Introduction to Division Algebras, Sphere Algebras and Twistors

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ABSTRACT

A very basic introduction is given to the roles of division algebras and the related sphere algebras concerning the structure of space-time in the dimensionalities $D = 3, 4, 6$ and $10$, with special emphasis on twistors transformations for light-likeness conditions and Hopf maps, together with some outlook for particle and string theory.

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This introduction, given the condition that it should fit into a single lecture, will be quite schematic. Many mathematical and technical details will be left out, for the purpose of giving a more intuitive overview of the subject. The intention is that the listener/reader should get a starting point for more detailed or extensive study. There will be nothing said about supersymmetry, although its incorporation in the context of twistors is very natural and interesting.

The basic object to start with is a null vector $P^\mu$ in Minkowski space. It can be thought of as the momentum for a massless particle. It fulfills

$$P^2 = 0.$$  

In four dimensions, $P^\mu$ may be expressed as a bilinear of spinors:

$$P^\mu = \frac{1}{2} \lambda \gamma^\mu \lambda.$$  

This is a "twistor transformation" [1]. This is most easily seen using the isomorphism $SO(1, 3) \approx SL(2; \mathbb{C})$ [1,2]. Then, a spinor is a two-component complex object

$$\lambda = \begin{bmatrix} \xi \\ \eta \end{bmatrix},$$

and a vector is a hermitean $2 \times 2$-matrix

$$P = \begin{bmatrix} \sqrt{2}p^+ & p^* \\ p & \sqrt{2}p^- \end{bmatrix} = "P^\mu \gamma^\mu".$$  

The gamma matrices are

$$\gamma^+ = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}, \quad \gamma^- = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{bmatrix}, \quad \gamma^I = \begin{bmatrix} 0 & e_I^* \\ e_I & 0 \end{bmatrix},$$

where $e_I$, $I = 0, 1$, forms a basis for the complex numbers. The spinor bilinear in (1)

$$P = \lambda \lambda^\dagger = \begin{bmatrix} \xi \xi^* & \xi \eta^* \\ \eta \xi^* & \eta \eta^* \end{bmatrix}$$

is easily seen to be light-like. In $SL(2; \mathbb{C})$ language, this reads

$$P^2 = P \text{tr} P.$$  

For which dimensionalities is it possible to generalize these statements? If we take the gamma matrices formally as in (2), where $\{e_I\}$ is a basis for the transverse space, we obtain two conditions:
• Existence of Clifford algebra $\implies$ Alternative algebra,
• Light-likeness of spinor bilinear $\implies$ Division algebra.

To clarify, a (normed) division algebra \[5\] is an algebra (not necessarily associative) where $|ab| = |a||b|$, and alternativity means that $[a, b, c] \equiv (ab)c - a(bc)$ is completely antisymmetric. There is a theorem \[3,4,5\], that the only (real) alternative division algebras are

\[\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\] the real numbers, the complex numbers, the quaternions, and the octonions \[6\].

I denote them $\mathbb{K}_\nu$, where $\nu = 1, 2, 4, 8$ are the dimensionalities. The uniqueness of these algebras is the fundamental reason for the well known statement

$$(\lambda_1 \gamma_\mu \lambda_2) \gamma^\mu \lambda_3 + \text{cyclic permutations} = 0$$

in $D = \nu + 2 = 3, 4, 6, 10$, which is a stronger version of the light-likeness of the spinor bilinear. These dimensionalities are the ones where twistor transforms exist \[1,7,8\].

Here follows some notation and properties of the division algebras:

Orthonormal basis: $\mathbb{K}_\nu \ni x = \sum_{I=0}^{\nu-1} x_I e_I$ \hspace{1em} ($e_0 = 1$).

Conjugation $(x \to x^*)$: $e_0 \to e_0$, $e_i \to -e_i$, $i = 1, \ldots, \nu - 1$.

Norm: $|x| = (x^*x)^{1/2}$.

Real part: $[x] = \frac{1}{2}(x + x^*)$.

Imaginary part: $\{x\} = \frac{1}{2}(x - x^*)$.

Multiplication table (illustrated for $\mathbb{O}$ in the figure below, where moving clockwise in the triangle gives positive sign, and the triangle may be rotated any integer multiple of $\frac{2\pi}{7}$):

$$e_i e_j = -\delta_{ij} + \sigma_{ijk} e_k,$$

$\sigma$ antisymmetric, $\sigma_{i,i+1,i+n} = 1$ where $\nu = 2^n$,

$$e_{i+n-1} = e_i.$$
\(\mathbb{R}, \mathbb{C}\) commutative, associative.
\(\mathbb{H}\) non-commutative, associative.
\(\mathbb{O}\) non-commutative, non-associative (but alternative);

\[ [e_i, e_j, e_k] = 2\rho_{ijkl}e_l, \quad \rho_{ijkl} = -({^*}\sigma)_{ijkl} = -\frac{1}{6}\epsilon_{ijklmnp}\sigma_{mnp}. \]

**Space-time and division algebras:**

The isomorphism \(SL(2; \mathbb{C}) \approx SO(1, 3)\) is generalized to \([2]\)

\[ SL(2; \mathbb{K}_\nu) \approx SO(1, \nu + 1). \]

Also the conformal algebras of \((\nu + 2)\)-dimensional space-time are naturally formulated as

\[ Sp(4; \mathbb{K}_\nu) \approx SO(2, \nu + 2). \]

There is a natural interpretation in terms of Jordan algebras \([9]\): A Jordan algebra has a symmetric product fulfilling

\[ X \odot (X \odot Y) = (X \odot X)(X \odot Y). \]

The \(n \times n\) hermitian matrices with division algebra entries is a basis for a Jordan
algebra for all \( n, \nu \neq 8 \), and for \( n \leq 3, \nu = 8 \). This means that the vectors

\[
V = \begin{bmatrix} \sqrt{2}v^+ & v^* \\ v & \sqrt{2}v^- \end{bmatrix}
\]

in \((\nu + 2)\)-dimensional Minkowski space with product \( U \circ V = \frac{1}{2}(UV + VU) \) form a Jordan algebra. The Lorentz group is its structure group, and the conformal group its conformal group.

**Light-like lines and projective spaces:**

The celestial sphere in \( \nu + 2 \) dimensions (the space of light-cone directions) is \( S^\nu \).

We have two ways of representing it:

- \( \{ P \mid P^2 = \text{tr} P \} \),
- \( \{ \lambda \text{ modulo transformation leaving } \lambda \lambda^\dagger \text{ invariant} \} \).

The first one is a well known representation of the projective space \( \mathbb{K}_\nu P^1 = S^\nu \).

To understand the second one, we must examine the invariance of \( \lambda \lambda^\dagger \).

\[
\begin{align*}
\delta(\xi \xi^*) = 0 \implies & \xi \rightarrow \xi \Omega, \quad |\Omega| = 1, \\
\delta(\eta \xi^*) = 0 \implies & \eta \rightarrow \eta \xi^* \Omega,
\end{align*}
\]

where \( a \circ b = (a(bX^*))X = (aX^*)(Xb) = X^*((Xa)b) \).
The set of transformations (parametrized by \( \Omega \)) obviously has the topology \( S^{\nu-1} \). For the cases \( \nu = 1, 2, 4 \) the groups are \( S^0 = \mathbb{Z}_2, S^1 = U(1) \) and \( S^3 = SO(3) \), realized by (right) multiplication by unit elements in \( \mathbb{K}_\nu \). For the case \( \nu = 8 \), we have the space \( S^7 \), which, due to non-associativity, is not a group.

However, the transformations (3) close to a “group”! With the infinitesimal version \( \delta_\alpha \xi = \xi \alpha, \delta_\alpha \eta = \eta \circ X_\alpha ([\alpha] = 0): \)

\[
[\delta_\alpha, \delta_\beta] = \delta_{[\alpha,\beta]} X, \tag{4}
\]

where \([a, b]_X = a \circ b - b \circ a\).

Explanation: There is a classification of “parallelizable” manifolds [10,11]. Parallelizable here means that there are as many globally defined orthonormal vector-fields as the dimension of the manifold.

Examples: \( S^1 \) is a trivial example, where the vector field is tangent to the circle. \( S^2 \), like any even-dimensional sphere, does not have any vector field on it – it can not be “combed”.

The only simply connected compact parallelizable manifolds are the Lie groups and \( S^7 \). If these vector fields exist one can use them to define parallel transport of vectors. Since transport around any closed curve gives back the same vector, the curvature of the corresponding connection vanishes. We can think of the manifold equipped with this connection as “flat”, and the transport as translation.

If the parallelizing connection is written \( \tilde{\Gamma} = \Gamma - T \), where \( \Gamma \) is the metric connection, the vielbeins will not be covariantly constant, but transport as \( \mathcal{D} e = T \) (\( T \) is torsion, and this can be taken as its definition). Then it follows from \([\mathcal{D}_m, \mathcal{D}_n] = R_{mn}\) that

\[
[\mathcal{D}_a, \mathcal{D}_b] = 2T_{ab}^c \mathcal{D}_c \quad (m, n \text{ space indices}, a, b \text{ tangent indices}).
\]

These are our \( S^7 \) transformations [12,13]. What distinguishes \( S^7 \) from the Lie groups is that its torsion (“structure constants”) vary over the space. Explicitly, one choice of the torsion tensor is the one in (4):

\[
T_{ijk}(X) = [e^*_i(e_j \circ e_k)].
\]

So, back to the spinor realization of the celestial sphere \( S^\nu \) ! We found that \( \lambda \lambda^\dagger \) is invariant under \( S^{\nu-1} \) transformations. The celestial sphere is the space of \( S^{\nu-1} \).
orbits in $\mathbb{R}^{2\nu}$ modulo a positive real scale, \textit{i.e.} in $S^{2\nu-1}$. The modding out of $S^{\nu-1}$ is the Hopf map [14].

$$
\begin{align*}
S^1 & \longrightarrow S^1 = \mathbb{R}P^1 \\
S^3 & \longrightarrow S^2 = \mathbb{C}P^1 \\
S^7 & \longrightarrow S^4 = \mathbb{H}P^1 \\
S^{15} & \longrightarrow S^8 = \mathbb{O}P^1.
\end{align*}
$$

Equivalently, the total space $S^{2\nu-1}$ is a (topologically non-trivial) fiber bundle over $S^{\nu}$ with $S^{\nu-1}$ as fiber.

The twistor transform is the translation between the Jordan algebra and homogeneous coordinates descriptions of $\mathbb{K}_\nu P^1$.

When formulated in terms of orbits in a two-component $\mathbb{K}_\nu$-valued object (the spinor) the maps look simple. The topology is quite non-trivial – we take the complex Hopf map as example. Any $S^1$ orbit is linked to any other. We refer to reference [15] for an illustration.

Particles:

One passes to twistor variables with the transform

$$
P = \lambda \lambda^\dagger \quad (P^\mu = \frac{1}{2} \lambda \gamma^\mu \lambda)$$

$$\omega = X \lambda \quad (\omega = X_\mu (\gamma^\mu \lambda))$$

The spinor $\omega$ is conjugate to $\lambda$. Counting the degrees of freedom:

$$
(X, P) : (\nu + 2) - 2 \cdot 1 = 2(\nu + 1)$$

$$
(\lambda, \omega) : 2 \cdot 2\nu - 2(\nu - 1) = 2(\nu + 1)
$$

The twistor variables form together

$$
Z = \begin{bmatrix} \lambda \\ \omega \end{bmatrix}
$$

which is a spinor of the conformal group, $Sp(4; \mathbb{K}_\nu)$. The conformal symmetry can be made manifest (it can in a space-time picture, too). From $Z$, $S^{\nu-1}$ generators can be formed.
Strings:
The same equations hold. If we want to treat left- and rightmovers separately, $P$ and $X$ are not independent – $P = \partial X$. This gives rise to constraints between $\lambda$ and $\omega$ (spoiling conformal invariance). These are much like the spinorial constraints for the superparticle or superstring: half first class, half second class [16]. The first class ones are the Virasoro constraint and an $S^{\nu-1}$ Kac-Moody constraint.

Counting degrees of freedom:

\[
X : (\nu + 2) - 2 \cdot 1 = \nu
\]

\[
(\lambda, \omega) : 2 \cdot 2\nu - 1 \cdot \frac{1}{2} \cdot 2\nu - 2 \cdot 1 - 2 \cdot (\nu - 1) = \nu
\]

The second class constraints pose problems for covariant quantization. The twistor formulation, that solves this problem for the superparticle, has just the same obstacle to covariant quantization in the bosonic sector!

There are different $D = 10$ twistor-like approaches [17 and references therein], where one takes a set of eight twistors instead of one. Then the KM symmetry is $\hat{SO}(8)$ instead of $\hat{S}^7$. I don’t think anyone has performed a canonical analysis on these, but the problem with second class constraints remain.

There is an alternative string twistor formulation [18], where left- and right-movers mix (off shell), and where all constraints are first class. It is reached by using both $P$ and $X$, and replacing $P$ by the sum $\lambda_1\lambda_1^\dagger + \lambda_2\lambda_2^\dagger$. Then, nothing can be said about $P$ except that it is light-like or time-like. Apart from the two $S^{\nu-1}$’s there is a $\nu$- dimensional mixture between the two $\lambda$’s. The manifolds of $\lambda$’s giving the same $P$ can be shown to be $U(1)$, $SO(3) \times U(1)$, $SO(5)$ and $SO(9)/G_2$. The last one is not parallelizable, so it is unclear if this can be used in $D = 10$.

Even though the problems with the second class constraints remain, the twistor formulation of superstrings has an interesting structure. The Kac-Moody algebra $\hat{S}^{\nu-1}$ is enlarged to an $N = \nu$ superconformal algebra. There is a theorem that SCA’s do not exist for $N > 4$, but this is circumvented by the field-dependence of the $S^7$ structure functions (the torsion tensor).

Generators:

|   | spin 2   |
|---|----------|
| 1 |          |

|   | spin 3/2 |
|---|----------|
| $\nu$ |          |

|   | spin 1   |
|---|----------|
| $\nu - 1$ |          |
An interesting thing is that these algebras are present also in the light-cone superstring [19], that seems to have nothing to do with twistors! One may hope, that it survives as a remnant of a gauge fixed symmetry, like the Virasoro algebra in the light-cone bosonic string.

The algebra $\tilde{S}^7$ [13] has some peculiar features that are not present in the lower-dimensional algebras. The anomalies may be field-dependent, but a redefinition of the current yields a central extension of the ordinary type, with one certain numerical coefficient disregarding the field content! For any field content, the BRST operator can be made to vanish quantum mechanically. We do not yet understand at what point the algebra can be used to make predictive statements about field content. We also lack a superfield formulation for the $N = 8$ superconformal algebra.

The Jordan algebra business can be carried on to $3 \times 3$ matrices of octonions, and to matrices of any dimension for the other division algebras [2,20]. The “exceptional” Jordan algebra $J_3(\mathbb{O})$ is of exceptional interest, since it has exceptional symmetry groups.

Analogies:

|            | $J_2(\mathbb{O})$ | $J_3(\mathbb{O})$ |
|------------|-------------------|-------------------|
| derivations: | $SO(9)$          | $F_4$             |
| structure:  | $SO(1,9)$         | $E_6$             |
| conformal:  | $SO(2,10)$        | $E_7$             |

The matrices $P \in J_3(\mathbb{O})$ with $P^2 = \text{Ptr}P$ modulo a real positive scale form the projective space $\mathbb{O}P^2$. The lack of homogeneous coordinates for $\mathbb{O}P^2$ seems to make a twistor transform impossible. We have however constructed objects in $\mathbb{O}^3$ that seem to be natural generalizations of homogeneous coordinates [13]. They may be useful for constructing a twistor transform for $J_3(\mathbb{O})$, and may have transformation properties under $E_6$ that are “almost spinorial”. Maybe (non-linear) superextensions of exceptional groups with $S^7$ as the remaining bosonic generators can be found, and maybe also a superconformal algebra in $D = 10$ can be formulated this way...
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