THE WILLMORE ENERGY AND THE MAGNITUDE OF EUCLIDEAN DOMAINS

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Abstract. We study the geometric significance of Leinster’s notion of magnitude for a compact metric space. For a smooth, compact domain $X$ in an odd-dimensional Euclidean space, we show that the asymptotic expansion of the function $M_X(R) = \text{Mag}(R \cdot X)$ at $R = \infty$ determines the Willmore energy of the boundary $\partial X$. This disproves the Leinster-Willerton conjecture for a compact convex body in odd dimensions.

Introduction

The notion of magnitude was introduced by Leinster [8, 9] as an extension of the Euler characteristic to (finite) enriched categories. Magnitude has been shown to unify notions of “size” like the cardinality of a set, the length of an interval or the Euler characteristic of a triangulated manifold, and it even relates to measures of the diversity of a biological system. See [10] for an overview.

Viewing a metric space as a category enriched over $[0, \infty)$, Leinster and Willerton proposed and studied the magnitude of metric spaces [9, 11]: If $(X, d)$ is a finite metric space, a weight function is a function $w : X \to \mathbb{R}$ which satisfies $\sum_{y \in X} e^{-d(x,y)} w(y) = 1$ for all $x \in X$. Given a weight function $w$, we define the magnitude of $X$ as $\text{Mag}(X) := \sum_{x \in X} w(x)$; this definition is independent of the choice of weight function. Beyond finite metric spaces, the magnitude of a compact, positive definite metric space $(X, d)$ was made rigorous by Meckes [12]:

$$\text{Mag}(X) := \sup \{ \text{Mag}(\Xi) : \Xi \subset X \text{ finite} \} .$$

Instead of the magnitude of an individual space $(X, d)$, it proves fruitful to study the magnitude function $M_X(R) := \text{Mag}(X, R \cdot d)$ for $R > 0$.

Compact convex subsets $X \subset \mathbb{R}^n$ provide a key example, surveyed in [10]. Motivated by properties of the Euler characteristic and computer calculations, Leinster and Willerton [11] conjectured a surprising relation to the intrinsic volumes $V_i(X)$, which would shed light on the geometric content of the magnitude function:

$$M_X(R) = \sum_{k=0}^{n} \frac{1}{k! \omega_k} V_k(X) R^k + o(1), \quad \text{as } R \to \infty .$$

Here, $\omega_k$ is the volume of the $k$-dimensional unit ball. This asymptotic expansion resembles the well-known expansion of the heat trace, with leading terms $V_n(X) = \text{vol}_n(X)$, $V_{n-1}(X) = \text{vol}_{n-1}(\partial X)$ [4]. The expansion coefficients for the heat trace, however, are not proportional to $V_k(X)$ for $k \leq n - 2$. 
The conjectured behavior (1) was disproved by Barceló and Carbery [1] for the unit ball $B_5 \subset \mathbb{R}^5$. They explicitly computed the rational function $\mathcal{M}_{B_5}$ and observed numerical disagreement of the coefficients of $R^k$. Their results were extended to balls in odd dimensions in [14].

In spite of this negative result, the authors were able to prove a variant of (1), with modified prefactors, which confirmed the close relation between magnitude and intrinsic volumes [2]: When $n = 2m - 1$ is odd and $X \subseteq \mathbb{R}^n$ is a compact domain with smooth boundary, there are coefficients $(c_j(X))_{j \in \mathbb{N}}$ such that

\[
\mathcal{M}_X(R) = \sum_{j=0}^{\infty} \frac{c_j(X)}{n! \omega_n} R^{n-j} + O(R^{-\infty}), \quad \text{as } R \to \infty,
\]

where

\[
c_0(X) = \text{vol}_n(X), \quad c_1(X) = m \text{vol}_{n-1}(\partial X), \quad c_2(X) = \frac{m^2}{2} (n-1) \int_{\partial X} H \, dS.
\]

Here, $H$ denotes the mean curvature of $\partial X$. Each coefficient $c_j$ is an integral over $\partial X$ computable from the second fundamental form of $\partial X$ and its covariant derivatives. For $j = 0, 1, 2$ and $X$ convex, the coefficient $c_j$ is proportional to the intrinsic volume $V_{n-j}(X)$, for $j = 0, 1, 2$. This proves that the Leinster-Willerton conjecture holds for modified universal coefficients up to $O(R^{n-3})$.

The following variant of the Leinster-Willerton conjecture therefore remained plausible. It would confirm the relation between magnitude and intrinsic volumes and, in particular, show that $c_n$ is proportional to the Euler characteristic $V_0$:

**Conjecture 1.** For $n > 0$, there are universal constants $\gamma_{0,n}, \gamma_{1,n}, \ldots, \gamma_{n,n}$ such that for any compact convex subset $X \subseteq \mathbb{R}^n$, $\mathcal{M}_X(R) = \sum_{k=0}^{n} \gamma_{k,n} V_k(X) R^k + o(1)$, as $R \to \infty$.

In this paper we prove that Conjecture 1 fails in all odd dimensions $n \geq 3$ and find unexpected geometric content in $c_3$. While the conjecture holds true for the terms of order $R^n$, $R^{n-1}$ and $R^{n-2}$, the $R^{n-3}$-term is not proportional to an intrinsic volume:

**Theorem 2.** Assume that $n \geq 3$ is odd and that $X \subseteq \mathbb{R}^n$ is a compact domain with smooth boundary. Then there is a dimensional constant $\lambda_n \neq 0$ such that

\[
c_3(X) = \lambda_n W(\partial X),
\]

where $W(\partial X) := \int_{\partial X} H^2 \, dS$ is the Willmore energy of the boundary of the hypersurface $\partial X$.

Building on [2], the proof reformulates the magnitude function in terms of an elliptic boundary value problem of order $n + 1$ in $\mathbb{R}^n \setminus X$, which is then studied using methods from semiclassical analysis. See Proposition 4 and Equation (5) below.

To see that Theorem 2 disproves Conjecture 1 in the fourth term, we observe that the Willmore energy is not an intrinsic volume: The only intrinsic volume with the same scaling property as the Willmore energy is $V_{n-3}$. For instance, if $n = 3$ then $V_{n-3}$ is the Euler characteristic while $\int_{\partial X} H^2 \, dS$ can be non-zero even when $\partial X$ has vanishing Euler characteristic (e.g. for a torus). In general dimension, for
a > 0 the solid ellipsoid
\[ X_a := \left\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |x'|^2 + \frac{|x_n|^2}{a^2} \leq 1 \right\}, \]
satisfies that \( W(\partial X_a) \to \infty \) as \( a \to 0 \). On the other hand, Hausdorff continuity of intrinsic volumes shows that \( V_{n-3}(X_a) \) converges to a finite number, namely the \( n - 3 \)-rd intrinsic volume of the \( n - 1 \)-dimensional unit ball. Therefore Theorem 2 implies the following.

**Corollary 3.** Assume that \( n \geq 3 \) is odd and that \( X \subseteq \mathbb{R}^n \) is a compact convex domain with smooth boundary. There are universal constants \( \gamma_{n-2,n}, \gamma_{n-1,n}, \gamma_{n,n} \) such that
\[ \mathcal{M}_X(R) = \sum_{k=-n}^{n} \gamma_{k,n} V_k(X) R^k + O(R^{n-3}), \quad \text{as } R \to \infty. \]
However, there is no constant \( \gamma_{n-3,n} \) such that \( \mathcal{M}_X(R) = \sum_{k=-n}^{n} \gamma_{k,n} V_k(X) R^k + O(R^{n-4}) \) as \( R \to \infty \). In particular, the Leinster-Willerton conjecture fails even with modified universal coefficients.

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**Background and notation**

We assume that \( X \subseteq \mathbb{R}^n \) is a compact domain with \( C^\infty \)-boundary, where \( n = 2m - 1 \) odd. Denote by \( \Omega := \mathbb{R}^n \setminus X \) the exterior domain. We use the Sobolev spaces \( H^s(\mathbb{R}^n) := (1 - \Delta)^{-s/2} L^2(\mathbb{R}^n) \) of exponent \( s \geq 0 \). Here, the Laplacian \( \Delta \) is given by \( \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \). The spaces \( H^s(X) \) and \( H^s(\Omega) \) are defined using restrictions. The Sobolev spaces \( H^s(\partial X) \) can be defined using local charts or as \( (1 - \Delta_{\partial X})^{-s/2} L^2(\partial X) \).

We use \( \partial_\nu \) to denote the Neumann trace of a function \( u \) in \( \Omega \). The operator \( \partial_\nu \) extends to a continuous operator \( H^s(\Omega) \to H^{s-3/2}(\partial X) \) for \( s > 3/2 \). Similarly, \( \gamma_0 : H^s(\Omega) \to H^{s-1/2}(\partial X) \) denotes the trace operator defined for \( s > 1/2 \).

For \( R > 0 \) we shall need the operators
\[ D_R^j := \begin{cases} \partial_\nu \circ (R^2 - \Delta)^{(j-1)/2} & \text{when } j \text{ is odd}, \\ \gamma_0 \circ (R^2 - \Delta)^{j/2} & \text{when } j \text{ is even}. \end{cases} \]
By the trace theorem, \( D_R^j \) is continuous as an operator \( D_R^j : H^s(\Omega) \to H^{s-j-1/2}(\partial X) \) for \( s > j + 1/2 \).

We recall a key observation from [1], in the reformulation presented in [2]:

**Proposition 4.** [2, Proposition 9] Suppose that \( h_R \in H^{2m}(\Omega) \) is the unique weak solution to the boundary value problem
\[ \begin{cases} (R^2 - \Delta)^m h_R = 0 & \text{in } \Omega \\ D_R^j h_R &= \begin{cases} R^j, & j \text{ even}, \\ 0, & j \text{ odd}. \end{cases}, \quad j = 0, \ldots, m - 1. \]
Then the following identity holds
\[ \mathcal{M}_X(R) = \frac{\text{vol}_n(X)}{n!} R^n - \frac{1}{n!} \sum_{m \leq j \leq m} \int_{\partial X} D^2_{j} \frac{1}{n!} R^n \omega_{R} dS. \]

The operators \( D^j_R \) define a matrix-valued Dirichlet-Neumann operator \( \Lambda(R) : \mathcal{H}_+ \to \mathcal{H}_- \) in the Hilbert space
\[
\mathcal{H} := \bigoplus_{j=0}^{m-1} H^{2m-j-1/2}(\partial X) \oplus \bigoplus_{j=m}^{n} H^{2m-j-1/2}(\partial X)
\]
as follows: \( \Lambda(R)(u_j)_{j=0}^{m-1} := (D^j_R u)_{j=m}^{n} \), where \( u \in H^{2m}(\Omega) \) is the unique weak solution to
\[
\begin{cases}
(R^2 - \Delta)^m u = 0 & \text{in } \Omega \\
D^j_R u = u_j & \text{for } j = 0, \ldots, m-1.
\end{cases}
\]
The operator \( \Lambda(R) \) is a parameter-dependent pseudodifferential operator on \( \partial X \). The parameter \( R \) enters like an additional co-variable, which allows us to compute the asymptotics of \( \mathcal{M}_X \) from Proposition 4. For the convenience of the reader we recall the salient features of the parameter-dependent pseudodifferential calculus, see for instance \([5, 6, 13]\) for further details. We restrict to parameters \( R \in \mathbb{R}_+ = (0, \infty) \).

**Definition 5.** A parameter-dependent pseudodifferential operator \( A \) of order \( s \) on \( \mathbb{R}^n \) is an operator on the Schwartz space of the form
\[
Af(x) := (2\pi)^{-n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} a(x, \xi, R) e^{i(x-y)\xi} f(y) dy d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n),
\]
where the full symbol \( a \) admits a polyhomogeneous expansion of order \( s \) in \( (\xi, R) \). That is, for \( k \in \mathbb{N} \) there are functions \( a_{s-k} \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+) \) with
\[
a_{s-k}(x, t\xi, tR) = t^{s-k} a_{s-k}(x, \xi, R), \quad \text{for } t \geq 1, \| (\xi, R) \| \geq 1,
\]
and \( a \) can be written as an asymptotic sum
\[
a \sim \sum_{k=0}^{\infty} a_{s-k}.
\]
We call \( a_s \) the principal symbol of \( A \). If \( a_s(x, \xi, R) \) is invertible for every \( (x, \xi, R) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \), we say that \( A \) is elliptic with parameter.

Definition 5 on \( \mathbb{R}^n \) extends by standard techniques, using coordinate charts, to define a pseudodifferential operator and its full symbol on a compact manifold, see for instance \([2, 5, 6, 13]\). The use of the parameter-dependent calculus is crucial to the work \([2]\) and the computations in this paper, including formulas for the symbol of a product of two pseudodifferential operators and the parametrix construction. In particular, if \( A \) is elliptic with parameter of order \( s \) on a compact manifold, it has a parametrix with parameter \( B \) of order \( -s \). The full symbol expansion
The computation proving Equation (4) follows from [13, Section 5.5].

For $R \to \infty$ the parameter-dependent calculus further allows to compute expectation values of the form $\int_M A(1) \, dx$ in terms of the symbol:

**Lemma 6.** Suppose that $A : C^\infty(M) \to C^\infty(M)$ is a parameter-dependent pseudodifferential operator of order $s$ acting on a compact manifold $M$ equipped with a volume density. Then there is an asymptotic expansion

$$
\int_M A(1) \, dx = \sum_{k=0}^\infty a_k R^{s-k} + O(R^{-\infty}),
$$

where the coefficients $a_k$ are computed as follows: Expand the full symbol of $A$ into terms homogeneous in $(\xi, R)$ as $\sigma_A(x, \xi, R) \sim \sum_{k=0}^\infty \sigma_{s-k}(A)(x, \xi, R)$ and set

$$
a_k := \int_M \sigma_{s-k}(A)(x, 0, 1) \, dx.
$$

For the proof of Lemma 6 we refer the reader to [2, Lemma 20] or [3, Lemma 2.24], but let us outline the main idea. The claimed asymptotics of Lemma 6 is coordinate invariant because $\int_M A(1) \, dx$ is coordinate invariant. It therefore suffices to compute the asymptotics for an operator $A$ on $\mathbb{R}^n$ as in Equation (3), assuming $a$ is compactly supported in the $x$-variable. In this case, $A(1) = a(x, 0, R)$, so that for $R \geq 1$

$$
\int_{\mathbb{R}^n} A(1) \, dx = \int_{\mathbb{R}^n} a(x, 0, R) \, dx = \sum_{k=0}^\infty \int_M \sigma_{s-k}(A)(x, 0, R) \, dx + O(R^{-\infty}) =
$$

$$
= \sum_{k=0}^\infty \int_M \sigma_{s-k}(A)(x, 0, 1) \, dx R^{s-k} + O(R^{-\infty}) = \sum_{k=0}^\infty a_k R^{s-k} + O(R^{-\infty}).
$$

The reader should note that the integrands $a_{s-k}(x, 0, R) = a_{s-k}(x, 0, 1) R^{m-k}$ are well defined because each $a_{s-k}$ is homogeneous in $(\xi, R)$, and not only in $\xi$.

From Proposition 4 and Lemma 6 we deduce a formula for the expansion coefficients $c_k$:

$$
c_k(X) := - \sum_{\frac{n-1}{2} \leq j \leq m} \sum_{0 \leq l \leq j} \int_{\partial X} (2j-2l-k)(\Lambda_{j-l-1,2l})(x, 0, 1) \, dS,
$$

for $k > 0$ where $\Lambda = (\Lambda_{j+m,l})_{j,l=0}^{m-1}$ and $\sigma_{2j-2l-k}(\Lambda_{2j-l-1,2l})$ the homogeneous part of order $2j - 2l - k$ in its symbol (with parameter). See [2, Proposition 20].

The full symbol of the parameter-dependent operator $\Lambda$ can be computed by adapting standard techniques in semiclassical analysis [6]. The operator $\Lambda$ is first computed using boundary layer potentials. To define these, we consider the function

$$
K(R; z) := \frac{\kappa_n}{R} e^{-R|z|}, \quad z \in \mathbb{R}^n.
$$

The constant $\kappa_n > 0$ is chosen such that

$$
(R^2 - \Delta)^m K = \delta_0
$$
in the sense of distributions on \( \mathbb{R}^n \). For \( l = 0, \ldots, n \), we define the functions

\[
K_l(R; x, y) := (-1)^l D_{R,y}^{n-l} K(R; x - y), \quad x \in \mathbb{R}^n, \ y \in \partial X.
\]

Here \( D_{R,y} \) denotes \( D'_R \) acting in the \( y \)-variable. We also consider the distributions

\[
K_{j,k}(R; x, y) := D_{R,y}^j K_k(R; x, y), \quad x \in \partial X.
\]

Each \( K_{j,k} \) defines a parameter-dependent pseudodifferential operator \( A_{j,k}(R) : C^\infty(\partial X) \to C^\infty(\partial X) \),

\[
A_{j,k}(R)f(x) := \int_{\partial X} K_{j,k}(R; x, y)f(y)dS(y), \quad x \in \partial X.
\]

The integral defining \( A_{j,k}(R) \) is understood in the sense of an exterior limit. These operators combine into a \( 2m \times 2m \)-matrix of operators \( \mathcal{A} := (A_{j,l})_{j,l=0}^m : \mathcal{H} \to \mathcal{H} \). It decomposes into matrix blocks

\[
\mathcal{A} = \begin{pmatrix}
\mathcal{A}_{++} & \mathcal{A}_{+-} \\
\mathcal{A}_{-+} & \mathcal{A}_{--}
\end{pmatrix} : \mathcal{H}_+ \oplus \mathcal{H}_- \to \mathcal{H}_+ \oplus \mathcal{H}_-,
\]

with \( \mathcal{A}_{pq} : \mathcal{H}_q \to \mathcal{H}_p \) for \( p, q \in \{+, -\} \). By integrating by parts as in [2, Proposition 12], one can show that if \( u \) solves Equation (2) then

\[
u_+ = \mathcal{A}_{++} u_+ + \mathcal{A}_{+-} u_-,
\]

where \( u_+ := (u_j)_{j=0}^{m-1} \) and \( u_- := (u_{m+j})_{j=0}^{m-1} \). Therefore, \( (1 - \mathcal{A}_{++})u_+ = \mathcal{A}_{+-} u_- \) and we can express the Dirichlet-Neumann operator \( \Lambda \) in terms of layer potentials as

\[
\Lambda = \mathcal{B}(1 - \mathcal{A}_{++}).
\]

Here \( \mathcal{B} = (B_{j+m,l})_{j,l=0}^{m-1} \) denotes a parametrix (with parameter) of \( \mathcal{A}_{+-} := (A_{j,l+m})_{j,l=0}^{m-1} \), See more in the proof of [2, Theorem 18].

The proof of Theorem 2 uses Equation (6) to compute components of the symbol of the Dirichlet-Neumann operator \( \Lambda \). The formula for \( c_3 \) then follows from (5).

**Proof of Theorem 2**

To prove Theorem 2 we note that we by Equation (5) only need to compute the third term \( \sigma_{2j-2l-3}(\Lambda_{2j-1,2l}) \) in the polyhomogeneous expansion

\[
\sigma(\Lambda_{2j-1,2l})(x, \xi, R) \sim \sum_{k=0}^\infty \sigma_{2j-2l-1-k}(\Lambda_{2j-1,2l})(x, \xi, R),
\]

in the range \( m/2 < j \leq m, \ 0 \leq l < m/2 \). In fact, we only need to compute the evaluation \( \sigma_{2j-2l-3}(\Lambda_{2j-1,2l})(x, 0, 1) \). Recall that we are using the parameter-dependent calculus, so that each \( \sigma_{2j-2l-1-k}(\Lambda_{2j-1,2l})(x, \xi, R) \) is homogeneous of degree \( -2j - 2l - 1 - k \) in \( (\xi, R) \).

For the convenience of the reader, we change to the notation \( (x', \xi', R) \in T^*\partial X \times \mathbb{R}_+ \) for coordinates and cotangent variables on the boundary \( \partial X \), as used in [2]. For an integer \( k \in \mathbb{Z} \), we use the notation

\[
\sigma_k(\mathcal{A}_{++}) := (\sigma_{j-l+k}(\mathcal{A}_{j,l}))_{j,l=0}^{m-1},
\]

\[
\sigma_k(\mathcal{A}_{+-}) := (\sigma_{j-l+k-m}(\mathcal{A}_{j,l+m}))_{j,l=0}^{m-1} \quad \text{and}
\]

\[
\sigma_k(\mathcal{B}) := (\sigma_{j+m-l+k}(\mathcal{B}_{j+m,l}))_{j,l=0}^{m-1}.
\]
Here we write $\sigma_{j-l+k}(A_{j,l})$ for the degree $j-l+k$ part of $a_{j,l}$ written as a symbol depending on the variable $(x',\xi',R) \in T^*\partial X \times \mathbb{R}_+$. The symbols $\sigma_k(\mathbb{H}_{++})$, $\sigma_k(\mathbb{H}_{+-})$ and $\sigma_k(\mathbb{B})$ relate to the (parameter-dependent) Douglis-Nirenberg calculus naturally appearing in the boundary reduction of boundary value problems [2, 5]. The reader should note the difference with the expressions appearing just after [2, Proposition 37] in that they are for symbols in the variables $(x', y', \xi', R)$. The process of going between these two symbol expressions is one of the difficulties in the computation ahead.

The reader can note that $\sigma_0(\mathbb{H}_{++})$, $\sigma_0(\mathbb{H}_{+-})$ and $\sigma_0(\mathbb{B})$ are the matrices of principal symbols of $\mathbb{H}_{++}$, $\mathbb{H}_{+-}$ and $\mathbb{B}$, respectively. In particular,

$$\sigma_0(\mathbb{B}) = \sigma_0(\mathbb{H}_{+-})^{-1}.$$ 

It follows from [2, Theorem 12] that $\sigma_0(\mathbb{B})$ does not depend on $x' \in \partial X$. Define the symbol

$$\mathbb{D} = (\delta_{j,k}(R^2 + |\xi|^2)^{j/2})_{j,k=0}^n.$$ 

By the computational result [2, Theorem 12], there are constant $m \times m$-matrices $C_0$, $C_1$, $C_2$, $C_3$ such that

$$\sigma_0(\mathbb{H}_{++}) = \mathbb{D}C_0\mathbb{D}^{-1}, \quad \sigma_0(\mathbb{H}_{+-}) = \mathbb{D}C_1\mathbb{D}^{-1},$$

$$\sigma_{-1}(\mathbb{H}_{++}) = H\mathbb{D}C_2\mathbb{D}^{-1}, \quad \sigma_{-1}(\mathbb{H}_{+-}) = H\mathbb{D}C_3\mathbb{D}^{-1}, \quad \text{and}$$

$$\sigma_0(\mathbb{B}) = \mathbb{D}C_1^{-1}\mathbb{D}^{-1},$$

where $H$ denotes the mean curvature of $\partial X$ and we in each identity embed $m \times m$-matrices in a suitable fashion into $2m \times 2m$-matrices.

From [2, Lemma 22, part a] and the $x'$-independence of $\sigma_0(\mathbb{B})$ we can from Equation (4) deduce that

$$\sigma_{-1}(\mathbb{B}) = -\sigma_0(\mathbb{B})\sigma_{-1}(\mathbb{H}_{+-})\sigma_0(\mathbb{B}) = H\mathbb{D}C_1^{-1}C_3C_1^{-1}\mathbb{D}^{-1},$$

as well as

$$\sigma_{-2}(\mathbb{B}) = -\sigma_0(\mathbb{B})\left(\sigma_{-2}(\mathbb{H}_{+-}) + \sum_{j=1}^{n-1} \partial_{\xi_j} \sigma_0(\mathbb{H}_{+-})\sigma_0(\mathbb{B})\partial_{x_j} \sigma_{-1}(\mathbb{H}_{+-}) - \sigma_{-1}(\mathbb{H}_{+-})\sigma_0(\mathbb{B})\sigma_{-1}(\mathbb{H}_{+-})\right)\sigma_0(\mathbb{B}).$$
Using [2, Lemma 22, part b], we write

\[
\sigma_{-2}(\Lambda) = \sigma_{-2}(\mathbb{B})(1 - \sigma_0(\mathbb{A}++) - \sigma_{-1}(\mathbb{B})\sigma_{-1}(\mathbb{A}++) - \sigma_0(\mathbb{B})\sigma_{-2}(\mathbb{A}++) + \\
+ i \sum_{j=1}^{n-1} \partial_{\xi_j} \sigma_{-1}(\mathbb{B}) \partial_{x_j} \sigma_{-1}(\mathbb{A}++) = \\
= - \sigma_0(\mathbb{B}) \left( \sigma_{-2}(\mathbb{A}++) - \sigma_{-1}(\mathbb{A}++) \sigma_0(\mathbb{B}) \sigma_{-1}(\mathbb{A}++) \right) \sigma_0(\mathbb{B})(1 - \sigma_0(\mathbb{A}++)) + \\
+ \sigma_0(\mathbb{B}) \sigma_{-1}(\mathbb{A}++) \sigma_0(\mathbb{B}) \sigma_{-1}(\mathbb{A}++) - \sigma_0(\mathbb{B}) \sigma_{-2}(\mathbb{A}++) - \\
- i \sum_{j=1}^{n-1} \partial_{\xi_j} (\sigma_0(\mathbb{B}) \sigma_{-1}(\mathbb{A}++) \sigma_0(\mathbb{B})) \partial_{x_j} \sigma_{-1}(\mathbb{A}++)
\]

Since all \(\sigma_0\)-occurrences only depend on \(R^2 + |\xi|^2\), all its \(\xi\)-derivatives will vanish at \(\xi = 0\), and therefore,

\[
\sigma_{-2}(\Lambda)(x', 0, R) = \\
= \left[ - \sigma_0(\mathbb{B}) \left( \sigma_{-2}(\mathbb{A}++) - \sigma_{-1}(\mathbb{A}++) \sigma_0(\mathbb{B}) \sigma_{-1}(\mathbb{A}++) \right) \sigma_0(\mathbb{B})(1 - \sigma_0(\mathbb{A}++)) + \\
+ \sigma_0(\mathbb{B}) \sigma_{-1}(\mathbb{A}++) \sigma_0(\mathbb{B}) \sigma_{-1}(\mathbb{A}++) - \sigma_0(\mathbb{B}) \sigma_{-2}(\mathbb{A}++) \right]_{\xi' = 0} = \\
= \left[ - \mathbb{D}C_1^{-1} \mathbb{D}^{-1} \left( \sigma_{-2}(\mathbb{A}++) \mathbb{D}C_1^{-1}(1 - C_0) \mathbb{D}^{-1} + \sigma_{-2}(\mathbb{A}++) \right) + \\
+ H^2 \mathbb{D}C_1^{-1} C_3 C_1^{-1} C_3 C_1^{-1}(1 - C_0) \mathbb{D}^{-1} + H^2 \mathbb{D}C_1^{-1} C_3 C_1^{-1} C_3 C_1^{-1} \mathbb{D}^{-1} \right]_{\xi' = 0}.
\]

Assume for now that \(\sigma_{-2}(\mathbb{A}++)(x', 0, R) = \sigma_{-2}(\mathbb{A}++)(x', 0, R) = 0\). Then this computation shows that indeed, there are universal constants \((d_{j+m,l})_{j,l=0}^{m-1}\) (independent of \(X\)) such that for \(\frac{m}{2} < j \leq m\) and \(0 \leq l < m/2\),

\[
\sigma_{2j-2l-2}(A_{2j-1,2l})(x, 0, 1) = d_{2j-1,2l} H(x)^2.
\]

In particular, we have shown that for a dimensional constant \(\lambda_n\), we have that \(c_3(X) = \lambda_n \int_{\partial X} H^2 dS\). It follows from [14] that \(\lambda_n \neq 0\) for \(n \geq 3\) odd.

It remains to show that \(\sigma_{-2}(\mathbb{A}++)(x', 0, R) = \sigma_{-2}(\mathbb{A}++)(x', 0, R) = 0\). Note that we do not claim that \(\sigma_{-2}(\mathbb{A}++) = \sigma_{-2}(\mathbb{A}++) = 0\) just that when restricting to \(\xi' = 0\) the symbols vanish. This last step in the proof relies on the technically involved computations in [2, Appendix A.2] and the process of going from “two-variable symbols” \(a(x, y, \xi, R)\) to “one-variable symbols” \(a(x, \xi, R)\), see [7, Theorem 7.13]. We pick local coordinates at a point on \(\partial X\). We can assume that this point is \(0 \in \mathbb{R}^n\) and that the coordinates are of the form \((x', S(x'))\), where \(x'\) belongs to some neighborhood of \(0 \in \mathbb{R}^{n-1}\) and \(S\) is a scalar function with \(S(0) = 0\) and
\[ \nabla S(0) = 0. \] We can express \( a_{jk} \) as
\[
a_{jk}(x', y', \xi', R) = b_{0,m-p-q}(R^2 + |\xi'|^2, S(x') - S(y')),
\]
when \( j = 2p, k = n - 2q \)
\[
a_{jk}(x', y', \xi', R) = b_{1,m-p-q}(R^2 + |\xi'|^2, S(x') - S(y')) +
\]
\[
(\xi' \cdot \nabla S(y'))b_{0,m-p-q}(R^2 + |\xi'|^2, S(x') - S(y')),
\]
when \( j = 2p + 1, k = n - 2q - 1 \)
\[
a_{jk}(x', y', \xi', R) = b_{2,m-p-q}(R^2 + |\xi'|^2, S(x') - S(y')) +
\]
\[
((\xi' \cdot \nabla S(y')) + (\xi' \cdot \nabla S(x')))b_{1,m-p-q}(R^2 + |\xi'|^2, S(x') - S(y')) +
\]
\[
(\xi' \cdot \nabla S(x'))(\xi' \cdot \nabla S(y'))b_{0,m-p-q}(R^2 + |\xi'|^2, S(x') - S(y')),
\]
when \( j = 2p + 1, k = n - 2q - 1 \),

where
\[
b_{r,N}(u, z) = \begin{cases} (-i \partial_z)^r (u - \partial_z^2)^{-N} \delta_{z=0}, & N \leq 0, \\ (-i \partial_z)^r \sum_{k=0}^{N-1} \hat{c}_{k,r,N} \frac{|z|^k}{(z + \epsilon)^{r+k+N}} & N > 0, \end{cases}
\]

for some coefficients \( \hat{c}_{k,r,N} \).

We need to verify that \( \sigma_{j-k-2}(A_{j,k})(x', 0, R) = 0 \) for any \( j \) and \( k \). The symbol \( \sigma_{j-k-2}(A_{j,k}) \) in \( x' = 0 \) is by [7, Theorem 7.13] given by the terms of order \( j - k - 2 \) in the expression
\[
a_{jk}(0, 0, \xi', R) - i \sum_{l=1}^{n-1} \frac{\partial^2 a_{jk}}{\partial \xi_l \partial y_l}(0, 0, \xi', R) - \frac{1}{2} \sum_{j,s=1}^{n-1} \frac{\partial^4 a_{jk}}{\partial \xi_l \partial y_l \partial y_s}(0, 0, \xi', R).
\]

Recall that \( S(0) = 0 \) and \( \nabla S(0) = 0 \) so there are several terms vanishing when setting \( x' = 0 \). Indeed, no term of order \( j - k - 2 \) in \( a_{jk}(0, 0, \xi', R) \) is non-zero. All non-zero terms of order \( j - k - 2 \) in \( \sum_{l=1}^{n-1} \frac{\partial^2 a_{jk}}{\partial \xi_l \partial y_l}(0, 0, \xi', R) \) are odd functions under the reflection \( \xi' \mapsto -\xi' \), so they vanish when restricting to \( \xi' = 0 \). Similar computations show that terms of order \( j - k - 2 \) in \( \frac{1}{2} \sum_{j,s=1}^{n-1} \frac{\partial^4 a_{jk}}{\partial \xi_l \partial y_l \partial y_s} \) all contain a factor of \( \xi_l \) or \( \xi_s \), so they vanish when restricting to \( \xi' = 0 \).

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