HIGH DIMENSIONAL REGRESSION AND MATRIX ESTIMATION WITHOUT TUNING PARAMETERS

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ABSTRACT. A general theory for Gaussian mean estimation that automatically adapts to unknown sparsity under arbitrary norms is proposed. The theory is applied to produce adaptively minimax rate-optimal estimators in high dimensional regression and matrix estimation that involve no tuning parameters.

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1. Abstract theory

Suppose that $Y \sim N_n(\mu, \sigma^2 I_n)$ for some $\mu \in \mathbb{R}^n$ and $\sigma \geq 0$, where $I_n$ denotes the $n \times n$ identity matrix, and $N_n(\mu, \sigma^2 I_n)$ is the $n$-dimensional Gaussian distribution with mean vector $\mu$ and covariance matrix $\sigma^2 I_n$. The statistical problem of Gaussian mean estimation is the problem of estimating the unknown mean vector $\mu$ using the observed data vector $Y$. Typically, the parameter $\sigma$ is also unknown. Given an estimator $\hat{\mu}$, the most common measure of risk is the risk under the quadratic loss, namely, the quantity $\mathbb{E} \| \hat{\mu} - \mu \|^2$, where $\| \cdot \|$ denotes the Euclidean norm on $\mathbb{R}^n$.

The data vector $Y$ is itself an unbiased estimator of $\mu$. Stein (1956) famously proved that the naive estimator $Y$ is inadmissible under quadratic loss, and an estimator that strictly dominates the naive estimator was produced by James and Stein (1961).

A surprising number of problems in mathematical statistics can be framed as Gaussian mean estimation problems, where the aim is to construct estimators of $\mu$ that perform well when the true $\mu$ satisfies some given conditions. For example, it may be known to the statistician that the true $\mu$ belongs to some convex set $C$. A reasonable estimate of $\mu$ in this case is the Euclidean projection of the data vector $Y$ onto the set $C$. There is a wealth of literature on the analysis of this estimator and its applications. The monographs of van der Vaart and Wellner (1996), van de Geer (2000), Massart (2007) and Bühlmann and van de Geer (2011) contain the essential references to the statistics literature on this topic. A precise approximation of the risk of this

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estimator under quadratic loss was recently obtained by Chatterjee (2014b). The problem has also received considerable attention in the signal processing literature; see Rudelson and Vershynin (2008), Stojnic (2009), Chandrasekaran, Recht, Parrilo and Willsky (2012), Oymak and Hassibi (2013), Chandrasekaran and Jordan (2013), Amelunxen, Lotz, McCoy and Tropp (2014), McCoy and Tropp (2014a,b) and Foygel and Mackey (2014) for current developments on this front.

Often, the convex set \( C \) that presumably contains the true \( \mu \) appears in the form of a ball of some radius \( r \) centered at the origin for some norm \( K \) on \( \mathbb{R}^n \). Many high dimensional estimation problems of contemporary interest fit into this framework. We will see examples in later sections.

To compute the projection estimator discussed in the preceding paragraph, the statistician needs to know the value of \( r \). If \( r \) is unknown, which is usually the case, then the projection estimator cannot be reliably defined. Using a wrong value of \( r \) for projection can lead to spurious outcomes.

The problem demands the construction of an adaptive estimator whose performance adapts to the unknown value of \( r \), in the sense that the performance of the estimator should become better as the \( K \)-norm of the true \( \mu \) decreases.

The goal of this section is to propose a solution to this general problem. Namely, if \( Y \sim N_n(\mu, \sigma^2 I_n) \), and \( K \) is a norm on \( \mathbb{R}^n \), to produce an estimator of \( \mu \) that has good performance whenever \( K(\mu) \) is small. The performance needs to get better as \( K(\mu) \) gets smaller, and no knowledge about the value of \( K(\mu) \) should be required to construct the estimator.

The theory presented in this section produces an estimator with the above property in this completely general setting. An application to high dimensional regression is given in Section 3 and an application to matrix estimation is given in Section 3. The abstract theory involves two main ideas. The first idea shows how in the above framework, any estimator of \( \sigma \) may be used to yield an estimate of the mean vector. This is the content of Theorem 1.1 stated below. Note that the reverse direction is easy: given an estimate \( \hat{\mu} \) of the mean vector \( \mu \), one can easily get a reliable estimator \( \hat{\sigma} \) as \( \hat{\sigma}^2 = \|Y - \hat{\mu}\|^2/n \).

Recall that for a norm \( K \), the dual norm \( K^\circ \) is defined as

\[
K^\circ(x) = \sup_{y \neq 0} \frac{x \cdot y}{K(y)},
\]

where \( x \cdot y \) denotes the standard inner product on \( \mathbb{R}^n \).

**Theorem 1.1.** Let \( K \) be a norm on \( \mathbb{R}^n \) and let \( K^\circ \) be the dual norm of \( K \). Take any \( \mu \in \mathbb{R}^n \), \( \sigma \geq 0 \), and let \( Y \sim N_n(\mu, \sigma^2 I_n) \). Let \( \hat{\sigma} \) be any random variable defined on the same probability space as \( Y \). Define

\[
\hat{\mu} := \arg\min \{ K(\nu) : \nu \in \mathbb{R}^n, \|Y - \nu\|^2 \leq n\hat{\sigma}^2 \}.
\]

Let \( Z \sim N_n(0, I_n) \). Then

\[
\frac{\mathbb{E}\|\hat{\mu} - \mu\|^2}{n\sigma^2} \leq \frac{16}{n\sigma} K(\mu) \mathbb{E}(K^\circ(Z)) + 2 \sqrt{\frac{2}{n}} + \frac{2 \mathbb{E}|\hat{\sigma}^2 - \sigma^2|}{\sigma^2},
\]

where \( K^\circ \) is the dual norm of \( K \).

The theorem says that if \( \hat{\sigma} \) is a good estimate of \( \sigma \), and \( K(\mu) \) is small, then \( \hat{\mu} \) is a good estimate of \( \mu \). In particular, if the value of \( \sigma \) is known, then we can just take \( \hat{\sigma} = \sigma \) and make the third term in the error bound equal to zero.

The proof of Theorem 1.1 is given in Section 5. A rough sketch of the idea behind the proof is as follows. For each \( L \geq 0 \), let \( B_L \) be the \( K \)-ball of radius \( L \) centered at the origin, and let \( \hat{\mu}_L \) be the Euclidean projection of \( Y \) on to \( B_L \). Now take any \( 0 \leq L' \leq L \), and consider the triangle formed by the vertices \( Y, \hat{\mu}_L \) and \( \hat{\mu}_{L'} \). It is a standard fact that if \( y \) is the projection of a point \( x \) on to a convex set \( C \), and \( z \) is any point in \( C \), then the angle between the rays \( yz \) and \( yx \) is an obtuse
Geometric considerations imply that in the triangle formed by the dotted lines, the angle at $\hat{\mu}_L$ is necessarily obtuse. This, in turn, implies that if the two rays emanating from $Y$ are approximately of equal length, then $\hat{\mu}_L$ must be close to $\hat{\mu}_{L'}$. Since the angle at $\hat{\mu}_L$ is an obtuse angle, an easy geometric argument shows that $\hat{\mu}_{L'}$ is close to $\hat{\mu}_L$ if and only if

$$\|Y - \hat{\mu}_{L'}\| \approx \|Y - \hat{\mu}_L\|,$$

where $a \approx b$ means that $a/b$ is close to 1.

The final step is to show that if $K(\mu)$ is small, and $L := K(\mu)$, then $\hat{\mu}_L$ is close to $\mu$. This is well known argument, often used in proving error bounds for penalized regression estimators. As a consequence, we get

$$\|Y - \hat{\mu}_L\|^2 \approx n\hat{\sigma}^2.$$

Combining, we get

$$\|Y - \hat{\mu}_L\| \approx \|Y - \hat{\mu}_{L'}\|.$$

The first step of the proof now implies that $\hat{\mu}_L$ is close to $\hat{\mu}_{L'}$. But $\hat{\mu} = \hat{\mu}_L$ and $\hat{\mu}_L$ is close to $\mu$. Thus, $\hat{\mu}$ is close to $\mu$.

Although Theorem 1.1 gives a prescription for constructing an estimator of $\mu$ using an estimator of $\sigma$, it does not solve the problem of estimating $\sigma$ if $\sigma$ is unknown. It turns out that it is possible to directly construct an estimator of $\sigma$ that has good performance whenever $K(\mu)$ is sufficiently small. This is second key idea of the theory presented in this section, which yields the next theorem.
Theorem 1.2. Take any \( n \geq 5 \). Suppose that \( K \) is a norm on \( \mathbb{R}^n \) and \( Y \sim N_n(\mu, \sigma^2 I_n) \) for some \( \mu \in \mathbb{R}^n \) and \( \sigma \geq 0 \). Let \( Z \sim N_n(0, I_n) \) be defined on the same probability space as \( Y \). Define

\[
\hat{\sigma} := \frac{K(Y)}{K(Z)}.
\]

Let \( K^\circ \) be the dual norm of \( K \). For each \( k \geq 1 \), let \( m_k := (\mathbb{E}(K^\circ(Z)^k)^{1/k} \). Let

\[
a := \sup\{\|v\| : v \in \mathbb{R}^n, K(v) \leq 1\}.
\]

Then

\[
\mathbb{E}(\hat{\sigma} - \sigma)^2 \leq \frac{K(\mu)^2 m_2^2}{(n-2)^2} + \frac{32\sqrt{2}\sigma^2 a^2 m_2^2}{(n-4)^2}.
\]

This result says, roughly speaking, that whenever \( K \) satisfies certain conditions and \( K(\mu) \) is small enough, then \( \hat{\sigma} \) is a good estimate of \( \sigma \). As usual, the proof is given in Section 5. A brief sketch of the idea behind the proof is as follows. Let \( \varepsilon := Y - \mu \). If \( K(\mu) \) is small, then by the triangle inequality for \( K \),

\[
K(Y) = K(\varepsilon + \mu) \approx K(\varepsilon).
\]

The second step is to observe that

\[
K(Z) = \sup\{Z \cdot v : v \in \mathbb{R}^n, K^\circ(v) \leq 1\},
\]

and therefore \( K(Z) \) is the maximum of a Gaussian field. There are general inequalities which show that under mild conditions, the fluctuations of the maximum of a Gaussian field are small compared to its expected value. If these conditions hold, then \( K(Z) \) would be close to \( \mathbb{E}(K(Z)) \) with high probability. By the same logic, since \( \varepsilon \sim N_n(0, \sigma^2 I_n) \), therefore

\[
K(\varepsilon) \approx \sigma \mathbb{E}(K(Z))
\]

with high probability. Combining this with the first step of the proof, we get

\[
\hat{\sigma} = \frac{K(Y)}{K(Z)} \approx \frac{K(\varepsilon)}{K(Z)} \approx \sigma.
\]

It is easy to see how a combination of Theorems 1.1 and 1.2 can give an estimator of \( \mu \) that has accuracy whenever \( K(\mu) \) is small enough. Moreover, two different norms can be used for the two parts. An example of a result that is obtained by combining Theorem 1.1 and Theorem 1.2 is the following.

Theorem 1.3. Take any \( n \geq 5 \). Suppose that \( K \) and \( \tilde{K} \) are two norms on \( \mathbb{R}^n \) and \( Y \sim N_n(\mu, \sigma^2 I_n) \) for some \( \mu \in \mathbb{R}^n \) and \( \sigma \geq 0 \). Let \( Z \sim N_n(0, I_n) \) be defined on the same probability space as \( Y \). Define

\[
\hat{\sigma} := \frac{\tilde{K}(Y)}{\tilde{K}(Z)}
\]

and

\[
\hat{\mu} = \arg\min\{K(\nu) : \nu \in \mathbb{R}^n, \|Y - \nu\|^2 \leq n\hat{\sigma}^2\}.
\]

For each \( k \geq 1 \), let \( m_k := (\mathbb{E}(\tilde{K}^\circ(Z)^k)^{1/k} \), where \( \tilde{K}^\circ \) is the dual norm of \( \tilde{K} \). Let

\[
a := \sup\{\|v\| : v \in \mathbb{R}^n, \tilde{K}^\circ(v) \leq 1\}.
\]
\[
\frac{E\|\hat{\mu} - \mu\|^2}{n\sigma^2} \leq \frac{16}{n\sigma} K(\mu) E(K^2(Z)) + 2\sqrt{\frac{2}{n}} + \frac{2K(\mu) m_2^2}{(n - 2)^2\sigma^2} + 64\sqrt{2}\alpha^2 m_3^2 \left(\frac{n - 4}{n - 2}\right) + 4\tilde{K}(\mu)m_2^2 \left(\frac{n - 2}{n\sigma}\right) + \frac{28am_4}{n - 4}.
\]

It is not clear whether the estimator \(\hat{\mu}\) given in Theorem 1.3 has any general optimality property under any set of conditions. However we will see in the following sections that in special cases of interest, \(\hat{\mu}\) indeed turns out to be minimax rate-optimal.

2. Application to high dimensional regression

Consider the familiar regression framework. Let \(n\) and \(p\) be two positive integers, and let \(X\) be an \(n \times p\) matrix with real entries, called the design matrix. Let \(\beta_0 \in \mathbb{R}^p\) be a vector of parameters, \(\sigma \geq 0\) be another parameter, and let
\[
Y = X\beta_0 + \varepsilon,
\]
where \(\varepsilon \sim N_n(0, \sigma^2 I_n)\). The vector \(\beta_0\) and the number \(\sigma\) are unknown to the statistician, who knows only the response vector \(Y\) and the design matrix \(X\). The objective is to estimate the vector \(\beta_0\) from the observed vector \(Y\), which is called the response vector.

The regression problem is called high dimensional if the number of covariates \(p\) is large, usually much larger than the number of data points \(n\). Regression problems where the number of covariates exceeds the number of responses have become increasingly important in the last twenty years. Statisticians have devised a number of penalized regression techniques for dealing with such problems, such as the Lasso by Tibshirani (1996), basis pursuit by Chen, Donoho and Saunders (1998), the SCAD algorithm of Fan and Li (2001), the LARS algorithm of Efron, Hastie, Johnstone and Tibshirani (2004), the elastic net by Zou and Hastie (2005) and the Dantzig selector by Candès and Tao (2007). Sophisticated variants of these methods have emerged over the years, such as the group Lasso by Yuan and Lin (2006), the adaptive Lasso by Zou (2006) and the square-root Lasso by Belloni, Chernozhukov and Wang (2011). Penalized regression is not the only approach; for example, methods of model selection by testing hypotheses have been proposed by Birgé and Massart (2001), Birge (2006), Abramovich, Benjamini, Donoho and Johnstone (2006) and Barber and Candès (2015).

There is now a large body of literature on the theoretical properties of the penalized regression techniques mentioned above. Perhaps the most widely studied among these methods is the Lasso. The theoretical analysis of basis pursuit by Chen, Donoho and Saunders (1998) already had substantial information in this direction because of the close connection between the Lasso and basis pursuit. Knight and Fu (2000) analyzed the Lasso when \(p\) is fixed and \(n \to \infty\). Later, theoretical results that accommodated \(p\) growing as fast as \(n\) or even faster, began to emerge. A notable early example is Greenshtein and Ritov (2004), where the influential notion of ‘persistence’ for measuring the efficacy of a high dimensional estimator was introduced. Meinshausen and Bühlmann (2006) gave a novel application of the Lasso to recovering dependency structures in a paper that contains some of the earliest theoretical techniques for analyzing the Lasso in a high dimensional setting. Bunea, Tsybakov and Wegkamp (2007a) were the first to prove oracle inequalities for \(\ell^1\)-penalized regression. The Dantzig selector was introduced by Candès and Tao (2007), who also introduced a number of novel ideas that spurred much of the subsequent research on the analysis of sparse regression techniques. Zhao and Yu (2007), introduced the ‘irrepresentability condition’ that guarantees consistent model selection by the Lasso. This work was significantly refined and extended by Zhang and Huang (2008), Meinshausen and Yu (2009) and Wainwright (2009).
The optimality of model selection by the Lasso under a wide range of conditions was established by Candès and Plan (2009). Theoretical analysis of the Lasso for generalized linear models was carried out by van de Geer (2008). Bickel, Ritov and Tsybakov (2009) constructed a unified theoretical framework that is capable of analyzing the Lasso, the Dantzig selector, and a variety of other techniques. Unification of oracle inequalities for a variety of high dimensional techniques, including the Lasso and the Dantzig selector, was achieved by van de Geer and Bühlmann (2009). A further unification with general high dimensional M-estimators was obtained by Negahban, Ravikumar, Wainwright and Yu (2012). Raskutti, Wainwright and Yu (2011), Bartlett, Mendelson and Neeman (2012) and Chatterjee (2014a,b) revisited prediction error bounds for the Lasso. A nice textbook reference for many of these developments is Bühlmann and van de Geer (2011).

Recently, researchers have started investigating the asymptotic distributional properties of penalized regression estimates and ways to build confidence intervals and carry out tests of hypotheses; see Wasserman and Roeder (2009), Meinshausen, Meier and Bühlmann (2009), Meinshausen and Bühlmann (2010), Minnier, Tian and Cal (2011), Tibshirani and Taylor (2012), van de Geer and Bühlmann (2014), Javanmard and Montanari (2014a,b), Zhang and Zhang (2014) and Lockhart, Taylor, Tibshirani and Tibshirani (2014) for the latest developments.

A common feature of the methods discussed above is that the user is required to supply the values of one or more ‘tuning parameters’. The tuning parameters are numbers or other objects that are chosen at the user’s discretion, and the user ‘tunes’ the values of these parameters to get optimal results. In practice, the tuning process usually involves the data, for example through cross-validation. The resultant estimates are invariably a highly complicated objects with very little theoretical understanding. The only existing papers that provide some level of theoretical justification for cross-validated estimates in high dimensional regression are those of Lecué and Mitchell (2012), Homrighausen and McDonald (2013a,b, 2014) and Chatterjee and Jafarov (2015).

There is a closely related technique, known as ‘aggregation’, that has a much more extensive theoretical foundation. The central goal of aggregation is to take a finite collection of predictors, and find a combination of them that optimizes some measure of performance. Aggregation was pioneered by Nemirovski (2000), and vastly developed by many authors, including Yang (2000, 2001, 2004), Gyorfi, Kohler, Krzyzak and Walk (2002), Wegkamp (2003), Tsybakov (2004), Catoni (2004), Bunea, Tsybakov and Wegkamp (2007b) and Rigollet and Tsybakov (2007, 2012). Aggregation, however, has its own tuning parameter that requires user intervention, namely, the choice of predictors that are aggregated.

The mathematical statistics literature cited above contains prescriptions for choosing optimal values of the tuning parameters. For example, if \( \sigma \) is known, a popular choice for the penalty parameter in ordinary Lasso is

\[
\lambda = \frac{4\|X^T w\|_\infty}{n},
\]

where \( X^T \) is the transpose of the design matrix \( X \), \( w \) is a vector of \( n \) i.i.d. \( N(0, \sigma^2) \) random variables, and \( \| \cdot \|_\infty \) denotes the \( \ell_\infty \) norm on \( \mathbb{R}^n \). See Negahban, Ravikumar, Wainwright and Yu (2012) and Chichignoud, Lederer and Wainwright (2014) for further details.

The main problem with such theoretical prescriptions is that they require a priori knowledge about the unknown parameter \( \sigma \). A variety of estimates for \( \sigma \) in high dimensional regression have been proposed in recent times, for example by Städler, Bühlmann and van de Geer (2010), Fan, Guo and Hao (2012), Sun and Zhang (2012), Dicker (2014) and Chatterjee and Jafarov (2015). The difficulty with using these estimates is that these estimates themselves involve tuning parameters, and moreover they usually need the regression to be performed before the estimate of \( \sigma \) can be produced. One regression technique where knowledge about \( \sigma \) is not required is the square-root
Lasso of Belloni, Chernozhukov and Wand (2011). Square-root Lasso has many nice properties, such as near-oracle performance when the number of nonzero components of $\beta_0$ is small. However, square-root Lasso is not completely free of tuning parameters: it has its own tuning parameter that needs to be calibrated by the user.

The main contribution of this section is a simple new regression procedure for high dimensional data that does not need the user to input anything other than the design matrix and the response vector. The estimator is obtained by specializing the abstract method proposed in Section 1 to the regression setting. A general error bound is provided in Theorem 2.1 below. The error bound needs to be calibrated by the user.

Given a response vector $Y$ and a design matrix $X$ related by equation (2), the proposed technique produces a randomized estimate $\hat{\beta}$ of $\beta_0$ and a randomized estimate $\hat{\sigma}$ of $\sigma$ through the following sequence of steps. The method requires no conditions on the design matrix, but seems to perform better in simulations if the design matrix is standardized.

The theorem stated below gives an upper bound on the expected mean squared prediction error of $\hat{\beta}$. The prediction error measures how well $X\hat{\beta}$ estimates the mean vector $X\beta_0$. It also gives an upper bound on the risk of $\hat{\sigma}$ under quadratic loss.

**Theorem 2.1.** Let all notation be as above. Suppose that $n \geq 8$ and $p \geq 8$. Let

$$r := \frac{|\beta_0|_1 \gamma}{\sigma} \sqrt{\frac{\log(p + n)}{n}}.$$

Then

$$\mathbb{E}\|X\hat{\beta} - X\beta_0\|^2 \leq Cr + Cr^2 + C \sqrt{\frac{\log(p + n)}{n}} + C \log(p + n)$$

and

$$\mathbb{E}\left(\frac{\hat{\sigma}}{\sigma} - 1\right)^2 \leq Cr^2 + \frac{C \log(p + n)}{n},$$

where $C$ is a universal constant.

The reason for dividing the risk by $n\sigma^2$ is to compare $X\hat{\beta}$ with the naive unbiased estimate of $X\beta_0$, which is simply the vector $Y$. The risk of the naive estimator is equal to $n\sigma^2$.

The proof of Theorem 2.1 is based on a straightforward application of Theorem 1.3 of Section 1. When $X$ has rank $n$, Theorem 1.3 implies Theorem 2.1 upon taking $K$ to be the norm

$$K(\mu) = \min\{|\beta|_1 : \beta \in \mathbb{R}^{p+n}, \mu = X\beta\}$$

and $K$ to be the norm

$$K(\mu) = \min\{|\beta|_1 : \beta \in \mathbb{R}^p, \mu = X\beta\}.$$
The reason for using two different norms is that the quantity $a$ in Theorem 1.3 is well-behaved for $\tilde{K}$ but may not be well-behaved for $K$. When the rank of $X$ is less than $n$, a slightly modified argument is required. The details of these arguments are given in Section 5.

The minimax rate-optimality of $\hat{\beta}$ for prediction error in $\ell^1$ balls is established using Theorem 3 in Raskutti, Wainwright and Yu (2011). This result says the following. Let all notation be as above. Take any $L \geq 0$. Suppose that there are constants $\kappa > 0$ and $c \geq 0$ such that for all $\beta \in \mathbb{R}^p$ with $|\beta|_1 \leq 2L$,

$$\frac{\|X\beta\|}{\sqrt{n}} \geq \kappa \|\beta\| - \kappa c L \sqrt{\log p \over n}.$$  \hfill (3)

Suppose further that $c_1, c_2 > 0$ and $\delta \in (0, 1)$ are constants such that

$$\frac{p}{L \sqrt{n}} \geq c_1 p^\delta \geq c_2.$$  \hfill (4)

Then Theorem 3 of Raskutti, Wainwright and Yu (2011) implies that

$$\inf_{\tilde{\beta}} \sup_{\beta_0: |\beta_0|_1 \leq L} \mathbb{E} \|X\tilde{\beta} - X\beta_0\|^2 \geq C L \sigma^2 \sqrt{\log p \over n}, \hfill (5)$$

where the infimum is taken over all possible estimators $\tilde{\beta}$ and $C$ is a constant that depends only on $\gamma$, $\kappa$, $c$, $c_1$, $c_2$ and $\delta$.

It was shown by Raskutti, Wainwright and Yu (2011) and Rudelson and Zhou (2013) that the condition (3) holds for a large class of design matrices. This includes, but is not limited to, matrices with i.i.d. Gaussian entries. The condition (4) is a mild growth condition on the dimensions of the design matrix which has nothing to do with the matrix entries. It is possible that the lower bound (5) holds under more general conditions than (3) and (4).

The square-root Lasso of Belloni, Chernozhukov and Wang (2011) also achieves minimax rate-optimality (and moreover, near-oracle performance) with a theoretically prescribed choice of the tuning parameter. However, this optimality holds when $\beta$ has a small number of nonzero entries, and not when $\beta$ has small $\ell^1$ norm.

Two other recently proposed adaptively rate-optimal estimators for high dimensional regression deserve mention. One, called SLOPE, was suggested by Bogdan, van den Berg, Sabatti, Su and Candès (2014). SLOPE was shown to be adaptively minimax rate-optimal by Su and Candès (2015) for a certain norm called the ‘sorted $\ell^1$ norm’. Another one, called AV$_\infty$, was suggested by Chichignoud, Lederer and Wainwright (2014) and shown to be adaptively rate-optimal for sup-norm sparsity.

There are several unresolved issues about the estimator proposed in this section. The foremost theoretical issue is that Theorem 2.1 requires the Gaussian error assumption. Although simulation results suggest that some version of the theorem should hold even for non-Gaussian errors, there is no mathematical proof. The main difficulty in extending the result to the non-Gaussian setting is that one has to show that the quantity $M_2$ remains relatively unchanged if $Z$ is replaced by a non-Gaussian vector with i.i.d. components that have zero mean and unit variance. This universality cannot be expected in full generality (for example, it fails when $X$ is the identity matrix), but some mild condition on $X$ may suffice. Homoskedasticity is another assumption that one should be able to drop, although removing this assumption will probably require a modification of the estimator. Another minor problem is that the proposed estimator is a randomized estimator. Although most estimators used in practice are randomized because the tuning parameters are chosen using randomized processes such as cross-validation, it would be nice to have a non-randomized estimator with properties similar to that of the proposed one.
3. Application to matrix estimation

Let $M = (\mu_{ij})_{1 \leq i \leq l \leq m}$ be an $l \times m$ matrix with real entries. The set of all such matrices will be denoted by $\mathbb{R}^{l \times m}$. Let $s_1, \ldots, s_k$ be the singular values of $M$, where $k = \min\{l, m\}$. Recall that the Hilbert–Schmidt norm of $M$ is defined as

$$
\|M\|_{\text{HS}} := \left( \sum_{i=1}^{l} \sum_{j=1}^{m} \mu_{ij}^2 \right)^{1/2} = \left( \sum_{i=1}^{k} s_i^2 \right)^{1/2},
$$

and that the nuclear norm (or trace norm) of $M$ is defined as $\|M\|_* := \sum_{i=1}^{k} s_i$.

Let $Y$ be an $l \times m$ random matrix with independent entries, whose $(i, j)$th entry $y_{ij}$ is normally distributed with mean $\mu_{ij}$ and variance $\sigma^2$. In other words, $Y$ is a noisy version of $M$, where the noise is Gaussian with equal variance for all entries. The goal is to estimate the entries of $M$ using the data $Y$, when both $M$ and $\sigma$ are unknown. Since we are interested in the values of the individual entries, it makes sense to use quadratic loss. That is, if $\hat{M}$ is an estimate of $M$, the risk

$$
\mathbb{E} \|\hat{M} - M\|_{\text{HS}}^2
$$

is used to measure the quality of the estimate.

The problem of estimating the entries of a large matrix from incomplete and/or noisy entries has received widespread attention in the last fifteen years. Early work using spectral analysis was done by a number of authors in the engineering literature, for example by Azar, Flit, Karlin, McSherry and Sala (2001) and Achlioptas and McSherry (2001). Recent papers on spectral methods for matrix completion include those of Cai, Candès and Shen (2010), Keshavan, Montanari and Oh (2010a,b), Candès, Sing-Long and Trzasko (2013), Nadakuditi (2014), Gavish and Donoho (2014) and Chatterjee (2015).

Early examples of non-spectral methods appeared in Fazel (2002), Rennie and Srebro (2005) and Rudelson and Vershynin (2007). Recently, there has been a surge of activity around non-spectral matrix completion and estimation, especially by nuclear norm penalization. The idea was popularized through the works of Candès and Recht (2009), Candès and Tao (2010) and Candès and Plan (2010). Notable recent papers on this topic include those of Mazumder, Hastie and Tibshirani (2010), Negahban and Wainwright (2011), Koltchinskii, Lounici and Tsybakov (2011), Rohde and Tsybakov (2011), Koltchinskii (2012), Donoho, Gavish and Montanari (2013), Donoho and Gavish (2014) and Davenport, Plan, van den Berg and Wootters (2014).

We will now use the general theory of Section 1 to construct an estimator of $M$ that is adaptively minimax rate-optimal for matrices with small nuclear norm. The proposed estimator is defined in three steps.

Step 1: Let $Z$ be an $l \times m$ matrix with i.i.d. $N(0,1)$ entries.

Step 2: Let $\hat{\sigma} := \|Y\|_*/\|Z\|_*.$

Step 3: Let $\hat{M} := \arg\min\{\|A\|_* : A \in \mathbb{R}^{l \times m}, \|Y - A\|_{\text{HS}}^2 \leq lm\hat{\sigma}^2\}$.

The following theorem gives an upper bound on the risk of $\hat{M}$ under quadratic loss.

**Theorem 3.1.** Let $M$, $\sigma$ and $\hat{M}$ be as above. Let

$$
s := \frac{\|M\|_*(\sqrt{l} + \sqrt{m})}{lm\sigma},
$$

Then

$$
\mathbb{E} \|\hat{M} - M\|_{\text{HS}}^2 \leq Cs + Cs^2 + C \sqrt{\frac{1}{lm}}
$$
and
\[ \mathbb{E}\left( \frac{\hat{\sigma}}{\sigma} - 1 \right)^2 \leq C s^2 + \frac{C}{lm}, \]
where \( C \) is a universal constant.

The purpose of dividing the error by \( lm\sigma^2 \) is to compare the risk of \( \hat{M} \) with the risk of the naive estimator \( Y \), which is equal to \( lm\sigma^2 \).

Just like Theorem 2.1, the proof of this result is a direct application of Theorem 1.3, by treating \( l \times m \) matrices as vectors in \( \mathbb{R}^{lm} \), and taking both \( K \) and \( \tilde{K} \) to be the nuclear norm. The details are in Section 5.

The next theorem shows that in regions where the nuclear norm is neither too small nor too large, \( \hat{M} \) is an adaptively minimax rate-optimal estimator.

**Theorem 3.2.** Let \( l \leq m \) be two positive integers. Take any \( \delta \geq 0 \) and let
\[ s := \frac{\delta (\sqrt{l} + \sqrt{m})}{lm\sigma}. \]
Suppose that \( 2/l \leq s \leq 1 \). Consider the setting of Theorem 3.1. Let \( \tilde{M} \) be any estimate of \( M \) based on \( Y \). Then there exists an \( l \times m \) matrix \( M \) with \( \|M\|_* \leq \delta \), such that if this is the true \( M \), then
\[ \frac{\mathbb{E}\|\tilde{M} - M\|_{HS}^2}{lm\sigma^2} \geq Cs, \]
where \( C \) is a positive universal constant.

The asymptotic minimax risk with respect to the nuclear norm in the Gaussian matrix estimation problem was evaluated by Donoho and Gavish (2014). In an earlier work, Donoho, Gavish and Montanari (2013) showed that matrix estimation by nuclear norm penalization achieves the minimax risk asymptotically if the penalty parameter is optimally tuned. This tuning, however, would require knowledge about \( \sigma^2 \). If the elements of \( Y \) are uniformly bounded (with a known bound) instead of Gaussian, the USVT estimator of Chatterjee (2015) is minimax rate-optimal with respect to the nuclear norm. However, for Gaussian entries with unknown \( \sigma^2 \), there exists no minimax rate-optimal estimator in the literature, other than the one proposed in this section. The proof of Theorem 3.2 is given in Section 5.

4. Simulation results

This section contains simulation results for the regression estimator proposed in Section 2. For simplicity, the entries of the design matrices were chosen to be i.i.d. standard Gaussian random variables. The value of \( \sigma \) was varied, as was the parameter vector \( \beta_0 \). The results were compared with the corresponding results from Lasso with 10-fold cross-validation. The output is tabulated in Table 1.

The table shows that the proposed estimator generally has higher prediction error than Lasso with 10-fold cross-validation. However, the estimator appears to be doing a better job at model selection than the Lasso: it returns a far smaller number of false positives, while detecting the true positives at a rate that is comparable with the Lasso. The large number of false positives returned by the Lasso is considered to be a problematic feature. In the examples that are tabulated in Table 1, two to four covariates were included in each model. The Lasso with 10-fold cross-validation typically selected 15 to 25 covariates, whereas the proposed algorithm typically selected less than seven covariates, and usually succeeded in selecting all or most of the relevant covariates. The most striking example
Table 1. Simulation results comparing the proposed estimator and Lasso with 10-fold cross-validation. The design matrices were constructed with i.i.d. standard Gaussian entries, to which an intercept term was added. The reported values are averages over 50 simulations in each case.

| n   | p   | $\sigma$ | $\mathbb{E}(y|x)$ | Proposed estimator | Lasso with cross validation |
|-----|-----|----------|-------------------|--------------------|-----------------------------|
|     |     |          |                   | Average            | Average                     |
|     |     |          |                   | # true positives   | # false positives           |
|     |     |          |                   | prediction error   | Average                     |
|     |     |          |                   | # true positives   | # false positives           |
|     |     |          |                   | prediction error   | error                       |
| 100 | 1000| 2        | $x_1 + x_2$       | 1.42 1.74 1.28     | 1.92 12.84 0.77             |
| 100 | 1000| 3        | $x_1 + x_2$       | 0.68 2.00 1.79     | 1.18 12.96 1.76             |
| 200 | 1000| 2        | $x_1 + x_2 - x_3$ | 2.72 0.66 1.51     | 3.00 22.62 0.56             |
| 200 | 1000| 3        | $x_1 + x_2 - x_3$ | 2.16 3.28 1.89     | 2.88 17.28 1.27             |
| 300 | 300 | 2        | $x_1 + 2x_2$      | 2.00 12.14 0.25    | 2.00 11.48 0.22             |
| 300 | 300 | 3        | $x_1 + 2x_2$      | 1.94 7.98 0.75     | 2.00 11.54 0.52             |
| 400 | 4000| 2        | $x_1 + x_2 + x_3 + x_4$ | 4.00 0.18 1.27 | 4.00 31.62 0.38 |
| 400 | 4000| 3        | $x_1 + x_2 + x_3 + x_4$ | 3.56 1.40 2.16 | 4.00 33.56 0.96 |

from the table is the following: $n = 400$, $p = 4000$, $\sigma = 2$, and $\mathbb{E}(y|x) = x_1 + x_2 + x_3 + x_4$. In this example, there are four relevant covariates. Simulations were run 50 times. In all instances, both the proposed estimator and the Lasso with 10-fold cross-validation succeeded in selecting the four relevant covariates. On the other hand, the proposed estimator rarely selected more than one or two irrelevant covariates, whereas the Lasso selected approximately 32 irrelevant covariates on average. Incidentally, the tendency of the Lasso and other high dimensional regression algorithms for selecting large numbers of irrelevant variables is well known among practicing statisticians; a recent paper where this has been noted is G’Sell, Hastie and Tibshirani (2013).

Recall that Theorem 2.1 is a result about the prediction error of the proposed estimator. The simulation results presented in Table 1 suggest that it would be interesting to have a counterpart of Theorem 2.1 that analyzes the model selection property of the estimator.

5. Proofs

5.1. Proof of Theorem 1.1. Let $K$ be a norm on $\mathbb{R}^n$ and $K^\circ$ be its dual norm. An easy consequence of the definition (11) of the dual norm is that for any $x$ and $y$,

$$x \cdot y \leq K^\circ(x)K(y).$$

In particular,

$$K(x)K^\circ(x) \geq \|x\|^2,$$

where $\|x\|$ is the Euclidean norm of $x$. An important result about the dual norm is that for any norm $K$,

$$K^\circ\circ = K.$$ (7)

For a proof, see Theorem 15.1 in Rockafellar (1970). Another standard result that we will use is the Hilbert projection theorem, which says that any point in $\mathbb{R}^n$ has a unique Euclidean projection on to a given closed convex set. Here ‘Euclidean projection’ means a point in the convex set that is closest to the given point.
Lemma 5.1. Let \( K \) be a norm on \( \mathbb{R}^n \). Take any \( x \in \mathbb{R}^n \) and for each \( L \geq 0 \), let \( w_L \) be the Euclidean projection of \( x \) on to the \( K \)-ball of radius \( L \) centered at zero. Then for any \( L \) and \( L' \),

\[
\|w_L - w_{L'}\|^2 \leq \|x - w_L\|^2 - \|x - w_{L'}\|^2.
\]

Proof. Take any \( L \geq L' \geq 0 \). Since \( K(w_{L'}) \leq L \) and \( w_L \) is the Euclidean projection of \( x \) on to the \( K \)-ball of radius \( L \) centered at zero, therefore for each \( t \in [0,1] \),

\[
\|x - (tw_{L'} + (1-t)w_L)\|^2 \geq \|x - w_L\|^2.
\]

This can be rewritten as

\[
t^2\|w_L - w_{L'}\|^2 + 2t(x - w_L) \cdot (w_L - w_{L'}) \geq 0.
\]

Dividing throughout by \( t \) and letting \( t \to 0 \) gives the inequality

\[
(w_{L'} - w_L) \cdot (x - w_L) \leq 0.
\]

Consequently,

\[
\|w_L - w_{L'}\|^2 = \|(x - w_{L'}) - (x - w_L)\|^2
\]

\[
= \|x - w_{L'}\|^2 + \|x - w_L\|^2 - 2(x - w_{L'}) \cdot (x - w_L)
\]

\[
= \|x - w_{L'}\|^2 - \|x - w_L\|^2 + 2(w_{L'} - w_L) \cdot (x - w_L)
\]

\[
\leq \|x - w_{L'}\|^2 - \|x - w_L\|^2.
\]

This completes the proof in the case \( L \geq L' \). The case \( L < L' \) is treated by exchanging \( L \) and \( L' \) in the above argument. \( \square \)

Proof of Theorem 1.1. For each \( L \geq 0 \), let \( \hat{\mu}_L \) be the Euclidean projection of \( Y \) on to the \( K \)-ball of radius \( \hat{L} \) centered at zero.

Suppose that for a particular realization of \( Y \), \( \hat{\mu} \neq 0 \). Let \( \hat{L} := K(\hat{\mu}) \). Take any \( \nu \) such that \( K(\nu) \leq \hat{L} \) and \( \|Y - \nu\|^2 < n\hat{\sigma}^2 \). Since \( \hat{\mu} \neq 0 \), therefore 0 is outside the closed Euclidean ball of radius \( \sqrt{n\hat{\sigma}} \) centered at \( Y \). On the other hand \( \nu \) lies in the interior of this ball. Therefore the chord connecting \( \nu \) and 0 contains a point \( \nu' \) that lies on the boundary of this ball. This point satisfies \( K(\nu') < \hat{L} \), which is impossible by the definition of \( \hat{L} \). Therefore, any \( \nu \) with \( K(\nu) \leq \hat{L} \) must satisfy \( \|Y - \nu\|^2 \geq n\hat{\sigma}^2 \). In other words,

\[
\hat{\mu} = \hat{\mu}_L.
\] (8)

The argument also shows that

\[
\|Y - \hat{\mu}\|^2 = n\hat{\sigma}^2,
\] (9)

for otherwise the chord connecting \( \hat{\mu} \) and 0 would contain a point that would give a contradiction to the definition of \( \hat{\mu} \).

Fix \( L = K(\hat{\mu}) \) for the rest of the proof. Let

\[
\hat{\sigma}^2_L := \frac{\|Y - \hat{\mu}_L\|^2}{n}.
\]

Then by Lemma 5.1 and the identities (8) and (9),

\[
\frac{\|\hat{\mu}_L - \hat{\mu}\|^2}{n} = \frac{\|\hat{\mu}_L - \hat{\mu}_L\|^2}{n}
\]

\[
\leq \frac{1}{n}\|Y - \hat{\mu}_L\|^2 - \frac{\|Y - \hat{\mu}\|^2}{n}
\]

\[
= \frac{1}{n}\|Y - \hat{\mu}_L\|^2 - \|Y - \hat{\mu}\|^2 = |\hat{\sigma}_L^2 - \hat{\sigma}^2|.
\]
Therefore,
\[
\frac{\|\hat{\mu}_L - \hat{\mu}\|^2}{n} \leq |\hat{\sigma}_L^2 - \sigma^2| + |\hat{\sigma}^2 - \sigma^2|.
\]
(10)

Note that we derived this inequality under the assumption that \(\hat{\mu} \neq 0\). Next, suppose that \(\hat{\mu} = 0\). Then \(\|Y\|^2 \leq n\hat{\sigma}^2\) and \(\hat{\mu} = \hat{\mu}_0\). Therefore by Lemma 5.1,
\[
\frac{\|\hat{\mu}_L - \hat{\mu}\|^2}{n} = \frac{\|\hat{\mu}_L - \hat{\mu}_0\|^2}{n} \leq \frac{1}{n} \left( \|Y - \hat{\mu}_L\|^2 - \|Y - \hat{\mu}_0\|^2 \right)
= \frac{\|Y\|^2}{n} - \hat{\sigma}_L^2 \leq \hat{\sigma}^2 - \hat{\sigma}_L^2.
\]

Thus, (10) holds even when \(\hat{\mu} = 0\). Let
\[
s^2 := \frac{\|Y - \mu\|^2}{n}.
\]

Then note that
\[
|\hat{\sigma}_L^2 - \sigma^2| \leq |\hat{\sigma}_L^2 - s^2| + |s^2 - \sigma^2|
= \frac{1}{n} (\|Y - \mu\|^2 - \|Y - \hat{\mu}_L\|^2) + |s^2 - \sigma^2|
= \frac{1}{n} (2(Y - \mu) \cdot (\hat{\mu}_L - \mu) - \|\mu - \hat{\mu}_L\|^2) + |s^2 - \sigma^2|
\leq \frac{2}{n} (Y - \mu) \cdot (\hat{\mu}_L - \mu) + |s^2 - \sigma^2|.
\]

Therefore,
\[
E(|\hat{\sigma}_L^2 - \sigma^2|) \leq \frac{2\sigma}{n} E \left( \sup_{\nu : K(\nu) \leq 2L} Z \cdot \nu \right) + \frac{\sqrt{2\sigma^2}}{\sqrt{n}}
= \frac{4\sigma K(\mu)}{n} E(K^\circ(Z)) + \frac{\sqrt{2\sigma^2}}{\sqrt{n}}.
\]

Combining this with (10) shows that
\[
E\frac{\|\hat{\mu}_L - \hat{\mu}\|^2}{n\sigma^2} \leq \frac{4}{n\sigma} K(\mu) E(K^\circ(Z)) + \sqrt{\frac{2}{n}} + \frac{E(|\hat{\sigma}^2 - \sigma^2|)}{\sigma^2}.
\]

(11)

Next, note that since \(K(\mu) = L\) and \(\hat{\mu}_L\) is the Euclidean projection of \(Y\) onto the \(K\)-ball of radius \(L\) centered at the origin,
\[
\|Y - \mu\|^2 \geq \|Y - \hat{\mu}_L\|^2
= \|Y - \hat{\mu} - \hat{\mu}_L\|^2 + \|\mu - \hat{\mu}_L\|^2 + 2(Y - \mu) \cdot (\mu - \hat{\mu}_L),
\]
which gives
\[
E\|\mu - \hat{\mu}_L\|^2 \leq E\left(2(Y - \mu) \cdot (\hat{\mu}_L - \mu)\right)
\leq 2\sigma E \left( \sup_{\nu : K(\nu) \leq 2L} Z \cdot \nu \right) = 4\sigma K(\mu) E(K^\circ(Z)).
\]

Combining this with (11) and using the inequality \(\|\hat{\mu} - \mu\|^2 \leq 2\|\hat{\mu} - \hat{\mu}_L\|^2 + 2\|\hat{\mu}_L - \mu\|^2\) completes the proof of the theorem.
5.2. Proofs of Theorems 1.2 and 1.3

Lemma 5.2. Let $K$ be a norm on $\mathbb{R}^n$ and $K^\circ$ be its dual norm. Let $Z \sim N_n(0, I_n)$. Then for any integer $k \in [1, n)$,

$$
\mathbb{E}(K(Z)^{-k}) \leq \frac{\mathbb{E}(K^\circ(Z)^k)}{(n-k)^k}.
$$

Proof. Recall that $\|Z\|^2$ is a $\chi^2$ random variable with $n$ degrees of freedom, which has probability density function

$$
f(x) = \frac{x^{n/2-1}e^{-x/2}}{\Gamma(n/2)2^{n/2}}
$$
on $[0, \infty)$. Consequently,

$$
\mathbb{E}(\|Z\|^k) = \int_0^\infty \frac{x^{(n+k)/2-1}e^{-x/2}}{\Gamma(n/2)2^{n/2}} \, dx = \frac{\Gamma((n+k)/2)2^{k/2}}{\Gamma(n/2)}.
$$

Similarly, if $1 \leq k < n$,

$$
\mathbb{E}(\|Z\|^{-k}) = \int_0^\infty \frac{x^{(n-k)/2-1}e^{-x/2}}{\Gamma(n/2)2^{n/2}} \, dx = \frac{\Gamma((n-k)/2)2^{-k/2}}{\Gamma(n/2)}.
$$

Therefore by (6) and the independence of $\|Z\|$ and $Z/\|Z\|$, we get

$$
\mathbb{E}(K(Z)^{-k}) \leq \mathbb{E}(\|Z\|^{-2k}K^\circ(Z)^k)
= \mathbb{E}(\|Z\|^{-k}K^\circ(Z/\|Z\|)^k)
= \mathbb{E}(\|Z\|^{-k})\mathbb{E}(K^\circ(Z/\|Z\|)^k)
= \frac{\mathbb{E}(\|Z\|^{-k})\mathbb{E}(\|Z\|^kK^\circ(Z/\|Z\|)^k)}{\mathbb{E}(\|Z\|^k)}
= \frac{\mathbb{E}(\|Z\|^{-k})\mathbb{E}(K^\circ(Z)^k)}{\mathbb{E}(\|Z\|^k)} = \frac{\Gamma((n-k)/2)2^{-k/2}\mathbb{E}(K^\circ(Z)^k)}{\Gamma((n+k)/2)}.
$$

By the identity $\Gamma(t) = (t-1)\Gamma(t-1)$, we get

$$
\Gamma((n+k)/2) \geq ((n-k)/2)\Gamma((n-k)/2),
$$

which completes the proof. \qed

Let $Z \sim N_n(0, I_n)$ and let $\mathcal{V}$ be any measurable subset of $\mathbb{R}^n$. Let $b := \sup_{v \in \mathcal{V}} \|v\|$ and assume that $b$ is finite. Let $M := \sup_{v \in \mathcal{V}} Z \cdot v$. The following concentration inequality for $M$ was proved by [Tsirelson, Ibragimov and Sudakov (1976)], although it follows with slightly worse constants from earlier works of [Sudakov and Tsirelson (1974)] and [Borell (1975)]. For any $t \geq 0$,

$$
P(|M - \mathbb{E}(M)| \geq t) \leq 2e^{-t^2/(2b^2)}.
$$

(12)

In the familiar version of this inequality, the set $\mathcal{V}$ is assumed to be finite. It is easy to pass to arbitrary bounded measurable $\mathcal{V}$ by approximating $M$ by maxima over finite subsets of $\mathcal{V}$ and observing that $M$ and $\mathbb{E}(M)$ can be recovered in the limit of such approximations.
We will later need upper bounds on the moments of \( M \). When \( \mathcal{V} \) is a finite set, it is easy to give general upper bounds, as follows.

**Lemma 5.3.** Let \( M, b \) and \( \mathcal{V} \) be as above. Let \( N \) be the size of the set \( \mathcal{V} \). Suppose that \( 3 \leq N < \infty \). Then for any integer \( k \in [1, 2 \log N] \),

\[
(E|M|^k)^{1/k} \leq 3b\sqrt{\log N}.
\]

**Proof.** Note that for any even integer \( k \geq 2 \),

\[
E(M^k) \leq \sum_{v \in \mathcal{V}} E(Z \cdot v)^k \leq N b^k (k-1)!!.
\]

Consequently, if \( m_k := (E|M|^k)^{1/k} \), then for any even \( k \),

\[
m_k \leq N^{1/k} b k^{1/2} \left( 1 - \frac{1}{k} \right)^{1/k} \left( 1 - \frac{3}{k} \right)^{1/k} \cdots \left( 1 - \frac{k-1}{k} \right)^{1/k}.
\]

The inequality \( 1-x \leq e^{-x} \) implies that the product on the right is bounded by \( e^{-1/4} \). On the other hand, by Hölder’s inequality, if \( k \leq l \) then \( m_k \leq m_l \). Therefore if \( k \leq 2 \log N \) and \( \log N \geq 1 \), then choosing \( l \) to be an even number between \( 2 \log N \) and \( 4 \log N \) (which exists because \( 2 \log N \geq 2 \)), the above inequality applied to \( m_l \) gives

\[
m_k \leq m_l \leq 2e^{1/4}b\sqrt{\log N} \leq 3b\sqrt{\log N}.
\]

This completes the proof of the lemma. \( \square \)

**Proof of Theorem 1.2.** Let \( Z' := (Y - \mu)/\sigma \). Note that

\[
|K(Y) - \sigma K(Z')| \leq K(\mu).
\]

Therefore

\[
|\hat{\sigma} - \sigma| = \frac{|K(Y) - \sigma K(Z)|}{K(Z)} \leq \frac{|K(Y) - \sigma K(Z')|}{K(Z)} + \sigma|K(Z') - K(Z)|
\]

\[
\leq \frac{K(\mu)}{K(Z)} + \frac{\sigma|K(Z') - K(Z)|}{K(Z)}.
\]

Thus,

\[
E(\hat{\sigma} - \sigma)^2 \leq 2K(\mu)^2 E(K(Z)^{-2}) + 2\sigma^2 E(K(Z)^{-2})(K(Z) - K(Z'))^2
\]

\[
\leq 2K(\mu)^2 E(K(Z)^{-2}) + 2\sigma^2 (E(K(Z)^{-4})E((K(Z) - K(Z'))^4))^{1/2}.
\]

By Lemma 5.2 this gives

\[
E(\hat{\sigma} - \sigma)^2 \leq \frac{2K(\mu)^2 m_2^2}{(n-2)^2} + \frac{2\sigma^2 m_2^2 (E((K(Z) - K(Z'))^4))^{1/2}}{(n-4)^2}.
\]

(13)

Now recall that by the identity \( \mathbf{7} \),

\[
K(Z) = K^{oo}(Z) = \sup_{v : K^{oo}(v) \leq 1} Z \cdot v.
\]

Therefore by the concentration of Gaussian maxima (inequality \( \mathbf{12} \)),

\[
P(|K(Z) - E(K(Z))| \geq t) \leq 2e^{-t^2/2\sigma^2}
\]
for each \( t \geq 0 \). Since \( Z' \sim N_n(0, I_n) \), this implies that
\[
\mathbb{P}(\|K(Z) - K(Z')\| \geq t) \leq \mathbb{P}(\|K(Z) - \mathbb{E}(K(Z))\| \geq t/2) + \mathbb{P}(\|K(Z') - \mathbb{E}(K(Z))\| \geq t/2)
\]
\[
\leq 4e^{-t^2/8a^2}.
\]
Thus,
\[
\mathbb{E}(\|K(Z) - K(Z')\|^4) \leq \int_0^\infty 16t^3e^{-t^2/8a^2}\,dt
\]
\[
= \int_0^\infty 8ue^{-u^2/8a^2}\,du = 512a^4.
\]
Substituting this in (13), we get the desired inequality.

\[\square\]

**Proof of Theorem 1.3.** Simply combine Theorems 1.1 and 1.2 and use the inequality
\[
|\hat{\sigma}^2 - \sigma^2| \leq (\hat{\sigma} - \sigma)^2 + 2|\hat{\sigma} - \sigma|
\]
and finally Hölder’s inequality to bound \( \mathbb{E}|\hat{\sigma} - \sigma| \leq (\mathbb{E}(\hat{\sigma} - \sigma)^2)^{1/2} \).

\[\square\]

5.3. **Proof of Theorem 2.1.** First, suppose that \( X \) has rank \( n \), so that \( Y' = Y \). In this case we will use Theorem 1.3 with \( \mu = X\beta_0 \). Define two functions on \( \mathbb{R}^n \) as
\[
K(x) := \min\{|\beta|_1 : \beta \in \mathbb{R}^p, x = X\beta\},
\]
\[
\tilde{K}(x) := \min\{|\beta|_1 : \beta \in \mathbb{R}^{p+n}, x = \tilde{X}\beta\}.
\]

It is easy to prove that these are norms, since \( X \) and \( \tilde{X} \) have rank \( n \). Take any \( v \in \mathbb{R}^n \) such that \( K(v) \leq 1 \). Then there exists \( \beta \in \mathbb{R}^p \) such that \( |\beta|_1 \leq 1 \) and \( v = X\beta \). Therefore,
\[
Z \cdot v = Z \cdot X\beta = \sum_{j=1}^p \beta_j(Z \cdot X_j) \leq |\beta|_1 \max_{1 \leq j \leq p} |Z \cdot X_j|.
\]

By the definition (1) of the dual norm \( K^* \), this shows that
\[
K^*(Z) \leq \max_{1 \leq j \leq p} |Z \cdot X_j|.
\]

Therefore by Lemma 5.3
\[
\mathbb{E}(K^*(Z)) \leq 3\gamma\sqrt{n \log p}
\]
provided that \( p \geq 3 \). Similarly, one has
\[
\tilde{K}^*(Z) \leq \max_{1 \leq j \leq p+n} |Z \cdot \tilde{X}_j|.
\]

Let \( \tilde{X}_j \) denote the \( j^{th} \) column of \( \tilde{X} \). By the construction of \( \tilde{X} \), \( \max_{1 \leq j \leq p+n} \|\tilde{X}_j\|/\sqrt{n} = \gamma \). This implies, by Lemma 5.3 that
\[
(\mathbb{E}(\tilde{K}^*(Z)^k))^{1/k} \leq 3\gamma\sqrt{n \log (p + n)}
\]
for every integer \( k \in [1, 2\log p] \).

Next, take any \( v \in \mathbb{R}^n \) such that \( \tilde{K}^*(v) \leq 1 \). Since \( \tilde{K}(\tilde{X}_j) \) is clearly \( \leq 1 \) for each \( j \), therefore \( |v \cdot \tilde{X}_j| \leq 1 \) for every \( j \) by the definition of the dual norm. The cases \( j = p+1, \ldots, p+n \) for this
inequality imply that the components of $v$ are all bounded by $(\sqrt{n\gamma})^{-1}$. Consequently, $\|v\| \leq \gamma^{-1}$.

Thus,
\[
\sup\{\|v\| : v \in \mathbb{R}^n, \tilde{K}(v) \leq 1\} \leq \gamma^{-1}.
\]  

(16)

Plugging in the estimates (14), (15) and (16) into Theorem 1.3 and observing that $K(X\beta_0)$ and $\tilde{K}(X\beta_0)$ are both bounded above by $|\beta_0|_1$, we get the statement of Theorem 2.1 when rank($X$) = $n$. Moreover, Theorem 1.2 gives the desired upper bound on $\mathbb{E}(\hat{\sigma} - \sigma)^2$.

Next, suppose that rank($X$) = $k < n$. Then there is a $k \times n$ matrix $A$ that maps the column space of $X$ onto $\mathbb{R}^k$ and preserves inner products. Let $X'' = AX$ and $Y'' = AY'$. Then $Y'' \sim N_k(X''\beta_0, \sigma^2 I_k)$. Moreover, $\|Y'' - X''\beta\| = \|Y' - X'\beta\|$ for any $\beta \in \mathbb{R}^p$. Therefore the definition of $\hat{\beta}$ implies that
\[
\hat{\beta} = \text{argmin}\{|\beta|_1 : \beta \in \mathbb{R}^p, \|Y'' - X''\beta\|^2 \leq k\hat{\sigma}^2\},
\]
where $\hat{\sigma} := \tilde{K}(Y)/\tilde{K}(Z)$. Define $K''$ on $\mathbb{R}^k$ as
\[
K''(x) := \min\{|\beta|_1 : \beta \in \mathbb{R}^p, x = X''\beta\}.
\]

Again, it is easy to prove that this is a norm since $X''$ has rank $k$. To complete the proof, apply Theorem 1.1 with $Y''$ and $K''$ in place of $Y$ and $K$ and use Theorem 1.2 to get a bound for $\mathbb{E}|\hat{\sigma}^2 - \sigma^2|$ using the estimates (15) and (16) obtained above. Lastly, note that $\|X''\hat{\beta} - X''\beta_0\| = \|X\hat{\beta} - X\beta_0\|$, and multiply the resulting inequality by $k/n$.

5.4. Proof of Theorem 3.1. We can put this problem into the setting of Theorem 1.3 by letting $n = lm$, and writing elements of $\mathbb{R}^n$ as $l \times m$ matrices by putting the first $l$ components as the first column, components $l + 1$ through $2l$ as the second column, and so on. For an element $x \in \mathbb{R}^n$ let $M(x)$ denote the corresponding matrix. Define $K(x)$ to be the nuclear norm of the matrix $M(x)$. It is easy to see that this is indeed a norm on $\mathbb{R}^n$. Taking this $K$ in Theorem 1.3 and $\tilde{K} = K$, it is easy to see that the estimator $\hat{M}$ is precisely the estimator prescribed by Theorem 1.3 in this setting.

Recall that the spectral norm of $M$ is defined as
\[
\|M\| := \max_{1 \leq i \leq k} s_i.
\]
Suppose that $A$ is an $l \times m$ matrix with $\|A\| \leq 1$. Then
\[
\|A\|_{HS}^2 \leq k\|A\|^2 \leq k.
\]
Combining this with the well-known fact that the spectral norm is the dual of the nuclear norm (see Horn and Johnson (1991), page 214), it follows that the quantity $a$ of Theorem 1.3 is bounded by $\sqrt{k}$.

Next, let $Z = (z_{ij})_{1 \leq i,j \leq m}$ be a matrix of i.i.d. $N(0, 1)$ entries. Then by Proposition 2.4 in Rudelson and Vershynin (2010), $(\mathbb{E}\|Z\|)^{1/r} \leq C(r)(\sqrt{l} + \sqrt{m})$ for every $r \geq 1$, where $C(r)$ is a constant that depends only on $r$. The proof is now easily completed by inserting these estimates into the error bounds from Theorem 1.2 and Theorem 1.3.

5.5. Proof of Theorem 3.2. Throughout this proof, $C$ will denote any positive universal constant, whose value may change from line to line. Let $k := \lfloor ls/2 \rfloor$. Then $1 \leq k \leq l$. Let $M = (\mu_{ij})_{1 \leq i \leq l, 1 \leq j \leq m}$ be an $l \times m$ random matrix whose first $k$ rows consist of i.i.d. Uniform$[-\sigma, \sigma]$ random variables. Declare the remaining rows, if any, to be zero. Then note that $M$ has rank $\leq k \leq ls/2$. Since $\|M\|_*$ is the sum of the singular values of $M$ and $\|M\|_{HS}^2$ is the sum of squares
of the singular values of $M$, and the number of nonzero singular values equals the rank of $M$, the Cauchy–Schwarz inequality gives
\[ \|M\|_* \leq (ls)^{1/2} \|M\|_{\text{HS}} \leq (ls/2)^{1/2}(\sigma^2 lms/2)^{1/2} \leq \delta. \]

Let $Y = (y_{ij})_{1 \leq i \leq l, 1 \leq j \leq m}$ be a matrix such that given $M$, the entries of $Y$ are independent, and $y_{ij} \sim N(\mu_{ij}, \sigma^2)$. Then it is not difficult to show that
\[ \mathbb{E}(\text{Var}(\mu_{ij} \mid Y)) \geq C\sigma^2. \]

On the other hand, since $\tilde{\mu}_{ij}$ is a function of $Y$, the definition of variance implies that
\[ \mathbb{E}((\tilde{\mu}_{ij} - \mu_{ij})^2 \mid Y) \geq \text{Var}(\mu_{ij} \mid Y). \]

Combining the last two displays, we get
\[ \mathbb{E}\|\tilde{M} - M\|_{\text{HS}}^2 \geq \sum_{i=1}^k \sum_{j=1}^m \mathbb{E}(\tilde{\mu}_{ij} - \mu_{ij})^2 \geq Ckm\sigma^2 \geq Clm\sigma^2 s, \]

which completes the proof of the theorem.

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