WEIGHTED COMPOSITION OPERATORS ON THE CLASS OF SUBORDINATE FUNCTIONS

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Abstract. In this article, we study the weighted composition operators preserving the class \( P_\alpha \) of analytic functions subordinate to \( \frac{1+\alpha z}{1-z} \) for \( |\alpha| \leq 1, \alpha \neq -1 \). Some of its consequences and examples for some special cases are presented.

1. Introduction

Let \( \mathcal{H}(D) \) denote the class of analytic functions defined on the unit disk \( D = \{ z \in \mathbb{C} : |z| < 1 \} \) with the (metrizable) topology of uniform convergence on compact subsets of \( D \) and we denote the boundary of \( D \) by \( T \). Weighted composition operator is a combination of multiplication and composition operators. These operators are mainly studied in various Banach spaces or Hilbert spaces of \( \mathcal{H}(D) \).

Recently, Arévalo et al. [1] initiated the study of weighted composition operator restricted to the Carathéodory class \( \mathcal{P}_1 \), which consists of all \( f \in \mathcal{H}(D) \) with positive real part and with a normalization \( f(0) = 1 \). Clearly the class \( \mathcal{P}_1 \) is not a linear space but it is helpful to solve some extremal problems in geometric function theory. See [8].

In this article, we generalize the recent work of Arévalo et al. [1] by considering weighted composition operators preserving the class \( \mathcal{P}_\alpha \). This class is connected with various geometric subclasses of \( \mathcal{H}(D) \) in the univalent function theory (see [6, 8, 12]). Since the class \( \mathcal{P}_\alpha \) is not a linear space, for a given map on \( \mathcal{P}_\alpha \), questions about operator theoretic properties are not meaningful. However, one can talk about special classes of self-maps of \( \mathcal{P}_\alpha \) and fixed points of those maps. This is the main purpose of this article.

The article is organized as follows. In Section 2, we introduce the class \( \mathcal{P}_\alpha \) and list down some basic properties about this class. In Section 3, we give characterization for weighted composition operators to be self-maps of the class \( \mathcal{P}_\alpha \) (see Theorem 1). The above situation is analyzed closely for various special cases of symbols in Section 4. In Section 5, we present some simple examples.

2. Some preliminaries about the class \( \mathcal{P}_\alpha \)

For \( f \) and \( g \in \mathcal{H}(D) \), we say that \( f \) is subordinate to \( g \) (denoted by \( f(z) \preceq g(z) \) or \( f \preceq g \)) if there exists an analytic function \( \omega : D \rightarrow D \) such that \( \omega(0) = 0 \) and...
f = g \circ \omega. If f(z) \prec z, then f is called Schwarz function. For |\alpha| \leq 1, \alpha \neq -1, define h_\alpha on \mathbb{D} by h_\alpha(z) = \frac{1+\alpha z}{1-z} and the half plane \mathbb{H}_\alpha is described by
\mathbb{H}_\alpha := h_\alpha(\mathbb{D}) = \{w \in \mathbb{C} : 2\text{Re}\{(1+\alpha)w\} > 1 - |\alpha|^2\}.
In particular, if \alpha \in \mathbb{R} and -1 < \alpha \leq 1, then
h_\alpha(\mathbb{D}) = \{w \in \mathbb{C} : \text{Re} w > (1 - \alpha)/2\}
so that \text{Re} h_\alpha(z) > (1 - \alpha)/2 in \mathbb{D}.
For |\alpha| \leq 1, \alpha \neq -1, it is natural to consider the class \mathcal{P}_\alpha defined by
\mathcal{P}_\alpha := \{f \in \mathcal{H}(\mathbb{D}) : f(z) \prec h_\alpha(z)\}.
It is worth to note that for every f \in \mathcal{P}_\alpha, there is an unique Schwarz function \omega such that
f(z) = \frac{1 + \alpha \omega(z)}{1 - \omega(z)}.
It is well-known [12, Lemma 2.1] that, if g is an univalent analytic function on \mathbb{D}, then f(z) \prec g(z) if and only if f(0) = g(0) and f(\mathbb{D}) \subset g(\mathbb{D}). In view of this result, the class \mathcal{P}_\alpha can be stated in an equivalent form as
\mathcal{P}_\alpha := \{f \in \mathcal{H}(\mathbb{D}) : f(0) = 1, f(\mathbb{D}) \subset \mathbb{H}_\alpha\}.
We continue the discussion by stating a few basic and useful properties of the class \mathcal{P}_\alpha.

**Proposition 1.** Suppose f \in \mathcal{P}_\alpha and f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n, then |a_n| \leq |\alpha + 1| for all n \in \mathbb{N}. The bound is sharp as the function h_\alpha(z) = 1 + \sum_{n=1}^{\infty}(1 + \alpha)z^n shows.

**Proof.** This result is an immediate consequence of Rogosinski’s result [13, Theorem X] (see also [6, Theorem 6.4(i), p. 195]) because h_\alpha(z) (and hence, (h_\alpha(z)−1)/(1+\alpha)) is a convex function. \qed

**Proposition 2.** (Growth estimate) Let f \in \mathcal{P}_\alpha. Then for all z \in \mathbb{D}, one has
\frac{1 - |\alpha z|}{1 + |z|} \leq |f(z)| \leq \frac{1 + |\alpha z|}{1 - |z|}.

**Proof.** This result trivially follows from clever use of classical Schwarz lemma and the triangle inequality. \qed

From the ‘growth estimate’ and the familiar Montel’s theorem on normal family, one can easily get the following result.

**Proposition 3.** The class \mathcal{P}_\alpha is a compact family in the compact-open topology (that is, topology of uniform convergence on compact subsets of \mathbb{D}).

Because the half plane \mathbb{H}_\alpha is convex, the following result is obvious.

**Proposition 4.** The class \mathcal{P}_\alpha is a convex family.
For \( p \in (0, \infty) \), the Hardy space \( H^p \) consists of analytic functions \( f \) on \( \mathbb{D} \) with

\[
\|f\|_p := \sup_{r \in (0,1)} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}}
\]

is finite and \( H^\infty \) denotes the set of all bounded analytic functions on \( \mathbb{D} \). We refer to [5] for the theory of Hardy spaces. By Littlewood’s subordination theorem [11, Theorem 2], it follows that if \( f \prec g \) and \( g \in H^p \) for some \( 0 < p \leq \infty \), then \( f \in H^p \) for the same \( p \). As a consequence we easily have the following.

**Proposition 5.** The class \( \mathcal{P}_\alpha \) is a subset of the Hardy space \( H^p \) for each \( 0 < p < 1 \).

**Proof.** Because \((1-z)^{-1} \in H^p \) for each \( 0 < p < 1 \), it follows easily that \( h_\alpha \in H^p \) for each \( 0 < p < 1 \) and for \( |\alpha| \leq 1, \alpha \neq -1 \). The desired conclusion follows. \( \square \)

**Remark 1.** Although \( \mathcal{P}_\alpha \) does not posses the linear structure, due to being part of \( H^p \), the results on \( H^p \) space, such as results about boundary behavior, are valid for functions in the class \( \mathcal{P}_\alpha \).

### 3. Weighted Composition on \( \mathcal{P}_\alpha \)

For an analytic self-map \( \phi \) of \( \mathbb{D} \), the composition operator \( C_\phi \) is defined by

\[
C_\phi(f) = f \circ \phi \quad \text{for} \quad f \in \mathcal{H}(\mathbb{D}).
\]

One can refer [4], for the study of composition operators on various function spaces on the unit disk. Throughout this article, \( \alpha \) denotes a complex number such that \( |\alpha| \leq 1, \alpha \neq -1 \), unless otherwise stated explicitly and \( \phi \) denotes an analytic self-map of \( \mathbb{D} \). The following result deals with composition operator when it is restricted to the class \( \mathcal{P}_\alpha \).

**Proposition 6.** The composition operator \( C_\phi \) induced by the symbol \( \phi \), preserves the class \( \mathcal{P}_\alpha \) if and only if \( \phi \) is a Schwarz function.

**Proof.** Suppose that \( C_\phi \) preserves the class \( \mathcal{P}_\alpha \). Then \( C_\phi(h_\alpha) \in \mathcal{P}_\alpha \), and thus

\[
\frac{1 + \alpha \phi(0)}{1 - \phi(0)} = 1.
\]

This gives that \( \phi(0) = 0 \) which implies that \( \phi \) is a Schwarz function. The converse part holds trivially. \( \square \)

For a given analytic self-map \( \phi \) of \( \mathbb{D} \) and analytic map \( \psi \) of \( \mathbb{D} \), the corresponding weighted composition operator \( C_{\psi,\phi} \) is defined by

\[
C_{\psi,\phi}(f) = \psi(f \circ \phi) \quad \text{for} \quad f \in \mathcal{H}(\mathbb{D}).
\]

If \( \psi \equiv 1 \), then \( C_{\psi,\phi} \) reduced to a composition operator \( C_\phi \) and if \( \phi(z) = z \) for all \( z \in \mathbb{D} \), then \( C_{\psi,\phi} \) reduced to a multiplication operator \( M_\psi \). For a given analytic map \( \psi \) of \( \mathbb{D} \), the corresponding multiplication operator \( M_\psi \) is then defined by

\[
M_\psi(f) = \psi f \quad \text{for} \quad f \in \mathcal{H}(\mathbb{D}).
\]
The characterization of $M_\psi$ that preserves the class $P_\alpha$ is given in Section 4.

Banach begun the study of weighted composition operators. In [2], Banach proved the classical Banach-Stone theorem, that is, the surjective isometries between the spaces of continuous real-valued functions on a closed and bounded interval are certain weighted composition operators. In [7], Forelli proved that the isometric isomorphism of the Hardy space $H^p$, ($p \neq 2$) are also weighted composition operators. The same result for the case of Bergman space is proved by Kolaski in [9].

The study of weighted composition operators can be viewed as a natural generalization of the well known field in the analytic function theory, namely, the composition operators. Moreover, weighted composition operators appear in applied areas such as dynamical systems and evolution equations. For example, classification of dichotomies in certain dynamical systems is connected to weighted composition operators, see [3].

In this section, we discuss weighted composition operator that preserves $P_\alpha$. Before, we do this, let us recall some useful results from the theory of extreme points.

**Lemma 1.** ([8, Theorem 5.7]) Extreme points of the class $P_\alpha$ consists of functions given by

$$f_\lambda(z) = \frac{1 + \alpha \lambda z}{1 - \lambda z}, \quad |\lambda| = 1.$$ 

A point $p$ of a convex set $A$ is called extreme point if $p$ is not a interior point of any line segment which entirely lies in $A$. We denote, the set of all extreme points of the class $P_\alpha$ by $E_\alpha$. That is, $E_\alpha = \{ f_\lambda : |\lambda| = 1 \}$. Now, we recall a well-known result by Krein and Milman [10].

**Lemma 2.** ([8, Theorem 4.4]) Let $X$ be a locally convex, topological vector space and $A$ be a convex, compact subset of $X$. Then, the closed convex hull of extreme points of $A$ is equal to $A$.

The original version of it is proved in [10]. On $\mathcal{H}(\mathbb{D})$, $f_n \ucrel \to f$. It is easy to see that $C_{\psi,\phi}(f_n) \ucrel \to C_{\psi,\phi}(f)$ whenever $f_n \ucrel \to f$. Thus, $C_{\psi,\phi}$ is continuous on $\mathcal{H}(\mathbb{D})$ (in particular on $P_\alpha$).

**Proposition 7.** Suppose that $C_{\psi,\phi}$ preserves the class $P_\alpha$. Then $\phi$ is a Schwarz function and there exists a Schwarz function $\omega$ such that

$$\psi = h_\alpha \circ \omega = \frac{1 + \alpha \omega}{1 - \omega}.$$ 

**Proof.** Suppose that $C_{\psi,\phi}$ preserves the class $P_\alpha$. Take $f \equiv 1$ to be a constant function, which belongs to $P_\alpha$. Thus, $C_{\psi,\phi}(f) = \psi \in P_\alpha$ and hence, there exists a Schwarz function $\omega$ such that

$$\psi = h_\alpha \circ \omega = \frac{1 + \alpha \omega}{1 - \omega}.$$ 

In particular, $\psi(0) = 1$.

Since $h_\alpha \in P_\alpha$, we have $\psi(0)(h_\alpha(\phi(0))) = 1$, which gives $\phi(0) = 0$. Hence $\phi$ will be a Schwarz function. \qed
In view of above result, from now on, we will assume that $\psi = h_\alpha \circ \omega = \frac{1 + \alpha \omega}{1 - \omega}$ and $\phi, \omega$ are Schwarz functions.

**Theorem 1.** Let $\phi, \omega$ and $\psi$ be as above. Then, $C_{\psi, \phi}$ preserves the class $P_\alpha$ if and only if

$$2Q(\omega)|\phi| < (1 - |\omega|^2) + P(\omega)|\phi|^2 \text{ on } \mathbb{D},$$

where, $P(\omega) = |\alpha \omega|^2 - |1 + (\alpha - 1)\omega|^2$ and $Q(\omega) = |(\alpha - 1)|\omega|^2 + \overline{\omega} - \alpha \omega|.$

**Proof.** At first we prove that, $C_{\psi, \phi}$ preserves the class $P_\alpha$ which is equivalent to the inclusion $C_{\psi, \phi}(E_\alpha) \subset P_\alpha$. To do this, we suppose that $C_{\psi, \phi}(E_\alpha) \subset P_\alpha$. Since $P_\alpha$ is a convex family, we obtain

$$C_{\psi, \phi}(\text{convex hull } (E_\alpha)) \subset P_\alpha.$$

Now, by Krein-Milman theorem and the fact that $C_{\psi, \phi}$ is continuous on a compact family $P_\alpha$, we see that $C_{\psi, \phi}(P_\alpha) \subset P_\alpha$. The converse part is trivial.

Next, we prove that $C_{\psi, \phi}(E_\alpha) \subset P_\alpha$ if and only if

$$2Q(\omega)|\phi| < (1 - |\omega|^2) + P(\omega)|\phi|^2 \text{ on } \mathbb{D}.$$

Assume that $C_{\psi, \phi}(E_\alpha) \subset P_\alpha$. This gives $\psi(f_\lambda \circ \phi) \in P_\alpha$ for all $|\lambda| = 1$. Thus, for all $|\lambda| = 1$, there exists a Schwarz function $\omega_\lambda$ such that $\psi(f_\lambda \circ \phi) = h_\alpha \circ \omega_\lambda$. That is,

$$\frac{1 + \alpha \omega}{1 - \omega} = \frac{1 + \alpha \omega_\lambda}{1 - \omega_\lambda}.$$

Solving this equation for $\omega_\lambda$, we get that

$$\omega_\lambda = \frac{\omega + \lambda \phi + (\alpha - 1)\lambda \omega \phi}{1 + \alpha \lambda \phi \omega}.$$

For each $|\lambda| = 1$, $\omega_\lambda$ is a Schwarz function if and only if

$$|\omega + \lambda \phi + (\alpha - 1)\lambda \omega \phi|^2 < |1 + \alpha \lambda \phi \omega|^2 \text{ for all } |\lambda| = 1,$$

which is equivalent to

$$2\Re (\lambda \phi (|\omega|^2 + \overline{\omega} - \alpha \omega)) < (1 - |\omega|^2) + |\phi|^2(|\alpha \omega|^2 - |1 + (\alpha - 1)\omega|^2),$$

for all $|\lambda| = 1$. By taking supremum over $\lambda$ on both sides, the last inequality gives (3.1). The converse part follows by repeating the above arguments in the reverse direction. \hfill \Box

**Remark 2.** Suppose that $\alpha = a + ib$ and $\omega(z) = u(z) + iv(z)$. Then,

$$P(\omega) = |\alpha \omega|^2 - |1 + (\alpha - 1)\omega|^2 = a(|\omega|^2 - 1) + (a - 1)|\omega - 1|^2 + 2bv.$$

Set $q(\omega) = (\alpha - 1)|\omega|^2 + \overline{\omega} - \alpha \omega$ so that $Q(\omega) = |q(\omega)|$. Upon simplifying, we get that

$$q(\omega) = (\alpha - 1)\overline{\omega}(\omega - 1) - 2iav = (\alpha - 1)(|\omega|^2 - \omega) - 2iv$$

and thus

$$q(\omega) = [(a - 1)(|\omega|^2 - u) + bv] + i[b(|\omega|^2 - u) - v(a + 1)].$$
Also, it is easy to see that
\begin{equation}
-q(\omega) = |1 - \omega|^2 \psi + (|\omega|^2 - 1) \quad \text{with} \quad \psi = \frac{1 + \alpha \omega}{1 - \omega}.
\end{equation}

4. Special cases

In this section, first we recall some familiar results on Hardy space $H^p$ which will help the smooth traveling of this article.

**Proposition 8.** ([5, Theorem 1.3]) *For every bounded analytic function $f$ on $D$, the radial limit $\lim_{r \to 1} f(re^{i\theta})$ exists almost everywhere (abbreviated by a.e.).*

In view of Proposition 8, every Schwarz function has radial limit a.e. and using the fact that the function $h_\alpha$ has radial limit a.e., it is easy to see that, every function $f \in P_\alpha$ has radial limit a.e. on $T$. Also, it is well-known that (see [5, Section 2.3])
\[
\sup_{|z| < 1} |f(z)| = \operatorname{ess sup}_{0 \leq \theta \leq 2\pi} |f(e^{i\theta})|,
\]
for every $f \in H^\infty$. Now, we will state a classical theorem of Nevanlinna.

**Proposition 9.** ([5, Theorem 2.2]) *If $f \in H^p$ for some $p > 0$ and its radial limit $f(e^{i\theta}) = 0$ on a set of positive measure, then $f \equiv 0$.***

Since every Schwarz function $f$ belongs to $H^\infty$ and every $f \in P_\alpha$ belongs to $H^p$ for $0 < p < 1$, the above result is valid for functions in the class $P_\alpha$ and for Schwarz functions.

An analytic function $f$ on $D$ is said to be an \textit{inner function} if $|f(z)| \leq 1$ for all $z \in D$ and its radial limit $|f(\zeta)| = 1$ a.e. on $|\zeta| = 1$.

**Theorem 2.** *Suppose that $\phi$ and $\omega$ are Schwarz functions, $\phi$ is inner and $\psi = \frac{1 + \alpha \omega}{1 - \omega}$. Then, $C_{\psi, \phi}$ preserves the class $P_\alpha$ if and only if $\psi \equiv 1$ (i.e., $\omega \equiv 0$).*

**Proof.** If $\psi \equiv 1$, then $C_{\psi, \phi}$ becomes a composition operator $C_\phi$ and thus, $C_{\psi, \phi}$ preserves the class $P_\alpha$, because $\phi$ is a Schwartz function.

Conversely, suppose that $C_{\psi, \phi}$ preserves the class $P_\alpha$. Then, by Theorem 1, one has the inequality
\[
2Q(\omega)|\phi| < (1 - |\omega|^2) + P(\omega)|\phi|^2 \quad \text{on} \quad D.
\]

With abuse of notation, we denote the radial limits of $\phi$, $\omega$ and $\psi$ again by $\phi$, $\omega$ and $\psi$, respectively. Also, let $\alpha = a + ib$ and $\omega(z) = u(z) + iv(z)$. By allowing $|z| \to 1$ in (3.1), we get that
\[
2Q(\omega) \leq (1 - |\omega|^2) + P(\omega) \quad \text{a.e. on} \quad T,
\]
which after computation is equivalent to
\[
Q(\omega) \leq (a - 1)(|\omega|^2 - u) + bv \quad \text{a.e. on} \quad T.
\]
In view of (3.2) in Remark 2 the above inequality can rewritten as
\[
|q(\omega)| \leq \Re[q(\omega)] \quad \text{a.e. on} \quad T,
which gives that \( \text{Im} [q(\omega)] = 0 \) a.e. on \( T \). Again by using (3.3) in Remark 2, we have
\[
|1 - \omega|^{2} \text{Im}(\psi) = 0 \text{ a.e. on } T.
\]

Analyzing the function \( \omega \) through the classical theorem of Nevanlinna (see Proposition 9), one can get that \( \text{Im} \psi = 0 \) a.e. on \( T \). Now the proof of \( \psi \equiv 1 \) is as follows:

Consider the analytic map \( f = e^{-i(\psi - 1)} \). Then,
\[
|f| = e^{\text{Im} \psi} = 1 \text{ a.e. on } T \quad \text{and} \quad 1 = f(0) \leq \sup_{|z|<1} |f(z)| = \text{ess sup}_{0 \leq \theta \leq 2\pi} |f(e^{i\theta})| = 1.
\]
Hence, by the maximum modulus principle, we get that \( f \equiv 1 \) which gives \( \psi \equiv 1 \). □

**Corollary 1.** \( M_{\phi} \) preserves the class \( \mathcal{P}_{\alpha} \) if and only if \( \psi \equiv 1 \).

**Proof.** The desired result follows if we set \( \phi(z) \equiv z \) in Theorem 2 □

**Theorem 3.** Suppose that \( \alpha \) is a real number, \( \phi \) and \( \omega \) are Schwarz functions, \( \omega \) is an inner function and \( \psi = \frac{1 + \alpha \omega}{1 - \omega} \). Then, \( C_{\psi, \phi} \) preserves the class \( \mathcal{P}_{\alpha} \) if and only if \( \phi \) is identically zero.

**Proof.** If \( \phi \equiv 0 \), then \( C_{\psi, \phi} \) becomes a constant map \( \psi \) and hence it preserves \( \mathcal{P}_{\alpha} \). Conversely, suppose that \( C_{\psi, \phi} \) preserves the class \( \mathcal{P}_{\alpha} \). Then, by Theorem 1
\[
2Q(\omega)|\phi| < (1 - |\omega|^{2}) + P(\omega)|\phi|^{2} \quad \text{on } D.
\]
By allowing \( |z| \to 1 \), we get that
\[
2|1 - \omega|^{2} |\psi| |\phi| \leq (2\text{Im} \alpha \text{Im} \omega + (\text{Re} \alpha - 1)|1 - \omega|^{2})|\phi|^{2} \text{ a.e. on } T,
\]
from which we obtain that
\[
|1 - \omega|^{2} |\psi| |\phi| \leq 0 \text{ a.e. on } T.
\]
By the hypothesis on \( \omega \) and \( \psi \), and the classical theorem of Nevanlinna, we find that \( \phi \equiv 0 \). □

Here is an easy consequence of Theorem 3

**Corollary 2.** Let \( \alpha \) be a real number, \( \phi \) and \( \omega \) are Schwarz functions and that \( \phi \neq 0 \). Suppose that \( C_{\psi, \phi} \) preserves the class \( \mathcal{P}_{\alpha} \), and
\[
E = \{ \zeta \in T : |\omega(\zeta)| = 1 \}.
\]
Then the Lebesgue arc length measure of the set \( E \) is zero, i.e., \( m(E) = 0 \).
5. Examples for special cases

In this section, we give specific examples of $\phi$ and $\psi$ so that $C_{\psi,\phi}$ preserves the class $P_\alpha$. For a bounded analytic function on $D$, we denote $\sup_{|z|<1} |f(z)|$ by $\|f\|$.

**Example 1.** Suppose that $\|\phi\| < 1$. If $\|\omega\| < 1 - \|\phi\|$, then $C_{\psi,\phi}$ preserves the class $P_\alpha$, for $\alpha \in [0, 1]$.

**Proof.** In view of Theorem 1, it suffices to verify the inequality (3.1). This inequality can be rewritten as

$$2(1 - \alpha) \overline{\omega}(\omega - 1) + 2i\alpha \text{Im } \omega |\phi| + (1 - \alpha)(|\omega|^2 + |1 - \omega|^2|\phi|^2 - 1) < \alpha(1 - |\omega|^2)(1 - |\phi|^2).$$

We may set $\|\omega\| = A$ and $\|\phi\| = B$. Thus, it is enough to check that

$$2(1 - \alpha)A(A + 1)B + 4\alpha AB + (1 - \alpha)(A^2 + (1 + A)^2B^2 - 1) < \alpha(1 - A^2)(1 - B^2)$$

which is equivalent to

$$[A + B + AB - 1][(1 - \alpha)(A + B + AB + 1) + \alpha(A + B - AB + 1)] < 0.$$ 

This yields the condition $A + B + AB - 1 < 0$. This means that $A < \frac{1 - B}{1 - \alpha}$ and the desired conclusion follows. $\square$

Since the condition $A + B + AB - 1 < 0$ gives $B < \frac{1 - A}{1 + \alpha}$, we have the following result.

**Example 2.** Suppose that $\|\omega\| < 1$. If $\|\phi\| < \frac{1 - \|\omega\|}{1 + \|\omega\|}$, then $C_{\psi,\phi}$ preserves the class $P_\alpha$, for $\alpha \in [0, 1]$.

**Example 3.** Suppose that $\phi(z) = z(az + b)$, where $a$ and $b$ are non-zero real numbers such that $|a| + |b| = 1$. Take $\omega(z) = cz + d$ with

$$c = -\frac{ab}{K} \quad \text{and} \quad d = \frac{1 - (a^2 + b^2)}{K} \quad \text{for } K > 2 + \sqrt{5}.$$

Then $C_{\psi,\phi}$ preserves the class $P_1$.

**Proof.** Clearly $|\phi(z)|^2 \leq a^2 + b^2 + 2abx$ for $z = x + iy$ and thus,

$$0 < 1 - (a^2 + b^2) - 2abx \leq (1 - |\phi|^2).$$

Also note that

$$|\text{Im } \omega| \leq |2cx + d| = \frac{1 - (a^2 + b^2) - 2abx}{K},$$

and

$$|\omega(z)| \leq |c| + |d| = \frac{1 - |ab| - (|a| - |b|)^2}{K} \leq \frac{1}{K}.$$

The inequality (3.1) for $\alpha = 1$ reduces to

$$4|\phi| |\text{Im } \omega| < (1 - |\omega|^2)(1 - |\phi|^2).$$
Since $4|\phi||\Im \omega| \leq 4|\Re \omega| \leq 4|2cx + d|$ and
\[
\left(1 - \frac{1}{K^2}\right) K|2cx + d| \leq (1 - |\omega|^2)(1 - |\phi|^2),
\]
to verify the inequality (3.1), it suffices to verify the inequality
\[
\frac{4}{K} < 1 - \frac{1}{K^2}, \text{ i.e., } K^2 - 4K - 1 > 0.
\]
This gives the condition $K > 2 + \sqrt{5}$ and the proof is complete. \qed

**Remark 3.** By letting $\alpha = 0$ in the Theorem 1, we see that $C_{\psi, \phi}$ preserves the class $\mathcal{P}_0$ if and only if $|1 - \omega||\phi| + |\omega| < 1$ on $\mathbb{D}$.

**Example 4.** If $|\phi| \leq |\omega| < \sqrt{2} - 1$ on $\mathbb{D}$, then $C_{\psi, \phi}$ preserves $\mathcal{P}_0$.

**Proof.** In view of Remark 3 and the assumption that $|\phi| \leq |\omega|$, it is enough to show that $|\omega||1 - \omega| < 1 - |\omega|$ which, by squaring and then simplifying, is seen to be equivalent to the inequality
\[
|\omega|^4 - 2\Re \omega|\omega|^2 + 2|\omega| - 1 < 0.
\]
In order to verify the last inequality, we observe that
\[
|\omega|^4 - 2\Re \omega|\omega|^2 + 2|\omega| - 1 \leq |\omega|^4 + 2|\omega|^3 + 2|\omega| - 1 = (|\omega|^2 + 1)(|\omega|^2 + 2|\omega| - 1)
\]
which is negative whenever $|\omega|^2 + 2|\omega| - 1 < 0$, i.e., $|\omega| < \sqrt{2} - 1$. The desired result follows. \qed

**Example 5.** If $|\phi| \leq |\omega| < s_0$ or $|\omega| \leq |\phi| < s_0$ on $\mathbb{D}$, then $C_{\psi, \phi}$ preserves $\mathcal{P}_a$ for every $\alpha$ with $-1 < \alpha < 0$, where $s_0 \approx 0.2648$ is the unique positive root of the polynomial $P(x) = 2x^4 + 8x^3 + 12x^2 - 1$.

**Proof.** Without loss of generality, we assume that $|\phi| \leq |\omega|$. In view of Remark 2 and the assumption that $\alpha \in (-1, 0)$, the inequality (3.1) can be rewritten as
\[
2|(1 - \alpha)\overline{\omega}(\omega - 1) + 2i\alpha \Im \omega||\phi| + (1 - \alpha)(|\omega|^2 + |1 - \omega||\phi|^2 - 1) - \alpha(1 - |\omega|^2)(1 - |\phi|^2) < 0.
\]
By setting $||\omega|| = A$ and $||\phi|| = B$ (so that $B \leq A$), it suffices to check that
\[
2(1 - \alpha)A(A + 1)B - 4\alpha AB + (1 - \alpha)(A^2 + (1 + A)^2B^2 - 1) - \alpha < 0,
\]
which is equivalent to
\[
-\alpha[(A + B + AB)^2 + 4AB] + (A + B + AB)^2 - 1 < 0.
\]
Since $B \leq A$ and $\alpha \in (-1, 0)$, the last inequality holds if
\[
(2A + A^2)^2 + 4A^2 + (2A + A^2)^2 - 1 = 2A^4 + 8A^3 + 12A^2 - 1 < 0.
\]
Clearly, the function $P(x) = 2x^4 + 8x^3 + 12x^2 - 1$ is strictly increasing on $(0, \infty)$ and thus, $P(x) < 0$ for $0 \leq x < s_0$, where $s_0$ is the unique positive root of $P(x)$. The desired result follows. \qed
6. Fixed points

In this section, we discuss the fixed points of weighted composition operators. It is time to recall a result due to Yu-Qing Chen [15, Theorem 2.1]. The modern way of writing it is as follows:

**Proposition 10.** Let $X$ be a metrizable topological vector space and $C$ be a convex compact subset of $X$. Then, every continuous mapping $T : C \to C$ has a fixed point in $C$.

We set $X = \mathcal{H}(\mathbb{D})$, $C = \mathcal{P}_\alpha$, $T = C_{\psi, \phi}$ and observe that every weighted composition operator on $\mathcal{P}_\alpha$ has a fixed point. Indeed, one has something more than this as we can see below.

**Theorem 4.** Let $\phi, \psi, \omega$ be as before and $\phi$ is not a rotation. Suppose that $C_{\psi, \phi}$ is a self-map of $\mathcal{P}_\alpha$. Then, $C_{\psi, \phi}$ has a unique fixed point which can be obtained by iterating $C_{\psi, \phi}$ for any $f \in \mathcal{P}_\alpha$. Furthermore, if $\phi$ is an inner function, then the fixed point is the constant function $1$.

**Theorem 5.** Let $\phi, \psi, \omega$ be as before and $\phi$ is a rotation. Suppose that $C_{\psi, \phi}$ is a self-map of $\mathcal{P}_\alpha$ and $F$ denotes the set of all fixed points of $C_{\psi, \phi}$. Then, there are three distinct cases:

1. If $\phi(z) \equiv z$, then $F = \mathcal{P}_\alpha$.
2. If $\phi(z) \equiv \lambda z$ and $\lambda^n \neq 1$ for every $n \in \mathbb{N}$, then $F = \{1\}$.
3. If $\phi(z) \equiv \lambda z$ and $\lambda^n = 1$ for some $n > 1$, then

$$F = \{f : f(z) = g(z^n)\} \text{ for some } g \in \mathcal{P}_\alpha.$$ 

The proofs of these two theorems follow from the lines of the proofs of the corresponding results of Section 4 of [1]. Moreover, the key tools for the proofs are from Section 6.1 of Shapiro’s book [14]. So we omit the details.

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