Instability and no-hair paradigm in d-dimensional charged-AdS black holes

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Is it possible the no-hair paradigm is violated when a black hole undergoes an instability? Employing massless charged scalar perturbations, we address this question within the context of conformally invariant Einstein-Maxwell theory for some allowed \( d \)-dimensional \((d = 4n + 4\) with conformal parameter \( n = 0, 1, 2, \ldots \)) topological small AdS black-holes. We provide numerical analyses that show the non trivial scalar hairy black hole solutions to include planar and spherical horizon topologies in higher dimensions with an even conformal parameter \( n \). It is also shown that the solutions presented here cannot be considered as scalar hairs for a Reissner-Nordstrom background. As a result, for \( d \)-dimensional small AdS black holes in the presence of a conformally invariant Maxwell source, the no-scalar hair paradigm seems to be supported.

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I. INTRODUCTION

The recent discovery of a supermassive black hole (BH) at the core of distant galaxy Messier 87* by the Event Horizon Telescope (EHT) [1] confirms that BHs are real astrophysical objects. LIGO [2–4] has already succeeded in detecting gravitational waves resulting from the merger of two BHs. These outcomes encourage one to take steps towards a more complete understanding of BH physics, not merely as abstract objects derived from vacuum solutions of Einstein field equations of general relativity (GR) but also as ubiquitous objects in the universe. Interesting questions such as the faith of information trapped inside a BH or their evaporation through various mechanisms have been raised and to some extent answered in the past. One such interesting phenomena associated with BHs is a classical process known as “superradiant scattering,” through which the BH will lose energy and might eventually disappear. In other words, an incident wave can be amplified as it scatters off the BH so that the additional energy radiated to infinity is drawn at the expense of the BH. The event horizon, a null one-way viscous membrane, as one of nontrivial features of a BH provides a dissipative mechanism in order to extract energy from vacuum even at the classical level.

Superradiant scattering as a radiation enhancing process is at the center of attention in various contexts. From an astrophysical point of view it is interesting since it allows astrophysical BHs to play the role of particle detectors [5]. In recent years, interest in superradiant BHs has also grown considerably among particle physicists because it is believed that this phenomena can be used to control dark matter candidates such as axion-like particles [6, 7]. Presently, there is not a strong belief that superradiance is the wave analogue of the Penrose process for particles \[^{1}[8, 9]\] where a perturbing field with the associated charge \( q \) and low-frequency \( \omega \), having possible spins \( s = 0, 1, 2 \) and relevant azimuthal quantum number \( m \), i.e. a scalar, electromagnetic or gravitational wave packet, is strengthened via extracting energy flux from a BH. Within the context of GR, this phenomenon might happen in the small amplitude limit for a BH including the Kerr horizon with angular velocity \( \Omega_h \) or the Reissner-Nordström (RN) horizon with electric potential \( \Phi_h \), if the conditions \( \omega < m\Omega_h \) and \( \omega < q\Phi_h \) are met respectively [10]. A very important point that should be noted is that in both types of horizons, the relevant area/entropy does not decrease but only the rotational and Coulomb energies are respectively extracted, see [11–13] for more details. This means that superradiance phenomena, in agreement with our classical intuition, will not lead to the extraction of information from a BH. Note that in the case of formation of a mechanism that might be able to trap such superradiant modes

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Although in both processes, we deal with energy extraction from BHs, they will occur under different conditions which indicate that the two phenomena are, in principle, separate from each other, see [10] for more details.
in and around the BH, background level instability is likely to appear. Interestingly, this instability is caused by the surrounding event horizon with a perfect reflective mirror causing repeated bouncing back and forth between the mirror and horizon which could grow without bound. This growth eventually reaches a point where the radiation pressure leads to the collapse of the mirror, sometimes known as the “BH bomb” [14–16]. Concerning the Kerr BH, it has been shown in [17] that the mass of perturbing field can play the role of a natural mirror, attracting a considerable amount of interest on the Kerr BH instability under a massive bosonic test field, see [18–23]. However, for a RN background perturbed by a massive charged scalar field, superradiance instability will not occur [24, 25]. In contrast, in the case of a massless charged scalar field with small amplitude, placing a reflecting mirror far from the event horizon will make superradiance instability in a RN background a possibility [26, 27]. Besides, given the fact that black holes are solutions to Einstein field equations, the question arises as to what happens to such phenomena in the context of alternative theories of gravity. Such questions have been explored and investigated in [28–35].

Looked at from a different angle, superradiant phenomena somehow determines the final fate of the BH involved and, as such, can lead to the confirmation or violation of the “no-hair theorem” (NHT) [36–38] which states that a perturbed BH should eventually end in a stationary state, signified by just a small number of parameters: mass $M$, charge $Q$ and angular momentum $J$. Historically, a series of NHTs appeared when search was begun to look for possible interaction of BHs with any kind of matter. An important consequence of this classical theorem is that the stable state of a BH, determined by the above mentioned parameters, ensures that information is not lost on the initial conditions of the BH. Despite the existence of some possibilities to violate this theorem in models such as Einstein-Yang-Mills (EYM) [39, 40] and Einstein-Skyrme (ES) [41, 42] with their diverse combinations with Higgs or dilaton fields [43, 44], it has not been conclusively proven that these models are counter examples to NHT [45]. It is noteworthy that the EHT data also recently opened a new window to analyze these models [46]. In [47], it is shown that in the case of non-abelian gauge fields coupled to gravity, a family of solutions with non-trivial $SU(2)$ YM fields exist in such a way as to make radial perturbations grow and thus becoming unstable. Concerning the Einstein-Skyrme model, the stability condition of the Skyrme hair solution is not known at the non-linear level. In general, it is believed that a perturbation field, such as massless/massive or a vector field etc, once violating the NHT, finally results in a new stable hair configuration. As an example, the authors in [48] have demonstrated that for a RN-BH, enclosed within a cavity and perturbed by a charged, massless scalar field, it is possible to have stable hair solutions. In [33], it has also been shown that such a possibility could exist in the framework of alternative theories of gravity, in particular $F(R)$ gravity. However, there are some NHTs proposed in [49–51] that do not admit such solutions. For a review involving scalar hairy solutions, see [52, 53].

A space-time well adapted to superradiance phenomena is that of anti-de Sitter (AdS) which provides a natural, rather than an artificial reflecting boundary. Such a scenario becomes all the more interesting due to the “AdS/CFT correspondence” [54–56]. The BH bomb as well as superradiant instability as two related issues, have been extensively studied within the RN-AdS and Kerr-AdS backgrounds [57–62]. It should be noted that in such a framework one is not only restricted to four-dimensional BHs. The stability of superradiant modes in higher-dimensional BHs with a RN-AdS horizon has been analytically as well as numerically studied in [63–67].

In this paper we are interested in focusing on superradiance phenomena beyond the common electrically charged-BHs which is based on the standard Maxwell theory (SMT). Indeed, the existence of some serious shortcomings such as: vacuum polarization in quantum electrodynamics (QED), infinite self-energy of point-like charges [68, 69] and also incompatibility with some theoretical evidence in the limit of strong electromagnetic fields [70] in SMT, motivates us to embark on such a study. Furthermore, in higher dimensions, the standard Maxwell action loses its most important property, namely conformal invariance\(^2\). It is believed that these issues can be addressed via extending SMT to include a non-linear Maxwell modification. One of the most popular non-linear extension of SMT is known as power Maxwell invariance which includes a non-linear term of the form $(F_{\alpha\beta} F^{\alpha\beta})^{n+1}$ so that if $n = \frac{d}{2} - 1$ (where $n = 0, 1, 2, ...$), conformal symmetry is restored [71–75]. Note that the underlying $d$-dimensional framework has two clear differences compared to the standard higher-dimensional counterpart. First, unlike the standard case, gravity is coupled to a non-linear Maxwell field. Second, the relevant electrodynamic source is a “conformally invariant Maxwell source” (CIMS).

Overall, our main purpose in this work is to evaluate the validity of NH paradigm in higher dimensions by investigating instability of a non-standard $d$-dimensional small RN-AdS BH, that is $r_h \ll L$ where $r_h$ and $L$ are the event horizon and AdS radius, perturbed by a charged, massless scalar field. More precisely, we deal with a new class of $d$-dimensional RN-AdS BH within the framework of a conformally invariant Einstein-Maxwell gravity. We

\(^2\) We note that in GR, due to non trivial interaction between geometry of spacetime and matter, the presence of a matter source possessing conformal symmetry leads to a simpler Einstein equation [71]. Conformal symmetry also plays a significant role in the study of the critical phenomena in statistical physics as well as AdS/CFT correspondence and string theory.
use numerical solutions to argue that there is the possibility of supporting the NH conjecture by CIMS in higher dimensions allowed by the theory.

II. SCALAR HAIRY BLACK HOLES IN d-DIMENSIONS WITH A CIMS SOURCE

A. The setup

We consider Einstein gravity in a \(d\)-dimensional AdS spacetime coupled to a CIMS source as well as a charged, massless scalar field, described by the action

\[
S[g_{\alpha\beta}, A_\alpha, \Psi, \Lambda] = \int d^d x \sqrt{-g} \left( R - 2\Lambda - \frac{1}{4} (F^\alpha_\beta F_{\alpha\beta})^{d/4} - \frac{1}{2} g^{\alpha\beta} (D_\alpha \Psi^* D_\beta \Psi) \right),
\]

with \(F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha\) and \(D_\alpha = \nabla_\alpha - iq A_\alpha\) where for \(d = 4\), the familiar Einstein-Maxwell action coupled to a scalar field \(\Psi\) is recovered. In the above action, \(\Lambda\) represents an AdS spacetime with a negative value given by \(-(d-1)(d-2)/2L^2\) for asymptotic solutions with \(R\) and \(L\) respectively denoting a scalar curvature and length scale. It is well known that the electro-magnetic part of the above action \(S[A_\alpha]\) is invariant under conformal transformation \(g_{\alpha\beta} \rightarrow \Omega^2 g_{\alpha\beta}\) and \(A_\alpha \rightarrow A_\alpha\). Also the conformal invariance of \(S[A_\alpha]\) is guaranteed via the traceless term \(T^\alpha_\alpha = 0\) for energy-momentum tensor which will only occur for \(d = 4n + 4\) with the non-linear parameter \(n = 0, 1, 2, ... \) [71–75]. By varying the above action with respect to \(g_{\mu\nu}\), the gauge field \(A_\mu\) and scalar field \(\Psi\), the following field equations are derived

\[
\begin{align*}
G_{\alpha\beta} + \Lambda g_{\alpha\beta} &= T_{\alpha\beta} = (T_{EM}^{\alpha\beta} + T_{\Psi}^{\alpha\beta}) ,
\frac{1}{\sqrt{-g}} \partial_\alpha \left( \sqrt{-g} F^{\alpha\beta} F_{\alpha\beta} \right)^n &= J^\beta ,
D_\alpha D^\alpha \Psi &= 0 ,
\end{align*}
\]

with

\[
\begin{align*}
T_{EM}^{\alpha\beta} &= (n + 1) F_{\sigma\tau} F^{\sigma\tau} (F^{\alpha\beta} F_{\alpha\beta})^{n-1} - \frac{1}{4} g_{\alpha\beta} (F^{\alpha\beta} F_{\alpha\beta})^{n+1} ,
T_{\Psi}^{\alpha\beta} &= D_\sigma (\Psi^* D_\beta \Psi) - \frac{1}{2} g_{\alpha\beta} (g^{\sigma\tau} D_\sigma (\Psi^* D_\tau \Psi)) ,
J^\beta &= \frac{iq}{2} \left( \Psi^* D^\beta \Psi - \Psi (D^\beta \Psi)^* \right) ,
\end{align*}
\]

where \(\nabla_\beta J^\beta = 0 = \nabla_\alpha T^{\alpha\beta}\). Note that due to \(U(1)\) symmetry in the underlying theory, the scalar field \(\Psi\) as well as vector potential \(A_\alpha\) enjoy “gauge freedom” which keeps \(F_{\alpha\beta}\) and \(D_\alpha \Psi\) invariant under the following gauge transformations

\[
\Psi \rightarrow \exp(i\chi) \Psi , \quad A_\alpha \rightarrow A_\alpha + q^{-1} \chi_\alpha .
\]

Here, \(\chi\) can be any real scalar field. Gauge freedom, in essence, allows us to work in any cost-effective gauge. In subsection C, one will see that the existence of this property which would enable us to extract static solutions in the non-linear regime, is very essential [48].

Neglecting the effects of scalar field in test approximation, the underlying theory described by (1) admits a spherically symmetric BH solution as follows

\[
ds^2 = -\eta(r) dt^2 + \frac{dr^2}{\eta(r)} + r^2 d\Omega_{4n+2}^2 ,
\]

with the laps function \(\eta(r)\), the line element \(d\Omega_{4n+2}^2\) of a \((4n + 2)\)-dimensional hypersurface and curvature constant \(k\) defined as

\[
d\Omega_{4n+2}^2 = \begin{cases} 
\begin{align*}
d\theta_1^2 + \sum_{i=2}^{4n+2} \prod_{j=1}^{i-1} \sin^2 \theta_j & d\theta_i^2 , \quad k = 1 \\
\sum_{i=1}^{4n+2} dx_i^2 & k = 0 \\
d\theta_1^2 + \sinh^2 \theta_1 d\theta_2^2 + \sinh^2 \theta_1 \sum_{i=3}^{4n+2} \prod_{j=2}^{i-1} \sin^2 \theta_j d\theta_i^2 & k = -1 
\end{align*}
\end{cases}
\]

...
Interestingly, the CIMS equation (3) in the absence of the scalar field implies that the electric field in \((4n+4)\)-dimensions still obeys the inverse-square law i.e. \(E_r = \frac{Q}{r^2}\). In other words, in the presence of a conformally invariant source the relevant expression of the electric field is the same as that for the RN solution in 4d, independent of the dimension. It is worth noticing that the root of this feature comes from the CIMS so that in the framework of a SMT based higher-dimensional theory, the electric field is dependent on the dimension. As a result, within the extended CIM setup and based on vanishing electromagnetic potential at the event horizon \(r_h\), we may take the following anzats for the gauge potential

\[
A_\alpha dx^\alpha = (\phi(r)dt, 0, 0, 0) \Rightarrow A_\alpha = \left( \frac{-Q}{r} + C, 0, 0, 0 \right),
\]

similar to its 4d counterpart. It is worth mentioning that the constant of integral \(C\) comes from AdS/CFT correspondence which would allow us to make the vector potential vanish on the event horizon, i.e. \(A_\alpha(r_h) = 0\) \([76, 77]\). As is shown in detail in \([59]\), the constant \(C\) only shifts the real part of the frequency, without any effect on the imaginary part, which is an essential characteristic in studying superradiant instability. In other words, \(U(1)\)-gauge transformation (8) guarantees that presence of \(C\) just makes the scalar field \(\Psi\) to undergo a frequency shift as \(\exp(iqCt)\) \([78]\). Therefore, to study superradiance phenomena, one takes \(C = 0\) in the linear approximation, just as was done in \([59, 64, 78]\). However, there is a point worth paying attention to; the solution for the scalar field in the background for which \(A_\alpha(r_h) = 0\) does not represent the same state as the one in the background that satisfies \(A_\alpha(r_h) \neq 0\) since they are related through a \(U(1)\) transformation which does not vanish at infinity. So these two backgrounds actually address states that obey different boundary conditions. This will become clear in the next subsection.

Now, taking the \((rr)\) component in Eq. (2), the lapse function \(\eta(r)\) reads

\[
\eta(r) = k - \frac{m}{r^{4n+1}} + \frac{2^n Q^{2n+2}}{r^{4n+2}} + \frac{r^2}{L^2},
\]

which can be written in a more useful form

\[
\eta(r) = k - \frac{2m(r)}{r^{4n+1}} + \frac{r^2}{L^2}, \quad \text{with} \quad m(r) = m + \frac{2^{n-1} Q^{2n+2}}{r},
\]

were, \(m\) is an integration constant representing the mass of BH. Note that the above laps function \(\eta(r)\) is quite different from that obtained in a higher-dimensional SMT-based space since, as is clear, the electric charge term goes as \(r^{-4n+2}\) while the standard Maxwell source gives \(r^{-(8n+2)}\). This means that the field arising from a CIMS falls off slower than that given by a standard Maxwell source. In the lapse function (11), \(k = 1, 0, -1\) represent solutions with spherical, planar and hyperbolic geometries for the horizon, respectively. In what follows, we will see the important role played by the topology of the horizon in describing the superradiance phenomena.

### B. Scalar wave scattering: linear regime

Let us now focus on the scattering of a scalar test field in a \(d\)-dimensional BH background with a CIMS. This will be done at the linear level which means that having a small amplitude for scalar field, we can ignore the changes induced by its fluctuations so that the spacetime geometry can still be considered as fixed. The spherical symmetry characteristic of the test scalar field \(\Psi\), suggest the following perturbation anzats

\[
\Psi = \frac{\psi(r)}{r^{2n+1}} \exp(-i\omega t),
\]

where \(\psi(r)\) is the radial part of the wave function and \(\omega\) is the frequency. Inserting the above expression in Klein-Gordon equation (4), in the presence of gauge potential (10), we arrive at the following equation

\[
\eta^2 \psi'' + \eta \eta' \psi' + U_{eff} \psi = 0,
\]

with

\[
U_{eff} = (\omega + qA_t)^2 - \left( \frac{n(4n+2)}{r^2} \eta^2 + \frac{2n+1}{r} \eta \eta' \right),
\]

where a prime denotes derivative with respect to \(r\). Using tortoise coordinate \(r_\ast\), \(\eta^{-1} = \frac{dr_\ast}{dr}\), equation (14) is re-expressed as a Schrodinger-like equation

\[
\frac{d^2 \psi}{dr_\ast^2} + U_{eff} \psi = 0,
\]
the potential $U_{\text{eff}}$, depending on the model at hand, describes the test field as well as the curvature of the background. For $r$ going to spatial infinity or the event horizon, the behavior of the radial wave function in these limits are

$$
\psi(r) \sim \exp\left(-i(\tilde{\omega} - \frac{qQ}{r_h})r\right), \quad \tilde{\omega} \equiv \omega + qC \quad r \to r_h,
$$

$$
\psi(r) \sim r^{-(2n+2)} \quad r \to +\infty,
$$

(17)

where the condition of no outgoing wave from the event horizon has been imposed. Here, it is interesting to note that invariance of equation (14) under transformations $q \to -q$ and $Q \to -Q$ means that, without loss of generality, we may fix the positive sign for $qQ$.

There is a subtlety regarding the above argument which merits a more detailed discussion at this point. One way to realize superradiance phenomena is by studying the energies as well as propagation of the scalar wave function. To have superradiance, it is necessary that signs of the group and phase velocities of the scalar field wave function become negative and positive respectively, that is $v_g < 0$ and $v_p > 0$. Also, the $(tr)$ component of the energy-momentum tensor should be nonzero. In what follows, by taking the gauge $C = 0$, we investigate the above statements to see if they are satisfied. Regarding the radial solution (17), one finds that the behavior of the perturbed scalar field near the horizon is as follows

$$
\Psi = \frac{\exp^{-i\omega t}\exp^{ikr}}{r^{2n+1}}\psi(r),
$$

(18)

where $k = -(\omega + qA_t(r_h))$ and $A_t(r_h) = -\frac{Q}{r_h}$. This means that the group velocity is $v_g = \frac{d\omega}{dk} = -1$, implying that information is passing into the BH which is in agreement with the boundary condition demanded at the horizon. As for the phase velocity, we also have

$$
v_p = \frac{\omega}{k} = \frac{\omega}{-(\omega + qA_t(r_h))} > 0.
$$

(19)

Here, the positivity of $v_p$ is guaranteed since in a superradiant regime, $\omega < -qA_t(r_h)$.

Positive phase velocity means that energy is extracted off the BH from the point of view of a horizon observer. From the near-horizon behavior of the scalar field, $\Psi(r) \propto \exp^{-i\omega t}\exp^{-(\omega + qA_t(r_h))r}$, a distant observer also recognizes that the wave is coming out of the BH provided that the superradiant condition, $\omega < -qA_t(r_h)$, is satisfied. In order to investigate the third quantity mentioned above, we calculate the $(tr)$ component of the energy-momentum tensor and find

$$
T_{tr} \propto \frac{(\omega + qA_t(r_h))k}{\eta} = \frac{k^2}{\eta} > 0,
$$

meaning that the energy flux at the horizon is non-zero.

As an alternative approach to the above arguments, one may study the effects of the scalar perturbation on energy extraction from the BH. To do so, we consider the conserved energy flux for the scalar field, $J_{\mu} = -T_{\mu\nu}\xi^\nu$, using the energy-momentum tensor $T_{\mu\nu}$ and Killing vector $\xi^\nu$, which is timelike at infinity [79]. By projecting the energy flux vector along the null horizon to generate the Killing vector of the BH background $\chi^\mu$, we obtain the following rate of energy extraction from the BH (further details can be found in [80])

$$
-\langle J_{\mu}\chi^\mu \rangle = \langle J_{\mu}n^\mu \rangle = \int dAn^\mu J_{\mu},
$$

(20)

where $n^\mu = -\chi^\mu$ is the appropriately directed normal to the horizon. Therefore, due to continuity and stationarity of the scalar field, the energy flux transferred to the horizon should be carried away by a conserved radial current $J_r = i(\Psi^*\Psi_{,r} - \Psi\Psi_{,r}^*)$ so that the integral of $F \equiv \int dAn^\mu J^\mu$ over the horizon represents the net radial energy flux extracted from the BH. Here, $n_{\mu} \sim \nabla_{\tau}r$ represents a radial normal vector. It is now clear that for superradiance to occur, i.e. having outward energy flow, it is necessary that $n_{\mu}J^\mu > 0$. Now for the complex scalar field perturbation near the horizon, given by (18), we have

$$
n_r J^r = \frac{2k}{r^{2n+2}}|\psi(r)|^2.
$$

(21)

It is clear that $n_rJ^r > 0$ provided that $k > 0$. This is not possible unless the superradiant condition ($\omega < \frac{qQ}{r_h}$) is met. Given the above arguments, one interesting point to note is that by transforming the scalar solution obtained in the gauge $A_t(r_h) \neq 0 \ (C = 0)$ to gauge $A_t(r_h) = 0 \ (C \neq 0)$, the scalar field acquires a frequency shift that results in
phase and group velocities with opposite signs on the horizon\(^3\). However, without resorting to a \(U(1)\) transformation, i.e. if from the beginning one works in the gauge \(A_\theta(r_h) = 0\), a frequency shift will no longer appear and as a result, the sign of phase and group velocities on the horizon will be the same. As a consequence, the background with \(A_\theta(r_h) = 0\) actually is free of superradiant instability. However, through numerical analysis, it will be shown below that the underlying background still undergoes an instability.

By imposing a natural reflecting AdS boundary that guaranties the vanishing of the scalar test field, \(\psi(r = L) = 0\), we will deal with a two-point boundary value problem that results in a discrete spectrum for \(\psi(r)\) in (14) with frequency modes \(\omega_m\) having real and imaginary parts, \(\omega_m = \omega_R + i\omega_I\). Here, the label \(m\) denotes the number of nodes related to scalar field modes trapped in the region \(r_h < r < L\). Given the fact that the relevant anzats for the scalar field (13) is time dependent, it is clear that if \(\omega_I > 0\) the amplitude grows exponentially and then in the background instability occurs. However if \(\omega_I < 0\), then due to exponential decay of the amplitude of the scalar field modes, instability will be absent. Note that the validity of these statements is independent of the sign of \(\omega_R\) as well as the dimension of the space-time so that without loss of generality it can be applicable to higher-dimensional BHs.

As a result, \(\omega_I\) is the parameter which determines the realization of instability. Therefore, focusing on specifying the sign of \(\omega_I\), in the rest of this subsection we will investigate the possibility of the realization of instability for RN-AdS BHs in the framework of a conformally invariant Einstein-Maxwell theory. Following the method adopted in [48], via integrating the radial perturbation equation (16) from \(r = r_h(1 + \epsilon)\) to \(r = L\) and fixing an initial value for \(\omega_m\) we numerically compute the spectrum of modes for the underlying BHs with different horizon topologies. Fig. 1 is a summary of the results obtained from the numerical solution. First, the numerical solution for the event horizon with open geometry \((k = -1)\) is not favored since for this topology the small BH approximation \((r_h \ll L)\) is not met. For planar topology \((k = 0)\), the numerical solution shows that by increasing the value of the conformal parameter \(n\) as well as scalar charge \(q\), the probability of occurrence of instability increases. However, as is clear from Fig. 1, for \(n = 0, 1\) \((d = 4, 8)\), increasing the scalar charge \(q\), we have \(\omega_I \leq 0\) but for \(n = 2\), the sign of \(\omega_I\) at \(L = 28\) with \(q = 0.8\) crosses over to positive values\(^2\). This may be interpreted as having an enhanced chance of occurrence of superradiance for large values of the scalar charge and AdS radius in \(d\)-dimensional AdS small BHs with flat horizon sourced by a CIM. Concerning the BHs with spherical geometry for the horizon \((k = +1)\), Fig. 1 tells us that despite the fact that by increasing the scalar charge \(q\), the instability becomes stronger, going to allowed higher dimensions, its probability of occurrence may become weaker. In other words, according to \(\tau = \frac{1}{1 + \epsilon}\) one can see by decreasing the value of \(\omega_I\), a longer time is required to achieve instability. Also we see that in allowed higher dimensions the starting point of instability \((\omega_I \sim 0\) corresponding to very long-lived quasi-normal modes [81, 82]) is located at distances away from the event horizon \((r_h)\). So, like in the previous case, here again increasing the scalar charge in higher dimensions allowed by CIMs, the instability is realizable.

Generally, by employing a numerical method and a linear approximation, we have so far shown that scattering of a scalar test field off a \(d\)-dimensional small BH (with topological indexes \(k = 0, +1\) as well as a CIMs), will lead to instability. As a supplementary point, one may regard the instability discussed above as a superconducting-like instability since it is shown that the gravitational dual of a holographic superconducting system is acquired via the coupling of AdS-gravity to a system consisting of Maxwell and charged scalar fields [83]. In other words, the superconducting phase transition is actually equivalent to a classical instability of an AdS-BH

\(^3\) When one imposes that the vector potential to vanish at the event horizon i.e. \(A_\theta(r_h) = 0\), it means that the scalar filed will undergo a frequency shift according to \(e^{\frac{i\omega t}{\sqrt{2} h}}\). The behavior of the scalar field perturbation near the event horizon is thus given by

\[
\Psi = e^{\frac{ikr_h}{(2n+1)\epsilon}} e^{-(\omega - \frac{qQ}{\sqrt{2} h})t} r \to r_h, \tag{22}
\]

where \(k = -\omega\). So the group velocity is \(v_g = \frac{d\omega}{d\epsilon} = -1\) while the phase velocity is positive

\[
v_p = \frac{\omega - \frac{qQ}{\sqrt{2} h}}{k} = \frac{-\omega - \frac{qQ}{\sqrt{2} h}}{\omega - \omega} > 0, \tag{23}
\]

when the superradiance condition, \(\omega < \frac{qQ}{\sqrt{2} h}\) is satisfied. This clearly shows that the sign of the group and phase velocities can be different even when one considers \(A_\theta(r_h) = 0\), so that the energy will be extracted off the black hole for this choice of the gauge.

\(^2\) Using a matching method for near and far region solutions of the radial equation (16) for small BHs, the authors in [59, 64] analytically found the frequency \(\omega_m\). Changing the gauge potential in [64] to (10), we used the results of [64] to fixe the initial value for \(\omega_m\).

\(^5\) Concerning the planar AdS horizon in four dimensions \((n = 0)\), superradiant instability is absent and instead we deal with another type of instability related to an AdS background, known as “near-horizon scalar condensation instability.” Unlike superradiance instability, this type of instability is suppressed in small BH approximation. A four dimensional AdS background with planar horizon does not satisfy small horizon approximation \(r_h \ll L\) and therefore superradiant instability is absent while the near-horizon scalar condensation instability is present, see [62, 85] for more details. Our numerical analysis however shows that this cannot be extended to \(n > 0\) since for \(n = 1, 2\) small BH approximation is restored, so that the near-horizon scalar condensation instability is suppressed and we only deal with superradiant instability.
perturbed by a charged scalar field [84].

FIG. 1: The imaginary part of the frequency plotted in terms of the length of AdS for different values of conformal parameter $n = 0, 1, 2$ ($d=4, 8, 12$) as well as different curvatures of the event horizon ($k = 0, k = 1$). We set the values $Q = 0.9$ and $m = 1$ for charge and mass of BH, respectively.

C. Static hairy solutions: non-linear regime

So far, using a standard spherically symmetric metric (9), we have studied the realization of instability at leading order, i.e. in a fixed background. However, in what follows, by going beyond linear approximation, we may extract a family of static BH solutions including charged scalar-field hair which is surrounded by the event horizon and a natural AdS boundary. To do so, we generalize metric (9) to

$$ds^2 = -\eta(r)\xi(r)dt^2 + \frac{dr^2}{\eta(r)} + r^2d\Omega^2_{4n+2},$$

(24)
which includes an additional lapse function $\xi(r)$ carrying the matter field backreaction effects on the spacetime geometry \[48\]. Before going any further, it should be noted that ansats (13) represents the fact that the scalar perturbations near critical frequency are not static but time dependent. By appealing to $U(1)$ symmetry one is able to extract static scalar hair solutions from time-dependent ansatz (13) at the non-linear level. Specifically speaking, the static scalar hair solutions are commonly obtain by using $\chi = \omega t$ in gauge transformation (8) \[48\]. This eliminates the time dependency of the scalar field as $\Psi \equiv \psi(r)$ and the time component of vector partial $A_t$ becomes

$$A_t = -\frac{Q}{r} + \frac{\omega}{q}.$$ \hspace{1cm} (25)

As a result, at the critical frequency $\omega_c = \frac{Q}{r_h}$, we automatically have $A(r_h) = 0$. Therefore, the gauge transformation (8) will render $A_t$ static, making the search for hairy static solutions near critical frequency in which the scalar field and BH are in equilibrium, a possibility. For $\omega < \omega_c$, the system gets out of equilibrium and makes the occurrence of instability possible \[48\]. One of the reasons for demanding $A(r_h) = 0$ in studying instability and extracting scalar hairy BH solutions, is the requirement of having a regular event horizon \[48\].

Now by introducing the gauge potential $A_t = A_0(r)$ due to gauge transformation (25) and fixing the generalized metric (24), the equations of motion (2)-(4) result in the following four equations

\[\eta(r)^2(1 + 2n)\xi'(r) - q^2 r A_0(r)^2 \psi'(r)^2 - \eta(r)^2 r \xi(r) \psi'(r)^2 = 0,\] \hspace{1cm} (26)

\[\eta'(r) + \frac{4n + 1}{r} (\eta(r) - k) + \frac{2n - 1}{\xi(r)(n + 1)} r A_0(r)^{2n + 2} - \frac{(4n + 3)}{r} \xi'(r) + \frac{\xi'(r)}{2} \eta(r) = 0,\] \hspace{1cm} (27)

\[(2n + 1) \eta(r) A_0'(r)^2 A_0''(r) + \left(\frac{(4n + 2)}{r} \eta(r) - \frac{\eta(r) \xi'(r)}{2 \xi(r)}\right) A_0'(r)^{2n + 1} - (-2)^n q^2 \xi(r) \eta(r)^2 A_0(r) = 0,\] \hspace{1cm} (28)

\[\eta(r) \psi''(r) + \left(\frac{\eta'(r)}{r} + \frac{4n + 2}{\eta(r)} + \frac{\eta(r) \xi'(r)}{2 \xi(r)}\right) \psi'(r) + \frac{q^2 A_0(r)^2 \psi(r)}{\eta(r) \xi(r)} = 0,\] \hspace{1cm} (29)

where $A_0'(r) = E$ with $E$ being the electric field. It seems that this set of equations cannot be solved analytically so in what follows, we numerically provide a possible family of solutions for the above coupled equations.

Let us first discuss the boundary conditions on the BH horizon, $r = r_h$ and the AdS boundary, $r = L$. Assuming regular horizon, that is, $\eta(r_h) = 0$ and $\eta'(r_h) > 0$ we have $m(r_h) = \frac{\lambda}{2} + (k + \frac{2}{L^2})$. For the value of the additional laps function $\xi(r)$ at the event horizon we set $\xi(r_h) = 1$ while the requirement to have finite physical quantities on the horizon imposes the conditions $A(r_h) \equiv A_h = 0 = \psi_h \equiv \psi'(r_h)$ for the field variables. As a result, employing a Taylor series expansion

\[m = m_h + m_h'(r - r_h) + ...,\] \hspace{1cm} (30)

\[A_0 = A_h + A_h'(r - r_h) + \frac{A_h''}{2}(r - r_h)^2 + ...,\] \hspace{1cm} (31)

\[\psi = \psi_h + \psi_h'(r - r_h) + \frac{\psi_h''}{2}(r - r_h)^2 + ...,\] \hspace{1cm} (32)

for the field variables in the near event horizon and substituting them into field equations (26)-(29), we arrive at

\[m_h' = 2^{n - 2} r_h^{4n + 2} E_h^{2n + 2},\] \hspace{1cm} (33)

\[\xi_h' = \frac{r_h^3 q^2 \psi_h^2 E_h^2}{(1 + 2n)\left(-2n - 1 r_h^2 r_h^{2n + 2} + (4n + 1) k + (4n + 3) r_h^2 E_h^2\right)^\frac{1}{2}},\] \hspace{1cm} (34)

\[A_h'' = \frac{(2n + 1)\left(2n - 1 r_h^2 E_h^{2n + 2} + (4n + 1) k + (4n + 3) r_h^2 E_h^2\right)}{(2n + 1)\left(-2n - 1 r_h^2 E_h^{2n + 2} + (4n + 1) k + (4n + 3) r_h^2 E_h^2\right)^\frac{1}{2}} + \left(\frac{\xi_h'}{2} - \frac{4n + 2}{r_h}\right) \frac{E_h}{2n + 1},\] \hspace{1cm} (35)

\[\psi_h'' = -\frac{q^2 \psi_h r_h^2 E_h^2}{2 \left(2n + 1\right)^\frac{1}{2}} + \left(\frac{4n + 3}{E_h^2} r_h^2 + (4n + 1) k - 2^{n - 1} r_h^2 E_h^{2n + 2}\right)^\frac{1}{2}.\] \hspace{1cm} (36)

Moreover the above boundary conditions imply that the Dirichlet boundary condition $\psi(r = L) = 0$ is satisfied by the test scalar field $\psi$ without imposing any constraint on other quantities $\eta(r), \xi(r)$ and $A(r)$ \[62\].

By numerically integrating the coupled differential equations (26)-(29) as well as applying the above series expansions as initial conditions evaluated at $r = r_h + \epsilon$ ($r_h = 1, \epsilon \sim 10^{-15} - 10^{-10}$) to circumvent singularity on the event horizon,
we are able to extract the static hairy BH solutions. Note that henceforward we just focus on flat and spherical geometries \((k = 0, \ + 1)\) since we have already found that for \(k = -1\), the condition of small BH is violated.

Let us first present the solutions for variables\(^6\) \(\eta(r), \xi(r), A(r), \psi(r)\) in Figs. 2 and 3 for \(k = 0\) and \(k = +1\), respectively. As can be seen, we are generally dealing with two types solutions: single and multiple radial nodes for the scalar field \(\psi(r)\). From a stability perspective, a solution with a single radial node at the AdS boundary is desirable in the sense that it addresses a scalar field solution enclosed between the horizon and AdS boundary which, due to being in the ground state, enjoys the classical and quantum mechanical stability. It should be stressed that although a single node scalar field solution can be a good candidate to provide a stable hairy BH, further investigation in the form of spherically symmetric perturbations is also needed. The solutions with more radial nodes represent the scalar field in exited states (depending on the number of nodes) which is expected to be classically unstable. Note that even in the event of classical stability of these exited solutions, there is yet the probability of their transition to the ground state via quantum tunneling [86]. As a result, the scalar field solutions with more radial nodes are not prone to displaying a stable hairy BH. The four solutions shown in Figs. 2 and 3 have been obtained for even conformal parameters \(n = 0\) or \(n = 2\) so that for \(n = 1\), there is no possibility of a numerical static BH solution.

![Graph of \(\eta(r), \xi(r), A(r), \psi(r)\) vs Radius](image)

**FIG. 2:** A sample plot for two laps functions \(\eta, \xi\) and two matter fields \(A, \psi\) for a particular static BH solution with flat topology and conformal parameter \(n = 2\). The numerical values used are \(q = 0.7, A' = 0.2, \psi_h = 0.3\) with AdS radius \(L = 12\). There is no numerical solution for static BH with conformal parameter \(n = 1\).

**Concerning the far field behavior of variables \(\eta(r), \xi(r), A(r)\),** it should be noted that due to lack of an asymptotically flat BH solution, such solutions include non trivial scalar hair [49, 50] so that it is not necessity for these three variables to have finite values as \(r \to \infty\).
FIG. 3: A sample plots of the laps functions $\eta$, $\xi$ and two matter functions $A, \psi$ for a particular static BH solution with spherical geometrical geometry and conformal parameters: $n = 0$ (top row) and $n = 2$ (down row). In the top row, we have used numerical values $q = 0.9$, $A'_h = 0.2$, $\psi_h = 0.25$ and $q = 0.06$, $A'_h = 0.3$, $\psi_h = 0.5$ for left and right panels, respectively. In the bottom row, we have used numerical values $q = 0.6$, $A'_h = 0.5$, $\psi_h = 0.4$ and $q = 0.655$, $A'_h = 0.55$, $\psi_h = 0.8$ for left and right panels, respectively. The value of AdS radius is taken as $L = 40$. There is no possibility of numerical static BH solution for the conformal parameter $n = 1$.

an AdS radius larger than what is expected for the case when $n = 0$. Overall, in spite of the existence of non-trivial scalar hairy BH solutions with planar and spherical symmetry horizons in solution space, it seems that such solutions are not able to form a scalar hairy BH and the system still remains in its primary status i.e., RN-AdS background.

FIG. 4: A plot of the scalar field $\psi$ as a function of radius for two horizon topologies: planar, left panel and spherical, right panel. We used numerical values $L = 10$, $A'_h = 0.5$, $\psi_h = 0.5$ and $L = 40, A'_h = 0.3$, $\psi_h = 0.1$. 
III. CONCLUDING REMARKS

It is an established fact that the embedding of a charged BH in an asymptotically AdS space-time may result in instability due to the role played by the timelike boundary of space-time acting as a natural reflecting boundary, the so-called Dirichlet wall. The nature of this instability is superradiant if one chooses the gauge $C = 0$, since in this gauge the superradiance appears for a particular set of states with certain boundary conditions at infinity. However, in the gauge $C \neq 0$ imposed directly by AdS-RN, we have encountered a superconducting-like instability for a set of states that obey different boundary conditions at infinity. We note that both sets of states are mapped into each other by means of $U(1)$ gauge transformations which do not vanish at infinity, implying that they are different states. This means that the instabilities are related, while addressing different physical phenomena.

In this paper, with the aim of investigating the no-scalar hair paradigm in higher dimensions, we have followed the possibility of occurrence of instability in a $d$-dimensional topological RN-AdS-small BH arising from Einstein-conformally invariant Maxwell gravity. In fact, the conformal invariance feature of the source does not allow arbitrary dimensions higher than four, but allows dimensions which are multiples of four, that is $d = 8, 12, ...$, equivalent to the conformal parameters $n = 1, 2, ...$

Employing a charged massless scalar perturbing test field with small amplitude, we have applied a numerical procedure to investigate instability at the level of linear approximation. Within the context of small black holes with $r_h \ll L$, the numerical analyses cannot tell us much about instability for $k = -1$. In contrast, for two modes $k = 0, +1$ it was shown that by increasing the scalar charge there is a possibility of higher dimensional instability, see Fig. 1. Next, by leaving linear approximation we have numerically extracted a family of static BH solutions, including a charged massless scalar-field hair which is enclosed between the event horizon (with planar and spherical horizon topologies) and the natural AdS boundary. Interestingly, for even conformal parameters, in particular $n = 2$, there are static solutions with scalar profiles $\psi(r)$ including single and multiple radial nodes which respectively address the ground and excited states with classical stability/instability. We found that for the relevant BH configuration with planar horizon topology there is no possibility of extracting a ground state scalar hairy solution, see the left panel in Fig. (4), while for its spherical counterpart, there is. Generally, for BH configurations with spherical horizon topology with two configurations there is a chance to have ground state scalar hairy profiles; for small scalar charge (right panel in Fig. 4) and also for AdS length smaller than what we expect from the standard case $n = 0$. However, none of these two cases are solid candidates to be identified as the ground state scalar hairy solutions. The former, as can be seen from the numerical solution shown in Fig. 1, suggests that instability will not occur for small values of the scalar charge. The latter is due to what we expect to see as the first node at the AdS at distances larger than that for $n = 0$, contrary to what is observed in Fig. 5. So, in spite of the existence of non-trivial scalar hairy BH solutions with planar and spherical horizon symmetries, because of their weak probability of occurrence, the system has more chance to remain in RN-AdS background. As a consequence, no-scalar hair theorem is supported in the sense that the possibility of interaction of a scalar field with a RN-AdS BH with the consequence of the emergence of a new quantity such as mass and charge, does not exist.
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