Orders of chaoticity of unitaries

Adrian Ortega, Andrew B Frigyik and Mátyás Koniorczyk

1 Department of Quantum Optics and Quantum Information, Institute for Solid State Physics and Optics, Wigner Research Centre for Physics, 1525 PO Box 49, Hungary
2 Óbuda University, 1081 Budapest Népszínház st. 8, Hungary
E-mail: koniorczyk.matyas@wigner.hu

Keywords: dynamical entropy, iterated quantum maps, measurement uncertainty

Abstract
We introduce the concept of $K$-th order chaoticity of unitaries, and analyze it for the case of two-level quantum systems. This property is relevant in a certain quantum random number generation scheme. We show that no unitaries exist with an arbitrary order of chaoticity.

1. Introduction

The quantification of the rate at which a signal source can produce information has always been one of the fundamental questions of information theory [1]. The celebrated result of Shannon states that a signal source that can be modeled with a sequence of independent identically distributed random variables (emitting letters one after another) produces information that amounts to the entropy of the variable. If the output of the source is encoded in long sequences, this is the amount of bits per letter minimally needed for encoding it, in an asymptotic sense. Entropy thus characterizes the lossless compressibility of the source’s output.

If the variables in the sequence are not independent, instead of the entropy, the entropy rate becomes relevant. This quantity takes into account the correlations between the letters. In order to use the source in question as a random signal generator we want it to have the maximal possible entropy rate since it is the entropy rate that characterizes the amount of generated randomness. The idea of entropy rate can be generalized to the continuous setting, leading to the concept of Kolmogorov-Sinai entropy [2, 3].

Considering quantum dynamical systems, the generalization of the Kolmogorov-Sinai entropy is not obvious and can be approached from different points of view. The generalizations to non-commutative dynamical systems by Connes, Narnhofer and Thirring [4] and the one by Alicki and Fannes [5] are mathematically sound, but their operational meaning is less transparent than what we have in the classical case. In addition, they both vanish for finite dimensional quantum systems. The possible operational meaning highly depends on certain aspects of the given problem that has to be addressed: the freedom of choosing the dynamics, protocols, and measurements introduces a significant amount of ambiguity into the issues surrounding application and interpretation. Slomczyński and Życzkowski [6], for instance, introduce the notion of ‘coherent states entropy’ in order to study a certain aspect of quantum chaos. Quantum dynamical entropy, used in the sense of ‘amount of uncertainty in measurement results’ has also been studied by many authors, e.g. [7–10].

Especially since some of the quantum entropy definitions are only reasonable for systems with infinite dimensional Hilbert spaces, there are not many contributions concerning protocols involving simple finite dimensional quantum systems in terms of quantum dynamical entropy. One of these is due to Alicki et al [11] who relate a partial entropy related to the Alicki-Fannes entropy but meaningful for finite dimensional systems to decoherence rate along with an illustration how it works on a particular simple model. Slomczyński and Szczepanek [12] discuss a simple protocol involving qubits or qutrits. They consider an iterated dynamics scheme: an application of a unitary and subsequently a measurement, in each step. This setup can also be considered as a theoretical model of a quantum random number generator. They calculate a certain kind of dynamical entropy in this setting which, very obviously, is related to the performance of a unitary operator in such a random number generator scheme. This entropy rate is characteristic to unitary maps.
A unitary that has the ability to implement a perfect random number generator—i.e., if it can be used, along with a suitably chosen measurement, to produce sequences of independent uniformly distributed random variables with maximal entropy rate—is termed ‘chaotic unitary’. Somewhat surprisingly, not all unitary qubit and qutrit operators are chaotic, even though there is a significant manifold of suitable operators including the most commonly considered ones.

Our present contribution aims to generalize the results of Słomczyński and Szczepanek to introduce a hierarchy of structural properties of unitaries: the notion of a unitary being ‘chaotic to the Kth order’. We will find that even if the random number generator is based on the repeated application of a qubit unitary and a subsequent measurement, the emerging hierarchy of randomness generation ability leads us to a nontrivial structure of the set of unitaries.

2. Method

2.1. PVM dynamical entropy

Consider some iterated discrete-time dynamics of a quantum system. The Hilbert space of the system is \( \mathcal{H} \), which we assume to be finite dimensional, that is \( \text{dim} \mathcal{H} = d < \infty \). The initial state of the system is described by the (Hermitian, positive semidefinite, unit-trace) density operator \( \rho^{(0)} \), and it evolves according to

\[
\rho^{(k+1)} = \mathcal{E}[\rho^{(k)}],
\]

where \( \mathcal{E} \) is a completely positive trace preserving map. The overall goal is to describe the ‘amount of randomness’ generated in such a process, or a related one eventually disturbed by measurements.

As was said before, the characterization of the randomness in information-theoretic terms is ambiguous. One possible choice is the quantity introduced by Słomczyński and Życzkowski \[6, 13\], and it has been calculated for unitary dynamics of 2 and 3 dimensional systems by Słomczyński and Szczepanek \[12\]. This will be our starting point.

This approach assumes that a measurement is made after each step of the evolution. Take a rank-1 POVM measurement \( M \) with outcomes 1...k on a \( d \)-dimensional quantum system which is characterized by the pure states \( |\phi_j\rangle \in \mathcal{H} \), \( j = 1, \ldots, k \) so that

\[
\sum_j |\phi_j\rangle \langle \phi_j| = \frac{k}{d} \mathbb{I}.
\]

For \( k = d \) we get a projector-valued measurement (PVM). The probability of obtaining the \( j \)-th measurement result if the system is in the state \( \rho \) is

\[
P_j = \frac{d}{k} |\langle \phi_j| \rho |\phi_j\rangle|
\]

and the system is left in the state \( |\phi_j\rangle \) after the measurement. Hence, the model is such that the iterated dynamics described by \( \mathcal{E} \) is interrupted by the measurement after each evolution step governed by \( \mathcal{E} \). Clearly as the measurement modifies the system’s state, the process will differ from the one defined in (1), yet this approach will finally lead to a quantity characteristic for \( \mathcal{E} \).

The sequence of measurement outcomes forms a stochastic process \( X = X_0, X_1, \ldots \), to which the approaches of information theory can be applied. In particular it is possible to calculate its entropy rate

\[
H(U, M) = H(X) = \lim_{l \to \infty} \frac{\eta(X_0 \ldots X_l)}{l}
\]

describing the asymptotic minimum of the bits required to encode a symbol of such a process when maximally compressed losslessly. This quantity depends on the unitary \( U \) and the measurement \( M \). Here \( \eta \) is the Shannon-entropy function

\[
\eta(X) = -\sum_x p(x) \log_2 p(x),
\]

where \( x \) runs through all possible values of \( X \), and \( p(x) \) is the probability of obtaining \( x \).

In [12] only unitary evolutions and PVM measurements were considered:

\[
\mathcal{E}(\rho) = U \rho U^\dagger,
\]

in which case we just obtain a classical Markov-chain with the probability transition matrix

\[
P_{i \to j} = |\langle \phi_j| U|\phi_i\rangle|^2.
\]
The PVM entropy rate is defined as

\[ H(U) = \max_{M \in \text{PVM}} H(U, M) \]  

(note that the measurements are restricted to PVMs) which, as derived in [12] on the basis of equations (4) and (7), calculates as

\[ H(U) = \frac{1}{d} \max_{dM \in \text{ONB}} \sum_{j=1}^{d} \eta((\langle \phi_j | U | \phi_j \rangle)^2) \]

\[ = \frac{1}{d} \max_{dV \in \text{U(d)}} \sum_{j,l=1}^{d} \eta((\langle (V^\dagger V)_{j,l} \rangle)^2), \]

where ONB stands for orthonormal basis, and \( U(d) \) is the set of \( d \)-dimensional unitaries.

This quantity is calculated analytically in [12] for all unitaries in \( d = 2 \) in closed form, and it is also studied in detail for and \( d = 3 \). For bigger systems it can also be calculated via a numerical optimization over the unitary group. In particular, according to [12], for a given \( 2 \times 2 \) unitary, writing it in its eigenbasis as

\[ U = \begin{pmatrix} \exp i\phi & 0 \\ 0 & \exp i\psi \end{pmatrix}, \]

with \( \phi, \psi \in [0, 2\pi] \) and introducing

\[ \theta = \min(|\phi - \psi|, 2\pi - |\phi - \psi|), \]

we have

\[ H(U) = \begin{cases} 1, & \theta \geq \frac{\pi}{2} \\ \eta(\cos^2(\frac{\theta}{2})) + \eta(\sin^2(\frac{\theta}{2})), & \theta < \frac{\pi}{2}, \end{cases} \]

with \( \eta(x) = -x \log_2 x \) for \( x > 0 \) and \( \eta(0) = 0 \). Regarding chaoticity, corollary 1 of [12] implies that \( U \) is chaotic if and only if

\[ |\text{tr}(U)| \leq \sqrt{2}, \]

or, equivalently, \( \theta \geq \pi/2 \). We will use these facts as the basis of our considerations.

### 3. Results

#### 3.1. \( K \)-sampled PVM dynamical entropies

In general the process of measurement in each time step of an iterated unitary evolution proposed in [12] can be considered as a first step towards a more general approach. Consider a modified protocol in which the measurement is performed after each \( K \)th iteration only. We define the \( K \)-th order dynamical entropy as the entropy of \( U^K \):

\[ H_K(U) = H(U^K). \]  

(The definition could be extended to more general completely positive trace preserving maps.) It is reasonable to ask whether there are evolutions for which \( H_K \) is nonzero (or even maximal) for some or all values of \( K \)? In what follows we will study this question for the case of 2-dimensional unitaries.

We will call a unitary \( U \) chaotic to the \( K \)-th order, if \( H_K(U) \) is maximal. For instance, the Pauli operators, for instance, are chaotic, but their square is the identity. So measuring in every other step generates no randomness at all. This also means that if we have a random number generator based on Pauli operators and measurements at all. We will study this question for the case of 2-dimensional unitaries.

We will call a unitary \( U \) chaotic to an arbitrary order, if \( H_K(U) \) is maximal for any \( K \). Such a unitary could be very useful in random generation for it does not create the need to carry out the measurement in every iterative step in order to obtain a proper generator.

The idempotent nature of a matrix is clearly a stronger property than nonchaoticity. Hence, the notion of an idempotent and a non-idempotent matrix will be useful for our study. For a unitary \( U \), if \( U^K = I \) for some \( K > 0 \) integer, and \( K \) is the lowest integer for which this property holds, then we say that \( U \) is idempotent of order \( K \). (If \( U \) is idempotent to the order \( K \), then trivially \( U^N K = I \) for arbitrary \( N \in \mathbb{Z} \).) We say that a non-idempotent matrix is a matrix such that for all \( K, U^K \neq I \).
As dynamical entropy and thus the notion of chaoticity is phase invariant, instead of idempotency, the notion of ‘phase idempotency’, that is, \( U^K = \exp(i\varphi)I \) for some \( K \) and \( \varphi \), could also be considered instead of idempotency. It is also a stronger property than nonchaoticity to the \( K \)-th order, also excluding the latter. Moreover as it is a more general notion of idempotency, it could reveal more non-chaotic unitaries. We leave its consideration to future studies and address idempotency in what follows. In case of \( SU(2) \) matrices, for instance, it is easy to see phase idempotency means

\[ |\text{tr}( U^K )| \leq \sqrt{2}. \]

(15)

Before addressing chaoticity to the \( K \)-th order as well as idempotency, let us first restrict ourselves to the group of unimodular unitaries, and discuss \( K = 1 \) in that case.

### 3.2. Eigenphase distribution of chaotic unitaries in the group \( SU(2) \)

In the rest of our considerations we restrict our discussion to the group \( SU(2) \). We will not lose generality as any unitary has a unimodular counterpart that leads to the same physical behavior up to an irrelevant global phase. Let us also remark that the elements of this subgroup of unitaries are readily implementable on some real quantum computers, such as the IBM, IonQ and Rigetti platforms [14–16].

The restriction to \( SU(2) \) leads to a significant simplification: for the two phases characterizing the unitary in the form as in equation (10),

\[ \phi + \psi = 2\pi m \]

(16)

holds, where \( m \) is a natural number. Hence, we have a single phase parameter instead of two, and an integer playing a simpler role. From equation (16) it follows that \( \phi = 2\pi m - \psi \); substituting it into equation (10), the condition of chaoticity in equation (15) reads:

\[ |\cos \psi| \leq 2^{-1/2}. \]

(17)

Note that the use of the single parameter \( \psi \) restricts the considered set of unitaries to a subgroup of \( SU(2) \); a great circle in \( S^3 \). Every other \( SU(2) \) element is conjugate to one of the elements of this subgroup [17]. Hence, using this parameter is consistent with the Haar measure of the group, justifying our discussion of distributions or probabilities.

This last inequality is the defining condition for \( SU(2) \) matrices to be chaotic, i.e. they are chaotic if \( \psi \in \left[ \pi/4, 3\pi/4 \right] \cup \left[ 5\pi/4, 7\pi/4 \right] \). The length of the intervals sum up to \( \pi \), which means that half of the \( SU(2) \) matrices defined through \( \psi \) are chaotic and half of them are non-chaotic. In other words, if we draw uniformly a \( \psi \) phase and construct the corresponding \( SU(2) \), with probability 1/2 it will be chaotic. Thus, the probability of obtaining \( k \) instances of chaotic matrices (of the first order) out of \( N \) trials has a binomial distribution (or a normal distribution in the limit as \( N \) goes to infinity).

### 3.3. Idempotency of arbitrary order and chaoticity of order \( K \)

Some of the prominent examples of chaotic unitaries found in [12], such as the Pauli operators, are idempotent of order two, i.e. their square is the identity operator, hence they cannot be chaotic to the second order. Idempotency of order \( n \) excludes chaoticity to the order of \( n \). Therefore, let us address the question of idempotency of order \( n \) where \( n \) is arbitrary. (This implies, by definition, that \( n \) is the lowest such value for which \( U^n = I \).) Based on equation (10), we can express a certain class of unitaries as

\[ U \propto \begin{pmatrix} e^{im_1 \pi/4} p_1 & 0 \\ 0 & e^{im_2 \pi/4} p_1 \end{pmatrix}, \]

(18)

where \( m_i, p_i (i = 1, 2) \) are integers and \( m_i \) and \( p_i \) are relative primes. With these conditions it is easy to produce examples with arbitrary order of idempotency, for instance,

\[ D^{(4)} = e^{i\pi/4} \text{Diag}(e^{i\pi/4}, e^{i3\pi/4}) \]

(19)

is idempotent of order 4 and

\[ D^{(8)} = e^{i3\pi/32} \text{Diag}(e^{i\pi/32}, e^{i17\pi/32}) \]

(20)

has idempotency of order 8. Furthermore, in the first case \( \theta = \pi \) while \( \theta = \pi/2 \) in the second case, and thus the latter has the same degree of chaoticity as \( \sqrt{\alpha_0} \)-gate (period 4). Note that one can always enforce an arbitrarily (or indefinitely) long period, provided that the \( p_i \)'s are very large different primes with \( m_i \ll p_i \) (preferably \( m_i = 1 \) for
The phase $\psi = i \pi K / p_x$ is determined by equation (21); if we choose it as in equation (18), then $m_1/p_1 = 2m - 1/p_2$. This is the condition to construct a unitary that is chaotic of a given order $K$. As an example, choose $K = 5$ and $p_2 = 2$. Then $\psi = \pi/2$, $\phi = 3\pi/2$ and

$$U^5 = \begin{pmatrix} e^{i5\pi/2} & 0 \\ 0 & e^{i5\pi/2} \end{pmatrix}$$

is chaotic with $\theta = \pi$.

Note that unitaries with rational frequencies as in equation (18) are the exceptions more than the rules in the group SU(2). Following the discussion of the previous section, rational frequencies $\psi = m_2/p_2$ form a subset of (Lebesgue) measure zero and thus in a random trial we are always going to draw a SU(2)-matrix without a rational frequency. Hence in principle this larger set of SU(2) matrices is non-idempotent, but it remains to be determined if a matrix in it is chaotic or not for a certain $K$. We shall investigate in the following this larger subset of SU(2) matrices.

### 3.4. Chaoticity to arbitrary order

Recall that unitaries chaotic to all orders are chaotic to the $K$-th order for any $K$, and thus they have to be non-idempotent. Figure 2 illustrates a circuit with such a $U$. The dashed lines illustrate that we can measure at any step of the circuit and the sequence of the so obtained measurement outcomes will have a maximal entropy. We shall discuss in the following if such chaotic unitaries exist.

First let us check the condition of non-idempotency. In order to see if one can obtain a non-idempotent matrix in the group SU(2), we set $\phi = x \pi$ and $\psi = y \pi$ in equation (16). Thus the real numbers $x, y$ must fulfill

$$x + y = 2m.$$  

If both $x$ and $y$ are rational numbers, then there will be a power such that if we raise the matrix to that power we get the identity. Hence, in order for an element of SU(2) to be non-idempotent at least one of the members of the pair $(x, y)$ must have been irrational. Meanwhile the condition in equation (23) also holds, hence, if one of them is irrational, so is the other, as their sum is an (even) integer.

From the arguments given so far, we can come to an important consequence: there exist no element that is chaotic to arbitrary order within the group SU(2). If such a matrix existed, it would have to be non-idempotent. If $x, y$ in equation (23) are irrationals, then by the Kronecker’s approximation theorem [18], $\exp(iK\pi y)$ will fill densely the unitary circle. This means that for some $K$’s, the condition equation (17) will be violated and, if we do

---

**Figure 1.** Chaoticity of order $K$: Using condition equation (21) we can construct a unitary $U$ such that its $K$-th power is chaotic: the measurements are performed in each $K$th step.

**Figure 2.** Chaoticity to arbitrary order. The dashed lines indicate potential measurements; any of them can be carried out or skipped. The result sequence is composed of the outcomes of the subsequent measurements that were actually carried out.

---
the measurement at this step, the matrix will be non-chaotic. In fact there will be infinitely many $K$-s for which $H(U^x)$ will be arbitrarily close to zero.

The question arises how to generate particular non-idempotent matrices that are candidates in search for matrices chaotic for given orders. Examples of phases with the property described in equation (23) can be drawn straightforwardly for real quantities such as $x = q_1 + \sqrt{r}$ and $x = q_2 - \sqrt{r}$, where $q_1, q_2 \in \mathbb{Q}, q_1 + q_2 = 2m$ and $\sqrt{r}$ is not an integer. In the following we use these type of numbers to show an interesting connection between our problem and the field of algebraic number theory. We describe two types of series of non-idempotent $SU(2)$ matrices. In general, in one of them the matrices converge to the identity as the parameter grows, while the other ‘travesses’ all $SU(2)$.

Let us give an example from the set consisting of the first kind of matrices, as a motivation. Assume that $x^{1/t} = (1 + \sqrt{5})/2$ and $y^{1/t} = (1 - \sqrt{5})/2$, where $t$ is a positive integer. This is the golden ratio and its conjugate, with $t$ a positive integer such that

$$ (x^{1/t} + y^{1/t}) = 2m. \quad (24) $$

The numbers defined above are the celebrated Lucas numbers [19]

$$ L_t = \lambda_+^t + \lambda_-^t, \quad (25) $$

with $\lambda_+ = (1 + \sqrt{5})/2$, where we are considering only those $t$ values for which $L_t$ is even. The first non trivial even Lucas number is $L_3 = 4$, which implies $\phi = (\lambda_+)^3 \pi \approx 0.7416$ and $\psi = (\lambda_-)^3 \pi \approx 5.5415$ but

$$ |e^{i\phi} + e^{i\psi}| \approx 1.4747, \quad (26) $$

and this is bigger than $\sqrt{2}$, which is the lower bound for a matrix to be chaotic. Yet, since $1.4747$ is close to $\sqrt{2}$, we may consider the unitary with above eigenphases as close to chaotic: the actual value of the dynamical entropy calculated using (12) is 0.9944, which is indeed close to the maximum value of 1. Note that for large enough $t$, $\lambda_+ \approx 2m$ from equation (23) and $|e^{i\phi} + e^{i\psi}| \approx |e^{i\phi} + 1| = 2$ will always yield a non-chaotic matrix.

After this illustration let us generalize our selection in order to show how to construct in a more general way series of non-idempotent $SU(2)$ elements. The process results in two series, one of which converges to the identity, and the other traverses the whole $SU(2)$. The numbers $\lambda_\pm$ that enter the definition of the Lucas numbers are solutions of the quadratic equation $x^2 - x - 1 = 0$. We can generalize easily this result in order to obtain a whole class of non-idempotent $SU(2)$ matrices. Let $\alpha, \beta$ be solutions of the quadratic equation

$$ x^2 + ax + b = 0, \quad (27) $$

where $a, b \in \mathbb{Z} - \{0\}$. Obviously its solutions are $\alpha = (-a + \sqrt{a^2 - 4b})/2, \beta = (-a - \sqrt{a^2 - 4b})/2$ (if the discriminant $D = a^2 - 4b$ is square-free (its prime factorization is a product of distinct primes where each factor is to the power of, at most, one) then $\alpha, \beta$ belong to the quadratic field $\mathbb{Q}(\sqrt{D})$ [20]). Note that $\alpha^t + \beta^t = m \quad (28)$

is always an integer. This last equation can be proven by induction taking into account that $\alpha + \beta = -a$ and $\alpha\beta = b$ are integers and thus, if $\alpha^t + \beta^t$ is an integer,

$$ \alpha^{t+1} + \beta^{t+1} = (\alpha^t + \beta^t)(\alpha + \beta) - \alpha\beta(\alpha^{t-1} + \beta^{t-1}) \quad (29) $$

is also an integer. In equation (23), set $x = \alpha$ and $y = \beta$, i.e. we want to have

$$ \alpha^t + \beta^t = \text{even}. \quad (30) $$

And this does hold: to prove it one can use induction again and notice that if $\alpha + \beta$ is even, using equation (29) yields that equation (30) is even, irrespective of the value of $\alpha\beta$ (c.f. $L_t$ is even for some values of $t$). For simplicity we consider from here onward the case when $a, b \leq 0$. Furthermore, in order to have non-idempotency we choose $a, b$ such that $\sqrt{a^2 + 4b}$ is not an integer. A non-idempotent $U$ is attained by choosing the algebraic numbers $x = \alpha$ and $y = \beta$ constructed as above. Here, $t$ is a parameter for which one can choose $|\cos \psi| \leq 2^{-1/2}$ to obtain a chaotic unitary, c.f. Section 3.2.

On to the other type of series: those that traverse the whole $SU(2)$, if we want to avoid $\beta' \rightarrow 0$ in the limit of large $t$, we need to require that $\beta' < -1$. For an example, choose $|a| = 2, |b| = 101$ (both primes) and $t = 8$; this yields $|\cos \psi| \approx 0.387 < 2^{-1/2}$ and thus $U$ is an $SU(2)$ non-idempotent matrix with eigenphases

$$ \phi = \pi (2 + \sqrt{2^2 + 4 \times 101})/2^5, \quad \psi = \pi (2 - \sqrt{2^2 + 4 \times 101})/2^5. $$

4. Discussion

In this section first, we focus on two immediate outcomes of the results presented in the previous section. Next we address the question of how a simple model of noise affects the chaoticity of a generic $SU(2)$ matrix.

A realistic implementation of a chaotic $SU(2)$ matrix will always be subject to noise coming from distinct sources. As of today, the sources of noise and errors in devices of the Noisy Intermediate-Scale Quantum (NISQ)
era are complex and diverse [21, 22]. In this section, we would like to understand what the impact of a small amount of noise on a chaotic SU(2) matrix is.

The following is a simple noise model: take a uniformly distributed random phase $\lambda$ from the interval $[-\epsilon \pi, \epsilon \pi]$ where $\epsilon$ is small and positive. Using equation (16) we modify the phases to

$$\tilde{\phi} + \tilde{\psi} = (\phi + \lambda) + (\psi - \lambda).$$

(31)

With this choice we still get an SU(2) matrix, albeit with different eigenphases that we can modify at each time step. Each time we draw a random number $\lambda$ from the interval $[-\epsilon \pi, \epsilon \pi]$ we will get an irrational value (again the rationals in these intervals have measure zero). Thus the modified eigenphases $\tilde{\phi}$ and $\tilde{\psi}$ will always be irrational in any step of the protocol. Aside from the non-idempotency added by $\lambda$, we also note from equation (17) that if $|\cos \psi|$ is close to the value of $2^{-1/2}$ and $\epsilon$ is "large enough", then in some trials we can alternate between chaotic and non-chaotic unitaries. Note that because of the way we defined the dynamics here, any multiplicative noise that rotates the unitary in question is irrelevant. In other words, $U$ is defined up to an arbitrary (noisy) unitary transformation.

5. Conclusion

We have introduced the notion of chaoticity of a unitary to the Kth order. We have discussed it in case of two-level quantum systems. We have also studied idempotency which plays an important role when it comes to chaoticity.

Our most important conclusion is that no unitaries exist that are chaotic to an arbitrary order. From the point of view of random number generation with the studied scheme, the situation is somewhat analogous to the classical Marsaglia-type pseudo random number generators [23]: they can have extremely long periods but they are periodic.

A possible further generalization of this work could be the study of the entropy rate of a process arising from the same iterated unitary evolution but making measurements in unevenly or even randomly distributed discrete time instants only. For instance, one could investigate the situation when the measurement is made after at most $K$ steps. This could be a model of a random generator in which the measurement is eventually skipped, e.g. due to failure, in a few consecutive steps. This may be a potential direction of future research.

In a future work, we would like to study the relationship between the chaotic unitaries defined here compared to the unitaries defined in Random Matrix Theory [24]. It would be interesting to see if extra symmetries, such as the one present in the Circular Unitary and Orthogonal Ensembles, affect in any way the chaoticity presented here.

Acknowledgments

This research was supported by the National Research, Development, and Innovation Office of Hungary under project numbers K133882 and K124351 and the Ministry of Culture and Innovation and the National Research, Development and Innovation Office within the Quantum Information National Laboratory of Hungary (Grant No. 2022–2.1.1-NL-2022–00004). The idea of the presented research raised while visiting and discussing with Igor Jex in Prague.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

ORCID iDs

Adrian Ortega © https://orcid.org/0000-0001-5816-7274
Andrew B Frigyik © https://orcid.org/0000-0002-4220-4680

References

[1] Cover T M and Thomas J A 2005 Elements of Information Theory (New York: Wiley) (https://doi.org/10.1002/047174882x)
[2] Martin N F and England JW 2011 Mathematical Theory of Entropy (Cambridge: Cambridge University Press) 12
[3] Walters P 2000 An Introduction to Ergodic Theory (United States of America: Springer Science & Business Media) vol. 79
[4] Connes A, Narnhofer H and Thirring W 1987 Commun. Math. Phys. 112 691–719
[5] Alicki R and Fannes M 1994 Lett. Math. Phys. 32 75–82
[6] Słomczyński W and Życzkowski K 1994 J. Math. Phys. 35 5674–700
[7] Srinivas M D 1978 J. Math. Phys. 19 1952–61
[8] Beck C and Graudenz D 1992 Phys. Rev. A 46 6265–76
[9] Crutchfield J P and Wiesner K 2008 Phys. Lett. A 372 375–80
[10] Kollár B and Koniorczyk M 2014 Phys. Rev. A 89 022338
[11] Alicki R, Łoziński A, Pakoński P and Życzkowski K 2004 J. Phys. A: Math. Gen. 37 5157–72
[12] Słomczynski W and Szczepanek A 2017 IEEE Trans. Inf. Theory 63 7821–31
[13] Słomczyński W and Życzkowski K 1995 J. Math. Phys. 36 5201–5201
[14] IBM Quantum https://quantum-computing.ibm.com/ 2023-02 https://quantum-computing.ibm.com/
[15] IonQ https://ionq.com/ accessed: 2022-06
[16] Rigetti https://www.rigetti.com/ 2023-02 https://www.rigetti.com/
[17] Faraut J 2008 Analysis on Lie groups Cambridge Studies in Advanced Mathematics (Cambridge: Cambridge University Press)
[18] Apostol T M 1990 Modular Functions and Dirichlet series in Number Theory (United States of America: Springer)
[19] Andrews G E 1994 Number Theory (United States of America: Dover Publications) rev ed
[20] Niven I, Zuckerman H S and Montgomery H L 1991 An Introduction to the Theory of Numbers (New York: Wiley) 5th edn
[21] Besch S and Karpuucu UR 2019 Benchmarking quantum computers and the impact of quantum noise arXiv https://arxiv.org/abs/1912.05546
[22] Bharti K et al 2022 Rev. Mod. Phys. 94 015004
[23] Marsaglia G 2003 J. Mod. Appl. Stat. Methods 2 2–13
[24] Guhr T, Müller–Groeling A and Weidenmüller H A 1998 Phys. Rep. 299 189