Don’t relax: early stopping for convex regularization

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Abstract

We consider the problem of designing efficient regularization algorithms when regularization is encoded by a (strongly) convex functional. Unlike classical penalization methods based on a relaxation approach, we propose an iterative method where regularization is achieved via early stopping. Our results show that the proposed procedure achieves the same recovery accuracy as penalization methods, while naturally integrating computational considerations. An empirical analysis on a number of problems provides promising results with respect to the state of the art.

Keywords: monotone inclusion, maximal monotone operator, operator splitting, cocoercive operator, composite operator, duality, stochastic errors, primal-dual algorithm

Mathematics Subject Classifications (2010): 47H05, 49M29, 49M27, 90C25

1 Introduction

Many machine learning problems require to estimate a quantity of interest based on random noisy data/measurements. Towards this end, a common approach is considering estimators defined by the minimization of an empirical objective, where a data fit term is penalized using a regularizer,
encoding prior information on the quantity to be estimated. From a modeling perspective, this latter approach can be seen as the relaxation of an ideal problem with equality constraints defined by exact data, whereas from a computational perspective it reduces in principle to the solution of a single optimization problem.

In practice however, the regularization parameter needs to be chosen, and hence the solution of multiple optimization problems is typically required. Moreover, computational and estimation aspects are usually considered separately, leading to potential dichotomy and trade-offs between estimation and computational aspects [11]. Indeed, these observations have recently motivated the development of techniques to compute solutions corresponding to different penalization levels (regularization path) [20, 23] as well as an interest on the interplay between estimation and computation [27].

In this paper, we investigate and apply iterative regularization techniques in the context of linear inverse problems modeling many machine learning problems. The key idea behind iterative regularization is that early stopping the iterative optimization of an empirical problem, performs a form of implicit regularization [21]. Iterative regularization algorithms are classical in inverse problems [21] and have been recently analyzed and applied in machine learning to find the minimal norm solution [1, 9, 28, 29, 33, 34]. These works show that iterative regularization methods typically share the same estimation properties of penalized methods, but are often advantageous from a computational perspective. Indeed, since the number of iterations becomes the regularization parameter, iterative regularization schemes have a built-in warm restart property that allows to easily compute a whole regularization path, if the involved regularizer is the squared norm one.

The main question we discuss in this work is how to derive and analyze fast iterative regularization schemes for large classes of regularizers. Indeed, flexibility in the choice of this latter functional is key for good estimation and has been the subject of much recent work. However, while how to exploit such penalties is clear using relaxation approaches, how to derive corresponding iterative regularization schemes is less obvious.

In this paper, we derive iterative regularization for a strongly convex regularizer, by considering the iterative minimization of this latter functional under equality constraints defined by the noisy data (rather than a relaxation). The iteration thus obtained does not converge to the desired solution, but can be shown to be robust to noise if suitably stopped. Indeed, a stability argument shows the noisy iteration deviates gradually from a noiseless iteration which in turns can be shown to converge to the ideal solution. Exploiting this latter result, an optimal stopping rule and the corresponding recovery results can be derived.

We explore this general idea considering two distinct iterations. The first is based on a dual gradient descent (a.k.a. mirror descent [5], and linearized Bregman iteration [12]), while the second corresponds to an accelerated variant [7]. While both methods are shown to lead to the same recovery guarantees, acceleration allows for more aggressive stopping rules with substantial computational gains.

The idea of considering iterative regularization and early stopping for convex regularizers is not new, we refer to [14] for an interesting survey on known results, open problems, and additional references. Some previous approaches [25] rely on Morozov discrepancy principle [21], other approaches are based on stability, see [13, 10]. However the existing studies do not analyze the
algorithms presented in this paper. More importantly, we are not aware of any previous results considering the regularization effect of accelerated iterations [24].

Our theoretical findings are complemented by empirical results on three different applications: variable selection, matrix completion, and image deblurring. The experiments confirm the theoretical results and show that the recovery properties of iterative regularization are comparable to penalization approaches with much lower computational costs.

The rest of the paper is organized as follows: in Section 2 we describe the setting and the main assumptions, in Section 3 we introduce the iterations we study, and in Section 4 we state the main results, discuss them, and provide the main elements of the proof. In Section 5 we present several experimental results on matrix completion, variable selection, and deblurring problems.

2 Problem setting

We consider a general problem of the form

$$y = Xw^\dagger,$$  \hspace{1cm} (2.1)

for a given matrix $X : \mathbb{R}^p \rightarrow \mathbb{R}^n$, an observation $y \in \mathbb{R}^n$, and a vector $w^\dagger \in \mathbb{R}^p$. Such formulation include for instance regression, feature selection, as well as many image/signal processing problems. In general, the solution of the above linear equation is not unique, and a selection principle is needed to choose an appropriate solution (e.g. in the high dimensional scenario, where $p > n$). In this paper we assume that the solution of interest $w^\dagger$ minimizes a function $R : \mathbb{R}^p \rightarrow [\rightarrow -\infty, +\infty]$ encoding some prior information on the problem at hand. We assume $R$ to be proper, lower semicontinuous, strongly convex, and we let $w^\dagger$ to be the unique solution of the optimization problem

$$\min_{y= Xw} R(w).$$  \hspace{1cm} (2.2)

In practice, one does not have access to $y$, but only to a noisy version $\tilde{y}$. In particular in this paper we consider a worst case scenario, where the noise is deterministic, i.e. $\|y - \tilde{y}\| \leq \delta$, for some $\delta > 0$. The goal is then to find a stable estimation of $w^\dagger$ only observing $X$ and $\tilde{y}$.

The classical way to achieve this goal is to relax the equality constraints, and use a Tikhonov regularization scheme:

$$\min_{w\in \mathbb{R}^p} \|\tilde{y} - Xw\|^2 + \lambda R(w).$$

A data fidelity term is added to the function $R$, multiplied by a regularization parameter $\lambda$. Such an approach usually requires two steps: first, the solution of a regularized problem for several values of the regularizing parameter, and second the model selection, where the best regularized solution is selected among the computed ones.

In this paper we avoid relaxation, and consider iterative regularization schemes. We define a sequence $(\hat{w}_k)_{k\in \mathbb{N}}$ derived by applying an appropriate minimization algorithm to the noisy problem

$$\min_{\tilde{y} = Xw} R(w).$$  \hspace{1cm} (2.3)

\footnote{For simplicity, the results are stated in finite dimensional euclidean spaces, but all the conclusions hold if $\mathbb{R}^p$ and $\mathbb{R}^n$ are replaced by Hilbert spaces $\mathcal{H}$ and $\mathcal{G}$.}
Such a procedure converges to a minimizer of the noisy problem (2.3), which is not the solution we are looking for, however a good solution can be achieved by early stopping. More precisely, we show that, depending on the noise level, we can select an element $\hat{w}_t$ of the sequence $(\hat{w}_t)_{t \in \mathbb{N}}$ which converges to $w^\dagger$ when the noise goes to zero. An intuition of why this procedure works can be derived from the proof's strategy. To analyze the behavior of the sequence $(\hat{w}_t)_{t \in \mathbb{N}}$ we define an auxiliary (regularizing) sequence $(w_t)_{t \in \mathbb{N}}$, that is the sequence obtained applying the same minimization algorithm devised for problem (2.3), to the ideal problem (2.2), which therefore converges to $w^\dagger$.

The choice of the stopping time will be derived by the following error decomposition

$$\|\hat{w}_t - w^\dagger\| \leq \|\hat{w}_t - w_t\| + \|w_t - w^\dagger\|.$$  

The term $\|w_t - w^\dagger\|$ is an optimization (or regularization) error. We will show that it vanishes for increasing $t$ and in fact will prove non asymptotic bounds. The term $\|\hat{w}_t - w_t\|$ measures stability to noise and we will see to increase with $t$ and $\delta$. Given data and knowledge of the noise level, our actual regularization procedure is specified by a suitable choice $t_\delta$ and this results in the explicit bound $\|\hat{w}_{t_\delta} - w^\dagger\| \leq c\delta^{1/2}$. Note that the dependence on the noise level $\delta$ is the same as in Tikhonov regularization [21]. In the rest of the paper, we develop the above idea providing all the details.

**Notation** In the following, the operator norm of the matrix $X$ is denoted by $\|X\|$.

## 3 Iterative regularization algorithm for a general penalty

In this section we begin presenting the iterative regularization procedures we study based on dual gradient descent (DGD) and accelerated dual gradient descent (ADGD). The first one is a basic algorithm, while the second is its accelerated version, requiring some additional steps. First, recall that the regularizing function $R$ in (2.3) is assumed to be strongly convex. This implies that there exists $\alpha \in [0, +\infty[$ and a proper, lower semicontinuous, and convex function $F: \mathbb{R}^p \to [0, +\infty[$ such that

$$R = F + \frac{\alpha}{2} \|\cdot\|^2. \quad (3.1)$$

Both DGD and ADGD belong to the class of first order methods, requiring only matrix and vector multiplications, and the computation of the proximity operator of $\alpha^{-1}F$, which is defined as

$$(\forall w \in \mathbb{R}^p) \quad \text{prox}_{\alpha^{-1}F}(w) = \arg\min_{u \in \mathbb{R}^p} \left\{ F(u) + \frac{\alpha}{2} \|u - w\|^2 \right\}. \quad (3.2)$$

The computation of the proximity operator involves a minimization problem, which can be solved explicitly in many relevant cases [19]. In particular, it reduces to the well-known soft-thresholding operator when $F$ is equal to the $\ell^1$ norm, and to a projection, when $F$ is the indicator function of a convex and closed set. We will show in the supplementary material that DGD reduces to a gradient descent on the dual of problem in (2.3). Its asymptotic minimization properties for the problem in (2.3), which is not the one we want to solve, have been studied in [18]. Note that this algorithm, up to a change of variables, is called linearized Bregman iteration in the series of papers [25, 2, 33, 12, 14]. The same algorithm is also called mirror descent in the optimization community [5]. By considering a Nesterov acceleration [24] of gradient descent, we derive ADGD,
that is the FISTA variant on the dual problem, which has been considered in [7, 32]. Algorithms DGD and ADGD can be seen as minimization algorithms applied to the dual of the original noise free problem in (2.2), in the presence of a nonvanishing error on the gradient.

### Dual Gradient Descent (DGD)

Let \( \hat{v}_0 = 0 \in \mathbb{R}^p \) and \( \gamma = \alpha \|X\|^{-2} \)

- For \( t = 0, 1, \ldots \) iterate
  
  \[
  \hat{w}_t = \text{prox}_{\alpha^{-1}F} \left( -\alpha^{-1}X^T \hat{v}_t \right) \\
  \hat{v}_{t+1} = \hat{v}_t + \gamma (X \hat{w}_t - \hat{y}) \\
  \hat{u}_t = \frac{1}{t+1} \sum_{k=0}^{t} \hat{w}_k
  \]

### Accelerated Dual Gradient Descent (ADGD)

Let \( \hat{v}_0 = \hat{v}_1 = \hat{z}_0 = 0 \in \mathbb{R}^p \), \( \gamma = \alpha \|X\|^{-2} \), and \( \theta_0 = 1 \)

- For \( t = 0, 1, \ldots \) iterate
  
  \[
  \hat{w}_t = \text{prox}_{\alpha^{-1}F} \left( -\alpha^{-1}X^T \hat{z}_t \right) \\
  \hat{r}_t = \text{prox}_{\alpha^{-1}F} \left( -\alpha^{-1}X^T \hat{v}_t \right) \\
  \hat{z}_t = \hat{v}_t + \gamma (X \hat{r}_t - \hat{y}) \\
  \theta_{t+1} = (1 + \sqrt{1 + 4\theta_t^2})/2 \\
  \hat{v}_{t+1} = \hat{z}_t + \frac{\theta_t - 1}{\theta_{t+1}} (\hat{z}_t - \hat{z}_{t-1})
  \]

Before studying the regularizing properties of the proposed procedures, we show that DGD is a generalization of the well-known Landweber iteration (see [21]).

**Remark 3.1 (Connections with Landweber iteration)** Consider Algorithm DGD in the special case \( F = 0 \). Noting that, for every \( w \in \mathbb{R}^p \), \( \text{prox}_{\alpha^{-1}F}(w) = w \), we derive

\[
\hat{w}_{t+1} = \hat{w}_t - \gamma \alpha^{-1}X^T (X \hat{w}_t - \hat{y}),
\]

which coincides with the Landweber iteration for solving Problem (2.2) studied in the context of regression in [33]. ADGD provides a FISTA variant of Landweber iteration, for which we prove here regularization properties.

The previous remark shows that the proposed algorithms are generalization of the Landweber iteration for a more general penalty term of the form in (3.1). While it is well known that early stopping of the Landweber iteration leads to stable approximations of the minimal norm solution of an inverse problem, here we generalize such result to obtain stable approximations of the solution defined by general regularizers. The presence of the additional term \( F \) in the regularization function introduces in the algorithm a (nonlinear) proximal operation.

### 4 Early stopping for strongly convex iterative regularization

In this section, we present and discuss the main results of the paper. We start with DGD.

**Theorem 4.1 (Dual gradient descent)** Let \( \delta \in [0, 1] \). Let \((\hat{u}_t)_{t \in \mathbb{N}}\) be the averaged sequence generated by DGD. Assume that there exists \( \bar{v} \in \mathbb{R}^p \) such that \(-X^T \bar{v} \in \partial R(v^1)\). Set \( a = 2\|X\|^{-1} \) and \( b = \|X\| \|v^1\| \alpha^{-1} \), where \( v^1 \) is a solution of the dual problem of (2.2). Then, for every \( t \in \mathbb{N} \),

\[
\|\hat{u}_t - v^1\| \leq a t^{1/2} \delta + b t^{-1/2}.
\]

In particular, choosing \( t_\delta = [c \delta^{-1}] \) for some \( c > 0 \), we derive

\[
\|\hat{u}_{t_\delta} - v^1\| \leq [a (c^{1/2} + 1) + bc^{-1/2}] \delta^{1/2}.
\]
Before discussing the above result, we state an analogous result for the accelerated variant.

**Theorem 4.2 (Accelerated dual gradient descent)** Let $\delta \in [0, 1]$ and let $(\hat{w}_t)_{t \in \mathbb{N}}$ be the sequence generated by ADGD. Assume that there exists $\hat{v} \in \mathbb{R}^p$ such that $-X^T\hat{v} \in \partial R(w^\dagger)$. Set $a = 4\|X\|^{-1}$ and $b = 2\|X\|\|v^\dagger\|/\alpha$, where $v^\dagger$ is a solution of the dual problem of (2.2). Then, for every $t \geq 2$,

$$\|\hat{w}_t - w^\dagger\| \leq a\delta + bt^{-1}. \quad (4.3)$$

In particular, choosing $t_\delta = \lceil c\delta^{-1/2} \rceil$ for some $c > 0$,

$$\|\hat{w}_t - w^\dagger\| \leq [a(c + 1) + bc^{-1}]\delta^{1/2}. \quad (4.4)$$

We first discuss the results and make a comparison with related work, and then give a sketch of the proof. The complete proof can be found in the supplementary material.

**Discussion and comparison with related work** As anticipated in Section 2, the bounds in (4.2) and (4.4) are derived by optimizing a stability plus regularization/optimization bound. Note in particular that the constants appearing in the regularization error are determined by the strong convexity constant and the norm of the operator $X$. The above results show that, given a noise level $\delta$, regularization is achieved computing a suitable number $t_\delta$ of iterations of DGD and ADGD.

The number of required iterations tends to infinity as the noise goes to zero. The definition of $t_\delta$ in Theorems 4.1 and 4.2 is an early stopping rule. The dependence of the noise that we get in Theorems 4.1 and 4.2 is optimal [21], and coincides with the Tikhonov regularization one. The difference between DGD and ADGD is on the computational aspect: indeed, to achieve the same recovery accuracy, a number of iterations of the order of $\delta^{-1}$ are needed for the basic scheme, and only $\delta^{-1/2}$ iterations are needed for the accelerated method. This kind of result resembles the behaviour of the $\nu$-method for the minimal norm solution [21].

The condition $-X^T\hat{v} \in \partial R(v^\dagger)$ can be interpreted as an abstract regularity condition on the subdifferential of $R$ [14]. When $R = \|\cdot\|^2/2$, and more generally when $R$ is real-valued, it is automatically satisfied under our assumptions, and it corresponds to what is called a source condition [21, 17].

**Remark 4.3 (Avoiding averaging)** For DGD, regularizing properties are proved for the averaged sequence. However, if sparsity properties of the solution are of interest, averaging is not appropriate. For the nonaveraged sequence $\hat{w}_t$ defined by DGD, we have that for every $\delta \in [0, 1]$ there exists $t_\delta = O(\delta^{-1})$ such that $\|\hat{w}_t - w^\dagger\| \leq (a + 2b)\delta^{1/2}$, with $a$ and $b$ defined as in Theorem 4.1. See Proposition A.2 and Theorem A.1 in the supplementary material for the proof.

**Remark 4.4 (Inexact prox)** In some interesting cases, the proximity operator is not available in closed form, but can be still computed at reasonable cost (see [30, 32] for a throughout discussion). The results in Theorem 4.1 and 4.2 hold also if the proximity operator is computed inexactly, at an increasing precision.

**Remark 4.5 (Beyond worst case)** While we considered a general regularization $R$ and obtained worst-case results, an interesting question is if these results can be improved under additional assumptions on $R$, e.g. assuming it is sparsity inducing. This will be the subject of future work, and we refer to [26] for some results in this direction.
We next compare our iterative regularization methods with related work. The case $R = \| \cdot \|^2$ is classic, see [21]. In [25] an iterative regularization procedure based on the so called Bregman iteration, is considered. An early stopping rule based on a discrepancy principle in the case of noisy data is also presented. There is one main difference with respect to our contribution. Each DGD or ADGD step does not require inner algorithms if the proximity operator is available in closed form, while Bregman iteration requires the solution of a nontrivial minimization problem at each step. Such step is computationally as costly as solving a Tikhonov regularized problem. A stability analysis for the Bregman iteration is presented in Theorem 4.2 in [13], while weak convergence without the strong convexity assumption is proved in [25]. A qualitative early stopping rule for the DGD algorithm has been considered in [2] for the total variation case. Finally, a related algorithm to the DGD is the one considered in [10]. The setting of [10] is more general than ours, but the obtained results are weaker: the stopping rule is of the form $O(\delta^{-2})$ and no quantitative bounds of $\| \hat{w}_{t_d} - w^\dagger \|$ are given.

**Sketch of the proof**  We now discuss the main elements of the proof. The complete argument can be found in the supplementary material. We start from the proof of Theorem 4.1 and then we will briefly comment on the proof of Theorem 4.2. The proof of Theorem 4.1 is based on a decomposition of the error to be estimated in two terms. The idea is to build an auxiliary sequence and to majorize the error with the sum of two quantities that can be interpreted as a stability and an optimization (regularization) error, respectively. Bounds on these two terms are then provided.

We first introduce the corresponding algorithm to solve the target problem in (2.2). This algorithm is not used in practice, but is needed only for the theoretical analysis, and is the noise free version of DGD, where $\hat{y}$ is replaced by $y$. Starting from $v_0 = 0$, the $t$-th iteration is defined by

$$w_t = \text{prox}_{\alpha^{-1}F} (-\alpha^{-1}X^Tv_t), \quad u_{t+1} = v_t + \gamma(Xw_t - y), \quad u_t = \sum_{k=0}^{t} w_t/(t+1) \quad (4.5)$$

for the gradient descent algorithm applied to the dual of problem (2.2) (see the supplementary material for its definition). The choice of $t_d$ is derived from the the following error decomposition

$$\| \hat{u}_t - w^\dagger \| \leq \| \hat{u}_t - u_t \| + \| u_t - w^\dagger \|. $$

The term $\| u_t - w^\dagger \|$ is called approximation, but also optimization or regularization error. It vanishes for increasing $t$ and in fact the following non asymptotic bound holds $\| u_t - w^\dagger \| \leq \|X\|\|v^\dagger\|\alpha^{-1}t^{-1/2}$. The term $\| \hat{u}_t - u_t \|$ measures stability and its behavior for fixed $t$ and noise level $\delta$ is $\| \hat{u}_t - u_t \| \leq 2\|X\|^{-1}\delta t^{1/2}$. The choice of $t_d$ is obtained optimizing the resulting bound with respect to $t \in \mathbb{N}$, that is $t_d = \arg\min_{t \in \mathbb{N}} (\|X\|\|v^\dagger\|\alpha^{-1}t^{-1/2} + 2\|X\|^{-1}\delta t^{1/2})$.

The stopping rule for ADGD follows analogously from a general result about convergence of proximal methods in the presence of computational errors [1].

5 Numerical experiments

In this section we compare our iterative regularization techniques (DGD and ADGD with early stopping) with Tikhonov regularization on three different problems: variable selection, matrix
completion, and image deblurring. The performance of the Tikhonov regularization scheme depends of course on the chosen algorithm to solve the regularized problems. We use state of the art techniques: accelerated proximal gradient descent with warm-restart [22]. The model selection phase is performed as follows: we first solve the regularized problem with a very large value $\lambda_0$, and then for the sequence $\lambda_i = 2^{-i}\lambda_0$. Since in practice the noise level is unknown, we choose $\lambda$ using holdout cross-validation keeping 1/10 of the available points for validation. For initializing the accelerated gradient descent on the regularized problem we use the warm-restarting trick, which is known (in practice) to dramatically accelerate the computation of the regularization path [8]. The comparison relies heavily on the stopping rule used for stopping the iteration computing the minimizer of the Tikhonov regularized functional. We used a very loose stopping rule for the algorithm for a given $\lambda_i$ to make Tikhonov regularization more competitive. More precisely the iterations were stopped when the distance between to successive iterations was less than $0.001 \cdot \delta$. Since accelerated proximal gradient descent involves steps with the same computational complexity to those of DGD and ADGD, the comparison between the three approaches is made in terms of number of iterations. The number of iterations for Tikhonov regularization is the total number of iterations for all different $\lambda$ values.

5.1 Variable selection

We consider a linear regression problem with $n = 500$ examples and $p = 2000$ variables. We assume that $\tilde{y} \in \mathbb{R}^{500}$ is obtained corrupting with a Gaussian noise of mean zero and variance $\delta/\sqrt{n}$ a measurement $Xw_*$, where $w_*$ is a vector having a small number of nonzero components (10, 30, or 60, respectively). In this example, the covariates are correlated with a random covariance matrix $\Sigma$ with $\Sigma = C^TC$, where $C$ is a random matrix with entries drawn independently at random from a gaussian distribution with standard deviation 0.1. To perform variable selection, and obtain a sparse estimator we apply our iterative regularization methods, DGD and ADGD, to the elastic net regularizing function $R(w) = \|w\|_1 + (\alpha/2)\|w\|_2^2$. We compared the number of iterations of DGD, ADGD, and Tikhonov regularization on 50 different realizations of sample points. The parameters were chosen using a validation set of 100 samples. The results are shown in Table 1. For Tikhonov regularization, we used a second least squares step on the selected variables to compute the validation score, requiring an extra computation load that we did not quantify here. It is worth noticing that iterative regularization does not require this further step. The results suggest that Tikhonov regularization and iterative regularization algorithms have very similar prediction and variable selection performances. DGD is approximately as fast as state of the art variational regularization, while ADGD is much faster.

5.2 Matrix completion

We consider the problem of recovering a low-rank data matrix $W \in \mathbb{R}^{n \times p}$ from a sampling of its entries. We denote by $\Omega$ the subset of indices corresponding to sampled entries. We find an approximate solution of this problem by minimizing a strongly convex relaxation [13] given by the sum of the nuclear norm with the squared Frobenius norm, that is:

$$\min_{W \in \mathbb{R}^{n \times p}} \|W\|_* + \frac{\alpha}{2}\|W\|_F^2,$$

(5.1)
Table 1: Performances of DGD, ADGD, and warm-started Tikhonov regularization with accelerated proximal gradient descent. False positives are the selected irrelevant variables. False negatives are the discarded relevant features. Prediction error is the average prediction error of the estimated solution in percent. The results are averaged over 50 trials with the standard deviation between parentheses.

| Noise Variables | Algorithm   | False Positive | False Negative | Prediction Error | Iterations |
|-----------------|-------------|----------------|----------------|-----------------|------------|
| 10              | DGD         | 0.10 (0.3)     | 0.53 (0.7)     | 3.7 (0.6)       | 890 (200)  |
|                 | ADGD        | 0.40 (0.9)     | 0.53 (0.7)     | 3.7 (0.6)       | 140 (30)   |
|                 | Tikhonov    | 0.62 (1)       | 0.28 (0.5)     | 3.6 (0.3)       | 580 (40)   |
| 0.1             | DGD         | 8.8 (5)        | 1.8 (1)        | 4.8 (0.4)       | 860 (90)   |
|                 | ADGD        | 5.0 (5)        | 1.8 (1)        | 4.6 (0.4)       | 110 (16)   |
|                 | Tikhonov    | 12 (9)         | 2.1 (1)        | 5.4 (0.6)       | 860 (140)  |
| 0.6             | DGD         | 49 (10)        | 5.2 (2)        | 8.1 (0.8)       | 940 (100)  |
|                 | ADGD        | 27 (10)        | 5.7 (2)        | 7.4 (0.7)       | 170 (30)   |
|                 | Tikhonov    | 53 (20)        | 5.7 (2)        | 7.4 (0.7)       | 1800 (400) |
| 1               | DGD         | 2.2 (3)        | 2.7 (1)        | 46 (2)          | 480 (100)  |
|                 | ADGD        | 1.1 (2)        | 2.8 (1)        | 45 (2)          | 92 (40)    |
|                 | Tikhonov    | 2.3 (2)        | 2.9 (1)        | 48 (4)          | 360 (90)   |
| 30              | DGD         | 17 (10)        | 14 (3)         | 65 (3)          | 560 (50)   |
|                 | ADGD        | 14 (10)        | 15 (3)         | 64 (3)          | 220 (3)    |
|                 | Tikhonov    | 8.0 (7)        | 15 (2)         | 63 (4)          | 990 (300)  |
| 60              | DGD         | 40 (10)        | 33 (4)         | 77 (3)          | 560 (10)   |
|                 | ADGD        | 40 (20)        | 33 (6)         | 77 (3)          | 220 (3)    |
|                 | Tikhonov    | 35 (30)        | 36 (8)         | 78 (2)          | 1700 (500) |

where $\hat{Y} \in \mathbb{R}^{n \times p}$, is such that, for every $(i,j) \notin \Omega$, $\hat{Y}_{i,j} = 0$, and $\mathcal{X}: \mathbb{R}^{n \times p} \to \mathbb{R}^{n \times p}$ is such that $(\mathcal{X}W)_{i,j} = W_{i,j}$ if $(i,j) \in \Omega$ and 0 otherwise. DGD applied to this problem is the Singular Value Thresholding (SVT) algorithm described in [15] and note that, interestingly, ADGD is its accelerated counterpart. The most expensive computational part is the proximal step, which requires an SVD decomposition [19]. While in [15] the authors apply the algorithm to noisy data, they then propose as an improvement a different relaxation [16]. Here we show that SVT with early stopping is indeed an efficient algorithm to deal with matrix completion of noisy data. We tested the performance on simulated data using a standard procedure described in [15]. We multiplied random gaussian matrices with independent entries and variance 1 of size $n \times r$ and $r \times p$ where $r$ is the chosen rank, and then we added an additive gaussian noise. We computed the Root Mean Square Error of the proposed approximation: $\text{RMSE}(\hat{W}) = (\sum_{(i,j) \in A} (W_{i,j} - \hat{Y}_{i,j})^2)^{1/2}/|A|$, where $A$ is the test set. As can be seen in Table 2 ADGD is comparable to state of the art Tikhonov regularization, with a significantly lower computational cost. In addition, we compare DGD with Tikhonov regularization (with accelerated proximal gradient+warm restart) on the MovieLens 100k dataset[4]. We averaged our results over five trials. We left out one tenth of the known entries at each trial and chose the best step/parameter via 2-fold cross validation. The mean RMSE for DGD and Tikhonov was 1.02. It required 250 iterations on average using DGD, and 550 iterations using Tikhonov.

[4]http://grouplens.org/datasets/movielens/
Table 2: Minimal achieved RMSE and associated cost of ADGD and Tikhonov approach solved with warm-starting, accelerated proximal gradient method, on simulated data with additive gaussian noise of standard deviation $\delta$. We used the ground truth to select the best parameter. The percentage of known entries (knowledge ratio) is 0.12, 0.39 and 0.57 for, respectively, ranks 10, 50 and 100. The matrices are of size $1000 \times 1000$. The results were averaged over 5 simulations with the standard deviation between parentheses. For ADGD, the iterative trials were capped at 500 iterations (for noise levels of 0.01) and 250 (noise of 0.1 and 1).

| Noise | Rank | RMSE ADGD | RMSE Tikhonov | Iterations ADGD | Iterations Tikhonov |
|-------|------|-----------|---------------|-----------------|---------------------|
| 0.01  | 10   | $2.1 \cdot 10^{-3} (3.8 \cdot 10^{-5})$ | $7.6 \cdot 10^{-3} (1.5 \cdot 10^{-4})$ | 500 (0) | 527 (3) |
|       | 50   | $3.2 \cdot 10^{-3} (2.2 \cdot 10^{-5})$ | $9.1 \cdot 10^{-3} (4.1 \cdot 10^{-5})$ | 500 (0) | 295 (1) |
|       | 100  | $4.7 \cdot 10^{-1} (2.9 \cdot 10^{-1})$ | $1.1 \cdot 10^{-1} (6.4 \cdot 10^{-1})$ | 500 (0) | 273 (1) |
| 0.1   | 10   | $0.23 (4.6 \cdot 10^{-4})$ | $0.75 (9.0 \cdot 10^{-5})$ | 250 (0) | 539 (3.7) |
|       | 50   | $0.35 (2.1 \cdot 10^{-4})$ | $0.95 (2.7 \cdot 10^{-5})$ | 250 (0) | 425 (0.49) |
|       | 100  | $0.48 (2.0 \cdot 10^{-4})$ | $1.1 (4.3 \cdot 10^{-1})$ | 190 (0) | 470 (0.4) |
| 1     | 10   | 27 (0.28) | 76 (1.1) | 191 (0) | 698 (3.6) |
|       | 50   | 41 (0.20) | 108 (0.20) | 205 (0.4) | 729 (3.6) |
|       | 100  | 55 (0.18) | 125 (0.11) | 210 (0.4) | 742 (2.8) |

Figure 1: From left to right: original Cameraman image, noisy blurred image, restored image with Tikhonov regularization, restored image with ADGD.

### 5.3 Image deblurring

Finally, we apply ADGD to an image processing problem, namely deblurring, with a strongly convex perturbation of total variation. More precisely, given an image $W \in \mathbb{R}^{256 \times 256}$, we consider the regularization function $R(W) = TV(W)_{1,2} + \frac{3}{2} ||W||^2$, where $TV$ is the discrete total variation. In this application the proximity operator of the total variation penalty is not available in closed form. In our experiments, this is computed at each iteration using 20 steps of accelerated dual forward backward on the denoising problems corresponding to (3.2), and by warm starting with the previous approximate proximal point. We assume to have access to a noisy image $\hat{y}$, obtained corrupting the original image with a Gaussian blur of one pixel and an additive Gaussian noise with variance 0.01. We compared the iterative regularization ADGD with early stopping with the solution obtained with the Tikhonov approach corresponding to the best regularization parameter on the cameramen image. The quality of an approximation of the original image is measured in terms of PSNR, and the best results are reported in Figure 1. On the computational side, for the Tikhonov approach we set $\lambda_0 = 10^5$, and then decreased it by multiplying it by 0.8 at each step. The best solution is obtained for $\lambda = 6.8$, while iterative regularization achieves the best results at the third iteration.
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A Supplementary Material

A.1 Derivation of the algorithm

We start showing that the proposed procedures DGD and ADGD are indeed a gradient and an accelerated gradient descent algorithm applied to the dual problem of the noisy minimization problem

\[
\min_{Xw=\hat{y}} R(w), \quad \text{with } R = F + \frac{\alpha}{2} \| \cdot \|^2. \tag{A.1}
\]

Let \( C \) be a convex and closed subset of \( \mathbb{R}^n \). With \( \delta_C \) we denote the indicator function of \( C \), which takes value 0 on \( C \) and \( +\infty \) otherwise. The optimization problem in (A.1) can be equivalently written as

\[
\min_{w \in \mathbb{R}^p} R(w) + \delta_{\hat{y}}(Xw). \tag{A.2}
\]

The above optimization problem is given by the sum of two convex, proper, and lower semicontinuous functions, where one of the two is composed with a linear operator. This is the suitable form to apply Fenchel-Rockafellar duality. First recall that the Fenchel conjugate of \( G : \mathbb{R}^p \to [-\infty, +\infty] \) is \( G^* : \mathbb{R}^p \to [-\infty, +\infty] \), such that, for every \( u \in \mathbb{R}^p \), \( G^*(v) = \sup_{w \in \mathbb{R}^p} \langle v, w \rangle - G(w) \). The dual of the problem in (A.2) is then (see e.g. [3, Definition 15.19])

\[
\min_{v \in \mathbb{R}^p} R^*(-X^T v) + \langle \hat{y}, v \rangle. \tag{A.3}
\]

Since \( R = F + (\alpha/2)\| \cdot \|^2 \), its conjugate is differentiable with Lipschitz continuous gradient and is given by (see [3, Example 13.4]),

\[
R^*(v) = \frac{1}{2\alpha} \| v \|^2 - \inf_{u \in \mathbb{R}^p} \left\{ F(u) + \frac{\alpha}{2} \| u - \frac{v}{\alpha} \|^2 \right\}.
\]

The second term on the right hand side is called Moreau envelope of \( F \), and we use the notation \( \alpha^{-1}F(v) = \inf_{u \in \mathbb{R}^p} \left\{ F(u) + \frac{\alpha}{2} \| u - \frac{v}{\alpha} \|^2 \right\} \). A formula for the gradient of \( \alpha^{-1}F \) is known [3, Proposition 12.29], and we derive

\[
\nabla R^*(v) = \alpha^{-1}v - \alpha^{-1}\nabla(\alpha^{-1}F)(\alpha^{-1}v) = \alpha^{-1}v - \alpha^{-1}\left( \alpha(\alpha^{-1}v - \text{prox}_{\alpha^{-1}F}(\alpha^{-1}v)) \right) = \text{prox}_{\alpha^{-1}F}(\alpha^{-1}v) \tag{A.4}
\]

This implies that one step of gradient descent applied to the problem in (A.3) can be written as

\[
v_{t+1} = v_t + \gamma (X \text{ prox}_{\alpha^{-1}F}(\alpha^{-1}X^T v_t) - \hat{y}),
\]

and this is the main iteration in DGD. The derivation of ADGD is analogous, simply the gradient descent method is replaced by FISTA acceleration [6].
A.2 Proofs and auxiliary results

In order to prove Theorems 4.1 and 4.2, we need some auxiliary results. We start with dual gradient descent.

Theorem A.1 Let \((w_t)_{t \in \mathbb{N}}\) be the sequence in \(\mathbb{R}^p\) generated by iteration (4.5) and define \(u_t = \sum_{k=0}^{t} w_k/(t + 1)\). Assume that there exists \(\bar{v} \in \mathbb{R}^p\) such that
\[-X^T \bar{v} \in \partial R(w^\dagger)\].

Then
\[
\|w_t - w^\dagger\| \leq \|X\| \|\bar{v}\| \alpha \sqrt{t}.
\] (A.5)

where \(v^\dagger\) is a solution of the dual problem.

Proof. Thanks to the assumption \(-X^T \bar{v} \in \partial R(v^\dagger)\), strong duality holds, namely the dual problem has a solution \(v^\dagger\), and the minimum of the problem in (A.3) is the same as the minimum of the problem in (2.3). For every \(v \in \mathbb{R}^p\), let \(D(v) = R^*( -X^T v) + \langle \hat{y}, v \rangle\). Then (see for instance [6, Theorem 3.1]) it holds
\[
D(v_t) - D(v^\dagger) \leq \|X\|^2 \|v_0 - v^\dagger\|^2 \frac{1}{2 \alpha t}.
\]

Next, strong convexity implies that
\[
\frac{\alpha}{2} \|w_t - w^\dagger\|^2 \leq D(v_t) - D(v^\dagger).
\]

Combining the two inequalities, and recalling that \(v_0 = 0\), we derive
\[
\|w_t - w^\dagger\| \leq \|X\| \|v^\dagger\| \frac{1}{\alpha \sqrt{t}}.
\]

The statement then follows from convexity of the norm. \(\Box\)

And now we are ready for the main stability result.

Proposition A.2 Let \((\hat{w}_t)_{t \in \mathbb{N}}, (\hat{u}_t)_{t \in \mathbb{N}}\) be the sequences generated by DGD. Let \((w_t)_{t \in \mathbb{N}}\) be defined as in (4.5) with \(v_0 = 0\), and define, for every \(t \in \mathbb{N}\), \(u_t = \sum_{k=0}^{t} w_k/(t + 1)\). Assume \(\delta < 1\). Then the following hold:

i) There exists \(t_\delta \in \{1/\delta, \ldots, 2[1/\delta]\}\) such that
\[
\|w_{t_\delta} - \hat{w}_{t_\delta}\| \leq 2\|X\|^{-1} \delta t_\delta^{1/2}.
\] (A.6)

ii) For every \(t \in \mathbb{N}\),
\[
\|u_t - \hat{u}_t\| \leq 2\|X\|^{-1} \delta t^{1/2}.
\] (A.7)
Proof.

i): For every $t \in \mathbb{N}$, using the firm nonexpansiveness of $\text{prox}_{\alpha^{-1}F}$ and the definition of $\gamma$

\[
\|\hat{v}_t - v_t + \gamma (X(\hat{w}_t - w_t))\|^2 = \|\hat{v}_t - v_t\|^2 + 2\gamma \langle \hat{v}_t - v_t, X(\hat{w}_t - w_t) \rangle + \gamma^2 \|X(\hat{w}_t - w_t)\|^2 \\
\leq \|\hat{v}_t - v_t\|^2 - 2\gamma \alpha \|\hat{w}_t - w_t\|^2 + \gamma^2 \|X(\hat{w}_t - w_t)\|^2 \\
\leq \|\hat{v}_t - v_t\|^2 - \gamma \alpha \|\hat{w}_t - w_t\|^2 \\
\leq \|\hat{v}_t - v_t\|^2. \tag{A.8}
\]

Consequently,
\[
\|\hat{v}_{t+1} - v_{t+1}\| = \|\hat{v}_t - v_t + \gamma X(\hat{w}_t - w_t) - \gamma (\hat{y} - y)\| \\
\leq \|\hat{v}_t - v_t\| + \gamma \delta
\]

and therefore
\[
\|\hat{v}_{t+1} - v_{t+1}\| \leq \gamma \delta (t + 1). \tag{A.9}
\]

Moreover,
\[
\|\hat{v}_t - v_t + \gamma X(\hat{w}_t - w_t)\|^2 = \|\hat{v}_{t+1} - v_{t+1} + \gamma (\hat{y} - y)\|^2 \\
= \|\hat{v}_{t+1} - v_{t+1}\|^2 + \gamma^2 \|y - \hat{y}\|^2 + 2\gamma \langle \hat{v}_{t+1} - v_{t+1}, \hat{y} - y \rangle \\
\geq \|\hat{v}_{t+1} - v_{t+1}\|^2 + \gamma^2 \|y - \hat{y}\|^2 - 2\gamma \delta \|\hat{v}_{t+1} - v_{t+1}\|.
\]

Hence, \text{(A.8)} and \text{(A.9)} yield
\[
\gamma \alpha \|\hat{w}_t - w_t\|^2 \leq \|\hat{v}_t - v_t\|^2 - \|\hat{v}_t - v_t + \gamma X(\hat{w}_t - w_t)\|^2 \\
\leq \|\hat{v}_t - v_t\|^2 - \|\hat{v}_{t+1} - v_{t+1}\|^2 + 2\gamma \delta \|\hat{v}_{t+1} - v_{t+1}\| \\
\leq \|\hat{v}_t - v_t\|^2 - \|\hat{v}_{t+1} - v_{t+1}\|^2 + 2\gamma^2 \delta^2 (t + 1). \tag{A.10}
\]

Summing the previous inequality for $t \in \{t_1, \ldots, T\}$, with $T \geq 2$ we derive
\[
\gamma \alpha \sum_{t=t_1}^{T} \|\hat{w}_t - w_t\|^2 \leq 4\gamma^2 \delta^2 T^2 \tag{A.11}
\]

Taking $t_1 = \lfloor 1/\delta \rfloor$ and $T = 2\lfloor 1/\delta \rfloor$ it follows that
\[
\sum_{t=t_1}^{T} \|\hat{w}_t - w_t\|^2 \leq 4\|X\|^{-2} \delta^2 \lfloor 1/\delta \rfloor^2.
\]

Thus there exists at least a $t_\delta \in \{\lfloor 1/\delta \rfloor, \ldots, 2\lfloor 1/\delta \rfloor\}$ such that
\[
\|\hat{w}_{t_\delta} - w_{t_\delta}\|^2 \leq 4\|X\|^{-2} \delta^2 \left[ \frac{1}{\delta} \right] \leq 4\|X\|^{-2} \delta^2 t_\delta.
\]

ii): Summing the inequalities in (A.10) for $t = 0, \ldots, T$ we derive:
\[
\gamma \alpha \sum_{t=0}^{T} \|\hat{w}_t - w_t\|^2 \leq 4\gamma^2 \delta^2 T^2 \tag{A.12}
\]
Convexity of $\| \cdot \|^2$, and the fact that $\hat{w}_0 = w_0$ imply

$$\| \hat{w}_T - w_T \|^2 \leq \frac{1}{T+1} \sum_{t=0}^{T} \| \hat{w}_t - w_t \|^2 \leq 4 \| X \|^{-2} \delta^2 T$$  \hspace{1cm} (A.13)

The following lemma characterizes the asymptotic behavior of the sequence $(\theta_t)_{t \in \mathbb{N}}$.

**Lemma A.3** Let $(\theta_t)_{t \in \mathbb{N}}$ be the sequence defined in ADGD. Then, for every $t \in \mathbb{N}$

$$\frac{t+1}{2} \leq \theta_t \leq t + 1$$

**Proof.** We prove the first inequality by induction. The case $t = 0$ is clear since $\theta_0 = 1$. Now suppose that the inequality is true for $t$. We derive

$$\theta_{t+1} = \frac{1 + \sqrt{1 + 4\theta_t^2}}{2} \geq \frac{1 + \sqrt{1 + (t+1)^2}}{2} \geq \frac{t + 2}{2}.$$

For the second inequality, the case $t = 0$ is also clear. Now suppose that the inequality is true for $t$. We derive

$$\theta_{t+1} = \frac{1 + \sqrt{1 + 4\theta_t^2}}{2} \leq \frac{1 + \sqrt{1 + 4(t+1)^2}}{2} \leq \frac{1 + 1 + 2(t+1)}{2} = t + 2.$$

The following theorem is obtained exploiting existing results on convergence of forward-backward algorithm in the presence of computational errors. In particular, the result is derived combining [1, Proposition 3.3] (see also [31, 32] for related results) with Lemma A.3 and the relationship between convergence of the dual objective function and the primal iterates.

**Theorem A.4** Let $(\hat{w}_t)_{t \in \mathbb{N}}$ be the sequence generated by ADGD. Then, for every $t \in \mathbb{N}$, $t \geq 1$,

$$\| \hat{w}_t - w^\dagger \| \leq 2 \| X \| \| v^\dagger \|/\alpha t + 4 \| X \|^{-1} \delta t.$$  \hspace{1cm} (A.14)

Theorem A.4 is a direct corollary of Theorem A.1 and Proposition A.2. Theorem 4.2 directly follows from Theorem A.4.

**Proof.** For every $v \in \mathbb{R}^p$, let $D(v) = R^*(\langle -X^Tv, \hat{y} \rangle + \langle \hat{y}, v \rangle)$. Then strong convexity yields

$$\forall t \in \mathbb{N} \quad \frac{\alpha}{2} \| \hat{w}_t - w^\dagger \|^2 \leq D(\hat{v}_t) - \min_{v \in \mathbb{R}^p} D(v).$$  \hspace{1cm} (A.15)

Proposition 3.3 in [1] and Lemma A.3 imply

$$D(\hat{v}_t) - \min_{v \in \mathbb{R}^p} D(v) \leq \frac{1}{2\gamma \delta^2} \left( \| v^\dagger \| + \gamma \delta \sum_{k=0}^{t} \theta_k \right)^2 \leq \frac{1}{\gamma(t+1)^2} \left( \| v^\dagger \| + \gamma \delta \frac{(t+2)(t+3)}{2} \right)^2$$
we derive
\[ \| \hat{w}_t - w^\dagger \| \leq \frac{2\|X\|\|v^\dagger\|}{\alpha t} + \frac{4}{\|X\|} \delta t \] (A.16)

We are now ready to prove the main results. Proof. [Proof of Theorem 4.1]. Theorem A.1 and Proposition A.2 imply
\[ \| \hat{u}_t - w^\dagger \| \leq a t^{1/2} \delta + b t^{-1/2}. \]

Since \( t_\delta = \lceil c\delta^{-1} \rceil \), we have \( c\delta^{-1} \leq t_\delta \leq c\delta^{-1} + 1 \), therefore
\[ \| \hat{u}_t - w^\dagger \| \leq a t^{1/2} \delta + b t^{-1/2} \leq a (c\delta^{-1} + 1)^{1/2} \delta + b c^{-1/2} \delta^{1/2}. \]
The statement follows noting that \( (c\delta^{-1} + 1)^{1/2} \leq (c^{1/2} + 1) \delta^{-1/2} \). \( \Box \)

Finally, we prove Theorem 4.2.

Proof. [Proof of Theorem 4.2]. Theorem A.4 yields
\[ \| \hat{w}_t - w^\dagger \| \leq a t \delta + b t^{-1}. \]

Since \( t_\delta = \lceil c\delta^{-1/2} \rceil \), we have \( c\delta^{-1/2} \leq t_\delta \leq c\delta^{-1/2} + 1 \), therefore
\[ \| \hat{u}_t - w^\dagger \| \leq a t \delta + b t^{-1} \leq a (c\delta^{-1/2} + 1) \delta + b c^{-1} \delta^{1/2}. \]
The statement follows noting that \( (c\delta^{-1/2} + 1) \leq (c + 1) \delta^{-1/2} \). \( \Box \)