Topological magnon bands for magnonics

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Topological excitations in magnetically ordered systems have attracted much attention lately. We report on topological magnon bands in ferromagnetic Shastry-Sutherland lattices whose edge modes can be put to use in magnonic devices. The synergy of Dzyaloshinskii-Moriya interactions and geometrical frustration is responsible for the topologically non-trivial character. Using exact spin wave theory we determine the finite Chern numbers of the magnon bands which give rise to chiral edge states. The quadratic band crossing point vanishes due the present anisotropies and the system enters a topological phase. We calculate the thermal Hall conductivity as an experimental signature of the topological phase. Different promising compounds are discussed as possible physical realizations of ferromagnetic Shastry-Sutherland lattices hosting the required antisymmetric Dzyaloshinskii-Moriya interactions. Routes to applications in magnonics are pointed out.

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Topological phases\cite{1,2} exist in both fermionic and bosonic systems and constitute a fast developing research area. While the theoretical understanding of fermionic topological systems has made impressive progress, topological bosonic excitations have gained considerable attention only in the last few years. Despite the increasing conceptual knowledge of topological matter only very few materials have been identified with topological properties compared to the large number of potential topological materials\cite{3}. Even less is known about potential applications. This is in particular true for topological bosonic signatures\cite{4}. Thus, it is a major challenge to theoretically predict and experimentally verify topological bosonic fingerprints in order to move towards useful applications.

In the research of topological properties in condensed matter, the magnetic degrees of freedom have increased in importance. Magnetic data storage is already a ubiquitous everyday’s technology\cite{5}. Recently, magnetic spin waves, so-called magnons, themselves are used to carry and to process information which is called ‘magnonics’\cite{6-8}. Adding topological aspects the field of magnonics\cite{9} considerably enhances the possibilities to build efficient devices for which we will make a proposal in this Letter.

The challenge in finding topological signatures in magnetically ordered spin systems are the small Dzyaloshinskii-Moriya (DM) interactions\cite{10,11} which induce only small Berry curvatures. The size of the DM terms relative to the isotropic coupling is roughly as large as $|g-2|/2$, i.e., the deviation of the $g$-factor from 2, because both result from spin-orbit coupling. Thus, the DM terms are generically too small to induce detectable topological effects. In strongly frustrated systems, however, the relative size of the DM terms can indeed be comparable to the isotropic couplings\cite{12}.

Another issue is the localization of edge modes. If we employ the wording of semiconductor physics, one must distinguish direct (at given momentum) and indirect gaps (allowing for changes of momentum). The existence of direct gaps throughout the Brillouin zone (BZ) is sufficient to separate bands so that their topological properties are well-defined. But the vanishing of the indirect gap generically implies that the edge states are not localized anymore, i.e., the bulk-boundary correspondence does not hold anymore\cite{13}.

In magnetic systems, three types of elementary excitations occur. Long-range ordered magnets display magnons (or spin waves)\cite{14}, valence bond crystals mostly feature triplons\cite{15} while quantum spin liquids may display fractional excitations\cite{16} for instance spinons\cite{17}. For triplons, topological behavior, i.e., non-zero Chern numbers\cite{18}, has been predicted\cite{19,20} and verified\cite{21} in Shastry-Sutherland lattices and in spin ladders\cite{22}. For ferromagnetically ordered systems, topological magnons have been theoretically suggested in kagome lattices\cite{23,24}, pyrochlore lattices\cite{25} and in honeycomb lattices\cite{26}. For antiferromagnets, they have been proposed in pyrochlore lattices\cite{27} and square and cubic lattices exploiting the Aharonov-Casher effect\cite{9}. In analogy to the quantum Hall effect\cite{28}, the magnon Hall effect\cite{4} as well as the triplon Hall effect\cite{19} arise since the topological Berry curvature acts analogously to a magnetic field. So far, only the magnon Hall effect has been observed\cite{19,29}. Topologically non-trivial spinons have been discussed in Mott insulators\cite{30,31} as well as quantum spin liquids\cite{23,32,33}.

The Shastry-Sutherland model\cite{34} is commonly studied with antiferromagnetic couplings\cite{35}. Including DM interactions induces topological properties\cite{19,20}. A transverse magnetic field acts as control parameter to tune topological phase transition. In view of the verification of the magnon Hall effect in ferromagnets it is indicated to study the Shastry-Sutherland lattice also with purely ferromagnetic couplings. This is one of the two main objectives of this article; the second one is to discuss compounds which are likely to realize this model and to point out possible applications.

We show by exact spin wave theory that the ferromagnetic Shastry-Sutherland model with DM couplings has topological bands with non-trivial Chern numbers. The occurrence of a ferromagnetic ground state repre-
sents the spontaneous breaking of time-reversal symmetry. In combination with the DM interactions a finite Berry curvature is induced which may lead to finite Chern numbers. The degeneracy at the quadratic band crossing point (QBCP) is lifted and a gap opens. The expected numbers. The degeneracy at the quadratic band crossing Berry curvature is induced which may lead to finite Chern try. In combination with the DM interactions a finite presents the spontaneous breaking of time-reversal symmetry.

Real materials are always three-dimensional (3D); so we look for the ferromagnetic Shastry-Sutherland model realized in various layers of 3D materials. If the interlayer is not too strong the 3D quantum Hall states can be considered as ensemble of layered 2D quantum Hall states so that it is appropriate to investigate 2D models.

Layers of the Shastry-Sutherland lattice are found in various insulating magnetic materials, since it is easily constructed from corner-sharing squares. The squares are not aligned parallel or perpendicular to one other so that dimers are formed, see Fig. 1(a). Due to the lack of inversion symmetry about the midpoints of the bonds DM interactions are possible and generically occur from spin-orbit interactions. To reach large values of the DM couplings it is indicated to include atoms with large atomic number for large electron velocities favoring relativistic effects. Moreover, the couplings should be ferromagnetic so that it is indicated to avoid linear bonds which would allow for antiferromagnetic superexchange according to the Goodenough-Kanamori rules. Hence the Shastry-Sutherland lattice depicted in Fig. 1 appears promising if superexchange via larger subgroups does not occur (this is what happens in SrCu

Hence the Shastry-Sutherland lattice depicted in Fig. 1 shows the ferromagnetic Shastry-Sutherland model comprises the Heisenberg couplings \( J \) and \( J' \) as well as the DM interaction \( D_z \). The unit cell is highlighted in green. The sequence of the spin in the term \( D_z \cdot S_i \times S_j \) is shown by the arrow pointing from \( i \) to \( j \). The DM couplings follow a clockwise rotation, see circular arrows.

We compute the four bands from the unit cell with four sites shown in Fig. 1(b) in green. The DM couplings of the Shastry-Sutherland lattice can be directed in-plane or out-of-plane\(^{19}\). Usually, however, the out-of-plane couplings dominate in 2D\(^{15,40}\). In order to focus on a minimal model we thus constrain the DM coupling to a uniform direction perpendicular to the plane \( \mathbf{D} = D_z \hat{e}_z \) as shown in Fig. 1(b). Obviously, this introduces a chiral orientation. Single-ion anisotropy (SIA) \( A^{\alpha\beta} (\alpha, \beta \in \{x, y, z\}) \) is typically present in ferromagnets with spins \( S \gtrsim 1/2 \). For the minimal model, we consider it to favor easy-axis alignment along the \( z \)-axis \( A^{zz} = A \geq 0 \). The SIA and the DM coupling compete because the latter profits from tilts away from the \( z \)-axis. To estimate if the fully polarized state remains the ground state we study two next-nearest neighbor spins coupled by \( J' \) as classical vectors of length \( S \) with polar angles \( \theta_1 \) and \( \theta_2 \) and relative azimuthal angle \( \varphi := \varphi_1 - \varphi_2 \) which takes the value \( \tan \varphi = d := D_z/J' \) at the energy minimum \( E \)

\[
2E/(J'S^2) = -a(x+1) - x - y - |x-y|\sqrt{1+d^2} \tag{1}
\]

where \( a := A/J' \), \( x := \cos(\theta_1 + \theta_2) \), and \( y := \cos(\theta_1 - \theta_2) \) with \( |x|, |y| \leq 1 \). The SIA term \( a \) is split into four parts \( a/4 \) because each site has four \( J' \) bonds. As long as \( 1 + a/2 \geq \sqrt{1+d^2} \) full polarization is optimum, i.e., a canted state can occur for \( d \geq \sqrt{a+a^2/4} \) only, which is a conservative estimate because the effects of \( J \) of quantum fluctuations, and of the geometric constraints in the lattice are not included. Hence, for small SIA and DM coupling the SIA wins and the fully polarized state is generic.

The complete Hamiltonian of the minimal model consists of three parts

\[
\mathcal{H} = \mathcal{H}_H + \mathcal{H}_{DM} + \mathcal{H}_{SIA} \tag{2a}
\]

\[
\mathcal{H}_H = -J \sum_{\langle ij \rangle} \left[ \frac{1}{2} (S_i^+ S_j^- + S_i^- S_j^+) + S_i^z S_j^z \right]
- J' \sum_{\langle\langle ij \rangle\rangle} \left[ \frac{1}{2} (S_i^+ S_j^- + S_i^- S_j^+) + S_i^y S_j^y \right] \tag{2b}
\]

\[
\mathcal{H}_{DM} = -\frac{iD_z}{2} \sum_{\langle\langle ij \rangle\rangle} (S_i^+ S_j^- - S_i^- S_j^+) \tag{2c}
\]

\[
\mathcal{H}_{SIA} = -A \sum_i (S_i^z)^2, \tag{2d}
\]
with ferromagnetic couplings $J, J' > 0$; $J$ serves as the energy unit henceforth. A pair of nearest neighbors and of next-nearest neighbors is denoted by $(ij)$ and by $(ijj)$, respectively.

We use the Dyson-Maleev representation of the spin operators\cite{41,42} which is exact as long as a single magnon above the fully polarized ground state is considered. But even for several magnons spin wave theory is well justified due to the large spins involved ($S \approx 4 - 5$ for \{RE=Gd, Dy, Ho, Er, Y\}). The bilinear Hamiltonian in momentum space reads

$$\mathcal{H} = \sum_{\mathbf{k}} \sum_{nm} b_{n,k}^\dagger \mathcal{M}_{nm}(\mathbf{k}) b_{m,k}$$

where $b_{n,k}^\dagger$ and $b_{n,k}$ are the bosonic creation and annihilation operators at the site $n \in \{1, 2, 3, 4\}$, see Fig. 1(b). The $4 \times 4$ Hamiltonian matrix is given by

$$\mathcal{M}(\mathbf{k}) = \begin{pmatrix} A & B^\dagger(k_a, k_b) \\ B(k_a, k_b) & A \end{pmatrix}$$

with the $2 \times 2$ matrices

$$A = \begin{pmatrix} JS + 4J'S + A(2S - 1) & -JS \\ -JS & JS + 4J'S + A(2S - 1) \end{pmatrix}$$

$$B = \begin{pmatrix} C(1 + e^{ik_a}) & -C^* (e^{ik_a} + e^{ik_a+k_b}) \\ -C^* (1 + e^{ik_b}) & -C (e^{ik_b} + e^{ik_a+k_b}) \end{pmatrix}$$

where $C := S(J' + iD_z)$. Diagonalizing $\mathcal{M}(\mathbf{k})$ yields four distinct magnon bands $\mathcal{H} = \sum_{n,k} \omega_n(\mathbf{k}) b_{n,k}^\dagger b_{n,k}$ depicted in Fig. 2. The four bands come in pairs $p$ of two bands which are degenerate on the boundary of the BZ. We strongly presume that this degeneracy is linked to the composite point symmetry of the Shastry-Sutherland lattice which consists of a vertical or horizontal translation shifting vertical dimers to horizontal ones and vice versa combined with a rotation by $90^\circ$. But we did not find an analytic proof. The whole lattice is $C_4$ symmetric considering rotations about the centers of the squares so that dispersions display the same symmetry.

Ferromagnetic Heisenberg models without spin anisotropic couplings such as SIA or DM coupling display gapless Goldstone bosons\cite{43} with a quadratic dispersion at low energies at the $\Gamma$ point. As soon as the SIA $A$ is turned on the continuous spin rotation symmetry is no longer broken spontaneously, but externally and a finite spin gap $A(2S - 1)$ appears; note the offset energy axis in the lower panel of Fig. 2. This stabilizes the fully polarized ground state since it becomes energetically isolated from the remaining spectrum.

For vanishing DM coupling, two magnon bands cross quadratically at the $\Gamma$ point at finite energy. Hence the model displays an unusual QBCP. Linear Dirac cones\cite{44} or variants of them\cite{19} are more standard. Generically, one can assign a Berry phase of $\pi$ (or multiples of $\pi$) to them\cite{45}. The QBCP is stable and can be interpreted as a pair of Dirac cones\cite{46} which are superimposed due to the $C_4$ symmetry\cite{45}. As a result, a QBCP can have a Berry flux of $0$ or $\pm 2\pi$. The QBCP can either be removed by breaking the $C_4$ symmetry which splits it into an even number of Dirac cones or by lifting its degeneracy, e.g., by opening a gap leading to topologically non-trivial bands. Turning on the DM interaction ($D_z \neq 0$) induces the latter scenario. But as shown in Fig. 2 the degeneracy of the upper pair of bands and of the lower pair of bands at the boundary of the BZ persists so that no Chern number of a single band can be defined. Hence one defines the Chern number of subspaces by taking the trace over the Berry curvature in each subspace\cite{20,47} which derives from the Berry phase of the determinants of unitary transformations along closed paths\cite{45}. Denoting the Chern number of a pair of bands by $C(p)$ where $p$ stands for ‘upper’ or ‘lower’ one has

$$C(p) = \frac{1}{2\pi} \int_{\text{BZ}} \int \sum_{n \in p} [F_{n,ab}(\mathbf{k})] \, dk_a \, dk_b ,$$

where $F_{n,ab}$ is the Berry curvature of band $n$ defined by

$$F_{n,ab}(\mathbf{k}) = \frac{\partial A_{n,b}(\mathbf{k})}{\partial k_a} - \frac{\partial A_{n,a}(\mathbf{k})}{\partial k_b}$$

with $A_{n,\mu}(\mathbf{k}) = \langle \mathbf{k}, n | \nabla_{k_\mu} | \mathbf{k}, n \rangle$. 

FIG. 2: One-magnon dispersions for $J = J'$ for two values of the DM coupling. The critical case is $D_z = 0$ (upper left panel and black lines in lower panel) where the QBCP at the $\Gamma$ point is clearly visible. The degeneracy of the quadratic bands is lifted for finite $D_z > 0$ (upper right panel and red lines in lower panel) so that distinct bands appear which show non-trivial topological Chern numbers $C = \pm 1$. 

The numerical robust calculation of the Berry curvature is done by discretization of the BZ\(^{49}\) avoiding the eigenstates precisely at the boundaries of the BZ. This is possible because the relevant curvature occurs in the vicinity of the \(\Gamma\) point anyway. The calculated Chern numbers of the pairs of magnon bands is \(C^{\text{upper/lower}} = \pm 1\) as shown in Fig. 2. Changing the sign of the \(D_z\) reverses the sign of the Chern numbers. The non-zero Chern numbers can be attributed to the complex hopping stemming from the DM coupling leading to fluxes of fictitious fields\(^{23,29}\). The direct gap between both pairs of bands occurs at \(\Gamma\) and is given by \(8D_zS\) as long as \(4D_z < J\). Otherwise the direct gap is located at the \(M\) point and takes the value \(2JS\).

![Diagram of strip geometry](image)

**Fig. 3:** (Upper panel) Eigen energies of a strip geometry, see lower right panel, with \(N = 50\), \(J = J'\), \(D_z = 0.2J\), and \(A = 0.2J\). (Lower left panel) Probability density \(|\psi(k_a = 0, r_b)|^2\) as function of the site \(r_b\) of both edge modes; the color of their curves corresponds to the color of the boundary sites in the lower right panel.

According to the bulk-boundary correspondence\(^{18,36}\) the existence of a non-trivial Chern numbers implies topologically protected edge states\(^{40}\). For verification, we analyze a finite strip of \(N = 50\) unit cells in \(b\)-direction and periodic boundaries in \(a\)-direction, see lower right panel in Fig. 3. The energy eigen values as function of the well-defined momentum \(k_a\) are depicted in the upper panel of Fig. 3. One can easily see two chiral edge states moving right and left according to the slope of their dispersion branches which connect the two continua shown in red. Additionally, the lower left panel illustrates the localization of these modes at the lower (yellow curve and sites) and upper (blue curve and sites) edge of the strip.

Next we address possible experimental signatures. Since magnons do not carry charge, usual electric conductivity measurements do not make sense. The thermal Hall effect offers a way to detect non-trivial Berry curvatures in real materials. The thermal Hall effect consists in a finite temperature gradient perpendicular to a heat current. The expression for the transversal heat conductivity \(\kappa_{ab}\)\(^{51}\) is given by

\[
\kappa_{ab} = -\frac{k_B^2 T}{\hbar} \sum_{n,k} c_2(\rho_n) F_{n,ab}(k)
\] (8)

where we sum over all magnon bands and set \(k_B = 1\) and \(\hbar = 1\). The weight \(c_2(\rho_n)\) is given by

\[
c_2(\rho) = \int_{\varepsilon_n}^{\infty} d\varepsilon \left(\frac{d\rho}{d\varepsilon}\right)_{\mu=0} \] (9a)

\[
= -2 \text{Li}_2(-\rho) + \rho \log^2(\rho^{-1} + 1) - \log^2(\rho + 1) + 2 \log(\rho + 1) \log(\rho^{-1} + 1), \] (9b)

where \(\rho\) is the Bose-Einstein distribution \((\exp(\beta\omega) - 1)^{-1}\) and \(\text{Li}_m\) is the dilogarithm for \(m = 2\) (Spence’s integral in general). Eq. (8) clearly shows that the transversal heat conductivity \(\kappa_{ab}\) depends directly on the Berry curvature, thus representing an ideal fingerprint of non-trivial topological properties. Fig. 4 displays the results of (8) as function of temperature for various values of \(D_z\). For the topological phase \((D_z \neq 0)\), the conductivity first slightly decreases to negative values before it strongly increases as function of temperature. For high temperature \(\kappa_{ab}\) approaches a finite value. In comparison, the topologically trivial bands for \(D_z = 0\) may have a finite curvature, but such that it cancels in the sum over the BZ so that \(\kappa_{ab}\) vanishes.

Since the magnetization generally decreases with increasing temperature till eventually the ferromagnetic phase ceases to exist at \(T_c\), \(\kappa_{ab}\) should also decreases until it disappears at \(T_c\). Improving the calculations by applying self-consistent spin wave theory the signature starts to decrease for higher temperature before no self-consistent solution is found anymore as depicted by the dashed lines (for details see Supplementary).

In conclusion, a finite thermal Hall conductivity \(\kappa_{ab}\) can serve as a smoking gun signature in experiments to verify topological properties of a material under study.

![Diagram of thermal Hall conductivity](image)

**Fig. 4:** Thermal Hall conductivity \(\kappa_{ab}\) as function of temperature for various values of the DM coupling \(D_z\) at \(J = J'\), \(A = 0.2J\), and \(S = 4\).

In view of the above findings, we suggest to characterize the magnetic properties of the putative
realization\textsuperscript{38,39} of ferromagnetic Shastry-Sutherland lattices in detail, for instance by inelastic neutron scattering. This will help to determine the relevant microscopic model which in turn will render the calculation of the Berry curvature possible. In parallel, measurements of the thermal Hall conductivity can provide evidence\textsuperscript{4,29} for finite Berry curvatures.

The intriguing next step towards an application will be to tailor the edges of strip by decorating them similar to what has been proposed and computed for fermionic models\textsuperscript{17,52}. In this way, largely different group velocities can be achieved depending on the direction in which signals of packets of magnons travel. The key is to structure the upper and the lower boundary of a strip in different manner so that the group velocity of the right and of the left moving packet is very different. Ideally, the group velocities should be tunable by moderate changes of the model controlled by external parameters such as magnetic fields or pressure. The realization of this phenomenon will pave the way to fascinating devices in magnonics such as delay lines and interference devices.

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Supplemental Material for ”Topological magnon bands for magnonics”

Self-consistent spin wave theory

The temperature dependency of the static calculation of the transversal heat conductivity $\kappa_{ab}$ stems only from the weight $c_2(\rho_n)$ and the prefactor. The Berry curvature $F_{n,ab}(k)$ remains static regarding the temperature. In order to make quantitative statements it is appropriate to improve the results. Here we use the Dyson-Maleev representation of the spin operators which leads to an exact description at quartic level and is given by

\begin{align*}
S_i^+ &= \sqrt{2S}(b_i - \frac{1}{2S} b_i^+ b_i) \\
S_i^- &= \sqrt{2S} b_i^+ \\
S_i^z &= b_i^+ b_i - S.
\end{align*}

The complete Hamiltonian in the bosonic representation is then described by

\begin{align*}
\mathcal{H} &= \mathcal{H}_H + \mathcal{H}_{DM} + \mathcal{H}_{SIA} \\
\mathcal{H}_H &= -J \sum_{\langle ij \rangle} S(b_i^+ b_j + b_i^+ b_j - b_i^+ b_j - b_j^+ b_i) \\
&\quad + J \sum_{\langle ij \rangle} \frac{1}{2} (b_i^+ b_i^+ b_i b_j + b_i^+ b_i^+ b_j b_i) - b_i^+ b_i^+ b_i b_j \\
&\quad - J' \sum_{\langle\langle ij \rangle\rangle} S(b_i^+ b_j + b_j^+ b_i - b_i^+ b_j - b_j^+ b_i) \\
&\quad + J' \sum_{\langle\langle ij \rangle\rangle} \frac{1}{2} (b_i^+ b_i^+ b_j b_j + b_j^+ b_j^+ b_i b_i) - b_i^+ b_i^+ b_j b_j \\
\mathcal{H}_{DM} &= -iD_z \sum_{\langle ij \rangle} S(b_i^+ b_j + b_j^+ b_i) \\
&\quad + \frac{iD_z}{2} \sum_{\langle\langle ij \rangle\rangle} (b_i^+ b_i^+ b_j b_j + b_j^+ b_j^+ b_i b_i) \\
\mathcal{H}_{SIA} &= -A \sum_i (2S - 1)b_i^+ b_i + b_i^+ b_i^+ b_i,
\end{align*}

where we neglected all constant terms. Applying a mean field decoupling reduces the quartic terms into bilinear terms. For this purpose we introduce the expectation values

\begin{align*}
n &= \langle b_i^i b_i \rangle \in \mathbb{R} \\
a &= \langle b_i^i b_j \rangle \in \mathbb{R} \quad \text{for } \langle ij \rangle \\
c &= \langle b_i^i b_j \rangle \in \mathbb{C} \quad \text{for } \langle\langle ij \rangle\rangle,
\end{align*}

with the bosonic creation $b_i^+$ and annihilation operators $b_n$ at the site $n \in \{1, 2, 3, 4\}$. The Hamiltonian becomes implicitly temperature dependent since the expectation values are depending on temperature. The $4 \times 4$ Hamilton matrix reads

\begin{equation}
\mathcal{M}(k) = \begin{pmatrix}
A & B(k_a, k_b) \\
B^t & A
\end{pmatrix}
\end{equation}

with the $2 \times 2$ matrices

\begin{equation}
A = \begin{pmatrix} A_{11} & A_{12} \\
A_{21} & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} B_{11} & B_{12} \\
B_{21} & B_{22} \end{pmatrix}
\end{equation}

FIG. S1: Magnetitation as a function of temperature $T$ for various $D_z$ at $J = J', A = 0.2J$, and $S = 4$. The horizontal dashed line indicates the value at which the spin gap closes.

\[D_z = 0.1J, \quad D_z = 0.2J, \quad D_z = 0.3J\]
\[ A_{11} = A_{22} = J(S - n + a) + 4J'(S - n + \text{Re}(c)) + A(2S - 4n - 1) + 4D_z \text{Im}(c) \]  
\[ A_{12} = A_{21} = -J(S - n + a) \]  
\[ B_{11} = -J'(S - n + c^*) - J'(S - n + c)e^{ik_a} - iD_z(S - n)(1 + e^{ik_a}) \]  
\[ B_{21} = -J'(S - n + c^*) - J'(S - n + c)e^{ik_b} + iD_z(S - n)(1 + e^{ik_b}) \]  
\[ B_{12} = -J'(S - n + c^*)e^{ik_a} - J'(S - n + c^*)e^{i(k_a + k_b)} + iD_z(S - n)(e^{ik_a} + e^{i(k_a + k_b)}) \]  
\[ B_{22} = -J'(S - n + c) e^{ik_b} - J'(S - n + c^*) e^{i(k_a + k_b)} - iD_z(S - n)(e^{ik_b} + e^{i(k_a + k_b)}) \]

By expressing the expectation values using the Bose-Einstein distribution we are able to determine self-consistently the renormalized dispersion and the corresponding magnetization \( m \) at a specific temperature. The magnetization is given by the simple relation \( m = S - n \). The renormalized spin gap \( \Delta \) is purely determined by the SIA being given by

\[ \Delta = A(2S - 4n - 1). \]  

\( \text{(S10)} \)

Obviously, the spin gap closes before the magnetization vanishes, so that in this approximation a Curie temperature cannot be determined. The spin gap closes for \( 2S - 4n - 1 = 0 \). For the magnetization this implies that the spin gap closes if the magnetization reaches the value \( m = (2S + 1)/4 \) as indicated by the horizontal dashed line in Fig. S1.

The self-consistently calculated magnetization shows the totally unexpected problem that no solution can be found even before the spin gap closes or the magnetization vanishes. It appears that the phase transition from the ordered phase induced by the SIA to the disordered phase cannot be captured at all by spin wave theory. This issue deserves further investigations, but it is beyond the scope of the present article.