BOUNDING EXT FOR MODULES FOR ALGEBRAIC GROUPS, FINITE GROUPS AND QUANTUM GROUPS

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Abstract. Given a finite root system Φ, we show that there is an integer $c = c(Φ)$ such that $\dim \text{Ext}_G^1(L, L') < c$, for any reductive algebraic group $G$ with root system $Φ$ and any irreducible rational $G$-modules $L, L'$. There also is such a bound in the case of finite groups of Lie type, depending only on the root system and not on the underlying field. For quantum groups, a similar result holds for $\text{Ext}^n$, for any integer $n \geq 0$, using a constant depending only on $n$ and the root system. Weaker versions of this are proved in the algebraic and finite group cases, sufficient to give similar results for algebraic and generic cohomology. The results both use, and have consequences for, Kazhdan-Lusztig polynomials. An appendix proves a stable version, needed for small prime arguments, of Donkin’s tilting module conjecture.

1. Introduction and discussion

In previous work with Edward Cline [12], we showed that there exists a bound on $\dim H^1(G, L)$—for a semisimple, simply connected algebraic group $G$ and an irreducible rational $G$-module $L$—by a constant $C$ that depends only on the root system of $G$, and not on the module $L$ or the underlying field. We also proved a similar result for the finite groups of Lie type and irreducible modules in the natural characteristic. This result represented the first general progress on a conjecture of Robert Guralnick [19] that there exists a universal bound on $\dim H^1(H, V)$, for all finite groups $H$ and for all faithful, absolutely irreducible representations $V$ over any field $k$. The space $H^1(H, V)$, which parametrizes conjugacy classes of complements to $V$ in the semidirect product $V \rtimes H$, is particularly important for maximal subgroup theory.

As is well-known, $H^1(H, V) \cong \text{Ext}_H^1(k, V)$, where $k$ is the 1-dimensional trivial module. For any pair $L, V$ of $kH$-modules, $\text{Ext}_H^1(L, V)$ parametrizes equivalence classes of short exact sequences $0 \to V \to E \to L \to 0$ of $kH$-modules. These sequences are of...
interest for all $L$, not just for the trivial module. Guralnick also expressed the view (privately) that the dimensions of $\text{Ext}^1$-groups between irreducible modules should also be quite small. We know that no universal constant bound is possible here (see Scott-Xi [31]); however, the asymptotic properties of the growth of these $\text{Ext}^1$-groups has considerable interest.

The first main result of this paper, see Theorem 5.1, establishes that, given a finite root system $\Phi$, there exists a constant $c = c(\Phi)$ such that if $G$ is a semisimple, simply connected algebraic group over an algebraically closed field $k$ of any characteristic $p$, and if $G$ has root system $\Phi$, then $\dim \text{Ext}^1_G(L, L') \leq c$, for all irreducible, rational $G$-modules $L, L'$. This result improves upon a similar partial result given in [12, Thm. 7.9]. As in [12, Thm. 7.10], it implies the same result for finite groups of Lie type associated with the root system $\Phi$ with $L, L'$ irreducible modules in the natural characteristic. See Corollary 5.3.

In this paper, as in [12], we attack the issue of a bound in the algebraic group case by first showing that, if the prime $p$ is fixed, there is a uniform bound for all $L, L'$. Some care is required when $p = 2$; see §3. With this analysis complete, take $p$ large, and, in particular, assume that the Lusztig character formula holds. We can then utilize the (Lusztig) quantum group $U_\zeta$ at a $p$th root of 1 associated to $G$. We also can assume that if $L = L(\lambda)$ and $L' = L(\mu)$ (for dominant weights $\lambda, \mu$), then $\lambda \not\equiv \mu \mod p$ and $\lambda < \mu$. In [12], we treated the case for $\lambda, \mu$ regular and showed that

$$\dim \text{Ext}^1_G(L(\lambda), L(\mu)) \leq \dim \text{Ext}^1_G(\Delta^\text{red}(\lambda), \nabla^\text{red}(\mu)) = \dim \text{Ext}^1_{U_\zeta}(L_\zeta(\lambda), L_\zeta(\mu)),$$

using the natural modules $\Delta^\text{red}(\lambda)$ and $\nabla^\text{red}(\mu)$ that arise from the irreducible modules $L_\zeta(\lambda)$ and $L_\zeta(\mu)$, respectively, for $U_\zeta$ by a standard "reduction mod $p$" process. Here, we treat the singular weight case by focusing more on the (ostensibly larger) group $\text{Ext}^1_G(\Delta(\lambda), L(\mu))$ and by using translation arguments to reduce to the regular case (an approach announced by us in [30]), maintaining the condition $\lambda \not\equiv \mu \mod p$. The bound is given in that case, as in [12], by an appropriate "top" coefficient $\mu(x, y)$ of a Kazhdan-Lusztig polynomial for the affine Weyl group of $\Phi$. A different way of treating the case $\lambda \not\equiv \mu \mod p$ is stated in Theorem 5.4, which bounds an infinite sum of $\text{Ext}^1$-dimensions with a single constant. The proof of this latter result is postponed to §7.

Section 6 studies the interaction of quantum group cohomology/representation theory with combinatorial considerations of Kazhdan-Lusztig polynomials. Let $U_\zeta$ be the (Lusztig) quantum enveloping algebra at an $l > h$ root of unity (where $h$ is the Coxeter number of $\Phi$). Corollary 6.7 shows that $\dim \text{Ext}^n_{U_\zeta}(L_\zeta(\lambda), L_\zeta(\nu))$ is bounded uniformly by a constant depending only on $n$ and $\Phi$. Actually, a stronger result, given in Theorem 6.5, shows that there is a uniform bound on the sum of all these dimensions, over all possible dominant weights $\nu$, when the weight $\lambda$ is fixed. The bound again depends only on $n$ and $\Phi$, and not on $\lambda$. In addition to homological applications, there are interesting consequences for Kazhdan-Lusztig polynomials; see Theorem 6.9. The
key to Theorem 6.5 is Lemma 6.3, which shows that the composition series length of all PIMs are uniformly bounded in the quantum case, depending only on $\Phi$. While this very quickly yields the boundedness property given in Theorem 6.5, the bounds thus obtained are quite crude. (The results of Section 6 lead—in a paper [28] in preparation—to a “complexity theory” for the quantum enveloping algebras $U_\zeta$, analogous to the classical cohomological complexity theory for finite groups and restricted enveloping algebras.)

The main result of §7, given in Theorem 7.1, provides a higher degree version of the main $\Ext^1$-result given in §5. Specifically, we show that, given nonnegative integers $e, m$, there exists a constant $c(\Phi, m, e)$ depending only on $\Phi$, $e$, and $m$ such that $\dim \Ext^2_{\Phi}(L(\lambda), L(\mu)) \leq c(\Phi, m, e)$, for all $p^{e+1}$-restricted dominant weights $\lambda$ and all dominant weights $\mu$. The proof is by induction on $m$. When $p \geq 2h - 2$, the projective covers of the irreducible modules for the $r$th infinitesimal subgroup $G_r$ of $G$ are known to have compatible $G$-module structures, and an essential step involves showing that there is a bound $C(\Phi, r)$ on the $G$-length of these modules which depends only on the root system $\Phi$ and the integer $r$; see Lemma 7.2. As mentioned above, the analogous result for the quantum PIM $Q_\zeta(\lambda)$ for all dominant $\lambda$ is established in §6. The difference between the two cases seems to hinge on the fact that in the algebraic group case $G/G_1 \cong G$, while in the quantum enveloping algebra case, if $u_\zeta$ is the little quantum group, then the quotient $U_\zeta/u_\zeta \cong U(\mathfrak{g})$ has a completely reducible finite dimensional representation theory. Another important ingredient makes use of the stability estimates proved in [13] for generic cohomology (viewed in terms of twists by powers of of the Frobenius endomorphism) of reductive groups. Section 7 concludes with the postponed proof of Theorem 5.4. An interesting consequence of this theorem shows that there are only a finite number of dominant weights $\lambda$ with $\lambda \neq 0 \mod p^2$ for which $H^1(G, L(\lambda)) \neq 0$. Further, given any dominant $\mu$, $\dim H^1(G, L(\mu))$ may be computed in terms of some $\dim H^1(G, L(\lambda))$ with $\lambda$ as above. See Remark 7.5(c).

Possibly changing $G$ when $p = 2$, we may even take $\lambda \neq 0 \mod p$.

It is important to mention that the results of Section 7 do not require any restriction on the size of the characteristic $p$. In an Appendix (Section 8), we develop the necessary machinery to handle the small prime cases. A long open problem in modular representation theory is the existence of a rational $G$-module structure on the PIMs for the infinitesimal subgroups $G_e$ of $G$. As mentioned above, when $p \geq 2h - 2$, the existence of a $G$-module structure is known to hold (by Jantzen and earlier work by Ballard). On the other hand, for all $p$, Donkin [16] has conjectured that, for any $e \geq 1$ and any $p^e$-restricted dominant weight $\lambda$, the $G_e$-PIM $Q_e(\lambda)$ is $G_e$-isomorphic to the restriction to $G_e$ of the tilting module $T(2(p^e - 1)p + w_0\lambda)$ of highest weight $2(p^e - 1)p + w_0\lambda$. We prove, for all characteristics, a “stable” version of Donkin’s conjecture; see Corollary

\footnote{We defer to [28] some of our previous (posted preprint) remarks relevant to [20] and possible behavior of higher degree cohomology of finite simple groups. Related comments in this paper may be found above Corollary 6.7.}
for a precise statement. Making use of this result, we construct in Corollary 8.3 a
substitute for the $G_r$-PIMs which can be used in Section 7 to complete the arguments
for all characteristics.

Our results on algebraic group cohomology for $G$ have the consequence of bounding
generic cohomology with irreducible coefficients in any given degree $m$ by a constant
depending only on $m$ and the root system; see the end of §7. If $m > 1$, it remains open
if there is such a bound for all finite groups of Lie type associated to the root system.
(The issue is that the rate of convergence in the limit $\lim_{d \to \infty} H^m(G(^d), V)$ depends on
the coefficient module $V$.)

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2. Some preliminaries

Let $\Phi$ be an irreducible (classical) finite root system spanning a Euclidean space $E$
with inner product $(u, v)$. The assumption that $\Phi$ is irreducible is only a convenience,
and the setting can easily be generalized to the case of a general finite root system. Fix a
set $\Pi = \{\alpha_1, \ldots, \alpha_{rk(G)}\}$ of simple roots and let $\Phi^+$ be the positive roots determined by
$\Pi$. Let $\alpha_0$ be the maximal short root in $\Phi^+$, and, for $\alpha \in \Phi$, put $\alpha^\vee = \frac{2}{(\alpha, \alpha)} \alpha$, the coroot
attached to $\alpha$. Let $Q = Q(\Phi)$ be the root lattice, i.e., $Q = \mathbb{Z} \alpha_1 \oplus \cdots \oplus \mathbb{Z} \alpha_{rk(G)} \subset E$.

Let $X \subset E$ be lattice of all integral weights: $\lambda \in \mathbb{E}$ belongs to $X$ if and only if
$(\lambda, \alpha^\vee) \in \mathbb{Z}$ for all $\alpha \in \Pi$. Thus, $Q \subseteq X$. Let $\varpi_1, \ldots, \varpi_{rk(G)} \in X$ be the fundamental
dominant weights defined by $(\varpi_i, \alpha_j^\vee) = \delta_{i,j}, 1 \leq i, j \leq rk(G)$. Then $X^+ := \mathbb{N} \varpi_1 \oplus \cdots \oplus \mathbb{N} \varpi_{rk(G)}$ is the set of dominant weights. Put $\rho = \varpi_1 + \cdots + \varpi_{rk(G)}$ (the Weyl
weight) and $h = (\rho, \alpha_0^\vee) + 1$ (the Coxeter number of $\Phi$).

For a positive integer $l$, let $X^+_l := \{\lambda \in X^+ \mid (\lambda, \alpha^\vee) < l, \forall \alpha \in \Pi\}$ be the set of
$l$-restricted dominant weights. More generally, if $e \geq 1$ is an integer, we set $X^+_{e,l} := \{\lambda \in X^+ \mid (\lambda + \rho, \alpha^\vee) < le, \forall \alpha \in \Pi\}$. When $l = p$, a prime, the set $X^+_p$ of $p$-restricted
dominant weights will be used in Sections 7 and 8.

Regard $E$ (or any subset) as a poset by setting $\lambda \leq \nu$ provided that $\nu - \lambda = \sum_{\alpha \in \Pi} n_\alpha \alpha$, where each $n_\alpha$ is a non-negative integer. Another partial ordering $\leq'$ is
sometimes useful: put $\lambda \leq' \nu$ provided that $\nu - \lambda = \sum_{\alpha \in \Pi} q_\alpha \alpha$, with each $q_\alpha \in \mathbb{Q}^+$. An ideal $\Omega$ of $\Gamma$ is a subset such that $\lambda \leq \nu$, with $\nu \in \Omega$ and $\lambda \in \Gamma$, implies $\lambda \in \Omega$.

The Weyl group $W$ is a Coxeter group with fundamental reflections $\{s_{\alpha_1}, \ldots, s_{\alpha_{rk(G)}}\}$, where, given $\alpha \in \Phi$, $s_\alpha : E \to E$, $x \mapsto s_\alpha(x) := x - (x, \alpha^\vee) \alpha$, $x \in E$. For $\alpha \in \Phi$, $n \in \mathbb{Z}$,
let $s_{\alpha,n} : E \to E$ be the affine transformation $x \mapsto s_{\alpha,n}(x) := x - ((x, \alpha^\vee) - n) \alpha$. Let
$W_\alpha$ be the group of affine transformations generated by the $s_{\alpha,n}, \alpha \in \Phi, n \in \mathbb{Z}$. Since

\[\text{This improves on an earlier posted preprint, which treated only the case } p > h \text{ in the presence of the Lusztig character formula. Actually, the argument there, using derived category "shifted standard filtrations," appears to be complete only when a bound on the exponent of both weights $\lambda$ and $\mu$ is given. It does, however, appear to give useful bounds in that case.}\]
s_a = s_{a,0} ∈ W_a, W is a subgroup of W_a; in fact, W_a ∼= W × Q, identifying Q = Q(Φ) with the subgroup of Aff(Φ) (= group of affine transformations of Φ) consisting of translations by elements of Q. Putting S_a := S ∪ \{s_{a,-1}\}, (W_a, S_a) is a Coxeter system. If s = s_a, n ∈ W_a, H_s ⊂ E is its fixed-point hyperplane.

For a positive integer l, W_a,l be the subgroup of W_a generated by the affine reflections s_{a,n} in which l divides n. There is an evident isomorphism ε_l : W_a ∼= W_a,l in which s_{a,n} ↦ s_{a,n,l}. We will use the “dot” action of W_a on E: for w ∈ W_a, x ∈ E, put w · x := w(x + ρ) − ρ. For a positive integer l, w · x := ε_l(w) · x. Setting S_{a,l} := S ∪ \{s_{a,-1}\}, (W_{a,l}, S_{a,l}) is a Coxeter system.

For x, y ∈ W_{a,l}, let P_{y,x} ∈ Z[q] be the Kazhdan-Lusztig polynomial in q = t^2 associated with the pair (y, x) ∈ W_{a,l} × W_{a,l}.\footnote{The integer l should always be clear from context when discussing P_{y,x}.} Necessarily, P_{y,x} ≠ 0 implies that y ≤ x in the Bruhat-Chevalley order on W_{a,l}. If ℓ : W_{a,l} → N is the length function (defined by S_{a,l}) and y < x, then P_{y,x} is a polynomial in q of degree ≤ (ℓ(x) − ℓ(y) − 1)/2. (Also, P_{x,x} = 1 for all x.) For y ≤ x, let μ(y, x) be the coefficient of q^{ℓ(x)−ℓ(y)−1}/2 in P_{y,x}. Thus, μ(y, x) = 0 unless x and y have opposite parity. In particular, μ(x, x) = 0. If x < y, set μ(y, x) := μ(y, x), and put μ(y, x) = 0 if x, y are not comparable.

Let C^-_i ⊂ E be the chamber defined by the hyperplanes H_s, s ∈ \{s_{a,1}, \ldots, s_{a,l}, s_{a,-1}\}. Its closure $\overline{C^-_i}$ is a fundamental domain for the dot action of W_{a,l} on E (or for the l-action of W_a on E). Let C^- := C^-_1 if l = 1.

We call w ∈ W_a dominant if w · C^- + ρ is contained in the dominant cone \{x ∈ E | (x, α_i^+) ≥ 0, 1 ≤ i ≤ r\} of E. (Similarly, an element ε_l(w) = w is l-dominant if w · C^-_l + ρ = w · C^- + ρ is contained in the dominant cone. Thus, w is dominant if and only if ε_l(w) is l-dominant.) Let W^+_a (resp., W^+_a(l)) be the set of dominant elements in W_a (resp., W_{a,l}). For any x ∈ W_a, the right coset W x of W in W_a contains a unique element x' of maximal length among all other elements in the right coset. Then W^+_a is simply the set of right coset representatives which have maximal length. (Equivalently, W^+_a consists of all elements w_0 x, where x is a “distinguished” (= minimal length) right coset representative of W in W_a, and w_0 ∈ W is the element of maximal length in W.) A similar description of W^+_a(l) also holds.

Let $X^+_a := \{λ ∈ X^+ \mid (λ + ρ, α^+) ≠ 0 \bmod l, ∀α ∈ Φ\}$ be the l-regular dominant weights. A weight that is not l-regular is called l-singular, or just singular. Thus, $X^+_a(l) ≠ \emptyset$ if and only if l ≥ h. Let λ = w · λ^- ∈ X^+, λ^- ∈ C^-_l and w ∈ W^+_a(l). Define

\begin{equation}
χ_{KL}(λ, l) = \sum_{y ∈ W^+_a(l)} (-1)^{ℓ(w)−ℓ(y)} P_{y,w}(1)χ(y · λ^-) ∈ ℤX.
\end{equation}
For $\nu \in X^+$, $\chi(\nu) = \sum_{w \in W} (-1)^{\ell(w)} e(w \cdot \nu) / \sum_{w \in W} (-1)^{\ell(w)} e(w \cdot 0)$ is the Weyl character in the integer group algebra $\mathbb{Z}X$. If $l$ is clear from context, write $\chi_{KL}(\lambda)$ for $\chi_{KL}(\lambda, l)$.

We work with several algebraic objects attached to the root system $\Phi$.

(1) $G$ denotes a simple, simply connected algebraic group over an algebraically closed field $k$ of positive characteristic with fixed Borel subgroup $B$ containing a maximal torus $T$. The Lie algebras of $G, T, B, \text{ etc.}$ will be denoted by the corresponding fraktur letters $\mathfrak{g}, \mathfrak{b}, \mathfrak{t}$, etc. We assume that $\Phi = \Phi(T)$ is the set of roots of $T$, so $X = X(\Phi) = X(T)$ and $Q = Q(\Phi) = Q(T)$. For $\lambda \in X^+$, let $L(\lambda)$ be the irreducible rational $G$-module of highest weight $\lambda$. Also, let $\Delta(\lambda)$ and $\nabla(\lambda)$ be the standard (Weyl) and costandard modules, respectively, of highest weight $\lambda$. Thus, $L(\lambda)$ is the socle (resp., head) of $\nabla(\lambda)$ (resp., $\Delta(\lambda)$). Also, $\Delta(\lambda)$ and $\nabla(\lambda)$ have equal characters given by Weyl’s formula, i.e., $\text{ch} \Delta(\lambda) = \text{ch} \nabla(\lambda) = \chi(\lambda)$.

Assume that $p \geq h$. A $p$-regular dominant weight $\lambda = w \cdot \lambda^-$, $\lambda^- \in C_p^-$ is said to satisfy the Lusztig character formula (LCF) if $\text{ch} L(\lambda) = \chi_{KL}(\lambda, p)$. Recall that the Jantzen region $\Gamma_{\text{Jan}} := \{ \lambda \in X^+ \mid (\lambda + \rho, \alpha_i) \leq p(p-h+2) \}$. For characteristic $\geq h$ sufficiently large (depending on $\Phi$) the LCF holds for all regular $\lambda \in \Gamma_{\text{Jan}}$; see [2] (and also the survey [35] §8) for references) as well as [18] where the methods of [2] are improved to give a specific bound for each root system. If $l \geq h$, then $\text{ch} L_{\zeta}(\lambda) = \chi_{KL}(\lambda, l)$ for any $\lambda \in X_{\text{reg}, l}^+$. (Suitably formulated, the LCF holds for all $\lambda \in X^+$, $l$-regular or not.) In particular, if $p \geq h$ and if the LCF holds on $\Gamma_{\text{Jan}}$, then given $\lambda \in X_{\text{reg}, l}^+$, $L(\lambda)$ is obtained by reduction mod $p$ from $L_{\zeta}(\lambda)$, $\zeta = \sqrt{-1}$.

We assume that $G$ as well as $B, T$ are defined and split over the prime field $\mathbb{F}_p$. Let $F : G \rightarrow G$ be the Frobenius morphism, and, for $e \geq 1$, let $G_e = \ker(F^e)$ be the $e$th infinitesimal subgroup. If $V$ is a rational $G$-module $V^{(e)}$ denotes the rational $G$-module obtained by pulling the action of $G$ on $V$ back through $G^e$. The set $X_{e,p}^+$ indexes the irreducible rational $G_e$-modules; given $\lambda \in X_{e,p}^+$, $L(\lambda)|_{G_e}$ is an irreducible $G_e$-module, and representatives of the distinct isomorphism classes of irreducible $G_e$-modules are given by these modules. (When $L(\lambda)$, or any $G$-module, is to be regarded as a $G_e$-module by restriction, we will often be somewhat informal, writing $L(\lambda)$ instead of $L(\lambda)|_{G_e}$.) For $\lambda \in X^+$, write

\begin{equation}
(2.0.2) \quad \lambda = \sum_{i=0}^{\infty} p^i \lambda_i, \quad (\lambda_i \in X_{1,p}^+), \quad \lambda^{(i)} = \sum_{j=i}^{\infty} p^{j-i} \lambda_j.
\end{equation}

For convenience, we often denote $\lambda^{(1)}$ more simply by $\lambda^\dagger$. Thus, given $\lambda \in X^+$, it has a unique decomposition $\lambda = \lambda_0 + p \lambda^\dagger$, for $\lambda_0 \in X_{p+1}^+, \lambda^\dagger \in X^+$. Also, $L(\lambda_0) \cong L(\lambda_0) \otimes L(\lambda^\dagger)^{(1)}$.

\footnote{Thoughtout this paper, $k$ will always denote an algebraically closed field. The assumption that $G$ be simple is only for convenience. All results hold if $G$ is only assumed to be semisimple.}

\footnote{It is useful to note that $\Gamma_{\text{Jan}}$ contains $X_{1,p}^+$ if and only if $p \geq 2h - 3$. Also, if $\sigma = \sigma_0 + p \sigma^\dagger \in \Gamma_{\text{Jan}}$ with $\sigma_0 \in X_{1,p}^+$, $\sigma^\dagger \in X^+$, then $\sigma^\dagger \in C_p$.}
(2) Let \( l \) be a positive integer, and let \( \zeta = \sqrt[l]{T} \) be a primitive \( l \)th root of unity in the complex numbers \( \mathbb{C} \). We will assume that \( l \) is an odd integer\(^8\) and if \( \Phi \) is of type \( G_2 \), in addition, that \((l, 3) = 1\). Let \( U_\zeta = U_{(\zeta = \sqrt{l})} \) be the (Lusztig) quantum enveloping algebra over the cyclotomic field \( \mathbb{Q}(\zeta) \) with “root system \( \Phi \). In the sequel, when discussing \( U_\zeta \), the above restriction on \( l \) (imposed by \( \Phi \)) will always be in force (though not usually mentioned). Also, \( U_\zeta \)-mod denotes the category of finite dimensional \( U_\zeta \)-modules which are integrable and type 1. For \( \lambda \in X^+ \), let \( L_\zeta(\lambda) \) (resp., \( \Delta_\zeta(\lambda) \), \( \nabla_\zeta(\lambda) \)) be the irreducible (resp., standard (Weyl) costandard) module of highest weight \( \lambda \). If \( l > h \) and \( \lambda \in X^+_{\text{reg},l} \), then \( \chi L_\zeta(\lambda, l) \)\(^9\).

Let \( u_\zeta \) be the “little” quantum group attached to \( U_\zeta \). It is a normal, Hopf subalgebra of \( U_\zeta \) such that \( U_\zeta//u_\zeta \cong U(\mathfrak{g}_\zeta) \) the universal enveloping algebra of the complex simple Lie algebra with root system \( \Phi \). If \( M \in U_\zeta\text{-mod} \), then the subspace \( M^{u_\zeta} \) of \( u_\zeta \)-fixed points is a locally finite (and completely reducible) \( U(\mathfrak{g}_\zeta) \)-module (i.e., a rational module for the complex algebraic group \( G_\zeta \) with Lie algebra \( \mathfrak{g}_\zeta \)). For \( M, N \in U_\zeta\text{-mod} \), and any integer \( n \), an elementary Hochschild-Serre spectral sequence argument gives:

\[
\text{Ext}^n_{U_\zeta}(M, N) \cong \text{Ext}^n_{u_\zeta}(M, N)^{U(\mathfrak{g}_\zeta)}.
\]

Let \( \text{Fr} : U_\zeta \to U_\zeta//u_\zeta \cong U(\mathfrak{g}_\zeta) \) be the quotient (Frobenius) map. Given a \( \mathfrak{g}_\zeta \)-module \( M, M^{(1)} \in U_\zeta\text{-mod} \) denotes the pullback of \( M \) through \( \text{Fr} \). For \( \lambda \in X^+ \), let \( L_\zeta(\lambda) \) be the irreducible \( \mathfrak{g}_\zeta \)-module of highest weight \( \lambda \).

3. Some cohomology results

Let \( G \) be as in §2(1). We will need the following result, due to Andersen [11, Thm. 4.5].

**Theorem 3.1.** Unless \( p = 2 \) and \( G \) has type \( C_r \),

\[
\text{Ext}^1_{G_1}(L, L) = 0
\]

for any rational irreducible \( G_1 \)-module.

Because this result fails when \( G \) has type \( C_r \) and \( p = 2 \), more attention is required to bound \( \text{Ext}^1_G \) in case \( p = 2 \). Until Proposition 3.2 \( k \) has characteristic 2 and \( G \) (resp., \( G' \)) is the simple, simply connected algebraic group over \( k \) of type \( C_r \) (resp., type \( B_r \)). Let \( T', \) etc. be the maximal torus, etc. of \( G' \). The group \( G' \) contains a closed subgroup \( G'' \) which is simple of type \( D_r \), viz., \( G'' \) is the closed subgroup of \( G' \) generated by \( T' \) and the root subgroups \( U_\alpha \) corresponding to long roots \( \alpha \) in \( \Phi' \). The torus \( T'' := T' \) is a maximal torus in \( G'' \). Also, \( G'' \) is simply connected because one easily checks that the index \( X(T'')/Q(T'') \) has order 4.

\(^8\)The assumption that \( l \) is odd can be avoided, using [25].

\(^9\)The requirement that \( \lambda \in X^+_{\text{reg},l} \) is not necessary, but requires more care in the definition of \( \chi_{\text{KL}}(\lambda, l) \) (2.0.1) and is not needed in this paper. Also, the assumption that \( l > h \) can sometimes be relaxed; see [35] §7.
The Euclidean space $\mathbb{E}$ (resp., $\mathbb{E}'$, $\mathbb{E}''$) contains $X(T)$ (resp., $X(T')$, $X(T'')$) and has orthonormal basis $\{\epsilon_1, \cdots, \epsilon_r\}$ (resp., $\{\epsilon'_1, \cdots, \epsilon'_r\}$, $\{\epsilon''_1, \cdots, \epsilon''_r\}$), chosen as in [N] pp. 267-273. Thus,

$$\Phi = \{\pm \epsilon \pm \epsilon_j | i \neq j, 1 \leq i, j \leq r\},$$

$$\Phi' = \{\pm \epsilon'_i \pm \epsilon'_j | i \neq j, 1 \leq i, j \leq r\},$$

$$\Phi'' = \{\pm \epsilon''_i \pm \epsilon''_j | i \neq j, 1 \leq i, j \leq r\},$$

describe the root systems of $G$, $G'$ and $G''$, respectively. Since $T'' = T'$, we can assume that $\epsilon'_i = \epsilon''_i$.

Identify $X^Q(T) := \mathbb{Q} \otimes X(T)$ with the $\mathbb{Q}$-span of $\epsilon_1, \cdots, \epsilon_r$ in $\mathbb{E}$, and make a similar convention for $X^Q(T')$ and $X^Q(T'')$. There is a special isomorphism $\varphi : X^Q(T') \to X^Q(T), \epsilon'_i \mapsto 2\epsilon_i, 1 \leq i \leq r$, as in [N] Def. 1, §18.2). It corresponds to the bijection\(^{10}\)

$$\Phi' \leftrightarrow \Phi, \begin{cases} \pm \epsilon'_i \pm \epsilon'_j \leftrightarrow \pm \epsilon_i \pm \epsilon_j; \\
\pm \epsilon'_i \leftrightarrow \pm \epsilon_i,
\end{cases}$$

for $1 \leq i, j \leq r$. For example, observe that

$$\varphi(\overline{\epsilon'_i}) = \begin{cases} 2\overline{\epsilon_i}, & 1 \leq i < r; \\
\overline{\epsilon_i}, & i = r.
\end{cases}$$

Under this correspondence long (resp., short) roots correspond to short (resp. long) roots. Also, if $\alpha' \leftrightarrow \alpha$ in (*) with $\alpha$ short (resp., long), then $\varphi(\alpha') = \alpha$ (resp., $\varphi(\alpha') = 2\alpha$).

The special isomorphism $\varphi$ corresponds to a (special) isogeny $\theta : G \to G'$. Similarly, there is a (special) isogeny $\theta' : G' \to G$ defined by the special isomorphism $\varphi' : X^Q(T) \to X^Q(T')$ which maps each $\epsilon_i$ to $\epsilon'_i, 1 \leq i \leq r$. Regarding $G''$ as a subgroup of $G'$, let $G''_1$ be the scheme-theoretic image of $G''_1$ in the infinitesimal subgroup scheme $G_1$ of $G$ under $\theta'$. Then $\overline{\theta'} = G''_1/\kappa$, where $\kappa$ is a closed (infinitesimal) subgroup of $T''$. Also, $G''_1 = \text{Ker}(\theta)$.

If $\lambda = \sum_{i=1}^r a_i \overline{\epsilon_i} \in X(T)$, let $\lambda_\sigma = \sum_{i=1}^{r-1} a_i \overline{\epsilon_i} \in X(T)$ and $\lambda_r = a_r \overline{\epsilon_r} \in X(T)$. There exists a unique $\tilde{\lambda}_\sigma \in X(T')^+$ (resp., $\tilde{\lambda}_r \in X(T')^+$) such that $\varphi(\tilde{\lambda}_\sigma) = 2\lambda_\sigma^{(1)}$ (resp., $\varphi(\tilde{\lambda}_r) = \lambda_r$). Define (somewhat abusing our previous notation)

$$\tilde{\lambda}^{(1)} := \tilde{\lambda}_\sigma + \tilde{\lambda}_r \in X(T')^+.$$

Now we return to the general simple group $G$ in the first part of the following result.

**Proposition 3.2.** Let $\lambda, \nu \in X^+$, and suppose that $\lambda_0 = \nu_0$.

\(a\) Ext$^1_G(L(\lambda), L(\nu)) \cong Ext^1_G(L(\lambda^{(1)}), L(\nu^{(1)})) \) (in the notation of \([2.0.2]\)) unless $p = 2$ and $G$ has type $C_r$.

\(^{10}\)Note that this bijection only agrees with $\varphi$ up to scalar multiples.
(b) Suppose \( G \) has type \( C_r \) and \( p = 2 \). If \( r > 2 \), then
\[
\text{Ext}^1_G(L(\lambda), L(\nu)) \approx \text{Ext}^1_G(L(\tilde{\lambda}(1)), L(\tilde{\nu}(1))).
\]

Proof. (a) is proved in [12, Lem. 7.1], where it is remarked that it essentially contained in [1]. We now show (b). If \( \lambda_0 = \nu_0 \), then \( \lambda_{\sigma,0} = \nu_{\sigma,0} \). Also, using [33, Thm. 11.1] as well as the tensor product theorem, \( L(\lambda) = L(\lambda_{\sigma}) \otimes L(\lambda_{\tau}) \) and \( L(\nu) = L(\nu_{\sigma}) \otimes L(\nu_{\tau}) \). Let \( M = L(\lambda)^* \otimes L(\nu) \). There is a Hochschild-Serre exact sequence
\[
(3.2.1) \quad 0 \rightarrow H^1(G_1', M)_{G/G_1'} \rightarrow H^1(G, M) \rightarrow H^1(G/G_1', M_{G_1'}) \rightarrow 0.
\]
But \( L(\lambda_r) \) is a trivial \( G_1' \)-module [33, Thm. 11.1]. If \( X := L(2\lambda^{(1)}_{\sigma})^{*} \otimes L(\lambda_{\tau})^{*} \otimes L(2\nu_{\sigma})^{*} \otimes L(\nu_{\tau}) \), then \( H^1(G_1', M) \cong \text{Ext}^1_{G_1'}(L(\lambda), L(\nu)) \cong \text{Ext}^1_{G_1'}(L(\lambda_{\sigma}, 0), L(\nu_{\sigma,0})) \otimes X = 0 \) by Theorem 3.1 since \( \lambda_{\sigma,0} = \nu_{\sigma,0} \). \( L(\lambda_{\sigma,0}) \) is an irreducible \( G_1' \)-module, and \( G'' \) is simply connected of type \( D_r \) with \( r > 2 \). Thus, \( \text{Ext}^1_G(L(\lambda), L(\nu)) \cong H^1(G/G_1', M_{G_1'}) \cong H^1(G', M_{G_1'}) \). However, \( M_{G_1'} \cong \text{Hom}_{G_1'}(L(\lambda), L(\nu)) \cong \text{Hom}(L(\tilde{\lambda}(1)), L(\tilde{\nu}(1))) \), so the result follows. \( \square \)

4. Connections with quantum enveloping algebras and Kazhdan-Lusztig polynomials

If \( x \in E \), the point-stabilizer \( (W_a)_x \) for the dot action of \( W_a \) on \( E \) is isomorphic to a finite parabolic subgroup of \( W_a \), so
\[
(4.0.2) \quad \max_{x \in E}|(W_a)_x| = |W| < \infty.
\]
Given \( x, y \in W_a, \mu(x, y) \) denotes, as in §2, the coefficient of \( q^{(\ell(\nu) - \ell(\lambda)-1)/2} \) in the Kazhdan-Lusztig polynomial \( P_{x,y} \) for the Coxeter system \( (W_a, S_a) \) if \( x < y \). If \( x > y \), \( \mu(x, y) := \mu(y, x) \), and if \( x = y \), then \( \mu(x, y) = 0 \).

Lemma 4.1. There exists a positive integer \( E(\Phi) \) such that \( \mu(x, y) \leq E(\Phi) \) for all \( x, y \in W_a^+ \).

For a representation theory proof, see [12, Lemma 7.6], and for a combinatoric proof, see [31]. In fact, [12] shows that \( E(\Phi) = h[\Phi](2h - 2\rho) \) works, where \( \Phi \) is the Kostant partition function. For another proof in a more general context, see §6 below.

The integers \( \mu(x, y) \) in the following result are computed in the Coxeter group \( W_{a,t} \).

Lemma 4.2. ([12, (1.5.3)]) Given \( \lambda, \nu \in X_{r,\text{reg},t}^+ \) which are \( W_{a,t} \)-conjugate (under the dot action), write \( \lambda = w \cdot \lambda^- \) and \( \nu = y \cdot \lambda^- \) for \( \lambda^- \in C_t^- \) and \( w, y \in W_{a,t} \). Then
\[
\left\{\begin{array}{l}
\dim \text{Ext}^1_{U_{\varsigma}}(L_{\varsigma}(x \cdot \lambda^-), L_{\varsigma}(y \cdot \lambda^-)) = \dim \text{Ext}^1_{U_{\varsigma}}(L_{\varsigma}(y \cdot \lambda^-), L_{\varsigma}(x \cdot \lambda^-)) = \mu(y, x); \\
\dim \text{Ext}^1_{U_{\varsigma}}(L_{\varsigma}(x \cdot \lambda^-), \nabla_{\varsigma}(y \cdot \lambda^-)) = \dim \text{Ext}^1_{U_{\varsigma}}(\Delta_{\varsigma}(y \cdot \lambda^-), L_{\varsigma}(x \cdot \lambda^-)) = \mu(y, x).
\end{array}\right.
\]
In case \( \lambda, \nu \) are not \( W_{a,t} \)-liked, these \( \text{Ext}^1 \)-groups all vanish.
Remark 4.3. More generally, [11] Thm. 3.5 proves that
\[
\dim \text{Ext}_{U_{\zeta}}^a(L_{\zeta}(x \cdot \lambda^-), L_{\zeta}(y \cdot \lambda^-)) = \\
\sum_{z \in W_{a,t}^+} \dim \text{Ext}_{U_{\zeta}}^a(L(x \cdot \lambda^-), \nabla_{\zeta}(z \cdot \lambda^-)) \dim \text{Ext}_{U_{\zeta}}^b(\Delta_{\zeta}(z \cdot \lambda^-), L_{\zeta}(x \cdot \lambda^-)).
\]
Of course, \(\text{Ext}_{U_{\zeta}}^a(L(x \cdot \lambda^-), \nabla_{\zeta}(z \cdot \lambda^-)) = 0\) unless \(z \leq x\), and \(\text{Ext}_{U_{\zeta}}^b(\Delta_{\zeta}(z \cdot \lambda^-), L_{\zeta}(x \cdot \lambda^-)) = 0\) unless \(z \leq y\). Letting
\[
p_{z,x} := \sum_{n \geq 0} \text{Ext}_{U_{\zeta}}^n(L_{\zeta}(x \cdot \lambda^-), \nabla_{\zeta}(z \cdot \lambda^-))t^n = \sum_{n \geq 0} \text{Ext}_{U_{\zeta}}^n(\Delta_{\zeta}(z \cdot \lambda^-), L_{\zeta}(z \cdot \lambda^-))t^n,
\]
t医疗(z)\(\bar{p}_{z,x}\) is the Kazhdan-Lusztig polynomial \(P_{z,x}\) (for \(W_{a,t}\) in \(q = t^2\). (Here \(\bar{p}_{z,x}\) is the polynomial in \(t^{-1}\) obtained by replacing each power \(t^i\) by \(t^{-i}\) in \(p_{z,x}\).)

Let \(G\) be as in §2(1). For \(\lambda \in X^+\), we will make use of four additional \(G\)-modules: \(\Delta^\text{red}(\lambda), \nabla^\text{red}(\lambda), \Delta^p(\lambda), \) and \(\nabla_p(\lambda)\) defined as follows. Let \(\lambda = \lambda_0 + p\lambda^\dagger\) as after (2.1.2). For \(l \geq h\), the module \(\Delta^\text{red}(\lambda)\) (resp., \(\nabla^\text{red}(\lambda)\)) is defined to be the reduction modulo \(p\) of the \(U_{\zeta}\)-irreducible module \(L_{\zeta}(\lambda)\) (for \(l = p\)) with respect to a minimal (resp., maximal) lattice. When the LCF formula holds for all \(p\)-restricted dominant weights,
\[
\begin{align*}
\Delta^\text{red}(\lambda) & \cong L(\lambda_0) \otimes \Delta(\lambda^\dagger)^{(1)}; \\
\nabla^\text{red}(\lambda) & \cong L(\lambda_0) \otimes \nabla(\lambda^\dagger)^{(1)}. 
\end{align*}
\]
(4.3.1)

It is not necessary to go into details here. In general, we define
\[
\begin{align*}
\Delta^p(\lambda) & = L(\lambda_0) \otimes \Delta(\lambda^\dagger)^{(1)}; \\
\nabla_p(\lambda) & = L(\lambda_0) \otimes \nabla(\lambda^\dagger)^{(1)}. 
\end{align*}
\]
(4.3.2)

It is easy to see that \(\Delta^\text{red}(\lambda)\) and \(\Delta^p(\lambda)\) have head \(L(\lambda)\), and \(\nabla^\text{red}(\lambda)\) and \(\nabla_p(\lambda)\) have socle \(L(\lambda)\).

Now [12] Thm. 5.4] gives

**Lemma 4.4.** Assume \(p > h\) and that the LCF holds for \(G\) on an ideal of \(p\)-regular weights containing the \(p\)-regular \(p\)-restricted weights.

(a) If \(\lambda = w \cdot \lambda^-, \nu = y \cdot \lambda^-,\) with \(\lambda^- \in C_p^-\) and \(w \leq w\) in \(W_{a,p}\), then
\[
\dim \text{Ext}_G^1(\Delta^\text{red}(\lambda), \nabla(\nu)) = \dim \text{Ext}_G^1(\Delta(\nu), \nabla^\text{red}(\lambda)) = \mu(x, y).
\]
Furthermore, if either \(\lambda, \nu\) are not \(W_{a,p}\)-conjugate or if they are conjugate but \(y \not\leq w\), then these \(\text{Ext}_G^1\)-groups vanish.

(b) If \(\lambda = w \cdot \lambda^-, \nu = y \cdot \lambda^-,\) with \(\lambda^- \in C_p^-\) and \(w, y \in W_{a,p}\), then
\[
\dim \text{Ext}_G^1(\Delta^\text{red}(\lambda), \nabla^\text{red}(\nu)) = \mu(x, y).
\]

\[\text{These modules were first introduced by Lusztig [24], and then studied by Lin [23].}\]
5. Proof of the Main \( \operatorname{Ext}^1 \)-Result

In this section we first prove the following result.

**Theorem 5.1.** There is, for any finite root system \( \Phi \), a constant \( c = c(\Phi) \) with the following property. If \( G \) is a simple, simply connected algebraic group with root system \( \Phi \) over an algebraically closed field \( k \) of arbitrary characteristic \( p > 0 \), then

\[
\dim \operatorname{Ext}_G^1(L, L') \leq c
\]

for any two irreducible, rational \( G \)-modules \( L, L' \).

It is elementary to reduce to the case in which \( G \) is simple, and thus that \( \Phi \) is irreducible. To begin with, if \( \Phi \) has type \( C_2 \) and \( p = 2 \), then we can quote [32, Prop. 2.3] which says that \( \dim \operatorname{Ext}_G^1(L(\lambda), L(\nu)) \leq 1 \) for all \( \lambda, \nu \in X^+ \). Thus, we assume that if \( p = 2 \), then \( G \) is not of type \( C_2 \). Then using Proposition 3.2 repeatedly (if necessary), we need only find a common bound (depending only on \( \Phi \)) for the spaces \( \operatorname{Ext}_G^1(L(\lambda), L(\nu)) \) for \( \lambda, \nu \in X^+ \) with \( \lambda_0 \neq \nu_0 \).

First, we find a bound for \( \Phi \) and the prime \( p \) fixed. Since \( \lambda_0 \neq \nu_0 \), the Hochschild-Serre exact sequence (see 3.2.1) implies that

\[
\begin{align*}
\operatorname{Ext}_G^1(L(\lambda), L(\nu)) &\cong \operatorname{Ext}_{G_1}^1(L(\lambda), L(\nu))^G/G_1.
\end{align*}
\]

The following result is proved in [12, Thm. 7.7], though the proof there contains some errors.

**Lemma 5.2.** Let \( \lambda, \nu \in X^+ \) satisfy \( \lambda \neq \nu_0 \). Then

\[
\dim \operatorname{Ext}_G^1(L(\lambda), L(\nu)) \leq p^{\|\Phi\|}(2(p-1)\rho).
\]

**Proof.** We can assume that \( \lambda < \nu \). Because \( \lambda_0 \neq \nu_0 \), a simple Hochschild-Serre spectral sequence argument shows that

\[
\operatorname{Ext}_G^1(L(\lambda), L(\nu)) \cong \operatorname{Ext}_{G_1}^1(L(\lambda), L(\nu))^G/G_1.
\]

Let \( \text{St} = L((p-1)\rho) \) be the Steinberg module. It is self-dual as a rational \( G \)-module so there exists a surjection \( \text{St} \otimes \text{St} \rightarrow L(0) = k \) of \( G \)-modules, and, therefore, tensoring with \( L(\lambda) \) and setting \( S := \text{St} \otimes \text{St} \otimes L(\lambda) \), we obtain an exact sequence

\[
0 \rightarrow N \rightarrow S \rightarrow L(\lambda) \rightarrow 0
\]

in \( G \)-mod.

If \( M \in G_1 \)-mod, let \( r_1(M) = \operatorname{rad}_{G_1}(M) \) denote the \( G_1 \)-radical of \( M \). If \( M \) is a \( G \)-module, then so is \( r_1(M) \). In particular, the inclusion \( N \hookrightarrow S \) implies that \( r_1(N) \subseteq r_1(S) \). Since \( L(\nu)|_{G_1} \) is completely reducible, there are natural maps

\[
\xymatrix{
\operatorname{Hom}_{G_1}(S, L(\nu)) \ar[r] & \operatorname{Hom}_{G_1}(N/r_1(N), L(\nu)) \ar[r] & \operatorname{Hom}_{G_1}((r_1(S) \cap N)/r_1(N), L(\nu)) \ar[r] & 0.
}
\]

[12] Sin [32] gives a precise determination of \( \dim \operatorname{Ext}_G^1(L(\lambda), L(\nu)) \) for arbitrary \( \lambda, \nu \); see also [36].
Since \((r_1(S) \cap N)/r_1(N) \subseteq N/r_1(N)\) (and the latter module is completely reducible as a \(G_1\)-module), \(\beta\) is surjective. Any \(G_1\)-map \(S \rightarrow L(\nu)\) vanishes on \(r_1(S)\), so \(\beta \circ \alpha = 0\). Finally, the cokernel \((N/(r_1(S) \cap N)) \cong (N + r_1(S))/r_1(S)\) of the inclusion \((r_1(S) \cap N)/r_1(N) \hookrightarrow N/r_1(N)\) is a \(G_1\)-direct summand of \(S/r_1(S)\). Thus, any \(G_1\)-map \(N/r_1(N) \rightarrow L(\nu)\) vanishing on \((r_1(S) \cap N)/r_1(N)\) lifts to a \(G_1\)-map \(S/r_1(S) \rightarrow N\), so \((5.2.2)\) is exact. Since \(St|_{G_1}\) is projective, \(S|_{G_1}\) is projective, so \(\text{Hom}_{G_1}(r(S) \cap N, L(\nu)) \cong \text{Ext}^1_{G_1}(L(\lambda), L(\nu))\) by \((5.2.1)\) and \((5.2.2)\). Taking \(G\)-fixed points, \(\text{Hom}_G(r(S) \cap N, L(\nu)) \cong \text{Ext}^1_G(L(\lambda), L(\nu))\). Thus, \(\dim \text{Ext}^1_G(L(\lambda), L(\nu)) \leq \dim S_\nu\), where \(S_\nu\) is the \(\nu\)-weight space in \(St \otimes St \otimes L(\lambda)\). Now repeat the argument in \([12]\) Lem. 7.6: if \(\tau\) is a weight in \(St \otimes St\), then \(\lambda < \nu\) implies that \(\dim L(\lambda)_{\nu-\tau} \leq \Psi(\lambda - (\nu - \tau)) \leq \Psi(\tau) \leq \Psi(2(p-1)\rho)\). Since \(\dim St = p|\Phi + 1|\), we get finally that \(\dim S_\mu \leq p|\Phi|\Psi(2(p-1)\rho)\). \(\square\)

At this point, when \(\Phi\) and the prime \(p\) are fixed, there exists an upper bound for all the dimensions \(\dim \text{Ext}^1_G(L(\lambda), L(\nu))\), \(\lambda, \nu \in X^+\). So, to get a uniform bound, not depending on \(p\), it is enough to treat uniformly all sufficiently large \(p\).

Thus, we assume \(p > h\) and that the LCF holds for \(G\) holds on an ideal of \(p\)-regular weights containing the \(p\)-regular \(p\)-restricted weights. We show the desired bound is \(F(\Phi) := |W/E(\Phi)|/2\), using the notation of \((1.0.2)\) and \((4.1)\).

Lemmas \(4.1\) and \(4.4\) imply that \(\dim \text{Ext}^1_G(\Delta(\lambda), \nabla_{\text{red}}(\nu)) \leq E(\Phi)\), when \(\lambda, \nu \in X^+_\text{reg}\).

If \(\lambda, \nu \not\in X^+_\text{reg}\), we can assume \(\lambda = w \cdot \lambda^-\) and \(\nu = y \cdot \lambda^-\), for some \(\lambda^- \in U^-\). Let \(T^a_b\) be the translation operator from the category of \(G\)-modules with irreducible \(W_{a,p}\)-conjugate to \(\lambda^-\) to the category of \(G\)-modules with highest weight linked to \(-2\rho\). Let \(T^a_b\) be its (left or right) adjoint. Then \(L(\nu_0) = T^a_b L(\tau_0)\), for some restricted \(p\)-regular weight \(\tau_0\). A standard argument (see \([27]\)) shows that \(\nabla_{\text{red}}(\nu) = T^a_b \nabla_{\text{red}}(\tau)\), where \(\tau = \tau_0 + p\nu^\tau\). Thus, \(\text{Ext}^1_G(\Delta(\lambda), \nabla_{\text{red}}(\nu)) \cong \text{Ext}^1_G(\Delta(\lambda), T^a_b \nabla_{\text{red}}(\tau)) \cong \text{Ext}^1_G(T^a_b \Delta(\lambda), \nabla_{\text{red}}(\tau))\). But \(T^a_b \Delta(\lambda)\) has a \(\Delta\)-filtration with sections of the form \(\Delta(v \cdot \lambda')\), \(v \in (W_{a,p})_x\) and \(\lambda \in X^+_{\text{reg}}\), where \(x\) belongs to the facet containing \(\lambda^-\). Each \(\Delta(v \cdot \lambda')\) occurs with multiplicity at most 1, and some \(p\)-regular \(\lambda'\). So dim \(\text{Ext}^1_G(\Delta(v \cdot \lambda'), \nabla_{\text{red}}(\tau)) \leq E(\Phi)\), by the above. Also, half of these sections satisfy dim \(\text{Ext}^1_G(\Delta(v \cdot \lambda'), \nabla_{\text{red}}(\tau)) = \text{Ext}^1_G(U_{\text{reg}}(\Delta(x \cdot \lambda'), L_{\xi}(\tau)) = 0\), for those \(v \in (W_{a,p})_x\) which have the same parity as \(v\). (The group \((W_{a,p})_x\) is generated by reflections\(^{13}\).

Thus, dim \(\text{Ext}^1_G(T^a_b \Delta(\lambda), \nabla_{\text{red}}(\nu)) \leq (|W_a,p)/2)(E(\Phi) \leq E(\Phi)\).

\(\textbf{3}\)Since \(\nu_0 \in X^+_1\), \(0 < (\nu_0 + \rho, \alpha^\vee) \leq p\), for \(\alpha \in \Pi\). Thus, if \(\nu_0\) is in the upper closure of an alcove containing \(\sigma \in X^+\), then \(0 < (\sigma + p, \alpha^\vee) < p\), for all \(\alpha \in \Pi\). This means that \(\sigma = \tau_0 \in X^+_1\).

\(\textbf{4}\)Here we are identifying regular weights with elements of \(W_{a,p}^+\), and “parity” refers to the parity of the corresponding Coxeter group elements. Recall that \(\mu(x, y) = 0\) unless \(x, y\) have opposite parity.
We can assume that $\lambda, \nu \in X^+$ have distinct $p$-restricted parts (i. e., $\lambda_0 \neq \nu_0$). Consider the following diagram

$$\begin{array}{c}
\text{Ext}^1_G(L(\lambda), L(\nu)) \\
\alpha \downarrow \\
\text{Ext}^1_G(\Delta(\lambda), L(\nu)) \longrightarrow \text{Ext}^1_G(\Delta(\lambda), \nabla_{\text{red}}(\nu))
\end{array}$$

where $\alpha$ is induced by the surjection $\Delta(\lambda) \to L(\lambda)$ and $\beta$ is induced by the injection $L(\nu) \hookrightarrow \nabla_{\text{red}}(\nu)$. By the long exact sequence of $\text{Ext}^*$, the kernel of $\beta$ is an image of $\text{Hom}_G(\Delta(\lambda), \nabla_{\text{red}}(\nu)/L(\nu))$. However, $\nabla_{\text{red}}(\nu) \cong L(\nu_0) \otimes \nabla(\nu^\dagger)^{(1)}$, so all the composition factors of $\nabla_{\text{red}}(\nu)$ have the form $L(\nu_0) \otimes L(\tau)^{(1)} \cong L(\nu_0) \otimes p\tau$ for some $\tau \in X^+$. Since $\lambda_0 \neq \nu_0$, there are no nonzero homomorphisms $\Delta(\lambda) \to \nabla_{\text{red}}(\nu)/L(\nu)$, so $\beta$ is an injection. Similarly, the kernel of $\alpha$ is an image of $\text{Hom}_G(\text{rad} \Delta(\lambda), L(\nu))$, which is also 0 since $\lambda < \nu$. Thus, $\alpha$ is an injection. Hence, $\dim \text{Ext}^1_G(L(\nu), L(\lambda)) = \dim \text{Ext}^1_G(L(\lambda), L(\nu)) \leq \dim \text{Ext}^1_G(\Delta(\lambda), \nabla_{\text{red}}(\nu)) \leq F(\Phi)$. This completes the proof of Theorem 5.1.

Just as in [12, Thm. 7.10], the following result for finite groups holds.

**Corollary 5.3.** There is a constant $c' = c'(\Phi)$ with the following property. Let $\sigma : G \to G$ be an endomorphism such that the group $G_\sigma$ of $\sigma$-fixed points is a finite group. Then $\dim \text{Ext}^1_{G_\sigma}(L, L') < c'$ for all irreducible $G_\sigma$-modules $L, L'$ over $k$.

Finally, we state the following further $\text{Ext}^1$-results. The proof will be given in §7.

**Theorem 5.4.** There exists a constant $\tilde{C}(\Phi)$ depending only on $\Phi$ such that if $G$ is any simple, simply connected algebraic group, then, for any $\lambda \in X^+$,

$$\sum_{\nu : \nu_0 \neq \lambda_0} \dim \text{Ext}^1_G(L(\lambda), L(\nu)) < \tilde{C}(\Phi).$$

**Corollary 5.5.** For every dominant weight $\lambda$, there are at most $\tilde{C}(\Phi)$ dominant weights $\mu$ with $\mu_0 \neq \lambda_0$ and $\text{Ext}^1_G(L(\lambda), L(\mu)) \neq 0$.

There is no bound when $\mu_0 = \lambda_0$. However, one can always reduce to the case of $\mu_0 \neq \lambda_0$ as in the proof of Theorem 5.1. See Remark 7.5(c) for a discussion of the $\lambda = 0$ case.

### 6. Further Kazhdan-Lusztig theory

Throughout this section, $\Phi$ is an irreducible root system as in §2. A proof of the following very elementary result is left to the reader.

**Lemma 6.1.** If $\lambda, \nu, \tau \in X^+$, the multiplicity $[L_C(\lambda) \otimes L_C(\nu) : L_C(\tau)]$ is at most $\dim L_C(\tau)$. Also, the inequality $\text{length}(L_C(\lambda) \otimes L_C(\nu)) \leq \min [\dim L_C(\lambda), \dim L_C(\nu)]$ on lengths holds.
This result fails in positive characteristic, e. g., let $G = SL_2(k)$ for characteristic $p > 0$. Identify $X^+$ with $\mathbb{N}$. For $r \geq 0$, put $V(r) = L(1) \otimes \text{St}(r)$, where $\text{St}(r) := \bigotimes_{i=0}^{r} L(p-1)^{(i)}$ is irreducible. Since $L(1)$ and $L(p-1)$ are isomorphic to the Weyl modules $\Delta(1)$ and $\Delta(p-1)$, respectively, $L(1) \otimes L(p-1)$ has a Weyl filtration with sections $\Delta(p)$ and $\Delta(p-2) \cong L(p-2)$ (Clebsch-Gordan). Thus, $V(r)$ has $V(r-1)^{(1)}$ and the irreducible module $S(r) := L(p-2) \otimes L(p-1)^{(1)} \otimes \cdots \otimes L(p-1)^{(r)}$ as subquotient modules. Continuing, we see $V(r)$ has $S(1)^{(r-1)}, S(2)^{(r-2)}, \ldots, S(r)$ among its irreducible composition factors. Now let $r \to \infty$.

If $x, y \in W_a$, and $m \in \mathbb{N}$, let

$$c_{x,y}^{[m]} = \text{coefficient of } t^m \text{ in the Kazhdan-Lusztig polynomial } P_{x,y} \text{ for } W_a.$$  

Thus, $c_{x,y}^{[m]} = 0$ unless $x \leq y$ in the partial ordering on $W_a$. If $x < y$, then $P_{x,y} \in \mathbb{N}[t]$ is a polynomial in $t$ of (even) degree $\leq \ell(y) - \ell(x) - 1$, $c_{x,y}^{[m]} = 0$ unless $0 \leq m \leq \ell(y) - \ell(x) - 1$.

The following result is a weak version (and corollary) of Theorem 6.9. However, its proof here is quite different and is potentially useful.

**Theorem 6.2.** Let $m \in \mathbb{N}$. There exists an integer $d(\Phi, m)$ such that if $x, y \in W_a^+$ with $x \leq y$, then

$$c_{x,y}^{[\ell(y) - \ell(x) - m]} \leq d(\Phi, m).$$

**Proof.** Pick an integer $l > h$, which is odd and not divisible by 3 if $\Phi$ has type $G_2$, and let $U_\zeta$ be as in §2(2). Using the isomorphism $\epsilon : W_a \to W_{a,t}$, it suffices to prove the result for $W_{a,t}$—that is, we can assume that $c_{x,y}^{[\ell(y) - \ell(x) - m]}$ is a coefficient in the Kazhdan-Lusztig polynomial $P_{x,y}$ for $W_{a,t}$. Let $C_1, \cdots, C_s$ be the $l$-restricted alcoves in $\mathbb{E}$. For $1 \leq i \leq s$, let $\lambda_i \in C_i$ be the unique dominant weight $W_{a,T}$-linked to $-2\rho \in C_T$. Let $M = \bigoplus_{i=1}^{s} L(\lambda_i)$, and consider the $U(\mathfrak{g}_C) = U_{\zeta}/u_\zeta$-module $M' := \text{Ext}_{U_\zeta}^m(M, M)^{(-1)}$. For dominant weights $\lambda = \lambda_0 + l\lambda^t$ and $\nu = \nu_0 + l\nu^t$ ($\lambda_0, \nu_0 \in X^+_l$, $\lambda^t, \mu^t \in X^+$),

$$\dim \text{Ext}_{U_\zeta}^m(L(\lambda), L(\nu)) \leq \dim \text{Hom}_{U_{\zeta}}(L(\lambda^t) \otimes L(\nu^t), M') \leq \dim M'$$

by Lemma 6.1, putting $\nu^{t*} := -w_0\nu^t$. Now let $\nu = x \cdot (-2\rho)$ and $\lambda = y \cdot (-2\rho)$. Then

$$c_{x,y}^{[\ell(x) - \ell(y) - m]} = \dim \text{Ext}_{U_\zeta}^m(\Delta(\nu), L(\lambda)) \leq \dim \text{Ext}_{U_\zeta}^m(L(\nu), L(\lambda)) \leq \dim M'$$

by Remark 6.3. So $d(\Phi, m) := \dim M'$ works.

We now work with $U_\zeta$ where $\zeta = \sqrt{T}$ as per §2(2). For $\lambda \in X^+$, let $Q_\zeta(\lambda)$ be the projective cover of $L(\lambda)$ in $U_\zeta$-mod.

**Lemma 6.3.** There is a constant $C(\Phi)$, such that, given any $\lambda \in X^+$,

$$\text{length}(Q_\zeta(\lambda)) \leq C(\Phi)$$

for any quantum enveloping algebra $U_\zeta$ of type $\Phi$ for $l$ odd, not divisible by 3 in case $\Phi$ has type $G_2$, and otherwise arbitrary.
Proof. We first show that there is a constant bounding the length of any $Q_\zeta(\lambda_0)$ for $\lambda_0 \in X_{\text{reg},l}^+ \cap X_{1,l}^+$ for all $l$. For fixed $l$, $|X_{1,l}^+| < \infty$, so it suffices to give is a bound that works universally for all $l \geq h$. It is known that $Q_\zeta(\lambda_0)$ has highest weight $2(l-1)\rho + w_0\lambda_0$. For $\nu \in X_{\text{reg},l}^+$, the multiplicity of $\Delta_\zeta(\nu)$ as a section in a $\Delta_\zeta$-filtration of $Q_\zeta(\lambda_0)$ equals, by Brauer-Humphreys reciprocity, the multiplicity $[\nabla_\zeta(\nu) : L_\zeta(\lambda_0)] = [\Delta_\zeta(\nu) : L_\zeta(\lambda_0)]$.

If this multiplicity $\neq 0$, necessarily $\nu \in X_{\text{reg},l}^+$ and $\nu \leq 2(l-1)\rho + w_0\lambda_0 \leq 2(l-1)\rho$. Thus, the number of possible $\nu$ is absolutely bounded by some integer independent of $l$. But $[\Delta_\zeta(\nu) : L_\zeta(\lambda_0)]$ is expressed in terms of the coefficients of inverse Kazhdan-Lusztig polynomials $Q_{x,y}$, $x, y \in W_a$ satisfying $x \cdot \lambda^- = \lambda_0$ and $y \cdot \lambda^- = \nu$, $\lambda^- \in C_l^-$. (For a discussion of the $Q_{x,y}$, see, e. g., [14, §7.3].) Since, independently of $l$, there are only a finite number of possible $x, y \in W_a$, these multiplicities are also bounded independently of $l$.

Suppose that $\lambda = \lambda_0 + l\lambda^t$ as after 240.2. Then $Q_\zeta(\lambda) = Q_\zeta(\lambda_0) \otimes L_\zeta(\lambda^t)^{(1)}$. If $\nu = \nu_0 + l\nu_1 \in X_{\text{reg},l}^+$ is so that $L_\zeta(\nu)$ is a composition factor of $Q_\zeta(\lambda_0)$, then $\nu_0 + l\nu_1 \leq 2(l-1)\rho + w_0\lambda_0 \leq (l-1)\rho$. Thus, $\nu_1 \leq \rho$, so that there is a bound on the possible $\dim L_\zeta(\nu_1)$. By Lemma 6.1, this integer bounds the number of composition factors $L(\tau)$ of $L_\zeta(\nu_1) \otimes L_\zeta(\lambda^t)$. For such a $\tau$, $L_\zeta(\nu_0) \otimes L_\zeta(\tau)^{(1)}$ is an irreducible $U_\zeta$-module. Hence, there is an absolute bound on the number of composition factors of any $Q_\zeta(\lambda)$.

Thus, the result holds for $Q_\zeta(\lambda)$ with $\lambda \in X_{\text{reg},l}^+$. However, if $\lambda \notin X_{\text{reg},l}^+$, then $Q_\zeta(\lambda)$ is a direct summand of the translate of some $Q_\zeta(\lambda^\#)$, with $\lambda^\# \in X_{\text{reg},l}^+$. Since translation operators from $l$-regular weights to $l$-singular weights preserve irreducible modules (or map them to zero), the length of $Q_\zeta(\lambda)$ is bounded by the length of $Q_\zeta(\lambda^\#)$, and the result is completely proved.

The following is an immediate consequence.

**Corollary 6.4.** For any $\lambda \in X^+$, length($\Delta_\zeta(\lambda)$) $\leq C(\Phi)$ for the standard modules $\Delta_\zeta(\lambda)$ for any quantum group $U_\zeta$ of type $\Phi$.

We next have the following application. The “sum formulation” here and in Theorem 5.4 is inspired by a somewhat analogous use of sums in [21].

**Theorem 6.5.** For a fixed $n$, there is a constant $C'(\Phi, n)$ such that, for all $\lambda \in X^+$,

$$(6.5.1) \quad \sum_{\nu} \dim \text{Ext}^n_{U_\zeta}(L_\zeta(\lambda), L_\zeta(\nu)) \leq C'(\Phi, n)$$

for any quantum group $U_\zeta$ of type $\Phi$ ($l$ arbitrary).

**Proof.** If $P_\ast \to L_\zeta(\lambda)$ is a minimal projective resolution, $\dim \text{Ext}^n_{U_\zeta}(L_\zeta(\lambda), L_\zeta(\nu)) = \dim \text{Hom}_{U_\zeta}(P_n, L_\zeta(\nu))$ equals the number of times $Q_\zeta(\nu)$ appears as a direct summand of $P_n$. The number of indecomposable summands of $P_0$ is $Q_\zeta(\lambda)$ is 1. For $P_1$ the number of indecomposable summands is (strictly) bounded by length($P_0$) $\leq C(\Phi)$. Then the number of indecomposable summands of $P_2$ is bounded by $C(\Phi)^2$, . . . , and, finally, the number of indecomposable summands of $P_n$ is bounded by $C(\Phi)^n$. Thus, $C'(\Phi, n) = C(\Phi)^n$ works. \qed
Remark 6.6. We briefly indicate further results which can be found in [28] and depend upon [26]. For regular dominant weights \( \lambda = x \cdot \lambda^- \), \( \nu = y \cdot \lambda^- \), write \( \mu(\lambda, \nu) := \mu(x, y) \).

By Theorem 6.5, there are only finitely many \( n \)-tuples \((\lambda_1, \cdots, \lambda_{n-1}, \nu)\) of dominant weights for which the non-negative integers \( \mu(\lambda, \lambda_1), \cdots, \mu(\lambda_{n-1}, \nu) \) are all nonzero. Also, the dimensions of the \( \text{Ext}^l_{U_\zeta}(L_\zeta(\lambda_i), L_\zeta(\lambda_{i+1})) \) (with \( \lambda_0 = \lambda \) and \( \lambda_n = \nu \) and \( 0 \leq i \leq n - 1 \)) are all uniformly bounded by an integer independent of the weights and \( l \). Thus, the right-hand side of

\[
\sum_\nu \dim \text{Ext}^n_{U_\zeta}(L_\zeta(\lambda), L_\zeta(\nu)) \leq \sum_{(\lambda_1, \cdots, \lambda_{n-1}, \nu)} \mu(\lambda, \lambda_1) \mu(\lambda_1, \lambda_2) \cdots \mu(\lambda_{n-1}, \nu) < \infty.
\]

This discussion suggests the question of determining

\[
R_\Phi := \text{Max}_{x \in W_\Phi^+} \left( \sum_{y \in W_\Phi^+} \mu(x, y) \right).
\]

Theorem 6.5 implies this maximum is finite depending on \( \Phi \), but the argument does not give a good bound (which remains an open problem). Theorem 6.5 gives an exponential bound \( \sum_\nu \dim \text{Ext}^n_{U_\zeta}(L_\zeta(\lambda), L_\zeta(\nu)) \leq R_\Phi \). However, dropping the sum over \( \nu \), [28] gives a polynomial growth bound \( \dim \text{Ext}^n_{U_\zeta}(L_\zeta(\lambda), L_\zeta(\nu)) \leq D(\Phi)n^{O(1)} \), with \( D(\Phi) \) a constant depending on \( \Phi \), but not on \( \lambda \) or \( \nu \).

Determination of \( \sum_{y \in W_\Phi^+} \mu(w_0, y) \) (or a good bound for it) is an open problem, related to bounding 1-cohomology (and the Guralnick conjecture). It is currently open whether \( \mu(w_0, y) \) is bounded over all \( \Phi \), with 3 the largest known value; see [29]. By [31], \( \mu(x, y) \to \infty \) with larger (type \( A \)) root systems. In particular, the constant \( D(\Phi) \) must depend on \( \Phi \) and tend to infinity as \( \Phi \) gets large. Conceivably, in the spirit of Guralnick’s conjecture, one might replace \( D(\Phi) \) by a universal constant if \( \lambda = 0 \) and \( \mu \) is allowed to be arbitrary.

Corollary 6.7. For a fixed \( n \), there is a constant \( C(\Phi, n) \) such that, for all \( \lambda, \nu \in X^+ \),

\[
\dim \text{Ext}^n_{U_\zeta}(L_\zeta(\lambda), L_\zeta(\nu)) \leq C(n, \Phi)
\]

for any quantum group \( U_\zeta \) of type \( \Phi \) (\( l \) arbitrary).

Corollary 6.8. There is a constant \( C''(\Phi, n) \) for any \( n \) such that, for any \( \lambda \in X^+ \),

\[
\sum_\nu \dim \text{Ext}^n_{U_\zeta}(L_\zeta(\lambda), \nabla_\zeta(\nu)) \leq C''(\Phi, n)
\]

for any quantum group \( U_\zeta \) of type \( \Phi \) (\( l \) arbitrary).

Proof. By adjoint associativity of translation functors, it suffices to consider only \( l \)-regular weights. By Remark 4.3 \( \dim \text{Ext}^n_{U_\zeta}(L_\zeta(\lambda), \nabla_\zeta(\nu)) \leq \dim \text{Ext}^n_{U_\zeta}(L_\zeta(\lambda), L_\zeta(\nu)) \). Now apply Theorem 6.5.

Using this corollary and Remark 4.3 again, we get

\[
\sum_\nu \dim \text{Ext}^n_{U_\zeta}(L_\zeta(\lambda), \nabla_\zeta(\nu)) \leq C''(\Phi, n)
\]
Theorem 6.9. Let $m \in \mathbb{N}$. Let $C^m(\Phi, m)$ be as in Corollary 6.8. If $\nu \in W_0^+$, then
\[
\sum_{x \leq y, x \in W_0^+} C^{[\ell(y)-\ell(x)-m]}_{x,y} \leq C^m(\Phi, m).
\]

7. Higher Ext$^n$ for algebraic groups

We first prove a higher degree version of Theorem 5.1. For $\nu \in X^+$, let $e_p(\nu)$ denote the exponent of the largest power of $p$ appearing in the $p$-adic expansion $\nu$. Equivalently, $e_p(\nu)$ is the smallest nonnegative integer $e$ such that $\nu \in X_{e+1, p}$, i.e., if $\nu = \sum a_i \omega_i$, then each $a_i < p^{e+1}$.

Theorem 7.1. Let $m, e$ be nonnegative integers. There exists a constant $c(\Phi, m, e)$ with the following property. If $G$ is a semisimple, simply connected algebraic group with root system $\Phi$ over an algebraically closed field $k$ of characteristic $p$, then, for $\lambda, \nu \in X^+$ with $e_p(\lambda) \leq e$,
\[
\dim \text{Ext}^m_G(L(\lambda), L(\nu)) = \dim \text{Ext}^m_G(L(\nu), L(\lambda)) \leq c(\Phi, m, e).
\]
In particular, $\dim \text{H}^m(G, L(\nu)) \leq c(\Phi, m, 0)$, $\forall \nu \in X^+$.

We can assume to start that $G$ is simple, i.e., it is as in §2(1). The proof requires two lemmas. Given $e \geq 1$ and $\tau \in X^+_{s, p}$, let $Q_e(\tau)$ be the projective cover of the irreducible $G_e$-module $L(\tau)|_{G_e}$. It is known that $Q_e(\tau)$ is the injective hull of $L(\tau)|_{G_e}$. When $s = 1$, so that $\tau \in X^+_{1, p}$, it will be sometimes convenient to denote $Q_1(\tau)$ by $Q_1(\tau)$. When $p \geq 2h - 2$, each $Q_e(\lambda)$, $e \geq 1, \tau \in X^+_{s, p}$, has a compatible structure as a rational $G$-module $[22, \S 11.11]$. In that case, writing $\tau = \tau_0 + p \tau_1 + \cdots + p^{e-1} \tau_{e-1}$ as per (2.0.2), $Q_e(\tau) \cong Q_1(\tau_0) \otimes Q_1(\tau_1)^{\otimes 1} \otimes \cdots \otimes Q_1(\tau_{e-1})^{\otimes (e-1)}$ as rational $G$-modules. In addition, $Q_e(\tau)$ has highest weight $2(p^e - 1)p + w_0(\tau)$. (This later statement is true for all $p$, if $Q_e(\nu)$ is regarded as a $G_eT$-module.)

Lemma 7.2. Let $f$ be a positive integer. There exists a constant $C^p(\Phi, f)$ satisfying the following condition. Let $G$ be a simple, simply connected algebraic group, having root system $\Phi$, over an algebraically closed field $k$ of characteristic $p \geq 2h - 2$. If $\nu \in X^+_{f, p}$, then $\text{length}(Q_f(\nu)) \leq C^p(\Phi, f)$. In addition, if $L(\omega)$ is a composition factor of $Q_f(\nu)$, then $e_p(\omega) \in X^+_{f+1, p}$.

Proof. We prove the last assertion first. Let $L(\omega)$ be a composition factor of $Q_f(\nu)$, so that $\omega \leq 2(p^f - 1)p + w_0\nu$. For any simple root $\alpha$, $(\omega, \alpha^\vee) \leq (\omega, \alpha^\vee_0) \leq (2(p^f - 1)p + w_0\nu) = 2(p^f - 1)(h - 1) \leq (p^f - 1)p < p^{f+1}$. Therefore, $\omega \in X^+_{f+1, p}$.

For a given prime $p$, there are only finitely many $\nu \in X^+$ satisfying $e_p(\nu) < f$, and hence only finitely many modules $Q_f(\nu)$, which collectively have a bounded length. Therefore, we need to find a uniform bound for the $Q_f(\nu)$ under the assumptions that $p \geq 3h - 3$ and that the LCF holds in the Jantzen region $\Gamma_{\text{Jan}}$. These assumptions will remain in effect for the remainder of the proof. (In that case, we will give an explicit formula for $C^p(\Phi, f)$ in terms of the bound $C(\Phi)$ of Lemma 6.3.) Let $\zeta = \sqrt[p]{1}$. For $\tau \in$
$X_{1,p}^+$, $Q_1(\tau)$ is obtained by “reduction mod $p$” from the $U_\zeta$-projective indecomposable module $Q_\zeta(\nu)$\footnote{Here $Q_1(\tau)$ is the projective $G$-module in the full subcategory $C'$ of $G$-mod with objects having composition factors $L(\nu)$ with $\nu \leq 2(p-1)\rho + w_0$ $\tau$. Thus, it is the reduction mod $p$ of some $U_\zeta$-module $Q^*_\zeta(\lambda_0)$ by \cite{17} 33, itself projective in an analogous category. However, it is easy to argue, from the validity of the LCF, together with the assumption that $p \geq 3h - 3$ so that $\dim Q_1(\tau) = \dim Q_\zeta(\tau)$ for each $\tau \in X_{1,p}^+$. Also, $Q_\zeta(\lambda_0)$ is also projective in the quantum version $C_\zeta$ of $C$. We conclude $Q_\zeta(\lambda_0) = Q^*_\zeta(\lambda_0)$.}. By Lemma 6.3, $Q_\zeta(\tau)$ has length at most $C(\Phi)$. The validity of the LCF implies that each composition factor of $Q_\zeta(\tau)$ reduces mod $p$ to an irreducible $G$-module. Therefore, $\text{length}(Q_\zeta(\tau)) = \text{length}(Q_1(\tau))$.

We claim that, given a non-negative integer $e$, there is a positive integer $C^0(\Phi, e)$ such that if $e_p(\nu) \leq e$, then $\text{length}(\Delta(\nu)) \leq C^0(\Phi, e)$. If $e = 0$, then $e_p(\nu) \leq e$ means that $\nu \in X_{1,p}$. Then $\text{length}(\Delta(\nu)) \leq \text{length}(Q_1(\nu)) \leq C(\Phi)$ by the above paragraph. We prove the claim by induction on $\Delta(\nu)$. We prove the claim by induction on $\nu$. Assume that $e_p(\nu) = e$. Since $\Delta(\nu)$ can be realized by reduction mod $p$ from $\Delta_\zeta(\nu)$, $\Delta(\nu)$ has a filtration with at most $\text{length}(\Delta(\nu)) \leq C(\Phi)$ sections each having character $\chi_{KL}(\tau) = \text{ch}(L(\tau_0) \otimes \Delta(\tau^\dag))(1)$ for $\tau, \tau^\dag \in X^+, \tau_0 \in X_{1,p}^+$. Again, $\text{ch}(\Delta(\tau^\dag))$ is a sum of at most $C(\Phi)$ characters $\chi_{KL}(\sigma) = \text{ch}(L(\sigma_0) \otimes \Delta(\sigma^\dag))(1)$, for $\sigma, \sigma^\dag \in X^+$ and $\sigma_0 \in X_{1,p}^+$. It follows that $\tau_0 + p\sigma_0 + p^2\sigma^\dag$ is a highest weight of a composition factor of $\Delta(\nu)$ and of $Q_{e+1}(\nu)$, and so has $e_p$-value at most $e+1$. Hence $e_p(\sigma^\dag) \leq e-1$. By induction, $\text{length}(\Delta(\sigma^\dag)) \leq C^0(\Phi, e-1)$. Applying the Steinberg tensor product theorem, it follows that $\text{length}(\Delta(\nu)) \leq C(\Phi)^2C^0(\Phi, e-1)$. The claim follows with $C^0(\Phi, e) = C(\Phi)^{2e+1}$ for all $e \geq 0$.

We now prove the lemma by (a new) induction on $f \geq 1$. If $f = 1$, $C^0(\Phi, f) = C(\Phi)$ works as already remarked. So fix $f > 1$ and write $Q_f(\nu) = Q_1(\nu_0) \otimes Q_{f-1}(\nu^\dag)(1)$. By induction, $Q_{f-1}(\nu^\dag)$ has length bounded by $C^0(\Phi, f-1)$, and hence has a $\Delta$-filtration with at most $C^0(\Phi, f-1)$-terms $\Delta(\tau)$ with $e_p(\tau) \leq f-1$. But $Q_1(\nu_0)$ has length at most $C(\Phi)$ with composition factors $L(\sigma_0) \otimes L(\sigma^\dag)(1) \cong L(\sigma_0) \otimes \Delta(\sigma^\dag)(1)$, $\sigma_0 \in X_{1,p}^+, \sigma^\dag \in X^+$. For $\alpha \in P$, $(\sigma^\dag, \alpha^\vee) \leq (\sigma^\dag, \alpha^\vee) \leq 2h - 2$, so that, independently of $p$, the possible $\sigma^\dag$ have the form $\sigma^\dag = \sum_i a_i \omega_i$, with each $a_i \leq 2h - 2$. Therefore, there is an integer $M$ (given by the Weyl dimension formula) bounding all possible $\dim \Delta(\sigma^\dag)$ (and independent of $p$). By Lemma 6.1 (and the first paragraph of this proof), for $\tau$ as above, $\Delta(\sigma^\dag) \otimes \Delta(\tau)$ has a $\Delta$-filtration with at most $\dim \Delta(\sigma^\dag)$-sections $\Delta(\tau')$ with $e_p(\tau') \leq f$. Thus, using the previous paragraph, $\text{length}(Q_f(\nu)) \leq C(\Phi)C^0(\Phi, f-1)MC^0(\Phi, f)$. In other words, $\text{length}(Q_f(\nu)) \leq C(\Phi)^{f+1}(f-1)^{f-1}M^{f-1}$. \hfill \qed

Lemma 7.3. Let $n$ be a non-negative integer. There exists an integer $f = f(\Phi, n)$ depending only on $\Phi$ and $n$ with the following property. If $G$ is a simple simply connected algebraic group over a field $k$ of positive characteristic $p$ and if $V$ is any finite dimensional rational $G$-module, then $H^n(G, V^{(s)}(\zeta)) \cong H^n(G, V^{(s')}(\zeta))$ for integers $s, s' \geq f(\Phi, n)$.

Proof. Let $\alpha_{\text{max}} = \sum n_i \alpha_i$ be the maximal root in $\Phi^+$ and let $c = \max\{n_1, \ldots, n_{\text{rk}(G)}\}$ be the maximal coefficient. Let $t(\Phi)$ be the torsion exponent of $X/Q$. For an integer $m$,
let $e(m) := \lceil \frac{m-1}{p-1} \rceil$, where $\lceil \cdot \rceil$ is largest integer function. Set $f(\Phi, n) := e(ct(\Phi)n) + 1$. Then [13, Thm. 6.6, Cor. 6.8] shows that, for $s, s' \geq f(\Phi, n)$, the cohomology spaces $H^q(G, V(s))$ and $H^q(G, V(s'))$ are isomorphic. $\square$

We now prove Theorem 7.1 by induction on $m$. If $m = 0$, take $c(\Phi, e, 0) = 1$ for all $e$. So suppose $m > 0$ and the theorem holds for smaller $m$. Our proof is modeled on the $p \geq 2h - 2$ case, so we consider that case first.

We first bound $\dim \operatorname{Ext}_G^m(L(0), L(\nu))$ for all primes $p \geq 2h - 2$ and all $\nu \in X^+$. If $\nu = 0$, then $\operatorname{Ext}_G^m(L(0), L(\nu)) = 0$, so assume that $\nu \neq 0$. By Lemma 7.3, we can assume that the first nonzero term in the $p$-adic expansion of $\nu$ occurs with $\nu_r \neq 0$, for $r \leq f(\Phi, m)$. Form the short exact sequence $0 \to R_{r+1}(0) \to Q_{r+1}(0) \to L(0) \to 0$ in $G$-module. Then $\operatorname{Ext}_G^*(Q_{r+1}(0), L(\nu)) = 0$, so that $\operatorname{Ext}_G^m(L(0), L(\nu)) \cong \operatorname{Ext}_G^{m-1}(R_{r+1}(0), L(\nu))$. Thus, by induction, plus Lemma 7.2

$$\dim \operatorname{Ext}_G^m(L(0), L(\nu)) = \dim \operatorname{Ext}_G^{m-1}(R_{r+1}(0), L(\nu))$$

$$\leq C^p(\Phi, f(\Phi, m)) - 1)c(\Phi, m - 1, f(\Phi, m) + 1).$$

Continuing with $p \geq 2h - 2$, consider $L(\lambda)$ with $e_\lambda(\lambda) \leq e$. Form the short exact sequence $0 \to R_{e+1}(\lambda) \to Q_{e+1}(\lambda) \to L(\lambda) \to 0$. Applying the Hochschild-Serre spectral sequence, we find that $\operatorname{Ext}_G^\lambda(Q_{e+1}(\lambda), L(\nu)) = 0$ unless $\operatorname{Hom}_G(Q_{e+1}(\lambda), L(\nu)) \neq 0$. In this later case, $\lambda = \nu_0 + p\nu_1 + \cdots + p^r\nu_e$ and $\operatorname{Ext}_G^m(Q_{e+1}(\lambda), L(\nu)) \cong \operatorname{Ext}_G^m(L(0), L(\nu'))$, where $\nu' = (\nu - \lambda)/p^{e+1}$. So

$$\dim \operatorname{Ext}_G^m(L(\lambda), L(\nu')) \leq (C^p(\Phi, e) - 1)c(\Phi, m - 1, e + 1)$$

$$+ C^p(\Phi, f(\Phi, m) - 1)c(\Phi, m - 1, f(\Phi, m) + 1),$$

competing the proof of Theorem 7.1 for $p \geq 2h - 2$.

Finally, it suffices now to give (by induction on $m$) a bound $c(\Phi, m, e)$ for any individual prime $p$.

We begin, as in the $p \geq 2h - 2$ case treated above, by bounding $\dim \operatorname{Ext}_G^m(L(0), L(\nu))$ for all $\nu \in X^+$. As before, we can assume $\nu \neq 0$ and $e_\nu(\nu) = f(\Phi, m)$. Set $r = f(\Phi, m)$ and let $Q(r+1, 0)$ be the $G$-module guaranteed in Corollary 8.5 below. Then $\operatorname{Ext}_{G^{r+1}}^*(Q(r+1, 0), L(\nu)) = 0$, and so $\operatorname{Ext}_G^*(Q(r+1, 0), L(\nu)) = 0$. That is, $Q(r+1, 0)$ behaves in this respect like $Q_{r+1}(0)$ in the $p \geq 2h - 2$ case. The number of composition factors $L(\omega)$, $\omega \in X^+$, and the values $e_\omega(\omega)$ are all bounded as a function of $r$ for our given $p$, provided we choose a single $Q(r+1, 0)$ in Corollary 8.5 for each $r$ and $p$. We now obtain a bound on $\dim \operatorname{Ext}_G^m(L(0), L(\nu))$ as in the $p \geq 2h - 2$ case, using property (2) of Corollary 8.5 namely, the fact that $L(0)$ is a $G$-quotient of $Q(r+1, 0)$. The number $C^p(\Phi, f(\Phi, m))$ is simply replaced by the length of $Q(r+1, 0)$, and $c(\Phi, m - 1, f(\Phi, m) + 1)$ is replaced by $c(\Phi, m - 1, s)$, where $s$ is the maximum value of all $e_\nu(\omega)$ with $L(\omega)$ a composition factor of $Q(r+1, 0)$.

For $0 \neq \nu \in X_{r+1,p}$, the argument is not as close to the $p \geq 2h - 2$ case, but still uses the modules from Corollary 8.5. Put $e' = (e + 1) + \lceil \log_p(2h - 2) \rceil$ with $\lceil \cdot \rceil$ the greatest
integer function. Any weight \( \gamma \in X^+ \) of \( L(-w_0 \lambda) \otimes L(\lambda) \) satisfies, for all \( \alpha \in \Pi \),
\[
(\gamma, \alpha^\vee) \leq (\gamma, \alpha_0^\vee) \leq (2(p^{e+1} - 1)\rho, \alpha_0^\vee) = (p^{e+1} - 1)(2h - 2) < p^{e+1}.
\]
So \( e_p(\gamma) \leq e' \). If \( \text{Hom}_{G_{e+1}}(L(\lambda), L(\nu)) = 0 \), then \( \text{Ext}_{G_{e+1}}^\bullet(Q(e' + 1, \lambda), L(\nu)) = 0 \) and \( \text{Ext}_{G}^\bullet(Q(e' + 1, \lambda), L(\nu)) = 0 \). In this case, \( \dim \text{Ext}_{G}^\bullet(L(\lambda), L(\nu)) \) is bounded as in the \( \lambda = 0 \) case, noting there are only finitely many \( \lambda \in X^+ \) with \( e_p(\lambda) \leq e \) for any fixed \( e \) and \( p \). Hence, we may assume that \( \text{Hom}_{G_{e+1}}(L(\lambda), L(\nu)) \neq 0 \). Thus, \( \nu = \lambda + p^{e+1} \nu' \), for some \( \nu' \in X^+ \), and \( L(\nu) = L(\lambda) \otimes L(\nu')^{(e+1)} \). We have
\[
\text{Ext}_{G}^m(L(\lambda), L(\nu)) = \text{Ext}_{G}^m(L(0), L(-w_0 \lambda) \otimes L(\lambda) \otimes L(\nu')^{(e+1)}).
\]

As noted above, all composition factors \( L(\gamma) \) of \( L(-w_0 \lambda) \otimes L(\lambda) \) satisfy \( e_p(\gamma) \leq e' \), so that \( L(\gamma) \otimes L(\nu')^{(e+1)} \) is irreducible. Thus, the number of composition factors of \( L(-w_0 \lambda) \otimes L(\lambda) \otimes L(\nu')^{(e+1)} \) is just the number of composition factors of \( L(-w_0 \lambda) \otimes L(\lambda) \) in this \( (\nu = \lambda + p^{e+1} \nu') \) case. Since there are only finitely many \( \lambda \) with \( e_p(\lambda) \leq e \) for a given \( e \) and \( p \), we obtain a bound from the \( \lambda = 0 \) case already treated the previous paragraph. This completes the proof of Theorem 7.1.

Remarks 7.4. (a) For some readers, the cohomology case \( \lambda = 0 \) in Theorem 7.1 may be the most interesting. However, our proof for that case requires also treatment of the nonzero \( \lambda \in X^+ \).

(b) For \( m = 1 \), the restriction \( e_p(\lambda) \leq e \) may be removed, and a bound independent of \( e \) given; see Theorem 5.4. We do not know if this can be done for \( m > 1 \).

Next, we prove Theorem 5.4. First, assume that \( p \geq 3h - 3 \) and that the LCF holds for all \( \lambda \in X^+_1 \cap X^+_{reg,p} \). For \( \lambda_0 \in X^+_1 \), the proof of Lemma 7.2 shows that \( Q_1(\lambda_0) \) has a composition series with at most \( C(\Phi) \)-terms \( L(\nu_0 + p\nu^\dagger) = L(\nu_0) \otimes \Delta(\nu^\dagger)^{(1)} \), where \( \nu_0 \in X^+_1 \) and \( \nu^\dagger \in X^+ \) satisfies \( (\nu^\dagger, \alpha^\vee) \leq 2h - 2 \) for all \( \alpha \in \Pi \). Furthermore, there is an integer \( M \) such that \( \dim \Delta(\nu^\dagger) \leq M \) for all such \( \nu^\dagger \) and all primes \( p \). Therefore, \( (\text{rad}_G Q_1(\lambda_0)) \otimes \Delta(\lambda^\dagger)^{(1)} \) has a filtration with \( C(\Phi) - 1 \) sections \( L(\nu_0) \otimes (\Delta(\nu^\dagger)^{(1)} \otimes \Delta(\lambda^\dagger))^{(1)} \). On the other hand, each \( \Delta(\nu^\dagger) \otimes \Delta(\lambda^\dagger) \) itself has a \( \Delta \)-filtration with at most \( M \) terms. It follows that \( (\text{rad}_G Q_1(\lambda_0)) \otimes \Delta(\lambda^\dagger)^{(1)} \) has a \( p \)-filtration with at most \( M(C(\Phi) - 1) \) sections of the form \( \Delta^p(\xi) \).

For any \( \tau \in X^+ \), \( \Delta^p(\tau) \) has head \( L(\tau) \). Thus,
\[
\sum_{\nu, \nu_0 \neq \lambda_0} \dim \text{Hom}_G ( (\text{rad}_G Q_1(\lambda_0)) \otimes \Delta(\lambda^\dagger)^{(1)}, L(\nu)) \leq \dim \text{head}( (\text{rad}_G Q_1(\lambda_0)) \otimes \Delta(\lambda^\dagger)^{(1)}) \leq M(C(\Phi) - 1).
\]

However, if \( \nu_0 \neq \lambda_0 \), a Hochschild-Serre spectral sequence argument shows that \( \text{Ext}_{G}^0(Q_1(\lambda_0) \otimes \Delta(\lambda^\dagger)^{(1)}, L(\nu)) = 0 \). This vanishing also holds if \( L(\nu) \) is replaced by \( L(I) := \bigoplus_{\nu \in I} L(\nu) \), where \( I \) is any finite set of dominant weights \( \nu \) with \( \nu_0 \neq \lambda_0 \).

(We could even take \( I \) to be infinite.) Observe that \( (\text{rad}_G Q_1(\lambda_0)) \otimes \Delta(\lambda^\dagger)^{(1)} \subseteq \text{rad}_G(Q_1(\lambda_0) \otimes \Delta(\lambda^\dagger)^{(1)}) \), while \( Q_1(\lambda_0) \otimes \Delta(\lambda^\dagger)^{(1)}/(\text{rad}_G Q_1(\lambda_0)) \otimes \Delta(\lambda^\dagger)^{(1)} \cong L(\lambda_0) \otimes \Delta(\lambda^\dagger)^{(1)} \).
\(\Delta(\lambda^i)^{(1)}\) is, as a \(G_1\)-module, a direct sum of copies of \(L(\lambda_0)|_{G_1}\). The same then holds for \(\text{rad}_G(Q_1(\lambda_0) \otimes \Delta(\lambda^i)^{(1)})/ (\text{rad}_G Q_1(\lambda_0) \otimes \Delta(\lambda^i)^{(1)})\), so that there are no non-trivial \(G\)-homomorphisms of this quotient module to \(L(I)\). Therefore, there is a containment

\[
\text{Hom}_G \left( \text{rad}_G(Q_1(\lambda_0) \otimes \Delta(\lambda^i)^{(1)}), L(I) \right) \hookrightarrow \text{Hom}_G \left( (\text{rad}_G Q_1(\lambda_0)) \otimes \Delta(\lambda^i)^{(1)}, L(I) \right),
\]

so that

\[
\sum_{\nu \in I} \dim \text{Ext}^1_G(L(\lambda), L(\nu)) = \dim \text{Ext}^1_G(L(\lambda), L(I)) 
\leq \dim \text{Hom}_G \left( \text{rad}_G(Q_1(\lambda_0) \otimes \Delta(\lambda^i)^{(1)}), L(I) \right) 
\leq \dim \text{Hom}_G \left( (\text{rad}_G Q_1(\lambda_0)) \otimes \Delta(\lambda^i)^{(1)}, L(I) \right) 
\leq M(C(\Phi) - 1).
\]

(7.4.1)

Since \(I\) may include any finite set of weights \(\nu\) with \(\nu \neq \nu_0\), the theorem follows in this large \(p\) case, using \((C(\Phi) - 1)M\) for \(\tilde{C}(\Phi)\).

It remains to treat the finitely many primes \(p\) for which the assumptions above do not hold, i. e., either \(p < 3h - 3\) or \(p \geq 3h - 3\) or the LCF does not hold. In place of \(Q_1(\lambda_0)\), use the \(Q(1, \lambda_0) \in G\)-mod defined in Corollary 5.4. The \(G\)-modules \(Q(1, \lambda_0)\) have a filtration with sections \(L(\nu_0) \otimes L(\nu^i)^{(1)}\), the latter a homomorphic image of \(L(\nu_0) \otimes \Delta(\nu^i)^{(1)}\) for \(\nu_0 \in X_1^{\pm}\). Now consider \(Q(1, \lambda_0) \otimes \Delta(\lambda^i)^{(1)}\). Observe \(\Delta(\nu^i) \otimes \Delta(\lambda^i)^{(1)}\) has a \(\Delta\)-filtration with sections of the form \(\Delta(\xi^i)\) with the total number of sections bounded by the maximum \(M\) of the \(\dim \Delta(\nu^i)\). Thus, the tensor product \(Q(1, \lambda_0) \otimes \Delta(\lambda^i)^{(1)}\) has a filtration with at most \(M\) sections, all homomorphic images of modules \(\Delta^p(\xi)\). Now we can argue as above, using the fact that \(Q(1, \lambda_0)\) has \(L(\lambda_0)\) as a \(G\)-homomorphism image. (The role of \((\text{rad}_G Q_1(\lambda_0)) \otimes \Delta(\lambda^i)^{(1)}\) is played by \((\text{rad}_G Q_1(\lambda_0)) \otimes \Delta(\lambda^i)^{(1)}\), while the role of \(\text{rad}_G(Q_1(\lambda_0) \otimes \Delta(\lambda^i)^{(1)}\) is played by the kernel of the map \(Q(1, \lambda_0) \otimes \Delta(\lambda^i)^{(1)} \to L(\lambda)\).) This completes the proof of Theorem 5.4.

Remarks 7.5. (a) Theorem 5.4 fails if the \(\nu_0 \neq \lambda_0\) condition is dropped. Just take \(\lambda = 0\) and \(\nu\) any nonzero weight with \(H^1(G, L(\nu)) \neq 0\). By 10, \(\text{Ext}^1_G(L(0), L(\nu)) \cong H^1(G, L(\nu)) \hookrightarrow H^1(G, L(p\nu)) \hookrightarrow H^1(G, L(p^2\nu)) \hookrightarrow \cdots\).

That is, \(\text{Ext}^1_G(L(\lambda), L(p^m\nu)) \neq 0\) for all \(m \geq 0\).

(b) A similar theorem holds, with essentially the same proof, if “\(L(\nu)\)” is replaced in the statement by “\(\nabla(\nu)\)” or “\(\nabla_{\text{red}}(\nu)\)”.

(c) According to 13 Thm. 7.1, the vector space \(H^1(G, L(\mu))\) is isomorphic to \(H^1(G, L(\mu'))\) if \(\mu = p\mu'\), except possibly when \(G\) has type \(C\), \(p = 2\), and \(\mu' \neq 0\) mod 2. Hence, we may always replace \(\mu\) with a weight \(\lambda\) (of the form \(p^{-r}\mu\)) with \(H^1(G, L(\mu))\) isomorphic to \(H^1(G, L(\lambda))\) and \(\lambda \neq 0\) mod \(p^2\). At this point, with \(\lambda \neq 0\) mod \(p^2\), we claim there are only finitely many \(\lambda\) with \(H^1(G, L(\lambda)) \neq 0\). If \(G\) is not of type \(C\), with \(p = 2\), then \(\lambda \neq 0\) mod \(p\) and Theorem 5.4 applies, or else \(\lambda = p\lambda'\), and Theorem 5.4 applies to \(H^1(G, L(\lambda')) \cong H^1(G, L(\lambda))\). In the exceptional case, where \(G\) has type
Lemma 5.2, replace \( \tilde{\lambda} \) finitely many \( 2 \), \( H_\lambda \) twisted groups by Avrunin [6]. In the split case, for any finite dimensional rational generic cohomology was first defined in the split case in [13] and then extended to the twisted groups by Avrunin [6]. In the split case, for any finite dimensional rational \( G \)-module \( V \) and positive integer \( m \), the generic cohomology of \( V \) in homological degree \( m \) is defined to be the common limit
\[
(7.5.1) \quad H^m_\text{gen}(G, V) := \lim_{d \to \infty} H^m(G(p^d), V) = \lim_{s \to \infty} H^m(G, V^s).
\]

The generic cohomology in the non-split case is defined similarly; see [6] and [12, §7]. The first limit is realized for \( d \) sufficiently large (ostensibly depending on \( V \)), and the second limit is realized for \( s \) sufficiently large. A sufficiently large \( s \) depending only on \( m \) and \( \Phi \) is described in Lemma 7.3. For 1-cohomology (\( m = 1 \)), this dependence on \( V \) can be avoided by appealing to Bendel, Nakano, and Pillen [7], and an upper bound on \( H^1(G(p^d), L(\lambda)) \) can be obtained as in [12]. However, for \( m > 1 \), such results are unavailable. In particular, though \( H^m(G(p^d), L(\lambda)) \) is eventually bounded for large \( d \) by a constant depending only on the root system \( \Phi \), we do not know if there exists such a universal bound (still depending on \( \Phi \)) when \( d \) and \( L(\lambda) \) are allowed to vary.

8. Appendix

Throughout this section, let \( G \) be as in §2(1). Thus, the characteristic of the algebraically closed field \( k \) is denoted by \( p \).

The following lemma generalizes slightly [15, Lem. 4.2] which treats the case \( m = 1 \). We prove it in general by reducing to that case. Note that \( m = 0 \) is allowed. For \( r \in \mathbb{N} \), write \( St_r = L((p^r - 1)\rho) \in G\text{-mod} \) and \( St = St_1 \).

**Lemma 8.1.** Let \( M_1, M_2 \in G\text{-mod} \) be finite dimensional and let \( m \geq 0 \) be an integer. Then, for all \( r \) sufficiently large,
\[
(8.1.1) \quad \text{Hom}_G(M_1 \otimes St_r^m, M_2 \otimes St_r^m) \cong \text{Hom}_{G_{r+m}}(M_1 \otimes St_r^m, M_2 \otimes St_r^m)
\]
via the natural restriction map.

**Proof.** Replace \( M_2 \) by \( M_{r}^* \otimes M_2 \) to assume that \( M_1 = k \). Since \( St_r^m \) is self-dual, the left-hand side of (8.1.1) can then be rewritten as \( \text{Hom}_G(St_r^m \otimes St_r^m, M_2) \), with a similar rearrangement on the right-hand side. On both sides, we may replace \( M_2 \) with its submodule \( M_{2}^{G_m} \) of \( G_m \)-fixed points. As a \( G \)-module, \( M_{2}^{G_m} = M_{m}^{(m)} \). The lemma thus reduces to showing, for \( r \gg 0 \), that \( \text{Hom}_G(St_r^m \otimes St_r^m, M^{(m)}) \cong \text{Hom}_{G_{r+m}}(St_r^m \otimes St_r^m, M^{(m)}) \), viz., \( \text{Hom}_G(St_r \otimes St_r, M) \cong \text{Hom}_{G_r}(St_r \otimes St_r, M) \). Also, this is equivalent to the \( m = 1 \)
case, \( \text{Hom}_G(\text{St}^{(1)} \otimes \text{St}^{(1)}, M^{(1)}) \cong \text{Hom}_{G_{r+1}}(\text{St}^{(1)} \otimes \text{St}^{(1)}, M^{(1)}) \), which is shown in [15, Lemma 4.2], to be an isomorphism for large \( r \) (with \( M^{(1)} \) replaced by any finite dimensional \( G \)-module).

**Theorem 8.2.** Let \( e \geq 1 \) be an integer. There exists an integer \( N = N(\Phi, e) \) with the following property. For \( \lambda \in X_{e,p}^+ \), if \( n \geq N \), the \( G_{n+e} \)-mod injective hull \( Q_{n+e}(p^n(p^n - 1)\rho + \lambda) \) of \( L(\lambda) \otimes \text{St}^{(e)}_n \) has a compatible rational \( G \)-module structure. Moreover, for sufficiently large \( N \), one such \( G \)-structure is that of the indecomposable tilting module \( T((p^n+p^e - 1)\rho + (p^e - 1)\rho + w_0\lambda) \) with highest weight \( (p^n+p^e - 1)\rho + (p^e - 1)\rho + w_0\lambda \).

**Proof.** We give the proof for the case \( e = 1 \), leaving the modifications for the general case to the reader. If \( p \geq 2h - 2 \), we can take \( N = 0 \). For the remaining primes it suffices to provide an \( N \) which works for any fixed prime \( p \) and fixed \( \lambda \in X_{1,p}^+ \).

Define \( \lambda' := (p-1)\rho + w_0\lambda \). For any finite dimensional vector space \( Y \) over \( k \), let \( \nu_n(Y) := [\dim Y / \dim \text{St}_n] \), where \( [\cdot] \) is the largest integer function. Since \( \nu_n(Y) \in \mathbb{N} \), it has a minimal value over all choices of \( n \) and \( G \)-quotients \( Y \) of \( \text{St} \otimes L(\lambda') \otimes \text{St}^{(1)}_n \) for which the composite

\[
(8.2.1) \quad L(\lambda) \otimes \text{St}^{(1)}_n \hookrightarrow \text{St} \otimes L(\lambda') \otimes \text{St}^{(1)}_n \cong \text{St}_{n+1} \otimes L(\lambda') \twoheadrightarrow Y
\]

is injective. The left-hand injection is induced by the inclusion \( L(\lambda) \hookrightarrow \text{St} \otimes L(\lambda') \) in \( G \)-mod. Fix a pair \((n, Y)\) which achieves this minimum value. Since \( \text{St}_{n+r} \cong \text{St}_n \otimes \text{St}^{(r)}_r \), it follows (after applying \(- \otimes \text{St}^{(n+1)}_r\) to (8.2.1)) that, for any integer \( r \geq 0 \), \((n + r, Y \otimes \text{St}^{(n+1)}_r)\) also achieves the minimum value.

Let \( \overline{Y} \) be the quotient of \( Y \) by any \( G \)-submodule not containing the image of \( L(\lambda) \otimes \text{St}^{(1)}_n \). Also, \( \nu_p(\overline{Y}) \leq \nu_p(Y) \), and \( \overline{Y} \) fits into a diagram like (8.2.1), so \( \nu_p(\overline{Y}) = \nu_p(Y) \). There is such a quotient \( \overline{Y} \) of minimum dimension, and we henceforth replace \( Y \) with \( \overline{Y} \). As a result, the \( G_{n+1} \)-socle of \( Y \) is now homogeneous. In fact, \( \text{Soc}_{G_{n+1}} Y \cong L(\lambda) \otimes \text{St}^{(1)}_n \otimes M^{(n+1)} \), where \( M \in G \)-mod. Explicitly, \( M \cong \text{Hom}_{G_{n+1}}(L(\lambda) \otimes \text{St}^{(1)}_n, Y)(-n-1) \) as a \( G/G_{n+1} \)-module.

We claim \( M \) is the trivial module \( k = L(0) \). Certainly, \( k \subseteq M \), and \( \dim \text{Hom}_G(k, M) = 1 \), since \( \dim \text{Hom}_G(L(\lambda) \otimes \text{St}^{(1)}_n, \text{St}_{n+1} \otimes L(\lambda')) = 1 \). Similarly (replacing \( n \) by \( n+r \)), we have \( \dim \text{Hom}_G(\text{St}_r, M \otimes \text{St}_r) = 1 \) for all \( r \geq 0 \). (Otherwise, \( Y \otimes \text{St}^{(n+1)}_r \) contains the \( G \)-submodule \( L(\lambda) \otimes (\text{St}_r \otimes M)^{(n+1)} \) which contains at least two copies of \( L(\lambda) \otimes \text{St}^{(1)}_{n+r} \), in its socle. One of these can then be factored out to give a pair \((n+r, Y')\) with with \( \nu_p(Y') < \nu_p(Y) \). This would contradict the minimality of the pair \((n, Y)\).)

For \( r \) sufficiently large, we have also \( 1 = \dim \text{Hom}_G(\text{St}_r, M \otimes \text{St}_r) \) by Lemma 8.1. Of course, \( M \otimes \text{St}_r \cong \text{St}_r \otimes ((M/k) \otimes \text{St}_r) \) in \( G_r \)-mod. If \( M/k \neq 0 \), let \( E \) be an irreducible \( G_r \)-submodule. We can assume that \( r \) is large enough that all \( G_r \)-composition factors of \( M \) have highest weights which are \( p^r \)-restricted, and so all \( G_r \)-composition factors of \( M \), such as \( E \), belong to \( G \)-mod. The \( G_r \)-module map \( E \otimes \text{St}_r \rightarrow M \otimes \text{St}_r \) is \( G_r \)-split, hence (by Lemma 8.1), the map

\[
E \otimes \text{St}_r \otimes \text{St}^{(r)}_s \rightarrow M \otimes \text{St}_r \otimes \text{St}^{(r)}_s
\]
is a $G$-map, and is $G$-split for all $s \gg 0$ (depending on $r \gg 0$). However, setting $q = \dim St$, so that $q^n = \dim St_n$, we have

$$\nu_{n+r+s}(\frac{Y \otimes St_r^{(n+1)} \otimes St_t^{(n+r+1)}}{L(\lambda) \otimes St_n^{(1)} \otimes E(n+1) \otimes St_t^{(n+1)} \otimes St_s^{(n+1+r)}})$$

$$= [(\dim Y - q^n \dim L(\lambda) \otimes E(n+1) q^{r+s})/q^{n+r+s}]$$

$$= [\dim Y/q^n - \dim L(\lambda) \otimes E(n+1)]$$

$$< [\dim Y/q^n] = \nu_n(Y).$$

This contradicts the minimality of $\nu_n(Y)$. So $M/k = 0$, proving the claim.

Thus, $\text{Soc}_{G_{n+1}} Y \cong L(\lambda) \otimes St_n^{(1)}$. Since $St \otimes L(\lambda') \otimes St_n^{(1)} \cong St_{n+1} \otimes L(\lambda')$ is $G_{n+1}$-injective,

$$L(\lambda) \otimes St_n^{(1)} \subseteq Q \subseteq St \otimes L(\lambda') \otimes St_n^{(1)}$$

where $Q := Q_{n+1}(\lambda + p(p^n - 1)\rho)$ is the $G_{n+1}$-injective hull of $L(\lambda) \otimes St_n^{(1)}$. The $G_{n+1}$-submodule $Q$ must map injectively to the $G$-quotient $Y$ of $St \otimes L(\lambda') \otimes St_n^{(1)}$. Since $Q$ is $G_{n+1}$-injective, there is a $G_{n+1}$-isomorphism $Y \cong Q \otimes X$, for some $X \in G_{n+1}$-mod. However, $\text{Soc}_{G_{n+1}} Y \cong L(\lambda) \otimes St_n^{(1)} \cong \text{Soc}_{G_{n+1}} Q$, so $\text{Soc}_{G_{n+1}} X = 0$. Thus, $X = 0$, and $Q = Y$ has a $G$-structure.

While this achieves one $G$-structure on the $G_{n+1}$-injective hull of $L(\lambda) \otimes St_n^{(1)}$, we may have to take $n$ larger to get the last assertion, regarding a tilting module $G$-structure. Temporarily, put $\lambda'' = (p^{n+1} - 1)\rho + (p-1)\rho + w_0\lambda$, and let $T(\lambda'')$ be the indecomposable $G$-tilting module having highest weight $\lambda''$. By [22 II.8], $T(\lambda'')|_{G_{n+1}}$ is injective. Therefore, $T(\lambda'')$ has a filtration as a $G_{n+1}$-module with sections “baby Verma modules” $\hat{Z}_{n+1}(\mu)$ for $G_{n+1}T$ (in the notation of [22 II.9]). Since $T(\lambda'')$ has a unique maximal weight, namely, $\tau_0 := (p^{n+1} - 1)\rho + (p-1)\rho + w_0\lambda$, it can be assumed (using [22 II,9.8]) that bottom section of the filtration is $\hat{Z}_{n+1}(\tau_0)$. On the other hand, $\hat{Z}_{n+1}(\tau_0)$ has $G_{n+1}$-socle $L(2(p^{n+1} - 1)\rho - \tau_0)\ast_{G_{n+1}} [22 II.9.6]$. But $L(2(p^{n+1} - 1)\rho - \tau_0)\ast \cong L(\lambda) \otimes St_n^{(1)}$ as $G_{n+1}$-modules. Thus, $(L(\lambda) \otimes St_n^{(1)})|_{G_{n+1}}$ is contained in the $G_{n+1}$-socle of $T(\lambda'')$, so there is a $G_{n+1}$-split injection $Q \hookrightarrow T(\lambda'')$. Tensoring with $St_u^{(n+1)}$ for $u \gg 0$ and applying Lemma 8.1 we obtain a $G$-split $G$-module injection

$$Q \otimes St_u^{(n+1)} \hookrightarrow T(\lambda'') \otimes St_u^{(n+1)}.$$  

By [22 II.9], the right-hand $G$-module is a tilting module, so the left-hand side is one also. The theorem now follows, after replacing $N$ by $N + u$ and $n$ by $n + u$.

\[\Box\]

Remark 8.3. Let $\lambda \in X^r_{\ast,p}$ and suppose that $n$ is large enough so that $Q_{n+e}(\lambda + p^e(p^n - 1)\rho) \cong T((p^{n+e} - 1)\rho + (p^e - 1)\rho + w_0\lambda)|_{G_{n+e}}$. Using the main result of [15] (generalized from $G_1$ to $G_r$, $r \geq 1$, using Lemma 8.1), any two compatible $G$-structures on $Q_{n+e}(\lambda + p^e(p^n - 1)\rho)$ become isomorphic in $G$-mod, after tensoring with $St_t^{(n+e)}$ for $r \gg 0$. Also, $Q_{n+e}(\lambda + p^e(p^n - 1)\rho) \otimes St_t^{(n+e)} \cong Q_{n+e+r}(\lambda + p^r(p^{n+r} - 1)\rho)$ (using [15].
Lemma §2, the fact that $Q_{n+e}(\lambda + p^e(p^n - 1)\rho)$ is a $G$-module and hence a $G_{n+r+s}$-module, and an Hochschild-Serre spectral sequence argument), it follows that any $G$-module structure on $Q_{n+e}(\lambda + p^e(p^n - 1)\rho)$ becomes isomorphic, after tensoring with $\text{St}_r^{(n+e)}$, to the tilting module $T((p^{n+r+e} - 1)\rho + (p^e - 1)\rho + w_0\lambda)$. 

A noted conjecture of Donkin [16, (2.2)] states that, for any characteristic $p$ and positive integer $e$, if $\lambda \in X_{e,p}^+$, then $Q_e(\lambda) \cong T_{|G_e}(2(p^e - 1)\rho + w_0\lambda)|_{G_e}$. The conjecture is true if $p \geq 2h - 2$ (and in some small rank examples). One interesting feature of our theorem above is that the restriction to $G_e$ of the $G_{n+e}$-injective hull $Q_{n+e}(\lambda + p^e(p^n - 1)\rho)$ is a direct sum of copies of the $G_e$-injective hull $Q_e(\lambda)$ of $L(\lambda)$. In this way, we have obtained a “stable” version of Donkin’s conjecture. We record this in the following corollary.

**Corollary 8.4.** Let $e \geq 1$ be an integer and let $\lambda \in X_{e,p}^+$. Then there is a positive integer $M$ such that $Q_e(\lambda)^{\oplus M} \cong T|_{G_e}$ for some $G$-module $T$. Moreover, $T$ can be chosen to be the indecomposable tilting module $T((p^{n+e} - 1)\rho + (p^e - 1)\rho + w_0\lambda)$ and $M = \dim \text{St}_n$ for any sufficiently large $n$.

**Proof.** We can take $T = T((p^{n+e} - 1)\rho + (p^e - 1)\rho + w_0\lambda)$ as in the statement of the theorem. Then $T_{|G_{n+e}}$ identifies with the injective hull in $G_{n+e}$-mod of the irreducible module $L(\lambda) \otimes \text{St}_n^{(e)}$. Therefore, $\text{Soc}_{G_{n+e}} T = L(\lambda) \otimes \text{St}_n^{(e)}$. Let $Z \in G_n$-mod be so that $Z^{(e)} = \text{Hom}_{G_e}(L(\lambda), T)$ in $G_{n+e}$-mod. It follows the $G_n$-socle of $Z$ must be $\text{St}_n$. Since $\text{St}_n$ is an injective $G_n$-module, $\text{St}_n$ divides $Z$, and so $Z = \text{St}_n$. Therefore, the $G_e$-socle of $T$ is $L(\lambda)^{\oplus M}$, for $M = \dim \text{St}_n$, so that $T_{|G_e} \cong Q_e(\lambda)^{\oplus M}$, as required. \hfill \Box

There is another useful way to choose a $G$-module isomorphic to a direct sum of copies of $Q_e(\lambda)$, as in Corollary 8.4. We state this result as a separate corollary. Observe that $Q(e, \lambda)$ below, when restricted to $G_e$, is necessarily a direct sum of copies of $Q_e(\lambda)$ by properties (1) and (2).

**Corollary 8.5.** Let $e \geq 1$ be an integer, and let $\lambda \in X_{e,p}^+$. Then there is a (rational, finite dimensional) $G$-module $Q(e, \lambda)$ such that:

1. $Q(e, \lambda)|_{G_e}$ is injective and projective.
2. $L(\lambda)$ is both a $G$-submodule and a $G$-quotient module of $Q(e, \lambda)$.
3. All irreducible $G_e$-submodules or irreducible $G_e$-quotient modules of $Q(e, \lambda)$ are isomorphic to $L(\lambda)$.

**Proof.** Just take $Q(e, \lambda) = T \otimes \text{St}_n^{(e)}$ in the proof of Corollary 8.4 \hfill \Box

We believe that many more $G_e$-modules $Q$ have the property that, for some positive integer $M$, $Q^{\oplus M}$ is the restriction to $G_e$ of a rational $G$-module. We hope to provide necessary and sufficient conditions in a later paper.

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