Reproducibility in Learning

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Abstract

We introduce the notion of a reproducible algorithm in the context of learning. A reproducible learning algorithm is resilient to variations in its samples — with high probability, it returns the exact same output when run on two samples from the same underlying distribution. We begin by unpacking the definition, clarifying how randomness is instrumental in balancing accuracy and reproducibility. We initiate a theory of reproducible algorithms, showing how reproducibility implies desirable properties such as data reuse and efficient testability. Despite the exceedingly strong demand of reproducibility, there are efficient reproducible algorithms for several fundamental problems in statistics and learning. First, we show that any statistical query algorithm can be made reproducible with a modest increase in sample complexity, and we use this to construct reproducible algorithms for finding approximate heavy-hitters and medians. Using these ideas, we give the first reproducible algorithm for learning halfspaces via a reproducible weak learner and a reproducible boosting algorithm. Interestingly, we utilize a connection to foams [KORW12] as a higher-dimension randomized rounding scheme. Finally, we initiate the study of lower bounds and inherent tradeoffs for reproducible algorithms, giving nearly tight sample complexity upper and lower bounds for reproducible versus nonreproducible SQ algorithms.

∗Our choice of the term “reproducibility” is inconsistent with current guidance from the Association for Computing Machinery regarding usage of the terms “reproducibility” and “replicability” [Ass20], of which we were regrettably unaware at the time of publication. In future work, we adopt more consistent terminology, using “replicability” to refer to the same stability notion termed “reproducibility” in this work.

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1 Introduction

Reproducibility is vital to ensuring scientific conclusions are reliable, and researchers have an obligation to ensure that their results are replicable. In the last twenty years, lack of reproducibility has been a major issue in nearly all scientific areas of study. For example, a 2012 Nature article by Begley and Ellis reported that the biotechnology company Amgen was only able to replicate 6 out of 53 landmark studies in haematology and oncology [BIE12]. In a 2016 Nature article, Baker published a survey of 1500 researchers, reporting that 70% of scientists had tried and failed to replicate the findings of another researcher, and that 52% believed there is a significant crisis in reproducibility [Bak16].

A key issue underlying the reproducibility crisis (as articulated in many articles, e.g., [Loa05]) is the fact that new data/publications are growing at an exponential rate, giving rise to an explosion of methods for data generation, screening, testing, and analysis, where, crucially, only the combinations producing the most significant results are reported. Such practices (also known as P-hacking, data dredging, and researcher degrees of freedom) can lead to erroneous findings that appear to be significant, but that don’t hold up when other researchers attempt to replicate them. Identifying and mitigating these problems is quite subtle. First, it is not easy to come up with an agreed-upon set of practices that guarantees reproducibility, and secondly, testing to determine whether or not a finding is statistically significant is a complex task.

Within the subfields of machine learning and data science, there are similar concerns about the reliability of published findings. The performance of models produced by machine learning algorithms may be affected by the values of random seeds or hyperparameters chosen during training, and performance may be brittle to deviations from the values disseminated in published results [HIB17, IHP17, LKM18]. To begin addressing concerns about reproducibility, several prominent machine learning conferences have begun hosting reproducibility workshops and holding reproducibility challenges, to promote best practices and encourage researchers to share the code used to generate their results [PVLS+20].

In this work, we aim to initiate the study of reproducibility as a property of algorithms themselves, rather than the process by which their results are collected and reported. We define the following notion of reproducibility, which informally says that a randomized algorithm is reproducible if two distinct runs of the algorithm on two sets of samples drawn from the same distribution, with internal randomness fixed between both runs, produces the same output with high probability.

Definition 1.1 (Reproducibility). Let $D$ be a distribution over a universe $\mathcal{X}$, and let $A$ be a randomized algorithm with sample access to $D$. $A(\vec{s})$ is $\rho$-reproducible if

$$\Pr_{\vec{s}_1, \vec{s}_2, r} [A(\vec{s}_1; r) = A(\vec{s}_2; r)] \geq 1 - \rho,$$

where $\vec{s}_1$ and $\vec{s}_2$ denote sequences of samples drawn i.i.d. from $D$, and $r$ denotes a random binary string representing the internal randomness used by $A$.

Our definition of reproducibility is inspired by the literature on pseudodeterministic algorithms, particularly the work of Grossman and Liu [GL19] and Goldreich [Gol19]. In the pseudodeterministic setting, the primary concern is reproducing the output of an algorithm given the same input, over different choices of the algorithm’s internal randomness. Our notion (Definition 1.1) is more suitable for the setting of machine learning, where it is desirable to reproduce the exact same output of an algorithm (with high probability) over different sample sets drawn from a distribution $D$.

We observe the following key properties of Definition 1.1.

Stability. Reproducibility is a strong stability property that implies independent parties can replicate previous results with high probability, so long as the randomness used to achieve these results is made public. For researchers solving machine learning and data analysis tasks, reproducibility allows researchers to verify published results with high probability, as long as the datasets are drawn from the same distribution.

Generalization. Reproducibility implies generalization. A reproducible learning algorithm, with high probability, outputs a hypothesis $h$ such that the difference between the risk of $h$ and the empirical risk of $h$ on the training set is small. Intuitively, reproducibility implies that $h$ is independent of the training set with high probability. Thus, a Hoeffding bound can be applied to bound the risk in terms of the empirical risk.

Privacy. Differential privacy (DP) is an important notion that requires small distance between the two distributions induced by an algorithm, when run on any two datasets that differ in a single element. Crucially,
it asks for the guarantees in the worst case over datasets. Reproducible algorithms guarantee a different form of privacy: If $A$ is reproducible, then what $A$ learns (for example, a trained classifier) is almost always the same; thus, $A$ is usually independent of the chosen training data. In this way, reproducible algorithms are prevented from memorizing anything that is specific to the training data, similar to differentially private algorithms. Reproducibility is weaker than differential privacy in the sense that reproducibility only applies to in-distribution samples, whereas differential privacy applies to any training set. On the other hand, reproducibility is stronger in the sense that its guarantee for in-distribution samples is global rather than local (for neighboring samples).

**Testability.** While differential privacy has become the standard for privacy-preserving computation, an important issue that is the subject of extensive research is testing and verifying differential privacy. As discussed in [CNP20], DP-algorithms and their implementations are usually analyzed by hand, and proofs of differential privacy are often intricate and prone to errors. Implementing such an algorithm in practice often gives rise to DP leaks, due to coding errors or assumptions made in the proof that do not hold on finite computers (such as the ability to sample from continuous distributions). Moreover, the complexity of verifying differential privacy is hard. Verification in the black-box setting (where the auditor has oracle access to the learning algorithm) was recently shown to be infeasible, as low query complexity implies high values of the the privacy parameters $\epsilon$ and $\delta$ [GM18]. In the white-box setting where $A$ is given to the tester, [CNP20] shows that testing for differential privacy is coNP$^P$-complete. This has led to an active research area aiming at developing automated as well as interactive testing and verification methods for differential privacy [NFPH15, GHH+13, RP10, AH18, BGA+15, BCK+21, FJ14, ZK17]. In contrast, reproducibility is a form of privacy that can be efficiently tested in (randomized) polynomial time (in the dimension of the data universe and $\rho$).

### 1.1 Our Main Results

#### 1.1.1 Reproducibility: Properties and Alternative Definitions

We discuss alternative definitions of reproducibility and show that they are all essentially equivalent. Then, we also prove some other nice properties of reproducible algorithms. (All formal statements and proofs are in Appendix A.)

1. **Alternative Definitions and Amplification.** We start by discussing two alternative definitions of reproducibility and relate them to our definition. First, we can generalize the definition to include algorithms $A$ that not only have access to internal randomness and to random samples from an underlying distribution $D$, but that also have access to extra non-random inputs. This more general definition captures both the original definition of pseudodeterministic algorithms as well as our definition of reproducible learning algorithms, and all of our results remain unchanged. Second, we discuss an alternative two-parameter definition, and show that the definitions are qualitatively the same. We show how to amplify the reproducibility parameter by a standard argument where the sample complexity is increased modestly.

2. **Public versus Private Randomness.** Recall that we define reproducibility as the probability that an algorithm returns the same answer when run twice using different random samples from $D$ but the same internal randomness. In [GL19], the authors define a related concept in which the internal randomness is divided into two pieces, public and private randomness, but the algorithm should return the same answer when just the public randomness is held fixed. We show that, without loss of generality, it suffices to use only public randomness.

3. **Reproducibility Implies Generalization.** Learning algorithms attempt to use finite samples to generate hypotheses on unknown, possibly complex distributions. The error of a hypothesis $h$ on the underlying distribution is called the generalization error. A reproducible algorithm outputs the same hypothesis with high probability, and thus the algorithm seldom draws distinctions between specific samples and the entire distribution.

4. **Connections to Data Reuse.** We explore the connection between reproducible algorithms and the adaptive data analysis model discussed in [DFH+15b] and [DFH+15a]. We show that reproducible
algorithms are strongly resilient against adaptive queries. Informally, with respect to reproducible algorithms, the sample complexity and accuracy of (reproducibly) answering m adaptively chosen queries behaves similarly to the sample complexity and accuracy of reproducibly answering m nonadaptively chosen queries.

1.1.2 Upper Bounds

Our main technical results are reproducible algorithms for some well-studied statistical query and learning problems that are used as building blocks in many other algorithms.

1. **Simulating SQ Algorithms.** In Section 2, we give a generic algorithm that reduces the problem of ρ-reproducibly estimating a single statistical query with tolerance τ and error δ to that of nonreproducibly estimating the same query within a smaller tolerance and error.

   **Theorem 1.2** (Theorem 2.3 Restated). Let ψ : X → {0, 1} be a statistical query. Then the sample complexity of ρ-reproducibly estimating ψ within tolerance τ and error δ is at most the sample complexity of (nonreproducibly) estimating ψ within tolerance τ′ = τρ and error δ′ = τδ.

The basic idea is to obtain an estimate of the statistical query with a smaller tolerance τ′ and then use a randomized rounding scheme where the interval [0, 1] is divided into intervals of size roughly τ/ρ. Then, every value in the interval is rounded to the midpoint of the region it occurs in. The partition into intervals is chosen with a random offset so that with high probability nearby points will lie in the same region.

2. **Heavy-hitters.** Using our simulation of SQ queries, in Section 3, we demonstrate the usefulness of reproducibility by giving a reproducible algorithm rHeavyHitters for identifying approximate ρ-heavy-hitters of a distribution, i.e. the elements in the support of the distribution with probability mass at least ρ.

   **Lemma 1.3** (Lemma 3.3 Restated). For all ε ∈ (0, 1/2), v ∈ (ε, 1 − ε), with probability at least 1 − ρ, rHeavyHittersρ,v,ε is ρ-reproducible, and returns a list of v'-heavy-hitters for some v' ∈ [v − ε, v + ε]. Furthermore, the sample complexity is bounded by O(ρ⁻²).

The high level idea of our algorithm is to first draw sufficiently many samples, s₁, Q₁ = |s₁|, so that with high probability all heavy-hitters are in s₁. In the second stage, we draw a fresh set s₂ of Q₂ many samples and use them to empirically estimate the density of each element in s₁, and remove those that aren’t above the cutoff v', where v' is chosen randomly from [v − ε, v + ε] to avoid boundary issues.

3. **Median Finding.** In Section 4, we design a reproducible algorithm for finding an approximate median in an arbitrary distribution over a finite domain. Approximate median finding is a fundamental statistical problem, and is also extensively studied in the privacy literature.

   **Theorem 1.4** (Theorem 4.8 Restated). Let τ, ρ ∈ [0, 1] and let δ = 1/3. Let D be a distribution over X, where |X| = 2^d. Then rMedianρ,τ,δ is ρ-reproducible, outputs a τ-approximate median of D with success probability 1 − δ, and has sample complexity

   \[\tilde{\Omega}(\left(\frac{1}{\tau^2 (\rho - \delta)^2}\right) \cdot \left(\frac{3}{\tau^2}\right)^{\log^* |X|})\]

To describe the key ideas in the algorithm, we first show how approximate-median finding is useful for turning many algorithms into reproducible ones. Consider any problem where the correct answers form an interval, and assume we start with a (not-necessarily) reproducible algorithm that is mildly accurate. Then we can run a reproducible approximate-median finding algorithm on the distribution of outputs of the original algorithm to construct a very accurate reproducible algorithm.

We will actually use this strategy recursively to reproducibly solve approximate median itself. Our algorithm recursively composes a mildly accurate reproducible median algorithm with a generic very
accurate non-reproducible median algorithm. This recursive technique is inspired by, but simpler than, previous algorithms in the privacy literature \cite{BNSV15, KLM20}, and like these algorithms, the sample complexity of our algorithm has a non-constant but very slowly growing dependence on the domain size.

4. **Learning Halfspaces.** In Section 5, we obtain a reproducible algorithm \texttt{rHalfspaceWkL} for weakly learning halfspaces. In Section 6, we transform it into a reproducible strong learner by way of a reproducible boosting algorithm \texttt{rBoost}. We stress that our algorithms for halfspaces are reproducible in the stronger distribution-free setting.

**Theorem 1.5** (Corollary 6.5 Restated). Let \( D \) be a distribution over \( \mathbb{R}^d \), and let \( f : \mathbb{R}^d \to \{\pm 1\} \) be a halfspace with margin \( \tau \) in \( D \). For all \( \rho, \epsilon > 0 \), Algorithm \texttt{rBoost} run with weak learner \texttt{rHalfspaceWkL} \( \rho \)-reproduces returns a hypothesis \( h \) such that, with probability at least \( 1 - \rho \), \( \Pr_{x \sim D}[h(x) = f(x)] \geq 1 - \epsilon \). Furthermore, the overall sample complexity is \( \tilde{O}(\frac{d^{19/9}}{\rho^9 \epsilon^2 \tau^{29/9}}) \).

In order to reproducibly learn halfspaces, we start with a simple weak learning algorithm for halfspaces \cite{Ser02} that takes examples \((\bar{x}_i, y_i) \in X \times \{\pm 1\}\), normalizes them, and returns the halfspace defined by vector \( \sum_i \bar{x}_i y_i \). We show a concentration bound on the sum of normalized vectors from a distribution, and then argue that all vectors within the concentration bound are reasonable hypothesis with non-negligible advantage.

Our randomized rounding scheme is a novel application of the randomized rounding technique developed in the study of foams \cite{KORW12}. The concentration bound together with the foams rounding scheme \cite{KORW12} yields a reproducible halfspace weak learner. We then obtain our reproducible strong learner for halfspaces by combining it with a (new) reproducible boosting algorithm. Our algorithm is sample efficient but inefficient with respect to runtime, due to the inefficiency of the foams rounding scheme. We also give another randomized rounding procedure that gives a polynomial-time strong reproducible halfspace learner, but with polynomially larger sample complexity.

1.1.3 **The Price of Reproducibility.**

In Section 7 we ask what is the cost of turning a nonreproducible algorithm into a reproducible one. We first show that a \( \tau \)-tolerant \( \rho \)-reproducible SQ algorithm \( \mathcal{A} \) for \( \phi \) implies a \( \rho \)-reproducible algorithm for the \( \tau \)-coin problem: given samples from a \( \rho \)-biased coin with the promise that either \( p \geq 1/2 + \tau \) or \( p \leq 1/2 - \tau \), determine which is the case. Our main result in this section is nearly tight upper and lower bound bounds of \( \Theta(\tau^{-2} \rho^{-2}) \) on the sample complexity of \( \rho \)-reproducibly solving the \( \tau \)-coin problem (for constant \( \delta \)), and thus the same bounds for \( \rho \)-reproducibly answering SQ queries. On the other hand, it is well-known that the nonreproducible sample complexity of the \( \tau \)-coin problem is \( \Theta(\tau^{-2} \log(1/\delta)) \) (see, e.g. \cite{Mou}). So the cost of guaranteeing \( \rho \)-reproducibility for SQ queries is a factor of \( \rho^{-2} \).

For upper bounds, our generic algorithm in Section 2 converts any SQ query into a reproducible one: if our end goal is a \( \rho \)-reproducible algorithm for estimating a statistical query with tolerance \( \tau \) and error \( \delta \), then the sample complexity is at most the sample complexity of \( \text{nonreproducibly} \) answering the query to within tolerance \( \tau' \), and success probability \( 1 - \delta' \) where \( \tau' = O(\tau \rho) \) and \( \delta' = O(\delta \tau) \), which has sample complexity \( O(\tau^{-2} \rho^{-2} \log(1/\delta')) \). The main result in this section is the following lower bound for \( \rho \)-reproducibly answering statistical queries.

**Theorem 1.6** (Theorem 7.1 Restated). Let \( \tau > 0 \) and let \( \delta < 1/16 \). Any \( \rho \)-reproducible algorithm for solving the \( \tau \)-coin coin problem with success probability at least \( 1 - \delta \) requires sample complexity \( \Omega(\tau^{-2} \rho^{-2}) \).

**Related Work.** A subset of these results \cite{ILS21} was presented at the TPDP 2021 workshop.

Our Definition 1.1 is inspired by the literature on pseudodeterministic algorithms \cite{GG11, GGR13, GG17, GGH18, GGMW19, GL19, Gol19}. In particular, \cite{GL19} and \cite{Gol19} define reproducibility in the context of pseudodeterminism. There, the input of a reproducible algorithm is a fixed string. In our setting, the input of a reproducible learning algorithm is a distribution, only accessible by randomly drawing samples.
Independently of our work, [GKM21] define a property equivalent to reproducibility, called “pseudo-global stability”. Their \((\alpha, \beta)\)-accurate \((\eta', \nu')\)-pseudo-global stability definition is equivalent to the \((\eta, \nu)\)-reproducibility definition discussed in Appendix A except that pseudo-global stability includes explicit parameters for correctness and sample complexity. In Appendix A, we show that these two definitions are equivalent to Definition L1 up to polynomial factors. [GKM21] gives pseudo-globally stable SQ algorithms, an amplification of the stability parameter, and an algorithm to find a heavy-hitter of a distribution. The authors use pseudo-global stability to show that classes with finite Littlestone dimension can be learned user-level privately, and they connect pseudo-global stability to approximate differential privacy. Pseudo-global stability is a generalization of global stability, introduced in [BLM20]. Those authors use global stability as an intermediate step to show that classes with finite Littlestone dimension can be learned privately, and they show how global stability implies generalization.

Our work is related to other notions of stability in machine learning which, like our definition, are properties of learning algorithms. In the supervised learning setting, stability is a measure of how much the output of a learning algorithm changes when small changes are made to the input training set. An important body of work establishes strong connections between the stability of a learning algorithm and generalization [DW79a, DW79b, KR99, BE02, SSSS10]. Distributional notions of stability which remain stable under composition and postprocessing, were defined and shown to be closely connected to differential privacy and adaptive data analysis (e.g., [BNS16, DFH15a]). In fact, the definition of differential privacy itself is a form of stability known as max-KL stability. Stability-based principles have also been explored in the context of unsupervised learning where model selection is a difficult problem since there is no ground truth. For example, a stable algorithm for clustering has the property that when the algorithm is applied to different data sets from the same distribution, it will yield similar outputs (e.g., [vL10]).

In all of these settings, stability depends on how close the outputs are when the inputs are close; what varies is the particular measure of closeness in input and output space. For example, closeness in the output can be with respect to function or parameter space; for distributional stability close means that the output distributions are close with respect to some metric over distributions. Our definition of reproducibility can be viewed as an extreme form of stability where the output is required to be identical almost all of the time, and not just similar. Thus reproducibility enjoys many of the nice properties of stable algorithms (e.g., postprocessing, composition) but has the advantage of being far easier to verify.

Open Questions and Future Work One motivation for examining reproducibility in algorithms is the “reproducibility crisis” in experimental science. Can we use reproducibility to create statistical methodologies that would improve reproducibility in published scientific work? A concrete step towards this would be to design reproducible hypothesis testing algorithms. We can view a null hypothesis as postulating that data will come from a specific distribution \(D\), and want algorithms that accept with high probability if the data comes from \(D\) (or a “close” distribution) and reject with good probability if the data distribution is “far” from \(D\). For example, the coin problem is a degenerate case in which the data are Boolean and the distance is the difference in the expected values. For different types of data and distance metrics, what is the optimal sample complexity of hypothesis testing, and how much more is that for reproducible hypothesis testing?

A related problem is that of learning under distributional shifts, or individual-based fair learning (where we want the learning algorithm to treat similar people similarly with respect to a similarity metric defining closeness). A key step in making algorithms reproducible is a randomized procedure to round the output of a standard empirical learner to a single hypothesis in a way that is independent of the underlying distribution. Can similar ideas be used to design learning algorithms robust to distributional shifts, or to give more informed performance metrics?

This work establishes that there exist reproducible algorithms for a variety of learning problems. However, we do not characterize exactly which learning algorithms can be made reproducible, or how reproducibility affects the required sample complexity. Is it possible to identify an invariant of concept classes which characterizes the complexity of reproducible learning, analogous to VC-dimension for PAC learning [VC71], representation dimension and one-way communication complexity for exact differential privacy [FX14, BNS13], and Littlestone Dimension for approximate differential privacy [BLM20]? A specific problem of interest is that of learning linear functions over finite fields. If the data has full dimension, the function can be solved for uniquely; so, designing reproducible algorithms when the data does not form a basis seems interesting.
Also, we described the first reproducible boosting algorithm. Are there natural conditions under which a boosting algorithm can always be made reproducible? Are the sample complexity upper bounds we obtain for our applications tight or close to tight? In particular, is there a reproducible algorithm for approximate median that has only \(\log^* |X|\) dependence on the domain size?

Reproducibility provides a distinctive type of privacy. Except with the small probability \(\rho\), a reproducible algorithm’s outputs are a function entirely of the underlying distribution and the randomness of the algorithm, not the samples. Thus, a reproducible algorithm seldom leaks information about the specific input data. We borrowed techniques from the study of private data analysis and differential privacy, and we hope that future work will formalize connections between reproducibility and private data analysis. We also hope that some applications of differential privacy will also be achievable through reproducibility.

# 2 Statistical Queries

We show how to use randomized rounding to reproducibly simulate any SQ oracle and therefore any SQ algorithm. The statistical query model introduced by [Kea98] is a restriction of the PAC-learning model introduced by [Val84]. We consider the statistical query oracle primarily in the context of unsupervised learning (e.g., see [Fel16]).

**Definition 2.1 (Statistical query oracle).** Let \(\tau \in [0, 1]\) and \(\phi : X \to [0, 1]\) be a query. Let \(D\) be a distribution over domain \(X\). A statistical query oracle for \(D\), denoted \(O_D(\tau, \phi)\), takes as input a tolerance parameter \(\tau\) and a query \(\phi\), and outputs a value \(v\) such that \(|v - E_{x \sim D}[\phi(x)]| \leq \tau\).

**Definition 2.2 (Simulating a statistical query oracle).** Let \(\rho \in [0, 1]\) and \(\tau, \phi, D\) be as above. Let \(O_D\) be a statistical query oracle for \(D\). Let \(\bar{s}\) denote an i.i.d. sample drawn from \(D\). We say that a routine \(\text{STAT}\) simulates \(O_D\) with failure probability \(\delta\) if for all \(\tau, \delta, \phi\), there exists an \(n_0 \in \mathbb{N}\) such that if \(n > n_0\), \(v \leftarrow \text{STAT}(\tau, \phi, \bar{s})\) satisfies \(|v - E_{x \sim D}[\phi(x)]| \leq \tau\) except with probability \(\delta\).

To denote a routine simulating a statistical query oracle for fixed parameters \(\tau, \phi\), and (optionally) \(\rho\), we write these parameters as subscripts.

**Algorithm 1 \(\text{rSTAT}_{\rho, \tau, \phi}(\bar{s})\)**

Parameters:
- \(\tau\) - tolerance parameter
- \(\rho\) - reproducibility parameter
- \(\phi\) - a query \(X \to [0, 1]\)

1. \(\alpha = \frac{2\tau}{\rho + 1 - 2\tau}\)
2. \(\alpha_{\text{off}} \leftarrow \frac{1}{\tau} [0, \alpha]\)
3. Split \([0, 1]\) in regions: \(R = \left\{ [0, \alpha_{\text{off}}), [\alpha_{\text{off}}, \alpha_{\text{off}} + \alpha), \ldots, [\alpha_{\text{off}} + i\alpha, \alpha_{\text{off}} + (i + 1)\alpha), \ldots, [\alpha_{\text{off}} + k\alpha, 1) \right\}\)
4. \(v \leftarrow \frac{1}{|\bar{s}|} \sum_{x \in \bar{s}} \phi(x)\)
5. Let \(r_v\) denote the region in \(R\) that contains \(v\)
6. **return** the midpoint of region \(r_v\)

**Theorem 2.3** upper bounds the sample complexity of \(\text{rSTAT}_{\tau, \phi, \rho}\). In Section 7 we show this upper bound is tight as a function of \(\rho\).

**Theorem 2.3 (\(\text{rSTAT}\) simulates a statistical query oracle).** Let \(\tau, \delta, \rho \in [0, 1]\), \(\rho > 2\delta\), and let \(\bar{s}\) be a sample drawn i.i.d. from distribution \(D\). Then if

\[
|\bar{s}| \in \tilde{O} \left( \frac{1}{\tau^2 (\rho - 2\delta)^2} \right)
\]

\(\text{rSTAT}_{\rho, \tau, \phi}(\bar{s})\) \(\rho\)-reproducingly simulates an SQ oracle \(O_D(\tau, \phi)\) with failure rate \(\delta\).
In Section 7, we will prove a near matching lower bound on the sample complexity of $\rho$-reproducibly estimating a statistical query with tolerance $\tau$ and success probability $1 - \delta$.

**Proof.** We begin by showing that $r\text{STAT}_{\rho,\tau,\phi}$ simulates an SQ oracle $O_{D,\tau,\phi}$ with failure rate $\delta$.

Let $\tau' = \frac{\tau(\rho - 2\delta)}{\rho + 1 - 2\delta}$. Recall $\alpha \overset{\text{def}}{=} \frac{2\tau}{\rho + 1 - 2\delta}$, so $\frac{2\alpha}{\rho} = \rho - 2\delta$. A Chernoff bound gives that

$$\left| \frac{1}{|s|} \sum_{x \in s} \phi(x) - \mathbb{E}_{x \sim D}[\phi(x)] \right| \leq \tau' = \frac{\tau(\rho - 2\delta)}{\rho + 1 - 2\delta}$$

except with failure probability $\delta$, so long as $|s| \geq \log(2/\delta)/(2\tau')^2$. Outputting the midpoint of region $r_v$ can further offset this result by at most $\alpha/2 = \frac{\tau}{\rho + 1 - 2\delta}$. Therefore

$$|v - \mathbb{E}_{x \sim D}[\phi(x)]| \leq \frac{\tau(\rho - 2\delta)}{\rho + 1 - 2\delta} + \frac{\tau}{\rho + 1 - 2\delta} = \tau,$$

except with probability $\delta$, so long as the sample $s$ satisfies

$$\log(2/\delta)/(2\tau')^2 = \frac{\log(2/\delta)(\rho + 1 - 2\delta)^2}{2\tau'^2(\rho - 2\delta)^2} \leq \frac{4\log(2/\delta)}{2\tau'^2(\rho - 2\delta)^2} \leq |s|.$$

We now show that $r\text{STAT}_{\rho,\tau,\phi}$ is $\rho$-reproducible by considering two invocations of $r\text{STAT}_{\rho,\tau,\phi}$ with common randomness $r$ on samples $s_1, s_2 \sim D$ respectively. The probability that either empirical estimate of $\mathbb{E}_{x \sim D}[\phi(x)]$ fails to satisfy tolerance $\tau$ is at most $2\delta$. Denote by $v_1$ and $v_2$ the values returned by the parallel runs $r\text{STAT}(s_1; r)$ and $r\text{STAT}(s_2; r)$ at line 4. Conditioning on success, values $v_1$ and $v_2$ differ by at most $2\tau'$. $r\text{STAT}$ outputs different values for the two runs if and only if $v_1$ and $v_2$ are in different regions of $R$, determined by the common randomness $r$. This occurs if some region’s endpoint is between $v_1$ and $v_2$; since $\alpha_{\text{off}}$ is chosen uniformly in $[0, \alpha]$, the probability that $v_1$ and $v_2$ land in different regions is at most $2\tau'/\alpha = \rho - 2\delta$. Accounting for the $2\delta$ probability of failure to estimate $\mathbb{E}_{x \sim D}[\phi(x)]$ to within tolerance, $r\text{STAT}_{\rho,\tau,\phi}(s)$ is $\rho$-reproducible.

\section{Heavy-hitters}

Next, we present our reproducible approximate heavy-hitters algorithm, analyzing its sample complexity and reproducibility. We will use this algorithm as a subroutine in later algorithms such as in the approximate-median algorithm. Also, we will show how to use this algorithm to give a generic way to boost reproducibility from constant $\rho$ to arbitrarily small $\rho$.

**Definition 3.1** (Heavy-Hitter). Let $D$ be a distribution over $X$. Then we say $x \in X$ is a $v$-heavy-hitter of $D$ if $\Pr_{x' \sim D}[x' = x] \geq v$.

**Definition 3.2** ((Approximate) Heavy-Hitter Problem). Let $L_v$ be the set of $x \in \text{supp}(D)$ that are $v$-heavy-hitters of $D$. Given sample access to $D$, output a set $L$ satisfying $L_{v+\epsilon} \subseteq L \subseteq L_{v-\epsilon}$.

Let $D$ be a distribution over $X$. The following algorithm reproducibly returns a set of $v'$-heavy-hitters of $D$, where $v'$ is a random value in $[v - \epsilon, v + \epsilon]$. Picking $v'$ randomly allows the algorithm to, with high probability, avoid a situation where the cutoff for being a heavy-hitter (i.e. $v'$) is close to the probability mass of any $x \in \text{supp}(D)$.
Algorithm 2 rHeavyHitters$\rho,v,\epsilon$

Input: samples $X_{\text{set}}$, $S$ from distribution $D$ over $X$ plus internal randomness $r$

Parameters: Target reproducibility $\rho$, target range $[v-\epsilon, v+\epsilon]$

Output: List of $v'$-heavy-hitters of $D$, where $v' \in [v-\epsilon, v+\epsilon]$

1. Set $X_{\text{set}} \leftarrow Q_1 \overset{\text{def}}{=} \left[ \frac{\ln(6/\rho \cdot (v-\epsilon))}{v-\epsilon} \right]$ examples from $D$ // Step 1: Find candidate heavy-hitters
2. Set $S \leftarrow Q_2 \overset{\text{def}}{=} \frac{\ln(Q_1/\rho) \cdot Q_1^2}{\rho v^2}$ fresh examples from $D$ // Step 2: Estimate probabilities
3. For all $x \in X_{\text{set}}$ do
   - $\hat{p}_x \leftarrow \Pr_{x \sim S}[x = x]$ // Estimate $p_x \overset{\text{def}}{=} \Pr_{x \sim D}[x = x]$
   - $v' \leftarrow [v-\epsilon, v+\epsilon]$ uniformly at random
   - Remove from $X_{\text{set}}$ all $x$ for which $\hat{p}_x < v'$.

Return $X_{\text{set}}$

Algorithm rHeavyHitters returns exactly the list of $v'$-heavy-hitters so long as the following hold:

1. In Step 1 of Algorithm 2 all $(v-\epsilon)$-heavy-hitters of $D$ are included in $X_{\text{set}}$.
2. In Step 2, the probabilities $\hat{p}_x$ for all $x \in X_{\text{set}}$ are correctly estimated to within error $\rho \epsilon/(3Q_1)$.
3. In Step 3, the randomly sampled $v'$ does not fall within an interval of width $\rho \epsilon/(3Q_1)$ centered on the true probability of a $(v-\epsilon)$-heavy-hitter of $D$.

We show that these 3 conditions will hold with probability at least $1 - \rho/2$, and so will hold for two executions with probability at least $1 - \rho$.

**Lemma 3.3.** For all $\epsilon \in (0,1/2)$, $v \in (\epsilon,1-\epsilon)$, with probability at least $1 - \rho$, rHeavyHitters is reproducible, returns a list of $v'$-heavy-hitters for some $v' \in [v-\epsilon, v+\epsilon]$, and has sample complexity $\tilde{O}\left(\frac{1}{\rho^2 (v-\epsilon)^2}\right)$.

**Proof.** We say Step 1 of Algorithm 2 succeeds if all $(v-\epsilon)$-heavy-hitters of $D$ are included in $X_{\text{set}}$ after Step 1. Step 2 succeeds if the probabilities for all $x \in X_{\text{set}}$ are correctly estimated to within error $\rho \epsilon/(3Q_1)$. Step 3 succeeds if the returned $X_{\text{set}}$ is exactly the set of $v'$-heavy-hitters of $D$. Quantities $Q_1$ and $Q_2$ are defined in the pseudocode of Algorithm 2.

In Step 1, an individual $(v-\epsilon)$-heavy-hitter is not included with probability at most $(1 - v + \epsilon)^Q_1$; union bounding over all $1/(v-\epsilon)$ possible $(v-\epsilon)$-heavy-hitters, Step 1 succeeds with probability at least $1 - \frac{1^Q_1}{v-\epsilon} > 1 - \rho/6$. Here, for clarity of presentation in the statement of Lemma 3.3, we make use of the inequality $v - \epsilon < \ln(1/(1 - v + \epsilon))$.

By a Chernoff bound, each $p_x$ is estimated to within error $\rho \epsilon/(3Q_1)$ with all but probability $\rho/(6Q_1)$ in Step 2. Union bounding over all $Q_1$ possible $x \in X_{\text{set}}$, Step 2 succeeds except with probability $\rho/6$.

Conditioned on the previous steps succeeding, Step 3 succeeds if the randomly chosen $v'$ is not within $\rho \epsilon/(3Q_1)$ of the true probability of any $x \in X_{\text{set}}$ under distribution $D$. A $v'$ chosen randomly from the interval $[v-\epsilon, v+\epsilon]$ lands in any given subinterval of width $\rho \epsilon/(3Q_1)$ with probability $\rho/(6Q_1)$, and so by a union bound, Step 3 succeeds with probability at least $1 - \rho/6$.

Therefore, Algorithm 2 outputs exactly the set of $v'$-heavy-hitters of $D$ with probability at least $1 - \rho/2$. If we consider two executions of Algorithm 2 both using the same shared randomness for choosing $v'$, output the set of $v'$-heavy-hitters of $D$ with probability at least $1 - \rho$, and so rHeavyHitters is $\rho$-reproducible.

The sample complexity is $Q_1 + Q_2 \in \tilde{O}\left(\frac{(\rho \epsilon (v-\epsilon))^{-2}}{\rho}\right)$. \hfill \Box

**Corollary 3.4.** If $\nu$ and $\epsilon$ are constants, then rHeavyHitters$\rho,v,\epsilon$ has sample complexity $\tilde{O}\left(\frac{1}{\rho^2}\right)$.

**Learning Heavy-hitters using Statistical Queries.** Next, we show that any statistical query algorithm for the $v$-heavy-hitters problem requires $\Omega(\log|X|/\log(1/r))$ calls to the SQ oracle. Since Algorithm 2 has a sample complexity independent of the domain size, this implies a separation between reproducible problems and problems solvable using only SQ queries.

Consider the ensemble $\{D_x\}_{x \in X}$ on $X$, where distribution $D_x$ is supported entirely on a single $x \in X$. 8
Claim 3.5 (Learning Heavy-hitters using Statistical Queries). Any statistical query algorithm for the \( v \)-heavy-hitters problem on ensemble \( \{D_x\}_{x \in X} \) requires \( \Omega(\log |X|/\log(1/\tau)) \) calls to the SQ oracle.

Proof. An SQ algorithm for the \( v \)-heavy-hitters problem must, for each distribution \( D_x \), output set \( \{x\} \) with high probability. An SQ oracle is allowed tolerance \( \tau \) in its response to statistical query \( \phi \). So, for any \( \phi \), there must be some distribution \( D_x \) for which the following holds: at least a \( \tau \)-fraction of the distributions \( D_{x'} \) in the ensemble satisfy \( |\phi(x') - \phi(x)| \leq \tau \). Thus, in the worst case, any correct SQ algorithm can rule out at most a \((1-\tau)\)-fraction of the distributions in the ensemble with one query. If \( X \) is finite, then an SQ algorithm needs at least \( \log_{1/\tau}(|X|) \) queries. \( \Box \)

4 Approximate Median

In this section, we design a reproducible algorithm for finding an approximate median in an arbitrary distribution over a finite domain. In addition to being a significant problem in its own right, and one studied extensively in the privacy literature, this is a key sub-routine for making many algorithms reproducible.

In particular, for any problem where the correct answers form an interval, and we have a (not-necessarily) reproducible algorithm that is correct strictly more than half the time, we can run the approximate median finding algorithm on the distribution of outputs of the original to construct a reliably correct and reproducible version. (In fact, we use this technique recursively within our reproducible median-finding algorithm itself.

Our algorithm \( r\text{MedianOfMedians} \) composes a mildly accurate reproducible median algorithm with a generic very accurate non-reproducible median algorithm.) We use a recursive technique inspired by but simpler than previous algorithms in the privacy literature \cite{BNSV15, KLM+20}, and like for these algorithms, the sample complexity of our algorithm has a non-constant but very slowly growing dependence on the domain size.

**Definition 4.1 (\( \tau \)-approximate median).** Let \( D \) be a distribution over a well-ordered domain \( X \). \( x \in X \) is a \( \tau \)-approximate median of \( D \) if \( \Pr_{x' \sim D}[x' \leq x] \geq 1/2 - \tau \) and \( \Pr_{x' \sim D}[x' \geq x] \geq 1/2 - \tau \).

4.1 Reproducible Approximate Median Algorithm

In this section, we present a pseudocode description of our \( \tau \)-approximate median algorithm \( r\text{Median} \) (Algorithm 3), and prove the following theorem.

**Theorem 4.2 (Reproducible Median).** Let \( \tau, \rho \in [0,1] \) and let \( \delta = \rho/2 \). Let \( D \) be a distribution over \( X \), where \( |X| = 2^d \). Then \( r\text{Median}_{\rho,d,\tau,\delta} \) (Algorithm 3) is \( \rho \)-reproducible, outputs a \( \tau \)-approximate median of \( D \) with all but probability \( \delta \), and has sample complexity

\[
 n \in \tilde{O}\left(\frac{1}{\tau^2(\rho - \delta)^2} \cdot \left(\frac{3}{\tau} \right)^{|X|}\right)
\]

As an introduction to the key ideas of Algorithm 3, we consider a weighted binary tree \( T \) based on distribution \( D \). Each internal node has two edges (a 0-edge and a 1-edge). Root-to-leaf paths represent binary representations of numbers. The weight of each internal node \( v \) is the probability that its associated binary prefix (induced by the root-to-\( v \) path) appears in an element drawn from \( D \). If within this tree we can find a node \( v \) with weight in \([1/4, 3/4]\), then we can use the associated prefix to return an approximate median of \( D \) with approximation parameter potentially much larger than \( \tau \).

To achieve a specified approximation parameter \( \tau \), rather than using \( D \) itself to construct the binary tree \( T \), we will use a distribution \( D^m \) over medians of \( D \). Specifically, we use a non-reproducible median algorithm to sample from \( \tau \)-approximate medians of \( D \). Identifying an approximate median of distribution \( D^m \) for even a very large approximation parameter then ensures we return a \( \tau \)-approximate median of \( D \).

The question remains of how to efficiently search \( T \) to find a node \( v \) of weight in \([1/4, 3/4]\) (under \( D^m \)). We perform this search recursively by using \( r\text{Median} \) to find a prefix length \( \ell \) such that the probability of sampling two elements from \( D^m \) agreeing on a prefix of length \( \ell \) is large. We can then restrict our search for \( v \) to nodes near level \( \ell \) in \( T \) (starting from the root). We apply the reproducible heavy-hitters
algorithm rHeavyHitters to find high weight nodes near level $\ell$ of $T$, and then exhaustively search the list of heavy-hitters to find an appropriate $v$.

We use the following non-reproducible approximate median algorithm, that returns the median of its sample $\tilde{s}$, as a subroutine of Algorithm 3.

**Lemma 4.3** (Simple Median Algorithm). Let sample $\tilde{s}$ be drawn from distribution $D$. Algorithm Median($\tilde{s}$) returns a $\tau$-approximate median on $D$ using $|\tilde{s}| = 3(1/2 - \tau)\ln(2/\delta)/\tau^2$ samples with success probability at least $1 - \delta$.

**Proof.** Algorithm Median($\tilde{s}$) fails when more than half of the elements in sample $\tilde{s}$ are either i) smaller than the $(1/2 - \tau)$-percentile element of $D$ or ii) larger than the $(1/2 + \tau)$-percentile element of $D$. Let event $E_i$ denote the first case and event $E_{ii}$ denote the second case. Since the elements in $\tilde{s}$ are drawn i.i.d., the first event can be bounded by a Chernoff bound. Let $X$ be a random variable denoting the number of elements in $\tilde{s}$ that are smaller than the $(1/2 - \tau)$-percentile element of $D$.

$$\Pr[E_i] = \Pr[X \geq (1 + \tau/(1/2 - \tau))\mathbb{E}[X]]$$

$$\leq \exp(-\tau/(1/2 - \tau)^2\mathbb{E}[X]/3)$$

$$\leq \exp\left(-\frac{\tau^2}{1/2 - \tau} \frac{|\tilde{s}|}{3}\right) = \exp(-\ln(2/\delta))$$

$$= \delta/2$$

The same argument can be used to bound the second event $E_{ii}$. By a union bound, the algorithm succeeds with probability at least $1 - \delta$. 

Before proceeding with the description of Algorithm 3, we fix some useful notation for its analysis.

- $n_m$ - sample complexity of Median$_{\tau,\delta_0}$
- $n_h$ - sample complexity of rHeavyHitters$_{\rho_0,v,\epsilon}$
- $n_{sq}$ - sample complexity of rSTAT$_{\tau,\rho_0,\phi}$
- $n_d$ - sample complexity of rMedian$_{\rho,d,\tau,\delta}$
- $D^m$ - Algorithm 3 takes as input a sample from distribution $D$ over $\mathcal{X}$, where $|\mathcal{X}| = 2^d$. We use $D^m$ to denote the distribution induced by sampling $n_m$ examples from $D$, computing Median$_{\tau,\delta_0}$ on these examples, and returning the output
- $D_{[\log d]}$ - We use $D_{[\log d]}$ to denote the distribution induced by sampling 2 examples from $D^m$ and returning the longest prefix $\ell$ on which the two medians agree. Note that this new distribution is over a new domain $\mathcal{X}'$ with $|\mathcal{X}'| = 2^{[\log d]} \in \Theta(d)$.
- $\rho_0 \in O(\rho/\log^* |X|)$
- $\delta_0 \in O\left(\left(\frac{\delta}{n_h + n_{sq}}\right)^{2\log^* |X|} \cdot \left(\frac{z^2}{3}\right)^{2\log^* |X|^2}\right)$
Lemma 4.4 (Termination). Algorithm 3 terminates after $T = \log^* |\mathcal{X}|$ recursive calls.

Proof. Algorithm 3 reaches its base case when invoked with parameter $d = 1$. At each successive recursive call (Line 13), the domain size $2^d$ is reduced to $2^{\lceil \log d \rceil} < 2d$, and so $d = 1$ after no more than $T = \log^* |\mathcal{X}|$
Lemma 4.5 (Sample Complexity). Let $\tau, \delta, \rho \in [0, 1]$. Let $D$ be a distribution over $X$, with $|X| = 2^d$. Then $r_{\text{Median}}_{\rho, d, \tau, \delta}$ has sample complexity
\[
n \in O \left( \frac{1}{\tau^2(\rho - \delta)^2} \cdot \left( \frac{3 \log(2/\delta_0)}{\tau^2} \right)^{\log^* |X|} \right)\]

Proof. We begin by arguing that, for $d > 1$, $r_{\text{Median}}_{\rho, d, \tau, \delta}$ has sample complexity $n_m(2n_{[\log, d]} + n_h + 4n_{sq})$. First, observe that Line 9 of Algorithm 3 is the only line that uses the sample \( s \), and it uses \( s \) to generate a sample \( \tilde{m} \) of size \( |\tilde{s}|/n_m \) from \( D_m \). The remaining subroutines use subsamples from \( \tilde{m} \). Therefore, if the sample complexity of the remaining subroutines is bounded by some value \( N \), then $r_{\text{Median}}_{\rho, d, \tau, \delta}$ will have sample complexity \( Nn_m \). We now consider the sequence of subroutines and their respective complexities.

1. Line 13 $r_{\text{Median}}_{\rho, [\log, d], \tau, \delta}$ requires $n_{[\log, d]}$ examples from $D_{[\log, d]}$. Line 11 generates an example from $D_{[\log, d]}$ from 2 examples from $D_m$, and so the call to $r_{\text{Median}}_{\rho, [\log, d], \tau, \delta}$ at Line 13 contributes $2n_{[\log, d]}$ to the sample complexity.

2. Line 14 and Line 19 $r_{\text{HeavyHitters}}_{\rho_0, v, e}$ requires $n_h$ examples from $D_m$.

3. Line 22 the at most 3 calls to $r_{\text{STAT}}_{\rho_0, \tau, \phi_0}$ require $3n_{sq}$ examples from $D_m$.

4. Line 23 $r_{\text{STAT}}_{\rho_0, \tau, \phi_0}$ requires $n_{sq}$ examples from $D_m$.

Therefore $r_{\text{Median}}_{\rho, d, \tau, \delta}$ uses $n = n_m(2n_{[\log, d]} + n_h + 4n_{sq})$ examples from $D$. In the base case, the entire contribution to the sample complexity comes from the call to $\text{Median}_{\tau, \delta_0}$, which requires $n_m$ examples from $D_1$. Unrolling the recursion, we have
\[
n \in O \left( (2n_m)^{\log^* |X|}(n_h + n_{sq}) \right) \\
\in O \left( \frac{1}{\tau^2(\rho - \delta)^2} \cdot \left( \frac{3 \log(2/\delta_0)}{\tau^2} \right)^{\log^* |X|} \right).
\]

Lemma 4.6 (Accuracy). Let $\rho, \tau, \delta \in [0, 1]$ and let $n$ denote the sample complexity proved in Lemma 4.5. Let $\tilde{s}$ be a sample of elements drawn i.i.d. from $D$ such that $|\tilde{s}| \in \Omega(n)$. Then $r_{\text{Median}}(\tilde{s})$ returns a $\tau$-approximate median of $D$ except with probability $\delta$.

Proof. First, we prove that $r_{\text{Median}}(\tilde{s})$ returns a $\tau$-approximate median of $D$, conditioned on the success of all recursive calls and subroutines. We proceed inductively. In the base case we have that $|X| = 2$, and therefore at least one of the two elements in $X$ must be a $\tau$-approximate median. The statistical query performed in line 6 of Algorithm 3 uses sample $\tilde{s}$ to estimate the fraction of $D_1$ supported on 0, to within tolerance $\tau/2$, so long as $|\tilde{s}| \geq n_m$. This holds from Lemma 4.5 and so a $\tau$-approximate median for $D_1$ is returned in the base case.

It remains to show that if a $\tau$-approximate median for $D_{[\log, d]}$ is returned at Line 13 of Algorithm 3 that a $\tau$-approximate median for $D$ is returned. We first note that, except with probability $\delta_0 \cdot |\tilde{s}|/n_m$, all elements of $\tilde{m}$ are $\tau$-approximate medians of $D$. To generate the sample supplied to $r_{\text{Median}}$ at Line 13, we pair up the elements of $\tilde{m}$ to obtain the $|\tilde{s}|/(2n_m)$, which denote the longest prefix on which a pair of elements from $\tilde{m}$ agree. Then $\tilde{s}_{\tau m}$ constitutes a sample of size $n_{[\log, d]}$ drawn i.i.d. from $D_{[\log, d]}$ and by inductive assumption the call to $r_{\text{Median}}$ at Line 13 returns a $\tau$-approximate median of $D_{[\log, d]}$. Therefore we have that $\Pr_{x_1, x_2 \sim D_m}[x_1|\ell = x_2|\ell] \geq 1/2 - \tau$ and $\Pr_{x_1, x_2 \sim D_m}[x_1|\ell + 1 = x_2|\ell + 1] < 1/2 + \tau$. It follows that there must exist a prefix $s$ of length $\ell$ such that $\Pr_{S \sim D}[x|\ell = s] \geq 1/4$. If $\ell = d$, then $x_1 = x$, and so any prefix $s$ such that $\Pr_{x \sim D}[x|\ell = s] \geq 1/4$ is a $3/8$-median of $D^m$ and therefore a $\tau$-median of $D$. In this case $s$ is returned at Line 21.
For the remainder of the proof, we assume \( \ell < d \). We argue that there must exist a prefix \( s \) of length \( \ell \) or \( \ell + 1 \) for which \( 1/4 \leq \Pr_{x \sim D^m}[x|_s = s] \leq 3/4 \). We already have that there exists a prefix \( s \) of length \( \ell \) such that \( \Pr_{x \sim D^m}[x|_s = s] \geq 1/4 \). Suppose that \( \Pr_{x \sim D^m}[x|_s = s] > 3/4 \). Now suppose that one of \(|s|0\) or \(|s|1\) had probability greater than 3/4 under \( D^m \). Then it must be the case that \( \Pr_{x_1, x_2 \sim D^m}[x|_{01} = x_{2|1}] > 9/16 \), and so \( \Pr_{x \sim D^{|\log d|}}[x|_{1} \leq s] < 1 - 9/16 = 7/16 \), contradicting that \( \ell \) is a \( \tau \)-approximate median of \( D_{|\log d|} \). So both \(|s|0\) and \(|s|1\) must have probability less than 3/4 under \( D^m \). Because \( s \) has probability at least 3/4, it follows that at least one of \(|s|0\) and \(|s|1\) must have probability at least 1/4 under \( D^m \), and so we have that there exists a prefix \( s' \) of length \( \ell + 1 \) such that \( 1/4 \leq \Pr_{x \sim D^m}[x|_{1\ell} = s'] \leq 3/4 \).

Now that we have the existence of such a prefix, we will argue that when the loop of Line 22 terminates, \( s \) is a prefix satisfying

\[
\frac{1}{4} - \tau \leq \Pr_{x \sim D^m}[x|_s = s] \leq \frac{3}{4} + \tau.
\]

Observe that the calls to \( r_{\text{HeavyHitters}} \) at Line 10 and Line 19 identify a common prefix \( s \) such that \( \Pr_{x \sim D^m}[x|_s = s] \geq 1/4 \). This follows from taking \( v = 5/16 \), \( \epsilon = 1/16 \), and the fact that the sample \( s_\ell \) and \( s_{\ell+1} \) constitute i.i.d. samples of size \( n_h \) drawn from \( D^m_\ell \) and \( D^m_{\ell+1} \) respectively (where we use \( D^m_\ell \) to indicate the distribution induced by sampling from \( D^m \) and returning only the first \( \ell \) bits). Then we have from the proof of Lemma 3.3 that all \( v - \epsilon = 1/4 \)-heavy hitters from \( D^m_\ell \) and \( D^m_{\ell+1} \) are contained in the set \( V \). The loop beginning at Line 22 will use reproducible statistical queries to estimate the probability of each \( v \in V \) under \( D^m_{\ell+1} \). If the estimated probability \( p_v \in [1/4, 3/4] \), then \( v \) is stored in \( s \), and so the last such string visited by the loop is the value of \( s \) upon termination.

Now we show that if \( s_0 = s|0 \cdots 0 \) is returned at Line 35 then it is a \( \tau \)-approximate median of \( D \), otherwise \( s_1 = s|1 \cdots 1 \) is a \( \tau \)-approximate median. Conceptually, we partition the domain \( X \) into three sets:

1. \( C_{s_0} = \{ x \in X : x < s_0 \} \)
2. \( C_{s} = \{ x \in X : s_0 \leq x \leq s_1 \} \)
3. \( C_{s_1} = \{ x \in X : x > s_1 \} \)

Because \( s \) satisfies \( 1/4 - \tau \leq \Pr_{x \sim D^m}[x|_s = s] \leq 3/4 + \tau \), it must be the case that \( D^m \) assigns probability mass at least \( 1/4 - \tau \) to the union \( C_{s_0} \cup C_{s_1} \). Then it holds that at least one of \( C_{s_0} \) and \( C_{s_1} \) is assigned probability mass at least \( 1/8 - \tau/2 \). The statistical query made at Line 33 estimates the probability mass assigned to \( C_{s_0} \) by \( D^m \) to within tolerance \( \tau \), so if \( s_0 \) is returned, it holds that \( \Pr_{x \sim D^m}|x < s_0| \geq 1/8 - 3\tau \). Because we know \( \Pr_{x \sim D^m}|x \in C_s| \geq 1/4 - \tau \), we then also have that \( \Pr_{x \sim D^m}|x \geq s_0| \geq 1/4 - \tau \). Because \( D^m \) is a distribution over \( \tau \)-approximate medians of \( D \), we have that \( s_0 \) is a \( \tau \)-approximate median of \( D \) as desired. If \( s_0 \) is not returned, then it must be the case that \( \Pr_{x \sim D^m}|x > s_1| \geq 1/8 - 3\tau \), and a similar argument shows that \( s_1 \) must be a \( \tau \)-approximate median of \( D \).

Finally, we argue that all recursive calls and subroutines are successful, except with probability \( \delta \). Failures can occur exclusively at the following calls.

- Line 9 the \( \log^* |X| \cdot |S|/(n_m) \) calls to \( \text{Median}_{\tau, \delta_0} \)
- Line 13 the \( \log^* |X| \) recursive calls to \( \text{Median}_{\rho, |\log d|, \tau, \delta} \)
- Line 16 and Line 19 the \( 2 \log^* |X| \) calls to \( \text{rHeavyHitters}_{\rho_0, v, \epsilon} \)
- Line 22 the \( (\text{at most}) 4 \log^* |X| \) calls to \( \text{rSTAT}_{\rho_0, \tau, \phi_v} \)
- Line 33 the \( \log^* |X| \) calls to \( \text{rSTAT}_{\rho_0, \tau, \phi_{\delta_0}} \)

Calls to \( \text{Median}_{\tau, \delta_0} \) dominate the total failure probability, and so taking \( \delta_0 \in O(\frac{\delta}{|S| \log^* |X|}) \) suffices to achieve failure probability \( \delta \).

**Lemma 4.7 (Reproducibility).** Let \( \rho, \tau, \delta \in [0, 1] \) and let \( n \) denote the sample complexity proved in Lemma 4.6. Let \( \tilde{s} \) be a sample of \( O(n) \) elements drawn i.i.d. from \( D \). Then \( \text{Median}_{\rho, d, \tau, \delta} \) is \( \rho \)-reproducible.
Proof. We prove the lemma by inductive argument. First, we observe that reproducibility of the value returned in the base case depends only on the value \(p_0 \leftarrow \text{rSTAT}_{\rho_0, \tau, \phi} (\vec{s})\) in Line 3. Therefore, reproducibility in the base case follows from the \(\rho_0\)-reproducibility of \(\text{rSTAT}_{\rho_0, \tau, \phi}\).

We now argue that if the \(i + 1\)th recursive call is \(\rho\)-reproducible, that the \(i\)th recursive call is \((\rho + 5\rho_0)\)-reproducible.

Two parallel executions of the \(i\)th level of recursion, given samples \(\vec{s}_1\) and \(\vec{s}_2\) drawn i.i.d. from the same distribution \(D\), will produce the same output so long as the following values are the same:

1. \(\ell \leftarrow \text{rMedian}_{\rho, d, \tau, \phi} (\vec{s}_{rm})\) at Line 18
2. \(V \leftarrow \text{rHeavyHitters}_{\rho_0, v, \epsilon} (\vec{s}_\ell)\) at Line 16
3. \(V \leftarrow V \cup \text{rHeavyHitters}_{\rho_0, v, \epsilon} (\vec{s}_{\ell + 1})\) at Line 19
4. \(s \leftarrow \text{rSTAT}_{\rho_0, \tau, \phi} (\vec{s}_{meds})\) when the loop at Line 22 terminates
5. \(p_{s_0} \leftarrow \text{rSTAT}_{\rho_0, \tau, \phi} (\vec{s}_{s_0})\) at Line 8

produce the same value. We have that 1 holds by inductive assumption.

Conditioning on 1, the calls to \(\text{rHeavyHitters}_{\rho_0, v, \epsilon}\) are made on samples drawn i.i.d. from the same distribution, and so the \(\rho_0\)-reproducibility of \(\text{rHeavyHitters}_{\rho_0, v, \epsilon}\) guarantees that \(V\) contains the same list of heavy-hitters in both runs except with probability \(2\rho\).

Conditioning on both 1 and 2, it follows that the loop at Line 22 iterates over the same strings \(V\), and so both runs make the same sequence of statistical queries \(\text{rSTAT}_{\tau, \rho_0, \phi}\). From conditioning on 2, and the values of \(v\) and \(\epsilon\), we have that \(|V| \leq 3\), and so the \(\rho_0\)-reproducibility of \(\text{rSTAT}_{\tau, \rho_0, \phi}\) gives us that sequence of values \(p_v \leftarrow \text{rSTAT}_{\rho_0, \tau, \phi} (\vec{s}_q)\) is the same in both runs, except with probability \(3\rho_0\).

Finally, conditioning on 1, 2, and 3, the values of \(s_0\) and \(s_1\) are the same across both runs, and so the same statistical query \(\text{rSTAT}_{\rho_0, \tau, \phi}\) is made in both runs. Whether \(s_0\) or \(s_1\) is returned depends only on the value \(r_{s_0} \leftarrow \text{rSTAT}_{\tau, \rho_0, \phi} (\vec{s}_{s_0})\), and so the \(\rho_0\)-reproducibility of \(\text{rSTAT}_{\tau, \rho_0, \phi}\) gives us that the same string is returned by both executions. A union bound over all failures of reproducibility then gives us that the \(i\)th recursive call will be \((\rho + 6\rho_0)\)-reproducible.

From Lemma 4.4 we have that no more than \(T = \log^* |\mathcal{X}|\) recursive calls are made by the algorithm. Therefore \(\text{rMedian}_{\rho, d, \tau, \phi}\) is reproducible with parameter \(\rho_0 + 5T\rho_0 \leq 6\rho_0 \log^* |\mathcal{X}| = \rho\).

Theorem 4.8 then follows as a corollary of Lemma 4.5, Lemma 4.6, and Lemma 4.7.

Theorem 4.8 (Reproducible Median). Let \(\tau, \rho \in [0, 1]\) and let \(\delta = \rho/2\). Let \(D\) be a distribution over \(\mathcal{X}\), where \(|\mathcal{X}| = 2^d\). Then \(\text{rMedian}_{\rho, d, \tau, \delta}\) (Algorithm 3) is \(\rho\)-reproducible, outputs a \(\tau\)-approximate median of \(D\) with all but probability \(\delta\), and has sample complexity

\[
n \in \tilde{O}\left(\frac{1}{\tau^2(\rho - \delta)^2} \cdot \frac{3}{\tau^2} \log^* |\mathcal{X}|\right)
\]

5 Learning Halfspaces

In Section 2 we saw how combining a concentration bound with a randomized rounding technique yielded a reproducible algorithm. Specifically, given a statistical query algorithm with an accuracy guarantee (with high probability) on the 1-dimensional space \([0, 1]\), we can construct a reproducible statistical query algorithm using randomized rounding. By sacrificing a small amount of accuracy, our reproducible statistical query algorithm can decide on a canonical return value in \([0, 1]\).

In this section, we extend this argument from \(\mathbb{R}\) to \(\mathbb{R}^d\), by way of an interesting application of a randomized rounding technique from the study of foams [KORW12]. Algorithm 1 in [KORW12] probabilistically constructs a tiling of \(\mathbb{R}^d\) such that every point is rounded to a nearby integer lattice point. This tiling has an additional property that the probability that two points are not rounded to the same point by a constructed tiling is at most linear in their \(l_2\) distance. In the usual PAC-learning setting, there is a simple weak learning algorithm for halfspaces that takes examples \((\vec{x}_i, y_i) \in \mathcal{X} \times \{-1, 1\}\), normalizes them, and returns the halfspace
defined by vector $\sum_i \vec{x}_i \cdot y_i$ [Ser02]. We show a concentration bound on the sum of normalized vectors from a distribution, and then argue that all vectors within the concentration bound are reasonable hypotheses with non-negligible advantage. The combination of this concentration bound and the foam-based rounding scheme yields a reproducible halfspace weak learner rHalfspaceWkL.

However, constructing this foam-based rounding scheme takes expected time that is exponential in the dimension $d$. We give an alternative rounding scheme that randomly translates the integer lattice and rounds points to their nearest translated integer lattice point. This construction yields another reproducible halfspace weak learner $rHalfspaceWkL_{\text{max}}$ with roughly an additional factor of $d$ in the sample complexity, but with polynomial runtime. In Section 6, we show how to combine these reproducible weak learners with a reproducible boosting algorithm, yielding a polynomial-time reproducible strong learner for halfspaces.

### 5.1 Reproducible Halfspace Weak Learner: An Overview

Let $D$ be a distribution over $\mathbb{R}^d$, and let $E_X$ be an example oracle for $D$ and $f$, where $f : \mathbb{R}^d \to \{\pm 1\}$ is a halfspace that goes through the origin. Let $\|\vec{x}\|$ denote the $l_2$ norm of vector $\vec{x}$. We assume that $D$ satisfies a (worst-case) margin assumption with respect to $f$.

**Definition 5.1.** [Margin] Let $D$ be a distribution over $\mathbb{R}^d$. We say $D$ has margin $\tau_f$ with respect to halfspace $f(\vec{x}) \overset{\text{def}}{=} \sign(\vec{w} \cdot \vec{x})$ if $\frac{f(\vec{x})}{\|\vec{x}\|} \geq \tau_f$ for all $x \in \text{supp}(D)$. Additionally, we say $D$ has (worst-case) margin $\tau$ if $\tau = \sup_f \tau_f$.

Our reproducible halfspace weak learner $rHalfspaceWkL$ uses its input to compute an empirical estimation $\vec{z}$ of the expected vector $E_{x \sim D}[\vec{x} \cdot f(x)]$. Then, $rHalfspaceWkL$ uses its randomness to construct a rounding scheme $R$ via Algorithm ConstructFoams. $R$ is used to round our (rescaled) empirical estimation $\vec{z}$, and the resulting vector defines the returned halfspace. The algorithm relies on the margin assumption to ensure that the weak learner’s returned hypothesis is positively correlated with the true halfspace $f$.

**Algorithm 4 rHalfspaceWkL($\vec{z}; r$) /* a $\rho$-reproducible halfspace weak learner */

**Parameters:**
- $\rho$ - desired reproducibility
- $d$ - dimension of halfspace
- $\tau$ - assumed margin
- $\alpha$ - a constant, $\alpha = .05$

**Input:** A sample $\vec{z}$ of $m = \left(\frac{\alpha \rho}{\tau_{2d}}\right)^{1/(1/2-a)}$ examples $(\vec{x}_i, y_i)$ drawn i.i.d. from distribution $D$

**Output:** A hypothesis with advantage $\tau/4$ on $D$ against $f$

\[
\begin{align*}
    k &\leftarrow \frac{1}{m} \frac{8\sqrt{d}}{\tau_{2d}} = 8 \cdot \left(\frac{\rho}{\alpha \rho}\right)^{1/(1/2-a)} \left(\frac{\alpha^2}{\sqrt{d}}\right)^{(1/2+a)/(1/2-a)} \quad /\text{ Scaling factor} \\
    \vec{z} &\leftarrow \sum_{i} \frac{\vec{x}_i}{\|\vec{x}_i\|} \cdot y_i \\
    R &\leftarrow \text{ConstructFoams}(d) \quad /\text{ Rounding scheme } R : \mathbb{R}^d \rightarrow \mathbb{Z}_d \\
    \vec{w} &\leftarrow R(k \cdot \vec{z}) \\
    \text{return} \quad \text{Hypothesis } h(\vec{z}) \overset{\text{def}}{=} \vec{z} \cdot \frac{\vec{w}}{\|\vec{w}\|}
\end{align*}
\]

The subroutine ConstructFoams previously appeared as Algorithm 1 in [KORWT2]. For completeness, we include a description below (Algorithm 5).

---

1. The parameter $\alpha$ is a constant, but we leave it in variable form for convenience in the analysis; we choose $\alpha = .05$ in this proof for clarity of presentation, but one could optimize the choice of $\alpha$ to yield a slightly better sample complexity.
Algorithm 5 ConstructFoams($d$) // Algorithm 1 in [KORW12]
Input: dimension $d$
Output: rounding scheme $R : \mathbb{R}^d \to \mathbb{Z}^d$

Let $f : [0, 1]^d \to \mathbb{R}$ s.t. $f(x_1, \ldots, x_d) \overset{\text{def}}{=} \prod_{i=1}^d (2\sin^2(\pi x_i))$
Let all points in $\mathbb{R}^d$ be unassigned

for stage $t = 1, 2, \ldots$ until all points are assigned do

Uniformly at random sample $Z_t, H_t$ from $[0, 1]^d \times (0, 2^d)$.
Let droplet $D_t$ be the set of points \( \{ x| x \in -Z_t + [0, 1]^d, f(x + Z_t) > H_t \} \).
Let $R$ map all currently unassigned points in $D_t$ to $(0, 0, \ldots, 0)$ and extend this assignment periodically to all integer lattice points.

return $R$

The following is the main result of this section.

Theorem 5.2. Let $D$ be a distribution over $\mathbb{R}^d$, and let $f : \mathbb{R}^d \to \{\pm 1\}$ be a halfspace with margin $\tau$ in $D$.
Then $\text{rHalfspaceWkL}(\vec{z}; \tau)$ is a $(\rho, \tau/4, \rho/4)$-weak learner for halfspaces. That is, Algorithm 4 reproducibly returns a hypothesis $h$ such that, with probability at least $1 - \rho/2$, $\frac{1}{2} \mathbb{E}_{\vec{x} \sim D} h(\vec{x}) f(\vec{x}) \geq \tau/4$, using a sample of size $m = \left( \frac{896\sqrt{\frac{\tau}{\rho}}}{\tau} \right)^{20/9}$.

Proof. Correctness (Advantage): We argue correctness in two parts. First, we show the expected weighted vector $\mathbb{E}_{\vec{x} \sim D} \left[ \frac{\vec{x} \cdot f(\vec{x})}{\|\vec{x}\|} \right]$ defines a halfspace with good advantage (see Lemma 5.8), following the arguments presented in Theorem 3 of [Ser02]. Then, we argue that rounding the empirical weighted vector $\vec{z}$ in Algorithm 4 only slightly rotates the halfspace. By bounding the possible loss in advantage in terms of the amount of rotation (Lemma 5.9), we argue that the rounded halfspace $\vec{w}/\|\vec{w}\|$ also has sizable advantage.

By Lemma 5.8 the expected weighted vector $\mathbb{E}_{\vec{x} \sim D} \left[ \frac{\vec{x} \cdot f(\vec{x})}{\|\vec{x}\|} \right]$ has advantage $\tau/2$ on $D$ and $f$. The martingale-based concentration bound in Corollary 5.11 implies that the distance between $\vec{z}$ and $\mathbb{E}[\vec{z}] = m \cdot \mathbb{E}_{\vec{x} \sim D} \left[ \frac{\vec{x} \cdot f(\vec{x})}{\|\vec{x}\|} \right]$ is less than $4m^{1/2+a}$ with probability at least $1 - e^{-m^{2a}/2}$ for any $a \in (0, 1/2)$ (chosen later). Then, the vector is scaled by $k$ and rounded. Any rounding scheme $R$ randomly generated by ConstructFoams always rounds its input to a point within distance $\sqrt{d}$ (Observation 5.3). Combining, the total distance between vectors $\frac{\vec{w}}{\|\vec{w}\|}$ and $\mathbb{E}[\vec{z}]/\|\mathbb{E}[\vec{z}]\|$ is at most $\frac{4m^{1/2+a} + \sqrt{d}/k}{\|\mathbb{E}[\vec{z}]\|}$.

As $D$ has margin $\tau$ with respect to $f$, for all $\vec{x} \in \text{supp}(D)$, $\frac{\vec{x}}{\|\vec{x}\|} \cdot f(\vec{x})$ has length at least $\tau$ in the direction of the expected weighted vector $\mathbb{E}_{\vec{x} \sim D} \left[ \frac{\vec{x} \cdot f(\vec{x})}{\|\vec{x}\|} \right]$. Thus, $\|\mathbb{E}[\vec{z}]\| \geq \tau m$, and the above quantity is at most $\frac{4m^{1/2+a} + \sqrt{d}/k}{\tau m}$. Simplifying, $\frac{4m^{1/2+a}}{\tau m} = \frac{4}{\tau m^{1/2-a}} = \frac{4\tau}{896\sqrt{\frac{\tau}{\rho}}/\tau} < \tau/8$ and $\frac{\sqrt{d}/k}{\tau m} = \frac{\sqrt{d}}{8\sqrt{\frac{\tau}{\rho}}/\tau} = \tau/8$. By applying Lemma 5.9 with $\theta = \tau/8 + \tau/8 = \tau/4$, we can conclude that $h$ has advantage at least $\tau/2 - \tau/4 = \tau/4$, as desired.

Reproducibility: Let $\vec{z}_1$ and $\vec{z}_2$ denote the empirical sums of vectors $\vec{x}_1 y_i$ from two separate runs of rHalfspaceWkL. It suffices to show that the rounding scheme $R$ constructed by ConstructFoams rounds $k \cdot \vec{z}_1$ and $k \cdot \vec{z}_2$ to the same vector $\vec{w}$ with high probability. The distance between $\vec{z}_1$ and $\vec{z}_2$ is at most $2 \cdot 4m^{1/2+a}$ with probability at least $1 - 2e^{-m^{2a}/2}$, by Corollary 5.11 the triangle inequality, and a union bound. 

\footnote{A dedicated reader may notice that the scaling factor $k$ is subconstant. A possible error may arise if the scaling factor is so small that the halfspace vector $\vec{z} \cdot k$ gets rounded to 0 by the rounding function $R$ (constructed by ConstructFoams). Fortunately, with our choice of parameters, this turns out to not be an issue. The empirical vector sum $\vec{z}$ has norm at least $\tau \cdot m$, where $\tau$ is the margin size and $m$ is the sample complexity. As we have chosen scaling factor $k$ such that $m \cdot k = 8\sqrt{\tau}/\tau^2$, the input given to $R$ has norm at least $8\sqrt{d}/\tau$. Every rounding function $R$ constructed by ConstructFoams rounds its input to a point at distance at most $\sqrt{d}$ away (Observation 5.3), so we can be sure that $R$ never rounds our vector to the zero vector.}
bound. After scaling by $k$, this distance is at most $8km^{1/2+a}$. By Lemma 5.3, the probability that $R$ does not round $k \cdot \vec{z}_1$ and $k \cdot \vec{z}_2$ to same integer lattice point is at most $7 \cdot 8km^{1/2+a}$. Altogether, the reproducibility parameter is at most

$$2e^{-m\alpha/2} + 56km^{1/2+a}.$$  

The second term satisfies $56km^{1/2+a} = 448 \cdot \frac{e}{896} \cdot 1 = \rho/2$, and the first term $2e^{-m\alpha/2} \leq \rho/2$ when $m \geq (2\ln(4/\rho))^{1/(2\alpha)}$. So, as long as $\alpha$ is chosen such that $m = \left(\frac{896\sqrt{2}}{\tau \rho}\right)^{1/(1/2-a)} \geq (2\ln(4/\rho))^{1/(2\alpha)}$, the algorithm is $\rho$-reproducible. This occurs if $\left(\frac{896}{\rho}\right)^{2/(1/2-a)} \geq 2\ln(4/\rho)$, which is true for all values of $\rho \in (0,1)$ when $\alpha = .05$.

**Failure rate:** The algorithm succeeds when the martingale concentration bound holds. So, the failure probability of rHalfspaceWL is at most $e^{-m\alpha/2} \leq \rho/4$.

**Sample complexity:** Plugging in $\alpha = .05$ in the expression $m = \left(\frac{896\sqrt{2}}{\tau \rho}\right)^{1/(1/2-a)}$ yields the conclusion.

\[\square\]

5.2 Reproducible Weak Halfspace Learner – Definitions and Lemmas

5.2.1 Foams-Based Rounding Scheme from [KORW12]

For completeness, we restate relevant results from [KORW12] for our construction.

**Lemma 5.3** (Combining Theorem 1 and Theorem 3 of [KORW12]). Let $R : \mathbb{R}^d \rightarrow \mathbb{Z}^d$ be the randomized rounding scheme constructed by Algorithm 5 (Algorithm 1 in [KORW12]). Let $x, y \in \mathbb{R}^d$, and let $\epsilon \overset{\text{def}}{=} d_2(x, y)$. Then $\Pr[R(x) = R(y)] \geq 1 - O(\epsilon)$, where the probability is over the randomness used in the algorithm.

**Proof.** Theorem 3 of [KORW12] states that $f(\vec{x}) = \Pi_{i=1}^d (2\sin^2(\pi x_i))$ is a proper density function and $f_{[0,1]^d}(|\nabla f, u|) \leq 2 \pi$ for all unit vectors $u$. Theorem 1 of [KORW12] states the following. Let $f$ be a proper density function, and points $x, y \in \mathbb{R}^d$ such that $y = x + \epsilon \cdot u$, where $\epsilon > 0$ and $u$ is a unit vector. Let $N$ denote the number of times the line segment $\overline{xy}$ crosses the boundary between different droplets (potentially mapping to the same integer lattice point) in an execution of ConstructFoams. Then $\mathbb{E}[N] \approx \epsilon \int_{[0,1]^d} |\nabla f, u|$, where the $\approx$ notation is hiding a $W\epsilon^2$ term, where $W > 0$ is a universal constant depending only on $f$. The authors refine this statement ([KORW12], page 24) to show that the $W\epsilon^2$ term can be made arbitrarily small. Combining, $\mathbb{E}[N] \leq 2\pi x + W\epsilon^2 < 6.3 \epsilon$. By Markov’s inequality, $\Pr[N = 0] < 1 - 6.3 \epsilon$.

\[\square\]

**Observation 5.4.** ConstructFoams always outputs a rounding scheme $R$ with the following property: the $l_2$ distance between any vector $\vec{v} \in \mathbb{R}^d$ and $R(\vec{v})$ is at most $\sqrt{d}$.

This follows from noticing that $R$ maps each coordinate of $\vec{v}$ to its floor or ceiling.

**Theorem 5.5** (Runtime of ConstructFoams; [KORW12], page 25). There are universal constants $1 < c < C$ such that Algorithm 6, when run with $f(\vec{x}) = \Pi_{i=1}^d (2\sin^2(\pi x_i))$, takes between $c^d$ and $C^d$ stages except with probability at most $c^{-d}$.

5.2.2 Weak Learning Definitions

**Definition 5.6** (Weak Learning Algorithm (in the Filtering Model)). Let $\mathcal{C}$ be a concept class of functions from domain $\mathcal{X}$ to $\{\pm 1\}$, and let $f \in \mathcal{C}$. Let $D$ be a distribution over $\mathcal{X}$. Let $\mathsf{WL}$ be an algorithm that takes as input a labeled sample $S = \{(x_i, f(x_i))\}_m$ drawn i.i.d. from $D$, and outputs a hypothesis $h : \mathcal{X} \rightarrow [-1,1]$. Then $\mathsf{WL}$ is a $(\gamma, \delta)$-weak learner for $\mathcal{C}$ with sample complexity $m$ if, for all $f, D$, with probability at least $1 - \delta$, $\mathsf{WL}(S)$ outputs a hypothesis $h : \mathcal{X} \rightarrow [-1,1]$ such that $\mathbb{E}_{x \sim D} f(x) h(x) \geq 2\gamma$, where $S$ is a sample of size $|S| = m$ drawn i.i.d. from $D$.

We say a $(\gamma, \delta)$-weak learner has advantage $\gamma$. Equivalently, if a hypothesis $h$ satisfies $\frac{1}{2} \mathbb{E}_{x \sim D} f(x) h(x) \geq \gamma$, then we say $h$ has advantage $\gamma$ (on $D$ and $f$).

\[3\]The constant $\alpha$ can be improved slightly if $\alpha$ is chosen as a function of $\rho$. 

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\textbf{Definition 5.7} (Reproducible Weak Learning Algorithm). Algorithm \texttt{rWkL} is a (\(\rho, \gamma, \delta\))-weaker if \texttt{rWkL} is \(\rho\)-reproducible and a \((\gamma, \delta)\)-weaker learner.

5.2.3 Halfspaces and Their Advantage

\textbf{Definition 5.1.} [Margin] Let \(D\) be a distribution over \(\mathbb{R}^d\). We say \(D\) has margin \(\tau_f\) with respect to halfspace \(f(\mathbf{x}) = \text{sign}(\mathbf{w} \cdot \mathbf{x})\) if \(\frac{\mathbf{w} (\mathbf{x} - \mathbf{w})}{\|\mathbf{w}\|} \geq \tau_f\) for all \(x \in \text{supp}(D)\). Additionally, we say \(D\) has (worst-case) margin \(\tau\) if \(\tau = \sup_f \tau_f\).

\textbf{Lemma 5.8} (Advantage of Expected Weighted Vector Hypothesis [Ser02]). Let \(f(\mathbf{x}) = \text{sign}(\mathbf{w} \cdot \mathbf{x})\) be a halfspace, and let \(D\) be a distribution over \(\mathbb{R}^d\) with margin \(\tau\) with respect to \(f\). Let \(\mathbf{z} = \mathbb{E}_{\mathbf{v} \sim D} \left[ \frac{\mathbf{v}}{\|\mathbf{v}\|} f(\mathbf{v}) \right]\). Then the hypothesis \(h_{\mathbf{z}}(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|} \cdot \mathbf{z}\) has advantage at least \(\tau/2\).

\textbf{Proof.} The advantage of \(h_{\mathbf{z}}\) is \(\frac{1}{2} \mathbb{E}_{\mathbf{v} \sim D} [h_{\mathbf{z}}(\mathbf{v}) f(\mathbf{v})] = \frac{1}{2} \mathbb{E}_{\mathbf{v} \sim D} \left[ \frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} f(\mathbf{v}) \right] = \frac{\mathbf{v}^2}{\|\mathbf{v}\|^2} = \|\mathbf{v}\| \geq \|\mathbf{w}\|\), by the Cauchy-Schwarz inequality. Vector \(\mathbf{z}\) is a convex combination of \(\frac{\mathbf{v}}{\|\mathbf{v}\|}\) terms, for \(\mathbf{v} \in \text{supp}(D)\). By the margin assumption, \(\frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \frac{\mathbf{w}}{\|\mathbf{w}\|} \geq \tau\) for all \(x \in \text{supp}(D)\). Thus, \(\frac{\mathbf{v}^2}{\|\mathbf{v}\|^2} \geq \tau\).

\textbf{Lemma 5.9} (Advantage of Perturbed Halfspaces). Consider a halfspace defined by unit vector \(\mathbf{w}\), and let \(h(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|} \cdot \mathbf{w}\). Assume \(h\) has advantage \(\gamma\), i.e. \(\frac{1}{2} \mathbb{E}_{x \sim D} f(x) h(x) \geq \gamma\). Let \(\mathbf{u}\) be any vector such that \(\|\mathbf{u}\| \leq \theta\), where \(\theta \in [0, \sqrt{3}/2]\). Let perturbed vector \(\mathbf{w}' = \frac{\mathbf{w} + \mathbf{u}}{\|\mathbf{w} + \mathbf{u}\|}\), and let \(h'(\mathbf{x}) = \frac{\mathbf{x}}{\|\mathbf{x}\|} \cdot \mathbf{w}'\). Then \(h'\) has advantage at least \(\gamma - \theta\).

\textbf{Proof.} First, we bound the maximum distance between \(\mathbf{w}\) and \(\mathbf{w}'\). Then, we apply Cauchy-Schwarz to bound the advantage loss. \(\mathbf{w}'\) is constructed by perturbing \(\mathbf{w}\) by a vector \(\mathbf{u}\), and then normalizing to norm 1. \(\mathbf{w}'\) is furthest away from \(\mathbf{w}\) when the vector \(\mathbf{w}'\) is tangent to the ball of radius \(\|\mathbf{u}\|\) around \(\mathbf{w}\). In this case, \(\|\mathbf{w}' - \mathbf{w}\|^2 = (1 - \sqrt{1 - \theta^2})^2 + \theta^2 = 2 - 2\sqrt{1 - \theta^2} < 4\theta^2\). So, \(\|\mathbf{w}' - \mathbf{w}\| < \sqrt{4\theta^2}\). The advantage of \(h'\) is

\[
\frac{1}{2} \mathbb{E}_{x \sim D} \left[ \frac{x}{\|x\|} \cdot \mathbf{w}' \cdot f(x) \right] = \frac{1}{2} \mathbb{E}_{x \sim D} \left[ \frac{x}{\|x\|} \cdot (\mathbf{w} + (\mathbf{w}' - \mathbf{w})) \cdot f(x) \right] = \gamma + \frac{1}{2} (\mathbf{w}' - \mathbf{w}) \cdot \mathbb{E}_{x \sim D} \left[ \frac{x}{\|x\|} \cdot f(x) \right]
\]

By Cauchy-Schwarz, the second term of the right-hand side has magnitude at most \(\sqrt{4\theta^2} \cdot 1/2\), so the advantage of \(h'\) is at least \(\gamma - \theta\).

5.2.4 Concentration Bound on Sum of Normalized Vectors

Let \(D\) be a distribution on \(\mathbb{R}^n\). Let \(\mathbf{v} = \{\mathbf{v}_1, \ldots, \mathbf{v}_T\} \in D^T\) be a random sample of \(T\) vectors from \(D\) with the following properties:

1. \(\mathbb{E}_{\mathbf{v} \in D^T} [\sum_{i=1}^T \mathbf{v}_i] - \mathbb{E}_{\mathbf{v} \in D}[\mathbf{v}] = 0\).
2. \(\forall \mathbf{v} \in D, \|\mathbf{v}\|_2 \leq c\).

\textbf{Lemma 5.10.} Let \(D, \mathbf{v} \in D^T\) satisfy properties (1) and (2) above, and let \(\mathbf{v}_{\leq T} = \sum_{i=1}^T \mathbf{v}_i\). Then for all \(\Delta > 0\),

\[
\mathbb{P}_{\mathbf{v}}[\|\mathbf{v}_{\leq T}\|_2 \geq \sqrt{T} (1 + c/2) + \Delta] \leq e^{-\Delta^2/2c^2T}.
\]

For a proof, see Appendix B.

\textbf{Corollary 5.11.} Let \(D\) be a distribution supported on the unit ball in \(d\) dimensions, and let \(f\) be a halfspace. Let \(S\) be a sample of \(T\) examples \((\mathbf{x}_i, f(\mathbf{x}_i))\) drawn i.i.d. from \(D\), and let \(\hat{z} = \sum_{i} \mathbf{x}_i \cdot f(\mathbf{x}_i)\). Let \(a \in (0, 1/2)\). Then \(\mathbb{P}_{S \sim D}[\|\hat{z} - \mathbb{E}_{\mathbf{v} \sim D}[\mathbf{v} f(\mathbf{v})]\| \geq 4T^{1/2 + a}] \leq e^{-T^{2a}/2}\).
Proof. In order to have $D$ satisfy the properties (1) and (2) above, we must translate $D$ by the expectation $E_{x \sim D} \hat{v}f(\hat{v})$. After this translation, the maximum length of a vector in the support is $c = 2$. Plugging in $\Delta = 2T^{1/2+a}$ and noting $2T^{1/2+a} \geq 2T^{1/2}$ yields the conclusion. \qed

5.3 Coordinate-Based Rounding Scheme

Algorithm $r$HalfspaceStL uses polynomial sample complexity and runs in polynomial time except for subroutine ConstructFoams, which runs in expected exponential time in the dimension $d$ (Theorem 5.5). Next, we consider a simpler rounding scheme that rounds points coordinate-by-coordinate to a randomly shifted integer lattice. This rounding scheme requires tighter concentration bounds, resulting in approximately another factor of $d$ in the sample complexity. In return, it can be constructed by ConstructBoxes and executed in linear time in sample complexity $m$ and dimension $d$.

Algorithm 6 ConstructBoxes($d$) // constructs coordinate-based rounding schemes
Input: dimension $d$
Output: rounding scheme $R : \mathbb{R}^d \rightarrow \mathbb{R}^d$

Uniformly at random draw $Z$ from $[0,1)^d$. Let box $B$ be the set of points \{x|\forall i \in [d], x_i \in [-1/2 + z_i, 1/2 + z_i]\} Let $R$ map all points in $B$ to point $Z$ and extend this assignment periodically by integer lattice points

return $R$

Lemma 5.12. Let $R : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the randomized rounding scheme constructed by ConstructBoxes. Let $\vec{x}, \vec{y} \in \mathbb{R}^d$, and let $\epsilon \overset{\text{def}}{=} d_{l_2}(\vec{x}, \vec{y})$. Then $\Pr[R(\vec{x}) = R(\vec{y})] \geq 1 - d\epsilon$.

Proof. We bound this probability by a crude $l_2$ to $l_1$ distance conversion. If $\vec{x}$ and $\vec{y}$ have $l_2$ distance $\epsilon$, then the distance between $x_i$ and $y_i$ is at most $\epsilon$ for all coordinates $i \in [d]$. The $i$'th coordinate of $\vec{x}$ and $\vec{y}$ are not rounded to the same point with probability $|x_i - y_i|$. By a union bound, $\Pr[R(\vec{x}) = R(\vec{y})] \geq 1 - d\epsilon$. \qed

Observation 5.13. ConstructBoxes always outputs a rounding scheme $R$ with the following property: the ($l_2$) distance between any vector $\vec{v} \in \mathbb{R}^d$ and $R(\vec{v})$ is at most $\sqrt{d}/2$.

This follows from noticing that $R$ maps each coordinate of $\vec{v}$ to value within distance $1/2$.

5.3.1 Reproducible Halfspace Weak Learner using Boxes

Algorithm 7 rHalfspaceWkL\text{box($\vec{z}$, $r$)} // a $\rho$-reproducible halfspace weak learner
Parameters: desired reproducibility $\rho$, dimension $d$, assumed margin $\tau$, constant $a = .1$
Input: A sample $S$ of $m = \left(\frac{64d^{5/2}}{\rho^{1+a}}\right)^{1/(1/2-a)}$ examples $(\vec{x}_i, y_i)$ drawn i.i.d. from distribution $D$
Output: A hypothesis with advantage $\gamma/4$ on $D$ against $f$

$k \leftarrow \frac{4\sqrt{d}}{m^{1/2}} = 4 \cdot \left(\frac{6^{1/2}}{d^{1/2} \rho^{1+a}}\right)^{1/(1/2-a)}$ \hspace{1cm} // Scaling factor
$\vec{z} \leftarrow \sum_{y_i} \vec{y}_i \ divided \text{by} \ |\vec{z}|$
$R \leftarrow \text{ConstructBoxes}(d)$ (Algorithm 6) \hspace{1cm} // Rounding scheme $R : \mathbb{R}^d \rightarrow \mathbb{R}^d$
$\vec{w} \leftarrow R(k \cdot \vec{z})$

return Hypothesis $h(\vec{x}) \overset{\text{def}}{=} \vec{x} \div \vec{w}$

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Theorem 5.14. Let D be a distribution over \(\mathbb{R}^d\), and let \(f : \mathbb{R}^d \to \{\pm 1\}\) be a halfspace with margin \(\tau\) in D. Then \(r_{\text{HalfspaceWKL}}(\delta; \tau)\) is a \((\rho/4, \rho/4)\)-weak learner for halfspaces. That is, Algorithm 3 \(\rho\)-reproducibly returns a hypothesis \(h\) such that, with probability at least \(1 - \rho/2\), \(\Pr_{x \sim D} h(f(x)) \geq \tau/4\), using a sample of size \(m = \left( \frac{64d^{3/2}}{\tau^2 \rho} \right)^{5/2}\).

Proof. Correctness (Advantage): The proof proceeds almost identically to the proof of Theorem 5.2. By Lemma 5.8, the expected weighted vector \(\mathbb{E}_{x \sim D} \left[ \frac{\delta f(x)}{\|f\|} \right]\) has advantage \(\tau/2\) on \(D\) and \(f\). Any rounding scheme \(R\) randomly generated by ConstructBoxes always rounds its input to a point within distance \(\sqrt{d}/2\), so the distance between vectors \(\frac{\delta f(x)}{k \|f\|}\) and \(\frac{\delta f(x)}{\|f\|}\) is at most \(4m^{1/2+\rho} + \sqrt{d}k\).

Simplifying, \(\frac{4m^{1/2+\rho}}{\tau m} = \frac{4}{\tau m^{1/2-\rho}} < \tau/8\) and \(\frac{\sqrt{d}k}{\tau m} = \frac{\sqrt{d}}{\sqrt{m}} = \tau/8\). By applying Lemma 5.9 with \(\theta = \tau/8 + \tau/8\), we can conclude that \(h\) has advantage at least \(\tau/2 - (\tau/8 + \tau/8) = \tau/4\), as desired.

Reproducibility: Let \(z_1\) and \(z_2\) denote the empirical sums of vectors \(\bar{x}_i y_i\) from two separate runs of \(r_{\text{HalfspaceWKL}}\). It suffices to show that the rounding scheme \(R\) constructed by ConstructBoxes rounds \(k \cdot z_1\) and \(k \cdot z_2\) to the same vector \(\bar{w}\) with high probability. The distance between \(z_1\) and \(z_2\) is at most \(2 \cdot 4m^{1/2+\rho}\) with probability at least \(1 - 2e^{-m^{2(1-\rho)/2}}\), by Corollary 5.11 the triangle inequality, and a union bound. After scaling by \(k\), this distance is at most \(8km^{1/2+\rho}\). By Lemma 5.12 the probability that \(R\) does not round \(k \cdot z_1\) and \(k \cdot z_2\) to same integer lattice point is at most \(d \cdot 8km^{1/2+\rho}\). Altogether, the reproducibility parameter is at most \(2e^{-m^{2(1-\rho)/2}} + 8dkm^{1/2+\rho}\).

The second term satisfies \(8dkm^{1/2+\rho} = 8d(km/m^{1/2-a}) = \rho/2\), and the first term \(2e^{-m^{2(1-\rho)/2}} \leq \rho/2\) when \(m \geq (2 \ln(4/\rho))^{1/(2a)}\). So, as long as \(a\) is chosen such that \(m = \left( \frac{64d^{3/2}}{\tau^2 \rho} \right)^{1/(2a)} \geq (2 \ln(4/\rho))^{1/(2a)}\), the algorithm is \(\rho\)-reproducible. This occurs if \(\left( \frac{64}{\rho} \right)^{2a/(1-2a)} \geq 2 \ln(4/\rho)\), which is true for all values of \(\rho \in (0, 1)\) when \(a = .07\). For simpler constants, we use \(a = .1\).

Failure rate: The algorithm succeeds when the martingale concentration bound holds. So, the failure probability of \(r_{\text{HalfspaceWKL}}\) is at most \(e^{-m^{2(1-\rho)/2}} \leq \rho/4\).

Sample complexity: Plugging in \(a = .1\) in the expression \(m = \left( \frac{64d^{3/2}}{\tau^2 \rho} \right)^{1/(2(1-\rho))}\) yields the conclusion. \(\square\)

6 Reproducible Boosting

In this section, we argue that a small modification of the boosting algorithm in [Ser03] is a reproducible boosting algorithm. Given access to a reproducible weak learner, this boosting algorithm \(\rho\)-reproducibly outputs a hypothesis. Boosting algorithms are a natural candidate for constructing reproducible algorithms — many boosting algorithms in the standard PAC-setting are deterministic, and the final classifier returned is often a simple function of the weak learner hypotheses (e.g., a majority vote). Combining this reproducible boosting algorithm with our reproducible halfspace weak learner from Section 5 yields a reproducible strong learner for halfspaces.

Specifically, we modify the smooth boosting algorithm described in [Ser03] in the batch setting, presenting it in the filtering setting [BS07]. This boosting algorithm has three main components, all of which can be made reproducible: (i) checking for termination (via a statistical query), (ii) running the weak learner (reproducible by assumption), and (iii) updating the weighting function (deterministic). The final classifier is a sum of returned weak learner hypotheses. With high probability over two runs, our boosting algorithm \(r_{\text{boost}}\) collects the exact same hypotheses \(h_1, \ldots, h_T\) from its reproducible weak learner.
6.1 Reproducible Boosting Algorithm: An Overview

In smooth boosting algorithms, a “measure” function \( \mu : \mathcal{X} \to [0, 1] \) determines a reweighting of distribution \( D \). The induced reweighted distribution, denoted \( D_{\mu} \), is defined by the probability density function \( D_{\mu}(x) = \mu(x) \cdot D(x)/d(\mu) \), where \( d(\mu) \) is a normalizing factor \( \mathbb{E}_{x \sim D}\mu(x) \). We refer to \( d(\mu) \) as the density of measure \( \mu \). A sample \( \vec{s} \) is drawn from \( D_{\mu} \) and passed to the weak learner \( \text{rWkL} \). Sampling from \( D_{\mu} \) using example oracle \( \text{EX} \) is done by rejection sampling — draw a sample \((x, y)\) from \( \text{EX} \) and a random \( b \in \epsilon [0, 1] \); if \( r \leq \mu(x) \), keep \((x, y)\); otherwise, reject \( x \) and loop until we keep \((x, y)\). On expectation, we require \( m/d(\mu) \) examples from \( D \) to sample \( m \) examples from \( D_{\mu} \).

At the beginning of the algorithm, \( \mu(x) = 1 \) for all \( x \in \text{supp}(D) \). Weak learner hypotheses \( h_t \) are used to modify update \( \mu \) (and thus \( D_{\mu} \)) for future weak learner queries. The algorithm terminates when the density \( d(\mu) \) drops below the desired accuracy parameter \( \epsilon \) — at this point, the majority vote hypothesis \( \mathbf{h} = \text{sign}(\sum \mu_t h_t) \) has accuracy at least \( 1 - \epsilon \) over \( D \).

More specifically, we define \( \mu_{t+1}(x) = M(g_t(x)) \) using a base measure function \( M : \mathbb{R} \to [0, 1] \) and score function \( g : \mathcal{X} \to \mathbb{R} \). As in \cite{Ser03}, we use a capped exponential function as our base measure function \( M(a) = \begin{cases} 1 & a \leq 0 \\ (1 - \gamma)^{a/2} & a > 0 \end{cases} \). The score function is \( g_t(x) = \sum_{i=1}^t(h_i(x)f(x) - \theta) \), where \( \theta < \gamma \) is chosen as a function of \( \gamma \).

\begin{algorithm}
\caption{\texttt{rBoost}\textsuperscript{rWkL}(\vec{s}; r) \-comment{a \( \rho \)-reproducible boosting algorithm}}
\begin{algorithmic}
\Statex \textbf{Input:} A sample \( \vec{s} \) of \( m \) examples \((\vec{x}_i, y_i)\) drawn i.i.d. from distribution \( D \).
\Statex Access to reproducible weak learner \( \text{rWkL} \) with advantage \( \gamma \) and sample complexity \( m_{\text{rWkL}} \).
\Statex Parameters: desired reproducibility \( \rho \), accuracy \( \epsilon \), constant \( \theta \) \( = \gamma/(2 + \gamma) \), round complexity \( T = O(1/\epsilon \gamma^2) \)
\Statex \textbf{Output:} A hypothesis \( \mathbf{h} = \text{sign}(\sum_{i=1}^T \mu_i h_i) \), where the \( h_i \)’s are weak learner hypotheses.
\Statex \begin{align*}
g_0(x) & \equiv 0 \\
\mu_1(x) & \equiv M(g_0) = 1 \\
t & \leftarrow 0
\end{align*}
\Statex \textbf{while} 1 \textbf{do}
\Statex \begin{align*}
t & \leftarrow t + 1 \\
D_{\mu_t}(x) & \equiv \mu_t(x) \cdot D(x)/d(\mu_t) \quad \text{ // Reweighted distribution}
\end{align*}
\Statex \begin{align*}
\vec{s}_1 & \leftarrow \widehat{\mathcal{O}}(m_{\text{rWkL}}/\epsilon) \text{ fresh examples from } \vec{s} \\
\vec{s}_{\text{rWkL}} & \leftarrow \text{RejectionSampler}(\vec{s}_1, m_{\text{rWkL}}, \mu; r_1) \\
\text{Hypothesis } h_t & \leftarrow \text{rWkL}(\vec{s}_{\text{rWkL}}; r_2) \\
g_t(x) & \equiv g_{t-1}(x) + h_t(x)f(x) - \theta \\
\mu_{t+1}(x) & \equiv M(g_t(x)) \quad \text{ // Reweight distribution using } h_t
\end{align*}
\Statex \begin{align*}
\vec{s}_2 & \leftarrow \widehat{\mathcal{O}}\left(\frac{1}{\rho \epsilon \gamma^2}\right) \text{ fresh examples from } \vec{s} \\
\text{if } \text{rSTAT}_{\tau, \rho, \phi}(\vec{s}_2; r_3) \leq 2\epsilon/3 \text{ then} \\
\quad \text{ Exit while loop}
\end{align*}
\Statex \textbf{return} \( \mathbf{h} \leftarrow \text{sign}(\sum_t \mu_t)$
\end{algorithmic}
\end{algorithm}
Algorithm 9 RejectionSampler($\bar{s}_{\text{all}}, m_{\text{target}}, \mu; r$)  // draw a sample from distribution $D_\mu$
Input: sample $\bar{s}_{\text{all}}$ drawn i.i.d. from distribution $D$, target size of output sample $m_{\text{target}} \in |\bar{s}_{\text{all}}|$, and description of measure function $\mu: \mathcal{X} \rightarrow [0, 1]$.
Output: $\bot$ or a sample $\bar{s}_{\text{kept}}$ of size $|\bar{s}_{\text{kept}}| = m_{\text{target}}$

$\bar{s}_{\text{kept}} \leftarrow \emptyset$

for $i = 1$ to $i = |\bar{s}_{\text{all}}|$ do

Use randomness $r$ to randomly pick a $b \in [0, 1]$

if $\mu(x_i) \geq b$ then

$\bar{s}_{\text{kept}} \leftarrow \bar{s}_{\text{kept}} \cup \{(x_i, y_i)\}$ // Reject $(x_i, y_i)$ w. p. $1 - \mu(x)$

if $|\bar{s}_{\text{kept}}| = m_{\text{target}}$ then

return $\bar{s}_{\text{kept}}$ // Add example $(x_i, y_i)$ to $\bar{s}_{\text{kept}}$

end if

end if

end for

return $\bot$ // Ran out of fresh samples in $\bar{s}_{\text{all}}$

A subtle note is that this boosting algorithm must precisely manage its sample $\bar{s}$ and random string $r$ when invoking subroutines. In order to utilize the reproducibility of subroutines (e.g. $rWkL$), the boosting algorithm needs to ensure that it uses random bits from the same position in $r$. A first-come-first-serve approach to managing $r$ (i.e. each subroutine uses only the amount of randomness it needs) fails immediately for $rBoost$ — the amount of randomness RejectionSampler needs is dependent on the sample, so the next subroutine (in this case, $rWkL$) may not be using the same randomness across two (same-randomness $r$) runs of $rBoost$.

If one can precisely upper bound the amount of randomness needed for each of $L$ subroutines, then $r$ can be split into chunks $r_1 || r_2 || \ldots || r_L$, avoiding any desynchronization issues. Alternatively, one can split $r$ into $L$ equally long random strings by only using bits in positions equivalent to $l$ mod $L$ for subroutine $l \in [L]$.

6.2 Analysis of $rBoost$ (Algorithm 8)

As before, function $f$ in concept class $C$ is a function from domain $\mathcal{X}$ to $\{\pm 1\}$. $D$ is a distribution over $\mathcal{X}$.

**Theorem 6.1** (Reproducible Boosting). Let $\epsilon > 0, \rho > 0$. Let $rWkL$ be a $(\rho r, \gamma, \delta_{rWkL})$-weak learner. Then $rBoost_{rWkL}(\bar{s}; r)$ is $\rho$-reproducible and with probability at least $1 - \rho$, outputs a hypothesis $h$ such that $Pr_{x \sim D}[h(x) = f(x)] \geq 1 - \epsilon$. $rBoost$ runs for $T = O(1/(\epsilon^2 \gamma_{rWkL}))$ rounds and uses $\tilde{O} \left( \frac{\text{sample complexity}}{\epsilon^2 \gamma_{rWkL}} + \frac{1}{\rho \epsilon \gamma_{rWkL}} \right)$ samples, where the $\tilde{O}$ notation hides $\log(1/(\rho \epsilon \gamma_{rWkL}))$ factors and $m_{rWkL}(\rho / (6T))$ denotes the sample complexity of $rWkL$ with reproducibility parameter $\rho / (6T)$.

For readability, we break the proof into components for round complexity, correctness, reproducibility, sample complexity, and failure probability.

**Proof.**

**Round Complexity:** Theorem 3 in Ser03 gives a $T = O(1/(\epsilon^2 \gamma_{rWkL}))$ round complexity bound for this boosting algorithm in the batch setting. Analogous arguments hold in the filtering setting, so we defer to Ser03 for brevity.

**Correctness:** Similarly, since reproducible weak learner $rWkL$ satisfies the definitions of a weak learner, the correctness arguments in Ser03 also hold. A small difference is the termination condition — rather than terminate when the measure satisfies $d(\mu) < \epsilon$, our algorithm terminates when the density estimated by $rSTAT$ is less than $2 \epsilon / 3$. We run $rSTAT$ on query $\phi(x) = \mu(x)$ with tolerance parameter $\epsilon / 3$. Thus, when the $rBoost$ terminates, $d(\mu) < \epsilon$.

**Reproducibility:** We show this boosting algorithm not only reproducibly outputs the same hypothesis $h$, but that each returned weak learner hypothesis $h_t$ is identical across two runs of the boosting algorithms (using the same randomness $r$) with high probability. The reweighted distribution $D_{\mu_t}$ depends only on the previous weak learner hypotheses $h_1, \ldots, h_{t-1}$, so the only possibilities for loss of reproducibility are: (i) returning $\bot$ while rejection sampling from $D_{\mu}$; (ii) running the reproducible weak learner; and (iii)
using a statistical query to decide to exit the while loop. We note that our choice of parameters adds non-reproducibility at most $\rho/(3T)$ for each and apply a union bound over at most $T$ rounds of boosting.

1. By Lemma 6.2, $O(\frac{m_{rWkL}(\rho/(6T))}{\epsilon} \cdot \log(T/\rho))$ examples suffice to guarantee $\text{RejectionSampler}$ outputs $\bot$ with probability at most $\rho/(6T)$. Union bounding over two runs, this is at most $\rho/(3T)$.

2. By Lemma 6.4 running $rWkL$ with reproducibility parameter $\rho/(6T)$ will add a $\rho/(3T)$ contribution to the non-reproducibility.

3. We run $rSTAT$ with reproducibility parameter $\rho/(3T)$.

**Sample Complexity:** There are two contributions to the sample complexity: samples used for the weak learner $rWkL$, and samples used by $rSTAT$ to estimate the density of measure $\mu$. Fresh samples are used for each of $T$ rounds of boosting. Together, by Theorem 6.3 and the definition of $s_1$ in Algorithm 8, the sample complexity is

$$O\left(T \cdot \left(\frac{m_{rWkL}(\rho/(6T))}{\epsilon} \cdot \log(T/\rho) + \frac{\log(T/\rho)}{(\epsilon^2)(\rho)^2}\right)\right) = \tilde{O}\left(\frac{m_{rWkL}(\rho/(6T))}{\epsilon^2\gamma^2} + \frac{1}{\rho^2\epsilon^2\gamma^2}\right)$$

where the $\tilde{O}$ notation hides $\log(1/(\rho\epsilon^2))$ factors and $m_{rWkL}(\rho/(6T))$ denotes the sample complexity of $rWkL$ with reproducibility parameter $\rho\gamma^2$.

**Failure Probability:** Assuming the weak learner returns correct hypotheses when it is reproducible, the boosting algorithm $rBoost$ is correct when it is reproducible, so the failure probability is bounded above by $\rho^3$. \hfill \qed

### 6.3 Rejection Sampling Lemmas

Next, we show that reproducibility composes well with rejection sampling throughout the execution of $rBoost$.

**Lemma 6.2 (Failure Rate of RejectionSampler).** Let measure $\mu$ have density $d(\mu) \geq \epsilon/3$. Let $s_{\text{all}}$ be a sample drawn i.i.d. from distribution $D$. If $|s_{\text{all}}| \geq \frac{3m_{\text{target}}}{\epsilon} \cdot \log(1/\delta)$, then $\text{RejectionSampler}(s_{\text{all}}, m_{\text{target}}, \mu; r)$ outputs $\bot$ with probability at most $\delta$.

**Proof.** The probability $\text{RejectionSampler}$ outputs $\bot$ is precisely the probability a binomial random variable $X \sim B(|s_{\text{all}}|, d(\mu))$ is at most $m_{\text{target}}$. By a Chernoff bound, $\Pr[X \leq (1-.5)|s_{\text{all}}| \cdot d(\mu)/8] \leq \exp(-|s_{\text{all}}| \cdot d(\mu)/24)$. Thus, $\Pr[X \leq m_{\text{target}}] \leq \delta$. \hfill \qed

**Remark 6.3.** The following is a justification of why we may assume $d(\mu) \geq \epsilon/3$ in the previous lemma.

When $\text{RejectionSampler}$ is first called in round 1 of $rBoost$, $\mu(x) = 1$ for all $x$, so $d(\mu) = 1$. In subsequent rounds $t \geq 2$, $\text{RejectionSampler}$ is only called if, in previous round $t-1$, $rSTAT$ estimated $d(\mu)$ to be at least $2\epsilon/3$. $rSTAT$ is run with tolerance $\epsilon/3$, so $d(\mu) \geq \epsilon/3$ whenever $rSTAT$ succeeds. Whenever we apply the above lemma, we are assuming the success of previous subroutines (by keeping track of and union bounding over their error).

The following lemma shows that rejection sampling before running a reproducible algorithm only increases the non-reproducibility $\rho$ by a factor of 2. To be precise, we let $p$ denote the probability the rejection sampler returns $\bot$. However, when we apply this Lemma in the proof of Theorem 6.1 we will have already accounted for this probability.

**Lemma 6.4 (Composing Reproducible Algorithms with Rejection Sampling).** Let $A(\bar{s}, r)$ be a $\rho$-reproducible algorithm with sample complexity in Let $\mu : X \rightarrow [0,1]$. Consider $B$, the algorithm defined by composing $\text{RejectionSampler}(\bar{s}, m, \mu; r')$ with $A(\bar{s}, r)$. Let $q$ be the probability that $\text{RejectionSampler}$ returns $\bot$. Then $B$ is a $2q + 2\rho$-reproducible algorithm.

\footnote{A more precise sample complexity statement in terms of the failure probability $\delta$ can be obtained by unboxing the error probabilities. The algorithm can fail if $\text{RejectionSampler}$ outputs $\bot$, if $rWkL$ fails, and if $rSTAT$ fails. Bounding each of these quantities by $\delta/(3T)$ ensures that the $rBoost$ has failure rate $\delta$.}
Proof. Since \( \mathcal{A} \) is \( \rho \)-reproducible, \( \Pr_{\vec{s}_1, \vec{s}_2,r} [\mathcal{A}(\vec{s}_1; r) = \mathcal{A}(\vec{s}_2; r)] \geq 1 - \rho \). However, the rejection sampling is done with correlated randomness, so \( \vec{s}_1 \) and \( \vec{s}_2 \) are not independent. Consider an imaginary third run of algorithm \( \mathcal{A}(\vec{s}_3; r) \), where \( \vec{s}_3 \) is drawn from \( D_\rho \) using separate randomness. We will use a triangle-inequality-style argument (and a union bound) to derive the conclusion. Conditioned on \( \text{RejectionSampler} \) not returning \( \perp \), algorithm \( \mathcal{B}(\vec{s}_1'; r)|||r) \) returns the same result as \( \mathcal{A}(\vec{s}_3, r) \) (when both algorithms use randomness \( r \) for the execution of \( \mathcal{A} \)) with probability at least \( 1 - \rho \). The same statement holds for the second run \( \mathcal{B}(\vec{s}_2'; r)|||r) \). Thus, 
\[
\Pr_{\vec{s}_1', \vec{s}_2',r'|||r} [\mathcal{B}(\vec{s}_1'; r)|||r) = \mathcal{B}(\vec{s}_2'; r)|||r)] \text{ neither output } \perp \geq 1 - 2\rho.
\]

Finally, \( \mathcal{B} \) may fail to be reproducible if either \( \text{RejectionSampler} \) call returns \( \perp \), so we union bound over this additional \( 2\rho \) probability. \( \square \)

6.4 Reproducible Strong Halfspace Learner

We give two reproducible strong learners for halfspaces by combining boosting algorithm \( \text{rBoost} \) with reproducible weak halfspace learners \( \text{rHalfspaceWkL} \) and \( \text{rHalfspaceWkLbox} \).

Corollary 6.5. Let \( D \) be a distribution over \( \mathbb{R}^d \), and let \( f : \mathbb{R}^d \rightarrow \{\pm 1\} \) be a halfspace with margin \( \tau \) in \( D \). Let \( \epsilon > 0 \). Then

- Algorithm \( \text{rBoost} \) run with weak learner \( \text{rHalfspaceWkL} \) \( \rho \)-reproducibly returns a hypothesis \( h \) such that, with probability at least \( 1 - \rho \), \( \Pr_{\vec{x} \sim D} [h(\vec{x}) = f(\vec{x})] \geq 1 - \epsilon \), using a sample of size \( \tilde{O} \left( \frac{d^{10/9}}{\tau^{76/9} \rho^{20/9} \rho^{28/9}} \right) \).
- Algorithm \( \text{rBoost} \) run with weak learner \( \text{rHalfspaceWkLbox} \) \( \rho \)-reproducibly returns a hypothesis \( h \) such that, with probability at least \( 1 - \rho \), \( \Pr_{\vec{x} \sim D} [h(\vec{x}) = f(\vec{x})] \geq 1 - \epsilon \), using a sample of size \( \tilde{O} \left( \frac{d^{15/4}}{\tau^{76/9} \rho^{20/9} \rho^{28/9}} \right) \).

Proof. For the first strong learner, we compose Theorem 6.1 with Theorem 6.2 \( \text{rHalfspaceWkL} \) has advantage \( \gamma = \tau/4 \), so \( \text{rBoost} \) has round complexity \( T = O(1/(\epsilon \gamma^2)) = O(1/(\epsilon \tau^2)) \). \( \text{rBoost} \) runs \( \text{rHalfspaceWkL} \) with parameter \( \rho_{\text{wkl}} = \rho/6T \), so the sample complexity \( m_{\text{rHalfspaceWkL}} \) is \( \tilde{O} \left( \frac{d^{10/9}}{\tau^{76/9} \rho^{20/9} \rho^{28/9}} \right) \). Thus, the sample complexity for the boosting algorithm is
\[
\tilde{O} \left( \frac{d^{10/9}}{\tau^{76/9} \rho^{20/9} \rho^{28/9}} \right).
\]

For the second strong learner, we compose Theorem 6.1 with Theorem 6.14 As before, \( \text{rBoost} \) has round complexity \( T = O(1/(\epsilon \tau^2)) \) and runs \( \text{rHalfspaceWkLbox} \) with parameter \( \rho_{\text{wkl}} = \rho/6T \). The sample complexity \( m_{\text{rHalfspaceWkLbox}} \) is \( \tilde{O} \left( \left( \frac{d^{3/2}}{\tau^2 \rho_{\text{wkl}}} \right)^{5/2} \right) = \left( \frac{d^{3/2}}{\tau^2 \rho} \right)^{5/2} \). Thus, the sample complexity for the boosting algorithm is
\[
\tilde{O} \left( \frac{m_{\text{rHalfspaceWkLbox}}}{\epsilon^2 \tau^2} + \frac{1}{\epsilon^2 \tau^2} \right) = \tilde{O} \left( \frac{1}{\epsilon^2 \tau^2} \left( \frac{d^{3/2}}{\tau^2 \rho} \right)^{5/2} \right) = \tilde{O} \left( \frac{d^{15/4}}{\tau^{76/9} \rho^{20/9} \rho^{28/9}} \right). \quad \square
\]

Remark 6.6. \( \text{rHalfspaceWkLbox} \) can be run in time polynomial in the input parameters, so the strong learner obtained by running \( \text{rBoost} \) with weak learner \( \text{rHalfspaceWkLbox} \) is a poly(\( 1/\epsilon, 1/\rho, 1/\tau, d \))-time algorithm. However, the other weak learner \( \text{rHalfspaceWkL} \) uses a foams construction subroutine from [KORWT2] that takes expected exponential in \( d \) runtime. The corresponding strong learner runs in time polynomial in \( 1/\epsilon, 1/\rho, \) and \( 1/\tau \), but exponential in \( d \).

6.5 Discussion

Algorithm \( \text{S} \) follows the smooth boosting framework of Servedio [Ser13], which also shows how to boost a weak halfspace learner under a margin assumption on the data. They show that their boosted halfspace learner obtains a hypothesis with good margin on the training data, and then apply a fat-shattering dimension argument to show generalization to the underlying distribution with sample complexity \( \tilde{O}(1/(\tau \epsilon^2)) \). Notably, this gives sample complexity independent of \( d \). Moreover, their smooth boosting algorithm is tolerant to malicious noise perturbing an \( \eta \in O(\tau \epsilon) \) fraction of its sample.
A generic framework for differentially private boosting was given in [BCS20], with an application to boosting halfspaces. Their boosting algorithm also follows the smooth boosting framework, but uses a variant of the round-optimal boosting algorithm given in [BHK09]. Their halfspace learner similarly requires a margin assumption on the data and tolerates random classification noise at a rate $\eta \in O(\tau^2)$. They give two generalization arguments for their halfspace learner, both of which are dimension-independent. The first follows from prior work showing that differential privacy implies generalization [BNS+16] and gives sample complexity $O\left(\frac{1}{\tau^2 \eta^2} + \frac{1}{\tau^2} + \alpha^{-2} + \varepsilon^{-2}\right)$ for approximate differential privacy parameters $(\alpha, \beta)$. The second follows from a fat-shattering dimension argument and gives a tighter bound of $\tilde{O}\left(\frac{1}{\tau^2 \eta^2}\right)$.

Boosting algorithms have been thoroughly studied over the past few decades, and there are many types of boosting algorithms (e.g. distribution-reweighting, branching-program, gradient boosting) with different properties (e.g. noise-tolerance, parallelizability, smoothness, batch vs. filtering). It would be interesting to see which of these techniques can be made reproducible, and at what cost.

## 7 SQ–Reproducibility Lower Bound

How much does it cost to make a nonreproducible algorithm into a reproducible one? In this section, we show a lower bound for reproducible statistical queries via a reduction from the coin problem.

**Theorem 7.1 (SQ–Reproducibility Lower Bound).** Let $\tau > 0$ and let $\delta \leq 1/16$. Let $\phi: \mathcal{X} \rightarrow [0, 1]$ be a statistical query. Let $A$ be a $\rho$-reproducible SQ algorithm for $\phi$ with tolerance less than $\tau$ and success probability at least $1 - \delta$. Then $A$ has sample complexity at least $m \in \Omega(1/(\tau^2 \rho^2))$.

Note that this nearly matches the reproducible statistical query upper bound in Theorem 2.3 in the case that $\delta \in \Theta(\rho)$.

Recall the coin problem: promised that a 0-1 coin has bias either $1/2 - \tau$ or $1/2 + \tau$ for some fixed $\tau > 0$, how many flips are required to identify the coin’s bias with high probability?

**Proof of Theorem 7.1.** A $\tau$-tolerant $\rho$-reproducible SQ algorithm $A$ for $\phi$ naturally induces a $\rho$-reproducible algorithm $B$ for the $\tau$-coin problem — $B$ runs $A$ (the results of the coin flips are the $\phi(x)$‘s), and $A$ accepts (outputs 1) if $A$’s output is $\geq 1/2$, otherwise rejects. The success probability of $B$ is at least that of $A$. As $A$ is reproducible for all distributions, $B$ also satisfies the assumption in Lemma 7.2 that $B$ is $\rho$-reproducible for coins with bias in $(1/2 - \tau, 1/2 + \tau)$. By Lemma 7.2, any reproducible algorithm solving the coin problem with these parameters has sample complexity $m \in \Omega(1/(\tau^2 \rho^2))$, implying the lower bound.

**Lemma 7.2 (Sample Lower Bound for the Coin Problem).** Let $\tau < 1/4$ and $\rho < 1/16$. Let $B$ be a $\rho$-reproducible algorithm that decides the coin problem with success probability at least $1 - \delta$ for $\delta = 1/16$. Furthermore, assume $B$ is $\rho$-reproducible, even if its samples are drawn from a coin $C$ with bias in $(1/2 - \tau, 1/2 + \tau)$. Then $B$ requires sample complexity $m \in \Omega(1/(\tau^2 \rho^2))$, i.e. $p \in \Omega(1/\tau \rho \sqrt{m})$.

**Proof.** Assume we have an algorithm $B(b_1..b_m; r)$ of sample complexity $m$ so that (i) if the $b_i$’s are chosen i.i.d. in $\{0, 1\}$ with bias $1/2 - \tau$, $B$ accepts with at most $\delta$ probability (over both random $r$ and the $b_i$’s), and (ii) if the $b_i$’s are drawn i.i.d. with bias $1/2 + \tau$, $B$ accepts with at least $1 - \delta$ probability.

Let $p \in [0, 1]$ denote the bias of a coin. Since $B$ is $\rho$-reproducible, $B$ is $\rho$-reproducible for any distribution on $p$. In particular, pick $p \in \mathcal{U}_{1/2 - \tau, 1/2 + \tau}$. Let $C_{-\tau}$ denote a coin with bias $1/2 - \tau$, and let $C_{+\tau}$ denote a coin with bias $1/2 + \tau$. By Markov’s inequality, each of the following is true with probability at least $1 - 1/4$ over choice of $r$:

- $\Pr_{b_1..b_m \sim \text{i.i.d.} C_{-\tau}}[B(b_1..b_m; r) \text{ accepts}] \leq 4\delta$
- $\Pr_{b_1..b_m \sim \text{i.i.d.} C_{+\tau}}[B(b_1..b_m; r) \text{ accepts}] \geq 1 - 4\delta$
- When $p$ is chosen between $1/2 - \tau$ and $1/2 + \tau$ uniformly, and then $b_1..b_m, b'_1..b'_m$ are sampled i.i.d. with expectation $p$, $\Pr[B(b_1..b_m; r) = A(b'_1..b'_m; r)] \geq 1 - 4\rho$

By a union bound, there exists an $r^*$ so that every above statement is true. Note that for any $p$, given $\sum b_i = j$, the samples $b_1..b_m$ are uniformly distributed among all Boolean vectors of Hamming weight $j$. Let
$a_j \overset{\text{def}}{=} \Pr[B(b_1..b_m; r^*) \text{ accepts } | \sum b_i = j]$. Then the probability $B$ accepts using $r^*$ on bits with bias $p$ is $\text{Acc}(p) = \sum_j a_j \binom{m}{j} p^j (1-p)^{m-j}$. In particular, this is a continuous and differentiable function.

Since $\text{Acc}(1/2 - \tau) < 4\delta < 1/4$ and $\text{Acc}(1/2 + \tau) > 1 - 4\delta > 3/4$, there is a $q \in (1/2 - \tau, 1/2 + \tau)$ with $\text{Acc}(q) = 1/2$. We show that $\text{Acc}(p)$ is close to $1/2$ for all $p$ close to $q$ by bounding the derivative $\text{Acc}'(p)$ within the interval $[1/4, 3/4]$, which contains $[1/2 - \tau, 1/2 + \tau]$.

By the standard calculus formulas for derivatives,

$$\text{Acc}'(p) = \sum_j a_j \binom{m}{j} (jp^j - (m-j)p^j/(1-p))$$

$$= \sum_j a_j \binom{m}{j} p^j (1-p)^{m-j} (j/p - (m-j)/(1-p))$$

$$= \sum_j a_j \binom{m}{j} p^j (1-p)^{m-j} (j - mp)/(p(1-p)).$$

Since $1/4 < p < 3/4$, $p(1-p) > 3/16 > 1/6$, and $0 \leq a_j \leq 1$. So this sum is at most

$$\sum_j \binom{m}{j} p^j (1-p)^{m-j} 6|j - mp| = 6\mathbb{E}_j[|j - mp|]$$

where the last expectation is over $j$ chosen as the sum of $m$ random Boolean variables of expectation $p$. This expectation is $O(m^{1/2})$ because the expectation of the absolute value of the difference between any variable and its expectation is at most the standard deviation for the variable.

Since the derivative is at most $O(\sqrt{m})$, there is an interval $I$ of length $\Omega(1/\sqrt{m})$ around $q$ so that $1/3 < \text{Acc}(p) < 2/3$ for all $p$ in this interval. Since $\text{Acc}(p) \notin (1/3, 2/3)$ at $p = 1/2 - \tau$ and $p = 1/2 + \tau$, interval $I$ is entirely contained in $(1/2 - \tau, 1/2 + \tau)$. So, there is an $\Omega(1/\tau \sqrt{m})$ chance that a random $p \in_U [1/2 - \tau, 1/2 + \tau]$ falls in interval $I$. For $p \in I$, there is a $2\text{Acc}(p)(1 - \text{Acc}(p)) > 4/9$ conditional probability of non-reproducibility for $B$. Therefore, $\rho \geq \Omega(1/\tau \sqrt{m})$ and $m \in \Omega(1/\tau^2 \rho^2)$.

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A Reproducibility: Alternative Definitions and Properties

In this section, we consider a few alternative criteria for reproducibility and show how they relate to our definition of reproducibility. We also demonstrate other robustness properties of reproducibility such as amplifying the parameters, as well as data/randomness reuse.

Alternative Definitions and Amplification. In the main body of the paper, we have chosen to define \( \mathcal{A} \) as have two sources of random inputs: samples \( \tilde{s} \) drawn from distribution \( D \) and internal randomness \( r \). \( \mathcal{A} \) has no additional inputs. However, we could more generally define \( \mathcal{A} \) to have additional, nonrandom inputs. In this more general definition, we define \( \mathcal{A}(x; \tilde{s}, r) \) where \( \tilde{s} \) and \( r \) are as defined previously, and \( x \) is an auxiliary input (or tuple of inputs). \( \mathcal{A}(x; \tilde{s}, r) \) is \( \rho \)-reproducible with respect to distribution \( D \) if for every input \( x \), \( \mathcal{A}(x; \tilde{s}, r) \) is \( \rho \)-reproducible. This definition generalizes both pseudodeterministic algorithms (in which there is no underlying distribution, so \( \tilde{s} \) is empty) as well as our definition of reproducible learning algorithms (in which there are no additional inputs, so \( x \) is empty).

Rather than parameterize reproducibility by a single parameter \( \rho \), one could use two variables \( (\eta, \nu) \).

Definition A.1 ((\( \eta, \nu \))-reproducibility). Let \( \mathcal{A}(x; \tilde{s}, r) \) be an algorithm, where \( \tilde{s} \) are samples from \( D \), and \( r \) is the internal randomness. We say that a particular random string \( r \) is \( \eta \)-good for \( \mathcal{A} \) on \( x \) with respect to \( D \) if there is a single “canonical” output \( Z_r \) such that \( \Pr[A(x; \tilde{s}, r) = Z_r] \geq 1 - \eta \). Then \( \mathcal{A} \) is \( (\eta, \nu) \)-reproducible with respect to \( D \) if, for each \( x \), the probability that a random \( r \) is \( \eta \)-good for \( \mathcal{A} \) (on \( x \) and \( D \)) is at least \( 1 - \nu \).

\((\eta, \nu)\)-reproducibility is qualitatively the same as \( \rho \)-reproducibility, but might differ by polynomial factors. If \( \mathcal{A} \) is \( (\eta, \nu) \)-reproducible, then \( \mathcal{A} \) is \( \rho \)-reproducible on \( D \), where \( \rho \leq 2\eta + \nu \). The probability that two runs of \( \mathcal{A} \), using the same internal randomness \( r \), output different results is at most \( \Pr[r \text{ not } \eta \text{-good}] \) plus the probability that at least one run is not the special output \( Z_r \) (conditioned on \( r \) being \( \eta \)-good). In the other direction, if \( \mathcal{A} \) is \( \rho \)-reproducible, then \( \mathcal{A} \) is \((\rho, \nu)\)-reproducible for any \( \rho \leq \nu < 1 \). Say there is a \( \nu \) probability that \( r \) is not \( \eta \)-good. Conditioned on picking a not \( \eta \)-good \( r \), there is a conditional (at least) \( \eta \) probability of the second run of \( \mathcal{A} \) returning something different than the first run\(^\text{3}\). Thus, \( \rho \geq \eta \nu \).

A similar definition called “pseudo-global stability”, developed independently to our work, appears in [GKM21]. That definition parametrizes by the sample complexity \( m \) and does not explicitly parametrize by auxiliary input \( x \). Additionally, their definition includes an \((\alpha, \beta)\)-accuracy guarantee on \( Z_r \), the very likely output. To keep both definitions consistent with their original conventions, we write \( \eta' \overset{\text{def}}{=} 1 - \eta \) and \( \nu' \overset{\text{def}}{=} 1 - \nu \).

Definition A.2 (Pseudo-global stability, Definition 15 in [GKM21]). A learning algorithm \( \mathcal{A} \) with sample complexity \( m \) is said to be \((\alpha, \beta)\)-accurate, \((\eta', \nu')\)-pseudo-globally stable if there exists a hypothesis \( h_r \), for every \( r \in \text{supp}(R) \) (depending on \( D \)) such that \( \Pr_{r \sim R}[\text{err}_D(h_r) \leq \alpha] \geq 1 - \beta \) and

\[
\Pr_{r \sim R}\Pr_{x \sim D}[\mathcal{A}(s; r) = h_r] \geq \eta'
\]

where \( \tilde{s} \) is a sample of \( m \) (labeled) examples \((x_i, y_i)\) drawn from distribution \( D \).

The final condition of Definition A.2 is equivalent to saying that i) a randomly chosen \( r \) is \( \eta' \)-good with probability at least \( \nu' \), and ii) for every \( r \), \( h_r \) is the output that witnesses the \( \eta' \)-goodness. Carrying the accuracy guarantee through the previous argument, an \((\alpha, \beta)\)-accurate \((\eta', \nu')\)-pseudo-globally-stable algorithm \( \mathcal{A} \) implies a \( (2(1 - \eta') + (1 - \nu'')) = (2\eta + \nu')\)-reproducible algorithm \( \mathcal{A} \) also with \((\alpha, \beta)\)-accuracy.

If we are willing to increase the sample complexity of \( \mathcal{A} \), we can make the connection stronger:

Theorem A.3 (Amplification of Reproducibility). Let \( 0 < \eta, \nu, \beta < 1/2 \) and \( m > 0 \). Let \( \mathcal{A} \) be an \((\eta, \nu)\)-reproducible algorithm for distribution \( D \) with sample complexity \( m \) and failure rate \( \beta \). If \( \rho > 0 \) and \( \nu + \rho < 3/4 \), then there is a \( \rho \)-reproducible algorithm \( \mathcal{A}' \) for \( D \) with sample complexity \( m' = O(m(\log 1/\beta)^3/\rho(1/2 - \eta)^2) \) and failure rate at most \( O(\beta + \rho) \). The construction of \( \mathcal{A}' \) does not depend on \( D \).

\(^3\)If \( 1 - \eta \geq 1/2 \), then the probability that two runs of \( \mathcal{A} \) using the same randomness returns the same result is at most \( (1 - \eta)^2 + \eta^2 \), i.e., when \( \mathcal{A} \) has only two possible outputs. This is less than \( 1 - \eta \), assuming \( 1 - \eta \geq 1/2 \). If \( 1 - \eta < 1/2 \), then there must be more than two outputs, and the probability of nonreproducibility is again larger than \( \eta \).
Proof. Set \( k = 3 \log 1/\beta \). For each random string \( r \), let \( D_r \) be the distribution on outputs of \( A(x; \vec{s}; r) \) (over random \( \vec{s} \)). Algorithm \( A' \) randomly picks \( k \)-many strings \( r_1, \ldots, r_k \), runs the reproducible heavy-hitters algorithm (Algorithm \ref{rhh}) on the distributions \( D_{r_1}, \ldots, D_{r_k} \), and outputs the first returned heavy-hitter (or \( \perp \) if each subroutine returns the empty list). We say there are \( k \) rounds of \( A' \), one per random string \( r \).

The reproducibility of \( \text{rHeavyHitters} \) implies the reproducibility of \( A' \). We show that a heavy-hitter in \( D_r \) for randomly chosen \( r \) is often a correct answer, except with probability comparable to \( \beta \).

By definition, \( r \) is \( \eta \)-good iff \( D_r \) has a \( 1 - \eta \) heavy-hitter. Since \( \eta < 1/2 \), this heavy-hitter will be unique, and there will be no other \( 1 - \eta > 1/2 \) heavy-hitters. Given any \( r \), we can draw from distribution \( D_r \) by running algorithm \( A \) with fresh samples \( \vec{s} \). Consider running the reproducible heavy-hitters algorithm with parameters \( v = (3/2 - \eta)/2, \epsilon = (1/2 - \eta)/2 \), and reproducibility \( \rho' = \rho/k \). These are chosen so that \( \nu + \epsilon = 1 - \eta \) and \( \nu - \epsilon = 1/2 \). If \( r \) is \( \eta \)-good, then \( \text{rHeavyHitters} \) will return the (unique) majority element for \( D_r \) with probability at least \( 1 - \rho/k \). If \( r \) is not \( 1/2 \)-good, the reproducible heavy-hitters algorithm with the same parameters will return the empty list with probability at least \( 1 - \rho/k \).

Next, we compute the conditional probability that the first element returned by \( \text{rHeavyHitters} \) is correct. The probability that \( \text{rHeavyHitters} \) produces an empty list in one round is at most \( \nu \) (when the randomly chosen \( r \) is not \( \eta \)-good) plus \( \rho/k \) (when \( r \) is \( \eta \)-good but the heavy-hitters algorithm fails). At most a \((2\beta)\)-fraction of random strings \( r \) satisfy both of the following two conditions: i) \( D_r \) has a majority element \( Z_r \), and ii) \( Z_r \) is an incorrect output. Thus, the conditional probability of outputting an incorrect answer, given \( \text{rHeavyHitters} \) produces a non-empty output, is at most \((2\beta + \rho/k)/(1 - \nu - \rho/k)\). By assumption, \( \nu + \rho/k < 3/4 \), so this is \( O(\beta + \rho) \).

So far, we have bounded the probability that \( A' \) returns an incorrect answer. \( A' \) could also fail if \( \text{rHeavyHitters} \) returns the empty list in each of \( k \) rounds. Since \( \nu + \rho/k < 3/4 \), this happens with probability at most \((3/4)^k \leq \beta \). So, the overall probability of error is at most \( O(\beta + \rho) \).

If two runs of \( A' \) use the same \( r_i \)'s and same randomness for each heavy-hitters call, they only produce different answers if a pair of \( \text{rHeavyHitters} \) calls produces different answers in the same round. By the reproducibility of \( \text{rHeavyHitters} \), this only happens with probability \( \rho/k \) each round, for a total non-reproducibility probability at most \( \rho \).

\( A' \) calls \( \text{rHeavyHitters} \) \( k = O(\log 1/\beta) \) times. Each example used by \( \text{rHeavyHitters} \) is created by running \( A \), which has sample complexity \( m \). By Lemma \ref{rhh1}, \( \text{rHeavyHitters}_{\rho',\nu,\epsilon} \) has sample complexity \( \overline{O} \left( \frac{m \log^3(1/\beta)}{\rho'(1/2 - \eta)^2} \right) \). Substituting in \( \rho' = \rho/k, \epsilon = (1/2 - \eta)/2, \) and \( \nu - \epsilon = 1/2 \), \( A \) has sample complexity \( km \cdot \overline{O} \left( \frac{k^2}{\rho^2(1/2 - \eta)^2} \right) = \overline{O} \left( \frac{m \log^3(1/\beta)}{\rho^2(1/2 - \eta)^2} \right) \).

\textbf{Corollary A.4.} Let \( \alpha > 0 \) and \( \rho < 1/4 - \alpha \). Let \( A \) be a \( \rho \)-reproducible algorithm using \( m \) samples is correct except with error at most \( \alpha \). Then for arbitrary \( \rho' \) satisfying \( \rho > \rho' > 0 \), there is a \( \rho' \)-reproducible algorithm \( A' \) with sample complexity \( m' = \overline{O} \left( \frac{m \log^3(1/\beta)}{\rho' \rho^2} \right) \) that is correct except with error at most \( O(\beta + \rho') \).

\textbf{Proof.} By the arguments immediately after Definition \ref{rhh1}, a \( \rho \)-reproducible algorithm implies a \((\rho/x,x)\)-reproducible algorithm. Choosing \( x = 1/2 - \alpha \) allows us to apply Theorem \ref{rhh1} for any \( \rho < 1/4 \). The \((1/2 - \eta) \) term in Theorem \ref{rhh1} simplifies to \( \alpha/(1 - 2\alpha) \) in this context. When \( \alpha \) can be chosen as a constant, the sample complexity simplifies to \( m' = \overline{O}(m \log^3(1/\beta)/\rho'^2) \).

\textbf{Public versus Private Randomness.} We define reproducibility as the probability that when run twice using the same (public) randomness, but with independently chosen data samples, the algorithm returns the same answer. In \cite{GL19}, the authors define a related concept, but divide up the randomness into two parts, where only the first randomness part gets reused in the second run of the algorithm. In their applications, there are no data samples, so re-running the algorithm using identical randomness would always give identical results; rather, they were trying to minimize the amount of information about the random choices that would guarantee reproducibility, i.e., minimize the length of the first part.

Similarly, we could define a model of reproducibility that involved two kinds of random choices. Define \( A(x; \vec{s}; r_{\text{pub}}, r_{\text{priv}}) \), \( \vec{s} = (s_1, \ldots, s_m) \) to be \( \rho \)-reproducible with respect to \( r_{\text{pub}} \) and \( D \) if for every \( x \), random

\text{\footnote{Since \( \eta < 1/2 \) by assumption, \( r \) being not \( 1/2 \)-good implies \( r \) is not \( \eta \)-good.}}
Lemma A.6 (Reproducibility Implies Generalization). Let \( s_1 \) and \( s_2 \) drawn from \( D^m \), and random \( r_{pub}, r_{priv}, r'_{priv} \):

\[
\Pr[A(x; s_1; r_{pub}, r_{priv}) = A(x; s_2; r_{pub}, r'_{priv})] \geq 1 - \rho.
\]

If we want to minimize the amount of information we need to store to guarantee reproducibility, keeping \( r_{priv} \) and \( r_{pub} \) distinct may be important. However, if all we want is to have a maximally reproducible algorithm, the following observation shows that it is always better to make the entire randomness public.

**Lemma A.5.** If \( A(x; \tilde{s}; r_{pub}, r_{priv}) \) is \( \rho \)-reproducible with respect to \( r_{pub} \) over \( D \), then \( A(x; \tilde{s}; r_{pub}, r_{priv}) \) is \( \rho \)-reproducible with respect to \((r_{pub}, r_{priv})\) over \( D \).

**Proof.** We show for each value of \( x \) and \( r_{pub} \):

\[
\Pr[A(x; s_1; r_{pub}, r_{priv}) = A(x; s_2; r_{pub}, r'_{priv})] \leq \Pr[A(x; s_1; r_{pub}, r_{priv}) = A(x; s_2; r_{pub}, r_{priv})].
\]

Fix \( x \) and \( r_{pub} \). For each possible value \( R \) of \( r_{priv} \) and each possible output \( Z \), let \( q_{R,Z} = \Pr[A((x; s_1; r_{pub}, R)] \equiv Z, \) and let \( \tilde{q}_R \) be the vector indexed by \( Z \) whose \( Z^{th} \) coordinate is \( q_{R,Z} \). Then

\[
\Pr[A((x; s_1; r_{pub}, R)] = A(x; s_2; r_{pub}, R)] = \sum_Z (q_{R,Z})^2 = \| \tilde{q}_R \|^2,
\]

and

\[
\Pr[A(x; s_1; r_{pub}, R)] = A(x; s_2; r_{pub}, R')] = \sum_Z (q_{R,Z} q_{R',Z}) = \langle \tilde{q}_R, \tilde{q}'_R \rangle.
\]

Thus,

\[
\Pr[A(x; s_1; r_{pub}, r_{priv}) = A(x; s_2; r_{pub}, r'_{priv})] = \mathbb{E}_{R,R'}[\langle \tilde{q}_R, \tilde{q}'_R \rangle] \\
\leq \mathbb{E}_{R,R'}[\| \tilde{q}_R \|_2 \| \tilde{q}'_R \|_2] \\
= (\mathbb{E}_R[\| \tilde{q}_R \|_2]^2) \\
\leq \mathbb{E}_R[\| \tilde{q}_R \|_2^2] \\
= \Pr[A(x; s_1; r_{pub}, r_{priv}) = A(x; s_2; r_{pub}, r_{priv})].
\]

\( \square \)

We will implicitly use this observation in the boosting algorithm section, since it will be convenient to think of the two runs of the boosting algorithm as picking samples each step independently, when using the same random string would create some correlation.

**Reproducibility Implies Generalization.** We show that a hypothesis generated by a reproducible algorithm has a high probability of having generalization error close to the empirical error. Let \( h \) be a hypothesis, \( c \) a target concept, and \( D \) a distribution. The risk (generalization error) of \( R(h) \) is defined as \( \Pr_{x \sim D}[h(x) \neq c(x)] \). If \( \tilde{s} \) is a sample drawn i.i.d. from \( D \), then the empirical risk \( \tilde{R}_x(h) \) is defined as \( \Pr_{x \in \tilde{s}}[h(x) \neq c(x)] \).

**Lemma A.6 (Reproducibility Implies Generalization).** Let sample \( \tilde{s} \sim D^n \), and let \( \delta > 0 \). Let \( h \) be a hypothesis output by \( \rho \)-reproducible learning algorithm \( A(\tilde{s}; r) \), where \( r \) is a random string. Then, with probability at least \( 1 - \rho - \delta \) over the choice of \( \tilde{s} \) and \( r \), \( R(h) \leq \tilde{R}_\tilde{s}(h) + \sqrt{\ln(1/\delta)/(2n)} \).

**Proof.** Consider running \( A(\tilde{s}_2; r) \), where \( \tilde{s}_2 \) is an independent sample of size \( m \) drawn from \( D \), but \( r \) is the same as before. Let \( h_2 \) denote the returned hypothesis. Since \( h_2 \) is independent of \( \tilde{s} \), \( \Pr_{x \sim D^n}[	ilde{R}_\tilde{s}(h_2) - R(h_2) \geq \epsilon] \leq \exp(-2m\epsilon^2) \) for \( \epsilon > 0 \) by Hoeffding’s inequality. By the reproducibility of \( A \), \( h_2 = h \) with probability at least \( 1 - \rho \). By a union bound, \( R(h) \geq \tilde{R}_\tilde{s}(h) + \sqrt{\ln(1/\delta)/(2n)} \) with probability at least \( 1 - \rho - \delta \). \( \square \)

In the above argument, we use the definition of reproducibility to create independence between \( \tilde{s} \) and \( h \), allowing us to use Hoeffding’s inequality.

**Connections to Data Reuse.** We consider the adaptive data analysis model discussed in [DFH+15b] and [DFH+15a], and we prove that reproducible algorithms are resilient against adaptive queries (Lemma A.7). The proof is via a hybrid argument.
Lemma A.7 (Reproducibility $\implies$ Data Reusability). Let $D$ be a distribution over domain $\mathcal{X}$. Let $\mathcal{M}$ be a mechanism that answers queries of the form $q : \mathcal{X} \to \{0,1\}$ by drawing a sample $S$ of $n$ i.i.d. examples from $D$ and returning answer $a$. Let $A$ denote an algorithm making $m$ adaptive queries, chosen from a set of queries $Q$, so that the choice of $q_i$ may depend on $q_j, a_j$ for all $j < i$. Denote by $[A, \mathcal{M}]$ the distribution over transcripts $\{q_1, a_1, \ldots, q_m, a_m\}$ of queries and answers induced by $A$ making queries of $\mathcal{M}$. Let $\mathcal{M}'$ be a mechanism that behaves identically to $\mathcal{M}$, except it draws a single sample $S'$ of $n$ i.i.d. examples from $D$ and answers all queries with $S'$.

If $\mathcal{M}$ answers all queries $q \in Q$ with $\rho$-reproducible procedures, then $SD_\Delta([A, \mathcal{M}], [A, \mathcal{M}']) \leq (m-1)\rho$, where $SD_\Delta(D_1, D_2)$ denotes the statistical distance between distributions $D_1$ and $D_2$.

Proof. For $i \in [m]$, let $[A, \mathcal{M}_i]$ denote the distribution on transcripts output by $A$’s interaction with $\mathcal{M}_i$, where $\mathcal{M}_i$ is the analogous mechanism that draws new samples $S_1, \ldots, S_i$ for the first $i$ queries, and reuses sample $S_i$ for the remaining $m-i$ queries. Note that $\mathcal{M}' = \mathcal{M}_1$ and $\mathcal{M} = \mathcal{M}_m$.

For $i \in [m-1]$, consider distributions $[A, \mathcal{M}_i]$ and $[A, \mathcal{M}_{i+1}]$. We will bound the statistical distance by a coupling argument. Let $S_1, \ldots, S_{i+1}$ denote random variables describing the samples used, and let $r$ denote the randomness used over the entire procedure. $[A, \mathcal{M}_i]$ can be described as running the entire procedure (with randomness $R$) on $S_1, \ldots, S_{i-1}, S_{i+1}, S_{i+1}, \ldots, S_{i+1}$, and $[A, \mathcal{M}_{i+1}]$ can be described as running the entire procedure (with randomness $R$) on $S_1, \ldots, S_{i-1}, S_i, S_{i+1}, S_{i+1}, \ldots, S_{i+1}$.

These distributions are identical for the first $i-1$ queries and answers, so the $i$’th query $q_i$ is identical, conditioned on using the same randomness. Both $S_i$ and $S_{i+1}$ are chosen by i.i.d. sampling from $D$, so $\Pr_{S_i, S_{i+1}, r}[A(q_i, S_{i+1}; r) = A(q_i, S_i; r)] \geq 1 - \rho$ by reproducibility. Conditioned on both transcripts including the same $(i+1)$’th answer $a_{i+1}$ (and continuing to couple $S_{i+1}$ and $r$ for both runs), the remaining queries and answers $q_{i+1}, a_{i+1}, \ldots, q_m, a_m$ is identical. Therefore, $SD_\Delta([A, \mathcal{M}_i], [A, \mathcal{M}_{i+1}]) \leq \rho$ for all $i \in [m-1]$.

Unraveling, $SD_\Delta([A, \mathcal{M}], [A, \mathcal{M}']) \leq (m-1)\rho$. \hfill $\square$

Remark A.8. This connection may be helpful for showing that reproducibility cannot be achieved efficiently in contexts where data reuse is not efficiently achievable.

B Concentration of Sum of Vectors

In this Section, we use Azuma’s inequality to prove a concentration bound on the sum of vectors from a distribution.

Let $D$ be a distribution on $\mathbb{R}^n$. Let $v = \{v_1, \ldots, v_T\} \in D^T$ be a random sample of $T$ vectors from $D$ with the following properties:

1. $\mathbb{E}_{v \in D^T}[\sum_{i=1}^T v_i] - \mathbb{E}_{v \in D}[v] = 0$.
2. $\forall v \in D$, $\|v\|_2 \leq c$.

The following lemma shows that the length of $v^1 + v^2 + \ldots + v^T$ is tightly concentrated.

Lemma 5.10. Let $D, v \in D^T$ satisfy properties (1) and (2) above, and let $v^{\leq T} = \sum_{i=1}^T v_i$. Then for all $\Delta > 0$,

$$\Pr_{v}[\|v^{\leq T}\|_2 \geq \sqrt{T}(1 + c/2) + \Delta] \leq e^{-\Delta^2/2c^2T}.$$

The intuition behind Lemma 5.10 is similar to the one-dimensional case, where $D$ is a distribution over $(-1,1)$, $v \in D^T$, and $\sum_{i=1}^T v_i$ is concentrated around zero, with standard deviation $\sqrt{T}$. Let $v^{\leq i}$ denote $\sum_{i=1}^T v_i$. In the one-dimensional case, we can prove concentration of $v^{\leq T}$ via a Chernoff or martingale argument since the expected value of $v^{\leq i}$ (the sum of the first $i$ numbers) is equal to $v^{\leq i-1}$. However for the higher dimensional case, $v^{\leq i}$ is now the sum of the first $i$ vectors, and it is in general not the case that the expected length of $v^{\leq i}$ is equal or even not much larger than the length of $v^{\leq i-1}$. However, if the length of $v^{\leq i-1}$ is sufficiently large (greater than $\sqrt{T}$), then $\mathbb{E}[\|v^{\leq i}\|_2 | v^{\leq i-1}]$ can be upper bounded (approximately) by $\|v^{\leq i-1}\|_2 + 1/\sqrt{T}$. Therefore, if we want to bound the probability that the length of $v^{\leq T}$ is large (at least $\sqrt{T} + \Delta$), there must be some time $t$ such that the vector $v^{\leq t}$ is outside of the ball of radius $\sqrt{T}$ around the origin, and never returns. So we can bound the probability that $\|v^{\leq t}\|_2 \geq \sqrt{T} + \Delta$, by considering the sequence of random variables $x^{\leq t}, \ldots, x^{\leq T}$ such that $x^{\leq t}$ is equal to the length of $v^{\leq t}$, and for each $t' \geq t$,
Proof. Since

We will show that $x^{t^i}$ is a supermartingale where $|x^{t^i+1} - x^{t^i}|$ is bounded by a constant, and then the concentration inequality will follow from Azuma’s Lemma.

Definition B.1. Let $D$ be a distribution over $\mathbb{R}^n$ satisfying the above two properties.

1. Let $v = \{v_1, \ldots, v_{T'}\} \in D^{T'}$ be a sequence of $T' \leq T$ random variables, and let $v_0 \in \mathbb{R}^n$ have length $\sqrt{T}$. For $0 \leq i \leq T'$, let $v_i = \sum_{j=1}^{T'} v_j$.

2. The stopping time $\tau \in [T']$ (with respect to $\{v_i\}$) is equal to:

$$\min\{|i| \in [T'] \mid ||v_i||_2 < \sqrt{T}\} \cup \{T'\}.$$ 

That is, $\tau$ is the first time $i$ such that the length of $v_i$ drops below $\sqrt{T} + \frac{i}{3\sqrt{T}}$ (and otherwise $\tau = T'$).

3. For each $i \in [T']$, we define the sequence of random variables $x^{i0}, x^{i1}, x^{i2}, \ldots, x^{iT'}$ where $x^{i0} = ||v_0||_2 = \sqrt{T}$, and for all $i \geq 1$, $x^{i1}$ will be the adjusted length of the first $i$ vectors, $||v^{i1}||$ with stopping condition $\tau$:

$$x^{i1} = \begin{cases} \sqrt{T} - \frac{ci}{2\sqrt{T}} & \text{if } \tau > i \\ x^{i\tau} & \text{otherwise} \end{cases}$$

Claim B.2. The sequence of random variables $x^{i1}, \ldots, x^{iT'}$ is a supermartingale.

Proof. We need to show that for every $i \in [T']$, $E[x^{i1} \mid x^{i1-1}] \leq x^{i1-1}$. Fix $i \in [T']$; if $\tau \leq i - 1$ then $x^{i1} = x^{i1-1}$ so the condition holds. Otherwise assume that $\tau \geq i$. Since

$$E[x^{i1} \mid x^{i1-1}] = E[x^{i1} \mid v^{i1} = v_1] = E[||v^{i1-1} + v_1||_2] - \frac{ci}{2\sqrt{T}}$$

and $x^{i1-1} = ||v^{i1-1}||_2 - \frac{ci(i-1)}{2\sqrt{T}}$, it suffices to show that $E[||v^{i1-1} + v_1||_2] \leq ||v^{i1-1}||_2 + \frac{c}{2\sqrt{T}}$.

To prove this, we can write $v_1 = v_{i1} + v_{i1}^\perp$ where $v_{i1}$ is the component of $v_1$ in the direction of $v^{i1-1}$, and $v_{i1}^\perp$ is the orthogonal component. Since the expected length of $v^{i1-1} + v_{i1}^\perp$ is equal to the length of $v^{i1-1}$ (by property 1), we just have to show that the expected length of $v^{i1-1} + v_{i1}^\perp$ is at most $\frac{c}{2\sqrt{T}}$. Since $v_{i1}$ has length at most $c$, so does $v_{i1}^\perp$, so we have:

$$E[||v^{i1-1} + v_{i1}^\perp||_2] \leq (||v^{i1-1}||_2 + c)^2 \leq \frac{c}{2\sqrt{T}}$$

where the last inequality holds since $\tau \geq i$ implies $||v^{i1-1}||_2 \geq \sqrt{T}$.

Claim B.3. For all $i$, $|x^{i1} - x^{i1-1}| \leq c$.

Proof. Since $v_1$ has length at most $c$ the absolute value of the difference between $||v^{i1}||_2$ and $||v^{i1-1}||_2$ is at most 2. The claim easily follows since $x^{i1} = ||v^{i1}|| + \frac{ci}{2\sqrt{T}}$.

The above two Claims together with Azuma’s inequality gives:

$$Pr[|x^{T'} - x^{0}| \geq \Delta] \leq e^{-\Delta^2/2c^2T}.$$ 

Proof. (of Lemma 5.10) In order for $v^{T'}$ to have length at least $\sqrt{T}(1 + c/2) + \Delta$, there must be some largest time $t \in [T]$ such that $||v^t||_2 \in (\sqrt{T}, \sqrt{T} + 1]$. That is, at all times $t' \geq t$ the vector $v^{t'}$ is outside the ball of radius $\sqrt{T}$. Thus by the above argument, the random variables $x^{t_i}$ are a supermartingale where the absolute value of the difference between successive variables is at most $c$, and by Azuma, $Pr[x^{T'} \geq \sqrt{T} + \Delta] \leq e^{-\Delta^2/2c^2T}$. Since $x^{T'} = ||v^{T'}||_2 - \frac{Fc}{2\sqrt{T}} = ||v^{T'}||_2 - \sqrt{T}c$, $Pr[||v^{T'}||_2 \geq \sqrt{T}(1 + c/2) + \Delta] \leq e^{-\Delta^2/2c^2T}$. 

\[\square\]