The Conjugate Post Correspondence Problem

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Abstract

We introduce a modification to the Post Correspondence Problem where (in the formulation using morphisms) we require the images to be conjugate words. This problem is then shown to be undecidable by reducing it to the word problem for a special type of semi-Thue systems.

Keywords: Conjugate words, Post Correspondence Problem, Decidability

1 Introduction

Two words $x$ and $y$ are conjugates if they can be written in the form $x = uv$ and $y = vu$ for some words $u$ and $v$. The problem of deciding whether there exist two conjugate words with the same images under a pair of morphisms is known as the circular Post Correspondence Problem, or CPCP for short. More formally CPCP asks given morphisms $h$ and $g$, whether there exist words $u$ and $v$ such that $h(uv) = g(vu)$ when $uv \neq \varepsilon$. The CPCP is known to be undecidable as was shown in [4], see [2] for a somewhat simpler proof. Here we give a new variant of the problem by requiring that the images are conjugate words with the same pre-image. We call this problem the conjugate-PCP and give it the following formal definition:

**Problem 1.** Given two morphisms $h, g : A^* \rightarrow B^*$, decide whether or not there exists a word $w \in A^+$ such that $h(w) = uv$ and $g(w) = vu$ for some words $u, v \in B^*$.

The behaviour of the instances of conjugate-PCP differ vastly from the more traditional variants of the PCP where a valid presolution (prefix of a possible
solution) can be verified by a matching of the images. Working out a possible solution to a conjugate-PCP instance is much less intuitive.

For example let us have some morphisms $h, g$ and guess that a solution $w$ begins with the letter $a$. Then the situation is the following:

$$h(w) = h(a) \cdot \cdot \cdot g(a) \cdot \cdot \cdot$$

$$g(w) = g(a) \cdot \cdot \cdot h(a) \cdot \cdot \cdot$$

The validity of the presolution $a$ cannot be verified because there is no matching that needs to happen between $h(a)$ and $g(a)$. Moreover the factorization of the images to $u$ and $v$ need not be unique even for minimal solutions:

**Example.** Let $h, g : \{a, b\}^* \rightarrow \{a, b\}^*$ be morphisms defined by

$$h(a) = aba, \quad g(a) = bab,$$

$$h(b) = b, \quad g(b) = a.$$ 

Now $ab$ is a minimal solution for the conjugate-PCP instance $(h, g)$ having two factorizations: $u = a, v = bab$ or $u = aba, v = b$.

In the next section we will prove that the conjugate-PCP is indeed also undecidable by reducing it to a word problem for a special type of semi-Thue systems.

## 2 The Proof of Undecidability

We recall the construction of the semi-Thue system $T_M$ in [3]. The construction used the structure of a given Turing machine $M$. We can simplify our presentation by acknowledging the existence of such a system and declaring that our new system has the same properties.

Let $T = (\Sigma, R)$ be our semi-Thue system with the following properties:

1. $\Sigma = A \cup \overline{A} \cup B \cup \overline{B}$ with pairwise disjoint alphabets $A, \overline{A}, B, \overline{B}$. Notably $A = \{a, b, L, R\}$ where $L, R$ are markers for the left and right border of the word, respectively.
2. \( T \) is \((B \cup \overline{B})\)-deterministic in the following way:

\[ R \subseteq (A^*BA^* \times A^*BA^*) \cup ( \overline{A^*BA^*} \times \overline{A^*BA^*}) \cup (A^*BA^* \times \overline{A^*BA^*}) \cup ( \overline{A^*BA^*} \times A^*BA^*) \]

(i) If there is a rule \( t \) in \( R \) where all symbols are non-overlined, then the corresponding overlined rule \( \overline{t} \), where all symbols are overlined, is also in \( \overline{R} \), and vice versa.

(ii) For all words \( w \in (A \cup \overline{A})(B \cup \overline{B})(A \cup \overline{A})^* \), if there is a rule in \( R \) giving \( w \rightarrow_T w' \) then the rule is unique.

(iv) There is a single rule from \( A^*BA^* \times \overline{A^*BA^*} \) and a single rule from \( \overline{A^*BA^*} \times A^*BA^* \), moreover these rules are such that they re-write everything between the \( L \) and \( R \) markers, namely if there are rules giving \( u \rightarrow_T \overline{w_0} \) and \( \overline{w} \rightarrow_T w_0 \) for a \( u \in A^*BA^* \) then the rules are \((u, \overline{w_0})\) and \((\overline{u}, w_0)\), respectively.

3. \( T \) has an undecidable circular word problem. In particular it is undecidable whether \( T \) has a circular derivation \( w_0 \rightarrow_T^* w_0 \) where \( w_0 \in A^*BA^* \) is the word appearing in rules of 2(iv). Note that \( w_0 \) and \( u \) in case 2(iv) are fixed words from the construction of the semi-Thue system \( T_M \) for a TM \( M \), and \( w_0 \neq u \).

The special \((B \cup \overline{B})\)-determinism of \( T \) can be interpreted as derivations being in two different phases: the normal phase and the overlined phase. Transitioning between phases is via the unique rules from 2/(iv). It is straightforward to see that all derivations do not go through phase changes and that the phase is changed more than once if and only if \( T \) has a circular derivation. The system considered is now clear from the context and we write the derivations omitting the index \( T \) simply as \( \rightarrow \).

We now add a few additional rules to \( T \): we remove the unique rule \((u, \overline{w_0})\) and replace it with one extra step by introducing rules \((u, s)\) and \((s, \overline{w_0})\) where \( s \) is a new symbol for the intermediate step. The corresponding overlined rules \((\overline{u}, \overline{s})\) and \((\overline{s}, w_0)\) are added also to replace the rule \((\overline{u}, w_0)\). These new rules are needed in identifying the border between words \( u \) and \( v \), and adding them has no effect on the behaviour of \( T \).

By theorem ?? we have the following lemma.

**Lemma 1.** Assume that the semi-Thue system \( T \) is constructed as above. Then \( T \) has an undecidable individual circular word problem for the word \( w_0 \).
We now reduce the individual circular word problem of the system $T$ to the conjugate-PCP.

Let $R = \{t_0, t_1, \ldots, t_{h-1}, t_h\}$, where the rules are pairs $t_i = (u_i, v_i)$. We denote by $l_x$ and $r_x$ the left and right desynchronizing morphisms defined by

$$l_x(a) = xa, \quad r_x(a) = ax$$

for all words $x$. In the following we consider the elements of $R$ as letters. Denote by $A_j$ the alphabet $A$ where letters are given subscripts $j = 1, 2$. Define morphisms $h, g : (A_1 \cup A_2 \cup \overline{A_1} \cup \overline{A_2} \cup \{\#, \#', I\} \cup R)^* \rightarrow \{a, b, d, e, f, \#, \$, £\}^*$ according to the following table:

|   | $h$                                                                 | $g$                                                                 |
|---|----------------------------------------------------------------------|----------------------------------------------------------------------|
| $I$          | $s\overline{d}_2(w_0\#)d$                                           | $\$ee,                                                             |
| $x_1$        | $ddx$                                                               | $xee, \quad x \in \{a, b\}$                                       |
| $x_2$        | $ddx$                                                               | $xee, \quad x \in \{a, b\}$                                       |
| $t_i$        | $d^{-1}\overline{d}_2(v_i)$                                        | $r_{e2}(u_i), \quad t_i \notin \{t_{h-1}, t_h\}$                   |
| $t_{h-1}$    | $dsff$                                                              | $r_{e2}(u\#)$                                                      |
| $t_h$        | $f\$£$l_{e2}(w_0\#)ee$                                             | $sffe£dd$                                                         |
| $\#$         | $dd\#d$                                                             | $\#ee$                                                            |
| $\overline{\pi}_1$ | $xee$                                                           | $xxd, \quad \pi \in \{\overline{\pi}, \overline{\pi}'\}$          |
| $\overline{\pi}_2$ | $exe$                                                           | $xxd, \quad \pi \in \{\overline{\pi}, \overline{\pi}'\}$          |
| $\overline{t}_i$ | $e^{-2}\overline{d}_2(v_i)e$                                      | $r_{d2}(u_i), \quad \overline{t}_i \notin \{\overline{t}_{h-1}, \overline{t}_h\}$ |
| $\overline{t}_{h-1}$ | $sf$                                                              | $r_{d2}(u\#)$                                                      |
| $\overline{t}_h$ | $ff\$£                                                             | $sffe\$                                                           |
| $\#$         | $e\#ee$                                                             | $\#dd$                                                            |

Here the re-writing rules are of the form $t_i = (u_i, v_i)$, for some $u_i, v_i$. The following rules play important roles:

$t_{h-1} = (u, s)$, where $u$ is the unique word such that $(u, w_0) \in R$, and $t_h = (s, w_0)$.

We begin by examining the forms of the images of $h$ and $g$. The morphisms are a modification of the ones in [3] with slight alterations made such that it is possible to have (finite) solutions to the conjugate-PCP instance with easily identifiable borders between the factors $u$ and $v$ using special symbols $\$ and £. The symbols $d, e$ and $f$ function as desynchronizing symbols. The desynchronizing symbols $d$ and $e$ make sure that in the solution $w$ the factors that
will represent the configurations of the semi-Thue system $T$ are of the correct form, that is of the form where the determinism is kept intact. This follows from the forms of $h$ and $g$: under $g$ all images are desynchronized by either $e^2$ (non-overlined letters) or $d^2$ (overlined letters). To get similarly desynchronized factors in the image under $h$ we note that in the pre-image the words between two $#-$symbols (similarly for overlined symbols $\overline{\cdot}$) are of the form $\alpha t\beta$ where $\alpha \in \{a_1, b_1\}$, $\beta \in \{a_2, b_2\}$ and $t \in R$ (with end markers $L$ and $R$ omitted from $\alpha$ and $\beta$). The symbol $f$ is not really used in desynchronizing but making sure that the change between phases is carried out correctly.

The following lemma is useful in our proof:

**Lemma 2.** Words $h(w)$ and $g(w)$ are conjugates if and only if $h(w_1)$ and $g(w_2)$ are conjugates for all conjugates $w_1$ and $w_2$ of $w$.

*Proof.* If $h(w_1)$ and $g(w_2)$ are conjugates for all conjugates $w_1$ and $w_2$ of $w$ then $h(w)$ and $g(w)$ are conjugates.

Assume then that $h(w)$ and $g(w)$ are conjugates and let $w_1$ and $w_2$ be conjugates of $w$. We may now write $w_1 = xwx^{-1}$ and $w_2 = ywy^{-1}$. Denote $w' = w^{-1}$ and $w'' = w''$. Now $h(w_1) = h(xw') = h(x)h(w')$ is a conjugate of $h(w')h(x) = h(w')$ and $g(w_2) = g(yw') = g(y)g(w')$ is a conjugate of $g(w'y)g(y) = g(w'y) = g(w)$. By our assumption also $h(w_1)$ and $g(w_2)$ are conjugates. \hfill $\square$

Next we will show that a circular derivation beginning from word $w_0$ exists in $T$ if and only if there is a solution to the conjugate-PCP instance $(h, g)$. We will prove the claim in the following two lemmata.

**Lemma 3.** If there is a circular derivation in $T$ beginning from $w_0$, then there exists a non-empty word $w$ such that $h(w) = uv$ and $g(w) = vu$ for some words $u$ and $v$.

*Proof.* Assume that a circular derivation exists. The derivation is of the form $w_0 = \alpha_1 \beta_1 \rightarrow \alpha_1 \beta_1 \alpha_2 \beta_2 \rightarrow \cdots \rightarrow u \rightarrow s \rightarrow w' = \alpha_1 \beta_1 \cdots w \rightarrow \overline{w} \rightarrow w_0$. This derivation can be coded into a word

$$w = Iw_1 \#w_2 \#w_3 \# \cdots \#t_h^{-1}t_h \overline{w_1} \#w_2 \#w_3 \# \cdots \#t_{h-1}t_h,$$

where $w_i = \alpha_i \beta_i$ for each $i$, where we recall that $t_i = (u_i, v_i)$ is the unique rewriting rule used in each derivation step. The rules $t_{h-1}$ and $t_h$ appear right before the transition to overlined derivation as they correspond to the final and
intermediate steps before the transition. Let us consider the images of $w$ under the morphisms $h$ and $g$:

$$h(w) = l_{d^2}(w_0\#\alpha_1v_1\beta_1\#\alpha_2v_2\beta_2\#\cdots\#s)$$

and

$$g(w) = r_{e^2}(\ell\alpha_1u_1\beta_1\#\alpha_2u_2\beta_2\#\cdots\#u\#s)$$

These images are indeed very similar. The images match at all positions that do not contain a desynchronizing symbol ($d$ or $e$) or a special symbol ($\ell$ or $\ell$). Thus, if we erased all of these non-matching symbols we would have images that are equal (and of the form $q^2$ for a word $q$). Also the non-matching symbols are such that $d$ is always matched with $e$ and $\ell$ is always matched with $\ell$. It is quite clear that the factors in both $h(w)$ and $g(w)$ beginning and ending in the same special symbol are the same, that is, the factors $u = \ell\cdots\ell$ and $v = \ell\cdots\ell$ appearing in both images are equal. It follows that $h(w) = uv$ and $g(w) = vu$, which proves our claim.

**Lemma 4.** If there exists a non-empty word $w$ such that $h(w) = uv$ and $g(w) = vu$ for some words $u$ and $v$, then there is a circular derivation in $T$ beginning from $w_0$.

**Proof.** Firstly we show that the factor $f^3$ must appear in $h(w)$ and hence $t_{h^{-1}}t_h$ or $t_{h^{-1}}l_h$ is a factor in $w$. Assume the contrary: there is no factor $f^3$ in $h(w)$.

From the construction of $g$ we know that also $h(w)$ is desynchronized so that between each letter there is either a factor $d^2$ or $e^2$. Conjugation of $g(w)$ does not break this property except possibly for the beginning and the end of $h(w)$ ($h(w)$ could start and end in a single desynchronizing symbol).

Take now the first letter $c$ of $w$. We can assume that it is a non-overlined letter as the considerations are similar for the overlined case. The letter $c$ cannot be $t_{h^{-1}}$ as it would have to be followed by $t_h$: $f^2$ does not appear as a factor under $g$ without $f^3$, and $t_{h^{-1}}t_h$ produces $f^4$, which is uncoverable by $g$. From the construction of $h$ we see that the letters following $c$ must also be non-overlined, otherwise the desynchronization would be broken. Thus the desynchronizing symbol is the same for all these following letters. But as we can see from the form of the morphisms $h$ and $g$, we have a different desynchronizing symbols under $g$ for $c$ and its successors. It is clear that $h(g)$ must contain both $d$ and $e$ and so $w$ must have both non-overlined and overlined letters. If there is a change in the desynchronizing symbol in $h(w)$ then it contradicts the form.
of the images under $g$. Hence we must have the factor $t_{h-1}t_h$ in $w$ to make the transition without breaking the desynchronization.

The images of the factor $t_{h-1}t_h$ are as follows:

$$h(t_{h-1}t_h) = dsf ff\$\ll_2(w_0\#)ee$$

and

$$g(t_{h-1}t_h) = r_{e^2}(u\#)sfff\$\ll d.$$  

As we can see the desynchronizing symbols do not match. Hence we also must have the overlined copy of this factor in $w$, that is the factor $\overline{t_{h-1}t_h}$, the images of which are as follows (the letter $I$ is a forced continuation to the overlined factor to account for the special symbols $\$\$ and $\ll$):

$$h(\overline{t_{h-1}t_h}) = sff f\$\ll_2(w_0\#)d$$

and

$$g(\overline{t_{h-1}t_h}) = r_{e^2}(u\#)sfff\$\ll e.$$  

One of either of these factors has one swap between symbol $d$ and $e$. From the above we concluded that we need an even number of these swaps as for every factor $t_{h-1}t_h$ we must also have the factor $\overline{t_{h-1}t_h}$ and vice versa. It is possible that $h(w)$ ends in the letter $f$. In this case the swap happens "from the end to the beginning", i.e., the prefix of a factor doing the swap is at the end of $w$ and the remaining suffix is at the beginning of $w$. The following proposition shows that we can in fact restrict ourselves to the case where the factors $t_{h-1}t_h$ and $\overline{t_{h-1}t_h}$ are intact, that is, the swap does not happen from the end to the beginning of $h(w)$ as a result of the conjugation between $h(w)$ and $g(w)$.

**Proposition.** It may be assumed that the first and last symbols of $h(w)$ are $\$\$ and $\ll$.

**Proof of proposition.** If $h(w)$ is not of the desired form then it has $\ll\$ as a factor (by above the symbols from $\overline{t_{h-1}t_h}$ are in $w$). Images of the letters under $h$ do not have $\$\$ as a factor so there is a factorization $w = w_1w_2$ such that $h(w_1)$ ends in $\ll$ and $h(w_2)$ begins with $\$\$ ($w_1$ ends in $\overline{t_h}$ and $w_2$ begins with $I$). By Lemma 2 $h(w)$ and $g(w)$ are conjugates if and only if $h(w_2w_1)$ and $g(w_2w_1)$ are, where now $h(w_2w_1)$ has $\$\$ as the first symbol and $\ll$ as the last symbol.
Now by the proposition we may assume that \( w \) begins with \( I \) and ends with \( \overline{t_h} \). From this it also follows that when \( h(w) = uv \) and \( g(w) = vu \) the word \( u \) has \( \$ \) as the first and the last symbol and \( v \) has \( \mathcal{L} \) as the first and the last symbol. It follows that \( w = I \cdots t_h \cdots \overline{t_h} \), where the border between \( u \) and \( v \) is in the image \( h(t_h) \):

\[
\begin{align*}
  h(w) &= \underbrace{\$l_{d2}(w_0\#)}_u d \cdots f \underbrace{\$l_{e2}(w_0\#)ee}_v \cdots f f \mathcal{L} \\
  g(w) &= \underbrace{L_{ee}}_v \cdots \underbrace{sf ff \mathcal{L}dd}_u \cdots \underbrace{sf ff \$}_u
\end{align*}
\]

Here the border between \( u \) and \( v \) need not be in the image of the same instance of \( t_h \). Nevertheless we know by above that in the image under \( g \) the word \( u \) begins with \( \$ \) and \( v \) has \( \mathcal{L} \) as the first and the last symbol. To get this image as a factor of \( g(w) \) we must have \( t_h \alpha_1 t_1 \beta_1 \# \) in \( w \), where \( t_1 = (u_1, v_1) \) is the first rewriting rule used and \( w_0 = \alpha_1 u_1 \beta_1 \). Now

\[
h(t_h \alpha_1 t_1 \beta_1 \#) = f \underbrace{\$l_{d2}(w_0\#)}_u d \underbrace{\alpha_1 v_1 \beta_1 \# ee}_v
\]

which shows that

\[
I \alpha_1 t_1 \beta_1 \# \alpha_2 t_2 \beta_2 \# \in w \tag{1}
\]

where by the \((B \cup \overline{B})\)-determinism of \( T \) rule \( t_2 \in \mathcal{R} \) is the unique rule and \( \alpha_1, \alpha_2 \in L\{a_1, b_1\}^* \cup \{\varepsilon\} \) and \( \beta_1, \beta_2 \in \{a_2, b_2\}^* R \cup \{\varepsilon\} \) are unique words such that \( g(\alpha_2 t_2 \beta_2) = r_{e2}(\alpha_2 v_2 \beta_2) = r_{e2}(\alpha_1 v_1 \beta_1) \). Again,

\[
h(I \alpha_1 t_1 \beta_1 \# \alpha_2 t_2 \beta_2 \#) = \underbrace{\$l_{d2}(w_0\#)}_u d \underbrace{\alpha_1 v_1 \beta_1 \# \alpha_2 v_2 \beta_2 \#}_v
\]

which as also a factor of \( g(w) \) gives that

\[
t_h \alpha_1 t_1 \beta_1 \# \alpha_2 t_2 \beta_2 \# \alpha_3 t_3 \beta_3 \# \in w \tag{2}
\]

for a unique \( t_3 \in \mathcal{R} \) and \( \alpha_3 \in L\{a_1, b_1\}^* \cup \{\varepsilon\} \), \( \beta_3 \in \{a_2, b_2\}^* R \cup \{\varepsilon\} \).

It is easy to see that the words given by this procedure beginning with \( I \) or \( t_h \) (as in \([1]\) and \([2]\) respectively) contain derivations of the system \( T \) starting from \( w_0 \) where configurations are represented as words between \#-symbols and consecutive configurations in these words are also consecutive in \( T \) (as is explained in the beginning of the proof), that is, we get from the former to the latter by a single derivation step.
From the finiteness of $w$ it follows that long enough factors of $w$ with the forms as in 1 and 2 represent cyclic computations: the configuration/symbol $s$ is reached eventually and from there we have the rule $(s, w_0)$ which starts a new cycle. We conclude that $T$ must have a cyclic computation starting from configuration $w_0$.

\[ \square \]

Lemmas 1, 3 and 4 together yield our main theorem:

**Theorem 1.** The conjugate-PCP is undecidable.

This result does not generalize using this same construction by say, adding more desynchronizing symbols and border markers for each element in the permutation. The generalization of the conjugate-PCP would be the following problem:

**Problem** (Image Permutation Post Correspondence Problem). Given two morphisms $h, g : A^* \to B^*$, does there exist a word $w \in A^+$ and an $n$-permutation $\sigma$ such that $h(w) = u_1u_2 \cdots u_n$ and $g(w) = u_{\sigma(1)}u_{\sigma(2)} \cdots u_{\sigma(n)}$ for some words $u_1,\ldots,u_n \in B^*$?

The reason the construction does not work for the general $n$-permutations is that allowing more factors to be permuted can make solutions that do not describe TM computations. This is because of special cases for different values of $n$ and $\sigma$, but also by the fact that the permuted factors may be single letters. In fact any solution $w$ that produces Abelian equivalent words $h(w)$ and $g(w)$ also has a permutation that makes one of the words into the other. A "simple" proof using the techniques in this chapter is for now deemed unlikely, and some other approach may prove to be more fruitful.

As a related result we note that PCP instances where one of the morphisms is a permutation of the other are undecidable. Indeed, it was shown by Halava and Harju in [1] that the PCP is undecidable for instances $(h, h\pi)$, where $h : A^* \to B^*$ is a morphism and $\pi : A^* \to A^*$ is a permutation.

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