Unification of twistors and Ramond vectors

A.A. Zheltukhin

\textsuperscript{a} Kharkov Institute of Physics and Technology, 61108 Kharkov, Ukraine,
\textsuperscript{b} Fysikum, AlbaNova, University of Stockholm, SE-10691 Stockholm, Sweden

Abstract

We generalize the idea of supertwistors and introduce a new supersymmetric object – the \( \theta \)-twistor which includes the composite Ramond vector [11] well known from the spinning string dynamics. The symmetries of the chiral \( \theta \)-twistor superspace are studied. It is shown that the chiral spin structure introduced by the \( \theta \)-twistor breaks the superconformal boost symmetry but preserves the scale symmetry and the super-Poincare symmetry. This geometrical effect of breaking correlates with the Gross-Wess effect of the conformal boost breaking for bosonic scattering amplitudes.

1 Introduction

Conformal invariance is one of the guiding principles both in theory of particles and condensed matter. The conception of scale invariance plays a great role in the description of classical and quantum phase transitions. In field theories scaling appears as a symmetry of Lagrangians of massles fields with dimensionless couplings or as an asymptotic high-energy symmetry of scattering amplitudes. The superconformal symmetry unifies strings in \( AdS_5 \times S^5 \) space and the supersymmetric Yang-Mills theory on its boundary [1]. Moreover, this symmetry is assumed to be a hidden symmetry of \( M \) theory [2].

One of widespread convictions is that the scale invariance implies the conformal invariance. However, the previous study of scale invariant amplitudes of scalar-scalar, scalar-spinor and scalar-photon scatterings has shown that the spin structures yield obstacles for the conformal symmetry realization [3]. Since the spin structure is built in the super-Poincare group, considered as a fundamental symmetry of the space-time, it is important to study the relationship between superconformal and scale symmetries encoded in the geometry of supersymmetric spaces. The twistor spaces [4] and their supersymmetric generalizations [5], [6] are important superspaces connected with superstring and super Yang-Mills theories [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18]. The supertwistor space is invariant under the superconformal symmetry generalizing the conformal symmetry of the twistor space.

Here we analyze the idea of supertwistor and introduce an alternative supersymmetric generalization of the Penrose twistor called the \( \theta \)-twistor. A new property of the \( \theta \)-twistor is the appearance of the composite Ramond vector (or alternatively a Grassmannian spinor) as the additional component of twistor instead of the Grassmannian scalar in the case of the supertwistor. This result follows from a hidden dual symmetry of the fundamental quadratic
form [3] defining the supertwistor space and transforming the scalar Grassmannian component of the supertwistor into the composite Ramond vector. The latter is known as the solution [11] of the Dirac constraint for the original Grassmannian Ramond vector [19] in the models of massless spinning particles and strings [20], [21], [22], [23], [24], [25]. We study the symmetry properties of the θ-twistor space and find that this space has all of the symmetries of the supertwistor space with the exception of the superconformal boosts. The breaking of the superconformal symmetry is a consequence of the change of the Grassmannian scalar by the Grassmannian vector or alternatively by spinor which takes into account a fine spin structure of the chiral θ-twistor superspace. This superspace mechanism of the (super)conformal symmetry breaking correlates with the Gross-Wess mechanism of the conformal symmetry breaking [3] just triggered by the substitution of vector or spinor particles for scalar particles in their scattering amplitudes. We find that because of the dual symmetry between the θ-twistor and supertwistor the invariant Cartan-Volkov differential forms in the supertwistor and θ-twistor spaces are also dual. These forms may be used for the construction of the dual Wess-Zumino terms and invariant actions of particles and strings in the θ-twistor space.

2 The supertwistor

A commuting Weyl spinor να belonging to the Penrose spinor doublet (να, νβxβα) is inert under the transformations of D = 4, N = 1 supersymmetry

\[ \delta \theta_\alpha = \varepsilon_\alpha, \quad \delta x_{\alpha\dot{\alpha}} = 2i(\varepsilon_\alpha \bar{\theta}_{\dot{\alpha}} - \theta_\alpha \varepsilon_{\dot{\alpha}}), \quad \delta \nu_\alpha = 0. \]  

To introduce the supertwistor [5] we consider the complex superspace (yαâ, θα, θâ) and the supersymmetric Cartan-Volkov differential form ωαâ associated with the superspace

\[ y_{\alpha\dot{\alpha}} \equiv x_{\alpha\dot{\alpha}} - 2i \theta_\alpha \bar{\theta}_{\dot{\alpha}}, \quad \omega_{\alpha\dot{\alpha}} = dy_{\alpha\dot{\alpha}} + 4i d\theta_\alpha \bar{\theta}_{\dot{\alpha}}. \]  

The invariant scalar form (ννννν) constructed from ωαâ [2] and να, ¯νâ may be presented as a supersymmetric differential form formed by the triplet ZA and its complex conjugate ZA

\[ (\nu \nu \nu) = s(Z, dZ) = -iZ_A d\bar{Z}^A. \]  

The triplets unify the spinors (να, ¯νâ) with the composite coordinates qα, qâ, η, ¯η

\[ Z_A \equiv (-i q_\alpha, \bar{v}_{\dot{\alpha}}, 2\eta), \quad \bar{Z}^A \equiv (\nu^\alpha, i \bar{q}_{\dot{\alpha}}, 2\eta), \]

\[ \eta \equiv \nu^\alpha \theta_\alpha, \quad \bar{q}_{\dot{\alpha}} = (q_\alpha)^* \equiv \nu^\alpha y_{\alpha\dot{\alpha}} = \nu^\alpha x_{\alpha\dot{\alpha}} - 2i \eta \bar{\theta}_{\dot{\alpha}}. \]  

The triplet components form a linear representation of the supersymmetry

\[ \delta \bar{q}_{\dot{\alpha}} = -4i\eta \varepsilon_{\dot{\alpha}}, \quad \delta \eta = \nu^\alpha \varepsilon_\alpha, \quad \delta \nu_\alpha = 0. \]  

The complex pair (ZA, ZA) defines the supertwistor introduced in [5] as a supersymmetric generalization of the projective Penrose twistor.

The supertwistor space may be equivalently defined as a complex projective superspace equipped with the invariant bilinear null form s(Z, Z')

\[ s(Z, Z') \equiv -iZ_A \bar{Z}'^A = -q_\alpha \nu^\alpha + \bar{v}_{\dot{\alpha}} \bar{q}_{\dot{\alpha}}' - 4i \eta \eta' = 0, \]  

2
where the complex conjugate triplet $\bar{Z}^A$ is given by (1) with $\nu'$ substituted for $\nu$

$$\bar{Z}^A \equiv (\nu'^{\alpha}, i\bar{q}^\alpha_{\dot{a}}, 2\eta'), \quad \bar{q}^\alpha_{\dot{a}} = \nu'^{\alpha}y_{\dot{a}a}, \quad \eta' = \nu'^{\alpha}\theta_\alpha. \quad (7)$$

The quadratic form (6) is invariant under the global superconformal symmetry as it was shown in [5]. The fermionic sector of the supertwistor (4) contains only the scalar projection $\eta$ of $\theta_\alpha$. It sets the question: whether it is possible to preserve the superconformal symmetry without the reduction of the $\theta$ components while the twistor supersymmetrization?

3 The $\theta$-twistor: A unification of Penrose twistor and Ramond vector

A characteristic point in the supertwistor construction is the unification of the chiral coordinates $(y_{\dot{a}a}, \theta_\alpha)$ with the Penrose spinor $\nu_\alpha$ having the same chirality as $\theta_\alpha$. The chirality coincidence permits to construct the left projection of the coordinates $(y_{\dot{a}a}, \theta_\alpha)$ on the spinor $\nu^{\alpha}$ transforming the complex vector $y_{\dot{a}a}$ into the spinor $\bar{q}^\alpha_{\dot{a}} = \nu^{\alpha}y_{\dot{a}a}$ and the spinor $\theta_\alpha$ into the complex scalar $\eta = \nu^\alpha\theta_\alpha$, i.e. making a step down in spins: $(1, \frac{1}{2}) \to (\frac{1}{2}, 0)$. This projection preserves the linear character of the supersymmetry transformations in the chiral space

$$\delta y_{\dot{a}a} = -4i\theta_\alpha\bar{\varepsilon}_{\dot{a}}, \quad \delta \theta_\alpha = \varepsilon_\alpha \quad (8)$$

and yields the linear realization (5) of the supersymmetry by the triplet $\bar{Z}^A$ (1).

An alternative supersymmetric triplet was proposed in [26], where the chiral coordinates $(y_{\dot{a}a}, \theta_\alpha)$ were unified with the c.c. Penrose spinor $\bar{\nu}_{\dot{a}}$ whose chirality has opposite sign to the $\theta_\alpha$ chirality. In that case one can not construct a projection of $\theta_\alpha$ on $\bar{\nu}_{\dot{a}}$ reducing the number of the fermionic coordinates. However, in the bosonic sector one can construct the new composite spinor $l_\alpha$ produced by the right projection of the chiral coordinate $y_{\dot{a}a}$ on $\bar{\nu}^\dot{a}$

$$l_\alpha \equiv y_{\dot{a}a}\bar{\nu}^{\dot{a}} = x_{\dot{a}a}\bar{\nu}^{\dot{a}} - 2i\theta_\alpha\bar{\eta}, \quad l_\alpha = q_\alpha - 4i\theta_\alpha\bar{\eta}. \quad (9)$$

Because the complex matrix $y_{\dot{a}a}$ is not a Hermitian its right and left contractions with spinors are not connected by complex conjugation

$$(\bar{q}_{\dot{a}})^* = l_\alpha + 4i\theta_\alpha\bar{\eta}, \quad \nu^{\alpha}l_\alpha = \bar{q}_{\dot{a}}\bar{\nu}^{\dot{a}}. \quad (10)$$

Fixation of $\theta_\alpha$ as a superpartner of $l_\alpha$ results in the nonlinear realization of the supersymmetry (1) by the three spinors

$$\delta l_\alpha = -4i\theta_\alpha(\bar{\nu}^{\dot{a}}\bar{\varepsilon}_{\dot{a}}), \quad \delta \theta_\alpha = \varepsilon_\alpha, \quad \delta \bar{\nu}_{\dot{a}} = 0 \quad (11)$$

which form the new complex spinor triplet $\Xi_A$ [26]

$$\Xi_A \equiv (-il_\alpha, \bar{\nu}^{\dot{a}}, \theta^{\alpha}), \quad \bar{\Xi}^A \equiv (\Xi_A)^* = (\nu^{\alpha}, i\bar{l}_\alpha, \bar{\theta}^{\dot{a}}). \quad (12)$$

In the $\Xi$-triplet space the supersymmetry generators take the form

$$Q^{\alpha} = \frac{\partial}{\partial \nu^{\alpha}} + 4i\nu^{\alpha}(\bar{\theta}^{\dot{a}}\frac{\partial}{\partial \bar{\nu}^{\dot{a}}}), \quad \bar{Q}^{\dot{a}} \equiv -(Q^{\alpha})^* = \frac{\partial}{\partial \bar{\nu}^{\dot{a}}} + 4i\bar{\nu}^{\dot{a}}(\theta^{\alpha}\frac{\partial}{\partial \theta^{\alpha}}) \quad (13)$$
with their anticommutator closed by the vector generator \( P^{\beta\alpha} = (\bar{\nu}^\beta \frac{\partial}{\partial \alpha} + \nu^\alpha \frac{\partial}{\partial \beta}) \)

\[
\{Q^\alpha, \bar{Q}^\beta\} = 4iP^{\beta\alpha}, \quad [Q^\gamma, P^{\beta\alpha}] = [\bar{Q}^\gamma, P^{\beta\alpha}] = \{Q^\gamma, Q^\beta\} = \{\bar{Q}^\gamma, \bar{Q}^\beta\} = 0. \tag{14}
\]

The quadratic form \( (6) \) expressed in terms of \( \Xi \) and \( \bar{\Xi} \) \( (12) \) becomes a non-linear form

\[
s(Z, \bar{Z}') \equiv -iZ_\alpha \bar{Z}'^\alpha = s(\Xi, \bar{\Xi}) \equiv -l_\alpha \nu^\alpha + \bar{\nu}^\beta \bar{l}^\alpha_\beta - 4i(\nu'\bar{\nu}_\alpha)\theta^\alpha \bar{\theta}_\alpha = 0 \tag{15}
\]

in the \( \Xi \)-triplet space. That nonlinearity is a consequence of the fixation of \( \theta_\alpha \) as the superpartner of \( l_\alpha \). However, this fixing is not the only possible. In fact, to construct the supertwistor \( Z \)-triple the spin \( \text{down} \) transition \((1, \frac{1}{2}) \rightarrow (\frac{1}{2}, 0)\) was used. But, there is an alternative way described by the spin transition \((1, \frac{1}{2}) \rightarrow (\frac{1}{2}, 1)\). This way proposes the complex vector \( (\theta_\alpha \bar{\nu}_\beta) \) as a superpartner of the spinor \( l_\beta \). With this observation we multiply the second equation in \((11)\) by \( \bar{\nu}^\beta \) and obtain Eqs. \((11)\) just rewritten in the desired linear form

\[
\delta l_\alpha = 4i(\theta_\alpha \bar{\nu}_\beta)\bar{\xi}^\beta, \quad \delta (\theta_\alpha \bar{\nu}_\beta) = \varepsilon_\alpha \bar{\nu}_\beta, \quad \delta \bar{\nu}_\alpha = 0, \tag{16}
\]

or equivalently in the explicit linear form as

\[
\delta l_\alpha = -4i(\sigma^m \bar{\eta})\alpha \bar{\eta}^m, \quad \delta \bar{\eta}_m = -\frac{1}{2}(\varepsilon \sigma^m \bar{\nu}), \quad \delta \bar{\nu}_\alpha = 0. \tag{17}
\]

The grassmannian vectors \( \bar{\eta}_m \) and \( \eta_m \) in \((17)\) are the composite Ramond vectors

\[
\eta_m \equiv -\frac{1}{2}(\nu \sigma_m \bar{\theta}), \quad \bar{\eta}_m = (\eta_m)^* = -\frac{1}{2}(\theta \sigma_m \bar{\nu}),
\]

\[
\nu^\beta \bar{\theta}_\alpha \equiv \eta_{\beta\alpha} = (\sigma^m)_{\beta\alpha} \eta_m, \quad \eta_m \eta_n + \eta_n \eta_m = 0. \tag{18}
\]

previously introduced in \((11)\) (see details in \((27)\)) to prove the equivalence between superparticles and spinning particles. In terms of the Ramond vectors the non-linear term \( 4i(\nu'\bar{\nu}_\alpha)\theta^\alpha \bar{\theta}_\alpha \) in \((15)\) is presented in the bilinear form

\[
-4i(\nu'_\alpha \bar{\nu}_\alpha)\theta^\alpha \bar{\theta}_\alpha \equiv 4i\eta' \bar{\eta} \equiv 2i(\bar{\nu} \sigma_m \theta)(\nu' \sigma^m \bar{\theta}) \equiv -8i\bar{\eta}_m \eta'_m. \tag{19}
\]

As a result, the quadratic form \((15)\) defining the supertwistor space becomes the new quadratic form

\[
s = s(Z, \bar{Z}') = s(\Xi, \bar{\Xi}) \equiv -i\Xi_\alpha \bar{\Xi}'^\alpha = -l_\alpha \nu^\alpha + \bar{\nu}^\beta \bar{l}^\alpha_\beta - 8i\bar{\eta}_m \eta'^m = 0 \tag{20}
\]

in the complex projective space of the \( \Xi \)-triplets including the composite Ramond vector

\[
\Xi_\alpha \equiv (-il_\alpha, \bar{\nu}^\alpha, 2\sqrt{2}\bar{\eta}_m), \quad \bar{\Xi}'^\alpha \equiv (\Xi_\alpha)^* = (\nu^\alpha, i\bar{l}^\alpha_\beta, 2\sqrt{2}\eta^m). \tag{21}
\]

So, the supertwistor \( Z \)-triple \((1)\) transforms into the \( \Xi \)-triple \((21)\) under the substitution \((q_\alpha, \bar{\eta}) \rightarrow (l_\alpha, \sqrt{2}\bar{\eta}_m)\) producing the new quadratic representations of the quadratic form \((6)\) but now in terms of the \( \Xi \)-triple. Thus, the \( Z \)-triple \((1)\) and the \( \Xi \)-triple \((21)\) occur to be equal in their own rights and it gives a reason to call the new \( \Xi \)-triple \((21)\) (or equivalently the \( \Xi \)-triple \((12)\)) the \( \theta \)-twistor to emphasize its difference from the supertwistor.

The \( \theta \)-twistor independently appears as a geometrical object alternative to the supertwistor from another point of view. The latter bases on the observation that the supertwistor
and the \(\theta\)-twistor are the general solutions of different supersymmetric constraints. The supertwistor solves the generalized chiral constraint in the superspace \((y_{a\dot{\alpha}}, \theta_{\alpha})\) complemented by the \emph{left} Weyl spinor \(\nu_{\alpha}\) and the new scalar operator \(\nu_{\alpha}D^{\alpha}\),

\[
\begin{align*}
\bar{D}^{\dot{\alpha}}F(x, \theta, \bar{\theta}) &= 0 \rightarrow F = F(y, \theta), \\
\nu_{\alpha}D^{\alpha}F(y, \theta, \nu) &= 0 \rightarrow F = F(\bar{Z}^{A}).
\end{align*}
\]

(22)

where \(F(\bar{Z}^{A})\) is the superfield depending on the triplet \(\bar{Z}^{A}\). In contrast, the \(\theta\)-twistor solves the supersymmetric constraints in the chiral space complemented by the \emph{right} Weyl spinor \(\bar{\nu}_{\dot{\alpha}}\) and the Dirac chiral operator \(\bar{\nu}_{\dot{\alpha}}\frac{\partial}{\partial x^{\dot{\alpha}}},\)

\[
\begin{align*}
\bar{D}^{\dot{\alpha}}F(x, \theta, \bar{\theta}) &= 0 \rightarrow F = F(y, \theta), \\
\bar{\nu}_{\dot{\alpha}}\frac{\partial}{\partial x^{\dot{\alpha}}}F(y, \theta, \nu) &= 0 \rightarrow F = F(\Xi_{A}).
\end{align*}
\]

(23)

It is easy to see that the additional Dirac constraint in (23), selecting the \(\Xi\)-triplet, may be rewritten in an equivalent form using the composite Ramond vector. It follows from the multiplication of the second of Eqs. (23) by \(\theta_{\alpha}\)

\[
\begin{align*}
\theta_{\alpha}\bar{\nu}_{\dot{\alpha}}\frac{\partial}{\partial x^{\dot{\alpha}}}F(y, \theta, \nu) &\equiv \bar{\eta}_{\dot{\alpha}\alpha}\frac{\partial}{\partial x^{\dot{\alpha}}}F(y, \theta, \nu) \equiv \bar{\eta}^{m}\partial_{m}F(y, \theta, \nu) = 0.
\end{align*}
\]

(24)

We see that the \(\theta\)-twistor is an object dual to the supertwistor but preserving all of the spin degrees of freedom encoded by the spinor \(\theta_{\alpha}\). Then our question about the superconformal symmetry transforms to the question whether all of the symmetries of supertwistor survive the transition to the \(\theta\)-twistor? In other words what is the payment for the restoration of the spin degrees of freedom lost by the supertwistor? Studying the \(\theta\)-twistor symmetries is necessary to answer the question.

It is easy to see the invariance of the nonlinear and quadratic representations (20), (15) under the supersymmetry (11), (17), the scale and phase transformations

\[
\begin{align*}
\nu'_{\beta} &= e^{\varphi}\nu_{\beta}, \quad \bar{\nu}'_{\dot{\beta}} = e^{\bar{\varphi}}\bar{\nu}_{\dot{\beta}}, \quad \nu_{\beta} = e^{-\varphi}\nu'_{\beta}, \quad \bar{\nu}_{\dot{\beta}} = e^{-\bar{\varphi}}\bar{\nu}'_{\dot{\beta}}, \quad \theta_{\beta} = e^{\varphi}\theta'_{\beta}, \quad \bar{\theta}_{\dot{\beta}} = e^{\bar{\varphi}}\bar{\theta}'_{\dot{\beta}},
\end{align*}
\]

(25)
described by the complex parameter \(\varphi = \varphi_{R} + i\varphi_{I}\), as well as under the \(\gamma_{5}\) rotations (26)

\[
\begin{align*}
\theta'_{\beta} &= e^{i\lambda}\theta_{\beta}, \quad \bar{\theta}'_{\dot{\beta}} = e^{-i\bar{\lambda}}\bar{\theta}_{\dot{\beta}}.
\end{align*}
\]

(26)

So, the \(\theta\)-twistor triplet is closed under these symmetries forming their representations similarly the \(Z\)-triplet. But \(Z\)-triplet also realizes the (super)conformal boosts (28) and forms a representation of the superconformal group. What is about the closure of the \(\Xi\) and \(\Xi\) triplets under the superconformal boosts \(S^{\alpha}, \bar{S}^{\dot{\alpha}}\)? We discuss the question below.

4 \(\theta\)-twistor and superconformal symmetry breaking

To answer the question whether the \(\theta\)-twistor forms a representation of the superconformal boosts, let us remind the superboost realization (26) in the chiral superspace \((y_{a\dot{\alpha}}, \theta_{\alpha})\) (29)

\[
\begin{align*}
\delta y_{a\dot{\alpha}} &= 4i\theta_{\alpha}(\xi^{\beta}y_{\beta\dot{\alpha}}), \quad \delta \theta_{\alpha} = -y_{a\dot{\alpha}}\bar{\xi}^{\dot{\alpha}} + 4i\theta_{\alpha}(\xi^{\beta}\theta_{\beta}).
\end{align*}
\]

(27)
The left multiplication of (27) by \(\nu^{\alpha}\) yields the variations of the supertwistor components

\[
\begin{align*}
\delta q_{\dot{\alpha}} = (\delta \nu^{\beta} + 4i\eta^{\beta})y_{\beta\dot{\alpha}}, \quad \delta \eta = (\delta \nu^{\beta} + 4i\eta^{\beta})\bar{\theta}_{\dot{\beta}} + (q^{\dot{\alpha}}\xi_{\dot{\alpha}}).
\end{align*}
\]

(28)
where dependence on \( y \) and \( \theta \) is eliminated by the choice of the transformation law for \( \nu^\beta \)

\[
\delta \nu^\beta = -4i\eta \bar{\xi}^\beta, \quad \delta \eta = (\bar{q}^\alpha \bar{\xi}_\alpha), \quad \delta \bar{q}_\beta = 0.
\]

(29)

Eqs. (29) define the variations of the supertwistor under the superconformal boosts.

A similar procedure, realized by multiplication of \( \delta y_{a\alpha} \) from (27) by \( \bar{v}^\beta \), yields the superconformal boost transformations of \( l_a \) but with the right index of \( y_{a\beta} \) occupied by \( \delta \bar{v}^\beta \)

\[
\delta l_a = y_{a\beta} \delta \bar{v}^\beta + 4i\theta_\alpha (\xi^\beta l_\beta), \quad \delta \theta_\alpha = -y_{a\beta} \bar{\xi}^\beta + 4i\theta_\alpha (\xi^\beta \theta_\beta).
\]

(30)

In the variation of \( \theta_\alpha \) (30) we also observe the right index of \( y_{a\beta} \) contracted with the parameter \( \bar{\xi}^\beta \) index. It means that the minimal complex dimension of the bosonic superpartner of \( \theta_\alpha \) needed for the superconformal boosts realization equals four and it prevents a reduction of \( y_{a\beta} \) into \( l_a \). In the supertwistor case the transmutation of \( y_{a\beta} \) into \( \bar{q}_\beta \) has been accompanied by a synchronous transmutation of \( \theta_\alpha \) into the Grassmannian scalar \( \eta \) whose component number is also half of the component number of the spinor \( \theta_\alpha \). Because of this double reduction there was not breaking in the superconformal symmetry realization by the \( \bar{Z} \)-triplet forming the antiholomorphic sector of the supertwistor space. On the contrary, in the case of \( \Xi \)-triplets we preserve the spinor \( \theta_\alpha \) from a reduction but transmute the complex vector \( y_{a\beta} \) into the spinor \( l_a \) whose number of components equals half of the component number of \( y_{a\beta} \). The reduction yields a deficit in the bosonic sector of \( \Xi_\mathcal{A} \). To compensate the deficit the components of \( \Xi_\mathcal{A} \) must be extended at least by one auxiliary spinor.

To check this observation we shall treat \( \nu_\alpha \) as one of the Newman-Penrose basis elements in the spinor space \[4\] and add an auxiliary spinor \( \bar{v}_\alpha \), defined by the relations

\[
\nu^\alpha \nu_\alpha = 1, \quad \nu_\alpha \nu^\beta - \nu^\alpha \nu^\beta = \delta_\alpha^\beta,
\]

(31)

which may be used as the second element of the spinor dyad. Taking into account the completeness condition from Eqs. (31) we present the variation of \( \theta_\alpha \) (30) in the form

\[
\delta \theta_\alpha = -l_a \bar{\xi} + m_\alpha \bar{\zeta} + 4i\theta_\alpha (\xi^\beta \theta_\beta),
\]

(32)

where \( \bar{\xi}, \bar{\zeta} \) are effective superboost parameters given by the projections of \( \bar{\xi}^\beta \) on the dyad

\[
\bar{\xi} = (\bar{\xi}^\beta v_\beta), \quad \bar{\zeta} = (\bar{\xi}^\beta v_\beta), \quad m_\alpha = (y \bar{v})_\alpha
\]

(33)

and the new independent spinor \( m_\alpha \) is the projection of \( y_{a\beta} \) on the spinor \( \bar{v}^\beta \). The superboost of \( l_a \) (30) can also be presented by a similar expansion in the spinors \( l_a, m_\alpha \) and \( \theta_\alpha \)

\[
\delta l_a = l_a (\bar{v}_\alpha \delta \bar{v}^\alpha) - m_\alpha (\bar{v}_\alpha \delta \bar{v}^\alpha) + 4i\theta_\alpha (\xi^\beta l_\beta).
\]

(34)

The expansions (32), (34) confirm the need to extend the holomorphic \( \Xi_\mathcal{A} \) triplet by the auxiliary spinor \( m_\alpha \) to realize the superconformal boosts.

Otherwise, it might be that the \( \Xi_\mathcal{A} \)-triplet could realize a half of the superconformal boosts. It requires the absence of \( m_\alpha \) in Eqs. (32), (34) that is equivalent to the conditions

\[
\bar{\zeta} = (\bar{\xi}^\alpha \bar{v}_\alpha) = 0, \quad (\bar{v}_\alpha \delta \bar{v}^\alpha) = 0
\]

(35)

whose solutions contain the spinor \( \nu_\alpha \) belonging to the \( \Xi_\mathcal{A} \)-triplet from the antiholomorphic sector of the \( \theta \)-twistor space

\[
\xi_\alpha = \nu_\alpha \xi, \quad \delta \nu_\alpha = \nu_\alpha \delta \varphi
\]

(36)
and consequently introduces the auxiliary spinor $\nu_a$ instead of $m_\alpha$. It proves that the holomorphic $\Xi_A$ triplet does not realize even the half of the superboosts parametrized by $\xi$.

However, the solution (36) points out on a possibility to realize the half of superconformal boosts as a symmetry of the whole $\theta$-twistor space which mixes its holomorphic $\Xi_A$ and antiholomorphic $\Xi^A$ triplets. The substitution of (36) in (34) yields the realization of the half of superboosts in the form

$$\delta \theta_\alpha = -l_\alpha \bar{\xi} - 4i \theta_\alpha (\nu^\beta \theta_\beta) \xi, \quad \delta l_\alpha = l_\alpha \delta \bar{\varphi} + 4i \theta_\alpha (\nu^\beta l_\beta) \xi, \quad \delta \bar{\nu}^\dot{\alpha} = \bar{\nu}^\dot{\alpha} \delta \varphi. \quad (37)$$

The variation $\delta \bar{\varphi}$ is fixed by the condition

$$\delta \bar{\varphi} = 4i (\nu^\beta \theta_\beta) \xi \rightarrow \delta \bar{\eta}_m = \frac{1}{2} (l_{\sigma_m} \bar{\nu}) \xi. \quad (38)$$

which confines the transformed Ramond vector $\bar{\eta}_m$ inside of the $\Xi_A$ triplet. The substitution of $\delta \bar{\varphi}$ (38) in Eqs. (37) transforms them to the form

$$\delta l_\alpha = 4i [ (\nu^\beta \theta_\beta) l_\alpha + (\nu^\beta l_\beta) \theta_\alpha ] \xi, \quad \delta \bar{\nu}^\dot{\alpha} = 4i \bar{\nu}^\dot{\alpha} (\nu^\beta \theta_\beta) \xi, \quad \delta \theta_\alpha = -l_\alpha \bar{\xi} - 4i \theta_\alpha (\nu^\beta \theta_\beta) \xi. \quad (39)$$

Eqs. (39) contain the spinor $\nu^\alpha$ from the antiholomorphic $\Xi^A$ triplet and the equations must be added by their complex conjugate to take into account the superboost of $\nu^\alpha$. It shows that the whole $\theta$-twistor space closes under the half of the superconformal boosts which mix the holomorphic and antiholomorphic sectors formed by the triplets $\Xi_A$ and $\Xi^A$ although each of them is not closed.

The problem of the superconformal boost realization more sharpens for the case of the $\Xi_A$ triplet (21) containing the Ramond vector $\bar{\eta}_m$ as the superpartner of $l_\alpha$ instead of $\theta_\alpha$.

Here we encounter with the problem of transmutation of $\theta_\alpha$ into $\bar{\eta}_m$ without an extension of the $\theta$-twistor space. It is seen from Eqs. (39) defining the superboosts of the constituents of the Ramond vector $(\theta_\alpha \bar{\nu}_\beta)$. After multiplication of $\delta \theta_\alpha$ by $\bar{\nu}^\dot{\beta}$ (39) and taking into account the relation $\nu^\beta l_\beta = \bar{q}_\beta \bar{\nu}^\dot{\beta}$ (11) we obtain the superboost realization including the desired Ramond vector $\bar{\eta}_m$

$$\delta l_\alpha = 4i [ \eta l_\alpha - (\sigma^m \bar{q})_\alpha \bar{\eta}_m ] \xi, \quad \delta \bar{\nu}^\dot{\alpha} = 4i (\nu \sigma^m)_\alpha \bar{\eta}_m \xi, \quad \delta \bar{\eta}_m = \frac{1}{2} (l_{\sigma_m} \nu) \xi. \quad (40)$$

In the variation $\delta l_\alpha$ (40) we observe the appearance of $\bar{q}_\alpha$ which belongs neither to the $\Xi_A$ nor to $\Xi^A$ triplets but to the supertwistor triplet $\tilde{Z}^A$ (1). It proves that the whole complex superspace formed by $\Xi_A$ and $\Xi^A$ triplets is not closed under the superconformal boosts and each of these triplets form representations of only the maximal subgroup of the superconformal group.

The superconformal symmetry breaking is explained by the difference of chiralities of the $\bar{\nu}^\dot{\alpha}$ and $\theta_\alpha$ spinors forming the $\Xi_A$ or $\Xi^A$ triplets. This effect correlates with the Gross-Wess mechanism associated with the nontrivial spin structure of the scattering amplitudes.

5 Dual Wess-Zumino terms and dual actions

The $\theta$-twistor introduces an alternative supersymmetric extension of the Penrose twistor and deserves of studying in various physical applications. It was previously shown that using the
chiral superspace associated with the $\theta$-twistors yields an infinite chain of higher spin chiral supermultiplets $(\frac{1}{2}, 1), (1, \frac{3}{2}), (\frac{3}{2}, 2), ..., (S, S + \frac{1}{2})$ generalizing the scalar supermultiplet $[26]$. These supermultiplets include the auxiliary $F$-field absent in the supertwistor description.

Another interesting problem is to study actions of supersymmetric models of particles, strings and branes in the $\theta$-twistor space. The actions may be constructed using the supersymmetric differential forms in the $\theta$-twistor space. It is illustrated by a simple example of the supersymmetric one-form $[3]$ which has two dual representations

$$ (\nu \omega \tilde{\nu}) = (Z, dZ) = -iZ_A d\tilde{Z}^A = -q_\alpha d\nu^\alpha + \tilde{\nu}^\dot{\alpha} d\tilde{q}_{\dot{\alpha}} - 4i\tilde{\eta} d\eta $$

as a consequence of the above discussed dual symmetry connecting $\theta$-twistor and supertwistor. The differential forms (11) may be presented in the equivalent form

$$ -iZ_A d\tilde{Z}^A = [\nu^\alpha d\eta_{\alpha} + \tilde{\nu}^\dot{\alpha} d\tilde{\eta}_{\dot{\alpha}} - 4i\tilde{\eta} d\eta] - d(\nu x \tilde{\nu}) = -i\Xi_A d\tilde{\Xi}^A = [\nu^\alpha d\eta_{\alpha} + \tilde{\nu}^\dot{\alpha} d\tilde{\eta}_{\dot{\alpha}} - 4i\tilde{\eta} d\eta] - d(\nu x \tilde{\nu}), $$

where the first terms in r.h.s of (12) are dual Wess-Zumino terms in the supertwistor and $\theta$-twistor spaces respectively. These terms are invariant under the supersymmetries (5) and (17) respectively up to the total differential of the variation of the scalar $(\nu \omega \tilde{\nu})$ which absorbs the space-time coordinates $x_m$. The Wess-Zumino terms in (12) may be used as the Lagrangians for two dual supersymmetric actions of a particle with spin. The $\theta$-twistor representation of the action is given by the integral with respect to the proper time of the particle

$$ S = \int d\tau \{ [\nu^\alpha \dot{\eta}_{\alpha} + \tilde{\nu}^\dot{\alpha} \dot{\tilde{\eta}}_{\dot{\alpha}} - 4i(\tilde{\eta} \dot{\eta} - \tilde{\eta} \dot{\tilde{\eta}})] + \lambda s(\Xi, \tilde{\Xi}) \}, $$

where $\dot{\tau} = \frac{d\tau}{d\tau}$ and $\lambda$ is the Lagrange multiplier fixing the constraint (20) $s(\Xi, \tilde{\Xi}) = 0$. The action (13) is invariant under the proper time reparametrizations and automatically introduces the correct kinetic term for the complex Ramond vector encoding the spin degrees of freedom of the massless spinning particle.

### 6 Conclusion

Stimulated by the Gross-Wess observation $[3]$ about the spin structures in scattering amplitudes of massless particles as the obstructions preventing the scale symmetry extension up to the conformal symmetry, we have addressed the same question to generalized superspaces with an inherent chiral spin structure. On this way the supersymmetric twistors called the $\theta$-twistors and dual to the well known supertwistors $[5], [6]$ were revealed. The fermionic constituent of the $\theta$-twistor is presented by the composite grassmannian Ramond vector $[11]$ or by the chiral superspace coordinate $\theta_\alpha$ $[26]$ contrarily to the scalar grassmannian constituent of the supertwistor. We established that the super-Poincare and scale covariant chiral structures, associated with the $\theta$-twistors, create some obstacles for the (super)conformal boost realization. As a result, the triplets creating the $\theta$-twistor superspace form representations of only the maximal subgroup of the superconformal group. In the case when the $\theta$-twistor is realized by three spinors the possibility appears to restore half of the (super)conformal

---

1 The Wess-Zumino terms linear in derivatives were previously considered in $[30]$.  

8
boosts. The partial restoration goes by means of mixing of the holomorphic and antiholomorphic triplets forming the $\theta$-twistor space. It is interesting to understand a possible role of this mixing in Yang-Mills theory for the description of scattering amplitudes different from the MHV amplitudes.

The $\theta$-twistor construction is automatically generalized to the case of extended supersymmetries just as the supertwistor construction \[^{[5]}\] and it is interesting to investigate the geometrical properties of the corresponding supermanifolds along the line developed in \[^{[31]}\].

Because the Ramond vector naturally appears as the $\theta$-twistor constituent it may shed a new light on the mystery of the GSO projection \[^{[33]}\]. In this connection it is interesting to construct new supersymmetric actions of particles, strings and branes in the $\theta$-twistor space and to understand more on the connections between diffeomorphisms, $\kappa$-transformations and non-linear realizations \[^{[2]}\], \[^{[11]}\], \[^{[12]}\], \[^{[32]}\], \[^{[35]}\], \[^{[36]}\]. We hope to study these issues elsewhere.

7 Acknowledgements

The author is grateful to Fysikum at the Stockholm University for kind hospitality and Ingemar Bengtsson and Steven Giddings for useful discussions. The work was partially supported by the grant of the Royal Swedish Academy of Sciences.

References

[1] J. Maldacena, Adv. Theor. Math. Phys. 2 (1980) 231.

[2] P. West, JHEP 0408 (2004) 007.

[3] D. Gross and J. Wess, Phys. Rev. D 2 (1970) 753.

[4] R. Penrose and W. Rindler, Spinor and twistor methods in space-time geometry. Cambridge Univ. Press, 1986.

[5] A. Ferber, Nucl. Phys. B 132 (1978) 55.

[6] E. Witten, Phys. Lett. B 77 (1978) 215.

[7] E. Witten, Commun. Math. Phys. 252 (2004) 189; Adv.Theor.Math.Phys. 8 (2004) 779.

[8] V.P. Nair, Noncommutative mechanics, Landau levels, twistors and Yang-Mills amplitudes, hep-th/0506120

[9] T. Shirafuji, Prog. Theor. Phys. 70 (1983) 18.

[10] I. Bengtsson and M. Cederwall, Nucl. Phys. B 302 (1988) 81.

[11] D.V. Volkov and A.A. Zheltukhin, JETP Lett. 48 (1988) 63; Lett. in Math. Phys. 17 (1989) 141; Nucl. Phys. B 335 (1990) 723.

\[^{2}\] Taking into account of the spontaneous vacuum transitions \[^{[34]}\] in the Veneziano and Neveu-Schwarz dual models has given an alternative mechanism of the tachyon elimination. In the Neveu-Schwarz model this mechanism reveals the existence of the broken symmetry group with an infinite number of generators containing the group $SU(2) \times SU(2) \times U(1) \times U(1) \ldots \times U(1)$... as a subgroup.
[12] D.P. Sorokin, V.I. Tkach, D.V. Volkov and A.A. Zheltukhin, Phys. Lett. B 216 (1989) 302; D.P. Sorokin, V.I. Tkach and D.V. Volkov, Mod. Phys. Lett. A 4 (1989) 901.

[13] A.A. Zheltukhin, Sov. J. Nucl. Phys. 48 (1988) 375; 51 (1990) 950; I.A. Bandos and A.A. Zheltukhin, Fortschr. Phys. 61 (1993) 619; Class. Quant. Grav. 12 (1995) 609.

[14] W. Siegel, Phys. Rev. D 52 (1995) 1042, hep-th/0404255.

[15] O.E. Gusev and A.A. Zheltukhin, JETP Lett. 64 (1996) 487.

[16] I. Bengtsson and A.A. Zheltukhin, Phys. Lett. B 570 (2003) 222.

[17] N. Berkovits, Phys. Rev. Lett. 93 (2004) 011601, hep-th/0402045.

[18] D.V. Uvarov and A.A. Zheltukhin, Phys. Lett. B 545 (2002) 183; B 570 (2003) 222; JHEP 08 (2002) 008; 03 (2004) 063; Mod. Phys. Lett. A 20 (2005) 769.

[19] P. Ramond, Phys. Rev. D 3 (1971) 2415.

[20] F.A. Berezin and M.S. Marinov, JETP Lett. 21 (1975) 678;

[21] L. Brink, P. Di Vecchia, P. Howe, S. Deser and B. Zumino, Phys. Lett. B 64 (1976) 435.

[22] L. Brink, P. Di Vecchia and P. Howe, Phys. Lett. B 65 (1976) 471.

[23] S. Deser and B. Zumino, Phys. Lett. B 65 (1976) 369.

[24] R. Casalbuoni, Nuovo Cimento A 33 (1976) 115; 389.

[25] A.A. Zheltukhin, Yad. Fiz. 42 (1985) 720; 46 (1987) 1791; Theor. Math. Phys. 65 (1985) 1072; Phys. Lett. B 168 (1986) 43.

[26] A.A. Zheltukhin, A new type of supersymmetric twistors and higher spin chiral multiplets, hep-th/0606234; The θ-twistor versus the supertwistor, math-ph/0612008.

[27] A.A. Zheltukhin, Mod. Phys. Lett. A 21 (2006) 2117, gr-qc/0512166.

[28] J. Wess and B. Zumino, Nucl. Phys. B 70 (1974) 39.

[29] J. Wess and J. Bagger, Supersymmetry and Supergravity. Princeton Univ. Press, 1992.

[30] D.V. Volkov, A.A. Zheltukhin and Yu.P. Bliokh, Sov. Phys. Solid State 13 (1971) 1396; D.V. Volkov, A.A. Zheltukhin, JETP 78 (1980) 1867; Solid State Com. 36 (1980) 733.

[31] S. Giddings and P. Nelson, Phys. Rev. Lett. 59 (1987) 2620.

[32] J. Gomis, K. Kaminura and P. West, Diffeomorphism, κ transformations and the theory of non-linear realizations, hep-th/0607104.

[33] F. Gliozzi, J. Sherk and D. Olive, Nucl. Phys. B 122 (1977) 253.

[34] D.V. Volkov, A.A. Zheltukhin and A.I. Pashnev, Yad. Fiz. 18 (1973) 902; 21 (1975) 1104; 22 (1975) 1236; 27 (1978) 243; JETP Lett. 20 (1974) 222; 21 (1975) 207.
[35] A.A. Zheltukhin, Phys. Lett. **B 116** (1982) 147; Theor. Math. Phys. **52** (1982) 666; **56** (1984) 785; Yad. Fiz. **33** (1981) 1723; **34** (1981) 562.

[36] I.A. Bandos and A.A. Zheltukhin, Sov. J. Part. Nucl. **25** (1992) 1065; Phys. Lett. **B 288** (1992) 77.