Injectivity of non-singular planar maps with disconnecting curves in the eigenvalues space

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Abstract

Fessler and Gutierrez [8, 10] proved that if a non-singular planar map has Jacobian matrix without eigenvalues in $(0, +\infty)$, then it is injective. We prove that the same holds replacing $(0, +\infty)$ with any unbounded curve disconnecting the upper (lower) complex half-plane. Additionally we prove that a Jacobian map $(P, Q)$ is injective if $P_x + Q_y$ is not a surjective function.

Keywords: Jacobian Conjecture, global injectivity, eigenvalue continuity

1 Introduction

Let us consider a map $F = (P, Q) \in C^1(\mathbb{R}^2, \mathbb{R}^2)$. Let

$$J_F(x, y) = \begin{pmatrix} P_x(x, y) & P_y(x, y) \\ Q_x(x, y) & Q_y(x, y) \end{pmatrix}$$

be the jacobian matrix of $F$ at $(x, y)$. We denote by $T(x, y)$ the trace of $J_F(x, y)$, i.e. the divergence of the vector field $F(x, y)$, by $D(x, y)$ its

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determinant and by $\Delta(x, y) = T(x, y)^2 - 4D(x, y)$ the discriminant of the eigenvalues equation. We say that $F(x, y)$ is a non-singular map if $D(x, y) \neq 0$ on all of $\mathbb{R}^2$, and that it is a Jacobian map if $D(x, y)$ is a non-zero constant on all of $\mathbb{R}^2$. We denote by $\Sigma_F(x, y)$ the spectrum of $J_F(x, y)$, i.e. the set of its eigenvalues. We set $\Sigma_F = \bigcup \{ \Sigma_F(x, y) : (x, y) \in \mathbb{R}^2 \}$.

The implicit function theorem gives the injectivity of a map in a neighbourhood of a point $(x^*, y^*) \in \mathbb{R}^2$ such that $D(x^*, y^*) \neq 0$. On the other hand, even if $D(x, y) \neq 0$ on all of $\mathbb{R}^2$ the map can be non-injective, as the exponential map $(e^y \cos x, e^y \sin x)$. The search for additional conditions ensuring the global injectivity of a locally invertible map is a classical problem. A fundamental result is Hadamard global inverse function theorem, which gives the global invertibility of a proper non-singular map $F \in C^1(\mathbb{R}^n, \mathbb{R}^n)$. In this field some old problems still resist the attempts to find a solution. The celebrated Jacobian Conjecture is concerned with polynomial maps $F : \mathbb{C}^n \to \mathbb{C}^n$ [13]. According to such a conjecture, a polynomial map with non-zero constant jacobian determinant is invertible, with polynomial inverse. Such a statement and its variants were studied in different settings, even replacing $\mathbb{C}^n$ with $\mathbb{R}^n$ or other fields, and several partial results were proved, but it is not yet proved or disproved even for $n = 2$ [11 16]. Another famous problem, known as the Global Asymptotic Stability Jacobian conjecture [14] was proved in the planar case to be equivalent to a global injectivity one. Such a conjecture was proved to be true in dimension 2 [8 9 10], false in higher dimensions [6].

In [8 10] the injectivity of a map with $D(x, y) > 0$ was proved under the additional assumption that for $(x, y) \notin K$, $K$ compact, the eigenvalues of $J_F(x, y)$ do not belong to $(0, +\infty)$. Such a result was extended in [7 11 12 15]. Other results proving injectivity with different additional conditions were obtained in [2 3 4 5].

In this paper we propose an approach based on the eigenvalues continuity. We prove that the injectivity can be proved by replacing the half-line $(0, +\infty)$ by any unbounded curve $\delta$ in the complex plane, provided $\delta$ disconnects the upper (lower) half-plane. This allows to prove the injectivity as a consequence of some suitable inequalities. Moreover we prove that a Jacobian map is injective if there exists $z \in (-\infty) \cup S^1 \cup (0, +\infty)$ such that $z$ is not an eigenvalue of $J_F(x, y)$, for $(x, y) \in \mathbb{R}^2$. As a consequence, if the function $T(x, y)$ is not surjective, then $F(x, y)$ is injective. We do not require $F(x, y)$
to be polynomial.

2 Results

We report next Lemma without proof, since it is a standard statement in finite dimensional spectral theory. We denote by \( Re(\lambda) \), \( Im(\lambda) \), resp. the real and imaginary part of the complex number \( \lambda \).

**Lemma 1.** Let \( F \in C^1(\mathbb{R}^2, \mathbb{R}^2) \). Then there exist functions \( \lambda_1, \lambda_2 \in C^0(\mathbb{R}^2, \mathbb{R}^2) \), such that for all \( (x, y) \in \mathbb{R}^2 \) \( \lambda_1(x, y) \) and \( \lambda_2(x, y) \) are the eigenvalues of \( J_F(x, y) \). Such functions can be taken such that \( Re(\lambda_1(x, y)) \geq 0 \) and \( Re(\lambda_2(x, y)) \leq 0 \).

Either such eigenvalues are real or complex conjugate. This implies that the set \( \lambda_1(\mathbb{R}^2) \cup \lambda_2(\mathbb{R}^2) \) is symmetric with respect to the x axis. For the reader’s convenience we report the main theorem proved in [8, 10], that will be applied in the following.

**Theorem 1.** (Fessler - Gutierrez) Let \( F \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) with \( D(x, y) > 0 \) for all \( (x, y) \in \mathbb{R}^2 \). Assume there exists a compact set \( K \subset \mathbb{R}^2 \) such that for all \( (x, y) \not\in K \) the eigenvalues of \( J_F(x, y) \) are not in \((0, +\infty)\). Then \( F \) is injective [8, 10].

When convenient, in the following we sometimes identify \( \mathbb{C} \) with the real plane \( \mathbb{R}^2 \). Let us set \( \mathbb{C}^+ = \{u + iv : v \geq 0\} \). We say that an unbounded curve \( \delta \in C^0([0, +\infty), \mathbb{C}^+) \) disconnects the half-plane \( \mathbb{C}^+ \) if \( \delta(0) = 0 + 0 \cdot i \equiv (0, 0) \), \( \delta \) has no other points on the real axis and there exist two connected subsets \( A, B \), such that \( \mathbb{C}^+ = A \cup B \), \( \partial A = \delta = \partial B \). We do not require \( A \) and \( B \) to be disjoint. Such a definition implies that the open real half-axes \((0, +\infty)\) and \((-\infty, 0)\) are not both contained in \( A \) or in \( B \).

We write \( K^c \) for the set-theoretical complement of a set \( K \).

**Theorem 2.** Let \( F \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) with \( D(x, y) > 0 \) for all \( (x, y) \in \mathbb{R}^2 \). Assume there exists a compact set \( K \subset \mathbb{R}^2 \) and a curve \( \delta \in C^0([0, +\infty), \mathbb{C}^+) \) disconnecting \( \mathbb{C}^+ \) and such that for all \( (x, y) \not\in K \) the eigenvalues of \( J_F(x, y) \) are not on \( \delta \). Then \( F \) is injective.
Proof. We prove that the hypotheses of theorem 1 either hold for $F(x, y)$ or for $-F(x, y)$. By absurd, assume that neither $F(x, y)$ nor $-F(x, y)$ satisfy them. Hence for every compact $K \subset \mathbb{R}^2$ there exist both $(x^+_K, y^+_K) \not\in K$, $(x^-_K, y^-_K) \not\in K$ and two eigenvalues $\lambda(x^+_K, y^+_K) \in (0, +\infty)$ and $\lambda(x^-_K, y^-_K) \in (-\infty, 0)$. This implies that also $\frac{D(x^+_K, y^+_K)}{\lambda(x^+_K, y^+_K)} \in (0, +\infty)$ and $\frac{D(x^-_K, y^-_K)}{\lambda(x^-_K, y^-_K)} \in (-\infty, 0)$ are eigenvalues. Hence one has $\lambda_i(x^+_K, y^+_K) \in (0, +\infty)$ and $\lambda_i(x^-_K, y^-_K) \in (-\infty, 0)$, $i = 1, 2$. By the compactness of $K$ there exists a curve $\gamma \in C^0([-1, 1], \mathbb{R}^2)$ with no points in $K$ and connecting $(x^+_K, y^+_K)$ to $(x^-_K, y^-_K)$: $\gamma(-1) = (x^-_K, y^-_K)$, $\gamma(1) = (x^+_K, y^+_K)$.

Let us consider the eigenvalue function $\lambda_1(x, y)$ with $\Re(\lambda_1(x, y)) \geq 0$, as in Lemma 1. Let us consider the curve $\lambda_1(\gamma(t))$. One has $\lambda_1(\gamma(-1)) \in (-\infty, 0)$, $\lambda_1(\gamma(1)) \in (0, +\infty)$. By hypothesis both points are not on $\delta$, and since $\delta(0) = (0, 0)$ one of them belongs to the set $A$, the other one to $B$. The curve $\lambda_1(\gamma(t))$ connects them, hence it has to cross the common boundary of $A$ and $B$, which is $\delta$. This contradicts the hypothesis that no eigenvalues are on $\delta$.

Curves disconnecting $\mathbb{C}^+$ may be very complex. In order to deduce simple conditions for injectivity we consider a simple class of separating curves $\delta$, i.e. the graphs of the functions $v = au^b$, $a, b > 0$, $u \geq 0$, or $v = a(-u)^b$, $a, b > 0 \geq u$.

Corollary 1. Let $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ with $D(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2$. Assume there exists a compact set $K \subset \mathbb{R}^2$ and $a, b \in \mathbb{R}$, $a, b > 0$, such that for all $(x, y) \not\in K$ one of the following condition holds:

i) if $T \geq 0$ then $\sqrt{|\Delta|} \neq aT^b$;

ii) if $T \leq 0$ then $\sqrt{|\Delta|} \neq a(-T)^b$;

then $F(x, y)$ is injective.

Proof. We prove only i), the statement ii) can be proved similarly.

Since $a, b > 0$, the curve $\delta$ of equation $v = au^b$, $u \geq 0$, starts at the origin and separates $\mathbb{C}^+$. Such a curve has no points on the $x$ axis, except the
origin, hence if an eigenvalue $\lambda$ belongs to $\delta$ its imaginary part is not zero. This implies that $\Delta = T^2 - 4D < 0$, hence $|\Delta| = 4D - T^2$. An eigenvalue $\lambda$ belongs to $\delta$ if and only if $\text{Re}(\lambda) > 0$ and

$$\text{Im}(\lambda) = a \text{Re}(\lambda)^b \iff \sqrt{|\Delta|} = \sqrt{4D - T^2} = aT^b. \quad (1)$$

By hypothesis this does not occur, hence the thesis.

We emphasize that corollary 1 contains two independent statements. Statement i) is not concerned with points where $T(x, y) < 0$; statement ii) is not concerned with points where $T(x, y) > 0$.

In next corollary we prove the injectivity under a suitable assumption on the ratio $\frac{T^2}{D}$.

**Corollary 2.** Let $F \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ with $D(x, y) > 0$ for all $(x, y) \in \mathbb{R}^2$. Assume there exists a compact set $K \subset \mathbb{R}^2$ and $c \in [0, 4]$ such that for all $(x, y) \notin K$ one has:

$$\frac{T^2}{D} \neq c,$$

then $F(x, y)$ is injective.

**Proof.** If $c = 0$, then $T(x, y)$ does not vanish, has constant sign and one can apply the theorems about the Global Asymptotic Stability Jacobian Conjecture [8, 9, 10].

If $c \in (0, 4)$, we take as $\delta$ the line of equation $\sqrt{4 - c} u - \sqrt{c} v = 0$. Assume by absurd that an eigenvalue $\lambda$ belongs to $\delta$. Then one has $\Delta < 0$ and

$$0 = \sqrt{4 - c} T - \sqrt{c} \sqrt{4D - T^2},$$

$$(4 - c)T^2 = 4cD - cT^2,$$

which implies $T^2 = cD$, contradiction.

If $c = 4$, then either $\Delta < 0$ on all of $K^c$, or $\Delta > 0$ on all of $K^c$. In the former case the eigenvalues are not real, hence theorem 1 applies. In the latter they are real and one can take the imaginary axis as $\delta$. 

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The situation is much simpler when dealing with real Jacobian maps. We can always reduce to the case \( D(x, y) \equiv 1 \), by possibly multiplying one component by a suitable non-zero constant. If \( D(x, y) \equiv 1 \), the eigenvalues are contained in the set

\[
\mathbb{G} = (-\infty, 0) \cup S^1 \cup (0, +\infty),
\]

where \( S^1 \) is the unit circle in \( \mathbb{C} \). Such eigenvalues appear in couples \( \lambda, \frac{1}{\lambda} \), if real, or \( u \pm iv \), if non-real. The set \( \Sigma_F(x, y) \) is symmetric w. r. t. the real axis, i.e. it coincides with its conjugate \( \Sigma_F(x, y) \). Disconnecting \( \mathbb{G} \) requires at most a couple of points. This is used in next statements in order to prove injectivity. For the reader’s convenience we report the main theorem proved in [15], that will be applied in the following.

**Theorem 3.** (Rabanal) Let \( F \in C^1(\mathbb{R}^2, \mathbb{R}^2) \). If there exists \( \varepsilon > 0 \) such that \( J_F(x, y) \) has no eigenvalues in \([0, \varepsilon)\), then \( F \) is injective.

**Theorem 4.** Let \( F \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) with \( D(x, y) \equiv 1 \) on all of \( \mathbb{R}^2 \). If one of the following conditions holds, then \( F \) is injective:

i) there exists a compact set \( K \subset \mathbb{R}^2 \) and \( z \in S^1 \) such that for all \( (x, y) \notin K \) one has \( z \notin \Sigma_F(x, y) \).

ii) there exists \( z \in \mathbb{R} \setminus \{-1, 0, 1\} \) such that for all \( (x, y) \in \mathbb{R}^2 \) one has \( z \notin \Sigma_F(x, y) \).

**Proof.**

i) Let \( \Omega_K \) be a closed disk large enough to have \( K \subset \Omega_K \). For all \( (x, y) \notin \Omega_K \) one has \( z \notin \Sigma_F(x, y) \). The continuous maps \( \lambda_i, i = 1, 2 \), map the connected set \( \Omega^c_K \) into connected subsets \( \lambda_i(\Omega^c_K), i = 1, 2 \), of \( \mathbb{G} \). We consider three cases.

i.1) If \( z = u + iv \in S^1, z \neq \pm 1 \), is not an eigenvalue, then also \( \overline{z} = u - iv \in S^1 \) is not an eigenvalue. The couple \( u \pm iv \) disconnects \( \mathbb{G} \). One has

\[
\mathbb{G} \setminus \{u - iv, u + iv\} = \mathbb{G}_- \cup \mathbb{G}_+,
\]
where $G_-$ and $G_+$ are connected and $(-\infty, 0) \subset G_-$, $(0, +\infty) \subset G_+$. If $\lambda_1(\Omega_K^c) \subset G_-$, then also $\lambda_2(\Omega_K^c) \subset G_-$, hence $J_F(x, y)$ has no eigenvalues in $(0, +\infty)$ for $(x, y) \notin \Omega_K$, thus proving the injectivity of $F$. Similarly, if $\lambda_1(\Omega_K^c) \subset G_+$, then also $\lambda_2(\Omega_K^c) \subset G_+$ and $J_F(x, y)$ has no eigenvalues in $(-\infty, 0)$ for $(x, y) \notin \Omega_K$, thus proving the injectivity of $-F$, hence that one of $F$.

i.2) $z = \frac{1}{z} = 1$. Then the number 1 disconnects $G$ and one can write

$$G \setminus \{1\} = G_+ \cup (0, 1) \cup (1, +\infty),$$

where we have set $G_+ = \left\{(x, y) \subset (0, +\infty) \cup \left((0, \frac{1}{z}) \cup S^1\right) \setminus \{1\}. \right\}$ If for some $(x, y) \in \Omega_K^c$ the matrix $J_F(x, y)$ has a positive eigenvalue, then both eigenvalues are positive and by the connectedness of $\lambda_i(\Omega_K^c), i = 1, 2$, one has

$$\lambda_1(\Omega_K^c) \cup \lambda_2(\Omega_K^c) \subset (0, 1) \cup (1, +\infty).$$

As a consequence $(-\infty, 0)$ contains no eigenvalues, so that $-F$ is injective.

On the other hand, if for some $(x, y) \in \Omega_K^c$ the matrix $J_F(x, y)$ has an eigenvalue in $G_-$, than both eigenvalues are in $G_-$ and by connectedness

$$\lambda_1(\Omega_K^c) \cup \lambda_2(\Omega_K^c) \subset G_.$$

Hence there are no eigenvalues in $(0, +\infty)$ and theorem 1 gives the injectivity of $F$.

i.3) $z = \frac{1}{z} = -1$. Similar to i.2).

ii) If $z \in \mathbb{R} \setminus \{-1, 0, 1\}$ is not an eigenvalue, then the numbers $z$ and $\frac{1}{z}$ have the same sign and disconnect the set $G$ into three connected sets. For instance, if $z \in (0, 1)$ we can write

$$G = (0, z) \cup \left(\frac{1}{z}, +\infty\right) \cup \left((z, \frac{1}{z}) \cup S^1\right).$$

Similarly, exchanging $z$ and $\frac{1}{z}$, if $z \in (1, +\infty)$. As in case i.2) at least one component is free of eigenvalues for $(x, y) \notin \Omega_K$. If $(0, z)$ does not contain eigenvalues, then one can apply theorem 3 with $\varepsilon = z$ in order to get
injectivity. Similarly if \( \left( \frac{1}{z}, +\infty \right) \) does not contain eigenvalues; such a case is equivalent to \((0, z)\) not containing eigenvalues. If \( \left( z, \frac{1}{z} \right) \cup S^1 \) does not contain eigenvalues, then \((-\infty, 0)\) is free of eigenvalues and applying either theorem 1 or theorem 3 to \(-F\) one proves the injectivity of \(F\).

\[ \Box \]

We can deduce a simple corollary from theorem 4.

**Corollary 3.** Let \( F \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) with \( D(x, y) \equiv 1 \) on all of \( \mathbb{R}^2 \). If there exists \( z \in (-\infty) \cup S^1 \cup (0, +\infty) \) which is not an eigenvalue of \( J_F(x, y) \), for any \((x, y) \in \mathbb{R}^2\), then \( F(x, y) \) is injective.

**Proof.** Under the above hypothesis either i) or ii) of theorem 3 hold on all of \( \mathbb{R}^2 \).

\[ \Box \]

The condition on the eigenvalues can be deduced from suitable conditions on \( T(x, y) \).

**Corollary 4.** Let \( F \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) with \( D(x, y) \equiv 1 \) on all of \( \mathbb{R}^2 \). If one of the following conditions holds, then \( F \) is injective:

i) there exists a compact set \( K \subset \mathbb{R}^2 \) and \( h \in [-2, 2] \) such that for all \((x, y) \notin K\) one has \( T(x, y) \neq h \).

ii) there exists \( h \in (-\infty, -2) \cup (2, +\infty) \) such that for all \((x, y) \in \mathbb{R}^2\), \( T(x, y) \neq h \).

**Proof.** i) \( \lambda \neq 0 \) is an eigenvalue if and only if for some \((x, y)\) one has

\[ \lambda^2 - T(x, y)\lambda + 1 = 0, \]

hence

\[ \lambda = \frac{T(x, y) \pm \sqrt{T(x, y)^2 - 4}}{2} \]

If there exists \( h \in \mathbb{R} \) such that \( T(x, y) \neq h \), then \( z_{1,2} = \frac{h \pm \sqrt{h^2 - 4}}{2} \) are not eigenvalues of \( J_F \). In case i) one has \( z_{1,2} \in S^1 \) and point i) of theorem 3 applies.

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In case ii) one has \( z_{1,2} \in \mathbb{G} \setminus S^1 \) and point ii) of theorem \( \square \) applies.

As a consequence we have the following corollary.

**Corollary 5.** Let \( F \in C^1(\mathbb{R}^2, \mathbb{R}^2) \) with \( D(x, y) \equiv 1 \) on all of \( \mathbb{R}^2 \). If \( T(x, y) \) is not surjective, then \( F(x, y) \) is injective.

**Proof.** If \( T(x, y) \) is not surjective, then either i) or ii) of corollary \( \square \) hold on all of \( \mathbb{R}^2 \).

The hypotheses of corollary \( \square \) do not apply to even-degree polynomial maps. In fact, if \( F(x, y) \) is an even-degree polynomial map, then \( T(x, y) \) is an odd-degree polynomial, hence it is surjective. On the other hand, odd-degree Jacobian maps with non-surjective \( T(x, y) \) do exist. An example of polynomial Jacobian map with non-surjective \( T(x, y) \) is given by \( F(x, y) = (x + y^3, y - x^3 - 3x^2y^3 - 3xy^6 - y^9) \). In this case one has \( T(x, y) = 2 - 9y^2(x + y^3)^2 \) which does not assume values greater than 2.

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