Ray trajectories for Alcubierre spacetime

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Abstract

The Alcubierre spacetime was simulated by means of a Tamm medium which is asymptotically identical to vacuum and has constitutive parameters which are continuous functions of the spatial coordinates. Accordingly, the Tamm medium is amenable to physical realization as a micro- or nanostructured metamaterial. A comprehensive characterization of ray trajectories in the Tamm medium was undertaken, within the geometric-optics regime. Propagation directions corresponding to evanescent waves were identified: these occur in the region of the Tamm medium which corresponds to the warp bubble of the Alcubierre spacetime, especially for directions perpendicular to the velocity of the warp bubble at high speeds of that bubble. Ray trajectories are acutely sensitive to the magnitude and direction of the warp bubble’s velocity, but rather less sensitive to the thickness of the transition zone between the warp bubble and its background. In particular, for rays which travel in the same direction as the warp bubble, the latter acts as a focusing lens, most notably at high speeds.

Keywords: Alcubierre warp drive, Tamm medium, metamaterial, ray tracing

1. Introduction

Metamaterials provide opportunities to study general-relativistic scenarios which would otherwise be either impractical or impossible to explore [1]. This may be achieved by exploiting the formal analogy that exists between light propagation in vacuum subjected to a gravitational field and light propagation in a certain nonhomogeneous bianisotropic medium, known as a Tamm medium [2–4]. The constitutive properties of this Tamm medium are determined by the symmetries of the spacetime metric which characterize the gravitational field. The practical creation of Tamm mediums is beginning to look like an increasingly realistic proposition, as rapid developments are made in the science of micro- and nanostructured metamaterials. In recent years, metamaterial-based simulations of black holes [5], cosmic strings [6], de Sitter spacetime [7, 8] and wormholes [9] have been proposed, for examples.

As a useful Gedankenexperiment, the spacetime of the Alcubierre metric has generated considerable interest since its introduction in 1994 [10–12]. The spacetime is characterized by a warp bubble which moves with respect to an asymptotically flat background at a speed \( v'_s c_0 \), where \( c_0 \) is the speed of light in the absence of a gravitational field and \( v'_s \geq 0 \). By effectively contracting the spacetime ahead of the warp bubble and expanding the spacetime behind it, arbitrary relative speeds \( v'_s \) may be achieved, in principle. In contrast, within the local environment of the warp bubble, its speed is necessarily subluminal, i.e. \( v'_s < 1 \).

Serious obstacles stand in the way of a physical realization of the Alcubierre spacetime, stemming from violations of various energy constraints [13, 14]. Even the subluminal regime is associated with negative energy densities [15]. However, negative energy densities are not unprecedented in astrophysics. For example, the emission of Hawking radiation by a black hole is accompanied by a flow of negative energy [16] and the construction of wormholes relies on negative energy density [17]. Nor is negative energy density unprecedented within the realm of metamaterials: propagation of (monochromatic) plane waves...
with negative phase velocity [18]—which is intimately related to the phenomenon of negative refraction [19] that certain metamaterials have been shown to support in experimental observations [20]—also involves negative energy density [21], at least in the absence of dissipation [22].

In this paper, we present a simulation of the Alcubierre spacetime, in the form of a Tamm medium which is physically realizable, in principle. Under the geometric-optics approximation, a comprehensive characterization of light ray trajectories through the Tamm medium is provided.

As regards notation, 3-vectors are underlined and unit 3-vectors are additionally distinguished by a caret, whereas \(\mathbf{3}\)-dyadics [23] are double underlined. The identity dyadic is unit-valued at the origin and decays uniformly to zero as \(r_s \to \infty\), where \(r_s'(t') = \nu_s'c_0t'\) and the time-dependent translated displacement

\[
r_s'(t') = \sqrt{\lvert x' - x'_s(t') \rvert^2 + y'^2 + z'^2}.
\]

The scalar parameter \(R > 0\) is a measure of the warp bubble radius while \(\sigma > 0\) provides a measure of the inverse thickness of the transition zone between the warp bubble and its background. For \((1/\sigma) \ll R\), the function \(f\) has an approximately top-hat profile which propagates along the positive \(x'\) axis relative speed \(\nu_s'\).

In order to eliminate the time dependence which enters via \(x'_s(t')\), let us introduce the spacetime coordinates [23]

\[
\Gamma = \nu \left( t' - \frac{x'_s}{c_0} \right), \quad x = v \left( x' - x'_s c_0 t' \right),
\]

\[
y = y', \quad z = z',
\]

with the scalar quantity

\[
v = \frac{1}{\sqrt{1 - (v'_s)^2}}.
\]

This coordinate change amounts to a Lorentz transformation [23]. The line element (1) may then be expressed as

\[
\begin{aligned}
\text{dr}^2 &= -c_0^2 dt^2 + \left[ dx' - v'_s f(r'_s) c_0 dt' \right]^2 + dy'^2 + dz'^2, \\
&= \frac{2 f(r_s) v'_s \left[ f(r_s) - 1 \right] (v'_s)^2 - 1}{1 - (v'_s)^2} c_0^2 \text{dr}^2,
\end{aligned}
\]

\[
+ \frac{2 f(r_s) v'_s \left[ f(r_s) - 1 \right] (v'_s)^2 - 1}{1 - (v'_s)^2} c_0 \text{dr} + \frac{1 + (v'_s)^2 \left[ f(r_s) f(r_s) (v'_s)^2 - 2 \right] - 1}{1 - (v'_s)^2} \text{dx} + \text{dy} + \text{dz},
\]

wherein

\[
r_s = \sqrt{\frac{x^2}{v'^2} + y^2 + z^2}
\]

is independent of \(t\).

We follow the well-established approach of Tamm [24–26], wherein the covariant Maxwell equations are expressed in noncovariant form, with the same spacetime coordinate being fixed as time throughout all spacetime. Thus, electromagnetic fields in curved spacetime may be described by the constitutive relations

\[
\begin{aligned}
D(r, t) &= \varepsilon_0 \gamma(r) E(r, t) - \sqrt{\varepsilon_0 \mu_0} \Gamma(r) \times B(r, t), \\
B(r, t) &= \sqrt{\varepsilon_0 \mu_0} \gamma(r) \times E(r, t) + \mu_0 \gamma(r) H(r, t),
\end{aligned}
\]

of an equivalent medium in flat spacetime, using SI units, with \(\ell = x + y \hat{y} + z \hat{z}\). The components of the \(3 \times 3\) dyadic \(\gamma(r)\) and the 3-vector \(\Gamma(r)\) are defined in indicial notation as

\[
\gamma_{\ell m} = \sqrt{-g_{\ell m}} \gamma_{\ell m}, \quad (\ell, m \in \{1, 2, 3\}),
\]

where \(g\) denotes the determinant of spacetime metric \(g_{\alpha\beta}\), \((\alpha, \beta \in \{0, 1, 2, 3\})\), prescribing the curved spacetime. The sign of the square root term in the definition of \(\gamma_{\ell m}\) is selected such that the metric for vacuous Minkowskian spacetime is represented by the dyadic \(\gamma = \mathbf{I}\). The fictitious Tamm medium represented by the constitutive relations (8) is spatiotemporally local and nonhomogeneous, provided that the spacetime represented is not flat. It is generally bianisotropic, i.e. \(D(r, t)\) is anisotropically coupled to \(E(r, t)\), and \(B(r, t)\) is anisotropically coupled to \(H(r, t)\), via the orthorhombictype dyadic \(\Gamma(r)\) [24].

Furthermore, the Tamm medium is not Lorentz reciprocal [26] in general, but it does satisfy the Post constraint [27].

For the case of Alcubierre spacetime characterized by the line element (6), the definitions (9) deliver

\[
\begin{aligned}
\gamma(\ell) &= \hat{x} \ell + \frac{1 - (v'_s)^2}{1 - f(r_s)} (y \hat{y} + z \hat{z}), \\
\Gamma(\ell) &= \frac{f(r_s) v'_s [1 - f(r_s)] (v'_s)^2}{1 - f(r_s)} \ell.
\end{aligned}
\]

As we are interested in a physically realizable metamaterial that represents the Alcubierre spacetime, the limits

\[
\lim_{\ell \to 0} \gamma(\ell) = \hat{x}, \quad \lim_{\ell \to \infty} \gamma(\ell) = I,
\]

and
bear considerable promise. Thus, the Tamm medium is like a gravitation-free vacuum for large values of |r_s| whereas its constitutive parameters remain bounded at small values of |r_s|.

The nontrivial constitutive parameters included in \( y_{22}(\equiv y_{33}) \) and \( \Gamma_{1} \), namely \( y_{22} \) and \( \Gamma_{1} \), are illustrated in figure 1 as functions of \( x \) and \( y \) for \( \sigma = 5 \), \( R = 1 \) and \( v'_s \in [0.3, 0.6, 0.9] \). The corresponding plots of \( y_{22} \) and \( \Gamma_{1} \) versus \( z \) are identical to those versus \( y \). The unit of \( \sigma \) (i.e. inverse length unit) is the reciprocal of that for \( R \) and \( r_s \) (i.e. length unit). Since the quantities \( \sigma, R \) and \( r_s \) only occur in the definitions \( \varphi(\equiv y_{33}) \) and \( \varphi(\equiv y_{1}) \) as the dimensionless product terms \( \sigma R \) and \( \sigma r_s \) (via the function \( f(r_s) \)), the results presented in figure 1 are independent of the particular length unit chosen for \( R \) and \( r_s \) (as well as \( x \) and \( y \)), and the inverse length unit chosen for \( \sigma \).

The constitutive parameters \( y_{22} \) and \( \Gamma_{1} \) are continuous functions of \( r \) for all values of \( v'_s \in [0, 1] \). Furthermore, the constitutive-parameter space is approximately partitioned into two disjoint regions with \( y_{22} \) and \( \Gamma_{1} \) being approximately constant-valued in each. That is, we have

(i) an inner region—which corresponds to the warp bubble of Alcubierre spacetime—wherein \( y_{22} \approx 1 - (v'_s)^2 \) and \( \Gamma_{1} \approx v'_s \), and

(ii) an outer region wherein \( y_{22} \approx 1 \) and \( \Gamma_{1} \approx 0 \).

The inner region is shaped like a prolate spheroid whose major axis is aligned parallel to the \( x \) axis. The prolate spheroid becomes increasingly elongated as the relative speed \( v'_s \) increases.

In the transition zone between the warp bubble and its background the constitutive parameters of the Tamm medium can vary strongly, depending upon the value of the parameter \( \sigma \). For the example represented in figure 1, the gradients of the constitutive parameters versus \( x \) and \( y \) at the transition zone are all less than approximately one per unit length in magnitude. This imposes a constraint on the wavelength for our ray-tracing study, since under the geometric-optics approximation the wavelength is required to be much shorter than the length scales over which the constitutive parameters vary significantly. Thus, if we consider electromagnetic waves in the optical regime then the transition zone is required to be at least 10 \( \mu \)m thick for the case represented in figure 1. We note that nonhomogeneous metamaterials with graded constitutive-parameter profiles at length scales less than 10 \( \mu \)m are within the limits of current technology [28].

For the presentation of ray trajectories in section 4, let us introduce the semi-major axis length \( a_m \) and semi-minor axis length \( a_m \) of the ellipse representing the inner region in the \( xy \) plane, defined via

\[
\hat{x} \cdot \Gamma_{0}(v'_s Industry) = \frac{1}{2} \hat{x} \cdot \Gamma_{0}(\hat{y} Industry) 
\]

3. Analysis of quasi-plane waves

As a precursor to our investigation of ray trajectories, we first consider a quasi-plane wave whose electric and magnetic fields are of the form [29]

\[
E(r, t) = \text{Re} \{ E_\nu(r) \exp[i(k_0 \cdot r - \omega t)] \}
\]

\[
H(r, t) = \text{Re} \{ H_\nu(r) \exp[i(k_0 \cdot r - \omega t)] \}
\]

The quantities \( E_\nu(r) \) and \( H_\nu(r) \) in equations (15) are spatially varying, complex-valued vectors; \( \omega \) is the angular frequency; and the wavenumber \( k_0 = \omega \sqrt{\epsilon_0 \mu_0} \). Within the quasi-plane-wave regime the relative wavevector \( k \) varies with \( r \), but it
is convenient to omit the dependence on \( \mathbf{r} \) in our notational representation of \( \hat{\mathbf{k}} \).

The source-free Maxwell curl postulates

\[
\nabla \times \mathbf{E}(\mathbf{r}, t) + \frac{1}{\varepsilon_0} \frac{\partial B(\mathbf{r}, t)}{\partial t} = 0 \\
\nabla \times \mathbf{H}(\mathbf{r}, t) - \frac{1}{\mu_0} \frac{\partial D(\mathbf{r}, t)}{\partial t} = 0
\]

in combination with the constitutive relations (8) and electromagnetic fields (15) yield the nonhomogeneous vector differential equations

\[
[\nabla (k \cdot \mathbf{r}) - \nabla \phi(\mathbf{r})] \times \mathbf{E}_0(\mathbf{r}) \\
- \nabla \times \frac{1}{\varepsilon_0} \phi(\mathbf{r}) \cdot \mathbf{H}_0(\mathbf{r}) = -\frac{1}{i k_0} \nabla \times \mathbf{E}_0(\mathbf{r}) \\
[\nabla (k \cdot \mathbf{r}) - \nabla \phi(\mathbf{r})] \times \mathbf{H}_0(\mathbf{r}) \\
+ \nabla \times \frac{1}{\mu_0} \phi(\mathbf{r}) \cdot \mathbf{E}_0(\mathbf{r}) = \frac{1}{i k_0} \nabla \times \mathbf{H}_0(\mathbf{r}).
\]

Under the geometric-optics approximation, the constitutive parameters are assumed to vary only very slowly over the distance of a wavelength. Thus, \( \nabla (k \cdot \mathbf{r}) \approx k \) and the right-hand sides of equations (17) are approximately null-valued. Hence equations (17) reduce to [30]

\[
\{[\det \gamma(x, y) - p \cdot \gamma(x, y) \cdot p] \mathbb{I} + pp \cdot \gamma(x, y) \} : \mathbf{E}_0(\mathbf{r}) = 0,
\]

wherein the vector \( p = k - \nabla \phi(x, y) \) is introduced. Since [23]

\[
\det[\gamma(x, y) - pp] \equiv \gamma(x, y) \cdot [p p + pp] = 0,
\]

we find that the existence of nonzero solutions to equation (18) imposes the condition

\[
\mathcal{H} \equiv \det \gamma(x, y) - p \cdot \gamma(x, y) \cdot p = 0.
\]

Equation (20) represents the dispersion relation from which the magnitude \( k \) of the relative wavevector \( \hat{k} \) may

Figure 2. Maps illustrating the directions of \( \hat{k} = k/k = \hat{k}_x \hat{k}_y + \hat{k}_z \hat{k}_z \) for which \( \text{Im}[k] = 0 \) (blue/dark shading) and \( \text{Im}[k] \neq 0 \) (red/light shading), at the coordinate origin for \( \nu \in [0.60, 0.62, 0.70, 0.90] \). Parameter values: \( \sigma = 5 \) and \( R = 1 \).
be extracted as follows. Writing \( \mathbf{k} = k(\sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}) \), we see that the left-hand side of equation (20) is quadratic in \( k \); hence

\[
k \equiv k^\pm(\theta, \phi) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},
\]

where the coefficients

\[
a = [1 - (v'_s)^2](\cos^2 \theta + \sin^2 \theta \sin^2 \phi) + [1 - f(r_s)]^2(v'_s)^2 \sin^2 \theta \cos^2 \phi \\
b = \frac{2k_0 v'_s f(r_s)[1 + [1 - f(r_s)]^2(v'_s)^2] \sin \theta \cos \phi}{1 - [1 - f(r_s)]^2(v'_s)^2},
\]

\[
c = \frac{k_0^2(1 + f(r_s)[2 - f(r_s)(v'_s)^2])[(v'_s)^2]}{1 - [1 - f(r_s)]^2(v'_s)^2}.
\]

Let us note that \( k^+(\pi - \theta, \pi + \phi) = -k^-(\theta, \phi) \), i.e. \( k^- \) is the wavenumber for a quasi-plane wave travelling in the opposite direction to the quasi-plane wave with wavenumber \( k^+ \). This observation is a manifestation of the unirefringence of vacuum [31].

The discriminant term \( b^2 - 4ac \) in equation (21) can have a negative value. Therefore the relative wavenumber \( k \) may be complex-valued with a nonzero imaginary part, in spite of the fact that all the coefficients of the dispersion relation (20) have real values. However, \( \text{Im}[k] \neq 0 \) is indicative of evanescent waves which we exclude from our study of ray trajectories.

The partition of the \( k \)-phase space into a propagating-wave regime and an evanescent-wave regime is illustrated in figure 2, wherein the directions of \( \mathbf{k} \) for which \( \text{Im}[k] \neq 0 \) are represented at the coordinate origin for \( v'_s \in \{0.6, 0.62, 0.7, 0.9\} \). We choose the parameter values \( \sigma = 5 \) and \( R = 1 \) for these plots. Only the directions in one octant of the unit sphere need be displayed, because of symmetry. At relative speeds \( v'_s \leq 0.6 \), the relative wavenumbers are wholly real-valued for all propagation directions. As \( v'_s \) increases just beyond 0.6, the relative wavenumbers with nonzero imaginary parts emerge for \( \mathbf{k} \) directed in the \( yz \) plane. As \( v'_s \) increases further, \( \text{Im}[k] \neq 0 \) occurs at increasingly larger values of
\( \hat{\mathbf{x}} \cdot \mathbf{k} \); in the limit \( v'_s \to 1 \), we find that Im\( \{k\} \neq 0 \) occurs for all directions of propagation. The same trend is observed at locations throughout the inner region of the constitutive-parameter space referred to in our discussion of figure 1. In the outer region, however, \( k \) is everywhere real-valued for all propagation directions.

Parenthetically, we remark that the emergence of evanescent waves in non-dissipative anisotropic and bianisotropic mediums for certain wavevector directions is not unprecedented. For example, non-dissipative Faraday chiral mediums—which are similar to the Tamm medium under consideration insofar as they are bianisotropic and not Lorentz reciprocal [24]—also support evanescent waves for certain wavevector directions [32].

4. Ray trajectories

Let us preface our study of ray trajectories by noting that the time-averaged Poynting vector for a general Tamm medium may be expressed as [30, 33]

\[
\langle \mathbf{P} \rangle_t = \rho \gamma(\mathbf{r}) \cdot \mathbf{p}, \tag{25}
\]

where the scalar \( \rho \) is positive-valued provided that \( \gamma(\mathbf{r}) \) is either positive- or negative-definite. Since it is clear from equation (10) that both distinct eigenvalues of \( \gamma(\mathbf{r}) \), namely \( 1 \) and \( \frac{1 - (v'_s)^2}{1 - (v'_s)^2(v_r)^2} \), are positive-valued, we see that \( \langle \mathbf{P} \rangle_t \) is parallel to \( \gamma(\mathbf{r}) \cdot \mathbf{p} \). Now, direct vector differentiation of the scalar function \( \mathcal{H} \) defined in equation (20) reveals that \( \nabla_k \mathcal{H} = 2 \gamma(\mathbf{r}) \cdot \mathbf{p} \), wherein the shorthand \( \nabla_q \equiv \hat{x} \partial/\partial q_x + \hat{y} \partial/\partial q_y + \hat{z} \partial/\partial q_z \) for \( q = q_x \hat{x} + q_y \hat{y} + q_z \hat{z} \) is employed. Therefore, \( \nabla_k \mathcal{H} \) lies parallel to the direction of energy flux and so \( \mathcal{H} \) provides a convenient Hamiltonian function for our ray-tracing study.

We parametrize the ray trajectories in terms of \( \tau \) via \( \mathbf{r}(\tau) \); similarly, the parametrization \( k(\tau) \) is used for the relative wavevector. The ray trajectories are thus governed by the coupled vector differential equations [34, 35]

\[
\frac{d\mathbf{r}}{d\tau} = \nabla_k \mathcal{H}, \quad \frac{dk}{d\tau} = -\nabla_r \mathcal{H}, \tag{26}
\]

with the direction of ray trajectories being given by \( \nabla_k \mathcal{H} \). Once appropriate initial conditions \( \mathbf{r}(0) \) and \( k(0) \) have been

Figure 4. As figure 3 except that \( x_0 = 28 \) and \( k(0) = -\hat{x} \).
specified, the system \( (26) \) can be solved for \( \mathbf{r}(\tau) \) and \( \mathbf{k}(\tau) \) using standard numerical methods, e.g. the Runge–Kutta method [35].

We begin our presentation of ray trajectories by considering an array of rays in the \( xy \) plane, initially parallel to the \( x \) axis and equally spaced. That is, we take \( \mathbf{r}(0) = x_0 \hat{x} + y_0 \hat{y} \) with \( x_0 \) fixed and \(-1.5a_m < y_0 < 1.5a_m\). As in figures 1 and 2, we set \( \sigma = 5 \) and \( R = 1 \). In figure 3, ray trajectories are shown for \( x_0 < 0, \mathbf{k}(0) = \hat{y} \) and \( v'_s \in \{0.3, 0.6, 0.9\} \). The inner region of the constitutive-parameter space referred to in our discussion of figure 1 is shown as a shaded (yellow) ellipse (with semi-major axis length \( a_m \) and semi-minor axis length \( a_s \)) per equations \((14)\)), which becomes more eccentric at larger values of \( v'_s \). The inner region is seen to have a focusing effect, with the focus lying on the \( +x \) axis. Furthermore, the focus shifts towards the coordinate origin as the relative speed \( v'_s \) increases.

That the Tamm medium is not reciprocal in the Lorentz sense [24] is vividly illustrated by a comparison of figures 3 and 4. The scenario represented in figure 4 is the same as that of figure 3 except that \( x_0 > 0 \) and \( \mathbf{k}(0) = -\hat{x} \). Quite unlike figure 3, there is no evidence of focusing by the inner region in figure 4. On the contrary, rays appear to diverge as they pass through the inner region at low values of \( v'_s \), while rays are almost entirely excluded from the inner region altogether at \( v'_s = 0.9 \).

Further insight into the absence of Lorentz-reciprocity of the Tamm medium is provided in figure 5 wherein ray trajectories initially parallel to the \( y \) axis and equally spaced are presented. The initial relative wavevector for these trajectories is \( \mathbf{k}(0) = \hat{y} \) and we have set \( \mathbf{r}(0) = x_0 \hat{x} + y_0 \hat{y} \) with fixed \( y_0 < 0 \) while \(-1.5a_M < x_0 < 1.5a_M\). As for figures 1–4, \( \sigma = 5 \) and \( R = 1 \). The plots in figure 5 are clearly asymmetric with respect to the \( y \) axis and the ray trajectories become progressively excluded from the inner region as \( v'_s \) increases.

Rays initially propagating in radial directions in the \( xy \) plane are represented in figure 6 for \( \sigma = 10 \) and \( R = 1 \). We track rays which emanate from point sources in the inner region (at the coordinate origin) and in the outer region at locations on the positive and negative \( x \) axis, i.e. \( \mathbf{r}(0) = x_0 \hat{x} \). The relative speed \( v'_s \in \{0.3, 0.6, 0.9\} \). Equally spaced angular directions for the initial relative wavevector \( \mathbf{k}(0) \) were considered. However, some initial directions in the inner region correspond to evanescent waves and these are not represented in figure 6. The proportion of directions which correspond to evanescent waves increases as \( v'_s \) increases for sources in the inner region. Indeed, for \( v'_s = 0.9 \) with the source at the coordinate origin, only 30% of the possible \( \mathbf{k}(0) \)
Figure 6. Trajectories for rays in the \( xy \) plane, emanating from sources on the \( x \) axis, i.e. \( r(0) = x_0 \hat{x} \), at equally spaced angular directions of \( k(0) \), for \( v'_s \in \{0.3, 0.6, 0.9\} \). Parameter values: \( \sigma = 10 \) and \( R = 1 \).

radial directions correspond to propagating rays. The general trends apparent in figures 3–5 are also apparent in figure 6. That is, the inner region has a focusing effect for sources located outside the inner region with \( x_0 < 0 \); for sources located outside the inner region with \( x_0 > 0 \) ray trajectories tend to be progressively excluded from the inner region as \( v'_s \) increases.

Let us now turn to the influence of the thickness of the transition zone between the inner and outer regions, as dictated by the parameter \( \sigma \) via the scalar function \( f \). In figure 7 ray trajectories are provided which correspond to the scenario of figure 6 but with \( \sigma = 1 \) and 25. We note that \( \sigma = 25 \) results in a more sharply defined top-hat profile with straighter sides for \( f \), whereas \( \sigma = 1 \) results in a profile with more rounded sides, as compared to \( \sigma = 10 \) which was used for figure 6. Comparing figures 6 and 7, we deduce that, although the change in the direction of rays at the boundary between the inner and outer regions becomes more pronounced as \( \sigma \) increases, the general pattern of ray trajectories remains largely unaffected.

For clarity of representation, ray trajectories restricted to the \( xy \) plane were considered in figures 3–7. Trajectories of the same form can be observed in the \(xz\) plane. The trajectories for the \( yz\) plane are likewise similar, albeit then the inner region of the constitutive-parameter space is obviously circular in shape, regardless of the relative speed \( v'_s \).

5. Closing remarks

A flat-spacetime representation of the Alcubierre spacetime has been established by means of a Tamm medium which is asymptotically identical to the vacuum and has constitutive parameters which are continuous functions of the spatial coordinates. Thus, the Tamm medium is amenable to physical realization as a micro- or nanostructured metamaterial. An alternative approach—which utilizes a Galilean transformation
instead of the Lorentz transformation (4)—gives rise to a Tamm medium which is not asymptotically identical to the vacuum and is accordingly less well-suited to physical realization [36].

Our geometric-optics study has revealed that ray trajectories are acutely sensitive to the relative speed $v'$ of the warp bubble, and to the direction of its velocity $v'$, but rather less sensitive to the thickness of the transition zone between the warp bubble and its background. In particular, for rays which travel in the same direction as $v'$, the warp bubble acts as a focusing lens, especially at large values of $v'$.

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