Hosotani model in closed-string theory

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Abstract

The Hosotani mechanism in the closed-string theory with current algebra symmetry is described by the (old covariant) operator method. We compare the gauge symmetry breaking mechanism in a string theory which contains $SU(2)$ symmetry originated from current algebra with the one in an equivalent compactified closed-string theory. We also investigate the difference between the Hosotani mechanism and the Higgs mechanism in closed-string theories by calculation of a four-point amplitude of ‘Higgs’ bosons at tree level.

1 Introduction

String theories [1] have made a great impact on the development of unification in forces and matters in nature. The original models of string theories as candidates for the unification were formulated in higher dimensional spacetime [2, 3]. In such models, we must consider that the extra dimensions except for four dimensions become invisible to us [4].

Compactification of extra spaces is utilised in such theories to generate gauge symmetries which may govern the forces in nature [5]. This scheme is the extension of the idea of Kaluza and Klein [6]. The gauge interaction can be unified with gravity in higher dimensions. A lot of work has been done on the generalised Kaluza-Klein theory, including cosmological considerations [7].

Recently, it has been clarified that large gauge symmetries are required in order to cancel quantum anomalies [3] in string theories. Thus, if we want to construct the unified theory from string theory in higher dimensions, we should consider symmetry-breaking mechanisms rather than symmetry generation at the stage of the compactification into four dimensions. In fact, the compactification onto the manifolds [8] (or orbifolds [9]) which have no, or few, continuous symmetries is considered in such string theories.

More recently, the investigation by fermionic construction of string model building [10] and other methods to formulate new string models [11] has resulted in the concept of four-dimensional string theory. Most recently, Gepner [12] showed that some of the four-dimensional models can be interpreted as
compactified theories on Calabi-Yau manifolds, which give rise to phenomenologically favoured low-energy aspects [13]. Thus it could be said that the geometrical interpretation of compactification ‘strikes back’ in string theories.

Now let us go back to the issue of gauge symmetry breaking. The mechanisms for geometrical symmetry breaking are known in field theory with extra dimensions. The so-called Wilson loop mechanism or Hosotani mechanism [14, 15] is one of them; the breakdown of symmetry is caused by, roughly speaking, the ‘vacuum expectation value’ of gauge fields on an extra, non-simply-connected space. These vacuum gauge fields play the role of the order parameter in the mechanism. The simplest example for a non-simply-connected manifold is a circle, $S^1$. The model with extra $S^1$ is originally investigated by Hosotani [15] and, because of its simplicity, many authors pursued the Hosotani model from various points of view [16, 17]. Of course the simplest model is not realistic, but it is expected that we can examine qualitative properties of the mechanism by the study of the model with the extra space $S^1$.

Particle physicists often regard the test of their models to be their explanation of early evolution in the universe. From a cosmological viewpoint, phase transition in the early universe is an interesting subject to investigate (see, for example [18]). The present author studied the one-loop free energy of the Hosotani model with several sorts of matter fields at zero and finite temperature. We concluded that no remarkable dependence on temperature was found in the model [17]. Similar analysis was made for a model in open-string theory and almost the same conclusion was drawn [19]. In the case of open strings, we anticipate the similarity to the ‘particle’ case, since the interaction with external gauge fields is restricted to the edges of the world sheet, which behave like tracks of moving particles.

We hope to look into the symmetry-breaking mechanism in closed-string theory. We expect purely ‘stringy’ effects in the closed-string model which cannot be obtained in the models mentioned earlier. Indeed the subject of current research concentrates on the closed-string theory, which is phenomenologically interesting [8]; for a review see [20]. This is another reason for investigating the mechanism in closed-string theory.

In this paper we will show a framework of the symmetry-breaking mechanism in closed-string theory. The treatment of the Hosotani mechanism is easy in the operator formalism [1]. We begin with the deformation of Virasoro generators and show the interpretation of symmetry breaking in the model. The order parameter is considered to be set by hand in this paper. The dynamical determination of the order parameter is left to be examined in future publications.

This paper is organised as follows. In §2 we review the old (fashioned) operator method [1] and we describe torus compactification in closed-string theory by this method. In §3, firstly we introduce a gauge symmetry into closed-string theory by current algebra in the usual way [1]. Secondly, using an analogy with the torus compactification, we formulate a symmetry-breaking mechanism which can be interpreted as the Higgs mechanism in the theory. The Hosotani mechanism in closed-string theory is described in §4. We give an
evaluation of a four-point amplitude of scalars in the model. We also calculate the same amplitudes in the Higgs model explained in §3. It is shown that the difference can be seen in the amplitude even if the masses of scalar bosons are identical in both models. The final section is devoted to discussion.

In this paper we consider bosonic strings only. We set $\alpha' = 1/2$ throughout this paper.

2 Torus compactification in closed-string theory

We begin with a review of the operator approach to string theory [1]. First we introduce the Virasoro generators which are constructed from the ‘harmonic oscillator’ operators associated with excitation of string modes:

$$L_n = \frac{1}{2} \sum_{m=\infty}^{\infty} : \alpha_{n-m} \cdot \alpha_m : .$$

(1)

Here the ‘normal order’ should be taken as

$$: \alpha_n^\mu \alpha_m^\nu : = \alpha_n^\mu \alpha_m^\nu \quad \text{when } n < m$$

$$= \alpha_m^\nu \alpha_n^\mu \quad \text{when } n > m.$$

(2)

By virtue of this treatment, $L_0$ is simply written as

$$L_0 = \frac{1}{2} P^2 + \sum_{m=1}^{\infty} : \alpha_{-m} \cdot \alpha_m :$$

(3)

where we rewrite $\alpha_0$ as $P$.

In closed string theory we must prepare a set of copies of Virasoro generators $\tilde{L}_n$, which are expressed in right moving oscillator modes $\tilde{\alpha}_m$.

Next we adopt a tachyonic state $|\cdot\rangle$ as usual. It satisfies

$$L_0 |\cdot\rangle = \tilde{L}_0 |\cdot\rangle = |\cdot\rangle.$$ 

(4)

Now, since we wish to investigate gauge symmetry breaking in subsequent sections, we ought to pay attention to the masses of the light fields.

The ‘mass operator’ is defined as

$$M^2 = 4(L_0 + \tilde{L}_0 - 2)$$

(5)

for closed strings. The quotation marks imply that this operator gives masses for any external string state whose momentum is set equal to zero.

In the closed strings, this mass operator is always accompanied with the following mass-matching condition:

$$(L_0 - \tilde{L}_0) |\cdot\rangle = 0$$

(6)

where $|\cdot\rangle$ is an external physical state. The physical states are expected to be made from the ‘creation’ operators $\alpha_{-n}$ ($n > 0$) applied to the tachyonic state.
To take a complete account for physical string states, it is necessary to mention the existence of spurious states [1]. However, we shall neglect that painful task in this paper and merely remark that the construction of spurious states is related to the Virasoro algebra.

Now consider compactification of one dimension to a circle, $S^1$. We denote the dimension as the $I$th direction and the other directions are labelled by the index $i$. The momenta of strings are discretised in the $I$th direction. For closed strings it is well known that there are windings on the torus; these are also specified by an integer. These quantum numbers are rearranged according to momenta of left- and right-moving sectors of strings:

$$P^I_L = \frac{\ell'}{2R} + mR \quad \text{and} \quad P^I_R = \frac{\ell'}{2R} - mR$$  \hspace{1cm} (7)

where $R$ is the radius of $S^1$. $\ell'$ and $m$ are integral and represent discreteness of momentum on $S^1$ and the winding number around $S^1$, respectively. The following two quantities will be important in later analysis:

$$\frac{1}{2} \{(P^I_L)^2 + (P^I_R)^2\} = \left(\frac{\ell'}{2R}\right)^2 + (mR)^2$$ \hspace{1cm} (8)

$$\frac{1}{2} \{(P^I_L)^2 - (P^I_R)^2\} = \ell' m$$ \hspace{1cm} (9)

Expression (8) plus the contribution from a number of string oscillators is just the mass of a certain external state. Expression (9) is closely connected with the mass-matching condition on each string excitation level.

Let us consider a linear transformation in $P_I$. For instance, suppose the following mixing:

$$\begin{pmatrix} P'_L \\ P'_R \end{pmatrix} = \begin{pmatrix} \cosh \alpha & \sinh \alpha \\ \sinh \alpha & \cosh \alpha \end{pmatrix} \begin{pmatrix} P^I_L \\ P^I_R \end{pmatrix}$$ \hspace{1cm} (10)

where $\alpha$ is a constant. Then the quantities (8) and (9) become

$$\frac{1}{2} \{(P^I'_{L})^2 + (P^I'_{R})^2\} = \left(\frac{\ell'}{2R}\right)^2 e^{2\alpha} + (mR)^2 e^{-2\alpha}$$ \hspace{1cm} (11)

and

$$\frac{1}{2} \{(P^I'_{L})^2 - (P^I'_{R})^2\} = \ell' m$$ \hspace{1cm} (12)

under the transformation (10).

As seen from (11), this transformation gives rise to the variation of the radius of $S^1$ according to

$$R \rightarrow R' = Re^{-\alpha}.$$ \hspace{1cm} (13)

Note that the combination (12) (or (9)) is invariant under mixing of this type. The exercise above can be regarded as a very primitive example of a general
lattice compactification considered by many authors [11]. The information of the momentum lattice is usually assumed to be carried by external states. However, there is another possible formulation to describe this.

We consider a perturbation on the Virasoro operator. In the case mentioned above, we define two operators \( L'_0 \) and \( \bar{L}'_0 \) as follows:

\[
L'_0 = L_0 - \frac{1}{2} (P^I_L)^2 + \frac{1}{2} (P^I'_L)^2
\]

\[
\bar{L}'_0 = \bar{L}_0 - \frac{1}{2} (P^I_R)^2 + \frac{1}{2} (P^I'_R)^2
\]

where \( P^I' \) is to be considered as an operator and is defined as (10). Thus the states do not have to carry all the information about the compactification. In the present case, it is sufficient that states have information of a fixed-radius compactification, say, \( R = 1/\sqrt{2} \). The value of the radius \( R \) is attributed to \( P^I' \) in \( L'_0 \) of (14).

The momentum in an extra space in string theory can be given in two equivalent ways described above. Eventually, the replacement \( P^I_L \to P^I'_L \) in \( L_n \) produces sets of Virasoro generators \( L'_n \) and \( \bar{L}'_n \) which obey the same Virasoro algebra as \( L_n \).

The method described in this section will be extended to the case with current-algebra symmetry in closed strings in the subsequent sections.

3 The Higgs mechanism

First of all, we introduce current algebra into the closed string theory [1]. The current which has an index of the adjoint representation can be analysed by the mode expansion on the world-sheet coordinate:

\[
J^a(z) = \sum_{-\infty}^{\infty} J^a_n z^{n-1}.
\]

(15)

The coefficients, as the operators, satisfy the commutation relations:

\[
[J^a_m, J^b_n] = i f^{abc} J^c_{m+n} + \frac{1}{2} k \delta^{ab} \delta_{m+n,0}
\]

(16)

where the level of the Kac-Moody algebra \( k \) is a constant. Gauge symmetry arises from this (affine) Kac-Moody algebra. The zero modes \( J^a_0 \) form a Lie algebra \( G \) with the relation of the generators

\[
[J^a_0, J^b_0] = i f^{abc} J^c_0
\]

(17)

where \( f^{abc} \) are the structure constants of \( G \). We equip another current \( \bar{J}(\bar{z}) \) for the right-moving modes. Therefore we can describe the gauge symmetry \( G \times \bar{G} \) in the low-energy sector of the theory. In this paper we consider only the case with \( \bar{G} \sim G \). The massless spectrum consists of not only the graviton,
antisymmetric tensor field and dilaton but also non-Abelian vector bosons and scalars. The string states \( J^\pm_a \bar{\alpha}^{-1}_i |0\rangle \) and \( \alpha^{-1}_i J^\pm_a |0\rangle \) correspond to the gauge bosons which transform as the \((\text{adj},1)\) and \((1,\text{adj})\) representation of \( G \times G \). \( J^\pm_a, J^\pm_b |0\rangle \) corresponds to a multiplet of scalars of the \((\text{adj},\text{adj})\) representations \([21, 22]\). Here, and in the following, we choose \( SU(2) \) as symmetry group \( G \) for simplicity. Moreover, we take the Cartan-Weyl basis, and then the adjoint index \( a = \{1, 2, 3\} \) is reformed to \( \{+, -, 3\} \). Namely, (17) yields \[ J^+ + 0, J^- 0 \] = 2 \( J^3 0 \), \[ J^3 0, J^+ 0 \] = \( J^+ 0 \), and so on. The (unperturbed) Virasoro operator is expressed as \[ 21, 22, 23, 24 \]

\[ L_0 = \frac{1}{2} (P_L)^2 + \frac{1}{2} \sum : \alpha_{-m} \cdot \alpha_{m} : + \frac{1}{K} \sum : J^-_n J^+_n :, \]  

where \( K = c_v + k \). \( c_v \) is defined by the relation \( f^{acd} f^{bcd} = c_v \delta^{ab} \). \( \bar{L}_0 \) is written in an expression similar to (18).

Now we try to examine gauge symmetry breaking of this model in analogy with the example illustrated in the preceding section. We are going to use the following principles. First, we use the Fourier components of original string modes and the Kac-Moody currents as building blocks of new Virasoro operators \( L'_0 \) and \( \bar{L}'_0 \). Second, the perturbed Virasoro operators \( L'_0 \) and \( \bar{L}'_0 \) should involve continuous parameters which indicate the deviation from the original \( L_0 \) and \( \bar{L}_0 \). Third, low-lying external states, at least, are found trivially. Other points will be clarified through the construction below.

The concept of perturbed operators has previously been considered \([25]\), but in a different context.

Similarly to the case with momenta on a circle, we consider the following transformation among zero modes of currents belonging to Cartan subalgebra:

\[ J^3 0' = \cosh \beta J^3 0 + \sinh \beta J^3 \]
\[ \bar{J}^3 0' = \cosh \beta \bar{J}^3 0 + \sinh \beta \bar{J}^3 \]

where, as previously

\[ J^3 0 = \frac{1}{2\pi i} \oint dz J^3(z) \]

and \( \beta \) is a parameter of perturbation.

Proceeding along this lines, we can construct new \( L'_0 \) and \( \bar{L}'_0 \):

\[ L'_0 = L_0 + A \{ (J^3 0')^2 - (J^3 0)^2 \} \]
\[ \bar{L}'_0 = \bar{L}_0 + A \{ (\bar{J}^3 0')^2 - (\bar{J}^3 0)^2 \} \]

where \( A \) is a constant to be determined. This choice of the new Virasoro operators makes the left-right level matching unchanged (for any \( A \)),

\[ L'_0 - \bar{L}'_0 = L_0 - \bar{L}_0 . \]  

The constant \( A \) is hence determined by consideration of other Virasoro generators. They are naturally defined as

\[ L'_n = L_n + B \{ J^3 0' J^3_n - J^3 0 J^3_n \} \]
\[ \bar{L}'_n = \bar{L}_n + B \{ \bar{J}^3 0' \bar{J}^3_n - \bar{J}^3 0 \bar{J}^3_n \} \]

\[ 6 \]
where $B$ is a constant.

We assume that the new Virasoro generators (20) obey the same form of the Virasoro algebra

$$[L_n', L_m'] = (n - m)L_{n+m'} + \text{the central term}.$$  

(23)

Then the followings identities are required:

$$A = 1/k \quad B = 2/k.$$  

(24)

Incidentally, the case with torus compactification is regained formally by supposing that the gauge group is Abelian and $k = 2$.

Now the mass-shifts for charged states are given by

$$\delta M_{op}^2 = \frac{4}{k}(\sinh \beta)^2(2(J_0^3)^2 + (\bar{J}_0^3)^2) + 4(\sinh \beta \cosh \beta)J_0^3 \bar{J}_0^3,$$  

(25)

where $J_0^3$ and $\bar{J}_0^3$ act as ‘charge’ operators. Obviously, gauge bosons $(J_{-1}^\pm \bar{\alpha}_{-1}^\pm |0\rangle$ and $J_{-1}^\pm \alpha_{-1}^\pm |0\rangle$) and scalar bosons $(J_{-1}^\pm \bar{J}_{-1}^\pm |0\rangle)$ are found by making (25) massive when $\beta$ is slightly different from zero.

The physical meaning of this symmetry breaking can be read as follows [26]. The vertex operator for a scalar boson at zero momentum is written in the form:

$$\phi^{ab} \sim \int dz d\bar{z} J^a(z) \bar{J}^b(\bar{z}).$$  

(26)

Therefore the second term proportional to $\sinh \beta \cosh \beta$ in (25) can be taken as the effect of condensation of zero-mode of $\phi^{33}$. The ‘vacuum parameter’ $\beta$ indicates the magnitude of the order parameter $\langle \phi^{33} \rangle$ in some non-linear manner. In this way, we can deal with a spontaneously broken ($SU(2)$) gauge theory with massive gauge and Higgs bosons.

It is well known, on the other hand, that a level one ($k = 1$) $SU(2) \times SU(2)$ Kac-Moody symmetry can be generated from the left- and right-moving modes of strings on a circle $S^1$ [27]. To see this, we only examine the left-moving algebra formed by the currents defined as [28]

$$J^+(z) = : \exp(i\sqrt{2}X^L_L) :$$

$$J^-(z) = : \exp(-i\sqrt{2}X^L_L) :$$

$$J^3(z) = (i/\sqrt{2})\partial_z X^L_L$$  

(27)

where $X^L_L$ is the left-moving mode of strings on $S^1$. If the radius of the circle $R$ is set to $1/\sqrt{2}$, they form $SU(2)$ Kac-Moody algebra. If the value of $R$ deviates from $1/\sqrt{2}$, the $SU(2)$ symmetry will become broken and only $U(1)$ symmetry will be left, which trivially exists in a model with a circle.

Alternatively, we can consider the background metric in path integral as

$$G_{\mu\nu} = (-1, 1, 1, \ldots, R^2).$$  

(28)
and we take the string coordinate $X^I (R = 1/\sqrt{2})$. This results in the same partition functions as before. Thus we can regard $R^2$ as a zero mode of Kaluza-Klein scalar field; but in the string case, this Kaluza-Klein scalar is interacting with the stringy excitations which form $SU(2)$-adjoint scalar, and vector and other massive fields. The deviation of $R$ from $1/\sqrt{2}$ can be interpreted as the expectation value of a scalar field.

Thus one can check the mass spectrum in the ‘Higgs’ mechanism by comparison with each of the other models. The inclusion of background fields is well described in the path integral approach [29, 30]. For our purpose, we consider the zero-mode part of the partition function in particular. The zero-mode piece of the bosonic string coordinate on $S^1$ is of the form

$$\bar{X}^I = 2\pi R(m\sigma_1 + \ell\sigma_2) \quad (29)$$

where $\sigma_1$ and $\sigma_2$ are coordinates of the world sheet and $m$ and $\ell$ are integers [29]. Here we take the notion $G_{\mu\nu} = (-1, 1, \ldots, 1)$ as in the former case above.

Then the integrand of the partition function is proportional to

$$\sum_{\ell m} e^{-S} = \frac{(\ell^2 / 2R^2)}{\sum_{\ell m} \exp \left( -\pi \tau_2 \left( \frac{\ell^2 + 2m^2 R^2}{2R^2} \right) + 2\pi i \tau_1 \ell m' \right) } \quad (30)$$

We can read the mass spectrum from this expression, i.e.,

$$M^2 = 2(\ell'^2 / 2R^2 + 2m^2 R^2 - 4) + \text{oscillators} \quad (31)$$

where $-4$ in the parentheses comes from the tachyon in bosonic strings. $\ell'$ and $m$ are integers. The last term including $\tau_1$ in (30) is concerned with the mass-matching constraint such as (9). In order to compare with our model, we define

$$Q_L \equiv (1/\sqrt{2})(\ell' + m) \quad Q_R \equiv (1/\sqrt{2})(\ell' - m) \quad (32)$$

and further, we set

$$Q'_L = \cosh \beta Q_L + \sinh \beta Q_R \quad Q'_R = \cosh \beta Q_R + \sinh \beta Q_L \quad (33)$$

where $\sqrt{2}R = \exp(-\beta)$. Using this set of variables, (31) is rewritten as

$$M^2 = 4\{(Q'_L)^2 + (Q'_R)^2 - 2\} + \text{oscillators} \quad (34)$$

Thus the mass shift is given by

$$\delta M^2 = 4[(\sinh \beta)^2((Q_L)^2 + (Q_R)^2) + 2(\sinh \beta \cosh \beta)Q_LQ_R] \quad (35)$$

It seems that we may identify $Q_L$ and $Q_R$ with the eigenvalues of charge operators $(2/k)^{1/2}J^3_3$ and $(2/k)^{1/2}J^3_0$ with $k = 1$, respectively. Indeed, one can easily check the case for massless external states. Namely, the gauge bosons $J^\pm_{-1}\bar{\alpha}_{-1}|0\rangle$ have $Q_L = \pm \sqrt{2}$ and $Q_R = 0$, whilst $J^\pm_{-1}\alpha_{-1}|0\rangle$ have $Q_L = 0$ and $Q_R = \pm \sqrt{2}$. The scalar Higgs bosons $J^\pm_{-1}J^\pm_{-1}|0\rangle$ also acquire masses. We can safely say that we can construct a modular invariant partition function with Higgs mechanism in $SU(2) \times SU(2)$ gauge theory, because of the correspondence with torus compactification when $k = 1$. 

8
4 The Hosotani model in closed strings

Let us now consider the Hosotani model in closed string theory. The model we consider has $SU(2) \times SU(2)$ symmetry at first, in the same fashion as discussed in §3, as well as a compact extra-space $S^1$. First of all, we give a specific example for a mixed transformation among internal momenta and charge operators. That is,

$$
\begin{pmatrix}
P_I^L \\
Q'_L \\
P'_R \\
Q'_R
\end{pmatrix} =
\begin{pmatrix}
\cosh \theta & 0 & 0 & \sinh \theta \\
0 & \cosh \theta & \sinh \theta & 0 \\
0 & \sinh \theta & \cosh \theta & 0 \\
\cosh \theta & 0 & 0 & \cosh \theta
\end{pmatrix}
\begin{pmatrix}
P_I^L \\
Q_L \\
P'_R \\
Q_R
\end{pmatrix}
$$

(36)

where $Q_L = (2/k)^{1/2} J_0^{3}$ and $Q_R = (2/k)^{1/2} \bar{J}_0^{3}$, and the notations are same as before, but the number of large dimensions is one less than that in the Higgs model treated in §3. Subsequently, we construct the Virasoro operators as follows:

$$L'_0 = L_0 + \frac{1}{2} ((Q'_L)^2 + (P'_L)^2 - (Q_L)^2 - (P_L)^2)$$

$$\bar{L}'_0 = \bar{L}_0 + \frac{1}{2} ((Q'_R)^2 + (P'_R)^2 - (Q_R)^2 - (P_R)^2).$$

(37)

This example is clearly a straightforward extension of the previous models in §§2 and 3. The Virasoro generators $L_n$ can also be constructed in a similar way.

Generally speaking, a transformation matrix $T$ which belongs to $SO(2,2)$ can be utilised in this type of model. Namely, if by such a matrix the relation

$$T \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \end{pmatrix} T^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \end{pmatrix}$$

is satisfied, then the condition $L'_0 - \bar{L}'_0 = L_0 - \bar{L}_0$ holds by use of $T$ as the transformation matrix in (36). As is known, the number of parameters which characterise a general $T$ is six. However, transformations which belong to a subgroup $SO(2)_L \times SO(2)_R$ for instance, the rotation

$$
\begin{pmatrix}
P'_L \\
Q'_L \\
P'_R \\
Q'_R
\end{pmatrix} =
\begin{pmatrix}
\cos \phi_L & -\sin \phi_L & 0 & 0 \\
\sin \phi_L & \cos \phi_L & 0 & 0 \\
0 & 0 & \cos \phi_R & -\sin \phi_R \\
0 & 0 & \sin \phi_R & \cos \phi_R
\end{pmatrix}
\begin{pmatrix}
P_L \\
Q_L \\
P'_R \\
Q_R
\end{pmatrix}
$$

(39)

leaves each Virasoro operator unchanged:

$$L'_0 = L_0 \quad \bar{L}'_0 = \bar{L}_0.$$  

(40)

In other words, the transformation such as (39) makes no change in the mass spectrum in the string model. As far as symmetry breaking is concerned the whole mass spectrum is governed by four parameters. Again, of these four
parameters two correspond to the mechanism considered before, in §§2 and 3, one is concerned with the variation of the size of $S^1$ and one is concerned with the Higgs mechanism. Thus the number of parameters which ‘describe’ the geometrical symmetry breaking is two. One of these two parameters is, say, $\theta$ in (36). The other will be discussed later, because we first wish to clarify the physical implications of the example.

To see the physical interpretation, we think about the difference in mass operators:

$$
(L'_0 + \bar{L}'_0) - (L_0 + \bar{L}_0) = (\sinh \theta)^2 \{(P_L^I)^2 + (P_R^I)^2 + (Q_L)^2 + (Q_R)^2\} + 2(\sinh \theta \cosh \theta)(P_L^I Q_R + Q_L P_R^I).
$$

(41)

In the limit of small $\theta$, this reduces to

$$
(2\theta)(2/k)^{1/2}(P_L^I J_0^3 + P_R^I J_0^3) \equiv -V_0.
$$

(42)

This expression can be regarded as the ‘zero-mode’ of the vertex, of vector bosons:

$$
(2/k)^{1/2}(\bar{J}(z)\partial X^I_L(z) + J(z)\partial X^I_R(z)) e^{ikX}.
$$

(43)

In addition, when we analyze the new propagator with $\theta \neq 0$, we can give the following expansion in terms of the original propagator ($\theta = 0$), up to the left-right matching constraint:

$$
\frac{1}{L_0 + L'_0 - 2} = \frac{1}{L_0 + L_0 - 2} + \frac{1}{L_0 + L_0 - 2} V_0 \frac{1}{L_0 + L_0 - 2} + \frac{1}{L_0 + L_0 - 2} V_0 \frac{1}{L_0 + L_0 - 2} + \ldots.
$$

(44)

This shows the insertion of interactions with zero-modes of gauge fields on $S^1$. Therefore we can say that the mechanism described above is a simplest example of the Hosotani mechanism in closed-string theory. It can be also said that in this example the scale of the vacuum gauge field is given by

$$
\langle A^3_I \rangle \sim (-2\theta)
$$

(45)
in the small $\theta$ limit. $A^3_I$ denotes a certain combination of zero modes of two gauge fields.

Now let us examine scattering amplitudes in this framework. A tree amplitude of four particles contains poles associated with the intermediate states [1]. Thus we can obtain much knowledge about the mass spectrum and we can compare the amplitude with that in other models such as the Higgs model. For simplicity we show a scattering amplitude of four doubly adjoint scalar bosons:

$$
\phi^{+3} + \phi^{33} \rightarrow \phi^{+3} + \phi^{33}.
$$

(46)

The superscripts indicate the left and right $SU(2)$ ‘charge’ classified as in §3. The external states are represented as

$$
\phi^{+3} \sim (2/k) J^+_m J^-_{-m} |0, P^I = 0\rangle
$$

(47)
and the $\phi^{33}$ emission vertex is determined to be

$$V^{33}(z = \bar{z} = 1) = \frac{2}{k} \left( \sum_n J_n^3 - J_0^3 + J_0^3' \right) \left( \sum_n \bar{J}_n^3 - \bar{J}_0^3 + \bar{J}_0^3' \right) e^{ikX(1)}$$ \hspace{1cm} (48)

by investigation of the ghost-decoupling condition [1].

Using these preparations, we can calculate the four-point amplitude of scalars and this reduces to:

$$A(\theta) = \frac{\kappa^2}{4} \times \left[ \left( \Gamma \left( -\frac{1}{8} s + \frac{1}{8} m^2 + 1 \right) \Gamma \left( -\frac{1}{8} t - 1 \right) \Gamma \left( -\frac{1}{8} u + \frac{1}{8} m^2 + 1 \right) \right) \left( \Gamma \left( \frac{1}{8} s - \frac{1}{8} m^2 \right) \Gamma \left( \frac{1}{8} t + 2 \right) \Gamma \left( \frac{1}{8} u - \frac{1}{8} m^2 \right) \right) + \frac{\Gamma \left( -\frac{1}{8} s + \frac{1}{8} m^2 + 1 \right) \Gamma \left( -\frac{1}{8} t - 1 \right) \Gamma \left( -\frac{1}{8} u + \frac{1}{8} m^2 - 1 \right) \Gamma \left( \frac{1}{8} s - \frac{1}{8} m^2 \right) \Gamma \left( \frac{1}{8} t \right) \Gamma \left( \frac{1}{8} u - \frac{1}{8} m^2 \right) + \frac{\Gamma \left( -\frac{1}{8} s + \frac{1}{8} m^2 - 1 \right) \Gamma \left( -\frac{1}{8} t - 1 \right) \Gamma \left( -\frac{1}{8} u + \frac{1}{8} m^2 + 1 \right) \Gamma \left( \frac{1}{8} s - \frac{1}{8} m^2 \right) \Gamma \left( \frac{1}{8} t \right) \Gamma \left( \frac{1}{8} u - \frac{1}{8} m^2 \right) + \frac{\Gamma \left( -\frac{1}{8} s + \frac{1}{8} m^2 + 1 \right) \Gamma \left( -\frac{1}{8} t + 1 \right) \Gamma \left( -\frac{1}{8} u + \frac{1}{8} m^2 - 1 \right) \Gamma \left( \frac{1}{8} s - \frac{1}{8} m^2 + 1 \right) \Gamma \left( \frac{1}{8} t \right) \Gamma \left( \frac{1}{8} u - \frac{1}{8} m^2 + 1 \right) \right) \right]$$ \hspace{1cm} (49)

where $\kappa^2$ is the coupling in closed strings and $m^2(\theta) = 8/k(\sinh \theta)^2$. In the amplitude (49), the exchange of tachyon and graviton appears in the first term in the first parentheses. This term has poles at $t = -8, 0, \ldots$. The gauge interaction and interaction among charged particles are contained in the second parentheses. For instance, the last term describes the contribution of massive gauge bosons to intermediate states. The term contains poles in the $s$ channel at $s = m^2, m^2 + 8, \ldots$. Note that the ratio of couplings of gauge boson and graviton exchange at tree level turns out to be

$$2/k(\cosh \theta)^2.$$ \hspace{1cm} (50)

The $k$ dependence has been mentioned by Ginsparg [31]; a new feature is the dependence of effective coupling on the ‘vacuum parameter’ $\theta$.

We can also carry out calculation of the same amplitude in the Higgs model introduced in §3. The result is

$$\frac{4}{k^2} A_{Higgs}(\beta) = \frac{4}{k^2} A(\theta \rightarrow \beta) + \frac{2}{k} (\sinh \beta)^2 \left( \frac{\Gamma \left( -\frac{1}{8} s + \frac{1}{8} m^2 \right) \Gamma \left( -\frac{1}{8} t - 1 \right) \Gamma \left( -\frac{1}{8} u + \frac{1}{8} m^2 \right) \Gamma \left( \frac{1}{8} s - \frac{1}{8} m^2 \right) \Gamma \left( \frac{1}{8} t \right) \Gamma \left( \frac{1}{8} u - \frac{1}{8} m^2 \right)} \right) + \frac{2}{k} (\sinh \beta)^2 (\cosh \beta)^2 \left( \frac{\Gamma \left( -\frac{1}{8} s + \frac{1}{8} m^2 \right) \Gamma \left( -\frac{1}{8} t + 1 \right) \Gamma \left( -\frac{1}{8} u + \frac{1}{8} m^2 \right) \Gamma \left( \frac{1}{8} s - \frac{1}{8} m^2 + 1 \right) \Gamma \left( \frac{1}{8} t \right) \Gamma \left( \frac{1}{8} u - \frac{1}{8} m^2 + 1 \right)} \right).$$
where \( m^2 = 8/k (\sinh \beta)^2 \). We find that the amplitudes differ slightly between the two models, even though the masses of gauge bosons are same. This difference is, of course, due to the ways of coupling to the background fields, i.e., gauge and scalar bosons. The last term in the additional terms in (51) contains two four-scalar couplings and two Higgs background fields \(<\phi^{33}>\) in the scattering of low-lying states.

So far we have studied the Hosotani model using an example (36) which includes only one vacuum parameter. We must return to investigate another freedom in modification of mass spectrum. Needless to say, two parameters originate from (the combinations of) two vacuum gauge fields associated with \( SU(2)_L \) and \( SU(2)_R \) symmetry. Again, we give another example for finding the parameter:

\[
\begin{pmatrix}
P'_{L} \\
Q'_{L} \\
P'_{R} \\
Q'_{R}
\end{pmatrix}
= 
\begin{pmatrix}
1 & B/2 & 0 & -B/2 \\
-B/2 & 1 & B/2 & 0 \\
0 & B/2 & 1 & -B/2 \\
-B/2 & 0 & B/2 & 1
\end{pmatrix}
\begin{pmatrix}
P_{L} \\
Q_{L} \\
P_{R} \\
Q_{R}
\end{pmatrix}
\]

where \( B \) is the new vacuum parameter. Though the matrix in (52) looks a little bizarre, this rotation matrix evidently obeys \( SO(2,2) \) symmetry. Also multiplication of the matrices of the same form defines a small group. The investigation of the mass operator with this parameter reveals a possibility that eight massless vector bosons emerge when \( B = 1 \). These bosons probably take the form of some mixture of \( SU(3) \) gauge fields. This enlargement of symmetry can be understood when \( k = 1 \), using the equivalent torus compactlification [27, 32]. In this case, we must employ a two-torus as an internal space. Let us assume that the background fields, that is to say, metric and antisymmetric tensor fields on the torus, are

\[
G_{mn} = 
\begin{pmatrix}
\cosh 2\theta & -\sinh 2\theta \\
-\sinh 2\theta & \cosh 2\theta
\end{pmatrix}
\]

and

\[
B_{mn} = 
\begin{pmatrix}
0 & B \\
-B & 0
\end{pmatrix}
\]

where \( m \) and \( n \) are the indices of the two-torus and \( \theta \) and \( B \) are constants. Note that \( \det G_{mn} = 1 \). When we consider the partition function of the compactified model, we can read the mass spectrum from the path-integral form. The comparison will be made in a manner similar to that in §3, but now we have two compactified dimensions; both radii of two scales are set to \( R = 1 \). First we can construct the eigenvalue of the ‘charge’ from a set of quantised numbers, of momentum and winding, in the direction of one dimension. Next, we interpret that another dimension will remain as a one-torus, that is, a circle. There the notation of the left-right momenta is maintained. In consequence, we obtain a model which can be compared with the Hosotani model in the operator formulation.
In this situation mentioned above, one can calculate the partition function
by the path-integral method and after some slightly tedious rearrangement using
Jacobi’s transformation, we find the mass spectrum:

$$M^2 = \frac{1}{2} \left\{ (P'_L)^2 + (P'_R)^2 + (Q'_L)^2 + (Q'_R)^2 \right\} + \text{(oscillators)}. \quad (55)$$

We have only to know how we can construct momenta $P'$ and charges $Q'$ at
finite $\theta$ and $B$ from these at $\theta = B = 0$ (denoted as $P$ and $Q$). The relation
turns out to be:

$$
\begin{pmatrix}
P'_L \\
Q'_L \\
P'_R \\
Q'_R
\end{pmatrix}
= 
\begin{pmatrix}
\cosh \theta & 0 & 0 & \sinh \theta \\
0 & \cosh \theta & \sinh \theta & 0 \\
0 & \sinh \theta & \cosh \theta & 0 \\
\sinh \theta & 0 & 0 & \cosh \theta
\end{pmatrix}
\times
\begin{pmatrix}
1 & B/2 & 0 & -B/2 \\
-B/2 & 1 & B/2 & 0 \\
0 & B/2 & 1 & -B/2 \\
-B/2 & 0 & B/2 & 1
\end{pmatrix}
\begin{pmatrix}
P'_L \\
Q_L \\
P'_R \\
Q_R
\end{pmatrix}. \quad (56)
$$

The correspondence of parameters is apparent. The symmetry enhancement
(e.g., when $\theta = 0$ and $B = 1$) can be explained by explicit construction of currents from bosonic coordinates similar to (27) in the compactified model. As is
well known the maximal symmetry group has rank equal to the dimension of the
torus. Note, however, that in our model in the operator method, the maximal
symmetry is a single $\text{SU}(3)$, not $\text{SU}(3) \times \text{SU}(3)$; the latter is attained by the
previous two-torus compactification. There is not a one-to-one correspondence
between the spectrum of the strings with currents and that of the naïve compactified string model. If we want exactly similar partition functions, we need
an appropriate projection in compactification of the background. Anyway, the
occurrence of symmetry enhancement is a purely ‘stringy’ effect.

This feature clearly originates from condensation of the antisymmetric tensor
field, as is manifestly shown in this parametrisation; although this is less obvious
when we take a general method of parametrisation of the vacuum parameters.

5 Discussion

In this paper we have shown the method describing the gauge symmetry breaking
mechanism, especially the Hosotani mechanism, in bosonic string theory by
introduction of modified Virasoro operators. Also, a comparison with the models
with symmetries which come from torus compactification has been made.

We have left two problems which require further study. One is on the derivation
of Virasoro generators from the ‘first principles’ such as the consideration of stress tensor in two-dimensional theory. The study of Wess-Zumino-Witten
models will help the investigation of some algebraic structure when background
fields exist. The progress in the study of the case with $k > 1$ will prove helpful
in this problem. It is necessary to research into the connection with the description of conformal field theories. Another problem, apart from the trivial extension to general symmetry groups, is the application to the supersymmetric model. We can examine the dynamical determination of vacuum parameters and finite temperature effects based on such a model, because the supersymmetric models do not suffer from the disease of tachyon. In superstring models, we have two further distinct points of interest. First, the enlargement of the gauge group is anticipated in the models as in the bosonic case. In the supersymmetric case, however, projections onto physical states are essential and then the symmetry enhancement mechanism may not have a straightforward generalisation. Second, we expect interesting aspects in the thermal property of the model. We hope that the consideration of symmetry-breaking (or enhancement) mechanisms in string theory brings a new perspective to string cosmology.

Finally, we wonder how the Hosotani model on a torus can be extended to the Wilson loop mechanism on general non-simply-connected manifolds. Perhaps an analogous construction of Wilson loops can be performed in terms of the description of strings on non-simply-connected Calabi-Yau manifolds by Gepner [12]. In such a model, if it is possible, vacuum parameters will no longer be continuous quantities but discrete ones.

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