Dark SU(N) glueball stars on fluid branes

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The glueball dark matter, in the pure SU(N) Yang-Mills theory, engenders dark SU(N) stars that comprise self-gravitating compact configurations of scalar glueball fields. Corrections to the highest frequency of gravitational wave radiation emitted by dark SU(N) star mergers on a fluid brane with variable tension, implemented by the minimal geometric deformation, are derived and their consequences analysed. Hence, dark SU(N) stars mergers on a fluid brane-world are shown to be better detectable by the LIGO and the eLISA experiments.

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I. INTRODUCTION

Dark matter and dark energy comprise new directions in gravity and high energy physics, towards theories that are beyond the General Relativity (GR) and which can explain such two puzzling phenomena. Among successful attempts to propose theories beyond GR, the method of geometrical deformation (MGD) consists of a suitable approach to derive new solutions of the effective Einstein field equations [1–3], encoding compact stellar distributions. These new solutions are complementary to other successful paradigms [4, 5]. The MGD comprises the brane tension (σ) as a free parameter, controlling the high energy regime of an inflationary brane-world scenario that has the GR as the low energy limit. In fact, during the cosmological evolution, the brane temperature has been severely modified. It varied from, for instance, \( T \sim 10^4 \text{ K} \) – when the matter density equaled the radiation density, around 5.6 \( \times 10^3 \) years after the Big Bang – to the current value of \( T \sim 2.73 \text{ K} \), in the CMB. An underlying setup can be, thus, implemented by a variable tension fluid brane [7], whereon compact self-gravitating systems can undergo the MGD [5].

The MGD was developed using Randall-Sundrum like models [47]. It has the bulk dark pressure and radiation as leading constituents of the stress-energy tensor, in the brane effective Einstein field equations [10]. Explicit solutions are compact stellar structures with a bulk Weyl fluid imprint [1–3, 11]. The MGD is a strong and robust procedure that has been recently endowed with observational and experimental precise bounds, from gravitational lensing effects [12] to the classical tests of GR [13].

On the other hand, gauge fields that are further away from the Standard Model of elementary particles were proposed as pure Yang-Mills dark fields [14, 15]. In fact, the Standard Model can be coupled to hidden sectors, governed by a pure Yang-Mills setup, in the low energy regime. The scalar glueball dark matter model implements such a SU(N) Yang-Mills sector [14, 16–18], as a self-interacting field with large cross section. When Standard Model particles and fields are forbidden to interact with the SU(N) scalar glueball, gravity can induce a self-gravitating system, manifesting Bose-Einstein condensation of glueballs into compact stellar objects. Ref. [17] discusses relevant elastic scatterings among glueballs, manifesting their feasibility as a self-interacting dark matter candidate. A refined and detailed analysis can be checked in Ref. [14], together with other general aspects [19].

Although dark SU(N) compact systems were studied [17], any realistic approach that provides observational and experimental signatures of dark SU(N) stars on inflationary scenarios lacks, still. Here the MGD is proposed as a procedure to implement dark SU(N) brane-world stars, in the context of the evolution of the Universe, as well as to refine the analysis of gravitational waves produced by dark SU(N) glueball star mergers, enhancing the window for current experiments to detect them.

This paper is organised as follows: Sect. II is devoted to a brief review, regarding the MGD of stellar distributions on a fluid brane, ruled by a variable tension that encodes the cosmological evolution. In Sect. III, corrections to the highest frequency of gravitational wave radiation, emitted by dark SU(N) star mergers due to the finite brane tension, are derived for both the \( \phi^4 \) self-interacting glueball potential and the glueball potential in the large \( N \) limit. These corrections make the gravitational wave radiation, emitted by dark SU(N) MGD stars mergers, to be more able to detect than in the GR limit setup. Hence, a larger spectrum of gravitational waves on fluid branes is expected, enhancing the window to be probed by the LIGO and the eLISA experiments. Sect. IV is dedicated to draw the conclusions and final comments.

II. THE MGD SETUP AND FLUID BRANES

The MGD procedure is able to derive high energy corrections to GR, when the vacuum in the outer region of a compact distribution is permeated by a 5-dimensional (5D) bulk Weyl fluid [1, 3, 20]. The codimension-1 brane...
that designates our Universe has tension (self-gravity) as a leading parameter, which varies as the temperature decreases, across the Universe inflation \([7, 21]\). The most useful and applicable brane-world scenarios in this context are implemented by fluid branes, evincing the Eötvös law that provides the dependence of the brane tension with the temperature \([7, 21]\).

The MGD setup has recently imposed the brane (variable) tension bounds \(\sigma \gtrsim 5.19 \times 10^9 \text{ MeV}^4\) (in the context of the classical tests of GR) \([13]\) and \(\sigma \gtrsim 3.18 \times 10^6 \text{ MeV}^4\) (regarding the Bose-Einstein condensation of weakly interacting gravitons into MGD black holes) \([22]\). The MGD represents a deformation of the Schwarzschild metric, implemented by bulk effects in the brane-world paradigm, whose low energy regime \(\sigma \to \infty\) recovers the Schwarzschild standard solution.

The 4D Einstein effective equations can be derived when the 5D bulk Einstein equations are projected onto the 4D brane, by the Gauss-Codazzi method (the convention \(8\hbar G = c = 1 = \hbar\) is going to be fixed\(^1\), where \(G = \hbar c/M_{pl}^2\) and \(M_{pl}\) denotes the Planck scale, and \(\mu, \nu = 0, 1, 2, 3\)), yielding \([10]\)

\[
R_{\mu\nu} + \left(\Lambda - \frac{1}{2}R\right)g_{\mu\nu} - T_{\mu\nu} = 0, \tag{1}
\]

where \(R_{\mu\nu}, R\), and \(\Lambda\) are, respectively, the Ricci tensor, the Ricci scalar, and the 4D cosmological constant. The effective stress tensor in Eq. (1) can be split into a sum, \(T_{\mu\nu} = T^{\mu\nu}_{\text{matter}} + \sigma^{-1}S_{\mu\nu}\), where the first component \(T^{\mu\nu}_{\text{matter}}\) denotes the brane matter stress tensor and \(S_{\mu\nu}\) encodes high-energetic corrections from the 5D Weyl fluid. The tensor \(S_{\mu\nu}\) encrypts Kaluza-Klein imprints from the bulk onto the brane \([4, 10]\).

Compact stellar structures that are solutions of the Einstein brane field equations (1) are usually obtained for static, spherically symmetric, metrics \((2)\),

\[
ds^2 = -A(r)dt^2 + B(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\varphi^2). \tag{2}
\]

The MGD procedure fixes the g\(_{tr}\) metric component in (2) and deforms the outer \(g_{rr}^{-1}(r) \equiv B(r)\) metric component \([2, 3, 11]\),

\[
B(r) = 1 - \frac{2M}{r} + \varsigma \exp\left(\int_R^r \frac{f(A(r))}{g(A(r))} dr\right), \tag{3}
\]

where \(f(A(r)) = \frac{3A}{4A''} + (\ln A)^2 + \frac{2}{r} \ln A - 1 + \frac{1}{\Lambda}\) and \(g^{-1}(A(r)) = \left(\frac{1}{2} \ln A\right)' + \frac{3}{2} \left(\frac{1}{2} \ln A\right)\), \(s\)' is defined as \(\frac{dA}{A}\), and \(R \equiv \int \rho(r) dr/\int r^2 \rho(r) dr\), where \(\rho\) is the energy density of the stellar matter distribution \([1]\). The parameter \(\varsigma\) in Eq. (3) encodes a bulk-induced deformation of the vacuum, at the compact distribution surface, comprising the necessary 5D (bulk) Weyl fluid data onto the brane \([13]\). It is significant to observe that the outer metric is defined in the region \(r > R\) \([1]\), whose deformation yields \([2]\)

\[
A(r) = 1 - \frac{2M}{r}, \tag{4a}
\]

\[
B(r) = \left[1 + \left(1 - \frac{3M}{2r}\right)^{-1}\right] \left(1 - \frac{2M}{r}\right), \tag{4b}
\]

where

\[
l \equiv \left(1 - \frac{2M}{R}\right)^{-1}\left(1 - \frac{3M}{2R}\right)R. \tag{5}
\]

The MGD black hole event horizons are \(r_1 = 2M\) and \(r_2 = -\varsigma l + \frac{3M}{2}\). The infinitely rigid brane limit \(\varsigma^{-1} \sim \sigma \to \infty\), that characterises the GR limit, yields \(r_1 \gg r_2\). Besides, \(\varsigma\) is a parameter that relies on the inherent compact star configuration. Refs. \([1, 8]\) show that the metric radial component (4a) can be written as

\[
B(r) = 1 - \frac{2M}{r} - \left(1 - \frac{2M_0}{r}\right) \left(1 - \frac{3M_0}{2r}\right)^{-1} l_0 \varsigma, \tag{6}
\]

thus exhibiting a part that is beyond the Schwarzschild solution, up to order \(O((\varsigma)^{-2})\), where \(M_0 = M|_{\sigma \to \infty}\) is the GR mass function and \(l_0 = l(M_0)\), concerning Eq. (5). Bulk imprints are highest at the star surface \(r = R\). This shall be a prominent information in the analysis of dark SU(N) stars on fluid branes, in the next section.

The less compact the star, the smaller the \(|\varsigma|\) parameter is \([1, 2]\). Moreover, the current experimental and observational data was used in Ref. \([13]\), to impose the strongest bound \(|\varsigma| \lesssim 6.1 \times 10^{-11}\) on the MGD parameter, using the classical tests of GR. This result, together with the most recent and strict bound on the variable brane tension \(\sigma \gtrsim 3.18 \times 10^6 \text{ MeV}^4\) \([22]\), justifies high order \(O((\varsigma)^{-2})\) terms to be dismissed, yielding

\[
\varsigma(\sigma, R) = -\frac{b_0}{R^2} \sigma^{-1}, \tag{7}
\]

where \(b_0 \sim 1.35\) \([13]\). The negativity of \(\varsigma\) compels the MGD star gravitational field to be weakened, as an effect of a finite brane tension, when compared to the GR regime \(\sigma \to \infty\).

Heretofore no condition on the variable brane tension has been imposed, although cosmological evidence drives the brane tension to fluctuate across the Universe inflation \([7, 23]\). On Eötvös fluid branes, the brane tension varies with respect to the Universe temperature \(T\). One associates the regime \(\sigma \approx T - \tau\) \([7]\), where \(\tau\) is some critical value that makes \(\sigma\) to assume only non-negative values after the Big Bang \([7, 21]\). This varying brane tension can eliminate any initial singularity at the early Universe. In fact, the brane Universe was created at a \(\tau\) temperature, corresponding to the scale factor value \(a_0\) that is defined by the coupling constants \([7, 23]\). Ref. \([7]\) derived the relationship between temperature and the

\(^1\) Obviously, the precise values of all involved parameters shall be suitably taken into account, in the calculations in Sect. III.
scale factor, $T(t) \approx \frac{1}{a(t)}$ [7]. This result yields a time-
dependent brane tension

$$\frac{\sigma(t)}{\sigma_0} = 1 - \frac{a_0}{a(t)}. \quad (8)$$

At the extremely hot early Universe, the brane tension was negligible ($\sigma \approx 0$). Subsequently, both the variable
brane tension and the 4D coupling parameter grew, as
the scale factor asymptotically increased, in the inflation-
ary brane-world scenario [24]. The time-dependent
brane tension expression yields $\frac{\Delta \sigma}{\sigma_0} = 1 - \frac{a_0}{a(t)} \left(1 - \frac{a_0}{a(t)}\right)$. This (dynamical) cosmological "constant" had attained
a huge negative value and achieved lowly positive values
[21]. It also engenders supplementary attraction [repu-
sion] at small [large] values of the scale factor, similarly to
the dark matter [dark energy]. This inflationary cosmology scenario emulates the (cosmological) standard model
of dark matter ($\Omega \approx 0$). Subsequently, both the variable
brane tension, to the highest frequency of the
gravitational waves, that is compatible with the current bounds of the
CMB, in an inflationary brane-world scenario.

The Klein-Gordon equation, that is derived from this action
by the Euler-Lagrange equations, couples to the Ein-
stein field equations and shall be analyzed, for both the
large $N$ glueball field potential and for the $\phi^4$ self-interacting
brane tension and the 4D coupling parameter grew, as
brane yields

$$\frac{\sigma(t)}{\sigma_0} = 1 - \frac{a_0}{a(t)}. \quad (8)$$

This regime $\Lambda \gg 1$ holds for the glueball dark matter
potential configurations can be derived, when the repulsive
interaction strength and the properties of dark SU($N$) stars
mass and radii variation on a fluid brane. Moreover, dark
SU($N$) stars features can be, in this context, analyzed
along the inflationary brane-world era.

### III. FLUID BRANE CORRECTIONS TO DARK
SU($N$) STARS

Hidden SU($N$) Yang-Mills sectors, that are further
away the Standard Model of elementary particles, can
be realized by the (scalar) glueball dark matter model
[14, 17]. The gravitational interactions among glueballs
engender a self-interacting system, having the glueball
mass ($m$) and the number ($N$) of colors as the main
underlying parameters. When $10$ eV $\lesssim m \lesssim 10$ KeV and
$10^3 \lesssim N \lesssim 10^6$, then the dark glueball field self-gravity

$$V(\phi) = \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4, \quad (10)$$

was shown to rule both the glueball dark matter self-interaction strength and the properties of dark SU($N$) stars as well. For $\lambda > 0$, stable dark SU($N$) star configura-
tions can be derived, when the repulsive $\lambda \phi^4$ self-interaction compensates gravity [17, 27].

The coupled system of Einstein equations and Klein-
Gordon ones was derived in [17]. Imposing time averaging
over the oscillation period $\frac{2\pi}{\omega}$ of the $\phi$ field, such
system yields

$$M'(x) - \frac{\Lambda}{6} \phi^4(x) + \left(1 + \frac{\Omega^2}{A(x)}\right) \phi^2(x) + \frac{\Phi^2(x)}{B(x)} x^2 = 0, \quad (11a)$$

$$\frac{(\ln A(x))'}{B(x)} + \frac{\Lambda}{48} x \Phi^4(x) + \frac{x}{4} \left(1 - \frac{\Omega^2}{A(x)}\right) \phi^2(x) - \frac{x \Phi^2(x)}{8 B(x)} + \frac{1}{x} \left(1 - \frac{1}{B(x)} - 1\right) = 0, \quad (11b)$$

$$\phi''(x) + \left(\frac{1}{2} \left(\ln \frac{A}{B}\right)' + \frac{2}{x}\right) \Phi(x) - \frac{\Lambda}{2} \phi^2(x) + \left(1 - \frac{\Omega^2}{A(x)}\right) B(x) \Phi(x) = 0, \quad (11c)$$

where [17, 27]

$$x = \frac{m \omega t}{\omega}, \quad (12a)$$

$$\Lambda = \frac{12 \lambda}{m^2}, \quad (12b)$$

$$\Omega = \frac{\omega^2}{m}. \quad (12c)$$

The regime $\Lambda \gg 1$ holds for the glueball dark matter
model [28], yielding Eq. (11c) to imply
\[
\Lambda^{-1} \left[ \phi''(x) + \left( \frac{2}{x} + \frac{1}{2} \ln \left( \frac{A}{B} \right) \right) \phi'(x) \right] - B(x)\phi(x) \left[ 1 - \frac{\Omega^2}{A(x)} - \frac{1}{2} \phi^2(x) \right] = 0 , \tag{13}
\]
where
\[
x = \frac{x}{\sqrt{\Lambda}}, \quad \phi = \sqrt{2\Lambda} \Phi, \quad M = \frac{M}{\sqrt{\Lambda}}. \tag{14}
\]
The \(\Lambda \gg 1\) limit can induce the first term in Eq. (13) to be dismissed, yielding
\[
\lim_{\Lambda \to 1^+} \left[ \phi(x) - \sqrt{2} \left( \frac{\Omega^2}{A(x)} - 1 \right)^{1/2} \right] = 0 , \tag{15}
\]
implies that
\[
M'(x) - x^2 \left[ \frac{1}{4} \left( \frac{\Omega^2}{B(x)} \right) + 1 \right] \phi^2(x) + \frac{3}{32} \phi^4(x) = 0 , \tag{16}
\]
\[
B(\ln A)'x^2 - 2M + \left[ 1 - \frac{\Omega^2}{A} \right] \phi^2 \left[ \phi^2 + \frac{3}{16} \phi^4 \right] x^3 = 0 , \tag{17}
\]
where \(x\) is the argument of all functions in Eqs. (16, 17). Similarly to Refs. [17, 27], these equations can be solved by numerical methods, with \(M(0) = 0\), for \(0 \leq x \leq x_R\).

The results obtained from using the self-interacting \(\phi^4\) glueball potential (10) are depicted in Figs. 1–3. In what follows, \(\sigma \sim 10^6\) MeV\(^4\) shall denote the current brane tension bound \(\sigma \gtrsim 3.18 \times 10^6\) MeV\(^4\) [22]. The same notation shall be used for the values \(\sigma \sim 10^9\) and \(10^{12} \sim \text{MeV}^4\), but these exact values shall be adopted in what follows, unless otherwise explicitly stated.

![FIG. 1. Dark SU(N) MGD star mass \(M(x_R)\), in the \(\phi^4\) scalar glueball potential setup, normalized by \(\sqrt{\frac{3\lambda M_{\phi4}}{m^*}}\), with respect to \(\Lambda(0)_{TP}\), for different values of the fluid brane tension \(\sigma = 10^{12}\) MeV\(^4\) (dashed line); \(\sigma = 10^9\) MeV\(^4\) (black line); \(\sigma \sim 10^6\) MeV\(^4\) (gray line).](image1)

![FIG. 2. Dark SU(N) MGD star radius \(x_R\), in the \(\phi^4\) scalar glueball potential setup, normalized by \(\sqrt{\frac{3\lambda M_{\phi4}}{m^*}}\), with respect to \(\Lambda(0)_{TP}\), for different values of the fluid brane tension \(\sigma = 10^{12}\) MeV\(^4\) (dashed line); \(\sigma = 10^9\) MeV\(^4\) (black line); \(\sigma \sim 10^6\) MeV\(^4\) (gray line).](image2)

![FIG. 3. Dark SU(N) MGD star ratio \(\frac{M(x_R)}{\phi_{\text{crit}}^3}\) in the \(\phi^4\) scalar glueball potential setup, normalized by \(\sqrt{\frac{3\lambda M_{\phi4}}{m^*}}\), with respect to \(\Lambda(0)_{TP}\), for different values of the fluid brane tension \(\sigma = 10^{12}\) MeV\(^4\) (dashed line); \(\sigma = 10^9\) MeV\(^4\) (black line); \(\sigma \sim 10^6\) MeV\(^4\) (gray line).](image3)

The gravitational balance of self-interacting scalar fields was studied [28], also in the context of stability bounds on compact objects [22, 29], represented by spherically symmetric boson star solutions. According to the \(g_{rr}\) metric component in Eq. (4b), the boundary condition \(M(x = 0) = 0\) implies that \(\lim_{r \to 0} B(r) = 1 + \frac{\zeta r^\frac{1}{2}}{r^\frac{1}{2}}\).

In Fig. 1, the mass spectrum (y-axis) between the point \(\Lambda(0)_{TP} = 0\) up to the critical (maximum) point in the plots cannot be attained. In fact, Ref. [27] showed that a dark SU(N) star accretes by seizing the surrounding dark matter. Thereafter, the dark SU(N) MGD star mass increases by accretion, up to a maximum, represented in the third column in Table I. The results are presented for different values of the brane tension:
The highest gravitational wave frequency emitted by dark SU(N) MGD star mergers, due to the brane tension. The current lower bound for the brane tension \( \sigma \gtrsim 3.18 \times 10^6 \text{MeV}^4 \) [22], in the last line of Eq. (24), provides a realistic fluid brane scenario, wherein the highest gravitational wave radiation frequency is up to \( \sim 26.2\% \) higher than the predictions in the GR limit.

Now, we can see that the highest gravitational wave frequency that is emitted by dark SU(N) MGD star mergers, in Eqs. (22), for the self-interacting \( \phi^4 \) glueball potential, can be better detected by the LIGO and the eLISA experiments [26], having a wider range than the spectrum of frequencies provided by Schwarzschild solutions [17]. This shall be clear in what follows, by analyzing the \( N - m \) parameter space.

Ref. [17] argued that the dark SU(N) stars have parameters in the ranges \( 100 \text{eV} \lesssim m \lesssim 10 \text{KeV} \) and \( 10^3 \lesssim N \lesssim 10^6 \), yielding a maximum \( 10^6 \mathcal{M}_\odot \lesssim M \lesssim 10^9 \mathcal{M}_\odot \) for the dark SU(N) MGD star mass, whereas the lowest dark SU(N) MGD star radius varies in the range \( 10^2 \lesssim R \lesssim 10^5 \), in unit of the Solar radius \( R_\odot \). Hence, the highest gravitational wave frequency that is emitted by dark SU(N) MGD stars mergers, given by Eqs. (22), for the \( \phi^4 \) glueball potential (10), can be better detectable by the LIGO and the eLISA experiments [26]. Moreover, dark SU(N) MGD star mergers have specific signatures that are quite different of Schwarzschild black hole mergers, due to the subsequent analysis of Table I, as well as Eqs. (24). In fact, since dark SU(N) MGD stars do not necessarily collapse to form a black hole, their gravitational wave frequency of emission has a distinct signature of the ones emitted by black hole mergers.

The highest frequency of gravitational wave radiation \( f_{\text{max}} \), emitted from dark SU(N) MGD stellar mergers, can be allocated in the range \( 30 \mu \text{Hz} \lesssim f_{\text{max}} \lesssim 100 \text{mHz} \), that can be further detected by the eLISA mission [30]. In addition, the LIGO experiment can probe the range of gravitational waves frequency \( 50 \text{Hz} \lesssim f_{\text{max}} \lesssim 1 \text{KHz} \), nowadays. Both these ranges are, respectively, represented by the light-gray and the gray bands in Figs. 4 and 5 below, that represent the \( N - m \) parameter space. The black band represent the self-interacting \( \phi^4 \) glueball dark matter. Fig. 4 is based on the \( \sigma \rightarrow \infty \) GR limit, whereas Fig. 5 takes into account the brane tension bound \( \sigma \gtrsim 3.18 \times 10^6 \text{MeV}^4 \).
The differences between Fig. 4 and Fig. 5, both for the self-interacting glueball dark matter, reside on the distinction between the GR limit and the MGD setup, respectively. Although the spectrum of frequencies detected by the LIGO and the eLISA experiments are slightly modified by the 5D Weyl fluid in the MGD setup, when one goes from Fig. 4 to Fig. 5, the self-interacting glueball dark matter (black band) in the N-m parameter space is considerably thickened.

In the next subsection, the glueball potential in the large N regime shall be employed, to derive similar corrections, due to a fluid brane variable tension.

### B. Large number of SU(N) colors

Regarding the large N limit regime, the scalar glueball potential associated with the SU(N) Yang-Mills dark sector has power counting \( \lambda_i^{i+2} \sim 1/N^i \), where \( i \in N \) for the cubic and higher order terms in Eq. (10) \[14, 15, 17\]. Considering all higher order terms, the dark glueball potential (10) in this regime reads \[14, 15, 17\],

\[
V(\phi) = \left( \frac{m^2 N}{4\pi} \right)^2 \sum_{i=2}^{\infty} \frac{1}{j!} \left( \frac{4\pi\phi}{Nm} \right)^j ,
\]  

which is the Taylor expansion of the exponential function of the argument \( 4\pi\phi/Nm \), when its two first terms are not taken into account \[17\]. Similarly to Eqs. (15 – 17), the coupled equations can be acquired \[17, 27\]:

\[
\frac{A(x)}{\Omega^2} = 2F_1^{-1} \left( 0.5; \{ 1, 1.5 \}; 4\pi\phi^2(x) \right) ,
\]

\[
M'(x) - x^2 \left( \frac{\Omega^2}{4A(x)} \phi^2(x) + \frac{I_0}{16\pi^2} \right) = 0 ,
\]

\[
\frac{\ln(A(x))'}{B(x)} - \frac{2M(x)}{x^2} - \frac{\Omega^2 x \phi^2(x)}{2A(x)} + \frac{xI_0}{8\pi^2} = 0 ,
\]

where \( I_0 = I_0(\phi(x) - 1) \), and the \( \Lambda \gg 1 \) regime is adopted \[17, 27\]. The symbol \( 2F_1 \) is the usual generalized hypergeometric function and \( I_0 \) denotes the (modified) Bessel function. These two functions are derived when the time averaging of the potential (25) is computed \[17\].

The results obtained from using the large N limit glueball potential in Eq. (25) are plot in Figs. 6 - 8.
FIG. 6. Dark SU(N) MGD star mass $M(x_R)$, in the scalar glueball potential (25), normalized by $M_0^2$, with respect to $\lambda M_0$, for different values of the fluid brane tension $\sigma = 10^{12}$ MeV$^4$ (dashed line); $\sigma = 10^9$ MeV$^4$ (black line); $\sigma \sim 10^6$ MeV$^4$ (gray line).

FIG. 7. Dark SU(N) MGD star radius $x_R$, in the scalar glueball potential (25), normalized by $\lambda M_0$, with respect to $\lambda M_0$, for different values of the fluid brane tension $\sigma = 10^{12}$ MeV$^4$ (dashed line); $\sigma = 10^9$ MeV$^4$ (black line); $\sigma \sim 10^6$ MeV$^4$ (gray line).

FIG. 8. Dark SU(N) MGD star ratio $M(x_R)/x_R^2$, in the scalar glueball potential (25), normalized by $\lambda M_0$, with respect to $\lambda M_0$, for different values of the fluid brane tension $\sigma = 10^{12}$ MeV$^4$ (dashed line); $\sigma = 10^9$ MeV$^4$ (black line); $\sigma \sim 10^6$ MeV$^4$ (gray line).

Figs. 6 - 8 take into account a finite brane tension, being smoother than their respective $\sigma \to \infty$ counterparts [17]. Comparing Figs. 1, 2 and 3 (the $\phi^4$ dark glueball potential) to, respectively, Figs. 4, 5, and 6 (the large $N$ limit dark glueball potential), one realizes that the dark SU(N) MGD stars have bigger radii and are more massive than their GR limit counterparts.

Analogously to the self-interacting $\phi^4$ glueball potential, in Fig. 6 the mass spectrum $(y_{\sigma})$ is always larger than a Schwarzschild black hole event horizon values, whatever the glueball potential is considered. In Fig. 7 the mass spectrum $(y_{\sigma})$ is again obtained for the maximum of same mass, a similar result to the one in Ref. [17] that considers the $\sigma \to \infty$ GR limit. Hence there is no collapse process of a dark SU(N) MGD star that originates a black hole. The highest frequency of the gravitational wave radiation is again obtained for the maximum value of the dark SU(N) MGD star mass. Both the MGD star highest effective radius and mass, respectively, $R = \frac{1}{2\lambda^2} \frac{M_{pl}}{N m^2} x_R$ and $M = \frac{1}{2\lambda^2} \frac{M_{pl}^2}{N m^2} M(x_R)$ are presented:

| Brane tension $\sigma$ | $M(x_R)/x_R^2$ | $\frac{M(x_R)}{x_R^2}$ |
|------------------------|-----------------|-----------------------|
| $\infty$ (GR limit)    | 0.319           | 0.74                  |
| $10^{12}$ (MeV$^4$)    | 0.322           | 0.74                  |
| $10^9$ (MeV$^4$)       | 0.327           | 0.73                  |
| $10^6$ (MeV$^4$)       | 0.372           | 0.72                  |

Table II show that the ratio $\frac{x_R}{M(x_R)}$ is always greater than 2 (in normalized units), irrespectively of the brane tension values, whatever the glueball potential is considered. It means that the radius of a dark SU(N) MGD star is always larger than a Schwarzschild black hole event horizon of same mass, a similar result to the one in Ref. [17] that considers the $\sigma \to \infty$ GR limit. Hence there is no collapse process of a dark SU(N) MGD star that originates a black hole. The highest frequency of the gravitational wave radiation is again obtained for the maximum value of the dark SU(N) MGD star mass. Both the MGD star highest effective radius and mass, respectively, $R = \frac{1}{2\lambda^2} \frac{M_{pl}}{N m^2} x_R$ and $M = \frac{1}{2\lambda^2} \frac{M_{pl}^2}{N m^2} M(x_R)$ are presented:

One then gets a highest gravitational wave frequency, that reads

$$f_{max} = \frac{m^2 N}{\sqrt{\pi M_{pl}}} \text{supp} \left( \frac{M(x_R)}{x_R^2} \right) \sim \beta_2(\sigma)$$ (50 Hz). (29)
where the function

\[ \beta_2(\sigma) \approx 123.4 \kappa \left( m^2 N \right) \text{GeV}^{-2}, \tag{30} \]

has a tuning factor \( \kappa \), that is a function of the variable brane tension, according to Eq. (29) and to the last column of Table II, given by

\[
\kappa = \begin{cases} 
1, & \text{for } \sigma \to \infty \text{ (GR limit)} \\
1.020, & \text{for } \sigma = 10^{12} \text{ MeV}^4 \\
1.088, & \text{for } \sigma = 10^9 \text{ MeV}^4 \\
1.114, & \text{for } \sigma \sim 10^6 \text{ MeV}^4 
\end{cases} \tag{31} \]

The parameter \( \kappa \) indicates the corrections to the unit (that corresponds to the \( \sigma \to \infty \) GR limit), for different values for the brane tension.

Similarly to what was accomplished in the Subsect. III. A, dark SU(N) stars have parameters in the ranges \( 100 \text{ eV} \lesssim m \lesssim 10 \text{ KeV} \) and \( 10^3 \lesssim N \lesssim 10^6 \), yielding a maximum \( 10^6 M_\odot \lesssim M \lesssim 10^9 M_\odot \) for the dark SU(N) MGD star mass, whereas the lowest dark SU(N) MGD star radius lies in the range \( 10^2 \lesssim R \lesssim 10^5 \). Hence, the highest frequency of the gravitational wave by dark SU(N) MGD star mergers, is ruled by Eqs. (29) for the large \( N \) limit glueball potential. The highest frequency of gravitational wave radiation \( f_{\text{max}} \), from dark SU(N) MGD stellar mergers, can be allocated in the range 30 \( \mu \text{Hz} \lesssim f_{\text{max}} \lesssim 100 \text{ mHz} \), that can be further detected by the eLISA mission [30]. Also, the LIGO experiment can probe the spectrum 50 Hz \( \lesssim f_{\text{max}} \lesssim 1 \text{ KHz} \). Both these spectra are, respectively, represented by the light-gray and the gray bands in Figs. 9 and 10 below, respectively, in the \( \sigma \to \infty \) GR limit and in the brane tension bound \( \sigma \gtrsim 3.18 \times 10^6 \text{ MeV}^4 \), for the large \( N \) glueball potential.
\[ \sigma \to \infty \] GR limit has a tiny intersection between the possibility of detection at the LIGO experiment and the black band that represents the self-interacting glueball dark matter, in Fig. 9. However, the finite brane tension makes this black band to be larger in Fig. 10. It represents an increased spectrum to be probed by the LIGO and eLISA experiments, that are themselves also slightly thickened by brane-world effects. It can be realized by respectively comparing the gray and the light-gray bands in Figs. 9 and 10.

IV. CONCLUDING REMARKS AND OUTLOOK

The pure SU(N) Yang-Mills glueball dark matter model was proposed to condense into dark SU(N) MGD stars on fluid branes, representing compact configurations of scalar glueball fields. The scalar glueball dark matter model may, eventually, decay into Standard Model elementary particles [14]. Figs. 5 and 10 shows that a fluid brane-world scenario, wherein the brane tension varies according to Eq. (8), provides a thickened band that represents the self-interacting glueball dark matter, for respectively both \( \phi^4 \) and the large \( N \) potentials that describe the glueball self-interaction. Since by self-interaction the glueballs can agglutinate and condense into dark SU(N) MGD stars, this band has a larger intersection with the LIGO (and the eLISA) experiment to detect gravitational waves.

Taking into account a finite brane tension makes the signature of dark SU(N) star mergers more susceptible to be detected by the LIGO experiment and by the future LISA/eLISA project. In fact, with the most recent and precise brane tension bound [22], Fig. 5 shows a bigger area of intersection, between the self-interacting glueball dark matter and both the experimental windows for detection, than its GR \( \sigma \to \infty \) limit depicted in Fig. 4. Analogously, for the large \( N \) potential, Fig. 10 also presents a larger range of intersection, between the large \( N \) self-interacting potential glueball dark matter and both the LIGO and the eLISA detectable spectra.

To summarize, Figs. 4 and 9 represent the GR \( \sigma \to \infty \) limit, respectively both \( \phi^4 \) and the large \( N \) potentials that describe the glueball self-interaction, representing a more improbable scenario to detect gravitational waves. Hence, dark SU(N) MGD stars, that have mass and radius corrected by the bulk 5D Weyl fluid, should be better detectable by the current LIGO experiment [26] and by the eLISA project [30].

Finally, the search for TeV-scale gravity signatures in the ATLAS detector at \( s = 13 \) TeV is being currently approached [31]. Black holes produced with a mass above the formation threshold evaporate in Higgs particles, leptons, particle jets and photons and are currently searched at the LHC. Moreover, signatures of TeV extra-dimensional models are encoded in the partners of the Z and W bosons that might be permitted to access the 5D bulk. These partners can be manifested as resonances in dilepton spectra and are more intricate to detect at the LHC. Kaluza-Klein partners of the Standard Model particles have not been observed still, pushing the lower mass limits beyond 13 TeV. To summarize, collider data indicate string theory magnitude extra dimensions, whose searches continue at the LHC and confirmation could happen in the next generation of colliders [32]. The search for evidences of extra dimensions can be, then, dislocated also to observational aspects, as accomplished in Sect III, by investigating models whose signatures increase the spectrum of the highest frequency of gravitational wave radiation by the LIGO and eLISA experiments.

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