The time-dependent \( CP \) asymmetry in
\[ B^0 \rightarrow K_{\text{res}}\gamma \rightarrow \pi^+\pi^-K_S^0\gamma \] decays

\[ a,b S. Akar, b E. Ben-Haim, c J. Hebinger, c E. Kou and d F.-S. Yu \]

\[ a \text{University of Cincinnati, Cincinnati, OH, United States} \]
\[ b \text{LPNHE, Sorbonne Université, Paris Diderot Sorbonne Paris Cité, CNRS/IN2P3, Paris, France} \]
\[ c \text{LAL, Université Paris-Sud, CNRS/IN2P3, Université Paris-Saclay, Orsay, France} \]
\[ d \text{School of Nuclear Science and Technology, Lanzhou University, Lanzhou 730000, China} \]

Abstract

The time-dependent \( CP \) asymmetry in \( B^0 \rightarrow K_{\text{res}}\gamma \rightarrow \pi^+\pi^-K_S^0\gamma \) is sensitive to the photon polarisation in the quark level process \( b \rightarrow s\gamma \). While this polarisation is predominantly left-handed in the standard model, it could be modified by the existence of new physics contributions that may possess different \( CP \) properties. In this paper, we derive the \( CP \) violation formulae for \( B^0 \rightarrow K_{\text{res}}\gamma \rightarrow \pi^+\pi^-K_S^0\gamma \) including the most dominant intermediate states. We propose a new observable that could be measured in a time-dependent amplitude analysis of \( B^0 \rightarrow \pi^+\pi^-K_S^0\gamma \) decays, providing a stringent constraint on the photon polarisation. We discuss the future prospects for obtaining such constraints from measurements at Belle II and LHCb.
1 Introduction

The exclusive $b \to s \gamma$ process is one of the most sensitive observables to new physics in $B$ physics: unlike many other $b$-hadron decays, it is described in the standard model (SM) by a single operator, the electro-magnetic type $\bar{s}\sigma_{\mu\nu}(1 + \gamma_5)bF^{\mu\nu}$, which minimises the uncertainties from hadronic effects. In the era of the LHC and the upgraded $B$-factory experiment, Belle II, an interesting opportunity opens to investigate the circular-polarisation of the photon in $b \to s \gamma$ process and gain additional insight into its nature. In the standard model, the photon polarisation of $b \to s \gamma$ is predicted to be predominantly left-handed ($\gamma_L$) due to the operator mentioned above. Several new-physics models contain new particles that couple differently from the SM, inducing an opposite chirality operator $\bar{s}\sigma_{\mu\nu}(1 - \gamma_5)bF^{\mu\nu}$; these models predict an enhanced right-handed photon contribution ($\gamma_R$). Examples of such new physics models are given in Refs. [1–4]. The photon polarisation in $b \to s \gamma$ transitions is therefore a fundamental property of the SM, and its experimental determination may provide information on physics beyond the SM.

Photon polarisation measurement is a challenge in $B$ physics, and much effort has been put into it in recent years. Two types of methods to determine photon polarisation have been proposed and carried out: measuring the angular distribution of the recoil particles (see [5–12] for theoretical proposals and [13,14] for experimental results), and measuring the time-dependent $CP$ asymmetry (Refs. [15–18] and [19–22] for theory and experiment, respectively). In this article, we discuss the second method.

Obtaining information on photon polarisation from the time-dependent $CP$ asymmetry measurement is illustrated with the promising mode $B \to K_{\text{res}}\gamma \to (n\pi)K^0_S\gamma$, where $K_{\text{res}}$ is a kaonic resonance and $n\pi$ designates $n$ pions, and for which $(n\pi)K^0_S$ forms a $CP$ eigenstate. The illustration is depicted in Fig. 1. The time-dependent $CP$ asymmetry originates from the interference of the $B \to K_{\text{res}}\gamma \to (n\pi)K^0_S\gamma$ and $B \to \bar{K}_{\text{res}}\gamma \to (n\pi)K^0_S\gamma$ amplitudes, one of which emerges as a result of $B-\bar{B}$ oscillation. The key point is that interference occurs only when photons coming from $B$ and $\bar{B}$ amplitudes are circularly polarised in

$$B^0 \quad c'(c) \quad \frac{\bar{K}_{\text{res}}\gamma_{L(R)}}{K_{\text{res}}\gamma_{L(R)}} \quad \frac{B^0}{(t = 0)} \quad \frac{\bar{B}^0}{(t = 0)} \quad \frac{f+(t)}{f-(t)} \quad \frac{(n\pi)K^0_S\gamma_{L(R)}}{(n\pi)K^0_S\gamma_{L(R)}}$$

Figure 1: Schematic description of the time-dependent $CP$ asymmetry of $B \to K_{\text{res}}\gamma \to (n\pi)K^0_S\gamma$. The factors $f_{-}(t)$ and $f_{+}(t)$ are the time-dependent oscillation and non-oscillation probabilities, respectively, of a $B^0$ or a $\bar{B}^0$ meson. The $q$ and $p$ are the $B-\bar{B}$ oscillation parameters, which correspond to $q/p = (V_{tb}V_{td})/(V_{tb}V_{td}) = e^{-2i\beta}$ in the SM. The coefficients $c$ and $c'$ represent the ratio of the the standard operator contribution $\bar{s}\sigma_{\mu\nu}(1 + \gamma_5)bF^{\mu\nu}$ and that of the non-standard one $\bar{s}\sigma_{\mu\nu}(1 - \gamma_5)bF^{\mu\nu}$, respectively. In the SM, $c'/c = m_s/m_b (\simeq 0)$ leading to an expected $CP$ asymmetry to be almost zero.
the same direction. Let us define the rate of $\bar{B}$ ($B$) mesons decaying into left- (right-) handed photons to be $c$, and the rate of $\bar{B}$ ($B$) into right- (left-) handed photons to be $c'$. The former process is induced by the standard operator contribution $\bar{c}\sigma\mu\nu(1 + \gamma_5)bF^\mu\nu$, and the latter is induced by the non-standard operator $\bar{c}\sigma\mu\nu(1 - \gamma_5)bF^\mu\nu$. Knowing that in the SM $c'/c = m_s/m_b \simeq 0$, that is, the left- (right-) handed photon is nearly forbidden for a $B$ ($\bar{B}$) meson decay, the interference of $B$ and $\bar{B}$ is expected to be nearly zero. Therefore, observation of non-zero $CP$ asymmetry signals new physics. Once non-zero $CP$ asymmetry is observed, one can further determine the “photon polarisation”, by measuring the ratio of $c'/c$ using as input the oscillation parameters $q/p$. In the SM $q/p = (V_{tb}^*V_{td})/(V_{tb}V_{td}^*) = e^{-2\beta}$. In this article, the notation used for the weak $CP$ mixing phase is $\beta$ rather than its equivalent, $\phi_1$. Its numerical value is obtained from $\sin 2\beta$ measurements determined in other modes $^{32}$, such as $B^0 \to J/\psi K^0_S$.

The simplest decay mode to study in this regard is $B^0 \to K_s^0(892)\gamma \to K_s^0\pi^0\gamma$, for which the first measurements of the mixing induced $CP$ violation were reported by the BABAR $^{19}$ and Belle $^{20}$ experiments: $S_{K^0\pi^0\gamma} = -0.03 \pm 0.29 \pm 0.03$ and $S_{K^0\pi^0\gamma} = -0.32_{-0.33}^{+0.05}$, respectively. As these measurements are statistically limited, they can be significantly improved by the Belle II experiment, which plans to accumulate a data sample 50 times as large as those accumulated by the first-generation $B$ factories.

In this paper, we discuss two methods to obtain information on the photon polarisation via the measurement of the mixing induced $CP$ violation in the decay $B^0 \to K_{res}\gamma \to \rho^0 K^0_S\gamma, \pi^+\pi^- K^0_S\gamma$. The main difficulty comes from the fact that the final state $\pi^+\pi^- K^0_S$ can originate not only from the $CP$ eigenstate $\rho^0 K^0_S$ but also from other intermediate states. In order to disentangle these contributions, such as $K^{\ast\pm}\pi\pi$, a detailed amplitude analysis is required. Such an analysis has been pioneered by the Belle collaboration $^{21}$ and extended by the BABAR collaboration $^{23}$.

In this paper, we revisit the method to obtain the mixing-induced $CP$ asymmetry in $B^0 \to K_{res}\gamma \to \pi^+\pi^- K^0_S\gamma$ decays to gain more insight on the photon polarisation. One essential ingredient of the method is the $CP$-eigenvalue that intervenes in the measurement, considering the dominant intermediate decay modes

$$K_1(1270), \quad K_1(1400) \quad (J^P = 1^+),$$
$$K^*(1410), \quad K^*(1680) \quad (J^P = 1^-),$$
$$K_2^*(1430) \quad (J^P = 2^+),$$

for the kaonic resonances and

$$B^0 \to K_{res}\gamma \to (\rho^0 K^0_S)\gamma \to K^0_S(\pi^+\pi^-)\gamma,$$
$$B^0 \to K_{res}\gamma \to (K^*\pi^-)\gamma \to (K^0_S\pi^+)\pi^-\gamma,$$
$$B^0 \to K_{res}\gamma \to ((K\pi)_0^+\pi^-)\gamma \to (K^0_S\pi^+)\pi^-\gamma,$$

for the $K\pi$ or $\pi\pi$ intermediate states. The notation $(K\pi)_0$ designates the $K\pi$ $S$-wave. A derivation of this $CP$-sign has not been demonstrated before.
In Sec. 2 we introduce the time-dependent $CP$ asymmetry formulae for $B^0 \rightarrow K_{res} \gamma \rightarrow \pi^+ \pi^- K_S^0 \gamma$ decays. In Sec. 3 we derive the $CP$-sign for the decay amplitudes with different intermediate states, which is required in order to extract the $CP$ asymmetry. Using these results we derive the time-dependent $CP$ asymmetry expression for $B^0 \rightarrow K_{res} \gamma \rightarrow \pi^+ \pi^- K_S^0 \gamma$ and $B^0 \rightarrow \rho^0 K_S^0 \gamma$ decays in Sec. 4. In Sec. 5 we present two methods to obtain information on the photon polarisation. Finally, in Sec. 6 we discuss the future prospects for these measurements at Belle II, and we conclude in Sec. 7.

2 Time-dependent $CP$ asymmetry for $B^0 \rightarrow \pi^+ \pi^- K_S^0 \gamma$

In the limit where the rate $\Gamma_{\pi^+ \pi^- K_S^0 \gamma}(t)$ comes only from the amplitude for $B^0 \rightarrow \rho^0 K_S^0 \gamma$ decays, we define the time-dependent $CP$ asymmetry as

$$\frac{\Gamma_{\rho^0 K_S^0 \gamma}(t) - \Gamma_{\rho^0 K_S^0 \gamma}(t)}{\Gamma_{\rho^0 K_S^0 \gamma}(t) + \Gamma_{\rho^0 K_S^0 \gamma}(t)} \equiv S_{\rho^0 K_S^0 \gamma} \sin(\Delta mt) - C_{\rho^0 K_S^0 \gamma} \cos(\Delta mt),$$

(1)

where

$$S_{\rho^0 K_S^0 \gamma} = \frac{2 \text{Im} \left( \sum_{\lambda=L,R} [M^{* \rho^0 K_S^0}_\lambda M^{\rho^0 K_S^0}_\lambda] \right)}{\sum_{\lambda=L,R} \left[ |M^{\rho^0 K_S^0}_\lambda|^2 + |M^{\rho^0 K_S^0}_\lambda|^2 \right]},$$

(2)

$$C_{\rho^0 K_S^0 \gamma} = -\frac{\sum_{\lambda=L,R} \left[ |M^{\rho^0 K_S^0}_\lambda|^2 - |M^{\rho^0 K_S^0}_\lambda|^2 \right]}{\sum_{\lambda=L,R} \left[ |M^{\rho^0 K_S^0}_\lambda|^2 + |M^{\rho^0 K_S^0}_\lambda|^2 \right]},$$

(3)

with $M^{\rho^0 K_S^0}_\lambda$ and $M^{\rho^0 K_S^0}_\lambda$ correspond to the $B^0$ and $\bar{B}^0$ decay amplitudes, respectively, with the left- and right-handed photon polarisation, designated by $\lambda = L, R$. Integration over the phase space is implicit here. In this article we adopt the convention $|CP|B^0\rangle = +|B^0\rangle$. The mass eigenstates are defined as $|B_{1/2}\rangle = p|B^0\rangle \pm q|\bar{B}^0\rangle$ with

$$\frac{q}{p} = +\sqrt{\frac{M_{12}^* - \frac{i}{2} \Gamma_{12}^*}{M_{12} - \frac{i}{2} \Gamma_{12}}},$$

(4)

and the mass difference is taken such that

$$\Delta m = M_2 - M_1 = -2 \text{Re} \left( \frac{q}{p} (M_{12} - \frac{i}{2} \Gamma_{12}) \right).$$

(5)

From the expression for $S_{\rho^0 K_S^0 \gamma}$ in Eq. (2), the numerator is zero, and no mixing induced $CP$ violation is expected, unless the $B^0$ and $\bar{B}^0$ can both decay into final states.

---

1 This convention is equivalent to $C|B^0\rangle = -|\bar{B}^0\rangle$. 

3
with the same photon polarisation $\lambda$. If the interference is non-zero, mixing induced CP violation may have observable effects. Determining $S_{\rho K^0_S}$ requires the knowledge of the CP signs inherited by the decay of the kaonic resonances. These are derived in Sec. 3.

As discussed in Sec. 1, other intermediate states than $\rho^0 K^0_S$ are expected in $B^0 \to K_{\text{res}} \gamma \to \pi^+\pi^- K^0_S \gamma$ decays, and these need to be carefully separated. Including all contributions, the time-dependent CP asymmetry expression becomes

$$\frac{\Gamma_{\pi^+\pi^- K^0_S \gamma}(t) - \Gamma_{\pi^+\pi^- K^0_S \gamma}(0)}{\Gamma_{\pi^+\pi^- K^0_S \gamma}(0) + \Gamma_{\pi^+\pi^- K^0_S \gamma}(t)} = S_{\pi^+\pi^- K^0_S \gamma} \sin(\Delta mt) - C_{\pi^+\pi^- K^0_S \gamma} \cos(\Delta mt), \quad (6)$$

with

$$S_{\pi^+\pi^- K^0_S \gamma} = \frac{2\text{Im} \left( \frac{2}{p} \sum_{\lambda=L,R} [M_\lambda \overline{M}_\lambda] \right)}{\sum_{\lambda=L,R} \left[ |\overline{M}_\lambda|^2 + |M_\lambda|^2 \right]}, \quad (7)$$

$$C_{\pi^+\pi^- K^0_S \gamma} = \frac{-\sum_{\lambda=L,R} \left[ |\overline{M}_\lambda|^2 - |M_\lambda|^2 \right]}{\sum_{\lambda=L,R} \left[ |\overline{M}_\lambda|^2 + |M_\lambda|^2 \right]}, \quad (8)$$

where the integration over the phase space is implicit. The $B^0$ and $\bar{B}^0$ decay amplitudes, $M_\lambda$ and $\overline{M}_\lambda$, respectively, are now sums over the three considered intermediate states

$$M_\lambda = M_{\rho^0 K^0_S} + M_{\pi^+\pi^-} + M_{(K\pi)_0^\pm} \pi^\mp, \quad (9)$$

$$\overline{M}_\lambda = \overline{M}_{\rho^0 K^0_S} + \overline{M}_{\pi^+\pi^-} + \overline{M}_{(K\pi)_0^\pm} \pi^\mp. \quad (10)$$

### 3 Relations between amplitudes

In this section, we derive the CP sign, establishing relations among the four amplitudes $M_L$, $M_R$, $\overline{M}_L$ and $\overline{M}_R$. The decay amplitude $M_\lambda$ ($\overline{M}_\lambda$) can be written as the sum of products $M_\lambda = \sum_i A_i \times \overline{A}_i$ ($\overline{M}_\lambda = \sum_i \overline{A}_i \times A_i$), where $A_i$ ($\overline{A}_i$) is the decay amplitude of $B^0$ ($\bar{B}^0$) to $K_{\text{res}} \gamma$ ($\overline{K}_{\text{res}} \gamma$), and $A_i$ ($\overline{A}_i$) is the decay amplitude\footnote{We keep $\lambda$ for the strong decay here though, as discussed later in this section, the squared amplitude does not depend on $\lambda$.} of $K_{\text{res}}$ ($\overline{K}_{\text{res}}$) to the intermediate state $i$:

$$M_\lambda = M_{\rho^0 K^0_S} + M_{\pi^+\pi^-} + M_{(K\pi)_0^\pm} \pi^\mp = A_\lambda \times (A_{\lambda}^{\rho^0 K^0_S} + A_{\lambda}^{K^0_{\pi^+\pi^-}} + A_{\lambda}^{(K\pi)_0^\pm} \pi^\mp), \quad (11)$$

$$\overline{M}_\lambda = \overline{M}_{\rho^0 K^0_S} + \overline{M}_{\pi^+\pi^-} + \overline{M}_{(K\pi)_0^\pm} \pi^\mp = \overline{A}_\lambda \times (\overline{A}_{\lambda}^{\rho^0 K^0_S} + \overline{A}_{\lambda}^{K^0_{\pi^+\pi^-}} + \overline{A}_{\lambda}^{(K\pi)_0^\pm} \pi^\mp). \quad (12)$$

#### 3.1 $B \to K_{\text{res}} \gamma$ amplitudes

First, we consider the $B$ decay part. In the SM, the $B(\bar{B}) \to K_{\text{res}} (\overline{K}_{\text{res}}) \gamma$ transition comes from the penguin diagram with a top quark and a $W$ boson in the loop. These
interactions can be written by the matrix elements

\[
\begin{align*}
\overline{A}_R &= \langle \overline{K}_{\text{res}} | H^- | \overline{B} \rangle, \\
\overline{A}_L &= \langle \overline{K}_{\text{res}} | H^+ | \overline{B} \rangle, \\
A_R &= \langle K_{\text{res}} | H^+ | B \rangle, \\
A_L &= \langle K_{\text{res}} | H^- | B \rangle,
\end{align*}
\]

where the effective Hamiltonians, at leading order in QCD, are

\[
\begin{align*}
H^+ &= -\frac{G_F}{\sqrt{2}} \frac{e}{16\pi^2} V_{tb} V_{ts}^* m_b c [\bar{s}P^\mu (1 + \gamma_5) b F_{\mu\nu}], \\
H^- &= -\frac{G_F}{\sqrt{2}} \frac{e}{16\pi^2} V_{tb} V_{ts}^* m_b c' [\bar{s}P^\mu (1 - \gamma_5) b F_{\mu\nu}].
\end{align*}
\]

and

\[
c = -\frac{1}{2} F_2(m_t), \quad c' = -\frac{1}{2} F_2(m_t) \frac{m_s}{m_b},
\]

where \( F_2 \) is the Inami-Lim function that includes the top quark loop contribution \[24\]. The \( c' \) contribution, which is proportional to a small factor \( m_s/m_b \), is often neglected in the literature.\footnote{Apart from the term that is proportional to \( m_s \), the right handed contribution, \( c' \), also receives some small contributions from the charm quark loop (see Refs. \[25,30\] for more details).}

Including the one-loop QCD correction to this contribution, \( c \) becomes simply the Wilson coefficient \( C_{7\gamma}^{(0)\text{eff}} \). By including the right-handed contributions from new physics, the \( c' \) coefficient can be affected as mentioned in the introduction. Note that for \( b \rightarrow s\gamma \) transitions, the Hamiltonians are given by the Hermitian conjugate

\[
\begin{align*}
H^{+\dagger} &= -\frac{G_F}{\sqrt{2}} \frac{e}{16\pi^2} V_{tb}^* V_{ts} m_b c^* [\bar{b}P^\mu (1 - \gamma_5) s F_{\mu\nu}], \\
H^{-\dagger} &= -\frac{G_F}{\sqrt{2}} \frac{e}{16\pi^2} V_{tb}^* V_{ts} m_b c^* [\bar{b}P^\mu (1 + \gamma_5) s F_{\mu\nu}].
\end{align*}
\]

We can also find the relations among the \( B \) decay amplitudes \( A_{L,R} \) and \( \overline{A}_{L,R} \) by applying the parity (\( P \)) and charge conjugation (\( C \)) operators. We first consider the case where \( K_{\text{res}} \) is a \( J^P = 1^+ \) state. Inserting the unit matrix \( P^\dagger P \) yields

\[
\begin{align*}
\overline{A}_R &= \langle \overline{K}_{\text{res}} | \gamma_R | \overline{B} \rangle, \\
&= \eta_F^R (B) \eta_F^R (K_{\text{res}}) \eta_F^R (\gamma) (-1)^{j_0 - j_1 - j_2} \left( \frac{c'}{c} \right) \langle \overline{K}_{\text{res}} | H^+ | \overline{B} \rangle \\
&= + \left( \frac{c'}{c} \right) \overline{A}_L,
\end{align*}
\]

where \( j_i \) is the total spin of the initial \((i = 0)\) and the final \((i = 1, 2)\) particles. Here we used \( P\sigma^{\mu\nu} (1 - \gamma_5) b F_{\mu\nu} P^\dagger = + \sigma^{\mu\nu} (1 + \gamma_5) b F_{\mu\nu} \). Similarly the relation between \( A_{L,R} \) and \( \overline{A}_{L,R} \) can be obtained by applying a \( C \) transformation. Inserting the unit matrix \( C^\dagger C \) yields

\[
\begin{align*}
\overline{A}_R &= \langle \overline{K}_{\text{res}} | \gamma_R | C^\dagger C H^- C^\dagger C | \overline{B} \rangle, \\
&= \eta_F^C (B) \eta_F^C (K_{\text{res}}) \eta_F^C (\gamma) \left( \frac{c'}{c^*} \right) \langle K_{\text{res}} | -H^+ | B \rangle \\
&= + \left( \frac{c'}{c^*} \right) A_R,
\end{align*}
\]
where we used $C\sigma_{\mu\nu}(1-\gamma_5)bF[^{\nu\mu}C^1] = -\bar{b}\sigma_{\mu\nu}(1-\gamma_5)sF[^{\mu\nu}$.

The phrase convention of the $C$ transformation of $K_{\text{res}}$ is chosen to be $C|K_{\text{res}}⟩ = -|K_{\text{res}}⟩$ throughout this article.

In summary, for the amplitudes of $B$ mesons decaying into $J^P = 1^+$ kaonic states:

\[
\begin{align*}
\mathcal{A}_R &= + \left( \frac{c}{c} \right) \mathcal{A}_L, & A_R &= + \left( \frac{\gamma}{\gamma} \right) A_L, \\
\overline{\mathcal{A}}_R &= + \left( \frac{c}{c} \right) \overline{\mathcal{A}}_L, & \overline{A}_R &= + \left( \frac{\gamma}{\gamma} \right) \overline{A}_L,
\end{align*}
\]

and for $J^P = 1^-$ and $J^P = 2^+$ kaonic states:

\[
\begin{align*}
\mathcal{A}_R &= - \left( \frac{c}{c} \right) \mathcal{A}_L, & A_R &= - \left( \frac{\gamma}{\gamma} \right) A_L, \\
\overline{\mathcal{A}}_R &= + \left( \frac{c}{c} \right) \overline{\mathcal{A}}_L, & \overline{A}_R &= + \left( \frac{\gamma}{\gamma} \right) \overline{A}_L.
\end{align*}
\]

### 3.2 $K_{\text{res}} \rightarrow \pi^+\pi^- K^0_S$ amplitudes

First, we find a relation between $\mathcal{A}_\lambda^i$ and $\overline{\mathcal{A}}^i$, which are related by the $C$ transformation. The amplitude $\overline{\mathcal{A}}^i_\lambda$ corresponds to the same point in phase space as $\mathcal{A}_\lambda^i$, as we are interested in interference between the two. To start, the amplitude of the decay $K_{\text{res}} \rightarrow \rho^0 K^0_S$ can be written in terms of the product of the matrix elements:

\[
\begin{align*}
\mathcal{A}_\lambda^{\rho^0 K^0_S} &= \langle \pi^+(p_1)\pi^- (p_2) |\mathcal{H}_s |\rho^0 \rangle \langle \rho^0 K^0_S(p_3) |\mathcal{H}_s | K_{\text{res}}⟩, \\
\overline{\mathcal{A}}_\lambda^{\rho^0 K^0_S} &= \langle \pi^+(p_1)\pi^- (p_2) |\mathcal{H}_s |\rho^0 \rangle \langle \rho^0 K^0_S(p_3) |\mathcal{H}_s | K_{\text{res}}⟩,
\end{align*}
\]

where $\mathcal{H}_s$ and $\mathcal{H}_s'$ are the Hamiltonians describing the two corresponding strong decays. Applying a $C$ transformation gives

\[
\begin{align*}
\mathcal{A}_\lambda^{\rho^0 K^0_S}(p_1, p_2, p_3) &= \langle \pi^+(p_1)\pi^- (p_2) |\mathcal{C}^1 \mathcal{C} \mathcal{H}_s' \mathcal{C}^1 |\rho^0 \rangle \langle \rho^0 K^0_S(p_3) |\mathcal{C}^1 \mathcal{C} \mathcal{H}_s' \mathcal{C}^1 | K_{\text{res}}⟩ \\
&= \langle \pi^-(p_1)\pi^+(p_2) |\mathcal{H}_s' |\rho^0 \rangle \langle \rho^0 K^0_S(p_3) |\mathcal{H}_s | K_{\text{res}}⟩ \\
&= -\langle \pi^-(p_2)\pi^+(p_1) |\mathcal{H}_s' |\rho^0 \rangle \langle \rho^0 K^0_S(p_3) |\mathcal{H}_s | K_{\text{res}}⟩,
\end{align*}
\]

where $\mathcal{H}_s$ and $\mathcal{H}_s'$ are invariant under charge conjugation. In this development we used $C|\rho^0⟩ = -|\rho^0⟩$, $C|K^0_S⟩ = -|K^0_S⟩$, related to the approximation $CP|K^0_S⟩ = |K^0_S⟩$, and $C|K_{\text{res}}⟩ = -|K_{\text{res}}⟩$, according to the convention given above. In the second line of Eq. (26), the $C$ transformation swaps the $\pi^+$ and $\pi^-$ momenta. Since $\rho^0 \rightarrow \pi^+\pi^-$ is a $P$-wave decay, interchanging $p_1$ and $p_2$ leads to an overall minus sign in the third line of Eq. (26). Thus, writing explicitly the momentum assignment of $\pi^+\pi^- K^0_S$ we obtain

\[
\begin{align*}
\mathcal{A}_\lambda^{\rho^0 K^0_S}(p_1, p_2, p_3) &= \overline{\mathcal{A}}_\lambda^{\rho^0 K^0_S}(p_2, p_1, p_3) \\
&= -\overline{\mathcal{A}}_\lambda^{\rho^0 K^0_S}(p_1, p_2, p_3).
\end{align*}
\]

The amplitude describing the $K_{\text{res}} \rightarrow K^*\pi$ process can be written as

\[
\begin{align*}
\mathcal{A}_\lambda^{K^*\pi^-} &= \langle K^0_S(p_3)\pi^+(p_1) |\mathcal{H}_s' |K^*⟩ \langle K^*\pi^- (p_2) |\mathcal{H}_s | K_{\text{res}}⟩, \\
\overline{\mathcal{A}}_\lambda^{K^*\pi^+} &= \langle K^0_S(p_3)\pi^- (p_2) |\mathcal{H}_s' |K^*⟩ \langle K^*\pi^+ (p_1) |\mathcal{H}_s | K_{\text{res}}⟩,
\end{align*}
\]

6
and applying the $C$ transformation results in
\[
A_{\Lambda}^{K^*\pi^-}(p_1, p_2, p_3) = \langle K_0^0(p_3)\pi^+(p_1)|C^\dagger C\mathcal{H}'\mathcal{C}^\dagger C|K^*\pi^- (p_2)\rangle \langle K^*\pi^- (p_2)\mathcal{C}\mathcal{H}_s\mathcal{C}^\dagger C|K_{\text{res}}\rangle \\
= \langle K_0^0(p_3)\pi^- (p_1)|\mathcal{H}_s'K^*\pi^- (p_2)\rangle \langle K^*\pi^- (p_2)|\mathcal{H}_s|K_{\text{res}}\rangle \\
= \overline{A}_{\Lambda}^{K^*\pi^-}(p_2, p_1, p_3),
\]
where the momentum assignment is explicitly written for clarity. This reflects the general strong-interaction dynamics, where no simple relation allows to interchange $p_1$ and $p_2$ when the $\pi^-$ and $\pi^-$ are swapped. Thus, the relation between $A_{\Lambda}^{K^*\pi^-}(p_1, p_2, p_3)$ and $\overline{A}_{\Lambda}^{K^*\pi^-}(p_1, p_2, p_3)$ is unknown, except for the trivial case where $p_1 = p_2$. A similar conclusion applies when considering the process $K_{\text{res}} \rightarrow (K\pi)_0\pi^\mp$.

It is possible to obtain relations between $A_{\Lambda}^i$ and $A_{\Lambda}^j$ and between $\overline{A}_{\Lambda}^i$ and $\overline{A}_{\Lambda}^j$. Indeed, since the decay of the kaonic resonances only depend on the strong interaction, the former and the latter relations are expected to be the same. Furthermore, since decays with left- and right-handed photons do not interfere, only relations between products of amplitudes with the same photon polarisation are needed. The explicit computation for three kaonic resonances shows that, after integrating over the decay angles, these products do not depend on the left or right polarisation of the resonances:
\[
A_{\Lambda}^iA_{\Lambda}^j = A_{\Lambda}^iA_{\Lambda}^j, \quad \overline{A}_{\Lambda}^i\overline{A}_{\Lambda}^j = \overline{A}_{\Lambda}^i\overline{A}_{\Lambda}^j,
\]
where $i, j = \rho^0 K_0^0, K^{*+/-}, (K\pi)_0^{+/-}$.

## 4 Expression of the time-dependent $CP$ asymmetry

The expressions of the mixing induced $CP$ violation parameters, $S_{\rho^0 K_0^0\gamma}$ and $S_{\pi^+\pi^- K_0^0\gamma}$, given in Eqs. (2) and (7), respectively, can be rewritten using the relations between the amplitudes describing $B^0 \rightarrow K_{\text{res}}\gamma \rightarrow \pi^+\pi^- K_0^0\gamma$ decays obtained in Sec. 3. The aim is to express $S_{\rho^0 K_0^0\gamma}$ and $S_{\pi^+\pi^- K_0^0\gamma}$ in terms of amplitudes corresponding to a single $B$-flavour (choosing $B^0$) and a single polarisation (choosing $\lambda = L$).

The squared $B^0$ and $\overline{B}^0$ amplitudes, $|M_\lambda|^2$ and $|\overline{M}_\lambda|^2$, respectively, are written as

\footnote{We use the shortened notation here but $|M_\lambda|^2$ and $|\overline{M}_\lambda|^2$ represent the amplitudes for given momenta, i.e. $|M_\lambda(p_1, p_2, p_3)|^2$ and $|\overline{M}_\lambda(p_1, p_2, p_3)|^2$, respectively.}
\[|M_\lambda|^2 = |M^{\rho K}\lambda|^2 + |M^{K\pi}\lambda|^2 + |M^{(K\pi)0}\lambda|^2 + 2\text{Re} \left( M^{\rho K}\lambda M^{K\pi}\lambda^* \right) + 2\text{Re} \left( M^{\rho K}\lambda M^{(K\pi)0}\lambda^* \right), \tag{31}\]

\[|\overline{M}_\lambda|^2 = |\overline{M}^{\rho K}\lambda|^2 + |\overline{M}^{K\pi}\lambda|^2 + |\overline{M}^{(K\pi)0}\lambda|^2 + 2\text{Re} \left( \overline{M}^{\rho K}\lambda \overline{M}^{K\pi}\lambda^* \right) + 2\text{Re} \left( \overline{M}^{\rho K}\lambda \overline{M}^{(K\pi)0}\lambda^* \right), \tag{32}\]

and the cross term by
\[M^{\rho K}_\lambda \overline{M}_\lambda = M^{\rho K}_\lambda \overline{M}^{\rho K}_\lambda + M^{K\pi}_\lambda \overline{M}^{K\pi}_\lambda + M^{(K\pi)0}_\lambda \overline{M}^{(K\pi)0}_\lambda + \left[ M^{\rho K}_\lambda \overline{M}^{K\pi}_\lambda + M^{K\pi}_\lambda \overline{M}^{\rho K}_\lambda \right] + \left[ M^{\rho K}_\lambda \overline{M}^{(K\pi)0}_\lambda + M^{(K\pi)0}_\lambda \overline{M}^{\rho K}_\lambda \right] + \left[ M^{K\pi}_\lambda \overline{M}^{(K\pi)0}_\lambda + M^{(K\pi)0}_\lambda \overline{M}^{K\pi}_\lambda \right]. \tag{33}\]

The next step consists in replacing, in Eqs. (32) and (33), the \(B^0\) decay amplitudes by those corresponding to the \(B^0\) decay. This is done by using the results obtained in Sec. 3 for the \(CP\) signs of the \(B\) decay part, in Eqs. (22) and (23), and the \(K_{\text{res}}\) decay part, in Eqs. (27) and (29). For \(\lambda = L\) we obtain
\[|\overline{M}_L(p_1, p_2, p_3)|^2 = \left| c \overline{c} \right|^2 \left[ |M_L^{\rho K}(p_2, p_1, p_3)|^2 + |M_L^{K\pi}(p_2, p_1, p_3)|^2 + |M_L^{(K\pi)0}(p_2, p_1, p_3)|^2 \right. + 2\text{Re} \left( M_L^{\rho K}(p_2, p_1, p_3) M_L^{K\pi}(p_2, p_1, p_3) \right) \]
\[\left. + 2\text{Re} \left( M_L^{\rho K}(p_2, p_1, p_3) M_L^{(K\pi)0}(p_2, p_1, p_3) \right) \right], \tag{34}\]

and
\[M^*_L(p_1, p_2, p_3) \overline{M}_L(p_1, p_2, p_3) = \left( \frac{c}{c^*} \right) M^*_L(p_1, p_2, p_3) M_L(p_2, p_1, p_3). \tag{35}\]

Furthermore, we express all the amplitudes in terms of one polarization, choosing \(\lambda = L\). To do so, we obtain relations between left and right amplitudes using Eq. (30), together with Eqs. (22) and (23)
\[M^*_R(p_1, p_2, p_3) \overline{M}_R(p_1, p_2, p_3) = M^*_L(p_1, p_2, p_3) \overline{M}_L(p_1, p_2, p_3), \]
\[|M_R(p_1, p_2, p_3)|^2 = \left| \frac{c}{c^*} \right|^2 |M_L(p_1, p_2, p_3)|^2, \]
\[|\overline{M}_R(p_1, p_2, p_3)|^2 = \left| \frac{c^*}{c} \right|^2 |\overline{M}_L(p_1, p_2, p_3)|^2. \tag{36}\]
Using these relations, Eq. (7) can be re-written as

\[
\mathcal{S}_{\pi^+\pi^-K_0^{0}\gamma} = 4\text{Im} \left( \frac{q}{p} \frac{c c'}{|c|^2 + |c'|^2} \right) \left( \frac{\int \sum_{i,j} [M_{L}^i(p_1,p_2,p_3)M_{L}^j(p_2,p_1,p_3)] dp}{\int \sum_{i,j} [M_{L}^i(p_1,p_2,p_3)M_{L}^j(p_2,p_1,p_3) + M_{L}^i(p_2,p_1,p_3)M_{L}^j(p_2,p_1,p_3)] dp} \right)
\]

\[
= 4\text{Im} \left( \frac{q}{p} \frac{c c'}{|c|^2 + |c'|^2} \right) \left( \frac{\int \sum_{i,j} [A^{*i}(p_1,p_2,p_3)A^{j}(p_2,p_1,p_3)] dp}{\int \sum_{i,j} [A^{*i}(p_1,p_2,p_3)A^{j}(p_1,p_2,p_3) + A^{*i}(p_2,p_1,p_3)A^{j}(p_2,p_1,p_3)] dp} \right),
\]

where \(i, j\) run over \(\rho^0K_0^{0}, K^*\pi^-, (K\pi)_0^{+}\pi^-\). Here we also used the fact that the weak decay part of the total amplitude can be factored out, as \(M_{L}^i(p_1,p_2,p_3) = A_{L}A^{*i}(p_1,p_2,p_3)\), and thus cancels in the ratio. The notation \(A^{*i}(p_1,p_2,p_3)\) corresponds to the strong part of the amplitude averaged over the \(K_{\text{res}}\) helicity states. Similarly, Eq. (8) can be re-written as

\[
C_{\pi^+\pi^-K_0^{0}\gamma} = \frac{\int \sum_{i} \left[ |A^{i}(p_1,p_2,p_3)|^2 - |A^{i}(p_2,p_1,p_3)|^2 \right] dp}{\int \sum_{i} \left[ |A^{i}(p_1,p_2,p_3)|^2 + |A^{i}(p_2,p_1,p_3)|^2 \right] dp}.\]

Expanding the sum over the hadronic amplitudes in the expression of \(\mathcal{S}_{\pi^+\pi^-K_0^{0}\gamma}\) in Eq. (37).
we obtain

\[ \sum_{i,j} [A^i(p_1, p_2, p_3)A^j(p_1, p_2, p_3)] = A^{\rho K^0_S}(p_1, p_2, p_3)A^{\rho K^0_S}(p_2, p_1, p_3) \]

\[ + A^{*K^+\pi^-}(p_1, p_2, p_3)A^{K^+\pi^-}(p_2, p_1, p_3) \]

\[ + A^{(K\pi)_0^+}(p_1, p_2, p_3)A^{(K\pi)_0^+}(p_2, p_1, p_3) \]

\[ + A^{*K^+\pi^-}(p_1, p_2, p_3)A^{*K^0_S}(p_2, p_1, p_3) \]

\[ + A^{*K^+\pi^-}(p_1, p_2, p_3)A^{(K\pi)_0^+}(p_2, p_1, p_3) \]

\[ + A^{(K\pi)_0^+}(p_1, p_2, p_3)A^{(K\pi)_0^+}(p_2, p_1, p_3) \]

\[ + A^{*K^+\pi^-}(p_1, p_2, p_3)A^{*K^0_S}(p_2, p_1, p_3) \]

\[ + A^{*K^+\pi^-}(p_1, p_2, p_3)A^{(K\pi)_0^+}(p_2, p_1, p_3) \]

\[ + A^{(K\pi)_0^+}(p_1, p_2, p_3)A^{(K\pi)_0^+}(p_2, p_1, p_3) \]  

\[ = - \left| A^{\rho K^0_S}(p_1, p_2, p_3) \right|^2 \]

\[ + A^{*K^+\pi^-}(p_1, p_2, p_3)A^{K^+\pi^-}(p_2, p_1, p_3) \]

\[ + A^{(K\pi)_0^+}(p_1, p_2, p_3)A^{(K\pi)_0^+}(p_2, p_1, p_3) \]

\[ - 2\text{Re} \left( A^{*\rho K^0_S}(p_1, p_2, p_3)A^{K^+\pi^-}(p_1, p_2, p_3) \right) \]

\[ - 2\text{Re} \left( A^{*\rho K^0_S}(p_1, p_2, p_3)A^{(K\pi)_0^+}(p_1, p_2, p_3) \right) \]

\[ + A^{*K^+\pi^-}(p_1, p_2, p_3)A^{(K\pi)_0^+}(p_2, p_1, p_3) \]

\[ + A^{(K\pi)_0^+}(p_1, p_2, p_3)A^{K^+\pi^-}(p_2, p_1, p_3) \]

and

\[ \sum_{i,j} [A^i(p_1, p_2, p_3)A^j(p_1, p_2, p_3)] = \left| A^{\rho K^0_S}(p_1, p_2, p_3) \right|^2 \]

\[ + \left| A^{K^+\pi^-}(p_1, p_2, p_3) \right|^2 \]

\[ + \left| A^{(K\pi)_0^+}(p_1, p_2, p_3) \right|^2 \]

\[ + 2\text{Re} \left( A^{*\rho K^0_S}(p_1, p_2, p_3)A^{K^+\pi^-}(p_1, p_2, p_3) \right) \]

\[ + 2\text{Re} \left( A^{*\rho K^0_S}(p_1, p_2, p_3)A^{(K\pi)_0^+}(p_1, p_2, p_3) \right) \]

\[ + 2\text{Re} \left( A^{K^+\pi^-}(p_1, p_2, p_3)A^{(K\pi)_0^+}(p_1, p_2, p_3) \right) \]

The minus signs appearing in Eq. (40) originate from the relation in Eq. (27). On the other hand, as shown in Sec. 3 there is no general symmetry relation between \(A^{K^+\pi^-}(p_1, p_2, p_3)\) and \(A^{K^+\pi^-}(p_2, p_1, p_3)\) and similarly for \((K\pi)_0\). Thus the second, third, sixth and seventh terms in Eq. (40) cannot be further simplified.

From Eqs. (40) and (41), it follows that in a time-dependent amplitude analysis, the \(CP\) asymmetry measurement can be directly related to the photon polarisation for the \(\rho^0 K^0_S\) amplitude as

\[ S^{\rho K^0_S} = -\frac{2\text{Im} \left( \frac{q_{cc'}}{p} \right)}{|c|^2 + |c'|^2} \]  

(42)

10
We note that $S_{\rho^0 K_S^0 \gamma} = -S_{\pi^+ K_S^0 \gamma}$ (see Appendix A), and emphasise once more that the expression of $S_{\pi^+ \pi^- K_S^0 \gamma}$ obtained in Eq. (37) is independent of the intermediate kaonic resonance. Thus, Eq. (37) can be used by experimental studies in two ways: i) if the different kaonic resonances can be distinguished experimentally, the amplitudes in Eq. (37) can be considered as those of a given kaonic resonance decaying into the corresponding isobars; ii) if the kaonic resonances are not distinguished experimentally, then the amplitudes can be considered as sums over all the kaonic resonances decaying into the corresponding isobars.

5 Proposed experimental strategies

In this section two methods are described to obtain information on photon polarisation from mixing-induced CP violation parameter measurements. The first, suitable in the context of a limited-size data samples, like those used by the BABAR and Belle collaborations, is described in Sec. 5.1. Then, in Sec. 5.2 we propose a novel method that is better suited to larger data-sample, as expected in the Belle II experiment.

5.1 Phase-space integrated analysis

Time-dependent amplitude analyses to extract the CP asymmetries for individual resonances is currently not feasible at the B factories due to the limited sizes of the available data samples. Indeed, only the CP asymmetry of the full $K\pi\pi$ system is measured \cite{21,22}. After integration over the total Dalitz plane of the $K\pi\pi$ system, the expressions of the time-dependent CP asymmetry parameters can be simplified as

\begin{align}
S_{\pi^+ \pi^- K_S^0 \gamma} &= \frac{2\text{Im} \left( \frac{q}{p} \epsilon e' \right) \int_{\text{tot}} \text{Re} \left( \mathcal{A}^* (p_1, p_2, p_3) \mathcal{A} (p_2, p_1, p_3) \right) dp}{|c|^2 + |c'|^2 \int_{\text{tot}} |\mathcal{A} (p_1, p_2, p_3)|^2 dp}, \quad (43) \\
C_{\pi^+ \pi^- K_S^0 \gamma} &= 0. \quad (44)
\end{align}

As shown in Fig. 2 and Fig. 3, the imaginary part of $\mathcal{A}^* (p_1, p_2, p_3) \mathcal{A} (p_2, p_1, p_3)$ cancels after integration over the whole Dalitz plane and thus does not appear in the numerator of Eq. (43). Furthermore, the factor 2 in the numerator of Eq. (43) originates from the fact that $\int_{\text{tot}} |\mathcal{A} (p_1, p_2, p_3)|^2 dp = \int_{\text{tot}} |\mathcal{A} (p_2, p_1, p_3)|^2 dp$. Then, the CP asymmetry of the $\rho^0 K_S^0$ mode, given in Eq. (42), is obtained via the dilution factor

\begin{align}
D \equiv \frac{S_{\pi^+ \pi^- K_S^0 \gamma}}{S_{\rho^0 K_S^0 \gamma}} &= \frac{\int_{\text{tot}} \text{Re} \left( \mathcal{A}^* (p_1, p_2, p_3) \mathcal{A} (p_2, p_1, p_3) \right) dp}{\int_{\text{tot}} |\mathcal{A} (p_1, p_2, p_3)|^2 dp}, \quad (45)
\end{align}

11
where

\[
\text{Re} \left( A^*(p_1, p_2, p_3) A(p_2, p_1, p_3) \right) = - \left| A^{\rho K^0_0}(p_1, p_2, p_3) \right|^2 (46)
\]

\[
-2\text{Re} \left( A^{\rho K^0_0}(p_1, p_2, p_3) A^{K^+\pi^-}(p_1, p_2, p_3) \right)
\]

\[
-2\text{Re} \left( A^{\rho K^0_0}(p_1, p_2, p_3) A^{(K\pi)_0^+(\pi^-}(p_1, p_2, p_3) \right)
\]

\[
+\text{Re} \left( A^{K^+\pi^-}(p_1, p_2, p_3) A^{K^+\pi^-}(p_2, p_1, p_3) \right)
\]

\[
+\text{Re} \left( A^{(K\pi)_0^+(\pi^-}(p_1, p_2, p_3) A^{(K\pi)_0^+\pi^-}(p_2, p_1, p_3) \right)
\]

\[
+\text{Re} \left( A^{(K\pi)_0^+\pi^-}(p_1, p_2, p_3) A^{K^+\pi^-}(p_2, p_1, p_3) \right)
\]

\[
|A(p_1, p_2, p_3)|^2 = \left| A^{\rho K^0_0}(p_1, p_2, p_3) \right|^2 (47)
\]

\[
+ \left| A^{K^+\pi^-}(p_1, p_2, p_3) \right|^2
\]

\[
+ \left| A^{(K\pi)_0^+\pi^-}(p_1, p_2, p_3) \right|^2
\]

\[
+2\text{Re} \left( A^{\rho K^0_0}(p_1, p_2, p_3) A^{K^+\pi^-}(p_1, p_2, p_3) \right)
\]

\[
+2\text{Re} \left( A^{\rho K^0_0}(p_1, p_2, p_3) A^{(K\pi)_0^+\pi^-}(p_1, p_2, p_3) \right)
\]

Note that the last term in Eq. (41), which represents the interference between the \(K^+\pi^-\) and \((K\pi)_0^+\pi^-\) amplitudes, does not appear anymore in Eq. (47). This is due to the fact that this term cancels after integration over the polar angles, as the interference between \(P\)- and \(S\)-waves are, respectively, a linear combination of odd and even order Legendre polynomials.

The measurement of the dilution factor of Eq. (45) can be performed via a time-integrated analysis. As proposed in Refs. [21, 22], in order to get the best sensitivity for \(D\), its value can be measured from an amplitude analysis of \(B^+ \rightarrow K^+\text{res}\gamma \rightarrow K^+\pi^-\pi^+\gamma\) decays\(^5\) assuming isospin symmetry. Indeed, a larger data sample is expected for the final state \(K^+\pi^-\pi^+\gamma\) comparing to the neutral isospin partner \(K^0\pi^-\pi^+\gamma\), due to a larger branching fraction, as well as better experimental reconstruction and selection efficiencies.

As mentioned in Sec. 4, the expression of \(S_{\pi^+\pi^-K^0_2\gamma}\), and hence of the dilution factor, are valid across the whole \(K\pi\pi\) phase space and can be integrated. However, the different amplitudes vary across the phase space. For instance, the \(K_1(1270)\) has a larger branching fraction to \(\rho^0 K^0_0\) than higher-spin resonances. Thus, an optimised set of cuts in the \(m_{K\pi\pi}\) spectrum needs to be considered when measuring the dilution factor.

We emphasise that the measurement of the dilution factor \(D\), which does not require the study of \(CP\) asymmetries but only that of the intermediate resonance amplitudes, can

\(^5\)Charge conjugation is implicit throughout the document.
be obtained independently, for instance from the LHCb experiment, benefiting from a larger data sample of $B^+ \rightarrow K^+\pi^−\pi^+\gamma$ decays comparing to the $B$ factories.

### 5.2 Time-dependent amplitude analysis

Considering that a larger data sample is available, as that expected in Belle II, we assume that a time-dependent amplitude analysis of $B^0 \rightarrow K_{res}\gamma \rightarrow \pi^+\pi^−K^0\gamma$ decays becomes feasible. In this section, we show that considering different regions of the $K\pi\pi$ Dalitz plane separately provides more information that significantly improves the sensitivity to new-physics contributions to the photon polarisation.

The expression of $S_{\pi^+\pi^-K^0\gamma}$ can be re-written with the integration being performed over a small region in the Dalitz plane, $\delta p$, such as

$$S_{\pi^+\pi^-K^0\gamma}^{\delta p} = 4\text{Im} \left( \frac{q}{p} \frac{\xi}{1 + |\xi|^2} \frac{\int_{\delta p} A^*_{123}A_{213} dp}{|A_{123}|^2 + |A_{213}|^2 dp} \right), \quad (48)$$

using, for simplicity, the conventions

$$A^*_{123}A_{213} = \sum_{i,j} \left[ A^*_{i} (p_1,p_2,p_3) A^j (p_2,p_1,p_3) \right],$$

$$|A_{123}|^2 + |A_{213}|^2 = \sum_{i,j} \left[ A^*_{i} (p_1,p_2,p_3) A^j (p_1,p_2,p_3) + A^*_{i} (p_2,p_1,p_3) A^j (p_2,p_1,p_3) \right],$$

$$\frac{\xi}{1 + |\xi|^2} = \frac{cc'}{|c|^2 + |c'|^2},$$

where we introduced the notation $\xi \equiv c'/c$ as the ratio of right- to left-handed amplitudes. Writing the hadronic decay amplitude as

$$A_{123} = |A_{123}| e^{i\delta_{123}}, \quad A_{213} = |A_{213}| e^{i\delta_{213}}, \quad (49)$$

the real and imaginary parts of the hadronic contribution in Eq. [48] can be expressed as

$$\int_{\delta p} A^*_{123}A_{213} dp \left/ \int_{\delta p} |A_{123}|^2 + |A_{213}|^2 dp \right| \equiv a^{\delta p}$$

$$\int_{\delta p} |A_{123}| |A_{213}| dp \int_{\delta p} \cos(\delta_{213} - \delta_{123}) dp \equiv a^{\delta p}$$

$$\int_{\delta p} |A_{123}| |A_{213}| dp \int_{\delta p} \sin(\delta_{213} - \delta_{123}) dp \equiv b^{\delta p}$$

$$\int_{\delta p} |A_{123}| |A_{213}| dp \int_{\delta p} \sin(\delta_{213} - \delta_{123}) dp \equiv b^{\delta p}, \quad (50)$$

13
so that Eq. (48) can be re-written as

\[ S^\delta p_{\pi^+\pi^-K_{S}^{0}\gamma} = 4\text{Im} \left( \frac{\xi}{p \cdot \rho + |\xi|^2} \right) a^\delta p + 4\text{Re} \left( \frac{\xi}{p \cdot \rho + |\xi|^2} \right) b^\delta p \]  

\[ = \frac{4}{1 + |\xi|^2} \left[ a^\delta p \left[ \text{Im} \xi \cos 2\beta - \text{Re} \xi \sin 2\beta \right] + b^\delta p \left[ \text{Re} \xi \cos 2\beta + \text{Im} \xi \sin 2\beta \right] \right) \]

\[ = \frac{4}{1 + |\xi|^2} \left( \text{Re} \xi \left[ b^\delta p \cos 2\beta - a^\delta p \sin 2\beta \right] + \text{Im} \xi \left[ a^\delta p \cos 2\beta + b^\delta p \sin 2\beta \right] \right) . \]

Thus, measuring the real and imaginary parts of the amplitudes for different regions in the Dalitz plane provides a more precise determination of \( \xi \). Indeed, in a similar way as in Ref. [31], it is possible to define symmetric regions in the Dalitz plane: \( I \) above the bisector line \( m_{13} - m_{23} \) and \( \bar{I} \) below. In these symmetric regions, the relations

\[ a^I = a^{\bar{I}}, \quad \text{and} \quad b^I = -b^{\bar{I}}, \]

hold, from which the following relations are obtained:

\[ S^+ = S^I_{\pi^+\pi^-K_{S}^{0}\gamma} + S^{\bar{I}}_{\pi^+\pi^-K_{S}^{0}\gamma} = \frac{8}{1 + |\xi|^2} \left( \text{Im} \xi \cos 2\beta - \text{Re} \xi \sin 2\beta \right) a^I, \]

\[ S^- = S^I_{\pi^+\pi^-K_{S}^{0}\gamma} - S^{\bar{I}}_{\pi^+\pi^-K_{S}^{0}\gamma} = \frac{8}{1 + |\xi|^2} \left( \text{Re} \xi \cos 2\beta + \text{Im} \xi \sin 2\beta \right) b^I. \]

From Eqs. (53) and (54) it follows that by measuring separately the time-dependent asymmetry in the regions \( I \) and \( \bar{I} \) it becomes possible to independently constrain the real and imaginary parts of \( \xi \). Note that Eq. (53) is strictly equivalent to Eq. (43), and that \( D = -2a^I \). Using Eqs. (53) and (54), \( \text{Re} \xi \) and \( \text{Im} \xi \) are expressed as

\[ \frac{\text{Re} \xi}{1 + |\xi|^2} = \frac{1}{8} \left( \frac{S^-}{b^I} \cos 2\beta - \frac{S^+}{a^I} \sin 2\beta \right), \]

\[ \frac{\text{Im} \xi}{1 + |\xi|^2} = \frac{1}{8} \left( \frac{S^-}{b^I} \sin 2\beta + \frac{S^+}{a^I} \cos 2\beta \right). \]

The partition scheme of the Dalitz plane must be optimised as a function of the amplitude content in the different regions and the available data sample. From the anti-symmetric relation shown in Eq. (27), it follows that the integrals of the real and imaginary parts of the \( \rho K_{S}^{0} \) amplitudes are real and independent of the integration region, with the values

\[ a^\delta p_{\rho K_{S}^{0}} = -\frac{1}{2}, \quad \text{and} \quad b^\delta p_{\rho K_{S}^{0}} = 0. \]

On the contrary, as shown in Fig. [2], the real and imaginary parts of the \( K^*\pi \) amplitude vary as a function of the Dalitz-plane position. Furthermore, it clearly appears that the real (imaginary) part of the \( K^*\pi \) amplitude exhibits a symmetric (anti-symmetric) distribution with respect to the Dalitz plane bisector. Similar behaviour is observed for the \((K\pi)_{0}\pi \) amplitude. As shown in Fig. [3], when including the amplitudes of all the intermediate states, these symmetry properties with respect to the Dalitz plane bisector remain.
Figure 2: Normalised distributions of the real (left) and imaginary (right) parts of the decay amplitude \( K_1(1270) \to K^*\pi \). While the real part of the amplitude \((\rho^0)\) is symmetric with respect to the Dalitz-plane bisector, the imaginary part \((b^0)\) exhibits an anti-symmetric pattern. The axes correspond to \( s_{13} = m_{\pi^+K^0_S}^2 \) and \( s_{23} = m_{\pi^-K^0_S}^2 \) in GeV\(^2\)/c\(^4\). A similar behaviour is observed for all kaonic resonances.

Figure 3: Normalised distributions of the real (left) and imaginary (right) parts of the decay amplitude \( K_1(1270) \to \pi^+\pi^-K^0_S \), including all the intermediate resonances. The axes correspond to \( s_{13} = m_{\pi^+K^0_S}^2 \) and \( s_{23} = m_{\pi^-K^0_S}^2 \) in GeV\(^2\)/c\(^4\). A similar behaviour is observed for all kaonic resonances.

6 Constraints on new physics and future prospects

Finally, the constraints on \( c'/c \), which can be obtained from the time-dependent measurement of \( B^0 \to \pi^+\pi^-K^0_S\gamma \) decays, are discussed. The common name of \( c'/c \) is \( C_7'/C_7 \); we use this notation hereafter.

Currently, the most stringent constraints on \( C_7'/C_7 \) are obtained from the time-dependent \( CP \) asymmetry in \( B \to K_S\pi^0\gamma \) decays, the angular coefficients of \( B \to K^*e^+e^- \).
decays at $q^2 \to 0$, $A_T^{(2)}$ and $A_T^m$, and the branching fraction of the inclusive $B \to X_s\gamma$ process. In Fig. 4, we show the constraints on $\text{Re}(C'_7/C_7)$ and $\text{Im}(C'_7/C_7)$ at the 3 standard deviations level, obtained from the available measurements of these observables. They are overlaid with the constraints that can be obtained with the expected precision at Belle II with datasets of 10 ab$^{-1}$ and 50 ab$^{-1}$ and at LHCb with datasets of 8 fb$^{-1}$ and 22 fb$^{-1}$. The expected constraints are obtained by using the current central values of the observables, and assuming the measurements to be limited by the statistical uncertainties.

As it is well known, $S_{\pi^0K_S^0\gamma}$ provides a precise determination of $C'_7/C_7$, even though it cannot disentangle its real part from its imaginary part.

![Figure 4: The light red region is the constraint on $\text{Re}(C'_7/C_7)$ and $\text{Im}(C'_7/C_7)$ at the 3 standard deviations level, as obtained from the current measurement of time-dependent CP asymmetry in $B^0 \to \pi^0K_S^0\gamma$ decays, $S_{\pi^0K_S^0\gamma} = -0.15 \pm 0.20 \pm 0.23$, overlaid with the expected precision (dark red) at Belle II with integrated luminosities of 10 ab$^{-1}$ (left) and 50 ab$^{-1}$ (right). The dashed green contour is the constraint obtained from the angular coefficients of $B \to K^*e^+e^-$ decays at $q^2 \to 0$, measured by LHCb $A_T^{(2)} = -0.23 \pm 0.24$ and $A_T^m = 0.14 \pm 0.23$, overlaid with the expected precision (full green line) at LHCb Run II (8 fb$^{-1}$) and Run III (22 fb$^{-1}$). The grey circular contour is the allowed region from the branching fraction measurement of the inclusive $B \to X_s\gamma$ processes with $B(B \to X_s\gamma)_{E_{\gamma}>1.6\text{GeV}} = 3.27 \pm 0.14 \pm 0.23$, and $B(B \to X_s\gamma)_{E_{\gamma}>1.6\text{GeV}} = 3.36 \pm 0.23$.](image_url)

Next, let us show the expected constraint from $S_{\pi^+\pi^-K_S^0\gamma}$, integrating over the whole Dalitz plane, as described in Sec. 5.1. The determination of $\text{Re}(C'_7/C_7)$ and $\text{Im}(C'_7/C_7)$ via $S_{\pi^+\pi^-K_S^0\gamma}$ depends on the value of the dilution factor, hence on the amplitudes of intermediate states and on the integration region. In Fig. 5, we show the constraints for different values of the dilution factor, $\mathcal{D} = \{1, 0.6, 0.3\}$. For the sake of demonstration, the central values are arbitrarily chosen as $S_{\pi^+\pi^-K_S^0\gamma} = \{0.15, 0.09, 0.05\}$, to facilitate the comparison with the constraints on $C'_7/C_7$ from $S_{\pi^0K_S^0\gamma}$ as shown in Fig. 4. The experimental uncertainties are obtained by scaling the statistical uncertainty, $\sigma(S_{\pi^+\pi^-K_S^0\gamma}) = 0.25$, from
Figure 5: Prospects for determining $\text{Re}(C_7'/C_7)$ and $\text{Im}(C_7'/C_7)$ from a measurement of $S_{\pi^+\pi^-K_S^n\gamma}$, integrating over the whole Dalitz plane. The colour changes from dark blue to light blue for decreasing values of the dilution factor $D = \{1, 0.6, 0.3\}$. For each of these, a central value of $S_{\pi^+\pi^-K_S^n\gamma}$ is chosen arbitrarily to be, respectively, $S_{\pi^+\pi^-K_S^n\gamma} = \{0.15, 0.09, 0.05\}$, and the current experimental uncertainty is scaled by the increase of integrated luminosity. The grey and green contours are described in Fig. 4.

Let us now consider the proposed observable $S^-$ of Eq. (54), representing the difference of the time-dependent $CP$ asymmetries measured in two regions of the Dalitz plane. The observable $S^+$ in Eq. (53) yields a similar constraint as that from the integrated analysis, which is shown in Fig. 5. On the contrary, $S^-$ leads to a different kind of constraint; an example is shown in Fig. 6 with the central values $S^- = \{0.30, 0.18, 0.10\}$, chosen to match those of $S_{\pi^+\pi^-K_S^n\gamma}$ in Fig. 5, and with the hadronic parameter values $b_I = \{-0.5, -0.3, -0.15\}$. The uncertainties on $S^-$ are obtained assuming $\sigma(S^I) = \sigma(S^T) = \sqrt{n} \sigma(S_{\pi^+\pi^-K_S^n\gamma})$, where $n = 2$ corresponds to the number of Dalitz-plane regions, and neglecting the correlations between $S^I$ and $S^T$. The uncertainties on $b^I$ are obtained assuming $\sigma(b^I) = \sigma(a^I) = \sigma(D) / \sqrt{n}$. The relation $\sigma(b^I) = \sigma(a^I)$ is obtained assuming the uncertainties on the hadronic decay amplitude magnitude and phase difference are similar.
The sets of central values used in Fig. 7 are \{S_n\}. In this paper, we derive the formula for the time-dependent CP asymmetry both Re(\(C'_7/C_7\)) and Im(\(C'_7/C_7\)) from the proposed observable \(S^-\). The colour changes from dark blue to light blue for increasing values of the hadronic parameter \(b^l = \{-0.5, -0.3, -0.15\}\). The corresponding values of \(S^- = \{0.30, 0.18, 0.10\}\) are chosen to match those used for \(S_{\pi^+\pi^-K_0^0\gamma}\) in Fig. 5 and we assume that its uncertainties are of the same order as that of \(S^+\). The uncertainties on the hadronic parameter are also taken into account. We assume that they are at the level of the current uncertainties on dilution factor, \(\sigma(a^l) = \sigma(b^l) = \sigma(D)/\sqrt{2}\). All uncertainties are further scaled according to the increase of integrated luminosity. The grey and green contours are described in Fig. 4.

![Graph showing the combination of constraints from Eqs. (53) and (54).

It is remarkable that the constraint on \(C'_7/C_7\) obtained from \(S^-\) is orthogonal to that from \(S^+\). From Eqs. (53) and (54), it is clear that this orthogonality does not depend on the values chosen for this demonstration. Thus, by combining the two observables it is possible to disentangle the real part of \(C'_7/C_7\) from its imaginary part. Unlike the time-dependent analysis of \(B^0 \rightarrow \pi^0K_0^0\gamma\) decays, that of \(B^0 \rightarrow \pi^+\pi^-K_0^0\gamma\) allows to obtain a simultaneous measurement of \(S^-\) and \(S^+\).

We finally show, in Fig. 7, an example of the combined constraints from Eqs. (53) and (54). As before, the central values of \(S^+, S^-, a^l\) and \(b^l\) are chosen arbitrarily, and the uncertainties are estimated from the \(B\) measurements of \(S_{\pi^+\pi^-K_0^0\gamma}\) and \(D\). The sets of central values used in Fig. 7 are \{\(S^+, S^-, a^l, b^l\)\} = \{0.17, 0.13, -0.5, -0.15\}, \{0.13, 0.04, -0.3, -0.3\} and \{0.13, -0.03, -0.15, -0.5\}. Even though the obtained constraints depend on the hadronic parameters, it is clear that combining the information from \(S^+\) and \(S^-\) measured in \(B^0 \rightarrow \pi^+\pi^-K_0^0\gamma\) decays allows to independently constrain both \(\text{Re}(C'_7/C_7)\) and \(\text{Im}(C'_7/C_7)\).

7 Conclusion

In this paper, we derive the formula for the time-dependent CP asymmetry of \(B^0 \rightarrow K_{\text{res}}\gamma \rightarrow [\rho^0K_0^0, K^{+}\pi^-, (K\pi)^{+}\pi^-]\gamma \rightarrow \pi^+\pi^-K_0^0\gamma\); it is the first time that this formula is
Figure 7: Prospect for the determination of \( \text{Re}(C'_7/C_7) \) and \( \text{Im}(C'_7/C_7) \) by combining the two observables \( S^+ \) and \( S^- \). The central value of \( S^- \) is chosen arbitrarily while its uncertainty is estimated by using the measurement from Ref. [22]. We also take into account the uncertainties on the hadronic parameters by assuming them to be at the same level as the current uncertainties on the dilution factor. As a demonstration, we chose three sets of values: \( \{S^+, S^-, a', b'\} = \{0.17, 0.13, -0.5, -0.15\} \) (blue), \( \{0.13, 0.04, -0.3, -0.3\} \) (red) and \( \{0.13, -0.03, -0.15, -0.5\} \) (green). The uncertainties are taken as \( \sigma(S^I) = \sigma(S^\bar{I}) = \sqrt{n}\sigma(S_{\pi^+\pi^-K^0_s\gamma}) \), and \( \sigma(a^I) = \sigma(b^I) = \sigma(D)/\sqrt{n} \), where \( n = 2 \) corresponds to the number of Dalitz-plane regions. All the uncertainties are further scaled according to the increase of integrated luminosity. The grey and green contours are described in Fig. 4.

derived including all these intermediate states. As it turns out, the formula is the same for all \( K_{\text{res}} \) states with \( J^P = (1^+, 1^-, 2^+) \). This allows to extract the time-dependent CP asymmetry \( S_{\rho^0K^0_s\gamma} \) by measuring the phase-space integrated \( S_{\pi^+\pi^-K^0_s\gamma} \) and the dilution factor \( D \). The constraint from this measurement on the \( C'_7/C_7 \) complex plane is similar to that obtained from the measurement of \( S_{\pi^0K^0_s\gamma} \); it corresponds to a diagonal band. The dilution factor can be obtained from the charged decay mode \( B^+ \rightarrow \pi^+\pi^-K^+\gamma \), which benefits from a higher branching fraction and a better detection efficiency compared to the neutral decay mode. In particular, the LHCb experiment is currently in the best position to provide additional information on \( D \).

We also show that performing a time-dependent amplitude analysis of \( B^0 \rightarrow \pi^+\pi^-K^0_s\gamma \) decays gives access to a new observable, \( S^- \), allowing to simultaneously constraint \( \text{Re}(C'_7/C_7) \) and \( \text{Im}(C'_7/C_7) \), when combined with \( S^+ \). This is the main result of this paper. Such an analysis is not currently feasible due to the limited size of the available data sample, whereas it will become accessible with the dataset expected from the Belle II experiment. We present prospects for the determination of \( C'_7/C_7 \) from a time-dependent analysis of \( B^0 \rightarrow \pi^+\pi^-K^0_s\gamma \) decays at Belle II, considering two approaches: a phase-space integrated analysis and an amplitude analysis using information from the \( K^0_s\pi^+\pi^- \)
Dalitz-plane.

The analysis of $B^0 \rightarrow \pi^+\pi^-K^0_S\gamma$ decays should provide stringent constraints on the photon polarisation in the upcoming years. In particular, the constraints on $C'_7/C_7$ from this measurement are complementary to those obtained from the time-dependent $CP$ asymmetry of $B^0 \rightarrow \pi^0K^0_S\gamma$ and the angular analysis of $B \rightarrow K^*e^+e^-$ at low $q^2$.

Acknowledgements

We would like to thank François Le Diberder, David London and Mike Sokoloff for fruitful discussions concerning this paper and their valuable comments and advice.
A The time-dependent CP asymmetry in $B^0 \rightarrow K^{*0}\gamma \rightarrow \pi^0 K^0_S\gamma$ and $B^0_s \rightarrow \phi\gamma \rightarrow K^+ K^-\gamma$

For comparison, we include the CP formulae for the $B^0 \rightarrow K^{*0}\gamma \rightarrow \pi^0 K^0_S\gamma$ and $B^0_s \rightarrow \phi\gamma \rightarrow K^+ K^-\gamma$ in this appendix.

Let us obtain the amplitude relations, as done in Sec. 3. The relations between the right and left handed amplitudes in Eq. (23) hold also in the cases of $B^0 \rightarrow K^{*0}\gamma$ and $B^0_s \rightarrow \phi\gamma$ decays. For the strong amplitude $K^{*0} \rightarrow \pi^0 K^0_S\gamma$, the charge transformation leads to

$$A_{\lambda}^{\pi^0 K^0_S} = \langle K^{*0}_S(p_1)\pi^0(p_2)|\mathcal{H}'_s|K^*\rangle$$
$$= \langle K^{*0}_S(p_1)\pi^0(p_2)|\mathcal{H}'_s|K^*\rangle,$$

where we assigned $C|K^{*0}\rangle = -|K^{*0}\rangle$ for consistency. For $\phi \rightarrow K^+ K^-$

$$A'_{\lambda}^{K^+ K^-} = \langle K^+(p_1)K^-(p_2)|\mathcal{H}'_s|\phi\rangle$$
$$= -\langle K^-(p_1)K^+(p_2)|\mathcal{H}'_s|\phi\rangle$$
$$= \langle K^-(p_2)K^+(p_1)|\mathcal{H}'_s|\phi\rangle,$$

where the last line is explained by the fact that the $\phi$ decays through a p-wave. For the parity transformation, Eq. (30) holds here as well. Using these relations, for $B^0 \rightarrow K^{*}\gamma$ and $B^0_s \rightarrow \phi\gamma$ decays we find

$$M^{\pi^0 K^0_S} M^{\pi^0 K^0_S} = \left(\frac{c}{c'}\right)^2 \left|M^{\pi^0 K^0_S}_L\right|^2,$$  \hspace{1cm} (58)
$$M^{K^+ K^-} M^{K^+ K^-} = \left(\frac{c}{c'}\right)^2 \left|M^{K^+ K^-}_L\right|^2,$$  \hspace{1cm} (59)

and Eq. (36) holds here as well. As a result, we find for $B^0 \rightarrow K^*\gamma$

$$C_{\pi^0 K^0_S\gamma} = 0, \quad S_{\pi^0 K^0_S\gamma} = \frac{2\text{Im}\left(\frac{q_p c'}{p c'}\right)}{|c|^2 + |c'|^2}.$$  \hspace{1cm} (60)

For the $B^0_s \rightarrow \phi\gamma \rightarrow K^+ K^-\gamma$ decay, Eq. (60) has an additional term due to the large $\Delta \Gamma_s$ with respect to $\Delta \Gamma_d$ such that

$$\frac{\Gamma(t) - \Gamma(t)}{\Gamma(t) + \Gamma(t)} = S_{K^+ K^-} \sin(\Delta m t) - C_{K^+ K^-} \cos(\Delta m t)$$
$$\frac{\cosh(\Delta m t)}{\cosh(\Delta m t)} - \mathcal{A}^\Delta_{K^+ K^-} \sinh(\Delta m t),$$  \hspace{1cm} (61)

where $\mathcal{A}^\Delta$ is given by

$$\mathcal{A}^\Delta_{K^+ K^-} = \frac{2\text{Re}\left(\frac{2}{p} \sum_{\lambda=L,R} [M_{\lambda} M_{\lambda}^*]\right)}{\sum_{\lambda=L,R} \left|M_{\lambda}\right|^2 + \left|M_{\lambda}\right|^2}.$$  \hspace{1cm} (62)
Then, we find

\[ C_{K^+K^-} = 0, \quad S_{K^+K^-} = \frac{2\text{Im}\left(\frac{q_s}{p_s}c_d c_d'\right)}{|c_d|^2 + |c_d'|^2}, \quad A_{K^+K^-}^S = \frac{2\text{Re}\left(\frac{q_s}{p_s}c_d c_d'\right)}{|c_d|^2 + |c_d'|^2}, \quad (63) \]

where \( q_s/p_s \) indicates the \( B^0_s - \bar{B}^0_s \) mixing phase and \( c_d \) indicates the coefficient of the \( b \to d\gamma \) transition amplitude. Integration over the whole phase space is implicit. The first measurement of \( A_{K^+K^-}^S \) was recently obtained by LHCb collaboration \cite{34}.

References

[1] Damir Becirevic, Emi Kou, Alain Le Yaouanc, and Andrey Tayduganov. Future prospects for the determination of the Wilson coefficient \( C_7' \gamma \). *JHEP*, 08:090, 2012.

[2] Emi Kou, Cai-Dian Lu, and Fu-Sheng Yu. Photon polarisation in the \( b \to s\gamma \) processes in the Left-Right Symmetric Model. *JHEP*, 12:102, 2013.

[3] Naoyuki Haba, Hiroyuki Ishida, Tsuyoshi Nakaya, Yasuhiro Shimizu, and Ryo Takahashi. Search for new physics via photon polarisation of \( b \to s\gamma \). *JHEP*, 03:160, 2015.

[4] Ayan Paul and David M. Straub. Constraints on new physics from radiative \( B \) decays. 2016.

[5] Michael Gronau, Yuval Grossman, Dan Pirjol, and Anders Ryd. Measuring the photon polarisation in \( B \to K\pi\pi\gamma \). *Phys. Rev. Lett.*, 88:051802, 2002.

[6] Michael Gronau and Dan Pirjol. Photon polarisation in radiative \( B \) decays. *Phys. Rev.*, D66:054008, 2002.

[7] E. Kou, A. Le Yaouanc, and A. Tayduganov. Determining the photon polarisation of the \( b \to s\gamma \) using the \( B \to K_1(1270)\gamma \to (K\pi\pi)\gamma \) decay. *Phys. Rev.*, D83:094007, 2011.

[8] E. Kou, A. Le Yaouanc, and A. Tayduganov. Angular analysis of \( B \to J/\psi K_1 \) towards a model independent determination of the photon polarisation with \( B \to K_1\gamma \). *Phys. Lett.*, B763:66–71, 2016.

[9] Fady Bishara and Dean J. Robinson. Probing the photon polarisation in \( B \to K^{*}\gamma \) with conversion. *JHEP*, 09:013, 2015.

[10] L. Oliver, J. C. Raynal, and R. Sinha. Note on new interesting baryon channels to measure the photon polarisation in \( b \to s\gamma \). *Phys. Rev.*, D82:117502, 2010.

[11] Frank Kruger and Joaquim Matias. Probing new physics via the transverse amplitudes of \( B^0 \to K^{*0}(\to K^-\pi^+)l^+l^- \) at large recoil. *Phys. Rev.*, D71:094009, 2005.
[12] Damir Becirevic and Elia Schneider. On transverse asymmetries in $B \to K^*\ell^+\ell^-$. *Nucl. Phys.*, B854:321–339, 2012.

[13] Roel Aaij et al. Angular analysis of the $B^0 \to K^{\ast 0}e^+e^-$ decay in the low-$q^2$ region. *JHEP*, 04:064, 2015.

[14] Roel Aaij et al. Observation of Photon polarisation in the $b\bar{s}$ Transition. *Phys. Rev. Lett.*, 112(16):161801, 2014.

[15] David Atwood, Michael Gronau, and Amarjit Soni. Mixing induced $CP$ asymmetries in radiative B decays in and beyond the standard model. *Phys. Rev. Lett.*, 79:185–188, 1997.

[16] David Atwood, Tim Gershon, Masashi Hazumi, and Amarjit Soni. Mixing-induced $CP$ violation in $B \to P(1)P(2)\gamma$ in search of clean new physics signals. *Phys. Rev.*, D71:076003, 2005.

[17] David Atwood, Tim Gershon, Masashi Hazumi, and Amarjit Soni. Clean Signals of $CP$-violating and $CP$-conserving New Physics in $B \to PV\gamma$ Decays at $B$ Factories and Hadron Colliders. 2007.

[18] Franz Muheim, Yuehong Xie, and Roman Zwicky. Exploiting the width difference in $B_s \to \phi\gamma$. *Phys. Lett.*, B664:174–179, 2008.

[19] Bernard Aubert et al. Measurement of Time-Dependent $CP$ Asymmetry in $B^0 \to K^0_S\pi^0\gamma$ Decays. *Phys. Rev.*, D78:071102, 2008.

[20] Y. Ushiroda et al. Time-Dependent $CP$ Asymmetries in $B^0 \to K^0_S\pi^0\gamma$ transitions. *Phys. Rev.*, D74:111104, 2006.

[21] J. Li et al. Time-dependent $CP$ Asymmetries in $B^0 \to K^0_S\rho^0\gamma$ Decays. *Phys. Rev. Lett.*, 101:251601, 2008.

[22] P. del Amo Sanchez et al. Time-dependent analysis of $B^0 \to K^0_S\pi^+\pi^-\gamma$ decays and studies of the $K^+\pi^-\pi^+$ system in $B^+ \to K^+\pi^-\pi^+\gamma$ decays. *Phys. Rev.*, D93(5):052013, 2016.

[23] S Akar. Study of $B \to K\pi\pi\gamma$ decays with the BABAR Experiment: the photon helicity and the resonant structure of the $K\pi\pi$ system. PhD thesis, Paris U., VI-VII, 2013.

[24] T. Inami and C. S. Lim. Effects of Superheavy Quarks and Leptons in Low-Energy Weak Processes $k(L) \to \mu^-\mu^+, K^+ \to \pi^+\nu\bar{\nu}$ and $K^0 \leftrightarrow \bar{K}^0$. *Prog. Theor. Phys.*, 65:297, 1981. [Erratum: Prog. Theor. Phys.65,1772(1981)].

[25] Benjamin Grinstein, Yuval Grossman, Zoltan Ligeti, and Dan Pirjol. The Photon polarization in $B \to X\gamma$ in the standard model. *Phys. Rev.*, D71:011504, 2005.

[26] Benjamin Grinstein and Dan Pirjol. The $CP$ asymmetry in $B^0(t) \to K_S\pi^0\gamma$ in the standard model. *Phys. Rev.*, D73:014013, 2006.
[27] A. Khodjamirian, R. Ruckl, G. Stoll, and D. Wyler. QCD estimate of the long distance effect in $B \to K^*\gamma$. *Phys. Lett.*, B402:167–177, 1997.

[28] Patricia Ball and Roman Zwicky. Time-dependent CP Asymmetry in $B \to K^*\gamma$ as a (Quasi) Null Test of the Standard Model. *Phys. Lett.*, B642:478–486, 2006.

[29] A. Khodjamirian, Th. Mannel, A. A. Pivovarov, and Y. M. Wang. Charm-loop effect in $B \to K^{(*)}\ell^+\ell^-$ and $B \to K^*\gamma$. *JHEP*, 09:089, 2010.

[30] M. Matsumori and A. I. Sanda. The Mixing-induced $CP$ asymmetry in $B \to K^*\gamma$ decays with perturbative QCD approach. *Phys. Rev.*, D73:114022, 2006.

[31] Anjan Giri, Yuval Grossman, Abner Soffer, and Jure Zupan. Determining gamma using $B^\pm \to DK^\pm$ with multibody D decays. *Phys. Rev.*, D68:054018, 2003.

[32] Y. Amhis et al. Averages of $b$-hadron, $c$-hadron, and $\tau$-lepton properties as of summer 2016. *Eur. Phys. J.*, C77(12):895, 2017.

[33] M. Misiak et al. Updated NNLO QCD predictions for the weak radiative B-meson decays. *Phys. Rev. Lett.*, 114(22):221801, 2015.

[34] Roel Aaij et al. First experimental study of photon polarization in radiative $B_s^0$ decays. *Phys. Rev. Lett.*, 118(2):021801, 2017. [Addendum: Phys. Rev. Lett.118,no.10,109901(2017)].