ON COMMUTING TONELLI HAMILTONIANS: AUTONOMOUS CASE

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Abstract. We show that the Aubry sets, the Mañé sets, Mather’s barrier functions are the same for two commuting autonomous Tonelli Hamiltonians. We also show the quasi-linearity of $\alpha$-functions from the dynamical point of view and the existence of common $C^{1,1}$ critical subsolution for their associated Hamilton-Jacobi equations.

1. Introduction

Let $M$ be a closed, connected $C^\infty$ Riemannian manifold. Let $TM$ and $T^*M$ be the tangent bundle and cotangent bundle of $M$, respectively. In local coordinates, we may express them as

$$TM = \{(q, \dot{q}) : q \in T_qM\}$$

and

$$T^*M = \{(q, p) : p \in T^*_qM\},$$

respectively. Let $pdq$ be the Liouville form. A $C^2$ function $H : T^*M \to \mathbb{R}$ is called Tonelli Hamiltonian if $H$ satisfies the following conditions:

- $H$ is fiberwise strictly convex, i.e., the fiberwise Hessian $\frac{\partial^2 H}{\partial p^2}$ is positively definite for every $(q, p) \in T^*M$.
- $H$ has superlinear growth, i.e., $H(q, p) |p| \to +\infty$ as $|p| \to +\infty$, where $|\cdot|$ is the norm induced by the Riemannian metric on $M$.

For a Tonelli Hamiltonian $H$, the dynamics of the Hamilton flow $\phi^t_H$ are well understood, thanks to the celebrated Mather theory [Man], [Mat1], [Mat2] and its weak KAM approach [Fa3].

Let $\{\cdot\}$ be the Poisson bracket. Recall that two Hamiltonians $H_1, H_2$ are commuting (in involution) if $\{H_1, H_2\} = 0$.

In this paper, we restrict ourselves to the relations in Mather theory between dynamics of two commuting Tonelli Hamiltonians. We show that so many things are same for two commuting Tonelli Hamiltonians. As a byproduct, we also show quasi-linearity of Mather’s $\alpha$-functions [Vi], from the view point of dynamics.

For a Tonelli Hamiltonian $H$, let $L_H$ be the Lagrangian associated to $H$ by Legendre transformation, i.e.,

$$L_H(q, \dot{q}) = p\dot{q} - H(q, p),$$

here $p$ and $\dot{q}$ are related by $\dot{q} = \frac{\partial H(q, p)}{\partial p}$. Throughout this paper, $L_H$ denotes the Legendre transformation from tangent bundle $TM$ to cotangent bundle $T^*M$, i.e.,

$$L_H(\dot{q}) = p \iff \dot{q} = \frac{\partial H(q, p)}{\partial p}.$$ 

For each cohomology class $c \in H^1(M, \mathbb{R})$, Mather’s $\alpha$-function is defined as follows:

$$\alpha_H(c) = -\min_\mu \int (L_H - \eta)d\mu,$$

where $\eta$ is a smooth (throughout this paper, smoothness means that $C^r, r \geq 2$) closed 1-form on $M$ with $[\eta] = c$ (throughout this article, $[\cdot]$ denotes de-Rham cohomology class of a closed 1-form); the minimum is taken over all invariant (under the Euler-Lagrange flow $\phi^t_{L_H}$ of $L_H$) Borel probability measures. We say that an invariant Borel probability measure $\mu$ is $c$-minimal if

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Note that the convergence of the limit is nontrivial, it follows from the convergence of Lax–Oleinik here \( \eta \). Let \( \theta : \mathbb{R} \times [0, 1] \to \mathbb{X} \) and any absolutely continuous curve \( \gamma : [0, T] \to \mathbb{X} \) have the projections of \( \eta \). Let \( \rho \) and it is also called Mañé set. Then, \( \dot{M} \) is a smooth closed 1-form on \( \mathbb{X} \). Let \( \pi \) be the projection of \( T^* \mathbb{X} \) or \( \mathbb{X} \) along the associated fibers onto \( \mathbb{X} \), according to the circumstance. The projection of \( \dot{M} \) (or \( \dot{M} \), equivalently) into \( \mathbb{X} \) is called projected Mather set. We denote the projected Mather set by \( \dot{M} = M \). 

For any \( \mathbb{R} \ni T > 0 \) and any closed 1-form, let 

\[
\hat{h}^T_{\mathbb{R}, \eta}(q_1, q_2) = \inf_{\gamma} \int_0^T (L_H - \eta + \alpha_H(\eta)) (\gamma(t), \dot{\gamma}(t)) dt,
\]

where minimum is taken over all absolutely continuous curve \( \gamma : [0, T] \to \mathbb{X} \) with \( \gamma(0) = q_1, \gamma(T) = q_2 \). Let 

\[
h_{\mathbb{R}, \eta}(q_1, q_2) = \lim_{T \to +\infty} \hat{h}^T_{\mathbb{R}, \eta}(q_1, q_2).
\]

Note that the convergence of the limit is nontrivial, it follows form the convergence of Lax–Oleinik semigroup in the time-independent case [Fa2]. Let 

\[
\rho_{\mathbb{R}}(q_1, q_2) = \rho_{\mathbb{R}, c}(q_1, q_2) = \rho_{\mathbb{R}}(q_1, q_2) = h_{\mathbb{R}, \eta}(q_1, q_2) + h_{\mathbb{R}, \eta}(q_2, q_1),
\]

here \( \eta \) is a smooth closed 1-form on \( \mathbb{X} \) with \( [\eta] = c \). Now projected Aubry set \( \dot{A}_{\mathbb{R}, c} = \{ q \in \mathbb{X} : \rho_{\mathbb{R}}(q, q) = 0 \} \). Then \( \rho_{\mathbb{R}} \) is a pseudo-metric on \( \dot{A}_{\mathbb{R}, c} \). Now define an equivalence relation \( \sim_{\rho_{\mathbb{R}, c}} \) on \( \dot{A}_{\mathbb{R}, c} \) by \( q_1 \sim_{\rho_{\mathbb{R}, c}} q_2 \) iff \( \rho_{\mathbb{R}}(q_1, q_2) = 0 \). Now let quotient Aubry set \( \dot{A}_{\mathbb{R}, c}, \rho_{\mathbb{R}, c} \) be the quotient metric space of \( \dot{A}_{\mathbb{R}, c} \) under the relation \( \sim_{\rho_{\mathbb{R}, c}} \).

We say that an absolutely continuous curve \( \gamma : \mathbb{R} \to \mathbb{X} \) is a c-minimizer, if for any interval \([a, b]\) and any absolutely continuous curve \( \gamma_1 : [a, b] \to \mathbb{X} \) such that \( \gamma_1(a) = \gamma(a) \) and \( \gamma_1(b) = \gamma(b) \), we have

\[
\int_a^b (L_H - \eta + \alpha_H(\eta)) (\gamma(t), \dot{\gamma}(t)) dt \leq \int_a^b (L_H - \eta + \alpha_H(\eta)) (\gamma(t), \dot{\gamma}(t)) dt,
\]

where \( \eta \) is a smooth closed 1-form on \( \mathbb{X} \) such that \( [\eta] = c \). We define Mañé set

\[
\dot{N}_{\mathbb{R}, c} = \dot{N}_{L, c} = \dot{N}_{c} = \cup \{ (\gamma(t), \dot{\gamma}(t)) : \gamma \text{ is a c-minimizer} \}.
\]

Thus \( \dot{N}_{c} \subset T \mathbb{X} \). Let

\[
\dot{N}_{\mathbb{R}, c} = \dot{N}_{L, c} = \dot{N}_{c} = \mathcal{L}_H \dot{N}_{H, c},
\]

and it is also called Mañé set. Then, \( \dot{N}_{c} \subset T^* \mathbb{X} \).

Let \( \gamma : \mathbb{R} \to \mathbb{X} \) be a c-minimizer. Let \( q \) be in \( \alpha \)-limit set and \( q' \) be in \( \omega \)-limit set of \( \gamma \). If \( \rho_{\mathbb{R}}(q, q') = 0 \), we say that \( \gamma \) is a regular c-minimizer. We define Aubry set

\[
\dot{A}_{H, c} = \dot{A}_{L, c} = \dot{A}_{c} = \cup \{ (\gamma(t), \dot{\gamma}(t)) : \gamma \text{ is a regular c-minimizer} \}.
\]

Clearly, \( \dot{A}_{c} \subset \dot{N}_{c} \). Let

\[
\dot{A}_{H, c} = \dot{A}_{L, c} = \dot{A}_{c} = \mathcal{L}_H \dot{A}_{H, c},
\]

and it is also called Aubry set.

Let projected Aubry set

\[
A_{H, c} = A_{L, c} = A_{c}
\]

and projected Mañé set

\[
\dot{N}_{H, c} = \dot{N}_{L, c} = \dot{N}_{c}
\]

be the projections of

\[
\dot{A}_{H, c} = \dot{A}_{L, c} = \dot{A}_{c} \text{ and } \dot{N}_{H, c} = \dot{N}_{L, c} = \dot{N}_{c}
\]

into \( \mathbb{X} \) respectively.

We have the following inclusions:

\[
\dot{M} = \dot{A}_{H, c} \leq \dot{A}_{c} \leq \dot{N}_{H, c}.
\]
Let $\eta$ be a smooth closed 1-form on $M$. We introduce two semigroups of nonlinear operators $(T_{H,\eta,t})_{t \geq 0}$ and $(T_{H,\eta,t}^+)_{t \geq 0}$ respectively. These semigroups are the so-called Lax-Oleinik semigroups. To define them, let us fix $u \in C^0(M, \mathbb{R})$ and $t \geq 0$. For $q \in M$, we set

$$T_{H,\eta,t}^-u(q) = \inf_{\gamma} \left\{ u(\gamma(0)) + \int_0^t (L_H - \eta + \alpha_H(\eta))(\gamma(s), \dot{\gamma}(s))ds \right\},$$

where the infimum is taken over all absolutely continuous curve $\gamma : [0, t] \to M$ such that $\gamma(t) = q$. Also, for $q \in M$, we set

$$T_{H,\eta,t}^+u(q) = \sup_{\gamma} \left\{ u(\gamma(t)) - \int_0^t (L_H - \eta + \alpha_H(\eta))(\gamma(s), \dot{\gamma}(s))ds \right\},$$

where the supremum is taken over all absolutely continuous curve $\gamma : [0, t] \to M$ such that $\gamma(0) = q$.

A function $u$ is a forward (resp. backward) weak KAM solution of Hamilton-Jacobi equation

$$H(q, \eta + d_q u) = \alpha_H(\eta)$$

if $T_{H,\eta,t}^-u(q)$ (resp. $T_{H,\eta,t}^+u(q)$) for any $t \geq 0$. Let $S_{H}^+(\eta)$ (resp. $S_{H}^-(\eta)$) be the set of all forward (resp. backward) weak KAM solutions of Hamiltonian Jacobi equation

$$H(q, \eta + d_q u) = \alpha_H(\eta),$$

where $\eta$ is a smooth closed 1-form on $M$. By weak KAM theory [Fa3], we have

$$\lim_{t \to \infty} T_{H,\eta,t}^+u \in S_{H}^+(\eta)$$

and

$$\lim_{t \to \infty} T_{H,\eta,t}^-u \in S_{H}^-(\eta)$$

for any $u \in C^0(M, \mathbb{R})$. Now we can state our main results as follows:

**Theorem 1.** Let $H_1, H_2$ be two Tonelli Hamiltonians $H_1, H_2$. If $\{H_1, H_2\} = 0$, then

$$T_{H_1,\eta,s}^-T_{H_2,\eta,r}^-u = T_{H_2,\eta,r}^-T_{H_1,\eta,s}^-u,$$

for any $u \in C^0(M, \mathbb{R})$, any smooth closed 1-form $\eta$ on $M$ and $\mathbb{R} \ni s, r \geq 0$.

**Remark 1.1.** When we complete writing down this paper, we learn to know that the result in Theorem 1 has appeared in [BT]. But the proof here is a more directly variational discussion, which is very different from [BT].

**Theorem 2.** Let $H_1, H_2$ be two Tonelli Hamiltonians. If $\{H_1, H_2\} = 0$, then $S_{H_1}^+(\eta) = S_{H_2}^+(\eta)$ and $S_{H_1}^-(\eta) = S_{H_2}^-(\eta)$, for any smooth closed 1-form $\eta$.

Now we recall the definitions of barrier functions [Ma2]. Let $\eta$ be a smooth closed 1-form with $[\eta] = c$, then the first barrier function

$$B_{H,c}(q) = h_{H,c}(q, q);$$

the second barrier function

$$b_{H,c}(q) = \min_{\xi, \zeta \in A_{H,c}} \left\{ h_{H,c}(\xi, q) + h_{H,c}(q, \zeta) - h_{H,c}(\xi, \zeta) \right\}.$$  

Clearly, $B_{H,c}$ and $b_{H,c}$ are independent of the choice of closed 1-form $\eta$ with $[\eta] = c$.

**Theorem 3.** Let $H_1, H_2$ be two Tonelli Hamiltonians. If $\{H_1, H_2\} = 0$, then Mathers barrier functions $B_{H_1,c}(q) = B_{H_2,c}(q)$ and $b_{H_1,c}(q) = b_{H_2,c}(q)$, for any cohomology class $c \in H^1(M, \mathbb{R})$.

**Theorem 4.** Let $H_1, H_2$ be two Tonelli Hamiltonians. If $\{H_1, H_2\} = 0$, then $\tilde{N}_{H_1,c} = \tilde{N}_{H_2,c}$ and $\tilde{N}_{H_1,c} = \tilde{N}_{H_2,c}$ for any cohomology class in $H^1(M, \mathbb{R})$.

**Theorem 5.** (Quasi-linearity of $\alpha$-function) Let $H_1, H_2$ be two Tonelli Hamiltonians. If $\{H_1, H_2\} = 0$, then $\alpha_{H_1+c}(c) = \alpha_{H_2}(c) + \alpha_{H_2}(c)$, for any cohomology class $c \in H^1(M, \mathbb{R})$.

**Remark 1.2.** In the case that $M = T^n$, the result in Theorem 5 has been obtained by Viterbo by his symplectic homogenization theory [V]. It should be mentioned that his result also covers the case of non-Tonelli Hamiltonians, where $\alpha$ functions are replaced by homogenized Hamiltonians.
For any \( c \in H^1(M, \mathbb{R}) \), let
\[
\Sigma_{H_1}(c) = \{(q, p) : H_1(q, p) \leq \alpha_{H_1}(c)\},
\]
and
\[
\Sigma_{H_2}(c) = \{(q, p) : H_2(q, p) \leq \alpha_{H_2}(c)\}.
\]
Let \( \Sigma(c) = \Sigma_{H_1}(c) \cap \Sigma_{H_2}(c) \). Now we have

**Theorem 6.** Given any smooth closed 1-form \( \eta \) with \( [\eta] = c \), there exists a \( C^{1,1} \) function \( u \) such that \( \eta + du \subset \Sigma(c) \).

**Remark 1.3.** This theorem implies that both Hamilton-Jacobi equations
\[
H_1(q, \eta + d_u) = \alpha_{H_1}(c)
\]
and
\[
H_2(q, \eta + d_u) = \alpha_{H_2}(c)
\]
have a common \( C^{1,1} \) subsolution, in the case that \( \{H_1, H_2\} = 0 \).

**Remark 1.4.** In a preprint [Cui], the first author has extended some results of this article to the time-periodic case.

**Note.** The first version of this paper appeared in July, 2009 and we submitted it to a journal on August 21, 2009. As Zavidovique pointed out, which version contained a big gap, although it is very easy to fix. In November, the corrected version appeared and on November 18, 2009, we put it on arXiv (arXiv:0911.3471). Shortly after, Zavidovique also put on (on November 19, 2009) a reprint [Zaj](arXiv:0911.3739), which contains similar results.

The results of this paper were also posted by the first author at a network meeting of Humboldt Foundation (November 24-26, 2009, Heidelberg), and the announcement of results was submitted to Humboldt Foundation by the first author on September 04, 2009.

2. **Proof of Theorem 1**

We prove the first equality, and the second equality in the theorem can be proved similarly. For any point \( q_0 \in M \), we will prove that
\[
T_{H_1, \eta, s}^- T_{H_2, \eta, r}^- u(q_0) = T_{H_2, \eta, r}^- T_{H_1, \eta, s}^- u(q_0).
\]
By the definition,
\[
T_{H_1, \eta, s}^- T_{H_2, \eta, r}^- u(q_0) = \min_{x \in M} (T_{H_2, \eta, r}^- u(x)) + h_{H_1, \eta}^s(x, q_0)
\]
\[
= \min_{x, y \in M} (u(y) + h_{H_2, \eta}^r(y, x) + h_{H_1, \eta}^s(x, q_0)).
\]
Clearly, there exist two points \( x_0, y_0 \) such that
\[
T_{H_1, \eta, s}^- T_{H_2, \eta, r}^- u(q_0) = u(y_0) + h_{H_2, \eta}^r(y_0, x_0) + h_{H_1, \eta}^s(x_0, q_0)).
\]
We assume that
\[
\gamma_1 : [0, r] \to M \text{ and } \gamma_2 : [r, r + s] \to M
\]
are two minimizers that reach \( h_{H_2, \eta}^r(y_0, x_0), h_{H_1, \eta}^s(x_0, q_0) \) respectively.

Now we have

**Lemma 2.1.** \( \mathcal{L}_{H_2}(\gamma_1(r)) = \mathcal{L}_{H_1}(\gamma_2(r)) \).

The proof of this lemma is just a standard variational discussion. Throughout this paper, we use * to denote the conjunction of curves or trajectories.

**Proof.** Let \( \Gamma(v, t) \) be an arbitrary variation of \( \gamma_1 \ast \gamma_2 \), here \( v \in (-\epsilon, \epsilon) (0 < \epsilon \in \mathbb{R}), t \in [0, r + s] \), \( \Gamma(v, 0) = \gamma_1(0), \Gamma(v, r + s) = \gamma_2(r + s) \), and
\[
\Gamma(0, t) = \begin{cases}
\gamma_1(t) & \text{when } 0 \leq t \leq r, \\
\gamma_2(t) & \text{when } r \leq t \leq r + s.
\end{cases}
\]
Then, for any fixed \( v \), we have
\[
\left. \frac{d}{dv} \right|_{v=0} \left( \int_0^r (L_{H_2} - \eta + \alpha_{H_2}([\eta]))(\Gamma(v, t), \frac{\partial \Gamma(v, t)}{\partial t}) dt \right.
+ \int_{r}^{r+s} (L_{H_1} - \eta + \alpha_{H_1}([\eta]))(\Gamma(v, t), \frac{\partial \Gamma(v, t)}{\partial t}) dt
\]
\[
\geq \left. \int_0^r (L_{H_2} - \eta + \alpha_{H_2}([\eta]))(\gamma_1(t), \dot{\gamma}_1(t)) dt \right.
+ \int_{r}^{r+s} (L_{H_1} - \eta + \alpha_{H_1}([\eta]))(\gamma_2(t), \dot{\gamma}_2(t)) dt.
\]

Then, we have
\[
\left. \frac{d}{dv} \right|_{v=0} \left( \int_0^r (L_{H_2} - \eta + \alpha_{H_2}([\eta])) + \int_{r}^{r+s} (L_{H_1} - \eta + \alpha_{H_1}([\eta]))(\Gamma(v, t), \frac{\partial \Gamma(v, t)}{\partial t}) dt \right.
= 0.
\]

Thus,
\[
0
= \left. \frac{d}{dv} \right|_{v=0} \left( \int_0^r (L_{H_2} - \eta + \alpha_{H_2}([\eta])) + \int_{r}^{r+s} (L_{H_1} - \eta + \alpha_{H_1}([\eta]))(\Gamma(v, t), \frac{\partial \Gamma(v, t)}{\partial t}) dt \right.
= \int_0^r \left( \frac{\partial L_{H_2}}{\partial q}(\Gamma(0, t), \frac{\partial \Gamma(0, t)}{\partial v}) \frac{\partial \Gamma(v, t)}{\partial v} \right|_{v=0} + \frac{\partial L_{H_2}}{\partial \dot{q}}(\Gamma(0, t), \frac{\partial \Gamma(0, t)}{\partial v}) \frac{\partial^2 \Gamma(v, t)}{\partial v \partial t} \bigg|_{v=0} dt
+ \int_{r}^{r+s} \left( \frac{\partial L_{H_1}}{\partial q}(\gamma_1(t), \dot{\gamma}_1(t)) \frac{\partial \Gamma(v, t)}{\partial v} \bigg|_{v=0} + \frac{\partial L_{H_1}}{\partial \dot{q}}(\gamma_1(t), \dot{\gamma}_1(t)) \frac{\partial^2 \Gamma(v, t)}{\partial \dot{q} \partial v} \bigg|_{v=0} \right) dt
+ \int_{r}^{r+s} \frac{\partial L_{H_1}}{\partial \dot{q}}(\gamma_2(t), \dot{\gamma}_2(t)) \frac{\partial \Gamma(v, t)}{\partial v} \bigg|_{v=0} + \int_{r}^{r+s} \frac{\partial L_{H_1}}{\partial \dot{q}}(\gamma_2(t), \dot{\gamma}_2(t)) \frac{\partial^2 \Gamma(v, t)}{\partial \dot{q} \partial v} \bigg|_{v=0} \right) dt
= \left. \frac{\partial L_{H_2}}{\partial q}(\gamma_1(t), \dot{\gamma}_1(t)) \frac{\partial \Gamma(v, t)}{\partial v} \bigg|_{v=0} \right)^2
+ \int_0^r \left( \frac{\partial L_{H_2}}{\partial q}(\gamma_1(t), \dot{\gamma}_1(t)) \frac{\partial \Gamma(v, t)}{\partial v} \bigg|_{v=0} \right)^2 dt
+ \int_{r}^{r+s} \left( \frac{\partial L_{H_1}}{\partial \dot{q}}(\gamma_2(t), \dot{\gamma}_2(t)) \frac{\partial \Gamma(v, t)}{\partial v} \bigg|_{v=0} \right)^2 dt
\]

where the second equality follows from direct calculation, together with the facts that \( \eta \) is closed, and the variation is taken to be fixed endpoints; the last equality follows from that \( \gamma_1 \) is a solution of Euler-Lagrange equation associated to \( L_{H_2} \), and \( \gamma_2 \) is a solution of Euler-Lagrange equation associated to \( L_{H_1} \). So, we have
\[
\frac{\partial L_{H_2}(\gamma_1(t), \dot{\gamma}_1(t))}{\partial \dot{q}} = \frac{\partial L_{H_1}(\gamma_2(t), \dot{\gamma}_2(t))}{\partial \dot{q}},
\]

since the above formula holds for any variation \( \Gamma \). In other words, \( L_{H_2}(\dot{\gamma}_1) = L_{H_1}(\dot{\gamma}_2) \), and Lemma 2.1 follows. \( \square \)

Let
\[
\mathcal{L}_{H_2}(\dot{\gamma}_1) = \mathcal{L}_{H_1}(\dot{\gamma}_2) := p^\circ.
\]
Hence, if we assume that $\mathcal{L}_{H_1}(\gamma_2(r + s)) = p_0$, then
\[(y_0, \mathcal{L}_{H_2}(\gamma_1(0))) = \phi_{H_2}^{-r} \phi_{H_1}^{-s}(q_0, p_0).\]
In this case,
\[
T_{H_1, \eta, s} T_{H_2, \eta, r} u(q_0) = u(y_0) + ((p - \eta) dq) \left( \phi_{H_2}(x_0, p^\circ)|_{-r, 0} \ast \phi_{H_1}(x_0, p^\circ)|_{0, s} \right)
\]
Here, and in the following, $\eta$ is regarded as a smooth section of $T^* M$ and $(p - \eta) dq$ is regarded as a smooth 1-form on $T^* M$. Note that the first equality follows from direct calculation; the second equality follows from that $\phi_{H_2}(q_0, p_0) = (x_0, p^\circ)$ and the fact that $H_2$ is constant on the trajectory of $\phi_{H_1}$.

Let $\gamma_4 : [s, s + r] \to M$ be the curve such that $\pi : \phi_{H_2}^{-t}|_{-r, 0}(q_0, p_0) = \gamma_4$, up to a time translation. Similarly, let $\gamma_3 : [0, s] \to M$ be the curve such that $\pi : \phi_{H_1}^{-t}|_{-s, 0}(\gamma_4(s), \mathcal{L}_{H_2}(\gamma_4(s))) = \gamma_3$, up to a time translation. Since $\{H_1, H_2\} = 0$, we have
\[
\pi \circ \phi_{H_1}^{-r} \phi_{H_2}^{-s}(q_0, p_0) = y_0.
\]
Hence,
\[
T_{H_2, \eta, r} T_{H_1, \eta, s} u(q_0) \leq u(\pi \circ \phi_{H_1}^{-s} \phi_{H_2}^{-r}(q_0, p_0)) + \int_0^s (L_{H_1} - \eta + \alpha_{H_1}(c))(\gamma_3(t), \gamma_3(t)) dt
\]
\[
+ \int_s^{s + r}(L_{H_2} - \eta + \alpha_{H_2}(c))(\gamma_4(t), \gamma_4(t)) dt
\]
Here, the first inequality follows from the definition of $T_{H_1, \eta, t}$, the first equality follows from the direct calculation. For the second equality, we should say some more words:
\[
((p - \eta) dq) \left( \phi_{H_2}(x_0, p^\circ)|_{0, s} \ast \phi_{H_2}(q_0, p_0)|_{-r, 0} \right)
\]
\[
- ((p - \eta) dq) \left( \phi_{H_2}(x_0, p^\circ)|_{-r, 0} \ast \phi_{H_1}(x_0, p^\circ)|_{0, s} \right)
\]
\[
= \langle dp \wedge dq, \phi_{H_1}|_{0, s} \phi_{H_2}(x_0, p^\circ)|_{0, s} \rangle
\]
\[
= 0,
\]
here, the first equality follows from Stokes' formula (recall that $\{H_1, H_2\} = 0$, the last inequality follows from
\[
\phi_{H_1}|_{0, s} \phi_{H_2}(x_0, p^\circ)|_{0, s}
\]
is isotropic.

Similarly, we can prove the opposite inequality, and hence Theorem 1 is proved.

Based on Theorem 1, we have the following propositions, which are crucial in the proof of Theorem 2.

**Proposition 2.1.** $S_{H_1, \eta}^\perp \cap S_{H_2, \eta}^\perp \neq \emptyset, S_{H_1, \eta}^\perp \cap S_{H_2, \eta}^\perp \neq \emptyset$.

**Proof.** We only prove the the first relation, and the second one can be proved similarly.
For any $u \in S_{H_1, \eta}$ and for any $s, r \in [0, \infty)$, we have
\[
T_{H_1, \eta, s} T_{H_2, \eta, r} u = T_{H_2, \eta, r} T_{H_1, \eta, s} u = T_{H_2, \eta, r} u.
\]
Now let $s \to \infty$, we have
\[
\lim_{s \to \infty} T_{H_1, \eta, s} T_{H_2, \eta, r} u = \lim_{s \to \infty} T_{H_2, \eta, r} u = T_{H_2, \eta, r} u \in S_{H_1, \eta}
\]
for any $r \in [0, \infty]$, by weak KAM theory [Fa3]. Now let $r \to \infty$, we have $T_{H_2, \eta, r} u$ converges uniformly to a function $u^* \in S_{H_2, \eta}$ [Fa3]. In fact, we also have $u^* \in S_{H_1, \eta}$. This follows from the stability of backward weak KAM solutions [Fa3], since $T_{H_2, \eta, r} u \in S_{H_1, \eta}$ for each $r$. Hence, we have $u^* \in S_{H_1, \eta} \cap S_{H_2, \eta}$ and the first relation is proved.

**Proposition 2.2.** $H_2|_{A_{H_1, [\eta]}} = \alpha_{H_2}([\eta]); H_1|_{A_{H_2, [\eta]}} = \alpha_{H_1}([\eta])$.

**Proof.** Throughout this paper, $\overline{\eta + du}$ denotes the closure of the set of
\[
\{(q, \eta + du)\mid u \text{ is differentiable at } q\}.
\]
Choose $u^* \in S_{H_1, \eta} \cap S_{H_2, \eta}$, we have $\alpha_{H_1}([\eta]) = H_2|_{\overline{\eta + du}^*}$. Since $A_{H_1, [\eta]} \subset \overline{\eta + du}^*$, the first equality holds. The second equality follows similarly. \hfill \Box

3. Proof of Theorem 2

Let $\eta$ be any smooth closed 1-form on $M$. Now, we will prove that if $u \in S_{H_1}(\eta)$, then $u \in S_{H_2}(\eta)$.

Firstly, we will show that if $u \in S_{H_1}(\eta)$, then $H_2|_{\overline{\eta + du}} = \text{constant}$. It follows from $u \in S_{H_1}(\eta)$ that $u$ is Lipschitz, hence $u$ is differentiable almost everywhere (with respect to Lebesgue measure). Since $\{H_1, H_2\} = 0$, $H_2$ is constant along this trajectory of $\phi_{H_1}$. Let $q$ be a differentiable point of $u$, then there exists an unique trajectory $(q(t), p(t)) : t \in (-\infty, 0)$ of the Hamilton flow $\phi_{H_1}^t$ such that $q(0) = q, p(0) = \eta|_q + du$ and the limit set of $(q(t), p(t)) : t \in (-\infty, 0)$ lies in $\mathcal{A}_{H_1, [\eta]}$. Hence, $H_2$ is constant on the closure of this trajectory, and so, $H_2$ is constant on some compact subset of $\mathcal{A}_{H_1, [\eta]}$. By Proposition 2.2, we have $H_2|_{\eta + du}$, hence $H_2|_{\overline{\eta + du}}$ is constant and the constant is $\alpha_{H_2}([\eta])$.

Next, we will show that $u \in S_{H_2}(\eta)$, by showing that $u$ is a viscosity solution of
\[
H_2(q, \eta + du) = \alpha_{H_2}([\eta]).
\]

Let us recall the definition of viscosity solution of Hamilton-Jacobi equation. Firstly, let us fix arbitrarily a smooth closed 1-form $\eta$ on $M$. A function $u : M \to \mathbb{R}$ is a viscosity subsolution of Hamilton-Jacobi equation
\[
H(q, \eta + du) = d
\]
if for every $C^1$ function $\phi : M \to \mathbb{R}$ and every point $q_0 \in M$ such that $u - \phi$ has a maximum at $q_0$, we have $H(q_0, \eta|_{q_0} + du, \phi) \leq d$. A function $u : M \to \mathbb{R}$ is a viscosity supersolution of Hamilton-Jacobi equation
\[
H(q, \eta + du) = d
\]
if for every $C^1$ function $\phi : M \to \mathbb{R}$ and every point $q_0 \in M$ such that $u - \phi$ has a minimum at $q_0$, we have $H(q_0, \eta|_{q_0} + du, \phi) \geq d$.

A function $u : M \to \mathbb{R}$ is a viscosity solution of Hamilton-Jacobi equation
\[
H(q, \eta + du) = d
\]
if it is both a subsolution and a supersolution. The set of viscosity solutions of
\[
H(q, \eta + du) = d
\]
is denoted by $S_H^\text{vis}(\eta)$.

For a Tonelli Hamilton $H$, the Hamilton-Jacobi equation
\[
H(q, \eta + du) = d
\]
has a viscosity solution if and only if $d = \alpha_H([\eta])$ [Fa3]. Moreover, a function $u$ is a viscosity solution of
\[
H(q, \eta + du) = \alpha_H([\eta])
\]
Lemma 3.1. There exists a constant 

Now we will prove that \( u \) is also a viscosity supersolution of

\[
H_2(q, \eta + d_\eta u) = \alpha_{H_2}(|\eta|). 
\]

We denote by \( D \) the set of lower differential of \( u \) at \( x_0 \).

Recall the definition of semi-concave function (with linear modulus) \([CS], [Fa3]\). Let us fix once for all a finite atlas \( \Phi \) of charts \( \phi : B_3 \to M \), where \( B_r \) is the open ball of radius \( r \) centered at zero in \( \mathbb{R}^d \). We assume that the sets \( \phi(B_1)(\phi \in \Phi) \) cover \( M \). A family \( \mathcal{F} \) of \( C^2 \) functions is said \( K \)-bounded if

\[
|d^2(u \circ \phi)_x| \leq K
\]

for all \( x \in B_1, \phi \in \Phi, u \in \mathcal{F} \).

Definition 3.2. A function \( u : M \to \mathbb{R} \) is called \( K \)-semi concave (with linear modulus) if there exists a \( K \)-bounded subset \( \mathcal{F}_u \) of \( C^2(M, \mathbb{R}) \) such that

\[
u = \inf_{f \in \mathcal{F}_u} f.
\]

The constant \( K \) is also called semi-concave constant. A function \( u : M \to \mathbb{R} \) is called \( K \)-semi-convex if \(-u \) is \( K \)-semi-concave.

The following lemma is due to Fathi \([Fa3]\), and we state it here with slight modifications:

**Lemma 3.1.** There exists a constant \( K > 0 \), such that \( u \) is \( K \)-semi-concave, for each \( u \in S_{H_1}(\eta) \) or \(-u \in S_{H_1}^+(\eta) \).

So, if \( u \) is a backward weak KAM solution of \( H_1 \), then \( u \) is semi-concave \([Fa3]\). Hence, \( u \) is differentiable at \( q \) if \( D_-u(q) \neq \emptyset \). So, we have that \( u \) is also a viscosity supersolution of

\[
H_2(q, \eta + d_\eta u) = \alpha_{H_2}(|\eta|),
\]

since \( H_2|_{\eta = \alpha_{H_2}(|\eta|)} = \alpha_{H_2}(|\eta|) \) \([Fa3]\). Now we have that \( u \) is a viscosity solution of

\[
H_2(q, \eta + d_\eta u) = \alpha_{H_2}(|\eta|).
\]

Hence, \( u \) is also a backward weak KAM solution to Hamilton-Jacobi equation

\[
H_2(q, \eta + d_\eta u) = \alpha_{H_2}(|\eta|),
\]

by weak KAM theory \([Fa3]\).

Similarly, if \( u \in S_{H_1}(\eta) \), then \( u \in S_{H_1}^+(\eta) \).

Thus, \( S_{H_1}(\eta) = S_{H_1}^+(\eta) \).

Now we will show that \( S_{H_1}^+(\eta) = S_{H_2}^+(\eta) \). Before we enter into the proof, we recall the definition of symmetrical Hamiltonian. Let \( H(q, p) \) be a Tonelli Hamiltonian, then the symmetrical Hamiltonian (with respect to \( \eta \)) is defined as \( H(q, \eta + p) = H(q, \eta - p) \).

**Lemma 3.2.** \( L_{H^1}(q, \dot{q}) - \eta(q) = L_{H}(q, -\dot{q}) - \eta(-\dot{q}) \).
Proof. In fact, we have

\[
L_H(q, \dot{q}) - \eta \dot{q} = (p - \eta) \dot{q} - \dot{H}(q, p) = p_1 \dot{q} - \dot{H}(q, \eta + p_1) \quad (p_1 = p - \eta).
\]

Hence, if \( u \) and \( H \) are related by \( \dot{q} = \frac{\partial H(q, p)}{\partial p} \) in the first equality. All the inequalities are obviously, except the last one. We must check the last equality, since where the Legendrian equality is used. Note that the relation can also be expressed as

\[
\dot{q} = \frac{\partial H(q, p)}{\partial p} = \frac{\partial H(q, -p + 2\eta)}{\partial (p + 2\eta)} = -\frac{\partial H(q, -p + 2\eta)}{\partial (-p + 2\eta)}.
\]

In other words,

\[
\dot{q} = \frac{\partial H(q, -p)}{\partial (-p)}.
\]

Hence, if \( \dot{q} = L_H(p) \), then \( -\dot{q} = L_H(-p) \). Thus, the lemma follows. \( \square \)

The Lagrangian \( \dot{L}_H := L_H \) is also called symmetrical Lagrangian to \( L_H \). By the fundamental result of weak KAM theorem, we have \( u \in S_H^+(\eta) \) if and only if \( -u \in S_H^{-}(\eta) \). Hence, we only need to show that \( S_H^{-} = S_H^{+} \), when \( \{H_1, H_2\} = 0 \). Clearly, \( \{H_1, H_2\} = 0 \), whenever \( \{H_1, H_2\} = 0 \). Hence, it follows from the above discussions that \( S_{H_1}^+(\eta) = S_{H_2}^+(\eta) \).

Thus, Theorem 2 is proved.

4. Proof of Theorem 3

Definition 4.1. For a Tonelli Hamiltonian \( H \) and a smooth closed 1-form \( \eta \), we say \( u_- \in S_H^{-}(\eta) \) and \( u_+ \in S_H^+(\eta) \) are conjugate with respect to \( H \) if \( u_- = u_+ \) on the projected Mather set \( M_{H,|\eta|} \). If \( u_- \) and \( u_+ \) are conjugate with respect to \( H \), we also denote this relation by \( u_- \sim_H u_+ \).

Based on this definition, we can express equivalent definitions [Fa3] of \( A_{H,c} \) and \( N_{H,c} \) as follows:

\[
A_{H,c} = \cap \left\{ q : u_-(q) = u_+(q), \text{ where } u_- \in S_H^{-}(\eta), u_+ \in S_H^+(\eta), u_- \sim_H u_+ \right\}
\]

and

\[
N_{H,c} = \cup \left\{ q : u_-(q) = u_+(q), \text{ where } u_- \in S_H^{-}(\eta), u_+ \in S_H^+(\eta), u_- \sim_H u_+ \right\}.
\]

Consequently, we also have [Fa3]

\[
\dot{A}_{H,c} = \cap \left\{ (q, p) : p = d_q u_- = d_q u_+ : u_- \in S_H^{-}(\eta), u_+ \in S_H^+(\eta), u_- \sim_H u_+ \right\}
\]

and

\[
\dot{N}_{H,c} = \cup \left\{ (q, p) : p = d_q u_- = d_q u_+ : u_- \in S_H^{-}(\eta), u_+ \in S_H^+(\eta), u_- \sim_H u_+ \right\}.
\]

Proposition 4.1. If \( \{H_1, H_2\} = 0 \), then \( \dot{M}_{H_1,c} \subseteq \dot{A}_{H_2,c} \), \( \dot{M}_{H_2,c} \subseteq \dot{A}_{H_1,c} \), for any cohomology class \( c \in H^1(M, \mathbb{R}) \).

Proof. We only need to show that \( \dot{M}_{H_1,c} \subseteq \dot{A}_{H_2,c} \), by the symmetry of \( H_1 \) and \( H_2 \). By weak KAM theory [Fa3], we have

\[
\dot{M}_{H_1,c} \subseteq \cap_{u_- \in S_H^{-}(\eta)} \{ \eta + du_- \} = \cap_{u_- \in S_H^{-}(\eta)} \{ \eta + du_- \}.
\]

By the result of Sorrentino [Sa], \( \dot{M}_{H_1,c} \) is also invariant under the flow of \( \phi^t_{H_2} \), since \( \{H_1, H_2\} = 0 \). Note that \( \dot{M}_{H_1,c} \) lies in the graph of \( \eta + du_- \), for any \( u_- \in S_H^{-}(\eta) = S_H^{-}(\eta) \).

In [Fa3], Fathi proved the following lemma:
Lemma 4.1. For any two points \( q_0, q_1 \), we have the following equality

\[
h_{H,q}(q_0, q_1) = \sup \left\{ u_-(q_1) - u_+(q_0) : u_+ \in S^+_H(\eta), u_- \in S^-_H(\eta), u_- \sim_H u_+ \right\}.
\]

Moreover, for any given \( q_0, q_1 \in M \), this supremum is actually attained.

As a consequence of this lemma, together with the definition of conjugate pair of weak KAM solutions, we have

Corollary 4.1.

\[
h_{H,q}(q_0, q_1) = \sup \left\{ u_-(q_1) - u_-(q_0) : u_- \in S^-_H(\eta) \right\}
\]

for any two points \( q_0, q_1 \in A_{H,c} \).

We also need the definition of dominated function. Let \( L_H \) be the Lagrangian associated to the Hamiltonian \( H \). Recall that a function \( f : M \to \mathbb{R} \) is dominated by \( L_H \) if for each absolutely continuous curve \( \gamma : [a, b] \to M \), we have

\[
 f(\gamma(b)) - f(\gamma(a)) \leq \int_a^b L_H(\gamma(t), \dot{\gamma}(t)) \, dt.
\]

If \( f \) is dominated by \( L_H \), we denote it by \( f \preceq L_H \).

Choose any \( (q_0, p_0) \in M_{H_{1,c}}, \) we will show that \( (q_0, p_0) \in \mathcal{A}_{H_{2,c}} \). Fix a smooth closed 1-form \( \eta \) with \( [\eta] = c \).

Firstly, we will show that \( \pi \circ \phi^1_{H_{2}}(q_0, p_0) \) is a \( c \)-minimizer with respect to \( L_{H_{2}} \). For any \( t_1 < t_2 \in \mathbb{R} \), and any absolutely continuous curve \( \gamma_1 : [t_1, t_2] \to M \) with

\[
 \gamma_1(t_1) = \pi \circ \phi^1_{H_{2}}(q_0, p_0), \gamma_1(t_2) = \pi \circ \phi^2_{H_{2}}(q_0, p_0),
\]

we have

\[
 \int_{t_1}^{t_2} (L_{H_{2}} - \eta + \alpha_{H_{2}}([\eta]))(\pi \circ \phi^t_{H_{2}}(q_0, p_0), \frac{d}{dt}(\pi \circ \phi^t_{H_{2}}(q_0, p_0))) \, dt
\]

\[
 = \int_{t_1}^{t_2} \left( (\pi \circ \phi^t_{H_{2}}(q_0, p_0)) - u_-(\pi \circ \phi^t_{H_{2}}(q_0, p_0)) \right)
\]

\[
 \leq \int_{t_1}^{t_2} (L_{H_{2}} - \eta + \alpha_{H_{2}}([\eta]))(\gamma_1(t), \dot{\gamma}_1(t)) \, dt,
\]

where \( u_- \in S^-_{H_{1}}(\eta) \); the first equality follows from the fact that \( H_{2}(\phi^t_{H_{2}}(q_0, p_0)) = \alpha_{H_{2}}([\eta]) \); the second equality follows from the fact that \( \phi^t_{H_{2}}(q_0, p_0) \) lies in \( \bigcap_{u_- \in S^-_{H_{1}}(\eta)} \{ \eta + du_- \} \); the inequality follows from the fact that \( u_- \preceq L_{H_{2}} - \eta + \alpha_{H_{2}}([\eta]) \). Thus,

\[
 M_{H_{2,c}} \subseteq \mathcal{N}_{H_{2,c}}.
\]

Hence, we only need to show that \( \rho_{H_{2,c}}(q_0, q_0) = 0 \), for any \( q_0 \) lies in the \( \alpha \)-limit set and \( q_0 \) lies in the \( \omega \)-limit set of \( \pi \circ \phi^t_{H_{2}}(q_0, p_0) \). Clearly, both \( q_0 \) and \( q_0 \) lie in \( A_{H_{2,c}} \), since \( \pi \circ \phi^t_{H_{2}}(q_0, p_0) \) is \( c \)-minimizer with respect to \( L_{H_{2}} \). Thus, we can use the formula in Corollary 4.1 to calculate \( \rho_{H_{2,c}}(q_0, q_0) \):

\[
 \rho_{H_{2,c}}(q_0, q_0) = h_{H,q}(q_0, q_0) + h_{H,q}(q_0, q_0)
\]

\[
 = \sup \left\{ u_-(q_0) - u_-(q_0) : u_- \in S^-_{H_{1}}(\eta) \right\}
\]

\[
 + \sup \left\{ v_-(q_0) - v_-(q_0) : v_- \in S^-_{H_{1}}(\eta) \right\}
\]

\[
 = \sup \left\{ u_-(q_0) - u_-(q_0) : u_- \in S^-_{H_{1}}(\eta) \right\}
\]

\[
 + \sup \left\{ v_-(q_0) - v_-(q_0) : v_- \in S^-_{H_{1}}(\eta) \right\}
\]

\[
 = h_{H_{1,c}}(q_0, q_0) + h_{H_{1,c}}(q_0, q_0)
\]

\[
 = \rho_{H_{1,c}}(q_0, q_0),
\]
Lemma 4.3. the completeness. Moreover, the infimum is attained for each $H$

Proof. Then $u$ and, moreover, the supremum is attained for each $H$

Lemma 4.2. Assume that $\{H_1, H_2\} = 0$. Let $u_-, u_+ \in \mathcal{S}_H^-(\eta) = \mathcal{S}_{H_2}^- (\eta) \cup \mathcal{S}_H^+(\eta) = \mathcal{S}_{H_2}^+ (\eta)$. Then $u_-$ and $u_+$ are conjugate with respect to $H_1$ if and only if $u_-$ and $u_+$ are conjugate with respect to $H_2$, i.e., $u_- \sim_{H_1} u_+ \iff u_- \sim_{H_2} u_+$.

Proof. It is a direct consequence of Proposition 4.1 and the fact [Fa3] that

$$A_{H,c} = \cap \{ q : u_-(q) = u_+(q), \text{ here } u_- \in \mathcal{S}_H^-(\eta), u_+ \in \mathcal{S}_H^+(\eta), u_- \sim_H u_+ \}.$$

In [Fa1, Fa3], Fathi showed that

Lemma 4.2.

$$B_{H,c}(q) = \sup \left\{ u_-(q) - u_+(q) : u_- \in \mathcal{S}_H^-(\eta), u_+ \in \mathcal{S}_H^+(\eta), u_- \sim_H u_+ \right\},$$

and, moreover, the supremum is attained for each $q$.

Recall that

$$b_{H,c}(q) = \inf_{\xi, \zeta \in A_{H,c}} \left\{ h_{H,\eta}(\xi, q) + h_{H,\eta}(q, \zeta) - h_{H,\eta}(\xi, \zeta) \right\}.$$

In fact, the following lemma also appeared in [Fa1]:

Lemma 4.3.

$$b_{H,c}(q) = \inf \left\{ u_-(q) - u_+(q) : u_- \in \mathcal{S}_H^-(\eta), u_+ \in \mathcal{S}_H^+(\eta), u_- \sim_H u_+ \right\}.$$

Moreover, the infimum is attained for each $q$.

Since we can not find an explicit proof of this lemma in the literature, we will give a proof for the completeness.
Proof. For any conjugate pair \( u_- \in S^-_H(\eta), u_+ \in S^+_H(\eta), u_- \sim_H u_+ \), we have

\[
b_{H,c}(q) = \min_{\xi, \zeta} \left\{ h_{H,\eta}(\xi, q) + h_{H,\eta}(q, \zeta) - h_{H,\eta}(\xi, \zeta) : \xi, \zeta \in A_{H,c} \right\}
\]

\[
\leq \min_{\xi, \zeta} \left\{ h_{H,\eta}(\xi, q) + h_{H,\eta}(q, \zeta) - (u_-(\xi) - u_+(\xi)) : \xi, \zeta \in A_{H,c} \right\}
\]

\[
= \min_{\xi, \zeta} \left\{ (u_-(\xi) + h_{H,\eta}(\xi, q)) - (u_+(\xi) - h_{H,\eta}(q, \zeta)) : \xi, \zeta \in A_{H,c} \right\}
\]

\[
= \min_{\xi \in A_{H,c}} \left\{ u_-(\xi) + h_{H,\eta}(\xi, q) \right\} - \max_{\zeta \in A_{H,c}} \left\{ u_+(\zeta) - h_{H,\eta}(q, \zeta) \right\}
\]

\[
= u_-(q) - u_+(q),
\]

where the inequality follows from the fact \[\text{Fa3}\] that

\[
u_-(\xi) - u_+(\xi) \leq h_{H,\eta}(\xi, \zeta),
\]

the last equality follows from the constructions of weak KAM solutions \[\text{CIS}, \text{Co}, \text{Pa3}\]:

\[
u_-(q) = \min_{\xi \in A_{H,c}} \left\{ u_-(\xi) + h_{H,\eta}(\xi, q) \right\}
\]

and

\[
u_+(q) = \max_{\zeta \in A_{H,c}} \left\{ u_+(\zeta) - h_{H,\eta}(q, \zeta) \right\}.
\]

Thus, we have

\[
b_{H,c}(q) = \inf \left\{ u_-(q) - u_+(q) : u_- \in S^-_H(\eta), u_+ \in S^+_H(\eta), u_- \sim_H u_+ \right\}.
\]

Next, we will show that

\[
b_{H,c}(q) = \inf \left\{ u_-(q) - u_+(q) : u_- \in S^-_H(\eta), u_+ \in S^+_H(\eta), u_- \sim_H u_+ \right\}.
\]

Let \( \xi_0, \zeta_0 \in A_{H,c} \) such that

\[
b_{\xi_0}(q) = h_{H,\eta}(\xi_0, q) + h_{H,\eta}(q, \xi_0) - h_{H,\eta}(\xi_0, \zeta_0).
\]

Then there exists a conjugate pair \( u_- \in S^-_H(\eta), u_+ \in S^+_H(\eta), u_- \sim_H u_+ \), such that

\[
b_c(q) = h_{H,\eta}(\xi_0, q) + h_{H,\eta}(q, \xi_0) - h_{H,\eta}(\xi_0, \zeta_0) \\
= \left( h_{H,\eta}(\xi_0, q) + h_{H,\eta}(q, \zeta_0) \right) - \left( u_-(\xi_0) - u_+(\xi_0) \right) \\
\geq u_-(q) - u_+(q),
\]

where the second inequality follows from Lemma 4.1 (where the supremum is obtained); the third equality follows from the fact that \( \xi_0, \zeta_0 \in A_{H,c} \) and \( u_- = u_+ \) on \( A_{H,c} \) for each conjugate pair \( u_- \) and \( u_+ \); and the inequality follows from the constructions of weak KAM solutions.

Thus, Lemma 4.3 follows.

Now Theorem 3 follows from Theorem 1, Proposition 4.2, Lemma 4.2, and Lemma 4.3.

**Corollary 4.2.** \( b_{H,c} \) is semi-concave.

**Proof.** By Lemma 3.1, we have \( u_- - u_+ \) is \( K \)-semi-concave for each conjugate pair

\[
u_- \in S^-_H(\eta), u_+ \in S^+_H(\eta), u_- \sim_H u_+.
\]

It should be stressed that \( K \) is independent of the choice of conjugate pair. The corollary follows from the fact that the infimum of a family of semi-concave functions with the same semi-concave constant is also semi-concave.

**Corollary 4.3.** \( c \to b_{H,c} \) is lower-semi-continuous. As a consequence, \( c \to N_{H,c} \) is upper-semi-continuous, as a set-valued function \[\text{Mat2}\].

**Proof.** It is a direct consequence of Lemma 4.3 and stability of viscosity solutions \[\text{Pa3}\].
Corollary 4.4. \((\overline{A}_{H_1,c},\rho_{H_1,c})\) and \((\overline{A}_{H_2,c},\rho_{H_2,c})\) are isometric, for any \(c \in H^1(M,\mathbb{R})\).

5. Proof of Theorem 4

Since
\[
\hat{A}_{H,c} = \cap \{(q,p) : p = d_qu_-, u_+ \in S^-_H(\eta)\},
\]
and
\[
\hat{N}_{H,c} = \cup \{(q,p) : p = d_qu_-, u_+ \in S^-_H(\eta)\}.
\]

Theorem 4 follows from Theorem 2 and Proposition 4.2.

6. Proof of Theorem 5

Firstly, note that \(\{H_1 + H_2, H_1\} = \{H_1 + H_2, H_2\} = 0\), if \(\{H_1, H_2\} = 0\). So, by Theorem 3, we have
\[
\hat{A}_{H_1 + H_2,c} = \hat{A}_{H_1,c} = \hat{A}_{H_2,c}.
\]

Now choose any point \((q_0,p_0) \in \hat{A}_{H_1 + H_2,c} \approx \hat{A}_{H_1,c} = \hat{A}_{H_2,c}\), then
\[
\alpha_{H_1 + H_2}(c) = (H_1 + H_2)(q_0,p_0) = H_1(q_0,p_0) + H_2(q_0,p_0) = \alpha_{H_1}(c) + \alpha_{H_2}(c).
\]

By the quasi-linearity of \(\alpha\)-function, we have

Remark 6.1. In Theorem 1, if we let \(s = r\), it is easy to verify that
\[
T_{H_1,\eta,t}^{-}T_{H_2,\eta,t}^{-} u = T_{H_2,\eta,t}^{+}T_{H_1,\eta,t}^{-} u = T_{H_1 + H_2,\eta,t}^{-} u,
\]
and
\[
T_{H_1,\eta,t}^{+}T_{H_2,\eta,t}^{+} u = T_{H_2,\eta,t}^{+}T_{H_1,\eta,t}^{+} u = T_{H_1 + H_2,\eta,t}^{+} u
\]
for any \(u \in C^0(M,\mathbb{R})\), any smooth closed 1-form \(\eta\) on \(M\) and \(\mathbb{R} \ni t \geq 0\).

7. Proof of Theorem 6

Let \(\eta\) be a smooth closed 1-form on \(M\) with \(\lbrack \eta \rbrack = c\). Choose \(u \in S^-_{H_1}(\eta) = S^-_{H_2}(\eta)\), we will prove that \(T_{H_1,\eta,s}^{-}T_{H_2,\eta,r}^{-} u\) is a \(C^{1,1}\) subsolution of both Hamilton-Jacobi equations:
\[
H_1(q,\eta + du) = \alpha_{H_1}(c)
\]
and
\[
H_2(q,\eta + du) = \alpha_{H_2}(c),
\]
provided that \(r\) and \(s\) are sufficiently small.

Since \(u\) is semi-convex, so there exists \(\epsilon_0 > 0\) such that both \(T_{H_1,\eta,s}^{-} u\) and \(T_{H_2,\eta,r}^{-} u\) are semi-convex and semi-concave functions, hence both functions are \(C^{1,1}\), in the case that \(s, r < \epsilon_0\) [Be].

So, by the same discussion, there exists \(\epsilon_1 > 0\) such that when \(s, r < \epsilon_1\), we have \(T_{H_1,\eta,s}^{-}T_{H_2,\eta,r}^{-} u\) is a \(C^{1,1}\) function.

In the following, we will show that \(T_{H_1,\eta,s}^{-}T_{H_2,\eta,r}^{-} u\) is a subsolution of both Hamilton-Jacobi equations. Since \(\{H_1, H_2\} = 0\), by the above lemma, we have \(T_{H_1,\eta,s}^{-}T_{H_2,\eta,r}^{-} u = T_{H_2,\eta,r}^{-}T_{H_1,\eta,s}^{-} u\).

Clearly, \(T_{H_2,\eta,r}^{-} u\) is a subsolution of \(H_2(q,\eta + du) = \alpha_{H_2}(c)\), and \(T_{H_1,\eta,s}^{-} u\) is a subsolution of \(H_1(q,\eta + du) = \alpha_{H_1}(c)\).

Now we need another useful lemma is proved in [Fl.]:

Lemma 7.1. Given a Lipschitz function \(u : M \to \mathbb{R}\), the following properties are equivalent:

- \(u\) is a subsolution of \(H(q,\eta + du) = \alpha_{H}(c)\).
- The function \([0, +\infty) \ni t \mapsto T_{H,t,u}^{-}(q)\) is non-decreasing for each \(q \in M\).
- The function \([0, +\infty) \ni t \mapsto T_{H,t,u}^{-}(q)\) is non-increasing for each \(q \in M\).
Now we prove that \( T_{H_1, \eta, s}^{-} T_{H_2, \eta, r}^{-} u \) is a subsolution of
\[
H_1(q, \eta + du) = \alpha H_1(c).
\]
Clearly, we only need to show that \( T_{H_1, \eta, s}^{-} T_{H_2, \eta, r}^{-} u \) is a subsolution of
\[
H_1(q, \eta + du) = \alpha H_1(c).
\]
By Lemma 7.1, we just need to show that \([0, +\infty) \ni s \to T_{H_1, \eta, s}^{-} T_{H_2, \eta, r}^{-} u \) is non-decreasing for each \( q \in M \) and \( r > 0 \). Since \( T_{H_1, \eta, s}^{-} T_{H_2, \eta, r}^{-} u = T_{H_2, \eta, r}^{-} T_{H_1, \eta, s}^{-} u \), it follows from the following two facts:
1. \([0, +\infty) \ni s \to T_{H_1, \eta, s}^{-} u \) is non-decreasing, since \( u \) is a subsolution of \( H_1(q, \eta + du) = \alpha H_1(c) \);
2. \( T_{H_2, \eta, r}^{-} \) has the monotony property, i.e., for each \( u, v \in C^0(M, \mathbb{R}) \) and all \( r > 0 \), we have
\[
  u \leq v \Rightarrow T_{H_2, \eta, r}^{-} u \leq T_{H_2, \eta, r}^{-} v.
\]
Similarly, we have that \( T_{H_1, \eta, s}^{-} T_{H_2, \eta, r}^{-} u \) is also a subsolution of
\[
H_2(q, \eta + du) = \alpha H_2(c).
\]
Theorem 6 follows.

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