Stabilization of 2D Quantum Gravity by branching interactions

Oscar Diego*

Instituto de Estructura de la Materia, CSIC
Serrano 123, 28006 Madrid
Spain

Abstract

In this paper the stabilization of 2D quantum gravity by branching interactions is considered. The perturbative expansion and the first nonperturbative term of the stabilized model are the same than the unbounded matrix model which define pure gravity, but it has new nonperturbative effects that survives in the continuum limit.
The topological expansions of 2D quantum gravity coupled to unitary conformal matter with $c \leq 1$, are non Borel summable\cite{1}, and therefore, is not possible extract from them nonperturbative definitions of 2D quantum gravity. But, if one found some quantum field theory with the same perturbative expansion than the topological expansion and with sensible instantonic configurations related to the non Borel summability, the quantum field theory would be the nonperturbative definition of 2D quantum gravity.

The perturbative series of matrix models are the same than the topological expansions of 2D quantum gravity\cite{2}, but unfortunately they are ill defined nonperturbatively because their potentials are unbounded from below. There are several well defined models with the same perturbative expansions than the matrix models, unfortunately not all of them verify the minimal conditions of every sensible quantum field theory:

a) Reality of the physical observables; for instance, analytical continuations gives complex observables\cite{3}.

b) Instantonic configurations related to the asymptotic behaviour of the topological expansion: the origin of nonperturbative behaviours unrelated to the perturbative expansions are arbitrary restrictions on the configuration space, for instance, in unitary matrix models the eigenvalues are restricted to the real positive axis\cite{4}, and in some nonperturbative definitions of $c = 1$ matrix model the eigenvalues are restricted by an infinite wall. And this restrictions have not physical meaning.

For gravity without matter only the stochastic stabilization verify the above two conditions\cite{5}, but unfortunately is very difficult to extend it to the case of 2D quantum gravity coupled to $c = 1$ matter.
In this paper a new stabilization method for 2D quantum gravity without matter is proposed, which verify conditions a) and b).

The new stabilized models are constructed by adding branching interactions to the unbounded potentials, for example:

\[ V = T r (\Phi^2 - g\Phi^3) + \frac{\alpha}{N} T r (\Phi^2)^2. \]  

(1)

For \( N \) fixed and finite, \( V \) is bounded from below if \( \alpha \) is positive, but in the planar limit the Hartree approximation is exact and the potential becomes:

\[ V_{\text{Hartree}} = T r (\Phi^2 - g\Phi^3) + \alpha \omega T r (\Phi^2) \]  

(2)

where \( \omega \) is the vacuum expectation value of \( \frac{1}{N} T r \Phi^2 \), therefore in the planar limit the effective potential is the same that the potential of the unbounded matrix model with a self consistent parameter, and this model belongs to the universality class of pure gravity. Of course matrix models with branching interactions can have several critical points\[6, 7, 8, 9\], but in each phase there are only one vacuum. Therefore, in some region of the \((g, \alpha)\) phase space, (1) defines the topological expansion of 2D quantum gravity.

This behaviour is very different from stabilization with trace terms, for example:

\[ V = T r (\Phi^2 - g\Phi^3 + \alpha \Phi^4) \]  

(3)

where \( V \), for \( \alpha \) positive, is also bounded from below. But in the planar limit the potential is bounded from below, and in the pure gravity phase there are almost two vacuum\[10\], one is the vacuum of pure gravity and the other has not sensible physical interpretation. In fact, the well defined vacuum is probably the new vacuum, and the pure gravity vacuum probably only exist in the perturbative expansion.
In[4, 8] has been claimed that, in the pure gravity phase, the matrix models with branching interactions are equivalents to the unbounded matrix models nonperturbatively. The argument is the following: first, the partition function (1) is rewritten:

\[ Z = \int d\Phi \int dx e^{N^2x^2/2} \exp\{-NTrV_x\} \]

\[ V_x = V + 2xTr\Phi^2 \]

(4)

then, the integration over \( \Phi \) is performed and after the double scaling limit the partition function becomes:

\[ Z = \int_{-\infty}^{\infty} dx \delta(x) Z_0(N^{4/5}(g_c(x) - g)) \]

(5)

where \( Z_0 \) is the partition function of the unbounded matrix model. Therefore in the double scaling limit both matrix models are equivalents. But this argument is wrong because the integration first over \( \Phi \) and then over \( x \) in (4) is ill defined for finite \( N \), while (1) is well defined, therefore the integrations \( dx \) and \( d\Phi \) do not commute for \( N \) finite.

I will show also that in the double scaling limit both integrations \( d\Phi \) and \( dx \) do not commute. There will be new nonperturbative contributions to the partition function when the integrations over \( \Phi \) and \( x \) are performed before the double scaling limit. This new contribution survives in the double scaling limit.

Even the new matrix models verify both conditions a) and b), they do not verify the Ward identities of matrix models (loops equations, string equation, KdV flows). In fact, it is not possible to found a stabilization that verify the reality condition a), and the loop equations, string equation and all the mathematical structures which arise in a perturbative analysis of the matrix
models. In fact, I will show that the new nonperturbative effects are related with the break down of the symmetries of the matrix model.

2. Let be the matrix model:

\[
Z = \int d\Phi \exp\{-NV\}
\]
\[
V = Tr\{\Phi^2 - \frac{2g}{3}\Phi^3\}
\]  \hspace{1cm} (6)

it is the simplest model which define 2D quantum gravity without matter and is unbounded from below.

Perturbatively:

\[
\int d\Phi \sum_a \frac{\partial}{\partial \Phi_{aa}}\{\exp(-NV)\} = 0
\]  \hspace{1cm} (7)

therefore

\[
\langle P(\Phi) \rangle = 0
\]
\[
P(\Phi) = Tr[g\Phi^2 - \Phi]
\]  \hspace{1cm} (8)

this is the first equation of a set of identities which are equivalents to the loops equations and Virasoro constraints.

Let us define the new matrix model

\[
Z_\alpha = \int d\Phi \exp\{-NW_\alpha\}
\]
\[
W_\alpha = V(\lambda) + \frac{\alpha}{N}\left[P(\Phi)\right]^2
\]  \hspace{1cm} (9)

where \(\alpha\) is positive.

Integrating over the angular variables the new matrix model becomes:

\[
Z_\alpha = \int \prod_{i=1}^N d\lambda_i \exp\{-NS_{eff}\}
\]
\[
S_{eff} = \sum_{i=1}^N V(\lambda_i) + \frac{1}{N} \sum_{i \neq j} \ln |\lambda_i - \lambda_j| + \frac{\alpha}{N} \left[\sum_{i=1}^N P(\lambda_i)\right]^2
\]  \hspace{1cm} (10)
For finite $N$, $S_{\text{eff}}$ is bounded from below: when $\lambda_i$ goes to infinity, the effective action increases as $\alpha g \lambda_i^4$. Hence, $Z_\alpha$ is well defined.

In the planar limit, $N \to \infty$, the branching interaction is replaced by the Hartree term

$$\frac{1}{N} (TrP)^2 \to \langle P \rangle TrP$$

and the eigenvalue density $\rho$ in the large $N$ limit is given by the saddle point condition:

$$- g\lambda^2 + \lambda (1 + 2\alpha g \langle P \rangle) - \alpha \langle P \rangle + \int d\mu \frac{\rho(\mu)}{\lambda - \mu} = 0$$

$$\int d\mu \rho(\lambda) (g\mu^2 - \mu) = \langle P \rangle$$

$$\int d\mu \rho(\mu) = 1$$

(12)

and from (8) there is one solution with $\langle P \rangle = 0$, and with the same planar eigenvalue density than the unbounded matrix model (8). This solution is unique because the above equations are the saddle-point equations for an unbounded matrix potential with only one local minimum. Therefore, in the planar limit, the stabilized matrix model is independent on $\alpha$, and the continuum limit is defined by a critical coupling constant $g_c$ independent on $\alpha$.

Following reference[7] the partition function $Z_\alpha$ can be rewritten as:

$$Z_\alpha = \int d\Phi \int dx exp\{-NS_{\text{eff}}\}$$

$$\tilde{S}_{\text{eff}} = \sum_{i=1}^{N} V(\lambda_i) + \frac{1}{N} \sum_{i \neq j} \ln |\lambda_i - \lambda_j| + 2x \sum_{i=1}^{N} P(\lambda_i) - \frac{N x^2}{\alpha}$$

(13)

because all terms in the effective action $\tilde{S}_{\text{eff}}$ are order $N$, the saddle point
condition is given by:
\[ \frac{\partial \bar{S}_{\text{eff}}}{\partial \lambda_k} = 0 \]
\[ \frac{\partial \bar{S}_{\text{eff}}}{\partial x} = 0 \]  \hspace{1cm} (14)
and is trivial to check that the above equations are the same than (12). The saddle point value for \( x \) is
\[ x_s = \alpha \langle P \rangle. \]  \hspace{1cm} (15)

At subleading order in the expansion \( 1/N \), \( Z_\alpha \) must depends on \( \alpha \), but in the double scaling limit only survives the dependence on \( \alpha \) through the value of the critical coupling constant \( g_c \), which is independent on \( \alpha \). Hence the new matrix model \( Z_\alpha \), has the same topological expansion than the unbounded matrix model (8).

Equation (8) is perturbative, nonperturbatively \( \langle P \rangle \) must be different from zero. This means that the symmetries of the matrix model must break down by nonperturbative corrections.

\( \langle P \rangle \) can be calculated as follows:
\[ \langle P \rangle = \int d\Phi P(\Phi) \exp(-NW_\alpha) \]  \hspace{1cm} (16)
the first nonperturbative contribution to \( \langle P \rangle \) is given by the instantonic configuration: \( N-1 \) eigenvalues are placed in the support of \( \rho \) and one eigenvalue \( \lambda_T \) is placed on the top of the effective potential for one eigenvalue:
\[ \Gamma_\alpha = -N\{\lambda^2 - 2\frac{g}{3}\lambda^3 + 2\alpha \langle P \rangle (g\lambda^2 - \lambda) + 2 \int d\mu \rho(\mu) \ln |\lambda - \mu| \} \]  \hspace{1cm} (17)
and only the isolated eigenvalue \( \lambda_T \) gives the first nonperturbative contribution to (16)
\[ \langle P \rangle = \frac{1}{N}(g\lambda_T^2 - \lambda_T) \exp(-N\Gamma_\alpha(\lambda_T)). \]  \hspace{1cm} (18)
The instantonic configuration $\lambda_T$ is then given by the condition

$$\frac{d}{d\lambda} (\Gamma_\alpha)_{\lambda=\lambda_T} = 0$$

(19)

and at leading order the above equation becomes:

$$\lambda - g\lambda^2 + \int d\mu \frac{\rho(\mu)}{\lambda - \mu} = 0$$

(20)

which is exactly the same equation for the instantonic configuration of the unbounded matrix model. Therefore, in the double scaling limit, the first nonperturbative correction is the same in both matrix models:

$$\delta Z^0_\alpha = \exp (-N\Gamma_{\alpha=0}(\lambda_T))$$

(21)

From (16), the instantonic configuration with one eigenvalue placed at the top of the effective potential gives the leading term depending on $\alpha$ that survives in the double scaling limit:

$$\delta Z^1_\alpha = \exp \left(-2\alpha (g\lambda_T^2 - \lambda_T) \exp(-N\Gamma_{\alpha=0}(\lambda_T)) \right).$$

(22)

This factor is not universal but is finite in the double scaling limit and depends on the renormalized coupling constant. Therefore the partition function in the double scaling limit is given by:

$$Z_\alpha = Z_0 + \delta Z^0_\alpha \delta Z^1_\alpha$$

(23)

where $Z_0$ is the perturbative part of the partition function and is independent on $\alpha$.

The same results can be obtained from the integral representation (13). But the integral over $x$ must be performed at the same time that integration
over the eigenvalues, otherwise the new correction $\delta Z^1_\alpha$ cannot be calculated from (13).

This behaviour is different from stabilization with trace terms, for instance:

$$V = Tr(\Phi^2 - 2g^2 \Phi^3 + \frac{\alpha}{N} \Phi^4)$$

(24)

where the planar limit is also independent of $\alpha$ and therefore the perturbative double scaling must be independent on $\alpha$. But the nonperturbative dependence on $\alpha$ which survives in the double scaling limit is

$$\delta Z^1_\alpha = \exp(\alpha \lambda^4_T)$$

(25)

which is independent on the renormalized coupling constant, this fact suggest that the scaling part of the partition function that depends on the cosmological constant does not depend on the stabilized term of the potential. Therefore even the model (24) is well defined for finite $N$, the double scaling limit is probably ill defined nonperturbatively.

3. The conclusions that follows from this work are:

The matrix model with branching interactions (4) defines the topological expansion and the first nonperturbative correction of unbounded matrix model (3). As in the stochastic stabilization, there are only one perturbative vacua.

The branching term gives a new nonperturbative correction to the partition function that survives in the double scaling limit but they are not universal. Therefore, there are an infinity number of well defined models that verify the two conditions a) and b). In order to fix only one model, additional assumptions are needed.
This new nonperturbative correction depends on the renormalized cosmological constant, and this support the hypothesis that the stabilized model is well defined in the double scaling limit because the scaling part of the partition function must depends on the stabilized term of the matrix potential.

The integral representation \((13)\) is ill defined and the integration over \(x\) and \(\Phi\) commute only perturbatively. Unfortunately the integral representation of\([7, 8]\) can not be used to perform more rigorous nonperturbative computation.

The new matrix model does not verify the usual loops equations and from the identities

\[
\int d\Phi \frac{\partial}{\partial \Phi_{ab}} [(\exp(L\Phi))_{ab} \exp(-NW_\alpha)] = 0 \tag{26}
\]

one can extract new loops equations:

\[
NV' \left( \frac{\partial}{\partial L} \right) \langle W(L) \rangle = \int_0^L dJ \langle W(J)W(L-J) \rangle - 2\alpha NP' \left( \frac{\partial}{\partial L} \right) \langle PW(L) \rangle. \tag{27}
\]

there is an open question the physical meaning of this new loops equations.

Therefore the stabilization with branching interactions may be a sensible definitions of 2D quantum gravity, and can be extended to the \(c = 1\) case, whereas the stochastic stabilization for \(c = 1\) matrix model is given by a non solvable model.

The results of this paper can be extended to arbitrary branching interactions, for instance:

\[
W_\alpha = V + \frac{\alpha}{N} Tr\Phi^2 Tr\Phi^2. \tag{28}
\]

This matrix model has several phases depending on the value of \(\alpha\), and there is one phase belonging to the universality class of pure gravity. Of course, in
this phase the critical coupling constant depends on $\alpha$.

Even for the pure gravity phase there is a new nonperturbative contribution to the partition function, which is not universal but is finite in the double scaling limit. This new term is given by the nonperturbative contribution to $\langle Tr\Phi^2\rangle$.

Acknowledgment

I would like to thank University of Santiago for the financial support and the Particle Physics Department for the hospitality during part of this work.
References

[1] P. Ginsparg and J. Zinn-Justin, *Phys. Lett.* **B255**, 189 (1991); *Action principle and large order behaviour of nonperturbative gravity* in Random Surfaces, Quantum Gravity and Strings, (Cargese 1990); B. Eynard and J. Zinn-Justin, *Phys. Lett.* **B302**, 396(1993).

[2] E. Brézin and V. A. Kazakov, *Phys. Lett.* **B236**, 144 (1990); M. Douglas and S. Shenker, *Nucl. Phys.* **B335**, 635 (1990); D. J. Gross and A. A. Migdal, *Phys. Rev. Lett.* **64**, 717 (1990); *Nucl. Phys.* **B340**, 333 (1990).

[3] F. David, *Mod. Phys. Lett.* **A5**, 1651 (1990); P. G. Sylvestrov and A. S. Yelkovsky, *Phys. Lett.* **B251**, 525 (1990); F. David, *Nucl. Phys.* **B348**, 507 (1991); M. Saadi and G. Zemba, *Int. Jour. Mod. Phys.* **A7**, 501 (1992).

[4] T. R. Morris, *Nucl. Phys.* **B356**, 703 (1991); S. Dalley, C. Johnson and T. Morris, *Nucl. Phys.* **B368**, 625 (1992); ibid. 655; *Phys. Lett.* **B291**, 11 (1992); *Nucl. Phys.* (Proc. Suppl) **25A**, 87 (1992).

[5] J. Greensite and M. Halpern; *Nucl. Phys.* **B242**, 167(1984); E. Marinari and G. Parisi, *Phys. Lett.* **B274**, 537 (1990); M. Karliner and A. Migdal, *Mod. Phys. Lett.* **A5**, 2565 (1990); J. Ambjørn and G. Greensite and S. Varsted, *Phys. Lett.* **B249**, 411 (1990); J. L. Miramontes, J. S. Guillén and M. A. H. Vozmediano, *Phys. Lett.* **B253**, 38 (1991); A. Dabholkar, *Nucl. Phys.* **B368**, 293 (1992); O. Diego, *Mod. Phys. Lett.*,**A9**, 2445 (1994).
[6] S. R. Das, A. Dhar, A. M. Sengupta and S. R. Wadia, *Mod. Phys. Lett.*, A5, 1041 (1990).

[7] I. R. Klebanov and A. Hashimoto, *Nucl. Phys.* B434, (1995) 264.

[8] S. Sawada and M. Ueda, *Mod. Phys. Lett.* A6, (1991) 3717.

[9] G. Korchensky, *Mod. Phys. Lett.* A7, (1992) 3081; *Phys. Lett.* B296, (1992) 323;
L. Alvarez-Gaumé, J. L. Barbón and C. Crnkovic, *Nucl. Phys.* B394 (1993) 383;
J. L. F. Barbón, K. Demeterfi, I. R. Klebanov and C. Schmidhuber, *Nucl. Phys.* B440, (1995) 189;
F. Sugino and O. Tsuchiya, *Mod. Phys. Lett.* A9, (1994) 3149.

[10] J. Jurkiewicz, *Phys. Lett.* B245, 178 (1990); G. M. Cicta, L. Molinori and E. Montaldi, *J. Phys.* A23, L421 (1990); G. Bhanot, G. Mandal and O. Narayan, *Phys. Lett.* B251, 388 (1990); K. Demeterfi, N. Deo, S. Jain and C. Tan, *Phys. Rev.* D 42, 4105 (1990); M. Sasaki and H. Suzuki, *Phys. Rev.* D43, 4015 (1991).

[11] F. David *Mod. Phys. Lett.* A5, 1019 (1990).