SURGERY OPERATIONS TO FOLD MAPS TO CONSTRUCT FOLD MAPS WHOSE SINGULAR VALUE SETS MAY HAVE CROSSINGS

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Abstract. Constructing Morse functions and their higher dimensional versions of fold maps is fundamental, important and challenging in investigating the topologies and the differentiable structures of differentiable manifolds via Morse functions, fold maps and more general generic maps. It is one of important and interesting branches of the singularity theory of differentiable maps and applications to geometry of manifolds.

In this paper we present fold maps with information of cohomology rings of their Reeb spaces. Reeb spaces are defined as the spaces of all connected components of all preimages, and in suitable situations inherit topological information such as homology groups and cohomology rings of the manifolds. Previously, the author demonstrated construction of fold maps in various cases: key methods are surgery operations to manifolds and maps and in this paper, we present more useful surgery operations and by them we construct new fold maps. More precisely, fold maps with singular value sets with crossings: the singular value set of a smooth map is the image of the set of all singular points and note that for fold maps, the set of all singular points are closed submanifolds without boundaries and the restrictions to them are immersions of codimension 1.

1. Introduction and fundamental notation and terminologies.

As the well-known classical theory of Morse functions shows, investigating the topologies and the differentiable structures of differentiable manifolds via Morse functions, fold maps and more general generic maps is one of important and interesting branches of the singularity theory of differentiable maps and applications to geometry of manifolds.

1.1. Fold maps. Fold maps are smooth maps regarded as higher dimensional versions of Morse functions.

For the explanation of the strict definition and fundamental properties of a fold map, we explain fundamental terminologies and notation related to singular points of smooth \((C^\infty)\) maps. They are used throughout the present paper. Throughout the paper, manifolds are assumed to be smooth (of class \(C^\infty\)) and maps between (smooth) manifolds are assumed to be smooth (of class \(C^\infty\)) unless otherwise stated: for example, in the presentation of a sketch of the proof of Proposition 4 and so on, piecewise smooth and PL manifolds which may not be smooth appear.

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A singular point of a smooth map $c$ is a point at which the rank of the differential of the map drops. A singular value of the map is a point which is a value at some singular point. The set $S(c)$ of all singular points is the singular set of the map. The singular value set is the image $c(S(c))$. The regular value set of $c$ is the complement of $c(S(c))$. A singular (regular) value of the map is a point in the singular (resp. regular) value set.

A diffeomorphism is always assumed to be smooth and the diffeomorphism group of a manifold is the group of all diffeomorphisms on the manifold.

Definition 1. Let $m > n \geq 1$ be integers. A smooth map between an $m$-dimensional smooth manifold with no boundary into an $n$-dimensional smooth manifold with no boundary is said to be a fold map if at each singular point $p$, the map is represented as

$$(x_1, \cdots, x_m) \mapsto (x_1, \cdots, x_{n-1}, \sum_{k=n}^{m-i} x_k^2 - \sum_{k=m-i+1}^{m} x_k^2)$$

for some coordinates and an integer $0 \leq i(p) \leq \frac{m-n+1}{2}$.

For a fold map, the following properties hold.

- For any singular point $p$, the $i(p)$ in Definition 1 is unique: $i(p)$ is the index of $p$.
- The set consisting of all singular points of a fixed index of the map is a smooth and closed submanifold of the source manifold with no boundary of dimension $n - 1$.
- The restriction to the singular set of the original map is a smooth immersion.

We define a crossing and a normal crossing of a family of smooth immersions.

Let $a$ be a positive integer and $\{c_j : X_j \to Y\}_{j=1}^a$ be a family of $a$ smooth immersions $c_j$ from $m_j$-dimensional smooth manifolds $X_j$ without boundaries into an $n$-dimensional smooth manifold $Y$ with no boundary. A crossing of the family of these smooth immersions is a point $y \in Y$ such that $\bigcup_{j=1}^a c_j^{-1}(y)$ has at least two points. A crossing is normal if the following properties hold.

For a smooth manifold $X$, we denote the tangent bundle by $TX$ and the tangent vector at $p \in X$ by $T_pX \subset TX$.

1. The disjoint union $\bigcup_{j=1}^a c_j^{-1}(y)$ of these preimages is a finite set consisting of exactly $b > 1$ points.
2. Let $\{p_j\}_{j=1}^b$ be the set just before. Let $n(j)$ be the number satisfying $p_j \in X_{n(j)}$: we can define uniquely. Consider the intersection $\bigcap_{j=1}^b dc_{n(j)}(T_{p_j}X_{n(j)})$ of the images of the differentials at all points and we denote the dimension of this by $i(d)$. Then $i(d) + \sum_{j=1}^b (n - m_{n(j)}) = n$.

We can define the notions for a single immersion (the case $a = 1$).

In the present paper, we only consider crossings such that preimages consist of exactly two points.

A stable fold map is a fold map whose restriction to the singular set is a smooth immersion such that the crossings of the restriction to the singular set of the original fold map are always normal. For systematic explanations on stable (fold) maps, see [1] for example. In this paper, we construct stable fold maps such that the restriction to the singular set of the original fold map may have normal crossings.
1.2. Reeb spaces. Reeb spaces are also fundamental and important tools in investigating the topologies of the source manifolds of smooth maps whose codimensions are negative.

Let $X$ and $Y$ be topological spaces. For $p_1, p_2 \in X$ and for a continuous map $c : X \to Y$, we define as $p_1 \sim_c p_2$ if and only if $p_1$ and $p_2$ are in a same connected component of $c^{-1}(p)$ for some $p \in Y$. Thus $\sim_c$ is an equivalence relation on $X$ and we denote the quotient space $X/\sim_c$ by $W_c$

Definition 2. We call $W_c$ the Reeb space of $c$.

We denote the induced quotient map from $X$ into $W_c$ by $q_c$. We can define $\tilde{c} : W_c \to Y$ uniquely so that the relation $c = \tilde{c} \circ q_c$ holds.

Proposition 1 ([17]). For stable fold maps, the Reeb spaces are polyhedra and the dimensions are equal to the dimensions of the target manifolds.

For Reeb spaces, see also [11] for example.

1.3. Reeb spaces of fold maps. We introduce a class of stable fold maps such that the Reeb spaces inherit much information of algebraic topological invariants of the manifolds in Proposition 2.

A simple fold map $f$ is a fold map such that the restriction $q_f|_{S(f)}$ is injective.

PID means a principal ideal domain throughout the present paper.

Proposition 2 ([16] ([3] and [4])). Let $m$ and $n$ be integers satisfying $m > n \geq 1$. Let $A$ be a commutative group. Let $M$ be a smooth, closed, connected and orientable manifold of dimension $m$ and $N$ be an $n$-dimensional smooth manifold with no boundary.

Then, for a simple fold map $f : M \to N$ such that preimages of regular values are always disjoint unions of standard spheres and that indices of singular points are always 0 or 1, the following three hold.

1. Three induced homomorphisms $q_f_* : \pi_j(M) \to \pi_j(W_f)$, $q_f^* : H_j(M; A) \to H_j(W_f; A)$, and $q_f^* : H^j(W_f; A) \to H^j(M; A)$ are isomorphisms for $0 \leq j \leq m - n - 1$.

2. Let $A$ be a commutative ring. Let $J$ be the set of all integers greater than or equal to 0 and smaller than or equal to $m - n - 1$ and if $\oplus_{j \in J} H^j(W_f; A)$ and $\oplus_{j \in J} H^j(M; A)$ are defined as algebras where the sums and the products are canonically induced from the cohomology rings $H^*(W_f; A)$ and $H^*(M; A)$ respectively and where the maximal degrees are $m - n - 1$, then $q_f$ induces an isomorphism between the algebras $\oplus_{j \in J} H^j(W_f; A)$ and $\oplus_{j \in J} H^j(M; A)$ and this is a restriction of $q_f^*$ to $\oplus_{j \in J} H^j(W_f; A)$.

3. Let $A$ be a PID and let $m = 2n$ be hold. In this situation, the rank of $H_n(M; A)$ is twice the rank of $H_n(W_f; A)$ and in addition if $H_{n-1}(W_f; A)$, which is isomorphic to $H_{n-1}(M; A)$, is a free module over $A$, then the first two modules over $A$ are also free.

Remark 1. Proposition 2 holds in cases where preimages of regular values may contain homotopy spheres obtained by gluing two copies of a standard closed disc by a diffeomorphism between their boundaries as connected components. The class of such homotopy spheres accounts for the class of all homotopy spheres except 4-dimensional homotopy spheres which are not standard spheres (such manifolds are still undiscovered). However, we only handle cases where the preimages are disjoint unions of standard spheres.
1.4. Explicit fold maps and their Reeb spaces. It is fundamental and important to construct explicit fold maps in applying geometric theory of Morse functions and fold maps to understanding of geometric properties of manifolds. However, even on (families of) manifolds which are not so complicated, it is difficult. We present known examples here.

For a topological space $X$, an $X$-bundle is a bundle whose fiber is $X$. For a smooth manifold $X$, a smooth $X$-bundle is an $X$-bundle whose structure group is (a subgroup of) the diffeomorphism group.

Example 1. (1) The class of special generic maps is a proper subset of that of simple fold maps in Proposition 2. A special generic map is a fold map such that the index of each singular point is 0. Standard spheres admit special generic maps into arbitrary Euclidean spaces whose dimensions are smaller than or equal to the dimensions of the spheres: canonical projections of unit spheres are simplest ones. In [12], [13], [15], [18] and so on, homotopy spheres which are not diffeomorphic to standard spheres do not admit special generic maps into sufficiently high dimensional Euclidean spaces (the dimensions of the Euclidean spaces are assumed to be smaller than those of the homotopy spheres). Moreover, as another fundamental and important property of special generic maps, the maximal degree $j = m - n - 1$ in Proposition 2 can be replaced by $j = m - n$.

Last, the Reeb space of a (stable) special generic map $f$ from a closed and connected manifold of dimension $m$ into $\mathbb{R}^n$ satisfying the relation $m > n \geq 1$ is homeomorphic to and regarded as an $n$-dimensional compact and connected manifold we can immerse into $\mathbb{R}^n$. The image is regarded as the image of a suitable immersion of the $n$-dimensional manifold. The boundary of the Reeb space and the image $q_f(S(f))$ of the singular set agree.

Conversely, for integers $m > n \geq 1$ and an $n$-dimensional compact manifold we can immerse into $\mathbb{R}^n$, we can construct a (stable) special generic map from a suitable closed and connected manifold of dimension $m$ into $\mathbb{R}^n$ whose Reeb space is diffeomorphic to the $n$-dimensional manifold.

Moreover, we have the following smooth or linear bundles for a general special generic map $f$ from a closed and connected manifold of dimension $m$ into $\mathbb{R}^n$ and we can construct the map just before so that the bundles are trivial.

(a) If we restrict the map $q_f$ to the preimage of the interior of the Reeb space, then it gives a smooth $S^{m-n}$-bundle over the interior of the Reeb space.

(b) If we restrict the map $q_f$ to the preimage of a small collar neighborhood of the boundary of the Reeb space and consider the composition of this with a canonical projection onto the boundary, then it gives a linear $S^{m-n+1}$-bundle over the boundary.

These facts can be seen in [12], in the articles [6] and [7] by the author, and so on.

(2) ([2], [3] and [5]) Let $l > 0$ be an integer. Let $m > n \geq 1$ be integers. We can construct a stable fold map on a manifold represented as an $l - 1$ connected sum of total spaces of smooth $S^{m-n}$-bundles over $S^n$ into $\mathbb{R}^n$ (a standard sphere if $l = 1$) satisfying the following properties.
1.5. Construction of explicit fold maps such that the restrictions to the singular sets may have crossings by new surgery operations (bubbling operations) and the organization of this paper. In this paper, we present further studies on construction in Example 1 (3). We introduce and use improved versions of bubbling operations. The organization of the paper is as the following. In the next section, we introduce a bubbling operation first introduced in [6] by extending the original definition to construct fold maps such that the restrictions to singular sets may have crossings. Defining this extended operation is also a new work in the present paper. The last section is devoted to explanations of simple and important examples and main results. We present construction of new families of explicit fold maps via operations before and investigate the cohomology rings of the Reeb spaces. We also explain properties on cohomology rings which fold maps obtained previously in [6] and [7] do not satisfy. Proposition 2 and so on are key tools in knowing the cohomology rings of the manifolds from Reeb spaces in suitable cases and we introduce an extended version of this last as Proposition 4. The proof is done based on the proofs of the related known results.

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2. Bubbling operations and fold maps such that preimages of regular values are disjoint unions of spheres.

2.1. Definitions and fundamental properties of Reeb spaces and bubbling operations. Throughout this paper, let $m > n \geq 1$ be integers, $M$ be a smooth, closed and connected manifold of dimension $m$, $N$ be a smooth manifold of dimension $n$ with no boundary, and $f : M \to N$ be a smooth map unless otherwise stated.

In addition, the structure groups of bundles such that their base spaces and fibers are manifolds are assumed to be (subgroups of) diffeomorphism groups except some cases in the presentation of a sketch of the proof of Proposition 4 and so on: the
bundles are smooth in a word except several cases. A linear bundle is a smooth bundle whose fiber is a $k$-dimensional unit disc (standard closed disc of a fixed diameter) or the $k$-dimensional unit sphere in $\mathbb{R}^{k+1}$ and whose structure groups are subgroups of the $k$-dimensional orthogonal group $O(k)$ and the $(k+1)$-dimensional one $O(k+1)$ acting canonically and linearly, respectively.

We introduce bubbling operations, first introduced in [6], referring to the article. We revise some notions and terminologies from the original definition and define an extended version.

**Definition 3.** For a stable fold map $f : M \to N$, let $P$ be a connected component of $(W_j - q_j(S(f))) \cap \tilde{f}^{-1}(N - f(S(f)))$, which we may regard as an open manifold diffeomorphic to $\bar{f}(P)$ in $N$.

Let $l > 0$ and $l' \geq 0$ be integers. Assume that there exist families $\{S_j\}_{j=1}^l$ and $\{N(S_j)\}_{j=1}^l$ of finitely many closed and connected manifolds and total spaces of linear bundles over these manifolds whose fibers are unit discs. We also denote by $N_j$ the image of the section obtained by taking the origin for each fiber diffeomorphic to a unit disc for each $N(S_j)$. Assume that the dimensions of $N(S_j)$ are all $n$. Assume also that immersions $c_j : N(S_j) \to P$ satisfying the following properties exist.

1. Crossings of the family $\{c_j|_{\partial N(S_j)} : \partial N(S_j) \to P\}_{j=1}^l$ are normal and $\bigcup_{j=1}^l c_j|_{\partial N(S_j)^{-1}(p)}$ consists of at most two points for each $p \in P$.
2. Crossings of $\{c_j|_{S_j} : S_j \to P\}_{j=1}^l$ are normal and $\bigcup_{j=1}^l c_j|_{S_j}^{-1}(p)$ consists of at most two points for each $p \in P$.
3. The set of all the crossings of the family $\{c_j|_{S_j} : S_j \to P\}_{j=1}^l$ of the immersions is a finite set.
4. Let $\{p_j\}_{j'=1}^{l'}$ be the set of all the crossings of the family $\{c_j|_{S_j} : S_j \to P\}_{j=1}^l$ of the immersions. The set of all crossings of the family $\{c_j|_{\partial N(S_j)} : \partial N(S_j) \to P\}_{j=1}^l$. For each $p_j'$, there exist one or two integers $1 \leq a(j'), b(j') \leq l$ and small standard closed discs $D_{2j'-1} \subset S_{a(j')}$ and $D_{2j'} \subset S_{b(j')}$ of finitely many closed and connected manifolds and total spaces of linear bundles over$S_a(j')$ and $S_{b(j')}$ satisfying the following properties (see also FIGURE 1).

(a) (The dimensions) $\dim D_{2j'-1} = \dim S_{a(j')}$ and $\dim D_{2j'} = \dim S_{b(j')}$.
(b) (The locations of $p_j'$) $p_j'$ is in the images of the immersions: $p_j' \in c_{a(j')}(\text{Int} D_{2j'-1})$ and $p_j' \in c_{b(j')}(\text{Int} D_{2j'})$.
(c) If $a(j') = b(j')$, then $D_{2j'-1}$ and $D_{2j'}$ do not intersect.
(d) If we restrict the bundle $N(S_{a(j')})$ to $c_{a(j')}(D_{2j'-1})$, restrict the bundle $N(S_{b(j')})$ to $c_{b(j')}(D_{2j'})$, and consider the total spaces, which are regarded as manifolds with corners, then they agree as subsets in $\mathbb{R}^n$.

(e) The set of all the crossings of the family $\{c_j|_{S_j} : S_j \to P\}_{j=1}^l$ is the disjoint union of the $l'$ corners of the subsets just before each of which corresponds to $1 \leq j' \leq l'$.

Then we call the family $\{(S_j, N(S_j), c_j : N(S_j) \to P)\}_{j=1}^l$ a normal system of submanifolds compatible with $f$.

In the situation of Definition 3, let $\{N'(S_j) \subset N(S_j)\}_{j=1}^l$ be the family of total spaces of subbundles of $\{N(S_j)\}_{j=1}^l$ over the manifolds whose fibers are standard closed discs and let the diameters of the fibers be all $0 < r < 1$. For a suitable $r$, same properties as presented in Definition 3 hold. In other words, we can obtain a family $\{(S_j, N'(S_j), c_j|_{N'(S_j)} : N'(S_j) \to P)\}_{j=1}^l$ and this is also regarded as a
normal system of submanifolds compatible with $f$, by identifying each fiber, which is a standard closed disc of diameter $r$ with a unit disc via the diffeomorphism defined by $t \mapsto \frac{1}{2}t$.

Definition 4. The family $\{(S_j, N(S_j), c_j : N(S_j) \to P)) \}^{l}_{j=1}$ is said to be a wider normal system supporting the normal system of submanifolds $\{((S_j, N'(S_j), c_j |_{N'(S_j)} : N'(S_j) \to P)) \}^{l}_{j=1}$ compatible with $f$.

We have the following immediately.

Definition 5. For a stable fold map $f : M \to N$ and an integer $l > 0$, let $P$ be a connected component of $|W_f - q_f(S(f))| \cap \bar{f}^{-1}(N - f(S(f)))$ and let $\{(S_j, N(S_j), c_j : N(S_j) \to P)) \}^{l}_{j=1}$ be a normal system of submanifolds compatible with $f$. Let $\{((S_j, N'(S_j), c'_j : N'(S_j) \to P)) \}^{l}_{j=1}$ be a wider normal system supporting this. Assume that we can construct a stable fold map $f'$ on an $m$-dimensional closed manifold $M'$ into $\mathbb{R}^n$ satisfying the following properties.

1. $Q$ is the preimage $f^{-1}(\bigcup^{l}_{j=1}c'_j(N'(S_j)))$.
2. $M - \text{Int}Q$ is realized as a compact submanifold of $M'$ of dimension $m$ by considering a suitable smooth embedding $\varepsilon : M - \text{Int}Q \to M'$.
3. $f|_{M-\text{Int}Q} = f' \circ \varepsilon|_{M-\text{Int}Q}$ holds.
4. $f'(S(f'))$ is the disjoint union of $f(S(f))$ and $\bigcup^{l}_{j=1}c_j(\partial N(S_j))$.

This enables us to define a procedure of constructing $f'$ from $f$. We call it a normal bubbling operation to $f$. Furthermore, the union $\bigcup^{l}_{j=1}c_j(S_j)$ of the images of the immersions and the family of the images $c_j(S_j)$ for all $c_j$ are called the generating normal systems of the normal bubbling operation: we call each manifold $S_j$ a generating manifold of the operation.

Ideas for the operations originate from some of [8], [9] and [10]. Especially, [8] and [9] are on bubbling surgeries introduced by Kobayashi: a bubbling surgery is the case where the generating normal system consists of exactly one point.

In the present paper, we consider normal bubbling operations whose generating manifolds are standard spheres. Moreover, essentially we consider only stable fold maps such that preimages of regular values are disjoint unions of almost-spheres (standard spheres) and that indices of singular points are 0 or 1. Proposition 2 is

\[\text{\textbullet}\]
for simple fold maps satisfying this. Moreover, the codimensions of fold maps are larger than 1 unless otherwise stated.

More precisely, we consider normal bubbling operations in situations satisfying the following properties.

Definition 6. For a stable fold map $f : M \to N$ on an $m$-dimensional closed and connected manifold into an $n$-dimensional manifold with no boundary satisfying $m - n > 1$ and an integer $l > 0$, let $P$ be a connected component of $(W_f - q_f(S(f))) \cap \bar{f}^{-1}(N - f(S(f)))$ and let $\{(S_j, N(S_j), c_j : N(S_j) \to P)\}_{j=1}^l$ be a normal system of submanifolds compatible with $f$ such that each $S_j$ is a point or a standard sphere. Suppose that we can perform a normal bubbling operation to $f$ to obtain $f'$ whose generating normal system is $\{c_j(S_j)\}$ satisfying the following properties.

1. The union $\bigcup_{j=1}^l c_j(N(S_j))$ of the images of the immersions are in an open set $U \subset P$ such that $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a trivial bundle whose fiber is a standard sphere.
2. The indices of points in the preimage of new connected components in the resulting singular value set are all 1.
3. For each regular value $p \in U$ of the resulting map, the preimages are disjoint unions of standard spheres.

We say that this operation is an admissible trivial operation with standard spheres or that this operation is ATSS.

FIGURE 2 shows for a case (where) some generating manifolds are circles and $n = 2$ using a same figure as that of FIGURE 1 or Definition 3. Numbers indicate the numbers of connected components of the preimages of regular values in the corresponding places.

We give an explanation of an explicit local fold map around $p_j'$ respecting the FIGURE 2 (or 1).

First in Example 1 (2), we explain a fold map for $l = 1$. We construct a product bundle $D^n \times S^{m-n}$ over $D^n$. We also set a Morse function $\tilde{f}_{m-n,[a,b]}$ on $S^{m-n} \times [-1,1]$ onto $[a,b] \subset (0, +\infty) \subset \mathbb{R}$ such that the preimage of the minimum is the boundary, that there exist exactly two singular points, and that singular points are in the interior. We glue the projection of the product bundle and the map $\tilde{f}_{m-n,[a,b]} \times \text{id}_{S^{m-n}} : [a, +\infty) \times S^{m-n}$ where we identify the base space $D^n$ of the product bundle with a standard closed disc of dimension $n$ whose diameter is $a$ (note that $S^0$ is a two point set with the discrete topology). By gluing suitably, we have a fold map from $S^n \times S^{m-n}$ onto the standard closed disc of dimension $n$ whose center is the origin and diameter is $b$ in $\mathbb{R}^n$. This is a desired map. More precisely, see [2] and [5].
In this construction, we replace \( \tilde{f}_{m-n,[a,b]} \) by \( \tilde{f}_{m-n,[a,b']} := \tilde{f}_{m-n,[a,b]}|f_{m-n,[a,b]}^{-1}[a,b'] \) where \( b' < b \) is sufficiently close to \( b \). We denote the resulting map onto the standard closed disc of dimension \( n \) whose center is the origin and diameter is \( b' \) in \( \mathbb{R}^n \) by \( \tilde{f}_{m,n,b'} \).

We consider the composition of \( \tilde{f}_{m-n+\dim D_{2j'-1},\dim D_{2j'-1},r} \) for \( r > 0 \) with a suitable diffeomorphism and thus we have a smooth map onto a sufficiently small standard closed disc \( D'_{2j'-1} \supset D_{2j'-1} \) of dimension \( \dim D_{2j'-1} \) satisfying \( S_{a(j')} \supset D'_{2j'-1} \supset \text{Int} D'_{2j'-1} \supset D_{2j'-1} \). We can take a sufficiently small standard closed disc \( D'_{2j'} \supset D_{2j'} \) of dimension \( \dim D_{2j'} \) satisfying \( S_{b(j')} \supset D'_{2j'} \supset \text{Int} D'_{2j'} \supset D_{2j'} \) and consider the product map of the previous smooth map and the identity map \( \text{id}_{D'_{2j'/}} \). We compose the resulting map with a suitable diffeomorphism respecting FIGURE 2 (or 1).

Last, we change the map into a desired map so that the resulting singular value set is as presented. We can restrict the map to a total space of a trivial \( D^{m-n} \)-bundle over the target space, identified with \( D'_{2j'-1} \times D'_{2j'} \). For the composition of \( \tilde{f}_{m-n+\dim D_{2j'-1},\dim D_{2j'-1},r'} \) for \( r' > 0 \) with a suitable embedding, we can restrict the map to a total space of a trivial \( D^{m-n} \)-bundle over the target space, diffeomorphic to a standard closed disc of dimension \( \dim D_{2j'} \). The total space is regarded as a submanifold of the domain of the original map and we restrict the composition of \( \tilde{f}_{m-n+\dim D_{2j'-1},\dim D_{2j'-1},r'} \) for \( r' > 0 \) with a suitable diffeomorphism to the complement of the submanifold in the domain, which is also a compact submanifold of dimension \( m-n+\dim D_{2j'} \) of the domain. We consider (the composition of) the product map of this restriction and the identity map \( \text{id}_{D'_{2j'-1}} \) (with a suitable diffeomorphism) We replace the original projection of the trivial \( D^{m-n} \)-bundle over the target space, identified with \( D'_{2j'-1} \times D'_{2j'} \), by this, in a suitable way. We have a desired map.

The map is said to be a local canonical fold map around a crossing for an ATSS operation.

We can see the following easily and we use this implicitly in various scenes of the present paper.

**Corollary 1.** Let \( f \) be a stable fold map \( f : M \to N \) on an \( m \)-dimensional closed and connected manifold into an \( n \)-dimensional manifold with no boundary satisfying \( m-n > 1 \). If an ATSS operation is performed to \( f \) and a new map \( f' \) is obtained, then \( W_f \) is a proper subset of \( W_{f'} \) such that for the map \( f' : W_{f'} \to N \), the restriction to \( W_f \) is \( f : W_f \to N \).

3. **Examples, Construction of New Families of Stable Fold Maps (and their Reeb Spaces) and a Relation between the Cohomology Rings of their Reeb Spaces and their Source Manifolds.**

We will present new examples of stable fold maps by applying ATSS operations to fundamental fold maps such as some special generic maps and investigate homology groups and cohomology rings of the resulting Reeb spaces.

Such studies were also demonstrated in [6] and [7]. However, obtained examples are simple fold maps. We obtain stable fold maps which may not be simple and calculate the cohomology rings of their Reeb spaces. We compare some of the new results to some of the known results in these articles.
We first obtain examples as Proposition 3. Throughout this section, for an integer \( i \geq 0 \), the \( i \)-th module of a graded commutative algebra \( A \) over a commutative ring \( R \) is the module consisting of all elements of degree \( i \) and the 0-th module is assumed to be \( R \) forgetting the ring structure where the action by \( R \) is defined in a canonical way.

Definition 7. For two graded commutative algebras \( A_1 \) and \( A_2 \) over a commutative ring \( R \), a graded commutative algebra \( A \) over \( R \) is obtained by defining the 0-th modules in a canonical way from the direct sum if the following properties hold.

1. For an integer \( i > 0 \), the \( i \)-th module is the direct sum of the \( i \)-th module of \( A_1 \) and the \( i \)-th module of \( A_2 \).
2. For a pair \((a_{i,j}, a_{i,j}) \in A_1 \oplus A_2 \) of elements of degree \( i_j > 0 \) for \( j = 1, 2 \), they are defined as elements of degree \( i_1 \) and \( i_2 \), respectively, and the product is \((a_{i_1,2}a_{i_2,2}) \in A_1 \oplus A_2 \) and an element of degree \( i_1 + i_2 \).
3. For \( r \in R \), which is an element of degree 0, and a pair \((a_{i,1}, a_{i,2}) \in A_1 \oplus A_2 \) of elements of degree \( i > 0 \), they are defined as elements of degree 0 and degree \( i \), respectively, and the product is \((ra_{i,1}, ra_{i,2}) \in A_1 \oplus A_2 \) and an element of degree \( i \).

Proposition 3. Let \( R \) be a PID having an identity element \( 1 \in R \) satisfying \( 1 \neq 0 \in R \). Let \( k \in R \) be represented as \( k_0 \in \mathbb{Z} \) times the identity element \( 1 \). Let \( f : M \rightarrow N \) be a stable fold map on an \( m \)-dimensional closed and connected manifold into an \( n \)-dimensional manifold with no boundary satisfying \( m - n > 1 \) and \( n \) be even. Let \( N \) be not closed. We also assume at least one of the following conditions.

1. \( n \) is divisible by 4.
2. For the identity element \( 1 \in R \), \( 1 + 1 = 0 \in R \).

Then by an ATSS operation to \( f \), we have a new fold map \( f' \) satisfying the following properties.

1. \( H_i(W_f; R) \) is isomorphic to \( H_i(W_f; R) \) for \( i \neq \frac{n}{2} \), and \( H_i(W_f; R) \oplus \mathbb{R} \) for \( i = \frac{n}{2} \).
2. The cohomology ring \( H^*(W_f; R) \) is isomorphic to an algebra obtained by defining the 0-th modules in a canonical way from the direct sum of the cohomology ring \( H^*(W_f; R) \) and a graded commutative algebra \( A_R \) over \( R \). We denote the \( i \)-th module by \( A_{R,i} \). \( A_R \) is zero for \( i \neq 0, \frac{n}{2} \), \( n \) and isomorphic to \( R \) for \( i = \frac{n}{2} \). The square of a generator of \( A_{R,\frac{n}{2}} \) is represented as \( k\nu_{R,n} \) where \( \nu_{R,n} \) is a generator of \( A_{R,n} \).

Proof. In the proof, we abuse notation in section 2: especially notation in Definition 3 and around this. Set \( l = 1 \) in Definition 3(6). Let us find a suitable normal system of submanifolds compatible with \( f \) \( \{(S_1, N(S_1), c_1 : N(S_1) \rightarrow P)\} \) such that \( S_1 \) is a standard sphere of dimension \( \frac{n}{2} \). We take an open disc \( U \) in the regular value set sufficiently close to the intersection of the image of a connected component of the singular set consisting of singular points of index 0 and the complement of the corner in the boundary of the image of the map: the image \( f(M) \) is regarded as a manifold which may have corners and smoothly embedded in the target manifold. Singular points of index 0 exist and we can take \( U \) as this. In fact, \( M \) is closed, \( N \) is not compact (closed) and there exists a singular point of index 0.

Let us assume \( k_0 \geq 0 \).

We can take \( \{(S_1, N(S_1), c_1 : N(S_1) \rightarrow P)\} \) so that the family \( \{c_1\} \) of immersions has exactly \( k_0 \geq 0 \) pairs of crossings (\( 2k_0 \) crossings), that the normal bundle of the
immersion is trivial and that \( c_1(N(S_1)) \subset U \). We can perform an ATSS operation whose generating system is \( c_1(N(S_1)) \) to obtain a new fold map \( f' \). We use a local canonical fold map around a crossing in section 2 around each crossing of \( c_1(S_1) \); around the remaining singular values and regular values, we construct products of Morse functions with exactly one singular point, which is of index 1, and identity maps on \((n-1)\)-dimensional manifolds and trivial \( S^{m-n} \)-bundles over \( n \)-dimensional manifolds and last glue all the local maps together. We can perform this construction thanks to the assumption that \( U \) is an open ball in the regular value set sufficiently close to the image of a connected component of the singular set consisting of singular points of index 0.

By the definition of a normal bubbling operation, \( W'_f \) is regarded as a space obtained by attaching a polyhedron \( A \) we can obtain by identifying exactly \( k_0 \)-pairs of disjointly embedded PL discs of dimension \( n \) of a smooth (PL) manifold \( S_1 \times S^\sharp \) to \( B := f^{-1}(c_1(N(S_1))) \subset W_f \). \( i_{B,f} \) and \( i_{B,A} \) denote the canonical inclusions of \( B \) into \( W_f \) and \( A \), respectively. We have the following exact sequence.

\[
\begin{array}{c}
\rightarrow & H_i(B; R) \xrightarrow{i_{B,f} \oplus i_{B,A}} H_i(W_f; R) \oplus H_i(A; R) \\
\rightarrow & H_i(W_f; R) \xrightarrow{i_{B,A} \ominus i_{B,f}} H_{i-1}(B; R) \\
\end{array}
\]

By observing the definition of a normal bubbling operation and the topologies of these spaces, the image of \( i_{B,f} \) is zero except for the case \( i = 0 \) and \( i_{B,A} \) is injective. \( H_i(W_f; R) \) is isomorphic to \( H_i(W_f; R) \) for \( i \neq \frac{n}{2} \) and \( H_i(W_f; R) \oplus R \) for \( i = \frac{n}{2} \). In the case \( i = \frac{n}{2} \), the summand \( R \) is seen to be generated by the class represented by \( \{\ast\} \times S^\sharp \) in the original manifold \( S_1 \times S^\sharp \) to obtain \( A \) where \( \ast \) is a suitable point in \( S_1 \). In the case \( i = n \), the summand \( R \) is seen to be generated by the class represented by \( A \), which is an \( n \)-dimensional polyhedron obtained from the original \( n \)-dimensional closed, connected and orientable manifold \( S_1 \times S^\sharp \).

We discuss the cohomology rings. By the construction, the resulting cohomology ring is represented as an algebra obtained by defining the 0-th modules in a canonical way from the direct sum of that of \( W_f \) and a new graded commutative algebra \( A_R \): we denote the \( i \)-th module of \( A_R \) by \( A_{R,i} \). This is zero except \( i = \frac{n}{2}, n \). In these two cases \( i = \frac{n}{2}, n \), the modules are isomorphic to \( R \). We investigate the product of two elements of degree \( \frac{n}{2} \).

We discuss the case where the number \( k_0 \geq 0 \) of the pairs of crossings is 1. In this case, the square of a generator of the module \( A_{R,\frac{n}{2}} \) can be a generator of \( A_{R,n} \) and also 0. We can define a cohomology class regarded as the dual of the homology class represented by \( \{\ast\} \times S^\sharp \) before: the value at this class is the identity element 1 \( \in R \) and the values at classes in \( H_{\frac{n}{2}}(W_f; R) \oplus \{0\} \subset H_{\frac{n}{2}}(W_f; R) \oplus R \) are zero. We evaluate the value of the square \( S_q \) at the class represented by the \( n \)-dimensional polyhedron \( A \): more precisely, the class obtained after the original manifold \( S_1 \times S^\sharp \) is deformed and attached to \( W_f \).

For the original manifold \( S_1 \times S^\sharp \), the class represented by \( S_1 \times \{\ast\} \subset S_1 \times S^\sharp \) can be mapped to the class represented by \( \{\ast\} \times S^\sharp \) in the original manifold deformed and attached to \( W_f \) and regarded as a subspace in \( W'_f \). Moreover, if we treat this sphere as an embedding, then via a suitable homotopy, we can change this to a PL homeomorphism onto a sphere regarded as the preimage of the image \( c_1(F) \) of a fiber \( F \) of the bundle \( N(S_1) \) by the map \( \tilde{f} : W_f \to N \). The class represented by \( S_1 \times \{\ast'\} \subset S_1 \times S^\sharp \) can be also mapped to zero if we perform a normal bubbling operation in another suitable way.
We explain about this argument on the class represented by \( S_1 \times \{ *' \} \subset S_1 \times S^{\mathbb{F}} \).
We consider one point in the pair \((p_1, p_2)\) of the crossing in \( c_1(S_1)\): take \( p_2 \). We can take \( D_1, D_2, D_3 \) and \( D_4 \) as in Definition 3. The key ingredient is which of the two connected components of \( \partial_c^{-1}(c_1(\partial D_4)) \) is a branched or a non-manifold point. We can consider another normal bubbling operation to \( f \) to exchange the types of the topologies around these two connected components without changing other parts including the structure around \( p_1 \) from the original \( f' \).

This yields the fact that the value of the square \( S_4 \) before at the class represented by the \( n \)-dimensional polyhedron \( A \) or the class obtained after the original manifold \( S_1 \times S^{\mathbb{F}} \) is deformed and attached to \( W_f \) can be a generator of \( A_{R,n} \). It can be also zero if we perform the operation in a suitable way.

For a general \( k_0 \geq 0 \) and also for \( k_0 < 0 \), we can argue similarly. For the case where the number \( k_0 \) is 0, see also the proofs of some propositions and theorems of [7]. This completes the proof.

\[ \square \]

This is obtained by applying Proposition 3 inductively.

**Theorem 1.** Let \( R \) be a PID having an identity element \( 1 \in R \) satisfying \( 1 \neq 0 \in R \). Let \( k_1 > 0 \) be an integer and \( \{ k_{2,j} \}_{j=1}^{k_1} \subset R \) be a sequence of elements of \( R \) such that \( k_{2,j} \) is represented as \( k_{2,j} \neq 0 \in \mathbb{Z} \) times the identity element 1. Let \( f : M \to N \) be a stable fold map on an \( m \)-dimensional closed and connected manifold \( M \) into an \( n \)-dimensional manifold \( N \) with no boundary satisfying \( m - n > 1 \) and \( n \) be even. Let \( N \) be not closed. We also assume at least one of the following conditions.

1. \( n \) is divisible by 4.
2. For the identity element \( 1 \in R, 1 + 1 = 0 \in R \).

Then by an ATSS operations to \( f \), we have a new fold map \( f' \) satisfying the following properties.

1. \( H_i(W_f; R) \) is isomorphic to \( H_i(W_f; R) \) for \( i \neq \frac{n}{2} \) and \( H_i(W_f; R) \oplus R^{k_1} \) for \( i = \frac{n}{2} \).
2. The cohomology ring \( H^*(W_f; R) \) is isomorphic to and identified with an algebra obtained by defining the 0-th modules in a canonical way from the direct sum of the cohomology ring \( H^*(W_f; R) \) and a graded commutative algebra \( A_R \) over \( R \). We denote the \( i \)-th module of \( A_R \) by \( A_{R,i} \). \( A_R \) is zero for \( i \neq 0, \frac{n}{2}, n \) and isomorphis \( R \) for \( i = 0 \) and \( R^{k_1} \) for \( i = \frac{n}{2}, n \). Take a suitable generator \( \{ a_j \}_{j=1}^{k_1} \) of \( \{ 0 \} \oplus A_{R,\frac{n}{2}} \subset H^{\frac{n}{2}}(W_f; R) \oplus A_{R,\frac{n}{2}} \) and a suitable generator \( \{ b_j \}_{j=1}^{k_1} \) of \( \{ 0 \} \oplus A_{R,n} \subset H^n(W_f; R) \oplus A_{R,n} \). The square of \( a_j \) is represented as \( k_{2,j} b_j \in \{ 0 \} \oplus A_{R,n} \) and the product of distinct two elements \( a_j \) is zero.

We compare Theorem 1 (Proposition 3) to known results in [6] and [7]. In these articles, we investigated the homology groups and the cohomology rings of Reeb spaces and summands playing same roles as \( A_R \) plays in Proposition 3 and Theorem 1. One of important properties in the situations of the previous results is that we can always take non-zero elements whose degrees are \( \frac{n}{2} \) having vanishing squares where \( n \) is even if non-zero elements of degree \( \frac{n}{2} \) exist. On the other hand, in the situations of Proposition 3 and Theorem 1, this property does not hold in general.

**Theorem 2.** Let \( R \) be a PID having an identity element \( 1 \in R \) satisfying \( 1 \neq 0 \in R \). Let \( k_1, k_2 > 0 \) be integers and \( \{ k_{2,j} \}_{j=1}^{(k_1+1)k_2} \subset R \) be a sequence of elements of \( R \) such
that $k_{2,j}$ is represented as $k_{2,j,0} \in \mathbb{Z}$ times the identity element 1. Let $f : M \to N$ be a stable fold map on an $m$-dimensional closed and connected manifold into an $n$-dimensional manifold with no boundary satisfying $m - n > 1$ and $n$ be even. Let $N$ be not closed. We also assume at least one of the following conditions.

1. $n$ is divisible by 4.
2. For the identity element $1 \in R$, $1 + 1 = 0 \in R$.

Then we can change $f$ in the following way into a stable fold map $f_0$ into $N$.

First we remove the interior of manifold $D$ represented as a disjoint union of $k_1$ copies of a product of a standard closed disc of dimension $\frac{n}{2} + 1$ and $(\frac{n}{2} - 1)$-dimensional standard standard sphere smoothly embedded in the regular value set sufficiently close to the intersection of the image of a connected component of the singular set consisting of singular points of index 0 and the complement of the corner in the boundary of the image of the map: the image $f(M)$ is regarded as a manifold which may have corners and smoothly embedded in the target manifold. After that we attach a product map of a height function on a unit disc of dimension $m - n + 1$ and the identity map on $\partial D$ instead by diffeomorphisms between the boundaries so that the resulting singular value set $f_0(S(f_0))$ is the disjoint union of the original one and an embedded manifold diffeomorphic to $\partial D$.

Moreover, by an ATSS operation to $f_0$, we have a new fold map $f'$ satisfying the following properties.

1. $H_i(W_f; R)$ is isomorphic to $H_i(W_{f'}; R)$ for $i \neq \frac{n}{2}$, $H_i(W_f; R) \oplus R^{k_1} \oplus R^{k_2}$ for $i = \frac{n}{2}$ and $H_n(W_f; R) \oplus R^{k_2}$ for $i = n$.
2. The cohomology ring $H^*(W_f; R)$ is isomorphic to and identified with an algebra obtained by defining the 0-th modules in a canonical way from the direct sum of the cohomology ring $H^*(W_f; R)$ and a graded commutative algebra $A_R$ over $R$. We denote the $i$-th module by $A_R,i$. Then this is zero for $i \neq 0$, $\frac{n}{2}$, $n$, isomorphic to $R^{k_1}$ for $i = 0$, isomorphic to $R^{k_1} \oplus R^{k_2}$ for $i = \frac{n}{2}$ and isomorphic to $R^{k_2}$ for $i = n$. Take a suitable generator $\{a_{1,j}\}^{k_1}_{j=1} \cup \{a_{2,j}\}^{k_2}_{j=1}$ of $\{0\} \oplus A_R,\frac{n}{2} \subset H^{\frac{n}{2}}(W_f; R) \oplus A_R,\frac{n}{2}$ and a suitable generator $\{b_{j}\}^{k_2}_{j=1}$ of $\{0\} \oplus A_R,n \subset H^n(W_f; R) \oplus A_R,n$. The product of distinct $a_{1,j}$’s is 0, the product of $a_{1,j}$ and $a_{2,j}$ is represented as $k_{2,(j_1-1)k_2+j_2}b_j \in \{0\} \oplus A_R,n$ and the square of $a_{2,j}$ is represented as $k_{2,k_1+k_2}b_j$.

We present a sketch of the proof for the case $k_1 = 1$. Rigorous explanations are needed in the last part or the discussion on the cohomology rings of the proof and they are explained in the proofs of propositions and theorems in [7]. Together with the case $k_1 > 0$, rigorous proofs are left to readers.

A sketch of the proof. We can see that $H_i(W_{f_0}; R)$ is isomorphic to $H_i(W_{f'}; R)$ for $i \neq \frac{n}{2}$, $H_i(W_f; R) \oplus R$ for $i = \frac{n}{2}$ and $H_n(W_f; R)$ for $i = n$. The cohomology ring is isomorphic to an algebra obtained by identifying the 0-th modules in a canonical way in the direct sum of $H^*(W_f; R)$ and an algebra $A_0,R$: we denote the $i$-th module by $A_{0,R,i}$ and this is zero for $i \neq 0$, $\frac{n}{2}$ and isomorphic to $R$ for $i = 0$, $\frac{n}{2}$. We explain about an ATSS operation to $f_0$.

We can choose $k_2$ immersions of $S^2$ into the regular value set sufficiently close to the new connected component of the singular value set $f_0(S(f_0))$ so that the images are disjoint as Proposition 3. We can use the $j$-th immersion such that the following properties hold.
(1) At a fundamental class of the domain of the immersion into the (interior) image \( f_0(M_0) \), which is regarded as a manifold which may have corners and smoothly embedded in the target manifold, the value of the homomorphism induced by the inclusion is \( k_2 \cdot j \) times a generator of the class represented by a sphere parallel to an \( \frac{n}{2} \)-dimensional sphere in the new connected component in the resulting singular value set \( f_0(S(f_0)) \) representing a generator of the \( \frac{n}{2} \)-th homology group of the connected component, diffeomorphic to \( S^{\frac{n}{2}-1} \times S^{\frac{n}{2}} \).

(2) The number of the crossings of the immersion is \( k_2, k_2 + j \) for \( n > 2 \) and larger than or equal to this for \( n = 2 \); this is based on fundamental properties of immersions and embeddings of curves into planes and surfaces.

By applying Proposition 3 one after another observing the changes of the topologies of the Reeb spaces and so on, we have a result: for more rigorous discussions, see also [7].

We explain about another important property in the situations of Theorems 1 and 2 (Proposition 3). The square of each element of degree \( \frac{n}{2} \) of an algebra playing roles \( A_R \) plays must be 0 or divisible by 2 in known results in [6] and [7]. By taking suitable numbers \( k_{2,j,0} \), we can obtain cases which do not satisfy this property.

Related to this observation, we have the following by virtue of Theorems 1 and 2 and their proofs.

**Corollary 2.** Let \( R \) be a PID having an identity element 1 \( \in R \) satisfying 1 \( \neq 0 \in R \) and assume also 1 + 1 = 0 \( \in R \). In the situations of Theorem 1 and 2, consider the ranks of the submodules of \( \{0\} \oplus A_{R,\frac{n}{2}} \subset H^{\frac{n}{2}}(W_f; R) \oplus A_{R,\frac{n}{2}} \) generated by the set of all elements whose squares vanish. In Theorem 1, the rank can be an arbitrary integer 0 \( \leq i_r \leq k_1 \) and in Theorem 2, it can be an arbitrary integer \( k_1 \leq i_r \leq k_1 + k_2 \).

Last we extend Proposition 2

**Proposition 4.** Proposition 2 holds even if we weaken the condition so that the restrictions to the singular sets may have crossings which are normal and each connected component of the preimage of each of which contains at most two singular points.

The **piecewise smooth category** is the category whose objects are smooth manifolds with canonically defined PL structures and whose morphisms are piecewise smooth maps between the manifolds. The category is known to be equivalent to the PL category.

**A sketch of the proof.** This is essentially due to the discussions in referred articles and [14]. More rigorous proof is left to readers.

We construct a manifold bounded by the original manifold collapsing to the Reeb space in the piecewise smooth category.

Note that for each point in the image and an \( n \)-dimensional small standard closed disc containing this in the interior, each connected component of the preimage is one of the following types as smooth manifolds (which may have corners),

1. A product of an \((m-n)\)-dimensional standard sphere and an \( n \)-dimensional standard closed disc.
(2) A product of an manifold obtained by removing the interior of the union of three disjointly and smoothly embedded standard closed discs in $S^{m-n+1}$ and an $(n-1)$-dimensional standard closed disc.

(3) A product of an manifold obtained by removing the interior of the union of four disjointly and smoothly embedded standard closed discs in $S^{m-n+1}$, a closed interval $I$ and an $(n-2)$-dimensional standard closed disc.

There exist three types of the topologies of small regular neighborhoods of points in the Reeb space. Each type of these types of the topologies corresponds to each case above. All points in the Reeb space of each type form manifolds whose dimensions are $n$, $n-1$, and $n-2$, respectively.

For each case, we can construct bundles whose fibers are as above in the piecewise linear category and we can construct bundles whose fibers are $D^{m-n+1}$, $D^{m-n+2}$, or $D^{m-n+2} \times I$ in the category whose subbundles obtained by restricting the fibers to suitable compact submanifolds of the boundaries of the discs $D^{m-n+1}$ or $D^{m-n+2}$ are the original bundles: note that the dimensions of the suitable compact submanifolds are same as those of the boundaries and that in the last case we restrict $D^{m-n+2} \times I$ to $\partial D^{m-n+2} \times I = S^{m-n+1} \times I$ first. We can locally construct these bundles and we can glue them in the piecewise smooth category (PL category).

This yields a desired $(m+1)$-dimensional compact PL manifold collapsing to $W_f$, which is an $n$-dimensional polyhedron. The resulting $(m+1)$-dimensional manifold is regarded as a manifold obtained by attaching handles whose indices are larger than or equal to $m-n$ to $M \times \{1\} \subset M \times [0,1]$ where we discuss in the PL category.

We can apply this to some explicit fold maps obtained in Theorems 1 and 2 and so on.

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