Analytic Approach To a Generalization of Chern Classes in Supergeometry

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Abstract

Some cohomology elements, called $\nu$-classes, as a supergeneralization of universal Chern classes, are introduced for canonical super line bundles over $\nu$-projective spaces, a novel supergeometric generalization of projective spaces. It is shown that these classes may be described by analytic representatives of elements of generalized de Rham cohomology.

Keywords: Chern classes, Super projective spaces, $\nu$-projective spaces, Canonical super line bundles.

1 Introduction

In this paper, we introduce an analytic approach to study the first $\nu$-classes. These classes are a novel generalization of the first Chern classes in supergeometry introduced in [20]. In addition, at the end, we introduce some cocycles in a de Rham complex which may be considered as representatives of $\nu$-classes of rank higher than 1 for universal super vector bundles.

In fact, among geometric concepts, Chern classes are important topological invariants associated to complex vector bundles. So it is important to obtain a proper generalization of Chern classes in supergeometry. Since 1980, many efforts have been made to generalize this concept which we now mention some of them.

In [3], [4], [5] and [12], supermanifolds are defined in the sense of De Witt [6] and Rogers [17]. Note that in this approach, a super vector bundle over a De Witt supermanifold does always carry a connection. In all of these references, for a given complex super vector bundle of rank $(r, s)$ over a De Witt supermanifold, $r$ even and $s$ odd Chern classes are defined and consequently, one may have even and odd total Chern classes. In addition, a representative of Chern classes in terms of the curvature form of a connection is given. Note that the second author has shown, in [21], that the even and odd Chern classes of a super vector bundle are the same as Chern classes of two canonically associated vector bundles. Since a category with pairs of vector bundles as its objects is not equivalent to the category of super vector bundles, this theory of Chern classes for super vector bundles is exactly the common one for the former category. For more details, see [21].

In the paper [15], based on the Leites [13] and Kostant [11] approach to supermanifolds, a left(right) connection is introduced and its associated curvature form is computed. Then, considering a locally free sheaf with a smooth left(right) connection and applying invariant polynomials to its curvature form, a globally defined closed differential form is obtained. It is shown that there is a representative of the Chern classes in terms of the introduced differential forms. But as it is mentioned in [15], the classes obtained in this way are nothing but the Chern classes of the reduced vector bundle. Indeed, in the smooth construction, the category of locally free sheaves on a supermanifold is equivalent to the
category of $\mathbb{Z}_2$-graded locally free sheaves on the reduced manifold. So the introduced Chern classes are just those of common geometry.

In [23], associated to a super vector bundle $\mathcal{E}$, it is considered a decomposition as $\mathcal{E}_{\text{red}} = \mathcal{E}_0 \oplus \mathcal{E}_1$ for the reduced vector bundle. Then, by an approach which coincides to that of Quillen([16]), a representative of Chern classes of $\mathcal{E}$ is given in terms of the Chern classes of $\mathcal{E}_0$ and $\mathcal{E}_1$. Since $\mathcal{E}_0$ and $\mathcal{E}_1$ are components of the reduced vector bundle, this representative do not have any information about the superstructure, i.e., this theory does not indicate the extent to which even and odd elements of a super vector bundle are associated to each other. Solving this problem is a motivation for defining $\nu$-classes.

This paper aims at describing $\nu$-classes by analytic representatives of elements of generalized de Rham cohomology. The idea of analytic approach to describe $\nu$-classes derives from the idea of analytic description of Chern classes in common geometry where there are different ways to approach Chern classes, such as homotopy or analytic approach. In homotopy approach, one can consider any vector bundle $\xi$ on a manifold $M$ as a pullback of the universal bundle along a map, say $f$, from $M$ to infinite Grassmannian, unique up to homotopy. Therefore, one can describe Chern classes of $\xi$ as the pullback of Chern classes of the universal bundle. In [1], this theory is generalized in supergeometry. Indeed, a canonical super vector bundle over $\nu$-grassmannian is constructed and it is shown that it has the property of the universal super vector bundle. So by having universal $\nu$-classes, one can obtain $\nu$-classes of rank higher than 1 for any super vector bundle.

## 2 Preliminaries

In this section, we deal with some basic definitions of supergeometry such as supermanifolds, $\nu$-domains, derivations and etc.

### 2.1 Supermanifolds

A super ringed space is a pair $(M, \mathcal{O})$ where $M$ is a topological space and $\mathcal{O}$ is a sheaf of supercommutative $\mathbb{Z}_2$-graded rings. So for any open subset $U \subset M$, the section $\mathcal{O}(U)$ is a supercommutative $\mathbb{Z}_2$-graded ring. We denote its even and odd parts by $\mathcal{O}^0(U)$ and $\mathcal{O}^1(U)$ respectively. Thus, one has

$$\mathcal{O}(U) = \mathcal{O}^0(U) \oplus \mathcal{O}^1(U)$$

with the property that $\mathcal{O}^i \mathcal{O}^j \subset \mathcal{O}^{(i+j) \mod 2}$.

We call each element of $\mathcal{O}^0$ and $\mathcal{O}^1$ a homogeneous element of even and odd parity respectively.

We denote the parity of a homogeneous element $a$ by $p(a)$ which equals 0 if it is even and equals 1 if it is odd.

Now, let $(M, \mathcal{O}_M)$ and $(N, \mathcal{O}_N)$ be two super ringed spaces. By a morphism of super ringed spaces from $(M, \mathcal{O}_M)$ to $(N, \mathcal{O}_N)$, we mean a pair $\psi = (\psi, \psi^\star)$ where $\psi : M \rightarrow N$ is a continuous map and
\[ \psi^*: \mathcal{O}_N \rightarrow \tilde{\psi}_*(\mathcal{O}_M) \] is a homomorphism between sheaves of \( \mathbb{Z}_2 \)-graded rings. Thus, for any open subset \( V \subset N \), we have

\[ \psi^*_V: \mathcal{O}_N(V) \rightarrow \mathcal{O}_M(\tilde{\psi}^{-1}(V)) \]

which commutes with the corresponding restriction morphisms and in addition satisfies the following:

\[ \psi^*_V(\mathcal{O}_N^0(V)) \subset \mathcal{O}_M^0(\tilde{\psi}^{-1}(V)), \quad \psi^*_V(\mathcal{O}_N^1(V)) \subset \mathcal{O}_M^1(\tilde{\psi}^{-1}(V)). \]

Let \( U \) be an open subset of \( \mathbb{C}^m \). A superdomain \( U^{m|n} \) is a super ringed space \((U, \mathcal{C}_U^{m|n})\) where \( \mathcal{C}_U^{m|n} \) is the sheaf of supercommutative rings such that for any open subset \( V \subset U \) we have

\[ \mathcal{C}_U^{m|n}(V) = \mathbb{C}^\infty(V)[e_1, \cdots, e_n] \]

where \( \{e_1, \cdots, e_n\} \) are anticommuting variables satisfying \( e_i e_j = -e_j e_i \) and \( \mathbb{C}^\infty(V) \) is the algebra of complex valued smooth functions on \( V \).

If there is no ambiguity, we write \( \mathcal{C}_U \) for \( \mathcal{C}_U^{m|n} \).

Suppose \( \{z_k\}_{1 \leq k \leq m} \) be a global coordinate system on \( \mathbb{C}^m \). Then, it is called even coordinates on \( U^{m|n} \).

In addition, \( \{e_1, \cdots, e_n\} \) introduced by (1) is called odd coordinates on \( U^{m|n} \).

A super ringed space \((M, \mathcal{O})\) is called a supermanifold of dimension \( m|n \) if it is locally isomorphic to \( \mathbb{C}^{m|n} = (\mathbb{C}^m, \mathcal{C}_m) \), namely, there exists an open cover \( \{V_\alpha\} \) for \( M \) such that for any \( \alpha \), there exists an isomorphism \( \psi_\alpha = (\tilde{\psi}_\alpha, \psi_\alpha^*) \) from \( (V_\alpha, \mathcal{O}|_{V_\alpha}) \) to \( \mathbb{C}^{m|n} \). Thus for an open subset \( V \subset \mathbb{C}^m \), \( \tilde{\psi}_\alpha: V_\alpha \rightarrow \mathbb{C}^m \) is a homeomorphism and \( \psi_\alpha^*: \mathcal{C}_m(V) \rightarrow \mathcal{O}(\tilde{\psi}_\alpha^{-1}(V)) \) is an isomorphism of \( \mathbb{Z}_2 \)-graded rings.

Clearly, the morphisms between supermanifolds are just morphisms between the corresponding super ringed spaces such that for any \( x \in V_\alpha \), \( \psi^*: \mathcal{C}_m, \tilde{\psi}(x) \rightarrow \mathcal{O}_x \) is local, i.e., \( \psi^*(m_{\tilde{\psi}(x)}) \subset m_x \), where \( m_x \) is the unique maximal ideal in \( \mathcal{O}_x \).

A super vector bundle of rank \( r|s \) over a supermanifold \((M, \mathcal{O})\) is a locally free sheaf of \( \mathcal{O} \)-modules of rank \( r|s \) over \( M \).

### 2.2 \( \nu \)-domains

By a \( \nu \)-domain \((U, \mathcal{C}_U, \nu)\), we mean a superdomain \((U, \mathcal{C}_U)\) with an odd involution \( \nu: \mathcal{C}_U \rightarrow \mathcal{C}_U \) when it is considered as a morphism between sheaves of \( \mathcal{C}_U^{\infty} \)-modules. In other words, one has

\[ \nu^2 = 1, \quad \nu(\mathcal{C}_U)^0 \subset (\mathcal{C}_U)^1, \quad \nu(\mathcal{C}_U)^1 \subset (\mathcal{C}_U)^0. \]

Hence, \( \mathcal{C}_U \) may be considered as a sheaf of \( \mathcal{C}_{\nu_0} \)-modules where \( \mathcal{C}_{\nu_0} = \mathbb{C}[\nu_0] \) is a ring generated by indeterminate \( \nu_0 \) with \( \nu_0^2 = 1 \). Indeed, one may define \( \nu_0 a := \nu(a) \), for each \( a \in \mathcal{C}_U \).

A morphism of \( \nu \)-domains is a morphism, say \( \psi = (\tilde{\psi}, \psi^*) \), from \((\mathbb{C}^m, \mathcal{C}_m, \nu)\) to \((\mathbb{C}^k, \mathcal{C}_k, \nu')\) where \( \psi^* \) preserves the \( \mathcal{C}_{\nu_0} \)-module structure of corresponding sheaves, i.e., for any open subset \( V \subset \mathbb{C}^k \), the morphism \( \psi^*_V: \mathcal{C}_k^{\nu}(V) \rightarrow \mathcal{C}_m(\tilde{\psi}^{-1}(V)) \) is a \( \mathcal{C}_{\nu_0} \)-module homomorphism.
2.3 Differential forms

Now, we introduce the notion of derivations and differential forms in supergeometry.

A homogeneous derivation of a superalgebra \( \mathcal{A} \) on a field \( k \) is a \( k \)-linear map \( D : \mathcal{A} \rightarrow \mathcal{A} \) such that

\[
D(ab) = (Da)b + (-1)^{p(D)p(a)}a(Db) \quad a, b \in \mathcal{A},
\]

where by \( p(D) \) we mean the parity of the derivation.

A map \( D : \mathcal{A} \rightarrow \mathcal{A} \) is a derivation of \( \mathcal{A} \) if any of its homogeneous components is a homogeneous derivation. The space of all derivations on \( \mathcal{A} \) is denoted by \( \text{Der}(\mathcal{A}) \).

Let \( (M, \mathcal{O}) \) be a supermanifold. In addition, let \( \text{Der}(\mathcal{O}(U)) \) be the space of all derivations of \( \mathcal{O}(U) \) for an open subset \( U \subset M \). Considering \( T(U) \) as the tensor algebra of \( \text{Der}(\mathcal{O}(U)) \) over \( \mathcal{O}(U) \) and \( T^r(U) \) as the homogeneous subspace corresponding to \( r \in \mathbb{N} \), the set \( \text{Hom}_{\mathcal{O}(U)}(T^r(U), \mathcal{O}(U)) \) contains all \( r \)-linear maps on \( \text{Der}(\mathcal{O}(U)) \) such that for any \( X_1, \cdots, X_r \in \text{Der}(\mathcal{O}(U)), f \in \mathcal{O}(U) \) and \( \omega \in \text{Hom}_{\mathcal{O}(U)}(T^r(U), \mathcal{O}(U)) \), we have

\[
\langle X_1, \cdots, fX_1, \cdots, X_r, \omega \rangle = (-1)^{p(f)\sum_{i=1}^{r} p(X_i)} f \langle X_1, \cdots, X_i, \cdots, X_r, \omega \rangle.
\]

**Definition 2.1.** The set of \( r \)-differential forms on \( U \) denoted by \( \Omega^r(U, \mathcal{O}) \) is the set of all maps \( \omega \in \text{Hom}_{\mathcal{O}(U)}(T^r(U), \mathcal{O}(U)) \) such that it satisfies the following condition:

\[
\langle X_1, \cdots, X_i, X_{i+1}, \cdots, X_r, \omega \rangle = (-1)^{1+p(X_i)\sum_{i=1}^{r} p(X_i)} \langle X_1, \cdots, X_{i+1}, X_i, \cdots, X_r, \omega \rangle.
\]

\( \Omega^r(U, \mathcal{O}) \) is an \( \mathcal{O}(U) \)-module with the property below

\[
\langle X_1, \cdots, X_r, \omega f \rangle = \langle X_1, \cdots, X_r, \omega \rangle f,
\]

In addition, \( \Omega^r(U, \mathcal{O}) \) is \( \mathbb{Z}_2 \)-graded. Thus, we have

\[
\langle X_1, \cdots, X_r, \omega \rangle \in \mathcal{O}^k(U),
\]

where \( k := (p(\omega) + \sum_{i=1}^{r} p(X_i)) \mod 2 \).

Set

\[
\Omega(U, \mathcal{O}) = \bigoplus_{r=0}^{\infty} \Omega^r(U, \mathcal{O}), \quad \Omega^0(U, \mathcal{O}) = \mathcal{O}(U)
\]

The correspondence \( U \rightarrow \Omega(U, \mathcal{O}) \) defines a sheaf of supercommutative superalgebras on \( M \) called the sheaf of differential forms.

**Lemma 2.1.** Let \( U \subset M \) be an arbitrary open set. There exists a unique even derivation \( d : \Omega(U, \mathcal{O}) \rightarrow \Omega(U, \mathcal{O}) \) of degree 1, satisfying the following properties:

1) On \( \Omega^0(U, \mathcal{O}) \), we have

\[
d : \Omega^0(U, \mathcal{O}) \rightarrow \Omega^1(U, \mathcal{O})
\]

\[
\langle Xdg \rangle = Xg, \quad \forall g \in \mathcal{O}(U) = \Omega^0(U, \mathcal{O}), \quad X \in \text{Der}(\mathcal{O}(U)).
\]

2) \( d^2 = 0 \).
3 $\nu$-projective spaces

In this section, first, we introduce a new generalization of projective spaces in supergeometry, different from superprojective spaces, called $\nu$-projective spaces which are constructed by gluing $\nu$-domains. At the end, we introduce a $1|0$-super line bundle over this space.

3.1 Construction of $\nu$-projective spaces

In this section, by introducing gluing morphisms, it is shown the way a $\nu$-projective space is constructed by gluing a number of copies of $\nu$-domains $(\mathbb{C}^m, \mathcal{C}_{\mathbb{C}^m}, \nu)$, where $\mathcal{C}_{\mathbb{C}^m} = C_{\mathbb{C}^m}^\infty[e_1, \ldots, e_{n+1}]$ and $C_{\mathbb{C}^m}^\infty$ is the sheaf of complex valued smooth functions.

For each $i \in \{1, \ldots, m + n + 1\}$, let $(U_i, \mathcal{O}_i) = (\mathbb{C}^m, \mathcal{C}_{\mathbb{C}^m})$ be a $\nu$-domain. We write $U_i$ instead of $(U_i, \mathcal{O}_i)$ for brevity.

The $\nu$-domain $U_i$ is called standard if $1 \leq i \leq m + 1$, otherwise it is called nonstandard. The reason for the adoption of this convention will be further clarified.

Corresponding to the index $i$, we label the $\nu$-domain $U_i$ by an even supermatrix, say $A_i$, of dimension $1|0 \times (m + 1)|n$ with the property that the $i$-th entry is equal to 1 if $1 \leq i \leq m + 1$, otherwise it is equal to $1\nu$ which is a formal symbol. Obviously, $A_i$ may be decomposed into two blocks $B_1$ and $B_2$ which are respectively called even and odd part of $A_i$ because of the parity of their elements. In addition, except the $i$-th entry the blocks $B_j$, $(j = 1, 2)$, are filled by even and odd coordinates from left to right respectively according to the order induced by their indices. In this process, a coordinate, say $w$, is replaced by $\nu(w)$ if it places in a part with opposite parity. We separate even and odd parts of $A_i$ by a vertical line called divider line.

Denote the $j$-th entry of $A_i$ by $M_j(A_i)$. It is clear that $M_j(A_i)$ is even if $1 \leq j \leq m + 1$, otherwise it is an odd entry of $A_i$. Obviously, $M_i(A_i) = 1$ for $i \leq m + 1$ and $M_i(A_i) = 1\nu$ for $i > m + 1$.

Now, we introduce $M_j'(A_i)$ which will be used in the subsequent sections.

For $j \leq m + 1$, set $M_j'(A_i) = M_j(A_i)$ and otherwise define $M_j'(A_i) = \nu(M_j(A_i))$. Clearly, $M_j'(A_i)$ always takes even values and $M_i'(A_i)$ is always equal to the identity. By convention $\nu^0 = id$, we may set $M_j'(A_i) = \nu^{p(j)}(M_j(A_i))$ for each $j$ where $p(j) = 0$ if $j \leq m + 1$, otherwise $p(j) = 1$.

3.1.1 Gluing morphisms of $\nu$-domains

Now, we are going to introduce gluing morphisms of $\nu$-domains.

Consider $\nu$-domains $(U_i, \mathcal{O}_i)$ and $(U_j, \mathcal{O}_j)$ labeled by $A_i$ and $A_j$ respectively. Now, let $U_{ji}$ be a set consisting of those points of the $\nu$-domain $U_j$ for which $M_j'(A_j)$ is invertible. The gluing morphism $g_{ij}$ on $U_{ji}$ is as follows:

$$g_{ij} = (g_{ij}, g_{ij}^*): (U_{ji}, \mathcal{O}_j|_{U_{ji}}) \rightarrow (U_{ij}, \mathcal{O}_i|_{U_{ij}})$$

(2)
where \( \bar{g}_{ij} : U_{ji} \rightarrow U_{ij} \) is a homeomorphism to be defined by lemma 3.1 and \( g_{ij}^* : \mathcal{O}_j|_{U_{ij}} \rightarrow \mathcal{O}_j|_{U_{ij}} \) is an isomorphism between sheaves determined by defining on each entry of \( D_iA_i \) as a rational expression which appears as the corresponding entry provided by the pasting equation (see [19]):

\[
D_i \left( (M_i'(A_j))^{-1}A_j \right) = D_iA_i \tag{3}
\]

where by \( D_i(A_i) \) we mean the supermatrix \( A_i \) in which the \( i \)-th entry, \( M_i(A_i) \), is omitted.

By [19], the uniqueness of the morphism \( g_{ij} \) is concluded.

**Proposition 3.1.** The morphisms \( \{g_{ij}^*\} \) defined as above, are gluing morphisms. In other words, the corresponding sheaves of \( \nu \)-domains \( U_i \) and \( U_j \) can be glued together through the morphism \( g_{ij}^* \).

**Proof.** By ([19], page 135), \( \{g_{ij}^*\} \) are gluing morphisms if and only if the following conditions hold:

1) \( g_{ii}^* = id \)

2) \( g_{ji}^* \circ g_{ij}^* = id \)

3) \( g_{ki}^* \circ g_{jk}^* \circ g_{ij}^* = id \)

1) By definition, we know that the morphism \( g_{ii}^* \) is defined by the equation

\[
D_i \left( (M_i'(A_j))^{-1}A_i \right) = D_iA_i \tag{3}
\]

Since \( M_i'(A_i) = 1 \), the equality holds.

2) Let \( i \leq m + 1 \) and \( j > m + 1 \). The morphism \( g_{ji}^* \) is obtained from the following:

\[
D_j \left( (M_j'(A_i))^{-1}A_j \right) = D_jA_j. \tag{4}
\]

To compute \( g_{ji}^* \circ g_{ij}^* \), it is required to replace the supermatrix \( A_j \) in (3) by the left hand side of (4). So one has

\[
D_i \left( (M_i'[Z^{-1}A_i])^{-1}[Z^{-1}A_i]_j \right) = D_iA_i \tag{5}
\]

where \( Z = M_j'A_i \) and \( [Z^{-1}A_i]_j \) is a supermatrix which is obtained from \( Z^{-1}A_i \) by substituting \( 1\nu \) for the \( j \)-th entry.

Representing entries of \( A_i \) by \( a_i \), one gets

\[
[Z^{-1}A_i]_j = [Z^{-1}a_1, \ldots, Z^{-1}a_m, Z^{-1}a_{m+1}, \ldots, Z^{-1}a_{m+n}] \]

where \( 1\nu \) places in the \( j \)-th position. Hence,

\[
[Z^{-1}A_i]_j = Z^{-1}a_1, \ldots, a_m, a_{m+1}, \ldots, Z(1\nu), \ldots, a_{m+n}. \tag{6}
\]

To have meaningful expressions, consider the following rule for \( 1\nu \):

\[ w.1\nu = \nu(w). \]
Thus \( Z(1\nu) = (\nu a_j)1\nu = a_j \) and accordingly one can rewrite (6) as below
\[
[Z^{-1}A_i]_j = Z^{-1}[a_1, \ldots, a_m|a_{m+1}, \ldots, a_j, \ldots, a_{m+n}] = Z^{-1}A_i
\]
So for the left hand side of (5) one has:
\[
D_i \left( \left[ M'_j[Z^{-1}A_i]_j \right]^{-1} \right) = D_i \left( \left[ M'_j(Z^{-1}A_i) \right]^{-1} \right) = \left( Z^{-1}(M'_j(A_i)) \right)^{-1} Z^{-1}D_i A_i
\]
\[
= \left( M'(A_i) \right)^{-1} ZZ^{-1}D_i A_i = D_i A_i
\]

hence \( g^*_j \circ g^*_i = \text{id} \) holds.

3) To prove the equality \( g^*_k \circ g^*_j \circ g^*_i = \text{id} \), it only suffices to show that \( g^*_j \circ g^*_k \) is obtained from the following:
\[
D_i \left( \left[ M'_j(A_k) \right]^{-1} A_k \right) = D_i A_i
\]
Note that the morphism \( g^*_j \) is defined by
\[
D_j \left( \left[ M'_j(A_k) \right]^{-1} A_k \right) = D_j A_j
\] (7)
To compute \( g^*_j \circ g^*_k \), it is required to replace the supermatrix \( A_j \) in (3) by the left hand side of (7). So we have
\[
D_i \left( \left[ M'_j[Z^{-1}A_k]_j \right]^{-1} \right) = D_i A_i
\] (8)
where \( Z = M'_j(A_k) \).
As for the second condition, one has \( Z^{-1}A_k = Z^{-1}A_k \). Thus the left hand side of (8) is as follows:
\[
D_i \left( \left[ M'_j[Z^{-1}A_k]_j \right]^{-1} \right) = D_i \left( \left[ M'_j(Z^{-1}A_k) \right]^{-1} \right) = \left( Z^{-1}(M'_j(A_k)) \right)^{-1} Z^{-1}D_i A_k
\]
\[
= \left( M'(A_k) \right)^{-1} ZZ^{-1}D_i A_k = \left( M'(A_k) \right)^{-1} D_i A_k.
\]
By substituting in (8), one gets
\[
\left( M'(A_k) \right)^{-1} D_i A_k = D_i A_i
\]
So one has the following:
\[
g^*_j \circ g^*_k = g^*_k
\]
This completes the proof.

Now, we are going to define \( \bar{g}_{ij} \) which is introduced by (2).

**Lemma 3.1.** Let \( g^*_i \) be the gluing morphism between sheaves of rings from \( O_i|_{U_{ij}} \) to \( O_j|_{U_{ij}} \) determined by pasting equation (3). Then the following morphism is induced by \( g^*_i \) on the corresponding reduced manifolds:
\[
(\bar{g}_{ij}, \bar{g}^*_i) : (U_{ij}, C^\infty(U_{ij})) \rightarrow (U_{ij}, C^\infty(U_{ij})).
\]
Proof. Let $\mathcal{J}_i$ be the sheaf of the nilpotent elements of $\mathcal{O}_i$. Since $g^*_ij(\mathcal{J}_i) \subset \mathcal{J}_j$, the following morphism is induced by $g^*_ij$:
\[
\mathcal{O}_i|_{U_{ij}} \rightarrow \mathcal{O}_j|_{U_{ji}}
\]
On the other hand, we have the following isomorphism:
\[
\tau_i : \mathcal{O}_i|_{\mathcal{J}_i} \rightarrow C^\infty(U_i)
\]
\[
s + \mathcal{J}_i \mapsto \tilde{s}
\]
where $\tilde{s}(z)$, $z \in U_i$, is a unique complex number so that $s - \tilde{s}(z)$ is not invertible in $\mathcal{O}_i(U)$ for any neighborhood $U$ of $z$.

So one has the following map between the sheaves of rings of smooth functions:
\[
\tilde{g}^*_ij : C^\infty(U_i) \rightarrow C^\infty(U_j)
\]
\[
\tilde{g}^*_ij := \tau_j \circ \mathcal{O}_j|_{\mathcal{J}_j} \circ \tau^{-1}_i.
\]
Therefore by ([19], Th. 4.3.1), there exists a smooth map $\tilde{g}_{ij} : U_j \rightarrow U_i$ such that for any $f \in C^\infty(U_i)$, $\tilde{g}^*_ij(f) = f \circ \tilde{g}_{ij}$.

\[\Box\]

### 3.2 The underlying space of a $\nu$-projective space

Here, we are going to identify the underlying space of a $\nu$-projective space $\nu P^{m|n}$.

**Proposition 3.2.** If $(X, \mathcal{O})$ be the ringed space obtained by gluing $\nu$-domains $(U_i, \mathcal{O}_i)$ through $g_{ij}$, then the corresponding reduced manifold, $(X, \tilde{\mathcal{O}})$, is diffeomorphic to $\mathbb{CP}^m$.

To prove the theorem, we need the following lemma:

**Lemma 3.2.** There exists an injective immersion from $X$ to $\mathbb{CP}^{m+n}$.

**Proof.** To define an injective immersion from $X$ to $\mathbb{CP}^{m+n}$, it is sufficient to define a family of smooth maps $\{\psi_{it} : U_i \rightarrow V_t\}$ where $U_i$ is a $\nu$-domain and $(V_t, \phi_t)$, $t \in \{1, \cdots, m + n + 1\}$, is the chart on $\mathbb{CP}^{m+n}$. Obviously, $V_t$ is in bijection with $\mathbb{C}^{m+n}$.

Now, suppose that $A_i$ is the label of $U_i$, previously introduced in 3.1. Define $\psi_{it} : U_i \rightarrow \mathbb{C}^{m+n}$ as a map with the component functions to be equal to the entries of the following matrix:
\[
D_t\left(\left(M_t(\nu^\alpha A_i)\right)^{-1}\nu^\alpha A_i\right)
\]
where $\nu^\alpha : \mathcal{O} \rightarrow \tilde{\mathcal{O}}$ is a map defined by $\nu^\alpha a = \nu^\phi(a)$ on each homogeneous element $a$. In addition, for row matrix $A_i = (a_{ij})$ we set $\nu^\alpha A_i = (\nu^\alpha a_{ij})$. 

Let $\tilde{g}_{ij}$ be a map introduced in lemma 3.1 and let $\theta_{ts}$ be the transition map from $\phi_t$ to $\phi_s$. Then one has
\[
\theta_{ts} \circ \psi_{it} = \psi_{js} \circ \tilde{g}_{ji}.
\]
Note that the component functions of $\psi_{it}$ are equal to the entries of (9). Thus $\theta_{ts} \circ \psi_{it}$ is a map whose component functions are equal to the entries of the following matrix:
\[
D_s \left[ \left( M_s (\nu'' A_i) \right)^{-1} \nu'' A_i \right]^{-1} \left[ (M_s (\nu'' A_i))^{-1} \nu'' A_i \right]
\]
which by the proof of the proposition 3.1, one gets
\[
D_s \left[ \left( M_s (\nu'' A_i) \right)^{-1} \nu'' A_i \right]^{-1} \left( M_s (\nu'' A_i) \right)^{-1} \nu'' A_i
\]
\[
= D_s \left[ \left( M_s (\nu'' A_i) \right)^{-1} \nu'' A_i \right]^{-1} \left( M_s (\nu'' A_i) \right)^{-1} \nu'' A_i
\]
\[
= D_s \left[ \left( M_s (\nu'' A_i) \right)^{-1} \nu'' A_i \right]^{-1} \nu'' A_i.
\]
On the other hand, the components of the map $\tilde{g}_{ij}$ are equal to the entries of the matrix $(M_j (\nu'' A_i))^{-1} \nu'' A_i$. Therefore, the components of $\psi_{js} \circ \tilde{g}_{ji}$ are the entries of the following matrix:
\[
D_s \left[ \left( M_s (M_j \nu'' A_i) \right)^{-1} \nu'' A_i \right]^{-1} \left( M_s (M_j \nu'' A_i) \right)^{-1} \nu'' A_i
\]
\[
= D_s \left[ \left( M_s (M_j \nu'' A_i) \right)^{-1} \nu'' A_i \right]^{-1} \left( M_s (M_j \nu'' A_i) \right)^{-1} \nu'' A_i
\]
\[
= D_s \left[ \left( M_s (M_j \nu'' A_i) \right)^{-1} \nu'' A_i \right]^{-1} \nu'' A_i.
\]
Hence, $\theta_{ts} \circ \psi_{it} = \psi_{js} \circ \tilde{g}_{ji}$.

Thus the maps \{\psi_{it}\} are coordinate representations of a map say $\psi$ from $X$ to $\mathbb{C}P^{m+n}$.

Now, we are going to show that $\psi : X \rightarrow \mathbb{C}P^{m+n}$ is an injective immersion.

The smoothness of each map $\psi_{it}$ results the smoothness of $\psi$. The left inverse of $\psi$ is a map with component functions which are equal to the entries of the following matrix:
\[
D_t \left[ (M_t \{Y\}_t)^{-1} \{Y\}_t \right]
\]
where $\{Y\}_t = [y_1, \ldots, y_{t-1}, 1, y_t, \ldots, y_m]$ whenever $Y = [y_1, \ldots, y_{t-1}, y_t, \ldots, y_m]$.

Using $\phi$ as the left inverse of $\psi$, one has
\[
\phi_{ti} \circ \psi_{it} = id_{U_i}.
\]
Thus, $\psi$ is an injective immersion.

\[\square\]

Proof of proposition 3.2. Consider a map $\Lambda : \mathbb{C}P^m \rightarrow \mathbb{C}P^{m+n}$ defined pointwise by $\Lambda(P) = \pi_1(P)$ where $\pi_1$ is a map induced by the following map:
\[
\pi : \mathbb{C}^m \rightarrow \mathbb{C}^{m+n}
\]
\[
(z_1, \ldots, z_m) \mapsto (z_1, \ldots, z_m, 0, \ldots, 0)
\]
Obviously, Λ is an imbedding and Λ(\mathbb{C}P^m) = \psi(\mathcal{X})$. Hence, there exists a unique diffeomorphism such as $\overline{\Lambda} : \mathcal{X} \rightarrow \mathbb{C}P^m$ with the property $\Lambda \circ \overline{\Lambda} = \psi$.

\[\square\]

### 3.3 Super line bundles

**Proposition 3.3.** There exists a canonical \(1|0\)-super line bundle over $\nu\mathcal{P}^m|\nu$.

**Proof.** On each neighborhood $U_i$, define a sheaf of $\mathcal{O}_i$-modules of rank $1|0$ as $\mathcal{O}_i \otimes_{\mathcal{C}} \langle A_i \rangle_{\mathcal{C}}$, where by $\langle A_i \rangle_{\mathcal{C}}$ we mean the super vector space generated by $A_i$ over $\mathcal{C}$. The sheaves may be glued together through the morphisms as below:

$$
\eta_{ij} : \mathcal{O}_i|_{U_{ij}} \otimes_{\mathcal{C}} \langle A_i \rangle_{\mathcal{C}} \rightarrow \mathcal{O}_j|_{U_{ji}} \otimes_{\mathcal{C}} \langle A_j \rangle_{\mathcal{C}}
$$

$$
a \otimes A_i \mapsto g_{ij}^*(a)(M_i'(A_j))^{-1} \otimes A_j.
$$

It can be shown that these morphisms satisfy the conditions of the proposition 3.1. Therefore one obtains a $1|0$-super line bundle over $\nu\mathcal{P}^m|\nu$ which we denote it by $\nu\gamma_1$.

\[\square\]

### 4 Analytic approach to $\nu$-classes

Here, we are going to generalize the exponential sheaf sequence in supergeometry, then by introducing a proper cocycle associated to $\nu\mathcal{P}^m|\nu$, we define $\nu$-classes. At the end, we generalize a part of de Rham theorem and provide differential forms representing $\nu$-classes.

#### 4.1 The generalized exponential sheaf sequence

Let $\mathcal{C}_{\nu_0}$ be the ring generated by an indeterminate $\nu_0$ with condition $\nu_0^2 = 1$. One may decompose $\mathcal{O} \otimes \mathcal{C}_{\nu_0}$ as follows:

$$
\mathcal{O} \otimes \mathcal{C}_{\nu_0} = (\mathcal{O}^0 \otimes \mathcal{C}_{\nu_0}) \oplus (\mathcal{O}^1 \otimes \mathcal{C}_{\nu_0})
$$

where $\mathcal{O}^0$ and $\mathcal{O}^1$ are respectively the even and odd parts of the sheaf $\mathcal{O}$.

**Lemma 4.1.** The following short sequence is exact.

$$
0 \rightarrow \frac{\mathbb{Z}}{2}\nu_0 \rightarrow \mathcal{O}^0 \oplus \nu_0 \mathcal{O}^1 \rightarrow E \rightarrow \frac{(\mathcal{O}^0 \otimes \mathcal{C}_{\nu_0})^*}{\{-1, +1\}} \rightarrow 0
$$

where $(\mathcal{O}^0 \otimes \mathcal{C}_{\nu_0})^*$ denotes the subsheaf of even invertible elements of $(\mathcal{O}^0 \otimes \mathcal{C}_{\nu_0})$. In addition the second map is defined as below

$$
p + q\nu_0 \mapsto p + q\nu(1)\nu_0
$$
in which \(p, q \in \frac{Z}{2}\), and the third map may be obtained by the following map:

\[
E' : f + \nu_0 g \mapsto \exp \left( 2\pi \sqrt{-1} (f + \nu_0 \nu g) \right)
\]

(10)

where \(f \in O^0, g \in O^1\) and \(\exp(h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} \). Note that \(\nu\) is an odd involution defined previously in subsection 2.2.

**Proof.** If \(p + q \nu(1) \nu_0 = 0\), then \(p = 0\) and \(q = 0\). Thus the first map is one-to-one.

Now, we show that the kernel of the third map is equal to the image of the second map, i.e., \(\text{ker}(E) = \{p + q \nu(1) \nu_0; \ p, q \in \frac{Z}{2}\}\). To this end, one should note that

\[
E'(f + \nu_0 g) = \exp \left( 2\pi \sqrt{-1} (f + \nu_0 \nu g) \right) = (\exp 2\pi \sqrt{-1} f) \left( \cos(2\pi \nu g) + \nu_0 \sqrt{-1} \sin(2\pi \nu g) \right)
\]

where \(f\) and \(\nu g\) are the elements of \(O^0\).

Hence, for any element in the form of \(p + q \nu(1) \nu_0\), in which \(p, q \in \frac{Z}{2}\), one has

\[
E'(p + q \nu(1) \nu_0) = \exp \left( 2\pi \sqrt{-1} (p + q \nu_0) \right) = \pm 1.
\]

Conversely, let \(E'(f + \nu_0 g) = \pm 1\). Set

\[
\nu g = g'.
\]

We have

\[
\exp \left( 2\pi \sqrt{-1} (f + \nu_0 g') \right) = \pm 1 \implies \exp \left( 2\pi \sqrt{-1} f \right) \exp \left( 2\pi \sqrt{-1} \nu_0 g' \right) = \pm 1.
\]

By definition of the exponential map, one obtains

\[
\exp \left( 2\pi \sqrt{-1} f \right) \left[ \cos(2\pi g') + \nu_0 \sqrt{-1} \sin(2\pi g') \right] = \pm 1
\]

which concludes

\[
\sin(2\pi g') = 0 \implies g' = \frac{k_1}{2} (k_1 \in \mathbb{Z}), \quad \implies \quad g = \frac{k_1}{2} \nu(1)
\]

Therefore,

\[
\exp(2\pi \sqrt{-1} f) = \pm 1 \implies \cos(2\pi f) + \sqrt{-1} \sin(2\pi f) = \pm 1
\]

which gives

\[
\sin(2\pi f) = 0 \implies f = \frac{k_2}{2} (k_2 \in \mathbb{Z}),
\]

Hence, by setting \(p = \frac{k_2}{2}, q = \frac{k_1}{2}\), one has \(\text{ker}(E) = \{p + q \nu(1) \nu_0; \ p, q \in \frac{Z}{2}\}\) which is equal to the image of the second map of the sequence.

To prove that \(E\) is a surjection, let \((f + \nu_0 g) \in (O^0 \otimes \mathbb{C}_{\nu_0})^*\). We should find an element of \(O^0 \otimes \nu_0 O^1\), say \(f' + \nu_0 g'\), such that

\[
E'(f' + \nu_0 g') = f + \nu_0 g.
\]
By definition of $E$, one gets the following equations:

\[
\exp(2\pi\sqrt{-1}f') \cos(2\pi\nu g') = f \\
\exp(2\pi\sqrt{-1}f') \sin(2\pi\nu g') = -\sqrt{-1}g
\]

which by considering a proper branch of logarithm, as the inverse of exponential function, one may solve these equations and find $f' + \nu_0 g'$.

4.2 Introduction to $\nu$-classes

The main purpose of this subsection is to introduce a proper 1-cocycle, $\{h_{ts}\}$, which represents an element of Cech cohomology on $\mathbb{C}P^m$. Then, we introduce a generalization of Chern classes in super-geometry called $\nu$-classes.

**Definition 4.1.** For arbitrary indices $i$ and $j$ define $h_{ij}$ as follows:

\[
h_{ij} \in (\mathcal{O}_{ij}^0 \otimes \mathcal{C}_{\nu_0})|_{\mathcal{U}_ij}, \\
h_{ij} = \nu_0^{(p(i)+p(j))} (M_i'(A_j))^{-1}
\]

Note that $M_i'(A_j)$ is defined previously in the subsection 3.1.

**Lemma 4.2.** $\{h_{ts}\}$ is a 1-cocycle corresponding to the open covering $\{U_i\}_{1 \leq i \leq m+n+1}$.

*Proof.* It is necessary to show that for arbitrary indices $i$, $j$ and $k$, the following equality holds on $U_{ijk}$

\[
h_{jk} h_{ik}^{-1} h_{ij} = id.
\]

Equivalently, one needs to prove the following equality on $U_{ijk}$:

\[
(M_j'A_k)^{-1} M_i'A_k = M_i'A_j
\]

(11)

For this we consider two cases as below

1) If $i \leq m + 1$ and $j, k \in \{1, \cdots, m + n + 1\}$, then

\[
M_i'A_k = M_iA_k, \quad M_i'A_j = M_iA_j.
\]

Considering the $i$-th entries of $A_j$ and $A_k$ in the pasting equation of $g_{j,k}^*$, i.e., $D_j((M_j'A_k)^{-1}A_k) = D_jA_j$, one gets

\[
(M_j'(A_k))^{-1} M_i(A_k) = M_i(A_j) \quad \implies \quad (M_j'(A_k))^{-1} M_i'(A_k) = M_i'(A_j).
\]
2) If \( i > m + 1 \) and \( j, k \in \{1, \cdots, m+n+1\} \), then
\[
M'_i(A_k) = \nu(M_i(A_k)), \quad M'_i(A_j) = \nu(M_i(A_j)).
\]

In the same way as in the first case, one has \((M'_jA_k)^{-1}(M_iA_k) = M_iA_j\). So by applying \( \nu \) on the two sides of this equation one gets
\[
(M'_jA_k)^{-1}\nu(M_iA_k) = \nu(M_iA_j) \quad \implies \quad (M'_jA_k)^{-1}M'_i(A_k) = M'_i(A_j)
\]
Note that for any arbitrary index \( k \), \( M'_k(A_k) = 1 \). So in the case \( i = k \), one has the following:
\[
(M'_j(A_k))^{-1}M'_k(A_k) = M'_i(A_j) \quad \implies \quad (M'_j(A_k))^{-1} = M'_k(A_j).
\]

Now, suppose that \( i, j, \text{and} k \) be arbitrary indices. Then we have following relations on \( U_{ijk} \):
\[
\begin{align*}
\lambda_{jk} &= \nu_0^{p(j)+p(k)}(M'_jA_i)^{-1} = \nu_0^{p(j)+p(k)}(M'_jA_i) \in \mathcal{O}_{\lambda_{ijk}} \\
\lambda_{ik} &= \nu_0^{p(k)+p(i)}(M'_iA_j)^{-1} = \nu_0^{p(k)+p(i)}(M'_iA_j) \in \mathcal{O}_{\lambda_{ijk}} \\
\lambda_{ij} &= \nu_0^{p(i)+p(j)}(M'_jA_i)^{-1} = \nu_0^{p(i)+p(j)}(M'_jA_i) \in \mathcal{O}_{\lambda_{ijk}}
\end{align*}
\]
Therefore we have
\[
\lambda_{jk} \lambda_{ik}^{-1} \lambda_{ij} = \left(\nu_0^{p(j)+p(k)}(M'_jA_i)^{-1}(M'_jA_i)\right) \left(\nu_0^{p(k)+p(i)}(M'_kA_i)^{-1}\right) \left(\nu_0^{p(i)+p(j)}(M'_jA_i)^{-1}\right)
\]
\[
= \nu_0^{2(p(i)+p(j)+p(k))} = id
\]

Let \( \eta \) denotes the 1-cocycle \( \{h_{ts}\} \). Note that \( \eta \) defines an element in Cech cohomology of \( CP^m \) with coefficients in \( \mathcal{O}^* \{\mathbb{C} \otimes \mathcal{O}_0\}^* \) which we denote it by \( \Gamma \). Hence,
\[
\Gamma \in H^1(CP^m, \mathcal{O}^* \{\mathbb{C} \otimes \mathcal{O}_0\}^*)
\]
The short exact sequence introduced in lemma 4.1 gives us the following long exact sequence of Cech cohomology groups:
\[
\cdots \to H^1(CP^m, \mathcal{O}^* \otimes \nu_0 \mathcal{O}^1) \to H^1(CP^m, \mathcal{O}^* \{\mathbb{C} \otimes \mathcal{O}_0\}^*) \xrightarrow{\delta} H^2(CP^m, \mathcal{O}_0^* \{\mathbb{Z}/2\nu_0\}) \to \cdots
\]

Define \( c := \delta(\Gamma) \) which is called \( \nu \)-class of \( \nu \gamma_1 \) and may be considered as an analogous of the universal Chern class in supergeometry. One may define this class for any super line bundle by a proper generalization of homotopy classification theorem in supergeometry.

**Notation:** By \( H^k(M, \mathbb{Z}/2[\nu_0]) \) we mean the \( k \)-th Cech cohomology group of \( M \) with coefficients in \( \mathbb{Z}/2[\nu_0] \).

**Theorem 4.1.** Associated to isomorphism class of each super line bundle on \( \mathcal{M} = (M, \mathcal{O}) \), there exists a homotopy class of a morphism \((\phi, \psi)\) from \( \mathcal{M} \) to \( \nu \mathcal{P}^m[n] \) for some pair of natural numbers, say \((m, n)\).
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Proof. See [22].

Hence, one may define $\nu$-class of $E$ as follows:

$$\nu c(E) = \phi^*(\nu c) \in H^2(M, \frac{Z}{2}[\nu_0])$$

4.3 The de Rham theorem

Here, we are going to generalize the de Rham theorem in supergeometry for a special case. Indeed, we show that the second Cech and de Rham cohomology groups, with coefficients in $C_{\nu_0}$, are isomorphic to each other. To this end, following the method used in ([7], Prop.1, p.141), first, we prove lemma 4.3 and the theorem 4.2.

Let $\mathcal{M} = (M, \mathcal{O})$ be a supermanifold and let $\Omega^i$ be a sheaf of $\mathbb{Z}_2$-graded super vector spaces of $i$-differential forms on the supermanifold $\mathcal{M}$ and $\Omega^i_d$ be a subsheaf of $\Omega^i$ whose elements are closed differential forms.

**Lemma 4.3.** The following short sequences are exact:

$$0 \to \mathbb{R} \to \Omega^0 \xrightarrow{d} \Omega^1_d \to 0 \quad (12)$$

$$0 \to \Omega^1_d \to \Omega^1 \xrightarrow{d} \Omega^2_d \to 0 \quad (13)$$

**Proof.** A sheaf sequence is exact if and only if for each $p \in M$, the corresponding sequence of stalks is exact as a sequence of super vector spaces. So for (12), it is sufficient to prove that for each $p \in M$ the sequence $0 \to \mathbb{R}_p \xrightarrow{i} \Omega^0_p \xrightarrow{d} \Omega^1_{d,p} \to 0$ is exact. Equivalently, one needs to prove the following equalities:

$$\text{Ker}(i) = 0,$$

$$\text{Im}(i) = \mathbb{R}_p = \text{Ker}(d),$$

$$\text{Im}(d) = \Omega^1_{d,p}.$$  

The first equality is obvious.

To prove the second equality, let $[f]_p \in \Omega^0_p$ denotes the germ of $f \in \mathcal{O}(U)$ such that $d[f]_p = [0] \in \Omega^1_{d,p}$. Then, there exists a small enough contractible neighborhood $V \subset U$ of $p$ such that it splits on $M$ and one has $df|_V = 0$. Therefore, by ([11], Th. 4.6), $f$ is constant, i.e., $[f]_p \in \mathbb{R}_p$. Conversely, let $[f]_p \in \mathbb{R}_p$ then $d[f]_p = 0$. So $\text{Im}(i) = \mathbb{R}_p = \text{Ker}(d)$.

By ([11], Th. 4.6), $d : \Omega^0_p \to \Omega^1_{d,p}$ is a surjective map, so $\text{Im}(d) = \Omega^1_{d,p}$.

Similarly, one may prove that (13) is an exact sequence.

**Theorem 4.2.** The following short sequences are exact:

$$0 \to \mathbb{R} \otimes C_{\nu_0} \to \Omega^0 \otimes C_{\nu_0} \xrightarrow{d} \Omega^1_d \otimes C_{\nu_0} \to 0 \quad (14)$$

$$0 \to \Omega^1_d \otimes C_{\nu_0} \to \Omega^1 \otimes C_{\nu_0} \xrightarrow{d} \Omega^2_d \otimes C_{\nu_0} \to 0 \quad (15)$$
Proof. The exactness of the sequence (12) at the right shows that the short sequence (14) is exact at the right. In addition, $\mathbb{R} \otimes \mathbb{C}_{v_0} \hookrightarrow \Omega^0 \otimes \mathbb{C}_{v_0}$ is an injection.

Similarly, one may prove the exactness of the sequence (15).

$\square$

This theorem leads us to generalize the de Rham theorem as below.

For an arbitrary sheaf $\mathcal{O}$ on a supermanifold $M$ and a locally finite open cover $\mathcal{U} = \{U_\alpha\}$, let $C^p(\mathcal{U}, \mathcal{O}) = \prod_{\alpha_0 \neq \alpha_1 \neq \cdots \neq \alpha_p} \mathcal{O}(U_{\alpha_0} \cap \cdots \cap U_{\alpha_p})$ denotes the set of $p$-cochains of $\mathcal{O}$. Then we have the following short exact sequence of cochain groups:

\[
0 \longrightarrow C^p(\mathcal{U}, \mathbb{C}_{v_0}) \xrightarrow{i} C^p(\mathcal{U}, \Omega^0 \otimes \mathbb{C}_{v_0}) \xrightarrow{d} C^p(\mathcal{U}, \Omega^1 \otimes \mathbb{C}_{v_0}) \longrightarrow 0
\]

which by [8], it gives rise to an associated long exact sequence of cohomology groups:

\[
0 \longrightarrow H^0(M, \mathbb{C}_{v_0}) \longrightarrow H^0(M, \Omega^0 \otimes \mathbb{C}_{v_0}) \longrightarrow H^0(M, \Omega^1 \otimes \mathbb{C}_{v_0})
\]

\[
\longrightarrow H^1(M, \mathbb{C}_{v_0}) \longrightarrow H^1(M, \Omega^0 \otimes \mathbb{C}_{v_0}) \longrightarrow H^1(M, \Omega^1 \otimes \mathbb{C}_{v_0})
\]

\[
\longrightarrow H^2(M, \mathbb{C}_{v_0}) \longrightarrow H^2(M, \Omega^0 \otimes \mathbb{C}_{v_0}) \longrightarrow H^2(M, \Omega^1 \otimes \mathbb{C}_{v_0}) \longrightarrow \cdots
\]

(16)

Note that we replace $\mathbb{R} \otimes \mathbb{C}_{v_0}$ by $\mathbb{C}_{v_0}$ because of the equality $\mathbb{R} \otimes \mathbb{C}_{v_0} = \mathbb{C}_{v_0}$.

In the same way, one has a long exact sequence of cohomology groups associated to the short exact sequence (15).

**Proposition 4.1.** The following map is an isomorphism:

\[
\frac{H^0(M, \mathbb{C}_{v_0})}{dH^0(M, \Omega^1 \otimes \mathbb{C}_{v_0})} \longrightarrow H^2(M, \mathbb{C}_{v_0})
\]

**Proof.** Since $\Omega^0 \otimes \mathbb{C}_{v_0}$ is a locally free sheaf, by ([14], page 188) it is fine and one has

\[
H^1(M, \Omega^0 \otimes \mathbb{C}_{v_0}) = 0, \quad H^2(M, \Omega^0 \otimes \mathbb{C}_{v_0}) = 0.
\]

Hence, a part of the sequence (16) reduces to the following exact sequence:

\[
0 \longrightarrow H^1(M, \Omega^1 \otimes \mathbb{C}_{v_0}) \xrightarrow{\delta_1} H^2(M, \mathbb{C}_{v_0}) \longrightarrow 0
\]

thus $\delta_1$ is an isomorphism.

Similarly, we have the following long exact sequence associated to the short exact sequence (15):

\[
0 \longrightarrow H^0(M, \mathbb{C}_{v_0}) \longrightarrow H^0(M, \Omega^1 \otimes \mathbb{C}_{v_0}) \xrightarrow{d} H^0(M, \Omega^2 \otimes \mathbb{C}_{v_0}) \xrightarrow{d_2} H^1(M, \mathbb{C}_{v_0}) \longrightarrow 0
\]

Because of the exactness, we have $\ker(d_2) = \text{Im}(d)$ which results that $\delta_2 : \frac{H^0(M, \Omega^2 \otimes \mathbb{C}_{v_0})}{dH^0(M, \Omega^1 \otimes \mathbb{C}_{v_0})} \longrightarrow H^1(M, \Omega^1 \otimes \mathbb{C}_{v_0})$ is an isomorphism.

So $\delta_1 \delta_2$ is an isomorphism and one has

\[
\frac{H^0(M, \mathbb{C}_{v_0})}{dH^0(M, \Omega^1 \otimes \mathbb{C}_{v_0})} \cong H^2(M, \mathbb{C}_{v_0})
\]

$\square$

**Corollary 4.1.** Up to isomorphism, we may consider $\nu c$ as an element of the de Rham cohomology.
4.4 Computing $\nu$-classes

Here, we are going to obtain a preimage of the cocycle $\eta = \{h_{ij}\}$ under the map $E'$ defined in lemma 4.1. In fact, we should find $(f_{ij} + \nu_0 g_{ij}) \in \mathcal{O}^0 \oplus \nu_0 \mathcal{O}^1$ such that

$$E'(f_{ij} + \nu_0 g_{ij}) = h_{ij}. \quad (17)$$

To this end, consider two cases as below:

1) If $1 \leq i, j \leq m + 1$ or $i, j > m + 1$, then $\nu_0^{p(i)+p(j)} = 1$, so we have

$$h_{ij} = (M'_i(A_j))^{-1}.$$ 

By substituting in (17), one has

$$E'(f_{ij} + \nu_0 g_{ij}) = (M'_i(A_j))^{-1}$$

which gives

$$\exp\left( (2\pi \sqrt{-1}) f_{ij} \right) \left( \cos(2\pi \nu g_{ij}) + \nu_0 \sqrt{-1} \sin(2\pi \nu g_{ij}) \right) = (M'_i(A_j))^{-1}. \quad (19)$$

Since there is no coefficient of $\nu_0$ on the right hand side, one concludes

$$\sin(2\pi \nu g_{ij}) = 0 \quad \Rightarrow \quad \nu g_{ij} = 0, \quad \cos(2\pi \nu g_{ij}) = 1.$$ 

So one has

$$\exp\left( (2\pi \sqrt{-1}) f_{ij} \right) = (M'_i(A_j))^{-1} \quad \Rightarrow \quad f_{ij} = \frac{1}{2\pi \sqrt{-1}} \log\left( (M'_i(A_j))^{-1} \right).$$

Hence,

$$f_{ij} + \nu_0 \nu (g_{ij}) = \frac{1}{2\pi \sqrt{-1}} \log\left( (M'_i(A_j))^{-1} \right). \quad (18)$$

2) If $1 \leq i \leq m + 1$ and $j > m + 1$, then $\nu_0^{p(i)+p(j)} = \nu_0$, so we have

$$h_{ij} = \nu_0 (M'_i(A_j))^{-1}.$$ 

By substituting in (17), one gets

$$\exp\left( (2\pi \sqrt{-1}) f_{ij} \right) \left( \cos(2\pi \nu g_{ij}) + \nu_0 \sqrt{-1} \sin(2\pi \nu g_{ij}) \right) = \nu_0 (M'_i(A_j))^{-1} \quad (19)$$

which yields the following relations:

$$\cos(2\pi \nu g_{ij}) = 0 \quad \Rightarrow \quad 2\pi \nu g_{ij} = \frac{\pi}{2} \quad \Rightarrow \quad g_{ij} = \frac{1}{4} \nu(1), \quad \sin(2\pi \nu g_{ij}) = 1.$$ 

So one may rewrite (19) as follows:

$$\exp\left( (2\pi \sqrt{-1}) f_{ij} \right) \nu_0 \sqrt{-1} = \nu_0 (M'_i(A_j))^{-1}$$
which concludes 

$$\exp \left( (2\pi \sqrt{-1}) f_{ij} \right) = -\sqrt{-1} (M'_i(A_j))^{-1}$$

$$\implies f_{ij} = \frac{1}{2\pi \sqrt{-1}} \log \left( -\sqrt{-1} (M'_i(A_j))^{-1} \right).$$

Hence,

$$f_{ij} + \nu_0 \nu(g_{ij}) = \frac{1}{2\pi \sqrt{-1}} \log \left( -\sqrt{-1} (M'_i(A_j))^{-1} \right) + \nu_0 \frac{1}{4} \nu(1). \quad (20)$$

Thus a right inverse of $E'$, say $L$, is as follows:

$$L(h_{ij}) = f_{ij} + \nu_0 \nu(g_{ij}),$$

where $f_{ij}$ and $\nu(g_{ij})$ are introduced by one of the equations (18) and (20).

Since $M'_i(A_j) \in \mathbb{C} \setminus \{0\}$, one may decompose $\nu$-domains $U_j$ by a decomposition of $\mathbb{C} \setminus \{0\}$.

Let $V^1$ (resp. $V^2$) be an open subset of $\mathbb{C}$ obtained by removing 0 and all positive (resp. negative) real numbers from the complex plane. So one may obtain the following decomposition of $U_j$:

$$U_j = U^1_j \cup U^2_j$$

where $U^1_j$ and $U^2_j$ are respectively the preimages of $V^1$ and $V^2$ under $h_{ij}$.

Now, one may compute $L(h_{ij})$ by considering the restriction $h_{ij}|_{U^k_j \cap U^l_j}$ and a proper branch of logarithm on $U^k_j \cap U^l_j$ where $k, l = 1, 2$.

Now, if we set

$$(\delta \eta)_{ijk} = L(h_{jk}) - L(h_{ik}) + L(h_{ij}),$$

then $\{(\delta \eta)_{ijk}\}$ is a representative of $\nu^c$.

**Example 4.1.** Consider the super line bundle over $\nu^c \mathbb{P}^{2|1}$. From the subsection 3.1, we know that $\nu^c \mathbb{P}^{2|1}$ is constructed by gluing $\nu$-domains $(U_i, \mathcal{O}_i) = (\mathbb{C}^m, \mathcal{C}^\infty)$, $(1 \leq i \leq 4)$. Let $U_i$ be labeled by $A_i$ where

$$A_1 = (1, z_1^{(1)}, z_2^{(1)}|e_1^{(1)})$$

$$A_2 = (z_1^{(2)}, 1, z_2^{(2)}|e_1^{(2)})$$

$$A_3 = (z_1^{(3)}, z_2^{(3)}, 1|e_1^{(3)})$$

$$A_4 = (z_1^{(4)}, z_2^{(4)}, \nu(e_1^{(4)})|1).$$

Note that $U_1, U_2$ and $U_3$ are standard super-domains and $U_4$ is a nonstandard super-domain.

By definition of $h_{ij}$, we have

$$h_{21} = \frac{1}{z_1^{(1)}}$$

$$h_{32} = \frac{1}{z_2^{(2)}}$$

$$h_{43} = \nu_0 \frac{1}{\nu e_1^{(3)}}$$

$$h_{14} = \nu_0 \frac{1}{z_1^{(4)}}.$$
1) \( \text{real}(z_1^{(1)}) < 0 \),

2) \( \text{Im}(z_1^{(1)}) \neq 0 \),

and similarly \( U_1^2 \) consists of those points of \( U_1 \) for which at least one of the relations \( \text{real}(z_1^{(1)}) > 0 \) and \( \text{Im}(z_1^{(1)}) \neq 0 \) holds.

In the same way, one may obtain such a decomposition for the \( \nu \)-domain \( U_2 \).

Considering the restriction of \( h_{21}|_{U_1 \cap U_2^1} \), one has \( h_{21} = \frac{1}{z_1^{(1)}} \) such that \( \text{real}(z_1^{(1)}) < 0 \) or \( \text{Im}(z_1^{(1)}) \neq 0 \) holds.

So by assuming \( 0 < \text{arg}(z_1^{(1)}) < 2\pi \), one has

\[
L(h_{21}) = \frac{1}{2\pi\sqrt{-1}} \log \left( \frac{1}{z_1^{(1)}} \right) = \frac{1}{2\pi\sqrt{-1}} \log \left( \frac{z_1^{(1)}}{|z_1^{(1)}|^2} \right)
= \frac{1}{2\pi\sqrt{-1}} \left( -\log |z_1^{(1)}| + \sqrt{-1}(2\pi - \text{arg}(z_1^{(1)})) \right).
\]

By similar computations, one may obtain a representative of \( \nu \)-c as follows:

\[
(\delta\eta)_{241} = L(h_{41}) - L(h_{21}) + L(h_{24}) = -\frac{1}{2}
\]

### 4.5 De Rham representative of the \( \nu \)-class for super line bundles

In this subsection, we are going to find a closed 2-form representing \( \nu \)-classes. In [7], it is proved that for a complex line bundle \( \gamma \) on a compact manifold \( M \) of dimension \( n \), there exists a representative of Chern classes in terms of the curvature form \( \Theta \) as follows:

\[
c = \left[ \frac{1}{2\pi\sqrt{-1}} \Theta \right] \in H^2_{DR}(M).
\]

Now, we introduce a curvature form \( \Theta \) in terms of \( \{h_{ij}\} \) by applying a partition of unity. Inspired by it, we obtain the desired closed 2-form representing \( \nu \)-classes.

Let \( M \) be a compact manifold of dimension \( n \) and \( \gamma \) denotes a complex line bundle on it. Let \( \{\rho_\alpha\} \) be a partition of unity subordinate to \( \{U_\alpha\} \), an open cover of \( M \). A connection 1-form \( \theta_\alpha \) may be defined as follows:

\[
\theta_\alpha s_\alpha = \nabla(s_\alpha) = \sum_\beta \rho_\beta \nabla^\beta (s_\alpha) = \sum_\beta \rho_\beta \nabla^\beta (h_{\alpha\beta}s_\beta) = \sum_\beta \rho_\beta d(h_{\alpha\beta})s_\beta,
\]

where \( s_\alpha \) is a section on \( U_\alpha \) such that \( s_\alpha(x) \) is a basis for the fiber of \( \gamma \) over \( x \). By restriction to \( U_\alpha \cap U_{\alpha'} \), one gets

\[
\theta_\alpha h_{\alpha\alpha'} s_{\alpha'} = \sum_\beta \rho_\beta d(h_{\alpha\beta})h_{\beta\alpha'}s_{\alpha'},
\]

hence,

\[
\theta_\alpha = \sum_\beta \rho_\beta d(\log(h_{\alpha\beta})) \quad (21)
\]
So we may obtain the curvature form \( \Theta \) as follows:

\[
\Theta = d\theta_\alpha = \sum_\beta d(\rho_\beta) d\left(\log(h_{\alpha \beta})\right).
\]

Note that the cocycle \( \eta \) defined in the subsection 4.2, does not correspond to any super line bundle. So by applying formula (21) for the cocycle \( \eta \) and substituting \( L(h_{ij}) \) instead of \( \log(h_{ij}) \), we may obtain local 1-forms which their exterior derivatives define a closed 2-form representing \( \nu_c \).

**Theorem 4.3.** For a super line bundle \( ^\nu \gamma_1 \), there exists a representation of \( \nu \)-class \( ^\nu \gamma_1 \) by a closed 2-form corresponding to the cocycle \( \eta \).

**Proof.** Inspired by the previous part, we may define a 1-form as \( \omega_i = \sum_j \rho_j d(L(h_{ij})) \) on \( U_i \) which its exterior derivative is the following 2-form:

\[
R_i = \sum_j d(\rho_j)d(L(h_{ij})) = \frac{1}{2\pi\sqrt{-1}} \sum_j \frac{1}{h_{ij}} d(\rho_j)d(h_{ij}) + \nu_0 \sum_j \frac{k_{ij}}{4} d(\rho_j)d(\nu(1))
\]

where \( k_{ij} = 0 \) if \( 1 \leq i, j \leq m + 1 \) or \( i, j > m + 1 \), otherwise \( k_{ij} = 1 \).

Now, we prove that \( R_i \) defines a global closed 2-form \( R \). To this end, we compute \( \omega_i - \omega_j \).

Let \( 1 \leq i \leq m + 1 \) and \( j > m + 1 \). So one has

\[
\omega_i - \omega_j = \sum_{k=1}^{m+n+1} \rho_k d(L(h_{ik})) - \sum_{k=1}^{m+n+1} \rho_k d(L(h_{jk}))
\]

\[
= \sum_{k=1}^{m+1} \rho_k d\left(\frac{1}{2\pi\sqrt{-1}} \log (M'_i(A_k))^{-1}\right) + \sum_{k=m+2}^{m+n+1} \rho_k d\left(\frac{1}{2\pi\sqrt{-1}} \log (-\sqrt{-1}(M'_i(A_k))^{-1} + \nu_0 \frac{1}{4} \nu(1))\right)
\]

\[
- \sum_{k=1}^{m+1} \rho_k d\left(\frac{1}{2\pi\sqrt{-1}} \log (\sqrt{-1}(M'_j(A_k))^{-1}) - \nu_0 \frac{1}{4} \nu(1)\right) - \sum_{k=m+2}^{m+n+1} \rho_k d\left(\frac{1}{2\pi\sqrt{-1}} \log (M'_j(A_k))^{-1}\right)
\]

\[
= \frac{1}{2\pi\sqrt{-1}} \sum_{k=1}^{m+1} \rho_k d\left(\log (-\sqrt{-1}(M'_i(A_k))^{-1}(M'_j(A_k)))\right) + \nu_0 \frac{1}{4} d(\nu(1)) \sum_{k=1}^{m+1} \rho_k
\]

\[
+ \frac{1}{2\pi\sqrt{-1}} \sum_{k=m+2}^{m+n+1} \rho_k d\left(\log (-\sqrt{-1}(M'_i(A_k))^{-1}(M'_j(A_k)))\right) + \nu_0 \frac{1}{4} d(\nu(1)) \sum_{k=m+2}^{m+n+1} \rho_k
\]

hence, by the equality (11) proved in lemma 4.2, we have

\[
\omega_i - \omega_j = \frac{1}{2\pi\sqrt{-1}} d\left(\log (-\sqrt{-1}(M'_i(A_j))^{-1})\right) + \nu_0 \frac{1}{4} d(\nu(1)) = d(L(h_{ij}))
\]

Therefore, \( \omega_i - \omega_j = d(L(h_{ij})) \) for any arbitrary \( i, j \).

From the relation \( \omega_i - \omega_j = d(L(h_{ij})) \), we deduce that \( (d\omega_i)|_{U_{ij}} = (d\omega_j)|_{U_{ij}} \), that is \( (R_i)|_{U_{ij}} = (R_j)|_{U_{ij}} \), which by the globality property of sheaves it proves that we have a global closed 2-form \( R \) defined by \( R_{|U_i} := R_i \).

Hence, by isomorphisms \( \delta_1 \) and \( \delta_2 \) defined in proposition 4.1, one has

\[
(\delta_2(R))_{j,1} = d(L(h_{ij})),
\]

\[
(\delta_1\delta_2(R))_{k,l,r} = \left(\delta_1 d(L(h_{ij}))\right)_{k,l,r}
\]

\[
= L(h_{lr}) - L(h_{kr}) + L(h_{kl})
\]
Hence,
\[
(\delta_1 \delta_2 (R))_{k,l,r} = (\delta_1 d(L(h_{ij})) )_{k,l,r} = L(h_{lr}) - L(h_{kr}) + L(h_{kl})
\]
Therefore, one may represent \( \nu^c \) in terms of the closed 2-form \( R \). 

\[\square\]

4.6 Representation of \( \nu \)-classes for super vector bundles

In the previous section, we show that the (first) \( \nu \)-class of the canonical 1|0-super line bundle over \( \nu \)-projective space may be represented by a closed 2-form in \( \Omega \otimes \mathbb{C}_{\nu_0} \). So one may deduce that \( \nu \)-classes of higher rank may be represented by higher degree closed forms. In this section, in a similar way of representing universal Chern classes in common geometry, for the canonical super vector bundle over \( \nu \)-grassmannian \( \nu \text{Gr}_{k|l}(m|n) \) [2], we introduce closed forms in \( \Omega_\varphi \otimes \mathbb{C}_{\nu_0} \) which may be considered as representatives of \( \nu \)-classes of rank higher than 1. By \( \Omega_\varphi \) we mean the structure sheaf of \( \nu \)-grassmannian. In [9], Chern classes of a complex vector bundle of rank \( k \) are represented by closed \( 2r \)-forms \((1 \leq r \leq k)\) which are polynomials in the curvature form of a connection on the bundle. One may obtain the curvature 2-form matrix \( \Theta \) for a certain connection by a method as in the previous subsection.

Let \( M \) denotes a manifold of dimension \( n \) and \( \xi^k \) a complex vector bundle over it. There exists a partition of unity \( \{ \rho_\alpha \} \) subordinate to an open covering \( \{ U_\alpha \} \) of \( M \). Let \( \{ h^{\alpha \beta} \} \) be coordinate transformations of \( \xi^k \) and \( \nabla \) be a connection defined as follows:

\[
\nabla(s^\alpha_i) = \sum_\beta \rho_\beta \nabla^\beta(s^\alpha_i) = \sum_\beta \rho_\beta \nabla^\beta(\sum_l h^{\alpha \beta}_{il} s^\beta_l) = \sum_l \sum_\beta \rho_\beta d(h^{\alpha \beta}_{il}) s^\beta_l = \sum_j \sum_\beta \rho_\beta d(h^{\alpha \beta}_{ij}) h^{\beta \alpha}_{ij} s^\alpha_j,
\]

where \( s^\alpha_i, (1 \leq i \leq k) \), are sections of \( \xi|_{U_\alpha} \) such that for each \( x \in U_\alpha \), \( \{ s^\alpha_i(x) \} \) is a basis for the fiber over \( x \).

Let \( \theta^\alpha = (\theta^\alpha_{ij}) \) be the connection form of \( \nabla \). Then, \( \nabla(s^\alpha_i) = \sum_j \theta^\alpha_{ij} s^\alpha_j \).

Therefore, we have

\[
\theta^\alpha_{ij} = \sum_l \rho_\beta d(h^{\alpha \beta}_{il}) h^{\beta \alpha}_{ij},
\]

or one may write

\[
\theta^\alpha = \sum_\beta \rho_\beta d(h^{\alpha \beta}) h^{\beta \alpha}
\]

By a straightforward calculation, one may prove that \( \Theta^\alpha = d\theta^\alpha - \theta^\alpha \wedge \theta^\alpha \) is the corresponding curvature 2-form matrix. So we have

\[
\Theta^\alpha = d\theta^\alpha - \theta^\alpha \wedge \theta^\alpha
\]

\[
= \sum_\beta d(\rho_\beta) d(h^{\alpha \beta}) h^{\beta \alpha} - \sum_\beta \rho_\beta d(h^{\alpha \beta}) d(h^{\beta \alpha}) - \sum_{\beta, \beta'} \rho_\beta d(h^{\alpha \beta}) h^{\beta \alpha} \wedge \rho_{\beta'} d(h^{\alpha \beta'}) h^{\beta' \alpha}
\]
Now, we apply this technique to obtain a supermatrix-valued 2-form for a canonical \( k|l \)-super vector bundle \( {}^\nu \gamma_{k|l} \) over \( {}^\nu \text{grassmannian } {}^\nu \text{Gr}_{k|l}(m|n) \). To obtain such a 2-form, we should define a cocycle \( \{ h^{\alpha\beta} \} \). By considering gluing morphisms of \( {}^\nu \gamma_{k|l} \) introduced in [2], one may define

\[
h^{\alpha\beta} = \nu_0^{p(\alpha)+p(\beta)}(M'_\alpha A_\beta)^{-1}
\]

where an index \( \alpha = I|J \) is a multi-index such that \( I \) and \( J \) are respectively ordered sets of \( \{1, \ldots, m\} \) and \( \{1, \ldots, n\} \) with the property that \( |I| + |J| = k + l \). In addition, \( p(\alpha) = 0 \) if \( |I| = k \), otherwise \( p(\alpha) = 1 \). Moreover, \( A_\alpha \) is a \( k|l \times m|n \) supermatrix associated to the \( \nu \)-domain \( U_\alpha \). By \( M'_\alpha A_\beta \) we mean a \( k|l \times k|l \) standard supermatrix which is obtained by applying certain modifications on columns of \( A_\beta \) with indices in \( \alpha \). For more information, see (2], section 2).

One can easily show that \( \{ h^{\alpha\beta} \} \) defined by (22) is a cocycle. Now, consider a partition of unity \( \{ \rho_\alpha \} \) subordinate to the open covering \( \{ U_\alpha \} \) of \( {}^\nu \text{Gr}_{k|l}(m|n) \). We define a \( (k|l \times k|l) \) supermatrix-valued 1-form as follows:

\[
\omega^\alpha = \sum_\beta \rho_\beta d(h^{\alpha\beta} h^{\beta\alpha})
\]

By setting \( R^\alpha = d\omega^\alpha - \omega^\alpha \wedge \omega^\alpha \), we obtain a \( (k|l \times k|l) \) supermatrix-valued 2-form as follows:

\[
R^\alpha = \sum_\beta d(\rho_\beta) d(h^{\alpha\beta}) h^{\beta\alpha} - \sum_\beta \rho_\beta d(h^{\alpha\beta}) d(h^{\beta\alpha}) - \sum_{\beta, \beta'} \rho_\beta d(h^{\alpha\beta}) h^{\beta\alpha} \wedge \rho_{\beta'} d(h^{\alpha'\beta'}) h^{\beta'\alpha'}.
\]

One may prove that \( \omega^{\alpha'} = d(h^{\alpha'\alpha})(h^{\alpha\alpha})^{-1} + (h^{\alpha'\alpha})\omega^\alpha(h^{\alpha\alpha})^{-1} \) which results \( R^{\alpha'} = (h^{\alpha'\alpha}) R^\alpha (h^{\alpha'\alpha})^{-1} \).

Therefore, to define global forms, it is sufficient to consider polynomials invariant under conjugation by general linear group. Hence, because of the multiplicative property of Berezinian(superdeterminant), one may consider \( \text{Ber}(I + z R^\alpha) \) (\( z \) is a complex variable) which has a power expansion as follows\([10]\):

\[
\text{Ber}(I + z R^\alpha) = \sum_{k=0}^{\infty} c_k(R^\alpha) z^k
\]

where \( c_k(R^\alpha) = Tr \wedge^k R^\alpha \) and by \( Tr \) we mean the supertrace.

Now, it is necessary to show that \( Tr \wedge^k R^\alpha \) is closed. To this end, note that we have

\[
\text{Ber}(I + z R^\alpha) = \exp \left( Tr \ln(I + z R^\alpha) \right) = \exp \left( z Tr(R^\alpha) - \frac{z^2}{2} Tr((R^\alpha)^2) + \frac{z^3}{3} Tr((R^\alpha)^3) - \cdots \right).
\]

A comparison with (25) shows that the coefficients \( c_k(R^\alpha) \) can be expressed as polynomials in \( s_k(R^\alpha) := Tr(R^\alpha)^k \). By \([10]\), \( c_k(R^\alpha) \) and \( s_k(R^\alpha) \) are connected by the following relation:

\[
c_{k+1} = \frac{c_k}{k+1}(s_1 c_k - s_2 c_{k-1} + \cdots + (-1)^k s_{k+1})
\]

Thus, we should show that \( Tr(R^\alpha)^k \) is closed. To this end, we prove the following proposition:

**Proposition 4.2.** Let \( \omega^\alpha \) and \( R^\alpha \) be supermatrices defined by (23) and (24) respectively. We have

\[
dR^\alpha = \omega^\alpha \wedge R^\alpha - R^\alpha \wedge \omega^\alpha = [\omega^\alpha, R^\alpha]
\]
Proof: Using the formula \( R^\alpha = d\omega^\alpha - \omega^\alpha \wedge \omega^\alpha \), we get
\[
dR^\alpha = dd\omega^\alpha - d\omega^\alpha \wedge \omega^\alpha + \omega^\alpha \wedge d\omega^\alpha = -(R^\alpha + \omega^\alpha \wedge \omega^\alpha) \wedge \omega^\alpha + \omega^\alpha \wedge (R^\alpha + \omega^\alpha \wedge \omega^\alpha)
\]
\[
= -R^\alpha \wedge \omega^\alpha + \omega^\alpha \wedge R^\alpha = [\omega^\alpha, R^\alpha].
\]

Hence, we have
\[
dTr((R^\alpha)^k) = \sum_{i+j=k-1} Tr((R^\alpha)^i (R^\alpha)^j) = \sum_{i+j=k-1} Tr((R^\alpha)^i [\omega^\alpha, R^\alpha](R^\alpha)^j) = Tr[\omega^\alpha, (R^\alpha)^k] = 0.
\]

One may consider \( c_k(R^\alpha) \) as a de Rham representative of the \( k \)-th \( \nu \)-class of the canonical super vector bundle \( ^\nu \gamma_k \). Since de Rham cohomology has finite dimension, there exists some index \( i \) such that the classes of \( c_k(R^\alpha) \), (for \( k > i \)), are equal to zero.

5 Conclusion

In this paper, via an analytic approach, the first \( \nu \)-class of a canonical 1|0-super line bundle over \( \nu \)-projective space is represented by a 2-cocycle as an element of generalized de Rham cohomology. At the end, in a similar way of representation of Chern classes in common geometry, by considering canonical super vector bundles over \( \nu \)-grassmannian, the closed forms \( c_k(R^\alpha) \) are introduced which may be considered as representatives of \( \nu \)-classes of rank higher than 1. In common geometry, for each Chern class, there is a representative in terms of Schubert cycles. It seems that a similar phenomenon may occur in supergeometry in the case of finding a proper generalization of Schubert cycles.

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