I begin to look at the growth of weak density inhomogeneities of nonrelativistic matter, in bimetric-MOND (BIMOND) cosmology. Far from making an exhaustive study, I concentrate on one attractive cosmological scenario, which employs matter-twin-matter-symmetric versions of BIMOND, and, furthermore, assumes that, on average, the universe is symmetrically populated in the two sectors. MOND effects are totally absent in an exactly symmetric universe, apart from the significant possible appearance of a cosmological constant, \( \Lambda \sim \left( \frac{a_0}{c} \right)^2 \). MOND effects—local and cosmological—do enter when density inhomogeneities that differ in the two sectors appear and develop. MOND later takes its standard form in systems that are islands dominated by pure matter, as are presumably the well scrutinized systems such as galaxies. I derive the nonrelativistic (weak-field-slow-motion) equations governing small-scale fluctuation growth. The equations split into two uncoupled systems, one for the sum, the other for the difference, of the fluctuations in the two sectors. The former is governed strictly by Newtonian dynamics, and describes standard growth of fluctuations. The latter is governed by MOND dynamics, which entails stronger gravity, and nonlinearity even for the smallest of perturbations. These cause the difference to grow faster than the sum, leading to anticorrelated perturbations, conducing to matter-twin-matter segregation (which continues for high overdensities). The nonlinearity also causes interaction between nested perturbations on different scales. Because matter and twin matter (TM) repel each other in the MOND regime, matter inhomogeneities grow not only by their own self gravity, but also through shepherding by flanking TM overdensities (and vice versa). The relative importance of gravity and pressure in the MOND system (analog of the Jeans criterion), depends also on the strength of the perturbation. The development of structure in the universe, in either sector, thus depends crucially on two initial fluctuation spectra: that of matter alone and that of the matter-TM difference. I also discuss the back reaction on cosmology of BIMOND effects that appear as “phantom matter” (interpreted by some as “dark matter”), resulting from inhomogeneity differences between the two sectors.

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I. INTRODUCTION

A full fledged treatment of cosmology and of structure formation in the MOND paradigm is still wanting. Structure formation on an expanding cosmological background, in nonrelativistic (NR) versions of MOND, has been studied with \( N \)-body calculations, based on various heuristic extensions of MOND theories designed to work for isolated NR systems, e.g., in [1–5]. Clearly, a definite treatment of the problem has to be based on a fully relativistic theory, encompassing both cosmology and structure formation. Much work along such lines has been done in the framework of TeVeS—a relativistic MOND theory propounded by Bekenstein [6] (Sanders’s stratified framework [7] is a precursor, and [8] is a recent review of TeVeS and its cosmology). For example, [9–14] considered various aspects of cosmology at large, the cosmic microwave background (CMB), and linear stages of structure formation in TeVeS-like theories. Although very encouraging, these results do not yet provide a fully satisfactory description of the phenomena at hand.

A new relativistic theory for MOND (called BIMOND, for “bimetric MOND”) was proposed recently based on a bimetric structure of space-time [15]. In this theory, gravity is described by two metrics instead of the one underlying general relativity (GR). Ordinary matter, to whose sector belongs our direct ken, couples, as in GR, only to one metric, call it the primary one. The other metric serves as an auxiliary to which the primary metric couples. Such a bimetric description of gravity has a long history, and, in particular, has been discussed recently in connection with some “dark substance” phenomena (see [16–19], and references therein). The novelty in BIMOND is in the special choice of interaction between the two metrics, which is particularly germane in the context of MOND, and which reproduces MOND phenomenology.

In [15], I made preliminary comments on cosmological solutions within BIMOND. Classes of cosmological solutions within certain versions of BIMOND were also considered in [20]. But, there is still much left to be done on this problem.

We are also still lacking any treatment of the important problem of the growth of small perturbations. Ideally, one should treat the problem in a fully relativistic framework, as done in [12–14], in the context of TeVeS-like theories. Here I take the lighter task of considering only density inhomogeneities of NR matter on scales smaller than cosmological. This limited-scope formalism, which governs the growth of perturbations under matter dominance, is still rather
illuminating, and shows a variety of phenomena potentially highly pertinent to structure formation in our universe.

I limit myself to a universes that on average is completely symmetric in matter and twin matter (TM). Twin matter is the putative matter that couples to the “auxiliary” metric as (ordinary) matter couples to the “primary” one. At present, the existence of TM is not required phenomenologically (in particular, it is not, in fact cannot be, a sort of “dark matter,” as assumed to explain galaxy dynamics); it is invoked mainly for aesthetic reasons: With TM we can obtain a simple and symmetric world picture that affords favorable cosmologies. In addition, as our present analysis shows, symmetric presence of matter and TM has very interesting consequences for structure growth. It leads to phenomena that are not only greatly different from those in GR (without dark matter), but also different from those in MOND without TM.

Of course, the existence of TM would add crucial unknowns to our cosmological world view. Its properties, composition, self-interactions, amounts, etc., are not known (see [15] for more on this). To minimize these unknowns I assume here that, with one exception, the TM sector is identical to the matter sector in all respects, having the same properties (masses, etc.), the same interactions within the sector (electromagnetic, strong, etc.), the same interaction with its metric (geodesic motion, etc.), and the same average amounts of its different components present in the universe. As shown in [15] this symmetry leads to simple cosmological solutions, identical with those of GR, albeit with a naturally appearing cosmological constant (CC) of order \( (a_0/c)^2 \), where \( a_0 \) is the MOND constant.\(^1\) The only difference between the two sectors I allow for, is in the amplitude and spectrum of the primordial inhomogeneities. These are putatively ascribed to quantum fluctuations in the early Universe. They could, thus, differ in the two sectors—at least in their exact distribution, if not in their statistical properties—even with all else being the same.

Some preliminaries of the problem discussed here are alluded to in [21], where I considered the problem of NR, small, isolated matter-TM systems. Here, this problem is extended to the evolution of perturbations on an otherwise homogeneous and matter-TM symmetric (MTMS), expanding universe.

In section II, I briefly describe the aspects of BIMOND and of its NR limit that we need in this paper. In section III, I derive the equations controlling the evolution of small departures from exact density equality in general NR systems. This is applied to cosmological small inhomogeneities in section IV. In section V, I describe succinctly how the density differences back react on the cosmological equations, introducing modifications that can act as cosmological “dark matter” and “dark energy” (beside the CC). Section VI is a discussion.

II. BIMOND RECAPTED

BIMOND is a class of relativistic theories of gravity that involve two metrics, \( g_{\mu\nu} \) and \( \hat{g}_{\mu\nu} \). They are governed by an action of the form

\[
I = -\frac{1}{16\pi G} \int \left[ \beta g^{1/2} R + \alpha \hat{g}^{1/2} \hat{R} - 2(\beta \hat{g})^{1/4} a_0^2 \mathcal{M} \right] d^4 x + I_M(g_{\mu\nu}, \psi) + \hat{I}_M(\hat{g}_{\mu\nu}, \hat{\psi}),
\]

where \( R \) and \( \hat{R} \) are the Ricci scalars of the two metrics as appear standardly in the Einstein-Hilbert action of GR, \( G \) is Newton’s constant, and I use units where \( c = 1 \). The interaction term \( \mathcal{M} \) is a dimensionless, scalar function of the two metrics and their first derivatives.

Bimetric theories have been much discussed before; the novelty and crux of BIMOND is in the choice of the interaction term. The key observation, in the MOND context, is that with two metrics and their first derivatives, we can construct tensors with the dimensions of acceleration:

\[
C^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \hat{\Gamma}^\alpha_{\beta\gamma},
\]

where \( \Gamma^\alpha_{\beta\gamma} \), \( \hat{\Gamma}^\alpha_{\beta\gamma} \) are the Levi-Civita connections of the two metrics. Within the MOND paradigm, having the acceleration constant, \( a_0 \), at our disposal, we can construct from these dimensionless scalars to serve as variables on which \( \mathcal{M} \) depends. These scalars are obtained by contracting the dimensionless tensors \( a_0^{-4} C^\alpha_{\beta\gamma} \).

It befits us to concentrate on scalars that are quadratic in the \( C^\alpha_{\beta\gamma} \), which I do. In particular, the scalars constructed from the quadratic tensor

\[
\Upsilon_{\mu\nu} \equiv C^\alpha_{\mu\lambda} C^\lambda_{\nu\gamma} - C^\alpha_{\nu\lambda} C^\lambda_{\mu\gamma},
\]

\(^1\) So the successes of standard Big Bang cosmology, such as nucleosynthesis, are retained. The symmetry assumption entails, for example, having the same inflation, and the same mechanism for baryogenesis in the two sectors.

\(^2\) In the units where \( c = 1 \), \( a_0 \) has dimensions of length\(^{-1}\). In standard units, \( a_0 \) is replaced everywhere by \( \ell^{-1} \equiv a_0/c^2 \). Then in the NR limit only \( a_0 \) appears. In standard units, \( C^0_{\beta\gamma} \) have dimensions of length\(^{-1}\); so the dimensionless tensors to be used are \( \ell C^0_{\beta\gamma} = (c^2/a_0)C^0_{\beta\gamma} \).
have particular appeal; this tensor is symmetric under the interchange of the two metrics. The scalars I use are then
\[ \Upsilon = g^{\mu\nu} \Upsilon_{\mu\nu}, \quad \hat{\Upsilon} = \hat{g}^{\mu\nu} \Upsilon_{\mu\nu}. \] (4)

These are obtained from each other under metric interchange, and are used symmetrically in constructing MTMS actions for BIMOND. The action terms \( I_M \) and \( \hat{I}_M \) are the matter actions for standard matter and for the putative TM (with their degrees of freedom denoted by \( \psi \) and \( \hat{\psi} \)), whose existence is suggested by the double metric nature of the theory. Just as standard matter couples only to the standard metric \( g_{\mu\nu} \), TM couples only to the second metric \( \hat{g}_{\mu\nu} \). There is no direct interaction between the two matter sectors. The dimensionless parameters \( \beta \) and \( \alpha \) permit us to use gravitational couplings in the two sectors, \( G' = G/\beta \) and \( G' = G/\alpha \), that differ from each other and from \( G \) itself. In the rest of the paper I shall concentrate on symmetric versions of the theory; this means taking \( \alpha = \beta \), and that the two metrics appear in a symmetric way in the interaction term. As explained in [15, 21], there are good reasons to take \( \beta \approx 1 \) (see below). However, for the sake of generality I shall keep a general value of \( \beta \) in some of the discussion. Note that without the interaction term the theory consists of two uncoupled copies of GR.

The resulting field equations are of the form
\[ \beta G_{\mu\nu} + S_{\mu\nu} = -8\pi G T_{\mu\nu}, \] (5)
\[ \beta \hat{G}_{\mu\nu} + \hat{S}_{\mu\nu} = -8\pi G \hat{T}_{\mu\nu}, \] (6)
where
\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \quad \hat{G}_{\mu\nu} = \hat{R}_{\mu\nu} - \frac{1}{2} \hat{g}_{\mu\nu} \hat{R} \] (7)
are the Einstein tensors, and \( T_{\mu\nu}, \hat{T}_{\mu\nu} \) are the energy-momentum tensors (EMTs) for the two sectors.

The equations of motion of matter in the two sectors, in the presence of gravity, are the standard ones since they are still controlled by the same matter actions as in standard physics.

The tensors \( S_{\mu\nu} \) and \( \hat{S}_{\mu\nu} \) fall from the variation of the interaction term with respect to \( g_{\mu\nu} \) and \( \hat{g}_{\mu\nu} \), respectively.\(^4\) The metrics appear in the interaction in three ways: (i) through their derivatives, in \( C^\lambda_{\mu\nu}, \) (ii) contracting the \( C^\lambda_{\mu\nu} \) in the quadratic scalars on which \( M \) depends, (iii) otherwise, such as in the volume density \( (gg)^{1/4} \), or in scalars such as \( g/\hat{g} \) or \( g^{\mu\nu} \hat{g}_{\mu\nu} \). As a result, \( S_{\mu\nu} \) and \( \hat{S}_{\mu\nu} \) have the schematic form
\[ S_{\mu\nu} = (Q(C)_{\mu\nu})_{,\lambda} + N(CC)_{\mu\nu} + a_2^Q P g_{\mu\nu}, \]
\[ \hat{S}_{\mu\nu} = (\hat{Q}(C)_{\mu\nu})_{,\lambda} + \hat{N}(CC)_{\mu\nu} + a_2^\hat{P} \hat{P} \hat{g}_{\mu\nu}, \] (8)
with the three term types corresponding, respectively, to the above types of appearance of the metrics. Here, \( (C)_{\mu\nu} \) denotes tensors linear in \( C^\lambda_{\mu\nu} \), and \( (CC)_{\mu\nu} \) denotes tensors quadratic in them;\(^5\) they are the same in the two equations. Also, \( \hat{\cdot} \) denotes covariant derivative with respect to \( \hat{g}_{\mu\nu} \). \( Q, N, P \), etc. are dimensionless, and presumably of order unity. They depend on the two metrics through the quadratic scalars, and possibly through scalars not containing the metric derivatives, such as \( g/\hat{g} \) or \( g^{\mu\nu} \hat{g}_{\mu\nu} \). \( Q \) and \( N \) are linear in derivatives of \( M \) with respect to the quadratic scalars, e.g., \( \Upsilon \) and \( \hat{\Upsilon} \) defined in Eq.(4), while \( P \) gets contributions from \( M \) itself and from its derivatives with respect to scalars containing the metrics alone. This classification of the terms helps understand some interesting special cases and limits of the theory. In what follows I refer to these contributions as \( Q, N, P \) terms, respectively.

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\(^3\) The special appeal in the choice of the tensor \( \Upsilon_{\mu\nu} \), and the two scalars constructed from it, is twofold [15]: First, it leads to a theory with a particularly simple NR limit. Second, this choice is the best equivalent to the choice of the Ricci tensor, \( R_{\mu\nu} \), and the Ricci scalar, \( R = g^{\mu\nu} R_{\mu\nu} \), in the Einstein-Hilbert Lagrangian of GR. In GR we have to use the full curvature tensor, which contains the second derivatives of the metric; this is both necessary (to get a scalar), and harmless (because the second derivatives in the Einstein-Hilbert Lagrangian do not affect the field equations). In fact, it is well known that we can replace \( R \) in the GR Lagrangian by \( g^{\mu\nu}(\Gamma^\alpha_{\mu\nu} \Gamma^\lambda_{\alpha\nu} - \Gamma^\alpha_{\mu\lambda} \Gamma^\lambda_{\nu\nu}) \), exactly analogous to our \( \Upsilon, \hat{\Upsilon} \) scalars. In our case we do not add the second derivatives; this would be both harmful (as it would lead to a higher-derivative theory), and unnecessary (as the first-derivative terms are tensor in themselves).

\(^4\) They satisfy, together, one set of four Bianchi-like differential identities, stemming from the invariance of the interaction term to general coordinate transformations (details in [15]).

\(^5\) Examples of such tensors are \( C^\lambda_{\mu\nu}, \delta^\lambda_\nu C^\alpha_{\mu\nu}, \) \( g^{\mu\nu} g^{\alpha\beta} C^\lambda_{\alpha\beta}, \) etc.

\(^6\) Examples of such tensors are \( C^\lambda_{\mu\nu}, C^\alpha_{\mu\nu} C^\lambda_{\alpha\nu}, \) \( \Upsilon_{\mu\nu}, \) etc.
From the symmetry of the problem we have
\[ \hat{Q}(g_{\mu\nu}, \hat{g}_{\mu\nu}) = -Q(g_{\mu\nu}, g_{\mu\nu}), \quad \hat{N}(g_{\mu\nu}, \hat{g}_{\mu\nu}) = N(g_{\mu\nu}, g_{\mu\nu}), \quad \hat{P}(g_{\mu\nu}, \hat{g}_{\mu\nu}) = P(g_{\mu\nu}, g_{\mu\nu}). \]  \tag{9}

BIMOND versions with \( \beta = 1 \) have a simple GR limit: Assume that in the limit \( a_0 \to 0 \), i.e., when the quadratic scalar arguments of \( \mathcal{M} \) go to infinity, \( \mathcal{M} \) becomes independent of them. Then, the \( Q \) and \( N \) terms vanish. The \( P \) terms then give CC-type terms that might still couple the two metrics, but which are rather small locally. It is also possible to choose \( \mathcal{M} \) such that the coupling disappears altogether in this limit, so the \( P \) terms in \( S_{\mu\nu} \) collect to \( a_0^2 P(\infty)g_{\mu\nu} \) [15]. The theory then decouples into two copies of GR, possibly with a cosmological constant, if \( \mathcal{M}(\infty) \neq 0 \). I have not yet been able to ascertain if there is an acceptable GR limit for theories with \( \beta \neq 1 \). This may be one reason to prefer theories with \( \beta = 1 \).

When the two metrics are equal, we have from Eq.\( (9) \) \( \hat{Q} = -Q \equiv -Q_0, \quad \hat{N} = N \equiv N_0, \quad \hat{P} = P \equiv P_0, \) where, furthermore, all these functions of the two metrics become constants. This is because we cannot construct (non-constant) scalars from a single metric and its first derivatives, so all the scalar arguments of these functions must become constants when \( \hat{g}_{\mu\nu} = g_{\mu\nu} \). Clearly, \( Q \) terms vanish linearly in the metric difference, and terms of type \( N \) vanish quadratically. We are then left only with \( P \) terms, which are of a cosmological constant type with \( \Lambda \propto a_0^2 P_0 \). From this follows that, in systems with identical matter and TM mass distributions, and the same initial and boundary conditions, the metrics in the two sectors are the same, and are a solution of the standard Einstein equations with a cosmological constant (CC) \( \Lambda \).\(^8,\)\(^9\)

### A. Cosmological background

Preliminary considerations of cosmology in BIMOND have been described in [21] and in [20]. As mentioned in the Introduction, here I assume all along the special, but attractive, case of an MTMS cosmology. This means that matter sources in cosmology are assumed to be identical in the two sectors, apart from differences in the initial seed inhomogeneities. If we neglect these differences to lowest order, the MTMS cosmology in BIMOND is identical to the Friedmann-Robertson-Walker (FRW) cosmology in GR, with a CC, \( \Lambda \propto a_0^2 P_0 \), with \( G' = G/\beta \) instead of \( G \), and with the standard matter content. All the standard results derived for the latter (Big Bang, Nucleosynthesis, etc.) apply here as well (with \( G' \) as gravitational constant).

To a higher order, the differences in the inhomogeneities in the two sectors engender departure of the background cosmology from FRW: the cosmological metric is no longer a solution of the Einstein equations. This is discussed briefly in section V. To some approximation the cosmology could still be considered an FRW one, but with non-standard, “phantom” contributions to the cosmological EMT in BIMOND. This sort of feedback goes beyond the feedback due to departure from smoothness, which has been discussed extensively respecting GR cosmology. Here it is specifically due to differences between the inhomogeneities in the two sectors. As long as the statistical properties of the inhomogeneities in the two sectors are the same, the cosmological background metrics are still the same, but they satisfy the BIMOND equations, instead of the Einstein equations.

The observed \(^4\)He abundance was used in [22] to derive a bound of \( |G'/G - 1| \leq 0.13 \), where \( G' \) is the value that applies at the formation time. This would imply, in the present context, \( |\beta - 1| \lesssim 0.13 \), lending further motivation for taking \( \beta = 1 \).

### B. The nonrelativistic limit

Consider first the NR limit of BIMOND as applied to a system that is small on cosmological scales (discussed in detail in [15]). Since we assume a symmetric cosmology, with the two metrics equal, there is a coordinate frame in which locally the two metrics are nearly Minkowskian. Because of local inequality of matter and TM mass distributions, the departure from Minkowski is not the same for the two metrics, which can thus be written as
\[ g_{\mu\nu} = \eta_{\mu\nu} - 2\phi \delta_{\mu\nu} + h_{\mu\nu}, \quad \hat{g}_{\mu\nu} = \eta_{\mu\nu} - 2\hat{\phi} \delta_{\mu\nu} + \hat{h}_{\mu\nu}, \]  \tag{10}

\(^7\) The NR limit of the theory does have a good Newtonian limit even for \( \beta \neq 1 \).

\(^8\) For example, a double Schwarzschild solution is a vacuum solution of BIMOND, which corresponds to a black hole of equal matter and TM masses. Similarly, standard GR gravitational waves are also vacuum solutions of BIMOND.

\(^9\) Pictorially, in a double membrane picture the Einstein-Hilbert actions for the two metrics may be viewed, as usual, as the elastic energies of the membranes—the cost of distorting them out of the flat. Then, the CC may be viewed as the contact energy density, remaining when \( \hat{g}_{\mu\nu} = g_{\mu\nu} \), and the rest of the interaction is the “cost of separating” the membranes.
with the potentials $\phi$, $\hat{\phi}$, $h_{\mu\nu}$, $\hat{h}_{\mu\nu}$ treated to lowest order in the weak-field approximation.\textsuperscript{10}

Note, importantly, that the weak-field approximation assumes that $\phi, \hat{\phi} \ll 1$ (in our units where $c = 1$). This, however, does not imply that whenever $\phi$ or $\hat{\phi}$ appear they raise the order in the perturbation: Unlike the case of GR, here we have an additional dimensioned constant, $a_0$, and quantities such as $\nabla \phi / a_0$ or $(\nabla \phi - \nabla \hat{\phi}) / a_0$ are not assumed small, and do not raise the order of terms in which they appear. In this light, $C_{\mu\nu}^\alpha$, as they appear in $\{C\}_{\mu\nu}$ and $\{CC\}_{\mu\nu}$ are first order in the small parameters; however, in their appearance in the arguments of $Q$, $N$, $P$, etc. $a_0^{-1}C_{\mu\nu}^\alpha$ should not be taken as small.\textsuperscript{11}

The $N$ terms are second order and can be neglected. The $P$ terms contribute to lowest order as a homogeneous CC, whose value we know is observationally negligible in local NR physics. As to the $Q$ terms, since the $\{C\}_{\mu\nu}$ are already first order we can equate the two metrics everywhere else with the background Minkowski metric (so, for example, covariant derivatives become normal derivatives). But note that this approximation applies only to the metrics themselves, not to their derivatives, which appear only in $a_0^{-1}C_{\mu\nu}^\alpha$, which are not assumed small.

It can be seen\textsuperscript{12} that to the lowest order $Q = -\hat{Q}$, and so $S_{\mu\nu} = -\hat{S}_{\mu\nu}$. Thus, the sum and difference of the two field equations are, in this approximation,

$$\beta(G_{\mu\nu} + \hat{G}_{\mu\nu}) = -8\pi G (\mathcal{T}_{\mu\nu} + \hat{T}_{\mu\nu}), \quad \beta(G_{\mu\nu} - \hat{G}_{\mu\nu}) + 2S_{\mu\nu} = -8\pi G (\mathcal{T}_{\mu\nu} - \hat{T}_{\mu\nu}).$$

(11)

The sum of the Einstein tensors is linear in the sums of the potentials, and their difference, as well as $S_{\mu\nu}$, depends only on the differences of the potentials.

In [15] I treated the NR limit where one further assumes that motions are slow, so that time derivatives are negligible compared with spatial ones. I then showed that with the choice of the quadratic scalar arguments given in Eq.(4), which I shall assume,\textsuperscript{13} the field equations imply, in a certain choice of gauge: $h_{\mu\nu} = \hat{h}_{\mu\nu} = 0$. The two metrics thus have the form of the first-order metric in GR, but the NR potentials, $\phi$ and $\hat{\phi}$, are determined not from the Poisson equation but from field equations derived from the action

$$\mathcal{L} = -\frac{1}{8\pi G} \left[ \beta(\nabla \hat{\phi})^2 + \beta(\nabla \phi)^2 - a_0^2 \mathcal{M}[\nabla \phi - \nabla \hat{\phi})^2] \right] + \rho \left( \frac{1}{2} \nabla^2 - \phi \right) + \rho \left( \frac{1}{2} \nabla^2 - \hat{\phi} \right).$$

(12)

This follows from the fact that to lowest order we have $\Upsilon = \hat{\Upsilon} \propto (\nabla \phi - \nabla \hat{\phi})^2$, and so the interaction term takes the form $a_0^2 \mathcal{M}[\nabla \phi - \nabla \hat{\phi})^2]$. (I use here the same symbol, $\mathcal{M}$, for a function of a single variable: the reduction of the relativistic interaction function, also denoted $\mathcal{M}$, which depends on more variables.) This action also governs the particle equations of motions in the two sectors, which take the standard form: $\mathbf{a} = -\nabla \phi$ and $\mathbf{a} = -\nabla \hat{\phi}$. Because the metric has the GR form in terms of a single potential: $g_{\mu\nu} = \eta_{\mu\nu} - 2\phi \delta_{\mu\nu}$, light bending by slowly moving masses is expressed in terms of the potential in the same way as in GR: photons “see” the same potential as massive particles.

The field equations for the potentials are

$$\Delta \phi = 4\pi G \beta^{-1} \rho + \beta^{-1} \nabla \cdot [(\nabla \phi - \nabla \hat{\phi}) \mathcal{M}] \equiv 4\pi G (\rho + \rho_p),$$

$$\Delta \hat{\phi} = 4\pi G \beta^{-1} \hat{\rho} - \beta^{-1} \nabla \cdot [(\nabla \phi - \nabla \hat{\phi}) \mathcal{M}] \equiv 4\pi G (\hat{\rho} + \hat{\rho}_p),$$

(13)

where $\rho_p$ and $\hat{\rho}_p$ play the role of “phantom matter” (PM) densities for the two sectors.

As detailed in [21], it is convenient to work with the two potentials $\phi$ and $\hat{\phi}$ such that

$$\phi = \zeta \phi + \hat{\phi}, \quad \hat{\phi} = \zeta \phi - \phi,$$

(14)

which satisfy

$$\Delta \hat{\phi} = 4\pi G (\rho + \hat{\rho}), \quad \nabla \cdot \left\{ \mu [(\nabla \phi / a_0) \nabla \hat{\phi}] \right\} = 4\pi G (\rho - \hat{\rho}),$$

(15)

\textsuperscript{10} The separation into $\phi$ and $h_{\mu\nu}$ is a mere convenience: One defines $\phi \equiv -(1 + g_{\alpha\alpha})/2$, and $h_{\alpha\beta} = 0$.

\textsuperscript{11} $c_0$ may be viewed as a proxy for a scale length $\ell \equiv c^2 / a_0$. Terms such as $\nabla \phi / a_0$ appear, in fact, as $\ell (\nabla \phi / c^2)$. The NR limit in the MOND context has to be understood as $c \to \infty$, $\ell \to \infty$, with $c^2 / \ell$ fixed.

\textsuperscript{12} Following from Eq.(9), noting that in this approximation $Q$ is symmetric to the interchange of the metrics, as its argument is quadratic in their difference.

\textsuperscript{13} The general form of the scalar was also discussed in [15].
with the boundary conditions at infinity $\tilde{\phi} \to 0$, while $\phi \propto \ln(r)$ when $M \neq \tilde{M}$, and $\tilde{\phi} \to 0$ when $M = \tilde{M}$; $M$ and $\tilde{M}$ are the total masses in matter and TM respectively. Here, $\zeta = (2\beta)^{-1}$, and $\tilde{\mu}(x) \equiv 2\beta - 4\tilde{M}'(4x^2)$.\(^{14}\) MOND and Newtonian behaviors for isolated, pure-matter systems, such as galaxies, dictate the asymptotic behaviors $\tilde{\mu}(x \ll 1) \approx x$, $\tilde{\mu}(x \gg 1) \approx (1-\zeta)^{-1}$. Thus $\tilde{\phi} = \phi^N + \phi^N$ is simply the sum of the matter and TM Newtonian potentials, and $\tilde{\phi}$ is a MOND potential satisfying the nonlinear Poisson equation proposed by [23] with $\rho - \tilde{\rho}$ as a source (and a modified value of $\tilde{\mu}$ at large arguments).

### III. FLUCTUATIONS IN AN ISOLATED MTMS SYSTEM

Consider first the development of weak fluctuations in a cosmologically small, isolated, NR system. The two matter components are treated as fluids. The time dependent problem is governed by the NR continuity equations in the two sectors

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \tilde{\rho}_t + \nabla \cdot (\tilde{\rho} \tilde{\mathbf{v}}) = 0,$$

and the respective Euler equations

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla \phi - \rho^{-1} \nabla p, \quad \tilde{\mathbf{v}}_t + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}} = -\nabla \tilde{\phi} - \tilde{\rho}^{-1} \nabla \tilde{p},$$

where $\rho$, $\tilde{\rho}$ are the pressures in the two sectors.

In a system made predominantly of matter—as we have, purportedly, in the solar system, galaxies, etc.—these equations lead to standard MOND dynamics as they have been applied extensively in the past. In this case, the MOND acceleration is generically larger than the Newtonian one, and when the acceleration is smaller than $a_o$, dynamics is controlled by the MOND limit.

However, as discussed in [21], things are rather different in systems where the distribution of matter and TM are close to each other (i.e., $|\rho - \tilde{\rho}| \ll \rho$). For example, when $\rho = \tilde{\rho}$ the dynamics is always Newtonian, no matter how small the acceleration is.\(^{15}\)

Consider then weak perturbations on an arbitrary, symmetric background flow with $\rho = \tilde{\rho} = \rho_o(\mathbf{r}, t)$, $\mathbf{v} = \tilde{\mathbf{v}} = \mathbf{v}_o(\mathbf{r}, t)$. Then, from Eq.(15), $\phi = 0$, and $\phi = \phi = \beta^{-1} \phi^N$, where $\phi^N$ is the Newtonian potential of $\rho_o$: we have a double Newtonian system with an effective $G_c = G/\beta$.\(^{16}\) If we seed the system with fluctuations that are the same in the two sectors, they will remain so, with $\phi = 0$ at all times, and the development remaining quasi-Newtonian in both sectors.

Taking the lowest order in the fluctuations we have

$$\delta \rho_t + \nabla \cdot (\delta \rho \mathbf{v}) + \nabla \cdot (\rho_o \delta \mathbf{v}) = 0, \quad \delta \tilde{\rho}_t + \nabla \cdot (\delta \tilde{\rho} \tilde{\mathbf{v}}) + \nabla \cdot (\rho_o \tilde{\delta} \mathbf{v}) = 0,$$

and the Euler equations

$$\delta \mathbf{v}_t + (\delta \mathbf{v} \cdot \nabla)\mathbf{v}_o + (\mathbf{v}_o \cdot \nabla)\delta \mathbf{v} = -\zeta \nabla \delta \phi - \nabla \tilde{\phi} - \rho_o^{-1} \nabla \delta p + \delta \rho_o^{-2} \nabla p_o,$$

$$\delta \tilde{\mathbf{v}}_t + (\delta \tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}}_o + (\tilde{\mathbf{v}}_o \cdot \nabla)\delta \tilde{\mathbf{v}} = -\zeta \nabla \delta \phi - \nabla \tilde{\phi} - \rho_o^{-1} \nabla \tilde{\delta} p + \delta \rho_o^{-2} \nabla p_o,$$

where $p_o$ is the pressure field in the unperturbed system (assumed to be the same in the two sectors, as we assume the same equation of state for the two). The potential fluctuations are obtained from:

$$\Delta \delta \tilde{\phi} = 4\pi G (\delta \rho + \delta \tilde{\rho}),$$

$$\nabla \cdot (\tilde{\mathbf{v}}_o \nabla \tilde{\phi}/a_o \nabla \phi) = 4\pi G (\delta \rho - \delta \tilde{\rho}).$$

If $|\delta \rho| \sim |\delta \tilde{\rho}| \sim |\delta \rho - \delta \tilde{\rho}|$, the sources for these equations are of similar magnitudes; but this need not be so.

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\(^{14}\) The factor 4 in the argument accounts for the fact that we use the variable $\tilde{\phi} = (\phi - \tilde{\phi})/2$.

\(^{15}\) This is the NR expression of the reduction of BIMOND to GR for completely symmetric systems.

\(^{16}\) This case is not to be confused with the Newtonian limit in a system made of pure matter or pure TM, in which case $G$ itself plays the role of the Newton constant.
It is useful to work with the sums and differences:
\[ u = \delta v + \delta \nu, \quad u_\nu = \delta v - \delta \nu, \quad q = \delta \rho + \delta \dot{\rho}, \quad \tilde{q} = \delta \rho - \delta \dot{\rho}. \] (21)

If we can assume that in both sectors \( p \) can be taken a function of \( \rho \); e.g., for a polytropic gas, or when the process is isentropic, we write \( \delta \rho = (dp/d\rho)(\rho_0)\delta \rho \equiv v_s^2(\rho_0)\delta \rho \) and get:
\[ q_\nu + \vec{\nabla} \cdot (q\nu) + \vec{\nabla} \cdot (\rho_0 u) = 0, \] (22)
\[ u_\nu + (u \cdot \nabla)v_\nu + (v_\nu \cdot \nabla)u = -2\zeta \vec{\nabla} \delta \phi - \rho_0^{-1} \vec{\nabla} (v_s^2 q) + q \rho_0^{-2} v_s^2 \vec{\nabla} \rho_0, \] (23)
\[ \Delta \delta \phi = 4\pi G q, \] (24)
\[ \tilde{q}_\nu + \vec{\nabla} \cdot (\tilde{q}v_\nu) + \vec{\nabla} \cdot (\rho_0 \tilde{u}) = 0, \] (25)
\[ \tilde{u}_\nu + (\tilde{u} \cdot \nabla)v_\nu + (v_\nu \cdot \nabla)\tilde{u} = -2\vec{\nabla} \tilde{\phi} - \rho_0^{-1} \vec{\nabla} (v_s^2 \tilde{q}) + \tilde{q} \rho_0^{-2} v_s^2 \vec{\nabla} \rho_0, \] (26)
\[ \vec{\nabla} \cdot (\tilde{\mu} (\tilde{\nabla} \phi/a_0) \vec{\nabla} \phi) = 4\pi G \tilde{q}. \] (27)

The barred and unbarred systems decouple, with the unbarred fluctuations developing according to standard Newtonian theory, while the barred fluctuations develop according to MOND. If we start with \( q \) and \( \tilde{q} \) fluctuations of a similar magnitude, and the barred fluctuations are in the MOND regime, \( \tilde{q} \) grows faster than \( q \). This leads in the limit to \( \delta \rho \approx -\delta \dot{\rho} \), namely segregation of the two sectors. This echoes the fact that matter and TM repel each other in the MOND regime [21].

IV. COSMOLOGICAL FLUCTUATION GROWTH

The use in the cosmological context of the above NR equations, derived for an isolated system on a double Minkowski background, can be justified for weak perturbations in NR matter, of scale length much smaller than cosmological scales: Expand the BIMOND field equations around their cosmological solution of the FRW equations with a CC and \( \tilde{\mu}_{\nu \mu} = q_{\mu \nu} \). The EMTs are expanded around their value for the homogeneous background with only the 00 components contributing. For scales much smaller than the (space-time) curvature radius the background metrics can be taken as nearly Minkowski. Then, our formalism above applies.

In this application, \( \rho_0(t) \) is the ambient, homogeneous cosmological density of the specific matter component we concentrate on, and \( \rho_b \) is its effective pressure.

I shall take in what follows \( \beta = 1 \).

As is standard, we express the space derivatives with respect to comoving coordinates \( \mathbf{R} \) instead of the proper-distance coordinates \( \mathbf{r} = \alpha(t) \mathbf{R} \), which appear in the above equations. We also put \( v_\nu = H(t) \mathbf{r}, \; \vec{\nabla} \cdot v_\nu = 3H, \) where \( H = \dot{a}/a \) is the Hubble parameter. Also, write the equations in terms of partial time derivatives at a fixed comoving position: for the field \( S, \; \dot{S} = S + (v_\nu \cdot \vec{\nabla})S \). The continuity equation for the background NR matter amounts to \( \rho_b \propto a^{-3}, \) so, \( \dot{\rho}_b = \partial \rho_b / \partial t = -3H \rho_b. \)

Define \( \eta = \delta \rho / \rho_b, \; \epsilon = \dot{\eta} = q / \rho_b, \; \tilde{\epsilon} = \dot{\eta} - \tilde{\eta} = \tilde{q} / \rho_b. \) For these we have:
\[ \dot{\epsilon} + a^{-1} \vec{\nabla}_R \cdot u = 0, \] (28)
\[ \dot{\tilde{u}} + (\dot{a}/a) u = -a^{-1} \vec{\nabla}_R \delta \tilde{\phi} - v_s^2 a^{-1} \vec{\nabla}_R \epsilon, \] (29)
\[ \Delta_R \delta \tilde{\phi} = 4\pi G a^2 \rho_b \epsilon, \] (30)
\[ \dot{\epsilon} + a^{-1} \vec{\nabla}_R \cdot \tilde{u} = 0, \] (31)
\[ \dot{\tilde{u}} + (\dot{a}/a) \tilde{u} = -2a^{-1} \vec{\nabla}_R \tilde{\phi} - v_s^2 a^{-1} \vec{\nabla}_R \tilde{\epsilon}, \] (32)
\[ \nabla_R (\tilde{\mu} \frac{\nabla_R \tilde{\phi}}{a a_0} \nabla_R \tilde{\phi}) = 4\pi G a^2 \rho_s \epsilon. \] (33)

The unbarred system is identical to that in standard gravity: the perturbation sum develops fully in a Newtonian manner. Combining the time derivative of the continuity equation with the \( \nabla_R \) divergence of the Euler equation we get as usual

\[ \dot{\epsilon} + 2(\dot{a}/a) \epsilon = 4\pi G \rho_c \epsilon + v^2 a^{-2} \Delta_R \epsilon, \] (34)

and for the space Fourier components defined by \( \epsilon = \int d^3 k \epsilon_k (t) e^{i k \cdot r}, \) we have the standard result

\[ \dot{\epsilon}_k + 2(\dot{a}/a) \dot{\epsilon}_k = (4\pi G \rho_s - v^2 k^2/a^2) \epsilon_k. \] (35)

The barred system, which is decoupled from the unbarred one, describes the evolution of the perturbation difference; it behaves as a single fluid governed by MOND, with the strength of the governing potential, \( \bar{\rho} \), doubled [as expressed by the factor 2 in front of \( \nabla_R \tilde{\phi} \) in Eq.(32)], and with a Newtonian limit for which the effective Newton constant is \( G. \)\(^\ast\)

We can still eliminate the velocity \( \bar{\mathbf{u}} \) to get

\[ \ddot{\epsilon} + 2(\dot{a}/a) \dot{\epsilon} - 2a^{-2} \Delta_R \bar{\phi} - v^2 a^{-2} \Delta_R \bar{\epsilon} = 0, \] (36)

but here \( \Delta_R \bar{\phi} \) is not simply expressible in terms of \( \bar{\epsilon} \). The problem is nonlinear even for the smallest of perturbations, and perturbations on different scales can affect each other’s growth (see below).

Given some initial \( \bar{\epsilon}(\mathbf{R}, t_0) \) and \( \bar{\mathbf{u}}(\mathbf{R}, t_0) \), we can calculate \( \bar{\epsilon}(\mathbf{R}, t) \) from the continuity equation, we then calculate \( \bar{\phi} \) from the potential equation, then calculate \( \dot{\bar{\epsilon}} \) from Eq.(36), and so propagate the system in time. The velocities are propagated in time in parallel, using the Euler equation.

### A. The deep-MOND regime

With matter and TM present, there are two senses in which we can understand the term “a deep-MOND system”: The weaker sense applies to systems where \( |\nabla \bar{\phi}| \ll a_0 \) so that \( \bar{\phi} \) is determined by deep-MOND physics (with its peculiar symmetries, etc.). If we have an inhomogeneity of proper length scale \( \lambda \) that can be considered in isolation, i.e., is not subject to an external-field effect (EFE) from a larger-scale inhomogeneity of which it is a part (see below), this amounts to

\[ |\delta \rho - \delta \bar{\rho}| G \lambda \ll a_0. \] (37)

In the stronger sense of the term, one requires, also, that \( |\nabla \bar{\phi}| \ll |\nabla \bar{\phi}| \) so that the full potentials \( \phi \) and \( \bar{\phi} \) are determined by deep-MOND physics. This requires further

\[ |\delta \rho + \delta \bar{\rho}| G \lambda \ll (a_0 |\delta \rho - \delta \bar{\rho}| G \lambda)^{1/2}. \] (38)

In past applications of MOND, where TM is absent, the second requirement follows automatically from the first. This is also the case in the present context if \( |\delta \rho - \delta \bar{\rho}| \sim \max(|\delta \rho|, |\delta \bar{\rho}|) \) (for example, if the fluctuations in the two sectors are uncorrelated), but not if \( |\delta \rho - \delta \bar{\rho}| \ll \max(|\delta \rho|, |\delta \bar{\rho}|) \).

Because here we have full decoupling of the sum and difference systems, with the latter being governed only by \( \bar{\phi} \), I shall use the term in the weaker sense, which is enough to ensure that the barred system is governed by deep-MOND physics.

In the deep-MOND limit, Eq.(33) is well approximated by

\[ \nabla_R (|\nabla_R \bar{\phi}||\nabla_R \bar{\phi}|) = 4\pi G \rho_s a_0 \epsilon, \] (39)

where \( \rho_s = a^3 \rho_s. \) This equation is conformally invariant [21, 24].

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\(^\ast\) Since \( \tilde{\mu}(\infty) = 2, \bar{\phi} \) is half the Newtonian potential sourced by \( \rho_s \epsilon \) in this limit; the 1/2 factor canceled by above mentioned factor 2.
B. Coupling of inhomogeneities on different scales

The implications of Eq. (33) for the dynamics in massive systems have been discussed extensively (e.g., in [23, 25]). One clear outcome of these studies concerns the way in which the dynamics of a larger, mother system couple to those in a smaller subsystem. When the two do not differ much in size and/or mass, the interaction between them, which results from the nonlinearity of the theory, is not easy to describe. However, when the subsystem is much smaller, and much less massive, than the mother system (such as a star in a galaxy, or a galaxy in a galaxy cluster), the interaction is simply described: The subsystem moves within the mother system as if it were a test particle, irrespective of its own intrinsic accelerations. In other words, the dynamics of the mother system is oblivious to the internal dynamics in its small subsystems, which can be treated as test particles moving in their combined mean field. On the other hand, the dynamics within the subsystem may be greatly affected by the acceleration, $g_{ex}$, with which it falls within the mother system, provided this acceleration dominates the internal ones, and is in the MOND regime ($g_{ex} \ll a_0$). The intrinsic dynamics is then quasi-Newtonian with an effective gravitational constant $G' \sim G a_0/g_{ex}$, and some mild anisotropy induced by the external field. This is known as the MOND external-field-effect (EFE).

In the present context, the mass source in the equation for the potential, $\rho_\epsilon \tilde{\epsilon}$, must exhibit, at any time, a continuum spectrum of fluctuations, and it is difficult to describe the mutual couplings of the different scales. We expect, however, that the evolution of a fluctuation in $\tilde{\epsilon}$ on a given scale, will be affected mainly by larger-scale perturbations within which it is nested, in the spirit of the EFE.

A difference fluctuation of magnitude $\tilde{\epsilon}$ and (proper) scale length $\lambda$, would have a characteristic intrinsic acceleration, when in isolation, $\sim (a_0 G \rho_\epsilon \tilde{\epsilon})^{1/2}$. Thus, if, e.g., $\tilde{\epsilon} \propto \lambda^{-\alpha}$, for $\alpha > 1$ the typical acceleration decreases with increasing scale, and we do not expect a pronounced EFE. For $\alpha < 1$, the largest scales determine the dynamics within smaller ones, as described roughly by the EFE. For $\alpha = 1$ the accelerations on all scales are of the same order.

C. Pressure vs gravity

For the unbarred, fluctuation-sum system, the standard (Jeans) criterion tells us when pressure becomes unimportant in impeding gravitational growth. This applies [e.g., per Eq. (35)], when $M_s \gg G^{-3/2} v_s^4 \rho_b^{-1/2}$, where $M_s = \rho_b \lambda^3$ is the background mass (in either sector) in a cube of side $\lambda$.

For the barred system, the Jeans criterion for gravity dominance is very different. We saw that because of the nonlinearity of this system we cannot consider separately the growth of a single perturbation with a given amplitude and scale length, as all modes are, in principle, coupled. I discuss two end cases. First, consider a perturbation of magnitude $\tilde{\epsilon}$ and (proper) scale $\lambda$ that is not affected by larger-scale overdensities (through an analog of the EFE). From Eq. (32), or Eq. (36), the Jeans criterion is still $|\nabla \phi| \gg v_s^1 |\tilde{\epsilon}|$. From Eq. (33) we now have in the MOND limit $|\nabla \phi|^2/\lambda a_0 \sim G \rho_\epsilon |\tilde{\epsilon}|$. Put together, these give the condition for gravity dominance

$$M_s \gg (G a_0)^{-1} v_s^4 |\tilde{\epsilon}|$$

(40)

(where gradients are substituted by division by $\lambda$; e.g., $|\nabla \tilde{\epsilon}| \sim |\tilde{\epsilon}|/\lambda$).

Note, in particular, the appearance of $|\tilde{\epsilon}|$ in the criterion, in addition to the expected characteristic, deep-MOND $M \propto v^4$ relation. This occurs because the pressure increment is linear in $\tilde{\epsilon}$, while the potential increment $\phi$ is proportional to $|\tilde{\epsilon}|^{1/2}$. This $|\tilde{\epsilon}|$ factor makes the criterion much easier to satisfy for weak perturbations.\(^{18}\)

If the particular perturbation at hand cannot be considered in isolation, but is subject to an EFE characterized by $g_{ex}$, the Jeans criterion takes the form $\rho_b G(a_0/g_{ex}) \lambda^2 \gg v_s^4$, or

$$M_s \gg G^{-3/2} (a_0/g_{ex})^{3/2} v_s^4 \rho_b^{-1/2}.$$  \(41\)

As a concrete example, consider a baryonic, difference perturbation, of magnitude $\tilde{\epsilon}$ and scale $\lambda$, that may be considered unaffected by an EFE. Taking the ambient temperature appropriate for the matter-dominance era $T \approx 3 \times 10^4 (a/10^{-3})^{-1}$K, and ambient density $\rho_b \approx 4 \times 10^{-22} (a/10^{-3})^{-3}$gr cm$^{-3}$, the condition (37) for the perturbation being in the deep-MOND regime can be written

$$M_b/M_\odot \ll 2 \times 10^7 \left( \frac{a}{10^{-3}} \right)^6 |\tilde{\epsilon}|^{-3}.$$  \(42\)

\(^{18}\) In [26] I gave the Jeans criterion for a MOND universe as $M_s \gg (G a_0)^{-1} v_s^4$. This was based on a pure matter world-picture, where it was assumed, tentatively, that the source for the MOND potential includes the background density, and so the potential increment is linear in the density increment. Here, with an MTMS background, the background MOND potential vanishes, so the perturbation alone sources $\phi$, and so $|\nabla \phi| \propto |\tilde{\epsilon}|^{1/2}$.
The condition (40) for gravity dominance over pressure is
\[ \frac{M_b}{M_\odot} \gg 3 \times 10^5 \left( \frac{a}{10^{-3}} \right)^{-4} |\bar{\epsilon}|. \]

In comparison, the Jeans criterion relevant for the sum of the perturbations is
\[ \frac{M_b}{M_\odot} \gg 10^6 \left( \frac{a}{10^{-3}} \right)^{-1.5}. \]

So, there is a wide range of scales for which pressure stymies the growth of $\epsilon$ but not of $\bar{\epsilon}$.

### D. Later stages

The analysis above highlights the way in which small-scale, NR density inhomogeneities grow when they are still small compared with the ambient density. We have learned much about the mechanisms that are at play and how they greatly differ from those underlying perturbation growth in standard dynamics. For example, we noted the shepherding of matter overdensities by flanking TM overdensities (and vice-versa): a matter overdensity grows faster when it sits in a TM depression. A related effect is the segregation of matter and TM, as evinced by the tendency of difference fluctuations to grow faster than the sum.

But, beyond these qualitative observations, a numerical attack on the problem is needed to follow the growth of even weak perturbations in detail, because of the inherent MOND nonlinearity. This is also, clearly, the case when we deal with structure buildup to its presently observed stage, where the perturbations are not small. As mentioned in the introduction there have been several N-body studies of structure formation in MOND, and these could be easily adapted to include TM.

Without numerical studies, it is difficult to tell exactly how, and to what extent, these trends continue to shape structure in the later stages. Some questions that naturally arise are: “Up to what scales are matter and TM effectively segregated today?” In [21], I conjectured that this happens on the supercluster-void scales; namely, that matter and TM have formed interlacing cosmic webs in which TM concentrations sit in the observed matter voids, and vice-versa, and matter and TM filaments avoid each other. Another important question is: “How rare are TM objects, such as galaxies, in matter territory, and what observable effects can they have?” If the emptiness of matter voids is an indication, we do not expect TM galaxies within large matter concentrations. Other questions are: “Does TM help empty matter voids more efficiently than in the CDM paradigm?” (For possible indications for unexpectedly large and empty voids see [27–33].) “Are higher very-large-scale velocities produced in the present picture than in the CDM paradigm?” (See [34–36] for possible indications of such unexpectedly high velocities.) “Are objects with high peculiar velocities more common in our world picture than in the CDM paradigm?” (See [37] for a discussion of the issue in connection with the bullet cluster, and [38] for a recent assessment, and for references to earlier work.)

We can get some guidance in pondering these questions by considering the interaction between well separated matter and TM bodies. This question has been considered in detail in [21], which deals, in a sense, with the opposite of the situation we treat here of small departures from well mixed configurations.

The basic result found there is that matter and TM bodies repel each other in the MOND regime, and (for $\beta = 1$) they do not interact in the high-acceleration regime. For example, using the conformal invariance of the deep-MOND limit of the theory, I calculated the force between two masses $M_1$ and $M_2$ in the deep-MOND regime to be (defined as negative for attraction)
\[ F = -\frac{2}{3} \left( \frac{a_0G}{r} \right)^{1/2} [(M_1 \pm M_2)^{3/2} - M_1^{3/2} - M_2^{3/2}], \]

where the plus sign applies when the two masses are in the same sector (in which case $F$ is always attractive), and the minus sign when they are different (in which case the force is always repulsive); $r$ is the separation. We see that for equal masses the repulsive force between matter and TM masses $M$ is $(4/3)(a_0G)^{1/2}M^{1/2}/r$, which is $\approx 2.4$ times larger than the attractive force between two masses $M$ in the same sector. This can give us an idea of how far from each other separated bodies of matter and TM can separate over the Hubble time.\(^{19}\)

The characteristic separation speed for matter and TM masses $M$ is $\sim 2.3(a_0GM)^{1/4} \sim 260(M/10^{10}M_\odot)^{1/4}\text{km s}^{-1}$. In a Hubble time this produces a separation of several megaparsecs for galactic mass objects and a few tens of

\(^{19}\) This is only a rough estimates, since such bodies are hardly ever isolated on the relevant scales.
V. PHANTOM MATTER AND ITS BACK REACTION ON COSMOLOGY

There has been much discussion recently regarding back reaction of inhomogeneities on cosmological evolution in GR. The issue concerns the extent to which we err by applying FRW theory (which assumes homogeneous energy density) to the mean cosmological energy density, instead of calculating cosmological evolution correctly with inhomogeneously distributed energy, and then averaging the outcome. (For recent discussion of this issue, with references to earlier work, see, e.g., [39, 40].) The exact importance of this effect is still moot.

Obviously, such effects exist in BIMOND as well, but they are not the subject here. In a MTMS universe governed by BIMOND there are added back-reaction effects of inhomogeneities, which, as we saw, induce the appearance of phantom matter components that should have influence on cosmology.

In a universe with different matter and TM inhomogeneities, the \( S_{\mu\nu} \) and \( \dot{S}_{\mu\nu} \) terms in the BIMOND field equations (5)(6) do not vanish; so, what is their effect on the cosmological background? Strictly speaking, we then have to solve the full BIMOND cosmological equations. However, to some approximation, we may treat the resulting departure from GR perturbatively, by viewing the effects of these terms as perturbations on the MTMS, FRW cosmology: In this approximation, the \( S_{\mu\nu} \) and \( \dot{S}_{\mu\nu} \) terms are viewed as the EMTs of MOND phantom matter that can effectively be added to \( T_{\mu\nu} \) and \( \dot{T}_{\mu\nu} \) in sourcing cosmic gravity. We saw that to zeroth order in the metric difference, all that remains from these terms is a CC term, which has to be reckoned with in zeroth-order cosmology. This cosmology is then used to calculate the development of perturbations in the two sectors, starting from some seed perturbations.

The next stage in such an approximation scheme would be to use the deduced local departures of the two metrics from the FRW metric, used in the lower order approximation, to calculate local contributions to \( S_{\mu\nu} \) and \( \dot{S}_{\mu\nu} \), and put them back as sources in the equations for cosmology. This program clearly deserves a thorough treatment, which, however, is beyond our scope here. In particular, the results of such an analysis is expected to depend on the details of

\[ (\mathbf{r}_k \cdot \mathbf{a}_k) = \frac{2}{3} (a_0 G m)^{1/2}. \]

20 If the bullet test mass starts at rest a distance \( r_0 \) from a TM mass \( M \), its speed at a distance \( r \) is, assuming the system is isolated, \( \nu(r) = (4a_0 GM)^{1/3} M(r/r_0) \approx 1.6 \times 10^9 (M/10^{14} M_\odot)^{1/4} (r/r_0) \text{km s}^{-1}. \)

21 Such a system does not satisfy a kinematic virial relation: since \( \mathbf{F}_k = m_k \mathbf{v}_k \), we find that \( d(\sum_k m_k \mathbf{r}_k \cdot \mathbf{v}_k)/dt > 0 \); so no steady state can be reached.
the specific BIMOND version at hand, and there is a large variety of versions to explore. Here I only describe briefly some general properties of the different term types defined in Eq.(8), as they would be fed back into the cosmological equations.

The zeroth order of the $P$ term (in the matter sector), $a_0^2 P_0 g_{\mu\nu}$ gives a genuine CC term [if $P_0 \neq 0$]. When matter and TM separate, $P$ becomes space-time dependent, and in this regard behaves as a “dark energy” field. In the weak-field approximation, we can substitute in the variables on which $P$ depends $g_{\mu\nu} \approx \hat{g}_{\mu\nu}$, but $\Upsilon \approx \hat{\Upsilon} \propto (\nabla \phi - \nabla \hat{\phi})^2$.

The contribution of the $P$ term is then

$$S^{P}_{\mu\nu} \approx a_0 P(z) g_{\mu\nu}, \quad z \equiv (\nabla \phi - \nabla \hat{\phi})^2/a_0^2.$$  \hspace{1cm} (49)

From Eq.(9) we see that $\hat{P} = P$ in this approximation. The value of $P$ in this expression varies in space-time between $P_0$ and $P(\infty)$. (As explained in [21], considerations of the GR limit of the theory, for $a_0 \to 0$, dictate that $P(\infty)$ is not infinite.) Today we have in much of the volume of the universe $\rho, \pi \approx 0$, and $\Lambda \approx \pi G$.

In a MTMS universe this density averages to zero over large cosmological volumes. This can be seen in two (independent) ways. First, it follows from the fact that under the interchange of matter and TM,

$$\rho \leftrightarrow \hat{\rho} \text{ leads to } \phi \leftrightarrow \hat{\phi},$$

which, in turns gives $\rho_p \to -\hat{\rho}_p$. The phantom density around a matter concentration is equal and opposite in sign to that around a similar TM concentration. In our world picture the large-scale averages of the universe are invariant to matter-TM interchange, hence the average phantom density, which changes sign, must vanish. This result holds also for a finite system with matter-TM symmetry. For example, in a system where matter-TM interchange can be achieved by a volume-preserving transformation (such as translation, reflection, rotation) the total phantom mass in a symmetric volume vanishes.

We can also argue differently, noting that the phantom density is a divergence of a vector $(\nabla \phi - \nabla \hat{\phi}) \mathcal{M}(z)$, whose absolute value is bounded (outside of point masses). As a result, its volume average vanishes in the limit of a large volume. This property of the phantom density follows from the form of the relativistic $Q$ term contribution: We see from Eq.(50) that the trace of the phantom EMT is a covariant divergence of a vector $V^\lambda \propto Q^{\mu\nu} \{C\}^\lambda_{\mu\nu}$. Its integral over a space-time volume is hence a surface integral.

Even if the large-scale average of the NR phantom density averages to zero it could still affect cosmology due to its inhomogeneity. In addition, there may be higher order contributions that do not average to zero.

The $Q$ terms may also have contributions of the form $(M' g^{-\alpha} C_{\alpha \beta \lambda})_{\mu\nu} g_{\lambda\mu}$, or $(M' g^{-\alpha} C_{\alpha \beta \lambda})_{\mu\nu} g_{\lambda\mu}$, which are of the form $V^\lambda \propto Q^{\mu\nu} \{C\}^\lambda_{\mu\nu}$: i.e., they contribute in cosmology as dark energy with oscillating, vanishing-average density.

The PM around matter concentrations can be (and is) observed, and is interpreted by most as “dark matter.” As discussed in [21], the PM around TM concentrations may be observed with weak lensing of matter photons (it does not produce strong lensing), on which it acts as a repelling mass.

For the $N$ terms, the contribution of lowest order in the departure from metric equality is the second order. In the NR limit their characteristic contribution to the phantom EMT is of the order of $(8\pi G)^{-1} z N(z) (a_0/c)^2 [z \text{ defined in Eq.(49)}]$. Using the well known proximity $a_0 \approx c H_0/2\pi$, we can write this contribution as $\sim 10^{-2} z N(z) \rho_c$, where $\rho_c$ is the critical density today. We saw above that in most of the cosmological volume today $z \lesssim 10^{-3}$. Also we have that $N(0)$ is of order unity (from considerations of the MOND limit, see [15, 21]), and $N(\infty) = 0$ (from considerations of the GR limit of BIMOND). $N$ is related to the extrapolating function of MOND and is constrained to remain of order unity, or below, for all arguments. So the contribution of the $N$ terms to the density can be at most of order $\rho_c$. 
(in systems where $|\vec{\nabla}\phi| \sim a_{\phi}$), but some orders of magnitude smaller than $\rho_s$ in most of the cosmic volume. It would be interesting to see what their tensorial form is, and to what extent they produce observable lensing effects.

Note, finally, that inasmuch as the statistical properties of matter in the two sectors are the same, so are the feedback effects discussed above. Namely, the effects of inhomogeneities, and the statistical properties of the PM (their cosmological mean densities, power spectra, etc.) are the same in the two sectors. The two mean background metrics thus remain the same.

VI. DISCUSSION

I have considered the dynamics of small, NR perturbations in an MTMS universe. An intriguing aspect of this world picture is that MOND dynamics, cosmological or local, are completely absent in a perfectly symmetric universe, apart from a possible appearance of a CC of order $(a_{\phi}/c)^2$. All other departure from standard physics encapsulated in MOND—as it appears in cosmology, in structure formation, and in bound systems, such as the solar system or galactic systems—is entirely an effect of “separation” between matter and TM, or separation between the two metrics. In a double-membrane picture of BIMOND, MOND effects are due to local separation of the membranes.

Twin matter plays two roles in our narrative: In the first, it provide a counteracting medium to matter; similar to the role of a neutralizing positive ion background to an electron gas. As a result of this presence, we could assume the very appealing symmetric cosmology, identical to that of GR with a CC. In addition, the presence of matter and TM in equal average quantities leads to a well-defined MOND dynamics, in which the MOND potential is sourced only by departures from uniformity, not by the bulk. To fulfill this role, TM only has to be present in the same amount as matter; it does not necessarily have to clump.

In its second role, TM plays an equivalent part to that of matter; so it can be likened to a positron component in a globally neutral electron-positron plasma (with mutual repulsion, and no annihilation). In this role it also takes part in shaping the large-scale structure of matter.

There are several qualitative features that make the growth of fluctuations in the proposed world picture different from the standard one. Among them: (i) Two initial fluctuation spectra, not one, determine the large-scale structure at later times. For the NR problem discussed here, we can think of these as the fluctuations in matter alone, and of those in the matter-TM density difference. This is important to appreciate, for example, because we have strong constraints on matter fluctuations at the time of matter-radiation decoupling, from the CMB anisotropies; but, we have none on the difference fluctuations $\bar{\epsilon}$ at that time. So, numerical simulations have to explore all possible options. (ii) The NR fluctuations in the sum and difference of the densities form two decoupled systems controlled by different types of gravitational physics. The sum system is underlaid by standard Newtonian gravity, while the difference system is underlaid by MOND gravity, which is stronger for the same source strength. (iii) In the difference system, the competition between gravity and pressure is strongly tilted in favor of gravity, with the Jeans criterion being much more lenient than in the sum system. (iv) As a result of (ii) and (iii), the difference fluctuations tend to grow faster than the sum fluctuation, leading towards a limit where $|\bar{\epsilon}| \gg |\epsilon|$, or $\delta\rho \approx -\hat{\delta}\rho$. This implies tendency towards anticorrelation of matter and TM overdensities, or to segregation of the two sectors. This applies not only when the perturbations are weak: the fact that distinct matter and TM bodies repel each other for low accelerations implies that the process of segregation continues during the stages where the overdensities are high. (v) There is an added enhancement, due to the fact that once the fluctuations in the two sectors are anticorrelated, with maxima of each residing in minima of the other, each overdensity grows not only by the pull of its own gravity, but also by shepherding by the repulsive action of the flanking overdensities in the other sector. (vi) There can be strong coupling between the perturbations on different scales due to the nonlinearity inherent in MOND.

There is much that remains to be studied, even within the narrow paradigm of the MTMS universe. An even richer prospect opens if one considers more general cases such as matter-TM asymmetric BIMOND versions, or asymmetric universes in which the two sectors are not equally populated, or do not have exactly the same properties.

For example, we do not know why a universe that presumably started with a vanishing baryon-number density has ended up having more baryons than antibaryons, with the difference resulting in the present-day baryon density. This baryogenesis, as it is called, which requires a breakdown of symmetries (CP and baryon-number conservation), is thought to have occurred during some early phase of thermal disequilibrium. If the corresponding conditions in the two sectors where slightly different, we could have ended up with different baryon densities in the two sectors. It

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22 Even if the initial fluctuations are different in the two sectors, we saw that their statistical properties are driven to equalization by the faster, MOND-like growth of the fluctuation difference.
would be interesting to study the effects of such departure from MTMS, to lowest order in the density difference, to see what effects it has on both the cosmological evolution, and on structure growth.

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