Algorithmic Instabilities of Accelerated Gradient Descent

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Abstract

We study the algorithmic stability of Nesterov’s accelerated gradient method. For convex quadratic objectives, Chen et al. (2018) proved that the uniform stability of the method grows quadratically with the number of optimization steps, and conjectured that the same is true for the general convex and smooth case. We disprove this conjecture and show, for two notions of algorithmic stability (including uniform stability), that the stability of Nesterov’s accelerated method in fact deteriorates exponentially fast with the number of gradient steps. This stands in sharp contrast to the bounds in the quadratic case, but also to known results for non-accelerated gradient methods where stability typically grows linearly with the number of steps.

1 Introduction

Algorithmic stability has emerged over the last two decades as a central tool for generalization analysis of learning algorithms. While the classical approach in generalization theory originating in the PAC learning framework appeal to uniform convergence arguments, more recent progress on stochastic convex optimization models, starting with the pioneering work of Bousquet and Elisseeff (2002) and Shalev-Shwartz et al. (2009), has relied on stability analysis for deriving tight generalization results for convex risk minimizing algorithms.

Perhaps the most common form of algorithmic stability is the so called uniform stability (Bousquet and Elisseeff, 2002). Roughly, the uniform stability of a learning algorithm is the worst-case change in its output model, in terms of its loss on an arbitrary example, when replacing a single sample in the data set used for training. Bousquet and Elisseeff (2002) initially used uniform stability to argue about the generalization of empirical risk minimization with strongly convex losses. Shalev-Shwartz et al. (2009) revisited this concept and studied the stability effect of regularization on the generalization of convex models. Their bounds were recently improved in a variety of ways (Feldman and Vondrak, 2018, 2019; Bousquet et al., 2020) and their approach has been influential in a variety of settings (e.g., Koren and Levy, 2015; Gonen and Shalev-Shwartz, 2017; Charles and Papailiopoulos, 2018). In fact, to this day, algorithmic stability is essentially the only general approach for obtaining tight (dimension free) generalization bounds for convex optimization algorithms applied to the empirical risk (see Shalev-Shwartz et al., 2009; Feldman, 2016).

Significant focus has been put recently on studying the stability properties of iterative optimization algorithms. Hardt et al. (2016) considered stochastic gradient descent (SGD) and gave the first bounds on its uniform stability for a convex and smooth loss function, that grow linearly with the number of optimization steps. As observed by Feldman and Vondrak (2018) and Chen et al. (2018), their arguments also apply with minor modifications to full-batch gradient descent (GD). Bassily et al. (2020) exhibited a significant gap in stability between the smooth and non-smooth cases, showing that non-smooth GD and SGD are inherently less stable than their smooth counterparts. Even further, algorithmic stability has also been used as an analysis technique in stochastic mini-batched iterative optimization (e.g., Wang et al., 2017; Agarwal et al., 2020), and has been proved crucial to the design and analysis of differentially private optimization algorithms (Wu et al., 2017; Bassily et al., 2019; Feldman et al., 2020), both of which focusing primarily on smooth optimization.

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Having identified smoothness as key to algorithmic stability of iterative optimization methods, the following fundamental question emerges: how stable are optimal methods for smooth convex optimization? In particular, what is the algorithmic stability of the celebrated Nesterov accelerated gradient (NAG) method (Nesterov, 1983)—a cornerstone of optimal methods in convex optimization? Besides being a basic and natural question in its own right, its resolution could have important implications to the design and analysis of optimization algorithms, as well as serve to deepen our understanding of the generalization properties of iterative gradient methods. Chen et al. (2018) addressed this question in the case of convex quadratic objectives and derived bounds on the uniform stability of NAG that grow quadratically with the number of gradient steps (as opposed to the linear growth known for GD). They conjectured that similar bounds hold true more broadly, but fell short of proving this for general convex and smooth objectives. Our work is aimed at filling this gap.

1.1 Our Results

We establish tight algorithmic stability bounds for the Nesterov accelerated gradient method (NAG). We show that, somewhat surprisingly, the uniform stability of NAG grows exponentially fast with the number of steps in the general convex and smooth setting. Namely, the uniform stability of $T$-steps NAG with respect to a dataset of $n$ examples is in general $\exp(\Omega(T^2/n))$, and in particular, after merely $T = O(\log n)$ steps the stability becomes the trivial $\Omega(1)$. This result demonstrates a sharp contrast between the stability of NAG in the quadratic case and in the general convex, and disproves the conjecture of Chen et al. (2018) that the uniform stability of NAG in the general convex setting is $O(T^2/n)$, as in the case of a quadratic objective.

Our results in fact apply to a simpler notion of stability—one that is arguably more fundamental in the context of iterative optimization methods—which we term initialization stability. The initialization stability of an algorithm $A$ (formally defined in Section 2 below) measures the sensitivity of $A$’s output to an $\epsilon$-perturbation in its initialization point. For this notion, we demonstrate a construction of a smooth and convex objective function such that, for sufficiently small $\epsilon$, the stability of $T$-steps NAG is lower bounded by $\exp(\Omega(T))\epsilon$. Here again, we exhibit a dramatic gap between the quadratic and general convex cases: for quadratic objectives, we show that the initialization stability of NAG is upper bounded by $O(T\epsilon)$.

For completeness, we also prove initialization stability upper bounds in a few relevant convex optimization settings: for GD, we analyze both the smooth and non-smooth cases; for NAG, we give bounds for quadratic objectives as well as for general smooth ones. Table 1 summarizes the stability bounds we establish compared to existing bounds in the literature. Note in particular the remarkable exponential gap between the stability bounds for GD and NAG in the general smooth case, with respect to both stability definitions. Stability lower bounds for NAG are discussed in Sections 3 and 4; initialization stability upper bounds for the various settings are given in Appendix D, and additional uniform stability bounds are detailed in Appendix E.

| Method | Setting | Init. Stability | Unif. Stability | Reference |
|--------|---------|----------------|----------------|-----------|
| GD     | convex, smooth | $\Theta(\epsilon)$ | $\Theta(T/n)$ | Hardt et al. (2016) |
| GD     | convex, non-smooth | $\Theta(\epsilon + \eta\sqrt{T})$ | $\Theta(\eta\sqrt{T} + \eta T/n)$ | Bassily et al. (2020) |
| NAG    | convex, quadratic | $O(T\epsilon)$ | $\Theta(T^2/n)$ | Chen et al. (2018) |
| NAG    | convex, smooth | $\exp(\Theta(T))\epsilon$ | $\exp(\Theta(T))T/n$ | (this paper) |

Table 1: Stability bounds introduced in this work (in bold) compared to existing bounds. For simplicity, all bounds in the smooth case are for $\eta = \Theta(1/\beta)$. The lower bounds for NAG are presented here in a simplified form and the actual bounds exhibit a fluctuation in the increase of stability; see also Fig. 2 and the precise results in Sections 3 and 4.

Finally, we remark that our focus here is on the general convex (and smooth) case, and we do not provide formal results for the strongly convex case. However, we argue that stability analysis in the latter case is not as compelling as in the general case. Indeed, a strongly convex objective admits a unique minimum, and so NAG will converge to an $\epsilon$-neighborhood of this minimum in $O(\log(1/\epsilon))$ steps from any initialization, at which point its stability becomes $O(\epsilon)$; thus, with strong convexity perturbations in initialization get quickly washed away as the algorithm rapidly converges to the unique optimum. (A similar reasoning also applies
1.2 Overview of Main Ideas and Techniques

We now provide some intuition to our constructions and highlight some of the key ideas leading to our results. We start by revisiting the analysis of the quadratic case which is simpler and better understood.

**Why NAG is stable for quadratics:** Consider a quadratic function $f$ with Hessian matrix $H \succeq 0$. For analyzing the initialization stability of NAG, let us consider two runs of the method initialized at $x_0, \tilde{x}_0$ respectively, and let $(x_t, y_t), (\tilde{x}_t, \tilde{y}_t)$ denote the corresponding NAG iterates at step $t$. Further, let us denote by $\Delta x_t \triangleq x_t - \tilde{x}_t$ the difference between the two sequences of iterates. Using the update rule of NAG (see Eqs. (1) and (2) below) and the fact that for a quadratic $f$, differences between gradients can be expressed as $\nabla f(x) - \nabla f(x') = H(x - x')$ for any $x, x' \in \mathbb{R}^d$, it is straightforward to show that the distance $\Delta x_t$ evolves according to

$$
\Delta x_{t+1} = (I - \eta H)((1 + \gamma_t)\Delta x_t - \gamma_t \Delta x_{t-1}).
$$

This recursion can be naturally put in matrix form, leading to:

$$
\begin{pmatrix}
\Delta x_{t+1} \\
\Delta x_t
\end{pmatrix} = \prod_{k=1}^{t} \begin{pmatrix}
(1 + \gamma_k)A & -\gamma_k A \\
I & 0
\end{pmatrix} \begin{pmatrix}
\Delta x_t \\
\Delta x_0
\end{pmatrix},
$$

where here $A = I - \eta H$. Thus, for a quadratic $f$, bounding the divergence $\|\Delta x_t\|$ between the two NAG sequences reduces to controlling the operator norm of the matrix product above, namely

$$
\left\| \prod_{k=1}^{t} \begin{pmatrix}
(1 + \gamma_k)A & -\gamma_k A \\
I & 0
\end{pmatrix} \right\|.
$$

Remarkably, it can be shown that this norm is $O(t)$ for any $0 \leq A \leq I$ and any choice of $-1 \leq \gamma_1, \ldots, \gamma_t \leq 1$. (This can be seen by writing the Schur decomposition of the involved matrices, as we show in Appendix D.A.)

As a consequence, the initialization stability of NAG for a quadratic objective $f$ is shown to grow only linearly with the number of steps $t$.

**What breaks down in the general convex case:** For a general convex (twice-differentiable and smooth) $f$, the Hessian matrix is of course no longer fixed across the execution. Assuming for simplicity the one-dimensional case, similar arguments show that the relevant operator norm is of the form

$$
\left\| \prod_{k=1}^{t} \begin{pmatrix}
(1 + \gamma_k)A_k & -\gamma_k A_k \\
I & 0
\end{pmatrix} \right\|,
$$

where $0 \leq A_1, \ldots, A_t \leq 1$ are related to Hessians of $f$ taken at suitable points along the optimization trajectory. However, if $A_k$ are allowed to vary arbitrarily between steps, the matrix product above might explode exponentially fast, even in the one-dimensional case! Indeed, fix $\gamma_k = 0.9$ for all $k$, and set $A_k = 0$ whenever $k \mod 3 = 0$ and $A_k = 1$ otherwise; then using simple linear algebra the operator norm of interest can be shown to satisfy

$$
\left\| \begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
1.9 & -0.9 \\
1 & 0
\end{pmatrix} \begin{pmatrix}
1.9 & -0.9 \\
1 & 0
\end{pmatrix} \right\|^{t/3} = \left\| \begin{pmatrix}
0 & 0 \\
2.71 & -1.71
\end{pmatrix} \right\| \geq 1.15^t.
$$

**How a hard function should look like:** The exponential blowup we exhibited above hinged on a worst-case sequence $A_1, \ldots, A_t$ that varies significantly between consecutive steps. It remains unclear, however, what does this imply for the actual optimization setup we care about, and whether such a sequence can be realized by Hessians of a convex and smooth function $f$. Our main results essentially answer the latter
question on the affirmative and build upon a construction of such a function \( f \) that directly imitates such a bad sequence.

Concretely, we generate a hard function inductively based on a running execution of NAG, where in each step we amend the construction with a “gadget” function having a piecewise-constant Hessian (that equals either 0 or the maximal \( \beta \)); see Fig. 1 for an illustration of this construct. The interval pieces are carefully chosen based on the NAG iterates computed so far in a way that a slightly perturbed execution would traverse through intervals with an appropriate pattern of Hessians that induces a behaviour similar to the one exhibited by the matrix products above, leading to an exponential blowup in the stability terms.

Fig. 2 shows a simulation of the divergence between the two trajectories of NAG on the objective function we construct, illustrating how the divergence fluctuates between positive and negative values, with its absolute value growing exponentially with time. More technical details on this construction can be found in Section 3.

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**From initialization stability to uniform stability:** Finally, we employ a simple reduction to translate our results regarding initialization stability to relate to uniform stability in the context of empirical risk optimization, where one uses (full-batch) NAG to minimize the (convex, smooth) empirical risk induced by

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1Chen et al. (2018) give an alternative argument based on Chebyshev polynomials.
a sample $S$ of $n$ examples. Concretely, we show that by replacing a single example in $S$, we can arrive at a scenario where after one step of NAG on the original and modified samples the respective iterates are $\epsilon = \Theta(1/n)$ away from each other, whereas in the remaining steps both empirical risks simulate our bad function from before. Thus, we again end up with an exponential increase in the divergence between the two executions, that leads to a similar increase in the algorithmic (uniform) stability: the latter becomes as large as $\Omega(1)$ after merely $T = O(\log n)$ steps of full-batch NAG. The formal details of this reduction can be found in Section 4.

1.3 Discussion and Additional Related Work

It is interesting to contrast our results with what is known for the closely related heavy ball method (Polyak, 1964). Classic results show that while for convex quadratic objectives the (properly tuned) heavy ball method attains the accelerated $O(1/T^2)$ convergence rate, for general convex and smooth functions it might even fail to converge at all (see Lessard et al., 2016). More specifically, it is known that there exists objectives for which heavy ball assumes a cyclic trajectory that never gets close to the optimum; it is then not hard to turn such a construction to an instability result for heavy ball, as a slight perturbation in the cyclic pattern can be shown to make the method converge to optimum.

Also related to our work is Devolder et al. (2014), that analyzed GD and NAG with inexact first-order information, namely, in a setting where each gradient update is contaminated with a bounded yet arbitrary perturbation. Interestingly, they showed that in contrast to GD, NAG suffers from an accumulation of errors—which appears analogous to the linear increase in initialization stability the latter experiences in the quadratic case. At the same time, in the general convex case their results might seem to be at odds with ours as we show that even a single perturbation at initialization suffices for extreme instabilities. However, note that they analyze the impact of perturbations on the convergence rate of NAG (in terms of objective value), whereas algorithmic stability is concerned with their effect on the actual iterates: specifically, initialization stability captures to what extent the iterates of the algorithm might stray away from their original positions as a result of a small perturbation in the initialization point.

Our work leaves a few intriguing open problems for future investigation. Most importantly, it remains unclear whether there exists a different accelerated method (one with the optimal $O(1/T^2)$ rate for smooth and convex objectives) that is also poly($T$)-stable. Bubeck et al. (2015) suggested a geometric alternative to NAG that comes to mind, and it could be interesting to check whether this method or a variant thereof is more stable than NAG. Another open question is to resolve the gap between our stability lower and upper bounds for NAG in the regime $\eta \ll 1/\beta$: while our lower bounds have an exponential dependence on $\eta$, the upper bounds do not. Finally, it could be interesting to determine whether the $O(T\epsilon)$ initialization stability bound we have for NAG in the quadratic case is tight (the corresponding uniform stability result is actually tight even for linear losses, but this may not be the case for initialization stability).

2 Preliminaries

In this work we are interested in optimization of convex and smooth functions over the $d$-dimensional Euclidean space $\mathbb{R}^d$. A function $f$ is said to be $\beta$-smooth (for $\beta > 0$) if its gradient is $\beta$-Lipschitz, namely, if for all $u,v \in \mathbb{R}^d$ it holds that $\|\nabla f(u) - \nabla f(v)\| \leq \beta\|u - v\|$.

2.1 Nesterov Accelerated Gradient Method

The Nesterov Accelerated Gradient (NAG) method (Nesterov, 1983) we consider in this paper takes the following form. Starting with $x_0$ and $y_0 = x_0$, it iterates for $t = 1, 2, \ldots$:

\begin{align*}
x_t &= y_t - 1 - \eta \nabla f(y_t) ; \quad (1) \\
y_t &= x_t + \gamma_t (x_t - x_{t-1}) , \quad (2)
\end{align*}

where $\gamma_t = \frac{t-1}{T^2}$ and $\eta > 0$ is a step-size parameter. For a $\beta$-smooth convex objective $f$ and $0 < \eta \leq 1/\beta$, this method exhibits the convergence rate $O(1/\eta T^2)$; for $\eta = 1/\beta$, this gives the optimal convergence rate for the class of $\beta$-smooth convex functions (see Nesterov (2003)). We remark that while NAG appears in several
other forms in the literature, many of these are in fact equivalent to the one given in Eqs. (1) and (2). For more details, see Appendix C.

Throughout, we use the notation \( \text{NAG}(f, x_0, t, \eta) \) to refer to the iterates \((x_t, y_t)\) at step \(t\) of NAG on \(f\) initialized at \(x_0\) with step size \(\eta\). We sometimes drop the step size argument and use the shorter notation \(\text{NAG}(f, x_0, t)\) when \(\eta\) is clear from the context.

We will use the following definitions and relations throughout. We introduce the following notation for the momentum term of NAG, for all \(t > 0\):

\[
m_t \triangleq \gamma_t (x_t - x_{t-1}).
\] (3)

Using this notation, we have that

\[
x_t = y_{t-1} - \eta \nabla f(y_{t-1}),
\] (4)

\[
y_t = x_t + m_t = y_{t-1} - \eta \nabla f(y_{t-1}) + m_t,
\] (5)

\[
m_t = \gamma_t(m_{t-1} - \eta \nabla f(y_{t-1})).
\] (6)

Here, Eq. (6) follows from Eqs. (3) to (5) via

\[
m_t = \gamma_t(x_t - x_{t-1}) \quad \text{(Eq. (3))}
\]

\[
= \gamma_t(y_{t-1} - \eta \nabla f(y_{t-1}) - x_{t-1}) \quad \text{(Eq. (4))}
\]

\[
= \gamma_t(x_{t-1} + m_{t-1} - \eta \nabla f(y_{t-1}) - x_{t-1}) \quad \text{(Eq. (5))}
\]

\[
= \gamma_t(m_{t-1} - \eta \nabla f(y_{t-1})).
\]

2.2 Algorithmic Stability

We consider two forms of algorithmic stability. The first is the well-known uniform stability (Bousquet and Elisseeff, 2002), while the second is initialization stability which we define here.

Uniform stability. Consider the following general setting of supervised learning. There is a sample space \(Z\) of examples and an unknown distribution \(D\) over \(Z\). We receive a training set \(S = (z_1, \ldots, z_n)\) of \(n\) samples drawn i.i.d. from \(D\). The goal is finding a model \(w\) with a small population risk:

\[
R(w) \triangleq \mathbb{E}_{z \sim D}[\ell(w; z)],
\]

where \(\ell(w; z)\) is the loss of the model described by \(w\) on an example \(z\). However, as we cannot evaluate the population risk directly, learning algorithms will be applied on the empirical risk with respect to the sample \(S\), given by

\[
R_S(w) \triangleq \frac{1}{n} \sum_{i=1}^{n} \ell(w; z_i).
\]

In this paper, our algorithm of interest in this context is full-batch NAG, namely, NAG applied to the empirical risk \(R_S\). We use the following notion of uniform stability.\(^2\)

**Definition 1 (uniform stability).** Algorithm \(A\) is \(\epsilon\)-uniformly stable if for all \(S, S' \in \mathbb{Z}^n\) such that \(S, S'\) differ in at most one example, the corresponding outputs \(A(S)\) and \(A(S')\) satisfy

\[
\sup_{z \in \mathbb{Z}} |\ell(A(S); z) - \ell(A(S'); z)| \leq \epsilon.
\]

We use \(\delta_A^{\text{unif}}(n)\) to denote the infimum over all \(\epsilon > 0\) for which this inequality holds.

\(^2\)We give here a definition suitable for deterministic algorithms, which suffices for the context of this paper. Similar definitions exist for randomized algorithms; see for example Hardt et al. (2016); Feldman and Vondrak (2018).
Initialization stability. A second notion of algorithmic stability that we define and discuss in this paper, natural in the context of iterative optimization methods, pertains to the stability of the optimization algorithm with respect to its initialization point. Initialization stability measures the sensitivity of the algorithm’s output to a small perturbation in its initial point; formally,

**Definition 2 (Initialization stability).** Let $A$ be an algorithm that when initialized at a point $x \in \mathbb{R}^d$, produces $A(x) \in \mathbb{R}^d$ as output. Then for $\epsilon > 0$, the initialization stability of $A$ at $x_0 \in \mathbb{R}^d$ is given as

$$
\delta_{A}^{\text{init}}(x_0, \epsilon) = \sup \{ \| A(\tilde{x}) - A(x_0) \| : \tilde{x} \in \mathbb{R}^d, \| \tilde{x} - x_0 \| \leq \epsilon \}.
$$

## 3 Initialization Stability of NAG

In this section we prove our first main result, regarding the initialization stability of NAG:

**Theorem 3.** Let $\epsilon, G, \beta > 0$ and $0 < \eta \leq 1/\beta$. Consider two initialization points $x_0 = 0$, $\tilde{x}_0 = \epsilon$. Then, there exists a convex, $\beta$-smooth, $G$-Lipschitz function $f$ that attains a minimum over $\mathbb{R}$, and universal constants $c_1, c_2 > 0$, such that the sequences $(x_t, y_t) = \text{NAG}(f, x_0, t, \eta)$ and $(\tilde{x}_t, \tilde{y}_t) = \text{NAG}(f, \tilde{x}_0, t, \eta)$ satisfy

$$
\delta_{\text{NAG}_i}^{\text{init}}(x_0, \epsilon) \geq |x_t - \tilde{x}_t| \geq \min \left\{ \frac{G}{\eta \beta}, c_2 e^{c_1 \eta \beta \epsilon} \right\}, \quad \forall t \in \{ \lceil \frac{10}{\eta \beta} \rceil (i + 2) : i = 1, 2, \ldots \}.
$$

Furthermore, for all $t > \lceil \frac{10}{\eta \beta} \rceil \left( \ln \frac{2G}{\eta \beta} + 3 \right)$ it holds that $\delta_{\text{NAG}_i}^{\text{init}}(x_0, \epsilon) \geq \frac{G}{2 \eta \beta}$.

In words, the theorem establishes an exponential blowup in the distance between the two trajectories $x_t$ and $\tilde{x}_t$ during the initial $O(1/\epsilon)$ steps, after which the (lower bound on the) distance reaches a constant and stops increasing. Notice that in the blowup phase, an increase in distance happens roughly every $\eta \beta$ steps; indeed, the actual behaviour of NAG on the function we construct exhibit fluctuations in the difference $x_t - \tilde{x}_t$, as illustrated in Fig. 2. We remark that a similar bound holds also for the $y_t$ sequence produced by NAG.

**Construction.** Throughout this section, we will assume without loss of generality that $0 < \epsilon < G/2\beta$. (When $\epsilon \geq G/2\beta$ our result holds simply for a constant function.) To lower bound the initialization stability and prove the theorem, we will rely on the following construction of functions $f_0, f_1, \ldots : \mathbb{R} \to \mathbb{R}$. Let the parameters $G, \beta, \eta, \epsilon > 0$ be given, and for all $i \geq 0$ define $n_i \triangleq \lceil 10/\eta \beta \rceil (i + 2)$. The construction proceeds as follows:

(i) Let $f_0(x) \triangleq -Gx$;

(ii) For $i \geq 1$:

- Let $(x_{n_i}, y_{n_i}) = \text{NAG}(f_{i-1}, 0, n_i, \eta)$ and $(\tilde{x}_{n_i}, \tilde{y}_{n_i}) = \text{NAG}(f_{i-1}, \epsilon, n_i, \eta)$;

- Define $f_i : \mathbb{R} \to \mathbb{R}$ as follows:

$$
f_i(x) \triangleq -Gx + \beta \int_{-\infty}^{x} \int_{-\infty}^{y} 1 \left( \exists j \leq i \text{ s.t. } z \in [y_{n_j}, y_{n_j}^\text{max}] \right) dzdy,
$$

where $y_{n_i}^\text{min} = \min \{ y_{n_i}, \tilde{y}_{n_i} \}$, $y_{n_i}^\text{max} = \max \{ y_{n_i}, \tilde{y}_{n_i} \}$;

(iii) Let $M = \sup \{ i \geq 0 : \max_x \nabla f_i(x) < -\frac{1}{2}G, \forall 0 \leq j \leq i \}$.

Note that the above recursion defines an infinite sequence of functions $f_0, f_1, f_2, \ldots : \mathbb{R} \to \mathbb{R}$. Ultimately, we will be interested in the functions $\{ f_i \}_{i \leq M}$ which we will analyze in order to prove the instability result. Further, note that $\max_x \nabla f_M(x) < -\frac{1}{2}G$, thus $M$ itself is well-defined, possibly $\infty$. The functions constructed above are not lower-bounded (and thus do not admit a minimum). Below we define a lower-bounded adaptation of $f_M$ (assuming $1 \leq M < \infty$, which is proved in Lemma 7 later on). Our modified version of $f_M$, termed $f$ is defined by a quadratic continuation of $f_M$ right of $p \triangleq y_{n_M}^\text{max}$, up to a plateau. This construction is defined formally as:

$$
f(x) \triangleq \begin{cases} f_M(x) & x \leq p; \\
f_M(p) + \nabla f_M(p)(x - p) + \frac{\beta}{2}(x - p)^2 & p < x \leq p - \frac{1}{\beta} \nabla f_M(p); \\
f_M(p) - \frac{1}{2\beta} \nabla f_M(p)^2 & \text{otherwise}.
\end{cases}
$$
Analysis. We start by stating a few lemmas we will use in the proof of our main theorem. Our focus is on the functions $f_i$ for $0 \leq i \leq M$, deferring the analysis of $f$ to after we establish that $M$ is finite. First, we show that the functions we constructed are indeed convex, smooth and Lipschitz.

Lemma 4. For all $0 \leq i \leq M$, the function $f_i$ is convex, $\beta$-smooth and $G$-Lipschitz.

Proof. The second derivative of $f_i$ is
\[
\nabla^2 f_i(x) = \beta \cdot 1 \left\{ \sum_{j \leq i} z \in [y_{n_j}^{\min}, y_{n_j}^{\max}] \right\} \in [0, \beta].
\]
Thus, $f_i$ is convex and $\beta$-smooth. We lower bound the first derivative by
\[
\nabla f_i(x) = -G + \beta \int_{-\infty}^{x} 1 \left\{ \sum_{j \leq i} z \in [y_{n_j}^{\min}, y_{n_j}^{\max}] \right\} dz \geq -G,
\]
and by the definition of $M$, $\nabla f_i(x) < -G/2$. Hence for all $x \in \mathbb{R}$ we have $\nabla f_i(x) \in [-G, -G/2)$, so $f_i$ is $G$-Lipschitz over $\mathbb{R}$.

Next, we analyze the iterations of NAG on $f_i$ for any given $0 \leq i \leq M$. Fix such index $i$ and consider $(x_t, y_t) = \text{NAG}(f_i, 0, t)$ and $(\tilde{x}_t, \tilde{y}_t) = \text{NAG}(f_i, \epsilon, t)$ for all $t \leq T$ for some $T \geq n_{i+1}$. We introduce the following compact notation for differences between the NAG terms related to the two sequences:
\[
\Delta_i^x \triangleq x_t - \tilde{x}_t, \quad \Delta_i^y \triangleq y_t - \tilde{y}_t, \quad \Delta_i^f \triangleq \nabla f_i(x_t) - \nabla f_i(\tilde{x}_t), \quad \Delta_i^m \triangleq m_t - \tilde{m}_t.
\]
From the update rules of NAG (Eqs. (4) to (6)), we have that
\[
\Delta_i^x = \Delta_{i-1}^x - \eta \Delta_{i-1}^f, \quad \Delta_i^y = \Delta_{i-1}^y + \Delta_{i-1}^m - \eta \Delta_{i-1}^f, \quad \Delta_i^m = \gamma_t \Delta_{i-1}^m - \eta \Delta_{i-1}^f.
\]
Our next lemma below describes the evolution of the differences $\Delta_i^f$ and $\Delta_i^m$ in terms of $\Delta_i^y$.

Lemma 5. For all $t \leq T$,
\[
\Delta_i^f = \begin{cases} \beta \Delta_i^y & \text{if } t \in \{n_j\}_{j=1}^i; \\ 0 & \text{otherwise}, \end{cases} \quad \text{and} \quad \Delta_i^m = \begin{cases} \gamma_t (\Delta_{i-1}^m - \eta \Delta_{i-1}^f) & \text{if } t \in \{n_j + 1\}_{j=1}^i; \\ \gamma_t \Delta_{i-1}^m & \text{otherwise}. \end{cases}
\]

The following lemma summarise the evolution of the distance between the sequences $y_t$ and $\tilde{y}_t$ at steps $t \in \{n_j\}_{j=1}^i$ and for $t > n_{i+1}$. The exponential growth is achieved by a balance between the difference in momentum terms and the difference between the sequences.

Lemma 6. For the difference terms $\Delta_i^y$, we have the following:
(i) For all $1 \leq j \leq i$, it holds that $\frac{1}{2} \eta \beta |\Delta_{n_j}^y| \leq |\Delta_{n_{j+1}}^m| \leq \frac{1}{6} \eta \beta |\Delta_{n_{j+1}}^y|$. 
(ii) For all $0 \leq j \leq i$, it holds that $|\Delta_{n_{j+1}}^y| = y_{n_{j+1}}^{\max} - y_{n_{j+1}}^{\min} \geq 3\epsilon$. 
(iii) For all $t > n_{i+1}$, it holds that $|\Delta_t^y| \geq |\Delta_{n_{i+1}}^y|$. 

Finally, we can show that $f$ is well-defined by proving that $M$ is finite in the following lemma. The bound of $M$ also indicate that after $O(\log \frac{1}{\epsilon})$ steps the two trajectories $y_t, \tilde{y}_t$ reach a constant distance.

Lemma 7. It holds that $1 \leq M \leq \ln \frac{3G}{2\beta \epsilon}$ (in particular, $M$ is finite), and $y_{n_{M+1}}^{\max} - y_{n_{M+1}}^{\min} \geq \frac{G}{3\beta}$. 

Now we can return to our $f$. First, we show that indeed posses the basic properties for Theorem 3.

Lemma 8. The function $f$ is convex, $\beta$-smooth, $G$-Lipschitz and attains a minimum $x^0 \in \arg \min_{x} f(x)$ s.t. $|x_0 - x^*| = O((G/\eta \beta^2) \log(G/3\epsilon^2))$ for $x_0 \in \{0, \epsilon\}$.

The final lemma we require shows that the distance between the two trajectories is the same for $f$ and $f_M$. This holds true since the two functions coincide for $x \leq p$, after which the iterates reach a plateau which induces similar stability dynamics as the linear part of $f_M$ at $x > p$. 

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**Lemma 9.** Let \((x_t, y_t) = \text{NAG}(f, 0, t, \eta)\) and \((\tilde{x}_t, \tilde{y}_t) = \text{NAG}(f, \epsilon, t, \eta)\) be the iterations of NAG on \(f\) from our initialization points. Similarly, for \(f_M\), let \((\tilde{x}_t, \tilde{y}_t) = \text{NAG}(f_M, 0, t, \eta)\) and \((\tilde{x}_t, \tilde{y}_t) = \text{NAG}(f_M, \epsilon, t, \eta)\). Then for all \(t\), we have that \(x_t - \tilde{x}_t = \tilde{x}_t - \tilde{x}_t\) and \(y_t - \tilde{y}_t = \tilde{y}_t - \tilde{y}_t\).

We defer the proofs of Lemmas 5 to 9 to Appendix A, and proceed to prove our main result.

**Proof of Theorem 3.** Based on Lemma 9, it suffices to show that the lower bound holds for the function \(f_M\). Let \(c_1 = \frac{1}{11}\ln(3), c_2 = \frac{4}{5}3^{-3}\). Let \(t = n_i\) for some \(i \geq 1\). The first case we will deal with is when \(i \leq M + 1\). We already established with Lemma 6 that \(|\Delta^y_{n_i}| \geq 3^{i-1}\epsilon\). Since \(t = n_i = (i+2)[10/\eta\beta]\), \(i \geq \eta\beta/11 - 2\). Hence,

\[
|\Delta^y_{n_i}| \geq 3^{i\eta\beta/11 - 3}\epsilon \implies |\Delta^y_{n_i}| \geq \frac{5}{4}c_2e^{c_1\eta\beta T}\epsilon.
\]

To relate to \(|x_t - \tilde{x}_t|\),

\[
|x_t - \tilde{x}_t| = |\Delta^x_{n_i}| \\
\geq |\Delta^x_{n_i} - |\Delta^m_{n_i}|| \\
\geq |\Delta^x_{n_i} - |\Delta^m_{n_i+1}|| \prod_{t=n_i+2}^{n_i} \gamma_t \quad \text{(Eq. (8))} \tag{Lemma 5 for } t = n_i, \ldots, n_i - 1 + 2 \]

\[
\geq |\Delta^x_{n_i} - |\Delta^m_{n_i+1}|| \quad \text{(Since } t \geq 1 \Rightarrow 0 \leq \gamma_t \leq 1) \tag{\text{Lemma 6}}
\]

\[
\geq |\Delta^x_{n_i}| (1 - \frac{\eta\beta}{5}) \geq \frac{4}{5}|\Delta^y_{n_i}| \geq c_2e^{c_1\eta\beta T}\epsilon. \quad \text{(\eta \leq \frac{1}{\beta})}
\]

Here, (*) follows from Lemma 6 if \(i > 1\) and the case of \(i = 1\) follows by combining Lemma 5 and \(\Delta^m_{n_i+1} = 0\) which implies that \(\Delta^m_{n_i+1} = 0\). If \(t > n_i + 1\) (includes the case of \(i > M + 1\) and \(t > [10/\eta\beta]\ln \frac{4d}{2\beta \epsilon} + 3\) from Lemma 7), since \(t - 1 \geq n_i + 1\),

\[
|x_t - \tilde{x}_t| = |\Delta^y_{t-1} - \eta\Delta^f_{t-1}| \\
\geq |\Delta^y_{t-1}| \quad \text{(Eq. (7))} \tag{Lemma 5} \]

\[
\geq |\Delta^y_{n_i+1}|. \quad \text{(Lemma 6)}
\]

And using Lemma 7 we conclude that \(|x_t - \tilde{x}_t| \geq \frac{G^2}{15}\). Hence, with Lemma 4, \(f_M\) holds all properties of Theorem 3 beside attaining a minimum, and using Lemmas 8 and 9, \(f\) posses all the properties needed for Theorem 3. \(\blacksquare\)

## 4 Uniform Stability of NAG

In this section we present our second main result, regarding the uniform stability of (full-batch) NAG. This is given formally in the following theorem.

**Theorem 10.** For any \(G, \beta, \eta\) and \(n \geq 4\) such that \(0 < \eta \leq 1/\beta\), there exists a loss function \(l(w; z)\) that is convex, \(\beta\)-smooth and \(G\)-Lipschitz in \(w\) (for every \(z \in Z\)) and universal constants \(c_3, c_4 > 0\), such that the uniform stability of \(T\)-steps full-batch NAG with step size \(\eta\) is

\[
\delta_{\text{NAG}_{T, \epsilon}}^\text{unif}(n) \geq \min\left\{ \frac{G^2}{3\epsilon}, c_4e^{c_3\eta G T}\beta^2 \eta G^2 \right\}, \quad \forall \ T \in \left\lfloor \frac{10}{\eta\beta} \frac{n}{n-3} \right\rfloor (i+2) : i = 1, 2, \ldots, \}

Furthermore, for all \(T \geq \left\lceil \frac{40}{9\beta} \left( \ln \frac{6G}{\beta \epsilon} + 3 \right) \right\rfloor\) it holds that \(\delta_{\text{NAG}_{T, \epsilon}}^\text{unif}(n) \geq \frac{G^2}{3\epsilon}\).

The comments following Theorem 3 regarding the exponential blowup and the fluctuating behaviour also apply here. Note also the perhaps surprising inverse dependence on \(\beta (\beta \eta^2\) is also \(O(1/\beta)\)). The dependence can be explained by the fact that smooth optimization over a highly non-smooth yet still \(G\)-Lipschitz function must have a small step size (with \(\eta \leq 1/\beta\)) which improves stability.
**Construction.** We denote the given parameters for the theorem with $\hat{G}, \hat{\beta}, \hat{\eta}, n$. We will use the construction from Section 3 with the properties of Theorem 3 in order to create a loss function and samples which will have the same optimization for $t \geq 1$. For the construction we define the following setting of $G, \beta, \eta, \epsilon$:

$$G = \hat{G}, \quad \beta = \hat{\beta}, \quad \eta = \frac{n - 3}{n} \hat{\eta}, \quad \epsilon = \frac{\beta \eta^2 G}{n - 3}. $$

Using these parameters, we obtain $f$ from the construction of Section 3. As we proved in the previous section, this is the function for which Theorem 3 holds. Note that for $T < n_1$, the lower bounds already holds even for quadratics, as we show in Appendix E.1. We also define the following functions,

$$\ell(w; 1) \triangleq 0, \quad \ell(w; 2) \triangleq -\beta \eta G w + \beta \int_{-\infty}^{w} \int_{-\infty}^{y} 1[z \in [0, \eta G]] dz dy, $$

and further let $\ell(w; 3) = -Gw$, $\ell(w; 4) = Gw$, and $\ell(w; 5) = f(w)$. Note that all are convex, $\beta$-smooth and $G$-Lipschitz. Let $\mathcal{Z} = \{1, 2, 3, 4, 5\}$ be our sample space, we consider the loss function $\ell(w; z)$ over $\mathbb{R} \times \mathcal{Z}$. Our samples of interest are $S = (1, 3, 4, 5, \ldots, 5) \in \mathcal{Z}^n$ and $S' = (2, 3, 4, 5, \ldots, 5) \in \mathcal{Z}^n$. Thus, the empirical risks $R_S, R_{S'}$ corresponding to $S, S'$ are given by

$$R_S(w) = \frac{1}{n} g_1(w) + \frac{n - 3}{n} f(w) = \frac{n - 3}{n} f(w), \quad R_{S'}(w) = \frac{1}{n} g_2(w) + \frac{n - 3}{n} f(w).$$

**Analysis.** The key lemma below shows that we constructed a scenario that reduces the problem of analyzing the uniform stability of full-batch NAG to analyzing its initialization stability on the function $f$ we constructed in Section 3.

**Lemma 11.** For all $t = 1, 2, \ldots$, we have that

$$\text{NAG}(f, 0, t, \eta) = \text{NAG}(R_S, 0, t, \hat{\eta}),$$

$$\text{NAG}(f, \epsilon, t, \eta) = \text{NAG}(R_{S'}, 0, t, \hat{\eta}).$$

Using this key lemma, Theorem 10 follows immediately by setting $c_3 = c_1/4$ and $c_4 = c_2/4$, for the samples $S, S'$ we defined and $z = 3$. As $\ell(w; 3) = -Gw$,

$$|\ell(x_T; 3) - \ell(\tilde{x}_T; 3)| = G|x_T - \tilde{x}_T|,$$

and by using the definitions of $G, \beta, \eta, \epsilon$ we get the two cases of Theorem 3 which lower bound $|x_T - \tilde{x}_T|$ as needed.

Below we give a proof sketch for the second equality of Lemma 11, deferring the full proofs of the lemma and Theorem 10 to Appendix B.

**Proof of Lemma 11 (sketch).** We sketch the proof of the second equality stated in the lemma; the proof of the first equality is simpler and follows similar lines (details are given in Appendix B).

The proof proceeds via induction over the iterations. Let $(x_t, y_t) = \text{NAG}(f, \epsilon, t, \eta)$ and $(\tilde{x}_t, \tilde{y}_t) = \text{NAG}(R_{S'}, 0, t, \hat{\eta})$ for $t \leq T$. For $t = 1$,

$$\tilde{x}_1 = \tilde{y}_0 - \hat{\eta}(\nabla R_{S'}(y_0)) = \frac{1}{n} - \hat{\eta} \frac{n - 3}{n} \nabla f(0) \quad (\text{Eq. (4)})$$

$$x_1 = x_0 + \eta (\nabla f(\epsilon) - \nabla f(0)). \quad (\text{Eq. (4)} \text{ and } y_0 = \epsilon)$$
In Claim 24 we show that $\nabla f(\epsilon) = \nabla f(0) = -G$, thus $\tilde{x}_1 = x_1$. Since $\gamma_1 = 0$, it follows from Eq. (5) that $\tilde{y}_1 = \tilde{x}_1$ and similarly $x_1 = y_1$, hence $\tilde{y}_1 = y_1$. For $t > 1$,

\[
\begin{align*}
\hat{x}_t &= \hat{y}_{t-1} - \hat{\eta} \frac{n-3}{n} \nabla f(\hat{y}_{t-1}) - \frac{1}{n} \nabla g_2(\hat{y}_{t-1}) \\
&\quad - \frac{1}{n} \nabla g_2(y_{t-1}) \\
&= x_t - \eta \frac{1}{n} \nabla g_2(y_{t-1}).
\end{align*}
\]

(Eq. (4))

We need to show that $\nabla g_2(y_{t-1}) = 0$. Since by our construction, $\nabla f(x) \leq 0$ for all $x \in \mathbb{R}$, we observe only negative gradients and the iterations always move in the positive direction. Hence, $y_{t-1} \geq y_1$. Since $y_{t-1} \geq y_1 = x_1 + \gamma_1(x_1 - x_0) = y_0 - \eta \nabla f(y_0) = \epsilon - \eta \nabla f(\epsilon) = \epsilon + \eta G$, it follows that

\[
\nabla g_2(y_{t-1}) = -\beta \eta G + \beta \int_{-\infty}^{y_{t-1}} 1[z \in [0, \eta G]]dz = -\beta \eta G + \beta \eta G = 0,
\]

hence $\hat{x}_t = x_t$. Since by the induction assumption, $\hat{x}_{t-1} = x_{t-1}$, it follows from Eq. (5) that $\tilde{y}_t = y_t$, and we finished our induction. \qed

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A Proofs for Section 3

For the proofs in this section, we require the following lemma that states that the iterates of NAG over consecutive functions \( f_{i-1} \) and \( f_i \) in the construction in Section 3 are identical up to iteration \( t = n_i \). Hence, for \( j \leq i \), the iterates of NAG over \( f_i \) and \( f_j \) are identical up to iteration \( t = n_{j+1} \).

**Lemma 12.** For all \( 1 \leq i \leq M \) and \( x_0 \in \{0, \epsilon\} \) we have

\[
\text{NAG}(f_i, x_0, t) = \text{NAG}(f_{i-1}, x_0, t), \quad \forall t \leq n_i.
\]
A.1 Proof of Lemma 12

We will need the following technical claims (proofs below).

**Claim 13.** For all $1 \leq i \leq M$ we have $y_{n_i}^{\max} - y_{n_i}^{\min} < \frac{G}{2\beta}$.

**Claim 14.** Let $f : \mathbb{R} \to \mathbb{R}$ a convex, $\beta$-smooth function such that for all $x \in \mathbb{R}$, $\nabla f(x) < -\frac{G}{2\beta}$. Let $(x_t, y_t) = \text{NAG}(f, x_0, t, \eta)$ for all $t \leq T$, for some $T$ and step size $\eta$. Then $y_t > y_{t-1}$ and if $t > [20/\eta \beta]$, then $y_t > y_{t-1} + \frac{2G}{\eta \beta}$.

We now proceed to prove Lemma 12.

**Proof.** Let $(x_t, y_t) = \text{NAG}(f, x_0, t)$ and $(\tilde{x}_t, \tilde{y}_t) = \text{NAG}(f_{t-1}, x_0, t)$ for all $t \leq n_i$. We first note that for all $x \in (-\infty, y_{n_i}^{\min})$,

$$\nabla f_i(x) = -G + \beta \int_{-\infty}^{x} \left[ \exists j \in [i] \text{ s.t. } z \in [y_{n_j}^{\min}, y_{n_j}^{\max}] \right] dz$$

$$= -G + \beta \int_{-\infty}^{x} \left[ \exists j \in [i-1] \text{ s.t. } z \in [y_{n_j}^{\min}, y_{n_j}^{\max}] \right] dz = \nabla f_{i-1}(x).$$

We will use induction over $t$. For $t = 0$, $y_0 = x_0 = \tilde{x}_0 = \tilde{y}_0$ is the starting point. For $t = 1$,

$$x_1 = y_0 - \eta \nabla f_i(y_0) = \tilde{y}_0 - \eta \nabla f_i(\tilde{y}_0) = \tilde{x}_1 = \tilde{y}_1.$$  

(Eq. (4))

From Claim 14, $y_{n_i}^{\min} > \frac{G}{2\beta}$, and given the assumption that $\epsilon < \frac{G}{2\beta}$, $\tilde{y}_0 \in (-\infty, y_{n_i}^{\min})$, thus,

$$\nabla f_i(\tilde{y}_0) = \nabla f_{i-1}(\tilde{y}_0).$$

This means that $x_1 = \tilde{x}_1$, and since $\gamma_1 = 0$,

$$y_1 = x_1 + \gamma_1(x_1 - x_0) = x_1 = \tilde{x}_1 = \tilde{y}_1.$$ 

For $t > 1$,

$$x_t = y_{t-1} - \eta \nabla f_i(y_{t-1}) = \tilde{y}_{t-1} - \eta \nabla f_i(\tilde{y}_{t-1}) = \tilde{x}_{t-1} = \tilde{y}_{t-1}.$$ 

(induction assumption)

(Eq. (4))

Again, we want to show that $\tilde{y}_{t-1} \in (-\infty, y_{n_i}^{\min})$. Combining Claim 14 with the fact that $\tilde{y}_{n_i} \in \{y_{n_i}^{\min}, y_{n_i}^{\max}\}$,

$$\tilde{y}_{t-1} \leq \tilde{y}_{n_i} - \frac{G}{2\beta} < \tilde{y}_{n_i} - \frac{G}{2\beta} \leq y_{n_i}^{\max} - \frac{G}{2\beta}.$$ 

and from Claim 13,

$$y_{n_i}^{\max} - \frac{G}{2\beta} < y_{n_i}^{\min}.$$ 

Thus, $\tilde{y}_t < y_{n_i}^{\min}$ and as before, $\nabla f_i(\tilde{y}_{t-1}) = \nabla f_{i-1}(\tilde{y}_{t-1})$, hence, $x_t = \tilde{x}_t$. Lastly, $y_t = \tilde{y}_t$ follows directly from the induction assumption and $x_t = \tilde{x}_t$ by Eq. (5).

Below we prove the claims used in the proof above.
Proof of Claim 13. From the definition of $M$, $\max_x \nabla f_i(x) < -G/2$. Specifically,

$$-G/2 \geq \nabla f_i(y_{n_i}^{\text{max}}) = -G + \beta \int_{-\infty}^{y_{n_i}^{\text{max}}} 1[\exists j \in [i] \text{ s.t. } z \in [y_{n_j}^{\text{min}}, y_{n_j}^{\text{max}}]] dz.$$

Thus,

$$\frac{G}{2\beta} > \int_{-\infty}^{y_{n_i}^{\text{max}}} 1[\exists j \in [i] \text{ s.t. } z \in [y_{n_j}^{\text{min}}, y_{n_j}^{\text{max}}]] dz$$

$$\geq \int_{-\infty}^{y_{n_i}^{\text{max}}} 1[z \in [y_{n_i}^{\text{min}}, y_{n_i}^{\text{max}}]] dz$$

$$= y_{n_i}^{\text{max}} - y_{n_i}^{\text{min}}.$$

In order to prove Claim 14 we need another technical result (proof in Appendix A.11).

Claim 15. Let $\gamma_t = \frac{t - 1}{t + 2}$ for $t \geq 1$. Then $\sum_{k=2}^{t} \prod_{j=k}^{t} \gamma_j \geq \frac{(t-1)^2}{4(t+2)}$.

Proof of Claim 14. Since for all $x \in \mathbb{R}$, $\nabla f(x) < -G/2$, from Eq. (6),

$$m_t = \gamma_t(m_{t-1} - \eta \nabla f(y_{t-1})) \geq \gamma_t\left(m_{t-1} + \frac{\eta G}{2}\right).$$

Unfolding the recursion until reaching $m_1 = \gamma_1(x_1 - x_0) = 0 \cdot (x_1 - x_0) = 0$, we obtain

$$m_t \geq \frac{\eta G}{2} \sum_{k=2}^{t} \prod_{j=k}^{t} \gamma_j \geq 0.$$

Hence,

$$y_t = y_{t-1} - \eta \nabla f(y_{t-1}) + m_t \quad \text{(Eq. (5))}$$

$$\geq y_{t-1} + \frac{\eta G}{2} > y_{t-1}.$$

If $t > \lceil 10/\eta \beta \rceil$, using Claim 15,

$$m_t \geq \frac{\eta G}{2} \sum_{k=2}^{t} \prod_{j=k}^{t} \gamma_j$$

$$\geq \frac{\eta G(t - 1)^2}{8(t + 2)}$$

$$\geq \frac{\eta G(20/\eta \beta)^2}{8(20/\eta \beta + 3)}$$

$$= \frac{G}{\beta} \frac{2.5}{1 + 0.15/\eta \beta} > \frac{2G}{\beta},$$

and we finish with

$$y_t = y_{t-1} - \eta \nabla f(y_{t-1}) + m_t \quad \text{(Eq. (5))}$$

$$> y_{t-1} + \frac{2G}{\beta}. \quad \blacksquare$$

A.2 Proof of Lemma 5

In order to prove Lemma 5 we need the following claim (proved below).
Proof of Lemma 5. The proof of the first property considers two different cases:

- If \( t = n_k \) for some \( k \in [i] \), we know from Lemma 12 that
  \[
  y_{n_k}^\text{min} = \min(y_{n_k}, \tilde{y}_{n_k}), \\
  y_{n_k}^\text{max} = \max(y_{n_k}, \tilde{y}_{n_k}).
  \]

  Using this property,
  \[
  \Delta f_{n_k}^i = \beta \int_{y_{n_k}}^y \left[ \exists j \in [i] \text{ s.t. } z \in [y_{n_j}^\text{min}, y_{n_j}^\text{max}] \right] dz = \beta \Delta y_{n_k}^i.
  \]

- If \( t \not\in \{n_j\}_j=1 \), then from Claim 16 we know that they are both inside an interval of the form \((y_{n_k}^\text{max}, y_{n_k}^\text{min})\) (or \((-\infty, y_{n_1}^\text{min})\) or \((y_{n_1}^\text{max}, \infty)\)), in which case they are inside an interval with second derivative of 0, and using the mean value theorem, there exists some \( y_{mid} \in (y_t, \tilde{y}_t) \), for which
  \[
  \Delta f_i^i = \Delta y_i \cdot \nabla^2 f_i(y_{mid}) = \Delta y_i \cdot 0 = 0.
  \]

The second property comes from the first together with Eq. (9),
\[
\Delta f_i^m = \gamma_i (\Delta f_i^{m-1} - \eta \Delta f_i^{f-1}),
\]
which completes the proof.

Proof of Claim 16. For some \( j \in [i] \) and \( t \in [T] \cup \{0\} \):

- If \( t < n_j \), then from Claim 14,
  \[
  y_t \leq y_{n_{j-1}} < y_{n_j} - \frac{G}{2\beta}.
  \]

  From Lemma 12, \( y_{n_j} \in \{y_{n_j}^\text{min}, y_{n_j}^\text{max}\} \), and combining with Claim 13,
  \[
  y_{n_j} - \frac{G}{2\beta} \leq y_{n_j}^\text{max} - \frac{G}{2\beta} \leq y_{n_j}^\text{min}.
  \]

- If \( t > n_j \), then similarly,
  \[
  y_t \geq y_{n_{j+1}} > y_{n_j} + \frac{G}{2\beta} \geq y_{n_j}^\text{min} + \frac{G}{2\beta} > y_{n_j}^\text{max}.
  \]

This concludes the proof.

### A.3 Proof of Lemma 6

The three parts of the lemma are defined separately in the following lemmas (proved separately afterwards). The first analyzes the evolution of the distance between the iterates \( y_t \) and \( \tilde{y}_t \) at steps \( t \in \{n_j\}_{1 \leq j \leq i+1} \) and relates it to the difference in momentum terms.

Lemma 17. For all \( 1 \leq j \leq i \),
\[
\frac{2}{3} \eta \beta |\Delta y_{n_j}| \leq |\Delta m_{n_j+1}| \leq \frac{1}{5} \eta \beta |\Delta y_{n_{j+1}}|.
\]

The second lemma, derived from the first, shows that the difference \( \Delta y_i \) exhibits an exponential blowup between steps \( t = n_j \) and \( t = n_{j+1} \).

Lemma 18. For all \( 0 \leq j \leq i \), it holds that \(|\Delta y_{n_{j+1}}| = y_{n_{j+1}}^\text{max} - y_{n_{j+1}}^\text{min} \geq 3^j \epsilon \).

In order to bound the iterations after \( t = n_{i+1} \), the following lemma shows that the distance between the iterates \( y_t \) and \( \tilde{y}_t \) after the exponential growth phase (when \( t > n_{i+1} \)) does not decrease.

Lemma 19. For all \( t > n_{i+1} \), it holds that \(|\Delta y_t| \geq |\Delta y_{n_{i+1}}|\).

The three lemmas are proved in the following sections.
A.4 Proof of Lemma 17

We will need the following claim (proof in Appendix A.11).

Claim 20. Let $\gamma_t = \frac{t-1}{t+2}$. Then for $n, m \geq 1$,

$$\sum_{k=n}^{m} \prod_{t=n}^{k-1} \gamma_{t+1} \geq \frac{n}{2} \left(1 - \frac{n^2}{(m+1)^2}\right).$$

Proof of Lemma 17. First let us assume that $2\eta \beta |\Delta^y_{nj}| \leq |\Delta^m_{nj+1}|$ for some $j \in [i]$. Thus,

$$\Delta^y_{nj+1} = \Delta^y_{nj+1-1} - \eta \Delta^f_{nj+1} + \Delta^m_{nj+1} \quad \text{(Eq. (8))}$$

$$= \Delta^y_{nj} - \eta \sum_{k=n_j}^{n_{j+1}-1} \Delta^f_k + \sum_{k=n_j+1}^{n_{j+1}} \Delta^m_k \quad \text{(Eq. (8) multiple times)}$$

$$= \Delta^y_{nj} - \eta \beta \Delta^y_{nj} + \sum_{k=n_j+1}^{n_{j+1}} \Delta^m_k \quad \text{(Lemma 5)}$$

$$= \Delta^y_{nj} (1 - \eta \beta) + \sum_{k=n_j+1}^{n_{j+1}} \Delta^m_k = (*).$$

Using Lemma 5 ($t \in [T]/\{n_j + 1\}_{j=1} \Rightarrow \Delta^m_t = \gamma_t \Delta^m_{t-1}$) recursively on $\Delta^m_k$,

$$(*) = \Delta^y_{nj} (1 - \eta \beta) + \Delta^m_{nj+1} \sum_{k=n_j+1}^{n_{j+1}} \prod_{t=n_j+1}^{k-1} \gamma_{t+1}.$$ 

Thus, since $\eta \beta \leq 1$ and for $t \geq 0$, $\gamma_{t+1} \geq 0$,

$$|\Delta^y_{nj+1}| \geq |\Delta^m_{nj+1}| \sum_{k=n_j+1}^{n_{j+1}} \prod_{t=n_j+1}^{k-1} \gamma_{t+1} - (1 - \eta \beta) |\Delta^y_{nj}|.$$ 

Using our assumption that $2\eta \beta |\Delta^y_{nj}| \leq |\Delta^m_{nj+1}|$,

$$|\Delta^y_{nj+1}| \geq \left( \sum_{k=n_j+1}^{n_{j+1}} \prod_{t=n_j+1}^{k-1} \gamma_{t+1} - \frac{3(1-\eta \beta)}{2\eta \beta} \right) |\Delta^m_{nj+1}|$$

$$\geq \left( \sum_{k=n_j+1}^{n_{j+1}} \prod_{t=n_j+1}^{k-1} \gamma_{t+1} - \frac{3(1-\eta \beta)}{2\eta \beta} \right) |\Delta^m_{nj+1}|.$$ 

Using Claim 20,

$$\sum_{k=n_{j+1}}^{n_{j+1}} \prod_{t=n_{j+1}}^{k-1} \gamma_{t+1} \geq \frac{n_{j+1} + 1}{2} \left(1 - \frac{(n_{j+1} + 1)^2}{(n_{j+1} + 2)^2}\right)$$

$$\geq \frac{30}{2\eta \beta} \left(1 - \frac{3^2}{4^2}\right) \quad \text{for } n_i = \lceil 10/\eta \beta \rceil (i + 2)$$

$$\geq \frac{13}{2\eta \beta}.$$ 

Plugging it gives us

$$|\Delta^y_{nj+1}| \geq (13 - 3(1 - \eta \beta)) \frac{|\Delta^m_{nj+1}|}{2\eta \beta}$$

$$\geq \frac{5}{\eta \beta} |\Delta^m_{nj+1}|. \quad \text{(\eta \beta \geq 0)}$$
and we concluded our inductive argument. Using Lemma 5,

\[ |\Delta_{n_{j+1}}^m| = \gamma_{n_{j+1}} |\Delta_{n_{j+1}}^m - \eta \Delta_{n_{j+1}}^f| \]

\[ = \gamma_{n_{j+1}} |\Delta_{n_{j+1}}^m - \eta \beta \Delta_{n_{j+1}}^y| \]

\[ = \gamma_{n_{j+1}} \prod_{k=2}^{n_j} |\gamma_k - \eta \beta \Delta_{n_{j+1}}^y| \]

\[ = \gamma_{n_{j+1}} |\eta \beta \Delta_{n_{j+1}}^y| \]

\[ \geq \frac{2}{3} \eta \beta |\Delta_{n_{j+1}}^y| \]

Now for a given \( j > 1 \),

\[ \Delta_{n_{j+1}}^m = \gamma_{n_{j+1}} (\Delta_{n_{j+1}}^m - \eta \Delta_{n_{j+1}}^f) \]

\[ = \gamma_{n_{j+1}} \Delta_{n_{j+1}}^m - \gamma_{n_{j+1}} \eta \beta \Delta_{n_{j+1}}^y, \]

and again using Lemma 5 (\( t \in [T]/\{n_j + 1\} \) \( \Rightarrow \Delta_{n_j}^m = \gamma_t \Delta_{n_{j-1}}^m \)),

\[ \Delta_{n_{j+1}}^m = \Delta_{n_{j-1}+1}^m \prod_{k=n_{j-1}+2}^{n_{j+1}} \gamma_k - \gamma_{n_{j+1}} \eta \beta \Delta_{n_{j+1}}^y. \]

Thus, since \( \gamma_t \geq 0 \) for \( t \geq 1 \),

\[ |\Delta_{n_{j+1}}^m| \geq \gamma_{n_{j+1}} \eta \beta |\Delta_{n_{j+1}}^y| - |\Delta_{n_{j+1}}^m| \prod_{k=n_{j+1}+2}^{n_{j+1}+1} \gamma_k. \]

By the induction assumption, \( \frac{2}{3} \eta \beta |\Delta_{n_{j-1}}^y| \leq |\Delta_{n_{j-1}+1}^m| \). Hence, \( |\Delta_{n_{j+1}}^m| \leq \frac{1}{3} \eta \beta |\Delta_{n_{j+1}}^y| \) as we showed in the beginning of the proof. Thus,

\[ |\Delta_{n_{j+1}}^m| \geq \gamma_{n_{j+1}} \eta \beta |\Delta_{n_{j+1}}^y| - \frac{1}{5} \eta \beta |\Delta_{n_{j+1}}^y| \prod_{k=n_{j+1}+2}^{n_{j+1}+1} \gamma_k \]

\[ = \left( \gamma_{n_{j+1}} - \frac{1}{5} \prod_{k=n_{j+1}+2}^{n_{j+1}+1} \gamma_k \right) \eta \beta |\Delta_{n_{j+1}}^y| \]

\[ \geq \left( \frac{9}{10} - \frac{1}{5} \right) \eta \beta |\Delta_{n_{j+1}}^y| \]

\[ = \frac{2}{3} \eta \beta |\Delta_{n_{j+1}}^y|, \]

and we concluded our inductive argument.

\[ \blacksquare \]

### A.5 Proof of Lemma 18

**Proof.** Let \( i \leq M \). Let \((x_t, y_t) = NAG(f_t, 0, t)\) and \((\hat{x}_t, \hat{y}_t) = NAG(f_t, \epsilon, t)\) for all \( t \leq n_{i+1} \). From Lemma 17 we know that for all \( j \in [i] \),

\[ |\Delta_{n_{j+1}}^v| \geq \frac{10}{3} |\Delta_{n_{j}}^v| \implies |\Delta_{n_{j+1}}^v| \geq 3 |\Delta_{n_{j}}^v|. \]

Using Lemma 5,

\[ \Delta_{n_{j+1}}^y = \Delta_{n_{j+1}}^v - \eta \Delta_{n_{j+1}}^f + \Delta_{n_{j+1}}^m \]

\[ = \Delta_{n_{j+1}}^v + \Delta_{n_{j+1}}^m \prod_{k=2}^{n_1} \gamma_k \]

\[ = \Delta_{n_{j+1}}^v. \]

\[ (m_1 = \bar{m}_1 = 0 \Rightarrow \Delta_{n_{j+1}}^m = 0) \]
Repeating this argument leads to $\Delta y_{n_i} = \Delta y_0 = -\epsilon$. Thus, $|\Delta y_{n_{i+1}}| \geq 3^j \epsilon$. Since $y_{n_{i+1}}^{\max} - y_{n_{i+1}}^{\min} = |\Delta y_{n_{i+1}}|$ as $y_{n_{i+1}}^{\max}$ and $y_{n_{i+1}}^{\min}$ are defined as the max and min of $\{y_{n_{i+1}}, y_{n_{i+1}}\}$,

$$|y_{n_{i+1}}^{\max} - y_{n_{i+1}}^{\min}| \geq 3^j \epsilon,$$

and since this is true for all $i \in [M]$, we showed our exponential growth. All is left to show is that for all $j \leq i$, $|\Delta y_{n_{j+1}}| = y_{n_{j+1}}^{\max} - y_{n_{j+1}}^{\min}$. If $j = i$, then this is immediate since $y_{n_{j+1}}^{\max}, y_{n_{j+1}}^{\min}$ are defined as the max and min of $y_{n_{j+1}}, y_{n_{j+1}}$. If $j < i$, then based on Lemma 12, the iterations over $f_j$ and over $f_{j+1}$ are the same up to $n_{j+1}$. Invoking Lemma 12 up to $i$ means that the iterations over $f_j$ and $f_i$ are the same up to $n_{j+1}$. Hence, for $j \leq i$, $|\Delta y_{n_{j+1}}| = y_{n_{j+1}}^{\max} - y_{n_{j+1}}^{\min}$.

### A.6 Proof of Lemma 19

**Proof.** We start by showing that

$$\sigma(\Delta y_{n_{i+1}}) = \sigma(\Delta m_{n_{i+1}}),$$

where $\sigma$ is the sign function. Using Eq. (8) recursively from $n_{i+1}$ to $n_i$,

$$\Delta y_{n_{i+1}} = \Delta y_{n_{i+1}} - \eta \Delta f_{n_{i+1}} + \Delta m_{n_{i+1}} \tag{Eq. (8)}$$

$$= \Delta y_{n_i} - \eta \sum_{k=n_i}^{n_{i+1}-1} \Delta f_{k} + \sum_{k=n_{i+1}}^{n_{i+1}} \Delta y_{k} \tag{Eq. (8) multiple times}$$

$$= \Delta y_{n_i} - \eta \beta \Delta m_{n_i} + \sum_{k=n_{i+1}}^{n_{i+1}} \Delta y_{k} \tag{Lemma 5}$$

$$= \Delta y_{n_i} (1 - \eta \beta) + \sum_{k=n_{i+1}}^{n_{i+1}} \Delta y_{k}.$$  

Using Lemma 5 ($t \in [T]/\{n_j + 1\}_{j=1}^i \Rightarrow \Delta m = \gamma(t \Delta m_{t-1})$) recursively on $\Delta m$,

$$\Delta y_{n_{i+1}} = \Delta y_{n_i} (1 - \eta \beta) + \Delta m_{n_{i+1}} + \sum_{k=n_{i+1}}^{n_{i+1}} \prod_{t=n_{i+1}}^{n_{i+1} - 1} \gamma_{t+1}.$$  

Now we use the fact that $|\Delta y_{n_{i+1}}| \geq 3|\Delta y_{n_i}|$ from Lemma 17,

$$3|\Delta y_{n_{i+1}}| \leq |\Delta y_{n_{i+1}}| = \sigma(\Delta y_{n_{i+1}}) \Delta y_{n_{i+1}}$$

$$= \sigma(\Delta y_{n_{i+1}}) \left( \Delta y_{n_i} (1 - \eta \beta) + \Delta m_{n_{i+1}} + \sum_{k=n_{i+1}}^{n_{i+1}} \prod_{t=n_{i+1}}^{n_{i+1} - 1} \gamma_{t+1} \right)$$

$$\leq |\Delta y_{n_i} (1 - \eta \beta)| + \sigma(\Delta y_{n_{i+1}}) \left( \Delta m_{n_{i+1}} + \sum_{k=n_{i+1}}^{n_{i+1}} \prod_{t=n_{i+1}}^{n_{i+1} - 1} \gamma_{t+1} \right),$$

hence, since $0 \leq \eta \beta \leq 1$,

$$\sigma(\Delta y_{n_{i+1}}) \left( \Delta m_{n_{i+1}} + \sum_{k=n_{i+1}}^{n_{i+1}} \prod_{t=n_{i+1}}^{n_{i+1} - 1} \gamma_{t+1} \right) \geq (2 + \eta \beta) |\Delta y_{n_i}| \geq 2|\Delta y_{n_i}| \geq 0$$

$$\implies \sigma(\Delta y_{n_{i+1}}) = \sigma(\Delta m_{n_{i+1}}),$$

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where the last equality follows from the fact that for \( t \geq 2, \gamma_t > 0 \). Thus, for \( t > n_i+1 \), again using Eq. (8) recursively,
\[
\Delta^y_i = \Delta^y_{i-1} - \eta \Delta^f_{i-1} + \Delta^m_i
\]
(Eq. (8))
\[
= \Delta^y_{n_i+1} - \eta \sum_{k=n_i+1}^{t-1} \Delta^f_k + \sum_{k=n_i+1}^t \Delta^m_k
\]
(Eq. (8) multiple times)
\[
= \Delta^y_{n_i+1} + \sum_{k=n_i+1}^t \Delta^m_k.
\]
(Lemma 5, \( k > n_i \implies \Delta^f_k = 0 \))

From Lemma 5, we know that for all \( k > n_i + 1 \),
\[
\sigma(\Delta^m_k) = \sigma(\Delta^m_{k-1}) = \cdots = \sigma(\Delta^m_{n_i+1}).
\]
Hence, since also \( \sigma(\Delta^y_{n_i+1}) = \sigma(\Delta^m_{n_i+1}) \),
\[
|\Delta^y_i| = |\Delta^y_{n_i+1} + \sum_{k=n_i+1}^{n_i+1} \Delta^m_k|
\]
\[
= |\Delta^y_{n_i+1}| + \sum_{k=n_i+1}^{n_i+1} |\Delta^m_k| \quad \text{(all terms share sign)}
\]
\[
\geq |\Delta^y_{n_i+1}|,
\]
and we conclude the proof. \( \blacksquare \)

### A.7 Proof of Lemma 7

**Proof.** First we will show that \( M \geq 1 \). We already know that \( \max_x \nabla f_0(x) = -G < -\frac{1}{2}G \). From the definition of \( f_1(x) \), \( \max_x \nabla f_1(x) = -G + \beta(y_{n_1}^{\max} - y_{n_1}^{\min}) \). Let \((\tilde{x}_t, \tilde{y}_t) = \text{NAG}(f_0, 0, t, \eta)\) and \((\tilde{x}_t, \tilde{y}_t) = \text{NAG}(f_0, \epsilon, 0, \eta)\) for \( t \leq n_1 \). Hence,
\[
y_{n_1}^{\max} - y_{n_1}^{\min} = |\Delta^y_{n_1}|
\]
(definition of \( y_{n_1}^{\max}, y_{n_1}^{\min} \))
\[
= |\Delta^y_{n_1-1} - \eta \Delta^f_{n_1-1} + \Delta^m_{n_1}|
\]
(Eq. (8))
\[
= |\Delta^y_{n_1-1}|
\]
(Lemma 5)
\[
= |\Delta^y_{n_1-1}|
\]
(repeating the two steps above)
\[
= \epsilon < \frac{G}{2\beta}
\]
(assumption about \( \epsilon \))

Therefore,
\[
\max_x \nabla f_1(x) < -G + \frac{1}{2}G = -\frac{1}{2}G,
\]
and \( M \geq 1 \). Now we move to upper-bounding \( M \). Fix some finite \( I \leq M \). From the definition of \( M \) we know that \( \max_x \nabla f_I(x) < -\frac{G}{2} \). Hence,
\[
-\frac{G}{2} > \max_x \nabla f_I(x)
\]
\[
= \max_x \left\{ -G + \beta \int_{-\infty}^x \left\{ \exists j \in [I] \text{ s.t. } z \in [y_{n_j}^{\min}, y_{n_j}^{\max}] \right\} dz \right\}
\]
\[
\geq -G + \beta(y_{n_I}^{\max} - y_{n_I}^{\min}),
\]
which implies that \( y_{n_I}^{\max} - y_{n_I}^{\min} < \frac{G}{2\beta} \). But from Lemma 18, \( y_{n_I}^{\max} - y_{n_I}^{\min} \geq \epsilon \beta^{I-1} \), thus, \( I \leq \log_3 \frac{3G}{2\beta\epsilon} \) and since we fixed an arbitrary \( I \leq M \), \( M \leq \log_3 \frac{3G}{2\beta\epsilon} \). In order to lower bound \( y_{n,M+1}^{\max} - y_{n,M+1}^{\min} \) we will
exploit the definition of $M$. We know that $\max_x \nabla f_{M+1}(x) \geq -\frac{G}{2}$ (this is the first function that violates the condition in the definition of $M$). Thus,

$$\frac{-G}{2} \leq \max_x \nabla f_{M+1}(x)$$

$$= \max_x \left\{ -G + \beta \int_{-\infty}^x 1 [\exists j \in [M+1] \text{ s.t. } z \in [y_{n_j}^{\min}, y_{n_j}^{\max}] ] dz \right\}$$

$$\leq -G + \beta \sum_{j=1}^{M+1} (y_{n_j}^{\max} - y_{n_j}^{\min}),$$

which implies, using Lemma 18, that

$$\sum_{j=1}^{M+1} |\Delta_{n_j}^y| = \sum_{j=1}^{M+1} (y_{n_j}^{\max} - y_{n_j}^{\min}) \geq \frac{G}{2\beta}.$$ 

On the other hand, since $|\Delta_{n_j+1}^y| \geq 3|\Delta_{n_j}^y|$ due to Lemma 17, we have

$$\frac{G}{2\beta} \leq \sum_{j=1}^{M+1} |\Delta_{n_j}^y| \leq \sum_{j=1}^{M+1} |\Delta_{n_{M+1}}^y| \cdot 3^{j-(M+1)} \leq \frac{3}{2} |\Delta_{n_{M+1}}^y|$$

$$\implies |\Delta_{n_{M+1}}^y| = |\Delta_{n_{M+1}}^y| \geq \frac{G}{3\beta}. \Box$$

A.8 Proof of Lemma 8

**Proof.** The first derivative of $f$ is

$$\nabla f(x) \triangleq \begin{cases} 
\nabla f_M(x) & x \leq p; \\
\nabla f_M(p) + \beta(x - p) & p < x \leq p - \frac{1}{\beta} \nabla f_M(p); \\
0 & \text{otherwise.}
\end{cases}$$

Hence, for all $x \in \mathbb{R}$, $|\nabla f(x)| \leq \max(|\nabla f_M(x)|, |\nabla f_M(p)|) \leq G$, where the last inequality comes from Lemma 4. Thus, $f$ is $G$-Lipschitz. The second derivative of $f$ is

$$\nabla^2 f(x) \triangleq \begin{cases} 
\nabla^2 f_M(x) & x \leq p; \\
\beta & p < x \leq p - \frac{1}{\beta} \nabla f_M(p); \\
0 & \text{otherwise.}
\end{cases}$$

Thus, from Lemma 4, for all $x \in \mathbb{R}$, $0 \leq \nabla^2 f(x) \leq \beta$. Hence, $f$ is convex and $\beta$-smooth. Let $x^* \triangleq p - \frac{\nabla f_M(p)}{\beta}$. Since $f$ is convex and $\nabla f(x^*) = 0$, $x^* \in \arg \min_x f(x)$. In order to bound the distance $|x_0 - x^*|$ we will first bound $p = y_{n_M}^{\max}$.

Since $M \leq \ln \frac{3G}{2\beta\epsilon}$ (Lemma 7),

$$n_M \leq \left\lceil \frac{10}{\eta\beta} \left( \ln \frac{3G}{2\beta\epsilon} + 2 \right) \right\rceil.$$ 

Since for NAG, $|m_t| \leq |m_{t-1}| + \eta |\nabla f(y_{t-1})|$ (Eq. (6)), for a $G$-Lipschitz function, $|m_t| \leq \eta G(t-1)$. Hence, for a sequence $(x_t, y_t)_{t=0}^T$ of NAG on a $G$-Lipschitz function,

$$|y_t| \leq |y_{t-1}| + \eta G(t-1) + \eta G$$

$$= |y_{t-1}| + \eta Gt$$

$$= |y_0| + \eta G \frac{t(t + 1)}{2}.$$
where the last inequality follows from Proof of Claim 23.

First we will show by induction that for all $1 \leq t \leq n_M + 1$, we have

\[ x_t - x_t^\star \leq |x_0| + |p| + \left| \nabla f_M(p) \right| \]

where the last transition is due to Proof of Lemma 9. For all $t > n_M$, $\nabla f(y_t) = 0$.

Proof of Lemma 9. For $t \leq n_M + 1$ the lemma follows from Lemma 21. We will show by induction that the lemma also follows for $t > n_M + 1$.

\[
\begin{align*}
  x_t - x_t^\star & = y_{t-1} - \eta \nabla f_M(y_{t-1}) - \dot{y}_{t-1} + \eta \nabla f_M(\dot{y}_{t-1}) \\
& = y_{t-1} - \dot{y}_{t-1} \\
& = \dot{y}_{t-1} - \dot{y}_{t-1} \\
& = \dot{y}_{t-1} - \eta \nabla f(\dot{y}_{t-1}) - \dot{y}_{t-1} + \eta \nabla f(\dot{y}_{t-1}) \\
& = x_t - x_t^\star.
\end{align*}
\]

The second equality follows immediately from the first with Eq. (4).

In order to prove Claim 22 we use the following claim.

Claim 23. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex, $\beta$-smooth function such that for all $x \in \mathcal{R}$, $\nabla f(x) \leq 0$ ($\nabla f(x) < 0$). Let $(x_t, y_t) = \text{NAG}(f, x_0, t, \eta)$ for all $t \leq T$, for some $T$ and step size $\eta$. Then $y_t \geq y_{t-1}$ ($y_t > y_{t-1}$).

Proof of Claim 23. First we will show by induction that for all $1 \leq t \leq T$, $m_t \geq 0$. For $t = 1$, $m_1 = \gamma_1(x_1 - x_0) = 0 \cdot (x_1 - x_0) = 0$. From Eq. (6), $1 < t \leq T$, $m_t = \gamma_t(m_{t-1} - \eta \nabla f(y_{t-1}))$, and since $m_{t-1}$ is non-negative and $\nabla f(y_{t-1})$ is non-positive, $m_t \geq 0$. In order to finish the claim we use Eq. (5),

\[
y_t = y_{t-1} - \eta \nabla f(y_{t-1}) + m_t \geq y_{t-1} - \eta \nabla f(y_{t-1}),
\]

where the last inequality follows from $m_t$ being non-negative. The claim follows in the respective case according to $\nabla f(x)$ being negative or non-positive.
Proof of Claim 22. We will prove the claim by showing that \( y_t > p - \frac{1}{\beta} \nabla f_M(p) \). Using Claim 23 it is enough to show that \( y_{n,M+1} > p - \frac{1}{\beta} \nabla f_M(p) \).

\[
p - \frac{1}{\beta} \nabla f_M(p) \leq p + \frac{G}{\beta} \tag{Lipschitz condition of \( f_M \)}
\]

\[
\leq y_{n,M}^{\min} + \frac{G}{2\beta} + \frac{G}{\beta} \tag{\( p = y_{n,M}^{\max} \) and Claim 13}
\]

\[
\leq y_{n,M}^{\min} + \frac{3G}{2\beta}
\]

\[
\leq y_{n,M} + \frac{3G}{2\beta} \tag{Claim 14 with Lemma 21}
\]

\[
\leq y_{n,M+1}.
\]

Hence, \( \nabla f(y_t) = 0 \). \( \blacksquare \)

A.10 Proof of Lemma 21

Proof of Lemma 21. Let \((x_t, y_t) = NAG(f_M, x_0, t)\) and \((\tilde{x}_t, \tilde{y}_t) = NAG(f, x_0, t)\) for \( t \leq n_M \). From Claim 23 we know that for all \( t \leq n_M \),

\[
y_t \leq y_{n,M} \leq y_{n,M}^{\max} = p \implies \nabla f_M(y_t) = \nabla f(y_t).
\]

We will continue by induction. At initialization, \( y_0 = x_0 = \tilde{x}_0 = \tilde{y}_0 \). At step \( t \leq n_M + 1 \),

\[
x_t = y_{t-1} - \eta \nabla f_M(y_{t-1}) = y_{t-1} - \eta \nabla f(y_{t-1}) = \tilde{y}_{t-1} - \eta \nabla f(\tilde{y}_{t-1}) = \tilde{x}_t. \tag{Eq. (4)}
\]

And using the induction assumption again, \( y_t = \tilde{y}_t \), using Eq. (5). \( \blacksquare \)

A.11 Proofs of Technical Claims

Here we prove Claims 15 and 20.

Proof of Claim 20. We observe that

\[
(*) \triangleq \sum_{k=n}^{m} \prod_{t=n}^{k-1} \frac{t}{t+3} = \left( 1 + \frac{n}{n+3} + \frac{n(n+1)}{(n+3)(n+4)} + \sum_{k=n+3}^{m} \frac{n(n+1)(n+2)}{k(k+1)(k+2)} \right) \\
\geq \left( 1 + \frac{n^3}{(n+1)^3} + \frac{n^3}{(n+2)^3} + \sum_{k=n+3}^{m} \frac{n^3}{k^3} \right) \\
= \sum_{k=n}^{m} \left( \frac{n}{k} \right)^3.
\]

We will lower bound the latter by integration,

\[
(*) \geq \sum_{k=n}^{m} \left( \frac{n}{k} \right)^3 \\
\geq n^3 \int_{n}^{m+1} x^{-3}dx \\
= n^3 \left( \frac{1}{n^2} - \frac{1}{(m+1)^2} \right) \\
= n \left( \frac{1}{n^2} - \frac{1}{(m+1)^2} \right).
\]
Proof of Claim 15. In the case of \( t = 1 \) both sides are 0. For \( t > 1 \),
\[
\sum_{k=2}^{t} \prod_{j=k}^{t} \gamma_j = \sum_{k=2}^{t} \prod_{j=k}^{t} \frac{j-1}{j+2}
\]
\[
= \frac{t-1}{t+2} + \frac{(t-2)(t-1)}{(t+1)(t+2)} + \frac{(t-2)(k-1)(k+1)}{t(t+1)(t+2)}
\]
\[
\geq \frac{(t-1)^3}{t^2(t+2)} + \frac{(t-2)^3}{t^2(t+2)} + \frac{(t-2)(k-1)^3}{t^2(t+2)}
\]
\[
= \frac{1}{t^2(t+2)} \sum_{k=2}^{t} (k-1)^3
\]
\[
= \frac{1}{t^2(t+2)} \sum_{k=1}^{t-1} k^3.
\]

Using the formula for \( \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} \),
\[
\sum_{k=2}^{t} \prod_{j=k}^{t} \gamma_j \geq \frac{(t-1)^2 t^2}{4(t+2)^2} = \frac{(t-1)^2}{4(t+2)}.
\]

B Proofs for Section 4

Here we prove Lemma 11 and Theorem 10.

B.1 Proof of Lemma 11

First we will need the following simple claim.

Claim 24. Given \( f_M \) and \( f \) from the construction in Section 3 with parameters \( G, \beta, \eta, \epsilon \),
\[
\nabla f_M(0) = \nabla f_M(\epsilon) = \nabla f(0) = \nabla f(\epsilon) = -G.
\]

Proof. Let \((x_t, y_t) = \text{NAG}(f_M, 0, t)\) and \((\tilde{x}_t, \tilde{y}_t) = \text{NAG}(f_M, \epsilon, t)\) for \( t \leq T \). From Claim 16, for all \( i \in [M] \), \( y_0 < y_{\text{min}}^i \) and \( \tilde{y}_0 < y_{\text{min}}^i \). Thus,
\[
\nabla f_M(0) = \nabla f_M(y_0) = -G + \beta \int_{-\infty}^{y_0} [\exists j \in [M] \text{ s.t. } z \in [y_{\text{min}}^i, y_{\text{max}}^{\text{max}}]] dz = -G,
\]
and similarly, \( \nabla f_M(\epsilon) = \nabla f_M(\tilde{y}_0) = -G \). Since \( p = y_{\text{max}}^{\text{max}} > \max\{y_0, \tilde{y}_0\} \) as we stated above, and from the definition of \( f \), for all \( x \in (-\infty, p] \), \( \nabla f(x) = \nabla f_M(x) \), then also \( \nabla f(0) = \nabla f(\epsilon) = -G \).

Proof of Lemma 11. Both parts are similar. We start with the first equality and prove by induction. Let \((x_t, y_t) = \text{NAG}(f, 0, t, \eta)\) and \((\tilde{x}_t, \tilde{y}_t) = \text{NAG}(R_S, 0, t, \tilde{\eta})\) for \( t \leq T \). For \( t = 1 \),
\[
\tilde{x}_1 = \tilde{y}_0 - \tilde{\eta} \nabla R_S(\tilde{y}_0) \quad \text{(Eq. (4))}
\]
\[
= y_0 - \eta \frac{n-3}{n} \nabla f(y_0) \quad \text{(Eq. (12))}
\]
\[
= y_0 - \eta \nabla f(0) \quad \text{(def of } \eta, \eta = \tilde{\eta} \frac{n-3}{n})
\]
\[
= x_1. \quad \text{(Eq. (4))}
\]
Since $\gamma_1 = 0,$
\[
\tilde{y}_1 = \tilde{x}_1 + \gamma_1(\tilde{x}_1 - \tilde{x}_0) = \tilde{x}_1,
\]
and similarly $x_1 = y_1,$ hence $\tilde{y}_1 = y_1.$ For $t > 1,$
\[
\tilde{x}_t = \tilde{y}_{t-1} - \tilde{\eta} \nabla R_{S'}(\tilde{y}_{t-1}) \\
= \tilde{y}_{t-1} - \tilde{\eta} \frac{n-3}{n} \nabla f(\tilde{y}_{t-1}) \tag{Eq. (4)}
\]
\[
= \tilde{y}_{t-1} - \eta \nabla f(\tilde{y}_{t-1}) \tag{Eq. (12)}
\]
\[
= y_{t-1} - \eta \nabla f(y_{t-1}) \tag{def of $\eta$, $\eta = \tilde{\eta} \frac{n-3}{n}$}
\]
\[
= x_{t-1}. \tag{induction assumption} \tag{Eq. (4)}
\]
We conclude with
\[
y_t = x_t + \gamma_t(x_t - x_{t-1}) \tag{Eq. (5)}
\]
\[
= \tilde{x}_t + \gamma_t(\tilde{x}_t - \tilde{x}_{t-1}) \tag{from above}
\]
\[
= \tilde{x}_t + \gamma_t(\tilde{x}_t - \tilde{x}_{t-1}) \tag{induction assumption} \tag{Eq. (5)}
\]
Now we move to prove the second equality, again by induction. Let $(x_t, y_t) = \text{NAG}(f, \epsilon, t, \eta)$ and $(\tilde{x}_t, \tilde{y}_t) = \text{NAG}(R_{S'}, 0, t, \tilde{\eta})$ for $t \leq T.$ For $t = 1,$
\[
\tilde{x}_1 = \tilde{y}_0 - \tilde{\eta} \nabla R_{S'}(y_0) \tag{Eq. (4)}
\]
\[
= 0 - \tilde{\eta} \frac{1}{n} \nabla g_2(0) \tag{\tilde{y}_0 = 0 and Eq. (13)}
\]
\[
= -\frac{n-3}{n} \nabla f(0) \tag{def of $\eta$, $\eta = \tilde{\eta} \frac{n-3}{n}$}
\]
\[
= \beta \frac{n^2}{n-3} - \eta \nabla f(0) \tag{\nabla g_2(x) = -\beta \eta G + \beta \cdot \int_{-\infty}^{y_{t-1}} 1[z \in [0, \eta G]]dz}
\]
\[
= \epsilon - \eta \nabla f(0) \tag{$\epsilon = \frac{\beta n^2 G}{n-3}$}
\]
\[
= \epsilon - \eta \nabla f(\epsilon) \tag{Claim 24}
\]
\[
= x_1. \tag{Eq. (4) and $y_0 = \epsilon$}
\]
Since $\gamma_1 = 0,$
\[
\tilde{y}_1 = \tilde{x}_1 + \gamma_1(\tilde{x}_1 - \tilde{x}_0) = \tilde{x}_1,
\]
and similarly $x_1 = y_1,$ hence $\tilde{y}_1 = y_1.$ Now for $t > 1.$ Repeating the steps we did for (1),
\[
\tilde{x}_t = \tilde{y}_{t-1} - \tilde{\eta} \nabla R_{S'}(\tilde{y}_{t-1}) \tag{Eq. (4)}
\]
\[
= \tilde{y}_{t-1} - \tilde{\eta} \frac{n-3}{n} \nabla f(\tilde{y}_{t-1}) - \tilde{\eta} \frac{1}{n} \nabla g_2(\tilde{y}_{t-1}) \tag{Eq. (13)}
\]
\[
= \tilde{y}_{t-1} - \eta \nabla f(\tilde{y}_{t-1}) - \tilde{\eta} \frac{1}{n} \nabla g_2(\tilde{y}_{t-1}) \tag{def of $\eta$, $\eta = \tilde{\eta} \frac{n-3}{n}$}
\]
\[
= y_{t-1} - \eta \nabla f(y_{t-1}) - \tilde{\eta} \frac{1}{n} \nabla g_2(y_{t-1}) \tag{induction assumption} \tag{Eq. (4)}
\]
\[
= x_t - \tilde{\eta} \frac{1}{n} \nabla g_2(y_{t-1}).
\]
We need to show that $\nabla g_2(y_{t-1}) = 0.$ Since
\[
\nabla g_2(y_{t-1}) = -\beta \eta G + \beta \cdot \int_{-\infty}^{y_{t-1}} 1[z \in [0, \eta G]]dz,
\]
it is enough to show that $y_{t-1} \geq \eta G$. From Claim 23, for $t > 1$, $y_{t-1} \geq y_1$. Thus,

$$y_{t-1} \geq y_1 = x_1 + \gamma_1(x_1 - x_0) = 0 \quad \text{(Eq. (5))}$$

$$= y_0 - \eta \nabla f(y_0) \quad \text{(Eq. (4))}$$

$$= \epsilon + \eta G. \quad \text{(Claim 24)}$$

So indeed $y_{t-1} \geq \eta G$, hence, $\hat{x}_t = x_t$. We conclude with

$$y_t = x_t + \gamma_t(x_t - x_{t-1}) \quad \text{(Eq. (5))}$$

$$= \hat{x}_t + \gamma_t(\hat{x}_t - x_{t-1}) \quad \text{(from above)}$$

$$= \hat{x}_t + \gamma_t(\hat{x}_t - \hat{x}_{t-1}) \quad \text{(induction assumption)}$$

$$= \bar{y}_t. \quad \text{(Eq. (5))}$$

\[\Box\]

**B.2 Proof of Theorem 10**

**Proof.** Let $c_3 = c_1/4$ and $d_4 = c_2/4$. In order to lower bound the uniform stability of algorithm $A$ we need to pick $S,S' \in \mathbb{Z}^n$ and $z \in \mathbb{Z}$ and lower bound $|\ell(A(S),z) - \ell(A(S'),z)|$. We use $S,S'$ which we defined at Section 4, with $z = 3 \in \mathbb{Z}$. We showed at Lemma 11 that we match the iterations on $f$ of our construction. For $z = 3$, since $f(w;3) = -Gw$,

$$|\ell(x_T,z) - \ell(\hat{x}_T,z)| = G|x_T - \hat{x}_T|.$$  

The first case is when $T \in \{[10n/\hat{\eta}\hat{\beta}(n-3)](i+2) : i = 1, 2, 3 \ldots\}$, for which

$$T \in \{[10/\eta\beta](i+2) : i = 1, 2, 3 \ldots\}. \quad (\eta = \frac{n-3}{n} \hat{\eta}, \hat{\beta})$$

So using Theorem 3, based on $S,S',z$ and our definitions for $G, \beta, \eta$ and $\epsilon$,

$$\delta_{\text{uniform}}^{\text{NAGT},\epsilon}(n) \geq G \min \left\{ \frac{G}{3\beta}, c_2e^{c_1n\eta\beta T} \epsilon \right\} \geq \min \left\{ \frac{G^2}{3\beta}, c_4e^{c_3n\gamma\beta T} \frac{\hat{G}^2G^2}{n} \right\}.$$ 

Above we used the fact that $\eta = \frac{n-3}{n} \hat{\eta} \geq \frac{3}{4}$ since $n \geq 4$. Now for the second case of $T > [40/\hat{\eta}\hat{\beta}] \left( \ln \frac{6n}{\hat{\eta}^2\hat{\beta}^2} + 3 \right)$, in which,

$$T > [40/\hat{\eta}\hat{\beta}] \left( \ln \frac{6n}{\hat{\eta}^2\hat{\beta}^2} + 3 \right)$$

$$\geq [10n/\hat{\eta}\hat{\beta}(n-3)] \left( \ln \frac{3n^2}{2\hat{\eta}^2\hat{\beta}^2(n-3)} + 3 \right) \quad (n \geq 4)$$

$$= [10/\eta\beta] \left( \ln \frac{3(n-3)}{2\eta^2\beta^2} + 3 \right) \quad (\eta = \frac{n-3}{n} \hat{\eta}, \hat{\beta})$$

$$= [10/\eta\beta] \left( \ln \frac{3G}{2\beta\epsilon} + 3 \right) \quad (\epsilon = \frac{\hat{G}^2G^2}{n-3})$$

Hence, using Theorem 3,

$$|\ell(x_T,z) - \ell(\hat{x}_T,z)| = \frac{G^2}{3\beta} = \frac{\hat{G}^2}{3\hat{\beta}} \implies \delta_{\text{uniform}}^{\text{NAGT},\epsilon}(n) \geq \frac{\hat{G}^2}{3\hat{\beta}}.$$  

\[\Box\]

**C Other Variants of NAG**

Here we show the equivalence between the version of NAG we analyze in this paper, and other versions that are common in the literature.
C.1 Variant I

First we consider a common variant that appears for example in (Allen-Zhu and Orecchia, 2017). Starting at $\tilde{z}_0 = \tilde{y}_0 = \tilde{x}_0$, this variant proceeds for $t = 1, 2, \ldots$ as:

$$\tilde{x}_{t+1} = \tau_t \tilde{z}_t + (1 - \tau_t) \tilde{y}_t; \quad \text{(14)}$$
$$\tilde{y}_{t+1} = \tilde{x}_{t+1} - \frac{1}{\beta} \nabla f(\tilde{x}_{t+1}); \quad \text{(15)}$$
$$\tilde{z}_{t+1} = \tilde{z}_t - \alpha_{t+1} \nabla f(\tilde{x}_{t+1}), \quad \text{(16)}$$

where $\alpha_{t+1} = \frac{t+2}{2t}$ and $\tau_t = \frac{2}{t+2}$.

Our next claim establishes that this variant is precisely equivalent to the NAG iterations considered in the paper (Eqs. (1) and (2)) with step size $\eta = 1/\beta$.

Claim 25. For all $t \in [T]$, $\tilde{x}_t = y_{t-1}$ and $\tilde{y}_t = x_t$.

Proof. We will prove using induction. For $t = 1$, $\tilde{x}_1 = y_0$ follows from

$$\tilde{x}_1 = \tau_0 \tilde{z}_0 + (1 - \tau_0) \tilde{y}_0 \quad \text{(Eq. (14))}$$
$$= \tau_0 \tilde{x}_0 + (1 - \tau_0) \tilde{x}_0 \quad \text{(initialization)}$$
$$= \tilde{x}_0$$
$$= x_0 \quad \text{(initialization)}$$

The second equality, $\tilde{y}_t = x_t$, is immediate for all $t \in [T]$ given $\tilde{x}_t = y_{t-1}$ since

$$\tilde{y}_t = \tilde{x}_t - \frac{1}{\beta} \nabla f(\tilde{x}_t) \quad \text{(Eq. (15))}$$
$$= y_{t-1} - \frac{1}{\beta} \nabla f(y_{t-1}) \quad \text{(from (1))}$$
$$= x_t \quad \text{(Eq. (4))}$$

We finish by showing that $\tilde{x}_{t+1} = y_t$,

$$\tilde{x}_{t+1} = \tilde{y}_t + \tau_t (\tilde{z}_t - \tilde{y}_t) \quad \text{(Eq. (14))}$$
$$= \tilde{y}_t + \frac{2}{t+2} (\tilde{z}_t - \tilde{y}_t) \quad (\tau_t = \frac{2}{t+2})$$
$$= \tilde{y}_t + \gamma_t \frac{2}{t-1} (\tilde{z}_t - \tilde{y}_t) \quad (\gamma_t = \frac{1}{t-1})$$
$$= \tilde{y}_t + \gamma_t \left( \frac{2}{t-1} \tilde{z}_t - \frac{2}{t-1} \tilde{y}_t \right)$$
$$= \tilde{y}_t + \gamma_t \left( \frac{2}{t-1} \tilde{z}_t - \frac{2}{t-1} \tilde{y}_t + \frac{t+1}{t-1} \tilde{y}_t - \frac{t+1}{t-1} \tilde{y}_t \right)$$
$$= \tilde{y}_t + \gamma_t \left( \tilde{y}_t + \frac{2}{t-1} \tilde{z}_{t-1} - \frac{t+1}{t-1} \tilde{x}_t + \left( \frac{t+1}{\beta(t-1)} - \frac{2\alpha_t}{t-1} \right) \nabla f(\tilde{x}_t) \right) \quad \text{(Eq. (15), Eq. (16))}$$

$$= \tilde{y}_t + \gamma_t \left( \frac{2}{t-1} \tilde{z}_{t-1} - \frac{t+1}{t-1} \frac{1 - \tau_t}{\beta(t-1)} \tilde{y}_{t-1} + \tau_t - \frac{t+1}{t-1} \tilde{y}_{t-1} \right) \quad \text{(Eq. (14))}$$
$$= \tilde{y}_t + \gamma_t \left( \tilde{y}_{t-1} - \frac{(1 - \tau_t)(\tilde{y}_{t-1} + \tau_t - \tilde{z}_{t-1})}{t-1} \right)$$
$$= \tilde{y}_t + \gamma_t \left( \tilde{y}_{t-1} - \frac{(1 - \tau_t)(\tilde{y}_{t-1} + \tau_t - \tilde{z}_{t-1})}{t-1} \right)$$
$$= \tilde{y}_t + \gamma_t (\tilde{y}_{t-1} - \tilde{z}_{t-1}) \quad \text{(induction assumption)}$$
$$= x_t + \gamma_t (x_t - x_{t-1}) \quad \text{(Eq. (5))}$$

$\blacksquare$
C.2 Variant II

Next, we consider a second variant of NAG that appears in, e.g., Lan (2012). Starting at $\hat{x}_1^{ag} = \hat{x}_1$, this version takes the form

$$\hat{x}_t^{md} = \tilde{\beta}_t^{-1}\hat{x}_t + (1 - \tilde{\beta}_t^{-1})\hat{x}_t^{ag}; \quad (\text{Eq. (17)})$$

$$\hat{x}_{t+1} = \hat{x}_t - \gamma_t \nabla f(\hat{x}_t); \quad (\text{Eq. (18)})$$

where $\tilde{\beta}_t = (t + 1)/2$ and $\gamma_t = (t + 1)/4\beta$.

The claim below establishes the equivalence between this variant and the version of NAG given in Eqs. (1) and (2) with step size $\eta = 1/2\beta$.

**Claim 26.** For all $t \in [T]$, $\hat{x}_t^{ag} (1) = x_{t-1}$ and $\hat{x}_t^{md} (2) = y_{t-1}$.

**Proof.** We will prove by induction. For $t = 1$, the first equality follows from the initialization,

$$\hat{x}_1^{ag} = \hat{x}_1 = x_0.$$  

The second equality follows from

$$\hat{x}_1^{md} = \tilde{\beta}_1^{-1}\hat{x}_1 + (1 - \tilde{\beta}_1^{-1})\hat{x}_1^{ag} \quad (\text{Eq. (17)})$$

$$= \tilde{\beta}_1^{-1}\hat{x}_1 + (1 - \tilde{\beta}_1^{-1})\hat{x}_1 \quad (\text{initialization})$$

$$= \hat{x}_1 \quad (\text{initialization})$$

$$= x_0.$$  

For $t > 1$, we first show that $\hat{x}_t^{ag} = x_{t-1}$,

$$\hat{x}_t^{ag} = \tilde{\beta}_{t-1}^{-1}\hat{x}_{t-1} + (1 - \tilde{\beta}_{t-1}^{-1})\hat{x}_{t-1}^{ag} \quad (\text{Eq. (19)})$$

$$= \tilde{\beta}_{t-1}^{-1}(\hat{x}_{t-1} - \tilde{\gamma}_{t-1}\nabla f(\hat{x}_{t-1})) + (1 - \tilde{\beta}_{t-1}^{-1})\hat{x}_{t-1}^{ag} \quad (\text{Eq. (18)})$$

$$= \hat{x}_{t-1}^{md} - \tilde{\beta}_{t-1}^{-1}\tilde{\gamma}_{t-1}\nabla f(\hat{x}_{t-1}) \quad (\text{Eq. (17)})$$

$$= \hat{x}_{t-1}^{md} - \eta \nabla f(\hat{x}_{t-1}) \quad (\eta = \frac{1}{2\beta})$$

$$y_{t-2} - \eta \nabla f(y_{t-2}) \quad (\text{induction assumption})$$

$$= x_{t-1}. \quad (\text{Eq. (4)})$$

We conclude by showing that $\hat{x}_t^{md} = y_{t-1}$ using the induction assumption and the equality $\hat{x}_t^{ag} = x_{t-1}$ we showed above,

$$\hat{x}_t^{md} = \tilde{\beta}_t^{-1}\hat{x}_t + (1 - \tilde{\beta}_t^{-1})\hat{x}_t^{ag} \quad (\text{Eq. (17)})$$

$$= \frac{\tilde{\beta}_t^{-1}}{\tilde{\beta}_t} \tilde{\beta}_{t-1}^{-1}\hat{x}_{t-1} + (1 - \tilde{\beta}_t^{-1})\hat{x}_t^{ag} \quad (\text{Eq. (19)})$$

$$= \frac{t}{t+1} \tilde{\beta}_{t-1}^{-1}\hat{x}_{t-1} + (1 - \tilde{\beta}_t^{-1})\hat{x}_t^{ag} \quad (\tilde{\beta}_t = \frac{t+1}{2})$$

$$= \frac{t}{t+1} (\hat{x}_{t-1}^{ag} + (1 - \tilde{\beta}_{t-1}^{-1})\hat{x}_{t-1}^{ag}) + (1 - \tilde{\beta}_t^{-1})\hat{x}_t^{ag} \quad (\text{Eq. (19)})$$

$$= \hat{x}_t^{ag} \left( \frac{t}{t+1} + 1 - \tilde{\beta}_t^{-1} \right) + \frac{t(1 - \tilde{\beta}_{t-1}^{-1})}{t+1} \hat{x}_t^{ag} \quad (\tilde{\beta}_t = \frac{t+1}{2})$$

$$= \hat{x}_t^{ag} \left( \frac{t}{t+1} + 1 - \frac{2}{t+1} \right) + \frac{t - 2}{t+1} \hat{x}_t^{ag} \quad (\tilde{\beta}_t = \frac{t+1}{2})$$

$$= \hat{x}_t^{ag} + \gamma_{t-1}(\hat{x}_t^{ag} - \hat{x}_t^{ag}) \quad (\gamma_{t-1} = \frac{t-2}{t+1})$$

$$= x_{t-1} + \gamma_{t-1}(x_{t-1} - x_{t-2}) \quad (\text{induction assumption})$$

$$= y_{t-1}. \quad (\text{Eq. (5)})$$
D Initialization Stability Upper Bounds

In this section we prove initialization bounds for GD in the convex and smooth setting and NAG in the setting of a quadratic objective.

D.1 Gradient Descent, Smooth Objectives

In this section we consider fixed step-size GD in the convex and \( \beta \)-smooth setting. The update rule of this version of GD is \( x_{t+1} = x_t - \eta \nabla f(x_t) \), where \( 0 < \eta \leq \frac{1}{\beta} \).

Claim 27. Let \( f \) be a convex, \( \beta \)-smooth function. Then for all \( x_0 \in \mathbb{R}^d \), \( \epsilon > 0 \), and \( T \geq 1 \),

\[
\delta_{GD,1}^{\text{init}} (x_0, \epsilon) \leq \epsilon.
\]

This bound is tight since for \( f \) \( \equiv 0 \), we have trivially that \( \delta_{GD,1}^{\text{init}} (x_0, \epsilon) = \epsilon \) for all \( T \). The proof of the above claim mostly follow arguments of Hardt et al. (2016) and is given here for completeness. First we state the well-known co-coercivity property of the gradient operator over smooth functions (e.g., Nesterov, 2003).

Lemma 28. Let \( f \) be a convex and \( \beta \)-smooth function on \( \mathbb{R}^d \). Then for any \( u, v \in \mathbb{R}^d \), we have

\[
(\nabla f(u) - \nabla f(v))^T (u - v) \geq \frac{1}{\beta} \| \nabla f(u) - \nabla f(v) \|^2.
\]

Below is a simple contractive property of GD based on this lemma.

Corollary 29. Let \( f \) be a convex and \( \beta \)-smooth function on \( \mathbb{R}^d \). Then for any \( u, v \in \mathbb{R}^d \) and \( \eta \leq \frac{2}{\beta} \), we have

\[
\|(u - \eta \nabla f(u)) - (v - \eta \nabla f(v))\|_2 \leq \|u - v\|_2.
\]

Proof. Write:

\[
\|(u - \eta \nabla f(u)) - (v - \eta \nabla f(v))\|_2^2 = \|u - v\|_2^2 + \eta^2 \| \nabla f(u) - \nabla f(v) \|^2 - 2\eta (u - v)^T (\nabla f(u) - \nabla f(v))
\]

\[
\leq \|u - v\|_2^2 + \eta^2 \| \nabla f(u) - \nabla f(v) \|^2 - \frac{2\eta}{\beta} \| \nabla f(u) - \nabla f(v) \|^2
\]

\[
\leq \|u - v\|_2^2.
\]

We can now prove our claim.

Proof of Claim 27. Let \( x_0, \tilde{x}_0 \) be our starting points such that \( \|x_0 - \tilde{x}_0\| \leq \epsilon \). Let \( (x_t)_{t=0}^{T-1} \) and \( (\tilde{x}_t)_{t=0}^{T-1} \) be the iterations of GD over \( f \) starting at \( x_0 \) and \( \tilde{x}_0 \) respectively. Thus, by Corollary 29,

\[
\|x_t - \tilde{x}_t\| = \| (x_{t-1} - \eta \nabla f(x_{t-1})) - (\tilde{x}_{t-1} - \eta \nabla f(\tilde{x}_{t-1})) \| \leq \| x_{t-1} - \tilde{x}_{t-1} \|.
\]

Invoking the same argument recursively,

\[
\|x_t - \tilde{x}_t\| \leq \| x_0 - \tilde{x}_0 \| \leq \epsilon.
\]

Hence, the initialization stability of GD is at most \( \epsilon \).

D.2 Gradient Descent, Non-smooth Objectives

In this section we consider GD with a constant step size in the convex and non-smooth setting. The update rule of this version of GD is

\[
x_{t+1} = \Pi_{\Omega} [x_t - \eta \nabla f(x_t)],
\]

where \( \Pi_{\Omega} [\cdot] \) is the Euclidean projection onto a compact convex set \( \Omega \subseteq \mathbb{R}^d \). Often, the final output of the algorithm is the average of the iterates. The claim below holds for both final and average versions.
Claim 30. Let \( f \) be a convex, \( G \)-Lipschitz function. Then for GD with \( T \) steps, for all \( x_0 \in \mathbb{R}^d \) and \( \epsilon > 0 \),

\[
\delta_{GD}^{\text{init}}(x_0, \epsilon) \leq \epsilon + 2G\eta \sqrt{T}.
\]

The proof is similar to the one of Bassily et al. (2020); we give it here for completeness.

**Proof.** Let \( \tilde{x}_0 \in \mathbb{R}^d \) such that \( \|\tilde{x}_0 - x_0\| \leq \epsilon \). Let \( \delta_t \triangleq \|x_t - \tilde{x}_t\| \). Then,

\[
\delta_{t+1}^2 = \|\Pi_{\Omega}(x_t - \eta \nabla f(x_t)) - \Pi_{\Omega}(\tilde{x}_t - \eta \nabla f(\tilde{x}_t))\|^2 \\
\leq \|(x_t - \eta \nabla f(x_t)) - (\tilde{x}_t - \eta \nabla f(\tilde{x}_t))\|^2 \\
= \delta_t^2 + \eta^2 \|\nabla f(x_t) - \nabla f(\tilde{x}_t)\|^2 + 2\eta \langle \nabla f(x_t) - \nabla f(\tilde{x}_t), x_t - \tilde{x}_t \rangle \\
\leq \delta_t^2 + \eta^2 \|\nabla f(x_t) - \nabla f(\tilde{x}_t)\|^2,
\]

where the last inequality follows from convexity,

\[
f(x_t) \geq f(\tilde{x}_t) + \langle \nabla f(\tilde{x}_t), x_t - \tilde{x}_t \rangle \\
\geq f(x_t) + \langle \nabla f(x_t), \tilde{x}_t - x_t \rangle + \langle \nabla f(\tilde{x}_t), x_t - \tilde{x}_t \rangle \\
\implies \langle \nabla f(x_t) - \nabla f(\tilde{x}_t), x_t - \tilde{x}_t \rangle \geq 0.
\]

From the Lipschitz condition,

\[
\|\nabla f(x_t) - \nabla f(\tilde{x}_t)\| \leq \|\nabla f(x_t)\| + \|\nabla f(\tilde{x}_t)\| \leq 2G,
\]

hence

\[
\delta_{t+1}^2 \leq \delta_t^2 + 4\eta^2 G^2.
\]

Invoking the argument above recursively,

\[
\delta_t \leq \sqrt{\delta_0^2 + 4\eta^2 G^2 t} \\
\leq \delta_0 + 2\eta G \sqrt{t} \\
\leq \epsilon + 2\eta G \sqrt{t}.
\]

Since this bound holds for all \( t = 0, \ldots, T - 1 \), the bound also holds after averaging,

\[
\left\| \frac{1}{T} \sum_{t=0}^{T-1} x_t - \frac{1}{T} \sum_{t=0}^{T-1} \tilde{x}_t \right\| \leq \epsilon + 2\eta G \sqrt{T}.
\]

Thus we proved our initialization stability bound.

We note that this bound is tight up to a constant factor: this can be shown using the same type of construction as Bassily et al. (2020) use for lower bounding the uniform stability of GD in the non-smooth case. The idea is to use initial points \( x_0 = 0, \tilde{x}_0 = (\epsilon/\sqrt{d}) \cdot (1, \ldots, 1) \), with the following objective function over the unit ball:

\[
f(x) = G \max\{0, x_1 - c, \ldots, x_d - c\},
\]

for \( c < \epsilon/\sqrt{d} \). The first trajectory will stay put as \( x_1 = 0 \) is a minimizer; for the second trajectory, the arguments as in (Bassily et al., 2020) show that at iteration \( i \) a valid sub-gradient will be \( Ge_i \) (\( e_i \) is the \( i \)'th standard basis element), so that \( \tilde{x}_i = x_0 - \sum_{i=1}^i Ge_i \) and we will have

\[
\|x_t - \tilde{x}_t\| = \|\tilde{x}_t\| = \Omega(G\eta \sqrt{t}).
\]
D.3 Accelerated Gradient Method, Convex Objectives

In this section we provide an exponential upper bound of initialization stability for NAG in the convex and
smooth setting.

**Claim 31.** Let \( f \) be convex and \( \beta \)-smooth function. Then for NAG with step size \( \eta \leq 1/\beta \) and \( T \) steps, for all \( x_0 \in \mathbb{R}^d \) and \( \epsilon > 0 \),

\[
\delta_{\text{NAG}}^{\text{init}}(x_0, \epsilon) \leq \epsilon + \eta \beta \epsilon 3^{T-1}.
\]

**Proof.** Let \( x_0, \tilde{x}_0 \) be our starting points s.t. \( \|x_0 - \tilde{x}_0\| \leq \epsilon \). Let us consider two runs of the method initialized
at \( x_0, \tilde{x}_0 \) respectively:

\[
(x_t, y_t) = \text{NAG}(f, x_0, t), \quad (\tilde{x}_t, \tilde{y}_t) = \text{NAG}(f, \tilde{x}_0, t), \quad \forall \ t \geq 0,
\]

We will show the bound by first proving that \( \|\Delta_y^T\| \leq \epsilon + \eta \beta \epsilon 3^t \) by induction. For \( t = 0 \) the claim is
immediate. Assuming the claim is correct for \( k = 0, \ldots, t \), we will prove for \( t + 1 \).

\[
\|\Delta^T_{t+1}\| = \|\Delta^T_t - \eta \Delta^T_t + \Delta^T_{t+1}\| \quad \text{(Eq. (8))}
\]

\[
\leq \|\Delta^T_t - \eta \Delta^T_t\| + \|\Delta^T_{t+1}\| \quad \text{(Corollary 29)}
\]

\[
\leq \|\Delta^T_t\| + \|\Delta^T_{t+1}\| + \eta \sum_{k=1}^{t} \|\Delta^T_k\| \quad \text{(Eq. (9) recursively with } \gamma_k \leq 1)\)
\]

\[
= \|\Delta^T_t\| + \eta \sum_{k=1}^{t} \|\Delta^T_k\| \quad \text{(\( \Delta^T_{t+1} = 0 \) since } \gamma_1 = 0)\)
\]

\[
\leq \|\Delta^T_t\| + \eta \beta \sum_{k=1}^{t} \|\Delta^T_k\| \quad \text{(smoothness)}
\]

\[
\leq \epsilon + \eta \beta \epsilon 3^t + \eta \beta \sum_{k=1}^{t} (\epsilon + \eta \beta \epsilon 3^k) \quad \text{(induction assumption)}
\]

\[
\leq \epsilon + \eta \beta \epsilon 3^t + \eta \beta \sum_{k=1}^{t} (\epsilon + 3^k) \quad (\eta \leq 1/\beta)
\]

\[
= \epsilon + \eta \beta \epsilon 3^t + \eta \beta \left( t + 3^{t+1} - 1 \right) \quad (t \geq 0 \implies 3^t \geq 2t)
\]

\[
\leq \epsilon + \eta \beta \epsilon \left( t + 3^t + 3 \right) \quad (t \geq 0 \implies 3^t \geq 2t)
\]

\[
\leq \epsilon + \eta \beta \epsilon \left( 2t + 3^t + 3 \right) \quad (t \geq 0 \implies 3^t \geq 2t)
\]

We finish with

\[
\|\Delta^T_t\| = \|\Delta^T_{t-1} - \eta \Delta^T_{t-1}\| \quad \text{(Eq. (7))}
\]

\[
\leq \|\Delta^T_{t-1}\| \quad \text{(Corollary 29)}
\]

\[
\leq \epsilon + \eta \beta \epsilon 3^{t-1}.
\]

D.4 Accelerated Gradient Method, Quadratic Objectives

The argument we give below is similar to the one presented by Chen et al. (2018) for bounding the uniform
stability of NAG for quadratic objectives and relies on some of their technical results, which we state and
prove here for completeness. The following claim bound the initialization stability of NAG for a quadratic
objective.
Claim 32. Let $f$ be a quadratic function with a positive semi-definite Hessian. Then for NAG with $T$ steps, for all $x_0 \in \mathbb{R}^d$ and $\epsilon > 0$,

$$\delta_{\text{NAG}}(x_0, \epsilon) \leq 4T\epsilon.$$ 

In order to prove Claim 32 we need the following technical claim (proof at Appendix D.5).

Lemma 33. Suppose $M_k = \begin{pmatrix} (1 + \gamma_k)A & -\gamma_k A \\ 1 & 0 \end{pmatrix}$, where $0 \leq A \leq 1$ and $-1 \leq \gamma_k \leq 1$. Then

$$\left\| \prod_{k=1}^{t} M_k \right\| \leq 2(t + 1).$$

Proof of Claim 32. Let $0 \leq H \leq \beta$ be the Hessian of $f$. Let $x_0, \tilde{x}_0$ be our starting points s.t. $\|x_0 - \tilde{x}_0\| \leq \epsilon$. Let us consider two runs of the method initialized at $x_0, \tilde{x}_0$ respectively:

$$(x_t, y_t) = \text{NAG}(f, x_0, t), \quad (\tilde{x}_t, \tilde{y}_t) = \text{NAG}(f, \tilde{x}_0, t), \quad \forall t \geq 0,$$

For $t \geq 1$ we can combine Eq. (4) and Eq. (5) to obtain

$$x_{t+1} = (1 + \gamma_t)x_t - \gamma_t x_{t-1} - \eta\nabla f((1 + \gamma_t)x_t - \gamma_t x_{t-1}).$$

Using our notation for $\Delta^x_t \triangleq x_t - \tilde{x}_t$ and the fact that for a quadratic $f$, differences between gradients can be expressed as $\nabla f(x) - \nabla f(x') = H(x - x')$ for any $x, x'$,

$$\Delta^x_{t+1} = (1 + \gamma_t)\Delta^x_t - \gamma_t \Delta^x_{t-1} - \eta H((1 + \gamma_t)\Delta^x_t - \gamma_t \Delta^x_{t-1}).$$

We can rewrite this matrix form,

$$\begin{pmatrix} \Delta^x_{t+1} \\ \Delta^x_t \end{pmatrix} = \begin{pmatrix} (1 + \gamma_t)(I - \eta H) & -\gamma_t(I - \eta H) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta^x_t \\ \Delta^x_{t-1} \end{pmatrix}.$$ 

Thus,

$$\begin{pmatrix} \Delta^x_{t+1} \\ \Delta^x_t \end{pmatrix} = \prod_{k=1}^{t} \begin{pmatrix} (1 + \gamma_k)(I - \eta H) & -\gamma_k(I - \eta H) \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta^x_0 \\ \Delta^x_{0} \end{pmatrix}.$$ 

We can bound the norm of $\Delta^x_t$,

$$\|\Delta^x_t\| = \|\Delta^x_0 - \eta \Delta^x_t\| = \|\Delta^x_0 - \eta \Delta^x_t\| \leq \|\Delta^x_0\|,$$

where the first equality comes from Eq. (7), the second in the initialization and the third is Corollary 29. Since $0 \leq H \leq \beta I$, $0 \leq I - \eta H \leq I$. Thus, using Lemma 33 and the triangle inequality we obtain

$$\|\Delta^x_{t+1}\| \leq \|\Delta^x_{t+1}, \Delta^x_t\| \leq 2(t + 1)\|\Delta^x_1, \Delta^x_0\| \leq 4(t + 1)\epsilon.$$ 

Thus, the initialization stability after $T$ iterations is upper bounded by $4T\epsilon$. ■

D.5 Proof of Lemma 33

First we prove the following claim.

Claim 34. Let $T = \begin{pmatrix} \lambda_1 & c \\ 0 & \lambda_2 \end{pmatrix}$, then for all $t > 0$, $T^t = \begin{pmatrix} \lambda_1^t & c \sum_{i=0}^{t-1} \lambda_1^i \lambda_2^{t-i-1} \\ 0 & \lambda_2^t \end{pmatrix}$.

Proof. By induction. For $t = 1$ the claim is immediate. Assuming the lemma is correct for $t - 1 > 0$, then

$$T^t = T^{t-1}T = \begin{pmatrix} \lambda_1^{t-1} & c \sum_{i=0}^{t-2} \lambda_1^i \lambda_2^{t-i-2} \\ 0 & \lambda_2^{t-1} \end{pmatrix} \begin{pmatrix} \lambda_1 & c \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1^t & c(\lambda_1^{t-1} + \lambda_2 \sum_{i=0}^{t-2} \lambda_1^i \lambda_2^{t-i-2}) \\ 0 & \lambda_2^t \end{pmatrix} = \begin{pmatrix} \lambda_1^t & c \sum_{i=0}^{t-1} \lambda_1^i \lambda_2^{t-i-1} \\ 0 & \lambda_2^t \end{pmatrix}.$$ 

■
Lemma 35. Suppose \( H_k = \begin{pmatrix} 1 + \gamma_k h & -\gamma_k h \\ 0 & 0 \end{pmatrix} \), where \( 0 \leq h \leq 1 \) and \(-1 \leq \gamma_k \leq 1\). Then
\[
\|\prod_{k=1}^t H_k\|_2 \leq 2(t + 1).
\]

Proof of Lemma 35. Let \( M_t = \prod_{k=1}^t H_k \). Thus,
\[
\|M_t\|_2 = \max \{x^T M_t y : x, y \in \mathbb{R}^2 \text{ with } \|x\|_2 = \|y\|_2 = 1\}.
\]

Let \( \tilde{x}_t, \tilde{y}_t \) be defined as
\[
\tilde{x}_t, \tilde{y}_t = \arg \max_{x,y \in \mathbb{R}^2 \text{ with } \|x\|_2 = \|y\|_2 = 1} (x^T M_t y).
\]

For a given \( h \), let \( f_t \) be defined as
\[
f_t(\gamma_1, \ldots, \gamma_t) \triangleq \|M_t\|_2 = \tilde{x}_t^T M_t \tilde{y}_t.
\]

Note that \( f \) is a multivariate linear function and as such attains its maximum in the extreme values of the variables. Thus,
\[
f_t(\gamma_1, \ldots, \gamma_t) \leq \max_{\forall i \leq t, \gamma_i \in \{-1, 1\}} f_t(\gamma_1, \ldots, \gamma_t).
\]

Using induction we will show that
\[
\max_{\forall i \leq t, \gamma_i \in \{-1, 1\}} f_t(\gamma_1, \ldots, \gamma_t) \leq 2(t + 1).
\]

For \( t = 0 \), \( f_0 = \|M_0\|_2 = \|I\|_2 = 1 \). For \( t = 1 \),
\[
M_1 = \left\{ \begin{pmatrix} 2h & -h \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & h \\ 1 & 0 \end{pmatrix} \right\},
\]

and it is easy to verify that \( \|M_1\| \leq 4 \). Now for \( t \geq 2 \). Let's assume that \( \gamma_1 = -1 \). Thus,
\[
f(\gamma_1, \ldots, \gamma_t) = \tilde{x}_t^T M_t \tilde{y}_t
\]
\[
= \tilde{x}_t^T \prod_{k=2}^t H_k \begin{pmatrix} 0 & h \\ 1 & 0 \end{pmatrix} \tilde{y}_t
\]
\[
\leq f_{t-1}(\gamma_2, \ldots, \gamma_t) \left\| \begin{pmatrix} 0 & h \\ 1 & 0 \end{pmatrix} \tilde{y}_t \right\|^{-1}_2.
\]

The last transition is due to the optimal values of \( \tilde{x}_{t-1} \) and \( \tilde{y}_{t-1} \) in the definition of \( f_{t-1} \). Since
\[
\left\| \begin{pmatrix} 0 & h \\ 1 & 0 \end{pmatrix} \tilde{y}_t \right\|^{-1}_2 \leq 1 \quad (0 \leq h \leq 1)
\]
we conclude using the induction assumption. Similarly, if \( \gamma_t = -1 \),
\[
f(\gamma_1, \ldots, \gamma_t) \leq \left\| \tilde{x}_t \begin{pmatrix} 0 & h \\ 1 & 0 \end{pmatrix} \right\|^{-1}_2 f_{t-1}(\gamma_1, \ldots, \gamma_{t-1}),
\]

and again we obtain our result with the induction assumption. If \( \gamma_k = -1 \) for \( k \in \{2, \ldots, t - 1\} \),
\[
H_{k+1} H_k H_{k-1} = \begin{pmatrix} (1 + \gamma_{k+1} h) & -\gamma_{k+1} h \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -h \\ 1 & 0 \end{pmatrix} \begin{pmatrix} (1 + \gamma_{k-1} h) & -\gamma_{k-1} h \\ 0 & 0 \end{pmatrix}
\]
\[
= h \begin{pmatrix} (1 - \gamma_{k+1} \gamma_{k-1}) h & -\gamma_{k+1} \gamma_{k-1} h \\ 0 & 0 \end{pmatrix},
\]

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Hence, \( \lambda \) where \( D \) is diagonal. Thus, \( H_k = \begin{bmatrix} 2h & -h \\ 1 & 0 \end{bmatrix} \), which does not depend on \( k \), so we will denote it with \( H \) for all \( k \). The Schur decomposition of \( H \) of the form \( QUQ^{-1} \) where \( Q \) is unitary and \( U \) is upper triangular, is

\[
H = Q \begin{pmatrix} \lambda_1 & -(2h\lambda_2 - h + 1) \\ 0 & \lambda_2 \end{pmatrix} Q^{-1},
\]

where \( \lambda_1 = h + \sqrt{h(h-1)} \) and \( \lambda_2 = h - \sqrt{h(h-1)} \) are the eigenvalues of \( H \) and \( Q = \frac{1}{\sqrt{1+h}} \begin{pmatrix} \lambda_1 & -1 \\ 1 & \lambda_2 \end{pmatrix} \).

Thus, taking \( H \) to the power of \( t \) and using Claim 34,

\[
H^t = Q \begin{pmatrix} \lambda_1^t & c \sum_{i=0}^{t-1} \lambda_1^i \lambda_2^{t-i-1} \\ 0 & \lambda_2^t \end{pmatrix} Q^{-1},
\]

for \( c = -(2h\lambda_2 - h + 1) \). Returning to \( f_t \),

\[
f_t(1, \ldots, 1) = \left\| \begin{pmatrix} \lambda_1^t & c \sum_{i=0}^{t-1} \lambda_1^i \lambda_2^{t-i-1} \\ 0 & \lambda_2^t \end{pmatrix} \right\|_2 \\
\leq \left\| \begin{pmatrix} \lambda_1^t & c \sum_{i=0}^{t-1} \lambda_1^i \lambda_2^{t-i-1} \\ 0 & \lambda_2^t \end{pmatrix} \right\|_F.
\]

Since \( |\lambda_1| = |\lambda_2| = h \leq 1 \),

\[
f(1, \ldots, 1) \leq \sqrt{2h^t + (|c|(t-1)h^{t-1})^2} \leq \sqrt{2 + |c|^2(t-1)^2}.
\]

\[
|c|^2 = |h - 1 - 2h^2 + i \cdot 2h\sqrt{h(1-h)}|^2 \\
= h^2 - 2h - 4h^3 + 1 + 4h^2 + 4h^4 + 4h^3 - 4h^4 \\
= 5h^2 - 2h + 1 \leq 4.
\]

Thus,

\[
f(1, \ldots, 1) \leq \sqrt{2 + 4(t-1)^2} \leq \sqrt{4(t+1)^2} \leq 2(t+1),
\]

and we conclude our lemma. \( \square \)

Finally we move to our proof.

**Proof of Appendix D.5.** Since \( A \) is symmetric, it can be written as \( A = QDQ^{-1} \) where \( Q \) is an orthogonal matrix and \( D \) is diagonal. Thus,

\[
M_k = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} (1 + \gamma_k)D & -\gamma_kD \\ I & 0 \end{pmatrix} \begin{pmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{pmatrix}.
\]

Hence,

\[
\prod_{k=1}^{t} M_k = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \prod_{k=1}^{t} \begin{pmatrix} (1 + \gamma_k)D & -\gamma_kD \\ I & 0 \end{pmatrix} \begin{pmatrix} Q^{-1} & 0 \\ 0 & Q^{-1} \end{pmatrix}.
\]
Let
\[ N_k = \begin{pmatrix} (1 + \gamma_k)D & -\gamma_k I \\ I & -\gamma_k D \end{pmatrix} \quad \text{and} \quad P_{k,i} = \begin{pmatrix} (1 + \gamma_k)D_{ii} & -\gamma_k D_{ii} \\ 1 & 0 \end{pmatrix}. \]

We will bound \( \| \prod_{k=1}^t N_k \|_2 \). Let \( \tilde{x} \in \mathbb{R}^{2d} \) be defined as
\[ \tilde{x} = \arg \max_{x: \|x\|_2 = 1} \| \prod_{k=1}^t N_k x \|_2, \]
and let \( y = \prod_{k=1}^t N_k \tilde{x} \). Note that since \( N_k \) is a block matrix where all the blocks are square and diagonal, for \( i \leq d \),
\[ y_i = \begin{pmatrix} \prod_{k=1}^t P_{k,i}(\tilde{x}_i, \tilde{x}_{i+d})^T \\ 1 \end{pmatrix} \quad \text{and} \quad y_{i+d} = \begin{pmatrix} \prod_{k=1}^t P_{k,i}(\tilde{x}_i, \tilde{x}_{i+d})^T \\ 2 \end{pmatrix}. \]

From Lemma 35, \( \| \prod_{k=1}^t P_{k,i} \|_2 \leq 2(t+1) \), hence,
\[ \| (y_i, y_{i+d}) \| \leq 2(t+1) \cdot \| (\tilde{x}_i, \tilde{x}_{i+d}) \|. \]

Thus,
\[ \| y \|^2 = \sum_{i=1}^d \| (y_i, y_{i+d}) \|^2 \leq \sum_{i=1}^d (2(t+1))^2 \| (\tilde{x}_i, \tilde{x}_{i+d}) \|^2 = (2(t+1))^2 \| \tilde{x} \|^2 = (2(t+1))^2. \]

And we conclude with \( 2(t+1) \geq \| y \|_2 = \| \prod_{k=1}^t N_k \|_2 = \| \prod_{k=1}^t M_k \|_2 \).

\section*{E Additional Uniform Stability Bounds}

\subsection*{E.1 Lower Bound for NAG, Quadratic Objectives}

In this section we show that the upper bound of the uniform stability of NAG for quadratic objectives established by Chen et al. (2018) of \( 4\eta G^2 T^2/n \) is tight up to a constant factor. Our sample space is \( \mathcal{Z} = \{1, 2\} \), the loss function is
\[ f(w; z) = \begin{cases} Gw & \text{if } z = 1; \\ -Gw & \text{if } z = 2. \end{cases} \]
and our training samples are \( S = (1, \ldots, 1) \) and \( S' = (2, 1, \ldots, 1) \). Thus,
\[ R_S(w) = Gw \quad \text{and} \quad R_{S'} = \frac{n-2}{n} Gw. \]

Let us consider two runs of the method on \( R_S \) and \( R_{S'} \) respectively:
\[ (x_t, y_t) = \text{NAG}(R_S, x_0, t), \quad (\tilde{x}_t, \tilde{y}_t) = \text{NAG}(R_{S'}, x_0, t), \quad \forall \ t \geq 0, \]

The difference between gradients at time \( t \) is
\[ \Delta_t^f = \nabla R_S(x_t) - \nabla R_{S'}(\tilde{x}_t) = G - \frac{G}{n} = \frac{2G}{n}. \]
We will now show by induction that $\Delta_t^m = \frac{t-1}{4} \cdot \frac{2G_\eta}{n}$. For $t = 1$,

$$\Delta_1^m = m_1 - \bar{m}_1 = \gamma_1(x_1 - x_0) - \gamma_1(\bar{x}_1 - \bar{x}_0)$$

(Eq. (3))

$$= 0. \quad (\gamma_1 = 0)$$

Assuming the claim for $t - 1$,

$$\Delta_t^m = \gamma_t(\Delta_{t-1}^m - \eta \Delta_{t-1}^f)$$

(Eq. (9))

$$= \frac{t-1}{t+2} \left( \frac{t-2}{4} \cdot \frac{2G_\eta}{n} - \eta \Delta_{t-1}^f \right)$$

(induction assumption)

$$= \frac{t-1}{t+2} \left( \frac{t-2}{4} \cdot \frac{2G_\eta}{n} + \frac{2G_\eta}{n} \right)$$

($\Delta_{t-1}^f = 2G/n$)

$$= \frac{t-1}{t+2} \left( \frac{t+2}{4} \cdot \frac{2G_\eta}{n} \right)$$

$$= \frac{t-1}{4} \cdot \frac{2G_\eta}{n},$$

and we finished our induction. Now for the full dynamics,

$$\Delta_T^x = \Delta_{T-1}^x - \eta \Delta_{T-1}^f$$

(Eq. (7))

$$= \Delta_{T-1}^x + \Delta_{T-1}^m - \eta \Delta_{T-1}^f$$

(Eq. (8))

$$= \Delta_{T-1}^x + \frac{t-2}{4} \frac{2G_\eta}{n} - \eta \frac{2G}{n}$$

$$= \Delta_{T-1}^x + \frac{t+2}{4} \frac{2G_\eta}{n}.$$

Repeating this argument recursively,

$$\Delta_T^x = \Delta_0^x + \frac{2G_\eta}{n} \sum_{t=1}^T \frac{t+2}{4}$$

$$= \frac{2G_\eta}{n} \sum_{t=1}^T \frac{t+2}{4}$$

($\Delta_0^x = x_0 - \bar{x}_0 = 0$)

$$= \frac{2G_\eta}{n} \cdot \frac{3T(T+2)}{8}$$

$$= \frac{3G_\eta T(T+2)}{4n}.$$

Thus, for both $z = 1$ and $z = 2$,

$$|f(x_T;z) - f(\bar{x}_T;z)| = G|\Delta_T^x|$$

$$= \frac{3}{4} \frac{G^2 \eta T(T+2)}{n}$$

$$= \Theta(\eta G^2 T^2 / n).$$

Hence, the upper bound provided by Chen et al. (2018) is tight.

### E.2 Upper Bound for NAG, Convex Objectives

In this section we provide an exponential upper bound of uniform stability for NAG in the convex and smooth setting.
Claim 36. Let \( t(\cdot, z) \) be convex and \( \beta \)-smooth function for all \( z \in \mathcal{Z} \). Then for NAG with step size \( \eta \leq 1/\beta \) and \( T \) steps, for all \( x_0 \in \mathbb{R}^d \) and \( \epsilon > 0 \),

\[
\delta_{\text{NAG}, \epsilon}(n) \leq \frac{2\eta G^2}{n}(3T^{-1} + 1).
\]

Proof. Let \( S, S' \) be our sample sets which differ in only one example. For better similarity to our arguments with initialization stability, let \( f(x) = R_S(x) \) and \( f'(x) = R_{S'}(x) \). Let us consider two runs of the method initialized at \( x_0 \) on \( f, f' \) respectively:

\[
(x_t, y_t) = \text{NAG}(f, x_0, t), \quad (\tilde{x}_t, \tilde{y}_t) = \text{NAG}(f', x_0, t), \quad \forall \ t \geq 0,
\]

Note now that our notation of \( \Delta^f_t \triangleq \nabla f(x_t) - \nabla f(\tilde{x}_t) \) does not suffice to describe the dynamics, as we now have two functions. Using the notation \( e_t \triangleq \nabla f(\tilde{x}_t) - \nabla f'(\tilde{x}_t) \), we have

\[
\nabla f(x_t) - \nabla f'(\tilde{x}_t) = \Delta^f_t + \nabla f(\tilde{x}_t) - \nabla f'(\tilde{x}_t) = \Delta^f_t + e_t.
\]

Using the notations of \( \Delta^x_t \), \( \Delta^y_t \), \( \Delta^m_t \) (Corollary 29) and \( \gamma_t \), \( m_t \), \( \bar{m}_t \), we have:

\[
\Delta^x_t = y_{t-1} - \eta \nabla f(y_{t-1}) - (\bar{y}_{t-1} - \eta \nabla f'(\bar{y}_{t-1}))
\]
\[
\Delta^y_t = x_t + m_t - (\tilde{x}_t + \bar{m}_t)
\]
\[
\Delta^m_t = \gamma_t(m_{t-1} - \eta \nabla f(y_{t-1})) - \gamma_t(\bar{m}_{t-1} - \eta \nabla f'(\bar{y}_{t-1}))
\]

Thus, our basic equations becomes (instead of Eqs. (7) to (9))

\[
\Delta^x_t = \Delta^y_{t-1} - \eta (\Delta^f_{t-1} + e_{t-1}), \tag{20}
\]
\[
\Delta^y_t = \Delta^x_t + \Delta^m_t = \Delta^y_{t-1} - \eta (\Delta^f_{t-1} + e_{t-1}) + \Delta^m_t, \tag{21}
\]
\[
\Delta^m_t = \gamma_t(\Delta^m_{t-1} - \eta (\Delta^f_{t-1} + e_{t-1})). \tag{22}
\]

Since \( f, f' \) are different only in one term, using the Lipschitz property,

\[
\|e_t\| \leq \frac{2G}{n}.
\]

We will show the bound by first proving that \( \|\Delta^f_t\| \leq \frac{2G}{n}3^t \) by induction. For \( t = 0 \) the claim is immediate. Assuming the claim is correct for \( k = 0, \ldots, t \), we will prove for \( t + 1 \).

\[
\|\Delta^f_{t+1}\| = \|\Delta^f_t - \eta (\Delta^f_t + e_t) + \Delta^m_t\| \tag{Eq. (21)}
\]
\[
\leq \|\Delta^f_t\| - \eta \|\Delta^f_t\| + \|\Delta^m_t\| + \eta \|e_t\| \tag{Corollary 29}
\]
\[
\leq \|\Delta^f_t\| + \|\Delta^m_t\| + \eta \|e_t\| \tag{Corollary 29}
\]
\[
\leq \|\Delta^f_t\| + \eta \|e_t\| + \eta \sum_{k=1}^{t} (\|\Delta^f_k\| + \|e_k\|) \tag{\Delta^m_t = 0 since \( \gamma_1 = 0 \)}
\]
\[
\leq \|\Delta^f_t\| + \frac{2\eta G}{n} + \eta \sum_{k=1}^{t} \left(\|\Delta^f_k\| + \frac{2G}{n}\right) \tag{\|e_t\| \leq 2G/n}
\]
\[
\leq \|\Delta^f_t\| + \frac{2\eta G}{n} + \eta \sum_{k=1}^{t} \|\Delta^m_k\| + \frac{2\eta G}{n} (t + 1) \tag{smoothness}
\]
\[
\leq \frac{2\eta G}{n} 3^t + \eta \beta \sum_{k=1}^{t} \left(\frac{2\eta G}{n} 3^k\right) + \frac{2\eta G}{n} (t + 1) \tag{induction assumption}
\]
\[
\leq \frac{2\eta G}{n} 3^t + \sum_{k=1}^{t} \left( \frac{2\eta G}{n} 3^k \right) + \frac{2\eta G}{n} (t + 1) \quad (\eta \leq 1/\beta)
\]
\[
\leq \frac{2\eta G}{n} \left( 3^t + 3 \frac{3^t - 1}{2} + t + 1 \right)
\]
\[
\leq \frac{2\eta G}{n} \left( 3^t + 3 \frac{3^t}{2} + t \right)
\]
\[
\leq \frac{2\eta G}{n} 3^{t+1}.
\]
\[(t \geq 0 \implies 3^t \geq 2t)\]

Hence,

\[
||\Delta_f^T|| = ||\Delta_f^{T-1} - \eta (\Delta_f^{T-1} + e^{T-1})||
\]
\[
\leq ||\Delta_f^{T-1}|| + \eta ||e^{T-1}||
\]
\[
\leq \frac{2\eta G}{n} 3^{T-1} + \frac{2\eta G}{n}.
\]

Using the Lipschitz condition, we obtain our bound.