Superconvergent Gradient Recovery for Virtual Element Methods

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Abstract. Virtual element methods is a new promising finite element methods using general polygonal meshes. Its optimal a priori error estimates are well established in the literature. In this paper, we take a different viewpoint. We try to uncover the superconvergent property of the virtual element methods by doing some local post-processing only on the degrees of freedom. Using linear virtual element method as an example, we propose a universal recovery procedure to improve the accuracy of gradient approximation for numerical methods using general polygonal meshes. Its capability of serving as a posteriori error estimators in adaptive methods is also investigated. Compared to the existing residual-type a posteriori error estimators for the virtual element methods, the recovery-type a posteriori error estimator based on the proposed gradient recovery technique is much simpler in implementation and asymptotically exact. A series of benchmark tests are presented to numerically illustrate the superconvergence of recovered gradient and validate the asymptotical exactness of the recovery-based a posteriori error estimator.

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1. Introduction. The idea of using polygonal elements can be tracked back to Wachspress [49]. After that, there has been tremendous interest in developing finite element/difference methods using general polygons, see the review paper [33] and the references therein. Famous examples includes the polygonal finite element methods [43,44], mimetic finite difference methods [9,30,31,41,42], hybrid high-order methods [19,20], polygonal discontinuous Galerkin methods [35], etc.

The virtual element methods evolve from the mimetic finite difference methods [9,30,31,41,42] within the framework of the finite element methods. It was first proposed for the Poisson equations [6]. Then, it has been developed to many other equations [2,8,14,14,16]. It generalizes the classical finite element methods on from simplexes to general polygons/polyhedrons including concave ones. This endows the virtual element methods with the capability of dealing with polygons(polyhedrons) with arbitrary numbers of edges (faces) and coping with more general continuity. This makes the virtual element methods handle hanging nodes naturally and simplifies the procedure of adaptive mesh refinement. Different from other polygonal finite element methods, the non-polynomial basis functions are never explicitly constructed and the evaluation of non-polynomial functions is totally unnecessary. As consequence, the only available data in virtual element methods is the degrees of freedom. The optimal convergence theory is well established in [65].

In many cases, gradient attracts much more attention than the solution itself. That is due to two different aspects: first, gradient has physical means like momentum, pressure, et. al; second, many problems like the free boundary value problems,
moving interface problems depend on the first derivatives of the solutions. For virtual element methods, like their predecessors: standard finite element methods, the gradient approximate accuracy is one order lower than the corresponding solution approximation accuracy. Thus, a more accurate approximate gradient is highly desirable in scientific and engineering computing.

For finite element methods on triangles or quadrilaterals, it is well-known gradient recovery is one of the most important post-processing procedures to reconstruct a more accurate approximation gradient than the finite element gradient. The gradient recovery methods are well developed for the classical finite element methods and there are a massive number of works in the literature, to name a few [5, 24, 25, 32, 37, 50, 54]. Famous examples include the simple/weighted averaging [52], superconvergent patch recovery [53, 54] (SPR), and the polynomial preserving recovery [36, 37, 51] (PPR). Right now, SPR and PPR become standard tools in modern scientific and engineering computing. It is evident by the fact that SPR is available in many commercial finite element software like ANSYS, Abaqus, and LS-DYNA, and PPR is included in COMSOL Multiphysics.

The first purpose of this paper is to introduce a gradient recovery as a post-processing procedure for the linear virtual element method and uncover its superconvergence property. To recover the gradient on a general polygonal mesh, the most straightforward idea is to take simple averaging or weighted averaging. But we will encounter two difficulties: first, the data of gradient is not computable in the linear virtual element method; second, the consistency of the simple averaging or weighted averaging methods depends heavily on the symmetric of the local patches and they are inconsistent even on some uniform meshes. To overcome the first difficulty, one may simply replace the virtual element gradient by its polynomial projection. Then, we are able to apply the simple averaging or weighted averaging methods to then projected virtual element gradients. But we may be at risk of introducing some additional error and computational cost. Similarly, if we want to generalize SPR to general polygonal meshes, we also have those two difficulties. The second difficulty is more severe since there is no longer any local symmetric property for polygonal meshes. To tackle those difficulties, we generalize the idea of PPR [51] to the general polygons, which only uses the degrees of freedom and the consistency on arbitrary polygonal meshes is guaranteed by the polynomial preserving property. We prove the polynomial preserving and boundedness property of the generalized gradient recovery operator. Moreover, the superconvergence of the recovered gradient using the interpolation of the exact solution is theoretically justified. We also numerically uncover the superconvergent property of the linear virtual element methods. The recovered gradient is numerically proven to be more accurate than the virtual element gradient. The post-processing procedure also provides a way to visualize the gradient filed which is not directly available in the virtual element methods.

Adaptive methods are essential tools in scientific and engineer computing. Since the pioneering work of Babuška and Rheinboldt [4] in 1970s, there was a lot effort devoted to both the theoretical development of adaptive algorithms and applications of adaptive finite element methods. For classical finite element methods, adaptive finite element methods has reached a stage of mature, see the monograph [11, 18, 40, 48] and the references therein. For adaptive finite element methods, one of the key ingredient is the a posteriori error estimators. In the literature, there are two types a posteriori error estimators: residual-type and recovery-type.

For virtual element methods, there are only a few works concerning on the a
posteriori error estimates and adaptive algorithms. In [10], Da Veiga and Manzini derived a a posteriori error estimator for $C^1$ virtual element methods. In [15], Cangiani et al. proposed a a posteriori error estimator for the $C^0$ conforming virtual element methods for solving second order general elliptic equations. In [11], Berrone and Borio derived a new a posteriori error estimator for the $C^0$ conforming virtual element methods using the projection of the virtual element solution. In [34], Mora et al. conducted a posteriori error analysis for a virtual element method for the Steklov eigenvalue problems. All the above a posteriori error estimators are residual-type. To the best of our knowledge, there is no recovery-type a posteriori error estimators for virtual element methods.

The second purpose of this paper is to present a recovery-type a posteriori error estimator for the linear virtual element methods. But for the virtual element methods, there is no explicit formulation for the basis functions. To construct a fully computable a posteriori error estimator, we propose to use the gradient of polynomial projection of virtual element solution subtracting the polynomial projection of the recovered gradient. Comparing the existing residual-type a posteriori error estimators [10, 11, 15], the error estimator only has one term and hence it is much simpler. The error estimator is numerically proven to be asymptotically exact, which makes it more favorable than other a posteriori error estimators for virtual element methods.

The rest of the paper is organized as follows. In Section 2, we introduce the model problem and notation. In Section 3, we present the construction of virtual element space and the definition of discrete finite element space. In Section 4, we propose the gradient recovery procedure and prove the consistency and boundedness of the proposed gradient recovery operator. The recovery-based a posteriori error estimator is constructed in Section 5. In Section 6, the superconvergent property of the gradient recovery operator and the asymptotical exactness of the recovery-based error estimator is numerically verified. Some conclusion is drawn in Section 7.

2. Model problems. Let $\Omega \subset \mathbb{R}^2$ a bounded polygonal domain with Lipschitz boundary $\partial \Omega$. Throughout this paper, we adopt the standard notations for Sobolev spaces and their associate norms given in [13, 18, 23]. For a subdomain $D$ of $\Omega$, let $W^{k,p}(D)$ denote the Sobolev space with norm $\| \cdot \|_{k,p,D}$ and seminorm $| \cdot |_{k,p,D}$. When $p = 2$, $W^{k,2}(D)$ is simply denoted by $H^k(D)$ and the subscript $p$ is omitted in its associate norm and seminorm. $(\cdot, \cdot)_D$ denote the standard $L^2$ inner product on $D$ and the subscript is ignored when $D = \Omega$. Let $\mathbb{P}_m(D)$ be the space of polynomials of degree less than or equal to $m$ on $D$ and $n_m$ be the dimension of $\mathbb{P}_m(D)$ which equals to $\frac{1}{2}(m + 1)(m + 2)$.

Our model problem is the following Poisson equation

\begin{align}
-\Delta u &= f \text{ in } \Omega, \\
u &= 0 \text{ on } \partial \Omega.
\end{align}

The homogeneous Dirichlet boundary condition is considered for the sake of clarity. Inhomogeneous Dirichlet and other types boundary conditions apply as well without substantial modification.

Define the bilinear form $a(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ as

\begin{equation}
a(u, v) = (\nabla u, \nabla v),
\end{equation}

for any $u, v \in H^1(\Omega)$. It is easy to see that $|v|^2_{1, \Omega} = a(v, v)$ and $| \cdot |_{1, \Omega}$ is a norm on $H^1_0(\Omega)$ by the Poincaré inequality.
The variational formulation of (2.1) and (2.2) is to find $u \in H^1_0(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall v \in H^1_0(\Omega).$$

(2.4)

Lax-Milgram theorem implies it admits a unique solution.

3. Virtual element method. Let $\mathcal{T}_h$ be a partition of $\Omega$ into non-overlapping polygonal elements $E$ with non self-intersecting polygonal boundaries. Let $h$ be the maximum diameter.

In this paper, we concentrate on the lowest order virtual method [6]. To define the virtual element space, we begin with defining local virtual element spaces on each element. For such purpose, let

$$\mathbb{B}(\partial E) := \{ v \in C^0(\partial E) : v|_e \in P_1(e), \quad \forall e \in \partial E \}. \quad (3.1)$$

Then, the local virtual element space $V^E$ on the element $E$ can be defined as

$$V(E) = \{ v \in H^1(\Omega) : v|_{\partial E} \in \mathbb{B}(\partial E), \quad \Delta v|_E = 0 \}. \quad (3.2)$$

The virtual element space is

$$V_h = \{ v \in H^1(\Omega) : v|_E \in V(E) \quad \forall E \in \mathcal{T}_h \} \quad (3.3)$$

The degrees of freedom in $V_h$ are only the values of $v_h$ at all vertices. Furthermore, let $V_{h,0} = V_h \cap H^1_0(\Omega)$ be the subspace of $V_h$ with homogeneous boundary condition.

The soul of virtual element methods is that the non-polynomial basis functions are never explicit constructed and needed. This is made possible by introducing the projection operator $\Pi^\nabla$. For any function $v_h \in V(E)$, its projection $\Pi^\nabla v_h$ is defined to satisfy the following orthogonality:

$$(\nabla p, \nabla (\Pi^\nabla v_h - v_h))_E = 0, \quad \forall p \in P_1(E). \quad (3.4)$$

Similarly, we can define the $L^2$ projection $\Pi^0$ as

$$(p, \Pi^0 v_h - v_h)_E = 0, \quad \forall p \in P_1(E). \quad (3.5)$$

For the linear virtual element method, those two projection are equivalent, i.e. $\Pi^\nabla = \Pi^0$. In the subsequence, we will make no distinction between two projections.

On each element $E \in \mathcal{T}_h$, we can define the following discrete bilinear form

$$a^E_h(u_h, v_h) = (\nabla \Pi^\nabla u_h, \nabla \Pi^\nabla v_h)_E + S^E(u_h - \Pi^\nabla u_h, v_h - \Pi^\nabla v_h) \quad (3.6)$$

for any $u_h, v_h \in V(E)$. The discrete bilinear form $S^E$ is symmetric and positive which is fully computable using only the degrees of freedom of $u_h$. The readers are referred to [3, 7] for the detail definition of $S^E$.

Then, we can define the discrete bilinear $a_h(\cdot, \cdot)$

$$a_h(u_h, v_h) = \sum_{E \in \mathcal{T}_h} a^E_h(u_h, v_h), \quad (3.7)$$

for any $u_h, v_h \in V_h$. The linear virtual element method for the model problem (2.1) is to find $u_h \in V_{h,0}$ such that

$$a_h(u_h, v_h) = (f, \Pi^0 v_h), \quad \forall v_h \in V_{h,0}. \quad (3.8)$$
4. Superconvergent Gradient recovery. In the section, we present a high-accuracy and efficient post-processing technique for the virtual element methods. Our idea is to generalize the polynomial preserving recovery [51] to general polygonal meshes. The generalized method works for a large class of numerical methods based on polygonal meshes including mimetic finite finite difference methods [9,41], polygonal finite element methods [44], and virtual element methods [6,8]. To illustrate the main idea, we use the virtual element methods as an example to present the proposed algorithm.

We focus on the linear virtual element methods. Let $V_h$ be the linear virtual element space on a general polygonal mesh $T_h$ as defined in the previous section. The set of all vertices and of all edges of the polygonal mesh $T_h$ is denoted by $N_h$ and $E_h$, respectively. Let $I_h$ be the index set of $N_h$.

The proposed gradient recovery is formed in three steps: (1) construct local patches of elements; (2) conduct local recovery procedure; (3) formulate the recovered data in a global expression.

To construct a local patch, we first construct a union of mesh elements around a vertex. For each vertex $z_i \in N_h$ and nonnegative integer $n \in \mathbb{N}$, define $L(z_i, n)$ as

$$L(z_i, n) = \begin{cases} z_i, & \text{if } n = 0, \\ \bigcup \{E : E \in T_h, \tau \cap L(z_i, 0) \neq \emptyset\}, & \text{if } n = 1, \\ \bigcup \{E : E \in T_h, \tau \cap L(z_i, n-1) \text{ is an edge in } E_h\}, & \text{if } n \geq 2. \end{cases}$$

(4.1)

It is easy to see that $L(z_i, n)$ consists of the mesh elements in the first $n$ layers around the vertex $z_i$. Then $\Omega_{z_i} = L(z_i, n_i)$ with $n_i$ be the smallest integer such that $L(z_i)$ satisfies the rank condition in the following sense

Definition 4.1. A local patch $\Omega_{z_i}$ is said to satisfy the rank condition if it admits a unique least-squares fitted polynomial $p_{z_i}$ in (4.2).

Remark 4.2. For virtual element methods, we are more interested in the case that $T_h$ consists of polygons with more than four vertices. In general, to guarantee the rank condition we need $n_i = 1$ for interior vertices and $n_i = 2$ for boundary vertices.

Remark 4.3. For boundary vertices, there are alternative ways to construct the local patch satisfying the rank condition in Definition 4.1. The readers are referred to [27].

To reconstruct the recovered gradient at a given vertex $z_i$, let $B_{z_i}$ be the set of vertices in $\Omega_{z_i}$ and $I_i$ be the indexes of the $B_{z_i}$. Using the vertices in $B_{z_i}$ as sampling points, we fit a quadratic polynomial $p_{z_i}$ at the vertex $z_i$ in the following least-squares sense

$$p_{z_i}(z) = \arg \min_{p \in \mathbb{P}_2(\Omega_{z_i})} \sum_{j \in I_i} |p(z_{i_j}) - u_{h,j}|^2.$$  

(4.2)

To avoid numerical instability in real numerical computation, let

$$h_i = \max \{|z_{ik} - z_{ij}| : i_k, i_j \in I_i\},$$

and define the local coordinate transform

$$F : (x, y) \rightarrow (\xi, \eta) = \frac{(x, y) - (x_i, y_i)}{h_i},$$

(4.3)

where $z = (x, y)$ and $\hat{z} = (\xi, \eta)$. All the computations are performed at the reference local element patch $\Omega_{z_i} = F(\Omega_{z_i})$. Then we can rewrite $p_{z_i}(z)$ as

$$p_{z_i}(z) = \hat{p}^T \hat{a} = \hat{p}^T \hat{a},$$

(4.4)
where
\[ \mathbf{p}^T = (1, x, y, x^2, xy, y^2), \quad \mathbf{p}^T = (1, \xi, \eta, \xi^2, \xi \eta, \eta^2), \]
\[ \mathbf{a} = (a_1, a_2, a_3, a_4, a_5, a_6), \quad \hat{\mathbf{a}} = (a_1, h_1 a_2, h_1 a_3, h_1^2 a_4, h_1^2 a_5, h_1^2 a_6). \]

Let \( \hat{z}_{ij} = F(z_{ij}) \). The coefficient \( \hat{a} \) is determined by solving the linear system
\[ (\hat{A}^T \hat{A}) \hat{a} = \hat{A}^T \mathbf{b}, \tag{4.5} \]

where
\[ \hat{A} = \begin{pmatrix} 1 & \xi_{i2} & \eta_{i2} & \xi_{i2}^2 & \xi_{i2} \eta_{i2} & \eta_{i2}^2 \\ 1 & \xi_{i2} & \eta_{i2} & \xi_{i2}^2 & \xi_{i2} \eta_{i2} & \eta_{i2}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_{I_i} & \eta_{I_i} & \xi_{I_i}^2 & \xi_{I_i} \eta_{I_i} & \eta_{I_i}^2 \end{pmatrix} \quad \text{and} \quad \mathbf{b}^T = \begin{pmatrix} (u_{h})_{i1} \\ \vdots \\ (u_{h})_{I_i} \end{pmatrix} \]
with \( |I_i| \) being the cardinality of the set \( I_i \).

**Remark 4.4.** As observed in [27], the least-squares fitting procedure will not improve the accuracy of the solution approximation. We can remove one degree of freedom in the least-squares fitting procedure by assuming
\[ \hat{p}_{z_i}(z) = u_{h,i} + \hat{a}_2 x + \hat{a}_3 y + \hat{a}_4 x^2 + \hat{a}_5 x y + \hat{a}_6 y^2. \]

To determine \( \hat{a} = (\hat{a}_2, \hat{a}_3, \cdots, \hat{a}_6)^T \), we only need to solve a 5 \times 5 linear system instead of a 6 \times 6 linear system.

Then the recovered gradient \( G_h u_h \) at the vertex \( z_i \) is defined as
\[ G_h u_h(z_i) = \nabla p_i(z_i) = \frac{1}{h_i} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}. \tag{4.6} \]

Once we obtain \( G_h u_h(z_i) \) for each \( i \in I_h \), the global recovered gradient can be interpolated as
\[ G_h u_h = \sum_{i \in I_h} G_h u_h(z_i) \phi_i \tag{4.7} \]

The recovery procedure is summarized in Algorithm 1. From Algorithm 1, we can clearly observe that we actually only use the information of degrees of freedom available in the linear virtual element method.

For theoretical analysis purpose, we can also treat \( G_h \) as an operator from \( V_h \) to \( V_h \times V_h \). It is easy to see that \( G_h \) is a linear operator.

To show the consistency of the gradient recovery operator, we begin with the following theorem:

**Theorem 4.5.** If \( u \) is a quadratic polynomial on \( \Omega_{z_i} \), then \( G_h u(z_i) = \nabla u(z_i) \) for each \( i \in I_h \).

**Proof.** By the definition of \( \nabla p_i(z_i) \), we only need to show
\[ \nabla p_i(z_i) = \nabla u(z_i) \tag{4.8} \]
for all \( u \in \mathbb{P}(\Omega_{z_i}) \). For easing the presentation, we consider the least-squares fittings on the domain \( \Omega_{z_i} \) instead of the local referred domain \( \Omega_{z_i} \). Suppose \{\( q_1(z), q_2(z), \cdots, q_6(z) \}\)
Algorithm 1 Superconvergent Gradient Recovery Procedure
Let polygonal mesh $T_h$ and the data (FEM solutions) $(u_{h,i})_{i \in I_h}$ be given. Then repeat steps (1) – (3) for all $i \in I_h$.

(1) For every $z_i$, construct a local patch of elements $\Omega_{z_i}$. Let $B_{z_i}$ be the set of vertices in $\Omega_{z_i}$, and $I_i$ be the indexes of the $B_{z_i}$.
(2) Construct reference local patch $\hat{\Omega}_{z_i}$ and reference set of vertices $\hat{B}_{z_i}$.
(3) Find a polynomial $\hat{p}_{z_i}$ over $\hat{\Omega}_{z_i}$ by solving the least squares problem

$$
\hat{p}_{z_i} = \arg \min_{\hat{p}} \sum_{j \in I_i} |\hat{p}(\hat{z}_i) - u_{h,j}|^2 \text{ for } p \in \mathbb{P}^2(\hat{\Omega}_{z_i}).
$$

(4) Calculate the partial derivatives of the approximated polynomial functions, then we have the recovered gradient at each vertex $z_i$

$$
G_h u_h(z_i) = \nabla p_{z_i}(z_i) = \frac{1}{h_i} \nabla \hat{p}_{z_i}(0,0).
$$

For the recovery of the gradient $G_h u_h$ on the whole domain $\Omega$, we propose to interpolate the values $G_h u_h(z_i)_{i \in I_h}$ by using the standard linear interpolation of the virtual element method.

is the monomial basis of $\mathbb{P}_2(\Omega_{z_i})$ and let $p = (q_1(z), q_2(z), \cdots, q_6(z))$. Then $p_{z_i}(z) = p^T a$ where $a$ is determined by the linear system

$$
A^T a = A^T b. \quad (4.9)
$$

To prove the polynomial preserving property, it is sufficient to show the equation (4.8) is true when $u = q_j(z)$ for $j = 1, 2, \cdots, 6$. Let $u = q_j(z)$. Then it implies that

$$
b^T = ((q_j(z_{i_1}))(q_j(z_{i_2}))(q_j(z_{i_3})) \cdots (q_j(z_{i_{|I_i|}}))). \quad (4.10)
$$

According to (4.9), it is easy to see that $A e_j = b$ where $e_j$ is the $j$th canonical basis function in $\mathbb{R}^6$. It implies that $e_j$ is the unique solution to the linear system (4.5). From (4.4), we can see that $p_{z_i}(z) = p(z)^T e_j = q_j(z)$ and hence $\nabla p_{z_i}(z_i) = \nabla u(z_i)$. Thus, for the quadratic polynomial $u$, we have $G_h u(z_i) = \nabla u(z_i)$.

Theorem 4.5 means $G_h$ preserves quadratic polynomials at $z_i$. Using the polynomial preserving property above, we can show the following Lemma:

**Lemma 4.6.** Suppose $u_h \in V_h$, then we have

$$
|G_h u_h(z_i)| \lesssim h^{-1} |u_h|_{1, \Omega_{z_i}}. \quad (4.11)
$$

**Proof.** According to (4.5) and (4.6), the recovered gradient $G_h u_h(z_i)$ can be expressed as

$$
G_h u_h(z_i) = \left(\begin{array}{c} G_{z_i}^T u_h(z_i) \\ G_{z_i}^h u_h(z_i) \end{array}\right) = \frac{1}{h_i} \left(\begin{array}{c} \sum_{j=1}^{|I_i|} c_{z_i}^1 u_{h,i} \\ \sum_{j=1}^{|I_i|} c_{z_i}^2 u_{h,i} \end{array}\right), \quad (4.12)
$$

where $c_{z_i}^k$ is independent of the mesh size. Setting $u \equiv u_{h,i}$ in Theorem 4.5 means

$$
G_h u_h(z_i) = \left(\begin{array}{c} 0 \\ 0 \end{array}\right), \quad (4.13)
$$
Combining the above two equations, we have

\[
G_h u_h(z_i) = \frac{1}{h_i} \left( \sum_{j=1}^{n_i} c_j^1 (u_{h,i_j} - u_{h,i}) \right). \tag{4.14}
\]

For any \( z_{ij} \), we can find \( z_i = z_{j_1}, z_{j_2}, \ldots, z_{j_{n_i}} = z_{ij} \) such that the line segment \( z_j z_{j+1} = e_{j\ell} \) is an edge of an element \( E \in \Omega_z \). Then we can rewrite \( G_h u_h(z_i) \) as

\[
G_h^\ell u_h(z_i) = \sum_{j=1}^{n_i} c_j^1 \sum_{\ell=1}^{n_j-1} \frac{u_{h,j\ell+1} - u_{h,j\ell}}{h_i}. \tag{4.15}
\]

Note that \( u_h \) is virtual element function. Then we have \( u_h|_{e_{j\ell}} \) is a linear polynomial and hence it holds that

\[
\frac{u_{h,j\ell+1} - u_{h,j\ell}}{|e_{j\ell}|} = \frac{\partial u_h}{\partial t_{j\ell}} \leq |\nabla u_h|_{0,\infty,e_{j\ell}}, \tag{4.16}
\]

where \( t_{j\ell} \) is the unit vector in the direction from \( z_{j\ell} \) to \( z_{j\ell+1} \). Substituting (4.16) into (4.15) gives

\[
|G_h^\ell u_h(z_i)| = \sum_{E \in \Omega_z} \sum_{e \in E} \frac{|e|}{h_i} |\nabla u_h|_{0,\infty,e}. \tag{4.17}
\]

Since \( u_h|_e \) is a linear polynomial, the inverse inequality \[13, 18\] is applicable, which implies

\[
|\nabla u_h|_{0,\infty,e} \lesssim |e|^{-\frac{3}{2}} |u_h|_{0,e}. \tag{4.18}
\]

By the trace inequality, we have

\[
|u_h|_{0,e} \lesssim h^{-\frac{1}{2}} |\nabla u_h|_{0,E} + h^{\frac{1}{2}} |\nabla u_h|_{1,E}. \tag{4.19}
\]

Combining the above estimates and noticing that \( \frac{|e|}{h_i} \) is bounded by a fixed constant, we have

\[
|G_h^\ell u_h(z_i)| \lesssim \sum_{E \in \Omega_z} \left( h^{-2} |\nabla u_h|_{0,E} + h^{-1} |\nabla u_h|_{1,E} \right) \lesssim h^{-2} |u_h|_{0,\Omega_z} + h^{-1} |u_h|_{1,\Omega_z}.
\]

Let \( \bar{u}_h = \frac{1}{|\Omega_z|} \int_{\Omega_z} u_h dz \). Setting \( u_h \equiv \bar{u}_h \) in Theorem 4.5 implies \( G_h \bar{u}_h(z_i) = (0,0)^T \). Replacing \( u_h \) by \( u_h - \bar{u}_h \) in the above estimate, we have

\[
|G_h^\ell u_h(z_i)| \leq |G_h^\ell (u_h - \bar{u}_h,E)(z_i)| \leq h^{-2} |u_h - \bar{u}_h,E|_{0,\Omega_z} + h^{-1} |u_h - \bar{u}_h,E|_{1,\Omega_z} \lesssim h^{-1} |u_h|_{1,\Omega_z},
\]

where we have used the scaled Poincaré-Friedrichs inequality in \[12\].
Similarly, we can establish the same error bound for $G_h u_h(z_i)$. Thus, the estimate (4.11) is true. \hfill \square

Based on the above lemma, we establish the local boundedness in $L_2$ norm:

**Theorem 4.7.** Suppose $u_h \in V_h$, then for any $E \in \mathcal{T}_h$, we have

$$
||G_h u_h||_{0,E} \lesssim |u_h|_{1,\Omega_E},
$$

(4.20)

where $\Omega_E = \bigcup_{z \in E \cap \mathcal{N}_h} \Omega_z$.

**Proof.** Let $I_E$ be the index set of $E \cap \mathcal{N}_h$. Since $\{\phi_i\}_{i \in I_h}$ is the canonical basis for $V_h$, we have

$$
||G_h u_h||_{0,E} \lesssim |E|^\frac{1}{2} \sum_{j=1}^{|I_E|} |G_h u_h(z_i)|
$$

$$
\lesssim \sum_{j=1}^{|I_E|} |E|^\frac{1}{2} h^{-1} |\nabla u_h|_{1,\Omega_{z_i}}
$$

$$
\lesssim \sum_{j=1}^{|I_E|} |\nabla u_h|_{1,\Omega_{z_i}}
$$

$$
\lesssim |u_h|_{1,\Omega_E},
$$

where we have used the fact that $|E|^\frac{1}{2} h^{-1}$ is bounded by a fixed constant. \hfill \square

As a direct consequence, we can prove the following corollary

**Corollary 4.8.** Suppose $u_h \in V_h$, then we have

$$
||G_h u_h||_{0,\Omega} \lesssim |u_h|_{1,\Omega}.
$$

(4.21)

Corollary 4.8 implies that $G_h$ is a linear bounded operator from $V_h$ to $V_h \times V_h$. Now, we are in the perfect position to present the consistency result of $G_h$.

**Theorem 4.9.** Suppose $u \in H^3(\Omega_E)$, then we have

$$
||G_h u - \nabla u||_{0,E} \lesssim h^2 ||u||_{3,\Omega_E}.
$$

**Proof.** Define

$$
\mathcal{F}(u) = ||G_h u - \nabla u||_{0,E}.
$$

By the boundedness of $G_h$, it is easy to see that

$$
\mathcal{F}(u) \leq ||G_h u||_{0,E} + ||\nabla u||_{0,E}
$$

$$
\lesssim |u|_{1,\Omega_E}.
$$

The polynomial property of the gradient recovery operator $G_h$ implies $G_h p = \nabla p$ for any $p \in P_2(\Omega_E)$. Thus we have

$$
\mathcal{F}(u + p) = \mathcal{F}(u).
$$

By the Brambler-Hilbert Lemma \cite{13,18}, we obtain

$$
\mathcal{F}(u) \lesssim h^2 ||u||_{3,\Omega_E}.
$$

\hfill \square

**Theorem 4.9** implies the gradient recovery operator is consistent in the sense that the recovered gradient using the exact solution is superconvergent to the exact gradient at rate of $O(h^2)$.  

9
5. Adaptive virtual element method. The adaptive virtual element method is just a loop of the following steps:

\[ \text{Solve} \rightarrow \text{Estimate} \rightarrow \text{Mark} \rightarrow \text{Refine} \]  \hspace{2cm} (5.1)

Starting from an initial polygonal mesh, we solve the equation by using the linear virtual element method. Once the virtual element solution is available, we need to design a fully computational a posteriori error estimator using only the virtual element solution. This step is vital for the adaptive virtual element method because it determines the performance of the adaptive algorithm. In this paper, we introduce a recovery-based a posteriori error estimator using the proposed gradient recovery method, which we elaborate in the coming subsection.

5.1. Recovery-based a posteriori error estimator. Once the recovered gradient is reconstructed, we are ready to present the recovery-type a posteriori error estimator for virtual element methods. However, for virtual element methods, their basis functions are not explicitly constructed which means that both $G_h u_h$ and $\nabla u_h$ are not computable quantities. To overcome the difficulty, we propose to use $\Pi_E^0 G_h u_h$ and $\nabla \Pi_E^0 u_h$, which are computable. We define a local a posteriori error estimator on each triangular element $\tau_{h,j}$ as

\[ \eta_{h,E} = \| \Pi_E^0 G_h u_h - \nabla \Pi_E^0 u_h \|_{0,E}, \]  \hspace{2cm} (5.2)

and the corresponding global error estimator as

\[ \eta_h = \left( \sum_{E \in T_h} \eta_{h,E}^2 \right)^{1/2}. \]  \hspace{2cm} (5.3)

To measure the performance of the a posteriori error estimator (5.2) or (5.3), we introduce the effective index

\[ \kappa_h = \frac{\| \Pi_E^0 G_h u_h - \nabla \Pi_E^0 u_h \|_{0,\Omega}}{\| \nabla u - \nabla \Pi_E^0 u_h \|_{0,\Omega}}, \]  \hspace{2cm} (5.4)

which are computable when the exact solution $u$ is provided.

For a posteriori error estimators, the ideal case we expect is the so-called asymptotical exactness.

**Definition 5.1.** The a posteriori error estimator (5.2) or (5.3) is said to be asymptotically exact if

\[ \lim_{h \to 0} \kappa_h = 1. \]  \hspace{2cm} (5.5)

A series of benchmark numerical examples in the next section indicate the recovery-based a posteriori error estimator (5.2) or (5.3) is asymptotically exact for the linear virtual element methods, which distinguish it from the residual-type a posteriori error estimators for virtual element methods in the literature $[10, 11, 15]$.

5.2. Marking Strategy. When a posteriori error estimator (5.2) is available, we pick up a set elements to be refined. This process is called marking. There are several different marking strategy. In this paper, we adopt the bulk marking strategy...
proposed by Dörfler \cite{22}. Given a constant $\theta \in (0, 1]$, the bulk strategy is to find $\mathcal{M}_h \subset \mathcal{T}_h$ such that
\begin{equation}
\left( \sum_{E \in \mathcal{M}_h} \eta^2_{h,E} \right)^{\frac{1}{2}} \leq \theta \left( \sum_{E \in \mathcal{T}_h} \eta^2_{h,E} \right)^{\frac{1}{2}} \quad (5.6)
\end{equation}
In general, the choice of $\mathcal{M}_h$ is not unique. We select $\mathcal{M}_h$ such that the cardinality of $\mathcal{M}_h$ is smallest.

5.3. Adaptive mesh refinement. One of the main advantages of virtual element methods is its flexibility in local mesh refinement. Virtual element methods allow that elements have arbitrary number of edges and two edges are collinear. Those advantages enable the virtual element methods to naturally handle hanging nodes. A polygon with a hanging node is just a polygon has an extra edge collinear with another edge. It avoids artificial refinement of the unmarked neighborhood in the classical adaptive finite element methods. Take the polygon in Figure 5.1 as example. It is a pentagon with five vertices $V_1, V_2, \cdots, V_5$. But there are three hanging nodes $V_6, V_7, V_8$ which are generated by the refinement of its neighborhood element. In the virtual element method, we can treat the pentagon with three hanging nodes as an octagon with eight vertices $V_1, V_2, \cdots, V_8$. Note that in the octagon, there are four edges are collinear which is allowed in the virtual element method.

In the paper, we adopt the same way to refine a polygon as in \cite{15, 47}. We divide a polygon into several sub-polygons by connecting its barycenter to each planar edge center. Note that two or more edge collinear to each other is treated as one planar edge. We take the polygon in Figure 5.1 as example again. The refinement of the polygonal is illustrated in the Figure 5.1 by the dashed lines. Note that the four edges are collinear to each other is viewed as one edge $V_5V_1$. Thus, in the refinement, we bisect $V_5V_1$ instead the four collinear edges.

6. Numerical Results. In this section, we present several numerical examples to demonstrate our numerical discovery. The first example is to illustrate the superconvergence of the proposed gradient recovery. The other examples are to numerical validate the asymptotical exactness of the recovery-based \textit{a posteriori} error estimator.

In the virtual element method, the basis functions are never explicitly constructed and the numerical solution is unknown inside elements. In the computational test, we shall use the the projection $\Pi_h u_h$ to compute different errors instead of using $u_h$. In addition, all the convergence rates are illustrated in term of the degrees of freedom (DOF). In two dimensional cases, $\text{DOF} \approx h^2$ and the corresponding convergence rates in term of the mesh size $h$ are doubled of what we plot in the graphs.

6.1. Test Case 1: Smooth problem. In this example, we consider the following homogeneous elliptic equation
\begin{equation}
- \Delta u = 2\pi^2 \sin(\pi x) \sin(\pi y), \quad \text{in } \Omega = (0, 1) \times (0, 1). \quad (6.1)
\end{equation}
The exact solution is $u(x, y) = \sin(\pi x) \sin(\pi y)$.

In this test, we adopt six different types of meshes to numerically show the superconvergence of the proposed gradient recovery method. The first level of each type meshes is plotted in Fig. 6.1. The first type meshes $\mathcal{T}_{h,1}$ are just the uniform square meshes. The second type meshes $\mathcal{T}_{h,2}$ are uniform hexagonal meshes. The third type meshes $\mathcal{T}_{h,3}$ are uniform nonconvex meshes. $\mathcal{T}_{h,4}$ are generated by adding
Fig. 5.1: Illustration of handling hanging note in the virtual element method and local refinement of polygonals with collinear edges

Table 6.1: Characteristics of six different types of meshes

| Mesh Type | Unstructured mesh | Concave element | Quadrilateral element | Random perturbation | Obtuse angle |
|-----------|-------------------|-----------------|------------------------|-------------------|-------------|
| $T_{h,1}$ | no                | no              | yes                    | no                | no          |
| $T_{h,2}$ | no                | no              | yes                    | no                | yes         |
| $T_{h,3}$ | no                | yes             | no                     | no                | yes         |
| $T_{h,4}$ | yes               | no              | yes                    | yes               | yes         |
| $T_{h,5}$ | yes               | no              | no                     | no                | yes         |
| $T_{h,6}$ | yes               | no              | no                     | no                | yes         |

random perturbation to the meshes $T_{h,1}$. The fifth type meshes $T_{h,5}$ are generated by applying the following coordinate transform

$$x = \hat{x} + \frac{1}{10} \sin(2\pi \hat{x}) \sin(2\pi \hat{y}),$$

$$y = \hat{y} + \frac{1}{10} \sin(2\pi \hat{x}) \sin(2\pi \hat{y});$$

to the uniform hexagonal meshes $T_{h,2}$. The sixth type meshes $T_{h,6}$ are smoothed Voronoi meshes generated by Polymesher [46]. The characterization of the six different types meshes are summarized in Table 6.1.

In addition to the discrete $H_1$ semi-error $\| \nabla u - \nabla \Pi_h^0 u_h \|_{0,\Omega}$ and the recovered error $\| \nabla u - \Pi_h^0 G_h u_h \|_{0,\Omega}$, we also consider the $\| \nabla u_h - \nabla u_I \|_{0,\Omega} = \sqrt{(u_h - u_I)^T A_h (u_h - u_I)}$ where $A_h$ is the stiffness matrix, $u_I$ is the interpolation of $u$ into the virtual element space $V_h$, and $u_h$ (or $u_I$) is a vector of value of $u_h$ (or $u_I$) on the degrees of freedom.
The error $\|\nabla u_h - \nabla u_I\|_{0,\Omega}$ plays an important role in the study of superconvergence for gradient recovery methods in the classical finite element methods [5,50]. We say the gradient of the numerical solution is superclose to the gradient of the interpolation of the exact solution if $\|\nabla u_h - \nabla u_I\|_{0,\Omega} \lesssim O(h^{1+\rho})$ for some $0 < \rho \leq 1$. The supercloseness result is a sufficient but not necessary condition to prove the superconvergence of gradient recovery methods [5,24,50,51].

We plot the rates of convergence for the above three difference errors in Fig. 6.2. As predicted in [6], the discrete $H_1$ semi-error decays at rate of $O(h)$ for all the above six different types of meshes. Concerning the the error $\|\nabla u_h - \nabla u_I\|_{0,\Omega}$, we can only observe the $O(h^2)$ supercloseness on two structured convex meshes and the transformed meshes $T_h$. It is not surprise since the supercloseness depend heavily on the symmetric of meshes even on triangular mesh, see [5,50]. But the recovered gradient is superconvergent to the exact gradient at rate of $O(h^2)$ on all the above meshed including the unstructured meshes and concave meshes. The above numerical observation is summarized in Table 6.2.

6.2. Test Case 2: L-shape domain problem. In this example, we consider the Laplace equation

$$-\Delta u = 0,$$

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Fig. 6.2: Sample erres for numerical tests: (a) on structured quadrilateral meshes; (b) on structured hexagonal meshes; (c) on structure concave meshes; (d) on unstructured quadrilateral meshes; (e) on unstructured hexagonal meshes; (f) on unstructured Voronoi meshes.
Table 6.2: Summary of Numerical Results on the six different types of meshes

| Mesh Type | $\|\nabla u - \Pi^0 \nabla u_h\|_{0, \Omega}$ | $\|\nabla u_h - \nabla u_I\|_{0, \Omega}$ | $\|\nabla u - \Pi^E G_h u_h\|_{0, \Omega}$ |
|-----------|---------------------------------|---------------------------------|---------------------------------|
| $T_{h,1}$ | $O(h)$                          | $O(h^2)$                        | $O(h^2)$                        |
| $T_{h,2}$ | $O(h)$                          | $O(h^2)$                        | $O(h^2)$                        |
| $T_{h,3}$ | $O(h)$                          | $O(h)$                          | $O(h^4)$                        |
| $T_{h,4}$ | $O(h)$                          | $O(h)$                          | $O(h^4)$                        |
| $T_{h,5}$ | $O(h)$                          | $O(h^2)$                        | $O(h^2)$                        |
| $T_{h,6}$ | $O(h)$                          | $O(h)$                          | $O(h^2)$                        |

Fig. 6.3: Meshes for Test Case 2: (a) Initial mesh; (b) Adaptively refined mesh.

on the L-shaped domain $\Omega = (-1, 1) \times (-1, 1) \setminus (0, 1) \times (-1, 0)$. The exact solution is $u = r^{2/3} \sin(2\theta/3)$ in polar coordinate. The corresponding boundary condition is computed from the exact solution $u$. Note the exact solution $u$ has a singularity at the original.

To resolve the singularity, we use the adaptive virtual element method described in Section 5. The initial mesh is plotted in Fig. 6.3a which a uniform meshes consisting of square elements. In Fig. 6.3b we show the corresponding adaptively refined mesh. It is not hard to see that the refinement is conducted near the singular point.

In Fig. 6.4a we graph the rates of convergence for discrete $H_1$ semi-error $\|\nabla u - \nabla \Pi_h^0 u_h\|_{0, \Omega}$ and the discrete recovery error $\|\nabla u - \Pi^E G_h u_h\|_{0, \Omega}$. From the plot, we can clearly observe $O(h)$ optimal convergence for the virtual element gradient and $O(h^2)$ superconvergence for the recovered gradient for the adaptive virtual element method. It means the recovery-based a posteriori error estimator (5.2) is robust. To quantify the performance the error estimator, we draw the effective index (5.4) in Fig. 6.4b. It shows that the effective index converges to 1 rapidly after a few iterations. It means the a posteriori error estimator is asymptotically exact as defined in Definition 5.1.
6.3. Test Case 3: Problem with two Gaussian surfaces. Consider the Poisson equation \[2.1\] on the unit square with the exact solution

\[u(x, y) = \frac{1}{2\pi\sigma} \left[ e^{-\frac{1}{2} \left( \frac{x-\mu_1}{\sigma} \right)^2} e^{-\frac{1}{2} \left( \frac{y-\mu_1}{\sigma} \right)^2} + e^{-\frac{1}{2} \left( \frac{x-\mu_2}{\sigma} \right)^2} e^{-\frac{1}{2} \left( \frac{y-\mu_2}{\sigma} \right)^2} \right]\]

as in \[47\]. In this test, the standard deviation is \(\sigma = \sqrt{10^{-3}}\) and the two means are \(\mu_1 = 0.25\) and \(\mu_2 = 0.75\).

The difficulty of this problem is the existence of two Gaussian surfaces, where the
Fig. 6.6: Numerical result for Test Case 3: (a) Numerical errors; (b) Effective index.

Fig. 6.7: Meshes for Test Case 4: (a) Initial mesh; (b) Adaptively refined mesh.

doesn’t resolve the Gaussian surfaces. In Fig. 6.5b, we show the adaptively refined mesh. Clearly, the mesh is refined near the location of the Gaussian surfaces. In Fig. 6.6a, we present the numerical errors. Similar to Test case 2, we can observe the desired optimal and superconvergent results. Moreover, the asymptotical exactness of the error estimator (5.2) is numerically verified in Fig. 6.6b by the fact the effective index is convergent to one.
6.4. Test Case 4: Problem with sharp interior layer. As in [10, 15], we consider the Poisson equation (2.1) and (2.2) on the unit square with a sharp interior layer. The exact solution is

$$u(x, y) = 16x(1 - x)y(1 - y) \arctan(25x - 100y + 25).$$

The initial mesh is the transformed hexagonal mesh $\mathcal{T}_{h, 5}$ as in Test Case 1, which is shown in Fig. 6.7a. It is a unstructured polygonal mesh. The interior sharp layer is totally unresolved by the initial which causes the main difficulty. Fig. 6.7b is the mesh generated by the adaptive virtual element method prescribed in Section 5. It is obvious that the mesh is refined to resolve the interior layer as expected.

In Fig. 6.8, we present the qualitative results. As anticipated, the desired $O(h)$ optimal convergence rate for the virtual element gradient and $O(h^2)$ superconvergence rate for the recovered gradient can be numerically observed. Also, the effective index is numerically proved to be one, which validate the asymptotical exactness of the error estimator (5.2).

7. Conclusion. In this paper, a superconvergent gradient recovery method for the virtual element method is introduced. The proposed post-processing technique uses only the degrees of freedom which is the only data directly obtained from the virtual element method. It generalizes the idea of polynomial preserving recovery [37,51] to general polygonal meshes. Theoretically, we prove the proposed gradient recovery method is bounded and consistent. It meets the standard of a good gradient recovery technique in [1]. Numerically, we validate the superconvergence of the recovered gradient using the virtual element solution on several different types of general polygonal meshes including concave and unstructured meshes. In the future, it would be interesting to present a theoretical proof of those superconvergence for the virtual element method.

Its capability of serving as a posteriori error estimator is also exploited. The asymptotical exactness of the recovery-based a posteriori error estimators is numer-
ically verified by three benchmark problems. To the best of our knowledge, it is the first recovery-based *a posteriori* error estimator for the virtual element methods. Compared to the existing residual type *a posteriori* error estimators, it has several advantages: First, it is simple in both the idea and implementation, which makes it is more realistic for practical problems; Second, the unique characterization of the error estimator is the asymptotical exactness, which prevails over all other *a posteriori* error estimators in the literature for the virtual element methods.

The application of gradient recovery is not limited to adaptive methods. It has also been applied to many other fields, like enhancing eigenvalues [26, 38, 39] and designing new numerical methods for higher order PDEs [17, 28, 29]. We will make use of those advantages of gradient recovery to study more interesting real application problems in the future work.

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