Transducer Descriptions of DNA Code Properties and Undecidability of Antimorphic Problems

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Abstract

This work concerns formal descriptions of DNA code properties, and builds on previous work on transducer descriptions of classic code properties and on trajectory descriptions of DNA code properties. This line of research allows us to give a property as input to an algorithm, in addition to any regular language, which can then answer questions about the language and the property. Here we define DNA code properties via transducers and show that this method is strictly more expressive than that of trajectories, without sacrificing the efficiency of deciding the satisfaction question. We also show that the maximality question can be undecidable. Our undecidability results hold not only for the fixed DNA involution but also for any fixed antimorphic permutation. Moreover, we also show the undecidability of the antimorphic version of the Post Corresponding Problem, for any fixed antimorphic permutation.

1 Introduction

The study of formal methods for describing independent language properties (widely known as code properties) provides tools that allow one to give a property as input to an algorithm and answer questions about this property. Examples of such properties include classic ones \cite{4,17,27,28} like prefix codes, bifix codes, and various error-detecting languages, as well as DNA code properties \cite{2,10,11,13–15,18–21,25} like \(\theta\)-nonoverlapping and \(\theta\)-compliant languages. A formal description method should be expressive enough to allow one to describe many desirable properties. Examples of formal methods for describing classic code properties are the implicational conditions method of \cite{16}, the trajectories method of \cite{5}, and the transducer methods of \cite{8}. The latter two have been implemented to some extent in the Python package FAdo \cite{9}. A formal method for describing DNA code properties is the method of trajectory DNA code properties \cite{6,21}.

Typical questions about properties are the following:
**Satisfaction problem:** given the description of a property and the description of a regular language, decide whether the language satisfies the property.

**Maximality problem:** given the description of a property and the description of a regular language that satisfies the property, decide whether the language is maximal with respect to the given property.

**Construction problem:** given the description of a property and a positive integer $n$, find a language of $n$ words (if possible) satisfying the given property.

In the above problems regular languages are described via (non-deterministic) finite automata (NFA). Depending on the context, properties are described via trajectory regular expressions or transducer expressions. The satisfaction problem is the most basic one and can be answered usually efficiently in polynomial time. The maximality problem as stated above can be decidable, in which case it is normally PSPACE-hard. For existing transducer properties, both problems can be answered using the online (formal) language server LaSer [24], which relies on FAdo. LaSer allows users to enter the desired property and language, and returns either the answer in real time (online mode), or it returns a Python program that computes the desired answer if executed at the user’s site (program generation mode). For the construction problem a simple statistical algorithm is included in FAdo, but we think that this problem is far from being well-understood.

The general objective of this research is to develop methods for formally describing DNA code properties that would allow one to express various combinations of such properties and be able to get answers to questions about these properties. While the satisfaction and construction questions are important from both the theoretical and practical viewpoints, the maximality question is at least of theoretical interest and a classic problem in the theory of codes. The contributions of this work are as follows:

1. The definition of a new simple formal method for describing many DNA code properties, called $\theta$-transducer properties, some of which cannot be described by the existing transducer and trajectory methods for classic code properties; see Sect. 3. These methods are closed under intersection of code properties. This means that if two properties can be described within the method then also the combined property can be described within the method. This outcome is important as in practice it is desirable that languages satisfy more than one property.

2. The demonstration that the new method of transducer DNA code properties is properly more expressive than the method of trajectories; see Sect. 4.

3. The demonstration that the maximality problem can be decidable for some transducer DNA code properties but undecidable for some others; see Sect. 5.

4. The demonstration that some classic undecidable problems (like PCP) remain undecidable when rephrased in terms of any fixed (anti-)morphic permutation $\theta$ of the alphabet, with the case $\theta = \text{id}$ corresponding to these classic problems, where $\text{id}$ is
the (morphic) identity; see Sect. 6. This contribution is mathematically relevant to the undecidability of the maximality problem for DNA-related properties, so it is natural to include it with the above contributions in one publication.

5. The presentation of a natural hierarchy of DNA properties which are all $\theta$-transducer properties; see Section 7. This hierarchy generalizes the concept of bond-free properties in [13, 18, 19].

Even though, our main motivation is the description of DNA-related properties, we follow the more general approach which considers properties described by transducers involving a fixed (anti-)morphic permutation $\theta$; again, the classical transducer properties are obtained by letting $\theta = id$. In the setting of DNA properties, we consider the alphabet $\Delta = \{A, C, G, T\}$ and $\theta = \delta$ being the involution (i.e., antimorphic permutation with $\delta^2 = id$) given by $\delta(A) = T$, $\delta(T) = A$, $\delta(C) = G$, and $\delta(G) = C$. As it turns out, in the case when $\theta$ is morphic all questions that we consider in this paper can be answered analogous to the solutions for the classical case where $\theta = id$. Therefore, we focus on the transducer properties involving antimorphic permutations in this paper.

2 Basic Notions and Background Information

In this section we lay down our notation for formal languages, (anti-)morphic permutations, transducers, and language properties. We assume the reader to be familiar with the fundamental concepts of language theory; see e.g., [12, 26]. Then, in Sect. 2.2 we recall the method of transducers for describing classic code properties, and in Sect. 2.3 we recall the method of trajectories for describing DNA-related properties.

2.1 Formal Languages and (Anti-)morphic Permutations

An alphabet $A$ is a finite set of letters; $A^*$ is the set of all words or strings over $A$; $\varepsilon$ denotes the empty word; and $A^+ = A^* \setminus \{\varepsilon\}$. A language $L$ over $A$ is a subset $L \subseteq A^*$; the complement $L^c$ of $L$ is the language $A^* \setminus L$. For an integer $m \in \mathbb{N}$ we let $A^{\leq m}$ denote the set of words whose length is at most $m$; i.e., $A^{\leq m} = \bigcup_{i \leq m} A^i$. The DNA alphabet is $\Delta = \{A, C, G, T\}$. Often it is convenient to consider the generic alphabet $A_k = \{0, 1, \ldots, k - 1\}$ of size $k$ rather than a general alphabet; note that $A_2 \subseteq A_3 \subseteq A_4 \subseteq \cdots$. Throughout this paper we only consider alphabets with at least two letters because our investigations would become trivial over unary alphabets.

Let $w \in A^*$ be a word. Unless confusion arises, by $w$ we also denote the singleton language $\{w\}$, e.g., $L \cup w$ means $L \cup \{w\}$. If $w = xyz$ for some $x, y, z \in A^*$, then $x, y, \text{ and } z$ are called prefix, infix (or factor), and suffix of $w$, respectively. For a language $L \subseteq A^*$, the set $\text{Pref}(L) = \{x \in A^* \mid \exists y \in A^*: xy \in L\}$ denotes the language containing all prefixes of words in $L$. If $w = a_1a_2\cdots a_n$ for letters $a_1, a_2, \ldots, a_n \in A$, then $|w| = n$ is the length of $w$; for $b \in A$, $|w|_b = |\{i \mid a_i = b, 1 \leq i \leq n\}|$ is the tally of $b$ occurring in $w$; the $i$-th
Let the letter of $w$ is $w[i] = a_i$ for $1 \leq i \leq n$; the infix of $w$ from the $i$-th letter to the $j$-th letter is $w[i:j] = a_ia_{i+1}\cdots a_j$ for $1 \leq i \leq j \leq n$; and the reverse of $w$ is $w^R = a_na_{n-1}\cdots a_1$.

Consider a generic alphabet $A_k$ with $k \geq 2$. The identity function on $A_k$ is denoted by $\text{id}_k$; when the alphabet is clear from the context, the index $k$ is omitted. For a permutation (or bijection) $\theta: A_k \rightarrow A_k$, the permutation $\theta^{-1}$ is the inverse of $\theta$ as usual; i.e., $\theta \circ \theta^{-1} = \text{id}_k$ (the composition of two functions $(g \circ h)(x) = g(h(x))$ for all $x$). For $i \in \mathbb{Z}$, the permutation $\theta^i$ is the $i$-fold composition of $\theta$; i.e., $\theta^0 = \text{id}_k$, $\theta^i = \theta \circ \theta^{i-1}$, and $\theta^{-i} = (\theta^i)^{-1}$ for $i > 0$. There exists a number $n$, called the order of $\theta$, such that $\theta^n = \text{id}_k$. An involution $\theta$ is a permutation of order 2; i.e., $\theta = \theta^{-1}$.

A permutation $\theta$ over $A_k$ can naturally be extended to operate on words in $A_k^*$ as (a) morphic permutation $\theta(uv) = \theta(u)\theta(v)$, or (b) antimorphic permutation $\theta(uv) = \theta(v)\theta(u)$, for $u, v \in A_k^*$. As before, the inverse $\theta^{-1}$ of the (anti-)morphic permutation $\theta$ over $A_k^*$ is the (anti-)morphic extension of the permutation $\theta^{-1}$ over $A_k^*$. Note that the composition of two antimorphic or two morphic permutations yields a morphic permutation, whereas the composition of a morphic and an antimorphic permutation yields an antimorphic permutation. Therefore, if $\theta$ is an antimorphic permutation, then $\theta^i$ is morphic if and only if $i$ is even. The identity $\text{id}_k$ always denotes the morphic extension of $\text{id}_k$ while the antimorphic extension of $\text{id}_k$, called the mirror image or reverse, is usually denoted by the exponent $R$.

**Example 1.** The DNA involution, denoted as $\delta$, is an antimorphic involution on $\Delta = \{A, C, G, T\}$ such that $\delta(A) = T$ and $\delta(C) = G$, which implies $\delta(T) = A$ and $\delta(G) = C$.

A language operator is any mapping $\text{Op}: 2^{A^*} \rightarrow 2^{A^*}$. The prefix function $\text{Pref}$ defined earlier is an example of a language operator. A transducer (see Sect. 2.2) can be viewed as a language operator. Any (anti-)morphic permutation, as well as any other function, $h: A^* \rightarrow A^*$ over words is extended to a language operator such that for $L \subseteq A^*$

$$h(L) = \bigcup_{x \in L} \{h(x)\}.$$ 

If $\text{Op}_1$ and $\text{Op}_2$ are language operators, then $(\text{Op}_1 \vee \text{Op}_2)$ is the language operator such that $(\text{Op}_1 \vee \text{Op}_2)(X) = \text{Op}_1(X) \cup \text{Op}_2(X)$, for all languages $X$.

### 2.2 Describing Classic Code Properties by Transducers

A (language) property $\mathcal{P}$ is any set of languages. A language $L$ satisfies $\mathcal{P}$, or has $\mathcal{P}$, if $L \in \mathcal{P}$. Here by a property $\mathcal{P}$ we mean an $(n)$-independence in the sense of [17]: there exists $n \in \mathbb{N} \cup \{|0\}$ such that a language $L$ satisfies $\mathcal{P}$ if and only if all nonempty subsets $L' \subseteq L$ of cardinality less than $n$ satisfy $\mathcal{P}$. A language $L$ satisfying $\mathcal{P}$ is maximal (with respect to $\mathcal{P}$) if for every word $w \in L^c$ we have $L \cup w$ does not satisfy $\mathcal{P}$—note that, for any independence $\mathcal{P}$, every language in $\mathcal{P}$ is a subset of a maximal language in $\mathcal{P}$ [17]. To our knowledge all code related properties in the literature, including DNA code properties, are independence properties. As we shall see further below the focus of this work is on $3$-independence properties that can also be viewed as independent with respect to a binary relation in the sense of [28].
A transducer $t$ is a non-deterministic finite state automaton with output; see e.g., [3,30]. In general, a transducer can have an output alphabet $B$ which is different from its input alphabet $A$; thus, defining a relation over $A^* \times B^*$. In this paper however, we only consider transducers where the input alphabet coincides with the output alphabet, $A = B$, which leads to the following simplified definition: a transducer is a quintuple $t = (Q, A, E, I, F)$, where $A$ is the input and output alphabet, $Q$ is a finite set of states, $E$ is a set of directed edges between states from $Q$ which are labeled by word pairs $(u, v) \in A^* \times A^*$, $I$ is a set of initial states, and $F$ a set of final states. For an edge label $(u, v)$ the word $u$ is called input, while the word $v$ is called output. The transducer $t$ realizes the set of all pairs $(x, y) \in A^* \times A^*$ such that $x$ is formed by concatenating the inputs, and $y$ is formed by concatenating the outputs of the labels in a path of $t$ from the initial to the final states. If $t$ realizes $(x, y)$ then we write $y \in t(x)$. We say that the set $t(x)$ contains all possible outputs of $t$ on input $x$. It is well known that for two regular languages $R_1, R_2$ there exists a transducer $t$ that realizes the relation $R_1 \times R_2$; i.e., $t$ realizes $(x, y)$ if and only if $x \in R_1$ and $y \in R_2$. The transducer $t^{-1}$ is the inverse of $t$; that is, $x \in t^{-1}(y)$ if and only if $y \in t(x)$ for all words $x, y$. Note that $t^{-1}$ is obtained from $t$ by simply swapping the input with the output word on each edge in $t$. For a language $L$ we naturally extend our notation such that

$$t(L) = \bigcup_{x \in L} t(x).$$

Thus, a transducer can be viewed as a language operator.

Let $\theta$ be an (anti-)morphic permutation and $t$ be a transducer which are both defined over the same alphabet $A$. The transducer $t$ is called $\theta$-input-preserving if for all $w \in A^+$ we have $\theta(w) \in t(w)$; $t$ is called $\theta$-input-altering if for all $w \in A^+$ we have $\theta(w) \notin t(w)$. We use the simpler terms input-altering and input-preserving $t$, respectively, when $\theta = \text{id}$. Note that $\theta(w) \in t(w)$ is equivalent to $w \in \theta^{-1}(t(w))$ as well as $t^{-1}(\theta(w)) \supseteq w$.

**Definition 2 ([8]).** An input-altering transducer $t$ describes the property that consists of all languages $L$ such that

$$t(L) \cap L = \emptyset.$$  \hspace{1cm} (1)

An input-preserving transducer $t$ describes the property that consists of all languages $L$ such that

$$w \notin t(L \setminus w), \text{ for all } w \in L.$$  \hspace{1cm} (2)

A property is called an input-altering (resp. input-preserving) transducer property, if it is described by an input-altering (resp. input-preserving) transducer.

Note that every input-altering transducer property is also an input-preserving transducer property. Input-altering transducers can be used to describe properties like prefix codes, bifix codes, and hypercodes. Input-preserving transducers are intended for error-detecting properties, where in fact the transducer plays the role of the communication channel. Figure 1 shows a couple of examples.

Many input-altering transducer properties can be described in a simpler manner by trajectory regular expressions [5,8], that is, regular expressions over $\{0, 1\}$. For example,
Figure 1: The left transducer is input-altering and describes the prefix codes: on input \(x\) it outputs any proper prefix of \(x\). The right transducer is input-preserving and describes the 1-substitution error-detecting languages: on input \(x\) it outputs either \(x\) or any word differing from \(x\) in exactly one position. Note: in this and the following transducer figures, an arrow with label \((a, a)\) represents a set of edges with labels \((a, a)\) for all \(a \in A\); and similarly for an arrow with label \((a, \varepsilon)\). An arrow with label \((a, b)\) represents a set of edges with labels \((a, b)\) for all \(a, b \in A\) with \(a \neq b\).

the expression \(0^*1^*\) describes prefix codes and the expression \(1^*0^*1^*\) describes infix codes. On the other hand, there are natural transducer properties that cannot be described by trajectory expressions [8].

2.3 Describing DNA-related Properties by Trajectories

In [2, 10, 11, 13–15, 18–21, 25] the authors consider numerous properties of languages inspired by reliability issues in DNA computing. We state three of these properties below. In Sect. 7 we present a hierarchy of DNA properties which generalizes some of the DNA properties presented in [13, 18, 19]. Let \(\theta\) be an antimorphic permutation over \(A_*^k\). Recall that in the DNA setting \(\theta = \delta\) is an involution, and therefore, we have \(\theta^2 = \text{id}\).

(A) A language \(L\) is \(\theta\)-nonoverlapping if \(L \cap \theta(L) = \emptyset\).

(B) \(L\) is \(\theta\)-compliant if \(\forall w \in \theta(L), x, y \in A_*^k: xwy \in L \implies xy = \varepsilon\).

(C) \(L\) is strictly \(\theta\)-compliant if it is \(\theta\)-nonoverlapping and \(\theta\)-compliant.

Many of the existing DNA-related properties can be modelled using the concept of a bond-free property, first defined in [21] and later rephrased in [6] in terms of trajectories. We follow the formulation in [6]. Let \(\bar{e} = (\bar{e}_1, \bar{e}_2)\), where \(\bar{e}_1\) and \(\bar{e}_2\) are two regular trajectory expressions. First, we define the following language operators.

\[
\Phi_{\bar{e}}(L) = (((L \prec \bar{e}_1 A^+) \cap A^+) \uplus_{\bar{e}_2} A^*) \cup (((L \prec \bar{e}_1 A^*) \cap A^+) \uplus_{\bar{e}_2} A^+). \quad (3)
\]

\[
\Phi^*_{\bar{e}}(L) = ((L \prec \bar{e}_1 A^*) \cap A^+) \uplus_{\bar{e}_2} A^*. \quad (4)
\]

The word operations \(\uplus_{\bar{e}_i}\) and \(\prec_{\bar{e}_i}\) are called shuffle (or scattered insertion) and scattered deletion, respectively, over the trajectory \(t\). A trajectory is any word over \(\{0, 1\}\). For any words \(x, w\) and trajectory \(t\) with \(|t|_0 = |x|\) and \(|t|_1 = |w|\), \(x \uplus_{\bar{e}_i} w\) is the set \(\{y\}\) such that the word \(y\) is of length \(|t|\) and results by the following process which scans the symbols of \(x\) left to right and also of \(w\) left to right. For each index \(i = 0, \ldots, |t| - 1\), \(y[i]\) is the next
symbol of $x$ if $t[i] = 0$, or the next symbol of $w$ if $t[i] = 1$. If $|t|_0 = |x|$ and $|t|_1 = |w|$ is not satisfied then $x \sqcup \sqcap w = \emptyset$. For example, $1122 \sqcup_{001010} = 113242$. The reader is referred to [6, 22] for more details. For any languages $X, W$ and trajectory expression $\bar{a}$, we have that

$$X \sqcup_{\bar{a}} W = \bigcup_{x \in X, w \in W, t \in L(\bar{a})} x \sqcup_t w.$$ 

For any words $x, w$ and trajectory $t$ with $|t| = |x|$ and $|t|_1 = |w|$, $x \leadsto_t w$ is either the set $\{y\}$ such that the word $y$ is of length $|t|_0 = |x| - |w|$ and satisfies $\{x\} = y \sqcup_t w$, or the empty set otherwise. For example, $113242 \leadsto_{001010} 34 = 1122$. The reader is referred to [6, 22] again for more details. For any languages $X, W$ and trajectory expression $\bar{a}$, we have that

$$X \leadsto_{\bar{a}} W = \bigcup_{x \in X, w \in W, t \in L(\bar{a})} x \leadsto_t w.$$ 

**Definition 3.** [6] Let $\theta$ be an involution and $\bar{e}_1, \bar{e}_2$ be two regular trajectory expressions. The **bond-free property described by** $(\bar{e}_1, \bar{e}_2)$ is

$$\mathcal{B}(\bar{e}_1, \bar{e}_2) = \{L \subseteq A^* | \theta(L) \cap \Phi_{\bar{e}_1, \bar{e}_2}(L) = \emptyset\}. \tag{5}$$

The **strictly bond-free property described by** $(\bar{e}_1, \bar{e}_2)$ is

$$\mathcal{B}^s(\bar{e}_1, \bar{e}_2) = \{L \subseteq A^* | \theta(L) \cap \Phi_{\bar{e}_1, \bar{e}_2}^s(L) = \emptyset\}. \tag{6}$$

A **regular $\theta$-trajectory property** is a bond-free property described by $(\bar{e}_1, \bar{e}_2)$, or a strictly bond-free property described by $(\bar{e}_1, \bar{e}_2)$, for some pair $(\bar{e}_1, \bar{e}_2)$.

**Example 4.** The $\theta$-compliant property is a regular $\theta$-trajectory property in $\mathcal{B}(1^*0^+1^*, 0^+)$: deleting $x$ and $y$ in any $xwy$ (according to $1^*0^+1^*$), where at least one symbol gets deleted, and then inserting nothing (according to $0^+$) cannot result into a word in $\theta(L)$. The $\theta$-nonoverlapping property is a regular $\theta$-trajectory property in $\mathcal{B}^s(0^+, 0^+)$: deleting nothing and then inserting nothing in any word $w$ cannot result into a word in $\theta(L)$. The strictly $\theta$-compliant property is a regular $\theta$-trajectory property in $\mathcal{B}^s(1^*0^+1^*, 0^+)$: deleting $x$ and $y$ in any $xwy$ (according to $1^*0^+1^*$) and inserting nothing (according to $0^+$) cannot result into a word in $\theta(L)$.

We note that the actual definitions of bond-free properties in [6] are given in terms of a pair $(T_1, T_2)$ of arbitrary sets of trajectories. However, here we only consider sets of trajectories that can be represented by regular expressions. Moreover, the second statement of Theorem 12, in Sect. 4, remains true if one uses $(T_1, T_2)$ instead of $(\bar{e}_1, \bar{e}_2)$, as the proof makes no use of the fact that the trajectory sets involved are regular.
3 New Transducer-based DNA-related Properties

A question that arises from the discussion in sections 2.2 and 2.3 is whether existing transducer-based properties include DNA-related properties. It turns out that this is not the case: for instance the $\delta$-nonoverlapping property, which seems to be the simplest DNA-related property, cannot be described by any input-preserving transducer; see Proposition 8. In this section, we define new transducer-based properties that are appropriate for DNA-related applications, we demonstrate Proposition 8, and discuss how existing DNA-related properties can be described with transducers. Then, in Sect. 4 we examine the relationship between the new transducer properties and the regular $\theta$-trajectory properties which were proposed in [6].

Definition 5. A transducer $t$ and an (anti-)morphic permutation $\theta$, defined over the same alphabet, describe 3-independent properties in two ways:

1.) strict $\theta$-transducer property ($S$-property): $L$ satisfies the property $S_{\theta,t}$ if

$$\theta(L) \cap t(L) = \emptyset$$

(7)

2.) weak $\theta$-transducer property ($W$-property): $L$ satisfies the property $W_{\theta,t}$ if

$$\forall w \in L: \theta(w) \notin t(L \setminus w)$$

(8)

Any of the properties $S_{\theta,t}$ or $W_{\theta,t}$ is called a $\theta$-transducer property.

The difference between $S$-properties and $W$-properties is that $S_{\theta,t}$ forbids that $L \in S_{\theta,t}$ contains a word $w$ such that any $\theta(w) \in t(w)$, while this case is allowed for $L \in W_{\theta,t}$. For fixed $t$, $\theta$, and $L$, Condition (7) implies that for all $w \in L$ we have $\theta(w) \cap t(L \setminus w) = \emptyset$ which is equivalent to Condition (8). In other words, if $L$ satisfies $S_{\theta,t}$, then $L$ satisfies $W_{\theta,t}$ as well. If $\theta = \text{id}$ and $t$ is input-altering, or input-preserving, then the above defined properties specialize to the existing ones stated in Definition 2.

Example 6. Consider the transducers in Fig. 2. For any word $xwy$, the left transducer $t_s$, say, can delete $x$, then keep $w$ (which has to be non-empty), and then delete $y$. Thus, $t_s(L) \cap \theta(L) = \emptyset$ if and only if $L$ is strictly $\theta$-compliant. Now let $xwy$ with $xy \neq \varepsilon$ and $w \neq \varepsilon$. If $y$ is nonempty, the right transducer $t$ can delete $x$, then keep $w$, and then delete $y$ using the upper path (containing state 1); and if $x$ is nonempty, $t$ can delete $x$, then keep $w$, and then delete $y$ using the lower path (containing state 2). Thus, $t(L) \cap \theta(L) = \emptyset$ if and only if $L$ is $\theta$-compliant. Using FAdo [9] format the left transducer can be specified by the following string, assuming alphabet \{a, b\}

@Transducer 2 * 0\n0 a @epsilon 0\n0 b @epsilon 0\n0 a a 1\n0 b b 1\n1 a a 1\n
0 b b 1\n1 a a 1\n1 b b 1\n1 @epsilon @epsilon 2\n2 a @epsilon 2
2 b @epsilon 2\n
8
As in the classic case where \( \theta = \text{id} \), also in the general case we have that \( \theta \)-input-altering transducers play an important role for \( \mathcal{S} \)-properties because only then the maximality question is decidable. We did not fully explore the usefulness of \( \theta \)-input-preserving for antimorphic permutations yet. For morphic \( \theta \), however, every transducer \( t \) can be modified to obtain a \( \theta \)-input-preserving transducer \( t' \) such that \( W_{\theta,t} = W_{\theta,t'} \); this concept can be utilized in order to efficiently decide the satisfaction problem; see Sect. 5.

Remark 7. Note that only \( S_{\theta,t} \) for a transducer \( t \) which is not \( \theta \)-input-altering can exclude specific words from all languages which satisfy the property \( S_{\theta,t} \). Otherwise, when \( t \) is \( \theta \)-input-altering, it must not realize \( (w, \theta(w)) \); and when we consider an \( \mathcal{W} \)-property, then \( \theta(w) \in t(w) \) is allowed for \( w \in L \). In particular, every singleton language \( L = \{w\} \) satisfies all properties \( W_{\theta,t} \), as well as, \( S_{\theta,t} \) if \( t \) is \( \theta \)-input-altering.

As input-altering transducer properties are a subset of input-preserving transducer properties, we only consider the case of input-preserving transducer properties in the next two results.

The next result demonstrates that existing transducer properties are not suitable for describing even simple DNA-related properties.

**Proposition 8.** The \( \delta \)-nonoverlapping property is not describable by any input-preserving transducer.

**Proof.** The singleton language \( L = \{\text{AT}\} \subseteq \Delta^* \) is not \( \delta \)-nonoverlapping, because the word \( \text{AT} = \delta(\text{AT}) \) is a \( \delta \)-palindrome. Analogously to Remark 7, a transducer property \( P_t = W_{\text{id},t} \), which is described by some input-preserving transducer \( t \), cannot exclude any singleton language. Therefore, we must have \( L \in P_t \). \( \square \)

The counter example language \( \{\text{AT}\} \) used to prove the previous result is rather artificial, as in practice code-related languages should have more than two elements. However, the
statement remains true even if we focus on languages containing more than one word. This case is handled in the next proposition.

**Proposition 9.** There is no input-preserving transducer $t$ that satisfies Equation (2) for all $\delta$-nonoverlapping languages $L$ having at least two elements.

**Proof.** Assume the contrary, that is, there is an input-preserving transducer $t$ such that for any DNA language $L \subseteq \Delta^*$ with at least two element we have

$$\delta(L) \cap L = \emptyset \quad \text{iff} \quad \forall u \in L : t(u) \cap (L \setminus u) = \emptyset.$$  

We can assume that $t$ is in normal form, that is, the label of every edge is of the form $(a, \varepsilon)$ or $(\varepsilon, a)$, for some $a \in \Delta$. Assume that $t$ has $n$ states, for some positive integer $n$, and let $m > n$. We have that $\{A^mC^n, G^mT^m\}$ is not $\delta$-nonoverlapping, so without loss of generality we have that $G^mT^m \in t(A^mC^n)$. Consider an accepting path $\pi$ of $t$ whose label is $(A^mC^n, G^mT^m)$ and say $\pi$ consists of $N$ consecutive edges, for some positive integer $N$. Then, these edges are $s_{i-1} \xrightarrow{(x_i,y_i)} s_i$, for $i = 1, \ldots, N$, so that the concatenation of the $x_i$’s is equal to $A^mC^n$ and the concatenation of the $y_i$’s is equal to $G^mT^m$. As $t$ is in normal form, we have $N = 4m$, and as $m > n$, there is a smallest integer $k \geq 1$ such that state $s_k$ is equal to a previous one, that is $s_k = s_j$ such that $j < k$. By the choice of $k$, we have $k \leq n < m$. Let $x = x_1 \cdots x_j$, $u = x_{j+1} \cdots x_k$, $x' = x_{k+1} \cdots x_N$, and $y = y_1 \cdots y_j$, $v = y_{j+1} \cdots y_k$, $y' = y_{k+1} \cdots y_N$. As $j - k > 0$ and $t$ is in normal form we have that

$$|u| > 0 \quad \text{or} \quad |v| > 0.$$  

Using a standard pumping argument for finite state machines, the path that results if we delete from $\pi$ the $k - j$ edges between $s_j$ and $s_k$ is also an accepting path whose label is $(xx', yy')$. As each $x_i$ and $y_i$ is of length 0 or 1, we have $|ux| \leq k < m$ and $|uy| < m$, and also $|u| \leq k - j$ and $|v| \leq k - j$. This implies $xx' = A^{m-|u|}C^m$ and $yy' = G^{m-|v|}T^m$. As $xx' \neq yy'$ and $yy' \in t(xx')$ we have that $\{xx', yy'\}$ is not $\delta$-nonoverlapping, which implies $xx' = \delta(yy')$, that is, $A^{m-|u|}C^m = A^{m-|v|}C^m$ and, therefore, $|u| = |v| = 0$ which contradicts (9).

\[\square\]

## 4 Expressiveness of Transducer-based Properties

In this section we examine the descriptive power of the newly defined transducer DNA-related properties, that is, the $\theta$-transducer properties. In Theorem 12 we show that these properties properly include the regular $\theta$-trajectory properties. On the other hand, in Proposition 10 we show that there is an independent DNA-related property that is not a $\theta$-transducer property.

**Proposition 10.** The $\theta$-free property (defined below) \cite{13} is not a $\theta$-transducer property.

(D) A language $L \subseteq A^*$ is $\theta$-free if and only if $L^2 \cap A^+\theta(L)A^+ = \emptyset$.  

| 10 |
Proof. First note that every \( \theta \)-transducer property is \( 3 \)-independent, so it is sufficient to show that, for \( \theta = \delta \) and \( A = \Delta \), the \( \theta \)-free property is not \( 3 \)-independent. Assume the contrary and consider the language

\[
K = \{\text{ACGT, CCAC, GTAA}\}.
\]

This is not \( \delta \)-free, as \( \text{ACGT} = \delta(\text{ACGT}) \) and \( \text{CCACGTAA} \in \Delta^+\text{ACGT}\Delta^+ \). On the other hand, one verifies that every nonempty subset of \( K \) of cardinality less than \( 3 \) is \( \delta \)-free, so by our assumption also \( K \) must be \( \delta \)-free, which is a contradiction.

The remainder of this section is devoted to Theorem 12. Recall the DNA alphabet is \( \Delta = \{A, C, G, T\} \). The following DNA language property is considered in Theorem 12

\[
\mathcal{H} = \{ L \subseteq \Delta^* | H(u, \theta(v)) \geq 2, \text{ for all } u, v \in L \},
\]

where \( H(\cdot, \cdot) \) is the Hamming distance function with the assumption that its value is \( \infty \) when applied on different length words. Note that \( \mathcal{H} \) is described by \( \delta \) and the transducer shown in Fig. 3.

![Figure 3: The transducer describing, together with \( \delta \), the \( S \)-property \( \mathcal{H} \).](image)

**Example 11.** The following DNA languages do not satisfy \( \mathcal{H} \):

- \( L_0 = \{\text{AGG, CCA}\} \), \( L_0' = \{\text{GAG, CCC}\} \).

For instance, \( H(\text{CCA}, \delta(\text{AGG})) = 1 \). The following languages satisfy \( \mathcal{H} \):

- \( L_1 = \{\text{ACG, GAT}\} \), \( L_2 = \{\text{CAC, GCT}\} \),

- \( L_3 = \{\text{AAA, CCT}\} \), \( L_4 = \{\text{AAA, CTC}\} \), \( L_5 = \{\text{AAA, TCC}\} \).

For instance, as \( \delta(\text{AAA}) = \text{TTT} \) and all words \( u \in L_3 \) contain at most one \( T \), it follows that \( H(u, \delta(\text{AAA})) \geq 2 \). Now using \( \delta(\text{CCT}) = \text{AGG} \), one verifies that \( H(u, \delta(\text{CCT})) \geq 2 \) for any \( u \in L_3 \). Thus, indeed \( L_3 \) satisfies \( \mathcal{H} \).

**Theorem 12.**

1. Let \( \theta \) be an antimorphic involution. Every regular \( \theta \)-trajectory property is a \( \theta \)-transducer property.
2. Property \( \mathcal{H} \) is a \( \delta \)-transducer property, but not a (regular) \( \delta \)-trajectory one.
Proof. We use the following notation: $\Phi^2_e$ for either of the operators $\Phi^e_+$ and $\Phi^e_-$, and $B^s(\bar{e})$ for either of the properties $B(e)$ and $B^a(e)$.

For the first statement, we show that given any trajectory regular expression $\bar{a}$, each of the following operators is a transducer operator

$$
\begin{align*}
\mathbf{t}_1^a(X) &= X \rightsquigarrow_{\bar{a}} A^+ \\
\mathbf{t}_2^a(X) &= X \llim_{\bar{a}} A^* \\
\mathbf{t}_3^a(X) &= X \rightsquigarrow_{\bar{a}} A^- \\
\mathbf{t}_4^a(X) &= X \rlim_{\bar{a}} A^-
\end{align*}
$$

The statement then would follow by noting that if $t$ and $s$ are transducer operators then also $(t \circ s)$ and $(t \lor s)$ are transducer operators [3], and if $a$ is an automaton, then one can construct the transducer $(s \uparrow a)$ such that $y \in (s \uparrow a)(x)$ if and only if $y \in s(x) \cap L(a)$ [23]. For example, for any pair $\bar{e} = (\bar{e}_1, \bar{e}_2)$, we have that

$$
\Phi^s_{\bar{e}}(L) = (t_2^{{\bar{e}_2}} \circ (t_1^{{\bar{e}_1}} \uparrow a_+))(L),
$$

where $a_+$ is any automaton accepting $A^+$. The claim about $t_4^a$ is already shown in [8]. For the claim about $t_3^a$, first note that $X \rlim_{\bar{a}} A^* = (X \rlim_{\bar{a}} A^+) \cup (X \rlim_{\bar{a}} \{\varepsilon\})$, so $t_3^a$ is equal to $(t_4^a \lor t_{a, id})$, where $t_{a, id}$ is a transducer with $t_{a, id}(x) = x \rlim_{\bar{a}} \{\varepsilon\}$ and defined as follows. First note that by definition, $y \in x \rlim_{\bar{a}} \{\varepsilon\}$ if and only if $y = x$ and 0$|x|$ $\in L(\bar{a})$. Let $a$ be an automaton with no empty transitions accepting $L(\bar{a})$. Then, $t_{a, id}$ is made based on $a$ as follows. Its set of transitions consists of all tuples $(p, a/a, q)$ such that $(p, 0, q)$ is a transition of $a$—we say that the latter is the corresponding transition of the former. The initial and final states of $t_{a, id}$ are those initial and final states, respectively, of $a$ that appear in the transitions of $t_{a, id}$. It follows that $t_{a, id}$ realizes a pair $(x, y)$ of words using some path $P$ of transitions, if and only if $x = y$ and the automaton $a$ accepts 0$|x|$ using a path consisting of the corresponding transitions that make the path $P$.

In [22] it is observed that $y \in (x \rightsquigarrow_t w)$ if and only if $x \in (y \llim_t w)$, for all words $x, y, w$ and trajectories $t$, which implies that $t_3^a$ and $t_4^a$ are simply the inverses of the transducers $t_1^a$ and $t_2^a$, respectively.

For the second statement we recall that $\mathcal{H}$ is described by $\delta$ and the transducer shown in Fig. 3. For the second part of the statement, we argue by contradiction, so we assume that there is a pair of trajectory regular expressions $\bar{e} = (\bar{e}_1, \bar{e}_2)$ such that

$$
\mathcal{H} = B^s(\bar{e}_1, \bar{e}_2).
$$

Using the definition of $\Phi^2$, one verifies that

$$
\Phi^2(a) \subseteq aA^*, \text{ for all } a \in A.
$$

Consider the DNA language $K = \{A, C\}$. One verifies that $K$ does not satisfy $\mathcal{H}$, but on the other hand $\delta(K) \cap \Phi^2_{\bar{e}}(K) = \emptyset$, which means that $K$ satisfies $B^s(\bar{e}_1, \bar{e}_2)$, which leads to the required contradiction. 

\qed
The counter example used to prove the second statement of Theorem 12 is a little artificial, as the language \( K = \{A, C\} \) consists of 1-letter words, which is of no practical value. The next result gives a stronger statement, as it requires that all words involved are of length at least 2.

**Proposition 13.** The following property

\[ \mathcal{H}_2 = \{ L \subseteq \Delta^* \mid |u| \geq 2 \text{ and } H(u, \theta(v)) \geq 2, \text{ for all } u, v \in L \} \]

is a \( \delta \)-transducer property but not a \( \delta \)-trajectory property.

The proof of this results require a couple of intermediate results, which we present next.

**Lemma 14.** Let \( x, y \) be any words and \( s, t \) be any trajectories. If \( y \in (x \sim_s A^*) \cap A^+ \) then

\[ |t| - |s| = |t|_1 - |s|_1 = |y| - |x| \quad \text{and} \quad |s|_1 < |x|. \]

**Proof.** The premise of the statement implies that \( y \in z \uplus \uplus \uplus \uplus \uplus \uplus \uplus \uplus \uplus w_2 \) and \( z \in ((x \sim_s w_1) \cap A^+) \) for some words \( z, w_1, w_2 \) with \( |z| > 0 \). Informally, this means that \( y \) results by deleting \( |w_1| \) symbols from \( x \), with \( |w_1| < |x| \), and then inserting \( |w_2| \) symbols. More formally as \( |t| = |y| \) and \( |s| = |x| \), we have that \( |t| - |s| = |y| - |x| \). Also as \( |z| = |x| - |w_1| = |s| - |s|_1 \), we have that \( |s| > |s|_1 \) and, therefore, \( |x| > |s|_1 \), as required. Now, we have

\[ |s|_1 = |w_1| = |x| - |z| = |x| - (|y| - |w_2|) = |x| - |y| + |t|_1 \]

and, therefore, \( |t|_1 - |s|_1 = |y| - |x| \).

**Lemma 15.** Let \( \bar{e} = (\bar{e}_1, \bar{e}_2) \) be a pair of trajectory regular expressions and assume that \( \mathcal{H} = \mathcal{B}^2(\bar{e}) - \) as we shall see further below this assumption leads to a contradiction.

1. There is no pair \( (s, t) \) of trajectories in \( L(\bar{e}_1) \times L(\bar{e}_2) \) such that \( |s| = |t| = 3 \) and \( |s|_1 = |t|_1 = 2 \).

2. If \( x, y \) are DNA words of length \( 3 \) and \( (s, t) \in L(\bar{e}_1) \times L(\bar{e}_2) \) such that \( x \neq \delta(y) \) and \( y \in ((x \sim_s \Delta^*) \cap \Delta^+) \uplus \Delta^* \) then \( |s| = |t| = 3 \) and \( |s|_1 = |t|_1 = 1 \).

3. We have that \( 010 \in L(\bar{e}_1) \) or \( 010 \in L(\bar{e}_2) \).

4. We have that \( (001, 001) \in L(\bar{e}_1) \times L(\bar{e}_2) \) or \( (100, 100) \in L(\bar{e}_1) \times L(\bar{e}_2) \).

**Proof.** We shall use some of the seven languages in Example 11.

For the first statement, assume for the sake of contradiction that the two trajectories have equal length and exactly two 1s each. By applying \( (AAA \sim_s \Delta^*) \cap \Delta^+ \) followed by \( \uplus \Delta^* \), the result is \( \Phi^2(AAA) \) and is equal to \( AAA \Delta \) or \( \Delta AAA \) or \( \Delta \Delta \), depending on whether \( t = 011 \) or \( t = 101 \) or \( t = 110 \), respectively. More specifically, if \( t = 011 \) then \( \Phi^2(AAA) \) contains \( \delta(\text{CCT}) \), which contradicts the fact that \( L_3 \) satisfies \( \mathcal{H} \). If \( t = 101 \) then \( \Phi^2(AAA) \) contains \( \delta(\text{CTC}) \), which contradicts the fact that \( L_4 \) satisfies \( \mathcal{H} \). If \( t = 110 \) then \( \Phi^2(AAA) \) contains \( \delta(\text{TCC}) \), which contradicts the fact that \( L_5 \) satisfies \( \mathcal{H} \).
For the second statement, Lemma 14 implies that $|s| = |t| = 3$ and $|s|_1 = |t|_1 \leq 1$, and $x \neq \delta(y)$ implies that $|s|_1 \neq 0$. Hence, $|s|_1 = |t|_1 = 1$, as required.

For the third statement, the fact that $L_0$ does not satisfy $H$ implies that there are words $u, v \in L_0$ such that $\delta(v) \in \Phi^2(u)$ and, therefore, there are words $w_1, w_2$ and $(s, t) \in L(\tilde{e}_1) \times L(\tilde{e}_2)$ such that

$$\delta(v) \in ((u \sim_s w_1) \cap \Delta^+) \sqcup_t w_2.$$  

By the previous statement, $|s| = |t| = 3$ and $|s|_1 = |t|_1 = 1$, which implies $|w_1| = |w_2| = 1$. For the sake of contradiction assume $s \neq 010$ and $t \neq 010$. Let $u = u_1u_2u_3$ with each $u_i$ being a symbol. There are four cases about the values of $s$ and $t$, all of which lead to contradictions. For example, if $s = 001$ and $t = 001$ then $\delta(v) = u_1u_2w_2$, which implies that $v = w_2\bar{u}_2\bar{u}_1$. By inspection, one verifies that $u_1u_2u_3, w_2\bar{u}_2\bar{u}_1$ cannot be both in $L_0$.

For the fourth statement, the fact that $L_0$ does not satisfy $H$ implies that there are words $u, v \in L_0$ such that $\delta(v) \in \Phi^2(u)$ and, therefore, there are words $w_1, w_2$ and $(s, t) \in L(\tilde{e}_1) \times L(\tilde{e}_2)$ such that

$$\delta(v) \in ((u \sim_s w_1) \cap \Delta^+) \sqcup_t w_2.$$  

By a previous statement, $|s| = |t| = 3$ and $|s|_1 = |t|_1 = 1$, which implies $|w_1| = |w_2| = 1$. Let $u = u_1u_2u_3$ with each $u_i$ being a symbol. The rest of the proof consists of four parts:

- $s = 010$ leads to a contradiction;
- $t = 010$ leads to a contradiction;
- $s = 001$ implies $t = 001$;
- $s = 100$ implies $t = 100$.

We demonstrate the first and fourth parts and leave the other two parts to the reader to verify. For the first part, if $s = 010$ then depending on whether $t = 001$ or $t = 010$ or $t = 100$, we have that $\delta(v) = u_1u_3w_2$ or $\delta(v) = w_2u_1u_3$ or $\delta(v) = w_2u_1w_3$, and hence, $v = \bar{w}_2\bar{u}_3\bar{u}_1$ or $v = \bar{w}_2\bar{u}_3\bar{u}_1$ or $v = \bar{u}_3u_1\bar{w}_2$. One verifies by inspection that, in any case, it is impossible to have $u, v \in L_0$. Finally for the last part, if $s = 100$ then, as $t$ cannot be 010, we have that $\delta(v) = u_2u_3w_2$ or $\delta(v) = w_2u_2u_3$ and hence, $v = \bar{w}_2\bar{u}_3\bar{u}_2$ or $v = \bar{u}_3u_2\bar{w}_2$. One verifies by inspection that, in either case, it is impossible to have $u, v \in L_0$.

**Proof.** (Of Proposition 13.) The fact that $H_2$ is a $\delta$-transducer $S$-property is established using the transducer in Fig. 4. For the second part of the statement, we argue by con-

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![Figure 4: The transducer describing, together with $\delta$, the $S$-property $H_2$.](image-url)
tradition, so we assume that there is a pair of trajectory regular expressions \((\bar{e}_1, \bar{e}_2)\) such that
\[
\mathcal{H}_2 = B^2(\bar{e}_1, \bar{e}_2).
\]
By Lemma 15, we have that 001 \(\in L(\bar{e}_2)\) or 100 \(\in L(\bar{e}_2)\), and that 001 \(\in L(\bar{e}_1)\) or 100 \(\in L(\bar{e}_1)\). Moreover, we can distinguish the following four cases, which all lead to contradictions. We also consider the languages \(L_1\) and \(L_2\) defined in Example 11.

**Case ‘010 \(\in L(\bar{e}_1)\) and 001 \(\in L(\bar{e}_2)\).** Then, \(\text{GCT}\) results into \(\text{GT}\), then into \(\text{GCT}\) and then into \(\text{CAC}\) using, respectively, the operations \(\sim_{010}, \sqcup_{001}\) and \(\delta\), which contradicts the fact that \(L_2\) satisfies \(\mathcal{H}\).

**Case ‘010 \(\in L(\bar{e}_1)\) and 100 \(\in L(\bar{e}_2)\).** Then, \(\text{GAT}\) results into \(\text{GT}\), then into \(\text{CGT}\) and then into \(\text{ACG}\) using, respectively, the operations \(\sim_{010}, \sqcup_{100}\) and \(\delta\), which contradicts the fact that \(L_1\) satisfies \(\mathcal{H}\).

**Case ‘001 \(\in L(\bar{e}_1)\) and 010 \(\in L(\bar{e}_2)\).** Then, \(\text{ACG}\) results into \(\text{AC}\), then into \(\text{ATC}\) and then into \(\text{GAT}\) using, respectively, the operations \(\sim_{001}, \sqcup_{010}\) and \(\delta\), which contradicts the fact that \(L_1\) satisfies \(\mathcal{H}\).

**Case ‘100 \(\in L(\bar{e}_1)\) and 010 \(\in L(\bar{e}_2)\).** Then, \(\text{CAC}\) results into \(\text{AC}\), then into \(\text{AGC}\) and then into \(\text{GCT}\) using, respectively, the operations \(\sim_{100}, \sqcup_{010}\) and \(\delta\), which contradicts the fact that \(L_2\) satisfies \(\mathcal{H}\).

\(\square\)

5 The Satisfaction and Maximal Problem

For \(\theta = \text{id}\) and for input-altering and -preserving transducers the satisfaction and maximality problems are decidable \([8]\). In particular, for a regular language \(L\) given via an automaton \(a\), Condition (1) can be decided in time \(O(|t| |a|^2)\), where the function \(|\cdot|\) returns the size of the machine in question (its number of edges plus the length of all labels on the edges). Condition (2) can be decided in time \(O(|t| |a|^2)\), as noted in Remark 16. The maximality problem is decidable, but PSPACE-hard, for both input-altering and -preserving transducer properties.

**Remark 16.** Let \(s = t \downarrow a \uparrow a\) be the transducer obtained by two product constructions: first on the input of \(t\) with \(a\); then, on the output of the resulting transducer with \(a\). In \([8]\) the authors suggest to decide whether or not \(L\) satisfies the input-preserving transducer property \(W_{\text{id}, t}\) by testing if the transducer \(s\) is functional \((|s(x)| \leq 1\) for all \(x \in A^*\)). However, deciding \(L \in W_{\text{id}, t}\) can be done by the cheaper test of whether or not \(s\) implements a (partial) identity function \((s(x) = \{x\}\) or \(s(x) = \emptyset\) for all \(x \in A^*\)). Using the identity test from \([1]\), we obtain that Condition (2) can be decided in time \(O(|t| |a|^2)\) when the alphabet is considered constant. Also note that the identity test does not require that \(t\) is input-preserving if \(\theta = \text{id}\). When \(\theta\) is antimorphic, however, the identity test does not work anymore and we have to resort to the more expensive functionality test for \(\theta\)-input-preserving transducers.

In this work we are interested in the case when \(\theta \neq \text{id}\) is antimorphic; furthermore, the \(\theta\)-input-altering or -preserving restrictions on the transducer are not necessarily present
in the definition of $W$-properties or $S$-properties. Table 1 summarizes under which conditions the satisfaction and maximality problems are decidable for regular languages. For the satisfaction problem, except for the case of non-restricted transducer $W$-properties, Conditions (7) and (8) can be tested similarly to Conditions (1) and (2). For the case of non-restricted transducer $W$-properties, we show decidability using a different method; see Sect. 5.1. The undecidability result holds for every fixed permutation $\theta$ over an alphabet with at least two letters, in particular, all results apply to the DNA-involution $\delta$. All maximality results are discussed in Sect. 5.2.

| Problem      | Property $S_{\theta,t}$         | Property $W_{\theta,t}$         |
|--------------|---------------------------------|---------------------------------|
| Satisfaction | decidable in $O(|t||a|^2)$ as in [8] | decidable as in [8]            |
|              | no restriction $t$ is $\theta$-i.-altering | $t$ is $\theta$-i.-preserving |
| Maximality   | undecidable Corollary 26         | decidable, PSPACE-hard         |
|              |                                  | Theorem 22, Corollary 23        |

Table 1: (Un-)decidability of the satisfaction and the maximality problems for a fixed antimorphic permutation $\theta$, a given transducer $t$, and a regular language $L$ given via an automaton $a$.

Remark 17. We note that deciding the satisfaction question for any $\theta$-trajectory property involves testing the emptiness conditions in (5) or (6), which requires time $O(|a|^2|a_1||a_2|)$, where $a_1, a_2$ are automata corresponding to $\bar{e}_1, \bar{e}_2$. Such a property can be expressed as $\theta$-transducer $S$-property (recall Theorem 12) using a transducer of size $O(|a_1||a_2|)$ and, therefore, the satisfaction question can still be solved within the same asymptotic time complexity.

5.1 The Satisfaction Problem for non-restricted $W$-properties

We establish the decidability of non-restricted transducer $W$-properties for regular languages. We do not concern the complexity of this algorithm; optimizing the algorithm and analyzing its complexity is part of future research. Let $t$ be a transducer, $\theta$ be an antimorphic permutation, and $L$ be a regular language over the alphabet $A$. Let $a_L$ and $a_{\theta(L)}$ be the NFAs accepting the languages $L$ and $\theta(L)$, respectively. Let $s = (Q_s, A, E_s, I_s, F_s) = t \downarrow a_L \uparrow a_{\theta(L)}$ be the product transducer such that $y \in s(x)$ if and only if $y \in t(x)$, $x \in L$, and $y \in \theta(L)$. We consider $s$ to be trim, i.e., every state in $Q_s$ lies on a path that leads from an initial state to a final state. Furthermore, $s$ is considered to be in normal form such that every edge is either labeled $(a, \varepsilon)$ or $(\varepsilon, a)$ for some letter $a \in A$. Thus, for any path $p \xrightarrow{\ldots, (x,y), \ldots} q$ of length $\ell$ (the path has $\ell$ edges) in $s$ we have $|xy| = \ell$.

Lemma 18. Let $L$ be a regular language, $t$ be a transducer, $\theta$ be an antimorphic involution, and $s = t \downarrow a_L \uparrow a_{\theta(L)}$ (all defined over $A$). The regular language $L$ satisfies $W_{\theta,t}$ if and
only if for all words $x, y \in A^+$

$$y \in s(x) \implies \theta(x) = y.$$  

**Proof.** We will prove the contrapositive: $L \notin W_{\theta,t}$ if and only if there exists $x, y \in A^+$ such that $y \in s(x)$ and $\theta(x) \neq y$. Recall that $L \notin W_{\theta,t}$ if and only if there exists $w \in L$ such that $\theta(w) \notin t(L \setminus w)$.

Assume that $L \notin W_{\theta,t}$ and, therefore, $w \in L$ exists such that $\theta(w) \in t(L \setminus w)$. Let $x \in L \setminus w$ such that $\theta(w) \in t(x)$ and $y = \theta(w) \in \theta(L)$. Clearly, we have $y \in s(x)$ and $y \neq \theta(x)$.

Conversely, assume that $x, y \in A^+$ exists such that $y \in s(x)$ and $y \neq \theta(x)$. Let $w = \theta^{-1}(y)$ and note that $w \in L$ (because $y \in \theta(L)$), $x \in L \setminus w$, and $\theta(w) \in t(x) \subseteq t(L \setminus w)$. Therefore, $L \notin W_{\theta,t}$. \hfill \Box

Let $T_s = \{(x_1, x_2, x_3) \in (A^*)^3 \mid |x_1x_2x_3| \leq |s|\}$ be a set of word triples. Note that the length restrictions for the words ensures that $T_s$ is a finite set. For each triple $t = (x_1x_2x_3) \in T_s$ we define a relation

$$R_t = \{(x_1(x_2)^kx_3, \theta(x_1(x_2)^kx_3)) \mid k \in \mathbb{N}\} \subseteq A^* \times A^*.$$  

Note that we allow that any word of $x_1, x_2, x_3$ is empty; in particular, if $x_2 = x_3 = \varepsilon$, then $R_t$ contains only one pair of words $(x_1, \theta(x_1))$.

**Lemma 19.** Let $L$ be a regular language, $t$ be a transducer, $\theta$ be an antimorphic involution, and $s = t \downarrow a_L \uparrow a_{\theta(L)}$ (all defined over $A$). The regular language $L$ satisfies $W_{\theta,t}$ if and only if the relation realized by $s$ satisfies

$$s \subseteq \bigcup_{t \in T_s} R_t.$$  

(10)  

**Proof.** Recall that for every $(x, y) \in R_t$ with $t \in T_s$ we have $\theta(x) = y$. If $s$ satisfies Equation (10), then for all $(x, y)$ which are realized by $s$, we have $\theta(x) = y$; and by Lemma 18 $L$ satisfies $W_{\theta,t}$.

Conversely, suppose that $L$ satisfies $W_{\theta,t}$, let $(x, y)$ be a pair of words that is realized by $s$, and note that $\theta(x) = y$ by Lemma 18. If $|x| \leq |s|$, then $(x, \theta(x)) = (x, y) \in R_t$ for $t = (x, \varepsilon, \varepsilon) \in T_s$.

Otherwise, every accepting path in $s$ that is labeled by $(x, \theta(y))$ contains more than $|s|$ edges, and therefore, must have a repeating state $p$

$$s \xrightarrow{(x_1,y_1)}^* p \xrightarrow{(x_2,y_2)}^* p \xrightarrow{(x_3,y_3)}^* f$$

such that $x = x_1x_2x_3$ and $\theta(x) = y_1y_2y_3$, $s \in I_s$, $f \in F_s$, $x_2y_2 \neq \varepsilon$, $|x_1x_2y_1y_2| \leq |s|$ (using the pigeonhole principle). By Lemma 18 for all $i \in \mathbb{N}$

$$x_1x_2^i x_3 = \theta^{-1}(y_1^i y_2 y_3) = \theta^{-1}(y_3)\theta^{-1}(y_2)\theta^{-1}(y_1).$$
Firstly note, that this implies $|x_2| = |y_2|$. Now, consider $i = 2|x|$. Because $|x_1x_2x_3| \geq |s| \geq |x_1x_2y_1y_2|$, we have that $\theta^{-1}(y_2)\theta^{-1}(y_1)$ is a suffix of $x_3$. Since $i$ is sufficiently large, the suffix $x_2x_3$ of $x_1x_2x_3$ cannot overlap with the prefix $\theta^{-1}(y_3)$ of $x_1x_2x_3$. Hence, there exists a suffix $u$ of $\theta^{-1}(y_2)$ and an integer $j \geq 2$ such that

$$x_2x_3 = u\theta^{-1}(y_2)^j\theta^{-1}(y_1).$$

Chose $v$ such that $\theta^{-1}(y_2) = vu$ and note that $x_2 = uv$ because $|x_2| = |y_2|$ (this argument is a special case of the well-known Fine and Wilf’s Theorem). Let $x_3' = u\theta^{-1}(y_1)$ and observe that $x_3 = u(vu)^{-1}x_2 = x_2^{-1}x_3'$. Furthermore, $|x_1x_2x_3'| \leq |x_1x_2y_1y_2| \leq |s|$. We conclude that $(x, \theta(x)) = (x_1x_2x_3', \theta(x_1x_2x_3')) \in R_t$ for $t = (x_1, x_2, x_3') \in T_s$. \hfill \Box

In order to test whether or not Equation (10) is satisfied, we perform two separate tests. Firstly, we test whether or not $s$ satisfies the weaker condition

$$s \subseteq \bigcup_{(x_1,x_2,x_3) \in T_s} (x_1x_2^*x_3) \times \theta(x_1x_2^*x_3).$$

(11)

Secondly, we ensure that

$$\forall x, y: y \in s(x) \implies |x| = |y|.$$  

(12)

Lemma 20. **Equation (10) is satisfied if and only if Equations (11) and (12) are satisfied.**

*Proof.* If Equation (10) is satisfied, then Equation (11) is satisfied because $R_{(x_1,x_2,x_3)} \subseteq (x_1x_2^*x_3) \times \theta(x_1x_2^*x_3)$ for $(x_1,x_2,x_3) \in T_s$. Also note that for all $(x,y) \in R_t$ with $t \in T_s$ we have $|x| = |y|$; therefore, Equation (10) implies Equation (12).

Conversely, assume that Equations (11) and (12) are satisfied. For all $(x,y)$ that are realized by $s$ we have there exists $(x_1,x_2,x_3) \in T_s$ and $i,j \in \mathbb{N}$ such that $x = x_1x_2x_3$ and $y = \theta(x_1x_2x_3)$. Since the equation $|x| = |y|$ must also be satisfied, it is clear that $i = j$ and, hence, $(x,y) \in R_{(x_1,x_2,x_3)}$. We conclude that Equations (11) and (12) imply Equation 10. \hfill \Box

Theorem 21. **Let $L$ be a regular language given as automaton, $t$ be a given transducer, and $\theta$ be a given antimorphic involution (all defined over $A$). It is decidable whether $L$ satisfies $W_{\theta,t}$ or not.**

*Proof.* According to Lemmas 19 and 20 we have to decide whether or not the two Equations (11) and (12) are satisfied for the transducer $s \vdash t \upharpoonright a_L \upharpoonright a_{\theta(L)}$. It is known that it is decidable whether or not a given transducer is included in a recognizable relation (that is a relation $\bigcup_{i=1}^n A_i \times B_i$ for regular $A_i, B_i$); see [3]. Therefore, the inclusion in Equation (11) is decidable.

The property in Equation (12) can be verified by an algorithm that assigns an integer to each state in $s$: the integer $i$ is assigned to $q \in Q_s$ if there exists a path $s \xrightarrow{(x,y)}^* q$ from a starting state $s \in I_s$ such that $i = |x| - |y|$. The test fails if a state is assigned two distinct integers or if a final state from $F_s$ is assigned an integer different from 0; otherwise,
the test is successful. Assigning the integers can be done by a simple depth-first traversal of $s$. We omit further details on the implementation of this algorithm as it can be done analogously to the test whether or not a given transducer implements a (partial) identity function which can be found in [1].

5.2 The Maximality Problem

Here we show how to decide maximality of a regular language $L$ with respect to a $\theta$-transducer property; see Theorem 22. This result only holds when we consider $\mathcal{W}$-properties or when we consider $\mathcal{S}$-properties for $\theta$-input-altering transducers. As in the case of existing transducer properties, it turns out that the maximality problem is PSPACE-hard; see Corollary 23. When we consider general $\mathcal{S}$-properties, the maximality problem becomes undecidable; see Corollary 26.

**Theorem 22.** For an antimorphic permutation $\theta$, a transducer $t$, and a regular language $L$, all defined over $A_k^*$, such that either

i.) $L \in \mathcal{W}_{\theta,t}$ or

ii.) $L \in \mathcal{S}_{\theta,t}$ and $t$ is $\theta$-input altering,

$L$ is maximal with property $\mathcal{W}_{\theta,t}$ (resp., $\mathcal{S}_{\theta,t}$) if and only if

$$L \cup \theta^{-1}(t(L)) \cup t^{-1}(\theta(L)) = A_k^*.$$ (13)

**Proof.** i.) Suppose $L \cup \theta^{-1}(t(L)) \cup t^{-1}(\theta(L)) = A_k^*$. For every word $w \in L^c$ we have $\theta(w) \in t(L)$ or $w \in t^{-1}(\theta(L))$. In the former case, we immediately obtain that $L \cup w$ does not satisfy $\mathcal{W}_{\theta,t}$. In the latter case, there exists $u \in L$ such that $\theta(u) \in t(w)$, and therefore, $L \cup w$ does not satisfy $\mathcal{W}_{\theta,t}$. We conclude that $L$ is maximal with respect to $\mathcal{W}_{\theta,t}$.

Conversely, suppose there exists a word $w$ such that $w \notin L \cup \theta^{-1}(t(L)) \cup t^{-1}(\theta(L))$. Clearly, $w \in L^c$. Furthermore, we must have $\theta(w) \notin t(L)$ and $\theta(w) \notin t(w)$ for all $u \in L$. Since $L \in \mathcal{W}_{\theta,t}$, we also have that $\theta(u) \notin t(L \setminus u)$ for all $u \in L$. Thus, we obtain that $\forall u \in (L \cup u): \theta(u) \notin t((L \cup u) \setminus u)$, and therefore, $L$ is not maximal with respect to $\mathcal{W}_{\theta,t}$.

ii.) Suppose $L \cup \theta^{-1}(t(L)) \cup t^{-1}(\theta(L)) = A_k^*$. For all $w \in L^c$ we have $\theta(w) \cap t(L) \neq \emptyset$ or $t(w) \cap \theta(L) \neq \emptyset$. Thus, $L \cup w$ does not satisfy $\mathcal{S}_{\theta,t}$ and $L$ is maximal with respect to $\mathcal{S}_{\theta,t}$.

Conversely, suppose there exists a word $w$ such that $w \notin L \cup \theta^{-1}(t(L)) \cup t^{-1}(\theta(L))$. Hence, $\theta(w) \cap t(L) = \emptyset$ and $t(w) \cap \theta(L) = \emptyset$. Furthermore, we have $\theta(L) \cap t(L) = \emptyset$ because $L \in \mathcal{W}_{\theta,t}$ and $\theta(w) \cap t(w) = \emptyset$ because $t$ is $\theta$-input-altering. We conclude that $L \cup w$ satisfies $\mathcal{W}_{\theta,t}$, and therefore, $L$ is not maximal with respect to $\mathcal{W}_{\theta,t}$. \qed

We note that it is PSPACE-hard to decide whether or not Equation (13) holds when $L$ is given as NFA because it is PSPACE-hard to decide universality of a regular language given as NFA ($L \subseteq A_k^*$ is universal if $L = A_k^*$) [29].

**Corollary 23.** For an antimorphic permutation $\theta$, a transducer $t$, and a regular language $L$ given as NFA, all defined over $A_k^*$, such that either
i.) $L \in \mathcal{W}_{\emptyset, \mathbf{t}}$ or

ii.) $L \in \mathcal{S}_{\emptyset, \mathbf{t}}$ and $\mathbf{t}$ is $\emptyset$-input altering,

it is PSPACE-hard to decide whether or not $L$ is maximal with property $\mathcal{W}_{\emptyset, \mathbf{t}}$ (resp., $\mathcal{S}_{\emptyset, \mathbf{t}}$).

**Proof.** According to Theorem 22 deciding maximality of $L$ with property $\mathcal{W}_{\emptyset, \mathbf{t}}$ (resp., $\mathcal{S}_{\emptyset, \mathbf{t}}$) is equivalent to deciding universality of $L \cup \emptyset^{-1}(\mathbf{t}(L)) \cup \mathbf{t}^{-1}(\emptyset(L))$. Let $\mathbf{t}_\emptyset$ be a transducer without final state which does not accept any pair of words. Now, $L$ is maximal with property $\mathcal{S}_{\emptyset, \emptyset}$ (resp., $\mathcal{W}_{\emptyset, \emptyset}$) if and only if $L$ is universal—a problem which is known to be PSPACE-hard. \qed

In the rest of this section we show that it is undecidable whether or not a transducer is $\emptyset$-input-preserving. This question relates directly to the maximality problem of the empty language $\emptyset$ with respect to the property $\mathcal{S}_{\emptyset, \emptyset}$, as stated in Corollary 26. We will reduce the famous, undecidable Post correspondence problem to the problem of deciding whether or not a given transducer is $\emptyset$-input-preserving.

**Definition 24.** The Post correspondence problem (PCP) is, given words $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \in \Sigma^+$ and $\beta_0, \beta_1, \ldots, \beta_{k-1} \in \Sigma^+$, decide whether or not there exists a non-empty sequence of integers $i_1, i_2, \ldots, i_n \in A_\ell = \{0, 1, \ldots, \ell-1\}$ such that

$$\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_n} = \beta_{i_1} \beta_{i_2} \cdots \beta_{i_n}.$$ 

It is well-known that the PCP is undecidable, even if $\Sigma = A_2$ is the binary alphabet.

**Theorem 25.** For every fixed antimorphic permutation $\emptyset$ over $A_2^k$ with $k \geq 2$ it is undecidable whether or not a given transducer is $\emptyset$-input-preserving.

**Proof.** Let $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \in \Sigma^+$ and $\beta_0, \beta_1, \ldots, \beta_{k-1} \in \Sigma^+$ be the PCP instance $\mathcal{A}$. We will define a transducer $\mathbf{t}_\mathcal{A}$ which accepts all pairs $(w, \emptyset(w))$ unless $w$ is a binary encoding of a word $uv$ where $u \in \Sigma^+$ and $v \in A_\ell^+$ such that $v$ describes an integer sequence $i_1, i_2, \ldots, i_n$ that is a solution of $\mathcal{A}$ and $u$ is the corresponding solution word. For the ease of notation, we assume that $\Sigma$ and $A_\ell$ are two disjoint alphabet and we let $\Gamma = \Sigma \cup A_\ell$ be their union. For $m = \lceil \log_2 |\Gamma|\rceil$, we let $h : \Gamma \to A_2^n$ be a morphic block code; i.e., an encoding of $\Gamma$ into binary words of length $m$ such that $h(a) = h(b)$ implies $a = b$ for all $a, b \in \Gamma$. Our goal is to define $\mathbf{t}_\mathcal{A}$ such that $\emptyset(w) \notin \mathbf{t}_\mathcal{A}(w)$ if and only if $w = h(uv)$ for $u \in \Sigma^+, v \in A_\ell^+$, $n = |v|$, and

$$u = \alpha_{v[n]} \alpha_{v[n-1]} \cdots \alpha_{v[1]} = \beta_{v[n]} \beta_{v[n-1]} \cdots \beta_{v[1]}.$$ 

The transducer $\mathbf{t}_\mathcal{A}$ will consist of 3 effectively constructable components $\mathbf{t}_R$, $\mathbf{t}_\alpha$, and $\mathbf{t}_\beta$. Each component can be seen as a fully functional transducer such that $\mathbf{t}_\mathcal{A}$ becomes the union of the three transducers; this implies that

$$y \in \mathbf{t}_\mathcal{A}(x) \iff y \in \mathbf{t}_R(x) \cup \mathbf{t}_\alpha(x) \cup \mathbf{t}_\beta(x).$$

Each transducer component “validates” a certain property of a word $w$, by accepting all word pairs $(w, \emptyset(w))$ which do not have that property:
1.) $t_R$ accepts $(w, \theta(w))$ if and only if $w \notin h(\Sigma^+ A_k^+));

2.) for $w \in h(uv)$ with $u \in \Sigma^+$ and $v \in A_k^+$, $t_\alpha$ accepts $(w, \theta(w))$ if and only if $u \neq \alpha v_{[n]} \alpha v_{[n-1]} \cdots \alpha v_{[1]}$; and

3.) for $w \in h(uv)$ with $u \in \Sigma^+$ and $v \in A_k^+$, $t_\beta$ accepts $(w, \theta(w))$ if and only if $u \neq \beta v_{[n]} \beta v_{[n-1]} \cdots \beta v_{[1]}$.

The first component ensures that every pair $(w, \theta(w))$ that is not accepted by $t_A$ must have the desired form $w \in h(uv)$ with $u \in \Sigma^+$ and $v \in A_k^+$. Components $t_\alpha$ and $t_\beta$ ensure that

$$\alpha v_{[n]} \alpha v_{[n-1]} \cdots \alpha v_{[1]} = u = \beta v_{[n]} \beta v_{[n-1]} \cdots \beta v_{[1]}$$

is the solution word that corresponds the integer sequence $v_{[n]}, v_{[n-1]}, \ldots, v_{[1]}$ if $(w, \theta(w))$ is not accepted by $t_A$. Therefore, every word pair $(w, \theta(w))$ which is not accepted by $t_A$ yields a solution for $\mathcal{A}$ and, vice versa, every solution for $\mathcal{A}$ yields a word pair $(w, \theta(w))$ that cannot be accepted by $t_A$. We conclude that $t_A$ is $\theta$-input-preserving if and only if the PCP instance $\mathcal{A}$ has no solution. This implies that for fixed antimorphic $\theta$ over $A_k^+$ with $k \geq 2$ it is undecidable whether or not a given transducer is $\theta$-input-preserving because the PCP is undecidable.

Now, let us describe the transducer component $t_R$ and recall that it has to work over the alphabet $A_k^+$. It is well known that for any two regular languages $R_1$ and $R_2$ there effectively exists a transducer which accepts the relation $R_1 \times R_2$. There is $t_R$ such that $t_R = (A_k^+ \setminus h(\Sigma^+ A_k^+)) \times A_k^+$. It is easy to observe that we have $t_R(w) = A_k^*$ if $w \notin h(\Sigma^+ A_k^+)$, and $t_R(w) = \emptyset$ if $w \in h(\Sigma^+ A_k^+)$. Therefore, we have $\theta(w) \notin t_R(w)$ if and only if $w \notin h(\Sigma^+ A_k^+)$. Note that this in particular implies that, if $\theta(w) \notin t_R(w)$, then $w \in h(\Gamma^+) \subseteq (A_k^*)^*$. The other two transducer components $t_\alpha$ and $t_\beta$ will only work over word pairs from $h(\Gamma^+) \times \theta(h(\Gamma^+))$.

![Diagram](image)

Figure 5: For $z \in \{\alpha, \beta\}$ the two transducers $t_\alpha$ and $t_\beta$ enforce that $w$ encodes a solution of the PCP instance $\mathcal{A}$ if $\theta(w) \notin (t_\alpha + t_\beta)(w)$ and $w \in h(\Sigma^+ A_k^+)$. Finally, we define the two transducers $t_\alpha$ and $t_\beta$ which are based on the words $\alpha_i$ and $\beta_i$, respectively. For $z \in \{\alpha, \beta\}$ we define $t_z$ as shown in Fig. 5. For a pair of words $(x, y) \in t_z$, it is easy to see that $x \in h(\Gamma^*)$ and $y \in \theta(h(\Gamma^*))$. Furthermore, the edges from the final state $f_z$ to itself ensure that if $(x, y) \in \theta$, then for all words $x' \in h(\Gamma^*)$ and $y' \in \theta(h(\Gamma^*))$,
we have \((xx', yy') \in t_z\) (we will not leave the final state anymore once it is reached, unless
the word pair is not defined over \(h(\Gamma^*) \times \theta(h(\Gamma^*)))\). There are three possibilities to switch
from state \(s_z\) to the final state \(f_z\):

1.) we read a word from \(h(A_\ell)\) in the first component and a words from \(\theta(h(A_\ell))\) in the
second component;

2.) we read a word from \(h(\Sigma)\) in the first component and a words from \(\theta(h(\Sigma))\) in the
second component; or

3.) we read the word \(\theta(h(i))\) with \(i \in A_\ell\) in the second component and in the first com-
ponent we read a word \(h(z')\) such that \(z'\) is not a prefix of \(z_i\) and \(z_i\) is not a prefix \(z'\)
because of the length restriction on \(z'\).

For \(x \in h(\Gamma^*)\) let \(u\) denote the longest word in \(\Sigma^*\) such that \(h(u)\) is a prefix of \(x\) (thus,
either \(x = h(u)\) or \(x = h(uix')\) for an integer \(i \in A_\ell\) and \(x' \in \Gamma^*\); and for \(y \in \theta(h(\Gamma^*))\) let
\(v\) denote the longest word in \(A^*_\ell\) such that \(\theta(h(v))\) is a prefix of \(y\) and let \(n = |v|\) (thus,
either \(y = \theta(h(v))\) or \(y = \theta(h(y'av)) = \theta(h(v))\theta(h(a))\theta(h(y'))\) for a symbol \(a \in \Sigma\) and
\(y' \in \Gamma^*\)). Because \(\theta(h(v_a))(\theta(h(v_{n-1})))\cdots(\theta(h(v_1)))\) is a prefix of \(y\) we obtain that the pair
\((x, y)\) is accepted by \(t_z\) if \(u \neq z_{v[n]}z_{v[n-1]}\cdots z_{v[1]}\). Conversely, if \(u = z_{v[n]}z_{v[n-1]}\cdots z_{v[1]}\),
then \((h(u), \theta(h(v)))\) labels a path from \(s_z\) to \(s_z\); since there is no edge from \(s_z\) which is labeled
\((h(i), \varepsilon), (\varepsilon, \theta(h(a)))\), or \((h(i), \theta(h(a)))\) for \(i \in A_\ell\) and \(a \in \Sigma\), we obtain that \((x, y)\) cannot
not be accepted by \(t_z\).

Suppose \(\theta(w) \notin t_z(w)\) and \(w \in h(uv)\) for words \(u \in \Sigma^+\) and \(v \in A^*_\ell\). Following our
notion from the previous paragraph, \(u\) is the longest word in \(\Sigma^*\) such that \(h(u)\) is a prefix of \(w\), and \(v\) is the longest word in \(A^*_\ell\) such that \(\theta(h(v))\) is a prefix of \(\theta(w)\). Therefore, we
obtain that \(u = z_{v[n]}\cdots z_{v[1]}\).

This leads to the undecidability of the maximality problem of a regular language \(L\)
with respect to a \(\theta\)-transducer-property \(S_{\theta, t}\).

**Corollary 26.** For every fixed antimorphic permutation \(\theta\) over \(A^*_k\) with \(k \geq 2\), it is undecid-
able whether or not the empty language \(\emptyset\) is maximal with respect to the property \(S_{\theta, t}\),
for a given transducer \(t\).

**Proof.** Clearly, the empty language satisfies \(S_{\theta, t}\). For a word \(w\), the language \(\{w\}\) satisfies
\(S_{\theta, t}\) if and only if \(\theta(w) \notin t(w)\). Therefore, \(\emptyset\) is maximal with property \(S_{\theta, t}\) if and only if \(t\)
is \(\theta\)-input-preserving. Theorem 25 concludes the proof.

**6 Undecidability of the \(\theta\)-PCP and the \(\theta\)-input-altering Transducer Problem**

Analogous to the undecidable PCP (see Definition 24), we introduce the \(\theta\) version of the
PCP and prove that it is undecidable as well; see Theorem 28. Further, we utilize the \(\theta\)
version of the PCP in order to show that it is undecidable whether or not a transducer is \( \theta \)-input-altering; see Corollary 29.

**Definition 27.** For a fixed antimorphic permutation \( \theta \) over \( A_k^* \), we introduce the \( \theta \)-Post correspondence problem (\( \theta \)-PCP): given words \( \alpha_0, \alpha_1, \ldots, \alpha_{\ell-1} \in A_k^+ \) and \( \beta_0, \beta_1, \ldots, \beta_{\ell-1} \in A_k^* \), decide whether or not there exists a non-empty sequence of integers \( i_1, \ldots, i_n \in A_\ell = \{0, 1, \ldots, \ell - 1\} \) such that

\[
\alpha_1 \alpha_2 \cdots \alpha_n = \theta(\beta_{i_1} \beta_{i_2} \cdots \beta_{i_n}).
\]

**Theorem 28.** For every fixed antimorphic permutation \( \theta \) over \( A_k^* \) with \( k \geq 2 \) the \( \theta \)-PCP is undecidable.

**Proof.** In order to prove that \( \theta \)-PCP is undecidable, we will state an effective reduction of any PCP instance \( \mathcal{A} \) over alphabet \( A_2 \) to a \( \theta \)-PCP instance \( \mathcal{T} \) over alphabet \( A_k \) such that \( \mathcal{A} \) has a solution if and only if \( \mathcal{T} \) has a solution. Let \( \alpha_0, \alpha_1, \ldots, \alpha_{\ell-1} \in A_2^+ \) and \( \beta_0, \beta_1, \ldots, \beta_{\ell-1} \in A_2^* \) be an instance of the PCP which we call \( \mathcal{A} \).

Note that \( \theta \) and \( \theta^{-1} \) are well-defined over \( A_2 \subseteq A_k \). We define two morphisms \( g, h \) on \( A_2^* \) such that

\[
g(0) = 00, \quad g(1) = 01, \quad h(0) = 10, \quad h(1) = 11.
\]

Note that for each pair of letters \( z \in A_2^* \) we have either \( z \in h(A_2) \) or \( z \in g(A_2) \). Moreover, we let

\[
\begin{align*}
g_j &= g(\alpha_j), & \delta_j &= \theta^{-1}(h(\beta^R_j)), \quad \text{for } j = 0, \ldots, \ell - 1, \\
g_\ell &= h(0), & \delta_\ell &= \theta^{-1}(g(0)), \\
g_{\ell+1} &= h(1), & \delta_{\ell+1} &= \theta^{-1}(g(1)).
\end{align*}
\]

be the \( \theta \)-PCP instance \( \mathcal{T} \).

![Figure 6: Transforming the solution \( i_1, i_2, \ldots, i_n \) of the PCP instance \( \mathcal{A} \) into the solution \( i_1, i_2, \ldots, i_n, i'_m, i''_{m-1}, \ldots, i'_1 \) of the \( \theta \)-PCP instance \( \mathcal{T} \); all variables are defined in the text.](image)

First, let us show that if \( \mathcal{A} \) has a solution than \( \mathcal{T} \) has a solution as well. Let \( i_1, i_2, \ldots, i_n \in A_\ell \) with \( n \geq 1 \) be a solution of the PCP instance \( \mathcal{A} \) and let \( w \) be the word corresponding to this solution; i.e.,

\[
w = \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_n} = \beta_{i_1} \beta_{i_2} \cdots \beta_{i_n}.
\]
Figure 6 illustrates the following construction. Let $m = |w|$. For $j = 1, \ldots, m$ we let $i_j' = \ell$ if $w_{[j]} = 0$ and $i_j' = \ell + 1$ if $w_{[j]} = 1$; these indices are chosen such that

$$
\gamma_{i_m} \gamma_{i_{m-1}} \cdots \gamma_{i_1} = h(w^R),
$$
$$
\delta_{i_m} \delta_{i_{m-1}} \cdots \delta_{i_1} = \theta^{-1}(g(w_{[m]})) \theta^{-1}(g(w_{[m-1]})) \cdots \theta^{-1}(g(w_{[1]})) = \theta^{-1}(g(w)).
$$

The integer sequence $i_1, i_2, \ldots, i_n, i_m', i_{m-1}', \ldots, i_1'$ is a solution of the $\theta$-PCP instance $f(\alpha)$ because

$$
\theta(\delta_1 \cdots \delta_n \delta_m' \cdots \delta_1') = \theta(\delta_1 \cdots \delta_n) \cdots \theta(\delta_1')
$$
$$
= \theta(\theta^{-1}(g(w))) \cdots \theta(\theta^{-1}(h(\beta_2^R))) \cdots \theta(\theta^{-1}(h(\beta_1^R)))
$$
$$
= g(w) \cdots h(\beta_2^R) \cdots h(\beta_1^R)
$$
$$
= g(\alpha_1) \cdots g(\alpha_n) \cdots h(w^R)
$$
$$
= \gamma_{i_1} \cdots \gamma_{i_n} \cdots \gamma_{i_m'} \cdots \gamma_{i_1'}.
$$

Vice versa, let $i_1, i_2, \ldots, i_n \in A_{d+2}$ with $n \geq 1$ be a solution of the $\theta$-PCP instance $T$ and let $w$ be the word corresponding to this solution, that is,

$$
w = \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_n} = \theta(\beta_1 \beta_2 \cdots \beta_n) = \theta(\beta_n) \cdots \theta(\beta_2) \theta(\beta_1).
$$

Recall that for every word $\gamma_{i_j}$ we have that either $\gamma_{i_j} \in g(A_2^+) \ (\text{in case } i_j < \ell)$ or $\gamma_{i_j} \in h(A_2) \ (\text{in case } i_j \geq \ell)$. Since $g(A_2)$ and $h(A_2)$ contain mutually distinct two-letter words, for every pair of letters $p = w_{[2r-1:2r]}$ with $r \in \mathbb{N}$: if $p \in g(A_2)$, then $p$ is covered by a factor $\gamma_{i_j}$ with $i_j < \ell$; and if $p \in h(A_2)$, then $p$ is covered by a factor $\gamma_{i_j}$ with $i_j \geq \ell$. Symmetrically, for $p = w_{[2r-1:2r]}$ with $r \in \mathbb{N}$: if $p \in h(A_2)$, then $p$ is covered by a factor $\theta(\delta_{i_j})$ with $i_j < \ell$; and if $p \in g(A_2)$, then $p$ is covered by a factor $\theta(\delta_{i_j})$ with $i_j > \ell$.

Figure 7: Transforming the solution $i_1, i_2, \ldots, i_n$ of the $\theta$-PCP instance $T$ into the solution $i_1, i_2, \ldots, i_n'$ of the PCP instance $A$; all variables are defined in the text.

Consider the case where $i_1 < \ell$. Figure 7 illustrates the following construction. In this case, $\gamma_{i_1} = g(\alpha_1)$ is a prefix of $w$ and $\theta(\delta_{i_1}) = h(\beta_1^R)$ is a suffix of $w$; thus, $w_{[1:2]} \in g(A_2)$ and $w_{[w-1:w]} \in h(A_2)$. Further, we obtain that $i_n \geq \ell$ because $\gamma_{i_n}$ has to cover $w_{[w-1:w]} \in h(A_2)$. There exists an integer $n'$ with $1 \leq n' < n$ such that $i_1, i_2, \ldots, i_{n'} < \ell$ but $i_{n'+1} \geq \ell$. We will show that the sequence $i_1, i_2, \ldots, i_{n'}$ is a solution of the PCP instance $A$ by comparing the longest prefix of $w$ which belongs to $g(A_{2}^+)$ with the longest suffix of $w$ which belongs to $h(A_{2}^+)$. Let $m$ be an even integer such that $w_{[1:m]} \in g(A_2^+)$.
but \( w_{[m+1;m+2]} \in h(A_2) \). Because \( i_{n'+1} \) has to match with the first letter pair in \( w \) which belongs to \( h(A_2) \), it is not difficult to see that
\[
w_{[1;m]} = \gamma_{i_2} \cdots \gamma_{i_{n'}} = g(\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{n'}}).
\]
Because \( w_{[1;m]} \in g(A_2^+) \) and \( w_{[m+1;m+2]} \in h(A_2) \), there exists an integer \( j < n \) such that
\[i_j, i_{j+1}, \ldots, i_n \geq \ell, \ i_{j-1} < \ell, \text{ and}\]
\[w_{[1;m]} = \theta(\delta_{i_j} \delta_{i_{j+1}} \cdots \delta_{i_n}) = \theta(\delta_{i_n}) \cdots \theta(\delta_{i_{j+1}}) \theta(\delta_{i_j}).\]
Due to the design of the word pairs \((\gamma_\ell, \delta_\ell)\) and \((\gamma_{\ell+1}, \delta_{\ell+1})\) and because
\[\theta(\delta_{i_n}) \cdots \theta(\delta_{i_{j+1}}) \theta(\delta_{i_j})\]
is a prefix of \( w \), we have that \( \gamma_{i_j} \gamma_{i_{j+1}} \cdots \gamma_{i_n} = h((\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{n'}})^R) \) is a suffix of \( w \). Since \( i_{j-1} < \ell \), we see that this suffix \( h((\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{n'}})^R) \) of \( w \) is preceded by a letter pair from \( g(A_2) \). This implies that the suffix \( \theta(\delta_{i_{n'}}) \cdots \theta(\delta_{i_2}) \theta(\delta_{i_1}) \) of \( w \) equals \( h((\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{n'}})^R) \). Therefore,
\[h((\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{n'}})^R) = \theta(\delta_{i_{n'}}) \cdots \theta(\delta_{i_2}) \theta(\delta_{i_1}) \]
\[= h(\beta_{i_{n'}}^R) \cdots h(\beta_{i_2}^R) h(\beta_{i_1}^R) \]
\[= h((\beta_{i_1} \beta_{i_2} \cdots \beta_{i_{n'}})^R).\]
We conclude that \( \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_{n'}} = \beta_{i_1} \beta_{i_2} \cdots \beta_{i_{n'}} \) and, therefore, \( i_1, i_2, \ldots, i_{n'} \) is a solution of the PCP instance \( \mathcal{A} \).

The case when \( i_1 \geq \ell \) can be treated analogously, where we compare the longest prefix of \( w \) which belongs to \( h(A_2^+) \) and the longest suffix of \( w \) which belongs to \( g(A_2^+) \). In this case, there exists \( n' \leq n \) such that \( i_{n'}, i_{n'+1}, \ldots, i_n \) is a solution of the PCP instance \( \mathcal{A} \).

We can utilize the \( \theta \)-PCP in order to prove that it is undecidable whether or not a transducer is \( \theta \)-input-altering, even for one-state transducers.

**Corollary 29.** For every fixed antimorphic permutation \( \theta \) over \( A_k^+ \) with \( k \geq 2 \) it is undecidable whether or not a given (one-state) transducer is \( \theta \)-input-altering.

**Proof.** Let \( \alpha_0, \alpha_1, \ldots, \alpha_{\ell-1} \in A_k^+ \) and \( \beta_0, \beta_1, \ldots, \beta_{\ell-1} \in A_k^+ \) be the \( \theta \)-PCP instance \( \mathcal{A} \). We let \( t_\mathcal{A} \) be the one-state transducer shown in Fig. 8. Clearly, we have \( y \in t_\mathcal{A}(x) \) if and only if there exists an integer sequence \( i_1, i_2, \ldots, i_n \in A_k \) such that \( x = \alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_n} \) and
\[y = \theta^2(\beta_{i_1}) \theta^2(\beta_{i_2}) \cdots \theta^2(\beta_{i_n}) = \theta^2(\beta_{i_1} \beta_{i_2} \cdots \beta_{i_n});\]
note that \( \theta^2 \) is always morphic, even if \( \theta \) is not.

Recall that it is allowed for \( \theta \)-input-altering transducers to accept the empty word pair \((\varepsilon, \varepsilon)\). We have \( w \in \theta^{-1}(t_\mathcal{A}(w)) \) for some word \( w \in A_k^+ \) if and only if there exists an integer sequence \( i_1, i_2, \ldots, i_n \) such that
\[\alpha_{i_1} \alpha_{i_2} \cdots \alpha_{i_n} = w = \theta^{-1}(\theta^2(\beta_{i_1} \beta_{i_2} \cdots \beta_{i_n})) = \theta(\beta_{i_1} \beta_{i_2} \cdots \beta_{i_n}).\]
Therefore, \( t_\mathcal{A} \) is \( \theta \)-input-altering if and only if the \( \theta \)-PCP instance \( \mathcal{A} \) has a solution. Theorem 28 concludes the proof.

∀i ∈ Aℓ: (α_i, θ^2(β_i))

Figure 8: t_A encodes the θ-PCP instance α_0, α_1, ..., α_{ℓ-1}, β_0, β_1, ..., β_{ℓ-1}.

7 A Hierarchy of DNA-related θ-transducer Properties

In [13,18,19] the authors consider numerous properties of languages inspired by reliability issues in DNA computing. Let θ be defined over A^* and assume that θ^2 = id since in the DNA setting θ = δ is an involution. The relationships between some of the defined 3-independent DNA-related properties are displayed in Fig. 9. All properties have in common that they forbid certain “constellations” of words. Consider a language L ⊆ A^+ and two words uwv, θ(xwy) ∈ A^+ with w ≠ ε as shown in the top property in Fig. 9. The same notation can be employed for all properties in the figure, where some properties require that x, y, u, or v are empty, e.g., for x = y = ε we obtain the θ-compliant property. In the case of θ-nonoverlapping all of x, y, u, v are empty and

(A) a language L is θ-nonoverlapping if for all w ∈ A^+, we have w ∉ L or θ(w) ∉ L. This is equivalent to require that L ∩ θ(L) = ∅.

For all properties, except θ-nonoverlapping, the language L has property P, if uwv ∈ L and θ(xwy) ∈ L implies that uwxy = ε. For example,

(B) a language L is θ-compliant if for all w ∈ A^+ and u, v ∈ A^*, we have uwv, θ(w) ∈ L ⇒ uv = ε; and

(C) a language L is θ-5'-overhang-free if for all w ∈ A^+ and u, y ∈ A^*, we have uw, θ(wy) ∈ L ⇒ uy = ε.

Previous papers considered the strict version only for some of the properties. Here, we generalize the concept of strict properties such that if uwv ∈ L and θ(xwy) ∈ L, then L does not satisfy the strict property P^s (even if uwxy = ε). For example,

(D) a language L is strictly θ-compliant if for all w ∈ A^+ and for all u, v ∈ A^*, we have uwv ∉ L or θ(w) ∉ L; and

(E) a language L is strictly θ-5'-overhang-free if for all w ∈ A^+ and u, y ∈ A^*, we have uw ∉ L or θ(wy) ∉ L.

Note that θ-nonoverlapping is actually a strict property while its “normal version” would be the property that is trivially satisfied by every language in A^+.

Furthermore, we introduce the weak version of a property which follows the concept of classic code properties like the (weakly) overlap-free property where it is allowed for a word to overlap with itself, but not with another word: for a language L which satisfied the weak
property $P^\theta$, if the words $uwv$ and $\theta(xwy)$ belong to $L$, then $uvxy = \varepsilon$ or $uwv = \theta(xwy)$. For example,

(F) a language $L$ is weakly $\theta$-5'-overhang-free if for all $w \in A^+$ and $u, y \in A^*$, we have $uw, \theta(wy) \in L$ implies $uy = \varepsilon$ or $uw = \theta(wy)$.

Note that for some properties, like $\theta$-compliant, the weak property $P^\theta$ coincides with the (normal) property $P$.

If a language $L$ satisfies the strict property $P^\theta$, then it also satisfies the corresponding (normal) property $P$; and if $L$ satisfies the (normal) property $P$, then it also satisfies the corresponding weak property $P^\theta$. Furthermore, there is a normal, strict, and weak hierarchy of properties which is shown in Fig. 9, where $\theta$-nonoverlapping only exists in the strict hierarchy. For all three hierarchies an arrow $P^x \rightarrow Q^x$ (for $x \in \{\varepsilon, s, w\}$) between two properties $P^x$ and $Q^x$ means that if a language $L$ satisfies property $P^x$, then it also
satisfies property $Q^x$.

Let us discuss how these properties can be described as $\theta$-transducer properties. The type of the property ($W$-property or $S$-property) and the type of the transducer (unrestricted, $\theta$-input-altering, $\theta$-input-preserving) is important when it comes to the complexity of the satisfaction problem and the decidability of the maximality problem; see Table 1. Firstly, observe that $L$ is $\theta$-nonoverlapping if $L$ satisfies the $\theta$-transducer property $S_{\theta,t_{id}}$, where $t_{id}$ is a transducer realizing the identity relation. Since any strict property, including $\theta$-nonoverlapping, is not satisfied by a singleton language $\{w\}$ that consists of one $\theta$-palindrome $w = \theta(w)$, strict properties cannot be described as $S$-properties by a $\theta$-input-altering transducer or as $W$-properties, according to Remark 7.

Figure 10 shows two families of transducers which are capable of describing any of the DNA-related properties that we introduced in this section. Depending on whether or not $u$ (resp., $v, x, y$) is empty one has to omit a set of edges in each transducer. The $S$-properties $S_{\theta,t_s}$ describe the strict properties, the $S$-properties $S_{\theta,t_w}$ describe the normal properties, and the $W$-properties $W_{\theta,t_w}$ describe the weak properties. If we omit red and orange edges (i.e., $xy = \varepsilon$), then $t_w$ is $\theta$-input-altering because the input word is strictly longer than the output word. Therefore, $S_{\theta,t_w} = W_{\theta,t_w}$, i.e., the normal property coincides with the corresponding weak property. The case when all blue and green edges are omitted is symmetric when input and output swap roles. We demonstrate this construction in Examples 30 and 31.

Example 30. Let $t_s^C$ and $t_w^C$ be the two transducers that are obtained by omitting all red and orange edges in $t_s$ and $t_w$ (Fig. 10), respectively. Then $S_{\theta,t_s^C}$ is the strict $\theta$-compliant property, whereas $S_{\theta,t_w^C}$ is the (normal) $\theta$-compliant property. Since $t_w^C$ is $\theta$-input-altering, $S_{\theta,t_w^C}$ is equal to $W_{\theta,t_w^C}$ and the properties $\theta$-compliant and weak $\theta$-compliant coincide.

Figure 10: The family of transducers which describes all properties shown in Fig. 9. Each of the two transducer families describes 16 different transducers: We can either omit or include each of the red, orange, blue and green edges. These edges are omitted depending on the property that is described, for example, omit all red edges if $x = \varepsilon$ in Fig. 9.
Example 31. Let $t_s^{5OF}$ and $t_w^{5OF}$ be the two transducers that are obtained by omitting all red and green edges in $t_s$ and $t_w$ (Fig. 10), respectively. Then $S_{\theta, t_s^{5OF}}$ is the strict $\theta$-$5'$-overhang-free property, $S_{\theta, t_w^{5OF}}$ is the (normal) $\theta$-$5'$-overhang-free property, and $\mathcal{W}_{\theta, t_w^{5OF}}$ is the weak $\theta$-$5'$-overhang-free property.

Observe that the word $z = \text{AACG}$ can have a $\theta$-$5'$-overhang with itself (as $x = \text{AA}$, $w = \theta(w) = \text{CG}$, and $y = \text{TT}$). As expected, $t_w^{5OF}$ does accept the word pair ($\text{AACG, CGTT}$) and, therefore, the singleton language $\{z\}$ does not satisfy the (normal) $\theta$-$5'$-overhang-free property $S_{\theta, t_w^{5OF}}$, however, $\{z\}$ does satisfy the weak $\theta$-$5'$-overhang-free property $\mathcal{W}_{\theta, t_w^{5OF}}$.

Lastly, note that the (strict, weak) $\theta$-overhang-free property is different from the other properties in Fig. 9 in so far that it forbids two word constellations: $\theta$-$5'$-overhangs and $\theta$-$3'$-overhangs. This property can be described by a transducer which contains two components, where one component covers the $\theta$-$5'$-overhangs and the other component covers the $\theta$-$3'$-overhangs.

8 Conclusions

We have defined a transducer-based method for describing DNA code properties which is strictly more expressive than the trajectory method. In doing so, the satisfaction question remains efficiently decidable. The maximality question for some types of properties is decidable, but it is undecidable for others. While some versions of the maximality question for trajectory properties are decidable, the case of any given pair of regular trajectories and any given regular language is not addressed in [6], so we consider this to be an interesting problem to solve.

The maximality questions are phrased in terms of any fixed antimorphic permutation. This direction of generalizing decision questions is also applied to the classic Post Correspondence Problem, where we demonstrate that it remains undecidable. A consequence of this is that the question of whether a given transducer is $\theta$-input-altering is also undecidable. It is interesting to note that if, instead of fixing $\theta$, we fix the transducer $t$ to be the identity, or the transducer defining the $\mathcal{S}$-property $\mathcal{H}$ (see Fig. 3 in Sect. 4), then the question of whether or not

$$\theta(L) \cap t(L) = \emptyset$$

is decidable (given any regular language $L$ and antimorphic permutation $\theta$).

The topic of studying description methods for code properties requires further attention. One important aim is the actual implementation of the algorithms, as it is already done for several classic code properties [9, 24]. An immediate plan is to incorporate in those implementations what we know about DNA code properties. Another aim is to increase the expressive power of our description methods. The formal method of [16] is quite expressive, using a certain type of first order formulae to describe properties. It could perhaps be further worked out in a way that some of these formulae can be mapped to transducers. We also note that if the defining method is too expressive then even the...
satisfaction problem could become undecidable; see for example the method of multiple sets of trajectories in [7].

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