LIMITING BEHAVIOR OF A CLASS OF HERMITIAN-YANG-MILLS METRICS

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Abstract. We study the limiting behavior of Hermitian-Yang-Mills metrics on a class of rank two slope-stable vector bundles over the product of two elliptic curves with a family of product metrics, which are flat and have areas $\epsilon$ and $\epsilon^{-1}$ on two factors respectively. The method is to construct a family of Hermitian metrics and then compare them with the normalized Hermitian-Yang-Mills metrics. We find that the metrics are close in $C^k$ to arbitrary order in $\epsilon$.

1. Introduction

A Calabi-Yau manifold is a compact Kähler manifold with zero first Chern class and vanishing first Betti number. Yau’s solution [30] to the Calabi Conjecture provides a unique Ricci-flat Kähler metric in each Kähler class of a Calabi-Yau manifold. Motivated by mirror symmetry and the SYZ conjecture [23], Gross and Wilson [11] initiated the study of the limiting behavior of Yau’s Ricci-flat metrics in a large complex structure limit. They considered a general K3 surface that is a hyper-Kähler rotation of an elliptic K3 surface with 24 singular fibers, and showed that its Ricci-flat metrics converge (collapse) to a metric on the base $S^2$ with singularities on the discriminant locus of 24 points. Later, several other papers (cf. [29, 31, 19, 26, 10]) studied the same question.

In this paper, we will study the Hermitian-Yang-Mills (HYM for brevity) version of the above question. Let $V$ be a slope stable holomorphic vector bundle over a compact Kähler manifold $X$ with a Kähler metric (form) $\omega$. By a theorem of Donaldson [2] and Uhlenbeck-Yau [27], $V$ admits an irreducible HYM metric $H$, which is unique up to multiplicative constant. Suppose $X$ is a Calabi-Yau manifold with a family of Kähler metrics $\omega_\epsilon$ approaching a large Kähler metric limit, and suppose $V$ is slope stable with each $\omega_\epsilon$, then we obtain a family of HYM metrics $H_\epsilon$.

Question. What is the limiting behavior of $H_\epsilon$, after normalization, when $\omega_\epsilon$ goes to a large Kähler metric limit?

The Kähler manifold $X$ we consider here is a product $T \times B$ of two copies of the complex one-torus $\mathbb{C}/\Gamma$, where $\Gamma = \mathbb{Z} + i\mathbb{Z}$. In this case, a family of product metrics $\omega_\epsilon$, which are flat and have areas $\epsilon$ and $\epsilon^{-1}$ on $T$ and $B$ respectively, approaches a large Kähler metric limit when $\epsilon \to 0$ (cf. [16]).

The holomorphic vector bundle $V$ over $X$ is constructed as follows (cf. [3, 4]). Let $T^*$ be the dual of $T$ and let $X^* = T^* \times B$. The product $X^* \times_B X$ is a smooth complex threefold. Let $Y$ be a compact (complex) curve of $X^*$ so that the induced projection $\phi : Y \to B$ is a two-sheet branched cover. Let

$$
\iota : Y \times_B X \longrightarrow X^* \times_B X, \quad p_1 : Y \times_B X \longrightarrow Y, \quad p_2 : Y \times_B X \longrightarrow X
$$
be the canonical inclusion and projections. Let $P$ be the Poincaré line bundle on $X^* \times_B X$. Then for any degree zero line bundle $F$ over $Y$, we can form a line bundle $N = K^1_Y \otimes \varphi^* K_B^{-1/2} \otimes F$ over $Y$ and a rank two vector bundle over $X$ 

$$V = p_{2*}(\iota^* P \otimes p_1^* N).$$

Its first Chern class vanishes. Moreover, by an adiabatic argument (cf. [1]), $V$ is $\omega_\varepsilon$-slope stable for small $\varepsilon$. Therefore there exists a family of irreducible HYM metrics $H_{1,\varepsilon}$ on $V$ with respect to $\omega_\varepsilon$. Because $c_1(V) = 0$, the associate curvatures $\Theta(H_{1,\varepsilon})$ satisfy

\begin{equation}
\Lambda \Theta(H_{1,\varepsilon}) \equiv \frac{i \Theta(H_{1,\varepsilon}) \wedge \omega_\varepsilon}{\omega_\varepsilon^2} = 0.
\end{equation}

The purpose of this paper is to investigate the above question for $H_{1,\varepsilon}$ when $\varepsilon \to 0$. We construct in section 5 explicitly a family of Hermitian metrics $H_{0,\varepsilon}$ on $V$. Equation (1.1) leads us first to establish

**Proposition 1.** For any positive integer $l$, there is a constant $C = C(l)$ depending only on $l$ and an open cover of $X$ such that for sufficiently small $\varepsilon > 0$, the associate curvatures $\Theta(H_{0,\varepsilon})$ of $H_{0,\varepsilon}$ satisfy

$$\| \Lambda \Theta(H_{0,\varepsilon}) \|_{C^0} \leq C^l.$$

We then normalize $H_{1,\varepsilon}$ with respect to $H_{0,\varepsilon}$ and compare them. The main result of this paper is

**Theorem 2.** For any non-negative integer $k$ and positive integer $l$, there is a constant $C = C(k,l)$ depending on $k$, $l$ and an open cover of $X$ such that for sufficiently small $\varepsilon > 0$,

$$\| (H_{0,\varepsilon})^{-1} H_{1,\varepsilon} - \text{Id} \|_{C^k} \leq C^l.$$

Here $\Lambda \Theta(H_{0,\varepsilon})$ and $(H_{0,\varepsilon})^{-1} H_{1,\varepsilon}$ are in $\text{End}(V)$ (cf. [2], p.4), where there is no natural $C^k$-norm. We use $H_{0,\varepsilon}$ to define a $C^k$-norm. That is, for a local trivialization of $V$, we choose a unitary frame field relative to $H_{0,\varepsilon}$ and define a $C^k$-norm on $\text{End}(V)$ to be the $C^k$-norm of the resulting matrix representations. The $C^k$-norm of a function is defined as in [3] p.53 which does not depend on $\varepsilon$. Hence, the metrics $H_{1,\varepsilon}$ and $H_{0,\varepsilon}$ are close in $C^k$ to arbitrary order in $\varepsilon$.

We will prove the above results in the last two sections (see Proposition 10 and Theorem 15). The key step to construct $H_{0,\varepsilon}$ is to construct a family of HYM metrics on $V$ over the product of a neighborhood of a branched point in $B$ and the fiber $T$. In section 3, we construct such metrics (3.8) and hence derive a PDE (4.1) depending on $\varepsilon$. This equation has a unique smooth solution and also a singular solution $\frac{1}{2} \ln r$. Moreover, according to Gidas-Ni-Nirenberg’s theorem in [2], it can be reduced to an ODE (4.2) on the interval $[0, 2r_0]$, which is a singular perturbed equation, the small parameter is $\varepsilon$. We estimate $C^k$-norm of the difference between the smooth solution and the singular solution on the interval $[r_0, 2r_0]$ in section 4. We find that they are close in $C^k$ to arbitrary order in $\varepsilon$. In section 5, we first use the Green function of a degree zero divisor on $B$ to construct a HYM metric on $V$, which is singular on $V$ over the fiber of every branched point. However, this singular metric is essentially the same as the metrics (3.8) when the PDE (4.1) takes the singular solution. Hence, we can glue this metric to the local smooth HYM metrics (3.8). The resulting metrics can be normalized conformally to obtain
a family of Hermitian metrics $H_{0, \epsilon}$ so that $\text{Tr} \Lambda \Theta(H_{0, \epsilon}) = 0$. This guarantees that $\det((H_{0, \epsilon})^{-1} H_{1, \epsilon})$ is a constant. Hence, $H_{1, \epsilon}$ can be normalized so that this constant is 1.

We believe that our method can be applied to the case of the elliptic $K3$ surface over $\mathbb{P}^1$ with a section if we can know its large Kähler metric limit more. Thus, we believe that it may have many important applications to mirror symmetry (cf. [13, 14, 17, 14, 24, 5, 6, 29]).

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2. A localization of $V$

In this section, we shall set up the notations that will be followed in this paper. Let $\Gamma = \mathbb{Z} + i\mathbb{Z}$ and $\Gamma^*$ be the dual of $\Gamma$. Let $T$ and $B$ be two copies of the complex one-torus $\mathbb{C}/\Gamma$ and let $X = T \times B$. Let $T^* = \mathbb{C}^*/\Gamma^*$ be the dual of $T$ and let $X^* = T^* \times B$. Set $z = x_1 + ix_2$, $w = y_1 + iy_2$ and $w^* = y_1^* + iy_2^*$ to be the complex coordinates of $B$, $T$ and $T^*$ respectively. We endow $X$ with a family of Kähler metrics depending on $\epsilon$:

$$\omega_{\epsilon} = \epsilon dy_1 \wedge dy_2 + \epsilon^{-1} dx_1 \wedge dx_2.$$  

By viewing $\Gamma$ as the transformation group of $\mathbb{C}$ and viewing $\Gamma^*$ as the transformation group of $\mathbb{C}^*$, $\mathbb{C}^* \times \mathbb{C}$ becomes the universal cover of $T^* \times T$ with the deck transformation group $\Gamma^* \times \Gamma$:

$$g_{(\lambda^*, \lambda)}(w^*, w) = (w^* + \lambda^*, w + \lambda).$$

Hence

$$\mathbb{C}^* \times \mathbb{C} \times B)/\Gamma^* \times \Gamma = X^* \times_B X.$$  

We recall the construction of the Poincaré line bundle. Start with the trivial line bundle $\mathcal{P}$ over $\mathbb{C}^* \times T$ with the standard flat connection along $\mathbb{C}^*$, and with connection which connection form along $T$ at $\{w^*\} \times T$ is

$$\theta = -\pi i (w^* d\overline{w} + \overline{w^*} dw).$$

Then we can lift the $\Gamma^*$ action on $\mathbb{C}^*$ to $\mathcal{P}$, if we denote by $\varepsilon_{(w^*, w)}$ its constant one global section,

$$g_{\lambda^*, \varepsilon_{(w^* + \lambda^*, w)}} = \exp(-\pi i (\lambda^* \overline{w} + \overline{\lambda^*} w) \varepsilon_{(w^*, w)}).$$

Thus, $\mathcal{P}$ can be reduced to a line bundle $\mathcal{P}$ over $T^* \times T$, which is called the Poincaré line bundle. Moreover, the curvature associated to $\theta$ is

$$\Theta = -\pi i (dw^* \wedge d\overline{w} + d\overline{w^*} \wedge dw).$$
which is a \((1, 1)\)-form. This makes \(\mathcal{P}\) a holomorphic line bundle with holomorphic frame

\[(2.4) \quad \tilde{\epsilon}_{(w^*, w)} = \exp(\pi iw^* \bar{w}) \tilde{\epsilon}_{(w^*, w)};\]

it transforms under \(\Gamma^* \times \Gamma\) via

\[
\begin{align*}
\tilde{\epsilon}^{(0, \lambda)}_{(w^* + \lambda^*, w)} &= \exp(\pi i w^* \lambda^*) \tilde{\epsilon}_{(w^*, w)}, \\
\tilde{\epsilon}^{(\lambda^*, 0)}_{(w^*, w^* + \lambda^*)} &= \exp(-\pi i \lambda^* w) \tilde{\epsilon}_{(w^*, w)}.
\end{align*}
\]

\(\mathcal{P}\) can be viewed as a line bundle over \(X^* \times B\) by pulling back. By (2.3), we have

\[(2.5) \quad c_1(\mathcal{P}) = \Theta - \frac{1}{2}(dw^* \wedge d\bar{w} + d\bar{w}^* \wedge dw).
\]

As in section 1, we take a (complex) curve \(Y\) in \(X^*\) such that the restricted map of \(Y\) to \(B\) is a 2 : 1 branched cover. Such a curve can be constructed as follows. Pick a curve \(Y_0 \subset \mathbb{P}^1 \times \mathbb{P}^1\) of class \([0 \times \mathbb{P}^1] + d[\mathbb{P}^1 \times 0]\).

This can be constructed as the graph of a degree \(d\) polynomial \(C : \mathbb{C} \to \mathbb{C}\). Fix a 2 : 1 branched cover \(T^* \to \mathbb{P}^1\) and \(B \to \mathbb{P}^1\). We define a curve \(Y \subset X^*\) be the pre-image of \(Y_0\) under the natural map \(X^* \to \mathbb{P}^1 \times \mathbb{P}^1\). Then \(\varphi : Y \to B\) is a 2 : 1 branched cover and the degree of \(\varphi : Y \to T^*\) is \(2d\).

Then as in section 1, we can use \(Y\) and \(\mathcal{P}\) to construct the rank two vector bundle \(V\) over \(X\). From section 7 of [4], we can see that

\[(2.6) \quad c_1(V) = p_{2*}(\iota^* c_1(\mathcal{P})),
\]

\[(2.7) \quad c_2(V) = \frac{1}{2} (p_{2*}(\iota^* c_1(\mathcal{P})))^2 - \frac{1}{2} p_{2*}(\iota^* c_1(\mathcal{P})^2),
\]

and \(c_1(V) = 0\). (Equation (2.7) can also be derived by curvatures in section 5.) Moreover, by (2.6), we have

\[\int_X c_2(V) = \left(\frac{i}{2}\right)^2 \int_X p_{2*}(dw^* \wedge d\bar{w}^* \wedge dw \wedge d\bar{w})\]

\[= \frac{i}{2} \int_Y q^*(dw^* \wedge d\bar{w}^*)\]

\[= \frac{i}{2} \int_{[q(Y)]} dw^* \wedge d\bar{w}^*\]

\[= \deg(q).
\]

Next we should simplify \(V\). Let

\[D_0 = \sum_{a=1}^{n} \xi_a
\]

be the branched locus on \(B\). By the Riemann-Hurwitz formula, the genus \(g(Y)\) of \(Y\) is bigger than 1 and \(n = 2(g(Y) - 1)\). Since the degree of \(K_Y\) is \(2(g(Y) - 1)\) and the degree of \(K_B\) is 0,

\[\deg(K_Y^{1/2} \otimes \varphi^* K_B^{-1/2}) = g(Y) - 1 = \frac{n}{2}.
\]
that is disjoint from the branched locus \( D \).
Therefore, \( N \) in the first section can be taken as \( \varphi^*(\mathcal{O}_B(D)) \otimes \mathcal{F}' \) for a degree zero line bundle \( \mathcal{F}' \). Without loss of generality, we assume that \( \mathcal{F}' \) is trivial. Otherwise, we can tensor a flat metric on \( \mathcal{F}' \) with the constructed Hermitian metrics on \( V \) in section 5. Thus,

\[
V = p_{2*}(\mathcal{L}) \quad \text{for} \quad \mathcal{L} = \iota^*\mathcal{P} \otimes (\varphi \circ p_1)^*(\mathcal{O}_B(D)).
\]

For our purpose, we will give a local trivialization of this vector bundle. We denote by \( d_B \) the distance on \( B \) induced from the Euclidean metric on \( \mathbb{C} \). Hence, \( d_B \) does not depend on \( \epsilon \). We pick a small \( r_0 > 0 \) so that the discs

\[
U_\alpha = \{ z \in B \mid d_B(z, \xi_\alpha) < 2r_0 \} \subset B, \quad \alpha = 1, \ldots, 5n/4,
\]

are disjoint. For such an \( \alpha \), we pick an analytic chart \( z_\alpha \) of \( U_\alpha \) so that \( z_\alpha(\xi_\alpha) = 0 \).

In the following, for convenience, we will denote \( \alpha = 0, 1, \ldots, 5n/4; \ a = 1, \ldots, n; \) and \( j = n + 1, \ldots, 5n/4 \).

We first localize \( \mathcal{L} \). Denote \( U_0^n = B \setminus D \). Then \( \{ U_0^n, U_{n+1}, \ldots, U_{5n/4} \} \) is an open cover of \( B \). We can give a local holomorphic frame \( e_0 \) of \( \mathcal{O}_B(D) \mid_{U_0^n} \) and \( e_j \) of \( \mathcal{O}_B(D) \mid_{U_j} \) such that over \( U_j \cap U_0^n \),

\[
e_j = z_j^{-1}e_0.
\]

Denote by \( \Omega_j \) the pre-image of \( U_j \) and by \( \Omega_0^n \) of \( U_0^n \) in \( Y \times_B X \) under the map \( \varphi \circ p_1 \).

Then

\[
\tilde{\nu}_j(w^*(z), w, z_j) = \tilde{\varepsilon}(w^*(z), w) \otimes e_j(z_j)
\]

forms a local holomorphic frame of \( \mathcal{L} \mid_{\Omega_j} \) and

\[
\tilde{\nu}_0(w^*(z), w, z_j) = \tilde{\varepsilon}(w^*(z), w) \otimes e_0(z_j)
\]

of \( \mathcal{L} \mid_{\Omega_0^n} \). They satisfy, over \( \Omega_j \cap \Omega_0^n \),

\[
\tilde{\nu}_j(w^*(z), w, z_j) = z_j^{-1}\tilde{\nu}_0(w^*(z), w, z_j).
\]

They also transform under \( \Gamma \) via the first formula in (2.5).

Now we can localize \( V \) over \( X \). We take \( U_0 = B \setminus (D_0 \cup D_1) \). Then \( U_0, U_a, U_j \) also form an open cover of \( B \) and their pre-images \( \bar{U}_0, \bar{U}_a, \bar{U}_j \) in \( X \) form an open cover of \( X \). We have a local holomorphic frame of \( V \) over \( \bar{U}_0 \)

\[
\tilde{\mu}_1^0(w, z) = p_{2*}\tilde{\nu}_0(w_1^*(z), w, z), \quad \tilde{\mu}_2^0(w, z) = p_{2*}\tilde{\nu}_0(w_2^*(z), w, z).
\]

Here \( w_1^*(z) \) and \( w_2^*(z) \) are the two local sections of \( \varphi : Y \to B \) when restricted to \( U_0 \). We caution that the two sections \( w_1^*(z) \) and \( w_2^*(z) \) only exist locally. But this will not confuse us to construct the Hermitian metrics in section 5. On the other hand, under our assumption, we can assume that \( w_1^*(z_j) \) and \( w_2^*(z_j) \) are well-defined on \( U_j \). Hence we have a local holomorphic frame of \( V \) over \( \bar{U}_j \)

\[
\tilde{\mu}_1^j(w, z_j) = p_{2*}\tilde{\nu}_j(w_1^*(z_j), w, z_j), \quad \tilde{\mu}_2^j(w, z_j) = p_{2*}\tilde{\nu}_j(w_2^*(z_j), w, z_j).
\]
Combining (2.9) and (2.10) with (2.8) gives the relations over \( U_0 \cap U_j \)

\[
\hat{\mu}_i = z_{ij}^{-1} \hat{\mu}_i, \quad \hat{\mu}_2 = z_{ij}^{-1} \hat{\mu}_2.
\]

We next look at \( U_\alpha \). Since \( \varphi : Y \to B \) is the two-to-one branched cover ramified at \( \xi_a \), we choose \( w_a^* \) so that over \( U_\alpha \) the curve \( Y \subset X^* \) is given by \( (w_a^*)_2 = z_a \). Hence the direction image sheaf \( \varphi_* \mathcal{O}_Y|_{U_\alpha} \) is a free \( \mathcal{O}_{U_\alpha} \)-module generated by 1 and \( w_a^* \). For \( V|_{U_\alpha} \), following (2.9) we can pick \( w_1^*(z_a) = \sqrt{z_a} \) and \( w_2^*(z_a) = -\sqrt{z_a} \), and set

\[
\hat{\mu}_1^a = \frac{1}{\sqrt{2}}(\hat{\mu}_1^0 + \hat{\mu}_2^0), \quad \hat{\mu}_2^a = \frac{\sqrt{z_a}}{\sqrt{2}}(\hat{\mu}_1^0 - \hat{\mu}_2^0).
\]

The sections \( \hat{\mu}_1^a \) and \( \hat{\mu}_2^a \) are well-defined holomorphic sections of \( V|_{U_\alpha} \) independent of the choice of single-valued branch of \( \sqrt{z_a} \); also the two sections \( \hat{\mu}_1^a \) and \( \hat{\mu}_2^a \) generate the holomorphic bundle \( V|_{U_\alpha} \). Thus we can and shall set them to be a holomorphic frame of \( V|_{U_\alpha} \). In other words, (2.12) gives the transition functions over \( U_0 \cap U_\alpha \) between the frames \( (\hat{\mu}_1^a, \hat{\mu}_2^a) \) and \( (\hat{\mu}_1^0, \hat{\mu}_2^0) \).

Similarly, we can also use \( \varepsilon_{(w_a^*, w_\alpha^*)} \) to define a smooth frame \( (\hat{\mu}_1^a, \hat{\mu}_2^a) \) of \( V|_{U_\alpha} \). They also satisfy the relations

\[
\hat{\mu}_1^a = \frac{1}{\sqrt{2}}(\hat{\mu}_1^0 + \hat{\mu}_2^0), \quad \hat{\mu}_2^a = \frac{\sqrt{z_a}}{\sqrt{2}}(\hat{\mu}_1^0 - \hat{\mu}_2^0), \quad \text{over } U_j \cap U_\alpha;
\]

\[
\hat{\mu}_1^a = \frac{1}{\sqrt{2}}(\hat{\mu}_1^0 + \hat{\mu}_2^0), \quad \hat{\mu}_2^a = \frac{\sqrt{z_a}}{\sqrt{2}}(\hat{\mu}_1^0 - \hat{\mu}_2^0), \quad \text{over } U_\alpha \cap U_0.
\]

Finally, by (2.11), the local holomorphic frames are related to the smooth frames:

\[
(\hat{\mu}_1^a, \hat{\mu}_2^a) = (\hat{\mu}_1^0, \hat{\mu}_2^0)A_\alpha,
\]

where

\[
A_\alpha = \begin{cases} 
\begin{pmatrix} \exp(\pi iw_1^*(z)) & 0 \\ 0 & \exp(\pi iw_2^*(z)) \end{pmatrix}, & \alpha = 0, j; \\
\begin{pmatrix} \cosh(\pi i \sqrt{z_a}) & \sqrt{z} \sinh(\pi i \sqrt{z_a}) \\ \sqrt{z} \sinh(\pi i \sqrt{z_a}) & \cosh(\pi i \sqrt{z_a}) \end{pmatrix}, & \alpha = a.
\end{cases}
\]

3. The system of HYM connections

In this section we first recall some definitions and notations on connections in Hermitian vector bundles as in Chapter 1 of [12]. (Hence our notations here differs from [12].) Let \( E \) be a rank \( r \) complex vector bundle over a Kähler manifold \( (M, \omega) \). Let \( D \) be a connection in \( E \). Let \( s_U = (s_1, \cdots, s_r) \) be a local frame of \( E \) over an open set \( U \subset M \). Then we can write

\[
Ds_i = \sum s_j \theta_i^j.
\]

The matrix 1-form \( \theta_U = (\theta_i^j) \) is called the connection form of \( D \) with respect to \( s_U \).

The curvature form \( \Theta_U \) of \( D \) relative to \( s_U \) is defined by

\[
\Theta_U = d\theta_U + \theta_U \wedge \theta_U.
\]

If \( s'_U = (s'_1, \cdots, s'_r) \) is another local frame over \( U \), which is related to \( s_U \) by

\[
s_U = s'_U A_U,
\]

then

\[
\Theta_U = \Theta_U + (A_U)^t \Theta_U A_U.
\]
where $A_U : U \to GL(r, \mathbb{C})$ is a matrix-valued function on $U$. Let $\theta'_U = (\theta'_{ij})$ and $\Theta'_U$ be the connection and curvature form of $D$ relative to $s'_U$. Then

$$\theta_U = A_U^{-1} \theta'_U A_U + A_U^{-1} dA_U,$$

and

$$\Theta_U = A_U^{-1} \Theta'_U A_U.$$

Let $H$ be a Hermitian metric on $E$. We set

$$h_{ij} = H(s_i, s_j)$$

and $H_U = (h_{ij})$. $H_U$ is a positive definite Hermitian matrix at every point of $U$. Under a change of frame given by (3.1), we have

$$H_U = (A_U)^t H_U A_U.$$

Here $(A_U)^t$ is denoted as the transpose of $A_U$.

Now if $E$ is a holomorphic vector bundle and $H$ is a Hermitian metric on $E$, then there exists a canonical connection $D_H$, which is called the Hermitian connection, defined as follows. Let $\tilde{s}_U = (\tilde{s}_1, \cdots, \tilde{s}_r)$ be a local holomorphic frame on $U$ and $\tilde{H}_U$ be the Hermitian matrix for $H$ with respect to $\tilde{s}_U$. Then the Hermitian connection with respect to $\tilde{s}_U$ is determined by

$$(\tilde{\theta}_U)^t = \partial \tilde{H}_U \tilde{H}_U^{-1}$$

and its curvature form is

$$\tilde{\Theta}^t = \partial (\partial \tilde{H}_U \tilde{H}_U^{-1}),$$

which is a matrix $(1, 1)$-form. Hence according to (3.3), the curvature form $\Theta'$ of the Hermitian connection with respect to any frame $s'_U$ is also a matrix $(1, 1)$-form.

We define

$$\Lambda \Theta' = \frac{m \cdot \Theta' \wedge \omega^{m-1}}{\omega^m},$$

where $m = \dim \mathcal{C} M$. If $c_1(V) = 0$, then $H$ is called a HYM metric if and only if

$$\Lambda \Theta' = 0.$$

In the following we shall derive the system of HYM connections of $V$ over $U_a$ for $1 \leq a \leq n$. Since $V|_{U_a}$ are essentially the same, we shall workout one of them in detail. For convenience, we shall drop the super(sub)-script $a$.

We endow $V|_{U}$ with a class of metrics. Let $u_x : U \to \mathbb{R}$ be a real function and set

$$(\hat{h}_x)^t = \left( \begin{array}{cc} e^{-u_x} & 0 \\ 0 & e^{u_x} \end{array} \right).$$

Since $u_x$ does not depend on the variable $w$, $\hat{h}_x$ gives a Hermitian metric $h_x$ so that it is the Hermitian matrix for $h_x$ in $(\hat{\mu}_1, \hat{\mu}_2)$. Thus, by (3.3) and (2.14),

$$\hat{h}_x = A^t \hat{h}_x A$$

gives the Hermitian matrix for $h_x$ in $(\hat{\mu}_1, \hat{\mu}_2)$, which depends on $w$. Hence the Hermitian connection also depends on $w$ (see below).

We let $D_{h_x}$ be the Hermitian connection on $(V|_{U}, h_x)$; let $\hat{\theta}_x$ and $\hat{\theta}_x$ be the connection forms of $D_{h_x}$ with respect to $(\hat{\mu}_1, \hat{\mu}_2)$ and $(\hat{\mu}_1, \hat{\mu}_2)$. Then, by (3.3),

$$(\hat{\theta}_x)^t = \partial \hat{h}_x \cdot \hat{h}_x^{-1}$$
and, by (3.2), $\hat{\theta}_e$ is related to $\hat{\theta}_e$ as

\begin{equation}
\hat{\theta}_e = A \hat{\theta}_e A^{-1} - dAA^{-1}.
\end{equation}

Inserting (3.9) into (3.10) and inserting the resulting equation into (3.11), we have

\[
\hat{\theta}_e = -\partial AA^{-1} + (\partial \hat{\theta}_e \cdot \hat{\theta}_e^{-1})^t + (\hat{h}_e \partial AA^{-1} \hat{h}_e^{-1})^t
\]

\[
= -\pi i \left( \begin{array}{cc} 0 & z \\ 1 & 0 \end{array} \right) d\bar{w} - \pi i \left( \begin{array}{cc} 0 & e^{2u_e} \\ z e^{-2u_e} & 0 \end{array} \right) dw + \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) \frac{\partial u_e}{\partial \bar{z}} dz.
\]

Therefore the associated curvature form is

\[
\hat{\Theta}(h_e) = \pi^2 (|z|^2 e^{-2u_e} - e^{2u_e}) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) d\bar{w} + \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \frac{\partial^2 u_e}{\partial \bar{z} \partial z} dz \land d\bar{z} + \text{other terms},
\]

and thus, by definition (3.11) with $m = 2$ and $\omega = \omega_e$ in (2.1),

\[
\Lambda \hat{\Theta}(h_e) = \left( \epsilon \frac{\partial^2 u_e}{\partial \bar{z} \partial z} + \epsilon^{-1} \pi^2 (|z|^2 e^{-2u_e} - e^{2u_e}) \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right).
\]

Based on this, we see that $h_e$ becomes HYM if $u_e$ satisfies the equation:

\begin{equation}
\frac{\partial^2 u_e}{\partial \bar{z} \partial z} = \pi^2 e^{-2} (\exp(2u_e) - |z|^2 \exp(-2u_e)).
\end{equation}

4. REDUCTION TO ODE

In this section, we shall study the solution to the Dirichlet problem:

\begin{equation}
\begin{cases}
\triangle u = 4\pi^2 e^{-2} (\exp(2u) - r^2 \exp(-2u)) & \text{in } B_{2r_0}(0), \\
\frac{1}{2} \ln(2r_0) & \text{on } \partial B_{2r_0}(0),
\end{cases}
\end{equation}

where $x = (x_1, x_2)$ is the standard coordinate of $B_{2r_0}(0)$, $r^2 = x_1^2 + x_2^2$, and $\triangle = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$.

**Theorem 3.** Equation (4.1) has a unique smooth and radially symmetric solution $u_e$ that satisfies the following estimates:

1. let $v_e(r) = u_e(r) - \frac{1}{2} \ln r$, $r \in [r_0, 2r_0]$, then for any positive integer $l$ and $k$ so that $l > k \geq 0$, there is a constant $C = C(r_0, l, k)$ such that for any $0 < \epsilon < 1/8$,

\[
\| v_e^{(k)} (r) \|_{C^l([r_0, 2r_0])} < C \epsilon^{l-k}.
\]

Here $v_e^{(k)}(r)$ is the $k$-th derivative of $v_e$ in $r$; and

2. for any integer $k \geq 0$ and any $R_k < 2r_0$, there exists a constant $C = C(r_0, R_k)$ such that for any $0 < \epsilon < 1/8$,

\[
\| u_e \|_{C^k(B_{R}(0))} \leq C \epsilon^{-k-2}.
\]

**Proof.** After substituting $\overline{\pi}$ for $2u - \ln(2r_0)$, $x_1$ for $\frac{x_1}{2r_0}$, $x_2$ for $\frac{x_2}{2r_0}$, $r^2$ for $\frac{r^2}{4r_0^2}$, and $\epsilon$ for $\frac{\epsilon}{4\sqrt{2\pi}r_0^{-\frac{3}{2}}}$, equation (4.1) becomes

\[
\begin{cases}
\triangle \pi = \epsilon^{-2} (\exp(-\pi) - r^2 \exp(-\pi)) & \text{in } B_1(0), \\
\pi = 0 & \text{on } \partial B_1(0).
\end{cases}
\]
The theorem then follows from Proposition \[4\] Lemma \[7\] and Proposition \[9\] below.

\[\square\]

**Proposition 4.** Equation (4.2) has a unique smooth and radially symmetric solution \( \pi_\epsilon \) that satisfies \( \frac{\partial}{\partial r} \pi_\epsilon > 0 \) for \( 0 < r < 1 \).

**Proof.** Because for each \( x = (x_1, x_2) \) the function \( \epsilon^{-2} \left( \exp \pi - r^2 \exp(\pi) \right) \) is a monotone increasing function of \( \pi \), according to \[21\] the boundary value problem (4.2) is uniquely solvable.

To prove the second part, we first use the maximum principle to prove that the solution \( \pi_\epsilon \) to (4.2) is negative. Let \( x_0 \in \overline{B}_1(0) \) be such that \( \pi_\epsilon(x_0) = \max_{x \in \overline{B}_1(0)} \pi_\epsilon \). In case \( \pi_\epsilon(x_0) \geq 0 \) and \( x_0 \notin \partial B_0 \), we have \( 2\pi_\epsilon(x_0) - |x_0|^2 > 0 \), and there is a neighborhood \( \Omega \subset B_1(0) \) of \( x_0 \) such that \( 2\pi_\epsilon(x) - |x|^2 > 0 \) in \( \Omega \). Therefore

\[
\Delta \pi_\epsilon = \epsilon^{-2} \left( \exp \pi_\epsilon - r^2 \exp(-\pi_\epsilon) \right) > 0, \quad x \in \Omega.
\]

Applying the strong maximum principle, we know that the maximum of \( \pi_\epsilon \) on \( \Omega \) can be achieved only on \( \partial \Omega \), contradicting to that \( x_0 \) is a local maximum of \( \pi_\epsilon \). This proves that \( \pi_\epsilon < 0 \) in \( B_1(0) \). After this, we can apply Corollary 1 of \[7, \text{p.227}\] to conclude that \( \pi_\epsilon \) is radially symmetric and \( \frac{\partial}{\partial r} \pi_\epsilon > 0 \) for all \( 0 < r < 1 \). \[\square\]

Because of this, we can reduce (4.2) to an ODE:

\[
(4.3) \quad \pi_\epsilon''(r) + r^{-1} \pi_\epsilon'(r) = \epsilon^{-2} \left( \exp \pi_\epsilon(r) - r^2 \exp(-\pi_\epsilon(r)) \right).
\]

Our next goal is to show that the solution \( \pi_\epsilon(r) \) is close to \( \ln r \) for \( r \in [\frac{1}{2}, 1] \) when \( \epsilon \to 0 \). We shall set \( \pi_\epsilon(r) = \pi_\epsilon(r) - \ln r \) and estimate \( \| \pi_\epsilon^{(k)}(r) \|_{C^0([\frac{1}{2}, 1])} \). Clearly, \( \pi_\epsilon(1) = 0 \) and \( \lim_{r \to 0} \pi_\epsilon(r) = +\infty \).

**Lemma 5.** When \( 0 < r < 1 \), \( \pi_\epsilon(r) \) satisfies

\[
\pi_\epsilon'(r) > 0, \quad \pi_\epsilon'(r) < 0, \quad \pi_\epsilon''(r) > 0 \quad \text{and} \quad \pi_\epsilon'''(r) < 0.
\]

**Proof.** By (4.4), \( \pi_\epsilon'(r) \) satisfies

\[
(4.4) \quad \pi_\epsilon'(r) + r^{-1} \pi_\epsilon(r) = 2\epsilon^{-2} r \sinh \pi_\epsilon(r).
\]

We first use the maximum principle to prove \( \pi_\epsilon(r) > 0 \). If it is not, let \( r_0 \) be the first point in \((0,1)\) such that \( \pi_\epsilon(r_0) = \min_{r \in (0,1)} \pi_\epsilon(r) \leq 0 \). Hence \( \pi_\epsilon'(r_0) = 0 \), \( \pi_\epsilon''(r_0) \geq 0 \). Therefore (4.4) implies \( \pi_\epsilon(r_0) = 0 \). Then we can assume that there exists a \( r_1 \in (r_0, 1) \) such that \( \pi_\epsilon(r_1) = \max_{r \in (r_0,1)} \pi_\epsilon(r) > 0 \). Thus, \( \pi_\epsilon'(r_1) = 0 \), \( \pi_\epsilon''(r_1) \leq 0 \). This contradicts to (4.4). Hence \( \pi_\epsilon(r) > 0 \) for all \( 0 < r < 1 \).

Now applying Theorem 3 in \[7\] to equation (4.4) gives \( \pi_\epsilon(r) < 0 \) for \( r \in [\frac{1}{2}, 1] \). We claim that this inequality holds for all \( r \in (0,1) \). Otherwise, there exists a \( r_2 \in (0, \frac{1}{2}) \) such that \( \pi_\epsilon'(r_2) = 0 \) and \( \pi_\epsilon''(r) < 0 \) for any \( r > r_2 \). Hence, \( \pi_\epsilon''(r_2) \leq 0 \) and therefore (4.4) implies \( \sinh \pi_\epsilon(r_2) \leq 0 \) or \( \pi_\epsilon(r_2) \leq 0 \). It is a contradiction.

The inequality for the second derivative follows directly from (4.4). Differentiating (4.4) with respect to \( r \) and using (4.4) again, we have

\[
(4.5) \quad \pi_\epsilon'''(r) = 2 \left( r^{-2} + \epsilon^{-2} r \cosh \pi_\epsilon(r) \right) \pi_\epsilon''(r).
\]

Hence \( \pi_\epsilon'''(r) < 0 \) follows. \[\square\]

For \( t \in (0,1) \), we set

\[
M_i(t) = \left\{ \begin{array}{ll}
\max_{r \in [t,1]} |\pi_\epsilon^{(i)}(r)|, & \text{for } i = 0, 1, 2; \\
\max_{r \in [t,1]} |\sinh \pi_\epsilon(r)|, & \text{for } i = 3.
\end{array} \right.
\]
Furthermore, $M_i(t)$ is strictly decreasing in $t \in (0, 1)$; and $M_0(t) < M_3(t)$.

We need the inequality

\[(4.6) \quad M_3(1/4) \leq 2^8 \epsilon^2.\]

Rewrite \[(4.3)\]:

\[(r \varpi'(r))' = 2 \epsilon^{-2} r^2 \sinh \varpi(r);\]

and integrate over $[0,1]$:

\[
\varpi_-(1) = \int_0^1 (r \varpi'(r))' dr = \int_0^1 2 \epsilon^{-2} r^2 \sinh \varpi(r) dr.
\]

On the other hand, Lemma \[5\] implies

\[
\varpi_-(1) = \lim_{r \to 1^{-}} \frac{\varpi(r) - \varpi(1)}{r - 1} \leq \lim_{r \to 1^{-}} \frac{\ln r - \ln 1}{r - 1} = 1.
\]

Hence,

\[(4.7) \quad \int_0^1 r^2 \sinh \varpi(r) dr \leq \epsilon^2/2.\]

As $\sinh \varpi(r)$ is strictly decreasing,

\[(1/8)^2 \sinh \varpi(1/4) < r^2 \sinh \varpi(r) \quad \text{for } r \in [1/8, 1/4].\]

Integrating over $[1/8, 1/4]$ and using \[(4.7)\], we obtain

\[(1/8)^3 \sinh \varpi(1/4) < \epsilon^2/2.\]

This proves \[(4.6)\].

We need more estimates on $M_i(t)$.

**Lemma 6.** For any $t,t' \in [1/4, 1/2]$ and for any $0 < \epsilon < 1/8$, we have
\[(1) \quad M_2(t) = \frac{2^8}{\epsilon} M_3(t) + \frac{1}{\epsilon} M_1(t);\]
\[(2) \quad M_1(t) < \frac{2}{\epsilon} M_3(t); \quad \text{and}\]
\[(3) \quad M_3(t') < \frac{2}{\epsilon} M_3(t), \quad \text{for } t' > t.\]

**Proof.** The first follows directly from \[(4.3)\] and Lemma \[5\]. We now prove \[(2)\]. For $1/4 \leq t \leq 1/2$ and $0 < \epsilon < 1/8$, the Taylor expansion of $\varpi(r)$ at $r = t$ is

\[\varpi(t + \epsilon) = \varpi(t) + \varpi'(t) \epsilon + \varpi''(t + \eta \epsilon) \epsilon^2/2, \quad 0 \leq \eta \leq 1.\]

Then, by Lemma \[5\] we estimate

\[0 > \varpi'(t) \epsilon = \varpi(t + \epsilon) - \varpi(t) - \varpi'(t + \eta \epsilon) \epsilon^2/2 > -\varpi(t) - \varpi''(t) \epsilon^2/2.\]

Hence,

\[M_1(t) < \epsilon^{-1} M_0(t) + (\epsilon/2) M_2(t) < \epsilon^{-1} M_3(t) + (\epsilon/2) M_2(t).\]

Substituting \[(1)\] into the above inequality, we obtain

\[M_1(t) < \epsilon^{-1} M_3(t) + t \epsilon^{-1} M_3(t) + (\epsilon/2) t M_1(t),\]

and therefore,

\[M_1(t) < \frac{1 + t}{\epsilon(1 - \epsilon/2)} M_3(t) \leq \frac{2}{\epsilon} M_3(t).\]

This proves \[(2)\].

For \[(3)\], we can rewrite \[(4.4)\] as

\[(r \varpi'(r))' = 2 \epsilon^{-2} r^2 \sinh \varpi(r).\]
Integrating over $[t, 1]$ and using Lemma 5, we get

\[(4.8) \quad 2e^{-2} \int_{t}^{1} r^2 \sinh \nu_\epsilon (r) dr = \nu_\epsilon \prime (1) - t \nu_\epsilon \prime (t) \leq t |\nu_\epsilon (t)| = t M_1 (t).\]

On the other hand, as in the proof of inequality (4.6), we have

\[(4.9) \quad 2e^{-2} \int_{t}^{t'} r^2 \sinh \nu_\epsilon (r) dr \geq 2e^{-2} \int_{t}^{t'} r^2 \sinh \nu_\epsilon (r) dr \geq 2e^{-2} t'^2 (t' - t) \sinh \nu_\epsilon (t'). \]

Combining (4.8) with (4.9) gives (3). \qed

We are ready to prove

**Lemma 7.** For any positive integer $l$ and non-negative integer $k$ so that $l > k \geq 0$, there exists a constant $C = C(l, k)$ such that for $0 < \epsilon < 1/8$,

\[\| \nu_\epsilon^{(k)} (r) \|_{C^0 ([\epsilon, 1])} \leq C \epsilon^{l-k}.\]

**Proof.** According to our definitions, \[\| \nu_\epsilon^{(k)} (r) \|_{C^0 ([\epsilon, 1])} = M_k (\frac{1}{2})\] for $k = 0, 1, 2$. We first look at the case for $k = 0$. By Lemma 6, we have

\[M_3 (t') \leq \frac{2^2}{t' - t} \epsilon M_3 (t), \text{ for } t' > t.\]

Then by the iterated method and by (4.6), we get

\[M_3 \left( \frac{1}{2} \cdot \frac{l - 1}{l} \right) \leq 2^3 l (l - 1) \epsilon M_3 \left( \frac{1}{2} \cdot \frac{l - 2}{l} \right) \leq (2^3)^{l-1} 2 \cdot 3^2 \cdot 2 \epsilon^{l-2} M_3 (1/4) \leq 2^{3l+1} (l!)^2 \epsilon^{l-1}.\]

Hence

\[M_0 (1/2) \leq M_3 (1/2) \leq M_3 \left( \frac{1}{2} \cdot \frac{l - 1}{l} \right) \leq 2^{3l+1} (l!)^2 \epsilon^{l-1}.\]

This proves the case for $k = 0$.

The case for $k = 1$ follows from Lemma 6

\[M_1 (1/2) < M_1 \left( \frac{1}{2} \cdot \frac{l - 1}{l} \right) \leq \frac{2}{3} \epsilon M_3 \left( \frac{1}{2} \cdot \frac{l - 1}{l} \right) \leq 2^{3l+1} (l!)^2 \epsilon^{l-1}.\]

The case for $k = 2$ follows from the first two cases and Lemma 6.

For the case $k \geq 3$, taking the derivatives to two sides of (4.8) and using the inductive method gives the result. \qed

Now we estimate \[\| \nu_\epsilon \|_{C^{1, k} (B_1 (0))},\] which will be used in the last section. In this time, for brevity set

\[F(\nu_\epsilon, r) = e^{-2} (\exp \nu_\epsilon - r^2 \exp (-\nu_\epsilon)).\]

We denote by $F_1$ and $F_2$ the derivatives of $F$ in the first variable and the second variable respectively; we also have the notations $F_{11}$, $F_{12}$, $F_{22}$ and so on. For example,

\[F_1 = e^{-2} (\exp \nu_\epsilon + r^2 \exp (-\nu_\epsilon)), \quad F_2 = -2r e^{-2} \exp (-\nu_\epsilon); \quad F_{11} = F, \quad F_{12} = -F_2, \quad F_{22} = -2e^{-2} \exp (-\nu_\epsilon) = r^{-1} F_2;\]
and
\[ F_{221} = -F_{22}, \quad F_{222} = 0. \]

**Lemma 8.** For any \( 0 < r < 1 \),
\( (1) \) \( 0 < F < \frac{1}{2r}, \quad 0 < F_1 < \frac{1}{2r}, \quad -\frac{2}{r^2} < F_2 < 0; \) and
\( (2) \) \( \| \mathcal{P}_e \|_{C^0} \leq \frac{1}{2r}, \quad \| \mathcal{P}_e' \|_{C^0} \leq \frac{1}{2r}, \quad \| \mathcal{P}_e'' \|_{C^0} \leq \frac{3}{2r^2}. \)

**Proof.** By Proposition 4 and Lemma 5,
\[ \ln r < \mathcal{P}_e(r) < 0 \quad \text{and} \quad 0 < \mathcal{P}_e'(r) < r - 1. \]
Hence the first two items in (1) are valid. The third item in (1) is from the derivative
\[ F_2' = -2e^{-2}(1 - r \mathcal{P}_e(r)) \exp(-\mathcal{P}_e) < 0. \]

As to the items in (2), we consider the inequality
\[ 0 < (r \mathcal{P}_e'(r))' = rF < r^{-2}. \]
Integration by parts gives
\[ 0 < \mathcal{P}_e'(r) < (r/2)e^{-2}. \]
Combined with the second inequality in (4.10), we see that when \( r > \epsilon \), \( 0 < \mathcal{P}_e'(r) < 1 \), and when \( r < \epsilon \), \( 0 < \mathcal{P}_e'(r) < 1/2 \). Hence, we get the second inequality in (2). The first item in (2) is from
\[ 0 < -\mathcal{P}_e(0) = \mathcal{P}_e(1) - \mathcal{P}_e(0) = \int_0^1 \mathcal{P}_e(r) dr \leq (2\epsilon)^{-1}. \]
Now by (4.3), the third inequality in (2) is direct from the first inequality in (1) and (4.11). \( \square \)

**Proposition 9.** For any ball \( B_R(0) \subset B_1(0) \) and any non-negative integer \( k \), there exists a constant \( C = C(R, k) \) such that for any positive \( \epsilon \) small enough,
\[ \| \mathcal{P}_e \|_{C^k(B_R(0))} \leq C\epsilon^{-k-2}. \]

**Proof.** We need to prove that for any \( k \geq 3 \),
\[ \| \nabla^k \mathcal{P}_e \|_{L^2(B_R(0))} \leq C\epsilon^{-k}. \]
We assume that \( \mathcal{P}_e \) has the compact support in \( B_1(0) \) and \( B_R(0) = B_1(0) \); otherwise one can use cut-off functions. The estimates for \( k = 3, 4 \) are obvious and we omit the proof. We first estimate \( \| \nabla^3 \mathcal{P}_e \|_{L^2(B_1(0))} \).

By a direct calculation, we have
\[ \triangle^2 \mathcal{P}_e = \triangle F = F(F_1 + \mathcal{P}_e^2) + 2F_{22}(1 - r \mathcal{P}_e') \]
and
\[ | \nabla \triangle^2 \mathcal{P}_e |^2 = |(F_1 \mathcal{P}_e + F_2)(F_1 + \mathcal{P}_e^2) + F(F_1 + F_{12} + 2\mathcal{P}_e \mathcal{P}_e') - 2F_{22} \mathcal{P}_e'(1 - r \mathcal{P}_e') - 2r FF_{22}|^2. \]
Then the above lemma implies
\[ | \nabla \triangle^2 \mathcal{P}_e |^2 \leq C\epsilon^{-10} + C\epsilon^{-6} \exp(-2\mathcal{P}_e) \]
for a generic constant $C$. But integration by parts and the above lemma yields
\[
\int_{B_1(0)} \exp(-2\pi r) dx_1 dx_2 = 2\pi \int_0^1 r \exp(-2\pi r) dr
\]
\[= \pi r^2 \exp(-2\pi r)|^1_0 + 2\pi \int_0^1 r^2 \exp(-2\pi r)|^1_0 dr \leq \pi + C\epsilon^{-1}.
\]
Thus,
\[
\|
\nabla^5 \pi_\epsilon \n\|_{L^2(B_1(0))}^2 \leq \int_{B_1(0)} u \Delta^5 u = \int_{B_1(0)} |
\nabla \Delta^2 \pi_\epsilon |^2 \leq C\epsilon^{-10}.
\]

In this way, we can prove (4.12) for any $k \geq 6$. The only trouble is to estimate $\int_0^1 r \exp(-2p\pi r) dr$ for any positive integer $p$. But this can be done by using integration by parts $m$ times.

Combined with Lemma (8), we get
\[
\|
\pi_\epsilon \n\|_{W^{k,2}(B_1(0))} \leq C\epsilon^{-k}.
\]

The Sobolev inequality [8, P.171] then gives, for any $0 < \delta < 1$
\[
(4.13) \quad \|
\pi_\epsilon \n\|_{C^{k,\delta}(B_1(0))} \leq C\epsilon^{-(k+2)}.
\]

This prove the proposition. □

5. Construction of a family of Hermitian metrics

In this section, if $H$ is a Hermitian metric on $V$, we will denote by $D_H$ the associated Hermitian connection, by $\Theta(H)$ and $\hat{\Theta}(H)$ the curvature forms of $D_H$ relative to the smooth frames $(\hat{\mu}_1^\alpha, \hat{\mu}_2^\alpha)$ and the holomorphic frames $(\tilde{\mu}_1^\alpha, \tilde{\mu}_2^\alpha)$.

Following the convention in section 2, $\xi_\alpha$ is a branched point on $B$ and $\xi_j$ is a point in the support of $D$. Let
\[
\hat{D} = \sum \xi_\alpha - 4 \sum \xi_j
\]
be a new divisor on $B$ which degree is zero. Let $G$ be the Green function of $\hat{D}$ [16 p.339-340] whose local expansion near $\xi_\alpha$ for $1 \leq \alpha \leq 5n/4$ has the form
\[
G(z_\alpha) = -c_\alpha \log |z| + 2g_\alpha(z_\alpha)
\]
for the constant $c_\alpha = 1$ or $c_j = -4$ and some harmonic function $g_\alpha$. We assume that $r_0$ is small enough so that $G|_{U_0}$ has the above local expansion.

We now construct a Hermitian metric on $V$ using the Green function $G$ and the HYM metric $h_\epsilon$, which is denoted as $h_\epsilon$ in section 3. Over $U_0$, we define $h_0$ to be the metric given by the Hermitian matrix valued function for $(\hat{\mu}_1^0, \hat{\mu}_2^0)$
\[
\hat{h}_0 = e^{\frac{1}{2}G} I,
\]
where $I$ is the $2 \times 2$ identity matrix. Thus, the ambiguity of choosing $(\hat{\mu}_1^0, \hat{\mu}_2^0)$ in section 2 is irrelevant. By (3.4) and the notation in (2.13), the Hermitian matrix for $h_0$ in $(\hat{\mu}_1^0, \hat{\mu}_2^0)$ is
\[
\hat{h}_0 = (A_0)^t \hat{h}_0 \overline{A}_0.
\]
Since $G$ is harmonic, direct calculation as in section 3 gives
\begin{equation}
\hat{\Theta}(h_0) = \hat{\Theta}(h_0) = -\pi i \left( \frac{\partial \omega_1(z)}{\partial \bar{z}} 0 \right) dz \wedge d\bar{w} + \pi i \left( \frac{\partial \omega_2(z)}{\partial \bar{z}} 0 \right) dw \wedge d\bar{z}.
\end{equation}
(5.1)

Hence, $h_0$ is a HYM metric on $V|\mathcal{U}_0$. For $n + 1 \leq j \leq 5n/4$, because of (2.13), the metric $h_0$ under $(\hat{\mu}_1, \hat{\mu}_2)$ over $\mathcal{U}_j \cap \mathcal{U}_0$ is given by the matrix valued function
\begin{equation}
\hat{h}_j = e^g I.
\end{equation}

In this way $h_0$ extends to a smooth metric over $\mathcal{U}_j$. But over $\mathcal{U}_a$’s, because of (2.14), the metric $h_0$ under $(\hat{\mu}_1, \hat{\mu}_2)$ has the form
\begin{equation}
\hat{h}_a = e^{g_a} \begin{pmatrix} |z|^{-\frac{1}{2}} 0 \\ 0 |z|^{\frac{1}{2}} \end{pmatrix}.
\end{equation}
Clearly, $h_0$ does not extend to the point $\xi_a$. However, in section 3 we have found a new HYM metric $\hat{h}_a^c$ of $V|\mathcal{U}_a$ that under the frame $(\hat{\mu}_1^a, \hat{\mu}_2^a)$ has the form
\begin{equation}
\hat{h}_a^c = \begin{pmatrix} e^{-u_a} 0 \\ 0 e^{u_a} \end{pmatrix}
\end{equation}
of which $u_a$ is the solution to the equation 4.1. We let $h_{a,c} = e^{g_a} h_a^c$; then $h_{a,c}$ is also a HYM metric on $V|\mathcal{U}_a$.

What we shall do is to interpolate the two metrics $h_0$ and $h_{a,c}$ over $\mathcal{U}_a$. We let
\begin{equation}
\rho : (0, (2r_0)^2) \rightarrow [0, 1]
\end{equation}
be a fixed $C^\infty$ cut-off function with $\rho(r^2) = 1$ for $r < r_0$, $\rho(r^2) = 0$ for $r \geq \frac{4}{3}r_0$. We then define
\begin{equation}
h_a|\mathcal{U}_a = (1 - \rho(|z|^2))h_0 + \rho(|z|^2)h_{a,c}.
\end{equation}
It is a smooth Hermitian metric on $V|\mathcal{U}_a$ that coincides with $h_0$ for $|z| \geq \frac{4}{3}r_0$ and coincides with $h_{a,c}$ for $|z| \leq r_0$. After working this out for all branched points, we obtain a global Hermitian metric $h_\ast$ that is $h_0$ on $V \cup \mathcal{U}_{\mathrm{in}}(\mathcal{U}_a(\frac{4}{3}r_0))$ and $h_{a,c}$ on $V|\mathcal{U}_{\mathrm{out}}(r_0)$. Here we denote by $\mathcal{U}_{\mathrm{in}}(r)$ the pre-image in $X$ of $U_{\mathrm{in}}(r)$, which is the disc in $B$ with center $\xi_a$ and radius $r$. From now on we denote $U_0 = B - (D \cup (\cup_{a=1}^{n} U_a(\frac{2}{3}r_0)))$, $U_0 = U_0 \times T^2$, and take the corresponding trivialization of $V$.

Hence, over $\mathcal{U}_0$ and $\mathcal{U}_j$, $\hat{\Theta}(h_\ast) = \hat{\Theta}(h_0)$. Over $\mathcal{U}_a$, direct calculation gives
\begin{equation}
\hat{\Theta}(h_a) = \pi^2 (|z|^2 \exp(\phi_1 - \phi_2) - \exp(\phi_1 - \phi_1)) \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) dw \wedge d\bar{w} \end{equation}
(5.2)
\begin{equation}
- \pi i \left( \frac{\partial^2 \phi_1}{\partial \bar{z} \partial \bar{z}} 0 \right) dz \wedge d\bar{z} - \pi i \left( \frac{\partial \phi_1}{\partial \bar{z}} \right) \left( \frac{\partial \phi_2 - \phi_2}{\partial \bar{z}} \right) 1 + \frac{\partial (\phi_1 - \phi_2)}{\partial \bar{z}} dz \wedge d\bar{w}
\end{equation}
\begin{equation}
- \pi i \left( \exp(\phi_1 - \phi_2)(1 + \frac{\partial (\phi_1 - \phi_2)}{\partial \bar{z}}) \right) \frac{\partial \phi_2 - \phi_1}{\partial \bar{z}} dz \wedge dw.
\end{equation}
where
\begin{equation}
\phi_1 = \ln((1 - \rho)r^{-\frac{1}{2}} + \rho \exp(-u_a)) \quad \text{and} \quad \phi_2 = \ln((1 - \rho)r^{\frac{1}{2}} + \rho \exp u_a).
\end{equation}
Notice that near the boundary of $\mathcal{U}_a$ the functions $\phi_1$ and $\phi_2$ reduces to $-\frac{1}{2}\ln r$ and $\frac{1}{2}\ln r$, and their sum $\phi_1 + \phi_2$ vanishes. Hence we can extend $\phi_1 + \phi_2$ to all $X$ by assigning zero to it away from all $\mathcal{U}_a$. This point will be used in the following normalization.

Now, over $\bar{\mathcal{X}} - \cup^m \mathcal{U}_a$, by (5.1) and the notations in section 2, we have
\begin{equation}
(5.3) \quad \text{Tr} \Theta(h_x) = -\pi ip_{2*}(dw^* \wedge d\bar{w} + dw \wedge d\bar{w})
\end{equation}
and
\begin{equation}
(5.4) \quad \text{Tr} (\hat{\Theta}(h_x) \wedge \hat{\Theta}(h_x)) = -2\pi^2 p_{2*}(dw^* \wedge d\bar{w} + dw \wedge d\bar{w}).
\end{equation}
Over $\mathcal{U}_a$ for $1 \leq a \leq n$, by (6.2), we have
\begin{equation}
(5.5) \quad \text{Tr} (\hat{\Theta}(h_x)) = -\partial \bar{\partial} (\phi_1 + \phi_2)
\end{equation}
and
\begin{equation}
\text{Tr} (\hat{\Theta}(h_x) \wedge \hat{\Theta}(h_x)) = -2\pi^2 \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} dz \wedge d\bar{z} \wedge dw \wedge d\bar{w},
\end{equation}
where $\varphi = |z|^2 \exp(\phi_1 - \phi_2) + \exp(\phi_2 - \phi_1)$. Moreover, when $r_0 \to 0$,
\begin{equation}
(5.6) \quad \int_{\mathcal{U}_a} \text{Tr} (\hat{\Theta}(h_x) \wedge \hat{\Theta}(h_x)) = 8\pi^2 \int_{\mathcal{U}_a} (\varphi''(r) + \frac{1}{r} \varphi'(r)) r dr d\theta = 16\pi^2 r_0 \to 0.
\end{equation}
Combining (5.3) with (5.5) and combining (5.4) with (5.6), we get (2.7) by (2.6).

We need to modify the metric $h_x$ conformally. From (5.2) we have
\begin{equation}
\text{Tr} (\Lambda \hat{\Theta}(h_x)) = -\epsilon \frac{\partial^2 (\phi_1 + \phi_2)}{\partial z \partial \bar{z}}.
\end{equation}
To make it vanish, we will normalize $h_x$ conformally by the factor $\exp(-\frac{1}{2}(\phi_1 + \phi_2))$:
\begin{equation}
H_{0,\epsilon} = \exp\left(-\frac{1}{2}(\phi_1 + \phi_2)\right) \cdot h_x.
\end{equation}
Consequently,
\begin{equation}
(5.7) \quad \text{Tr} (\Lambda \hat{\Theta}(H_{0,\epsilon})) = 0.
\end{equation}
Moreover, by our construction, $\Lambda \hat{\Theta}(H_{0,\epsilon}) = 0$ over $\mathcal{U}_0$, $\mathcal{U}_f$ and $\mathcal{U}_a(r_0)$; and
\begin{equation}
(5.8) \quad \Lambda \hat{\Theta}(H_{0,\epsilon}) = \psi \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right)
\end{equation}
over $\mathcal{U}_a \setminus \mathcal{U}_a(r_0)$ for the function
\begin{equation}
(5.9) \quad \psi = \frac{1}{\epsilon} \pi^2 (|z|^2 \exp(\phi_1 - \phi_2) - \exp(\phi_2 - \phi_1)) - \frac{\epsilon}{2} \frac{\partial^2 (\phi_1 - \phi_2)}{\partial z \partial \bar{z}}.
\end{equation}
On the other hand, by the first part of Theorem 3, the function $\psi$ satisfies, for any $l$ and $0 < \epsilon < 1/8$,
\begin{equation}
(5.10) \quad \| \psi \|_{C^0([r_0, 2r_0])} \leq C(l, r_0) \epsilon^{l-1}.
\end{equation}
Therefore we have the following desired estimates immediately
\begin{equation}
(5.11) \quad \| \Lambda \hat{\Theta}(H_{0,\epsilon}) \|_{C^0(\mathcal{U}_a)} \leq C(l, r_0) \epsilon^{l-1}.
\end{equation}
If we use $(\bar{\mu}_1, \bar{\mu}_2)$, since by (5.3) $\hat{\Theta}(H_{0,\epsilon}) = A^{-1}_a \hat{\Theta}(H_{0,\epsilon}) A_a$ and $A_a$ is fixed which does not depend on $\epsilon$, we also have,
\begin{equation}
(5.12) \quad \| \Lambda \hat{\Theta}(H_{0,\epsilon}) \|_{C^0(\mathcal{U}_a)} \leq C(l, r_0) \epsilon^{l-1}.
\end{equation}
Finally, by our construction, $(\hat{\mu}_{1}^{\alpha}, \hat{\mu}_{2}^{\alpha})$ is orthogonal for $H_{0,\epsilon}$. It can be normalized to a unitary frame $(\hat{\mu}_{1}^{\alpha}, \hat{\mu}_{2}^{\alpha})$:

\[(5.13) \quad (\hat{\mu}_{1}^{\alpha}, \hat{\mu}_{2}^{\alpha}) = (\hat{\mu}_{1}^{\alpha}, \hat{\mu}_{2}^{\alpha})N_{\alpha}\]

where

\[(5.14) \quad N_{\alpha} = \begin{cases} e^{\frac{k}{2}G_{I}}, & \text{when } \alpha = 0, \\ e^{\frac{k}{2}g_{-I}}, & \text{when } \alpha = j, \\ e^{\frac{k}{2}g_{\alpha}} \left( \begin{smallmatrix} \kappa^{-1} & 0 \\ 0 & \kappa \end{smallmatrix} \right), & \text{when } \alpha = a; \end{cases}\]

where

\[(5.15) \quad \kappa = \left( \frac{(1 - \rho) r^{\frac{1}{2}} + \rho e^{-u_{i}}}{(1 - \rho) r^{-\frac{1}{2}} + \rho e^{-u_{i}}} \right)^{\frac{1}{2}}.\]

Combining (5.13) with (5.14), we have

\[(5.16) \quad (\hat{\mu}_{1}^{\alpha}, \hat{\mu}_{2}^{\alpha}) = (\hat{\mu}_{1}^{\alpha}, \hat{\mu}_{2}^{\alpha})B_{\alpha}, \quad B_{\alpha} = N_{\alpha}A_{\alpha}.\]

If we denote by $\hat{\Theta}(H_{0,\epsilon})$ the curvature form of $D_{H_{0,\epsilon}}$ relative to $(\hat{\mu}_{1}^{\alpha}, \hat{\mu}_{2}^{\alpha})$, then (5.17) gives

\[(5.17) \quad \hat{\Theta}(H_{0,\epsilon}) = N_{\alpha}\hat{\Theta}(H_{0,\epsilon})N_{\alpha}^{-1}.\]

Hence, $N_{\alpha}$ and $\hat{\Theta}(H_{0,\epsilon})$ being diagonal makes

\[(5.18) \quad \Lambda \hat{\Theta}(H_{0,\epsilon}) = \hat{\Theta}(H_{0,\epsilon}).\]

Thus, by (5.11), we get Proposition 1:

**Proposition 10.** For any positive integer $l$, there is a constant $C = C(l, r_{0})$ such that for any $0 < \epsilon < 1/8$,

\[\| \hat{\Lambda}(H_{0,\epsilon}) \|_{C^{l}(U_{\alpha})} < C^{l-1}.\]

6. **Limiting behavior of HYM metrics**

In this section, when we are working with a single frame, we often drop the superscript and subscript $\alpha$.

Because $V$ is stable, following the work of Donaldson [2] and of Uhlenbeck-Yau [27], $V$ admits a HYM metric $H_{1,\epsilon}$, which is unique up to scale, with respect to the Kähler metric $\omega_{\epsilon}$. For this metric, denote by $\hat{H}_{1,\epsilon}$ and $\hat{H}_{1,\epsilon}$ the Hermitian matrices relative to $(\hat{\mu}_{1}, \hat{\mu}_{2})$ and $(\hat{\mu}_{1}, \hat{\mu}_{2})$. We will compare $H_{1,\epsilon}$ with $H_{0,\epsilon}$. The method is to estimate $\| \hat{H}_{1,\epsilon} - I \|_{C^{l}(U_{\alpha})}$.

As $H_{1,\epsilon}$ and $H_{0,\epsilon}$ are Hermitian metrics on $V$, there exists an element $H_{\epsilon} \in \text{End}(V)$ such that

\[H_{0,\epsilon}(H_{\epsilon}, \cdot) = H_{1,\epsilon}(\cdot, \cdot).\]

$H_{\epsilon}$ is denoted by $(H_{0,\epsilon})^{-1}H_{1,\epsilon}$ in section 1 (cf. [2] p.4]). We write $H_{\epsilon}(\hat{\mu}_{1}, \hat{\mu}_{2}) = (\hat{\mu}_{1}, \hat{\mu}_{2})\hat{H}_{\epsilon}$ and $H_{\epsilon}(\hat{\mu}_{1}, \hat{\mu}_{2}) = (\hat{\mu}_{1}, \hat{\mu}_{2})\hat{H}_{\epsilon}$. Consequently,

\[(6.1) \quad \hat{H}_{1,\epsilon} = (H_{\epsilon})^{t} \quad \text{and} \quad \hat{H}_{1,\epsilon} = (H_{\epsilon})^{t} \cdot \hat{H}_{0,\epsilon}.\]

We also have

\[(6.2) \quad \hat{H}_{1,\epsilon} = B^{t}(\hat{H}_{\epsilon})^{t} \hat{B},\]
which will be used in the estimates for higher order derivatives. By [5,10], we prove this equality as follows:

\[(\hat{H}_{1,e})_{ij} = H_{1,e}(\hat{\mu}_i, \hat{\mu}_j) = H_{0,e}(H_e(\hat{\mu}_i), \hat{\mu}_j) = H_{0,e}(H_e(b_{ki}\tilde{\mu}_k), b_{lj}\hat{\mu}_l) = b_{ki}\tilde{\mu}_j H_{0,e}(\hat{H}_e)_m\hat{\mu}_m, \hat{\mu}_t) = b_{ki}\tilde{\mu}_j (\hat{H}_e)_{ik}.\]

As \(H_{1,e}\) is the HYM metric, by [3,6] and the second equality in [6,1], direct computation as in [27, p.S264] yields

\[
0 = \Lambda\tilde{\Theta}(H_{1,e}) = \Lambda\bar{\partial}(\partial\hat{H}_1\cdot(\hat{H}_{1,e})^{-1})^t
\]

\[
= \Lambda\bar{\partial}(\hat{H}_e^{-1}\partial H_e) + \Lambda\hat{H}_e^{-1}\tilde{\Theta}(H_{0,e})H_e
\]

\[
- \Lambda\hat{H}_e^{-1}\cdot\tilde{\Theta}(\hat{H}_e^{-1})^t\cdot\hat{H}_e
\]

\[
- \Lambda\hat{H}_e^{-1}\cdot(\partial\hat{H}_0\cdot(\hat{H}_0,e)^{-1})^t\cdot\partial\hat{H}_e.
\]

Taking the trace of the above system and combining with \(\text{Tr}(\Lambda\tilde{\Theta}(H_{0,e})) = 0\), which is equivalent to [5,7] by [5,3], we have

\[\triangle \ln \det \hat{H}_e = 0.\]

Hence det \(\hat{H}_e = \text{const.}\) We normalize \(H_{1,e}\) such that det \(\hat{H}_e = 1\).

We first do \(C^0\)-estimates. In order to control \(H_e\), we should estimate \(\text{Tr} H_e\). From [22, p.876], we have

\[
\triangle \text{Tr} \hat{H}_e \leq \text{Tr} \hat{H}_e \cdot |\Lambda\tilde{\Theta}(H_{0,e})|,
\]

which is actually derived from [5,3]. We need

**Lemma 11.** There is a function \(I(e)\) depending only on \(e\) with \(I(e) \geq Ce^{2}\), where \(C\) is a constant, such that for any function \(f\) on \(X\),

\[
\|df\|_2^2 \geq I(e)(\|f\|_4^2 - \|f\|_2^2).
\]

**Proof.** We shall follow the proof in [11]. First, we comment that the lemma is about the estimate of the Sobolev constants. To begin with, because \(X\) has volume one and dimension four, following the notation of [18] Lemma 2], for any arbitrary function \(f\) over \(X\),

\[
\|df\|_2^2 \geq D(4)C_2(\|f\|_4^2 - \|f\|_2^2).
\]

By [18], \(D(4)\) is an absolute constant, \(C_2 = D(4)C_0^2\), \(2C_1 \geq C_0 \geq C_1\), and \(C_1\) is the constant given by the isoperimetric inequality

\[
C_1(\min\{\text{vol}(M_1), \text{vol}(M_2)\})^3 \leq \text{vol}(N)^4
\]

of which \(N\) runs through all codimension one submanifolds dividing \(X\) into two components \(M_1\) and \(M_2\). Because \(X\) is flat and diam(\(X\)) = \(\sqrt{2}/e\), [11] Thm 13 implies

\[
C_1 \geq C_4 \left(\int_0^{\text{diam}(X)} r^3 dr\right)^{-5} = C_5 e^{20}
\]

for constants \(C_4\) and \(C_5\) independent of \(e\). Henceforth, \(C_0 \geq C_6 e^{20}\); and for \(I(e)\):

\[
I(e) = \min\{D(4), 1\}C_2 \geq Ce^{10}.
\]

□
Then we have

**Proposition 12.** For any positive integer $l$, there is a constant $C(l, r_0)$ such that for any $0 < \epsilon < \frac{3}{16}$,

$$\text{Tr} \hat{H}_\epsilon < 2 + C(l, r_0) \epsilon^{\frac{1}{l-1}}.$$

*Proof.* Let $\tau = \text{Tr} \hat{H}_\epsilon$, then from (6.4) and (5.12), we have

$$\Delta \tau \leq C_1 \epsilon^{l-1} \tau,$$

where $C_1$ is a constant depending only on $l$ and $r_0$. Hence we have

$$\int_X \tau^{2p-1} \Delta \tau \leq C_1 \epsilon^{l-1} \int_X \tau^{2p} \text{ for } p \geq 1. \quad (6.5)$$

Because

$$\int_X \tau^{2p-1} \Delta \tau = (2p - 1)p^{-2} \int_X |\nabla \tau^p|^2,$$

then from (6.5),

$$\int_X |\nabla \tau^p|^2 p^2 (2p - 1)^{-1} C_1 \epsilon^{l-1} \int_X \tau^{2p}. \quad (6.6)$$

Combined with Lemma 11 we obtain

$$\| \tau \|_{2p}^2 \leq (1 + p^2 (2p - 1)^{-1} C_2 \epsilon^{l-1}) \| \tau \|_{2p}^2 \leq (1 + C_3 \epsilon^{l-1} p) \| \tau \|_{2p}^2.$$

If we set $p = 2^m$, then

$$\| \tau \|_{2^{m+2}}^2 \leq (1 + C_3 \epsilon^{l-1} 2^m) \| \tau \|_{2^{m+1}}^2.$$

Iterating the inequality, we obtain

$$\| \tau \|_{2\infty}^2 \leq \prod_{m=0}^\infty (1 + C_3 \epsilon^{l-1} 2^m) \| \tau \|_{2}^2. \quad (6.7)$$

It is easy to see that there is a constant $C_4$ such that

$$\prod_{m=0}^\infty (1 + C_3 \epsilon^{l-1} 2^m) \| \tau \|_{2}^2 < \exp(C_4 \epsilon^{\frac{1}{l-1}}). \quad (6.8)$$

It remains to estimate $\| \tau \|_{2}^2$. First we prove that there exists a point $x_0$ in $X$ such that $\tau(x_0) = 2$. Such a point $x_0$ may depend on $\epsilon$. Otherwise $\tau(x) > 2$ for every $x$ in $X$ since we have normalized $H_1, e$ such that $\det H_1 = 1$. Hence two eigenvalues $\lambda_1(x)$ and $\lambda_2(x) = \lambda_1^{-1}(x)$ are not equal for every $x \in X$ and define two smooth functions on $X$. Thus, there are only two different eigenvalues in $V_x$ up to constant. Normalizing them forms two smooth sections of $V$. Therefore, $V$ as a complex vector bundle splits into two trivial line bundles. Consequently, $c_1(V) = c_2(V) = 0$, which contradicts to (2.7).

Now we assume that $\tau(x_0) = 2$. Because $X$ is a flat torus, for any $x \in X$, and $x_0$ can be joined by a minimal geodesic $\gamma(x)$ (where $x$ is not the cut point of $x_0$, the geodesic is unique). Thus, we have

$$\tau(x) \leq \tau(x_0) + \int_{\gamma(x)} \partial_\tau \leq 2 + \int_{\gamma(x)} |\nabla \tau|.$$

Hence

$$\tau^2 \leq 4 + 4 \int_{\gamma(x)} |\nabla \tau| + \left( \int_{\gamma(x)} |\nabla \tau| \right)^2 \leq 4 + 4 \int_{\gamma(x)} |\nabla \tau| + \int_{\gamma(x)} |\nabla \tau|^2.$$
Using (6.6) for \( p = 1 \), then,
\[
\| \tau \|_2^2 = \int_X \tau^2 \leq 4 + 4 \int_X \int_{\gamma(x)} |\nabla \tau| + \int_X \int_{\gamma(x)} |\nabla \tau|^2
\]
\[
\leq 4 + 4 \text{diam}(X) \int_X |\nabla \tau| + \text{diam}(X) \int_X |\nabla \tau|^2
\]
\[
\leq 4 + C_6 \varepsilon^l \tau \leq 4 + C_7 \varepsilon^l \tau^2
\]
and thus,
\[
(6.9) \quad \| \tau \|_2^2 \leq \frac{4}{1 - C_7 \varepsilon^l \tau^2}.
\]

Now from (6.7), (6.8) and (6.9), we have
\[
\| \tau \|_\infty \leq 4 \exp(C_4 \varepsilon^l \tau^2) \leq 4(1 + C_8 \varepsilon^l \tau^2)
\]
or
\[
\| \tau \|_\infty \leq 2(1 + C_9 \varepsilon^l \tau^2) \leq 2 + C_{10} \varepsilon^l \tau^2.
\]

Now we are in position to prove \( C^0 \)-estimates.

**Theorem 13.** For any positive integer \( l \), there is a constant \( C(l,r_0) \) such that for any \( 0 < \varepsilon < 1/8 \),
\[
\| \tilde{H}_{1,\varepsilon} - I \|_{C^0(U_\varepsilon)} < C(l,r_0) \varepsilon^l \tau^2.
\]

**Proof.** By the first equality in (6.1),
\[
\| \tilde{H}_{1,\varepsilon} - I \|_2^2 = \| \tilde{H}_{1,\varepsilon} - I \|_2^2 = \text{Tr}(\tilde{H}_{1,\varepsilon} - I)^2.
\]
On the other hand, \( \text{det} \tilde{H}_{1,\varepsilon} = \text{det} \tilde{H}_{1,\varepsilon} = 1 \) and \( \text{Tr} \tilde{H}_{1,\varepsilon} = \text{Tr} \tilde{H}_{1,\varepsilon} \). Then, by the above proposition, direct calculation proves the theorem.

Next we estimate \( \| \tilde{H}_{1,\varepsilon} - I \|_{C^0(U_\varepsilon)} \) for \( k \geq 1 \). As \( \tilde{H}_{1,\varepsilon} = (\tilde{H}_{1,\varepsilon})^t \), we only need to estimate \( \| \tilde{H}_{1,\varepsilon} - I \|_{C^k(U_\varepsilon)} \). Our starting point is (6.12). By this equality, since \( \tilde{H}_{1,\varepsilon} \) is the HYM metric, formula (6.6) gives
\[
0 = \Lambda \tilde{\Theta}(\tilde{H}_{1,\varepsilon}) = \Lambda(\partial(\partial^t(\tilde{F}_{1,\varepsilon}^t B)(B^t(\tilde{F}_{1,\varepsilon}^t B)))^{-1})^t,
\]
which is equivalent to
\[
(\tilde{H}_{1,\varepsilon})^t B \cdot \Lambda(\partial(\partial^t(\tilde{F}_{1,\varepsilon}^t B)(B^t(\tilde{F}_{1,\varepsilon}^t B)))^{-1})^t \cdot B^{-1} = 0.
\]
On the other hand, (6.10) implies \( \tilde{H}_{0,\varepsilon} = B^t B \). Hence formula (6.6) also gives
\[
\Lambda \tilde{\Theta}(\tilde{H}_{0,\varepsilon}) = \Lambda(\partial(\partial^t(\tilde{B}^t B)(B^t B)^{-1}))^t,
\]
or
\[
(6.11) \quad B \cdot \Lambda \tilde{\Theta}(\tilde{B}^t B)(B^t B)^{-1})^t \cdot B^{-1} \tilde{H}_{1,\varepsilon} = B \cdot \Lambda \tilde{\Theta}(\tilde{H}_{0,\varepsilon}) \cdot B^{-1} \tilde{H}_{1,\varepsilon}.
\]
Subtracting (6.12) from (6.11), expanding the LHS of the resulting equation, and adapting suitably some terms, therefore we obtain

\[ 0 = i\Lambda \bar{\partial} \partial \mathcal{H}_e - i\Lambda \bar{\partial} \partial \mathcal{H}_e \cdot \bar{H}_e \wedge \partial \mathcal{H}_e \\
- i\Lambda \bar{H}_e \cdot \bar{\partial} \log B \cdot \bar{H}_e^{-1} \wedge \partial \mathcal{H}_e \\
- i\Lambda \bar{\partial} \mathcal{H}_e \cdot \bar{H}_e^{-1} \cdot \overline{(\partial \log B)^t} \cdot \bar{H}_e \\
- i\Lambda \partial \mathcal{H}_e \wedge \bar{\partial} \log B - i\Lambda (\partial \log B)^t_1 \wedge \partial \mathcal{H}_e \\
- i\Lambda \bar{\partial} \mathcal{H}_e \partial (\bar{\partial} \log B) + i\Lambda \bar{\partial} (\bar{\partial} \log B) \cdot \mathcal{H}_e \\
- i\Lambda (\partial \log B)^t_1 \cdot \mathcal{H}_e \wedge \bar{\partial} \log B \\
+ i\Lambda \bar{H}_e \cdot \bar{\partial} \log B \wedge (\partial \log B)^t_1 \cdot \bar{H}_e \\
+ i\Lambda \bar{\partial} \Theta(\bar{H}_{0,e}) \cdot B^{-1} \bar{H}_e. \tag{6.13} \]

Here for brevity, we have introduced the notations \( \mathcal{H}_0 = \bar{H}_0 - I \), \( \bar{\mathcal{H}}_0 = \bar{H}_0^{-1} - I \), and \( \bar{\partial} \log B = \partial B \cdot B^{-1} \).

We introduce

\[ x_i, e = e^{-1/2} x_i, \quad y_i, e = e^{1/2} y_i, \quad \text{for} \quad i = 1, 2; \]

and

\[ z_e = e^{-1/2} z, \quad w_e = e^{1/2} w. \tag{6.14} \]

Then (2.1) can be rewritten as

\[ (6.15) \quad \omega_e = dy_{1,e} \wedge dy_{2,e} + dx_{1,e} \wedge dx_{2,e}. \]

This is the Euclidean metric. We will use \( \nabla^k \Delta \) and \( C^k \) to denote the \( k \)-th covariant derivatives, the Laplacian and the \( C^k \)-norm with respect to these new coordinates.

The system (6.13) can be rewritten as

\[ I_2 = I_1 + I_0, \]

where

\[ I_2 = \frac{\partial^2 \mathcal{H}_e}{\partial z_e \partial z_e} + \frac{\partial^2 \mathcal{H}_e}{\partial w_e \partial w_e}, \]

\[ I_1 = \frac{\partial \mathcal{H}_e}{\partial z_e} \bar{H}_e \frac{\partial \mathcal{H}_e}{\partial z_e} + \frac{\partial \mathcal{H}_e}{\partial w_e} \bar{H}_e \frac{\partial \mathcal{H}_e}{\partial w_e} \\
+ \frac{\partial \log B}{\partial z_e} \bar{H}_e^{-1} \frac{\partial \mathcal{H}_e}{\partial z_e} + \frac{\partial \log B}{\partial w_e} \bar{H}_e^{-1} \frac{\partial \mathcal{H}_e}{\partial w_e} \\
+ \frac{\partial \mathcal{H}_e}{\partial z_e} \bar{H}_e^{-1} \left( \frac{\partial \log B}{\partial z_e} \right)^t \bar{H}_e + \frac{\partial \mathcal{H}_e}{\partial w_e} \bar{H}_e^{-1} \left( \frac{\partial \log B}{\partial w_e} \right)^t \bar{H}_e \\
- \frac{\partial \mathcal{H}_e}{\partial z_e} \frac{\partial \log B}{\partial z_e} - \frac{\partial \mathcal{H}_e}{\partial w_e} \frac{\partial \log B}{\partial w_e} \\
- \left( \frac{\partial \log B}{\partial z_e} \right)^t \frac{\partial \mathcal{H}_e}{\partial z_e} - \left( \frac{\partial \log B}{\partial w_e} \right)^t \frac{\partial \mathcal{H}_e}{\partial w_e}. \]
and

\[
I_0 = - \mathcal{H}_e \frac{\partial^2 \log B}{\partial z \partial \bar{z}} - \mathcal{H}_e \frac{\partial^2 \log B}{\partial w \partial \bar{w}} + \frac{\partial^2 \log B}{\partial z \partial \bar{z}} \mathcal{H}_e + \frac{\partial^2 \log B}{\partial w \partial \bar{w}} \mathcal{H}_e \\
+ \hat{\mathcal{H}}_e \frac{\partial \log B}{\partial z} \hat{\mathcal{H}}_e \left( - \frac{\partial \log B}{\partial z} \right) \hat{\mathcal{H}}_e + \hat{\mathcal{H}}_e \frac{\partial \log B}{\partial w} \hat{\mathcal{H}}_e \left( - \frac{\partial \log B}{\partial w} \right) \hat{\mathcal{H}}_e \\
- \hat{\mathcal{H}}_e \partial \log B \left( - \frac{\partial \log B}{\partial z} \right) \hat{\mathcal{H}}_e - \hat{\mathcal{H}}_e \partial \log B \left( - \frac{\partial \log B}{\partial w} \right) \hat{\mathcal{H}}_e \\
+ \left( \frac{\partial \log B}{\partial z} \right) \hat{\mathcal{H}}_e \partial \log B \hat{\mathcal{H}}_e + \left( \frac{\partial \log B}{\partial w} \right) \hat{\mathcal{H}}_e \partial \log B \hat{\mathcal{H}}_e \\
+ i \Lambda B \tilde{\Theta}(H_{0,e}) B^{-1} \hat{\mathcal{H}}_e.
\]

\( I_1 \) contains the terms with the first order derivatives of \( \mathcal{H}_e \) and \( I_0 \) contains the terms with no derivative of \( \mathcal{H}_e \). In \( I_0 \) all but the last term have the factor \( \hat{\mathcal{H}}_e \) or \( \hat{\mathcal{H}}_e \), which, by Theorem 3 and (5.14), (5.15), are very small:

(6.16) \( \| \mathcal{H}_e \|_{C^0} \leq C(l, r_0) e^{\frac{l-1}{14}} \), \( \| \hat{\mathcal{H}}_e \|_{C^0} \leq C(l, r_0) e^{\frac{l-1}{14}} \).

We should first deal with the last term.

Combining (3.6) with (2.15) and using (5.16), (5.17) and (5.18), we have

\[
\Lambda B \tilde{\Theta}(H_{0,e}) B^{-1} = \Lambda B A^{-1} \tilde{\Theta}(H_{0,e}) A B^{-1} = \Lambda N \tilde{\Theta}(H_{0,e}) N^{-1} = \Lambda \hat{\Theta}(H_{0,e}).
\]

Hence by (6.11),

(6.17) \( \| \Lambda B \tilde{\Theta}(H_{0,e}) B^{-1} \|_{C^0} \leq C(l, r_0) e^{\frac{l-1}{14}} \).

Moreover, we recall (6.8):

\[
\Lambda \hat{\Theta}(H_{0,e}) = \psi \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right),
\]

where \( \psi \) is defined in (5.9) and is zero when restricted to \( U_0, U_j, \) and \( U_a(r_0) \). Then the first part of Theorem 3 implies

(6.18) \( \| \Lambda B \tilde{\Theta}(H_{0,e}) B^{-1} \|_{C^k(U_a)} \leq e^{k/2} \| \Lambda B \tilde{\Theta}(H_{0,e}) B^{-1} \|_{C^k(U_a)} \leq C(l, r_0, k) e^{l-1-k/2} \).

Next we estimate the terms coming from \( \partial \log B \) and \( \partial \bar{\partial} \log B \). The most complicated case is over \( U_0 \). (Note we have shrunk \( U_0 \) in section 5.) Hence we will omit the other cases and only estimate for this case. By (5.10) and (5.14),

\[
B = e^{-g_a} \left( \begin{array}{cc} \kappa^{-1} & 0 \\ 0 & \kappa \end{array} \right) A,
\]

where \( \kappa \) is defined in (5.14), which can be written as

\[
\kappa = \left\{ \begin{array}{ll} \left( \frac{1}{e^{g_a}} \right)^{\frac{1}{2}} \left( \frac{1 - e^{\rho \exp(g_a, A) \ln r}}{1 - e^{\rho \exp(g_a, A) \ln r}} \right)^{\frac{1}{2}} & \text{when } r \in [r_0, 2r_0] \\
\end{array} \right.
\]

when \( r \in [0, r_0] \).
Direct calculation yields
\[
\frac{\partial \log B}{\partial z} = \frac{1}{2} \frac{\partial g_\alpha}{\partial z} + \frac{\partial \log \kappa}{\partial z} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \frac{\partial \log B}{\partial w} = \pi i \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix},
\]
\[
\frac{\partial^2 \log B}{\partial z \partial \overline{z}} = \frac{\partial^2 \log \kappa}{\partial z \partial \overline{w}} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \frac{\partial^2 \log B}{\partial z \partial \overline{w}} = \pi i \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \frac{\partial^2 \log B}{\partial w \partial \overline{w}} = 0.
\]
Consequently,
\[
\| \frac{\partial \log B}{\partial \overline{w}} \|_{c^0(\mathcal{U})} \leq C \epsilon^{- \frac{1}{2}}, \quad \| \frac{\partial \log B}{\partial \overline{w}} \|_{c^1(\mathcal{U})} \leq C,
\]
and for \( k \geq 2 \),
\[
\| \frac{\partial \log B}{\partial \overline{w}} \|_{c^k(\mathcal{U})} = 0.
\]
Moreover, by Theorem 3 if we discuss the \( C^k \)-norm of \( \frac{\partial \log B}{\partial \overline{w}} \) on the domain \( \mathcal{U} \setminus \mathcal{U}(r_0) \) and \( \mathcal{U}(r_0) \) respectively, we see that
\[
\| \frac{\partial \log B}{\partial \overline{w}} \|_{c^k(\mathcal{U})} \leq C(r_0, k) \epsilon^{- \frac{k-1}{2}}.
\]
By (6.16) (6.17) (6.19) (6.21), we obtain
\[
\| I_0 \|_{c^0} \leq C(l, r_0) \epsilon^{\frac{l-1}{2}}.
\]
Having made above preparations, we begin to estimate \( \| \nabla^k \mathcal{H}_\epsilon \|_{L^2(\mathcal{U})} \). The approach is standard, so we will not mention basic inequalities such as Young’s and Hölder’s inequality. We only need to be very careful to deal with \( \epsilon \). We assume that \( \mathcal{H}_\epsilon \) has a compact support in \( \mathcal{U}_\alpha \) otherwise we can shrink the open sets \( \mathcal{U}_\alpha \) to \( \mathcal{U}_\alpha' \) such that they still form an open cover of \( X \) and then use cut-off functions. Here we note that we can shrink \( \mathcal{U}_\alpha \) to \( \mathcal{U}_\alpha' \) by shrinking \( U_\alpha' \) in \( B \) and hence the cut-off functions can be taken only defined on \( B \). Thus \( \| \frac{\partial}{\partial \overline{z}} \| = \epsilon^{-1} \| \frac{\partial}{\partial \overline{w}} \| \), which is good enough for us to estimate. We will omit the domain \( \mathcal{U} \) of integration. We will take \( C \) as the generic constant which depends on \( l, k, r_0 \). Remember that \( \mathcal{H}_\epsilon \) is Hermitian symmetric.

We first do
\[
\int | \nabla \mathcal{H}_\epsilon |^2 = - \int \mathcal{H}_\epsilon \nabla \cdot \mathcal{H}_\epsilon = - \int \mathcal{H}_\epsilon \cdot (I_1 + I_0) \leq C \| \mathcal{H}_\epsilon \|_{c^0} \int | \nabla \mathcal{H}_\epsilon |^2 + C \| \mathcal{H}_\epsilon \|_{c^0} \| I_0 \|_{c^0}.
\]
When \( \epsilon \) is small enough, by (6.10), \( \| \mathcal{H}_\epsilon \|_{c^0} \) is very small and hence the first term of the RHS can be controlled by the LHS. Combining this with (6.10) and (6.22) gives
\[
\| \nabla \mathcal{H}_\epsilon \|_{L^2} \leq C(l, r_0) \epsilon^{\frac{l-1}{2}}.
\]
Next we estimate
\[
\int | \nabla^2 \mathcal{H}_\epsilon |^2 \leq \int \nabla \mathcal{H}_\epsilon \nabla \mathcal{H}_\epsilon \leq 2 \int |I_1|^2 + \int |I_0|^2 \leq C \int | \nabla \mathcal{H}_\epsilon |^4 + C \epsilon^{-5} \int | \nabla \mathcal{H}_\epsilon |^2 + C \epsilon^{l-21}.
\]
We need the following Gagliardo-Nirenberg inequality.
Lemma 14. \[20\] Let \( f \in L^p(\mathbb{R}^n), \ D^m f \in L^q(\mathbb{R}^n), \ l \leq p, q \leq +\infty. \) Then for any \( i \) \((0 \leq i \leq m)\), there exists a constant \( C \) such that

\[
\| \ D^i f \|_{L^r(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)}^{1 - \frac{i}{m}} \| D^m f \|_{L^q(\mathbb{R}^n)}^{\frac{i}{m}},
\]

where

\[
\frac{1}{r} = (1 - \frac{i}{m}) \frac{1}{p} + \frac{i}{m} \frac{1}{q}.
\]

We use this lemma for the case: \( n = 2, \ i = 1, \ r = 4, \ m = 2, \ q = 2, \) and \( p = +\infty, \)

\[
\int | \nabla_\epsilon \mathcal{H}_\epsilon |^4 \leq C \| \mathcal{H}_\epsilon \|_{C^0}^2 \int | \nabla_\epsilon^2 \mathcal{H}_\epsilon |^2.
\]

Hence, when \( \epsilon \) is small enough, in \((6.24)\), the first term of the RHS can be controlled by the LHS. Thus, by \((6.23)\), we get

\[
(6.25) \quad \| \nabla_\epsilon^2 \mathcal{H}_\epsilon \|_{L^2} \leq C \epsilon^{\frac{1 - 2l}{2}}.
\]

Now we use the inductive method to estimate \( \| \nabla_\epsilon^k \mathcal{H}_\epsilon \|_{L^2} \). By observation, we find that

\[
M_0(k) \triangleq \int | \nabla_\epsilon^k \mathcal{H}_\epsilon |^2 \leq C \sum M_i,
\]

where

\[
M_1 = \sum \int | \nabla_\epsilon^{i_1} \mathcal{H}_\epsilon |^2 | \nabla_\epsilon^{i_2} \mathcal{H}_\epsilon |^2 | \nabla_\epsilon^{i_3} \mathcal{H}_\epsilon |^2,
\]

for \( i_1, i_2 > 0, \ i_3 \geq 0, \ i_1 + i_2 + i_3 = k; \)

\[
M_2 = \sum \| \frac{\partial \log B}{\partial z_e} \|_{C^{k-1-k_1}}^2 \int | \nabla_\epsilon^{i_1} \mathcal{H}_\epsilon |^2 | \nabla_\epsilon^{i_2} \mathcal{H}_\epsilon |^2 | \nabla_\epsilon^{i_3} \mathcal{H}_\epsilon |^2,
\]

for \( i_1, i_2 > 0, \ i_3 \geq 0, \ i_1 + i_2 + i_3 = k_1 \leq k - 1; \)

\[
M_3 = \sum \| \frac{\partial \log B}{\partial z_e} \|_{C^{k-1-k_1}}^2 \int | \nabla_\epsilon^{k_1} \mathcal{H}_\epsilon |^2, \quad \text{for} \ 0 \leq k_1 \leq k - 1;
\]

\[
M_4 = \sum \| \frac{\partial \log B}{\partial z_e} \|_{C^{j_1}}^2 \| \frac{\partial \log B}{\partial z_e} \|_{C^{j_2}}^2 \int | \nabla_\epsilon^{i_1} \mathcal{H}_\epsilon |^2 | \nabla_\epsilon^{i_2} \mathcal{H}_\epsilon |^2 | \nabla_\epsilon^{i_3} \mathcal{H}_\epsilon |^2,
\]

for \( i_1, i_2 > 0, \ i_3 \geq 0, \ i_1 + i_2 + i_3 = k_1 \leq k - 2, \ j_1 + j_2 = k - k_1 - 2; \)

\[
M_5 = \sum \| \frac{\partial \log B}{\partial z_e} \|_{C^{j_1}}^2 \| \frac{\partial \log B}{\partial z_e} \|_{C^{j_2}}^2 \int | \nabla_\epsilon^{k_1} \mathcal{H}_\epsilon |^2,
\]

for \( 0 \leq k_1 \leq k - 2, \ j_1 + j_2 = k - k_1 - 2; \)

\[
M_6 = \sum \| \Lambda \tilde{\Theta}(H_{0, \epsilon}) B^{-1} \|_{C^{k-2-k_1}}^2 \int | \nabla_\epsilon^{k_1} \mathcal{H}_\epsilon |^2, \quad \text{for} \ 0 < k_1 \leq k - 2;
\]

\[
M_7 = \| \Lambda \tilde{\Theta}(H_{0, \epsilon}) B^{-1} \|_{C^{k-2}}^2.
\]

We first deal with \( M_1 \). When \( i_1 < k_1 \), Lemma \[14\] implies

\[
\left( \int | \nabla_\epsilon^{i_1} \mathcal{H}_\epsilon |^{\frac{2k_1}{k_1}} \right)^{\frac{k_1}{2k_1}} \leq C \| \mathcal{H}_\epsilon \|_{C^0}^{1 - \frac{i_1}{m}} \left( \int | \nabla_\epsilon^{k_1} \mathcal{H}_\epsilon |^2 \right)^{\frac{k_1}{2k_1}}.
\]
Then the Hölder inequality and this inequality implies that the summands of $M_1$ for $i_3 > 0$ are less than

$$
\left( \int |\nabla^i \mathcal{H}_\epsilon |^\frac{p}{q} \right)^{\frac{q}{p}} \left( \int |\nabla^{i_2} \mathcal{H}_\epsilon |^2 \right)^{\frac{1}{2}} \left( \int |\nabla^{i_1} \mathcal{H}_\epsilon |^2 \right)^{\frac{1}{2}} \leq C \| \mathcal{H}_\epsilon \|^{\frac{q}{p}}_{C^0} \int |\nabla^k \mathcal{H}_\epsilon |^2;
$$

as the same reason, the summands of $M_1$ for $i_3 = 0$ are less than

$$
C \| \mathcal{H}_\epsilon \|^{\frac{q}{p}}_{C^0} \int |\nabla^k \mathcal{H}_\epsilon |^2.
$$

Hence when $\epsilon$ is small enough, $M_1$ can be controlled by $M_0(k)$.

As the above discussion, we see that $M_2$ and $M_4$ can be controlled by $M_3$ and $M_5$ respectively. Thus we get

$$
M_0(k) \leq C(M_3 + M_5 + M_6 + M_7).
$$

If we let $M_0(k_1) \leq C\epsilon f(k_1)$ for any $k_1 \leq k - 1$, then by (6.21) and (6.16),

the summand of $M_3 \leq \left\{ \begin{array}{ll}
C\epsilon^{k+k_1-4+f(k_1)}, & \text{for } 0 < k_1 \leq k - 1, \\
C\epsilon^{l-k-15}, & \text{for } k_1 = 0;
\end{array} \right.$

the summand of $M_5 \leq \left\{ \begin{array}{ll}
C\epsilon^{k+k_1-8+f(k_1)}, & \text{for } 0 < k_1 \leq k - 2, \\
C\epsilon^{l-k-19}, & \text{for } k_1 = 0;
\end{array} \right.$

the summand of $M_6 \leq C\epsilon^{2l-k+k_1+f(k_1)}$, for $0 < k_1 \leq k - 2$;

$M_7 \leq C\epsilon^{2l-k-2}$.

Clearly, $M_0$ and $M_7$ can be controlled by $M_3$. Hence $M_0 \leq C(M_3 + M_5)$. Moreover, (6.23) and (6.24) imply $f(1) = l - 16$ and $f(2) = l - 21$. If we let $f(k_1) = l - 11 - 5k_1$ for any integer $1 \leq k_1 \leq k - 1$, then we see that

$$
M_0(k) \leq C\epsilon^{-5+f(k-1)} = C\epsilon^{l-11-5k}.
$$

Therefore by the inductive method, $f(k) = l - 11 - 5k$, i.e.,

$$
\int |\nabla^k \mathcal{H}_\epsilon |^2 \leq C\epsilon^{l-11-5k}.
$$

By the Sobolev inequality, we get

$$
\| \mathcal{H}_\epsilon \|_{C^{k,\delta}(\mathcal{U})} \leq C\epsilon^{\frac{l-15-5k}{2}},
$$

which is transferred, by (6.13), to

$$
\| \mathcal{H}_\epsilon \|_{C^{k,\delta}(\mathcal{U})} \leq C\epsilon^{\frac{l-15-6k}{2}}.
$$

Thus we obtain, since $\mathcal{H}_\epsilon = \tilde{H}_{1,\epsilon} - \tilde{I}$,

**Theorem 15.** For any positive integers $l$ and $k$ so that $l > 6k + 15$, there is a constant $C = C(l, k, \mathcal{U}_\epsilon)$ depending on $l$, $k$ and an open cover $\{\mathcal{U}_\epsilon\}$ of $X$ which is smaller than $\{\mathcal{U}_\epsilon\}$ such that for any positive $\epsilon$ small enough,

$$
\| \tilde{H}_{1,\epsilon} - \tilde{I} \|_{C^k(\mathcal{U}_\epsilon)} \leq C\epsilon^{\frac{l-15-6k}{2}}.
$$

Now we can prove Theorem 2.
Proof. For the unitary frame $(\tilde{\mu}_1, \tilde{\mu}_2)$ associated to $H_{0,\epsilon}$, the resulting matrix representations of $(H_{0,\epsilon})^{-1}H_{1,\epsilon}$ are $\tilde{H}_{1,\epsilon}$. Moreover, we can replace $\frac{15\epsilon^{-6}}{2}$ by $l'$. Thus the above theorem implies Theorem 2. \hfill \Box

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