On cubics and quartics through a canonical curve

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Abstract

We construct families of quartic and cubic hypersurfaces through a canonical curve, which are parametrized by an open subset in a Grassmannian and a Flag variety respectively. Using G. Kempf’s cohomological obstruction theory, we show that these families cut out the canonical curve and that the quartics are birational (via a blowing-up of a linear subspace) to quadric bundles over the projective plane, whose Steinerian curve equals the canonical curve.

1 Introduction

Let \( C \) be a smooth nonhyperelliptic curve of genus \( g \geq 4 \), which we consider as an embedded curve \( \iota_\omega : C \hookrightarrow \mathbb{P}^{g-1} \) by its canonical linear series \(|\omega|\). Let \( I = \bigoplus_{n \geq 2} I(n) \) be the graded ideal of the canonical curve. It was classically known (Noether-Enriques-Petri theorem, see e.g. [ACGH] p. 124) that the ideal \( I \) is generated by its elements of degree 2, unless \( C \) is trigonal or a plane quintic.

It was also classically known how to construct some distinguished quadrics in \( I(2) \). We consider a double point of the theta divisor \( \Theta \subset \text{Pic}^{g-1}(C) \), which corresponds by Riemann’s singularity theorem to a degree \( g-1 \) line bundle \( L \) satisfying \( \dim |L| = \dim |\omega L^{-1}| = 1 \) and we observe that the morphism \( \iota_L \times \iota_{\omega L^{-1}} : C \longrightarrow C' \subset |L|^* \times |\omega L^{-1}|^* = \mathbb{P}^1 \times \mathbb{P}^1 \) (here \( C' \) denotes the image curve) followed by the Segre embedding into \( \mathbb{P}^3 \) factorizes through the canonical space \(|\omega|^*\), i.e.,

\[
\begin{align*}
C & \quad \hookrightarrow |\omega|^* \\
\downarrow & \quad \downarrow \pi \\
\mathbb{P}^1 \times \mathbb{P}^1 & \quad \hookrightarrow \mathbb{P}^3,
\end{align*}
\]

where \( \pi \) is projection from a \((g-5)\)-dimensional vertex \( \mathbb{P}V^\perp \) in \(|\omega|^*\). We then define the quadric \( Q_L := \pi^{-1}(\mathbb{P}^1 \times \mathbb{P}^1) \), which is a rank \( \leq 4 \) quadric in \( I(2) \) and coincides with the projectivized tangent cone at the double point \([L] \in \Theta\) under the identification of \( H^0(C, \omega)^* \) with the tangent space \( T_{[L]} \text{Pic}^{g-1}(C) \). The main result, due to M. Green [Gr], asserts that the set of quadrics \( \{Q_L\} \), when \( L \) varies over the double points of \( \Theta \), linearly spans \( I(2) \). From this result one infers a constructive Torelli theorem by intersecting all quadrics \( Q_L \)— at least for \( C \) general enough.

The geometry of the theta divisor \( \Theta \) at a double point \([L]\) can also be exploited to produce higher degree elements in the ideal \( I \) as follows: we expand in a suitable set of coordinates a local equation \( \theta \) of \( \Theta \) near \([L]\) as \( \theta = \theta_2 + \theta_3 + \ldots \), where \( \theta_i \) are homogeneous forms of degree \( i \). Having seen that \( Q_L = \text{Zeros}(\theta_2) \), we denote by \( S_L \) the cubic \( \text{Zeros}(\theta_3) \subset |\omega|^* \), the osculating cone of \( \Theta \) at \([L]\). The cubic \( S_L \) has many nice geometric properties: under the blowing-up of the vertex
\( \mathbb{P}^4 \subset S_L \), the cubic \( S_L \) is transformed into a quadric bundle \( \tilde{S}_L \) over \( \mathbb{P}^1 \times \mathbb{P}^1 \) and it was shown by G. Kempf and F.-O. Schreyer \([KS]\) that the Hessian and Steinerian curves of \( \tilde{S}_L \) are \( C' \subset \mathbb{P}^1 \times \mathbb{P}^1 \) and \( C \subset |\omega|^* \) respectively, which gives another proof of Torelli’s theorem.

In this paper we construct and study distinguished cubics and quartics in the ideal \( I \) by adapting the methods of \([KS]\) to rank-2 vector bundles over \( C \). Our construction basically goes as follows (section 2): we consider a general 3-plane \( W \subset H^0(C, \omega) \) and define the rank-2 vector bundle \( E_W \) as the dual of the kernel of the evaluation map in \( \omega \) of sections of \( W \). The bundle \( E_W \) is stable and admits a theta divisor \( \Theta_W \) in \( JC \). This surprising analogy with the Hessian and Steinerian curves are the plane curve \( \Gamma \), image under the projection with center \( \mathbb{P}^W \) as follows (section 2): we consider a general 3-plane \( W \subset H^0(C, \omega) \) and \( C \subset |\omega|^* \) respectively, which gives another proof of Torelli’s theorem.

Our main tool to study the tangent cones \( F_W \) is G. Kempf’s cohomological obstruction theory \([KL, K2, KS]\) which in our set-up leads to a simple criterion (Proposition 4.1) for \( b \in \mathbb{P} T_O JC = |\omega|^* \) to belong to \( F_W \). We deduce in particular from this criterion that the cubic polar \( P_x(F_W) \) of \( F_W \) with respect to a point \( x \in W^\perp \) also contains the canonical curve. Here \( W^\perp \) denotes the annihilator of \( W \subset H^0(\omega) \). We therefore obtain a rational map from the flag variety \( Fl(3, g - 1, H^0(\omega)) \) parametrizing pairs \((W, x)\) to the ideal of cubics \( |I(3)| \)

\[
F_3 : Fl(3, g - 1, H^0(\omega)) \rightarrow |I(3)|, \quad (W, x) \mapsto P_x(F_W).
\]  

(1) Like the cubic osculating cones \( S_L \), the quartic tangent cones \( F_W \) transform under the blowing-up of the vertex \( \mathbb{P}^W \subset F_W \) into a quadric bundle \( \tilde{F}_W \rightarrow \mathbb{P}^W = \mathbb{P}^2 \). Their Hessian and Steinerian curves are the plane curve \( \Gamma \), image under the projection with center \( \mathbb{P}^W \), \( \pi : C \rightarrow \Gamma \subset \mathbb{P}^W \), and the canonical curve \( C \subset |\omega|^* \) (Theorem 4.8). This surprising analogy with the osculating cones \( S_L \) remains however unexplained.

(2) Let us denote by \( |F_4| \subset |I(4)| \) and \( |F_3| \subset |I(3)| \) the linear subsystems spanned by the quartics \( F_W \) and the cubics \( P_x(F_W) \) respectively. Then we show (Theorem 6.1) that both base loci of \( |F_4| \) and \( |F_3| \) coincide with \( C \subset |\omega|^* \), i.e., the quartics \( F_W \) (resp. the cubics \( P_x(F_W) \)) cut out the canonical curve.

The starting point of our investigations was the question asked by B. van Geemen and G. van der Geer (\([vGvG]\) page 629) about “these mysterious quartics” which arise as tangent cones to \( 2\theta \)-divisors in the Jacobian having multiplicity \( \geq 4 \) at the origin. In that paper the authors implicitly conjectured that the base locus of \( |F_4| \) equals \( C \), which was subsequently proved by G. Welters \([W]\). Our proof follows from the fact that \( |F_4| \) contains all squares of quadrics in \( |I(2)| \).

This paper leaves many questions unanswered (section 7), like e.g., finding explicit equations of the quartics \( F_W \), their syzygies, the dimensions of \( |F_3| \) and \( |F_4| \). The techniques used here also apply when replacing \( |\omega|^* \) by Prym-canonical space \( |\omega\alpha|^* \), and generalizing rank-2 vector bundles to symplectic bundles.

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2 Some constructions for rank-2 vector bundles with canonical determinant

In this section we briefly recall some known results from [BV], [vGI] and [PP] on rank-2 vector bundles over $C$.

2.1 Bundles $E$ with $\dim H^0(C, E) \geq 3$

Let $W \subseteq H^0(C, \omega)$ be a 3-plane. We denote by $[W] \in \text{Gr}(3, H^0(\omega))$ the corresponding point in the Grassmannian and by $B \subseteq \text{Gr}(3, H^0(\omega))$ the codimension 2 subvariety consisting of $[W]$ such that the net $\mathbb{P}W \subset |\omega|$ has a base point. For $[W] \notin B$ we consider (see [vGI] section 4) the rank-2 vector bundle $E_W$ defined by the exact sequence

$$0 \to E_W^* \to \mathcal{O}_C \otimes W \xrightarrow{ev} \omega \to 0. \quad (2.1)$$

Here $E_W^*$ denotes the dual bundle of $E_W$. We have $E_W = \omega$ and $W^* \subseteq H^0(C, E_W)$. We denote by $D$ the effective divisor in $|\mathcal{O}_{Gr}(g-2)|$ defined by the condition

$$[W] \in D \iff \dim H^0(C, E_W) \geq 4.$$  

Moreover $B \subseteq D$ and $E_W$ is stable if $[W] \notin D$.

Let $W^0 \subseteq H^0(\omega)^* = H^1(\mathcal{O})$ denote the annihilator of $W \subseteq H^0(\omega)$. We call the projective subspace $\mathbb{P}W^0 \subset |\omega|^*$ the vertex and denote by

$$\pi : |\omega|^* \dashrightarrow \mathbb{P}W^0, \quad \pi : C \to \Gamma \subset \mathbb{P}W^0,$$

the projection with center $\mathbb{P}W^0$. If $[W] \notin B$, then $C \cap \mathbb{P}W^0 = \emptyset$ and $\pi$ restricts to a morphism $C \to \mathbb{P}W^0$. Its image is a plane curve $\Gamma$ of degree $2g - 2$. We note that $E_W = \pi^*(T(-1))$, where $T$ is the tangent bundle of $\mathbb{P}W^* = \mathbb{P}^2$.

Conversely any bundle $E$ with $\det E = \omega$ and $\dim H^0(C, E) \geq 3$ is of the form $E_W$.

2.2 Bundles $E$ with $\dim H^0(C, E) \geq 4$

Following [BV] (see also [PP] section 5.2) we associate to a bundle $E$ with $\dim H^0(C, E) = 4$ a rank $\leq 6$ quadric $Q_E \subseteq |I(2)|$, which is defined as the inverse image of the Klein quadric under the dual $\mu^*$ of the exterior product map

$$\mu^* : |\omega|^* \to \mathbb{P}(\Lambda^2 H^0(E)^*) \supset \text{Gr}(2, H^0(E)^*), \quad Q_E := (\mu^*)^{-1}(\text{Gr}).$$

Composing with the previous construction, we obtain a rational map

$$\alpha : D \dashrightarrow |I(2)|, \quad \alpha([W]) = Q_{E_W}.$$  

Moreover given a $Q \in |I(2)|$ with $\text{rk} Q \leq 6$ and $\text{Sing} Q \cap C = \emptyset$, it is easily shown that

$$\alpha^{-1}(Q) = \{[W] \in D \mid \mathbb{P}W^0 \subset Q\}.$$  

If $\text{rk} Q = 6$, then $\alpha^{-1}(Q)$ has two connected components, which are isomorphic to $\mathbb{P}^3$.

2.1 Lemma. We have $[W] \notin D$ if and only if the linear map induced by restricting quadrics to the vertex $\mathbb{P}W^0$

$$\text{res} : I(2) \to H^0(\mathbb{P}W^0, \mathcal{O}(2))$$

is an isomorphism.

Proof. It is enough to observe that the two spaces have the same dimension and that a nonzero element in $\ker \text{res}$ corresponds to a $Q \in |I(2)|$ with $\text{rk} Q \leq 6$. \qed
2.3 Definition of the quartic $F_W$

We will now define the main object of this paper. Given $[W] \notin \mathcal{B}$, we consider the $2\theta$-divisor $D(E_W) \subset JC$ (see e.g. [BV], [vGI], [PP]), whose set-theoretical support equals

$$D(E_W) = \{ \xi \in JC \mid \dim H^0(C, \xi \otimes E_W) > 0 \}. $$

Since $\text{mult}_\mathcal{O} D(E_W) \geq \dim H^0(C, E_W) \geq 3$ and since any $2\theta$-divisor is symmetric, the first nonzero term of the Taylor expansion of a local equation of $D(E_W)$ at the origin $\mathcal{O}$ is a homogeneous polynomial $F_W$ of degree 4. The hypersurface in $|\omega|^* = \mathbb{P}TCJC$ associated to $F_W$ is also denoted by $F_W$. Here we restrict attention to the case $\dim H^0(C, E_W) = 3$ or 4. We have

$$F_W := \text{Cone}_\mathcal{O}(D(E_W)) \subset |\omega|^*.$$ 

The study of the quartics $F_W$ for $[W] \in \text{Gr}(3, H^0(\omega)) \setminus \mathcal{D}$ is the main purpose of this paper. If $[W] \in \mathcal{D}$, the quartics $F_W$ have already been described in [PP] Proposition 5.12.

2.2 Proposition. If $\dim H^0(C, E_W) = 4$, then $F_W$ is a double quadric

$$F_W = Q^2_{E_W}. $$

Since $|I(2)|$ is linearly spanned by rank $\leq 6$ quadrics (see [PP] section 5), we obtain the following fact, which will be used in section 6.

2.3 Proposition. The linear subsystem $|F_4|$ contains all squares of quadrics in $|I(2)|$.

Although we will not use that fact, we mention that the rational map [11] is given by a linear subsystem $\mathbb{P} \Gamma \subset |\mathcal{J}_B(g-1)|$, where $\mathcal{J}_B$ is the ideal sheaf of the subvariety $\mathcal{B}$. If $g = 4$, the inclusion is an equality (see [OPP] section 6). If $g > 4$, a description of $\mathbb{P} \Gamma$ is not known.

3 Kempf’s cohomological obstruction theory

In this section we outline Kempf’s deformation theory [K1] and apply it to the study of the tangent cones $F_W$ of the divisors $D(E_W)$.

3.1 Variation of cohomology

Let $\mathcal{E}$ be a vector bundle over the product $C \times S$, where $S = \text{Spec}(A)$ is an affine neighbourhood of the origin of $JC$. We restrict attention to the case

$$\mathcal{E} = \pi_C^* E_W \otimes \mathcal{L},$$

for some 3-plane $W$, and recall that Kempf’s deformation theory was applied [K1], [K2], [KS] to the case $\mathcal{E} = \pi_C^* M \otimes \mathcal{L}$, for a line bundle $M$ over $C$. The line bundle $\mathcal{L}$ denotes the restriction of a Poincaré line bundle over $C \times JC$ to the neighbourhood $C \times S$. The fundamental idea to study the variation of cohomology, i.e. the two upper-semicontinuous functions on $S$

$$s \mapsto h^0(C \times \{s\}, \mathcal{E} \otimes_A \mathbb{C}_s), \quad s \mapsto h^1(C \times \{s\}, \mathcal{E} \otimes_A \mathbb{C}_s),$$

where $\mathbb{C}_s = A/\mathfrak{m}_s$ and $\mathfrak{m}_s$ is the maximal ideal of $s \in S$, is based on the existence of an approximating homomorphism.
3.1 Theorem (Grothendieck, [K1] section 7). Given a family $\mathcal{E}$ of vector bundles over $C \times S$, there exist two flat $A$-modules $F$ and $G$ of finite type and an $A$-homomorphism $\alpha : F \to G$ such that for all $A$-modules $M$, we have isomorphisms

$$H^0(C \times S, \mathcal{E} \otimes_A M) \cong \ker (\alpha \otimes_A id_M), \quad H^1(C \times S, \mathcal{E} \otimes_A M) \cong \coker (\alpha \otimes_A id_M).$$

By considering a smaller neighbourhood of the origin, we may assume the $A$-modules $F$ and $G$ to be locally free (Nakayama’s lemma). Moreover ([K1] Lemma 10.2) by restricting further the neighbourhood, we may find an approximating homomorphism $\alpha : F \to G$ such that $\alpha \otimes \mathbb{C}_0 : F \otimes_A A/m_0 \to G \otimes_A A/m_0$ is the zero homomorphism.

We apply this theorem to the family $\mathcal{E} = \pi_1^* E_W \otimes L$, for $[W] \notin D$. Since by Riemann-Roch $\chi(\mathcal{E} \otimes \mathbb{C}_s) = \chi(E_W \otimes L_s) = 0, \forall s \in S$, and since $h^0(C, E_W) = 3$, the local equation $f$ of the divisor $D(E_W)_S = \{ s \in S \mid h^0(C \times \{s\}, E_W \otimes L_s) > 0 \}$ is given at the origin $\mathcal{O}$ by the determinant of a $3 \times 3$ matrix of regular functions $f_{ij}$ on $S$, with $1 \leq i, j \leq 3$, which vanish at $\mathcal{O}$, i.e., the $A$-modules $F$ and $G$ are free and of rank 3. Hence

$$f = \det (f_{ij}).$$

The linear part of the regular functions $f_{ij}$ is related to the cup-product as follows ([K1] Lemma 10.3 and Lemma 10.6): let $m = m_0$ be the maximal ideal of the origin $\mathcal{O} \in S$ and consider the exact sequence of $A$-modules

$$0 \to m/m^2 \to A/m^2 \to A/m \to 0.$$

After tensoring with $\mathcal{E}$ over $C \times S$ and taking cohomology, we obtain a coboundary map

$$H^0(C, E_W) = H^0(C \times \{s\}, \mathcal{E} \otimes_A A/m) \xrightarrow{\delta} H^1(C \times \{s\}, \mathcal{E} \otimes_A m/m^2) = H^1(C, E_W) \otimes m/m^2,$$

where $m/m^2$ is the Zariski cotangent space at $\mathcal{O}$ to $JC$. Note that we have a canonical isomorphism $(m/m^2)^* \cong H^1(\mathcal{O})$ and that a tangent vector $b \in H^1(\mathcal{O})$ gives, by composing with the linear form $l_b : m/m^2 \to \mathbb{C}$, a linear map $\delta_b : H^0(E_W) \to H^1(E_W)$. As in the line bundle case [K1], one proves

3.2 Lemma. For any nonzero $b \in H^1(\mathcal{O}) = T_\mathcal{O} JC$, we have

1. The linear map $\delta_b : H^0(E_W) \to H^1(E_W)$ coincides with the cup-product ($\cup b$) with the class $b$, and is skew-symmetric after identifying $H^1(E_W)$ with $H^0(E_W)^*$ (Serre duality).

2. The coboundary map $\delta : H^0(E_W) \to H^1(E_W) \otimes m/m^2$ is described by a skew-symmetric $3 \times 3$ matrix $(x_{ij})$, with $x_{ij} \in H^1(\mathcal{O})^*$. Moreover the linear form $x_{ij}$ coincides with the differential $(df_{ij})_0$ of $f_{ij}$ at the origin $\mathcal{O}$.

The coboundary map $\delta$ induces a linear map

$$\Delta : H^1(\mathcal{O}) \to \Lambda^2 H^0(E_W)^*, \quad b \mapsto \delta_b,$$

which coincides with the dual of the multiplication map of global sections of $E_W$. Moreover

$$\ker \Delta = W^\perp = \{ x_{12} = x_{13} = x_{23} = 0 \}.$$

Using a flat structure [K2] we can write the power series expansion of the regular functions $f_{ij}$ around $\mathcal{O}$

$$f_{ij} = x_{ij} + q_{ij} + \cdots,$$
where $x_{ij}$ and $q_{ij}$ are linear and quadratic polynomials respectively. We easily calculate the expansion of $f$: by skew-symmetry its cubic term is zero, and its quartic term equals

$$F_W : q_{11}x_{23}^2 + q_{22}x_{13}^2 + q_{33}x_{12}^2 + x_{12}x_{23}(q_{13} + q_{31}) - x_{12}x_{23}(q_{12} + q_{21}) - x_{12}x_{13}(q_{23} + q_{32}).$$

We straightforwardly deduce from this equation the following properties of $F_W$.

**3.3 Proposition.** 1. The quartic $F_W$ is singular along the vertex $\mathbb{P}W^\perp$.

2. For any $x \in W^\perp$, the cubic polar $P_x(F_W)$ is singular along the vertex $\mathbb{P}W^\perp$.

### 3.2 Infinitesimal deformations of global sections of $E_W$

We first recall some elementary facts on principal parts. Let $V$ be an arbitrary vector bundle over $C$ and let $\text{Rat}(V)$ be the space of rational sections of $V$ and $p$ be a point of $C$. The space of principal parts of $V$ at $p$ is the quotient

$$\text{Prin}_p(V) = \text{Rat}(V)/\text{Rat}_p(V),$$

where $\text{Rat}_p(V)$ denotes the space of rational sections of $V$ which are regular at $p$. Since a rational section of $V$ has only finitely many poles, we have a natural mapping

$$\text{pp} : \text{Rat}(V) \to \text{Prin}(V) := \bigoplus_{p \in C} \text{Prin}_p(V), \quad s \mapsto (s \mod \text{Rat}_p(V))_{p \in C}. \quad (3.1)$$

Exactly as in the line bundle case ([K1] Lemma 3.3), one proves

**3.4 Lemma.** There are isomorphisms

$$\ker \text{pp} \cong H^0(C, V), \quad \text{coker} \text{pp} \cong H^1(C, V).$$

In the particular case $V = \mathcal{O}$, we see that a tangent vector $b \in H^1(\mathcal{O}) = T_CJC$ can be represented by a collection $\beta = (\beta_p)_{p \in I}$ of rational functions $\beta_p \in \text{Rat}(\mathcal{O})$, where $p$ varies over a finite set of points $I \subset C$. We then define $\text{pp}(\beta) = (\omega_p)_{p \in I} \in \text{Prin}(\mathcal{O})$, where $\omega_p$ is the principal part of $\beta_p$ at $p$. We denote by $[\beta] = b$ its cohomology class in $H^1(\mathcal{O})$. Note that we can define powers of $\beta$ by $\beta^k := (\beta^k_p)_{p \in I}$.

For $i \geq 1$, let $D_i$ be the infinitesimal scheme $\text{Spec}(A_i)$, where $A_i$ is the Artinian ring $\mathbb{C}[\epsilon]/\epsilon^{i+1}$. As explained in [K2] section 2, a tangent vector $b \in H^1(\mathcal{O})$ determines a morphism

$$\exp_{t,b} : D_t \to JC,$$

with $\exp_{t,b}(x_0) = \mathcal{O}$, where $x_0$ is the closed point of $D_t$. Let $\mathbb{L}_{i+1}(b)$ denote the pull-back of the Poincaré sheaf $\mathcal{L}$ under the morphism $\exp_{t,b} \times \text{id}_C$. Note that we have the following exact sequences

$$D_1 \times C : \quad 0 \to \epsilon \mathcal{O} \to \mathbb{L}_2(b) \to \mathcal{O} \to 0, \quad (3.2)$$

$$D_2 \times C : \quad 0 \to \epsilon^2 \mathcal{O} \to \mathbb{L}_3(b) \to \mathbb{L}_2(b) \to 0. \quad (3.3)$$

The second arrows in each sequence correspond to the restriction to the subschemes $\{x_0\} \times C \subset D_1 \times C$ and $D_1 \times C \subset D_2 \times C$ respectively. As above we choose a representative $\beta$ of $b$. Following
section 2, one shows that the space of global sections $H^0(C \times D_1, \mathbb{L}_{i+1}(b) \otimes E)$, with $E = E_W$ and $[W] \not\in D$, is isomorphic to the $A_i$-module

$$V_i(\beta) = \{f = f_0 + \cdots + f_i \epsilon^i \in \text{Rat}(E) \otimes A_i \text{ such that } f \exp(\epsilon \beta) \text{ is regular } \forall p \in C\}. \quad (3.4)$$

An element $f \in V_i(\beta)$ is called an $i$-th order deformation of the global section $f_0 \in H^0(E)$. In the case $i = 2$, the condition $f \in V_i(\beta)$ is equivalent to the following three elements,

$$f_0, \quad f_1 + f_0 \beta, \quad f_2 + f_1 \beta + f_0 \frac{\beta^2}{2}, \quad (3.5)$$

being regular at all points $p \in C$ — for $i = 1$, we consider the first two elements. Alternatively this means that their classes in $\text{Prin}(E)$ are zero. We note that, given two representatives $\beta = (\beta_p)_{p \in I}$ and $\beta' = (\beta'_p)_{p \in I}$ with $[\beta] = [\beta']$, the two subspaces $V_i(\beta)$ and $V_i(\beta')$ of $\text{Rat}(E) \otimes A_i$ are different and that any rational function $\varphi \in \text{Rat}(\mathcal{O})$ satisfying $\text{pp}(\varphi) = \text{pp}(\beta' - \beta)$ induces an isomorphism $V_i(\beta) \cong V_i(\beta')$.

We consider a class $b \in H^1(\mathcal{O}) \setminus W^1$ and a representative $\beta$ such that $[\beta] = b$. By taking cohomology of (3.2) tensored with $E$, we observe that a first order deformation of $f_0$, i.e., a global section $f = f_0 + f_1 \epsilon \in V_1(\beta) \cong H^0(C \times D_1, \mathbb{L}_2(b) \otimes E)$ always exists. Since $\text{rk}(\cup b) = 2$, the global section $f_0$ is uniquely determined up to a scalar

$$f_0 \cdot \mathcal{C} = \ker (\cup b : H^0(E) \longrightarrow H^1(E)).$$

Moreover any two first order deformations of $f_0$ differ by an element in $\epsilon H^0(E)$.

We now state a criterion for a tangent vector $b = [\beta]$ to lie on the quartic tangent cone $F_W$ in terms of a second order deformation of $f_0 \in H^0(E)$.

**3.5 Lemma.** A cohomology class $b = [\beta] \in H^1(\mathcal{O}) \setminus W^1$ is contained in the cone over the quartic $F_W$ if and only if there exists a global section

$$f = f_0 + f_1 \epsilon + f_2 \epsilon^2 \in V_2(\beta) \cong H^0(C \times D_2, \mathbb{L}_2(b) \otimes E).$$

**Proof.** The proof is similar to [KS] Lemma 4. We work over the Artinian ring $A_4$, i.e., $\epsilon^5 = 0$. By Theorem 3.1 applied to the family $\mathbb{L}_2(b) \otimes E$ over $C \times D_4$, there exists an approximating homomorphism of $A_4$-modules

$$A_4^{\otimes 3} \xrightarrow{\varphi} A_4^{\otimes 3}, \quad (3.6)$$

such that $\ker \varphi|_{D_2} \cong H^0(C \times D_2, \mathbb{L}_2(b) \otimes E)$, $\text{coker} \varphi|_{D_2} \cong H^1(C \times D_2, \mathbb{L}_2(b) \otimes E)$, and $\varphi \otimes \mathbb{C}_0 = 0$. We denote by $\varphi|_{D_2}$ the homomorphism obtained from (3.6) by projecting to $A_2$. Note that any $A_4$-module is free. The matrix $\varphi$ is equivalent to a matrix

$$M := \begin{pmatrix} \epsilon^u & 0 & 0 \\ 0 & \epsilon^v & 0 \\ 0 & 0 & \epsilon^w \end{pmatrix}.$$ 

Since $\varphi \otimes \mathbb{C}_0 = 0$, we have $u, v, w \geq 1$. Moreover we can order the exponents so that $1 \leq u \leq v \leq w$. It follows from the definition of $D(E_W)$ as a determinant divisor that the pull-back of $D(E_W)$ by $\exp_4 : D_4 \longrightarrow JC$ is given by the equation (in $A_4$)

$$\det M = \epsilon^{u+v+w}.$$
We immediately see that \( b \in F_W \) if and only if \( u + v + w \geq 5 \). Let us now restrict \( \varphi \) to \( D_1 \), i.e., we project \( (3.6) \) to \( A_1 \). Since we assume \( b \notin W^\perp = \ker \Delta \), the restriction \( \varphi_{|D_1} \) is nonzero and by skew-symmetry of rank 2, i.e., \( u = v = 1 \) and \( w \geq 2 \). Hence \( b \in F_W \) if and only if \( w \geq 3 \).

On the other hand the \( A_2 \)-module \( \ker \varphi_{|D_2} \cong H^0(C \times D_2, \mathbb{L}_q(b) \otimes E) \) has length \( 2 + w \). Let \( \mu \) be the multiplication by \( \epsilon^2 \) on this \( A_2 \)-module. Then by \( (3.4) \) the \( A_2 \)-module \( \ker \mu \) is isomorphic to the \( A_1 \)-module \( H^0(C \times D_1, \mathbb{L}_q(b) \otimes E) \), which is of length 4, provided \( b \notin W^\perp \). Hence we obtain that \( w \geq 3 \) if and only if there exists an \( f \in H^0(C \times D_2, \mathbb{L}_q(b) \otimes E) \) such that \( \mu(f) = \epsilon^2 f_0 \). This proves the lemma.

\[
\square
\]

## 4 Study of the quartic \( F_W \)

In this section we prove geometric properties of the quartic \( F_W \).

### 4.1 Criteria for \( b \in F_W \)

We now show that the criterion of Lemma \( 3.5 \) simplifies to a criterion involving only a first order deformation \( f = f_0 + f_1 \epsilon \in V_1(\beta) \) of \( f_0 \). As above we assume \( b \notin W^\perp \).

First we observe that the rational differential form \( f_1 \wedge f_0 \) is independent of the choice of the representative \( \beta \), i.e., \( f_1 \wedge f_0 \) only depends on the cohomology class \( b = [\beta] \): suppose we take \( \beta' = (\beta_p \cdot \varphi)_{p \in I} \), where \( \varphi \in \text{Rat}(\omega) \). Then \( f_0 \) and \( f_1 \) transform into \( f'_0 = f_0 \) and \( f'_1 = f_1 + \varphi f_0 \), from which it is clear that \( f'_1 \wedge f'_0 = f_1 \wedge f_0 \).

Secondly one easily sees that \( f_0 = \pi(b) \) (section 2.1) and that, under the canonical identification \( \Lambda^2 W^* = \Lambda^2 H^0(E) = W \), the \( 2 \)-plane \( H^0(E) \wedge f_0 \) coincides with the intersection \( V_b := H_b \cap W \), where \( H_b \) denotes the hyperplane determined by \( b \in H^1(\mathcal{O}) \).

It follows from these two remarks that, given \( b \) and \( W \), the form \( f_1 \wedge f_0 \) is well-defined up to a regular differential form in \( V_b \subset W \).

### 4.1 Proposition. We have the following equivalence

\[
b \in F_W \iff f_1 \wedge f_0 \in H_b.
\]

**Proof.** Since \( f_1 \wedge f_0 \) does not depend on \( \beta \), we may choose a \( \beta \) with simple poles at the points \( p \in I \). By Lemma \( 3.5 \) and relation \( (3.5) \) we see that \( b \in F_W \) if and only if the cohomology class \( [f_1 \beta + f_0 \beta^2] \) is zero in \( H^1(E)/\text{im} (\cup b) \) — we recall that \( f_1 \) is defined up to \( H^0(E) \).

First we will prove that \( [f_0 \beta^2] \in \text{im} (\cup b) \). The commutativity of the upper right triangle of the diagram (see e.g. [K1])

\[
\begin{array}{c}
H^0(E) \\
\downarrow \beta^2 \\
H^0(E(2I)) \\
\cap \\
\cap \\
\cap \\
\text{Rat}(E) \rightarrow \text{Prin}(E)
\end{array}
\]

\[
\begin{array}{c}\leftarrow H^0(E(2I)) \rightarrow E(2I)|_{2I} \rightarrow H^1(E)
\end{array}
\]
implies that \([f_0 \beta^2] = f_0 \cup [\beta^2]\). Moreover the skew-symmetric cup-product map \(\cup b\)

\[
\cup b = \wedge \overline{b} : H^0(E) = W^* \to H^1(E) = W = \Lambda^2 W^*
\]

identifies with the exterior product \(\wedge \overline{b}\), where \(\overline{b} = \pi(b) \in W^*\). It is clear that \(\text{im} (\cup b) = \text{im} (\wedge \overline{b}) = \ker (\wedge \overline{b})\), where \(\wedge \overline{b}\) also denotes the linear form

\[
\wedge \overline{b} : \Lambda^2 W^* \to \Lambda^3 W^* \cong \mathbb{C}.
\] (4.1)

As already observed, we have \(f_0 = \overline{b}\). Denoting by \(c \in W^*\) the class \(\pi([\beta^2])\), we see that the relation \((f_0 \wedge c) \wedge \overline{b} = \overline{b} \wedge c \wedge \overline{b} = 0\) implies that \(f_0 \cup [\beta^2] \in \ker (\wedge \overline{b}) = \text{im} (\cup b)\).

Therefore the previous condition simplifies to \([f_1 \beta] \in \text{im} (\cup b)\). We next observe that the linear form \(\wedge \overline{b}\) on \(H^1(E)\) identifies with the exterior product map \(H^1(E) \to H^1(\omega) \cong \mathbb{C}\).

Since we have a commutative diagram

\[
f_1 \in H^0(E(I)) \xrightarrow{\beta} \text{Prin}(E) \xrightarrow{} H^1(E)
\]

\[
\downarrow \wedge f_0 \quad \downarrow \wedge f_0
\]

\[
f_1 \wedge f_0 \in H^0(\omega) \xrightarrow{\beta} \text{Prin}(\omega) \xrightarrow{} H^1(\omega),
\]

and since \(f_1 \wedge f_0 \in H^0(\omega) \subset \text{Rat}(\omega)\), we easily see that the condition \([f_1 \beta] \in \text{im} (\cup b)\) is equivalent to \(f_1 \wedge f_0 \in H_0 = \ker (\cup b : H^0(\omega) \to H^1(\omega))\).

\[
\square
\]

In the following proposition we give more details on the element \(f_1 \wedge f_0 \in H^0(\omega)\). We additionally assume that \(\pi(b) \notin \Gamma\), which implies that the global section \(f_0 \in H^0(E)\) does not vanish at any point and hence determines an exact sequence

\[
0 \to \mathcal{O} \to f_0 \to E \wedge f_0 \to \omega \to 0.
\] (4.2)

The coboundary map of the associated long exact sequence

\[
\cdots \to H^0(\omega) \xrightarrow{\cup e} H^1(\mathcal{O}) \to \cdots
\] (4.3)

is symmetric and coincides (e.g. [K1] Corollary 6.8) with cup-product \(\cup e\) with the extension class \(e \in \mathbb{P} H^1(\omega^{-1}) = [\omega^2]^*\). Moreover \(\cup e\) is the image of \(e\) under the dual of the multiplication map

\[
H^1(\omega^{-1}) = H^0(\omega^2)^* \hookrightarrow \text{Sym}^2 H^0(\omega)^*, \quad e \mapsto \cup e.
\] (4.4)

We note that \(\text{corank}(\cup e) = 2\) and that \(\ker (\cup e) = V_b\). Hence \((f_1 \wedge f_0) \cup e\) is well-defined.

**4.2 Proposition.** If \(\pi(b) \notin \Gamma\), then \(f_1 \wedge f_0 \notin \ker (\cup e)\) and we have (up to a nonzero scalar)

\[
(f_1 \wedge f_0) \cup e = b \in H^1(\mathcal{O}).
\]
Proof. We keep the notation of the previous proof. The condition \( f_1 \wedge f_0 \in V_b \) implies that \( f_1 \) is a regular section and, by \([3.5]\), that \( f_0 \) vanishes at the support of \( b \), i.e., \( \pi(b) \in \Gamma \). As for the equality of the proposition, we introduce the rank-2 vector bundle \( \hat{E} \) which is obtained from \( E \) by (positive) elementary transformations at the points \( p \in I \) and with respect to the line in \( E_p \) spanned by the nonzero vector \( f_0(p) \). Then we have \( E \subset \hat{E} \subset E(I) \) and \( \hat{E} \) fits into the exact sequence

\[
0 \rightarrow E \rightarrow \hat{E} \rightarrow \mathcal{O}_I \rightarrow 0.
\]

Moreover \( f_1 \in H^0(\hat{E}) \), which follows from condition \([3.5]\). We also have the following exact sequences

\[
0 \rightarrow \mathcal{O}(I) \rightarrow \hat{E} \xrightarrow{\wedge f_0} \omega \rightarrow 0 \quad (\hat{e})
\]

\[
0 \rightarrow \mathcal{O} \xrightarrow{f_0} E \xrightarrow{\wedge f_0} \omega \rightarrow 0 \quad (e),
\]

and the extension class \( \hat{e} \in H^1(\omega^{-1}(D)) \) is obtained from \( e \) by the canonical projection \( H^1(\omega^{-1}) \rightarrow H^1(\omega^{-1}(I)) \). Taking the associated long exact sequences, we obtain

\[
f_1 \in H^0(\hat{E}) \xrightarrow{\wedge f_0} H^0(\omega) \xrightarrow{\wedge \hat{e}} H^1(\mathcal{O}(I))
\]

\[
H^0(E) \xrightarrow{\wedge f_0} H^0(\omega) \xrightarrow{\wedge e} H^1(\mathcal{O}),
\]

where the two squares commute. This means that

\[
\pi_I ((f_1 \wedge f_0) \cup e) = (f_1 \wedge f_0) \cup \hat{e} = 0.
\]

Since \( f_1 \wedge f_0 \) does not depend on \( \beta \) (nor on \( I \)), the latter relation holds for any \( I \) with \( I = \text{supp} \beta \). Hence, denoting by \( \langle I \rangle \) the linear span in \( |\omega|^* \) of the support \( I \) of \( \beta \), we obtain

\[
(f_1 \wedge f_0) \cup e \in \bigcap_{I=\text{supp} \beta} \ker \pi_I = \bigcap_{b \in \langle I \rangle} \langle I \rangle = b.
\]

\( \square \)

4.2 Geometric properties of \( F_W \)

4.3 Proposition. For any \([W] \notin \mathcal{D} \) we have the following

1. The quartic \( F_W \) contains the canonical curve \( C \), i.e., \( F_W \in |I(4)| \).

2. The quartic \( F_W \) contains the secant line \( \overline{pq} \), with \( p \neq q \), if and only if \( \overline{pq} \cap \mathbb{P}W^\perp \neq \emptyset \) or \( \dim W \cap H^0(\omega(-2p - 2q)) > 0 \).

3. Let \( \Sigma \) be the set of points \( p \) at which the tangent line \( T_p(C) \) intersects the vertex \( \mathbb{P}W^\perp \). Then \( \Sigma \) is empty for general \([W] \) and finite for any \([W] \). Moreover any point \( p \in C \setminus \Sigma \) is smooth on \( F_W \) and the embedded tangent space \( T_p(F_W) \) is the linear span of \( T_p(C) \) and \( \mathbb{P}W^\perp \).

Proof. All statements are easily deduced from Proposition \([4.1]\). Given a point \( p \in C \) we denote by \( p_p \in \text{Prin}_p(\mathcal{O}) \) the principal part supported at \( p \) of a rational function with a simple pole at \( p \). Then the class \([p_p] \in H^1(\mathcal{O})\) is proportional to \( i_\omega(p) = \mathbb{P}H^1(\mathcal{O}) \) and the section \( f_0 \) vanishes
at $p$. Hence $f_0 p_\pi \in \text{Prin}(E)$ is everywhere regular and we may choose $f_1 = 0$. This proves part 1. See also [PP].

As for part 2, we introduce $\beta_{\lambda,\mu} = \lambda p_\pi + \mu q_\pi \in \text{Prin}(O)$ for $\lambda, \mu \in \mathbb{C}$ and denote by $s_{p_\pi}$ and $s_{q_\pi}$ the global sections $\pi([p_\pi])$ and $\pi([q_\pi])$, which vanish at $p$ and $q$ respectively. Then one checks that $f_0 = \lambda s_{p_\pi} + \mu s_{q_\pi} \in \ker (\text{pr}[\beta_{\lambda,\mu}])$ and $\text{pr}(f_1) = \lambda \mu (s_{q_\pi} p_\pi + s_{p_\pi} q_\pi) \in \text{Prin}(E)$. With this notation the condition of Proposition 4.1 transforms into

$$0 = \lambda_\mu (f_0 \wedge f_1) = \lambda \mu (\lambda_\mu (f_0 \wedge f_1)) = \lambda \mu (\lambda^2 \gamma_p + \mu^2 \gamma_q), \quad \text{(4.5)}$$

where $\lambda_\mu$ is the linear form defined by $[\beta_{\lambda,\mu}] \in H^1(O)$. The scalars $\gamma_p$ and $\gamma_q$ are the values of the section $s_{p_\pi} \wedge s_{q_\pi} \in W \cap H^0(\omega(-p - q))$ at $p$ and $q$ respectively. We now conclude noting that $s_{p_\pi} \wedge s_{q_\pi} = 0$ if and only if $\overline{pq} \cap \mathbb{P} W_\perp \neq \emptyset$.

As for part 3, we first observe that the assumption $\Sigma = C$ implies that the restriction $\pi|_C : C \to \mathbb{P} W^*$ contracts $C$ to a point, which is impossible. Next we consider the tangent vector $t_q$ at $p$ given by the direction $q$. By putting $\lambda = 1$ and $\mu = \epsilon$, with $\epsilon^2 = 0$, into equation (4.5) we obtain that $t_q \in T_p(F_W)$ if and only if $\epsilon \gamma_p = 0$, i.e., $\pi(q) \in T_{\pi(p)}(\Gamma)$. Hence $T_p(F_W) = \pi^{-1}(T_{\pi(p)}(\Gamma))$, which proves part 3.

\begin{proof}
\end{proof}

4.3 The cubic polar $P_x(F_W)$

Firstly we deduce from Propositions 4.1 and 4.2 a criterion for $b \in P_x(F_W)$, with $x \in W_\perp$. Let $H_x$ be the hyperplane determined by $x \in H^1(O)$. As above we assume $b \notin W_\perp$ and $\pi(b) \notin \Gamma$, i.e., the pencil $V = V_b$ is base-point-free.

4.4 Proposition. We have the following equivalence

$$b \in P_x(F_W) \iff f_1 \wedge f_0 \in H_x.$$

\begin{proof}
We recall from section 4.1 that $\cup e$ induces a symmetric isomorphism $\cup e : (V_\perp)^* \cong V_\perp$ and we denote by $Q^* \subset \mathbb{P}(V_\perp)^*$ and $Q \subset \mathbb{P} V_\perp$ the two associated smooth quadrics. Note that $Q$ and $Q^*$ are dual to each other. Combining Propositions 4.1, 4.2 and 3.3 (1) we see that the restriction of the quartic $F_W$ to the linear subspace $\mathbb{P} V_\perp \subset |\omega|^*$ splits into a sum of divisors

$$(F_W)|_{\mathbb{P} V_\perp} = 2\mathbb{P} W_\perp + Q.$$ 

We also observe that $Q$ only depends on $V$ (and on $W$) and not on $b$. Taking the polar with respect to $x \in W_\perp$, we obtain

$$(P_x(F_W))|_{\mathbb{P} V_\perp} = 2\mathbb{P} W_\perp + P_x(Q).$$

Finally we see that the condition $b \in P_x(Q)$ is equivalent to $f_0 \wedge f_1 = (\cup e)^{-1}(b) \in H_x$. \qed

We easily deduce from this criterion some properties of $P_x(F_W)$.

4.5 Proposition. The cubic $P_x(F_W)$ contains the canonical curve $C$, i.e., $P_x(F_W) \in |I(3)|$.

\begin{proof}
We first observe that the two closed conditions of Proposition 4.4 are equivalent outside $\pi^{-1}(\Gamma)$. Hence they coincide as well on $\pi^{-1}(\Gamma)$ and we can drop the assumption $\pi(b) \notin \Gamma$. Now, as in the proof of Proposition 4.3 (1), we may choose $f_1 = 0$. \qed

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4.6 Proposition. We have the following properties

\[ \bigcap_{x \in W^\perp} P_x(F_W) = S_W \cup \mathbb{P}W^\perp \cup \bigcup_{n \geq 2} \Lambda_n, \]

\[ F_W \cap S_W = C \cup \Lambda_1, \quad \text{and} \quad \bigcup_{n \geq 0} \Lambda_n \subset F_W, \]

where \( S_W \) is an irreducible surface. For \( n \geq 0 \), we denote by \( \Lambda_n \) the union of \((n + 1)\)-secant \( \mathbb{P}^n \)'s to the canonical curve \( C \), which intersect the vertex \( \mathbb{P}W^\perp \) along a \( \mathbb{P}^{n-1} \). If \( W \) is general, then \( \Lambda_n = \emptyset \) for \( n \geq 2 \) and \( \Lambda_1 \) is the union of \( 2(g - 1)(g - 3) \) secant lines.

Proof. We consider \( b \) in the intersection of all \( P_x(F_W) \) and we first suppose that \( \pi(b) \notin \Gamma \). Then by Propositions 4.5 and 3.3(2) we have

\[ f_0 \land f_1 \in \bigcap_{x \in W^\perp} H_x = W. \]

Hence we obtain that \( \mathbb{P}W^\perp \cap \bigcap_{x \in W^\perp} P_x(F_W) \) is reduced to the point \((\cup e)(W) \in \mathbb{P}V^\perp\). On the other hand a standard computation shows that \( S_W \) is the image of \( \mathbb{P}^2 \) under the linear system of the adjoint curves of \( \Gamma \). Hence \( S_W \) is irreducible.

If \( \pi(b) \in \Gamma \), we denote by \( p_1, \ldots, p_{n+1} \in C \) the points such that \( \pi(p_i) = \pi(b) \). Then \( f_0 \) vanishes at \( p_1, \ldots, p_{n+1} \). Since \( f_1 \land f_0 \) does not depend on the support of \( b \), we can choose \( \text{supp } b \) such that \( p_i \notin \text{supp } b \). Then \( f_1 \) is regular at \( p_i \) and we deduce that \( f_1 \land f_0 \in H^0(\omega(-\sum p_i)) \cap W = V_b \). Now any rational \( f_1 \) satisfying \( f_1 \land f_0 \in V_b \) is \( \text{im } (\land f_0) \) is regular everywhere, which can only happen when \( f_0 \) vanishes at the support of \( b \). By uniqueness we have \( \text{supp } b \subset \{ p_1, \ldots, p_{n+1} \} \) and \( b \in \Lambda_n \). Note that \( \Lambda_0 = C \). This proves the first equality.

If \( b \in F_W \cap S_W \), we have \( f_1 \land f_0 \in W \cap H_b = V_b \) and we conclude as above. Note that \( \Lambda_1 \) is contained in \( S_W \) and is mapped by \( \pi \) to the set of ordinary double points of \( \Gamma \).

For any \([W] \in \text{Gr}(3, H^0(\omega)) \setminus \mathcal{D}\) we introduce the subspace of \( I(3) \)

\[ L_W = \{ R \in I(3) \mid R \text{ is singular along the vertex } \mathbb{P}W^\perp \}. \]

Then Propositions 4.5 and 3.3(2) imply that \( P_x(F_W) \in L_W \). More precisely, we have

4.7 Proposition. The restriction of the polar map of the quartic \( F_W \) to its vertex \( \mathbb{P}W^\perp \)

\[ \mathbf{P} : \ W^\perp \to L_W, \quad x \mapsto P_x(F_W), \]

is an isomorphism.

Proof. First we show that \( \dim L_W = g - 3 \). We choose a complementary subspace \( A \) to \( W^\perp \), i.e. \( H^0(\omega)^\ast = W^\perp \oplus A \), and a set of coordinates \( x_1, \ldots, x_{g-3} \) on \( W^\perp \) and \( a_1, a_2, a_3 \) on \( A \). This enables us to expand a cubic \( F \in S^3H^0(\omega) \)

\[ F = F_3(x) + F_2(x)G_1(a) + F_1(x)G_2(a) + G_3(a), \quad F_i \in \mathbb{C}[x_1, \ldots, x_{g-3}], \quad G_i \in \mathbb{C}[a_1, a_2, a_3], \]

with \( \deg F_i = \deg G_i = i \). Let \( \mathcal{S}_A \) denote the subspace of cubics singular along \( \mathbb{P}A \), i.e. \( G_2 = G_3 = 0 \). We consider the linear map

\[ \alpha : I(3) \to \mathcal{S}_A, \quad F \mapsto F_3(x) + F_2(x)G_1(a). \]
Since by Lemma 2.1 any monomial \(x_i x_j \in H^0(\mathbb{P} W^\perp, \mathcal{O}(2))\) lifts to a quadric \(Q_{ij} \in I(2)\), we observe that the monomials \(x_i x_j x_k\) and \(x_i x_j a_t\), which generate \(S_A\), also lift e.g. to \(Q_{ij} x_k\) and \(Q_{ij} a_t\) in \(I(3)\). Hence \(\alpha\) is surjective and \(\text{dim } L_w = \text{dim ker } \alpha\) is easily calculated. One also checks that this computation does not depend on \(A\).

In order to conclude, it will be enough to show that \(P\) is injective. Suppose that the contrary holds, i.e., there exists a point \(x \in W^\perp\) with \(P_x(F_W) = 0\). Given any base-point-free pencil \(V \subset W\) and any \(b \in V^\perp\), we obtain by Proposition 4.4 that \(f_0 \wedge f_1 \in H_x\). Since \(\cup e : (V^\perp)^* \xrightarrow{\sim} V^\perp\) is an isomorphism, we see that for \(b \notin (\cup e)^{-1}(H_x)\) the element \(f_0 \wedge f_1\) must be zero. This implies that \(b \in \Lambda\) and since \(b\) varies in an open subset of \(|\omega|^*\), we obtain \(\Lambda = |\omega|^*\), a contradiction. \(\square\)

4.4 The quadric bundle associated to \(F_W\)

Let \(\mathbb{P}^{g-1}_W \to |\omega|^*\) denote the blowing-up of \(|\omega|^*\) along the vertex \(\mathbb{P} W^\perp \subset |\omega|^*\). The rational projection \(\pi : |\omega|^* \dashrightarrow \mathbb{P}^2 = \mathbb{P} W^*\) resolves into a morphism \(\tilde{\pi} : \mathbb{P}^{g-1}_W \to \mathbb{P}^2\). Since \(F_W\) is singular along \(\mathbb{P} W^\perp\) (Proposition 3.4 (2)), the proper transform \(\tilde{F}_W \subset \mathbb{P}^{g-1}_W\) admits a structure of a quadric bundle \(\tilde{\pi} : \tilde{F}_W \to \mathbb{P}^2\).

The contents of Propositions 4.3 and 4.5 can be reformulated in a more geometrical way.

4.8 Theorem. For any \([W] \in \text{Gr}(3, H^0(\omega)) \setminus D\), the quadric bundle \(\tilde{\pi} : \tilde{F}_W \to \mathbb{P}^2\) has the following properties

1. Its Hessian curve is \(\Gamma \subset \mathbb{P}^2\).
2. Its Steinerian curve is the (proper transform of the) canonical curve \(C \subset |\omega|^*\).
3. The rational Steinerian map \(\text{St} : \Gamma \dashrightarrow C\), which associates to a singular quadric its singular point, coincides with the adjoint map \(\text{ad}\) of the plane curve \(\Gamma\). Moreover the closure of the image \(\text{ad}(\mathbb{P}^2)\) equals \(S_W\).

4.9 Remark. We note that Theorem 4.8 is analogous to the main result of [KS] (replace \(\mathbb{P}^2\) with \(\mathbb{P}^1 \times \mathbb{P}^1\)). In spite of this striking similarity and the relation between the two parameter spaces \(\text{Sing} \Theta\) and \(\text{Gr}(3, H^0(\omega))\) (see [PP]), we were unable to find a common frame for both constructions.

5 The cubic hypersurface \(\Psi_V \subset \mathbb{P}^{g-3}\) associated to a base-point-free pencil \(\mathbb{P} V \subset |\omega|\)

In this section we show that the symmetric cup-product maps \(\cup e \in \text{Sym}^2 H^0(\omega)^*\) (see (4.3)) arise as polar quadrics of a cubic hypersurface \(\Psi_V\), which will be used in the proof of Theorem 6.1.

Let \(V\) denote a base-point-free pencil of \(H^0(\omega)\). We consider the exact sequence given by evaluation of sections of \(V\)

\[
0 \longrightarrow \omega^{-1} \longrightarrow \mathcal{O}_C \otimes V \xrightarrow{ev} \omega \longrightarrow 0. \tag{5.1}
\]

Its extension class \(v \in \text{Ext}^1(\omega, \omega^{-1}) \cong H^1(\omega^{-2}) \cong H^0(\omega^3)^*\) corresponds to the hyperplane in \(H^0(\omega^3)\), which is the image of the multiplication map

\[
\text{im } (V \otimes H^0(\omega^2) \longrightarrow H^0(\omega^3)). \tag{5.2}
\]
We consider the cubic form $\Psi_V$ defined by

$$
\Psi_V : \text{Sym}^3 H^0(\omega) \xrightarrow{\mu} H^0(\omega^3) \xrightarrow{\bar{v}} \C,
$$

where $\mu$ is the multiplication map and $\bar{v}$ the linear form defined by the extension class $v$. It follows from the description (5.2) that $\Psi_V$ factorizes through the quotient

$$
\Psi_V : \text{Sym}^3 \V \longrightarrow \C,
$$

where $\V := H^0(\omega)/V$. We also denote by $\Psi_V \subset \P V$ its associated cubic hypersurface.

A 3-plane $W \supset V$ determines a nonzero vector $w$ in the quotient $\V = H^0(\omega)/V$ and a general $w$ determines an extension (1.2) — recall that $W^* \cong H^0(E)$. Hence we obtain an injective linear map $\V \hookrightarrow H^1(\omega^{-1})$, $w \mapsto e$, which we compose with (4.4)

$$
\Phi : \V \hookrightarrow H^1(\omega^{-1}) = H^0(\omega^2)^* \hookrightarrow \text{Sym}^2 H^0(\omega)^*, \quad w \mapsto e \mapsto \cup_e.
$$

Since $V \subset \ker(\cup_e)$, we note that $\text{im } \Phi \subset \text{Sym}^2 \V^*$.

We now can state the main result of this section.

5.1 Proposition. The linear map $\Phi : \V \rightarrow \text{Sym}^2 \V^*$ coincides with the polar map of the cubic form $\Psi_V$, i.e.,

$$
\forall w \in \V, \quad \Phi(w) = P_w(\Psi_V).
$$

Proof. This is straightforwardly read from the diagram obtained by relating the exact sequences (5.1) and (2.1) via the inclusion $V \subset W$. We leave the details to the reader. \qed

We also observe that, by definition of the Hessian hypersurface (see e.g. [DK] section 3), we have an equality among degree $g - 2$ hypersurfaces of $\P V = \P^{g-3}$

$$
\text{Hess}(\Psi_V) = \mathcal{D} \cap \P \V,
$$

where we use the inclusion $\P \V \subset \text{Gr}(3, H^0(\omega))$.  

5.2 Remark. We recall (see [DK] (5.2.1)) that the Hessian and Steinerian of a cubic hypersurface coincide and that the Steinerian map is a rational involution $i$. In the case of the cubic $\Psi_V$, the involution

$$
i : \text{Hess}(\Psi_V) \dashrightarrow \text{Hess}(\Psi_V)
$$

corresponds to the involution of [BV] Propositions 1.18 and 1.19, i.e., $\forall w \in \mathcal{D} \cap \P \V$, the bundles $E_w$ and $E_{i(w)}$ are related by the exact sequence

$$
0 \longrightarrow E_{i(w)}^* \longrightarrow \mathcal{O}_C \otimes H^0(E_w) \xrightarrow{ev} E_w \longrightarrow 0.
$$

Since we will not use that result, we leave its proof to the reader.

6 Base loci of $|F_3|$ and $|F_4|$

Let us denote by $|F_3| \subset |I(3)|$ and $|F_4| \subset |I(4)|$ the linear subsystems spanned by the image of the rational maps $F_3$ and $F_4$ respectively. Then we have the following

6.1 Theorem. The base loci of $|F_3|$ and $|F_4|$ coincide with the canonical curve $C \subset |\omega|^*$. 

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Proof. Let $b \in Bs|F_3|$ and let us suppose that $b \notin C$. We consider a base-point-free pencil $V \subset H_b$. With the notation of section 5, we introduce the rational map

$$r_b : \mathbb{P} V \dashrightarrow \mathbb{P} V, \quad w \mapsto r_b(w) = w', \quad \text{with } \tilde{\Psi}_V(w, w', \cdot) = b,$$

where $\tilde{\Psi}_V$ is the symmetric trilinear form of $\Psi_V$. We note (Proposition 4.2) that, for $w \in P(V)$, the element $r_b(w)$ is collinear with the nonzero element $f_0 \wedge f_1 \mod V$ and that $r_b$ is defined away from the hypersurface $\text{Hess}(\Psi_V)$, which we assume to be nonzero. Since $b \in Bs|F_3|$ we obtain by Proposition 4.4 that

$$r_b(w) = \left( \bigcap_{x \in W} H_x \right) \mod V = W \mod V = w.$$

Hence $r_b$ is the identity map (away from $\text{Hess}(\Psi_V)$). This implies that $\tilde{\Psi}_V(w, w, \cdot) = b$ for any $w \in P(V)$, hence $\Psi_V = x_0^3$, where $x_0$ is the equation of the hyperplane $\mathbb{P}(H_b/V)$. This in turn implies that $\text{Hess}(\Psi_V) = 0$, i.e., $\mathbb{P}V \subset D$. Since for a general $[W] \in \text{Gr}(3, H^0(\omega))$ the pencil $V = W \cap H_b$ is base-point-free, we obtain that a general $[W]$ lies on the divisor $D$, which is a contradiction.

As for $|F_4|$, we recall that the fact $Bs|F_4| = C$ follows from [We]. Alternatively, it can also be deduced by noticing (see Proposition 2.3) that $Bs|F_4| \subset Bs|I(2)|$. Hence, if $C$ is not trigonal nor a plane quintic, we are done. In the other cases, the result can be deduced from Proposition 4.3 — we leave the details to the reader.

7 Open questions

7.1 Dimensions

The projective dimensions of the linear systems $|F_3|$ and $|F_4|$ are not known for general $g$. The known values of $\dim |F_4|$ for a general curve $C$ are given as follows (see [PP]).

| $g$  | 4   | 5   | 6   | 7   |
|------|-----|-----|-----|-----|
| $\dim |F_4|$ | 4   | 15  | 40  | 88  |

The examples of [PP] section 6 show that $\dim |F_4|$ depends on the gonality of $C$. Moreover it can be shown that $|F_4| \neq |I(4)|$.

7.2 Prym-canonical spaces and symplectic bundles

The construction of the quartic hypersurfaces $F_W$ admit various analogues and generalizations, which we briefly outline.

(1) Let $P_\alpha := \text{Prym}(C_\alpha/C)$ denote the Prym variety of the étale double cover $C_\alpha \to C$ associated to the nonzero 2-torsion point $\alpha \in JC$. Given a general 3-plane $Z \subset H^0(C, \omega\alpha)$, we associate the rank-2 vector bundle $E_Z$ defined by

$$0 \to E_Z^* \to \mathcal{O}_C \otimes Z \xrightarrow{ev} \omega\alpha \to 0.$$

By [IP] Proposition 4.1 we can associate to $E_Z$ the divisor $\Delta(E_Z) \in |2\Xi|$, where $\Xi$ is a symmetric principal polarization on $P_\alpha$. Its projectivized tangent cone at the origin $0 \in P_\alpha$ is a quartic hypersurface $F_Z$ in the Prym-canonical space $\mathbb{P}T_0P_\alpha \cong |\omega\alpha|^*$. Kempf’s obstruction theory equally applies to the quartics $F_Z$. We note that $F_Z$ contains the Prym-canonical curve $i_{\omega\alpha}(C) \subset |\omega\alpha|^*$. 

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Let $W$ be a vector space of dimension $2n + 1$, for $n \geq 1$. We consider a general linear map
\[ \Phi : \Lambda^2 W^* \longrightarrow H^0(C, \omega). \]
By taking the $n$-th symmetric power $\text{Sym}^n \Phi$ and using the canonical maps $\text{Sym}^n(\Lambda^2 W^*) \to \Lambda^{2n} W^* \cong W$ and $\text{Sym}^n H^0(\omega) \to H^0(\omega^\otimes n)$, we obtain a linear map
\[ \alpha : W \longrightarrow H^0(\omega^\otimes n), \]
which we assume to be injective. We then define the rank $2n$ vector bundle $E_\Phi$ by
\[ 0 \longrightarrow E^*_\Phi \longrightarrow \mathcal{O}_C \otimes W \xrightarrow{\text{ev}} \omega^\otimes n \longrightarrow 0. \]
The bundle $E_\Phi$ carries an $\omega$-valued symplectic form and the projectivized tangent cone at $O \in JC$ to the divisor $D(E_\Phi)$ is a hypersurface $F_\Phi$ in $|\omega|^*$ of degree $2n + 2$. Moreover $F_\Phi \in |I(2n + 2)|$.

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