ABSTRACT. Consideration here is a generalized \( \mu \)-type integrable equation, which can be regarded as a generalization to both the \( \mu \)-Camassa-Holm and modified \( \mu \)-Camassa-Holm equations. It is shown that the proposed equation is formally integrable with the Lax-pair and the bi-Hamiltonian structure and its scale limit is an integrable model of hydrodynamical systems describing short capillary-gravity waves. Local well-posedness of the Cauchy problem in the suitable Sobolev space is established by the viscosity method. Existence of peaked traveling-wave solutions and formation of singularities of solutions for the equation are investigated. It is found that the equation admits a single peaked soliton and multi-peakon solutions. The effects of varying \( \mu \)-Camassa-Holm and modified \( \mu \)-Camassa-Holm nonlocal nonlinearities on blow-up criteria and wave breaking are illustrated in detail. Our analysis relies on the method of characteristics and conserved quantities and is proceeded with a priori differential estimates.

Keywords: \( \mu \)-Camassa-Holm equation, modified \( \mu \)-Camassa-Holm equation, integrable system, blow-up, wave breaking, peakons.

AMS Subject Classification (2000): 35B30,35G25

1. INTRODUCTION

In this paper, we are concerned with the following new integrable partial differential equation

\[
m_t + k_1 \left( 2\mu(u)u_u - u_x^2 \right) + k_2 \left( 2mu_x + um_x \right) + \gamma u_x = 0,
\]

where \( u(t, x) \) is a function of time \( t \) and a single spatial variable \( x \), and

\[
m = \mu(u) - u_{xx}, \quad \mu(u) = \int_S u(t, x) \, dx,
\]

with \( S = \mathbb{R}/\mathbb{Z} \) which denotes the unit circle on \( \mathbb{R}^2 \). Eq.(1.1) is reduced to the \( \mu \)-Camassa-Holm (CH) equation [30]

\[
m_t + 2mu_x + um_x + \gamma u_x = 0,
\]

for \( k_1 = 0 \) and \( k_2 = 1 \), and the modified \( \mu \)-CH equation [42]

\[
m_t + \left( 2\mu(u)u - u_x^2 \right) + \gamma u_x = 0,
\]

for \( k_2 = 0 \) and \( k_1 = 1 \), respectively.

It is known that the CH equation [30] of the following form

\[
m_t + um_x + 2u_xm + \gamma u_x = 0, \quad \text{with} \quad m = u - u_{xx}
\]

was proposed as a model for the unidirectional propagation of the shallow water waves over a flat bottom (see also [15, 28]), with \( u(x, t) \) representing the height of the water's free surface in terms of non-dimensional variables. It is completely integrable with a bi-Hamiltonian structure and an infinite number of conservation laws [30]. It is of interest to note that the CH equation can also be derived by tri-Hamiltonian duality from the KdV equation (a number of additional examples of dual integrable systems derived applying the
method of tri-Hamiltonian duality can be found in [24, 39]). The CH equation has two remarkable features: existence of peakon and multi-peakons [1, 4, 5] (when \(\gamma = 0\)) and breaking waves, i.e., wave profile remains bounded while its slope becomes unbounded in finite time [9, 10, 11, 12, 13, 33]. Geometrically, the CH equation describes the geodesic flows on the Bott-Virasoro group for the case \(\gamma \neq 0\) [37, 45] and on the diffeomorphism group of the unit circle under \(H^1\) metric for the case \(\gamma = 0\) [31], respectively. Note that the two geometric descriptions are not analogous: for \(\gamma = 0\) (on the diffeomorphism group) the Riemannian exponential map is a local chart, but this is not the case for \(\gamma \neq 0\) (on the Bott-Virasoro group) cf. the discussion in [14]. The CH equation also arises from a non-stretching invariant planar curve flow in the centro-equiaffine geometry [7, 38]. Well-posedness and wave breaking of the CH equation were studied extensively, and many interesting results have been obtained, see [9, 11, 12, 13, 33], for example. The \(\mu\)-CH equation (1.3) was first introduced by Khesin, Lenells and Misiołek [30]. They verified that the \(\mu\)-CH equation is bi-Hamiltonian and admits cusped and smooth traveling wave solutions. In [32], it was shown that the \(\mu\)-CH equation also admits single peakon and multi-peakons. Interestingly, the \(\mu\)-CH equation describes a geodesic flow on diffeomorphism group over \(S\) with certain metric. Its well-posedness and blow up were investigated in [23, 30].

Another well-known integrable equation admitting peakons with quadratic nonlinearities is the Degasperis-Procesi (DP) equation [17], which takes the form

\[
m_t + um_x + 3u_xm = 0, \quad m = u - u_{xx}.
\]

(1.6)

It is regarded as a model for nonlinear shallow water dynamics, which can also be obtained from the governing equations for water waves [15]. Its asymptotic accuracy is the same as for the Camassa-Holm shallow water equation. More interestingly, it admits the shock peakons in both periodic [20] and non-periodic settings [35]. Wave breaking phenomena and global existence of solutions for the Degasperis-Procesi equation were investigated in [8, 18, 19, 20, 35, 36]. The \(\mu\)-version of the DP equation was introduced by Lenells, Mi-siolek and Tiğlay [32], which takes the form (1.6) with \(m\) replaced by \(m = \mu(u) - u_{xx}\).

Its integrability, well-posedness, blow up and existence of peakons were investigated in [23, 30].

It is noted that all nonlinear terms in the CH and DP equations are quadratic. So it is of great interest to find such integrable equations with cubic and higher-order nonlinear terms. The CH equation can be obtained by the tri-Hamiltonian duality approach from the bi-Hamiltonian structure of the KdV equation [39]. Similarly, the approach applied to the modified KdV equation then yields the so-called modified CH equation [39, 22, 40]

\[
m_t + \left((u^2 - u_x^2)m\right)_x = 0, \quad m = u - u_{xx},
\]

(1.7)

which has cubic nonlinearities, and is integrable with the Lax-pair and the bi-Hamiltonian structure. It was shown that the modified CH equation has a single peaked soliton and multi-peakons with a different character than those of the CH equation [27], and it also has new features of blow-up criterion and wave breaking mechanism. The \(\mu\)-version of the modified CH equation, called the modified \(\mu\)-CH equation

\[
m_t + \left((2u\mu(u) - u_x^2)m\right)_x = 0, \quad m = \mu(u) - u_{xx},
\]

(1.8)

was introduced in [42]. It is also formally integrable with the bi-Hamiltonian structure and the Lax-pair, and arises from a non-stretching planar curve flows in \(\mathbb{R}^2\) [42]. On the other hand, its local well-posedness, wave breaking, existence of peakons and their stability were discussed in [42, 34]. As an extension of both the CH and modified CH equations, an integrable equation with both quadratic and cubic nonlinearities has been introduced by
Fokas [22], which takes the form

\[ m_t + k_1 \left((u^2 - u_x^2)\right)_x + k_2 (2u_x m + um_x) = 0, \quad m = u - u_{xx}. \]  

(1.9)

It was shown by Qiao et al [41] that Eq. (1.9) is integrable with the Lax-pair and the bi-Hamiltonian structure. Its peaked solitons were also obtained in [41]. Indeed, Eq. (1.9) can be obtained by the tri-Hamiltonian duality approach from the bi-Hamiltonian structure of the Gardner equation

\[ m_t + k_1 u^2 u_x + k_2 u u_x = 0. \]  

(1.10)

It was noticed in [27] and [42] that the short-pulse equation as an approximation of the Maxwell equation [44],

\[ v_{xt} + \frac{1}{2} (v^2 v_x)_x + v = 0 \]  

(1.11)

is a scale limit of both the modified CH equation (1.7) and the modified \( \mu \)-CH (1.8) equation.

The goal of the present paper is to investigate whether or not the generalized \( \mu \)-version (1.1) with two different nonlocal nonlinearities has similar remarkable properties as the CH equation (1.5) and the modified CH equation (1.7) as well as their \( \mu \)-versions. As is well known, in order for persistent structures and properties to exist in such equations, there must be a delicate nonlinear/nonlocal balance, even in the absence of linear dispersion with \( \gamma = 0 \). Indeed, it is shown in Theorem 4.1 that Eq. (1.1) admits a single peaked soliton and multi-peakon solutions even the parameters \( k_1 \) and \( k_2 \) have a different sign.

On the other hand, it is found that those two parameters \( k_1 \) and \( k_2 \) corresponding to the modified \( \mu \)-CH cubic and \( \mu \)-CH quadratic nonlinearities respectively play an important role in the effects of varying nonlocal nonlinearities on the finite blow-up of solutions, or wave breaking. It is well known that wave breaking relies crucially on strong nonlinear and nonlocal dispersion. Indeed, in the case of the CH equation with the quadratic nonlinearity, the wave cannot break in finite time when the momentum potential \( m = (1 - \partial_x^2)u \) keeps positive initially [9]. The breaking wave, however, occurs for the modified CH equation with the cubic nonlinearity, even \( m \) is positive initially [27]. Those of key different features of wave breaking make our investigation on the wave breaking to the generalized \( \mu \)-version (1.1) more interesting but the analysis more challenging.

In view of the transport theory, the key issue to obtain the wave breaking result (Theorem 7.2) is to derive a priori differential estimate so that the slope \( k_1 u_x m + k_2 u_x \) is unbounded below. Rather, the nonlocal nature and mixed strong nonlinearities of the problem prohibit the use of classical blow-up approaches. To this end, we shall adopt the method of characteristics with conservative quantities and the new mechanism of a priori differential inequalities to control higher nonlocal nonlinearities. This new approach is expected to have more applications to deal with wave breaking of the other nonlinear dispersive model wave equations with higher nonlocal nonlinearities.

The outline of the paper is as follows. In Section 2, integrability of Eq. (1.1) is illuminated. Section 3 is devoted to some basic facts and results, which will be used in the subsequent sections. In Section 4, existence of the single peakon and multi-peakons of Eq. (1.1) are demonstrated. Local well-posedness in the Sobolev space to the Cauchy problem associated with Eq. (1.1) is then obtained in Section 5. Blow-up criterion for strong solutions is established in Section 6. Finally in Section 7, with a new blow-up mechanism, sufficient conditions for wave breaking of strong solutions in finite time are described in detail.
2. INTEGRABILITY AND CONSERVATION LAWS

In this section, we study integrability of Eq. (1.1). On one hand, Eq. (1.1) admits a bi-Hamiltonian structure, and it is formally integrable. Indeed, it can be written in the bi-Hamiltonian form

\[ m_t = J \frac{\delta H_1}{\delta m} = K \frac{\delta H_2}{\delta m}, \]

where

\[ J = -k_1 \partial_x m \partial_{x^{-1}} m \partial_x - k_2 (m \partial_x + \partial_x m) - \frac{1}{2} \gamma \partial_x, \quad \text{and} \quad K = -\partial A = \partial_x^3, \]

are compatible Hamiltonian operators, while

\[ H_1 = \frac{1}{2} \int_S m u \, dx, \quad \text{and} \quad H_2 = \int_S \left( \mu^2(u) u^2 + \mu(u) u u_x^2 - \frac{1}{12} u_x^4 \right) \, dx + k_2 \int_S (\mu(u) u + \frac{1}{2} u u_x^2) \, dx \]

are the corresponding Hamiltonian functionals. According to Magri’s theorem [38], the associated recursion operator

\[ R = J \cdot K - \frac{1}{2} \]

produces a hierarchy of commuting Hamiltonian functionals and bi-Hamiltonian flows.

On the other hand, Eq. (1.1) admits the following Lax-pair

\[ \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_x = U(m) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}_t = V(m, u, \lambda) \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \]

where \( U \) and \( V \) are given by

\[ U(m, \lambda) = \begin{pmatrix} 0 & \lambda m \\ -k_1 \lambda m - k_2 \lambda & 0 \end{pmatrix}, \]

and

\[ V(m, u, \lambda) = \begin{pmatrix} -\frac{1}{2} u_x & -\mu(u) - k_1 \lambda (2 \mu(u) u - u_x^2) m - k_2 \lambda u m \\ G & \frac{1}{2} k_2 u_x \end{pmatrix}, \]

for \( \gamma = 0 \) with

\[ G = k_2 \left( \frac{1}{2} \lambda + \lambda k_1 u \right) + \frac{1}{2} k_1 \mu(u) + k_2 \lambda (2 \mu(u) - u_x^2) m + k_1 k_2 \lambda (2 u \mu(u) - u_x^2 - u m), \]

and

\[ U(m, \lambda) = \begin{pmatrix} -\sqrt{-\frac{1}{2} \gamma \lambda} & \lambda m \\ -k_1 m - k_2 \lambda & \sqrt{-\frac{1}{2} \gamma \lambda} \end{pmatrix}, \]

and

\[ V(m, u, \lambda) = \begin{pmatrix} A & B \\ G & -A \end{pmatrix}, \]

for \( \gamma \neq 0 \) with

\[ A = \frac{1}{4} \sqrt{-2 \gamma} \left( \lambda^{-1} + 2 \lambda k_1 (2 u \mu(u) - u_x^2) + 2 \lambda k_2 u \right) - \frac{1}{2} k_2 u_x, \]

\[ B = -\mu \frac{2}{2} + \frac{1}{2} \sqrt{-2 \gamma} u_x - k_1 \lambda (2 u \mu(u) - u_x^2) m - \lambda k_2 u m. \]

Let us now consider a scale limit of Eq. (1.1). Applying the scaling transformation

\[ x \mapsto \epsilon x, \quad t \mapsto \epsilon^{-1} t, \quad u \mapsto \epsilon^2 u \]
to Eq.\((1.1)\) produces
\[
\begin{align*}
(e^2\mu(u) - u_{xx})_t + k_1((2e^2u\mu(u) - u_x^2)(e^2\mu(u) - u_{xx}))_x \\
+ k_2(2u_x(e^2\mu(u) - u_{xx}) + u(e^2\mu(u) - u_{xx}))_x + \gamma u_x = 0.
\end{align*}
\] (2.2)

Expanding
\[
u(t, x) = u_0(t, x) + \epsilon u_1(t, x) + \epsilon^2 u_2(t, x) + \cdots
\]
in powers of the small parameter \(\epsilon\), the leading order term \(u_0(t, x)\) satisfies
\[-u_{0,xx} + k_1(u_0^2 u_{0,xx})_x - k_2(u_{0,xx} u_{0,xx} + u_0 u_{0,xxx}) + \gamma u_{0,x} = 0.
\]

It follows that \(v = u_{0,x}\) satisfies the following integrable equation
\[v_x - k_1 v_x^2 v_{xx} + k_2(vv_{xx} + \frac{1}{2}v_x^2) - \gamma v = 0,
\]
which describes asymptotic dynamics of a short capillary-gravity wave \([21]\), where \(v(t, x)\) denotes the fluid velocity on the surface.

3. PRELIMINARIES

In this section, we are concerned with the following Cauchy problem of Eq.\((1.1)\) with \(\gamma = 0\)
\[
\begin{align*}
&\left\{ \begin{array}{ll}
m_t + k_1(2\mu(u)u - u_x^2)m_x + k_2(2mu_x + um_x) = 0, & t > 0, \quad x \in \mathbb{R}, \\
u(0, x) = u_0(x), & m = \mu(u) - u_{xx}, \quad x \in \mathbb{R}, \\
u(t, x + 1) = u(t, x), & t \geq 0, \quad x \in \mathbb{R}.
\end{array} \right.
\end{align*}
\] (3.1)

In the following, for a given Banach space \(Z\), we denote its norm by \(\| \cdot \|_Z\). Since all space of functions are over \(\mathbb{S}\), for simplicity, we drop \(\mathbb{S}\) in our notations of function spaces if there is no ambiguity. Various positive constants will be uniformly denoted by common letters \(c\) or \(C\), which are only allowed to depend on the initial data \(u_0(x)\).

The notions of strong (or classical) and weak solutions will be needed.

**Definition 3.1.** If \(u \in C([0, T]; H^s) \cap C^1([0, T]; H^{s-1})\) with \(s > \frac{5}{2}\) and some \(T > 0\) satisfies \((3.1)\), then \(u\) is called a strong solution of \((1.1)\) on \([0, T]\). If \(u(t, x)\) is a strong solution on \([0, T]\) for every \(T > 0\), then it is called a global strong solution.

Plugging \(m\) in terms of \(u\) into Eq.\((1.1)\) results in the following fully nonlinear partial differential equation:
\[
u_t + k_1 [(2u\mu(u) - \frac{1}{3}u_x^2)u_x + \partial_x A^{-1}(2\mu^2(u)u + \mu(u)u_x^2) + \frac{1}{3}u_x^3] + k_2 [uu_x + A^{-1}\partial_x (2u\mu(u) + \frac{1}{2}u_x^2)] = 0,
\] (3.2)
where \(A = \mu - \partial_x^2\) is a linear operator, defined by \(A(u) = \mu(u) - u_{xx}\), \(A^{-1}\) is its inverse operator. Recall that \([30]\)
\[u = A^{-1}m = g \ast m,
\] (3.3)
where \(g\) is the Green function of the operator \(A^{-1}\), given by
\[g(x) = \frac{1}{2}(x - \frac{1}{2})^2 + \frac{23}{24}.
\] (3.4)
Its derivative can be assigned to zero at \(x = 0\), so one has \([32]\)
\[g_x(x) \overset{\text{def}}{=} \begin{cases} 
0, & x = 0, \\
x - \frac{1}{2}, & 0 < x < 1.
\end{cases}
\] (3.5)

The above formulation allows us to define a weak solution as follows.
Definition 3.2. Given the initial data \( u_0 \in W^{1,3} \), a function \( u \in L^\infty([0, T], W^{1,3}) \) is said to be a weak solution to the Cauchy problem (3.1) if it satisfies the following identity:

\[
\int_0^T \int_S \left[ u \varphi_t + k_1 (u) u^2 \varphi_x + \frac{1}{2} u_3^2 \varphi - g_x \ast (2 u_2 (u) + \mu(u) u_x^2) \varphi \\
- \frac{1}{2} \mu(u_x^2) \varphi + k_2 \left( \frac{1}{2} u^2 \varphi_x - g_x \ast \left( 2 u \mu(u) + \frac{1}{2} u_x^2 \right) \varphi \right) \right] \, dx \, dt
\]

(3.6)

for any smooth test function \( \varphi(t, x) \in C^\infty_c([0, T) \times S) \). If \( u \) is a weak solution on \([0, T)\) for every \( T > 0 \), then it is called a global weak solution.

The local well-posedness to the Cauchy problem (3.1) will be established in the next section.

Denote

\[ \mu_0 = \mu(u(t)) = \int_S u(t, x) \, dx, \quad \mu_1 = \left( \int_S u_x^2(t, x) \, dx \right)^{\frac{1}{2}}. \]

(3.7)

It is easy to see from equation (3.8) and the conservation law \( H_1 \) that \( \mu(u) \) and \( \mu_1 \) are conserved. Furthermore, we have the following result.

Lemma 3.1. [23] Let \( u_0 \in H^s, s > 5/2, \) and \( u \in C([0, T); H^s) \cap C^1([0, T); H^{s-1}) \) is a solution of the Cauchy problem (3.1). Then the following inequality holds:

\[ \| u(t, \cdot) - \mu(u(t, \cdot)) \|_{L^\infty} \leq \frac{\sqrt{3}}{6} \mu_1. \]

(3.8)

Lemma 3.2. [29] Let \( r, t \) be real numbers such that \(-r < t \leq r\). Then

\[
\| fg \|_{H^r} \leq c \| f \|_{H^r} \| g \|_{H^r}, \quad \text{if} \quad r > 1/2,
\]

\[
\| fg \|_{H^{r+1/2}} \leq c \| f \|_{H^r} \| g \|_{H^r}, \quad \text{if} \quad r < 1/2,
\]

where \( c \) is a positive constant depending on \( r \) and \( t \).

Lemma 3.3. [2] [16] If \( r > 0 \), then \( H^r \cap L^\infty \) is a Banach algebra. Moreover

\[
\| fg \|_{H^r} \leq c (\| f \|_{L^\infty} \| g \|_{H^r} + \| f \|_{H^r} \| g \|_{L^\infty}),
\]

where \( c \) is a constant depending only on \( r \).

Lemma 3.4. [2] [16] If \( r > 0 \), then

\[
\| [\Lambda^r, f] g \|_{L^2} \leq c (\| \partial_x f \|_{L^\infty} \| \Lambda^{r-1} g \|_{L^2} + \| \Lambda^r f \|_{L^2} \| g \|_{L^\infty}),
\]

where \( c \) is a constant depending only on \( r \).

The following estimates for solutions to the one-dimensional transport equation have been used in [16, 26].

Lemma 3.5. [2] [16] Consider the one-dimensional linear transport equation

\[
\partial_t f + v \partial_x f = g, \quad f|_{t=0} = f_0.
\]

(3.9)

Let \( 0 \leq \sigma < 1 \), and suppose that

\[
f_0 \in H^\sigma, \quad g \in L^1([0, T]; H^\sigma),
\]

\[
v_x \in L^1([0, T]; L^\infty), \quad f \in L^\infty([0, T]; H^\sigma) \cap C([0, T]; S').
\]

Then \( f \in C([0, T]; H^\sigma) \). More precisely, there exists a constant \( C \) depending only on \( \sigma \) such that, for every \( 0 < t \leq T \),
\[ \|f(t)\|_{H^s} \leq e^{CV(t)} \left( \|f_0\|_{H^s} + C \int_0^t \|g(\tau)\|_{H^s} d\tau \right) \quad \text{with} \quad V(t) = \int_0^t \|\partial_x v(\tau)\|_{L^\infty} d\tau. \]

As for the periodic case, thanks to Lemma 3.9 in [6], we know that for every \( u \in H^1(S) \)
\[ \|u\|_2^2 \leq \|u\|_{H^1}^2 \leq 3 \|u\|_\mu^2 \]
where
\[ \|u\|_\mu^2 = ((\mu - \partial_x^2)u, u)_{L^2} = \mu^2(u) + \int_S u_x^2 dx, \]
which implies that the norm \( \| \cdot \|_\mu \) is equivalent to the norm \( \| \cdot \|_{H^1} \).

4. PEAKED SOLUTIONS

It is known that a remarkable property to the CH and \( \mu \)-CH equations is the existence of single and multi-peakons. In this section, we are concerned with the existence of single and multi-peakons to Eq. (1.1). Recall first that the single peakon and multi-peakons for \( \mu \)-CH equation and modified \( \mu \)-CH equation. Their single peakons are given respectively by [30]
\[ u(t, x) = \frac{12}{13} cg(x - ct), \quad (4.1) \]
and [42]
\[ u(t, x) = \frac{2\sqrt{3}c}{5} g(x - ct), \quad (4.2) \]
where
\[ g(x) = \frac{1}{2}(x - [x] - \frac{1}{2})^2 + \frac{23}{24} \]
with \([x]\) denoting the largest integer part of \( x \). Their multi-peakons are given by
\[ u(t, x) = \sum_{i=1}^N p_i(t) g(x - q^i(t)), \quad (4.3) \]
where \( p_i(t) \) and \( q^i(t) \) satisfy the following ODE system respectively for the \( \mu \)-CH equation [32]
\[ \dot{p}_i(t) = - \sum_{j=1}^N p_ip_jg_x(q^i - q^j), \]
\[ \dot{q}^i(t) = \sum_{j=1}^N p_jg(q^i(t) - q^j(t)), \quad i = 1, 2, \ldots, N, \]
and for the modified \( \mu \)-CH equation (1.1) [42]
\[ \dot{p}_i(t) = 0, \]
\[ \dot{q}^i(t) = \frac{1}{12} \left[ \sum_{j,k \neq i} (p_j + p_k)^2 + 25p_i^2 \right] + p_i \left[ \sum_{j \neq i} p_j \left( (q^i - q^j + \frac{1}{2} \lambda_{ij})^2 + \frac{49}{12} \right) \right] + \sum_{j < k, j,k \neq i} p_jp_k(q^j - q^k + \epsilon_{jk})^2, \]
where \( g_x \) is defined by (3.5), and
\[
\lambda_{ij} = \begin{cases} 
1, & i < j \\
-1, & i > j 
\end{cases},
\]
\[
\epsilon_{jk} = \begin{cases} 
1, & k - j \geq 2 \\
0, & k - j \leq 1 
\end{cases}.
\] (4.4)

Recall that the single peakon of the CH equation (1.5) and the modified CH equation (1.7) are given respectively by [4]
\[
u = ce^{-|x-ct|}
\]
and [27]
\[
u = \sqrt{\frac{3c}{2}}e^{-|x-ct|}.
\]
Their periodic peakons are given respectively by [10, 43]
\[
u = \frac{c}{\cosh\left(\frac{1}{2}\right)} \cosh \left(\frac{1}{2} - (x - ct) + [x - ct]\right)
\]
and [43]
\[
u = \sqrt{\frac{3c}{1 + 2\cosh^2(\frac{1}{2})}} \cosh \left(\frac{1}{2} - (x - ct) + [x - ct]\right).
\]
Their multi-peakons are given by [4, 5]
\[
u = \sum_{i=1}^{N} p_i(t) e^{-|x - q_i(t)|},
\]
where \( p_i(t) \) and \( q_i(t) \) satisfy the system respectively for the CH equation (1.5) with \( \gamma = 0 \) [5]
\[
\dot{p}_i = \sum_{j=1}^{N} p_i p_j \text{sign}(q_i - q_j) e^{-|q_i - q_j|},
\]
\[
\dot{q}_i = \sum_{j=1}^{N} p_j e^{-|q_i - q_j|},
\]
and the system for the modified CH equation (1.7) [27]
\[
\dot{p}_i = 0,
\]
\[
\dot{q}_i = \frac{2}{3} p_i^2 + 2 \sum_{j=1}^{N} p_i p_j e^{-|q_i - q_j|} + 4 \sum_{1 \leq j, k \leq N : j < i} p_k p_i e^{-|q_i - q_k|}.
\]

The existence of the single peakons of Eq. (1.1) is governed by the following result.

**Theorem 4.1.** For any \( c \geq -\frac{169k_2^2}{1200k_1} \), Eq. (1.1) admits the peaked periodic-one traveling wave solution \( u_c = \phi_c(\xi) \), \( \xi = x - ct \), where \( \phi_c(\xi) \) is given by
\[
\phi_c(\xi) = a\left[\frac{1}{2}(\xi - \frac{1}{2})^2 + \frac{23}{24}\right]
\]
with
\[
a = \begin{cases} 
-13k_2 \pm \sqrt{169k_2^2 + 1200k_1}, & k_1 \neq 0, \\
\frac{12c}{13k_2}, & k_1 = 0, \ k_2 \neq 0
\end{cases}.
\] (4.6)
for \( \xi \in [-1/2, 1/2]\) and \( \phi(\xi) \) is extended periodically to the real line.

**Proof.** In view of (4.5), we assume that the periodic peakon of (1.1) is given by

\[
    u_c(t, x) = a \left[ \frac{1}{b}(\xi - [\xi] - \frac{1}{2})^2 + \frac{23}{12} \right],
\]

where \( a \) is a constant. According to Definition 3.2, \( u_c \) satisfies the equation

\[
    \sum_{j=1}^{6} I_j := \int_0^T \int_S u_{c,t} \varphi dx dt + k_1 \int_0^T \int_S (2\mu(u_c)u_{c,x} - \frac{1}{3}u_{c,x}^3) \varphi dx dt
\]

\[
    + k_1 \int_0^T \int_S g_x * (\mu(u_c)(2\mu(u_c)u_c + u_{c,x}^2)) \varphi dx dt + \frac{1}{3} k_1 \int_0^T \int_S \mu(u_{c,x}^3) \varphi dx dt
\]

\[
    + k_2 \int_0^T \int_S g_x * (2\mu(u_c)u_c + \frac{1}{2}u_{c,x}^2) \varphi dx dt + k_2 \int_0^T \int_S u_{c,xx} \varphi dx dt = 0,
\]

for some \( T > 0 \) and every test function \( \varphi(t, x) \in C^\infty_c([0, T) \times S) \). For any \( x \in S \), one finds that

\[
    \mu(u_c) = a \int_0^{ct} \left[ \frac{1}{2}(x - ct + \frac{1}{2})^2 + \frac{23}{12} \right] dx + a \int_{ct}^1 \left[ \frac{1}{2}(x - ct - \frac{1}{2})^2 + \frac{23}{12} \right] dx = a,
\]

\[
    \mu(u_{c,x}^3) = a^3 \int_0^{ct} (x - ct + \frac{1}{2})^3 dx + a^3 \int_{ct}^1 (x - ct - \frac{1}{2})^3 dx = 0.
\]

To compute \( I_j, j = 1, \ldots, 6 \), we need to consider two possibilities: (i) \( x > ct \), and (ii) \( x \leq ct \). For \( x > ct \), we have

\[
    2\mu(u_c)u_c - \frac{1}{3}u_{c,x}^2 = \frac{4}{3}a^2(\xi - \frac{1}{2})^2 + \frac{23}{12}a^2.
\]

On the other hand,

\[
    g_x * (2\mu(u_c)u_c + u_{c,x}^2)
\]

\[
    = \int_S \left( x - y - [x - y] - \frac{1}{2} \right) \left( 2a^2(y - ct - [y - ct] - \frac{1}{2})^2 + \frac{23}{12}a^2 \right) dy
\]

\[
    = \int_0^{ct} \left( x - y - \frac{1}{2} \right) \left( 2a^2(y - ct + \frac{1}{2})^2 + \frac{23}{12}a^2 \right) dy
\]

\[
    + \int_x^{ct} \left( x - y - \frac{1}{2} \right) \left( 2a^2(y - ct - \frac{1}{2})^2 + \frac{23}{12}a^2 \right) dy
\]

\[
    + \int_x^1 \left( x - y + \frac{1}{2} \right) \left( 2a^2(y - ct - \frac{1}{2})^2 + \frac{23}{12}a^2 \right) dy = \frac{1}{3}a^2\xi(\xi - 1)(1 - 2\xi),
\]

\[
    g_x * (2\mu(u_c)u_c + \frac{1}{2}u_{c,x}^2)
\]

\[
    = \int_S \left( x - y - [x - y] - \frac{1}{2} \right) \left( \frac{3}{2}a^2(y - ct - [y - ct] - \frac{1}{2})^2 + \frac{23}{12}a^2 \right) dy
\]

\[
    = \int_0^{ct} \left( x - y - \frac{1}{2} \right) \left( \frac{3}{2}a^2(y - ct + \frac{1}{2})^2 + \frac{23}{12}a^2 \right) dy
\]

\[
    + \int_x^{ct} \left( x - y - \frac{1}{2} \right) \left( \frac{3}{2}a^2(y - ct - \frac{1}{2})^2 + \frac{23}{12}a^2 \right) dy
\]

\[
    + \int_x^1 \left( x - y + \frac{1}{2} \right) \left( \frac{3}{2}a^2(y - ct - \frac{1}{2})^2 + \frac{23}{12}a^2 \right) dy = \frac{1}{4}a^2\xi(\xi - 1)(1 - 2\xi),
\]
It follows that
\[
I_1 = \int_0^T \int_S u_{c,t} \varphi dx dt = -ca \int_0^T \int_S (\xi - \frac{1}{2}) \varphi(x,t) dx dt,
\]
\[
I_2 = k_1 a^3 \int_0^T \int_S \left[ \frac{2}{3} (\xi - \frac{1}{2})^3 + \frac{23}{24} (\xi - \frac{1}{2}) \right] \varphi(x,t) dx dt,
\]
\[
I_3 = \frac{1}{3} k_1 a^3 \int_0^T \int_S (-2\xi^3 + 3\xi^2 - \xi) \varphi(x,t) dx dt, \quad I_4 = 0,
\]
\[
I_5 = \frac{1}{4} k_2 a^2 \int_0^T \int_S (-2\xi^3 + 3\xi^2 - \xi) \varphi(x,t) dx dt,
\]
\[
I_6 = k_2 a^2 \int_0^T \int_S \left[ \frac{1}{2} (\xi - \frac{1}{2})^3 + \frac{23}{24} (\xi - \frac{1}{2}) \right] \varphi(x,t) dx dt.
\]
Plugging above expressions into (4.8), we deduce that for any \( \varphi(x,t) \in C^\infty_c([0,T] \times S) \)
\[
\sum_{j=1}^6 I_j = \int_0^T \int_S a \left( \frac{25}{12} k_1 a^2 + \frac{13}{12} k_2 a - c \right) (\xi - \frac{1}{2}) \varphi(x,t) dx dt.
\]
A similar computation yields for \( x \leq ct \) that
\[
2\mu(u_c) u_c - \frac{1}{2} u_{c,x}^2 = \frac{2}{3} a^2 (\xi + \frac{1}{2})^2 + \frac{23}{12} a^2,
\]
\[
2\mu(u_c) u_c + \frac{1}{2} u_{c,x}^2 = \frac{3}{2} a^2 (\xi + \frac{1}{2})^2 + \frac{23}{12} a^2,
\]
and
\[
g_x * (2\mu(u_c) u_c + u_{c,x}^2) = \frac{1}{3} a^2 \xi (\xi - 1)(1 - 2\xi),
\]
\[
g_x * (2\mu(u_c) u_c + \frac{1}{2} u_{c,x}^2) = \frac{1}{4} a^2 \xi (\xi + 1)(1 + 2\xi),
\]
This allows us to compute
\[
\sum_{j=1}^4 I_j = \int_0^T \int_S a \left( \frac{25}{12} k_1 a^2 - ca \right) (\xi - \frac{1}{2}) \varphi(x,t) dx dt,
\]
\[
I_5 = \frac{1}{4} k_2 a^2 \int_0^T \int_S (2\xi^3 + 3\xi^2 + \xi) \varphi(x,t) dx dt,
\]
\[
I_6 = k_2 a^2 \int_0^T \int_S \left[ \frac{1}{2} (\xi + \frac{1}{2})^2 + \frac{23}{24} (\xi + \frac{1}{2}) \right] \varphi(x,t) dx dt,
\]
whence we arrive at
\[
\sum_{j=1}^6 I_j = \int_0^T \int_S a \left( \frac{25}{12} k_1 a^2 + \frac{13}{12} k_2 a - c \right) (\xi + \frac{1}{2}) \varphi(x,t) dx dt.
\]
Since \( \varphi \) is arbitrary, both cases imply that \( a \) fulfills the equation
\[
\frac{25}{12} k_1 a^2 + \frac{13}{12} k_2 a - c = 0.
\]
Clearly, its solutions are given by (4.6). Thus the theorem is proved. \( \square \)
Furthermore, one can show that Eq. (4.1) admits the multi-peakons of the form (4.3), where \( p_i(t) \) and \( q^i(t) \), \( i = 1, 2, \ldots, N \), satisfy the following ODE system

\[
\begin{aligned}
\dot{p}_i + k_2 \sum_{j=1}^{N} p_i p_j (q^i - q^j - \frac{1}{2}) &= 0, \\
\dot{q}_i - k_1 \left[ \frac{1}{12} (23 \sum_{j,k \neq i} (p_j + p_k)^2 + 25p_i^2) \\
&- p_i \left( \sum_{j \neq i} p_j (q^i - q^j)^2 + \frac{1}{2} \lambda_{ij}^2 + \frac{49}{12} \right) \\
&- \sum_{j<k,j,k \neq i} p_j p_k (q^j - q^k + \epsilon_{jk})^2] \\
&- k_2 \sum_{j=1}^{N} p_j \left( \frac{1}{2} (q^i - q^j)^2 - \frac{1}{2} |q^i - q^j| + \frac{13}{12} \right) = 0,
\end{aligned}
\]

where \( \lambda_{ij} \) and \( \epsilon_{jk} \) are given by (4.4).

In particular, when \( N = 2 \), system (4.9) can be solved explicitly, which yields

\[
\begin{aligned}
p_1 &= \frac{ae^{b(\tau - t_0)}}{1 + e^{b(\tau - t_0)}}, \\
p_2 &= \frac{a}{1 + e^{b(\tau - t_0)}}, \\
q^1 &= \frac{-k_1 a^2}{b} \left( \frac{1}{12} + (\frac{1}{2} - a_1)^2 \right) \frac{1}{1 + e^{b(\tau - t_0)}} \\
&+ \frac{a}{12} (23k_1 a + 6a_1(a_1 - 1)k_2 + 13k_2)(t-t_0) \\
&+ \frac{a}{6b} (k_1 a - 3a_1(a_1 - 1)k_2) \ln(1 + e^{b(\tau - t_0)}) + c_1, \\
q^2 &= q^1 + a,
\end{aligned}
\]

where \( a, a_1, b > 0 \) and \( t_0 \) are some constants.

5. The local well-posedness

In this section, we shall discuss the local well-posedness of the Cauchy problem (3.1). Our result is the following theorem:

\textbf{Theorem 5.1.} Suppose that \( u_0 \in H^s(S) \) for some constant \( s > 5/2 \). Then there exists \( T > 0 \), which depends only on \( \|u_0\|_{H^s} \), such that problem \((3.1)\) has a unique solution \( u(t,x) \) in the space \( \dot{C}([0,T];H^s(S)) \cap \dot{C}^1([0,T];H^{s-1}(S)) \). Moreover, the solution \( u \) depends continuously on the initial data \( u_0 \) in the sense that the mapping of the initial data to the solution is continuous from the Sobolev space \( H^s \) to the space \( \dot{C}([0,T];H^s(S)) \cap \dot{C}^1([0,T];H^{s-1}(S)) \).

In the remaining part of this section, we will concentrate on proving this theorem. The proof is approached by the viscosity method, which has been applied to establish the local well-posedness for the KdV equation \((3)\) and the CH equation \((33)\).

Firstly, we consider the Cauchy problem for a regularized version of the first equation of (3.1),

\[
\begin{aligned}
m_t - \varepsilon m_{xxt} &= -k_1 ((2\mu(u)u - u_x^2)m)_x - k_2 (2mu_x + um_x), \quad t > 0, \quad x \in \mathbb{R}, \\
u(t,x+1) &= u(t,x), \quad t > 0, \quad x \in \mathbb{R}, \\
u(0,x) &= u_0(x) \in H^s, \quad x \in \mathbb{R}, \quad s \geq 1.
\end{aligned}
\]
Here $\varepsilon$ is a constant and $0 < \varepsilon < 1/16$. We start by inverting the linear differential operator on the left hand side. For any $0 < \varepsilon < 1/16$ and any $s$, the integral operator
\[
D = (1 - \varepsilon \partial_x^2)^{-1}(\mu - \partial_x^2)^{-1} : H^s \rightarrow H^{s+4}
\] defines a bounded linear operator on the indicated Sobolev spaces.

To prove the existence of a solution to the problem in (5.1), we use the operator (5.2) to both sides of the equation in (5.1) and then integrate the resulting equation with respect to time $t$. This leads to the following equation
\[
u(t, x) = u_0(x) - \int_0^t D \left\{ k_1 \left[ 2\mu^2(u)u_x - \mu(u)\partial_x^2(u^2) + \mu(u)\partial_x(u_x^2) + \frac{1}{3}\partial_x(u_x^3) \right] 
   + k_2 \left[ 2\mu(u)u_x - \frac{1}{2}\partial_x^2(u^2) + \frac{1}{2}\partial_x(u_x^2) \right] \right\} (\tau, x) \, d\tau.
\]
A standard application of the contraction mapping theorem leads to the following existence result which is stated without the detailed proof.

**Theorem 5.2.** For each initial data $u_0 \in H^s$ with $s \geq 1$, there exists a $T > 0$ depending only on the norm of $u_0$ in $H^s$ such that there corresponds a unique solution $u(t, x) \in C([0, T]; H^s)$ of the Cauchy problem (5.1) in the sense of distribution. If $s \geq 2$, the solution $u(t, x) \in C^\infty([0, \infty); H^s)$ exists for all time. In particular, when $s \geq 4$, the corresponding solution is a classical globally defined solution of (5.1).

The global existence result follows from the conservation law
\[
\mu^2(u)(t) + \int_R (u_x^2 + \varepsilon u_{xx}^2) \, dx = \mu^2(u)(0) + \int_R (u_0^2 + \varepsilon u_{0xx}^2) \, dx,
\]
admitted by (5.1) in its integral form.

Now we establish the a priori estimates for solutions of (5.1) using energy estimates.

**Theorem 5.3.** Suppose that for some $s \geq 4$, the function $u(t, x)$ is the solution of (5.1) corresponding to the initial data $u_0 \in H^s$. For any real number $q \in (1/2, s - 2]$, there exists a constant $c$ depending only on $q$ such that
\[
\int_S (\Lambda^{q+2}u)^2 \, dx \leq \int_S \left[ (\Lambda^{q+2}u_0)^2 + \varepsilon(\Lambda^{q+3}u_0)^2 \right] \, dx + c \int_0^t \left( \int_S (\Lambda^{q+2}u)^2 \, dx + 1 \right)^2 \, d\tau.
\]  

For any $q \in [0, s - 2]$, there exists a constant $c$ such that
\[
\|u_t\|_{H^{q+1}} \leq c\|u\|_{H^{q+2}}^2 \left( 1 + \|u\|_{H^{q+2}} \right). \tag{5.4}
\]

**Proof:** First of all, we recall that the operator $\Lambda = (1 - \partial_x^2)^{1/2}$ and $\|v\|_{H^s} = \|\Lambda^sv\|_{L^2}$. Note that $\mu(u)$ is free of $x$, then the equation in (5.1) is equivalent to the following equation:
\[
\mu_t(u) - u_{xxx} + \varepsilon u_{xxxxx} = -k_1(2\mu^2(u)u_x - 4\mu(u)u_xu_{xx} + 2u_xu_{xx}^2 + u_x^2u_{xxx} - 2\mu(u)uu_{xxx}) 
   - k_2(2\mu(u)u_x - 2u_xu_{xx} - uu_{xxx}).
\]
For any \( q \in (1/2, s - 2] \), applying \((\Lambda^q u)\Lambda^q\) to both sides of the above equation and integrating with respect to \( x \), we have the following estimate

\[
\frac{1}{2} \frac{d}{dt} \left( \mu^2(\Lambda^q u) + \|\Lambda^q u_x\|_{L^2}^2 + \epsilon \|u_{xx}\|_{H^q}^2 \right) = (2k_1 \mu(u) + k_2) \int_S \left( \frac{3}{2} \Lambda^q u_x \Lambda^q u^2_x + \Lambda^q u_x u^2_x - \Lambda^q u_x \Lambda^q u^2_x \right) dx
\]

Using the Cauchy-Schwartz inequality, \(|\mu(u)| \leq c\|u\|_{L^2} \leq c\|u\|_{H^{q+2}}\), we deduce that

\[
\frac{1}{2} \frac{d}{dt} \left( \mu^2(\Lambda^q u) + \|\Lambda^q u_x\|_{L^2}^2 + \epsilon \|u_{xx}\|_{H^q}^2 \right) \leq c(\|u\|_{H^{q+2}}^4 + \|u\|_{H^{q+2}}^3). \tag{5.5}
\]

Next, we differentiate the equation in (5.1) with regard to \( x \) and then obtain:

\[
-\frac{d}{dt} \left( \mu^2(\Lambda^q u) + \|\Lambda^q u_x\|_{L^2}^2 + \epsilon \|u_{xx}\|_{H^q}^2 \right) \equiv J_1 + J_2 + J_3, \tag{5.6}
\]

where

\[
J_1 = (4k_1 \mu(u) + 2k_2) \int_S \Lambda^q u_x \Lambda^q (u_x u_{xx}) dx
\]

\[
J_2 = (2k_1 \mu(u) + k_2) \int_S \left( \Lambda^{q+1} u_x \Lambda^{q+1} u^2_x - \Lambda^q u_x \Lambda^q u^2_x \right) dx,
\]

\[
J_3 = -k_1 \int_S \Lambda^q u_x \Lambda^q (2u_x u_{xx}^2 + u_{xx}^2) dx = k_1 \int_S \Lambda^q u_{xx} \Lambda^q (u_x^2 u_{xx}) dx
\]

Applying Lemma 3.4, the Cauchy-Schwartz inequality and observing the fact that \( H^q \) is a Banach algebra for \( q > 1/2 \), we obtain

\[
J_1 \leq c(1 + \|u\|_{H^{q+2}}^3) \|u\|_{H^{q+2}}^3 \leq c(\|u\|_{H^{q+2}}^4 + \|u\|_{H^{q+2}}^3),
\]

and

\[
J_2 \leq c(1 + \|u\|_{H^{q+2}}^3) \|u\|_{H^{q+2}}^3 \leq c(\|u\|_{H^{q+2}}^4 + \|u\|_{H^{q+2}}^3),
\]

and

\[
J_3 \leq c(1 + \|u\|_{H^{q+2}}^3) \|u\|_{H^{q+2}}^3 \leq c(\|u\|_{H^{q+2}}^4 + \|u\|_{H^{q+2}}^3),
\]

and

\[
J_3 \leq c(1 + \|u\|_{H^{q+2}}^3) \|u\|_{H^{q+2}}^3 \leq c(\|u\|_{H^{q+2}}^4 + \|u\|_{H^{q+2}}^3),
\]

Applying Lemma 3.4, the Cauchy-Schwartz inequality and observing the fact that \( H^q \) is a Banach algebra for \( q > 1/2 \), we obtain

\[
J_1 \leq c(1 + \|u\|_{H^{q+2}}^3) \|u\|_{H^{q+2}}^3 \leq c(\|u\|_{H^{q+2}}^4 + \|u\|_{H^{q+2}}^3),
\]
Collecting above expressions, we have

\[ J_2 \leq c(1 + \|u\|_{H^{q+2}}^2) (\|u\|_{H^{q+2}}^2 (\|u_x\|_{L^\infty} \|u_{xx}\|_{H^{q}} + \|u\|_{H^{q+1}} \|u_{xx}\|_{L^\infty}) + 3\|u\|_{H^{q+2}}^3), \]

\[ \leq c(\|u\|_{H^{q+2}}^2 + \|u\|_{H^{q+2}}^3), \]

\[ J_3 \leq c (\|u\|_{H^{q+1}} (2\|u_x\|_{L^\infty} \|u_{xx}\|_{H^{q}} + \|u\|_{H^{q+1}} \|u_{xx}\|_{L^\infty}) + 3\|u\|_{H^{q+2}}^4) \]

\[ \leq c(\|u\|_{H^{q+2}}^2 + \|u\|_{H^{q+2}}^3). \]

Collecting above expressions, we have

\[ \frac{1}{2} \frac{d}{dt} (\mu^2 (\Lambda^q u_x) + \|\Lambda^q u_{xx}\|_{L^2}^2 + \varepsilon \|u_{xxx}\|_{H^q}^2) \leq c(\|u\|_{H^{q+2}}^4 + \|u\|_{H^{q+2}}^3). \quad (5.7) \]

Whence from (5), 5 and 5.7, one finds that

\[ \frac{d}{dt} (\mu^2 (\Lambda^q u) + \|\Lambda^q u_x\|_{L^2}^2 + \|\Lambda^q u_{xx}\|_{L^2}^2 + \varepsilon \|u_{xx}\|_{H^q}^2 + \varepsilon \|u_{xxx}\|_{H^q}^2) \]

\[ \leq c(\|u\|_{H^{q+2}}^2 + \|u\|_{H^{q+2}}^3). \]

Integrating the above inequality from 0 to t yields

\[ \mu^2 (\Lambda^q u(t)) + \|\Lambda^q u_x(t)\|_{L^2}^2 + \|\Lambda^q u_{xx}(t)\|_{L^2}^2 + \varepsilon \|u_{xx}(t)\|_{H^q}^2 + \varepsilon \|u_{xxx}(t)\|_{H^q}^2 \]

\[ \leq \mu^2 (\Lambda^q u_0) + \|\Lambda^q u_{0,x}\|_{L^2}^2 + \|\Lambda^q u_{0,xx}\|_{L^2}^2 + \varepsilon \|u_{0,xx}\|_{H^q}^2 + \varepsilon \|u_{0,xxx}\|_{H^q}^2 \]

\[ + \int_0^t (1 + \|u(\tau)\|_{H^{q+2}}^2)^2 \, d\tau. \]

In view of the equivalence of the norms \( \| \cdot \|_\mu \) and \( \| \cdot \|_{H^1} \), we get the result of (5.3).

In order to estimate the norm of \( u_t \), applying the operator \( \Lambda^q u_t \Lambda^q \) to the equation in 5.1 and integrating the result with respect to \( x \) lead to

\[ \int_S \Lambda^q u_t \left( \Lambda^q (\mu - \partial_x^2) u_t + \varepsilon \Lambda^q u_{xxxxx} \right) \, dx = \mu^2 (\Lambda^q u_t) + \|\Lambda^q u_{xx}\|_{L^2}^2 + \varepsilon \|\Lambda^q u_{xxxt}\|_{L^2}^2, \]

\[ |4k_1 \mu(u) + 2k_2| \int_S \Lambda^q u_t \Lambda^q (u_{xx}u_{xx}) \, dx | \leq c(1 + \|u\|_{H^{q+1}}) \|u_t\|_{H^{q+1}} \|u\|_{H^{q+1}}, \]

\[ | -2 \mu(u) k_1 \mu(u) + k_2 | \int_S \Lambda^q u_t \Lambda^q (u_{xx}) \, dx | \leq c(1 + \|u\|_{H^{q+1}}) \|u_t\|_{H^{q+1}} \|u\|_{H^{q+1}}, \]

\[ | (2k_1 \mu(u) + k_2) | \int_S \Lambda^q u_t \Lambda^q (u_{xx}) \, dx | \]

\[ = | - (2k_1 \mu(u) + k_2) | \int_S \left( \Lambda^q u_{tx} \Lambda^q (u_{xx}) + \Lambda^q u_t \Lambda^q (u_{xx}) \right) \, dx | \]

\[ \leq c \|u_{tx}\|_{H^q} \left( \|u\|_{H^{q+2}}^3 + \|u\|_{H^{q+2}}^2 \right), \]

and

\[ | -k_1 \int_S \Lambda^q u_t \Lambda^q (2u_x u_{xx} + u_x^2 u_{xx}) \, dx | = | k_1 \int_S \Lambda^q u_t \Lambda^q (u_{xx}^2 u_{xx}) \, dx | \leq c \|u_{tx}\|_{H^q} \|u\|_{H^{q+2}}^3. \]

Similarly, in view of the equivalence of the norms \( \| \cdot \|_\mu \) and \( \| \cdot \|_{H^1} \), it follows that

\[ \|u_t\|_{H^{q+1}}^2 \leq \mu^2 (\Lambda^q u_t) + \|\Lambda^q u_{xx}\|_{L^2}^2 + \varepsilon \|\Lambda^q u_{xxxt}\|_{L^2}^2 \leq c \|u_t\|_{H^{q+1}} \left( \|u\|_{H^{q+2}}^3 + \|u\|_{H^{q+2}}^2 \right). \]

Applying the Cauchy inequality, we deduce that

\[ \|u_t\|_{H^{q+1}} \leq \frac{1}{2} \|u_t\|_{H^{q+1}}^2 + c^2 \left( \|u\|_{H^{q+2}}^3 + \|u\|_{H^{q+2}}^2 \right). \]
Thus, the above inequality implies that
\[ \|u_t\|_{H^{q+1}}^2 \leq C (\|u\|_{H^{q+2}}^2 + \|u\|_{H^{q+2}}^2)^2. \]

This completes the proof of this theorem. □

To show the existence of the solutions to the initial value problem (5.1), we regularize its initial data \( u_0 \). For any fixed real number \( s > 5/2 \), suppose that the function \( u_0 \in H^s \), and let \( u_0^\varepsilon \) be the convolution \( u_0^\varepsilon = \phi^\varepsilon \ast u_0 \) of the functions \( \phi^\varepsilon(x) = \varepsilon^{-1/16} \hat{\phi}(\varepsilon^{-1/16}x) \) such that the Fourier transform \( \hat{\phi} \) of \( \phi \) satisfies \( \hat{\phi} \in C_0^\infty \), \( \hat{\phi}(\xi) \geq 0 \) and \( \hat{\phi}(\xi) = 1 \) for any \( \xi \in (-1, 1) \). Then it follows from Theorem 5.3 that for each \( \varepsilon \) with \( 0 < \varepsilon < 1/16 \) the Cauchy problem

\[
\begin{align*}
\mu_t(u) - u_{xxt} + \varepsilon u_{xxxxxt} &= -k_1 ((2\mu(u)u - u_x^2)u_x) - k_2 (2mu_xu + um_x), \quad t > 0, \quad x \in \mathbb{R}, \\
u(0, x) &= u_0^\varepsilon(x), \quad x \in \mathbb{R},
\end{align*}
\]

which is equivalent to the following problem:

\[
\begin{align*}
\mu_t(u) - u_{xxt} &= - (2k_1 \mu(u)u + k_2 u_x)u_x + \frac{k_1}{3} (u_x^3 + \mu(u_x^3)) \\
&\quad - \partial_x(\mu - \partial_x^2)^{-1} [k_1(2\mu^2(u)u + \mu(u)x^2u_x^2) + k_2(2\mu(u)u + \frac{1}{2}u^2_x)] , \\
u(0, x) &= u_0^\varepsilon(x), \quad x \in \mathbb{R},
\end{align*}
\]

and has a unique solution \( u^\varepsilon(x, t) \in C^\infty(0, \infty); H^\infty) \). To show that \( u^\varepsilon \) converges to a solution of the problem (3.1), we first demonstrate the properties of the initial data \( u_0^\varepsilon \) in the following theorem. The proof is similar to that of Lemma 5 in [3].

**Lemma 5.1.** Under the above assumptions, there hold

\[
\begin{align*}
\|u_0^\varepsilon\|_{H^q} &\leq c, & q \leq s, \quad (5.9) \\
\|u_0^\varepsilon\|_{H^q} &\leq c\varepsilon^{(s-q)/16}, & q > s, \quad (5.10) \\
\|u^\varepsilon - u_0\|_{H^q} &\leq c\varepsilon^{(s-q)/16}, & q \leq s, \quad (5.11) \\
\|u_0^\varepsilon - u_0\|_{H^s} &\equiv o(1), \quad (5.12)
\end{align*}
\]

for any \( \varepsilon \) with \( 0 < \varepsilon < 1/16 \), where \( c \) is a constant independent of \( \varepsilon \).

Combining the estimates in Lemma 5.1 and the a priori estimates in Theorem 5.3 we shall evaluate norms of the function \( u^\varepsilon \) in the following theorem, which will be used to show the convergence of \( \{u^\varepsilon\} \).

**Lemma 5.2.** There exist constants \( c_1, c_2 \) and \( M \) such that the following inequalities hold for any \( \varepsilon \) sufficiently small and \( t < 1/(cM) \):

\[
\begin{align*}
\|u^\varepsilon\|_{L^s} &\leq \frac{c_1}{(1 - cMt)^{c_2}}, \\
\|u^\varepsilon\|_{L^{s+p}} &\leq \frac{c_1\varepsilon^{-p/16}}{(1 - cMt)^{c_2}}, & p > 0, \\
\|u_t^\varepsilon\|_{L^{s+p}} &\leq \frac{c_4\varepsilon^{-3(p+1)/16}}{(1 - cMt)^{c_3}}, & p > -2.
\end{align*}
\]

**Proof.** Choose a fixed number \( q = s - 2 \). It follows from (5.3) that

\[
\int_{\mathbb{R}} (\Lambda^s u^\varepsilon)^2 \, dx \leq \int_{\mathbb{R}} \left( (\Lambda^s u_0^\varepsilon)^2 + \varepsilon (\Lambda^s u_0^\varepsilon_x)^2 \right) \, dx + c \int_0^t \left( \int_{\mathbb{R}} (\Lambda^s u^\varepsilon)^2 \, dx + 1 \right)^2 \, d\tau.
\]
Then the following inequality
\[ \| u^\varepsilon \|_{H^s}^2 = \int_S (\Lambda^s u^\varepsilon)^2 \, dx \leq \frac{M_s + 1}{1 - ct(M_s + 1)} \leq \frac{M}{1 - cMt}, \]
holds for any \( t \in [0, 1/(cM)) \), where
\[ M_s = \int_S \left( (\Lambda^s u_0^\varepsilon)^2 + \varepsilon (\Lambda^s u_{0,x}^\varepsilon)^2 \right) \, dx, \quad \text{and} \quad M = 1 + M_s. \]

Let \( q = s + p - 2 \). In a similar way, applying Lemma 5.1 to (5.3), one may obtain the inequality
\[ \| u^\varepsilon \|_{H^{s+p}} \leq \frac{c_1 (1 + \varepsilon - p/8 + \varepsilon^{1-(p+1)/8})^{\frac{1}{2}}}{(1 - cM t)^{c_2}} \leq \frac{c_1 \varepsilon^{-p/16}}{(1 - cM t)^{c_3}}, \]
for some constant \( c_1 \). Then applying the above results and Lemma 5.1 to (5.4) with \( q = s + p - 1 \), we deduce that
\[ \| u^\varepsilon \|_{H^{s+p}} \leq \frac{c_4 \varepsilon^{-3(p+1)/16}}{(1 - cM t)^{c_3}}, \]
for some constant \( c_4 \). Thus this completes the proof of the theorem.

We now show that \( \{ u^\varepsilon \} \) is a Cauchy sequence. Let \( u^\varepsilon \) and \( u^\delta \) be solutions of (5.8), corresponding to the parameters \( \varepsilon \) and \( \delta \), respectively, with \( 0 < \varepsilon < \delta < 1/16 \), and let \( w = u^\varepsilon - u^\delta \), \( f = u^\varepsilon + u^\delta \). Then \( w \) satisfies the following problem
\[
\begin{cases}
  w_t - \varepsilon w_{xxt} + (\delta - \varepsilon) u^\varepsilon_{xxt} \\
  = -2k_1 \mu(w) w^\varepsilon u^\varepsilon_x - \left( 2k_1 \mu(u^\delta) + k_2 \right) \left( w u^\varepsilon_x + u^\delta w_x \right) \\
  + \frac{k_1}{3} w_x \left( (u^\varepsilon_x)^2 + u^\varepsilon w^\delta_x + (u^\delta_x)^2 \right) - \frac{k_1}{3} \mu \left( w_x \left( (u^\varepsilon_x)^2 + u^\varepsilon w^\delta_x + (u^\delta_x)^2 \right) \right) \\
  - \partial_x (\mu - \delta^2) \left\{ k_1 \left[ 2 \mu(w) \mu(f) u^\varepsilon + 2 \mu^2(u^\delta) w + \mu(w) (u^\varepsilon_x)^2 + \mu(u^\delta) w_x f_x \right] \\
  + k_2 \left[ 2 \mu(w) u^\varepsilon + 2 \mu(u^\delta) w + \frac{1}{2} w_x f_x \right] \right\}, \\
  w(0, x) = u_0^\varepsilon(x) - u_0^\delta(x), \quad x \in \mathbb{R}.
\end{cases}
\] (5.13)

Theorem 5.4. There exists \( T > 0 \), such that \( \{ u^\varepsilon \} \) is a Cauchy sequence in the space \( C([0, T]; H^s), s > 5/2 \).

Proof. For a constant \( q \) with \( 1/2 < q < \min\{1, s - 2\} \), multiplying \( \Lambda^q w \Lambda^q \) to both sides of the equation in (5.13) and then integrating with respect to \( x \) over \( S \), we obtain
\[
\int_S \Lambda^q w \left( \Lambda^q (w_t - \varepsilon w_{xxt}) \right) \, dx = \frac{1}{2} \frac{d}{dt} \left( \| \Lambda^q w \|_{L^2}^2 + \varepsilon \| \Lambda^q w_x \|_{L^2}^2 \right),
\]
\[
\int_S \Lambda^q w \Lambda^q u^\varepsilon_{xxt} \, dx = \int_S \Lambda^q w_x \Lambda^q u^\varepsilon_{xxt} \, dx \leq c \| u^\varepsilon \|_{H^{q+1}} \| w \|_{H^{q+1}}.
\]

Similar to the proof of (5.3), for the right hand side of the equation in (5.13) we obtain
\[
\left| \int_S \Lambda^q w \Lambda^q \left\{ \frac{k_1}{3} w_x \left( (u^\varepsilon_x)^2 + u^\varepsilon w^\delta_x + (u^\delta_x)^2 \right) - \frac{k_1}{3} \mu \left( w_x \left( (u^\varepsilon_x)^2 + u^\varepsilon w^\delta_x + (u^\delta_x)^2 \right) \right) \right\} \, dx \right|
\leq c \| w \|_{H^{q+1}}^2 \left( \| u^\varepsilon \|_{H^{q+1}}^2 + \| u^\delta \|_{H^{q+1}}^2 \right),
\]
and
\[
\left| \int_S \Lambda^q w \Lambda^q \left\{ \partial_x (\mu - \partial_x^2)^{-1} \left( k_1 \left[ 2\mu(w)\mu(f)u^\varepsilon + 2\mu^2(u^\delta)w + \mu(w)(u^\varepsilon)_x^2 + \mu(u^\delta)w_x f_x \right] \\
+ k_2 \left[ 2\mu(w)u^\varepsilon + 2\mu(u^\delta)w + \frac{1}{2} w_x f_x \right] \right) \right\} \right| \, dx \leq c \|w\|_{H^q} \left( \|u^\varepsilon\|_{H^{q+1}}^2 + \|u^\delta\|_{H^{q+1}}^2 + \|\varepsilon\|_{H^{q+1}} + \|\delta\|_{H^{q+1}} \right).
\]

Therefore, we deduce that for any \( \hat{T} \in (0, 1/(cM)) \), there is a constant \( c \) depending on \( \hat{T} \) such that
\[
\frac{1}{2} \frac{d}{dt} \left( \|\Lambda^q w\|_{L^2}^2 + \varepsilon \|\Lambda^q w_x\|_{L^2}^2 \right) \leq c \|w\|_{H^{q+1}}^2 \left( \|u^\varepsilon\|_{H^{q+1}}^2 + \|u^\delta\|_{H^{q+1}}^2 + \|\varepsilon\|_{H^{q+1}} + \|\delta\|_{H^{q+1}} \right). \tag{5.14}
\]

Then, differentiating the equation in (5.13) with respect to \( x \), applying \( \Lambda^q w_x \Lambda^q \) to both sides of the resulting equation and integrating with respect to \( x \) over \( S \), one finds that
\[
\left| \int_S \Lambda^q w_x \Lambda^q u^\delta_{xxt} \, dx \right| \leq c \|u^\delta\|_{H^{q+1}} \|w\|_{H^{q+1}},
\]
and
\[
\left| \int_S \Lambda^q w_x \Lambda^q \left\{ \partial_x^2 (\mu - \partial_x^2)^{-1} \left( k_1 \left[ 2\mu(w)\mu(f)u^\varepsilon + 2\mu^2(u^\delta)w + \mu(w)(u^\varepsilon)_x^2 + \mu(u^\delta)w_x f_x \right] \\
+ k_2 \left[ 2\mu(w)u^\varepsilon + 2\mu(u^\delta)w + \frac{1}{2} w_x f_x \right] \right) \right\} \right| \, dx \leq c \|w_x\|_{H^q} \left( \|u^\varepsilon\|_{H^{q+1}}^2 + \|u^\delta\|_{H^{q+1}}^2 + \|\varepsilon\|_{H^{q+1}} + \|\delta\|_{H^{q+1}} \right).
\]

For the rest terms of the right hand side, we just estimate one term
\[
\left| \int_S \Lambda^q w_x \Lambda^q \left( (u^\delta)_x^2 \right)_x \, dx \right| = \left| \int_S \Lambda^{q+1} \left( (u^\delta)_x^2 \right)_x \, dx \right| f_x + \left( u^\delta_x^2 \right) w_x + \left( u^\delta_x^2 \Lambda^{q+1} w_x \right) \, dx - \int_S \Lambda^q w \Lambda^q \left( (u^\delta)_x^2 \right)_x \, dx. \]
Then applying the Cauchy-Schwartz inequality and Lemma 5.4, we obtain
\[
\left| \int_S \Lambda^q w_x \Lambda^q \left( (u_x^\delta)^2 w_x \right)_x \ dx \right| \\
\leq c \|w\|_{H^{q+1}} \left( \|2u_x^\delta u_{xx}^\delta\|_{L^\infty} \|\Lambda^q w_x\|_{L^2} + \|\Lambda^{q+1}(u_x^\delta)^2\|_{L^2} \|w_x\|_{L^\infty} \right) \\
+ \|2u_x^\delta u_{xx}^\delta\|_{L^\infty} \|w\|_{H^{q+1}}^2 + c \|w\|_{H^{q+1}} \|u_x^\delta\|_{H^{q+1}}^2 \\
\leq c \|w\|_{H^{q+1}}^2 \|u_x^\delta\|_{H^{q+2}}^2.
\]
Therefore, one finds that for any $\hat{T} \in (0, 1/(cM))$, there is a constant $c$ depending on $\hat{T}$ such that
\[
\frac{1}{2} \frac{d}{dt} \left( \|\Lambda^q w_x\|_{L^2}^2 + \epsilon \|w_{xx}\|_{H^q}^2 \right) \\
\leq c \|w\|_{H^{q+1}}^2 \left( \|w_x^\epsilon\|_{H^{q+2}}^2 + \|u_x^\delta\|_{H^{q+2}}^2 + \|u_x^\epsilon\|_{H^{q+1}} + \|u_x^\delta\|_{H^{q+1}} \right) \\
+ (\delta - \epsilon) \|u_t^\delta\|_{H^{q+3}} \|w\|_{H^{q+1}},
\]
for any $t \in [0, \hat{T})$. Altogether, we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|\Lambda^q w_x\|_{L^2}^2 + \|\Lambda^q w_x\|_{L^2}^2 + \epsilon \|w_x\|_{H^q}^2 + \epsilon \|w_{xx}\|_{H^q}^2 \right) \\
\leq c \|w\|_{H^{q+1}}^2 \left( \|w_x^\epsilon\|_{H^{q+2}}^2 + \|u_x^\delta\|_{H^{q+2}}^2 + \|u_x^\epsilon\|_{H^{q+1}} + \|u_x^\delta\|_{H^{q+1}} \right) \\
+ (\delta - \epsilon) \|w\|_{H^{q+1}} \left( \|u_t^\delta\|_{H^{q+3}} + \|u_t^\delta\|_{H^{q+3}} \right).
\]
With Lemma 5.1 and Lemma 5.2 in hand and in view of the fact that $1/2 < q < \min\{1, s - 2\}$, integrating the above inequality with respect to $t$ leads to the estimate
\[
\|w\|_{H^{q+1}}^2 \leq 2 \|w_0\|_{H^{q+1}}^2 + 2 \epsilon \|w_0\|_{H^{q+2}}^2 + 2 c \int_0^t (\delta^\gamma \|w\|_{H^{q+1}} + \|w\|_{H^{q+1}}^2) \ d\tau,
\]
for any $t \in [0, \hat{T})$, where $\gamma = 1$ if $s \geq q + 3$, and $\gamma = (4 + 3s - 3q)/16$ if $s < q + 3$. It follows from Gronwall’s inequality and (5.11) in Lemma 5.1 that
\[
\|w\|_{H^{q+1}} \leq (2 \|w_0\|_{H^{q+1}}^2 + 2 \epsilon \|w_0\|_{H^{q+2}}^2)^{1/2} e^{c t} + \delta^\gamma (e^{c t} - 1) \\
\leq c \delta^{(s-q-1)/16} e^{c t} + \delta^\gamma (e^{c t} - 1)
\]
for some constant $c$ and $t \in [0, \hat{T})$.

Next, multiplying $\Lambda^{s-1} w \Lambda^{s-1}$ to both sides of the equation in (5.13) and then integrating with respect to $x$, following the way of the estimate (5.14), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left( \|\Lambda^{s-1} w\|_{L^2}^2 + \epsilon \|\Lambda^{s-1} w_x\|_{L^2}^2 \right) \\
\leq c \|w\|_{H^s}^2 \left( \|w_x^\epsilon\|_{H^s}^2 + \|u_x^\delta\|_{H^s}^2 + \|u_x^\epsilon\|_{H^s} + \|u_x^\delta\|_{H^s} \right) + (\delta - \epsilon) \|u_t^\delta\|_{H^s} \|w\|_{H^s}.
\]
In the following, differentiating the equation in (5.13) with respect to $x$, applying $\Lambda^{s-1} w_x \Lambda^{s-1}$ to both sides of the resulting equation and integrating with respect to $x$ over $S$, we note that
\[
\left| \int_S \Lambda^{s-1} w_x \Lambda^{s-1} \left( (u_x^\delta)^2 w_x \right)_x \ dx \right| \\
= \left| \int_S \Lambda^s \left( \Lambda^s, (u_x^\delta)^2 \right) w_x + (u_x^\delta)^2 \Lambda^s w_x \ dx \right| - \int_S \Lambda^{s-1} w \Lambda^{s-1} \left( (u_x^\delta)^2 w_x \right) \ dx \\
\leq c \left( \|w\|_{H^s} \|u_x^\delta\|_{H^s}^2 + \|w\|_{H^s} \|w\|_{H^{q+1}} \|u_x^\delta\|_{H^{q+1}}^2 \right).
\]

Therefore, following the way of the estimate in (5.15), we deduce that
\[
\frac{1}{2} \frac{d}{dt} \left( \| \Lambda^s w \|_{L^2}^2 + \varepsilon \| w \|_{H^{s+1}}^2 \right) \\
\leq c \| w \|_{H^s}^2 \left( \| w^\varepsilon \|_{H^s}^2 + \| w^\delta \|_{H^s}^2 + \| u^\varepsilon \|_{H^s} + \| u^\delta \|_{H^s} \right) \\
+ c \| w \|_{H^s} \| w \|_{H^{s+1}} \| u^\varepsilon \|_{H^s} \| u^\delta \|_{H^{s+1}} + (\delta - \varepsilon) \| u_t^\varepsilon \|_{H^{s+2}} \| w \|_{H^s},
\]
for some constant \( c \) and \( t \in [0, \bar{T}) \). All in all, it follows from the above estimates, and the inequality in (5.16) and Lemma 5.2 that there exists a constant \( c \) depending on the \( \bar{T} \in (0, 1/(cM)) \) such that
\[
\frac{1}{2} \frac{d}{dt} \left( \| w \|_{H^s}^2 + \varepsilon \| w \|_{H^{s+1}}^2 \right) \\
\leq c \| w \|_{H^s}^2 \left( \| u^\varepsilon \|_{H^s}^2 + \| u^\delta \|_{H^s}^2 + \| u^\varepsilon \|_{H^s} + \| u^\delta \|_{H^s} \right) \\
+ c \| w \|_{H^s} \| w \|_{H^{s+1}} \| u^\varepsilon \|_{H^s} \| u^\delta \|_{H^{s+1}} + (\delta - \varepsilon) \| u_t^\varepsilon \|_{H^{s+2}} \| w \|_{H^s} \\
\leq c \left( \delta^\alpha \| w \|_{H^s} + \| w \|_{H^s}^2 \right),
\]
where \( \alpha = \min \{7/16, (s - q - 2)/16 \} > 0 \). Therefore, integrating the above inequality with respect to \( t \) leads to the estimate
\[
\| w \|_{H^s}^2 \leq 2 \| w_0 \|_{H^s}^2 + 2c \| w_0 \|_{H^{s+1}}^2 + 2c \int_0^t (\delta^\alpha \| w \|_{H^s} + \| w \|_{H^s}^2) \, dt.
\]
By Gronwall’s inequality and (5.10) in Theorem 5.1, we have
\[
\| w \|_{H^s} \leq (2 \| w_0 \|_{H^s}^2 + 2c \| w_0 \|_{H^{s+1}}^2)^{1/2} e^{ct} + \delta^\alpha (e^{ct} - 1) \\
\leq c(\| w_0 \|_{H^s} + \delta^{7/16}) e^{ct} + \delta^\alpha (e^{ct} - 1).
\]
Making use of (5.12) in Lemma 5.1 and the above equality, we deduce that \( \| w \|_{H^s} \to 0 \) as \( \varepsilon, \delta \to 0 \).

At last, we prove convergence of \( \{ u_t^\varepsilon \} \). Multiplying both sides of the equation in (5.13) by \( \Lambda^{s-1} w_t \Lambda^s \), and integrating the resulting equation with respect to \( x \), one obtains the equation using the integration by parts
\[
\| u_t^\varepsilon \|_{H^{s-1}}^2 + \varepsilon \| u_t^\varepsilon \|_{H^s}^2 \\
\leq c \| u_t^\varepsilon \|_{H^{s-1}} \| u \|_{H^s} \left( \| u^\varepsilon \|_{H^s}^2 + \| u^\delta \|_{H^s}^2 + \| u^\varepsilon \|_{H^s} + \| u^\delta \|_{H^s} \right) \\
+ (\delta - \varepsilon) \| u_t^\varepsilon \|_{H^{s+1}} \| w_t \|_{H^{s-1}} \\
\leq \delta^\alpha \| u_t^\varepsilon \|_{H^{s-1}}^2 + c \| w_t \|_{H^{s-1}} \| w \|_{H^s},
\]
for some constant \( c \) and \( t \in [0, \bar{T}) \). Hence,
\[
\| u_t \|_{H^{s-1}} \leq c(\delta^\alpha + \| w \|_{H^s}).
\]
Then it follows that \( u_t \to 0 \) as \( \varepsilon, \delta \to 0 \) in \( H^{s-1} \) norm. This implies that both \( \{ u^\varepsilon \} \) and \( \{ u_t^\varepsilon \} \) are Cauchy sequences in the space \( C([0, \bar{T}); H^s) \) and \( C([0, \bar{T}); H^{s-1}) \), respectively. Let \( u(t, x) \) be the limit of the sequence \( \{ u^\varepsilon \} \). Taking the limit on both sides of the equation in (5.1) as \( \varepsilon \to 0 \), one shows that \( u \) is the solution of the problem (3.1). \( \square \)

The verification for the uniqueness of the solution \( u \) follows the technique to obtain the norm \( \| w \|_{H^0} \) in Theorem 5.4.

**Theorem 5.5.** Suppose that \( u_0 \in H^s, s > 5/2 \). Then there exists a \( T > 0 \), such that the problem (3.1) has a unique solution \( u(t, x) \) in the space \( C([0, T); H^s) \cap C^1([0, T); H^{s-1}) \).
Proof. Suppose that $u$ and $v$ are two solutions of the problem (3.1) corresponding to the same initial data $u_0$ such that $u, v \in L^2([0, T); H^s)$. Then $w = u - v$ satisfies the following Cauchy problem

$$
\begin{cases}
\frac{d}{dt} w_t = -2k_1 \mu(w) uu_x - (2k_1 \mu(v) + k_2) (wu_x + vw_x) \\
\quad + \frac{k_1}{3} w_x (u_x^2 + u_x v_x + v_x^2) - \frac{k_1}{3} \mu(w_x (u_x^2 + u_x v_x + v_x^2)) \\
\quad - \partial_x (\mu - \partial_x^2)^{-1} \left[ k_1 \left( 2\mu(w) \mu(f) u + 2\mu^2(v) w + \mu(w) u_x^2 + \mu(v) w_x f_x \right) \\
\quad + k_2 \left( 2\mu(w) u + 2\mu(v) w + \frac{1}{2} w_x f_x \right) \right],
\end{cases}
$$

(5.17)

$w(0, x) = 0, \ x \in \mathbb{R}$.

where $f = u + v$. For any $1/2 < q < \min\{1, s - 2\}$. Applying the operator $\Lambda^q$ to both sides of the above equation and then multiplying the resulting expression by $\Lambda^q w$ to integrate with respect to $x$, one obtains the following equation

$$
\frac{1}{2} \frac{d}{dt} \| \Lambda^q w \|_{L^2}^2 = - \int_{\mathbb{R}} \Lambda^q w \Lambda^q \left\{ -2k_1 \mu(w) uu_x - (2k_1 \mu(v) + k_2) (wu_x + vw_x) \\
\quad + \frac{k_1}{3} w_x (u_x^2 + u_x v_x + v_x^2) - \frac{k_1}{3} \mu(w_x (u_x^2 + u_x v_x + v_x^2)) \\
\quad - \partial_x (\mu - \partial_x^2)^{-1} \left[ k_1 \left( 2\mu(w) \mu(f) u + 2\mu^2(v) w + \mu(w) u_x^2 + \mu(v) w_x f_x \right) \\
\quad + k_2 \left( 2\mu(w) u + 2\mu(v) w + \frac{1}{2} w_x f_x \right) \right] \right\} \ dx.
$$

Similar to the proof of the estimate (5.14), there is a constant $c$ such that

$$
\frac{d}{dt} \| \Lambda^q w \|_{L^2}^2 \leq c \| w \|_{H^{q+1}}^2 \left( \| u \|_{H^{q+2}}^2 + \| v \|_{H^{q+2}}^2 + \| u \|_{H^{q+1}} + \| v \|_{H^{q+1}} \right).
$$

Next, differentiating the equation in (5.17), applying $\Lambda^q w_x \Lambda^q$ to both sides of the resulting equation and integrating over $\mathbb{R}$ with regard to $x$, one finds that

$$
\frac{1}{2} \frac{d}{dt} \| \Lambda^q w_x \|_{L^2}^2 = - \int_{\mathbb{R}} \Lambda^q w_x \Lambda^q \left\{ -2k_1 \mu(w) uu_x - (2k_1 \mu(v) + k_2) (wu_x + vw_x) \\
\quad + \frac{k_1}{3} w_x (u_x^2 + u_x v_x + v_x^2) - \frac{k_1}{3} \mu(w_x (u_x^2 + u_x v_x + v_x^2)) \\
\quad - \partial_x (\mu - \partial_x^2)^{-1} \left[ k_1 \left( 2\mu(w) \mu(f) u + 2\mu^2(v) w + \mu(w) u_x^2 + \mu(v) w_x f_x \right) \\
\quad + k_2 \left( 2\mu(w) u + 2\mu(v) w + \frac{1}{2} w_x f_x \right) \right] \right\} \ dx.
$$

Similarly to the proof of the inequality (5.15), we deduce that there exists a constant $c$ such that

$$
\frac{d}{dt} \| \Lambda^q w_x \|_{L^2}^2 \leq c \| w \|_{H^{q+1}}^2 \left( \| u \|_{H^{q+2}}^2 + \| v \|_{H^{q+2}}^2 + \| u \|_{H^{q+1}} + \| v \|_{H^{q+1}} \right).
$$

Combining the above two expressions, we deduce that

$$
\frac{d}{dt} \left( \| \Lambda^q w \|_{L^2}^2 + \| \Lambda^q w_x \|_{L^2}^2 \right) \\
\leq c \| w \|_{H^{q+1}}^2 \left( \| u \|_{H^{q+2}}^2 + \| v \|_{H^{q+2}}^2 + \| u \|_{H^{q+1}} + \| v \|_{H^{q+1}} \right).
$$
In view of $1/2 < q < \min\{1, s - 2\}$, then Gronwall’s inequality and the boundedness of $\|u\|_{H^{s+2}}$ and $\|v\|_{H^{q+2}}$ lead to

$$\|w\|_{H^{q+1}} \leq \|w_0\|_{H^{q+1}} e^{\bar{c}t} = 0,$$

for some constant $\bar{c}$ and any $t \in (0, T)$. Hence, $\|w\|_{H^{q+1}} = 0$ and thus $w = 0$ i.e. $u = v$.

Next, multiplying the equation in (5.17) by $\Lambda^{s-1}u_t\Lambda^{-1}$, and integrating the resulting equation with respect to $x$, one obtains the inequality using the integration by parts

$$\|w_t\|_{H^{s-1}} \leq c \|w\|_{H^s} \left(\|u\|_{H^s}^2 + \|v\|_{H^s}^2\right).$$

In view of $w = 0$ and the boundedness of $\|u\|_{H^s}$ and $\|v\|_{H^s}$, it follows that $\|w_t\|_{H^{s-1}} = 0$ and thus $w_t = 0$ i.e. $u_t = v_t$. Hence the uniqueness of the solution to the problem (3.1) is proved. The continuous dependency of solutions on initial data can be verified by using a similar technique used for the KdV equation by Bona and Smith [3]. This completes the proof of Theorem 5.1. \qed

6. Blow-up Scenarios and a Global Conservative Property

In this section, attention is now turned to the blow-up phenomena. We firstly present the following precise blow-up scenario.

**Theorem 6.1.** Let $u_0 \in H^{s+2}$, $s > 1/2$ be given and assume that $T$ is the maximal existence time of the corresponding solution $u(t, x)$ to the initial value problem (3.1) with the initial data $u_0$. Assume that $T_{u_0}^* > 0$ is the maximum time of existence. Then

$$T_{u_0}^* < \infty \Rightarrow \int_0^{T_{u_0}^*} \|m(\tau)\|_{L^\infty}^2 d\tau = \infty. \quad (6.1)$$

**Remark 6.1.** The blow-up criterion in (6.1) implies that the lifespan $T_{u_0}^*$ does not depend on the regularity index $s$ of the initial data $u_0$. Indeed, let $u_0$ be in $H^s$ for some $s > 5/2$ and consider some $s' \in (5/2, s)$. Denote by $u_s$ (resp., $u_{s'}$) the corresponding maximal $H^s$ (resp., $H^{s'}$) solution given by the above theorem. Denote by $T_{u_0}^*$ (resp., $T_{u_s}^*$, $T_{u_{s'}}^*$) the lifespan of $u_s$ (resp., $u_{s'}$). Since $H^s \hookrightarrow H^{s'}$, uniqueness ensures that $T_{u_0}^* \leq T_{u_s}^*$ and that $u_s \equiv u_{s'}$ on $[0, T_{u_0}^*)$. Now, if $T_{u_0}^* < T_{u_s}^*$, then we must have $u_{s'} \in C((0, T_{u_0}^*); H^{s'})$. Hence, $u_{s'} \in L^2((0, T_{u_0}^*); L^\infty)$, which contradicts the above blow-up criterion (6.1). Therefore, $T_{u_0}^* = T_{u_s}^*$.

**Proof.** We shall prove Theorem 6.1 by an inductive argument with respect to the index $s$. This will be carried out by three steps.

**Step 1.** For $s \in (1/2, 1)$, applying Lemma 3.5 to the equation in (3.1), i.e.,

$$m_t + (k_1(2\mu_0 u - u_x^2) + k_2 u)m_x = -2k_1 u_x m^2 - 2k_2 m u_x, \quad (6.2)$$

we arrive at

$$\|m(t)\|_{H^s} \leq \|m_0\|_{H^s} + C \int_0^t \|\partial_x (k_1(2\mu_0 u - u_x^2) + k_2 u)(\tau)\|_{L^\infty} \|m(\tau)\|_{H^s} d\tau$$

$$+ 2C \int_0^t \|(u_x m^2 + m u_x)(\tau)\|_{H^s} d\tau \quad (6.3)$$

for all $0 < t < T_{u_0}^*$. Owing to the Moser-type estimate in Lemma 3.3, one has

$$\|u_x m^2\|_{H^s} \leq C(\|u_x\|_{H^s}\|m\|_{L^\infty}^2 + \|u_x\|_{L^\infty}\|m\|_{L^\infty}\|m\|_{H^s}). \quad (6.4)$$

An application of the Young inequality then implies

$$\|u_x\|_{L^\infty} \leq \|g_x\|_{L^1} \cdot \|m\|_{L^\infty} \leq \|m\|_{L^\infty}.$$
Step 2. Applying Lemma 3.5 to (6.10) yields the proof of Theorem 6.1 for By the Moser-type estimates in Lemma 3.3, one has and which contradicts the assumption on the maximal existence time ≤ 1. Thus, the inequality (6.8) implies that This estimate together with the fact \(\|u_x\|_{H^s} \leq C\|m\|_{H^s}\) and (6.4), gives rise to and

\[
\|\partial_x (k_1(2\mu_0u - u^2_x) + k_2u)\|_{L^\infty} \leq C (1 + \|m\|_{L^\infty}) \|u_x\|_{L^\infty} \\
\leq C\|m\|_{L^\infty} (1 + \|m\|_{L^\infty}).
\]

Plugging (6.6) and (6.5) into (6.3) leads to

\[
\|m(t)\|_{H^s} \leq \|m_0\|_{H^s} + C \int_0^t \|m(\tau)\|_{H^s} \|m(\tau)\|_{L^\infty} (1 + \|m(\tau)\|_{L^\infty}) d\tau,
\]

which, by Gronwall's inequality, yields

\[
\|m(t)\|_{H^s} \leq \|m_0\|_{H^s} e^{C \int_0^t \|m(\tau)\|_{L^\infty} (1 + \|m(\tau)\|_{L^\infty}) d\tau}.
\]

Therefore, if the maximal existence time \(T_{u_0}^* < \infty\) satisfies

\[
\int_0^{T_{u_0}^*} \|m(\tau)\|_{L^\infty}^2 d\tau < \infty,
\]

then

\[
\int_0^{T_{u_0}^*} (? + \|m(\tau)\|_{L^\infty}^2) d\tau \leq \int_0^{T_{u_0}^*} \left( \frac{1}{2} + \frac{3}{2} \|m(\tau)\|_{L^\infty}^2 \right) d\tau < \infty.
\]

Thus, the inequality (6.8) implies that

\[
\limsup_{t \to T_{u_0}^*} \|m(t)\|_{H^s} < \infty,
\]

which contradicts the assumption on the maximal existence time \(T_{u_0}^* < \infty\). This completes the proof of Theorem 6.1 for \(s \in (1/2, 1)\).

Step 2. For \(s \in [1, 2)\), by differentiating the equation (6.2) with respect to \(x\), we have

\[
\partial_t (m_x) + (k_1(2\mu_0u - u^2_x) + k_2u)\partial_x (m_x) = -3k_1u_x(m^2)_x - 2k_1u_xm^2 - 2k_2u_xm - 3k_2m_xu_x.
\]

(6.10)

Applying Lemma 3.3 to (6.10) yields

\[
\|m_x(t, \cdot)\|_{H^{s-1}} \leq \|m_0|_{H^{s-1}} + C \int_0^t \|\partial_x (k_1(2\mu_0u - u^2_x) + k_2u)\|_{L^\infty} \|\partial_x m\|_{H^{s-1}} d\tau \\
+ C \int_0^t \|3k_1u_x(m^2)_x + 2k_1u_xm^2 + 2k_2u_xm + 3k_2m_xu_x\|_{H^{s-1}} d\tau.
\]

(6.11)

By the Moser-type estimates in Lemma 3.3 one has

\[
\|u_x(m^2)_x\|_{H^{s-1}} \leq C\|m\|_{L^\infty}^2 \|m\|_{H^s}.
\]

(6.12)

and

\[
\|u_xm^2\|_{H^{s-1}} \leq C\|m\|_{L^\infty}^2 \|m\|_{H^{s-1}}.
\]

(6.13)

Using (6.6), (6.12), and (6.13) in (6.11), and combining with (6.7), we conclude that, for \(1 \leq s \leq 2\), (6.7) holds.

Repeating the same argument as in Step 1, we see that Theorem 6.1 holds for \(1 \leq s < 2\).
Step 3. Suppose $2 \leq k \in \mathbb{N}$. By induction, we assume that (6.1) holds when $k-1 \leq s < k$, and prove that it holds for $k \leq s < k+1$. To this end, we differentiate (6.2) $k$ times with respect to $x$, producing
\[
\partial_t \partial_x^km + \left(k_1(2\mu_0u - u_x^2) + k_2u\right)\partial_x^{k-1}m \\
= -\sum_{\ell=0}^{k-1} C_k^\ell \partial_x^{k-\ell} \left(k_1(2\mu_0u - u_x^2) + k_2u\right)\partial_x^{\ell+1}m \\
- 2k_1 \partial_x^k(u_xm^2) - 2k_2 \partial_x^k(mu_x).
\]

Applying Lemma 3.5 to the above equation again yields that
\[
\|\partial_x^km(t)\|_{H^{s-k}} \leq \|\partial_x^km_0\|_{H^{s-k}} + C \int_0^t \|\partial_x^km(\tau)\|_{H^{s-k}} \|(2k_1mu_x + k_2u_x)(\tau)\|_{L^\infty} d\tau \\
+ C \int_0^t \left| \sum_{\ell=0}^{k-1} C_k^\ell \partial_x^{k-\ell} \left(k_1(2\mu_0u - u_x^2) + k_2u\right)\partial_x^{\ell+1}m(\tau) \right|_{H^{s-k}} d\tau \\
+ C \int_0^t \left(2k_1 \partial_x^k(u_xm^2) + 2k_2 \partial_x^k(u_xm)\right)(\tau) \|_{H^{s-k}} d\tau.
\]

Using the Moser-type estimate in Lemma 3.3 and the Sobolev embedding inequality, we have
\[
\left| \sum_{\ell=0}^{k-1} C_k^\ell \partial_x^{k-\ell} \left(k_1(2\mu_0u - u_x^2) + k_2u\right)\partial_x^{\ell+1}m \right|_{H^{s-k}} \\
\leq C \sum_{\ell=0}^{k-1} C_k^\ell \|k_1(2\mu_0u - u_x^2) + k_2u\|_{H^{s-k}} \|\partial_x^m\|_{L^\infty} \\
+ \|\partial_x^{k-\ell} \left(k_1(2\mu_0u - u_x^2) + k_2u\right)\|_{L^\infty} \|m\|_{H^{s-k+\ell+1}} \\
\leq C \|m\|_{H^s} \|m\|_{H^{k-\frac{1}{2}+\epsilon_0}} \left(1 + \|m\|_{H^{k-\frac{1}{2}+\epsilon_0}} \right),
\]
where the genius constant $\epsilon_0 \in (0, \frac{1}{4})$ so that $H^{k+\epsilon_0} \hookrightarrow L^\infty$ holds. Substituting these estimates into (6.14), we obtain
\[
\|m(t)\|_{H^s} \leq \|m_0\|_{H^s} + C \int_0^t \left(1 + \|m(\tau)\|_{H^{k-\frac{1}{2}+\epsilon_0}} \right) \|m(\tau)\|_{H^{k-\frac{1}{2}+\epsilon_0}} d\tau,
\]
where we used the Sobolev embedding theorem $H^{k-\frac{1}{2}+\epsilon_0} \hookrightarrow L^\infty$ with $k \geq 2$. Applying Gronwall’s inequality then gives
\[
\|m(t)\|_{H^s} \leq \|m_0\|_{H^s} \exp\{C \int_0^t \|m(\tau)\|_{H^{k-\frac{1}{2}+\epsilon_0}} \left(1 + \|m(\tau)\|_{H^{k-\frac{1}{2}+\epsilon_0}} \right) d\tau\}. \tag{6.16}
\]
In consequence, if the maximal existence time $T_{u_0}^\ast < \infty$ satisfies
\[
\int_0^{T_{u_0}^\ast} \|m(\tau)\|_{L^\infty}^2 d\tau < \infty,
\]
thanks to the uniqueness of solution in Theorem 5.1, we then find that $\|m(t)\|_{H^{k-\frac{1}{2}+\epsilon_0}}$ is uniformly bounded in $t \in (0, T_{u_0}^\ast)$ by the induction assumption, which along with (6.16)
The solution to the above initial-value problem is given by the corresponding strong solution. For every $T' \in (0, T)$, we can improve the result of Theorem 6.1 as follows: Remark 6.2. In fact, we can improve the result of Theorem 6.1 as follows:

3. Applying the maximum principle to the transport equation (6.2), we immediately get

$$\|m(t)\|_{L^\infty} \leq \|m_0\|_{L^\infty} + C \int_0^t \| (k_1 mu_x + k_2 u_x) (\tau)\|_{L^\infty} \|m(\tau)\|_{L^\infty} d\tau. \tag{6.17}$$

Then, Gronwall’s inequality applied to the above inequality yields

$$\|m(t)\|_{L^\infty} \leq \|m_0\|_{L^\infty} \exp \left( C \int_0^t \| (k_1 mu_x + k_2 u_x) (\tau)\|_{L^\infty} d\tau \right), \tag{6.18}$$

which along with the result of Theorem 6.1 gives rise to (6.17).

In order to demonstrate a conservative property, let us consider the trajectory equation

$$\left\{ \begin{array}{l}
\frac{dq}{dt} = (k_1 (2\mu_0 u - u_x^2) + k_2 u)(t, q(t, x)) \\
q(0, x) = x.
\end{array} \right. \tag{6.19}$$

Lemma 6.1. Let $u_0 \in H^s$, $s > 5/2$, and let $T > 0$ be the maximal existence time of the corresponding strong solution $u$ to (3.1). Then (6.18) has a unique solution $q \in C^1([0, T] \times \mathbb{S})$ such that the map $q(t, \cdot)$ is an increasing diffeomorphism over $\mathbb{S}$ with

$$q_x(t, x) = \exp \left( \int_0^t (2k_1 mu_x + k_2 u_x)(s, q(s, x)) ds \right) > 0, \text{ for all } (t, x) \in [0, T] \times \mathbb{S}. \tag{6.20}$$

Furthermore,

$$m(t, q(t, x)) = m_0(x) \exp \left( -2 \int_0^t (k_1 mu_x + k_2 u_x)(s, q(s, x)) ds \right) \tag{6.21}$$

for all $(t, x) \in [0, T] \times \mathbb{S}$.

Proof. Since $u \in C^1([0, T), H^{s-1}(\mathbb{S}))$ and $H^s \hookrightarrow C^1$, both $u(t, x)$ and $u_x(t, x)$ are bounded, Lipschitz in the space variable $x$, and of class $C^1$ in time. Therefore, by well-known classical results in the theory of ordinary differential equations, the initial value problem (6.18) has a unique solution $q(t, x) \in C^1([0, T] \times \mathbb{S})$.

Differentiating (6.18) with respect to $x$ yields

$$\left\{ \begin{array}{l}
\frac{d}{dt} q_x = (2k_1 mu_x + k_2 u_x)(t, q) q_x, \\
q_x(0, x) = 1,
\end{array} \right. \tag{6.22}$$

for all $(t, x) \in [0, T] \times \mathbb{S}$. The solution to the above initial-value problem is given by

$$q_x(t, x) = \exp \left( \int_0^t (2k_1 mu_x + k_2 u_x)(s, q(s, x)) ds \right), \text{ for all } (t, x) \in [0, T] \times \mathbb{S}. \tag{6.23}$$

For every $T' < T$, it follows from the Sobolev embedding theorem that

$$\sup_{(s,x) \in [0,T'] \times \mathbb{S}} |(2k_1 mu_x + k_2 u_x)(s, x)| < \infty.$$
We infer from the expression of \( q_x \) that there exists a constant \( K > 0 \) such that \( q_x(t, x) \geq e^{-Kt} \), \((t, x) \in [0, T) \times \mathbb{S}\), which implies that the map \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{S} \) before blow-up with
\[
q_x(t, x) = \exp \left( \int_0^t (2k_1mu_x + k_2u_x)(s, q(s, x)) \, ds \right) > 0, \quad \text{for all} \quad (t, x) \in [0, T) \times \mathbb{S}.
\]

On the other hand, from (6.2) we have
\[
\frac{d}{dt} m(t, q(t, x)) = m_t(t, q) + \left( k_1(2\mu_0u - u_x^2) + k_2u \right)(t, q(t, x))m_x(t, q(t, x)) = -2 \left( k_1mu_x + k_2u_x \right)(t, q(t, x))m(t, q(t, x)).
\]

Therefore, solving the equation with regard to \( m(t, q(t, x)) \) leads to
\[
m(t, q(t, x)) = m_0(x) \exp \left( -2 \int_0^t (k_1mu_x + k_2u_x)(s, q(s, x)) \, ds \right).
\]

This completes the proof of Lemma 6.1.

\[ \square \]

**Remark 6.3.** Lemma 6.1 shows that, if \( m_0 = (\mu - \partial_x^2)u_0 \) does not change sign, then \( m(t, x) \) will not change sign for any \( t \in [0, T) \).

**Remark 6.4.** Since \( q(t,\cdot): \mathbb{S} \to \mathbb{S} \) is a diffeomorphism of the line for every \( t \in [0, T) \), the \( L^\infty \)-norm of any function \( v(t, \cdot) \in L^\infty \) is preserved under the family of diffeomorphisms \( q(t, \cdot) \), that is,
\[
\|v(t, \cdot)\|_{L^\infty} = \|v(t, q(t, \cdot))\|_{L^\infty}, \quad t \in [0, T).
\]

The following blow-up criterion implies that wave-breaking depends only on the infimum of \( k_1mu_x \) or \( k_2u_x \).

**Theorem 6.2.** Let \( u_0 \in H^s, \ s > 5/2 \) be as in Theorem 5.1 Then the corresponding solution \( u \) to (5.1) blows up in finite time \( T^*_{u_0} > 0 \) if and only if
\[
\liminf_{t \uparrow T^*_{u_0}} \left( \inf_{x \in \mathbb{S}} (k_1mu_x(t, x) + k_2u_x(t, x)) \right) = -\infty. \tag{6.21}
\]

**Proof.** In view of the equation in (6.20), if \( \inf_{x \in \mathbb{S}} \{k_1mu_x + k_2u_x\} \geq -C_1 \) for \( 0 \leq t \leq T^*_{u_0} \), then it implies that
\[
\|m(t)\|_{L^\infty}^2 = \|m(t, q(t, x))\|_{L^\infty}^2 = \|m_0(x)\|_{L^\infty}^2 \exp \left( -4 \int_0^t (k_1mu_x + k_2u_x)(s, q(s, x)) \, ds \right)
\]
\[
\leq e^{4C_1} \|m_0(x)\|_{L^\infty}^2.
\]

Thanks to Theorem 6.1 it ensures that the solution \( m(t, x) \) does not blow up in finite time. On the other hand, if
\[
\liminf_{t \uparrow T^*_{u_0}} \left( \inf_{x \in \mathbb{S}} (k_1mu_x(t, x) + k_2u_x(t, x)) \right) = -\infty,
\]
by Theorem 5.1 for the existence of local strong solutions and the Sobolev embedding theorem, we infer that the solution will blow up in finite time. The proof of the theorem is then complete.

\[ \square \]

**Corollary 6.1.** Let \( u_0 \in H^s \) be as in Theorem 6.1 with \( s > 5/2 \). Then the corresponding solution \( u \) to (5.1) blows up in finite time \( T^*_{u_0} > 0 \) if and only if
\[
\liminf_{t \uparrow T^*_{u_0}} \left( \min_{x \in \mathbb{S}} \{k_1mu_x(t, x), k_2u_x(t, x)\} \right) = -\infty. \tag{6.22}
\]
Proof. It is observed that for any $t \in [0, T)$
\[
\inf_{x \in \mathbb{S}} (k_1 m u_x(t, x) + k_2 u_x(t, x)) \geq 2 \min \{ \inf_{x \in \mathbb{S}} (k_1 m u_x(t, x)), \inf_{x \in \mathbb{S}} (k_2 u_x(t, x)) \}.
\]
If there exists a positive constant $C$ such that
\[
\min \{ \inf_{x \in \mathbb{S}} (k_1 m u_x(t, x)), \inf_{x \in \mathbb{S}} (k_2 u_x(t, x)) \} \geq -C,
\]
then
\[
\inf_{x \in \mathbb{S}} (k_1 m u_x(t, x) + k_2 u_x(t, x)) \geq -2C.
\]
This thus implies from Theorem 6.2 that the solution $u(t, x)$ does not blow up in finite time.

On the other hand, similar to the proof of Theorem 6.2, if
\[
\lim \inf_{t \to T^*} \left( \min \{ \inf_{x \in \mathbb{S}} (k_1 m u_x(t, x)), \inf_{x \in \mathbb{S}} (k_2 u_x(t, x)) \} \right) = -\infty,
\]
then the solution will blow-up in finite time. \hfill \square

7. WAVE-BREAKING MECHANISM

In this section, we give some sufficient conditions for the breaking of waves for the initial-value problem (3.1). In the case of $\mu_0 = \mu(u_0) = 0$, we have the following blow-up result.

**Theorem 7.1.** Let $k_1 > 0$, $k_2 > 0$ and $u_0 \in H^s$, $s > 5/2$, $u_0 \neq 0$ with $\mu_0 = 0$. Let $T > 0$ be the maximal time of existence of the corresponding solution $u(t, x)$ of (3.1) with the initial value $u_0$. Then the solution $u$ blows up in finite time $T^* < \infty$.

**Proof.** In view of Theorem 6.1 with the density argument, it suffices to consider the case $s \geq 3$. Since $\mu(u)$ is conserved, so we have $\mu(u) = 0$. Thus the equation in (6.2) is reduced to
\[
u_{xxt} - k_1 (u_x^2 u_{xx})_x + k_2 (2 u_x u_{xx} + uu_{xxx}) = 0.
\]
(7.1)
Integrating it with respect to $x$ from 0 to $x$, we arrive at
\[
u_{xt} + (-k_1 u_x^2 + k_2 u) u_{xx} + \frac{k_2}{2} u_x^2 = -\frac{k_2}{2} \mu_1^2.
\]
(7.2)
Notice that $u \in C([0, T); H^s)$ is a periodic function of $x$. Then there exists a $x_0 \in \mathbb{S}$ such that $u_{0,x}(x_0) < 0$. Consider the flow (6.18) with $\mu_0 = 0$, i.e., the flow $q(t, x)$ governed by $-k_1 u_x^2 + k_2 u$,
\[
n \frac{dq}{dt} = (-k_1 u_x^2 + k_2 u)(t, q(t, x)), \quad q|_{t=0} = x.
\]
(7.3)
Along with the flow $q(t, x)$, we deduce from (7.2) that
\[
n \frac{du_x(t, q(t, x))}{dt} = u_{xt} + (-k_1 u_x^2 + k_2 u) u_{xx} = -\frac{k_2}{2} (u_x^2 + \mu_1^2).
\]
Let $w(t) = k_2 u_x(t, q(t, x_0))$. It follows from the above equation that
\[
n \frac{dw}{dt} = -\frac{1}{2} (w^2 + k_2^2 \mu_1^2).
\]
Solving it we get
\[
w(t) = k_2 u_x(t, q(t, x_0)) = \mu_1 k_2 \frac{u_{0,x}(x_0) - \mu_1 \tan(\frac{t}{2} k_1 k_2 t)}{\mu_1 + u_{0,x}(x_0) \tan(\frac{t}{2} k_1 k_2 t)}.
\]
It is thereby inferred that
\[
\inf_{x \in \mathbb{R}} (k_2 u_x(t,x)) \leq k_2 u_x(t,q(t,x_0)) \to -\infty, \quad \text{as } t \to T^* \leq t^*,
\]
where \( t^* = -2 \arctan(\frac{\mu_1}{u_0(x_0)})/(\mu_1 k_2) < \infty \).

Let \( U(t) = k_1 u_{xx}(t,q(t,x_0)) \). It follows from (7.1) and (7.3) that
\[
\frac{dU}{dt} = k_1 u_{xxt} + (-k_1 u_x^2 + k_2 u) k_1 u_{xxx} = 2u_x(U^2 - k_2 U).
\]
Solving it we get
\[
U(t) = k_1 u_{xx}(t,q(t,x_0)) = \frac{k_2 U(0)}{U(0) - (U(0) - k_2) \left( \cos(\frac{\mu_1}{2} k_2 t) + \frac{u_0(x_0)}{\mu_1} \sin(\frac{\mu_1}{2} k_2 t) \right)^4}.
\]
Accordingly, we have
\[
k_1 (mu_x)(t,q(t,x_0)) = -k_1 (u_x u_{xx})(t,q(t,x_0))
\]
\[
= -\frac{\mu_1 k_1 k_2 u_{0,xx}(x_0)(u_{0,x}(x_0) - \mu_1 \tan(\frac{\mu_1}{2} k_2 t))}{(\mu_1 + u_{0,x}(x_0) \tan(\frac{\mu_1}{2} k_2 t)) J_1},
\]
where
\[
J_1 = k_1 u_{0,xx}(x_0) - (k_1 u_{0,xx}(x_0) - k_2) \cos^4(\frac{1}{2} \mu_1 k_2 t) \left( 1 + \frac{u_{0,x}(x_0)}{\mu_1} \tan(\frac{1}{2} \mu_1 k_2 t) \right)^4.
\]
It is thereby inferred that
\[
k_1 (mu_x)(t,q(t,x_0)) \to +\infty \quad \text{as } t \to T^* \leq t^*.
\]
Moreover, a simple computation shows that for any constant \( a \)
\[
(k_1 u_x m + ak_2 u_x)(t,q(t,x_0))
\]
\[
= \frac{\tan \alpha}{J_2} \left[ k_1 (1 - a) \tan^4 \alpha + 2k_1 (1 - a) \tan^2 \alpha + k_1 \right. \left. + ak_2 \mu_1 (1 + \tan^2 \theta)^2 \tan \theta \left( u_{0,x}(x_0) \right)^{-1} + ak_1 (\tan^4 \theta + 2 \tan^2 \theta) \right],
\]
where
\[
J_2 = \frac{k_1}{k_2 \mu_1} \left[ (\tan^4 \alpha + 2 \tan^2 \alpha) - (1 + \tan^2 \theta)^2 \tan \theta \left( u_{0,x}(x_0) \right)^{-1} - (\tan^4 \theta + 2 \tan^2 \theta) \right],
\]
\[
\alpha = \frac{1}{2} k_2 \mu_1 t - \theta, \quad \theta = \arctan \left( \frac{u_{0,x}(x_0)}{\mu_0} \right).
\]
When \( a > 1 \), it then follows from (7.4) that
\[
(k_1 u_x m + ak_2 u_x)(t,q(t,x_0)) \to -\infty, \quad \text{as } t \to T^* \leq t^*,
\]
which implies the desired result.

To deal with blow-up solution with \( \mu_0 \neq 0 \), the following two lemmas will be useful.

**Lemma 7.1.** Let \( u_0 \in H^s, s \geq 3 \) and \( T > 0 \) be the maximal time of existence of the corresponding solution \( u(t,x) \) to (3.1) with the initial data \( u_0 \). Then \( M = mu_x, P = u_x \) and \( \Gamma = (k_1 m + k_2 u_x) \) satisfy
\[
M_t + (k_1 (2 \mu_0 u - u_x^2) + k_2 u) M_x
\]
\[
= -2k_1 M^2 - \frac{5}{2} k_2 u_x M + k_1 \left[ 2 \mu_0^2 (u - \mu_0) - \mu_0 (u_x^2 + \mu_1^2) \right] m
\]
\[
+ k_2 \left[ 2 \mu_0 (u - \mu_0) - \frac{1}{2} \mu_1^2 \right] m
\]
\[
(7.5a)
\]
\[ P_t + (k_1(2\mu_0 u - u^2_x) + k_2 u) P_x = k_1 [2\mu_0^2(u - \mu_0) - \mu_0(u_x^2 + \mu_1^2)] + k_2 [2\mu_0(u - \mu_0) - \frac{1}{2}(u_x^2 + \mu_1^2)], \]

and
\[ \Gamma_t + (k_1(2\mu_0 u - u^2_x) + k_2 u) \Gamma_x = -2\Gamma^2 + \left(\frac{3}{2}k_1 - k_1\mu_0\right)u_x \Gamma 
+ (k_1 m + k_2) \left[ k_1 (2\mu_0^2(u - \mu_0) - \mu_0\mu_1^2) + k_2 (2\mu_0(u - \mu_0) - \frac{1}{2}\mu_1^2) \right]. \]

**Proof.** We will give a proof of the last estimate (7.5c) only, since the other cases can be obtained similarly. In view of (6.2), a direct computation shows
\[
(\mu - \partial_x^2) [u_t + (k_1(2\mu_0 u - u^2_x) + k_2 u) u_x] = m_t - k_1 \mu(u_x^2) - [k_1(2\mu_0 u - u^2_x)u_{xxx} + 2k_1 u_x^2 m + k_2 (uu_{xx} + u_x^2)] 
= k_1 (-2\mu_0^2 u_x - 2\mu_0 u_x u_{xx} + 4u_x^2 u_{xx} + 2u_x^2 u_{xxx} - \mu(u_x^3) 
- k_2(2\mu_0 u_x + u_x u_{xx}),
\]

which implies
\[
0 = \left[ k_1 (2\mu_0^2(u - \mu_0) - \mu_0(u_x^2 - \mu_1^2) - 2u_x^2 u_{xx} - \mu(u_x^3) 
- k_2(2\mu_0 u_x + u_x u_{xx}) \right].
\]

Differentiating it with respect to \( x \) leads to
\[
u_{xx} + k_1 ((2\mu_0 u - u^2_x)u_{xx} + 2u_x^2 m) + k_2 (uu_{xx} + u_x^2) 
= k_1 [2\mu_0^2(u - \mu_0) + \mu_0(u_x^2 - \mu_1^2) - 2u_x^2 u_{xx}] 
+ k_2 [2\mu_0(u - \mu_0) + \frac{1}{2}(u_x^2 - \mu_1^2)]. \]

Clearly,
\[
u_{xx} m_t + (k_1(2\mu_0 u - u^2_x) + k_2 u) m_x u_x = -2k_1 u_x^2 m^2 - 2k_2 m u_{xx}. \]

Multiplying (7.7) by \( m \) and adding the resulting one with (7.8), we arrive at
\[
M_t + (k_1(2\mu_0 u - u^2_x) + k_2 u) M_x 
= -4k_1 M^2 - 3k_2 u_x M + k_1 [2\mu_0^2(u - \mu_0) + \mu_0(u_x^2 - \mu_1^2) - 2u_x^2 u_{xx}] m \]
\[ + k_2 [2\mu_0(u - \mu_0) + \frac{1}{2}(u_x^2 - \mu_1^2)] m. \]

Combining (7.7) and (7.9), we obtain
\[
\Gamma_t + (k_1(2\mu_0 u - u^2_x) + k_2 u) \Gamma_x 
= -2k_1 M^2 - \frac{3}{2}k_1k_2 u_x M - \frac{k_1 k_2}{2} u_{xx}^2 + k_1^2 [2\mu_0^2(u - \mu_0) - \mu_0(u_x^2 + \mu_1^2)] m \]
\[ + k_2 [2\mu_0(u - \mu_0) - \frac{1}{2}\mu_1^2] m + k_1 k_2 [2\mu_0^2(u - \mu_0) - \mu_0(u_x^2 + \mu_1^2)] 
+ k_2^2 [2\mu_0(u - \mu_0) - \frac{1}{2}\mu_1^2]. \]

A simply computation then gives the desired result. \( \square \)

**Lemma 7.2.** Let \( T > 0 \) be the maximal time of existence of the solution \( u(t, x) \) to the Cauchy problem (3.1) with initial data \( u_0 \in H^s, s \geq 3 \). Assume \( k_1 \geq 0, k_2 \geq 0 \) and \( m_0(x) \geq 0 \) for all \( x \in S \). Then there hold
\[
|u_x(t, x)| \leq u(t, x), \Gamma_t + (k_1(2\mu_0 u - u^2_x) + k_2 u) \Gamma_x \leq -2\Gamma^2 + C_1 u_x \Gamma + C_2(k_1 m + k_2),
\]

(7.11)
where the constants \( C_1 \) and \( C_2 \) are given by
\[
C_1 = \frac{3}{2}k_2 - k_1\mu_0, \\
C_2 = k_1\mu_0\mu_1\left(\frac{\sqrt{3}}{3}\mu_0 - \mu_1\right) + k_2\mu_1\left(\frac{\sqrt{3}}{3}\mu_0 - \frac{1}{2}\mu_1\right).
\]

**Proof.** Since \( m_0 \geq 0 \), Lemma 6.1 implies that \( m(t, x) \geq 0 \) for any \( t > 0, x \in \mathbb{S} \). According to (3.3), there holds
\[
u(t, x) = g * m = \int_{\mathbb{S}} \left[ \frac{1}{2}(x - y - \frac{1}{2})^2 + \frac{23}{24}\right] m(t, y) dy > 0,
\]
for any \( t > 0, x \in \mathbb{S} \). Note that \( \mu_0 = \mu(u) > 0 \), and \( u_x = \int_{\mathbb{S}} (x - y - \frac{1}{2}) m(t, y) dy \). It follows that
\[
u + \nu_x = \int_{\mathbb{S}} \left[ \frac{1}{2}(x - y + \frac{1}{2})^2 + \frac{11}{24}\right] m(t, y) dy > 0, \\
\nu - \nu_x = \int_{\mathbb{S}} \left[ \frac{1}{2}(x - y - \frac{3}{2})^2 + \frac{11}{24}\right] m(t, y) dy > 0.
\]
This implies
\[
|\nu_x(t, x)| \leq \nu(t, x),
\]
for all \((t, x) \in [0, T) \times \mathbb{S}\). By Lemma 3.1 we have
\[
\|\nu - \nu_0\|_{L^\infty} \leq \frac{\sqrt{3}}{6} \mu_1.
\]
(7.12)

In view of (7.5c), it follows that
\[
\Gamma_x + \left( k_1 (2\mu_0 \nu - \nu_x^2) + k_2 \nu \right) \Gamma_x \\
= -2\Gamma^2 + \left( \frac{2}{3}k_2 - k_1\mu_0 \right) \nu_x \Gamma \\
+ (k_1 m + k_2) \left[ k_1 (2\mu_0^2 - \mu_0\mu_1) - \mu_0\mu_1^2 \right] + k_2 \left[ 2\mu_0 (u - \mu_0) - \frac{1}{2}\mu_1^2 \right] \\
\leq -2\Gamma^2 + \left( \frac{2}{3}k_2 - k_1\mu_0 \right) \nu_x \Gamma \\
+ (k_1 m + k_2) \left[ k_1\mu_0\mu_1 \left( \frac{\sqrt{3}}{3}\mu_0 - \mu_1 \right) + k_2\mu_1 \left( \frac{\sqrt{3}}{3}\mu_0 - \frac{1}{2}\mu_1 \right) \right].
\]

This completes the proof of Lemma 7.2. \( \square \)

In the following, we use the notations \( \tau = \frac{t}{b}, b = \sqrt{\frac{1}{2k_2 C_2}}, n(\tau, x) = \frac{1}{m(\tau, x)} \), \( n_0(x) = n(0, x) \), and \( \Gamma = k_1 \mu u_x + k_2 u_x \) for convenience of the certain complicated computations.

In view of signs of \( C_1 \) and \( C_2 \), we need to consider four cases: (i). \( C_1 \leq 0, C_2 \leq 0 \); (ii). \( C_1 \leq 0, C_2 > 0 \); (iii). \( C_1 > 0, C_2 > 0 \); (iv). \( C_1 > 0, C_2 < 0 \). For the case (i), it is obvious to see from (7.11) that the solution blows up at finite time. Here we only consider the case (ii). The other cases can be discussed in a similar manner. For the case (ii), we have the following blow-up result, which demonstrates that wave-breaking depends on the infimum of \( \Gamma = k_1 \mu u_x + k_2 u_x \) for time \( t \in [0, T) \).

**Theorem 7.2.** Suppose \( k_1 > 0, k_2 > 0, C_1 \leq 0, C_2 > 0 \) and \( u_0 \in H^s \), with \( s > 5/2 \). Let \( T^* > 0 \) be the maximal time of existence of the the corresponding solution \( m(t, x) \) of (3.1) with the initial value \( m_0 \). Assume \( m_0(x) = (\mu - \partial_x^2)u_0 \geq 0 \) for any \( x \in \mathbb{S} \) and \( m_0(x) > 0 \) for some \( x_0 \in \mathbb{S} \) and that
\[
(1) \quad u_{0,x}(x_0) = -\frac{1}{2bk_2}, \quad \text{or} \quad (7.13a) \\
(2) \quad u_{0,x}(x_0) < \min \left\{ -\frac{1}{2bk_2}, -\frac{\sqrt{k_2 + 2k_1 m_0(x_0)}}{2\sqrt{k_2(k_1 m_0(x_0) + k_2)}} \right\}. \quad (7.13b)
\]
Then the solution blows up at finite time $T^* \leq t^*$, where

$$t^* = \begin{cases} 
\frac{1}{2k_2C_2} \ln \left( \frac{k_2}{k_1m_0(x_0)} + 1 \right), & \text{for (1)}, \\
\frac{1}{2k_2C_2} \ln \left( \frac{k_2}{k_1m_0(x_0)} + \sqrt{n_2(x_0) - n_3(x_0) - \frac{k_2}{k_1}m_0(x_0)} \right), & \text{for (2)} 
\end{cases}$$

with

$$n_\tau(0, x_0) = 2b(k_1 + \frac{k_2}{m_0(x_0)})u_{0,x}(x_0).$$

**Proof.** Again, by Theorem 6.1 with the density argument, it suffices to consider the case $s \geq 3$. Recall $\Gamma = k_1mu_x + k_2u_x$. Thanks to (6.18) and (7.5c), along the flow in (6.18) we have

$$\Gamma_x + (k_1(2\mu_0u - u_x^2) + k_2u) \frac{\Gamma}{x} \leq -2\Gamma^2 + C_1u_x\Gamma + C_2(k_1m + k_2).$$

Similarly, one can see from (3.1) that

$$\frac{dm}{dt}(t, q(t, x_0)) = -2m\Gamma(t, q(t, x_0)).$$

Notice that $3k_2 - 2\mu_0k_1 \leq 0$, which implies that

$$-\frac{d^2}{dt^2}n(\tau, q(t, x_0)) = \frac{d}{dt}\left( \frac{1}{m^2} \frac{dm(\tau, q(t, x_0))}{dt} \right) = -2\frac{d}{dt}\left( \frac{\Gamma}{m} \right)(\tau, q(t, x_0))$$

$$\geq (2k_1\mu_0 - 3k_2)u_x^2(k_1 + \frac{k_2}{m})$$

$$- 2[k_1\mu_0\mu_1(\sqrt{\frac{3}{2}}\mu_0 - \mu_1) + k_2\mu_1(\frac{\sqrt{3}}{2}\mu_0 - \frac{1}{2}\mu_1)](k_1 + \frac{1}{m}k_2),$$

$$\geq -2C_2(k_1 + k_2n)(\tau, q(t, x_0)).$$

(7.15)

It is found from (7.15) that

$$(1 - \frac{d^2}{dt^2})n(\tau, q(t, x_0)) \geq \frac{k_1}{k_2}. \quad (7.16)$$

Multiplying (7.16) by $e^\tau$ and integrating the resulting equation from 0 to $\tau$, we obtain

$$e^\tau \left( n(\tau, q(t, x_0)) - n_\tau(\tau, q(t, x_0)) \right) \geq n_0(x_0) - n_\tau(0, x_0) - \frac{k_1}{k_2}(e^\tau - 1). \quad (7.17)$$

Furthermore, multiplying (7.17) by $e^{-2\tau}$ and integrating it from 0 to $\tau$ yields

$$0 \leq n(\tau, q(t, x_0))$$

$$\leq \frac{1}{2}e^\tau \left( n_0(x_0) + n_\tau(0, x_0) + \frac{k_1}{k_2} \right) - \frac{k_1}{k_2} + \frac{1}{2}e^{-\tau} \left( n_0(x_0) - n_\tau(0, x_0) + \frac{k_1}{k_2} \right). \quad (7.18)$$

In view of the assumptions of the theorem, we now consider two possibilities.

(1) If $x_0$ satisfies (7.13a), then we have

$$n_0(x_0) - n_\tau(0, x_0) + \frac{k_1}{k_2} = \frac{2\Gamma(0, x_0)}{m_0(x_0)} + \frac{k_1}{k_2} = 0.$$

(2) If $x_0$ satisfies (7.13b), then we have

$$n_0(x_0) - n_\tau(0, x_0) + \frac{k_1}{k_2} = \frac{2\Gamma(0, x_0)}{m_0(x_0)} + \frac{2k_1}{k_2}.$$

It implies that

$$n_0(x_0) - n_\tau(0, x_0) + \frac{k_1}{k_2} = \frac{2}{m_0(x_0)} + \frac{2k_1}{k_2}.$$
From (7.18) and above expressions, we have

\[ 0 \leq \frac{1}{m(t, q(t, x_0))} \leq -e^{-\tau} \left( \frac{k_1}{k_2} e^{\tau} - \frac{1}{m_0(x_0)} - \frac{k_2}{k_1} \right). \]

It implies that one may find a time \( 0 < T^* \leq T^* = \sqrt{\frac{1}{2b} \ln(1 + \frac{k_1}{k_2} m_0(x_0))} \) such that

\[ m(t, q(t, x_0)) \to +\infty, \quad \text{as} \quad t \to T^*. \]

In terms of (7.17), we also have

\[ \Gamma(t, q(t, x_0)) = -\frac{m_t(t, q(t, x_0))}{2m(t, q(t, x_0))} = \frac{1}{2b} \frac{n_{\tau}(\tau, q(t, x_0))}{n(\tau, q(t, x_0))} \leq \frac{1}{2b} \left[ n(\tau, q(t, x_0)) + e^{-\tau} \left( \frac{k_1}{k_2} (e^{\tau} - 1) - n_0(x_0) + n_{\tau}(0, x_0) \right) \right]. \]

(7.19)

\[ \Gamma(t, q(t, x_0)) \leq 1 + e^{-\tau} \left( \frac{k_1}{k_2} (e^{\tau} - 1) - n_0(x_0) + n_{\tau}(0, x_0) \right) m(t, q(t, x_0)). \]

As discussed above, we see that

\[ \frac{k_1}{k_2} e^{\tau} - 2n_0(x_0) - \frac{2k_1}{k_2} \leq -(n_0(x_0) + \frac{k_1}{k_2}), \quad \text{for} \quad t \leq T^*. \]

Thus we deduce from (7.19) that

\[ \liminf_{t \to T^*} \left( \inf_{x \in \mathbb{S}} \Gamma(t, x) \right) = -\infty \quad \text{as} \quad t \to T^*. \]

(2). If the initial data satisfies (7.13b), it then follows that

\[ n_0(x_0) + n_{\tau}(0, x_0) + \frac{k_1}{k_2} = (1 + 2b k_2 u_{0, x}(x_0)) \left( \frac{k_1}{k_2} + \frac{1}{m_0(x_0)} \right) < 0, \]

and

\[ 4b^2 (k_1 m_0(x_0) + k_2)^2 u_{0, x}(x_0) > 1 + \frac{2k_1}{k_2} m_0(x_0). \]

A direct computation yields

\[ n_{\tau}(0, x_0) = \frac{2b \Gamma(0, x_0)}{m_0(x_0)} = 2b \left( k_1 + \frac{k_2}{m_0(x_0)} \right) u_{0, x}(x_0). \]

Combining above expressions, we find

\[ n_{\tau}^2(0, x_0) - n_0^2(x_0) - \frac{k_1}{k_2} n_0(x_0) > 0. \]

and

\[ \sqrt{n_{\tau}^2(0, x_0) - n_0^2(x_0) - 2\frac{k_1}{k_2} n_0(x_0)} > -n_0(x_0) - n_{\tau}(0, x_0). \]

So the equation

\[ \frac{1}{2} \left( n_0(x_0) + n_{\tau}(0, x_0) + \frac{k_1}{k_2} \right) z^2 - \frac{k_1}{k_2} z + \frac{1}{2} \left( n_0(x_0) - n_{\tau}(0, x_0) + \frac{k_1}{k_2} \right) = 0 \]  (7.20)

has one positive root

\[ z_1 = \frac{\frac{k_1}{k_2} - \sqrt{n_{\tau}^2(0, x_0) - n_0^2(x_0) - 2\frac{k_1}{k_2} n_0(x_0)}}{n_0(x_0) + n_{\tau}(0, x_0) + \frac{k_1}{k_2}} \]

and one negative root

\[ z_2 = \frac{\frac{k_1}{k_2} + \sqrt{n_{\tau}^2(0, x_0) - n_0^2(x_0) - 2\frac{k_1}{k_2} n_0(x_0)}}{n_0(x_0) + n_{\tau}(0, x_0) + \frac{k_1}{k_2}}. \]
It is then inferred from (7.18) that
\[
0 \leq \frac{1}{m(t, q(t, x))} \leq \frac{1}{2} e^{-\gamma} (n_0(x_0) + n_\tau(0, x_0) + \frac{k_1}{k_2})(e^{\gamma} - z_1)(e^{\gamma} - z_2),
\]
Note that \(z_1 > 1\) and \(z_2 < 0\). Then one may find a time \(0 < T^* \leq t^* = b \ln z_1\) such that
\[
m(t) \to +\infty \text{ as } t \to T^*.
\]
For any \(t \leq T^*\), we have
\[
n_\tau(0, x_0) + \frac{k_1}{k_2}(e^{\gamma} - 1) - n_0(x_0)
\]
\[
\leq \frac{k_1}{k_2} - \sqrt{n_0^2(0, x_0) - n_\tau^2(0, x_0) - 2 \frac{k_1}{k_2} n_0(x_0)} - n_0(x_0) + n_\tau(0, x_0) - \frac{k_1}{k_2}
\]
\[
= \sqrt{n_0^2(0, x_0) - n_\tau^2(0, x_0) - 2 \frac{k_1}{k_2} n_0(x_0) - 2 \frac{k_1}{k_2} n_0(x_0) - \frac{k_1}{k_2}} < 0.
\]
Making use of (7.19), we deduce that
\[
\liminf_{t \uparrow T^*} \left( \inf_{x \in S} \Gamma(t, x) \right) = -\infty \text{ as } t \to T^*.
\]
This completes the proof of the theorem. \(\square\)

**Remark 7.1.** In the case of \(C_1 > 0\), \(C_2 \geq 0\), replacing \(C_2\) by
\[
\tilde{C}_2 = (3k_2 - 2\mu_0 k_1)\mu_0^2 + \frac{5\sqrt{3}}{3} k_2 \mu_0 \mu_1 - \frac{1}{12}(9k_2 + 26k_1 k_0) \mu_0^2,
\]
Theorem 7.2 still holds.

**Remark 7.2.** In the case of \(C_1 > 0\) and \(C_2 < 0\), Theorem 7.2 holds just by changing \(C_2\) to
\[
\tilde{C}_2 = (3k_2 - 2\mu_0 k_1)\mu_0^2 + \frac{\sqrt{3}}{6} \mu_1^2.
\]
Recall that \(\mu_0 = \mu(u(t)) = \int S u(t, x) \, dx\) and \(m_0 = (\mu - \partial_x^2)u_0\).

**Lemma 7.3.** Let \(T > 0\) and \(v \in C^1([0, T); H^2(\mathbb{R}))\). Then for every \(t \in [0, T)\), there exists at least one point \(\xi(t) \in \mathbb{R}\) with
\[
I(t) := \inf_{x \in \mathbb{R}} (v_x(t, x)) = v_x(t, \xi(t)).
\]
The function \(I(t)\) is absolutely continuous on \((0, T)\) with
\[
\frac{dI(t)}{dt} = v_{tx}(t, \xi(t)), \text{ a.e. on } (0, T).
\]

The following result demonstrates that wave-breaking could depend on the infimum of \(u_x\).

**Theorem 7.3.** Assume \(k_1 > 0\), \(k_2 > 0\) and \(m_0 > 0\). Let \(u_0 \in H^s, s > 5/2\) and \(T^* > 0\) be the maximal time of existence of the corresponding solution \(u(t, x)\) to (3.1) with the initial data \(u_0\). If \(C_2 > 0\) (defined in Lemma 7.2) and
\[
\inf_{x \in \mathbb{R}} (u_{0,x}(x)) < -\sqrt{\frac{2k_1 \mu_0 \mu_1 (\sqrt{3} \mu_0 - 3 \mu_1) + k_2 \mu_1 (2\sqrt{3} \mu_0 - 3 \mu_1)}{3k_2 + 6\mu_0 k_1}} := -K,
\]
then the corresponding solution \(u(t, x)\) to (3.1) blows up in finite time \(T^*\) with
\[
0 < T^* \leq \frac{4}{(k_2 + 2k_1 \mu_0)(\inf_{x \in \mathbb{R}} u_{0,x}(x) + \sqrt{-K \inf_{x \in \mathbb{R}} u_{0,x}(x)})},
\]
such that
\[\liminf_{t \uparrow T^*} \left( \inf_{x \in \mathbb{R}} (u_x(t, x)) \right) = -\infty.\]

**Proof.** The proof of the theorem is approached by applying Lemma \[7.3\]. In fact, if we define \[w(t) = u_x(t, \xi(t)) = \inf_{x \in \mathbb{R}} [u_x(t, x)],\] then for all \( t \in [0, T^*), \ u_{xx}(t, \xi(t)) = 0. \) Thus one finds that
\[
\frac{d}{dt} w(t) \leq -\left( \frac{1}{2} k_2 + k_1 \mu_0 \right) \left( w^2(t) - K^2 \right).
\]

Applying the assumptions of Theorem \[7.3\] it can be easily deduced that if
\[
w(0) < -\sqrt{\frac{2k_1 \mu_0 \mu_1 (\sqrt{3} \mu_0 - 3 \mu_1) + k_2 \mu_1 (2 \sqrt{3} \mu_0 - 3 \mu_1)}{3k_2 + 6 \mu_0 k_1}},
\]
then \( T^* \) is finite and \( \liminf_{t \uparrow T^*} \left( \inf_{x \in \mathbb{R}} u_x(t, x) \right) = -\infty. \) This completes the proof of the theorem. \( \square \)

The following result shows that wave-breaking may depend on the infimum of \( m u_x. \)

**Theorem 7.4.** Suppose \( u_0 \in H^s(S), s > 5/2 \) with \( k_2 > 0, \ k_1 > 0 \) and \( C_2 > 0 \) (defined in Lemma \[7.2\]). Let \( T^* > 0 \) be the maximal time of existence of the corresponding solution \( u(t, x) \) to \( (3.1) \) with the initial data \( u_0. \) Assume that for some \( x_0 \in S \) with \( m_0(x_0) > 0 \) and
\[
u_0, x_0(x_0) < -\sqrt{C_2 (k_1 m_0(x_0) + k_2)} \]
\[\frac{1}{k_1 m_0(x_0)} \]
(7.21)

Then the solution \( u(t, x) \) blows up at finite time
\[
T^* \leq t^* = -\frac{k_1 m_0(x_0) u_0, x_0(x_0)}{C_2 (k_1 m_0(x_0) + k_2)} - \sqrt{\left( \frac{k_1 m_0(x_0) u_0, x_0(x_0)}{C_2 (k_1 m_0(x_0) + k_2)} \right)^2 - \frac{1}{C_2 (k_1 m_0(x_0) + k_2)}}. \]

Moreover, when \( T^* = t^* \), the following estimate of the blow-up rate holds
\[
\liminf_{t \uparrow T^*} \left( (T^* - t) \inf_{x \in S} m u_x(t, x) \right) \leq -\frac{1}{2k_1}. \]

**Proof.** As in the proof of Theorem \[6.1\] it suffices to consider the case \( s \geq 3. \) Thanks to \[7.3a\] and \[6.18\], we have
\[
\frac{d}{dt} M(t, q(t, x_0)) \leq -2k_1 M^2(t, q(t, x_0)) + C_2 m(t, q(t, x_0)).
\]
And recall that
\[
\frac{d}{dt} m(t, q(t, x_0)) = -2M(t, q(t, x_0)) (k_1 m + k_2).
\]

Denote \( \overline{m}(t) := 2 \left( m(t, q(t, x_0)) + k_2/k_1 \right), \) then
\[
\frac{d}{dt} M(t, q(t, x_0)) \leq -2k_1 M^2(t, q(t, x_0)) + C_2 \left( \frac{\overline{m}}{2} - \frac{k_2}{k_1} \right) \leq -2k_1 M^2(t, q(t, x_0)) + \frac{C_2}{2} \overline{m}.
\]
Applying the above two equations, we deduce that
\[ \frac{d}{dt} \left( \frac{1}{w(t)} \right) = -2k_1 \frac{d}{dt} \left( \frac{1}{w(t)} M \right) \]
\[ = \frac{2k_1}{w(t)^2} \left( -w(t) \frac{d}{dt} M(t) + M(t) \frac{d}{dt} w(t) \right) \geq -k_1 C_2. \]

Integrating it from 0 to \( t \) leads to
\[ \frac{1}{w(t)^2} \frac{d}{dt} w(t) \geq C_3 - k_1 C_2 t, \]
with
\[ C_3 := \frac{w(0)'}{w(0)^2} = -\frac{2k_1 M(0)}{w(0)} = -\frac{k_1^2 (m_0 u_{0,x})(x_0)}{k_1 m_0(x_0) + k_2}. \]
and hence
\[ \frac{1}{w(t)} \leq \frac{1}{2} \left( k_1 C_2 t^2 - 2C_3 t + \frac{k_1}{k_1 m_0(x_0) + k_2} \right). \]

Let
\[ t^* := \frac{C_3}{k_1 C_2} - \sqrt{\left( \frac{C_3}{k_1 C_2} \right)^2 - \frac{1}{C_1(k_1 m_0(x_0) + k_2)}}, \]
and
\[ t_* := \frac{C_3}{k_1 C_2} + \sqrt{\left( \frac{C_3}{k_1 C_2} \right)^2 - \frac{1}{C_2(k_1 m_0(x_0) + k_2)}}. \]
Thus,
\[ 0 \leq \frac{1}{w(t)} \leq \frac{1}{2} k_1 C_2 (t - t^*)(t - t_*). \]
Therefore,
\[ \inf_{x \in S} m u_x(t, x) \leq M(t) \rightarrow -\infty, \quad \text{as} \quad t \rightarrow T^* \leq t^*, \]
This implies that the solution \( u(t, x) \) blows up at the time \( T^* \).

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