ON SPECIAL GENERIC MAPS OF RATIONAL HOMOLOGY SPHERES INTO EUCLIDEAN SPACES

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ABSTRACT. Special generic maps are smooth maps between smooth manifolds with only definite fold points as their singularities. The problem of whether a closed n-manifold admits a special generic map into Euclidean p-space for 1 ≤ p ≤ n was studied by several authors including Burlet, de Rham, Porto, Furuya, Eliašberg, Saeki, and Sakuma. In this paper, we study rational homology n-spheres that admit special generic maps into \( \mathbb{R}^p \) for \( p < n \). We use the technique of Stein factorization to derive a necessary homological condition for the existence of such maps for odd n. We examine our condition for concrete rational homology spheres including lens spaces and total spaces of linear \( S^3 \)-bundles over \( S^4 \), and obtain new results on the (non-)existence of special generic maps.

1. INTRODUCTION

Let \( f : M^n \to \mathbb{R}^p \), 1 ≤ p ≤ n, be a smooth map of a connected closed n-dimensional smooth manifold M into Euclidean p-space. A point \( x \in M \) is called a definite fold point of \( f \) if there exist local coordinates \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_p)\) centered at \( x \) and \( f(x) \), respectively, such that \( f \) takes the form

\[
y_i \circ f = x_i, \quad 1 \leq i \leq p - 1, \\
y_p \circ f = x_p^2 + \cdots + x_n^2.
\]

The map \( f \) is called a special generic map if every singular point of \( f \) is a definite fold point. In this paper, we study special generic maps of rational homology spheres, i.e., closed manifolds with the rational homology groups of a sphere.

The notion of special generic maps first appeared in a paper of Calabi \cite{calabi} under the name of quasisurjective mappings. We note that special generic maps \( M^n \to \mathbb{R} \) are the same as Morse functions of \( M \) with only maxima and minima as their critical points, and can thus only exist when \( M^n \) is homeomorphic to the standard n-sphere \( S^n \) by a well-known result of Reeb \cite{reeb}. Special generic maps \( M^n \to \mathbb{R}^2 \) were studied for \( n = 3 \) by Burlet and de Rham \cite{burlet}, and for \( n > 3 \) by Porto and Furuya \cite{porto}, and Saeki \cite{saeki}. Moreover, Sakuma \cite{sakuma} and Saeki \cite{saeki} studied special generic maps \( M^n \to \mathbb{R}^3 \) under various assumptions on the source manifold \( M^n \). Hara \cite{hara} studied the existence of special generic maps \( M^n \to \mathbb{R}^p \) for \( p \leq n/2 \) by

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using $L^2$-Betti numbers of $M^n$. Eliashberg [6] studied special generic maps $M^n \to \mathbb{R}^n$ for orientable $M^n$.

For a given source manifold $M^n$, Saeki posed the problem to determine the set $S(M^n)$ of all integers $p \in \{1, \ldots, n\}$ for which there exists a special generic map $M^n \to \mathbb{R}^p$ (see Problem 5.3 in [14]). Saeki observed that $S(M^n) = \{1, \ldots, n\}$ if and only if $M^n$ is diffeomorphic to the standard $n$-sphere $S^n$. Moreover, we note that if $S(M^n) \not\subset \{n\}$, then $M$ is (oriented) nullcobordant by Corollary 3.3 in [14].

Nishioka [11] determined the dimension set $S(M^2)$ for any simply connected closed 5-manifold $M$. In [20], the author determined the dimension set $S(\Sigma^7)$ for 14 of Milnor’s exotic 7-spheres $\Sigma^7$.

In this paper, we study the dimension set $S(M^n)$ of a rational homology sphere $M$ of odd dimension $n$ by using the technique of Stein factorization of special generic maps (see Section 2.2). Previously, Saeki [14] obtained the following characterization of homotopy spheres in terms of Stein factorization.

**Theorem 1.1** (Proposition 4.1 in [14]). Let $f: M^n \to \mathbb{R}^p$ ($1 \leq p < n$) be a special generic map. Then $M^n$ is a homotopy sphere if and only if the Stein factorization $W_f$ is contractible.

In Section 3, we show the following homological version of Theorem 1.1 for any coefficient ring $R \neq 0$ with identity.

**Theorem 1.2.** Let $f: M^n \to \mathbb{R}^p$ ($1 \leq p < n$) be a special generic map. Suppose that $M^n$ is orientable. If $M^n$ is an $R$-homology $n$-sphere (see Definition 2.1), then the Stein factorization $W_f$ is an $R$-homology $p$-ball. The converse holds under the additional assumption that $R$ is a principal ideal domain (for example, $R = \mathbb{Z}$ or $R = k$ a field).

We observe that Theorem 1.1 is a consequence of Theorem 1.2 for $R = \mathbb{Z}$. In fact, this follows from the homological version of the Whitehead theorem (see Corollary 4.33 in [9, p. 367]) by noting that $M^n$ is simply connected if and only if the Stein factorization $W_f$ is simply connected (see Proposition 3.9 in [14]).

As an application of our Theorem 1.2, we show in Proposition 4.2 that if a rational homology sphere $M$ of odd dimension $n = 2k + 1 \geq 5$ admits a special generic map into $\mathbb{R}^p$ for some $1 \leq p < n$, then the cardinality of the finite abelian group $H_k(M; \mathbb{Z})$ is the square of an integer. However, this is in general not a sufficient condition for the existence of special generic maps on $M$ (see Remark 4.3).

We point out that Proposition 4.2 can be seen as a torsion analog of the fact that a closed manifold which admits a special generic map into Euclidean $p$-space for some $1 \leq p < n$ has even Euler characteristic (see e.g. Corollary 3.8 in [14]). As shown in Proposition 4.5 the square of a positive integer can always be realized as $|H_k(M; \mathbb{Z})|$ for some highly connected rational homology sphere $M$ of suitable odd dimension $n = 2k + 1 \geq 5$ that admits a special generic map into $\mathbb{R}^p$ for some $1 \leq p < n$. On the other hand, there are plenty of rational homology $n$-spheres $M$ for which $|H_k(M; \mathbb{Z})|$ is not the square of an integer, so that $M$ admits no special generic maps into $\mathbb{R}^p$ for any $1 \leq p < n$. For instance, we show that this is the case for many lens spaces whose dimension is congruent to 3 (mod 4) (see Example 4.7), and many total spaces of linear $S^3$-bundles over $S^4$ (see Example 4.8).

**Notation.** The cardinality of a set $X$ is denoted by $|X|$. The symbol $\cong$ either means diffeomorphism of smooth manifolds or isomorphism of groups. The singular
locus of a smooth map \( f \) between smooth manifolds will be denoted by \( S(f) \). Let \( D^p = \{ x = (x_1, \ldots, x_p) \in \mathbb{R}^p; x_1^2 + \cdots + x_p^2 \leq 1 \} \) denote the closed unit ball in Euclidean \( p \)-space, and \( S^{p-1} := \partial D^p \) the standard \((p - 1)\)-sphere.

2. Preliminaries

In preparation of the proofs of our results, we review in this section several results on homology spheres and homology balls (see Section 2.1), and the Stein factorization of special generic maps (see Section 2.2).

2.1. Homology spheres and homology balls. Let \( R \neq 0 \) be a commutative ring with identity.

**Definition 2.1.** A closed \( R \)-orientable topological \( n \)-manifold \( P^n \) is called an \( R \)-homology \( n \)-sphere if \( \tilde{H}_n(P; R) \cong \tilde{H}_n(S^n; R) \) (where note that \( \tilde{H}_n(S^n; R) \cong R \) and \( \tilde{H}_i(S^n; R) = 0 \) for \( i \neq n \)). A compact \( R \)-orientable topological \( p \)-manifold \( Q^p \) with boundary is called an \( R \)-homology \( p \)-ball if \( H_*(Q; R) \cong H_*(D^p; R) = 0 \).

**Remark 2.2.** Let \( P^n \) be a closed topological manifold of dimension \( n > 0 \) such that \( H_n(P; R) \cong H_n(S^n; R) (\cong R) \). If \( 2R \neq 0 \), then \( P \) is automatically \( R \)-orientable. In fact, if such a manifold \( P^n \) was not \( R \)-orientable, then it would follow from Theorem 3.26(b) in [9] that \( H_n(P; R) \cong \{ r \in R; 2r = 0 \} \), which cannot be isomorphic to \( R \) as \( R \) has elements of order \( > 2 \) by assumption.

**Remark 2.3.** If a closed topological \( n \)-manifold \( P^n \) is orientable, then \( P^n \) is \( R \)-orientable for all \( R \). Conversely, if \( 2 \neq 0 \) in \( R \), then \( R \)-orientability of \( P^n \) implies that \( P^n \) orientable (see [9], p. 235).

**Proposition 2.4.** Suppose that \( R \) is a principal ideal domain (for example, \( R = \mathbb{Z} \) or \( R = k \) a field). If \( Q^p \) is an \( R \)-homology \( p \)-ball of dimension \( p \geq 1 \), then \( \partial Q \) is an \( R \)-homology \((p - 1)\)-sphere.

**Proof.** As \( Q^p \) is a compact \( R \)-orientable \( p \)-manifold, its boundary \( \partial Q \) is a closed \( R \)-orientable \((p - 1)\)-manifold. In order to show that \( \tilde{H}_*(\partial Q; R) \cong \tilde{H}_*(S^{p-1}; R) \), we remove the interior of a \( p \)-disk \( D^p \) embedded in the interior of \( Q \) to obtain a compact \( R \)-orientable \( p \)-manifold \( Q' \) with boundary the closed \( R \)-orientable \((p - 1)\)-manifold \( \partial Q' = \partial Q \sqcup S^{p-1} \). Using excision and the homotopy axiom for homology, we note that \( H_*(Q', S^{p-1}; R) \cong \tilde{H}_*(Q; R) = 0 \) because \( Q \) is an \( R \)-homology \( p \)-ball. Since \( R \) is a principal ideal domain, we can apply the universal coefficient theorem as stated on the bottom of p. 196 in [9] to \( G = R \) and the augmented chain complex \( C : \cdots \to C_0(Q', S^{p-1}) \otimes \mathbb{Z} R \to R \to 0 \) of the pair \((Q', S^{p-1})\) to conclude that \( H^*(Q', S^{p-1}; R) = 0 \). Then, \( H_*(Q', \partial Q; R) = 0 \) by Poincaré-Lefschetz duality (see Theorem 3.43 in [9] p. 254). Finally, from the reduced homology long exact sequences of the pairs \((Q', \partial Q)\) and \((Q', S^{p-1})\) we then see that \( \tilde{H}_*(\partial Q; R) \cong \tilde{H}_*(Q'; R) \cong \tilde{H}_*(S^{p-1}; R) \). \( \square \)

In Section 4 we are eventually concerned with the case \( R = \mathbb{Q} \), in which we replace the term “\( R \)-homology” by “rational homology”.

**Proposition 2.5.**  
(a) If \( P^n \) is an \( R \)-homology \( n \)-sphere, then \( \tilde{H}_i(P; \mathbb{Z}) \), \( i < n \), are finite abelian groups. If \( P^n \) is in addition orientable, then \( P^n \) is a rational homology \( n \)-sphere.
We define an equivalence relation \( \sim \) on \( X \) as follows. Two points \( x_1, x_2 \in X \) are called equivalent, \( x_1 \sim x_2 \), if there is a point \( y \in Y \) such that \( x_1 \) and \( x_2 \) are contained in the same connected component of the fiber \( f^{-1}(y) \). The equivalence relation \( \sim \) on \( X \) gives rise to a unique factorization of \( f \) of the form

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{q_f} & & \downarrow{q_f} \\
W_f, & \xrightarrow{f} & W_f,
\end{array}
\]

where \( W_f := X/\sim_f \) is the quotient space equipped with the quotient topology, \( q_f: X \to W_f \) is the continuous quotient map, and the map \( f: W_f \to Y \) is continuous. The diagram (2.1), or sometimes the space \( W_f \), is called the Stein factorization of \( f \).

Let \( f: M^n \to \mathbb{R}^p, 1 \leq p < n \), be a special generic map of a connected closed smooth \( n \)-manifold \( M \) into Euclidean \( p \)-space. In the following, we recall from [14] some important properties of the Stein factorization of \( f \).

As explained in [14] p. 267, the Stein factorization \( W_f \) of \( f \) can be equipped with the structure of a compact parallelizable smooth \( p \)-manifold with boundary in such a way that the quotient map \( q_f: M \to W_f \) is a smooth map which satisfies

(b) If \( Q^n \) is an \( R \)-homology \( p \)-ball, then \( \tilde{H}_i(Q;\mathbb{Z}), i \in \mathbb{Z} \), are finite abelian groups. If \( Q^n \) is in addition orientable, then \( Q^n \) is a rational homology \( p \)-ball.

Proof. Since \( P^n \) and \( Q^n \) are compact manifolds, their integral homology groups are finitely generated in every degree by Corollary A.8 and Corollary A.9 in [9] p. 527. Therefore, by applying the universal coefficient theorem for homology as stated in Theorem 3A.3 in [9] p. 264 to the augmented chain complex \( C: \cdots \to C_0(P) \to \mathbb{Z} \to 0 \) of \( P \), we conclude from \( H_i(C; \mathbb{R}) = \tilde{H}_i(P; \mathbb{R}) = 0 \) for \( i < n \) that \( \tilde{H}_i(P; \mathbb{Z}) = \tilde{H}_i(C) = 0 \) for \( i < n \) because \( R \neq 0 \). Similarly, we conclude from \( \tilde{H}_i(Q; \mathbb{R}) = 0 \) for \( i \in \mathbb{Z} \) that \( \tilde{H}_i(Q; \mathbb{Z}) = 0 \) for \( i \in \mathbb{Z} \). Thus, \( \tilde{H}_i(P; \mathbb{Z}) \), \( i < n \), and \( \tilde{H}_i(Q; \mathbb{Z}) \), \( i \in \mathbb{Z} \), are finite abelian groups. Finally, if \( P^n \) and \( Q^n \) are in addition orientable, then, using \( H_*(X; \mathbb{Q}) \cong H_*(X; \mathbb{Z}) \otimes \mathbb{Q} \) for any space \( X \) (see Corollary 3A.6(a) in [9] p. 266), it follows that \( P^n \) is a rational homology \( n \)-sphere, and \( Q^n \) is a rational homology \( p \)-ball.

Remark 2.6. If a closed \( R \)-orientable topological \( n \)-manifold \( P^n \) of dimension \( n > 0 \) satisfies \( \tilde{H}_i(P; R) = 0 \) for \( i < n \), then \( P \) is an \( R \)-homology \( n \)-sphere. In fact, analogously to the proof of Proposition 2.5(a) we can show that \( \tilde{H}_0(P; \mathbb{Z}) = 0 \) because \( n > 0 \). Thus, we have \( \tilde{H}_0(P; \mathbb{Z}) \cong \mathbb{Z} \), and obtain

\[
R \cong \text{Hom}(\tilde{H}_0(P; \mathbb{Z}), R) \cong H^0(P; R) \cong H_n(P; R)
\]

by the universal coefficient theorem for cohomology (Theorem 3.2 in [9] p. 195) and Poincaré duality for \( P \) (Theorem 3.30 in [9] p. 241). All in all, \( H_*(X; \mathbb{Q}) \cong H_*(X; \mathbb{Z}) \otimes \mathbb{Q} \) for any space \( X \) (see Corollary 3A.6(a) in [9] p. 266), it follows that \( P^n \) is a rational homology \( n \)-sphere, and \( Q^n \) is a rational homology \( p \)-ball.

2.2. Stein factorization of special generic maps. First, let us recall the notion of Stein factorization of an arbitrary continuous map.

Definition 2.7. Let \( f: X \to Y \) be a continuous map between topological spaces. We define an equivalence relation \( \sim_f \) on \( X \) as follows. Two points \( x_1, x_2 \in X \) are called equivalent, \( x_1 \sim_f x_2 \), if there is a point \( y \in Y \) such that \( x_1 \) and \( x_2 \) are contained in the same connected component of the fiber \( f^{-1}(y) \). The equivalence relation \( \sim_f \) on \( X \) gives rise to a unique factorization of \( f \) of the form

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{q_f} & & \downarrow{q_f} \\
W_f, & \xrightarrow{f} & W_f,
\end{array}
\]
\[ q_f^{-1}(\partial W_f) = S(f), \] and restricts to a diffeomorphism \( S(f) \cong \partial W_f \). Moreover, it is shown in the proof of Proposition 2.1 in \cite{14} that \( M \setminus S(f) \) is the total space of a smooth (not necessarily linear) \( S^{n-p} \)-bundle \( \pi: M \setminus S(f) \to W_f \setminus \partial W_f \) over the interior of \( W_f \). Furthermore, it is shown there that \( M \) is homeomorphic to \( \partial \tilde{E} \), where \( \tilde{E} \) is the total space of the topological \( D^{n-p+1} \)-bundle \( \rho: \tilde{E} \to W \) associated with the \( S^{n-p} \)-bundle \( \pi: \pi^{-1}(W) \to W \) that is the restriction of \( \pi \) over the closure \( W = W_f \setminus C \) of \( W_f \setminus C \) in \( W_f \) for a sufficiently small collar neighborhood \( C \cong \partial W_f \times [0,1] \) of \( \partial W_f \) in \( W_f \) (compare Proposition 3.1 in \cite{14}).

Let \( R \) be a commutative ring with identity. Since \( \tilde{E} \) is homotopy equivalent to \( W \), and \( W \cong W_f \) by construction, we have

\[
(2.2) \quad H_*(\tilde{E}; R) \cong H_*(W; R) \cong H_*(W_f; R)
\]

and

\[
(2.3) \quad H^*(\tilde{E}; R) \cong H^*(W; R) \cong H^*(W_f; R).
\]

From now on, let us assume that \( M \) is orientable. Then, the sphere bundle \( \pi \) is orientable in the sense of \cite{9} p. 442, and the associated disk bundle \( \rho \) is orientable as well. We conclude that the total space \( \tilde{E} \) of \( \rho \) is an orientable compact topological \( (n+1) \)-manifold. Since the manifolds \( M \), \( \tilde{E} \), and \( W_f \) are all \( R \)-orientable by Remark 2.3 Poincaré-Lefschetz duality (see Theorem 3.43 in \cite{9} p. 254) implies

\[
(2.4) \quad H_*(M; R) \cong H^{n-*}(M; R),
\]

\[
(2.5) \quad H_*(\tilde{E}, \partial \tilde{E}; R) \cong H^{n+1-*}(\tilde{E}; R),
\]

\[
(2.6) \quad H_*(W_f, \partial W_f; R) \cong H^{p-*}(W_f; R),
\]

and

\[
(2.7) \quad H_*(\partial W_f; R) \cong H^{p-1-*}(\partial W_f; R).
\]

Analogously to Proposition 3.10 in \cite{14}, we have a long exact sequence of the form

\[
(2.8) \quad \ldots \to H_{q+1}(M; R) \to H_{q+1}(W_f; R) \to H^{n-q}(W_f; R) \to H_q(M; R) \to H_q(W_f; R) \to H^{n-q+1}(W_f; R) \to \ldots
\]

\[
\ldots \to H_1(M; R) \to H_1(W_f; R) \to H^n(W_f; R) \to 0.
\]

(In order to derive \(2.8\), we start with the homology long exact sequence of the pair \( (\tilde{E}, \partial \tilde{E}) \). Then, we make the replacements \( H_q(\partial \tilde{E}; R) \cong H_q(M; R) \) by using that \( M \) is homeomorphic to \( \partial \tilde{E} \), \( H_q(\tilde{E}; R) \cong H_q(W_f; R) \) by \(2.2\), and \( H_q(\tilde{E}, \partial \tilde{E}; R) \cong H^{n+1-*}(\tilde{E}; R) \cong H^{n+1-*}(W_f; R) \) by using \(2.5\) and \(2.3\). The right end of \(2.8\) has the claimed form because \( M \) and \( W_f \) are both connected so that the map \( H_0(\partial \tilde{E}; R) \to H_0(\tilde{E}; R) \) is an isomorphism.)
Next, we note that
\begin{equation}
H^p(W_f; R) \cong H_{p-q}(W_f, \partial W_f; R) = 0, \quad q \geq p, \tag{2.9}
\end{equation}
where $H_0(W_f, \partial W_f; R) = 0$ holds because $W_f$ is connected as the image of the connected space $M$ under the surjective Stein factorization $q_f: M \to W_f$. Thus, using (2.9) and 2.8, we conclude that
\begin{equation}
H_q(M; R) \cong H_q(W_f; R), \quad q \leq n - p. \tag{3.2}
\end{equation}

3. PROOF OF THEOREM 1.2

Our proof is almost identical to the proof of Proposition 4.1 in [14]. However, we have to assure that it still works under the weaker assumptions (we use coefficients in $R$ instead of $\mathbb{Z}$, and make no assumptions about fundamental groups).

Let us first suppose that the Stein factorization $W_f$ of $f$ is an $R$-homology $p$-ball, with $R$ being a principal ideal domain. Since $M$ is orientable by assumption and $W_f$ is orientable as a parallelizable manifold, the total space $\tilde{E}$ of the disk bundle $\rho$ is an $R$-orientable compact topological $(n+1)$-manifold. Moreover, we have
\begin{equation}
H_*(\tilde{E}; R) \cong H_*(W_f; R) \cong H_*(D^p; R) \cong H_*(D^{n+1}; R). \tag{2.2}
\end{equation}

Thus, $\tilde{E}$ is an $R$-homology $(n+1)$-ball. Hence, using that $R$ is a principal ideal domain, we conclude from Proposition 2.4 that $\partial \tilde{E}$ is an $R$-homology $n$-sphere. As $M$ is homeomorphic to $\partial \tilde{E}$, $M$ is an $R$-homology $n$-sphere as well.

Conversely, we suppose that $M$ is an $R$-homology $n$-sphere, so that
\begin{equation}
H_*(M; R) \cong H_*(S^n; R) = 0, \quad q < n. \tag{3.1}
\end{equation}

Combining (3.1) with the long exact sequence (2.8), we obtain
\begin{equation}
H_q(W_f; R) \cong H^{n-q+1}(W_f; R), \quad 0 < q < n. \tag{3.2}
\end{equation}

Next, we observe that
\begin{equation}
\bar{H}_q(W_f; R) = 0, \quad q \leq n - p + 1, \tag{3.3}
\end{equation}
which follows for $q \leq n - p$ from (2.10) and (3.1), and for $q = n - p + 1 (> 0)$ (provided that $q < n$) from (3.2) and (2.9).

In the following, we show by induction on $q$ that $\bar{H}_q(W_f; R) = 0$ for all $q \leq p$. (Then, it follows immediately that the compact $R$-orientable $p$-manifold $W_f$ is an $R$-homology $p$-ball, where note that $W_f$ is orientable as a parallelizable manifold.) For this purpose, we fix $n - p + 2 \leq q \leq p$, and suppose that we have already shown
\begin{equation}
\bar{H}_i(W_f; R) = 0, \quad i \leq q - 1. \tag{3.4}
\end{equation}

In the following, we have to show that $H_q(W_f; R) = 0$. (Note that (3.3) is the basis $q = n - p + 2$ of the induction.) Since $0 < q < n$, we have
\begin{equation}
H_q(W_f; R) \cong H^{n-q+1}(W_f; R) \cong H_{p-n+q-1}(W_f, \partial W_f; R) \cong \bar{H}_{p-n+q-2}(W_f; R), \tag{3.5}
\end{equation}
where the last isomorphism is a connecting homomorphism in the reduced homology long exact sequence of the pair $(W_f, \partial W_f)$, and is an isomorphism because $\bar{H}_{p-n+q-1}(W_f; R) = 0 = \bar{H}_{p-n+q-2}(W_f; R)$ by induction hypothesis (3.4) (note
that \( p - n + q - 1 \leq q - 1 \) because \( p < n \). If \( q = n - p + 2 \), then we obtain as desired

\[
(3.6) \quad H_q(W_f; R) \overset{\text{[5.5]}}{=} \overline{H}_0(\partial W_f; R) = 0,
\]

where the last equality holds because the number of connected components of \( \partial W_f = \partial S(f) \) is at most \( 1 + \text{rank } H_{p-1}(M; \mathbb{Z}) = 1 \) by Proposition 3.15 in [14], where note that \( M \) is orientable. (Here, \( \text{rank } H_{p-1}(M; \mathbb{Z}) = 0 \) holds by Proposition [2.3] because \( 1 < p < n \).) If \( q > n - p + 2 \), then

\[
(3.7) \quad H_q(W_f; R) \overset{\text{[3.3]}}{=} H_{p-n+q-2}(\partial W_f; R) \overset{\text{[2.7]}}{=} H^{n-q+1}(\partial W_f; R).
\]

Since \( \partial W_f \cong S(f) \), and \( S(f) \) is a closed subset of the closed orientable \( n \)-manifold \( M^n \), we have \( H^{n-q+1}(\partial W_f; R) \cong H_{q-1}(M, M \setminus S(f); R) \) by Poincaré-Lefschetz duality as stated in Corollary 8.4 in [11, p. 352], where note that the Čech cohomology can be replaced by singular cohomology because \( S(f) \) is a manifold. Next, we observe that \( H_{q-1}(M, M \setminus S(f); R) \cong H_{q-2}(M \setminus S(f); R) \), which is a connecting morphism in the reduced homology long exact sequence of the pair \( (M, M \setminus S(f)) \), and is an isomorphism because \( \overline{H}_{q-1}(M; R) = 0 = \overline{H}_{q-2}(M; R) \) by (3.1), where \( q - 1 < n \) because \( q \leq p < n \). Altogether, we have shown that

\[
(3.8) \quad H_q(W_f; R) \overset{\text{[3.3]}}{=} H^{n-q+1}(\partial W_f; R) \cong H_{q-2}(M \setminus S(f); R).
\]

Now we recall that there is a smooth \( S^{n-p} \)-bundle \( \pi: M \setminus S(f) \to \text{int } W_f \) over the interior of the Stein factorization \( W_f \). We also recall that \( \pi \) is an orientable sphere bundle in the sense of [9, p. 442] because \( M^n \) is orientable. Let \( D(\pi): E' \to \text{int } W_f \) be the orientable topological \( D^{n-p+1} \)-bundle associated with \( \pi \). Then, the Thom isomorphism yields \( H^{s}(W_f; \mathbb{Z}) \cong H^{s+(n-p+1)}(E', M \setminus S(f); \mathbb{Z}) \) (see Corollary 4D.9 in [9, p. 441]). Consequently, \( H_{\pi}(W_f; \mathbb{Z}) \cong H_{\pi+(n-p+1)}(E', M \setminus S(f); \mathbb{Z}) \) by Corollary 3.3 in [9, p. 196]. Then, Corollary A.4 in [9, p. 264] yields

\[
(3.9) \quad H_{\pi}(W_f; R) \cong H_{\pi+(n-p+1)}(E', M \setminus S(f); R).
\]

Let us consider the following part of the homology long exact sequence of the pair \((E', M \setminus S(f))\):

\[
(3.10) \quad H_{q-1}(E', M \setminus S(f); R) \to H_{q-2}(M \setminus S(f); R) \to H_{q-2}(E'; R).
\]

Since \( E' \) is homotopy equivalent to \( W_f \), we have \( H_{q-2}(E'; R) \cong H_{q-2}(W_f; R) = 0 \) by induction hypothesis (3.4), where note that \( q - 2 > 0 \) because \( q > n - p + 2 > 2 \). Furthermore, we have

\[
H_{q-1}(E', M \setminus S(f); R) \overset{\text{[3.9]}}{=} H_{p-n+q-2}(W_f; R) \overset{\text{[3.4]}}{=} 0,
\]

where we can apply the induction hypothesis because \( 0 < q - n + p - 2 < q \). All in all, we obtain

\[
H_q(W_f; R) \overset{\text{[3.8]}}{=} H_{q-2}(M \setminus S(f); R) \overset{\text{[3.10]}}{=} 0.
\]

This completes the proof of Theorem 1.2.
4. AN APPLICATION IN ODD DIMENSIONS

For the proof of our application Proposition 4.2 we need the following

Lemma 4.1. Let \( \cdots \to A_{i-1} \to A_i \to A_{i+1} \to \cdots \) be a long exact sequence of finite abelian groups such that \( A_i = 0 \) for almost all \( i \in \mathbb{Z} \). If \( |A_{-i}| = |A_i| \) for all \( i \in \mathbb{Z} \), then \( |A_0| = k^2 \) for some integer \( k \).

Proof. Every map \( \alpha_i : A_i \to A_{i+1} \) of the given long exact sequence gives rise to a short exact sequence

\[ 0 \to \ker(\alpha_i) \to A_i \to \text{im}(\alpha_i) \to 0 \]

of finite abelian groups. Thus, we have

\[ (4.1) \quad |A_i| = |\ker(\alpha_i)| \cdot |\text{im}(\alpha_i)|, \quad i \in \mathbb{Z}. \]

Using that \( \ker(\alpha_i) = \text{im}(\alpha_{i-1}) \) by exactness of the given sequence, we have

\[ \prod_{i \text{ odd}} |A_i| = \prod_{i \text{ odd}} |\ker(\alpha_{i-1})| \cdot |\ker(\alpha_{i+1})| = \prod_{i \text{ even}} |\ker(\alpha_i)| \cdot |\ker(\alpha_i)| \]

\[ = \prod_{i \text{ even}} |A_i|, \]

where note that the products are finite because \( A_i = 0 \) for almost all \( i \in \mathbb{Z} \).

If \( |A_{-i}| = |A_i| \) for all \( i \in \mathbb{Z} \), then we can write

\[ |A_0| = \prod_{i \text{ odd}} |A_i| = \left( \prod_{j \geq 0} |A_{2j+1}| \right)^2 \left( \prod_{j \geq 1} |A_{2j}| \right)^2. \]

Thus, the positive integers \( a = |A_0|, x = \prod_{j \geq 0} |A_{2j+1}| \) and \( y = \prod_{j \geq 1} |A_{2j}| \) satisfy \( ay^2 = x^2 \). By comparing the exponents of prime numbers in the prime factorizations of \( a, x, \) and \( y \), we conclude that \( |A_0| = k^2 \) for some integer \( k \). \( \square \)

For the rest of this paper, let \( M \) be a connected closed smooth manifold of dimension \( n \geq 1 \).

The main result of this section is the following

Proposition 4.2. Let \( f : M^n \to \mathbb{R}^p \) (\( 1 \leq p < n \)) be a special generic map. If \( M^n \) is a rational homology n-sphere of odd dimension \( n = 2k + 1 \geq 5 \), then the cardinality of the finite abelian group \( H_k(M; \mathbb{Z}) \) is the square of an integer.

Proof. Since rational homology spheres are orientable by Remark 2.3, we conclude from Theorem 1.2 that the Stein factorization \( W_f \) is a rational homology \( p \)-ball. By Proposition 2.5 \( H_i(M; \mathbb{Z}), i < n, \) and \( \widetilde{H}_i(W_f; \mathbb{Z}), i \in \mathbb{Z}, \) are finite abelian groups. Hence, taking \( C \) to be the augmented chain complexes of \( M \) and \( W_f \) in Corollary 3.3 in [9, p. 196], we obtain

\[ (4.2) \quad \widetilde{H}^i(M; \mathbb{Z}) \cong \widetilde{H}_{i-1}(M; \mathbb{Z}), \quad i < n, \]

and

\[ (4.3) \quad \widetilde{H}^i(W_f; \mathbb{Z}) \cong \widetilde{H}_{i-1}(W_f; \mathbb{Z}), \quad i \in \mathbb{Z}, \]

respectively. Poincaré duality for \( M^n \) yields

\[ (4.4) \quad H_i(M; \mathbb{Z}) \cong H^{n-i}(M; \mathbb{Z}) \cong H_{n-i-1}(M; \mathbb{Z}), \quad 1 \leq i \leq n - 2. \]
In view of $H^{n-1}(W_f; \mathbb{Z}) = 0$ (see (2.9) applied for $q = n - 1 \geq p$) and (4.3), the long exact sequence (2.8) takes for $n \geq 5$ and $R = \mathbb{Z}$ the form

(4.5) $$0 = H_{n-2}(W_f; \mathbb{Z}) \rightarrow H_2(W_f; \mathbb{Z}) \rightarrow H_{n-3}(M; \mathbb{Z}) \rightarrow H_{n-3}(W_f; \mathbb{Z}) \rightarrow H_3(W_f; \mathbb{Z}) \rightarrow \ldots$$

$$\ldots$$

$$\ldots \rightarrow H_{q+1}(M; \mathbb{Z}) \rightarrow H_{q+1}(W_f; \mathbb{Z}) \rightarrow H_{n-q-1}(W_f; \mathbb{Z}) \rightarrow H_{q}(M; \mathbb{Z}) \rightarrow H_{q}(W_f; \mathbb{Z}) \rightarrow H_{n-q}(W_f; \mathbb{Z}) \rightarrow \ldots$$

$$\ldots$$

$$\ldots \rightarrow H_2(M; \mathbb{Z}) \rightarrow H_2(W_f; \mathbb{Z}) \rightarrow H_{n-2}(W_f; \mathbb{Z}) = 0.$$ 

By assumption, $n = 2k + 1$ is odd. Writing $\cdots \rightarrow A_{i-1} \rightarrow A_i \rightarrow A_{i+1} \rightarrow \ldots$ with $A_0 = H_k(M; \mathbb{Z})$ for the above exact sequence (4.5), our claim will follow from Lemma 4.1 once we show that $|A_{-i}| = |A_i|$ for all integers $i > 0$. If $i \equiv 1 \pmod{3}$, say $i = 3s + 1$, then we see that $A_i = H_{k-s}(W_f; \mathbb{Z}) = A_{-i}$ for $k-s \geq 2$, and $A_i = 0 = A_{-i}$ for $k-s < 2$. If $i \equiv 2 \pmod{3}$, say $i = 3s + 2$, then we see that $A_i = H_{k+1+s}(W_f; \mathbb{Z}) = A_{-i}$ for $k+1+s \leq n-2$, and $A_i = 0 = A_{-i}$ for $k+1+s > n-2$. If $i \equiv 0 \pmod{3}$, say $i = 3s$, then we see by means of (4.4) that $A_i = H_{k-s}(M; \mathbb{Z}) \cong H_{k+s}(M; \mathbb{Z}) = A_{-i}$ for $k-s \geq 2$, and $A_i = 0 = A_{-i}$ for $k-s < 2$. All in all, we have shown that $|A_{-i}| = |A_i|$ for all integers $i > 0$, which completes the proof of Proposition 4.2. 

**Remark 4.3.** Our homological condition in Proposition 4.2 is in general not sufficient for a rational homology sphere $M$ of odd dimension $n \geq 5$ to admit a special generic map into $\mathbb{R}^p$ for some $1 \leq p < n$. In fact, the real projective space $\mathbb{R}P^5$ satisfies $H_2(\mathbb{R}P^5; \mathbb{Z}) = 0$, but there does not exist a special generic map $f: \mathbb{R}P^5 \rightarrow \mathbb{R}^p$ for any $1 \leq p < 5$. (Otherwise, the universal cover $\pi: S^5 \rightarrow \mathbb{R}P^5$ would induce a 2-sheeted covering $W_{f\pi} \rightarrow W_f$ of Stein factorizations of the special generic maps $f \circ \pi$ and $f$ by Proposition 2.6 in [5]. Thus, the space $W_{f\pi}$ would have even Euler characteristic $\chi(W_{f\pi}) = 2\chi(W_f)$ while being contractible by Theorem 1.1.)

**Remark 4.4.** Suppose that $n = 4l + 1$ for some integer $l \geq 1$. After Seifert [17], the linking form $b: TH_{2l}(N; \mathbb{Z}) \times TH_{2l}(N; \mathbb{Z}) \rightarrow \mathbb{Q}/\mathbb{Z}$ on the torsion subgroup of the homology group $H_{2l}(N; \mathbb{Z})$ of a closed oriented topological $n$-manifold $N$ is a nondegenerate skew-symmetric bilinear form. Wall has shown (see Theorem 3 in
Theorem 1.2 that \( M \) is a simply connected rational homology \( p \)-manifold with boundary. If \( n > p > 0 \), then by Proposition 2.1 in [14], there exists a special generic map \( f: M^n \to \mathbb{R}^p \), where the closed smooth \( n \)-manifold \( M \) is diffeomorphic to the boundary of the product \( W \times D^{n-p+1} \) (after smoothing the corners), and the Stein factorization of \( f \) is diffeomorphic to \( W \).

Now, we suppose in addition that \( n = 2k+1 \) for some integer \( k > 1 \), and that \( W \) is a simply connected rational homology \( p \)-sphere with only non-vanishing integral homology group in positive degree is \( H_k(W;\mathbb{Z}) \cong \mathbb{Z}/m\mathbb{Z} \). Then, it follows from Theorem 1.2 that \( M \) is a rational homology \( n \)-sphere, and \( M \) is simply connected by Proposition 3.9 in [14]. Moreover, using the assumptions on the homology of \( W \) in the long exact sequence (4.6) from the proof of Proposition 4.2, we see that \( H_q(M;\mathbb{Z}) = 0 \) for \( 2 \leq q \leq k-1 \), and that there is a short exact sequence

\[
0 \to H_k(W;\mathbb{Z}) \to H_k(M;\mathbb{Z}) \to H_k(W;\mathbb{Z}) \to 0.
\]

Thus, \( M \) is \((k-1)\)-connected by the Hurewicz theorem, and we have \( |H_k(M;\mathbb{Z})| = |H_k(W;\mathbb{Z})|^2 = m^2 \). This shows that \( M \) will have all the desired properties.

Thus, it remains to construct a manifold \( W \) having all of the above properties. For this purpose, we consider a finite connected simplicial complex \( K \) embedded in some \( \mathbb{R}^a \), and whose only non-vanishing integral homology group in positive degree is \( H_1(K;\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \). (Such a simplicial complex \( K \) can be obtained by an embedded 3-dimensional lens space \( L(m,1) = L_m(1,1) \) (compare Example 4.7 below) with a small open 3-disk removed.) Then, by taking \( r \)-fold suspension, we obtain a finite connected simplicial complex \( L \) embedded in \( \mathbb{R}^{a+r} \) whose only non-vanishing integral homology group in positive degree is \( H_{r+1}(L;\mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \). It is well-known that \( L \) is the deformation retract of a regular neighborhood \( V \) in \( \mathbb{R}^{a+r} \) that is a compact smoothly embedded \((a+r)\)-manifold (which is in particular parallelizable). Then, by choosing \( r > 0 \) so large that \( n = 2k+1 > p \) with \( k = r+1 \) and \( p = a+r \), the manifold \( W = V \) will have all of the desired properties. (In particular, note that \( W \) is simply connected by the Freudenthal suspension theorem.)

This completes the proof of Proposition 4.5. \( \square \)

Remark 4.6. Concerning the choice of an embedded simplicial complex \( K \subset \mathbb{R}^a \) in the proof of Proposition 4.5, we note that \( a = 5 \) is sufficient because any orientable
closed 3-manifold can be embedded in $\mathbb{R}^5$ according to a result of Hirsch [10]. Moreover, by results of Zeeman [21] and Epstein [5], the 3-dimensional punctured lens space $L(m,l) \setminus \text{pt}$ can be embedded into $\mathbb{R}^4$ if and only if $m$ is odd. Consequently, in Proposition 4.5 we can realize all values $k \geq 4$, and also $k = 3$ when $m$ is odd. We do not know if there exists a special generic map $M^7 \to \mathbb{R}^p$, $1 \leq p < 7$, where $M$ is a rational homology 7-sphere such that $|H_3(M;\mathbb{Z})|$ is even.

We conclude with applications of Proposition 4.2 to determine the dimension sets of some rational homology spheres.

**Example 4.7 (lens spaces).** For an integer $m > 1$ and integers $l_1, \ldots, l_{k+1}$ ($k \geq 0$) relatively prime to $m$, the lens space $L_m(l_1, \ldots, l_{k+1})$ (see e.g. Example 2.43 in [9, p. 144]) is a closed smooth $(2k+1)$-manifold whose integral homology groups are given by

$$H_i(L_m(l_1, \ldots, l_{k+1});\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } i = 0, 2k+1, \\ \mathbb{Z}/m\mathbb{Z} & \text{for } i \text{ odd, } 0 < i < 2k+1, \\ 0 & \text{otherwise.} \end{cases}$$

If $k \geq 3$ is odd, and $m = |H_k(L_m(l_1, \ldots, l_{k+1});\mathbb{Z})|$ is not the square of an integer, then Proposition 4.2 implies that $L_m(l_1, \ldots, l_{k+1})$ does not admit a special generic map into $\mathbb{R}^p$ for any $1 \leq p < 2k+1$. Furthermore, Říha [6] has shown that there is a special generic map $L_m(l_1, \ldots, l_{k+1}) \to \mathbb{R}^{2k+1}$ if and only if $L_m(l_1, \ldots, l_{k+1})$ is stably parallelizable. Hence, we have $S(L_m(l_1, \ldots, l_{k+1})) = \{2k+1\}$ if $L_m(l_1, \ldots, l_{k+1})$ is stably parallelizable, and $S(L_m(l_1, \ldots, l_{k+1})) = \emptyset$ else. If $m$ is an odd prime and $1 \leq l_i \leq m-1$ for all $i$, then it follows from [7] that $S(L_m(l_1, \ldots, l_{k+1})) = \{2k+1\}$ if and only if $k < m$ and $l_1^2j + \cdots + l_{k+1}^2j$ is divisible by $m$ for $j = 1, \ldots, [k/2]$, where $[x]$ denotes the biggest integer $\leq x$ for a real number $x$.

**Example 4.8 (linear $S^3$-bundles over $S^4$).** As explained in [4], fiber bundles over $S^4$ with fiber $S^3$ and structure group $SO(4)$ are classified by elements of $\pi_3(SO(4)) \cong \mathbb{Z} \oplus \mathbb{Z}$. Moreover, the nontrivial integral homology groups of the total space $M_{m,n}$ corresponding to $(m,n) \in \pi_3(SO(4))$ are $H_0(M_{m,n};\mathbb{Z}) \cong H_7(M_{m,n};\mathbb{Z}) \cong \mathbb{Z}$ and $H_3(M_{m,n};\mathbb{Z}) \cong \mathbb{Z}/n\mathbb{Z}$. We note that $M_{m,n}$ is a rational homology 7-sphere for $n \neq 0$. Hence, by Proposition 4.2 $M_{m,n}$ does not admit a special generic map into $\mathbb{R}^p$ for any $1 \leq p < 7$ whenever $|n|$ is not the square of an integer. Moreover, Říha [6] has shown that there is a special generic map $M_{m,n} \to \mathbb{R}^7$ if and only if $M_{m,n}$ is stably parallelizable. According to Wilkens [19], this is equivalent to the vanishing of an obstruction $\hat{\beta} \in H^4(M_{m,n};\pi_3(SO)) \cong \mathbb{Z}/n\mathbb{Z}$. This obstruction has been determined to be $\hat{\beta} = \frac{1}{m}(M_{m,n}) \equiv 2m \pmod{n}$ in [4, p. 365]. All in all, if $|n|$ is not the square of an integer, then

$$S(M_{m,n}) = \begin{cases} \{n\}, & n \mid 2m, \\ \emptyset, & \text{else.} \end{cases}$$

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**References**

1. G.E. Bredon, *Topology and Geometry*, Springer-Verlag New York, Graduate Texts in Mathematics 139 (1993). DOI: 10.1007/978-1-4757-6848-0
2. O. Burlet, G. de Rham, *Sur certaines applications génériques d’une variété close à trois dimensions dans le plan*, Enseign. Math. 20 (1974), 275–292.
3. E. Calabi, *Quasi-surjective mappings and a generalization of Morse theory*, Proc. U.S.-Japan Seminar in Differential Geometry, Kyoto (1965), 13–16.
4. D. Crowly, C.M. Escher, *A classification of $S^4$-bundles over $S^4$*, Differential Geometry and its Applications 18 (2003), 363–380. DOI: 10.1016/S0926-2245(03)00012-3
5. D.B.A. Epstein, *Embedding punctured manifolds*, Proc. Amer. Math. Soc. 16 (1965), 175–176.
6. J.M. Eliashberg, *On singularities of folding type*, Math. USSR-Izv. 4 (1970), 1119–1134.
7. J. Ewing, S. Moolgavkar, L. Smith, R.E. Stong, *Stable parallelizability of lens spaces*, Journal of Pure and Applied Algebra 10 (1977), 177–191.
8. Y. Hara, *Special generic maps and $L^2$-Betti numbers*, Osaka J. Math. 34 (1997), 151–167.
9. A.E. Hatcher, *Algebraic topology*, Cambridge Univ. Press, 2002.
10. M.W. Hirsch, *The Imbedding of Bounding Manifolds in Euclidean Space*, Annals of Mathematics 74 (1961), 494–497.
11. M. Nishioka, *Special generic maps of 5-dimensional manifolds*, Rev. Roumaine Math. Pures Appl. 60 (2015), 507–517.
12. P. Porto, Y.K.S. Furuya, *On special generic maps from a closed manifold into the plane*, Topology Appl. 35 (1990), 41–52.
13. G. Reeb, *Sur certaines propriétés topologiques des variétés feuilletées*, Actualités Scientifiques et Industrielles 1183 (Hermann, Paris, 1952), 91–154.
14. O. Saeki, *Topology of special generic maps of manifolds into Euclidean spaces*, Topology Appl. 49 (1993), 265–293.
15. O. Saeki, *Topology of special generic maps into $\mathbb{R}^3$*, in: Workshop on Real and Complex Singularities (São Carlos, 1992), Mat. Contemp. 5 (1993), 161–186.
16. K. Sakuma, *On special generic maps of simply connected $2n$-manifolds into $\mathbb{R}^3$*, Topology Appl. 50 (1993), 249–261.
17. H. Seifert, *Verschlingungsinvarianten*, Sitzungsber. Preu. Akad. Wiss., Phys.-Math. Kl. 1933, No. 26–29 (1933), 811–828.
18. C.T.C. Wall, *Quadratic forms on finite groups, and related topics*, Topology 2 (1963), 281–298.
19. D.L. Wilkens, *Closed $(s - 1)$-connected $(2s + 1)$-manifolds, $s = 3, 7$*, Bull. Lond. Math. Soc. 4 (1972), 27–31.
20. D.J. Wrazidlo, *Standard special generic maps of homotopy spheres into Euclidean spaces*, Topology Appl. 234 (2018), 348–358.
21. E.C. Zeeman, *Twisting spun knots*, Trans. Amer. Math. Soc. 115 (1965), 471–495.

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