A mixing operator $T$ for which $(T, T^2)$ is not disjoint transitive

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Abstract

Using a result from ergodic Ramsey theory, we answer a question posed by Bès, Martin, Peris and Shkarin by showing a mixing operator $T$ on a Hilbert space such that the tuple $(T, T^2)$ is not disjoint transitive.

KEYWORDS: mixing operators, disjoint transitive operators

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1 Introduction

Let $X$ be a separable topological vector space. Denote the set of bounded linear operators on $X$ by $\mathcal{L}(X)$. From now on $T$ is considered in $\mathcal{L}(X)$. An operator $T$ is called hypercyclic provided that there exists a vector $x \in X$ such that its orbit $\{T^n x : n \geq 0\}$ is dense in $X$ and $x$ is called a hypercyclic vector for $T$. Hypercyclic operators are one of the most studied objects in linear dynamics, see [9] and [1] for further information concerning concepts, results and a detailed account on this subject. More generally, a tuple of operators $(T_1, \ldots, T_N)$ is said to be disjoint hypercyclic (d-hypercyclic for short) if

$$\{(T_1^n x, \ldots, T_N^n x) : n \in \mathbb{N}\}$$

is dense in $X^N$ for some vector $x \in X$.

If $X$ is an $F$-space, thanks to Birkhoff’s theorem [1], $T$ is hypercyclic if and only if $T$ is topologically transitive, i.e. for every non-empty open sets $U, V$ of $X$, the return set $N(U, V) := \{n \geq 0 : T^n(U) \cap V \neq \emptyset\}$ is non-empty. If $N(U, V)$ is cofinite for every non-empty open sets $U$ and $V$ then $T$ is said to be mixing.

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The notion of disjoint transitivity, a strengthening of transitivity is defined as follows: a tuple of operators \((T_1, \ldots, T_N)\) is disjoint transitive (d-transitive for short), if for any \((N+1)\)-tuple \((U_i)_{i=0}^N\) of non-empty open sets

\[
N_{U_1, \ldots, U_N; U_0} := \{ k \geq 0 : T_1^{-k}(U_1) \cap \cdots \cap T_N^{-k}(U_N) \cap U_0 \neq \emptyset \}
\]

is non-empty. In particular, if the set \(N_{U_1, \ldots, U_N; U_0}\) happens to be cofinite for any \((N+1)\)-tuple \((U_i)_{i=0}^N\) of non-empty open sets, then \((T_1, \ldots, T_N)\) is said to be disjoint mixing (d-mixing for short). Connection between d-hypercyclicity and d-transitivity can be found in [6].

Bès, Martin, Peris and Shkarin [5] showed the following: if \(T\) is an operator on \(X\) satisfying the Original Kitai Criterion, then the tuple \((T_1, \ldots, T_r)\) is d-mixing, for any \(r \in \mathbb{N}\). As a consequence, any bilateral weighted shift \(T\) on \(l^p(\mathbb{Z})\), \((1 \leq p < \infty)\) or \(c_0(\mathbb{Z})\) is mixing if and only if \((T_1, \ldots, T_r)\) is d-mixing, for any \(r \in \mathbb{N}\). Nevertheless, they remarked that this phenomenon does not occur beyond the weighted shift context, by providing an example of a mixing Hilbert space operator \(T\) so that \((T, T^2)\) is not d-mixing. This result is a partial answer to the following question posed by the authors in the same paper [5].

**Question 1.1.** Does there exist a mixing continuous linear operator \(T\) on a separable Banach space, such that \((T, T^2)\) is not d-transitive?

Our aim is to give a positive answer to Question 1.1 (Theorem 1.6 below).

### 1.1 Preliminaries and main results

Let \(A \subseteq \mathbb{N}\), \(|A|\) stands for the cardinality of \(A\). Let \(\mathcal{F}\) be a set of subsets of \(\mathbb{N}\), we say that \(\mathcal{F}\) is a family on \(\mathbb{N}\) provided (I.) \(|A| = \infty\) for any \(A \in \mathcal{F}\) and (II.) \(A \subset B\) implies \(B \in \mathcal{F}\), for any \(A \in \mathcal{F}\). From now on \(\mathcal{F}\) will be a family on \(\mathbb{N}\).

In a natural way we generalize the notion of disjoint transitivity by introducing what we call \(\mathcal{F}\)-disjoint transitivity (or d-\(\mathcal{F}\) for short).

**Definition 1.2.** The tuple of sequence of operators \((T_{1,n_k}, \ldots, T_{N,n_k})_k\) is said to be d-\(\mathcal{F}\) if for any \((N+1)\)-tuple \((U_i)_{i=0}^N\) of non-empty open sets we have

\[
\{ k \geq 0 : T_{1,n_k}^{-1}(U_1) \cap \cdots \cap T_{N,n_k}^{-1}(U_N) \cap U_0 \neq \emptyset \} \in \mathcal{F}.
\]

In particular, if \(T_{i,n_k} = T_i^k\), for any \(k \in \mathbb{N}\), \(1 \leq i \leq N\) in the above definition, then the \(N\)-tuple of operators \((T_1, \ldots, T_N)\) is said to be d-\(\mathcal{F}\).

Observe that in particular whenever \(\mathcal{F}\) is the family of non-empty sets or the family of cofinite sets, we obtain the notion of disjoint transitivity and disjoint mixing respectively. On the other hand, if \(N = 1\) we obtain the \(\mathcal{F}\)-transitivity notion. More specifically, an operator \(T\) is called \(\mathcal{F}\)-transitive.
operator (or $\mathcal{F}$-operator for short) whenever the set $N(U,V) := \{ n \geq 0 : T^n(U) \cap V \neq \emptyset \}$ is in $\mathcal{F}$. This notion was introduced and studied in [8].

Recall that an operator $T$ is said to be \textit{chaotic} if it is hypercyclic and has a dense set of periodic points ($x \in X$ is a \textit{periodic point} of $T$ if $T^k x = x$ for some $k \geq 1$).

An operator $T$ is said to be \textit{reiteratively hypercyclic} if there exists $x \in X$ such that for any non-empty open set $U$ in $X$, the set $N(x,U) = \{ n \geq 0 : T^n x \in U \}$ has positive upper Banach density, where the upper Banach density of a set $A \subseteq \mathbb{N}$ is given by

$$\overline{Bd}(A) = \lim_{s \to \infty} \frac{\alpha^s}{s},$$

and $\alpha^s = \limsup_{k \to \infty} |A \cap [k+1, k+s]|$, for any $s \geq 1$. Reiteratively hypercyclic operators have been studied in [7] and [12].

It is known that there exists a reiteratively hypercyclic operator which is not chaotic, see [1]. However, concerning the converse we have the following result due to Menet.

\textbf{Theorem 1.3.} \textit{Theorem 1.1 [11]} Any chaotic operator is reiteratively hypercyclic.

Recall that a set $A \subseteq \mathbb{N}$ is \textit{syndetic} set if $A$ has bounded gaps, i.e. if $A$ is enumerated increasingly as $(x_n)_n = A$, then $\max_n x_{n+1} - x_n < M$, for some $M > 0$.

In [5], the authors show that there exists a mixing operator $T$ on a Hilbert space such that $(T, T^2)$ is not $d$-mixing. We show that the same operator satisfies more specific properties.

\textbf{Theorem 1.4.} There exists $T \in \mathcal{L}(l^2)$ such that $T$ is mixing, chaotic and $(T, T^2)$ is not $d$-syndetic.

So, our result improves the result of [5] already mentioned but still does not answer Question [1]. In answering the question, we will see Szemerédi’s famous theorem unexpectedly playing an important role. Indeed, using a result of ergodic Ramsey theory due to Bergelson and McCutcheon [3], which is in fact a kind of Szemerédi’s theorem for generalized polynomials we obtain the following result.

\textbf{Theorem 1.5.} \textit{If} $T$ \textit{is reiteratively hypercyclic then} $(T, \ldots, T^r)$ \textit{is} $d$-syndetic or not $d$-transitive, \textit{for any} $r \in \mathbb{N}$.

Now, by Theorem [1.3] Theorem [1.4] and Theorem [1.5] we can deduce our main result which gives a positive answer to Question [1.1].

\textbf{Theorem 1.6.} \textit{There exists a mixing and chaotic operator} $T$ \textit{in} $\mathcal{L}(l^2)$ \textit{such that} $(T, T^2)$ \textit{is not} $d$-transitive.
2 Proof of Theorem 1.6

As already mentioned, in order to prove Theorem 1.6 it is enough to prove Theorem 1.4 and Theorem 1.5.

2.1 Proof of Theorem 1.4

In Theorem 3.8 [5] the authors show an example of a mixing Hilbert space operator $T$ such that $(T, T^2)$ is not $d$-mixing. We will show that in addition $T$ is chaotic and $(T, T^2)$ is not $d$-syndetic. So in particular it is not $d$-mixing. We follow exactly the same sketch of proof of Theorem 3.8 [5] introducing minor modifications for our convenience. Nevertheless we describe here all the details for the sake of completeness.

Let $1 \leq p < \infty, -\infty < a < b < +\infty$ and $k \in \mathbb{N}$. Recall that the Sobolev space $W^{k,p}[a, b]$ is the space of functions $f \in C^{k-1}[a, b]$ such that $f^{(k-1)}$ is absolutely continuous and $f^{(k)} \in L^p[a, b]$. The space $W^{k,p}[a, b]$ endowed with the norm

$$||f||_{W^{k,p}[a,b]} = \left( \int_a^b \left( \sum_{j=0}^k |f^{(j)}(x)|^p \right) dx \right)^{1/p}$$

is a Banach space isomorphic to $L^p[0, 1]$. Now, $W^{k,2}[a, b]$ is a separable infinite-dimensional Hilbert space for each $k \in \mathbb{N}$. The family of operators to be considered lives on separable complex Hilbert spaces and is built from a single operator. Let $M \in \mathcal{L}(W^{2,2}[-\pi, \pi])$ be defined by the formula

$$M : W^{2,2}[-\pi, \pi] \to W^{2,2}[-\pi, \pi], \quad Mf(x) = e^{ix}f(x). \quad (2.1)$$

Denote $\mathcal{H} = W^{2,2}[-\pi, \pi]$ and $M^*$ the dual operator. Then, $M^* \in \mathcal{L}(\mathcal{H}^*)$. For each $t \in [-\pi, \pi], \delta_t \in \mathcal{H}^*$, where $\delta_t : \mathcal{H} \to \mathbb{C}, \delta_t(f) = f(t)$. Furthermore, the map $t \to \delta_t$ from $[-\pi, \pi]$ to $\mathcal{H}^*$ is norm-continuous. For a non-empty compact subset $K$ of $[-\pi, \pi]$, denote

$$X_K = \overline{\text{span}\{\delta_t : t \in K\}}$$

where the closure of $\text{span}\{\delta_t : t \in K\}$ is taken with respect to the norm of $\mathcal{H}^*$.

Now, the functionals $\delta_t$ are linearly independent, $X_K$ is always a separable Hilbert space and $X_K$ is infinite dimensional if and only if $K$ is infinite. The following condition holds

$$M^*\delta_t = e^{it}\delta_t, \quad \text{for each } t \in [-\pi, \pi].$$

Hence, each $X_K$ is an invariant subspace for $M^*$, which allows us to consider the operator

$$Q_K \in \mathcal{L}(X_K), \quad Q_K = M^*|_{X_K}.$$ The following is taken from [5] and tells us when $Q_K$ is mixing or non-transitive, we omit the proof.
Proposition 2.1. Proposition 3.9 [5]

Let $K$ be a non-empty compact subset of $[-\pi, \pi]$. If $K$ has no isolated points, then $Q_K$ is mixing. If $K$ has an isolated point, the $Q_K$ is non-transitive.

Now, consider the set

$$K = \left\{ \sum_{n=1}^{\infty} 2\pi \epsilon_n \cdot \frac{1}{2^n} : \epsilon \in \{0, 1\}^\mathbb{N} \right\}.$$  \hspace{1cm} (2.2)

then the operator $Q_K \in \mathcal{L}(X_K)$ is the one pointed out in [5] to be mixing such that $(Q_K, Q_K^2)$ is not $d$-mixing. In addition, we show that $Q_K$ is chaotic and that $(Q_K, Q_K^2)$ is not $d$-syndetic. This is the content of the next results.

Lemma 2.2. Let $K$ be the compact subset of $[-\pi, \pi]$ defined in (2.2), then $Q_K$ is chaotic.

Proof. The operator $Q_K$ is mixing by Proposition 2.1; hence it remains to show that it has a dense set of periodic points. Denote by $\text{Per}(Q_K)$ the set of periodic points of $Q_K$.

We would like to recall that $Q_K^n \delta_t = e^{int} \delta_t$, for any $n \in \mathbb{Z}_+$ and $t \in K$, the details can be found in the proof of Proposition 3.9 [5].

Consider the set $A = \left\{ \sum_{n=1}^{k} 2\pi \epsilon_n / 2^n : \epsilon \in \{0, 1\}^{1, \ldots, k}, k \in \mathbb{N} \right\}$.

Observe that $\sum_{n=1}^{k} 2\pi \epsilon_n / 2^n = 2\pi m / 2^n$, for some $m$ and any $\epsilon \in \{0, 1\}^{1, \ldots, k}$. So, clearly $\{\delta_t : t \in A\} \subseteq \text{Per}(Q_K)$. Moreover, if $r_1 = 2\pi m_1 / 2^{n_1} \in A$, $r_2 = 2\pi m_2 / 2^{n_2} \in A$, then $Q_K^{n_1} \cdot \cdot \cdot Q_K^{n_2} (\alpha_1 \delta_{r_1} + \alpha_2 \delta_{r_2}) = \alpha_1 \delta_{r_1} + \alpha_2 \delta_{r_2}$ for any $\alpha_1, \alpha_2 \in \mathbb{C}$, so span$\{\delta_t : t \in A\} \subseteq \text{Per}(Q_K)$.

On the other hand, since $A$ is dense in $K$, we deduce that $\{\delta_t : t \in A\} = \{\delta_t : t \in K\}$. Indeed, for any $r \in K$ there exists a sequence $(r_n) \subseteq A$ such that $r_n$ tends to $r$. Hence, $\|\delta_t - \delta_{r_n}\| = \sup_{\|f\|=1} |f(r) - f(r_n)|$ tends to 0.

Hence, $X_K = \text{span}\{\delta_t : t \in K\} = \text{span}\{\delta_t : t \in A\} = \text{span}\{\delta_t : t \in A\} \subseteq \text{Per}(Q_K)$. So, Per$(Q_K)$ is dense in $X_K$. \hfill \Box

The set $A \subset \mathbb{N}$ is thick if $A$ contains arbitrarily long intervals, i.e. for every $L > 0$ there exists $n \geq 1$ such that $\{n, n + 1, \ldots, n + L\} \subset A$.

Now, in order to obtain a mixing operator $T$ such that $(T, T^2)$ is not $d$-syndetic, it will be enough to show that the sequence of operators $(2Q_K^{a_n} - Q_K^{2a_n})_n$ is non-transitive along a thick set $A = (a_n)$. We have the following result.

Proposition 2.3. Let $K$ be the compact subset of $[-\pi, \pi]$ defined in (2.2), then the sequence $(2Q_K^{a_n} - Q_K^{2a_n})_n$ of continuous linear operators on $X_K$ is non-transitive, where $k_{n,r} = 2^n - r$ with $0 \leq r \leq n, n \in \mathbb{N}$.

Now we are in position to prove Theorem 1.4.
We adopt the same sketch of proof of Theorem 3.8 [5], still we expose here all the details. We need to show a mixing and chaotic operator $T$ such that $(T, T^2)$ is not $d$-syndetic.

Let $K$ be the compact set defined in (2.2). By Proposition 2.1 and Lemma 2.2 $Q_K$ is mixing and chaotic operator on the separable infinite dimensional Hilbert space $X_K$. On the other hand, by Proposition 2.3 $(2Q_K^n - Q_K^{2n})_{n \in \mathbb{N}}$ is non-transitive for some thick set $A$ written increasingly as $A = (a_n)$. Hence, there exists non-empty open sets $U, V$ in $X_K$ such that $(2Q_K^n - Q_K^{2n})(U) \cap V = \emptyset$, for any $n \in \mathbb{N}$. In other words,

$$\{ n \in \mathbb{N} : (2Q_K^n - Q_K^{2n})(U) \cap V \neq \emptyset \} \cap A = \emptyset,$$

i.e. the set $\{ n \in \mathbb{N} : (2Q_K^n - Q_K^{2n})(U) \cap V \neq \emptyset \}$ cannot be syndetic. In particular, $(Q_K, Q_K^2)$ is not $d$-syndetic. Indeed, pick a non-empty open set $V_0$ such that $2V_0 - V_0 \subseteq V$ (denote $B(x; r)$ the open ball centered at $x$ in $X_K$ with radius $r$). Pick $x \in X_K, r \in \mathbb{R}_+$ such that $B(x; r) \subseteq V$, then set $V_0 := B(x; r/3)$. Hence,

$$\{ n \in \mathbb{N} : U \cap Q_K^{-n}(V_0) \cap Q_K^{-2n}(V_0) \neq \emptyset \} \subseteq \{ n \in \mathbb{N} : (2Q_K^n - Q_K^{2n})(U) \cap V \neq \emptyset \}.$$

Consequently, $\{ n \in \mathbb{N} : U \cap Q_K^{-n}(V_0) \cap Q_K^{-2n}(V_0) \neq \emptyset \}$ cannot be a syndetic set and then $(Q_K, Q_K^2)$ is not $d$-syndetic. Since all separable infinite dimensional Hilbert spaces are isomorphic to $l^2$, there is a mixing and chaotic $T \in L(l^2)$ such that $(T, T^2)$ is not $d$-syndetic. This concludes the proof of Theorem 1.4.

In order to close this subsection, we need to prove Proposition 2.3 which follows exactly the same sketch of proof of Proposition 3.10 [5], except that instead of using Lemma A.3 [5] as the authors of [5] did, we just use Lemma 2.6 (below) in an analogous way. So it suffices to give the proof of Lemma 2.6.

Now, in order to prove Lemma 2.6 we need to quote another two lemmas proved in [5] that we state without proof.

**Lemma 2.4.** Lemma A.1 [5]

Let $f \in W^{2,2}[-\pi, \pi], f(-\pi) = f(\pi), f'(-\pi) = f'(\pi), c_0 = \|f\|_{L^\infty[-\pi, \pi]}$ and $c_1 = \|f''\|_{L^2[-\pi, \pi]}$. Then $\|f\|_{W^{2,2}[-\pi, \pi]} \leq \sqrt{3c_0^2 + c_1^2}$.

**Lemma 2.5.** Lemma A.2 [5]

Let $-\infty < \alpha < \beta < \infty$ and $a_0, a_1, b_0, b_1 \in \mathbb{C}$. Then there exists $f \in C^2[\alpha, \beta]$ such that

$$f(\alpha) = a_0, \quad f'(\alpha) = a_1, \quad f(\beta) = b_0, \quad f'(\beta) = b_1,$$

$$\|f\|_{L^\infty[\alpha, \beta]} \leq |a_0 + b_0|/2 + |a_0 - b_0|/2 + (\beta - \alpha)(|a_1| + |b_1|)/5,$$

$$\|f''\|_{L^2[\alpha, \beta]}^2 \leq \frac{24|a_0 - b_0|^2}{(\beta - \alpha)^3} + \frac{12 \cdot (|a_1|^2 + |b_1|^2)}{\beta - \alpha}.$$
Lemma 2.6. There exists a sequence \( (f_{2^n - r})_{n \in \mathbb{N}, 0 \leq r \leq n} \) of \( 2\pi \)-periodic functions on \( \mathbb{R} \) such that \( f_{2^n - r} \) is bounded and \( f_{2^n - r}(x) = e^{i(2^n - r)x} - e^{2i(2^n - r)x} \) whenever \( |x - \frac{2\pi m}{2^n}| \leq 2/(2^n)^5 \), for some \( m \in \mathbb{Z} \) and every \( n \in \mathbb{N}, 0 \leq r \leq n \).

Proof. We obtain the proof of this lemma doing convenient slight modifications in the proof of Lemma A.3 [5].

For \( n \in \mathbb{N}, 0 \leq r \leq n \), let \( k_{n,r} = 2^n - r \) and \( h_{k_{n,r}} = e^{i k_{n,r} x} - e^{2i k_{n,r} x} \).

Note that \( h_{k_{n,r}} \) is periodic with period \( 2\pi / k_{n,r} \). Let also \( \alpha_{n,r} = \frac{2}{2^n} \) and \( \beta_{n,r} = -2/(2^n)^5 \). By Lemma 2.5 there is \( g_{k_{n,r}} \in C^2[\alpha_{n,r}, \beta_{n,r}] \) such that

\[
g_{k_{n,r}}(\alpha_{n,r}) = h_{k_{n,r}}(2/(2^n)^5), \quad g_{k_{n,r}}(\beta_{n,r}) = h_{k_{n,r}}(-2/(2^n)^5),
\]

\[
\|g_{k_{n,r}}\|_{L^2[\alpha_{n,r}, \beta_{n,r}]} \leq \max\{|h_{k_{n,r}}(2/(2^n)^5)|, |h_{k_{n,r}}(-2/(2^n)^5)|\}
\]

\[
+ \frac{(\beta_{n,r} - \alpha_{n,r})}{5} \left( |h_{k_{n,r}}'(2/(2^n)^5)| + |h_{k_{n,r}}'(-2/(2^n)^5)| \right), \tag{2.4}
\]

\[
\|g''_{k_{n,r}}\|^2_{L^2[\alpha_{n,r}, \beta_{n,r}]} \leq 24 \left| h_{k_{n,r}}(2/(2^n)^5) - h_{k_{n,r}}(-2/(2^n)^5) \right|^2
\]

\[
+ 12 \frac{|h_{k_{n,r}}'(2/(2^n)^5)|^2 + |h_{k_{n,r}}'(-2/(2^n)^5)|^2}{(\beta_{n,r} - \alpha_{n,r})^3}. \tag{2.5}
\]

The equalities (2.3) imply that there is a unique \( f_{k_{n,r}} \in C^1(\mathbb{R}) \) such that \( f_{k_{n,r}} \) is periodic with period \( 2\pi/2^n \), \( f_{k_{n,r}} |_{(\alpha_{n,r}, \beta_{n,r})} = g_{k_{n,r}} \) and \( f_{k_{n,r}} |_{[\beta_{n,r}, \alpha_{n,r} + 2\pi/2^n]} = h_{k_{n,r}} \).

Periodicity of \( f_{k_{n,r}} \) with period \( 2\pi/2^n \) and the equality \( f_{k_{n,r}} |_{[\beta_{n,r}, \alpha_{n,r} + 2\pi/2^n]} = h_{k_{n,r}} \) imply that \( f_{k_{n,r}}(x) = e^{i(2^n - r)x} - e^{2i(2^n - r)x} \) whenever \( |x - \frac{2\pi m}{2^n}| \leq 2/(2^n)^5 \), for every \( m \in \mathbb{Z} \) with \( |m| \leq 2^n \) and every \( n \in \mathbb{N}, 0 \leq r \leq n \).

Since \( f_{k_{n,r}} \) is piecewise \( C^2 \), \( f_{k_{n,r}} |_{[-\pi, \pi]} \in W^{2,2}[-\pi, \pi] \). It remains to verify that the sequence \( (\|f_{k_{n,r}}\|_{W^{2,2}[-\pi, \pi]})_{n,r} \) is bounded.

Using the inequality \( |e^{it} - e^{is}| \leq |t - s| \) for \( t, s \in \mathbb{R} \), we have

\[
|h_{k_{n,r}}'(2/(2^n)^5)| = |h_{k_{n,r}}'(-2/(2^n)^5)| \leq 2(2^n - r)^2 \cdot 2/(2^n)^5.
\]

Hence by (2.4),

\[
\|f_{k_{n,r}}\|_{L^\infty_{(\alpha_{n,r}, \beta_{n,r})}} \leq 3 + 5^{-1} \left( \frac{2\pi}{2^n} - \frac{4}{(2^n)^5} \right) \cdot \frac{8(2^n - r)^2}{(2^n)^5} < 9.
\]

Since \( \|h_{k_{n,r}}\|_{L^\infty_{[\beta_{n,r}, \alpha_{n,r} + 2\pi/2^n]}} \leq 3 \) and \( f_{k_{n,r}} \) is \( 2\pi/2^n \)-periodic, we obtain,

\[
\|f_{k_{n,r}}\|_{L^\infty_{[-\pi, \pi]}} \leq \max\{3, 9\} = 9. \tag{2.6}
\]
Next,
\[
|h_{k,n,r}(2/(2^6)^5) - h_{k,n,r}(-2/(2^6)^5)| =
\left|2(e^{i(2^6 - r)/(26^n)^5} - e^{i(2^6 - r)(-2)/(26^n)^5}) - (e^{2i(2^6 - r)/(26^n)^5} - e^{2i(2^6 - r)(-2)/(26^n)^5})\right| =
\]
\[
4\sin\left(2 \cdot \frac{(2^6 - r)}{(26^n)^5}\right) - 2\sin\left(4 \cdot \frac{(2^6 - r)}{(26^n)^5}\right) =
4\sin \left(2 \cdot \frac{(2^6 - r)}{(26^n)^5}\right) \left(1 - \cos \left(2 \cdot \frac{(2^6 - r)}{(26^n)^5}\right)\right) =
16\sin^3 \left(\frac{(2^6 - r)}{(26^n)^5}\right) \cos \left(\frac{(2^6 - r)}{(26^n)^5}\right) \leq
16 \left(\frac{(2^6 - r)}{(26^n)^5}\right)^3 \leq \frac{16}{(26^n)^{12}}.
\]

On the other hand,
\[
\left|\frac{h'_{k,n,r}(2/(2^6)^5)}{(\beta_{n,r} - \alpha_{n,r})}\right|^2 + \left|\frac{h'_{k,n,r}(-2/(2^6)^5)}{(\beta_{n,r} - \alpha_{n,r})}\right|^2 \leq \frac{32}{(26^n)^3} \left(\frac{2\pi}{26^n} - \frac{4}{(26^n)^5}\right) \leq
\frac{32}{(26^n)^3} \frac{2\pi(26^n)^5 - 4 \cdot 26^n}{2\pi(26^n)^5 - 4(26^n)^5} \leq \frac{16}{(26^n)^3}.
\]

Hence by (2.5),
\[
\|f''_{k,n,r}\|_{L^2[\alpha_{n,r}, \beta_{n,r}]}^2 \leq 24 \cdot \frac{16}{(2^6)^{12}} \frac{2\pi}{2^n} - \frac{4}{(2^6)^3} \frac{16}{(2^6)^5} \leq
24 \cdot \frac{16^2 \cdot (2^6)^{-24}}{\left(\frac{2\pi}{2^6} - \frac{4}{2^6}\right)^3} + 12 \cdot \frac{16}{(2^6)^5} \leq
24 \cdot \frac{16^2 \cdot (2^6)^{-24}}{\left(\frac{2\pi}{2^6} - \frac{4}{2^6}\right)^3} + 12 \cdot \frac{16}{(2^6)^5} \leq
\frac{24 \cdot 16^2}{8 \cdot (2^6)^{21}} + \frac{12 \cdot 16}{(2^6)^5} \leq \frac{960}{(2^6)^5}.
\]

Since \(|h''_{k,n,r}(x)| \leq 6(\pi/26^n)^2\) for \(x \in [\beta_{n,r}, \alpha_{n,r} + 2\pi/26^n]\) we have,
\[
\|f''_{k,n,r}\|_{L^2[\alpha_{n,r}, \alpha_{n,r} + 2\pi/26^n]}^2 \leq 36 \cdot (2^6 - r)^4 \cdot \frac{4}{(2^6)^5} \leq \frac{144}{26^n}.
\]

Hence,
\[
\|f''_{k,n,r}\|_{L^2[\alpha_{n,r}, \alpha_{n,r} + 2\pi/26^n]}^2 \leq \frac{960}{(26^n)^3} + \frac{144}{26^n} \leq \frac{1104}{26^n}.
\]
Since $f''_{kn,r}$ is $2\pi/2^n$-periodic then
\[
\|f''_{kn,r}\|_{L^2[-\pi,\pi]}^2 = 2^{6n} : \|f''_{kn,r}\|_{L^2[\alpha_{n,r},\alpha_{n,r}+2\pi/2^n]}^2 \leq 1104. \tag{2.7}
\]
Now, by Lemma 2.4 and using (2.7) and (2.6) we obtain
\[
\|f_{kn,r}\|_{W^{2,2}[-\pi,\pi]} \leq \sqrt{3} \cdot 1104 + 92 < 64,
\]
for each $n \in \mathbb{N}$, $0 \leq r \leq n$. \hfill \Box

2.2 Proof of Theorem 1.5

The main ingredient of the proof of Theorem 1.5 is a result due to Bergelson and McCutcheon concerning essential idempotent s of $\beta \mathbb{N}$ (the Stone-Čech compactification of $\mathbb{N}$), and Szemerédi’s theorem for generalized polynomials [3]. So, we need first some background on $\beta \mathbb{N}$.

Recall that a filter is a family that is invariant by finite intersections, i.e. $\mathcal{F}$ is a family such that for any $A \in \mathcal{F}, B \in \mathcal{F}$ implies $A \cap B \in \mathcal{F}$. The collection of all maximal filters (in the sense of inclusion) is denoted by $\beta \mathbb{N}$. Elements of $\beta \mathbb{N}$ are known as ultrafilters; endowed with an appropriate topology, $\beta \mathbb{N}$ becomes the Stone-Čech Compactification of $\mathbb{N}$. Each point $i \in \mathbb{N}$ is identified with a principal ultrafilter $\mathcal{U}_i := \{A \subseteq \mathbb{N} : i \in A\}$ in order to obtain an embedding of $\mathbb{N}$ into $\beta \mathbb{N}$. For any $A \subseteq \mathbb{N}$ and $p \in \beta \mathbb{N}$, the closure of $A$, $clA$ in $\beta \mathbb{N}$ is defined as follows, $p \in clA$ if and only if $A \in p$. Given $p, q \in \beta \mathbb{N}$ and $A \subseteq \mathbb{N}$, the operation $(\mathbb{N}, +)$ can be extended to $\beta \mathbb{N}$ in such a way as to make $(\beta \mathbb{N}, +)$ a compact right topological semigroup. The extended operation can be defined as $A \in p + q$ if and only if $\{n \in \mathbb{N} : -n + A \in q\} \in p$. Now, according to a famous theorem of Ellis, idempotents (with respect to $+$) exist. Let $E(\mathbb{N}) = \{p \in \beta \mathbb{N} : p = p+p\}$ be the collection of idempotents in $\beta \mathbb{N}$. For further details see [10]. Given a family $\mathcal{F}$, the dual family $\mathcal{F}^*$ consists of all sets $A$ such that $A \cap F \neq \emptyset$, for every $F \in \mathcal{F}$. The following is a well-known result.

Lemma 2.7. (1) If $\mathcal{F}$ is an ultrafilter, then $\mathcal{F}^* = \mathcal{F}$.

(2) If $\mathcal{F} = \cup_{\alpha} \mathcal{F}_{\alpha}$, then $\mathcal{F}^* = \cap_{\alpha} \mathcal{F}_{\alpha}^*$.

In particular, whenever $\mathcal{F}$ is a union of some collection of ultrafilters, then $\mathcal{F}^*$ is the intersection of the same collection of ultrafilters.

The collection of essential idempotents is commonly referred to in the literature as $\mathcal{D}$.

The collection $\mathcal{D}$ (of $D$-sets) is the union of all idempotents $p \in \beta \mathbb{N}$ such that every member of $p$ has positive upper Banach density. Accordingly, $\mathcal{D}^*$ is the intersection of all such idempotents.

The following is a result of ergodic Ramsey theory due to Bergelson and McCutcheon [3]. It is indeed a sort of Szemerédi’s theorem stated originally for generalized polynomials and it will be crucial for proving Theorem 1.5.
Theorem 2.8. Theorem 1.25 [3] Let $F \subset \mathbb{N}$ have positive upper Banach density and $g_1, \ldots, g_r$ be polynomials, then

$$\left\{ k \in \mathbb{N} : \overline{d}(F \cap (F - g_1(k)) \cap \cdots \cap (F - g_r(k))) > 0 \right\} \in \mathcal{D}^*.$$

We can now prove Theorem 1.5.

Fix $r \in \mathbb{N}$. Let $T$ be reiteratively hypercyclic, then there exists $x \in X$ such that $\overline{d}(N(x, U)) > 0$, for any non-empty open set $U$ in $X$. First, let us see that

$$N_T(U, \ldots, U; U) = \{ k \geq 0 : T^{-k}U \cap \cdots \cap T^{-r}U \cap U \neq \emptyset \} \in \mathcal{D}^* \quad (2.8)$$

for any non-empty open set $U$ in $X$. Let $U$ non-empty open set, then

$$A_U := \{ k \geq 0 : \overline{d}\left(N(x, U) \cap (N(x, U) - k) \cap \cdots \cap (N(x, U) - rk)\right) > 0 \}$$

$$\subseteq \{ k \geq 0 : T^{-k}U \cap \cdots \cap T^{-rk}U \cap U \neq \emptyset \}.$$ 

In fact, let $k \in A_U$, then there exists a set $A$ with positive upper Banach density such that for any $n \in A$ it holds $T^{n+ik}x \in U$, for any $i \in \{0, \ldots, r\}$. Consequently, $T^n x \in T^{-k}U \cap \cdots \cap T^{-rk}U \cap U$. Now, by Theorem 2.8 it follows that $A_U \in \mathcal{D}^*$. Thus condition (2.8) holds.

Next, let $(U_j)_{j=0}^r$ be a finite sequence of non-empty open sets in $X$. Now, suppose that $(T, \ldots, T^r)$ is $d$-transitive, we must show that $N_T(U_1, \ldots, U_r; U_0)$ is a syndetic set. In fact, there exists $n \in \mathbb{N}$ such that

$$V_n := T^{-n}U_1 \cap \cdots \cap T^{-rn}U_r \cap U_0 \neq \emptyset.$$ 

Thus $V_n$ is open, then pick $O_1, O_2$ non-empty open sets such that $O_1 + O_2 \subset V_n$, then

$$T^{jn}(O_1 + O_2) \subset U_j, \quad \text{for any } j \in \{0, \ldots, r\}. \quad (2.9)$$

It is known that $\mathcal{D}^*$ is a filter. Now, by (2.8) we have

$$A := N_{O_1, \ldots, O_1; O_1} \cap N_{O_2, \ldots, O_2; O_2} \in \mathcal{D}^*.$$ 

In addition, it is well known that each set in $\mathcal{D}^*$ is indeed syndetic [4]. Hence, $A$ is syndetic. Let us show that $A + n \subseteq N(U_1, \ldots, U_r; U_0)$, then we are done because $A + n$ is syndetic, since the collection of syndetic sets is shift invariant.

In fact, let $t \in A + n$, then $t - n \in A$, which means

$$T^{-t}T^n(O_1) \cap \cdots \cap T^{-rt}T^n(O_1) \cap O_1 \neq \emptyset$$

$$T^{-t}T^n(O_2) \cap \cdots \cap T^{-rt}T^n(O_2) \cap O_2 \neq \emptyset.$$ 

By the linearity of $T$ we obtain

$$T^{-t}(T^n(O_1 + O_2)) \cap \cdots \cap T^{-rt}(T^n(O_1 + O_2)) \cap (O_1 + O_2) \neq \emptyset.$$ 

Then we conclude by (2.9), i.e.

$$T^{-t}U_1 \cap \cdots \cap T^{-rt}U_r \cap U_0 \neq \emptyset.$$ 

This concludes the proof of Theorem 1.5.
3 Tuple of powers of a weighted shift

In linear dynamics recurrence properties are frequently studied first in the context of weighted backward shifts.

Each bilateral bounded weight \( w = (w_k)_{k \in \mathbb{Z}} \) induces a bilateral weighted backward shift \( B_w \) on \( X = c_0(\mathbb{Z}) \) or \( l^p(\mathbb{Z})(1 \leq p < \infty) \), given by \( B_w e_k := w_k e_{k-1} \), where \((e_k)_{k \in \mathbb{Z}}\) denotes the canonical basis of \( X \).

Analogously, each unilateral bounded weight \( w = (w_n)_{n \in \mathbb{Z}^+} \) induces a unilateral weighted backward shift \( B_w \) on \( X = c_0(\mathbb{Z}^+) \) or \( l^p(\mathbb{Z}^+)(1 \leq p < \infty) \), given by \( B_w e_n := w_n e_{n-1}, n \geq 1 \) with \( B_w e_0 := 0 \), where \((e_n)_{n \in \mathbb{Z}^+}\) denotes the canonical basis of \( X \).

As previously mentioned, the authors of [5] proved that for any weighted shift \( B_w \), the following holds:

\[ B_w \text{ is mixing if and only if } (B_w, \ldots, B_rw) \text{ is } d\text{-mixing for any } r \in \mathbb{N}. \]

The aim of this section is to show that this result extends to those families on \( \mathbb{N} \) frequently studied in Ramsey theory.

Let us summarize some families commonly used in Ramsey theory.

- \( \mathcal{I} = \{ A \subseteq \mathbb{N} : A \text{ is infinite} \} \);
- \( \Delta = \{ A \subseteq \mathbb{N} : B - B \subseteq A, \text{for some infinite set } B \} \);
- \( \mathcal{IP} = \{ A \subseteq \mathbb{N} : \exists (x_n)_{n \in \mathbb{N}}, \sum_{n \in F} x_n \in A, \text{for any finite set } F \} \);
- The set \( A \) is piecewise syndetic (\( A \in \mathcal{PS} \) for short) if \( A \) can be written as the intersection of a thick and a syndetic set.

It is known that \( \mathcal{I}^* \) (family of cofinite sets), \( \Delta^*, \mathcal{IP}^* \) and \( \mathcal{PS}^* \) are filters. In addition, \( \mathcal{I}^* \not\subset \Delta^* \subset \mathcal{IP}^* \subset \mathcal{PS}^* \subset \mathcal{S} \), where \( \mathcal{S} \) denote the family of syndetic sets. For a rich source on this subject we refer the reader to [10].

The main result of this section is the following.

**Theorem 3.1.** Let \( \mathcal{F} \) be the family \( \Delta^*, \mathcal{IP}^*, \mathcal{PS}^* \) or \( \mathcal{S} \) then for any \( r \in \mathbb{N} \) the following are equivalent:

(i) \( T \) is an \( \mathcal{F} \)-operator;
(ii) \( T \oplus \cdots \oplus T^r \) is an \( \mathcal{F} \)-operator on \( X^r \).

In particular, a bilateral (unilateral) weighted backward shift \( B_w \) on \( c_0 \) or \( l^p(1 \leq p < \infty) \) is an \( \mathcal{F} \)-operator if and only if \( (B_w, \ldots, B_rw) \) is \( d\mathcal{F} \).

**Remark 3.2.** Obviously, mixing operators are \( \Delta^\ast \)-operators, but the converse is not true as exhibited in [8] and the example is a weighted shift. Therefore, the conclusion of Theorem 3.1 concerning weighted shifts does not necessarily follows from the statement: \( B_w \) is mixing if and only if \( (B_w, \ldots, B_rw) \) is \( d\mathcal{F} \), for any \( r \in \mathbb{N} \), shown in [5].

In order to prove Theorem 3.1 we will need the following results.

Recall that any tuple of powers of a fixed backward weighted shift on \( c_0 \) or \( l^p \) is \( d \)-transitive if and only if it is \( d \)-hypercyclic. This follows by Theorem 2.7 [6] and Theorem 4.1 [6]. Now, combining Theorem 4.1 [6] and
Theorem 2.5 [13] in its bilateral (unilateral) version, we obtain the following two propositions below.

**Proposition 3.3.** Let \( X = c_0(\mathbb{Z}) \) or \( l^p(\mathbb{Z})(1 \leq p < \infty) \), \( w = (w_j)_{j \in \mathbb{Z}} \) a bounded bilateral weight sequence, \( \mathcal{F} \) a filter on \( \mathbb{N} \) and \( r_0 = 0 < 1 \leq r_1 < \cdots < r_N \), then the following are equivalent:

(i) \( (B_{r_1}^1, \ldots, B_{r_N}^N) \) is \( d \)-\( \mathcal{F} \),

(ii) \( \oplus_{0 \leq s < l \leq N} B_{s}^{(r_l-r_s)} \) is \( \mathcal{F} \)-operator on \( X^{N(N+1)/2} \),

(iii) for any \( M > 0, j \in \mathbb{Z} \) and \( 0 \leq s < l \leq N \) it holds

\[
\left\{ m \in \mathbb{N} : \prod_{i=j+1}^{j+m(r_l-r_s)} |w_i| > M \right\} \in \mathcal{F},
\]

\[
\left\{ m \in \mathbb{N} : \frac{1}{\prod_{i=j-m(r_l-r_s)+1}^{j+m(r_l-r_s)} |w_i|} < M \right\} \in \mathcal{F}.
\]

**Proposition 3.4.** Let \( X = c_0(\mathbb{Z}_+) \) or \( l^p(\mathbb{Z}_+)(1 \leq p < \infty) \), \( w = (w_n)_{n \in \mathbb{Z}_+} \) a bounded unilateral weight sequence, \( \mathcal{F} \) a filter on \( \mathbb{N} \) and \( r_0 = 0 < 1 \leq r_1 < \cdots < r_N \),then the following are equivalent:

(i) \( (B_{r_1}^1, \ldots, B_{r_N}^N) \) is \( d \)-\( \mathcal{F} \),

(ii) \( \oplus_{0 \leq s < l \leq N} B_{s}^{(r_l-r_s)} \) is \( \mathcal{F} \)-operator on \( X^{N(N+1)/2} \),

(iii) for any \( M > 0, j \in \mathbb{Z}_+ \) and \( 0 \leq s < l \leq N \) it holds

\[
\left\{ m \in \mathbb{N} : \prod_{i=j+1}^{j+m(r_l-r_s)} |w_i| > M \right\} \in \mathcal{F}.
\]

The following results of Ramsey theory concern the preservation of certain notions of largeness in products.

**Proposition 3.5.** Corollary 2.3 [2] Let \( l \in \mathbb{N} \) and \( I \) be a subsemigroup of \( \mathbb{N}^l \),

a) if \( B \) is an \( IP^* \) set in \( \mathbb{N} \), then \( B^l \cap I \) is an \( IP^* \) set in \( I \)

b) if \( B \) is an \( \Delta^* \) set in \( \mathbb{N} \), then \( B^l \cap I \) is an \( \Delta^* \) set in \( I \).

**Proposition 3.6.** Corollary 2.7 [2] Let \( l \in \mathbb{N} \) and \( I \) be a subsemigroup of \( \mathbb{N}^l \),

a) if \( B \) is an \( PS^* \) set in \( \mathbb{N} \), then \( B^l \cap I \) is an \( PS^* \) set in \( I \).

We are now finally able to prove Theorem 3.1.

**Proof of Theorem 3.1.**

If \( T \oplus \cdots \oplus T^r \) is \( \mathcal{F} \)-operator on \( X^r \) for some \( r \in \mathbb{N} \), obviously \( T \) is \( \mathcal{F} \)-operator. Conversely, let \( T \) an \( \mathcal{F} \)-operator, \( r \in \mathbb{N} \) and \( U, V \) non-empty open sets, we need to show that \( N(U, V) \in t\mathcal{F} \), for any \( t = 1, \ldots, r \).

Denote,

\[
A = \{ m, 2m, \ldots, rm : m \in \mathbb{N} \} \cap (N(U, V) \times \cdots \times N(U, V)).
\]

(3.1)
By Proposition 3.5, we have that if $N(U,V)$ is $IP^*$-set ($\Delta^*$-set) in $\mathbb{N}$, then $A$ is $IP^*$-set ($\Delta^*$-set) in $\{m, 2m, \ldots, rm : m \in \mathbb{N}\}$. Analogously, by Proposition 3.6 we have that if $N(U,V)$ is $PS^*$-set in $\mathbb{N}$, then $A$ is $PS^*$-set in $\{m, 2m, \ldots, rm : m \in \mathbb{N}\}$.

Denote $\prod_i$ the projection onto the $i$-th coordinate. It is not difficult to see that $\prod_1(A) \in F$, for $F = \Delta^*, IP^*, PS^*$. Then, (3.1) is equivalent to say $B = \{m \in \mathbb{N} : tm \in N(U,V)\} \in F$, for any $t = 1, \ldots, r$.

Hence, $tB \subseteq N(U,V)$ and $B \in F$. Then $N(U,V) \subseteq tF$ for any $t = 1, \ldots, r$. Since $F = \Delta^*, IP^*, PS^*$; it is a filter, then it is not difficult to see that $T \oplus \cdots \oplus T^r$ is indeed an $F$-operator on $X^r$.

If $B_w$ is a weighted shift on $c_0$ or $l^p$ and $F = \Delta^*, IP^*, PS^*$; by Proposition 3.3 (Proposition 3.4), we have $B_w$ is an $F$-operator if and only if $(B_w, \ldots, B_w^r)$ is $d-F$ for any $r \in \mathbb{N}$.

Finally, let $F$ be the family of syndetic sets. Just recall that $T$ is syndetic operator if and only if $T$ is $PS^*$-operator [8]. Hence $T$ is syndetic operator if and only if $T \oplus \cdots \oplus T^r$ is $PS^*$-operator on $X^r$ for any $r \in \mathbb{N}$. If $B_w$ is a weighted shift then $B_w$ is syndetic operator if and only if $(B_w, \ldots, B_w^r)$ is $d-PS^*$ for any $r \in \mathbb{N}$. This concludes the proof of Theorem 3.1.

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