The Power of Nekrasov Functions

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Abstract

The recent AGT suggestion [1] to use the set of Nekrasov functions [2] as a basis for a linear decomposition of generic conformal blocks works very well not only in the case of Virasoro symmetry, but also for conformal theories with extended chiral algebra. This is rather natural, because Nekrasov functions are introduced as expansion basis for generalized hypergeometric integrals, very similar to those which arise in expansion of Dotsenko-Fateev integrals in powers of alpha-parameters. Thus, the AGT conjecture is closely related to the old belief that conformal theory can be effectively described in the free field formalism, and it can actually be a key to clear formulating and proof this long-standing hypothesis. As an application of this kind of reasoning we use knowledge of the exact hypergeometric conformal block for complete proof of the AGT relation for a restricted class of external states.

1 Introduction

Along with the matrix-model τ-functions [3], the Nekrasov functions [2], being coefficients of character expansions of the former ones [4], are very important new special functions, badly needed for developing a quantitative string theory [5]. They originally appeared in the framework of the instanton expansion of Seiberg-Witten quasi-classical (Whitham) τ-functions [6] and provide a kind of a quantization of Seiberg-Witten prepotential [7]. These functions with the theory of Hurwitz-Kontsevich functions [8], describing combinatorics of ramified Riemann surfaces, an essential subject for perturbative and non-perturbative string theory [9]. A new important application was recently suggested in [1]: generic Virasoro conformal blocks [10]-[20] can be nicely represented as linear combinations of Nekrasov functions for the U(2) quiver models [21]. For technical details of this AGT relation see [22, 23]. The AGT suggestion has a number of natural generalizations, the first in the line being that to SU(N) models, which, on the conformal side, should correspond to theories with the conformal algebra extended from Virasoro to W(N). The most straightforward idea in this direction would be to simply decompose certain W(N) conformal blocks as combinations of U(N)-quiver Nekrasov functions [22]. This idea works perfectly well for the "perturbative" (quasiclassical) Nekrasov functions which coincide not only with the DOZZ triple vertices [24] in Liouville model (U(2)-case), but also with the Fateev-Litvinov vertices [25] in affine Toda models (U(N)-case). It is now checked up to level two (order x^2) in the W(3) conformal blocks [26], where calculation depends on some knowledge about W(3) conformal blocks (we refer for details to a dedicated elementary-level summary in [27]).

In this short note we want to attract attention to another aspect of the AGT proposal. The Nekrasov integrals can be considered as an appropriate analytical continuation of expansion coefficients of the Dotsenko-Fateev integrals ("screening charges") [13] in powers of α-parameters. Thus, an apparent success of the AGT conjecture in description of generic conformal blocks can be considered as a strong support of the old hypothesis that the free field formalism can indeed be used to describe generic conformal theories. This is well established in particular distinguished examples [13, 18, 16], but in the check of the AGT relations in [11, 22, 23, 26] one actually works without any reference to particular model, only to its conformal properties. In what follows, we briefly list a small set of examples, which show how expansion in Nekrasov functions generalizes the standard hypergeometric series to the ones needed in description of the Dotsenko-Fateev integrals. Transition to generic conformal blocks still looks mysterious. However, now it can be formulated in a very clear and general form, and further work on the AGT relations would presumably clarify the old mysteries of representation theory of chiral algebras and make the subject much more transparent and understandable.

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2 Conformal blocks and their expansions

Conformal blocks often have a pronounced form of hypergeometric series.

Free field conformal block:

\[ B(x) = (1 - x)^{-2\vec{a}_1 \cdot \vec{a}_3} = (1 - x)^{-A} = 1 + Ax + \frac{A(A + 1)}{2} x^2 + \frac{A(A + 1)(A + 2)}{6} x^3 + \ldots \]  

Fateev-Litvinov conformal block for one special and one maximally degenerate states at external lines: In [25] it was shown that generic hypergeometric series is represented by subset of conformal blocks of the \( SL(N) \) Toda model with restricted external states:

\[ B(x) = N F_{N-1}(A_1, ..., A_N; B_1, ..., B_{N-1}; x) = 1 + x \frac{A_1 \ldots A_N}{B_1 \ldots B_{N-1}} + \frac{x^2}{2} \frac{A_1(A_1 + 1) \ldots A_N(A_N + 1)}{B_1(B_1 + 1) \ldots B_{N-1}(B_{N-1} + 1)} + \ldots \]  

The free field case is reproduced when \( A_i = B_i, A_N = A \).

Thus, an arbitrary hypergeometric function is some conformal block. However, inverse is not true: the generic conformal block does not have this simple hypergeometric structure.

Dotsenko-Fateev integral\(^1\):

\[ B(x) = \left\langle e^{\alpha_1 \phi(1)} : e^{\alpha_2 \phi(0)} : e^{\alpha_3 \phi(x)} : e^{\alpha_4 \phi(\infty)} : \int e^{\epsilon \phi(z)} dz \right\rangle \sim \]

\[ \sim (1 - x)^{-2\alpha_1 \alpha_3} \int_0^1 z^{-2\alpha_2 \epsilon - (1 - z)^{-2\alpha_3 \epsilon} - (z - x)^{2\epsilon - \epsilon} - 2\epsilon} dz \sim (1 - x)^{-A} \int_0^1 z^{-B} (1 - z)^{-C (z - x)^{-D} dz} \sim \]

\[ \sim (1 - x)^{-A} \sum_n \frac{x^n}{n!} \frac{\Gamma(D + n) \Gamma(B + C + D - 1 + n)}{\Gamma(B + D + n)} = (1 - x)^{-A} \cdot \text{2F1}(D, B + C + D - 1; B + D; x) \]  

Multiple Dotsenko-Fateev integrals: when several screenings are inserted, one obtains instead of \( \text{2F1} \)\(^2\)

\[ B(x) = (1 - x)^{-A} \int \ldots \int \prod_i z_i^{-B_i (1 - z_i)^{-C_i (z_i - x)^{-D_i} \prod_{i < j} (z_i - z_j)^{E_{ij}} \prod_i dz_i \]  

In variance with [23] such integrals are no longer hypergeometric, but they are similar in some respects and are often called "generalized hypergeometric integrals", see for example [28]. Still, if we are interested in expansion bases, the terminological does not help: such \( B(x) \) is not an ordinary hypergeometric series of the type \( \text{2F1} \), unless the matrix \( E_{ij} \) is very special (roughly, \( E_{ij} \sim \epsilon_i \delta_{i+1,j} \), see the last paper in [28]). It also deserves mentioning that there is a certain difference between these integrals for a single free field and for multiple \( (r = N - 1) \) fields: the less fields, the more relations between the numerous parameters \( A, B_i, C_i, D_i, E_{ij} \): they are all made from \( \epsilon_i \) and four \( (N - 1) \)-component vectors \( \vec{a}_1, \ldots, \vec{a}_4 \). Increasing \( N \), one actually enlarges class of the Fateev-Dotsenko integrals, they span the entire space of generalized hypergeometric series only for \( N \to \infty \).

Virasoro case without external fields [23]:

\[ 1 + x^\Delta 2 + \frac{x^2}{2} \frac{\Delta (8 \Delta^3 + (c + 8) \Delta^2 + (2c - 8) \Delta + c)}{2 (16 \Delta^2 + (2c - 10) \Delta + c)} + \frac{x^3}{6} \frac{\Delta(\Delta + 2)(8 \Delta^3 + (c + 18) \Delta^2 + (3c - 14) \Delta + 2c)}{4 (16 \Delta^2 + (2c - 10) \Delta + c)} + \ldots \]

\( ^1 \) We denote the screening charge parameter \( \epsilon_- \) rather than \( \epsilon_+ \), because after rescaling of dimensions [23] \( \Delta \to \Delta/(\epsilon_+ \epsilon_-) \) the Gamma-functions acquire the form \( \Gamma \left( \frac{2\alpha_3 + \epsilon_+ + \epsilon_-}{\epsilon_+ \epsilon_-} \right) \sim 2\alpha_3 (2\alpha_3 + \epsilon_+) \ldots (2\alpha_3 + (n - 1) \epsilon_+), \) which should be compared with the chiral Nekrasov function [13]. The screenings with \( \epsilon_+ \) correspond to anti-chiral functions.

\( ^2 \) In Dotsenko-Fateev approach, the dependence on the fifth vector \( \vec{a} \) is presumably restored from a sophisticated analytical continuation of the answer with arbitrary number of screening insertions. An exact relation between the multiple Fateev-Dotsenko integrals and the Virasoro or W-conformal blocks is believed to exist, but remains uncovered. The AGT relation could be a key to resolve it.
In particular, for $c = 1$:

$$B(x) = 1 + x \cdot \frac{\Delta}{2} + x^2 \cdot \frac{(8\Delta^3 + 9\Delta^2 - 6\Delta + 1)}{2(4\Delta - 1)^2} + x^3 \cdot \frac{2\Delta}{6} \cdot \frac{\Delta(\Delta + 2)(8\Delta^3 + 19\Delta^2 - 11\Delta + 2)}{4(4\Delta - 1)^2} + \ldots$$

(5)

$W^{(3)}$ case, general central charge $c$, two states $\tilde{\alpha}_1$ and $\tilde{\alpha}_3$ are special [26].

$$1 + x \cdot \frac{D\Delta D_{12}D_{34} - \frac{w}{3}(D_{12}W_{34} + W_{12}D_{34}) + \frac{2a}{9}W_{12}W_{34}}{D\Delta^2 - w^2} + O(x^2)$$

(6)

$D_{12} = \Delta + \Delta_1 - \Delta_2$, \hspace{1cm} $D_{34} = \Delta + \Delta_3 - \Delta_4$

$W_{12} = w + 2w_1 - w_2 + \frac{3w_1}{\Delta_1}(\Delta - \Delta_1 - \Delta_2)$, \hspace{1cm} $W_{34} = w + w_3 + w_4 - \frac{3w_3}{\Delta_3}(\Delta + \Delta_3 - \Delta_4)$

(7)

Note numerous differences between $W_{34}$ and $W_{12}$ in signs and coefficients, see [27]. The parameter $D = \Delta + \frac{3w^2}{4}$ and the central charge $c = 2(1 - 12e^2)$.

$W^{(3)}$ case, no external states, $c = 2$ [28].

$$B(x) = 1 + x \cdot \frac{\Delta}{2} \cdot \frac{(\Delta^3 - \frac{8}{3}w^2)}{2(\Delta^3 - w^2)} + x^2 \cdot \frac{81\Delta^4(\Delta - 1)^4(8\Delta^3 + 9\Delta^2 - 6\Delta + 1) + G_2w^2 + G_4w^4 + G_6w^6}{2 \cdot 81 \cdot (\Delta^3 - w^2)^2} + \ldots$$

(8)

$$G_2 = -72\Delta(\Delta - 1)^2(25\Delta^5 + 20\Delta^4 - 25\Delta^3 + 23\Delta^2 - 8\Delta + 1),$$

$$G_4 = 1664\Delta^5 + 608\Delta^4 - 2688\Delta^3 + 2704\Delta^2 - 608\Delta + 48,$$

$$G_6 = -128\Delta(4\Delta + 7)$$

For $w = 0$ this expression reduces to [5], note that this happens despite $c = 2$ in [8], while $c = 1$ in [5]. For $c \neq 2$ the level one term becomes $x \cdot \frac{\Delta}{2} \left( \frac{(\Delta^3 - \frac{8}{3}w^2)}{2(\Delta^3 - w^2)} \right)$, a similar deformation occurs in the second term, see [28].

Formulas [5] and [8] do not have the free field limit, because for these values of central charge in the free field model the intermediate states have vanishing $\Delta$ and $w$ whenever all the external momenta are zero.

To summarize, ordinary hypergeometric functions $pF_q$ are not sufficient to describe arbitrary conformal blocks (at least, the poorly studied hypergeometric integrals [28] are needed), and it is a natural question what should be a reasonable extension of this class of functions to serve these purposes. The AGT conjecture is actually a claim that the Nekrasov functions can be an answer to this challenge.

### 3 Nekrasov functions for ordinary Young diagrams

The Nekrasov functions are defined as coefficients of the character-like expansion of Nekrasov integrals, very similar to those in [4]. They are defined for the $N$-plets of Young diagrams. For our purposes we consider only 1-point quiver functions associated with $N_F = 2N$ fundamentals. Extension to other representations and quivers can be immediately provided. In the next section, we will also put $\epsilon \equiv \epsilon_+ + \epsilon_- = 0$ and $\epsilon_+ = -\epsilon_- = 1$. According to the AGT rules this corresponds to putting $c = 1$ and $c = 2$ in the Virasoro and $W^{(3)}$ cases respectively. In this section we, however, keep $\epsilon$ arbitrary.

The general definition of the Nekrasov function in the $SU(N)$ case is as follows

$$Z(Y_1, \ldots, Y_N) = \frac{\prod_i \eta(a_i, Y_i)}{\prod_{i,j} \xi(a_i - a_j, Y_i, Y_j)}$$

(9)

where

$$\eta(a, Y) = \prod_{(p, q) \in Y} P\left(a + (p - 1)\epsilon_+ + (q - 1)\epsilon_-ight)$$

(10)
for a polynomial \( P(a) = \prod_{j=1}^{N_f=2N} (a_i + \mu_f) \), and \( \xi \) is a comprehensive deformation of the hook formula for \( a_Y^{-2} = \xi(0, Y, Y) \) with \( \epsilon_-= -\epsilon_+ = -1 \):

\[
\xi(a, Y_1, Y_2) = - \prod_{(p,q) \in Y_1} \left( a + \epsilon_+ \left( k_j^T(Y_1) - i + 1 \right) - \epsilon_-(k_i(Y_2) - j) \right) \left( a + \epsilon_+ \left( k_j^T(Y_1) - i \right) - \epsilon_-(k_i(Y_2) - j + 1) \right)
\]

(11)

Here \( k_i(Y) \) is the height of the \( i \)-th column and \( k_j^T(Y) \) is the length of the \( j \)-th row of the diagram \( Y \), which is also denoted as \( Y = [k_1 k_2 \ldots] \). Of course \( k_1 \geq k_2 \geq \ldots \geq 0 \) and \( k_1^T \geq k_2^T \geq \ldots \geq 0 \). Note that the product runs over the first diagram only, while the second diagram enters just through the coefficient \( k_i(Y_2) \) in the product. Therefore, it is not surprising that when the second diagram is empty, this expression simplifies to \( \xi(a, \emptyset, Y) = 1 \), while the first diagram is empty, the expression reduces to

\[
\xi(a, Y, \emptyset) = \prod_{(p,q) \in Y} \left( a + (p - 1)\epsilon_+ + (q - 1)\epsilon_- \right) \left( a + p\epsilon_+ + q\epsilon_- \right)
\]

(12)

so that for fixed \( i \)

\[
\prod_{j \neq i} \xi(a_i - a_j, Y_i, \emptyset) = \prod_{(p,q) \in Y_i} Q_i(a_i + (p - 1)\epsilon_+ + (q - 1)\epsilon_-)
\]

(13)

which is very similar to (10), only with \( P(a) \) of degree \( N_f = 2N \) substituted by the polynomial

\[
Q_i(x) = \prod_{j \neq i} (x - a_j)(x - a_j + \epsilon)
\]

(14)

of degree \( 2N - 2 \). Further simplification occurs when one restricts \( Y \) to be either a line or a column.

These **chiral** Nekrasov functions have a clear form of the hypergeometric series terms:

\[
Z(\emptyset \ldots [n] \ldots \emptyset) = \frac{1}{\epsilon_-(n)!} P(a_i)P(a_i + \epsilon_+) \ldots P(a_i + (n - 1)\epsilon_+)
\]

(15)

where the diagram \( Y_i = [1^n] \) stands on the \( i \)-th place in the \( N \)-plet and the additional \( \epsilon_- \)-dependent factor comes from \( \xi(0, Y_i, Y_j) \).

The **anti-chiral** functions \( Z(\emptyset \ldots [n] \ldots \emptyset) \) have exactly the same form with \( \epsilon_+ \rightarrow \epsilon_- \).

## 4 AGT relations

### 4.1 Fateev-Litvinov conformal block via Nekrasov functions

The polynomial \( P(a) \) can be adjusted so that only one diagram contributes at each level:

\[
P(a_j) = 0 \quad \text{for} \quad j \neq i \quad \text{and} \quad P(a_i + \epsilon_+) = 0
\]

(16)

this fixes \( N \) out of \( N_f = 2N \) parameters \( \mu_f \). The first condition is needed to eliminate all \( N \)-plets of Young diagrams, where any non-empty diagram stands at any position, different from \( i \), including all mixed diagrams, where at least two non-trivial diagrams are present in the \( N \)-plet. The second condition eliminates in addition all diagrams except for the single-row \([1^n]\), standing in the \( i \)-th position in the \( N \)-plet. Both kind of requirements should be clear from a look at (10).

In this way, one can describe various hypergeometric series by the Fateev-Litvinov conformal blocks \([2]\) which depend on exactly \( 2N \) free parameters: \( \alpha_2, \tilde{\alpha}_4, \alpha_1 \) and \( \epsilon \), all combined in a sophisticated way into \( A_1, \ldots, A_N, B_1, \ldots, B_{N-1} \). Similarly to the free field case, the intermediate-state momentum \( \vec{\alpha} \) for these specific external states is almost (up to \( N \) possible values) dictated by the external momenta due to severe selection rules of the Toda-chain model). **This argument provides a complete proof of the AGT relation in this restricted setting** (such a possibility has been also anticipated in \([22]\)).

![Figure 1](image1.png)
4.2 Deviations from chirality and hypergeometricity

The non-chiral SU(2) Nekrasov functions can be used to introduce the needed corrections for the Virasoro conformal block, which deviates it from the hypergeometric form. For the Dotsenko-Fateev integrals it is almost obvious, if one recalls that the Nekrasov integrals (2) \((P(x)\) was defined in s.3)

\[
Z^{SU(2)}_k(N) = \int \prod_{i=1}^k \frac{dz_i}{2\pi i k_i!} \left( \frac{\epsilon}{\epsilon_+ \epsilon_-} \right)^k \frac{\Delta(0)\Delta(c)}{\Delta(\epsilon_+)\Delta(\epsilon_-)} P(z_i),
\]

\[
\Delta(x) \equiv \prod_{i<j} \left( (z_i - z_j)^2 - x^2 \right), \quad Q(x) \equiv \prod_i (x - a_i)(x - a_i + \epsilon),
\]

are nothing but the generalized hypergeometric integrals \([28]\) which appeared in \([1]\). It is not that clear what they have to do with the series like \([5]\) and \([8]\) – but they do, as successful tests of the AGT relation clearly demonstrate, and this is in perfect agreement with beliefs in conformal field theory and representation theory of loop algebras, where conformal blocks are thought to be somehow expandable in analytically continued Dotsenko-Fateev integrals. As it is now obvious, the further character-like expansion of integrals in the Nekrasov theory of loop algebras, where conformal blocks are thought to be somehow expandable in analytically continued Dotsenko-Fateev integrals converts the problem into the very clearly formulated AGT conjecture, and hopefully provides a key for its final resolution.

4.3 SU(2)/Virasoro case

The SU(2) case illustrates nicely that including the non-chiral Nekrasov function allows one to extend hypergeometric series into an interesting direction: for example, to describe the Virasoro conformal blocks.

Let us see how it works at the first two levels \([23]\):

\[
Z^{SU(2)}(x) = 1 + x \left( Z(\Box, \Box) + Z(\Box, \Box) \right) + x^2 \left( Z(\Box\Box, \Box) + Z(\Box, \Box\Box) + Z(\Box, \Box, \Box) + Z(\Box, \Box) \right) + \ldots =
\]

\[
= 1 + x \cdot Z_1^{SU(2)} + x^2 \cdot Z_2^{SU(2)} + \ldots = 1 + x \left( \frac{P(a)}{Q(2a)} + \frac{P(-a)}{Q(-2a)} \right) +
\]

\[
+ x^2 \left( \frac{P(a)P(a+1)}{4Q(2a)Q(2a+1)} + \frac{P(-a)P(-a+1)}{4Q(-2a)Q(-2a+1)} + \frac{P(a)P(a-1)}{4Q(2a)Q(2a-1)} + \frac{P(-a)P(-a-1)}{4Q(-2a)Q(-2a-1)} \right)
\]

Here \(a_2 = -a_1 = -a\) and we denote \(Q_1(a_1) = Q(2a), Q_2(a_2) = Q(-2a)\). Underlined and double-underlined are the chiral and anti-chiral contributions, non-underlined remains the mixing contribution from the pair of diagrams. At level one, there is no difference between the chiral and anti-chiral diagrams.

Matching with \([5]\) is achieved if one takes \(P(a) = a^4, Q(2a) = (2a)^2\):

\[
Z = 1 + x \cdot \frac{a^2}{2} + x^2 \cdot \left( \frac{2a^4(1+a)^4}{16a^2(2a+1)^2} + \frac{2a^4(1-a)^4}{16a^2(2a-1)^2} + \frac{a^8}{(2a-1)^2(2a+1)^2} \right) + \ldots =
\]

\[
= 1 + x \cdot \frac{a^2}{2} + x^2 \cdot \frac{a^2((a+1)^4(2a-1)^2 + (a-1)^2(2a+1)^2 + 8a^6)}{4(4a^2-1)^2} + \ldots =
\]

\[
= 1 + x \cdot \frac{a^2}{2} + x^2 \cdot \frac{a^2(8a^6 + 9a^4 - 6a^2 + 1)}{2(4a^2-1)^2} + \ldots
\]

If one now identifies \(\Delta = a^2\), then this expression reproduces \([5]\). It is straightforward to generalize this calculation to arbitrary central charges and external states, to level 3 \([23]\) and further \([1]\).

4.4 SU(3)/\(W(3)\) case

Instead of \([19]\) one now has

\[
Z^{SU(3)} = 1 + x \left( Z(\Box, \Box, 0) + Z(\Box, \Box, 0) + Z(\Box, \Box, 0) \right) + x^2 \left( Z(\Box\Box\Box, \Box, \Box) + Z(\Box, \Box\Box, 0) + Z(\Box, \Box, \Box) + \ldots = 1 + xZ_1 + x^2Z_2 + \ldots
\]
\[ \Delta = \alpha^2 + \beta^2, \quad w = \alpha(\alpha^2 - 3\beta^2) \] (22)

and the Nekrasov parameters \( a_1, a_2, a_3 = -a_1 - a_2 \). As usual, in terms of the \( \alpha \)-parametrization, the AGT is a linear relation (defined modulo rotations and Weyl reflections):

\[
\text{AGT:} \quad a_1 = \frac{\alpha}{\sqrt{3}} - \beta, \quad a_2 = \frac{\alpha}{\sqrt{3}} + \beta, \quad a_3 = -\frac{2\alpha}{\sqrt{3}} \] (23)

Then,

\[
\sum_{i=1}^{3} \frac{a_i^6}{\prod_{j \neq i} (a_i - a_j)^2} = \frac{(\alpha^2 + \beta^2)(\alpha^6 + 75\alpha^4\beta^2 - 45\alpha^2\beta^4 + 9\beta^6)}{2\beta^2(\beta^2 - 3\alpha^2)^2} = \frac{\Delta(\Delta^3 - \frac{5}{3}w^2)}{2(\Delta^3 - w^2)} \] (24)

i.e. the level one term in (21) nicely reproduces the one in (8) with \( P(a) = a^6 \), as claimed in [22, 26].

Now we can substitute the same values of \( a_i \) and \( P(a) \) into the second level term in (21) and express the result back through \( \Delta \) and \( w \) with the help of (22). Surprisingly or not, the latter step turns out possible. Moreover, the answer [26] is literally the same as in [8], in perfect accordance with the AGT conjecture. We do not need to include any \( U(1) \) factors in this simplified calculation, because we keep external momenta zero. Switching on \( \epsilon \neq 0 \ (c \neq 2) \) and external momenta is a more difficult calculation, requiring also some knowledge from representation theory of \( W(3) \) algebras (summarized for this purpose in [27]), but this tedious job [26] gives nothing new: the AGT relation remains to be true. Now there seem to be no room for doubt that the AGT conjecture is valid, time is now to learn lessons from it and apply it to resolution of old and new problems.

### 5 Some other comments on non-Virasoro AGT relations

To conclude this note, we briefly comment on another kind of problem [22] with the \( W(3) \) conformal blocks, which appears for non-vanishing external momenta.

First, unlike the Virasoro case, the triple vertices for \( W(3) \)-descendants are not fully defined by the \( W \)-symmetry alone: all the triple correlators of the form \( \langle V_1 V_2 (W_{-1}^k V_3) \rangle \) remain free parameters. One can resolve this problem in two ways: either via specifying a concrete conformal model, like free fields or affine Toda, or imposing restrictions on possible choice of the external states, like the speciality conditions of [25], requiring two of the four external states \((m - 2) \) in an \( m \)-point conformal block) to be the \( W \)-null-vectors at level one.

Second, there is a mismatch between the number of free parameters in conformal block and in the \( SU(3) \) Nekrasov function. The structure of this mismatch is better seen if one considers an arbitrary \( N > 2 \). In the \( SU(N) \) case, there are \( 3N \) parameters in the Nekrasov function \((N - 1 a's plus 2N\mu's plus 2 \epsilon's minus 1 common rescaling)\) and \( 5N - 4 \) parameters in the conformal block \( (4 \) external and \( 1 \) internal \( N - 1 \)-component momenta plus 1 central charge). Thus the mismatch is: extra \( 2(N - 2) \) parameters on the CFT side.

If one tries to resolve the problem by considering the free field model, then one external and one internal momentum are fully defined by the 3 external momenta, so that the number of parameters on the CFT side decreases to \( 3(N - 1) + 1 = 3N - 2 \) and this is a slight overplay: there are extra 2 parameters on the Nekrasov side.

An exact matching in the number of parameters is achieved if one restricts to the special states [22]. For \( N = 3 \) this subtracts 2 out of \( 5N - 4 = 11 \) parameters on the CFT side what brings this number down to \( 9 = 3N \), exactly the same as needed for the \( SU(3) \) Nekrasov functions.
The problem, however, persists. Not only a selected set of conformal blocks, but all of them can be one day calculated in a given conformal model, as they can, for example, in the model of $N - 1$ free fields. The $W^{(N)}$-symmetry does not fix them unambiguously, but in a given model they are all well defined. Nothing forbids these conformal blocks to have more free parameters than there are available on the Nekrasov side of the AGT relation. This does not happen to free fields, as we saw, but a mismatch seems to exist already in the affine-Toda model. Even for the free fields there is an open problem: the Nekrasov functions describe this case under speciality conditions, but what happens if they are lifted? In any case, this mismatch seems certain to occur in a generic model with the $W^{(N)}$-symmetry.

It is a very interesting and conceptually important question, if the Nekrasov functions would still provide a basis? If not, should their set be somehow extended? Do they really provide an exhaustive basis for expansion of arbitrary generalized hypergeometric integrals from [28]? What the mismatch, if any should mean from the point of view of the Dotsenko-Fateev approach? Further work on the AGT relation will hopefully provide answers to all these puzzles.

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