ASYMPTOTIC LIKELIHOOD OF CHAOS FOR SMOOTH FAMILIES OF CIRCLE MAPS

HIROKI TAKAHASI

ABSTRACT. We consider a smooth two-parameter family \( f_{a,L} : \theta \mapsto \theta + a + L\Phi(\theta) \) of circle maps with a finite number of critical points. For sufficiently large \( L \) we construct a set \( A_L^{(\infty)} \) of \( a \)-values of positive Lebesgue measure for which the corresponding \( f_{a,L} \) exhibits an exponential growth of derivatives along the orbits of the critical points. Our construction considerably improves the previous one of Wang and Young for the same class of families, in that the following asymptotic estimate holds: the Lebesgue measure of \( A_L^{(\infty)} \) tends to full measure in \( a \)-space as \( L \) tends to infinity.

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1. Introduction

In the study of one-dimensional dynamical systems, one important question is: how often are dynamics chaotic? Here, ”often” should be understood in the sense of Lebesgue measure in parameter space and ”chaotic dynamics” corresponds to maps which have absolutely continuous invariant probability measures (acip for short).

In this direction, there is a huge gap between a general belief and the existing theory. For the quadratic family \( x \mapsto 1 - ax^2 \), for instance, it is believed, and also suggested by rigorous computations \cite{19} that the set of parameters corresponding to acips should have large Lebesgue measure. Meanwhile, what is presently known at best is that this set has positive, yet very small Lebesgue measure \cite{6}. The aim of this paper is to narrow this gap, for certain smooth families of maps on the circle.

Let \( \Phi : S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R} \) be a Morse function. We consider a two-parameter family of circle maps of the form

\[
f_{a,L} : \theta \mapsto \theta + a + L\Phi(\theta) \quad a \in [0, 1), L > 0.
\]

The family with \( \Phi(\theta) = \sin(2\pi\theta) \) was introduced by Arnol’d \cite{11} and played an important role in the creation of KAM theory. For small \( L \), the map is a diffeomorphism, and this case was intensively studied for its connection with quasi-periodic motions.
on invariant tori in conservative systems. We explore dynamics at the other end of the spectrum, namely, the case with sufficiently large \( L \). Then the map has a finite number of critical points. Its graph has large slopes outside of a small neighborhood of the critical points.

The family of circle maps with large \( L \) becomes important in the theory of \textit{rank-one strange attractors}, developed by Wang and Young \cite{20} \cite{21}, based on the fundamental works of Jakobson \cite{6}, Benedicks and Carleson \cite{3}, Mora and Viana \cite{12}, and others. In brief terms, the theory indicates that dynamics of strange attractors in certain physically relevant multi-dimensional systems may be partially understood by analyzing the above circle family with large \( L \). Indeed, the existence of strange attractors in certain periodically forced ODEs with fully stochastic behaviors was rigorously proved along this line \cite{21}.

For large \( L \), a positive measure set of values of \( a \) was constructed in \cite{21} corresponding to maps with a unique acip. However, their construction seems far from optimal in that \( \liminf_{L \to \infty} \text{Leb} \left( \{ a \in [0,1) : f_{a,L} \text{ has an acip } \} \right) > 0 \) does not follow. Meanwhile, it is physically relevant to consider what happens in the asymptotic case \( L \to \infty \). An intuition is that parameters with acip are in abundance.

The reason for this deficiency is that their construction has to start with very small parameter intervals containing ”good parameters”, and the dependence of the sizes of the intervals on \( L \) is unclear. In this paper we develop another argument and show that \( \lim_{L \to \infty} \text{Leb} \left( \{ a \in [0,1) : f_{a,L} \text{ has a unique acip } \} \right) = 1 \). A key ingredient is to notice that for sufficiently large \( L \) it is possible to carry out an inductive construction taking the whole parameter space \( [0,1) \) as a start-up interval.

\subsection{Statement of the result.}

Let \( C \) denote the set of critical points of \( f_{a,L} \), which does not depend on \( a \). Since all the critical points of \( \Phi \) are non-degenerate, the cardinality of \( C \) is constant for all large \( L \). For each \( c \in C \), all \( a \in [0,1) \) and \( i \geq 0 \), write \( c_i(a) \) for \( f_{a,L}^{i+1}c \). Let \( | \cdot | \) denote the one-dimensional Lebesgue measure.

Our main theorem states the abundance of parameters for which the derivatives along orbits of all the critical points grow exponentially fast under iteration. It is well-known \cite{4} that this growth condition implies the existence of acips.

\textbf{Main Theorem.} \textit{For the above \((f_{a,L})\) there exists \( \lambda > 0 \) such that for all sufficiently large \( L \) there exists a set \( A_{L}^{(\infty)} \) in \([0,1)\) with positive Lebesgue measure such that for all \( a \in A_{L}^{(\infty)} \) and each \( c \in C \), \( |(f_{a,L}^{n})'c_0| \geq L^{\lambda n} \) holds for every \( n \geq 0 \). Moreover \( \lim_{L \to \infty} |A_{L}^{(\infty)}| = 1 \) holds.}

For parameters in \( A_{L}^{(\infty)} \), following \cite{20} \cite{21} it is possible to construct a unique acip \( \mu \) for which Lebesgue almost every \( \theta \in S^1 \) is generic, that is,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f_{a,L}^{i} \theta) = \int \varphi d\mu \quad \text{for all continuous } \varphi : S^1 \to \mathbb{R}.
\]

In particular, parameters corresponding to maps with periodic attractors are contained in the complement of \( A_{L}^{(\infty)} \). The same statement also follows from directly showing that all periodic points are hyperbolic repelling for parameters in the theorem.
Our construction yields an explicit measure estimate in terms of $L$ (Proposition 6.1), and as a by-product gives a bound on the speed of the convergence $|A_L^{(\infty)}| \to 1$ as follows: the measure of the complement of $A_L^{(\infty)}$ decreases to zero as $L \to \infty$, faster than any power of $L^{-1}$.

1.2. Outline of a proof. In the context of one-dimensional maps with critical points, the existence of acips for a positive measure set of parameters was first proved by Jakobson [6]. See [2] [5] [13] [3] [17] [18] [10] [15] [22] [14] for alternative arguments and generalizations.

An outline of the proof of the theorem is similar in spirit to that of [6], and (hence) to those of all the subsequent papers. The positive measure set $A_L^{(\infty)}$ is constructed by induction: at each step we get rid of undesirable parameters for which the corresponding maps may not have acips. In doing this we bring together ideas from Benedicks and Carleson [2] [3], Tsujii [17] [18], and develop them further.

Key constants are $\sigma, \lambda, \alpha, N, L$, chosen in this order. we have $\sigma, \alpha \ll 1$ and $N, L \gg 1$. The choice of them are made explicit afterwards.

Our induction scheme is divided into two parts. The first part consists of finite steps $0, 1, 2, \cdots, N$. The second part consists of the remaining steps $N+1, N+2, \cdots$. At each step $n$ we construct a set $A^{(n)}$. The parameter set in the main theorem is given by $A^{(\infty)} = \bigcap_{n \geq 0} A^{(n)}$.

For $0 \leq n \leq N$, let $A^{(n)}$ denote the set of all $a \in [0, 1)$ such that:

\[(1) \quad d(c_i(a), C) \geq \sigma \text{ for every } i \in [0, n] \text{ and } c \in C.\]

It turns out that parameters in $A^{(N)}$ enjoy a uniformly expanding property outside of a small neighborhood of the critical points of fixed size (Corollary 4.9). Since our primary interest is an exponential growth of derivatives, this property permits us to concentrate on returns of critical orbits to the inside of this small neighborhood.

Condition (1) for every $n$ is satisfied only for parameters in a set with zero Lebesgue measure. To get a set of positive measure we need to relax this condition. For each $n \geq N$ and $c \in C$ we introduce two conditions:

\[(X)_{c,n} \quad |(f_a^{-i})' c_i| \geq L \cdot \min\{\sigma, L^{-\alpha i}\} \text{ for every } 0 \leq i < j \leq n;\]
\[(Y)_{c,n} \quad |(f_a')' c_0| \geq L^{\alpha i} \text{ for every } 0 \leq i \leq n.\]

We say $f_a$ satisfies $(X)_n$ if $(X)_{n,c}$ holds for each $c \in C$. The meaning of $(Y)_n$ is analogous. There two conditions are taken as assumptions of induction at step $n$.

To recover the assumptions of induction at the next step $n+1$, we exclude from further consideration all parameters in $A^{(n)}$ for which some analytic condition leading to $(X)_{n+1}$ $(Y)_{n+1}$ fails. This condition is introduced in section 5. The remaining parameters constitute $A^{(n+1)}$.

This paper is organized as follows. In section 2 we prove three lemmas which will be frequently used later. In section 3 we perform the first part of the inductive steps $0, 1, \cdots, N$ and estimate the measure of $A^{(0)}$, $A^{(1)}, \cdots, A^{(N)}$. In section 4 we establish a common technique, a recovery of expansion, which will be used to estimate the measure of $A^{(n)}$ for $n > N$. In section 5 we introduce condition $W_n$ which defines the set $A^{(n)}$ for $n > N$. In section 6 we estimate the measure of $A^{(\infty)}$.

Unless otherwise stated, we always assume that $L$ is sufficiently large.
2. FUNDAMENTAL LEMMAS

We prove three lemmas which will be frequently used later. Lemma 2.2 gives distortion bounds for iterations of one fixed map. Lemma 2.6 gives distortion bounds for critical values for different parameters. Proofs are similar, by virtue of Lemma 2.3 which asserts a similarity between space and parameter derivatives, allowing us to transfer estimates in phase space to parameter space.

Write $f$ for $f_{a,L}$. Let $C_\varepsilon$ denote the $\varepsilon$-neighborhood of $C$. There exists $K_0 \geq 1$ depending only on $\Phi$ and small $\varepsilon > 0$ such that for all large $L$ we have:

$$K_0^{-1}L|c - \theta|^2 \leq |f(c) - f(\theta)| \leq K_0L|c - \theta|^2 \quad \text{for } c \in C \text{ and } \theta \in C_\varepsilon;$$

$$K_0^{-1}L|c - \theta| \leq |f'(\theta)| \leq K_0L|c - \theta| \quad \text{for } c \in C \text{ and } \theta \in C_\varepsilon;$$

$$|f'|, |f''| \leq K_0L.$$

2.1. Distortion in phase space. For $\theta \in S^1$, $n \geq 1$ and $i \in [0, n-1]$, define

$$d_i(a, \theta) = |(f_a^i)[\varphi]|^{-1} \cdot |f_a(I)\psi|,$$

when it makes sense. Fix $\beta \in (\frac{3}{2}, 2)$ and define

$$D_n(a, \theta) = L^{-\beta} \cdot \left[ \sum_{0 \leq i \leq n-1} d_i^(-1)(a, \theta) \right]^{-1}.$$

Put

$$K = \exp(2K_0L^{1-\beta}).$$

Note that $K \to 1$ as $L \to \infty$.

**Lemma 2.2.** If $\theta \in S^1$, $n \geq 1$ and $f_{a,\theta} \not\in C$ holds for every $0 \leq i \leq n-1$, then

$$\frac{|(f_a^n)[\varphi]|}{|(f_a^n)[\psi]|} \leq K \quad \text{for all } \varphi, \psi \in [\theta - D_n(a, \theta), \theta + D_n(a, \theta)].$$

**Proof.** Write $f$, $d_i$, $D_n$ for $f_{a,\theta}$, $d_i(a, \theta)$, $D_n(a, \theta)$ correspondingly. Let $I = [\theta - D_n, \theta + D_n]$. It suffices to prove the following for $j = 0, \ldots, n-1$:

$$|f^jI| \sup_{f^jI} \frac{|f'|^{n-j}}{|f'|} \leq \log K \cdot d_j^{-1} \left[ \sum_{0 \leq i \leq n-1} d_i^{-1} \right]^{-1}.$$

Indeed, summing this over all $j = 0, 1, \ldots, n-1$ gives

$$\sup_{\theta, \psi \in I} \log \frac{|(f_a^n)[\varphi]|}{|(f_a^n)[\psi]|} \leq \sum_{0 \leq j \leq n-1} \sup_{\theta, \psi \in I} \log \frac{|f'(f_a^j\theta)|}{|f'(f_a^j\psi)|} \leq \sum_{0 \leq j \leq n-1} |f^jI| \sup_{f^jI} \frac{|f'|^{n-j}}{|f'|} \leq \log K.$$
Hence (3) over all \( j = 0, 1, \ldots, k - 1 \) implies \( \frac{\| (f^k)' \psi \|}{\| f \|} \leq K \) for all \( \varphi, \psi \in I \).

Hence
\[
|f^k| \leq K|f'(f^k)\theta|I|
\]
\[
= KD_n d_k^{-1} |f'(f^k)\theta|I|
\]
\[
\leq KL^{-\beta}|f'(f^k)\theta|d_k^{-1} (\sum d_i^{-1})^{-1}
\]
by the definition of \( D_n \)
\[
\leq KL^{-\beta}|f'(f^k)\theta|.
\]
Hence, for all \( \psi \in I \) we have
\[
|f'(f^k)\theta - f'(f^k)\theta| \leq K_L|f^k| \leq KK_0 L^{1-\beta}|f'(f^k)\theta|.
\]
This yields \( |f'| \geq (1 - KK_0 L^{1-\beta})|f'(f^k)\theta| \) on \( f^k I \), and therefore
\[
|f^k| \sup_{f^k I} \frac{|f'|}{|f'|} \leq \frac{KK_0 L^{1-\beta}}{1 - KK_0 L^{1-\beta}} \cdot d_k^{-1} (\sum d_i^{-1})^{-1}
\]
\[
\leq 2K_0 L^{-\beta} \cdot d_k^{-1} (\sum d_i^{-1})^{-1},
\]
where the last inequality is because of \( K \to 1 \) as \( L \to \infty \). \( \square \)

2.3. Transversality. For \( c \in C \) and \( i \geq 0 \), write \( c'_i(a) = \frac{d a_i}{d a}(a) \).

Lemma 2.4. Let \( f_a \) satisfy \( (Y)_{n,c} \). Then we have
\[
1 - L^{-\lambda/2} \leq \frac{|c'_n(a)|}{|c'_n(a)/(c_0(a))|} \leq 1 + L^{-\lambda/2}.
\]

Proof. Since \( c'_n(a) = 1 + f'_a(c_{n-1}(a))c'_{n-1}(a) \), we have
\[
c'_n(a) = 1 + f'_a(c_{n-1}) + f'_a(c_{n-1})f'_a(c_{n-2}) + \cdots + f'_a(c_{n-1})f'_a(c_{n-2}) \cdots f'_a(c_1)f'_a(c_0).
\]
Dividing both sides by \( (f'_{n-1})(c_0) = f'_a(c_{n-1})f'_a(c_{n-2}) \cdots f'_a(c_1)f'_a(c_0) \) gives
\[
\frac{c'_n(a)}{(f^n_{n-1})(c_0)} = 1 + \sum_{i=1}^n \frac{1}{(f^i)(c_0)}.
\]
Hence
\[
1 - \sum_{i=1}^n \frac{1}{|f^i c_0|} \leq \frac{|c'_n(a)|}{|c'_n(a)/(c_0)|} \leq 1 + \sum_{i=1}^n \frac{1}{|f^i c_0|}.
\]
\( (Y)_{n,c} \) yields the desired inequality. \( \square \)

2.5. Distortion in parameter space. We transfer the distortion estimate in Lemma 2.2 to parameter space. For \( c \in C \) and \( n \geq 1 \), Define
\[
\Delta_n(a, c) = [a - D_n(a_0, c_0(a)), a + D_n(a_0, c_0(a))].
\]
Let
\[
K' = \exp \left( L^{-\frac{\lambda}{2}} + 3 \right).
\]

Lemma 2.6. Let \( f_a \) satisfy \( (X)_{n,c} \) and \( (Y)_{n,c} \). Then
\[
\frac{|c'_n(a)|}{|c'_n(b)|} \leq K' \quad \text{for all } a, b \in \Delta_n(a_0, c).
\]
Proof. Write \( d_t, \ D_n, \ \hat{\Delta}_n \) for \( d_t(c_0(a_*)), \ D_n(c_0(a_*)), \ \hat{\Delta}_n(a_*, c) \) correspondingly. We argue by induction on \( k \in [0, n - 1] \), with the assumption that \( \frac{|c'_j(a)|}{|c_j(a)|} \leq K' \) holds for all \( a, b \in \Delta \) and \( j = 0, 1, \cdots, k \). Note that this assumption for \( k = 0 \) is trivially satisfied by \( c'_0 = 1 \).

**Sublemma 2.7.** For all \( j = 0, 1, \cdots, k \) we have

\[
\log \frac{|(f'_a)c_j(a)|}{|(f'_b)c_j(b)|} \leq \frac{2K_0K'L^{1-\beta}}{1 - 2K_0K'L^{1-\beta}} \cdot d_j^{-1} \cdot d_j^{-1} \left[ \sum_{0 \leq i \leq n-1} d_i^{-1} \right]^{-1} \quad \text{for all } a, b \in \hat{\Delta}_n(a_*, c).
\]

Proof. We have

\[
|f'_a c_j(a) - f'_b c_j(b)| \leq |f'_a c_j(a) - f'_b c_j(a)| + |f'_b c_j(a) - f'_b c_j(b)|
\]

\[
= |f'_b c_j(a) - f'_b c_j(b)| \quad \text{since } f'_a \theta - f'_b \theta = 0
\]

\[
\leq K_0 L |c_j(\Delta)| \quad \text{since } |f''| \leq K_0 L.
\]

Using the assumption of induction and Lemma 2.4,

\[
|c_j(\Delta)| \leq 2K' |(f'_{a_*})' c_0(a_*)| d_j d_j^{-1} D_n
\]

\[
= 2K' L^{-\beta} |(f'_{a_*})' c_j(a_*)| d_j^{-1} \left( \sum d_i^{-1} \right)^{-1}
\]

\[
\leq 2K' L^{-\beta} |(f'_{a_*})' c_j(a_*)|.
\]

These two inequalities imply the desired inequality. \( \square \)

If \( j \geq 1 \), Lemma 2.3 and \( (Y)_{n,c} \) for \( f_{a_*} \) give

\[
|c_j'(a)| \geq K'^{-1} |c_j'(a_*)| \geq \frac{K'^{-1}}{2} |(f'_{a_*})' c_0(a_*)| \geq L^{-\frac{\Delta}{2}}.
\]

Since \( |c'_0| = 1 \), the same inequality remains valid if \( j = 0 \). We have

\[
\left| \frac{c_{j+1}(a)}{c_j(a)} - (f'_a)c_j(a) \right| = \left| \frac{1}{c_j(a)} \right| \leq L^{-\frac{\Delta}{2}}.
\]

The first factor in the right hand side of Sublemma 2.7 goes to 0 as \( L \to 0 \). Using this and \( (X)_{n,c} \) give

\[
| (f_a)' c_j(a) | \geq \frac{1}{2} | (f_a)' c_j(a_*) | \geq \frac{L}{2} \cdot \min \{ \sigma, L^{-\alpha_j} \}.
\]

Hence we have

\[
\left| \frac{c_{j+1}(a)}{c_j(a)} \right| \geq \frac{L}{2} \cdot \min \{ \sigma, L^{-\alpha_j} \} - L^{-\frac{\Delta}{2}} \geq \frac{1}{3} L^{-\alpha_j}.
\]

These three inequalities imply

\[
\left| \log \frac{|c_{j+1}(a)|}{|c_j(a)|} - \log |(f_a)' c_j(a)| \right| \leq L^{-\frac{\Delta}{2}}.
\]

By Sublemma 2.7,

\[
\left| \log \frac{|c_{j+1}(a)|}{|c_j(a)|} - \log |(f_a)' c_j(a)| \right| \leq L^{-\frac{\Delta}{2}} + \frac{3}{2} d_j \cdot \left( \sum d_i^{-1} \right)^{-1}.
\]
Thus, for all \(a, b \in \hat{\Delta}_n\),
\[
\left| \log \frac{|c'_{j+1}(a)|}{|c'_{j+1}(b)|} - \log \frac{|c'_j(a)|}{|c'_j(b)|} \right| \leq 2L^{-\frac{\beta}{2}} + 3d^{-1} \left( \sum d_i^{-1} \right)^{-1}.
\]

Summing this over all \(j = 1, \cdots, k\) implies
\[
|c'_{k+1}(a)| - |c'_{k+1}(b)| \leq K',
\]
which restores the assumption of the induction. \(\square\)

3. Parameter exclusion: special steps

In this section we perform the first part of the induction from step 0 to \(N\). Recall that the parameter sets \(A^{(n)}\) for \(0 \leq n \leq N\) are given by (1), where we set \(\sigma = K_0 L^{-1+\frac{\beta}{2}}\).

We estimate the measure of \(A^{(n)}\) for \(n = 0, 1, \cdots, N\).

3.1. Expansion. We consider orbits of critical points which stay outside of \(C_{\sigma}\). Lemma 2.4 transmits the expansion along these orbits to parameter space. The next lemma asserts that this expansion is large enough for critical values to completely wrap \(S^1\).

Lemma 3.2. Let \(n \leq N\), \(a \in [0, 1)\) and \(c \in C\). If \(c_i(a) \notin C_{\sigma}\) holds for \(i = 0, 1, \cdots, n - 1\), then \(c_n(\hat{\Delta}_n(a, c)) = [0, 1)\).

Proof. We have
\[
|c_n(\hat{\Delta}_n(a, c))| \geq \frac{K'}{1 + L} |(f_a^n)'c_0| |\hat{\Delta}_n(a, c)| \quad \text{by Lemma 2.6}
\]
\[
\geq \frac{K' - 1}{1 + L} |(f_a^n)'c_0| |\hat{\Delta}_n(a, c)| \quad \text{by Lemma 2.4}
\]
The assumption gives \(|(f_a^n)'c_0| \geq (L\sigma)^{n-i} |(f_a^i)'c_0|\), which yields \(|(f_a^n)'c_0| d_i \geq (L\sigma)^{n-i} L\sigma\). Therefore
\[
|(f_a^n)'c_0|^{-1} |\hat{\Delta}_n(a, c)|^{-1} = L^\beta \sum_{0 \leq i \leq n-1} |(f_a^n)'c_0|^{-1} d_i^{-1}
\]
\[
\leq \sigma^{-1} L^{-1+\beta} \sum_{i \geq 1} (L\sigma)^{-i}
\]
\[
\leq \frac{\sigma^{-1} L^{-1+\beta}}{L\sigma - 1}.
\]

Substituting this into the above inequalities we obtain the desired inequality. \(\square\)

3.3. Amending the definition of parameter intervals. We now introduce parameter intervals which are central to our scheme. Let \(c \in C\), \(n \geq 1\), and let \(f_a\) satisfy \((X)_{n,c}\), \((Y)_{n,c}\). As a result of Lemma 2.4, the map \(a \in \hat{\Delta}_n(a, c) \mapsto c_n(\hat{\Delta}_n(a, c))\) may not be injective (we have just shown that this is indeed the case if \(n \leq N\)). This causes silly combinatorial problems. To deal with this we shorten the interval
as follows. For a compact interval \( I \) centered at \( a \) and \( r \in (0,1) \), let \( r \cdot I \) denote the interval centered at \( a \) with length \( r \cdot |I| \). Define

\[
\Delta_n(a,c) = \begin{cases} 
\hat{\Delta}_n(a,c) & \text{if } |c_n(\hat{\Delta}_n(a,c))| \leq \frac{1}{3}; \\
\frac{1}{9|c_n(\frac{1}{2}a,a,c)|} \cdot \Delta_n(a,c) & \text{otherwise}.
\end{cases}
\]

By definition, \( c_n(\Delta_n(a,c)) \) is strictly contained in the half circle centered at \( c_n(a) \).

**Proposition 3.4.** For any sufficiently large \( N \) there exists \( L_0 \) such that if \( L \geq L_0 \), then for \( n = 0, 1, \cdots, N \) we have \( |A^{(n)}| \geq (1 - 3\sqrt{\sigma})^{n+1} \).

**Proof.** The inequality for \( n = 0 \) follows from the identity \( f_\alpha \theta - f_\beta \theta = a - b \). We argue by induction on \( n \). For \( (c, \bar{c}) \in C \times C \), let

\[
B_n(c, \bar{c}) = \{ a \in A^{(n-1)} \setminus A^{(n)} : d(c_n(a), \bar{c}) \leq \sigma \}.
\]

We have \( A^{(n-1)} \setminus A^{(n)} = \bigcup B_n(c, \bar{c}) \), where the union runs over all \( (c, \bar{c}) \in C \times C \).

**Lemma 3.5.** Let \( a, b \in B_n(c, \bar{c}) \) and assume that \( (i) \) \( \Delta_n(a,c) \cap \Delta_n(b,c) \neq \emptyset \); \( (ii) \) \( b \notin \Delta_n(a,c) \). Then we have \( \Delta_n(a,c) \subset \Delta_n(b,c) \).

**Proof.** Since \( a \in B_n(c, \bar{c}) \) we have \( |c_n(a) - \bar{c}| \leq \sigma \). This and Lemma 3.2 together imply the existence of a unique parameter \( \bar{a} \in \sqrt{\sigma} \cdot \Delta_n(a,c) \) such that \( c_n(\bar{a}) = \bar{c} \). For the same reason, there exists a unique \( \bar{b} \in \sqrt{\sigma} \cdot \Delta_n(b,c) \) such that \( c_n(\bar{b}) = \bar{c} \). Since the map \( \bar{a} \in \Delta_n(a,c) \cup \Delta_n(b,c) \rightarrow c_n(\bar{a}) \) is injective (by the above amendment), we have \( \bar{a} = \bar{b} \). This and the assumption \( (ii) \) implies that one of the connected component of \( \Delta_n(a,c) - \sqrt{\sigma} \cdot \Delta_n(a,c) \) is contained in \( \sqrt{\sigma} \cdot \Delta_n(b,c) \), and thus the inclusion holds.

Let \( a_1 \in B_n(c, \bar{c}) \). We define a finite sequence \( a_1, a_2, \cdots \) in \( B_n(c, \bar{c}) \) inductively as follows. Given \( a_1, \cdots, a_k \), if \( B_n(c, \bar{c}) \subset \bigcup_{j=1}^k \Delta_n(a_j,c) \), then we complete the definition. Otherwise, choose \( a_{k+1} \in B_n(c, \bar{c}) - \bigcup_{j=1}^k \Delta_n(a_j,c) \) arbitrarily. By (11), the lengths of the intervals \( \{ \Delta_n(a_j,c) \} \) are uniformly bounded from below, and thus the definition makes sense.

By Lemma 3.5, any two of the intervals thus defined are either disjoint or nested and altogether cover \( B_n(c, \bar{c}) \). Moreover, by Lemma 2.6 and Lemma 3.2, the set \( \Delta_n(a_j,c) - \sqrt{\sigma} \cdot \Delta_n(a_j,c) \) does not intersect \( B_n(c, \bar{c}) \). Hence

\[
|B_n(c, \bar{c})| \leq \sqrt{\sigma} \cdot \sum \Delta_n(a_j,c),
\]

where \( \sum \) denotes the summation over the maximal set of the subscript \( j \) for which the corresponding intervals are pairwise disjoint. Since \( |\Delta_n(a_j,c)| \leq L^{-\beta} \),

\[
\sum \Delta_n(a_j,c) \leq \frac{|A^{(n-1)}|}{|A^{(n-1)}|} (1 + 2L^{-\beta})
\]

\[
\leq (1 + 2L^{-\beta}) \frac{|A^{(n-1)}|}{(1 - 3\sqrt{\sigma})^n}
\]

\[
\leq 2|A^{(n-1)}|,
\]
where the last inequality holds for sufficiently large $L$ depending on $N$. This yields $|B_n(c, \tilde{c})| \leq 2\sqrt{\sigma}|A^{(n-1)}|$. Hence we have

$$|A^{(n)} \setminus A^{(n)}| \leq (\#C)^2 \cdot 2\sqrt{\sigma}|A^{(n-1)}| \leq 3\sqrt{\sigma}|A^{(n-1)}|,$$

or $|A^{(n)}| \geq (1 - 3\sqrt{\sigma})|A^{(n-1)}|$. This restores the assumption of the induction. \(\square\)

4. Recovering expansion

For large $L$, the dynamics is uniformly expanding in most of the phase space, outside of a small neighborhood of the critical set $C$. Meanwhile, returns of typical orbits to the inside of this neighborhood are inevitable. In this section we deal with the loss of derivatives associated with these returns to keep the further evolution of derivatives in track.

4.1. Constants. We introduce several constants. Let

$$\lambda_0 = \frac{1}{2} - \frac{\beta}{4} \quad \text{and} \quad \lambda = \frac{\lambda_0}{9}. \tag{5}$$

Fix $\alpha \ll \lambda$ and let

$$\delta_0 = L^{-1+\lambda_0} \quad \text{and} \quad \delta = L^{-\alpha N}. \tag{6}$$

We use three different sizes of neighborhoods of critical points, given by $\sigma, \delta, \delta_0$. We have $\delta / \delta_0, \delta_0 / \sigma \to 0$ as $L \to \infty$. The last convergence follows from $\beta < 2$.

4.2. Bound periods. Let $c \in C$, and let $f$ satisfy $(X)_{n,c}$, $(Y)_{n,c}$. For $p \in [1, n-1]$, let

$$I_p(c) = \left( c + \sqrt{L^{-1}D_{p+1}(c_0)}, c + \sqrt{L^{-1}D_p(c_0)} \right).$$

Let $I_{-p}(c)$ denote the interval which is the mirror image of $I_p(c)$ with respect to $c$. We call $p$ a bound period for $\varphi \in I_p(c) \cup I_{-p}(c)$. According to Lemma \ref{lem:period_bound} the orbit of $f(\varphi)$ shadows the orbit of $c_0$ for $p$ iterates, with bounded distortion.

We claim that for all $a \in A^{(N)}$, the intervals $\{I_p(c) : 1 \leq |p| \leq N, c \in C\}$ altogether cover $C_{\delta_0} - C_{\delta}$. Indeed we have

$$L^{-1}D_N(c_0) \leq |(f^{N-1})' c_0|^{-1} |f' c_{N-1}| \quad \text{by the definition of } D_N(c_0) \leq (K_0^{-1}L\sigma)^{-N+1}K_0L \quad \text{by } \ref{lem:asymptotic_approx}(1)$$

$$< \delta^2 \quad \text{by } \ref{lem:asymptotic_approx}(1) \text{ and } \ref{lem:asymptotic_approx}(3).$$

On the other hand,

$$\delta_0^2 = L^{-\beta+2\lambda_0} \sigma^2 \quad \text{by } \ref{lem:asymptotic_approx}(4)$$

$$\leq L^{-\beta} \sigma \quad \text{by } \ref{lem:asymptotic_approx}(5)$$

$$\leq L^{-1}D_1(c_0) \quad \text{by } \kappa > 1 \text{ and } d(c_0, C) \geq \sigma \text{ in } \ref{lem:asymptotic_approx}(1).$$

Hence $\sqrt{L^{-1}D_N(c_0)} < \delta < \delta_0 < \sqrt{L^{-1}D_1(c_0)}$, which implies the claim.

Lemma 4.3. Let $f$ satisfy $(X)_{n,c}$, $(Y)_{n,c}$ for some $n \geq N$ and $c \in C$. Then for $p \in [1, n-1]$ and $\varphi \in I_p(c) \cup I_{-p}(c)$ we have:

(a) $\log |c - \varphi|^{-\lambda \log L} \leq p \leq \log |c - \varphi|^{-\lambda/2 \log L}$;

(b) $|(f^{p+1})' \varphi| \geq \max\{|c - \varphi|^{-1 + 2\tau}, L^{\lambda(p+1)/3}\}$;
(c) if $p \leq N$, then $|(f^{p+1})'\varphi| \geq L^\alpha(p+1)/3$. 

**Proof.** We use the notation $\mathcal{O}(1)$ to denote all constants which stay bounded and bounded away from zero as $L \to \infty$.

Since $|c_0 - f\varphi| \leq K_0L|c - \varphi|^2$, we have 

$$|c_p - f^{p+1}\varphi| \leq \mathcal{O}(1)L|\theta - \varphi|^2|(f^p)'c_0|$$

by Lemma 2.2 

$$\leq \mathcal{O}(1)D_p(c_0)|(f^p)'c_0| \text{ since } \varphi \in I_\varepsilon(c) \cup I_p(c).$$

To estimate the right hand side we need 

**Sublemma 4.4.** We have $|(f^n)'\theta|D_n(\theta) \leq K^2\beta L^{2-\beta}$. 

**Proof.** We have 

$$|(f^n)'\theta|d_{n-1}d_{n-1}^{-1}D_n \leq L^{-\beta}|(f^n)'\theta|d_{n-1}$$

$$= L^{-\beta}|f'(f^{n-1}\theta)|^2$$

$$\leq K^2\beta L^{2-\beta}.$$

Hence we have $|c_p - f^{p+1}\varphi| \leq \mathcal{O}(1)L^{2-\beta}$. On the other hand, since $|c_0 - f\varphi| \geq K_0L|c - \varphi|^2$ we have 

$$|c_p - f^{p+1}\varphi| \geq \mathcal{O}(1)L|c - \varphi|^2|(f^p)'c_0|$$

$$\geq \mathcal{O}(1)L|c - \varphi|^2L^\lambda \text{ by (Y)$_{n,c}$}.$$ 

Putting these two inequalities together and rearranging yields $|c - \varphi|^2L^\lambda \leq 1$, which implies the upper estimate in (a).

**Sublemma 4.5.** We have $|c_p - f^{p+1}\varphi| \geq \mathcal{O}(1)L^{2-\beta-3\alpha p}$. 

**Proof.** We have 

$$|c_p - f^{p+1}\varphi| \geq \mathcal{O}(1)|(f^p)'c_0|D_{p+1}(c_0)$$

$$= \mathcal{O}(1)L^{-\beta}\left[\sum_{i=0}^{p}|(f^p)'c_0|^{-1}d_i^{-1}\right]^{-1}.$$ 

For the sum in the square bracket, $|(f^p)'c_0|d_i \geq L^2(\min\{\sigma, L^{-\alpha}\})^2$ which follows from $(X)_{n,c}$ gives 

$$\sum_{i=0}^{p}|(f^p)'c_0|^{-1}d_i^{-1} \leq L^{-2}\sigma^{-2} \cdot \frac{-\log \sigma}{\alpha \log L} + L^{-2}\sum_{i=0}^{p}L^{2\alpha i}$$

$$\leq L^{-2+3\alpha p}.$$ 

This implies the desired inequality. 

Sublemma 4.5 and $|f'| \leq K_0L$ give 

$$L^{-3\alpha p} \leq |c_p - f^{p+1}\varphi| \leq \mathcal{O}(1)L|c - \varphi|^2(K_0L)^p.$$ 

Rearranging this yields the lower estimate of $p$ in (a).
We have
\[ |(f^{p+1})'\varphi| \geq O(1)L|(f^p)'c_0||c - \varphi| \geq O(1)|(f^p)'c_0|D_{p+1}(c_0)||c - \varphi|^{-1}. \]
Sublemma 4.3 gives \(|(f^p)'c_0|D_{p+1}(c_0)| \geq L^{2-\beta-3\alpha p}. Hence \]
\[ |(f^{p+1})'\varphi| \geq O(1)L^{2-\beta-3\alpha p}|c - \varphi|^{-1}. \]
Substituting into this the upper estimate of \(p\) in (a) gives \[ |(f^{p+1})'\varphi| \geq L^{2-\beta}|c - \varphi|^{-1+\frac{6\delta}{\alpha p}}. \]
Substituting \(|c - \varphi|^{-1} \geq L^{\lambda p/2}\) into this which follows from (a),
\[ |(f^{p+1})'\varphi| \geq L^{2-\beta}|c - \varphi|^{-1+\frac{6\delta}{\alpha p}} \geq L^{(p+1)/3}. \]
The last inequality is a consequence of \(2 - \beta > \lambda_0\). This proves (b). A proof of (c) is analogous to that of (b).

4.6. **Decompositions into ”bound” and ”free” segments.** We introduce some useful language along the way. For \(\theta \in S^1\) such that \(f^i\theta \notin C\) for all \(i \geq 1\), let
\[
n_1 < n_1 + p_1 + 1 \leq n_2 < n_2 + p_2 + 1 \leq \cdots
\]
be defined as follows: \(n_1\) is the smallest \(j \geq 0\) such that \(f^j\theta \in C_\delta\). For \(k \geq 1\), let \(p_k\) be the bound period of \(f^k\theta\), and let \(n_{k+1}\) be the smallest \(j \geq n_k + p_k + 1\) such that \(f^j\theta \notin C_\delta\). (Note that an orbit may return to \(C_\delta\) during its bound periods, i.e. \(n_i\) are not the only return times to \(C_\delta\).) This decomposes the orbit of \(\theta\) into segments corresponding to time intervals \((n_k, n_k + p_k]\) and \([n_k + p_k + 1, n_{k+1}]\), during which we describe the orbit of \(\theta\) as being ”bound” and ”free” states respectively; \(n_k\) are called times of *free returns*. For orbits which return to \(C_{\delta_0}\) but not to \(C_\delta\), we similarly define bound and free states using \(C_{\delta_0}\) instead of \(C_\delta\).

The next lemma asserts that no return to \(C_{\delta_0}\) occurs during these bound periods.

**Lemma 4.7.** Let \(\theta\) make a free return at time \(n\) to \(C_{\delta_0} \setminus C_\delta\), with \(p\) the corresponding bound period. Then \(f^i\theta \notin C_{\delta_0}\) holds for every \(i \in [n+1, n+p+1]\).

**Proof.** Let \(c\) denote the critical point to which \(f^n\theta\) is bound. Lemma 4.2 gives \(|f^i\theta - f^{i-n}c| \leq KK_0|(f^{i-n})'(c_0)|D_p(c_0) \leq L^{-\beta}. Hence \]
\[
d(f^i\theta, C) \geq d(f^{i-n}c, C) - |f^i\theta - f^{i-n}c| \\
\geq \sigma - L^{-\beta} \quad \text{by } i-n \leq N \quad \text{and } (1) \\
> \delta_0.
\]
This yields the claim.

4.8. **Exponential growth outside of critical neighborhoods.** The next corollary asserts an exponential growth of derivatives outside of \(C_\delta\). Our derivation of this consists of two explicit parts: the obvious exponential growth outside of \(C_{\delta_0}\) for free segments and the recovered expansion in Lemma 4.3 for bound segments associated with returns to \(C_{\delta_0}\).

**Corollary 4.9.** For any \(a \in A^{(N)}\), \(f = f_a\) satisfies the following:
(a) if \(n \geq 1\) and \(\theta, f_a\theta, \cdots, f_a^{n-1}\theta \notin C_\delta\), then \(|(f_a^n)'\theta| \geq L^{1-3\lambda}\delta L^{3\lambda n};
\]
(b) if moreover \( f_a^n \theta \in C_\delta \), then \( |(f_a^n)'\theta| \geq L^{3\lambda n} \).

**Proof.** If the orbit of \( \theta \) makes no return to \( C_\delta \) in the first \( n-1 \) iterates, then the assertions are obvious. Otherwise, let \( 0 < n_1 < \cdots < n_s \leq n-1 \) denote the sequence of all free returns to \( C_\delta \) in the first \( n-1 \) iterates of \( \theta \), with \( p_1, \ldots, p_s \) the corresponding bound periods. We have

\[
|(f_n^{n_{s+1}})'f_{n_1}\theta| = \prod_{1 \leq k \leq s-1} |(f_{n_k+1}^{n_{k+1}-n_k-p_k-1})'(f_{n_k+1}^{n_k+1})'f_{n_k}\theta|.
\]

For each bound segment, (c) in Lemma 4.3 gives \( |(f_{p_k+1}^{p_k+1})'f_{n_k}\theta| \geq L^{(p_k+1)\lambda_0/3} \). For each free segment, we clearly have \( |(f_{n_{k+1}-n_k-p_k-1})'(f_{n_k+1}^{n_k+1})'f_{n_k}\theta| \geq L^{\lambda_0(n_{k+1}-n_k-p_k-1)} \).

Hence

\[
|(f_n^{n_{s+1}})'f_{n_1}\theta| \geq \prod_{1 \leq k \leq s-1} L^{\lambda(n_{k+1}-n_k)} \geq L^{\frac{3n}{3}}(n_{s+1} - n_1).
\]

Since \( |(f^1)'\theta| \geq L^{\lambda n_1} \) we obtain \( |(f^s)'\theta| \geq L^{\frac{3n}{3}} \).

For the remaining factor, by Lemma 4.7 we have \( f^i \theta \notin C_\delta \) for \( n_i + 1 \leq i \leq n-1 \). This and \( d(f_n^s, C) \geq \delta \) gives

\[
|(f_{n-s}^n)'f_{n_1}\theta| \geq K^{-1}_{n} L \delta L^{\lambda_0(n-s-1)} \geq \delta L^{\lambda_0(n-s-1)} \times L^{1-\lambda}.
\]

Consequently we obtain (a). If \( f^n \in C_\delta \), then Lemma 4.7 gives \( n \geq n_s + p_s + 1 \), and thus \( |(f_{n-s}^n)'f_{n_1}\theta| \geq L^{\frac{3n}{3}}(n-s) \). This proves (b). \( \square \)

5. **Recovering inductive assumptions**

We have already defined the sets \( A^{(0)}, \ldots, A^{(N)} \) and estimated their measure. At step \( n \geq N \), we exclude from \( A^{(n)} \) all parameters for which \( X_{n+1} \) or \( Y_{n+1} \) may fail. We introduce condition \( W_n \) which determines a rule of exclusion. Parameters have to satisfy this condition to be selected. In other words, for \( n \geq N \) we define

\[
A^{(n+1)} = \{ a \in A^{(n)} : W_n \text{ holds} \}.
\]

5.1. **Condition \( W_n \).** Let \( f_a \) satisfy \( X_n, Y_n \). We say \( f_a \) satisfies \( W_{n,c} \) if

\[
(7) \quad \sum_{\text{free return}} - \log d(c_i(a), C) \leq \frac{\alpha k \log L}{3} \quad \text{for every } k \in [0, n].
\]

The sum runs over all \( i \) at which the orbit of \( c_0 \) makes a free return to \( C_\delta \). We say \( f_a \) satisfies \( (W)_n \) if it satisfies \( (W)_{n,c} \) for every \( c \in C \).

The next proposition implies that the assumptions of the induction are recovered for parameters in \( A^{(n+1)} \).

**Proposition 5.2.** Let \( f_a \) satisfy \( W_{n,c} \). Then \( X_{n+1,c} \) and \( Y_{n+1,c} \) hold for \( f_a \).

**Proof.** We begin with the particular case in which \( c_0 \) makes no return to \( C_\delta \) in the first \( n \) iterates. (11) gives \( \delta |(f^N)'c_0| \geq \delta (K_0^{-1}L\sigma)^N \geq L^{3\lambda N} \). Hence

\[
|((f^{n+1})'c_0| = |((f^{n+1-N})'c_N)|(f^N)'c_0| \geq \delta L^{3\lambda(n+1-N)} |(f^N)'c_0| \quad \text{by Corollary 4.9} \geq L^{3\lambda(n+1)},
\]

which determines a rule of exclusion. Parameters have
which proves $Y_{n+1,c}$.

To check $X_{n+1,c}$ we only need to consider inequalities which are not covered by $X_{n,c}$. They are:

\begin{equation}
|f^{n+1-i}'c_i| \geq L \cdot \min \{\sigma, L^{-ai}\} \quad \text{for } 0 \leq i \leq n.
\end{equation}

Suppose that $i \geq N$. Corollary 4.9 and the definition of $\delta$ in (8) give

\begin{equation}
|f^{n+1-i}'c_i| \geq L\delta \geq LL^{-ai}.
\end{equation}

Suppose that $i < N$. Since $c_i, c_{i+1}, \ldots, c_{i+N} \notin C_\delta$, Corollary 4.9 gives $|f^Nc_i| \geq \delta L^{\lambda N}$ and $|f^{n+1-i-N}'c_{N+i}| \geq L\delta$. Therefore

\begin{equation}
|f^{n+1-i}'c_i| = |f^{n+1-i-N}'c_{N+i}||f^Nc_i| \geq L,
\end{equation}

where the last inequality follows from the definition of $\delta$ and $\alpha \ll \lambda$.

Proceeding to the general case, let $0 < n_1 < \cdots < n_s \leq n$ denote all the free returns to $C_\delta$ in the first $n$-iterates of $c_0$, with $p_1, \ldots, p_s$ the corresponding bound periods. Using Lemma 4.3 and then $W_{n,c}$,

\begin{equation}
\sum_{k=1}^{s} p_k \leq \frac{2}{\lambda \log L} \sum_{k=1}^{s} - \log d(c_{n_k}, C) \leq \frac{2\alpha n}{3\lambda \log L}.
\end{equation}

The chain rule gives

\begin{equation}
|f^{n_s+p_s+1}'c_0| = |f^{n_1}'c_0| \cdot \prod_{k=1}^{s-1} |f^{n_{k+1}-n_k-p_k-1}'c_{n_k+p_k+1}| \prod_{k=1}^{s} |f^{p_k+1}'c_{n_k}|.
\end{equation}

For the first term, $\lambda \ll \alpha$ and $n_1 \geq N$ give $L^{\lambda n_1} \geq \delta^{-1}$. This and Corollary 4.9 give $|f^{n_1}'c_0| \geq L^{\lambda n_1} L^{2\lambda n_1} \geq \delta^{-1} L^{2\lambda n_1}$. Using Lemma 4.3 for each term in the products, we obtain

\begin{equation}
|f^{n_s+p_s+1}'c_0| \geq \delta^{-1} L^{2\lambda(n_s+p_s+1-\sum_{k=1}^{s} p_k)}.
\end{equation}

The rest of the argument splits into two cases. First, suppose that $n_s+p_s \geq n$. Using $|f'| \leq K_0L$ and $p_s \leq \frac{2\alpha n}{3\lambda \log L}$ in (9) we have

\begin{equation}
\frac{|f^{n_s+p_s+1}'c_0|}{|f^{n+1}'c_0|} \leq (K_0L)^{p_s} \leq L^{\alpha n}.
\end{equation}

Hence

\begin{equation}
|f^{n+1}'c_0| \geq L^{2\lambda(n_s+p_s+1-\sum_{k=1}^{s} p_k)} \geq L^{2\lambda(n+1-\sum_{k=1}^{s} p_k)}.
\end{equation}

Next, suppose that $n_s+p_s < n$. Since $n_s$ is the last free return, Corollary 4.9 gives $|f^{n+1-n_s-p_s}'c_{n_s+p_s+1}| \geq K_0^{-1} \delta L^{3\lambda(n-n_s-p_s)}$. Combining this with (10) gives

\begin{equation}
|f^{n+1}'c_0| \geq L^{2\lambda(n+1-\sum_{k=1}^{s} p_k)},
\end{equation}

yielding the same inequality as in the previous case. In either of these two cases, substituting the upper estimate of the sum of the bound periods in (9) into the exponent of (12) yields the desired inequality.

Proof of $X_{n+1,c}$. We deal with four cases separately.
Case (i): \( i, n + 1 \notin \bigcup_{1 \leq k \leq s}[n_k + 1, n_k + p_k] \). We have
\[
|(f^{n+1-i})'\theta_i| \geq \delta L^{\lambda(n+1-i)} \quad \text{by Corollary 4.9 and Lemma 4.3}
\]
\[
\geq \delta L^{\alpha(n+1)-\alpha i} \quad \text{since } \alpha < \lambda
\]
\[
\geq L^{1-\alpha i} \quad \text{since } \delta L^{\alpha(n+1)} \geq L\delta L^{\alpha N} \geq L.
\]
Case (ii): \( i \notin \bigcup_{1 \leq k \leq s}[n_k + 1, n_k + p_k] \) and \( n + 1 \in [n_s + 1, n_s + p_s] \). Let \( \bar{c} \) denote the critical point to which \( c_{n_s} \) is bound. We have:
\[
d(c_{n_s}, C) \geq L^{-\frac{\alpha i}{3}} \quad \text{by } W_{n,c},
\]
\[
|(f^{n-s})'c_{n_s+1}| \geq K^{-1}L^{\lambda(n-s)} \quad \text{by Lemma 2.2 and } Y_{n,\bar{c}},
\]
\[
|(f^{n-s})'c_i| \geq L^{\frac{\delta}{3}(s_i-n_s)} \quad \text{since } i, n_s \text{ are free and } n_s \text{ is a return.}
\]
Combining these altogether gives
\[
|(f^{n+1-i})'c_i| \geq |(f^{n-s})'c_{n_s+1}|K_0^{-1}Ld(c_{n_s}, C)|(f^{n-s})'c_i|
\]
\[
\geq K_0^{-1}K^{-1}L^{\frac{\delta}{3}(n-i)+1}L^{-\alpha n/3}
\]
\[
\geq K_0^{-1}K^{-1}L^{\alpha(n-i)/3+1}L^{-\alpha n/3}
\]
\[
\geq L^{1-\alpha i/2},
\]
where the last inequality holds because \( i \geq 1 \) and \( K \to 1 \) as \( L \to \infty \).

Case (iii): \( i, n + 1 \in [n_k + 1, n_k + p_k] \) for some \( k \in [1, s] \). Let \( \bar{c} \) denote the critical point to which \( c_{n_k} \) is bound. If \( \sigma \geq L^{-\alpha(i-n_k-1)} \), then
\[
|(f^{n+1-i})'c_i| \geq K^{-1}|(f^{n+1-i})'\bar{c}_{i-n_k-1}| \quad \text{by Lemma 2.2}
\]
\[
\geq K^{-1}L \cdot L^{-\alpha(i-n_k-1)} \quad \text{by } X_{n,\bar{c}}
\]
\[
\geq L \cdot L^{-\alpha i}.
\]
Suppose that \( \sigma < L^{-\alpha(i-n_k-1)} \). Then the definition of \( \sigma \) in (1) gives \( i-n_k \leq 2\alpha^{-1} \). If \( n-i \leq \alpha N \), then the bound period \( p_k \) for \( n_k \) remains in effect at time \( n \), because all bound periods are \( \geq \alpha N \) (Lemma 4.3). Hence
\[
|(f^{n+1-i})'c_i| \geq K^{-1}|(f^{n+1-i})'\bar{c}_{i-n_k-1}|
\]
\[
\geq 1 \quad \text{by (1)}.
\]
It is left to consider the subcase \( n-i \geq \alpha N \). The proof of (ii) and \( n_k < i \) imply
\[
|(f^{n+1-n_k})'c_{n_k}| \geq L^{\frac{\delta}{3}(n-n_k)}L^{1-\alpha n_k/2} \geq L^{\frac{\delta}{3}(n-i)}L^{1-\alpha i/2}.
\]
Combining this with \( |(f^{i-n_k})'c_{n_k}| \leq (K_0L)^{-2\alpha^{-1}} \) we obtain
\[
|(f^{n+1-i})'c_i| = |(f^{n+1-n_k})'c_{n_k}||(f^{i-n_k})'c_{n_k}|^{-1}
\]
\[
\geq L^{\frac{\delta}{3}N}L^{1-\alpha i/2}(K_0L)^{-2\alpha^{-1}}
\]
\[
\geq L^{1-\alpha i/2}.
\]
Case (iv): \( i \in [n_k + 1, n_k + p_k] \) for some \( k \in [1, s] \). We may assume \( n_k + p_k < n + 1 \), for otherwise the case is covered by (iii). We have
\[
|\left( f^{n-n_k-p_k}\right)'c_{n_k+p_k+1}| \geq L^{1-\alpha(n_k+p_k+1)/2} \quad \text{by (ii)}
\geq L^{1-2\alpha n_k/3} \quad \text{by } p_k \leq \alpha n_k \text{ from } W_{n,c}.
\]

To estimate the remaining factor, we consider two cases separately as before. Suppose that \( \sigma < L^{-\alpha(i-n_k-1)} \), then \( i - n_k \leq 2\alpha^{-1} \).
\[
|\left( f^{n_k+p_k+1-i}\right)'c_i| = |\left( f^{i-n_k}\right)'c_i| \quad \text{by Lemma 2.2}
\geq L^\lambda p(K_0 L)^{-2\alpha^{-1}}
\geq 1 \quad \text{since } p_k \geq \alpha N.
\]
Combining this with the previous inequality implies the desired one.

Suppose that \( \sigma \geq L^{-\alpha(i-n_k-1)} \). Let \( \tilde{c} \) denote the critical point to which \( c_{n_k} \) is bound. Then
\[
|\left( f^{n_k+p_k+1-i}\right)'c_i| \geq K^{-1}|\left( f^{n_k+p_k+1-i}\right)'\tilde{c}_{i-n_k-1}| \quad \text{by Lemma 2.2}
\geq K^{-1}L^{1-\alpha(i-n_k-1)} \quad \text{by } X_{n,\tilde{c}}
\]
Combining these two inequalities,
\[
|\left( f^{n+1-i}\right)'c_i| \geq L^{2-\alpha+\alpha+\alpha\alpha n_k/3} \geq L^{1-\alpha i}.
\]
This completes the proof of Proposition 5.2. \(\square\)

6. Parameter exclusion: General steps

Set \( A(\infty) = \bigcap_{n \geq 0} A^{(n)} \). In this last section we prove

**Proposition 6.1.** For all large integer \( N \) there exists \( L_0 \) such that for all \( L \geq L_0 \),
\[
\left| A_L^{(\infty)} \right| \leq \left( 1 - L^{-\alpha N} \right) \left( 1 - \frac{3}{\sqrt{\sigma}} \right)^N.
\]

We prove this by combining the estimate of \( |A^{(N)}| \) in Proposition 3.4 with a new estimate which shows that the measure of \( A^{(n)} \setminus A^{(n+1)} \) relative to \( |A^{(N)}| \) decreases exponentially in \( n \).

6.2. Expansion in parameter space. We begin with a main technical estimate. Let \( c \in C, a \in A^{(n)} \) and suppose that \( c_0(a) \) makes a free return to \( C_\delta \) at time \( \nu \leq n + 1 \). We say \( \nu \) is an essential return time if
\[
\sum_{i+1 \leq j \leq \nu} \text{free return} 2 \log d(c_j(a), C) \leq \log d(c_i(a), C) \text{ for every free return } i \leq [0, \nu - 1].
\]
The sum ranges over all \( j \in [i + 1, \nu] \) at which \( c_0(a) \) makes a free return to \( C_\delta \).

**Lemma 6.3.** Let \( a \in A^{(n)} \) and \( c \in C \). For every essential return \( \nu \) in the first \( n + 1 \) iterates of \( c_0(a) \) we have
\[
D_\nu(a, c_0(a)) \cdot |(f_\nu)'c(a)| \geq \sqrt{d(c_\nu, C)}.
\]
In particular, for all \( b \in \Delta_\nu(a, c) - 5\sqrt{d(c_\nu(a), C)} \cdot \Delta_\nu(a, c) \) we have
\[
|c_\nu(a) - c_\nu(b)| \geq 4\sqrt{d(c_\nu(a), C)}.
\]
Proof. We finish the proof of the second assertion assuming the first one. The assumption on $b$ gives $b - a \geq \frac{1}{2} \cdot 5 \cdot \sqrt{d(c_\nu(a), C)} \cdot |\Delta_\nu(a, c)|$. Since $|\Delta_\nu(a, c)| \geq \frac{|\Delta_\nu(a, c)|}{|\Delta_\nu(a, c)|}$ and $\mid c_n(\Delta_\nu(a, c)) \mid \leq O(1)L^{2-\beta}$ by Sublemma 4.4 we have

$$b - a \geq O(1)L^{\beta-2} \cdot 5 \sqrt{d(c_\nu(a), C)} \cdot |\Delta_\nu(a, c)| \geq 4 \sqrt{d(c_\nu(a), C)} \mu(a, c_0(a)),$$

where the last inequality is because of $d(c_\nu(a), C) \leq \delta$. Hence we have

$$|c_\nu(a) - c_\nu(b)| \geq K^{-1} |c_\nu'(a)| |a - b| \quad \text{by Lemma 2.6}$$

$$\geq O(1)|f_\nu'(0)| |a - b| \quad \text{by Lemma 2.4}$$

$$\geq O(1)L^{-4}|f_\nu'(0)| |D_\nu(a, c_0(a))| D^2 d(c_\nu(a), C) \geq 4 \sqrt{d(c_\nu(a), C)},$$

by the first assertion.

Let $0 < n_1 < \cdots < n_t < \nu$ denote all the free returns in the first $\nu$ iterates of $c_0(a)$, with $p_1, \cdots, p_t$ the corresponding bound periods. Let

$$S_{n_k} = \sum_{i=n_k}^{n_{k+1}-1} d_i^{-1}(c_0) \text{ and } S_0 = \sum_{i=0}^\nu d_i^{-1}(c_0) - \sum_{k=1}^t S_{n_k}.$$  

Sublemma 6.4. We have $S_0 |(f^\nu)'c_0|^{-1} \leq \frac{1}{\sqrt{\delta}}$.

Proof. Let $i \notin \cup_{1 \leq k \leq t}[n_k, n_k + p_k]$. Suppose $c_i \notin C_{\sqrt{\delta}}$. We have $|f_\nu c_i| \geq K_0^{-1} L \sqrt{\delta}$. Split the itinerary from time to $i$ to $\nu$ into bound and free segments. Using Lemma 4.3 to each bound segment and Corollary 4.9 to each free segment, we have $|(f^\nu)'c_i| \geq L^{\lambda(\nu-i)/3}$. Hence

$$|(f^\nu)'c_i| d_i = |(f^\nu)'c_i| |f_\nu' c_i| \geq L^{\lambda(\nu-i)/3} K_0^{-1} \sqrt{\delta}.$$  

Suppose $c_i \notin C_{\sqrt{\delta}}$. Let $p$ denote the corresponding bound period. By $c_i \notin C_\delta$ and Lemma 4.7 no bound return follows $p$. Thus, we can estimate $|(f^\nu)'c_i|$ by split the itinerary of $c_0$ into free and bound states and argue in the same way as the previous case. Thus we have

$$|(f^\nu)'c_i| |f_\nu' c_i| \geq (L^{\lambda/3})^{\nu-i-p-1} |(f^{p+1})'c_i| |f_\nu' c_i|.$$  

Lemma 4.3 gives

$$|(f^{p+1})'c_i| |f_\nu' c_i| = 10 \sqrt{|(f^{p+1})'c_i|} \times 10 \sqrt{|(f^{p+1})'c_i|} \geq 4 \sqrt{\delta} \cdot L^{\lambda(p+1)/10}.$$  

These inequalities altogether yield $|(f^\nu)'c_i| d_i \geq 4 \sqrt{\delta} L^{\lambda(\nu-i)/10}.$

Combining the above two estimates give

$$S_0 |(f^\nu)'c_0|^{-1} \leq \frac{1}{4 \sqrt{\delta}} \sum_{i=0}^{\nu-1} L^{-\frac{\nu-i}{10}} + \frac{1}{\sqrt{\delta}} \sum_{i=0}^{\nu-1} L^{-\lambda(\nu-i)/3} \leq \frac{1}{\sqrt{\delta}}.$$  

Sublemma 6.5. For every $1 \leq k \leq t$ we have $S_{n_k} |(f^{n_k+p_k+1})'c_0|^{-1} \leq |d(c_{n_k}, C)|^{-\frac{2\nu}{5}}.$
Proof. Let $\tilde{c}$ denote the critical point to which $c_n$ is bound. By Lemma 2.2 for $n_k + 1 \leq i \leq n_k + p_k$ we have

$$
|(f^{n_k+p_k+1}c_0)d_0| \geq K^{-4} |(f^{n_k+p_k}c_0)\cdots(f^{i+1}c_0)| - (f^{i}c_0) |(f^{i-n_k}c_0)\cdots(14)|

Using $(X)_{n,\tilde{c}}$ to estimate the two fractions we have

$$
|f^{n_k+p_k+1}c_0|d_0| \geq K^{-4} L^2 \left( \min \{ \sigma, L^{-\alpha(n-k)} \} \right)^2.
$$

Proposition 4.3 gives $|f^{n_k+p_k+1}c_0|d_0| \geq |d(c_n, C)|^{\tilde{c}}$, and therefore

$$
S_n |(f^{n_k+p_k+1}c_0)|^{-1} \leq |d(c_n, C)|^{\tilde{c}} + K^4 L^{-2} \sum_{i=n_k+1}^{n_k+p_k} \left( \min \{ \sigma, L^{-\alpha(n-k)} \} \right)^{-2}
$$

$$
\leq |d(c_n, C)|^{\tilde{c}} - \sigma^{-1} \alpha^{-1} \log \sigma + \sum_{i=0}^{p_k} L^\alpha.
$$

For the last inequality, we have used the upper estimate of $p_k$ in Lemma 4.3 to estimate the last term.

Write $\nu = n_{t+1}$. Since $n_{t+1}$ is an essential return we have

$$
|d(c_n, C)|^{-1} \leq \prod_{k+1 \leq j \leq t+1} |d(c_n, C)|^{-2} \quad \text{for every } 1 \leq k \leq t.
$$

Substituting this into the inequality in Sublemma 6.5 gives

$$
S_n |(f^{n_k+p_k+1}c_0)|^{-1} \leq \prod_{k+1 \leq j \leq t+1} |d(c_n, C)|^{\tilde{c}}.
$$

Meanwhile, for $1 \leq k \leq t - 1$ we have

$$
|(f^{n_k+p_k+1}c_0)| \geq \prod_{1 \leq j \leq t} |(f^{n_j+p_j-1}c_0)| \cdot \prod_{k+1 \leq u \leq t} |(f^{p_j}c_0)|
$$

Multiplying $(14)$ with the above inequality gives

$$
S_n |(f^{n_k+p_k+1}c_0)|^{-1} \leq |d(c_n, C)|^{\tilde{c}} \prod_{k+1 \leq j \leq t} \frac{1}{|d(c_n, C)|^{\tilde{c}} |(f^{p_j}c_0)|}
$$

by Lemma 4.3

Summing this over all $1 \leq k \leq t - 1$ and $(14)$ for $k = t$ gives

$$
\sum_{1 \leq k \leq t} S_n |(f^{n_k+p_k+1}c_0)|^{-1} \leq |d(c_n, C)|^{\tilde{c}} \left( 1 + \sum_{k=1}^{t-1} \delta^{(t-k)/2} \right) \leq 2 |d(c_n, C)|^{\tilde{c}}.
$$

$$
Combining this with Sublemma 6.3 we obtain
\[ |(f^\nu)'c_0|^{-1} D_{\nu}^{-1} = L^\beta \left( \sum_{k=1}^{I} S_{nk} |(f^\nu)'c_0|^{-1} + S_0 |(f^\nu)'c_0|^{-1} \right) \leq \frac{1}{\sqrt{d(c_\nu, C)}}. \]

This completes the proof of Lemma 6.3. □

6.6. **Strategy.** The estimate of the measure of \( A^{(n)} \setminus A^{(n+1)} \) consists of three steps. We first decompose the set into a finite number of subsets which are characterized by combinatorial data describing the recurrence to the critical points. Then we estimate the measure of each of the subsets. Finally we put these estimates together, counting the total number of combinations.

6.7. **Decomposition of the parameter set.** For \( c \in C, \, q \geq 1 \) and \( n \geq N \), let
\[ B_q(c) = \{ a \in A^{(n)} \setminus A^{(n+1)} : c_0(a) \text{ makes exactly } q \text{ e.f.r.s in the first } n \text{ iterates} \}. \]

We have \( A^{(n)} \setminus A^{(n+1)} \leq \bigcup_{c \in C} B_q(c) \). We further decompose the right hand side as follows. For a \( q \)-tuple of pairs of positive integers \( X = ((\nu_1, \tau_1), \ldots, (\nu_q, \tau_q)) \), let \( B_X(c) \) denote the set of all \( a \in B_q(c) \) such that \( c_0(a) \) makes essential returns exactly at \( 0 < \nu_1 < \ldots < \nu_q = n \), with \( |c_{\nu_i}(a) - c^{\nu_i}| \leq \delta \) for every \( 1 \leq i \leq q \). In other words, \( c_{\nu_i}(a) \) is bound to \( c^{\nu_i} \). We write \( B_X(c) = \bigcup \{ B_R^c(c) \}, \) where
\[ B_R^c(c) = \left\{ x \in B_X(c) : \sum_{i=1}^{q} \lfloor -\log d(c_{\nu_i}(a), C) \rfloor = R \right\}, \]

where the square bracket denotes the integer part. For each partition \( Y = (r_1, \ldots, r_q) \) of \( R \) into \( q \) positive integers, define
\[ B_{X,Y}^c(c) = \{ a \in B_R^c(c) : \lfloor -\log d(c_{\nu_i}(a), C) \rfloor = r_i \text{ for every } 1 \leq i \leq q \}. \]

This set is a finite union of intervals. We clearly have \( B_q(c) = \bigcup_X \bigcup_R \bigcup_Y B_{X,Y}^c(c) \).

6.8. **Hierarchical covering.** Let \( \hat{r}_i = \exp \left( -\frac{r_i+1}{5} \right) \). In order to estimate the measure of \( B_{X,Y}^c(c) \), we construct a hierarchical covering of it, which consists of for each \( i = 1, 2, \ldots, q \) a finite collection of pairwise disjoint intervals \( \{ I_j^{(i)} \} \) intersecting \([0, 1]\) such that:

(i) \( B_{X,Y}^c(c) \subset \bigcup_j \hat{r}_i \cdot I_j^{(i)} \);

(ii) for any \( j \) there exists \( k \) such that \( I_j^{(i)} \subset \hat{r}_{i-1} \cdot I_k^{(i-1)} \).

Then we clearly have \( \sum_j |I_j^{(i)}| \leq \hat{r}_{i-1} \sum_j |I_j^{(i-1)}| \) for each \( i = q, q-1, \ldots, 2 \), and thus \( \sum_j |I_j^{(q)}| \leq \hat{r}_1 \hat{r}_2 \cdots \hat{r}_{q-1} \sum_j |I_j^{(1)}| \). (i) gives \( |B_{X,Y}^c(c)| \leq \hat{r}_q \sum_j |I_j^{(q)}| \), and thus \( |B_{X,Y}^c(c)| \leq \hat{r}_1 \hat{r}_2 \cdots \hat{r}_q \sum_j |I_j^{(1)}| \). Since the intervals \( \{ I_j^{(1)} \} \) are pairwise disjoint, intersect \([0, 1]\), and \( \leq L^{-\beta} \) in length, we obtain
\[ |B_{X,Y}^c(c)| \leq \exp \left( -\frac{R + q}{5} \right) \cdot \Delta_n(a, c, r). \]

For the construction of the hierarchical covering we need a couple of lemmas. Let
\[ \Delta_n(a, c, r) = \exp \left( -\frac{r}{5} \right) \cdot \Delta_n(a, c). \]
Lemma 6.9. Let \( a, b \in B^R_{X,\gamma}(c) \) and suppose \( \nu_i < \nu_j \). If \( \Delta_{\nu_i}(a, c, r_i) \cap \Delta_{\nu_j}(b, c, d) \neq \emptyset \) holds for some \( d \geq -\log \delta \), then \( \Delta_{\nu_j}(b, c) \subset \Delta_{\nu_i}(a, c, r_i - 1) \).

Proof. By Lemma 6.3 there exists \( \hat{a} \subset \Delta_{\nu_i}(a, c, r_i) \) such that \( c_{\nu_i}(\hat{a}) \in C \). Proposition 2.6 and \( \nu_i < \nu_j \) implies \( \hat{a} \notin \Delta_{\nu_j}(b, c) \), for otherwise \( \text{Dist}(\gamma_{\nu_j}, \Delta(b, \nu_j, 0)) \) is unbounded. This implies that one of the connected components of \( \Delta_{\nu_j}(b, c) - \Delta_{\nu_j}(b, c, d) \) is contained in \( \Delta_{\nu_i}(a, c, r_i) \). This implies

\[
2^{-1}(1 - e^{-d/\delta})|\Delta_{\nu_j}(b, c)| \leq |\Delta_{\nu_i}(a, c, r_i)|,
\]

and hence the inclusion holds. \( \square \)

Lemma 6.10. Let \( a, b \in B^R_{X,\gamma}(c) \). Assume that \( \Delta_{\nu_i}(a, c) \cap \Delta_{\nu_j}(b, c) \neq \emptyset \) and \( b \notin \Delta_{\nu_i}(a, c) \). Then we have \( \Delta_{\nu_i}(a, c) \subset \Delta_{\nu_j}(b, c) \).

Proof. Analogous to the proof of Lemma 3.5. \( \square \)

We are in position to construct a hierarchical covering of \( B^R_{X,\gamma}(c) \). First of all we claim that for all \( a \in B^R_{X,\gamma}(c) \) and each \( i = 1, 2, \ldots, q-1 \) there exists a finite sequence \( a_1, a_2, \ldots \) in \( B^R_{X,\gamma}(c) \cap \Delta_{\nu_i}(a, c) \) such that the corresponding intervals \( \{\Delta_{\nu_i+1}(a_j, c)\} \) are (a) pairwise disjoint, (b) contained in \( \Delta_{\nu_i}(a, r_i - 1) \), and (c) altogether cover \( B^R_{X,\gamma}(c) \cap \Delta_{\nu_i}(a, c) \).

This claim is proved as follows. We define the finite sequence in the statement inductively as follows. Choose some \( a_1 \in B^R_{X,\gamma}(c) \cap \Delta_{\nu_i}(a, c) \). Given \( a_1, \ldots, a_k \), we are done if \( B^R_{X,\gamma}(c) \cap \Delta_{\nu_i}(a, c) \subset \bigcup_{j=1}^k \Delta_{\nu_i+1}(a_j, c) \). Otherwise we choose some \( a_{k+1} \in B^R_{X,\gamma}(c) \cap \Delta_{\nu_i}(a, c) \cap \bigcup_{j=1}^k \Delta_{\nu_i+1}(a_j, c) \). Since the length of the intervals \( \{\Delta_{\nu_i+1}(a_j, c)\} \) are bounded from below, this definition makes sense and the resultant finite number of intervals altogether cover \( B^R_{X,\gamma}(c) \cap \Delta_{\nu_i}(a, c) \). By Lemma 6.10 any two of them are either disjoint or nested. This proves (a) (c).

By Lemma 6.3 the set \( \Delta_{\nu_i}(a, c) - \Delta_{\nu_i}(a, c, r_i) \) does not intersect \( B^R_{X,\gamma}(c) \). Since \( a_j \in B^R_{X,\gamma}(c) \) and \( a_j \in \Delta_{\nu_i}(a, c) \) we have \( a_j \in \Delta_{\nu_i}(a, c, r_i) \), which gives \( \Delta_{\nu_i+1}(a_j, c, r_i+1) \cap \Delta_{\nu_i}(a, c, r_i) \neq \emptyset \). This and Lemma 6.9 yield (b).

We are in position to define the intervals \( \{I^{(i)}_j\} \) for each \( i = 1, 2, \ldots, q \). Lemma 6.10 implies the existence of a finite sequence \( a^{(1)}_1, a^{(1)}_2, \ldots \) in \( B^R_{X,\gamma}(c) \) such that the corresponding intervals \( \Delta_{\nu_i}(a^{(1)}_1, c), \Delta_{\nu_i}(a^{(1)}_2, c), \ldots \) are pairwise disjoint and altogether cover \( B^R_{X,\gamma}(c) \). We set \( I^{(1)}_j = \Delta_{\nu_i}(a^{(1)}_j, c) \). Lemma 6.3 gives \( B^R_{X,\gamma}(c) \subset \bigcup_j \hat{r}_j \cdot I^{(1)}_j \).

Let \( i \in [1, q - 1] \). Having defined \( I^{(i)}_k = \Delta_{\nu_i}(a^{(i)}_k, c) \) for \( k = 1, 2, \ldots \), we define the intervals \( \{I^{(i+1)}_j\} \) which are contained in \( \hat{r}_i \cdot I^{(i)}_k \) as follows. According to the above claim, there exists a finite set \( a^{(i+1)}_1, a^{(i+1)}_2, \ldots \) in \( B^R_{X,\gamma}(c) \cap I^{(i)}_k \) such that any two of the corresponding intervals \( \{\Delta_{\nu_i+1}(a^{(i+1)}_j, c)\} \) are pairwise disjoint, contained in \( \hat{r}_i \cdot I^{(i)}_k \), and altogether cover \( B^R_{X,\gamma}(c) \cap I^{(i)}_k \). We set \( \Delta_{\nu_i+1}(a^{(i+1)}_j, c) = I^{(i+1)}_j \). By construction we have \( B^R_{X,\gamma}(c) \subset \bigcup_j \hat{r}_i \cdot I^{(i+1)}_j \), where the union runs over all intervals \( I^{(i+1)}_j \) defined in this way. Lemma 6.3 gives \( B^R_{X,\gamma}(c) \subset \bigcup_j \hat{r}_i+1 \cdot I^{(i+1)}_j \). This completes the construction of the hierarchical covering. \( \square \)
6.11. Conclusion. We are in position to finish the estimate of the measure of $A^{(\infty)}$. We begin by counting all feasible $R$, $Y$, $X$, $q$.

The next lemma asserts that the sum of essential return depths has a definite proportion, and as a result gives a lower bound for $R$.

**Lemma 6.12.** $\sum_{i=1}^{q} r_i \geq \alpha n \log L/2$.

*Proof.* We call a free return *inessential* if it is not an essential return. Let $\mu \in (0, n)$ be an inessential return. Let $i(\mu)$ denote the smallest $k \leq \mu - 1$ such that

$$2 \log d(c_j, C) > \log d(c_k, C).$$

No essential return occurs during the period $[i(\mu) + 1, \mu]$. We claim that there exists a set $F \subset (0, n)$ of inessential returns such that the intervals $[i(\mu) + 1, \mu]$ for $\mu \in F$ are mutually disjoint and cover all the inessential return times in $(0, n)$. Indeed, it suffices to define $F = \{\mu_1 > \mu_2 > \cdots > \mu_s\}$ as follows: let $\mu_1$ denote the largest inessential return in $(0, n)$. Given $\mu_k \in F$, let $\mu_{k+1}$ denote the largest inessential one which is $< i(\mu_k) + 1$.

Summing (16) over all $\mu \in F$ gives

$$\sum_{i=1}^{q} r_i = \sum_{0 \leq j \leq \mu \text{ inessential}} - \log d(c_j, C) \geq \frac{1}{2} \sum_{0 \leq i \leq n \text{ free}} \log d(c_i, C) \geq \frac{\alpha n \log L}{2}.$$  

Therefore

$$\sum_{i=1}^{q} r_i = \sum_{0 \leq j \leq \mu \text{ essential}} - \log d(c_j, C) \geq -\frac{1}{2} \sum_{0 \leq i \leq n \text{ free}} \log d(c_i, C) \geq \frac{\alpha n \log L}{2}.$$  

□

Let $p_i$ denote the bound period for the orbit of $c$ at the free return $\nu_i$. Since $c$ is free at time $n$, we have

$$\eta(q - 1) \leq \sum_{i=1}^{q-1} p_i \leq n.$$  

The upper estimate of the bound period in (a) Lemma 4.3 gives

$$\eta q \leq \sum_{i=1}^{q} p_i \leq \frac{2R}{\lambda \log L}.$$  

Since all the free returns under consideration are to the inside of $C_\delta$, the lower estimate in (a) of Lemma 4.3 gives $\eta \geq \alpha N$. Hence, for sufficiently large $N$ we have

$$q/n, q/R \leq \frac{2}{\alpha N}.$$  

By Stirling’s formula for factorials, the number of all feasible $Y = (r_1, \cdots, r_q)$ is bounded by the total number of combinations of dividing $R$ objects into $q$ groups, which is $\binom{R+q}{q} \leq e^{c(N)R}$, where $c(N) \to 0$ as $N \to \infty$. Similarly, the number of all feasible $X = (\nu_1, \tau_1, \cdots, \nu_q, \tau_q)$ is $\leq e^{c'(N)n} \cdot (\xi C)^q \leq e^{c''(N)n}$, where $c''(N) \to 0$ as $N \to \infty$.  

For \( n \geq N \) we have
\[ |A^{(n)} \setminus A^{(n+1)}| \leq \sum_{c \in C} \sum_{q} \sum_{R \geq \alpha n \log L/2} \sum_{X, Y} |B_{X, Y}^{R}(c)| \]
\[ \leq \sum_{c \in C} \sum_{q} \sum_{R \geq \alpha n \log L/2} \exp \left( c(N)R + c''(N)n - \frac{R + q}{5} \right) \left( 1 + 2L^{-\beta} \right) \]
\[ \leq \sum_{R \geq \alpha n \log L/2} \exp \left( \frac{c(N)R + c''(N)n - R}{6} \right) \left( 1 + 2L^{-\beta} \right) \]
\[ \leq \sum_{R \geq \alpha n \log L/2} \exp \left( -\frac{R}{7} \right), \]
where \( c''(N) \to 0 \) as \( N \to \infty \). Substituting the estimate of \( |A^{(N)}| \) in Proposition 3.4 gives
\[ \frac{|A^{(n)} \setminus A^{(n+1)}|}{|A^{(N)}|} \leq \left( 1 - 3 \sqrt{\sigma} \right)^{-N-1} \sum_{R} \exp \left( -\frac{R}{7} \right) \leq L^{-\alpha n/18}. \]

Hence we obtain
\[ |A^{(\infty)}| = |A^{(N)}| - \sum_{n \geq N} |A^{(n)} \setminus A^{(n+1)}| \]
\[ \geq |A^{(N)}| \left( 1 - \sum_{n \geq N} L^{-\frac{2n}{3}} \right) \]
\[ \geq (1 - 3 \sqrt{\sigma})^{N+1} \left( 1 - \sum_{n \geq N} L^{-\frac{2n}{3}} \right). \]

This completes the proof of Proposition 6.1 and hence that of the main theorem. \( \square \)

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Department of Mathematics, Kyoto University, Kyoto 606-8502 Japan
E-mail address: takahasi@math.kyoto-u.ac.jp