QCD parameters and $f_{B_c}$ from heavy quark sum rules*

Stephan Narison a,**

aLaboratoire Univers et Particules, CNRS-IN2P3, Case 070, Place Eugène Bataillon, 34095 - Montpellier Cedex 05, France.

Abstract

We report results of our recent works [1, 2] where we where the correlations between the $c, b$-quark running masses $\bar{m}_{c,b}$, the gluon condensate $\langle \alpha_sG^2 \rangle$ and the QCD coupling $\alpha_s$ in the $\overline{\text{MS}}$-scheme from an analysis of the charmonium and bottomium spectra and the $B_s$-meson mass. We use optimized ratios of relativistic Laplace sum rules (LSR) evaluated at the $\mu$-subtraction stability point where higher orders PT and $D \leq 6 \sim 8$-dimensions non-perturbative condensates corrections are included. We obtain [1] $\alpha_s(2.85) = 0.262(9)$ and $\alpha_s(9.50) = 0.180(8)$ from the (pseudo)scalar $M_{b(mb)} - M_{b(b)}$ mass-splittings at $\mu = 2.85(9.50)$ GeV. The most precise result from the charm channel leads to $\alpha_s(M_c) = 0.318(15)$ and $\alpha_s(M_Z) = 0.1183(19)(3)$ in excellent agreement with the world average: $\alpha_s(M_Z) = 0.1181(11)[3, 4]$. Updated results from a global fit of the (axial)-vector and (pseudo)scalar channels using Laplace and Moments sum rules @ N2LO [1] combined with the one from $M_B$ [2] lead to the new tentative QCD spectral sum rules (QSSR) average: $\bar{m}_c(\overline{m}_c)_{\text{average}} = 1266(6)$ MeV and $\bar{m}_b(\overline{m}_b)_{\text{average}} = 4196(8)$ MeV. The values of the gluon condensate $\langle \alpha_sG^2 \rangle$ from the (axial)-vector charmonium channels combined with previous determinations in Table 1, leads to the new QSSR average: $[1] \langle \alpha_sG^2 \rangle_{\text{average}} = (6.35 \pm 0.35) \times 10^{-2}$ GeV$^4$. Our results clarify the (apparent) discrepancies between different estimates of $\langle \alpha_sG^2 \rangle$ from $J/\psi$ sum rule but also shows the sensitivity of the sum rules on the choice of the $\mu$-subtraction scale. As a biproduct, we deduce the $B_c$-decay constants $f_{B_c} = 371(17)$ MeV and $f_{B_c}(2S) \leq 139(6)$ MeV.

Keywords: QCD spectral sum rules, Perturbative and Non-Perturbative calculations, Hadron and Quark masses, Gluon condensates, QCD coupling $\alpha_s$.

1. Introduction

Besides the importance of the QCD coupling $\alpha_s$ and the running heavy quark masse $\bar{m}_{c,b}$, the non-perturbative gluon condensates introduced by SVZ [5–7] play important rôle in gluodynamics and in the QCD spectral sum rules (QSSR) analysis where they enter as high-dimension operators in the OPE of the hadronic correlators. In particular, this is the case for the heavy quark systems and the pure Yang-Mills gluonia/glueball channels [8–10] where the light quark loops and condensates are absent to leading order. The heavy quark condensate contribution can be absorbed into the gluon one through the relation [5, 6]:

$$\langle \bar{Q}Q \rangle = -\langle \alpha_sG^2 \rangle/(12\pi M_Q) + \ldots$$

(1)

where a similar relation holds for the mixed heavy quark-gluon condensate $\langle \bar{Q}GQ \rangle$. $G$ is the short hand notation for the gluon field strength $G_{\mu\nu}^a$ and $M_Q$ is the pole mass. The SVZ original value [5, 6]:

$$\langle \alpha_sG^2 \rangle \approx 0.04 \text{ GeV}^4$$

(2)

extracted (for the first time) from charmonium sum rules [5, 6] has been challenged by different authors (for reviews, see e.g [11–14] and Table 1). One can see in Table 1 that the results from standard SVZ and FESR sum rules for heavy and light quark systems vary in a large range but all of them are positive numbers, while the
ones from analysis of the modified $\tau$-decays moments allow negative values. However, one should notice from the original QCD expression of the $\tau$-decay rate [53, 54] that the $\langle \alpha_s G^2 \rangle$ gluon condensate contribution is absent to leading order indicating that the original $\tau$-decay rate is a bad place for extracting a such quantity [55]. The presence of $\langle \alpha_s G^2 \rangle$ in the analysis of [43–46] is only an artifact of the high-moments where the systematic errors needs to be better controlled. Earlier lattice calculations indicate a non-zero positive value of $\langle \alpha_s G^2 \rangle$ [56–59] while recent estimates in Table 1 give positive values but about 2-7 times higher than the phenomenological estimates. However, the subtraction of the perturbative contribution in the lattice analysis which is scheme dependent is not yet well-understood [52] such that a direct comparison of the lattice results obtained at large orders of PT series with the ones from the truncated PT series used in the phenomenological analysis is quite delicate. These previous results indicate that $\langle \alpha_s G^2 \rangle$ is not yet well determined and motivate a reconsideration of its estimate.

A first step for the improvement of the estimate of the gluon condensate was the recent direct determination of the ratio of the dimension-six gluon condensate $\langle g^6 f_{abc} G^3 \rangle$ over the dimension-four one $\langle \alpha_s G^2 \rangle$ from the heavy quark systems with the value [22, 31, 60]:

$$\rho \equiv \frac{\langle g^6 f_{abc} G^3 \rangle}{\langle \alpha_s G^2 \rangle} = (8.2 \pm 1.0) \text{ GeV}^2,$$

which differs significantly from the instanton liquid model estimate [61–63] and may question the validity of the instanton liquid model approximation. Earlier lattice results in pure Yang-Mills found: $\rho \approx 1.2 \text{ GeV}^2$ [56–59] such that it is important to have new lattice results for this quantity. Note however, that the value given in Eq. 3 might also be an effective value of the unknown high-dimension condensates not taken into account in the analysis of [22, 31, 60] when requiring the fit of the data by the truncated OPE at that order in the extreme case where the OPE does not converge. We shall see that the effect of the $\langle g^6 f_{abc} G^3 \rangle$ term is a small correction at the stability region where the optimal results are extracted.

In this paper, we pursue a such program by reconsidering the extraction of the lowest dimension QCD parameters from the (axial-)vector and (pseudo)scalar charmonium and bottomonium spectra taking into account the correlations between $\alpha_s$, the gluon condensate $\langle \alpha_s G^2 \rangle$, and the $c, b$-quark running masses. We shall use these parameters for predicting the known masses of the (pseudo)scalar heavy quarkonia ground states and also re-extract $\alpha_s$ and $\langle \alpha_s G^2 \rangle$ from the mass-splittings $M_{\chi_b(1P)} - M_{\psi(2S)}$. In so doing, we shall work with the example of the QCD Laplace sum rules (LSR) where the corresponding Operator Product Expansion (OPE) in terms of condensates is more convergent than the moments evaluated at small momentum.

2. The heavy quarkonia Laplace sum rules (LSR)

Table 1: Selected determinations of $\langle \alpha_s G^2 \rangle$ in units of $10^5 \text{ GeV}^4$ from charmonium, bottomium and light quark sum rules (SR). The numbers marked with * are not included in the average. This average take into account the new results obtained in [1] from Exponential / Laplace Sum Rules (LSR). Estimates from variants of the SVZ sum rules using some weight functions are not considered here. The ones from high-moments of $\tau$-decays and from the lattices are only mentioned for comparisons.

| Sources | $\langle \alpha_s G^2 \rangle$ | References |
|---------|-----------------|------------|
| Vector Charmonium SR | $q^2 = 0$-moments | 4 ± 2 | SVZ 79 [5, 6] (guessed error) |
| | $q^2 \neq 0$-moments | 5.3 ± 1.2 | RRY 81-85 [17] |
| | | 9.2 ± 3.4 | Miller-Olsson 82 [18] |
| | | | Broadhurst et al. 94 [19] |
| | | | Ioffe-Zyablyuk 07 [20, 21] |
| | | | Narison 12a [22] |
| | LSR | 12 ± 2 | Bell-Bertlmann 82 [23–29] |
| | | | Narrow et al. 87 [30] |
| | | | Narison 12b [31] |
| Vector Bottomium SR | | Non-rel. vector mom. | 5.5 ± 3.0 | Yndurain 99 [32] |
| Other Charmonium and Bottomonium SR | LSR $M_{\psi} - M_{\eta_c}$ | 10 ± 4 | Narison 96 [33, 34] |
| | | $- M_{\psi} - M_T$ | 6.5 ± 2.5 | Narison 96 [33, 34] |
| | | $- M_{\psi} - M_{\psi(1S)}$ | 8.5 ± 3.0 | Narison 18 [1] |
| | | $- M_{\chi_b(1P),\rho} - M_{\eta_c,\rho}$ | 6.39 ± 0.35 | Narison 18 [1] |
| | $e^+e^- \rightarrow 1\text{=1} \text{ Hadrons SR}$ | LSR | 0.9 ± 6.6 | Eidelman et al. 79 [35] |
| | Ratio of LSR | 4 ± 1 | Launer et al. 84 [36] |
| | FESR | 13 ± 6 | Bertlmann et al. 88 [37, 38] |
| | Infinite norm | 1 ± 30* | Cause-Mennesser [39] |
| | $\tau$-like decay | 7 ± 1 | Narison 95 [40, 41] |
| | $\tau$-decay SR | Axial spectral function | 6.9 ± 2.6 | Dominguez-Sola 88 [42] |
| | SR Average 2018 | 6.35 ± 0.35 | |
| $\tau$-decay with high moments SR | ALEPH collaboration | 6.3 ± 1.2 | Duflot 95 [43] |
| | CLEO II collaboration | 2.4 ± 1.0 | Duflot 95 [43] |
| | OPAL collaboration | 0.9 ± 4 | Ackermann et al. 99 [44] |
| | ALEPH collaboration | $- 5 \pm 6$ | Schael et al. 05 [45] |
| | ALEPH collaboration | $- 12 \pm 0.6$ | Davier et al. 14 [46] |
| Lattice | O($\alpha_s^2$) | $= 13$ | Rakow 05 [47–49] |
| | O($\alpha_s^3$) | $= 27$ | Bali-Pineda 15 [50, 51] |
| | Average plaquette | $= 44$ | Lee 14 [52] |
• Form of the sum rule

We shall work with the Finite Energy version of the QCD Laplace sum rules (LSR) and their ratios:

$$\mathcal{L}_n^{\Pi}(\tau) = \int_0^{\infty} dt \int^{\infty}_{t_{c}} e^{-t\tau} \text{Im} \Pi_{V(A)}(t), \quad \mathcal{R}_n(\tau) = \frac{\mathcal{L}_{n+1}(\tau)}{\mathcal{L}_n(\tau)}, \quad (4)$$

where $\tau$ is the LSR variable, $t_c$ is the threshold of the “QCD continuum” which parametrizes, from the discontinuity of the Feynman diagrams, the spectral function $\text{Im} \Pi_{V(A)}(t, m_Q^2, \mu)$ associated to the transverse part $\Pi_{V(A)}(q^2, m_Q^2, \mu)$ of the two-point correlator:

$$\Pi^\nu_{V(A)}(q^2) = \frac{1}{i} \int d^4x e^{-i q \cdot x} \langle 0 \mid [J_{V(A)}^\nu(x), J_{V(A)}^\nu(0)] \mid 0 \rangle,$$

where $J_{V(A)}^\nu(x) = \bar{Q} \gamma^\mu \not{q} \not{D}^\nu \Psi(x)$ is the heavy quark vector (axial-vector) current. In the (pseudo)scalar channel associated to the local current $J_{S(P)} = \bar{Q} i(\gamma_5) \Psi(x)$, we work with the correlator:

$$\Psi_{S(P)}(q^2) = \frac{1}{i} \int d^4x e^{-i q \cdot x} \langle 0 \mid [J_{S(P)}(x), J_{S(P)}(0)] \mid 0 \rangle, \quad (6)$$

which is related to the longitudinal part $\Pi^{00}_{V(A)}(q^2)$ of the (axial-)vector one through the Ward identity [11, 12, 64]:

$$q^2 \Pi^{00}_{V(A)}(q^2) = \Psi_{PS}(q^2) - \Psi_{PS}(0). \quad (7)$$

Working with $\Psi_{PS}(q^2)$ is safe as $\Psi_{PS}(0)$ should affect the $Q^2$-moments and the exponential sum rules derived from $\Pi^{00}_{V(A)}(q^2)$ which is not accounted for in e.g. [17, 26, 30].

Originally named Borel sum rules by SVZ because of the appearance of a factorial suppression factor in the non-perturbative condensate contributions into the OPE, it has been shown by [65] that the PT radiative corrections satisfy instead the properties of an inverse Laplace sum rule though the present given name here.

• Parametrisation of the spectral function

$\text{Im} \Pi_{V}(t)$ is related to the ratio $R_{e^{-}e^{-}}$ of the total cross-section of $\sigma(e^{+}e^{-} \rightarrow \text{hadrons})$ over $\sigma(e^{+}e^{-} \rightarrow \mu^{+}\mu^{-})$ through the optical theorem. Expressed in terms of the leptonic widths and meson masses, it reads in a narrow width approximation (NWA):

$$R_{e^{-}e^{-}} = \frac{12\pi \text{Im} \Pi_{V}(t)}{9\pi \sum M_{V} \Gamma_{V \rightarrow e^{-}e^{-}} \delta(t - M_{V}^2)}, \quad (8)$$

where $M_{V}$ and $\Gamma_{V \rightarrow e^{-}e^{-}}$ are the mass and leptonic width of the $J/\psi$ or $\Upsilon$ mesons; $Q_{V} = 2/3(-1/3)$ is the charm (bottom) electric charge in units of $e$; $\alpha = 1/133$ is the running electromagnetic coupling evaluated at $M_{V}^2$. We shall use the experimental values of the $J/\psi$ and $\Upsilon$ parameters compiled by PDG [4]. We include the contributions of the $\psi(3097)$ to $\psi(4415)$ and $\Upsilon(9460)$ to $\Upsilon(11020)$ within NWA. The high-energy part of the spectral function is parametrized by the “QCD continuum” from a threshold $t_c$ (we use $\sqrt{t_c} = 4.6$ GeV and $\sqrt{t} = 11.098$ GeV just above the last resonance).

In the case of the axial-vector and (pseudo)scalar channels where there are no complete data, we use the duality ansatz:

$$\text{Im} \Pi(t); \Psi(t) \approx f_{H} M_{H}^{0}\delta(t - M_{H}^{2}) + \Theta(t - t_{c})^{\gamma} \text{QCDD continuum}, \quad (9)$$

where $M_{H}$ and $f_{H}$ are the lowest ground state mass and coupling analogue to $f_{e}$ and $f_{\pi}$. This implies:

$$\mathcal{R}_n \equiv \mathcal{R} \sim M_{H}^{2}. \quad (10)$$

indicating that the ratio of moments appears to be a useful tool for extracting the masses of hadrons [11–14]. We shall work with the lowest ratio of moments $\mathcal{R}_0$. Exponential sum rules have been used successfully by SVZ for light quark systems [5, 6, 11–14] and extensively by Bell and Bertlmann for heavy quarkonia in their relativistic and non-relativistic versions [23–30, 33, 34].

• QCD Perturbative expressions @N2LO

The perturbative QCD expression of the vector channel is deduced from the well-known spectral function to order $\alpha_s$ within the on-shell renormalization scheme [66, 67]. The one of the axial-vector current has been obtained in [17, 68–70]. To order $\alpha_s^2$ (N2LO), the spectral functions are usually parametrized as:

$$R^{(2)} \equiv C^{2}_{A} R^{(2)}_{A} + C^{2}_{F} R^{(2)}_{F} + C^{2}_{T} T \omega R^{(2)}_{T} + C^{2}_{G} G R^{(2)}_{G}, \quad (11)$$

where $C_{A}, C_{F}, C_{T}$ and $C_{G}$ are the usual SU(3) group factors and $\omega_n$ is the number of light quarks. We use the (approximate) but complete result in the on-shell scheme given by [71] for the abelian and non-abelian contributions. The one from light quarks comes from [72–74]. The one from heavy fermion internal loop comes from [75] for the vector current while the one from the
axial current is (to our knowledge) not available. The singlet one due to double triangle loop comes from [76]. The one from the gluonic double-bubble reconstructed from massless fermions comes from [72, 73, 75]. The previous on-shell expressions are transformed into the MS scheme through the relation between the on-shell $M_0$ and running $\bar{m}_0(\mu)$ quark masses [11–14, 77–86] @N2LO:

$$ M_0 = \bar{m}_0(\mu) \left[ 1 + \frac{4}{3} a_s + (16.2163 - 1.0414 n_\tau) a_s^2 + \ln \left( a_s + (8.8472 - 3.611 n_\tau) a_s^2 \right) + \ln^2 (1.7917 - 0.0833 n_\tau) a_s^2 + \cdots \right], \tag{12} $$

for $n_\tau$ light flavours where $\mu$ is the arbitrary subtraction point and $a_s = \alpha_s/\pi$, $\ln \equiv \ln(\mu/M_0)^2$.

- **QCD Non-Perturbative expressions @LO**

Using the OPE à la SVZ, the non-perturbative contributions to the two-point correlator can be parametrized by the sum of higher dimension condensates:

$$ \text{Im}\Pi(t) = \sum_{n=1,2} C_{2n}(m^2, \mu^2) (O_{2n}) $$

where $C_{2n}$ are Wilson coefficients calculable perturbatively and $(O_{2n})$ are non-perturbative condensates. In the exponential sum rules, the order parameter is the sum rule variable $\tau$ while for the heavy quark systems the relevant condensate contributions at leading order in $\alpha_s$ are the gluon condensate $\langle \alpha_s G^2 \rangle$ of dimension-four [5, 6], the dimension-six gluon $\langle g^4 f_{abc} G^3 \rangle$ and light four-quark $\alpha_s \langle au \rangle^2$ condensates [15, 16]. The condensates of dimension-8 entering in the sum rules are of seven types [61]. They can be expressed in different basis depending on how each condensate is estimated (vacuum saturation [61] or modified vacuum saturation [87]). Our estimate of these $D=8$ condensates is the same as in [22]. For the vector channel, we use the analytic expressions of the different condensate contributions given by Bertlmann [26]. We shall not include the eventual $D = 2$ coperator induced by a tachyonic gluon mass [88, 89] as it is dual to the contribution of large order terms [90], which we estimate using a geometric growth of the PT series. In various examples, its contribution is numerically negligible [91].

- **Initial QCD input parameters**

In the first iteration, we shall use the following QCD input parameters (mass in units of MeV):

$$\alpha_s(M_Z) = 0.325^{+0.008} \_{-0.010} \times (\alpha_s G^2) = (0.07 \pm 0.04) \ \text{GeV}^4, $$

$$\bar{m}_0(\bar{m}_0) = (1261 \pm 17), \ \bar{m}_0(\bar{m}_0) = (4177 \pm 11), \tag{14}$$

The central value of $\alpha_s$ comes from $\tau$-decay [55, 92]. The range covers the one allowed by PDG [3, 4] (lowest value) and the one from our determination from $\tau$-decay (highest value) [55]. The values of $\bar{m}_0(\bar{m}_0)$ are the average from our recent determinations from charmonium and bottomonium sum rules [22, 60]. The value of $\langle \alpha_s G^2 \rangle$ almost covers the range from different determinations mentioned in Table 1 and reviewed in [11, 12, 33, 34]. We shall use the ratio of condensates given in Eq. 3. For the light four-quark condensate, we shall use the value:

$$\alpha_s \langle au \rangle^2 = (5.8 \pm 1.8) \times 10^{-4} \ \text{GeV}^6, \tag{15}$$

obtained from the original $\tau$-decay rate [55] where the gluon condensate does not contribute to LO [53, 54] and by some other authors from the light quark systems [11, 12, 36, 93–95] where a violation by a factor about 3–4 of the vacuum saturation assumption has been found.

### 3. Charmonium Ratio of LSR Moments $R_j(\chi_{c1})$

- **Convergence of the PT series**

![Figure 1: Behaviour of the ratio of moments $R$ versus $\tau$ in GeV$^{-2}$ at different orders of perturbation theory. The input and the meaning of each curve are given in the legends: a) $J/\psi$ and b) $\chi_{c1}$.](image-url)
In so doing, we shall work with the renormalized (but non-resummed renormalization group) perturbative (PT) expression where the subtraction point \( \mu \) appears explicitly. We include the known N2LO terms. The \( D = 8(6) \) condensates contributions are included for the (axial-)vector current. The value of \( \sqrt{t_c} = 4.6 \) GeV is chosen just above the \( \psi(4040) \) mass for the vector current where the sum of all lower mass \( \psi \) state contributions are included in the spectral function. For the axial current, we use (as mentioned) the duality ansatz and leave \( t_a \) as a free parameter which we shall fix after an optimisation of the sum rule. We evaluate the ratio of moments at \( \mu = 2.8 \) GeV and for a given value of \( t_c = 20 \) GeV\(^2 \) for the \( \chi_{c1} \) around which they will stabilize (as we shall show later on). The analysis is illustrated in Fig. 1. On can notice the importance of the N2LO contribution which is dominated by the abelian and non-abelian contributions. The N2LO effects go towards the good direction of the values of the experimental masses.

- **LSR variable \( \tau \)-stability and Convergence of the OPE**

The OPE is done in terms of the exponential sum rule variable \( \tau \). We show in Fig. 2 the effects of the condensates of different dimensions. One can notice that the presence of condensates are vital for having \( \tau \)-stabilities which are not there for the PT-terms alone. The \( \tau \)-stability is reached for \( \tau \approx 0.6 \) GeV\(^{-2} \). At a given order of the PT series, the contributions of the \( D = 8 \) condensates are negligible at the \( \tau \)-stability region while the \( D = 6 \) contribution goes again to the right track compared with the data.

- **Continuum threshold \( t_c \)-stability for \( \mathcal{R}_{\chi_{c1}} \)**

We show the analysis in Fig. 3 where the curves correspond to different \( t_c \)-values. We find nice \( t_c \)-stabilities where we take the value:

\[
 t_c \approx (17 \sim 22) \text{ GeV}^2, \quad (16)
\]

where the lowest value corresponds to the phenomenological estimate \( M_{\psi}(2P) - M_{\chi_{c1}}(1P) \approx M_{\psi}(2S) - M_{\psi}(1S) \) while the higher one corresponds to the beginning of \( t_c \)-stability. This range of \( t_c \)-values induces an error of about 8 MeV in the meson mass determination.

- **Subtraction point \( \mu \)-stability**

The subtraction point \( \mu \) is an arbitrary parameter. It is popularly taken between 1/2 and 2 times an “ad hoc” choice of scale. However, the physical observables should be not quite sensitive to \( \mu \) even for a truncated PT series. In the following, like in the previous case of external (unphysical) variable, we shall fix its value by looking for a \( \mu \)-stability point if it exists at which the observable will be evaluated. This procedure has been used recently for improving the LSR predictions on molecules and four-quark charmonium and bottoming states [96–100]. Taking here the example of the ratios of moments, we show in Fig. 4 their \( \mu \)-dependence. We notice that \( \mathcal{R}_\psi \) is a smooth decreasing function of \( \mu \) while \( \mathcal{R}_{\chi_{c1}} \) presents a slight stability at:

\[
 \mu = (2.8 \sim 2.9) \text{ GeV}, \quad (17)
\]

at which we shall evaluate the two ratios of moments. On can notice that at a such higher scale, one has a better convergence of the \( \alpha_s(\mu) \) PT series.

- **Correlations of the QCD parameters**

Once fixed these preliminaries, we are now ready to study the correlation between \( \alpha_s \), the gluon condensate \( \langle \alpha_s G^2 \rangle \), and the \( \psi \)-quark running masses \( \overline{m}_J(\overline{\rho}_J) \). In so doing we request that the \( \sqrt{\mathcal{R}_\psi} \) sum rule reproduces within (2-3) MeV accuracy the experimental measurement, while the \( \chi_{c1} \) mass is reproduced within (8–10)
Figure 3: Behaviour of the ratio of moments $R_{c_1}$ versus $\tau$ in GeV$^{-2}$. The input and the meaning of each curve are given in the legend.

![Graph showing the behaviour of $R_{c_1}$ versus $\tau$.](image)

Figure 4: Behaviour of the ratio of moments $R_{J/\psi}$ and $R_{c_1}$ versus $\mu$ for $t_0 = 20$ GeV$^2$. The inputs and the meaning of each curve are given in the legends.

![Graph showing the behaviour of $R_{J/\psi}$ and $R_{c_1}$ versus $\mu$.](image)

One can see from Fig. 5 that within the alone $J/\psi$ sum rule the values of $\langle \alpha_s G^2 \rangle$ and $\overline{m}_c(\overline{m}_c)$ cannot be strongly constrained\(^1\). Once the constraint from the $\chi_{c1}$ sum rule is introduced, one obtains a much better selection. Taking as a conservative result the range covered by the change of $\mu$ in Eq. 17, one deduces:

$\langle \alpha_s G^2 \rangle = (8.5 \pm 3.0) \times 10^{-2}$ GeV$^4$, 
$\overline{m}_c(\overline{m}_c) = (1256 \pm 30)$ MeV. \hspace{1cm} (18)

We improve this determination by including the N3LO PT\(^{[103]}\) corrections and NLO $\langle \alpha_s G^2 \rangle$ gluon condensate (using the parametrization in \([20, 21]\)) contributions. The effects of these quantities on $\sqrt{R_{J/\psi}}$ and $\sqrt{R_{\chi_{c1}}}$ is about (1 to 2) MeV at the optimization scales which induces a negligible change such that the results quoted in Eq. 18 remain the same @N3LO PT and @NLO gluon condensate approximations. This value of $\langle \alpha_s G^2 \rangle$ is in good agreement with the one $(7.5 \pm 2.0) \times 10^{-2}$ GeV$^4$ from our previous analysis of the charmonium Laplace sum rules using resummed PT series\(^{[31]}\) indicating the self-consistency of the results. However, these results do not favor lower ones quoted in Table 1. Taking the weighted average of different sum rule determinations given in Table 1 with the new result in Eq. 18, we obtain the sum rule average:

$\langle \alpha_s G^2 \rangle_{\text{average}} = (6.30 \pm 0.45) \times 10^{-2}$ GeV$^4$, \hspace{1cm} (19)

where the error may be optimistic but comparable with the one of the most precise predictions given in Table 1. These results agree within the errors within our recent estimates of $\langle \alpha_s G^2 \rangle$ and $\overline{m}_c(\overline{m}_c)$\(^{[22, \hspace{1cm} 31, \hspace{1cm} 60]}\) obtained from the moments and their ratios subtracted at finite $Q^2 = n \times 4m_c^2$ with $n = 0, 1, 2$, and from the heavy quark mass-splittings\(^{[33, \hspace{1cm} 34]}\). Hereafter, we shall use the value of $\langle \alpha_s G^2 \rangle$ in Eq. 19.

\(^{1}\)Similar relations from vector moments have been obtained\([20, \hspace{1cm} 21]\) while the ones between $\alpha_s$ and $\overline{m}_c$ have been studied in\([101, \hspace{1cm} 102]\).
4. Bottomium Ratios of Moments $\mathcal{R}_\Upsilon(\chi_b)$

- $\tau$ and $t_c$-stability and test of convergences

The analysis is very similar to the previous $J/\psi$ sum rule. The relative perturbative and non-perturbative contributions are very similar to the curves in Figs. 1 to 2. We use the value: $\mu = 9.5$ GeV which we shall justify later on. However, it is informative to show in Fig.6 the $\tau$-behaviour of $\mathcal{R}_\Upsilon$ for different truncation of the OPE where $\tau$-stability is obtained at $\tau \approx 0.22$ GeV$^{-2}$. In Fig. 7, we show the $\tau$-behaviour of $\mathcal{R}_\Upsilon$ for different values of $t_c$ from which we deduce a stability at $\tau \approx 0.28$ GeV$^{-2}$ and $t_c$-stability which we shall take to be $\sqrt{t_c} \approx 11$ GeV. A much better convergence of the $\alpha_s$ series is observed as the sum rule is evaluated at a higher scale $\mu$. The OPE converges also faster as $\tau$ is smaller here.

- $\mu$-stability

The two sum rules are smooth decreasing functions of $\mu$ but does not show $\mu$-stability. Instead, their difference presents $\mu$-stability at:

$$\mu \approx (9 \sim 10) \text{ GeV},$$

as shown in Fig.8 at which we choose to evaluate the two sum rules.

- Mass of $\chi_b(1^{++})$ from $\mathcal{R}_\Upsilon$

Using the previous value of the QCD parameters, we predict from the ratio of $\chi_b$ moments:

$$M_{\chi_b} \approx 9677(26)_{a_s}(11)_{\alpha_s}(9)_{\alpha_s} m_b(99)_{\mu} \text{ MeV,}$$

which is (within the error) about 100 MeV lower than the experimental mass $M_{\chi_b}^{ex} = 9893$ MeV. The agreement between theory and experiment may be improved when more data for higher states are available or/and by including Coulombic corrections shown to be small for the vector current (see e.g. [60]) and not considered here.

- Correlation between $\alpha_s(\mu)$ and $\overline{m}_b(\overline{m}_b)$ from $\mathcal{R}_\Upsilon$

From the previous analysis, one can notice that the $\chi_b$ channel cannot help from a precise study of the correlation between $\alpha_s$ and $\overline{m}_b(\overline{m}_b)$. We show in Fig. 9 the result of the analysis from the $\Upsilon$ channel by requiring that the experimental value of $\sqrt{\mathcal{R}_\Upsilon}$ is reproduced.
within \((1 \sim 2)\) MeV accuracy. First, one can notice that the error due to the gluon condensate with the value given in Eq. 18 is negligible. Given the range of \(\alpha_s\) quoted in Eq. 14, one can deduce the prediction:

\[
\overline{m}_0(\overline{m}_0) = 4192(15)(8)_{\text{cou}} \text{ MeV},
\]

where we have added in Eq. 22 an error of about 8 MeV from Coulombic corrections as estimated in [31]. The previous result in Eq. 22 corresponds to:

\[
\alpha_s(M_{T}) = 0.321(12) \implies \alpha_s(M_{Z}) = 0.1186(15)(3) \quad (23)
\]
given by the range in Eq. 14. The running from \(M_T\) to \(M_Z\) due to the choice of the thresholds induces the last error (3).

- **Updated average value of \(\overline{m}_0(\overline{m}_0)\) from QSSR**

The result in Eq. 22 is consistent with the ones from LSR with RG resummed PT expressions [31]:

\[
\overline{m}_0(\overline{m}_0) = 4212(32) \text{ MeV},
\]

and the average of the ones from moments sum rules quoted in Eq. 14 and updated in [104]:

\[
\overline{m}_0(\overline{m}_0) = 4188(8) \text{ MeV},
\]

Taking the average of the LSR and updated moments determinations, we obtain the final estimate:

\[
\overline{m}_0(\overline{m}_0)_{\text{average}} = (4190 \pm 8) \text{ MeV},
\]

where the errors come from the most precise determination.

![Figure 10: Behaviour of \(M_0\) versus \(\tau\) for different values of \(t_c\).](image)

**5. (Pseudo)scalar charmonium**

In these channels, we shall work with the ratio of sum rules associated to the two-point correlator \(\Psi_{PS}(q^2)\), defined in Eq. 6 which is not affected by \(\Psi_{PS}(0)\). We shall use the PT expression known @N2LO [72–76], the contribution of the gluon condensates of dimension 4 and 6 to LO [15, 16].

![Figure 11: Behaviour of \(M_{\eta_c}\) versus \(\tau\) for different values of \(t_c\).](image)

- **\(\eta_c\) and \(\chi_{c0}\) masses**

The \(\eta_c\) sum rule shows a smooth decreasing function of \(\mu\) but does not present a \(\mu\)-stability. Then, we choose the value of \(\mu\) given in Eq. 17 for evaluating it. We show in Fig. 10 the \(\tau\)-behaviour of the \(\eta_c\)-mass for different values of \(t_c\) which we take from 10 GeV\(^2\) [around the mass squared of the \(\eta_c(2P)\) and \(\eta_c(3P)\)] until 13 GeV\(^2\) \((t_c\text{-stability})\). Similar analysis is done for the \(\chi_{c0}\) associated to the scalar current \(\bar{Q}(i)Q\) which is shown in Fig.11, where we take \(t_c = (16 - 24)\) GeV\(^2\). Using the averaged values of \((\alpha_s G^2)\) and \(\overline{m}_c(\overline{m}_c)\) in Eqs. 19 and 30, we deduce the optimal result in units of MeV:

\[
M_{\eta_c} = 2979(5)(11)\langle 11\rangle_{\text{ln}}(11)_{\text{ln}}(30)_{\text{ln}}(10)_{\text{cg}},
\]

\[
M_{\chi_{c0}} = 3411(1)_{\text{ln}}(17)_{\text{ln}}(26)_{\text{ln}}(30)_{\text{ln}}(20)_{\text{cg}},
\]

in good agreement within the errors with the experimental masses: \(M_{\eta_c} = 2984\) MeV and \(M_{\chi_{c0}} = 3415\) MeV but not enough accurate for extracting with precision the QCD parameters.

- **Correlation between \(\overline{m}_c(\overline{m}_c)\) and \((\alpha_s G^2)\)**

We study the correlation between \(\overline{m}_c(\overline{m}_c)\) and \((\alpha_s G^2)\) by requiring that the sum rules reproduce the masses of the \(\eta_c\) and \(\chi_{c0}\) within the error induced by the choice of \(t_c\) respectively 11 and 17 MeV. We show the result of the analysis in Fig. 12 keeping only the strongest constraint from \(M_{\eta_c}\). We deduce:

\[
\overline{m}_c(\overline{m}_c) = 1266(16) \text{ MeV},
\]

in good agreement with the one in [105] from pseudoscalar moments.

- **Updated average value of \(\overline{m}_c(\overline{m}_c)\) from QSSR**

We combine our determinations in Eqs. 18 and 28 with the updated determination [104]:

\[
\overline{m}_c(\overline{m}_c)_{\text{average}} = (1264 \pm 6) \text{ MeV},
\]
of two ones [22, 60] from vector moments sum rules quoted in Eq. 14. As a final result, we quote the updated average from exponential and moment sum rules from a global fit of the quarkonia spectra:

$$\overline{m}_\chi(\overline{m}_\chi)_{\text{average}} = (1264 \pm 6) \text{ MeV},$$

(30)

which is dominated by the most precise prediction quoted in Eq. 29. It is remarkable that this value agrees with the original SVZ estimate [5, 6] of the euclidian mass.

6. (Pseudo)scalar bottomium

- \(\eta_b\) and \(\chi_{b0}\) masses

The masses of the \(\eta_b(0^{-+})\) and \(\chi_{b0}(0^{++})\) are extracted in a similar way using the value of \(\mu\) in Eq. 20 and the parameters in Eqs. 19 and 26. We take the range \(\sqrt{r} = (9.5 \sim 12)\) GeV for the \(\eta_b\) [resp. \(\chi_{b0}\)] channels, as shown in Figs 13 and 14 from which we deduce in units of MeV:

$$M_{\eta_b} = 9394(16)_{\mu}(30)_{\mu}(7)_{s}(16)_{s}(8)_{s}G^2,$$

$$M_{\chi_{b0}} = 9844(7)_{\mu}(35)_{\mu}(6)_{s}(17)_{s}(29)_{s}G^2,$$

(31)
in good agreement with the data \(M_{\eta_b} = 9399\) MeV and \(M_{\chi_{b0}} = 9859\) MeV.

- Correlation between \(\overline{m}_{\eta_b}(\overline{m}_{\eta_b})\) and \(\langle s, G^2\rangle\)

The analysis done for charmonium is repeated here where we request that the sum rule reproduces the \(\eta_b\) and \(\chi_{b0}\) masses with the error induced by the choice of \(\tau_c\). Unfortunately, this constraint is too weak and leads to \(\overline{m}_{\eta_b}(\overline{m}_{\eta_b})\) with an accuracy of about 40 MeV which is less interesting than the estimate from the vector channel in Eq. 22.

7. \(\alpha_s\) and \(\langle s, G^2\rangle\) from \(M_{\chi_{b0}} - M_{\eta_b}\)

As the sum rules reproduce quite well the absolute masses of the (pseudo)scalar states, we can confidently use their mass-splittings for extracting \(\alpha_s\) and \(\langle s, G^2\rangle\). We shall not work with the Double Ratio of LSR [11, 12, 106, 107]

as each sum rule does not optimize at the same points. We check that, in the mass-difference , the effect of the choice of the continuum threshold is reduced and induces an error from 6 to 14 MeV instead of 11 to 35 MeV in the absolute value of the masses. The effect due to \(\overline{m}_{\chi,b}\) in Eqs. 30 and 26 and to \(\mu\) in Eqs. 17 and 20 induce respectively an error of about \(1\sim2\) MeV and \(1\sim8\) MeV. The largest effects are due to the changes of \(\alpha_s\) and \(\langle s, G^2\rangle\). We show their correlations in Fig 15 where we have runned the value of \(\alpha_s\) from \(\mu = 2.85\) GeV to \(M_{\tau}\) in the charm channel and from \(\mu = 9.5\) GeV to \(M_{\tau}\) in the bottom one where the values of \(\mu\) correspond to the scales at which the sum rules have been evaluated:

$$\alpha_s(2.85) = 0.262(9) \sim \alpha_s(M_{\tau}) = 0.318(15)$$

$$\sim \alpha_s(M_{\tau}) = 0.1183(19)(3),$$

$$\alpha_s(9.50) = 0.180(8) \sim \alpha_s(M_{\tau}) = 0.312(27)$$

$$\sim \alpha_s(M_{\tau}) = 0.1175(32)(3),$$

(32)
where the last error is due to the running procedure. We have requested that the method reproduces within the errors the experimental mass-splittings by about 2-3 MeV. These values are compared in Fig. 16 with the running at different \( \mu \) of the world average \([3, 4]\):

\[
\alpha_s(M_Z) = 0.1181(11) .
\]

(33)

With the central values given in Eqs. 19 and 23, the allowed region from both charmonium and bottomium channels leads to our final predictions:

\[
\alpha_s(M_Z) = 0.318(15) \quad \rightarrow \quad \alpha_s(M_Z) = 0.1183(19)(3) ,
\]

\[
\langle \alpha_s G^2 \rangle = (6.34 \pm 0.39) \times 10^{-2} \text{ GeV}^4 ,
\]

(34)

Adding into the analysis the range of input \( \alpha_s \) values given in Eq. 14 (light grey horizontal band in Fig. 15), one can deduce stronger constraints on the value of \( \langle \alpha_s G^2 \rangle \):

\[
\langle \alpha_s G^2 \rangle = (6.39 \pm 0.35) \times 10^{-2} \text{ GeV}^4 .
\]

(35)

Combining the previous values in Eqs. 18, 34 and 35 with the ones in Table 1, one obtains the \textit{new sum rule average}:

\[
\langle \alpha_s G^2 \rangle_{\text{average}} = (6.35 \pm 0.35) \times 10^{-2} \text{ GeV}^4 ,
\]

(36)

where we have retained the error from the most precise determination in Eq. 35 instead of the weighted error of 0.23. This result definitely rules out some eventual lower and negative values quoted in Table 1.

8. Correlated values of \( \overline{m}_{c,b}(\overline{m}_{c,b}) \) from \( M_{B_{c}} \)

We extend the previous analysis to the case of the \( B_{c} \)-meson \([2]\). We determine simultaneously \( \overline{m}_{c}(\overline{m}_{c}) \) and \( \overline{m}_{b}(\overline{m}_{b}) \) from the ratio of sum rules requested to reproduce the \( B_{c} \)-mass.

Figure 16: Comparison with the running of the world average \( \alpha_s(M_Z) = 0.1181(11)(3, 4) \) of our predictions at three different scales: \( M_{t} \) for the original low moment \( \tau \)-decay width \([55] \) (open circle), 2.85 GeV for \( M_{t,a} - M_{t,b} \) (full triangle) and 9.5 GeV for \( M_{t,a} - M_{t,b} \) (full square) \([1]\).

- First, we study the \( \tau \) and \( \tau \)-stability of the analysis for given values of \( \mu \) and \( \overline{m}_{c,b}(\overline{m}_{c,b}) \) determined previously. The result is shown in Fig. 17 where \( \tau \) and \( \tau \) stabilities are reached for \( \tau = 0.3 \text{ GeV}^{-2} \) and \( \tau = 50 - 70 \text{ GeV}^2 \).

- Second, we study the \( \mu \)-dependence of the result on Fig. 18 by fixing \( \overline{m}_{c,b}(\overline{m}_{c,b}) \) and varying the output values of \( \overline{m}_{c}(\overline{m}_{c}) \) versus \( \mu \) where we find a stability for \( \mu = (7.5 \pm 0.1) \text{ GeV} \).

- Third, given the previous optimal values of \( \tau \) and \( \mu \), we study the correlation between \( \overline{m}_{c}(\overline{m}_{c}) \) and \( \overline{m}_{b}(\overline{m}_{b}) \) by demanding that the sum rule reproduces the \( B_{c} \)-mass. The result of the analysis is shown in Fig. 19. We deduce from \( M_{B_{c}} \) and from the intersection region allowed by the charmonium and bottomium sum rules:

\[
\overline{m}_{c}(\overline{m}_{c}) = 1286(16) \text{ MeV}
\]

\[
\overline{m}_{b}(\overline{m}_{b}) = 4202(7) \text{ MeV} .
\]

(37)

Combined with previous estimates from heavy quarkonia, we deduce the new QCD Spectral Sum Rules (QSSR) tentative average:

\[
\overline{m}_{c}(\overline{m}_{c}) = 1266(6)_{\text{average}} \text{ MeV}
\]

Figure 17: \( M_{B_{c}} \) as function of \( \tau \) for different values of \( \tau \) and for \( \mu = 7.5 \text{ GeV} \).
where the recent experimental mass $M_{B_{c}(2S)} = 6872(1.5)$ MeV from CMS [109] has been used.

**Figure 20:** $f_{B_{c}}$ as function of $\tau$ for different values of $t_{c}$, for $\mu=7.5$ GeV and for $[\overline{m}_{c}(\overline{m}_{b}),\overline{m}_{b}(\overline{m}_{b})] = [1264, 4188]$ MeV.

**Figure 21:** $f_{B_{c}}$ as function of $\mu$ for $\tau \approx 0.3$ GeV$^{-2}$ and for and for $[\overline{m}_{c}(\overline{m}_{b}),\overline{m}_{b}(\overline{m}_{b})] = [1264, 4188]$ MeV.

\[ m_{c}(m_{b}) = 4186(8) \text{ MeV}. \]  
(38)

### 9. Decay constants $f_{B_{c}}, f_{B_{c}(2S)}$

Using the previous correlated values of $\overline{m}_{c,b}(\overline{m}_{c,b})$, we use the LSR $L_{S}^{c}$ in Eq. 4 for extracting the decay constant $f_{B_{c}}$ of the $B_{c}$ meson. The searches for the $(\tau, t_{c})$ and $\mu$-stabilities are respectively shown in Figs.20 and 21. We obtain:

\[ f_{B_{c}} = 371(17) \text{ MeV}, \]  
(39)

where the largest errors come from the higher order PT corrections. This result confirms previous QSSR results disagrees with the lattice one quoted in Table 3 of Ref. [108].

Using the positivity of the QCD continuum contribution in the spectral function, an upper bound on the $B_{c}(2S)$ decay constant has been also derived in Ref. [2]:

\[ f_{B_{c}(2S)} \leq 139(6) \text{ MeV}, \]  
(40)

### 10. Summary and Conclusions

- We have explicitly studied (for the first time) the correlations between $\alpha_{s}, \langle \alpha_{s}G^{2} \rangle$ and $\overline{m}_{c,b}$ using ratios of Laplace sum rules @N3LO of PT QCD and including the gluon condensate $\langle \alpha_{s}G^{2} \rangle$ of dimension 4 @NLO and the ones of dimension 6-8 @LO in the (axial-)vector charmonium and bottomonium channels. We have used the criterion of $\mu$-stability in addition to the usual sum rules stability ones (sum rule variable $\tau$ and continuum threshold $t_{c}$) for extracting our optimal results. They are given in Eqs. 18 to 30 and in Eqs. 22 and 26.

- We have extended the analysis to the (pseudo)scalar channels where the experimental masses of the lowest ground states are reproduced quite well. The $\eta_{c}$ sum rule also leads to an alternative prediction of $\overline{m}_{c}$ in Eq. 28. Updated average values of the charm and bottom running quark masses from relativistic QCD spectral sum rules (QSSR) including the new results from $M_{B_{c}}$ can be respectively found in Eq. 38.
The values have been used to extract the value of $f_B$ quoted in Eq.39 which confirms some previous results quoted in Table 3 of Ref. [108] but disagrees with some of them namely the one from Lattice calculations. Upper bound for the $B_c(2S)$ decay constant has been also derived in Eq. 40.

The $\chi^2$ – $\Delta$ mass-splittings lead to improved values of the gluon condensate $\langle \pi G^2 \rangle$ in Eqs. 34 and 35. The new sum rule average is given in Eq. 36 and Table 1.

Such mass-splittings also provide new predictions of $\alpha_s(q^2)$ at two different scales quoted in Eqs. 32 and 34 from Fig.15, which are in good agreement with the running of the world average quoted in Eq. 33 shown in Fig. 16. The most precise prediction given in Eq. 34, which we consider as a final estimate from QSSR, comes from the (pseudo)scalar charmonium mass-splittings.

References

[1] S. Narison, Int. J. Mod. Phys. A33 (2018) no. 10, 1850045; S. Narison. Addendum: Int. J. Mod. Phys. A33 (2018) no.10, 1850045.
[2] S. Narison, Phys. Lett. B802 (2020) 135221.
[3] For a review, see e.g. S. Bethke, Nucl. Part. Phys. Proc. 282-284 (2011) 149.
[4] PDG, C. Patrignani et al. (Particle Data Group), Chin. Phys. C40, 100001 (2016) and 2017 update.
[5] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B147 (1979) 385.
[6] M.A. Shifman, A.I. Vainshtein and V.I. Zakharov, Nucl. Phys. B147 (1979) 448.
[7] For a review, see e.g.: V.I. Zakharov, talk given at the Sakurai’s Fall Conference, 1979.
[8] S. Narison and G. Veneziano, Int. J. Mod. Phys. A4 (1989) no. 11, 2751.
[9] S. Narison, Nucl. Phys. B509 (1998) 312.
[10] For a review, see e.g.: S. Narison, Cambridge Monogr. Part. Phys. Nucl. Phys. Cosmol. 17 (2004) 1-778 [hep-ph/0205006].
[11] For a review, see e.g.: S. Narison, World Sci. Lect. Notes Phys. 26 (1989) 1.
[12] For a review, see e.g.: S. Narison, Phys. Rept. 84 (1982) 263.
[13] S. Narison, Acta Phys. Pol. B 26 (1995) 687.
[14] S.N. Nikolaev and A.V. Radyushkin, Nucl. Phys. B213 (1983) 285.
[15] S.N. Nikolaev and A.V. Radyushkin, Phys. Lett. B110 (1983) 476.
[16] L. I. Reinders, H. Rubinstein and S. Yazaki, Phys. Rept. 127 (1985) 1 and references therein.
[17] K.J. Miller and M.G. Olsson, Phys. Rev. D25 (1982) 1247.
[18] D. J. Broadhurst, P. A. Baikov, V. A. Ilyin, J. Fleischer, O. V. Tarasov and V. A. Smirnov, Phys. Lett. B329 (1994) 103.
[19] B.L. Ioffe and K.N. Zyaublyuk, Eur. Phys. J. C 27 (2003) 229.
[20] B.L. Ioffe, Prog. Part. Nucl. Phys. 56 (2006) 232.
[21] S. Narison, Phys. Lett. B706 (2012) 412.
[22] J.S. Bell and R.A. Bertlmann, Nucl. Phys. B177, (1981) 218.
[23] J.S. Bell and R.A. Bertlmann, Nucl. Phys. B187, (1981) 285.
[24] R.A. Bertlmann, Acta Phys. Austrica 53, (1981) 305.
[25] R.A. Bertlmann, Nucl. Phys. B204, (1982) 387.
[26] R.A. Bertlmann, Non-perturbative Methods, ed. Narison, World Scientific (1985).
[27] R.A. Bertlmann, Nucl. Phys. (Proc. Suppl.) B23 (1991) 307.
[28] R. A. Bertlmann and H. Neufeld, Z. Phys. C27 (1985) 437.
[29] J. Mawrho, J. Parker and G. Shaw, Z. Phys. C37 (1987) 103.
[30] S. Narison, Phys. Lett. B707 (2012) 259.
[31] F.J. Yndurain, Phys. Rept. 320 (1999) 287 [arXiv hep-ph/9903457].
[32] S. Narison, Phys. Lett. B387 (1996) 162.
[33] S. Narison, Nucl. Phys. (Proc. Suppl) A54 (1997) 238.
[34] S.I. Eidelman, L.M. Kurudzie, A.I. Vainshtein, Phys. Lett. B82 (1979) 278.
[35] G. Launer, S. Narison and R. Tarrach, Z. Phys. C26 (1984) 433.
[36] R.A. Bertlmann, G. Launer and E. de Rafael, Nucl. Phys. B250 (1985) 61.
[37] R.A. Bertlmann et al., Z. Phys. C39 (1988) 231.
[38] M.B. Causse and G. Mennesser, Z. Phys. C47 (1990) 611.
[39] S. Narison, Phys. Lett. B300 (1993) 293.
[40] S. Narison, Phys. Lett. B361 (1995) 121.
[41] C. Dominguez and J. Sola, Z. Phys. C40 (1988) 63.
[42] L. Duflot, Nucl. Phys. (Proc. Suppl.) B40 (1995) 37.
[43] The OPAL collaboration, K. Ackerstaff et al., Eur. Phys. J. C8 (1999).
[44] The ALEPH collaboration, S. Schael et al., Phys. Rept. 421, 191 (2005).
[45] M. Davier et al., Eur. Phys. J. C 74, (2014) 381.
[46] P.E. Rakow, arXiv:hep-lat/0510016.
[47] G. Burgio, F. Di Renzo, G. Marchesini and E. Onofri, Phys. Lett. B422 (1998) 219.
[48] R. Horley, P.E.L. Rakow and G. Schierholz, Nucl. Phys. (Proc. Suppl.) B 106 (2002) 870.
[49] G. S. Bali, C. Bauer and A. Pineda, Phys. Lett. Rev. Lett. 113 (2014) 092001.
[50] G. S. Bali, C. Bauer and A. Pineda, AIP Conf. Proc. 1701 (2016) 03010.
[51] T. Lee, Nucl. Part. Phys. Proc. 258-259 (2015) 181.
[52] E. Braaten, S. Narison and A. Pich, Nucl. Phys. B373 (1992) 381.
[53] S. Narison and A. Pich, Phys. Lett. B211 (1988) 183.
[54] S. Narison, Phys. Lett. B673 (2009) 30.
[55] A. Di Giacomo, Non-perturbative Methods, ed. Narison, World Scientific (1985).
[56] M. Campostrini, A. Di Giacomo, Y. Gunduc Phys. Lett. B225 (1989) 393.
[57] A. Di Giacomo and G.C. Rossi, Phys. Lett. B100 (1981) 481.
[58] M. Della, A. Di Giacomo and E. Meggiorolo, Phys. Lett. B408 (1997) 315.
[59] S. Narison, Phys. Lett. B693 (2010) 559, erratum ibid 705 (2011) 544.
[60] S.N. Nikolaev and A.V. Radyushkin, Phys. Lett. B 124 (1983) 243.
[61] T. Schafer and E.V. Shuryak, Rev. Mod. Phys. 70 (1998) 323.
[62] B.L. Ioffe and A.V. Samsonov, Phys. At. Nucl. 63 (2000) 1448.
[63] C. Becchi, S. Narison, E. de Rafael and F.J. Yndurain, Z. Phys. C8 (1981) 335.
[64] S. Narison and E. de Rafael, Phys. Lett. B103 (1981) 57.
[65] A.O.G. Kallen and A. Sabry, Kong. Dan. Vid. Sei. Mat. Fys. Med. 29N17 (1955) 1.
[66] J. Schwinger, Particles sources and fields, ed. Addison-Wesley.
