On the notions of singular domination and (multi-)singular hyperbolicity

To the Memory of Professor Shantao Liao

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Abstract  The properties of uniform hyperbolicity and dominated splitting have been introduced to study the stability of the dynamics of diffeomorphisms. One meets difficulties when trying to extend these definitions to vector fields and Shantao Liao has shown that it is more relevant to consider the linear Poincaré flow rather than the tangent flow in order to study the properties of the derivative. In this paper, we define the notion of singular domination, an analog of the dominated splitting for the linear Poincaré flow which is robust under perturbations. Based on this, we give a new definition of multi-singular hyperbolicity which is equivalent to the one recently introduced by Bonatti and da Luz (2017). The novelty of our definition is that it does not involve the blow-up of the singular set and the rescaling cocycle of the linear flows.

Keywords  multi-singular hyperbolicity, singular domination, star vector field, linear Poincaré flow

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1 Introduction

The dynamical properties of a system (a flow or diffeomorphism) which are robust, i.e., which persist under $C^1$ small perturbations, are often associated with invariant structures in the tangent bundle. A famous example of such a property is the $C^1$ structural stability which has been characterized [16,20,24] by the uniform hyperbolicity of the tangent dynamics.

Among other robust properties that have been studied one can mention Liao’s star property [19] (the robust hyperbolicity of all the periodic orbits) and the robust transitivity (the robust existence of a dense orbit). Various forms of hyperbolicity have been introduced to investigate them, such as the dominated splittings that we discuss below.

These concepts are in general easier to understand in the case of diffeomorphisms (see, for example, [28] for a survey). The transposition to flows leads to difficulties due to the presence of singularities: the
direct generalizations may not exist, or may not persist under small perturbations, and new definitions adapted to that setting are necessary.

A famous example of a flow whose behavior differs from diffeomorphisms is the Lorenz system [22] and its geometrical models [1, 15]: it exhibits a robustly transitive attractor where periodic orbits accumulate on a singularity in a robust fashion. Studying the robust transitivity in dimension three, the notion of singular hyperbolicity has been coined in [26]. Recently Bonatti and da Luz [6] have introduced a more general property called multi-singular hyperbolicity which allows to manage systems in higher dimension where singularities with different stable dimensions are linked by the dynamics. They have proved that the star property is generically equivalent to multi-singular hyperbolicity. Thus the long standing problem of characterizing the star flows, proposed by Liao and Mañé in the early 1980s, eventually obtained a generic answer, formulated in terms of the new notion of multi-singular hyperbolic sets.

Bonatti-da Luz’s definition involves dealing with several technical difficulties and working in an abstract setting. One of the main objectives of this text is to give an alternative, and easier to introduce, definition of the multi-singular hyperbolicity. This new definition involves only the well-known notions of stable and unstable manifolds of the singularities and the classical linear Poincaré flow introduced by Liao [21]. To that purpose, we first define and investigate weaker structures of the tangent space, over which the singular domination is robustly transitive in dimension three, the notion of multi-singular hyperbolicity. This new definition involves only the well-known notions of stable and multi-singular hyperbolicities are defined. In particular, we introduce as well the definition of the star property, proposed by Liao and Mañé in the early 1980s, eventually obtained a more adapted.

Throughout this paper, we consider the set $\mathcal{X}^1(M)$ of $C^1$ vector fields on a closed manifold $M$. Given $X \in \mathcal{X}^1(M)$, we denote by $\text{Sing}(X)$ the set of singularities, by $(\varphi_t)_{t \in \mathbb{R}}$ the flow generated by $X$ and by $(D\varphi_t)_{t \in \mathbb{R}}$ the linear flow on the tangent bundle $TM$ induced by the derivative. A vector field $X \in \mathcal{X}^1(M)$ is star if for any vector field $Y$ in a $C^1$ neighborhood of $X$, all the periodic orbits and singularities of $Y$ are hyperbolic. It is robustly transitive if for any vector field $Y$ in a $C^1$ neighborhood of $X$, there exists an orbit which is dense in $M$.

1.1 Dominated splitting

One of the weakest forms of hyperbolicity is the dominated splitting that first appeared in the works of Liao [20], Mañé [23] and Pliss [27] about the stability conjecture. It may be defined in the general setting of linear flows as follows.

Let us consider an invariant set $\Lambda \subset M$, a linear bundle $B \rightarrow \Lambda$ over a space $\Lambda$ and a linear flow $(A_t)_{t \in \mathbb{R}}$ on $B$ which extends the flow $(\varphi_t)_{t \in \mathbb{R}}$ on $\Lambda$: the map $A_t$ sends the fiber $B(\varphi_t(x))$ above a point $x$ to the fiber $B(\varphi_t(x))$, is linear, and satisfies the relation $A_{s+t} = A_s \circ A_t$.

An invariant splitting $B = E \oplus F$ into linear sub-bundles with constant dimension is dominated for $(A_t)_{t \in \mathbb{R}}$ if there exist $\eta, T > 0$ such that

$$\|A_t |_{E(x)}\| \cdot \|A_{-t} |_{F(\varphi_t(x))}\| < e^{-\eta t} \text{ for any } x \in \Lambda \text{ and } t > T.$$  

The dimension $i = \dim(E)$ is called the index of the splitting. To emphasize the roles of $\eta$ and $T$, one also says that the splitting $E \oplus F$ above is an $(\eta, T)$-dominated splitting.

The bundle $E$ is uniformly contracted by $(A_t)_{t \in \mathbb{R}}$ if there are $\eta, T > 0$ such that

$$\|A_t |_{E(x)}\| < e^{-\eta t} \text{ for any } x \in \Lambda \text{ and } t > T.$$  

$F$ is uniformly expanded if there are $\eta, T > 0$ such that $\|A_{-t} |_{F(x)}\| < e^{-\eta t}$ for any $x \in \Lambda$, $t > T$.

For diffeomorphisms a dominated splitting of the tangent bundle occurs once the system is robustly transitive [7]. In the flow case this is no longer true at the level of the tangent flow (see the example in [3] and in Proposition 3.4), but another linear flow, defined by Liao [21] and that we recall now, is more adapted.

Let $N'_{M \setminus \text{Sing}(X)}$ be the normal bundle, i.e., the quotient of the tangent bundle by the flow direction $\mathbb{R} X$ on the regular set $M \setminus \text{Sing}(X)$. The tangent flow induces on the quotient a flow $(\Psi_t)_{t \in \mathbb{R}}$ that is called the linear Poincaré flow (see also Subsection 2.2).
With this new tool, the property of robustly transitive diffeomorphisms is generalized to flows: the linear Poincaré flow of robustly transitive flows admits a dominated splitting [10].

1.2 Singular dominated splitting

Any dominated splitting for a continuous linear flow over a compact base persists under perturbations [8, Appendix B.1]. The existence of a dominated splitting for the tangent flow is thus a robust property. In the case of the linear Poincaré flow this may not be the case (since it is not defined in the singularities and the base is therefore not compact): a dominated splitting over a set containing singularities may not be preserved after small perturbations. For that reason, we propose to introduce the following stronger notion.

Definition 1.1. Let \( X \in \mathcal{X}^1(\mathcal{M}) \) and \( \Lambda \) be an invariant compact set. A decomposition of the normal bundle \( \mathcal{N}|_{\Lambda \setminus \text{Sing}(X)} = \mathcal{N}_1 \oplus \mathcal{N}_2 \) which is invariant by the linear Poincaré flow \((\Psi_t)_{t \in \mathbb{R}}\) is a singular dominated splitting of index \( i \) if

1. \( \mathcal{N}_1 \oplus \mathcal{N}_2 \) is a dominated\(^1\) splitting of index \( i \);
2. at each singularity \( \sigma \in \Lambda \setminus \text{Sing}(X) \),

- either there exists a dominated splitting of the form \( T_\sigma M = E^{ss} \oplus F \) for \((D\varphi_t)_{t \in \mathbb{R}}\) such that \( E^{ss} \) is uniformly contracting,
  \[ \dim(E^{ss}) = \dim(\mathcal{N}_1) \quad \text{and} \quad W^{ss}(\sigma) \cap \Lambda = \{ \sigma \}, \]
- or there exists a dominated splitting of the form \( T_\sigma M = E \oplus E^{uu} \) for \((D\varphi_t)_{t \in \mathbb{R}}\) such that \( E^{uu} \) is uniformly expanding,
  \[ \dim(E^{uu}) = \dim(\mathcal{N}_2) \quad \text{and} \quad W^{uu}(\sigma) \cap \Lambda = \{ \sigma \}. \]

The links between singular domination of the linear Poincaré flow and domination of the tangent flow are discussed in Subsection 3.1: the existence of a dominated splitting for the tangent flow corresponds to the special case where one of the bundles \( \mathcal{N}_1 \) or \( \mathcal{N}_2 \) is uniformly contracted or expanded (see Propositions 3.1 and 3.6).

As expected, the existence of a singular domination is a robust property.

Theorem A. Let \( X \in \mathcal{X}^1(\mathcal{M}) \) and \( \Lambda \) be a compact invariant set admitting a singular dominated splitting of index \( i \). Then there exist a \( C^1 \) neighborhood \( \mathcal{U} \) of \( X \) and a neighborhood \( U \) of \( \Lambda \) such that for any \( Y \in \mathcal{U} \), the maximal invariant set in \( U \) admits a singular dominated splitting of index \( i \).

Conversely we will see that Definition 1.1 is very natural once one looks for a robust domination. Let us introduce before the following definition.

Definition 1.2. Let \( \Lambda \) be an invariant compact set. A singularity \( \sigma \in \Lambda \) is active in \( \Lambda \) if it is hyperbolic and both sets \( W^s(\sigma) \cap (\Lambda \setminus \{ \sigma \}) \) and \( W^u(\sigma) \cap (\Lambda \setminus \{ \sigma \}) \) are non-empty.

For a dense and open set of \( X \in \mathcal{X}^1(\mathcal{M}) \), all the singularities are hyperbolic \([17,30]\). For such a system, any invariant compact set \( \Lambda \) decomposes as the union of an invariant compact set \( \Lambda' \) whose singularities are active in \( \Lambda' \), and a finite set of singularities linked to \( \Lambda \) by either their stable or unstable manifolds to \( \Lambda' \) (but not both). Consequently the non-trivial dynamics inside \( \Lambda \) is supported on \( \Lambda' \). The assumption that all the singularities are active in \( \Lambda \) is generically a very mild assumption. In particular it is always satisfied by chain-transitive sets of generic flows.

Proposition 3.9 below shows that if a vector field robustly admits a domination in a compact region \( U \) for the linear Poincaré flow, then the definition of singular domination holds on any invariant compact set contained in \( U \) and whose singularities are all active.

1.3 Uniform and singular hyperbolicities

The classical notion of uniform hyperbolicity, introduced by Anosov \([2]\) and Smale \([31]\), plays an important role in dynamics since it implies the stability of the system.

\(^1\) Since \( \Lambda \setminus \text{Sing}(X) \) is not compact, one needs to specify a metric on \( \mathcal{N}|_{\Lambda \setminus \text{Sing}(X)} \) for which the definition of domination holds: one considers here the quotient metric induced by any Riemannian metric on \( \mathcal{M} \).
Definition 1.3. Let $X \in \mathcal{X}^1(M)$. A compact set $\Lambda$ is uniformly hyperbolic if

1. $\Lambda$ admits a dominated splitting $TM|_\Lambda = E^s \oplus (R^X) \oplus E^u$ for the tangent flow;
2. $E^s$ is uniformly contracted and $E^u$ is uniformly expanded.

When $\Lambda$ is reduced to a singularity $\sigma$, the splitting becomes $TM|_\Lambda = E^s \oplus E^u$; the dimension of the stable space $E^s$ is then called the index of $\sigma$.

Definition 1.4. Let $X \in \mathcal{X}^1(M)$. An invariant compact set $\Lambda$ is singular hyperbolic if

- either $\Lambda$ admits a dominated splitting $TM|_\Lambda = E^{cu} \oplus E^{uu}$ for the tangent flow, $E^s$ is uniformly contracted and $E^{cu}$ is sectionally expanded: there are $\eta, T > 0$ such that
  $$|\text{Jac}(D\varphi_{-t}|_x)| < e^{-\eta t} \quad \text{for any } x \in \Lambda, \text{ any } t > T \text{ and any 2-plane } F \subset E^{cu}(x);$$
- or $\Lambda$ admits a dominated splitting $TM|_\Lambda = E^{cs} \oplus E^u$ such that $E^{cs}$ is sectionally contracted and $E^u$ is uniformly expanded.

Robustly transitive sets (not reduced to a point) in dimension three (not reduced to a point) satisfy additional structures. For example, [26] shows that their singularities satisfy the following property:

Definition 1.5. A singularity is Lorenz-like if there exists an invariant splitting

$$T_\sigma M = E^{ss} \oplus E^c \oplus E^{uu}$$

with $\dim(E^c) = 1$ such that the largest negative Lyapunov exponent $\lambda^{ss}$ along $E^{ss}$, the smallest Lyapunov exponent $\lambda^{uu}$ and the center Lyapunov exponent $\lambda^c$ satisfy

$$0 < |\lambda^c| < \min(-\lambda^{ss}, \lambda^{uu}).$$

These properties have been intensively studied; we refer for example to [4, 5].

1.4 Multi-singular hyperbolicity

In higher dimension, the singular hyperbolicity is not compatible with the coexistence of singularities with different stable dimensions. Such a difficulty appeared in [12] where da Luz built an open set of star flows in dimension 5 that exhibit singularities of indices 2 and 3 interacting with the same periodic orbits. In order to characterize star systems, it was thus necessary to find a more general property: it has been introduced recently by Bonatti and da Luz [6] and is called multi-singular hyperbolicity.

Bonatti-da Luz’s definition is presented in Section 5. It involves blow-ups at the singularities, the extension of the linear Poincaré flow (defined by [18]) and rescalings by certain dynamical cocycles. We propose the following alternative definition without using cocycles and blow-ups.

Definition 1.6. Let $X \in \mathcal{X}^1(M)$. An invariant compact set $\Lambda$ is multi-singular hyperbolic if

1. $\Lambda$ admits a singular dominated splitting $\mathcal{N}^s \oplus \mathcal{N}^u$
2. there exist $\eta, T > 0$ and a compact isolating neighborhood 2) $V$ of $\Lambda \cap \text{Sing}(X)$ such that
   $$\|\Psi_t|_{\mathcal{N}^s(x)}\| < e^{-\eta t} \quad \text{and} \quad \|\Psi_{-t}|_{\mathcal{N}^u(\varphi_t(x))}\| < e^{-\eta t}, \quad \text{whenever } x, \varphi_t(x) \in \Lambda \setminus V \text{ and } t > T;$
3. each singularity $\sigma \in \Lambda \cap \text{Sing}(X)$ is Lorenz-like with a splitting $T_\sigma M = E^{ss} \oplus E^c \oplus E^{uu}$ satisfying
   $$\dim(E^{ss}) = \dim(\mathcal{N}^s), \quad \dim(E^{uu}) = \dim(\mathcal{N}^u).$$

The dimension of $\mathcal{N}^s$ is uniquely defined and is called the index of $\Lambda$.

2) A neighborhood $V$ of a compact set $\Lambda$ is isolating if the maximal invariant set in $V$ coincides with $\Lambda$. 

Remark 1.7. (1) The multi-singular hyperbolicity we define here is literally different from the notion defined in [6]. In fact, we will see in Section 5 that our definition is a bit stronger, but under a mild assumption (all the singularities in Λ are active) the two notions coincide (see Theorems D and E). So we choose to keep the same name.

(2) Definition 1.6(3) is often a consequence of the two first: Proposition 4.3 shows that this is the case when each singularity σ ∈ Λ is hyperbolic and is accumulated by points of Λ which do not belong to the local stable nor local unstable manifold of σ.

(3) For singularities σ ∈ Λ such that \(E^c(σ)\) is attracting and \(W^u(σ) \cap Λ \setminus \{σ\} \neq \emptyset\), singular domination implies \(W^{ss}(σ) \cap Λ = \{σ\}\). (This is an immediate consequence of Definition 1.1.) An analogous property holds when \(E^c(σ)\) is expanding.

The multi-singular hyperbolicity is an open property. This is proved in [6] for the Bonatti-da Luz definition. In Section 4, we give an independent proof for Definition 1.6.

Theorem B. Let \(X \in \mathcal{X}^1(M)\) and Λ be a multi-singular hyperbolic set. Then there exist a \(C^1\) neighborhood \(U\) of \(X\) and a neighborhood \(U\) of Λ such that the maximal invariant set of \(Y \in U\) in \(U\) is multi-singular hyperbolic.

One naturally defines a multi-singular hyperbolic vector field as a vector field such that each chain-recurrent class is either multi-singular hyperbolic or a hyperbolic singularity.

From Theorem B, the set of multi-singular hyperbolic vector fields is \(C^1\)-open. Moreover, the definition implies easily that each periodic orbit is hyperbolic (see Proposition 4.2). As a consequence, any multi-singular hyperbolic vector field \(X\) has the star property.

From Remark 1.7(1), for an open and dense subset of \(\mathcal{X}^1(M)\) both definitions of multi-singular vector fields coincide. It allows us to restate the results from [6] by using Definition 1.6.

Theorem 1.8 (See [6]). The set of multi-singular hyperbolic vector fields is an open set which is dense in the space of star vector fields (for the \(C^1\) topology).

The following question remains open.

Question 1.9 (See [6, Question 1]). Is any star vector field multi-singular hyperbolic?

1.5 Link with uniform and singular hyperbolicities

We end this discussion on the multi-singular hyperbolicity by showing that Definition 1.6 generalizes the uniform hyperbolicity and singular hyperbolicity in the following sense (the first property goes back to [13, Proposition 1.1]).

Theorem C. Let \(X \in \mathcal{X}^1(M)\) and Λ be an invariant compact set such that each singularity σ ∈ Λ is active (i.e., σ is hyperbolic, \(W^s(σ) \cap Λ \setminus \{σ\} \neq \emptyset\) and \(W^u(σ) \cap Λ \setminus \{σ\} \neq \emptyset\)). Then

(1) Λ is uniform hyperbolic if and only if Λ is multi-singular hyperbolic and does not contain any singularity.

(2) Λ is singular hyperbolic if and only if Λ is multi-singular hyperbolic and all the singularities have the same index.

2 Preliminaries

This section collects classical notions and properties used in this paper.

2.1 Chain recurrence

Consider a continuous flow \((φ_t)_{t ∈ \mathbb{R}}\) on a compact metric space \((K, d)\). For \(ε > 0\), a sequence \(x_1, \ldots, x_n\) in \(K\) is an \(ε\)-pseudo orbit if for each \(1 ≤ i ≤ n − 1\), there exists \(t_i ≥ 1\) such that

\[d(φ_{t_i}(x_i), x_{i+1}) < ε.\]
One says that \( x \) is chain-attainable from \( y \) if for any \( \varepsilon > 0 \), there exists an \( \varepsilon \)-pseudo orbit \( \{x_i\}_{i=0}^n \) with \( x_0 = y, x_n = x \) and \( n \geq 1 \). A set \( \Lambda \) is chain-transitive if for any pair \((x,y) \in \Lambda \times \Lambda\), the first point \( x \) is chain-attainable from the second point \( y \).

A point \( x \) is chain-recurrent if \( x \) is chain-attainable from itself. The set of chain-recurrent points is denoted as \( \mathcal{R} \). For \( x \in \mathcal{R} \), we define the chain recurrence class of \( x \) as the union of the chain-transitive sets containing \( x \). By definition, the chain recurrence classes define a partition of the chain-recurrent sets into invariant compact sets.

### 2.2 Linear Poincaré flow and its extension

Given a vector field \( X \in \mathcal{X}^1(M) \), one defines the normal bundle \( \mathcal{N} \) on the complement of the singular set \( \text{Sing}(X) \) in the following way:

\[
\mathcal{N}|_{M \setminus \text{Sing}(X)} = \bigcup_{x \in M \setminus \text{Sing}(X)} \{ v \in T_xM : \langle v, X(x) \rangle = 0 \},
\]

where \( \langle \cdot, \cdot \rangle \) denotes the inner product. One then defines the linear Poincaré flow \( (\Psi_t)_{t \in \mathbb{R}} \) in the following way: for each vector \( v \in \mathcal{N}(x) \) with \( x \in M \setminus \text{Sing}(X) \) and any \( t \in \mathbb{R} \),

\[
\Psi_t(v) = D\varphi_t(v) - \frac{\langle D\varphi_t(v), X(\varphi_t(x)) \rangle}{\|X(\varphi_t(x))\|^2} \cdot X(\varphi_t(x)).
\]

Following [18], the linear Poincaré flow may be compactified at the singularities as a linear flow \( (\hat{\Psi}_t)_{t \in \mathbb{R}} \) called the extended linear Poincaré flow. Consider the projective bundle

\[
\mathcal{G}^1 = \{ L \subset T_xM : x \in M \text{ and } L \text{ is a one-dimensional linear space in } T_xM \}.
\]

We denote by \( \mathcal{G}^1(x) \) the fiber at the point \( x \in M \) for the bundle \( \mathcal{G}^1 \). The set \( M \setminus \text{Sing}(X) \) embeds naturally in \( \mathcal{G}^1 \) by the map \( x \mapsto \mathbb{R}X(x) \). The tangent flow induces a continuous flow \( (\hat{\varphi}_t)_{t \in \mathbb{R}} \) on \( \mathcal{G}^1 \) which extends \( (\varphi_t) \): for \( L = \mathbb{R}u \) in \( \mathcal{G}^1 \) one defines

\[
\hat{\varphi}_t(\mathbb{R}u) = \mathbb{R}D\varphi_t(u).
\]

One introduces a normal bundle over \( \mathcal{G}^1 \) which extends \( \mathcal{N}|_{M \setminus \text{Sing}(X)} \): for \( L \in \mathcal{G}^1(x) \),

\[
\mathcal{N}(L) = \{ v \in T_xM : v \text{ is orthogonal to the linear space } L \}.
\]

One defines \( (\hat{\Psi}_t)_{t \in \mathbb{R}} \) on \( \mathcal{N} \) in the following way: for each \( L = \mathbb{R}u \in \mathcal{G}^1(x), v \in \mathcal{N}(L) \) and \( t \in \mathbb{R} \),

\[
\hat{\Psi}_t(v) = D\varphi_t(v) - \frac{\langle D\varphi_t(v), D\varphi_t(u) \rangle}{\|D\varphi_t(u)\|^2} \cdot D\varphi_t(u).
\]

When \( x \) is a regular point and \( L = \mathbb{R}X(x) \), then for any \( v \in \mathcal{N}(L) = \mathcal{N}(x) \) and \( t \in \mathbb{R} \), one has \( \hat{\Psi}_t(v) = \Psi_t(v) \).

### 2.3 Lyapunov exponents

Consider \( X \in \mathcal{X}^1(M) \) and an invariant probability measure \( \mu \). The measure is regular if \( \mu(\text{Sing}(X)) = 0 \).

We recall the Oseledec theorem. For \( \mu \)-almost every \( x \in M \), there are \( k = k(x) \) numbers \( \lambda_1(x) < \lambda_2(x) < \cdots < \lambda_k(x) \) and a splitting

\[
T_xM = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_k(x)
\]

such that for any \( 1 \leq i \leq k \) and any unit vector \( v \in E_i(x) \),

\[
\lim_{t \to \pm \infty} \frac{1}{t} \log \|D\varphi_t(v)\| = \lambda_i.
\]
If $\mu$ is ergodic, then $k$ and $\lambda_1, \ldots, \lambda_k$ are constants on a full $\mu$-measure set.

A regular ergodic measure is hyperbolic if it has only one vanishing Lyapunov exponent (which is given by the flow direction). Equivalently, there exists a measurable splitting of the normal bundle $\mathcal{N} = N_1 \oplus \cdots \oplus N_\ell$ defined on a set with full $\mu$-measure which is invariant under the linear Poincaré flow and non-zero numbers $\chi(\ell) < \cdots < \chi(x)$ such that for $\mu$-almost every point $x$, any $1 \leq j \leq \ell$ and any unit vector $v \in N_j(x)$, the quantity $\log \| \Psi_t(v) \|$ converges to $\chi_j$ as $t \to \pm \infty$. The numbers $\chi_j$ coincide with the non-zero Lyapunov exponents $\lambda_i$ of $\mu$.

2.4 Dynamics above hyperbolic singularities

The following fundamental result allows to exile the strong stable manifold of a singularity from a compact invariant set containing the singularity. It comes from [18].

**Proposition 2.1.** Consider $X \in \mathcal{A}^1(M)$, a hyperbolic singularity $\sigma$ and a $(\widehat{\phi}_t)_{t \in \mathbb{R}}$-invariant compact set $\widehat{\Lambda}$ in the projective space $\mathbb{P}^1(\sigma)$. If $\widehat{\Lambda}$ admits a dominated splitting $\widehat{\mathcal{N}}|_{\widehat{\Lambda}} = N_1 \oplus N_2$ for $(\widehat{\phi}_t)_{t \in \mathbb{R}}$ of index $i < \text{ind}(\sigma)$ and if $\widehat{\Lambda}$ intersects the projective space of $\mathcal{E}^s(\sigma)$, then

- $E^s(\sigma)$ has a finer dominated splitting $E^{s*}(\sigma) \oplus E^{cs}(\sigma)$ for $(D\phi_t)_{t \in \mathbb{R}}$ with $\dim(E^{s*}(\sigma)) = i$;
- any line $L \subset E^{s*}(\sigma) \oplus E^u(\sigma)$ which is not contained in $E^{s*}(\sigma) \cup E^u(\sigma)$ is disjoint from $\widehat{\Lambda}$.

**Proof.** Up to changing the metric, one can assume that the bundles in the hyperbolic splitting $E^s(\sigma) \oplus E^u(\sigma)$ are orthogonal to each other. In particular over points of $\widehat{\Lambda}$ contained in the projective space of $E^s(\sigma)$, the bundle $N_2$ contains $E^u(\sigma)$.

Consider $L \in \widehat{\Lambda}$ which is contained in the projective space of $E^u(\sigma)$. As $E^s(\sigma)$ is orthogonal to the one-dimensional linear space given by $L$, one has $\widehat{\phi}_t(L)|_{E^s(\sigma)} = D\phi_t|_{E^s(\sigma)}$ for any $t \in \mathbb{R}$. By the domination $N_1 \oplus N_2$, the space $E^s(\sigma)$ splits into two dominated sub-bundles $E^{s*} = N_1$ and $E^{cs}$ over the orbit of $L$.

Since the cocycle $(\widehat{\phi}_t(L)|_{E^s(\sigma)})_{t \in \mathbb{R}}$ is constant over the orbit of $L$, these bundles are constant as well. This implies that $E^s(\sigma)$ admits a dominated splitting $E^{s*}(\sigma) \oplus E^{cs}(\sigma)$ for $(D\phi_t)_{t \in \mathbb{R}}$.

Up to changing the metric, we will further assume that $E^{s*}(\sigma)$ is orthogonal to $E^{cs}(\sigma)$.

In order to prove the second property, one will suppose by contradiction that there exist a non-vanishing vector $v^{s*} \in E^{s*}(\sigma)$ and a non-vanishing vector $v^u \in E^u(\sigma)$ such that the line $L_0 = \mathbb{R}(v^{s*} + v^u)$ belongs to $\widehat{\Lambda}$. Then $v^{s*} \cdot \|v^u\|^2 - v^u \cdot \|v^{s*}\|^2$ belongs to $\mathcal{N}_{L_0}$. Now, we look at the orbit of $v^{s*} \cdot \|v^u\|^2 - v^u \cdot \|v^{s*}\|^2 \in \mathcal{N}_{L_0}$ under the extended linear Poincaré flow. By definition,

\[
\begin{align*}
\widehat{\phi}_t(v^{s*} \cdot \|v^u\|^2 - v^u \cdot \|v^{s*}\|^2) &= D\phi_t(v^{s*})\|v^u\|^2 - D\phi_t(v^u)\|v^{s*}\|^2 \\
&= \frac{\|D\phi_t(v^{s*})\|^2\|v^u\|^2 - \|D\phi_t(v^u)\|^2\|v^{s*}\|^2}{\|D\phi_t(v^{s*})\|^2 + \|D\phi_t(v^u)\|^2} \cdot (D\phi_t(v^{s*}) + D\phi_t(v^u)) \\
&= \frac{\|v^{s*}\|^2 + \|v^u\|^2}{\|D\phi_t(v^{s*})\|^2 + \|D\phi_t(v^u)\|^2} \cdot (\|D\phi_t(v^{s*})\|^2 D\phi_t(v^{s*}) - \|D\phi_t(v^u)\|^2 D\phi_t(v^u)).
\end{align*}
\]

Let us first assume that $v^{s*} \cdot \|v^u\|^2 - v^u \cdot \|v^{s*}\|^2$ belongs to $\mathcal{N}_{L_0}$. As $\widehat{\phi}_t(L_0)$ accumulates on a subset $\alpha$ of the projective space of $E^{s*}(\sigma)$ when $t$ tends to $-\infty$, the invariance of the bundle $N_1$ implies that $\mathbb{R} \cdot \widehat{\phi}_t(v^{s*} \cdot \|v^u\|^2 - v^u \cdot \|v^{s*}\|^2)$ accumulates in $N_1|_\alpha$; however from (2.1), the lines $\mathbb{R} \cdot \widehat{\phi}_t(v^{s*} \cdot \|v^u\|^2 - v^u \cdot \|v^{s*}\|^2)$ accumulate on a linear subspace in $E^{s*}(\sigma)$ when $t$ tends to $-\infty$. This leads to a contradiction since $\mathcal{N}_{L_0}$ contains $E^u(\sigma)$ and intersects $N_2|_\alpha$ trivially.

We are now reduced to the case where $v^{s*} \cdot \|v^u\|^2 - v^u \cdot \|v^{s*}\|^2$ does not belong to $\mathcal{N}_{L_0}$. When $t$ tends to $+\infty$, the lines $\widehat{\phi}_t(L_0)$ accumulate on a subset $\omega$ of the projective space of $E^u(\sigma)$ and (from the domination) $\mathbb{R} \cdot \widehat{\phi}_t(v^{s*} \cdot \|v^u\|^2 - v^u \cdot \|v^{s*}\|^2)$ accumulates inside a linear space in $\mathcal{N}_2|_\omega$; however from (2.1), the lines $\mathbb{R} \cdot \widehat{\phi}_t(v^{s*} \cdot \|v^u\|^2 - v^u \cdot \|v^{s*}\|^2)$ accumulate in the linear space $E^{s*}(\sigma)$ which leads to a contradiction. This proves the second item.

\[\square\]
2.5 Star vector fields

They satisfy the following uniform property.

**Theorem 2.2** (See [19]). For any star vector field \( X \in \mathcal{X}^1(M) \), there exist \( \eta, T > 0 \) and a \( C^1 \) neighborhood \( \mathcal{U} \) of \( X \) with the following properties. For any \( Y \in \mathcal{U} \) and any periodic orbit \( \gamma \) of \( Y \) with period \( \pi(\gamma) \) larger than \( T \), let us denote by \( \mathcal{N}_\gamma = N^s \oplus N^u \) the hyperbolic splitting of the linear Poincaré flow \( (\Psi^Y_t)_t \) associated with \( Y \). Then for each \( p \in \gamma \), one has

\[
\|\Psi^Y_t|_{\mathcal{N}^s(\gamma)}\| \cdot \|\Psi^Y_t|_{\mathcal{N}^u(\gamma)}\| < e^{-2\eta t} \quad \text{for each } t \geq T,
\]

\[
\prod_{i=0}^{[\pi(\gamma)/T]-1} \|\Psi^Y_T|_{\mathcal{N}^s(\gamma)}\| \leq e^{-\eta \pi(\gamma)} \quad \text{and} \quad \prod_{i=0}^{[\pi(\gamma)/T]-1} \|\Psi^Y_{-T}|_{\mathcal{N}^u(\gamma)}\| \leq e^{-\eta \pi(\gamma)}.
\]

2.6 Connecting lemma

The connecting lemma was first obtained by Hayashi [16] for solving the stability conjecture for flows. In this paper, we use the connecting lemma mainly for finding (after perturbations) regular orbits in some maximal invariant sets which approach singularities. For our convenience, we state the following flow version which comes from [32, 33].

**Theorem 2.3.** Let \( X \in \mathcal{X}^1(M) \). For any \( C^1 \) neighborhood \( \mathcal{U} \) of \( X \), there exist \( T > 0 \), \( \rho \in (0, 1) \) and \( d_0 > 0 \) such that for any point \( x \in M \) which is non-periodic and non-singular under the flow \( (\varphi^X_t)_t \) generated by \( X \), one has the following property.

For any \( d \in (0, d_0) \), and any points \( p, q \notin \Delta_T(x, d) := \bigcup_{t \in [1, T]} \varphi^X_t(B_d(x)) \), if the forward orbit of \( p \) and the backward orbit of \( q \) intersect \( B_{\rho_d}(x) \), then there exists \( Y \in \mathcal{U} \) such that \( q \) is on the forward orbit of \( p \) under the flow \( (\varphi^Y_t)_t \). Moreover, \( Y(z) = X(z) \) for \( z \in M \setminus \Delta_T(x, d) \).

3 Singular domination

In this section, we discuss the notion of singular domination introduced in the introduction and prove that it is a robust property (see Theorem A). We also build a robust example of a flow with no dominated splitting of the tangent bundle (see Subsection 3.1) which shows that the linear Poincaré flow is more adapted than the tangent flow for studying the dynamics of vector fields. Finally, we motivate the definition of singular domination by proving that a robust dominated splitting of the linear Poincaré flow satisfies Definition 1.1.

3.1 Dominated splitting of the tangent flow

We first discuss the domination for the tangent flow: the next statement (an improved version of [9, Theorem B]) shows that it constraints the tangent behavior.

**Proposition 3.1.** Let \( X \in \mathcal{X}^1(M) \) and \( \Lambda \) be a chain-transitive invariant compact set such that

- all the singularities in \( \Lambda \) are hyperbolic;
- \( \Lambda \) admits a dominated splitting \( T_{\Lambda} M = E \oplus F \) for the tangent flow.

Then either \( E \) is uniformly contracted, or \( F \) is uniformly expanded.

**Remark 3.2.** Theorem B in [9] is stated for a chain recurrence class with domination for the tangent flow of a \( C^1 \) generic flow on 3-manifolds.

Let us recall the notion of cone fields. Given a continuous splitting \( TM|_\Lambda = E \oplus F \) over a compact set \( \Lambda \), one defines at each point \( x \in \Lambda \) a cone field around \( F \) of angle \( \alpha > 0 \) as

\[
C^F_\alpha(x) = \{ v \in T_x M : v = v^E + v^F, v^E \in E, v^F \in F \text{ and } \|v^E\| \leq \alpha \|v^F\| \}.
\]
Sketch of the proof of Proposition 3.1. Let \( x_0 \) be a regular point in \( \Lambda \). Without loss of generality, one assumes \( X(x_0) \notin E(x_0) \). Hence there exists \( \alpha > 0 \) such that \( X(x_0) \in C^{\alpha}_F(x_0) \). One extends the cone field \( C^\alpha_F \) continuously to a neighborhood \( U \) of \( \Lambda \). By domination, up to shrinking \( U \), there exist \( T > 0 \) and \( \lambda \in (0,1) \) such that \( D\varphi_t(C^\alpha_F(y)) \subset C^\lambda_{x_0}(\varphi_t(y)) \) for any \( t \geq T \) and any \( y \in \bigcup_{s \in [0,t]} \varphi^{-s}(U) \). Note that in the definition of chain recurrence, there is no loss of generality if one only considers pseudo orbits, whose times \( t_s \) between the jumps are larger or equal to \( T \) (see Subsection 2.1).

**Lemma 3.3.** For any regular point \( y \in \Lambda \setminus \text{Sing}(X) \), one has \( X(y) \in C^\alpha_F(y) \).

**Proof.** Fix \( y_0 \in \Lambda \setminus \text{Sing}(X) \) and a small neighborhood \( V \subset U \) of \( \text{Sing}(X) \cap \Lambda \) such that \( x_0, y_0 \notin V \) and \[ \bigcup_{s \in [-2T,2T]} \varphi_s(V) \subset U. \]

As all the singularities in \( \Lambda \) are hyperbolic, for each singularity \( \sigma \in \Lambda \), one fixes fundamental domains \( \Delta^s(\sigma) \subset V \) and \( \Delta^u(\sigma) \subset V \) of the stable and unstable manifolds of \( \sigma \), respectively. For \( \varepsilon_0 > 0 \) small enough, let us denote by \( \Delta^s_{\varepsilon_0}(\sigma) \) and \( \Delta^u_{\varepsilon_0}(\sigma) \) the \( \varepsilon_0 \)-neighborhoods of \( \Delta^s(\sigma) \) and \( \Delta^u(\sigma) \) such that

1. the sets \( \{ \Delta^s_{\varepsilon_0}(\sigma), \Delta^u_{\varepsilon_0}(\sigma) \}_{\sigma \in \Sigma(\Lambda) \cap \Lambda} \) are pairwise disjoint and disjoint from \( \text{Sing}(X) \);
2. for any \( x_1 \) and \( x_2 \) in the same \( \Delta^s_{\varepsilon_0}(\sigma) \), if \( X(x_1) \in C^\alpha_{x_0}(x_1) \) and \( d(x_1, x_2) < 4\varepsilon_0 \) then \( X(x_2) \in C^\alpha_{x_0}(x_2) \).

By the inclination lemma, the following property holds:

1. for any \( \sigma \in \Lambda \cap \text{Sing}(X) \) and any points \( x_1 \in \Delta^s_{\varepsilon_0}(\sigma) \), \( x_2 \in \Delta^u_{\varepsilon_0}(\sigma) \), there exist \( \hat{x}_1 \in \Delta^s_{\varepsilon_0}(\sigma) \), \( \hat{x}_2 \in \Delta^u_{\varepsilon_0}(\sigma) \) and \( t > T \) such that \( d(x_1, \hat{x}_1) < 2\varepsilon_0 \) and \( d(x_2, \hat{x}_2) = |\varphi_t(\hat{x}_1)| \).

As each singularity \( \sigma \in \Lambda \) is hyperbolic, there exist \( \varepsilon_1 \) small enough and a neighborhood \( W_\sigma \) of \( \sigma \) such that

1. for any \( x \in W_\sigma \) and any \( \varepsilon_1 \)-pseudo orbit \( \{ x_i \}_i \) with jumps at times \( \{ t_i \}_i \) such that \( t_i \geq 2T \),
   a. there exist \( 0 \leq i_0 < k \) and \( 0 \leq t = t_{i_0} \) such that \( \varphi(t_i_0) \in \Delta^u_{\varepsilon_0/2}\sigma \);
   b. if \( z_i \) is a pseudo orbit of \( \{ x_i \}_i \) connecting \( x_0 \) to \( y_0 \), with times \( \{ t_i \}_i \), then \( \varphi_{t_i}(z_i) \in \Delta^u_{\varepsilon_0/2}\sigma \).

Let \( W := \bigcup_{\sigma \in \Sigma(\Lambda) \cap \Lambda} W_\sigma \). By continuity of \( C^\alpha_F \), \( \mathbb{R}X \) and \( \langle \varphi_t \rangle \) in \( \mathbb{R} \), there exist \( \varepsilon_2, \varepsilon_3 > 0 \) such that

1. for any \( x, z \in U \setminus W \), if \( d(x, z) < \varepsilon_2 \) and \( X(x) \in C^\alpha_F(x) \), then \( X(z) \in C^\alpha_F(z) \);
2. for any \( x, y, z \in M \), if \( d(x, z) < \varepsilon_3 \), then \( d(\varphi(x), \varphi(z)) < \varepsilon_2 \) for any \( |s| < T \).

As \( \Lambda \) is chain-transitive, for \( \varepsilon < \frac{1}{2} \min \{ \varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 \} \) there exists an \( \varepsilon \)-pseudo orbit \( \{ x_i \}_i \) connecting \( x_0 \) to \( y_0 \), with times \( \{ t_i \}_i \), larger than \( 2T \). Let \( I \subset \{ 0, \ldots, l \} \) be the set of the possible integers \( i \) such that \( z_i \in W \). By (c) and (d), for each \( i \in I \), there exist \( 0 \leq i' < i \leq i'' < l \) and times \( 0 \leq t' \leq t_i' \) such that \( \varphi_{t'}(z_{i''}) \in \Delta^s_{\varepsilon_0} \sigma \) and \( \varphi_{t'}(z_{i''}) \in \Delta^u_{\varepsilon_0} \sigma \) for some singularity \( \sigma \). Then, there exist \( y_i \in \Delta^s_{\varepsilon_0}(\sigma) \) and \( t_i > T \) such that

\[ \max \{ d(y_i, \varphi_{t'}(z_{i''})), d(\varphi_{t'}(y_i), \varphi_{t''}(z_{i''})) \} < 2\varepsilon_0 \] and \( \varphi_{t''}(y_i) \in \Delta^u_{\varepsilon_0}(\sigma) \).

Now, one replaces the pseudo orbit segment from \( \varphi_{t'}(z_{i''}) \) to \( \varphi_{t''}(z_{i''}) \) by the true orbit segment \( \{ \varphi_i(y_i) \}_{i \in I} \), for each \( i \in I \). If \( t'' < T \), one replaces the pseudo orbit segment between \( \varphi_{-T+t'} \circ \varphi_{t'-1}(z_{i''}-1) \) and \( \varphi_{t'-1}(z_{i''}-1) \) by the orbit segment between \( \varphi_{-T+t''}(z_{i''}) \) and \( z_{i''} \).

By the choice of \( V \) and (f), one obtains in this way a new pseudo orbit connecting \( x_0 \) to \( y_0 \) such that

1. the times between the jumps are larger than \( T \);
2. all the jumps avoid \( W \);
3. each jump avoiding the sets \( \Delta^s_{\varepsilon_0} \sigma \) has size smaller than \( \varepsilon_2 \);
4. each jump in \( \Delta^s_{\varepsilon_0} \sigma \) has size smaller than \( 4\varepsilon_0 \).

By (b) and (e), the contraction of the cone field \( C^\alpha_F \) gives \( Y(y) \in C^\alpha_F(y) \).

We can now end the proof of the proposition. The contraction of the cone field \( C^\alpha_F \) implies that \( X(y) \in F(y) \) for any \( y \in \Lambda \). In particular for any singularity \( \sigma \in \Lambda \), the space \( F(\sigma) \) meets the stable space of \( \sigma \) so that \( E(\sigma) \) is contracted by forward iterations.

One then concludes as in [9]. Let us consider an arbitrary point \( x \in \Lambda \). Either \( x \) belongs to the stable set of singularity \( \sigma \) and the behavior of \( E(x) \) copies the behavior of \( E(\sigma) \), or the forward orbit of \( x \) accumulates on a regular point: there exist times \( t_n \to +\infty \) such that \( \| D\varphi_{t_n}(X(x)) \| \) is bounded. Since
$X(x) \in F(x)$, the domination implies that the sequence $\|D\varphi_{t_n}(x)\|_E$ decreases to zero. A compactness argument over $\Lambda$ concludes the uniform contraction of the bundle $E$. \hfill \qed

As a consequence, one builds robustly transitive flows whose tangent bundle does not admit any domination, which contrasts with [7]. This shows that the dominated splittings should be rather searched for the linear Poincaré flow than the tangent flow.

**Proposition 3.4.** There exists an open set of $C^1$ vector fields with no singularity on a manifold of dimension 5:

- whose dynamics is robustly transitive,
- whose tangent flow does not admit any dominated splitting.

**Remark 3.5.** Such an example also appeared in [3]. Since the construction is short, for completeness, we describe a construction here.

**Proof.** Bonatti and Viana [11] built a robustly transitive diffeomorphism $f$ on $\mathbb{T}^4$ such that

- $f$ admits a dominated splitting of the form $TT^4 = E \oplus F$ where $\dim(E) = \dim(F) = 2$;
- $E$ is neither uniformly contracting nor uniformly expanding, and so is $F$;
- neither $E$ nor $F$ can be split into non-trivial dominated sub-bundles.

In particular, $E \oplus F$ is the only dominated splitting for $f$. One considers the suspension of $f$ and gets a $C^1$ vector field $X_f$ which is also robustly transitive, and has no singularities. If the tangent flow of $X_f$ admits a dominated splitting $E_1 \oplus E_2$, then the robust transitivity and Proposition 3.1 imply that $E_1$ or $E_2$ is hyperbolic, which in return implies that for $f$, the bundle $E$ or $F$ is uniformly hyperbolic (see [13, Proposition 1.1]). In summary, the tangent flow of $X_f$ does not admit domination. \hfill \qed

Proposition 3.1 shows that a dominated splitting of the tangent flows implies a dominated splitting of the linear Poincaré flow. The next proposition (due to [14, Lemma 2.13]) gives a criterion to obtain a dominated splitting on the tangent flow when the linear Poincaré flow is dominated.

**Proposition 3.6.** Let $\Lambda$ be an invariant compact set for $X \in \mathcal{X}^1(M)$ with a dominated splitting $\mathcal{N}|_{\Lambda \setminus \text{Sing}(X)} = \mathcal{N}_1 \oplus \mathcal{N}_2$ of index $i$ for the linear Poincaré flow. Assume furthermore that

- $\Lambda \setminus \text{Sing}(X)$ is dense in $\Lambda$,
- there exist $\eta, T > 0$ such that $\|\Psi_1|_{\mathcal{N}_1}\| \leq e^{-\eta t} \frac{\|X(\varphi_t(x))\|}{\|x\|}$ for all $x \in \Lambda$ and $t > T$.

Then the tangent flow over $\Lambda$ admits a dominated splitting $TM|_{\Lambda} = E^s \oplus F$ of index $i$ and $E^s$ is uniformly contracted.

### 3.2 Robustness of singular domination: Proof of Theorem A

Due to the lack of compactness of the linear Poincaré flow over $M \setminus \text{Sing}(X)$, the robustness of singular dominated splittings is not a direct consequence of the robustness of dominated splittings for continuous linear cocycles over compact spaces. We now state and prove a more precise version of Theorem A.

**Theorem 3.7.** Let $X \in \mathcal{X}^1(M)$, $\eta, T > 0$ and $\Lambda$ be a compact invariant set admitting a singular dominated splitting $\mathcal{N}|_{\Lambda \setminus \text{Sing}(X)} = \mathcal{N}_1 \oplus \mathcal{N}_2$ of index $i$ which is $(\eta, T)$-dominated.

Then there exist a $C^1$ neighborhood $U$ of $X$ and a neighborhood $U$ of $\Lambda$ such that for each $Y \in U$, the maximal invariant set $\Lambda_Y$ of $Y$ in $U$ admits a singular dominated splitting of index $i$ which is $(\eta, T)$-dominated.

Before proving the robustness of a singular domination, we need some preparation.

**Lemma 3.8.** Let $X \in \mathcal{X}^1(M)$ and let $K \subset \Lambda$ be two invariant compact sets. Assume that

- $K$ admits a partially hyperbolic splitting $T_KM = E^s \oplus F$ for the tangent flow of index $i$;
- for any $x \in K$, one has $W^s(x) \cap \Lambda = \{x\}$.

Then there exist neighborhoods $U$ of $X$, $V$ of $K$ and $U$ of $\Lambda$ such that for any $Y \in U$,

- the maximal invariant set $K_Y$ in $V$ admits a partially hyperbolic splitting of index $i$;
- the maximal invariant set $\Lambda_Y$ in $U$ satisfies $W^s(x) \cap \Lambda_Y = \{x\}$ for any $x \in K_Y$.

**Proof.** Since the partial hyperbolicity is robust, the first item holds for a $C^1$ neighborhood $U$ of $X$ and a neighborhood $W$ of $K$. Let us assume by contradiction that the second item does not hold and that
there exist a sequence of vector fields $Y_n \in \mathcal{U}$, neighborhoods $V_n \subset W$ of $K$, neighborhoods $U_n$ of $\Lambda$ and points $x_n$ such that:

- $Y_n$ tends to $X$ in the $C^1$ topology, $\bigcap_{n \in \mathbb{N}} V_n = K$ and $\bigcap_{n \in \mathbb{N}} U_n = \Lambda$;
- the maximal invariant sets $K_n$ and $\Lambda_n$ of $Y_n$ in $V_n$ and $U_n$ satisfy $x_n \in K_n$ and

$$ \left(W^{ss}(x_n) \setminus \{x_n\}\right) \cap \Lambda_n \neq \emptyset. $$

Each point $x$ in $K_n$ has local strong stable manifolds which vary continuously in the $C^1$ topology with $x$ and $Y_n$. Up to replacing $x_n$ by iterates, one can consider constants $\varepsilon_0 > \varepsilon_1 > 0$ such that

$$ W_{\varepsilon_0}(x_n) \setminus W_{\varepsilon_1}(x_n) \cap \Lambda \neq \emptyset. $$

Up to taking a subsequence, $(x_n)$ converges to a point $x \in K$ and $W_{\varepsilon_0}(x) \setminus W_{\varepsilon_1}(x) \cap \Lambda \neq \emptyset$ which leads to the contradiction.

**Proof of Theorem 3.7.** We first address the singularities. Let us denote by $S_-$ the set of singularities $\sigma \in \Lambda$ admitting a dominated splitting $T_{\sigma} M = E^{ss} \oplus F$ for the tangent flow with $\dim(E^{ss}) = i$ and $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$. Analogously, one denotes by $S_+$ the set of singularities $\sigma \in \Lambda$ admitting a dominated splitting $T_{\sigma} M = E \oplus E^{uu}$ for the tangent flow with $\dim(E^{uu}) = \dim(M) - i - 1$ and $W^{uu}(\sigma) \cap \Lambda = \{\sigma\}$.

Since local strong stable and unstable manifolds vary continuously with respect to the points, each $\sigma \in S_-$ (resp. $\sigma \in S_+$) admits a neighborhood $U_\sigma$ with $U_\sigma \cap \text{Sing}(X) \cap \Lambda \subset S_-$ (resp. $\subset S_+$). As $\text{Sing}(X) \cap \Lambda$ is compact, there exist two open sets $V^-$ and $V^+$ such that

- $\text{Sing}(X) \cap \Lambda \subset V^- \cup V^+$;
- $V^- \cap \text{Sing}(X) \cap \Lambda \subset S_-$ and $V^+ \cap \text{Sing}(X) \cap \Lambda \subset S_+$.

Applying Lemma 3.8 to $V^- \cap \text{Sing}(X) \cap \Lambda$ (resp. $V^+ \cap \text{Sing}(X) \cap \Lambda$), one gets a $C^1$ neighborhood $U_0$ of $X$, a neighborhood $U_0$ of $\Lambda$ and an open subset $V$ of $V^- \cup V^+$ such that for each $Y \in U_0$,

- $\text{Sing}(Y) \setminus U_0 \subset V$;
- any singularity $\sigma$ in $\text{Sing}(Y) \cap V^-$ (resp. $\text{Sing}(Y) \cap V^+$) admits a splitting $E^{ss} \oplus F$ (resp. $E \oplus E^{uu}$) with $\dim(E^{ss}) = i$ (resp. $\dim(E^{uu}) = \dim(M) - i - 1$);
- the maximal invariant set $\Lambda^0_{\sigma}$ of $Y$ in $U_0$ satisfies $W^{ss}(\sigma) \cap \Lambda^0_{\sigma} \subset \{\sigma\}$ for each $\sigma$ in $\text{Sing}(Y) \cap V^-$ and $W^{uu}(\sigma) \cap \Lambda^0_{\sigma} \subset \{\sigma\}$ for each $\sigma$ in $\text{Sing}(Y) \cap V^+$.

This gives the second item of Definition 1.1 for the maximal invariant set $\Lambda^0_{\sigma}$.

We then compactify the sets $\Lambda^0_{\sigma} \setminus \text{Sing}(Y)$ for $Y \in U_0$. For each $\sigma \in \text{Sing}(Y) \cap V^-$, we denote by $K^-(\sigma)$ the projective space of the linear space $F(\sigma)$. Analogously for $\sigma \in \text{Sing}(Y) \cap V^+$ we denote by $K^+(\sigma)$ the projective space of the linear space $E(\sigma)$. We then define

$$ K_Y = \bigcup_{x \in \Lambda^0_{\sigma} \setminus \text{Sing}(Y)} R Y(x) \cup \bigcup_{\sigma \in \text{Sing}(Y) \cap \Lambda^0_{\sigma} \cap V^-} K^-(\sigma) \cup \bigcup_{\sigma \in \text{Sing}(Y) \cap \Lambda^0_{\sigma} \cap V^+} K^+(\sigma). $$

**Claim.** $K_Y$ is a compact $(\mathcal{C}_t^\gamma)_{t \in \mathbb{R}}$-invariant set which varies upper semi-continuously with respect to the vector fields $Y \in \mathcal{U}_0$.

Now, we give the proof of the claim. By definition, $K_Y$ is $(\mathcal{C}_t^\gamma)_{t \in \mathbb{R}}$-invariant. In order to prove the compactness and the semi-continuity, it suffices to prove that, for any sequence $Y_n \to Y$ and for any sequence of points $x_n \subset \Lambda^0_{Y_n} \setminus \text{Sing}(Y_n)$ converging to $\sigma \in \text{Sing}(Y) \cap V^-$ (resp. $\sigma \in \text{Sing}(Y) \cap V^+$), each limit of $R Y_n(x_n)$ belongs to $K^-(\sigma)$ (resp. $K^+(\sigma)$). We only consider the case where $x_n$ tends to $\sigma \in \text{Sing}(Y) \cap V^-$ since the other case is analogous.

Assume by contradiction, that $R Y_n(x_n)$ converges to a line $L$ which is not contained in $F(\sigma)$. By using the domination $T_{\sigma} M = E^{ss} \oplus F$ over $\sigma$ (and up to considering a subsequence), there exists $y_n$ in the backward orbit of $x_n$ such that $R Y_n(y_n)$ converges to a line $L' \subset E^{ss}(\sigma)$. On a small open neighborhood $V$ of $\sigma$, there exists a continuous $(D \varphi t)_{t \in \mathbb{R}}$-invariant cone field $C^{ss}$ such that any vector in $C^{ss}$ is uniformly expanded by $(D \varphi t)_{t \in \rho}$. For $n$ large enough, $Y_n(y_n)$ is tangent to $C^{ss}$: this implies that the backward orbit of $y_n$ escapes from $V$. Let $t_n > 0$ be the smallest number such that $\varphi_{-t_n}(y_n)$ is not in $V$ and let $z$ be an accumulation point of $\varphi_{-t_n}(y_n)$. The backward invariance of the cone field shows that $z$ belongs
to $W^{ss}(\sigma) \setminus \{\sigma\}$. It also belongs to $\Lambda$, and this contradicts the fact that $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$. This ends the proof of the claim.

Since the existence of a dominated splitting for continuous linear cocycles over compact spaces is robust, there exists an $(\eta, T)$-dominated splitting of index $i$ for $K_Y$ for the extended linear Poincaré flow for any $Y$ in a $C^1$ neighborhood $U$ of $X$. In particular, the linear Poincaré flow over $\Lambda_Y$ admits an $(\eta, T)$-dominated splitting for any $Y \in U$.

3.3 Robust dominated splitting implies singular dominated splitting

We now prove a converse statement to Theorem A. Note that the second assumption is very mild (it is satisfied once $X$ is Kupka-Smale and $\Lambda$ is chain-transitive).

**Proposition 3.9.** Let $X \in \mathcal{A}^1(M)$ and $\Lambda$ be a compact invariant set such that

- $\Lambda$ admits a robust dominated splitting of index $i$: there exist $\eta, T > 0$ and neighborhoods $U$ of $X$ and $\mathcal{U}$ of $\Lambda$, such that, for any $Y \in \mathcal{U}$, the maximal invariant set in $U$ admits an $(\eta, T)$-dominated splitting of index $i$ for the linear Poincaré flow;
- each $\sigma \in \Lambda \cap \text{Sing}(X)$ is hyperbolic; moreover $W^s(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset$ and $W^u(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset$.

Then the set $\Lambda$ admits a singular dominated splitting of index $i$.

**Proof.** Let $\bar{\Lambda}$ be the closure of $\{\mathbb{R}X(x) : x \in \Lambda \setminus \text{Sing}(X)\}$ in $G^1$. The extended linear Poincaré flow is a continuous cocycle over the compact space $G^1$. As a consequence the dominated splitting over $\Lambda \setminus \text{Sing}(X)$ for the linear Poincaré flow can be extended over $\bar{\Lambda}$ for the extended linear Poincaré flow. Consider a singularity $\sigma \in \Lambda$. Without loss of generality, one can assume $\dim(E^s(\sigma)) \geq i$. By assumption, there exists a line $L \in \bar{\Lambda}$ which is contained in $E^u(\sigma)$. Then, by the first item of Proposition 2.1, there exists a dominated splitting $E^s(\sigma) = E^{ss}(\sigma) \oplus E^{cs}(\sigma)$ with $\dim(E^{ss}(\sigma)) = i$.

It remains to show that $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$. This is proved by contradiction: we assume that there exists a point $y \in W^{ss}(\sigma) \cap \Lambda \setminus \{\sigma\}$.

**Claim.** There exist a sequence $X_n \to X$ in $\mathcal{A}^1(M)$ and a sequence $x_n \to \sigma$ in $M$ such that

- $\mathbb{R}X_n(x_n)$ converges to some line in $E^{ss}(\sigma) \oplus E^u(\sigma) \setminus \{E^{ss}(\sigma) \cup E^u(\sigma)\}$;
- $x_n$ belongs to the maximal invariant set of $X_n$ in $U$.

Now, we give the proof of the claim. We consider four cases.

**Case 1.** The $\alpha$-limit set of $y$ intersects $W^u(\sigma) \setminus \{\sigma\}$ at a point $z$. The connecting lemma (see Theorem 2.3) gives $C^1$ perturbations $X'$ of $X$ which coincide with $X$ on

$$\{\varphi_t(y), t > 0\} \cup \{\varphi_{-t}(z), t > 0\} \cup \{\sigma\}$$

and such that the orbits of $y$ and $z$ for $X'$ coincide and are contained in $U$.

One can consider a small chart $\psi_\sigma : T_{\sigma}M \to V$ on a neighborhood of $\sigma$ such that $\psi(0) = \sigma$ and such that $\psi^{-1}(W^{ss}(\sigma))$ and $\psi^{-1}(W^{ss}_{\text{loc}}(\sigma))$ coincide with the linear spaces $E^{ss}$ and $E^u$. By an arbitrarily small $C^1$ perturbation (in a neighborhood of $\sigma$), one can furthermore assume that $X'$ is linear on a neighborhood of 0. Another small $C^1$ perturbation near $y$ and $z$ gives a vector field $X''$ such that

- $X'' = X'$ on a small neighborhood of $\sigma$;
- $y$ and $z$ are on the same periodic orbit in $U$ under $X''$ which contains a piece of orbit of the linear vector field which is included in $E^{ss}(\sigma) \oplus E^u(\sigma)$ with points close to $\sigma$.

One deduces that there exists a point $x$ on the periodic orbit of $y$ and $z$ under $X''$ which is close to $\sigma$ such that

$$\frac{X''(x)}{\|X''(x)\|} \in E^{ss}(\sigma) \oplus E^u(\sigma) \setminus \{E^{ss}(\sigma), E^u(\sigma)\}.$$

**Case 2.** There exists a point $z \in W^u(\sigma) \cap \Lambda \setminus \{\sigma\}$ whose $\omega$-limit set intersects $W^{ss}(\sigma) \setminus \{\sigma\}$. This case is analogous to Case 1.

**Case 3.** $\alpha(y) \cap W^u(\sigma) = \emptyset$ and there exists $z \in W^u(\sigma) \cap \Lambda \setminus \{\sigma\}$ such that $\omega(z) \cap W^s(\sigma) = \emptyset$. As in the first case, one considers a chart $\psi_\sigma$ and a $C^1$ close vector field $X'$ which is linear near 0. Another small $C^1$ perturbation near $y$ and $z$ gives a vector field $X''$ such that
Lemma 4.1. Let us consider a continuous $\varphi_t : X \to X$ on a neighborhood of $\sigma$, on the positive orbit of $z$ and negative orbit of $y$.

• $z$ belongs to the positive orbit of $y$ and the orbit segment from $y$ to $z$;
• there is a point $x$ in the orbit segment from $y$ to $z$ which contains a piece of orbit of $x$ and the linear vector field $X$ included in $E^{ss}(\sigma) \oplus E^u(\sigma)$ with points close to $\sigma$.

Case 4. $\alpha (y) \cap W^u(\sigma) = \emptyset$ and there is $z \in W^u(\sigma) \cap \Lambda \setminus \{\sigma\}$ such that $\omega (z) \cap W^s(\sigma) \setminus W^{ss}(\sigma) \neq \emptyset$.

Then by the connecting lemma, one gets a $C^1$ perturbation $\tilde{X}$ of $X$ such that

• $\tilde{X}$ coincides with $X$ on $\{\varphi_t (y)\}_{t \in \mathbb{R}}$ and $y$ belongs to $W^{ss}(\sigma)$;
• There exists a point $z' \in W^u(\sigma) \cap W^s(\sigma)$.

One then concludes as in Case 3. This ends the proof of the claim.

4 Multi-singular hyperbolicity

In this section, we prove that multi-singular hyperbolicity is robust (see Theorem B), we discuss the Lorenz-like property of the singularities and we compare with uniform hyperbolicity and singular hyperbolicity (see Theorem C).

4.1 Preparation

We first state a basic result which will be used in this paper.

Lemma 4.1. Let us consider a continuous flow $\varphi_t : X \to X$ on a compact metric space $X$, a one-parameter family $\{a_t \}_{t \in \mathbb{R}}$ of continuous functions, $K \to \mathbb{R}$ and numbers $\{c_t \}_{t \geq 0}$ satisfying

$$a_t + s(x) \leq a_s (x) + a_t (\varphi_s (x)) \quad \text{for any } x \in K \text{ and any } t, s \in \mathbb{R},$$

$$\sup_{s \in [-t, t]} a_s (x) < c_t \quad \text{for any } x \in K \text{ and any } t \geq 0.$$

Then for any $T > 0$ and any orbit segment $\{\varphi_s (x)\}_{s \in [0, T]}$ with $t \geq 3T$, one has

$$a_t (x) \leq 3c_T + \frac{1}{T} \int_0^t a_T (\varphi_s (x)) ds.$$
Proof. By our assumptions \( a_0(x) = 0 \) for any \( x \in K \). Hence \( c_t > 0 \) for any \( t \geq 0 \). Given \( T > 0 \), for any \( t \geq 3T \) and any \( s \in [0, T] \), one has

\[
a_t(x) \leq a_s(x) + \sum_{i=0}^{\lfloor \frac{t}{T} \rfloor} a_T(\varphi_{s+iT}(x)) + a_{t-s-(\lfloor \frac{t}{T} \rfloor+1)T}(\varphi_{s+(\lfloor \frac{t}{T} \rfloor+1)T}(x)) \leq 2c_T + \sum_{i=0}^{\lfloor \frac{t}{T} \rfloor} a_T(\varphi_{s+iT}(x)).
\]

Then one integrates over the interval \([0, T] \) and divides it by \( T \):

\[
a_t(x) \leq 2 \cdot c_T + \frac{1}{T} \int_0^T \sum_{i=0}^{\lfloor \frac{t}{T} \rfloor} a_T(\varphi_{s+iT}(x))ds \leq 2 \cdot c_T + \frac{1}{T} \int_0^{(\lfloor \frac{t}{T} \rfloor+1)T} a_T(\varphi_s(x))ds
\]

\[
\leq 3 \cdot c_T + \frac{1}{T} \int_0^t a_T(\varphi_s(x))ds.
\]

This completes the proof. □

Proposition 4.2. Let \( X \in \mathcal{X}^1(M) \) and \( \Lambda \) be a multi-singular hyperbolic set. Then each regular measure \( \mu \) supported on \( \Lambda \) is hyperbolic, and its hyperbolic splitting for \((\Psi_t)_{t \in \mathbb{R}}\) coincides with the singular domination \( \mathcal{N}^s|_{\Lambda \smallsetminus \text{Sing}(X)} = \mathcal{N}^s \oplus \mathcal{N}^u \). Moreover, there exists \( \eta > 0 \) such that for any regular invariant measure \( \mu \) supported on \( \Lambda \) and for any \( T > 0 \) large enough,

\[
\frac{1}{T} \int \log \|\Psi_T|_{\mathcal{N}^s}\| d\mu < -\eta \quad \text{and} \quad \frac{1}{T} \int \log \|\Psi^{-T}|_{\mathcal{N}^u}\| d\mu < -\eta.
\]

Proof. Let \( \mathcal{N}^s|_{\Lambda \smallsetminus \text{Sing}(X)} = \mathcal{N}^s \oplus \mathcal{N}^u \) be singular domination over \( \Lambda \) for \((\Psi_t)_{t \in \mathbb{R}}\), let \( V \) be the closed neighborhood of \( \text{Sing}(X) \cap \Lambda \) and let \( \eta_0, T_0 > 0 \) be the numbers as in Definition 1.6.

Given a regular ergodic measure \( \mu \) supported on \( \Lambda \), since the maximal invariant set in \( V \) is \( \Lambda \cap \text{Sing}(X) \), there exists an open set \( U \) which is disjoint from \( V \) and satisfies \( \mu(U) > 0 \). Now, by the Oseledec theorem and the Poincaré recurrence theorem, one can choose \( x \in U \cap \Lambda \) such that

- \( \lim_{t \to +\infty} \frac{1}{t} \log \|\Psi_t|_{\mathcal{N}^s(x)}\| \) is the maximal Lyapunov exponent of \( \mu \) along \( \mathcal{N}^s \);
- there exists \( t > 0 \) arbitrarily large such that \( \varphi_t(x) \in U \).

The second item above and Definition 1.6 give that there exists \( t > T_0 \) arbitrarily large such that \( \|\Psi_t|_{\mathcal{N}^s(x)}\| < e^{-\eta_0 t} \), and thus the maximal Lyapunov exponent of \( \mu \) along \( \mathcal{N}^s \) is no larger than \(-\eta_0 \). Analogously, one can show that the minimal Lyapunov exponent of \( \mu \) along \( \mathcal{N}^u \) is no less than \( \eta_0 \), and hence \( \mu \) is hyperbolic. The moreover part comes from the dominated convergence theorem and the sub-additive ergodic theorem. □

4.2 Robustness of multi-singular hyperbolicity: Proof of Theorem B

By Theorem A, there exist a \( C^1 \) neighborhood \( U_0 \) of \( X \) and a closed neighborhood \( U_0 \) of \( \Lambda \) such that the maximal invariant set \( \Lambda_0^0 \) in \( U_0 \) for any \( Y \in U_0 \) admits a singular domination. This gives Definition 1.6(1). The singularities of \( X \) in \( \Lambda \) are hyperbolic and will be denoted by \( \sigma_1, \ldots, \sigma_\ell \). Up to reducing \( U_0 \), one can assume that each singularity of \( Y \in U_0 \) in \( U_0 \) is the continuation of some \( \sigma_i \). In particular, Definition 1.6(3) holds for \( Y \in U_0 \) and \( \Lambda_0^0 \). Up to changing the metric, one can also assume that the invariant spaces corresponding to splitting \( E^s \oplus E^c \oplus E^u \) over each singularity \( \sigma_i \) for \( X \) are orthogonal to each other.

Let \( V \) be the neighborhood of \( \{\sigma_1, \ldots, \sigma_\ell\} \) and \( \eta, T > 0 \) be the numbers given in Definition 1.6. Since \( V \) is compact, \( V \) remains an isolating neighborhood of the continuation of singularities \( \{\sigma_1, \ldots, \sigma_\ell\} \) for the \( C^1 \)-close vector fields. One only needs to check that there exist \( T_0 > T, \eta_0 \in (0, \eta) \) and a small enough open neighborhood \( U \) of \( \Lambda \) such that for each vector field \( Y \) which is \( C^1 \)-close to \( X \), the second property of the definition holds for the points in the maximal invariant set of \( Y \) in \( U \) with respect to the neighborhood \( V \) and the numbers \( \eta_0, T_0 \). In the following we consider the bundle \( \mathcal{N}^s \). The bundle \( \mathcal{N}^u \) can be handled in a similar way.

The proof is proceeded by contradiction. We assume that there exist
• a sequence \((X_n)\) which converges to \(X\) in \(\mathcal{A}^t(M)\),
• a sequence of positive numbers \(t_n \to +\infty\),
• a sequence of points \((x_n)\),
which satisfy
• the closure of \(\text{Orb}(x, \varphi_{X_n}^n(x_n))\) is contained in the \(1/n\)-neighborhood of \(\Lambda\);
• \(x_n, \varphi_{X_n}^n(x_n) \notin V\) and \(|\Psi_{X_n}^n|_{\mathcal{N}_t}|| \geq e^{-\frac{1}{n}t_n}\).

Let us denote \(L_n = \mathbb{R}X_n(x_n)\) and let \(\Lambda\) be the limit set of the orbits \(\{\varphi_{X_n}^n(L_n)\}_{s \in \mathbb{R}}\) in \(\mathcal{G}\) as \(n \to +\infty\): it is an invariant compact set which projects on an invariant subset of \(\Lambda\). Up to taking subsequences, there exists a probability invariant measure \(\tilde{\mu}\) on \(\Lambda\) which projects on a measure \(\mu\) on \(M\), such that

\[
\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \delta_{\varphi_{X_n}^n(x_n)} ds = \mu \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \delta_{\varphi_{X_n}^n(L_n)} ds = \tilde{\mu}.
\]

**Claim 1.** \(\frac{1}{\tau} \int \log \|\tilde{\Psi}_\tau|_{\mathcal{N}_\tau}\| d\tilde{\mu} \geq 0\) for any \(\tau > 0\).

Now, we give the proof of the claim. By the definition of the extended linear Poincaré flow, \(\tilde{\Psi}_\tau^X|_{\mathcal{N}_\tau(\varphi_{X_n}(x_n))} = \tilde{\Psi}_\tau^X|_{\mathcal{N}_\tau(\varphi_{X_n}(L_n))}\) for any \(s, t \in \mathbb{R}\). Since \(X_n\) converges to \(X\) in the \(C^1\) topology and the norm of the time \(t\)-map of a linear Poincaré flow is bounded by the norm of the time \(t\)-map of the tangent flow, for each \(t > 0\) there exists \(c_t > 0\) such that

\[
\sup_{n \in \mathbb{N}} \sup_{L \in \mathcal{B}^1} \sup_{s \in [-t, t]} \log \|\tilde{\Psi}_s^X(L)\| < c_t.
\]

Applying Lemma 4.1 to the family of the continuous functions \(\{\log \|\tilde{\Psi}_s^X|_{\mathcal{N}_\tau}\|\}_{t > 0}\), one gets

\[
\log \|\tilde{\Psi}_t^X|_{\mathcal{N}_\tau(L_n)}\| \leq 3c_t + \frac{1}{\tau} \int_0^{t_n} \log \|\tilde{\Psi}_s^X|_{\mathcal{N}_\tau(\varphi_{X_n}(L_n))}\| ds \quad \text{for any} \quad \tau > 0.
\]

The functions \(\frac{1}{\tau} \log \|\tilde{\Psi}_t^X|_{\mathcal{N}_\tau}\|\) converge to \(\frac{1}{\tau} \log \|\tilde{\Psi}_t^X|_{\mathcal{N}_\tau}\|\) as \(n \to +\infty\), uniformly in \(t\). The choice of orbit segment \(\{\varphi_{X_n}^n(L_n)\}_{s \in [0, t_n]}\) gives

\[
\lim_{n \to \infty} \sup_{t_n} \frac{1}{t_n} \log \|\tilde{\Psi}_t^X|_{\mathcal{N}_\tau(L_n)}\| \leq \frac{1}{\tau} \int \log \|\tilde{\Psi}_\tau|_{\mathcal{N}_\tau}\| d\tilde{\mu}.
\]

As \(\|\tilde{\Psi}_t^X|_{\mathcal{N}_\tau(L_n)}\| = \|\tilde{\Psi}_t^X|_{\mathcal{N}_\tau(x_n)}\| \geq e^{-\frac{1}{n}t_n}\), one gets the announced inequality. This ends the proof of the claim.

One can decompose \(\tilde{\mu}\) as the barycenter of three invariant probability measures

\[
\tilde{\mu} = \alpha \cdot \tilde{\nu} + \beta \cdot \tilde{\nu}^+ + \gamma \cdot \tilde{\nu}^-,
\]

where \(\tilde{\nu}, \tilde{\nu}^+, \tilde{\nu}^-\) projects to measures \(\nu, \nu^+, \nu^-\) on \(M\) such that \(\nu\) is regular and \(\nu^+\) (resp. \(\nu^-\)) is supported on the set of singularities \(v\) such that \(W^{ss}(v) \cap \Lambda \setminus \{\sigma\} = \emptyset\) (resp. \(W^{uu}(v) \cap \Lambda \setminus \{\sigma\} = \emptyset\)). We study independently each of these measures.

**Measure \(\tilde{\nu}\).** The Proposition 4.2 implies \(\frac{1}{\tau} \int \log \|\tilde{\Psi}_\tau|_{\mathcal{N}_\tau}\| d\tilde{\nu} < 0\) for any \(\tau > 0\) large enough.

**Measure \(\tilde{\nu}^+\).** Let us consider a singularity \(\sigma \in \Lambda\) in the support of \(\nu^+\). Since \(W^{ss}(\sigma) \cap \Lambda \setminus \{\sigma\} = \emptyset\), its preimage in \(\Lambda\) is contained in the projectivization \(K^+(\sigma)\) of the space \(E^c(\sigma) \oplus E^{uu}(\sigma)\). Since \(E^s(\sigma)\) is orthogonal to \(E^c(\sigma) \oplus E^{uu}(\sigma)\), the bundle \(\mathcal{N}^s\) above \(K^+(\sigma)\) coincides with the space \(E^{ss}(\sigma)\). This implies

\[
\frac{1}{\tau} \int \log \|\tilde{\Psi}_\tau|_{\mathcal{N}_\tau}\| d\tilde{\nu}^+ < 0 \quad \text{for any} \quad \tau > 0 \quad \text{large enough}.
\]

**Measure \(\tilde{\nu}^-\).** We then consider a singularity \(\sigma \in \Lambda\) in the support of \(\nu^-\). Since \(W^{uu}(\sigma) \cap \Lambda \setminus \{\sigma\} = \emptyset\), and since \(\sigma\) is accumulated by points \(\varphi_{X_n}^n(x_n)\) with \(0 < s_n < t_n\), the center direction \(E^c(\sigma)\) has to be unstable. Note also that the preimage of \(\sigma\) in \(\Lambda\) is contained in the projectivization \(K^- (\sigma)\) of the space \(E^{ss}(\sigma) \oplus E^c(\sigma)\) and that above \(K^- (\sigma)\) the bundle \(\mathcal{N}^s\) is contained in \(E^{ss}(\sigma) \oplus E^c(\sigma)\). As \(\sigma\) is
Lorenz-like, there exist \( \eta_\sigma > 0 \) and \( T_\sigma > 0 \) such that \( \| D\varphi_t |_{\mathcal{L}} \cdot \| \hat{\Psi}_t |_{\mathcal{N}^\tau (L)} \| < e^{-2\eta_\sigma t} \) for any \( t > T_\sigma \) and \( L \in K^{-}(\sigma) \).

Let us fix \( \varepsilon > 0 \) and \( \tau > T_\sigma \). There exists an open neighborhood \( V_\sigma \) of \( K^{-}(\sigma) \) such that

(a) \( \tilde{\mu}(V_\sigma \setminus K^{-}(\sigma)) \cdot \max(\| \tilde{\Psi}_\tau \|) < \varepsilon \),

(b) for any \( L \in V_\sigma \), one has \( \| D\varphi_t |_{\mathcal{L}} \cdot \| \tilde{\Psi}_t |_{\mathcal{N}^\tau (L)} \| < e^{-2\eta_\sigma \tau} \).

This in particular implies that for \( n \) large enough and any \( s \in [0, t_n] \) with \( \varphi_s^{X_n}(L_n) \in V_\sigma \), one has

\[
\| D\varphi_t |_{\mathcal{L}} \cdot \| \tilde{\Psi}_t |_{\mathcal{N}^\tau (L_n)} \| < e^{-\eta_\sigma \tau}.
\]

Let us fix \( \delta > 0 \) small enough. For each \( n \), one can introduce finitely many intervals \( I_{n1}, \ldots, I_{nm} \) that are the connected components of the set \( \{ s \in [0, t_n], \varphi_s^{X_n}(L_n) \in V_\sigma \} \) such that \( \hat{\varphi}_{I_{nk}}(L_n) \) meets the \( \delta \)-neighborhood of \( \sigma \). From (a), for \( n \) large enough one has

\[
\left| \frac{1}{t_n} \sum_{i=1}^{m_n} \int_{s \in I_{nk}} \log \| \hat{\Psi}_s |_{\mathcal{N}^\tau (\varphi_s^{X_n}(L_n))} \| ds - \int \log \| \hat{\Psi}_\tau |_{\mathcal{N}^\tau (\varphi_s^{X_n}(L_n))} \| ds \right| < 2\varepsilon.
\]

**Claim 2.** If \( \delta \) is small enough, then for \( n \) large enough \( \frac{1}{t_n} \sum_{i=1}^{m_n} \int_{s \in I_{nk}} \log \| D\varphi_t |_{\mathcal{L}} \cdot \| X_n(\varphi_s(x_n)) \| ds > -\varepsilon \).

Now, we give the proof of the claim. Let us denote \( I_{nk} = (a_{nk}, b_{nk}) \). Then one has

\[
\frac{1}{t_n} \sum_{i=1}^{m_n} \int_{s \in I_{nk}} \log \| D\varphi_t |_{\mathcal{L}} \cdot \| X_n(\varphi_s(x_n)) \| ds
\]

\[
= \frac{1}{t_n} \sum_{i=1}^{m_n} \int_{a_{nk}}^{b_{nk}} \log \| X_n(\varphi_{s+\tau}(x_n)) \| - \log \| X_n(\varphi_s^{X_n}(x_n)) \| ds
\]

\[
\leq \frac{1}{t_n} \sum_{i=1}^{m_n} \left( \int_{a_{nk}}^{b_{nk}} \log \| X_n(\varphi_s(x_n)) \| ds - \int_{a_{nk}}^{a_{nk}+\tau} \log \| X_n(\varphi_s^{X_n}(x_n)) \| ds \right).
\]

Since \( \varphi_{a_{nk}}(x_n), \varphi_{b_{nk}}^{X_n}(x_n) \in \partial V_\sigma \), there exists \( c > 0 \) such that

\[
\sup_{n \in \mathbb{N}} \sup_{x \in \partial V_\sigma, s \in (-\tau, \tau)} \log \| X_n(\varphi_s(x)) \| < c.
\]

If \( \delta > 0 \) is small, then \( |I_{nk}| \) is arbitrarily large, so that

\[
\frac{1}{|I_{nk}|} \left( \int_{a_{nk}}^{b_{nk}} \log \| X_n(\varphi_s(x_n)) \| ds - \int_{a_{nk}}^{a_{nk}+\tau} \log \| X_n(\varphi_s^{X_n}(x_n)) \| ds \right)
\]

is arbitrarily close to 0. This ends the proof of the claim.

By (4.1), (4.2) and the previous claim, for \( n \) large enough, one has

\[
\int \log \| \hat{\Psi}_\tau |_{\mathcal{N}^\tau} \cdot \| d(\tilde{\varphi}^-) |_{\mathcal{K}^{-}(\sigma)} \| ds < 0.
\]

This proves that \( \int \log \| \hat{\Psi}_\tau |_{\mathcal{N}^\tau} \cdot \| d(\tilde{\varphi}^-) |_{\mathcal{K}^{-}(\sigma)} \| ds < 0 \) for \( \tau > 0 \) large enough. As there are only finitely many singularities in \( \Lambda \), this gives \( \int \log \| \hat{\Psi}_\tau |_{\mathcal{N}^\tau} \cdot \| d(\tilde{\mu}) < 0 \) for \( \tau > 0 \) large enough.

To summarize, there exists \( \tau > 0 \) arbitrarily large such that \( \frac{1}{\tau} \int \log \| \hat{\Psi}_\tau |_{\mathcal{N}^\tau} \cdot \| d(\tilde{\mu}) < 0 \) which contradicts Claim 1. Theorem B is now proved.
4.3 A criterion for Lorenz-like singularities

In the definition of multi-singular hyperbolicity we require the singularities to be Lorenz-like. This is often a consequence of the other properties of the definition.

**Proposition 4.3.** Let \( X \in \mathcal{X}^1(M) \), let \( \Lambda \) be an invariant compact set satisfying (1) and (2) in Definition 1.6 and let \( \sigma \) be a hyperbolic singularity in \( \Lambda \). If there exist a sequence \( (y_n)_{n \in \mathbb{N}} \) in \( \Lambda \) and a neighborhood \( V_\sigma \) of \( \sigma \) such that

- \( y_n \) tends to \( \sigma \),
- the forward and backward orbit of \( y_n \) intersects \( M \setminus V_\sigma \),

then \( \sigma \) is Lorenz-like.

**Proof.** Let \( \mathcal{N}_{|\Lambda \setminus \text{Sing}(X)} = N^s \oplus N^u \) be singular domination over \( \Lambda \) for \( (\Psi_t)_{t \in \mathbb{R}} \) and let \( i \) be its index. Let \( \sigma \in \Lambda \) be a singularity as in the assumption. Then \( W^s(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset \) and \( W^u(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset \).

By the definition of singular domination, one gets that

- if \( \text{ind}(\sigma) > i \), then \( E^s(\sigma) = E^{ss}(\sigma) \oplus E^{su}(\sigma) \) with \( \dim(E^{ss}(\sigma)) = i \) and \( W^{ss}(\sigma) \cap \Lambda = \{\sigma\} \);
- if \( \text{ind}(\sigma) \leq i \), then \( E^u(\sigma) = E^{cu}(\sigma) \oplus E^{cu}(\sigma) \) with \( \dim(E^{cu}(\sigma)) = \dim(M) - 1 - i \) and \( W^{cu}(\sigma) \cap \Lambda = \{\sigma\} \).

Without loss of generality, from now on, we assume that \( \sigma \) admits a partially hyperbolic splitting for \( (D\varphi_t)_{t \in \mathbb{R}} \) of the form \( E^{ss} \oplus E^{cs} \oplus E^u \) where \( \dim(E^{ss}) = i \) and \( W^{ss}(\sigma) \cap \Lambda = \{\sigma\} \).

Up to changing the Riemannian metric, one can assume that each bundle in the splitting \( E^{ss} \oplus E^{cs} \oplus E^u \) is orthogonal to the other. Let \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \) be all the Lyapunov exponents of \( \sigma \) along \( E^{cs} \oplus E^u \), where \( k = \dim(E^{cs} \oplus E^u) \). It remains to prove that \( E^{cs} \oplus E^u \) is sectionally expanding under \( (D\varphi_t)_{t \in \mathbb{R}} \), i.e., \( \lambda_1 + \lambda_2 > 0 \), which in return implies that \( \dim(E^{cs}) = 1 \).

Let \( \lambda = \lambda_1 + \lambda_2 \). For \( \varepsilon > 0 \), consider a small neighborhood \( U_\sigma \subset V_\sigma \) of \( \sigma \) where one can define a cone field \( C^{cu} \) with respect to \( E^{cs} \oplus E^u \) such that

- \( C^{cu} \) is \( (D\varphi_t)_{t \geq T} \)-invariant for some constant \( T > 0 \);
- there exists \( c > 1 \) such that for any \( x \in U_\sigma \), any \( t > T \) satisfying \( \{\varphi_s(x)\}_{s \in [0,t]} \subset U_\sigma \) and any \( \dim(E^{cs} \oplus E^u) \)-dimensional linear space \( F \subset C^{cu}(x) \), there exists a 2-plane \( P \subset F \) such that

\[ |\det(D\varphi_t | P)| \leq c \cdot e^{(\lambda + \varepsilon)t}. \]

Moreover since \( W^{ss}(\sigma) \cap \Lambda = \{\sigma\} \), we can assume that \( \mathbb{R}X(x) \oplus N^u(x) \subset C^{cu}(x) \) for any \( x \in \Lambda \cap U_\sigma \).

Consider the sequence of points \( y_n \) as in the assumption. Let \( x_n \) be the first intersection of the backward orbit of \( y_n \) with the boundary of \( U_\sigma \) and let \( t_n > 0 \) be the first time that the forward orbit of \( x_n \) intersects the boundary of \( U_\sigma \). In particular,

\[ x_n, \varphi_{t_n}(x_n) \in \partial U_\sigma \quad \text{and} \quad y_n \in \{\varphi_s(x_n)\}_{s \in (0,t_n)} \subset U_\sigma. \]

Hence there exists a 2-plane \( P \subset \mathbb{R}X(x_n) \oplus N^u(x_n) \) such that

\[ |\det(D\varphi_{t_n} | P)| \leq c \cdot e^{(\lambda + \varepsilon)t_n}. \quad (4.3) \]

Let \( V \) be the compact isolating neighborhood in Definition 1.6(2). Then there exists \( l > 0 \) such that for any \( x \in \partial U_\sigma \) and \( t > 0 \) with \( \varphi_t(x) \in \partial U_\sigma \) and \( \{\varphi_s(x)\}_{s \in (0,l)} \subset U_\sigma \), the forward orbit of \( \varphi_t(x) \) and the backward orbit of \( x \) leave \( U_\sigma \) in time smaller than \( l \). Therefore, there exist \( s_n, \tau_n \in (-l, l) \) such that \( \varphi_{-s_n}(x_n) \in \partial V \) and \( \varphi_{t_n + \tau_n}(x_n) \in \partial V \). From Definition 1.6(2), there exist \( c, \eta > 0 \) such that

\[ |\det(D\varphi_{t_n + s_n + \tau_n} | P)| \geq c \cdot e^{n(t_n + s_n + \tau_n)} \quad \text{for any 2-plane } P \subset \mathbb{R}X(\varphi_{-s_n}(x_n)) \oplus N^u(\varphi_{-s_n}(x_n)). \quad (4.4) \]

As \( s_n \) and \( \tau_n \) are uniformly bounded, by (4.3) and (4.4), one has \( \lambda + \varepsilon \geq \eta \). The arbitrariness of \( \varepsilon \) implies that \( \lambda \geq \eta > 0 \).
4.4 Uniform and singular hyperbolicities: Proof of Theorem C

(a) We prove Theorem C(1). Let us consider a uniformly hyperbolic set $\Lambda$ such that for each $\sigma \in \text{Sing}(\Lambda)$ both $W^s(\sigma) \cap \Lambda \setminus \{\sigma\}$ and $W^u(\sigma) \cap \Lambda \setminus \{\sigma\}$ are non-empty. We first prove that $\text{Sing}(X) = \emptyset$ since the uniform hyperbolicity along regular orbits in $W^s(\sigma)$ and $W^u(\sigma)$ gives incompatible stable dimension at $\sigma$. The restriction of the splittings $E^s \oplus \mathbb{R}X$ and $\mathbb{R}X \oplus E^u$ to the normal bundle induces a dominated splitting for the linear Poincaré flow which satisfies the definition of multi-singular hyperbolicity.

(b) Conversely, if $\Lambda$ is a multi-singular hyperbolic set which does not contain any singularity, Definition 1.6(2) shows that the bundles $\mathcal{N}^s$ and $\mathcal{N}^u$ are respectively uniformly contracted and uniformly expanded by the linear Poincaré flow, whereas the action of the differential on the bundle $\mathbb{R}X$ remains bounded. Then Proposition 3.6 implies that the tangent bundle over $\Lambda$ admits dominated splittings $TM|_{\Lambda} = E^s \oplus F = E^c \oplus E^u$ with $\dim(E^s) = \dim(\mathcal{N}^s)$, $\dim(E^c) = \dim(\mathcal{N}^u)$. The existence of a finest dominated splitting (see [8, Appendix B.1]) gives a dominated splitting $TM|_{\Lambda} = E^s \oplus E^c \oplus E^u$ with $\dim(E^c) = 1$. Since the invariant bundle $\mathbb{R}X$ remains bounded, it remains in uniform cones transverse to $E^s$ and $E^u$. The invariance and the domination then give $E^c = \mathbb{R}X$, proving that $\Lambda$ is uniformly hyperbolic. The proof of the first item is completed.

(c) We now turn to Theorem C(2) and consider an invariant compact set $\Lambda$ which is singular hyperbolic. We will assume for instance that it has a dominated splitting of the form $TM|_{\Lambda} = E^{ss} \oplus E^{cu}$, as in Definition 1.4.

We first notice that at each point $x \in \Lambda$ we have $X(x) \in E^{cu}(x)$. Indeed if one assumes by contradiction that $x$ is regular and satisfies $X(x) \notin E^{cu}(x)$, the backward orbit of $x$ remains uniformly transverse to the bundle $E^{cu}$ and avoids a neighborhood of the singularities. The $\alpha$-limit set of $x$ is thus non-singular and (by domination), the restriction of the vector field $X$ is tangent to $E^{ss}$. This leads to a contradiction since for any probability measure on $\alpha(x)$, the Lyapunov exponent in the direction of the flow is not negative.

Since $X \in E^{cu}$, the linear Poincaré flow also admits a dominated splitting $\mathcal{N} = \mathcal{N}_1 \oplus \mathcal{N}_2$ of index $\dim(E^{ss})$, obtained by intersecting $E^{ss}(x) \oplus \mathbb{R}X(x)$ and $E^{cu}(x)$ with $\mathcal{N}(x)$ at each regular point $x$. Moreover for each singularity $\sigma \in \Lambda$, we also have $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$. Consequently $\Lambda$ has a singular domination of index $\dim(E^{ss})$. Since $E^{ss}$ is uniformly contracted, the bundle $\mathcal{N}_1$ is uniformly contracted by the linear Poincaré flow.

Let $V$ be a neighborhood of $\text{Sing}(X) \cap \Lambda$. For any regular point $x$, any unit vector $v \in \mathcal{N}_1$ and $t \in \mathbb{R}$, the volume growth under the tangent flow along the plane spanned by $v$ and $X(x)$ is equal to

$$\|\Psi_t(x).v\| \frac{\|X(\varphi_t(x))\|}{\|X(x)\|};$$

by singular hyperbolicity of $\Lambda$, there exist $T_0, \eta > 0$ such that for any $x \in \Lambda$ and $t > T_0$,

$$\|\Psi_t(x).v\| \frac{\|X(\varphi_t(x))\|}{\|X(x)\|} > \exp(2\eta t).$$

When $x$ and $\varphi_t(x)$ are outside $V$, the quotient $\frac{\|X(\varphi_t(x))\|}{\|X(x)\|}$ is bounded away from 0 by a constant $1/C$. Choose $T > T_0$ such that $\exp(\eta T) > C$ which implies $\|\Psi_t(x).v\| \geq \exp(\eta t)$. This concludes Definition 1.6(2).

Since each singularity is hyperbolic and $W^s(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset$, there exists a stable direction inside $E^{cu}(\sigma)$. The singular hyperbolicity implies that $E^{cu}(\sigma)$ decomposes as $E^{cu}(\sigma) = E^c \oplus E^u$ with $\dim(E^c) = 1$. We have thus proved that each singularity is Lorenz-like. This ends the proof that $\Lambda$ is multi-singular hyperbolic.

(d) Finally, we consider a multi-singular hyperbolic set $\Lambda$ with a singular dominated splitting $\mathcal{N} = \mathcal{N}^s \oplus \mathcal{N}^u$ as in Definition 1.6. We assume that at any singularity $\sigma \in \Lambda$:

- both $W^s(\sigma) \cap \Lambda \setminus \{\sigma\}$ and $W^u(\sigma) \cap \Lambda \setminus \{\sigma\}$ are non-empty,
- there exists a dominated splitting

$$T_\sigma M = E^{ss} \oplus E^c \oplus E^u,$$
where $E^{ss}$ and $E^u$ have the same dimensions as $N^s$ and $N^u$ and $E^c$ is a stable line.

The case where $E^c(\sigma)$ is an unstable line for all singularity $\sigma$ can be handled analogously. Up to changing the metric, the splitting can be assumed orthogonal at each space $T_xM$.

Singular domination implies that at each singularity either $W^{ss}(\sigma) \cap \Lambda \setminus \{\sigma\}$ or $W^{uu}(\sigma) \cap \Lambda \setminus \{\sigma\}$ is empty. Since $E^c$ is contracting and $W^{uu}(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset$, one concludes that

$$W^{ss}(\sigma) \cap \Lambda \setminus \{\sigma\} = \emptyset.$$  

(4.5)

We then prove the uniform contraction of the bundle $N^s$ under the flow $(\Psi_t)_{t \in \mathbb{R}}$. We choose $\eta_0, T_0 > 0$ and a small open neighborhood $V$ of $\text{Sing}(X) \cap \Lambda$ as in Definition 1.6(2). Let us consider any regular $x \in \Lambda$ and any $t > 0$.

If $x$ and $\varphi_t(x)$ do not belong to $V$ and $t \geq T_0$, one has $\|\Psi_t|_{N^s}(x)\| \leq \exp(-\eta_0 t)$.

If the orbit $(\varphi_s(x))_{s \in [0, t]}$ is contained in $V$, the property (4.5) implies that $\mathbb{R}X(\varphi_s(x))$ is close to a line in $E^c \oplus E^{uu}$. Then the dominated splitting $N^s \oplus N^u$ and the fact that $E^{ss}$ is orthogonal to $E^c \oplus E^u$ imply that $N^s(\varphi_s(x))$ is close to $E^{ss}(\sigma)$. Consequently, there exist $\eta_1, T_1 > 0$ such that if $t \geq T_1$, then

$$\|\Psi_t|_{N^s}(x)\| \leq \exp(-\eta_1 t).$$

We choose $C > 0$ such that for any piece of orbit of length $t \leq \max(T_0, T_1)$, we have $\|\Psi_t|_{N^s}(x)\| \leq C$. We also set $\eta = \min\{\eta_0, \eta_1\}$.

If the orbit segment $(\varphi_s(x))_{s \in [0, t]}$ is not entirely contained in $V$, we consider the largest interval $[t_1, t_2] \subset [0, t]$ such that $\varphi_{t_1}(x) \notin V$ (we take $t_2 = 0$ provided that $x \in V$). Then the previous estimates give

$$\|\Psi_t|_{N^s}(x)\| \leq C^3 \cdot \exp(3\eta(T_1 + T_0)) \cdot \exp(-\eta t).$$

This shows that $N^s$ is uniformly contracted by the linear Poincaré flow. Note that since the singularities $\sigma \in \Lambda$ are hyperbolic and since $W^s(\sigma) \cap \Lambda \setminus \{\sigma\}$ and $W^u(\sigma) \cap \Lambda \setminus \{\sigma\}$ are non-empty, $\Lambda \setminus \text{Sing}(X)$ is dense in $\Lambda$. By Proposition 3.6, there exists a dominated splitting $TM|_{\Lambda} = E^{ss} \oplus F$ with $\dim(E^{ss}) = \dim(N^s)$.

Any ergodic measure $\mu$ on $\Lambda$ is

- either supported on a Lorenz-like singularity $\sigma$: by definition the sum of the two smallest Lyapunov exponents along $F(\sigma) = E^{uu}$ is positive,
- or a regular measure having one vanishing Lyapunov exponent along $X$ and other positive Lyapunov exponents along $F$ due to Proposition 4.2.

This implies that for the tangent flow above $\Lambda$, the volume along 2-planes contained in $F$ grows exponentially and the set $\Lambda$ is singular hyperbolic.

The proof of Theorem C is now completed.

\[ \Box \]

5 Multi-singular hyperbolicity and rescaling

In this section, we compare Definition 1.6 with the definition given by Bonatti and da Luz [6]. We first recall some terminology about extended flows.

5.1 Extended maximal invariant set

Let $X \in A^1(M)$, and $\Lambda$ be an invariant compact set. Let $\sigma \in \Lambda$ be a hyperbolic singularity and consider the finest dominated splitting for $(D\varphi_t)_{t \in \mathbb{R}}$:

$$T_xM = E_k^s \oplus \cdots \oplus E_k^c \oplus \cdots \oplus E_i^c.$$ 

Let $i$ be the smallest integer such that the strong stable manifold of $\sigma$ tangent to $E_k^s \oplus \cdots \oplus E_k^c$ intersects $\Lambda$ only at $\sigma$. The space $E_{\sigma, \Lambda}^ss := E_k^s \oplus \cdots \oplus E_i^s$ is called the escaping stable space of $\sigma$ in $\Lambda$. Analogously, we define the escaping unstable space of $\sigma$ in $\Lambda$, and we denote it as $E_{\sigma, \Lambda}^{uu} := E_j^u \oplus \cdots \oplus E_i^u$. 
Now, the center space at $\sigma$ is defined as

$$E_{\sigma, A}^c = E_{s,-1}^s \oplus \cdots \oplus E_1^s \oplus E_1^u \oplus \cdots \oplus E_{j-1}^u.$$  

We will denote by $\mathbb{P}_{\sigma, A}^c$ the projective space of the center space $E_{\sigma, A}^c$.

**Definition 5.1.** Let $X \in \mathcal{X}^1(M)$ and $\Lambda$ be an invariant compact set whose singularities are all hyperbolic. The extended invariant set of $\Lambda$ is the compact subset of $G^1$ defined by

$$B(X, \Lambda) = \{ \mathbb{R}X(x) : x \in \Lambda \setminus \text{Sing}(X) \} \cup \bigcup_{\sigma \in \text{Sing}(X) \cap \Lambda} \mathbb{P}_{\sigma, A}^c.$$  

**Proposition 5.2** (See [6, Proposition 38]). Let $X \in \mathcal{X}^1(M)$ and $U$ be a compact set such that all the singularities in $U$ are hyperbolic. Let $\Lambda_{X,U}$ be the maximal invariant set of $X$ in $U$. Then there exists a $C^1$ neighborhood $U$ of $X$ where the map $Y \in U \mapsto B(Y, \Lambda_{Y,U})$ is upper semi-continuous.

### 5.2 Rescaling cocycle associated with a hyperbolic singularity

Let $X \in \mathcal{X}^1(M)$ and let $\sigma$ be a hyperbolic singularity. A cocycle $(h^t)_{t \in \mathbb{R}}$ over the flow $(\varphi_t)_{t \in \mathbb{R}}$ is a rescaling cocycle at $\sigma$ if

- there exists a neighborhood $U_\sigma$ of $\sigma$ and $C > 1$ such that for any $x \in U_\sigma$, $L \in G^1(x) \cap \tilde{M}_X$ and $t \in \mathbb{R}$ satisfying $\varphi_t(x) \in U_\sigma$, one has

$$C^{-1} \frac{h^t(L)}{\|D\varphi_t(L)\|} < C;$$

- for any small neighborhood $W$ of $\sigma$, there exists $C_W > 1$ such that for any $x \in M \setminus W$, $L \in G^1(x) \cap \tilde{M}_X$ and $t \in \mathbb{R}$ satisfying $\varphi_t(x) \in M \setminus W$, one has $C_W^{-1} < h^t(L) < C_W$.

At any hyperbolic singularity $\sigma$, there exists a rescaling cocycle, and it is unique up to multiplication by a cocycle bounded away from 0 and $+\infty$ (see [6, Theorem 1]).

The following property appears in [6, Corollary 63] and justifies the rescaling by the cocycle $(h^t)_{t \in \mathbb{R}}$.

**Proposition 5.4.** Let $X \in \mathcal{X}^1(M)$, let $\sigma$ be a Lorenz-like singularity with splitting

$$T_\sigma M = E^s \oplus E^c \oplus E^u$$

and let $\mathbb{P}E_{\sigma}^s$ denote the projective space of $E^s \oplus E^c$, so that the extended linear Poincaré flow admits a dominated splitting $N^s \oplus N^u$ with $\dim(N^s) = \dim(E^s)$ over $\mathbb{P}E_{\sigma}^s$.

If $(h^t)_{t \in \mathbb{R}}$ is a rescaling cocycle at $\sigma$, then the cocycle $(h^t \cdot \hat{\varphi}_t|_{N^s})_{t \in \mathbb{R}}$ contracts uniformly.

### 5.3 Bonatti-da Luz’s definition

We can now recall the definition introduced in [6].

**Definition 5.5.** Let $X \in \mathcal{X}^1(M)$. An invariant compact set $\Lambda$ is multi-singular hyperbolic (in the sense of Bonatti-da Luz) if

1. the singularities $\sigma \in \Lambda$ are hyperbolic, and we fix a rescaling cocycle $(h^t_\sigma)$ at each $\sigma$;
2. the extended linear Poincaré flow admits a dominated splitting $N^s \oplus N^u$ over $B(X, \Lambda)$;
(3) there exists a subset $S_+ \subset \text{Sing}(X) \cap \Lambda$ such that the cocycle $(h^t_0 \cdot \tilde{\Psi}_t |_{\mathcal{N}^s})_{t \in \mathbb{R}}$ is uniformly contracting, where $h^t_0 = \prod_{\sigma \in S_+} h^t_0$ for any $t \in \mathbb{R}$;

(4) there exists a subset $S_- \subset \text{Sing}(X) \cap \Lambda$ such that the cocycle $(h^t_- \cdot \tilde{\Psi}_t |_{\mathcal{N}^u})_{t \in \mathbb{R}}$ is uniformly expanding, where $h^t_- = \prod_{\sigma \in S_-} h^t_0$ for any $t \in \mathbb{R}$.

One says that $X$ is multi-singular hyperbolic in a compact set $U$ if the maximal invariant set of $X$ in $U$ is multi-singular hyperbolic.

**Remark 5.6.** Under the assumption that $W^s(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset$ and $W^u(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset$ for all the singularities $\sigma \in \Lambda$, the set $S_+$ (resp. $S_-$) has to coincide with the set of singularities whose stable dimension is $\dim(\mathcal{N}^s) + 1$ (resp. $\dim(\mathcal{N}^u)$) (see the proof of Proposition 5.7).

We then compare Definitions 1.6 and 5.5. We first show that the first implies the second.

**Theorem D.** Let $X \in \mathcal{X}^1(M)$ and let $\Lambda$ be an invariant compact set. If $\Lambda$ satisfies Definition 1.6, then it satisfies Definition 5.5.

**Proof.** Let $\mathcal{N}^s \oplus \mathcal{N}^u$ be the dominated splitting for the linear Poincaré flow over $\Lambda \setminus \text{Sing}(X)$ as in Definition 1.6. It extends to the closure in $\mathcal{G}^1$, hence to $\{\mathbb{R} X(x) : x \in \Lambda \setminus \text{Sing}(X)\}$. Let $i$ be its index. Let us consider a singularity $\sigma \in \Lambda$, and let us assume that it has the splitting

$$T_x M = E^{ss} \oplus E^c \oplus E^u$$

with $\dim(E^{ss}) = i$ and $\dim(E^c) = 1$, and $W^{ss}(\sigma) \cap \Lambda \setminus \{\sigma\} = \emptyset$ (the other cases can be addressed analogously). Then $\mathbb{P}^e_{\sigma, \Lambda}$ is contained in the projective space associated to $E^c \oplus E^u$. By Proposition 5.4, there exists a dominated splitting of index $i$ for the extended normal flow over $\mathbb{P}^e_{\sigma, \Lambda}$. Hence the dominated splitting can be extended to $B(X, \Lambda)$.

We now check Definition 5.5(3) (Definition 5.5(4) is checked analogously). The set $S_+$ is the set of singularities in $\Lambda$ with a dominated splitting

$$T_x M = E^s \oplus E^c \oplus E^{uu}$$

and let $(h^t_+ \cdot \tilde{\Psi}_t |_{\mathcal{N}^s})_{t \in \mathbb{R}}$ be the associated cocycle. In order to prove that $(h^t_+ \cdot \tilde{\Psi}_t |_{\mathcal{N}^s})$ is uniformly contracting, we have to prove that for any ergodic probability measure $\tilde{\mu}$ on $B(X, \Lambda)$, there exists $T > 0$ such that

$$\int \log h^T_+ + \log \| \tilde{\Psi}_T |_{\mathcal{N}^s} \| d\tilde{\mu} < 0. \quad (5.1)$$

Proposition 5.4 proves that it is the case for the measures supported on the invariant sets $\mathbb{P}^e_{\sigma, \Lambda} \subset \mathbb{P}E^c_{\sigma}$ associated with the singularities $\sigma \in S_+$. For the singularities $\sigma \in S_-$, $\mathbb{P}^e_{\sigma, \Lambda}$ is contained in the projective space $\mathbb{P}E^u_{\sigma}$, above which the cocycle $(\tilde{\Psi}_t |_{\mathcal{N}^s})_{t \in \mathbb{R}}$ is uniformly contracting; since $(h^t_+ \cdot \tilde{\Psi}_t |_{\mathcal{N}^s})_{t \in \mathbb{R}}$ is bounded, the property (5.1) holds for measures supported on $\mathbb{P}^e_{\sigma, \Lambda}$ also in this case.

It remains to consider ergodic measures $\tilde{\mu}$ whose projection on $\Lambda$ is a regular measure $\nu$. For each $T > 0$ we have

$$\int \log h^T_+ + \log \| \tilde{\Psi}_T |_{\mathcal{N}^s} \| d\tilde{\mu} = \int \log h^T_+ + \log \| \Psi_T |_{\mathcal{N}^s} \| d\nu. \quad (5.2)$$

By Theorem B, $X$ satisfies the star property on a neighborhood of $\Lambda$. By the proof of Theorem 5.6 in [29], each regular ergodic measure $\nu$ supported on $\Lambda$ is accumulated by periodic measures $\delta_{\gamma_n}$ supported on periodic orbits $\gamma_n$ contained in a small neighborhood of $\Lambda$. Hence, there exists a sequence of periodic orbits $\gamma_n$ such that $\delta_{\gamma_n}$ tends to $\nu$. Notice that singular domination over $\Lambda$ can be extended continuously to the maximal invariant set of $X$ in $U$ by Theorem A, which implies

$$\int \log h^T_+ + \log \| \Psi_T |_{\mathcal{N}^s} \| d\nu = \lim_{n \to \infty} \int \log h^T_+ + \log \| \Psi_T |_{\mathcal{N}^s} \| d\delta_{\gamma_n}. \quad (5.3)$$

On $\gamma_n$, the cocycle $(h^t_+)_{t \in \mathbb{R}}$ is bounded away from $0$ and $+\infty$. The Birkhoff ergodic theorem gives

$$\int \log h^T_+ d\delta_{\gamma_n} = \int \lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} \log h^T_+ (\varphi_{iT}(p)) d\delta_{\gamma_n} = \int \lim_{k \to \infty} \frac{1}{k} h^T_+ (p) d\delta_{\gamma_n} = 0. \quad (5.4)$$
The star property and Theorem 2.2 give $T > 0$ and $\eta > 0$ so that

$$
\int \log \|\Psi_T|_{\mathcal{N}^*}\|d\gamma_n = \int \lim_{k \to \infty} \frac{1}{k} \log \prod_{i=0}^{k-1} \|\Psi_T(\varphi_i T(p))|_{\mathcal{N}^*}\|d\gamma_n \leq -\eta. \quad (5.5)
$$

The equations (5.2)–(5.5) together imply (5.1) for the regular measure $\mu$. \hfill \Box

Then we show the converse under a mild condition.

**Theorem E.** Let $X \in \mathcal{X}^l(M)$ and $\Lambda$ be an invariant compact set whose singularities $\sigma$ satisfy $W^s(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset$ and $W^u(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset$. If $\Lambda$ satisfies Definition 5.5, then it also satisfies Definition 1.6.

We need an auxiliary result before proving Theorem E.

**Proposition 5.7.** Let $X \in \mathcal{X}^l(M)$ and $\Lambda$ be an invariant compact set satisfying Definition 5.5. Then any singularity $\sigma \in \Lambda$ such that $W^s(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset$ and $W^u(\sigma) \cap \Lambda \setminus \{\sigma\} \neq \emptyset$ is Lorenz-like. Moreover if $\sigma$ has the splitting $T_\sigma M = E^{ss} \oplus E^{cs} \oplus E^{u}$, then $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$.

*Proof.* Let $\mathcal{N}^s \oplus \mathcal{N}^u$ be the domination over $B(X, \Lambda)$ for the extended linear Poincaré flow as in Definition 5.5 and let $\sigma \in U$ be a singularity as in the statement of the proposition. By assumption, the center space $\mathbb{P}^c(\sigma, U)$ contains lines $L_s \subset E^s(\sigma)$ and $L_u \subset E^u(\sigma)$. Without loss of generality, one assumes that $\dim(E^s(\sigma)) > \dim(\mathcal{N}^s)$. By applying Proposition 2.1 to the domination over $\mathbb{P}^c(\sigma, U)$ for the extended linear Poincaré flow, there exists a dominated splitting

$$
T_\sigma M = E^{ss} \oplus E^{cs} \oplus E^{u}
$$

for the tangent flow with $\dim(E^{ss}) = \dim(\mathcal{N}^s)$.

We claim that $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$. Let us assume by contradiction, that this is not the case. Then there exists a line $L^s \subset E^{ss}$ which belongs to $\mathbb{P}^c_{\sigma, \Lambda}$; by the definition of $\mathbb{P}^c_{\sigma, \Lambda}$, there exists a line in $E^{ss} \oplus E^u$ which is not contained in $E^{ss} \cup E^u$ and belongs to $\mathbb{P}^c_{\sigma, \Lambda} \subset B(X, \Lambda)$. This contradicts the second item of Proposition 2.1.

In particular $L_s \subset E^{cs}(\sigma)$. Up to changing the metric, one can assume that the splitting

$$
T_\sigma M = E^{ss} \oplus E^{cs} \oplus E^u
$$

is orthogonal. Then $E^{cs}(\sigma) \subset \mathcal{N}^u(L_u)$. In order to satisfy Definition 5.5(4), the bundle $\mathcal{N}^u$ over $\mathbb{P}^c_{\sigma, \Lambda}$ has to be renormalized by the cocycle $(h^t_{\sigma})_{t \in \mathbb{R}}$, proving that $\sigma$ belongs to $S_\Lambda$.

From the first item of Definition 5.3 and Definition 5.5(4), $(\|D\varphi_t|_{L_s}\| \cdot \Psi_t|_{\mathcal{N}^u(L_s)})_{t \in \mathbb{R}}$ is uniformly expanding along the orbit of $L_s \in \mathbb{P}^c_{\sigma, \Lambda}$. This implies that $E^{cs} = L_s$ is one-dimensional and that the Lyapunov exponent $\lambda^c$ along $E^{cs}$ and the smallest Lyapunov exponent $\lambda^u$ along $E^u$ satisfy $\lambda^c + \lambda^u > 0$. Hence $\sigma$ is Lorenz-like. \hfill \Box

*Proof of Theorem E.* Let $\mathcal{N}^s \oplus \mathcal{N}^u$ be the dominated splitting given by Definition 5.5 and let $i$ be its index. By Proposition 5.7, each singularity in $\Lambda$ has a dominated splitting

$$
T_\sigma M = E^{ss} \oplus E^c \oplus E^{uu}
$$

with $\dim(E^c) = 1$, $\dim(E^{ss}) = i$ and either $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$ or $W^{uu}(\sigma) \cap \Lambda = \{\sigma\}$. This proves that $\Lambda$ admits a singular domination of index $i$.

For each singularity $\sigma \in \Lambda$, let $(h^t_{\sigma})_{t \in \mathbb{R}}$ be the renormalization cocycle at $\sigma$ and consider a small closed neighborhood $V_\sigma$ of $\sigma$ such that

- the maximal invariant set of $(\varphi_t)_{t \in \mathbb{R}}$ in $V_\sigma$ is $\sigma$;
- $V_\sigma$ is contained in the neighborhood of $\sigma$ given in the first item of Definition 5.3.

Taking the neighborhoods $V_\sigma$ small enough, one can assume that they are pairwise disjoint and let $C(\sigma) > 1$ be the constant associated to $V_\sigma$ by the second item in Definition 5.3. Take

$$
C = \prod_{\sigma \in \text{Sing}(X) \cap \Lambda} C(\sigma).
$$
By Definition 5.5, there exists $\eta > 0$ such that for each $L \in B(X, \Lambda)$, one has for any $t > 0$ large enough
\[
\| h_t^+ (L) \cdot \widehat{\Psi}_t |_{\mathcal{N}^c(L)} \| < e^{-2\eta t} \quad \text{and} \quad \| h_t^- \cdot \widehat{\Psi}_t |_{\mathcal{N}^u(L)} \| < e^{-2\eta t}.
\]
Fix $T$ large enough such that $C \cdot e^{-\eta T} < 1$. Now, for any $x \in \Lambda$ and $t > T$ such that $x$ and $\varphi_t(x)$ are disjoint from
\[
V := \bigcup_{\sigma \in \text{Sing}(X) \cap U} V_{\sigma},
\]
denoting $L = R_X(x)$, one has
\[
\| \Psi_t |_{\mathcal{N}^c(x)} \| < C \cdot \| h_t^+ (L) \cdot \widehat{\Psi}_t |_{\mathcal{N}^c(L)} \| \leq e^{-\eta t}.
\]
Similarly, one can show $\| \Psi_t^- |_{\mathcal{N}^u(x)} \| < e^{-\eta t}$ for $t > T$. 

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