Transition Between Ground State and Metastable States in Classical 2D Atoms

Minghui Kong, B. Partoens, and F. M. Peeters
Departement Natuurkunde, Universiteit Antwerpen (UIA)
Universiteitsplein 1, B-2610 Antwerpen, Belgium

Structural and static properties of a classical two-dimensional (2D) system consisting of a finite number of charged particles which are laterally confined by a parabolic potential are investigated by Monte Carlo (MC) simulations and the Newton optimization technique. This system is the classical analog of the well-known quantum dot problem. The energies and configurations of the ground and all metastable states are obtained. In order to investigate the barriers and the transitions between the ground and all metastable states we first locate the saddle points between them, then by walking downhill from the saddle point to the different minima, we find the path in configurational space from the ground state to the metastable states, from which the geometric properties of the energy landscape are obtained. The sensitivity of the ground-state configuration on the functional form of the inter-particle interaction and on the confinement potential is also investigated.

PACS numbers: 45.05.+x, 61.46.+w, 73.22.-f

I. INTRODUCTION

Wigner suggested in 1934 that a liquid to solid phase transition should occur in a three-dimensional (3D) Fermi system at low densities [1]. The quest for the observation of such a Wigner crystal has been the object of very intense and not stagnant work. After the first discovery of Wigner crystallization of electrons on the surface of liquid helium [2], there has been considerable theoretical and experimental progress in the study of the mesoscopic system consisting of a finite number of charged particles, which are laterally confined by a parabolic potential and repel each other through a Coulomb potential. This system is the classical analog of the well-known quantum dot problem. These quantum dots are atomic-like structures which have interesting optical properties and may be of interest for single electron devices. These systems and their configurations have been observed experimentally, and are important in solid-state physics, plasma physics as well as in atomic physics.

The classical approach is valid for quantum dots in high magnetic fields where the kinetic energy of the electrons is quenched, or for other classical systems, such as laser cooled ions in a trap [3] which are realized by electric and magnetic fields, trapped ions cooled by laser techniques [4], ions in a radio-frequency (RF) trap (Paul trap) [5,6] or a Penning trap [7,8] which can also serve as an illustration of 3D Coulomb clusters [9,10]. Very large Coulomb clusters have been created recently in strongly coupled RF dusty plasmas [11,12] which are like a two-dimensional (2D) layered system. Examples of 2D Coulomb clusters are electrons on the surface of liquid helium [13] and electrons in quantum dots [14]. The vortex clusters in an isotropic superfluid [15], vortices in superfluid He$^4$ [16,17], vortices in a Bose-Einstein condensate stirred with a laser beam [18] and in superconducting grains [20] have many common features with those of 2D charged particles [19]. Colloidal particles dissolved in water [21,22] and placed between two glass plates are another example of an experimental system where classical particles exhibit Wigner crystallization [23]. Very recently, macroscopic 2D Wigner islands, consisting of charged metallic balls above a plane conductor were studied and ground state, metastable states and saddle point configurations were found experimentally [24].

In a finite system there is a competition between the bulk triangular lattice and the circular confinement potential which tries to force the particles into a ring like configuration. Those configurations were systematically investigated in Ref. [25] and a Mendeleev-type of table for these classical atomic-like structures was constructed. The spectral properties of the ground state configurations were presented in Refs. [26,27] and generalized to screened Coulomb [28,29] and logarithmic [30,31] inter-particle interactions.

In the present paper we want to go one major step further and calculate not only all the different metastable states but also the saddle points between those local energy minima and the path followed by the particles to transit between those energy minima. The present work is motivated by recent experimental work [20] where it was found that: i) some of the configurations did not agree with the previous theoretical published one, and ii) they were able to observe some of the saddle points which are the key configurations for transition between different stable (ground or metastable) states. Therefore, we also investigated the stability of the ground state configurations against the functional form of the confinement potential and the exact form of the inter-particle interaction potential.

The present paper is organized as follows. In Sec. 2, we describe the model system. In Sec. 3, our numerical technique, used to obtain the ground and metastable states, is outlined. The technique we used to find the saddle points is similar to the Cerjan-Miller algorithm [32]. After the saddle points are found, we connect the saddle point to the global minimum or a local minimum by the ‘walking downhill’ method. Sec. 4 is devoted to the structural and static properties of the ground and metastable states for $N = 1 \sim 40$. The configurations are analyzed and com-
pared with available experimental data and the results of previous theoretical approaches. The phase diagram for 9 and 16 particles in the ground state with different functional forms of confinement potential and interparticle interaction is also calculated. The discussion on the saddle point is presented in Sec. 5, and the connecting path from the ground state to the metastable states is found, and we investigate the completely geometric properties of the energy landscape. Our conclusions are presented in Sec. 6.

II. MODEL SYSTEM

The model system consists of identical charged particles interacting through a Coulomb repulsive interacting and moving in a 2D plane where they are confined by a parabolic potential

$$H = \frac{q^2}{\varepsilon} \sum_{i>j} \frac{1}{|\vec{r}_i - \vec{r}_j|} + \sum_i V(\vec{r}_i).$$

The confinement potential $V(\vec{r}) = \frac{1}{2}m^*\omega_0^2 r^2$ is taken circular symmetric and parabolic, where $m^*$ is the effective mass of the particles, $q$ is the particle charge, $\omega_0$ is the radial confinement frequency and $\varepsilon$ is the dielectric constant of the medium the particles are moving in. Note that for the quantum dot problem an additional term appears in Eq. (1) which is the kinetic energy of the particles which is absent in our statical classical problem. Here the motion of the particles is restricted to the $(x,y)$ plane. To exhibit the scaling of the system, we introduce the characteristic scales in the problem: $r_0 = (2q^2/m\varepsilon\omega_0^2)^{1/3}$ for the length and $E_0 = (m\varepsilon\omega_0^2/q^4/2\omega_0^2)^{1/3}$ for the energy. After the scaling transformations $(r \rightarrow r/r_0, E \rightarrow E/E_0)$, the Hamiltonian can be rewritten in a simple dimensionless form as

$$H = \sum_{i>j} \frac{1}{|\vec{r}_i - \vec{r}_j|} + \sum_i V(\vec{r}_i),$$

with $V(\vec{r}) = x^2 + y^2$ and which only depends on the number of particles $N$. The numerical values for the parameters $\omega_0, r_0, E_0$ for some typical experimental systems were given in Ref. [27].

III. NUMERICAL APPROACH

Most of the previous works have treated the quantum mechanical problem of a small number of electrons. In the present paper, we consider only the classical system. Although a classical approach for the description of the behavior of electrons in quantum dots, in principle, is not applicable, it is possible that certain features of the classical system may survive in a quantum system.

Due to the presence of confinement energy and electron-electron Coulombic interaction, a complete description of the cluster system is complicated and can’t be obtained analytically. The Monte Carlo simulation technique [27] is relatively simple and rapidly convergent and it provides a reliable estimation of the total energy of the system in cases when relatively small number of Metropolis steps is sufficient. However, the accuracy of this method in calculating the explicit states is poor in certain cases. It becomes more difficult for clusters with a large number of particles, which have significantly more metastable states. To circumvent this problem we employ the numerical technique of Newton optimization which was outlined and compared with the standard Monte Carlo technique in Ref. [28]. In this way, we are able to obtain not only the ground state but also metastable states. It also yields the eigenfrequencies and the eigenmodes of the ground state configuration. Now only a small number of calculation steps is needed to obtain the same accuracy. Moreover, using the modified Newton approach, we can explore the stability of the system in its ground-state configuration through its spectrum.

By studying the characteristics of the energy landscape and the energy barrier between the different local minima, we are able to find the saddle point configurations which are very important and are the key configurations for transition between different stable states. The technique we used to find the saddle point is explained in more detail in Ref. [37], and is similar to the Cerjan-Miller algorithm [33]. After the saddle points are found, we connect the saddle point to the global minimum or a local minimum by the ‘walking downhill’ method. In this algorithm the direction of the steepest gradient is followed to force the system to transit from the saddle point state to the local minimum state. Which minimum is finally reached depends on the initial step, therefore we repeat this procedure several times to determine both minima which the saddle point state connects. Thus the connecting path followed by the particles to transit between those energy minima is found, from which the geometric properties of the energy landscape are obtained.

IV. GROUND STATE AND METASTABLE STATE

In Table I we list for $N=1, 2,..., 40$ the energy per particle $E/N$ in the ground state and in the metastable states, where we also list the energy difference with the ground state $\Delta E/N$. The configuration is indicated by the number of particles in the different rings, the position of the center of the ring and the radius of the different rings, the width of the ring which is defined as the difference of the maximum radius and minimum radius in the same ring, and the energy of the lowest three normal mode frequencies of the ground state are also given in Table I. This table is rather exhaustive and should be compared
patterns which show concentric shells at small
which leads to clusters with interesting self-organized
types of ordering: ordering into a triangular-lattice struc-
ally disappears in the center and the triangular Wigner
column. Note that with increasing
metastable and the ground state is given in the third col-
sible values for $E/N$
with a similar one published in Ref. [33] for a logarithmic
exactly located in the center of the parabolic potential well.
not always) the widths of the rings for metastable config-
metastable configurations increases and in general (but
increasing
largest width. The width of the rings increases with in-
experimentally observed ones. Consequently, an alterna-
tive explanation for the difference with the experiment
is that the experimental configuration got stuck in the
metastable configuration.

V. SADDLE POINTS

Between metastable states and the ground state there
are potential barriers. The system will prefer to trans-
fer over the lowest potential barrier, which is the saddle
point configuration between these energy minima, in or-
der to transit from one stable configuration to the other.
We plot in Fig. 4 the trajectories of the particles for the
system making a transition from the ground state
($N = 5$) to the metastable state (1, 4) and the saddle point
connecting them. The trajectories of the particles can
also be obtained by moving one of the particles to the
center of the system.

For 6 particles, the ground state (1, 5) and the
metastable state (6), corresponding to the hexagonal con-
figuration, are obtained. Moreover, the unstable equilib-
ria associated to saddle point configurations are also ob-
tained, and the energy landscape is shown schematically
in Fig. 5. There are two saddle points for this case, one
of them is very close to the metastable state in both en-
ergy and configuration, and will therefore be hard to see
experimentally [24]. In Fig. 5, the insets show the ar-
angement of the particles for the different states. Using
the ‘walking downhill method’, we found the central par-
ticle slowly moving to the periphery of the cluster. We would like to stress that the configuration with 6 particles on a perfect ring is a saddle point state in contrast to the claim made in Ref. [83]. This can be understood from the following simple model calculation: if 3 particles are placed on a circle with radius $A$, on the corners of an equilateral triangle, and the other 3 particles on another equilateral triangle’s corners with radius $B$ rotated over 60°, the energy is

$$E(c) = \frac{9}{2} \left( \frac{1 + c^2}{36} \right)^{1/3} \left( \frac{1}{1 + c} + \frac{1 + c}{\sqrt{3}c} + \frac{2}{\sqrt{1 - c + c^2}} \right)^{2/3}$$

where $c = A/B$. This function is shown in Fig. 6. It is clear that the perfect circle configuration i.e. $c = A/B = 1$ is a saddle point, and that the minimum is obtained if 3 particles move a bit to the center, and the other 3 particles move away from the center (see the insets in Fig. 7). Both shown metastable states are just connected by a rotation over 120°. The two minima in Fig. 6 correspond to the same configuration in which inner and outer ring are interchanged. Comparing our results with the Fig. 2 (‘$N = 6$: ground state, saddle point configuration and the hexagonal metastable state’) of Ref. [26], we see that the other saddle point is observed experimentally.

A list of the saddle point energies up to 20 particles is given in Table 1. From this table, we notice that there is only one saddle point state for $N = 3, 4, 5$ particles. But, on the other hand it is well-known that there are $(k - 1)$ saddle points when there are $k$ minima. For $N = 3$ and 4 one saddle point is found, although there is no metastable configuration. The reason is that the saddle point state connects two equilateral ground state configurations which can be obtained from each other by a simple rotation. For the simple case of 3 particles, we show the energy surface and the corresponding configurations schematically in Fig. 8. Notice that there are always more saddle points than minima for $N > 6$. With increasing the number of particles, more saddle point states are obtained and the energy landscape gets more complicated. For example for 9 particles, we obtain three saddle points and one metastable state. The results for the trajectories and energy landscape are shown in Fig. 9. Again, the ground state configurations corresponding with the black and the white dot are connected by a simple rotation, i.e. a symmetry operation.

**VI. CONCLUSION**

We presented the results of a numerical calculation of the configurations of the ground and all metastable states and their energies, the system consisting of classical 2D charged particles that are confined in a parabolic confinement potential for $N = 1, ..., 40$. These artificial atoms undergo configurational changes when the system transits from the ground state to the different metastable states, or between the different metastable states. Such transitions move through the lowest energy barrier connecting those states, i.e. through a saddle point. The connecting path from the ground state to all metastable states is found and the geometric properties of the energy landscape were discussed.

Sensitivity of the configuration on the form of the confinement potential and the interparticle interaction is investigated and a phase diagram was obtained. This sensitivity on e.g. the form of the confinement potential is probably the explanation why the experimental configuration [26] differs from our simulation results.

**VII. ACKNOWLEDGMENTS**

B. Partoens is a post-doctoral researcher of the Flemish Science Foundation (FWO-Vlaanderen). Stimulating discussions with Dr. J. Shi and M. Milošević are gratefully acknowledged. This work is supported by the Flemish Science Foundation (FWO-VI), the Belgian Inter-University Attraction Poles (IUAP-VI), the “Onderzoeksaan van de Universiteit Antwerpen” (GOA), the EU Research Training Network on “Surface Electrons on Mesoscopic Structures”, and INTAS.
[13] H. Thomas, G. E. Morfill, V. Demmel, J. Goree, B. Feuerbacher and D. M. öhlmann, Phys. Rev. Lett. 73, 652 (1994).
[14] Wen-Tau Juan, Zen-Hong Huang, Ju-Wang Hus, Yin-Ju Lai, and Lin I, Phys. Rev. E 58, 6947 (1998).
[15] P. Leiderer, W. Ebner, and V. B. Shikin, Surf. Sci. 113, 405 (1987).
[16] Nanostructure Physics and Fabrication, eds. M. A. Reed and W. P. Kirk (Academic Press, Boston, 1989).
[17] Y. Kondo, J.S. Korhonen, M. Krusius, V. V. Dmitriev, E. V. Thuneberg, and G. E. Volovik, Phys. Rev. Lett. 68, 3331 (1992).
[18] E. J. Yarmachuk and R. E. Packard, J. Low Temp. Phys. 46, 479 (1982).
[19] W. I. Galberson and K. W. Schwarz, Physics Today 40, 54 (1987).
[20] F. Chevy, K. W. Madison, and J. Dalibard, Phys. Rev. Lett. 85, 2223 (2000).
[21] D. Reedman and H. B. Brom, Physica C 183, 212 (1991).
[22] G. E. Volovik and U. Parts, Pis'ma Zh. Eksp. Teor. Fiz. 58, 826 (1993) [JETP Lett. 58, 774 (1993)].
[23] R. Bubeck, C. Bechinger, S. Neser, and P. Leiderer, Phys. Rev. Lett. 82, 3364 (1999).
[24] I. V. Schweigert, V. A. Schweigert, and F. M. Peeters, Phys. Rev. Lett. 84, 4381 (2000).
[25] C. A. Murray and D. M. Winkle, Phys. Rev. Lett. 58, 1200 (1987).
[26] M. Saint Jean, C. Even, and C. Guthmann, cond-mat/0101283.
[27] V. M. Bedanov and F. M. Peeters, Phys. Rev. B 49, 6667 (1994).
[28] V. A. Schweigert and F. M. Peeters, Phys. Rev. B 51, 7700 (1995).
[29] F. M. Peeters, V. A. Schweigert, and V. M. Bedanov, Physica B, 212, 237 (1995).
[30] Ying-Ju Lai and Lin I, Phys. Rev. E 60, 4743 (1999).
[31] L. Candido, J. P. Rino, N. Studart, and F. M. Peeters, J. Phys.: Condens. Matter 10, 11627 (1998).
[32] B. Partoens and F. M. Peeters, J. Phys.: Condens. Matter 9, 5383 (1997).
[33] L. J. Campbell and R. M. Ziff, Phys. Rev. B 20, 1886 (1979).
[34] P. Cheung, M. F. Choi, and P. M. Hui, Solid Stat. Commun. 103, 357 (1997).
[35] C. J. Cerjan and W. H. Miller, J. Chem. Phys. 75 (6), 2800, (1981).
[36] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. M. Teller, and E. Teller, J. Chem. Phys. 21, 1087 (1953).
[37] V. A. Schweigert and F. M. Peeters, Phys. Rev. Lett. 83, 2490 (1999).
[38] G. Date, M. V. N. Murthy, and R. Vathsan, J. Phys.: Condens. Matter 10, 5875 (1998).

Tables

TABLE I. The ground state and the metastable states for N=1,..., 40 Coulombic particles confined in a 2D parabolic well. We give the energies (E/N), ∆E/N, the shell structure (N_1, N_2,...), the radius and width of the shell, and the lowest three normal mode frequencies of the ground state configuration.

TABLE II. The energies of the ground state, the metastable states and the saddle point states for different number of particles (N).

Figure captions

FIG. 1. The lowest eigenfrequency and ∆E/N as function of the number of particles.

FIG. 2. The width of the different shells (logarithmic scale) as function of the number of particles.

FIG. 3. The phase diagram for the ground state of 9 particles. The dependence on the form of the confinement potential and the interparticle interaction is shown.

FIG. 4. The phase diagram for the ground state of 16 particles. The dependence on the form of the confinement potential and the interparticle interaction is shown.

FIG. 5. The trajectories of the particles making a transition from the ground state to the metastable state and the saddle point connecting them for 5 particles.

FIG. 6. The energy landscape and transition between the ground state to the metastable states for 6 particles.

FIG. 7. Part of the energy landscape and corresponding configurations near the metastable state for 6 particles.

FIG. 8. Schematic view of the energy surface and projection of the energy and the corresponding configurations for 3 particles.

FIG. 9. The energy landscape and transition from ground state to metastable states for 9 particles.
| N | E/N | $\Delta E/N$ | Configuration | Radius of the ring | Width of ring | Lowest Eigenfrequency |
|---|-----|--------------|---------------|-------------------|--------------|----------------------|
|   |     |              |               |                   |              | 1        | 2        | 3        |
| 1 | –   | –            | 1             | –                 | –            |          |          |          |
| 2 | 0.75| –            | 2             | 0.25000           | 0            | 1.41421  | 2.44949  |
| 3 | 1.31037| –        | 3             | 0.43679           | 0            | 1.41421  | 1.73205  | 2.44949  |
| 4 | 1.83545| –         | 4             | 0.61182           | 0            | 1.25189  | 1.41421  | 1.86483  |
| 5 | 2.33845| –          | 5             | 0.79483           | 0            | 1.02866  | 1.41421  | 1.94009  |
|    |     |              |               |                   |              |          |          |          |
|    | 2.36556| 0.02711     | 1.4           | (0)               | 0.98565      | 0        |          |          |
| 6 | 2.80456| –          | 1.5           | (0)               | 1.12182      | 0        |          |          |
|    |     |              |               |                   |              |          |          |          |
|    | 2.82476| 0.0202      | 6             | 0.94159           | 0.13974      |          |          |          |
| 7 | 3.23897| –          | 1.6           | (0)               | 1.25960      | 0        |          |          |
| 8 | 3.6689 | –            | 1.7           | (0)               | 1.39768      |          |          |          |
| 9 | 4.08812| –           | 2.7           | 0.18691           | 0.02355      | 0.12681  | 0.7754   | 0.91669  |
|    |     |              |               |                   |              |          |          |          |
|    | 4.09426| 0.00614     | 1.8           | (0)               | 1.53535      | 0        |          |          |
| 10| 4.48494| –           | 2.8           | 0.18189           | 0.47320      | 0.0891   | 0.97483  | 0.9852   |
|    |     |              |               |                   |              |          |          |          |
|    | 4.48816| 0.00322     | 3.7           | 0.32295           | 0.39275      | 0.02451  | 0.73727  | 0.83817  |
| 11| 4.86467| –           | 3.8           | 0.31407           | 0.61625      | 0.53084  | 0.89022  | 1.15011  |
|    |     |              |               |                   |              |          |          |          |
|    | 5.23894| –           | 3.9           | 0.31290           | 0.42935      | 0.6002E-4| 0.75236  | 0.75546  |
|    |     |              |               |                   |              |          |          |          |
|    | 5.24204| 0.0031      | 4.8           | 0.44386           | 0.32645      | 0.04940  | 0.79446  | 0.84578  |
| 12| 5.60114| –           | 4.9           | 0.43824           | 0.36449      | 0.45989  | 0.68255  | 0.75956  |
|    |     |              |               |                   |              |          |          |          |
|    | 5.95898| –           | 4.10          | 0.43499           | 0.42444      | 0.55826  | 0.66241  | 0.75956  |
|    |     |              |               |                   |              |          |          |          |
|    | 5.9629| 0.00371     | 5.9           | 0.56868           | 0.24511      | 0.12681  | 0.7754   | 0.91669  |
| 13| 6.30758| –           | 5.10          | 0.56408           | 0.30545      | 0.45989  | 0.68255  | 0.75956  |
|    |     |              |               |                   |              |          |          |          |
|    | 6.31554| 0.00796     | 1.5,9         | (-2.8E-2, -6.7E-3)| 0.07668      | 0.07668  | 0.41117  |          |
| 14| 6.6499 | –           | 1.5,10        | (0)               | 0.49686      | 0.49237  | 0.66241  | 0.92179  |
|    |     |              |               |                   |              |          |          |          |
|    | 6.65235| 0.00245     | 5.11          | 0.55826           | 0.43949      | 0.05416  | 0.54796  | 0.55927  |
| 15| 6.9829 | –           | 1.6,10        | (0)               | 0.49686      | 0.49237  | 0.66241  | 0.92179  |
|    |     |              |               |                   |              | 0.05126  | 0.54796  | 0.55927  |
| 16| 6.98433| 0.00143     | 1.5,11        | (-2.7E-4, -5.3E-3)| 0.58075      | 0.58075  | 0.58075  |          |
| 17| 7.30814| –           | 1.6,11        | (1.9E-3, -2.3E-3)| 0.39138      | 0.39138  | 0.39138  |          |
|    |     |              |               |                   |              |          |          |          |
|    | 7.31522| 0.00708     | 1.7,10        | (0)               | 0.09771      | 0.09771  | 0.09771  |          |
| 18| 7.32316| 0.01502     | 6.12          | 0.68919           | 0.61198      | 0.00614  | 0.65492  | 0.6782   |
|    |     |              |               |                   |              | 0.00614  | 0.65492  | 0.6782   |
|   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 19 | 7.63193 | – | 16,12 | (0,0) | 0.92663 | 3.56465 | 0 | 0.51668 | 0.66759 | 0.70531 | 1.09892 |
| 20 | 7.6328 | 8.7E-4 | 17,11 | (0.0001, 0) | 1.04683 | 3.72848 | 0.00673 | 0.21144 | 1.031E-4 | 0.62728 | 0.6926 |
| 21 | 7.94961 | – | 17,12 | (0,0) | 1.03672 | 3.81170 | 0.01262 | 0.28287 | 0.00317 | 0.63738 | 0.71452 |
| 22 | 8.26588 | – | 17,13 | (0.0005, 0.0013) | 1.02997 | 3.89626 | 0.20026 | 0.53913 | 0.29341 | 0.4018 | 0.5569 |
| 23 | 8.26645 | 5.7E-4 | 27,12 | 0.15456 | 1.27726 | 4.05127 | 0.02849 | 0.44673 | 0.12867 | 0.4083 | 0.58505 |
| 24 | 8.26756 | 0.00168 | 18,12 | (0,0) | 1.16048 | 4.04909 | 0.40055 | 0.44872 | 0.01262 | 0.28287 | 0.01487 |
| 25 | 8.57418 | – | 28,12 | 0.15569 | 1.39275 | 4.28533 | 0.62765 | 0.42346 | 0 | 0.67265 | 0.42346 |
| 26 | 8.57568 | – | 27,13 | 0.15132 | 1.26862 | 4.13118 | 0.43922 | 0.74099 | 0.01262 | 0.28287 | 0.01487 |
| 27 | 8.87758 | – | 28,13 | 0.15121 | 1.37540 | 4.36583 | 9.141E-4 | 0.55788 | 0.12867 | 0.4083 | 0.58505 |
| 28 | 8.87859 | 0.00101 | 38,12 | 0.25859 | 1.62401 | 4.52511 | 0.05289 | 0.66983 | 0.01262 | 0.28287 | 0.01487 |
| 29 | 9.17590 | – | 38,13 | 0.25419 | 1.60496 | 4.60039 | 0.00329 | 0.48021 | 0.01262 | 0.28287 | 0.01487 |
| 30 | 9.17756 | 0.00166 | 39,12 | 0.26352 | 1.73670 | 4.74997 | 0.56748 | 0.41325 | 0.01262 | 0.28287 | 0.01487 |
| 31 | 9.47079 | – | 39,13 | 0.25718 | 1.71084 | 4.82724 | 0.01301 | 0.51012 | 0.11377 | 0.5032 | 0.52713 |
| 32 | 9.47292 | 0.00213 | 38,14 | 0.25286 | 1.59306 | 4.67414 | 0.00985 | 0.43153 | 0.11377 | 0.5032 | 0.52713 |
| 33 | 9.47485 | 0.00406 | 48,13 | 0.36056 | 1.83873 | 4.83115 | 0.02597 | 0.42038 | 0.11377 | 0.5032 | 0.52713 |
| 34 | 9.76273 | – | 39,14 | 0.25454 | 1.69487 | 4.89949 | 0.00247 | 0.46781 | 0.10409 | 0.56681 | 0.61503 |
| 35 | 9.76383 | 0.0011 | 49,13 | 0.35766 | 1.93603 | 5.05885 | 0.04274 | 0.42849 | 0 | 0.37880 | 0.56307 |
| 36 | 10.0509 | – | 49,14 | 0.35397 | 1.91746 | 5.12748 | 0.01222 | 0.38138 | 0.01311 | 0.37880 | 0.56307 |
| 37 | 10.0527 | 0.0018 | 50,13 | 0.35930 | 2.04083 | 5.27916 | 0.07311 | 0.50131 | 0.01311 | 0.37880 | 0.56307 |
| 38 | 10.3356 | – | 50,14 | 0.35490 | 2.01900 | 5.34667 | 0.03475 | 0.46783 | 0.05682 | 0.17410 | 0.47677 |
|    |    |    |    |    | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
|----|----|----|----|----|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 29 | 10.6181 | 4,10,15 | 0.35181 | 2.00193 | 0.0022 | 5.19711 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 10.6193 | 5,10,14 | 0.46218 | 2.23618 | 0.0022 | 5.57000 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 10.6204 | 4,11,14 | 0.36293 | 2.12559 | 0.0022 | 5.59935 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 30 | 10.8973 | 5,10,15 | 0.45853 | 2.21795 | 0.0022 | 5.63340 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 10.8985 | 4,11,15 | 0.36734 | 2.10775 | 0.0022 | 5.69516 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 10.8999 | 4,10,16 | 0.35100 | 1.98987 | 0.0022 | 5.48103 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 10.9000 | 1,5,10,14 | (-2.3E-2, 1.7E-2) | 0.68914 | 2.45675 | 0.01628 | 5.78471 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 31 | 11.1739 | 5,11,15 | 0.45640 | 2.31542 | 0.0022 | 5.84745 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 11.1743 | 4E-4 | 1,5,10,15 | (0.0) | 0.68175 | 0.01628 | 2.43355 | 0.01628 | 5.84826 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 11.1754 | 5,11,14 | (-0.08, 0.1) | 0.68601 | 2.55743 | 0.01628 | 5.99410 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 11.1756 | 5,10,16 | 0.45469 | 2.20105 | 0.01628 | 5.69982 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 32 | 11.4466 | 1,5,11,15 | (-1.3E-2, 2.12E-4) | 0.67693 | 2.52730 | 0.01628 | 6.06078 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 11.4479 | 5,11,16 | 0.45336 | 2.29882 | 0.01628 | 5.90984 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 11.4481 | 1,6,10,15 | (-1.41E-4, -1.1E-2) | 0.77284 | 2.64856 | 0.01628 | 6.06602 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 33 | 11.7156 | 1,6,11,15 | (-3.2E-3, 2.6E-4) | 0.76976 | 2.73747 | 0.01628 | 6.27609 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 11.717 | 1,5,11,16 | (4.6E-3, 4.1E-3) | 0.67207 | 2.50792 | 0.01628 | 6.12124 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 34 | 11.9826 | 1,6,12,15 | (0.0) | 0.77212 | 0.01628 | 2.83337 | 0.01628 | 6.47800 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
| 11.9829 | 3E-4 | 1,6,11,16 | (5.4E-3, -1.5E-3) | 0.77212 | 0.01628 | 2.83337 | 0.01628 | 6.47800 | 0.01628 | 0.35006 | 0.58122 | 0.03407 | 0.44070 | 0.57886 | 0.03911 | 0.12706 | 0.57988 |
|               |   |                  |                  |                  |
|----------------|---|------------------|------------------|------------------|
|               |   | 0.76308          | 2.71525          | 6.33500          |
| 11.9856       | 0.003 | 1,7,11,15 | (-4.4 E-3, -1.5 E-3) | 0.01639          |
|               |   | 0.76569          | 2.90687          | 6.53777          |
| 35            | 12.2469 | (0.0) | 0.00196          | 0.43662          |
|               |   | 0.86132          | 2.92300          | 6.54591          |
| 12.2500       | 0.0031 | 1,7,11,16 | (2.2 E-3, -3.6 E-3) | 0.04942          |
|               |   | 0.77386          | 2.93587          | 6.67458          |
| 12.251        | 0.0041 | 1,6,13,15 | (-1.3 E-2, -7.3 E-3) | 0.18235          |
|               |   | 0.77386          | 2.93587          | 6.67458          |
| 36            | 12.5108 | (7 E-4, 1.7 E-3) | 0.00271          | 0.53950          |
|               |   | 0.76905          | 2.78682          | 6.59542          |
| 12.5111       | 3E-4 | 1,7,12,16 | (2.9 E-3, 2.9 E-3) | 0.04137          |
|               |   | 0.76430          | 2.90429          | 6.73797          |
| 12.5124       | 0.0016 | 1,6,13,16 | (-2.5 E-3, 4.5 E-3) | 0.06647          |
|               |   | 0.76430          | 2.90429          | 6.73797          |
| 12.5135       | 0.0027 | 1,7,13,15 | (-0.041, -0.023) | 0.20913          |
|               |   | 0.86813          | 3.14480          | 6.88005          |
| 12.5139       | 0.0031 | 1,7,11,17 | (-2.31 E-4, 3.8 E-3) | 0.03027          |
|               |   | 0.85528          | 2.90243          | 6.60309          |
| 37            | 12.7719 | (7 E-4, 1.7 E-3) | 0.00321          | 0.26342          |
|               |   | 0.85232          | 2.98912          | 6.80497          |
| 12.7724       | 5E-4 | 1,7,13,16 | (-2.21E-3, 3.84 E-4) | 0.06034          |
|               |   | 0.85858          | 3.10390          | 6.94783          |
| 12.7732       | 0.0013 | 1,6,13,17 | (0.013, -0.013) | 0.15906          |
|               |   | 0.76718          | 2.88302          | 6.79140          |
| 38            | 13.0304 | (3.3 E-5, 5.8 E-5) | 0.00613          | 0.16459          |
|               |   | 0.85267          | 3.07918          | 7.00319          |
| 13.0325       | 0.0021 | 2,8,12,16 | (2.21E-3, 3.84 E-4) | 0.22850          |
|               |   | 0.76718          | 2.88302          | 6.79140          |
| 13.0327       | 0.0023 | 1,7,12,18 | (1 E-3, 4.1 E-4) | 0.02664          |
|               |   | 0.84689          | 2.96973          | 6.86200          |

4
| E     | r     | ω     | 2,7,12,17 | 0.05029 | 1.05842 | 3.19102 | 7.00696 | 0.07477 | 0.43210 | 0.68153 | 0.85628 |
|-------|-------|-------|-----------|---------|---------|---------|---------|---------|---------|---------|---------|
| 13.0328 | 0.0024 | 2,7,12,17 | 0.08029 | 1.05842 | 3.19102 | 7.00696 | 0.07477 | 0.43210 | 0.68153 | 0.85628 |
| 13.0331 | 0.0027 | 1,7,14,16 | (8.98 E-4, 5.33E-4) | 0.88305 | 3.20222 | 7.13834 | 0.09294 | 1.01212 | 0.71661 |
| 39     | 13.2879 | 2,8,12,17 | 0.18669 | 1.9667 | 3.37497 | 7.19592 | 0.86305 | 4.18807 | 0.79679 |
| 13.2881 | 2E-4   | 2,7,13,17 | 0.13162 | 1.06169 | 3.28010 | 7.20053 | 0.09294 | 1.01212 | 0.71661 |
| 13.2882 | 3E-4   | 1,7,13,18 | (9 E-4, 0.0033) | 0.84868 | 3.05915 | 7.05762 | 0.05232 | 0.67493 | 0.83314 |
| 13.2885 | 6E-4   | 1,7,14,17 | (2 E-4, -5 E-4) | 0.85627 | 3.17395 | 7.19535 | 0.04100 | 0.86456 | 0.82641 |
| 13.2894 | 0.0015 | 2,7,14,16 | 0.13300 | 1.07631 | 3.41100 | 7.32553 | 0.0028 | 0.53723 | 1.50092 | 0.77203 |
| 40     | 13.5419 | 2,8,13,17 | 0.13522 | 1.15191 | 3.47914 | 7.40261 | 0.03091 | 0.62836 | 1.00580 | 0.60815 |
| 13.5423 | 4E-4   | 2,8,14,16 | 0.14575 | 1.15790 | 3.61776 | 7.51663 | 0.04100 | 0.86456 | 0.82641 |
| 13.5429 | 0.001  | 2,7,14,17 | 0.12997 | 1.06258 | 3.37175 | 7.39235 | 0.01194 | 0.46342 | 1.29363 | 0.95848 |
| 13.5434 | 0.0015 | 1,2,8,13,17 | (0.165, 0.214) | 0.39651 | 1.27319 | 3.47211 | 7.39190 | 0.01194 | 0.46342 | 1.29363 | 0.95848 |

E, r, ω are given in units of E₀, r₀, ω₀/√2

Table I. Minghui Kong et al.
This figure "fig2.png" is available in "png" format from:

http://arxiv.org/ps/cond-mat/0106395v1
| Number of Particles (N) | Energy of Ground state | Energy of Metastable state | Energy of Saddle point |
|-------------------------|------------------------|---------------------------|------------------------|
| 1                       | -                      | -                         | -                      |
| 2                       | 0.75000                | -                         | -                      |
| 3                       | 1.31037                | -                         | 1.46201                |
| 4                       | 1.83545                | -                         | 1.92064                |
| 5                       | 2.33845                | 2.36556                   | 2.36829                |
| 6                       | 2.80456                | 2.82476                   | 2.82689                |
| 7                       | 3.23897                | -                         | 3.27913                |
| 8                       | 3.66890                | -                         | 3.68738                |
| 9                       | 4.08812                | 4.09426                   | 4.08813                |
| 10                      | 4.48494                | 4.48816                   | 4.48495                |
| 11                      | 4.86467                | -                         | 4.87829                |
| 12                      | 5.23894                | 5.24204                   | 5.23955                |
| 13                      | 5.60114                | -                         | 5.61202                |
| 14                      | 5.95898                | 5.96269                   | 5.96335                |
| 15                      | 6.30758                | 6.31554                   | 6.30832                |
| 16                      | 6.64990                | 6.65235                   | 6.65117                |
| 17                      | 6.98290                | 6.98433                   | 6.98614                |
| 18                      | 7.30814                | 7.31522, 7.32316          | 7.31522                |
|    |        |        | 7.32154 | 7.32390 |
|----|--------|--------|---------|---------|
| 19 | 7.63193| 7.63280| 7.63358 | 7.63516 |
|    |        |        | 7.64155 | 7.64258 |
|    |        |        | 7.64266 | 7.64563 |
| 20 | 7.94961| 7.95623| 7.95637 | 7.95638 |
|    |        |        | 7.95639 | 7.95640 |
|    |        |        | 7.95641 | 7.95654 |
|    |        |        | 7.95693 | 7.95695 |
|    |        |        | 7.95709 | 7.95725 |
|    |        |        | 7.95739 | 7.95817 |
|    |        |        | 7.96316 |         |

*E is given in unit of $E_0$*

Table II. Minghui Kong *et al.*
This figure "fig3.png" is available in "png" format from:

http://arxiv.org/ps/cond-mat/0106395v1
This figure "fig4.png" is available in "png" format from:

http://arxiv.org/ps/cond-mat/0106395v1
This figure "fig5.png" is available in "png" format from:

http://arxiv.org/ps/cond-mat/0106395v1
This figure "fig6.png" is available in "png" format from:

http://arxiv.org/ps/cond-mat/0106395v1
This figure "fig7.png" is available in "png" format from:

http://arxiv.org/ps/cond-mat/0106395v1
This figure "fig8.png" is available in "png" format from:

http://arxiv.org/ps/cond-mat/0106395v1
This figure "fig9.png" is available in "png" format from:

http://arxiv.org/ps/cond-mat/0106395v1