An Approximate Generalization of the Okamur-Seymour Theorem

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Abstract—We consider the problem of multicommodity flows in planar graphs. Okamura and Seymour showed that if all the demands are incident on one face, then the cut-condition is sufficient for routing demands. We consider the following generalization of this setting and prove an approximate max flow-min cut theorem: for every demand edge, there exists a face containing both its end points. We show that the cut-condition is sufficient for routing $\Omega(1)$-fraction of all the demands. To prove this, we give a $L_1$-embedding of the planar metric which approximately preserves distance between all pair of points on the same face.

Index Terms—multicommodity flows, approximation algorithms, metric embedding

I. INTRODUCTION

Given a graph $G$ with edge capacities and multiple source-sink pairs, each with an associated demand, the multicommodity flow problem is to route demands simultaneously without violating edge capacities. The problem was first formulated in the context of VLSI routing in the 70’s and since then it has seen a long and impressive line of work.

The demand graph, $H$, is the graph obtained by including an edge $(s_i, t_i)$ for a demand with source-sink $s_i, t_i$. A necessary condition for the flow to be routed is that the capacity of every cut exceeds the demand across the cut. This condition is known as the cut-condition and is known to be sufficient when $G$ is planar and all the source-sink pairs are on one face [11] or when $G + H$ is planar [14]. However, one can construct small instances where the cut-condition is not sufficient for routing flow. When $G$ is series-parallel, if every cut has capacity at least twice the demand across it, then flow is routable [3], [6]. The flow-cut gap of a certain graph class is the smallest $\alpha$ such that flow is routable when capacity of every cut is at least $\alpha$ times the demand across it. Thus, for series-parallel graphs, the flow-cut gap is 2. For general graphs, the flow-cut gap is $\Theta(\log k)$ [10], where $k$ is the number of demand pairs.

The flow-cut gap for planar graphs ($G$ planar, $H$ arbitrary) is $O(\sqrt{\log n})$ [12] and is conjectured to be $O(1)$ [6]. Chekuri et al. [4] showed a flow-cut gap of $2^{O(k)}$ for $k$-outerplanar graphs. Lee et al. [8] made progress towards this conjecture by showing an $O(\log h)$ bound on the flow-cut gap, where $h$ is the number of faces on which source-sink vertices are incident. Filtser [5] further improved his bound by showing a flow-cut gap of $O(\sqrt{\log h})$, when all the source-sink vertices are incident on $h$ faces. In this paper, we consider instances where the source and sink of every demand lie on the same face, but all source-sink pairs don’t necessarily lie on a single face, and show that the flow-cut gap of such instances is $O(1)$. It is well known that the cut-condition is not sufficient for such instances (see Figure I). A common approach to establish bounds on the flow-cut gap is to bound the $L_1$ distortion incurred in embedding an arbitrary metric on the graph $G$ into a normed space. This, for instance, has been the method used to establish flow-cut gaps for general graphs [10], series-parallel graphs [3], [6] and planar graphs [12]. We too build on this technique to prove our results (see Theorem 5 and 6).

II. DEFINITIONS AND PRELIMinARIES

Let $G = (V,E)$ be a simple graph with edge capacities $c : E \rightarrow R_{\geq 0}$. We call this the supply graph. Let $H = (V,D)$ be a simple graph with demands on edges $d : D \rightarrow R_{\geq 0}$. We call this the demand graph. The objective of the multicommodity flow problem is to find paths between the end points of demand edges in the supply graph such that the following hold: for every demand edge $e \in D$, $d(e)$ paths are picked in the supply graph and every supply edge $e \in E$ is present in at most $c(e)$ paths.

We say that an instance is feasible if paths satisfying the above two conditions can be found. We also call such an
instance integrally feasible. If there exists an assignment of positive real numbers to paths such that total flow for every demand edge \( e \in E \) is \( d_e \) and total value of paths using a supply edge \( e \in E \) is at most \( c_e \), then we say that the instance is (fractionally) feasible. A cut \( S \subseteq V \) is a partition of the vertex set \((S, V \setminus S)\). The number of edges of \( G \) going across the cut is denoted by \( \delta_G(S) \). Similarly, \( \delta_H(S) \) denotes the number of demand edges going from \( S \) to \( V \setminus S \). One necessary condition for routing the flow (fractionally or integrally) is as follows: for every \( S \subseteq V \), \( \delta_G(S) \geq \delta_H(S) \). In other words, across every cut, total supply should be at least the total demand. This condition is also known as the cut-condition.

In general, the cut-condition is not sufficient for a feasible routing. We can ask for the following relaxation: given an instance for which the cut-condition is satisfied, what is the maximum value of \( f \), such that \( f \) fraction of every demand can be routed? The number \( f^{-1} \) is known as the flow-cut gap of the instance. There is an equivalent definition of the flow-cut gap: given an instance \((G, H)\) satisfying the cut condition, the smallest number \( k \), such that \((kG, H)\) is feasible, where \( kG \) denotes the graph with every edge capacity multiplied by \( k \) (see Figure I for an illustration). The following two classic results identify settings where the cut-condition is also sufficient for routing demands in planar graphs. We will be invoking these to prove our results.

**Theorem 1 (Okamura-Seymour [11]):** If \( G \) is a planar graph, all the edges of \( H \) are restricted to a face and \( G + H \) is eulerian, then the cut-condition is necessary and sufficient for integral routing of all the demands.

**Theorem 2 (Seymour [14]):** If \( G + H \) is planar and eulerian, then the cut-condition is necessary and sufficient for integral routing of all the demands.

Note that if eulerian condition is not satisfied, we get a half integral flow. Both the results can be converted into algorithms which run in polynomial time. In this paper, we will not be concerned with (half) integral flows and focus on proving flow-cut gap for instances that generalize the setting of the above stated results (see Theorem 5).

From now on, we assume a fixed planar embedding of \( G \). Without loss of generality, one can assume that \( G \) is 2-vertex connected. If there is a cut vertex \( v \) and \( ab \) is a demand separated by removal of \( v \), then replacing \( ab \) by \( av, vb \) maintains the cut-condition. By doing this for every cut vertex and demand separated by them, we get separate smaller instances for each 2-vertex connected component. Hence, every vertex is a part of cycle corresponding to some face. Hence, by our assumption, for every demand edge there exists a face such that both its end points lie on that face. Hence, we can associate every demand with a face. We abuse notation and use \( f \) to also denote the edges and vertices associated with the cycle of face \( f \). Given a set \( S \), we denote the subgraph induced by vertices in \( S \) as \( G[S] \). We call a subset \( A \subseteq V \) central if both \( G[A] \) and \( G[V - A] \) are connected. The following is well-known.

**Lemma 1 ([13]):** \((G, H)\) satisfies the cut-condition if and only if all the central sets satisfy the cut-condition.

The set of all faces of \( G \) will be denoted by \( F \). The dual of a planar graph \( G^D = (V^D, E^D) \) is defined as follows: \( V^D = F \) and if \( f_i, f_j \in F \) share an edge in \( G \), then \( (f_i, f_j) \in E^D \). It is a well known fact that edges of a central cut in \( G \) correspond to a simple circuit in \( G^D \) and vice versa. Given a graph \( G = (V, E) \) with edge length \( l : E \rightarrow \mathbb{R}_{\geq 0} \), we use \( d_G(u, v) \) to denote the shortest path distance between \( u \) and \( v \) in \( G \) w.r.t \( l \). We now describe the connection between flow-cut gap and embedding vertices into normed space.

**A. Embedding Metrics into \( L_1 \)**

Given an edge weighted graph \( G = (V, E) \) with edge length \( l : E \rightarrow \mathbb{R}_{\geq 0} \), associated shortest path metric \( d_G \), a graph \( H = (V, F) \) and an embedding \( f \) of vertices \( V \) into \( L_1 \), the contraction and expansion of \( f \) are the smallest \( \alpha, \beta \) respectively, such that \( ||f(u) - f(v)||_1 \geq d_G(u, v)/\alpha \) for all \( (u, v) \in E \) and \( ||f(u) - f(v)||_1 \leq \beta \cdot d_G(u, v) \) for all \( (u, v) \in E \). The distortion of the embedding, \( \text{dist}(G, H, f) \), is \( \alpha \cdot \beta \). Given \( G, H \), we are generally interested in finding an embedding with low distortion. If \( H \) is a clique, we refer to \( \text{dist}(G, H, f) \) simply as \( \text{dist}(G, f) \) and call it the distortion of \( G \) with respect to \( f \). Linial et al. [10] built on the result of Bourgain [1], and gave a polynomial time algorithm that embeds any graph on \( n \) vertices into \( L_1 \) with distortion \( O(\log n) \). Furthermore, this result is asymptotically the best possible, as there exist instances for which any embedding into \( L_1 \) has distortion \( \Omega(\log n) \). There is a rich literature on finding low distortion embeddings for special graph classes. It is well known that a tree can be embedded into \( L_1 \) with distortion 1, outerplanar graphs with distortion 1 [11], \( k \)-outerplanar graphs with distortion \( 2^O(k) \) [4], series-parallel graphs with distortion 2 [3]. Rao [12] showed that any planar metric can be embedded into \( L_1 \) with distortion \( O(\sqrt{\log n}) \) and there has essentially been no improvement upon this in the last two decades. It is conjectured that any planar graph can be embedded into \( L_1 \) with distortion \( O(1) \) [6] (also known as the GNRS-Conjecture). In this paper, we make progress towards this conjecture. Given a drawing of a planar graph in the plane, let \( H \) be the set of all pairs of vertices \( (u, v) \) such that \( u \) and \( v \) lie on the same face. We show the existence of a polynomial time computable \( f : V \rightarrow L_1 \) such that \( \text{dist}(G, H, f) = O(1) \). In this paper, we will work exclusively with the 1-norm, so we drop the subscript and denote \( ||f(u) - f(v)||_1 \) simply as \( ||f(u) - f(v)|| \). Also, if \( \beta = 1 \), we say that the embedding is non-expansive. By scaling, we may convert any \( L_1 \) embedding into a non-expansive one.

**B. Flow-Cut Gap and Embedding into \( L_1 \)**

Flow-cut gaps and embedding graphs into \( L_1 \) are intimately related. This connection was first observed by Linial et al. [10], who used it to prove flow-cut gap results for arbitrary graphs. We describe this connection formally now. Let \( G = (V, E), H = (V, F) \) be fixed graphs and \( c : E \rightarrow \mathbb{R}_{\geq 0}, d : F \rightarrow \mathbb{R}_{\geq 0} \) and \( l : E \rightarrow \mathbb{R}_{\geq 0} \) be the capacity, demand and length functions on respective
Theorem 3 (Okamura-Seymour [11]): Let $G = (V, E)$ be a planar graph with edge length $l : E \to \mathbb{R}_{\geq 0}$ and $f \in F$ be one of its faces. There exists an embedding of $V$ into $L_1$ such that for all $u, v \in f^2, |l(u, v)| = d_G(u, v)$ and for all $(u, v) \in E, |l(u, v)| \leq d_G(u, v) = l(u, v)$.

Theorem 4 (Seymour [14]): Let $G = (V, E)$ be a planar graph with edge length $l : E \to \mathbb{R}_{\geq 0}$ and $H = (V, T)$ be a demand graph such that $G$ is planar. Then there exists an embedding of $V$ into $L_1$ such that for all $(u, v) \in T, |f(u) - f(v)| = d_G(u, v)$ and for all $(u, v) \in E, |l(u, v)| \leq d_G(u, v) = l(u, v)$.

C. Cut Metrics and $L_1$ Embedding

Suppose we have a set of cuts with non-negative weights $C = \{(C_1, w_1), \ldots, (C_k, w_k)\}$. Define $\delta_C(u, v)$ to be $w_i$ if exactly one of $u, v$ is contained in $C_i$ and 0 otherwise. Let $\delta_C(u, v) = \sum_{i=1}^k \delta_C(u, v)$. It is easy to verify that $\delta_C$ induces a metric on $V$. We refer to $\delta_C$ as the distance induced by the cuts in $C$ or a cut-metric $C$. One can construct $f : V \to \mathbb{R}^k$ such that $\forall u, v \in V, |f(u) - f(v)| = \delta_C(u, v)$ as follows: for any vertex $u$, define $f(u) = (u_1, u_2, \ldots, u_k)$ where $u_i = w_i$ if $u \in C_i$, 0 otherwise. In fact, the converse of above is also true: given any embedding of vertices into $L_1$, there exists a set of weighted cuts $C$ such that the distance metric induced by $C$ is equal to the distance metric induced by the $L_1$ embedding (see Lemma 15.2 of [15] for a proof). Hence, to show a low distortion $L_1$ embedding of a metric, it is equivalent to show a collection of cuts which preserve distances with low distortion. Using the aforementioned equivalence of the cut-metric and $L_1$ embedding, we us them interchangeably from now on. Given a scalar $\alpha$ and a collection of weighted cuts $C$, $\alpha$-C denotes the the same collection of cuts with the weight of all cuts scaled by a (multiplicative) factor of $\alpha$.

III. OUR CONTRIBUTION

We generalize the result of Okamura and Seymour [11] and prove the following approximate max flow-min cut theorem:

Theorem 5: Let $G$ be an edge-capacitated planar graph and $H$ be a set of demand edges such that for each $(u, v) \in H$, there exists a face $f$ containing both $u$ and $v$. If the cut-condition is satisfied, then there exists a feasible routing of $\Omega(1)$-fraction of all the demands.

Using the congestion-distortion theorem, Theorem 5 can be stated in terms of metric embedding as follows:

Theorem 6: Let $G = (V, E)$ be a planar graph with edge length $l : E \to \mathbb{R}_{\geq 0}$ and $f$ be a set of vertices $(u, v)$ such that both $u$ and $v$ lie on the same face. Then there exists a constant $c > 1$ and an embedding $g : V \to \mathbb{R}$ such that $\|g(u) - g(v)\| \geq d_G(u, v)/c$ for $(u, v) \in T$ and $\|g(u) - g(v)\| \leq l(u, v)$ for $(u, v) \in E$.

We now give a brief overview of our approach. As mentioned before, we work with a fixed embedding of the given planar graph in the plane. We call a face $f \in F$ geodesic if for all $u, v \in f$, $d_G(u, v)$ is equal to the shortest path distance between $u, v$ using only the edges of the cycle corresponding to $f$. A face which is not geodesic is called non-geodesic. Let $F_G$ and $F_N$ denote the set of all the geodesic and non-geodesic faces. Given a face $f$, let $G_f$ be the subgraph enclosing minimal area in the plane that supports the metric on the vertices of face $f$. Let $S_f$ be the minimal area cycle bounding $G_f$ in the plane. Given a cycle $S$, let $R(S) \subseteq \mathbb{R}^2$ be the open region contained inside $S$.

If $f$ is a geodesic face, then $G_f$ is exactly the cycle bounding the face, i.e. $G_f = S_f$. In Section IV, we first show that the set $\{R(S_f) | f \in F\}$ forms a laminar or non-crossing set system. This implies that there is a face $f$ with minimal $R(S_f)$, i.e. for any $f' \neq f$ either $R(S_f) \subseteq R(S_{f'})$ or $R(S_f) \cap R(S_{f'}) = \emptyset$. We then go on to show that removing the edges of $(G_f \setminus S_f)$ doesn’t modify the metric on any of the non-geodesic faces other than $f$. We then go on to find a suitable embedding of the graph $(G \setminus G_f) \cap S_f$ inductively and show how to extend the embedding to include all the faces contained in $R(S_f)$. This extension argument turns out to be non-trivial and forms the core of our proof. To do this extension, we need to develop several new tools. In Section IV, we give an algorithm to modify the original length function so as to allow a nice inductive decomposition. We call such length functions $\alpha$-good and believe that this could be a useful tool in proving flow-cut gaps for other planar instances as well. In Section V, we come up with an embedding for all geodesic pairs of vertices, i.e. pairs of vertices for which there exists a shortest path using only the edges on the corresponding face. In Section VI, we develop a low distortion embedding for all pairs of vertices whose shortest path uses a fixed vertex of the graph. We believe that this problem is interesting in its own right and could prove to be useful in other settings. In Sections VII and VIII, we combine the tools developed in previous sections to complete the inductive step.

IV. LAMINAR STRUCTURE OF FACE SUPPORTS

We partition the faces of a planar graph $G$ into two sets: geodesic and non-geodesic. A face $f$ is called geodesic if for all $u, v \in f$, there exists a shortest path between $u$ and $v$ using only the vertices of the cycle associated with $f$. A face which is not geodesic is called a non-geodesic face. Let $f_I$ be
the infinite face in the planar embedding of $G$. Let $F_N$ denote the set of all the non-geodesic faces except $f_1$. Let $F_G$ be the set of all the geodesic faces.

Let $f$ be a face of $G$. A set of edges $E' \subseteq E$ is called a **support** of $f$ if for all $u, v \in f$, $d_{G}(u, v) = d_{E'}(u, v)$, where $G' = (V, E')$. In other words, restricting to $E'$ doesn’t change the shortest path metric on $f$. A support of $f$ is called minimal if deleting any edge from it changes the shortest path metric on the vertices of $f$. Given a cycle $C$, let $R(C)$ (resp. $R(C)$) denote the open (resp. closed) region contained inside the cycle $C$ in the planar embedding of $G$. We choose $R(C)$ and $R(C)$ such that it does not contain the infinite face $f_1$. Let $S_f$ be a cycle such that $R(S_f)$ is the inclusion wise minimal region containing a support of $f$ (i.e. there exists a support $E_f$ of $f$ such that $u, v \in R(S_f)$ for all $(u, v) \in E_f$). Note that if $f \in F_G$, then $S_f = f$. Also, for any $f \in F$, $R(f) \subseteq R(S_f)$. Note that given a fixed planar embedding of $G$, $S_f$ is a function of the edge-lengths.

**Lemma 2:** Let $f_1, f_2 \in F_N$. Then one of the following must hold: $R(S_{f_1}) \cap R(S_{f_2}) = \emptyset$ or $R(S_{f_1}) \subseteq R(S_{f_2})$ or $R(S_{f_2}) \subseteq R(S_{f_1})$.

**Proof.** For the sake of contradiction, assume that there exist $f_1, f_2 \in F_N$ such that $R(S_{f_1}) \cap R(S_{f_2}) \neq \emptyset$ and $R(S_{f_2}) \cap R(S_{f_1}) \neq \emptyset$. Then either $R(f_1) \in R(S_{f_2})$ or $R(f_2) \in R(S_{f_1})$. We present the argument for the case when $R(f_1) \in R(S_{f_2})$, an analogous argument works for the second case as well. Suppose $R(f_1) \in R(S_{f_2})$. Minimality of $R(S_{f_2})$ implies that there must exist $u_1, v_1 \in f_1$ such that any shortest path from $u_1$ to $v_1$ exits and enters the region $R(S_{f_2})$ at least once. Let $P(u_1, v_1)$ be the portion of the path between $x_1$ and $y_1$ on the cycle $S_{f_2}$. Note that $P(x_1, y_1)$ is a path on the boundary of $R(S_{f_2})$. Since $R(S_{f_2})$ defines the minimal region containing the support of the metric on $f_2$, there must exist $u_2, v_2 \in f_2$ such that some shortest path between $u_2$ and $v_2$ uses an edge on the path $P(x_1, y_1)$. Let $P(u_2, v_2)$ be such a shortest path and let $x_2, y_2$ be the first and last point of intersection of $P(u_2, v_2)$ with $P(u_1, v_1)$. Then replacing the portion of the path $P(u_1, v_1)$ between $x_2$ and $y_2$ by portion of the path $P(u_2, v_2)$ between $x_2$ and $y_2$, we obtain a shortest $u_1, v_1$ path contained inside $S_{f_2}$. This contradicts our assumption on the minimality of $R(S_{f_2})$. We can use an analogous argument at a contradiction in case $R(f_1) \cap R(S_{f_2}) = \emptyset$. Hence such $f_1, f_2$ can’t exist and it must be true that $R(S_{f_1}) \cap R(S_{f_2}) = \emptyset$ or $R(S_{f_2}) \subseteq R(S_{f_1})$ or $R(S_{f_1}) \subseteq R(S_{f_2})$ for any $f_1, f_2 \in F_N$. ■

Given a cycle $C$, let $I(C)$ be the set of vertices contained in $R(C)$. Note that $I(C)$ doesn’t contain vertices of $C$. Let $\alpha > 1$ be a given constant. Given a face $f \in F_N$ and its support cycle $S_f$, we say that $S_f$ is **$\alpha$-loose** if the following holds: for any $u, v \in f$, the length of any path between $u$ and $v$ using only the vertices in $I(S_f) \cup \{u, v\}$ is at least $\alpha \cdot d_G(u, v)$.

To make the induction argument work, only the laminar property of face supports is not sufficient and we need a stronger structure. We now describe this property in more detail. By Lemma 2, the set of regions $\{R(S_f) | f \in F_N\}$ forms a laminar structure. This implies that there exists a face $f \in F_N$ such that for any $f' \in F_N$, either $R(S_f) \subseteq R(S_{f'})$ or $R(f') \cap R(S_f) = \emptyset$. We call such faces **innermost**. Note that there could be multiple innermost faces. We say that a length function $l: E \to \mathbb{R}^+$ is **$\alpha$-good** if (i) $F_N = \emptyset$, i.e. the infinite face is (possibly) the only non-geodesic face (ii) if $f \in F_N$ is an innermost face, then the graph obtained by removing all the edges with both end-points inside $R(S_f)$ is $\alpha$-good and $S_f$ is $\alpha$-loose. We now show now that any length function can be converted into a $\alpha$-good one by modifying the edge lengths by a factor of at most $\alpha$.

**Theorem 7:** Let $G = (V, E)$ be a planar graph with $f_1$ as the infinite face, edge length $l: E \to \mathbb{R}_{\geq 0}$ and $\alpha > 1$ be a constant. Then there exists a new edge length $l': E \to \mathbb{R}^+$ such that (i) $G$ is $\alpha$-good with respect to $l'$ (ii) $l'(u, v) = l(u, v)$ for $u, v \in f_1$ (iii) $l(e)/\alpha \leq l'(e) \leq l(e)$ for $e \in E$.

**Proof.** We prove the theorem by induction on the number of faces in the graph. If $|F| = 1$ or $f_1$ is the only non-geodesic face, then the statement of the theorem follows trivially by setting $l' = l$. Suppose that $G$ has at least one non-geodesic face other than $f_1$, w.r.t the length function $l$. By Lemma 2, regions in the set $\{R(S_f) | f \in F_N\}$ form a laminar structure. A face $f \in F_N$ is called maximal if for any other face $f'$, either $R(S_{f'}) \subseteq R(S_f)$ or $R(S_f) \subseteq R(S_{f'}) = \emptyset$. Let $f_1, f_2, \ldots, f_k$ be the maximal faces of $G$ w.r.t length function $l$. Let $G_i$ be the graph consisting of vertices and edges contained completely inside $R(S_{f_i})$. If $k = 1$ and $R(S_{f_1}) \neq R(f_1)$ or $k \geq 2$, then each $G_i$ has strictly less number of faces than $G$. By induction, for each $G_i$, we have length functions $l_i$ satisfying the conditions of the theorem. We construct the new length function $l'$ as follows: for an edge $(u, v)$ not contained in any of the $G_i$, we set $l'(u, v) = l(u, v)$. If an edge $(u, v)$ is contained in $G_i$, we set $l'(u, v) = l_i(u, v)$. Note that if an edge $(u, v)$ is contained in $G_i$ and $G_j$, then it must be present on the infinite face of $G_i$, $G_j$ and $l_i(u, v) = l_j(u, v) = l(u, v)$ (by property (ii)). Hence $l'$ is a valid length function. Property (ii) also ensures that gluing together different length functions for $G_1, G_2, \ldots, G_k$ does not change the distance between any pair of vertices contained inside one of the $G_i$’s. Hence, $G$ is $\alpha$-good w.r.t $l'$.

Suppose that $k = 1$ and $R(S_{f_1}) = R(f_1)$. We consider two cases depending on whether face $f_1$ is $\alpha$-loose or not. Suppose that the face $f_1$ is not $\alpha$-loose. By the definition of a $\alpha$-loose face, there must exist $u, v \in f_1$ such that the shortest $u, v$ path using no edges on $f_1$ has length $\beta \cdot d_G(u, v)$, for some $\beta \leq \alpha$. Let $P(u, v) = \{u, u_1, \ldots, u_{|P|}, v\}$ be such a path with minimum value of $\beta$. Note that $\beta = 1$ is also possible. Let $l_1$ be the length function obtained by setting $l_1(e) = l(e)/\beta$ for $e \in P(u, v)$ and $l_1(e) = l(e)$ otherwise.

**Claim 1:** For any $x, y \in f_1$, $l_1(x, y) = l(x, y)$.

**Proof.** Suppose that the statement of the claim is false. By construction, $l_1(x, y) \leq l(x, y)$ for any $x, y \in f_1$. Let $x, y \in f_1$.
be such that \( l(x, y) > l_1(x, y) \) and \( l_1(x, y) \) is minimum. Let \( P_1(x, y) \) be the minimum length \( x, y \) path w.r.t \( l_1 \). By minimality of \( l_1(x, y) \), it follows that \( P_1(x, y) \cap f_1 = \{x, y\} \), i.e. \( P_1(x, y) \) uses only the vertices in \( \{x, y\} \cup I(f_1) \). Let the length of path \( P_1(x, y) \) w.r.t \( l \) be \( l(P_1(x, y)) \). Since \( l_1 \) is constructed by modifying the length of a subset of edges in \( P_1(x, y) \) by a factor of \( \beta \), we have that \( l(P_1(x, y)) \leq \beta \cdot l_1(x, y) \leq \beta \cdot l(x, y) \). This contradicts the minimality of \( P(u, v) \) and hence such a \( x, y \in f_1 \) cannot exist.

The path \( P(u, v) \) divides \( R(f_1) \) into two closed regions, say \( R_1 \) and \( R_2 \). Let \( G_1 \subseteq G \) and \( G_2 \subseteq G \) be the graphs contained inside \( R_1 \) and \( R_2 \) respectively. Since \( P \) is a shortest \( u, v \) path w.r.t \( l_1 \) (by Claim 1), it follows that for any \( f \in G_1 \), \( R(S_f) \subseteq R_i \) for \( i = 1, 2 \). Hence, if \( u, v \in G_i \), then \( d_{G_i}(u, v) = d_G(u, v) \) for \( i = 1, 2 \). Therefore we can compute length functions for \( G_1, G_2 \) separately by using induction and combine them as before. Using induction, we find length functions \( d_1, d_2 \) satisfying the conditions of the theorem and set \( l'(e) = d_i(e) \) depending on whether \( e \in G_1 \) or \( e \in G_2 \). If \( e \) belongs to both to \( G_1 \) and \( G_2 \), then \( e \) is on the infinite face of \( G_1 \) and \( G_2 \) and by the statement of the theorem, \( d_i(e) = d_2(e) \). Hence \( l' \) is a valid length function. By using the induction hypothesis and Claim 1, it immediately follows that the condition (i) and (ii) of the theorem is also satisfied. Hence, \( G \) is \( \alpha \)-good w.r.t \( l' \).

Suppose that face \( f_1 \) is \( \alpha \)-loose. Then the statement of the theorem holds for non-geodesic face \( f_1 \). If there are no non-geodesic faces contained strictly inside \( R(f_1) \), then the statement of the theorem follows by setting \( l' = l \). Otherwise, let \( g_1, g_2, \ldots, g_k \) be the maximal non-geodesic faces contained strictly inside \( R(f_1) \). By the laminar structure of the regions in \( \{R(S_f) | f \in F_N\} \), we have \( R(S_{g_i}) \cap R(S_{g_j}) = \emptyset \) for \( i \neq j \).

Let \( G_i \) be the graph consisting of vertices and edges contained completely inside \( R(S_{g_i}) \). By induction hypothesis, we have length functions \( l_1, \ldots, l_k \) such that each one of them satisfies the statement of the theorem. We set \( l'(e) = l(e) \) for any edge not contained inside any of the \( G_i \)'s and set \( l'(e) = l_i(e) \) if \( e \in G_i \). By induction hypothesis, for any \( u, v \in S_{g_i} \), \( l'(u, v) = l_i(u, v) \). This implies that we do not create any new non-geodesic face in \( R(f_1) \) \( \cup \cup_{i=1}^k R(S_{g_i}) \), and the metric on \( f_1 \) w.r.t \( l \) and \( l' \) is the same. Hence \( R(S_{f_1}) = R(f_1) \) and \( f_1 \) is \( \alpha \)-loose w.r.t \( l' \). This completes the proof of the theorem.

V. EMBEDDING FOR THE GEODESIC PAIRS

Let \( G = (V, E) \) be a planar graph and \( F \) be the set of its faces and \( u, v \in V \) be a pair of vertices on the same face (i.e. there exists a face \( f \in F \) such that \( u, v \in f \)). Furthermore, suppose that there exists a shortest path between \( u \) and \( v \) using only the edges of \( f \). We call \( (u, v) \) a geodesic pair. Let \( T \) be the set of all the geodesic pairs in \( G \). In Theorem 9, we show that there exists an embedding of \( V \) into \( L_1 \) which preserves the distances between all the geodesic pairs within a constant factor. Using the congestion-distortion theorem, the following result follows directly from Theorem 12 of Kumar [9].

Theorem 9: Let \( G = (V, E) \) be a planar graph with edge length \( l : E \to \mathbb{R}_{\geq 0} \) and \( T \) be pairs of vertices \( (u, v) \) such that both \( u \) and \( v \) lie on the same face. Let \( T_f \subseteq T \) denote the set of pairs of vertices incident on face \( f \). For each \( f \in F \), there exists disjoint set of vertices \( X_1, Y_1, X_2, Y_2, \ldots, X_k, Y_k \subseteq f \) such that \( X_i, Y_i \) appear contiguously on \( f \) in clockwise order and for each \( (u, v) \in T_f, u \in X_j, v \in Y_j \) for some \( j \in \{1, 2, \ldots, k\} \). Then there exists an embedding \( h : V \to L_1 \) such that \( ||h(u) - h(v)|| \leq d_G(u, v) \) for all \( (u, v) \in E \) and \( ||h(u) - h(v)|| \leq d_G(u, v)/3 \) for all \( (u, v) \in T \).

Proof: We first construct a set of geodesic pairs \( T_T \subseteq T \) as follows. Initially \( T_T = \emptyset \). Let \( f \in F \) and \( V(f) = \{v_1, v_2, \ldots, v_n, v_{n+1} = v_1\} \) be the vertices of \( f \) in clockwise order. We partition \( V(f) \) into contiguous sets of vertices as follows: let \( v_k \) be the largest \( k \) such that \( (v_1, v_k) \) is a geodesic pair. We set \( T_T = T_T \cup \{v_1, v_k\} \) and \( S_f = (v_k, v_{k+2}, \ldots, v_{n+1}) \). We then start from \( v_{k+1} \) and find the largest index \( i \) such that \( (v_{k+1}, v_i) \) is a geodesic pair, set \( T_T = T_T \cup \{v_{k+1}, v_i\} \) and \( S_f = (v_{k+1}, v_{k+2}, \ldots, v_i) \), and continue with this process until all the vertices in \( C(f) \) have been exhausted, i.e. \( C(f) = \cup_{i \geq 0} S_f \). We do the above for all the faces of \( G \). By construction, the graph \( G \cup T_T \) is planar. Therefore by Theorem 4, we have an embedding, say \( h : V \to L_1 \), that preserves the distance between the end points of vertices in \( T_T \) exactly. In other words, \( ||h(u) - h(v)|| = d_G(u, v) \) for all \( (u, v) \in T_T \) and \( ||h(u) - h(v)|| = d_G(u, v) \) for all \( (u, v) \in E \).

Let \( T_T' = \{f(u, v) = g(u, v) \in S_f \in T_T \} \). By construction, the first and the last vertex (w.r.t the numbering above) of each of the set \( S f \) form a geodesic pair, hence for all \( u, v \in T_T' \), we have that \( ||h(u) - h(v)|| = d_G(u, v) \).

Claim 2: The set \( T \setminus T_T' \) can be partitioned into \( T_0, T_1, T_2, T_3, T_4, T_5 \) such that each of the \( T_i \)’s satisfies the condition of Theorem 8.

Proof: It is sufficient to show that the geodesic pairs incident on each face can be partitioned into \( T_i \) for \( i = 0, 1, 2, 3, 4, 5 \) such that each \( T_i \) satisfies the condition of Theorem 8. Let \( f \subset F \) and \( S_f^0, S_f^1, \ldots, S_f^5 \) be the partition of \( C(f) \) created in the first step. Suppose there exists \( (u, v) \in T \setminus T_T' \) such that \( u \in S_f^0, v \in S_f^{t+1} \) for some \( t \geq 3 \), where \( [p] = p \) if \( p \leq k \) and \( p \leq k \) otherwise. Then our procedure in the first step would not have created separate partitions for \( S_f^0 \) and \( S_f^{t+1} \) and hence such a geodesic pair \( (u, v) \) can’t exist. Therefore for all \( u, v \in C(f) \) and \( (u, v) \in T \setminus T_T' \), one of the following must hold: \( u \in S_f^0, v \in S_f^{t+1} \) (called type I) or
\[ u \in S_1^l, v \in S_2^{l+2} \] (called type 2) for some \( l \).

Let \((u,v)\) be a geodesic pair \((u,v)\) such that \( u \in S_1^l \) and \( v \in S_2^l \). We say that \((u,v)\) belongs to class \( i \) if \( \min(i,j) \equiv 3 = i \). Let \( T_0^l, T_1^l, T_2^l \) be the geodesic pairs of class 0,1,2 of type \( i \) incident on the face \( f \) and \( T_3^l, T_4^l, T_5^l \) be the geodesic pairs of class 0,1,2 of type 2 incident on the face \( f \).

We set \( T_i = \bigcup_{f \in F} T_i^f \) for \( i = 0,1,2,3,4,5 \). The geodesic pairs in each of the \( T_i \)'s satisfy the condition of Theorem 8 and this completes the proof of the claim. ■

VI. SINGLE SOURCE SHORTEST PATH EMBEDDINGS

We next show how to find an embedding such that the distance between all pairs of vertices whose shortest path uses a fixed vertex \( v \) is approximately preserved. To prove this result, we make use of a well known result of Klein, Plotkin and Rao [7] on small diameter decomposition. Let \((X,D)\) be a finite metric space. A distribution \( \mu \) over (vertex) partitions of \( X \) is called \((\beta,\Delta)\)-lipschitz if every partition \( P \) in the support of \( \mu \) satisfies \( S \in P \Rightarrow \text{diam}_X(S) \leq \Delta \) and moreover for all \( x,y \in X \), \( \sum_{P \in \mu} [P(x) \neq P(y)] \leq \beta \cdot \text{diam}_X(x,y) \). Klein, Plotkin and Rao [7] showed that there exists a \((c,\Delta)\)-lipschitz partition of a planar metric where \( \Delta > 0 \) is arbitrary and \( c \) is an absolute constant.

**Theorem 10 (Klein, Plotkin and Rao [7]):** Let \((X,D)\) be a planar metric and let \( \Delta \in \mathbb{R}_{\geq 0} \) be a given number. Then there exists a polynomial-time computable \((c,\Delta)\)-lipschitz partition of \((X,D)\), where \( c \) is some absolute constant.

**Theorem 11:** Let \( G = (V,E) \) be a planar graph with edge-length \( l : E \rightarrow \mathbb{R}_{\geq 0} \) and \( v \in V \). Let \( T = \{(s_i,t_i)\}_{i=1}^k \) be the set of pair of vertices such that \( d(s_i,t_i) = d(u,v) + d(v,t_i) \) for \( i = 1,2,\ldots,k \). Then there exists an embedding \( g : V \rightarrow l_1 \) and a constant \( \beta > 1 \) such that \( d(s_i,t_i)/\beta \leq ||g(s_i)-g(t_i)|| \leq d(s_i,t_i) \) for all \( i \). Furthermore such an embedding can be computed in polynomial time.

**Proof.** We first prove the theorem for the special case when \( d(v,s_i) = d(v,t_i) \) for all \( (s_i,t_i) \in T \). Let \( B_i = \{x|d(v,x) \leq 2^i+1\} \) for \( i \geq 0 \). Since \( d(v,s_i) = d(v,t_i) \) for each \( (s_i,t_i) \in T \), there exists an embedding \( g : V \rightarrow l_1 \) and a constant \( \beta > 1 \) such that \( d(s_i,t_i)/\beta \leq ||g(s_i)-g(t_i)|| \leq d(s_i,t_i) \) for all \( i \). Furthermore such an embedding can be computed in polynomial time.

**Claim 5:** Let \( C = \bigcup_{i \geq 2} C_i \). For any \((u,w)\in G_i\), we have \( \delta_C(u,w) \leq 3 \cdot d_G(u,w) \). Furthermore, for any \((s_j,t_j)\in T_i\), we have \( \delta_C(s_j,t_j) \geq \frac{d_G(s_j,t_j)}{16c} \). ■

**Proof.** By Theorem 10, if an edge has length \( 0 \) in \( G_i \), then it is not separated by any cut in \( C_i \). Furthermore, any edge \((u,w)\in E\) can be a part of at most three \( G_i \)'s. Hence, using Claim 4, we obtain \( \delta_C(u,w) \leq 3 \cdot d_G(u,w) \). By Claim 4, for any \((s_j,t_j)\in T_i\), \( \delta_C(s_j,t_j) \geq d_G(s_j,t_j)/4c \). Using Claim 3, we have \( \delta_C(s_j,t_j) \geq d_G(s_j,t_j)/4c \). Hence, \( \delta_C(s_j,t_j) \geq \frac{d_G(s_j,t_j)}{16c} \). ■
Claim 6: For any \((s_j, t_j) \in T, \; \delta_C(s_j, t_j) \geq \frac{d_G(s_j, t_j)}{16 \cdot c} - 3|d_G(v, s_j) - d_G(v, t_j)|).\)

Proof. Without loss of generality, we may assume that \(d_G(v, s_j) \geq d_G(v, t_j).\) Let \(s'_j \) be the vertex on a shortest path from \(v \) to \(s_j\) such that \(d_C(v, s'_j) = d_G(v, t_j).\) By Claim 5, we have \(\delta_C(s'_j, t_j) \geq \frac{d_G(s'_j, t_j)}{16c}.\) All the cuts in \(C\) which contain exactly one of \(s'_j\) and \(t_j\) contribute to \(\delta_C(s'_j, t_j)\) and they can be partitioned into two groups: one in which \(s'_j\) and \(s_j\) are in the same partition and the other in which \(s_j\) and \(t_j\) are in the same partition. Let’s call them \(\delta_C(s'_j, s_j, t_j)\) and \(\delta_C(s'_j, s_j, t_j)\) respectively. Observe that, \(\delta_C(s'_j, s_j, t_j) = \delta_C(s'_j, s_j, t_j) + \delta_C(s'_j, s_j, t_j) - \delta_C(s'_j, s_j, t_j) < \delta_C(s'_j, s_j, t_j).\) Hence \(\delta_C(s'_j, t_j) \geq \delta_C(s', s_j) - \delta_C(s_j, t_j).\)

By Claim 5, we have \(\delta_C(s'_j, t_j) \geq \frac{d_G(s'_j, s_j)}{16c} - 3|d_G(v, s_j) - d_G(v, t_j)|.\) All the cuts \(\delta_C(s'_j, t_j) \geq \frac{d_G(s'_j, t_j)}{16c} - 3|d_G(v, s_j) - d_G(v, t_j)|.\)

We augment \(C\) by the following single cuts: let \(V = \{v_1, v_2, \ldots, v_n\}\) such that \(d_G(v, i) \leq d_G(v, i) \leq \ldots \leq d_G(v, v_n).\) Let \(R_i = \{v_1, v_2, \ldots, v_i\}\) and \(w(R_i) = 3 \cdot (d_G(v, i+1) - d_G(v, i))\) for \(i = 1, 2, 3, \ldots, n.\) Let \(R = \{R_1, R_2, \ldots, R_n\}\) and \(C'' = C \cup R.\)

Claim 7: For any \((u, w) \in G,\) we have \(\delta_C(u, w) \leq 6 \cdot d_G(u, w).\) Furthermore, for any \((s_j, t_j) \in T,\) we have \(\delta_C(s_j, t_j) \geq \delta_C(s_j, t_j) / 16c.\)

Proof. Observe that \(\delta_R(u, w) = 3 \cdot (d_G(u, v) - d_G(v, v)) \leq 3 \cdot d_G(u, v) \leq 3 \cdot d_G(u, v) + 3 \cdot d_G(u, v) = 6 \cdot d_G(u, v).\) Using Claim 6, for any \((u, w) \in E,\) we have \(\delta_C(u, w) = \delta_C(u, v) + \delta_R(u, w) \leq 3 \cdot d_G(u, v) + 3 \cdot d_G(u, v) = 6 \cdot d_G(u, v).\) Using Claim 6, for any \((s_j, t_j) \in T\) we have \(\delta_C(s_j, t_j) = \delta_C(s_j, t_j) + \delta_R(s_j, t_j) \leq d_G(s_j, t_j) / 16c - 3|d_G(v, s_j) - d_G(v, t_j)| + 3|d_G(v, s_j) - d_G(v, t_j)| = d_G(s_j, t_j) / 16c.\)

Let \(C'' = C'' / 6.\) Then for any \((u, w) \in G,\) we have \(\delta_C(u, w) \leq d_G(u, w)\) and for any \((s_j, t_j) \in T,\) we have \(\delta_C(s_j, t_j) \geq d_G(s_j, t_j) / 96c.\) By using the equivalence between the cut-metric and \(L_1\)-embedding, we have the desired result.

VII. CONSTRAINED EMBEDDING

Let \(G = (V, E)\) be a planar graph and \(f\) be its infinite face. Let \(V(f) = \{v_1, v_2, \ldots, v_n\}\) be the vertices on the cycle of \(f\) in clockwise ordering. Suppose that we know the cut-metric \(C = \{(C_1, w_1), (C_2, w_2), \ldots, (C_m, w_m)\}\) w.r.t \(V(f)\) such that each cut \(C_i \subseteq V(f)\) corresponds to a contiguous subset of vertices on cycle \(f.\) We say that \(C\) is a cut-metric \(C = \{C_1, C_2, \ldots, C_m\}\) w.r.t \(V(f)\) such that each cut \(C_i \subseteq V(f)\) corresponds to a contiguous subset of vertices on cycle \(f.\) We say that \(C\) is a cut-metric \(C = \{C_1, C_2, \ldots, C_m\}\) w.r.t \(V(f)\) such that each cut \(C_i \subseteq V(f)\) corresponds to a contiguous subset of vertices on cycle \(f.\)

Theorem 12: Let \(G = (V, E)\) be a planar graph with edge-length \(l: E \to [0, \infty)\) and \(F\) be its set of faces. Furthermore, let \(S\) be an \(\alpha\)-loose cycle and \(f_2 \in F\) be the unique non-geodesic face contained inside \(R(S).\) Let \(G_1 = (V_1, E_1) \subseteq G\) be the graph induced by the vertices and edges in \(R(S)\) and \(F_1\) be the set of faces of \(G_1.\) Let \(C\) be a cut-metric \(C = \{C_1, C_2, \ldots, C_m\}\) w.r.t \(V(f)\) such that each cut \(C_i \subseteq V(f)\) corresponds to a contiguous subset of vertices on cycle \(f.\) We say that \(C\) is a cut-metric \(C = \{C_1, C_2, \ldots, C_m\}\) w.r.t \(V(f)\) such that each cut \(C_i \subseteq V(f)\) corresponds to a contiguous subset of vertices on cycle \(f.\)

4Note that \(S, f_2 \in F_1\) and \(R(S)\) is the closed region bounded by cycle \(S.\)

5\(\beta\) is the constant from Theorem 11.
Proof. As mentioned before, we abuse notation and let $S$ also denote the set of vertices and edges incident on the cycle $S$. Consider two copies of $G_1$, say $H_1 = (V_1,E_1)$ and $H_2 = (V_1,E_1)$ with length functions $l_1,l_2$ defined as follows: $l_1(u,v) := l(u,v)$ for all $(u,v) \in S$ and $(l(u,v)/\alpha)$ otherwise; $l_2(u,v) := 0$ for all $(u,v) \in S$ and $(l(u,v))$ otherwise. By definition of a $\alpha$-loose cycle, it immediately follows that for any $u,v \in S$, $d_{H_1}(u,v) \geq d_{G_1}(u,v)$. Lemma 3 shows that the cuts in $C$ can be extended to $H_1$ such that $\delta_C(u,v) \leq d_{H_1}(u,v)$ for each $(u,v) \in E_1$. Let this cut-metric be $C'$. For notational convenience, we create an (equivalent) $L_1$ embedding $h : V_1 \rightarrow L_1$ from $C'$ by forming a new boundary for each $C \in C'$ and setting $h(u) = 0$ if $u \in C$ and $w_C$ otherwise.

Consider the graph $H_2$ with metric $l_2$. Since the length of all the edges in $S$ have been set to zero in $l_2$, we may treat all the vertices on the cycle $S$ as a single node, say $v_S$. If $d_{H_2}(u,v) \neq d_{G_1}(u,v)$ for some $u,v \in V_1$, then it must be the case that $d_{H_2}(u,v) = d_{H_2}(u,v_S) + d_{H_2}(v_S,v)$. Let $T_1$ be the set of all pairs of vertices $(u,v)$ in $H_2$ such that the shortest path between $u$ and $v$ in $H_2$ uses the vertex $v_S$. We use Theorem 11 to find an embedding $g_1 : V_1 \rightarrow L_1$ such that $d_{H_2}(u,v) \beta \leq ||g_1(u) - g_1(v)|| \leq d_{H_2}(u,v)$ for all $(u,v) \in T_1$.

Let $T_2$ be the set of all geodesic pairs in $G_2$ (see Section V for the definition of geodesic pairs). In this case, we use Theorem 9 to find an embedding $g_2 : V_1 \rightarrow L_1$ such that $d_{H_2}(u,v) \beta \leq ||g_2(u) - g_2(v)|| \leq d_{H_2}(u,v)$ for all $(u,v) \in T_2$. Let $T_3$ be the set of all pairs of vertices $(u,v)$ such that $u,v \in f_2$ i.e. the set of all pairs of points on the non-geodesic face $f_2$. We use Theorem 3 to find an embedding $g_3 : V_1 \rightarrow L_1$ such that $||g_3(u) - g_3(v)|| = d_{H_2}(u,v)$ for all $(u,v) \in T_3$ and $||g_3(u) - g_3(v)|| \leq d_{H_2}(u,v)$ for all $(u,v) \in E_1$. Let $g = (g_1 + g_2 + g_3)/3$. Since $||g(u) - g(v)|| \leq d_{H_2}(u,v)$ for $i = 1, 2, 3$, we have $||g(u) - g(v)|| \leq d_{H_2}(u,v)$ for $(u,v) \in E_1$. Let $T_4$ be the set of all pair of vertices $u,v$ which lie on the same face, i.e. $T = \{(u,v) | u,v \in f \}$ for some $f \in F_1$. Since $f_2$ is the only non-geodesic face in $F_1 \setminus S$, for any $u,v \in T$, the shortest path between $u,v \in H_2$ either goes through $v_S$ or $(u,v) \in E_1$ is a geodesic pair or $u,v \in f_2$. Hence, $T \subseteq T_1 \cup T_2 \cup T_3$. The statement of the theorem follows by noting that $\beta = \max\{3,\beta \}$, where $\beta = 3\beta \geq 21$.

Claim 8: $||z(u) - z(v)|| \leq l(u,v)$ for $(u,v) \in E_1$.

Proof. Consider an edge $(u,v) \in S$. Since $l_1(u,v) = l(u,v)$ and $l_2(u,v) = 0$, we have $||z(u) - z(v)|| = |||h(u) - h(v)|| + \alpha \frac{1}{\alpha} \cdot ||g(u) - g(v)|| \leq l_1(u,v) + l_2(u,v) = l(u,v)$.

Now consider an edge $(u,v) \in E_1 \setminus S$. Since $l_1(u,v) = \frac{l(u,v)}{\alpha}$ and $l_2(u,v) = l(u,v)$, we have $||z(u) - z(v)|| = |||h(u) - h(v)|| + \alpha \frac{1}{\alpha} \cdot ||g(u) - g(v)|| \leq \frac{1}{\alpha} \cdot l(u,v) + \frac{1}{\alpha} \cdot l(u,v)$. 

Claim 9: $||z(u) - z(v)|| \geq d_{G_1}(u,v) / \alpha$ for $(u,v) \in T$.

Proof. Let $(u,v) \in T$. We consider two cases depending on whether the shortest $u,v$ path in $H_2$ uses the vertex $v_S$. If the shortest $u,v$ path doesn't use the vertex $v_S$, then we have $d_{G_1}(u,v) = d_{H_2}(u,v)$ and $||z(u) - z(v)|| \geq \frac{1}{\alpha}$. Similarly, if the shortest $u,v$ path uses the vertex $v_S$, then $d_{G_1}(u,v) / \alpha \geq d_{G_1}(u,v) / \alpha$. Therefore, $||z(u) - z(v)|| \geq \frac{1}{\alpha}$. 

Using the equivalence of cut-metric and $L_1$-embedding, we can construct a cut-metric $Z$ from $z : V_1 \rightarrow L_1$ satisfying the conditions of the theorem.

VIII. PUTTING EVERYTHING TOGETHER

Theorem 13: Let $G = (V,E)$ be a planar graph with length function $l : E \rightarrow \mathbb{R}_{>0}$. If its set of faces and $T = \{(u,v) | u,v \in f \in F \}$. Then there exists a $z : V \rightarrow L_1$ such that $||z(u) - z(v)|| \leq l(u,v)$ for $(u,v) \in E$ and $||z(u) - z(v)|| \geq d_{G_1}(u,v) / \alpha$ for $(u,v) \in T$, where $\beta = 144/\alpha$.

Proof. We prove the theorem by using induction on the number of vertices. We first compute a $\alpha$-good length function $l'$ by setting $\alpha = 12 \beta$ in Theorem 7. If there are no non-geodesic faces w.r.t $l'$, then we use Theorem 9 to get an $L_1$ embedding with distortion at most $21$. If there exists a non-geodesic face w.r.t $l'$, we first embed $G$ in the plane such that the infinite face is geodesic (such a face always exists). We then use the decomposition guaranteed by $\alpha$-good length function and find an innermost non-geodesic $\alpha$-loose face. Let $G_1 = (V_1,E_1)$ be the graph obtained by removing all the vertices in $I(S)$. Since $f$ is non-geodesic w.r.t $l'$, there exists a vertex in $I(S)$. Hence, the number of vertices in $G_1$ is strictly smaller than $G$ and we inductively compute an embedding $z_1 : V \rightarrow L_1$ satisfying the conditions of the theorem. Using the equivalence between $L_1$-embedding and cut-metric, we compute a cut-metric equivalent to $z_1$, say $Z_1$. Let $Z_1'$ be the cut-metric induced by $Z_1$ on the cycle $S_f$. We use Theorem 12 to compute a cut-metric which extends $Z_1'$ to vertices in $G \setminus G_1$, say $Z_f$. We obtain the final cut-metric by setting $Z = (Z_1 \cup Z_1')$ and $Z_2$. Let $z : V \rightarrow L_1$ be the equivalent embedding to $Z$. By induction hypothesis and statement of Theorem 12, $||z(u) - z(v)|| \leq l'(u,v)$ for $(u,v) \in E$ and $||z(u) - z(v)|| \geq d_{G_1}(u,v) / \alpha$ for $(u,v) \in T$, where $d_{G_1}$ is the shortest path metric on $G$ w.r.t $l'$. The statement of the theorem then follows by noting that $l'$ is constructed by reducing the length of edges w.r.t $l$ by a factor of at most $\alpha$.

IX. CONCLUSIONS

In this paper, we proved a O(1) flow-cut gap when $G$ is planar and both end points of every demand edge is incident on one of the faces. Although our result does not directly imply any bounds on the (half)-integral flow-cut gap, we believe that it should be possible to exploit the laminar structure of flows
in such instances to prove such a bound. Inductive arguments have been used successfully for proving better flow-cut gaps for planar instances, for example series-parallel graphs [3] and $k$-outer planar graphs [4]. We believe that the techniques developed in this paper could be useful for extending such an approach to a more general setting.

REFERENCES

[1] Jean Bourgain. On lipschitz embedding of finite metric spaces in hilbert space. Israel Journal of Mathematics, 52(1):46–52, 1985.

[2] Amit Chakrabarti, Lisa Fleischer, and Christophe Weibel. When the cut condition is enough: A complete characterization for multflow problems in series-parallel networks. In Proceedings of the forty-fourth annual ACM Symposium on Theory of Computing, pages 19–26. ACM, 2012.

[3] Amit Chakrabarti, Alexander Jaffe, James R Lee, and Justin Vincent. Embeddings of topological graphs: Lossy invariants, linearization, and 2-sums. In Foundations of Computer Science, 2008. FOCS’08. IEEE 49th Annual IEEE Symposium on, pages 761–770. IEEE, 2008.

[4] Chandra Chekuri, Anupam Gupta, Ilan Newman, Yuri Rabinovich, and Alistair Sinclair. Embedding $k$-outerplanar graphs into $l_1$. SIAM Journal on Discrete Mathematics, 20(1):119–136, 2006.

[5] Arnold Filtser. A face cover perspective to $l_1$ embeddings of planar graphs. In Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 1945–1954. SIAM, 2020.

[6] Anupam Gupta, Ilan Newman, Yuri Rabinovich, and Alistair Sinclair. Cuts, trees and $l_1$-embeddings of graphs. Combinatorica, 24(2):233–269, 2004.

[7] Philip Klein, Serge A Plotkin and Satish Rao. Excluded minors, network decomposition, and multicommodity flow. In Proceedings of the twenty-fifth annual ACM Symposium on Theory of Computing, pages 682–690, 1993.

[8] Robert Krauthgamer, James R Lee, and Havana Rika. Flow-cut gaps and face covers in planar graphs. In Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, pages 525–534. SIAM, 2019.

[9] Nikhil Kumar. Multicommodity flows in planar graphs with demands on faces. In 31st International Symposium on Algorithms and Computation (ISAAC 2020).

[10] Nathan Linial, Eran London, and Yuri Rabinovich. The geometry of graphs and some of its algorithmic applications. Combinatorica, 15(2):215–245, Jun 1995.

[11] Haruko Okamura and Paul D Seymour. Multicommodity flows in planar graphs. Journal of Combinatorial Theory, Series B, 31(1):75–81, 1981.

[12] Satish Rao. Small distortion and volume preserving embeddings for planar and euclidean metrics. In Proceedings of the fifteenth Annual Symposium on Computational Geometry, pages 300–306. ACM, 1999.

[13] Alexander Schrijver. Combinatorial optimization: polyhedra and efficiency, volume 24. Springer Science and Business Media, 2003.

[14] Paul D Seymour. On odd cuts and plane multicommodity flows. Proceedings of the London Mathematical Society, 3(1):178–192, 1981.

[15] David P Williamson and David B Shmoys. The design of approximation algorithms. Cambridge university press, 2011.