Minimal warm inflation with complete medium response

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Abstract

If a homogeneous field evolves within a medium, with the latter gradually picking up a temperature, then the friction felt by the field depends on how its evolution rate compares with medium time scales. We suggest a framework which permits to incorporate the contributions from all medium time scales. As an example, we illustrate how warm axion inflation can be described by inputting the retarded pseudoscalar correlator of a thermal Yang-Mills plasma. Adopting a semi-realistic model for the latter, and starting the evolution at almost vanishing temperature, we show how the system heats up and then enters the “weak” or “strong” regime of warm inflation. Previous approximate treatments are scrutinized.
1. Introduction

A general empirical observation from interacting multiparticle systems is that they tend to equilibrate fast, often after a handful of elementary scatterings. In the context of inflation, such a “reheating” is expected to take place after a period of exponential expansion (cf., e.g., ref. [1]). At the same time, the inflationary expansion itself is normally assumed to be non-thermal [2]. One reason is that a thermal mass generated for the inflaton field would in general compromise the desired flatness of the potential.

The problem with large thermal corrections can, however, be evaded with certain types of interactions. In the present paper we are concerned with the example of axion inflation [3], which at its late stage can be described by

\[ \mathcal{L} \supset \frac{1}{2} (\partial^\mu \varphi \partial_\mu \varphi - m^2 \varphi^2) - \frac{\varphi X}{f_a}, \quad X \equiv \frac{e^{\nu \rho \sigma} g^2 F_{\mu \nu}^c F_{\rho \sigma}^c}{64 \pi^2}, \quad c \in \{1, \ldots, N_c^2 - 1\}, \quad (1.1) \]

where \( F_{\mu \nu}^c \) is the Yang-Mills field strength, \( N_c \) is the number of colours, and \( f_a \) is the axion decay constant. The thermal mass generated by this interaction vanishes to all orders in perturbation theory. It does not vanish non-perturbatively, however the value is very small at temperatures above the confinement scale (cf., e.g., refs. [1,5] and references therein).\(^1\)

\(^1\)Axion inflation has been proposed to evade the steepness of the potential from other considerations as well, restricting to an Abelian gauge field and studying its dynamics on a classical level rather than letting it thermalize [6] (see, e.g., ref. [7] for recent work and references). Thermalized cases with non-Abelian gauge fields have also been considered (cf., e.g., refs. [8,13] and references therein), however the thermal friction coefficient did not have the form that is believed to dominate at high temperatures (see below).
Recently, it has been stressed that the system in eq. (1.1) can lead naturally to warm inflation [18-19]. Apart from the absence of a thermal mass correction, an essential ingredient for this is a thermal friction coefficient, which slows down the inflaton motion and keeps the Yang-Mills field at a finite temperature. The additional thermal friction may permit for successful inflation in spite of largish slow-roll parameters, thereby possibly helping to evade so-called swampland concerns [20,21].

In refs. [14-17], the value of the thermal friction coefficient was adopted from the domain of small frequencies, $\omega \sim \alpha^2 T$ (here $\alpha \equiv g^2/(4\pi)$ is the Yang-Mills coupling), where it is related to the so-called Chern-Simons diffusion rate [22], known from numerical simulations at temperatures far above the confinement scale [23]. However, at early stages of warm inflation, when the temperature is low, the rate of change of $\varphi$ is relatively speaking larger, $\omega \sim \pi T$ or even $\omega \gg \pi T$. The purpose of the current study is to present a framework permitting to study all of these domains. This should include an initial state with $T \approx 0$ and, if appropriate, a crossover from Hubble friction to thermal friction dominated dynamics.

2. Rates and frequency scales

The system described by eq. (1.1) is sensitive to a number of scales: $m, f_a$, as well as the confinement scale of the Yang-Mills plasma, $\Lambda_{\text{MS}}$. The thermal environment brings in the temperature, $T$; cosmology brings in the Planck mass, $m_{\text{Pl}} \approx 1.221 \times 10^{19}$ GeV; and in addition we need to insert an initial value for the inflaton field, $\varphi(0)$. The initial $\dot{\varphi}(0)$ is chosen so that we are close to a slow-roll regime. For simplicity we assume that after thermalization, $T \gg \Lambda_{\text{MS}}$ and $\alpha \ll 1$, however our theoretical ingredients also apply in a strongly coupled regime, even if the practical implementation would be much harder there.

A key quantity for warm inflation is a thermal friction coefficient, denoted traditionally by $\Upsilon$. Letting $H$ be the Hubble rate, a “strong regime” of warm inflation refers to $Q \equiv \Upsilon/(3H) \gg 1$, a “weak regime” to $Q \lesssim 1$. The total friction felt by $\varphi$ is $3H(1 + Q)$.

In chaotic inflation [24], where $f_a \to \infty$ and $Q = 0$, a sufficient number of $e$-folds is obtained if $\varphi(0) \gtrsim (\text{a few}) \times m_{\text{Pl}}$, and the COBE normalization of temperature fluctuations requires $m \ll m_{\text{Pl}}$. It is argued that in the strong regime of minimal warm inflation, it is sufficient to have $\varphi(0) \lesssim m_{\text{Pl}}$, as the additional friction slows down the evolution of the scalar field [14]. In the weak regime, thermal effects play a role only towards the end of the inflationary period, nevertheless they have been argued to shift the spectral tilt $n_s$ in a favourable direction [17].

As the operator $\varphi \chi/f_a$ added in eq. (1.1) is non-renormalizable, certain constraints need

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2Only ref. [15] considered exactly the setup of eq. (1.1), and then not for inflation but rather for early dark energy. Ref. [14] considered “hybrid inflation”, implying that eq. (1.1) is supplemented by a constant during the inflationary period, which rapidly switches off as inflation ends. Ref. [16] had an exponential potential, ref. [17] the periodic $\sim \Lambda^2[1 + \cos(\varphi/f)]$, which is equivalent to eq. (1.1) around the end when $\varphi \approx \pi f$.
to be satisfied in order for the framework to be self-consistent. If we consider on-shell wave modes with $\omega^2 \sim m^2$, then there is an ultraviolet divergence to the self-energy, of magnitude $\sim \alpha^2 m^4/f_a^2$ (cf. eq. (5.2)). This should be a small correction, so we require

$$\alpha^2 m^2 \ll f_a^2,$$

(2.1)

implying that $m$ can be at most modestly larger than $f_a$.

On the other hand, assuming a cold initial state, the effects from $\varphi \chi/f_a$ turn out to be insignificant far from the regime $\alpha^2 m^2 \sim f_a^2$. Once we approach this regime, eq. (2.2) implies that the width of the scalar field is of the same order as its mass. Then the inflaton decays very fast, generating thereby an ensemble of Yang-Mills bosons, which can subsequently thermalize.

As far as $\Upsilon$ is concerned, the damping rate of the scalar field depends strongly on the frequency scale at which it is evolving. In the limit of vacuum decays we have (cf. eq. (5.2))

$$\Upsilon_{\text{UV}} \sim \frac{\alpha^2 m^3}{f_a^2},$$

(2.2)

whereas for slow thermal processes the rate is (cf. eq. (5.6))

$$\Upsilon_{\text{IR}} \sim \frac{\alpha^5 T^3}{f_a^2}.$$

(2.3)

The comparison of these rates to $H$ determines whether we are in the weak or strong regime.

A further quantity playing a key role is what may be referred to as the thermalization rate of the Yang-Mills plasma,

$$\Delta \sim \alpha^2 T.$$

(2.4)

Only if the system is probed with a frequency $\omega \sim m < \Delta$, does eq. (2.3) represent the correct interaction rate. For $\omega \gg \Delta$, i.e. particularly at the beginning, we should rather use eq. (2.2). This setup guarantees that thermal physics plays a substantial role only in a domain where the assumption of a sufficient thermalization time is self-consistent.

Now, the value of $T$, and subsequently those of $\Delta$ and $\Upsilon$, are determined by the dynamics of the solution, i.e. by the parameters $m$, $H$, and also by the values of $\Delta$ and $\Upsilon$ themselves. This non-linear dependence implies that we cannot fix $\Upsilon$ in advance, but that all eventualities need to be accounted for by the basic equations. This makes our setup, introduced in sec. 3, more complicated than those in refs. [14–17], which adopted eq. (2.3) for all frequencies.

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3For completeness, we recall that in a strongly coupled system, such as chiral perturbation theory, the requirement $m \ll f_a/\alpha$ is replaced by $m_\pi \ll 4\pi f_\pi$.  

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3. Scalar field equation of motion

For a quantitative study, we need to establish the equation of motion satisfied by \( \phi \) in the presence of a heat bath\(^4\). We start by recalling the general linear response argument for how the dynamics of a weakly coupled field and a heat bath are connected to each other. We first proceed “blindly”, assuming that all integrals are well-defined; subsequently, short-distance singularities that are inevitable in quantum field theory are incorporated. The logic bears certain resemblances e.g. to the classic ref. \([22]\) even if we go beyond it in many ways.

According to the equivalence principle, we are free to choose the frame in which to derive the equation of motion. For thermal field theory and statistical physics, it is convenient to operate in the medium rest frame, such that its four-velocity is \( u = (1, 0) \) and temperature appears in its textbook form. It is also helpful that all parameters are, to a good approximation, time-independent. For these reasons we first consider a locally Minkowskian frame. Subsequently the equation of motion is written in a covariant form, and we can then easily incorporate an expanding Friedmann-Lemaître-Robertson-Walker metric.

Consider a theory defined by

\[
\mathcal{L} = \frac{1}{2} \left( \partial^\mu \phi \partial_\mu \phi - m^2 \phi^2 \right) - \phi J + \mathcal{L}_\text{bath},
\]

where \( J \) is a gauge-invariant composite operator (\( J = \chi/f_a \) in the case of eq. (1.1)) and \( \mathcal{L}_\text{bath} \) is the Lagrangian for the heat bath degrees of freedom. If we assume that \( \phi \) is spatially constant and evolves slowly, it should satisfy a classical equation of motion derived from eq. (3.1),

\[
\ddot{\phi} + m^2 \phi + \langle J(t) \rangle = 0.
\]

The last term is important because the average value of the composite operator depends on the slowly evolving value of the \( \phi \)-background.

To determine \( \langle J(t) \rangle \), we can inspect how the density matrix of the heat bath evolves with time in the presence of a \( \phi \)-background. Let us write the heat bath Hamiltonian as \( \hat{H} = \hat{H}_\text{bath} + \phi \hat{J} \), and assume that the initial density matrix \( \hat{\rho}(0) \) is an equilibrium state, i.e. \( [\hat{H}_\text{bath}, \hat{\rho}(0)] = 0 \). We now solve the evolution equation \( i\partial_t \hat{\rho}(t) = [\hat{H}(t), \hat{\rho}(t)] \) to first order,

\[
\hat{\rho}(t) = \hat{\rho}(0) - i \int_0^t dt' [\hat{H}(t'), \hat{\rho}(0)] + \ldots,
\]

and note that here \( \hat{H}(t') \) can be replaced with \( \phi(t')\hat{J}(t') \), because \( \hat{H}_\text{bath} \) commutes with \( \hat{\rho}(0) \).

\(^4\)If \( \phi \) evolves instead within a non-equilibrium ensemble, there may be additional contributions to the friction coefficient (cf., e.g., ref. \([3]\)). This could be important particularly if the ensemble consists of Abelian gauge fields, which are less likely to thermalize, and generate no sphaleron contribution from their thermal modes.
Inserting this solution we can compute
\[
\langle \hat{J}(t) \rangle \equiv \text{Tr} [\hat{\rho}(t) \hat{J}(t)] = \langle \hat{J}(0) \rangle_0 - \int_0^t \! \! dt' \varphi(t') C_R(t-t') + \mathcal{O}(J^3) ,
\]
where the retarded correlator is defined as
\[
C_R(t-t') \equiv \theta(t-t') \langle [\hat{J}(t), \hat{J}(t')] \rangle_0 ,
\]
and the expectation value \( \langle \ldots \rangle_0 \) is taken with respect to \( \hat{\rho}(0) \). As the time evolution of the Heisenberg operator \( \hat{J}(t) \) is given by \( \hat{H}_{\text{bath}} \), we have used \( \langle \hat{J}(t) \rangle_0 = \langle \hat{J}(0) \rangle_0 \) in eq. (3.4).

Assuming \( \langle \hat{J}(0) \rangle_0 = 0 \) because \( \hat{J} \) is odd under discrete symmetries, and re-expressing the time integration domain, we end up with
\[
\ddot{\varphi}(t) + m^2 \varphi(t) - \int_0^\infty \! \! dt' C_R(t-t') \varphi(t') = 0 , \quad t \geq 0 . \tag{3.6}
\]

At this point, we note from eq. (5.22) that the Fourier transform of \( C_R \) grows like \( \sim \omega^4 \) at large frequencies (with a logarithmically divergent coefficient). This implies that \( C_R(t-t') \) diverges like \( \sim 1/(t-t')^5 \) at short times. Then neither the convolution integral in eq. (3.6), nor the Fourier transform of \( C_R \), are well-defined.

Let us assume that, as is usual in quantum field theory, the divergences can be taken care of by introducing some regularization and subsequently cancelling them by local counterterms. Given the degree of divergence, this requires that we modify eq. (3.6) into
\[
\delta Z \varphi^{(4)}(t) + \ddot{\varphi}(t) + m^2 \varphi(t) - \int_0^\infty \! \! dt' C_{R,B}(t-t') \varphi(t') = 0 , \tag{3.7}
\]
where \( C_{R,B} \) now stands for a “bare” correlator. The regularization prescription is left implicit.

We now analyse eq. (3.7) through 1-sided Fourier transforms (or Laplace transforms). Defining
\[
\tilde{\varphi}(\omega) \equiv \int_0^\infty \! \! dt \, e^{i\omega t} \varphi(t) , \tag{3.8}
\]
and assuming that \( \varphi(t) \) grows at most power-like at large \( t \) (physically, its absolute value decreases at large \( t \)), we note that \( \varphi \) is analytic in the upper half of the complex plane (\( \text{Im} \omega > 0 \)). The inverse transform is a 2-sided one,
\[
\varphi(t) \theta(t) = \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{\varphi}(\omega) . \tag{3.9}
\]
Similarly, \( C_R(\omega) = \int_0^\infty \! \! dt \, e^{i\omega t} C_R(t) \) is analytic in the upper half-plane.

Inserting the Fourier transforms and making use of the convolution theorem, we find
\[
[ -\delta Z \omega^4 + \omega^2 - m^2 + C_{R,B}(\omega) ] \tilde{\varphi}(\omega) = \mathcal{G}[\omega, \varphi^{(n)}(0)] ,
\]
\footnote{Or, concretely, multiplying eq. (3.7) by \( e^{i(\omega+\Theta t)} \), integrating over \( t \geq 0 \), evaluating \( \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{C_{R,B}(\omega') \tilde{\varphi}(\omega')}{\omega' - \omega + i\Theta} \) with the Cauchy theorem, again assuming the presence of regularization, which cuts off \( C_{R,B} \) at large \( |\omega'| \).}
where $\mathcal{G}$ contains all terms related to initial conditions,

$$
\mathcal{G}[\omega, \varphi^{(n)}(0)] = \delta Z \left[ -\varphi^{(3)}(0) + i\omega \varphi(0) + \omega^2 \dot{\varphi}(0) - i\omega^3 \varphi(0) \right] - \dot{\varphi}(0) + i\omega \varphi(0). \tag{3.11}
$$

Eq. (3.10) is immediately solved and transformed back as

$$
\varphi(t) = \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{2\pi} \frac{1}{-\delta Z \omega^4 + \omega^2 - m^2 + C_{R,B}(\omega) G[\omega, \varphi^{(n)}(0)]} \, d\omega. \tag{3.12}
$$

We assume in the following that $\delta Z$ cancels the divergences of $C_{R,B}(\omega)$, and therefore omit both $\delta Z$ and the divergences from $C_{R,B}$, returning to the notation $C_R$. It should also be mentioned that the strange-looking eq. (3.11) will not be needed in practice, however it illustrates that initial conditions are subject to renormalization as well.

Now, the idea is to consider a “macroscopic” $t > 0$ and then to deform the integration contour in eq. (3.12) into the lower half-plane. The deeper we can go, the faster is the exponential fall-off of the solution. Hence, the slowest dynamics of $\varphi$ can be identified by searching for the singularities closest to the real axis. We note in passing that $G[\omega, \varphi^{(n)}(0)]$ has zeros, however their locations depend on initial conditions, and therefore cannot coincide in general with the roots of the denominator, which are independent of initial conditions.

An alert reader may worry about the need to know $C_R$ in the lower half-plane. Indeed many text-book relations, such as the spectral representation

$$
C_R(\omega + i0^+) = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\rho(\omega')}{\omega' - \omega - i0^+}, \tag{3.13}
$$

where $\rho(\omega) \equiv \text{Im} C_R(\omega + i0^+)$ is called the spectral function, concern the side of the upper half-plane. Furthermore, $0^+$ cannot be made negative, since $1/(\omega' - z)$ is discontinuous across the real axis: $1/(\omega' - \omega - i0^+) - 1/(\omega' - \omega + i0^+) = 2\pi i \delta(\omega' - \omega)$. That said, $C_R$ is still defined in the lower half-plane; it is just not analytic there, but must have singularities in the following we assume that the singularity structure of $C_R$ is known in the lower half-plane, and return in sec. 5 to examples for how it could look like (there may be poles and cuts).

With a given $C_R$, let us search for the roots of the denominator. Concretely, we inspect

$$
\dot{\varphi} + \Upsilon \varphi + m_2^2 \varphi = -\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\omega^2 + \Upsilon i\omega - m_2^2}{\omega^2 - m^2 + C_R(\omega)} e^{-i\omega t} G[\omega, \varphi^{(n)}(0)]. \tag{3.14}
$$

The parameters $\Upsilon$ and $m_2^2$ are to be chosen such that the leading singularities are lifted. We note that the “literal” initial conditions encoded in $G[\omega, \varphi^{(n)}(0)]$ play no role, as the equation of motion obtained by cancelling the poles is valid only at large times.

\footnote{A possible way to determine $C_R$ is to solve the Cauchy-Riemann differential equations, taking $C_R(\omega + i0^+)$ as the initial condition. This system can be rephrased as a 2-dimensional Laplace equation. It is known that the solution of the Laplace equation with Cauchy boundary conditions is unstable, reflecting the singularities."}
Given that $C_R$ is generated by a non-renormalizable interaction, the whole setup is consistent only to the extent that $C_R$ is treated as a small correction. In this situation, we can solve for the roots iteratively. At tree-level, the roots are at $\omega = \pm m$. The symmetries $\text{Re} C_R(-m) = \text{Re} C_R(m)$ and $\text{Im} C_R(-m) = - \text{Im} C_R(m)$ imply that we can restrict to the root at $\omega = +m$. Denoting

$$C_R(m) = \text{Re} C_R + i \text{Im} C_R,$$

the desired parameters are given by

$$\Upsilon \approx \frac{\text{Im} C_R}{m}, \quad m_T^2 \approx m^2 - \text{Re} C_R.$$  \hfill (3.15)

In the limit of slow evolution ($m \ll \alpha^2 T$), the real part of $C_R$ gives the correction

$$\delta m^2_{\text{IR}} \equiv - \text{Re} C_R(0).$$  \hfill (3.16)

The negative sign appears because we are in Minkowskian spacetime; a Wick rotation to Euclidean spacetime inserts an imaginary unit to $\chi$, and then eq. (3.16) corresponds to the topological susceptibility that is being measured in lattice simulations [4]. For $m \ll \alpha^2 T$, the imaginary part of $C_R$ produces a thermal friction coefficient in accordance with refs. [26, 27],

$$\Upsilon_{\text{IR}} \equiv \lim_{\omega \to 0} (-i) C'_R(\omega).$$  \hfill (3.17)

More generally, $\text{Re} C_R$ and $\text{Im} C_R$ are evaluated at $m > 0$, and can have different values.

Having derived the basic equation in locally Minkowskian spacetime, we can promote the result originating from eq. (3.14) to a general coordinate system,

$$\varphi^\mu \; ;_{\mu} + \Upsilon u^\mu \varphi_{,\mu} + m_T^2 \varphi \simeq 0.$$  \hfill (3.18)

Restricting finally to a homogeneous field, yields our final scalar equation of motion,

$$\ddot{\varphi} + (3H + \Upsilon) \dot{\varphi} + m_T^2 \varphi \simeq 0.$$  \hfill (3.19)

The form agrees with refs. [14–17], however $\Upsilon$ obtained from eq. (3.16) interpolates between eqs. (2.2) and (2.3), and $m_T^2$ may contain a non-trivial correction as well.

4. Heat bath equation of motion

Usually, the mass parameter $m_T^2$ is assumed to be temperature-independent, and we have argued below eq. (1.1) that this is the case in the present system to a good approximation.

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It should be possible to construct renormalizable models of warm inflation as well (cf., e.g., ref. [25]). Our discussion applies to these cases provided that the inflaton is coupled weakly to the plasma degrees of freedom.
However, taken literally, $m^2_T$ from eq. (3.16) may show temperature dependence. This implies that the free energy density carried by the scalar field, $V \equiv m^2_T \varphi^2/2$, also contributes to the entropy density, $s = s_r - \partial_T V$, where $s_r$ is the contribution of thermal radiation. Physically, storing free energy in $\varphi$, whose value carries definite information, decreases the total entropy. Then we should take care that the total entropy does not decrease with time.

Let us denote the partial derivatives of the effective potential by

$$ \partial_\varphi V \equiv m^2_T \varphi , \quad \partial_T V \equiv \frac{(\partial_T m^2_T) \varphi^2}{2}, \quad (4.1) $$

and the energy density and pressure of the Yang-Mills plasma by $e_r$ and $p_r$, respectively. In equilibrium, the corresponding total variables are $e = e_r + V - T \partial_T V$, $p = p_r - V$. The condition of entropy increase can now be imposed as

$$ T \partial_t \left[ (s_r - \partial_T V) a^3 \right] = a^3 \Upsilon \dot{\varphi}^2(t). \quad (4.2) $$

Making use of $e_r = T s_r - p_r$ and $s_r = \partial_T p_r$, this can equivalently be rewritten as an equation for the energy density,

$$ \dot{e}_r + 3H(e_r + p_r - T \partial_T V) - T \partial_t (\partial_T V) = \Upsilon \varphi^2(t). \quad (4.3) $$

We note that eq. (4.3) can alternatively be obtained by writing the energy-momentum tensor as a sum of the radiation and field parts, and imposing its overall conservation [28].

To close the system, the equations above are supplemented by the Friedmann equation,

$$ H = \sqrt{\frac{8\pi}{3} \frac{e_r + V - T \partial_T V + \dot{\varphi}^2/2}{m_{p1}}}, \quad (4.4) $$

where we have assumed a spatially flat universe. Eqs. (4.3), (4.4) apply for any type of a radiation equation of state, however for our numerical solution in sec. 6 we adopt the simple conformal form

$$ p_r = \frac{g_s \pi^2 T^4}{90}, \quad T \gg \Lambda_{MS}, \quad (4.5) $$

with $g_s$ constant; $e_r$ and $s_r$ follow from this as stated above eq. (4.3).

If $\partial_T m_T^2 = 0$, eqs. (4.3) and (4.4) reduce to the standard form employed in refs. [14][17]. We also find that the effects from $\partial_T m_T^2$ are numerically small for the benchmarks of sec. 6 as illustrated by the nearly constant values of $m_T$ in fig. 2.

### 5. Retarded pseudoscalar correlator

Having suggested a formalism depending on $C_R$, the remaining step is to specify its form. In this section we motivate a simple but semi-realistic approximation for $C_R$. 

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In a weakly coupled thermal system, $C_R$ and the corresponding spectral function, $\rho(\omega) = \text{Im} C_R(\omega + i0^+)$, contain a lot of structure. At $\omega \gg \pi T$, $C_R$ is dominated by a vacuum part, up to power-suppressed corrections $[29]$; at $\omega \sim \pi T$, it develops substantial thermal modifications $[30]$; at $\omega \sim gT$, features originate from collective plasma excitations and Debye screening $[30]$. For very small frequencies, $\omega \ll \alpha^2 T$, yet another behaviour takes over, dominated by the non-perturbative dynamics of colour-magnetic fields $[23]$. A numerical evaluation of the contribution of the scales $\omega \sim gT$ and $\pi T$ $[30]$ suggests that it is overshadowed by the contribution from very small frequencies. Therefore, we adopt a model in which $C_R$ only contains a vacuum part from $\omega \gg \pi T$ and an infrared part from $\omega < \sim \alpha^2 T$. Furthermore, the “tails” of the two contributions are numerically small outside their domains of validity, so we can establish an interpolation simply by adding the parts together,

$$C_R(\omega) \simeq C_R^{\text{vac}}(\omega) + C_R^{\text{IR}}(\omega).$$

(5.1)

Starting with $C_R^{\text{vac}}(\omega)$, which is known up to NLO $[31,32]$ and even beyond it, we restrict here to the leading-order part,

$$\frac{C_{R,B}^{\text{vac}}(\omega)}{16d_A c_\chi^2} \approx \frac{g^4 \omega^4}{(4\pi)^2 f_a^2} \left\{ \frac{1}{\epsilon} + 2 \ln \left( \frac{i\mu}{\omega} \right) - 1 \right\}, \quad d_A \equiv N_c^2 - 1, \quad c_\chi \equiv \frac{1}{64\pi^2}. \quad \text{(5.2)}$$

Here $D = 4 - 2\epsilon$ is the space-time dimension, $g^2$ denotes the renormalized coupling,

$$g^2(\bar{\mu}) \approx \frac{1}{2b_0 \ln(\bar{\mu}/\Lambda_{\text{MS}})}, \quad b_0 \equiv \frac{11N_c}{3(4\pi)^2}, \quad \text{(5.3)}$$

and $\bar{\mu}$ is the renormalization scale of the $\overline{\text{MS}}$ scheme.

Before turning to further technical specifications, let us elaborate on the physical meaning of eq. (5.2). Because of the logarithm, $C_R$ has an imaginary part along the positive real axis ($\text{Im} \ln i = \pi/2$). This corresponds to the decay width of the scalar particle into gauge bosons ($\phi \to gg$), and yields the vacuum contribution to $\Upsilon$, cf. eq. (3.16). The real part of $C_R$ yields a mass correction, cf. eq. (3.16), but also includes a divergence, which necessitates the introduction of a counterterm $\delta Z$ in eq. (3.7). Even though the presence of a divergence may sound like a formal issue, it is closely related to the existence of the logarithm, and can to some extent be viewed as a consequence of the presence of a physical decay width.

Returning to technical details, we note the presence of the scale parameter $\bar{\mu}$ in two different locations, eqs. (5.2) and (5.3). The scale parameter is just an auxiliary quantity, and must cancel from physical results. However, the way in which this happens in the two occurrences is conceptually different. The $1/\epsilon$ divergence and $\ln \bar{\mu}$ in eq. (5.2) are reflections of the fact that eq. (1.1) is non-renormalizable; the divergence can be cancelled by a counterterm, denoted by $\delta Z$ in eq. (3.7). As a result the combination $1/\epsilon + \ln \bar{\mu}^2$ gets replaced with a physical logarithm $\ln \Lambda^2$, where $\Lambda$ characterizes the UV completion of the theory. From the
low-energy perspective, $\Lambda$ is an additional free parameter. Its contribution is supposed to be numerically small compared with the tree-level term $m^2$ within the domain of eq. (2.1), and we set $\Lambda \equiv f_a$ from now on. We note that if $\omega \sim m \sim f_a/\alpha$, then $\text{Re} C_R < 0$, yielding a positive mass correction according to eq. (3.16).

In contrast, the scale parameter appearing in eq. (5.3) is related to renormalizable physics of the Yang-Mills plasma, and gets cancelled by higher-order contributions. We can incorporate their main effects simply by choosing $\bar{\mu}$ according to the physical scales that appear, thereby eliminating large logarithms. As a qualitatively reasonable recipe we set

$$\bar{\mu} = \sqrt{(2\pi \Lambda_{\text{MS}})^2 + (2\pi T)^2 + |\omega|^2}, \quad \min(2\pi T, |\omega|) \gg 2\pi \Lambda_{\text{MS}}. \quad (5.4)$$

Let us then turn to the IR part of $C_R$. The benchmark information here is that around origin, the spectral function goes over into a “transport coefficient”,

$$\lim_{\omega \to 0} 2T \text{Im} C_R(\omega) = \frac{\Gamma_{\text{diff}}}{f_a^2}, \quad (5.5)$$

where $\Gamma_{\text{diff}}$ refers to the Chern-Simons diffusion rate. The Chern-Simons diffusion rate dominates the result in the domain $\omega \ll \Delta$, where $\Delta$ is from eq. (2.4). The determination of $\Gamma_{\text{diff}}$ requires a Monte Carlo simulation. For us it is sufficient to transcribe the numerical result from ref. [23] into a form similar to that in eq. (5.2),

$$\text{Im} C_R(\omega) \omega \ll \Delta \Rightarrow \frac{\Gamma_{\text{IR}}}{\omega} \sim \frac{12g^4(g^2 N_e T)^3}{16\pi f_a^2} \left( \frac{m_E}{\gamma} + 3.041 \right), \quad (5.6)$$

where $\gamma$ is a solution of the equation

$$\gamma = \frac{g^2 N_e T}{4\pi} \left( \ln \frac{m_E}{\gamma} + 3.041 \right), \quad (5.7)$$

and $m_E^2 \equiv g^2 N_e T^2 / 3$ is the Debye mass squared of Yang-Mills plasma.

Concerning the full $C_R$, there is neither a numerical determination nor a good understanding concerning its $\omega$-dependence. However, there is a belief that the Chern-Simons number should undergo diffusive motion; if so, it may be described by the Langevin equation, which in turn gives rise to a Lorentzian spectral shape (cf., e.g., ref. [27]). Therefore we assume in the following that

$$C_R^{\text{IR}}(\omega) \omega \ll \Delta \Rightarrow \frac{\omega \Delta \Gamma_{\text{IR}}}{\omega + i\Delta}. \quad (5.8)$$

Here $\text{Im} C_R^{\text{IR}}$ is chosen to match eq. (5.6) for $|\omega| \ll \Delta$, and the constant part of $\text{Re} C_R^{\text{IR}}$ is chosen so that $C_R^{\text{IR}}(0) = 0$, in accordance with the discussion below eq. (1.1). For $\omega \neq 0$, eq. (5.8) yields $\text{Re} C_R < 0$, corresponding to a positive mass correction according to eq. (3.16).

The value of $\Delta$ is unknown, so we assume

$$\Delta = c \left( \frac{g^2 N_e}{4\pi} \right)^2 T, \quad c \simeq 10. \quad (5.9)$$
The large coefficient is chosen in order to make eq. (5.8) as flat as possible, being therefore maximally welcoming to the approximation adopted in refs. [14, 17].

6. Examples of numerical solutions

Given \( C_R \) from sec. 5, we can solve for \( \Upsilon \) and \( m^2_T \) from eq. (3.16). Subsequently eqs. (3.20), (4.3) and (4.4) can be integrated with given initial conditions.

For representing the medium, we adopt the QCD-like values \( N_c = 3, g_* = 2d_{\text{axion}} + 1 = 17, \Lambda_{\text{MS}} = 0.2 \text{ GeV} \). We stress, however, that this serves purposes of illustration only: we do not otherwise constrain the parameters through QCD-like axion physics. In the real world, the gauge group could rather be a unified one, and the scale parameter could be different. However, a small \( \Lambda_{\text{MS}} \) simplifies the analysis, because then \( \alpha \) is small (numerically \( \alpha \sim 0.015 \)) and we can use a conformal equation of state for thermal radiation in eq. (4.5), as long as the initial temperature is \( \gg \Lambda_{\text{MS}} \), e.g. \( T(0) = 10^{-10} m_{\text{Pl}} \).

The key parameters affecting the solution are \( m, f_a, \) and \( \varphi(0) \). We have adopted two benchmarks to illustrate the dynamics, one leading to the weak \( (Q \lesssim 1) \) and the other to the strong regime \( (Q \gg 1) \) of warm inflation:

\[
\begin{align*}
m &= 7 \times 10^{-7} m_{\text{Pl}}, & f_a &= 8 \times 10^{-7} m_{\text{Pl}}, & \varphi(0) &= 4 m_{\text{Pl}}, & \text{(weak regime)} \quad (6.1) \\
m &= 7 \times 10^{-7} m_{\text{Pl}}, & f_a &= 2 \times 10^{-7} m_{\text{Pl}}, & \varphi(0) &= 2 m_{\text{Pl}}, & \text{(strong regime)} \quad (6.2)
\end{align*}
\]

In both cases eq. (2.1) is rather marginally satisfied, and this turns out to be essential for the dynamics: if \( f_a \) is further decreased, the vacuum part of \( C_R \) starts to dominate but the framework becomes theoretically inconsistent; if \( f_a \) is increased, the effects from \( C_R \) disappear, and we return to usual (cold) chaotic inflation.

As far as phenomenology goes, the amplitude of scalar perturbations, \( A_s \), can always be chosen to match the observed value, by tuning \( m/m_{\text{Pl}} \). Currently the most stringent test comes from whether the spectral tilt, \( n_s \), matches the Planck result \[33\]. A challenge here is that as the solution may interpolate between the weak and strong regimes, the corresponding predictions need to be adopted from numerical work, which is typically specific to a particular model or parametric form of \( \Upsilon \) (cf., e.g., refs. [34, 37] and references therein). In any case, according to ref. [17], the weak regime could lead to a phenomenologically viable value of \( n_s \), whereas ref. [14] found that the strong regime only works by adding a constant to the

\[8\]The initial \( \dot{\varphi} \) is unimportant as the slow-roll solution is an attractor; we take \( \dot{\varphi}(0) \approx m^2 \varphi(0)/[3H(0)]. \]

\[9\]As far as other predictions go, the tensor-to-scalar ratio \( r \) is argued to be small in warm inflation, as scalar perturbations are increased by thermal fluctuations but tensor perturbations supposedly not, even if we note that thermal fluctuations of a weakly coupled scalar field do yield a substantial contribution to the gravitational wave production rate [38]. Non-Gaussianities are argued to offer for a characteristic signature of warm inflation [39]. However, for both of these observables measurements only give upper bounds for now.
potential, i.e. by considering hybrid rather than chaotic inflation. Here we have added no constant, and suspect that our benchmark points do not produce the correct $n_s$. Nevertheless, we hope that they can clearly illustrate general features of the dynamics.

Our numerical results for $\phi$, $T$ and the number of $e$-folds $N$ are plotted in fig. 1. As outlined in sec. 2, the key rates and frequency scales governing the nature of the solution are $\Upsilon$, $H$, $m_T$ and $\Delta$, and they are plotted in fig. 2. These are conveniently scaled to the (approximate) initial Hubble rate,

$$H_{\text{ref}} \equiv \sqrt{\frac{4\pi}{3} \frac{m\phi(0)}{m_{\text{pl}}}} .$$

We observe from fig. 2(left) that at the initial stage of a weak-regime solution, $\Delta \ll m_T$. Then the vacuum term in eq. (2.2) is the dominant component of $\Upsilon$. At a certain moment, the temperature increases and the system rapidly moves into a domain in which $\Delta > m_T$, so that eq. (2.3) dominates. Simultaneously, $\Upsilon$ becomes larger than the Hubble rate.

Decreasing $f_a$ moderately, according to eq. (6.2), boosts both the vacuum and thermal $\Upsilon$ by an order of magnitude. Then the temperature reaches its maximal value already at the beginning, as shown in fig. 1(right). Now $\Delta > m_T$ and $\Upsilon > H$ according to fig. 2(right), and the dynamics takes place in the strong regime. However, this is only achieved if the vacuum part of $\Upsilon$ (cf. eq. (2.2)) is sufficiently large at the beginning.
Figure 2: Left: the frequency scales $m_T$, $\Delta$ (cf. eqs. (3.16) and (5.9)) and the rates $H$, $\Upsilon$, and $\Upsilon_{\text{naive}}$ for the parameters in eq. (6.1), normalized to $H_{\text{ref}}$ from eq. (6.3). Here $\Upsilon_{\text{naive}} \equiv \kappa_5 T^3 / f_a^2$ is the $\omega$-independent thermal width employed in refs. [14–17], shown for illustration but not affecting our dynamics (we have inserted $\kappa = 10^2$ and $\alpha = 0.015$). Right: the same for the parameters in eq. (6.2). The conclusions drawn from these plots are discussed around the end of sec. 6.

7. Conclusions and outlook

There has been renewed interest in models of warm axion inflation, as it was noted that adopting a more realistic thermal friction coefficient than in early works changes the nature of the solution, possibly rendering a phenomenologically viable scenario and simultaneously addressing theoretical issues such as the swampland problem [14–17]. However, as duly acknowledged in these works, the new friction coefficient was still not fully realistic, as it was specific to a certain frequency domain, which may or may not be realized by the actual solution. The purpose of this paper has been to present a theoretical framework which permits to eliminate this approximation.

Numerical benchmark results from our framework are shown in figs. 1 and 2. We find that reaching the strong regime of warm inflation requires fine-tuning, as the vacuum contribution from the non-renormalizable operator, eq. (2.2), needs to be substantial at early times, yet not so large that it would render the framework inconsistent. A weak regime exists more broadly, as increasing $f_a$ decreases the significance of the non-renormalizable operator and connects us to the case of cold inflation. On the qualitative level, we confirm the existence of scenarios as proposed in refs. [14–17], even if we find strong dependence on the vacuum
contribution in eq. (2.2), which was omitted in those works. As a consequence, the range of admissible values of $f_a$ is more tightly constrained. We stress again that eq. (2.3) employed in refs. [14–17] is not active around the beginning, when $m \gg \alpha^2 T$.

There are a number of directions in which our work could be expanded. One concerns the retarded pseudoscalar correlator $C_R$, which determines the value of the vacuum or thermal friction coefficient. Here we adopted a reasonable model, but more information and NLO corrections could be added in certain frequency domains [30]. One obstacle is, however, that the infrared part of $C_R$ is currently poorly understood, with only the transport coefficient in eq. (5.5) estimated, but the frequency dependence subject to argumentation (cf. sec. 5). Knowing the frequency dependence would be essential, as it determines the parameter $\Delta$ that dictates whether the vacuum or thermal value of $\Upsilon$ should be used.

Another extension would be to carry out parameter scans with various potentials, mapping the possibly fine-tuned domains that may be phenomenologically most viable. In principle there is no obstacle to attacking this challenge, apart from the fact that predictions rely on an interpolating function for the spectrum of scalar fluctuations, whose determination involves model dependences. Before embarking on scans, this function might deserve an independent re-evaluation (cf., e.g., ref. [40]).

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