PORTFOLIO OPTIMIZATION USING A NEW PROBABILISTIC RISK MEASURE

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ABSTRACT. In this paper, we introduce a new portfolio selection method. Our method is innovative and flexible. An explicit solution is obtained, and the selection method allows for investors with different degree of risk aversion. The portfolio selection problem is formulated as a bi-criteria optimization problem which maximizes the expected portfolio return and minimizes the maximum individual risk of the assets in the portfolio. The efficient frontier using our method is compared with various efficient frontiers in the literature and found to be superior to others in the mean-variance space.

1. Introduction. Portfolio selection models are of great practical significance to investors around the world. The way risk is defined and measured will lead to different optimal portfolios. Markowitz [10] laid the foundation for this line of research with the well-known mean-variance (M-V) model in a single period case. In Markowitz’s model, the portfolio variance was used as a measure of risk. Since then, many other risk definitions have been proposed.

One such measure is the semi-variance. Whilst the semi-variance was first proposed as a downside-risk measure by Markowitz [11], it is only in recent years that Li and Wu [8] presented an analytical solution to an optimal asset allocation problem in dynamic downside-risk framework.

Konno [5] and Konno and Yamazaki [7] used the mean absolute deviation as their risk measure while Konno and Suzuki [6] considered both variance and skewness as the measure of risk. The mean absolute deviation risk measure corresponds to the $l_1$ risk function, while the variance risk measure corresponds to the $l_2$ risk function.

It is widely established that variance is not a good risk measure both in theory and in practice [3, 9, 15] since variance penalizes returns both below and above the mean without discrimination. Researchers have been continually seeking for new risk measures for portfolio selection and risk management. One popular risk measure employed and researched widely is the Value-at-Risk (VaR). The VaR of a portfolio can be defined as the maximum possible loss a portfolio can sustain over a given time horizon under a specific confidence level.

As a measure of risk, VaR has its limitation. It lacks subadditivity and convexity, and is not a coherent risk measure [1]. VaR disregards the loss beyond VaR, and portfolio optimization based on a VaR risk measure is computationally difficult [3].

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Another form of risk measure falls in the category of minimax type models. Cai, Teo, Yang and Zhou [2] introduced a risk model based on the mean absolute deviation. Their risk measure is defined as a $l_\infty$ risk function. In their model, investors minimize the maximum of individual risk which is measured using the mean absolute deviation. Similarly, Teo and Yang [13] minimized the average of maximum individual risks over a number of time periods. Another minimax model, proposed by Deng, Li and Wang [4], maximizes the worst (minimally) possible expected ratios of returns on portfolio. Taking into consideration of the recent extreme economic developments, Polak, Rogers and Sweeney [12] develop the optimal portfolio with risk measured as the worst-case return. These authors argue that worst-case analyses are relevant since investors may become more conservative following the financial collapses of many international firms in 2008.

Our minimax portfolio selection model is different from all other in the literature. We introduce a probabilistic risk measure, with allowance to cater for investors with different degree of risk aversion. The portfolio selection problem is formulated as a bi-criteria optimization problem to maximize the expected portfolio return and minimize the maximum individual risk of the assets in the portfolio. We show that the bi-criteria optimization problem is equivalent to a linear programming problem using the monotonic property of the risk function. As a result, we are able to derive a simple analytical solution without requiring the computation of the covariances. With this analytical solution, investors can determine the percentages they should hold in each of the selected risky assets in the portfolio. In addition, we show that our model is superior to existing models in the mean-variance space, giving a frontier closest to Markowitz’s efficient frontier.

The remainder of the paper is organized as follows. Section 2 describes the problem, and introduces the new probabilistic risk measure. Section 3 develops the analytical solution to the problem. In Section 4, the efficient frontier is compared to the efficient frontiers of $l_2$, $l_1$ and $l_\infty$ models. Section 5 concludes the paper.

2. A new probabilistic risk measure. In this section, assume that an investor has a positive initial wealth of $M_0$, which is going to be invested in $N$ possible risky assets $S_j$, $j = 1, \ldots, N$. Let $x_j$, $j = 1, \ldots, N$, be the percentage of initial fund invested in asset $S_j$. Note that $\sum_{j=1}^{N} x_j = 1$. Moreover, we assume that short selling of the risky assets is not allowed, i.e., $x_j \geq 0$. Thus, define

$$\mathcal{X} = \{ \mathbf{x} = [x_1, \ldots, x_N]^T \in \mathbb{R}^N : \sum_{j=1}^{N} x_j = 1, \ x_j \geq 0, \ j = 1, \ldots, N \}, \quad (2.1)$$

where $R_j$ is the return of asset $S_j$. We assume that $R_j$ follows normal distribution with mean $r_j$ and standard deviation $\sigma_j$. Note that $r_j$ represents the expected return of the $j^{th}$ asset calculated averaging the returns over $T$ periods.

$$r_j = \frac{1}{T} \sum_{i=1}^{T} R_{ji}, \quad (2.2)$$

where $R_{ji}$ denotes the actual return of asset $j$ for the $i^{th}$ time period.

Thus, the expected return of the portfolio $\mathbf{x} = [x_1, \ldots, x_N]^T$ is given by
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\[ r(x_1, \ldots, x_N) = \mathbb{E}\left\{ \sum_{j=1}^{N} R_j x_j \right\} = \sum_{j=1}^{N} \mathbb{E}\{R_j\} x_j = \sum_{j=1}^{N} r_j x_j. \quad (2.3) \]

We introduce a probabilistic risk measure, which allows to cater for investors with different degree of risk aversion, defined as follows:

\[ w_p(x) = \min_{1 \leq j \leq N} \Pr\{|R_j x_j - r_j x_j| \leq \theta \varepsilon\}, \quad (2.4) \]

where \( \theta \) is a constant to adjust the risk level, and \( \varepsilon \) denotes the average risk of the entire portfolio, which is calibrated by the function below.

\[ \varepsilon = \frac{1}{N} \sum_{j=1}^{N} \sigma_j. \quad (2.5) \]

While it is not necessary to compute \( \varepsilon \), we introduce it to arrive at a reasonable risk measure for our numerical example. Assume that the investors are rational and risk-averse, the portfolio selection problem can be formulated as a bi-criteria optimization problem. The objective is to maximize both (2.4) and the expected return of the portfolio (2.3), which is formally stated as follows.

\[
\begin{align*}
\max & \left( \min_{1 \leq j \leq N} f(x_j), \sum_{j=1}^{N} r_j x_j \right), \\
\text{s.t.} & \quad x \in \mathcal{X},
\end{align*}
\]

(2.6a)

(2.6b)

where \( f(x_j) = \Pr\{|R_j x_j - r_j x_j| \leq \theta \varepsilon\} \).

The following definition is due to Theorem 3.1 in [14].

**Definition 2.1.** A solution \( x^* = [x_1^*, \ldots, x_N^*]^T \in \mathcal{X} \) is said to be a Pareto-minimal solution for the bi-criteria optimization problem (2.6) if there does not exist a solution \( \tilde{x} = [\tilde{x}_1, \ldots, \tilde{x}_N]^T \in \mathcal{X} \) such that

\[ \min_{1 \leq j \leq N} f(x_j^*) \leq \min_{1 \leq j \leq N} f(\tilde{x}_j), \text{ and } \sum_{j=1}^{N} r_j x_j^* \leq \sum_{j=1}^{N} r_j \tilde{x}_j, \]

for which at least one of the inequalities holds strictly.

We can transform the above problem into a bi-criteria optimization problem by adding another variable \( y \), and \( N \) constraints.

\[
\begin{align*}
\max & \left( y, \sum_{j=1}^{N} r_j x_j \right), \\
\text{s.t.} & \quad y \leq f(x_j), \quad j = 1, \ldots, N, \\
& \quad x \in \mathcal{X},
\end{align*}
\]

(2.7a)

(2.7b)

(2.7c)

where \( y \leq f(x_j) \) is the \( j^{(th)} \) probabilistic constraint. Since the optimization process will push the value of \( y \) to be equal to \( \min_{1 \leq j \leq N} f(x_j) \), it is clear that the optimization problem (2.7) is equivalent to the optimization problem (2.6).
3. Analytical solution to the problem. As mentioned in Section 2, the return, $R_j$, of asset $S_j$, follows the normal distribution with mean $r_j$ and standard deviation $\sigma_j$. Thus, $R_j - r_j$ is also normally distributed with mean 0 and standard deviation $\sigma_j$.

Let $q_j = R_j - r_j$. Then it follows that

$$f(x_j) = \Pr\{ |R_j x_j - r_j x_j| \leq \theta \varepsilon \} = \Pr\{ |R_j - r_j| \leq \frac{\theta \varepsilon}{x_j} \}$$

$$= 2 \int_{0}^{\frac{\theta \varepsilon}{x_j}} \frac{1}{\sqrt{2\pi} \sigma_j} \exp \left\{ - \frac{q_j^2}{2 \sigma_j^2} \right\} dq_j. \quad (3.1)$$

Clearly, $f(x_j)$ is a monotonically decreasing function with respect to $x_j$.

For the optimization problem (2.7), it clear that during the process of maximizing the objective function, the value of $y$ is to be pushed to reach $\min_{1 \leq j \leq N} f(x_j)$. Thus, if we choose $y$ to be an arbitrary but fixed real number between 0 and 1, using the monotonic property of $f(x_j)$, we can find an upper bound for $x_j$. We denote this upper bound by $U_j$. Define

$$U_j = \min\{1, \hat{x}_j\}, \quad (3.2)$$

where $\hat{x}_j = f^{-1}(y)$.

Consequently, for a fixed value of $y$, it is easy to show that the optimization problem (2.7) is equivalent to the linear programming problem stated as given below:

$$\max \sum_{j=1}^{N} r_j x_j, \quad (3.3a)$$

$$\text{s.t.} \sum_{j=1}^{N} x_j = 1, \quad (3.3b)$$

$$0 \leq x_j \leq U_j, \quad j = 1, \ldots, N. \quad (3.3c)$$

To obtain the analytical solution of problem (3.3), we first sort the assets in such an order that $r_1 \geq r_2 \geq \ldots \geq r_N$. Moreover, assume that there are no two distinct assets in the portfolio that have the same level of expected return as well as standard deviation, i.e., there exist no $i$ and $j$ such that $i \neq j$, but $r_i = r_j$, and $\sigma_i = \sigma_j$.

**Theorem 3.1.** Let the assets be sort in such an order that $r_1 \geq r_2 \geq \ldots \geq r_N$. Then, there exists an integer $k \leq N$ such that

$$\sum_{j=1}^{k-1} U_j < 1, \quad \text{and} \quad \sum_{j=1}^{k} U_j \geq 1,$$

and

$$x_j^* = \begin{cases} U_j, & j=1,\ldots,k-1 \\ 1 - \sum_{j=1}^{k-1} U_j, & j=k \\ 0, & j>k \end{cases} \quad (3.4)$$

is an optimal solution to problem (3.3).
Proof. Recall that $U_j = \min \{1, \hat{x}_j \} \leq 1, j = 1, \ldots, N$, $U_j$ is the upper bound of $x_j$. Therefore, from (3.3b) and (3.3c), we have $\sum_{j=1}^{k-1} U_j \leq 1$. Clearly, there exists an integer $k$, $1 \leq k \leq N$, such that $\sum_{j=1}^{k-1} U_j < 1$ and $\sum_{j=1}^{k} U_j \geq 1$.

Now, we shall show that $x^*$ is an optimal solution to the LP problem (3.3).

For the LP problem (3.3), $\hat{x} = [\hat{x}_1, \ldots, \hat{x}_N]^T$ is an optimal solution if and only if there exist $u \in \mathbb{R}$, $v \in \mathbb{R}^N$, and $w \in \mathbb{R}^N$ such that

\[
\begin{aligned}
\sum_{j=1}^{N} x_j &= 1, \\
r_j &= u - v_j + w_j, \\
v_j &\geq 0, w_j \geq 0, j = 1, \ldots, N, \\
v_j x_j &= 0, \\
w_j (x_j - U_j) &= 0,
\end{aligned}
\]

Let

\[
u^*_j = \begin{cases} 0, & j = 1, \ldots, k, \\ r_k - r_j, & j > k, \end{cases}
\]

and

\[
w^*_j = \begin{cases} r_j - r_k, & j = 1, \ldots, k - 1, \\ 0, & j > k - 1. \end{cases}
\]

By simple calculation, it is derived that $x^*, u^*, v^*$, and $w^*$ satisfy (3.5). Thus, $x^*$ is an optimal solution to the LP problem (3.3).

4. Efficient frontier analysis. In this section, we trace out the efficient frontiers in the traditional M-V space. We compare the former risk measures (i.e., $l_1$, $l_2$, and $l_\infty$ risk measures) with our probabilistic risk measure. 50 real stocks’ daily price data are used for the calculation.

4.1. Review of $l_1$, $l_2$, and $l_\infty$ risk measures.

$l_2$ risk measure. The classical M-V model chooses portfolio variance as the risk measure. The efficient frontier is drawn by minimizing the risk of the portfolio under a given level of return. Return of the portfolio takes values from $\min_{1 \leq j \leq N} r_j$ to $\max_{1 \leq j \leq N} r_j$. Markowitz formulated the portfolio optimization problem as a quadratic programming problem:

\[
\begin{aligned}
\min &\quad \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} x_i x_j \\
\text{s.t.} &\quad \sum_{j=1}^{N} r_j x_j \geq \rho, \\
&\quad \sum_{j=1}^{N} x_j = 1, \\
&\quad 0 \leq x_j \leq u_j, j = 1, \ldots, N,
\end{aligned}
\]
where $\sigma_{ij} = E[(R_i - r_i)(R_j - r_j)]$, $\rho$ is a parameter representing the minimum rate of return required, which is adjusted to construct the efficient frontier. And $u_j$ is the maximum percentage of capital that can be invested into asset $S_j$. Note that $u_j$ is set to be $\infty$ in our case.

**$l_1$ risk measure.** For the $l_1$ risk measure, it uses the mean absolute deviation as the risk measure of the portfolio. The major difference from $l_2$ risk measure is that with $l_1$ risk measure, the portfolio optimization problem is formulated as a linear programming problem given below.

$$
\min w(x) \quad (4.2a)
$$

s.t. $\sum_{j=1}^{N} r_j x_j \geq \rho \quad (4.2b)$

$$
\sum_{j=1}^{N} x_j = 1, \quad (4.2c)
$$

$$
0 \leq x_j \leq u_j, \quad j = 1, \ldots, N, \quad (4.2d)
$$

where it is shown in [6] that

$$
w(x) = E\left\{ \left| \sum_{j=1}^{N} R_j x_j - E\left[ \sum_{j=1}^{N} R_j x_j \right] \right| \right\} = \sqrt{\frac{2}{\pi}} \sigma(x),
$$

$$
\sigma(x) = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{ij} x_i x_j}.
$$

The risk function $w(x)$ is approximated by the returns of all selected assets over time.

$$
E\left\{ \left| \sum_{j=1}^{N} R_j x_j - E\left[ \sum_{j=1}^{N} R_j x_j \right] \right| \right\} = \frac{1}{T} \sum_{t=1}^{T} \left| \sum_{j=1}^{N} (r_{jt} - r_j) x_j \right|.
$$

Thus, the model can be stated as a linear programming optimization problem as follows.

$$
\min \sum_{t=1}^{T} y_t / T \quad (4.3a)
$$

s.t. $y_t + \sum_{j=1}^{N} a_{j,t} x_j \geq 0, \quad t = 1, \ldots, T, \quad (4.3b)$

$$
y_t - \sum_{j=1}^{N} a_{j,t} x_j \geq 0, \quad t = 1, \ldots, T, \quad (4.3c)
$$

$$
\sum_{j=1}^{N} r_j x_j \geq \rho, \quad (4.3d)
$$

$$
\sum_{j=1}^{N} x_j = 1, \quad (4.3e)
$$

$$
0 \leq x_j \leq u_j, \quad j = 1, \ldots, N, \quad (4.3f)
$$
where $a_{jt} = r_{jt} - r_j$, $j = 1, \ldots, N$, $t = 1, \ldots, T$. Now the problem becomes a $(T+N)$ dimensional linear programming optimization problem.

**$l_\infty$ risk measure.** The $l_\infty$ risk models solve a minimax optimization problem. The objective is set to maximize the return of the portfolio and as the same time minimize the biggest individual risk amount all the assets. The main contribution of this method is that it is possible to derive an analytical solution to the problem, which is obviously much more useful to investors. The model is stated as follows.

$$
\min \left( \max_{1 \leq j \leq N} q_j x_j, - \sum_{j=1}^{N} r_j x_j \right)
$$

(4.4a)

s.t. $x \in \mathcal{X}$,

(4.4b)

where $q_j = \mathbb{E}[|R_j - r_j|]$, and $\mathcal{X}$ is defined as in (2.1).

The minimax optimization problem (4.4) is finally transformed into a parametric optimization problem:

$$
\min F_\lambda(x, y) = \lambda y + (1 - \lambda) \left( - \sum_{j=1}^{N} r_j x_j \right)
$$

(4.5a)

s.t. $q_j x_j \leq y$, $j = 1, \ldots, N$

(4.5b)

$x \in \mathcal{X}$,

(4.5c)

where $\lambda \in (0, 1)$ is a parameter.

For the exact form of the analytical solution to (4.5), please refer to [2].

### 4.2. Comparison.

We compare the performances of the $l_1$, $l_\infty$ models and our new probabilistic risk model with Markowitz’s $l_2$ risk model. We trace out the efficient frontier in the traditional M-V space using 50 real stocks’ daily price data for 2480 days.

For $l_2$ and $l_1$ models in (4.1) and (4.3), we take $\rho M_0 \in [0, \max_{1 \leq j \leq N} r_j]$, while in $l_\infty$ model (4.5), $\lambda$ takes values in $[0, 1]$. In our probabilistic risk model, we take the values of $y$ from $[0, 1]$, and set the risk adjustment parameter $\theta$ in (2.4) to be equal to 1.

Figure 4.1 shows the efficient frontier plotted in the mean-variance space (Note that $P_r$ denotes the probabilistic risk measure.). All models perform better in the mean-variance space when compared to the $l_1$ model which uses the mean absolute deviation as the risk measure. The $l_\infty$ measure which is the result of a minimax model and which uses the mean absolute deviation as the measure of risk, is seen to deviate more from Markowitz’s frontier compared to the $P_r$ measure which we propose. In fact, for returns higher than 15%, our methodology is seen to coincide with Markowitz’s $l_2$ risk model. However, unlike Markowitz’s model, our methodology does not require the covariance information of the assets involved. Estimating covariances between securities requires time and much data collection. However, there are many traders and researchers who may not be convinced that covariance is the proper risk measure, as it is very sensitive to estimation errors [4]. On the other hand, the probability of an asset return deviating from its historical expected as in our $P_r$ model is something investors can identify with. In addition, inclusion of additional assets is easily implemented under our model since we have an analytical solution for the optimal investment strategy. Moreover, in the construction of the efficient frontier in Figure 4.1, the $P_r$ frontier is plotted assuming the investor’s risk
level is the average risk of the entire portfolio. In practice, investors have different levels of risk aversion, and this is easily allowed for in our model.

![M-V Efficient Frontier Comparison](image)

**Figure 4.1. Frontier Comparison**

5. **Conclusion.** Our portfolio selection method offers various advantages over other methods. It is flexible, and allows for an explicit solution. The new probabilistic risk measure has the flexibility to cater for investors with different degrees of risk aversion. The simplicity of the analytical solution means that the time involved in arriving at the optimal mix of assets to hold is short. Comparison of our efficient frontier with various efficient frontiers in the literature in the mean-variance space shows that our frontier is closest in performance to Markowitz’s model.

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