A Note on Reflexive Banach Algebras

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August 26, 2022

Abstract

We inspect the properties of reflexive Banach algebras, which are related to the pointwise products of its weakly null sequences.

1 Introduction

We had studied the properties of the space $wpL(A)$ for a Banach algebra $A$ in a previous article [5]. When $A$ is reflexive as a Banach space, $wpL(A)$ is comprised of almost periodic functionals. This simple observation in [5] led us to decompose a reflexive amenable Banach algebra as a direct sum of two ideals, one of which is finite dimensional and the other admits no nonzero almost periodic functionals. Our main tool is the set $N$ of all weak limit points of the pointwise products of the weakly null sequences. One of the results (Corollary 3.6) we obtain is an equivalent statement of the conjecture by Gale, Ransford, and White on reflexive amenable Banach algebras [4].

In Section 3, we study the properties of the primitive ideals and the maximal left ideals that contain $N$, as well as their implications for reflexive amenable Banach algebras. Inspired by Corollary 3.6, in Section 4 we provide some properties of reflexive unital Banach algebras, which are spanned by $N$.

2 Preliminaries and Notation

Let $A$ be a Banach algebra and $E$ be a Banach left $A$-module. The multiplication operator $L_a : E \to E$ is defined by $L_a x = ax$. Moreover, one defines

$$fa(x) = f(ax) = xf(a)$$

for all $a \in A$, $x \in E$ and $f \in E^*$. The maps $T_f : A \to E^*$ and $S_f : E \to A^*$ are defined by $T_f a = fa$ and $S_f x = xf$.

When $E$ is a Banach $A$-bimodule, the right multiplication operator $R_a : E \to E$ is defined by $R_a x = xa$. Further,

$$af(x) = f(xa) = fx(a)$$
for all $a \in A$, $x \in E$ and $f \in E^*$. The maps $\sigma_f : A \to E^*$ and $\tau_f : E \to A^*$ are
defined by $\sigma_f a = af$ and $\tau_f x = fx$. Clearly, $T_f = \tau_f$ and $S_f = \sigma_f$ when $E = A$.

The center of an $A$-bimodule $E$ is

$$Z(A, E) = \{ m \in E : am = ma \quad \forall a \in A \}$$

If $E = E_1 \oplus E_2$ as a direct sum of $A$-bimodules, then

$$Z(A, E) = Z(A, E_1) \oplus Z(A, E_2)$$

as $Z(A)$-modules. In fact, for any $M \in Z(A, E)$ there exist unique $m_i \in E_i \ (i = 1, 2)$ such that $M = m_1 + m_2$. Thus, for all $a \in A$

$$0 = aM - Ma = (am_1 - m_1a) + (am_2 - m_2a)$$

where, obviously, $(am_i - m_ia) \in E_i \ (i = 1, 2)$. Thus, $am_i = m_ia \ (i = 1, 2)$.

We follow the notation and terminology in [10] for the tensor products. For two Banach spaces $X$ and $Y$, $X \otimes_Y Y$ and $X \hat{\otimes}_Y Y$ denote the projective and injective tensor products, respectively. $X \hat{\otimes}_Y Y^*$ is isomorphic to the space of approximable operators $A(X, Y)$, and $(X \hat{\otimes}_Y Y^*)^*$ is isomorphic to the space of all bounded linear operators $B(X, Y^*)$. If $X$ or $Y$ has the metric approximation property (AP), then the canonical map $X \hat{\otimes}_Y Y \to (X^* \hat{\otimes}_Y Y^*)^*$ is an isometric embedding [10, Theorem 4.14]. If $X$ and $Y$ are reflexive (thus have the Radon-Nikodym property) and have AP, then $X \hat{\otimes}_Y Y = (X^* \hat{\otimes}_Y Y^*)^*$ [10, Theorem 5.33]. Thus

$$(X \hat{\otimes}_Y Y)^{**} = (X \hat{\otimes}_Y Y) \oplus (X^* \hat{\otimes}_Y Y^*)^⊥$$

as Banach spaces, where $(X^* \hat{\otimes}_Y Y^*)^⊥$ is the set of all $M \in (X \hat{\otimes}_Y Y)^{**}$ such that $M(X^* \hat{\otimes}_Y Y^*) = \{0\}$. When $A$ is a Banach algebra, $A \hat{\otimes}_Y A$ and $A \hat{\otimes}_Y A$ are Banach $A$-bimodules, e.g., the appendix in [9].

For a Banach algebra $A$, $\text{rad}(A)$ denotes the Jacobson radical of $A$. For a subset $S \subseteq A$, its hull $h(S)$ is the set of all primitive ideals of $A$ that contain $S$. If $I$ is the ideal generated by $S$, then $h(S) = h(I)$. We refer the reader to [2, 6, 7] for an extensive treatment of the general theory of Banach algebras.

**Lemma 2.1.** Let $A$ be a Banach algebra, $E$ be a Banach left $A$-module, and $f \in E^*$. Then, the following are equivalent.

i. If $(a_n)$ in $A$ and $(x_n)$ in $E$ are weakly null sequences, then $f(a_n x_n) \to 0$.

ii. If $(a_n)$ is a weakly null (resp. weakly Cauchy) sequence in $A$ and $(x_n)$ is a weakly Cauchy (resp. weakly null) sequence in $E$, then $f(a_n x_n) \to 0$.

iii. $T_f$ maps weakly precompact sets onto $L$-sets.

ii'. $S_f$ maps weakly precompact sets onto $L$-sets.

If $E$ is a Banach $A$-bimodule, then the following are also equivalent.
iv. If \((a_n)\) in \(A\) and \((x_n)\) in \(E\) are weakly null sequences, then \(f(x_na_n)\to 0\).

v. If \((a_n)\) is a weakly null (resp. weakly Cauchy) sequence in \(A\) and \((x_n)\) is a weakly Cauchy (resp. weakly null) sequence in \(E\), then \(f(x_na_n)\to 0\).

vi. \(\tau_f\) maps weakly precompact sets onto \(L\)-sets.

vi’. \(\sigma_f\) maps weakly precompact sets onto \(L\)-sets.

The proof of Lemma 2.1 is verbatim similar to the proof of [5, Lemma 3.1].

**Definition 2.1.** Let \(A\) be a Banach algebra and \(E\) be a Banach left \(A\)-module.

a. \(ap(A,E)\) (resp. \(wap(A,E)\)) is the set of all \(f\in E^*\) such that \(T_f\) is compact (resp. weakly compact),

b. \(wpL(A,E)\) is the set of all \(f\in E^*\) such that \(T_f\) maps weakly precompact sets onto \(L\)-sets,

c. \(lcc(A,E)\) (resp. \(rcc(A,E)\)) is the set of all \(f\in E^*\) such that \(T_f\) (resp. \(S_f\)) is completely continuous,

and \(cc(A,E) = lcc(A,E) \cap rcc(A,E)\). When \(A = E\), we drop \(E\) and simply write \(wpL(A), lcc(A), \) etc.

Clearly, \(ap(A,E) \subseteq cc(A,E) \subseteq lcc(A,E) \cup rcc(A,E) \subseteq wpL(A,E)\). If both \(A\) and \(E\) are reflexive as Banach spaces, then \(ap(A,E) = cc(A,E) = wpL(A,E)\).

**Proposition 2.2.** Let \(A\) be a unital Banach algebra and \(E\) be a Banach left \(A\)-module. If \(wpL(A) = \{0\}\) then \(wpL(A,E) = \{0\}\).

**Proof.** Let \(f \in wpL(A,E)\). Then, it is not difficult to show that \(xf \in wpL(A)\) for every \(x \in E\) by Lemma 2.1. Consequently, \(f(E) = f(AE) = \{0\}\). Thus, \(f = 0\). \(\Box\)

We could weaken the assumption that \(A\) is unital. The conclusion of Proposition 2.2 remained intact if we merely assumed that the closure of \(AE\) is equal to \(E\).

**Lemma 2.3.** Let \(A\) be a unital Banach algebra such that \(wpL(A) = \{0\}\), and let \(E\) be a Banach \(A\)-bimodule. Then, no nonzero right multiplication operator \(R_a : E \to E\) is compact.

**Proof.** \(wpL(A,E) = \{0\}\) by Proposition 2.2. Suppose there exists \(a \neq 0\) such that \(R_a\) is compact. Since \(A\) is unital, there exists \(f \in A^*\) such that \(af \neq 0\) by Hahn-Banach theorem. Therefore, \(S_af = S_fR_a\) is compact, i.e., \(af \in wpL(A,E)\). Contradiction. \(\Box\)
3 The Kernel of \( \wp L(\mathcal{A}) \)

**Definition 3.1.** Let \( \mathcal{A} \) be a reflexive Banach algebra.

\[
W = \wp L(\mathcal{A})^\perp = \bigcap \{ \ker f : f \in \wp L(\mathcal{A}) \}
\]

and \( N \subseteq \mathcal{A} \) is set of all \( z \in A \) such that \( x_n y_n \to z \) weakly for two weakly null sequences \( (x_n), (y_n) \) in \( A \).

**Lemma 3.1.** Let \( \mathcal{A} \) be a reflexive Banach algebra, and \( N \) be defined as in Definition 3.1. For \( l = 1, \ldots, n \) let \( (x_k^l)_{k \in \mathbb{N}} \) be weakly null sequences such that \( x_k^1 \cdots x_k^n \to z \) weakly. Then, \( z \in N \).

**Proof.** The proof is by induction. The base step \( n = 2 \) is true by definition. Suppose the hypothesis is true for \( n = m - 1 > 2 \). Let \( (x_k^l)_{k \in \mathbb{N}}, \ l = 1, \ldots, m \) be weakly null sequences such that \( x_k^1 \cdots x_k^n \to z \) weakly. For brevity, let \( w_k = x_k^2 \cdots x_k^n \). Since \( \mathcal{A} \) is reflexive, \((w_k)\) has a weakly convergent subsequence, say \( w_k \to z_2 \) weakly. By the intermediary step, \( z_2 \in N \). Clearly, \((z_k^1(w_k - z_2))\) is a product of two weakly null sequences, which converges weakly to \( z \). Thus, \( z \in N \).

**Lemma 3.2.** Let \( \mathcal{A} \) be a reflexive Banach algebra, and \( N, W \) be defined as in Definition 3.1. Then,

1. \( AN \subseteq N, NA \subseteq N \). If \( \mathcal{A} \) is unital, then \( AN = N = NA \).
2. \( W \) is a closed two-sided ideal and \( A/W \) has jwsc multiplication.
3. \( \wp L(\mathcal{A}) = N^\perp \). Thus, \( W \) is the closed linear span of \( N \).
4. \( W = \bigcap \{ \ker T_f : f \in \wp L(\mathcal{A}) \} = \bigcap \{ \ker S_f : f \in \wp L(\mathcal{A}) \} \)

**Proof.**
1. Let \( a \in \mathcal{A} \) and \( z \in N \). Let \((x_n), (y_n)\) be two weakly null sequences in \( \mathcal{A} \) such that \( x_n y_n \to z \) weakly. Clearly, \((ax_n)\) is weakly null and \( ax_n y_n \to az \) weakly. Thus, \( az \in N \). Similarly, \( za \in N \).
2. Let \( x \in \mathcal{A} \). \( fx, xf \in \wp L(\mathcal{A}) \) for every \( f \in \wp L(\mathcal{A}) \). Thus, \( f(xW) = fx(W) = \{0\} \) and \( f(Wx) = xf(W) = \{0\} \) for all \( f \in \wp L(\mathcal{A}) \). Hence, \( xW \cup Wx \subseteq W \).
3. Second, since \( W \) is a two-sided ideal, then \( ap(A/W) = ap(A) \cap W^\perp = ap(A) \) by [3, Corollary 4.3]. Thus, \( A/W^* = ap(A/W) \).
4. \( \wp L(\mathcal{A}) \subseteq N^\perp \) by [5, Lemma 3.1]. Conversely, if \( f \notin \wp L(\mathcal{A}) \), then there exist \( r > 0 \), weakly null sequences \((x_n), (y_n)\) such that \( |f(x_n y_n)| \geq r \) for all \( n \in N \). By the weak compactness of the unit ball, there is a subsequence \((x_{n_k} y_{n_k})\) weakly converging to some \( z \in N \). Thus, \( |f(z)| \geq r \) so \( f(N) \neq \{0\} \).
5. Second, \( W = \wp L(\mathcal{A})^\perp = N^\perp \). Thus, \( W \) is the closed linear span of \( N \).
6. \( W \subseteq \ker T_f \subseteq \ker f \) for any \( f \in \wp L(\mathcal{A}) \). In fact, for any \( x \in W \)

\[
f(xA) = f(xA) \subseteq f(W) = \{0\}
\]

Consequently, \( W \subseteq \bigcap \{ \ker T_f : f \in \wp L(\mathcal{A}) \} \subseteq \bigcap \{ \ker f : f \in \wp L(\mathcal{A}) \} = W \). Second equality is obtained similarly. \( \square \)
**Theorem 3.3.** Let $A$ be a reflexive unital Banach algebra, and $N$ be defined as in Definition 3.1. Let $I$ be a maximal (closed) left ideal and $P$ be a primitive ideal of $A$. Then,

a. there exists $f \in A^*$ such that $I = \ker S_f$ and $S_f(A)$ is closed in $A^*$.

b. $I \supseteq N$ if and only if $I$ has finite codimension.

c. $P \supseteq N$ if and only if $P$ has finite codimension.

**Proof.**

a. There exists a nonzero $f \in A^*$ such that $I \subseteq \ker f$ by Hahn-Banach theorem. Since $I$ is a left ideal, then $I \subseteq \ker S_f$. Thus, $I = \ker S_f$ by maximality.

Let $X = S_f(A)$ in $A^*$. Clearly, $X$ is reflexive and the map $\pi : A \to B(X)$ defined by $\pi(a)x f = ax f$ is an irreducible representation. Thus, $X = S_f(A) \cong A/I$, and $\pi$ is equivalent to the reduction of the left regular representation of $A$ on $A/I$, e.g., by [6, Theorem 4.2.21].

b. If $I \supseteq N$, then $W \subseteq \ker f$, so $f \in \wp L(A) = \ap (A)$. Thus, $S_f$ is a finite dimensional operator, being compact and having a closed range.

Conversely, if $I$ is finite codimensional, then there exists $f \in A^*$ such that $I = \ker f$. Thus, $I = \ker S_f$ and $S_f$ is a finite dimensional operator. Hence, $f \in \ap (A) = \wp L(A)$ and so $N \subseteq \ker S_f = I$ by Lemma 3.2.

c. Every primitive ideal is of the form $\{a \in A : aA \subseteq I\}$ for some maximal left ideal $I$, and in this case, $P$ is the largest two-sided ideal contained in $I$, e.g., see [6, Theorem 4.1.8].

If $N \subseteq P = \{a \in A : aA \subseteq I\} \subseteq I$, then the maximal left ideal $I$ has finite codimension by (b). Clearly, $P = \ker \pi$ where $\pi : A \to B(A/I)$ is the finite dimensional representation defined by $\pi(a)(x + I) = ax + I$. Thus, $P$ has finite codimension.

Conversely, if $P$ has finite codimension, so does $I$. Thus, $I \supseteq W \supseteq N$. Since $P$ is the largest ideal contained in $I$, then $P \supseteq W \supseteq N$.

**Corollary 3.4.** Let $A$ be a reflexive unital Banach algebra, and $N$ be defined as in Definition 3.1. Then, the following are equivalent.

i. Every irreducible representation is finite-dimensional.

ii. Every primitive ideal has finite codimension.

iii. Every maximal left ideal has finite codimension.

iv. $N \subseteq \rad (A)$.

If $A$ is also amenable, then (i-iv) is equivalent to

v. $A$ is finite-dimensional and semisimple.

**Proof.** (i $\iff$ v) is [4, Corollary 2.3]. (ii $\iff$ iv $\iff$ iii) by Theorem 3.3.

**Theorem 3.5.** Suppose $A$ is a reflexive amenable Banach algebra. Let $N, W$ be defined as in Definition 3.1. Then,

a. $A/W$ is semisimple and finite-dimensional.
b. $\text{wpL}(A)$ is a finite dimensional subspace of $A^*$.

c. The hull $h(N)$ is a finite set, and every $P \in h(N)$ is of the form $P = Az$ for some central idempotent $z \in A$.

d. $W$ has finite codimension, and $W = Ae$ for some central idempotent $e \in A$.

e. $W$ is a reflexive amenable Banach algebra such that $\text{wpL}(W) = \{0\}$.

**Proof.**

a. By [4, Corollary 2.3] since $A/W$ is a reflexive amenable Banach algebra such that every primitive ideal has finite codimension.

b. Clearly $(A/W)^* = \text{wpL}(A)$ and $W^* = A^*/\text{wpL}(A)$. Since $A/W$ is finite dimensional, so is $\text{wpL}(A)$.

c. $h(N)$ is finite by (a.). Each $P \in h(N)$ has finite codimension, so complemented in $A$. Thus, $P$ has an identity, say $z \in P$. $zx = zxz = xz$ for every $x \in A$, i.e., $z$ is a central idempotent. Thus, $P = Az$.

d. Write $h(N) = \{Az_1, \ldots, Az_n\}$. Since $A/W$ is semisimple, $W = \bigcap_{k=1}^{n} A_{z_k} = A_{z_1}z_2 \ldots z_n$.

Clearly, $e = z_1z_2 \ldots z_n$ is a central idempotent.

e. Clearly $W$ is reflexive. Being an ideal, $W$ is amenable if and only if it has a bounded approximate identity. Clearly, $e(xe) = (xe)e = xe$ for every $x \in A$. Thus, $e$ is the identity of $W$, and $W$ is amenable.

Third, given $w \in N$, let $(x_n), (y_n)$ be weakly null sequences in $A$ such that $x_ny_n \to w$ weakly. Then, $(x_ne), (y_ne)$ are weakly null sequences in $W$ such that $x_ney_ne = x_ny_ne \to we = w$ weakly. Thus, $f \in \text{wpL}(W)$ if and only if $N \subseteq \ker f$ if and only if $f = 0$. □

If $W$ in Theorem 3.5 is finite dimensional, then $W = \{0\}$, which is the case precisely when the central idempotent $e = 0$. In this case, $A$ is semisimple and finite dimensional. On the other hand, if $W \neq \{0\}$, then both $W$ and $A$ are infinite dimensional. Consequently,

**Corollary 3.6.** The following two statements are equivalent.

i. Every reflexive amenable Banach algebra is finite dimensional.

ii. If $A$ is a reflexive amenable Banach algebra for which $\text{wpL}(A) = \{0\}$, then $A = \{0\}$.

### 4 Some Implications of $\text{wpL}(A) = \{0\}$

**Proposition 4.1.** Let $A$ be a reflexive unital Banach algebra such that $\text{wpL}(A) = \{0\}$, and $N$ be defined as in Definition 3.1. Then, there exist $m \in \mathbb{N}$ and a finite subset $F = \{z_1, \ldots, z_m\} \subset N$ satisfying the following.

a. $1 = z_1 + \cdots + z_m$, thus $A = N + \cdots + N$ the sum of $m$ copies of $N$. 
b. The sum of \( m-1 \) copies of \( N \) is not dense in \( A \) when \( m > 1 \).

c. \( F \) is a linearly independent set.

**Proof.** Since \( \wp L(A) = \{0\} \), then \( A = \text{span} N \) by Lemma 3.2. Thus, there exists \( u \in \text{span} N \) with \( \|1-u\| < 1 \). \( u \) is a unit, so \( 1 \in u^{-1}\text{span} N \subset \text{span} N \) by Lemma 3.2. Let

\[
m = \min\{|F|: F \subseteq N \text{ and } 1 = \sum_{z \in F} z\}.
\]

and \( F = \{z_1, \ldots, z_m\} \subset N \) with \( 1 = z_1 + \cdots + z_m \). Then, \( A = A_{z_1} + \cdots + A_{z_m} \subseteq N + \cdots + N \) a sum of \( m \) copies of \( N \).

If the sum of \( m-1 \) copies of \( N \) were dense in \( A \), then there existed a unit \( u = w_1 + \cdots + w_{m-1} \in A \), where \( w_k \in N \). The set \( E = u^{-1}\{w_1, \ldots, w_{m-1}\} \subset N \) satisfies \( |E| < m \) and \( 1 = \sum_{z \in E} z \), which contradicts with (4).

Third, for a given \( z \in F \), let \( L_z \) be a maximal left ideal that contains \( \sum_{w \in F \setminus \{z\}} Aw. \ z \notin L_z \) since otherwise \( A = \sum_{w \in F} Aw \subseteq Az + L_z = L_z \). There exists \( \gamma_z \in A^* \) such that \( \gamma_z(z) = 1 \) and \( L_z \subseteq \ker \gamma_z \). As a result, \( F \) is a linearly independent set.

The following corollary of Theorem 3.5 improves [8, Proposition 3.4] by removing the approximation property from the hypotheses.

**Corollary 4.2.** Let \( A \) be a simple, reflexive, amenable Banach algebra which possess a non-zero \( f \in \text{ap}(A) \). Then \( A \) is finite-dimensional.

**Proof.** Since \( A \) is reflexive, then \( \wp L(A) = \text{ap}(A) \neq \{0\} \), thus \( W \) is a proper ideal. Since \( A \) is simple, \( W = \{0\} \). Consequently, \( A \) is finite-dimensional by Theorem 3.5.

Corollary 4.2 could be phrased equivalently as: if \( A \) is a simple, reflexive, amenable Banach algebra such that \( \wp L(A) \neq \{0\} \), then \( A \) is finite-dimensional.

**Theorem 4.3.** Let \( A \) be a reflexive unital Banach algebra for which \( \wp L(A) = \{0\} \). Then, \( A \) contains no minimal idempotents.

**Proof.** Suppose \( p \in A \) is a minimal idempotent. Since \( p \notin \text{rad}(A) \), then there exists a maximal left ideal \( L \subset A \) such that \( p \notin L \). Since \( \wp L(A) = \{0\} \), then the Banach left \( A \)-module \( E = A/L \) is infinite dimensional by Theorem 3.3. Clearly, since \( A \) is reflexive as a Banach space, so is \( E \).

Let \( \pi : A \to B(E) \) be the reduction of the left regular representation. Then, \( \pi \) is an irreducible representation of \( A \) (e.g. [6]), \( \pi(p) \neq 0 \) and \( \pi(p) \) is a minimal idempotent of \( \pi(A) \). Thus, \( \pi(A) \) contains the space of approximable operators \( A(E) \) by [1, Theorem 3]. On the other hand, since \( \pi(A) \) is reflexive as a Banach space, so is \( A(E) \). Thus, \( E \) must be finite dimensional. Contradiction.
Thus, \( Z = (A \otimes \pi A)^* \) is a linear isomorphism and

\[
(A \otimes \pi A)^* = (A \otimes \pi A) \oplus (A^* \otimes \pi A^*)
\]
as Banach spaces by (2). This direct sum is, indeed, a direct sum of A-bimodules. Moreover, by (1)

\[
Z(A, (A \otimes \pi A)^*) = Z(A, A \otimes \pi A) \oplus Z(A, (A^* \otimes \pi A^*)^*).
\]

**Theorem 4.4.** Suppose \( A \) is a reflexive unital Banach algebra and \( \text{wpL}(A) = \{0\} \). Then, \( Z(A, A \otimes \pi A) \subseteq \ker J \). If \( A \) also has AP, then \( Z(A, A \otimes \pi A) = \{0\} \).

**Proof.** Let \( 1 \) denote the identity of \( A \). Let \( m = \sum_{i=1}^{\infty} x_i \otimes y_i \in A \otimes A \) such that \( am = ma \) for all \( a \in A \). For a given \( \phi \in A^* \), define \( T : A \to A \) by

\[
Ta = \sum_{i=1}^{\infty} \phi(y_i) x_i,
\]
and let \( v = T1 \). \( T \) is compact, since it is the norm-limit of finite dimensional operators. Second, for each \( \psi \in A^* \),

\[
\psi(Ta) = \sum_{i=1}^{\infty} \psi(x_i) \phi(y_i) a = \psi \otimes \phi(ma) = \psi \otimes \phi(am) = \psi \left( \sum_{i=1}^{\infty} ax_i \phi(y_i) \right) = \psi(av)
\]

Thus, \( T = R_v \) is a compact right multiplication operator on \( A \). Hence, \( v = 0 \) by Lemma 2.3.

Consequently, \( \psi \otimes \phi(m) = \sum_{i=1}^{\infty} \psi(x_i) \phi(y_i) = 0 \) for each \( \phi, \psi \in A^* \). In particular, if \( A \) has AP, we conclude that \( m = 0 \).

As a final note, suppose \( A \) is a reflexive amenable Banach algebra \( A \) with AP, and let \( W \) be defined as in Definition 3.1. Every reflexive amenable Banach algebra \( A \) has an identity, so \( W \) is a reflexive unital Banach algebra such that \( \text{wpL}(W) = \{0\} \) by Theorem 3.5. \( W \) also has AP. In fact, if \( e \) is the identity of \( W \), \( K \subseteq W \) a compact subset, and \( T : A \to A \) a finite dimensional operator, then \( eTe : W \to W \) is a finite dimensional operator and

\[
\sup_{x \in K} \|x - (eTe)x\| \leq \sup_{x \in K} \|e(x - Tx)\| \leq \|e\| \sup_{x \in K} \|x - Tx\|.
\]

Thus, \( Z(W, W \otimes \pi W) = \{0\} \) by Theorem 4.4. Consequently, every virtual diagonal \( M \) of \( W \) annihilates \( W^* \otimes \pi W^* \).

**Corollary 4.5.** Let \( A \) be a reflexive amenable Banach algebra with AP. If \( \mathcal{B}(A, A^*) = \mathcal{K}(A, A^*) \), then \( A \) is finite dimensional.

**Proof.** Let \( W \subseteq A \) be as in Definition 3.1. Since \( W \) has finite codimension in \( A \) by Theorem 3.5, then \( \mathcal{B}(W, W^*) = \mathcal{K}(W, W^*) \). Also \( W \) is a reflexive amenable Banach algebra with AP such that \( \text{wpL}(W) = \{0\} \) by Theorem 3.5, and \( Z(W, W \otimes \pi W) = \{0\} \) by Theorem 4.4. Every virtual diagonal \( M \) of \( W \) annihilates \( W^* \otimes \pi W^* = \mathcal{K}(W, W^*) = \mathcal{B}(W, W^*) = (W \otimes \pi W)^* \), so \( M = 0 \). Thus, \( W = \{0\} \), so \( A \) is finite dimensional.
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