GEODESICS, DISTANCE, AND THE CAT(0) PROPERTY FOR THE MANIFOLD OF RIEMANNIAN METRICS

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Abstract. Given a fixed closed manifold $M$, we exhibit an explicit formula for the distance function of the canonical $L^2$ Riemannian metric on the manifold of all smooth Riemannian metrics on $M$. Additionally, we examine the (metric) completion of the manifold of metrics with respect to the $L^2$ metric and show that there exists a unique minimal path between any two points. This path is also given explicitly. As an application of these formulas, we show that the metric completion of the manifold of metrics is a CAT(0) space.

1. Introduction

In this paper, we give explicit formulas for the distance between Riemannian metrics, as measured by the canonical $L^2$ Riemannian metric on the manifold of all metrics $\mathcal{M}$ over a given closed manifold $M$. We also show that geodesics are unique and, at least on the metric completion of the manifold of metrics, there exists a geodesic—also explicitly given—connecting any two given points. We then apply these formulas to give the main result of this paper (Theorem 5.6): that the completion of the manifold of metrics is nonpositively curved in a metric sense.

Theorem. The metric completion of $\mathcal{M}$ with respect to its $L^2$ Riemannian metric, $(\overline{\mathcal{M}}, d)$, is a CAT(0) space.

Fix any closed manifold $M$ of dimension $n$, and consider the space $\mathcal{M}$ of all $C^\infty$-smooth Riemannian metrics on $M$. This space carries a canonical weak Riemannian metric known as the $L^2$ metric (defined in Sect. 2.2). The $L^2$ metric has many interesting local and infinitesimal properties. This local geometry is well understood due to the work of Freed–Groisser [FG89] and Gil-Medrano–Michor [GMM91], who have shown that its sectional curvature is nonpositive, and that the geodesic equation on $\mathcal{M}$ is explicitly solvable. The $L^2$ metric has also found numerous applications, for example in the study of moduli spaces, where it naturally gives rise to the well-known Weil-Petersson metric. Indeed, the results of this paper stand in clear analogy with similar results on the Weil-Petersson metric, which is geodesically convex [Wol87] and CAT(0) [Yam04].

In our own work [Cla10, Cla, Cla11], we have focused on the global geometry of the $L^2$ metric on $\mathcal{M}$ and submanifolds, studying the distance it induces between Riemannian metrics on $M$. This approach, in joint work with Rubinstein [CRT11], has led to the formulation of criteria for the existence of Kähler–Einstein metrics and for the convergence of the Kähler–Ricci flow on Fano manifolds. In this paper, we deepen the understanding of this global
approach with the above-mentioned explicit formulas for the $L^2$ distance and geodesics, as well as by establishing the CAT(0) property for the metric completion. (Whenever we refer to a completion in this paper, we mean the metric completion.) Roughly speaking, a metric space is CAT(0) if (i) geodesics (i.e., distance-minimizing paths) exist between any two points, and (ii) every geodesic triangle is “thinner” than a triangle in the Euclidean plane with the same side lengths, in the sense that the edges of the triangle are closer together than in Euclidean space. (Cf. [BH99]; precise definitions are also given in [5.1]) A metric space is called nonpositively curved if it is locally CAT(0).

At this point, it is important to note that even though, as mentioned above, the sectional curvature of the $L^2$ metric is nonpositive, this infinitesimal result does not show that the manifold of metrics is a nonpositively curved metric space, as it would in the finite-dimensional setting. Indeed, as a weak Riemannian metric on an infinite-dimensional manifold, many theorems from finite-dimensional Riemannian geometry fail to hold. For instance, given any point $g_0 \in \mathcal{M}$, there exist other points at arbitrarily close distances to $g_0$ that are not in the image of the exponential mapping at $g_0$. (This last point is directly implied by Theorem 4.16 in this paper, though it is also easy to see from the work of Gil-Medrano–Michor [GMM91, Rmk. 3.5].) In particular, geodesics do not necessarily exist between points in $\mathcal{M}$, even locally (i.e., in a small metric ball), and so the first criterion for $\mathcal{M}$ to be a nonpositively curved metric space fails. Thus, it is important to consider the metric completion of $\mathcal{M}$, where geodesics between any two points do indeed exist—note that for any point of $\mathcal{M}$, there exist arbitrarily close metrics for which the geodesic connecting the two runs through degenerate metrics, i.e., points not in $\mathcal{M}$. (Nevertheless, as a corollary of the CAT(0) property for the completion, we do have that geodesic triangles in $\mathcal{M}$ satisfy the CAT(0) inequality, cf. Theorem 5.6.)

The CAT(0) property has important implications, for instance, on the existence of generalized harmonic maps [Jos94, Jos95, Jos96, Jos97a, Jos97b, KS93, KS97] and on actions of groups of isometries [GKM08]. We plan to explore these in future work.

We wish to also briefly mention the relevance of the global approach to the geometry of the $L^2$ metric to questions related to the convergence of Riemannian manifolds. In fact, Anderson [And92] has used the $L^2$ metric to study spaces of Einstein metrics. (He refers to the distance function of the $L^2$ metric on $\mathcal{M}$ as the extrinsic $L^2$ metric, because he considers the distance obtained by infima of lengths of paths that are allowed to travel through $\mathcal{M}$, as opposed to restricting paths to the submanifold of Einstein metrics.) One appeal of the $L^2$ metric in this context is that it provides a very weak notion of convergence. Another appeal is that we have previously shown that convergence in the $L^2$ metric implies a strong convergence of the induced measures [Cla11 Cor. 4.11]—which hints that it could be suited for studying convergence of metric measure spaces. Unfortunately, when considered on the full space $\mathcal{M}$, convergence in the $L^2$ metric is perhaps too weak—it does not imply any more synthetic-geometric notion of convergence, such as Gromov–Hausdorff convergence. (For proofs and a more detailed discussion of these facts, we refer to [Cla11 Sect. 4.3].) However, as Anderson’s work showed, restricted to spaces of Einstein metrics, convergence in the $L^2$ metric in fact does imply Gromov–Hausdorff convergence (or stronger). An open question is what other subspaces of $\mathcal{M}$ might have this desirable property.

The paper is organized as follows. In Section 2, we set up the necessary preliminaries, both on $L^2$ metrics on spaces of sections in general, as well as on the $L^2$ metric on $\mathcal{M}$ in particular. In Section 3, we find a simplified description for the $L^2$ distance between metrics,
which transforms the problem from finding the infimum of lengths of paths in the infinite-dimensional space $M$ into a tractable finite-dimensional problem (Theorem 3.9). In Section 4, we show that there exists a unique geodesic connecting any two given metrics in the completion of $M$. We also write down an explicit formula for this geodesic, which in turn allows us to make the formula for the $L^2$ distance between metrics explicitly computable (Theorem 4.16). Again, Theorem 3.9 turns this infinite-dimensional problem into a finite-dimensional one. In Section 5, we use the formulas for geodesics and distance obtained in the previous sections to show the CAT(0) property for the metric completion of $M$, again by extrapolation from a finite-dimensional problem. Finally, in Section 6 we outline some open problems regarding the $L^2$ metric that we find to be of interest.

Acknowledgements. I would like to thank Jacob Bernstein, Guy Buss, and Michael Kapovich for helpful discussions during the preparation of this manuscript, as well as Yanir A. Rubinstein for comments on an earlier version.

2. Preliminaries

2.1. $L^2$ metrics. In this subsection dealing with general $L^2$ metrics on spaces of sections, we follow the definitions and results of Freed–Gromov [FG89, Appendix].

Let $M$ be a smooth, closed manifold. Let $\pi : E \to M$ be a smooth fiber bundle, and denote the fiber over $x \in M$ by $E_x$. Suppose we are given

1. a smooth volume form $\mu$ on $M$ with $\text{Vol}(M, \mu) = 1$, and
2. a smooth Riemannian metric $g$ defined on vectors in the vertical tangent bundle $T^vE$.

Let $\mathcal{E}$ denote a space of sections of $E$, where we allow the possibilities

1. $\mathcal{E} = \Gamma^s(E)$, the space of Sobolev sections $M \to E$ with $L^2$-integrable weak derivatives up to order $s$. Here we require $s > n/2$ if $E \to M$ is not a vector bundle.
2. $\mathcal{E} = \Gamma(E)$, the space of smooth sections $M \to E$.

By standard results on mapping spaces, $\mathcal{E}$ is a manifold in either of these cases [Pal68, Ham82, Example 4.1.2], (In case (1), it is a separable Hilbert manifold, and in case (2), it is a Fréchet manifold.) With this data, we can define an $L^2$-type Riemannian metric on $\mathcal{E}$ as follows. The tangent space at $\sigma \in \mathcal{E}$ is identified with the space of vertical vector fields “along $\sigma$”, that is, with the space of sections of the pulled-back bundle $\sigma^*T^vE$. Now, for $X, Y \in T_\sigma \mathcal{E}$, define the $L^2$ metric by

$$(X, Y)_\sigma := \int_M g(\sigma(x))(X(x), Y(x)) \, d\mu(x).$$

We denote by $d$ the distance function induced by $(\cdot, \cdot)$ on $\mathcal{E}$, and by $d_x$ the distance function induced by $g$ on $E_x$. Then $d$ is a pseudometric and $d_x$ is a metric (in the sense of metric spaces). Note that $(\cdot, \cdot)$ is in general a weak Riemannian metric on $\mathcal{E}$, that is, for any $\sigma$, the topology induced by $(\cdot, \cdot)_\sigma$ on $T_\sigma \mathcal{E}$ is weaker than the manifold topology. In this case, it is in principle possible that $d$ is not a metric in that it could fail to separate points. There are known examples of this due to the work of Michor–Mumford [MM06, MM05], where weak Riemannian metrics are constructed for which the induced distance between any two points is always zero. However, Theorem 2.1 below will show that $L^2$ metrics as we have defined them do not suffer from this pathology.
Thinking of $E$ as a bundle of metric spaces $\cup_{x \in M}(E_x, d_x)$ over $M$, we can define an $L^p$ metric $\Omega_p$ on $E$ by

\begin{equation}
\Omega_p(\sigma, \tau) := \left( \int_M d_x(\sigma(x), \tau(x))^p \, d\mu(x) \right)^{1/p}.
\end{equation}

Note that $\Omega_p$ is indeed a metric (in the sense of metric spaces) on $E$. All the required properties are easily implied from those of $d_x$. For example, $\Omega_p$ is positive definite because, as a Riemannian metric on a finite-dimensional manifold, $d_x$ is positive definite for each $x$, and two unequal elements $\sigma, \tau \in E$ necessarily differ over a set of positive $\mu$-measure. Only the triangle inequality is not immediately obvious—but this inequality follows, as in the case of an $L^p$ norm, from the triangle inequality for $d_x$ and Hölder’s inequality.

The following theorem gives a positive lower bound for the distance, with respect to $d$, between distinct elements of $E$. In the proof, and throughout the rest of the paper, a prime will denote the partial derivative in the variable $t$.

**Theorem 2.1.** The following inequality holds for any path $\sigma_t$, $t \in [0, 1]$, in $E$:

\begin{equation}
L_{(\cdot, \cdot)}(\sigma_t)^2 \geq \int_M L_g(\sigma_t(x))^2 \, d\mu,
\end{equation}

where on the left-hand side, we measure the length in $E$ with respect to $(\cdot, \cdot)$, while on the right-hand side, we measure the length in $E_x$ with respect to $g$. In particular, for any $\sigma, \tau \in E$, we have $d(\sigma, \tau) \geq \Omega_2(\sigma, \tau)$, and so $d$ is a metric on $E$.

**Proof.** Without loss of generality, suppose that $\sigma_t$ is parametrized proportionally to $(\cdot, \cdot)$-arc length. In this case, we have $L_{(\cdot, \cdot)}(\sigma_t)^2 = E_{(\cdot, \cdot)}(\sigma_t)$, where $E_{(\cdot, \cdot)}$ denotes the energy of the path with respect to $(\cdot, \cdot)$. On the other hand, we have

\[
E_{(\cdot, \cdot)}(\sigma_t) = \int_0^1 \int_M g(\sigma_t(x)(\sigma_t(x), \sigma_t'(x)) \, d\mu \, dt = \int_M \int_0^1 g(\sigma_t(x)(\sigma_t'(x), \sigma_t'(x)) \, dt \, d\mu
\]

\[
= \int_M E_g(\sigma_t(x)) \, d\mu \geq \int_M L_g(\sigma_t(x))^2 \, d\mu.
\]

Here, we have used Fubini’s theorem followed by a well-known application of Hölder’s inequality which gives $E_g(\sigma_t(x)) \geq L_g(\sigma_t(x))^2$ for any $x \in M$. As above, $E_g$ denotes the energy of the path with respect to $g$. This proves (2.2), from which $d(\sigma, \tau) \geq \Omega_2(\sigma, \tau)$ follows directly.

Now that we have set up the situation for a general $L^2$ metric, we turn to the main focus of this paper, when $E$ is the space of smooth Riemannian metrics.

**2.2. Preliminaries on the manifold of metrics.** For any point $x$ in our closed base manifold $M$, let $S_x := S^2 T_x^* M$ denote the vector space of symmetric $(0, 2)$-tensors based at $x$, and let $S := \Gamma(S^2 T^* M)$ denote the space of smooth, symmetric $(0, 2)$-tensor fields. Similarly, denote by $M_x := S^2 T_x^* M$ the vector space of positive-definite, symmetric $(0, 2)$-tensors at $x$, and by $M := \Gamma(S^2 T^* M)$ the space of smooth sections of this bundle. Thus, $M$ is the space of smooth Riemannian metrics on $M$. In the notation of the previous section, we have $E = S^2 T^* M$, $E_x = M_x$, and $E = M$. Thus we see that $M$ is a Fréchet manifold, and since $M$ is an open subset of $S$, we have a canonical identification of the tangent space $T_g M$ with $S$ for any $g \in M$. (Similarly, the tangent space to $M_x$ at any $a \in M_x$ is identified with $M_x$; thus we have $T^*_a(S^2 T^* M) \cong S_x$.)
Any element \( \tilde{g} \in \mathcal{M} \) gives rise to a natural scalar product on \( T_{\tilde{g}}\mathcal{M} \cong \mathcal{S} \) as follows. For \( h, k \in \mathcal{S} \), the canonical scalar product that \( \tilde{g} \) induces on \((0,2)\)-tensors is

\[
\text{tr}_{\tilde{g}}(hk) = \text{tr}(\tilde{g}^{-1}h\tilde{g}^{-1}k) = \tilde{g}^{ij}h_{a}g_{lm}k_{jm},
\]

where by expressions like \( \tilde{g}^{-1}h \) we of course mean the \((1,1)\)-tensor obtained by raising an index of \( h \) using \( \tilde{g} \). Then \( \text{tr}_{\tilde{g}}(hk) \) is a function on \( \mathcal{M} \), and by integrating it with respect to the volume form \( \mu_{\tilde{g}} \) of \( \tilde{g} \), we get a scalar product

\[
(h, k)_{\tilde{g}} := \int_{\mathcal{M}} \text{tr}_{\tilde{g}}(hk) \, d\mu_{\tilde{g}}.
\]

This \( L^2 \) scalar product fits into the framework of the last subsection as follows. For the rest of the paper, we fix some arbitrary reference metric \( g \in \mathcal{M} \) that has total volume \( \text{Vol}(\mathcal{M}, g) = 1 \). Given a tensor field \( h \in \mathcal{S} \) or a tensor \( b \in \mathcal{S}_{x} \), denote by the capital letter the \((1,1)\)-tensor obtained by raising an index using \( g \), i.e., \( H = g^{-1}h \) and \( B = g(x)^{-1}b \). For each \( x \in \mathcal{M} \) and \( a \in \mathcal{M}_{x} \), define a scalar product on \( T_{a}\mathcal{M}_{x} \) (vertical vectors) by

\[
\langle b, c \rangle_{a} := \text{tr}_{a}(bc)\sqrt{\text{det} A},
\]

where \( b, c \in T_{a}\mathcal{M}_{x} \). Thus, \( \langle \cdot, \cdot \rangle \) gives a Riemannian metric on \( \mathcal{M}_{x} \). For the remainder of the paper, we denote by \( \mu := \mu_{g} \) the volume form of \( g \). Then the scalar product \((2.3)\) is given by the \( L^2 \) metric (in the sense of the last section)

\[
(h, k)_{\tilde{g}} = \int_{\mathcal{M}} \langle h(x), k(x) \rangle_{\tilde{g}(x)} \, d\mu.
\]

As in the last subsection, we denote by \( d \) and \( d_{x} \) the distance functions of \( \langle \cdot, \cdot \rangle_{\tilde{g}} \) and \( \langle \cdot, \cdot \rangle_{a} \), respectively. By Theorem 2.1 it is immediate that \( d \) is a metric on \( \mathcal{M} \), a fact that we already proved in a less elegant way in \([\text{Cla}10]\) Thm. 18.

For \( \tilde{g} \in \mathcal{M} \) and \( a \in \mathcal{M}_{x} \), we will denote the norms associated with \( \langle \cdot, \cdot \rangle_{\tilde{g}} \) and \( \langle \cdot, \cdot \rangle_{a} \) by \( \| \cdot \|_{\tilde{g}} \) and \( \| \cdot \|_{a} \), respectively, throughout the remainder of the paper.

In \([\text{Cla}]\), we determined the completion of \((\mathcal{M}, d)\), which we will denote in the following by \( \overline{\mathcal{M}} \). We will summarize the relevant details of this here.

Let \( \tilde{g} : \mathcal{M} \to S^{2}T^{*}M \) be any measurable section that induces a positive semidefinite scalar product on each tangent space of \( \mathcal{M} \). We call such a section a measurable semimetric. A measurable semimetric induces a measurable volume form (and hence a measure) on \( \mathcal{M} \) using the usual formula \( \mu_{\tilde{g}} := \sqrt{\text{det} \tilde{g}} \, dx^{1} \wedge \cdots \wedge dx^{n} \) in local coordinates. We denote by \( \mathcal{M}_{f} \) the set of all measurable semimetrics on \( \mathcal{M} \) that have finite volume, i.e., with \( \int_{\mathcal{M}} d\mu_{\tilde{g}} < \infty \). We also introduce an equivalence relation on \( \mathcal{M}_{f} \) by saying \( g_{0} \sim g_{1} \) if and only if the following statement holds almost surely (with respect to the measure \( \mu \)) on \( \mathcal{M} \): \( g_{0}(x) \) fails to be positive definite if and only if \( g_{1}(x) \) fails to be positive definite. We then have the following theorem.

**Theorem 2.2** ([Cla] Thm. 5.17). There is a natural bijection between \( \overline{\mathcal{M}} \) and \( \overline{\mathcal{M}}_{f} := \mathcal{M}_{f}/\sim \).

In the following, we will make use of some consequences of this theorem that we have worked out in previous papers. Though it will not play a direct role here, for completeness we describe the bijection mentioned in Theorem 2.2. This requires a definition.
Definition 2.3. Let \( \{g_k\} \) be a sequence in \( \mathcal{M} \), and let \([g_0] \in \widehat{\mathcal{M}}_f\). Define
\[
X_{g_0} := \{ x \in M \mid \mu_{g_0}(x) = 0 \} \quad \text{and} \quad D_{\{g_k\}} := \{ x \in M \mid \lim_{k \to \infty} \mu_{g_k} = 0 \}.
\]
We say that \( \{g_k\} \) \( \omega \)-converges to \([g_0]\) if for every representative \(g_0 \in [g_0]\), the following holds:
1. \( \{g_k\} \) is \(d\)-Cauchy,
2. \( X_{g_0} \) and \( D_{\{g_k\}} \) differ at most by a \(\mu\)-nullset,
3. \( g_k(x) \to g_0(x) \) for \(\mu\)-a.e. \( x \in M \setminus D_{\{g_k\}} \), and
4. \( \sum_{k=1}^{\infty} d(g_k, g_{k+1}) < \infty \).

We call \([g_0]\) the \(\omega\)-limit of the sequence \(\{g_k\}\).

The bijection of Theorem 2.2 is given by showing that: (i) For any Cauchy sequence \(\{g_k\} \subset \mathcal{M}\), there exists an \(\omega\)-convergent subsequence; (ii) Two \(\omega\)-convergent subsequences \(\{g_k^0\} \) and \(\{g_k^1\} \) have the same \(\omega\)-limit if and only if they represent the same point in \(\overline{\mathcal{M}}\), i.e., if and only if \( \lim_{k \to \infty} d(g_k^0, g_k^1) = 0 \); and (iii) For each element \([g_0] \in \overline{\mathcal{M}}_f\), there exists a sequence in \(\mathcal{M}\) \(\omega\)-converging to \([g_0]\).

At this point, we would like to point out that we will retain the notation \(d\) for the metric induced on the completion \(\overline{\mathcal{M}}\) from \((\mathcal{M}, d)\). It will also be convenient to use the bijection of Theorem 2.2 to see \(d\) as a metric on \(\mathcal{M}_f\), and as a pseudometric on \(\mathcal{M}_f\). Of course, for \(g_0, g_1 \in \mathcal{M}_f\), we have \(g_0 \sim g_1\) if and only if \(d(g_0, g_1) = 0\).

In what follows, we will also be concerned with special subsets of \(\mathcal{M}\) that have convenient properties. They are essentially subsets that are, in a pointwise sense, uniformly bounded away from infinity and the boundary of \(\mathcal{M}\).

Definition 2.4. For \(\tilde{g} \in \mathcal{M}\) and \(x \in M\), let \(\lambda_{\min}^\tilde{g}(x)\) denote the minimal eigenvalue of \(\tilde{G}(x) = g(x)^{-1}\tilde{g}(x)\). A subset \(\mathcal{U} \subset \mathcal{M}\) is called amenable if it is of the form
\[
\mathcal{U} = \{ \tilde{g} \in \mathcal{M} \mid \lambda_{\min}^\tilde{g}(x) \geq \zeta \text{ and } |\tilde{g}(x)|_{g(x)} \leq C \text{ for all } \tilde{g} \in \mathcal{U} \text{ and } x \in M \}
\]
for some constants \(C, \zeta > 0\).

We denote the closure of \(\mathcal{U}\) in the \(L^2\) norm \(\| \cdot \|_g\) by \(\mathcal{U}^0\); it consists of all measurable, symmetric \((0, 2)\)-tensors \(\tilde{g}\) satisfying the bounds of (2.4) \(\mu\)-a.e.

Remark 2.5.

1. Note that the preceding definition differs from that in our previous works (cf. [Cla, Def. 3.1], [Cla11, Def. 2.11]). The above definition is coordinate-independent and therefore more satisfying. Additionally, the results we need from those previous works are valid for the definition here because of the following equivalence: If \(\mathcal{U} \subset \mathcal{M}\) is amenable in this new sense, then there exist \(\mathcal{U}', \mathcal{U}'' \subset \mathcal{M}\) that are amenable in the old sense, and such that \(\mathcal{U}' \subset \mathcal{U} \subset \mathcal{U}''\).

2. If \(\mathcal{U} \subset \mathcal{M}\) is amenable, then \(\mathcal{U}^0\) is pointwise convex, by which we mean the following. Let \(g_0, g_1 \in \mathcal{U}^0\), and let \(\rho\) be any measurable function on \(M\) taking values between 0 and 1. Then \(\rho g_0 + (1 - \rho) g_1 \in \mathcal{U}^0\). This is straightforward to see by the concavity of the function mapping a matrix to its minimal eigenvalue, and the convexity of the norm \(| \cdot |_{g(x)}\).

The following lemma was originally proved in [Cla, Lem. 3.3] for amenable subsets, but the same proof (which is more or less self-evident) works for \(L^2\) closures of amenable subsets.
Lemma 2.6. Let $U$ be an amenable subset. Then there exists a constant $K > 0$ such that for all $\tilde{g} \in U^0$,
\begin{equation}
\frac{1}{K} \leq \left( \frac{\mu_\tilde{g}}{\mu} \right) \leq K,
\end{equation}
where by $(\mu_\tilde{g}/\mu)$ we denote the unique measurable function on $M$ such that $\mu_\tilde{g} = (\mu_\tilde{g}/\mu)\mu$.

To end this subsection, we have a somewhat unexpected and extremely useful result that bounds the distance between two semimetrics uniformly based on the intrinsic volume of the subset on which they differ.

Proposition 2.7 ([Cla11] Prop. 2.20). Let $g_0, g_1 \in \mathcal{M}_f$ and $A := \text{carr}(g_1 - g_0)$. Then
\[ d(g_0, g_1) \leq C(n) \left( \sqrt{\text{Vol}(A, g_0)} + \sqrt{\text{Vol}(A, g_1)} \right), \]
where $C(n)$ is a constant depending only on $n = \dim M$.

3. $d = \Omega_2$ on $\mathcal{M}$

In this section, we show that the distance function of the $L^2$ Riemannian metric is exactly given by the $L^2$-type metric $\Omega_2$ that we defined in (2.1).

3.1. Paths of degenerate metrics and Riemannian distances. If $g_0, g_1 \in \mathcal{M}$ and $g_t$ is a piecewise differentiable path in $\mathcal{M}$ between them, then $d(g_0, g_1) \leq L(g_t)$. The goal of this subsection is to prove a similar inequality for certain paths of semimetrics in $\mathcal{M}_f$.

We first have to be precise about what $L(g_t)$ should mean if $g_t \in \mathcal{M}_f$. We denote by $\mathcal{M}_c \subset \mathcal{M}_f$ the set of all continuous Riemannian metrics on $M$. By $\mathcal{S}_c$, we denote the closure of $\mathcal{S}$ in the $C^0$ norm. For $\tilde{g} \in \mathcal{M}_f$, denote by $\mathcal{S}^0_{\tilde{g}}$ the set of measurable $(0,2)$-tensor fields $h$ such that $h(x) = 0$ whenever $\tilde{g}(x)$ is not positive definite, and such that the quantity
\[ \|h\|_{\tilde{g}} := \left( \int_{M \setminus X_{\tilde{g}}} \text{tr}_{\tilde{g}}(h^2) \sqrt{\det G} d\mu \right)^{1/2} \]
is finite, where in the above $X_{\tilde{g}} \subseteq M$ denotes the set on which $\tilde{g}$ is not positive definite.

We will consider paths of (semi-)metrics $g_t$, $t \in [0,1]$, in both $\mathcal{M}_c$ and $\mathcal{M}_f$. We will call such a path $g_t$ differentiable in $\mathcal{M}_c$ (resp. $\mathcal{M}_f$) if, for each $x \in M$, $g_t(x)$ is a differentiable path in $\mathcal{M}_x$ and, additionally, $g'_t$ is contained in $\mathcal{S}_c$ (resp. $\mathcal{S}^0_{g_t}$) for all $t \in [0,1]$.

Definition 3.1. For $E \subseteq M$, we call a path $g_t$, $t \in [0,1]$, in $\mathcal{M}_f$ continuous on $E$ if $g_t(x)|_E$ is continuous in $x$ for all $t$. If $E = M$, we call $g_t$ simply continuous.

To avoid confusion, we emphasize that a continuous path is one that is continuous in $x$ for each $t$, and a differentiable path is one that is differentiable in $t$ for each $x$.

Let $g_t$, $t \in [0,1]$, be a path in $\mathcal{M}_f$ or $\mathcal{M}_c$ that is piecewise differentiable. We denote by $L(g_t)$ the length of $g_t$ as measured in the naive “Riemannian” way:
\[ L(g_t) = \int_0^1 \|g'_t\|_{g_t} \, dt. \]
When we refer to the length $L(a_t)$ of a path $a_t$ in $\mathcal{M}_x$, we implicitly mean the length with respect to $\langle \cdot, \cdot \rangle$.

It is easy to see (cf. also the proof of [Cla09] Cor. 3.16]) that the $C^0$ topology on $\mathcal{M}_c$ is stronger than the Riemannian $L^2$ topology. Let $g_t$ be a piecewise differentiable path in $\mathcal{M}_c$. 

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Lemma 3.4. We also choose continuous functions connecting two continuous metrics \( g_0 \) and \( g_1 \). It is intuitive, but perhaps not immediately clear, that using smooth approximations, one could show \( d(g_0, g_1) \leq L(g_t) \) as in the case of smooth metrics. We formalize this in the following lemma. The proof is straightforward, but we include some details for those readers unfamiliar with regularization of tensors on manifolds.

Lemma 3.2. Let \( g_0, g_1 \in \mathcal{M}_c \), and suppose that \( g_t, t \in [0, 1] \), is a piecewise differentiable path in \( \mathcal{M}_c \) connecting them. Then \( d(g_0, g_1) \leq L(g_t) \).

Proof. Let \( \{ U_\alpha, \varphi_\alpha \} \) be a finite atlas of charts \( \varphi_\alpha : U_\alpha \to \mathbb{R}^n \) for \( M \). Choose a partition of unity \( p_\alpha \) subordinate to this atlas. We denote the push-forward of \( g_t \) via \( \varphi_\alpha \) by \( g_\alpha^t \) and, by an abuse of notation, denote the locally-defined tensor obtained from restricting \( g_t \) to \( U_\alpha \) by the same. We will regularize these metrics by convolution in local coordinates, letting \( \phi \) be any function on \( \mathbb{R}^n \) that has norm \( 1 \) in \( L^1(\mathbb{R}^n) \) and that vanishes outside the unit ball. Defining \( \phi_\epsilon(x) := \epsilon^{-n} \phi(x/\epsilon) \), we have that for all \( i, j, \) and \( \alpha \), the convolutions \( (g_\alpha^{t,\epsilon})_{ij} := \phi_\epsilon * (g_\alpha^t)_{ij} \) and \( (g_\alpha^{t,\epsilon})_{ij} := (\phi_\epsilon * g_\alpha^t)_{ij} = \phi_\epsilon * (g_\alpha^t)_{ij} \) (the prime, as usual, denotes the partial derivative w.r.t. \( t \)) are smooth functions converging in the \( C^0 \) norm to \( (g_\alpha^t)_{ij} \) and \( (g_\alpha^t)'_{ij} \), respectively, as \( \epsilon \to 0 \). Furthermore, since we are dealing with a finite number of indices, for any given \( t \) we can choose \( \epsilon > 0 \) small enough that \( (g_\alpha^{t,\epsilon})_{ij} \) and \( (g_\alpha^{t,\epsilon})'_{ij} \) are uniformly \( C^0 \)-close to \( g_\alpha^t \) and \( (g_\alpha^t)' \), respectively.

One easily sees that for each \( \epsilon > 0 \), \( \|\phi_\epsilon\|_{L^1(\mathbb{R}^n)} = 1 \), and also that this implies that convolution with \( \phi_\epsilon \) has operator norm \( 1 \) when viewed as a linear operator on \( C^0(\mathbb{R}^n) \). From this, and the compactness of the time interval on which \( g_t \) is defined, one can straightforwardly conclude that \( \epsilon > 0 \) can be chosen small enough that \( (g_\alpha^{t,\epsilon})_{ij} \) and \( (g_\alpha^{t,\epsilon})'_{ij} \) are uniformly \( C^0 \)-close to \( g_\alpha^t \) and \( (g_\alpha^t)' \), respectively, independently not just of \( i, j \), and \( \alpha \), but also \( t \).

Finally, using our partition of unity, we define \( g_\ast_t := \sum_\alpha p_\alpha g_\alpha^{t,\epsilon} \) to get a path of smooth Riemannian metrics connecting \( g_0 \) and \( g_1 \). We have \( (g_\ast_t)' = \sum_\alpha p_\alpha (g_\alpha^{t,\epsilon})' \), and so one sees that with \( \epsilon \) small enough, \( L(g_\ast_t) \) is arbitrarily close to \( L(g_t) \). By the above-mentioned fact that the \( C^0 \) topology on \( \mathcal{M} \) is stronger than the Riemannian \( L^2 \) topology, we also have that \( d(g_0, g_\ast_t) \) and \( d(g_1, g_\ast_t) \) can be made arbitrarily small, from which the desired result follows.

Using the above result on paths of continuous metrics, we can prove what we need about paths in \( \mathcal{M}_f \). We first briefly set up some notation, and then state the result in a lemma.

Definition 3.3. Let \( E \subseteq M \) be any subset. We denote by \( \chi(E) \) the characteristic (or indicator) function of \( E \). The characteristic function of its complement is denoted by \( \overline{\chi}(E) := \chi(M \setminus E) \).

Lemma 3.4. Let \( g_0, g_1 \in \mathcal{M}_c \), and let \( g_t, t \in [0, 1] \), be any smooth path in \( \mathcal{M}_c \) from \( g_0 \) to \( g_1 \). Furthermore, let \( E \subseteq M \) be any measurable subset.

We define \( \tilde{g}_t := \overline{\chi}(E)g_0 + \chi(E)g_t \); in particular \( \tilde{g}_1 = \overline{\chi}(E)g_0 + \chi(E)g_1 \). Then

\[
d(g_0, \tilde{g}_1) \leq L(\tilde{g}_t).
\]

Proof. For each \( k \in \mathbb{N} \), choose an open set \( U_k \) and a closed set \( Z_k \) such that \( Z_k \subseteq E \subseteq U_k \), and such that \( \mu(U_k \setminus Z_k) < \frac{1}{k} \). (This is possible because the Lebesgue measure is regular.) We also choose continuous functions \( f_k \) with the properties that

1. \( 0 \leq f_k \leq 1 \),
2. if \( x \notin U_k \), then \( f_k(x) = 0 \), and
3. if \( x \in Z_k \), then \( f_k(x) = 1 \).
For each $t \in [0,1]$, let \( \tilde{g}_t^k := f_kg_t + (1-f_k)g_0 \). Thus, we have that in particular, \( \tilde{g}_t^k \) coincides with \( g_1 \) on \( Z_k \) and with \( g_0 \) on \( M \setminus U_k \). Our goal is to show that \( \lim_{k \to \infty} L(\tilde{g}_t^k) \leq L(\tilde{g}_t) \) and \( d(\tilde{g}_t^k, \tilde{g}_1) \to 0 \) as \( k \to \infty \), as we can then conclude from the triangle inequality and Lemma \ref{lem:2.7} that
\[
d(\tilde{g}_t^k, \tilde{g}_1) \leq d(\tilde{g}_t^k, \tilde{g}_0) + d(\tilde{g}_0, \tilde{g}_1) \leq L(\tilde{g}_t^k) + d(\tilde{g}_0, \tilde{g}_1).
\]
The statement of the lemma then follows by passing to the limit on the right.

We begin with the claim that \( \lim_{k \to \infty} L(\tilde{g}_t^k) \leq L(\tilde{g}_t) \). Since \( M \) and \([0,1]\) are compact, we have that
\[
N := \max_{x \in M, t \in [0,1]} |g_t'(x)|^2_{g_t(x)} < \infty.
\]
(Recall that by the definition of a smooth path in \( \mathcal{M}_c \), we have \( g_t' \in \mathcal{S}_c \) for all \( t \).) Therefore, noting that \( (\tilde{g}_t^k)' = g_t' \) on \( Z_k \) and \( (\tilde{g}_t^k)' \equiv 0 \) on \( M \setminus U_k \), we may estimate
\[
\| (\tilde{g}_t^k)' \|_{\tilde{g}_t^k}^2 = \int_{Z_k} |g_t'|^2_{g_t} \, d\mu + \int_{U_k \setminus Z_k} |f_kg_t'|^2_{f_kg_t} \, d\mu
\]
\[
= \| \chi(Z_k) g_t' \|_{g_t}^2 + \int_{U_k \setminus Z_k} \text{tr}_{f_kg_t} ((f_kg_t')^2) \sqrt{\det(f_kG_t)} \, d\mu
\]
\[
= \| \chi(Z_k) g_t' \|_{g_t}^2 + \int_{U_k \setminus Z_k} f_k^{n/2} \text{tr}_{g_t} ((g_t')^2) \sqrt{\det G_t} \, d\mu
\]
\[
\leq \| \chi(E) g_t' \|_{g_t}^2 + \int_{U_k \setminus Z_k} N \, d\mu.
\]
The first term in the last line is just \( \| \tilde{g}_t^k \|_{\tilde{g}_t}^2 \), since \( \tilde{g}_t = g_t \) on \( E \). The second term is just \( N \cdot \mu(U_k \setminus Z_k) < N/k \), which converges to zero as \( k \to \infty \). Thus, we have that
\[
\| \tilde{g}_t^k \|_{\tilde{g}_t} \leq \| \tilde{g}_t \|_{\tilde{g}_t} + N/k \text{ for each } t \in [0,1],
\]
which implies the claim that \( \lim_{k \to \infty} L(\tilde{g}_t^k) \leq L(\tilde{g}_t) \).

We now move on to the claim that \( \lim_{k \to \infty} d(\tilde{g}_t^k, \tilde{g}_1) = 0 \). Since \( g_0 \) and \( g_1 \) are continuous metrics, it is clear that we can find an amenable subset \( \mathcal{U} \) such that \( g_0, g_1 \in \mathcal{U}^0 \). But we also know that at each point, \( \tilde{g}_t^k \) and \( \tilde{g}_1 \) are linear combinations of \( g_0 \) and \( g_1 \) with coefficients between zero and one. Hence, by the pointwise convexity of \( L^2 \) closures of amenable subsets (cf. Remark \ref{rem:2.5} \cite{2}), \( \tilde{g}_t^k, \tilde{g}_1 \in \mathcal{U}^0 \) for all \( k \in \mathbb{N} \). Thus, by Lemma \ref{lem:2.6}, there exists a constant \( K \) such that
\[
(\mu_{\tilde{g}_t^k}/\mu) \leq K \text{ for all } k \in \mathbb{N} \text{ and } (\mu_{\tilde{g}_1}/\mu) \leq K.
\]
Using this, Proposition \ref{prop:2.7} and the fact that \( \tilde{g}_t^k \) and \( \tilde{g}_1 \) differ only on \( U_k \setminus Z_k \), we can conclude
\[
d(\tilde{g}_t^k, \tilde{g}_1) \leq C(n) \left( \sqrt{\text{Vol}(U_k \setminus Z_k, \tilde{g}_t^k)} + \sqrt{\text{Vol}(U_k \setminus Z_k, \tilde{g}_1)} \right) \leq 2C(n) \sqrt{\frac{K}{k}}.
\]
This proves the second claim and so, as noted above, the statement of the lemma follows. \( \square \)

In what follows, we will have to deal with reparametrizations of paths. Given a path in \( \mathcal{M}_f \) (or in any space of sections), one can reparametrize globally, in that one replaces \( g_t \), \( t \in [0,1] \), with \( g_{\varphi(t)} \), for some appropriate \( \varphi : [0,1] \to [0,1] \). One can also “reparametrize” pointwise, in that one uses \( g_{\varphi_x(t)} \), where for each \( x \in M \), \( \varphi_x : [0,1] \to [0,1] \) is a function with the appropriate properties. Of course, the latter changes the image of the path in \( \mathcal{M}_f \), but for our purposes it can do so in advantageous ways. The next definition deals with the specific reparametrizations we will need.
**Definition 3.5.** Let \( g_t, t \in [0, 1], \) be a path in \( \mathcal{M}_f \). By the pointwise reparametrization of \( g_t \) proportional to arc length, we mean the path in \( \tilde{g}_t = g_t \) in \( \mathcal{M}_f, t \in [0, 1], \) where for each \( x \in M, \) \( \tilde{g}_t(x) \) is the path obtained from \( g_t(x) \) by reparametrization proportional to \( \langle \cdot, \cdot \rangle \)-arc length.

Given this definition, the following lemma is essentially self-evident.

**Lemma 3.6.** Let \( g_0, g_1 \in \mathcal{M}_f \), and let \( g_t \) be a piecewise differentiable path in \( \mathcal{M}_f \) connecting \( g_0 \) and \( g_1 \). Suppose \( g_t \) fails to have a two-sided derivative at times \( t_0 < t_1 < \cdots < t_k = 1 \). If \( g_t \) is continuous on \( E \subseteq M \) for all \( t \in [0, 1] \), and \( L(g_t(x) | [(t_i, t_{i+1})]) \) is continuous as a function of \( x \) on \( E \) for all \( i = 0, \ldots, k-1 \), then the path obtained from \( g_t \) via pointwise reparametrization proportional to arc length is continuous on \( E \).

**3.2. Proof that \( d = \Omega_2 \).** We now get into the heavy lifting of this section. We will need two rather technical lemmas to get from the restricted situation of Lemma 3.4 to the desired general result. In the following, we will always denote by \( B_\delta(x) \) the closed geodesic ball around \( x \in M \) with radius \( \delta \) (w.r.t. the fixed reference metric \( g \)).

**Lemma 3.7.** Let any \( g_0, g_1 \in \mathcal{M} \) and \( \epsilon > 0 \) be given. Then there exists a \( \delta = \delta(\epsilon, g_0, g_1) > 0 \) with the property that for any \( x_0 \in M \), we can find a path \( g_{x_0,t} \) in \( \mathcal{M}_f \), for \( t \in [-\epsilon, 1+\epsilon] \), from \( g_0 \) to \( \Delta_i(\mathcal{B}_\delta(x_0))g_0 + \chi(\mathcal{B}_\delta(x_0))g_1 \) such that for each \( x \in B_\delta(x_0) \), we have

\[
| g'_{x_0,t}(x) |_{g_{x_0,t}} < \begin{cases} 1, & t \in [-\epsilon, 0] \cup [1, 1+\epsilon], \\ d_x(g_0(x), g_1(x)) + 3\epsilon, & t \in [0, 1]. \end{cases}
\]

Furthermore, for each \( t \), \( g_{x_0,t} \) is constant on \( M \setminus B_\delta(x_0) \) and is continuous on \( B_\delta(x_0) \).

**Proof.** For a given \( x_0 \in M \), we may choose a smooth path \( a_{x_0,t}, t \in [0, 1], \) in \( \mathcal{M}_{x_0} \) connecting \( g_0(x_0) \) and \( g_1(x_0) \) that has length

\[
L(a_{x_0,t}) < d_{x_0}(g_0(x_0), g_1(x_0)) + \epsilon.
\]

Furthermore, this path can be chosen in such a way that there exist constants \( \zeta, \tau > 0 \), depending on \( g_0, g_1, \) and \( \epsilon \) but not on \( x_0 \), such that

\[
a_{x_0,t} \in \mathcal{M}_{x_0}^{\zeta, \tau} := \{ a \in \mathcal{M}_{x_0} \mid \det A \geq \zeta \text{ and } |a|_{g(x_0)} \leq \tau \text{ for all } 1 \leq i, j \leq n \}
\]

for all \( x_0 \in M \) and \( t \in [0, 1] \). (This relies on the fact that \( g_0 \) and \( g_1 \) are smooth metrics, and so are contained in a common compact subset of \( S^2 T^* M \).)

For the rest of the proof, when we refer to geodesics and the Levi-Civita connection \( \nabla \) on \( M \), we mean those belonging to our fixed reference metric \( g \). Now, let \( \delta \) be small enough that for any \( x_0 \in M \) and any \( x \in B_\delta(x_0) \), there exists a unique minimal geodesic (up to reparametrization) from \( x_0 \) to \( x \).

The Levi-Civita connection \( \nabla \) can be extended to all tensor fields, and in particular to \( T^* M \otimes T^* M \). A brief calculation shows that if \( h \in S^2 T^* M \) and \( X \) is a vector field on \( M \), then \( \nabla_X h \in S^2 T^* M \). (That is, symmetry is preserved.) Therefore \( \nabla \) induces a connection \( \nabla \) on the vector bundle \( S^2 T^* M \).

For each \( x_0 \in M \) and \( x \in B_\delta(x_0) \), we denote by \( P_{x_0,x} \) the parallel transport with respect to \( \nabla \) along the minimal geodesic from \( x_0 \) to \( x \). In local vector bundle coordinates for \( S^2 T^* M \), the parallel transport of an element of \( S_{x_0} = S^2 T^* x_0 M \) is the solution of a first-order linear ODE with coefficients depending smoothly upon \( x_0 \) and \( x \). We know that \( P_{x_0,x} a \) is a linear isometry (w.r.t. the scalar product induced by \( g \)) between \( S_{x_0} \) and \( S_x \), so \( P_{x_0,x} a \) depends smoothly on \( a \in s_{x_0} \). Furthermore, since solutions of ODEs behave continuously under
perturbations of the coefficients, the mapping \((x_0, x) \mapsto P_{x_0, x}\) is continuous [CL55, Ch. 1, Thm. 7.4].

We next let \(\tilde{a}_{x_0, t}(x)\) be the path in \(S^2T_x^*M\) obtained from \(a_{x_0, t}\) by parallel transport, i.e., \(\tilde{a}_{x_0, t}(x) = P_{x_0, x}(a_{x_0, t})\). By the discussion above on the smoothness/continuity of parallel transport, \(\tilde{a}_{x_0, t}(x)\) is smooth in the \(t\) variable and continuous in the \(x\) variable.

Now, since \((x_0, x, a) \mapsto P_{x_0, x}(a)\) is continuous, this mapping is uniformly continuous when restricted to the compact space

\[
\bigcup_{x_0 \in M} \left( \{x_0\} \times \bigcup_{x \in B_\delta(x_0)} \mathcal{M}^\xi_\tau \right).
\]

Thus we may (by shrinking \(\delta\) if necessary) assume that \(P_{x_0, x}(\mathcal{M}^\xi_\tau) \subset \mathcal{M}_x\) for each \(x_0 \in M\) and \(x \in B_\delta(x_0)\).

Since we have assumed that each path \(a_{x_0, t}\) is contained in \(\mathcal{M}^\xi_\tau\), this implies that \(\tilde{a}_{x_0, t}(x)\) is a path in \(\mathcal{M}_x\), running from \(P_{x_0, x_0}(g_0(x_0))\) to \(P_{x_0, x_0}(g_1(x_0))\). Again by continuity of parallel transport, by shrinking \(\delta\) we may assume that

\[
d_x(g_0(x), P_{x_0, x_0}(g_0(x_0))) \leq \eta \quad \text{and} \quad d_x(g_1(x), P_{x_0, x_0}(g_1(x_0))) \leq \eta
\]

for any choice of \(\eta > 0\), uniformly in \(x_0\) and \(x\). Furthermore, since the differential of a linear transformation is again the transformation itself, we have \(\tilde{a}'_{x_0, t}(x) = P_{x_0, x}(a'_{x_0, t})\). Thus, by the above-mentioned continuity of \(P_{x_0, x}(a)\), one easily sees that we can just as well shrink \(\delta\) to get the following bound, uniform in \(x_0\) and \(x\):

\[
L(\tilde{a}_{x_0, t}(x)) < L(a_{x_0, t}) + \epsilon < d_{x_0}(g_0(x_0), g_1(x_0)) + 2\epsilon.
\]

Finally, we note that \(d_x(g_0(x), P_{x_0, x_0}(g_0(x_0)))\), \(d_x(g_1(x), P_{x_0, x_0}(g_1(x_0)))\), and \(L(\tilde{a}_{x_0, t}(x))\) are all continuous in \(x\), since all of the quantities involved in their computation are continuous.

For any \(x_0 \in M\), \(x \in B_\delta(x_0)\) and \(\alpha \in \{0, 1\}\), we let \(\sigma^\alpha_{x_0, x_0, t}, t \in [0, 1]\), be the geodesic in \(\mathcal{M}_x\) connecting \(g_\alpha(x)\) and \(P_{x_0, x_0}(g_\alpha(x_0))\). We assume that this geodesic is parametrized proportionally to arc length; note that in this case, \(\sigma^\alpha_{x_0, x_0, t}\) varies continuously in \(x\) on \(B_\delta(x_0)\) for fixed \(\alpha\), \(x_0\), and \(t\). Referring to (3.5), we see that for given \(x\) and \(\alpha\), if \(\eta > 0\) is small enough, then this geodesic indeed exists and is unique. In fact, such a positive \(\eta\) can be found independently of \(x\) and \(\alpha\) since \(g_0(x)\) and \(g_1(x)\) lie in the compact region \(\bigcup_x \mathcal{M}^\xi_\tau \subset S^2T_x^*M\). We may shrink \(\delta\) if necessary to insure that (3.5) is satisfied for this \(\eta\).

Define metrics \(\hat{g}_0\) and \(\hat{g}_1\) by

\[
\hat{g}_0(x) := \begin{cases} g_0(x) & \text{if } x \not\in B_\delta(x_0), \\ P_{x_0, x_0}(g_0(x_0)) & \text{if } x \in B_\delta(x_0), \end{cases} \quad \text{and} \quad \hat{g}_1(x) := \begin{cases} g_1(x) & \text{if } x \not\in B_\delta(x_0), \\ P_{x_0, x_0}(g_1(x_0)) & \text{if } x \in B_\delta(x_0). \end{cases}
\]

(Note that both metrics equal \(g_0\) on \(M \setminus B_\delta(x_0)\).) We consider the paths

\[
g^0_{x_0, t}(x) := \begin{cases} g_0(x) & \text{if } x \not\in B_\delta(x_0), \\ \sigma^0_{x_0, x_0, t} & \text{if } x \in B_\delta(x_0), \end{cases} \quad \text{and} \quad g^1_{x_0, t}(x) := \begin{cases} g_1(x) & \text{if } x \not\in B_\delta(x_0), \\ \sigma^1_{x_0, x_0, t} & \text{if } x \in B_\delta(x_0). \end{cases}
\]

Then these are smooth paths in \(\mathcal{M}_f\) that are continuous on \(B_\delta(x_0)\). The path \(g^0_{x_0, t}\) connects \(g_0\) and \(\hat{g}_0\), while \(g^1_{x_0, t}\) connects \(\chi(B_\delta(x_0))g_0 + \chi(B_\delta(x_0))\hat{g}_1\) and \(\hat{g}_1\). Furthermore, by shrinking \(\delta\) to obtain \(\eta < \epsilon\), we have by (3.5) that for each \(x_0 \in M\) and \(x \in B_\delta(x_0)\), we have

\[
L(g^\alpha_{x_0, t}(x)) = d_x(g_\alpha(x), P_{x_0, x}(g_\alpha(x_0))) < \epsilon
\]
for $\alpha = 0, 1$. We also have that on $B_\delta(x_0)$, $L(g_{x_0,t}^{\alpha}(x)) = d_x(g_\alpha(x), P_{x_0,x}(g_\alpha(x)))$ is continuous in $x$, as noted after (3.6).

We define a path $\tilde{g}_{x_0,t}$ by

$$
(3.8) \quad \tilde{g}_{x_0,t}(x) := \begin{cases} 
g_0(x) & \text{if } x \notin B_\delta(x_0), \\
\tilde{a}_{x_0,t}(x) & \text{if } x \in B_\delta(x_0).
\end{cases}
$$

Then $\tilde{g}_{x_0,t}$ is a path from $\tilde{g}_0$ to $\tilde{g}_1$ as noted above, both $\tilde{a}_{x_0,t}(x)$ and $L(\tilde{a}_{x_0,t}(x))$ vary continuously with $x$ on $B_\delta(x_0)$.

Thus, by the following concatenation,

$$
(3.9) \quad \tilde{g}_{x_0,t} := g^0_{x_0,t} \ast \tilde{g}_{x_0,t} \ast (g^1_{x_0,t})^{-1},
$$

we get a piecewise smooth path in $M_f$ from $g_0$ to $\overline{\bigcup(B_\delta(x_0))}g_0 + \chi(B_\delta(x_0))g_1$ that in continuous on $B_\delta(x_0)$. (Here, $(g^1_{x_0,t})^{-1}$ denotes the path $g^1_{x_0,t}$ run through in reverse.) Let us assume that $\tilde{g}_{x_0,t}$ is parametrized such that it runs through $g^0_{x_0,t}$ for $t \in [-\epsilon, 0]$, then $g_{x_0,t}$ for $t \in [0, 1]$, and finally $(g^1_{x_0,t})$ for $t \in [1, 1 + \epsilon]$.

Denote by $g_{x_0,t}$ the path obtained from $\tilde{g}_{x_0,t}$ by pointwise reparametrizing each portion of the concatenation (3.9) proportionally to arc length. Then by Lemma 3.4 and the statements following (3.6), (3.7), and (3.8), $g_{x_0,t}$ is a piecewise smooth path in $M_f$ that is continuous when restricted to $B_\delta(x_0)$, and by construction $g_{x_0,t}(x) = g_0(x)$ for all $t \in [-\epsilon, 1 + \epsilon]$ if $x \notin B_\delta(x_0)$. For $x \in B_\delta(x_0)$, the estimates (3.6) and (3.7) give

$$
|g_{x_0,t}(x)|_{g_{x_0,t}} < \begin{cases} 
1, & t \in [-\epsilon, 0) \cup (1, 1 + \epsilon], \\
(d_x(g_0(x), g_1(x)) + 2\epsilon, & t \in [0, 1).
\end{cases}
$$

Finally, since $g_0$ and $g_1$ are smooth metrics, the function $x \mapsto d_x(g_0(x), g_1(x))$ is continuous. Therefore, we may assume that there is small enough that $d_{x_0}(g_0(x_0), g_1(x_0)) < d_x(g_0(x), g_1(x)) + \epsilon$ for all $x \in B_\delta(x_0)$. This and the above inequality show that $g_{x_0,t}$ has all the desired properties.

**Lemma 3.8.** Let any $g_0, g_1 \in M$ and $\epsilon > 0$ be given, and let $\delta = \delta(\epsilon, g_0, g_1) > 0$ be as in Lemma 3.7. Consider a finite collection of closed subsets $\{F_i \mid i = 1, \ldots, N\}$ with the property that for each $i$, there exists $x_i \in F_i$, such that $F_i \subseteq B_\delta(x_i)$ for some $0 < \delta' < \delta$, and such that $F_i \cap F_j = \emptyset$ for all $i \neq j$. We denote

$$
F := \bigcup_{i=1, \ldots, N} F_i.
$$

Then there exists a path $\tilde{g}_t$, for $t \in [-\epsilon, 1 + \epsilon]$, from $g_0$ to $\tilde{g}_1 := \overline{\bigcup(F)}g_0 + \chi(F)g_1$ satisfying

$$
L(\tilde{g}_t)^2 < (1 + 2\epsilon) \left[ \Omega_2(g_0, g_1)^2 + 6\epsilon \Omega_1(g_0, g_1) + 9\epsilon^2 + 2\epsilon \right].
$$

Furthermore, $\tilde{g}_t$ satisfies the assumptions of Lemma 3.4, and so also

$$
d(g_0, \tilde{g}_1)^2 < (1 + 2\epsilon) \left[ \Omega_2(g_0, g_1)^2 + 6\epsilon \Omega_1(g_0, g_1) + 9\epsilon^2 + 2\epsilon \right].
$$

**Proof.** For each $i \in \mathbb{N}$, let $g_{i,t} := g_{x_i,t}$ be the path from $g_0$ to $\overline{\bigcup(B_\delta(x_i))}g_0 + \chi(B_\delta(x_i))g_1$ guaranteed by Lemma 3.7. Then for each $x \in B_\delta(x_i)$, we have

$$
|g_{x_0,t}(x)|_{g_{x_0,t}} < \begin{cases} 
1, & t \in [-\epsilon, 0) \cup (1, 1 + \epsilon], \\
(d_x(g_0(x), g_1(x)) + 3\epsilon, & t \in [0, 1).
\end{cases}
$$

Additionally, $g_{i,t}(x)$ is constant in $t$ for $x \notin B_\delta(x_i)$. 

Since the sets $F_i$ are pairwise disjoint and closed, we can find $\eta > 0$ such that the closed subsets

$$B_\eta(F_i) = \{ x \in M \mid \text{dist}_g(x, F_i) \leq \eta \}$$

are still pairwise disjoint. (Here, dist$_g$ denotes the distance function of the Riemannian metric $g$ on $M$.) Since $F_i \subseteq B_{\delta'}(x_i)$ for some $0 < \delta' < \delta$, we may also choose $\eta$ small enough that $B_\eta(F_i) \subseteq B_\delta(x_i)$ for all $i$.

Now, for each $i$, we define a continuous function for $x \in B_\eta(F_i)$ by

$$s_i(x,t) := \left( \frac{\eta - \text{dist}_g(x, F_i)}{\eta} \right) (t + \epsilon) - \epsilon,$$

so that $s_i(x,-\epsilon) \equiv -\epsilon$. Furthermore, $s_i(x,t) = t$ for all $x \in F_i$ and $t \in [-\epsilon, 1 + \epsilon]$, and $s_i(x,t) = -\epsilon$ for all $t$ if $x \in \partial B_\eta(F_i)$. We define a smooth path in $\mathcal{M}_\epsilon$ as follows:

$$\tilde{g}_t(x) := \begin{cases} 
  g_{i,t}(x), & x \in F_i, \\
  g_{i,s_i(x,t)}(x) \in B_\eta(F_i), & x \in B_\eta(F_i), \\
  g_0(x), & x \notin \cup_i B_\eta(F_i),
\end{cases} \quad \text{for } t \in [-\epsilon, 1 + \epsilon].$$

With this definition, we can see that the path $\tilde{g}_t := \tilde{x}(F)g_0 + \chi(F)\tilde{g}_t$ satisfies the assumptions of Lemma 3.4, and hence $d(g_0, \tilde{g}_t) \leq L(\tilde{g}_t)$. We claim that (3.10) and hence (3.11) hold as well.

To see this, note that $|\tilde{g}'_t(x)|_{\tilde{g}_t(x)} = |g'_{i,t}(x)|_{g_{i,t}}$ for all $x \in F$. Therefore, using (3.12), we can estimate

(3.13)

$$L(\tilde{g}_t)^2 \leq (1 + 2\epsilon)E(\tilde{g}_t) = (1 + 2\epsilon)\sum_{i=1}^N \int_{-\epsilon}^{1+\epsilon} \int_{F_i} |\tilde{g}'_t(x)|_{\tilde{g}_t(x)}^2 \, d\mu \, dt$$

$$< (1 + 2\epsilon) \left[ \sum_{i=1}^N \int_0^1 \int_{F_i} (d_x(g_0(x), g_1(x)) + 3\epsilon)^2 \, d\mu \, dt + \sum_{i=1}^N \int_{[-\epsilon,0] \cup [1,1+\epsilon]} \int_{F_i} \, d\mu \, dt \right]$$

$$= (1 + 2\epsilon) \left[ \sum_{i=1}^N \int_{F_i} d_x(g_0(x), g_1(x))^2 \, d\mu 
+ 6\epsilon \cdot \sum_{i=1}^N \int_{F_i} d_x(g_0(x), g_1(x)) \, d\mu + (9\epsilon^2 + 2\epsilon) \sum_{i=1}^N \int_{F_i} \, d\mu \right]$$

$$\leq (1 + 2\epsilon) \left[ \Omega_2(g_0, \tilde{g}_1)^2 + 6\epsilon \Omega_1(g_0, \tilde{g}_1) + 9\epsilon^2 + 2\epsilon \right].$$

The last line follows by the formulas for $\Omega_1$ and $\Omega_2$, as well as the fact that $\text{Vol}(M, \mu) = 1$.

Finally, we note that $\Omega_1(g_0, \tilde{g}_1) \leq \Omega_1(g_0, g_1)$ and $\Omega_2(g_0, \tilde{g}_1) \leq \Omega_2(g_0, g_1)$ since $\tilde{g}_1$ equals $g_1$ on $F$ and $g_0$ everywhere else. Thus (3.13) in fact implies (3.10).

We now have all the pieces necessary to prove the main result of this section.

**Theorem 3.9.** $d(g_0, g_1) = \Omega_2(g_0, g_1)$ for all $g_0, g_1 \in \mathcal{M}$.

**Proof.** We have already shown in Theorem 2.1 that $d(g_0, g_1) \geq \Omega_2(g_0, g_1)$, so it only remains to show the reverse inequality.

Let any $\epsilon > 0$ be given, and let $\delta = \delta(\epsilon, g_0, g_1)$ be the number guaranteed by Lemma 3.7.
Choose $0 < \delta' < \delta$ and a finite number of points $x_i \in M$, $i = 1, \ldots, N$, such that $B_i := \text{int}(B_{\delta}(x_i))$ cover $M$. (Here, int denotes the interior of a set.) For each $i$, we choose $0 < \delta_i < \delta'$ such that

$$\max\{\text{Vol}(B_i \setminus B_{\delta_i}(x_i), g_0), \text{Vol}(B_i \setminus B_{\delta_i}(x_i), g_1)\} < \frac{\epsilon}{2^{N-1}}.$$  

(3.14)

We then let $F_i := B_{\delta_i}(x_i)$. For each $i = 2, \ldots, N$, define

$$F_i := B_{\delta_i}(x_i) \setminus \bigcup_{j<i} B_j.$$  

We wish to see that the sets $F_i$ cover $M$ up to a set of measure $\epsilon$, intrinsically with respect to both $g_0$ and $g_1$. By (3.14), for $\alpha = 0, 1$,

$$\text{Vol}(F_i, g_\alpha) \geq \text{Vol}(B_1, g_\alpha) - \frac{\epsilon}{2^{N-1}}.$$  

To estimate $\text{Vol}(F_1 \cup F_2, g_\alpha)$, note that

$$F_1 \cup F_2 = B_{\delta_1}(x_1) \cup (B_{\delta_2}(x_2) \setminus B_1) = B_{\delta_1}(x_1) \cup (B_{\delta_2}(x_2) \setminus (B_1 \setminus B_{\delta_1}(x_1))).$$  

The first set in the union on the right-hand side is completely contained in $B_1$, and the second set is completely contained in $B_2$. Furthermore, they are disjoint. Therefore, again using (3.14),

$$\text{Vol}(F_1 \cup F_2, g_\alpha) = \text{Vol}(B_{\delta_1}(x_1), g_\alpha) + \text{Vol}(B_{\delta_2}(x_2) \setminus (B_1 \setminus B_{\delta_1}(x_1)))$$

$$\geq \left(\text{Vol}(B_1, g_\alpha) - \frac{\epsilon}{2^{N-1}}\right) + \left(\text{Vol}(B_{\delta_2}(x_2)) - \text{Vol}(B_1 \setminus B_{\delta_1}(x_1), g_\alpha)\right)$$

$$\geq \left(\text{Vol}(B_1, g_\alpha) - \frac{\epsilon}{2^{N-1}}\right) + \left(\text{Vol}(B_{\delta_2}(x_2)) - \frac{\epsilon}{2^{N-1}}\right)$$

$$\geq \left(\text{Vol}(B_1, g_\alpha) - \frac{\epsilon}{2^{N-1}}\right) + \left(\text{Vol}(B_2, g_\alpha) - 2 \cdot \frac{\epsilon}{2^{N-1}}\right)$$

$$\geq \text{Vol}(B_1 \cup B_2, g_\alpha) - \left(1 + 2\right)\frac{\epsilon}{2^{N-1}}.$$  

(3.15)

If we continue in this way, we find that for $F := \bigcup_i F_i$,

$$\text{Vol}(F, g_\alpha) \geq \text{Vol}(M, g_\alpha) - \left(\sum_{j=0}^{N-1} 2^j\right)\frac{\epsilon}{2^{N-1}} = \text{Vol}(M, g_\alpha) - \epsilon,$$

(3.16)

where we recall that $\bigcup_i B_i = M$.

Now, note that as defined, the sets $F_i$ satisfy the assumptions of Lemma 3.8. Let $\tilde{g}_i$ and $\tilde{g}_1$ be as in the lemma. Then we have that

$$d(g_0, \tilde{g}_1)^2 < (1 + 2\epsilon) \left[\Omega_2(g_0, g_1)^2 + 6\Omega_1(g_0, g_1) + 9\epsilon^2 + 2\epsilon\right].$$  

(3.17)

On the other hand, $\tilde{g}_1$ and $g_1$ differ only on $M \setminus F$, where $\tilde{g}_1 = g_0$. Thus, by (3.16) and Proposition 2.7, we have

$$d(\tilde{g}_1, g_1) \leq C(n) \left(\sqrt{\text{Vol}(M \setminus F, g_0)} + \sqrt{\text{Vol}(M \setminus F, g_1)}\right) < 2C(n)\sqrt{\epsilon}.$$  

(3.18)

Applying the triangle inequality to (3.17) and (3.18), we obtain

$$d(g_0, g_1) < \sqrt{(1 + 2\epsilon) \left[\Omega_2(g_0, \tilde{g}_1)^2 + 6\Omega_1(g_0, \tilde{g}_1) + 9\epsilon^2 + 2\epsilon\right] + 2C(n)\sqrt{\epsilon}}.$$
Sending $\epsilon \to 0$ then gives the desired result, $d(g_0, g_1) \leq \Omega_2(g_0, g_1)$.

Since the completion of $(\mathcal{M}, d)$ is already known, the previous theorem implies that the $L^2$ completion of $\mathcal{M}$ (in the sense discussed following (2.1)) is given by $\hat{\mathcal{M}}_f$.

4. Minimal paths in $\mathcal{M}$ and $\mathcal{M}_x$

In this section, we use the formula $d = \Omega_2$, together with an analysis of the geometry of the fiber spaces $(\mathcal{M}_x, \langle \cdot, \cdot \rangle)$, to obtain results about geodesics in $\mathcal{M}$. (We will use the terms geodesic and minimal path interchangeably to mean a path whose length minimizes the distance globally, i.e., between any two of its points. If we are referring to geodesics in the sense of Riemannian geometry, we will refer to Riemannian geodesics.) Of course, as the completion of the path metric space $(\mathcal{M}, d)$, $\mathcal{M}$ is itself a complete path metric space [BH99, 3.6(3)]. (By path metric space, we mean that the distance between points is equal to the infimum of the lengths of paths between those points. Some authors refer to this as an inner or intrinsic metric space.) However, since $\mathcal{M}$ is not locally compact, completeness is no guarantee that minimal paths exists between arbitrary points—even in the Riemannian case, this does not hold, as an example of McAlpin shows [McA65, Sect. I.E] (see also [Lan95, Sect. VIII.6]).

Nevertheless, we will show that a unique geodesic exists between any two points, and we can give an explicit and easily computable formula for this geodesic. Our analysis of the geometry of $\mathcal{M}_x$ builds upon the foundation set up by Freed–Groisser [FG89] and Gil-Medrano–Michor [GMM91].

To begin this section, we give some estimates for the distance function $d_x$ of $(\mathcal{M}_x, \langle \cdot, \cdot \rangle)$, and determine the completion $(\mathcal{M}_x, d_x)$. Following that, we summarize the work of Freed–Groisser and Gil-Medrano–Michor mentioned above. We use that work to help determine minimal paths between points of $(\mathcal{M}_x, d_x)$, and show that these exist and are unique. Following that, especially for use in Section 5, we show that these minimal paths in $\mathcal{M}_x$ vary continuously with their endpoints. Finally, we use the preceding results and Theorem 3.9 to give explicit formulas for the distance function and minimal paths of $\mathcal{M}$.

4.1. The metric $d_x$ on $\mathcal{M}_x$. We begin our investigation of $d_x$ by giving estimates of it from below and above, which allows us to determine the completion of $(\mathcal{M}_x, d_x)$.

Given a tensor $a \in S_x$, we have as before $A = g(x)^{-1}a$. For positive semi-definite $a$, we will denote by $\sqrt{A}$ the square root of the determinant of $A$. Similarly, $\sqrt[n]{\det A}$ simply denotes $\sqrt[n]{\det A}$. Note that these quantities are coordinate-independent since $A$ is an endomorphism of $T_xM$.

Our first result bounds $d_x$ from below based on the determinants of two given elements, and is the pointwise analog of [Cla10, Lemma 12]. It will come in useful when showing that given paths are minimal in $\mathcal{M}_x$.

**Lemma 4.1.** Let $a_0, a_1 \in \mathcal{M}_x$. Then

$$d_x(a_0, a_1) \geq \frac{4}{\sqrt{n}} \left| \sqrt[n]{A_1} - \sqrt[n]{A_0} \right|.$$  

In particular the functions $a \mapsto \det A$ and $a \mapsto \sqrt[n]{A}$ are continuous and Lipschitz continuous, respectively, on $(\mathcal{M}_x, d_x)$. 
Proof. The proof is essentially the same as [Cla10, Lemma 12], but for completeness we include it here.

First, let \( a \in \mathcal{M}_x \), and suppose that \( b \in T_a \mathcal{M}_x \cong S_x \). Let \( b_1 \) be the pure-trace part of \( b \) \((b_1 = \frac{1}{n} \text{tr}_a b)\) and \( b_0 \) be the trace-free part \((b_0 = b - b_1)\). It is easy to see that \( \text{tr}_a(b_0b_1) = 0 \), and therefore

\[
\text{tr}_a(b^2) = \text{tr}_a(b_0^2) + \text{tr}_a(b_1^2) = \text{tr}_a(b_0^2) + \frac{1}{n} (\text{tr}_a b)^2.
\]

Since \( \text{tr}_a(b_0^2) \geq 0 \), we can conclude that \( (\text{tr}_a b)^2 \leq n \text{tr}_a(b^2) \).

Let \( a_t, t \in [0, 1] \), be any path connecting \( a_0 \) and \( a_1 \). We can estimate

\[
\partial_t \sqrt{A_t} = \frac{1}{4} (\det A_t)^{-3/4} \text{tr}_a(a'_t)(\det A_t) = \frac{1}{4} \left( \text{tr}_a(a'_t)^2 \sqrt{A_t} \right)^{1/2} 
\leq \frac{1}{4} \left( n \text{tr}_a((a'_t)^2) \sqrt{A_t} \right)^{1/2} = \frac{\sqrt{n}}{4} \|a'_t\|_{a_t},
\]

where we have used the inequality of the last paragraph. Integrating this last estimate gives

\[
\sqrt{A_1} - \sqrt{A_0} = \int_0^1 \partial_t \sqrt{A_t} dt \leq \frac{\sqrt{n}}{4} \int_0^1 \|a'_t\|_{a_t} dt = \frac{\sqrt{n}}{4} L(a_t).
\]

Since this holds for any path \( a_t \) between \( a_0 \) and \( a_1 \), and we can just as easily exchange \( a_0 \) and \( a_1 \), the statement of the lemma is proved. \( \square \)

Now, let \( a_1 \in \mathcal{M}_x \) and consider the path \( a_t = ta_1, t \in (0, 1] \). It is straightforward to compute that \( L(a_t) = \frac{4}{\sqrt{n}} \sqrt{A_1} \), and therefore we can think of the zero tensor (denoted simply by 0) as representing a point in the completion of \((\mathcal{M}_x, d_x)\). (We will make this more precise in Proposition 4.3 below.) The metric \( d_x \) naturally extends to a metric on the completion (which we again denote by \( d_x \)). By Lemma 4.1, \( a_t \) is minimal, so we have \( d_x(a_1, 0) = \frac{4}{\sqrt{n}} \sqrt{A_1} \) for any \( a_1 \in \mathcal{M}_x \). Thus, by an application of the triangle inequality, we have the following “converse” of Lemma 4.1. It is a pointwise analog of Proposition 2.7.

**Lemma 4.2.** Let \( a_0, a_1 \in \mathcal{M}_x \). Then

\[
d_x(a_0, a_1) \leq \frac{4}{\sqrt{n}} \left( \sqrt{A_0} + \sqrt{A_1} \right).
\]

We now wish to determine the completion of \((\mathcal{M}_x, d_x)\), which we will do by comparison with another metric. Consider the Riemannian metric \( \langle \cdot, \cdot \rangle^0 \) on \( \mathcal{M}_x \) given by

\[
\langle b, c \rangle^0_a = \text{tr}_a(bc).
\]

This metric turns \( \mathcal{M}_x \) into a complete Riemannian manifold—indeed, into a symmetric space (see [Ebi70, Thm. 8.9] or [Cla, Prop. 4.9]). Since the scalar product \( \langle \cdot, \cdot \rangle_a \) differs from \( \langle \cdot, \cdot \rangle_a^0 \) only by the factor \( \sqrt{A} \), one reasonably suspects that the only points that could be missing from the completion of \((\mathcal{M}_x, \langle \cdot, \cdot \rangle)\) are those with determinant zero. The next proposition confirms this hunch and makes it rigorous.

**Proposition 4.3.** The completion of \((\mathcal{M}_x, \langle \cdot, \cdot \rangle)\) can be identified with

\[
\overline{\mathcal{M}_x} \cong \text{cl}(\mathcal{M}_x)/\partial \mathcal{M}_x,
\]

where \( \text{cl}(\mathcal{M}_x) \) denotes the topological closure of \( \mathcal{M}_x \) as a subspace of \( S_x \), and \( \partial \mathcal{M}_x \) denotes the boundary in \( S_x \).
The topology is given by the following. Given a sequence \( \{a_k\} \subset \overline{\mathcal{M}_x} \), it converges to \( a_0 \in \mathcal{M}_x \) if and only if it does so in the manifold topology, and it converges to \([0] \in \mathcal{M}_x \) (the equivalence class of the zero tensor) if and only if \( \det A_k \to 0 \). In fact, \( d_x(a, [0]) = \frac{4}{\sqrt{n}} \sqrt{A} \) for any \( a \in \overline{\mathcal{M}_x} \).

Proof. By the standard construction of the completion of a Riemannian manifold, we must consider all piecewise differentiable paths of the form \( a_t, t \in [0, 1] \), in \( \mathcal{M}_x \) that have finite length with respect to \( \langle \cdot, \cdot \rangle \) and show two facts. First, either \( \lim_{t \to 1} a_t \in \mathcal{M}_x \) (in the topology of \( \mathcal{S}_x \)) or \( \lim_{t \to 1} \det A_t = 0 \). Second, if \( \lim_{t \to 1} \det A_t = 0 \) and \( \tilde{a}_t, t \in [0, 1] \), is another path in \( \mathcal{M}_x \) satisfying \( \lim_{t \to 1} \det \tilde{A}_t = 0 \), then \( a_t \) and \( \tilde{a}_t \) are equivalent in the sense that \( \lim_{t \to 1} d_x(a_t, \tilde{a}_t) = 0 \). (From these facts, the statements about the topology on \( \overline{\mathcal{M}_x} \) follow immediately.)

The second fact, however, is immediate from Lemma 4.2. So to prove the first fact, suppose we do not have \( \lim_{t \to 1} \det A_t = 0 \). By Lemma 4.1, one can easily see that \( \det A_t \) must nevertheless converge to some limit \( \eta > 0 \). Furthermore, Lemma 4.1 implies that there exists \( \epsilon > 0 \) such that \( \eta/2 < \det A_t < 3\eta/2 \) for all \( t \in [1 - \epsilon, 1) \). But since \( \langle \cdot, \cdot \rangle \) is equivalent to \( \langle \cdot, \cdot \rangle^0 \) on the subset \( \{a \in \mathcal{M}_x \mid \eta/2 < \det A < 3\eta/2\} \), the completeness of \( \langle \cdot, \cdot \rangle^0 \) implies that \( \lim_{t \to 1} a_t \in \mathcal{M}_x \), as desired.

The formula for \( d_x(a, [0]) \) holds by the discussion following Lemma 4.1. \( \square \)

As in the case of \( \overline{\mathcal{M}} \), the completion \( \overline{\mathcal{M}_x} \) together with the metric induced from \( d_x \) (which we will again denote by \( d_x \)) is a path metric space. Furthermore, given Proposition 4.3, we see that Lemmas 4.1 and 4.2 continue to hold if \( a_0, a_1 \in \overline{\mathcal{M}_x} \), so from now on we will assume the lemmas are stated as such.

4.2. Riemannian geodesics in \( \mathcal{M}_x \). In this subsection, we recall some results on geodesics and the exponential mappings for \( (\mathcal{M}_x, \langle \cdot, \cdot \rangle) \).

As explained in [FG89 Appendix], the formulas for geodesics on \( \mathcal{M}_x \) follow directly from those on \( \mathcal{M} \) determined by Freed–Griesser [FG89 Thm. 2.3] and Gil-Medrano–Michor [GMM91 Thm. 3.2]. Namely, we have that \( g_t \) is a geodesic in \( (\mathcal{M}, \langle \cdot, \cdot \rangle) \) if and only if \( g_t(x) \) is a geodesic in \( (\mathcal{M}_x, \langle \cdot, \cdot \rangle) \). Therefore, in the following we will quote formulas that Freed–Griesser and Gil-Medrano–Michor formulated for \( \mathcal{M} \), translated into the result for \( \mathcal{M}_x \).

For the remainder of the paper, we denote by \( b_T := b - \frac{1}{n} (\text{tr} a_0 b) a_0 \) the traceless part of any \( b \in T_{a_0} \mathcal{M}_x \cong \mathcal{S}_x \). Furthermore, in all that follows, exp without a subscript denotes the usual exponential of a matrix or linear transformation, while \( \exp_{a_0} \) denotes the Riemannian exponential mapping of \( a_0 \in \mathcal{M}_x \).

**Theorem 4.4.** Let \( a_0 \in \mathcal{M}_x \) and \( b \in T_{a_0} \mathcal{M}_x \cong \mathcal{S}_x \). Define

\[
q(t) := 1 + \frac{t}{4} \text{tr} a_0 (b), \quad r(t) := \frac{t}{4} \sqrt{n \text{tr} a_0 (b_T^2)}.
\]

Then the geodesic starting at \( a_0 \) with initial tangent \( a'_0 = b \) is given by

\[
a_t = \begin{cases} 
(q(t)^2 + r(t)^2)^{\frac{3}{2}} a_0 \exp \left( \frac{4}{\sqrt{n \text{tr} (b_T^2)}} \arctan \left( \frac{r(t)}{q(t)} \right) b_T \right) & \text{if } b_T \neq 0, \\
q(t)^{4/n} a_0 & \text{if } b_T = 0.
\end{cases}
\]
In particular, the change in the volume element $\sqrt{A_t}$ is given by
\begin{equation}
\sqrt{A_t} = (g(t)^2 + r(t)^2)\sqrt{A_0}.
\end{equation}

For precision, we specify the range of arctan in the above. At a point where $\text{tr}_{a_0} b \geq 0$, it assumes values in $(-\frac{\pi}{2}, \frac{\pi}{2})$. At a point where $\text{tr}_{a_0} b < 0$, arctan$(r(t)/q(t))$ assumes values as follows, with $t_0 := -\frac{4}{\text{tr}_{a_0} b}$:

1. in $[0, \frac{\pi}{2})$ if $0 \leq t < t_0$,
2. in $(\frac{\pi}{2}, \pi)$ if $t_0 < t < \infty$,

and we set arctan$(r(t)/q(t)) = \frac{\pi}{2}$ if $t = t_0$.

Finally, the geodesic is defined on the following domain. If $b_T = 0$ and $\text{tr}_{a_0} b < 0$, then the geodesic is defined for $t \in [0, t_0)$. Otherwise, the geodesic is defined on $[0, \infty)$.

We note here that if, in the above, $b_T = 0$, i.e., the initial tangent vector of the geodesic is pure-trace, then the geodesic is a certain parametrization of a straight ray for as long as it is defined.

Gil-Medrano–Michor also performed a detailed analysis of the exponential mapping of $M$. We quote here a portion of their results, translated into the pointwise result for $M_x$.

**Theorem 4.5** ([GMM91 §3.3, Thm. 3.4]). Let $a_0 \in M_x$ and $U := S_x \setminus (-\infty, -4/n]a_0$. Then $U$ is the maximal domain of definition of $\exp_{a_0}$, and $\exp_{a_0}$ is a diffeomorphism between $U$ and
\begin{equation}
V := \exp_{a_0}(U) = \left\{ a_0 \exp(a_0^{-1}b) \Big | \text{tr}_{a_0}(b_T^2) < \frac{(4\pi)^2}{n} \right\}.
\end{equation}

The inverse of $\exp_{a_0}$ is given by the following. For $b \in S_x$, define
\begin{equation}
\psi(b) := \begin{cases} 
\frac{4}{n} \left( \exp \left( \frac{\text{tr}_{a_0} b}{4} \right) \cos \left( \frac{\sqrt{n \text{tr}_{a_0}(b_T^2)}}{4} \right) - 1 \right) a_0 & \text{if } b_T \neq 0 \\
+ \frac{4}{n} \left( \exp \left( \frac{\text{tr}_{a_0} b}{4} \right) \sin \left( \frac{\sqrt{n \text{tr}_{a_0}(b_T^2)}}{4} \right) b_T \right) a_0 & \text{if } b_T = 0.
\end{cases}
\end{equation}

Now, if $a_1 \in V$, write (uniquely) $a_1 = a_0 \exp(a_0^{-1}b)$ for some $b \in S_x$. Then $\exp_{a_0}^{-1}(a_1) = \psi(b)$.

In what follows, we will also require some facts about the exponential mapping of certain submanifolds of $M_x$, as defined below.

**Definition 4.6.** Let $a \in M_x$. We define
\begin{equation}
M_{x, \sqrt{A}} := \{ b \in M_x \mid \sqrt{B} = \sqrt{A} \}.
\end{equation}

Note that since the derivative of the map $b \mapsto \sqrt{B}$ is $c \mapsto \frac{1}{2} \text{tr}_b(c)\sqrt{B}$ [Bes08, Prop. 1.186] (cf. the errata in the previous reference), we have
\begin{equation}
T_b M_{x, \sqrt{A}} = \{ c \in S_x \mid \text{tr}_b c = 0 \}
\end{equation}
for all $b \in M_{x, \sqrt{A}}$. Also, since for any $b \in M_x$, there exists a number $\lambda > 0$ such that $\sqrt{\lambda B} = \sqrt{A}$, we have $M_x \cong \mathbb{R}_+ \times M_{x, \sqrt{A}}$. This decomposition is orthogonal with respect to $\langle \cdot, \cdot \rangle$, since if $b = \lambda b$, $\lambda \in \mathbb{R}$, is tangent to $\mathbb{R}_+ \cdot b$ and $k \in T_{b} M_{x, \sqrt{A}}$, then $\text{tr}_k(hk) = \lambda \text{tr}_b k = 0$. We call vectors tangent to $\mathbb{R}_+ \cdot b$ pure-trace, and those tangent to $T_b M_{x, \sqrt{A}}$ traceless.
By [Ebi70] Thm. 8.9 (see also [FG89, Prop. 1.13]), \( \mathcal{M}_{x,\sqrt{A}} \) is (non-canonically) isometric to the symmetric space \( SL(n)/SO(n) \). Furthermore, we have the following formula for the exponential mapping \( \exp^0_b \) of \( \mathcal{M}_{x,\sqrt{A}} \) (where we have again translated the result to a pointwise one):

**Theorem 4.7.** Let \( b \in \mathcal{M}_{x,\sqrt{A}} \) and \( c \in T_b \mathcal{M}_{x,\sqrt{A}} \). Then

\[
\exp^0_b(c) = b \exp(b^{-1}c)
\]

and \( \exp^0_b \) is a diffeomorphism from \( T_b \mathcal{M}_{x,\sqrt{A}} \) to \( \mathcal{M}_{x,\sqrt{A}} \).

### 4.3. Existence and uniqueness of geodesics in \( \overline{\mathcal{M}}_x \)

We now turn to the proof of existence and uniqueness of geodesics in the completion \( \overline{\mathcal{M}}_x \). These geodesics will turn out to have a relatively simple form: they are either Riemannian geodesics, or they are the concatenation of the geodesic from the initial point to the singular point \([0]\) of \( \mathcal{M}_x \), followed by the geodesic from \([0]\) to the terminal point. As we saw in §4.1 such a geodesic is consequently a concatenation of straight segments.

On the other hand, note that \( \overline{\mathcal{M}}_x \) is not locally compact, as by Proposition 4.3, the closed ball around \([0]\) of radius \( r > 0 \) is the noncompact set

\[
B(r,[0]) = \{[0]\} \cup \left\{ a \in \mathcal{M}_x \mid \frac{4}{\sqrt{n}} \sqrt[4]{A} \leq r \right\}.
\]

For this reason, existence of geodesics in \( \overline{\mathcal{M}}_x \) does not follow from general theory, e.g., the Hopf–Rinow–Cohn–Vossen Theorem [BH99, Prop. 1.3.7]. Therefore, it falls upon us to prove this existence directly.

We begin this subsection with a general result on non-minimality of paths, which we will then use to prove that Riemannian geodesics in \( \mathcal{M}_x \), which are unique by Theorem 4.5, minimize as long as they exist. Following that, we analyze the boundary of the image of exponential mapping of \( \mathcal{M}_x \), which will allow us to show the full existence and uniqueness result.

#### 4.3.1. Non-minimal paths

At this point, we require a fundamental result in Riemannian geometry, adapted to a low-regularity situation. We begin with two technical lemmas.

**Lemma 4.8.** Let \([a,b]\) be an interval in \( \mathbb{R} \), and let \( r : [a,b] \to \mathbb{R} \) be \( C^1 \). For \( t \in [a,b] \), define

\[
\hat{r}(t) := \min \left( \max_{s \in [0,t]} r(s), 1 \right).
\]

Then \( \hat{r} : [a,b] \to \mathbb{R} \) is absolutely continuous. In particular, \( \hat{r} \) is a.e.-differentiable (with respect to Lebesgue measure), \( \hat{r}' \) is integrable, and

\[
\hat{r}(b) - \hat{r}(a) = \int_a^b \hat{r}'(t) \, dt.
\]

**Proof.** We note that \( \hat{r} \) is a continuous, monotone function. Thus, by [Rud87, Thm. 7.18], if we can show that \( \hat{r} \) maps sets of measure zero to sets of measure zero, then \( \hat{r} \) is absolutely continuous, and the other properties follow.

Now, for any \( t \in [a,b] \), either \( \hat{r}(t) = r(t) \), \( \hat{r}(t) = 1 \), or there exists a critical point \( t_0 \) of \( r \) with \( t_0 < t \) and \( \hat{r}(t) = r(t_0) \). Let \( \text{Crit}(r) \) denote the set of critical points of \( r \); by Sard’s Theorem, \( r(\text{Crit}(r)) \) has measure zero. But if \( N \subseteq [a,b] \) is any Lebesgue null set, then \( r(N) \)
is also a null set, and by the above discussion, \( \hat{r}(N) \subseteq r(N) \cup r(\text{Crit}(N)) \cup \{ 1 \} \), so \( \hat{r}(N) \) has measure zero, as desired.

\[ \square \]

**Lemma 4.9.** Let \((N, \gamma)\) be a finite-dimensional Riemannian manifold, and let \( f : [0, 1] \to \mathbb{R} \) be an absolutely continuous function. Let \( p \in N \) and \( v \in T_p N \) be such that \( f(t)v \) lies in the domain of definition of \( \exp_p \) for all \( t \). Then the radial path \( \sigma(t) := \exp_p(f(t)v) \) is an absolutely continuous curve in \( N \). Furthermore, \( \sigma \) is a.e.-differentiable, \( |\sigma'(t)|_\gamma = |f'(t)||v|_\gamma \) for a.e. \( t \), and

\[
L(\sigma) = \int_0^1 |\sigma'(t)|_\gamma \, dt = |v|_\gamma \int_0^1 |f'(t)| \, dt.
\]

**Proof.** Since \( f(t)v \) lies within the domain of definition of \( \exp_p \) for all \( t \), we may use the exponential mapping to apply classical results about absolutely continuous functions.

In particular, since \( \exp_p \) is smooth and \( f \) is absolutely continuous, \( \sigma \) is absolutely continuous, and hence a.e.-differentiable. By the Gauss Lemma [Kli95, Lem. 1.9.1], the differential of \( \exp_p \) maps \( f'(t)v \) isometrically into \( \sigma'(t) \) (wherever these exist), so \( |\sigma'(t)|_\gamma = |f'(t)||v|_\gamma \). Finally, the formula for the length of \( \sigma \) follows from [Rud87, p. 159]. \( \square \)

The following theorem is the result we need—it is a modification and extension of [Kli95, Thm. 1.9.2].

**Theorem 4.10.** Let \((N, \gamma)\) be a finite-dimensional Riemannian manifold, \( p \in N \), \( v \in T_p N \) nonzero. Let \( \tilde{\rho}(t), \, t \in [0, 1], \) be a differentiable curve in \( T_p M \) from \( 0 \) to \( v \) that lies within the domain of definition of \( \exp_p \) for all \( t \). Assume \( \tilde{\rho}(t) \neq 0 \) for \( t > 0 \) and write \( \tilde{\rho}(t) \) in polar coordinates,

\[
\tilde{\rho}(t) = r(t)w(t), \quad w(t) := \tilde{\rho}(t)/|\tilde{\rho}(t)|_\gamma.
\]

Define

\[
\hat{r}(t) := \min \left( \max_{s \in [0, t]} r(s), 1 \right),
\]

as well as \( \tilde{\sigma}(t) := \tilde{\rho}(t)v/|v|_\gamma \) and \( \tilde{\tau}(t) := tv \) for \( t \in [0, 1] \). Finally, define \( \rho(t) := \exp_p(\tilde{\rho}(t)), \quad \sigma(t) := \exp_p(\tilde{\sigma}(t)), \) and \( \tau(t) := \exp_p(\tilde{\tau}(t)) \).

Then \( \rho, \sigma, \) and \( \tau \) are paths in \( N \) from \( p \) to \( q := \exp_p(v) \). The path \( \sigma \) is differentiable for a.e. \( t \in [0, 1] \), and \( L(\sigma) = \int_0^1 |\sigma'(t)|_\gamma \, dt \). For every \( t \) where this is well defined, we have \( |\sigma'(t)|_\gamma \leq |\rho'(t)|_\gamma \). Furthermore, \( L_\gamma(\tau) = L_\gamma(\sigma) \leq L_\gamma(\rho) \).

Suppose the differential \( D_{\tilde{\rho}(t)} \exp_p \) of the exponential mapping is of maximal rank for all \( s, t \in [0, 1] \). Then if and only if \( \tilde{\rho} \) is a reparametrization of \( \tilde{\tau} \), the following equalities hold: \( L_\gamma(\rho) = L_\gamma(\tau) = L_\gamma(\sigma) \) and \( |\sigma'(t)|_\gamma = |\rho'(t)|_\gamma \) for a.e. \( t \). In particular, if \( \exp_p \) is a diffeomorphism from \( U \subseteq T_p N \) to \( V \subseteq N \), then \( \tau \) is of minimal length among all paths in \( V \) from \( p \) to \( q \), and it is unique (up to reparametrization) with respect to this property.

**Proof.** We first note that by Lemma 4.8, \( \hat{r} \) is absolutely continuous, so Lemma 4.9 applies to give that \( \sigma \) is a.e.-differentiable, \( |\sigma'(t)|_\gamma = |\hat{r}'(t)| \) wherever these are defined, and \( L(\sigma) = \int_0^1 |\sigma'(t)|_\gamma \, dt \).

Let \( \alpha := |v|_\gamma \). For small \( \epsilon > 0 \), we define a map

\[
F : [0, 1] \times (\epsilon, 1] \to N; \quad (s, t) \mapsto \exp_p(asw(t)).
\]
Then we have $\rho(t) = F(r(t)/a, t)$ and $\rho'(t) = r'(t) \frac{\partial F}{\partial s}(\frac{r(t)}{a}, t) + \frac{\partial F}{\partial t}(\frac{r(t)}{a}, t)$. Furthermore, by the Gauss Lemma [Kl95, Lem. 1.9.1], $u_1 := \partial F(s, t)/\partial s$ is orthogonal to $u_2 := \partial F(s, t)/\partial t$, and $|\partial F(s, t)/\partial s|_\gamma = |aw(t)|_\gamma = a$. Therefore, for a.e. $t$,

\begin{equation}
|r'(t)|^2_\gamma = |r'(t)|^2 + \left|u_2 \left(\frac{r(t)}{a}, t\right)\right|^2 \geq |r'(t)|^2 \geq |\rho'(t)|^2 = |\sigma'(t)|^2_\gamma.
\end{equation}

This shows that $L_\gamma(\sigma) \leq L_\gamma(\rho)$. Furthermore, we note that $\sigma$ is just a reparametrization of $\tau$, since $\sigma(t) = \tau(\hat{r}(t)/|v|_\gamma)$, and $\hat{r}$ is a continuous, monotone function. Therefore $L_\gamma(\tau) = L_\gamma(\sigma)$ follows (cf. [BH99, Prop. I.1.20]).

If equality holds in (4.4), then $r'(t) = \hat{r}'(t) \geq 0$. Additionally, we must have

$$0 = \frac{\partial F}{\partial t}(s, t)\bigg|_{(s, t) = (r(t)/r, t)} = D_{rsu(t)}\exp_p(rsw(t))\bigg|_{(s, t) = (r(t)/r, t)}.$$

Since $r(t) \neq 0$ for $t > 0$, if $D_{s\hat{r}(t)}\exp_p$ is of maximal rank for all $s, t \in [0, 1]$, then $u'(t) = 0$. Thus, if equality holds in (4.4) for a.e. $t$, then $w(t) \equiv w(1) = v/|v|_\gamma$. This shows that $\rho$ is a reparametrization of $\tau$.

Finally, using (4.4), we have

\begin{equation}
L_\gamma(\rho) = \int_0^1 |\rho'(t)|_\gamma \, dt \geq \int_0^1 |r'(t)|_\gamma \, dt \geq \int_0^1 r'(t) \, dt = r(1) = L_\gamma(\tau).
\end{equation}

So if $L_\gamma(\tau) = L_\gamma(\rho)$, then by (4.4) and (4.5), $r'(t) \geq 0$ for all $t$, and similarly to the previous paragraph, if $D_{s\hat{r}(t)}\exp_p$ is of maximal rank for all $s, t \in [0, 1]$ we have $u'(t) \equiv 0$, so again $\rho$ is a reparametrization of $\tau$. \hfill $\square$

The above general theorem has the following consequence in our setting. Let $a_t$ be a piecewise differentiable path in $M_x$, and write $a_t = \lambda_t a_0 \exp(a_0^{-1} b_t)$, $t \in [0, 1]$, with $\lambda_t \in \mathbb{R}_+$ and $tr_{a_0} b_t = 0$ for all $t$. (Note that by the discussion following Definition 4.6, this is always possible.) First, we claim that the path $\tilde{a}_t := a_0 \exp(a_0^{-1} b_t)$ is the projection of $a_t$ onto $M_{x, \sqrt{A_0}}$, since

$$\sqrt{A_t} = \sqrt{A_0} \sqrt{\det \exp(a_0^{-1} b_t)} = \sqrt{A_0} \sqrt{\exp(tr_{a_0} b_t)} = \sqrt{A_0}.$$

Write $b_t$ in polar coordinates with respect to $\langle \cdot, \cdot \rangle_{\tilde{a}_t}$; that is,

$$b_t = \beta_t c_t, \quad c_t := \frac{b_t}{|b_t|_{\tilde{a}_t}}.$$

As in Theorem 4.10 define

$$\beta_t := \min \left(\max \{\beta_t, 1\} \right)$$

and $\hat{a}_t := \exp^0_0(\beta_t b_1)$, where, as in Theorem 4.7, $\exp^0$ denotes the exponential mapping of $M_{x, \sqrt{A_0}}$. With this notation, we have the following.

**Lemma 4.11.** Define $\tilde{a}_t := \lambda_t \hat{a}_t$. Then $\tilde{a}_t$ is a rectifiable path from $a_0$ to $a_1$, and $L(\tilde{a}_t) \leq L(a_t)$, with equality if and only if $\tilde{a}_t$ is a reparametrization of the radial geodesic (of $M_{x, \sqrt{A_0}}$), $a_0^0 := \exp^0_0(t b_1)$. 

Proof. Since \( \tilde{a}_t \) and \( \hat{a}_t \) are the projections onto \( M_{x, \sqrt{x_0}} \) of \( a_t \) and \( \tilde{a}_t \), respectively, we have
\[
(4.6) \quad \text{tr}_{a_t}(\bar{a}_t') = \lambda_t^{-2} \text{tr}_{\tilde{a}_t}(\tilde{a}_t') = 0,
\]
and similarly for \( \hat{a}_t \).

Now, note that \( \tilde{a}_t = \exp_{a_0}(b_t) \) and \( \hat{a}_t = \exp_{a_0}(\hat{b}_t) \). Thus, the hypotheses of Theorem \ref{minimality} are satisfied with \( N = M_{x, \sqrt{x_0}}, v = b_1, \rho(t) = \tilde{a}_t, \) and \( \sigma(t) = \hat{a}_t \). In particular, we have \( |\bar{a}_t|_{\tilde{a}_t} \leq |\tilde{a}_t|_{\tilde{a}_t} \) for all \( t \), with equality if and only if \( \tilde{a}_t \) is a reparametrization of \( a_t^0 \).

Let us again consider the paths \( a_t = \lambda_t \tilde{a}_t \) and \( \hat{a}_t = \lambda_t \hat{a}_t \). We have
\[
a_t' = \lambda_t' \tilde{a}_t + \lambda_t \tilde{a}_t', \quad \bar{a}_t' = \lambda_t' \tilde{a}_t + \lambda_t \tilde{a}_t',
\]
and similarly \( \tilde{a}_t' = (\lambda_t'/\lambda_t)\tilde{a}_t + \lambda_t \tilde{a}_t'. \) By (4.6) and the discussion following Definition 4.6 these decompositions are orthogonal, since the first term in each is pure-trace and the second is traceless. Thus we have
\[
|a_t|^2_{a_t} = \left[ \text{tr}_{a_t} \left( \left( \frac{\lambda_t'}{\lambda_t} a_t \right)^2 \right) + \lambda_t \text{tr}_{\tilde{a}_t}((\lambda_t \tilde{a}_t')^2) \right] \sqrt{\lambda_t \bar{A}_t} = \lambda_t^{n/2} \left[ n \left( \frac{\lambda_t'}{\lambda_t} \right)^2 + |\tilde{a}_t'|_{\tilde{a}_t}^2 \right] \sqrt{\bar{A}_0},
\]
and since \( |\bar{a}_t|_{\bar{a}_t} \leq |\tilde{a}_t|_{\tilde{a}_t} \), with equality if and only if \( \tilde{a}_t \) is a reparametrization of \( a_t^0 \),
\[
|\tilde{a}_t|_{\tilde{a}_t}^2 = \lambda_t^{-n/2} \left[ n \left( \frac{\lambda_t'}{\lambda_t} \right)^2 + |\tilde{a}_t'|_{\tilde{a}_t}^2 \right] \sqrt{\bar{A}_0} \leq |a_t|^2_{a_t}.
\]
The result of the lemma now follows.

\( \Box \)

4.3.2. Minimality of Riemannian geodesics. The results of the last subsection allow us to prove, with one additional estimate, the minimality of Riemannian geodesics in \( M_x \).

Theorem 4.12. Let \( a_0, a_1 \in M_x \). Suppose that \( a_1 = a_0 \exp(a_0^{-1}b_1) \), where \( \text{tr}_{a_0}(b_1^2) < (4\pi)^2/n \). Let \( a_t \) be the Riemannian geodesic between \( a_0 \) and \( a_1 \) as given in Theorems \ref{minimality} and \ref{minimality}. Then \( a_t \) is the unique minimal path (up to reparametrization) in \( M_x \) between \( a_0 \) and \( a_1 \).

Proof. Since, by Theorem 4.5, \( \exp_{a_0} \) is a diffeomorphism onto its image, Theorem \ref{minimality} implies that \( a_t \) is the unique minimal path among the class of paths that lie completely within the image of \( \exp_{a_0} \). Therefore, we must show that there are no shorter paths that exit \( V := \text{im}(\exp_{a_0}) \).

By Proposition 4.3 and Theorem 4.5, the boundary of \( V \) in \( M_x \) is
\[
\partial V = \{ [0] \} \cup \left\{ c \in M_x \mid c = a_0 \exp(a_0^{-1} \gamma), \text{ tr}_{a_0}(\gamma^2_T) = \frac{(4\pi)^2}{n} \right\}.
\]

So first, suppose that there exists a path \( \alpha_t = \lambda_t a_0 \exp(a_0^{-1}c_t) \) in \( M_x \) between \( a_0 \) and \( a_1 \) with \( L(\alpha_t) \leq L(a_t) \), \( \text{tr}_{a_0}(c_t) \equiv 0 \), \( \lambda_t \in \mathbb{R} \), and \( \text{tr}_{a_0}(c_t^2) \geq (4\pi)^2/n \) for some \( t_0 \in (0, 1) \). Since \( \text{tr}_{a_0}(b_1^2) < (4\pi)^2/n \), we may deduce from Lemma 4.11 that there exists a path \( \tilde{a}_t \) lying completely in \( V \) with \( L(\tilde{a}_t) < L(\alpha_t) \leq L(a_t) \), a contradiction to the minimality of \( a_t \) among paths lying in \( V \).

Now, consider any path \( \alpha_t \) from \( a_0 \) to \( a_1 \) that passes through \([0]\). By Proposition 4.3 we have that \( d_x[a_0, [0]] = \frac{4}{\sqrt{n}} \sqrt{\bar{A}_0} \), and similarly for \( a_1 \). Therefore \( L(a_t) \geq \frac{4}{\sqrt{n}} \sqrt{\bar{A}_0 + \sqrt{\bar{A}_1}} \).

Thus, if we can show that \( L(a_t) < \frac{4}{\sqrt{n}} (\sqrt{\text{tr}_a(a_0)} + \sqrt{\text{tr}_a(a_1)}) \), the theorem will be proved.
To show this, let $\psi$ be as in Theorem 4.5 so that $a_1 = \exp_{a_0}(\psi(b))$. Since the case $b_T = 0$ is trivial, we simply estimate $L(a_t) = |\psi(b)|_{a_0}$ for $b_T \neq 0$. We first have

$$
\text{tr}_{a_0}(\psi(b)^2) = \frac{16}{n^2} \left[ \exp \left( \frac{\text{tr}_{a_0} b}{2} \right) \cos^2 \left( \frac{\sqrt{n \text{tr}_{a_0}(b_T^2)}}{4} \right) - 2 \exp \left( \frac{\text{tr}_{a_0} b}{4} \right) \cos \left( \frac{\sqrt{n \text{tr}_{a_0}(b_T^2)}}{4} \right) + 1 \right] \text{tr}_{a_0}(a_0^2)
$$

(4.8)

$$
+ \frac{16}{n \text{tr}_{a_0}(b_T^2)} \exp \left( \frac{\text{tr}_{a_0} b}{2} \right) \sin^2 \left( \frac{\sqrt{n \text{tr}_{a_0}(b_T^2)}}{4} \right) \text{tr}_{a_0}(b_T^2)
$$

$$
= \frac{16}{n} \left( \exp \left( \frac{\text{tr}_{a_0} b}{2} \right) - 2 \exp \left( \frac{\text{tr}_{a_0} b}{4} \right) \cos \left( \frac{\sqrt{n \text{tr}_{a_0}(b_T^2)}}{4} \right) + 1 \right).
$$

On the other hand, using the formula $\exp(\text{tr}_{a_0}(b)) = \text{det}(\exp(a_0^{-1}b))$, we have $\exp(\frac{\text{tr}_{a_0} b}{2}) = \sqrt{\text{det} \exp(a_0^{-1}b)} = \frac{\sqrt{A_1}}{\sqrt{A_0}}$. Therefore,

$$
|\psi(b)|_{a_0}^2 = \text{tr}_{a_0}(\psi(b)^2) \sqrt{A_0}
$$

$$
= \frac{16}{n} \left( \frac{\sqrt{A_1}}{\sqrt{A_0}} - 2 \frac{\sqrt{A_1}}{\sqrt{A_0}} \cos \left( \frac{\sqrt{n \text{tr}_{a_0}(b_T^2)}}{4} \right) + 1 \right) \sqrt{A_0}
$$

(4.9)

$$
= \frac{16}{n} \left( \sqrt{A_0} - 2 \sqrt{A_0} \sqrt{A_1} \cos \left( \frac{\sqrt{n \text{tr}_{a_0}(b_T^2)}}{4} \right) + \sqrt{A_1} \right).
$$

Since $\text{tr}_{a_0}(b_T^2) < (4\pi)^2/n$, the argument of cosine in the above equation lies strictly between 0 and $\pi$, and therefore we can estimate

$$
|\psi(b)|_{a_0}^2 < \frac{16}{n} \left( \sqrt{A_0} + \sqrt{A_1} \right)^2,
$$

(4.10)

as was to be shown. \qed

4.3.3. Geodesics in $\overline{M}_x$. Theorem 4.12 will now allow us to determine that, for any element $a \in M_x$, the singular point $[0]$ is the unique closest point to $a$ on the boundary of the image of $\exp_a$. This will help to find geodesics between points that do not have a Riemannian geodesic connecting them.

**Lemma 4.13.** Let $a_0 \in M_x$, and let $a_1 \in \partial V \subset \overline{M}_x$, where $V$ denotes the image of $\exp_{a_0}$ (cf. Theorems 4.5 and 4.7). Then

$$
d_x(a_0, a_1) = \frac{4}{\sqrt{n}} \left( \sqrt{A_1} + \sqrt{A_0} \right).
$$

(4.11)

In particular, $d_x(a_0, \partial V) = \frac{4}{\sqrt{n}} \sqrt{A_0}$, and this distance is realized uniquely for $[0] \in \partial V$.

**Proof.** If $a_1 = [0]$, then by Proposition 4.3 $d_x(a_0, a_1) = \frac{4}{\sqrt{n}} \sqrt{A_0}$. Thus (4.11) is proved in this case.

If $a_1 \neq [0]$, then we write $a_1 = a_0 \exp(a_0^{-1}b)$ with $\text{tr}_{a_0}(b_T^2) = (4\pi)^2/n$. Let $c_k = a_0 \exp(a_0^{-1}b_k)$ be a sequence of elements of $V$ with $d_x(c_k, a_1) \rightarrow 0$; in particular, $d_x(a_0, c_k) \rightarrow d_x(a_0, a_1)$. By
Proposition 4.3. \(\{c_k\}\) (resp. \(\{b_k\}\)) converges in the standard topology on \(\mathcal{M}_x\) to \(a_1\) (resp. \(b\)). In particular, \(\text{tr}_{a_0}(b_k^2) \to (4\pi)^2/n\). Since \(c_k \in V\), we may conclude
\[
d_x(a_0, a_1)^2 = \lim_{k \to \infty} d_x(a_0, c_k)^2 = \lim_{k \to \infty} |\psi(b_k)|^2_{a_0} = \frac{16}{n} \left( \sqrt{A_1} + \sqrt{A_0} \right)^2,
\]
where we have used (4.9) in the last equality.

We are now in a position to prove the main result of this section.

**Theorem 4.14.** There exists a unique (up to reparametrization) minimal path between any two points \(a_0, a_1 \in \overline{\mathcal{M}}_x\). If there exists \(b \in S_x\) such that \(a_1 = a_0 \exp(a_0^{-1}b)\), then this minimal path is given by

1. the Riemannian geodesic connecting \(a_0\) and \(a_1\) (cf. Theorems 4.4 and 4.5), if we have \(\text{tr}_{a_0}(b^2) < (4\pi)^2/n\);
2. the concatenation of the straight segments from \(a_0\) to \([0]\) and from \([0]\) to \(a_1\), if we have \(\text{tr}_{a_0}(b^2) \geq (4\pi)^2/n\).

Otherwise either \(a_0\) or \(a_1\) is \([0]\), and the minimal path is the straight segment between the two.

**Proof.** We have already shown statement [1] in Theorem 4.12.

Let us now show that the unique minimal path between \(a_0 \in \mathcal{M}_x\) and \([0]\) is the straight segment \(a_t = ta_0\). In the discussion preceding Lemma 4.2, we remarked that this segment is minimal, so it remains to show uniqueness. Let \(\tilde{a}_t = a_0 \exp(a_0^{-1}b_t)\), \(t \in [0, 1]\), be a differentiable path with \(\text{tr}_{a_0} b_t = 0\) for all \(t\). Furthermore, let \(\lambda_t\) be a differentiable family of nonnegative real numbers with \(\lambda_1 = 0\), and define a path \(\tilde{a}_t := \lambda_t \tilde{a}_t\) from \(a_0\) to \([0]\). (Note that any path from \(a_0\) to \([0]\) can be written in this way.) As in the proof of Lemma 4.11 we have
\[
|\tilde{a}_t'|_{a_t}^2 = \left[ n \left( \frac{\lambda_t'}{\lambda_t} \right)^2 + |\tilde{a}_t'|_{\tilde{a}_t}^2 \right].
\]
This quantity is minimized when (i) \(|\tilde{a}_t'|_{a_t} = 0\), i.e., when \(b_t = 0\), for all \(t\), and (ii) \(\lambda_t' \leq 0\) for all \(t\). But then \(\tilde{a}_t\) is simply a reparametrization of \(a_t = ta_0\), as desired.

Finally, to show [2], we note that any path from \(a_0\) to \(a_1\) must necessarily pass through \(\partial V\) (cf. (1.7)). By the above argument and Lemma 4.13 the unique minimal path between \(a_0\) and \(\partial V\) is the straight segment between \(a_0\) and \([0]\). If additionally \(W := \text{im}(\exp_{a_1})\), then we also see that any path from \(a_1\) to \(a_0\) must pass through \(\partial W \subset \overline{\mathcal{M}}_x\). Since the unique minimal path from \(a_1\) to \(\partial W\) is the straight segment from \(a_1\) to \([0]\), and the concatenation of these straight segments is continuous and rectifiable, this concatenation is the unique minimal path connecting \(a_0\) and \(a_1\).

In the two following subsections, we work out the consequences of this theorem: the continuous dependence of geodesics in \(\overline{\mathcal{M}}_x\) on their endpoints, and formulas for geodesics and distance between elements of the completion \(\overline{\mathcal{M}}\) of the manifold of Riemannian metrics.

4.4. **Continuous dependence of geodesics.** The goal of this subsection is to prove the following theorem, which we will require to show the CAT(0) property for \(\overline{\mathcal{M}}_x\).

**Theorem 4.15.** Geodesics in \(\overline{\mathcal{M}}_x\) vary continuously with their endpoints.
Continuous dependence means the following: Let \( \alpha_k, \beta_k, \alpha, \beta \in \mathcal{M}_x \), with \( \alpha_k \xrightarrow{d} \alpha \) and \( \beta_k \xrightarrow{d} \beta \). Let \( \gamma_k, \gamma : [0,1] \to \mathcal{M}_x \) be the geodesics, parametrized proportionally to arc length, connecting \( \alpha_k \) to \( \beta_k \) and \( \alpha \) to \( \beta \), respectively. Then \( \gamma_k \xrightarrow{d} \gamma \) in \( C^0([0,1], \mathcal{M}_x) \), where the metric on \( \mathcal{M}_x \) used to define \( C^0([0,1], \mathcal{M}_x) \) is, of course, \( d_c \). Furthermore, considering the mapping \( \mathcal{M}_x \times \mathcal{M}_x \to C^0([0,1], \mathcal{M}_x) \) given by mapping two points to the geodesic between them, one sees that it suffices to prove continuity when \( \alpha_k \) is fixed, since this mapping is continuous if and only if it is continuous on each factor of the domain.

The remainder of this subsection will constitute the proof of Theorem 4.15. Let \( a_0, a_1 \in \mathcal{M}_x \) be given arbitrarily. We must show that if \( \{a_k\} \) is any sequence with \( d_x(a_k, a_1) \to 0 \), then the geodesics \( a_{k,t} \) connecting \( a_0 \) and \( a_k \) converge in the \( C^0 \) topology to the geodesic \( a_t \) connecting \( a_0 \) and \( a_1 \). We will thus use the description of geodesics in Theorem 4.14. Note also that by Proposition 4.3, if \( a_1 \neq [0] \), then \( \{a_k\} \) converges in the standard topology on \( \mathcal{M}_x \).

We will adopt the convention that if \( a_k \neq [0] \), then we write \( a_k = a_0 \exp(a_0^{-1}b_k) \) for some \( b_k \in \mathcal{S}_x \). Furthermore, let \( c_k := b_k - \frac{1}{n} \text{tr}_{a_0}(b_k) \) be the \( a_0 \)-traceless part of \( b_k \). Similarly, if \( a_1 \neq [0] \), we may write \( a_1 = a_0 \exp(a_0^{-1}c) \), with \( c \) the \( a_0 \)-traceless part of \( b \).

We now break the proof into several cases.

4.4.1. The case \( a_1 = [0] \). In this case, assume that \( a_k \neq [0] \) for \( k \) large enough, since otherwise the desired result is trivial. Then by Proposition 4.3, \( \sqrt{A_k} \to 0 \), so \( \exp \left( \frac{1}{4} \text{tr}_{a_0}(b_k) \right) \to 0 \) as well, since \( \sqrt{A_k} = \sqrt{A_0} \left( \det \exp(a_0^{-1}b_k) \right)^{1/4} = \sqrt{A_0} \exp \left( \frac{1}{4} \text{tr}_{a_0}(b_k) \right) \). Thus, we first see by Lemma 4.2 that if \( \text{tr}_{a_0}(c_k^2) \geq (4\pi)^2/n \) and \( a_k \) is \( d_x \)-close to \( [0] \), then \( a_{k,t} \) —the concatenation of the straight segments from \( a_0 \) to \( [0] \) and \( [0] \) to \( a_k \) —is close to \( a_t \)—the straight segment from \( a_0 \) to \( [0] \). Second, if \( \text{tr}_{a_0}(c_k^2) < (4\pi)^2/n \) and \( a_k \) is close to \( [0] \), then by (4.3), \( \exp_{a_0}^{-1}(b_k) \) is close to \(-\frac{1}{n}a_0 \). Since Riemannian geodesics vary continuously with their initial tangent vector, we thus have that \( a_{k,t} \) is \( C^0 \)-close to \( a_t = \exp_{a_0}(-\frac{1}{n}ta_0) \), which is nothing but a certain parametrization of the straight segment between \( a_0 \) and \( a_1 \). The statement thus follows for \( a_1 = [0] \).

4.4.2. The case \( a_1 \neq [0], \text{tr}_{a_0}(c^2) < (4\pi)^2/n \). In this case (and all subsequent cases), for \( k \) large enough, we must have \( a_k \neq [0] \). Thus, for \( k \) large enough, we can say \( \text{tr}_{a_0}(c_k^2) < (4\pi)^2/n \) as well, and so \( a_1 \) (resp. \( a_{k,t} \)) is the Riemannian geodesic connecting \( a_0 \) and \( a_1 \) (resp. \( a_k \)). Since Riemannian geodesics vary continuously with their endpoints, the statement holds in this case.

4.4.3. The case \( a_1 \neq [0], \text{tr}_{a_0}(c^2) > (4\pi)^2/n \). Here again, for \( k \) large enough, \( \text{tr}_{a_0}(c_k^2) > (4\pi)^2/n \) as well. Since the straight segments between \( a_k \) and \( [0] \) converge to the straight segment between \( a_1 \) and \( [0] \), the statement also holds here.

4.4.4. The case \( a_1 \neq [0], \text{tr}_{a_0}(c^2) = (4\pi)^2/n \). This final case is the most involved, and requires some analysis of the limiting behavior of Riemannian geodesics.

First, we note that \( a_t \) is the concatenation of the straight segments from \( a_0 \) to \( [0] \) and \( [0] \) to \( a_1 \). Therefore, if \( a_k \) is close to \( a_1 \) and \( \text{tr}_{a_0}(c_k^2) \geq (4\pi)^2/n \), then the geodesic between \( a_0 \) and \( a_k \) is \( C^0 \)-close to the geodesic between \( a_0 \) and \( a_1 \), by the same argument as in the last case.
Therefore, we must only worry about the case that \( \text{tr}_{a_0}(c_k^2) < (4\pi)^2/n \). For simplicity, we assume that this holds for all \( k \) large enough, and the general case follows by combining the previous paragraph with the following argument.

Let \( \tilde{a}_{k,t} := a_{k,1-\cdot} \) be the geodesic from \( a_k \) to \( a_0 \). It suffices to show that for all \( \delta, \epsilon > 0 \), there exist \( \rho, \sigma < 0 \) such that for \( k \) large enough, we have \( \min_{t \in [0,1]} \sqrt{A_{k,t}} \leq \delta, \left| a_{k,t}' - \rho a_0 \right|_{a_0} < \epsilon, \) and \( \left| \tilde{a}_{k,0} - \sigma a_k \right|_{a_k} < \epsilon \). For in this case, by Theorem 4.4 and the continuous dependence of Riemannian geodesics on initial data, up to the first time \( t_0 \) (resp. \( t_1 \)) when \( \sqrt{A_{k,t}} = \delta \) (resp. \( \sqrt{A_{k,t}} = \delta \)), we will have that \( a_{k,t} \) is arbitrarily \( C^0 \)-close to the straight segment between \( a_0 \) (resp. \( a_k \)) and \([0]\). Furthermore, we can make \( k \) large enough that \( \sqrt{A_{t_0}}, \sqrt{A_{t_1}} \leq 2\delta \). Thus, by Lemma 4.2 for \( t \in (t_0, t_1) \) we have

\[
d_x(a_{k,t}, a_t) \leq d_x(a_{k,t}, [0]) + d_x([0], a_{k,t_0}) + d_x(a_{k,t_0}, a_{t_0}) + d_x(a_{t_0}, [0]) + d_x([0], a_t)
\]

\[
< \frac{4}{\sqrt{n}} \delta + \frac{4}{\sqrt{n}} \delta + d_x(a_{k,t_0}, a_{t_0}) + 2 \cdot \frac{4}{\sqrt{n}} \delta + 2 \cdot \frac{4}{\sqrt{n}} \delta,
\]

which is arbitrarily small. (Recall from Theorem 4.4 that the change in the volume element is quadratic, so we have \( \sqrt{A_{k,t}} \leq \delta \) if \( t \in (t_0, t_1) \). Also, it is clear from the form of \( a_t \) that \( \sqrt{A_t} < 2\delta \) for \( t \in (t_0, t_1) \).

We first show the statement about \( a_{k,0}' \). To start, note that \( b_k \to b \), and in particular \( \text{tr}_{a_0}(b_k) \to \text{tr}_{a_0} b \) and \( \text{tr}_{a_0}(c_k^2) \to \text{tr}_{a_0}(c^2) = (4\pi)^2/n \). From (4.3), we see that

\[
a_{k,0}' = \exp_{a_0}^{-1}(a_k) = \frac{4}{n} \left( \exp \left( \frac{\text{tr}_{a_0} b_k}{4} \right) \cos \left( \frac{\sqrt{n} \text{tr}_{a_0}(c_k^2)}{4} \right) - 1 \right) a_0
\]

\[
+ \frac{4}{\sqrt{n} \text{tr}_{a_0}(c_k^2)} \exp \left( \frac{\text{tr}_{a_0} b_k}{4} \right) \sin \left( \frac{\sqrt{n} \text{tr}_{a_0}(c_k^2)}{4} \right) c_k.
\]

By the above arguments, the factors of cosine and sine in (4.12) converge to \(-1 \) and \( 0 \), respectively. Thus, \( a_{k,0}' \to \rho a_0 \) for some \( \rho < 0 \), as desired.

Similarly, note that \( a_0 = a_k \exp(-a_0^{-1}b_k) = a_k \exp(a_k^{-1}(-a_k a_0^{-1}b_k)) \). We also have

\[
\text{tr}_{a_k}(-a_k a_0^{-1}b_k) = -\text{tr}_{a_0}(b_k), \quad \text{tr}_{a_k}(-a_k a_0^{-1}c_k) = \text{tr}_{a_0}(-c_k) = 0,
\]

\[
\text{tr}_{a_k}((-a_k a_0^{-1}c_k)^2) = \text{tr}_{a_0}(c_k^2).
\]

Thus, in this case, (4.3) gives

\[
\tilde{a}_{k,0}' = \exp_{a_k}^{-1}(a_0) = \frac{4}{n} \left( \exp \left( - \frac{\text{tr}_{a_0} b_k}{4} \right) \cos \left( \frac{\sqrt{n} \text{tr}_{a_0}(c_k^2)}{4} \right) - 1 \right) a_k
\]

\[
- \frac{4}{\sqrt{n} \text{tr}_{a_0}(c_k^2)} \exp \left( - \frac{\text{tr}_{a_0} b_k}{4} \right) \sin \left( \frac{\sqrt{n} \text{tr}_{a_0}(c_k^2)}{4} \right) a_k a_0^{-1} c_k.
\]

The same argument as for \( a_{k,0}' \) shows that there exists \( \sigma < 0 \) for which \( \tilde{a}_{k,0}' \) is arbitrarily close to \( \sigma a_k \) for \( k \) large enough.

To conclude the proof of this case, we show the statement about \( \sqrt{A_{k,t}} \). Recall from Theorem 4.4 that \( \sqrt{A_{k,t}} = (q_k(t)^2 + r_k(t)^2) \sqrt{A_0} \), where \( q_k(t) = 1 + \frac{t}{4} \text{tr}_{a_0}(a_{k,0}') \) and \( r_k(t) = \frac{t}{4} \sqrt{n} \text{tr}_{a_0}((a_{k,0}')^2) \). Note that \( \min_{t \in [0,1]} q_k(t)^2 = 0 \) for all \( k \), and that by the above arguments,
tr_{a_k}(a_{k,0}^2) \to 0 \text{ for } k \to \infty. \text{ These facts combine to show that } \min_{t \in [0,1]} \sqrt{A_{k,t}} \leq \delta \text{ for } k \text{ large enough.}

Thus, Theorem 4.13 is proved.

4.5. Geodesics in \( \mathcal{M} \). With minimal paths in \((\mathcal{M}_x,d_x)\) determined, we can explicitly determine \(d_x\) and thus, by Theorem 3.9 \(d\). Furthermore, since there exists a unique minimal path between any two elements in \(\mathcal{M}_x\), one sees that there is a unique minimal path between any two elements \(g_0,g_1 \in \mathcal{M}\): the path \(g_t\) that gives the minimal path \(g_t(x)\) between \(g_0(x)\) and \(g_1(x)\) for each \(x \in M\). We thus are now able to combine these results into a theorem for \(\mathcal{M}\), as well as summarize the explicit realizations of geodesics and distance that we have determined up to this point, reformulating them for \(\mathcal{M}\). (Note that to get the statement of the theorem we use, in addition to the theorems of the preceding subsections, the formula (4.9).)

**Theorem 4.16.** There exists a unique minimal path \(g_t, t \in [0,1]\), between any two points \(g_0,g_1 \in \mathcal{M}\), given by the following. Let \(k\) be a measurable, symmetric \((0,2)\)-tensor field on \(M\), defined on the subset \(N\) where neither \(g_0\) nor \(g_1\) is zero, such that \(g_1 = g_0 \exp(g_0^{-1}k)\) on this subset. Denote by \(P\) the subset of \(M\) where \(\text{tr}_{g_0}(k_T^2) < (4\pi)^2/n\) (here \(k_T\) denotes the traceless part of \(k\)). Write \(h := \psi(k)\), where \(\psi(k)(x)\) is given as in Theorem 4.5. Finally, let \(q_t\) and \(r_t\) be one-parameter families of functions on \(N\) given by

\[
q_t := 1 + \frac{t}{4} \text{tr}_{g_0}(h), \quad r_t := \frac{t}{4} \sqrt{n \text{tr}_{g_0}(h_T^2)}.
\]

Then at points \(x \in N \cap P\) where \(h_T(x) \neq 0\), we have

\[
g_t(x) = (q_t^2(x) + r_t^2(x))^\frac{2}{n} g_0(x) \exp\left(\frac{4}{\sqrt{n \text{tr}_{g_0}(h_T^2)}} \arctan\left(\frac{r_t(x)}{q_t(x)}\right) g_0^{-1}(x) h_T(x)\right).
\]

At points \(x \in N \cap P\) where \(h_T(x) = 0\), we have

\[
g_t(x) = q_t^{4/n} g_0(x) = \left(1 + \frac{\sqrt[4]{G_1} - \sqrt[4]{G_0}}{\sqrt[4]{G_0}} t\right) g_0(x)
\]

In both of the above cases \(g_t(x)\) does not intersect \([0]\). Additionally, the range of \(\arctan\) is given by the following. At a point where \(\text{tr}_{g_0} h \geq 0\), it assumes values in \((-\frac{\pi}{2}, \frac{\pi}{2})\). At a point where \(\text{tr}_{g_0} h < 0\), \(\arctan(r(t)/q(t))\) assumes values as follows, with \(t_0 := -\frac{4}{\text{tr}_{g_0} h}\):

1. in \([0, \frac{\pi}{2})\) if \(0 \leq t < t_0\),
2. in \((\frac{\pi}{2}, \pi)\) if \(t_0 < t < \infty\),

and we set \(\arctan(r(t)/q(t)) = \frac{\pi}{2}\) if \(t = t_0\).

At all other points of \(M\), \(g_t(x)\) passes through \([0]\), and we have

\[
g_t(x) = \begin{cases} 
1 - \frac{4}{G_0(x)} + \frac{4}{G_1(x)} t \frac{1}{G_0(x)} g_0(x), & t \in \left[0, \frac{4}{G_0(x) + \sqrt[4]{G_1(x)}}\right], \\
\frac{4}{G_0(x) + \sqrt[4]{G_1(x)}} t - \frac{4}{G_0(x) + \sqrt[4]{G_1(x)}} g_1(x), & t \in \left[\frac{4}{G_0(x) + \sqrt[4]{G_1(x)}}, 1\right].
\end{cases}
\]
The distance induced by the $L^2$ Riemannian metric between $g_0$ and $g_1$ is given by

$$d(g_0, g_1) = \Omega_2(g_0, g_1) = \left( \int_M d_x(g_0(x), g_1(x))^2 d\mu \right)^{1/2},$$

with

$$d_x(g_0(x), g_1(x)) = \begin{cases} 
|\psi(k)(x)|^2_{g_0(x)}, & x \in N \cap P, \ k_T \neq 0, \\
\frac{4}{\sqrt{n}} \left( \sqrt{G_1(x)} - \sqrt{G_0(x)} \right), & x \in N \cap P, \ k_T = 0, \\
\frac{4}{\sqrt{n}} \left( \sqrt{G_0(x)} + \sqrt{G_1(x)} \right), & x \notin N \cap P.
\end{cases}$$

Here, $|\psi(k)(x)|^2_{g_0(x)}$, for $k_T \neq 0$, is given explicitly by

$$|\psi(k)(x)|^2_{g_0(x)} = \frac{4}{\sqrt{n}} \left( \sqrt{G_0(x)} - 2 \sqrt{G_0(x)} \sqrt{G_1(x)} \cos \left( \frac{\sqrt{n} \text{tr}_{g_0}(k_T(x)^2)}{4} \right) + \sqrt{G_1(x)} \right)^{1/2}$$

As noted in the introduction, the existence of geodesics in $\overline{M}$ is in stark contrast to the situation for the incomplete space $M$, where the image of the geodesic mapping at a point contains no open $d$-ball.

5. The CAT(0) property

In this section, we show the CAT(0) property for $(\overline{M}, d)$. We will require all of the main results that we have shown so far.

The strategy is to first show that $(\overline{M}, d)$ is CAT(0), which will follow from several general theorems on CAT(0) spaces, as well as a further technical lemma using the results of the last section to show that the interiors of certain geodesic triangles in $\overline{M}_x$ are totally geodesic.

Following that, we will apply a concise argument to show that the formula $d = \Omega_2$ implies that $\overline{M}$ is a CAT(0) space if $\overline{M}_x$ is for each $x \in M$. This will give the main result.

5.1. Background and notation. We introduce the following conventions and notation in this section. For any $a, b \in M_x$, denote by $v_{a,b}$ the unique element of $T_aM_x \cong S_x$ such that $b = a \exp(a^{-1}v_{a,b})$. Let $u_{a,b} := v_{a,b} - \frac{1}{n}(\text{tr}_a v_{a,b})a$ denote the traceless part of $v_{a,b}$.

If $a, b \in \overline{M}_x$, then we denote by $[a, b]$ the unique (unparametrized) geodesic connecting $a$ and $b$.

Let us now discuss some facts from metric geometry, in particular the definition of a CAT(0) space. Our main reference here is [BH99].

Let $(X, \delta)$ be a path metric space. Let $x, y, z \in X$, and let $\Delta xyz$ be a geodesic triangle, that is, a triangle composed of geodesics (globally distance-minimizing paths) $[x, y]$, $[y, z]$, and $[z, x]$. A comparison triangle $\Delta \tilde{x}\tilde{y}\tilde{z}$ for $\Delta xyz$ is a triangle in Euclidean space, $\mathbb{E}^2$, with side lengths equal to the side lengths of $\Delta xyz$. The points $\tilde{x}$, $\tilde{y}$, and $\tilde{z}$ are called comparison points. More generally, for any $w \in \Delta xyz$—say $w \in [x, y]$—a comparison point $\tilde{w} \in \Delta \tilde{x}\tilde{y}\tilde{z}$ for $w$ is the unique point on $[\tilde{x}, \tilde{y}]$ with $|\tilde{w} - \tilde{x}| = \delta(x, w)$ (and consequently, $|\tilde{w} - \tilde{y}| = \delta(y, w)$).

We say that $X$ is CAT(0) if two conditions hold. First, we require that there exists a geodesic (minimal path) between any two points in $X$. Second, if $x, y, z \in X$ and $\Delta xyz$ is a geodesic triangle, then for any $w_1, w_2 \in \Delta xyz$ with comparison points $\tilde{w}_1, \tilde{w}_2 \in \Delta \tilde{x}\tilde{y}\tilde{z}$, we have

$$d(w_1, w_2) \leq \delta(\tilde{w}_1, \tilde{w}_2).$$
That is, triangles in $X$ are “no thicker” than in Euclidean space.

The space $X$ is said to have nonpositive curvature if it is locally a CAT(0) space, that is, if for each $x \in X$ there exists an open metric ball $B(x, r_x)$, $r_x > 0$, such that $B(x, r_x)$ with the induced metric is a CAT(0) space.

There is also a characterization, due to Alexandrov, of CAT(0) spaces in terms of angles. Let $x \in X$, and let $\gamma_1, \gamma_2 : [0,1] \to X$ be geodesics, parametrized proportional to arc length, with $\gamma_1(0) = x = \gamma_2(0)$. Define the Alexandrov (or upper) angle between $\gamma_1$ and $\gamma_2$ to be

$$\angle(\gamma_1, \gamma_2) := \limsup_{s,t \to 0} \overline{Z}_x(\gamma_1(s), \gamma_2(t)),$$

where $\overline{Z}_x(\gamma_1(s), \gamma_2(t))$ denotes the angle in a comparison triangle for $\triangle x \gamma_1(s) \gamma_2(t)$ at the vertex corresponding to $x$. Similarly, if $\triangle xyz \subseteq X$ is a geodesic triangle, then the Alexandrov angle at the vertex $x$ is the Alexandrov angle, as defined above, between the geodesics $[x, y]$ and $[x, z]$.

We then have the following alternative characterization of CAT(0) spaces.

**Theorem 5.1** ([BH99, Thm. II.1.7]). Let $(X, \delta)$ be a metric space such that geodesics exist between any two points. Then the following are equivalent:

1. $X$ is a CAT(0) space.
2. The Alexandrov angle at any vertex of any geodesic triangle in $X$ with distinct vertices is no greater than the angle at the corresponding vertex of a comparison triangle in $\mathbb{E}^2$.

It is this second characterization of CAT(0) spaces that we will use to show that $\overline{M}_x$ is CAT(0). Roughly, it says that a space is CAT(0) if geodesics diverge at least as quickly as lines in $\mathbb{E}^2$.

We will also need a technical lemma that says that if two triangles have Alexandrov angles no greater than a comparison triangle, and they can be glued along one side to form a larger triangle, then this larger triangle also has Alexandrov angles no greater than a comparison triangle.

**Lemma 5.2** (Gluing Lemma for Triangles [BH99, Lem. II.4.10]). Let $X$ be a metric space in which every pair of points can be connected by a geodesic. Let $\triangle pq_1q_2$ be a triangle in $X$ with distinct vertices, and let $r \in [q_1, q_2]$ be distinct from $q_1$ and $q_2$.

If, for both $i = 1, 2$, the angles of $\triangle pqr$ are no greater than the corresponding angles of a comparison triangle in $\mathbb{E}^2$, then the angles of $\triangle pq_1q_2$ are also no greater than those of a comparison triangle in $\mathbb{E}^2$.

To conclude the background we need, we note that if $a \in M_x$, then the Alexandrov and Riemannian angles between geodesic rays based at $a$ coincide [BH99, Cor. II.1A.7]. Therefore, in this case, we shall simply refer to the angle between two geodesics.

### 5.2. The CAT(0) property for $(\overline{M}_x, d_x)$

We first wish to show that $(\overline{M}_x, d_x)$ is a space of nonpositive curvature in the sense of Alexandrov. Since the sectional curvature of $(\overline{M}_x, \langle \cdot, \cdot \rangle)$ is nonpositive, the Riemannian space $(M_x, d_x)$ has nonpositive curvature in the sense of Alexandrov [BH99, Thm. II.1A.6]. Therefore, it only remains to be shown that there is a neighborhood of $[0]$ in $(\overline{M}_x, d_x)$ that is a CAT(0) space. For this, it suffices to show that any triangle intersecting $[0]$ has (Alexandrov) angles no greater than the angles in a comparison
triangle. We also need only consider nontrivial triangles, where none of the vertices lie on any of the edges, since such a triangle has Alexandrov angles zero at each vertex.

Consider a triangle \( \triangle abc \subset \mathcal{M}_x \), with (Alexandrov) angles \( \alpha, \beta, \) and \( \gamma \) at the (distinct) vertices \( a, b, \) and \( c \), respectively. Assume that \([0] \in \triangle abc\). By renaming the vertices if necessary, \( \triangle abc \) then falls into one of the following five distinct cases:

1. None of \( a, b, c \) are equal to \([0]\), and \( \text{tr}_a(u_{a,b}^2), \text{tr}_b(u_{b,c}^2), \text{tr}_c(u_{c,a}^2) \geq (4\pi)^2/n \). (See the beginning of §5.1 for the definitions of \( u_{a,b}, \) etc.)
2. None of \( a, b, c \) are equal to \([0]\), and \( \text{tr}_a(u_{a,b}^2) < (4\pi)^2/n \), but \( \text{tr}_b(u_{b,c}^2), \text{tr}_c(u_{c,a}^2) \geq (4\pi)^2/n \).
3. None of \( a, b, c \) are equal to \([0]\), and \( \text{tr}_a(u_{a,b}^2), \text{tr}_b(u_{b,c}^2) < (4\pi)^2/n \), but \( \text{tr}_c(u_{c,a}^2) \geq (4\pi)^2/n \).
4. We have \( c = [0] \) and \( \text{tr}(u_{a,b}^2) \geq (4\pi)^2/n \).
5. We have \( c = [0] \) and \( \text{tr}(u_{a,b}^2) < (4\pi)^2/n \).

We wish to show that all of these cases are either trivial or reduce to case [5]. The cases are depicted in Figure 1, where curves with one dash represent the geodesic \([a, b]\), with two dashes \([b, c]\), and with three \([c, a]\).

Let \( \triangle \bar{abc} \) be a comparison triangle in \( \mathbb{E}^2 \) for \( \triangle abc \), with comparison angles \( \bar{\alpha}, \bar{\beta}, \) and \( \bar{\gamma} \).

In case [1], we immediately see that all of the angles of \( \triangle abc \) are zero. Therefore, we trivially have \( \alpha \leq \bar{\alpha}, \beta \leq \bar{\beta}, \gamma \leq \bar{\gamma} \).

In case [2], we note that \( \gamma = 0 \) and the angles \( \alpha \) and \( \beta \) are the same as those in \( \triangle ab[0] \). On the other hand, we have that \( d_x(a, [0]) < d_x(a, c) \) and \( d_x(b, [0]) < d_x(b, c) \), so the law of cosines (see also Figure 2) implies that if the Alexandrov angles at the vertices \( a \) and \( b \) of \( \triangle ab[0] \) are smaller than the corresponding angles in \( \triangle \bar{ab}[0] \), then the same holds for \( \triangle abc \) and \( \triangle \bar{abc} \). Thus this case reduces to case [5].

\[ \text{Figure 1. Triangles intersecting } [0] \]
In case (3), we may apply the Gluing Lemma for Triangles [5, 2] with \( p = b, q_1 = a, q_2 = c, \)
and \( r = [0] \) (see Figure 3) to see that if the angles of any triangle as in (5) are smaller than those of a comparison triangle, then so are those of \( \triangle abc. \)

In case (4), the (Alexandrov) angles in \( \triangle abc \) are zero, as in case (1), so the angle inequalities are again trivially satisfied. (Note that this triangle actually is trivial, since \([0]\) lies on the edge \([a, b], \) but we include this case for the sake of clarity.)

Finally, we must deal with case (5). For this, we require a lemma.

**Lemma 5.3.** Let \( a \) and \( b \) be as in case (5) above. Then \( \triangle ab[0] \subset P, \) where \( P \subset \mathcal{S}_x \) is the plane

\[
P := \{ y_1 a \exp(y_2 a^{-1} u_{a,b}) \mid (y_1, y_2) \in \mathbb{R}^2 \}.
\]

Furthermore, the interior \( O_{a,b} \subset P \) of \( \triangle ab[0] \) is a totally geodesic local submanifold of \( \mathcal{M}_x. \)

**Proof.** It is clear that \([a, [0]] \) and \([b, [0]] \) lie in \( P, \) since these are segments of the lines defined by \( y_2 = 0 \) and \( y_2 = 1, \) respectively. Furthermore, one sees from (4.3) that the traceless part of \( \exp^{-1}_a(b) = \psi(v_{a,b}) \) is proportional to \( u_{a,b}. \) Therefore, referring to (4.1), it is apparent that the geodesic \([a, b]\) is of the following form. We can find a pair of functions \( (f_1, f_2) : [0, 1]^2 \to \mathbb{R}_+ \times \mathbb{R} \) such that it is given by \( f_1(t) a \exp(f_2(t) u_{a,b}). \) Hence \([a, b] \subset P, \) so \( \triangle ab[0] \subset P, \) as desired. That \( O_{a,b} \) is a local submanifold of \( \mathcal{M}_x \) follows immediately from the fact that it is an open subset in the submanifold \( P \cap \mathcal{M}_x. \)

Finally, we show that \( O_{a,b} \) is totally geodesic. Let \( \tilde{a} = y_1 a \exp(y_2 a^{-1} u_{a,b}) \) and \( \hat{a} = z_1 a \exp(z_2 a^{-1} u_{a,b}) \) for some \( (y_1, y_2), (z_1, z_2) \in \mathbb{R}_+ \times \mathbb{R}. \) If we assume \( \tilde{a}, \hat{a} \in O_{a,b}, \) then we have \( 0 < y_2, z_2 < 1. \)

Now we write \( a = y_1^{-1} \tilde{a} \exp(-y_2 a^{-1} u_{a,b}) \) and

\[
\hat{a} = \frac{z_1}{y_1} \tilde{a} \exp((z_2 - y_2) a^{-1} u_{a,b}) = \frac{z_1}{y_1} \tilde{a} \exp(a^{-1}((z_2 - y_2) \tilde{a} a^{-1} u_{a,b})).
\]

Note that \( \operatorname{tr}_a(\tilde{a} a^{-1} u_{a,b}) = \operatorname{tr}_a(u_{a,b}) = 0. \) Furthermore,

\[
\operatorname{tr}_{\hat{a}}(\hat{a} a^{-1} u_{a,b}) = \operatorname{tr}_a(u_{a,b}) < \frac{(4\pi)^2}{n},
\]

since \( z_2 < 1, y_2 > 0, \) and by assumption, \( \operatorname{tr}_a(u_{a,b}^2) < (4\pi)^2/n. \) Thus, \( \hat{a} \in \operatorname{im}(\exp_{\tilde{a}}) \).

Let \( \psi := \exp_{\tilde{a}}^{-1}, \) and let \( c := \psi(\hat{a}). \) From the above discussion, and recalling (4.3), we note that \( c_T \) is proportional to \( \tilde{a} a^{-1} u_{a,b}, \) say \( c_T = \lambda \tilde{a} a^{-1} u_{a,b}. \) Let \( \gamma_t \) be the geodesic segment between \( \tilde{a} \) and \( \hat{a}. \) As above, we can find a pair of functions \( (f_1, f_2) : [0, 1]^2 \to \mathbb{R}_+ \times \mathbb{R} \) such

![Figure 2. Case (2).](image1)

![Figure 3. Case (3).](image2)
Theorem 5.5. \(\gamma_t = f_1(t)\bar{a} \exp(f_2(t)\bar{a}^{-1}ct) = f_1(t)\bar{a} \exp(\lambda f_2(t)\bar{a}^{-1}(\bar{a}a^{-1}u_{a,b}))\) 
\[= y_1 f_1(t)a \exp((y_2 + \lambda f_2(t))a^{-1}u_{a,b}).\]

Thus \(\gamma_t\) is contained in \(P\).

To see that \(\gamma_t\) is contained in \(O_{a,b}\), we first note that minimality of \(\gamma_t\) and Lemma 4.11 imply that \(f_2\) is a monotone function. Thus, if \(\gamma_t\) exits \(O_{a,b}\), then there must exist \(t_0 \in (0,1)\) with \(\gamma_{t_0} \in [a,b]\). Since distinct geodesics cannot intersect tangentially, there must be \(t_1 > t_0\) with \(\gamma_{t_1} \in [a,b]\), and \(\gamma_t \not\in [a,b]\) for \(t \in (t_0, t_1)\). But minimality of \(\gamma_t\) and \(\gamma_t\) then imply the existence of two distinct minimal paths between \(\gamma_{t_0}\) and \(\gamma_{t_1}\), contradicting Theorem 4.14. This completes the proof.

We now return to case (5). Since \(O_{a,b}\) is a totally geodesic local submanifold of a manifold with nonpositive Riemannian curvature, it is itself a space of nonpositive curvature in the sense of Alexandrov [BH99, Thm. II.1A.6]. By Theorem 4.14 and Lemma 5.3, there exists a unique minimal path between any two points in \(O_{a,b}\). Furthermore, by Theorem 4.15, minimal paths between points in \(O_{a,b}\) vary continuously with their endpoints. Thus, we may apply a theorem of Alexandrov [BH99, Thm. II.4.9] to see that \(O_{a,b}\) is a CAT(0) space.

Consider the completion (w.r.t. \(d_x\)), \(\bar{O}_{a,b} = O_{a,b} \cup \triangle ab[0]\). By [BH99, Cor. II.3.11], the completion of a CAT(0) space is CAT(0), so \(\bar{O}_{a,b}\) is CAT(0). In particular, \(\triangle ab[0] \subset \bar{O}_{a,b}\) satisfies the CAT(0) inequality, and so has Alexandrov angles no greater than a comparison triangle \(\triangle \bar{ab}[0]\), as was to be shown.

This completes all possible cases, and so we have the following.

**Theorem 5.4.** \((\mathcal{M}_x, d_x)\) is a space of nonpositive curvature in the sense of Alexandrov.

Furthermore, by Theorem 4.14, there exists a unique minimal geodesic between any two points in \(\mathcal{M}_x\), and by Theorem 4.15, these geodesics vary continuously with their endpoints. Thus, we may again apply the aforementioned theorem of Alexandrov [BH99, Thm. II.4.9] to immediately obtain:

**Theorem 5.5.** \((\mathcal{M}_x, d_x)\) is a CAT(0) space.

5.3. The CAT(0) property for \(\mathcal{M}\). We now claim that Theorems 4.16 and 5.5 imply that \((\mathcal{M}, d)\) is a CAT(0) space as well. Theorem 4.16 gives the existence of geodesics, so it remains to show that triangles in \(\mathcal{M}\) satisfy the CAT(0) inequality.

To see this, consider the product bundle \(M \times \mathbb{E}^2\), where \(\mathbb{E}^2 = (\mathbb{R}^2, |\cdot|)\), as above, denotes Euclidean space with its standard norm. We denote the fiber over \(x\) of this bundle by \(E_x\). Our reference volume form \(\mu\) on \(M\) allows us to define a Hilbert space \(L^2(M \times \mathbb{E}^2, \mu)\) of square-integrable sections of this bundle. As a Hilbert space, it is flat; in particular, it is CAT(0) [BH99, p. 167]. Thus, to show that triangles in \(\mathcal{M}\) satisfy the CAT(0) inequality, it suffices to take a comparison triangle in \(L^2(M \times \mathbb{E}^2, \mu)\) rather than in \(\mathbb{E}^2\).

Let \(g, h, k \in \mathcal{M}\), and let \(\ell_1, \ell_2 \in \Delta ghk\) be any points. Assume, without loss of generality, that \(\ell_1 \in [g, h]\), the geodesic between \(g\) and \(h\), and \(\ell_2 \in [g, k]\). By Theorem 4.16, when evaluated at the point \(x \in M\), the geodesic \([g, h]\) in \(\mathcal{M}\) agrees with the geodesic \([g(x), h(x)]\) in \(\mathcal{M}_x\). Thus \(\ell_1(x) \in [g(x), h(x)]\) for all \(x\), and similarly for \(\ell_2(x)\). Define \(\tilde{g}, \tilde{h}, \tilde{k}, \tilde{\ell}_1, \tilde{\ell}_2 \in L^2(M \times \mathbb{E}^2, \mu)\) by the following. For all \(x\), let \(\tilde{g}(x) := (0,0)\) and \(\tilde{h}(x) := (d_x(g(x), h(x)), 0)\). Let \(\tilde{k}(x)\) be the unique point in the closed upper half-plane with \(|\tilde{k}(x) - \tilde{g}(x)| = d_x(g(x), k(x))\) and

\[= y_1 f_1(t)a \exp((y_2 + \lambda f_2(t))a^{-1}u_{a,b}).\]
Finally, define $\bar{\triangle}$ convergence in the $L^2$ Ricci flow converges w.r.t. $d$ dimensions. This is because, as one sees from Theorem 4.16, there is no $d$ nonpositive curvature in the sense of Alexandrov—i.e., locally CAT(0)—as we could in finite $D$.

The metric completion of $\bar{\triangle}$, the group of orientation-preserving diffeomorphisms of $\triangle$, is a CAT(0) space. It is weak $2$-ball of positive radius around any point such that geodesics between points in this ball exist and remain in this triangle satisfies the CAT(0) inequality. $(\text{space [Bou75].})$

Thus, $\bar{\triangle}$ satisfies the CAT(0) inequality with respect to the comparison triangle $\bar{\triangle}$. Since the points $g,h,k \in \bar{\triangle}$ were arbitrary, we have proved the following.

**Theorem 5.6.** The metric completion of $\triangle$ with respect to its $L^2$ Riemannian metric, $(\bar{\triangle}, d)$, is a CAT(0) space.

In particular, if $\bar{\triangle} \subset \triangle$ is any geodesic triangle consisting of smooth metrics, then this triangle satisfies the CAT(0) inequality.

Note that even though $\bar{\triangle}$ is a CAT(0) space, we cannot infer that $\triangle$ is a space of nonpositive curvature in the sense of Alexandrov—i.e., locally CAT(0)—as we could in finite dimensions. This is because, as one sees from Theorem 4.16, there is no $d$-ball of positive radius around any point such that geodesics between points in this ball exist and remain in $\triangle$.

6. Outlook

There are a number of open questions concerning the $L^2$ metric. As mentioned in the introduction, one may consider harmonic maps with $(\triangle, d)$ or its completion as a target, or consider groups of isometries of $\triangle$. Another is to find submanifolds of $\triangle$ on which convergence in the $L^2$ metric implies a more synthetic-geometric notion of convergence—e.g., Gromov–Hausdorff convergence, or convergence as a metric-measure space [Gro07, Sect. 3.12]. On a related note, we point out that in [CR11 Cor. 6.7], it has been shown that if the Kähler-Ricci flow converges w.r.t. $d$, then it converges smoothly—so at least for these special paths, convergence in the $L^2$ metric already implies stronger convergence.

Other open questions that seem difficult to solve concern the moduli space of Riemannian metrics, also called the space of Riemannian structures or, sometimes, superspace. If $\mathcal{D}$ is the group of orientation-preserving diffeomorphisms of $\triangle$, then this space is the quotient $\triangle/\mathcal{D}$, where $\mathcal{D}$ of course acts by pulling back metrics. It is not hard to see that the $L^2$ metric is invariant under this action, and so it projects to the quotient, which is a stratified space $[\text{Bou75}].$

From the metric-geometric standpoint, the first question one must ask is whether the $L^2$ metric induces a metric space structure on the quotient. Because the $L^2$ metric is weak and the quotient is singular, we cannot a priori exclude the situation that two orbits of the diffeomorphism group become arbitrarily close to one another. The very weak nature of convergence with respect to the $L^2$ metric makes finding a lower bound for the distance
between orbits, either directly or through some geometric invariants, a difficult proposition (cf. [Cla11, Sect. 4.3] for more on this).

Assuming the previous question is answered in the affirmative, another question one could ask about the moduli space is what its completion with respect to the $L^2$ metric is. Potentially, some of the very pathological degenerations that lead to losing all regularity of a limit metric in the completion of $(\mathcal{M}, d)$ come from degenerations along a diffeomorphism orbit.

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