1/J corrections to semiclassical AdS/CFT states from quantum Landau-Lifshitz model

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Abstract

One way to relate semiclassical string states and dual gauge theory states is to show the equivalence between their low-energy effective 2d actions. The gauge theory effective action, which is represented by an effective Landau-Lifshitz (LL) model, was previously found to match the string theory world-sheet action up to the first two orders in the effective parameter $\tilde{\lambda} = \lambda/J^2$, where $\lambda$ is the ‘t Hooft coupling and $J$ is the total $R$-charge. Here we address the question if quantizing the effective LL action reproduces the subleading $1/J$ corrections to the spin chain energies as well as the quantum corrections to the string energies. We demonstrate that this is indeed the case provided one chooses an appropriate regularization of the effective LL theory. Expanding near the BPS vacuum, we show that the quantum LL action gives the same $1/J$ corrections to energies of BMN states as found previously on the gauge theory and string theory sides. We also compute the subleading $1/J^2$ corrections and show that these too match with corrections computed from the Bethe ansatz. We also compare the results from the LL action with a more direct computation from the spin chain. We repeat the same computation for the $\beta$-deformed LL action and find that the quantum LL result is again equal to the $1/J$ correction computed from the $\beta$-deformed Bethe ansatz equations. We also quantize the LL action near the rotating circular and folded string solutions, generalizing the known gauge/string results for $1/J$ corrections to the classical energies. We emphasize the simplicity of this effective field theory approach as compared to the full quantum string computations.

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1 Introduction

Comparing semiclassical string states [1, 2, 3, 4] to “long” gauge theory operators [5, 6, 7] has turned out to be a fruitful approach to exploring AdS/CFT duality (for reviews and references see [8, 9, 10, 11, 12, 13]). A very simple and clear way of establishing the correspondence between “fast” strings and low-energy “coherent” spin chain states representing dual gauge-theory operators was suggested in [14] and further clarified and developed in [15, 16, 17] (for a review see [10]; various extensions were considered in [18, 19, 20, 21, 22, 23, 24]).

In the simplest nontrivial sector, the $SU(2)$ sector which has operators of the form $\text{Tr}(\Phi_1^J \Phi_2^J) + \ldots$, the corresponding low-energy effective action is derived from the thermodynamic limit ($J = J_1 + J_2 \gg 1$) of the ferromagnetic spin chain, where the Hamiltonian is the gauge-theory dilatation operator. These operators are dual to strings moving in the $R \times S^3$ subspace of $AdS_5 \times S^5$, where in the “fast string” limit of the classical string action one can reduce to the same classical action. This Landau-Lifshitz (LL) type action serves as an intuitive bridge between the gauge-theory and string-theory pictures, suggesting, in particular, how a continuous string action and string picture may appear from gauge theory, as well as suggesting that quantum string theory may have a microscopic spin chain description.

Viewed as an effective low-energy action that emerges from the two quantum “microscopic” theories – the gauge-theory spin chain and the quantum superstring$^1$ – the LL action should not be expected to lead to a well-defined quantum theory. Yet, supplemented with an appropriate UV cutoff or regularization prescription (as well as with relevant higher-derivative counterterms) it may still be able to capture part of the quantum corrections to these “microscopic” theories.

Provided the limits of applicability of this quantum LL model are understood, it may be very useful from both a conceptual and a technical point of view. On the conceptual side, the possibility of reproducing certain quantum $1/J$ corrections to both string and gauge-theory energies from the same effective LL action would continue to serve as an appealing way of understanding their matching.

On the technical side, the computation of quantum corrections in the LL framework is simpler than the full spin chain (e.g., Bethe ansatz) computation of finite-size corrections. It is also much simpler than the full superstring computation of quantum $\alpha'$ corrections to energies of string states for the obvious reason that here one does not include the contributions of the bosonic and fermionic modes which are “outside” the given $SU(2)$ sector, i.e. which are absent in the LL action. Omitting these other string modes is obviously not supposed to be correct in general, but in some simple cases it may happen that the role of these extra modes may be just to provide a particular UV regularization of the quantum LL result. This was first suggested in [25] on the

$^1$The limits on the two sides are, in general, different: (i) small $\lambda$, then expansion in large $J$, and (ii) large $\lambda$ for fixed $\tilde{\lambda} \equiv \frac{\lambda}{J^2}$, or large $J$, and then expansion in $\tilde{\lambda}$, see below.
example of the simplest circular string solution of \[4, 26\] (using results of earlier work in \[27, 28, 29, 30\]). Several new examples and non-trivial extensions (like \(1/J^2\) corrections to BMN energies or \(1/J\) corrections to the folded string energy) will be discussed below.

The quantum LL model may then provide a short-cut to some non-trivial spin chain or quantum string results which would be much harder to find by more direct computations. This is important since any new data about subleading quantum corrections to various semiclassical string energies is crucial for testing all-order conjectures about the structure of the quantum \(AdS_5 \times S^5\) string spectrum \[31, 32, 33, 34\]. It is also important to be able to go beyond the leading \(1/J\) correction, since the next order contains details about the lattice nature of the spin chain. Being able to compare the gauge theory and string theory results might provide clues to how the lattice nature of the spin chain manifests itself in the string theory.

Being an effective field theory, the quantum LL model should be supplemented with a regularization prescription, and which regularization one is to use should depend on the particular microscopic theory one is trying to approximate. For example, expanding near the ferromagnetic ground state, which is a BPS state of both gauge theory and string theory, one needs to assume a normal ordering prescription for the LL Hamiltonian so that the vacuum-state energy is not shifted. Choosing a regularization at higher orders is \textit{a priori} an open question. Still, taking into account this regularization ambiguity by introducing free parameters, one may able to make non-trivial predictions about the dependence of the energies on the quantum numbers of the fluctuation states. As we shall discuss below, the normal ordering prescription appears to be the right one to match both spin chain and string theory results up to quartic oscillator order (and also the right one to match spin chain result in six-order term in the oscillator Hamiltonian).

The \(\zeta\)-function regularization is a natural regularization for computing the leading \(1/J\) correction to the classical energy near a non-trivial solitonic LL state representing a macroscopic spinning string \[25\].\(^2\) The utility of the \(\zeta\)-function regularization in similar ground-state energy computations is well known; its use should be justified by additional global (space-time) properties that the 2-d theory should describe.\(^3\) Which regularization to use beyond the first order correction remains to be understood, but again starting with a solution depending on several parameters (like winding numbers and spins) one may still get non-trivial predictions about the quantum corrections to

\(^2\)The LL model defined on \(R_t \times S^2_a\) has no logarithmic UV divergencies, so the \(\zeta\)-function regularization is equivalent to introducing an explicit cutoff \(\sum_{n=1}^{\infty} e^{-\epsilon n}\) and dropping all terms which are singular in the limit \(\epsilon \to 0\), i.e. is a rather universal regularization prescription.

\(^3\)For example, the use of \(\zeta\)-function regularization in bosonic \(D = 26\) string theory in computing the ground state energy \[35\] which gives the standard value for the tachyon mass can be justified by the requirement of having consistency between the string mass values and gauge symmetries. A similar remark applies to the use of \(\zeta\)-function regularization in the computation of the Born-Infeld action in open bosonic string theory \[36\].
its energy. Comparing to other solutions in various limits may allow one to fix the required value of the regularization parameters.

Let us now describe the contents of this paper. In section 2 we first review the derivation of the $SU(2)$ LL action from both gauge theory (sect. 2.1) and string theory (sect. 2.2), emphasizing the limits and approximations involved. We then present the LL action in several equivalent forms making explicit its phase space structure which allows one to apply the standard operator quantization procedure.

In section 3 we expand the LL action near its trivial vacuum (corresponding to the ferromagnetic vacuum of the spin chain or a point-like BMN geodesic of the string theory) and compute quantum corrections to the fluctuation spectrum. At quadratic fluctuation order we get the leading (order $\tilde{\lambda}$) term in the BMN spectrum (sect. 3.1). In sect. 3.2 we consider the quartic fluctuation term in the LL Hamiltonian and using a normal ordering prescription obtain the leading $1/J$ correction to the $\tilde{\lambda}$ term in the BMN spectrum which matches the value from the gauge theory [5] or the full super-string computation [37, 38]. In sect. 3.3 we compute the next order $\tilde{\lambda}/J^2$ correction. At this order there are contributions from higher order corrections to the Hamiltonian as well as a second order perturbation theory correction coming from the first order correction to the Hamiltonian. Using $\zeta$-function regularization on this latter contribution and a normal ordering prescription on the former, we find agreement with results computed from the Bethe ansatz. We interpret the fact that one is able to reproduce the $1/J^2$ spin chain result by quantizing a continuous 2d action as an indication that the gauge theory and string theory results (obtained in different limits) may continue to match at $\tilde{\lambda}/J^2$ order. In sect. 3.4 we extend the computation to $\tilde{\lambda}^2/J$ order by including the 4-derivative (2-loop on the gauge side) term in the LL action. The results found from the quantum LL model are compared to the order $\tilde{\lambda}^2/J$ gauge-theory and string theory results in sect. 3.5 and complete agreement is found.

In section 4 we compute the Hamiltonian for the quantum fluctuations directly from the spin chain. Here we find that the Hamiltonian is quartic and automatically normal ordered, and that one can obtain the $1/J$ corrections which are consistent with the Bethe ansatz. However, we also encounter a subtlety in that the eigenstates of the quadratic piece of the Hamiltonian are not precisely in the Hilbert space. Instead, in order to develop perturbation theory, it is necessary to do the perturbative expansion around states which are not precisely eigenstates of the quadratic Hamiltonian. However, we then can perform a similarity transformation on the Hamiltonian, such that the transformed Hamiltonian will have interaction terms of all orders, but the states will have the usual Fock space form. The advantage of building the Hamiltonian this way is that there are no ambiguities about normal ordering or regularization.

In section 5 we generalize the LL computation of the $\tilde{\lambda}/J$ correction to the BMN spectrum to the case of the $\beta$-deformed version of the AdS/CFT [39, 40, 41] (for real deformation parameter $\beta$). We demonstrate that the corresponding $SU(2)_{\beta}$ anisotropic (XXZ) LL model [40] gives the same $1/J$ correction to the analog of the BMN spectrum.
as follows directly from the exact spin chain Bethe ansatz equations of ref. [40]. This new non-trivial result for the leading $1/J$ correction in the $\beta$-deformed theory (which was not yet found directly from the corresponding superstring theory) may be used to check a consistency of the corresponding “string Bethe ansatz” for non-zero deformation parameter $\beta$ (cf. [31, 40, 41]).

In section 6 we use the quantum LL model approach to compute $1/J$ corrections to the energy of a circular $J_1 = J_2 = J/2$ rotating string solution [41, 42] which corresponds to the simplest static solitonic state of the $SU(2)$ LL model. We first review the result of [25] about matching the $\zeta$-function regularized expression for the leading 1-loop correction to the soliton energy and the corresponding finite-size correction [25, 42] to the thermodynamic-limit spin chain result, which is also equal to the leading term in the exact 1-loop string-theory expression found in [28, 29]. In sect. 6.2 we extend the computation to the next sub-subleading $\lambda/J$ order using second order quantum-mechanical perturbation theory for the LL Hamiltonian (the $\lambda/J$ correction was not yet computed from either the gauge-theory spin chain or the string). We point out the existence of the regularization ambiguity which remains to be fixed: it is no longer clear that the $\zeta$-function regularization should continue to correspond to either of the two microscopic theories – spin chain or superstring. In sect. 6.3 we include the “gauge-theory 2-loop” $\lambda^2$ term in the classical LL action and again compute the leading 1-loop correction to the classical energy. The result matches the second order term in the formal expansion of the exact finite one-loop string correction to the spinning string energy [28, 29], provided one uses the $\zeta$-function regularization to define the formal expression for the string-theory coefficient (this prescription is the one consistent to the given $\lambda^2$ order with the Bethe ansatz for a similar $SL(2)$ case [31, 43]).

In section 7 we attempt to repeat what was done in section 6 in the case of a more complicated solitonic LL solution representing a folded 2-spin $(J_1, J_2)$ string [41, 46] rotating in $S^5$. Here the LL fluctuation Lagrangian explicitly depends on (elliptic functions of) the spatial coordinate $\sigma$, and computing its spectrum exactly appears difficult. Instead, we use the “short string” expansion in the parameter $\alpha = J_2/J$ ($J = J_1 + J_2$) and compute the two coefficients in the small $\alpha$ expansion of the 1-loop correction to the folded string energy. The corresponding results for both the string or the spin chain remain to be obtained, and we expect them to match the result of the quantum LL computation.

In Appendix A we compute the energy of $M$-impurity near-BMN state up to $1/J^2$ order in the $SU(2)$ sector directly from the Bethe ansatz. In Appendix B we give some technical details for the evaluation of sums in sect. 6.2. Appendix C contains a computation of the numerical coefficient of the $\alpha^2$ term in the $1/J$ correction to the folded string energy. In Appendix D we present the results of the similar computation for the $1/J$ correction to the energy of $(S, J)$ folded string in the $SL(2)$ sector.

There are a number of obvious open problems, including the range of applicability of the quantum LL model and the choice of regularization. There are several computations
similar to the ones described in this paper that would be useful to carry out. It would be interesting to repeat the computation of the $1/J^2$ correction to the energy of circular string in section 5 for a similar circular $(S,J)$ solution \cite{20} in the $SL(2)$ sector. This latter solution is stable and thus the result could be consistently compared to the Bethe ansatz one for the sub-subleading correction which should follow from a generalization of the analysis of the 1$/J$ correction in \cite{25,12,34}.

It is possible to repeat similar computations in the $SU(3)$ sector, and compare the results with string theory \cite{38} and gauge theory \cite{45}. Another sector to consider is the $SU(1|1)$ sector \cite{24,46} where results for 1$/J$ corrections should be easier to obtain. One could also find the quantum LL corrections to the fluctuations near the non-trivial $(J,J,J)$ vacuum of the $SU(3)_{\beta}$ sector of the $\beta$-deformed version of AdS/CFT \cite{11}. Another computation worth doing is for the $\tilde{\lambda}^2/J^2$ corrections to BMN states. This can be done on one hand by using quantum LL, and on the other hand by using Bethe ansatz. The two computations are expected to agree, as one expects full agreement between the gauge and the string theory up to two loops (i.e. at orders $\lambda$ and $\lambda^2$).

2 Landau-Lifshitz action in the $SU(2)$ sector

Let us start with recalling the derivation of the effective Landau-Lifshitz action on both the gauge theory (spin chain) and the string theory sides \cite{14,15,10}.

2.1 LL action from gauge theory

The planar 1-loop dilatation operator of the $\mathcal{N} = 4$ SYM theory coincides with the Hamiltonian of the ferromagnetic Heisenberg XXX$_{1/2}$ model \cite{5} ($\lambda = g_{YM}^2 N$)

$$H = \frac{\lambda}{(4\pi)^2} \sum_{l=1}^{J} (I - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}) . \quad (2.1)$$

To describe a subsector of eigenstates that correspond to “semiclassical” low-energy part of the spectrum it is useful to use the coherent states which are products of spin coherent states at each site with the characteristic property $\langle \vec{n} | \vec{\sigma} | \vec{n} \rangle = \vec{n}$, $\vec{n}^2 = 1$. In general, one can rewrite the usual phase space path integral as an integral over the overcomplete set of coherent states:

$$Z = \int [dU] \ e^{iS[U]} , \quad S = \int dt \left( \langle U | i \frac{d}{dt} | U \rangle - \langle U | H | U \rangle \right) . \quad (2.2)$$

The first (“Wess-Zumino”) term in the action $\sim iU^* \frac{d}{dt} U$ is the analog of the usual $\dot{p}\dot{q}$ term in the phase-space action. Applying this to the case of the Heisenberg spin chain Hamiltonian (2.1) one ends up with with the following action for the coherent state
variables $\vec{n}_l(t)$ at sites $l = 1, \ldots, J$ ($U^\dagger \vec{\sigma} U = \vec{n}$):

$$S = \int dt \sum_{l=1}^J \left[ \vec{C}(n_l) \cdot \partial_t \vec{n}_l - \frac{\lambda}{2(4\pi)^2} (\vec{n}_{l+1} - \vec{n}_l)^2 \right] . \quad (2.3)$$

Here $dC = \epsilon^{ijk} n_i dn_j \wedge dn_k$, i.e. $\vec{C}$ is a monopole potential on $S^2$. In local coordinates (at each site $l$) one has $\vec{n} = (\sin 2\psi \cos 2\varphi, \sin 2\psi \sin 2\varphi, \cos 2\psi)$, $\vec{C} \cdot d\vec{n} = \cos 2\psi \, d\varphi$. So far, no approximation was made. If we now consider the large $J$ limit and concentrate on low-energy excitations of the spin chain then $n_i$ should change slowly from site to site and it is natural to take the continuum limit by introducing the 2-d field $\vec{n}(t, \sigma) = \{\vec{n}(t, \frac{2\pi}{J} l)\}, l = 1, \ldots, J$. Then the action becomes ($\partial_1 = \partial_\sigma$)

$$S = J \int dt \int_0^{2\pi} d\sigma \left[ \vec{C} \cdot \partial_t \vec{n} - \frac{1}{8} \tilde{\lambda} (\partial_1 \vec{n})^2 + \ldots \right] , \quad \tilde{\lambda} \equiv \frac{\lambda}{J^2} , \quad (2.4)$$

where dots stand for higher derivative terms suppressed by $1/J$. The leading correction scales as $\frac{1}{J^2}(\partial_1^2 n)^2$. Indeed,

$$\vec{n}_{l+1} - \vec{n}_l = \frac{2\pi}{J} \partial_1 \vec{n} + \frac{1}{2} \left( \frac{2\pi}{J} \right)^2 \partial_1^2 \vec{n} + \frac{1}{6} \left( \frac{2\pi}{J} \right)^3 \partial_1^3 \vec{n} + \ldots ,$$

i.e.

$$\frac{\lambda}{2(4\pi)^2} \sum_{l=1}^J (\vec{n}_{l+1} - \vec{n}_l)^2 \to \frac{\lambda}{J} \left[ (\partial_1 \vec{n})^2 - \frac{\pi^2}{3J^2}(\partial_1^2 \vec{n})^2 + \ldots \right] . \quad (2.6)$$

Observing that $J$ appears in front of the action and thus plays the role of the inverse Planck constant, we may expect that the classical Landau-Lifshitz (LL) action (2.4) with the equations of motion

$$\partial_t n_i = \frac{1}{2} \tilde{\lambda} \epsilon_{ijk} n_j \partial_1^2 n_k$$

(2.7)

should be describing the low-energy part of the spectrum with energies scaling as $J\lambda$ to leading order in the quantum $1/J$ expansion. Since the first subleading term in (2.6) scales as $1/J^2$ one may expect that order $1/J$ corrections to the energies of the corresponding low-energy states can be found by quantizing just the continuous LL action. However, to capture $1/J^2$ and higher order corrections to the energies as described by the discrete Heisenberg Hamiltonian one would need to add higher-derivative terms omitted in taking the continuum limit. This will be discussed in detail below.

As we shall see, extending the observation in [25], the semiclassical quantization of the LL action does allow one to reproduce the $1/J$ corrections (as found, e.g., from the discrete Bethe ansatz) in a very simple way provided one uses an appropriate UV regularization. As usual in an effective field theory approach, the underlying microscopic UV finite theory (spin chain) dictates a particular choice of a regularization. There is no a priori choice of this regularization within the continuous effective theory, unless one uses some additional conditions like that energies of some BPS states should not be changed by $1/J$ corrections. We shall provide examples of this in what follows.
2.2 LL action from string theory

The same LL action appears [14, 15] as an effective action on the string theory side too where one also considers a (different) semiclassical limit. One concentrates on a sector of states for which large $J$ expansion is equivalent to quantum string (inverse string tension) expansion. One first takes $\lambda$ large, or, equivalently (for given sector of states), $J$ large to suppress quantum corrections and then expands the classical string action in the inverse of the effective semiclassical parameter $\tilde{\lambda} \equiv \frac{1}{J}$. The derivation goes through the following steps [14, 15, 17]: (i) one isolates a “fast” coordinate $\alpha$ whose momentum $p_\alpha$ is large for given class of string configurations; (ii) one gauge-fixes $t = \tau$ and $p_\alpha = J$ (or $\tilde{\alpha} = J\sigma$ where $\tilde{\alpha}$ is “T-dual” to $\alpha$); (iii) one expands the action in derivatives of “slow” coordinates, or equivalently, in $\sqrt{\tilde{\lambda}} = \frac{1}{J}$. In the $SU(2)$ sector of string states carrying two large spins in $S^5$, with string motions restricted to $S^3$ part of $S^5$, the relevant part of the $AdS_5 \times S^5$ metric is $ds^2 = -dt^2 + dX_i dX_i^*$, with $X_i X_i^* = 1$.

Setting $X_1 = X_1 + iX_2 = U_1 e^{i\alpha}$, $X_2 = X_3 + iX_4 = U_2 e^{i\alpha}$, $U_a U_a^* = 1$, (2.8)

we identify $\alpha$ as a coordinate associated to the total spin in the two planes and $U_i$ as “slow” coordinates determining the “transverse” string profile. Then

$$dX_a dX_a^* = (d\alpha + C)^2 + DU_a DU_a^*,$$

$$C = -iU_a^* dU_a,$$

$$DU_a = dU_a - iCU_a.$$ (2.9)

Introducing $\bar{n} = U^\dagger \bar{\sigma} U$, $U = (U_1, U_2)$ we get

$$dX_a dX_a^* = (D\alpha)^2 + \frac{1}{4} (d\bar{n})^2,$$

$$D\alpha = d\alpha + C(n).$$ (2.10)

Writing the resulting string sigma model action in phase space form, one may fix the gauge $t = \tau$, $p_\alpha = \text{const} = J$. Making the key assumption that the evolution of $U_a$ in $t$ is slow, i.e. the time derivatives are suppressed (which can be implemented by rescaling $t$ by $\tilde{\lambda}$ and expanding in powers of $\tilde{\lambda}$), we find, to the leading order in $\tilde{\lambda}$,

$$S = J \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} L,$$

$$L = -iU_a^* \partial_t U_a - \frac{1}{2} \tilde{\lambda} |D_1 U_a|^2 + O(\tilde{\lambda}^3).$$ (2.11)

This becomes the same as the $CP^1$ Landau-Lifshitz action (2.4) when written in terms of $\bar{n}$. The agreement between the low-energy effective actions on the spin chain and one the string side implies the matching of energies of the coherent states representing configurations with two large spins (and also the matching of near-by fluctuations).

This agreement between the effective LL actions extends also to the next $\tilde{\lambda}^2$ order [15]. To get the $\tilde{\lambda}^2$ term in (2.11) one is to do a field redefinition to trade time derivatives for spatial ones; the result is a generalization of (2.4)

$$L = \bar{C} \cdot \partial_t \bar{n} - \frac{\tilde{\lambda}}{8} (\partial_1 n_i)^2 + \frac{\tilde{\lambda}^2}{32} \left[ (\partial_1^2 n_i)^2 - \frac{3}{4} (\partial_1 n_i)^4 \right] + O(\tilde{\lambda}^3).$$ (2.12)
The same action is found on the spin chain side by adding to the dilatation operator \( \mathcal{O}_2 \) the 2-loop term \([47]\), \( H_2 = \frac{\lambda^2}{(4\pi)^2} \sum_{i=1}^{J} (\vec{\sigma}_i \cdot \vec{\sigma}_{i+1} - \vec{\sigma}_i \cdot \vec{\sigma}_{i+2}) \), taking coherent state expectation value and also including a quantum correction \([15]\).

This agreement between the effective actions is rather remarkable, given that the limits taken on the two sides of the duality are different \([48, 49]\): on gauge theory side we first take \( \lambda \) small and then expand in large \( J \) isolating contributions depending on \( \tilde{\lambda} = \frac{\lambda}{J^2} \), while on string side we first take \( J \) large with \( \tilde{\lambda} = \) fixed to suppress quantum corrections and then expanded in \( \tilde{\lambda} \).

A natural question is if this matching continues at subleading \( 1/J \) order, i.e. if corrections to thermodynamic limit on the 1-loop spin chain side are the same as the leading 1-loop corrections on the string theory side to the same linear order in \( \tilde{\lambda} \). This matching was found on several explicit examples: near BPS (BMN) states \([37, 38]\) and circular strings \([28, 29, 25]\) (see also \([30, 42, 50]\)). A simple way to understand why this happens was suggested in \([25]\) by computing the leading quantum correction to the energy of circular string state directly at the level of the effective LL model. From spin chain perspective, quantizing LL action should indeed correctly capture the leading \( 1/J \) correction to the energy, provided one uses an appropriate regularization equivalent to the one built into the discrete spin chain (Bethe ansatz) computation.

On the string theory side, the full 1-loop correction to the energy \([27, 28, 29]\) contains the contribution of not only the 2 “transverse” fluctuations described by the LL action but also 2 other \( S^5 \) fluctuations outside of \( S^3 \), 4 \( AdS_5 \) fluctuations and also of the fermionic fluctuations that are crucial for finiteness of the result. Remarkably, it was observed in \([25]\) that the leading \( 1/J \) (order \( \tilde{\lambda} \)) contribution of “external” bosonic and fermionic fluctuations has a trivial “counterterm”-type form, i.e. the full string result can be correctly reproduced by quantizing only the two “internal” LL fluctuations and using a specific \( (\zeta\text{-function}) \) regularization.

Thus, as on the spin chain side, the usual effective field theory ideology seems to apply: the full string theory can be interpreted as a UV finite microscopic theory which contains (when expanded near a particular circular string background with \( J \) large) “light” and “heavy” fluctuation modes, with the “light” modes described by the effective LL action, and the role of the “heavy” modes being to provide a regularization prescription for the quantum effective field theory. While the two microscopic theories – the spin chain and the string theory – are very different, both lead to the same LL action in the classical limit, and, moreover, to the same quantum version of it with the same regularization prescription.

Below we would like to explore other examples when this matching of the quantum corrections continues to happen. One motivation for starting directly with a quantum LL Hamiltonian is technical simplicity: both spin chain and full string theory computations of subleading corrections are rather involved, while the quantum LL framework provides simple framework for model computations and checking conjectures about structure of quantum corrections.
To prepare for the discussion of particular cases let us first present the explicit form of the LL Hamiltonian and its quantization.

2.3 Canonical structure of the LL Lagrangian

Let us start with rewriting the LL Lagrangian (2.4) or (2.11) in terms of two independent fields. Solving the constraint \( n_i n_i = 1 \) as \( n_3 = \sqrt{1 - n_1^2 - n_2^2} \) we get the following \( SO(2) \) invariant expression for the Lagrangian in terms of \( n_1 \) and \( n_2 \) \((a, b = 1, 2; n^2 = n_a n_a)\)

\[
L = h^2(n) \epsilon_{ab} \dot{n}_a n_b - H(n) ,
\]

\[
h^2(n) = \frac{1 - \sqrt{1 - n^2}}{2n^2} = \frac{1}{4} + \frac{1}{16} n^2 + \frac{1}{32} n^4 + ... ,
\]

\[
H(n) = \frac{\tilde{\lambda}}{8} \left[ n_a^2 + \frac{1}{1 - n^2} (n_a n'_a)^2 \right] ,
\]

where we use dot and prime for time and space derivatives. We have added and subtracted a total derivative term \( \epsilon_{ab} \dot{n}_a n_b = \frac{1}{2} \frac{\partial}{\partial \tau} \left( \arctan \frac{2n_a}{n_3} \right) \) to make the function \( h \) have regular expansion near \( n_a = 0 \). Thus (2.13) may be interpreted as a phase-space Lagrangian with, say, \( n_1 \) being a coordinate and \( n_2 \) related to its momentum.

In what follows we shall expand the LL action near particular solutions and quantize. To simplify the quantization it is useful to put \( L \) into the standard canonical form. This can be done by the field redefinition \( n_a \rightarrow z_a \) (which is regular at the origin)

\[
z_a = h(n) n_a , \quad n_a = 2\sqrt{1 - z^2} z_a .
\]

Then we get \((z^2 = z_a z_a)\)

\[
L = \epsilon_{ab} \dot{z}_a z_b - H(z) , \quad H(z) = \frac{\tilde{\lambda}}{2} \left[ (1 - z^2) z'^2_a + \frac{2 - z^2}{1 - z^2} (z_a z'_a)^2 \right] .
\]

Note that the LL Hamiltonian \( H(n) \) or \( H(z) \) is the same as for a sigma model on a sphere \( S^2 \) written in different coordinates.

Having the Lagrangian in the standard \( L = p \dot{q} - H(p, q) \) form the quantization is straightforward: we are to promote \( z_a \) to operators, impose the canonical commutation relation \([z_1(t, \sigma), z_2(t, \sigma')] = i\pi \delta(\sigma - \sigma')\) and decide how to define the quantum Hamiltonian \( H(z) \), i.e. how to order the “coordinate” and “momentum” operators in it. We will discuss this on explicit examples below.

Let us mention also another explicit parametrization of the LL Lagrangian in terms of angles \( \psi, \varphi \). If we set

\[
U_1 = \cos \psi e^{i \varphi} , \quad U_2 = \sin \psi e^{-i \varphi} , \quad \vec{n} = (\sin 2\psi \cos 2\varphi, \sin 2\psi \sin 2\varphi, \cos 2\psi) ,
\]

\[
\tilde{n} = (2\psi \cos 2\varphi) , \quad n = \frac{\gamma}{2} , \quad \alpha = \frac{\gamma}{2} .
\]

\[\text{In terms of global angular coordinates of } S^5 \text{ with the metric } ds^2 = dt^2 + d\gamma^2 + \cos^2 \gamma \, d\varphi_3^2 + \sin^2 \gamma \left( d\psi^2 + \cos^2 \psi \, d\varphi_1^2 + \sin^2 \psi \, d\varphi_2^2 \right) \text{ we have } \varphi = \frac{\varphi_1 - \varphi_2}{2}, \text{ and } \alpha = \frac{\varphi_1 + \varphi_2}{2}.
\]
then
\[ L = \cos 2\psi \frac{\dot{\phi}}{\sin^2 2\phi} + \frac{\tilde{\lambda}}{2} \left( \psi'^2 + \sin^2 2\psi \varphi'^2 \right). \] (2.19)

Setting
\[ \xi = \frac{1}{2} \cos 2\psi \] (2.20)
we get
\[ L = 2\xi' \frac{\dot{\phi}}{\sin^2 2\phi} + \frac{\tilde{\lambda}}{2} \left[ \frac{\xi'^2}{1 - 4\xi^2} + (1 - 4\xi^2)\varphi'^2 \right]. \] (2.21)

This form of the LL Lagrangian is useful for expansion around any particular solution with \( \psi \neq 0 \); near the solution with \( \psi = 0 \) or \( 1 - 4\xi^2 = 0 \) we get the usual polar-angle type of singularity and should use instead the “cartesian” form of \( L \) in (2.17) which is regular at the origin, i.e. near \( n_a = 0 \).

### 3 Quantization near the BPS vacuum: 1/J and 1/J^2 corrections to BMN spectrum

#### 3.1 Generalities and BMN spectrum

Let us now try to reproduce 1/J corrections to the leading terms in the BMN spectrum of fluctuations near the vacuum solution
\[ X_1 = e^{iJt}, \quad X_2 = 0, \quad \psi = 0, \quad \varphi = 0, \quad \text{i.e.} \quad \vec{n} = (0, 0, 1), \] (3.1)
by quantizing the above LL action. These corrections can be found from the Bethe ansatz on the spin chain \([5, 51]\) or from direct superstring quantization \([37, 38]\), but the derivation from the LL action turns out to be much simpler.

Expanding near this vacuum corresponds to expansion near \( n_a = 0 \) in (2.13) or \( z_a = 0 \) in (2.17). Observing that the factor \( J \) in front of the LL action (2.4), (2.11) plays the role of the inverse Planck constant, it is natural to rescale \( z_a \) as
\[ z_1 = \frac{1}{\sqrt{J}} f, \quad z_2 = \frac{1}{\sqrt{J}} g, \] (3.2)
so that powers of 1/J will play the role of coupling constants in the non-linear LL Hamiltonian for the fluctuations. To sixth order in the fluctuation fields \( f, g \) we get
\[ S = \int dt \int_0^{2\pi} d\sigma \frac{2\pi}{2\pi} \left( 2\dot{f}g - H \right), \quad H = H_2 + H_4 + H_6 + \ldots, \] (3.3)
\[ H_2 = \frac{1}{2} \tilde{\lambda}(f^2 + g^2), \] (3.4)
\[ H_4 = \frac{\tilde{\lambda}}{2J} \left[ 2(ff' + gg')^2 - (f^2 + g^2)(f'^2 + g'^2) \right], \quad (3.5) \]

\[ H_6 = \frac{\tilde{\lambda}}{2J^2} (f^2 + g^2)(ff' + gg')^2. \quad (3.6) \]

Let us first consider the quadratic approximation. The linearized equations of motion for fluctuations are

\[ \dot{f} = -\frac{\tilde{\lambda}}{2} g'', \quad \dot{g} = \frac{\tilde{\lambda}}{2} f'', \quad (3.7) \]

and their solution may be written as \((f, g)\) are real

\[ f(t, \sigma) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (a_n e^{-i\omega_n t + in\sigma} + a_n^\dagger e^{i\omega_n t - in\sigma}), \quad \omega_n = \frac{1}{2} \tilde{\lambda} n^2, \quad (3.8) \]

\[ g(t, \sigma) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-ia_n e^{-i\omega_n t + in\sigma} + ia_n^\dagger e^{i\omega_n t - in\sigma}). \quad (3.9) \]

For each solution of LL equations of motion one needs also to impose an extra constraint that the total momentum in \(\sigma\)-direction is zero \([14, 17]\)

\[ P = -i \int_0^{2\pi} d\sigma U^*_a U_a = \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos 2\psi \varphi' = 0. \quad (3.10) \]

Expanding near the vacuum, we obtain the constraint on fluctuations

\[ P = 2 \int_0^{2\pi} \frac{d\sigma}{2\pi} f'g = - \sum_{n=-\infty}^{\infty} na_n^* a_n = 0. \quad (3.11) \]

Upon quantization \([3.7]\) become the equations of motion for the operators \(f, g\)

\[ \dot{f} = i[H_2, f], \quad \dot{g} = i[H_2, g], \quad H_2 \equiv \int_0^{2\pi} \frac{d\sigma}{2\pi} H_2, \quad (3.12) \]

provided we use the canonical commutation relations

\[ [f(t, \sigma), f(t, \sigma')] = 0, \quad [g(t, \sigma), g(t, \sigma')] = 0, \quad [f(t, \sigma), g(t, \sigma')] = i\pi \delta(\sigma - \sigma'). \quad (3.13) \]

Then the coefficients in \([3.8], [3.9]\) satisfy

\[ [a_n, a_m^\dagger] = \delta_{n-m}, \quad (3.14) \]

i.e. \(a_n, a_n^\dagger\) can be interpreted as annihilation and creation operators, with the vacuum state \(|0\rangle\) defined by \(a_n|0\rangle = 0, \ n = 0, \pm 1, \ldots\). A general oscillator state is

\[ |\Psi> = \prod_{n=-\infty}^{\infty} \frac{(a_n^\dagger)^{k_n}}{\sqrt{k_n!}} |0\rangle. \quad (3.15) \]
The integrated Hamiltonian $\tilde{H}_2$ becomes

$$\tilde{H}_2 = \frac{\lambda}{4} \sum_{n=-\infty}^{\infty} n^2 (a_n a_n^\dagger + a_n^\dagger a_n) = \frac{\lambda}{2} \sum_{n=-\infty}^{\infty} n^2 a_n^\dagger a_n + e_0 , \quad (3.16)$$

$$e_0 = \frac{\lambda}{2} \sum_{n=-\infty}^{\infty} n^2 . \quad (3.17)$$

At this point we should add the requirement that the vacuum energy should be zero:

$$e_0 = 0 .$$

We know that the BMN vacuum is a BPS state in both gauge theory and string theory. This amounts to normal ordering prescription for the quadratic Hamiltonian or use of a regularization (e.g., the $\zeta$-function one) in which $e_0$ is set to zero. We stress that this condition is an additional constraint one needs to impose to make quantum LL theory consistent with “microscopic” spin chain or string theory. Similar conditions will be needed at higher orders to fix the regularization ambiguity present in quantum LL theory.

The momentum condition (3.11) becomes the constraint on physical states:

$$\sum_{n=-\infty}^{\infty} a_n^\dagger a_n |\Psi\rangle = 0 , \quad \sum_{n=-\infty}^{\infty} n k_n = 0 . \quad (3.18)$$

Let us consider $M$-impurity states as oscillator states with $k_n = 1$:

$$|M\rangle = a_n^\dagger ... a_M^\dagger |0\rangle . \quad (3.19)$$

For simplicity we shall consider states with all $n_j$ being different; computations for more general states with several equal $n_j$ are similar, at least for first order corrections in $1/J$.

The zero-momentum condition (3.18) gives

$$\sum_{j=1}^{M} n_j = 0 , \quad (3.20)$$

which is also the condition on BMN states present in both string and gauge theory.

The leading term in the energy of an $M$-impurity state is then

$$\langle M | \tilde{H}_2 | M \rangle = \frac{\lambda}{2} \sum_{j=1}^{M} n_j^2 , \quad (3.21)$$

which is the standard magnon energy on the spin chain side or the leading term in the BMN excitation energy on the string side.
Let us also compute the difference of spins

$$J_1 - J_2 = \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos 2\psi \hat{\alpha} \quad (3.22)$$
onumber

on the fluctuations around the vacuum $\psi = 0, \varphi = 0$. Here $\alpha = Jt$ is the “fast” coordinate, and $\cos 2\psi = \sqrt{1 - n_a^2} = 1 - z_a^2$ so that (to all orders in fluctuations)

$$J_1 - J_2 = J - 2 \sum_{n=-\infty}^{\infty} a_n^\dagger a_n , \quad (3.23)$$

where we again assumed normal ordering.\(^{5}\) Applied to $M$-impurity state the above relation gives $J_1 - J_2 = J - 2M$. Since $J_1 + J_2 = J$ we have

$$J_1 = J - M , \quad J_2 = M . \quad (3.24)$$

The corresponding gauge-theory states are $\text{Tr}(\Phi_1^J \Phi_2^J) + \ldots$, and $J$ plays the role of the length of spin chain and $M$ is the number of magnons.

### 3.2 $\tilde{\lambda}/J$ correction

To compute the $1/J$ correction to the energy of $M$-impurity state one needs to include the quartic term in the Hamiltonian \((3.5)\) or in $\tilde{H}_4 \equiv \int_0^{2\pi} \frac{d\sigma}{2\pi} H_4$ and use the standard quantum mechanical perturbation theory. Written in terms of the creation and annihilation operators it has the form

$$\tilde{H}_4 = \frac{\tilde{\lambda}}{4J} \sum_{n,m,k,l} nk \left[ \frac{1}{2} \delta_{n-k+m-l} (a_n a_k^\dagger a_m^\dagger a_l^\dagger + a_n^\dagger a_k a_m^\dagger a_l) 
- \frac{1}{2} \delta_{n-m+k+l} (a_n a_m^\dagger a_k^\dagger a_l^\dagger + a_n^\dagger a_m a_k^\dagger a_l^\dagger) 
- \delta_{n-m-k-l} (a_n^\dagger a_m^\dagger a_k a_l + a_n a_m a_k^\dagger a_l^\dagger) \right]. \quad (3.25)$$

Here we have omitted the time dependent phases ($a_n \to e^{-i\omega t} a_n$) in the interacting Hamiltonian $\tilde{H}_{\text{int}} = \tilde{H}_4 + \tilde{H}_6 + \ldots$ since they can be removed by a unitary transformation with the quadratic Hamiltonian $\tilde{H}_2$. Here and below the summations over $n, m, \ldots$ are from $-\infty$ to $\infty$. One should decide about the regularization, i.e. about the ordering of the operators $a_n$ and $a_n^\dagger$. The natural choice is again the normal ordering.\(^{6}\) The part

\(^{5}\)Here the normal ordering is again equivalent to the $\zeta$-function regularization with $\sum_{n=-\infty}^{\infty} n^s = 0$, $s = 0, 1, 2, \ldots$.

\(^{6}\)It can be justified by the requirement that both the vacuum state $|0\rangle$ and the 1-impurity state $(a_n^\dagger + a_{-n}^\dagger)|0\rangle$ should not receive subleading corrections to their energies.
of the Hamiltonian relevant for computing its expectation value in a state satisfying
the momentum constraint is then
\[
\bar{H}_4 = -\tilde{\lambda} \sum_{n,m} nma_n^†a_m^†a_ma_n .
\] (3.26)

Let us note that using the ζ-function regularization we would instead obtain from
(3.25)
\[
\bar{H}_4' = -\tilde{\lambda} J \left( \sum_{n,m} nma_n^†a_m^†a_ma_n + \sum_n n^2 a_n^†a_n \right) .
\] (3.27)

Thus at quartic oscillator order the ζ-function regularization is not equivalent to normal
ordering; we would still need then to discard the \( \sum n^2 a_n^†a_n \) term that would shift the
energy of the 1-impurity state. From full string theory computation perspective such
term should be cancelled by the contribution of fermions (cf. [37, 38]).

Using (3.26) the correction to the energy of \( M \)-impurity state is found to be\(^7\)
\[
\langle M|\bar{H}_4|M \rangle = \frac{\tilde{\lambda}}{J} \sum_{j=1}^{M} n_j^2 .
\] (3.28)

3.3 \( \tilde{\lambda}/J^2 \) correction

Let us now consider the \( O(1/J^2) \) correction to the energy of \( M \)-impurity state. Within
the standard quantum-mechanical perturbation theory for the continuum LL Hamilton-
ian it should be given by the the sum of the first order perturbation term for \( \bar{H}_6 \) in
(3.6) and the second order perturbation term for \( \bar{H}_4 \) in (3.26). The latter is
\[
\langle M|(\bar{H}_4)^{(2)}|M \rangle = \sum_{M \neq M'} \frac{\langle M|\bar{H}_4|M' \rangle \langle M'|\bar{H}_4|M \rangle}{E_M - E_{M'}}
\] (3.29)

where \( |M' \rangle \) is any possible intermediate state, and \( |M \rangle = a_1^†...a_{M'}^†|0 \rangle \).

Since \( \bar{H}_4 \) in (3.26) contains only terms of the form \((a_j^†)^2a_j^2\), the only non-trivial
intermediate states can be the \( M' \)-particle ones of the form \( a_{n_1}^†...a_{n_{M'}}^†|0 \rangle \) with \( M' = M \).

Then in order for the matrix element \( \langle M|\bar{H}_4|a_{n_1}^†...a_{n_{M'}}^†|0 \rangle \) to be non-zero, there should
be a \( j \) and \( k \) such that \( n_j' = n_j + q \) and \( n_k' = n_k - q \), with all other \( n_i' = n_i \), \( i \neq j, k \). In

\(^7\)There is a degeneracy since there are states with \( \sum_{j=1}^{M} n_j = 0 \) that have the same energy
\( \frac{\tilde{\lambda}}{J} \sum_{j=1}^{M} n_j^2 \). The sets of states \( \{n_{i=1,...,M} \} \) and \( \{-n_{i=1,...,M} \} \) satisfying this condition are permuta-
tions of one another, but since in this paper we assume that \( n_i \) are all different, these are the same
states. The only remaining degeneracy is a double degeneracy of states \( \{n \} \) and \( \{-n \} \). One can see
that the Hamiltonian expectation value matrix is diagonal. This remains true also when computing
next order corrections.
order for $|M\rangle$ to be distinct from $|M'\rangle$, we require that $0 \neq q \neq n_k - n_j$. With these conditions, we then find that

\[
\langle M|\tilde{H}_4|M'\rangle = -\frac{\tilde{\lambda}}{J} [n_j n_k + (n_j + q)(n_k - q)] = -\frac{\tilde{\lambda}}{J} [2n_j n_k + q(n_k - n_j - q)] , \tag{3.30}
\]

if $n_k \neq n_j$ and where $n_j + q$ and $n_k - q$ are not equal to one of the other $n_i$’s. The energy difference is

\[
E_M - E_{M'} = \frac{\tilde{\lambda}}{2} [n_j^2 + n_k^2 - (n_j + q)^2 - (n_k - q)^2] = \tilde{\lambda} q(n_k - n_j - q) . \tag{3.31}
\]

If $n_j + q = n_l$, and so $|M'\rangle$ has two impurities with the same momenta, then the matrix element is

\[
\langle M|\tilde{H}_4|M'\rangle = -\frac{\sqrt{2} \tilde{\lambda}}{J} [n_j n_k + n_l(n_j + n_k - n_l)] , \tag{3.32}
\]

and the energy difference is

\[
E_M - E_{M'} = \tilde{\lambda}(n_l - n_j)(n_k - n_l) . \tag{3.33}
\]

Hence, we find that

\[
\langle M| \langle \tilde{H}_4 \rangle^{(2)} |M\rangle \tag{3.34}
\]

\[
= \frac{\tilde{\lambda}}{J^2} \sum_{j<k} \left( \frac{1}{2} \sum_{q=-\infty}^{\infty} \frac{[2n_j n_k + q(n_k - n_j - q)]^2}{q(n_k - n_j - q)} + \sum_{\substack{l \neq j \\ell \neq k}} \frac{[n_j n_k + n_l(n_j + n_k - n_l)]^2}{(n_l - n_j)(n_k - n_l)} \right)
\]

where the factor of $\frac{1}{2}$ compensates for a double counting over $M'$ and the sum over $l$ compensates for a missing contribution from $M'$ states with oscillators with the same momenta.

The sum over $q$ in (3.34) is divergent and needs to be regularized. To this end, we can write the sum over $q$ as

\[
\sum_{q=-\infty}^{\infty} \frac{(2n_j n_k + q(n_k - n_j - q))^2}{q(n_k - n_j - q)}
\]

\[
= \sum_{q=-\infty}^{\infty} \left[ q(n_k - n_j - q) + 4n_j n_k + \frac{4n_j^2 n_k^2}{q(n_k - n_j - q)} \right] . \tag{3.35}
\]

Using the $\zeta$-function regularization, the first term inside the square brackets gives zero after summing over $q$, while the second term gives

\[
\sum_{q=-\infty}^{\infty} 4n_j n_k = 4n_j n_k [2\zeta(0) - 1] = -8n_j n_k . \tag{3.36}
\]
The last term inside the brackets gives a finite contribution\(^8\)

$$\sum_{q=-\infty}^{\infty} \sum_{0 \neq q \neq n_k - n_j} \frac{4n_j^2n_k^2}{q(n_k - n_j - q)} = \frac{4n_j^2n_k^2}{n_k - n_j} \sum_{q=-\infty}^{\infty} \left( \frac{1}{q} - \frac{1}{q - n_k + n_j} \right) = -\frac{8n_j^2n_k^2}{(n_k - n_j)^2}. \tag{3.37}$$

The sum over \(l\) in (3.34) can be symmetrized with the sum over \(j\) and \(k\), leading to the relation

$$\sum_{j<k} \sum_{l \neq k} \frac{(n_jn_k + n_l(n_j + n_k - n_l))^2}{(n_l - n_j)(n_k - n_l)} = -\sum_{j<k<l} (n_j^2 + n_k^2 + n_l^2 - n_jn_k - n_kn_l - n_ln_j)$$

$$= -\frac{(M-1)(M-2)}{2} \sum_j n_j^2 + (M-2) \sum_{j<k} n_jn_k = -\frac{M(M-2)}{2} \sum_j n_j^2, \tag{3.38}$$

where we made use of the zero-momentum constraint in (3.20). Putting (3.36), (3.37) and (3.38) into (3.34), we find that

$$\langle M|\langle \hat{H}_4 \rangle^{(2)}|M \rangle = -\frac{2\tilde{\lambda}}{J^2} \sum_{j \neq k} \frac{n_jn_k(n_j^2 - n_jn_k + n_k^2)}{(n_j - n_k)^2} - \frac{\tilde{\lambda}M(M-2)}{2J^2} \sum_j n_j^2. \tag{3.39}$$

Next, let us consider the expectation value of the six-order term in the LL Hamiltonian (3.6). Expressing it in terms of creation and annihilation operators using (3.8), (3.9) we face the question of regularization or how to order the operators \(a_n\) and \(a_n^\dagger\). It is no longer obvious that normal ordering is the right prescription: one could also keep the terms like \(a^\dagger a^\dagger a^\dagger a^\dagger\) and still satisfy the requirement that the energy of the vacuum and 1-impurity states is not shifted. For example, if we use the \(\zeta\)-function prescription we get

$$\hat{H}_6 = -\frac{\tilde{\lambda}}{2J^2} \left[ \sum_{n,m,k} (nm - n^2)a_n^\dagger a_m^\dagger a_k^\dagger a_na_ma_k + c \sum_{n,m} 2(nm - n^2)a_n^\dagger a_m^\dagger a_na_m \right], \tag{3.40}$$

where \(c = 1\). On the other hand, using normal ordering would give \(c = 0\), i.e. the second \(a^\dagger a^\dagger a^\dagger a^\dagger\) term would be absent. It is unclear \textit{a priori} which should be the right value of \(c\), i.e. if there are additional global conditions like protection of the energy of BPS states that should be imposed to match either the gauge theory or the string theory results (which were not yet proven to be equal at this order). We will see below that \(c = 0\) (\textit{i.e.} normal ordering) reproduces the correct spin-chain result, but for the moment let us keep \(c\) arbitrary.

---

\(^8\)While we have assumed that \(n_j \neq n_k\), it is clear that we can still regularize the first term in (3.34) if \(n_j = n_k\) (although (3.35) will have additional symmetry factors). In this case, the expression in (3.37) would be $$-\frac{4\pi^2n_j^4}{3}.$$
For the $M$-impurity state we have

\begin{align}
\langle M | \sum_{n,m,k} n m a_n^\dagger a_m^\dagger a_n a_m a_k | M \rangle &= -(M - 2) \sum_{j=1}^{M} n_j^2 , \quad (3.41) \\
\langle M | \sum_{n,m,k} n^2 a_n^\dagger a_m^\dagger a_n a_m a_k | M \rangle &= (M - 1)(M - 2) \sum_{j=1}^{M} n_j^2 , \quad (3.42) \\
\langle M | \sum_{n,m} n m a_n^\dagger a_m^\dagger a_n a_m | M \rangle &= - \sum_{j=1}^{M} n_j^2 , \quad (3.43) \\
\langle M | \sum_{n,m} n^2 a_n^\dagger a_m^\dagger a_n a_m | M \rangle &= (M - 1) \sum_{j=1}^{M} n_j^2 , \quad (3.44)
\end{align}

where we used the momentum constraint in (3.20). Then we find the following expectation value

\begin{equation}
\langle M | \bar{H}_6 | M \rangle = \tilde{\lambda} \frac{2}{J^2} M(M - 2 + 2c) \sum_{j=1}^{M} n_j^2 . \quad (3.45)
\end{equation}

The sum of (3.39) and (3.45) is not yet the full result. The original spin chain is discrete, so accordingly the coherent state (LL) action in the continuum limit will include also higher derivative terms that are suppressed by powers of $J$. In particular, at order $\lambda/J^4$ there is a higher derivative term in (2.6)

\begin{equation}
\Delta L = \tilde{\lambda} \pi \frac{2}{24J^2} (\vec{n}^2)^2 . \quad (3.46)
\end{equation}

It should then be added as a quantum counterterm to the LL action in order to match the discrete spin chain result. Similar terms should also appear on the string-theory side, representing the effective contributions of other bosonic and fermionic modes that are not included directly in the LL action, although details of how such terms can arise are presently unclear.

Written in terms of the fields $z_a$ or rescaled fields $f, g$ (3.2) the leading higher-derivative correction to the LL Hamiltonian is simply quadratic

\begin{align}
\Delta H = -\Delta L = -\frac{\tilde{\lambda} \pi^2}{6J^2} (f''^2 + g''^2) + O\left(\frac{\tilde{\lambda}}{J^4}\right) , \\
\Delta \bar{H}_2 = -\frac{\tilde{\lambda} \pi^2}{6J^2} \sum_n n^4 a_n^\dagger a_n , \quad (3.47)
\end{align}

where we are again assuming normal ordering to avoid shifting the vacuum energy. Its expectation value on the $M$-impurity state is found to be

\begin{equation}
\langle M | \Delta \bar{H}_2 | M \rangle = -\frac{\tilde{\lambda} \pi^2}{6J^2} \sum_{j=1}^{M} n_j^4 . \quad (3.48)
\end{equation}
Combining the results of (3.39), (3.45) and (3.48), we find that the total $\tilde{\lambda}/J^2$ correction is

$$E_M^{(2)} = -\frac{\tilde{\lambda}\pi^2}{6J^2} \sum_{j=1}^{M} n_j^4 - \frac{2\tilde{\lambda}}{J^2} \sum_{j \neq k} n_j n_k (n_j^2 - n_j n_k + n_k^2) \frac{2\tilde{\lambda}}{J^2} \sum_{j=1}^{M} n_j^2 \sum_{j \neq k} (n_j - n_k)^2 + \frac{\tilde{\lambda}cM}{J^2} \sum_{j=1}^{M} n_j^2$$  \hspace{1cm} (3.49)

In Appendix A we will compute the $\tilde{\lambda}/J^2$ correction to the spin-chain energy using directly the Bethe ansatz; we will find that eq. (3.49) agrees with (A.11) if $c = 0$. This means that the normal ordering of the six-order interaction term $\bar{H}_6$ is the correct regularization for reproducing spin-chain results.

### 3.4 $\tilde{\lambda}^2/J$ correction

Let us now include the next order $\tilde{\lambda}^2$ term in the classical LL action (2.12) and compute the leading $1/J$ correction to the energy of oscillator states. Putting again the LL Lagrangian in the form (2.17) and rescaling the fields as in (3.2) we obtain the following fluctuation Lagrangian to quartic order

$$L = 2\dot{f}g - \frac{1}{2} \tilde{\lambda} \left( f'^2 + g'^2 + \frac{1}{J} \left[ 2(gf' - fg')(f^2 + g^2)(f^2 + g^2) \right] \right) + \frac{1}{8} \tilde{\lambda}^2 \left( f''^2 + g''^2 + \frac{1}{J} \left[ (f'^2 + g'^2)^2 + 4(ff'' + gg'')^2 \right] \right) + 8(ff'' + gg'')(f'^2 + g'^2) - (f^2 + g^2)(ff''' + gg''') \right) \right).$$  \hspace{1cm} (3.50)

The quadratic part of the Hamiltonian is

$$\bar{H}_2 = \frac{\tilde{\lambda}}{2} \int_0^{2\pi} \frac{d\sigma}{2\pi} (f'^2 + g'^2) - \frac{\tilde{\lambda}^2}{8} \int_0^{2\pi} \frac{d\sigma}{2\pi} (f''^2 + g''^2),$$  \hspace{1cm} (3.51)

and the corresponding equations of motion are

$$\dot{g} = \frac{\tilde{\lambda}}{2} f'' + \frac{\tilde{\lambda}^2}{8} f''' , \hspace{1cm} \dot{f} = -\frac{\tilde{\lambda}}{2} g'' - \frac{\tilde{\lambda}^2}{8} g''' .$$  \hspace{1cm} (3.52)

Their solution is the same as in (3.8), (3.9), now with

$$\omega_n = \frac{1}{2} \tilde{\lambda} n^2 - \frac{1}{8} \tilde{\lambda}^2 n^4 ,$$  \hspace{1cm} (3.53)

which is of course the expansion of the BMN frequency $\omega_n = \sqrt{1 + \tilde{\lambda} n^2} - 1$ (see also [15]). The normal-ordered quadratic Hamiltonian is then

$$\bar{H}_2 = \sum_{n=-\infty}^{\infty} \omega_n a_n^\dagger a_n ,$$  \hspace{1cm} (3.54)
i.e. the leading term in the energy of the M-impurity state is

$$\langle M | \hat{H}_2 | M \rangle = \sum_{j=1}^M \omega n_j = \frac{\tilde{\lambda}}{2} \sum_{j=1}^M n_j^2 - \frac{\tilde{\lambda}^2}{8} \sum_{j=1}^M n_j^4 \ . \quad (3.55)$$

To find the $1/J$ correction we need to consider the quartic term

$$\tilde{H}_4 = (\tilde{H}_4)^{(1)} + (\tilde{H}_4)^{(2)} \ ,$$

where the first term is the contribution of the order $\tilde{\lambda}$ term (3.5) which was already discussed above in (3.28) while the second term is the order $\tilde{\lambda}^2$ contribution

$$(\tilde{H}_4)^{(2)} = -\frac{\tilde{\lambda}^2}{8J} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[ (f'^2 + g'^2)^2 + 4(f f'' + g g'')^2 + 8(f f'' + g g'')(f'^2 + g'^2) - (f^2 + g^2)(f f''' + g g''') \right] . \quad (3.56)$$

Using the normal-ordering prescription which at the quartic order is again equivalent to the condition that the energy of the vacuum and 1-impurity states is not shifted we finish with

$$(\tilde{H}_4)^{(2)} = \frac{\tilde{\lambda}^2}{J} \sum_{n,m}^3 n^3 m a_n^i a_m^j a_n a_m \ . \quad (3.57)$$

As a result, the subleading $\tilde{\lambda}^2$ contribution to the energy of an M-impurity state is

$$\langle M | (\tilde{H}_4)^{(2)} | M \rangle = -\frac{\tilde{\lambda}^2}{J} \sum_{j=1}^M n_j^4 \ . \quad (3.58)$$

### 3.5 Comparison with gauge theory and string theory results

Collecting the results (3.28),(3.49),(3.58) from the previous subsections we find the following expression for the energy of the M-impurity state \(\{n_j\}\) satisfying (3.20),

$$E(M) = J + \frac{1}{2} \tilde{\lambda} \left[ \left( 1 + \frac{2}{J} + \frac{2cM}{J^2} \right) \sum_{j=1}^M n_j^2 \right. - \frac{1}{J} \left( 4 \sum_{j \neq k} n_j n_k (n_j^2 - n_j n_k + n_k^2) + \frac{\pi^2}{3} \sum_{j=1}^M n_j^4 \right) + O\left( \frac{1}{J^3} \right) \right]$$

$$- \frac{1}{2} \tilde{\lambda}^2 \left[ \left( \frac{1}{4} + \frac{2}{J} \right) \sum_{j=1}^M n_j^4 + O\left( \frac{1}{J^2} \right) \right] + O(\tilde{\lambda}^3) . \quad (3.59)$$

Let us compare this expression with the gauge theory (exact spin chain) results. Here \(J\) corresponds to the number of spin chain sites, i.e. the spin chain length. Let us
first consider the case of two impurities \((n_1 = -n_2 = n)\). In the \(SU(2)\) sector of SYM theory the two-loop expression for the anomalous dimension for \(M = 2\) operators is known exactly for finite value of \(J\) \[17\]

\[ E_{\text{gauge}}(M = 2) = J + \frac{\lambda}{\pi^2} \sin^2 \frac{\pi n}{J - 1} - \frac{\lambda^2}{\pi^4} \sin^4 \frac{\pi n}{J - 1} \left( \frac{1}{4} + \frac{\cos^2 \frac{\pi n}{J - 1}}{J - 1} \right) + O(\lambda^3) . \]

Expanding in large \(J\) one gets

\[ E_{\text{gauge}}(M = 2) = J + \tilde{\lambda} \left[ \left( 1 + \frac{2}{J} \right) n^2 - \frac{\pi^2}{3J^2} n^4 + \ldots \right] \]

\[ - \tilde{\lambda}^2 \left[ \left( \frac{1}{4} + \frac{2}{J} \right) n^4 + \ldots \right] + O(\tilde{\lambda}^3) . \]

We have already seen that the results match for the one-loop terms if the 6-order term in the oscillator Hamiltonian is normal ordered, corresponding to \(c = 0\). Comparing \[3.61\] with the LL result \[3.59\] for \(M = 2\) we also see that the \(\tilde{\lambda}^2/J\) terms match.

The string theory results for \(1/J\) corrections with \(M = 2\) are known to agree with gauge theory at orders \(\tilde{\lambda}\) and \(\tilde{\lambda}^2\) \[38\] and thus they are also reproduced by our LL computation. The \(1/J^2\) (order \(\tilde{\lambda}/J^2\)) term has not yet been computed directly on the string-theory side.

For arbitrary \(M\), the \(1/J\) correction to the leading one loop (order \(\tilde{\lambda}\)) energy in the \(SU(2)\) sector was found from the Bethe ansatz in \[5\]. This expression was extended to higher orders in \(\tilde{\lambda}\) in \[49, 31\]. To second order in \(\tilde{\lambda}\) the spin chain result is (for different mode numbers \(n_j\))

\[ E_{\text{gauge}}(M) = J + \frac{1}{2} \tilde{\lambda} \sum_{j=1}^{M} \left( 1 + \frac{2}{J} \right) n_j^2 - \frac{1}{2} \tilde{\lambda}^2 \sum_{j=1}^{M} \left( \frac{1}{4} + \frac{2}{J} \right) n_j^4 + O(\tilde{\lambda}^3) . \]

An equivalent \(M\)-impurity result was found on the string theory side in \[52\]. Comparing to \[3.59\] we conclude that the equal order \(\tilde{\lambda}\) and \(\tilde{\lambda}^2\) gauge and string expression for \(1/J\) corrections is thus reproduced by the quantum LL computation for any number of impurities \(M\).\[10\]

The order \(\tilde{\lambda}/J^2\) corrections for the \(M\)-impurity states in \[8.49\], obtained from the quantum LL Hamiltonian, and \[A.11\], obtained from the Bethe ansatz, is a new result.

\[9\]The finite \(J\) version of \(M\)-impurity energy was derived recently in the \(SU(1|1)\) sector \[32\].

\[10\]The string theory result of \[52\] was presented in the form

\[ E = J' + M + \frac{\lambda'}{2} \left( 1 - \frac{2M - 2}{J} + \ldots \right) \sum_{j=1}^{M} n_j^2 - \frac{\lambda'^2}{4} \left( \frac{1}{2} - \frac{2M - 4}{J} + \ldots \right) \sum_{j=1}^{M} n_j^4 + \ldots \]

where \(\lambda' = \lambda/J^2\). To compare to the above LL result one needs to replace \(J = J' + M\) and then this expression becomes equivalent to \[3.59\] to the leading \(1/J\) order.
A pressing issue is to see if, and if, how, string theory reproduces it. If one restricts the Hamiltonian used in the full string theory [38] for obtaining the $\tilde{\lambda}/J$ corrections, to the $SU(2)$ sector, one basically obtains the same Hamiltonian as (3.25). This explains the agreement found for $\tilde{\lambda}/J$ corrections as obtained from the full string theory and the quantum LL approach. Generalizing to the $\tilde{\lambda}/J^2$ corrections we would expect that many of the details would be similar to those in quantum LL computation, although $\zeta$-function regularization would not be necessary because of explicit cancelations of infinities due to supersymmetry (for preliminary work in this direction see [53]).

Let us stress that getting (3.62) from the LL action did not involve any non-trivial ambiguity apart from a normal ordering assumption. For the quartic term in the interacting Hamiltonian, the normal ordering is required in order to protect the BPS states. It would be interesting to see if a corresponding principle can be found for the six oscillator term.

Another interesting computation that can be done is to find the $\tilde{\lambda}^2/J^2$ corrections. This can be done using both the quantum LL Hamiltonian and the Bethe ansatz, and we again expect to obtain the matching.

4 The interacting Hamiltonian directly from the spin chain

In this section we consider the Hamiltonian for interacting magnons obtained directly from the Heisenberg spin chain Hamiltonian. Hence, we will skip the intermediary step of finding a long wave-length Lagrangian before quantizing. By doing this we will encounter some subtleties in computing $1/J$ corrections.

In terms of the spins, the Hamiltonian (2.1) for the one-loop $SU(2)$ sector is

$$H = \frac{\lambda}{8\pi^2} \sum_{\ell=1}^{J} \left( \frac{1}{2} - 2\tilde{S}_\ell \cdot \tilde{S}_{\ell+1} \right)$$

(4.1)

We can now write the spin operators in terms of auxiliary oscillators [54], $a_\ell$ and their conjugates $a_\ell^\dagger$, as

$$S_\ell^3 = \frac{1}{2} - a_\ell^\dagger a_\ell, \quad S_\ell^+ = a_\ell, \quad S_\ell^- = a_\ell^\dagger(1 - a_\ell^\dagger a_\ell),$$

(4.2)

where we assume the usual commutation relations $[a_\ell, a_{\ell'}^\dagger] = \delta_{\ell,\ell'}$. In terms of the oscillators, the Hamiltonian becomes

$$H = \frac{\lambda}{8\pi^2} \sum_{\ell=1}^{J} \left[ (a_{\ell+1}^\dagger - a_\ell)(a_{\ell+1} - a_\ell) + (a_{\ell+1}^\dagger - a_\ell^\dagger)^2 a_{\ell+1}a_\ell \right].$$

(4.3)

The Hilbert space for the spin chain is not the full Fock space of the oscillators: clearly, each site can be at most singly occupied. Also, the quartic term in (4.3) is not
Hermitian, nor are $S^+$ and $S^-$ conjugate to each other under the usual conjugation rules for the oscillators. In order to have $H$ be Hermitian under the usual rules, we can add terms to $H$ that are nonzero only for states that have multiply occupied sites, which are outside of our Hilbert space.

In particular, let us first choose $H$ to be

$$H = \frac{\lambda}{8\pi^2} \sum_{\ell=1}^J \left( (a^\dagger_{\ell+1} - a^\dagger_{\ell})(a_{\ell+1} - a_{\ell}) - 2a^\dagger_{\ell+1}a^\dagger_{\ell+1}a_{\ell+1}a_{\ell} + a^\dagger_{\ell+1}a^\dagger_{\ell+1}a_{\ell+1}a_{\ell} + a^\dagger_{\ell}a^\dagger_{\ell+1}a_{\ell+1}a_{\ell} 
+ a^\dagger_{\ell+1}a^\dagger_{\ell}a_{\ell+1}a_{\ell} + a^\dagger_{\ell}a^\dagger_{\ell+1}a_{\ell+1}a_{\ell} - a^\dagger_{\ell+1}a^\dagger_{\ell+1}a_{\ell+1}a_{\ell+1} - a^\dagger_{\ell}a^\dagger_{\ell}a_{\ell}a_{\ell} \right), \quad (4.4)$$

where the last four terms in (4.4) will not affect empty or singly occupied sites. We can now transform this to momentum space by defining

$$a_n = \sum_{\ell=1}^J e^{2\pi i \ell/J} a^\dagger_{\ell}, \quad (4.5)$$

and thus getting the expression

$$H = H^{(2)} + H^{(4)} = \frac{\lambda}{2\pi^2} \sum_{n=-J/2}^{J/2} \sin^2\left(\frac{\pi n}{J}\right) a^\dagger_n a_n 
- \frac{\lambda}{\pi^2 J} \sum_{n_1,n_2,n_3=-J/2}^{J/2} \sin\left(\frac{\pi n_1}{J}\right) \sin\left(\frac{\pi n_3}{J}\right) \cos\left(\frac{\pi (n_1 - n_3)}{J}\right) a^\dagger_{n_1} a^\dagger_{n_2} a_{n_3} a_{n_1+n_2-n_3}. \quad (4.6)$$

This Hamiltonian is exact. If we take the large $J$ limit then we find

$$H = \frac{\tilde{\lambda}}{2} \sum_{n=-\infty}^{\infty} n^2 a^\dagger_n a_n - \frac{\tilde{\lambda}}{J} \sum_{n_1,n_2,n_3=-\infty}^{\infty} n_1 n_2 n_3 a^\dagger_{n_1} a^\dagger_{n_2} a_{n_3} a_{n_1+n_2-n_3}. \quad (4.7)$$

We see that the quadratic part in (4.7) is consistent with (3.16), but the quartic part is not quite the same as (3.25). Beyond this order, $H$ in (4.7) is obviously different from the LL Hamiltonian since the latter will have interaction terms of all orders, as is required by integrability.

The reason for these differences is that the oscillator states here are not quite the same as those in the LL case of the previous section. Since the Hamiltonian $H$ in (4.7) is automatically normal ordered, it will give the correct energy for the one magnon state, $a^\dagger_n|0\rangle$, up to first order in $1/J$. However, the second and higher order results will have corrections that depend on the lattice size. If we next consider the two magnon state, $a^\dagger_n a^\dagger_{-n}|0\rangle$, we find that the quartic piece in (4.7) gives no $1/J$ correction, which is in conflict with (3.26) as well as the exact answer given in [51, 5]. The problem is that the two magnon state as written above is not in the Hilbert space. It turns out

\footnote{We temporarily ignore the momentum constraint.}
that there is a small overlap with states that have sites that are double occupied. Instead, we should choose for the two magnon state

$$|n, -n\rangle \equiv \frac{1}{\sqrt{1 - \frac{2}{J}}} \left( a_n^\dagger a_{-n}^\dagger - \frac{1}{J} \sum_{n' = -\infty}^{\infty} a_{n'}^\dagger a_{-n'}^\dagger \right) |0\rangle,$$  \hspace{1cm} (4.8)

which has no overlap with states having multiply occupied sites. With this definition of the two magnon state and using the exact \( H \) in (4.6), we get

$$\langle n, -n|H^{(2)}|n, -n\rangle = \frac{\lambda}{\pi^2} \left[ \frac{1}{1 - 2/J} \sin^2 \frac{\pi n}{J} + \frac{2}{1 - 2/J} \frac{1}{J^2} \sum_{m = -J/2}^{J/2} \sin^2 \frac{\pi m}{J} \right]. \hspace{1cm} (4.9)$$

Performing the sum over the \( J \) modes we find

$$\langle n, -n|H^{(2)}|n, -n\rangle = \frac{\lambda}{\pi^2} \left[ \frac{1}{1 - 2/J} \sin^2 \frac{\pi n}{J} + \frac{1/J}{1 - 2/J} \right], \hspace{1cm} (4.10)$$

where we see that the last term is of order \( J^\lambda \). We could have also approximated \( \sin \frac{\pi m}{J} \approx \frac{\pi m}{J} \) and done \( \zeta \)-function regularization of the sum where we would have found that the last term was absent. To find the contribution from \( H^{(4)} \) in (4.6), we note that

$$\langle 0|a_m a_{-m} H^{(4)} a_{n}^\dagger a_{-n}^\dagger |0\rangle = -\frac{4\lambda}{\pi^2 J} \sin^2 \frac{\pi m}{J} \sin^2 \frac{\pi n}{J}, \hspace{1cm} (4.11)$$

so that

$$\langle n, -n|H^{(4)}|n, -n\rangle = \frac{\lambda}{\pi^2 (1 - 2/J)} \left[ -4 \sin^4 \frac{\pi n}{J} + 4 \sin^2 \frac{\pi n}{J} \sum_{m = -J/2}^{J/2} \sin^2 \frac{\pi m}{J} \right. \left. - \frac{4}{J^3} \left( \sum_{m = -J/2}^{J/2} \sin^2 \frac{\pi m}{J} \right)^2 \right]. \hspace{1cm} (4.12)$$

This then gives

$$\langle n, -n|H^{(4)}|n, -n\rangle = \frac{\lambda}{\pi^2 (1 - 2/J)} \left( -4 \sin^4 \frac{\pi n}{J} + 2 \sin^2 \frac{\pi n}{J} \frac{1}{J} \right). \hspace{1cm} (4.13)$$

Notice that the \( \zeta \)-function regularization of (4.12) would have removed the last two terms in (4.13).

Combining (4.10) with (4.13), we arrive at

$$\langle n, -n|H^{(4)}|n, -n\rangle = \frac{\lambda}{\pi^2 (1 - 2/J)} \left( \sin^2 \frac{\pi n}{J} - 4 \sin^4 \frac{\pi n}{J} \right) \hspace{1cm} (4.14)$$

\(^{12}\)Note that one way to avoid going outside the Hilbert space is to use two fermionic creation and annihilation operators at each site (c.f. [9]). However, it is not clear how to compare the resulting Hamiltonian to the LL Hamiltonian.
which is the desired result up to order $\tilde{\lambda}/J$. If we have used the $\zeta$-function regularization, we would have canceled out the unwanted terms of order $J\tilde{\lambda}$, but we would have gotten the wrong $\tilde{\lambda}/J$ term.

One can repeat the same for more than two impurities, although the computation gets rather tedious. Instead, we will do the analysis in a different way, which will also allow us to compare with the results in section 3. We note that to order $1/J$, the state $|n, -n\rangle$ can be written also as

$$|n, -n\rangle = U a_n^\dagger a_{-n}^\dagger |0\rangle,$$  \hspace{1cm} (4.15)

where $U$ is defined as

$$U = 1 - \frac{1}{2J} \left( \sum_{p,q} a_p^\dagger a_q^\dagger a_q a_p + \sum_{p,q} a_p^\dagger a_q^\dagger a_p a_q \right).$$  \hspace{1cm} (4.16)

The operator $U$ combines a projector with another factor that properly normalizes the state. In fact, this same operator can be used on all multi-impurity states to order $1/J$. We can then do a similarity transformation on $H$ and define a new Hamiltonian

$$\tilde{H} = U^\dagger H U, \quad U = U^\dagger. \hspace{1cm} (4.17)$$

Clearly, $\tilde{H}$ has the same action on the ground state as $H$. It is also evident that $\tilde{H}$ is not normal ordered. If we now expand $\tilde{H}$ to quartic order while normal ordering the operators, we find

$$\tilde{H}^{(4)} = \frac{\tilde{\lambda}}{2J} \sum_{p,q} (p^2 + q^2) a_p^\dagger a_q^\dagger a_q a_p - \frac{\tilde{\lambda}}{J} \sum_{p,q} pr a_p^\dagger a_q^\dagger a_q a_p a_{p+q} - O\left( \frac{1}{J^2} \right), \hspace{1cm} (4.18)$$

where in the intermediate steps, the sums are carried out assuming that the oscillator momenta run from $-J/2$ to $J/2$. If we now consider the expectation value of $\tilde{H}^{(4)}$ for a multi-impurity state we find

$$\langle M | \tilde{H}^{(4)} | M \rangle = \frac{\tilde{\lambda}}{J} \sum_{i<j}^M (n_i^2 + n_j^2) - \frac{\tilde{\lambda}}{J} \sum_{i<j}^M (n_i + n_j)^2 = -\frac{2\tilde{\lambda}}{J} \sum_{i<j}^M n_in_j. \hspace{1cm} (4.19)$$

Using the momentum constraint $\sum_i n_i = 0$, we find the same result as (3.28).

Still, $\tilde{H}$ is not the same as in (3.25), again suggesting that there is a nontrivial map between the oscillators in section 3 and those used here. Moreover, $\tilde{H}$ has a nonlocal part, indicating that the transformation itself is nonlocal.

In any case, while this method of constructing the Hamiltonian is clearly more cumbersome than constructing it directly from the LL action, there are no ambiguities in normal ordering or regularization. Computing the next-order correction is, in principle, doable and would provide a useful check on the results in (3.45).
5 1/J correction to energy of near BPS states in the β-deformed theory

Let us now apply the approach of section 3 of quantizing the LL Hamiltonian to obtain a new result: the 1/J correction to the energy of BMN-type states in the β-deformed version of the AdS/CFT which relates an exactly marginal superconformal deformation of SYM theory to string theory in the \( AdS_5 \times (S^5)_\beta \) background constructed using T-dualities and coordinate shifts [39] (we shall consider the case of real deformation parameter \( \beta \)). The corresponding string theory and the spin chain Hamiltonian were discussed in detail in [40], where the \( \beta \neq 0 \) analog of the LL action (2.4), (2.11) was derived both from the spin chain and from the string action. The analog of the \( SU(2) \) sector here contains the operators built out of two chiral scalars, \( \text{Tr}(\Phi_1^j \Phi_2^k) + ... \) and the BPS vacuum is again the \((J,0)\) state. The spectrum of the corresponding BMN states was discussed in [55, 39, 40].

5.1 Quantum LL Hamiltonian approach

Here our starting point will be the “β-deformed” LL action found in [40] which generalizes (2.19)

\[
L = \cos 2\psi \dot{\varphi} - \frac{\tilde{\lambda}}{2} \left[ \psi'^2 + \sin^2 2\psi \left( \varphi' + \frac{1}{2} \bar{\beta} \right)^2 \right], \quad \bar{\beta} \equiv \beta J. \tag{5.1}
\]

Here we ignored \( O(\tilde{\lambda}^2) \) terms and assumed that in addition to \( \tilde{\lambda} \) the parameter \( \bar{\beta} \) is also fixed in the large \( J \) limit (see [40] for details). This Lagrangian can be rewritten also in terms of the “cartesian” fields \( n_a \) in (2.13) as

\[
L = L_{\beta=0}(n) - \frac{1}{8} \bar{\lambda} \bar{\beta}^2 n^2 + \frac{1}{4} \bar{\lambda} \bar{\beta} \epsilon_{abc} n'_a n'_b , \tag{5.2}
\]

where \( L_{\beta=0} \) is the undeformed Lagrangian in (2.13)–(2.15).

We would like to expand (5.2) near the ground state with \( n_a = 0 \) and compute the leading 1/J correction to the fluctuation spectrum. We shall then compare this to the result that can be found directly from the Bethe ansatz equations given in [40]. The full string-theory computation of this 1/J correction appears to be rather complicated, but it should be fairly clear by now that it should be correctly captured (to the given leading order) by the quantum LL theory, so the string and gauge theory \( \tilde{\lambda}/J \) corrections should also match in the \( \beta \)-deformed theory.

The basis for this confidence is that the LL computation of the \( \tilde{\lambda}/J \) correction involves only the quadratic and quartic oscillator terms in the LL Hamiltonian and in this case the choice of normal ordering prescription appears to be essentially the unique regularization option, leaving no ambiguity. This is confirmed by matching to the exact spin chain results.
Changing the variables \( n_a \to z_a \to (f, g) \) as in (2.10), (3.2) and expanding near the \( f = g = 0 \) vacuum we obtain the fluctuation Lagrangian up to the quartic order

\[
L = L_{\beta=0}(f, g) - \frac{\tilde{\lambda} \beta^2}{2} (f^2 + g^2) + \frac{\tilde{\lambda} \beta^2}{2J} (f^2 + g^2)^2 - \frac{\tilde{\lambda} \beta}{J} (g f' - f g')(f^2 + g^2),
\]

(5.3)

where \( L_{\beta=0}(f, g) \) is the undeformed fluctuation Lagrangian in (3.3). The linearized equations of motion (cf. (3.7))

\[
\dot{g} = \frac{\tilde{\lambda}}{2} f'' - \tilde{\lambda} \beta g' - \frac{\tilde{\lambda}}{2} \beta^2 f, \quad \dot{f} = -\frac{\tilde{\lambda}}{2} g'' - \tilde{\lambda} \beta f' + \frac{\tilde{\lambda}}{2} \beta^2 g,
\]

(5.4)

are solved by (3.8), (3.9) with

\[
\omega_n = \frac{1}{2} \tilde{\lambda} (n + \bar{\beta})^2.
\]

(5.5)

As discussed in [40], the momentum constraint is unchanged from its \( \beta = 0 \) form (3.11).

The quadratic Hamiltonian may be written as

\[
\bar{H}_2 = \frac{1}{2} \tilde{\lambda} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[ (f' - \beta g)^2 + (g' + \beta f)^2 \right],
\]

(5.6)

and upon quantization it becomes

\[
\bar{H}_2 = \sum_{n=-\infty}^{\infty} \omega_n a_n^\dagger a_n.
\]

(5.7)

As in the \( \beta = 0 \) case here we used normal ordering to preserve the BPS ground state.

Applying this Hamiltonian to a physical \( M \)-impurity state we obtain for its energy

\[
E = \frac{\tilde{\lambda}}{2} \sum_{j=1}^{M} (n_j + \bar{\beta})^2 = \frac{\tilde{\lambda}}{2} \sum_{j=1}^{M} n_j^2 + \frac{\tilde{\lambda}}{2} \beta^2 M,
\]

(5.8)

where we used that \( \sum_{j=1}^{M} n_j = 0 \) as follows from the momentum constraint. As a result, the \( \beta \)-dependent part of the energy is sensitive only to the number of impurities but not to their detailed distribution. Note also that the zero-mode states with \( n_j = 0 \) get non-trivial shifts. The same remarks will apply to the \( 1/J \) correction to the leading spectrum (5.8).

The quartic Hamiltonian is

\[
\bar{H}_4 = (\bar{H}_4)_{\beta=0} - \frac{\tilde{\lambda}}{2J} \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[ \beta^2 (f^2 + g^2)^2 - 2\beta (f^2 + g^2)(f' g - g' f) \right],
\]

(5.9)

where \( (\bar{H}_4)_{\beta=0} \) is given by the integral of (3.5). Assuming normal ordering, we find that \( \bar{H}_4 \) can be written as (cf. (3.26))

\[
\bar{H}_4 = -\frac{\tilde{\lambda}}{J} \sum_{n,m} (n + \bar{\beta})(m + \bar{\beta}) a_n^\dagger a_m^\dagger a_n a_m.
\]

(5.10)
As a result, the leading quantum correction to the energy of the $M$-impurity state is found to be

$$\langle M|\hat{H}_4|M\rangle = \frac{\tilde{\lambda}}{J} \sum_{j=1}^{M} n_j^2 - \frac{\tilde{\lambda}\beta^2}{J} (M^2 - M) \ . \quad (5.11)$$

The total energy can be written then as ($\sum_{j=1}^{M} n_j = 0$)

$$E = J + \frac{\tilde{\lambda}}{2} \sum_{j=1}^{M} (n_j + \tilde{\beta})^2 \left( 1 + \frac{2}{J} \right) - \frac{\tilde{\lambda}\beta^2}{J} M^2 + O(\frac{1}{J^2}, \tilde{\lambda}^2) \ . \quad (5.12)$$

### 5.2 Bethe ansatz approach

Let us now show that the same expression can be found directly from the Bethe ansatz equations [40] for the corresponding anisotropic XXZ spin chain describing the 1-loop dilatation operator [56] of the $\beta$-deformed gauge theory in the $SU(2)\beta$ sector. The Bethe ansatz equations given in [56, 40] are

$$e^{-2\pi i\beta J} \left( \frac{u_k + i/2}{u_k - i/2} \right)^{J} = \prod_{j\neq k=1}^{M} \frac{u_k - u_j + i}{u_k - u_j - i} \ , \quad (5.13)$$

$$e^{-2\pi i\beta M} \prod_{k=1}^{M} \frac{u_k + i/2}{u_k - i/2} = 1 \ , \quad E_1 = \frac{\lambda}{8\pi^2} \sum_{j=1}^{M} \frac{1}{u_j^2 + 1/4} \ , \quad (5.14)$$

where $u_j$ are magnon rapidities and $J$ corresponds to the length of the chain. To take the large $J$ limit one rescales the the rapidities

$$u_k = Jx_k$$

and expands the logarithm of the Bethe equations in powers of $1/J$. This gives to the leading order in $1/J$ [40]

$$\frac{1}{x_k} = \frac{2}{J} \sum_{j=1}^{M} \frac{1}{x_k - x_j} + 2\pi(n_k + \tilde{\beta}) \ , \quad (5.15)$$

$$\sum_{k=1}^{M} \frac{1}{x_k} = 2\pi(mJ + \tilde{\beta}M) \ , \quad E_1 = \frac{\tilde{\lambda}}{8\pi^2} \sum_{j=1}^{M} \frac{1}{x_k^2} \ , \quad \tilde{\lambda} = \frac{\lambda}{J^2} \ , \quad (5.16)$$

where we introduced again $\tilde{\beta} = \beta J$. In the BMN limit we are interested in $J$ is taken to infinity while $M$ held is fixed, so that the integer $m$ should be set equal to zero. Then we find that the perturbative solution of (5.15) for $x_k$ generalizing the leading-order one in [40] is

$$x_k = \frac{1}{2\pi(n_k + \tilde{\beta})} \left( 1 - \frac{2}{J} \sum_{j\neq k}^{M} \frac{n_j + \tilde{\beta}}{n_j - n_k} \right) + O(\frac{1}{J^2}) \ , \quad (5.17)$$
with (5.16) implying that
\[ \sum_{j=1}^{M} n_j = 0 . \]  
As a result, the leading order correction to the dimension of the corresponding M-impurity BMN operators is found to be\(^{13}\)
\[ E_1 = \frac{\tilde{\lambda}}{2} \sum_{k=1}^{M} (n_k + \bar{\beta})^2 \left( 1 + \frac{4}{J} \sum_{j \neq k} \frac{n_j + \bar{\beta}}{n_j - n_k} \right) + O(\frac{\tilde{\lambda}}{J^2}) . \]  
(5.19)
Using (5.18) we get
\[ E_1 = \frac{\tilde{\lambda}}{2} \sum_{k=1}^{M} n_k^2 \left( 1 + \frac{2}{J} \right) + \frac{1}{2} \tilde{\lambda} \bar{\beta}^2 M \left[ 1 - \frac{2(M-1)}{J} \right] \]
\[ = \frac{\tilde{\lambda}}{2} \sum_{k=1}^{M} (n_k + \bar{\beta})^2 \left( 1 + \frac{2}{J} \right) - \frac{\tilde{\lambda} \bar{\beta}^2}{J} M^2 , \]  
(5.20)
which is exactly the expression found from the LL action (5.12). This provides a nice illustration of the power of the approach based on the quantum LL action. We expect that one can generalize the arguments for the $1/J^2$ corrections in sec. 3.3 and Appendix A to the case of nonzero $\beta$.

### 6 \(1/J\) corrections to the energy of circular rotating string solution

In the previous sections we used the LL action to compute energies of fluctuations near the (constant) vacuum solution $\vec{n} = (0, 0, 1)$ corresponding to the BPS vacuum $\text{Tr}\Phi^I_1$ in gauge theory and the $S^5$ geodesic in string theory.

In this section we extend the analysis of section 3 to expansions near a non-BPS state represented by the simplest static solitonic solution of the LL equations of motion
\[ \psi = \frac{\pi}{4} , \quad \varphi = m \sigma , \quad \text{i.e.} \quad \vec{n} = (\cos 2m \sigma, \sin 2m \sigma, 0) . \]  
(6.1)
This LL solution corresponds to the leading (order $\tilde{\lambda}$) term $\tilde{\lambda}$ in the circular rotating string solution of $[4, 26]$, i.e. $X_1 = \frac{1}{\sqrt{2}} e^{i(w+im)\tau}$, $X_1 = \frac{1}{\sqrt{2}} e^{i(w-im)\tau}$, with $w = \mathcal{J} = \frac{J}{\sqrt{\lambda}}$ and $J_1 = J_2 = J/2$. Here $m$ is an integer winding number which we shall assume to be positive.\(^{14}\) The classical energy is
\[ E_0 = J + \frac{\lambda m^2}{2J} + O(\lambda^2) = J \left[ 1 + \frac{1}{2} \tilde{\lambda} m^2 + O(\tilde{\lambda}^2) \right] , \]  
(6.2)
\(^{13}\)Note that this expression cannot be obtained from the $\beta = 0$ one by the shift $n_k \rightarrow n_k + \bar{\beta}$: while the solution for $x_k$ can be generated this way, the momentum constraint must remain the same: $\sum_j n_j = 0$.
\(^{14}\)Note that the limit of $m = 0$ brings us back to the ($SO(3)$ rotated) vacuum solution $\vec{n} = (1, 0, 0)$.
i.e. is the leading term in the $\tilde{\lambda}$ expansion of the energy $E = \sqrt{J^2 + \lambda m^2}$ of the full circular string solution.

Our aim will be to compute the quantum $1/J$ corrections to the energy of this non-constant ground state of the LL model (which in this section we shall denote as $|0_m\rangle$). While in the vacuum case of section 3 the correction to the ground state energy was absent and we concentrated on computing corrections to energies of near-by fluctuation modes (i.e. BMN states) here the question about the form of $1/J$ corrections to this non-BPS ground state energy is already a non-trivial one.

To compute the quantum $1/J$ corrections to the ground state energy (6.2) one should expand the LL action (2.19) near the solution (6.1) and quantize the Hamiltonian for the fluctuations. A convenient starting point is the LL action in the parametrization given in (2.21). Introducing the two fluctuation fields ($f, g$) as

$$\varphi = m\sigma + \frac{1}{\sqrt{J}}f, \quad \xi = -\frac{1}{2} \sin(2(\psi - \pi/4)) = \frac{1}{\sqrt{J}}g,$$

we find the following expression for the fluctuation Lagrangian up to the quartic order (cf. (3.3)–(3.6))

$$L = 2g\dot{f} - \tilde{\lambda}^2 (f'^2 + g'^2 - 4m^2 g^2) + \frac{4m\tilde{\lambda}}{\sqrt{J}} f'g^2 + \frac{2\tilde{\lambda}}{J} g^2 (f'^2 - g'^2) + ...$$

\[= 2g\dot{f} - (H_2 + H_3 + H_4 + ...) . \tag{6.4}\]

### 6.1 Leading $\tilde{\lambda}$ correction

To compute the leading “one-loop” correction to the ground state energy one should consider the quadratic order in fluctuations and sum over the corresponding characteristic frequencies [25]. The linearized equations of motion are

$$\dot{f} = -\frac{\tilde{\lambda}}{2} g'' - 2\tilde{\lambda} m^2 g, \quad \dot{g} = \frac{\tilde{\lambda}}{2} f'' , \tag{6.5}$$

and they are solved again as in (3.8), (3.9) where now the characteristic frequency is

$$\omega_n = \frac{\tilde{\lambda}}{2} n\sqrt{n^2 - 4m^2} \equiv \pm \frac{\tilde{\lambda}}{2} n^2 w_n , \quad w_n \equiv \sqrt{1 - \frac{4m^2}{n^2}} , \quad n = 0, \pm 1, \pm 2, ... . \tag{6.6}$$

We see that this solution has unstable (imaginary frequency) fluctuation modes with $n = \pm 1, ..., \pm 2m$ which is a manifestation of an instability of the full homogeneous circular string solution [4, 26]. In what follows we shall formally ignore this instability issue (see [28, 29, 25] for discussions) by formally defining all the expressions by analytic continuation from the region $m < 1/2$. Most of what follows can be readily repeated for a very similar stable $(S, J)$ solution of [26] appearing in the $SL(2)$ sector (with
the corresponding LL action derived in \[19, 22\] for which the full 1-loop quantum string correction was found in \[29\] (see also \[34\]). The discussion of the simpler $SU(2)$ solution has an advantage of being more explicit.

Imposing the commutation relations (3.13) as dictated by (6.4) we find that in terms of canonically normalized creation and annihilation operators with respect to our nontrivial ground state $|0_m\rangle$

\[
[a_n, a_k^\dagger] = \delta_{n-k},
\]

we have (cf. (3.8), (3.9))

\[
f(t, \sigma) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \sqrt{w_n} (a_n e^{-i\omega_n t + in\sigma} + a_n^\dagger e^{i\omega_n t - in\sigma}),
\]

(6.7)

\[
g(t, \sigma) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{w_n}} (-ia_n e^{-i\omega_n t + in\sigma} + ia_n^\dagger e^{i\omega_n t - in\sigma}).
\]

(6.8)

The quadratic Hamiltonian in (6.4) becomes

\[
\bar{H}_2 = \int_0^{2\pi} \frac{d\sigma}{2\pi} H_2 = \frac{1}{2} \sum_{n=-\infty}^{\infty} |\omega_n| (a_n a_n^\dagger + a_n^\dagger a_n) = \sum_{n=-\infty}^{\infty} |\omega_n| a_n a_n^\dagger + \frac{1}{2} \sum_{n=-\infty}^{\infty} |\omega_n|.
\]

(6.9)

The leading quantum correction to the energy of our solitonic solution is then given by

\[
E_1 = \langle 0_m | \bar{H}_2 | 0_m \rangle = \frac{1}{2} \sum_{n=-\infty}^{\infty} |\omega_n| = \frac{\tilde{\lambda}}{2} \sum_{n=1}^{\infty} n \sqrt{n^2 - 4m^2}.
\]

(6.10)

Here we should not of course use the normal ordering prescription since it simply ignores the shift of the vacuum energy. Instead, as common at quadratic oscillator level, it is natural to define the divergent sum in (6.10) using the $\zeta$-function regularization; this gives the following finite result \[25\]

\[
E_1 = \frac{\tilde{\lambda}}{2} \left[ m^2 + \sum_{n=1}^{\infty} (n \sqrt{n^2 - 4m^2} - n^2 + 2m^2) \right].
\]

(6.11)

This is exactly the same expression as found from the full string-theory 1-loop computation \[27, 28, 29\] as well as from the Bethe ansatz on the spin chain side formally applied to the corresponding one-cut Bethe root distribution \[25, 50\]. From the string theory point of view, the additional terms in (6.11), as compared to the unregularized expression in (6.10), come from contributions of other bosonic and fermionic modes which make the total string result finite.

### 6.2 Subleading $\tilde{\lambda}/J$ correction

Subleading $\tilde{\lambda}/J$ corrections to the circular string energy (2-loop correction on the string side and $1/J^2$ correction on the spin chain side) were not computed before. This computation is, however, straightforward in the present approach based on standard perturbation theory for the quantum LL Hamiltonian in (6.4). Obviously, $\langle 0_m | \bar{H}_3 | 0_m \rangle = 0,$
so to compute the $\tilde{\lambda}/J$ correction to the ground state energy we need to consider the second order perturbation theory term for $\tilde{H}_3$ and first order term for $\tilde{H}_4$.

Written in terms of creation and annihilation operators $\tilde{H}_3$ takes the form\(^{15}\)

$$\tilde{H}_3 = \frac{im\tilde{\lambda}}{2\sqrt{J}} \sum_{n,k,l=-\infty}^{\infty} \sqrt{\frac{w_k}{w_nw_l}} k \left( a_n^\dagger a_k^\dagger \delta_{n+l+k} - a_n^\dagger a_l^\dagger \delta_{n+l-k} + a_n^\dagger a_l^\dagger \delta_{n+l+k} - a_n^\dagger a_k^\dagger \delta_{n+l-k} \right),$$

where $w_n$ was defined in \((6.6)\). The second order perturbative correction $\tilde{H}_3$ receives contribution only from the 3-particle intermediate states

$$|1_n1_k1_l> = a_n^\dagger a_k^\dagger a_l^\dagger |0_m>.$$

The contribution vanishes if $n = k = l$. When the values of $n, k, l$ are all different we obtain

$$\mathcal{E}_1 \equiv -\frac{1}{3!} \sum_{n\neq k\neq l} \frac{\langle 0_m|\tilde{H}_3|1_n1_k1_l>\langle 1_l1_k1_n|\tilde{H}_3|0_m\rangle}{\omega_n + \omega_k + \omega_l}$$

$$= -\frac{m^2\tilde{\lambda}}{3J} \sum_{n\neq k\neq l} \delta(n + k + l) \left( \frac{1}{\sqrt{w_nw_k}} + n\sqrt{\frac{w_n}{w_kw_l}} + k\sqrt{\frac{w_k}{w_nw_l}} \right)^2,$$

where the sums are from $-\infty$ to $\infty$. When two values among $n, k, l$ are equal the contribution is

$$\mathcal{E}_2 \equiv -\sum_{n\neq k} <0_m|\tilde{H}_3|1_n2_k><2_k1_n|\tilde{H}_3|0_m>$$

$$= -\frac{\tilde{\lambda}m^2}{J} \sum_{n\neq k} \delta(n + 2k) \left( \frac{1}{\sqrt{w_n}} + n\sqrt{\frac{w_n}{w_k}} \right)^2.$$

To compute $\langle 0_m|\tilde{H}_4|0_m\rangle$ where $\tilde{H}_4$ is the integrated quartic term in \((6.4)\) we need to use that

$$\int_0^{2\pi} \frac{d\sigma}{2\pi} g^2 f^2 = + \frac{1}{16J^2} \sum_{n,k=-\infty}^{\infty} \left[ nk(2a_n^\dagger a_k^\dagger a_n a_k + 2a_n a_k a_n^\dagger a_k^\dagger + a_n^\dagger a_k^\dagger a_n a_k^\dagger) + a_n a_k a_n^\dagger a_k^\dagger + a_n^\dagger a_k^\dagger a_n a_k + a_n^\dagger a_k^\dagger a_n^\dagger a_k^\dagger \right]$$

\(^{15}\)Here again we remove the time dependent exponential factors by a unitary transformation.
\[
\int_0^{2\pi} \frac{d\sigma}{2\pi} g^2 g'^2 = -\frac{1}{16J^2} \sum_{n,k=-\infty}^{\infty} \left[ \frac{nk}{wnwk} (2a_n^\dagger a_k a_n a_k + 2a_n a_k a_n^\dagger a_k^\dagger - a_n a_k a_n^\dagger a_k^\dagger) - \frac{k^2}{wnwk} (a_n a_n^\dagger a_k^\dagger a_k) + a_n a_n^\dagger a_k^\dagger a_k^\dagger + a_n a_n^\dagger a_k^\dagger a_k + a_n^\dagger a_n a_k^\dagger a_k^\dagger \right],
\]
so that
\[
\mathcal{E}_3 \equiv \langle 0_m | \hat{H}_4 | 0_m \rangle = -\frac{\tilde{\lambda}}{8J} \left( \sum_{n=-\infty}^{\infty} \frac{n^2}{w_n^2} - \sum_{n,k=-\infty}^{\infty} \frac{k^2}{wnwk} + 3 \sum_{n=-\infty}^{\infty} n^2 + \sum_{n=-\infty}^{\infty} \frac{1}{wn} \sum_{k=-\infty}^{\infty} k^2 w_k \right)
\]
where again \( w_n = \sqrt{1 - \frac{4m^2}{n^2}} \).

As a result, the \( \tilde{\lambda}/J \) correction to the energy is the sum of (6.13), (6.14), (6.17)
\[
E_2 = \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3.
\]

The divergent sum \( \mathcal{E}_3 \) can be again defined using the \( \zeta \)-function regularization, but in contrast to quadratic oscillator case here it is no longer clear if this is the regularization that actually corresponds to the UV finite microscopic theories we are interested in – the string theory or the spin chain (which we still expect to agree at this order).

The total energy to this order is then given by the sum of (6.2), (6.11) and (6.18), i.e.
\[
E = J \left[ 1 + \frac{1}{2} m^2 \tilde{\lambda} \left( 1 + \frac{c_1}{J} + \frac{c_2}{J^2} + O\left( \frac{1}{J^3} \right) \right) + O(\tilde{\lambda}^2) \right].
\]

The coefficients \( c_1 \) and \( c_2 \) given by regularized sums may be evaluated numerically. Taking \( m = 1 \) (and ignoring imaginary contributions of unstable modes) we found that
\[
c_1 = -0.892, \quad c_2 = 11.04.
\]

Some details of evaluation of the corresponding sums are discussed in Appendix B.

### 6.3 \( \tilde{\lambda}^2 \) correction

Let us now repeat the computation of the leading correction to the classical energy from section 6.1 by starting with the LL action (2.12) containing an order \( \tilde{\lambda}^2 \) correction which represents the 2-loop contribution on the gauge theory side or the subleading term in the expansion of the string action in the large \( J \) limit. The solution (6.14) remains the solution of the LL equations corrected by higher-derivative \( \tilde{\lambda}^2 \) terms, while the classical energy (6.2) gets an additional term \( -\frac{1}{8} J \tilde{\lambda}^2 m^4 \). Written in angular parametrization
the Lagrangian (2.12) has the form

\[
L = \cos 2\psi \dot{\varphi} - \frac{\tilde{\lambda}}{2}(\psi'^2 + \sin^2 2\psi \varphi'^2) + \frac{\tilde{\lambda}^2}{8}\left[\psi'^4 + \psi''^2\right] + (9 + 7 \cos 4\psi)\psi'^2\varphi'^2 + \frac{1}{2}(5 + 3 \cos 4\psi) \sin^2 2\psi \varphi'^4 + \sin^2 2\psi \varphi''^2 \\
- 2 \sin 4\psi \varphi'(\psi''\varphi' - 2\psi'\varphi'')
\]  
(6.21)

Expanding it to quadratic order in fluctuations, we get

\[
L = 2g\dot{f} - \frac{1}{2}\tilde{\lambda}(f'^2 + g'^2 - 4m^2g^2) + \frac{1}{8}\tilde{\lambda}^2(f'^2 + g'^2 + 6m^2f'^2 - 6m^2g'^2 + 8m^4g^2) 
\]  
(6.22)

The corresponding equations of motion are

\[
\dot{f} = -\frac{\tilde{\lambda}}{2}g'' - 2\tilde{\lambda}m^2g - \frac{\tilde{\lambda}^2}{8}g'''' - \frac{3\tilde{\lambda}^2}{4}m^2 g'' - \tilde{\lambda}^2 m^4 g 
\]  
(6.23)

\[
\dot{g} = \frac{\tilde{\lambda}}{2}f'' + \frac{\tilde{\lambda}^2}{8}f'''' - \frac{3\tilde{\lambda}^2}{4}m^2 f'' 
\]  
(6.24)

Their solution is again given by (6.7),(6.8) where now the characteristic frequencies are found to be\textsuperscript{16}

\[
\omega_n = \pm \frac{\tilde{\lambda}}{2}\sqrt{(n^2 - 4m^2 + \frac{3\tilde{\lambda}^2}{2}n^2m^2 - \frac{\tilde{\lambda}^4}{4}n^4 - 2\tilde{\lambda}m^4)(n^2 - \frac{3\tilde{\lambda}^2}{2}n^2m^2 - \frac{\tilde{\lambda}^4}{4}n^4)} 
\]  
(6.25)

or, expanded in \( \tilde{\lambda} \) to the order we consider,

\[
\omega_n = \frac{\tilde{\lambda}}{2}n\sqrt{n^2 - 4m^2 - \frac{\tilde{\lambda}^2}{8}n(n^2 + 2m^2)\sqrt{n^2 - 4m^2}} + O(\tilde{\lambda}^3) 
\]  
(6.26)

where again we formally assume that \( n^2 > 4m^2 \). These are the same characteristic frequencies (the part that belongs to the SU(2) sector) as found from the full string analysis of fluctuations in \textsuperscript{27, 26, 28}.

The leading quantum correction to the classical energy to order \( \tilde{\lambda}^3 \) is then

\[
E_1 = \langle 0_m|\tilde{H}_2|0_m\rangle = \frac{1}{2} \sum_{n=-\infty}^{\infty} |\omega_n| \\
= \frac{\tilde{\lambda}}{2} \sum_{n=1}^{\infty} n\sqrt{n^2 - 4m^2} - \frac{\tilde{\lambda}^2}{8} \sum_{n=1}^{\infty} n(n^2 + 2m^2)\sqrt{n^2 - 4m^2} 
\]  
(6.27)

\textsuperscript{16}Comparing to (6.7),(6.8) we set again \( \omega_n \equiv \frac{\tilde{\lambda}}{2}n^2w_n \).
The sum in the $\tilde{\lambda}^2$ term here is divergent and needs to be regularized. A natural regularization choice is again, as in [25], to subtract and add the divergent part and use the $\zeta$-function regularization for the latter. This gives a generalization of (6.11):

$$E_1 = \frac{\tilde{\lambda}}{2} \left[ m^2 + \sum_{n=1}^{\infty} \left( n\sqrt{n^2 - 4m^2} - n^2 + 2m^2 \right) \right] - \frac{\tilde{\lambda}^2}{8} \left[ 3m^4 + \sum_{n=1}^{\infty} \left( n(n^2 + 2m^2)\sqrt{n^2 - 4m^2} - n^4 + 10m^4 \right) \right] . \quad (6.28)$$

Let us now compare this regularized expression with the result from string theory [28]. An immediate problem is that while the expansion of the full finite string 1-loop result in powers of $\tilde{\lambda}$ produces the convergent expression (6.11) at order $\tilde{\lambda}$, the coefficient of the $\tilde{\lambda}^2$ term happens to be given by a divergent sum over $n$ (this has to do with peculiar properties of the string 1-loop expression as a function of $\tilde{\lambda}$). The string result is given by the zero-mode term and the infinite sum of string-mode contributions; by formally expanding both parts in powers of $\tilde{\lambda}$ we get:

$$E_{\text{string}} = E_{\text{zero}} + E_{\text{non-zero}}, \quad E_{\text{zero}} = \frac{\tilde{\lambda}}{2} m^2 - \frac{5\tilde{\lambda}^2}{8} m^4 + O(\tilde{\lambda}^3) , \quad (6.29)$$

$$E_{\text{non-zero}} = \frac{\tilde{\lambda}}{2} \sum_{n=1}^{\infty} \left( n\sqrt{n^2 - 4m^2} - n^2 + 2m^2 \right) - \frac{\tilde{\lambda}^2}{8} \sum_{n=1}^{\infty} \left( n(n^2 + 2m^2)\sqrt{n^2 - 4m^2} - n^4 + 10m^4 \right) + O(\tilde{\lambda}^3) . \quad (6.30)$$

Compared to (6.27) here we get extra polynomial terms which represent contributions of other string modes “external” to the $SU(2)$ sector.\footnote{In essence, the reason why we are able to match the string and LL results using a particular regularization is that contributions of all “external” modes happen to be simply polynomial in $n$.} The second, order $\tilde{\lambda}^2$, sum over $n$ is divergent, which is an artifact of the naive expansion of the UV finite sum over $n$ in powers of $\tilde{\lambda}$. A procedure for extracting the coefficient of the $\tilde{\lambda}^2$ term turns out to be equivalent to the $\zeta$-function regularization prescription for (6.30).\footnote{We are grateful to N. Beisert for an important explanation related to this point. The $\zeta$-function regularization procedure does not apply at higher orders in $\tilde{\lambda}$.} We then finish with exactly the same expression (6.28) as found in the $\zeta$-function regularized LL model. As was shown in [34] for a similar $SL(2)$ sector solution, using the $\zeta$-function regularized expression instead of formally divergent $\tilde{\lambda}^2$ term in the 1-loop string energy\footnote{Ref.\[34\] considered the “string” Bethe ansatz equations [31], but to given order $\tilde{\lambda}^2$ they are the same as the gauge theory Bethe ansatz equations.} one matches the result that one finds from the Bethe ansatz\footnote{Ref.\[34\] considered the “string” Bethe ansatz equations [31], but to given order $\tilde{\lambda}^2$ they are the same as the gauge theory Bethe ansatz equations.} at the 1/$J$ order; the same statement formally applies also in the $SU(2)$ sector.
To conclude, the above discussion indicates that at quadratic oscillator level the $\zeta$-function regularization of the quantum LL model is the right regularization prescription needed to reproduce the gauge theory and string theory results. The ability to obtain both the string and gauge theory results from the quantum LL Hamiltonian gives an “explanation” of their matching. The question of which regularization should be used at quartic and higher interaction order still remains nontrivial. It is not a priori clear if the regularizations dictated by the string theory and the gauge theory sides will be the same, but we expect that the subleading $\tilde{\lambda}/J$ and $\tilde{\lambda}^2/J$ corrections will continue to agree between the two sides.

7 $1/J$ correction to the energy of folded string

The circular solution discussed in the previous section was homogeneous: derivatives of the background fields were constant, and as a result the coefficients in the fluctuation Lagrangian were also constant, leading to simple algebraic equations for the characteristic frequencies. The quantization of more general static but inhomogeneous solutions like the folded rotating $(J_1, J_2)$ string \[44, 6, 7\] presents a challenge since here finding the spectrum of the fluctuation operator appears to be technically complicated (cf. \[3\]). Considering only the LL sector of fluctuations leads to a simplification, and, in view of the above discussion, should be enough (at least to leading order) for computing the 1-loop correction to the energy of such an inhomogeneous solution.

7.1 Classical solution and fluctuations near it

Starting with the LL action \[2.19\], the LL solution we are going to consider here is the order $\tilde{\lambda}$ part of the full $(J_1, J_2)$ folded string background, i.e. \[14\] 20

$$\varphi = -wt, \quad \psi = \psi(\sigma), \quad (7.1)$$

where $\psi$ is a solution of the 1-d sine-Gordon equation:

$$\psi'' + 2w \sin 2\psi = 0, \quad w \equiv \frac{w}{\tilde{\lambda}}, \quad (7.2)$$

i.e.

$$\psi'^2 = 2w(\cos 2\psi - \cos 2\psi_0), \quad (7.3)$$

20Recall that $\varphi = (\varphi_1 - \varphi_2)/2$, where $\varphi_1 = w_1 t$, $\varphi_2 = w_2 t$, so that $w = (w_2 - w_1)/2 > 0$ (we shall assume that $w_2 > w_1$). The solution describes a string located at the center of AdS, while it is folded and stretched along a big circle in $S^5$, and rotates about its center of mass with frequency $w_2$. Its center of mass also moves (with frequency $w_1$) along an orthogonal big circle of $S^5$. 

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where $\psi$ changes from $-\psi_0$ to $\psi_0$. The solution can be expressed in terms of the Jacobi elliptic functions as:

$$
\sin \psi(\sigma) = \sqrt{q} \sin(C\sigma, q), \quad \cos \psi(\sigma) = \text{dn}(C\sigma, q),
$$

(7.4)

$$
q = \sin^2 \psi_0, \quad \sqrt{w} = \frac{1}{\pi} K(q), \quad C = \frac{2}{\pi} K(q), \quad K(q) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - q \sin^2 x}}.
$$

(7.5)

Here $\psi_0$, $w$, and $A$ are functions of the parameter $q$ which itself is related to the ratio of the two spins $J_1$, $J_2$ through the integral [14]:

$$
\frac{J_1 - J_2}{J} = \int_0^{2\pi} \frac{d\sigma}{2\pi} \cos 2\psi, \quad J = J_1 + J_2.
$$

(7.6)

The order $\tilde{\lambda}$ term in the classical energy can be expressed as [7]

$$
E = J \left[ 1 + \tilde{\lambda} F_1 \left( \frac{J_2}{J} \right) + O(\tilde{\lambda}^2) \right],
$$

(7.7)

$$
F_1 \left( \frac{J_2}{J} \right) = \frac{2}{\pi^2} K(q) \left[ E(q) - (1 - q) K(q) \right], \quad q = q \left( \frac{J_2}{J} \right), \quad \frac{E(q)}{K(q)} = 1 - \frac{J_2}{J},
$$

(7.8)

where $E(q) = \int_0^{\pi} dx \sqrt{1 - q \sin^2 x}$. Expanding in small $q$ (small $\psi_0$, i.e. small string size) or small $J_2/J$ we get [11 14]

$$
\alpha \equiv \frac{J_2}{J} = \frac{q}{2} + \frac{q^2}{16} + O(q^3), \quad q = 2\alpha - \frac{\alpha^2}{2} + O(\alpha^3).
$$

(7.9)

$$
E_0 = J \left[ 1 + \frac{1}{2} \tilde{\lambda} \alpha \left( 1 + \alpha + \frac{3\alpha^2}{8} + O(\alpha^3) \right) + O(\tilde{\lambda}^2) \right].
$$

(7.10)

To find the leading $1/J$ correction to the classical energy we need as in (6.10) to sum up the characteristic frequencies of fluctuations near the above solution. Starting from (2.19) or (2.21), the quadratic fluctuation Lagrangian is found to be

$$
L = 2g \dot{f} - \frac{1}{2} \tilde{\lambda} \left[ f'^2 + g'^2 - V_1(\sigma)f^2 - V_2(\sigma)g^2 \right],
$$

(7.11)

and the corresponding equations for fluctuations are

$$
\dot{f} = -\frac{1}{2} [g'' + V_2(\sigma)g], \quad \dot{g} = \frac{1}{2} [f'' + V_1(\sigma)f],
$$

(7.12)

where the potentials depend on the elliptic function $\cos \psi(\sigma) = \text{dn}(A\sigma, q)$ in [124]:

$$
V_1(\sigma) = 12w \cos 2\psi - 4w \cos 2\psi_0, \quad V_2(\sigma) = 4w \cos 2\psi.
$$

(7.13)

For notational simplicity we have rescaled the time by a factor of $\tilde{\lambda}$; this factor is easy to restore in the expressions for the characteristic frequencies.
Since the potentials do not depend on time, to find the characteristic frequencies
\[ g \sim A_1(\sigma)e^{i\omega t} + c.c. , \quad f \sim A_2(\sigma)e^{i\omega t} + c.c. , \quad \text{(7.14)} \]
one is to solve the corresponding $2 \times 2$ matrix Schrödinger equation on a circle, i.e. to
find the spectrum of the operator
\[ Q = \begin{pmatrix} 0 & -\frac{d^2}{d\sigma^2} - V_2(\sigma) \\ \frac{d^2}{d\sigma^2} + V_1(\sigma) & 0 \end{pmatrix}. \quad \text{(7.15)} \]
The integrability of the LL model suggests that this problem should have a systematic
solution. Being unable at present to find the spectrum of $Q$ exactly, we shall resort
to perturbation theory in string length $\psi_0$ or $q$, or equivalently, in the “filling fraction”
$\alpha \equiv \frac{\psi_0}{\pi}$.

7.2 Short string expansion

Let us start with the extreme short string limit $\psi_0 \ll 1$, i.e. $\alpha \to 0$, $q \to 0$. Then
$\sin \psi \approx \psi$ and the sine-Gordon equation (7.2) becomes linear
\[ \psi'' + 4w\psi = 0 , \quad \text{(7.16)} \]
with the solution
\[ \psi = \sqrt{q} \sin m\sigma \left[ 1 + O(q) \right] , \quad w = \frac{1}{4}m^2 , \quad \text{(7.17)} \]
where $m$ is integer. The condition on $w$ follows from the periodicity in $\sigma$. The integer $m$
represents the number of folds of the string. Then the potentials in (7.13) are constant
\[ V_1 = V_2 = m^2 + O(q) , \quad \text{(7.18)} \]
and we readily find the characteristic frequencies
\[ \omega_n = \pm \frac{1}{2} \tilde{\lambda} |n^2 - m^2| + O(q) . \quad \text{(7.19)} \]
Note that up to an $n$-independent term these are the same as the BMN frequencies.
Since these frequencies depend only on $\tilde{\lambda} = \frac{\lambda}{J_2^2}$ and not on $J_2$, this limit represents a
nearly point-like string and the correction to the ground-state energy should vanish.
This is indeed what one finds using the $\zeta$-function regularization and observing that

\[ ^{21}\text{The general classical finite gap solution of the LL model is known in terms of } \theta\text{-functions }^{57, 58}. \]

\[ \text{Linearizing it near the folded string solution one should be able to extract, in principle, the spectrum of the operator } Q. \]
since the Hamiltonian in (7.11) with $V_1, V_2$ in (7.18) is not positive definite, here the contributions of the frequencies of the $n^2 < m^2$ modes to the vacuum energy enter with a negative sign. This follows also from the general prescription for the vacuum energy in terms of characteristic frequencies of a mixed system of oscillators in [60] which was used in [29]:

$$E_1 = \frac{1}{2} \sum_{p=1}^{N} \text{sign}(C_p) \omega_{p,0} + \frac{1}{2} \sum_{n=1}^{\infty} \sum_{I=1}^{2N} \text{sign}(C^{(n)}_I) \omega_{I,n},$$  

(7.20)

where

$$C_p = 2m_{11}(\omega_{p,0})\omega_{p,0} \prod_{q \neq p}(\omega_{p,0}^2 - \omega_{q,0}^2), \quad C^{(n)}_I = m_{11}(\omega_{I,n}) \prod_{j \neq I}(\omega_{I,n} - \omega_{j,n}).$$  

(7.21)

Here $F^T(\omega_{I,n}, n) = F(-\omega_{I,n}, -n)$ is the matrix whose zero determinant condition is used to find characteristic frequencies and $m_{11}$ is the minor of $F$, i.e. the determinant of the matrix obtained from $F$ by removing the first row and first column. Since here the minor is essentially $m_{11} \sim (n^2 - m^2)$ and its sign depends on $n$, it is important to use the above general prescription with sign factors. As a result, $E_1 = 0 + O(q)$.

Let us now consider subleading corrections. Setting $m = 1$ we get from the small $q$ expansion of the elliptic functions in the general solution (7.4)

$$\sin \psi(\sigma) = \sqrt{q} \sin \sigma \left[ 1 + \frac{1}{4}q \cos^2 \sigma + O(q^2) \right],$$  

(7.22)

$$K(q) = \frac{\pi}{2} \left( 1 + \frac{q}{4} + \frac{9q^2}{64} \right) + O(q^3), \quad w = \frac{1}{4} \left( 1 + \frac{q}{2} + \frac{11q^2}{32} \right) + O(q^3).$$  

(7.23)

The potentials in (7.13) become

$$V_1 = 1 + \frac{9q}{2} - 6q \sin^2 \sigma + O(q^2), \quad V_2 = 1 + \frac{q}{2} - 2q \sin^2 \sigma + O(q^2).$$  

(7.24)

Then the eigenvalue problem for the fluctuation operator (7.15) or the set of equations for the characteristic frequencies becomes (see (7.14))

$$2i\omega A_1 = A''_2 + (1 + \frac{9q}{2} - 6q \sin^2 \sigma)A_2 + O(q^2),$$

$$-2i\omega A_2 = A''_1 + (1 + \frac{q}{2} - 2q \sin^2 \sigma)A_1 + O(q^2).$$  

(7.25)

\[^{22}\text{It is important to stress that non-positivity of the quadratic fluctuation Hamiltonian here does not imply an instability: the characteristic frequencies are real. The folded string solution is definitely stable for small string length. The reason why we do not have an instability as compared to, say, the inverted harmonic oscillator is that here both the "p\,^2n" and "q\,^2n" terms in the canonical Hamiltonian (f and g play the role of momentum and coordinate in (7.11)) may have opposite signs, i.e. one may have a "ghost tachyon", and this just means that the sign of the energy changes, but energy remains real.}\]
Combining these we obtain the fourth-order differential equation
\[ UA_1 = 4\omega^2 A_1 , \quad U = U_0 + qU_1 + O(q^2) , \] (7.26)
where
\[ U_0 = \frac{d^4}{d\sigma^4} + 2 \frac{d^2}{d\sigma^2} + 1 , \quad U_1 = (5 - 8 \sin^2 \sigma) \frac{d^2}{d\sigma^2} - 4 \sin 2\sigma \frac{d}{d\sigma} + 1 . \] (7.27)
Since \( A_1 \) must be periodic we can use the Fourier expansion
\[ A_1 = \sum_n c_n e^{in\sigma} , \]
where the coefficients then satisfy
\[ (n^4 - 2n^2 + 1 - 4\omega^2)c_n = 0 , \] (7.29)
which gives of course the same leading-order expression as found in (7.19) for \( m = 1 \), i.e. \( \omega = \pm \frac{1}{2}(n^2 - 1) \) (here we ignore the factor of \( \tilde{\lambda} \) which was absorbed in \( t \)).

To compute the order \( q \) correction to \( \omega \) we may use perturbation theory. The unperturbed \((q = 0)\) operator is \( U_0 \) in (7.27) and the unperturbed eigenvectors are \( v_n = (...)0,0,1,0,0,... \) with 1 at position \( n \), with coordinate \( \sigma \)-space eigenfunctions \( \langle \sigma|v_n \rangle = e^{in\sigma} \). Then the diagonal matrix element of the perturbation operator \( U_1 \) in (7.27) in this basis gives\(^{23} \) \( \langle v_n|U_1|v_n \rangle = -(n^2 - 1) \). As a result, we find
\[ 4\omega^2 = (n^2 - 1)^2 - q(n^2 - 1) , \] (7.30)
so that to the linear order in \( q \)
\[ \omega_{n \neq \pm 1} = \pm \frac{1}{2} \left( n^2 - 1 - \frac{1}{2}q + O(q^2) \right) , \quad \omega_{\pm 1} = 0 . \] (7.31)
Using these frequencies we find for the \( \zeta \)-function regularized leading quantum correction to the energy at order \( O(q) \):
\[ E_1 = \frac{1}{4} \tilde{\lambda} \left[ q + O(q^2) \right] = \frac{1}{2} \tilde{\lambda} \left[ \alpha + O(\alpha^2) \right] , \] (7.32)
where we used (7.9) and took into account the sign factors in the general expression for the vacuum energy in (7.20).

\(^{23}\)Each eigenvalue of the unperturbed operator (with the exception of \( n = 0 \) one) is double degenerate, i.e. the corresponding eigenfunctions are \( e^{i\sigma n}, e^{-i\sigma n} \). The \( 2 \times 2 \) matrix on these eigenvalues is diagonal.
Combining this quantum correction with the expression for the classical energy (7.10) we get for the leading $1/J$ correction to the energy of the folded string:

$$E = J + \frac{\lambda}{2J} \left[ \frac{J_2}{J} \left( 1 + \frac{1}{J} + O \left( \frac{1}{J^2} \right) \right) + \frac{1}{2} \left( \frac{J_2}{J} \right)^2 \left( 1 + \frac{a_2}{J} + O \left( \frac{1}{J^2} \right) \right) + O \left( \frac{J_2}{J} \right)^3 \right] + O \left( \frac{\lambda^2}{J^3} \right), \quad (7.33)$$

or, equivalently,

$$E = J + \frac{\lambda J_2}{2} \left( 1 + \frac{2 + J_2}{2J} + O \left( \frac{J_2}{J} \right) \right) + O(\lambda^2). \quad (7.34)$$

Note that this formally matches the expression for the near-BMN correction in (3.62) if we set $L = J$, $M = J_2 = 2$

$$E = J + \tilde{\lambda} \left( 1 + \frac{2}{J} + O \left( \frac{1}{J^2} \right) \right) + O(\tilde{\lambda}^2). \quad (7.35)$$

This should be expected since a short folded string should be very close to a BMN state. This correspondence with the BMN spectrum was known at the classical level where $J_2$ plays the role of the number of impurities $M$ (assuming, e.g., that all mode numbers $n_j$ are equal to 1); remarkably, it holds also at the quantum $1/J$ level.

We compute the $1/J$ correction at the next $(J_2/J)^2$ order in Appendix C. Using again the $\zeta$-function regularization we find the following expression

$$a_2 = \frac{1}{2} - \frac{\pi^2}{3} \approx -2.79. \quad (7.36)$$

The discussion of the folded string solution in the $SU(2)$ sector can be repeated for the folded string $(S, J)$ solution in the $SL(2)$ sector; we give some details of this case in Appendix D.

It would be interesting to compare these results for the leading $1/J$ correction to the energy of spinning strings found from the LL model with the direct Bethe ansatz and string theory computations. We expect that as in the circular string case discussed in [25], the two will match, with the $\zeta$-function regularized LL expression providing an “explanation” of their matching.

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Appendix A: $\tilde{\lambda}/J^2$ corrections from the Bethe ansatz

In this appendix we compute the energies of an $M$-impurity state in the $SU(2)$ sector up to and including $1/J^2$ corrections. Starting with the Bethe equations

$$\left(\frac{u_i + i/2}{u_i - i/2}\right)^J = \prod_{j \neq i}^{M} \frac{u_i - u_j + i}{u_i - u_j - i}, \quad (A.1)$$

we can write $\text{(A.1)}$ up to order $1/J^2$ accuracy as

$$\exp\left[\left(\frac{i}{u_i} - \frac{i}{12u_i^3}\right)J\right] = \exp\left[\sum_{j \neq i}^{M} \frac{2i}{u_i - u_j}\right]. \quad (A.2)$$

If we rewrite $1/u_i$ as

$$\frac{1}{u_i} = \frac{2\pi n_i}{J} + \Delta_i, \quad (A.3)$$

then we find the equation

$$J\Delta_i - \frac{(2\pi n_i)^3}{J^2} = \frac{4\pi}{J} \sum_{j \neq i}^{M} \frac{n_in_j}{n_j - n_i} + 2\sum_{j \neq i}^{M} \frac{(\Delta_i n_j^2 - \Delta_j n_i^2)}{(n_j - n_i)^2}. \quad (A.4)$$

We will assume that all $n_i$ are different. Up to the desired order, we can write $\Delta_i = \Delta_i^{(1)} + \Delta_i^{(2)}$, where

$$\Delta_i^{(1)} = \frac{4\pi}{J^2} \sum_{j \neq i}^{M} \frac{n_in_j}{n_j - n_i}, \quad (A.5)$$

and

$$\Delta_i^{(2)} = \frac{(2\pi n_i)^3}{12J^3} + \frac{8\pi}{J^3} \left[\sum_{k \neq j \neq i}^{M} \frac{n_in_jn_k}{(n_j - n_i)(n_k - n_i)} + \sum_{j \neq i}^{M} \frac{n_in_j(n_i^2 + n_j^2)}{(n_j - n_i)^3}\right]. \quad (A.6)$$

Now the energy is given by

$$E = \frac{\lambda}{8\pi^2} \sum_{i}^{M} \frac{1}{u_i^2 + 1/4} \approx \frac{\lambda}{8\pi^2} \sum_{i}^{M} \left(\frac{2\pi n_i}{J}\right)^2 \left[1 + \frac{2\Delta_i J}{2\pi n_i} + \frac{\Delta_i^2 J^2}{(2\pi n_i)^2}\right] \left(1 - \left(\frac{\pi n_i}{J}\right)^2\right). \quad (A.7)$$
Hence as an expansion in $1/J$ we find

$$E^{(0)} = \frac{\lambda}{2} \sum_{i} n_{i}^{2},$$  \hspace{1cm} (A.8)

$$E^{(1)} = \frac{j \lambda}{2\pi} \sum_{i} \Delta_{i}^{(1)} n_{i} = -\frac{\lambda}{j} \sum_{j \neq i} n_{i}n_{j} = \frac{\lambda}{j} \sum_{i} n_{i}^{2},$$  \hspace{1cm} (A.9)

where we used the momentum constraint (3.20) in the last step, and

$$E^{(2)} = -\frac{(2\pi)^{2}\lambda}{8J^{2}} \sum_{i} n_{i}^{4} + \frac{j \lambda}{2\pi} \sum_{i} \Delta_{i}^{(2)} n_{i} + \frac{j^{2}\lambda}{8\pi^{2}} \sum_{i} (\Delta_{i}^{(1)})^{2}$$

$$= -\frac{\pi^{2}\lambda}{6J^{2}} \sum_{i} n_{i}^{4} + \frac{4\lambda}{J^{2}} \left[ \sum_{j \neq i \neq k} \frac{n_{i}^{2}n_{j}n_{k}}{(n_{j} - n_{i})(n_{k} - n_{i})} + \sum_{j \neq i} \frac{n_{i}^{2}n_{j}(n_{i}^{2} + n_{j}^{2})}{(n_{j} - n_{i})^{3}} \right]$$

$$+ \frac{2\lambda}{J^{2}} \sum_{j \neq i} \sum_{k \neq j} \frac{n_{j}^{2}n_{k}}{(n_{k} - n_{i})(n_{j} - n_{i})}. \hspace{1cm} (A.10)$$

Symmetrizing the sums, and splitting the last term into a piece where $k = j$ and another piece where $k \neq j$, we find

$$E^{(2)} = -\frac{\pi^{2}\lambda}{6J^{2}} \sum_{i} n_{i}^{4} - \frac{2\lambda}{J^{2}} \sum_{j \neq i} \frac{n_{i}n_{j}(n_{i}^{2} - n_{i}n_{j} + n_{j}^{2})}{(n_{j} - n_{i})^{2}}. \hspace{1cm} (A.11)$$

**Appendix B: Evaluation of sums for circular string**

Here we shall provide some details of the computation of the coefficient $c_{2}$ in (6.20). We rewrite the sums $\mathcal{E}_{1}, \mathcal{E}_{2}$ in (6.13),(6.14) so that the summations are from 0 to $\infty$. To avoid contributions from unstable modes we shall formally consider only terms with $n > 2m$ and take the real part of the series. These complications are absent in the similar $SL(2)$ case where the solution is stable, but the $SU(2)$ case is useful for illustrating the computational procedure. Then

$$\mathcal{E}_{1} = -\frac{2\lambda m^{2}}{3J} \left[ \sum_{n \neq k > 2m} a(n,k) + \sum_{n,k > 2m} b(n,k) \right], \hspace{1cm} (B.1)$$

with

$$a(n,k) = \frac{k \sqrt{w_{n}/w_{n+k}} + n \sqrt{w_{n}/w_{n+k}} - (n+k) \sqrt{w_{n+k}/w_{n+k}}}{n^{2}w_{n} + k^{2}w_{k} + (n+k)^{2}w_{n+k}}, \hspace{1cm} (B.2)$$

$$b(n,k) = \frac{-k \sqrt{w_{n}/w_{n-k}} + n \sqrt{w_{n}/w_{n-k}} - (n-k) \sqrt{w_{n-k}/w_{n-k}}}{n^{2}w_{n} + k^{2}w_{k} + (n-k)^{2}w_{n-k}}. \hspace{1cm} (B.3)$$
Expanding the above coefficients for large \( n \) or large \( k \) we find that they go as \( 1/n^2 \), and \( 1/k^2 \), so that the series are convergent. Numerical computation gives, for \( m = 1 \),
\[
\sum_{n \neq k > 2} a(n, k) \simeq 0.33 .
\]

To extract the real part of the second series \( \sum b \) we split the sum into two sums with \( n > k + 2m \) and \( k > n + 2m \). Then one can show that
\[
\sum_{k, n > k + 2m} b(n, k) + \sum_{n, k > n + 2m} b(n, k) = 2 \sum_{n, k > 2m} a(n, k) .
\]
The sum \( E_2 \) in (6.14) can be rewritten as
\[
E_2 = -4m^2 \tilde{\lambda} \sum_{k > 2m} \left( \frac{1}{\sqrt{w_{2k}}} - \frac{\sqrt{w_{2k}}}{w_k} \right)^2 = -2m^2 \tilde{\lambda} \sum_{n > 2m} a(n, n) .
\]
Thus the convergent sum \( E_1 + E_2 \) is
\[
E_1 + E_2 = -\frac{2m\tilde{\lambda}}{J} \left[ \sum_{n, k > 2m} a(n, k) + \frac{4}{3} \sum_{n > 2m} a(n, n) \right] .
\]
Numerical evaluation for \( m = 1 \) gives
\[
E_1 = -0.76 \frac{\tilde{\lambda}}{J} , \quad E_2 = -0.147 \frac{\tilde{\lambda}}{J} .
\]
The sum \( E_3 \) in (6.17) is divergent; by using the \( \zeta \)-function regularization we get
\[
E_3 = -\frac{\tilde{\lambda}}{4J} \left[ S_1 - 5 - 10m^2 - 2(S_2 - \frac{5}{2})(S_3 - S_4 - 10m^2) \right] ,
\]
where the convergent sums \( S_k \) are defined by
\[
S_1 = \sum_{n > 2m} \frac{16m^4}{n^2 - 4m^2} , \quad S_2 = \sum_{n > 2m} \left( \frac{n}{\sqrt{n^2 - 4m^2}} - 1 \right) ,
\]
\[
S_3 = \sum_{n > 2m} \left( \frac{n^3}{\sqrt{n^2 - 4m^2}} - n^2 - 2m^2 \right) , \quad S_4 = \sum_{n > 2m} \left( n\sqrt{n^2 - 4m^2} - n^2 + 2m^2 \right) .
\]
Numerical evaluation for \( m = 1 \) gives
\[
E_3 = 6.43 \frac{\tilde{\lambda}}{J} .
\]
Combining (B.8) and (B.10) we get
\[
E_2 = E_1 + E_2 + E_3 = 5.52 \frac{\tilde{\lambda}}{J} ,
\]
leading to the value of \( c_2 \) in (6.20).
Appendix C: Subleading term in folded string energy

Here we extend the computation of quantum correction to the folded string energy in section 7.2 to the next $q^2$ or $\alpha^2 = \frac{1}{32}$ order. To this order the potentials in \ref{eq:7.13} are

\begin{align*}
V_1 &= 1 + \frac{9q}{2} - 6q\sin^2 \sigma + \frac{75q^2}{32} - 3q^2\sin^2 \sigma \left(1 + \cos^2 \sigma\right) + O(q^3), \\
V_2 &= 1 + \frac{q}{2} - 2q\sin^2 \sigma + \frac{11q^2}{32} - q^2\sin^2 \sigma \left(1 + \cos^2 \sigma\right) + O(q^3),
\end{align*}

and so the $O(q^2)$ term in \ref{eq:7.26} is found to be

\begin{equation}
U_2 = \left(\frac{43}{16} - 4\sin^2 \sigma \left(1 + \cos^2 \sigma\right)\right) \frac{d^2}{d\sigma^2} - 2\sigma \sin 2\sigma \left(1 + \cos 2\sigma\right) \frac{d}{d\sigma} + \frac{15}{16}. \quad \text{(C.3)}
\end{equation}

We need to use the second-order perturbation theory in $q$ for the operator \ref{eq:7.26}, i.e. to combine the first order term for $U_2$ with second-order term for $U_1$. Let us denote the eigenvalues as

\begin{equation}
W_n = W_n^{(0)} + qW_n^{(1)} + q^2(W_n^{(2)} + W_n^{n(2)}) + O(q^3), \quad \text{(C.4)}
\end{equation}

where, as found in section 6.2, $W_n^{(0)} = (n^2 - 1)^2$, $W_n^{(1)} = -(n^2 - 1)$, and $W_n^{(2)}$ comes from second-order term in $U_1$ and $W_n^{n(2)}$ from $U_2$. Noting that the unperturbed eigenvalues are double degenerate, we find that the second-order perturbation theory corrections $W_n^{(2)}$ are found by solving the zero-determinant condition of the matrix

\begin{equation}
\begin{pmatrix}
\sum_{k\neq n,-n} \frac{n[U_1|k><k|U_1|n]}{W_n^{(0)} - W_k^{(0)}} - W_n^{(2)} & \sum_{k\neq n,-n} \frac{n[U_1|k><k|U_1|n]}{W_n^{(0)} - W_k^{(0)}} \\
\sum_{k\neq n,-n} \frac{n[U_1|k><k|U_1|n]}{W_n^{(0)} - W_k^{(0)}} & \sum_{k\neq n,-n} \frac{n[U_1|k><k|U_1|n]}{W_n^{(0)} - W_k^{(0)}} - W_n^{(2)}
\end{pmatrix} \quad \text{(C.5)}
\end{equation}

This matrix is diagonal, so the degeneracy is not lifted also in the second-order perturbation theory. Computing $W_n^{(2)}$ we obtain:

\begin{equation}
W_n^{(2)} = -\frac{n^2}{n^2 - 1}, \quad W_1^{(2)} = -\frac{3}{4}. \quad \text{(C.6)}
\end{equation}

To find $W_n^{n(2)}$ we consider the extension of \ref{eq:7.28} to the $q^2$ order

\begin{equation}
0 = \frac{n^4 - n^2(2 + q) + 1 + q - 4\omega^2}{c_n} - 2q(n - 2)(n - 1)c_{n-2} \\
- 2q(n + 2)(n + 1)c_{n+2} - q^2 \left[\frac{3n^2 - 15}{16}\right] c_n + (n - 2)(n - 1)c_{n-2} \\
+ (n + 2)(n + 1)c_{n+2} + \frac{1}{4}(n - 4)(n - 2)c_{n-4} + \frac{1}{4}(n + 4)(n + 2)c_{n+2} + O(q^3). \quad \text{(C.7)}
\end{equation}

Proceeding as at leading order in $q$, we find that

\begin{equation}
W_n^{n(2)} = -\frac{3n^2 - 15}{16}. \quad \text{(C.8)}
\end{equation}
Combining the above expressions, we get that $$\omega = 0$$ for $$n = \pm 1$$ while for $$n \neq \pm 1$$

$$4\omega^2 = (n^2 - 1)^2 - q(n^2 - 1) - q^2\left[\frac{n^2}{n^2 - 1} + \frac{1}{16}(3n^2 - 15)\right] + O(q^3) , \quad (C.9)$$
i.e. the characteristic frequencies are

$$\omega_{n \neq \pm 1} = \pm \frac{1}{2} \left[n^2 - 1 - \frac{1}{2} q - \frac{1}{32} q^2 \frac{3n^4 + 2n^2 + 11}{(n^2 - 1)^2} + O(q^3)\right]. \quad (C.10)$$

Computing their sum, we get for the quantum correction to the energy (restoring the $$\tilde{\lambda}$$ factor)

$$E_1 = \tilde{\lambda} \left[\frac{q}{4} - \frac{11q^2}{128} - \frac{q^2}{2} \sum_{n=2}^{\infty} \frac{3n^4 + 2n^2 + 11}{32(n^2 - 1)^2} + O(q^3)\right] + O(\tilde{\lambda}^2) . \quad (C.11)$$

The sum here is divergent. Applying again the $$\zeta$$-function regularization ($$\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$$), the sum can be easily computed\(^\text{24}\) and using the relation (7.9), we end up with the following expression for the 1-loop quantum correction to order $$O(\alpha^2)$$

$$E_1 = \frac{1}{2} \tilde{\lambda} \left[\alpha + \frac{1}{2} a_2 \alpha^2 + O(\alpha^3)\right] , \quad (C.12)$$

$$a_2 = \frac{1}{2} - 2\zeta(2) = \frac{1}{2} - \frac{\pi^2}{3} , \quad (C.13)$$

which leads to the expression in (7.33).

**Appendix D: Folded string solution in SL(2) sector**

Here we shall repeat the discussion of section 7 for a folded string solution in the SL(2) sector. The SL(2) sector describes strings rotating in AdS$_3$ part of AdS$_5$ and whose center of mass is moving along big circle of S$^5$, i.e. their energy is parametrized by the two spins ($S, J$). The fast string limit corresponds to $J$ being large with $S/J$ and $\tilde{\lambda} = \frac{1}{2\sqrt{J}}$ being fixed \(^3\).

The corresponding Landau-Lifshitz Lagrangian \(^{19}\) is similar to the one in the SU(2) case \(^2\, \text{II}\) with $U_a \rightarrow V_r$ ($V_r^* V_r = -1$, $V_r = \eta^{rs} V_s$, $\eta_{rs} = (-1, 1)$)

$$L = -i V_r^* \partial_0 V_r - \tilde{\lambda} \frac{1}{2} |D_1 V_r|^2 + O(\tilde{\lambda}^2) . \quad (D.1)$$

In the parametrization

$$V_0 = \cosh \rho \, e^{i\eta}, \quad V_1 = \sinh \rho \, e^{-i\eta} , \quad (D.2)$$

\(^{24}\)We thank N. Beisert for this suggestion.
where $\rho$ is the radial $AdS_5$ coordinate and $\eta = \frac{1}{2}(t - \phi)$ the above LL Lagrangian (D.1) becomes
\[
L = - \cosh 2\rho \dot{\eta} - \frac{\bar{\lambda}}{2}(\rho'^2 + \sinh^2 2\rho \, \eta'^2) .
\] (D.3)

The folded string solution [4] describes a string which is stretched in the radial direction $\rho$, rotates in a plane in $AdS_5$ about its center of mass and also moves along a big circle in $S^5$. The string solution is given by $t = \kappa \tau$, $\phi = \omega_1 \tau$, $\varphi_3 = w_3 \tau$, and $\rho = \rho(\sigma)$. To leading order in the $1/J$ expansion, the corresponding solution of the LL equations is $\eta = -w \tau$, $w = \omega_1 - \kappa^2$ (we shall assume $\omega_1 > \kappa$), and $\rho$ satisfies the equation:
\[
\rho'' + 2w \sinh 2\rho = 0 , \quad w = \frac{w}{\bar{\lambda}} ,
\] (D.4)
i.e.
\[
\rho^2 = 2w(\cosh 2\rho_0 - \cosh 2\rho) ,
\] (D.5)
with $\rho$ changing from 0 to $\rho_0$. As discussed in [7, 26], this folded $SL(2)$ sector solution is related to the folded solution (7.4) in the $SU(2)$ sector by the following analytic continuation
\[
\rho \to i\psi, \quad \eta \to \varphi, \quad \kappa \to w_1, \quad \omega_1 \to w_2, \quad w_3 \to \kappa .
\]
Under this transformation the equation (D.4) becomes (7.2), and also the LL Lagrangian (D.3) becomes (2.19) up to an overall sign.

As in the $SU(2)$ case we may consider quadratic LL fluctuations near the folded string solution (we again rescale the time coordinate by $\bar{\lambda}$)
\[
\dot{f} = -\frac{1}{2} \left[ g'' + 4w(3 \cosh 2\rho - 2 \cosh 2\rho_0) \, g \right] , \quad \dot{g} = \frac{1}{2} \left( f'' + 4w \cosh 2\rho \, f \right) .
\] (D.6)
These equations are exactly the same as in the $SU(2)$ sector (7.12),(7.13) with $\rho \to i\psi$.

Here the short string limit corresponds to $\rho_0 \to 0$. The solution of (D.4) in the small $\rho_0$ limit is (for 1-fold case $m = 1$):
\[
\sinh \rho = -\sqrt{-q} \sin \sigma \left( 1 + \frac{q}{4} \cos^2 \sigma + O(q^2) \right) , \quad q \equiv -\sinh^2 \rho_0 .
\] (D.7)
Then the potentials in the fluctuation equations are the same as in the $SU(2)$ case in (7.24). The computation of the leading quantum correction to the classical energy follows the same steps as in the $SU(2)$ case. The only difference is that $q = -\sinh^2 \rho_0$ is now negative. We get as in (C.11)
\[
E_1 = \bar{\lambda} \left[ \frac{q}{4} - \frac{11q^2}{128} - \frac{q^2}{2} \sum_{n=2}^{\infty} \frac{3n^4 + 2n^2 + 11}{32(n^2 - 1)^2} + O(q^3) \right] .
\] (D.8)
To leading $\bar{\lambda}$ order the expression for the $AdS_3$ spin $S$ is [29]
\[
S = J \int_0^{2\pi} d\sigma \sinh^2 \rho = -\frac{1}{2} q J \left[ 1 + \frac{1}{16} q + O(q^2) \right] ,
\] (D.9)
where we have given the expansion for the folded solution in the small $S/J$ limit. Then
\[
q = -2\alpha - \frac{1}{2} \alpha^2 + O(\alpha^3) , \quad \alpha \equiv \frac{S}{J} . \tag{D.10}
\]
Taking into account the expansion for the classical string energy in the short string limit \[4\]
\[
E_0 = J + S + \frac{\lambda \alpha}{2J} \left[ 1 - \frac{\alpha}{2} + \frac{3\alpha^2}{8} + O(\alpha^3) \right] + O(\lambda^2) , \tag{D.11}
\]
and adding the regularized expression for $E_1$ in (D.8) we finish with (cf. \(7.33\))
\[
E = J + S + \frac{\lambda}{2J} \left[ \alpha \left( 1 - \frac{1}{J} + O\left(\frac{1}{J^2}\right) \right) - \frac{1}{2} \alpha^2 \left( 1 - \frac{a_2}{J} + O\left(\frac{1}{J^2}\right) \right) + O(\alpha^3) \right] + O\left(\frac{\lambda^2}{J^3}\right) , \tag{D.12}
\]
where again
\[
a_2 = \frac{1}{2} - \frac{\pi^2}{3} . \tag{D.13}
\]
As in the $SU(2)$ sector, if we formally set $S = 2$ we get
\[
E = J + 2 + \frac{\lambda}{J} \left( 1 - \frac{2}{J} + O\left(\frac{1}{J^2}\right) \right) + ... \tag{D.14}
\]
which matches again the near-BMN two-impurity result \[38\].
Note that a similar expression for the energy with the $1/J$ quantum corrections computed from regularized quantum LL Hamiltonian can be readily obtained \[25\] in the case of circular $(S, J)$ solution (for which the full string 1-loop correction was found in \[29\]). In the small string limit one gets
\[
E = J + S + \frac{\lambda m^2}{2J} \left[ \alpha \left( 1 - \frac{1}{J} + O\left(\frac{1}{J^2}\right) \right) + O(\alpha^2) \right] + O\left(\frac{\lambda^2}{J^3}\right) , \quad \alpha = \frac{S}{J} . \tag{D.15}
\]
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