A REMARK ON THE EVOLVING SURFACE FINITE ELEMENT METHOD

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Abstract. We solve an advection-diffusion equation on a moving surface by using finite elements. In contrast to the evolving surface finite element method (ESFEM) as introduced in [3] and [4] we do not consider a moving grid (which could degenerate). Instead we propose the alternative that all numerical schemes and proofs are with respect to a fixed surface. We consider two cases, firstly we assume that the surface evolution is given by an explicit family of diffeomorphisms of \( \mathbb{R}^{n+1} \) and secondly we assume that this family is defined by the flow of some explicitly given vector field.

Key words. finite elements, evolving surface, linear heat equation

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1. Introduction. In many applications it is important to consider PDEs which are defined on surfaces and not in Euclidean space, especially in the case of parabolic equations it is of interest to assume that these surfaces (where the equation is defined) evolve with respect to time in a certain prescribed way.

In [3] the so-called evolving surface finite element method (ESFEM) is proposed and introduced to solve an advection-diffusion equation on an evolving surface. This setting models e.g. the transport of an insoluble surfactant on the interface between two flowing fluids or pattern formation on the surfaces of growing organisms modeled by reaction-diffusion equations, cf. [3, Section 1.4] for further and a more detailed exposition of applications.

The idea of ESFEM is to consider at every time \( t \) for the finite element approximation an interpolating polyhedral surface \( \Gamma_h(t) \) consisting of triangles the vertices \( X_j(t), j = 1, \ldots, N, \) of which lie on the prescribed moving surfaces \( \Gamma(t) \) and also move according to the law of evolution prescribed for \( \Gamma(t) \). (E.g. assume the motion of the surfaces is given by a family of diffeomorphisms \( \Phi(t, \cdot) : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1} \) then the vertices satisfy \( X_j(t_2) = \Phi(t_2, \cdot) \circ \Phi(t_1, \cdot)^{-1}(X_j(t_1)) \)). Then piecewise linear globally continuous basis function \( \varphi_i(\cdot, t) \) with \( \varphi_i(t, X_j(t)) = \delta_{ij} \) are chosen which are defined on \( \Gamma_h(t) \). As written in [3, Section 7.2] one drawback of this method is the possibility of degenerating grids, i.e. the prescribed motion may lead to the effect that the triangulation \( \Gamma_h(t) \) is distorted.

To circumvent this problem in the present paper we analyze the alternative where we fix the grid by reformulating the problem on a stationary surface. Then we solve this problem which leads to a finite element solution \( \tilde{u}_h = \tilde{u}_h(x, t) \) of the problem on the fixed surface. Then a transformation of \( \tilde{u}_h \) by the evolution law of the surfaces, e.g. \( \Phi \) in the above notation, gives an approximate solution of the original problem. We remark that a ’bad’ \( \Phi \) which degenerates the grid in ESFEM, of course, also appears in our equation but not related to the grid generation. If the diffusion coefficient is the identity which is a special case of relevance then even the stiffness matrix is constant in time in our case.

Let us state the problem in our approach more precisely. Let \( \Gamma_0 \) be a smooth, compact, connected and oriented hypersurface in \( \mathbb{R}^{n+1}, n = 1, 2 \). Let \( \Phi(\cdot, t) : \Gamma_0 \rightarrow \mathbb{R}^{n+1} \)
\( \mathbb{R}^{n+1}, t \in [0, T_0], T_0 > 0, \) be a family of diffeomorphisms, \( \Phi \) smooth, \( \Gamma(t) = \Phi(\Gamma_0, t) \) the moving surfaces.

We define the set
\[
G_{T_0} = \bigcup_{t \in [0, T_0]} \Gamma(t) \times \{t\}
\]
and consider there the advection-diffusion equation
\[
\dot{u} + u \nabla \Gamma(t) \cdot v - \nabla \Gamma(t) \cdot (D_0 \nabla \Gamma(t) u) = 0
\]
where \( \dot{u} \) denotes the material derivative, \( v(p_t, t) = \frac{\partial}{\partial t}(\Phi(\cdot, t)^{-1}(p_t), t) \) the velocity of the moving surface in a point \((p_t, t) \in G_{T_0}\) and hence \( \nabla \Gamma(t) \cdot v \) its tangential divergence which we assume to be nonnegative in order to have uniqueness of a solution. We remark that it is an interesting case when this expression vanishes. Furthermore, the diffusion coefficient \( D_0 : G_{T_0} \to \text{Mat}(n + 1, \mathbb{R}) \) is so that it vanishes on the normal space of the moving surfaces, i.e. \( D_0(p_t, t) \nu = 0 \) for all \( \nu \in N_{p_t} \) where \( N_{p_t} \) is the normal space of \( \Gamma(t) \) in \( p_t \).

The problem under consideration is a finite element approximation of the unique \( u : G_{T_0} \to \mathbb{R} \) solving equation (1.2) and satisfying \( u = 0 \) on \( \partial \Gamma(t) \) if not empty and \( u(\cdot, 0) = u_0 \).

In [3] a semi-discrete finite element approximation of the solution of this problem is introduced (the above mentioned ESFEM) and convergence is proved, later in [4] the corresponding \( L^2 \)-error estimate is proved. In both papers a moving mesh is used with vertices lying on \( \Gamma(t) \) as explained above and some rather technical auxiliary estimates as the error estimate for the material derivative of the Ritz projection are proved which are not needed in our case.

We remark that the semi-discrete ESFEM from [3] and [4] is fully discretized (in the case of the identity as diffusion coefficient) in [6] and [7].

2. Hypersurfaces in \( \mathbb{R}^{n+1} \). We recall some facts and notations of embedded hypersurfaces in \( \mathbb{R}^{n+1} \) from [5]. We consider smooth embedded hypersurfaces \( M \subset \mathbb{R}^{n+1} \). Let \( F : \Omega \to \mathbb{R}^{n+1} \) with \( \Omega \subset \mathbb{R}^n \) open, be an embedding. For \( p \in \Omega \) the coordinate tangent vectors \( \partial_i F(p) = \frac{\partial F}{\partial p_i}(p), 1 \leq i \leq n \), provide a basis of the tangent space \( T_x M \) at \( x = F(p) \).

The metric on \( M \) is given by
\[
g_{ij} = \partial_i F \cdot \partial_j F
\]
for \( 1 \leq i, j \leq n \), the inverse metric by \( (g^{ij}) = (g_{ij})^{-1} \).

The tangential gradient of a function \( h : M \to \mathbb{R} \) is defined by
\[
\nabla^M h = g^{ij} \partial_j h \partial_i F
\]
where we sum over repeated indices.

For a smooth tangent vectorfield \( X = X^i \partial_i F = g^{ij} X_j \partial_i F \) on \( M \) (note that \( X_i = X \cdot \partial_i F \)) we define the covariant derivative tensor by
\[
\nabla^M_i X^j = \partial_i X^j + \Gamma^j_{ik} X^k
\]
where the Christoffel symbols \( \Gamma^k_{ij} \) are given by
\[
\Gamma^k_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).
\]
The tangential divergence of $X$ on $M$ is defined by
\begin{equation}
\nabla^M \cdot X = \text{div}_M X = \nabla^M_i X^i
\end{equation}
and the Laplace-Beltrami operator of $h$ on $M$ by
\begin{equation}
\Delta_M h = \text{div}_M \nabla^M h = \nabla^M \cdot (\nabla^M h).
\end{equation}

For a smooth vectorfield $X : M \to \mathbb{R}^{n+1}$ which is not necessarily tangent on $M$ we can also define the divergence with respect to $M$ by
\begin{equation}
\nabla^M \cdot X = \text{div}_M X = g^{ij} \partial_i X \cdot \partial_j F
\end{equation}
which reduces to the above expression if $X$ is tangent on $M$. We remark that, of course, the divergence in (2.7) is (and transforms like) a scalar function (when changing local coordinates of $M$). Furthermore, the tangential gradient transforms like a scalar function when changing coordinates in $M$.

3. Reformulation with fixed mesh. We propose an alternative approach in which the problem is reformulated as a linear parabolic equation on a fixed surface. This kind of problem is numerically well-understood and error estimates are available so nothing new has to be proven, especially the above mentioned possibility of degenerating grids is circumvented.

Assume that $u$ solves (1.2). Set
\begin{equation}
\tilde{u}(x, t) = u(\Phi(x, t), t),
\end{equation}
\begin{equation}
c(x, t) = \nabla \Gamma_0 \cdot \frac{\partial \Phi}{\partial t}
\end{equation}
and
\begin{equation}
\tilde{D}_0(x, t) = D_0(\Phi(x, t), t)
\end{equation}
for $(x, t) \in \Gamma_0 \times [0, T_0]$. Then there holds
\begin{equation}
\frac{\partial \tilde{u}}{\partial t} + c\tilde{u} - \nabla \Gamma_0 \cdot (\tilde{D}_0 \nabla \Gamma_0 \tilde{u}) = 0
\end{equation}
in $\Gamma_0 \times [0, T_0]$. This also holds vice versa, e.g. let $\tilde{u}$ satisfy (3.4) then
\begin{equation}
u(p_t, t) = \tilde{u}(\Phi(\cdot, t)^{-1}(p_t), t)
\end{equation}
satisfies (1.2).

In order to obtain an approximate solution $u_h$ of the original equation (1.2) on $\Gamma_{T_0}$ we propose to calculate a finite element solution of (3.4) and then to define
\begin{equation}
\tilde{u}_h(p_t, t) = \tilde{u}_h(\Phi(\cdot, t)^{-1}(p_t), t).
\end{equation}

The known convergence of $\tilde{u}_h$ to $\tilde{u}$ and the corresponding $L^2$- and $H^1$-error estimates immediately carry over (using transformation rule for integrals) to the convergence of $\tilde{u}_h$ to $u$. We denote that $\tilde{u}_h$ is not a linear function but can be calculated
easily at arbitrary points. Let $N_h$ denote the vertices of the mesh used on $\Gamma_0$. Noting for each $p \in N_h$ the points $(\Phi(p, t), t) \in G_{T_0}$ and corresponding values

$$(3.7) \quad \bar{u}_h(\Phi(p, t), t) = \bar{u}_h(p, t)$$

leads to a very simple calculation of $\bar{u}_h$ in the points of $G_{T_0}$ which are the vertices in the discretization used in [3]. Furthermore, in order to have a piecewise linear approximation of $u$ one can consider – of course – a continuous, piecewise linear function which interpolates the calculated values of $\bar{u}_h$ in the points $(\Phi(p, t), t)$. The error estimate carries over to this interpolating function.

We remark that in our approach we do not need to update the stiffness matrix if $D_0$ is constant in time which is an important special case since all the computations in [3, 4, 6, 7] are performed for this special case.

4. Moving surface as flow of a given vector field. In some applications the function $\Phi$ might not be prescribed explicitly but is given by the flow $y = y(x, t)$ of an initial hypersurface $\Gamma_0$ under a given smooth vectorfield $V : \mathbb{R}^{n+1} \times [0, T_0] \to \mathbb{R}^{n+1}$, i.e.

$$(4.1) \quad \frac{\partial y}{\partial t}(x, t) = V(y(x, t), t), \quad y(x, 0) = x$$

and

$$(4.2) \quad \Phi = y$$

describes the movement of our hypersurfaces as in the previous section.

In order to obtain a fully discrete finite element approximation of the equation (1.2) and given $V$ one can proceed as in [6, 7] where moving meshes are considered and the discretization of (4.1) leads to systems of ODEs for the mesh nodes.

We follow here our approach described in the previous section and consider the discretizations of (4.1) and (1.2) separately, e.g. we firstly solve (4.1) by a numerical method with some accuracy $h$ and then insert our approximate movement of $\Gamma$ into our above formulation on a fixed mesh. So we have to look how this discretization error resulting from approximation of (4.1) affects the total error. For simplicity we assume $D_0 \equiv 1$ as in [6, 7] and also the example computations in [3, 4] are in the special case that the diffusion coefficient is a constant multiple of the identity.

Instead of the coefficient $c$ in (3.4) there is only an approximation $c_h$ of $c$ with error $O(h)$ available so that we solve

$$(4.3) \quad \frac{\partial \bar{w}}{\partial t} + c_h \bar{w} - \Delta_{\Gamma_0} \bar{w} = 0$$

instead of (3.4). This yields to an error $e = \bar{u} - \bar{w}$ which satisfies

$$(4.4) \quad \frac{\partial e}{\partial t} + c e - \Delta_{\Gamma_0} e = (c_h - c)u.$$ 

Multiplication by $e$ and integration over $[0, T_0]$ yields

$$(4.5) \quad \|e\|_{L^2([0, T_0), H^1(\Gamma_0))} = O(h).$$
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