STRENGTHENING KAZHDAN’S PROPERTY (T) BY BOCHNER METHODS

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Abstract. In this paper, we propose a property which is a natural generalization of Kazhdan’s property (T) and prove that many, but not all, groups with property (T) also have this property.

Let $\Gamma$ be a finitely generated group. One definition of $\Gamma$ having property (T) is that $H^1(\Gamma, \pi, \mathcal{H}) = 0$ where the coefficient module $\mathcal{H}$ is a Hilbert space and $\pi$ is a unitary representation of $\Gamma$ on $\mathcal{H}$. Here we allow more general coefficients and say that $\Gamma$ has property $F \otimes H$ if $H^1(\Gamma, \pi_1 \otimes \pi_2, F \otimes \mathcal{H}) = 0$ if $(F, \pi_1)$ is any representation with $\dim(F) < \infty$ and $(\mathcal{H}, \pi_2)$ is a unitary representation.

The main result of this paper is that a uniform lattice in a semisimple Lie group has property $F \otimes H$ if and only if it has property (T). The proof hinges on an extension of a Bochner-type formula due to Matsushima-Murakami and Raghunathan. We give a new and more transparent derivation of this formula as the difference of two classical Weitzenböck formula’s for two different structures on the same bundle. Our Bochner-type formula is also used in our work on harmonic maps into continuum products [10, 12]. Some further applications of property $F \otimes H$ in the context of group actions will be given in [11].

1. Introduction and Statements of Results

Property (T), introduced by Kazhdan in 1966 in [19], plays a fundamental role in the study of discrete subgroups of Lie groups and more general finitely generated groups. In this paper we propose a stronger property, which we call property $F \otimes H$, which is a direct strengthening of one equivalent definition of property (T). It follows from work of Delorme and Guichardet that property (T) is equivalent to the statement that $H^1(\Gamma, \pi, \mathcal{H}) = 0$ whenever $\mathcal{H}$ is a Hilbert space and $\pi$ is a continuous unitary representation. This is easily seen to be equivalent to the statement that any continuous affine isometric $\Gamma$ action on a Hilbert space has a fixed point, commonly called property $FH$.

Definition 1.1. Let $D$ be a topological group, we say $D$ has property $F \otimes H$ if for every

1. $F$ a finite dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$,
2. continuous homomorphism $\pi : D \to GL(F)$,

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Hilbert space $\mathcal{H}$, and
continuous unitary representation $\pi_2 : D \to \mathcal{U}(\mathcal{H})$,
the first continuous cohomology group $H^1(\Gamma, \pi_1 \otimes \pi_2, F \otimes \mathcal{H}) = 0$.

We refer to representations $\pi_1 \otimes \pi_2$ as in the definition above as finite dimensional by unitary representations. Throughout this paper, cohomology for topological groups will be continuous cohomology. It is easy to show that property $F \otimes H$ is equivalent to having any continuous affine action of $D$ with finite dimensional by unitary linear part has a fixed point. The notation for property $F \otimes H$ evokes both the fixed point property $FH$ generalized here and the tensor product in the finite dimensional by unitary representations that appear in the definition.

Throughout this paper a semisimple Lie group will be a connected semisimple Lie group with finite center. We will not consider the case of semisimple Lie groups which are not algebraic, see the end of §2 for discussion of related examples. We will also always assume that $G$ is simply connected as an algebraic group, i.e. that every cover of $G$ which is algebraic is isomorphic to $G$. With this assumption, $G$ is the direct product of its simple factors [22, Proposition I.1.4.10]. As it is obvious that property $F \otimes H$ passes to quotients, there is no loss of generality in this assumption. The main result of this paper is the following:

**Theorem 1.2.** Let $D$ be a semisimple Lie group or a cocompact lattice in such. Then $D$ has property $F \otimes H$ if and only if it has property $(T)$.

**On Theorem 1.2:**

1. For $D$ a connected semisimple Lie group and $\pi_2$ irreducible, or more generally admissible, the desired vanishing follows from known results on relative Lie algebra cohomology. It does not appear possible to deduce the general case from this, mainly because those results do not seem to yield estimates on the size of solutions to cohomological equations, see [5, 1].
2. For a general locally compact group $D$ and a cocompact lattice $\Lambda < D$, it follows from work of Blanc that if $D$ has $F \otimes H$ then $\Lambda$ has $F \otimes H$ [1]. The converse (shown to hold for property $(T)$ by Kazhdan) is not immediate.
3. We prove Theorem 1.2 for cocompact lattices and prove it for the ambient group from that. The proof does not proceed by deducing $F \otimes H$ from $(T)$, but rather uses the fact that $G$ and $\Gamma$ have $(T)$ as long as $G$ has no factors locally isomorphic to $SO(1, n)$ or $SU(1, n)$.
4. It is not true in general that property $F \otimes H$ is equivalent to property $(T)$. The easiest example is $SL(n, \mathbb{Z}) \rtimes \mathbb{Z}^n$ for $n > 2$, see [2] below.
5. $F \otimes H$ for cocompact lattices was announced in [12, Theorem 1.8].
6. The issue of whether non-uniform lattices have $F \otimes H$ seems quite subtle and will be addressed elsewhere.

As mentioned above, the main step is proving Theorem 1.2 for cocompact lattices. We deduce $F \otimes H$ for $G$ from $F \otimes H$ for irreducible cocompact lattices in $G \times G$. 
Our proof also depends on a use of the Margulis-Corlette-Gromov-Schoen superrigidity theorems, which classify all finite dimensional representations of the lattices we consider [4, 15, 21, 22].

For cocompact lattices we translate the question into one concerning the de Rham cohomology of the associated locally symmetric space with coefficients in a flat $F \otimes \mathcal{H}$ bundle. We then use a Bochner-Matsushima-type formula to prove an estimate on the first eigenvalue of the Laplacian on one forms that implies vanishing of first cohomology. In order to define a Laplacian one requires a choice of metric on $F \otimes \mathcal{H}$, and our choice here is similar to the one made by Matsushima and Murakami in [25]. In fact, the work here is very close to the work in that paper, and we eventually reduce to an estimate on eigenvalues of the same finite dimensional matrix as they do. These estimates are obtained by Raghunathan in [29]. It is possible to derive the formula we use by closely following the derivation in [25]; however, we choose a different method. From our point of view, the estimate follows by subtracting two standard Weitzenböck formulas for different bundle structures on the same vector bundle. The use of two bundle structures is also a key technique in [25], but the relation of the computations there to standard differential geometric computations is not clear. One benefit of our point of view is that it makes it immediately clear why the negative Ricci curvature of $M$ does not spoil the computations: it appears in both formulas, and cancels when they are subtracted. It is worthwhile to compare our differential geometric interpretation of the formula from [25] to a well known differential geometric interpretation of an earlier Bochner-type formula of Matsushima for trivial bundles [23]. This well-known differential geometric interpretation is due to Calabi, was first described in print in detail by Dodziuk [7], and played a key role in the development of geometric superrigidity [18, 27]. As in those works, our Bochner formula will be useful in non-linear settings, see [10, 12].

In §4.4.6 we deduce some vanishing results for $H^i(\Gamma, \pi_1 \otimes \pi_2, F \otimes \mathcal{H})$ for more restricted choices of $F$. These results are probably not optimal. A reader familiar with the literature on cohomology of Lie groups and lattices might wonder why we do not pursue an analysis based on relative Lie algebra cohomology and methods closer to those of [2, 34, 44]. Approaches of this kind, whether directly to show vanishing for $G$ cohomology, or indirectly to show vanishing of $\Gamma$ cohomology via Matsushima-type formulas, encounter analytic difficulties. We discuss the issues that arise in [13]. We remark that our work gives a new interpretation of the Matsushima-Murakami formula for the Laplacian on vector bundles of the form $(K \backslash G \times F)\Gamma$ which is used in the proof of Matsushima’s formula. Our more geometric interpretation of this formula may be of interest even to those primarily interested in traditional applications of Matsushima’s formula.

In addition, the estimate on the Laplacian proved here using the Bochner method is also used in our work on harmonic maps into continuum products [10, 12].

In [11], using results in [9] we prove a local rigidity theorem for many geometric actions of groups with property $F \otimes H$ on compact manifolds. This can be seen as
a generalization of the theorem of the first author and Margulis on local rigidity of isometric actions of groups with property \((T)\) \cite{Fis}. Parts of this work were supported by visits to the Graduate Center of the City University of New York, Indiana University and Rice University. We thank these institutions for their hospitality and support. We would also like to thank Nicolas Monod for suggesting the name property \(F \otimes H\).

2. Other groups and property \(F \otimes H\)

Other than the groups that arise in Theorem 1.2, other groups known to have property \((T)\) fall into a few distinct classes. In this section we discuss which of these have or do not have property \(F \otimes H\).

Let \(H\) be a Lie group with Levi decomposition \(H = L \ltimes R^n\) and assume that \(L\) is semisimple with no compact factors and property \((T)\) and that the \(L\) representation on \(R^n\) has no invariant vectors. Then \(H\) is known to have property \((T)\), see e.g. \cite{Bou, Fis, Mon}. In many cases, it is obvious that \(H\) contains lattices, e.g. \(SL(n, \mathbb{Z}) \ltimes \mathbb{Z}^n < SL(n, \mathbb{R}) \ltimes \mathbb{R}^n\). For a more detailed discussion of \(H\)'s containing lattices, see \cite{Mon}. In any case, we have:

**Proposition 2.1.** If \(H\) is as in the preceding paragraph, then both \(H\) and any lattice in \(H\) fail to have property \(F \otimes H\).

**Proof.** It is clear that \(H\) admits an action on \(R^n\) with no fixed points, by letting \((l, v) \cdot w = lw + v\) where \(l \in L\) acts on \(w\) by the representation defining the semidirect product. It is also clear that this action has no fixed points when restricted to a lattice in \(H\). This can be viewed as providing non-trivial cohomology classes in \(H^1(H, R^n)\) or \(H^1(\Gamma, R^n)\). \(\square\)

This observation extends to more general semidirect products \(L \ltimes U\) with \(U\) unipotent, we leave the construction of more examples to the reader. For all other groups we know of with property \((T)\), the key fact seems to be the following simple lemma.

**Lemma 2.2.** Let \(D\) be a group with property \((T)\) such that every continuous finite dimensional representation of \(D\) over \(\mathbb{R}\) or \(\mathbb{C}\) is unitary. Then \(D\) has property \(F \otimes H\).

The lemma allows us to deduce that many other known examples of groups with property \((T)\) also have property \(F \otimes H\) simply by verifying the hypotheses of Lemma 2.2 for those groups. This is clear for higher rank semisimple algebraic groups defined over local fields other than \(\mathbb{R}\) or \(\mathbb{C}\). For lattices in those groups, it follows from Margulis’ superrigidity theorems \cite{Mar}. For quotients of lattices in \(SP(1, n)\) and \(F_{4-20}\) by infinite index normal subgroups it follows from Corlette’s superrigidity theorem \cite{Cor}. For any group where property \((T)\) derives from an action on a simplicial complex with a spectral condition on links, it can be deduced from work of Wang, Schoen-Wang or Izeki-Nayatani \cite{Wang, Schoen-Wang, Izeki-Nayatani}. It seems to us that for most models of random groups it is at least the case that \((T)\) implies \(F \otimes H\) with probability one. This is
most clear in Zuk’s model of random group where the proof of property (T) amounts to proving that the random group has a properly discontinuous cocompact action on a simplicial complex with an appropriate spectral condition on links. For a longer discussion/comparison between models of random groups, we recommend [28].

We believe that non-uniform lattices in semisimple Lie groups with property (T) will also have property FH. While the case of \( \mathbb{Q} \)-rank 1 lattices seems quite approachable, the general case involves some significant difficulties. We hope to pursue this elsewhere.

There is a class of groups with property (T) for which we currently do not know if property \( F \otimes H \) holds. If \( G \) is a semisimple Lie group (as above), there can exist infinite, non-split central extensions of \( G \) by \( \mathbb{Z}^d \). The simplest examples of these occur when \( G \) is the isometry group of an irreducible Kähler symmetric space and the universal cover \( \tilde{G} \) is a central extension of \( G \) by \( \mathbb{Z} \). Other examples also exist. It is known that these groups and their lattices have property (T) by an argument due to Serre, exposed in [3]. We do not know if these groups have property \( F \otimes H \). Serre’s argument does not adapt easily to our setting, so it appears that a new idea is needed for this case.

3. Reductions to cohomology of cocompact lattices

In this section we begin the proof of Theorem 1.2 by reducing to the following:

**Theorem 3.1.** Let \( G \) be a semisimple Lie group with property (T) of Kazhdan and no compact factors, and \( \Gamma \) a torsion-free, irreducible, cocompact lattice in \( G \). Let \( \pi_1 : G \to GL(F) \) be a non-trivial, finite dimensional, real representation that does not contain the trivial representation and \( \pi_2 : \Gamma \to U(\mathcal{H}) \) be a (real) unitary representation on a separable Hilbert space. Then \( H^1(\Gamma, \pi_1|_\Gamma \otimes \pi_2, F \otimes \mathcal{H}) = 0 \).

This theorem is proved in section 4 below.

3.1. First reductions on \( G \) and \( \Gamma \). In this section, we show how to reduce to the case of \( \Gamma \) an irreducible torsion-free lattice in a non-compact semisimple group or \( G \) a simple Lie group. The results are not difficult and we leave the proofs to the reader.

**Lemma 3.2.** Any compact group has property \( F \otimes H \).

Hence, it suffices to consider the case where \( G \) is noncompact. By a lemma of Selberg, every lattice contains a torsion-free normal subgroup of finite index [36]. We also have the following standard lemma.

**Lemma 3.3.** Let \( \Gamma' \) be a finite index normal subgroup of \( \Gamma \) and \( \rho \) any representation of \( \Gamma \) on a real or complex vector space. Then \( H^p(\Gamma', \rho|_{\Gamma'}) = 0 \) implies that \( H^p(\Gamma, \rho) = 0 \).

Therefore, we assume from now on that any lattice is torsion free and pass to finite index subgroups without further comment.
Recall that a lattice in a semisimple Lie group is irreducible if it projects densely into all simple factors. If $\Gamma$ is not irreducible then, possibly after passing to a subgroup of finite index in $\Gamma$, we have $G = G_1 \times G_2$, $\Gamma = \Gamma_1 \times \Gamma_2$, where $\Gamma_i$ is a lattice in $G$ [22, Definition II.6.5 and Theorem II.6.7]. So if $\Gamma$ is not irreducible, after passing to a subgroup of finite index, there is a (split) short exact sequence of groups $0 \to \Gamma_1 \to \Gamma \to \Gamma_2 \to 0$. The resulting long exact sequence in cohomology contains a portion of the form

$$
\cdots \to H^1(\Gamma_2, \rho) \to H^1(\Gamma, \rho) \to H^1(\Gamma_1, \rho) \to \cdots .
$$

Hence, vanishing of $H^1$ for both $\Gamma_1$ and $\Gamma_2$ implies $H^1(\Gamma, \rho) = 0$. Induction on the number of irreducible factors reduces us to the case of irreducible lattices.

When proving Theorem 1.2 for Lie groups, an argument like the one just given allows us to reduce to the case of $G$ a simple non-compact Lie group.

### 3.2. Decompositions of representations.

In this subsection we show how to reduce to the case $\pi_1$ does not contain a unitary subrepresentation and when all representations are real. Recall that a representation which is $\mathbb{C}$ linear is also $\mathbb{R}$ linear.

**Lemma 3.4.** Let $\rho$ be any continuous complex linear representation of a topological group $D$ on a complex Hilbert space $W$ and let $\rho|_{\mathbb{R}}$ be the associated real representation, then $H^1(\rho, W) = H^1(D, \rho|_{\mathbb{R}}, W)$ as real vector spaces.

The next observation is a special case of a general decomposition into irreducibles for representations of the type we consider. We do not state the general version here, simply to avoid a discussion of direct integrals. Again, the lemma stated is not difficult.

**Lemma 3.5.** Let $D$ be a topological group and $\pi$ a continuous finite dimensional by unitary representation of $D$. Assume $\pi_1$ is completely reducible and let $(\pi_1, F) = \bigoplus (\pi_i^1, F_i)$ be a decomposition into irreducible representations, then

$$H^1(D, \pi_1 \otimes \pi_2, F \otimes \mathcal{H}) = \sum_i H^1(D, \pi_i^1 \otimes \pi_2, F_i \otimes \mathcal{H}).$$

In particular, if $D$ has property $(T)$, and any $F_i$ is a unitary $D$ representation, then $\pi_i^1 \otimes \pi_2$ is unitary and $H^1(\Gamma, \pi_i^1 \otimes \pi_2, F_i \otimes \mathcal{H}) = 0$.

**On decomposing $\pi_2$:** It is also possible to decompose $\pi_2$ into irreducible representations using a direct integral. In this case, the analogue of Lemma 3.5 is not obvious, since solving a cohomological equation in every irreducible integrand of a direct integral does not necessarily solve the integral of the cohomological equation. This is already true for infinite direct sums, see §5 for more discussion.

**Lemma 3.6.** Let $D$ be either a countable discrete group or a topological group containing a countable dense subset. Then to prove $H^1(D, \pi) = 0$ for all continuous finite dimensional by unitary representations $\pi$ it suffices to consider $\pi = \pi_1 \otimes \pi_2$ where $\pi_2$ is a representation on a separable Hilbert space.
Proof. We prove the lemma first for $D$ a countable discrete group and then explain the modifications necessary for when $D$ is not discrete.

Let $\pi = \pi_1 \otimes \pi_2$ be a finite dimensional by unitary representation of $D$ on $F \otimes \mathcal{H}$ for $\mathcal{H}$ arbitrary and $c$ a cocycle in $H^1(D, \pi_1 \otimes \pi_2, F \otimes \mathcal{H})$. We will show that $c$ takes values in $F \otimes \mathcal{H}_c$ where $\mathcal{H}_c$ is a separable Hilbert space and $F \otimes \mathcal{H}_c$ is $\pi$ invariant. It then follows that $c$ is a coboundary for $\pi$ if and only if it is a coboundary for $\pi|_{F \otimes \mathcal{H}_c}$.

Consider the set $D \cdot c = \{\pi(d_1)c(d_2)|d_1, d_2 \in D\}$. This set is countable. If $\pi$ were a unitary representation, we would take the span of the closure of $D \cdot c$ and consider the restriction of $\pi$ to this, necessarily separable, subspace. In our setting, a little more care is required. Since $F$ is finite dimensional, we can pick a finite basis, $\{f_1, \ldots, f_k\}$ for $F$. Every element $\phi$ of $D \cdot c$ can now be written as $\phi = \sum_{i=1}^{k} f_i \otimes \psi_i^\phi$ where $\psi_i^\phi \in \mathcal{H}$.

It is clear that $(D \cdot c)^\mathcal{H} = \{\psi_i^\phi|\phi \in D \cdot c, 1 \leq i \leq k\}$ is a countable subset of $\mathcal{H}$. Let $\mathcal{H}_c$ be the closure of the span of $(D \cdot c)^\mathcal{H}$, then it is clear that $\mathcal{H}_c$ is a closed separable Hilbert subspace of $\mathcal{H}$, that $\mathcal{H}_c$ is $D$ invariant and that $c$ takes values in $F \otimes \mathcal{H}_c$. If $c$ is a coboundary, then $c(d) = \pi(d)v - v$. If such a $v$ exists, it clearly can be chosen in $F \otimes \mathcal{H}_c$. Therefore $c$ is a coboundary as a cocycle over $\pi$ if and only if $c$ is a coboundary as a cocycle over $\pi|_{F \otimes \mathcal{H}_c}$.

If $D$ is a topological group with countable dense subset $D_0$, we need only replace $D \cdot c$ by the countable set $D_0 \cdot c = \{\pi(d_1)c(d_2)|d_1, d_2 \in D_0\}$ and the same argument works. Continuity of $\pi$ and $c$ imply that, in this case, the closure of the span $D_0 \cdot c$ equals the closure of the span of $D \cdot c$. This then implies that $F \otimes \mathcal{H}_c$ constructed as above is a $D$ invariant closed, separable subspace of $F \otimes \mathcal{H}$.

3.3. Reductions via superrigidity theorems. In this subsection we recall a form of the Margulis-Corlette-Gromov-Schoen superrigidity theorems. Given a lattice $\Gamma$ in $G$, and a representation $\pi : \Gamma \rightarrow GL(F)$, we say that $\pi$ almost extends to a continuous representation of $G$ if there exist representations $\pi_1 : G \rightarrow GL(F)$ and $\pi_2 : \Gamma \rightarrow GL(F)$, where $\pi_2$ has bounded image and the images of $\pi_1$ and $\pi_2$ commute so that $\pi(\gamma) = \pi_1(\gamma)\pi_2(\gamma)$.

**Theorem 3.7.** Let $G$ be a semisimple Lie group with no compact factors and property (T) of Kazhdan. Let $\Gamma$ in $G$ be a lattice and $\pi : \Gamma \rightarrow GL(F)$ a finite dimensional representation. Then $\pi$ almost extends to a continuous representation of $G$.

**Proof.** This is a consequence of the strongest form of the superrigidity theorems. All the ingredients needed are assembled by Starkov in [37], though the theorem is not stated in quite this generality there.

For $G$ of higher rank and $\Gamma$ irreducible, this is proved in [22], though it is not stated explicitly there. As noted in [13], it follows easily from Lemma VII.5.15 and Theorems VII.5.15 and VII.6.16 of [22]. However, when $G$ contains rank one factors and $\Gamma$ (or some finite index subgroup) projects to a lattice in a rank one factor one needs to use Corlette’s work [5] to prove the analogue of [22, Theorem VII.5.15]. The proof of Theorem VII.6.16 is even more involved, as one needs to first
prove arithmeticity using work of Corlette and Gromov-Schoen and then deduce the theorem from cohomology vanishing theorems for finite dimensional representations of $\Gamma$. As noted in [37] this follows by assembling results from [29] and [31]. At various points in the argument, one needs to pass to finite index subgroups, but this is not required in the statement of results by use of [22] Lemma VII.5.1].

**Corollary 3.8.** Let $G$ and $\Gamma$ be as in Theorem 3.7. Let $\pi = \pi_1 \otimes \pi_2$ be any $\Gamma$ representation as in Definition 1.1. Then we can assume $\pi_1$ extends to $G$.

**Proof.** It is easy to check that if a representation $\pi$ of $\Gamma$ can be written as $\pi(\gamma) = \pi'(\gamma)\pi''(\gamma)$ where $\pi'$ and $\pi''$ commute, then $\pi$ can be realized as a tensor product of representations $\pi_0' \otimes \pi_0''$. So, if $\pi_1$ does not extend to $G$, then as a consequence of Theorem 3.7 we can write $\pi_1 = \pi_1' \otimes \pi_1''$ where $\pi_1'$ extends and $\pi_1''$ has bounded image. Regrouping $\pi$ as $\pi_1' \otimes (\pi_2' \otimes \pi_2)$ proves the corollary. $\square$

### 3.4. Reduction to cocompact lattices.

The purpose of this subsection is to show how one reduces the case of connected groups to the case of cocompact lattices. In the following lemma, we call a lattice in a product $G_1 \times G_2$ weakly irreducible if both projections on factors are dense.

**Lemma 3.9.** Let $G$ be a locally compact group and assume $G \times G$ admits a cocompact weakly irreducible lattice $\Gamma$. Then if $G$ has property $F \otimes H$ so does $G$.

**Proof.** Let $\Gamma$ be a weakly irreducible lattice in $G \times G$ and $(\pi, F \otimes H)$ a finite dimensional by unitary representation of $G$. If $p_1$ is the projection of $\Gamma$ on the first factor of $G \times G$, then $\pi \circ p_1$ is also clearly a finite dimensional by unitary $\Gamma$ representation. We will show that $H^1(\Gamma, \pi \circ p_1, F \otimes H) = 0$ implies $H^1(G, \pi, F \otimes H) = 0$. Any continuous cocycle $c : G \to F \otimes H$ restricts to a $p_1(\Gamma)$ cocycle. A coboundary for $\Gamma$ is a map of the form $b(\gamma) = \pi \circ p_1(\gamma)v - v$. Note that continuity of $\pi$ and the formula for $b$ immediately imply that $b$ is continuous in the induced topology on $p_1(\Gamma) < G$ and so $b$ extends continuously to a $G$ coboundary $b(g) = \pi(g)v - v$. If $H^1(\Gamma, \pi \circ p_1, F \otimes H) = 0$, then $c \circ p_1(\gamma) = b(\gamma)$. Since both sides extend continuously to functions on $G$, they are equal on $G$ as well, so $c(g) = b(g)$ and $H^1(G, \pi, F \otimes H) = 0$ as desired.

It is also possible to write this proof in terms of fixed points for affine actions. From this point of view, the idea is that for a continuous affine $G$ action, a fixed point for a dense subgroup is necessarily a fixed point for $G$. $\square$

**On Lemma 3.9:**

1. The lemma is a fact about $H^1$ and has no obvious analogue for higher degree cohomology. The key point is that in a continuous $G$ representation, a 1-coboundaries for a dense subgroup extends continuously to a 1-coboundary for $G$. This is not clear, and seems unlikely, for $k$-coboundaries when $k > 1$.
2. We recall that for every group $G$ as in Theorem 1.2 there is a weakly irreducible cocompact lattice in $G \times G$. This follows from our definition of irreducible lattice (note that we do not require that the projection to other
quotients of $G \times G$ be dense) and Borel’s construction of lattices in \([3]\). To construct the lattice, one can simply construct an irreducible lattice (in the standard sense) in $G_i \times G_i$ for each simple factor $G_i$ of $G$. Due to prior reductions, for our argument we need only construct irreducible cocompact lattices in $G \times G$ where $G$ is simple.

3.5. Proof that Theorem 3.1 implies Theorem 1.2.

Proof. By Lemma 3.2, Lemma 3.3 and the discussion around Equation 3.1, we see that we may assume that $G$ has no compact factors and that $\Gamma$ is irreducible and torsion free. By Lemma 3.4 it suffices to consider the case where $\pi = \pi_1 \otimes \pi_2$ where $\pi_1$ is a real linear representation and $\pi_2$ is a unitary representation over $\mathbb{R}$. By Lemma 3.6, it suffices to consider the case where $\mathcal{H}$ is a separable Hilbert space.

By Theorem 3.7, it suffices to prove Theorem 1.2 for representations which are of the form $\pi_1 \otimes \pi_2$ where $\pi_1 : G \rightarrow GL(F)$ is a real linear representation and $\pi_2$ is a unitary representation of $G$ or $\Gamma$. By Lemma 3.5 and the fact that $G$ representations are completely reducible, it suffices to consider the case where $\pi_1$ does not contain the trivial representation.

Lemma 3.9 and the second remark following that lemma show how Theorem 1.2 for cocompact lattices implies Theorem 1.2 for connected groups. \qed

4. Vanishing theorems for uniform lattices

In this section we prove Theorem 3.1. Actually, we prove a more general result than Theorem 3.1 which includes some cases where $G$ has factors of the form $SO(1,n)$ or $SU(1,n)$, see \([1,3]\). For this reason, until specified further, in this section $G$ is a semisimple Lie group with no compact factors and $\Gamma \subset G$ is a cocompact lattice. We also derive some results on higher cohomology groups in the case that the finite dimensional representation is complex instead of real. See \([4,4.6]\) for specific statements. For this reason, much of the development in this section treats $p$th cohomology groups rather than just $H^1$.

The main technical tool is an estimate on the Laplacian on smooth differential forms with values in a vector bundle. Assume that $\Gamma$ is torsion free and let $K$ be a maximal compact subgroup of $G$. We build the locally symmetric Riemannian manifold $M = \Gamma \backslash G/K$ and the locally constant vector bundle $\mathcal{E}(\pi) \rightarrow M$ associated to the representation $\pi$ and the principal $\Gamma$-bundle $\Gamma \rightarrow G/K \rightarrow M$. Note that the fiber of this bundle is the infinite dimensional vector space $E = F \otimes \mathcal{H}$.

We construct an inner product in the fibers of $\mathcal{E}(\pi)$, and therefore inner products $(\cdot, \cdot)$ on $\bigwedge^p T^*M \otimes \mathcal{E}(\pi)$, the spaces of smooth $\mathcal{E}(\pi)$-valued differential $p$-forms. Using these inner products, we define an adjoint operator $\delta$ to the exterior derivative operation $d$ and hence a Laplace operator $\Delta = d\delta + \delta d$. The estimate we obtain is the following.
Theorem 4.1. Let $G$, $\Gamma$ and $\pi$ be as in §3.1. There is a positive constant $C = C(G, \pi_1)$ such that for any smooth differential 1-form, $\eta$, with values in $\mathcal{E}(\pi)$ we have

$$((\triangle \eta, \eta) \geq C(\eta, \eta).$$

We will need this result itself for our geometric cocycle superrigidity theorem. It will be translated into an energy estimate and will play a crucial role by controlling a type of heat flow on a special set of mappings \cite{10, 12}.

We now outline the rest of the section. In §4.1 we discuss some cohomology isomorphisms which reduce the problem to considering de Rham cohomology of smooth $p$-forms with values in a vector bundle. In §4.2 we introduce the necessary operations on forms to define the Laplacian on a vector bundle. In §4.3 we discuss how vanishing cohomology can be deduced from an estimate on this Laplacian. The material in these three sections is mostly standard, though perhaps not well-known in our setting.

In §4.4, we derive a useful Bochner-type formula for the Laplacian on our particular vector bundle. While our final estimate is similar to that of Matsushima and Murakami, the derivation follows a different conceptual outline. We then use this formula and some Lie-theoretic computations of Raghunathan to obtain the required estimates and deduce vanishing results. We discuss first cohomology in §4.4.5 and higher degrees in §4.4.6. In §4.5, we give a direct computation of the flat Laplacian.

4.1. Some Cohomology Isomorphisms. Let $K$ be a maximal compact subgroup of $G$. Let $M = \Gamma \backslash G/K$, a compact locally symmetric manifold, and form the locally constant vector bundle $\mathcal{E}(\pi) \to M$ associated to the representation $\pi$ and the principal $\Gamma$-bundle $\Gamma \to G/K \to M$. Note that the canonical fiber of this bundle is $E = F \otimes \mathcal{H}$. Let $\mathcal{E}(\pi)_{loc}$ denote the Čech cohomology of $M$ with coefficients in the sheaf $\mathcal{E}$ of locally constant sections of $\mathcal{E}(\pi)$.

**Proposition 4.2.** $H^*(\Gamma, \pi) = H^*(M, \mathcal{E}(\pi)_{loc})$

This isomorphism holds on the level of cochains \cite{4}. Let $\check{H}^*(M, \mathcal{E})$ denote the Čech cohomology of $M$ with coefficients in the sheaf $\mathcal{E}$ of locally constant sections of $\mathcal{E}(\pi)$.

**Proposition 4.3.** $H^*(M, \mathcal{E}(\pi)_{loc}) = \check{H}^*(M, \mathcal{E})$

One can see this by inspecting the definitions. For more on sheaf cohomology, see for example \cite{11}. Let $H_{de \text{Rham}}^*(M, \mathcal{E}(\pi))$ denote the de Rham cohomology of smooth $\mathcal{E}(\pi)$-valued differential forms on $M$. The exterior derivative $d$ on $\mathcal{E}(\pi)$-valued $p$-forms is defined as follows. For a form of the type $\omega = \eta \otimes e$, where $\eta$ is an ordinary $p$-form on $M$ and $e$ is a section of $\mathcal{E}(\pi)$, we set

$$d\omega = d\eta \otimes e,$$

and extend linearly. Then one can define de Rham cohomology in the usual fashion.

**Proposition 4.4.** $\check{H}^*(M, \mathcal{E}) \cong H_{de \text{Rham}}^*(M, \mathcal{E}(\pi))$. 

This is discussed by Mok [26]. The key point is that a version of the Poincaré lemma holds. Using these isomorphisms, we restrict ourselves to the study of de Rham cohomology.

4.2. The Laplacian on a Euclidean Vector Bundle. In this section we recall some facts about Riemannian vector bundles. First, we recall the definitions of some standard operations which lead to a definition of the inner product and the Laplacian on the space of bundle-valued $p$-forms. Then we recall the Bochner-Weitzenböck formula for bundle-valued differential forms. A reference for this material is [8].

Throughout this section, we consider a vector bundle $V$ over a compact, oriented, Riemannian manifold $N$. We assume also that this bundle is Riemannian, that is, $V$ is equipped with a connection $\nabla$ and a fiberwise Euclidean structure $(\cdot,\cdot)$ which are compatible in the sense that for any two sections $a,b$ of $V$ and any continuous vector field $X$ on $N$,

$$X \cdot (a,b) = (\nabla_X a, b) + (a, \nabla_X b).$$

We denote by $A^p(V)$ the space of smooth $p$-forms on $N$ with values in $V$, i.e. the space of smooth sections of the bundle $\bigwedge^p T^* N \otimes V$ over $N$. Given the pair $V, \nabla$, one can define the exterior derivative operator $d$ for $\sigma \in A^p(V)$ by

$$d\sigma(X_1,\ldots,X_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} \nabla_{X_i} (\sigma(X_1,\ldots,\hat{X}_i,\ldots,X_{p+1})) + \sum_{i<j} (-1)^{i+j} \sigma([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_{p+1}),$$

where the hat means that that term is omitted. Note that $d^2$ is not necessarily zero. In fact, $d^2 = 0$ if and only if the connection $\nabla$ is flat.

In the case of $\mathcal{E}(\pi)$, the differential $d$ which computes de Rham cohomology is constructed by this process, $d^2 = 0$ and the bundle is flat.

4.2.1. The inner product on $V$-valued forms. The Hodge star operation for ordinary $p$-forms on $N$ induces a similar operation on $V$-valued forms which we shall also denote by $\ast$. For $\omega = \sum \eta_I \otimes e_I$ where $I$ is a multi-index of length $p$, $\eta_I$ is an ordinary $p$-form and $e_I$ is a section of $V$, we have

$$\ast \omega = \sum (\ast \eta_I) \otimes e_I.$$

Let $V^*$ be the dual vector bundle to $V$. The Euclidean structure in the fibers of $V$ defines an isomorphism $\#$ : $V \to V^*$ which is determined by the condition that for $u_x, v_x$ in the fiber of $V$ over $x \in N$

$$\langle u_x, \# v_x \rangle = (u_x, v_x)_x,$$

where $\langle \cdot, \cdot \rangle$ denotes the natural pairing of $V$ and $V^*$. We note that the operations $\ast$ and $\#$ commute.
We define a wedge product for $V$-valued forms. Let $\theta = \sum_{|I|=p} \theta_I \otimes e_I$ be an $V$-valued $p$-form and $\eta = \sum_{|J|=q} \eta_J \otimes f_J$ an $V^*$-valued $q$-form. We define an ordinary $(p+q)$-form, $\theta^t \wedge \eta$, on $N$ as follows
\[
\theta^t \wedge \eta = \sum_I \sum_J \langle e_I, f_J \rangle \theta_I \wedge \eta_J.
\]
The transpose notation is used since the forms live in different bundles and this is not the standard wedge product.

We now define the inner product of a pair of $V$-valued $p$-forms. For two such forms $\omega_1, \omega_2$, we set
\[
(\omega_1, \omega_2) = \int_N \omega_1^t \wedge \# \omega_2 = \int_N \omega_1^t \wedge \# \ast \omega_2.
\]
We use the same symbol here for the inner product of forms, the inner product in the fibers of $V$ and the induced inner products in various associated bundles. Which structure we mean will be clear from context.

The integrand can be evaluated in local coordinates. Pick a point $x \in N$. In a sufficiently small neighborhood $U$ of $x$, we can choose a basis of one forms $\theta_1, \ldots, \theta_n$ on $N$ which are orthonormal at each point of $U$ and positively oriented. Write $V$-valued $p$-forms $\omega_1, \omega_2$, in coordinates
\[
\omega_1 = \sum_{i_1 < \ldots < i_p} \theta^{i_1} \wedge \ldots \wedge \theta^{i_p} \otimes a_{i_1, \ldots, i_p},
\omega_2 = \sum_{i_1 < \ldots < i_p} \theta^{i_1} \wedge \ldots \wedge \theta^{i_p} \otimes b_{i_1, \ldots, i_p}
\]
where $a_{i_1, \ldots, i_p}$ and $b_{i_1, \ldots, i_p}$ are sections of $V$. Then at all points $z$ of $U$,
\[
(\omega_1^t \wedge \# \omega_2)(z) = \sum_{i_1 < \ldots < i_p} (a_{i_1, \ldots, i_p}, b_{i_1, \ldots, i_p})(z).
\]

4.2.2. The operators $\delta$ and $\Delta$. We now define the adjoint of exterior differentiation and the Laplacian. For any $V$-valued $p$-form $\theta$ on $N$, we define
\[
\delta \theta = (-1)^p \ast^{-1} \#^{-1} d \# \ast \theta.
\]

**Proposition 4.5.** The operator $\delta$ is the formal adjoint of $d$ with respect to the inner product $(\cdot, \cdot)$ on $V$-valued $p$-forms. That is, for two $V$-valued $p$-forms $\theta$ and $\omega$:
\[
(\delta \theta, \omega) = (\theta, d\omega).
\]

For a smooth $V$-valued $p$-form on $N$ we define the Laplacian by:
\[
\Delta \theta = d \delta \theta + \delta d \theta.
\]
This is a self-adjoint elliptic differential operator of second order on (sections of) $V$. We note that it is also a non-negative operator, since $(\Delta \theta, \theta) = (d \theta, d \theta) + (\delta \theta, \delta \theta) \geq 0$.

The formal adjoint $\nabla^*$ of $\nabla$ is defined by the property that $(\nabla \eta, \theta) = (\eta, \nabla^* \theta)$. 

4.2.3. The traditional Bochner-Weitzenböck formula. Here we recall the Bochner-Weitzenböck formula for $V$-valued $p$ forms. For a development from first principles, see [8, pages 3-13].

Let $R^V$ denote the curvature tensor of the connection $\nabla$ on $V$ and $R$ the Riemannian curvature tensor of $N$. We define the Ricci operator, $S$, on $V$-valued $p$-forms as follows. Let $X_1, \ldots, X_p$ be continuous vector fields on $N$, $x$ a point in $N$, and $e_1, \ldots, e_n$ an orthonormal basis of $T_xN$. Then for a $V$-valued $p$-form $\sigma$, $S(\sigma)$ is the $V$-valued $p$-form given at $x$ by

$$
S(\sigma)_x(X_1, \ldots, X_p) = \begin{cases} 
0, & \text{if } p = 0, \\
\sum_{k=1}^p \sum_{s=1}^n (-1)^k (R^V(e_s, X_k)\sigma)(e_s, X_1, \ldots, \hat{X}_k, \ldots, X_p), & \text{if } p \geq 1.
\end{cases}
$$

Theorem 4.6 (Eells-Lemaire, p. 13). Let $\sigma$ be a $V$-valued $p$-form on $N$. Then at each point $x$ of $N$ we have

$$
\frac{1}{2} \Delta(\sigma, \sigma)(x) = (\Delta\sigma, \sigma)(x) - (\nabla\sigma, \nabla\sigma)(x) - (S(\sigma), \sigma)(x).
$$

This is a pointwise formula, the inner products are the Euclidean product in the fiber over $x$ taken at the values of the forms at $x$. Integrating over $N$ yields:

Corollary 4.7.

$$
\int_N (\Delta\sigma, \sigma)dvol_N = \int_N (\nabla^*\nabla\sigma, \sigma)dvol_N + \int_N (S(\sigma), \sigma)dvol_N.
$$

The corollary follows as the left hand side of equation 4.4 integrates to zero by the divergence theorem, and $\nabla^*$ is the formal adjoint of $\nabla$.

4.3. A Sufficient Condition for Vanishing. In this section we show that an estimate of the type in Theorem 4.1 implies vanishing of cohomology. The argument is close to one due to Mok, compare to the proof of [26, Proposition 1.3.1].

Proposition 4.8. Suppose that there exists a positive constant $C$ such that for all smooth $E(\pi)$-valued $p$-forms $\theta$ on $M$

$$
(\Delta\theta, \theta) \geq C(\theta, \theta).
$$

Then $H_{de \text{Rham}}^p(M, E(\pi)) = 0$.

Proof. In order to give the proof, we need a more general viewpoint on our forms to employ some functional analysis. Let $A^p(M, E(\pi))$ be the Hilbert space of $E(\pi)$-valued differential forms $\eta$ on $M$ for which $\eta, d\eta$ and $\delta\eta$ satisfy an $L^2$ condition with respect to our inner product $\langle \cdot, \cdot \rangle$. Smooth $p$-forms are a dense in $A^p(M, E(\pi))$. By a simple approximation argument, condition 4.6 of Proposition 4.8 holds for all forms in $A^p(M, E(\pi))$ if it holds for all smooth forms. We need the following lemma from functional analysis.
Lemma 4.9 (Hörmander [16], Lemma 4.1.1, page 78). Let \( \phi : H_1 \to H_2 \) be a densely defined linear operator between two Hilbert spaces with adjoint \( \phi^* \). Suppose that \( H_3 \subset H_2 \) is a closed linear subspace which contains the range of \( \phi \). Then the range of \( \phi \) is dense in \( H_3 \) if and only if for some constant \( C > 0 \)

\[
||v||_{H_2} \leq C||\phi^*v||_{H_1}
\]

for all \( v \) in the intersection of \( H_3 \) with the domain of \( \phi^* \).

We apply this result to the exterior derivative map

\[
d : A^{p-1}(\mathcal{E}(\pi)) \to \ker d \subset A^p(\mathcal{E}(\pi)).
\]

Note that for an element \( \eta \) of ker \( d \) we have \( (\Delta \eta, \eta) = (\delta \eta, \delta \eta) \), so equation 4.7 follows from equation 4.6.

Since any smooth closed \( \mathcal{E}(\pi) \)-valued \( p \)-form \( \eta \) lies in the kernel of \( d \), we see that there is a solution to the equation \( du = \eta \). It remains to be show that \( u \) can be chosen smooth. This is true for the solution having minimal \( L^2 \)-norm.

To see that a solution of minimal norm exists, let \( t = \inf \{||u||_2 \mid du = \eta\} \) and consider a sequence \( u_i \) of elements of \( A^p(M, \mathcal{E}(\pi)) \) with \( du_i \) approaching \( t \). The ball of \( L^2 \)-forms having norm bounded by \( 2t \) is weak-* compact, so we may extract a weak-* subsequential limit of the \( u_i \). By lower semicontinuity of the \( L^2 \) norm, we see that

\[
||u||_2 \leq \lim \inf ||u_i||_2 = t.
\]

We claim that this limit \( u \) is also a solution to \( du = \eta \). Let \( \psi \) be a smooth test form. We have \( (\eta, \psi) = (du_i, \psi) = (u_i, \delta \psi) \to (u, \delta \psi) = (du, \psi) \). Hence \( du = \eta \).

We now show that \( \delta u = 0 \). Suppose that \( \delta u = g \) is not zero. Then there exists an \( h \) such that \( (g, h) = (\delta u, h) = (u, dh) =: b > 0 \). Notice this is also true when we replace \( h \) by \( rh \) where \( r \) is a positive real number. Let \( a = (dh, dh) > 0 \) and pick a positive real number \( r < 2a/b \). Let \( w = u - rh \). Then \( dw = du = f \), and \( (w, w) = (u, u) - 2r(u, dh) + r^2(dh, dh) = (u, u) - r(2a - br) < (u, u) \). This contradicts the minimality of the norm of \( u \), so we conclude that \( \delta u = 0 \).

Now note that the form \( \nu = \delta \eta \) is a smooth form, and by construction \( u \) satisfies

\[
\Delta u = (d\delta + \delta d)u = \delta \eta = \nu.
\]

So the elliptic regularity theorem implies that \( u \) is a smooth form. Thus we see that every smooth closed \( \mathcal{E}(\pi) \)-valued \( p \)-form bounds.

\[\Box \]

4.4. A variant of the Matsushima-Murakami Bochner formula. Our goal in this section is to develop a variant of the Bochner-Weitzenböck formula for \( \mathcal{E}(\pi) \)-valued \( p \)-forms on \( M \) which is analogous to the one found by Matsushima and Murakami [25]. As with all formulae of this type, this formula will be a computation of the difference between two second order operators on \( \mathcal{E}(\pi) \)-valued \( p \)-forms and the difference is a zeroth order (algebraic) operator related to curvature. Then we analyze this algebraic operator to prove Theorem 4.1.
One could give a development directly mirroring that of [25]. Instead, we give an alternate derivation which we feel is more intuitive and makes the relation to other Bochner-type formulae more explicit.

The key observation of Matsushima and Murakami in [25] is that the relevant vector bundle can be constructed in two distinct ways, that is, there are two (isomorphic) natural vector bundles associated to the given data. This allows us to compare the natural differential operators arising from the two constructions.

In §4.4.1 we exhibit two distinct vector bundle structures on \( E(\pi) \). In §4.4.2 we define a preferred Riemannian metric on \( E(\pi) \). We discuss the computation of the connections compatible with the metric in the two bundle structures in §4.4.3 and then, in §4.4.4 we derive our version of the Matsushima-Murakami Bochner formula in terms of the representation \( \pi \) and the structure of \( G \) (or, rather, its Lie algebra \( \mathfrak{g} \)). We deduce the required estimates and the vanishing theorems which follow in §4.4.5 and §4.4.6.

4.4.1. Some bundle isomorphisms. Recall that \( \Gamma < G \) is a torsion free lattice and \( K < G \) is a maximal compact subgroup of \( G \). We are working with the situation where \( K \) acts on \( G \) by right multiplication, \( \Gamma \) acts on \( G \) by left multiplication and our compact manifold is \( M = \Gamma \backslash G/K \). In fact, \( G \) is a right principal \((\Gamma \times K)\)-bundle over \( M \), where \( \Gamma \times K \) acts by the rule

\[
g \cdot (\gamma, k) = \gamma^{-1} g k.
\]

We denote the quotient map by \( q : G \to M \).

On Equation 4.8. Equation 4.8 may surprise readers not familiar with standard conventions on principal bundles. The action described is a right action, or equivalently, an action of the opposite group. This convention is used in the literature on connections on principal bundles and we use it here to remain consistent with the [20]. This convention is not typically used in the theory of flat bundles. However, since the opposite group is canonically isomorphic to the original group, proving the vanishing result for the opposite group of \( \Gamma \) is equivalent to the same result for \( \Gamma \).

We are interested in two representations \( \sigma \) and \( \sigma' \) of \( \Gamma \times K \) on \( E = F \otimes \mathcal{H} \), defined as follows.

\[
\sigma(\gamma, k)v = (\pi_1(\gamma) \otimes \pi_2(\gamma))v,
\]

\[
\sigma'(\gamma, k)v = (\pi_1(k) \otimes \pi_2(\gamma))v.
\]

We build the vector bundles over \( M \) associated to the right principal bundle \( G \to M \) and these representations, respectively \( E(\sigma) \) and \( E(\sigma') \). Recall that \( E(\sigma) \) is the quotient of \( G \times E \) by the right \( \Gamma \times K \)-action

\[
(g, v) \cdot (\gamma, k) = (\gamma^{-1} g k, \sigma(\gamma, k)^{-1} v) = (\gamma^{-1} g k, (\pi_1(\gamma)^{-1} \otimes \pi_2(\gamma)^{-1}) v),
\]
and $E(\sigma')$ is the quotient of $G \times E$ by the $(\Gamma \times K)$-action

$$(g, v) \cdot (\gamma, k) = (\gamma^{-1} g k, \sigma'(\gamma, k)^{-1} v) = (\gamma^{-1} g k, (\pi_1(k)^{-1} \otimes \pi_2(\gamma)^{-1}) v).$$

We shall denote the quotient maps here by $Q : G \times E \to E(\sigma)$ and $Q' : G \times E \to E(\sigma')$. Since $\sigma(\gamma, k) = \pi(\gamma)$ for all $(k, \gamma) \in K \times \Gamma$, we have:

**Lemma 4.10.** As vector bundles over $M$, $\mathcal{E}(\pi) \cong E(\sigma)$.

The following observation is a more general version of [25, Proposition 3.1].

**Lemma 4.11.** As vector bundles over $M$, $E(\sigma) \cong E(\sigma')$.

**Proof.** Consider the mapping $A : G \times E \to G \times E$ defined by

$$A(g, v) = (g, (\pi_1(g)^{-1} \otimes \text{Id}_H)v).$$

This is a bijective mapping which intertwines the $\Gamma \times K$ actions inducing the quotient mappings $Q$ and $Q'$. For each $g \in G$ the map $v \to A(g, v)$ is a linear isomorphism $E \to E$, so $A$ induces a vector bundle isomorphism $A' : E(\sigma) \to E(\sigma')$. \(\square\)

We will use these isomorphisms as identifications without further comment.

### 4.4.2. The Euclidean structure on $\mathcal{E}(\pi)$

We now construct a Euclidean structure on $\mathcal{E}(\pi)$. We begin with a second important observation of Matsushima and Murakami in [25]. Let $\mathfrak{g}$ be the Lie algebra of $G$ with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ relative to $K$. We abuse notation and use $\pi_1$ to denote both the representation of $G$ on $\mathfrak{f}$ and the induced representation of $\mathfrak{g}$ on $F$.

**Proposition 4.12** ([25], Lemma 3.1). There is an inner product $(\cdot, \cdot)_F$ on $F$ which satisfies

- $(\pi_1(X)u, v)_F = -(u, \pi_1(X)v)_F$ for $X \in \mathfrak{k}$, and
- $(\pi_1(X)u, v)_F = (u, \pi_1(X)v)_F$ for $X \in \mathfrak{p}$.

The first condition means that $(\cdot, \cdot)_F$ is invariant under $\pi_1(K)$.

Since $\pi_2$ is a unitary representation of $\Gamma$ on $\mathcal{H}$, each $\pi_2(\gamma)$ preserves the Hilbert product $(\cdot, \cdot)_\mathcal{H}$ on $\mathcal{H}$. We form an inner product on $E = F \otimes \mathcal{H}$ as the tensor product of the above two structures. That is, let $(a \otimes c, b \otimes d)_E = (a, b)_F \cdot (c, d)_\mathcal{H}$ and extend linearly. By the properties in Proposition 4.12 this inner product is clearly invariant by the action inducing the quotient $Q' : G \times E \to E(\sigma')$, and so induces a Euclidean structure on $E(\sigma') = \mathcal{E}(\pi)$ which we denote hereafter by $(\cdot, \cdot)$ and call the canonical metric on $\mathcal{E}(\pi)$.

### 4.4.3. Two linear connections

The principal bundle $G \to M$ carries a natural connection. The subgroup $K \subset G$ induces a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of $G$, where $\mathfrak{k}$ is the Lie subalgebra corresponding to $K$. We make a choice of horizontal subspace $H_g \subset T_g G$ by $H_g = dL_g(\mathfrak{p})$, where $L_g$ denotes left multiplication by $g$. This family $\{H_g\}_{g \in G}$ defines a connection in $G \to M$. 

This connection in $G \to M$ induces connections $\nabla$ and $\nabla'$ in the vector bundles $E(\sigma)$ and $E(\sigma')$. Note that we have associated representations, bundles and connections for all types of tensors over $M$ with values in one of these two bundles by the same methods.

Recall the computation of such an induced connection ([20] page 116). We state the result for $E(\sigma)$ but the analogous procedure works for $E(\sigma')$. There is a bijective correspondence between sections $f : M \to E(\sigma)$ and $E$-valued functions $f^* : G \to E$ such that

\[ f^*(g \cdot (\gamma, k)) = \sigma(\gamma, k) f^*(g). \]

This correspondence is given in one direction by lifting $f$ to a section of the trivial bundle $G \times E \to G$ and taking the second coordinate. Conversely, a function satisfying equation (4.9) defines the second coordinate of a section $G \to G \times E$ which descends via the quotient map $Q : G \times E \to E(\sigma)$ since it is $\Gamma \times K$ invariant.

Now we have the following result.

**Proposition 4.13.** Let $f : M \to E(\sigma)$ be a section with corresponding function $f^* : G \to E$. Let $X$ be a continuous vector field on $M$ with a horizontal lift $X^*$ to $G$. Then $\nabla_X f$ is the section which corresponds to the function $X^* f^* = \mathcal{L}_{X^*} f^*$, where $\mathcal{L}_{X^*}$ denotes the Lie derivative.

This same procedure works for all of the bundles in which we have interest. Note that the two quotient mappings $Q, Q'$ give rise to different classes of $E$-valued functions, so we get two distinct connections, $\nabla$ on $E(\sigma)$ and $\nabla'$ on $E(\sigma')$. It follows from elementary properties of the Lie derivative that these connections are compatible with our Euclidean structure.

As described in section 4.2 these connections (together with the canonical metric) induce the standard differential operators: exterior differential, Laplacian, and their adjoints. We use the standard notations to refer to the objects associated to $\nabla$ and we use primes to refer to the objects associated to $\nabla'$. The following result is immediate from the definitions.

**Lemma 4.14.** The exterior differential $d$ on $\mathcal{E}(\pi)$-valued $p$-forms associated to $\nabla$ as in section 4.2 is the usual de Rham cohomology coboundary map.

**4.4.4. An explicit formula.** We now apply Corollary 4.7 to $\mathcal{E}(\pi)$ twice, once for each pair of connection and Laplacian. Taking the difference of the resulting equations and recalling the notation of section 4.2, we obtain the following result.

**Proposition 4.15.** For an $\mathcal{E}(\pi)$-valued $p$-form $\sigma$ on $M$ we have that

\[ (\triangle \sigma, \sigma) - (\triangle' \sigma, \sigma) = (\nabla^* \nabla \sigma, \sigma) - ((\nabla')^* \nabla' \sigma, \sigma) + (S(\sigma) - S'(\sigma), \sigma). \]

Our goal is now to compute workable expressions for the terms on the right hand side of this equation in terms of the group $G$ and the representation $\pi$. 
Let \( \eta \) be an \( E(\pi) \)-valued \( p \)-form on \( M \). We lift \( \eta \) to the corresponding \( E \)-valued \( p \)-form \( \eta^* \) on \( G \) through the quotient map \( Q' \). Thus \( \eta^* \) satisfies the following properties:

- **P–I** \( \eta^* \circ L_\gamma = (\text{Id}_F \otimes \pi_2(\gamma))\eta^* \) for \( \gamma \in \Gamma \), where \( L_\gamma \) denotes left multiplication,
- **P–II** \( i(X)\eta^* = 0 \) for \( X \in \mathfrak{k} \),
- **P–III** \( \eta^* \circ R_k = (\pi_1(k)^{-1} \otimes \text{Id}_H)\eta^* \), for \( k \in K \), where \( R_k \) denotes right multiplication, and
- **P–IV** \( \mathcal{L}_X\eta^* = (-\pi_1(X) \otimes \text{Id}_H)\eta^* \) for \( X \in \mathfrak{k} \), where \( \mathcal{L}_X \) denotes the Lie derivative with respect to \( X \).

The fourth property is the linearization of the \( K \)-equivariance in property P–III.

As usual, we identify elements of the Lie algebra \( \mathfrak{g} \) with left-invariant vector fields on \( G \). Also, we fix a basis of \( \mathfrak{g} \) as follows. Let \( N = \dim M = \dim p \) and choose a basis \( X_1, \ldots, X_N \) of \( p \) which is orthonormal with respect to the restriction of the Killing form of \( \mathfrak{g} \) to \( p \). Let \( m = \dim \mathfrak{k} \), \( n = N + m \) and \( X_{N+1}, \ldots, X_n \) be an orthonormal basis of \( \mathfrak{k} \) with respect to the negative of the restriction of the Killing form of \( \mathfrak{g} \) to \( \mathfrak{k} \).

Let \( \omega^1, \ldots, \omega^n \) be the 1-forms on \( G \) which are dual to the left invariant vector fields \( X_1, \ldots, X_n \). These are determined by the conditions \( \omega^j(X_i) = \delta^j_i \). Note that this means that for \( 1 \leq k, j \leq N \), we have \( X_k \omega^j = 0 \). We choose our locally symmetric metric on \( M \) to be normalized so that \( G \rightarrow M \) is a Riemannian submersion for the metric \( \sum_{i=1}^n \omega^i \omega^i \) on \( G \). The induced volume form of \( K \subseteq G \) is given by \( \omega^{N+1} \wedge \cdots \wedge \omega^{N+m} \), where the forms are restricted to the compact Lie group \( K \). Also choose a countable complete orthonormal spanning set \( \{v_\alpha\}_{\alpha \in A} \) for our Hilbert space \( \mathcal{H} \).

We write our \( E \)-valued \( p \)-form \( \eta^* \) on \( G \) as

\[
\eta^* = \sum_\alpha \eta^\alpha \otimes v_\alpha,
\]

where the \( v_\alpha \in \mathcal{H} \) and the \( \eta^\alpha \) are \( F \)-valued \( p \)-forms on \( G \).

In fact, we identify \( \eta^* \) with the system of functions \( \eta^\alpha_{i_1, \ldots, i_p} \) on \( G \) (with values in \( E \)) determined by

\[
\eta^*(X_{i_1}, \ldots, X_{i_p}) = \sum_\alpha \eta^\alpha_{i_1, \ldots, i_p} \otimes v_\alpha.
\]

Note that we may now write

\[
\eta^* = \sum_{\alpha \in A} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \otimes \eta^\alpha_{i_1, \ldots, i_p} \otimes v_\alpha.
\]

We compute an expression for the terms on the right hand side of Corollary 4.15 in terms of the functions \( \eta^\alpha_{i_1, \ldots, i_p} \).
First, we shall need an expression for the inner product of two forms. Suppose that we have two $\mathcal{E}(\pi)$-valued $p$-forms $\eta, \theta$ on $M$ with lifts

$$
\eta^* = \sum_{\alpha \in A} \sum_{i_1 < \cdots < i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \otimes \eta^\alpha_{i_1, \ldots, i_p} \otimes v_\alpha
$$

$$
\theta^* = \sum_{\beta \in A} \sum_{j_1 < \cdots < j_p} \omega^{j_1} \wedge \cdots \wedge \omega^{j_p} \otimes \theta^\beta_{j_1, \ldots, j_p} \otimes v_\beta
$$

Let $c^{-1}$ be the Haar volume of the maximal compact subgroup $K$ of $G$ and let $D \subset G$ be a fundamental domain for the action of $\Gamma$ on $G$ by left multiplication. We have the following modification of a result of [25].

**Proposition 4.16.** For $\mathcal{E}(\pi)$-valued $p$-forms $\eta, \theta$ as above, their inner product is

$$
(\eta, \theta) = \frac{c}{p!} \sum_{\alpha \in A} \sum_{i_1, \ldots, i_p} \int_D (\eta^\alpha_{i_1, \ldots, i_p}, \theta^\alpha_{i_1, \ldots, i_p})_E \, dvol_G.
$$

**Proof.** This argument should be compared with [25, Proposition 5.1].

Recall the quotient mappings $q : G \to M$ and $Q' : G \times E \to E(\sigma) = \mathcal{E}(\pi)$. For $y \in G$, let $Q'_y(v)$ be the point $Q'(y, v)$ in $\mathcal{E}(\pi)$. Then $v \mapsto Q'_y(v)$ is a linear isomorphism of $E$ onto the fiber $\mathcal{E}(\pi)_{q(y)}$. So for each point $y \in G$ and any tangent vectors $Z_1, \ldots, Z_p$ to $G$ at $y$ we have

$$
\eta^*_y(Z_1, \ldots, Z_p) = Q'^{-1}_y(\eta_{q(y)}(q_*Z_1, \ldots, q_*Z_p)).
$$

In particular, this means that

$$
\sum_{\alpha \in A} \eta^\alpha_{1, \ldots, i_p} \otimes v_\alpha = Q'^{-1}_y(q_{q(y)}(q_*X_1, \ldots, q_*X_p)).
$$

Now fix $y \in G$ and let $x = q(y) \in M$. By our choices, the vectors

$$q_*X_1(y), \ldots, q_*X_N(y)$$

form an orthonormal basis of $T_xM$. In a sufficiently small neighborhood $U$ of $x$, we may choose an orthonormal basis of 1-forms $\theta^1, \ldots, \theta^N$ on $M$ such that

$$\theta^i_x(q_*X_j(y)) = \delta^i_j.$$

As in section 4.2, we express $\eta$ and $\theta$ on $U$ as

$$
\eta = \sum_{i_1 < \cdots < i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \otimes a_{i_1, \ldots, i_p},
$$

$$
\theta = \sum_{j_1 < \cdots < j_p} \omega^{j_1} \wedge \cdots \wedge \omega^{j_p} \otimes b_{j_1, \ldots, j_p},
$$

where $a_{i_1, \ldots, i_p}$ and $b_{j_1, \ldots, j_p}$ are sections of $\mathcal{E}(\pi)$. By the last paragraph, we see that

$$Q'_y \left( \sum_{\alpha \in A} \eta^\alpha_{i_1, \ldots, i_p} \otimes v_\alpha \right) = \eta_{q(y)}(q_*X_{i_1}, \ldots, q_*X_{i_p}) = a_{i_1, \ldots, i_p}(x),$$

where $y \in G$. By our choices, the vectors $X_1, \ldots, X_N$ form an orthonormal basis of $T_xM$. In a sufficiently small neighborhood $U$ of $x$, we may choose an orthonormal basis of 1-forms $\theta^1, \ldots, \theta^N$ on $M$ such that

$$\theta^i_x(q_*X_j(y)) = \delta^i_j.$$
and

\[ Q_y \left( \sum_{\beta \in A} \theta_{i_1,\ldots,i_p}^\beta \otimes v_\beta \right) = \theta_q(y)(q_*X_{i_1}, \ldots, q_*X_{i_p}) = b_{j_1,\ldots,j_p}(x). \]

By equation (4.12) we have

\[ \eta^i \wedge \# \theta(x) = \sum_{i_1 < \ldots < i_p} (a_{i_1,\ldots,i_p}(x), b_{j_1,\ldots,j_p}(x))^{\text{can}}(x), \]

and by the construction of the canonical metric in the fibers of \( E(\pi) \),

\[
\eta^i \wedge \# \theta(x) = \sum_{\alpha, \beta} \sum_{i_1 < \ldots < i_p} \left( \eta_{i_1,\ldots,i_p}^\alpha(y), \theta_{j_1,\ldots,j_p}^\beta(y) \right)_E(v_\alpha, v_\beta)_H
\]

\[ = \sum_{\alpha} \sum_{i_1 < \ldots < i_p} \left( \eta_{i_1,\ldots,i_p}^\alpha(y), \theta_{j_1,\ldots,j_p}^\alpha(y) \right)_E : \tag{4.10} \]

The reader should note the disappearance of terms coming from \( H \) in the last line of the computation and compare it to considerations in [26]. It is essential for all of our computations that no terms arise from the “flat, Hilbertian” part of either \( F \otimes H \) bundle over \( M \). The quotient map \( q : G \to M \) decomposes as \( q = q_1 \circ q_2 \) where \( q_2 : G \to \Gamma \backslash G \) and \( q_1 : \Gamma \backslash G \to M \). As noted as [25, Lemma 5.3], with our conventions, if \( f \) is a continuous function on \( M \), then \( \int_M f \, d\text{vol}_M = c \int_{\Gamma \backslash G} f \circ q_1 \, d\text{vol}_{\Gamma \backslash G} \). By lifting one more step up through \( q_2 \) we obtain that \( \int_M f \, d\text{vol}_M = c \int_D f \circ q \, d\text{vol}_G \). A simple application of this last equation to equation (4.10) completes the proof of the Proposition. \( \square \)

We now compute the terms on the right hand side of (4.15) in terms of the functions \( \eta_{i_1,\ldots,i_p}^\alpha \). We begin by working with the connection in \( E(\sigma') \).

Fix \( y \) in \( G \), and let \( x = q(y) = \Gamma y K \in M \). The vectors \( X_1(y), \ldots, X_N(y) \) project to an orthonormal basis \( \{ q_*X_i(y) \} \) of \( T_xM \). On a sufficiently small neighborhood \( U \) of \( x \) in \( M \), we extend this basis to a synchronous framing \( \{ Y_i \} \) over \( U \) such that \( Y_i(x) = q_*X_i(y), \nabla_{Y_i} Y_i(x) = 0, \) and \( (Y_j, Y_k) = \delta_{jk} \) on \( U \). Recall that such a framing is constructed by parallel transport of the frame \( \{ q_*X_i(y) \} \) radially outward from \( x \) along geodesics.

Let \( \theta^i \) be the basis for ordinary 1-forms on \( U \) which is dual to the framing \( \{ Y_i \} \). That is, at each point of \( U \), \( \theta^i(Y_j) = \delta^i_j \). Note that this means that the \( \theta^i \)’s form an orthonormal basis for the 1-forms on \( U \). As above, we decompose \( \eta \) in terms of this basis.

\[ \eta = \sum_{i_1 < \ldots < i_p} a_{i_1,\ldots,i_p} \otimes \theta^{i_1} \wedge \ldots \wedge \theta^{i_p}. \]
So we compute
\[ \nabla_{Y_k} \eta = \sum_{i_1 < \cdots < i_p} \nabla_{Y_k} a_{i_1, \ldots, i_p} \otimes \theta^{i_1} \wedge \cdots \wedge \theta^{i_p} \]
\[ + \sum_{i_1 < \cdots < i_p} \sum_{s=1}^p a_{i_1, \ldots, i_p} \otimes \theta^{i_1} \wedge \cdots \wedge \nabla_{Y_k} \theta^{i_s} \wedge \cdots \wedge \theta^{i_p}. \]

Note that because we are working in a synchronous frame, we have
\[ \nabla_{Y_k} \theta^i(x) = 0, \quad \text{for all } i \text{ and } k. \]

Hence the second sum vanishes and we need to compute \( \nabla_{Y_k} a_{i_1, \ldots, i_p}(x) \).

Since \( a_{i_1, \ldots, i_p} \) is a section of \( E(\sigma) \) we may use the Lie derivative description from section 4.4.3 Choose horizontal lifts \( Y^*_k \) of the vector fields \( Y_k \). By definition of the lifting procedure, the lift \( a^*_{i_1, \ldots, i_p} \) of \( a_{i_1, \ldots, i_p} \) is
\[ a^*_{i_1, \ldots, i_p} = Q_y^{-1}(\eta(Y_{i_1}, \ldots, Y_{i_p})) = \eta^*(Y^*_{i_1}, \ldots, Y^*_{i_p}). \]

So that
\[ a^*_{i_1, \ldots, i_p} = \sum_{\alpha \in A} \sum_{j_1 < \cdots < j_p} \eta^\alpha_{j_1, \ldots, j_p} \otimes v_\alpha \cdot \omega^{j_1}(Y^*_{i_1}) \cdots \omega^{j_p}(Y^*_{i_p}) \]

Hence the function \( G \to E \) corresponding to \( \nabla_{Y_k} a_{i_1, \ldots, i_p} \) is
\[ L(y) = Y^*_k(y) a^*_{i_1, \ldots, i_p}(y) \]
\[ = \sum_{\alpha \in A} \sum_{j_1 < \cdots < j_p} Y^*_k \eta^\alpha_{j_1, \ldots, j_p} \otimes v_\alpha \cdot \omega^{j_1}(Y^*_{i_1}) \cdots \omega^{j_p}(Y^*_{i_p}) \]
\[ + \sum_{\alpha \in A} \sum_{j_1 < \cdots < j_p} \sum_{s=1}^p \eta^\alpha_{j_1, \ldots, j_p} \otimes v_\alpha \cdot \omega^{j_1}(Y^*_{i_1}) \cdots \{Y^*_k(\omega^{j_s}(Y^*_{i_s}))\} \cdots \omega^{j_p}(Y^*_{i_p}) \]

Now, by the definition of the Lie derivative, to compute \( Y^*_k(\omega^{j_s}(Y^*_{i_s})) \) we only need to know \( \omega^{j_s}(Y^*_{i_s}) \) along a curve \( \gamma(t) \) in \( G \) which passes through \( \gamma(0) = y \) in the direction \( \gamma'(0) = X_k(y) \). The assignment \( t \mapsto y \exp(tX_k(e)) \) is easily checked to be such a curve. In fact, this curve is horizontal since \( \gamma'(t) \in H_{\gamma(t)} \) for all \( t \). So by the definition of parallel translation, we see that
\[ Y^*_i(\gamma(t)) = X_i(\gamma(t)). \]

This gives that, at a point \( \gamma(t) \), \( \omega^{j_r}(Y^*_{i_r}) = \delta^{j_r}_{i_r} \) and hence along \( \gamma(t) \) we have
\[ Y^*_k(\omega^{j_s}(Y^*_{i_s})) = 0. \]

Using this and that \( Y^*_i(y) = X_i(y) \), we see that
\[ L(y) = X_k(y) a^*_{i_1, \ldots, i_p}(y) = \sum_{\alpha \in A} X_k \eta^\alpha_{i_1, \ldots, i_p} \otimes v_\alpha. \]

From this argument, we deduce the following lemma.
Lemma 4.17. $\left( \nabla'_{Y_k} \eta \right)_{i_1, \ldots, i_p}^\alpha (y) = X_k \eta_{i_1, \ldots, i_p}^\alpha (y)$

Our next step is to compute the adjoint of $\nabla'_{Y_k}$ at $y$. We need the following result of Weil, which is a form of integration by parts.

**Proposition 4.18** (Weil, [42]). Let $f$ be a smooth function on $\Gamma \backslash G$, then

$$\int_{\Gamma \backslash G} X_i f \, d\text{vol}_{\Gamma \backslash G} = 0.$$ 

Lifting this result to the fundamental domain $D$ for the $\Gamma$-action on $G$ we have:

\[
\left( \eta, \nabla'_{Y_k} \theta \right) = -\sum_{k=1}^{N} X_k \eta_{i_1, \ldots, i_p}^\alpha (y).
\]

Using these results together, we can now find an expression for one of the terms of the right hand side of Proposition 4.15.

**Proposition 4.20.** For an $E(\pi)$ valued $p$-form $\eta$,

$$\left( \nabla' \nabla' \eta \right)_{i_1, \ldots, i_p}^\alpha = -\sum_{k=1}^{N} X_k^2 \eta_{i_1, \ldots, i_p}^\alpha,$$

and hence

$$\left( \nabla' \nabla' \eta, \eta \right) = \frac{c}{p!} \sum_{\alpha \in A} \sum_{i_1, \ldots, i_p=1}^{N} \int_{D} \left( -\sum_{k=1}^{N} X_k^2 \eta_{i_1, \ldots, i_p}^\alpha, \eta_{i_1, \ldots, i_p}^\alpha \right) \, d\text{vol}_{G}.$$
Proof. The second equality follows directly from the first and Proposition 4.16. For the first note that at \( x \) we have

\[
\nabla'^\alpha \nabla' \eta(x) = -\sum_k \nabla'^\alpha \nabla'_{Y_k} \eta(x).
\]

This is a standard computation—see for example [33, page 73] for the case of ordinary forms, the proof for vector bundle valued forms is analogous. The proposition now follows by an application of Lemmas 4.17 and 4.19. \( \square \)

We make similar computations for the connection \( \nabla \) in \( E(\sigma) \). As much is the same, we highlight differences and state results.

The main difference is that to use the Lie derivative description of \( \nabla \) one must instead lift into the bundle \( E(\sigma) \), while the functions \( \eta_{i_1,\ldots,i_p}^\alpha \) are lifts into \( E(\sigma') \). We remedy this by using the isomorphism between the two bundles given by the map \( A : G \times E \to G \times E \) defined in Lemma 4.10. Horizontal lifts of vectors tangent to \( M \) are performed in exactly the same way, so no change is required on this point.

We pull functions back by \( A \), take their appropriate Lie derivatives and push forward by \( A \). Inspecting the definition of \( A \), we see that the relevant functions in the \( E(\sigma) \) lift are then \( y \mapsto \pi_1(y)\eta_{i_1,\ldots,i_p}^\alpha(y) \). We then take the Lie derivative

\[
X_k(\pi_1(y)\eta_{i_1,\ldots,i_p}^\alpha(y)) = (L_{X_k}(\pi_1))(y)\eta_{i_1,\ldots,i_p}^\alpha(y) + \pi_1(y)X_k\eta_{i_1,\ldots,i_p}^\alpha(y)
= \pi_1(y)\pi_1(X_k)\eta_{i_1,\ldots,i_p}^\alpha(y) + \pi_1(y)X_k\eta_{i_1,\ldots,i_p}^\alpha(y),
\]

where \( (L_{X_k}(\pi_1)) \) is the Lie derivative of \( \pi \) along \( X_k \) and \( \pi_1(X_k) \) denotes the image of \( X_k \in \mathfrak{p} \) by the differential of the homomorphism \( \pi_1 \) at the identity. Applying \( A \) again, we deduce the following formula.

**Lemma 4.21.** \( (\nabla'_{Y_k} \eta)_{i_1,\ldots,i_p}^\alpha(y) = (X_k + \pi_1(X_k))\eta_{i_1,\ldots,i_p}^\alpha(y) \)

The adjoint of \( \nabla'_{Y_k} \) is computed as before, using the additional observation that for \( X_k \in \mathfrak{p} \), the linear map \( \pi_1(X_k) \) is symmetric with respect to \( (\ ,\ )_F \). The corresponding formula is:

**Lemma 4.22.** \( (\nabla'_{Y_k}^* \eta)_{i_1,\ldots,i_p}^\alpha(y) = -(X_k - \pi_1(X_k))\eta_{i_1,\ldots,i_p}^\alpha(y) \)

Combining the last two lemmas, we have:

**Lemma 4.23.** For an \( \mathcal{E}(\pi) \)-valued \( p \)-form \( \eta \),

\[
(\nabla' \nabla \eta)_{i_1,\ldots,i_p}^\alpha = -\sum_{k=1}^N (X_k^2 - \pi(X_k)^2)\eta_{i_1,\ldots,i_p}^\alpha,
\]

and hence

\[
(\nabla' \nabla \eta, \eta) = \frac{c}{p!} \sum_{\alpha \in A_{i_1,\ldots,i_p}} \sum_{A_{i_1,\ldots,i_p}=1}^N \int_D \left( -\sum_{k=1}^N (X_k^2 - \pi(X_k)^2)\eta_{i_1,\ldots,i_p}^\alpha, \eta_{i_1,\ldots,i_p}^\alpha \right)_F dvol_G.
\]
Proof. This lemma is proved in the same way as Proposition 4.20, using Lemmas 4.21 and 4.22 in place of Lemmas 4.17 and 4.19.

Combining Lemmas 4.20 and 4.23 yields:

Lemma 4.24.

\[
(\nabla^* \nabla \eta, \eta) - ( (\nabla')^* \nabla' \eta, \eta ) = \frac{c}{p!} \sum_{\alpha \in A_{i_1, \ldots, i_p=1}} \sum_{k=1}^{N} \int_{D} \left( \sum_{k=1}^{N} \pi(X_k)^2 \eta_{i_1, \ldots, i_p}, \eta_{i_1, \ldots, i_p} \right) \, dvol_G.
\]

It remains only to compute the curvature term from Proposition 4.15. By definition, the curvature terms are zero if \( p = 0 \), so we assume \( p \geq 1 \). In the definition of the Ricci tensor \( S \) of a vector bundle, one must use the curvature tensor \( R^V \) of the connected bundle. The Leibniz property for \( R^V \) for vector fields \( X, Y, Y_1, \ldots, Y_p \) is

\[
(R^V(X,Y)\eta)(Y_1, \ldots, Y_p) = R^V(X,Y)(\eta(Y_1, \ldots, Y_p))
\]

\[
- \sum_{i=1}^{p} \eta(Y_1, \ldots, R^N(X,Y)Y_i, \ldots, Y_p).
\]

Note that the sum in the second term is independent of the connection in the bundle: it depends only on the structure of the Riemannian manifold. Hence, the difference of the corresponding terms in Proposition 4.15 is zero. Also, the \( \Gamma \)-bundle connection is flat by construction, so the first term in this case is zero. This leaves us only one term. For a point \( x \in M \) and vectors \( Y_1, \ldots, Y_p \) at \( x \) and an orthonormal basis \( e_i \) of \( T_x M \),

\[
(S(\eta) - S'(\eta))_x(Y_1, \ldots, Y_p) = \sum_{k=1}^{p} \sum_{s=1}^{n} (-1)^{k+1} R'(e_s, Y_k)(\eta(e_s, Y_1, \ldots, \hat{Y}_k, \ldots, Y_p)).
\]

Recall that the \( Y_i \)'s form an orthonormal basis in a neighborhood \( U \) of \( x \). By the definition of curvature as an iterated covariant derivative and Lemma 4.17 we see that

\[
(S(\eta) - S'(\eta))_{i_1, \ldots, i_p} = \sum_{k=1}^{p} \sum_{s=1}^{n} (-1)^{k+1} (-X_sX_{i_k} + X_{i_k}X_s) \eta_{s,i_1,\ldots,\hat{i_k},\ldots,i_p}^\alpha
\]

\[
= \sum_{k=1}^{p} \sum_{s=1}^{N} (-1)^{k+1} [X_{i_k}, X_s] \eta_{s,i_1,\ldots,\hat{i_k},\ldots,i_p}^\alpha
\]

By property P-IV of our forms, since \([X_h, X_{i_k}] \in \mathfrak{g}\), we obtain

\[
(S(\eta) - S'(\eta))_{i_1, \ldots, i_p} = \sum_{k=1}^{p} \sum_{s=1}^{N} (-1)^{k+1} [X_{i_k}, X_s] \pi_1([X_{i_k}, X_s]) \eta_{s,i_1,\ldots,\hat{i_k},\ldots,i_p}^\alpha
\]
Applying the formula for inner products of forms, we obtain

\[ \langle S(\eta) - S'(\eta), \eta \rangle = \frac{c}{p!} \sum_{\alpha \in A} \sum_{i_1, \ldots, i_p = 1}^N \int_D \left( \langle (S(\eta) - S'(\eta))_{i_1, \ldots, i_p}, \eta_{i_1, \ldots, i_p} \rangle_F \right) \, dvol_G. \]

Combining the last two equations and doing a small bit of re-indexing yields the last piece of Proposition 4.15.

**Lemma 4.25.** The final term of the right hand side of Proposition 4.15 is

\[ \langle S(\eta) - S'(\eta), \eta \rangle \]

\[ = \frac{c}{(p-1)!} \sum_{\alpha \in A} \sum_{i_1, \ldots, i_p = 1}^N \int_D \left( \sum_{s=1}^N \pi_1([X_h, X_s]) \eta_{i_1, \ldots, i_p}^\alpha, \eta_{i_1, \ldots, i_p}^\alpha \right) \, dvol_G. \]

Assembling the results of Corollary 4.7 and Lemmas 4.20, 4.23 and 4.25, we deduce our variant of the Bochner formula.

**Proposition 4.26.** Let \( p \geq 1 \). Let \( \eta \) be an \( E(\pi) \)-valued \( p \)-form on \( M \) with \( E(\sigma') \) lift

\[ \eta^* = \sum_{\alpha \in A} \sum_{i_1 < \cdots < i_p} \eta_{i_1, \ldots, i_p}^\alpha \otimes v_\alpha \otimes \omega_1 \wedge \cdots \wedge \omega_p \]

as above. Then

\[ \langle \Box \eta, \eta \rangle - \langle \Box' \eta, \eta \rangle = \]

\[ = \frac{c}{p!} \sum_{\alpha \in A} \sum_{i_1, \ldots, i_p = 1}^N \int_D \left( \sum_{k=1}^N \pi_1([X_h, X_s]) \eta_{i_1, \ldots, i_p}^\alpha, \eta_{i_1, \ldots, i_p}^\alpha \right) \, dvol_G \]

\[ + \frac{c}{(p-1)!} \sum_{\alpha \in A} \sum_{i_1, \ldots, i_p = 1}^N \int_D \left( \sum_{k=1}^N \pi_1([X_h, X_s]) \eta_{i_1, \ldots, i_p}^\alpha, \eta_{i_1, \ldots, i_p}^\alpha \right) \, dvol_G. \]

4.4.5. **Estimate on the Laplacian.** In this section, we use results of Raghunathan to complete the proof of Theorems 3.1 and 4.1. In fact, we obtain more general statements in both cases, which depend upon more detailed knowledge of the representation.

First, it will be useful to rearrange the result of Proposition 4.26 slightly. We have the following equivalent formulation.

\[ \langle \Box \eta, \eta \rangle - \langle \Box' \eta, \eta \rangle = \frac{c}{(p-1)!} \sum_{\alpha \in A} \int_D \sum_{i_1, \ldots, i_p = 1}^N \left( T^p_{\pi_1} \eta_{i_1, \ldots, i_p}^\alpha, \eta_{i_1, \ldots, i_p}^\alpha \right)_F \, dvol_G, \]

where the function \( T^p_{\pi_1} \eta_{i_1, \ldots, i_p}^\alpha : G \to F \) is defined by

\[ T^p_{\pi_1} \eta_{i_1, \ldots, i_p}^\alpha = \frac{1}{p} \sum_{k=1}^N \pi_1([X_k]) \eta_{i_1, \ldots, i_p}^\alpha + \sum_{k=1}^N \pi_1([X_k]) \eta_{i_1, \ldots, i_p}^\alpha. \]
We introduce some terminology which will bring us closer to the language in [29, 30, 32].

Given fixed $p \geq 1$, $y \in G$, $\alpha \in A$ we shall consider $\theta = \eta^\alpha_y$ as an element of the finite dimensional vector space $R_p := \bigwedge^p p^* \otimes F$. We define an inner product $I_p$ on $R_p$ as follows: for $\theta^1, \theta^2 \in R_p$, put

$$I_p(\theta^1, \theta^2) = \sum_{i_1, \ldots, i_p = 1}^N (\theta^1(X_{i_1}, \ldots, X_{i_p}), \theta^2(X_{i_1}, \ldots, X_{i_p}))_F.$$

Also, the mapping $T^p_{\pi_1} : R_p \to R_p$ defined by

$$T^p_{\pi_1} \theta(Y_1, \ldots, Y_p) = \sum_{k=1}^N \left\{ \frac{1}{p} \pi_1(X_k)^2 \theta(Y_1, \ldots, Y_p) + \pi_1([Y_1, X_k])\theta(X_k, Y_2, \ldots, Y_p) \right\}$$

is a linear endomorphism of $R_p$. It is not difficult to check that this is symmetric with respect to $I_p$, so all of its eigenvalues are real.

**Proposition 4.27.** To check the hypothesis of Proposition 4.8, it suffices to show that the quadratic form

$$\theta \to I_p(T^p_{\pi_1} \theta, \theta)$$

on $R_p$ is positive definite.

**Proof.** Since $R_p$ is finite dimensional, positive-definiteness is equivalent to having all the eigenvalues of $T^p_{\pi_1}$ positive. Then one can choose $C$ to be the minimal eigenvalue of $T^p_{\pi_1}$ and obtain $I_p(T^p_{\pi_1} \theta, \theta) \geq C \cdot I_p(\theta, \theta)$ for every $\theta \in R_p$. It is important to note that $C$ depends only upon $p$ and $\pi_1$.

For each $y \in G$ and for each $\alpha \in A$, we think of $\eta^\alpha_y$ as an element of $R_p$. Using the definitions of $T^p_{\pi_1}$ and $I_p$ along with equations 4.19 and 4.20 and Proposition 4.16 we deduce that

$$(\triangle \eta, \eta) \geq (\triangle \eta, \eta) - (\triangle' \eta, \eta)$$

$$= \frac{c}{(p-1)!} \sum_{\alpha \in A} \int_D \sum_{i_1, \ldots, i_p = 1}^N \left( T^p_{\pi_1} \eta^\alpha_{i_1, \ldots, i_p}(y), \eta^\alpha_{i_1, \ldots, i_p}(y) \right)_F dvol_G$$

$$= \frac{c}{(p-1)!} \sum_{\alpha \in A} \int_D I_p(T^p_{\pi_1} \eta^\alpha_y, \eta^\alpha_y) dvol_G$$

$$\geq C \cdot \frac{c}{(p-1)!} \sum_{\alpha \in A} \int_D I_p(\eta^\alpha_y, \eta^\alpha_y) dvol_G$$

$$= C \cdot p \cdot (\eta, \eta).$$

Relabel $Cp$ as $C$ to complete the proof. \(\square\)
Let $\mathfrak{g}_C$ be the complexification of $\mathfrak{g}$. For a representation $\pi$ of $\mathfrak{g}$ on a real vector space, $F$, we let $\pi_C$ be the extension of $\pi$ to a representation of $\mathfrak{g}_C$ on $F_C = F \otimes \mathbb{R} \mathbb{C}$. Denote by $\Lambda_\pi$ the highest weight of $\pi_C$.

If $\mathfrak{g} = \mathfrak{so}_0(n, 1)$ or $\mathfrak{su}(n, 1)$ we denote by $\tau^N$ the natural representation on $\mathbb{R}^{n+1}$ or $\mathbb{R}^{2(n+1)}$, respectively, and let $\mu_N$ be the highest weight of $\tau^N_C$.

**Theorem 4.28** (Raghunathan [29, Theorem 1’, page 106]). Let $\mathfrak{g}$ be a semisimple Lie algebra and $\pi$ a non-trivial irreducible representation on a real vector space $F$. Let $\mathfrak{g} = \sum_i \mathfrak{g}^i$ be the decomposition of $\mathfrak{g}$ into simple factors and let $\pi^i = \pi|_{\mathfrak{g}^i}$. Suppose that the following conditions hold:

- There is an $i$ such that $\mathfrak{g}^i$ is non-compact and $\pi^i$ is non-trivial.
- No pair $(\mathfrak{g}^i, \Lambda_{\pi^i})$ is of the form $(\mathfrak{so}_0(n, 1), m \cdot \mu_N)$ or $(\mathfrak{su}(n, 1), m \cdot \mu_N)$, where $m$ is an integer.

Then $T^1_\pi$ is positive definite.

Combining this with the results above yields the following strengthening of Theorem 3.1.

**Theorem 4.29.** Let $G$ be a connected, semisimple real Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \sum_i \mathfrak{g}^i$ be the decomposition of $\mathfrak{g}$ into simple factors, and let $\mathfrak{g}^i_C$ be the complexification of $\mathfrak{g}^i$.

Let $\pi_1 : G \to \text{GL}(F)$ be a finite dimensional representation on a real vector space, $F$. Denote by $\pi_1$ also the induced representation of $\mathfrak{g}$ on $F$. Let $\pi^i = \pi_1|_{\mathfrak{g}^i}$, and let $\Lambda_i$ be the highest weight of the extension of $\pi^i$ to a representation of $\mathfrak{g}^i_C$ on $F_C = F \otimes \mathbb{R} \mathbb{C}$.

Let $\Gamma$ be an irreducible cocompact lattice in $G$, and let $\pi_2$ be a non-trivial unitary representation of $\Gamma$ on a separable Hilbert space $\mathcal{H}$. Form the representation $\pi = \pi_1|_\Gamma \otimes \pi_2$.

Suppose that

- There is an $i$ such that $\mathfrak{g}^i$ is non-compact and $\pi^i$ is non-trivial.
- No pair $(\mathfrak{g}^i, \Lambda_{\pi^i})$ is of the form $(\mathfrak{so}_0(n, 1), m \cdot \mu_N)$ or $(\mathfrak{su}(n, 1), m \cdot \mu_N)$ for an integer $m$, where $\mu_N$ denotes the highest weight of the complexification of the natural representation on $\mathbb{R}^{n+1}$ or $\mathbb{R}^{2(n+1)}$, respectively.

Then $H^1(\Gamma, \pi) = 0$.

Note that in the course of the proof, we have also proved the following extension of Theorem 4.28.

**Theorem 4.30.** Assume the hypotheses and notation of Theorem 4.29. Also assume that $\Gamma$ is torsion free. Form the locally symmetric Riemannian manifold $M = \Gamma \backslash G/K$, where $K$ is a maximal compact subgroup of $G$, and form the vector bundle $\mathcal{E}(\pi)$ over $M$ associated to the principal $\Gamma$-bundle $G/K \to M$ and the representation $\pi$ of $\Gamma$ on $F \otimes \mathcal{H}$.
Then there exists a positive constant $C$ such that for any smooth $\mathcal{E}(\rho)$-valued differential 1-form $\eta$ on $M$ we have

$$(\Delta \eta, \eta) \geq C(\eta, \eta).$$

**On Pointwise Formulas:** Though we do not need it in our applications, we point out that the computations above yield pointwise formulas as well. The main point is to subtract the two versions of equation 4.4 rather than the two versions of equation 4.5. The first point is that the left hand sides of equation 4.4 cancel, since the Laplacians on functions defined by our two connections are equal. Once one observes that, it is easy to see that all of the computations in 4.2 and 4.3 yield pointwise formulas. This is explicit in the statements of most of the lemmas above and easy to check in the remaining cases.

4.4.6. Vanishing in higher degrees. We obtain results for vanishing in higher degrees in the case where $\pi_1$ is a representation on a complex vector space $F$ using the computations in 30–32. Note that none of the development in sections 4.1 through 4.4.5 depends upon whether $F$ is a real or complex space. The only change is that $R_p$ is a complex vector space and $I_p$ is a Hermitian product.

To state the result, we introduce some more notation. Let $\theta$ denote the Cartan involution of $g$ associated to $\mathfrak{k}$, as well as its extension to $\mathfrak{g}_C$. Let $\mathfrak{p}_C$ be the orthogonal complement of $\mathfrak{k}_C$ with respect to the Killing form $\kappa$ of $\mathfrak{g}_C$. Let $\mathfrak{h}_\mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$, and $\mathfrak{h} \supset \mathfrak{h}_\mathfrak{k}$ a Cartan subalgebra of $\mathfrak{g}$. Let $\mathfrak{h}_C$ (respectively $\mathfrak{h}_{\mathfrak{k}C}$) be the $\mathbb{C}$-span of $\mathfrak{h}$ (respectively $\mathfrak{h}_\mathfrak{k}$).

Let $\Delta$ be the root system of $\mathfrak{g}_C$ with respect to $\mathfrak{h}_C$. For $\alpha \in \Delta$, let $H_\alpha$ be the unique element of $\mathfrak{h}_C$ such that $\kappa(H_\alpha, H_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}_C$. Let $\mathfrak{h}_\mathfrak{k}^* = \sum_{\alpha \in \Delta} \mathbb{R}H_\alpha = i\mathfrak{h}_{\mathfrak{k}C} \oplus (\mathfrak{h}_C \oplus \mathfrak{p}_C)$. We partition the set of roots $\Delta$ into three subsets. Let

$$
A = \{ \alpha \in \Delta \mid \theta(E_\alpha) = E_\alpha \} \\
B = \{ \alpha \in \Delta \mid \theta(\alpha) \neq \alpha \} \\
C = \{ \alpha \in \Delta \mid \theta(E_\alpha) = -E_\alpha \},
$$

where $E_\alpha$ denotes a root vector corresponding to $\alpha$. Then we have that $\Delta = A \cup B \cup C$, $\theta$ stabilizes the sets $\mathfrak{h}_C$, $\mathfrak{h}_C \cup \mathfrak{p}_C$ and $B$, and $\theta(\alpha) = \alpha$ if $\alpha \in A \cup C$.

Following Raghunathan, we say that an ordering on the real dual of $\mathfrak{h}^*$ is admissible when it is described by the following process: Let $H_1, \ldots, H_m$ be an orthonormal basis of $\mathfrak{h}^*$ chosen so that the initial elements $H_1, \ldots, H_l$ form a basis of $i\mathfrak{h}_\mathfrak{k}$, and declare that $\alpha$ in the real dual of $\mathfrak{h}^*$ is positive if the first nonvanishing $\alpha H_i$ is positive. If $O$ is an admissible order, then for a finite subset $E$ of the dual of $\mathfrak{h}^*$ we shall write $E^+(O)$ for the subset of positive elements in $E$.

Now, for an irreducible representation $\pi$ of $G$, let $\Lambda_\pi(O)$ denote the highest weight of $\pi$ with respect to the order $O$. Let $\Sigma_2(O) = C^+(O) \cup \{ \alpha \in B^+(O) \mid \alpha > \theta(\alpha) \}$ and $\Sigma_\pi(O) = \{ \alpha \in \Sigma_2(O) \mid \kappa(\Lambda_\pi(O), \alpha) \neq 0 \}$. With all of this notation, we can now state the following result.
**Theorem 4.31** (Raghunathan [32, 29]). Let \( \pi \) be an irreducible finite dimensional representation of \( G \). Then if \( \Sigma_\pi(O) \) contains (strictly) more than \( q \) elements for every admissible \( O \), then the Hermitian quadratic form \( \eta \mapsto I_p(T^p_\pi \eta, \eta) \) is positive definite for \( p \leq q \).

The same line of argument as above allows us to deduce the following result.

**Theorem 4.32.** Assume the notations and hypotheses of Theorems 4.29 and 4.30, with the exceptions that \( F \) is a complex vector space and \( \pi_1 \) is a complex linear representation, and \( \Gamma \) is torsion free. If Raghunathan’s condition on admissible orders is satisfied, then for each \( 1 \leq p \leq q \) there is a constant \( C_p \) such that every \( E(\rho) \)-valued \( p \)-form \( \eta \) on \( M \) satisfies

\[
(\Delta \eta, \eta) \geq C_p(\eta, \eta).
\]

Furthermore, \( H^p(\Gamma, \rho) = 0 \) for \( 1 \leq p \leq q \). This last assertion holds even if \( \Gamma \) contains elements of finite order.

### 4.5. Direct Computation of the flat Laplacian.

As it will be useful for the discussion to follow in §5, we now give a more direct computation of the Laplacian \( \Delta \) associated to the flat connection \( \nabla \). The main point is that we may realize the action of the Laplacian as the action of the Casimir operator of our Lie algebra on the family of functions \( \eta_{\alpha_i}^{i_1, \ldots, i_p} \). To do so, we continue with the notations and setup of this section. In particular, we make use of Lemma 4.21.

Recall that the exterior differential \( d \) may be defined as the anti-symmetrization of the connection. So for our \( E(\rho) \)-valued \( p \)-form \( \eta \) on \( M \) and vector fields \( Z_1, \ldots, Z_{p+1} \) on \( M \) we have

\[
(d\eta)(Z_1, \ldots, Z_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} (\nabla_{Z_i} \eta)(Z_1, \ldots, \hat{Z}_i, \ldots, Z_{p+1}).
\]

So, in terms of the family of functions we have defined above, using Lemma 4.21 we see that

\[
(d\eta)_{i_1, \ldots, i_{p+1}}^{\alpha} = \sum_{k=1}^{p+1} (-1)^{k+1} (X_i + \pi_1(X_i)) \eta_{i_1, \ldots, i_{k-1}, k, \ldots, i_{p+1}}^{\alpha}.
\]

A simple argument (using Weil’s integration by parts (equation 4.11) and the fact that \( \pi_1(X) \) is symmetric with respect to the inner product on \( F \) for \( X \in \mathfrak{p} \) (Lemma 4.12)) computes the adjoint of \( d \) to be as follows.

**Lemma 4.33.** \((\delta\eta)_{i_1, \ldots, i_{p+1}}^{\alpha} = - \sum_{k=1}^{p+1} (X_i - \pi_1(X_i)) \eta_{i_1, \ldots, i_{k-1}, k, \ldots, i_{p+1}}^{\alpha} \).
Then we compute directly.

\[(\delta d\eta)^\alpha_{i_1,\ldots,i_p} = - \sum_{k=1}^{N} (X_k - \pi_1(X_k))(d\eta)_{k,i_1,\ldots,i_p}^\alpha\]

\[= - \sum_{k=1}^{N} (X_k - \pi_1(X_k))(X_k + \pi_1(X_k))\eta_{i_1,\ldots,i_p}^\alpha\]

\[- \sum_{k=1}^{N} \sum_{j=1}^{p} (-1)^j (X_k - \pi_1(X_k))(X_{ij} + \pi_1(X_{ij}))\eta_{k,i_1,\ldots,i_j,\ldots,i_p}^\alpha,\]

\[(d\delta\eta)^\alpha_{i_1,\ldots,i_p} = \sum_{j=1}^{p} (-1)^{j+1}(X_{ij} + \pi_1(X_{ij}))(d^\ast\eta)_{i_1,\ldots,i_j,\ldots,i_p}^\alpha\]

\[= \sum_{j=1}^{p} (-1)^{j+1}(X_{ij} + \pi_1(X_{ij}) \left\{ - \sum_{k=1}^{N} (X_k - \pi_1(X_k))\eta_{k,i_1,\ldots,i_j,\ldots,i_p}^\alpha \right\}\]

\[= \sum_{k=1}^{N} \sum_{j=1}^{p} (-1)^j (X_{ij} + \pi_1(X_{ij}))(X_k - \pi_1(X_k))\eta_{k,i_1,\ldots,i_j,\ldots,i_p}^\alpha,\]

We add these equations to compute the Laplacian.

\[(\triangle\eta)^\alpha_{i_1,\ldots,i_p} = - \sum_{k=1}^{N} (X_k^2 - \pi_1(X_k)^2)\eta_{i_1,\ldots,i_p}^\alpha\]

\[= \sum_{k=1}^{N} \sum_{j=1}^{p} (-1)^j ([X_{ij}, X_k] - \pi_1([X_{ij}, X_k]))\eta_{k,i_1,\ldots,i_j,\ldots,i_p}^\alpha,\]

(4.21)

where we have used the equality

\[(X_i + \pi_1(X_i))(X_k - \pi_1(X_k)) - (X_k - \pi_1(X_k))(X_i + \pi_1(X_i)) = [X_i, X_k] - \pi_1([X_i, X_k])\]

\[p\] times to simplify the expression.

We may then proceed in the same way as Matsushima and Murakami do [25, page 385] to write the Laplacian in a more algebraic form. Where Matsushima and Murakami use their equation (4.10), we use the equivalent statement which follows from property P-IV of our lifted form \(\eta^\ast\). This yields:

\[(\triangle\eta)^\alpha_{i_1,\ldots,i_p} = -C\eta_{i_1,\ldots,i_p}^\alpha + \pi_1(C)\eta_{i_1,\ldots,i_p}^\alpha,\]

where \(C = \sum_{k=1}^{N} X_k^2 - \sum_{a=N+1}^{n} X_a^2\) is the Casimir operator of \(\mathfrak{g}\) and \(\pi_1(C) = \sum_{k=1}^{N} \pi_1(X_k)^2 - \sum_{a=N+1}^{n} \pi_1(X_a)^2\) is the Casimir operator of the representation \(\pi_1\).

In terms of the \(E = F \otimes \mathcal{H}\)-valued functions \(\eta_{i_1,\ldots,i_p}\) on \(G\), we conclude that

(4.22)

\[(\triangle\eta)^\alpha_{i_1,\ldots,i_p} = -C\eta_{i_1,\ldots,i_p} + (\pi_1(C) \otimes \text{Id}_\mathcal{H})\eta_{i_1,\ldots,i_p}^\alpha,\]
5. Obstructions to the approach via relative Lie algebra cohomology

In this section, we outline briefly what would be required to pursue an approach to our results via relative Lie algebra cohomology (also called \((g, K)\)-cohomology). We first discuss direct approaches for connected groups, and then turn to using Matsushima’s formula for cocompact lattices.

5.1. Connected Groups. Given the fact, mentioned following Theorem 1.2, that vanishing of \(H^1(G, \pi \otimes \pi_2, F \otimes \mathcal{H})\) is known for \(\pi_2\) irreducible or admissible, it is perhaps surprising that one cannot deduce Theorem 1.2 from this. We describe the scheme when \(\pi_2\) is an infinite direct sum, the case of direct integrals is essentially the same. If we have \((\pi_2, \mathcal{H}) = (\oplus \pi_{2,i}, \oplus \mathcal{H}_i)\) then:

\[
H^1(G, \pi_1 \otimes \pi_2, F \otimes \mathcal{H}) = \bigoplus_i Z_1(G, \pi_1 \otimes \pi_{2,i}, F \otimes \mathcal{H}_i) / \bigoplus_i B_1(G, \pi_1 \otimes \pi_{2,i}, F \otimes \mathcal{H}_i),
\]

where we know that the natural map \(B_1(G, \pi_1 \otimes \pi_{2,i}, F \otimes \mathcal{H}_i) \to Z_1(G, \pi_1 \otimes \pi_{2,i}, F \otimes \mathcal{H}_i)\) has an inverse \(p_i\) for every \(i\). It is straightforward to check that vanishing of the group \(H^1(D, \pi_1 \otimes \pi_2, F \otimes \mathcal{H})\) is equivalent to a uniform bound on the norms of all \(p_i\).

The existing proofs that cohomology vanishes for \(\pi_2\) admissible use quite involved homological arguments first to translate the question to one of relative Lie algebra cohomology and then to prove vanishing of relative Lie algebra cohomology. It is not clear that the required bound can be produced by this method.

One can try to bound \(p_i\) a posteriori by direct computation, using that \(G\) has property \((T)\) and that \(\pi_1\) is fixed. This direction does not seem fruitful.

5.2. Lattices. We now discuss the possibility of a proof for cocompact lattices using (an extension of) Matsushima’s formula, a topic on which we are more optimistic. Though it has a different guise here than above, the problem is again one of uniformity of estimates. For this section let \(G\) be any semisimple Lie group with finite center. For \(\pi : \Gamma \to GL(F)\) any finite dimensional representation which almost extends to \(G\), Matsushima’s formula [24] says:

\[
H^k(\Gamma, F) = \bigoplus \pi n_{\pi} H^k(g, \mathfrak{g}, V_{\pi,0} \otimes F),
\]

where the sum is over the irreducible \(G\) representations \(\pi\) appearing in \(L^2(G/\Gamma)\) and \(V_{\pi,0}\) is the space of \(K\)-finite vectors in the representation space \(V_{\pi}\) for \(\pi\). The Hodge theorem and/or more formal variants are used in the course of the proof. In particular, the isomorphism in equation (5.1) above is deduced by computing cohomology by computing harmonic forms.

In proofs of Matsushima’s formula, there is a step at which there is no isomorphism at the level of chain complexes, but only an isomorphism between spaces of the harmonic forms in the two complexes. In geometric language, the transition is between computing cohomology using the complex of smooth forms and computing
cohomology using the complex of $L^2$ forms. Since $L^2$ harmonic forms are smooth, for finite dimensional vector bundles, the Hodge theorem implies that one can compute cohomology by computing the space of $L^2$ harmonic forms. The passage to $L^2$ forms is important since it allows one to bring to bear various results from representation theory in the computations. For a clear general exposition of Matsushima’s formula, see section 2 of Schmid’s article [34].

Since the Hodge theorem fails in general for infinite dimensional vector bundles, it is no longer possible in our context to compute cohomology using harmonic forms. However, we can produce a formula along the lines of Matsushima’s result which controls the allowable harmonic forms in our more general bundle.

Let $G$ be a semisimple Lie group, $\Gamma < G$ a cocompact lattice and $\pi$ a finite dimensional by unitary representation of $\Gamma$. To avoid discussion of direct integrals, we assume that the unitary representation $\pi_2$ is irreducible. Let $\mathcal{H}^k(\Gamma, \pi, F \otimes \mathcal{H})$ be the space of smooth harmonic $k$ forms in the bundle $\Gamma \backslash (G/K \times \pi F \otimes \mathcal{H})$. Following the proof of Matsushima’s formula (with equation 4.22 in place of Matsushima’s computation of the similar object), we obtain an isomorphism:

$$(5.2) \quad \mathcal{H}^k(\Gamma, \pi, F \otimes \mathcal{H}) = \bigoplus \eta H^k(g, \mathfrak{k}, V_\eta, 0 \otimes F),$$

where the sum is over the irreducible $G$ representations $\eta$ appearing in $I^G_\Gamma(\pi_2)$ and $V_{\eta,0}$ is the space of $K$-finite vectors in the representation space $V_\eta$ for $\eta$. The expression above is correct because $H^k(g, \mathfrak{k}, V_{\pi,0} \otimes F)$ can be computed as the cohomology of a finite dimensional complex where a Hodge theorem is available for purely formal reasons.

Combined with the theorem of Borel-Wallach, Zuckerman and Schmid (precisely, the statement of [34 Theorem 2.13]), equation 5.2 implies that $\mathcal{H}^1(\Gamma, \pi, F \otimes \mathcal{H}) = 0$ in the situation of Theorem 3.1, so it is possible to control the part of the cohomology which consists of harmonic forms.

It should be possible to give an alternate proof of Theorem 3.1 using an argument close to the one given by Mok in [26]. We briefly sketch the idea to point out the difficulties. Given a bundle where an estimate of the type proven in Theorem 4.8 fails, Mok’s idea is to take a sequence of forms with $< \triangle f_i, f_i > \to 0$ and $\|f_i\|_2 = 1$ and use a renormalizing and limiting argument to construct another bundle over $\Gamma \backslash G/K$ where there are non-vanishing harmonic forms. Mok does this for flat bundles associated to unitary representations, and shows that the bundle constructed in the limit has the same form. In order to have a result applicable in our setting, we would need to repeat Mok’s argument with a flat bundles associated to a finite dimensional by unitary representation and prove that the limiting representation is finite dimensional by unitary. We believe this is possible and hope to pursue it in a later paper, where we would also give a detailed account of equation 5.2 and it’s generalizations. These two results would imply better vanishing results for higher degree cohomology than those in §4.4.6. In fact, in combination with results of Vogan-Zuckerman, we expect
this will yield optimal vanishing theorems for higher degree cohomology of cocompact lattices \[^{33}\]. However, due to the failure of Lemma \[^{39}\] for \(H^k\) when \(k > 1\), this will not yield vanishing theorems for the ambient group.

\textbf{References}

1. Philippe Blanc, \textit{Sur la cohomologie continue des groupes localement compacts}, Ann. Sci. École Norm. Sup. (4) \textbf{12} (1979), no. 2, 137–168. MR \textbf{MR543215} (80k:22009)

2. A. Borel and N. Wallach, \textit{Continuous cohomology, discrete subgroups, and representations of reductive groups}, second ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, RI, 2000. MR \textbf{MR1721403} (2000j:22015)

3. Armand Borel, \textit{Compact Clifford-Klein forms of symmetric spaces}, Topology \textbf{2} (1963), 111–122. MR \textbf{MR0146301} (26 #3823)

4. Kenneth S. Brown, \textit{Cohomology of groups}, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. MR \textbf{MR1324339} (96a:22009)

5. Kevin Corlette, \textit{Archimedean superrigidity and hyperbolic geometry}, Ann. of Math. (2) \textbf{135} (1992), no. 1, 165–182. MR \textbf{MR1147961} (92m:57048)

6. Pierre de la Harpe and Alain Valette, \textit{La propriété (T) de Kazhdan pour les groupes localement compacts (avec un appendice de Marc Burger)}, Astérisque (1989), no. 175, With an appendix by M. Burger. MR \textbf{MR1023471} (90m:22001)

7. Jozef Dodziuk, \textit{Vanishing theorems for square-integrable harmonic forms}, Proc. Indian Acad. Sci. Math. Sci. \textbf{90} (1981), no. 1, 21–27. MR \textbf{MR653943} (83h:58006)

8. James Eells and Luc Lemaire, \textit{Selected topics in harmonic maps}, CBMS Regional Conference Series in Mathematics, vol. 50, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1983. MR \textbf{MR703510} (85g:58030)

9. David Fisher, \textit{First cohomology and local rigidity of group actions}, preprint.

10. David Fisher and Theron Hitchman, \textit{Harmonic maps to continuum products and cocycle superrigidity}, in preparation.

11. \textit{Local rigidity via cohomology vanishing}, in preparation.

12. David Fisher and Theron Hitchman, \textit{Co-cycle superrigidity and harmonic maps with infinite-dimensional targets}, Int. Math. Res. Not. (2006), 72405, 1–19. MR \textbf{MR2211160}

13. David Fisher and G. A. Margulis, \textit{Local rigidity for cocycles}, Surveys in differential geometry, Vol. VIII (Boston, MA, 2002), Surv. Differ. Geom., VIII, Int. Press, Somerville, MA, 2003, pp. 191–234. MR \textbf{MR2039990} (2004m:22032)

14. David Fisher and Gregory Margulis, \textit{Almost isometric actions, property (T), and local rigidity}, Invent. Math. \textbf{162} (2005), no. 1, 19–80. MR \textbf{MR2198325}

15. Mikhail Gromov and Richard Schoen, \textit{Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one}, Inst. Hautes Études Sci. Publ. Math. (1992), no. 76, 165–246. MR \textbf{MR1215595} (94e:58032)

16. Lars Hörmander, \textit{An introduction to complex analysis in several variables}, revised ed., North-Holland Publishing Co., Amsterdam, 1973, North-Holland Mathematical Library, Vol. 7. MR \textbf{MR0344507} (49 #9246)

17. Hiroyasu Izeki and Shin Nayatani, \textit{Combinatorial harmonic maps and discrete-group actions on Hadamard spaces}, Geom. Dedicata \textbf{114} (2005), 147–188. MR \textbf{MR2174098}

18. Jürgen Jost and Shing-Tung Yau, \textit{Harmonic maps and superrigidity}, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 245–280. MR \textbf{MR1216587} (94m:58060)
19. D. A. Každan, *On the connection of the dual space of a group with the structure of its closed subgroups*, Funkcional. Anal. i Priložen. 1 (1967), 71–74. MR MR0209390 (35 #288)
20. Shoshichi Kobayashi and Katsumi Nomizu, *Foundations of differential geometry. Vol. I*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1996, Reprint of the 1963 original, A Wiley-Interscience Publication. MR MR1393940 (97c:53001a)
21. G. A. Margulis, *Discrete groups of motions of manifolds of nonpositive curvature*, Proceedings of the International Congress of Mathematicians (Vancouver, B.C., 1974), Vol. 2, Canad. Math. Congress, Montreal, Que., 1975, pp. 21–34. MR MR0492072 (58 #11226)
22. ____, *Discrete subgroups of semisimple Lie groups*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 17, Springer-Verlag, Berlin, 1991. MR MR1090825 (92h:22021)
23. Yozô Matsushima, *On the first Betti number of compact quotient spaces of higher-dimensional symmetric spaces*, Ann. of Math. (2) 75 (1962), 312–330. MR MR0158406 (28 #1629)
24. ____, *A formula for the Betti numbers of compact locally symmetric Riemannian manifolds*, J. Differential Geometry 1 (1967), 99–109. MR MR0222908 (36 #5958)
25. Yozô Matsushima and Shingo Murakami, *On vector bundle valued harmonic forms and automorphic forms on symmetric riemannian manifolds*, Ann. of Math. (2) 78 (1963), 365–416. MR MR0153028 (27 #2997)
26. Ngaiming Mok, *Harmonic forms with values in locally constant Hilbert bundles*, Proceedings of the Conference in Honor of Jean-Pierre Kahane (Orsay, 1993), no. Special Issue, 1995, pp. 433–453. MR MR1364901 (97c:58008)
27. Ngaiming Mok, Yum Tong Siu, and Sai-Kee Yeung, *Geometric superrigidity*, Invent. Math. 113 (1993), no. 1, 57–83. MR MR1223224 (94h:53079)
28. Yann Ollivier, *A January 2005 invitation to random groups*, Ensaios Matemáticos [Mathematical Surveys], vol. 10, Sociedade Brasileira de Matemática, Rio de Janeiro, 2005. MR MR2205306
29. M. S. Raghunathan, *On the first cohomology of discrete subgroups of semisimple Lie groups*, Amer. J. Math. 87 (1965), 103–139. MR MR0173730 (37 #2898)
30. ____, *Vanishing theorems for cohomology groups associated to discrete subgroups of semisimple Lie groups*, Osaka J. Math. 3 (1966), 409–424; ibid. (2) 87 (1967), 279–304. MR MR0227313 (37 #2898)
31. ____, *Cohomology of arithmetic subgroups of algebraic groups. I, II*, Ann. of Math. (2) 86 (1967), 409–424; ibid. (2) 87 (1967), 279–304. MR MR0227313 (37 #2898)
32. ____, *Corrections to: “Vanishing theorems for cohomology groups associated to discrete subgroups of semisimple Lie groups” (Osaka J. Math. 3 (1966), 243–256)*, Osaka J. Math. 16 (1979), no. 1, 295–299. MR MR527032 (80d:22015)
33. Steven Rosenberg, *The Laplacian on a Riemannian manifold*, London Mathematical Society Student Texts, vol. 31, Cambridge University Press, Cambridge, 1997, An introduction to analysis on manifolds. MR MR1462892 (98k:58206)
34. Wilfried Schmid, *Vanishing theorems for Lie algebra cohomology and the cohomology of discrete subgroups of semisimple Lie groups*, Adv. in Math. 41 (1981), no. 1, 78–113. MR MR625335 (82h:17009)
35. Richard Schoen and Mu tao Wang, *A fixed point theorem on metric spaces of non-positive curvature*, preprint, 2001.
36. Atle Selberg, *On discontinuous groups in higher-dimensional symmetric spaces*, Contributions to function theory (internat. Colloq. Function Theory, Bombay, 1960), Tata Institute of Fundamental Research, Bombay, 1960, pp. 147–164. MR MR0130324 (24 #A188)
37. A. N. Starkov, *Vanishing of the first cohomologies for lattices in lie groups*, J. Lie Theory 12 (2002), no. 2, 449–460. MR MR1923777 (2003e:22006)
38. Alain Valette, *Group pairs with property (T), from arithmetic lattices*, Geom. Dedicata **112** (2005), 183–196. MR MR2163898 (2006d:22014)

39. David A. Vogan, Jr. and Gregg J. Zuckerman, *Unitary representations with nonzero cohomology*, Compositio Math. **53** (1984), no. 1, 51–90. MR MR762307 (86k:22040)

40. P. S. Wang, *On isolated points in the dual spaces of locally compact groups*, Math. Ann. **218** (1975), no. 1, 19–34. MR MR0384993 (52 #5863)

41. Frank W. Warner, *Foundations of differentiable manifolds and Lie groups*, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York, 1983, Corrected reprint of the 1971 edition. MR MR722297 (84k:58001)

42. André Weil, *On discrete subgroups of Lie groups. II*, Ann. of Math. (2) **75** (1962), 578–602. MR MR0137793 (25 #1242)

43. Robert J. Zimmer, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984. MR MR776417 (86j:22014)

44. Gregg J. Zuckerman, *Continuous cohomology and unitary representations of real reductive groups*, Ann. of Math. (2) **107** (1978), no. 3, 495–516. MR MR496844 (81c:22025)

45. A. Żuk, *Property (T) and Kazhdan constants for discrete groups*, Geom. Funct. Anal. **13** (2003), no. 3, 643–670. MR MR1995802 (2004m:20079)

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