Quantized Temperatures Spectra in Curved Spacetimes

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Abstract

We consider the thermal properties of a scalar field theory on curved spacetimes. In particular, we argue for the existence in the de Sitter, Kruskal and Rindler manifolds of a discrete spectrum of allowed temperatures (the odd multiples of a fundamental one). For each temperature we give an explicit construction of the relative two point function in terms of the lowest temperature one. These results are actually valid for a wider class of static metrics with bifurcate Killing horizons, originally studied by Sewell. Some comments on the interpretation of our results are given.
It is well known that thermal effects occur when quantizing a field theory on certain kinds of curved spacetimes, the relevant cases being the Rindler, de Sitter and Kruskal manifolds \[1-4\]. In all these cases the presence of a dimensional parameter in the theory defines a fundamental temperature \(T_0 \neq 0\) which characterizes the ground state of the field when restricted to a suitable region of the manifold (wedge) bounded by horizons. Inside this region one can identify a “time” variable, associated with a suitable time-like Killing field and the thermal behaviour displays itself through the properties of the analytic continuation of two point functions to complex “time” in that the ground state of the field satisfies the Kubo-Martin-Schwinger (KMS) condition with respect to the “time” evolution at temperatures given by \(T_0 = \frac{a \bar{h}}{2\pi c k}, T_0 = \frac{h c}{2\pi R k}\) and \(T_0 = \frac{h c^3}{8\pi M k}\), for the Rindler, de Sitter and Kruskal cases respectively. In these formulas \(a\) is the proper acceleration of the Rindler observer, \(R\) is the radius which characterizes the de Sitter manifold and \(M\) is the mass of the eternal Schwarzschild black-hole. Actually, as shown by Sewell \[5\], the existence of a fundamental temperature can be inferred under suitable conditions for a wide class of static metrics, of which the above ones are special cases.

The question arises as to which thermal equilibrium states of the field exist at temperatures other than the fundamental one. It is well known that in the flat minkowskian case and with respect to the time coordinate of an inertial observer, the fundamental temperature is zero, the ground state being the usual Minkowski vacuum, and that there is a KMS state at any temperature \(T\).

In this paper we argue for the fact that the only thermal states which arise on a manifold of the kind considered and with respect to the timelike Killing coordinate in the wedge are at temperatures which are the odd multiples of the fundamental one, and we give an explicit construction of the corresponding two point functions. Our results are understood in terms of the compact character of the time orbits in the complexified extension of the manifolds.
We spell out our construction in the case of the de Sitter (dS) manifold. Such a manifold is the maximally symmetric Lorentz manifold given by the hyperboloid with equation \(\sum_i (x^i)^2 - (x^0)^2 = R^2\) embedded in the five-dimensional Minkowski ambient space \(\mathbb{R}^5\) from which it inherits the metric and the isometry group, i.e. the Lorentz group \(O(4,1)\). The world line of a freely falling observer moving through the point \((0,0,0,0,R)\) and contained in the \((x^0,x^4)\)-plane is the geodesic parameterized as \(\gamma = \{x^0 = R \sinh \frac{t}{R}, x^1 = x^2 = x^3 = 0, x^4 = R \cosh \frac{t}{R}\}\). The set of all events of dS which can be connected with the observer by two light-signals (one future and one past directed) is called right wedge (denoted \(M^+\)); it is parametrized by the coordinate map: \(x(t,\underline{x}) = \{x^0 = \sqrt{R^2 - r^2} \sinh \frac{t}{R}, (x^1, x^2, x^3) = \underline{x}, x^4 = \sqrt{R^2 - r^2} \cosh \frac{t}{R}; r \equiv |\underline{x}| < R\}\). The left wedge \(M^-\) is defined as above with the replacement \(x^4 \rightarrow -x^4\). The boundary of \(M^+ \cup M^-\) is the union of the null surfaces \(H^\pm = \{x \in dS, x^0 = \pm x^4\}\). These are respectively the future/past horizon of the geodesic observer. Moreover, call \(\mathcal{V}^\pm = \{x \in dS : \pm x^0 > |x^4|\}\). To complete the covering of the manifold requires the additional maps (for \(\mathcal{V}^\pm\)) \(x^0 = \pm \sqrt{r^2 - R^2} \cosh \frac{t}{R}, x^1, x^2, x^3\) as before, \(x^4 = \sqrt{r^2 - R^2} \sinh \frac{t}{R}, r > R\). In terms of the given coordinates the dS metric reads \(d\tau^2 = [1 - (\frac{r}{R})^2]dt^2 - [1 - (\frac{r}{R})^2]^{-1}dr^2 - r^2(d\theta^2 + \sin^2 \theta d\varphi^2)\), where \(\theta\) and \(\varphi\) are the polar angles of \(\underline{x}\). The vector \(\partial_t\) is a Killing field which is timelike in the two wedges and only therein. Despite the fragmented structure, the \(t\) coordinate labels a single one-parameter isometry group \(\mathcal{T}\) of dS, namely the one induced by the Lorentz boosts in the \((x^0,x^4)\) plane of the ambient space, and for \(s \in \mathbb{R}\) one has

\[\mathcal{T}_s[x(t,\underline{x})] = x(t + s,\underline{x}) \equiv x^s.\] (1)

The orbits of \(\mathcal{T}\) are branches of hyperbolas or half lines (on the Killing horizons). We stress that \(\mathcal{T}\) is an isometry defined on the whole manifold, and moreover it can be defined uniquely for complex values too as an isometry \(\mathcal{T}_{t+i\sigma}\) in the de Sitter complexified manifold \(dS^C\). Since the parameter \(t\) can be interpreted as the proper time of the observer sitting on

\[1\]In the sequel we use units such that \(\hbar = c = G = k = 1\).
the geodesic $\gamma$, one can call $\mathcal{T}$ the “time-translation group relative to $\gamma$” (see [3] for a more detailed discussion).

We now discuss the quantum theory of a (quasi)-free hermitian bosonic scalar field $\Phi$ on dS. In this case, the whole content of the theory is contained in the two point function $\mathcal{W}(x_1, x_2) = \langle \Omega | \Phi(x_1)\Phi(x_2)\Omega \rangle$. This kind of theory has been studied repeatedly in the literature. As usual when one quantizes a field theory on a curved spacetime, there is no natural prescription for defining the “vacuum” state $\Omega$ of the field. In particular, the requirement of invariance under the full dS isometry group $O(4,1)$ selects in the massive case a two-parameter family of possible “vacua” [7]. Different criteria have been given to single out among these a distinguished state, which has been called the “euclidean” vacuum [3,7,8]. This state is characterized by the local Hadamard condition and is non-singular on the full Killing horizon [9] (there is one and only one vacuum of this kind for each mass $m > 0$).

More recently the special status of this theory has been described in terms of the analyticity properties of the $\mathcal{W}$’s [9]. The thermal properties of the euclidean vacuum $|\Omega\rangle$ are encoded in the $2\pi i R$ periodic analytic behaviour of the time translated correlation function $\langle \Omega | \phi(x_1)\phi(x'_2)\Omega \rangle = \mathcal{W}(x_1, x'_2) \equiv \mathcal{W}_{12}(t)$, and of its permuted $\langle \Omega | \phi(x'_2)\phi(x_1)\Omega \rangle = \mathcal{W}(x'_2, x_1) \equiv \mathcal{W}'_{12}(t)$, along the complexified orbits of $\mathcal{T}$. Precisely, $\mathcal{W}_{12}(t)$ satisfies the KMS condition at inverse temperature $\frac{1}{T_0} = \beta_0 = 2\pi R$ in the sense that, distributionally, $\mathcal{W}_{12}(t+i2\pi R-i0) = \mathcal{W}'_{12}(t-i0)$. Since the theory is quasifree, the KMS property $\langle \Omega | B_{t+i2\pi R-i0} | \Omega \rangle = \langle \Omega | B_{t+i0} A | \Omega \rangle$ extends immediately to any two fields observables $A$ and $B$ supported in bounded regions $O_1$ and $O_2$ contained in the right wedge. The question as to which thermal states exist at temperatures other than $T_0$ and defined on the field algebra on the whole dS manifold, is addressed by the following theorem.

**Theorem 1** Let $|\Omega_3\rangle$ be a local state for a hermitian scalar bosonic quantum field $\Phi$ on the dS manifold which is KMS in the right wedge at inverse temperature $\beta$ w.r.t. the evolution $\mathcal{T}$. Then $\beta$ must be of the form $\beta_0/(2L+1)$, where $L$ is a nonnegative integer and $\beta_0 = 2\pi R$. 
Proof. We express the proof using the smeared field \( \Phi(f) = \int_{dS} dx \sqrt{g(x)} f(x) \Phi(x) \), where \( f(\bullet) \) is an infinitely differentiable real function on dS with compact support. We consider the smeared two point function \( W(t) = \int dx dy \sqrt{g(x)g(y)} f(x)h(y) \langle \Omega_\beta | \Phi(x)\Phi(y')|\Omega_\beta \rangle \) and its permuted \( W'(t) \). By the KMS condition \( W(t) \) and \( W'(t) \) are boundary values \( W(t+i0) \) and \( W'(t-i0) \) of functions \( W(z) \) and \( W'(z) \) which are holomorphic in the strips \( 0 < \text{Im} \ z < \beta \) and \( -\beta < \text{Im} \ z < 0 \), \( \text{Re} \ z = t \) respectively, continuous on the boundary and such that \( W(t+i\beta-i0) = W'(t-i0) \). We prove first that \( \beta \leq \beta_0 \). Indeed choose \( f \) and \( h \) such that their respective supports \( O_1 \) and \( O_2 \) lie in the right wedge and are causally disjoint (see Fig. 1). Then the microcausality axiom and the hermiticity of the field imply that there exists a maximal non empty open interval \( I \), containing the origin, such that \( W(t) = W'(t) = W(t^*) \) if \( t \in I \). Furthermore, \( I \) is bounded since if we let \( t \) increase (or decrease) the support of \( h(x-t) \) will eventually intersect the causal domain of \( f \) and the function \( W(t) \) will cease to be real (otherwise the fields would always commute). Since an analytic function on the real axis which is real on a segment is real on the whole axis, this implies a breaking of analyticity. Then, because of the periodic structure of the manifold in the complex \( z \) time, the function \( W(z) \) must have a break of analitycity on the \( \text{Im} \ z = 2\pi R = \beta_0 \) line: indeed, this line must be identified with the real axis on which \( W(z) \) is not analytical. Then, if we had \( \beta \geq \beta_0 \) the function \( W(z) \) would be holomorphic in the whole strip \( 0 \leq \text{Im} \ z \leq \beta \), leading to a contradiction. Now, since \( W(z) \) assumes real values with continuity on the segment \( I \) of the real axis, by the Schwarz reflection principle we can analytically continue it across the segment itself by the mere definition \( W(z) \equiv W^*(z^*) \), \( \text{Im} \ z \leq 0 \). Similarly, we can continue \( W(z) \) above the line \( i\beta \) across the “gate” \( \text{Re} \ z \in I \). Iterating this procedure, we finally obtain an holomorphic function in the domain \( U = \{ \text{Im} \ z \neq n\beta, \ n \ \text{integer} \} \cup \{ \text{Im} \ z = n\beta, \ \text{Re} \ z \in I \} \). Moreover, since \( W(z) \) takes the same values at every “gate”, it is necessarily a periodic function with period \( i\beta \) (at \( \text{Im} \ z = n\beta \) \( W(z) \) is continuous across the gates \( \text{Re} \ z \in I \), and discontinuous across the cuts \( \text{Re} \ z < \text{Inf} \ I, \ \text{Re} \ z > \text{Sup} \ I \)). Besides, due to the angular character of the imaginary part
of \( z \), \( \mathcal{W}(z) \) must be also \( i\beta_0 \)-periodical. Then, since \( \beta \leq \beta_0 \) and since, in particular, \( \mathcal{W}(z) \) is holomorphic in the strips \( n\beta < \text{Im} \ z < (n+1)\beta \), \( n \) positive integer, it must be \( \frac{\beta_0}{\beta} = M \), \( M \) positive integer.

Now note that the function \( \mathcal{W}(t+i\frac{\beta_0}{2}) \) is real for all values of \( t \), since the shift \( t \rightarrow t+i\frac{\beta_0}{2} \) is a parity transformation which maps a wedge in the opposite one, and the two wedges are spacelike separated. Hence \( M \) must be odd, otherwise the line \( i\beta_0 \) would have to be identified with the real axis on which \( \mathcal{W}(z) \) is not real outside the gate \( I \).

Note that the theorem does not assert that such states \( |\Omega_\beta \rangle \) exist, but it only rules out all temperatures which are not odd multiples of \( T_0 \). We complete the proof by giving a construction of the KMS state at inverse temperature \( \beta = \beta_0/(2L + 1) \), for \( L = 1, 2, \ldots \) (if it exists), in terms of the state \( |\Omega_{\beta_0} \rangle \) at the fundamental temperature \( \beta_0 \). Let \( p_1, p_2 \in dS \).

Since \( t \) is a Killing coordinate we have \( \mathcal{W}_\beta(p_1, p_2) = \int_{-\infty}^{\infty} d\omega e^{-i\omega(t_1-t_2)} g_\beta(\omega; x_1, x_2) \) and a similar formula for \( \mathcal{W}_\beta' \). Also \( g_\beta(\omega) - g_\beta'(\omega) = c(\omega) \), independent of \( \beta \), since the commutator is a c-number. In terms of Fourier transforms, the KMS condition writes \( g_\beta'(\omega) = e^{-\beta\omega} g_\beta(\omega) \), so that \( g_\beta(\omega) = c(\omega)/(1-e^{-\beta\omega}) \) and \( g_\beta'(\omega) = c(\omega)/(e^{\beta\omega} - 1) \) (in the flat case we can perform the limit of zero temperature to get \( g(\omega) = \theta(\omega)c(\omega), \ g'(\omega) = -\theta(-\omega)c(\omega) \)). Now, if \( \beta = \beta_0/(2L + 1) \), \( L \) a positive integer, we have the identity

\[
\frac{1}{1-e^{-\beta\omega}} = \sum_{n=0}^{L} \frac{e^{-n\beta\omega}}{1-e^{-\beta\omega}} + \sum_{n=1}^{L} \frac{e^{n\beta\omega}}{e^{\beta\omega} - 1}.
\]  

Then, it follows from the above that \( \mathcal{W}_\beta \) is given by

\[
\mathcal{W}_\beta(p_1, p_2) = \sum_{n=0}^{L} \mathcal{W}_{\beta_0}(T_{-in\beta}p_1, p_2) + \sum_{n=1}^{L} \mathcal{W}_{\beta_0}'(T_{in\beta}p_1, p_2) .
\]  

The minkowskian limit can now be easily performed by letting \( R \to \infty \) with \( \beta \) fixed, namely \( L \to \infty \) and hence \( \beta_0 = (2L + 1)\beta \to \infty \), to get the usual formula for the minkowskian case.

The higher temperature states \( |\Omega_\beta \rangle \) are of course not fully symmetric even if the fundamental state \( |\Omega_{\beta_0} \rangle \) is a maximally symmetric one. In the flat Minkowskian case the vacuum state \( |\Omega_M \rangle \) is Poincaré invariant, whereas a thermal state \( |\Omega_{M,\beta} \rangle \) at \( T > 0 \) is invariant under
space rotations and spacetime translations but not under a Lorentz boost (Doppler dipole
anisotropy). In the dS case the states $|\Omega_\beta\rangle$ are invariant under space rotations and of course
under the “time” translation (1), but invariance under those dS transformations which as
$R \rightarrow \infty$ contract into the group of space translations is lost. This is easily understandable
if we note that as long as $R \neq \infty$ the generators $P_i$ of such transformations do not commute
among each other and their commutator contains the generators of local Lorentz boosts
(geodesic deviation) so that additional invariance under the $P_i$’s would imply invariance
under the full dS group, a property which is enjoyed only by the fundamental state $|\Omega_\beta_0\rangle$.

Since the proof of the theorem makes reference only to the existence of two wedges,
bounded by horizons and causally disconnected, in which $t$ is a Killing time which is periodic
on the imaginary axis in the complex extension of the manifold, the theorem extends without
modification to the Kruskal [3] and Rindler [4] manifolds, which possess the same relevant
structure. Indeed the dS, Schwarzschild and Rindler metrics are all particular cases of a class
of static metrics studied by Sewell [5] and for which the author has given sufficient conditions
for the existence of a temperature. The underlying manifold is of the form $\mathcal{M} = X \times Y$
(pointwise $x = (\eta, \xi; y)$), where $X$ and $Y$ are two-dimensional and $X$ is an open submanifold
of $\mathbb{R}^2$. The corresponding Lorentz metric is given by the formula

$$d\tau^2 = A(\eta^2 - \xi^2, y)(d\eta^2 - d\xi^2) - B(\eta^2 - \xi^2, y)d\sigma^2(y) ,$$

where $A$ and $B$ are positive valued, smooth functions and $d\sigma^2(y)$ is a positive metric on
$Y$. We define the following submanifolds of $\mathcal{M}$: $\mathcal{M}^\pm = \{\pm \xi > |\eta|\}$, $\mathcal{V}^\pm = \{\pm \eta > |\xi|\}$,
$\mathcal{H}^\pm = \{\eta = \pm \xi\}$. $\mathcal{M}^\pm$ and $\mathcal{V}^\pm$ can be parametrized by coordinates $\rho$ and $s$ as

$$\eta = \rho \sinh s, \quad \xi = \pm \rho \cosh s$$

for $\mathcal{M}^\pm$, and interchanging the definitions of $\eta$ and $\xi$ for $\mathcal{V}^\pm$. In terms of such coordinates
the metric writes

$$d\tau^2 = \pm A(-\rho^2, y)(\rho^2 ds^2 - d\rho^2) - B(-\rho^2, y)d\sigma^2(y) ,$$
where + refers to $\mathcal{M}^+$ and $\mathcal{M}^-$ are respectively the right and left wedge, and they are bounded by the bifurcate Killing horizon $\mathcal{H}^+ \cup \mathcal{H}^-$. We see from (3) that $s$ is a timelike Killing coordinate in $\mathcal{M}^\pm$. Moreover, it is seen from (3) that $\mathcal{M}$ is periodic along the imaginary $s$ axis with period $2\pi$. Therefore, under these conditions the theorem extends to $\mathcal{M}$ and the uniformly accelerated particle detector moving along the world line $\rho = \text{const}$, $y = \text{const}$ will thermalize in the field at either the temperature given by the Tolman relation

$$T_0(\rho, y) = \frac{1}{2\pi \rho A(-\rho^2, y)^2},$$

or at any of the odd multiples of $T_0(\rho, y)$. Note the dependence of $T_0$ on $y$. For example, if $y = (\theta, \varphi)$, $d\sigma^2(y) = d\theta^2 + \sin^2\theta d\varphi^2$, $T_0$ depends on the direction as is to be expected for a static non isotropic metric. Specializing the above formula to the dS and Schwarzschild cases one obtains $T_0(r) = \frac{1}{2\pi R - r^2}$ for dS and $T_0(r) = \frac{1}{8\pi M} \sqrt{1 - \frac{2M}{r}}$ for Schwarzschild. These temperatures describe an Unruh type effect experienced by a uniformly accelerated observer in the dS and Schwarzschild spacetimes, superimposed to the thermal one purely due to curvature. The dS and Hawking temperatures $\frac{1}{2\pi R}$ and $\frac{1}{8\pi M}$ are obtained in the limit of inertial motion $r = 0$ (dS) and $r = \infty$ (Schwarzschild). In particular, the formula in the dS case has been recently established in [11].

Note that our result is based on the assumption that the thermal state be defined on the whole manifold, thereby being defined even through the event horizons and indeed the quantized temperatures arise from the relation (3) between the local map and the whole manifold. In other words, we require the horizons not to act as physical barriers; however we strictly do not require local definiteness [11][12]. On the other hand, when the proper analyticity requirements are dispensed with, and the state is considered only within a single wedge every thermal behaviour may be displayed and no constraint at all on the temperatures is found. In conclusion, quantized temperatures spectra are expected for a quasi–free field theory on a given Lorentz manifold whenever an imaginary time Killing orbit is topologically equivalent to a circle (the compactness of the imaginary time orbit discretizes temperatures in the same way that periodicity in real time gives a quantization of energy). Note also that in order for thermal states to arise it is necessary that the metric be static (in the wedge). If the metric is stationary but not static, thermal effects are destroyed by inertial...
dragging. For example, Kay and Wald have shown [9] that no KMS Hadamard state exists for a quasifree field propagating in the Kerr background.

We conclude with some remarks. Our thermal states represent the static equilibrium configurations between the fixed background manifold and the thermal bath of the field quanta; indeed our states correspond to the Hartle-Hawking ones [3,4] of the Schwarzschild case. One may wonder whether this structured thermal behaviour reflects itself also in the non equilibrium and dynamical cases, namely when Unruh states [13] are considered in place of equilibrium states or when the backreaction is properly taken in account and the manifold is free to fluctuate. We cannot address this question here, but it seems sensible to believe that like the Hawking temperature arises both in the static and dynamical case this should happen to be true for the whole temperature spectrum as well. If so, and presumably by taking into account the self–interaction of the field, the thermal radiation expected, for example from a black hole, could be spread over different black body spectra; a structured feature in the radiation emitted from a quantum black hole, although in a quite different formulation, has been recently pointed out in [14], where a line emission spectrum is predicted. One may speculate that also in the dynamical dS case a composite emission could be relevant in connection with the cosmological description of the origin of the cosmic background radiation and the different inflationary scenarios.

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FIG. 1. The “time” evolution of the observable $h$ enters the causal domain of observable $f$. 
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