Abstract

Automated decision-making tools increasingly assess individuals to determine if they qualify for high-stakes opportunities. A recent line of research investigates how strategic agents may respond to such scoring tools to receive favorable assessments. While prior work has focused on the short-term strategic interactions between a decision-making institution (modeled as a principal) and individual decision-subjects (modeled as agents), we investigate interactions spanning multiple time-steps. In particular, we consider settings in which the agent’s effort investment today can accumulate over time in the form of an internal state—impacts both his future rewards and that of the principal. We characterize the Stackelberg equilibrium of the resulting game and provide novel algorithms for computing it. Our analysis reveals several intriguing insights about the role of multiple interactions in shaping the game’s outcome: First, we establish that in our stateful setting, the class of all linear assessment policies remains as powerful as the larger class of all monotonic assessment policies. While recovering the principal’s optimal policy requires solving a non-convex optimization problem, we provide polynomial-time algorithms for recovering both the principal and agent’s optimal policies under common assumptions about the process by which effort investments convert to observable features. Most importantly, we show that with multiple rounds of interaction at her disposal, the principal is more effective at incentivizing the agent to accumulate effort in her desired direction. Our work addresses several critical gaps in the growing literature on the societal impacts of automated decision-making—by focusing on longer time horizons and accounting for the compounding nature of decisions individuals receive over time.

1 Introduction

Automated decision-making tools increasingly assess individuals to determine whether they qualify for life-altering opportunities in domains such as lending [27], higher education [32], employment [41], and beyond. These assessment tools have been widely criticized for the blatant disparities they produce through their scores [43, 3]. This overwhelming body of evidence has led to a remarkably active area of research into understanding the societal implications of algorithmic/data-driven automation. Much of the existing work on the topic has focused on the immediate or short-term societal effects of automated decision-making. (For example, a thriving line of work in Machine Learning (ML) addresses the unfairness that arises when ML predictions inform high-stakes decisions [18, 22, 31, 8, 1, 16, 11] by defining it as a form of predictive disparity, e.g., inequality in false-positive rates [22, 3] across social groups.) With the exception of several noteworthy recent articles (which we discuss shortly), prior work has largely ignored the processes through which algorithmic decision-making systems can induce, perpetuate, or amplify undesirable choices and behaviors.
Our work takes a long-term perspective toward modeling the interactions between individual decision subjects and algorithmic assessment tools. We are motivated by two key observations: First, algorithmic assessment tools often provide predictions about the latent qualities of interest (e.g., creditworthiness, mastery of course material, or job productivity) by relying on imperfect but observable proxy attributes that can be directly evaluated about the subject (e.g., past financial transactions, course grades, peer evaluation letters). Moreover, their design ignores the compounding nature of advantages/disadvantages individual subjects accumulate over time in pursuit of receiving favorable assessments (e.g., debt, knowledge, job-related skills). To address how individuals respond to decisions made about them through modifying their observable characteristics, a growing line of work has recently initiated the study of the strategic interactions between decision-makers and decision-subjects (see, e.g., [15, 26, 36, 30, 21]). This existing work has focused mainly on the short-term implications of strategic interactions with algorithmic assessment tools—e.g., by modeling it as a single round of interaction between a principal (the decision-maker) and agents (the decision-subjects) [30]. In addition, existing work that studies interactions over time assumes that agents are myopic in responding to the decision-maker’s policy [4, 42, 38, 15]. We expand the line of inquiry to multiple rounds of interactions, accounting for the impact of actions today on the outcomes players can attain tomorrow.

**Our multi-round model of principal-agent interactions.** We take the model proposed by Kleinberg and Raghavan [30] as our starting point. In Kleinberg and Raghavan’s formulation, a principal interacts with an agent once, where the interaction takes the form of a Stackelberg game. The agent receives a score \( y = f(\theta, o) \), in which \( \theta \) is the principal’s choice of assessment parameters, and \( o \) is the agent’s observable characteristics. The score is used to determine the agent’s merit with respect to the quality the principal is trying to assess. (As concrete examples, \( y \) could correspond to the grade a student receives for a class, or the FICO credit score of a loan applicant.) The principal moves first, publicly announcing her assessment rule \( \theta \) used to evaluate the agent. The agent then best responds to this assessment rule by deciding how to invest a fixed amount of effort into producing a set of observable features \( o \) that maximize his score \( y \). Kleinberg and Raghavan characterize the assessment rules that can incentivize the agent to invest in specific types of effort (e.g., those that lead to real improvements in the quality of interest as opposed to gaming the system). We generalize the above setting to \( T > 1 \) rounds of interactions between the principal and the agent and allow for the possibility of certain effort types rolling over from one step to the next. Our key finding is that longer time horizon provides the principal additional latitude in the range of effort sequences she can incentivize the agent to produce. To build intuition as to why repeated interactions lead to the expansion of incentivizable efforts, consider the following stylized example:

**Example 1.1.** Consider the classroom example of Kleinberg and Raghavan where a teacher (modeled as a principal) assigns a student (modeled as an agent) an overall grade \( y \) based on his observable features; in this case test and homework scores. Assume that the teacher chooses an assessment rule and assigns a score \( y = \theta_{TE}TE + \theta_{HW}HW \), where \( TE \) is the student’s test score \( HW \) is his homework score, and \( \theta_{TE}, \theta_{HW} \in \mathbb{R} \) are the weight of each score in the student’s overall grade. The student can invest effort into any of three activities: copying answers on the test, studying, and looking up homework answers online. In a one-round setting where the teacher only evaluates the student once, the student may be more inclined to copy answers on the test or look up homework answers online, since these actions immediately improve the score with relatively lower efforts. However, in a multiple-round setting, these two actions do not improve the student’s knowledge (which impacts the student’s future grades as well), and so these efforts do not carry over to future time steps. When there are multiple rounds of interaction, the student will be incentivized to invest effort into studying, as knowledge accumulation over time takes less effort in the long-run compared to cheating every time. We revisit this example in further detail in Appendix A.

**Summary of our findings and techniques.** We formalize settings in which the agent’s effort investment today can accumulate over time in the form of an internal state—impacting both his future rewards and that of the principal. We characterize the Stackelberg equilibrium of the resulting game and provide novel algorithmic techniques for computing it. We begin by establishing that for the principal, the class of all linear assessment policies remains as powerful as the larger class of all monotonic assessment policies. In particular, we prove that if there exists an assessment policy that can incentivize the agent to produce a particular sequence of effort profiles, there also exists a linear assessment policy which can incentivize the exact same effort sequence.
We then study the equilibrium computation problem, which in general involves optimizing non-convex objectives. Despite the initial non-convexity, we observe that when the problem is written as a function of the agent’s incentivized efforts, the principal’s non-convex objective becomes convex. Moreover, under a common assumption on agent’s conversion mapping from efforts to observable features, the set of incentivizable effort policies is also convex. Given this structure, we provide a polynomial-time algorithm that directly optimizes the principal’s objective over the set of incentivizable effort policies, which subsequently recovers agent’s and principal’s equilibrium strategies. Even though prior work [39, 40] has also taken this approach for solving other classes of non-convex Stackelberg games, our work has to overcome an additional challenge that the agent’s set of incentivizable efforts is not known a-priori. We resolve this challenge by providing a membership oracle (that determines whether a sequence of agent efforts can be incentivized by any assessment policy), which allows us to leverage the convex optimization method due to Kalai and Vempala [28].

Our analysis reveals several intriguing insights about the role of repeated interactions in shaping the long-term outcomes of decision-makers and decision subjects: For example, we observe that with multiple rounds of assessments, both parties can be better off employing dynamic/time-sensitive strategies as opposed to static/myopic ones. Crucially, perhaps our most significant finding is that by considering the effects of multiple time-steps, the principal is significantly more effective at incentivizing the agent to accumulate effort in her desired direction (as demonstrated in Figure 1 for a stylized teacher-student example). In conclusion, our work addresses two critical gaps in the growing literature on the societal impacts of automated decision-making—by focusing on longer time horizons and accounting for the compounding nature of decisions individuals receive over time.

1.1 Related work

A growing line of work at the intersection of computer science and social sciences investigates the impacts of algorithmic decision-making models on people (see, e.g., [25, 44, 34, 15]). As we outline below, significant attention has been devoted to settings in which decision-subjects are strategic and respond to the decision-maker’s choice of assessment rules. Liu et al. [34] and Kannan et al. [29] study how a utility-maximizing decision-maker may respond to the predictions made by a predictive rule (e.g., the decision-maker may interpret/utilize the predictions a certain way or decide to update the model entirely.) Mouzannar et al. [37] and Heidari et al. [23] propose several dynamics for how individuals within a population may react to predictive rules by changing their qualifications. Dong et al. [15], Hu et al. [26], Milli et al. [36] address strategic classification—a setting in which decision subjects are assumed to respond strategically and potentially untruthfully to the choice of the predictive model, and the goal is to design classifiers that are robust to strategic manipulation. Generalizing strategic classification, Perdomo et al. [38] propose a risk-minimization framework for performative predictions, which broadly refers to settings in which the act of making a prediction influences the prediction target. Incentive-aware learning [45, 2] is another generalization that, at a high-level, seeks to characterize the conditions under which one can train predictive rules that are robust to training data manipulations.

Two additional topics that are conceptually related to our work but differ in their motivating problems and goals are adversarial prediction and strategy-proof regression. The adversarial prediction prob-
As mentioned earlier, many of our modeling choices closely follow Kleinberg and Raghavan [30]. Below, we provide a summary of Kleinberg and Raghavan’s results and briefly mention some of the recent contributions following their footsteps. While much of prior work on strategic classification views all feature manipulation as undesirable [15, 26, 36], Kleinberg and Raghavan made a distinction between feature manipulation via \textit{gaming} (investing effort to change observable features in a way that has no positive impact on the quality the principal is trying to measure) and feature manipulation via \textit{improvement} (investing effort in such a way that the underlying characteristics the principal is trying to measure are improved). Their model consists of a single round of interaction between a principal and an agent, and their results establish the optimality and limits of linear assessment rules in incentivizing desired effort profiles. Several papers since then have studied similar settings (see, e.g., Miller et al. [35], Frankel and Kartik [19]) with goals that are distinct from ours. (For example, Frankel and Kartik find a fixed-point assessment rule that improves accuracy by under-utilizing the observable data and flattening the assessment rule.)

Finally, we mention that principle-agent games [33] are classic economic tools to model interactions in which a self-interested entity (the agent) responds to the policy/contract enacted by another (the principal) in ways that are contrary to the principle’s intentions. The principal must, therefore, choose his/her strategy accounting for the agent’s strategic response. Focusing on linear strategies is a common practice in this literature [24, 7, 17]. For simplicity, we present our analysis for linear assessment rules, but later show that the class of all linear assessment policies is equally as powerful as the class of all monotone assessment policies (Theorem 3.4).

2 Problem formulation

In our \textit{stateful} strategic regression setting, a principal interacts with the same agent over the course of $T$ time-steps, modeled via a Stackelberg game.\footnote{To improve readability, we adopt the convention of referring to the principal as she/her and the agent as he/him throughout the paper.} The principal moves first, announcing an \textit{assessment policy}, which consists of a sequence of assessment rules given by parameters $\{\theta_t\}_{t=1}^T$. Each $\theta_t$ is used for evaluating the agent at round $t = 1, \cdots, T$. The agent then best responds to this assessment rule by investing effort in different activities, which in turn produces a series of observable features $\{o_t\}_{t=1}^T$ that maximize his overall score. Through each assessment round $t \in \{1, \cdots, T\}$, the agent receives a score $y_t = f(\theta_t, o_t)$, where $\theta_t$ is the principal’s assessment parameters for round $t$, and $o_t$ is the agent’s observable features at that time. Following Kleinberg and Raghavan, we focus on monotone assessment rules.

\textbf{Definition 2.1 (Monotone assessment rules).} A assessment rule $f(\theta, \cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$ is monotone if $f(\theta, o) \geq f(\theta, o')$ for $o_k \geq o'_k \; \forall k \in \{1, \ldots, n\}$. Additionally, $\exists k \in \{1, \ldots, n\}$ such that strictly increasing $o_k$ strictly increases $f(\theta, o)$.

For convenience, we assume the principal’s assessment rules are linear, that is, $y_t = f(\theta_t, o_t) = \theta_t^\top o_t$. Later we show that the linearity assumption is without loss of generality. We also restrict $\theta_t$ to lie in the $n$-dimensional probability simplex $\Delta^n$. That is, we require each component of $\theta_t$ to be at least 0 and the sum of the $n$ components equal 1.

\textbf{From effort investments to observable features and internal states.} The agent can modify his observable features by investing effort in various activities. While these effort investments are private to the agent and the principal cannot directly observe them, they lead to features that the principal can observe. In response to the principal’s assessment policy, the agent plays an \textit{effort policy}, consisting of a sequence of effort profiles $\{e_t\}_{t=1}^T$, where each individual coordinate of $e_t$ (denoted by $e_{t,j}$) is a function of the principal’s assessment policy $\{\theta_t\}_{t=1}^T$. Specifically, the agent chooses his policy $\{e_1, \cdots, e_T\}$, so that it is a best-response to the the principal’s assessment policy $\{\theta_1, \cdots, \theta_T\}$.

Next, we specify how effort investment translates into observable features. We assume an agent’s observable features in the first round take the form $o_1 = o_0 + \sigma_W(e_1)$, where $o_0 \in \mathbb{R}^n$ is the initial value of the agent’s observable features before any modification, $e_1 \in \mathbb{R}^d$ is the effort the agent
100% than from Appendix G. We assume that the agent’s choice of effort \( e \) (e.g., in the \( j \)-th effort coordinate, \( e_j \)) rolls over from one time step to the next, and \( s_0 \) is the agent’s initial “internal state”. An agent’s observable features are, therefore, a function of both the effort he expends, as well as his internal state. Specifically, \( \alpha_t = \sigma_W(s_t + e_t) \) (here \( \sigma_W(s_0) \) is analogous to \( \alpha_0 \) in the single-shot setting). Note that while for simplicity, we assume the accumulating effort types are socially desirable, our results apply as well to settings where undesirable efforts can similarly accumulate.

**Utility functions for the agent and the principal.** Given the above mapping, the agent’s goal is to pick his effort profiles so that the observable features they produce maximize the sum of his scores over time, that is, the agent’s utility \( Y_t = \sum_{t=1}^{T} y_t = \sum_{t=1}^{T} \theta_t^T \alpha_t \). Our focus on the sum of scores over time is a conventional choice and is motivated by real-world examples. (A small business owner who applies for multiple loans cares about the cumulative amount of loans he/she receives. A student taking a series of exams cares about his/her average score across all of them.)

The principal’s goal is to choose his assessment rules over time so as to maximize cumulative effort investments according to her preferences captured by a matrix \( \Lambda \). Specifically, the principal’s utility \( \max_{\theta_t} \sum_{t=1}^{T} \theta_t^T \Lambda \sum_{t=1}^{T} \theta_t \). The principal’s utility can be thought of as a weighted \( \ell_1 \) norm of the agent’s cumulative effort, where \( \Lambda = \Omega + \Omega \) is a diagonal matrix with the element \( \Lambda_{jj} \) determines how much the principal wants to incentivize the agent invest in effort component \( e_j \).

**Constraints on agent effort.** As was the case in the single-shot setting of Kleinberg and Raghavan, we assume that the agent’s choice of effort \( e_t \) at each time \( t \) is subject to a fixed budget \( B \) (with respect to the \( \ell_1 \) norm). Without loss of generality, we consider the case where \( B = 1 \). We explore the consequences of an alternative agent effort formulation – namely a quadratic cost penalty – in Appendix G.

**Proposition 2.2.** It is possible to incentivize a wider range of effort profiles by modeling the principal-agent interaction over multiple time-steps, compared to a model which only considers one-shot interactions. See Appendix A for an example which illustrates this phenomena.

3 **Equilibrium characterization**

The following optimization problem captures the expression for the agent’s best-response to an arbitrary sequence of assessment rules.\(^2\) (Recall that \( d \) refers to the dimension of effort vectors (\( e_t \)’s), and \( n \) refers to the number of observable features, i.e., the dimension of \( \alpha_t \)’s.)

The set of agent best-responses to a linear assessment policy, \( \{e_t\}_{t=1}^{T} \), is given by the following optimization procedure:

\[
\{e_t\}_{t=1}^{T} = \arg \max_{e_1, \ldots, e_T} \sum_{t=1}^{T} \theta_t^T \sigma_W \left( s_0 + \Omega \sum_{i=1}^{t-1} e_i + e_t \right), \quad \text{s.t. } e_{t,j} \geq 0, \quad \sum_{j=1}^{d} e_{t,j} \leq 1 \forall t, j
\]

The goal of the principal is to pick an assessment policy \( \{\theta_t\}_{t=1}^{T} \) in order to maximize the total magnitude of the effort components she cares about, i.e.

\(^2\) Note that while we only consider diagonal \( \Omega \in \mathbb{R}^{d \times d} \), our results readily extend to general \( \Omega \in \mathbb{R}^{d \times d} \). By focusing on diagonal matrices we have a one-to-one mapping between state and effort components. Non-diagonal \( \Omega \) corresponds to cases where different effort components can contribute to multiple state components.

\(^3\) Throughout this section when it improves readability, we denote the dimension of matrices in their subscript (e.g., \( X_{n \times b} \) means \( X \) is an \( n \times b \) matrix).
While the optimization in Proposition 3.1 is nonconvex in general, we provide polynomial-time algorithms when an effort policy is not dominated, all of these bounds hold with equality and, because of this, where we abuse notation by treating $e^*_t$ as a function of $(\theta_1, \ldots, \theta_T)$. Substituting the agent’s optimal effort policy into the above expression, we obtain the following formalization of the principal’s assessment policy:

**Proposition 3.1 (Stackelberg Equilibrium).** Suppose the principal’s strategy space consists of all sequences of linear monotonic assessment rules. The Stackelberg equilibrium of the stateful strategic regression game, $\{(\theta^*_{e_t})_{i=1}^T, (e^*_t)_{t=1}^T\}$, can be specified as the following bilevel multiobjective optimization problem. As is standard throughout the literature, we assume that the agent breaks ties in favor of the principal. Moving forward, we omit the constraints on the agent and principal action space for brevity.

$$\left\{\theta^*_t\right\}_{t=1}^T = \arg\max_{\theta_1, \ldots, \theta_T} \left\{ \Lambda \sum_{t=1}^T e^*_t(\theta_t, \ldots, \theta_T) \right\}_{1}, \quad \text{s.t.} \quad \theta_t \in \Delta^n \forall t,$$

where we abuse notation by treating $e^*_t$ as a function of $(\theta_1, \ldots, \theta_T)$. Substituting the agent’s optimal effort policy into the above expression, we obtain the following formalization of the principal’s assessment policy:

**Theorem 3.4.** For any effort policy $\{e_t\}_{t=1}^T$ that is not dominated by another effort policy, there exists a linear assessment policy that can incentivize it.

See Appendix C for the complete proof. We characterize whether an effort policy $\{e_t\}_{t=1}^T$ is dominated or not by a linear program, and show that a subset of the dual variables correspond to a linear assessment policy and can incentivize it. Kleinberg and Raghavan present a similar proof for their setting, defining a linear program to characterize whether an effort profile $e_t$ is dominated or not. They then show that if an effort profile is not dominated, the dual variables of their linear program correspond to a linear assessment rule which can incentivize it. While the proof idea is similar, their results do not extend to our setting because our linear program must include an additional constraint for every time-step to ensure that the budget constraint is always satisfied. We show that by examining the complementary slackness condition, we can upper-bound the gradient of the agent’s cumulative score with respect to a subset of the dual variables $\{\lambda_t\}_{t=1}^T$ (where each upper bound depends on the “extra” term $\gamma_t$ introduced by the linear budget constraint for that time-step). Finally, we show that when an effort policy is not dominated, all of these bounds hold with equality and, because of this, the subset of dual variables $\{\lambda_t\}_{t=1}^T$ satisfy the definition of a linear assessment policy which can incentivize the effort policy $\{e_t\}_{t=1}^T$.

## 4 Equilibrium computation for linear effort conversions

While the optimization in Proposition 3.1 is nonconvex in general, we provide polynomial-time algorithms for settings in which the agent’s effort conversion function can reasonably be viewed as being linear, i.e. $\sigma_W = W$, where $W \in \mathbb{R}^{n \times d}$ is the agent’s effort conversion matrix. Each component $w_{ij}$ of $W$ is a nonnegative term which represents how much an increase in observable

$$\left\{\theta^*_t\right\}_{t=1}^T = \arg\max_{\theta_1, \ldots, \theta_T} \left\{ \Lambda \sum_{t=1}^T e^*_t(\theta_t, \ldots, \theta_T) \right\}_{1}, \quad \text{s.t.} \quad \theta_t \in \Delta^n \forall t,$$
which dominates it. We show that if this is the case, then

\[
\{\theta_t\}_{t=1}^{T} = \arg \max \alpha_t \left( \theta_t^T W + \sum_{i=1}^{T-t} \theta_{t+i}^T \Omega \right) \alpha_t
\]

is incentivizable. Since \(\{\theta_t\}_{t=1}^{T}\) is not incentivizable, there exists some effort policy \(\{z_t\}_{t=1}^{T}\) which dominates it. We show that if this is the case, then \(\{z_t\}_{t=1}^{T}\) must dominate either \(\{x_t\}_{t=1}^{T}\) or \(\{y_t\}_{t=1}^{T}\). This is a contradiction, since both are incentivizable. See Appendix E.1 for the full proof.

**Proposition 4.2.** The set of incentivizable effort policies is convex if the effort conversion function is linear.

**Proof Sketch.** In order to show that the set of incentivizable effort policies is convex, we assume that it is not and proceed via proof by contradiction. We construct an effort policy \(\{z_t\}_{t=1}^{T}\) by taking the element-wise average of two incentivizable effort policies \(\{x_t\}_{t=1}^{T}\) and \(\{y_t\}_{t=1}^{T}\), and assume it is not incentivizable. Since \(\{z_t\}_{t=1}^{T}\) is not incentivizable, there exists some effort policy \(\{\zeta_t\}_{t=1}^{T}\) which dominates it. We show that if this is the case, then \(\{\zeta_t\}_{t=1}^{T}\) must dominate either \(\{x_t\}_{t=1}^{T}\) or \(\{y_t\}_{t=1}^{T}\). This is a contradiction, since both are incentivizable. See Appendix E.1 for the full proof.

Note that the linear program from Theorem 3.4 can serve as a membership oracle for this set. To see this, note that given an effort policy \(\{e_t\}_{t=1}^{T}\), the LP returns a value of \(T\) if and only if \(\{e_t\}_{t=1}^{T}\) is incentivizable. We now show how to leverage this membership oracle to recover the principal’s optimal assessment policy in polynomial time.

Define \(\text{CvxOracle}(f, \mathcal{M}, R, r, \alpha_0, \epsilon, \delta)\) to be the membership oracle method of Kalai and Vempala [28], which, for a convex set \(\mathcal{C}\), takes a linear function \(f\) over the convex set \(\mathcal{C}\), membership oracle \(\mathcal{M}\) to the convex set \(\mathcal{C}\), initial point \(\alpha_0\) inside of \(\mathcal{C}\), radius \(R\) of a ball containing \(\mathcal{C}\), and a radius \(r\) of a ball contained in \(\mathcal{C}\) and centered at \(\alpha_0\) as input, and returns a member of the convex set which minimizes \(f\) up to some \(O(\epsilon)\) term, with probability at least \(1 - \delta\). We now present an informal version of their main theorem, followed by our algorithm.
Theorem 4.3 (Main Theorem of Kalai and Vempala [28] (Informal)). For any convex set $C \subseteq \mathbb{R}^n$, given a membership oracle $\mathcal{M}$, starting point $\alpha_0$, upper bound $R$ on the radius of the ball containing $C$, and lower bound $r$ on the radius of the ball containing $C$, the algorithm of Kalai and Vempala [28] returns a point $\alpha_T$ such that $f(\alpha_T) - \min_{\alpha \in C} f(\alpha) \leq \epsilon$ with probability $1 - \delta$, where the number of iterations is $I = \mathcal{O}(\sqrt{n \log(Rn/\epsilon \delta)})$, and $\mathcal{O}(n^4)$ calls to the membership oracle are made at each iteration.

Algorithm 1: Assessment Policy Recovery

Result: An assessment policy $\{\theta_t^*\}_{t=1}^T$

- Define $C$ to be the set of incentivizable effort policies;
- Let $f(\{e_t\}_{t=1}^T) = - \| \Lambda \sum_{t=1}^T e_t \|_1$, where $\{e_t\}_{t=1}^T$ is an incentivizable effort policy;
- Define $\mathcal{M}$ to be the linear program from Theorem 3.4;
- Fix an arbitrary assessment policy $\{\theta_{t,0}\}_{t=1}^T$. Solve for initial incentivizable effort policy $\{e_{t,0}\}_{t=1}^T$ as in Proposition 1;
- Set $R = \frac{T^{(d-1)}(d-1)!}{2T^{(d-1)+1}}$;
- $\{e_{t,1}\}_{t=1}^T = \text{CvxOracle}(f, \mathcal{M}, R, r, \{e_{t,0}\}_{t=1}^T, \epsilon, \delta)$;
- Set the primal variables of $\mathcal{M}$ equal to $\{e_{t,1}\}_{t=1}^T$, and solve a system of linear equations to recover the dual variables $\{\theta_t^*\}_{t=1}^T$;

Theorem 4.4 (Optimal Assessment Policy). Let $C$ be the set of incentivizable effort policies. Assuming that $C$ contains a ball with radius at least $r$ centered at $\{e_{t,1}\}_{t=1}^T$, the assessment policy $\{\theta_t^*\}_{t=1}^T$ recovered by Algorithm 1 is an $\epsilon$-optimal assessment policy, with probability at least $1 - \delta$.

Before proceeding the proof sketch for Theorem 4.4, we remark that the assumption of $C$ containing a ball of radius $r$ is commonplace within the membership oracle-based convex optimization literature, both in theory [20, 28], and practice (e.g., [5]). The assumption implies that if it is possible to incentivize an agent to play effort policy $\{e_{t,1}\}_{t=1}^T$, then it is also possible to incentivize them to play other effort policies within a small margin of $\{e_{t,1}\}_{t=1}^T$.

Proof Sketch. The proof consists of several steps. First, note that the agent’s effort policy consists of $T$ $d$-dimensional probability simplexes, which is a $T(d-1)$-dimensional simplex. The circumradius (i.e., the minimum radius of a ball containing the $T(d-1)$-dimensional simplex) is $R = \frac{T^{(d-1)}(d-1)!}{2T^{(d-1)+1}}$.

Next, we observe that we can use the linear program defined in the proof of Theorem 3.4 as a membership oracle to the set of incentivizable effort policies. Finally, we observe that the function we are trying to optimize is linear and that it is possible to identify an initial point $\{e_{t,0}\}_{t=1}^T$ within the convex set $C$. We can then use a membership oracle-based convex optimization procedure such as Kalai and Vempala [28] to recover the incentivizable effort policy which is most desirable to the principal (up to some $\mathcal{O}(\epsilon)$ term, with high probability) in polynomial time. Given this effort policy, we can use the complementary slackness conditions of our membership oracle linear program to recover the corresponding dual variables, a subset of which will correspond to an assessment policy which can incentivize the agent to play this effort policy. See Appendix E for full details.

The existence of such a membership oracle-based method shows that tractable algorithms exist to recover the principal’s optimal assessment policy, and heuristics need not be resorted to under a large class of settings, despite the bilevel multiobjective optimization problem which must be solved.

4.1 How many rounds are necessary to implement a desired effort profile?

In the classroom example, we saw that a wider range of effort profiles can be incentivized by extending the fixed budget setting of Kleinberg and Raghavan to multiple time-steps. But how long does the time horizon have to be in order to incentivize a desired effort profile if the principal can pick the time horizon? Additionally, what conditions are sufficient for an effort profile to be incentivizable? We formalize the notion of $(T, t)$-Implementability in the linear effort conversion function setting with these questions in mind.

Definition 4.5 ($(T, t)$-Implementability). A basis vector $b_j$ is said to be $(T, t)$-implementable if a rational agent can be motivated to spend their entire effort budget on $b_j$ for all times $1 \leq t' \leq t$. 

Figure 2: Contribution of studying vs $T - t$. We plot $\Omega_S$ (x-axis) vs $T - t$ (y-axis) for our classroom example in Appendix A. Note that $\Omega_S$ is assumed to be 1 in the appendix. Lighter colors indicate settings in which the student has more incentive to cheat. As long as $\Omega_S > 0$, there exists some time horizon under which the student is incentivized to study. As $\Omega_S$ increases, the number of extra time-steps required to incentivize studying decreases.

Theorem 4.6. If $T \geq t + \max_c \min_m \max_{i,j} \left\{ 0, W_{mc} - W_{mj} \right\}$ and $\Omega_{jj} W_{mj} - \Omega_{cc} W_{mc} > 0$, then $b_j$ is $(T,t)$-implementable. See Appendix F for the full derivation. This bound shows that any basis vector is incentivizable if it accumulates faster than other effort profiles. In the worst case, the space of incentivizable effort profiles is the same as in Kleinberg and Raghavan (just set $T = 1$). However, if an effort component accumulates faster than other effort components, there will always exist a time horizon $T$ for which it can be incentivized. In our classroom example, as long as the student retains some knowledge from studying, there always will exist a time horizon for which it is possible to incentivize the student to study (see Figure 2). Note that while the principal may be interested in incentivizing more than just basis vectors, there does not appear to be a closed-form lower bound for $T$ for non-basis effort profiles.

5 Concluding discussion

We proposed a simple and tractable model in which a principal assesses an agent over a series of timesteps to steer him in the direction of investment in desirable but unobservable types of activities. Our work addresses three crucial gaps in the existing literature, stemming from restricted focus on (1) short-term interactions, (2) with myopic agents, (3) ignoring the role of earlier effort investments (i.e., the state) on future rewards. We observe that within our stateful strategic regression setting, the principal is capable of implementing a more expansive space of average effort investments. Our main results consisted of algorithms for computing the equilibrium of the principal-agent interactions, and characterizing several interesting properties of the equilibrium. There are several natural extensions and directions for future work suggested by our basic model and findings.

Alternative cost functions. Following Kleinberg and Raghavan [30], we assumed throughout our analysis that the agent has a fixed effort budget in each round. One natural extension of our model is to explore alternative cost formulations for the agent. In Appendix G, we provide the analysis for one natural alternative—that is, a cost term which scales quadratically with the total effort expended. Our findings generally remain unaltered. The main qualitative difference between the equilibria of the fixed budget vs. quadratic cost is the following: While under the fixed budget setting, the agent’s optimal effort policy is a sequence of basis vectors and the principal’s optimal assessment policy generally is not, we find that the opposite is true under the quadratic cost setting. We believe the case-study of quadratic costs provides reassuring evidence for the robustness of our results to the choice of the cost function, however, we leave a more systematic study of equilibrium sensitivity to agent cost function as an interesting direction for future work.

Bounded rationality. While we assumed the principal and the agent in our model respond rationally and optimally to each other’s strategies, in real-world scenarios, people and institutions are often not fully rational. Therefore, it would be interesting to consider models where our players’ rationality is bounded, e.g., by designing assessment policies that are robust to suboptimal effort policies and are capable of implementing desired investments despite the agent’s bounded rationality.

Unknown model parameters & learning. We assumed the fundamental parameters of our model (e.g., $\sigma, \Omega, \Lambda$, and $T$) are public knowledge. It would be interesting to extend our work to settings
where not all these parameters are known. Can we design learning algorithms that allow the players to learn their optimal policy over time as they interact with their counterparts?

Other simplifying assumptions. Finally, we made several simplifying assumptions to gain the insights offered by our analysis. In particular, our algorithms for recovering the optimal principal and agent policies relied on the agent having a linear effort conversion function. It would be interesting to explore algorithms which work for a wider range of effort conversion functions. Additionally, we assumed that effort expended towards some action was time-independent (e.g., one hour spent studying today is equivalent to one hour spent studying yesterday). It would be interesting to relax this assumption and study settings in which the accumulation of effort is subjected to a discount factor.

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Figure 3: Comparison between the single-step and multi-step scenarios in the hypothetical classroom setting. The single-step formulation does not account for changes in the student’s internal state over time. In the multi-step formulation, effort put towards studying accumulates in the form of knowledge. Modeling this effort accumulation allows the teacher to incentivize the student to study across a wider range of parameter values. The agent can invest effort in 3 actions: cheating on the test (CT), studying (S), and cheating on the homework (CH). $W$ values denote how much one unit of effort translates to the two observable features, test score (T) and homework score (HW). The student’s score ($y_t$) at each time-step is a weighted average of these two observable features. In the multi-step setting, $s_t$ denotes the student’s internal knowledge state at time $t$.

A Formalizing the classroom example

Example A.1. We demonstrate this by revisiting the classroom example. Recall that a teacher assigns a student an overall grade $y = \theta_T T + \theta_H H$, where $T$ is the student’s test score $H$ is their homework score, and $\theta_T$ & $\theta_H$ are the weight of each score in the student’s overall grade. The student can invest effort into any of three activities: copying answers on the test (CT), improves(165,600),(996,650), studying (S), and cheating on the homework (CH). $W$ values denote how much one unit of effort translates to the two observable features, test score (T) and homework score (HW). The student’s score ($y_t$) at each time-step is a weighted average of these two observable features. In the multi-step setting, $s_t$ denotes the student’s internal knowledge state at time $t$.

Now consider a multi-step interaction between a teacher and student in which effort invested in studying carries over to future time-steps in the form of knowledge accumulation. The relationships between observable features and effort expended are now defined as

$$TE_t = TE_0 + W_{CT} CT_t + W_{ST} S_t$$

and

$$HW_t = HW_0 + W_{SH} S_t + W_{CH} CH_t$$

where $TE_0$ and $HW_0$ are the test and homework scores the student would receive if they did not expend any effort. If $W_{CT} = W_{CH} = 3$ and $W_{ST} = W_{SH} = 1$, there is no combination of $\theta_T, \theta_H$ values the teacher can deploy to incentivize the student to study, because the benefit of cheating is just too great. (See [30] for more detail.)

Now consider a multi-step interaction between a teacher and student in which effort invested in studying carries over to future time-steps in the form of knowledge accumulation. The relationships between observable features and effort expended are now defined as

$$TE_t = TE_0 + W_{CT} CT_t + W_{ST} S_t$$

and

$$HW_t = HW_0 + W_{SH} S_t + W_{CH} CH_t$$

where $s_t = \sum_{i=1}^{t} S_i$ is the agent’s internal knowledge state. Instead of assigning students a single score $y_t$, the teacher assigns the student a score $y_t$ at each round by picking $\theta_{T,T}, \theta_{H,H}$ at every step. The student’s grade is then the summation of all scores across time. Suppose $T \geq 3$, where $T$ is the number of rounds of interaction. Consider $W_{CT} = W_{CH} = 3, W_{ST} = W_{SH} = 1$, and $TE_0 = HW_0 = 0$. Unlike in the single-round setting, it is easy to verify that students can now be incentivized to study by picking $\theta_{T,T} = \theta_{H,H} = 0.5 \forall t$. 

15
B Equilibrium derivations

B.1 Agent’s best-response effort sequence

A rational agent solves the following optimization to determine his best-response effort policy:

\[
\left\{ e_t^* \right\}_{t=1}^T = \arg \max_{e_1, \ldots, e_T} \sum_{t=1}^T (y_t = f_t(e_1, \ldots, e_t)) \\
\text{s.t.} \ e_{t,j} \geq 0 \ \forall t, j, \ \sum_{j=1}^d e_{t,j} \leq 1 \ \forall t
\]

Recall that the agent’s score \( y_t \) at each time-step is a function of \( (e_1, \ldots, e_t) \), the sequence of effort expended by the agent so far. Replacing the score \( y_t \) and observable features \( o_t \) with their respective equations, we obtain the expression

\[
\left\{ e_t^* \right\}_{t=1}^T = \arg \max_{e_1, \ldots, e_T} \sum_{t=1}^T \theta_t^\top \sigma W (s_t + e_t) \\
\text{s.t.} \ e_{t,j} \geq 0 \ \forall t, j, \ \sum_{j=1}^d e_{t,j} \leq 1 \ \forall t
\]

where the agent’s internal state \( s_t \) at time \( t \) is a function of the effort he expends from time 1 to time \( t-1 \). Replacing \( s_t \) with the expression for agent state, we get

\[
\left\{ e_t^* \right\}_{t=1}^T = \arg \max_{e_1, \ldots, e_T} \sum_{t=1}^T \theta_t^\top \sigma W \left( s_0 + \Omega \sum_{i=1}^{t-1} e_i + e_t \right) \\
\text{s.t.} \ e_{t,j} \geq 0 \ \forall t, j, \ \sum_{j=1}^d e_{t,j} \leq 1 \ \forall t
\]

C Proof of Theorem 3.4

Proof. Let \( \kappa \) be the optimal value of the following linear program:

\[
V(\{e_t\}_{t=1}^T) = \min_{a_{1}, a_{2}, \ldots, a_{T}} \sum_{t=1}^{T} \|a_t\|_1 \\
\text{s.t.} \ W \left( \Omega \sum_{i=1}^{t-1} a_i + a_t \right) \geq W \left( \Omega \sum_{i=1}^{t-1} e_i + e_t \right), \ a_i \geq 0_d, \ \|a_t\|_1 \leq 1, \ \forall t
\]

Optimization 2 can be thought of as trying to minimize the total effort \( \{a_t\}_{t=1}^T \) the agent spends across all \( T \) time-steps, while achieving the same or greater feature values at every time \( t \) compared to \( \{e_t\}_{t=1}^T \). Let \( \{a_t^*\}_{t=1}^T \) denote the set of optimal effort profiles for Optimization 2. If \( \{e_t\}_{t=1}^T \in \{a_t^*\}_{t=1}^T \), a value of \( \kappa = T \) is obtained. A dominated effort policy is formally defined as follows:

Lemma C.1 (Dominated Effort Policy). An effort policy \( \{e_t\}_{t=1}^T \) is dominated by another effort policy if \( \kappa < T \).

The Lagrangian of Optimization 2 can be written as

\[
L = \sum_{t=1}^{T} \|a_t\|_1 + \sum_{t=1}^{T} \lambda_t W \left( \Omega \sum_{i=1}^{t-1} (e_i - a_i) + e_t - a_t \right) + \gamma_t (\|a_t\|_1 - 1) - \mu_t^\top a_t,
\]

where \( \lambda_t \geq 0_n, \ \mu_t \geq 0_d, \ \forall t \)
In order for stationarity to hold, \( \nabla_{a_t} L(a^*, \lambda^*, \mu^*, \gamma^*) = 0_d \forall t \), where \( x^* \) denotes the optimal values for variable \( x \). Applying the stationarity condition to Lagrangian function, we obtain

\[
1_d - W^T \lambda_t^* - \sum_{i=t+1}^{T} \Omega^T W^T \lambda_i^* + \gamma_t^* \cdot 1_d - \mu_t^* = 0_d, \; \forall t
\]  

(3)

Because of dual feasibility, \( \mu_t^* \geq 0_d \forall t \). By rearranging Equation 3 and using this fact, we can obtain the following bound on \( W^T \lambda_t^* + \sum_{i=t+1}^{T} \Omega^T W^T \lambda_i^* \):

\[
W^T \lambda_t^* + \sum_{i=t+1}^{T} \Omega^T W^T \lambda_i^* \leq (1 + \gamma_t^*) \cdot 1_d, \; \forall t
\]  

(4)

Next we look at the complementary slackness condition. For complementary slackness to hold, \( \mu_t^* a_i^* \neq 0 \forall t \). If \( \kappa = T \), then \( \{e_i\}_{t=1}^{T} \in \{a_i^*\}_{t=1}^{T} \) and therefore \( \{e_t\}_{t=1}^{T} \) is not dominated. If \( \{e_t\}_{t=1}^{T} \) is not dominated, \( \mu_t^* e_t = 0 \forall t \). This means that if \( e_{t,j} > 0 \), \( \mu_{t,j} = 0 \forall t, j \). This, along with Equation 3, implies that

\[
\left[W^T \lambda_t^* + \sum_{i=t+1}^{T} \Omega^T W^T \lambda_i^*\right]_j = 1 + \gamma_t^*
\]

for all \( t, j \) where \( e_{t,j} > 0 \).

Switching gears, consider the set of linear assessment policies \( L \) for which \( \{e_t\}_{t=1}^{T} \) is incentivizable. The set of linear assessment policies for which \( \{e_t\}_{t=1}^{T} \) is incentivizable is the set of linear assessment policies for which the derivative of the total score with respect to the agent’s effort policy is maximal at the coordinates which \( \{e_t\}_{t=1}^{T} \) has support on. Denote this set of coordinates as \( S \), and the set of coordinates which \( e_t \) has support on as \( S_t \). Formally,

\[
L = \left\{ \{\theta_t\}_{t=1}^{T} \left| \nabla_{\theta_t} \sum_{i=1}^{T} (y_i = f(\{a_i\}_{t=1}^{T}, \{\theta_t\}_{t=1}^{T}))\right|_{S_t} = \max_{j} \left( \nabla_{a_t} \sum_{i=1}^{T} y_i \right) \cdot 1_{|S_t|}, \; \forall t \right\}
\]

Recall that \( \sum_{t=1}^{T} y_t = \sum_{t=1}^{T} \theta_t^T W (s_0 + \Omega \sum_{i=1}^{t-1} a_i + a_t) \). Therefore, the gradient of \( \sum_{t=1}^{T} y_t \) with respect to \( a_t \) can be written as

\[
\nabla_{a_t} \sum_{t=1}^{T} y_t = W^T \theta_t + \sum_{i=t+1}^{T} \Omega^T W^T \theta_i, \; \forall t
\]

Note that the form of \( \nabla_{a_t} \sum_{t=1}^{T} y_t \) is the same as the LHS of Equation 4. We know that if \( \{e_t\}_{t=1}^{T} \in \{a_t^*\}_{t=1}^{T} \) is incentivizable, the inequality in Equation 4 will hold with equality for all coordinates for which \( \{e_t\}_{t=1}^{T} \) has positive support. Therefore, the derivative is maximal at those coordinates since it is bounded to be at most \( 1 + \gamma_t^* \), \( \forall t \) (due to the KKT conditions for the dominated effort policy linear program). Because of this, \( \{\lambda_t^*\}_{t=1}^{T} \) is in \( L \), which means that \( \{e_t\}_{t=1}^{T} \) can be incentivized using a linear mechanism.

\( \Box \)
D Equilibrium characterization for fixed budget setting

D.1 Agent effort policy

**Lemma D.1.** Under linear assessment policy \(\{\theta_1, \ldots, \theta_T\}\), a budget constrained agent will play an effort profile from the following set at round \(t\):

\[
e_t^* = \arg \max_{e_t} \left( \theta_t^T W + \left( \sum_{i=1}^{T-t} \theta_{t+i}^T \right) W \Omega \right) e_t
\]

\[
\text{s.t. } e_{t,j} \geq 0, \sum_{j=1}^T e_{t,j} \leq 1 \quad \forall j
\]

**Proof.** The agent’s score at each time \(y_t\) is a function of \((e_1, \ldots, e_t)\). We can replace \(y_t\), \(o_t\), and \(s_t\) with their respective equations to get an expression for the agent’s optimal effort policy \(\{e_t^*\}_{t=1}^T\) that depends on just \(\{\theta_t\}_{t=1}^T\), \(s_0\), \(W\), and \(\Omega\):

\[
\{e_t^*\}_{t=1}^T = \arg \max_{e_1, \ldots, e_T} \sum_{t=1}^T \theta_t^T W \left( s_0 + \Omega \sum_{i=1}^{t-1} e_i + e_t \right)
\]

\[
\text{s.t. } e_{t,j} \geq 0, \sum_{j=1}^T e_{t,j} \leq 1 \quad \forall t, j
\]

After expanding the outer sum over the principal assessment rules \(\{\theta_t\}_{t=1}^T\), factoring based on the agent’s effort at each \(t\), and dropping the initial state terms (as they don’t depend on \(\{e_1, \ldots, e_T\}\)), we get

\[
\{e_t^*\}_{t=1}^T = \arg \max_{e_1, \ldots, e_T} \left( \theta_1^T W + \left( \sum_{i=1}^{T-1} \theta_{i+1}^T \right) W \Omega \right) e_1 + \left( \theta_2^T W + \left( \sum_{i=1}^{T-2} \theta_{i+2}^T \right) W \Omega \right) e_2 + \ldots + \theta_T^T W e_T
\]

\[
\text{s.t. } e_{t,j} \geq 0, \sum_{j=1}^T e_{t,j} \leq 1 \quad \forall t, j
\]

(5)

Note that the optimization step in (5) is linear in the agent effort policy and can be split into \(T\) separate optimization problems, one for each \(e_t\). Thus, the agent can optimize each effort profile \(e_t\) separately by breaking the objective into \(T\) parts, each of which is given by the optimization in Lemma D.1.

Since the above objective function is linear in \(e_t\), the optimal solution for the agent consists of putting his entire effort budget on the highest-coefficient element of \(\theta_1^T W + \left( \sum_{i=1}^{T-t} \theta_{t+i}^T \right) W \Omega\). In the classroom setting (Example 1.1), this corresponds to a situation in which the student only cheats or only studies during each evaluation period. More precisely, let \(m\) denote the maximal element(s) of \(\theta_1^T W + \sum_{i=1}^{T-t} \theta_{t+i}^T W \Omega\). We then characterize the set of optimal agent effort profiles at each time-step as \(e_t^* = \mathbb{1}\{j = m\}\) \((1 \leq j \leq d)\). We assume that agents are rational and therefore play an effort policy \(\{e_t\}_{t=1}^T \in \{e_t^*\}_{t=1}^T\).
D.2 Principal assessment policy

The goal of the principal is to pick an assessment policy \( \{ \theta_t \}_{t=1}^T \) in order to maximize the total magnitude of the agent’s cumulative effort in desirable directions (parameterized by \( \Lambda \)), subject to the constraint that \( \theta_t \) lie in the \( n \)-dimensional probability simplex, i.e.

\[
\{ \theta_t^* \}_{t=1}^T = \arg \max_{\theta_1, \ldots, \theta_T} \left\| \Lambda \sum_{t=1}^T \theta_t \right\|_1
\]

such that \( \theta_t \in \Delta^n \ \forall t \)

From Lemma D.1, we know the form of a rational agent’s effort \( e_t \) will take for every \( t \in \{1, \ldots, T\} \). Substituting this into Equation 6, we obtain the following characterization of the principal’s assessment policy:

\[
\{ \theta_t^* \}_{t=1}^T = \arg \max_{\theta_1, \ldots, \theta_T} \left\| \Lambda \sum_{t=1}^T \theta_t \right\|_1
\]

such that \( \theta_t \in \Delta^n, e_t \geq 0, \sum_{j=1}^T e_{t,j} \leq 1 \ \forall t, j \)

E Proof of Theorem 4.4

E.1 The set of incentivizable effort policies is convex

**Proof.** Let the set of incentivizable effort policies be denoted by \( I = \{ \{a_t\}_{t=1}^T, \{y_t\}_{t=1}^T \} \). In order to show that \( I \) is convex, it suffices to show that for all effort policies \( \{x_t\}_{t=1}^T \) and \( \{y_t\}_{t=1}^T \) in \( I \), their element-wise average \( \{z_t\}_{t=1}^T \) also belongs to the set \( I \). Let the sets of all possible solutions for \( V(\{x\}_{t=1}^T) \) and \( V(\{y\}_{t=1}^T) \) be denoted by \( \{\{e_{x,t}\}_{t=1}^T\} \subseteq I \) and \( \{\{e_{y,t}\}_{t=1}^T\} \subseteq I \). Since \( \{x_t\}_{t=1}^T \in \{\{e_{x,t}\}_{t=1}^T\} \) and \( \{y_t\}_{t=1}^T \in \{\{e_{y,t}\}_{t=1}^T\} \), we use \( \{x_t\}_{t=1}^* \) and \( \{y_t\}_{t=1}^* \) as the solutions to \( V(\{x\}_{t=1}^T) \) and \( V(\{y\}_{t=1}^T) \) without loss of generality. Let the agent’s observable features at time \( t \) when playing effort policy \( \{a_t\}_{t=1}^T \) be denoted by \( g_t(\{a_t\}_{t=1}^T) \). If \( z_t = \frac{x_t + y_t}{2} \) for all \( t \), we know that \( 2g_t(\{z\}_{t=1}^T) = g_t(\{x\}_{t=1}^T) + g_t(\{y\}_{t=1}^T) \) for all \( t \), due to the linearity of agent feature values. Moreover, this holds for any combination of effort policies from \( \{\{e_{x,t}\}_{t=1}^T\} \) and \( \{\{e_{y,t}\}_{t=1}^T\} \).

Suppose that the effort policy \( \{z\}_{t=1}^T \) is not incentivizable. By definition, this must mean that there exists some other effort policy \( \{\xi_t\}_{t=1}^T \) such that an agent can achieve the same feature values at every time-step as he would have received if he had played effort policy \( \{z\}_{t=1}^T \), while expending less total effort at at least one time-step \( s \), i.e.

\[
g_t(\{\zeta_t\}_{t=1}^T) = g_t(\{z_t\}_{t=1}^T), \ \forall t
\]

and

\[
\|\zeta_s\|_1 < \|z_s\|_1, \ s \in \{1, \ldots, T\}.
\]

By linearity, \( z_s \)’s contribution to the agent’s feature values at time \( s \) is equal to the average of \( x_s \) and \( y_s \)’s contributions to the agent’s feature values at time \( s \). This means that \( 2W\zeta_s = Wz_s + Wy_s \). Let \( \zeta^*_s \) equal \( \zeta_s \) rescaled such that \( \|\zeta^*_s\|_1 = 1 \). \( W\zeta^*_s \succ W\zeta_s \) and there exists an index \( \ell \) such that \( |W\zeta^*_\ell| > |W\zeta_\ell| \) (since we assume the effort conversion matrix \( W \) is monotonic). Therefore, \( 2W\zeta^*_s \succ Wx_s + Wy_s \) and \( |W\zeta^*_\ell| > |Wx_\ell + Wy_\ell| \). Denote the effort policy with the rescaled version of \( \zeta_s \) as \( \{\zeta^*_t\}_{t=1}^T = \{\zeta^*_t\}_{t=1} \cup \zeta_s \cup \zeta^*_s \). It now follows that \( 2g_s(\{\zeta_t\}_{t=1}^T) \succ g_s(\{x_t\}_{t=1}^T) + g_s(\{y_t\}_{t=1}^T) \) and \( 2g_s(\{\zeta^*_t\}_{t=1}^T) \succ g_s(\{x_t\}_{t=1}^T) + g_s(\{y_t\}_{t=1}^T) \), which means that \( \zeta^*_s \) must dominate either \( x_s \) or \( y_s \). This is a contradiction, since the effort policies \( \{x_t\}_{t=1}^T \) and \( \{y_t\}_{t=1}^T \) are both incentivizable. Therefore, the set of incentivizable effort policies \( I \) must be convex. \( \square \)
E.2 Membership oracle-based optimization

Now that we have shown that the set of incentivizable effort policies is convex, we can proceed with our membership oracle-based optimization procedure. Our goal is to find the incentivizable effort policy which is most desirable to the principal. Therefore, the function we are trying to minimize is

\[ f(\{a_t\}_{t=1}^T) = -\|A\sum_{t=1}^T a_t\|_1, \]

where \(a_t\) is an incentivizable effort policy and \(A\) is a diagonal matrix where the element \(A_{jj}\) denotes how much the principal wants to incentivize the agent to invest in effort component \(e_j\). Note that this function is linear, as no element of \(\{a_t\}_{t=1}^T\) can be negative. We also need a membership oracle to the convex set of incentivizable effort policies. Fortunately, Optimization 2 gives us such an oracle. In particular, if a given effort policy \(\{e_t\}_{t=1}^T\) is incentivizable, \(V(\{e_t\}_{t=1}^T)\) will equal \(T\). If \(\{e_t\}_{t=1}^T\) is not incentivizable, \(V(\{e_t\}_{t=1}^T)\) will be some value strictly less than \(T\). Armed with these tools, all we need is an initial point \(\{e_{t,0}\}_{t=1}^T\) inside the set of incentivizable effort policies to use a membership oracle-based convex optimization procedure such as [28] to recover the agent effort policy which is most desirable to the principal. We can obtain such a point by fixing an arbitrary assessment policy \(\{\theta_{t,0}\}_{t=1}^T\) and solving the agent’s optimization in Optimization 1 to recover \(\{e_{t,0}\}_{t=1}^T\).

Now that we’ve found the incentivizable agent effort policy that is (approximately) most desirable to the principal, we need to find the assessment policy which incentivizes it. Optimization 2 can help us here as well. Recall that if an effort policy \(\{e_t\}_{t=1}^T\) is incentivizable, a subset of the dual variables of Optimization 2 correspond to a linear assessment policy which can incentivize it. So given the incentivizable effort policy which is most desirable to the principal, we can use the complementary slackness conditions of Optimization 2 to recover the assessment policy which can incentivize it.

F (T, t)-Implementability

Proof. From Proposition 3.1, we know that the agent’s effort profile \(e_t\) at time \(t\) will be a basis vector with weight 1 on the maximal element of \(\theta_t^T W + \left(\sum_{i=1}^{T-t} \theta_{t+i}^T\right) W\). Therefore, if \(b_j\) is the effort profile induced at time \(t\), then

\[
\sum_{k=1}^{n} \left(\theta_{t,k} + \Omega_{jj} \sum_{i=1}^{T-t} \theta_{t+i,k}\right) W_{kj} \geq \sum_{k=1}^{n} \left(\theta_{t,k} + \Omega_{zz} \sum_{i=1}^{T-t} \theta_{t+i,k}\right) W_{kz}, \text{ for } 1 \leq z \leq d
\]

(7)

Since we are interested in deriving an upper bound on \(T\), we can consider just assessment policies of the form \(\theta_t = \theta \forall t - \) that is, we limit the principal to employ the same assessment rule across all time-steps. After making this assumption and collecting terms, Equation 7 becomes

\[
\sum_{k=1}^{n} \theta_k ((W_{kj} - W_{kz}) + (T-t) (\Omega_{jj} W_{kj} - \Omega_{zz} W_{kz})) \geq 0, \text{ for } 1 \leq z \leq d
\]

By solving for \(T\), we obtain

\[
T \geq t + \frac{\sum_{k=1}^{n} \theta_k (W_{kz} - W_{kj})}{\sum_{k=1}^{n} \theta_k (\Omega_{jj} W_{kj} - \Omega_{zz} W_{kz})}, \text{ for } 1 \leq z \leq d
\]

(8)

Since the principal employs the same assessment rule across all time-steps, it is optimal for the principal to play \(\theta_{t,k} = \mathbb{1}\{k = m\}\) \(\forall t\), where \(m\) is the (non-unique) index of \(\theta\) which incentivizes \(b_j\) the most. In other words, \(m\) is the index that minimizes the RHS of Equation 8 while still satisfying \(\Omega_{jj} W_{kj} \geq \Omega_{zz} W_{kz}\) for all \(1 \leq z \leq d\). Equation 8 now becomes

\[
T \geq t + \min_k \frac{(W_{kz} - W_{kj})}{(\Omega_{jj} W_{kj} - \Omega_{zz} W_{kz})}, \text{ for } 1 \leq z \leq d
\]

(9)

Note that if \(\Omega_{jj} W_{mj} - \Omega_{zz} W_{mz} \leq 0\) for some \(z\), then \(b_j\) will never be incentivizable at some generic time \(t\), since this means an undesirable effort component accumulates at a rate faster than
effort component $j$. While this claim only holds for static $\theta$-policies, a similar condition holds for the general case - namely the denominator of the bound in Equation 8 must be greater than 0 for all $z$ in order for an effort profile to be incentivizable. In the classroom example, this would correspond to (the somewhat unrealistic) situation in which a student gains knowledge by cheating faster than he does from studying.

Finally, picking the $z$ index which maximizes the RHS of Equation 9 suffices for Equation 9 to hold for $1 \leq z \leq d$. Since $T \geq t$ must hold, the numerator be at least 0.

$$T \geq t + \max_z \min_m \max \left\{ 0, W_{mz} - W_{mj} \right\} \frac{1}{(\Omega_{jj} W_{mj} - \Omega_{zz} W_{mz})}$$

$\square$

G Alternative agent cost formulation

While we assume that each agent selects their action according to a fixed effort budget at every time-step, another common agent cost model within the strategic classification literature is that of a quadratic cost penalty. We now explore the use of such a cost formulation in our stateful setting.

G.1 Agent’s best-response effort sequence

Under the quadratic cost setting, a rational agent selects his effort policy in order to maximize his total score minus the quadratic cost of exerting the effort over all time steps. Next, we obtain a close-formed expression for the agent’s best-response to an arbitrary sequence of assessment rules under a linear effort conversion function.

Proposition G.1. If the effort conversion function has the form $\sigma_W = W$, the set of agent best-responses to a sequence of linear, monotonic assessment rules, $\{\theta_t\}_{t=1}^T$, is $e^*_t = W^\top \theta_t + (W\Omega)^\top \sum_{i=1}^{T-1} \theta_{t+i} \forall t$.

Proof. The agent solves the following optimization to determine his best-response effort policy:

$$\{e^*_t\}_{t=1}^T = \arg \max_{e_1, \ldots, e_T} \sum_{t=1}^T (y_t = f_t(e_1, \ldots, e_t)) - \frac{1}{2} \left\|e_t\right\|_2^2$$

s.t. $e_{t,j} \geq 0 \forall t, j$

Recall that the agent’s score $y_t$ at each time-step is a function of $(e_1, \ldots, e_t)$, the cumulative effort expended by the agent so far. Replacing the score $y_t$ and observable features $o_t$ with their respective equations, we obtain the expression

$$\{e^*_t\}_{t=1}^T = \arg \max_{e_1, \ldots, e_T} \sum_{t=1}^T \theta_t^\top W(s_t + e_t) - \frac{1}{2} \left\|e_t\right\|_2^2$$

s.t. $e_{t,j} \geq 0 \forall t, j$

where the agent’s internal state $s_t$ at time $t$ is a function of the effort he expends from time 1 to time $t - 1$. Replacing $s_t$ with the expression for agent state, we get

$$\{e^*_t\}_{t=1}^T = \arg \max_{e_1, \ldots, e_T} \sum_{t=1}^T \theta_t^\top W \left( s_0 + \Omega \sum_{i=1}^{t-1} e_i + e_t \right) - \frac{1}{2} \left\|e_t\right\|_2^2$$

s.t. $e_{t,j} \geq 0 \forall t, j$

Our goal is to separate the above optimization into $T$ separate optimization problems for computational tractability. As a first step towards this goal, we expand the sum over the principal’s assessment policy, obtaining the following form:
\[ \{e^*_t\}_{t=1}^{T} = \arg \max_{e_1, \ldots, e_T} \theta_1^T W (s_0 + e_1) + \theta_2^T W (s_0 + \Omega e_1 + e_2) + \ldots + \theta_T^T W \left( s_0 + \Omega \sum_{i=1}^{T-1} e_i + e_T \right) \]
\[ - \frac{1}{2} (\|e_1\|^2_2 + \|e_2\|^2_2 + \ldots + \|e_T\|^2_2) \]

\[ \text{s.t.} \quad e_{t,j} \geq 0 \ \forall t, j \]

Next, we factor the above based on \( e_t \)'s. Additionally, we drop the \( s_0 \) terms, since they do not depend on any \( e_t \).

\[ \{e^*_t\}_{t=1}^{T} = \arg \max_{e_1, \ldots, e_T} \left( \theta_1^T W + \sum_{i=1}^{T-1} \theta_{i+1}^T \Omega W \right) e_1 - \frac{1}{2} \|e_1\|^2_2 + \left( \theta_2^T W + \sum_{i=1}^{T-2} \theta_{i+2}^T \Omega W \right) e_2 \]
\[ - \frac{1}{2} \|e_2\|^2_2 + \ldots + \theta_T^T W e_T - \frac{1}{2} \|e_T\|^2_2 \]
\[ \text{s.t.} \quad e_{t,j} \geq 0 \ \forall t, j \]

Now Equation 10 can be separated based on agent effort profile at each time step \( t \). In particular, for \( e_t \) we have:

\[ e^*_t = \arg \max_{e_t} \left( \theta_t^T W + \sum_{i=1}^{T-t} \theta_{t+i}^T \Omega W \right) e_t - \frac{1}{2} \|e_t\|^2_2 \]
\[ \text{s.t.} \quad e_{t,j} \geq 0 \ \forall j \]

Finally, we can get a closed-form solution for each \( e^*_t \) by taking the gradient with respect to \( e_t \) and setting it equal to 0. Our final expression for \( e^*_t \) is

\[ e^*_t = W^T \theta_t + (W \Omega)^T \sum_{i=1}^{T-t} \theta_{t+i} \] (11)

**Corollary G.2.** The set of effort profiles the agent can play as a best-response to some linear assessment policy at each time step \( t \) grows as the time horizon \( T \) increases.

**Proof.** Fix any time horizon \( T \) and time step \( t \leq T \), the set of effort profiles the agent can play as a best response is a polytope:

\[ S_t(T) = \left\{ W^T \theta_t + (W \Omega)^T \sum_{i=1}^{T-t} \theta_{t+i} \mid \theta_t, \theta_{t+1}, \ldots, \theta_T \in \Delta_n \right\} \]

The corollary then follows from the fact that \( S_t(T) \subset S_t(T + 1) \). \( \square \)

### G.2 Principal’s equilibrium assessment policy

Next, given the form of the agent’s best response to an arbitrary assessment policy, we can derive the principal’s equilibrium strategy as follows:

**Theorem G.3** (Stackelberg Equilibrium). Suppose the principal’s strategy space consists of all sequences of linear monotonic assessment rules. The Stackelberg equilibrium of the stateful strategic regression game, \( \{\{\theta^*_t\}_{t=1}^{T}, \{e^*_t\}_{t=1}^{T}\} \), can be specified as follows:

\[ \forall t : e^*_t = W^T \theta^*_t + (W \Omega)^T \sum_{i=1}^{T-t} \theta^*_{t+i} \]
\[ \theta^*_t = \mathbb{1} \{k = \arg \max \| \Lambda (I + (t-1) \Omega^T) W^T \|_1 \} \].

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We can now solve a separate linear program for each $\theta$. As was the case with the agent’s optimal effort policy, we would like to separate the optimization for different weights in order to maximize a student’s knowledge, or a bank designing a sequence of loans over time in order to encourage good business practices.

Theorem G.4. The principal’s optimal assessment policy $\{\theta_t^*\}_{t=1}^T$ can contain $n$ distinct assessment rules.

Proof. Proposition G.1 already calculates the agent’s best response an arbitrary assessment policy. It only remains to characterize the principal’s best response to the agent.

The principal’s goal is to maximize the value of the agent’s internal state at time $T$. Writing this as an optimization problem, we have

$$\{\theta_t^*\}_{t=1}^T = \arg \max_{\theta_1, \ldots, \theta_T} \left\| \Lambda \sum_{t=1}^T e_t^*(\theta_1, \ldots, \theta_T) \right\|_1$$

subject to $\theta_t \in \Delta^n \forall t$

The sequence $\{\theta_t^*\}_{t=1}^T$ could correspond to a teacher designing a sequence of (test, homework) pairs with different weights in order to maximize a student’s knowledge, or a bank designing a sequence of evaluation metrics to determine the amount a loan applicant receives when applying for a sequence of loans over time in order to encourage good business practices.

From Equation 11 we know the form of the effort profile at each time for a rational agent. Substituting this into Equation 12, we obtain

$$\{\theta_t^*\}_{t=1}^T = \arg \max_{\theta_1, \ldots, \theta_T} \left\| \Lambda \sum_{t=1}^T (W^\top \theta_t + (W \Omega)^\top \sum_{i=1}^{T-t} \theta_{t+i}) \right\|_1$$

subject to $\theta_t \in \Delta^n \forall t$

As was the case with the agent’s optimal effort policy, we would like to separate the optimization for the principal’s optimal assessment policy into $T$ separate optimization problems. The current form can be separated based on $\theta$ because we have closed-form solutions for each $e_t^* (1 \leq t \leq T)$, which are all linear in the principal’s assessment policy $\{\theta_t\}_{t=1}^T$:

$$\{\theta_t^*\}_{t=1}^T = \arg \max_{\theta_1, \ldots, \theta_T} 1_d^\top \Lambda W^\top \theta_1 + 1_d^\top \Lambda (I + \Omega^\top) W^\top \theta_2 + \ldots + 1_d^\top \Lambda (I + (T-1) \Omega^\top) W^\top \theta_T$$

subject to $\theta_t \in \Delta^n \forall t$

We can now solve a separate linear program for each $\theta_t$:

$$\theta_t^* = \arg \max_{\theta_t} 1_d^\top \Lambda (I + (t-1) \Omega^\top) W^\top \theta_t$$

subject to $\theta_t \in \Delta^n$

Our final solution for $\theta_t^*$ has the form $\theta_t^* = 1 \{k = m\}$, where $m$ denotes the maximal element of $1_d^\top \Lambda (I + (t-1) \Omega^\top) W^\top$.

G.3 The dynamicity of equilibrium policies

Given our characterization above, one might wonder if the optimal solution for the principal is to simply play a fixed $\theta$ for all $t \in \{1, \ldots, T\}$. We show that this is generally not the case—specifically, due to the role of $t$ in determining the maximal component of vector $1_d^\top \Lambda (I + (t-1) \Omega^\top) W^\top$.

Theorem G.4. The principal’s optimal assessment policy $\{\theta_t^*\}_{t=1}^T$ can contain $n$ distinct assessment rules.

The general idea of the proof is as follows. The optimization problem for principal’s assessment rule at each time $t$ (Equation 13) is linear with respect to $t$, so any assessment rule $\theta$ which was optimal at some time $t' < t$ but is no longer optimal at time $t$ will never again be optimal at any time $t'' > t$. (This is because $1_d^\top \Lambda (I + (t-1) \Omega^\top) W^\top$ is growing at rate $1_d^\top \Lambda \Omega^\top W^\top$ with respect to $t$, so an element which was maximal at some time $t'$ but is not maximal anymore must have a smaller rate of change than the current maximal element, and will therefore never be maximal again.) So we can conclude that the number of optimal solutions of Equation 13 is at most $n$, since each assessment rule $\theta_t$ in the assessment policy is a basis vector with dimensionality $n$. 

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Figure 4: Left: Comparison of the two terms in each component of vector $V$. The first term decreases as $\frac{1}{k}$, while the second term asymptotically approaches some value as $k$ increases. Right: A scaled version of vector $V$ evaluated for different values of $t$. The blue circles denote the maximum component of $V$ for each time $t$. Elements of $V$ become maximal one-after-another over time.

Next, we provide an example for which there are exactly $n$ optimal solutions. In order to construct such an example, we pick $W$, $\Omega$, and $\Lambda$ to be square, diagonal matrices so that Equation 13 is separable into two terms: one that linearly depends on $t$ and one which has no dependence on $t$. Equation 13 now takes the form $\arg\max_\theta V^\top \theta$, where the $k$th element of $V$ takes the form $W_{kk} \Omega_{kk} + (t - 1) W_{kk} \Omega_{kk}^2$. Equation 13 is linear in $\theta$, so $\theta$ will be a basis vector with a 1 at the index where $V^\top$ is maximal and zeros elsewhere. We pick constants $\{W_{kk}\}_{k=1}^n$ and $\{\Omega_{kk}\}_{k=1}^n$ such that each element $V^{(k)} \in V$ becomes maximal one-after-one over time. Figure ?? shows how the two terms of $V^{(k)}$ change with $k$. Figure ?? shows how different indices of $V$ can be maximal for different times.

Next we provide the full proof for the claim that the principal’s assessment policy contains $n$ distinct assessment rules.

**Proof.** (Theorem G.4) To show that Equation 13 can have up to $n$ optimal solutions throughout time, it suffices to provide a specific example for which this is the case. Let $\theta, e \in \mathbb{R}^n$, $\Omega = \Lambda = \mathbb{R}^{n \times n}$, and $W = \mathbb{R}^{n \times n}$, where $W$ is a diagonal matrix. This corresponds to the case where effort invested in one action corresponds to a change in exactly one observable feature. Under this setting, Equation 13 simplifies to

$$
\theta_t = \arg\max_\theta \left[ \Omega_{11} W_{11} + (t - 1) \Omega_{11}^2 W_{11}, \ldots, \Omega_{kk} W_{kk} + (t - 1) \Omega_{kk}^2 W_{kk}, \ldots, \\
\Omega_{nn} W_{nn} + (t - 1) \Omega_{nn}^2 W_{nn} \right]^\top \theta \\
s.t. \quad \theta_t \in \Delta^n
$$

Equation 14 becomes

$$
\theta_t = \arg\max_\theta V^\top \theta \\
s.t. \quad \theta_t \in \Delta^n
$$

Now let $W_{kk} = \frac{1}{(k+1)^2}$ and $\Omega_{kk} = \frac{k}{100n^3}$ ($1 \leq k \leq n$). Equation 14 becomes

$$
\theta_t = \arg\max_\theta V^\top \theta \\
s.t. \quad \theta_t \in \Delta^n
$$

where

$$
V = \begin{bmatrix}
\frac{1}{400n^3} \left( 1 + (t - 1) \frac{1}{100n^3} \right), & \frac{2}{900n^3} \left( 1 + (t - 1) \frac{2}{100n^3} \right), & \ldots, & \frac{1}{100n^2(n+1)^2} \left( 1 + (t - 1) \frac{1}{100n^2} \right)
\end{bmatrix}
$$
Since Equation 15 is linear in \( \theta \), \( \theta \) will be a basis vector with support on the element of \( V^\top \) which is maximal. It is therefore sufficient to show that each element of \( V^\top \) is maximal at some point in time. We show via proof by induction that there exists some time \( t \in \mathbb{N} \) for which each element of \( V^\top \) is maximal.

**Base case:** \( V_1 \) is the maximal value of \( V \) when \( t = 1 \):

\[
V^\top = \left[ \frac{1}{100n^4}, \frac{2}{500n^4}, \ldots, \frac{1}{100n^4(n+1)^2} \right]^\top.
\]

**Inductive step:** Assume there is some time \( t_k > 1 \) such that the \( k \)th element of \( V \) is maximal. To show that element \( k + 1 \) is maximal at some time \( t_k + \tau_k \) \((\tau_k > 0)\), it suffices to show that there exist some \( \tau_k \) values such that \( V_k < V_{k+1} \) and \( V_{k+1} > V_{k+2+m} \) for all \( m \geq 0 \). It suffices to show this because if \( V_k \) is maximal at time \( t_k \), \( V_{k+1} \) will never be optimal for times \( t_k + \tau_k > t_k \) due to the linearity of the problem.

We first outline the condition for \( V_k < V_{k+1} \):

\[
\frac{k}{100n^3(k+1)^2} \left( 1 + (t_k + \tau_k - 1) \frac{k}{100n^3} \right) < \frac{(k+1)}{100n^3(k+2)^2} \left( 1 + (t_k + \tau_k - 1) \frac{k+1}{100n^3} \right)
\]

Next we solve for \( \tau_k \) and simplify:

\[
\tau_k > \frac{100n^3(k^2 + k - 1)}{2k^2 + 4k + 1} - (t_k - 1) \quad (16)
\]

We outline a similar condition for \( V_{k+1} > V_{k+2+m} \), for all \( m \geq 0 \):

\[
\frac{k+1}{100n^3(k+2)^2} \left( 1 + (t_k + \tau_k - 1) \frac{k+1}{100n^3} \right) > \frac{k+2+m}{100n^3(k+3+m)^2} \left( 1 + (t_k + \tau_k - 1) \frac{k+2+m}{100n^3} \right)
\]

We then solve for \( \tau_k \):

\[
\tau_k < \frac{100n^3 \left( (k+1) (k+3+m)^2 - (k+2+m) (k+2)^2 \right)}{(k+2+m)^2 (k+2)^2 - (k+1)^2 (k+3+m)^2} - (t_k - 1) \quad (17)
\]

Since Equation 17 needs to hold for all \( m \geq 0 \), it suffices to show that it holds for the value of \( m \) which makes the RHS of Equation 17 maximal. To find this \( m \) value, we take the derivative of Equation 17 with respect to \( m \) and set it equal to 0. We find that the RHS of Equation 17 is minimized when \( m \) is negative. However, \( m \geq 0 \), so within the constraints of \( m \), the RHS of Equation 17 is minimized when \( m = 0 \). Setting \( m = 0 \) and simplifying, we obtain

\[
\tau_k < \frac{100n^3 \left( k^2 + 3k + 1 \right)}{2k^2 + 8k + 7} - (t_k - 1) \quad (18)
\]

We now have sufficient conditions for \( V_k < V_{k+1} \) (Equation 16) and \( V_{k+1} > V_{k+2+m} \) (Equation 18). Writing the two inequalities together, we see that

\[
\frac{k^2 + k - 1}{2k^2 + 4k + 1} < \frac{k^2 + 3k + 1}{2k^2 + 8k + 7}
\]

which holds for all values of \( k \geq 1 \). Therefore, \( V_{k+1} \) will be the maximal element of \( V \) at time \( t_k + \tau_k \), where

\[
\frac{100n^3 \left( k^2 + k - 1 \right) }{2k^2 + 4k + 1} - (t_k - 1) < \tau_k < \frac{100n^3 \left( k^2 + 3k + 1 \right) }{2k^2 + 8k + 7} - (t_k - 1) \quad (19)
\]

\( \tau_k \) will be strictly greater than 0 for all values of \( k \), since \( \tau_n > 1 \). (This is a sufficient condition for \( \tau_k > 0 \) \( \forall k \) because \( \tau_k \) decreases as \( k \) increases.) We can see this by subtracting the LHS of Equation 19 from the RHS at \( k = n \) to obtain
100n^3 \left( n^2 + 3n + 1 \right) - (t_n - 1) - \left( \frac{100n^3 \left( n^2 + n - 1 \right)}{2n^2 + 4n + 1} - (t_n - 1) \right) = \frac{n^5 + 4n^4 + 4n^3}{(2n^2 + 8n + 7) \left( 2n^2 + 4n + 1 \right)}

which is greater than 1 for all values of \( n \geq 1 \).

Now we characterize a sufficiently long time period for \( V^T \) to switch to all \( n \) values. From Equation 19, we know that

\[
T = t_n - 1 + \tau_n - 1 > 1 + \frac{100n^3 \left( (n-1)^2 + (n-1) - 1 \right)}{2 (n-1)^2 + 4 (n-1) + 1}
\]

Therefore, picking a time horizon such that \( T > 100n^3 \) is a sufficient condition for the optimal solution of Equation 13 to switch to all \( n \) basis vectors.

**G.4 Optimality of linear assessment policies**

So far, for convenience we have focused on linear assessment policies for the principal. We next show that this restriction is without loss of generality, that is, linear assessment policies are at least as powerful as the larger class of Lipschitz assessment policies with constant \( K \leq 1 \), where the comparison is in terms of the effort policies each class can incentivize the agent to play.

**Theorem G.5.** Suppose \( K \leq 1 \) is constant and \( f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) is a \( K \)-Lipschitz function. For any effort policy \( \{e_t\}_{t=1}^T \), if there exists a sequence of assessment rules \( \{f(\theta_t, \cdot)\}_{t=1}^T \) to which \( \{e_t\}_{t=1}^T \) is the agent’s best-response, then there exists a linear assessment policy \( \{\theta_t\}_{t=1}^T \) to which \( \{e_t\}_{t=1}^T \) is also a best-response.

Here is the proof sketch. In order to show that linear assessment policies are optimal, we re-derive the optimal effort policy a rational agent will play for some arbitrary assessment policy \( \{f(\theta_t, \cdot)\}_{t=1}^T \). We find that an agent’s optimal effort policy is linear in \( \{\nabla \alpha_i, f(\theta_t, \cdot)\}_{t=1}^T \), the gradient of the assessment policy with respect to the agent’s observable features. Therefore, picking each decision rule to be \( f(\theta_t, o_t) = \theta_t^\top o_t \) is optimal, assuming no restrictions on \( \theta_t \). However, since we restrict each linear decision rule \( \theta_t \) to lie in the probability simplex \( \Delta^n \), playing the optimal \( \{\theta_t\}_{t=1}^T \) is at least as good as any assessment policy in the set of Lipschitz continuous assessment policies with Lipschitz constant \( K \leq 1 \).

**Proof.** Recall that

\[
\{a_t^*\}_{t=1}^T = \arg \max_{a_1, \ldots, a_T} \sum_{t=1}^T y_t - \frac{1}{2} \|a_t\|_2^2
\]

s.t. \( a_t^{(j)} \geq 0 \ \forall t, j \)

This is the generic optimization problem for the agent’s optimal effort policy \( \{a_t^*\}_{t=1}^T \) from Section G.1. However, instead of specifying the score \( y_t \) achieved at each time step to be a linear function of the agent’s observable features \( o_t \), we leave the relationship between observable features and score as some generic function \( y_t = f(\theta_t, o_t) \), parameterized by \( \theta_t \). We can still obtain an expression for \( a_t^* \) by taking the gradient of Equation 20 with respect to \( a_t \) and setting it equal to \( 0 \). By applying the chain rule, we obtain

\[
a_t^* = \nabla a_t^* \sum_{t=1}^T y_t = \sum_{t=1}^T \nabla a_t f(\theta_t, o_t) = \sum_{t=1}^T \nabla a_t W \left( s_0 + \Omega \sum_{k=1}^{i-1} a_k + i \right) \cdot \nabla \alpha_i f(\theta_t, o_t)
\]

\[
a_t^* = W^\top \nabla \alpha_i f(\theta_t, o_t) + \Omega^\top W^\top \sum_{i=t+1}^T \nabla \alpha_i f(\theta_t, o_t)
\]
Figure 5: Left: $T$ as a function of $E$. Larger $\Omega_{jj}$ and $W_{mj}$ terms correspond to fewer time-steps to incentivize $E$ units of effort. Right: $T$ as a function of $E$. While $T$ is inversely proportional to both $\Omega_{jj}$ and $W_{mj}$, increasing $\Omega_{jj}$ decreases the time required to incentivize $E$ units of effort more than an equal increase in $W_{mj}$.

The goal of the principal is to maximize the agent’s internal state at time $T$, $\Lambda^T \sum_{t=1}^T e_t$. Assuming the agent is rational and plays $e_t = \alpha^*_t$, $\forall t$, we can plug Equation 21 into this expression and simplify to obtain

$$\left\| \Lambda^T \sum_{t=1}^T e_t \right\|_1 = \left\| \Lambda W^T \sum_{t=1}^T \nabla_{\alpha_t} f(\theta_t, o_t) \right\|_1 + \left\| \Lambda \Omega^T W^T \sum_{t=1}^T \sum_{i=t+1}^T \nabla_{\alpha_i} f(\theta_i, o_i) \right\|_1$$

Due to the linearity of the problem, the optimal $\nabla_{\alpha_t} f(\theta_t, o_t)$ will be basis vectors for all $t$. Since we restrict $\theta_t$ to be in $\Delta^n$, $f(\theta_t, \cdot) = \theta_t$ is at least as optimal as all Lipschitz continuous functions with Lipschitz constant $K \leq 1$.

Note that while linear optimality does not hold across the set of all assessment policies, this is a result of our restrictions on $\theta_t$ and not due to some suboptimality of linear mechanisms. For example, if we chose to restrict our choice of assessment rules to lie within a probability simplex rescaled by $\Gamma \in \mathbb{R}^+$, then there would exist a linear assessment policy which would be at least as optimal as all Lipschitz functions with Lipschitz constant $K \leq \Gamma$.

**G.5 What levels of effort can be incentivized within $T$ rounds?**

According to Corollary G.2, we know that longer time horizons always expand the set of implementable effort sequences. In what follows, we characterize the number of rounds sufficient for reaching a cumulative effort level of $E$ in a designated effort component.

**Definition G.6** $(T, E)$-Incentivizability. An effort component $j$ is $(T, E)$-incentivizable if a rational agent can be motivated to expend at least $E$ units of effort in the direction of $j$ over $T$ rounds.

**Theorem G.7.** Let $W_{mj}$ denote the maximal element in the $j$th column of $W$. Then if

$$T = \left\lfloor \frac{1}{2} - \frac{1}{\Omega_{jj}} \right\rfloor + \frac{1}{2} \left( \frac{2}{\Omega_{jj}} - 1 \right)^2 + \frac{8E}{\Omega_{jj}W_{mj}} \right\rfloor,$$

effort component $j$ is $(T, E)$-incentivizable for $\Omega_{jj} > 0$. 

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Proof. The relationship between total effort $E$ and the minimum time horizon $T$ required to induce an agent to expend $E$ units of effort in the direction of effort component $j$ can be written as

$$\min_{\theta_1, \ldots, \theta_T} T \quad \text{s.t.} \quad E \leq \sum_{t=1}^{T} a^*_t(j), \quad \theta_t \in \Delta^n \forall t, \ T > 1 \quad (23)$$

where $a^*_t(j) = \sum_{k=1}^{n} (\theta^{(k)}_t + \Omega_{jj}(\sum_{s=1}^{T} \theta^{(s)}_t) W_{kj}$ (see Equation 11). Since we only care about the effort accumulated in coordinate $j$ at each time-step, the principal’s optimal assessment policy is to pick the assessment rule $\theta_t$ that maximizes the effort the agent expends in coordinate $j$ at time $t$. This translates to picking $\theta^{(k)}_t = \mathbb{1}\{W_{kj} = W_{mj}\} \forall t$, where $W_{mk} = \max_k W_{kj}$. In other words, the principal wants to play the same basis vector at every time-step, which will have weight on the observable feature that effort component $j$ contributes the most to. Plugging in this expression for $\theta^{(k)}_t$, the constraint in Equation 23 simplifies to

$$E \leq \sum_{t=1}^{T} (1 + \Omega_{jj} (T - t)) W_{mj} = \left(T + \frac{\Omega_{jj}}{2} (T^2 - T)\right) W_{mj}$$

Note that this will hold with equality if $E = \sum_{t=1}^{T} a^*_t(j)$. After solving for $T$ and simplifying, we get

$$T \geq \frac{1}{2} - \frac{1}{\Omega_{jj}} + \frac{1}{2} \sqrt{\left(\frac{2}{\Omega_{jj}} - 1\right)^2 + \frac{8E}{\Omega_{jj} W_{mj}}} \quad (24)$$

Since we want the time horizon to be as small as possible but need $T$ to be an integer, we take the ceiling of Equation 24 to get our final time horizon value. \hfill \Box

Note that the time horizon $T$ scales as $\sqrt{E}$ because $a^*_t$, the optimal agent effort profile at time $t$, has a linear dependence on $T - t$, and the total effort $E$ expended by the agent is proportional to $\sum_{t=1}^{T} a^*_t(j)$. Intuitively, this can be seen as the agent choosing to put in most of the work “up front” in order to reap the benefits of his effort across a longer period of time.

Note that the bound on $T$ is tight for $(T, E)$ pairs where $E = \sum_{t=1}^{T} e^{(j)}_t$. For example, let $j = 1$ and $\Omega = W = I \in \mathbb{R}^{2 \times 2}$. If we pick $\theta_t = [1 \ 0]^\top$, then $e^{(1)}_t = 1 + (T - t)$, from which it is straightforward to see that with 2 total time-steps, the cumulative effort in the direction of $j$ will be 3. By setting $E = 3$ in Equation 22, we get $T \geq 2$, showing that our lower bound on $T$ is indeed tight for this example.

A natural question is if we can recover a similar definition of $(T, E)$-incentivizability if we want to incentivize some arbitrary subset of effort $e_S$ over time. While we can obtain a bound for incentivizing one index $j \in S$ using the above formulation, obtaining a tighter characterization may require playing different assessment rules over time. Determining these optimal assessment rules requires solving an optimization problem, so a closed-form bound for this setting is not easy to obtain.

G.6 Discussion: comparing the fixed budget and quadratic cost models

While the principal is able to incentivize a wider range of effort profiles under both the fixed budget and quadratic cost setting, there are several differences in the optimal policies recovered in each setting. In the fixed budget setting, the optimal agent effort policy under linear effort conversion function is to play a basis vector at every time-step (see Proposition 4.1), while the principal’s optimal decision rules are generally not basis vectors. Somewhat surprisingly, in the quadratic cost setting the roles are exactly reversed. The principal’s optimal linear assessment policy is to play a sequence of basis vectors, while the agent’s effort policy will generally involve spending effort in different directions at the same time-step. While in settings such as our classroom example it may be desirable to incentivize agents to play basis vectors (e.g. only study), the choice of constraint on agent effort...
is problem-specific and should be chosen based on what is most realistic under the specific setting being studied.

Another difference between the two settings is the computational complexity of recovering the optimal linear policies for the principal and agent. In the fixed budget setting, we can recover the agent’s optimal effort policy by solving a sequence of linear programs, and we can recover the principal’s optimal assessment policy by using a membership oracle-based method. On the other hand, we have a simple closed-form solution for the agent’s optimal effort policy and can recover the principal’s optimal linear assessment policy by solving a sequence of linear programs under the quadratic cost formulation.