GLOBAL STABILITY AND LOCAL BIFURCATIONS IN A TWO-FLUID MODEL FOR TOKAMAK PLASMA

D. ZHELYAZOV *, D. HAN-KWAN †, AND J.D.M. RADEMACHER ‡

Abstract. We study a two-fluid description for high and low temperature components of the electron velocity distribution in an idealized tokamak plasma evolving on a cylindrical domain, and taking into account curvature effects only. We refine previous results from [9] on the laminar steady state stability and include diffusion. We take the temperature difference as our primary parameter and show that linear instabilities and bifurcations occur within a finite interval and for small enough diffusion only, while the steady state is globally stable for parameters sufficiently far outside the interval. We find that primary instabilities always stem from the lowest spatial harmonics for aspect ratios of poloidal vs. radial extent below some value larger than 2. Moreover, we show that any codimension-one bifurcation of the laminar state is a supercritical Andronov-Hopf bifurcation, which yields periodic solutions in the form of traveling waves. In the degenerate case, where the instability region in the temperature difference is a point, we prove that the bifurcating periodic orbits form an arc of stable periodic solutions. We also provide numerical simulations to illustrate and corroborate our analysis, and find additional bifurcations of the travelling waves.

1. Introduction. In this paper we analyze the linear and nonlinear stability, and local bifurcations of a laminar steady state in a two-fluid model for high and low temperature phases of a tokamak plasma near the plasma edge. Existence of coherent states and stability are important issues in tokamak plasma theory and application that have been studied, e.g., in [3,6,9,15–17]. We consider a simple two-dimensional model that appears in [19] and is a viscous variant of the model derived in [9], which captures nonlinear effects due to electric drift and the electron temperature gradient instability. Due to its simplicity, the mathematical stability and bifurcation analysis presented in this paper can be carried further than so far for the Vlasov-Maxwell or Vlasov-Poisson models considered in, e.g. [3,17].

Our model equations for the miscible phases $\rho^\pm$ of ‘hot’ and ‘cold’ plasma with constant temperatures $T^+ > T^- > 0$, and the electric potential $V$ read

$$
\begin{align*}
\partial_t \rho^+ &= T^+ \partial_{x_2} \rho^+ - E^+ \cdot \nabla \rho^+ + \nu \nabla^2 \rho^+, \\
\partial_t \rho^- &= T^- \partial_{x_2} \rho^- - E^- \cdot \nabla \rho^- + \nu \nabla^2 \rho^-, \\
E &= -\nabla V, \\
-\nabla^2 V &= \rho^+ + \rho^- - 1,
\end{align*}
$$

(1.1)

where $\nu > 0$, $E^\pm = (E_2, -E_1)^T$. The equations are posed on the cylindrical domain

$$
x = (x_1, x_2) \in [0, L_1] \times \mathbb{R}/L_2 \mathbb{Z},
$$

subject to the Dirichlet boundary conditions

$$
V(0, x_2, t) = V(L_1, x_2, t) = 0, \\
\rho^+(0, x_2, t) = \rho_{ss}^+(0), \quad \rho^+\left(L_1, x_2, t\right) = \rho_{ss}^+(L_1),
$$

(1.2)

*Centrum Wiskunde & Informatica, Science Park 123, 1098 XG Amsterdam. FOM Institute DIFFER - Dutch institute for fundamental energy research, Association EURATOM-FOM, P.O. Box 1207, 3430 BE Nieuwegein, the Netherlands
†École Normale Supérieure, Département de Mathématiques et Applications, 45 rue d’Ulm, 75005 Paris, France
‡Universität Bremen, Fachbereich 3 – Mathematik, Postfach 33 04 40, 28359 Bremen, Germany, rademach@math.uni-bremen.de
that respect the 'laminar' steady state
\[ \rho_{ss} = (\rho_{ss}^+, \rho_{ss}^-), \quad \rho_{ss}^+(x_1) := 1 - \frac{x_1}{L_1}, \quad \rho_{ss}^-(x_1) := \frac{x_1}{L_1} \] of (1.1) for which the electric potential and field vanish.

For \( \nu = 0 \), and hence without the Dirichlet boundary conditions on \( \rho^\pm \), this system has been derived in [9], as discussed below. Here we remark that the nonlinear terms in (1.1) stem from the electric field driving \( \rho^\pm \) via the electric drift, or ‘\( E \times B \) drift’ of all charged particles. The addition of viscous terms in (1.1) on the one hand allow to model additional physics by adding diffusion or dissipation. On the other hand, it changes the system from hyperbolic to parabolic, whose bifurcations are easier to analyze. It turns out that \( \nu > 0 \) allows for more variety in destabilization scenarios. In order to relate our results with the hyperbolic system, we include an analysis of the case of small \( \nu > 0 \). For the benefit of a significant simplification of the analysis, we restrict to the case of equal viscosity for \( \rho^\pm \), which is also a physically relevant choice.

System (1.1) derives from a drift kinetic model, describing the interaction between drifts for hot plasma in a slab with one periodic direction. See [9, (1.5)] and (up to a reflection) equivalently [19, (1)]. Here the configuration space is 2D in position and 1D in velocity and we refer the reader to [18], [19] for a discussion of the physics. Concerning equations for the potential we refer to the introduction of [10].

There are at least two possibilities to obtain the two-fluid model from the kinetic equations. The first one, following [9], approximates the distribution function of electrons by two Dirac-distributions in 1D velocity space, that is,
\[ f(x,v,t) \approx \rho^+(x,t) \frac{\delta(|v| - \sqrt{2T^+})}{2\pi \sqrt{2T^+}} + \rho^-(x,t) \frac{\delta(|v| - \sqrt{2T^-})}{2\pi \sqrt{2T^-}}, \] which yields a hyperbolic two-fluid system. Adding viscosity terms models the dissipation and gives the system (1.1).

The second possibility, following [19], is to consider a certain moment closure of the fluid hierarchy. The resulting equations have the form of a two-fluid system with equal viscosity coefficients, which agrees with our choice: the fluid equations in (1.1) and [19, (6)] are equivalent.

Indeed, for certain parameters oscillatory loss of stability is numerically observed in [19] and in the present paper we identify these mathematically as supercritical Andronov-Hopf bifurcations that lead to spatio-temporal oscillations in the form of periodic travelling waves. Away from threshold, or at degenerate points, our results indicate that further bifurcations can yield more complicated dynamics, which may lead to the ‘interchange turbulent transport’ in [19].

A well-known model for resistive drift wave turbulence near the plasma edge are the Hasegawa-Wakatani equations [20]. While there is a similarity to our model, in particular in the terms from \( E \times B \) drift, the Hasegawa-Wakatani model concerns mode coupling and the equations [20, (5)-(6)] differ from our model in various linear terms; for a direct comparison the formulation (1.6) below is convenient.

In the context of shear flow and shear induced stability there are also relations to (1.1) or (1.6). Again the model equations in the literature that we are aware of differ
significantly, e.g., [8], and more recently [25]. Interestingly, the instability via spatio-
temporal oscillations for low shear in [23] bears some qualitative analogy to our results.

There is a similarity between the two-fluid system (1.1) and Navier-Stokes equation
for 2D incompressible flow in vorticity form [2]. If we interpret the deviations
of hot and cold electron densities from the laminar state (see also (2.1) below) as
vorticities $\omega$, the velocity vector field is \((\nabla(\nabla^2)^{-1}(\omega_1 + \omega_2))^\perp\) and our system can be
interpreted as 2D Navier-Stokes with two vorticities, vorticity sources and an advec-
tion term generated by $\Delta T$.

Concluding the discussion of model relations, let us briefly compare our model to
Saltzman’s two-fluid model [21] of Rayleigh-Bénard convection. The latter comes from
the Boussinesq approximation of incompressible fluid between two parallel plates kept
at a fixed temperature difference by external heating. In contrast, our model has two
fluid species with prescribed temperatures, and the temperature of the surrounding
plasma is given at the (vertical) sides. It is instructive to determine the difference
between the model equations as it is a single term: passing to the co-moving frame
$x_2 \rightarrow x_2 - (T^+ + T^-)t/2$, changing variables via
\[
\rho^+ = \frac{1 + \nabla^2\psi + \theta}{2}, \quad \rho^- = \frac{1 + \nabla^2\psi - \theta}{2}, \quad V = -\psi, \tag{1.5}
\]
and denoting $x_1 = z$ and $x_2 = x$ system (1.1) becomes
\[
\partial_t \nabla^2 \psi + [\psi, \nabla^2 \psi] - \frac{\Delta T}{2} \partial_x \theta - \nu \nabla^4 \psi = 0,
\]
\[
\partial_t \theta + [\psi, \theta] - \frac{\Delta T}{2} \partial_x \nabla^2 \psi - \nu \nabla^2 \theta = 0, \tag{1.6}
\]
where the Poisson bracket is \([f_1, f_2] = \partial_x f_1 \partial_x f_2 - \partial_x f_2 \partial_x f_1\). We observe that (1.6)
differs from [21] eqns. (16’) and (17’) by the third term in the second equation only:
in [21] (17’) this reads $-\Delta T_0 \frac{\partial \psi}{\partial x}$, while we have $-\frac{\Delta T}{2} \partial_x \nabla^2 \psi$. Here $\Delta T_0$ is the tem-
perature difference and $H$ the distance between the plates. (For the full equation
comparison replace $\Delta T$ by $g\varepsilon$, where in [21] $g$ is the gravity acceleration and $\varepsilon$
the volume expansion coefficient.)

This difference indeed has an impact: the periodic boundary conditions are essential
for the bifurcation scenarios of (1.1), and there are no stationary bifurcations of the
laminar steady state, which is in contrast to the behaviour of the Saltzman model
(see, e.g., [13]).

Concerning the boundary conditions (1.2), Dirichlet conditions on $\rho^\pm$ are suitable
in this context and helpful for our analysis, though in other physical settings these
may not be the right choice. Notably, the boundary conditions respect the steady
state $\rho_{ss}$, whose relevance for the system was noted in [9] for $\nu = 0$. If $\nu > 0$, it is in
fact the only steady state in the flow invariant subspace of $x_2$-independent functions.
In this paper, we present a detailed analysis of its stability and bifurcations for $\nu > 0$.

It was found in [9] for $\nu = 0$ and $L_1 = L_2 = L$ that the laminar state is unstable in
a bounded interval $[\Delta T_1, \Delta T_2]$ of the parameter $\Delta T = T^+ - T^-$, see Fig. [1.1]. Note that
the parameter $\Delta T$ arises in the comoving variable $x_2 \rightarrow x_2 - T^- t$, which removes $T^-$. On the other hand, the laminar state is linearly stable for large enough temperature
difference, \( \Delta T > \Delta T_2 \) and also for small enough (including negative) temperature difference \( \Delta T < \Delta T_1 \). Here linear stability implies local nonlinear stability of the laminar steady state. Not surprisingly, this structure persists for moderate viscosity with modified thresholds \( \Delta T_1 = \mathcal{O}(\nu^2) \), \( \Delta T_2 = 4L/(5\pi^2) + \mathcal{O}(\nu^2) \), while for large viscosity the laminar state is linearly stable for all \( \Delta T \).

In addition to studying linear stability and instability, it was shown in [9] for \( \nu = 0 \) and \( L_1 = L_2 = L \) that the laminar steady state is actually globally stable for \( \Delta T < 0 \) and \( \Delta T > \Delta T_* = 4L/\pi^2 \) (in the sense of \( L^2 \)-space convergence). Here \( \Delta T_* \) is an estimate for a global stability threshold, which may not be sharp – in contrast to the linear stability thresholds. One of the original motivations for the present paper was to explain the necessity of a difference between the local and a global stability threshold through subcritical bifurcations at \( \Delta T_2 \). In such a case, the global stability threshold must be larger than the linear one due to the presence of non-zero amplitude solutions that do not converge to the laminar state. We prove, however, that the bifurcations are always supercritical. Of course, a difference can still be due to other nonzero amplitude solutions that the local bifurcation analysis cannot reveal and indeed, we present numerical evidence for such solutions in §8.2.

We also gain some insight into the threshold relation by studying nontrivial aspect ratios, that is, \( L_2 \neq L_1 \), which is also motivated from the modelling point of view as a thin stripe near the plasma edge, where \( 0 < L_1 \ll L_2 \). We show that for \( \nu = 0 \) the upper linear stability threshold \( \Delta T_2 \) of the lowest spatial harmonics and the estimated global stability threshold become

\[
\Delta T_2 = \frac{4L_1L_2^2}{\pi^2(L_2^2 + 4L_1^2)} = \frac{4L_1^2L_2}{(4 + \ell^2)\pi^2}, \quad \Delta T_* = \frac{4L_1}{\pi^2},
\]

so that \( \Delta T_* \) is in fact independent of \( L_2 \). Now we notice that the estimate of the global stability threshold \( \Delta T_* \) is related to the upper linear stability threshold via

\[
\lim_{\ell \to \infty} \Delta T_2 = \Delta T_*,
\]

so that the discrepancy is smaller on thinner domains. Note that as a (estimated) global stability threshold, \( \Delta T_* \) is always an upper bound for a linear instability in \( \Delta T \).

In addition, we find that \( \nu > 0 \) and \( L_1 \neq L_2 \) allow for much richer destabilization scenarios than \( L_1 = L_2 \), namely simultaneous purely imaginary eigenvalues, which suggests multiple Andronov-Hopf bifurcations. Nevertheless, we prove that all codimension-one bifurcations of the laminar state are of generic supercritical Andronov-Hopf type; and destabilizing ones for \( \ell \leq 2\sqrt{2} \) are always codimension-one.

Coming back to the model origins, the sign of \( \Delta T \) can be related to the region within the tokamak that is modelled by (1.1): ‘good curvature’ (negative \( \Delta T \)) and ‘bad curvature’ (positive \( \Delta T \)) regions, which is consistent with the different stability properties for positive and negative \( \Delta T \) as noted in [9] – the model captures the electron temperature gradient instability. Our rigorous results suggest, that the instability is present for moderate temperature gradients on the good curvature side (L-mode) only, and is suppressed for large enough gradients, but the modelling and physical relations to L-H transition (see [22]) remain to be understood. “Clearly, the model selection criteria, apart from the sound physics behind them, should be based on their
Fig. 1.1. (a) Schematic illustration of the main case of a primary 1-instability region in the stability analysis of the steady state \( \rho_{ss} \) when including viscosity. The estimate of the global stability threshold \( \Delta T^* \) is larger than the linear stability threshold, even at \( \nu = 0 \). However, in the limit \( \nu \to 0 \) the lower thresholds coincide, and if in addition \( \ell \to \infty \), then also the upper linear thresholds tend to the global ones. (b) Sketch of local bifurcation diagram of the steady state \( u = 0 \) with supercritical branches of stable limit cycles. Solid line represents stable solutions and dashed lines unstable ones.

capability to reproduce key experimental facts such as spontaneous L-H transitions, characteristic intermediate regimes (such as dithering), or hysteresis” [15].

In this paper, we pursue a mathematical analysis that may serve as a basis to investigate further the model assessment and relations to physical phenomena. The main results that we roughly outlined above may be summarized as follows, see Fig. 1.1 for illustration.

**Global stability (Theorem 7.3).** The steady state \( \rho_{ss} \) is globally exponentially \( L^2 \)-stable for \( \Delta T < 0 \) and \( \Delta T > \Delta T^* \). The global stability threshold is sharp as an \( L^2 \)-independent bound in the sense that it is realized as \( L^2 \to \infty \).

**Local bifurcations (Theorems 4.1, 4.3).** For parameter sets including \( L_2/L_1 < 2\sqrt{2} \approx 2.8 \) the following holds. At the stability thresholds \( \Delta T_j^* \), \( j = 1, 2 \), the critical modes are spatially the lowest harmonics, and the system undergoes generic supercritical Andronov-Hopf bifurcations corresponding to periodic travelling wave bifurcations with velocity \((T^+ + T^-)/2\).

The local unfolding of the degenerate case \( \Delta T_1 = \Delta T_2 \) proves that the two branches of periodic orbits are connected (in \( \Delta T \)), and form an arc of stable periodic solutions. We numerically corroborate that, further away from this degeneracy, secondary instabilities occur along the arc. See Figure 8.4.

In case \( L_2 \gg L_1 \), the primary instabilities can also be higher spatial harmonics, even simultaneously (Remark 3, Corollary 3.5, Theorem 4.4). We thus suspect rich dynamics already at onset for ‘thin’ domains, but a detailed analysis is beyond the scope of this paper. It is also possible that, as \( \Delta T \) increases, a sequence of destabilization and restabilization occur through different harmonics (Lemma 3.7).

This paper is organized as follows. In §2 we reformulate the problem for a subsequent bifurcation analysis. Section 3 concerns the spectrum of the linearized operator in the steady state \( \rho_{ss} \). In §4 we discuss the center manifold reduction, reduced vector fields and prove the main bifurcation results. In §5 we explain the relation to travelling wave bifurcations, and briefly consider pattern formation in case of an infinite strip. In §6 we discuss nonlinear instability for \( \nu > 0 \) in the linearly unstable region. §7 contains the global stability result. Finally, section 8 contains numerical computations, illustrating the results.

**Acknowledgement.** This work, supported by the European Communities under the contract of Association between EURATOM/FOM, was carried out within the
framework of the European Fusion Programme with financial support from NWO. The views and opinions expressed herein do not necessarily reflect those of the European Commission. D.H.-K. is grateful to the CWI, where this work was initiated, for its hospitality. J.R. has been supported in part by NWO cluster NDNS+ and thanks his previous employer, CWI, where most of the work has been done. D.Z. gratefully acknowledges hospitality at the University of Bremen. The authors thank Hugo de Blank for his comments and suggestions on a draft version, and the reviewers for their useful comments, helping us to improve the presentation of the paper.

2. Reformulation and setting. For the bifurcation study it is convenient to formulate (1.1) through the deviation \( u = (u_1, u_2) \) from \( \rho_{ss} \).

\[
\rho^+ = u_1 + \rho_{ss}^+, \quad \rho^- = u_2 + \rho_{ss}^-. \tag{2.1}
\]

In terms of \( u \), and in the comoving variable \( x_2 \rightarrow x_2 + T^* t \), system (1.1) reads

\[
\begin{align*}
\partial_t u_1 &= \Delta T \partial_{x_2} u_1 + \frac{E_2}{L_1} - E^\perp \cdot \nabla u_1 + \nu \nabla^2 u_1, \\
\partial_t u_2 &= -\frac{E_2}{L_1} - E^\perp \cdot \nabla u_2 + \nu \nabla^2 u_2,
\end{align*}
\]

\[
E = -\nabla V, \quad -\nabla^2 V = u_1 + u_2, \quad x \in [0, L_1] \times \mathbb{R}/L_2 \mathbb{Z}, \quad t \geq 0.
\]

subject to (periodic b.c. in \( x_2 \)) and homogeneous Dirichlet boundary conditions

\[
\begin{align*}
u_1(0, x_2, t) &= u_2(0, x_2, t) = V(0, x_2, t) = 0, \\
u_1(L_1, x_2, t) &= u_2(L_1, x_2, t) = V(L_1, x_2, t) = 0.
\end{align*}
\]

Remark 1. We want to briefly point out a peculiarity of the Poisson bracket nonlinearity in (1.1) and equivalently (2.1): viewed on complexified phase space, each eigenspace of the laplacian is flow invariant and the dynamics is purely linear. Indeed, take an eigenfunction \( e \) with eigenvalue \( -\lambda \) of the laplacian and set \( u_j = \alpha_j e \) with \( \alpha_j \in \mathbb{C} \) so that \( E = (\alpha_1 + \alpha_2)/\lambda \nabla e \). Hence, \( E^\perp \cdot \nabla u_j = 0 \) so that (2.1) is in fact linear. Noting that \( E_2 \) is a multiple of an eigenfunction implies the claim. (For the explicit formulas of \( e \) see \( g_k \) below.) In particular, \( x_2 \)-independent \( u_1, u_2, V \) form an invariant subspace, which follows also immediately from the translation symmetry in the \( x_2 \)-direction. The reduced evolution on this space given by the uncoupled heat equations \( \partial_t u_j = \nu \partial^2_{x_2} u_j, \quad j = 1, 2 \) together with the then trivial \(-\partial^2_{x_1} V = u_1 + u_2 \). However, this is the only case of such flow invariant (eigen)spaces for the real equations since all other eigenvalues and eigenspaces of the linear part of (2.1) are complex (see Lemma 3.1 below). Co-moving frames do not generate real eigenspaces due to the asymmetric advection terms. and the previous argument is incorrect for linear combinations of complex conjugates. It is by center manifold reduction in §4 that we find (real) invariant manifolds near bifurcation points with nontrivial dynamics.

Remark 2. Concerning boundary conditions, the spectral analysis of the linear part of (2.1) in §3 below also covers zero-flux boundary conditions on one side of the domain: doubling the domain extends the sine basis functions in \( x_1 \)-direction to the homogeneous Dirichlet case of (2.1). However, the image of an eigenfunction under the nonlinearity violates such boundary conditions.
Next we choose a simple functional analytic setting for a formulation of (2.1) as a parabolic problem by solving the Poisson equation. This is convenient for the center manifold reduction, but also gives a simple well-posedness setting.

Let \( \Omega := [0, L_1] \times [0, L_2] \) and denote the Sobolev spaces \( H^j = H^j([0, L_1] \times \mathbb{R}/L_2 \mathbb{Z}) \) as well as

\[
X := H^1_0([0, L_1] \times \mathbb{R}/L_2 \mathbb{Z}),
\]

\[Y := \{ f \in H^2 : f(0, x_2) = f(L_1, x_2) = 0 \},\]

\[Z := \{ f \in H^3 : f(0, x_2) = f(L_1, x_2) = 0 \},\]

which incorporate the boundary conditions. We shall use standard notation: for \( f_1, f_2 \in L^2([0, L_1] \times \mathbb{R}/L_2 \mathbb{Z}) \) we denote the scalar product by \( \langle f_1, f_2 \rangle = \int_{\Omega} f_1(x) f_2(x) dx \) and for \( f_j = (f_j, f_{j,2}) \in (L^2([0, L_1] \times \mathbb{R}/L_2 \mathbb{Z}))^2 \) \( j = 1, 2 \) by \( \langle f_1, f_2 \rangle_2 = \langle f_{1,1}, f_{2,1} \rangle + \langle f_{1,2}, f_{2,2} \rangle \).

Thanks to these boundary conditions we can solve the Poisson equation in (2.1); see also \[3\] for explicit solutions. We thus obtain \( E \) via the bounded operators \( A_j : Z \to H^j \) defined by

\[A_j f := \partial_{x_j} \nabla^2 f^{-1} f, \quad j = 1, 2,\]

\[Af = (A_1 f, A_2 f)^T,\]

\[A^\perp f = (A_2 f, -A_1 f)^T.\]

Notably, \( A_2 \) in fact maps into \( Z \), because \( E_2 = \partial_{x_2} V \) vanishes for \( x_1 = 0, L_1 \) due to the Dirichlet boundary conditions.

In order to apply standard results on parabolic equations, let us write (2.1) equivalently in the standard form

\[
\frac{du}{dt} = \mathbf{L} u + R(u),
\]

so that solutions of this and (1.1) are in 1-to-1 correspondence. Here

\[
\mathbf{L} = \left( \Delta \partial_{x_2} u_1 + \frac{1}{\nu} A_2(u_1 + u_2) + \nu \nabla^2 u_1, \right.
\]

\[
\left. -\frac{1}{\nu} A_1(u_1 + u_2) + \nu \nabla^2 u_2 \right),
\]

\[
R(u) = \left( -A^\perp(u_1 + u_2) \cdot \nabla u_1, \right.
\]

\[
\left. -A^\perp(u_1 + u_2) \cdot \nabla u_2 \right).\]

Note that \( \mathbf{L} \in \mathcal{L}(Z \times Z, X \times X) \) is the linearization of (1.1) in \( \rho_{ss} \). We have that \( R : Z \times Z \to Y \times Y \) since \( \nabla u_j \in H^2 \times H^2 \) and \( -A^\perp(u_1 + u_2) \cdot \nabla u_j \) vanish at \( x_1 = 0, L_1 \), and \( H^2 \) is a Banach algebra; \( R \) is in fact analytic in \( u \). See also \[4\]. Moreover, the imbeddings \( Z^2 \hookrightarrow Y^2 \hookrightarrow X^2 \) are dense and the uniformly elliptic operator \( -\mathbf{L} : Z \times Z \subset X \times X \to X \times X \) is a sectorial operator, generating an analytic semigroup, and so (2.1) admits mild and classical solutions \( u(t) \) for any initial condition \( u(0) \in Y \times Y \). The sectoriality is a consequence of the fact that the laplacian is sectorial in \( Y \) with domain \( L^2 \) of the cylinder \[12\], and this is robust under addition of the lower order terms in \( \mathbf{L} \). It thus also possesses a square root, which then provides an isomorphism from \( L^2 \) to \( X \). Hence, \( \mathbf{L} \) is also sectorial on \( Z \) with domain \( X \). Note also that \( \mathbf{L} \) has a compact resolvent and thus discrete spectrum accumulating at \( -\infty \). We discuss its spectrum in detail in the next section.
3. Spectrum of the linearization. For the bifurcation analysis, we distinguish the stable spectrum of $L$, $\sigma_-(L) := \{ \lambda \in \sigma(L) : \Re \lambda < 0 \}$, its neutral spectrum $\sigma_0(L) = \{ \lambda \in \sigma(L) : \Re \lambda = 0 \}$ and its unstable spectrum $\sigma_+(L) = \{ \lambda \in \sigma(L) : \Re \lambda > 0 \}$.

The next Lemma characterizes the spectrum and is the basis for the identification of bifurcations. While this concerns the comoving variable of system (2.1), the spectrum for the original system is the same up to a scaling of the imaginary parts. See [3]. Recall that $\ell = L_2/L_1$ is the domain aspect ratio. In the following we set $N_* = \mathbb{N} \setminus \{0\}$ for clarity.

**Lemma 3.1.** The spectrum $\sigma(L)$ of $L$ consists of the eigenvalues

$$
\lambda_k^\pm = i \frac{k_2 \Delta T}{\ell L_1} - \pi^2 \nu \frac{1}{L_1^2} \left( k_1^2 + \frac{4k_2^2}{\ell^2} \right) \pm \sqrt{D_k}, \quad k \in N_* \times \mathbb{Z},
$$

where

$$
D_k = \frac{k_2^2 \Delta T}{\ell^2 L_1} \left( \frac{4}{k_1^2 + 4(k_2/\ell)^2} - \frac{\pi^2 \Delta T}{L_1^2} \right).
$$

In particular, $\lambda_k^- \in \sigma_-(L)$, and if $D_k \leq 0$ then $\lambda_k^+ \in \sigma_-(L)$. Moreover, $\Re(\lambda_{(k_1,k_2)}^+) < \Re(\lambda_{(1,1)}^+)$. We will start to discuss the relevance and implications of this result after the proof. In preparation of the proof, choose the orthogonal basis of $X$ given by

$$
g_k(x) := \sin \left( \frac{k_1 \pi x_1}{L_1} \right) \exp \left( \frac{2i \pi k_2 x_2}{L_2} \right),
$$

where $k \in N_* \times \mathbb{Z}$. In order to express the operator $A$, denote

$$
\phi_k(x) := \cos \left( \frac{k_1 \pi x_1}{L_1} \right) \exp \left( \frac{2i \pi k_2 x_2}{L_2} \right).
$$

Indeed, if $f \in X$, the explicit solution to the Poisson equation $-\nabla^2 V = f$ in terms of this basis reads

$$
V(x) = \frac{2}{\pi^2} \sum_{k \in N_* \times \mathbb{Z}} \frac{1}{L_1^2 k_1^2 + 4 \frac{L_2}{\ell^2} k_2^2} \left( \int_{\Omega} f(y) g_k(y) dy \right) g_k(x).
$$

We therefore get the explicit formula for $A$:

$$
Af(x) = -\frac{2}{\pi} \sum_{k \in N_* \times \mathbb{Z}} \left< f, g_k \right> \frac{L_2}{L_1} \frac{k_1^2}{k_1^2 + 4 \frac{L_1}{\ell^2} k_2^2} \left( \frac{k_1 \phi_k(x)/L_1}{2ik_2 g_k(x)/L_2} \right).
$$

**Proof.** [Lemma 3.1] Consider functions of the form $q g_k(x)$, $k \in N_* \times \mathbb{Z}$, where $q \in \mathbb{C}^2$ is an arbitrary constant vector. Since

$$
A_2 g_k(x) = -\frac{2L_1}{\pi} \frac{k_2}{L_1^2 k_1^2 + 4 \frac{L_2}{\ell^2} k_2^2} g_k(x),
$$

the action of $L$ on such functions is

$$
(L q g_k)(x) = M_k q g_k(x),
$$
where

\[ M_k := \begin{pmatrix} C_1(k)\Delta T - C_2(k) - C_3(k) & -C_3(k) \\ C_2(k) & C_2(k) - C_3(k) \end{pmatrix}, \tag{3.8} \]

with

\[ C_1(k) := \frac{2\pi k_2}{L_2}, \quad C_2(k) := \frac{2i}{\pi} \frac{k_2}{L_2} \left( \frac{k_2}{L_1} k_1^2 + \frac{4\ell_2}{L_2^2} \right), \quad C_3(k) := \nu \pi^2 \left( \frac{k_2^2}{L_1^2} + \frac{4k_2^2}{L_2^2} \right). \]

The eigenvalues of \( M_k \) are readily computed to be \( \lambda^\pm(k) \). The claims on the real parts of \( \lambda^\pm(k) \) immediately follow from inspecting (3.1) — in particular \( D_k \) monotonically decreases in \( k_1 \). □

Note that the proof also implies that eigenfunctions of \( \mathbf{L} \) have the form

\[ \zeta_k(x) := \xi_k g_k(x) \in \mathbb{Z} \times \mathbb{Z}, \tag{3.9} \]

with \( \xi_k \) a eigenvector of \( M_k \).

The last statement in Lemma 3.1 means that only \( D_k > 0 \) and \( \lambda^+_k \) with \( k_2 \in \mathbb{Z} \backslash \{0\} \) allow for destabilization, and the real part in this case is given by

\[ \Re(\lambda^+_k) = -\pi^2 \frac{\nu}{L_2^2} \left( 1 + \frac{4k_2^2}{\ell^2} \right) + \sqrt{ \frac{k_2^2 \Delta T}{\ell^2 L_1} \left( \frac{4}{1 + 4(k_2/\ell)^2} - \pi^2 \frac{\Delta T}{L_1} \right) }. \tag{3.10} \]

Note that this is a function of the three parameters \( \nu/L_1^2, \Delta T/L_1, (k_2/\ell)^2 \). As expected, increasing viscosity always stabilizes, with increasing impact for increasing \( (k_2/\ell)^2 \). However, the dependence of the real part on \( k_2/\ell \) is not necessarily monotone, which allows for intricate destabilization scenarios.

The imaginary part, \( \Im(\lambda^+_k) \), is never zero, which means that all bifurcations are non-stationary and we generically expect Andronov-Hopf bifurcations, where \( k_2 \) determines the wavenumber of bifurcating solutions.

We consider the temperature difference \( \Delta T \) as the primary bifurcation parameter and therefore focus on the location of instabilities as \( \Delta T \) varies, as well as on the wavenumber of destabilizing modes determined by \( k_2 \).

\[ \begin{array}{c}
\text{(a)} \\
\text{(b)}
\end{array} \]

**Fig. 3.1.** Real parts of eigenvalues as functions of \( \Delta T \) for \( L_1 = L_2 = 1 \). The unstable eigenvalues with \( k \in \{1, \ldots, 10\} \times \{-10, \ldots, 10\} \) are plotted for (a) \( \nu = 10^{-3} \), (b) \( \nu = 4 \cdot 10^{-3} \). Eigenmodes with higher wavenumber have smaller location of maxima in order. The figure illustrates linear instabilities occur in a finite interval for \( \Delta T \), that in this parameter regime eigenmodes with wavenumber one are most unstable, and that increasing \( \nu \) stabilizes the spectrum globally in \( \Delta T \).
In Figure 3.1 we plot sample computations of spectrum as $\Delta T$ varies, illustrating the stabilizing effect of the viscosity. Crossings of eigenvalue curves at zero real part can occur, which is expected to generate rich bifurcations. However, in this paper we focus on simple Andronov-Hopf bifurcations.

Recall the spectral conditions at a primary Andronov-Hopf bifurcation

(i) There is a constant $\gamma > 0$ s.t. $\sup \{ \Re \lambda : \lambda \in \sigma_-(L) \} < -\gamma$,

(ii) $\sigma_0(L) = \{ \pm i \omega \}, \omega > 0$ and $\pm i \omega$ are simple eigenvalues, \hspace{1cm} (3.11)

(iii) $\sigma_+(L) = \emptyset$,

and in the nondegenerate case, the critical eigenvalues transversely cross the imaginary axis upon parameter variation.

It turns out that we can characterize a large part of parameter space, where critical eigenvalues have $k_2 = 1$, that is, $k = k_c := (1, 1)$. We therefore define the following particular case of (3.11).

CONDITION 1. It holds that $\Re \lambda^+_{k_2} = \Re \lambda^-_{k_2} = 0$ and there is $\gamma > 0$ such that $\Re \lambda^\pm_k < -\gamma$ for $k \in \mathbb{N} \times \mathbb{Z} \setminus \{ k_c, \bar{k}_c \}$.

Here and in the following we denote $\kappa = (\kappa_1, -\kappa_2)$ for $\kappa \in \mathbb{R}^2$.

Rearranging sign conditions on (3.10) and squaring, we readily compute that the sign of $\Re(\lambda^+_{(1, k_2)})$, for $k_2 = k^+_2$ is the sign of

$$d(\Delta T, \kappa_2) = \frac{4L_1^3}{4\kappa_2 + \ell^2} \Delta T - \frac{L_2^2 \pi^2}{\ell^2} \Delta T^2 - \nu^2 \pi^4 \frac{(4\kappa_2 + \ell^2)^2}{\kappa_2 \ell^4},$$ \hspace{1cm} (3.12)

which is somewhat simpler to handle. In particular, zeros of $d$ are the critical eigenvalues for bifurcations. This yields the following a priori bounds on $\Delta T$ for linear instability. Recall $\Delta T^* = \frac{4L_1^3}{\pi^2}$.

LEMMA 3.2. For all $\nu, \ell$ and $\kappa_2 > 0$, the real roots of $d(\cdot, \kappa_2)$ lie in $(0, \Delta T^*)$. Moreover, the real roots approach the endpoints in the limit $\ell \to \infty$ if $\nu = o(\ell^{-1})$.

Proof. Since $d(0, \kappa_2) \leq 0$ and $\partial_{\Delta T} d(0, \kappa_2) > 0$ the lower bound holds. For the upper bound, observe that $d(4L_1/\pi^2, \kappa_2) < 0$ and $\partial_{\Delta T} d(4L_1/\pi^2, \kappa_2) < 0$, which proves the claim since the quadratic coefficient of $\Delta T$ is negative. The statement on the limits readily follows from (3.12) upon multiplication by $\ell^2$. \[\square\]

Note that $d(\cdot, \kappa_2)$, as a quadratic polynomial in $\Delta T$, has two real roots $\Delta T_1(\kappa_2) \leq \Delta T_2(\kappa_2)$ if and only if the viscosity is sufficiently small,

$$\nu \leq \nu_{\text{crit}}(\kappa_2) := \frac{\sqrt{\pi} \ell L_1^2}{(4\kappa_2 + \ell^2)^{3/2}},$$ \hspace{1cm} (3.13)

with a double root at equality. Hence, this is a necessary and sufficient condition for the occurrence of critical eigenvalues $\lambda^+_{(1, \sqrt{\pi}^2)}$ as $\Delta T$ varies. However, it is subtle to determine when the critical eigenvalues destabilize the equilibrium as this requires to exclude unstable eigenvalues for all other $k_2$.

Nevertheless, the location of these parabola’s maxima in $\Delta T$ is at

$$\Delta T = \frac{2\ell^2 L_1}{(4\kappa_2 + \ell^2)^2 \pi^2},$$ \hspace{1cm} (3.14)
which is strictly decreasing in $\kappa_2$. Therefore, the $k_2$-value of these parabola in $\Delta T$ can be identified by the relative location of their maxima.

**Remark 3.** From (3.13) we infer that for fixed $\nu > 0$, increasing aspect ratio $\ell$ implies increasing wavenumber $k_2$ ($k_2^2 = \kappa_2$). Specifically, for fixed $L_2$ and $L_1 \to 0$ satisfying (3.13) requires that $k_2 = O(L_1^{-3})$. For fixed $L_1$ and $L_2 \to \infty$, the requirement is $k_2 = O(L_2)$. Concerning the upper destabilization threshold $\Delta T_2$, it then follows from (3.14) that in the first case $\Delta T_2 = O(L_1^3)$, while in the second case $\Delta T_2 \to \Delta T_* = 4L_1/\pi^2$ as $L_2 \to \infty$ in accordance with Lemma 3.2.

**Remark 4.** For $\kappa_2 = 1$ the roots satisfy $\Delta T_1 = O(\nu^2)$ and $\Delta T_2 = \frac{4\nu^2 L_1}{\pi^2(\nu^2 + 1)} + O(\nu^2)$, which was already illustrated in Figure 1.1.

The geometric nature of bifurcating solutions is determined by the $k_2$-value of critical and destabilizing eigenvalues as $\Delta T$ in- or decreases from outside $[0, \Delta T_*]$. We thus define

**Definition 3.3.** For given $L_1, \ell, \nu$, we say that $L$ possesses a $k_2$-instability region, if $d(\cdot, k_2)$ has two positive roots $\Delta T_1(k_2^2) \leq \Delta T_2(k_2^2)$. We call a $k_2$-instability region locally primary, if there is a neighbourhood $S \subseteq \mathbb{R}$ of $\Delta T_j := (\Delta T_1(k_2^2), \Delta T_2(k_2^2))$, s.t. the steady state $u = 0$ is stable for $\Delta T \in S \setminus J$ and $\Delta T_j(k_2^2) \neq \Delta T_j(\kappa_2)$ for $\kappa_2 \neq k_2^2$, $j = 1, 2$. Moreover, we say that the $k_2$-instability region is primary, if it is locally primary and $S = \mathbb{R}$.

To ease notation, we simply write $\Delta T_j$ for $\Delta T_j(1)$, $j = 1, 2$.

As a first step to understand the nature of destabilizing $k_2$-instability regions, we consider the case $k_2 = 1$ and in preparation define the following condition.

**Condition 2.** For some given $L_1, \ell, \nu > 0$ we have

\[
\frac{\Delta T}{\nu^2 \pi^2} > \frac{(4 + \ell^2)(4k_2^4 + \ell^2)}{16\ell^4 L_1 k_2^2} (\ell^4 - 16k_2^4) \tag{3.15}
\]

and $\nu < \nu_{\text{crit}}(1)$ for $\Delta T \in \{\Delta T_1, \Delta T_2\}$, and all $k_2 \in \mathbb{N}_*, \kappa_2 \geq 2$. Note that Condition \(2\) requires a ratio of threshold temperature difference and viscosity to dominate a ratio involving domain geometry and linear mode harmonics.

**Theorem 3.4.**

1. A 1-instability region of $L$ is locally primary if and only if Condition \(2\) holds. If it holds, then Condition \(1\) is satisfied at $\Delta T = \Delta T_j$, $j = 1, 2$. The critical eigenvalues are $\lambda_j = \pm \omega_j$ with $\omega_j = \pi \Delta T_j/(\ell L_1)$.

2. For $0 < \ell \leq 2\sqrt{2} \approx 2.8$ any 1-instability region of $L$ is primary and Condition \(2\) is satisfied at $\Delta T = \Delta T_j$, $j = 1, 2$.

The point of the theorem is that it provides conditions under which the destabilizing mode for increasing and decreasing $\Delta T$ is known, namely the lowest spatial harmonic. Note the bound on $\ell$ in item 2 is not sharp, but it in particular includes the case $\ell = 1$ considered in \(0\).

**Proof.**

1. A direct calculation gives

\[
\frac{d(\Delta T, 1) - d(\Delta T, \kappa_2)}{\kappa_2 - 1} = \frac{16\Delta T_{\ell^2 L_1^2}^4 - (4 + \ell^2)(2\kappa_2 + \ell^2)(\ell^4 - 16\kappa_2)\nu^2 \pi^4}{\kappa_2 \ell^4 (4 + \ell^2)(2\kappa_2 + \ell^2)}.
\]

In particular, Condition \(2\) is indeed equivalent to a 1-instability region being locally primary. The claims on $\Delta T$ follow readily from inspection of the zeros of $d$. 

11
2. This is the trivial observation that the right hand side in condition 3.15 is strictly negative for these values of $\ell$, while the left hand side is positive at all possible real roots $\Delta T$ of $d(\cdot, \kappa_2)$ on account of Lemma 3.2.

**Remark 5.** The critical frequencies in the original $x_2$-variable of (1.1) are in fact

$$\omega_j = \frac{T_j^+ + T_j^-}{L_2} \pi.$$  

The following corollary guarantees that other destabilization scenarios also occur.

**Corollary 3.5.** Let $\kappa_2 > 1$ and let $\ell$ be the unique positive solution $\ell = \ell_{\kappa_2}$ of

$$\ell^6 - \kappa_2 \ell^4 - 80 \kappa_2 \ell^2 - 64 \kappa_2 (2 + \kappa_2) = 0. \quad (3.17)$$

Then for $\nu = \nu_{\text{crit}}(1)$ the 1-instability region is a point, $\Delta T_1 = \Delta T_2$, that coincides with $\Delta T_2(\kappa_2)$. Notably, $\ell_{\kappa_2}$ is strictly increasing in $\kappa_2$.

**Proof.** Substituting the critical $\nu^2 = \frac{4\ell^6 L_1^4}{(4 + \ell^2)^2 \pi^2}$ from (3.13) and the corresponding critical value of $\Delta T = \frac{2\ell^4 L_1}{4(4 + \ell^2)^{3/2}}$ at the double root into the nominator of the right hand side of (3.16) gives

$$\frac{4\ell^6 L_1^4}{(4 + \ell^2)^3 \pi^2} (64 \kappa_2 (2 + \kappa_2) + 80 \kappa_2 \ell^2 + 4 \kappa_2 \ell^4 - \ell^6),$$

where $\kappa_2 = k_2^2$. The first factor is positive and roots of the second factor, which we denote by $q$, precisely solve (3.17). We have

$$\partial_q q = 80 \kappa_2 + 8 \kappa_2 \ell^2 - 3 \ell^4,$$

which is positive at $\ell = 0$ so that the cubic $q$ with negative cubic coefficient has a unique positive root. In addition, this implies that $\partial_q q < 0$ at this root so that together with

$$\partial_{\kappa_2} q = 4(32 + 32\kappa_2 + 20 \ell^2 + \ell^4) > 0$$

we infer from implicit differentiation that the location of this root strictly increases with $\kappa_2$. □

This means that the 1-instability region is not primary. In fact, it is also not primary for nearby parameter values that produce $\Delta T_2(4) > \Delta T_2(1)$. The solution to (3.17) for $\kappa_2 = 4$ is $\ell_4 \approx 5.37$, and for $\kappa_2 = 9$ it is $\ell_9 \approx 7.22$. See Figure 3.2. For $\ell$ between these values (and slightly above $\ell_9$), we numerically find that the 2-instability region is primary. In general, for any given $k_2$ Condition 2 is violated for sufficiently large $\ell$ (with $\nu, L_1$ fixed), since $\Delta T$ is bounded (Lemma 3.2).
Remark 6. It is possible to show that for $\nu$ small enough, there is a primary 1-instability region, if $\ell < \ell^* \approx 4.053$, where $\ell^*$ is the unique positive root of the polynomial $16(\ell^2 + 4)^3 - (\ell^2 - 8)(\ell^2 + 8)(\ell^2 + 16) = 0$.

For the case of small viscosity, we have the following corollary of Theorem 3.4.

Corollary 3.6. As $\nu \to 0$, $\mathbf{L}$ has $k_2$-instability regions for $k_2 \to \infty$ with $\Delta T_1(k_2^2) < \Delta T_2(k_2^2)$. For sufficiently small $\nu$, Condition 7 is satisfied at $\Delta T_2(1)$, and this is an instability threshold.

Proof. The presence of all $k_2$-instability regions clearly holds at $\nu = 0$ in view of 3.13. In addition, from 3.16 we infer at $\nu = 0$ that

$$d(\Delta T, 1) - d(\Delta T, \kappa_2) > 0,$$

so that the critical eigenfunction at the right endpoint of the instability interval has mode number $k_2 = \pm 1$. This persists for sufficiently small $\nu > 0$, since the thresholds depend continuously on $\nu$, and again from 3.16 we see that for each $\nu > 0$ there is only a finite range of $\kappa_2$ values, for which $d(\Delta T, 1) - d(\Delta T, \kappa_2) < 0$ is possible.

Lastly, we point out the possibility of multiple disjoint primary $k_2$-instability regions, where changing $\Delta T$ destabilizes and stabilizes multiple times. The following lemma shows that there can be at most two such regions since $\kappa = k_2^2$ is discrete. On the infinite cylinder the lemma implies a Turing-Hopf instability, see Lemma 5.1 in §5 below.

Lemma 3.7. Fix $L_1, L_2, \nu$ and consider $d(\Delta T, \kappa)$ as map $d : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$. Then $d$ has a unique critical point, which is a global maximum $(\Delta T_{\text{max}}, \kappa_{\text{max}})$ with $\kappa_{\text{max}} > 0$ and $\partial^2 d(\Delta T_{\text{max}}, \kappa_{\text{max}}) < 0$. Moreover, for any $\Delta T$, the function $d(\Delta T, \cdot)$ has a unique positive maximum.

Proof. For fixed $\Delta T$ we compute

$$\partial^2_\kappa d(\Delta T, \kappa) = \frac{2^7 L_1^3 \Delta T}{(4 \kappa + \ell_2)^2} - \frac{2 \nu^2 \pi^2}{\kappa^3}$$

so that a change in convexity (in $\kappa$-direction) for $\kappa > 0$ requires

$$4(L_1 \Delta T^{(1/3)} - \nu^2 \pi^2)\kappa = \ell^2,$$

which has at most one solution. Since $d(\Delta T, \kappa)$ as a function of $\kappa$ is convex near $\kappa = 0$ and $d(\Delta T, \kappa) \to -\infty$ as $\kappa \to \infty$, this implies a unique critical point in the convex region, which is therefore a global maximum for $\kappa > 0$. 

13
Putting this together with the fact that $d(\Delta T, \kappa)$ is a convex quadratic polynomial in $\Delta T$ we infer the lemma statement. □

In Figure 3.3 we plot eigenvalue curves, where two $k_2$-instability regions consist of a point. Parameters $\nu = \nu_{\text{crit}}(k_2) = \nu_{\text{crit}}(k'_2)$ and $\ell$ that produce such scenarios can be readily computed from (3.13); here we take $k_2 = 1$, $k'_2 = 4$. For perturbed $\nu < \nu_{\text{crit}}(1)$ the instability regions become disjoint open intervals.

**Remark 7.** On account of (3.13), for decreasing $\nu$ and also for increasing $\ell$, there is an increasingly long sequence of secondary instabilities of Andronov-Hopf type as $\Delta T$ increases from zero, with higher and higher spatial harmonics, and another reverse sequence as $\Delta T$ is moved towards $\Delta T_{\text{crit}}$. See Figure 3.1.

4. Center manifold reduction. In this section, we consider the vicinity of parameters with critical $\Delta T = \Delta T_j$ for $j = 1$ or $j = 2$ and assume that no other eigenvalues lies on the imaginary axis. The main example is a primary 1-instability region. For the unfolding of the bifurcation in the generic case $\Delta T_1 < \Delta T_2$ we introduce the parameter $\mu_1$ by $\Delta T = \Delta T_j + \mu_1$. In the degenerate case $\Delta T_1 = \Delta T_2$, where $\nu = \nu_{\text{crit}}(1)$, we additionally unfold with $\mu_2$ defined by $\nu = \nu_{\text{crit}}(1) - \mu_2^2$. For readability we frequently suppress the index $j$.

At bifurcation, the critical eigenvalues are then $\pm i\omega$ and we denote the associated eigenfunctions by $\zeta(x) := \zeta_{c_k}(x)$, $\zeta(x)$, see (3.9). Then $L$ possesses a two-dimensional real central subspace $E_c := \text{span} \{ \Re \zeta, \Im \zeta \} \subset Z^2$ and we will show that there is a locally invariant 2D center manifold

$$W_c = \{ u_0 + \psi(u_0, \mu) : u_0 \in O_{E_c} \} \subset O_{Z^2}, \mu \in O_{R^2},$$

with $\psi : O_{E_c} \to E^1_c, E_c \oplus E^2_c = Z^2$, and neighbourhoods $O_{Z^2}$ of $\mu = 0$, and $O_{E_c}, O_{Z^2}$ of $0 \in Z^2$. In case of a primary bifurcation the center manifold is also locally exponentially attracting.

Since we consider $k = k_c = (1,1)$, it is not surprising that the coefficients $C_m(k_c)$ defined in (3.8) show up. It turns out that following modifications are convenient.

$$c_1 := \frac{\pi \Delta T}{L_2}, c_2 := \frac{2}{\pi (\frac{4L_1^2}{L_2} + \frac{4L_2}{L_1})}, c_3 := \nu \pi^2 \left( \frac{1}{L_1^2} + \frac{4}{L_2^2} \right). \quad (4.1)$$

We give an overview of the main results in this section and postpone the lengthy proofs of Theorem 4.1 Corollary 4.2 and Theorem 4.3 to 4.1. We first consider the generic case of (3.13), where the unfolding goes by $\mu_1$ only. See Fig. 1.1 for a partial
Theorem 4.1. Suppose that Condition 1 holds for a fixed parameter set for which $\Delta T_1 < \Delta T_2$. Then the steady state $u = 0$ of system (2.5) possesses a locally exponentially attracting and locally invariant 2D center manifold near $u = 0$ with the reduced dynamics

$$\frac{dz}{dt} = i\omega z + \mu_1 az + b|z|^2 + \mathcal{O}(|\mu| + |z|^2),$$  \hspace{1cm} (4.2)

where

$$\omega = \frac{\pi \Delta T}{L_2},$$

$$a = \frac{2\pi c_2^2}{L_2(c_1 - c_3 i)(c_1 i - 2c_2 i - c_3)},$$

$$b = \frac{L_2^4 c_3^2 + c_2^2}{4\pi^2 \nu L_1^2 + 4L_1}.$$  \hspace{1cm} (4.3)

The following corollary proves the nature of the resulting bifurcations.

Corollary 4.2. Assume the hypotheses of Theorem 4.1. Then the steady state $u = 0$ of system (2.5) undergoes a generic supercritical Andronov-Hopf bifurcations as $\mu_1$ varies. Specifically, the reduced vector field coefficients satisfy $b < 0$, $\Im(a) = \mathcal{O}(\nu^2)$, and $\sgn(\Re(a)) = -(-1)^j$ at $\Delta T = \Delta T_j$.

In particular, near the stability thresholds there exist heteroclinic connections between the unstable steady state and the stable limit cycle.

As $\nu \downarrow 0$, the radius of the limit cycles, $|z(t)|$, scales near $\Delta T_1$ as $|z(t)| \propto \nu^{-1}\sqrt{\Delta T_1 - \Delta T}$, and near $\Delta T_2$ as $|z(t)| \propto \sqrt{\Delta T_2 - \Delta T}$.

The (temporal) heteroclinic connections in this theorem always exist for generic Andronov-Hopf bifurcations (see e.g. [14]). Their relevance is that every solution starting sufficiently close to the unstable steady state will converge to the stable limit cycle, and this is used in the numerical computations of §8.

Next, we formulate the result for unfolding the codimension-2 case $\Delta T_1 = \Delta T_2$, where the critical eigenvalues do not transversely cross the imaginary axis.

Theorem 4.3. Suppose that Condition 1 holds for a fixed parameter set for which $\Delta T_1 = \Delta T_2$. Then the steady state $u = 0$ of system (2.5) possesses a locally exponentially attracting and locally invariant 2D center manifold near $u = 0$ with the reduced dynamics

$$\frac{dz}{dt} = i(\omega + a_0 \mu_1)z + a_1 \mu_1 (a_2 \mu_2 - a_3 \mu_1) z + b|z|^2 + \mathcal{R}$$

$$\mathcal{R} = \mathcal{O}(\mu_2^2 + |\mu_1 \mu_2|^2) + |z|(|\mu| + |z|^2),$$  \hspace{1cm} (4.4)

where $a_j \in \mathbb{R}$, $j = 0, 1, 2, 3$, are given by $a_0 = \Im(a)$,

$$a_1 = \frac{\Re(a) \pi}{L_2(c_2 - c_1)}, \quad a_2 = \frac{1}{\sqrt{\pi L_1 L_2}}, \quad a_3 = \frac{\pi}{L_2^2},$$

and $a$, $b$ are the constants from Theorem 4.1.
In particular, for $0 < |\mu_2| \ll 1$, there exists a branch of stable periodic orbits, that is parametrized by $\mu_1$ and that terminates in supercritical Andronov-Hopf bifurcations at $\Delta T + \mu_1 = \Delta T_j$, $j = 1, 2$.

**Remark 8.** As Corollary 4.2 implies, under the conditions of Theorem 4.3, there are heteroclinic connections between the unstable steady state and the stable limit cycle.

The following Theorem shows, that the bifurcation results 4.1, 4.3 can be generalized to instabilities caused by higher spatial harmonics.

**Theorem 4.4.** Assume (3.11) holds with critical wavenumber $k_2$, so that $\lambda_{1n}^2(k_2) = \i \omega$. If $\Delta T_1(k_2^2) < \Delta T_2(k_2^2)$ then the statements of Theorem 4.1 and Corollary 4.2 hold with $\Delta T_j$ replaced by $\Delta T_j(k_2^2)$, and $L_j$ replaced by $L_j/k_2$ for the coefficients on the center manifold. If $\Delta T_1(k_2^2) = \Delta T_2(k_2^2)$ then the statement of Theorem 4.3 holds for the same modifications.

**Proof.** Under condition (3.11) the center manifold theorem applies as in the first parts of the proofs of Theorems 4.1 and 4.3. This yields a stable locally invariant manifold with reduced dynamics of Hopf normal form. The only remaining question is the sign of the coefficients.

If $k_2$ is the critical wavenumber in $x_2$-direction on the domain $[0, L_1] \times [0, L_2]$ then 1 is this wavenumber on the domain $[0, L_1] \times [0, L_2/k_2]$ so that Condition 1 holds there. Hence, on this domain and with the modifications in the claim, Theorem 4.1, Corollary 4.2 and Theorem 4.3 hold fully.

The theorem now follows since the bifurcating branches imbed into the original domain. 

**Remark 9.** Recall that there is a sequence of secondary Andronov-Hopf instabilities as noted in Remark 7. Whenever these occur with a simple pair of complex conjugate eigenvalues, analogous center manifold reduction results hold for an unstable 2D manifold. The reduced vector fields are of the same form with coefficients given analogous to the above results, but to be computed at different $k_2$ and other parameters.

### 4.1. Proofs of Theorem 4.1, Corollary 4.2 and Theorem 4.3

**Proof.** [Theorem 4.1] For the unfolding with $\mu_1$, we modify the definition of $R$ in (2.5) by adding the term $\mu_1 \partial_2 u_1$ in the first component and denote the result by $R(u; \mu_1)$. For the resulting bifurcation problem, we verify the hypotheses of the center manifold theorem [11, Theorem 3.3, p. 46].

As noted after (2.5), $L \in \mathcal{L}(Z^2, X^2)$ is sectorial so that Hypothesis 2.7 in that theorem holds, using [11, Remark 2.18 p. 37] Hypotheses 3.1(i) and 2.4 hold on account of Theorem 3.4. It remains to show Hypotheses 3.1(ii): smoothness of $R$. From (2.5) we explicitly compute

\[
\begin{align*}
DR(u; \mu_1)v &= \left( \mu_1 \partial_2 v_1 - A^+(v_1 + v_2) \cdot \nabla u_1 - A^+(u_1 + u_2) \cdot \nabla v_1 \right), \\
D^2R(u; \mu_1)[v, w] &= - \left( A^+(v_1 + v_2) \cdot \nabla w_1 + A^+(w_1 + w_2) \cdot \nabla v_1 \right),
\end{align*}
\]

Note that $A^+(v_1 + v_2) \in H^3 \times H^1$ and $\nabla w \in H^2 \times H^2$. Since $H^2$ is a Banach algebra (see for instance [1, Theorem (4.39)])], there is a constant $C_0 > 0$, such that

\[
\|A^+(v_1 + v_2) \cdot \nabla w\|_{H^3} \leq C_0\|A^+(v_1 + v_2)\|_{H^3 \times H^1}\|\nabla w\|_{H^2 \times H^2}.
\]
Also \( A^+(v_1 + v_2) \cdot \nabla w \) vanishes at \( x_1 = 0, L_1 \), hence \( \| D^2R(u; \mu_1)[v, w]\|_{Y^2} \leq C\|v\|_{Z^1}\|w\|_{Z^2} \), that is, \( R(u; \mu_1) \in C^2(Z^1, Y^2) \). Moreover all the higher derivatives are identically 0, hence \( R \) is analytic. This establishes the existence of the 2D center manifold and smoothness of \( \psi \) as needed below, and for which the reduced dynamics has the normal form \( (4.2) \). Here the critical frequency is \( \omega = \pi \Delta T/L_2 \) due to Theorem 3.4. In order to analyze the coefficients of the reduced equation, we write functions in the central subspace as

\[
u_0(t) = z(t)\zeta + \bar{z}(t)\bar{\zeta}, \quad z(t) \in \mathbb{C}.
\]

Using the expressions in [11, p. 125] we have

\[
a = \langle R_{11}(\zeta) + 2R_{20}(\zeta, \psi_{001}), \zeta^* \rangle_2, \tag{4.5}
\]

\[
b = \langle 2R_{20}(\zeta, \psi_{110}) + 2R_{20}(\bar{\zeta}, \psi_{001}) + 3R_{20}(\zeta, \bar{\zeta}, \zeta^*) \rangle_2. \tag{4.6}
\]

The quantities in these expressions are defined as follows: \( \zeta^* \) is the adjoint eigenvector to \( \zeta \), the operators \( R_{ik} \) are given by, see [11, p. 95-96],

\[
R_{01} := \partial_{\mu_1} R(0; 0) = 0,
\]

\[
R_{20}[v, w] := \frac{1}{2} D^2R(0; 0)[v, w] = \frac{1}{2} \left( A^+(v_1 + v_2) \cdot \nabla v_1 + A^+(w_1 + w_2) \cdot \nabla v_1 \right),
\]

\[
R_{11} v := \partial_{\mu_1} DR(0; 0) v = \left( \partial_{x_2} v_1 \quad \nabla^2 v_1 \right), \quad R_{30} = \frac{1}{3!} D^3R = 0,
\]

and the functions \( \psi_{ijk} \), from the expansion of \( \psi \), are the unique solutions to

\[
-L \psi_{001} = R_{01}, \quad (2oi - L) \psi_{200} = R_{20}(\zeta, \bar{\zeta}), \quad -L \psi_{110} = 2R_{20}(\bar{\zeta}, \bar{\zeta}). \tag{4.8}
\]

**Computation of \( a \).** Since \( R_{01} = 0 \) and \( \text{ker}(L) = \{0\} \), \( -L \psi_{001} = R_{01} \) implies \( \psi_{001} = 0 \). For this result the parameter \( \mu_2 \) is held fixed at zero so that, using (4.5), the coefficient \( a \) of the reduced system \( (4.2) \) is

\[
a = \langle R_{11}(\zeta), \zeta^* \rangle_2 = \frac{2\pi \xi^{11}}{L_2} ((g_1, 1, 0)^T, \zeta^*)_2, \tag{4.9}
\]

where \( \zeta^* \) is the adjoint eigenfunction, satisfying

\[
L^* \zeta^* = -i\omega \zeta^*, \quad \langle \zeta, \zeta^* \rangle_2 = 1. \tag{4.10}
\]

with the adjoint operator of \( L \) given by (using integration by parts)

\[
L^* v = \begin{pmatrix}
-\Delta T \partial_{x_2} v_1 + \frac{4i}{L_1} Bv + \nu \nabla^2 v_1 \\
\frac{1}{L_1} Bv + \nu \nabla^2 v_2
\end{pmatrix}, \quad v \in Y_2
\]

\[
Bv(x) = \frac{4i}{L_2} \sum_{k \in \mathbb{N} \times 2} \frac{k_2}{L_1^2 k_1^2 + \frac{4i}{L_1} k_2^2} (v_1 - v_2, g_k) g_k(x).
\]
The critical adjoint eigenfunction $\zeta^*$, as any eigenfunction of $L^*$, has the form $\zeta^*(x) = \eta g_m(x)$, where $\eta = (\eta^1, \eta^2) \in \mathbb{C}^2$ is an eigenvector of $M_m^*$ derived from (3.8). If $m \neq (1, 1)$, then $\langle \zeta, \zeta^* \rangle_2 = 0$, therefore $m = (1, 1)$, and hence $M_{1,1}^* \eta = -i \omega \eta$ so that from $\langle g_{11}, g_{11} \rangle = L_1 L_2 / 2$ and (4.9) we infer

$$a = \pi \xi_1 \eta L_1 i.$$  \hspace{1cm} (4.11)

Due to (3.7), there is $\xi \in \mathbb{C}^2$ such that

$$M_{1,1} \xi = i \omega \xi, \xi = (\xi_1^1, \xi_2^1)^T,$$  \hspace{1cm} (4.12)

and using (4.1) at the bifurcation points $\Delta T = \Delta T_j, j = 1, 2$, we have

$$c_3^2 = c_1 (2c_2 - c_1).$$  \hspace{1cm} (4.13)

Putting this together with equation (4.10) we readily check that

$$(M_{1,1} - i \omega) \xi = \left( c_1 i - c_2 i - c_3 \begin{array}{c} -c_2 i \\ c_2 i \\ -c_1 i + c_2 i - c_3 \end{array} \right) \xi = 0$$

$$(M_{1,1}^* + i \omega) \eta = \left( -c_1 i + c_2 i - c_3 \begin{array}{c} -c_2 i \\ c_2 i \\ c_1 i - c_2 i - c_3 \end{array} \right) \eta = 0$$  \hspace{1cm} (4.14)

Due to (4.13), the eigenvectors can be chosen as

$$\xi = \left( c_1 i - c_2 i - c_3 \begin{array}{c} c_2 i \\ c_1 i - c_2 i - c_3 \end{array} \right), \quad \eta = \delta \left( c_1 i - c_2 i - c_3 \right)$$

$$\delta = \frac{2}{L_1 L_2} \left( c_1 i - c_3 \right) (c_1 i - 2c_2 i - c_3)^*$$.  \hspace{1cm} (4.15)

where $\delta \neq 0$ provides the normalization. Therefore, (4.11) yields $a = \pi c_2^2 L_1 \delta i$ as claimed.

**Computation of $b$.** We first show $\psi_{200} = 0$; recall (4.8). Thanks to (4.12), $\zeta(x) = \xi g_{11}(x)$ and for $k \in \mathbb{N} \times \mathbb{Z}$ we have

$$A_1 g_k(x) = -\frac{L_2}{\pi} \frac{k_1}{L_1^2 k_1^2 + 4 L_2^2 k_2^2} \phi_k(x).$$  \hspace{1cm} (4.16)

A direct calculation yields $R_{20}(\zeta, \zeta) = 0$. Since $\text{ker}(2\omega - L) = \{0\}$ on account of Theorem 3.4, the equation for $\psi_{200}$ from (4.8) implies $\psi_{200} = 0$. Together with $R_{30} = 0$ and (4.6), this means

$$b = \langle 2R_{20}(\zeta, \psi_{110}), \zeta^* \rangle_2.$$  \hspace{1cm} (4.17)

Next, we compute $\psi_{110}$ using (4.8). From $\zeta = \xi g_{11}$ and (2.4), (3.6), (3.5) as well as (4.16), straightforward calculations give

$$-L \psi_{110} = 2R_{20}(\zeta, \zeta) = \frac{2i}{L_2^2 + 4 L_2^2} \bar{\tau} g_{2,0}.$$
Since the eigenvectors \((g_k)_{k \in \mathbb{N} \times \mathbb{Z}}\) of \(L\) are mutually orthogonal and \(M_{2,0}\) is a multiple of the identity, we have that \(\psi_{110} = \alpha g_{2,0}\), where

\[
\alpha = \frac{L_1^2 i}{2\pi^2 \nu} \frac{\xi^1 + \xi^2}{L_1^2 + 4 \nu L_2}.
\]

It follows, after straightforward calculations, that \(R_{20}(\zeta, \psi_{110}) = \beta g_{1,1}\phi_{2,0}\), where

\[
\beta = \alpha \left( \frac{2i(\xi^1 + \xi^2)}{L_2} + \frac{L_2}{L_1} \xi - i \frac{L_1(\xi^1 + \xi^2)}{2L_2} \right).
\]

Substitution into (4.17) yields

\[
b = \langle 2R_{20}(\zeta, \psi_{110}), \zeta \rangle = 2 \beta \cdot \eta \langle g_{1,1}, \phi_{2,0} \rangle = -\frac{L_1 L_2}{2} \beta \cdot \eta.
\]

Finally, we use that \(\xi \cdot \eta = 0\), see (4.15), and together with \(\xi \cdot \eta = \frac{2 \pi L_2}{L_1 L_2}\) we obtain

\[
b = -\frac{L_1^2}{4\pi^2 \nu} \frac{c_2^3 + c_4^3}{L_1^2 + 4L_1},
\]

which concludes the proof. \(\square\)

We now turn to the proof of Corollary 4.2.

**Proof.** From (4.3) and (4.1) we readily check \(b < 0\).

Writing (4.3) in terms of \(c_j\) and using (4.15), a straightforward calculation gives

\[
\Re(a) = \frac{4\pi}{L_2} c_2^3 c_3 (c_1 - c_3) (c_1 i - 2c_2 i - c_3) (c_2 - c_1).
\]

Thanks to \(c_j > 0\), \(j = 1, 2, 3\), all factors in this expression are positive, except possibly the last one, and therefore the sign of \(\Re(a)\) is the sign of \(c_2 - c_1\). Note that

\[
\frac{2\pi}{L_2} (c_2 - c_1) = \partial_{\Delta T} d(\Delta T, 1)/L_1^4,
\]

and that the quadratic polynomial \(d(\cdot, 1)\) has negative quadratic coefficient. Therefore, \(c_1 < c_2\) at \(\Delta T = \Delta T_1\) and so \(\Re(a) > 0\), while at \(\Delta T = \Delta T_2\) we have \(c_2 < c_1\), hence \(\Re(a) < 0\). We readily compute that \(3(a) = c_3^3 \Re(a)/(c_3 (c_2 - c_1)) = O(\nu^2)\).

In conclusion, there are generic supercritical Andronov-Hopf bifurcations at both endpoints of the instability region. As usual, the local invariance of the center manifold from Theorem 4.1 implies the existence of the claimed heteroclinic orbit between the unstable steady state and the stable limit cycle, contained in the center manifold.

Now consider the behaviour of \(a\) and \(b\) for small viscosity \(0 < \nu \ll 1\). With \(c_4 := \frac{c_4}{\nu}\) we get \(c_2, c_4 = O(1)\), and

\[
a = \frac{2\pi c_2^3}{L_2 (c_1 + \nu c_4) ((c_1 - 2c_2) i - \nu c_4)},
\]

\[
b = -\frac{L_1^2 c_2^3 + \nu^2 c_4^3}{4\pi^2 \nu L_2^2 + 4L_1}.
\]

(4.20)
Left endpoint of the instability region: $\Delta T_1$. Inspecting the formula for $d_{\Delta T}$ we find $c_1 = c_2 - \sqrt{c_2^2 - \nu^2 c_4^2}$, where $c_2^2 - \nu^2 c_4^2 > 0$ by (3.13). Hence,

$$c_1 = \frac{c_2^2}{2c_2} \nu^2 + O(\nu^4),$$

and we obtain

$$a = \frac{\pi c_2}{L_2 c_4} \nu + O(\nu), \quad b = -\frac{L_1 c_2^3}{4\pi^2(L_1^2 + L_2^2)} \nu + O(\nu^3).$$

Therefore the radius of the stable limit cycle $|z(t)|$ for sufficiently small $\mu_1$ is

$$\frac{2\pi}{L_1^2 c_4} \left( \frac{\pi (4L_1^2 + L_2^2) c_2}{L_2 c_4} \right)^{\frac{1}{2}} \frac{1}{\nu} \mu_1^{\frac{1}{2}} + O(\nu^{\frac{1}{2}}).$$

Right endpoint of the instability region: $\Delta T_2$. Here $c_1 = c_2 + \sqrt{c_2^2 - \nu^2 c_4^2}$, therefore

$$a = -\frac{c_2^2 \pi}{c_4^2 L_2} \nu + O(\nu), \quad b = \frac{L_1 c_2^3}{4\pi^2(L_1^2 + L_2^2)} \nu + O(\nu)$$

hence the radius of the stable limit cycle for small $-\mu_1$ is

$$\frac{\pi}{L_1^2 c_4} \left( \frac{4L_1^2 + L_2^2}{L_2} \right)^{\frac{1}{2}} \left( -\mu_1 \right)^{\frac{1}{2}} + O(\nu^{\frac{1}{2}}).$$

This concludes the proof. \(\square\)

We finally provide the proof of Theorem 4.3.

**Proof.** [Theorem 4.3] In order to unfold in $\mu_2$, we cannot cite a center manifold theorem from [11] verbatim. The reason is that $\mu_2$ modifies the second order derivative terms, but the results in [11] are formulated for parameter dependence of lower order terms only. However, as pointed out in [11] Remark 3.7, there is no problem if the domain of $L$ is independent of the parameter. This is the case here as long as $\nu = \nu_{\text{crit}}(1) - \mu_2^2 > 0$, which is valid for the purpose of unfolding from $\nu = \nu_{\text{crit}}$. Moreover, the precise, the proof of [11] Theorem 3.3, p. 46], which considers the phase space extended by the unfolding parameter space, applies as follows for $\nu_{\text{crit}}(1) > \mu_2^2$ due to the linearity in $\mu_2^2$. Set $\mu = (\mu_1, \mu_2^2)$, $\tilde{u} = (u, \mu)$ and $L\tilde{u} = (L + \mu_1 \partial_{\xi_2}(u_1, 0)^T - \mu_2^2 \nabla^2 u, 0)$ as well as $R(\tilde{u}) = (R(u), 0)$. (We use $\mu_2^2$ as the parameter instead of $\mu_2$ only to get more pleasant reduced equations.) For the extended problem, the parameter-free center manifold theorem applies [11] Theorem 2.9].

Therefore, as in the first part of the proof of Theorem 3.4, we obtain existence of the center manifold and the coefficient $b$ is unchanged. Let $A$ denote the real coefficient of $z$ in the vector field on the center manifold. It remains to derive the claimed $a_j$-dependent form

$$A = a_1 \mu_1 (a_2 \mu_2 - a_3 \mu_1) + O(\mu_2^2 + |\mu_1 \mu_2^1|).$$

For this we simply note that in the present case, (4.9) is replaced by the more general form

$$A = \langle R_{11}(\zeta) \mu, \zeta^* \rangle_2 = \mu_1 a - \mu_2^2 \langle \nabla^2 \zeta, \zeta^* \rangle_2,$$
where \( \zeta = \xi g_{1,1} \). Using \( \nabla^2 g_{1,1} = -\pi^2 \left( \frac{1}{L_1^2} + \frac{4}{L_2^2} \right) g_{1,1} \) as well as \( \langle \zeta, \zeta^* \rangle_2 = 1 \), we obtain

\[
A = \mu_1 a + \mu_2^2 \pi^2 \left( \frac{1}{L_1^2} + \frac{4}{L_2^2} \right) = \mu_1 a + \mathcal{O}(\mu_2^2),
\]

with \( a \) from Theorem 4.1 whose dependence on \( \mu_2 \) is considered next. Recall that \( \nu = \nu_{\text{crit}} - \mu_2^2 \), with \( \mu_2 = 0 \) giving equality in (3.13). Hence,

\[
\nu_{\text{crit}} = \frac{2\pi}{\nu} \left( \frac{1}{L_1^2} + \frac{4}{L_2^2} \right) \tilde{c}_3,
\]

with \( \tilde{c}_3 \) stems from writing

\[
d(\Delta T, 1)/L_1^4 = (2\tilde{c}_1 - \tilde{c}_2 \Delta T) \Delta T - \nu^2 \tilde{c}_3^2,
\]

with suitably defined \( \tilde{c}_j, j = 1, 2, 3 \) (note the relation to \( c_j \) in (4.1)). Then

\[
d(\Delta T, 1) = 0 \quad \text{gives} \quad \Delta T_{\text{crit}} = \tilde{c}_1 + \frac{\tilde{c}_3}{2} \mu_2 \sqrt{2\nu_{\text{crit}} - \mu_2^2}.
\]

Using (4.18), (4.19) with \( \Delta T = \Delta T_{\text{crit}} + \mu_1 \) then yields

\[
a = a_1 \left( \tilde{c}_3 \mu_2 \sqrt{2\nu_{\text{crit}} - \mu_2^2} - 2\tilde{c}_2 \mu_1 \right).
\]

The above formula for \( \nu_{\text{crit}} \) and expansion in \( \mu_2 = 0 \) gives claimed form of \( A \), when substituting the resulting \( a \) into (4.21).

The bifurcation scenario can be immediately read off the reduced vector field as follows: Let us fix \( \mu_2 \) with \( 0 < |\mu_2| \ll 1 \). Then the reduced vector field shows that at the stability thresholds we have supercritical Andronov-Hopf bifurcations. Let us introduce the polar form \( z = \rho e^{i\theta} \). In the instability region of the equilibrium \( \rho = 0 \) the stable limit cycle radii are

\[
\rho = \sqrt{a_1 \mu_1 (a_3 \mu_1 - a_2 \mu_2)},
\]

which gives the connected branch of periodic orbits with endpoints at the bifurcations.

## 5. Travelling wave bifurcation

As mentioned in the introduction, due to the translation symmetry in \( x_2 \), the Andronov-Hopf bifurcations correspond to periodic travelling wave bifurcations. Specifically, each periodic orbit is a steady state in a comoving frame \( y_2 = x_2 - st \) for certain \( s \). While this relation is generally known in the pattern formation community, for completeness we give some details. The converse is clear: periodic travelling wave bifurcations imply Andronov-Hopf bifurcations.

First note that the effect of the co-moving variable is the introduction of an advection term \( s \partial_{y_2} \) on the right hand side of the first two equations in (1.1). Therefore, the linearization \( M_k \) is replaced by

\[
M_{k,s} = M_k + s C_1(k) \text{Id},
\]

where \( C_1(k) = 2\pi ik_2/L_2 \). Hence, if \( \lambda_k \) is an eigenvalue of \( M_k \) then \( \lambda_k + s C_1(k) \) is an eigenvalue of \( M_{k,s} \), and choosing critical \( k_2 = +1 \), the frequency at bifurcation \( \omega \) is replaced by \( \omega + s 2\pi / L_2 \). The reduced equation on the center manifold then reads

\[
\dot{z} = i(\omega + s 2\pi / L_2) + \mu_1 a + b z |z|^2,
\]
where \(a\) and \(b\) are unmodified since the matrices made of \(c_j\) in (4.14) do not depend on \(s\). Hence, for \(s = s_* := -\omega L_2/2\pi\) we find steady state supercritical pitchfork bifurcations. Note the choice \(s_2 = -1\) reverses the sign of \(\omega\), simply leading to the complex conjugate equation.

Since the spectrum of the modified \(L\) possesses a double zero eigenvalue at \(s = s_*\), the coefficients on the center manifold are not immediately given by the Andronov-Hopf case used above. However, the reduced vector field on the 2D center manifold of the double zero eigenvalue reduces further to a scalar equation, undergoing a pitchfork bifurcation, precisely due to the translation symmetry. In polar coordinates of the Hopf normal form, this is due to detuning the trivial angular equation, co-rotation with velocity \(sC_1(1)\), to stationarity. Such reductions due to continuous symmetry also hold in more abstract contexts, see, e.g., Theorem 2.18 of [11], where an additional reflection symmetry is assumed.

In the context of travelling waves, let us briefly take the perspective of pattern formation, for which the infinite strip \(x \in [0, L_1] \times \mathbb{R}\) is the natural domain here. The linear stability analysis of the laminar steady state in this case involves the eigenvalues \(\lambda^s_k\) from (3) with \(k = (k_1, k_2)\) where \(k_2\) is continuous and rescaled: these are now eigenmodes in the essential spectrum given by \(\lambda^s_k\) with \(k = (k_1, L_2k_2)\), \((k_1, k_2) \in \mathbb{N}_+ \times \mathbb{R}\).

We are then lead to search for pattern-forming instabilities, and indeed, the system easily allows for the analogue of Turing-Hopf instabilities from reaction-diffusion systems, which is also well known in fluid dynamics, for instance Rayleigh-Bénard convection. The spectral configuration of a Turing-Hopf instabilities in general is such that the essential spectrum touches the imaginary axis at some \(\lambda = \lambda(k_1, k_2)\) with \(\Im(\lambda) \neq 0\), \((k_1, k_2) \neq 0\), and crosses it upon parameter variation. Due to Lemma 3.1 critical eigenmodes must have \(k_1 = 1\). More precisely, we have the following

**Lemma 5.1.** For any fixed \(L_1 > 0\) there exists a unique \(\nu_{TH} > 0\) such that the laminar steady state is stable for \(\nu > \nu_{TH}\) and all \(\Delta T\). For \(0 < \nu < \nu_{TH}\) it undergoes Turing-Hopf instabilities at precisely two values of \(\Delta T\), and these lie in \((0, \Delta_{TS})\). The critical eigenmodes have wavenumber \(k_1 = 1\) and some \(k_2 = k_2(\Delta T)\), and the imaginary part of the critical eigenmode (the frequency) is \(\Delta T \pi k_2(\Delta T)\).

For fixed \(L_1\) and any \(\Delta T \in (0, \Delta_{TS})\), a Turing-Hopf instability occurs at a unique \(\nu\), and there is no instability in \(\nu\) for \(\Delta T < 0\) or \(\Delta T \geq \Delta_{TS}\).

**Proof.** Recall that for bounded rectangular domains, the function \(d(\Delta T, k_2^2)\) from (3.12) has the sign of \(\Re(\lambda_1(k_1, k_2))\). This followed from (3.10), which in the case of a cylinder has \(k_2\) replaced by \(k_2 L_2\), where \(k_2\) is now the continuous wavenumber. Then \(\ell\) is replaced by \(1/L_1\) in (3.10) so that \(d(\Delta T, k_2^2)\) gives the sign of critical modes for the infinite strip when setting \(\ell = 1/L_1\). The scaling also implies that given an instability, the claimed critical frequency follows from (3.1) with \(k_2\) replaced by \(L_2 k_2\).

Lemma 3.2 shows that instabilities can occur for \(\Delta T \in [0, \Delta_{TS})\) only, and \(\Delta T = 0\) requires \(\nu = 0\). Due to Lemma 3.7 for any \(\nu \geq 0\), the graph of \(d: \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}\) is parabolic. In particular, there is a unique maximum of \(d(\Delta T, k_2^2)\), and it has \(k_2 > 0\). Hence, the maximal real part of the spectrum lies at a unique \((\Delta T, k_2)\), and for the lemma it now suffices (by continuity) to find values of \(\nu\) where the maximum of \(d(\Delta T, k_2^2)\) is positive and where it is negative.

For \(\nu = 0\) the maximum is positive since then (3.13) holds, and it is bounded as a function of \(k_2^2\) (with maximum at \(k_2 = 0\)). For \(\nu > 0\) the additional term in \(d(\Delta T, k_2^2)\)
is the negative contribution

\[-\nu^2 \pi^4 \left( \frac{4\kappa_2 + \ell^2}{\kappa_2 \ell^4} \right)^2\]

which is bounded from above as a function of \( \kappa_2 \) (with maximum at \( \kappa_2 = \ell^2/4 \)). Hence, \( \nu \) can be chosen so large, that \( d(\Delta T, \kappa_2) < 0 \) for all \( \Delta T, \kappa_2 \geq 0 \).

A numerically computed example is given in Figure 5.1 which is derived from that in Figure 3.2(a). Here the critical modes at onset of the instability on the infinite strip have wavenumber near 0.75. The periodic solutions of §4, alias wavetrains, are a signature of the bifurcating continuum of periodic solutions. The fact that these are supercritical implies that the Turing-Hopf instabilities are in fact supercritical: with \( L_2 = 2\pi/k_{TH} \) the mode with critical Turing-Hopf wavenumber is the lowest spatial harmonic and the supercritical Andronov-Hopf bifurcation implies positive real part of the Landau-coefficient in the Complex-Grinzbarg-Landau amplitude equations. For the general theory of modulation equations for pattern forming instabilities we refer to [4,23].

![Fig. 5.1. A Turing-Hopf instability for parameters as in Figure 3.2(a): real parts of eigenvalues \( \lambda^\pm_{k,2} \), \( k = (1,k_2) \), as functions of \( k_2 \) for \( \Delta T = 0.05 \) (stable), \( \Delta T = 0.08 \) (near bifurcation) and \( \Delta T = 0.2 \) (unstable).](image)

6. Nonlinear Instability. In this short paragraph, we give some details on the fact that the linear instability of the laminar state \( \rho_{ss} \) is indeed an instability for the nonlinear equation uniformly in \( \nu \). Roughly speaking, this means that in \((\Delta T_1, \Delta T_2)\), there are initial data which are arbitrarily close to the steady state and which get “far” from it exponentially quickly. We thus assume \( \Delta T_1 < \Delta T_2 \) and take \( \Delta T \in (\Delta T_1, \Delta T_2) \). This means that \( \sigma_\nu(L) \neq \emptyset \) (see the proof of Theorem 3.4).

In the parabolic formulation [23], the sectoriality of \( L \) allows to apply the well known nonlinear instability results from [12] for spectrum in the right half plane. However, this heavily relies on \( \nu > 0 \) and the following does not. Furthermore, the result given for the specific case here is actually stronger than the general ones in [12].

As in [9] Theorem 6.1, the following instability result holds for \( \nu \geq 0 \):

**Theorem 6.1.** Suppose \( \Delta T \in (\Delta T_1, \Delta T_2) \). There exist constants \( \delta_0, \eta_1, \eta_2 > 0 \) such that for any \( 0 < \delta < \delta_0 \) and any \( s \geq 0 \) there exists a solution \( (\rho^\pm, E) \) to (1.1) with \( \|\rho^\pm(0) - \rho^\pm_{ss}\|_{H^s} \leq \delta \) but such that:

\[\|\rho^\pm(t_\delta) - \rho^\pm_{ss}\|_{L^2} \geq \eta_1 \text{ and } \|E(t_\delta)\|_{L^2} \geq \eta_2,\]

with \( t_\delta = O(\|\log \delta\|) \).
In order to pass from linear to nonlinear instability, a natural candidate is the solution \((\rho^\pm, E)\) to (1.1) with initial datum
\[
\rho^\pm(0) = \rho^\pm_{ss} + \delta h^\pm,
\]
where \((h^+, h^-)\) is an eigenfunction associated to an eigenvalue \(\lambda_1\) with maximal real part for the linearized operator. We expect the solution \(\rho^\pm\) to behave like the solution of the linearized equations:
\[
\rho^\pm_{lin}(t) = \rho^\pm_{ss} + \delta e^{\lambda_1 t} h^\pm.
\]
Since the nonlinear part of (1.1) involves derivatives, controlling the error between \(\rho^\pm\) and the approximation \(\rho^\pm_{lin}\) is not straightforward; the main idea is to apply the method of Grenier [7], whose principle is to build a more precise approximation of \(\rho^\pm\), which allows to get a better control on the nonlinearities. Since the proof is identical to that of Theorem 6.1 in [9], we refer to this paper for details.

7. Global Nonlinear Stability. Let us now investigate the stability of the steady state \(\rho_{ss}\), outside of \([\Delta T_1, \Delta T_2]\). We first state the results and then give the proofs.

The key point is the following energy identity:

**Lemma 7.1.** For any initial data \(\rho_0 \in L^\infty\), we have the following estimate for the energy of the system (1.1)
\[
\mathcal{E}(t) := \|\rho - \rho_{ss}\|_2^2 - \frac{2}{L_1 \Delta T} \int_\Omega |\nabla V|^2 \, dx
+ 2 \nu \int_0^t \left[ \frac{2}{L_1 \Delta T} \|\rho^+ + \rho^- - 1\|_2^2 + \|\nabla (\rho - \rho_{ss})\|_2^2 \right] \, ds \quad (7.1)
\]
with \(\|\rho - \rho_{ss}\|_2^2 = \|\rho^+ - \rho^+_{ss}\|_2^2 + \|\rho^- - \rho^-_{ss}\|_2^2\), \(\|\nabla (\rho - \rho_{ss})\|_2^2 = \|\nabla (\rho^+ - \rho^+_{ss})\|_2^2 + \|\nabla (\rho^- - \rho^-_{ss})\|_2^2\), and \(\nabla V = \nabla (\nabla^2)^{-1} (\rho^+ + \rho^- - 1)\).

**Remark 10.** We take the opportunity to point out an error in the energy of [8] Theorem 5.1: in equations (5.1) and (5.2) of this paper, there is a factor 2 which is missing in front of \(\int_\Omega |\nabla V|^2 \, dx\).

We shall use in the following Poincaré type inequalities:

**Lemma 7.2.** With the same notations as before, we have, for any \(t \geq 0\):
\[
\|\nabla V\|_2^2 \leq \frac{2L_2^2}{\pi^2} \|\rho - \rho_{ss}\|_2^2, \quad (7.2)
\]
\[
\|\rho - \rho_{ss}\|_2^2 \leq \frac{L_2^2}{\pi^2} \|\nabla (\rho - \rho_{ss})\|_2^2. \quad (7.3)
\]
As a consequence of the energy identity, we can prove \(L^2\)-return to equilibrium, with exponential (and explicit) speed, for negative or large enough \(\Delta T\).

**Theorem 7.3.** If \(\Delta T < 0\) or \(\Delta T > \Delta T_* := \frac{4L_2^2}{\pi^2} \), then the steady state \(\rho_{ss}\) is globally asymptotically stable in \(L^2\), with exponential convergence rate \(-2\pi^2 \nu / \Delta T\).

**Remark 11.** Notably, the constants in all these results are independent of \(L_2\), and the convergence rate is larger on thinner domains (with smaller \(L_1\)), but also balancing with viscosity.
Recall that by Lemma 3.2, for fixed \( \ell = L_2/L_1 \) the upper instability threshold \( \Delta T_2 \) satisfies \( \Delta T_2 < \Delta T_* \), but that \( \lim_{\ell \to +\infty} \Delta T_2 = \Delta T_* \) if \( \nu = o(\ell^{-1}) \). Hence, the global threshold estimate \( \Delta T_* \) is also linked to linear instability.

Let us now prove Lemma 7.1, Lemma 7.2 and Theorem 7.3.

Proof. [Lemma 7.1] The proof follows from computations that are similar to those that can be found in [9], for the model without viscosity (that is \( \nu = 0 \)). We keep the notations of Section 2.

Taking the scalar product with \( u := (u_1, u_2) \) in the transport equations satisfied by \( u_1 \) and \( u_2 \) in (2.1), and integrating with respect to \( x \) entails:

\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 = \int_{\Omega} \frac{E_2}{L_1} u_1 dx - \int_{\Omega} \frac{E_2}{L_1} u_2 dx + \nu \int_{\Omega} u_1 \nabla^2 u_1 dx + \nu \int_{\Omega} u_2 \nabla^2 u_2 dx. \tag{7.4}
\]

Note indeed that due the periodicity with respect to \( x_2 \), the following contribution vanishes:

\[
\int_{\Omega} \partial_{x_2} u_1 u_1 dx = \int_{\Omega} \frac{1}{2} \partial_{x_2} u_1^2 dx = 0 = \int_{\Omega} \partial_{x_2} u_2 u_2 dx.
\]

Likewise, with Green’s Formula, using \( \text{div} E \perp = 0 \) and \( E_2 = -\partial_{x_2} V = 0 \) on \( x_1 = 0, L_1 \), we have (for \( i = 1, 2 \)):

\[
\int_{\Omega} E_\perp \cdot \nabla u_i u_i dx = \frac{1}{2} \int_{\Omega} E_\perp \cdot \nabla (u_i^2) dx = 0.
\]

Recall an identity proved in [9, Lemma 5.1]: for any \( t > 0 \), there holds

\[
\int_{\Omega} E_2 u_1 dx = - \int_{\Omega} E_2 u_2 dx. \tag{7.5}
\]

For the sake of completeness, we quickly reproduce the proof. Observe that

\[
\int_{\Omega} E_2 (u_2 - u_1) dx = \int_{\Omega} E_2 (u_1 + u_2 - 2u_1) dx = \int_{\Omega} E_2 (-\nabla^2 V - 2u_1) dx = -2 \int_{\Omega} E_2 u_1 dx. \tag{7.6}
\]

Indeed, relying on the periodicity in the \( x_2 \) direction and since \( \partial_{x_2} V = 0 \) on \( x_1 = 0, L \), we get:

\[
\int_{\Omega} \partial_{x_2} V \nabla^2 V dx = - \int_{\Omega} \partial_{x_2} \nabla V \cdot \nabla V dx + \int_{\Omega} \text{div} (\partial_{x_2} V \nabla V) dx = 0.
\]

This completes the proof of (7.5). Therefore, we have:

\[
\int_{\Omega} \frac{E_2}{L_1} u_1 dx + \int_{\Omega} \frac{E_2}{L_1} u_2 dx = -2 \int_{\Omega} \frac{E_2}{L_1} u_1 dx.
\]
Now compute, using the equations satisfied by \((u_1, u_2, V)\):
\[
\int_\Omega E_2u_1 \, dx = \int_\Omega V \partial_{x_2} u_1 \, dx - \left( \int_\Omega \text{div}(V u_1 e_2) \, dx \right)_{t=0} \quad = \frac{1}{\Delta T} \int_\Omega V \left( \partial_t u_1 + E^1 \cdot \nabla u_1 - \frac{E_2}{L_1} \right) \, dx - \frac{\nu}{\Delta T} \int_\Omega V \nabla^2 u_1 \, dx \\
= \frac{1}{\Delta T} \int_\Omega V \left( \partial_t (u_1 + u_2) + E^1 \cdot \nabla (u_1 + u_2) \right) \, dx \\
+ \frac{1}{\Delta T} \int_\Omega -\nabla V \partial_{x_2} u_2 \, dx - \frac{\nu}{\Delta T} \int_\Omega V \nabla^2 (u_1 + u_2) \, dx \\
= \frac{1}{\Delta T} \int_\Omega \left( \partial_t \nabla^2 V - E^1 \cdot \nabla (\nabla^2 V) \right) \, dx - \frac{\nu}{\Delta T} \int_\Omega V \nabla^2 (u_1 + u_2) \, dx.
\]
(7.7)

Observe that by Green’s formula:
\[
\int_\Omega -V \left( \partial_t \nabla^2 V - E^1 \cdot \nabla (\nabla^2 V) \right) \, dx = \frac{d}{dt} \left( \int_\Omega |\nabla V|^2 \, dx \right).
\]

Finally, using again (7.5), we have
\[
\int_\Omega \frac{E_2}{L_1} u_1 \, dx - \int_\Omega \frac{E_2}{L_1} u_2 \, dx = \frac{1}{L_1(T^+ - T^-)} \frac{d}{dt} \left( \int_\Omega |\nabla V|^2 \, dx \right) - \frac{2\nu}{L_1(T^+ - T^-)} \int_\Omega V \nabla^2 (u_1 + u_2) \, dx.
\]

Note that using Green’s formula and the Poisson equation satisfied by \(V\), we have the identities:
\[
\int_\Omega V \nabla^2 (u_1 + u_2) \, dx = \int_\Omega \nabla^2 V (u_1 + u_2) \, dx = -\|u_1 + u_2\|_{L^2}^2.
\]

and
\[
\nu \int_\Omega u_1 \nabla^2 u_1 \, dx + \nu \int_\Omega u_2 \nabla^2 u_2 \, dx = -\nu \int_\Omega |\nabla u_1|^2 \, dx - \nu \int_\Omega |\nabla u_2|^2 \, dx.
\]

Gathering all pieces together, we have proved that \(\frac{d}{dt} \mathcal{E}(t) = 0\). □

Let us now prove the Poincaré inequalities of Lemma 7.2.

**Proof** [Lemma 7.2] We prove (7.2) only since (7.3) can be treated similarly. Using the orthogonal basis \(\{3,3\}\), we write:
\[
\sum_{k_1, k_2 \in \mathbb{Z}} a_{k_1, k_2} g_k.
\]

Recall the Poisson equation satisfied by \(V\):
\[
-\nabla^2 V = u_1 + u_2.
\]

This yields:
\[
V = \sum_{k_1, k_2 \in \mathbb{Z}} \frac{1}{\pi^2 \left( \frac{k_1^2}{L_1^2} + \frac{4k_2^2}{L_2^2} \right)} a_{k_1, k_2} g_k,
\]
\[
\nabla V = \sum_{k_1, k_2 \in \mathbb{Z}} \frac{1}{26 \pi^2 \left( \frac{k_1^2}{L_1^2} + \frac{4k_2^2}{L_2^2} \right)} a_{k_1, k_2} \left( \frac{k_1}{2L_1} \phi_k + \frac{k_2}{L_2} g_k \right).
\]
Therefore,
\[ \int_{\Omega} |\nabla V|^2 \, dx \leq \frac{L_2^2}{\pi^4} \left( \|u_1\|^2_{L^2} + \|u_2\|^2_{L^2} \right), \]
which proves (7.2). □

Gathering all pieces together, we can now prove Theorem 7.3.

Proof. [Theorem 7.3] Using the energy identity (7.1) and applying the Poincaré inequality (7.2) we get:
\[ \|\rho - \rho_{ss}\|^2_{L^2} \leq \|\rho(0) - \rho_{ss}\|^2_{L^2} + \frac{2}{L_1 \Delta T} \int_{\Omega} |\nabla V|^2 \, dx \]
\[ - 2\nu \int_0^t \left[ - \frac{4}{L_1 \Delta T} \|\rho - \rho_{ss}\|^2_{L^2} + \|\nabla(\rho - \rho_{ss})\|^2_{L^2} \right] \, ds \]
\[ \leq \|\rho(0) - \rho_{ss}\|^2_{L^2} + \frac{1}{L_1 \Delta T} \frac{4 L_1^2}{\pi^2} \|\rho - \rho_{ss}\|^2_{L^2} \]
\[ - 2\nu \int_0^t \left[ - \frac{4}{L_1 \Delta T} \|\rho - \rho_{ss}\|^2_{L^2} + \|\nabla(\rho - \rho_{ss})\|^2_{L^2} \right] \, ds. \]

Hence, using the Poincaré inequality (7.3),
\[ \left( 1 - \frac{4 L_1}{\pi^2 \Delta T} \right) \|\rho - \rho_{ss}\|^2_{L^2} \]
\[ \leq \|\rho(0) - \rho_{ss}\|^2_{L^2} + 2\nu \left( \frac{4}{L_1 \Delta T} - \frac{\pi^2}{L_1^2} \right) \int_0^t \|\rho - \rho_{ss}\|^2_{L^2} \, ds. \]
As a consequence, by Gronwall inequality, we obtain L^2-stability and L^2-return to equilibrium, provided that
\[ \Delta T < 0, \text{ or } \Delta T > \frac{4 L_1}{\pi^2}, \]
which in particular implies \( \frac{\pi^2}{L_1^2} - \frac{4}{L_1 \Delta T} > 0. \) More specifically, we have:
\[ \|\rho - \rho_{ss}\|^2_{L^2} \leq \|\rho(0) - \rho_{ss}\|^2_{L^2} \exp \left( -\gamma t \right), \quad (7.9) \]
with \( \gamma := 2\nu \left( \frac{\pi^2}{L_1^2} - \frac{4}{L_1 \Delta T} \right) \left( 1 - \frac{4 L_1}{\pi^2 \Delta T} \right)^{-1} = 2\nu \frac{\pi^2}{L_1^2} > 0. \) \( \square \)

8. Numerical results. For illustration of the analytical results given in the previous sections we next discuss some numerical computations. In the first subsection we present numerical time integration using harmonic discretization and slow parameter ramping in order to approximate and track attractors. In the second subsection we show results of a complementary approach by numerical continuation, which allows to track stable and unstable branches more accurately, and also detect bifurcation points. In doing this, we mainly focus on the parameter set
\[ \nu = 9 \cdot 10^{-4}, \ L_1 = L_2 = 2, \quad (8.1) \]
which has \( L_2 < 2\sqrt{2} L_1 \) so that a primary 1-instability region \( (\Delta T_1, \Delta T_2) \) exists (Theorem 3.4), and the endpoints of this are supercritical Hopf-bifurcations to travelling waves (Corollary 4.2). For parameters (8.1) we readily compute that \( \Delta T_1 \approx 3 \cdot 10^{-4}, \Delta T_2 \approx 0.162. \)
8.1. Time integration and parameter ramping. Here we discretized (2.1) with a finite-dimensional spectral decomposition

\[ u_l(x,t) = \sum_{k_1=1}^{N_{x_1}} \sum_{k_2=-N_{x_2}}^{N_{x_2}} C_{k_1,k_2,l}(t) g_{k_1,k_2}(x), \ l = 1, 2, \]  

(8.2)

with harmonics \( g_k \) as defined in (3.3). We integrated the resulting system of ODEs for \( C_{k_1,k_2,l} \) using a semi-implicit Crank-Nicolson scheme, where the linear part only is implicit\(^1\). All the simulations were made with parameter values (8.1) and \( N_{x_1} = N_{x_2} = 32 \), and we selectively checked with \( N_{x_1} = N_{x_2} = 64 \).

Fig. 8.1. Contour plots of \( u_1(t_2) = \rho_1(t_2) - \rho_{ss} \) with \( t_2 \approx 1000 \), \( \Delta T \in (\Delta T_1, \Delta T_2) \) and parameters as in (8.1). (a) the dynamics is a translation in the periodic \( x_2 \)-direction, \( \Delta T = 0.159291 \) is close to the upper instability threshold, and (b) \( \Delta T = 0.146122 \), the dynamics resembles a modulated travelling wave, moving in the \( x_2 \) direction.

In Figure 8.1 we plot one solution near the upper stability threshold \( \Delta T_2 \approx 0.162 \) and another further inside the nonlinear regime as can be seen by the locus of parameters in Figure 8.2(a). The ‘weakly nonlinear’ solution for \( \Delta T \approx \Delta T_2 \) closely resembles the unstable eigenfunction and appears to be a periodic travelling wave as predicted by the theory. The solution further inside the nonlinear regime has a clear nonlinear structure and does not appear to be close to a travelling wave: it has additional oscillations superimposed to the drift. Indeed, the numerical results of §8.2 indicate that the travelling waves in this parameter region are unstable.

Here we located and tracked the solutions emerging from the supercritical Andronov-Hopf bifurcation at \( \Delta T = \Delta T_1 \), by a simple parameter ramping: for \( \Delta T \) near the bifurcation at \( \Delta T_1 \), we simulate an initial condition close to \( \rho_{ss} \) and after a transient, \( t = t_1 \approx 400 \), we compute the maximum over the interval \([t_1, 1000]\). Next, we slightly increase \( \Delta T \) and repeat this step with the initial condition being the solution at the final time of the previous step. In this way we obtain the ‘ramping diagram’ in Figure 8.2(a), for which we used 100 ramping steps with smaller stepsize near the endpoints. As predicted by Theorem 4.1, the slope of the ramping curve is larger near the left endpoint of the instability region than near the right endpoint. The numerical instability thresholds of the laminar state are in good agreement with the analytical ones with better accuracy near the right endpoint. Further away from these

\(^1\)We modified a code by Jean-Christophe Nave - MIT Department of Mathematics, jcnave@mit.edu for Navier-Stokes equations in vorticity formulation.
thresholds, solutions are no longer travelling waves as noted above and corroborated by the time trace of the domain mid-point plotted in Figure 8.2(b). However, the solution still turns into the near harmonic periodic solution bifurcating from the right endpoint \( \Delta T_2 \). Compare Figure 8.1(a).

\[ \Delta T \]

\[ \|
\]

\[ \nu \]

\[ \approx \]

\[ 1000 \]

\[ (x_1,x_2) = (L_1/2,L_2/2) \]

\[ \nu \approx 0.01 \]

\[ \nu < \nu_{\text{crit}} \approx 0.01 \]

\[ \nu \approx 9 \cdot 10^{-4} \]

\[ \nu \approx 9 \cdot 10^{-3} \]

\[ \nu \approx 9 \cdot 10^{-5} \]

\[ \Delta T = 0.08177 \]

\[ t \]

\[ u_1(l_1/2,l_2/2,t) \]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]

\[ \|
\]
Fig. 8.3. (a) Branch of travelling waves for $L_1 = L_2 = 2$ for $\nu = 9 \cdot 10^{-3}$; all solutions are stable. (b) Contour plot of the $u_1$-component of the travelling wave from (a) at $\Delta T = 0.07$. (c) Branch as in (a) but with $\nu = 9 \cdot 10^{-4}$; thick lines mark stable solutions, thin lines unstable ones and circles mark bifurcation points.

Fig. 8.4. Left panel: Bifurcation diagram as in Figure 8.3(c) with additional branches (blue) emerging from the bifurcation points of the primary branch, notably extending to $\Delta T > \Delta T_2$. Right upper panel: Magnification of left panel near $\Delta T = 0$. Right lower panel: $u_1$-component of the small amplitude solution on the blue branch near $\Delta T = 0.1$ showing the wavevector $k = (2, 1)$.

Here. The rightmost of these extends beyond $\Delta T_2$, turns around and connects to a fully unstable branch of solutions emerging from the laminar state with wavevector $k = (2, 1)$, which also connects to the leftmost of the additional branches. We also mention that along the unstable branches there appear to be further destabilizations by Hopf-bifurcations and thus much richer bifurcation diagrams than our present numerical computations can reveal.

In conclusion, the numerical results nicely corroborate our rigorous local analysis concerning stability, location and supercritical nature of bifurcations from the laminar state. Moreover, these results suggest that many other bounded solutions exist for small $\nu$, in particular also for values of $\Delta T \in (\Delta T_2, \Delta T_*)$ between the upper linear and the estimated global stability thresholds. The latter would explain the necessity of a difference between the local and global stability thresholds of the laminar state.

REFERENCES

[1] A.R. Adams, J.J.F. Fournier, *Sobolev spaces*, second ed., Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003.
[2] G.K. Batchelor, *An introduction to fluid dynamics*, Cambridge, 2000.
[3] P. Braasch, G. Rein, J. Vukadinovic, *Nonlinear stability of stationary plasmas—an extension of the energy-Casimir method*, SIAM J. Appl. Math. 59 (1999) no. 3, 831–844.
[4] M.C. Cross, P.C. Hohenberg, *Pattern formation outside equilibrium*, Rev. Mod. Phys. 65 (1993) 851–1112.
[5] T. Dohnal, J.D.M. Rademacher, H. Uecker and D. Wetzel. pde2path-V2: multi-parameter continuation and periodic domains. Preprint 2014.
[6] H. Goedbloed, S. Poedts, Principles of Magnetohydrodynamics: With Applications to Laboratory and Astrophysical Plasmas. Cambridge University Press, Cambridge, 2004.
[7] E. Grenier, On the nonlinear instability of Euler and Prandtl equations, Comm. Pure Appl. Math. 53 (2000), no. 9, 1067–1091.
[8] P. N. Guzdar, R. G. Kleva, and Liu Chen. Shear flow generation by drift waves revisited. Phys. Plasma 8 (2001)
[9] D. Han-Kwan, On the confinement of a tokamak plasma, SIAM J. Math. Anal. 42 (2010), no. 6, 2337–2367.
[10] D. Han-Kwan, Quasineutral Limit of the Vlasov-Poisson System with Massless Electrons, Communications in Partial Differential Equations, Volume 36, Issue 8, 2011, 1385-1425.
[11] M. Haragus and G. Iooss, Local bifurcations, center manifolds, and normal forms in infinite-dimensional dynamical systems, Universitext, Springer-Verlag London Ltd., London, 2011.
[12] D. Henry, Geometric theory of semilinear parabolic equations, Lecture Notes in Mathematics, vol. 840, Springer-Verlag, Berlin, 1981.
[13] P. Hirschberg and E. Knobloch, Mode interaction in large aspect ratio convection, Nonlinear Science no. 7, 1997, 537–556.
[14] Y. Kuznetsov, Elements of applied bifurcation theory, second ed., Springer-Verlag 1998.
[15] M.A. Malkov and P.H. Diamond, Weak hysteresis in a simplified model of the L-H transition, Phys. Plasmas 16, 012504, 2009.
[16] A. Mouton, Expansion of a singularly perturbed equation with a two-scale converging convection term. arXiv preprint arXiv:1404.4262 (2014).
[17] T. Nguyen, W. Strauss, Linear stability analysis of a hot plasma in a solid torus, Archive for Rational Mechanics and Analysis, 211(2), 2014, 629-671.
[18] Y. Sarazin, V. Grandgirard, E. Fleurence, X. Garbet, Ph. Ghendrih, P. Bertrand and G. Depret, Kinetic features of interchange turbulence, Plasma Phys. Control. Fusion 47 1817, 2005.
[19] Y. Sarazin, V. Grandgirard, G. DiF-Pradalier, E. Fleurence, X. Garbet, Ph. Ghendrih, P. Bertrand, N. Besse, N. Crouseilles, E. Sonnendrücker, G. Latu and E. Violard, Impact of large scale flows on turbulent transport, Plasma Phys. Control. Fusion 48 B1, 2006.
[20] A. Hasegawa and M. Miura, Plasma edge turbulence, Phys. Rev. Lett., 50 (1983) 682–686.
[21] B. Saltzman, Finite amplitude free convection as an initial value problem-I, Journal of atmospheric sciences 19, (1962) 329–341.
[22] F. Wagner, A quarter-century of H-mode studies, Plasma Phys. Control. Fusion 49 B1, 2007.
[23] I.S. Aranson, L. Kramer, The world of the complex Ginzburg-Landau equation. Rev. Mod. Phys. 74 (2002) 99–143.
[24] H. Uecker, D. Wetzel, and J.D.M. Rademacher. pde2path – a Matlab package for continuation and bifurcation in 2D elliptic systems. Num. Math.: Th. Meth. Appl. 7 (2014) 58-106. Software at www.staff.uni-oldenburg.de/hannes.uecker/pde2path/
[25] I. U. Uzun-Kaymak, P. N. Guzdar, R. Clary, R. F. Ellis, A. B. Hassam, and C. Teodorescu. Analysis and modeling of edge fluctuations and transport mechanism in the Maryland Centrifugal Experiment. Physics of Plasmas 15 (2008) 112308