l}} K(t,\tau)d\tau
\]
Therefore, the proof is complete.

\[ + \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} d\tau \int_0^\tau \frac{\partial^n}{\partial \tau^n} K(\tau, s) d\tau \]

\[ = \frac{\partial^n}{\partial \tau^n} \left( \int_0^t (t-\tau)^{n-\alpha-1} d\tau \right) \int_0^\tau \frac{\partial^n}{\partial \tau^n} K(\tau, s) d\tau \]

\[ = \frac{\partial^n}{\partial \tau^n} \sum_{l=1}^n \frac{d^{l-1}}{dt^{l-1}} \lim_{\tau \to 0+ \tau} \frac{\partial^{n-l}}{\partial \tau^{n-l}} K(t, s) \]

\[ + \frac{1}{\Gamma(n-\alpha)} \int_0^t \int_0^\tau (t-\tau)^{n-\alpha-1} d\tau d\sigma \int_0^\tau \partial^n K(\tau, s) d\tau \]

\[ = \frac{\partial^n}{\partial \tau^n} \sum_{l=1}^n \frac{d^{l-1}}{dt^{l-1}} \lim_{\tau \to 0+ \tau} \frac{\partial^{n-l}}{\partial \tau^{n-l}} K(t, s) \]

Therefore, the proof is complete. \( \Box \)

However, the fractional Leibniz integral rule for Caputo derivative of order \( 0 < \alpha \leq 1 \) is different from the general case which is given in the relation \( [13] \).

**Theorem 4.3.** Let the function \( K : \mathbb{J} \times \mathbb{J} \to \mathbb{R} \) be such that the following assumptions are fulfilled.

(a) For every fixed \( t \in \mathbb{J} \), the function \( K(t, s) = \frac{\partial^n}{\partial s^n} K(t, s) \) is measurable on \( \mathbb{J} \) and integrable on \( \mathbb{J} \) with respect to some \( t' \in \mathbb{J} \).

(b) The partial derivative \( \frac{\partial^n}{\partial t^n} K(t, s) \) exists for every interior point \( (t, s) \in \mathbb{J} \times \mathbb{J} \).

(c) There exists a non-negative integrable function \( g \) such that \( |\frac{\partial^n}{\partial t^n} K(t, s)| \leq g(s) \) for every interior point \( (t, s) \in \mathbb{J} \times \mathbb{J} \).

Then, the Caputo fractional derivative under Lebesque integration coincides with the fractional differentiation of an integral in Riemann-Liouville sense for \( 0 < \alpha \leq 1 \):
Corollary 4.2. If we have \( K(t, s) = f(t - s)g(s), \) \( t_0 = 0, \) and assumptions of Theorem 4.3 are fulfilled, then following equality holds true for convolution operator in Caputo’s sense of order \( \alpha \in (n - 1, n], \) where \( n \geq 2: \)

\[
\mathcal{C}_0^D_t^\alpha \int_{t_0}^t K(t, s)ds = \lim_{s \to t_0} t^{\alpha - 1}_t K(t, s) + \int_{t_0}^t R^\alpha_L \mathcal{C}_t^\alpha K(t, s)ds, \quad t \in \mathbb{J}. \tag{4.6}
\]

**Proof.** In accordance the formula (2.6), it is obvious for \( \alpha \in (0, 1]: \)

\[
\left( \mathcal{C}_0^D_t^\alpha g \right)(t) = \left( R^\alpha_L \mathcal{C}_0^D_t^\alpha g \right)(t) - \frac{(t - t_0)^{-\alpha}}{\Gamma(1 - \alpha)} g(t_0), \quad t \in \mathbb{J}. \tag{4.7}
\]

Since the formula (4.7), we can attain that

\[
\mathcal{C}_0^D_t^\alpha \int_{t_0}^t K(t, s)ds = R^\alpha_L \mathcal{C}_t^\alpha \int_{t_0}^t K(t, s)ds - \left[ \int_{t_0}^t K(t, s)ds \right]_{t=t_0} \times \frac{(t - t_0)^{-\alpha}}{\Gamma(1 - \alpha)}
\]

\[
= R^\alpha_L \mathcal{C}_t^\alpha \int_{t_0}^t K(t, s)ds, \quad t \in \mathbb{J}.
\]

Therefore, fractional Leibniz integral rule for Caputo derivative is identical with the Riemann-Liouville one whenever \( 0 < \alpha \leq 1: \)

\[
\mathcal{C}_0^D_t^\alpha \int_{t_0}^t K(t, s)ds = \lim_{s \to t_0} t^{\alpha - 1}_t K(t, s) + \int_{t_0}^t R^\alpha_L \mathcal{C}_t^\alpha K(t, s)ds, \quad t \in \mathbb{J}.
\]

It is important to introduce Caputo fractional derivative of convolution operator in general sense which is an accurate tool for testing particular solution of Caputo type multi-term FDEs.

**Corollary 4.2.** If we have \( K(t, s) = f(t - s)g(s), \) \( t_0 = 0, \) and assumptions of Theorem 4.3 are fulfilled, then following equality holds true for convolution operator in Caputo’s sense of order \( \alpha \in (n - 1, n], \) where \( n \geq 2: \)

\[
\mathcal{C}_0^D_t^\alpha \int_{t_0}^t f(t-s)g(s)ds = t^{\alpha - 1}_t \left\{ \sum_{l=1}^n \lim_{s \to t_0} \left( \frac{\partial^{n-l}}{\partial t^{n-l}} \frac{d^{l-1}}{dt^{l-1}} \lim_{s \to t_0} g(s) \right) \right\} + \int_{t_0}^t \mathcal{C}_s^\alpha f(\tau-s)g(s)ds, \quad t > 0. \tag{4.8}
\]

**Proof.** If we write \( f(t-s)g(s) \) instead of \( K(t, s) \) in (4.8), then we acquire

\[
\mathcal{C}_0^D_t^\alpha \int_{t_0}^t f(t-s)g(s)ds = t^{\alpha - 1}_t \left\{ \sum_{l=1}^n \frac{d^{l-1}}{dt^{l-1}} \lim_{s \to t_0} \frac{\partial^{n-l}}{\partial t^{n-l}} f(t-s)g(s) \right\}
\]

\[
+ \int_{t_0}^t \mathcal{C}_s^\alpha f(t-s)g(s)ds
\]

\[
= t^{\alpha - 1}_t \left\{ \sum_{l=1}^n \frac{d^{l-1}}{dt^{l-1}} \left( \lim_{s \to t_0} \frac{\partial^{n-l}}{\partial t^{n-l}} f(t-s) \lim_{s \to t_0} g(s) \right) \right\}
\]

\[
+ \int_{t_0}^t \mathcal{C}_s^\alpha f(t-s)g(s)ds
\]

\[
= t^{\alpha - 1}_t \left\{ \sum_{l=1}^n \lim_{s \to t_0} \frac{\partial^{n-l}}{\partial t^{n-l}} f(t-s) \frac{d^{l-1}}{dt^{l-1}} \lim_{s \to t_0} g(s) \right\}
\]

\[
= \sum_{l=1}^n \frac{d^{l-1}}{dt^{l-1}} \left( \lim_{s \to t_0} \frac{\partial^{n-l}}{\partial t^{n-l}} f(t-s) \lim_{s \to t_0} g(s) \right).
\]
Thus, the proof is complete.

**Corollary 4.3.** If we have \( K(t, s) = f(t - s)g(s), t_0 = 0, \) and assumptions of Theorem 4.3 are fulfilled, then following equality holds true for convolution operator in Caputo's sense for \( \alpha \in (0, 1): \)

\[
\int_0^t C^\alpha_0 \int_0^t f(t - s)g(s)ds = \lim_{s \to t} t^{1 - \alpha} f(t - s) \lim_{s \to t} g(s) + \int_0^t C^\alpha_0 s \int_0^t f(t - s)g(s)ds, \quad t > 0. \quad (4.9)
\]

**Proof.** If we make use of the substitution \( K(t, s) = f(t - s)g(s) \) in the relation (4.6), the proof is straightforward. So, we omit it here.

**Theorem 4.4.** The relationship between Leibniz integral rule for Riemann-Liouville and Caputo fractional differentiation operators of order \( n - 1 < \alpha \leq n, n \geq 2 \) holds true:

\[
\int_0^t \frac{d^n}{dt^n} K(t, s)ds = \frac{d^n}{dt^n} \int_0^t K(t, s)ds + \sum_{i=1}^{n-1} \left[ \frac{d^{n-1}}{dt^{n-1}} \lim_{s \to t} \frac{\partial^{i-1}}{\partial t^{i-1}} K(t, s) \right] \times \frac{(t - t_0)^{i-\alpha}}{\Gamma(i - \alpha + 1)}, t \in \mathbb{J}.
\]

(4.10)

**Proof.** Using the relationship between Riemann-Liouville and Caputo fractional derivatives [2.6], we get

\[
\int_0^t \frac{d^n}{dt^n} K(t, s)ds = \int_0^t \frac{d^n}{dt^n} K(t, s)ds + \sum_{i=1}^{n-1} \left[ \frac{d^{n-1}}{dt^{n-1}} \lim_{s \to t} \frac{\partial^{i-1}}{\partial t^{i-1}} K(t, s) \right] \times \frac{(t - t_0)^{i-\alpha}}{\Gamma(i - \alpha + 1)}, t \in \mathbb{J}.
\]

(4.11)

**Corollary 4.4.** If we replace \( K(t, s) \) with \( f(t - s)g(s) \) and consider \( t_0 = 0 \) for lower bound of the integral in Theorem 4.4, the relationship between Riemann-Liouville and Caputo type Leibniz integral rules for convolution operator of the functions \( f \) and \( g \) holds true for \( n - 1 < \alpha \leq n, n \geq 2 \):

\[
\int_0^t f(t - s)g(s)ds = \frac{t^{\alpha}}{\Gamma(1 - \alpha)} \left[ \lim_{s \to t} \frac{\partial^{i-1}}{\partial t^{i-1}} f(t - s) \lim_{s \to t} g(s) \right] \times \frac{(t - t_0)^{i-\alpha}}{\Gamma(i - \alpha + 1)}, t > 0. \quad (4.12)
\]

**Corollary 4.5.** The fractional Leibniz rule for Riemann-Liouville and Caputo type fractional differential operators coincides for \( 0 < \alpha \leq 1 \):

\[
\int_0^t \frac{d^\alpha}{dt^\alpha} K(t, s)ds = \frac{d^\alpha}{dt^\alpha} \int_0^t K(t, s)ds, \quad t \in \mathbb{J}, \quad (4.13)
\]

\[
\int_0^t f(t - s)g(s)ds = \frac{d^\alpha}{dt^\alpha} \int_0^t f(t - s)g(s)ds, \quad t > 0. \quad (4.14)
\]
5. Fractional Green’s function method

The Laplace transform is a convenient technique for solving the Cauchy problem associated with multi-term FDEs with constant coefficients. For instance, let us consider linear in-homogeneous FDE with multi-orders in Caputo’s sense and constant coefficients:

\[
\{ C_0 D_t^{\alpha_n} + \lambda_1 C_0 D_t^{\alpha_{n-1}} + \lambda_2 C_0 D_t^{\alpha_{n-2}} + \ldots + \lambda_{n-1} C_0 D_t^{\alpha_1} + \lambda_n \} y(t) = g(t), \quad t > 0,
\]

(5.1)

under the homogeneous initial conditions:

\[ y^{(k)}(0) = 0, \quad k = 0, 1, \ldots, n - 1, \]

(5.2)

where \( C_0 D_t^{\alpha_i} y(\cdot), \ i = 1, 2, \ldots, n, \) are the Caputo fractional differentiation operators of orders \( i - 1 \leq \alpha_i \leq i, \) \( \lambda_i \in \mathbb{R} \) for \( i = 1, 2, \ldots, n \) denote constants and \( g \in C([0, \infty), \mathbb{R}) \) is the continuous force or input function.

The classical analogue of the same problem is considered by Miller in [24]. Let us consider IVP for \( n \)-th order linear differential equation with constant coefficients:

\[
\{ D^n + \lambda_1 D^{n-1} + \lambda_2 D^{n-2} + \ldots + \lambda_{n-1} D + \lambda_n \} y(t) = 0, \quad t > 0,
\]

(5.3)

with zero initial conditions

\[ D^k y(0) = 0, \quad 0 \leq k \leq n - 1. \]

(5.4)

The fractional Green’s function is a very useful and applicable practical as well as theoretical tool for solving the IVP for multi-order FDE.

If we let

\[ P(x) = x^n + \lambda_1 x^{n-1} + \lambda_2 x^{n-2} + \ldots + \lambda_{n-1} x + \lambda_n \]

(5.5)

be a polynomial which is related to the equation (5.3), then according to the Laplace transform method, the unique solution of the following differential system:

\[
\begin{align*}
P(D)y(t) &= g(t), \quad t > 0, \\
D^k y(0) &= 0, \quad k = 0, 1, \ldots, n - 1,
\end{align*}
\]

can be represented in terms of a convolution integral

\[ y(t) = \int_0^t H(t - s) g(s) ds, \quad t > 0, \]

(5.6)

where \( H(\cdot) \) is the Green or weight function associated with the differential operator \( P(D) \) that is evaluated by taking inverse Laplace transform of the transfer function.

5.1. Applications of fractional Leibniz rules

In this subsection, we study applications of the fractional Leibniz integral rule in Riemann-Liouville and Caputo sense using the generalized Bagley-Torvik equations. Moreover, we have used Leibniz integral rule for checking candidate solution of the oscillator equation in classical sense.

In the following cases, to obtain analytical representation of solutions for the Cauchy problem we will apply fractional Green’s function method as we mentioned in Section 5.

Case 1: We consider the IVP for generalized Bagley-Torvik equations with Riemann-Liouville fractional derivatives of order \( 1 < \alpha \leq 2 \) and \( 0 < \beta \leq 1 \) in the form of:

\[
\begin{align*}
\{ (RL_0 D_t^\alpha) y(t) \} - \mu RL_0 D_t^\beta y(t) - \lambda g(t) &= g(t), \quad t > 0, \\
\theta_0 I_t^{2-\alpha} y(t)|_{t=0} = \theta_0 I_t^{1-\alpha} g(t)|_{t=0} &= 0, \quad \lambda, \mu \in \mathbb{R}.
\end{align*}
\]

(5.7)
Theorem 5.1. A unique solution $y \in C^2([0, \infty), \mathbb{R})$ of the Cauchy problem (5.7) has the following formula:

$$y(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)\alpha, \mu(t-s)^{\alpha-\beta})g(s)ds. \quad (5.8)$$

Proof. We assume that (5.7) has a unique solution $y(t)$ and $g(t)$ is continuous on $[0, \infty)$ and exponentially bounded, then $y(t), \left(\frac{RL_0}{\alpha}D_t^\alpha y\right)(t), \text{and} \left(\frac{RL_0}{\alpha}D_t^\alpha y\right)(t)$ are exponentially bounded, thus their Laplace transform exist.

Applying Laplace integral transform for Riemann-Liouville fractional derivative using the formula (2.7) to the both sides of (5.7) yields:

$$(s^\alpha - \mu s^\beta - \lambda) Y(s) = G(s). \quad (5.9)$$

Then we solve (5.9) with respect to $Y(s)$,

$$Y(s) = \frac{G(s)}{s^\alpha - \mu s^\beta - \lambda}. \quad (5.10)$$

Taking inverse Laplace transform of (5.10) and applying Lemma 2.2 we find an explicit representation of solution to (5.7):

$$y(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)\alpha, \mu(t-s)^{\alpha-\beta})g(s)ds. \quad (5.11)$$

Verification by substitution. Having found explicit form for $y(t)$, it remains to confirm that $y(t)$ is an analytical solution of (5.7) indeed. Firstly, for make the use of checking by substitution, we apply fractional Leibniz integral rule in Riemann-Liouville sense for the first and second terms of (5.7). Then the first term will be as follows:

$$\left(\frac{RL_0}{\alpha}D_t^\alpha y\right)(t) = \frac{RL_0}{\alpha}D_t^\alpha \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)\alpha, \mu(t-s)^{\alpha-\beta})g(s)ds$$

$$= \sum_{l=1}^{2} \lim_{s \to t^-} \frac{RL_0}{\alpha}D_t^{\alpha-l}(t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)\alpha, \mu(t-s)^{\alpha-\beta})g(s)\lim_{s \to t^-} \frac{d^{\alpha-l-1}}{dt^{\alpha-l-1}} g(s)$$

+ \int_0^t \frac{RL_0}{\alpha}D_t^{\alpha}(t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)\alpha, \mu(t-s)^{\alpha-\beta})g(s)ds

$$= \lim_{s \to t^-} \frac{RL_0}{\alpha}D_t^{\alpha-2}(t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)\alpha, \mu(t-s)^{\alpha-\beta})g(s)\lim_{s \to t^-} \frac{d^{\alpha-2}}{dt^{\alpha-2}} g(s)$$

+ \int_0^t \frac{RL_0}{\alpha}D_t^{\alpha}(t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)\alpha, \mu(t-s)^{\alpha-\beta})g(s)ds

$$= \lim_{s \to t^-} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)\alpha, \mu(t-s)^{\alpha-\beta})g(s)$$

+ \int_0^t \frac{RL_0}{\alpha}D_t^{\alpha}(t)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)\alpha, \mu(t-s)^{\alpha-\beta})g(s)ds$$

+ \int_0^t \frac{RL_0}{\alpha}D_t^{\alpha}(t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)\alpha, \mu(t-s)^{\alpha-\beta})g(s)ds.$$
From now on, we apply Pascal’s rule for binomial coefficients to the first term of above expression and the limit of the second term is equal to zero as \( s \to t - 0 \), we obtain

\[
\left( RL^\alpha_0 D^\alpha_t y \right) (t) = RL^\alpha_0 D^\alpha_t \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha-\beta, \alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) \, ds \\
= g(t) + \lambda \lim_{s \to t-0} (t-s)^\alpha E_{\alpha, \alpha-\beta, \alpha+1}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) \lim_{s \to t-0} g(s) \\
+ \mu \lim_{s \to t-0} (t-s)^{\alpha-\beta} E_{\alpha, \alpha-\beta, \alpha-\beta}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) \lim_{s \to t-0} g(s) \\
+ \int_0^t RL^\alpha_0 D^\alpha_t (t-s)^{\alpha-1} E_{\alpha, \alpha-\beta, \alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) \, ds
\]

Now, using Lemma 2.4 and again applying Pascal’s rule for the expression under above integral, we attain

\[
\int_0^t RL^\alpha_0 D^\alpha_t (t-s)^{\alpha-1} E_{\alpha, \alpha-\beta, \alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) \, ds \\
= \int_0^t (t-s)^{-1} E_{\alpha, \alpha-\beta, 0}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) \, ds \\
= \int_0^t \sum_{l=1}^{\infty} \sum_{k=0}^{\infty} \left( \frac{l+k}{l+k(\alpha-\beta)} \right) g(s) \, ds \\
= \int_0^t \sum_{l=0}^{\infty} \sum_{k=0}^{l} \left( \frac{l+k}{l+k(\alpha-\beta)} \right) g(s) \, ds \\
+ \int_0^t \sum_{l=0}^{\infty} \sum_{k=0}^{l} \left( \frac{l+k}{l+k(\alpha-\beta)} \right) g(s) \, ds
\]

Therefore, we have

\[
\left( RL^\alpha_0 D^\alpha_t y \right) (t) = g(t) + \lambda \int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha-\beta, \alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) \, ds
\]
\[ + \mu \int_{0}^{t} (t-s)^{\alpha-\beta-1} E_{\alpha,\alpha-\beta,\alpha} (\lambda(t-s)^{\alpha}, \mu(t-s)^{\alpha-\beta}) g(s) ds. \] (5.12)

Similarly, the second term of (5.7) will be

\[
\left( RL_0^\beta \right)^2 \left( \frac{RL_0^\beta}{RL_0^\beta} \right)^t \left( RL_0^\beta \right) (t) = RL_0^\beta \int_{0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha} (\lambda(t-s)^{\alpha}, \mu(t-s)^{\alpha-\beta}) g(s) ds
\]
\[
= \int_{0}^{t} (t-s)^{\alpha-\beta-1} E_{\alpha,\alpha-\beta,\alpha} (\lambda(t-s)^{\alpha}, \mu(t-s)^{\alpha-\beta}) g(s) ds. \] (5.13)

Taking linear combination of (5.12) and (5.13) together with (5.8), we get the desired result.

**Case 2:** We consider the Cauchy problem for generalized Bagley-Torvik equations which is the special case of Caputo type fractional multi-term differential equations with constant coefficients of order 1 < \( \alpha \leq 2 \) and 0 < \( \beta \leq 1 \) in the form of:

\[
\begin{cases}
\left( C_0^\alpha D_t^\alpha \right) y(t) - \mu \left( C_0^\beta D_t^\beta \right) y(t) = g(t), & t > 0, \\
y(0) = y'(0) = 0, & \lambda, \mu \in \mathbb{R}.
\end{cases} \] (5.14)

**Theorem 5.2.** A unique solution \( y \in C^2([0, \infty), \mathbb{R}) \) of the Cauchy problem (5.14) has the following formula:

\[ y(t) = \int_{0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha} (\lambda(t-s)^{\alpha}, \mu(t-s)^{\alpha-\beta}) g(s) ds. \] (5.15)

**Proof.** We assume that (5.14) has a unique solution \( y(t) \) and \( g(t) \) is continuous on \([0, \infty)\) and exponentially bounded, then \( y(t), \left( C_0^\alpha D_t^\alpha y \right)(t), \) and \( \left( C_0^\beta D_t^\beta y \right)(t) \) are exponentially bounded, thus their Laplace transform exist.

Applying the formula of Laplace transform for Caputo fractional derivative (2.8) to the both sides of (5.14) yields:

\[ (s^\alpha - \mu s^\beta - \lambda) Y(s) = G(s). \] (5.16)

Then we solve (5.16) with respect to \( Y(s) \),

\[ Y(s) = \frac{G(s)}{s^\alpha - \mu s^\beta - \lambda}. \] (5.17)

Taking inverse Laplace transform of (5.17) and applying Lemma 2.2, we find an explicit representation of solution to (5.14):

\[ y(t) = \int_{0}^{t} (t-s)^{\alpha-1} E_{\alpha,\alpha} (\lambda(t-s)^{\alpha}, \mu(t-s)^{\alpha-\beta}) g(s) ds. \] (5.18)

**Remark 5.1.** Since initial conditions equal to zero, in accordance with the formula (5.6), analytical solutions should be coincide with each other for the Cauchy problems in Riemann-Liouville (5.7) and Caputo (5.14) senses.

**Verification by substitution.** Having found explicit form for \( y(t) \), it remains to confirm that \( y(t) \) is an analytical solution of (5.14) indeed.
Now, we again make use of checking by substitution via Caputo fractional Leibniz integral rule. In this case, we apply first Pascal’s rule before applying fractional Leibniz rule since \( C_0 D_t^\alpha \frac{t-s}^{\alpha-1} \) is undefined in accordance with (2.12). Then according to the formula (2.3), we obtain

\[
(C_0 D_t^\alpha y)(t) = C_0 D_t^\alpha \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds
\]

and

\[
= g(t) + \lambda C_0 D_t^\alpha \int_0^t (t-s)^{2\alpha-1} E_{\alpha,\alpha-\beta,2\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds
\]

\[
+ \mu C_0 D_t^\alpha \int_0^t (t-s)^{2\alpha-\beta-1} E_{\alpha,\alpha-\beta,2\alpha-\beta}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds.
\]

Then using the formula for fractional Leibniz integral rule in Caputo sense (5.8) of order \( 1 < \alpha \leq 2 \), we have

\[
C_0 D_t^\alpha \int_0^t (t-s)^{2\alpha-1} E_{\alpha,\alpha-\beta,2\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds
\]

\[
= \frac{t}{\alpha t^2} \left[ \lim_{s \to t-0} \sum_{l=1}^2 \frac{\partial^{2-l}}{\partial t^{2-l}} (t-s)^{2\alpha-1} E_{\alpha,\alpha-\beta,2\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) \right] \frac{d^{\alpha-1} \lim_{s \to t-0} g(s)}{dt^{\alpha-1}}
\]

\[
+ \int_0^t C_0 D_t^\alpha (t-s)^{2\alpha-1} E_{\alpha,\alpha-\beta,2\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds
\]

\[
= \frac{t}{\alpha t^2} \left[ \lim_{s \to t-0} \frac{\partial}{\partial t} (t-s)^{2\alpha-1} E_{\alpha,\alpha-\beta,2\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) \lim_{s \to t-0} g(s) \right]
\]

\[
+ \int_0^t C_0 D_t^\alpha (t-s)^{2\alpha-1} E_{\alpha,\alpha-\beta,2\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) \frac{d}{dt} \lim_{s \to t-0} g(s) ds
\]

\[
= \frac{t}{\alpha t^2} \left[ \lim_{s \to t-0} (t-s)^{2\alpha-2} E_{\alpha,\alpha-\beta,2\alpha-1}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) \lim_{s \to t-0} g(s) \right]
\]

\[
+ \int_0^t C_0 D_t^\alpha (t-s)^{2\alpha-2} E_{\alpha,\alpha-\beta,2\alpha-1}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) \frac{d}{dt} \lim_{s \to t-0} g(s) ds
\]

\[
+ \int_0^t C_0 D_t^\alpha (t-s)^{2\alpha-1} E_{\alpha,\alpha-\beta,2\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds
\]

\[
= \frac{t}{\alpha t^2} \left[ \lim_{s \to t-0} (t-s)^{2\alpha-1} E_{\alpha,\alpha-\beta,2\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) \lim_{s \to t-0} g(s) \right]
\]

\[
+ \int_0^t C_0 D_t^\alpha (t-s)^{2\alpha-1} E_{\alpha,\alpha-\beta,2\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds
\]

\[
= \int_0^t C_0 D_t^\alpha (t-s)^{2\alpha-1} E_{\alpha,\alpha-\beta,2\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds.
\]

Thus, by Lemma 2.3 we get

\[
C_0 D_t^\alpha \int_0^t (t-s)^{2\alpha-1} E_{\alpha,\alpha-\beta,2\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds
\]

\[
= \int_0^t C_0 D_t^\alpha (t-s)^{2\alpha-1} E_{\alpha,\alpha-\beta,2\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds
\]
Similarly, by applying the formula (4.8), we also have
\[
\beta C + C = \lim_{s \to t-0} \int_0^s \left( (t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) \right) ds. \tag{5.20}
\]

On the other hand, since $0 < \beta \leq 1$, we will apply the formula (4.9) for second term of the equation (5.14):
\[
\begin{align*}
C_0 D_t^\beta \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds \\
= \lim_{s \to t-0} t I_t^{1-\beta} (t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) \\
+ \int_0^t R L_t D_t^\beta (t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds \\
= \lim_{s \to t-0} (t-s)^{\alpha-\beta} E_{\alpha,\alpha-\beta,\alpha-\beta+1}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) \\
+ \int_0^t R L_t D_t^\beta (t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds \\
= \int_0^t (t-s)^{\alpha-\beta-1} E_{\alpha,\alpha-\beta,\alpha-\beta}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds. \tag{5.22}
\end{align*}
\]

where
\[
t I_t^{1-\beta} \left( \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)} \right) = \frac{(t-t_0)^{\alpha-\beta}}{\Gamma(\alpha-\beta + 1)}, \quad t > 0.
\]

Next we plug (5.20) and (5.21) into (5.19), we therefore get
\[
\begin{align*}
C_0 D_t^\alpha \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds \\
= g(t) + \lambda \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha-\beta,\alpha}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds \\
+ \mu \int_0^t (t-s)^{\alpha-\beta-1} E_{\alpha,\alpha-\beta,\alpha-\beta}(\lambda(t-s)^\alpha, \mu(t-s)^{\alpha-\beta}) g(s) ds.
\end{align*}
\]
Taking linear combination of above equation together with (5.15) and (5.22), we arrive at
\[
\left( c^n_0 D_t^n y \right) (t) - \mu \left( c^n_0 D_t^n y \right) (t) - \lambda y(t) = g(t), \quad t > 0.
\]

**Remark 5.2.** Kilbas et al. [27] have obtained the analytical solutions of the Cauchy problems (5.7) and (5.14) in terms of Fox-Wright functions below:

\[
y(t) = \int_0^t (t-s)^{\alpha-1} H_{\alpha,\beta;\lambda,\mu}(t-s)g(s)ds, \quad t > 0,
\]

where
\[
H_{\alpha,\beta;\lambda,\mu}(t) := \sum_{l=0}^{\infty} \frac{\lambda^l t^l}{l!} E_{\alpha,\beta}^{(l)} \left[ \frac{(l+1,1)}{l(\alpha+\alpha-\beta)} \left| \mu t^{\alpha-\beta} \right| \right].
\]

**Proof.** Using the definition of Fox-Wright function [45, 46], we arrive at

\[
y(t) = \int_0^t \sum_{l=0}^{\infty} \frac{\lambda^l t^l}{l!} E_{\alpha,\beta}^{(l)} \left[ \frac{(l+1,1)}{l(\alpha+\alpha-\beta)} \right] g(s)ds
\]

\[
= \int_0^t \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(l+k)!}{k!} \frac{\lambda^l \mu^k (t-s)^{l+k(\alpha-\beta)+\alpha-1}}{\Gamma(l(\alpha+k(\alpha-\beta)+\alpha))} g(s)ds
\]

\[
= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\beta;\lambda,\mu}(t-s)^{\alpha-1}(t-s)^{\alpha-1} \mu(t-s)^{\alpha-\beta}g(s)ds, \quad t > 0.
\]

Therefore, our solution in terms of univariate version of bivariate Mittag-Leffler type functions coincide with the solution by means of Fox-Wright type functions shown in [27].

**Remark 5.3.** Podlubny [8] have attained the analytical solutions of the Cauchy problems (5.7) and (5.14) in terms of \( l \)-th derivative of two-parameter Mittag-Leffler functions below:

\[
y(t) = \int_0^t (t-s)^{\alpha-1} H_{\alpha,\beta;\lambda,\mu}(t-s)g(s)ds, \quad t > 0,
\]

where
\[
H_{\alpha,\beta;\lambda,\mu}(t) := \sum_{l=0}^{\infty} \frac{\lambda^l t^l}{l!} E_{\alpha,\beta;\lambda,\mu}^{(l)} \left[ \mu t^{\alpha-\beta} \right].
\]

**Proof.** Using the definition of \( l \)-th derivative of two-parameter Mittag-Leffler function, we arrive at

\[
y(t) = \int_0^t \sum_{l=0}^{\infty} \frac{\lambda^l t^l}{l!} E_{\alpha,\beta;\lambda,\mu}^{(l)} \left[ \mu(t-s)^{\alpha-\beta} \right] g(s)ds
\]

\[
= \int_0^t \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(l+k)!}{k!} \frac{\lambda^l \mu^k (t-s)^{l+k(\alpha-\beta)+\alpha-1}}{\Gamma(l(\alpha+k(\alpha-\beta)+\alpha))} g(s)ds
\]

\[
= \int_0^t (t-s)^{\alpha-1} E_{\alpha,\beta;\lambda,\mu}(t-s)^{\alpha-1}(t-s)^{\alpha-1} \mu(t-s)^{\alpha-\beta}g(s)ds, \quad t > 0.
\]
Therefore, our solution in terms of univariate version of bivariate Mittag-Leffler type functions coincide with the solution by means of $l$-th derivative of two-parameter Mittag-Leffler type functions shown in [8].

**Case 3:** In special case, we substitute $\alpha = 2$ and $\beta = 1$ in [5.7] and [5.14], then we get the following Cauchy problem for the classical second order linear differential equation - the oscillator equation with constant coefficients:

$$\begin{align*}
\begin{cases}
y''(t) - \mu y'(t) - \lambda y(t) &= g(t), & t > 0, \\
y'(0) &= y(0) = 0, & \lambda, \mu \in \mathbb{R},
\end{cases}
\end{align*}$$

(5.25)

**Theorem 5.3.** A unique solution $y \in C^2([0, \infty), \mathbb{R})$ of the Cauchy problem (5.25) has the following formula:

$$y(t) = \int_0^t (t-s) E_{2,1,2}(\lambda(t-s)^2, \mu(t-s)) g(s) ds, \quad t > 0.$$  

(5.26)

**Proof.** By using verification by substitution, we have

$$\frac{d^2}{dt^2} \int_0^t (t-s) E_{2,1,2}(\lambda(t-s)^2, \mu(t-s)) g(s) ds$$

$$= \sum_{l=1}^2 \lim_{s \to t-0} \frac{\partial^{2-l}}{\partial t^{2-l}} (t-s) E_{2,1,2}(\lambda(t-s)^2, \mu(t-s)) \frac{d^{l-1}}{dt^{l-1}} \lim_{s \to t-0} g(s)$$

$$+ \int_0^t \frac{\partial^2}{\partial t^2} (t-s) E_{2,1,2}(\lambda(t-s)^2, \mu(t-s)) g(s) ds$$

$$= \lim_{s \to t-0} \frac{\partial}{\partial t} (t-s) E_{2,1,2}(\lambda(t-s)^2, \mu(t-s)) \lim_{s \to t-0} g(s)$$

$$+ \lim_{s \to t-0} (t-s) E_{2,1,2}(\lambda(t-s)^2, \mu(t-s)) \frac{d}{dt} \lim_{s \to t-0} g(s)$$

$$+ \int_0^t \frac{\partial^2}{\partial t^2} (t-s) E_{2,1,2}(\lambda(t-s)^2, \mu(t-s)) g(s) ds$$

$$= \lim_{s \to t-0} E_{2,1,1}(\lambda(t-s)^2, \mu(t-s)) \lim_{s \to t-0} g(s)$$

$$+ \int_0^t (t-s)^{-1} E_{2,1,0}(\lambda(t-s)^2, \mu(t-s)) g(s) ds$$

$$= g(t) + \lambda \lim_{s \to t-0} (t-s)^2 E_{2,1,3}(\lambda(t-s)^2, \mu(t-s)) g(s) ds$$

$$+ \mu \lim_{s \to t-0} (t-s) E_{2,1,2}(\lambda(t-s)^2, \mu(t-s)) g(s) ds$$

$$+ \int_0^t (t-s)^{-1} E_{2,1,0}(\lambda(t-s)^2, \mu(t-s)) g(s) ds$$

$$= g(t) + \int_0^t (t-s)^{-1} E_{2,1,0}(\lambda(t-s)^2, \mu(t-s)) g(s) ds.$$  

(5.27)

Applying Pascal’s rule for binomial coefficients for the last term of above equality, we get

$$\int_0^t (t-s)^{-1} E_{2,1,0}(\lambda(t-s)^2, \mu(t-s)) g(s) ds$$
\[
\int_0^t \frac{(t-s)^{-1}}{\Gamma(0)} g(s) ds + \lambda \int_0^t (t-s) E_{2,1,2} (\lambda(t-s)^2, \mu(t-s)) g(s) ds \\
+ \mu \int_0^t E_{2,1,1} (\lambda(t-s)^2, \mu(t-s)) g(s) ds \\
= \lambda \int_0^t (t-s) E_{2,1,2} (\lambda(t-s)^2, \mu(t-s)) g(s) ds \\
+ \mu \int_0^t E_{2,1,1} (\lambda(t-s)^2, \mu(t-s)) g(s) ds.
\] (5.28)

Next, we have

\[
\frac{d}{dt} \int_0^t (t-s) E_{2,1,2} (\lambda(t-s)^2, \mu(t-s)) g(s) ds \\
= \int_0^t \frac{\partial}{\partial t} (t-s) E_{2,1,2} (\lambda(t-s)^2, \mu(t-s)) g(s) ds \\
= \int_0^t E_{2,1,1} (\lambda(t-s)^2, \mu(t-s)) g(s) ds.
\] (5.29)

Again taking linear combination of above equations (5.28) and (5.29) together with (5.26), we prove the desired result.

6. Conclusions and future work

The theory of Leibniz integral rule allows us to study particular solutions of classical and fractional multi-term differential equations. To the best of our knowledge, we derive explicit analytical solutions of well-known Bagley-Torvik and oscillator equations in terms of bivariate Mittag-Leffler functions via the technique of fractional Green’s function, since this theory has not been presented in recent literature.

The major contributions of our research work are as below:

- we have proposed a Leibniz rule for higher order derivatives in classical sense which is more productive tool for testing solutions of multi-order differential equation;
- we have introduced fractional Leibniz rule for Riemann-Liouville and Caputo type fractional differentiation operators;
- we have investigated differentiation of convolution operator which is more crucial in theory of differential equations with constant coefficients of classical and fractional-order derivatives;
- analytical explicit solutions of the generalized Bagley-Torvik and oscillator equations are derived in terms of univariate version of bivariate Mittag-Leffler type functions in accordance with the method of Laplace integral transform;
- We have showed that our analytical solutions are coincide with Fox-Wright type and \( l \)-th derivative of two-parameter Mittag-Leffler type functions;
- we tested the candidate solutions of Cauchy problems for Bagley-Torvik equations with fractional-order sense and oscillator equation with classical-order one via our new fractional Leibniz integral rules.
There are a number of potential directions in which the results acquired here can be extended. Our future work will proceed to study the Leibniz integral rule results for $\psi$-Hilfer and Hadamard type fractional derivatives and the analytical explicit solutions of multi-term fractional differential equations in terms of natural extensions of Mittag-Leffler type functions.

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