Harmonic functions via restricted mean-value theorems

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Abstract

Let $f$ be a function on a bounded domain $\Omega \subseteq \mathbb{R}^n$ and $\delta$ be a positive function on $\Omega$ such that $B(x, \delta(x)) \subseteq \Omega$. Let $\sigma(f)(x)$ be the average of $f$ over the ball $B(x, \delta(x))$. The restricted mean-value theorems discuss the conditions on $f$, $\delta$, and $\Omega$ under which $\sigma(f) = f$ implies that $f$ is harmonic. In this paper, we study the stability of harmonic functions with respect to the map $\sigma$. One expects that, in general, the sequence $\sigma^n(f)$ converges to a harmonic function. Among our results, we show that if $\Omega$ is strongly convex (respectively $C^{2, \alpha}$-smooth for some $\alpha \in [0, 1]$), the function $\delta(x)$ is continuous, and $f \in C^0(\Omega)$ (respectively, $f \in C^{2, \alpha}(\Omega)$), then $\sigma^n(f)$ converges to a harmonic function uniformly on $\Omega$.

1 Introduction

Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty domain, $n \geq 1$. A function $\delta : \Omega \to \mathbb{R}$ is called admissible if $\delta(x) > 0$ and $B(x, \delta(x)) \subseteq \Omega$ for all $x \in \Omega$, where $B(z, r)$ is the open ball of radius $r$ centered at $z$. If $f$ is a harmonic function (i.e. $f \in C^2(\Omega)$ and $\Delta f = 0$ on $\Omega$), then it satisfies the mean value property in every ball
within $\Omega$. In other words, the average of $f$ over any ball in $\Omega$ is equal to the value of $f$ at the center of the ball. It is well known that a locally bounded measurable function on $\Omega$ that satisfies the mean value property for all balls in $\Omega$ is harmonic; see \cite{5}. It turns out that under certain conditions on $\delta$ or $f$, a restricted mean value property would still imply that $f$ is harmonic. To be more precise, we call $f$ a $(\lambda, \delta)$-median function on $\Omega$ if

$$f(x) = \frac{1}{\lambda(B_x)} \int_{B_x} f(z) d\lambda(z) \quad \forall x \in \Omega,$$  

(1)

where $\lambda$ refers to the Lebesgue measure and $B_x = B(x, \delta(x))$. Hansen and Nadirashvili proved that

**Theorem 1.** Let $\delta$ be an admissible function on $\Omega$ and let $f$ be a $(\lambda, \delta)$-median function on $\Omega$ such that $|f| \leq h$ for some harmonic function $h$ on $\Omega$. Assume that $f$ is continuous or that $\delta$ is locally bounded away from zero. Then $f$ is harmonic.

Let $\sigma$ be the averaging function defined by

$$\sigma(f)(x) = \frac{1}{\lambda(B_x)} \int_{B_x} f(z) d\lambda(z) \quad x \in \Omega; \quad \sigma(f)(x) = f(x) \quad x \in \partial \Omega.$$  

(2)

In light of Theorem 1, we know that continuous fixed points of the averaging function $\sigma$ are harmonic. It is then natural to study the stability of the set of harmonic functions on $\Omega$ under the averaging function $\sigma$. In other words, given an initial function $f$ on $\Omega$, we would like to study the limit of the iterations $\sigma(f), \sigma(\sigma(f)), \sigma(\sigma(\sigma(f))), \ldots$, in $L^\infty$. In order for $\sigma$ to be an automorphism of $C^0(\Omega)$, we also need to assume that $\delta$ is a continuous function on $\Omega$ (see Lemma 4).

In section 3, we consider $C^{2,\alpha}$ smooth domains and $C^{2,\alpha}$ smooth functions, where $\alpha \in [0, 1]$ is arbitrary. A domain $\Omega$ is called $C^{2,\alpha}$ smooth, or of class $C^{2,\alpha}$, if at every $x \in \partial \Omega$ there exists a ball $B = B(x)$ and a one-to-one mapping $\psi$ of $B$ onto a domain $D \subset \mathbb{R}^n$ such that

$$\psi(B \cap \Omega) \subset \mathbb{R}^n_+; \quad \psi(B \cap \partial \Omega) \subset \partial \mathbb{R}^n_+; \quad \psi \in C^{2,\alpha}(B), \quad \psi^{-1} \in C^{2,\alpha}(D),$$

where $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n): x_n \geq 0\}$.

**Theorem 2.** Suppose $\Omega$ is a bounded domain of class $C^{2,\alpha}$ in $\mathbb{R}^n$, where $n \geq 1$ and $\alpha \in [0, 1]$. Suppose $\delta$ is a continuous and admissible function on $\Omega$. If $f \in C^{2,\alpha}(\Omega)$, then the averaging sequence $\sigma^n(f)$ is uniformly convergent on $\Omega$ to a harmonic function $u \in C^{2,\alpha}(\Omega)$ with $u = f$ on $\partial \Omega$.
In section 4, we consider strongly convex domains. We call a domain $\Omega \subset \mathbb{R}^n$ strongly convex, if any nontrivial convex linear combination of points in $\overline{\Omega}$ belongs to $\Omega$.

**Theorem 3.** Let $\Omega$ be a strongly convex bounded domain in $\mathbb{R}^n$ and $\delta$ be a continuous and admissible function on $\Omega$. If $f \in C^0(\overline{\Omega})$, then the averaging sequence $\sigma^n(f)$ is uniformly convergent to a harmonic function $u \in C^0(\overline{\Omega})$ with $u = f$ on $\partial \Omega$.

2 The averaging function

Throughout this section, we assume that $\Omega$ is a nonempty bounded domain in $\mathbb{R}^n$, $n \geq 1$, and $\delta$ is continuous and admissible, i.e. $\delta > 0$ and $B_x = B(x, \delta(x)) \subset \Omega$. Then we can extend $\delta$ to $\overline{\Omega}$ by zero. This extension is still continuous, since $\delta(x) \to 0$ as $x \to \partial \Omega$. Recall that $\sigma(f)(x)$ is the average of $f$ over $B_x = B(x, \delta(x))$, given by the equation 2. Also let $w_n$ be the volume of the $n$-dimensional unit ball and $\delta_x = \delta(x)$. Here and throughout, $L^1(\Omega)$ denotes the space of Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ such that $\int_\Omega |f|d\lambda < \infty$, and $L^\infty(\Omega)$ denotes the space of functions $f$ on $\Omega$ such that $|f| \leq K$, almost everywhere, for some $K > 0$. In this paper, we denote the norms in $L^1(\Omega)$ and $L^\infty(\Omega)$ by $\| \cdot \|$ and $\| \cdot \|_\infty$.

**Lemma 4.** If $f \in L^1(\Omega) \cap L^\infty(\Omega)$, then $\sigma(f)$ is continuous on $\Omega$ and $\|\sigma(f)\|_\infty \leq \|f\|_\infty$. Moreover, if $y \in \partial \Omega$ and $\lim f(x) = L$ as $x \to y \in \partial \Omega$, then $\lim \sigma(f)(x) = L$ as $x \to y$.

**Proof.** Fix $x \in \Omega$. By continuity of $\delta$, there exists $\epsilon$ such that if $d(x, y) < \epsilon$ then $|\delta_y - \delta_x| < \delta_x/2$. Let $y \in \Omega$ such that

$$d(x, y) < \min\{\epsilon, \frac{1}{2}\delta_x\}. \tag{3}$$

Then

$$|\sigma(f)(x) - \sigma(f)(y)| \leq \frac{1}{w_n \delta_x^n} \int_{B_x \oplus B_y} |f(z)|dz + \left| \frac{1}{w_n \delta_x^n} - \frac{1}{w_n \delta_y^n} \right| \int_{B_y} |f(z)|dz. \tag{4}$$
If $z \in B_y \setminus B_x$, then by the triangle inequality: $d(z, y) \geq d(x, z) - d(x, y) \geq \delta_x - d(x, y)$. It follows that

$$
\lambda(B_y \setminus B_x) \leq w_n \delta_y^n - w_n (\delta_x - d(x, y))^n \leq C_1 (\delta_y - \delta_x + d(x, y)) \delta_x^{n-1},
$$

where $C_1$ is a constant that depends only on $n$. This together with the similar inequality for $B_x \setminus B_y$ implies that

$$
\lambda(B_x \oplus B_y) \leq 2C_1 (|\delta_x - \delta_y| + d(x, y)) \delta_x^{n-1}, \tag{5}
$$

On the other hand,

$$
\left| \frac{1}{\delta_x^n} - \frac{1}{\delta_y^n} \right| = \frac{|\delta_x^n - \delta_y^n|}{\delta_x^n \delta_y^n} \leq C_2 \delta_x^{-n} |\delta_x - \delta_y|, \tag{6}
$$

where $C_2$ depends only on $n$. We use the estimates (5) and (6) and continue from (4) to conclude that:

$$
|\sigma(f)(x) - \sigma(f)(y)| \leq C \|f\|_{\infty} \delta_x^{n-1} (|\delta_x - \delta_y| + d(x, y)) \tag{7}
$$

for all $y$ satisfying (3), where $C$ is a constant that depends only on $n$. It follows that $\sigma(f)$ is continuous at each $x \in \Omega$.

For $x \in \Omega$, we have

$$
|\sigma(f)(x)| \leq \frac{1}{w_n \delta_x^n} \int_{B_x} |f(y)| dy \leq \|f\|_{\infty},
$$

and so $\|\sigma(f)\|_{\infty} \leq \|f\|_{\infty}$. Finally, suppose $y \in \partial \Omega$ such that $\lim f(x) = L$ as $x \to y$. Then for every $a > 0$ there exists $b > 0$ such that

$$
d(x, y) \leq b \Rightarrow |f(x) - L| \leq a.
$$

Suppose $z$ is close enough to $y$ such that $\delta_z \leq d(z, y) \leq b/2$. It follows that for all $x \in B_z$, we have $d(x, y) \leq d(x, z) + d(z, y) \leq b$, and so

$$
|\sigma(f)(z) - L| \leq \frac{1}{w_n \delta_z^n} \int_{B_z} |f(x) - L| d\lambda(x) \leq a,
$$

which proves that $\lim \sigma(f)(x) = L$ as $x \to y$. \hfill \Box
We define the averaging sequence of $f$ by setting:

$$f_0 = f, \ f_{n+1} = \sigma(f_n), \ \forall n \geq 0.$$  \hspace{1cm} (8)

We would like to show that this sequence is uniformly convergent to a harmonic function on $\Omega$. In the next sections, we prove this claim under certain conditions on $f$ and $\Omega$. We will make use of (7) which implies that the sequence $\sigma^n(f)$ is equicontinuous on $\Omega$. Our main task is to show that in fact the sequence $\sigma^n(f)$ is equicontinuous on $\Omega$ and then use Ascoli’s Theorem to derive a convergent subsequence. Finally we need to show that such a convergent subsequence converges to a harmonic function and subsequently show that the averaging sequence itself will converge uniformly to the same limit.

3 Smooth domains and smooth functions

Suppose $\Omega$ is a bounded domain of class $C^{2,\alpha}$ in $\mathbb{R}^n$, $\alpha \in [0,1]$. Then by Kellogg’s Theorem \cite{2}, Th. 6.14, for $f \in C^{2,\alpha}(\Omega)$, there exist a function $u \in C^{2,\alpha}(\Omega)$ such that

$$\Delta u = 0 \text{ in } \Omega, \ u = f \text{ on } \partial \Omega.$$ \hspace{1cm} (9)

Moreover, again by Kellogg’s Theorem, there exists a function $h \in C^{2,\alpha}(\Omega)$ such that

$$\Delta h = -1 \text{ in } \Omega, \ h = 0 \text{ on } \partial \Omega.$$ \hspace{1cm} (10)

By the Maximum Principle \cite{2}, we have $h > 0$ on $\Omega$.

**Lemma 5.** Let $\Omega, f, u,$ and $h$ be as above. Then there exists a positive constant $K$ such that

$$|f - u| \leq K h.$$  

**Proof.** Since $f - u \in C^{2,\alpha}(\Omega)$ and $f - u = 0$ on $\partial \Omega$, there exists a constant $C$ such that

$$|(f - u)(x)| \leq C \rho(x), \ \forall x \in \Omega,$$

where $\rho(x)$ is the distance from $x$ to $\partial \Omega$. On the other hand, we show that the function

$$q(x) = \frac{h(x)}{\rho(x)}$$
has a positive lower bound on $\Omega$. It is sufficient to show that $q$ has a continuous extension to $\overline{\Omega}$ which is positive at every $y \in \partial \Omega$. Clearly $q$ is continuous and positive at every $y \in \Omega$. If $y \in \partial \Omega$, then
\[
\lim_{x \to y} q(x) = \frac{\partial h}{\partial \nu}(y)
\]
is the inward unit normal derivative at $y$. Since $h \in C^{2,\alpha}(\overline{\Omega})$, this limit exists and gives a continuous extension of $q$ to $\partial \Omega$. Finally $\partial h/\partial \nu$ is positive at every $y \in \partial \Omega$ by Lemma 3.4 of [2].

Proof of Theorem 2. Let $u$ be the unique solution to the equations (9). Since $u$ satisfies the mean-value property in $\Omega$, we can assume without loss of generality that $f = 0$ on $\partial \Omega$. We first show that the averaging sequence is equicontinuous on $\Omega$. Lemma 4 (particularly equation (7)) implies that the sequence $f_n = \sigma^n(f)$ is equicontinuous at every $x \in \Omega$. We need the following lemma in order to show that the averaging sequence is equicontinuous at every $x \in \partial \Omega$.

Lemma 6. Let $h$ be the unique function satisfying equations (10). Then $|f_n| \leq Kh$ on $\overline{\Omega}$ for all $n \geq 0$, where $K$ is the positive constant guaranteed by Lemma 5.

Proof. Proof is by induction on $n \geq 0$. For $n = 0$, we have $|f_0| = |f| \leq Kh$ by Lemma 5. Suppose $|f_n| \leq Kh$ on $\overline{\Omega}$. For any $y \in \overline{\Omega}$, we show that $f_{n+1}(y) \leq Kh(y)$. If $y \in \partial \Omega$, then $f_{n+1}(y) = 0 = h(y)$. Thus, suppose $y \in \Omega$. Then
\[
f_{n+1}(y) = \frac{1}{w_n \delta^u_y} \int_{B_y} f_n(z) d\lambda(z) \leq \frac{K}{w_n \delta^u_y} \int_{B_y} h(z) d\lambda(z) \leq Kh(y),
\]
since $h$ is concave down on $\overline{\Omega}$. Similarly $f_{n+1}(y) \geq -Kh(y)$ and the lemma follows.

Since $h$ is continuous at $x \in \partial \Omega$, for any $a > 0$ there exists $b > 0$ such that
\[
y \in \overline{\Omega} \, , \, d(x, y) < b \Rightarrow |h(y)| < a,
\]
It follows that if $d(x, y) < b$, then $|f_n(y)| < Ka$, which implies the equicontinuity of $f_n$.  

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Next, let $\alpha = \inf \|f_n\|_\infty$. It follows from Ascoli’s Theorem that there exists a subsequence $f_{i_n}$ that is uniformly convergent to a function $F$ on $\bar{\Omega}$ and $\lim \|f_{i_n}\|_\infty = \alpha = \|F\|_\infty$. Moreover, since $f_n = 0$ on $\partial\Omega$, we have $F = 0$ on $\partial\Omega$. We will show, using the lemma below, that $F = 0$ on $\Omega$.

**Lemma 7.** Either $\alpha = 0$ or there exists $m$ such that $\|\sigma^m(F)\|_\infty < \|F\|_\infty$.

**Proof.** Since $\|\sigma^m(F)\|_\infty \leq \|F\|_\infty$ for all $m$, we assume $\|\sigma^m(F)\|_\infty = \|F\|_\infty = \alpha > 0$ for all $m$ and derive a contradiction. Let

$$\Lambda_i = \{z \in \bar{\Omega} : \sigma^i(F)(z) = \alpha\}, \ i \geq 0.$$ 

Then $d(\Lambda_i, \partial\Omega) > 0$, since $\sigma^i(F) = 0$ on $\partial\Omega$ and $\alpha \neq 0$. Also $\Lambda_i$ is nonempty and $\Lambda_{i+1} \subseteq \Lambda_i$, for all $i \geq 0$. Hence, there exists $\beta > 0$, independent of $i$, such that $\delta(z) > \beta$ for all $z \in \Lambda_i$. Since each $\Lambda_{i+1} \subseteq \Lambda_i$ and each $\Lambda_i$ is compact and nonempty, the intersection $\Lambda = \bigcap_{i=0}^\infty \Lambda_i$ is nonempty. On the other hand, if $z \in \Lambda$, then $B(z, \beta) \subset \Lambda$. This is a contradiction, since $\Lambda$ is bounded. \[\square\]

If $\alpha \neq 0$, then it follows from Lemma 7 that $\|\sigma^m(F)\|_\infty < \alpha$. Since $\sigma^m(F)$ is the uniform limit of $f_{i_n+m}$ as $n \to \infty$, we have $\lim \|f_{i_n+m}\|_\infty < \alpha$ which contradicts the property of $\alpha$. Hence, we have $\alpha = 0$ and $F = 0$, i.e. $f_{i_n}$ converges uniformly to zero. Since $\|f_{i+1}\|_\infty \leq \|f_i\|_\infty$, it follows that $f_n$ converges to zero uniformly on $\bar{\Omega}$. \[\square\]

### 4 Strongly convex domains and continuous functions

In this section, we prove Theorem 3. Recall that a domain $\Omega$ is called strongly convex, if any nontrivial convex linear combination of points in $\bar{\Omega}$ belongs to $\Omega$. Hence a disk in the plane is strongly convex, while a square is not.

**Proof of Theorem** Let $H(f)$ denote the convex hull of the graph of $f$ in $\mathbb{R}^{n+1}$. By the Carathéodory’s theorem in convex geometry, we have

$$H(f) = \left\{ \sum_{i=1}^{n+2} \alpha_i f(x_i) : \alpha_i \geq 0, \sum_{i=1}^{n+2} \alpha_i = 1, \ x_i \in \bar{\Omega} \right\}.$$
We first show that $H(\sigma(f)) \subseteq H(f)$. It is sufficient to show that $\sigma(f) \subseteq H(f)$. In other words, we need to show that, for every $x \in \overline{\Omega}$, we have $(x, \sigma(f)(x)) \in H(f)$. If $x \in \partial \Omega$, then $(x, \sigma(f)(x)) = (x, f(x)) \in H(f)$. Thus, suppose $x \in \Omega$. Let $\epsilon > 0$ and choose $z_i \in \Omega$, $i = 1, 2, \ldots, k$, such that

$$\left| \frac{1}{w_n \delta_n} \int_{B_x} f(z) dz - \frac{1}{k} \sum_{i=1}^{k} f(z_i) \right| < \epsilon, \quad \left| x - \frac{1}{k} \sum_{i=1}^{k} z_i \right| < \epsilon.$$

By definition of $H$, we have $(\sum z_i/k, \sum f(z_i)/k) \in H(f)$. Since $\epsilon$ is arbitrary and $H(f)$ is closed, we conclude that $(x, \sigma(f)(x)) \in H(f)$.

Next, we show that

**Lemma 8.** Let $x \in \partial \Omega$. Then for every $a > 0$ there exists $b > 0$ such that if $(z, t) \in H(f)$ and $d(z, x) < b$, then $|f(x) - t| < a$.

**Proof.** Proof is by contradiction. Suppose on the contrary that, there exists some $a > 0$ such that for every $n > 0$ there exists $p_n = (z_n, t_n) \in H(f)$ with $d(z_n, x) < 1/n$ but $|f(x) - t_n| \geq a$. Derive a subsequence $p_{i_n}$ such that $t_{i_n} \to t$ as $n \to \infty$, for some $t$ with

$$|f(x) - t| \geq a > 0. \tag{11}$$

Then $p_{i_n} \to (x, t)$ as $n \to \infty$. Since $H(f)$ is closed, we have $(x, t) \in H(f)$. By definition of $H$, there should exist $\alpha_i \geq 0$ with $\sum \alpha_i = 1$ such that $(x, t) = \sum \alpha_i (z_i, f(z_i))$ for some $z_i \in \overline{\Omega}$, $i = 1, \ldots, n + 2$. It follows that $x = \sum \alpha_i z_i$. Since $\Omega$ is strongly convex, all of the $\alpha_i$’s must be zero except one, say $\alpha_1 = 1$. But then $t = f(z_1) = f(x)$ which contradicts (11). \qed

**End of proof of Theorem 3** Lemma 8 implies that the sequence $f_n = \sigma^n(f)$ is equicontinuous at every $x \in \partial \Omega$. On the other hand, by (11), the sequence $f_n$ is equicontinuous at every $x \in \Omega$. Now let $h \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ be the unique harmonic function on $\Omega$ with $h|_{\partial \Omega} = f|_{\partial \Omega}$. Such a harmonic function exists because $\Omega$, being a strongly convex domain, satisfies the exterior sphere condition [2, Th. 6.13]. Now, the sequence $f_n - h$ is equicontinuous on $\overline{\Omega}$. Let $\alpha = \inf \|f_n - h\|_\infty$. By the equicontinuity of the sequence $f_n - h$, there exists a subsequence that converges uniformly to some continuous function $F$ on $\overline{\Omega}$ and $\|F\| = \alpha$. Moreover, $F = 0$ on $\partial \Omega$. It follows from Lemma 4 and the argument therein that $\alpha = 0$ and $f_n - h \to 0$ uniformly on $\overline{\Omega}$. \qed
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