Existence of nonnegative solutions of nonlinear fractional parabolic inequalities

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Abstract. We study the existence of nontrivial nonlocal nonnegative solutions \( u(x, t) \) of the nonlinear initial value problems

\[
\begin{align*}
(\partial_t - \Delta)^\alpha u & \geq u^\lambda & \text{in } \mathbb{R}^n \times \mathbb{R}, \ n \geq 1 \\
u & = 0 & \text{in } \mathbb{R}^n \times (-\infty, 0)
\end{align*}
\]

and

\[
\begin{align*}
C_1 u^\lambda & \leq (\partial_t - \Delta)^\alpha u \leq C_2 u^\lambda & \text{in } \mathbb{R}^n \times \mathbb{R}, \ n \geq 1 \\
u & = 0 & \text{in } \mathbb{R}^n \times (-\infty, 0),
\end{align*}
\]

where \( \lambda, \alpha, C_1, \) and \( C_2 \) are positive constants with \( C_1 < C_2 \). We use the definition of the fractional heat operator \((\partial_t - \Delta)^\alpha\) given in Taliaferro [J Math Pures Appl 133:287–328, 2020] and compare our results in the classical case \( \alpha = 1 \) to known results.

1. Introduction

In this paper, we study the existence of nontrivial nonlocal nonnegative solutions \( u(x, t) \) of the nonlinear initial value problems

\[
\begin{align*}
(\partial_t - \Delta)^\alpha u & \geq u^\lambda & \text{in } \mathbb{R}^n \times \mathbb{R}, \ n \geq 1 \\
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\end{align*}
\]

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\begin{align*}
C_1 u^\lambda & \leq (\partial_t - \Delta)^\alpha u \leq C_2 u^\lambda & \text{in } \mathbb{R}^n \times \mathbb{R}, \ n \geq 1 \\
u & = 0 & \text{in } \mathbb{R}^n \times (-\infty, 0),
\end{align*}
\]

where \( \lambda, \alpha, C_1, \) and \( C_2 \) are positive constants with \( C_1 < C_2 \).

For a discussion of where the nonlocal fractional heat operator \((\partial_t - \Delta)^\alpha\), \( \alpha > 0 \), arises naturally in applications, please see [7,10].

For each of the problems (1.1), (1.2) and (1.3), (1.4), we compare our results in the classical case \( \alpha = 1 \) to known results. Specifically, our result Theorem 2.1 in Sect. 2.1 for the problem (1.1), (1.2) implies

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(i) when \( \lambda > 1 \), the existence of a critical exponent \( \lambda_0(n, \alpha) > 1 \) for the nonexistence of nontrivial nonnegative solutions of the problem (1.1), (1.2), which agrees with the well-known Fujita exponent \( \lambda_F = 1 + 2/n \) when \( \alpha = 1 \) (see Remark 2.1) and

(ii) when \( 0 < \lambda < 1 \), a nonexistence result for nontrivial nonnegative solutions of the problem (1.1), (1.2) which when \( \alpha = 1 \) is similar to a result in [1] (see Sect. 2.1).

Similarly, our result Theorem 2.2 in Sect. 2.2 for the problem (1.3), (1.4) implies when \( \lambda > 1 \) the existence of a critical exponent \( \lambda_1(n, \alpha) > 1 \) for the existence of nontrivial nonnegative solutions of the problem (1.3), (1.4) which agrees when \( \alpha = 1 \) with a critical exponent in [5] for the existence of self-similar solutions of the problem

\[
\begin{cases}
(\partial_t - \Delta)^\alpha u = u^\lambda & \text{in } \mathbb{R}^n \times \mathbb{R}, \ n \geq 1 \\
u = 0 & \text{in } \mathbb{R}^n \times (-\infty, 0)
\end{cases}
\] (1.5)

when \( \alpha = 1 \) (see Sect. 2.2).

In order to complement our results for the two problems (1.1), (1.2) and (1.3), (1.4), we recall in Sect. 2.3 our result in [11] dealing with the existence of nontrivial nonlocal nonnegative solutions of the initial value problem

\[
\begin{cases}
0 \leq (\partial_t - \Delta)^\alpha u \leq u^\lambda & \text{in } \mathbb{R}^n \times \mathbb{R}, \ n \geq 1 \\
u = 0 & \text{in } \mathbb{R}^n \times (-\infty, 0).
\end{cases}
\] (1.6)

We refer to the four problems (1.1, 1.2), (1.3, 1.4), (1.5, 1.6), and (1.7, 1.8) as the superproblem, approximate problem, exact problem, and subproblem, respectively. We state in Sect. 2.4 in what sense nonlocal nonnegative solutions of these problems converge as \( \alpha \to 1 \) to local distributional solutions of the corresponding problems with \( \alpha = 1 \).

As in [11], we define the fully fractional nonlocal heat operator

\[
(\partial_t - \Delta)^\alpha : Y^p_\alpha \to X^p
\] (1.9)

for

\[
\left( p > 1 \text{ and } 0 < \alpha < \frac{n + 2}{2p} \right) \text{ or } \left( p = 1 \text{ and } 0 < \alpha \leq \frac{n + 2}{2p} \right)
\] (1.10)

as the inverse of the operator

\[
J_\alpha : X^p \to Y^p_\alpha
\] (1.11)

where

\[
X^p := \bigcap_{T \in \mathbb{R}} L^p(\mathbb{R}^n \times \mathbb{R}_T), \quad \mathbb{R}_T := (-\infty, T),
\] (1.12)

\[
J_\alpha f(x, t) := \int \int_{\mathbb{R}^n \times \mathbb{R}_T} \Phi_\alpha(x - \xi, t - \tau) f(\xi, \tau) \ d\xi \ d\tau
\] (1.13)
and
\[ Y^p_\alpha := J_\alpha (X^p). \] (1.14)

By (1.12), we mean \( X^p \) is the set of all measurable functions \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) such that
\[ \|f\|_{L^p (\mathbb{R}^n \times \mathbb{R}_T)} < \infty \quad \text{for all } T \in \mathbb{R}. \]

In the definition (1.13) of \( J_\alpha \),
\[ \Phi_\alpha (x, t) := \frac{t^{\alpha - 1}}{\Gamma (\alpha)} \frac{1}{(4 \pi t)^{n/2}} e^{-|x|^2/(4t)} \chi_{(0, \infty)} (t) \] (1.15)
is the fractional heat kernel.

When \( p \) and \( \alpha \) satisfy (1.10), it was shown in [11] that the operator (1.11) has the following properties:

(P1) it makes sense because \( J_\alpha f \in L^p_{\text{loc}} (\mathbb{R}^n \times \mathbb{R}) \) for \( f \in X^p \),
(P2) it is one to one and onto, and
(P3) if \( f \in X^p \) and \( u = J_\alpha f \), then \( f = 0 \) in \( \mathbb{R}^n \times (-\infty, 0) \) if and only if \( u = 0 \) in \( \mathbb{R}^n \times (-\infty, 0) \).

By properties (P1) and (P2), we can indeed define (1.9) as the inverse of (1.11) when \( p \) and \( \alpha \) satisfy (1.10). Property (P3) will be needed to handle the initial condition \( u = 0 \) in the above initial value problems.

Motivation for the above definition of (1.9) along with some more of its properties can be found in [11].

Stinga and Torrea [10] (see also Nyström and Sande [7]) gave an alternate definition of the fractional nonlocal heat operator
\[ (\partial_t - \Delta)^\alpha : U \to V, \]
which agrees with our definition (1.9) on the intersection \( U \cap Y^p_\alpha \) of their domains. Functions \( u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) in \( U \) are required to be bounded and sufficiently smooth. Their definition, unlike ours, is well suited for studying the Dirichlet problem for
\[ (\partial_t - \Delta)^\alpha u = f \quad \text{in } \Omega \times (0, T) \] (1.16)
where \( \Omega \subset \mathbb{R}^n \) is a bounded domain. However, our definition seems more suited for studying (1.16) when \( \Omega = \mathbb{R}^n \) and \( T = \infty \) because functions in \( Y^p_\alpha \) can be discontinuous and locally unbounded, which allows for a greater variety of solutions of (1.16).

The operator (1.9) is a fully fractional heat operator as opposed to time fractional heat operators in which the fractional derivatives are only with respect to \( t \), and space fractional heat operators, in which the fractional derivatives are only with respect to \( x \). Except for [2] and [11], we know of no results for nonlinear PDEs containing the
fully fractional heat operator \((\partial_t - \Delta)^{\alpha}\). However, results for linear PDEs containing this operator, including in particular

\[(\partial_t - \Delta)^{\alpha}u = f,\]

where \(f\) is a given function, can be found in \([7,9,10]\).

2. Statement and relevance of results

In this section, we state our results and relate them to results in \([1,4,5,11]\). In order to do this, we first note that for each fixed \(p \geq 1\), the open first quadrant of the \(\lambda\alpha\)-plane is the union of the following pairwise disjoint sets which are graphed in Fig. 1:

\[
A := \{ (\lambda, \alpha) : 0 < \lambda < 1 \text{ and } \alpha > 0 \},
\]

\[
B := \{ (\lambda, \alpha) : \lambda = 1 \text{ and } \alpha > 0 \},
\]

\[
C := \{ (\lambda, \alpha) : \lambda > 1 \text{ and } \alpha \geq \frac{n+2}{2p} \left(1 - \frac{1}{\lambda}\right) \},
\]

\[
D := \{ (\lambda, \alpha) : \lambda > 1 \text{ and } \frac{n+2}{2p} \left(1 - \frac{1}{\lambda}\right) \leq \alpha < \frac{n+2}{2} \left(1 - \frac{1}{\lambda}\right) \},
\]

\[
E := \{ (\lambda, \alpha) : \lambda > 1 \text{ and } 0 < \alpha < \frac{n+2}{2p} \left(1 - \frac{1}{\lambda}\right) \}.
\]

Note that if \(p = 1\), then \(D = \emptyset\).

2.1. The superproblem

Our result for the superproblem (1.1), (1.2) is the following.
**Theorem 2.1.** Suppose $\alpha$ and $p$ satisfy (1.10) and $\lambda > 0$. Then, the superproblem (1.1), (1.2) has a nontrivial nonnegative solution $u \in Y_\alpha^p$ if and only if

$$(\lambda, \alpha) \in B \cup D \cup E.$$ 

An immediate consequence of Theorem 2.1 is the following corollary.

**Corollary 2.1.** Suppose $\alpha$ and $p$ satisfy (1.10) and $0 < \lambda < 1$. Then, the only nonnegative solution $u \in Y_\alpha^p$ of the superproblem (1.1), (1.2) is the trivial solution $u \equiv 0$.

A result similar to Corollary 2.1 when $\alpha = 1$ was proved in [1] for mild nonnegative supersolutions of the initial value problem

$$\begin{align*}
(\partial_t - \Delta)u &= u^\lambda \quad \text{in } \mathbb{R}^n \times (0, \infty) \\
u(x, 0) &= u_0(x) \quad \text{for } x \in \mathbb{R}^n,
\end{align*}$$

where $0 < \lambda < 1$.

Since (1.10) holds when $p = 1$ and $0 < \alpha \leq (n + 2)/2$ and since $(\lambda, \alpha) \in D \cup E$ if and only if

$$0 < \alpha < \frac{n + 2}{2} \quad \text{and} \quad \lambda > \lambda_0(n, \alpha) := 1 + \frac{2\alpha}{n + 2 - 2\alpha}$$

(see Fig. 1), we obtain also from Theorem 2.1 the following result.

**Corollary 2.2.** Suppose $0 < \alpha \leq (n + 2)/2$ and $\lambda > 1$. Then, the superproblem (1.1), (1.2) has a nontrivial nonnegative solution $u \in Y_\alpha^1$ if and only if

$$\alpha \neq (n + 2)/2 \quad \text{and} \quad \lambda > \lambda_0(n, \alpha).$$

For comparison with Corollary 2.2, we recall the famous Fujita result [4], the following improved version of which appears in [8, Theorem 18.1].

**Theorem A.** If $1 < \lambda \leq 1 + 2/n$, then the only nonnegative solution $u \in L^\lambda_{\text{loc}}(\mathbb{R}^n \times (0, \infty))$ of the inequality

$$\begin{align*}
(\partial_t - \Delta)u &\geq u^\lambda \quad \text{in } \mathcal{D}'(\mathbb{R}^n \times (0, \infty))
\end{align*}$$

is the trivial solution $u \equiv 0$.

**Remark 2.1.** Since $\lambda_0(n, 1) = 1 + 2/n$, we see that Corollary 2.2 can be viewed as a fractional nonlocal version of Theorem A and $\lambda_0(n, \alpha)$ for $0 < \alpha < (n + 2)/2$ can be viewed as the critical exponent of Fujita type for nonnegative solutions $u \in Y_\alpha^1$ of the superproblem (1.1), (1.2).

For the proof of Theorem 2.1, we will need the following lemma of independent interest which gives in particular conditions for the nonexistence of nontrivial nonnegative solutions of

$$\begin{align*}
(\partial_t - \Delta)^m u &\geq u^\lambda \quad \text{in } \mathcal{D}'(\mathbb{R}^n \times (0, \infty))
\end{align*}$$

where $m$ is a positive integer.
**Lemma 2.1.** Suppose $m$ is a positive integer,

$$K > 0, \quad (\lambda, \alpha) \in C, \quad \alpha \leq m,$$

and $u \in L^\lambda_{\text{loc}}(\mathbb{R}^n \times (0, \infty))$ is a nonnegative solution of

$$(\partial_t - \Delta)^m u \geq (K(t + 1)^{-(m-\alpha)} u)^\lambda \text{ in } \mathcal{D}'(\mathbb{R}^n \times (0, \infty))$$

such that

$$(\partial_t - \Delta)^j u \in L^1_{\text{loc}}(\mathbb{R}^n \times (0, \infty)), \quad j = 1, 2, \ldots, m-1$$

and

$$(\partial_t - \Delta)^j u \geq 0, \quad j = 1, 2, \ldots, m-1.$$ 

Then,

$$u = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n \times (0, \infty)).$$

**Remark 2.2.** Since $(\lambda, 1) \in C$ if and only if $1 < \lambda \leq 1 + 2/n$, we see that a consequence of Lemma 2.1 with $m = \alpha = K = 1$ is Theorem A.

### 2.2. The approximate and exact problems

If $u \in Y_p^\alpha$, where $\alpha$ and $p$ satisfy (1.10), is a solution of the exact problem (1.5), (1.6), then for all $\beta > 0$ so is

$$u_\beta(x, t) := \beta^{-\frac{2n}{p-1}} u(x/\beta, t/\beta^2).$$

If, in addition, $u$ is self-similar, that is $u_\beta = u$ for all $\beta > 0$, then substituting $\beta = \sqrt{t}$, $t > 0$, in (2.6), we see that

$$u(x, t) = \begin{cases} t^{-\frac{2n}{p-1}} u(x/\sqrt{t}, 1) & \text{for } (x, t) \in \mathbb{R}^n \times (0, \infty), \\ 0 & \text{for } (x, t) \in \mathbb{R}^n \times (-\infty, 0]. \end{cases}$$

Moreover, any function $u$ satisfying (2.7) is self-similar. Inspired by [5], we will seek in this section solutions of the approximate problem (1.3), (1.4) of the form (2.7).

We have no results for solutions $u \in Y_p^\alpha$ of the approximate problem (1.3), (1.4) when $p > 1$ and $(\lambda, \alpha)$ lies in the curve

$$\alpha = \frac{n + 2}{2p} \left(1 - \frac{1}{\lambda}\right), \quad 1 < \lambda < \infty,$$ 

which is graphed in Fig. 1. Otherwise, we have the following result.

**Theorem 2.2.** Suppose $\lambda > 0$ and either

(i) $\alpha$ and $p$ satisfy (1.10)$_2$, or

(ii) $\alpha$ and $p$ satisfy (1.10)$_1$ and the point $(\lambda, \alpha)$ does not lie on the curve (2.8).
Then, there exist positive constants $C_1$ and $C_2$ such that the approximate problem (1.3), (1.4) has a nontrivial nonnegative solution $u \in Y^p_\alpha$ if and only if $$(\lambda, \alpha) \in E.$$ In this case, such a solution is given by

$$u(x, t) = \begin{cases} t^{-\frac{\alpha}{n-1}} w_\alpha(x/\sqrt{t}) & \text{for } (x, t) \in \mathbb{R}^n \times (0, \infty) \\ 0 & \text{for } (x, t) \in \mathbb{R}^n \times (-\infty, 0] \end{cases}$$ (2.9)

where

$$w_\alpha(z) = e^{-\frac{|z|^2}{4}} (|z|^2 + 1)^{-\left(\frac{n+2}{2} - \frac{\alpha}{n-1}\right)}$$ for $z \in \mathbb{R}^n$. (2.10)

An immediate consequence of Theorem 2.2 is the following corollary.

**Corollary 2.3.** Suppose

$$p \geq 1, \quad 0 < \alpha < \frac{n+2}{2p}, \quad \text{and} \quad \lambda > \frac{n+2}{n+2-2p\alpha}. \quad \text{(2.11)}$$

Then, there exist positive constants $C_1$ and $C_2$ such that a nontrivial nonnegative solution $u \in Y^p_\alpha$ of the approximate problem (1.3), (1.4) is given by (2.9) where $w_\alpha$ is defined in (2.10).

Since the conditions (2.11) hold if

$$\alpha = 1, \quad 1 \leq p < \frac{n+2}{2}, \quad \text{and} \quad \lambda > \frac{n+2}{n+2-2p},$$

we see that Corollary 2.3 can be viewed as an $\alpha \neq 1$ version of the following $\alpha = 1$ result in [5] for the exact problem (1.5), (1.6).

**Theorem B.** [5] Suppose

$$1 \leq p < \frac{n+2}{2} \quad \text{and} \quad \frac{n+2}{n+2-2p} < \lambda < \begin{cases} \frac{n+2}{n-2} & \text{if } n \geq 3 \\ \infty & \text{if } n = 1 \text{ or } 2. \end{cases}$$

Then, a nontrivial nonnegative solution $u \in Y^p_1$ of the exact problem (1.5), (1.6) with $\alpha = 1$ is given by

$$u(x, t) = \begin{cases} t^{-\frac{1}{n-1}} w(x/\sqrt{t}) & \text{for } (x, t) \in \mathbb{R}^n \times (0, \infty) \\ 0 & \text{for } (x, t) \in \mathbb{R}^n \times (-\infty, 0] \end{cases}$$ (2.12)

for some positive radial function $w : \mathbb{R}^n \to \mathbb{R}$ such that

$$\frac{w(z)}{w_1(z)}$$ is bounded between positive constants on $\mathbb{R}^n$

where $w_\alpha$ is defined in (2.10).
We have no results for the exact problem, but Corollary 2.3 and Theorem B motivate the following open question.

**Open question.** Suppose $\alpha$ and $p$ satisfy (1.10) so that the operator (1.9) is defined. For what $\lambda > 0$ does there exist a nontrivial nonnegative solution $u \in Y^p_\alpha$ of the exact problem (1.5), (1.6)?

Since any solution of the exact problem (1.5), (1.6) is also a solution of the approximate problem (1.3), (1.4), it follows from Theorem 2.2 that a necessary condition on $\lambda$ is that either $p > 1$ and $(\lambda, \alpha)$ lies on the curve (2.8) or $(\lambda, \alpha) \in E$ (see Fig. 1).

### 2.3. The subproblem

We have no results for solutions $u \in Y^p_\alpha$ of the subproblem (1.3), (1.4) when $p \geq 1$ and the point $(\lambda, \alpha)$ lies on the curve (2.8). Otherwise, we have the following result from [11].

**Theorem 2.3.** Suppose $\alpha$ and $p$ satisfy (1.10), $\lambda > 0$, and the point $(\lambda, \alpha)$ does not lie on the curve (2.8). Then, the subproblem (1.3), (1.4) has a nontrivial nonnegative solution $u \in Y^p_\alpha$ if and only if

$$(\lambda, \alpha) \in A \cup E.$$ 

### 2.4. Convergence of nonlocal solutions to local solutions

In this section, we state in what sense nonlocal nonnegative solutions of the super-, sub-, approximate, and exact problems in Sect. 1 converge as $\alpha \to 1$ to local distributional solutions of the corresponding problems with $\alpha = 1$.

To do this, let $\lambda \in (0, \infty)$ be fixed. Choose

$$p \in \left[ 1, \frac{n + 2}{2} \right]$$

such that

$$0 \leq \frac{1}{p} - \frac{1}{q} < \frac{2}{n + 2}$$

where

$$q = \max\{1, \lambda\}.$$ (2.15)

It is straightforward to check that there exists such a $p$ and by (2.13), the condition (1.10) on $\alpha$ for the operator (1.9) to be defined holds for all $\alpha$ in some neighborhood of 1.

With regard to the super- and subproblems, we have the following result which clearly implies corresponding results for the approximate and exact problems.

**Theorem 2.4.** Let $\lambda$, $p$, and $q$ be as stated above. Suppose $\alpha_j \to 1$ as $j \to \infty$ and $u_j \in Y^p_{\alpha_j}$ is a nonnegative solution of the nonlocal super (sub)-problem

$$\begin{align*}
(\partial_t - \Delta)^{\alpha_j} u_j &\geq (\leq) u_j^\lambda & &\text{in } \mathbb{R}^n \times \mathbb{R} \\
u_j &= 0 & &\text{in } \mathbb{R}^n \times (-\infty, 0).
\end{align*}$$

(2.16)
If \((\partial_t - \Delta)^{\alpha_j} u_j\) is a convergent\(^1\) nonnegative sequence in \(X^p\), then
\[
u_j \in L^q_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}) \quad \text{for } j = 1, 2, \ldots
\]
and some subsequence of \(u_j\) converges in \(L^q_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})\) to a solution \(u \in L^q_{\text{loc}}(\mathbb{R}^n \times \mathbb{R})\) of the local problem
\[
(\partial_t - \Delta)u \geq (\leq) u^\lambda \quad \text{in } D'(\mathbb{R}^n \times \mathbb{R}) \\
u = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0). \tag{2.18}
\]

3. \(J_\alpha\) version of results

We define the \(J_\alpha\) versions of the super-, approximate, exact, and subproblems in Sect. 1 to be, respectively, the problems
\[
\begin{cases}
f \geq (J_\alpha f)^\lambda & \text{in } \mathbb{R}^n \times \mathbb{R} \\
f = 0 & \text{in } \mathbb{R}^n \times (-\infty, 0); \tag{3.1}
\end{cases}
\]
\[
\begin{cases}
C_1 f \leq (J_\alpha f)^\lambda \leq C_2 f & \text{in } \mathbb{R}^n \times \mathbb{R} \\
f = 0 & \text{in } \mathbb{R}^n \times (-\infty, 0); \tag{3.3}
\end{cases}
\]
\[
\begin{cases}
f = (J_\alpha f)^\lambda & \text{in } \mathbb{R}^n \times \mathbb{R} \\
f = 0 & \text{in } \mathbb{R}^n \times (-\infty, 0); \tag{3.5}
\end{cases}
\]
and
\[
\begin{cases}
0 \leq f \leq (J_\alpha f)^\lambda & \text{in } \mathbb{R}^n \times \mathbb{R} \\
f = 0 & \text{in } \mathbb{R}^n \times (-\infty, 0); \tag{3.7}
\end{cases}
\]
which we will refer to, respectively, as the super-\(J_\alpha\) problem, approximate \(J_\alpha\) problem, exact \(J_\alpha\) problem, and sub-\(J_\alpha\) problem (or collectively as the \(J_\alpha\) problems).

If \(\alpha\) and \(p\) satisfy (1.10) and \(\lambda > 0\), then by properties (P1)–(P3) in Sect. 1 of \(J_\alpha\) and the definition of the fractional heat operator (1.9), \(u\) is a nonnegative solution in \(Y^p_\alpha\) of the super (approximate, exact, sub)-problem in Sect. 1 if and only if
\[
f := (\partial_t - \Delta)^\alpha u
\]
is a nonnegative solution in \(X^p\) of the super (approximate, exact, sub)-\(J_\alpha\) problem in this section. (However, the positive constants \(C_1\) and \(C_2\) in (1.3), (1.4) may be different than the positive constants \(C_1\) and \(C_2\) in (3.3), (3.4).)

\(^1\)A sequence \(\{f_j\} \subset X^p\) converges to \(f\) if and only if, for all \(T > 0\), \(\|f_j - f\|_{L^p(\mathbb{R}^n \times (-\infty, T))} \to 0\) as \(j \to \infty\). See [11, Remark 2.2].
Since we will only consider solutions $f$ of the $J_\alpha$ problems which are nonnegative on $\mathbb{R}^n \times \mathbb{R}$, $J_\alpha f$ in these problems will always be a well-defined nonnegative extended real-valued function on $\mathbb{R}^n \times \mathbb{R}$ even when the condition (1.10) is replaced with the weaker condition that
\[ p \in [1, \infty) \quad \text{and} \quad \alpha > 0. \quad (3.9) \]
Hence, in this section we study the $J_\alpha$ problems with condition (1.10) replaced with (3.9). However, our results in this section for the $J_\alpha$ problems will only yield corresponding results for the original versions of these problems in Sect. 1 when (1.10) holds, for otherwise the fractional heat operators in these original problems are not defined. (For a more detailed discussion of the properties of $J_\alpha$ when (1.10) does not hold, see [11, Section 4].)

Under the equivalence discussed above of the $J_\alpha$ problems and original versions of these problems in Sect. 1, Theorems 3.1–3.3, when restricted to the case that (1.10) holds, clearly imply Theorems 2.1–2.3, respectively.

**Theorem 3.1.** Suppose $p$ and $\alpha$ satisfy (3.9) and $\lambda > 0$. Then, the super-$J_\alpha$ problem (3.1), (3.2) has a nontrivial nonnegative solution $f \in X^p$ if and only if
\[ (\lambda, \alpha) \in B \cup D \cup E. \]

**Theorem 3.2.** Suppose $\lambda, \alpha > 0$ and either

(i) $p = 1$ or

(ii) $p \in (1, \infty)$ and the point $(\lambda, \alpha)$ does not lie of the curve (2.8).

Then, there exist positive constants $C_1$ and $C_2$ such that the approximate $J_\alpha$ problem (3.3), (3.4) has a nontrivial nonnegative solution $f \in X^p$ if and only if
\[ (\lambda, \alpha) \in E. \]
In this case, such a solution is given by $f = u^\lambda$, where $u$ is defined by (2.9).

**Theorem 3.3.** Suppose $\alpha$ and $p$ satisfy (3.9), $\lambda > 0$, and the point $(\lambda, \alpha)$ does not lie on the curve (2.8). Then, the sub-$J_\alpha$ problem (3.7), (3.8) has a nontrivial nonnegative solution $f \in X^p$ if and only if
\[ (\lambda, \alpha) \in A \cup E. \]

Before giving in Sect. 6, the rigorous proofs of Theorems 3.1 and 3.2—we proved Theorem 3.3 in [11]—we will now show heuristically via scaling and/or iteration how the curves
\[ \alpha = \frac{n + 2}{2} \left( 1 - \frac{1}{\lambda} \right) \quad (3.10) \]
and
\[ \alpha = \frac{n + 2}{2p} \left( 1 - \frac{1}{\lambda} \right), \quad (3.11) \]
which are graphed in Fig. 1, arise naturally when studying the existence of nontrivial nonnegative solutions of the above $J_\alpha$ problems.

The curve (3.10). To show how the curve (3.10) arises, suppose $f \in X_p$ is a nontrivial nonnegative solution of the super-$J_\alpha$ problem (3.1), (3.2) where $p$ and $\alpha$ satisfy (3.9) and $\lambda > 1$. Then, since $X_p \subset L_{\text{loc}}^1(\mathbb{R}^n \times \mathbb{R})$, it follows from (3.1) that

$$u := J_\alpha f \in L_{\text{loc}}^1(\mathbb{R}^n \times \mathbb{R})$$

(3.12)

and letting $H_\alpha = J_{\alpha^{-1}} = (\partial_t - \Delta)^{\alpha}$, we obtain from (3.1), (3.2) that $u$ is a nontrivial nonnegative solution of

$$H_{\alpha} u \geq u^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R}$$

(3.13)

$$u = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0).$$

(3.14)

Choose a smooth function $\varphi: \mathbb{R}^n \times \mathbb{R} \to [0, 1]$ such that $\text{supp } \varphi = B_1(0) \times [-1, 1]$ and for $R > 0$ let $\psi_R(x, t) = \varphi\left( \frac{x}{R}, \frac{t}{R^2} \right)^m$ where $m = \frac{2\alpha}{\lambda - 1}$. Also, letting $H_\alpha^*$ be the formal adjoint of $H_\alpha$ and using the case $\alpha$ is a positive integer as motivation we heuristically assume

$$|H_{\alpha}^* \psi_R(x, t)| \leq CR^{-2\alpha} \varphi \left( \frac{x}{R}, \frac{t}{R^2} \right)^{m-2\alpha} = CR^{-2\alpha} \psi_R(x, t)^{1/\lambda}$$

where $C > 0$ is a constant independent of $R$. Then, from (3.13) and (3.14) we see that

$$\int_{\mathbb{R}^n \times \mathbb{R}} u^\lambda \psi_R \, dx \, dt \leq \int_{\mathbb{R}^n \times \mathbb{R}} (H_\alpha u) \psi_R \, dx \, dt = \int_{\mathbb{R}^n \times \mathbb{R}} u H_\alpha^* \psi_R \, dx \, dt$$

$$\leq CR^{-2\alpha} \int_0^R \int_{|x| < R} u \psi_R^{1/\lambda} \, dx \, dt$$

$$\leq CR^{-2\alpha} (R^n + 2)^{-1-1/\lambda} \left( \int_{\mathbb{R}^n \times \mathbb{R}} u^\lambda \psi_R \, dx \, dt \right)^{1/\lambda}$$

by Hölder’s inequality. It therefore follows from (3.12) that

$$\int_{\mathbb{R}^n \times \mathbb{R}} u^\lambda \psi_R \, dx \, dt \leq CR^{n+2-\frac{2\alpha}{1-1/\lambda}}.$$

Hence, since $u$ is nontrivial, we find by sending $R$ to $\infty$ that the point $(\lambda, \alpha)$ lies on or below the curve (3.10).

The curve (3.11). To show how the curve (3.11) arises, we will use the following lemma which follows easily from Young’s inequality. See [11, Lemma 7.2].

Lemma 3.1. Suppose $p, q \in [1, \infty], \alpha, T \in (0, \infty)$, and

$$0 \leq \delta := \frac{1}{p} - \frac{1}{q} < \frac{2\alpha}{n + 2} < 1.$$
Then, for 

\[ f \in X^p \text{ and } f = 0 \text{ in } \mathbb{R}^n \times (-\infty, 0) \]

we have

\[ \| J_\alpha f \|_{L^q(\mathbb{R}^n \times (0, T))} \leq C \| f \|_{L^p(\mathbb{R}^n \times (0, T))} \]

for some constant \( C = C(n, \alpha, T, \delta) > 0 \).

To explain why the curve (3.11) is critical with regard to the question of the existence of nontrivial nonnegative solutions of the sub-\( J_\alpha \) problem, suppose

\[ \lambda \in (1, \infty), \quad \alpha \in (0, \infty), \quad p \in [1, \infty), \]

and the point \((\lambda, \alpha)\) lies above the curve (3.11). Let \( f \in X^p \) be a nonnegative solution of the sub-\( J_\alpha \) problem (3.7), (3.8) and let \( T > 0 \). Then,

\[ f \in L^p(0, T) \quad (3.15) \]

and

\[ f \leq (J_\alpha f)^\lambda \text{ in } \mathbb{R}^n \times (0, T). \quad (3.16) \]

By virtue of the fact that for \( 0 < \alpha_1 < \alpha_2 \) we have

\[ J_{\alpha_2} f \leq C J_{\alpha_1} f \text{ in } \mathbb{R}^n \times (0, T), \]

where \( C = C(T, \alpha_1, \alpha_2) \) is a positive constant, we can assume after scaling away the constant that (3.16) holds with the point \((\lambda, \alpha)\) lying above the curve (3.11) and below the horizontal line \( \alpha = (n + 2)/2 \). That is—see Fig. 1:

\[ \lambda \in (1, \infty) \quad \text{and} \quad \frac{n + 2}{2p} \left(1 - \frac{1}{\lambda}\right) < \alpha < \frac{n + 2}{2}. \quad (3.17) \]

If

\[ f \in L^\infty(\mathbb{R}^n \times (0, T)), \quad (3.18) \]

then we showed in [11, proof of Theorem 4.1] that \( f \equiv 0 \) in \( \mathbb{R}^n \times (0, T) \) and since (3.8) holds and \( T > 0 \) is arbitrary that \( f \equiv 0 \) in \( \mathbb{R}^n \times \mathbb{R} \). So to complete our discussion of how the curve (3.11) arises, we outline the proof by iteration of (3.18).

By (3.17), we can choose \( \varepsilon > 0 \) such that

\[ \frac{n + 2}{2p} \left(1 - \frac{1}{\lambda}\right) < \alpha - \varepsilon < \frac{n + 2}{2}. \quad (3.19) \]

If \( \frac{1}{p} \leq \frac{2(\alpha - \varepsilon)}{n + 2} \), then taking \( q = \infty \) in Lemma 4.1 we obtain (3.18). Suppose \( \frac{1}{p} > \frac{2(\alpha - \varepsilon)}{n + 2} \). Then, there exists a unique \( q \in [1, \infty) \) such that

\[ \frac{1}{p} - \frac{1}{q} = \frac{2(\alpha - \varepsilon)}{n + 2} \quad (3.20) \]
and from Lemma 3.1 we see that

\[ J_\alpha f \in L^q(\mathbb{R}^n \times (0, T)) \]

which by (3.7) implies

\[ f \in L^{q/\lambda}(\mathbb{R}^n \times (0, T)). \]  \hspace{1cm} (3.21)

Also, from (3.20) and (3.19) we find that

\[ \frac{1}{p} - \frac{\lambda}{q} = \frac{2\lambda}{n+2} \left( \alpha - \varepsilon - \frac{n+2}{2p} \left(1 - \frac{1}{\lambda}\right) \right) > 0 \]

which implies \( q/\lambda > p \). Iterating the process of going from (3.15) to (3.21), we find that \( f \in L^r(\mathbb{R}^n \times (0, T)) \) for larger and larger values of \( r \) and after a finite number of these iterations that (3.18) holds.

4. Preliminary results

In this section, we provide some remarks and lemmas needed for the proofs of our results in Sect. 3 dealing with solutions of the super-\( J_\alpha \) problem (3.1), (3.2) and the approximate \( J_\alpha \) problem (3.3), (3.4).

Lemma 4.1. Suppose \( \alpha, \beta \in (0, \infty), \ x \in \mathbb{R}^n, \ and \ 0 < \tau < t. \ Then,

\[ \int_{\mathbb{R}^n} \Phi_\alpha(x - \xi, t - \tau) \Phi_\beta(\xi, \tau) \ d\xi = \frac{(t-\tau)^{\alpha-1} \tau^{\beta-1}}{\Gamma(\alpha) \Gamma(\beta)} \Phi_1(x, t). \]  \hspace{1cm} (4.1)

Proof. Denote the left side of (4.1) by \( h(x, t, \tau) \). Using the convolution theorem and the well-known fact that the Fourier transform with respect to \( x \) of \( \Phi_\alpha(x, t) \) is given by

\[ \hat{\Phi}_\alpha(\cdot, t)(y) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-t|y|^2} \quad \text{for} \ t > 0 \ \text{and} \ y \in \mathbb{R}^n, \]  \hspace{1cm} (4.2)

we find for \( 0 < \tau < t \) that

\[ \hat{h}(\cdot, t, \tau)(y) = \left( \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} e^{-(t-\tau)|y|^2} \right) \left( \frac{\tau^{\beta-1}}{\Gamma(\beta)} e^{-\tau|y|^2} \right) \]

\[ = \frac{(t-\tau)^{\alpha-1} \tau^{\beta-1}}{\Gamma(\alpha) \Gamma(\beta)} e^{-t|y|^2} \]

\[ = \frac{(t-\tau)^{\alpha-1} \tau^{\beta-1}}{\Gamma(\alpha) \Gamma(\beta)} \hat{\Phi}_1(\cdot, t)(y) \]

which proves (4.1). \( \square \)

Lemma 4.2. Suppose \( \lambda, \alpha, T \in (0, \infty), \ p \in [1, \infty], \ and \)

\[ f(x, t) = g(x, t + T) \chi_{[0, \infty)}(t) \quad \text{for} \ (x, t) \in \mathbb{R}^n \times \mathbb{R} \]
where $g : \mathbb{R}^n \times \mathbb{R} \to [0, \infty)$ is a measurable function such that
\[
\|g\|_{L^p(\mathbb{R}^n \times (T, \hat{T}))} < \infty \quad \text{for all } \hat{T} > T \tag{4.3}
\]
and
\[
g \geq (J_\alpha g)^\lambda \quad \text{in } \mathbb{R}^n \times \mathbb{R}. \tag{4.4}
\]

Then, $f \in X^p$ and $f$ is a solution of the super-$J_\alpha$ problem (3.1), (3.2).

Proof. For $t > 0$, we have
\[
\|f\|_{L^p(\mathbb{R}^n \times (-\infty, t))} = \|f\|_{L^p(\mathbb{R}^n \times (0, t))} = \|g\|_{L^p(\mathbb{R}^n \times (T, t+T))} < \infty
\]
by (4.3). Thus, $f \in X^p$.

Clearly, $f$ satisfies (3.2). Since (3.1) clearly holds in $\mathbb{R}^n \times (-\infty, 0]$, it remains only to prove (3.1) holds in $\mathbb{R}^n \times (0, \infty)$.

For $(x, t) \in \mathbb{R}^n \times (0, \infty)$, we find from (4.4) that
\[
(J_\alpha f)(x, t) = \int_0^t \int_{\mathbb{R}^n} \Phi_\alpha(x - \xi, t - \tau)g(\xi, \tau + T) \, d\xi \, d\tau
\]
\[
= \int_0^\hat{t} \int_{\mathbb{R}^n} \Phi_\alpha(x - \xi, \hat{t} - \hat{\tau})g(\xi, \hat{\tau}) \, d\xi \, d\hat{\tau} \quad \text{where } \hat{t} = t + T \text{ and } \hat{\tau} = \tau + T
\]
\[
\leq \int_{-\infty}^\hat{t} \int_{\mathbb{R}^n} \Phi_\alpha(x - \xi, \hat{t} - \hat{\tau})g(\xi, \hat{\tau}) \, d\xi \, d\hat{\tau}
\]
\[
= (J_\alpha g)(x, \hat{t})
\]
\[
\leq g(x, \hat{t})^{1/\lambda}
\]
\[
= g(x, t + T)^{1/\lambda} = f(x, t)^{1/\lambda}.
\]
Thus, (3.1) holds in $\mathbb{R}^n \times (0, \infty)$. \hfill \qed

Remark 4.1. Note for use in Lemma 4.2 that if
\[
0 \leq g(x, t) \leq \psi(t) \Phi_\beta(x, t) \quad \text{for } (x, t) \in \mathbb{R}^n \times \mathbb{R}
\]
where $\beta \in (0, \infty)$ and $\psi : \mathbb{R} \to [0, \infty)$ is a continuous function, then for $0 < T < \hat{T} < \infty$ we have
\[
\|g\|_{L^\infty(\mathbb{R}^n \times (T, \hat{T}))} < \infty
\]
and for $p \in [1, \infty)$
\[
\|g\|_{L^p(\mathbb{R}^n \times (T, \hat{T}))}^p \leq \int_T^{\hat{T}} \psi(t)^p \left( \frac{t^{\beta - 1 - n/2}}{\Gamma(\beta)(4\pi)^{n/2}} \right)^p \left( \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} \, dx \right) \, dt
\]
\[
\leq (\hat{T} - T) \max_{T \leq t \leq \hat{T}} \psi(t)^p \left( \frac{t^{\beta - 1 - n/2}}{\Gamma(\beta)(4\pi)^{n/2}} \right)^p \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4t}} \, dx < \infty.
\]
Thus, $g$ satisfies (4.3).
Our proof of Lemma 2.1 is a modification of [8, Proof of Theorem 18.1(i)] and in particular requires the following lemma. See [8, pages 101-102] for its proof.

**Lemma 4.3.** Suppose \( v, f \in L^1_{\text{loc}}(\mathbb{R}^n \times (0, \infty)) \) are nonnegative functions such that
\[
(\partial_t - \Delta)v \geq f \quad \text{in } D'((\mathbb{R}^n \times (0, \infty)).
\]
Let
\[
\varphi \in C_0^\infty(B_1(0)) \quad \text{and} \quad \psi \in C_0^\infty((-1, 1))
\]
be nonnegative functions such that
\[
\varphi = 1 \text{ in } B_{1/2}(0), \quad \psi = 1 \text{ in } [0, 1/2), \quad \text{and } \varphi, \psi < 1.
\]
For \( R > 1, \beta > 2, \) and \( t_0 > 0, \) define
\[
\varphi_R(x) = \varphi \left( \frac{x}{R} \right)^\beta \quad \text{for } x \in \mathbb{R}^n
\]
and
\[
\psi_R(t) = \psi \left( \frac{t - t_0}{R^2} \right)^\beta \quad \text{for } t \geq t_0.
\]
Then,
\[
\int_{t_0}^\infty \int_{\mathbb{R}^n} f \varphi_R \psi_R \, dx \, dt \leq \frac{C}{R^2} \int \int_{Q_R} v(\varphi_R \psi_R)^{\frac{\beta - 2}{\beta}} \, dx \, dt
\]
where \( C > 0 \) does not depend on \( R \) and
\[
Q_R = (B_R(0) \times (t_0, t_0 + R^2)) \setminus (B_{R/2}(0) \times (t_0, t_0 + R^2/2)).
\]

**Proof of Lemma 2.1.** For \( R > 1, \gamma > 2m, \) and \( t_0 > 0, \) define
\[
\varphi_R(x) = \varphi \left( \frac{x}{R} \right)^\gamma \quad \text{and} \quad \psi_R(t) = \psi \left( \frac{t - t_0}{R^2} \right)^\gamma
\]
for \( x \in \mathbb{R}^n \) and \( t \geq t_0 \) where \( \varphi \) and \( \psi \) are as in Lemma 4.3. Let \( f_0(x, t) \) be the function on the right side of (2.2).

For \( j = 1, \ldots, m, \) we claim that
\[
\int_{t_0}^\infty \int_{\mathbb{R}^n} f_0 \varphi_R \psi_R \, dx \, dt \leq \left( \frac{C}{R^2} \right)^j \int \int_{Q_R} (H^{m-j} u)(\varphi_R \psi_R)^{\frac{\gamma - 2j}{\gamma}} \, dx \, dt
\]
where \( H = \partial_t - \Delta, \) \( C > 0 \) does not depend on \( R, \) and \( Q_R \) is defined in (4.5).

Inequality (4.6) holds for \( j = 1 \) by (2.2)–(2.4) and Lemma 4.3 with \( v = H^{m-1} u \) and \( f = f_0. \) Suppose inductively that (4.6) is true for some integer \( j \in [1, m - 1]. \) Let
\[
\tilde{\varphi}_R(x) := \varphi_R(x)^{\frac{\gamma - 2j}{\gamma}} = \varphi \left( \frac{x}{R} \right)^{\gamma - 2j} \quad \text{for } x \in \mathbb{R}^n
\]
and
\[
\hat{\psi}_R(t) := \psi_R(t) \left(\frac{t - t_0}{R^2}\right)^{\gamma - 2j} \quad \text{for } t \geq t_0.
\]
Then, using the inductive assumption, (2.3), (2.4), and Lemma 4.3 with
\[ f = H^{m-j}u, \quad v = H^{m-j-1}u, \quad \text{and } \beta = \gamma - 2j, \]
we find that
\[
\int_{t_0}^{\infty} \int_{\mathbb{R}^n} f_0 \varphi_R \psi_R \, dx \, dt \leq \left(\frac{C}{R^2}\right)^j \int_{t_0}^{\infty} \int_{\mathbb{R}^n} (H^{m-j}u) \hat{\varphi}_R \hat{\psi}_R \, dx \, dt
\]
\[
\leq \left(\frac{C}{R^2}\right)^j \left[ \left(\frac{C}{R^2}\right) \int_{Q_R} (H^{m-j-1}u)(\hat{\varphi}_R \hat{\psi}_R) \left(\frac{t - t_0}{R^2}\right)^{\gamma - 2j} \, dx \, dt \right]
\]
\[
= \left(\frac{C}{R^2}\right)^{j+1} \int_{Q_R} (H^{m-(j+1)}u)(\varphi_R \psi_R) \left(\frac{t - t_0}{R^2}\right)^{\gamma - 2(j+1)} \, dx \, dt
\]
which completes the inductive proof of (4.6) for \( j = 1, 2, ..., m \).

Taking \( j = m \) in (4.6), defining \( \gamma > 2m \) and \( \lambda' > 1 \) by
\[
\frac{2m}{\gamma} = \frac{1}{\lambda'} = 1 - \frac{1}{\lambda},
\]
and using Hölder’s inequality, we have
\[
\int_{t_0}^{\infty} \int_{\mathbb{R}^n} ((t + 1)^{-m+\alpha})^\lambda \varphi_R \psi_R \, dx \, dt
\]
\[
\leq \frac{C}{R^{2m}} \int_{Q_R} u(\varphi_R \psi_R)^{\frac{1}{\lambda'}}
\]
\[
= \frac{C}{R^{2m}} \int_{Q_R} (t + 1)^{m-\alpha} (t + 1)^{-m+\alpha} u(\varphi_R \psi_R)^{\frac{1}{\lambda'}} \, dx \, dt
\]
\[
\leq \frac{C}{R^{2m}} I(R) \left( \int_{Q_R} ((t + 1)^{-m+\alpha})^\lambda \varphi_R \psi_R \, dx \, dt \right)^{1/\lambda'}
\]
where
\[
I(R) = \left( \int_{t_0}^{t_0+R^2} \int_{|x|<R} (t + 1)^{\lambda'(m-\alpha)} \, dx \, dt \right)^{1/\lambda'}
\]
\[
\leq C(R^{n+2} R^{2\lambda'(m-\alpha)})^{1/\lambda'} \quad \text{by } (2.1)
\]
\[
= C R^{\frac{n+2}{\lambda'} + 2(m-\alpha)}.
\]
Hence,
\[
\int_{t_0}^{\infty} \int_{\mathbb{R}^n} ((t + 1)^{-m+\alpha})^\lambda \varphi_R \psi_R \, dx \, dt
\]
\[
\leq C R^{\frac{n+2}{\lambda'} - 2\alpha} \left( \int_{Q_R} ((t + 1)^{-m+\alpha})^\lambda (\varphi_R \psi_R) \, dx \, dt \right)^{1/\lambda}
\quad (4.7)
which implies
\[ \int_{t_0}^{\infty} \int_{\mathbb{R}^n} ((t + 1)^{-(m-\alpha)}u)^{\lambda} \varphi R_{\lambda} \psi R_{\lambda} \, dx \, dt \leq C R^{(\frac{n+2}{\lambda} - 2\alpha)\lambda}. \]  (4.8)

By (2.1), \(2\alpha \geq (n + 2)/\lambda'.\) Hence, sending \(R \to \infty\) in (4.8) we find that
\[ \int_{t_0}^{\infty} \int_{\mathbb{R}^n} ((t + 1)^{-(m-\alpha)}u)^{\lambda} \, dx \, dt < \infty, \]
which implies the integral on the right side of (4.7) tends to zero as \(R \to \infty\). Thus, sending \(R \to \infty\) in (4.7) yields \(u = 0\) in \(\mathbb{R}^n \times (t_0, \infty)\). Hence, since \(t_0 > 0\) was arbitrary, we see that (2.5) holds. \(\square\)

**Lemma 4.4.** Suppose \(p \in [1, \infty)\) and \(f \in X^p\) is a nonnegative function satisfying (3.2). Then,
\[ J_\alpha f \in L^1_{\text{loc}}(\mathbb{R}^n \times (0, \infty)) \text{ for } \alpha > 0, \]  (4.9)
\[ H J_\alpha f = J_{\alpha-1} f \text{ in } \mathcal{D}'(\mathbb{R}^n \times (0, \infty)) \text{ for } \alpha > 1, \]  (4.10)
and
\[ H J_1 f = f \text{ in } \mathcal{D}'(\mathbb{R}^n \times (0, \infty)), \]  (4.11)
where \(H = \partial_t - \Delta\).

**Proof.** We will need the following easily verified and/or well-known facts:
(i) \[ \int \int_{\mathbb{R}^n \times \mathbb{R}} \Phi_1(x - \xi, t - \tau) H^* \varphi(x, t) \, dx \, dt = \varphi(\xi, \tau) \]  (4.12)
for \(\varphi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})\) where \(H^* = \partial_t + \Delta;\) and
(ii) \[ \int_0^b \int_{B_b(0)} \Phi_\alpha(x - \xi, t - \tau) \, dx \, dt \in L^q(\mathbb{R}^n \times (0, b)) \]  (4.13)
for \(b, \alpha \in (0, \infty)\) and \(q \in [1, \infty].\)

To prove (4.9), let \(b, \alpha \in (0, \infty)\). Then, by (4.13) and Hölder’s inequality we have
\[ \int_0^b \int_{B_b(0)} J_\alpha f \, dx \, dt \]
\[ = \int_0^b \int_{B_b(0)} \int_0^b \int_{\mathbb{R}^n} \Phi_\alpha(x - \xi, t - \tau) f(\xi, \tau) \, dx \, dt \, d\xi \, d\tau \]
\[ = \int_0^b \int_{\mathbb{R}^n} f(\xi, \tau) \left( \int_0^b \int_{B_b(0)} \Phi_\alpha(x - \xi, t - \tau) \, dx \, dt \right) \, d\xi \, d\tau < \infty. \]
Thus, (4.9) holds.

To prove (4.10), suppose \(\beta, \gamma \in (0, \infty), \beta + \gamma = \alpha > 1,\) and \(\varphi \in C_0^\infty(\mathbb{R}^n \times (0, \infty)).\)
Then, assuming we can interchange the order of integration in the following calculation
(we will justify this after the calculation) and using the fact that $\Phi_\beta \ast \Phi_\gamma = \Phi_\alpha$ (see [11, Lemma 5.1]), we find that

$$
(HJ_\alpha f)\varphi = J_\alpha f(H^*\varphi) = (\Phi_\alpha \ast f)(H^*\varphi)
$$

$$
= (\Phi_\beta \ast \Phi_\gamma \ast f)(H^*\varphi)
$$

$$
= \iint \iint \Phi_\gamma(\eta - \xi, \zeta - \tau) f(\xi, \tau) \times \iint \Phi_\beta(x - \eta, t - \zeta) H^*\varphi(x, t) \, dx \, dt \, d\xi \, d\tau \, d\eta \, d\zeta
$$

$$
= \iint J_\gamma f(\eta, \zeta) \left( \iint \Phi_\beta(x - \eta, t - \zeta) H^*\varphi(x, t) \, dx \, dt \right) \, d\eta \, d\zeta.
$$

(4.14)

Taking $\beta = 1$ and $\gamma = \alpha - 1$ in (4.14) and using (4.12), we find that

$$
(HJ_\alpha f)\varphi = (J_{\alpha-1} f)(\varphi).
$$

Hence, (4.10) holds provided we justify the calculation (4.14) by verifying

$$
\iint (\Phi_\alpha \ast f)|H^*\varphi| < \infty.
$$

(4.15)

To do this, choose $b > 0$ such that $\text{supp} \varphi \subset B_b(0) \times (0, b)$ and repeat the calculation (4.14) with $\beta = \alpha - 1$ and $\gamma = 1$, and with $H^*\varphi$ replaced with $|H^*\varphi|$ to obtain

$$
\iint (\Phi_\alpha \ast f)|H^*\varphi| = \int_0^b \int_{\mathbb{R}^n} J_1 f(\eta, \zeta) \times \left( \int_0^b \int_{B_b(0)} \Phi_{\alpha-1}(x - \eta, t - \zeta) |H^*\varphi(x, t)| \, dx \, dt \right) \, d\eta \, d\zeta.
$$

(4.16)

Choose $p' \in (1, \infty)$ such that $0 < \frac{1}{p} - \frac{1}{p'} < \frac{2}{n+2}$. Then, by [11, Lemma 7.2], $J_1 f \in L^{p'}(\mathbb{R}^n \times (0, b))$. Hence, (4.15) follows from (4.13), (4.16), and Hölder’s inequality.

To prove (4.11), let $\varphi \in C_0^\infty(\mathbb{R}^n \times (0, \infty))$. Then, using (4.12) and assuming we can interchange the order of integration we see that

$$
H(J_1 f)\varphi = (J_1 f)(H^*\varphi) = (\Phi_1 \ast f)(H^*\varphi)
$$

$$
= \iint \left( \iint \Phi_1(x - \xi, t - \tau) H^*\varphi(x, t) \, dx \, dt \right) f(\xi, \tau) \, d\xi \, d\tau
$$

$$
= f(\varphi).
$$

Thus, (4.11) holds because interchanging the order of integration is validated by using (4.13) and Hölder’s inequality as in the proof of (4.15).
**Lemma 4.5.** [11, Lemma 7.4] Suppose $x \in \mathbb{R}^n$ and $t, \tau \in (0, \infty)$ satisfy
\[
|x|^2 < t \quad \text{and} \quad \frac{t}{4} < \tau < \frac{3t}{4}.
\] (4.17)

Then,
\[
\int_{|\xi|^2 < \tau} \Phi_1(x - \xi, t - \tau) \, d\xi \geq C(n) > 0
\]
where $\Phi_\alpha$ is defined by (1.15).

**Lemma 4.6.** Suppose $f, g : \mathbb{R}^n \times (0, T) \to \mathbb{R}$ are measurable functions, where $T \in (0, \infty)$. Define $h : \mathbb{R}^n \times (0, T) \to \mathbb{R}$ by
\[
h(x, t) = \int_0^t \int \mathbb{R}^n |g(x - \xi, t - \tau) f(\xi, \tau)| \, d\xi \, d\tau.
\]

Then,
\[
\|h\|_{L^q(\mathbb{R}^n \times (0, T))} \leq \|g\|_{L^r(\mathbb{R}^n \times (0, T))} \|f\|_{L^p(\mathbb{R}^n \times (0, T))}
\]
provided $p, q, r \in [1, \infty)$ and $\frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{r}$.

**Proof.** We can assume without loss of generality that $f$ and $g$ are nonnegative. Let $\bar{f}$ and $\bar{g}$ be the extensions of $f$ and $g$ to $\mathbb{R}^n \times \mathbb{R}$ by 0. Then,
\[
h = \bar{g} \ast \bar{f} \quad \text{in} \quad \mathbb{R}^n \times (0, T)
\]
where $\ast$ is the convolution operator in $\mathbb{R}^n \times \mathbb{R}$. Thus, by Young’s inequality
\[
\|h\|_{L^q(\mathbb{R}^n \times (0, T))} = \|\bar{g} \ast \bar{f}\|_{L^q(\mathbb{R}^n \times \mathbb{R})}
\leq \|\bar{g}\|_{L^r(\mathbb{R}^n \times \mathbb{R})} \|\bar{f}\|_{L^p(\mathbb{R}^n \times \mathbb{R})}
\leq \|\bar{g}\|_{L^r(\mathbb{R}^n \times (0, T))} \|\bar{f}\|_{L^p(\mathbb{R}^n \times (0, T))}.
\]

\[\square\]

5. **Proof of Theorem 2.4**

In this section, we prove Theorem 2.4.

**Proof of Theorem 2.4.** By (2.14), there exists $\varepsilon \in (0, 1)$ and
\[
r \in \left[1, \frac{n + 2}{n + 2\varepsilon}\right]
\] (5.1)
such that
\[
0 \leq \frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{r} < \frac{2 - 2\varepsilon}{n + 2}.
\] (5.2)
From (5.1), we see that
\[-(r - 1)n/2 > -\frac{n}{2} \left( \frac{n + 2}{n} - 1 \right) = -1\]
and for \(1 - \varepsilon \leq \alpha \leq 2\) that
\[r(\alpha - 1) - (r - 1)n/2 \geq -\varepsilon r - (r - 1)n/2 = -\frac{1}{2}(n + 2\varepsilon)r - n\]
\[> -\frac{n + 2 - n}{2} = -1.\]

Therefore, using the equation
\[\int_{\mathbb{R}^n} e^{-r|x|^2/(4t)} \, dx = \left( \frac{4\pi t}{r} \right)^{n/2},\]
we find for \(T \in (0, \infty)\) that
\[\|\Phi_\alpha\|_{L^r(\mathbb{R}^n \times (0,T))} = \frac{(4\pi)^{(r-1)n/2}}{r^{n/2}} \int_0^T \left| \frac{\alpha - 1}{\Gamma(\alpha)} \right|^r t^{-(r-1)n/2} \, dt\]
\[\leq C(n, r, \varepsilon, T) \quad \text{for } 1 - \varepsilon \leq \alpha \leq 2 \quad (5.3)\]
and
\[\|\Phi_\alpha - \Phi_1\|_{L^r(\mathbb{R}^n \times (0,T))} = \frac{(4\pi)^{(r-1)n/2}}{r^{n/2}} \int_0^T \left| \frac{\alpha - 1}{\Gamma(\alpha)} - 1 \right| t^{-(r-1)n/2} \, dt \rightarrow 0 \quad (5.4)\]
as \(\alpha \rightarrow 1\) by the dominated convergence theorem.

Letting
\[f_j = (\partial_t - \Delta)^{\alpha_j} u_j \quad (5.5)\]
and
\[u = J_1 f \quad \text{where } f \in X^p \quad (5.6)\]
is the limit in \(X^p\) as \(j \rightarrow \infty\) of \(f_j\), we find from (2.17) that
\[f_j = f = u = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0) \quad (5.7)\]
and thus using (5.2), (5.3), (5.4), and Lemma 4.6, we obtain for all \(T > 0\) that
\[\|u_j\|_{L^q(\mathbb{R}^n \times (-\infty,T))} = \|J_{\alpha_j} f_j\|_{L^q(\mathbb{R}^n \times (0,T))}\]
\[\leq \|\Phi\|_{L^r(\mathbb{R}^n \times (0,T))} \|f_j\|_{L^p(\mathbb{R}^n \times (0,T))}\]
\[\leq C(n, r, \varepsilon, T) \|f_j\|_{L^p(\mathbb{R}^n \times (0,T))} < \infty \quad \text{for } j = 1, 2, \ldots\]
and
\[\|u_j - u\|_{L^q(\mathbb{R}^n \times (-\infty,T))} = \|J_{\alpha_j} f_j - J_1 f\|_{L^q(\mathbb{R}^n \times (0,T))}\]
\[\leq \|J_{\alpha_j} f_j - J_{\alpha_j} f\|_{L^q(\mathbb{R}^n \times (0,T))} + \|J_{\alpha_j} f - J_1 f\|_{L^q(\mathbb{R}^n \times (0,T))}\]
\[\leq \|\Phi\|_{L^r(\mathbb{R}^n \times (0,T))} \|f_j - f\|_{L^p(\mathbb{R}^n \times (0,T))}\]
\[+ \|\Phi\|_{L^r(\mathbb{R}^n \times (0,T))} \|f\|_{L^p(\mathbb{R}^n \times (0,T))}\]
\[\rightarrow 0 \quad \text{as } j \rightarrow 0.\]
Thus,

\[ u \in L^q(\mathbb{R}^n \times (-\infty, T)) \quad \text{for all } T > 0 \]

and by a standard diagonalization argument, for some subsequence of \( u_j \), which we again denote by \( u_j \), we have

\[ u_j \to u \quad \text{in } X^q \hookrightarrow L^q_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}) \]

which by (2.15) and Hölder’s inequality implies

\[ u^\lambda_j \to u^\lambda \quad \text{in } L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}). \quad (5.8) \]

We next show

\[ Hu = f \quad \text{in } \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}) \quad (5.9) \]

where \( H = \partial_t - \Delta \) is the pointwise heat operator in \( \mathbb{R}^n \times \mathbb{R} \). To do this, let \( \varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) \) and \( H^* = -\partial_t - \Delta \) be the adjoint of \( H \). Then, recalling that \( H \Phi_1 = \delta \) in \( \mathcal{D}'(\mathbb{R}^n \times \mathbb{R}) \) and assuming we can interchange the order of integration in the following calculation (we will justify this after the calculation), we obtain from (5.6) that

\[
(Hu)\varphi = \iint u H^* \varphi \, dx \, dt = \iint (J_1 f) H^* \varphi \, dx \, dt
= \iint \left( \iint \Phi_1(x - \xi, t - \tau) H^* \varphi(x, t) \, dx \, dt \right) f(\xi, \tau) \, d\xi \, d\tau
= \iint \varphi(\xi, \tau) f(\xi, \tau) \, d\xi \, d\tau = (f)\varphi. \quad (5.10)
\]

To justify this calculation, choose \( R, T > 0 \) such that

\[ \text{supp } \varphi \subset B_R(0) \times (-T, T). \]

Then, from (5.7) we see that

\[
\iint_{\mathbb{R}^n \times \mathbb{R}} (J_1 |f|)(x, t) |H^* \varphi|(x, t) \, dx \, dt
= \iint_{B_R(0) \times (0, T)} (J_1 |f|)(x, t) |H^* \varphi|(x, t) \, dx \, dt < \infty
\]

because, by (5.2), (5.3), (5.6), and Lemma 4.6,

\[ \| J_1 |f| \|_{L^q(\mathbb{R}^n \times (0, T))} < \infty \]

which implies \( \| J_1 |f| \|_{L^1(B_R(0) \times (0, T))} < \infty \). Therefore, by Fubini’s theorem, we find that (5.10) and hence (5.9) hold.

It remains only to prove (2.18). To do this, let \( \varphi \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}) \) be nonnegative. Then, since \( f_j \to f \) in \( X^p \) and hence in \( L^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}) \) we obtain from (5.9), (5.5),
(2.16), and (5.8) that

\[(Hu)\varphi = (f)\varphi = \lim_{j \to \infty} (f_j)\varphi = \lim_{j \to \infty} ((\partial_t - \Delta)\alpha_j u_j)\varphi \geq (\leq) \lim_{j \to \infty} (u^\lambda_j)\varphi = (u^\lambda)\varphi\]

which proves (2.18). □

6. Proof of Theorems 3.1 and 3.2

In this section, we prove Theorems 3.1 and 3.2.

Theorem 3.1 is a consequence of Theorems 6.1–6.4 because Theorem 6.1 (6.2, 6.3, 6.4) guarantees under the assumption on \(p, \alpha, \lambda\) in Theorem 3.1 the nonexistence (existence, nonexistence, existence) of nontrivial nonnegative solutions \(f \in X^p\) of the super-\(J_\alpha\) problem (3.1), (3.2) when \((\lambda, \alpha) \in A(B, C, D \cup E)\).

**Theorem 6.1.** Suppose \(f : \mathbb{R}^n \times \mathbb{R} \to [0, \infty)\) is a measurable solution of the super-\(J_\alpha\) problem (3.1), (3.2) where \(\lambda \in (0, 1)\) and \(\alpha \in (0, \infty)\) are constants. Then, either

\[f = 0 \text{ a.e. in } \mathbb{R}^n \times \mathbb{R} \tag{6.1}\]

or there exists \(a \in [0, \infty)\) such that

\[f = 0 \text{ a.e. in } \mathbb{R}^n \times (-\infty, a), \tag{6.2}\]

\[f(x, t) \geq (M(t - a)^\alpha)^{\frac{\lambda}{1 - \lambda}} \text{ a.e. in } \mathbb{R}^n \times (a, \infty), \tag{6.3}\]

and

\[(J_\alpha f)(x, t) \geq (M(t - a)^\alpha)^{\frac{1}{1 - \lambda}} \text{ a.e. in } \mathbb{R}^n \times (a, \infty) \tag{6.4}\]

where

\[M = M(\lambda, \alpha) = \frac{\Gamma \left( \frac{\lambda \alpha}{1 - \lambda} + 1 \right)}{\Gamma \left( \alpha + \frac{\lambda \alpha}{1 - \lambda} + 1 \right)} \tag{6.5}\]

where \(\Gamma\) is the Gamma function.

**Proof.** Let

\[a = \sup \{ t \in \mathbb{R} : \| f \|_{L^\infty(\mathbb{R}^n \times (-\infty, t))} = 0 \}. \tag{6.6}\]

Then, (3.2) implies \(a \geq 0\). If \(a = \infty\), then (6.1) holds. Hence, we can assume

\[a \in [0, \infty). \tag{6.7}\]
It follows from (6.6) and (6.7) that (6.2) holds and it remains only to prove (6.3) and (6.4). To do this, we first prove
\[
f(x, t) \geq (N_0(t - a)^{\alpha})^{\frac{\lambda}{1 - \lambda}} \quad \text{a.e. in } \mathbb{R}^n \times (a, \infty)
\]  
for some positive constant $N_0 = N_0(n, \lambda, \alpha)$. Let $T > 0$ and $x_0 \in \mathbb{R}^n$ be fixed. To prove (6.8), it suffices to prove
\[
f(x, t) \geq (N_0(t - a)^{\alpha})^{\frac{\lambda}{1 - \lambda}} \quad \text{for } (x, t) \in \Omega(x_0, a, T),
\]
and some positive constant $N_0 = N_0(n, \lambda, \alpha)$ where
\[
\Omega(x_0, t_0, T) := \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : |x - x_0|^2 < (t - t_0) < T\}
\]
because
\[
\mathbb{R}^n \times (a, \infty) = \bigcup_{x_0 \in \mathbb{R}^n, T > 0} \Omega(x_0, a, T).
\]

Let
\[
t_0 \in (a, T + a)
\]
be fixed. Then, to prove (6.9), and hence (6.8), it suffices to prove
\[
f(x, t) \geq (N_0(t - t_0)^{\alpha})^{\frac{\lambda}{1 - \lambda}} \quad \text{for } (x, t) \in \Omega(x_0, t_0, T)
\]
and some positive constant $N_0 = N_0(n, \lambda, \alpha)$ because then sending $t_0$ to $a$ in (6.11) we get (6.9). Define
\[
g : \mathbb{R}^n \times \mathbb{R} \rightarrow [0, \infty)
\]
by
\[
g(x, t) = f(x + x_0, t + t_0).
\]
Then, $f$ satisfies (6.11), and hence (6.8), if and only if $g$ satisfies
\[
g(x, t) \geq (N_0t^{\alpha})^{\frac{\lambda}{1 - \lambda}} \quad \text{for } (x, t) \in \Omega(0, 0, T).
\]

It follows from (6.6) and (6.10) that $J_{\alpha} f$ is bounded below by a positive constant on bounded subsets of $\mathbb{R}^n \times (t_0, \infty)$, in particular on $\Omega(x_0, t_0, T)$. Hence, by (3.1) and (6.12) we see that $g$ is bounded below by a positive constant on $\Omega(0, 0, T)$. Thus, there exists a constant $b_0 > 0$ such that
\[
g(x, t) \geq (b_0T^{\alpha})^{\frac{\lambda}{1 - \lambda}} \geq (b_0t^{\alpha})^{\frac{\lambda}{1 - \lambda}} \quad \text{for } (x, t) \in \Omega(0, 0, T).
\]
(Note, however, that $b_0$ may depend not only on $n$, $\lambda$, and $\alpha$ but also on $x_0$, $t_0$, and $T$.)
Also, for \((x, t) \in \mathbb{R}^n \times \mathbb{R}\) we find from (6.12) and (3.1) that
\[
\frac{g(x, t)}{1/\lambda} = f(x + x_0, t + t_0) \geq (J_\alpha f)(x + x_0, t + t_0)
\]
\[
= \int_{-\infty}^{t+t_0} \int_{\mathbb{R}^n} \Phi_\alpha(x + x_0 - \xi, t + t_0 - \tau) f(\xi, \tau) \, d\xi \, d\tau
\]
\[
= \int_{-\infty}^{t} \int_{\mathbb{R}^n} \Phi_\alpha(x - \tilde{\xi}, t - \tilde{\tau}) f(\tilde{\xi} + x_0, \tilde{\tau} + t_0) \, d\tilde{\xi} \, d\tilde{\tau}
\]
\[
= J_\alpha g(x, t).
\]
Thus,
\[
g \geq (J_\alpha g)^\lambda \text{ in } \mathbb{R}^n \times \mathbb{R}. \quad (6.15)
\]
Let \(\beta := \frac{\lambda \alpha}{1 - \lambda}\). Then, for \((x, t) \in \Omega(0, 0, T)\) we obtain from (6.15), (6.14), and Lemma 4.5 that
\[
g(x, t)^{1/\lambda} \geq J_\alpha g(x, t) \geq b_1^\beta \alpha \int_{\Omega(0,0,T)} \Phi_\alpha(x - \xi, t - \tau) \tau^{\beta} \, d\xi \, d\tau
\]
\[
\geq b_0^\beta \alpha \int_{1/4}^{3/4} (t - \tau)^{\alpha - 1} \tau^{\beta} \frac{\Phi_1(x - \xi, t - \tau) d\xi}{\Gamma(\alpha)} \, d\tau
\]
\[
\geq b_0^\beta \alpha C(n) \int_{1/4}^{3/4} (t - \tau)^{\alpha - 1} \tau^{\beta} \, d\tau
\]
\[
= b_0^\beta \alpha C(n) \frac{\int_{1/4}^{3/4} (1 - s)^{\alpha - 1} s^{\beta} \, ds}{\Gamma(\alpha)} \tau^{\frac{\alpha}{1 - \lambda}}.
\]
Thus, letting
\[
N_0 = N_0(n, \lambda, \alpha) = C(n) \frac{\int_{1/4}^{3/4} (1 - s)^{\alpha - 1} s^{\beta} \, ds}{\Gamma(\alpha)} \tau^{\frac{\alpha}{1 - \lambda}}
\]
and \(b_1 = b_0^\beta N_0^{1 - \lambda}\) we have
\[
g(x, t) \geq (b_1 t^{\alpha})^{\frac{\lambda}{1 - \lambda}} \text{ for } (x, t) \in \Omega(0, 0, t) \quad (6.16)
\]
where
\[
\frac{b_1}{N_0} = \left( \frac{b_0}{N_0} \right)^{\lambda}.
\]
Iterating the method we used to derive (6.16) from (6.14), we inductively obtain a sequence \(\{b_j\}_{j=0}^\infty \subset (0, \infty)\) such that for \(j = 1, 2, \ldots\), we have
\[
\frac{b_j}{N_0} = \left( \frac{b_{j-1}}{N_0} \right)^{\lambda} \quad (6.17)
\]
and
\[
g(x, t) \geq (b_j t^{\alpha})^{\frac{\lambda}{1 - \lambda}} \text{ for } (x, t) \in \Omega(0, 0, T). \quad (6.18)
\]
Since $\lambda \in (0, 1)$, it follows from (6.17) that
\[
\lim_{j \to \infty} \frac{b_j}{N_0} = 1.
\]
Consequently, sending $j$ to $\infty$ in (6.18) we obtain (6.13) and hence also (6.8).

Using (3.1), (6.2), (6.8), and (6.5) and making the change of variables $\bar{t} = t - a$, $\bar{\tau} = \tau - a$, we obtain for $(x, t) \in \mathbb{R}^n \times (a, \infty)$ that
\[
f(x, t)^{1/\lambda} \geq J_\alpha f(x, t) \geq N_0^{\frac{\lambda}{1-\lambda}} \int_a^t \frac{(t - \tau)^{\alpha - 1}}{\Gamma(\alpha)} \frac{\tau^{\frac{\alpha}{1-\lambda}}}{M^{\frac{\alpha}{1-\lambda}}} d\tau
\]
\[
= N_0^{\frac{\lambda}{1-\lambda}} \int_0^t \frac{(\bar{t} - \bar{\tau})^{\alpha - 1}}{\Gamma(\alpha)} \frac{\bar{\tau}^{\frac{\alpha}{1-\lambda}}}{M^{\frac{\alpha}{1-\lambda}}} d\bar{\tau}
\]
\[
= N_0^{\frac{\lambda}{1-\lambda}} M \bar{t}^{\frac{\alpha}{1-\lambda}} = (N_0^{\lambda} M^{1-\lambda} (t - a)\alpha)^{\frac{1}{1-\lambda}}.
\]
Thus, for $(x, t) \in \mathbb{R}^n \times (a, \infty)$ we have
\[
f(x, t) \geq (J_\alpha f(x, t))^\lambda \geq (N_1(t - a)\alpha)^{\frac{1}{1-\lambda}}
\]
where
\[
\frac{N_1}{M} = \left( \frac{N_0}{M} \right)^{\lambda}.
\]
Iterating the method we used to derive (6.19) from (6.8), we inductively obtain a sequence $\{N_j\}_{j=0}^\infty \subset (0, \infty)$ such that for $j = 1, 2, ..., we have
\[
\frac{N_j}{M} = \left( \frac{N_{j-1}}{M} \right)^{\lambda}
\]
and
\[
f(x, t) \geq (J_\alpha f(x, t))^\lambda \geq (N_j(t - a)\alpha)^{\frac{1}{1-\lambda}} \text{ for } (x, t) \in \mathbb{R}^n \times (a, \infty).
\]
Since $\lambda \in (0, 1)$, it follows from (6.20) that
\[
\lim_{j \to \infty} \frac{N_j}{M} = 1.
\]
Consequently, sending $j$ to $\infty$ in (6.21) we obtain (6.3) and (6.4). \[\square\]

**Theorem 6.2.** Suppose $\alpha, T \in (0, \infty)$ and $p \in [1, \infty]$. Then, there exists $a = a(\alpha) > 0$ such that a solution
\[
f \in C^\infty(\mathbb{R}^n \times [0, \infty)) \cap X^p
\]
of the super-$J_\alpha$ problem

$$f \geq J_\alpha f \quad \text{in } \mathbb{R}^n \times \mathbb{R}$$

$$f = 0 \quad \text{in } \mathbb{R}^n \times (-\infty, 0)$$

is

$$f(x, t) = e^{a(t+T)} \Phi_1(x, t + T) \chi_{[0, \infty)}(t). \quad (6.22)$$

Proof. For all $a > 0$, the function $f$ given by (6.22) is in $C^\infty(\mathbb{R}^n \times [0, \infty))$. Thus, to prove Theorem 6.2, it suffices by Lemma 4.2, to show there exists $a = a(\alpha) > 0$ such that the function $g : \mathbb{R}^n \times \mathbb{R} \to [0, \infty)$ defined by

$$g(x, t) = e^{at} \Phi_1(x, t) \quad (6.23)$$

satisfies (4.3) and

$$g \geq J_\alpha g \quad \text{in } \mathbb{R}^n \times \mathbb{R}. \quad (6.24)$$

By Remark 4.1, $g$ satisfies (4.3) for all $a > 0$. Hence, it remains only to show there exists $a = a(\alpha) > 0$ such that $g$ satisfies (6.24)

The inequality (6.24) holds in $\mathbb{R}^n \times (-\infty, 0]$ because $g = 0$ there. On the other hand, for $(x, t) \in \mathbb{R}^n \times (0, \infty)$, it follows from (6.23) and Lemma 4.1 that

$$J_\alpha g(x, t) = \int_0^t \left( \int_{\mathbb{R}^n} \Phi_\alpha(x - \xi, t - \tau) \Phi_1(\xi, \tau) d\xi \right) e^{a\tau} d\tau$$

$$= \Phi_1(x, t) \int_0^t \frac{(t - \tau)^{\alpha-1}}{\Gamma(\alpha)} e^{a\tau} d\tau. \quad (6.25)$$

However, for $t, a > 0$ we have

$$e^{-at} \int_0^t (t - \tau)^{\alpha-1} e^{a\tau} d\tau \leq \int_{-\infty}^t (t - \tau)^{\alpha-1} e^{-a(t-\tau)} d\tau$$

$$= \frac{1}{a^\alpha} \int_0^\infty \zeta^{\alpha-1} e^{-\zeta} d\zeta \to 0 \quad \text{as } a \to \infty. \quad (6.25)$$

It follows therefore from (6.25) and (6.23) that there exists $a = a(\alpha) > 0$ such that $g$ satisfies (6.24) in $\mathbb{R}^n \times (0, \infty)$. \qed

Theorem 6.3. Suppose

$$f \in X^p \quad (6.26)$$

is a nonnegative solution of the super-$J_\alpha$ problem (3.1), (3.2) where $p \in [1, \infty)$ and

$$(\lambda, \alpha) \in C \quad (6.27)$$

where $C$ is the region defined in Sect. 2 and graphed in Fig. 1. Then,

$$f = J_\alpha f = 0 \quad \text{a.e. in } \mathbb{R}^n \times \mathbb{R}. \quad (6.28)$$
Proof. By (3.1) and (3.2), $f$ satisfies (6.28) a.e. in $\mathbb{R}^n \times (-\infty, 0]$. Hence, it suffices to prove
\[
f = J_\alpha f = 0 \text{ a.e. in } \mathbb{R}^n \times (0, \infty).
\] (6.29)

Let $u = J_m f$ where $m$ is the positive integer satisfying $\alpha \leq m < \alpha + 1$. By Lemma 4.4, $u$ satisfies (2.3) and (2.4) and
\[
H^m u(x, t) = f(x, t) \geq \left( J_\alpha f(x, t) \right)^{\lambda} = \left( \frac{\Gamma(m)}{\Gamma(\alpha)} (t + 1)^{\alpha - m} u(x, t) \right)^{\lambda}.
\] (6.29)

Hence, (6.29) follows from (6.27) and Lemma 2.1. □

Theorem 6.4. Suppose
\[
\lambda > 1, \quad 0 < \alpha < \frac{n + 2}{2} \left( 1 - \frac{1}{\lambda} \right), \quad p \in [1, \infty) \text{ and } T > 0.
\] (6.30)

Then,
\[
\beta := \frac{n + 2}{2} - \frac{\lambda \alpha}{\lambda - 1} = \frac{n + 2}{2} - \frac{\alpha}{1 - \frac{1}{\lambda}} > 0
\] (6.31)

and a solution
\[
f \in C^\infty(\mathbb{R}^n \times [0, \infty)) \cap X^p
\]
of the super-$J_\alpha$ problem (3.1), (3.2) is
\[
f(x, t) = A \Phi_\beta(x, t + T) \chi_{[0, \infty)}(t)
\] (6.32)
where
\[
A = A(n, \lambda, \alpha) = \left( \frac{(4\pi)^{(\lambda-1)n/2} \Gamma(\alpha + \beta)^{\lambda}}{\Gamma(\beta)} \right)^{\frac{1}{\lambda-1}}.
\] (6.33)

Proof. Inequality (6.31) follows from (6.30) and $f$ given by (6.32) is clearly in $C^\infty(\mathbb{R}^n \times [0, \infty))$. Thus, to prove Theorem 6.4, it suffices by Lemma 4.2 to show that the function $g : \mathbb{R}^n \times \mathbb{R} \to [0, \infty)$ defined by
\[
g(x, t) = A \Phi_\beta(x, t)
\]
satisfies (4.3) and (4.4).

By Remark 4.1, $g$ satisfies (4.3). Hence, it remains only to show $g$ satisfies (4.4). The inequality (4.4) holds in $\mathbb{R}^n \times (-\infty, 0]$ because $g = 0$ there. On the other hand,
for \((x, t) \in \mathbb{R}^n \times (0, \infty)\), it follows from (6.31) and Lemma 4.1 that

\[
J_\alpha g(x, t) = A \int_0^t \left( \int_{\mathbb{R}^n} \Phi_\alpha(x - \xi, t - \tau) \Phi_\beta(\xi, \tau) d\xi \right) d\tau = A \Phi_1(x, t) \int_0^t \frac{(t - \tau)^{\alpha-\beta-1}}{\Gamma(\alpha)\Gamma(\beta)} d\tau = A \Phi_1(x, t) \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} = A \Phi_\alpha + \beta(x, t)
\]

and thus

\[
\frac{(J_\alpha g(x, t))^\lambda}{g(x, t)} = A^{\lambda-1} \frac{\Gamma(\beta)^{\lambda\alpha+(\lambda-1)(\beta-\frac{n+2}{2})}e^{(1-\lambda)|x|^2/(4t)}}{(4\pi)^{(\lambda-1)n/2}\Gamma(\alpha+\beta)^\lambda} \leq A^{\lambda-1} \frac{\Gamma(\beta)}{(4\pi)^{(\lambda-1)n/2}\Gamma(\alpha+\beta)^\lambda} = 1
\]

by (6.30), (6.31), and (6.33). The proof of Theorem 6.4 is now complete.

Since any solution of the approximate \(J_\alpha\) problem is, after scaling, also a solution of the super- and sub-\(J_\alpha\) problems, it follows from Theorems 3.1 and 3.3 that if \(\lambda, \alpha, \) and \(p\) satisfy the conditions in Theorem 3.2 and \((\lambda, \alpha) \in A \cup B \cup C \cup D\), then there do not exist positive constants \(C_1\) and \(C_2\) such that the approximate \(J_\alpha\) problem has a nontrivial nonnegative solution \(f \in X_p\). Hence, Theorem 3.2 follows from the following theorem.

**Theorem 6.5.** Suppose the constants \(\lambda\) and \(\alpha\) satisfy

\[
\lambda > 1 \text{ and } 0 < \alpha < \frac{n+2}{2} \left( 1 - \frac{1}{\lambda} \right).
\]

Define \(f : \mathbb{R}^n \times \mathbb{R} \to [0, \infty)\) by

\[
f(x, t) = \begin{cases} 
    t^{-\frac{\alpha\lambda}{2}} \bar{w}(x/\sqrt{t}) & \text{for } (x, t) \in \mathbb{R}^n \times (0, \infty) \\
    0 & \text{for } (x, t) \in \mathbb{R}^n \times (-\infty, 0] 
\end{cases}
\]

where

\[
\bar{w}(z) = e^{-\frac{z^2}{4}} \left( |z|^2 + 1 \right)^{-\lambda} \left( \frac{n+2}{2} - \frac{\alpha\lambda}{2} - 1 \right) \text{ for } z \in \mathbb{R}^n.
\]

Then,

\[
f \in C^\infty(\mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}) \cap X^1
\]

and \(f\) is a solution of the approximate \(J_\alpha\) problem (3.3), (3.4) for some positive constants \(C_1\) and \(C_2\) depending only on \(n, \lambda, \) and \(\alpha\).

Moreover, if \(p \in [1, \infty)\), then \(f \in X_p\) if and only if

\[
\alpha < \frac{n+2}{2p} \left( 1 - \frac{1}{\lambda} \right).
\]
Proof. We first prove the last sentence of the theorem. Let \( p \in [1, \infty) \). Then, for all \( t > 0 \) we find under the change of variables \( x = \sqrt{\tau} z \) that

\[
\int_\mathbb{R}^n \int_{(-\infty, t]} f(x, \tau)^p \, dx \, d\tau = \int_0^t \tau^{-\frac{\alpha \lambda p}{\lambda - 1}} \left( \int_{\mathbb{R}^n} w \left( \frac{x}{\sqrt{\tau}} \right)^p \, dx \right) \, d\tau \\
= \int_0^t \tau^{\frac{n - \alpha \lambda p}{\lambda - 1}} \left( \int_{\mathbb{R}^n} w(z)^p \, dz \right) \, d\tau.
\]

Hence, since \( \int_{\mathbb{R}^n} w(z)^p \, dz < \infty \), we see that \( f \in X^p \) if and only if

\[
\frac{n}{2} - \frac{\alpha \lambda p}{\lambda - 1} > -1
\]

which is equivalent to (6.38).

It follows from (6.34) and the last sentence of the theorem that \( f \in X^1 \). One easily checks that \( f \in C^\infty(\mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\}) \). Hence, \( f \) satisfies (6.37).

We now complete the proof of the theorem by proving that \( f \) satisfies the inequalities (3.3). Let \( (x, t) \in \mathbb{R}^n \times (0, \infty) \) and let \( \bar{x} = x/\sqrt{t} \). Then,

\[
f(x, t) = t^{-\frac{\alpha \lambda}{\lambda - 1}} w(\bar{x})
\]

and under the variables \( \tau = t \bar{\tau} \) and \( \xi = \sqrt{t} \xi \), we get

\[
\Phi_\alpha(x - \xi, t - \tau) = t^{\alpha - 1 - n/2} \Phi_\alpha(\bar{x} - \bar{\xi}, 1 - \bar{\tau}).
\]

Thus,

\[
J_\alpha f(x, t) = \int_0^t \int_{\mathbb{R}^n} \Phi_\alpha(x - \xi, t - \tau) \tau^{-\frac{\alpha \lambda}{\lambda - 1}} w(\xi/\sqrt{\tau}) \, d\xi \, d\tau \\
= t^{\alpha - \frac{\alpha \lambda}{\lambda - 1}} \int_0^1 \int_{\mathbb{R}^n} \Phi_\alpha(\bar{x} - \bar{\xi}, 1 - \bar{\tau}) \bar{\tau}^{-\frac{\alpha \lambda}{\lambda - 1}} w(\bar{\xi}/\sqrt{\bar{\tau}}) \, d\bar{\xi} \, d\bar{\tau} \\
= t^{-\frac{\alpha \lambda}{\lambda - 1}} J_\alpha f(\bar{x}, 1).
\]

Hence, letting

\[
I(x) = \Gamma(\alpha)(4\pi)^{n/2} J_\alpha f(x, 1)
\]

we obtain from (6.39) that

\[
\frac{J_\alpha f(x, t)}{f(x, t)^{1/\lambda}} = \frac{J_\alpha f(\bar{x}, 1)}{w(\bar{x})^{1/\lambda}} = \frac{I(x)}{\Gamma(\alpha)(4\pi)^{n/2} w(\bar{x})^{1/\lambda}}
\]

for \( (x, t) \in \mathbb{R}^n \times (0, \infty) \). Thus, since (3.3) clearly holds in \( \mathbb{R}^n \times (-\infty, 0] \), in order to prove (3.3) it suffices to prove

\[
0 < C_1 \leq \frac{I(x)}{w(x)^{1/\lambda}} \leq C_2 \quad \text{for} \quad x \in \mathbb{R}^n
\]

(6.41)
where $C_1$ and $C_2$ depend only on $n$, $\lambda$, and $\alpha$ and from (6.40)

$$I(x) = \int_0^1 (1 - \tau)^{\alpha - 1 - n/2} \tau^{-\frac{\alpha \lambda}{\lambda - 1}} \int_{\mathbb{R}^n} e^{-\frac{|x - \xi|^2}{4(1-\tau)}} w(\xi/\sqrt{\tau}) d\xi d\tau. \quad (6.42)$$

To do this, we will need the identity

$$(g_a * g_b)(x) = \left(\frac{\pi ab}{a + b}\right)^{n/2} g_{a+b}(x) \quad \text{for } x \in \mathbb{R}^n \text{ and } a, b > 0, \quad (6.43)$$

where $g_a : \mathbb{R}^n \to (0, \infty)$ is defined by

$$g_a(x) = e^{-|x|^2/a}.$$  

This identity can be proved in a straightforward way using the convolution theorem for the Fourier transform and the well-known transform

$$\hat{g}_a(y) = (\pi a)^{n/2} e^{-\frac{|y|^2}{4a}}$$

to show that the left and right sides of (6.43) have the same Fourier transform.

We first prove the upper bound in (6.41). Since (6.34) implies

$$\delta := \frac{n + 2}{2} - \frac{\alpha \lambda}{\lambda - 1} > 0 \quad (6.44)$$

, it follows from (6.36) that

$$w(z) \leq e^{-\frac{\lambda |z|^2}{4}} \quad \text{for } z \in \mathbb{R}^n.$$  

Hence, for $x \in \mathbb{R}^n$ we obtain from (6.42) that

$$I(x) \leq \int_0^1 (1 - \tau)^{\alpha - 1 - n/2} \tau^{-\frac{\alpha \lambda}{\lambda - 1}} h_\tau(x) d\tau$$

where

$$h_\tau(x) = \int_{\mathbb{R}^n} e^{-\frac{|x - \xi|^2}{4(1-\tau)}} e^{-\frac{\lambda |\xi|^2}{4\tau}} d\xi.$$  

Defining

$$\sigma := 1 - 1/\lambda \in (0, 1) \quad (6.45)$$

by (6.34) and using (6.43), we find that

$$h_\tau(x) = \left(\frac{4\pi (1 - \tau) \tau}{\lambda (1 - \sigma \tau)}\right)^{n/2} e^{-\frac{|x|^2}{4(1-\tau)}} \leq (4\pi (1 - \tau)^{n/2} e^{-\frac{|x|^2}{4(1-\tau)}} \text{ for } 0 < \tau < 1 \text{ and } x \in \mathbb{R}^n.$$
Thus,
\[ I(x) \leq C \int_0^1 (1 - \tau)^{\alpha-1} \tau^{\delta-1} e^{-\frac{|x|^2}{4(1-\tau)}} d\tau, \quad (6.46) \]
where \( \delta \) is defined in (6.44).

Case I. Suppose \(|x| \leq 1\). Then, by (6.46), (6.45), (6.44), and (6.36),
\[ I(x) \leq C \int_0^1 (1 - \tau)^{\alpha-1} \tau^{\delta-1} d\tau = C w(x)^{1/\lambda}. \]
That is the upper bound in (6.41) holds.

Case II. Suppose \(|x| > 1\). Then, by (6.46), (6.45), and (6.44) we have
\[
e^{\frac{|x|^2}{4}} I(x) \leq C \int_0^1 (1 - \tau)^{\alpha-1} \tau^{\delta-1} e^{-\frac{(1}{1-\sigma \tau})} d\tau
\leq C \int_0^1 (1 - \tau)^{\alpha-1} \tau^{\delta-1} e^{-b \tau} d\tau \quad \text{where} \ b := \frac{\sigma |x|^2}{2} > \frac{\sigma}{4} > 0
\leq C \int_0^b (1 - \frac{s}{b})^{\alpha-1} \left( \frac{s}{b} \right)^{\delta-1} e^{-s} \frac{1}{b} ds \quad \text{where} \ s = b \tau
= C b^{-(\alpha+\delta-1)} (I_1(b) + I_2(b))
\]
where
\[ I_1(b) := \int_0^{b/2} (b - s)^{\alpha-1} s^{\delta-1} e^{-s} ds \leq C b^{\alpha-1} \int_0^{b/2} s^{\delta-1} e^{-s} ds \leq C b^{\alpha-1}\]
and
\[ I_2(b) := \int_{b/2}^b (b - s)^{\alpha-1} s^{\delta-1} e^{-s} ds \leq e^{-b/2} \int_0^b (b - s)^{\alpha-1} s^{\delta-1} ds
= e^{-b/2} b^{\alpha+\delta-1} \int_0^1 (1 - \tau)^{\alpha-1} \tau^{\delta-1} d\tau \quad \text{where} \ s = b \tau
= C (e^{-b/2} b^{\delta}) b^{\alpha-1} \leq C b^{\alpha-1}. \]
Hence, from (6.44) we obtain
\[ e^{\frac{|x|^2}{4}} I(x) \leq C b^{-\delta} = C (|x|^2)^{-\frac{(\frac{\alpha+\delta}{2})}{2-\frac{\alpha}{\lambda}}}. \]
Thus, the upper bound in (6.41) holds when \(|x| > 1\).
To complete the proof of the theorem, we now prove the lower bound in (6.41). Since \( w \) is a positive continuous function on \( \mathbb{R}^n \), we find from (6.42) and for \( x \in \mathbb{R}^n \) that

\[
I(x) \geq C \int_0^1 (1 - \tau)^{\alpha - 1 - n/2} \tau^{-\frac{\alpha}{x^2}} \int_{|\xi| < \sqrt{\tau}} e^{-\frac{|x - \xi|^2}{4(1-\tau)}} \, d\xi \, d\tau
\]

and thus from (6.44) and for \( |x| \leq 2 \), we have

\[
I(x) \geq C \int_0^1 (1 - \tau)^{\alpha - 1 - n/2} \tau^{-\frac{\alpha}{x^2}} e^{-\frac{9}{4(1-\tau)}} \, d\tau = C \geq Cw(x)^{1/\lambda}.
\]

Hence, it remains only to prove the lower bound in (6.41) when

\[
|x| > 2.
\]

Since for \( x \in \mathbb{R}^n \) and \( \tau > 0 \), the expression

\[
\frac{|B_{|x|}(x) \cap B_{\sqrt{\tau}}(0)|}{|B_{\sqrt{\tau}}(0)|} =: V\left(\frac{|x|}{\sqrt{\tau}}\right)
\]

is an increasing function of \( |x|/\sqrt{\tau} \), we have

\[
V\left(\frac{|x|}{\sqrt{\tau}}\right) \geq V(1) = \frac{|B_1(e) \cap B_1(0)|}{|B_1(0)|} > 0 \quad \text{for } 0 < \sqrt{\tau} < |x|
\]

where \( e := (1, 0, ..., 0) \in \mathbb{R}^n \). It follows therefore from (6.47) and (6.48) that

\[
I(x) \geq C \int_0^1 (1 - \tau)^{\alpha - 1 - n/2} \tau^{-\frac{\alpha}{x^2}} \int_{\xi \in B_{|x|}(x) \cap B_{\sqrt{\tau}}(0)} e^{-\frac{|x - \xi|^2}{4(1-\tau)}} \, d\xi \, d\tau
\]

\[
\geq C \int_0^1 (1 - \tau)^{-\mu} \tau^{\delta - 1} e^{-\frac{a}{\tau}} \, d\tau
\]

where \( \delta \) is defined in (6.44),

\[
\mu := \frac{n + 2}{2} - \alpha, \quad \text{and} \quad a := \frac{|x|^2}{4} > 1
\]

by (6.48).

Next making the change of variables \( s + a = a/(1 - \tau) \) in (6.49) and using (6.50), we obtain

\[
I(x) \geq C \int_0^\infty \left( \frac{a}{s + a} \right)^{-\mu} \left( \frac{s}{s + a} \right)^{\delta - 1} e^{-(s+a)} \frac{a}{(s+a)^2} \, ds
\]

\[
= C a^{1-\mu} e^{-a} \int_0^\infty (s+a)^{\mu-\delta-1} s^{\delta-1} e^{-s} \, ds
\]

\[
= C a^{-\delta} e^{-a} \int_0^\infty \left( 1 + \frac{s}{a} \right)^{\mu-\delta-1} s^{\delta-1} e^{-s} \, ds
\]

\[
\geq C e^{-\frac{|x|^2}{4}} (\frac{n+2}{2} - \frac{\alpha}{x^2})
\]

\[
\geq Cw(x)^{1/\lambda}
\]
because

\[ 1 < 1 + \frac{s}{a} < 1 + s \quad \text{for} \quad s > 0 \]

by (6.50). Thus, the lower bound in (6.41) holds when \(|x| > 2\). □

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**REFERENCES**

[1] J. Aguirre, M. Escobedo, A Cauchy problem for \( u_t - \Delta u = u^p \) with \( 0 < p < 1 \). Asymptotic behaviour of solutions. Ann. Fac. Sci. Toulouse Math. 8 (1986/87) 175–203.

[2] I. Athanasopoulos, L. Caffarelli, E. Milakis, On the regularity of the non-dynamic parabolic fractional obstacle problem, J. Differential Equations 265 (2018) 2614–2647.

[3] M. Escobedo, M. A. Herrero, Boundedness and blow up for a semilinear reaction-diffusion system, J. Differential Equations 89 (1991) 176–202.

[4] H. Fujita, On the blowing up of solutions of the Cauchy problem for \( u_t = \Delta u + u^{1+\alpha} \), J. Fac. Sci. Univ. Tokyo Sect. I, 13 (1966), 109–124.

[5] A. Haraux, F. Weissler, Nonuniqueness for a semilinear initial value problem. Indiana Univ. Math. J. 31 (1982) 167–189.

[6] E. Mitidieri, S. I. Pokhozhaev, A priori estimates and the absence of solutions of nonlinear partial differential equations and inequalities. (Russian) Tr. Mat. Inst. Steklova 234 (2001), 1–384; translation in Proc. Steklov Inst. Math. 2001, no. 3(234), 1–362.

[7] K. Nyström, O. Sande, Extension properties and boundary estimates for a fractional heat operator, Nonlinear Anal. 140 (2016) 29–37.

[8] P. Quittner, P. Souplet, Superlinear parabolic problems. Blow-up, global existence and steady states, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser Verlag, Basel, 2007.

[9] S. G. Samko, Hypersingular integrals and their applications, Analytical Methods and Special Functions, 5, Taylor & Francis, Ltd., London, 2002.

[10] P. R. Stinga, J. L. Torrea, Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation, SIAM J. Math. Anal. 49 (2017) 3893–3924.

[11] S. Taliaferro, Pointwise bounds and blow-up for nonlinear fractional inequalities, J. Math. Pures Appl. 133 (2020) 287-328.

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