Connectivity properties of the adjacency graph of $\text{SLE}_\kappa$ bubbles for $\kappa \in (4, 8)$

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Abstract

We study the adjacency graph of bubbles—i.e., complementary connected components—of an $\text{SLE}_\kappa$ curve for $\kappa \in (4, 8)$, with two such bubbles considered to be adjacent if their boundaries intersect. We show that this adjacency graph is a.s. connected for $\kappa \in (4, \kappa_0]$, where $\kappa_0 \approx 5.6158$ is defined explicitly. This gives a partial answer to a problem posed by Duplantier, Miller and Sheffield (2014). Our proof in fact yields a stronger connectivity result for $\kappa \in (4, \kappa_0]$, which says that there is a Markovian way of finding a path from any fixed bubble to $\infty$. We also show that there is a (non-explicit) $\kappa_1 \in (\kappa_0, 8)$ such that this stronger condition does not hold for $\kappa \in [\kappa_1, 8)$.

Our proofs are based on an encoding of $\text{SLE}_\kappa$ in terms of a pair of independent $\kappa/4$-stable processes, which allows us to reduce our problem to a problem about stable processes. In fact, due to this encoding, our results can be re-phrased as statements about the connectivity of the adjacency graph of loops when one glues together an independent pair of so-called $\kappa/4$-stable looptrees, as studied, e.g., by Curien and Kortchemski (2014).

The above encoding comes from the theory of Liouville quantum gravity (LQG), but the paper can be read without any knowledge of LQG if one takes the encoding as a black box.

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1 Introduction

1.1 Overview

Let $\kappa \in (4, 8)$ and let $\eta$ be a chordal Schramm-Loewner evolution (SLE$_{\kappa}$) curve [Sch00], say from 0 to $\infty$ in the upper half-plane $\mathbb{H}$. A bubble of $\eta$ is a connected component of $\mathbb{H} \setminus \eta$. We declare that two such bubbles are adjacent if their boundaries have non-empty intersection. In this paper we will study the adjacency graph of SLE$_{\kappa}$ bubbles for $\kappa \in (4, 8)$. (The analogous graph for $\kappa \in (0, 4] \cup [8, \infty)$ is uninteresting since SLE$_{\kappa}$ has only two complementary connected components for $\kappa \in (0, 4]$ and is space-filling for $\kappa \geq 8$ [RS05]).

A natural first question to ask about the adjacency graph of bubbles is whether it is connected, i.e., whether any two bubbles can be joined by a finite path in the graph. This question appears as [DMS14, Question 13.2] and is the SLE analogue of a well-known open problem for Brownian motion, which asks whether the adjacency graph of complementary connected components of a planar Brownian motion (say, stopped at some fixed time) is connected; see, e.g., [Bur] or [MP10, Open Problem (4)].

Intuitively, one expects that it is easier for the adjacency graph to be connected when $\kappa$ is closer to 4, since for smaller $\kappa$ the bubbles tend to be larger and the curve itself is “thinner”, e.g., in the sense that it has smaller Hausdorff dimension [Bef08] and a larger set of cut points [MW17].

However, due to the fractal nature of the SLE$_{\kappa}$ curve, it is not clear a priori whether the adjacency graph should be connected for any value of $\kappa \in (4, 8)$, even at a heuristic level. For instance, the set $S$ of points on the curve which do not lie on the boundary of any bubble has full Hausdorff dimension: indeed, by SLE duality [Zha08, Zha10, Dub09, MS16b, MS17], the dimension of the boundary of each bubble is equal to the dimension of SLE$_{16/\kappa}$, which is strictly less than the dimension of SLE$_{\kappa}$ [Bef08]. If $S$ contained a non-trivial connected subset, then no path of bubbles in the adjacency graph would be able to cross this subset (c.f. Corollary 1.2). One could also worry that there exist pairs of macroscopic bubbles separated by an infinite “cloud” of small bubbles, so that no finite path of bubbles can join them. Figure 1 shows a simulation of an SLE curve, which may help the reader to visualize these geometric features.

In this paper we will give an affirmative answer to the above question for an explicit range of values of $\kappa$. With $\psi(x) = \Gamma'(x)/\Gamma(x)$ denoting the digamma function, we have the following.

**Theorem 1.1.** For each fixed $\kappa \in (4, \kappa_0]$, the adjacency graph of bubbles of a chordal SLE$_{\kappa}$ curve is almost surely connected, where $\kappa_0 \approx 5.6158$ is the unique solution of the equation $\pi \cot(\pi \kappa/4) + \psi(2 - \kappa/4) - \psi(1) = 0$ on the interval $(4, 8)$.

We will prove Theorem 1.1 by proving an stronger condition (Theorem 2.8), which, roughly speaking, asserts that each bubble of the SLE$_{\kappa}$ curve is “connected to infinity” via an infinite path of bubbles in the adjacency graph which are chosen in a Markovian manner with respect to a natural parametrization of SLE that we introduce in Section 2. We also show that this stronger condition fails for $\kappa$ sufficiently close to 8 (Theorem 2.9). See Section 6 for some heuristic discussion concerning the values of $\kappa$ for which various connectivity properties hold.

As alluded to earlier, Theorem 1.1 tells us that for $\kappa \in (4, \kappa_0]$, there cannot be non-trivial connected subsets of the SLE$_{\kappa}$ curve which do not intersect the boundary of any bubble.

**Corollary 1.2.** For $\kappa \in (4, \kappa_0]$, the set of points on a chordal SLE$_{\kappa}$ curve which do not lie on the boundary of any bubble is almost surely totally disconnected.

**Proof.** Let $\eta$ be a chordal SLE$_{\kappa}$ curve and let $\tau_1$ and $\tau_2$ be forward and reverse stopping times of $\eta$, respectively, with $\tau_1 < \tau_2$ almost surely. By the reversibility of SLE$_{\kappa}$ [MS16a] and the domain
Markov property, the conditional law of \( \eta|_{[\tau_1, \tau_2]} \) conditioned on \( \eta|_{[0, \tau_1] \cup [\tau_2, \infty)} \) is that of an SLE\( _\kappa \) curve from \( \eta(\tau_1) \) to \( \eta(\tau_2) \) in the appropriate connected component \( D = D(\tau_1, \tau_2) \) of \( \mathbb{H} \setminus \eta([0, \tau_1] \cup [\tau_2, \infty)) \). Theorem 1.1 applied to this latter SLE curve implies that, almost surely, there does not exist a connected subset of \( \eta \) which does not intersect the boundary of any bubble of \( \eta \) and which disconnects the interior of \( D \), since such a set would disconnect the adjacency graph of bubbles of \( \eta|_{[\tau_1, \tau_2]} \).

We can choose a countable collection \( \mathcal{T} \) of random pairs of times \( (\tau_1, \tau_2) \) such that \( \tau_1 < \tau_2 \) a.s., \( \tau_1 \) (resp. \( \tau_2 \)) is a forward (resp. reverse) stopping time for \( \eta \), and the projection of \( \mathcal{T} \) onto its first and second coordinates are each dense (e.g., we could conformally map to \( \mathbb{D} \), parametrize \( \eta \) by Minkowski content [LS11, LZ13, LR15], then let \( \mathcal{T} \) be the set of pairs of ordered positive rational times). If \( X \) is a connected subset of \( \eta \) with more than one point and we choose \( (\tau_1, \tau_2) \in \mathcal{T} \) such that \( \tau_1 \) (resp. \( \tau_2 \)) is sufficiently close to the first (resp. last) time that \( \eta \) hits \( X \), then \( X \) will disconnect the interior of the domain \( D \) above. Hence the corollary follows from a union bound over all \( (\tau_1, \tau_2) \in \mathcal{T} \).

We also mention the recent related work [AS18], which studies the two-valued local sets of the Gaussian free field—a two-parameter family of random sets constructed from collections of SLE\(_4\)-type curves. Among other things, the authors determine the parameter values for which the adjacency graph of complementary connected components of these sets are connected, using very different techniques from those of the present paper.

### 1.2 Approach and outline

The key tool in our proof is a pair of independent \( \kappa/4 \)-stable processes \((L, R)\) with only downward jumps, first introduced in [DMS14, Corollary 1.20], which encode the geometry of the SLE\(_\kappa\) curve. The existence of these processes reduces our problem to analyzing stable processes rather than SLE\(_\kappa\).
We will give the definition of \((L, R)\) in Section 2.2. The definition uses the theory of Liouville quantum gravity (LQG): roughly speaking, \(L_t\) (resp. \(R_t\)) for \(t \geq 0\) gives the LQG length of the left (resp. right) outer boundary of \(\eta([0, t])\) minus the LQG length of the interval to the left (resp. right) of 0 which is disconnected from \(\infty\) by \(\eta([0, t])\), when \(\eta\) is parametrized by quantum natural time with respect to a certain GFF-type distribution. The downward jumps of \(L\) and \(R\) correspond to times at which \(\eta\) forms bubbles. We will review the aspects of LQG theory which are necessary to understand the definition in Section 2.1. The reader who is not familiar with LQG can take the existence of \((L, R)\) as a black box throughout the rest of the paper.

In Section 2.3 we use the process \((L, R)\) to formulate a condition for the adjacency graph of \(\text{SLE}_\kappa\) bubbles which implies connectedness. We will then state Theorems 2.8 and 2.9, which assert that this stronger condition holds for the range of \(\kappa\) considered in Theorem 1.1, but fails for \(\kappa\) sufficiently close to 8. The remaining sections of the paper will be devoted to proving Theorems 2.8 and 2.9.

In Section 3, we explain how to use the Markov and scaling properties of \((L, R)\) to reduce each of Theorems 2.8 and 2.9 to determining whether the expected logarithm of a certain quantity defined in terms of \((L, R)\) is positive or negative. The remainder of the paper contains the (somewhat tricky) Lévy process arguments needed to estimate these expectations. Theorem 2.8 (which implies Theorem 1.1) is proven in Section 4 and Theorem 2.9 is proven in Section 5.

Section 6 discusses some open problems related to various connectivity properties of the adjacency graph of \(\text{SLE}_\kappa\) bubbles.

1.3 Looptree interpretation

Due to the encoding discussed in Section 1.2, Theorem 1.1 can be re-phrased as a statement about the topological space obtained by gluing together a pair of so-called \(\kappa/4\)-stable looptrees, as studied, e.g., in [CK14]. We will not directly use looptrees in our proof, so a reader who only wants to see the proof of our results for \(\text{SLE}_\kappa\) can safely skip this subsection.

Stable looptrees are obtained from stable Lévy trees (as defined, e.g., in [DL05]) by replacing each branch point (corresponding to the jumps of the Lévy process) by a circle of perimeter equal to the magnitude of the jump. In the case of \(\kappa/2\)-stable processes, this construction is equivalent to the construction of the so-called forested wedge of weight \(\gamma^2 / 2\) (here \(\gamma = 4 / \sqrt{\kappa}\)) in [DMS14, Figure 1.18, Line 3], except that in the looptree definition the interiors of the disks are not included. The definition of looptrees/forested wedges is explained in Figure 2.

\textbf{Corollary 1.3.} Let \((L, R)\) be a pair of i.i.d. \(\kappa/4\)-stable processes with only downward jumps and let \(\mathcal{G}\) be the topological space obtained by gluing the looptrees \(\mathcal{T}^L\) and \(\mathcal{T}^R\) associated with \(L\) and \(R\) together according to the natural length measure along their boundaries which arises from the time parametrizations of \(L\) and \(R\), as described in Figure 2. Say that two loops of either \(\mathcal{T}^L\) and \(\mathcal{T}^R\) are adjacent if the corresponding subsets of \(\mathcal{G}\) intersect. If \(\kappa \in (4, \kappa_0]\), then the adjacency graph of loops is a.s. connected.

\textit{Proof.} Let \(\eta\) be an \(\text{SLE}_\kappa\) curve. By a slight abuse of notation, we also denote the range of \(\eta\) by \(\eta\). It follows from [DMS14, Theorem 1.19] that \((\mathbb{H}, \eta)\) is equivalent, as a topological space with a distinguished subset, to the topological space obtained from \(\mathcal{G}\) by filling in each of the loops of \(\mathcal{T}^L\) and \(\mathcal{T}^R\) by a copy of the disk, with the loop as its boundary, decorated by \(\mathcal{G}\). Here we use the above-mentioned equivalence between looptrees and forested wedges. Equivalently, \(\eta\), viewed as a topological space, is homeomorphic to \(\mathcal{G}\) via a homeomorphism under which boundaries of bubbles of \(\eta\) correspond to loops of \(\mathcal{T}^L\) or \(\mathcal{T}^R\). Consequently, the corollary follows from Theorem 1.1. \(\blacksquare\)
Figure 2: An illustration of the gluing of two independent $\kappa/4$-stable looptrees described in Corollary 1.3. **Left:** We begin with a pair $(L, R)$ of independent $\kappa/4$-stable processes with only negative jumps. We can choose a large $C > 0$ such that the graphs of $L_t$ and $C - R_t$ do not intersect in some time interval of interest (the particular value of $C$ is unimportant). **Middle:** For each jump of $L_t$, we draw a black curve underneath the graph of $L_t$ with the same endpoints as those of the jump, and which intersects each horizontal line only once. (The particular geometry of the curves chosen will not affect the topology of the resulting tree.) We similarly draw curves corresponding to jumps of $C - R_t$. We then identify pairs of points of the square if they lie on the same horizontal (green) segment that lies below the curve; and similarly for $C - R_t$. This produces a pair of independent forested wedges of weight $\gamma^2 - 2$. To glue the two forested wedges, we draw vertical (red) segments joining the two graphs, and we connect points on the two graphs that lie on the same vertical segment or on the same jump segment. **Right:** The resulting quotient is a pair of forested wedges with outer boundaries identified. The parts of the forested wedges colored in blue correspond to running minima of $L_t$ and $C - R_t$; or, equivalently, points of $L_t$ and $R_t$ which lie on horizontal green segments that intersect the rays $(-\infty, 0)$ and $(C, \infty)$ on the $y$-axis, colored in blue in the middle figure. If we remove the gray interior regions, we obtain a pair of $\kappa/4$-stable looptrees with their outer boundaries identified.

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2 A $\kappa/4$-stable process description of SLE$\kappa$ for $\kappa \in (4, 8)$

2.1 Liouville quantum gravity definitions

In order to define the pair of $\kappa/4$-stable processes which encode the geometry of $\eta$, we will need some definitions from the theory of Liouville quantum gravity (LQG). We will not state these definitions precisely here (instead referring to the cited papers), since the only feature of these definitions which is needed in the present paper is Theorem 2.1 below.

Let $\gamma := 4/\sqrt{\kappa} \in (\sqrt{2}, 2)$. If $D \subset \mathbb{C}$ is an open set and $h$ is a random distribution (generalized
function) on $D$ which behaves locally like the Gaussian free field on $D$ (see [She07,SS13,MS16b,MS17] for more on the GFF) then the $\gamma$-LQG surface associated with $h$ is, formally, the random Riemannian surface with Riemann metric tensor $e^{\gamma h(z)} (dx^2 + dy^2)$, where $dx^2 + dy^2$ denotes the Euclidean metric tensor. This definition does not make literal sense since $h$ is a distribution, not a pointwise-defined function, so we cannot exponentiate it. However, certain objects associated with $\gamma$-LQG surfaces can be defined rigorously using regularization procedures.

For example, Duplantier and Sheffield [DS11] constructed the volume form associated with a $\gamma$-LQG surface, which is a measure $\mu_h$ that can be defined as the limit of regularized versions of $e^{\gamma h(z)} dz$ (where $dz$ denotes Lebesgue measure). In a similar vein, one can define the $\gamma$-LQG length measure $\nu_h$ on certain curves in $\overline{D}$, including $\partial D$ and SLE$_{\kappa}$-type curves for $\kappa = \gamma^2$ (or equivalently the outer boundaries of SLE$_{\kappa}$-type curves, by SLE duality [Zha08,Zha10,Dub09,MS16b,MS17]) which are independent from $h$. The $\gamma$-LQG length measure can be defined in various ways, e.g., using semi-circle averages of a GFF on a domain with smooth boundary and then conformally mapping to the complement of an SLE$_{\kappa}$ curve [DS11,She16] or directly as a Gaussian multiplicative chaos measure with respect to the Minkowski content of the SLE curve [Ben17]. See also [RV14,Ber15] for surveys of a more general theory of regularized measures of this form, which dates back to Kahane [Kah85].

Also relevant for our purposes is the natural $\gamma$-LQG parametrization of an SLE$_{\kappa}$ curve that is independent from $h$; we call this parametrization quantum natural time. Parametrizing by quantum natural time is, roughly speaking, the same as parametrizing by “quantum Minkowski content”. It is the quantum analogue of the so-called natural parametrization of SLE [LS11,LZ13]. The precise definition of quantum natural time can be found in [DMS14, Definition 7.14].

In this paper, we will always take $D = \mathbb{H}$ to be the upper half-plane and $h$ to be the GFF-type distribution corresponding to the so-called $\frac{4}{\gamma} - \frac{2}{\gamma}$ - (equivalently, weight-$\frac{3\gamma^2}{2}$) quantum wedge, which is defined precisely in [DMS14, Definition 4.4]. Roughly speaking, $h$ is obtained from $\tilde{h} = \left(\frac{4}{\gamma} - \frac{2}{\gamma}\right) \log |\cdot|$, for $\tilde{h}$ a GFF on $\mathbb{H}$ with Neumann boundary conditions, by “zooming in near the origin” and then re-scaling so that the $\gamma$-LQG mass of $D \cap \mathbb{H}$ remains of constant order [DMS14, Proposition 4.6(ii)].

2.2 Definition of $(L, R)$

Let us now suppose that $h$ is the distribution corresponding to a $\frac{4}{\gamma} - \frac{2}{\gamma}$-quantum wedge ($\gamma = 4/\sqrt{\kappa}$), as above, and our SLE$_{\kappa}$ curve $\eta$ is sampled independently from $h$ and then parametrized by $\gamma$-quantum natural time with respect to $h$. To define the $\kappa/4$-stable processes $(L, R)$, consider for each $t > 0$ the hull generated by $\eta([0,t])$ (i.e., the closure of the set of points it disconnects from $\infty$) and let $x_t$ and $y_t$ denote the infimum and supremum, respectively, of the set of points where this hull intersects the real line. We define the left boundary length $L_t$ of $\eta$ at time $t$ to be the $\gamma$-LQG length of the boundary arc of the hull from $\eta(t)$ to $x_t$, minus the $\gamma$-LQG length of the segment $[x_t, 0]$. Similarly, we define the right boundary $R_t$ of $\eta$ at time $t$ to be the $\gamma$-LQG length of the boundary arc of the hull from $\eta(t)$ to $y_t$, minus the $\gamma$-LQG length of the segment $[0, y_t]$. See Figure 3 for an illustration. We note that the definition of $(L, R)$ is the continuum analogue of the so-called horodistance process for peeling processes on random planar maps, as studied, e.g., in [Cur15,GM17].

The following is [DMS14, Corollary 1.20], and is the only fact from LQG theory which we will need in this paper.

**Theorem 2.1.** The processes $L_t$ and $R_t$ are i.i.d. totally asymmetric $\frac{\kappa}{4}$-stable Lévy processes with only negative jumps.
Remark 2.2. Theorem 2.1 only specifies the law of \((L, R)\) up to a constant re-scaling of time, 
\((L_t, R_t) \mapsto (L_{ct}, R_{ct})\) for a constant \(c > 0\) (or equivalently \((L_t, R_t) \mapsto c^{4/κ}(L_t, R_t)\)). The properties of \((L, R)\) which we will be interested in do not depend on this scaling, so one can make an arbitrary choice of \(c\). In Section 5, we will fix the scaling in a particularly convenient way.

Theorem 2.1 is quite powerful because the behavior of these two Lévy processes neatly encode a lot of the geometry of the SLE\(_κ\) curve \(η\); the following set of examples illustrates this connection and will be used repeatedly in the proof of our main results. (The equivalences described in these examples are direct consequences of the theorem.)

Example 2.3.  
1. The time that a bubble of \(η\) is formed corresponds to a downward jump in either \(L_t\) or \(R_t\). For convenience, we call a bubble a left bubble or right bubble if it corresponds to a downward jump in \(L_t\) or \(R_t\), respectively.

2. For \(x > 0\), let \(ρ_x > 0\) be chosen so that the \(γ\)-LQG length of \([0, ρ_x]\) is \(x\). The time at which \(η\) disconnects \(ρ_x\) from \(∞\)—or, equivalently, the time the bubble with \(ρ_x\) on its boundary is formed—is equal to the first time that the process \(R_t\) jumps below \(−x\). The analogous result holds with \(L\) in place of \(R\) and with LQG lengths along the negative real axis in place of LQG lengths along the positive real axis.

3. If \(η\) forms a left bubble at a time \(τ > 0\), then for \(t \in [0, τ]\) the point \(η(t)\) lies on the boundary of this bubble if and only if \(\inf\{s > t : L_s \leq L_t\} = τ\), i.e., the time reversed process \(L_{τ−t}\) attains a running minimum at time \(τ−t\). The analogous result holds for right bubbles.

Before introducing one last example describing the geometry of \(η\) in terms of \((L, R)\), we recall some definitions from the theory of SLE.

Definition 2.4. We say that \(t ≥ 0\) is a local cut time of \(η\), and \(η(t)\) a local cut point, if \(η([0, t)) \cap η((t, t + \epsilon)) = \emptyset\) for some \(\epsilon > 0\). We call \(t\) a global cut time and \(η\) a global cut point if \(η([0, t)) \cap η((t, ∞)) = \emptyset\). Since in this paper we will usually want to consider local rather than global cut points, we will refer to local cut points and local cut times simply as cut points and cut times, respectively.

Cut points of \(η\) are precisely the points where two complementary connected components of \(η\) intersect. In other words, cut points correspond to edges of the adjacency graph of bubbles. (See Figure 4 below for an illustration.)

Example 2.5. In terms of the left and right boundary processes, cut times are times \(t\) for which there exists \(ε > 0\) with \(L_s > L_t\) and \(R_s > R_t\) for each \(s \in (t, t + \epsilon]\); and global cut times are cut times \(t\) such that the processes \(L\) and \(R\) achieve record minima when they first jump below \(L_t\) and
respectively, after time $t$. The processes $L$ and $R$ also identify the two bubbles whose boundaries share a given cut point: if $t$ is the cut time, then the two bubbles are formed at the first times after $t$ that the processes jump below $L_t$ and $R_t$, respectively. This characterization of cut points immediately implies that the adjacency graph of bubbles is bipartite, since a left bubble can be connected only to a right bubble, and vice versa. Finally, we note that, if $t$ is a global cut time, then the union of the two corresponding bubbles $b, b'$ disconnects the set of bubbles formed before time $t$ from all other bubbles in the adjacency graph.

2.3 $(L, R)$-Markovian paths to infinity

We now use this Lévy process description of SLE$_\kappa$ for $\kappa \in (4, 8)$ to define a “Markovian path to infinity” in the adjacency graph of SLE bubbles.

**Definition 2.6.** For $\kappa \in (4, 8)$, an $(L, R)$-Markovian path to infinity in the adjacency graph of bubbles of $\eta$ is an infinite increasing sequence of stopping times $\tau_1 < \tau_2 < \tau_3 < \cdots$ for $(L, R)$ such that almost surely

- $\tau_k \to \infty$,
- $\eta$ forms a bubble $b_k$ at each time $\tau_k$ (equivalently, either $L$ or $R$ has a downward jump at time $\tau_k$), and
- $b_k$ and $b_{k+1}$ are connected in the adjacency graph (i.e., $\partial b_k \cap \partial b_{k+1} \neq \emptyset$) for each $k$.

Note that an $(L, R)$-Markovian path to infinity is a random path defined for almost every realization of the SLE$_\kappa$ curve.

The existence of $(L, R)$-Markovian paths to infinity is a sufficient condition for connectivity of the adjacency graph of bubbles.

**Lemma 2.7.** Let $\kappa \in (4, 8)$, and suppose that, for every stopping time $\zeta$ for $(L, R)$ at which $\eta$ forms a bubble almost surely, the adjacency graph of bubbles admits an $(L, R)$-Markovian path to infinity with $\tau_1 = \zeta$. Then the adjacency graph is connected almost surely.

**Proof.** The event that the adjacency graph is connected can be expressed as the countable union over all pairs of times $t_1, t_2 \in \mathbb{Q} \cap [0, \infty)$ and all $N \in \mathbb{N}$ of the event that $b_1$ and $b_2$ are joined by a path in the adjacency graph, where for $j \in \{1, 2\}$, $b_j$ is the first bubble formed after time $t_j$ that corresponds to a jump of either $L$ or $R$ of magnitude at least $1/N$. Fix such a triple $(t_1, t_2, N)$, and let $\zeta_1$ and $\zeta_2$ be the times at which $\eta$ forms the bubbles $b_1$ and $b_2$, respectively. Since $\eta$ a.s. has infinitely many global cut points (see, e.g., [MW17, Theorem 1.2]), we can a.s. choose a cut point $\eta(s)$ with $s > \zeta_1, \zeta_2$. The point $\eta(s)$ lies on the boundary of two bubbles $b_3$ and $b_4$ (adjacent to each other) that, as noted in Example 2.5 above, together disconnect the set of bubbles formed up to time $s$ from all other bubbles in the adjacency graph. Hence, the $(L, R)$-Markovian paths started at each of $\zeta_1$ and $\zeta_2$ must each pass through one of $b_3$ or $b_4$, which yield finite paths from each of $b_1$ and $b_2$ to either $b_3$ or $b_4$.\]

In light of Lemma 2.7, Theorem 1.1 will be an immediate consequence of the following theorem.

**Theorem 2.8.** Suppose $\kappa \in (4, \kappa_0)$, with $\kappa_0 \approx 5.6158$ defined as in Theorem 1.1. If $\zeta$ is a stopping time of $(L, R)$ such that $\eta$ forms a bubble at time $\zeta$ almost surely, then the adjacency graph of bubbles admits an $(L, R)$-Markovian path to infinity with $\tau_1 = \zeta$.\]
The first two bubbles in the path of bubbles defined in the proof of the first half of Proposition 3.2. The curve \( \eta \) on the interval \([0, \tau_2]\) is contained in the regions shaded in gray. The cut point at time \( \sigma_1 \) corresponds to the edge of the adjacency graph connecting the bubbles \( b_1 \) and \( b_2 \). The random variables \( X_1 \) and \( X_2 \) defined in (3.2) give the \( \gamma \)-LQG lengths of the yellow and purple arcs, respectively.

The \((L,R)\)-Markovian path appearing in Theorem 2.8 is defined explicitly in the proof of Proposition 3.2 below. The times \( \tau_k \) can be taken to be stopping times for \( \eta \) as well as for \((L,R)\).

Theorem 2.9. There exists \( \kappa_1 \in (\kappa_0, 8) \) such that for \( \kappa \in [\kappa_1, 8) \), the adjacency graph of bubbles does not admit an \((L,R)\)-Markovian path to infinity (with any choice of starting time).

Our proof of Theorem 2.9 is based on the fact that a \( \kappa/4 \)-stable process converges in law to Brownian motion as \( \kappa \) increases to 8 (Proposition 5.1), and does not give an explicit formula for \( \kappa_1 \).

3 Reducing to an estimate for a single bubble

To prove Theorems 2.8 and 2.9, we first reduce the task of proving the existence or nonexistence of an \((L,R)\)-Markovian path to infinity (Definition 2.6) to computing an expectation involving a single bubble. We first introduce some notation that we will use repeatedly throughout the paper.

Notation 3.1. For a time \( t > 0 \), we denote by \( \sigma(t) \) the smallest \( s \in [0, t) \) such that \( L_r \geq L_s \) and \( R_r \geq R_s \) for all \( r \in [s, t) \).

Proposition 3.2. Let \( \kappa \in (4, 8) \) and let \( \eta \) and \((L,R)\) be as above. Let \( \tau \) be the first time that \( R \) jumps below \(-1\) and let \( \sigma = \sigma(\tau) \) (see Notation 3.1). Equivalently (as noted in Example 2.3), let \( \tau \) be the first time that \( \eta \) absorbs the point on the positive real axis at \( \gamma \)-LQG length 1 from the origin, and let \( \sigma \) be the time of the first cut point of \( \eta|_{[0, \tau]} \) which lies on the boundary of a bubble of \( \eta \) formed after time \( \tau \). If

\[
\mathbb{E} \log(L_{\tau} - L_\sigma) \geq 0,
\]

then for each stopping time \( \zeta \) for \((L,R)\) at which \( \eta \) forms a bubble almost surely, there is an \((L,R)\)-Markovian path to infinity with \( \tau_1 = \zeta \).
Conversely, let $\mathcal{M}$ denote the set of times in $[0, \tau]$ at which $L$ achieves a record minimum, and suppose that
\[\mathbb{E} \log \left( \sup_{t \in \mathcal{M}} (L_t - L_{\sigma(t)}) \right) < 0.\] (3.1)
Then the adjacency graph of bubbles of $\eta$ does not admit an $(L, R)$-Markovian path to infinity.

**Remark 3.3.** It should be possible to estimate the values of $\kappa$ for which each of the conditions of Proposition 3.2 holds by simulating stable processes numerically. However, the times $\sigma(t)$ of Notation 3.1 are not continuous functionals of $(L, R)$ with respect to the Skorohod topology. We expect that these times still converge for suitable approximations of $(L, R)$ (see [GMS17, Section 1.5] for related discussion concerning the analogous times for correlated Brownian motions), but the rate of convergence is likely rather slow, which may complicate attempts at simulations.

**Proof of Proposition 3.2.** First, suppose that $\mathbb{E} \log (L_{\tau} - L_{\omega}) > 0$ and suppose we are given a stopping time $\zeta$ for $(L, R)$ at which $\eta$ a.s. forms a bubble. We will construct a sequence of stopping times $\zeta = \tau_1 < \tau_2 < \tau_3 < \cdots$ of $(L, R)$ that constitute an $(L, R)$-Markovian path to infinity. We set $\tau_1 = \zeta$. We then define the times $\tau_k$ for $k \geq 2$ inductively as follows. Suppose that we have defined the time $\tau_k$, and that $\eta$ forms a bubble $b_k$ at time $\tau_k$; then we set $\sigma_k = \sigma(\tau_k)$ and
\[
\tau_{k+1} := \begin{cases} 
\inf \{ s > \tau_k : L_s < L_{\sigma_k} \}, & \text{if } b_k \text{ is a right bubble} \\
\inf \{ s > \tau_k : R_s < R_{\sigma_k} \}, & \text{if } b_k \text{ is a left bubble}. 
\end{cases}
\]
Equivalently, by Examples 2.3 and 2.5, $\sigma_k$ is the time of the first cut point of $\eta|_{[0, \tau_k]}$ on the boundary of $b_k$ which also lies on the boundary of a bubble formed after $b_k$, and we choose the next bubble $b_{k+1}$ to be the bubble (other than $b_k$) which has $\eta(\sigma_k)$ on its boundary. See Figure 4.

By definition, $\eta$ forms a bubble at each time $\tau_k$, and the bubbles formed at times $\tau_k$ and $\tau_{k+1}$ are adjacent for each $k$. So, to prove $\tau_1 < \tau_2 < \tau_3 < \cdots$ is an $(L, R)$-Markovian path to infinity, we just need to check that $\tau_k \to \infty$ almost surely as $k \to \infty$. Set
\[X_k := \begin{cases} R_{\tau_k} - R_{\sigma_k} & \text{if } b_k \text{ is a left bubble} \\
L_{\tau_k} - L_{\sigma_k} & \text{if } b_k \text{ is a right bubble}. \end{cases}\] (3.2)
If $b_k$ is a right bubble, then by definition $\tau_{k+1}$ is the first time after $\tau_k$ that $L - L_{\sigma_k}$ jumps below $-X_k$. The same is true if $b_k$ is a left bubble with $L$ replaced by $R$. Hence $X_{k+1}/X_k$ is obtained from the process $X_k^{-1}(L_{\tau_k} - L_{\sigma_k}, R_{\tau_k} - R_{\sigma_k})$ in the same manner that $L_{\tau} - L_{\omega}$ is obtained from $(L, R)$, except possibly with the roles of $L$ and $R$ interchanged. By the strong Markov property, the $\kappa/4$-stable scaling property of $L$ and $R$, and the symmetry between $L$ and $R$, the random variables $X_{k+1}/X_k$ for $k \in \mathbb{N}$ are i.i.d., with the same law as $L_{\tau} - L_{\omega}$. If $\mathbb{E} \log (L_{\tau} - L_{\omega}) > 0$, then the strong law of large numbers implies that a.s. $\limsup_{k \to \infty} \sum_{j=1}^k \log(X_{j+1}/X_j) = \infty$ and therefore $\limsup_{k \to \infty} X_k = \infty$. Since $\log(L_{\tau} - L_{\omega})$ is not identically zero, an elementary argument using the Kolmogorov zero-one law shows that the same is true if $\mathbb{E} \log (L_{\tau} - L_{\omega}) = 0$. Since $\max_{s \in [0, t]} \left( |L_s| + |R_s| \right) < \infty$ for each $t > 0$, this implies that a.s. $\tau_k \to \infty$ as $k \to \infty$.

Conversely, suppose that (3.1) holds. Let $\tau_1 < \tau_2 < \tau_3 < \cdots$ be a sequence of stopping times of $(L, R)$ with $\eta = \tau_1$, such that $\eta$ a.s. forms a bubble $b_k$ at each time $\tau_k$, and $b_k$ and $b_{k+1}$ are connected in the adjacency graph for each $k$.

We claim that $\tau_k$ almost surely does not tend to infinity as $k \to \infty$. To prove this claim, we first set $\sigma_k = \sigma(\tau_k)$ and define $X_k$ as in (3.2). For each $k \in \mathbb{N}$, $\tau_{k+1}$ is a stopping time greater than $\tau_k$ such that, at time $\tau_{k+1}$, the curve $\eta$ a.s. forms a bubble whose boundary shares a cut point with $b_k$. By Example 2.3, we can characterize $\tau_{k+1}$ in terms of $(L, R)$ as follows: if $b_k$ is a right bubble,
then at time \( \tau_{k+1} \), \( L_t \) a.s. jumps below \(-x\) for some random \( x \in [L_{\sigma_k}, L_{\tau_k}] \) for the first time after \( \tau_k \).

In the special case that \( x = L_{\sigma_k} \) almost surely, the bubble \( b_{k+1} \) is the bubble with the cut point \( \eta(\sigma_k) \) on its boundary.) Equivalently, the process \( t \mapsto L_t - L_{\tau_k} \) defined for \( t > \tau_k \) achieves a record minimum at \( t = \tau_{k+1} \), and \( \min_{t \in [0, \tau_{k+1}]} (L_t - L_{\tau_k}) \geq -X_k \). The same is true if \( k \) is a left bubble with \( L \) replaced by \( R \). We deduce from the scaling and Markov properties of \( L \) and \( R \) that \( X_{k+1}/X_k \) is stochastically dominated by \( \sup_{t \leq M} (L_t - L_{\sigma(t)}) \). Since (3.1) holds, the strong law of large numbers implies that a.s. \( \lim_{k \to \infty} \sum_{j=1}^k \log(X_{j+1}/X_j) = -\infty \) and therefore that \( \lim_{k \to \infty} X_k = 0 \).

Now, unlike in the first part of the proof, we cannot immediately conclude that \( \tau_k \) almost surely does not tend to infinity as \( k \to \infty \). The statement \( X_k \xrightarrow{a.s.} 0 \) says that some measure of boundary length of the bubbles \( b_k \) is tending to zero; we want to deduce from this that the path of bubbles must remain in some compact subset of \( \mathbb{H} \).

To see this, we observe that Example 2.5 implies that on the event that \( \tau_k \to \infty \), it must be the case that for each global cut point \( t \) of \( \eta \) with \( t > \tau_k \), the sequence of bubbles \( \{b_k\}_{k \in \mathbb{N}} \) must include one of the bubble with \( \eta(t) \) on its boundary. By Lemma 3.4 below, we can choose a further subsequence of bubbles \( b_{n_k} \) such that the corresponding random variables \( X_{n_k} \) are uniformly bounded from below. Since \( X_n \to 0 \) almost surely, we deduce that \( \tau_k \) almost surely does not tend to infinity, as desired.

We now state and prove Lemma 3.4, the missing ingredient we needed to prove Proposition 3.2.

**Lemma 3.4.** Let \( \eta \) be an \( SLE_\kappa \) curve for \( \kappa \in (4, 8) \), and let \( C > 0 \) be a constant. Almost surely, there are infinitely many global cut points of \( \eta \) such that, if \( \tau_1 \) and \( \tau_r \) are the times \( \eta \) forms the left and right bubbles whose boundaries share this cut point, then

\[
(R_{\tau_l} - R_{\sigma(\tau_l)}) \land (L_{\tau_r} - L_{\sigma(\tau_r)}) \geq C.
\]

**Proof.** We define times

\[
r_1 < s_1 < t_1 < r_2 < s_2 < t_2 < r_3 < \cdots
\]

inductively as follows. Set \( r_0 = s_0 = t_0 = 0 \). Inductively, let \( r_k \) be the first time \( t > t_{k-1} \) such that \( R \) attains a running minimum at \( t \) and \( L_t - \min_{s \in [0, t]} L_s \geq 1 \).

\[
1 \text{ Let } s_k \text{ the first global cut time of } R \text{ after time } r_{k-1}; \text{ such a cut time exists a.s. since } \eta \text{ a.s. has infinitely many global cut points (see, e.g. [MW17, Theorem 1.2]). Finally, let}
\]

\[
t_k = \inf \{ t > s_k : L_t < L_{s_k} \} \lor \inf \{ t > s_k : R_t < R_{s_k} \},
\]

i.e., \( t_k \) is the larger of the two times at which \( \eta \) forms a bubble whose boundary contains the cut point \( \eta(t_k) \).

Using Example 2.5, each \( r_k \) and each \( t_k \) is a stopping time for \( (L, R) \). By Example 2.3 the random variable of (3.3) associated to the cut point \( s_k \) is a.s. determined by \( (L, R)|_{[0, t_k]} \).

We claim that this sequence stochastically dominates an i.i.d. sequence of random variables. If we can prove this claim, then the lemma will follow directly from applying Kolmogorov’s 0-1 law. To show why this claim is true, we first recall how we defined global cut times in terms of

\footnote{It is not hard to see that such a time always exists: Since the running minimum process of \( R \) is a subordinator (Lemma VIII.1 on page 218 of [Ber96]), we can find infinitely many disjoint time intervals that are uniformly large (by the regenerative property of subordinators) and whose endpoints are times at which \( R \) attains running minima. The restrictions of \( L \) to these time intervals are conditionally independent given \( R \), so the 0-1 law implies that, on at least one of these time intervals, the value of \( L \) at the right endpoint of the interval will exceed its minimum on that interval by at least one.}
(L, R) in Example 2.5. In our setting, since \( r_k \) is a stopping time, we can similarly characterize the conditional distribution of \( s_k - r_k \) given \( L|_{[0, r_k]} \): the law of \( s_k - r_k \) is equal to the law of the first global cut time of \((L, R)\) such that the record minimum that \( L \) achieves at the first time \( \eta \) hits \((-\infty, 0]\) after this global cut time is \( \leq L_{r_k} - \min_{s \in [0, r_k]} L_s \). Since \( L_{r_k} - \min_{s \in [0, r_k]} L_s \geq 1 \), we deduce by the scaling property of \((L, R)\) that the random variable of (3.3) associated to the cut point \( s_k \) stochastically dominates an a.s. positive random variable defined independently of \( k \), namely, the random variable (3.3) associated to the first global cut time of \((L, R)\) such that the record minimum that \( L \) achieves after this global cut time is \( \leq -1 \). This proves our claim, and hence the lemma.

Proposition 3.2 implies that, to prove Theorems 2.8 and 2.9, it is enough to prove the following estimates for a single bubble of an SLE\( _\kappa \) curve:

**Theorem 3.5.** Fix \( \kappa \in (4, \kappa_0] \), where \( \kappa_0 \approx 5.6158 \) is defined as in Theorem 1.1. Let \( \tau \) be the first time that \( R \) jumps below \(-1\) and \( \sigma = \sigma(\tau) \). Then \( \mathbb{E} \log(L_\tau - L_\sigma) \geq 0 \).

**Theorem 3.6.** There exists \( \kappa_1 \in (\kappa_0, 8) \) such that for \( \kappa \in [\kappa_1, 8) \), the following is true. Let \( \mathcal{M} \) denote the set of times \( \leq \tau \) at which \( L \) achieves a record minimum. Then

\[
\mathbb{E} \log \left( \sup_{t \in \mathcal{M}} (L_t - L_{\sigma(t)}) \right) < 0.
\]

The next section is devoted to proving Theorem 3.5; we will prove Theorem 3.6 in Section 5.

### 4 Proof of Theorem 3.5

In this section we prove Theorem 3.5. In terms of \( \eta \), the time \( \tau \) in the theorem statement is the first time that \( \eta \) absorbs the point \( \rho_1 \) on the positive real axis at \( \gamma \)-LQG length 1 from the origin, and \( \sigma \) is the first cut point incident to both the bubble formed at time \( \tau \) and some bubble formed at a later time. In our proof of Theorem 3.5, we will also refer to the time \( \xi \) at which the process \( R \) achieves its minimum on \([0, \tau]\)—or, equivalently, the last time \( \eta \) hits the positive real axis before time \( \tau \). Figure 5 illustrates the definitions of the three times \( \xi, \sigma \) and \( \tau \) in terms of both \( \eta \) and the pair of processes \((L, R)\).

Our proof of Theorem 3.5 consists of three main steps.

1. **Showing that** \( L_\tau - L_\sigma \) **stochastically dominates** \( R_{\tau^-} - R_\sigma \). Since the definition of \( \sigma \) is tied closely to that of \( \tau \), which depends on \( R \) but not on \( L \), it is technically easier to study the random variable \( R_{\tau^-} - R_\sigma \) instead of \( L_\tau - L_\sigma \). So, we begin by showing that \( L_\tau - L_\sigma \) stochastically dominates \( R_{\tau^-} - R_\sigma \) (Proposition 4.1), which reduces the task of proving Theorem 3.5 to showing that \( \mathbb{E} \log(R_{\tau^-} - R_\sigma) \geq 0 \).

2. **Characterizing the law of** \((L, R)\) **run backwards from** \( \tau \) **to** \( \xi \). Since \( \sigma \) is most easily described in terms of the time-reversed processes \( L_{(\tau-t)^-} \) and \( R_{(\tau-t)^-} \), we next determine the joint law of these time-reversed processes. In Proposition 4.6, we show that if we run \( L \) and \( R \) backward from time \( \tau \) until the time \( \xi \) at which \( R \) reaches its minimum on \([0, \tau]\), then, conditional on \( \{R_{\tau^-} - R_\xi = r\} \), the law of this pair of time-reversed processes is the same (up to a vertical translation) as that of \((-L, -R)\) run until \(-R \) hits the level \(-r \). It follows (Corollary 4.8) that the regular conditional distribution of \( R_{\tau^-} - R_\sigma \) given \( \{R_{\tau^-} - R_\xi = r\} \) is equal to the law of the value of \( R \) at the time \( \theta_r \) of the last simultaneous running supremum of
Figure 5: The times $\xi$, $\sigma$ and $\tau$, defined in terms of $\eta$ (left) and in terms of $(L, R)$ (middle and right). Theorem 3.5 asserts the $\gamma$-LQG length of the yellow boundary arc—or, equivalently, the size of the increment in $L$ colored yellow in the middle graph—has nonnegative log expectation. The first step of our proof of Theorem 3.5 shows that this quantity stochastically dominates the $\gamma$-LQG length of the blue boundary arc—or, equivalently, the size of the increment in $R$ colored blue in the right graph.

$$(L, R)$$ before $R$ hits the level $r$. By the scaling property of stable processes, this implies that the expectation of $\log(R_{\tau^*} - R_\sigma)$ is equal to the sums of the expectations of $\log(R_{\tau^*} - R_\xi)$ and $\log(R_{\theta_1})$ (equation (4.8) below).

3. Computing the expectations of $\log(R_{\tau^*} - R_\xi)$ and $\log(R_{\theta_1})$. By the previous step, to prove Theorem 3.5, it is enough to show that the sum of the expectations of $\log(R_{\tau^*} - R_\xi)$ and $\log(R_{\theta_1})$ is positive. The first term is easy to handle: we derive the law of $R_{\tau^*} - R_\xi$ directly from a result in [DK06]. To analyze the law of $\log(R_{\theta_1})$, we use the fact from [DMS14] that the law of $(L, R)$ is equal to a time reparametrization of a pair $(\tilde{L}, \tilde{R})$ of correlated Brownian motions to express the law of $R_{\theta_1}$ as that of $\tilde{R}_{\tilde{\theta}_1}$, where $\tilde{\theta}_1$ is the last simultaneous running supremum of $(\tilde{L}, \tilde{R})$ before $\tilde{R}$ hits the level $r$. It follows from results in [Eva85] and [HP74] that the set of running suprema of a planar Brownian motion has the law of the range of a subordinator whose index we can compute explicitly; hence, we can deduce the law of $R_{\theta_1}$ from the arcsine law for subordinators [Ber99].

The next three subsections of the paper are devoted to the proofs of these three main steps. Some of the steps will use a discrete approximation of $(L, R)$, so before presenting those proofs, we recall the following consequence of the stable functional central limit theorem.

Let $\{X_j\}_{j \in \mathbb{N}}$ be an i.i.d. sequence of centered random variables with

$$P(X_1 = 1) = 1 - c \quad \text{and} \quad P(X_1 \leq -m) = cm^{-\kappa/4} \text{ for } m \in \mathbb{N}, \quad (4.1)$$

where the constant $c$ is chosen so that $EX_1 = 0$, and let $S_n = \sum_{i=1}^{n} X_i$ be the associated heavy-tailed random walk. Then, for some constant $C > 0$ (recall Remark 2.2), the rescaled walk

$$W_t^{(n)} := Cn^{-4/\kappa}S_{[nt]} \quad (4.2)$$

converges in distribution to $L$ in the space of cadlag functions $\mathcal{D}([0, \infty), \mathbb{R})$ with respect to the Skorohod topology (see, e.g., [JS03]).

4.1 Showing that $L_{\tau^*} - L_\sigma$ stochastically dominates $R_{\tau^*} - R_\sigma$.

We now begin with the first step of the proof, which is summarized in the following proposition.
Proposition 4.1. The random variable \( L_\tau - L_\sigma \) stochastically dominates \( R_\tau - R_\sigma \), i.e.,

\[ \mathbb{E}(g(L_\tau - L_\sigma)) \geq \mathbb{E}(g(R_\tau - R_\sigma)) \]

for all non-decreasing functions \( g \).

To prove Proposition 4.1, we want to characterize the regular conditional distributions of \( L \) and \( R \) on \([\sigma, \tau]\) given that \( \tau - \sigma = t \) and \( R_\sigma + 1 = r \). Intuitively, we should get (up to vertical translation) a pair of Lévy processes started at zero and conditioned to stay positive until time \( t \), with the second process jumping below \(-r\) at time \( t \). In the proof that follows, we will precisely define this laws of these two processes, and show that the law of the second process is equal to the law of the first process weighted by a decreasing function of its value at time \( t \) (Lemma 4.4). By a general probability result (Lemma 4.5), this property implies that the first process dominates the second, which is exactly the result we want to prove.

Before delving into the proofs of Lemmas 4.4 and 4.5, which will together imply Proposition 4.1, we introduce some definitions and results from the literature that we will use in the proofs of these two lemmas.

First, to analyze stochastic processes restricted to bounded intervals as random variables with values in \( D([0, \infty), \mathbb{R}) \), we introduce the following convention: if \( X : [0, \infty) \to \mathbb{R} \) is a càdlàg stochastic process and \( a < b \) are positive real numbers, then we define the process \( X \) on the interval \([a, b]\) as the process \( Y : [0, \infty) \to \mathbb{R} \) with \( Y_t = X_{t+a} \) for \( t \in [0, b-a) \) and \( Y_t = 0 \) for \( t \geq b-a \). Similarly, we define the process \( X \) on the interval \([a, b]\) as the process \( Y : [0, \infty) \to \mathbb{R} \) with \( Y_t = X_{t+a} \) for \( t \in [0, b-a] \) and \( Y_t = X_b \) for \( t \geq b-a \).

Second, our proof of Lemma 4.4 below uses two approximation procedures: the discrete approximations of Lévy processes by random walks given by (4.2), and an approximation of the condition that the processes stay positive by a condition that they stay at least \( \epsilon \) below their starting point. To take the necessary limits of the associated regular condition distributions, we will repeatedly use the following lemma, whose proof follows from the definition of regular conditional distributions:

Lemma 4.2. Let \( (X_n, Y_n) \) be a sequence of pairs of random variables with values in a product of separable metric spaces \( E_X \times E_Y \) with the Borel \( \sigma \)-algebra. Let \((X, Y)\) be another such pair of random variables with \((X_n, Y_n) \to (X, Y)\) in law. Suppose further that \( \nu_n : E_Y \times \mathcal{B}(E_X) \to [0, 1] \) is the regular conditional distribution of \( X_n \) given \( Y_n \), and that \( \nu_n(y, \cdot) \Rightarrow \nu(y, \cdot) \) weakly as measures (for \( Y \)-a.a. \( y^2 \)) for some function \( \nu : E_Y \times \mathcal{B}(E_X) \to [0, 1] \) that is a probability measure in its second argument and a measurable function in its first argument. Then \( \nu \) is the regular conditional distribution of \( X \) given \( Y \).

Finally, in order to take the \( \epsilon \to 0 \) limit of the processes conditioned to stay at least \( \epsilon \) below their starting point, we will need to know that the law of a Lévy process on \([0, t]\) started at \( \epsilon \) and conditioned to stay positive on \([0, t]\) tends to a unique limit (in the Skorohod topology) as \( \epsilon \to 0 \). This is the content of the following lemma, which appears as Lemma 4 in [CD10]:

Lemma 4.3. The law of a Lévy process on \([0, t]\) started at \( \epsilon \) and conditioned to stay positive on \([0, t]\) tends to a unique limit \( L_{t|t}^+ \) (in the Skorohod topology) as \( \epsilon \to 0 \); we call this limiting process the meander with length \( t \).

We can now characterize precisely the regular conditional distributions of \( L \) and \( R \) on \([\sigma, \tau]\) given that \( \tau - \sigma = t \) and \( R_\sigma + 1 = r \).

\footnote{We will omit this detail in future since a regular conditional probability \( \nu' : E_Y \times \mathcal{B}(E_X) \to [0, 1] \) of \( X \) given \( Y \) is defined only up to equality for \( Y \)-a.a. \( y \).}
Lemma 4.4. The regular conditional distributions of $L_{\sigma^+} - L_\sigma$ and $R_{\sigma^+} - R_\sigma$ on $[0, \tau - \sigma)$ given \( \{\tau - \sigma = t\} \cap \{R(\sigma) + 1 = r\} \) are given, respectively, by the law of the meander $L^+_{t|t}$ and the law of the meander $L^+_{t|t}$ weighted by
\[
\frac{(L^+_{t|t} + r)^{-\kappa/4}}{\mathbb{E}\left((L^+_{t|t} + r)^{-\kappa/4}\right)}.
\] (4.3)

Proof. Let $L^{(n)}$ and $R^{(n)}$ be independent copies of the rescaled walk $W^{(n)}$ of (4.2). Also, for fixed $r, \epsilon > 0$, let $L^{(n,r,\epsilon)}$ and $R^{(n,r,\epsilon)}$ be obtained from the independent processes $L^{(n)} + \epsilon$ and $R^{(n)} + \epsilon$ by conditioning both processes to stay positive until the first time $\tau^{(n,r,\epsilon)}$ that the process $R^{(n,r,\epsilon)}$ hits the level $-r$. We define the processes $L^{(r,\epsilon)}$ and $R^{(r,\epsilon)}$ and the stopping time $\tau^{(r,\epsilon)}$ analogously with $(L, R)$ in place of $(L^{(n)}, R^{(n)})$. Since we are conditioning on a positive probability event,
\[
(L^{(n,r,\epsilon)}, R^{(n,r,\epsilon)}, \tau^{(n,r,\epsilon)}) \xrightarrow{\mathcal{L}} (L^{(r,\epsilon)}, R^{(r,\epsilon)}, \tau^{(r,\epsilon)})
\] (4.4)

By the choice of step distribution in (4.1) and Bayes’ rule, the following two distributions are equal:

(I) the regular conditional distribution of $L^{(n,r,\epsilon)}$ on the interval $[0, \tau^{(n,r,\epsilon)} - 1/n)$ given $\{\tau^{(n,r,\epsilon)} = t\}$, weighted by \( \left(L^{(n,r,\epsilon)}_{t-1/n} + r\right)^{-\kappa/4}/\mathbb{E}\left((L^{(n,r,\epsilon)}_{t-1/n} + r)^{-\kappa/4}\right) \);

(II) the regular conditional distribution of $R^{(n,r,\epsilon)}$ on the interval $[0, \tau^{(n,r,\epsilon)} - 1/n)$ given $\{\tau^{(n)} = t\}$.

To prove the lemma, we would like to use this equality in distribution and take the limit as $n \to \infty$ and $\epsilon \to 0$. The $n \to \infty$ limit is fairly straightforward: applying (4.4) and Lemma 4.2, we see that the following distributions are also equal:

(III) the regular conditional distribution of $L^{(r,\epsilon)}$ on the interval $[0, \tau^{(r,\epsilon)})$ given $\{\tau^{(r,\epsilon)} = t\}$, weighted by \( \left(L^{(r,\epsilon)}_{t-} + r\right)^{-\kappa/4}/\mathbb{E}\left((L^{(r,\epsilon)}_{t-} + r)^{-\kappa/4}\right) \).

(IV) the regular conditional distribution of $R^{(r,\epsilon)}$ on the interval $[0, \tau^{(r,\epsilon)})$ given $\{\tau^{(r,\epsilon)} = t\}$.

Next, we would like to take $\epsilon \to 0$. By Lemma 4.3, the regular conditional distribution of $L^{(r,\epsilon)}$ on $[0, \tau^{(r,\epsilon)})$ given $\{\tau^{(r,\epsilon)} = t\}$ converges weakly as $\epsilon \to 0$ to the meander $L^+_{t|t}$ with length $t$. By the equality of the laws (III) and (IV), Lemma 4.3 also implies that (IV) converges weakly as $\epsilon \to 0$.

Taking $\epsilon \to 0$ in (III) and (IV), we deduce that

(V) the law of $L^+_{t|t}$, weighted by (4.3)

is equal to

(VI) the weak limit of (IV) as $\epsilon \to 0$.

\[3\text{To apply the lemma, it is enough to show that the regular conditional distribution of } L^{(n,r,\epsilon)} \text{ on the interval } [0, \tau^{(n,r,\epsilon)} - 1/n) \text{ given } \{\tau^{(n,r,\epsilon)} = t\} \text{ converges weakly as } n \to \infty \text{ for a.a. } t. \text{ Indeed, the regular conditional distribution of } L^{(n,r,\epsilon)} \text{ given } \{\tau^{(n,r,\epsilon)} = t\} \text{ is, for a.a. } t, \text{ the distribution of the process } L^{(n)} \text{ started at } \epsilon \text{ and conditioned to stay positive until time } t. \text{ The latter converges in law as } n \to \infty \text{ to the law of the Lévy process } L \text{ started at } \epsilon \text{ and conditioned to stay positive until time } t. \]
So, to prove the lemma, it is enough to prove the following claim:

**Claim.** The regular conditional distributions of $L_{\sigma+} - L_{\sigma} - R_{\sigma+} - R_{\sigma}$ on $[\sigma, \tau]$ given $\{\tau - \sigma = t\} \cap \{R(\sigma) + 1 = r\}$ are given, respectively, by the law of $L^+_t$ and (VI) with $r = 1 + R_{\sigma}$.

Fix $s, \delta > 0$. For $(\mathcal{L}, \mathcal{R}) \in \mathcal{D}([0, s + \delta], \mathbb{R}^2)$, the regular conditional distribution of $(L_{\sigma+} - L_{\sigma}, R_{\sigma+} - R_{\sigma})$ given that $\sigma = s$ and $(L, R)_{|[0, \sigma + \delta]} = (\mathcal{L}, \mathcal{R})$ (when these conditions are compatible) is that of a pair of independent Lévy processes conditioned to stay above $L_s - L_{s+\delta}$ and $R_s - R_{s+\delta}$, respectively, until the first time the second process jumps below $-1 - R_{s+\delta}$. Hence, considering the processes $L$ and $R$ separately, we have the following.

- The regular conditional distribution of $L_{\sigma+} - L_{\sigma}$ given $\{\sigma = s\}$, $\{\tau = w\}$, and $\{(L, R)_{|[0, \sigma + \delta]} = (\mathcal{L}, \mathcal{R})\}$ (when these conditions are compatible) is that of a Lévy process conditioned to stay above $L_s - L_{s+\delta}$ until time $w - s - \delta$. By scaling, this is the same is the law of a Lévy process conditioned to stay above $(L_s - L_{s+\delta})(w - s - \delta)^{4/\kappa}$ until time $w - s$.

Taking $\delta \to 0$ and applying Lemmas 4.3 and 4.2, we deduce that the regular conditional distribution of $L_{\sigma+} - L_{\sigma}$ given $\{\sigma = s\}$, $\{\tau = w\}$, and $\{(L, R)_{|[0, \sigma]} = (\mathcal{L}, \mathcal{R})\}$ is the law of a Lévy meander $L^+_t$ with length $w - s$.

- The regular conditional distribution of $R_{\sigma+} - R_{\sigma}$ given $\{\sigma = s\}$, $\{\tau = w\}$, and $\{(L, R)_{|[0, \sigma + \delta]} = (\mathcal{L}, \mathcal{R})\}$ (when these conditions are compatible) is that of a Lévy process conditioned to stay above $R_s - R_{s+\delta}$ until jumping below $-1 - R_{s+\delta}$ at time $w - s - \delta$. By scaling, this is the same is the law of a Lévy process conditioned to stay above $(R_s - R_{s+\delta})(w - s - \delta)^{4/\kappa}$ until jumping below $-1 - R_{s+\delta}$ at time $w - s$.

Vertically translating by $R_{s+\delta} - R_s$ yields exactly the law (IV) with $\epsilon, r$ and $t$ given by $(R_{s+\delta} - R_s)(w - s - \delta)^{4/\kappa}$, $(1 + R_s)(w - s - \delta)^{4/\kappa}$, and $w - s$, respectively.

Taking $\delta \to 0$ and applying Lemma 4.3, we deduce that the regular conditional distribution of $R_{\sigma+} - R_{\sigma}$ on $[0, w - s]$ given $(L, R)_{|[0, s]}$, $\{\sigma = s\}$, and $\{\tau = w\}$ is given by (VI) with $r$ and $t$ replaced by $1 + R_{\sigma}$ and $w - s$, respectively.

This proves the claim, and hence the lemma.

Hence, the result of Proposition 4.1 is a simple application of the following probability fact (setting $X = L^+_t$ and $f(x) = C(x + 1)^{-\kappa/4}$ where $C$ is the appropriate constant).

**Lemma 4.5.** Let $X$ be a nonnegative random variable, let $f : [0, \infty) \to [0, \infty)$ be a non-increasing function with $\mathbb{E}f(X) = 1$, and let $g : [0, \infty) \to \mathbb{R}$ be a non-decreasing function. Then $\mathbb{E}(f(X)g(X)) \leq \mathbb{E}g(X)$.

**Proof.** We write

$$\mathbb{E}(f(X)g(X)) - \mathbb{E}(g(X)) = \mathbb{E}((f(X) - 1)1_{f(X) > 1}g(X)) - \mathbb{E}((1 - f(X))1_{f(X) \leq 1}g(X)) \tag{4.5}$$

Observe that

$$\mathbb{E}((f(X) - 1)1_{f(X) > 1}) = \mathbb{E}((1 - f(X))1_{f(X) \leq 1}) \tag{4.6}$$

Hence, the result of Proposition 4.1 is a simple application of the following probability fact (setting $X = L^+_t$ and $f(x) = C(x + 1)^{-\kappa/4}$ where $C$ is the appropriate constant).
since the difference of the two expectations is \( E(f(X) - 1) \), which equals zero by hypothesis. On the other hand, since \( f \) is non-increasing and \( g \) is non-decreasing, \( g(x) \leq g(y) \) for any \( x, y \) with \( f(x) > 1 \) and \( f(y) \leq 1 \). Combining this with (4.6) shows that

\[
\mathbb{E}((f(X) - 1)1_{f(X) > 1}g(X)) \leq \mathbb{E}((1 - f(X))1_{f(X) \leq 1}g(X))
\]

which proves the lemma due to (4.5).

\[\square\]

### 4.2 Characterizing the law of \((L, R)\) run backwards from \(\tau\) to \(\xi\).

Recall that \(\xi\) is the time at which \(R\) attains its minimum on \([0, \tau]\), equivalently the time of the last running minimum of \(R\) before time \(\tau\). The result of Proposition 4.1 reduces the task of proving of Proposition 3.5 from showing that \(\mathbb{E} \log(L_\tau - L_\sigma) > 0\) to showing that \(\mathbb{E} \log(R_\tau - R_\sigma) > 0\). The latter is a more tractable quantity since the definition \(\sigma\) is, in some sense, more closely tied to the process \(R\). If we could reduce the problem even further to analyzing the (closely related) random variable \(R_\tau - R_\xi\), then we will have eliminated the dependence on the process \(L\) entirely. In this subsection, we will relate the quantities \(R_\tau - R_\sigma\) and \(R_\tau - R_\xi\) by deriving an expression for the regular conditional distribution of \((L, R)\) run backward from \(\tau\) to \(\xi\) given \(\{R_{\tau-} - R_\tau = \tau\}\). This expression will directly yield the formula (4.8) below for \(\mathbb{E} \log(R_\tau - R_\sigma)\) in terms of \(\mathbb{E} \log(R_\tau - R_\xi)\).

By Example 2.3, \(\tau - \sigma\) is the time of the last simultaneous running infimum of the processes \(L_{(\tau-t)\tau}\) and \(R_{(\tau-t)\tau}\) on \([0, \tau]\). Equivalently, \(\sigma\) is the last simultaneous running supremum of \(\{L_{\tau-t} - L_{(\tau-t)\tau}\}_{t \in [0, \tau-\xi]}\) and \(\{R_{\tau-t} - R_{(\tau-t)\tau}\}_{t \in [0, \tau-\xi]}\). The joint law of these processes has a neat description when conditioned on the quantity \(R_{\tau-} - R_\xi\).

**Proposition 4.6.** The regular conditional joint distribution of the processes \(\{L_{\tau-t} - L_{(\tau-t)\tau}\}_{t \in [0, \tau-\xi]}\) and \(\{R_{\tau-t} - R_{(\tau-t)\tau}\}_{t \in [0, \tau-\xi]}\) given \(\{R_{\tau-} - R_\xi = \tau\}\) is equal to the law of \((L, R)\) stopped at the first time the process \(R\) hits level \(\tau\).

To prove the proposition, we first note that, since \(\tau\) and \(\xi\) are independent of \(L\), the process \(\{L_{\tau-t} - L_{(\tau-t)\tau}\}_{t \in [0, \tau-\xi]}\) has the law of a \(\frac{1}{\xi}\)-stable Lévy process with only negative jumps on \([0, \tau-\xi]\), independent of \(R\). So we can restrict our attention to the process \(\{R_{\tau-t} - R_{(\tau-t)\tau}\}_{t \in [0, \tau-\xi]}\). To study this process, it helps to approximate \(R\) by a process \(R^{(n)}\) with the law of the re-scaled walk from (4.2).

Let \(\tau^{(n)}\) denote the first time that the process \(R^{(n)}\) hits the level \(-1\), and let \(\xi^{(n)}\) be the last time \(R^{(n)}\) achieves its minimum on \([0, \tau^{(n)}]\). Since the limiting process \(R\) achieves its minimum on \([0, \tau]\) at the unique time \(\xi\), we deduce from the stable FCLT that

\[
\{R^{(n)}_{t+\xi^{(n)}}\}_{t \in [0, \tau^{(n)}-\xi^{(n)}]} \rightarrow \{R_t+\xi\}_{t \in [0, \tau-\xi]} \quad \text{in law w.r.t. the Skorokhod topology.} \hspace{1cm} (4.7)
\]

We are interested in the cadlag-modified time reversal of this limiting process (vertically translated to start from zero and reflected about the \(x\)-axis), namely, the process \(\{R^{(n)}_{(\tau^{(n)}-t)} - R^{(n)}_{(\tau^{(n)}-t)\tau}\}_{t \in [0, \tau^{(n)}-\xi^{(n)}]}\). If we consider instead the corresponding time reversal \(\{R^{(n)}_{(\tau^{(n)}-t)} - R^{(n)}_{(\tau^{(n)}-t)\tau}\}_{t \in [0, \tau^{(n)}-\xi^{(n)}]}\) of the approximating walk \(\{R^{(n)}_{t+\xi^{(n)}}\}_{t \in [0, \tau^{(n)}-\xi^{(n)}]}\), then the law of the resulting process follows directly from the following lemma.

**Lemma 4.7.** Suppose that \(\{X_j\}_{j \in \mathbb{N}}\) is an i.i.d. sequence of random variables with distribution (4.1) and \(S_n = \sum_{i=1}^n X_i\) is the associated heavy-tailed random walk. Let \(\tau(b)\) be the first time that \(S_t = b\) for \(b > 0\) or \(S_t \leq b\) for \(b < 0\). Then, for any fixed \(r > 0 > y\), the law of \((X_1, \ldots, X_{\tau(b)})\) is equal to the conditional law of \((X_{\tau(y)-1}, \ldots, X_2, X_1)\) given that \(S_n > 0\) for all \(1 \leq n < \tau(y)\) and \(S_{\tau(y)-1} = r\).
Proof. Both laws are supported on the set
\[
\mathcal{X} = \bigcup_{n \in \mathbb{N}} \left\{ (x_1, \ldots, x_n) : \sum_{j=1}^n x_j = r \text{ and } \sum_{j=1}^m x_j < r \text{ for all } 1 \leq m < n \right\} \subset \bigcup_{n \in \mathbb{N}} (\mathbb{Z}_{\leq 1})^n.
\]
For \( x \in \mathbb{Z}_{\leq 1} \), write \( p(x) := \mathbb{P}[X_1 = x] \). Since \( \tau(r) \) and \( \tau(y) \) are stopping times for \( S_n \), we have for each \( (x_1, \ldots, x_n) \in \mathcal{X} \)
\[
\mathbb{P}\left[(X_1, \ldots, X_{\tau(r)}) = (x_1, \ldots, x_n)\right] = \mathbb{P}\left[(X_1, \ldots, X_n) = (x_1, \ldots, x_n)\right] = p(x_1) \cdots p(x_n)
\]
and
\[
\mathbb{P}\left[(X_{\tau(y)-1}, \ldots, X_1) = (x_1, \ldots, x_n) | S_n > 0 \text{ for all } 1 \leq n < \tau(y) \text{ and } S_{\tau(y)-1} = r \right]
= C_{r,y} \mathbb{P}[\{X_1, \ldots, X_n\} = (x_n, \ldots, x_1), X_{n+1} \leq -(r+y)]
= C_{r,y} \mathbb{P}[X_1 \leq -(r+y)] p(x_1) \cdots p(x_n).
\]
for some constant \( C_{r,y} \). Summing over all elements of \( \mathcal{X} \) yields \( C_{r,y} \mathbb{P}[X_1 \leq -(r+y)] = 1 \), proving the result. \( \square \)

Proof of Proposition 4.6. The conditional law of \( R^{(n)}|_{[\xi^{(n)}, \tau^{(n)}]} \) given \( \xi^{(n)} \) is that of a re-scaled heavy-tailed random walk started from 0 and conditioned to stay positive until it jumps below level \(-1\). From Lemma 4.7, applied to the above conditional law, we deduce that the conditional law \( \{R^{(n)}|_{[\tau^{(n)} \wedge \xi^{(n)}]} - R^{(n)}|_{[\tau^{(n)} \wedge \xi^{(n)}]}\} \) given \( \{R^{(n)}|_{[\tau^{(n)} \wedge \xi^{(n)}]} - R^{(n)}|_{[\tau^{(n)} \wedge \xi^{(n)}]} = r\} \) is the same as the law of \( R^{(n)} \) run until the first time that it hits the level \( r \). By this, (4.7), and Lemma 4.2, the conditional law \( \{R^{\wedge} - R^{(\wedge)}\} \) given \( \{R^{\wedge} - R^{\wedge} = r\} \) is that of a \( \frac{\kappa}{4} \)-stable Lévy process with only negative jumps stopped at the first time the process hits level \( r \). This concludes the proof. \( \square \)

Proposition 4.6 immediately implies the following corollary.

Corollary 4.8. The regular conditional distribution of \( R^{\wedge} - R^{\wedge} \) given \( \{R^{\wedge} - R^{\wedge} = r\} \) is equal to the law of the value of \( R \) at the time \( \theta^{\wedge} \) of the last simultaneous running supremum of \( (L, R) \) before \( R \) hits the level \( r \). In particular, since \( R^{\theta^{\wedge}} \overset{d}{=} r R_{\theta^{\wedge}} \) by scaling,
\[
\mathbb{E} \log(R^{\wedge} - R^{\wedge}) = \mathbb{E} \log(R^{\wedge} - R^{\wedge}) + \mathbb{E} \log(R_{\theta^{\wedge}}).
\]

4.3 Computing the expectations of \( \log(R^{\wedge} - R^{\wedge}) \) and \( \log(R_{\theta^{\wedge}}) \).

To finish the proof of Theorem 3.5, we compute the right-hand side of (4.8) and show it is non-negative for \( \kappa \in (4, \kappa_0) \). We treat the two terms separately.

Lemma 4.9. One has \( \mathbb{E} \log(R^{\wedge} - R^{\wedge}) = \pi \cot(\pi \kappa/4) \).

Proof. The law of \( \log(R^{\wedge} - R^{\wedge}) \) is given explicitly in the literature: [DK06, Example 7] gives the explicit joint density\(^4\)
\[
\mathbb{P}(-1 - R^{\wedge} \in dv, R^{\wedge} + 1 \in dv, R^{\wedge} + 1 \in dy) = \frac{\kappa}{4} \frac{\sin(\pi \kappa/4)}{\pi} \frac{(1 - y)^{\kappa - 2}}{(v + u)^{\kappa + 1}} dv \, du \, dy \quad (4.9)
\]
\(^4\)The formula in [DK06] uses the positivity parameter \( \rho \) defined as \( \mathbb{P}(R_1 \geq 0) \). For a \( \kappa/4 \)-stable process with only negative jumps, \( \rho = 1 - 4/\kappa \) (page 218 of [Ber96]).
for $u > 0$, $y \in [0, 1]$, and $v \geq y$. Substituting $v = y + w$ and integrating out $u$ gives

$$\mathbb{P}(R_{\tau^+} - R_{\xi} \in dw, R_{\xi} + 1 \in dy) = \left(1 - \frac{\kappa}{4}\right) \frac{\sin(\pi\kappa/4)}{\pi} \frac{(1 - y)^{\kappa/4 - 2}}{(y + w)^{\kappa/4}} dw dy$$

This last density has antiderivative

$$\mathbb{P}(R_{\tau^+} - R_{\xi} \in dw) = -\frac{\sin(\pi\kappa/4)}{\pi} \frac{w^{1 - \kappa/4}}{1 + w} dw.$$

(4.10)

Therefore, using the well-known identities for the Beta function $B(p, q)$ (see, e.g., Section 15.02 of [JJ99])

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p + q)} = \int_0^1 x^{p-1}(1 - x)^{q-1} dx, \quad p, q > 0$$

(4.11)

and

$$B(p, 1 - p) = \frac{\pi}{\sin(p\pi)} \quad 0 < p < 1$$

(4.12)

we get

$$\mathbb{E} \log(R_{\tau^+} - R_{\xi}) = -\frac{\sin(\pi\kappa/4)}{\pi} \int_0^1 \log(w) \frac{w^{1 - \kappa/4}}{1 + w} dw$$

$$= \frac{\sin(\pi\kappa/4)}{\pi} \frac{\partial}{\partial \beta} \left(\int_0^\infty \frac{w^{1-\beta}}{1 + w} dw\right)\bigg|_{\beta = \kappa/4}$$

$$= \frac{\sin(\pi\kappa/4)}{\pi} \frac{\partial}{\partial \beta} \left(\int_0^1 (1 - v)^{1-\beta} v^{\beta-2} dv\right)\bigg|_{\beta = \kappa/4} \quad \text{substituting } v = (1 + w)^{-1}$$

$$= \frac{\sin(\pi\kappa/4)}{\pi} \frac{\partial}{\partial \beta} \left(\frac{1}{\sin(\pi\beta)}\right)\bigg|_{\beta = \kappa/4} \quad \text{by (4.11)}$$

$$= -\sin(\pi\kappa/4) \frac{\partial}{\partial \beta} \left(\frac{1}{\sin(\pi\beta)}\right)\bigg|_{\beta = \kappa/4} \quad \text{by (4.12)}$$

$$= \pi \cot(\pi\kappa/4).$$

We now turn to analyzing the second term in (4.8).

**Lemma 4.10.** One has $\mathbb{E} \log R_{\theta_1} = \psi(2 - \kappa/4) - \psi(1)$, where, as mentioned in the introduction, $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ denotes the digamma function.

We will first compute the law of $R_{\theta_1}$.

**Lemma 4.11.** The law of $R_{\theta_1}$ is given by the generalized arcsine distribution,

$$\mathbb{P}(R_{\theta_1} \in dx) = \frac{\sin\pi(2 - \kappa/4)}{\pi} x^{1 - \kappa/4}(1 - x)^{\kappa/4 - 2} dx.$$ 

(4.13)

**Proof.** We will deduce the lemma from the arcsine law for a certain stable subordinator. Recall that $\theta_1$ is defined as the time of the last simultaneous running supremum of $(L, R)$ before $R$ hits the level $r$. The simultaneous running suprema of $(L, R)$ are easier to analyze by expressing the law as $(L, R)$ in terms of a pair of correlated Brownian motions with a particular subordination.

Suppose that $(\tilde{L}, \tilde{R})$ is a planar Brownian motion with $\var(\tilde{L}_1) = \var(\tilde{R}_1) = \frac{1}{2} - \frac{p}{2}$ and $\cov(\tilde{L}_1, \tilde{R}_1) = \frac{p}{2}$, where $p = -\cos(4\pi/\kappa)/(1 - \cos(4\pi/\kappa))$. For times $0 < s < t$, if $\tilde{L}_r > \tilde{L}_s$ and
Thus, \( \tilde{R}_r > \tilde{R}_s \) for all \( r \in (s,t] \), then we say that \( s \) is an ancestor of \( t \). A time \( t \) that does not have an ancestor is called ancestor free. The set of ancestor free times is an uncountable set and has zero Lebesgue measure by \cite[Lemma 1]{Shi85}.

Using standard Brownian motion techniques, it is shown in \cite[Proposition 12.3]{DMS14} that we can define a nondecreasing cadlag process \( \ell_t \) which is adapted to the filtration of \( (\tilde{L}_t, \tilde{R}_t) \) and which measures the local time for \( (\tilde{L}_t, \tilde{R}_t) \) at the ancestor-free times. Moreover, if \( T_u = \inf\{ t \geq 0 : \ell_t > u \} \) is the right-continuous inverse of \( \ell_t \), then the range of \( u \mapsto T_u \) is the set of ancestor free times and the pair \( (\tilde{L}_{T_u}, \tilde{R}_{T_u}) \) has the same joint law as the pair of \( \kappa/4 \)-stable processes \( -(\tilde{L}, \tilde{R}) \) (which have only upward jumps), modulo a deterministic scaling factor (see Remark 2.2).

In particular, the random variable \( R_{\theta_1} \) has the same law as \( -\tilde{R}_{\theta_1} \), where \( \theta_1 \) is the time of the last simultaneous running infimum of the correlated planar Brownian motion \( (\tilde{L}, \tilde{R}) \) before \( \tilde{R} \) hits the level \(-1\).

The set of values of \( -\tilde{R} \) at the simultaneous running infima of \( (\tilde{L}, \tilde{R}) \) is clearly regenerative; by scale invariance, it has the law of a stable subordinator. We claim that the index of this subordinator is \( 2 - \kappa/4 \). Once this is established, the arcsine law for subordinators \cite[Proposition 3.1]{Ber99} shows that the law of \( -\tilde{R}_{\theta_1} \leq R_{\theta_1} \) is given by the right side of (4.13), which concludes the proof.

To determine the index of the above subordinator, it is enough to compute the a.s. Hausdorff dimension of its range. First, we recall the following definition.

**Definition 4.12.** A \( \pi/2 \)-cone time of an \( \mathbb{R}^2 \)-valued process \((X,Y)\) is a time \( t \) for which, for some choice of \( \epsilon > 0 \), we have \( X_s > X_t \) and \( Y_s > Y_t \) for all \( s \in (t-\epsilon, t) \). The largest such interval \((t-\epsilon, t)\) is called a \( \pi/2 \)-cone interval of \((X,Y)\).

The set \( \mathcal{R} \) of times of the simultaneous running suprema of \( (\tilde{L}, \tilde{R}) \) is precisely the set of \( \pi/2 \)-cone times of \( (\tilde{L}, \tilde{R}) \) with the property that 0 is contained in the corresponding cone interval. Thus, \cite[Theorem 1]{Eva85} (applied to a linear transformation of \((\tilde{L}, \tilde{R})\) chosen so that the coordinates are independent) implies that the Hausdorff dimension of \( \mathcal{R} \) is \( 1 - \kappa/8 \) almost surely. On the other hand, \( \tilde{R}(\mathcal{R}) = S^{-1}(\mathcal{R}) \), where for \( r \geq 0 \), \( S_r := \inf\{ t > 0 : \tilde{R}_t = -r \} \). Since \( S \) is a 1/2-stable subordinator, \cite[Theorem 4.1]{HP74} implies that \( \dim(\mathcal{R}(\mathcal{R})) = 2 \dim \mathcal{R} = 2 - \kappa/4 \). Hence the set of values of \( -\tilde{R} \) at the simultaneous running infima of \((\tilde{L}, \tilde{R})\) is an index \( 2 - \kappa/4 \) subordinator.

**Proof of Lemma 4.10.** Using Lemma 4.11, we compute

\[
\mathbb{E} \log R_{\theta_1} = \frac{1}{B(2 - \kappa/4, \kappa/4 - 1)} \int_0^1 \log x \cdot x^{1-\kappa/4}(1-x)^{\kappa/4-2} \, dx \quad \text{by (4.12)} \\
= \frac{1}{B(2 - \kappa/4, \kappa/4 - 1)} \int_0^1 \frac{\partial}{\partial \beta} \left( x^{\beta-1}(1-x)^{\kappa/4-2} \right) \bigg| \beta = 2 - \kappa/4 \\
= \frac{1}{B(2 - \kappa/4, \kappa/4 - 1)} \left. \frac{\partial B(\beta, \kappa/4 - 1)}{\partial \beta} \right| \beta = 2 - \kappa/4 \quad \text{by (4.11)} \\
= \frac{\partial \log B(\beta, \kappa/4 - 1)}{\partial \beta} \bigg| \beta = 2 - \kappa/4 \\
= \frac{\partial \log \Gamma(\beta)}{\partial \beta} \bigg| \beta = 2 - \kappa/4 - \frac{\partial \log \Gamma(\beta + \kappa/4 - 1)}{\partial \beta} \bigg| \beta = 2 - \kappa/4 \quad \text{by (4.11)} \\
= \psi(2 - \kappa/4) - \psi(1). \quad \blacksquare
\]

**Proof of Theorem 3.5.** Plugging Lemmas 4.9 and 4.10 into (4.8) gives

\[
\mathbb{E} \log(R_{r-} - R_s) = \mathbb{E} \log(R_{r-} - R_t) + \mathbb{E} \log(R_{\theta_1}) = \pi \cot(\pi\kappa/4) + \psi(2 - \kappa/4) - \psi(1).
\]
The latter is a monotonically decreasing function of $\kappa$, and equals zero for $\kappa \approx 5.6158$. Combining this with Proposition 4.1 proves Theorem 3.5.

5 Proof of Theorem 3.6

To prove Theorem 3.6, we first characterize the limiting law of $L$ in the Skorohod topology as $\kappa$ tends to 8. To do this, we first need to specify the exact law of $L$. Recall from Remark 2.2 that we have thus far only specified the law of $L$ up to a multiplicative constant. Since changing this constant does not change the law of the random variable $\log\left(\sup_{t \in \mathcal{M}}(L_t - L_{\sigma(t)})\right)$, we may assume without loss of generality that $L$ is chosen to have characteristic function

$$E e^{i\lambda L_t} = e^{t|\lambda|^{\alpha}\left[\cos\frac{\pi \kappa}{8} + i \text{sgn}(\lambda) \sin\frac{\pi \kappa}{8}\right]},$$

so that

$$E e^{\lambda L_t} = e^{t\lambda^{\kappa/4}}$$

for $\lambda \geq 0$ [BDP08]. For this choice of $L$, we have the following convergence result:

**Proposition 5.1.** The process $L$ defined by (5.1) converges to $\sqrt{2}B$ in the Skorohod topology, where $B$ is a standard Brownian motion.

**Proof.** Convergence of finite-dimension distributions can be seen immediately from the expression (5.1) for the characteristic function of $L$ (since $L$ has independent and stationary increments). So we just need to check that, for any sequence of numbers in $(4, 8)$ tending to 8, the sequence of Lévy processes corresponding to those values of $\kappa$ is tight. In other words, we must show that, for each fixed $\epsilon, \epsilon' > 0$, there exists $N \in \mathbb{N}$ and $\kappa^* \in (4, 8)$ such that for $\kappa \in [\kappa^*, 8)$,

$$\mathbb{P}\left(\max_{j=1, \ldots, N} \max_{s \in [(j-1)/N, j/N]} \left| L_s - L_{(j-1)/N} \right| > \epsilon \right) < \epsilon'$$

If we are given $\epsilon, \epsilon' > 0$, choose $N \in \mathbb{N}$ large enough so that

$$\frac{1}{\epsilon} e^{-e^{2N}} < \epsilon'/(2N)^{1/2} \quad \text{and} \quad e^{1-\epsilon\sqrt{N}} < \epsilon'/N$$

We will show that

$$\mathbb{P}\left(\max_{j=1, \ldots, N} \min_{s \in [(j-1)/N, j/N]} \left| L_s - L_{(j-1)/N} \right| > \epsilon \right) < \epsilon'$$

and

$$\mathbb{P}\left(\max_{j=1, \ldots, N} \max_{s \in [(j-1)/N, j/N]} \left| L_s - L_{(j-1)/N} \right| > \epsilon \right) < \epsilon'.$$

when $\kappa$ is sufficiently close to 8.

First, Theorem 1 in [BDP08] derives the following power series representation of the density of $\min_{s \in [0, 1]} L_s$:

$$\mathbb{P}\left(\min_{s \in [0, 1]} L_s \in dx\right) = x^{-2} \sum_{n=1}^{\infty} \frac{x^{nk/4}}{\Gamma(nk/4 - 1)\Gamma(1 - n + 4/k)}, \quad x > 0.$$
Thus, we can compute its antiderivative term-by-term, from which we deduce that

\[ \text{min}_{s \in [0,1]} L_s > C \]  

for each fixed \( C \). Since the latter series (with \( C \) fixed) converges uniformly in \( \kappa \in [\kappa', 8] \) for each \( \kappa' > 4 \), the expression (5.6) converges to

\[ 1 - \sum_{n=1}^{\infty} \frac{C^{2n-1}}{(2n-1)\Gamma(2n-1)\Gamma(1-n+1/2)} = 1 - \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{C^{2n-1}}{(-4)^{n-1}(2n-1)(n-1)!} = \frac{1}{\sqrt{\pi}} \int_C^\infty e^{-x^2/4} dx \leq \frac{\sqrt{2}}{C} e^{-C^2} \]

as \( \kappa \to 8 \), where in the first equality we use the identities \( \Gamma(x+1) = x\Gamma(x) \) and \( \Gamma(1/2) = \sqrt{\pi} \) (the latter identity can be derived directly from (4.11) and (4.12)), which imply that \( \Gamma(1-n+1/2) = (-4)^{n-1}\sqrt{\pi}(n-1)!/(2n-2)! \). Hence, we can choose \( \kappa_s = \kappa_s(\epsilon, \epsilon') \) such that, for \( \kappa \in [\kappa_s, 8] \), we have

\[ \text{P} \left( \min_{s \in [0,1]} L_s > \epsilon N^{1/2} \right) < \frac{\sqrt{2}}{\epsilon N^{1/2}} e^{-\epsilon^2 N} \]

and therefore

\[ \text{P} \left( \min_{s \in [0,1]} L_s > \epsilon \right) = \text{P} \left( \min_{s \in [0,1]} L_s > \epsilon N^{4/\kappa} \right) \leq \text{P} \left( \text{min}_{s \in [0,1]} L_s > \epsilon N^{1/2} \right) < \frac{\sqrt{2}}{\epsilon N^{1/2}} e^{-\epsilon^2 N} < \epsilon'/N, \]

so that (5.4) follows from a union bound.

To prove (5.5), observe first that (5.2) implies that \( e^{\lambda \tau/L_t - t \lambda} \) is a martingale, so that applying the optional stopping theorem and dominated convergence yields

\[ \mathbb{E} \left( e^{-\lambda T(x)}, T(x) < \infty \right) = e^{-x \lambda^{4/\kappa}}, \quad \forall \lambda \geq 0 \]

where \( T(x) = \inf\{s \geq 0 : \max_{s \in [0,t]} L_s > x\}. \)

\[ \text{P} \left( \max_{s \in [0,t]} L_s > x \right) = \text{P} (T(x) < t) \leq e^{\lambda t} \mathbb{E} \left( e^{-\lambda T(x)}, T(x) < \infty \right) = e^{\lambda t} e^{-x \lambda^{4/\kappa}} \]

for all \( \lambda > 0 \). In particular, setting \( \lambda = 1/t = N \) gives

\[ \text{P} \left( \max_{s \in [0,1/N]} L_s > x \right) \leq e^{1-\epsilon\sqrt{N}} < \epsilon'/N, \]

and (5.5) follows from a union bound.

Proposition 5.1 allows us to show that \( \sup_{t \in \mathcal{M}} (L_t - L_{\sigma(t)}) \) converges to zero in distribution as \( \kappa \to 8 \), since the intervals \( [\sigma(t), t] \) are all degenerate in the \( \kappa \to 8 \) limit by well-known properties of Brownian motion. Formally, we have the following corollary:

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6This last step is described in greater detail in the proof of Theorem VII.3 in [Ber96].
Corollary 5.2. The random variable

\[ \max_{t \in \mathcal{M}} (t - \sigma(t)) \]

converges to zero in law as \( \kappa \to \infty \).

Proof. By Proposition 5.1, the law of \((L, R)\) converges as \( \kappa \to 8 \) to \((\sqrt{2}B_1, \sqrt{2}B_2)\), where \( B_1 \) and \( B_2 \) are independent standard Brownian motions. By Skorohod’s representation theorem, we can represent the distributions of \((L, R)\) for \( \kappa \in (4, 8) \) on the same probability space so that this convergence occurs almost surely. By the continuous mapping theorem, \( \tau \) converges to a limit almost surely as \( \kappa \to 8 \). Thus, if we assume for contradiction that \( \max_{t \in \mathcal{M}} (t - \sigma(t)) \) does not tend to zero as \( \kappa \to 8 \), we can choose a subsequence \( \kappa_n \) tending to 8 and, for each \( n \), an element \( t_n \) in the set \( \mathcal{M} \) corresponding to \( \kappa = \kappa_n \), such that the intervals \([\sigma(t_n), t_n]\) converge to an interval \([a, b]\) with \( a < b \) as \( n \to \infty \). By the almost sure convergence of the processes \( L \) in the Skorohod topology, the continuity of the limiting process \((\sqrt{2}B_1, \sqrt{2}B_2)\), and the definition of \( \sigma(t_n) \) (Notation 3.1) the interval \([a, b]\) is a \( \frac{\pi}{2} \)-cone interval for \((\sqrt{2}B_1, \sqrt{2}B_2)\) (Definition 4.12), which is a contradiction since an uncorrelated planar Brownian motion a.s. does not have any \( \frac{\pi}{2} \)-cone times [Shi85, Theorem 1]. \( \Box \)

Proposition 5.1 together with Corollary 5.2 implies that \( \sup_{t \in \mathcal{M}} (L_t - L_{\sigma(t)}) \) converges to zero in distribution as \( \kappa \to 8 \). Hence, for each fixed \( K > 0 \),

\[ \log \left( \sup_{t \in \mathcal{M}} (L_t - L_{\sigma(t)}) \right) \lor (-K) \to -K \]

in distribution as \( \kappa \to 8 \). So, to prove that the expectation of \( \log(\sup_{t \in \mathcal{M}} (L_t - L_{\sigma(t)})) \) is negative for \( \kappa \) sufficiently close to 8, it suffices to check the following uniform integrability result:

Lemma 5.3. For each fixed \( K > 0 \) and \( \kappa' \in (4, 8) \), the set of random variables \( \max_{s \in [0, \tau]} \log |L_s| \lor (-K) \) for \( \kappa \in [\kappa', 8) \) is uniformly integrable.

Proof. To prove uniform integrability, it suffices to show that the expectation of

\[ \varphi\left( \left| \max_{s \in [0, \tau]} \log |L_s| \lor (-K) \right| \right) \]

is bounded uniformly in \( \kappa \in [\kappa', 8) \), where \( \varphi(x) = e^{qx} \) for some \( q > 0 \). Proving this, in turn, reduces to showing that the expectation of

\[ \max_{s \in [0, \tau]} |L_s|^q \]

is bounded uniformly in \( \kappa \in [\kappa', 8) \) for some \( q > 0 \). We will prove such a bound using moment bounds on \( L_1 \) and \( \tau \).

First, simplifying equation (8.26) on page 292 of [Pao07] for \( \alpha = \kappa/4, \beta = -1 \) and \( X = -\cos(\pi \kappa/4)L_1 \) yields\(^7\)

\[ \mathbb{E}(|L_1|^r) = \frac{\Gamma(1 - 4r/\pi)}{\Gamma(1 - r)} \left( -\cos \left( \frac{\pi \kappa}{8} \right) \right)^{-r + 4r/\kappa} \]

The random variable \( X \) has characteristic function given by equation (8.8) on page 281 of [Pao07] with \( c = 1 \); comparing this characteristic function with that of \( L_1 \) yields the correct scaling \( X = -\cos(\pi \kappa/4)L_1 \).

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\(^7\)The random variable \( X \) has characteristic function given by equation (8.8) on page 281 of [Pao07] with \( c = 1 \); comparing this characteristic function with that of \( L_1 \) yields the correct scaling \( X = -\cos(\pi \kappa/4)L_1 \).
The latter is bounded uniformly in $\kappa \in [\kappa', 8)$ for each fixed $r < \kappa'/4$. As for $\tau$, [Pes08] derives the following series representation for the density $f_\tau$ of $\tau$:

$$f_\tau(t) = \frac{1}{\pi t^{2-4/\kappa}} \sum_{n=1}^{\infty} \left[ (-1)^{n-1} \sin(4\pi/\kappa) \frac{\Gamma(n-4/\kappa)}{\Gamma(n\kappa/4-1) t^{n-1}} \right.$$

$$\left. - \sin \left( \frac{4n\pi}{\kappa} \right) \frac{\Gamma(1+4n/\kappa)}{n!} \frac{1}{t^{4(n+1)/\kappa-1}} \right], \quad \forall t > 0.$$ 

Therefore, for $t \geq 1$ and $\kappa \in [\kappa', 8)$,

$$|f_\tau(t)| \leq \frac{1}{\pi t^{2-4/\kappa}} \sum_{n=1}^{\infty} \left[ \frac{\Gamma(n-4/\kappa)}{\Gamma(n\kappa/4-1) n!} + \frac{\Gamma(1+4n/\kappa)}{n!} \right]$$

$$\leq \frac{1}{\pi t^{2-4/\kappa}} \sum_{n=1}^{\infty} \left[ \frac{(n-1)!}{n\kappa'-4-2} + \frac{|4n/\kappa'!|!}{n!} \right] \leq \frac{C_{\kappa'}}{t^{2-4/\kappa}}.$$

Hence, for any choice of $0 < q < \kappa'/4 - 1$, the quantity $\mathbb{E}(\tau^{4q/\kappa})$ is bounded uniformly in $\kappa \in [\kappa', 8)$. Thus, fixing $0 < q < \kappa'/4 - 1$ and $1 < r < \kappa'/4$, we have

$$\mathbb{E}\left( \max_{s \in [0, 1]} |L_s|^{q} \right) = \mathbb{E}(\tau^{4q/\kappa}) \mathbb{E}\left( \max_{s \in [0, 1]} |L_s|^{q/\kappa} \right) \text{ (by scaling, since $\tau, L$ are independent)}$$

$$= \mathbb{E}(\tau^{4q/\kappa}) \mathbb{E}\left( \max_{s \in [0, 1]} |L_s|^{r} \right)^{q/r}$$

$$= \mathbb{E}(\tau^{4q/\kappa}) \left( \frac{r}{1-r} \right)^{q/r} \left( \mathbb{E}(|L_1|^{r})^{q/r} \right), \quad \text{by Doob's inequality}$$

which is bounded uniformly in $\kappa \in [\kappa', 8)$. This completes the proof. \qed

6 Open problems

Consider the following three properties the adjacency graph of bubbles of the SLE$_{\kappa}$ curves $\eta$:

(I) The graph is a.s. connected, i.e., there a.s. exists a finite path joining any pair of bubbles.

(II) Almost surely, there exists a path of bubbles from any fixed bubble to $\infty$ (i.e., only finitely many bubbles in the path intersect any given compact subset of $\mathbb{H}$).

(III) There exists an $(L, R)$-markovian path started at any stopping time $\zeta$ for $(L, R)$ at which $\eta$ forms a bubble (Definition 2.6).

Property (III) is clearly stronger than (II); the proof of Lemma 2.7 in fact shows that (II) is stronger than (I). In Theorem 2.8, we showed that (III) (and hence also (II) and (I)) hold for $\kappa \in (4, \kappa_0]$, and in Theorem 2.9 we showed that (III) fails for $\kappa$ sufficiently close to 8.

It is of interest to determine the exact set of values of $\kappa \in (4, 8)$ for which each of the above three properties hold. As mentioned in the introduction, our intuition suggests that it is easier for the adjacency graph to be connected when $\kappa$ is closer to 4. This means that for each of the above three properties, there should exist a critical $\kappa^* \in [\kappa_0, 8]$ for which the property holds for $\kappa \in (4, \kappa^*)$ but fails for $\kappa \in (\kappa^*, 8)$. 

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For property (III), one might guess that $\kappa^* = 6$, since this is the only “special” value of $\kappa$ in the range $(\kappa_0, 8)$ and our proof of Theorem 2.8, which gives $\kappa_0 \approx 5.6158$, seems to be reasonably close to optimal. But, we would not be surprised if this does not turn out to be true. It would be somewhat odd if there exists values of $\kappa$ for which (II) holds but (III) fails, since this would mean that there exist paths to infinity in the adjacency graph but that such paths cannot be found in a Markovian way. Hence $\kappa^* = 6$ might also be a reasonable guess for the critical value for property (II). For condition (I), we are not sure if $\kappa^* = 8$ (i.e., the graph is connected for all $\kappa$) or if $\kappa^* < 8$; we would not be surprised either way. Our results indicate that it might be difficult to prove connectedness for $\kappa$ close to 8 (if this is indeed true) since one would have to find a way of producing paths which is not Markovian with respect to $(L, R)$.

References

[AS18] J. Aru and A. Sepúlveda. Two-valued local sets of the 2D continuum Gaussian free field: connectivity, labels, and induced metrics. ArXiv e-prints, January 2018, 1801.03828.

[BDP08] V. Bernyk, R. C. Dalang, and G. Peskir. The law of the supremum of a stable Lévy process with no negative jumps. Ann. Probab., 36(5):1777–1789, 2008, 0706.1503. MR2440923

[Bef08] V. Beffara. The dimension of the SLE curves. Ann. Probab., 36(4):1421–1452, 2008, math/0211322. MR2435854 (2009e:60026)

[Ben17] S. Benoist. Natural parametrization of SLE: the Gaussian free field point of view. ArXiv e-prints, August 2017, 1708.03801.

[Ber96] J. Bertoin. Lévy processes, volume 121 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1996. MR1406564 (98e:60117)

[Ber99] J. Bertoin. Subordinators: examples and applications. In Lectures on probability theory and statistics (Saint-Flour, 1997), volume 1717 of Lecture Notes in Math., pages 1–91. Springer, Berlin, 1999. MR1746300 (2002a:60001)

[Ber15] N. Berestycki. An elementary approach to Gaussian multiplicative chaos. ArXiv e-prints, June 2015, 1506.09113.

[Bur] K. Burdzy. My favorite open problems. https://sites.math.washington.edu/~burdzy/open_mathjax.php.

[CD10] L. Chaumont and R. A. Doney. Invariance principles for local times at the maximum of random walks and Lévy processes. Ann. Probab., 38(4):1368–1389, 2010. MR2663630

[CK14] N. Curien and I. Kortchemski. Random stable looptrees. Electron. J. Probab., 19:no. 108, 35, 2014, 1304.1044. MR3286462

[Cur15] N. Curien. A glimpse of the conformal structure of random planar maps. Comm. Math. Phys., 333(3):1417–1463, 2015, 1308.1807. MR3302638

[DK06] R. A. Doney and A. E. Kyprianou. Overshoots and undershoots of Lévy processes. Ann. Appl. Probab., 16(1):91–106, 2006, math/0603210. MR2209337

[DL05] T. Duquesne and J.-F. Le Gall. Probabilistic and fractal aspects of Lévy trees. Probab. Theory Related Fields, 131(4):553–603, 2005, math/0501079. MR2147221 (2006d:60123)
[DMS14] B. Duplantier, J. Miller, and S. Sheffield. Liouville quantum gravity as a mating of trees. ArXiv e-prints, September 2014, 1409.7055.

[DS11] B. Duplantier and S. Sheffield. Liouville quantum gravity and KPZ. Invent. Math., 185(2):333–393, 2011, 1206.0212. MR2819163 (2012f:81251)

[Dub09] J. Dubédat. Duality of Schramm-Loewner evolutions. Ann. Sci. Éc. Norm. Supér. (4), 42(5):697–724, 2009, 0711.1884. MR2571956 (2011g:60151)

[Eva85] S. N. Evans. On the Hausdorff dimension of Brownian cone points. Math. Proc. Cambridge Philos. Soc., 98(2):343–353, 1985. MR795899 (86j:60185)

[GM17] E. Gwynne and J. Miller. Convergence of percolation on uniform quadrangulations with boundary to SLE$_6$ on $\sqrt{8/3}$-Liouville quantum gravity. ArXiv e-prints, January 2017, 1701.05175.

[GMS17] E. Gwynne, C. Mao, and X. Sun. Scaling limits for the critical Fortuin-Kasteleyn model on a random planar map I: cone times. Annales de l’Institut Henri Poincaré, to appear, 2017, 1502.00546.

[HP74] J. Hawkes and W. E. Pruitt. Uniform dimension results for processes with independent increments. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete, 28:277–288, 1973/74. MR0362508 (50 #14948)

[JJ99] H. Jeffreys and B. S. Jeffreys. Methods of mathematical physics. Cambridge University Press, Cambridge, 1999. Reprint of the third (1956) edition. MR1744997

[JS03] J. Jacod and A. N. Shiryaev. Limit theorems for stochastic processes, volume 288 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, second edition, 2003. MR1943877

[Kah85] J.-P. Kahane. Sur le chaos multiplicatif. Ann. Sci. Math. Québec, 9(2):105–150, 1985. MR829798 (88h:60099a)

[LR15] G. F. Lawler and M. A. Rezaei. Minkowski content and natural parameterization for the Schramm-Loewner evolution. Ann. Probab., 43(3):1082–1120, 2015, 1211.4146. MR3342659

[LS11] G. F. Lawler and S. Sheffield. A natural parametrization for the Schramm-Loewner evolution. Ann. Probab., 39(5):1896–1937, 2011, 0906.3804. MR2884877

[LZ13] G. F. Lawler and W. Zhou. SLE curves and natural parametrization. Ann. Probab., 41(3A):1556–1584, 2013, 1006.4936. MR3098684

[MP10] P. Mörters and Y. Peres. Brownian motion. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2010. With an appendix by Oded Schramm and Wendelin Werner. MR2604525 (2011i:60008)

[MS16a] J. Miller and S. Sheffield. Imaginary geometry III: reversibility of SLE$_\kappa$ for $\kappa \in (4,8)$. 184(2):455–486, 2016, 1201.1498.

[MS16b] J. Miller and S. Sheffield. Imaginary geometry I: interacting SLEs. Probab. Theory Related Fields, 164(3-4):553–705, 2016, 1201.1496. MR3477777

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[MS17] J. Miller and S. Sheffield. Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees. *Probab. Theory Related Fields*, 169(3-4):729–869, 2017, 1302.4738. MR3719057

[MW17] J. Miller and H. Wu. Intersections of SLE Paths: the double and cut point dimension of SLE. *Probab. Theory Related Fields*, 167(1-2):45–105, 2017, 1303.4725. MR3602842

[Pao07] M. S. Paolella. *Intermediate probability: a computational approach*. John Wiley & Sons, Ltd., Chichester, 2007.

[Pes08] G. Peskir. The law of the hitting times to points by a stable Lévy process with no negative jumps. *Electron. Commun. Probab.*, 13:653–659, 2008. MR2466193

[RS05] S. Rohde and O. Schramm. Basic properties of SLE. *Ann. of Math. (2)*, 161(2):883–924, 2005, math/0106036. MR2153402 (2006f:60093)

[RV14] R. Rhodes and V. Vargas. Gaussian multiplicative chaos and applications: A review. *Probab. Surv.*, 11:315–392, 2014, 1305.6221. MR3274356

[Sch00] O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000, math/9904022. MR1776084 (2001m:60227)

[She07] S. Sheffield. Gaussian free fields for mathematicians. *Probab. Theory Related Fields*, 139(3-4):521–541, 2007, math/0312099. MR2322706 (2008d:60120)

[She16] S. Sheffield. Conformal weldings of random surfaces: SLE and the quantum gravity zipper. *Ann. Probab.*, 44(5):3474–3545, 2016, 1012.4797. MR3551203

[Shi85] M. Shimura. Excursions in a cone for two-dimensional Brownian motion. *J. Math. Kyoto Univ.*, 25(3):433–443, 1985. MR807490 (87a:60095)

[SS13] O. Schramm and S. Sheffield. A contour line of the continuum Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):47–80, 2013, math/0605337. MR3101840

[Zha08] D. Zhan. Duality of chordal SLE. *Invent. Math.*, 174(2):309–353, 2008, 0712.0332. MR2439609 (2010f:60239)

[Zha10] D. Zhan. Duality of chordal SLE, II. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(3):740–759, 2010, 0803.2223. MR2682265 (2011i:60155)