Stationary, isotropic and homogeneous two-dimensional turbulence: a first non-perturbative renormalization group approach

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Abstract
We study the statistical properties of stationary, isotropic and homogeneous turbulence in two-dimensional (2D) flows, focusing on the direct cascade, that is on large wave-numbers compared to the integral scale, where both energy and enstrophy are provided to the fluid. Our starting point is the 2D Navier–Stokes equation in the presence of a stochastic forcing, or more precisely the associated field theory. We unveil two extended symmetries of the Navier–Stokes action which were not yet identified, one related to time-dependent (or time-gauged) shifts of the response fields and existing in both 2D and 3D, and the other to time-gauged rotations and specific to 2D flows. We derive the corresponding Ward identities, and exploit them within the non-perturbative renormalization group formalism, and the large wave-number expansion scheme developed in Tarpin et al (2018 Phys. Fluids 30 055102). We consider the flow equation for a generalized $n$-point correlation function, and calculate its leading order term in the large wave-number expansion. At this order, the resulting flow equation can be closed exactly. We solve the fixed point equation for the 2-point function, which yields its explicit time dependence, for both small and large time delays in the stationary turbulent state. On the other hand, at equal times, the leading order term vanishes, so we compute the next-to-leading order term. We find that the flow equations for
equal-time $n$-point correlation functions are not fully constrained by the set of extended symmetries, and discuss the consequences.

Keywords: turbulence, Navier-Stokes equation, non-perturbative renormalization group

1. Introduction

The two-dimensional (2D) Navier–Stokes equation is a relevant description for large scale atmospheric and oceanic flows, rotating fluids, or magnetically forced stratified turbulence [1, 2]. In 2D, not only the energy, but also the enstrophy (squared vorticity) is conserved, which yields the existence of a double cascade, spreading on two distinct inertial ranges, as originally predicted by Kraichnan [3], Leith [4] and Batchelor [5]. The energy flows from the integral scale $L$, where both energy and enstrophy are injected, towards the large scales where it is dissipated by some mechanism, such as an Ekman friction at scale $L_0$, while the enstrophy flows towards the small scales, where it is dissipated by viscosity at the Kolmogorov scale $\eta$. In the forced-dissipative stationary regime, the inverse cascade of energy is characterized by Kolmogorov scalings, with a $k^{-5/3}$ decay of the energy spectrum. In the direct cascade of enstrophy, the scalings were deduced by Kraichnan and Leith, following Kolmogorov types of arguments, which yield a $k^{-3}$ decay of the energy spectrum [3, 4]. The expression of the spectrum was later corrected by Kraichnan himself, who suggested the presence of logarithmic corrections of the form $k^{-3} \ln(L_k)^{-1/3}$ to ensure the constancy of the enstrophy flux [6]. Moreover, exact relations for the equal-time three-point correlations, analogous to the $-4/5$ law in three-dimensional (3D) turbulence, can also be derived in 2D, exploiting the conservation of both energy and enstrophy in their respective inertial ranges [7, 8].

In many respects, the understanding of 2D turbulence appears more advanced than its 3D counterpart. For instance, exact bounds on the exponents of the structure functions are known [8]. The small-scale statistics of the vorticity in the direct cascade was investigated by Falkovich and Lebedev [9, 10]. They found that the Kolmogorov-like exponents of the $2n$-point structure functions of the vorticity are not modified by intermittency effects, but that the power-laws are corrected by logarithms, following $\langle \omega^n(\vec{r}_1)\omega^n(\vec{r}_2) \rangle \propto \ln(L/|\vec{r}_1 - \vec{r}_2|)^{2n/3}$. This result received support from experimental measurements in electromagnetically forced conducting fluid layers turbulence [11] and in flowing soap films [12]. These works all indicate that there is no substantial intermittency in the small-scale statistics of 2D turbulence, at least in the absence of an Ekman friction. In the presence of such a term, the exponents of the structure functions are changed by intermittency corrections, which depend on the friction coefficient [13–15]. Let us also mention that perturbative renormalization group (RG) techniques have been applied to study 2D turbulence in the presence of a power-law forcing [16–19]. We refer the reader to reviews on 2D turbulence for a more exhaustive account [1, 2, 8, 20].

Despite these results, the complete characterization of the statistical properties of 2D turbulence remains a fundamental quest. In particular, the previous results concentrate on the structure functions, which are equal-time quantities, but the time dependence of velocity or vorticity correlations is also of fundamental interest. In this respect, recent theoretical works have shown that the time dependence of correlation (and response) functions could be calculated within the non-perturbative (also named functional) renormalization group (NPRG) formalism [21, 22]. This framework allows one to compute statistical properties of turbulence from ‘first principles’, in the sense that it is based on the forced Navier–Stokes (NS)
equation and does not require phenomenological inputs. It was exploited in 3D to obtain the exact time dependence of n-point generalized velocity correlation functions at leading order at large wave-numbers at non-equal times [22]. The case of equal times is much more involved and its complete analysis in 3D is still lacking.

The purpose of this paper is to use the NPRG formalism to investigate 2D turbulence. The outcome is three-fold. (1) We identify two new extended symmetries of the 2D NS action and derive the associated Ward identities. (2) At leading order in wave-numbers, we show that the flow equation for a generic n-point correlation function in the stream formulation can be closed exactly as in the velocity formulation in 3D, and we derive the fixed-point solution for the 2-point correlation. The corresponding predictions can be tested in experiments or numerical simulations. (3) We present a first step in the analysis of equal-time correlations in 2D. We compute the next-to-leading order (NLO) term in the large wave-number expansion, in order to probe the presence of intermittency corrections at equal times. Interestingly, almost all the terms are controlled by (extended) symmetries. These controlled terms turn out to vanish at equal times, which means that, as the leading terms, they cannot generate intermittency effects for equal-time correlations. However, the symmetries do not seem to be sufficient to completely constrain the flow equation at NLO, and the only remaining term could be non-zero and be responsible for intermittency. Our analysis suggests that the corresponding effects, if any, are anyhow weaker than in 3D, since almost all the terms are controlled and vanish in 2D because of the specific time-gauged rotation symmetry, which does not occur in 3D. This is in accordance with standard observations, which support weak or no intermittency in the direct cascade in 2D turbulence.

The remainder of the paper is organised as follows. The field theory associated with the 2D NS equation in the presence of a stochastic forcing is revisited in section 2, where its extended symmetries are analyzed in details. The NPRG formalism to study this field theory, developed in [22, 23, 24], is briefly presented in section 3, and the Ward identities related to the extended symmetries are derived in this framework. We then consider the flow equation for a generalized n-point correlation function in the stream formulation, and explain the principles of the large wave-number expansion. In section 4, the leading order term in this expansion is calculated exactly, and the corresponding fixed-point solution is obtained for the two-point function, yielding the general form of its time dependence. The leading order term vanishes at equal time, as in 3D. In section 5, we calculate the NLO term in the large wave-number expansion, focusing on coinciding times, and discuss the results.

2. Field theory for the stream function and its symmetries

We consider the NS equation in the presence of an external stirring force \( f_\alpha^{\text{inj}} \) and of an energy damping force \( f_\alpha^{\text{damp}} \), in order to sustain a stationary turbulent regime

\[
\partial_t v_\alpha + v_\beta \partial_\beta v_\alpha = \nu \partial^2 v_\alpha - \partial_\alpha p + f_\alpha^{\text{damp}} + f_\alpha^{\text{inj}},
\]

where \( \nu \) is the kinematic viscosity and \( p \) is the pressure divided by the density of the fluid, and where the velocity, pressure and force fields depend on the space-time coordinate \((t, \vec{x})\). This equation is supplemented by the incompressibility constraint

\[
\partial_\alpha v_\alpha = 0.
\]

\[\text{Let us point out that from the mathematical point of view, it is implicitly assumed in the present manuscript that, for reasonable boundary conditions, there exists a unique weak solution of the forced Navier-Stokes equation, except possibly for some configurations of the forcing with zero probability.}\]
Since in 2D the energy is transferred towards the large scales, the energy damping force is necessary to provide a dissipation mechanism at the largest scales. This damping can be achieved by a linear Ekman friction, \( \vec{f}^\text{damp} = \alpha \vec{v} \), which models the friction exerted on the bulk by \( \vec{L} \) local generalization of this term with a characteristic length scale \( L \gg L_0 \). This term can be interpreted as an effective friction acting only at the boundaries of the fluid, while not affecting the small scales, contrary to the Ekman friction.

For the derivation of the associated field theory and since universality is expected with respect to the precise form of the stirring force (as long as its characteristic distance scale \( L \) is \( L \gg \eta \)), one usually chooses a stochastic forcing with Gaussian distribution of variance

\[
\langle f_\alpha^{(\text{in})}(\vec{x},t) f_\beta^{(\text{in})}(\vec{x}',t') \rangle = D_{\alpha\beta}(t - t', \vec{x} - \vec{x}') = 2 \delta_{\alpha\beta} \delta(t - t') N_{L^{-1}}(|\vec{x} - \vec{x}'|).
\]

(4)

The mapping to a field theory is then achieved using the Martin-Siggia–Rose–Janssen–de Dominicis (MSRJD) response-field formalism, developed in [25–27]. Whereas in most derivations in the context of turbulence, the profile \( N \) has to be a power-law [17, 28–31], it can within the NPRG be shaped as a realistic large distance-scale forcing. The function \( N_{L^{-1}} \) is hence chosen such that its Fourier transform is smooth, is peaked at the scale \( L \), and is zero at vanishing wave-number and decays exponentially at large wave-number. The MSRJD formalism with the non-local terms (3) and (4) is presented in [22, 23, 24]. It yields the partition function for the velocity, pressure, response fields, under the form

\[
Z[\vec{J}, \vec{J}, K, K] = \int \mathcal{D}[\vec{v}, \vec{v}, p, \bar{p}] \exp\left(-\mathcal{S}_\text{NL}[\vec{v}, \bar{p}] - \Delta \mathcal{S}[\vec{v}, \bar{p}]\right) \mathcal{I}[\vec{J} \bar{J} + \vec{J} K + K \bar{p} + \bar{p} p],
\]

(5)

with the notation \( x \equiv (t, \vec{x}) \) and \( \int_x = \int d^3 x \; dt \), and similarly in the following \( p \equiv (\omega, \bar{p}) \) and \( \int_p = \int \frac{d^3 \omega}{(2\pi)^3} \; \frac{d\bar{p}}{2\pi} \). In the MSRJD formalism, the response velocity \( \vec{v} \) and response pressure \( \bar{p} \) are introduced as the Lagrange multipliers of the equation of motion and of the incompressibility constraint respectively, and \( \vec{J}, \vec{J}, K, K \) are the sources for the four fields. The NS action for the velocity is obtained as

\[
\mathcal{S}_\text{NL}[\vec{v}, \bar{p}] = \int_x \left\{ \bar{v}_\alpha(x) \left[ \partial_t v_\alpha(x) - \nu \nabla^2 v_\alpha(x) + v_\beta(x) \partial_\gamma v_\alpha(x) + \frac{1}{\rho} \partial_\alpha p(x) \right] + \bar{p}(x) \partial_\alpha v_\alpha(x) \right\}
\]

\[
\Delta \mathcal{S}[\vec{v}, \bar{p}] = \int_{x, x'} \left\{ \bar{v}_\alpha(t, \vec{x}) R_{L^{-1}}(|\vec{x} - \vec{x}'|) v_\alpha(t, \vec{x}') - \bar{v}_\alpha(t, \vec{x}) N_{L^{-1}}(|\vec{x} - \vec{x}'|) \bar{v}_\alpha(t, \vec{x}') \right\};
\]

(6)

Note that one does not need to impose the forcing to be solenoidal, as the incompressibility is explicitly enforced along each realization of the flow. As a consequence, it can be chosen proportional to the identity in component space without loss of generality.

Note that the standard derivation of the MSRJD action for NS implicitly assumes existence and unicity of weak solutions of the Navier–Stokes equation, which is a delicate issue from a mathematical point of view [32]. The assumption is actually a little weaker than strict uniqueness, since for a typical set of initial conditions, there may exist a set of velocity configurations spoiling unicity, as long as they are of zero measure with respect to the realizations of the forcing.
where the quadratic non-local part has been separated for latter purposes. These expressions hold in a generic dimension $d$. We now specialize to two dimensions, and introduce the stream function formulation.

### 2.1. Action for the stream function

In 2D, the incompressibility constraint allows one to express the velocity as $v_\alpha = \epsilon_{\alpha\beta}\partial_\beta \psi$ where $\psi$ is a pseudo-scalar field, called the stream function, and $\epsilon_{\alpha\beta}$ are the components of the antisymmetric tensor with two indices and with $\epsilon_{12} = 1$, which satisfies the identity $\epsilon_{\alpha\gamma}\epsilon_{\beta\gamma} = \delta_{\alpha\beta}$. The stream function is related to the vorticity field through a Laplacian:

$$\omega = \epsilon_{\alpha\beta}\partial_\alpha v_\beta = \epsilon_{\alpha\beta}\epsilon_{\beta\gamma}\partial_\alpha \partial_\gamma \psi = -\partial^2 \psi. \quad (7)$$

From the field theory (5) and (6), setting the sources $K$ and $\tilde{K}$ to zero, integrating over the pressure fields $p$ and $\bar{p}$, and using the resulting incompressibility constraints for $\bar{v}$ and $\bar{\psi}$, the NS action for the velocity can be expressed as an action for the stream function, as

$$S_\psi[\psi, \bar{\psi}] = \int x \partial_\alpha \bar{\psi}(x) \left[ \partial_t \partial_\alpha \psi(x) - v \nabla^2 \partial_\alpha \psi(x) + \epsilon_{\beta\gamma} \partial_\beta \partial_\gamma \psi(x) \right]$$

$$\Delta S_\psi[\psi, \bar{\psi}] = \int_{t,\tilde{t},x} \left\{ \partial_t \bar{\psi}(t, \bar{x}) R_{\alpha\beta}(\|x - \bar{x}\|) \partial_\alpha \psi(t, x) - \partial_\alpha \bar{\psi}(t, \bar{x}) N_{L-1}(\|x - \bar{x}\|) \partial_\alpha \psi(t, x) \right\}, \quad (8)$$

where the response stream is related to the response velocity through $\bar{v}_\alpha = \epsilon_{\alpha\beta}\partial_\beta \bar{\psi}$. One can then introduce a source for the stream function (resp. for the response stream) $J = -\epsilon_{\alpha\beta}\partial_\beta J_\alpha$ (resp. $\bar{J} = -\epsilon_{\alpha\beta}\partial_\beta \bar{J}_\alpha$) in the partition function, to obtain the moments of $\psi$ (resp. $\bar{\psi}$) as functional derivatives with respect to $J$ (resp. $\bar{J}$). Let us point out that this action is often obtained by taking the curl of the NS equation before casting it into a functional integral [17, 18]. Here, this operation comes as the consequence of the incompressibility constraint for $\bar{v}$. This shows that in 2D, the velocity field action (6) and the stream function one (8) are equivalent.

### 2.2. Symmetries and extended symmetries

The action (8) possesses several symmetries. In this work, we consider not only the exact symmetries of this action, but also its extended symmetries. We define an extended symmetry as a change of variables in the partition function which does not leave the action strictly invariant, but which induces a variation of the action linear in the fields. The key point is that one can derive from these extended symmetries Ward identities which are more general than their non-extended versions. Typically, the extended symmetries considered below are time-dependent generalizations of the original exact symmetries. As a consequence, the corresponding Ward identities are valid for arbitrary frequency instead of holding only at zero frequency. These identities are very useful in general within a field-theoretical framework, and in particular within the NPRG. Let us list all the symmetries and extended symmetries of the action (8), denoting generically $\eta$ or $\bar{\eta}$ their (scalar or vectorial) parameter:

\[
\begin{align*}
(a) \quad & \delta \psi = \eta(t), & (\bar{a}) \quad & \delta \bar{\psi} = \bar{\eta}(t), \\
(b) \quad & \delta \psi = 0, & & \delta \bar{\psi} = x_\alpha \bar{\eta}_\alpha(t), \\
(c) \quad & \delta \psi = 0, & & \delta \bar{\psi} = \frac{x^2}{2} \bar{\eta}(t), \\
(d) \quad & \delta \psi = \epsilon_{\alpha\beta} x_\alpha \bar{\eta}_\beta(t) + \eta_\alpha(t) \partial_\alpha \psi, & & \delta \bar{\psi} = \eta_\alpha(t) \partial_\alpha \bar{\psi}, \\
(e) \quad & \delta \psi = -\bar{\eta}(t) \frac{x^2}{2} + \eta(t) \epsilon_{\alpha\beta} x_\alpha \partial_\beta \psi, & & \delta \bar{\psi} = \eta(t) \epsilon_{\alpha\beta} x_\beta \partial_\alpha \bar{\psi}. \quad (9)
\end{align*}
\]
The symmetries \((a)\) and \((\bar{a})\) are exact symmetries which just follow from the definitions of \(\bar{\psi}\) and \(\bar{\psi}\). Indeed, they are determined up to a constant function of time, and the functional integral does not fix this gauge invariance. The symmetries \((b)\) and \((d)\) correspond to known extended symmetries of the velocity action: \((d)\) is the time-gauged (or time-dependent) Galilean symmetry [30, 33–35], and \((b)\) is a time-gauged shift of the response fields, first unveiled in [23]. On the other hand, the symmetries \((c)\) and \((e)\) are extended symmetries that were not, to the best of our knowledge, identified yet. The symmetry \((c)\) corresponds to a different time-gauged shift of the response field, which is also an extended symmetry of the 3D NS action (see below), while the symmetry \((e)\) can be interpreted as a time-gauged rotation, which is only realized in 2D.

Let us expound in more details these extended symmetries. For the time-gauged Galilean symmetry \((d)\), since the action (8) is invariant under global Galilean transformation, the only non-zero variation stems from the time derivative acting on the parameter of the transformation, and one can check that it vanishes. Interestingly, the time-gauged Galilean symmetry is an exact symmetry in the stream formulation, whereas it is an extended one for the velocity action, because the gauge degree of freedom is fixed in the latter.

Let us now consider the symmetries \((b)\) and \((c)\). A general space-time shift of the response field \(\psi(t, \bar{x}) \rightarrow \psi(t, \bar{x}) + \bar{\eta}(t, \bar{x})\) yields the first order variation of the action

\[
\delta(S_\psi + \Delta S_\psi) = -\int x \hat{\eta}(x) \left\{ \partial^2 (\partial_t - \nu \partial^2) \psi + \partial_\alpha \partial_\beta (\epsilon_{\alpha\beta\gamma} \partial_\gamma \psi \partial_\eta) + \int \left[ R_{\alpha\gamma}^{-1}(|\bar{x} - \bar{x}'|) \partial^2 \psi(t, \bar{x}') - 2N_{\alpha\gamma}^{-1}(|\bar{x} - \bar{x}'|) \partial^2 \bar{\psi}(t, \bar{x}') \right] \right\}. \tag{10}
\]

As in the velocity formulation, the non-linear term of this variation, stemming from the interaction, may vanish for some particular space dependence of \(\bar{\eta}\). The choice \(\bar{\eta}(x) = \eta(t)\) is simply the gauge-invariance \((\bar{a})\). The choice \(\bar{\eta}(x) = x_\alpha \bar{\eta}_\alpha(t)\) corresponds to the known time-gauged shift of the response fields (velocity and pressure) in the velocity formulation [23]. For this choice \(\delta(S_\psi + \Delta S_\psi)\) also vanishes, which means that the time-gauged shift \((b)\) is an exact symmetry in the stream formulation, while it is an extended one in the velocity formulation, as for the time-gauged Galilean symmetry.

We here uncover another transformation which leads to the new extended symmetry \((c)\): \(\bar{\eta}(t, \bar{x}) = \frac{\bar{\eta}}{2} \bar{\eta}(t)\). For this choice, the variation stemming from the interaction cancels by antisymmetry of \(\epsilon_{\alpha\beta\gamma}\). Let us emphasize that this symmetry is not specific to 2D, and can be expressed also in 3D in the velocity formulation, where it corresponds to a shift linear in space: \(\delta \bar{v}_\alpha = \epsilon_{\alpha\beta\gamma} x_\beta \eta_\gamma(t), \delta \bar{\rho} = \bar{v}_\alpha \epsilon_{\alpha\beta\gamma} x_\beta \eta_\gamma(t)\), where the \(\epsilon_{\alpha\beta\gamma}\) are the components of the fully antisymmetric tensor with three indices. One can indeed check that the corresponding variation reads

\[
\delta(S_\psi + \Delta S_\psi) = \int x \epsilon_{\alpha\beta\gamma} x_\beta \eta_\gamma(t) \left\{ \partial_\alpha \bar{v}_\eta + \int \left[ R_{\alpha\gamma}^{-1}(|\bar{x} - \bar{x}'|) \bar{v}_\eta(t, \bar{x}') - 2N_{\alpha\gamma}^{-1}(|\bar{x} - \bar{x}'|) \bar{v}_\eta(t, \bar{x}') \right] \right\},
\]

and is linear in the fields. This transformation is thus an extended symmetry of the NS velocity action in both 2D and 3D. This new symmetry yields Ward identities, given below in the 2D stream formulation, that could be useful in the study of the 3D NS turbulence as well. We did not find higher order space dependence of \(\bar{\eta}(t, \bar{x})\) which induces a linear variation of the action in the fields.
Finally, we now consider the extended symmetry \( (e) \), which is specific to 2D. This new symmetry can be interpreted as a time-gauged rotation in the same way as extended Galilean symmetry is a time-gauged translation in space. Using the anti-symmetry of \( \epsilon^{\alpha\beta} \), one can check that the variation under the transformation \( (e) \) is indeed linear in the field

\[
\delta(S_\psi + \Delta S_\psi) = 2 \int_x \eta(t) \left[ \partial_t \bar{\psi} - \partial_x \bar{\psi} \int_\beta R_{\kappa_\mu}([x - x']) \right].
\]  

(11)

This symmetry can be expressed in the velocity formulation as well, at the cost of introducing a non-local shift of the pressure.

These extended symmetries can be translated into Ward identities. As they are exploited in the present work within the NPRG framework, we derive them in terms of the effective average action (EAA) \( \Gamma_\kappa \) defined in the next section. They essentially coincide with the Ward identities for the standard effective action \( \Gamma \) usually defined in field theory, but for the terms associated with the variation of the non-local quadratic parts of the action. These terms are subtracted in the NPRG formalism (see appendix A.1), but they can be straightforwardly included to deduce the Ward identities in terms of \( \Gamma \) [23]. Let us now briefly introduce the EAA, and then establish these Ward identities.

3. NPRG formalism and Ward identities

3.1. Formulation of the NPRG

The NPRG is a modern implementation of Wilson’s original idea of the RG [36], conceived to efficiently average over fluctuations, even when they develop at all scales, as in standard critical phenomena [37–39]. It is a powerful method to compute the properties of strongly correlated systems, which can reach high precision levels [40, 41] and yield fully non-perturbative results, at equilibrium [42–44] and also out of equilibrium [45–49], restricting to a few classical statistical physics applications. The NS field theory was first studied using NPRG methods in [24, 50, 51], and we here follow the formalism developed in [22, 24]. The core idea of the NPRG is to organize the integration of the fluctuations by adding to the action a non-local quadratic term, called the regulator and noted \( \Delta S_\kappa \). The role of this regulator is to suppress, in a smooth way, the integration over momentum modes below the renormalization scale \( \kappa \).

By varying \( \kappa \) from the ultraviolet (UV) cutoff of the theory, where the regulator ensures that mean-field is exact, to its infrared (IR) cutoff, one smoothly integrates over the fluctuations of the fields.

It turns out that in the Navier–Stokes action (8), terms which can play the role of regulators are already present for physical reasons. Indeed, the functions \( N_{L^{-1}} \) and \( R_{L^{-1}} \) associated with forcing and large-scale dissipation satisfy all the requirements to act as regulators of the theory. Their Fourier transform are smooth functions, which vanish exponentially for large wave-numbers compared to \( L^{-1} \) or \( L^{-1}_0 \), and which regularize the fluctuating fields for small wave-numbers (see [24, 37]). Thus, identifying their typical scale, \( L^{-1} \) and \( L^{-1}_0 \), with the RG scale \( \kappa \) yields a regulator \( \Delta S_\kappa \) for the NS field theory. Since we are interested in the direct cascade where wave-numbers are larger than both \( L^{-1} \) and \( L^{-1}_0 \), we simply set \( L^{-1} = L^{-1}_0 = \kappa \). To study the inverse cascade, which corresponds to wave-numbers between \( L^{-1}_0 \) and \( L^{-1} \), the scale \( L \) should be kept fixed while setting \( L^{-1}_0 = \kappa \) (this is left for future work).

In the presence of the regulator \( \Delta S_\kappa \), the generating functional \( Z \) of the correlation functions becomes scale dependent
The average of the stream function (and response stream) can be obtained through functional derivatives of $\mathcal{W}_\kappa = \ln Z_\kappa$ with respect to the sources as

$$\Psi(x) = \langle \psi(x) \rangle = \frac{\delta \mathcal{W}_\kappa}{\delta J(x)}, \quad \Psi(x) = \langle \tilde{\psi}(x) \rangle = \frac{\delta \mathcal{W}_\kappa}{\delta J(x)}.$$  \hspace{1cm} (13)

When the renormalization scale $\kappa$ varies, $\mathcal{W}_\kappa$ evolves according to the following exact flow equation (which is similar to the Polchinski equation [52]):

$$\partial_\kappa \mathcal{W}_\kappa = -\frac{1}{2} \int_{xy} \partial_\kappa [\mathcal{R}_\kappa]_{ij}(x-y) \left\{ \frac{\delta^2 \mathcal{W}_\kappa}{\delta J_i(x) \delta J_j(y)} + \frac{\delta \mathcal{W}_\kappa}{\delta J_i(x)} \frac{\delta \mathcal{W}_\kappa}{\delta J_j(y)} \right\},$$  \hspace{1cm} (14)

where $i, j \in \{1, 2\}$ with $j_1 = J$ and $j_2 = J$. The EAA $\Gamma_\kappa$ is defined as the Legendre transform of $\mathcal{W}_\kappa$, up to the regulator term:

$$\Gamma_\kappa[\Psi, \tilde{\Psi}] + \mathcal{W}_\kappa[J, J] = \int_x \left\{ J \Psi + J \tilde{\Psi} \right\} - \Delta S_\kappa[\Psi, \tilde{\Psi}].$$  \hspace{1cm} (15)

The flow of $\Gamma_\kappa$ with the RG scale $\kappa$ is given by the Wetterich equation [55]

$$\partial_\kappa \Gamma_\kappa = \frac{1}{2} \int_{xy} \partial_\kappa [\mathcal{R}_\kappa]_{ij}(x-y) \left[ \Gamma^{(2)}_\kappa + [\mathcal{R}_\kappa]^{-1}_{ji} \right](y,x),$$  \hspace{1cm} (16)

where $\Gamma^{(2)}_\kappa$ is the Hessian of $\Gamma_\kappa$ and the regulator matrix $[\mathcal{R}_\kappa]_{ij}$ is defined as

$$[\mathcal{R}_\kappa]_{ij}(x-y) = \frac{\delta^2 \Delta S_\kappa}{\delta \phi_i(x) \delta \phi_j(y)},$$  \hspace{1cm} (17)

with $i, j \in \{1, 2\}$ and $\phi_1 = \Psi, \phi_2 = \tilde{\Psi}$. The RG flow equation (16) is also exact. Its initial condition corresponds to the ‘microscopic’ model, which is $S_\psi$ in (8). The flow is hence initiated at a very large wave-number $\Lambda$ at which the continuous description of the fluid dynamics in terms of NS equation starts to be valid. At this scale, one can show that $\Gamma_\Lambda$ identifies with the bare action $\Gamma_\Lambda = S_\psi$, since no fluctuation is yet incorporated. When $\kappa \to L^{-1}$, the regulator reaches its original value and one obtains the actual properties of the model, when all fluctuations up to the physical IR cutoff have been integrated over. Equation (16) provides the exact interpolation between these two scales. Although equation (16) is functional in fields, it is a pure first order differential equation in the RG scale $\kappa$. One thus expects that its solution for a given initial condition exists and is unique. This is indeed verified in most practical applications [37, 39].

When some form of scale invariance is physically realized in a system, this corresponds to a fixed point of the RG flow (see [22] for details). An essential property of RG flow equations is that smooth and globally defined fixed points are typically isolated, and hence unique in their basin of attraction [37, 39, 53, 54]. It was observed that the NPRG fixed point found for NS in different approximations indeed fulfills this property [24, 50, 51]. In the following, we assume that the fixed point describing a stationary turbulent flow exists and is unique beyond any approximation, and we derive below exact properties in its vicinity. For turbulence, the scaling behavior in the inertial range, including intermittency corrections, is observed to be universal. This strongly suggests that these properties are controlled by the fixed point, as in standard critical phenomena. It turns out that the NS fixed point has very peculiar properties, termed as ‘non-decoupling’, see [22, 24], which are indeed responsible for violations of

\[\text{Equation (12)}\]
standard scale invariance (and emergence of an explicit dependence in the integral scale), and are thus a potential source of intermittency effects. This is the scenario explored in this work.

3.2. Definition of generalized correlation functions

The functional \( \mathcal{W}_\kappa \) is the generating functional of connected correlation functions, which correspond to the cumulants for a field theory [53, 54]. The \( n \)-point generalized connected correlation functions can be obtained as functional derivatives of \( \mathcal{W}_\kappa \) with respect to the sources

\[
G^{(n)}_{i_1,\ldots,i_n} ([x_i]_{1 \leq i \leq n}; j) = \frac{\delta^n \mathcal{W}_\kappa}{\delta j_{i_1} (x_1) \ldots \delta j_{i_n} (x_n)}. \tag{18}
\]

where \( i_k = 1, 2 \) with \( j_1 = J \) and \( j_2 = J \) as before. They are called generalized because they include derivatives with respect to response fields, which are related to correlations with the forcing [24]. Note that in this definition, \( G^{(n)}_{i_1,\ldots,i_n} \) is still a functional of the sources, which is materialized by the square brackets and the explicit \( j \) dependency. Let us also introduce the notation \( G^{(m,n)} [x_1, \ldots, x_{m+n}] \) where the \( m \) first derivatives are with respect to \( J \) and the \( m \) last with respect to \( J \). We indicate that a correlation function is evaluated at zero sources using the notation

\[
G^{(n)}_{i_1,\ldots,i_n} ([x_i]_{1 \leq i \leq n}; j = 0) \tag{19}
\]

(and accordingly for \( G^{(m,n)} \)). The Fourier transforms of these functions are defined as:

\[
\tilde{G}^{(n)} ([p_\ell]_{1 \leq \ell \leq n}) = \int [x_i] \frac{G^{(n)} ([x_i]_{1 \leq \ell \leq n}) e^{-i \sum_{k=1}^n k \cdot x_k - \omega_k p_\ell}}{(2\pi)^d 1 \delta \sum_{k=1}^n \omega_k} \tag{20}
\]
or similarly extracting the delta function of conservation of the total wave-vector and frequency

\[
\tilde{G}^{(n)} ([p_\ell]_{1 \leq \ell \leq n}) = (2\pi)^d 1 \delta \sum_{k=1}^n \omega_k G^{(n)} ([p_\ell]_{1 \leq \ell \leq n-1}). \tag{21}
\]

The EAA is the generating functional of one particle-irreducible (1-PI) correlation functions, also called vertex functions. This means that any vertex functions can be obtained by taking functional derivatives of \( \Gamma_\kappa \) with respect to the average fields \( \Psi \) and \( \bar{\Psi} \). The \( n \)-point vertex (1-PI) functions are defined using the same conventions as for the connected correlation functions:

\[
\Gamma^{(n)}_{i_1,\ldots,i_n} (x_1, \ldots, x_n; \varphi) = \frac{\delta^n \Gamma_\kappa}{\delta \varphi_{i_1} (x_1) \ldots \delta \varphi_{i_n} (x_n)}, \tag{22}
\]

where \( i_k = 1, 2 \) with \( \varphi_1 = \Psi \) and \( \varphi_2 = \bar{\Psi} \), or alternatively \( \Gamma^{(m,n)} [x_1, \ldots, x_{m+n}; \varphi] \). Accordingly, we define \( \Gamma^{(n)}_{i_1,\ldots,i_n} (x_1, \ldots, x_n) \) and \( \Gamma^{(m,n)} (x_1, \ldots, x_{m+n}) \) as the previous vertex functions evaluated at zero fields. Finally we define the Fourier transforms before and after extracting the delta function of conservation of wave-vector and frequency, \( \tilde{\Gamma}^{(n)} ([p_\ell]_{1 \leq \ell \leq n}) \) and \( \tilde{\Gamma}^{(n)} ([p_\ell]_{1 \leq \ell \leq n-1}) \), respectively. The knowledge of the set of connected correlation functions or of the set of vertex functions is equivalent.

3.3. Ward identities for the vertex functions in the stream formulation

In sections 4 and 5, we present calculations within the large wave-number expansion. A key ingredient in these calculations is the existence of Ward identities for the vertex functions.
We hence give below the Ward identities associated with the extended symmetries of the NS action in terms of the vertex functions, and within the NPRG framework. Ward identities for the connected correlation functions can be derived in the same way.

The list (9) of the extended symmetries in the stream function formulation only contains continuous changes of variables which are at most linear in the field, and so are the corresponding variations of the action. In this case, they can be translated readily into Ward identities that the EAA must verify along the RG flow. These identities simply express that the EAA corresponding variations of the action. In this case, they can be translated readily into Ward identities for the connected correlation functions can be derived in the same way.

We hence give below the Ward identities associated with the extended symmetries of the NS fields. In the Fourier space, they read:

\[
\begin{align*}
(a) & \quad \int \frac{\delta \Gamma_\kappa}{\delta \tilde{\Psi}(x)} = 0, \\
(b) & \quad \int \frac{\partial \delta \Gamma_\kappa}{\partial \tilde{\Psi}(x)} = 0 \\
(c) & \quad \int \frac{\chi^2}{2} \frac{\delta \Gamma_\kappa}{\delta \tilde{\Psi}(x)} = -2 \int \partial_t \tilde{\Psi} \\
(d) & \quad \int \left\{ \left( -\epsilon_{\beta \alpha} x_\beta \partial_t + \partial_\alpha \tilde{\Psi} \right) \frac{\delta \Gamma_\kappa}{\delta \tilde{\Psi}(x)} + \partial_\alpha \tilde{\Psi} \frac{\delta \Gamma_\kappa}{\delta \tilde{\Psi}(x)} \right\} = 0 \\
(e) & \quad \int \left\{ \left( \frac{\chi^2}{2} \partial_t + \epsilon_{\alpha \beta} x_\beta \partial_\alpha \tilde{\Psi} \right) \frac{\delta \Gamma_\kappa}{\delta \tilde{\Psi}(x)} + \epsilon_{\alpha \beta} x_\beta \partial_\alpha \tilde{\Psi} \frac{\delta \Gamma_\kappa}{\delta \tilde{\Psi}(x)} \right\} = 2 \int \partial_t^2 \tilde{\Psi}.
\end{align*}
\]

From these functional identities, one can derive a hierarchy of identities for the vertex functions \( \Gamma^{(m,n)} \), by taking the corresponding functional derivatives and evaluating them at zero fields. In the Fourier space, they read:

\[
\begin{align*}
(a), (\bar{a}) & \quad \Gamma^{(m,n)}(\ldots, \tilde{\omega}, \tilde{q}, \ldots) |_{\tilde{q}=0} = 0 \\
(b) & \quad \frac{\partial}{\partial \tilde{q}^2} \Gamma^{(m+1,n)}_\kappa (\{ p_\ell \}_{1 \leq \ell \leq m}, \tilde{\omega}, \tilde{q}; \{ p_\ell \}_{1 \leq \ell \leq m-1}) |_{\tilde{q}=0} = 0 \\
(c) & \quad \frac{\partial^2}{\partial \tilde{q}^2} \Gamma^{(m+1,n)}_\kappa (\{ p_\ell \}_{1 \leq \ell \leq m}, \tilde{\omega}, \tilde{q}; \{ p_\ell \}_{1 \leq \ell \leq m-1}) |_{\tilde{q}=0} = 0 \\
& \quad \text{except } \frac{\partial^2}{\partial \tilde{q}^2} \Gamma^{(1,1)}_\kappa (\omega, \tilde{q}) |_{\tilde{q}=0} = -4i \omega \\
(d) & \quad \frac{\partial}{\partial \tilde{q}^2} \Gamma^{(m+1,n)}_\kappa (\omega, \tilde{q}; \{ p_\ell \}_{1 \leq \ell \leq m+1}) |_{\tilde{q}=0} = i \epsilon_{\alpha \beta} \hat{D}_\rho (\omega, \tilde{q}) \Gamma^{(m,n)}_\kappa (\{ p_\ell \}) \\
(e) & \quad \frac{\partial^2}{\partial \tilde{q}^2} \Gamma^{(m+1,n)}_\kappa (\omega, \tilde{q}; \{ p_\ell \}_{1 \leq \ell \leq m+1}) |_{\tilde{q}=0} = \hat{R}(\omega) \Gamma^{(m,n)}_\kappa (\{ p_\ell \}),
\end{align*}
\]

where we have introduced the two operators \( \hat{D}_\alpha (\omega) \) and \( \hat{R}(\omega) \) defined as:

\[
\begin{align*}
\hat{D}_\alpha (\omega) F(\{ p_\ell \}_{1 \leq \ell \leq n}) & \equiv - \sum_{k=1}^n p_k^\alpha \int \frac{F(\{ p_\ell \}_{1 \leq \ell \leq k-1}, \omega_k + \omega, p_k; \{ p_\ell \}_{k+1 \leq \ell \leq n}) - F(\{ p_\ell \})}{\omega} \\
\hat{R}(\omega) F(\{ p_\ell \}_{1 \leq \ell \leq n}) & \equiv 2i \epsilon_{\alpha \beta} \sum_{k=1}^n p_k^\alpha \int \frac{F(\{ p_\ell \}_{1 \leq \ell \leq k-1}, \omega_k + \omega, p_k; \{ p_\ell \}_{k+1 \leq \ell \leq n}) - F(\{ p_\ell \})}{\omega}.
\end{align*}
\]
3.4. Large wave-number expansion of the flow equations

Let us explain the principles of the large wave-number expansion. The flow equation (16) is exact. However, the flow equation for a generic $n$-point function $\Gamma^{(n)}$, which can be deduced by taking the corresponding functional derivatives of (16), is not closed as it involves the $(n+1)$ and $(n+2)$ vertex functions. As such, one has to consider an infinite hierarchy of flow equations. For example, the flow equation for the two-point function is given in the Fourier space by

$$
\partial_s \bar{\Gamma}^{(2)}_{mn}(p) = -\frac{1}{2} \bar{\Gamma}^{(4)}_{lmno}(q, -q, p) + \bar{\Gamma}^{(3)}_{lmno}(q, p) \bar{G}^{(2)}_{st}(q + p) \bar{\Gamma}^{(3)}_{tnl}(q + p, -p) \bar{G}^{(2)}_{li}(q),
$$

which depends on the 3- and 4-point vertices. The right-hand side (rhs) is represented diagrammatically in figure 1, where the dashed circles are the vertex functions, the thick lines are propagators and the cross is the derivative of the regulator. The rhs involves the integrated, or internal, wave-vector and frequency $q$ circulating in the loops besides the external wave-vector and frequency $p$ at which the vertex function on the left-hand side (lhs) is evaluated.

In most applications, this hierarchy is closed by simply truncating higher-order vertices, or proposing an ansatz for $\Gamma^{(n)}$. An alternative strategy, pioneered in [41, 56] and called the BMW approximation scheme, consists in expanding these vertices in the internal wave-vector $\vec{q}$. This approximation relies on the two following properties of the regulator: on the one hand, its insertion in the integration loop on the rhs of (16) cuts off the internal wave-number $|\vec{q}|$ to values $|\vec{q}| \lesssim \kappa$. As a consequence, if the system is probed at a wave-number scale $|\vec{p}|$ much larger than the renormalization scale, $p \gg \kappa$, there is a clear separation of scales in the flow equations: $q/p \ll 1$. On the other hand, the presence of the regulator ensures that the vertex functions are smooth at any finite scale $\kappa > 0$, which allows one to perform a Taylor expansion in powers of $\vec{q}$. The underlying idea is that, close to a fixed point, the vertex functions are expected to depend on the internal wave-number only through ratio of the type $q/p$, which means that the expansion at $q \simeq 0$ is expected to be equivalent to an expansion at $p \to \infty$. This expansion becomes exact in the limit of infinite wave-numbers, and the error at finite but large $p$ is small. In fact, this expansion was found to be a reliable approximation for arbitrary momenta [41, 56].

The BMW strategy has turned out to be very successful in the context of turbulence, since the expanded flow equations can be closed at zero fields thanks to the Ward identities, whereas it generically requires to keep a whole dependence in background fields. This was first noticed.

**Figure 1.** Diagrammatic representation of the flow of $\bar{\Gamma}^{(2)}$. The derivation of the identities $(d)$ and $(e)$ is reported in appendices A.2 and A.3 respectively.
in [24] for the two-point function, and generalized in [22] where the exact leading order term in the large wave-number expansion of the flow equation of an arbitrary \( n \)-point correlation function was obtained in 3D. The striking feature of these flow equations is that they do not exhibit the decoupling property usually expected for flow equations, e.g. in standard critical phenomena. The decoupling means that the non-linear (loop) part of the flow equation is negligible with respect to the linear one at large wave-numbers. This property ensures that the existence of a fixed point entails standard scale invariance for \( \kappa \) much smaller than any non-exceptional wave-number (see [22]). In 3D turbulence, the violation of the decoupling property yields a breaking of standard scale invariance, which is manifest in the time dependence of generic correlation functions. This breaking was related in [21, 22], at least for small time delays, to the sweeping effect. The latter is the random advection of small-scale velocities by large-scale structures in the turbulent flow, and is well-known phenomenologically [57, 58].

4. Time dependence of generic correlation functions

In this section, we express the flow equation for a generic generalized \( n \)-point correlation function in the stream formulation, and we compute its exact leading term in the large wave-number expansion for non-equal time delays. We then derive the corresponding fixed-point solutions for the two-point function, and show that its time dependence explicitly breaks scale invariance.

4.1. Flow equation for generic correlation functions at leading order

The flow equation for a generic connected correlation function of the stream and response stream functions \( G_{\psi}^{(n)} \) is obtained by taking \( n \) functional derivatives of (14) with respect to the sources \( j_{\alpha} \), \( 1 \leq k \leq n \), which yields

\[
\partial_\kappa G_{\psi_1 \cdots \psi_n}^{(n)}[(\{x_i\})_{1 \leq i \leq n}]; j] = -\frac{1}{2} \int_{y_1} y_2 \partial_\kappa [R_{\kappa}](y_1 - y_2) \left\{ G_{\psi_0 \psi_1 \cdots \psi_n}^{(n+2)}[y_1, y_2, \{x_i\}; j] + \sum_{\{i_1, i_2\} \in \{1, 2\}} G_{\psi_{i_1} \psi_{i_2} \psi_{\tilde{i}_1 \tilde{i}_2}}^{(n+1)}[y_1, \{x_i\}; j] G_{\psi_{i_1 \tilde{i}_1} \psi_{i_2 \tilde{i}_2}}^{(n+1)}[y_2, \{x_i\}; j] \right\}
\]

(27)

where the indices \( i_\ell \in \{1, 2\} \) stand for the sources \( J \) or \( \bar{J} \), and \( \{\{i_1\}, \{i_2\}\} \) indicates all the possible bipartitions of the \( n \) indices \( \{i_1\}_{1 \leq i \leq n} \), and \( \{\{x_i\}\}, \{\{x_i\}\} \) the corresponding bipartition in coordinates. Finally, \( c_1 \) and \( c_2 \) are the cardinals of \( \{i_1\} \) (resp. \( \{i_2\} \)).

We now consider the large wave-number expansion of (27). The calculation of the leading order term is formally the same as in 3D in the velocity formulation [22]. We refer the reader to [59] for details on the derivation in the stream formulation. One can first show, as in 3D, that in the limit of large wave-numbers, that is when all the \( \bar{p}_{i_\ell}, 1 \leq \ell \leq n \) are large compared to \( \kappa \), as well as all their partial sums, the flow equation (27) reduces to

\[
\partial_\kappa G_{\psi_0 \psi_1 \cdots \psi_n}^{(n)}[(\{p_i\})_{1 \leq i \leq n}]; j] = \frac{1}{2} \int_{q_1, q_2} \partial_\kappa G_{\psi, \psi_0}^{(2)}(-q_1, -q_2) \left[ \frac{\partial^2}{\partial \varphi(q_1) \partial \varphi(q_2)} \bar{G}_{\psi_0 \psi_1 \cdots \psi_n}^{(n)}[(\{p_i\}); j] \right]_{\varphi = 0} ,
\]

(28)

up to terms tending to zero faster than any power of the \( p_{i_\ell} \). In the velocity formulation, the leading order term is obtained by setting \( q_1, q_2 \) to zero in the equivalent term in brackets. In the stream formulation, because of the gauge symmetry \( (a) \) in (9), there is no information at

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this order. Indeed, the symmetry \( (a) \) implies that the vertex functions are zero if one of the wave-numbers is set to zero, according to (24), and thus
\[
\left[ \frac{\delta^2}{\delta \varphi_i(q_1) \delta \varphi_j(q_2)} \tilde{G}_{\psi,i_1 \ldots i_n}^{(n)}(\{p_i\}_{1 \leq i \leq n}) \right]_{q_1,q_2 \rightarrow 0} = 0. \tag{29}
\]
As a consequence, the first non-zero contribution comes from the second order term in the \( q_1, q_2 \) expansion (since the odd terms of this expansion vanishes by parity of \( \tilde{G}_{\psi,ij}^{(2)}(-q_1,-q_2) \)). It reads
\[
\partial_\kappa \tilde{G}_{\psi,i_1 \ldots i_n}^{(n)}(\{p_i\}_{1 \leq i \leq n}) = \frac{1}{2} \int_{q_1,q_2} \partial_\kappa \tilde{G}_{\psi,ij}^{(2)}(-q_1,-q_2) \frac{q_{ij}^a q_{ij}^b}{2} \frac{\delta^2}{\delta q_{ij}^a \delta q_{ij}^b} \tilde{G}_{\psi,i_1 \ldots i_n}^{(n)}(\{p_i\}_{1 \leq i \leq n}) \Bigg|_{q_1,q_2 \rightarrow 0}, \tag{30}
\]
where \( a, b \) take value in \( \{1,2\} \). As in the velocity formulation, one can then show that this flow equation can be closed using the Ward identities, which yields
\[
\partial_\kappa \tilde{G}_{\psi,i_1 \ldots i_n}^{(n)}(\{p_i\}_{1 \leq i \leq n}) = \frac{1}{2} \int_{q_1,q_2} \partial_\kappa \tilde{G}_{\psi,ij}^{(2)}(-q_1,-q_2) \tilde{D}_\mu(\varpi_1) \tilde{D}_\nu(\varpi_2) \tilde{G}_{\psi,i_1 \ldots i_n}^{(n)}(\{p_i\}_{1 \leq i \leq n}). \tag{31}
\]
Finally, one can check, using the correspondence
\[
\tilde{G}_{\psi,ij}^{(n)}(\{\omega_k,p_k\}) = (i)\epsilon_{k,t_1} \psi_1^{t_1} \cdots \epsilon_{k,t_n} \psi_n^{t_n} \tilde{G}_{\psi,i_1 \ldots i_n}^{(n)}(\{\omega_k,p_k\}) \tag{32}
\]
that the same result as in the velocity formulation, obtained in [22], is recovered.

4.2. Fixed-point and conservation of enstrophy in the direct cascade

Our goal is to describe the universal properties of the direct enstrophy cascade in the forced-dissipative stationary regime. This regime, characterized by some form of scale invariance, corresponds to a fixed point in the RG framework. In order to study the fixed point, it is convenient to work with dimensionless variables, denoted with a hat symbol, using the RG scale \( \kappa \) as unit of wave-numbers. We introduce the dimensionless forcing profile through
\[
N_\kappa(\tilde{q}) = D_\kappa |\tilde{q}|/\kappa \tilde{n}|\tilde{q}|/\kappa, \tag{33}
\]
where \( D_\kappa \) is a scale dependent forcing amplitude, and \( \tilde{n} \) is the specific forcing profile, which fulfills the requirements stated in section 3 (smoothness and fast decay at large wave-numbers). Similarly, the dimensionless effective friction can be defined through
\[
R_\kappa(\tilde{q}) = \nu_\kappa \tilde{q}^2 \tilde{\kappa}(|\tilde{q}|/\kappa), \tag{34}
\]
with \( \nu_\kappa \) a scale dependent coefficient and \( \tilde{\kappa} \) is also a smooth and fast decaying function. As the action (8) is dimensionless, one deduces that \( \psi \tilde{\psi} \) is also dimensionless, and that dimensionless frequencies can be defined according to \( \omega = \kappa^2 \nu_\kappa \tilde{\kappa} \). One then obtains that the dimensionless response stream is given by \( \psi = \kappa(D_\kappa^{-1} \nu_\kappa)^{1/2} \tilde{\psi} \), and the dimensionless stream function by \( \psi = \kappa^{-1}(D_\kappa \nu_\kappa^{-1})^{1/2} \tilde{\psi} \). At the fixed point, the coefficients \( D_\kappa \) and \( \nu_\kappa \) are generically expected to behave as power-laws \( D_\kappa \sim \kappa^{-\eta_\kappa} \) and \( \nu_\kappa \sim \kappa^{-\nu_\kappa} \), where the exponents are the fixed-point values of the functions.
\[ \eta_D(\kappa) = -\kappa \partial_\kappa \ln D_\kappa, \quad \eta_\nu(\kappa) = -\kappa \partial_\kappa \ln \nu_\kappa. \] (35)

One deduces that the dynamical critical exponent \( \gamma \), which characterizes the scaling between space and time as \( \omega \sim |\vec{p}|^\gamma \), is given by \( z = 2 - \eta_\nu^* \). The two exponents \( \eta_D^* \) and \( \eta_\nu^* \) are not independent. Their relation follows from the Galilean invariance. Let us temporarily introduce a coupling \( \lambda \) in front of the non-linear advection term in the action (8). The Galilean invariance entails that this coupling is not renormalized, that is \( \partial_\lambda \lambda = 0 \), or equivalently, the flow equation for the dimensionless coupling, defined through \( \lambda = kD_\kappa^{1/2} \nu_\kappa^{3/2} \lambda_\kappa \), is exactly

\[ \kappa \partial_\kappa \lambda_\kappa = -\frac{1}{2} \lambda_\kappa (2 + \eta_D(\kappa) - 3\eta_\nu(\kappa)). \] (36)

This implies that at any non-Gaussian fixed point \( (\lambda_* \neq 0) \), the exponent \( \eta_\nu^* \) is given by

\[ \eta_\nu^* = -(2 + \eta_D^*)/3. \] (37)

In 3D, the value of the exponent \( \eta_D^* \) is fixed by requiring that the mean rate of energy injection (and dissipation) \( \varepsilon \) is constant [24]. In 2D, this constraint only gives some bounds on the exponent, and one is led to also analyze the mean rate of enstrophy injection \( \varepsilon_\omega \) (and dissipation) [24]. The latter can be expressed as [20]

\[ \varepsilon_\omega = \langle (\nabla \times \vec{f})(t, \vec{x}) \cdot \vec{\omega}(t, \vec{x}) \rangle = \lim_{\delta t \to 0^+} \int_{\omega \vec{q}} q^4 N(\vec{q}) G^{(1,1)}_\psi(\omega, \vec{q}) e^{-\omega \delta t} \]

\[ = D_\kappa \kappa^4 \lim_{\delta t \to 0^+} \int_{\omega \vec{q}} q^6 \hat{n}(\vec{q}) \hat{G}^{(1,1)}_\psi(\omega, \vec{q}) e^{-\omega \delta t}. \] (38)

The properties of \( \hat{n} \) ensures that the integral is both UV and IR finite, and we denote \( \Omega^{-1} \) the value of this integral, which is non-universal. To obtain a constant (\( \kappa \) independent) mean enstrophy injection rate thus imposes \( \eta_D^* = 4 \). The identity (37) then yields \( \eta_\nu^* = 2 \). As a consequence, the leading scaling between space and time vanishes, that is \( z = 0 \), which is very peculiar and occurs only in \( d = 2 \). This scaling is thus governed by sub-leading logarithms, as \( \omega \sim (\ln(|\vec{p}|/\Lambda))^{\delta} \), where \( \Lambda \) is for instance the UV scale, and \( \delta \) an exponent to be determined.

In order to account for this sub-leading behavior, one is led to include logarithmic corrections in the scale-dependent coefficients, such that they behave at a fixed point as

\[ \nu_\kappa \sim \kappa^{-\eta_\nu^*} (\ln(\kappa/\Lambda))^{\gamma_\nu}, \quad D_\kappa \sim \kappa^{-\eta_D^*} (\ln(\kappa/\Lambda))^{\gamma_D}. \] (39)

This sub-leading behavior modifies the flow equation (36) near the fixed point as follows

\[ \kappa \partial_\kappa \lambda_\kappa = -\frac{1}{2} \lambda_\kappa (2 + \eta_D^* - 3\eta_\nu^*) + \frac{1}{2} (\gamma_D^* - 3\gamma_\nu^*) (\ln(\kappa/\Lambda))^{-1} \lambda_\kappa. \] (40)

Since the corrections to the fixed point are expected to decay faster than a logarithmic, we require that \( \gamma_D^* - 3\gamma_\nu^* = 0 \) at a non-Gaussian fixed point, which yields \( \gamma_D^* = 3\gamma_\nu^* = 3\gamma \). The value of \( \gamma \) is not a priori fixed, and should be computed by integrating the flow equations. In fact, if one assumes that there is no intermittency correction, that is if dimensional scalings are not modified, then \( \gamma \) can be determined by consistency, following Kraichnan’s argument. This is done in section 4.1. In the following, we make no assumptions on the presence or not of intermittency, and keep an undetermined exponent \( \gamma \).

In order to define dimensionless correlation functions, let us first express the forcing amplitude at the integral scale \( D_{\kappa^{-1}} \) as a function of the enstrophy dissipation rate using (38) evaluated at \( \kappa = L^{-1} \), \( \varepsilon_\omega = D_{\kappa^{-1}} L^{-3} \Omega^{-1} \). Assuming the log-corrected power-law behavior (39) on the whole inertial range, one can relate the coefficient \( D_\kappa \) to its value at the integral scale as
\[ D_\kappa = D_L \cdot (\kappa L)^{-4} \left( \frac{\ln(\kappa/\Lambda)}{\ln(L^{-1}/\Lambda)} \right)^{3/2} \approx \varepsilon_\omega \Omega \kappa^{-4} \left| \frac{\ln(\kappa/\Lambda)}{\ln(\Lambda)} \right|^{3/2} \equiv D_0 \varepsilon_\omega \kappa^{-4} s^{3/2}, \]

where we have introduced the ‘RG time’ \( s = \ln(\kappa/\Lambda) \) and the non-universal constant \( D_0 = \Omega |\ln(\Lambda)|^{-1} \). The coefficient \( \nu_\kappa \) can be related in the same way to its value at the dissipative scale \( \eta = (\nu^3/\varepsilon_\omega)^{1/6} \) as

\[ \nu_\kappa = \nu_\eta^{-1}(\eta)^{-2} \left( \frac{\ln(\kappa/\Lambda)}{\ln(\eta^{-1}/\Lambda)} \right)^{3/2} \approx \varepsilon_\omega^{1/3} \kappa^{-2} \left| \frac{\ln(\kappa/\Lambda)}{\ln(\eta)} \right|^{3/2} \equiv \nu_0 \varepsilon_\omega^{1/3} \kappa^{-2} s^{3/2}, \]

where we have identified \( \nu \approx \nu_\eta^{-1} \), i.e. neglected its evolution between the microscopic scale \( \Lambda \) and the dissipative one \( \eta^{-1} \), and defined \( \nu_0 = |\ln(\eta\Lambda)|^{-3/2} \).

### 4.3. Solution at the fixed point for the two-point function

In this section, we derive the expression of the two-point correlation function in the stream formulation \( C_\psi \equiv G^{(2,0)}_\psi \), obtained as the solution of the leading order flow equation at the fixed point. The flow equation at leading order for the mixed time-wavevector two-point correlation function can be deduced from the general flow (31) by performing the inverse Fourier transform on the frequency. It reads

\[ \kappa \partial_\kappa C_\psi(t, \vec{p}) = -\frac{1}{2} p^2 C_\psi(t, \vec{p}) \int_\omega \frac{\cos(\omega t) - 1}{\omega^2} J_\kappa(\omega), \]

where \( J_\kappa(\omega) \) can be expressed as

\[ J_\kappa(\omega) = -\int_q \hat{\partial}_\omega \hat{C}(\omega, \vec{q}) = -2 \int_q \left\{ \kappa \delta_{\kappa} N_\kappa(\vec{q}) |\hat{G}(\omega, \vec{q})|^2 - \kappa \delta_{\kappa} R_\kappa(\vec{q}) \hat{C}(\omega, \vec{q}) \text{Re}[\hat{G}(\omega, \vec{q})] \right\}, \]

with \( \hat{C} \) and \( \hat{G} \) the transverse parts of the two-point correlation and response function of the velocity, \( G_{\psi}^{(2,0)}(\vec{p}) = P_{\psi}^{\alpha}(\vec{p}) C(\vec{p}) \) and \( G_{\psi}^{(1,1)}(\vec{p}) = P_{\mu}^{\perp}(\vec{p}) G(\vec{p}) \). The remarkable feature of this equation is that \( \kappa \delta_{\kappa} C_\psi/C_\psi \) does not vanish at large wave-numbers, which means that there is no decoupling of the large wave-numbers. As shown in [22, 24], this leads to a violation of standard scale invariance, which is manifest in the solutions derived below.

This flow equation can be simplified in both the regime of large and small time delays as [22]

\[ \kappa \partial_\kappa C_\psi(t, \vec{p}) = C_\psi(t, \vec{p}) \times \begin{cases} \frac{\nu^3}{t} p^2 + \mathcal{O}(p_{\text{max}}) & t \ll 1 \\ \frac{\nu^3}{\kappa^6} t^2 p^2 + \mathcal{O}(p_{\text{max}}) & t \gg 1 \end{cases} \]

where \( \mathcal{O}(p_{\text{max}}) \) explicitly indicates the contributions beyond the leading order neglected in this flow equation. In order to study the fixed point, and using the dimensional analysis of section 4.2, we define the dimensionless correlation function

\[ C_\psi(t, \vec{p}) = \frac{D_0^{2/3} \varepsilon_\omega^{2/3} s^{3/2}}{\nu_0 \varepsilon_\omega^{1/3} s^3} \hat{C}_1 \left( t = \nu_0 \varepsilon_\omega^{1/3} s^3 \right) \]

such that one obtains the dimensionless flow equation

\[ C_\psi(t, \vec{p}) = \frac{D_0^{2/3} \varepsilon_\omega^{2/3} s^{3/2}}{\nu_0 \varepsilon_\omega^{1/3} s^3} \hat{C}_1 \left( t = \nu_0 \varepsilon_\omega^{1/3} s^3 \right). \]
\[
\left[ \partial_t - 6 + \frac{2\gamma}{s} - \beta \partial_s + \frac{2\gamma}{s} \partial_i \right] \tilde{C}_s(i, \tilde{p}) = \tilde{C}_s(i, \tilde{p}) \times \left\{ \hat{\alpha}_s^0 \tilde{p}^2 \tilde{r} + O(\tilde{p}_{\max}) \right\} \quad \tilde{p}_{\max} \ll 1, \quad \gamma \gg 1.
\]

where \( \hat{\alpha}_s^{0,\infty} = D_0 \nu_0^{-3} \tilde{p}_s^{0,\infty} / 4 \), with the dimensionless integrals \( \tilde{I}_s^{0,\infty} = D_0^{-1} \nu_0^{0,\infty} \) and \( \tilde{I}_s^{\infty} = \kappa^2 D_0^{-1} \nu_0^{\infty} \). In order to get rid of the explicit \( s \) dependence, we search for a solution of the form

\[
\tilde{C}_s(i, \tilde{p}) = \frac{1}{\tilde{p}^2} \ln(\tilde{p})^{-2} \tilde{C}_s(i = \ln(\tilde{p})^{-\gamma}, \tilde{p}),
\]

which satisfies the flow equation

\[
\left[ \partial_t + 2\gamma \left( \frac{1}{s} + \frac{1}{\ln(p/L)} - s \right) - \beta \partial_s + \gamma \left( \frac{1}{s} + \frac{1}{\ln(p/L) - s} \right) \tilde{p} \partial_i \right] \tilde{C}_s(i, \tilde{p}) = \tilde{C}_s(i, \tilde{p}) \times \left\{ \hat{\alpha}_s^0 \tilde{p}^2 \tilde{r} + O(\tilde{p}_{\max}) \right\} \quad \tilde{p}_{\max} \ll 1, \quad \gamma \gg 1.
\]

When approaching the fixed point, in the limit \( s \to -\infty \), the term \( \ln(p/L) \) is negligible compared to \( s \) for any given external wave-number \( p \), such that the terms proportional to \( \gamma \) vanish. Moreover, the fixed point corresponds by definition to \( \partial_t \tilde{C}_s = 0 \), and dimensionless quantities reaching a constant value \( \hat{\alpha}_s^0 \to \hat{\alpha}_s^0 \) and \( \hat{\alpha}_s^\infty \to \hat{\alpha}_s^\infty \). The fixed-point solution can be obtained by integrating the resulting fixed-point equation as

\[
\ln(\tilde{C}_s(i, \tilde{p})) = \begin{cases} -\gamma \hat{\alpha}_s^0 \int_0^\infty x \ln(x)^2 \, dx + \tilde{F}_0(i) + O(\tilde{p}_{\max}) & \tilde{p} \ll 1, \\
-\gamma \hat{\alpha}_s^\infty \int_0^\infty x \ln(x)^2 \, dx + \tilde{F}_\infty(i) + O(\tilde{p}_{\max}) & \tilde{p} \gg 1. \end{cases}
\]

where \( \tilde{F}_0, \tilde{F}_\infty \) are universal scaling functions, that can not be computed from the large wave-number regime alone, but which can be determined by (numerical) integration of the complete flow equation. One then deduces the dimensional physical two-point correlation function

\[
C_{s}(t, \tilde{p}) = \frac{C_0}{\tilde{p}^6} \ln(\tilde{p}L)^{-2} \tilde{F}_{0,\infty} \left( \tilde{\nu}_0 e^{1/3} \ln(\tilde{p}L)^{-\gamma} \right) 
\times \begin{cases} \exp(-\beta_0^s L^2 \tilde{\nu}_0 \int_0^\infty x \ln(x)^2 \, dx + O(\tilde{p}_{\max} L)) & \tilde{p} \ll 1, \\
\exp(-\beta_\infty^s L^2 \tilde{\nu}_0 \int_0^\infty x \ln(x)^2 \, dx + O(\tilde{p}_{\max} L)) & \tilde{p} \gg 1. \end{cases}
\]

with \( C_0 = D_0 \nu_0^{-1} \ln(LA)^{2\gamma}, \tilde{\nu}_0 = \nu_0 \ln(LA)^{\gamma}, \beta_0^s = \hat{\epsilon}_0^{1/3} L^2 \tilde{p}_0 \hat{\alpha}_s^0, \beta_\infty^s = \hat{\epsilon}_\infty^{1/3} \nu_0 L^2 \hat{\alpha}_s^\infty \), and \( \tilde{F}_{0,\infty} = \exp(\tilde{F}_{0,\infty}) \). The leading order term in the exponential, typically of order \( p^2 \), is exact. It provides the decorrelation time of the two-point function in the two regimes of small and large time-delays. This term involves an explicit dependence in the integral scale \( L \), and thus breaks standard scale invariance. In 3D, one obtains a Gaussian dependence in \( \ln(p) \) for large \( p \) and small \( t \) (no logarithms), which is usually interpreted as a consequence of the sweeping effect [22]. The solution (50) shows that the effect of sweeping takes a modified form in 2D, where it is corrected by a logarithmic. Moreover, it indicates that a crossover to a \( \ln(\tilde{p}) \) dependence occurs at long time delays, as was predicted in 3D [22].

The term \( \tilde{F}_{0,\infty}/\tilde{p}^6 \) in (50) corresponds to the solution that would be obtained assuming standard scale invariance (corrected by the sub-leading logarithm in 2D). As clear in equation (50), it is not exact in this calculation, since the contribution \( O(\tilde{p}_{\max} L) \) of the neglected sub-leading terms in the flow equation could modify this scaling solution, and possibly generate intermittency corrections. This has to be assessed by computing the NLO term in the
flow equation, which is the purpose of section 5. We first briefly discuss the generalization to n-point correlations.

4.4. Time-dependence of n-point functions at the fixed point

The flow equation (30) for a n-point function at leading order in the large wave-number expansion can be expressed in a time-wavevector representation as

$$
\partial_s G^{(n)}_{\psi, \ldots, \lambda}(t_1, \bar{p}_1, \ldots, t_{n-1}, \bar{p}_{n-1}) = \frac{1}{2} G^{(n)}_{\psi, \ldots, \lambda}(t_1, \bar{p}_1, \ldots, t_{n-1}, \bar{p}_{n-1}) \\
\times \sum_{k, l} \bar{p}_k \cdot \bar{p}_l \int J_k(\bar{p}) \frac{e^{i(\bar{p} \cdot \bar{r})} - e^{i(\bar{p} \cdot \bar{r})}}{\bar{p}^2}. 
$$

This equation can be simplified in both the limits of small and large time delays, as [22]

$$
\partial_s G^{(n)}_{\psi, \ldots, \lambda}(\{t_i, \bar{r}_i\}) = G^{(n)}_{\psi, \ldots, \lambda}(\{t_i, \bar{r}_i\}) \times \left\{ \begin{array}{ll}
\frac{1}{2} \left[ \sum_{k, l} t_k t_l \bar{p}_k \cdot \bar{p}_l \right]^2 + O(p_{\text{max}}) & t_k \ll 1 \\
\left( \sum_{k, l} \bar{p}_k \cdot \bar{p}_l (|t_k| + |t_l| - |t_k - t_l|) \right) + O(p_{\text{max}}) & t_k \gg 1
\end{array} \right. 
$$

In order to find the fixed point solution, one introduces the dimensionless n-point function, specifying the number m of ψ and m of ¯ψ fields, as

$$
G^{(m, \bar{m})}_{\psi, \ldots, \lambda + n}(\{t_i, \bar{r}_i\}) = \left( \frac{D_0}{v_0} \right)^{\frac{m - \bar{m}}{2}} \frac{s^{\frac{(m - \bar{m})}{3}}}{\kappa^{(m - \bar{m})}} G^{(m, \bar{m})}_{\psi, \ldots, \lambda + n}(\{t_i = v_0 s^{\frac{1}{3}} t, \bar{r}_i = \bar{r}/\kappa\}). 
$$

The dimensionless function hence satisfies the flow equation

$$
\left[ \partial_s - 4m + 2 + (m - \bar{m}) \frac{\gamma}{\kappa} - \sum_{k=1}^{m+\bar{m}-1} \bar{p}_k \bar{r}_k + \frac{\gamma}{\kappa} \sum_{k=1}^{m+\bar{m}-1} \bar{r}_k \partial_{\bar{r}_k} \right] G^{(m, \bar{m})}_{\psi, \ldots, \lambda + n} \\
= \hat{G}^{(m, \bar{m})}_{\psi, \ldots, \lambda + n} \times \left\{ \begin{array}{ll}
\frac{\hat{G}_0}{\kappa^2} \left[ \sum_{k, l} t_k t_l \bar{p}_k \cdot \bar{p}_l \right]^2 + O(p_{\text{max}}) & t_k \ll 1 \\
\frac{\hat{G}_\infty}{\kappa^2} \sum_{k, l} \bar{r}_k \cdot \bar{r}_l (|\bar{r}_k| + |\bar{r}_l| - |\bar{r}_k - \bar{r}_l|) + O(p_{\text{max}}) & t_k \gg 1
\end{array} \right. 
$$

As for the two-point function, the explicit s dependence can be removed by searching for a solution of a particular form. The corresponding fixed-point solutions, at unequal times, will have a similar behavior as the solutions obtained in 3D in [22], modified by the logarithmic corrections. We leave for future work their explicit derivation, and rather focus on the equal time correlation functions in the following.

5. N-point correlation function at equal times

The NLO term in the large wave-number expansion of the flow equation (30) is calculated in section 5.2. Before studying this term, let us assume that it decouples at equal time, which means that the O(p_{\text{max}}L) term in (52) is also zero at equal time, and expound the consequences for the two-point function.

5.1. Logarithmic corrections assuming no intermittency

In this section, we hence focus on the two-point function C_{\psi}, and assume decoupling at equal time, which means that there is no intermittency, and the exponent of the power-law in (50) (corresponding to Kolmogorov–Kraichnan scaling) is exact. In this case, as suggested by
Kraichnan [6], logarithmic corrections are needed in order to ensure consistency with the hypothesis of a constant enstrophy flux in the inertial range. Let us unfold Kraichnan’s argument within the present formalism, which will fix the value of the exponent γ of the logarithm. We then give explicitly the logarithmic corrections in the time-dependence of the two-point function and in the equal time n-point functions.

For this, we first compute the energy spectrum, assuming that the (connected) equal-time two-point function $C_\psi(0, \vec{x}) = \langle \psi(t, \vec{x}) \psi(t, 0) \rangle$, has no intermittency correction. Using the previous scaling analysis, one deduces that

$$C_\psi(0, \vec{p}) \sim |\vec{p}|^{-6} (\ln(|\vec{p}|L))^{-2\gamma}$$

(55)

and obtains for the energy spectrum

$$E(p) = 2\pi p^3 C_\psi(0, \vec{p}) \sim p^{-3} (\ln(pL))^{-2\gamma}.$$  

(56)

We now establish the expression of the flux of enstrophy. Let $T(p)$ be the rate of energy transfer owing to the nonlinear interactions in NS equation. The nonlinear transfer of enstrophy is then given by $p^3 T(p)$. The flux of enstrophy $Z(p)$ is defined as the nonlinear transfer across a scale $p$ as $Z(p) = \int_p^\infty p^2 T(p') dp'$. In the direct-cascade range of wave-numbers, the enstrophy flux is estimated to be [2, 6, 20]

$$Z(p) \sim \omega_p p^3 E(p) \quad \text{with} \quad \omega_p^2 \sim \int_0^p dp' p'^2 E(p')$$

(57)

where $\omega_p$ is the characteristic frequency of the distortion of eddies at scale $1/p$ and $p_{\text{max}} \sim 1/L$ is the lowest turbulent wave-number. In our framework, one obtains $\omega_p^2 \sim (\ln(pL))^{-2\gamma + 1}$, which then yields $Z(p) \sim (\ln(pL))^{-3\gamma+1}$. Requiring a constant enstrophy flux $Z(p) \equiv \varepsilon_\omega$ for wave-numbers in the direct cascade thus fixes $\gamma = 1/6$. As expected, this value corresponds to the log-corrected spectrum predicted by Kraichnan

$$E(p) \sim p^{-3} (\ln(pL))^{-1/3}.$$  

(58)

Let us emphasize that this reasoning does not prove the existence of the log-corrections, but simply deduces their form under the assumption of absence of intermittency. With this value of the exponent, the integral $\int_0^{pL} x(\ln x)^{\mu_\infty} \, dx$, with $\mu_0 \equiv 2\gamma$ and $\mu_\infty \equiv \gamma$, in equation (50) behaves at large $p$ as $(pL)^2 \ln(pL)^{\mu_\infty}$, with possible superimposed oscillations. Hence one obtains in the exponential

$$C_\psi(t, \vec{p}) \sim \begin{cases} \exp(-\beta_0^0 t^2 p^2 \ln(pL)^{-1/3}) & t \ll 1 \\ \exp(-\beta_{\infty}^0 |t| p^2 \ln(pL)^{-1/6}) & t \gg 1 \end{cases}$$

(59)

where numerical constants and a factor $L^2$ has been absorbed in the $\beta_0^0$. It would be very interesting to test this prediction in numerical simulations or experiments. If the exponent of the logarithm in the time dependence can be precisely determined (which is certainly difficult), this would constitute another test of the existence of intermittency in the 2D direct cascade.

To make connection with other existing results, let us express the equal-time two-point correlation function of the vorticity

$$C_{\omega}(0, \vec{p}) = p^4 C_\psi(0, \vec{p}) \sim p^{-2} \ln(pL)^{-1/3}.$$  

(60)

In real space, one obtains
\[ C_\omega(0, \vec{r}) = \int_0^\pi d\theta \int \frac{dp}{p} \ln(pL)^{-1/3} \exp(i\vec{p}|\vec{r}| \cos \theta) = \int \frac{dp}{p} \ln(pL)^{-1/3} J_0(pr) \]  

(61)

where \( J_0 \) is a Bessel function. The integral on \( p \) is cut in the IR by \( 1/L \). In the UV, the Bessel function is dominated by values \( p \lesssim 1/r \) since it rapidly oscillates around 0 at large \( p \), which suppresses the integrand. One thus obtains \( C_\omega^{(2,0)}(0, \vec{r}) \sim \ln(L/|\vec{r}|)^{2/3} \), which corresponds to the Falkovich and Lebedev prediction \[9, 10\].

Extending this comparison to higher-order equal time \( n \)-point correlations of the vorticity requires further work. Let us just give the result in the Fourier space. The flow equation (54) reduces at the fixed point, for equal times \( t_e = 0 \), and expressed for the \( m = 2n \) \((m = 0)\) vorticity correlation as

\[ -4n + 2 + \frac{n}{3s} - \sum_{k=1}^{2n-1} \hat{p}_k \partial \hat{p}_k \hat{G}_{\omega, j_1, \ldots, j_n} = 0. \]  

(62)

One deduces the general solution of this equation as

\[ \hat{G}_{\omega, j_1, \ldots, j_n}^{(2n,0)}(0, \hat{\vec{p}}_1, \ldots, 0, \hat{\vec{p}}_{2n-1}) = \left( \prod_{k=1}^{2n-1} \hat{p}_k^{-2} (\ln \hat{p}_k)^{-1/6} \right) \ln i\hat{\vec{p}}_1 + \cdots + \hat{\vec{p}}_{2n-1}|^{-1/6} \hat{f}^{(2n)}, \]  

(63)

where \( \hat{f}^{(2n)} \) is a scaling function. The obtained logarithmic corrections have an overall behavior compatible with Falkovich–Lebedev prediction in real space \( \langle \omega^m(\vec{r}_1)\omega^n(\vec{r}_2) \rangle \propto \ln(L/|\vec{r}_1 - \vec{r}_2|)^{2n/3} \). However, in order to make the statement precise, one needs to perform the multi-dimensional inverse Fourier transforms of (63). This requires to take into account the different integration sectors with great care. We leave the corresponding analysis for further work.

5.2. \( n \)-point correlation functions at equal times

At equal time, the exact leading term at large wave-number of the flow equation for a generic \( n \)-point correlation function, given by the rhs of equation (30), vanishes. This can be read off directly from (27), or equivalently in the frequency space from (31). Indeed, the equal-time correlation function is obtained by integrating over all the external frequencies

\[ \tilde{G}_\psi^{(n)}(\{t = 0, \hat{\vec{p}}_t\}_{1 \leq \ell \leq n}) = \int_{\omega_{j_1, \ldots, j_n}} \tilde{G}_\psi^{(n)}(\{\omega_t, \hat{\vec{p}}_t\}_{1 \leq \ell \leq n}). \]  

(64)

Since the operator \( \tilde{D}(\psi) \) in (31) acts as a finite difference, the integrated rhs vanishes upon absorbing the related shifts by a change of variable in the external frequencies. At equal times, the first non-trivial contribution hence comes from the NLO term in the large-wave number expansion, which is the fourth order term in the \( q_1, q_2 \) expansion

\[ \partial_\alpha \int_{\omega_{l_1}} \tilde{G}_\psi^{(n)}(\{p_l\}_{1 \leq \ell \leq n}) = \frac{1}{2} \int_{q_1, q_2} \partial_\alpha \tilde{G}_\psi^{(2)}(-q_1, -q_2) \times \int_{\omega_{l_1}} \frac{q_1^a q_2^b q_1^c q_2^d}{4!} \frac{\partial^4}{\partial q_1^a \partial q_1^b \partial q_2^c \partial q_2^d} \left[ \frac{\delta^2}{\delta \psi_1(q_1) \delta \psi_2(q_2)} \tilde{G}_\psi^{(n)}(\{p_l\}; j) \right]_{\rho=0} |_{q_1=q_2=0}, \]  

(65)

where as before \( a, b, c, d \) take value in \( \{1, 2\} \). The detailed calculation of this term is reported in appendix B, we summarize below the main steps.
First, one can show that among all the different combinations of \( \vec{q}_1 \) and \( \vec{q}_2 \) derivatives, only the ones with two \( \vec{q}_1 \) and two \( \vec{q}_2 \) survive after the integration over the external frequencies. The terms with four derivatives with respect to \( \vec{q}_1 \) vanish when evaluating at \( \vec{q}_2 = 0 \) because of the identity (a) in (24) related to the gauge symmetry, and similarly for \( \vec{q}_2 \). The terms with only one \( \vec{q}_1 \) derivative (and similarly only one \( \vec{q}_2 \)) vanish as well. The reason is that this derivative yields an overall \( \mathcal{D} \) operator as at leading order (see appendix B), and this contribution vanishes when integrating over the external frequencies. Only the terms with two derivatives of \( \vec{q}_1 \) and \( \vec{q}_2 \) remain, and they can be written, using space translation and rotation invariance of \( \partial_\kappa \tilde{G}_\psi^{(2)} \), as

\[
\partial_\kappa \int_{\omega_1, \omega_2} \tilde{G}_\psi^{(n)}(\{p_\kappa\}_{1\leq \kappa \leq n}) = \frac{1}{2} \int_{\omega_1, \omega_2} \tilde{K}_\psi(\omega_1, \omega_2) \times \int_{\omega_1} \left( \frac{\partial^4}{\partial q_1^\alpha \partial q_2^\beta \partial q_2^\gamma \partial q_2^\delta} + 2 \frac{\partial^4}{\partial q_1^\beta \partial q_2^\alpha \partial q_2^\gamma \partial q_2^\delta} \right) \left[ \frac{\delta^2}{\delta \varphi_(\mathbf{q}_1) \delta \varphi_(\mathbf{q}_2)} \tilde{G}_\psi^{(n)} (\{p_\ell\}_{1\leq \ell \leq n}; j) \right]_{\varphi=0} \bigg|_{\vec{q}_1=\vec{q}_2=0} \tag{66}
\]

with

\[
\tilde{K}_\psi(\omega_1, \omega_2) = \frac{1}{32} \int_{\mathbf{q}} \tilde{\varphi}_\psi(\mathbf{q}) (-\omega_1 - \omega_2, q^2)^2. \tag{67}
\]

Appendix B is devoted to show that

\[
\frac{\partial^4}{\partial q_1^\alpha \partial q_2^\beta \partial q_2^\gamma \partial q_2^\delta} \left[ \frac{\delta^2}{\delta \varphi_(\mathbf{q}_1) \delta \varphi_(\mathbf{q}_2)} \tilde{G}_\psi^{(n)} (\{p_\ell\}_{1\leq \ell \leq n}; j) \right]_{\varphi=0} = \delta_{\vec{q}_1} \delta_{\vec{q}_2} \tilde{\varphi}(\omega_1) \tilde{\varphi}(\omega_2) \tilde{G}_\psi^{(n)} (\{p_\ell\}), \tag{68}
\]

which means that the contribution with the uncrossed derivatives is completely controlled by the extended symmetries and can be closed exactly using the corresponding Ward identities. It turns out that this term vanishes after integration over frequencies by conservation of angular momentum.

\[
\int_{\omega_1} \tilde{\varphi}(\omega_1) \frac{2i\epsilon_{\alpha\beta\gamma}^n}{\omega_2} \sum_{k=1}^n \frac{\partial}{\partial p_k} \tilde{G}_\psi^{(n)} (\{\omega_k + \omega_2, \vec{p}_k\}, \ldots) \tag{69}
\]

\[
= \int_{\omega_1} \tilde{\varphi}(\omega_1) \frac{2i\epsilon_{\alpha\beta\gamma}^n}{\omega_2} \sum_{k=1}^n \frac{\partial}{\partial p_k} \left[ \tilde{G}_\psi^{(n)} (\{\omega_k + \omega_2, \vec{p}_k\}, \ldots) - \tilde{G}_\psi^{(n)} (\{\omega_k, \omega_2, \vec{p}_k\}, \ldots) \right] = 0,
\]

and thus, it gives no contribution at equal times.

To summarize, beyond the technical details, one finds that all the terms which are controlled by the extended symmetries of the NS action vanish after integration over external frequencies. The only remaining term, which is not controlled by symmetries is the one with the crossed derivatives

\[
\partial_\kappa \int_{\omega_1, \omega_2} \tilde{G}_\psi^{(n)}(\{p_\ell\}_{1\leq \ell \leq n}) \times \int_{\omega_1, \omega_2} \frac{\partial^4}{\partial q_1^\alpha \partial q_2^\beta \partial q_2^\gamma \partial q_2^\delta} \left[ \frac{\delta^2}{\delta \varphi_(\mathbf{q}_1) \delta \varphi_(\mathbf{q}_2)} \tilde{G}_\psi^{(n)} (\{p_\ell\}; j) \right]_{\varphi=0} \bigg|_{\vec{q}_1=\vec{q}_2=0} \tag{70}
\]
This term is \textit{a priori} non zero, and could be a source of intermittency. However, the effect can be expected to be much weaker than in 3D, since the time-gauged rotation does not hold in 3D and the corresponding terms do not vanish \textit{a priori}.

It is possible that the crossed contribution (70) turns out to be proportional to the uncrossed one (68), and thus vanishes, at least in some specific wave-vector configurations, but we have not been able to prove it. If this were case, this would imply that there is no intermittency in the direct cascade of 2D turbulence at equal times. It is instructive to consider the flow of the two-point function to further comment on this. The function appearing in square bracket in the rhs of the flow equation (66) for $n = 2$ is a function of $\vec{p}, \vec{q}_1, \text{ and } \vec{q}_2$, and the corresponding frequencies:

$$
\left[ \frac{\delta^2}{\delta \varphi_i(q_1) \delta \varphi_j(q_2)} \hat{G}^{(2)}_{\psi,\sigma,\mu,\nu}[\vec{p},\vec{q}] \right]_{q=0} \equiv F(\omega, \vec{p}, \vec{q}_1, \vec{q}_2, \vec{q}).
$$

(71)

The wave-vector part of $F$ involves only five independent scalars in 2D, which, considering the symmetry of exchange $\vec{q}_1 \leftrightarrow \vec{q}_2$ can be chosen as

$$
F(\vec{p}, \vec{q}_1, \vec{q}_2) = F(\rho^2, \vec{p} \cdot (\vec{q}_1 + \vec{q}_2), \vec{q}_1 \cdot \vec{q}_2, q_1^2, (\vec{p} \cdot \vec{q}_1)(\vec{p} \cdot \vec{q}_2)),
$$

(72)

omitting the frequencies, which play no role for evaluating the $q$ derivatives. One deduces that

$$
\frac{\partial F}{\partial q_1^\mu \partial q_1^\nu \partial q_2^\rho \partial q_2^\sigma} \bigg|_{\vec{q}_1 = \vec{q}_2 = 0} = \rho^4 \left[ F^{(0,4,0,0,0)} + 4 F^{(0,2,0,0,1)} + 2 F^{(0,0,0,1,0)} \right]
$$

$$
+ \rho \left[ 8 F^{(0,2,0,1,0)} + 4 F^{(0,2,1,0,0)} + 2 F^{(0,0,1,1,0)} \right] + 16 F^{(0,0,0,2,0)} + 4 F^{(0,0,2,0,0)}
$$

$$
\frac{\partial F}{\partial q_1^\mu \partial q_1^\nu \partial q_2^\rho \partial q_2^\sigma} \bigg|_{\vec{q}_1 = \vec{q}_2 = 0} = \rho^4 \left[ F^{(0,4,0,0,0)} + 4 F^{(0,2,0,0,1)} + 2 F^{(0,0,0,1,0)} \right]
$$

$$
+ \rho \left[ 4 F^{(0,2,0,1,0)} + 6 F^{(0,2,1,0,0)} + 6 F^{(0,0,1,1,0)} \right] + 8 F^{(0,0,0,2,0)} + 6 F^{(0,0,2,0,0)}.
$$

(73)

where all the derivatives of $F$ are evaluated at $(\rho^2, 0,0,0,0)$. Furthermore, one can exploit that within the flow equation,

$$
\frac{\partial^4}{\partial q_1^\mu \partial q_1^\nu \partial q_2^\rho \partial q_2^\sigma} F(\vec{p}, \vec{q}_1, \vec{q}_2) \bigg|_{\vec{q}_1 = \vec{q}_2 = 0} = 0.
$$

(74)

which yields the additional relations $F^{(0,4,0,0,0)} = F^{(0,2,0,0,1)} = F^{(0,0,0,2,0)} = 0$. Similarly, from the vanishing of the contribution

$$
\frac{\partial^4}{\partial q_1^\mu \partial q_1^\nu \partial q_2^\rho \partial q_2^\sigma} F(\vec{p}, \vec{q}_1, \vec{q}_2) \bigg|_{\vec{q}_1 = \vec{q}_2 = 0} = 0
$$

(75)

stems that $F^{(0,2,0,0,1)} = F^{(0,0,0,1,0)} = F^{(0,0,1,1,0)} = 0$. It follows that

$$
\frac{\partial^4}{\partial q_1^\mu \partial q_1^\nu \partial q_2^\rho \partial q_2^\sigma} F(\vec{p}, \vec{q}_1, \vec{q}_2) \bigg|_{\vec{q}_1 = \vec{q}_2 = 0} = \frac{3}{2} \frac{\partial}{\partial q_1^\mu \partial q_1^\nu \partial q_2^\rho \partial q_2^\sigma} F(\vec{p}, \vec{q}_1, \vec{q}_2) \bigg|_{\vec{q}_1 = \vec{q}_2 = 0} - \rho^4 F^{(0,0,0,0,2)}
$$

(76)

which shows that the crossed term is almost proportional to the uncrossed one, but for the last term, which is hence the only remaining one in the flow equation. This analysis suggests that a significant part of the crossed term turns out to be proportional to the uncrossed term, and hence vanishes as well. The remaining contribution in the flow equation is then expected to be very small, although we could not prove that it is zero. Moreover, the two terms would be strictly proportional for instance if the function $F$ depends only on the moduli of the wave-vectors, but not on their relative angles. For the structure functions, which involve only two
space points, it is not clear which configurations of wave-vectors dominate, and whether this relation could be fulfilled. In any case, this provides strong hints that the intermittency corrections for equal-time correlation functions are very small in 2D. The detailed analysis of structure functions is left for future investigations.

6. Conclusion and perspectives

In this paper, we investigated 2D forced turbulence, using field theoretical techniques. We unveiled two extended symmetries of the NS field theory that were not identified yet. One, related to time-gauged shifts of the response field, exists in both 3D and 2D, while the other one, related to time-gauged rotations, is only realized in 2D. These symmetries bring new exact relations between the correlation functions of the theory through Ward identities, which can be useful in general.

We then exploited these Ward identities in the framework of the NPRG, within the large-wave number expansion scheme developed in [22, 24], to compute some properties of the correlation functions of 2D isotropic and homogeneous turbulence in the direct cascade. The leading order term of this expansion can be closed exactly, and allowed us to obtain the time dependence of the 2-point correlation function in the stream formulation at both small and large time delays. This prediction could be tested in numerical simulations or in experiments. The generalization for \( n \)-point function is left for future work. This exact leading order contribution explicitly breaks standard scale invariance.

At equal times, the leading order term vanishes, and one is left with log-corrected power-laws. If one assumes that there is no intermittency, then one recovers Kraichnan’s logarithms, by unrolling a similar argument within the NPRG formalism. To assess the presence or not of intermittency in equal-time quantities, we calculated the NLO term in the large-wave number expansion. We found that almost all the terms are controlled by the symmetries, and that these terms vanish at equal times, and hence cannot generate intermittency. Nevertheless, there remains one contribution, which is not constrained by the symmetries. This contribution could lead to intermittency correction. However, this correction can be reasonably expected to be much weaker than in 3D, since in 3D many other contributions remain. Moreover, the unconstrained contribution could turn out to vanish in some specific wave-vector configurations, as the ones involved in the calculation of structure functions. Further works are in progress to approximate this contribution and estimate the related intermittency correction. Let us also emphasize that the techniques developed in the present work could be useful to study other hydrodynamical systems, such as passively advected quantities [60–72], which is underway [73].

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Appendix A. Extended symmetries and Ward identities

In this appendix, we explain the general derivation of Ward identities within the NPRG framework, and then give details on the derivation of the ones associated to the time-gauged Galilean and rotation symmetries in the stream formulation.
A.1. Ward identities in the NPRG framework

Let us illustrate the derivation of a Ward identity on a generic field theory for a field \( \phi \) which can possess multiple components. Within the NPRG framework, in the presence of the infrared regulator \( \Delta S_\kappa \), the associated partition function is

\[
Z_\kappa[j] = \int D[\phi] e^{-S[\phi] - \Delta S_\kappa[\phi] + j \phi}.
\]

(A.1)

Let us consider an infinitesimal transformation, \( \delta \phi = \delta_x \phi = O(\epsilon) \), which is at most linear in the fields

\[
\phi_i(x) \rightarrow \phi'_i(x) = \phi_i(x) + \delta_i \phi_i(x), \quad \delta_x \phi_i(x) = \epsilon \left[ \int_y A_{ij}(x, y) \phi_j(y) + B_i(x) \right].
\]

(A.2)

where \( A \) is an operator acting on \( \phi \). It follows that \( \langle \delta_x \phi \rangle = \delta_x \Phi \), with \( \Phi \equiv \langle \phi \rangle \). At linear order in \( \epsilon \), since \( Z_\kappa \) is unchanged by this change of variables, one obtains the identity

\[
\langle \delta_x S \rangle_j + \langle \delta_x \Delta S_\kappa \rangle_j + j \cdot \delta_x \Phi = 0,
\]

(A.3)

where \( \delta_x X \) is the part of \( \delta_x \Phi \) linear in \( \epsilon \). Defining \( \mathcal{R}^x_{ij}(x, y) \equiv \frac{\delta^2 S}{\delta \phi_i(x) \delta \phi_j(y)} \), one obtains from equation (A.3) the Ward identity

\[
\delta_x \Gamma_\kappa[\Phi] = \langle \delta_x S \rangle_j + \epsilon \int_{xyz} \mathcal{R}^x_{ij}(x, y) A_{ij}(x, z) G_j(y, z).
\]

(A.4)

where \( \delta_x \Gamma_\kappa[\Phi] = \int_x \frac{\delta \Gamma_\kappa}{\delta \Phi(x)} \delta_x \Phi(x) \). The second term vanishes if the regulator term is invariant under the transformation. Let us notice that, because of the definition of the Legendre transform, the variation of the regulator under the shift part of the change of variable (A.2) never enters the Ward identity in this formalism.

For exact symmetries of the action and of the regulator, equation (A.4) simply translates into \( \delta_x \Gamma_\kappa[\Phi] = 0 \), which means that \( \Gamma_\kappa \) also possesses these symmetries. For extended symmetries, where the variations of the action and of the regulator are non-zero but linear in the fields, the mean and the variation commute and the Ward identity reads \( \delta_x \Gamma_\kappa[\Phi] = \delta_x \mathcal{S}[\Phi] \), which means that the variation of the EAA \( \Gamma_\kappa \) is equal to the mean of the variation of \( \mathcal{S} \). This provides non-renormalization theorems which fix a sector of \( \Gamma_\kappa \) to its bare value. The transformations considered in this paper are all pure shifts of the fields, except the extended Galilean symmetry and the extended rotations. However, as the regulator is invariant under space translations, rotations, and is instantaneous (delta-correlated in time), it is invariant as well under time-gauged translations and time-gauged rotations, and thus it does not enter the corresponding Ward identities.

A.2. Ward identity for the time-gauged Galilean symmetry

The set of Ward identities related to the time-gauged Galilean symmetry for generic vertex functions are derived in [22, 24] in the velocity formulation. We present in this appendix their derivation in the stream formulation. The functional Ward identity for time-gauged Galilean transformation, equation (23) \( (d) \), reads

\[
\int_S \left\{ \left( -\epsilon_{\gamma\beta} x_\gamma \partial_\gamma + \partial_\beta \Psi \right) \frac{\delta \Gamma_\kappa}{\delta \Psi(x)} + \partial_\beta \Psi \frac{\delta \Gamma_\kappa}{\delta \Psi(x)} \right\} = 0.
\]

(A.5)
To deduce the Ward identity for a generic vertex function $\Gamma^{(m,n)}_\kappa$, one takes $m$ functional derivatives of (A.5) with respect to the stream function $\Psi(x)$ and $n$ with respect to the response stream $\bar{\Psi}(x)$, and then set the fields to zero, which yields, after multiplying by $e_{\alpha \beta}$ and in the Fourier space,

$$\frac{\partial}{\partial q^0} \Gamma^{(m,n)}_\kappa(\mathbf{q}, \mathbf{q}_{\nu}, \{\mathbf{p}_k\}_{1 \leq k \leq m+n}) = -i e_{\alpha \beta} \sum_{k=1}^{m+n} \frac{\partial}{\partial \mathbf{q}} \Gamma^{(m,n)}_\kappa(\{\mathbf{p}_k\}_{1 \leq k \leq m+n})$$

$$\equiv i e_{\alpha \beta} \mathcal{D}_\beta(\mathbf{q}) \Gamma^{(m,n)}_\kappa(\{\mathbf{p}_k\}_{1 \leq k \leq m+n}).$$

The corresponding expression for $\Gamma^{(m+1,n)}_\kappa$ can be deduced from (A.6) and reads

$$\frac{\partial}{\partial q^0} \Gamma^{(m+1,n)}_\kappa(\mathbf{q}, \mathbf{q}_{\nu}, \{\mathbf{p}_k\}_{1 \leq k \leq m+n-1}) = i e_{\alpha \beta} \mathcal{D}_\beta(\mathbf{q}) \Gamma^{(m,n)}_\kappa(\{\mathbf{p}_k\}_{1 \leq k \leq m+n-1}),$$

where the operator $\mathcal{D}_\beta(\mathbf{q})$ is now defined by

$$\mathcal{D}_\alpha(\mathbf{q}) F(\{\mathbf{p}_k\}_{1 \leq k \leq m+n}) = -\sum_{k=1}^n \widetilde{\Gamma}_{\kappa}^{(m+1,n)}(\{\mathbf{p}_k\}_{1 \leq k \leq m+n-1}) = F(\{\mathbf{p}_k\}_{1 \leq k \leq m+n-1}).$$

The expression shows explicitly the regularity of the limit $\mathbf{q} \to 0$.

### A.3. Ward identity for the time-gauged rotation symmetry

The functional Ward identity associated with the time-gauged rotation, equation (23) (e), is given by

$$\int \left\{ \frac{\chi^2}{2} \partial_\alpha + e_{\alpha \beta} x_\beta \partial_\alpha \Psi \Gamma^{(1,1)}_\kappa \right\} = 2 \int \partial_\beta \bar{\Psi}.$$

Let us first derive an identity for $\Gamma^{(1,1)}_\kappa$. Taking one derivative with respect to $\Psi(x')$, setting the fields to zero, and noting that $\partial_\alpha \Gamma^{(0,1)}_\kappa(x) = 0$ by translational invariance, one obtains in the Fourier space

$$\frac{\partial^2}{\partial p^2} \Gamma^{(1,1)}_\kappa(\mathbf{p}, \mathbf{p}') \bigg|_{p=0} = -4i \omega \delta^d(\mathbf{p}') \delta(\omega + \omega').$$

This result can be interpreted as the non-renormalization of the kinetic term in the bare action.

We now derive the identity for a generic vertex function. Taking $m$ functional derivatives of (A.9) with respect to $\Psi$ and $n$ with respect to $\bar{\Psi}$, one obtains in the Fourier space

$$\frac{\partial}{\partial q^0} \Gamma^{(m+1,n)}_\kappa(\mathbf{q}, \mathbf{q}_{\nu}, \{\mathbf{p}_k\}_{1 \leq k \leq m+n}) = i e_{\alpha \beta} \mathcal{D}_\beta(\mathbf{q}) \Gamma^{(m+1,n)}_\kappa(\{\mathbf{p}_k\}_{1 \leq k \leq m+n-1}),$$

where the identities (a) and (d) of equation (24) have been used. In contrast with the case of Galilean symmetry, this expression does not explicitly show that the limit $\mathbf{q} \to 0$ is
Substracting it from (A.12), one finally obtains an expression in terms of finite differences as

\[
\frac{\partial^2}{\partial q^2} \tilde{\Gamma}^{(m+1)n}_\kappa (\omega, \vec{q}, \{p_t\}_{1 \leq t \leq m+n+1}) \bigg|_{q=0} = \mathcal{R}(\omega) \tilde{\Gamma}^{(m,n)}_\kappa (\{p_t\}_{1 \leq t \leq m+n-1})
\]

with

\[
\mathcal{R}(\omega) F(\{p_t\}_{1 \leq t \leq n}) \equiv 2i\epsilon_{\alpha\beta} \sum_{k=1}^n p_k^\alpha \frac{\partial}{\partial p_k^\beta} \tilde{\Gamma}^{(m,n)}_\kappa (\{p_t\}_{1 \leq t \leq m+n-1})
\]

where the regularity of the limit \(\omega \to 0\) is now manifest.

Let us note that similar subtleties arise when passing from the \(\tilde{\Gamma}^{(m,n)}_\kappa\) to the \(\tilde{\Gamma}^{(m,n)}_\kappa\) for the symmetries (b) and (c) of equation (23), but the derivation of the corresponding identities is straightforward since the rhs is always zero. We finally recapitulate the list of Ward identities for the vertex function \(\tilde{\Gamma}^{(m,n)}\), which are the ones used in the derivation of the flow equations in the large wave-number expansion

(a) \(\tilde{\Gamma}^{(m,n)}(\omega, \vec{q}, \ldots) \bigg|_{q=0} = 0\)

(b) \(\frac{\partial}{\partial q^k} \tilde{\Gamma}^{(m,n+1)}_\kappa (\{p_t\}_{1 \leq t \leq m}, \omega, \vec{q}, \{p_t\}_{1 \leq t \leq n}) \bigg|_{q=0} = 0\)

(c) \(\frac{\partial^2}{\partial q^k \partial q^l} \tilde{\Gamma}^{(m,n+1)}_\kappa (\{p_t\}_{1 \leq t \leq m}, \omega, \vec{q}, \{p_t\}_{1 \leq t \leq n}) \bigg|_{q=0} = 0\)

except \(\frac{\partial^2}{\partial q^k} \tilde{\Gamma}^{(1,1)}_\kappa (\omega', \vec{q}', \omega, \vec{q}) \bigg|_{q=0} = 4i\omega \delta(\vec{q}') \delta(\omega + \omega')\)

(d) \(\frac{\partial}{\partial q^k} \tilde{\Gamma}^{(m+1,n)}_\kappa (\omega, \vec{q}, \{p_t\}_{1 \leq t \leq m+n+1}) \bigg|_{q=0} = i\epsilon_{\alpha\beta} D_\beta(\omega) \tilde{\Gamma}^{(m,n)}_\kappa (\{p_t\}_{1 \leq t \leq m+n})\)

(e) \(\frac{\partial^2}{\partial q^k \partial q^l} \tilde{\Gamma}^{(m+1,n)}_\kappa (\omega, \vec{q}, \{p_t\}_{1 \leq t \leq m+n+1}) \bigg|_{q=0} = \mathcal{R}(\omega) \tilde{\Gamma}^{(m,n)}_\kappa (\{p_t\}_{1 \leq t \leq m+n+1})\)

except \(\frac{\partial^2}{\partial q^k \partial q^l} \tilde{\Gamma}^{(1,1)}_\kappa (\omega, \vec{q}, \omega', \vec{q}') \bigg|_{q=0} = -4i\omega \delta(\vec{q}') \delta(\omega + \omega')\)

with the two following definitions for the operator \(\hat{D}_\alpha(\omega)\) and \(\mathcal{R}(\omega)\):

\[
\hat{D}_\alpha(\omega) F(\{p_t\}_{1 \leq t \leq n}) \equiv -\sum_{k=1}^n p_k^\alpha \frac{\partial}{\partial p_k^\alpha} F(\{p_t\}_{1 \leq t \leq k-1}, \omega_k + \omega, \vec{p}_k, \{p_t\}_{k+1 \leq t \leq n})
\]

\[
\mathcal{R}(\omega) F(\{p_t\}_{1 \leq t \leq n}) \equiv 2i\epsilon_{\alpha\beta} \sum_{k=1}^n p_k^\alpha \frac{\partial}{\partial p_k^\beta} F(\{p_t\}_{1 \leq t \leq k-1}, \omega_k + \omega, \vec{p}_k, \{p_t\}_{k+1 \leq t \leq n}).
\]
Appendix B. Flow equation at NLO in the large wave-number expansion at equal times

As mentioned in the main text, the leading order term of the large wave-number expansion in the flow equation vanishes when all times are equal (i.e. when integrated over all the external frequencies). In this appendix, we hence calculate the NLO term, focusing on equal-times correlations, and show that all the terms which are controlled by the extended symmetries vanish.

The NLO term of the flow equation for a generalized correlation function is given by equation (65). The four wave-number derivatives can be classified according to the respective number of $\bar{q}_1$ and $\bar{q}_2$ derivatives. Using the same argument as in section 5.2, if the four derivatives are $\bar{q}_1$ (resp. four $\bar{q}_2$), this contribution is zero because the vertex function with the wave-number $\bar{q}_2$ (resp. $\bar{q}_1$) vanishes when this wave-number is set to zero. Thus we consider separately the two remaining cases: three $\bar{q}_1$ and one $\bar{q}_2$ (and equivalently one $\bar{q}_1$ and three $\bar{q}_2$), and finally two $\bar{q}_1$ and two $\bar{q}_2$. The calculations to establish this general proof are lengthy but quite straightforward. We only report below the main steps, eluding the details which can be found in [59] and relies on similar ideas as in [22].

B.1. Contributions 1–3 and 3–1

Although no Ward identity exists for the third wave-number derivative of a vertex function, the time-gauged Galilean Ward identity can still be used on the leg with one $q$ derivative and the proof for one $\psi$ derivative (expounded in [22, 59]) can be carried through to show that one obtains an operator $\bar{D}$ acting on the external legs of the whole diagram

$$\partial_\kappa \int_{\{\omega\}} \tilde{G}^{(n)}_{\mu\nu\ldots}(\{p_i\}_{1\leq i \leq n})_{\text{NLO,1-3}} = \frac{1}{2} \int_{q_1,q_2} \partial_{\nu} G^{(2)}_{\kappa j}(\bar{q}_1, -\bar{q}_2) \times \int_{\{\omega\}} \frac{q_1^\mu q_1^\nu q_2^\rho q_2^\sigma}{3!} \partial^3 \partial_{q_1^\mu} \partial_{q_1^\nu} \partial_{q_2^\rho} \partial_{q_2^\sigma} \left\{ i e_{\nu\mu\kappa} \bar{D} \bar{G}^{(n)}_{\mu\nu\ldots}(\{p_i\}; j) \right\} \varphi = 0 \ , \ \bar{q}_1 = 0 \ .$$

(B.1)

Distributing the $\bar{q}_2$ derivatives, one obtains two types of terms: either all $\bar{q}_2$ derivatives act on the term in square bracket or one of them acts on the operator $\bar{D}$

$$\int_{\{\omega\}} \frac{q_1^\mu q_1^\nu q_1^\rho q_2^\sigma}{3!} \partial^3 \partial_{q_1^\mu} \partial_{q_1^\nu} \partial_{q_1^\rho} \partial_{q_2^\sigma} \left\{ i e_{\nu\mu\kappa} \bar{D} \bar{G}^{(n)}_{\mu\nu\ldots}(\{p_i\}; j) \right\} \varphi = 0 \ , \ \bar{q}_1 = 0 \ .$$

$$= \int_{\{\omega\}} \frac{q_1^\mu q_1^\nu q_1^\rho q_2^\sigma}{3!} i e_{\nu\mu\kappa} \bar{D} \bar{G}^{(n)}_{\mu\nu\ldots}(\{p_i\}; j) \left. \frac{\partial^3}{\partial q_1^\mu \partial q_1^\nu \partial q_1^\rho} \left[ \frac{\delta}{\delta \varphi_1}(q_1) \bar{G}^{(n)}_{\mu\nu\ldots}(\{p_i\}; j) \right] \right|_{\bar{q}_1 = 0} \varphi = 0 \ ,$$

$$+ \int_{\{\omega\}} \frac{q_1^\mu q_1^\nu q_1^\rho q_2^\sigma}{2} i e_{\nu\mu\kappa} \bar{D} \bar{G}^{(n)}_{\mu\nu\ldots}(\{p_i\}; j) \left. \frac{\partial^2}{\partial q_1^\mu \partial q_1^\nu} \left[ \frac{\delta}{\delta \varphi_1}(q_1) \bar{G}^{(n)}_{\mu\nu\ldots}(\{p_i\}; j) \right] \right|_{\bar{q}_1 = 0} \varphi = 0 \ ,$$

$$= \int_{\{\omega\}} \frac{q_1^\mu q_1^\nu q_1^\rho q_2^\sigma}{3!} i e_{\nu\mu\kappa} \bar{D} \bar{G}^{(n)}_{\mu\nu\ldots}(\{p_i\}; j) \frac{1}{\omega_1} \left. \frac{\partial^2}{\partial q_1^\mu \partial q_1^\nu} \left[ \frac{\delta}{\delta \varphi_1}(q_1 + \omega_1 q_2) \bar{G}^{(n)}_{\mu\nu\ldots}(\{p_i\}; j) \right] \right|_{\varphi = 0} \bar{q}_1 = 0 \ .$$

(B.2)

In the first term of the first equality, $\bar{D} \{q_1\} = 0$ shifts only the frequencies associated to $\bar{p}_1$ and $\bar{p}_2$, thus this term is zero due to the conservation of wave-number of the object in square bracket and the integration in frequency. The $\bar{q}_2$ derivative on $\bar{D}$ selects the frequency shift on the $q_1$ leg, which does not vanish. However, this term is proportional to $\epsilon_{\nu\mu\sigma} q_1^\nu q_2^\rho$. Hence, within equation (B.1), the conservation of wave-number of $G^{(2)}_\kappa(-q_1,-q_2)$ gives $\bar{q}_1 + \bar{q}_2 = 0$, and thus $\epsilon_{\nu\mu\sigma} q_1^\nu q_2^\rho = -\epsilon_{\nu\mu\sigma} q_2^\nu q_1^\rho = 0$, and this term gives no contribution either to the flow
equation. Hence, all the contributions with three $q_1$ derivatives and one $q_2$, or three $q_2$ derivatives and one $q_1$ vanish

$$\partial_\kappa \int_{(\omega)} \tilde{G}^{(n)}_{i_1\ldots i_n}(\{p_\ell\}_{1 \leq \ell \leq n})|_{\text{NLO,1-3}} = 0. \quad (B.3)$$

### B.2. Contributions 2–2

The only remaining contribution in the NLO term of the flow equation involves an equal number of $q_1$ and $q_2$ derivatives

$$\partial_\kappa \int_{(\omega)} \tilde{G}^{(n)}_{i_1\ldots i_n}(\{p_\ell\}_{1 \leq \ell \leq n})|_{\text{NLO,2-2}} = \frac{1}{2} \int_{q_1, q_2} \partial_\kappa \tilde{G}^{(2)}_{\bar{\omega}}(-q_1, -q_2) \int_{(\omega)} \frac{\partial^4}{\partial q_1^\mu \partial q_2^\nu \partial q_1^\rho \partial q_2^\sigma} \left[ \delta_{\phi_1(q_1) \delta_{\phi_2}(q_2)} \tilde{G}^{(n)}_{i_1\ldots i_n}(\{p_\ell\}; j) \right] |_{q_1 = q_2 = 0}. \quad (B.4)$$

Using the invariance under space translation and rotation in the $\tilde{q}_1, \tilde{q}_2$ integrals, one can write

$$\int_{\tilde{q}_1, \tilde{q}_2} \partial_\kappa \tilde{G}^{(2)}_{\bar{\omega}}(-q_1, -q_2) \frac{\partial^4}{\partial q_1^\mu \partial q_2^\nu \partial q_1^\rho \partial q_2^\sigma} \left[ \delta_{\phi_1(q_1) \delta_{\phi_2}(q_2)} \tilde{G}^{(n)}_{i_1\ldots i_n}(\{p_\ell\}; j) \right] = (\delta_{\mu \nu} \delta_{\rho \sigma} + \delta_{\mu \sigma} \delta_{\rho \nu} + \delta_{\mu \rho} \delta_{\nu \sigma}) \bar{K}_{ij}(\omega_1, \omega_2),$$

with $\bar{K}_{ij}$ given by equation (67), and where the notation $\bar{G}$ indicates that the delta of conservation has been extracted for the wave-numbers only, and not for the frequencies. The expression (B.4) hence comprises the two contributions given in equation (66), referred to as the uncrossed and the crossed ones, according to whether the $\tilde{q}_1$ derivative is contracted with the other $\tilde{q}_1$ derivative or with the $\tilde{q}_2$ derivative.

We show in the next sections, first on the example of the two-point function, and then for a generic $n$-point function, that the uncrossed contribution can be closed exactly using the Ward identities associated with the new symmetries

$$\partial_\kappa \tilde{G}^{(n)}_{i_1\ldots i_n}(\{p_\ell\}_{1 \leq \ell \leq n})|_{\text{uncrossed}} = \frac{1}{2} \int_{\varpi_1, \varpi_2} \tilde{K}_{ij}(\varpi_1, \varpi_2) \frac{\partial^4}{\partial q_1^\mu \partial q_2^\nu \partial q_1^\rho \partial q_2^\sigma} \left[ \delta_{\phi_1(q_1) \delta_{\phi_2}(q_2)} \tilde{G}^{(n)}_{i_1\ldots i_n}(\{p_\ell\}; j) \right] |_{q_1 = q_2 = 0}$$

$$= \frac{1}{2} \int_{\varpi_1, \varpi_2} \tilde{K}_{\psi \psi}(\varpi_1, \varpi_2) \tilde{R}(\varpi_1) \tilde{R}(\varpi_2) \tilde{G}^{(n)}_{i_1\ldots i_n}(\{p_\ell\}). \quad (B.6)$$

where $\tilde{R}$ is defined in equation (A.17). It follows that this contribution also vanishes when integrated over the external frequencies. However, the crossed contribution is not controlled by these Ward identities, and we have not been able to further constrain this last remaining term.

### B.3. Uncrossed derivatives contribution in the flow of the two-point function

In this section, we show that the uncrossed contribution to the flow of $G^{(2)}_{\psi \psi}(p_1, p_2)$ can be closed exactly. The detailed derivation is reported in [59]. The first step consists in showing
that the $q$-derivatives acting on the diagram in square bracket can be expressed as the operator $\tilde{R}$ acting on the external legs of the original diagram as follows

\[
\frac{\partial}{\partial q_1} \int_{k_{1},k_{2}} \tilde{G}^{(2)}_{\nu\nu}(p_{1},p_{2}) = \frac{1}{2} \int_{\nu_{1},\nu_{2}} \tilde{K}_{\nu}(\nu_{1},\nu_{2}) \int_{\nu_{1},\nu_{2}} \frac{\partial^4}{\partial q\partial q_{1}^2 \partial q_{2}^2 \partial q_{3}^2} \frac{\delta}{\delta q} \left[ - \tilde{G}^{(4)}_{\nu\nu}(q_{1},q_{2},k_{1},k_{2}) \right] 
\]

\[
+ \int_{k_{1},k_{2}} \tilde{R}_{\nu\nu}(q_{1},k_{1},k_{3}) \tilde{G}^{(2)}_{\nu\nu}(p_{1},-k_{1}) \tilde{G}^{(2)}_{\nu\nu}(p_{2},-k_{2}) \left[ - \tilde{R}(\nu_{1}) \tilde{R}(\nu_{2}) \tilde{G}^{(2)}_{\nu\nu}(k_{1},k_{2}) \right] 
\]

\[
\frac{\partial}{\partial q_{1}} \int_{\nu_{1},\nu_{2}} \tilde{G}^{(2)}_{\nu\nu}(p_{1},-k_{1}) \tilde{G}^{(2)}_{\nu\nu}(p_{2},-k_{2}) \left[ - \tilde{R}(\nu_{1}) \tilde{R}(\nu_{2}) \tilde{G}^{(2)}_{\nu\nu}(k_{1},k_{2}) + (\nu_{1} \leftrightarrow \nu_{2}) \right].
\]

(B.7)

Since $\tilde{G}^{(2)}$ and $\Gamma^{(2)}$ are inverse of each other, the second term of (B.7) can be rewritten as two combinations of $\tilde{G}^{(2)} \tilde{R}(\nu_{1}) \tilde{G}^{(2)}$ attached by a $\tilde{\Gamma}^{(2)}$. For each of them, one has

\[
\int_{k_{1},k_{2}} \tilde{G}^{(2)}_{\nu\nu}(p_{1},-k_{1}) \tilde{R}(\nu_{1}) \tilde{\Gamma}^{(2)}_{\nu\nu}(k_{1},k_{3}) \tilde{G}^{(2)}_{\nu\nu}(-k_{3},-k_{5})
\]

\[
= \frac{2i
u\alpha\beta}{\nu_{1}} \int_{k_{1},k_{3}} \tilde{G}^{(2)}_{\nu\nu}(p_{1},-k_{1}) \left[ k_{1}^{\alpha} \frac{\partial}{\partial k_{1}^{\beta}} \tilde{G}^{(2)}_{\nu\nu}(\nu_{1} + \nu_{2},k_{1},k_{3}) 
\right] + k_{3}^{\alpha} \frac{\partial}{\partial k_{3}^{\beta}} \tilde{G}^{(2)}_{\nu\nu}(k_{1},\nu_{3} + \nu_{1},k_{3}) \right] \tilde{G}^{(2)}_{\nu\nu}(-k_{3},-k_{5})
\]

\[
= -k_{5}^{\alpha} \frac{\partial}{\partial k_{5}^{\beta}} \tilde{G}^{(2)}_{\nu\nu}(p_{1},-\nu_{5} + \nu_{1},-k_{5}) - \frac{\partial}{\partial \nu_{1}} \tilde{G}^{(2)}_{\nu\nu}(\omega_{1} + \omega_{2},\bar{p}_{1},-\bar{k}_{5})
\]

\[
= -\tilde{R}(\nu_{1}) \tilde{G}^{(2)}_{\nu\nu}(p_{1},-k_{5}).
\]

(B.8)

We have thereby established that $\tilde{R}$ possesses the property:

\[
\frac{\partial^2}{\partial q\partial q^2} \left. \frac{\delta}{\delta q} \tilde{G}^{(2)}_{\nu\nu}(p_{1},p_{2}) \right|_{q=0} = \tilde{R}(\nu) \tilde{G}^{(2)}_{\nu\nu}(p_{1},p_{2}),
\]

(B.9)

which is analogous to the one satisfied by $\tilde{D}$ [22]. Inserting this result in the last line of (B.7) yields

\[
\frac{\partial}{\partial q_{1}} \int_{\nu_{1},\nu_{2}} \tilde{G}^{(2)}_{\nu\nu}(p_{1},p_{2}) \left|_{\nu_{1},\nu_{2}} \right. \int_{k_{1},k_{2}} \tilde{K}_{\nu\nu}(\nu_{1},\nu_{2}) \int_{k_{1},k_{2}} \left[ - \tilde{G}^{(2)}_{\nu\nu}(p_{1},-k_{1}) \tilde{G}^{(2)}_{\nu\nu}(p_{2},-k_{2}) \tilde{R}(\nu_{1}) \tilde{R}(\nu_{2}) \tilde{G}^{(2)}_{\nu\nu}(k_{1},k_{2}) 
\right] 
\]

\[
+ \tilde{R}(\nu_{1}) \tilde{G}^{(2)}_{\nu\nu}(p_{1},-k_{1}) \tilde{G}^{(2)}_{\nu\nu}(p_{2},-k_{2}) \tilde{R}(\nu_{2}) \tilde{G}^{(2)}_{\nu\nu}(k_{1},k_{2}) + (\nu_{1} \leftrightarrow \nu_{2}).
\]

(B.10)

In fact, this structure is the same as the one appearing with the operator $\tilde{D}$ for the leading order at unequal times [22]. It generalises as well for any correlation functions, as shown in the next section. Let us examine separately the two terms in square brackets. The first term reads
Inserting the expressions (B.11) and (B.12) into (B.10) finally leads to the expected result where we used integration by parts. For the second term, one has

\[
\int_{\mathbf{k}_1, \mathbf{k}_2} \mathcal{R}(\mathbf{w}_1) G_{\psi \psi}^{(2)}(\mathbf{p}_1, -\mathbf{k}_1) G_{\psi \psi}^{(2)}(\mathbf{p}_2, \mathbf{k}_2) \mathcal{R}(\mathbf{w}_2) G_{\psi \psi}^{(2)}(\mathbf{p}_2, -\mathbf{k}_2) = -4\epsilon_{\mathbf{v}_1 \mathbf{v}_2 \mathbf{p}_1 \mathbf{p}_2} \int_{\mathbf{k}_1, \mathbf{k}_2} \frac{\partial}{\partial \mathbf{k}_1} \left[ k_1^\nu \frac{\partial}{\partial \mathbf{p}_1} G_{\psi \psi}^{(2)}(\mathbf{p}_1, -\mathbf{k}_1) \right] \frac{\partial}{\partial \mathbf{p}_2} G_{\psi \psi}^{(2)}(\mathbf{p}_2, -\mathbf{k}_2) 
\]

(B.11)

and

\[
\int_{\mathbf{k}_1, \mathbf{k}_2} \frac{\partial}{\partial \mathbf{k}_2} G_{\psi \psi}^{(2)}(\mathbf{p}_2, -\mathbf{k}_2) \frac{\partial}{\partial \mathbf{p}_2} G_{\psi \psi}^{(2)}(\mathbf{p}_2, -\mathbf{k}_2) = -4\epsilon_{\mathbf{v}_1 \mathbf{v}_2 \mathbf{p}_1 \mathbf{p}_2} \int_{\mathbf{k}_1, \mathbf{k}_2} \frac{\partial}{\partial \mathbf{p}_2} G_{\psi \psi}^{(2)}(\mathbf{p}_1, -\mathbf{k}_1) \frac{\partial}{\partial \mathbf{k}_2} G_{\psi \psi}^{(2)}(\mathbf{p}_2, -\mathbf{k}_2) 
\]

(B.12)

Inserting the expressions (B.11) and (B.12) into (B.10) finally leads to the expected result

\[
\left. \partial_\nu \hat{G}_{\psi \psi}^{(2)}(\mathbf{p}_1, \mathbf{p}_2) \right|_{\text{uncrossed}} = \frac{1}{2} \int_{\mathbf{w}_1, \mathbf{w}_2} K_{\psi \psi}(\mathbf{w}_1, \mathbf{w}_2) 
\]

where we have used integration by parts. For the second term, one has

\[
\int_{\mathbf{k}_1, \mathbf{k}_2} \frac{\partial}{\partial \mathbf{k}_1} \left[ k_1^\nu \frac{\partial}{\partial \mathbf{p}_1} G_{\psi \psi}^{(2)}(\mathbf{p}_1, -\mathbf{k}_1) \right] \frac{\partial}{\partial \mathbf{p}_2} G_{\psi \psi}^{(2)}(\mathbf{p}_2, -\mathbf{k}_2) = -4\epsilon_{\mathbf{v}_1 \mathbf{v}_2 \mathbf{p}_1 \mathbf{p}_2} \int_{\mathbf{k}_1, \mathbf{k}_2} \frac{\partial}{\partial \mathbf{p}_1} G_{\psi \psi}^{(2)}(\mathbf{p}_1, -\mathbf{k}_1) \frac{\partial}{\partial \mathbf{k}_2} G_{\psi \psi}^{(2)}(\mathbf{p}_2, -\mathbf{k}_2) 
\]

(B.13)

\[
\frac{\partial^4}{\partial q_{\mathbf{v}_1}^2 \partial q_{\mathbf{v}_2}^2 \partial q_{\mathbf{p}_1}^2 \partial q_{\mathbf{p}_2}^2} \hat{G}_{\psi \psi}^{(n)}(\{\mathbf{p}_1\}, \{\mathbf{p}_2\}) \bigg|_{q_{\mathbf{v}_1} = q_{\mathbf{v}_2} = 0} = \delta_{\mathbf{v}_1 \mathbf{v}_2} \mathcal{R}(\mathbf{w}_1) \mathcal{R}(\mathbf{w}_2) \hat{G}_{\psi \psi}^{(n)}(\{\mathbf{p}_1\}, \{\mathbf{p}_2\}).
\]

(B.14)
The generalized correlation functions $\tilde{G}^{(n)}_{\ell_1,\ldots,\ell_n}(\{\mathbf{p}_i\})$ can be expressed as the sum over all trees whose edges are the propagators $\tilde{G}^{(2)}$, whose vertices are the vertex functions $\tilde{\Gamma}^{(1)}$ and with external legs carrying momenta (i.e. wave-vectors and frequencies) and indices matching the indices of the correlation function: $(\{\ell_1, \mathbf{p}_1\})$. Symbolically,

$$\tilde{G}^{(n)}_{\ell_1,\ldots,\ell_n}(\{\mathbf{p}_i\}) = \sum_{\text{trees}} \alpha_T \tilde{T}^{(n)}_{\ell_1,\ldots,\ell_n}(\{\mathbf{p}_i\}) \quad \text{with} \quad \tilde{T}^{(n)}_{\ell_1,\ldots,\ell_n}(\{\mathbf{p}_i\}) = \int_k \prod_{i=1}^n \mathcal{E}_i^T(\{\mathbf{p}_i\}, \{\mathbf{k}_i\}),$$

(B.15)

where $\alpha_T$ is a combinatorial factor, the $\mathcal{E}_i^T$ are the vertex functions and propagators entering the composition of the tree $T$ and the integration is done over all the internal momenta of the diagram. The $\{\mathbf{p}_i\}$, which are not empty form a partition of the external momenta $\{\mathbf{p}_i\}_{\ell_i \in \ell_n}$ and the internal momenta $\{\mathbf{k}_i\}$, are chosen such that when a propagator is attached to a vertex function, the sum of the momenta of the propagator and of the vertex function at the link is zero. Finally, the internal indices of the theory—here $\ell_i \in \{\psi, \tilde{\psi}\}$—have been omitted on $\mathcal{E}_i^T$ to alleviate notations but follow straightforwardly from the partition of momenta.

Let us first examine the action of only one functional derivative and subsequent two wave-number derivatives applied to a tree $\tilde{T}^{(n)}$ composing $\tilde{G}^{(n)}_{\ell_1,\ldots,\ell_n}(\{\mathbf{p}_i\})$. Using the property (B.9) demonstrated in the previous section, one obtains

$$\frac{\partial^2}{\partial \mathbf{q}_u^2} \delta \tilde{T}^{(n)}_{\ell_1,\ldots,\ell_n}(\{\mathbf{p}_i\})_{\mathbf{p}_i=0} \bigg|_{\mathbf{q}_u=0} = \delta_{u,0} \int_k \prod_{i=1}^n \mathcal{E}_i^T(\{\mathbf{p}_i\}, \{\mathbf{k}_i\}) \tilde{R}(\omega_u) \mathcal{E}_u^T(\{\mathbf{k}_i\}_{1\ell_i<n}).$$

(B.16)

Thus, one only needs to show that the operator $\tilde{R}(\omega_u)$ verifies the Leibniz rule.

$$\tilde{R}(\omega_u) \int_k \mathcal{E}_1^T(\{\mathbf{k}_i\}_{1\ell_i<n}, -\mathbf{k}) \mathcal{E}_2^T(\mathbf{k}, \{\mathbf{k}_i\}_{\ell_i=n}) = \int_k \left[ \tilde{R}(\omega_u) \mathcal{E}_1^T(\{\mathbf{k}_i\}_{1\ell_i<n}, -\mathbf{k}) \mathcal{E}_2^T(\mathbf{k}, \{\mathbf{k}_i\}_{\ell_i=n}) 
+ \mathcal{E}_1^T(\{\mathbf{k}_i\}_{1\ell_i<n}, -\mathbf{k}) \tilde{R}(\omega_u) \mathcal{E}_2^T(\mathbf{k}, \{\mathbf{k}_i\}_{\ell_i=n}) \right].$$

(B.17)

This can be checked by inspection of the rhs, which reads

$$\int_k \left[ \tilde{R}(\omega_u) \mathcal{E}_1^T(\{\mathbf{k}_i\}_{1\ell_i<n}, -\mathbf{k}) \mathcal{E}_2^T(\mathbf{k}, \{\mathbf{k}_i\}_{\ell_i=n}) + \mathcal{E}_1^T(\{\mathbf{k}_i\}_{1\ell_i<n}, -\mathbf{k}) \tilde{R}(\omega_u) \mathcal{E}_2^T(\mathbf{k}, \{\mathbf{k}_i\}_{\ell_i=n}) 
+ \mathcal{E}_1^T(\{\mathbf{k}_i\}_{1\ell_i<n}, -\mathbf{k}) \mathcal{E}_2^T(\mathbf{k}, \{\mathbf{k}_i\}_{\ell_i=n}) + \delta_{u,0} \tilde{T}^{(n)}_{\ell_1,\ldots,\ell_n}(\{\mathbf{p}_i\}) \right].$$

(B.18)

Integrating by part in $\tilde{k}$ and shifting the associated frequency, the two last terms cancel out, proving (B.17). One concludes that

$$\frac{\partial^2}{\partial \mathbf{q}_u^2} \delta \tilde{T}^{(n)}_{\ell_1,\ldots,\ell_n}(\{\mathbf{p}_i\})_{\mathbf{p}_i=0} \bigg|_{\mathbf{q}_u=0} = \tilde{R}(\omega_u) \tilde{T}^{(n)}_{\ell_1,\ldots,\ell_n}(\{\mathbf{p}_i\}).$$

(B.19)

The remaining task to prove (B.14) is to show that the same property holds for two functional derivatives and their subsequent wave-number derivatives. As for the leading order at unequal time [22], distributing the two $\tilde{q}_2$ derivatives and setting $\tilde{q}_2$ to zero, one then applies
the property (B.17) to show that the resulting $\mathcal{R}(\varpi_2)$ can be pulled out of the remaining diagram
\[
\frac{\partial^4}{\partial q_1^4} \frac{\partial^2}{\partial q_1^2 \partial q_2^2} \frac{\partial^2}{\partial \varphi(q_1) \partial \varphi(q_2)} \mathcal{R}(\varpi_2) \bigg|_{\varpi_2=0} = \delta_{\varpi_2}(\varpi_2) \bigg|_{\varpi_2=0}.
\]
Expressing $\mathcal{R}(\varpi_2)$ explicitly and distributing the $\partial_2$ derivative gives
\[
\frac{\partial^4}{\partial q_1^4} \frac{\partial^2}{\partial q_1^2 \partial q_2^2} \frac{\partial^2}{\partial \varphi(q_1) \partial \varphi(q_2)} \mathcal{R}(\varpi_2) \bigg|_{\varpi_2=0} = \delta_{\varpi_2}(\varpi_2) \bigg|_{\varpi_2=0} + \sum_{k=1}^{n} \rho_k \frac{\partial}{\partial q_k} \left[ \int_{\text{int}} \sum_{l=1}^{m} \left( \prod_{k \neq l} \mathcal{E}^{T}_{\varpi_2} \left( \left( \{ p_{l} \}_{l}, \{ k_{l} \}_{l} \right) \right) \right) \mathcal{R}(\varpi_2) \bigg|_{\varpi_2=0} \right]_{\varpi_2=0} + \sum_{k=1}^{n} \rho_k \frac{\partial}{\partial q_k} \left[ \int_{\text{int}} \sum_{l=1}^{m} \left( \prod_{k \neq l} \mathcal{E}^{T}_{\varpi_2} \left( \left( \{ p_{l} \}_{l}, \{ k_{l} \}_{l} \right) \right) \right) \mathcal{R}(\varpi_2) \bigg|_{\varpi_2=0} \right]_{\varpi_2=0} = \delta_{\varpi_2}(\varpi_2) \bigg|_{\varpi_2=0} + \sum_{k=1}^{n} \rho_k \frac{\partial}{\partial q_k} \left[ \int_{\text{int}} \sum_{l=1}^{m} \left( \prod_{k \neq l} \mathcal{E}^{T}_{\varpi_2} \left( \left( \{ p_{l} \}_{l}, \{ k_{l} \}_{l} \right) \right) \right) \mathcal{R}(\varpi_2) \bigg|_{\varpi_2=0} \right]_{\varpi_2=0} = \delta_{\varpi_2}(\varpi_2) \mathcal{R}(\varpi_2) \bigg|_{\varpi_2=0}.
\]
where the first term in the second equality vanishes by antisymmetry of $\epsilon_{\rho\sigma}$. This proves the property (B.14) and (B.6) follows.

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