EQUIVARIANT K-THEORY APPROACH TO $\varpi$-QUANTUM GROUPS

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Abstract. Various constructions for quantum groups have been generalized to $\varpi$-quantum groups. Such generalization is called $\varpi$-program. In this paper, we fill one of parts in the $\varpi$-program. Namely, we provide an equivariant K-theory approach to $\varpi$-quantum groups associated to the Satake diagram in (1), which is the Langlands dual picture of that constructed in [BKLW14], where a geometric realization of the $\varpi$-quantum group is provided by using perverse sheaves. As an application of the main results, we prove Li’s conjecture [L18] for the special cases with the satake diagram in (1).

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1. Introduction

1.1. Quantum symmetric pairs. Let $\mathfrak{g}$ be a Lie algebra, and $\theta : \mathfrak{g} \to \mathfrak{g}$ be an involution on $\mathfrak{g}$. Let $\mathfrak{g}^\theta$ be the fixed point subalgebra of $\mathfrak{g}$. The pair $(\mathfrak{g}, \mathfrak{g}^\theta)$ is called a symmetric pair, and the quantization of their universal enveloping algebra, denoted by $(U_q(\mathfrak{g}), U_q(\mathfrak{g}^\theta))$, is called a quantum symmetric pair. The algebra $U_q(\mathfrak{g}^\theta)$ itself is called an $\varpi$-quantum group.

Due to É. Cartan, $\varpi$-quantum groups can be classified in term of Satake diagrams [Ar62], which are bicolor Dynkin diagrams with black or white vertices and a diagram involution fixing all black vertices. Quantum symmetric pairs are systemically studied by Letzter [Le02, Le99] for finite types, and by Kolb [Ko14] for Kac-Moody cases. In general, the $\varpi$-quantum group, $U_q(\mathfrak{g}^\theta)$, is a coideal subalgebra of $U_q(\mathfrak{g})$. In the following two cases, $\varpi$-quantum groups are degenerated to the ordinary quantum groups. The first one is when vertices in Satake diagrams are all black. The second one is so called quasi-split case with diagonal type, see the definition in [LW19].

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1.2. $\tau$-program. From the above point of view, $\tau$-quantum groups are highly nontrivial generalization of the ordinary quantum groups. Generalizing various constructions for quantum groups to $\tau$-quantum groups is called $\tau$-program, which is proposed by Bao and Wang in [BW18a]. Let us recall some constructions for quantum groups and their generalization to $\tau$-quantum groups.

Canonical basis theory for quantum groups, introduced by Lusztig in [Lu90, Lu91], is a milestone in representation theory. The generalization of canonical bases and R-matrix to $\tau$-quantum groups were given by Bao and Wang in [BW18a, BW18b], in which $\tau$-canonical bases were defined, and by Balagovic and Kolb in [BK19], respectively.

In [BKLW14] (resp. [FL15], [FLLLW1]), authors considered the convolution algebras on double flag varieties associated to the algebraic group $O_{2r+1}(\mathbb{F}_q)$ (resp. $SO_{2r}(\mathbb{F}_q)$, $Sp_{2r}(\mathbb{F}_q((t))))$, which are $q$-Schur type algebras. The geometric realizations of the corresponding $\tau$-quantum groups are hence obtained. We shall call this approach BLM realization since it was firstly given by Belinson, Lusztig and MacPherson in [BLM90] for the case of type A.

It has been realized by Deng, Du, Parshall, and Wang that BLM construction has an algebraic counterpart. The construction for type A was given in [DDPW]. In [FLLLW2], a Hecke algebraic approach to $\tau$-quantum groups and the corresponding affine $q$-Schur algebras was developed, which is the algebraic counterpart of the geometric approach studied in [FLLLW1]. The algebraic approach to other types has been studied in [LL15, LW17, LL18].

A categorification of quantum groups was given independently by Khovanov and Lauda in [KL09, KL10], and Rouquier in [R08] by using KLR algebra. This work had been generalized to the $\tau$-quantum group $U^b$ (denoted by $U^\tau$ therein) by Bao, Shan, Wang and Webster in [BSWW].

In [Ri90], Ringel constructed a Hall algebra associated to a Dynkin quiver and shown that it was isomorphic to the half part of the corresponding quantum group. In [B13], Bridgeland realized the entire quantum groups by using Hall algebra of two-step complexes. Recently, Lu and Wang provided a Hall algebra approach to $\tau$-quantum groups for finite types in [LW19]. In [L18], Li defined $\sigma$-quiver varieties for $\tau$-quantum groups, which are analogies of Nakajima quiver varieties.

1.3. Langlands reciprocity. It is well-known that affine quantum groups have two different algebraic presentations. One is called Drinfeld-Jimbo realizations, the other one is called Drinfeld new realizations. The geometric counterpart for the case of affine type A also has two different approaches. The first one is using perverse sheaves on double flag varieties associated to the corresponding algebraic group $GL(\mathbb{F}_q((t)))$, and the second one is using equivariant K-theory on Steinberg varieties associated to $GL(\mathbb{C})$.

The second construction yields a classification of irreducible finite dimensional $U_q(\mathfrak{sl}_n)$-module as well as their character formulas. Ginzburg, Reshetikhin and Vassrot further provided a geometric realization of Schur-Weyl duality of affine type A in [GRW94] by using equivariant K-theory. For Hecke algebras, Iwahori and Matsumoto gave a BLM-type construction of the affine Hecke algebra associated to algebraic group $G$. The equivariant K theory construction of affine Hecke algebra was first done by Kazhdan and Lusztig [Lu85, KL85, KL87], and then improved by Ginzburg [G87]. Namely, there exists
an equivariant K-theory approach to the affine Hecke algebra by using the Steinberg variety associated with \( ^L G \), where \( ^L G \) is the Langlands dual of \( G \). This suggests that a mysterious link exists between the BLM realization and the equivariant K-theory realization. Such relationship is so called Langlands reciprocity for affine quantum group of type \( A_n \) in [GV93]. In light of \( \ast \)-program, the BLM realization of the \( \ast \)-quantum group \( U^\ast \) (denoted by \( U^{\mathbb{J}} \) therein), and the equivariant K-theoretic approach to affine quantum groups, they strongly suggest the existence of equivariant K-theoretic approach to the \( \ast \)-quantum group \( U^\ast \), which is expected by Wang and Li earlier. This is the original motivation of this project.

1.4. Main results. Since it is hard to find the Drinfeld new realization of \( U^\ast \), we shall consider a toy case in this paper, the \( \ast \)-quantum groups \( U^c \) and \( U^b \), whose satake diagram is as follows.

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\]

This paper devote to study \( \ast \)-quantum group \( U^c \) in term of equivariant K-groups on Steinberg variety of type C, including its generators, relations and the coideal structure. More precisely, we have the following theorem.

**Theorem** [Theorems 3.4 and 3.5] Let \( Z^c \) (resp. \( Z^a \)) be the Steinberg variety of type \( C \) (resp. type \( A \)). Then we have the following two algebra homomorphisms.

(a) \( U^b \longrightarrow K^{C^\ast \times Sp_d(C)}(Z^c)_K \).

(b) \( \Delta : C_\alpha \otimes K^{C^\ast \times Sp_d(C)}(Z^c) \longrightarrow C_\alpha \otimes K^{C^\ast \times GL_{d_1}}(Z^b_{d_1}) \otimes K^{C^\ast \times Sp_{d_2}(C)}(Z^a_{d_2}), \)

where \( d_1 + d_2 = d \) and \( C_\alpha \) is a 1-dimensional representation depending on a semisimple element \( \alpha \in C^\ast \times Sp_d(C) \), see details in Section 3.4.

Although \( U^c \) is isomorphic to \( U^b \) in [BKLW14], we note that \( K^{C^\ast \times G_b}(Z^b)_K \) is not isomorphic to \( K^{C^\ast \times G}(Z^c)_K \) in general. From this point of view, \( U^b \) has two different approaches. Although these two constructions are quite similar, the polynomial representations are different, see Theorem 3.2 and Proposition 4.2.

In [L18], Li constructs \( \sigma \)-quiver variety \( \mathcal{M}(v, w)^\sigma \), which is the fixed point subvariety of Nakajima quiver variety by \( \sigma \) and conjectures that there is an algebra homomorphism from the universal enveloping algebra, \( U(g^\theta) \), of the fixed point Lie subalgebra of \( g \) to the top degree Broel-Moore homology \( H_{\text{top}}(Z) \) if \( g \) is of finite types, where \( Z \) is the Steinberg type \( \sigma \)-quiver variety. See [L18] for more details. As an application of the main results, we prove Conjecture 5.3.4 in [L18] for the special cases with the Satake diagram in (1).
this paper. The proof of Proposition 3.3 is combinatorially involved, so we move the
detailed proof of it to the appendix.

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2. Convolution algebra in equivariant K-theory

2.1. Equivariant K-theory. Let $G$ be a linear algebraic group. Given a $G$-variety $X$, let $\text{Coh}^G(X)$ be the category of $G$-equivariant coherent sheaves on $X$, and $K^G(X)$ be the complexified Grothendieck group of $\text{Coh}^G(X)$. For any coherent sheaf $\mathcal{F} \in \text{Coh}^G(X)$, denote by $[\mathcal{F}]$ its equivalent class in $K^G(X)$.

In the special case that $X = \{pt\}$, a $G$-equivariant sheaf on $X$ is just a $G$-vector space. Hence $\text{Coh}^G(\{pt\}) = \text{Rep}(G)$, the category of representations of $G$, and $K^G(\{pt\}) = R(G)$, the complexified representation ring of $G$. Moreover, for any $G$-variety $X$, the external tensor product defines a $R(G)$-module structure on $K^G(X)$.

Given three smooth $G$-varieties $M_1$, $M_2$, $M_3$, let

$$ p_{ij} : M_1 \times M_2 \times M_3 \rightarrow M_i \times M_j $$

be the obvious projection maps. Let $Z_{12} \subseteq M_1 \times M_2$ and $Z_{23} \subseteq M_2 \times M_3$ be $G$-stable closed subvarieties. We denote

$$ (2) \quad Z_{12} \circ Z_{23} = p_{13}(p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})). $$

If the restriction of $p_{13}$ to $p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23})$ is a proper map, then we define the convolution product as follows.

$$ \ast : \quad K^G(Z_{12}) \otimes K^G(Z_{23}) \rightarrow K^G(Z_{12} \circ Z_{23}), $$

$$ [\mathcal{F}_1] \otimes [\mathcal{F}_2] \mapsto Rp_{13*}(p_{12*}[\mathcal{F}_1] \otimes p_{23*}[\mathcal{F}_2]), $$

where $\otimes$ is the derived tensor product. We note that if the properness condition doesn’t hold, then derived direct image $Rp_{13*}$ is not well-defined in general.

In the case that $M_1 = M_2 = M_3 = M$, the convolution product defines an algebra structure on $K^G(M \times M)$.

2.2. Flag variety of type C. Let $V = \mathbb{C}^{2d}$ with a nondegenerate skew-symmetric bilinear form $(,)$, and $G = Sp(V)$. To simplify notations, we set once for all

$$ N = 2n + 1. $$

Let

$$ \Lambda_\iota = \{ v = (v_i) \in \mathbb{N}^N \mid v_i = v_{N+1-i}, \quad \sum_{i=1}^N v_i = 2d \}. $$

For any $W \subseteq V$, let $W^\perp = \{ x \in V \mid (x, y) = 0, \forall y \in W \}$. For any $v \in \Lambda_\iota$, we further set

$$ \mathcal{F}_v = \{ F = (0 = V_0 \subset V_1 \subset \cdots \subset V_N = V) \mid V_i = V_{N-i}^\perp, \dim(V_i/V_{i-1}) = v_i, \forall i \}. $$
The componentwise action of $G$ on $\mathcal{F}_v$ is transitive. For any $F \in \mathcal{F}_v$, let $P_v$ be the stabilizer of $F$ in $G$. Therefore, we have $G/P_v \simeq \mathcal{F}_v$. Let $\mathcal{F} = \sqcup_{v \in \Lambda_v} \mathcal{F}_v$. Group $G$ naturally acts on $\mathcal{F}$.

2.3. Steinberg variety and the convolution algebra. Let $\mathcal{M} = T^* \mathcal{F}$, the cotangent bundle of $\mathcal{F}$. More precisely, $\mathcal{M}$ can be written into

$$\mathcal{M} = \{(F, x) \in \mathcal{F} \times \mathfrak{sp}_{2d} \mid x(F_i) \subseteq F_{i-1}, \forall i\}.$$ 

The conjugate action of $\mathcal{M}$ on $\mathcal{F}$ induces a $G$-action on $\mathcal{M}$. We define a $C^* \times G$-action on $\mathcal{M}$ by

$$(z, g) \cdot (F, x) = (gF, z^{-2}gxg^{-1}), \quad \forall (z, g) \in C^* \times G.$$ 

Let $\mathcal{N}$ be the nilpotent variety of $\mathfrak{sp}_{2d}$. By the definition of $\mathcal{M}$, we have $\mathcal{M} \subseteq \mathcal{F} \times \mathcal{N}$. Let $\pi : \mathcal{M} \to \mathcal{N}$, $(F, x) \mapsto x$ be the second projection map. It is clear that $\pi$ is a proper map. Moreover, one can define a $C^* \times G$-action on $\mathcal{N}$ such that $\pi$ is a $C^* \times G$-equivariant map.

Let $\mathcal{Z} = \mathcal{M} \times_{\mathcal{N}} \mathcal{M}$ be the generalized Steinberg variety of type $C$, which is the union of conormal bundles of $G$-orbits in $\mathcal{F} \times \mathcal{F}$. If we want to emphasize the type, we shall use $\mathcal{Z}^c$ instead of $\mathcal{Z}$. Group $C^* \times G$ acts on $\mathcal{Z}$ diagonally. Since $R(C^*) = \mathbb{C}[q, q^{-1}]$ and $\mathcal{Z} \subseteq \mathcal{M} \times \mathcal{M}$, by Section 2.1, $(K^{C^* \times G}(\mathcal{Z}), \ast)$ is a $\mathbb{C}[q, q^{-1}]$ algebra with unit.

2.4. Generators of the convolution algebra. In this section, we shall study generators of the algebra $K^{C^* \times G}(\mathcal{Z})$. Let us recall some notations from [BKLW14]. Let $\Xi_d = \{A = (a_{ij}) \in \text{Mat}_{N \times N}(\mathbb{N}) \mid \sum_{i,j} a_{ij} = 2d, \ a_{ij} = a_{N+1-i, N+1-j}, \forall i, j\}$. It has been shown in [BKLW14] Section 6 that the set $\Xi_d$ parameterizes $G$-orbits of $\mathcal{F} \times \mathcal{F}$. For each $A = (a_{ij}) \in \Xi_d$, denote

$$\text{ro}(A) = (\sum_j a_{ij})_{i=1,2,\ldots,N} \text{ and } \text{co}(A) = (\sum_i a_{ij})_{j=1,2,\ldots,N}.$$ 

Moreover, for any $v, w \in \Lambda$, the set $\Xi_d(v, w)$ parameterizes $G$-orbits of $\mathcal{F}_v \times \mathcal{F}_w$, where

$$\Xi_d(v, w) = \{M \in \Xi_d \mid \text{ro}(M) = v, \ \text{co}(M) = w\}.$$ 

Let $W_\xi = Z_{2d}^d \ltimes S_d$ be the Weyl group of type $C_d$, which has a natural action on set $\{1, 2, \ldots, 2d\}$. For $v \in \Lambda$, we set $\bar{v}_i = \sum_{r=1}^i v_r$ and $[v]_i = [1 + \bar{v}_{i-1}, \bar{v}_i] \subseteq \mathbb{N}$. Then $[v] = ([v]_1, [v]_2, \ldots, [v]_N)$ forms a partition of the set $\{1, 2, \ldots, 2d\}$. To a matrix $A \in \Xi_d$, we also associate a partition of the set $\{1, 2, \ldots, 2d\}$ as follows

$$[A] = ([A]_{11}, \ldots, [A]_{1N}, [A]_{21}, \ldots, [A]_{NN}),$$

where $[A]_{ij} = \sum_{(h,k) < (i,j)} a_{hk} + 1, \sum_{(h,k) < (i,j)} a_{hk} + a_{ij} \subseteq \mathbb{N}$, and $< \text{ is the left lexicographical order}$, i.e.,

$$(h, k) < (i, j) \iff h < i \text{ or } (h = i \text{ and } k < j).$$

For any partition $[v] = ([v]_1, [v]_2, \ldots, [v]_N)$ of the set $\{1, 2, \ldots, 2d\}$, we define a partition of the set $\{1, 2, \ldots, d\}$ as follows,

$$[v]^c = ([v]_1, [v]_2, \ldots, [v]_n, [v]_{n+1}^c) \text{ and } [v]_{n+1}^c = [\bar{v}_n + 1, d].$$
It is clear that $|\mathbf{v}|_{n+1}^2 = \frac{1}{2} |\mathbf{v}|_{n+1}$, where $| \cdot |$ is the cardinality of the set. Similarly, for any partition $[A]$ of the set $\{1, 2, \cdots, 2d\}$ in (3), we define a partition, denoted by $[A]^c$, of the set $\{1, 2, \cdots, d\}$,

$$[A]^c = ([A]_{11}, \cdots, [A]_{1N}, [A]_{21}, \cdots, [A]_{n+1}, [A]_{n+1, n+1}),$$

where $|A|_{n+1, n+1} = \sum_{h<k} (h, k) a_hk + 1, d$. To a partition $[v]$, we associate a subgroup $W_{[v]}$ of $W_c$.

$$W_{[v]} = S_{[v]}_1 \times S_{[v]}_2 \times \cdots \times S_{[v]}_n \times (\mathbb{Z}_2^{v_i} \times S_{[v]}_{n+1}),$$

where $S_{[v]}_i$ is the subgroup of $S_d$ consisting of all permutations which preserve $[v]_i$.

It is well known that there is also a bijection between the double coset $W_{[v]} W_c W_{[w]}$ and the $G$-orbits on $\mathcal{F}_v \times \mathcal{F}_w$. The precise correspondence is given by

$$W_{[v]} W_c W_{[w]} \rightarrow \Xi_d(v, w), \quad W_{[v]} \sigma W_{[w]} \mapsto m^\sigma = (m^\sigma_{ij}).$$

Here $m^\sigma_{ij} = \# \{a \in [v], |\sigma(a) \in [w]\}_j$.

We can define an order $\preceq$ on $\Xi_d$ as follows. For any $A = (a_{ij}), B = (b_{ij}) \in \Xi_d$, $A \preceq B$ if and only if

$$(4) \quad \text{ro}(A) = \text{ro}(B), \ \text{co}(A) = \text{co}(B), \ \text{and} \ \sum_{r \leq i; s \geq j} a_{rs} \leq \sum_{r \leq i; s \geq j} b_{rs}, \ \forall i < j.$$

This order is compatible with the Bruhat order on $W_c$ via the above bijection. By Lemma 3.8 in [BKLW14], we have the following proposition.

**Proposition 2.1.** (a) For any $A, B \in \Xi_d$, $\mathcal{O}_A \subseteq \mathcal{O}_B$ if $A \preceq B$.

(b) For any $A, B \in \Xi_d$, let $M(A, B)$ be the set consisting of all matrices $C \in \Xi_d$ such that $\mathcal{O}_C \subseteq p_{13}(p_{12}^{-1})^2 \mathcal{O}_A \cap p_{23}^{-1} \mathcal{O}_B$. We have

$$p_{13}(p_{12}^{-1})^2 \mathcal{O}_A \cap p_{23}^{-1} \mathcal{O}_B = \bigcup_{C \in M(A, B)} \mathcal{O}_C.$$

Moreover, there exists a unique matrix, denoted by $A \circ B$, such that for any $C \in M(A, B)$, we have $C \preceq A \circ B$.

(c) For any $A, A', B, B' \in \Xi_d$ with $A' \preceq A \preceq B' \preceq B$, we have

$$C' \preceq A \circ B, \ \forall \ C' \in M(A', B').$$

Let $\mathbf{R} = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_d^{\pm 1}]$. We shall define a natural action of Weyl group $W_c$ on $\mathbf{R}$ as follows, which induces a well-defined action of $W_{[v]}$ (resp. $W_{[A]}$) on $\mathbf{R}$. For any permutation $\sigma \in S_d$, the action of $\sigma$ on $\mathbf{R}$ is given by

$$\sigma : \mathbf{R} \rightarrow \mathbf{R}, \quad f(x_1^{\pm 1}, x_2^{\pm 1}, \cdots, x_d^{\pm 1}) \mapsto f(x_{\sigma(1)}^{\pm 1}, x_{\sigma(2)}^{\pm 1}, \cdots, x_{\sigma(d)}^{\pm 1}).$$

We now define a $\mathbb{Z}_2^d$-action on $\mathbf{R}$. We shall denote by $[1]_m = (\delta_{im})_{i=1, 2, \cdots, d}$ the $m$-th generator of $\mathbb{Z}_2^d$, $\forall m = 1, \cdots, d$. The action of $[1]_m$ on $\mathbf{R}$ is defined by

$$[1]_m : \mathbf{R} \rightarrow \mathbf{R}, \quad f(x_1^{\pm 1}, \cdots, x_{m-1}^{\pm 1}, x_m^{\pm 1}, x_{m+1}^{\pm 1}, \cdots, x_d^{\pm 1}) \mapsto f(x_1^{\pm 1}, \cdots, x_{m-1}^{\pm 1}, x_m^{\mp 1}, x_{m+1}^{\pm 1}, \cdots, x_d^{\pm 1}).$$
For any partition $I = (I_1, I_2, \cdots, I_{2d})$ of $2d$, denote by $R^{W_I}$ the subring of $R$ fixed by $W_I$. If $J = (J_1, J_2, \cdots, J_{2r+1})$ is another partition with $W_J \subseteq W_I$. We define a map

$$W_I/W_J: R^{W_I} \to R^{W_J}, \quad f \mapsto \sum_{\sigma \in W_I/W_J} \sigma(f).$$

**Proposition 2.2.** For any $v, v_1, v_2 \in \Lambda$, $A \in \Xi_d(v_1, v_2)$.

(a) There exist $C$-algebra isomorphisms $K^G(F_v) \cong R^{W[v]}$ and $K^G(O_A) \cong R^{W[A]}$.

(b) The first projection map $p_1: O_A \to F_{v_1}$ is a smooth fibration. Moreover, if $O_A$ is closed, then the direct image morphism $R_{p_1,A^*}$ is given by

$$R_{p_1,A^*}(f) = W_{[v_1]}/W_{[A]}(f \prod_{(s,t)} (1 - x_s/x_t)^{-1} \prod_{(l,m)} (1 - x_l/x_m)^{-1} (1 - x_lx_m)^{-1} \prod_{(p,q)} (1 - x_p/x_q)^{-1}),$$

where the product runs over the following ranges.

$$(s,t) \in [A]_{ja} \times [A]_{jb}, \quad 1 \leq a < b \leq 2n + 1, \quad j \in [1, n];$$

$$(l,m) \in [A]_{n+1,a} \times ([A]_{n+1,b} \cup [A]_{n+1,n+1}), \quad a, b \in [1, n] \quad \text{and} \quad a < b;$$

$$(p,q) \in [A]_{n+1,a} \times [A]_{n+1,a}, \quad p \leq q, \quad 1 \leq a \leq n.$$

**Proof.** Fix $A \in \Xi_d(v, w), (F, F') \in O_A$. There exists a decomposition $V = \oplus_{i,j \leq N} Z_{ij}$ such that

$$F_a = \oplus_{i \leq a} Z_{ij} \quad \text{and} \quad F'_b = \oplus_{j \leq b} Z_{ij}, \quad \forall a, b \in [1, N].$$

Let $P_F$ and $P_{F'}$ be the stabilizer of $F$ and $F'$ in $G$, respectively. Since $O_A = \{g \cdot (F, F') \mid g \in G\}$, the induction property [CG, Section 5.2.16] implies that

$$K^G(O_A) \cong K^G(G/(P_F \cap P_{F'})) \cong K^{P_{F} \cap P_{F'}}(pt) \cong R(P_F \cap P_{F'}),$$

where $R(P_F \cap P_{F'})$ is the representation ring of $P_F \cap P_{F'}$. Moreover, $\text{Sp}(Z_{n+1,n+1}) \times \prod_{(i,j) \leq (n+1,n+1)} GL(Z_{ij})$ is isomorphic to the Levi subgroup of $P_F \cap P_{F'}$. By [CG, Theorem 6.1.4] again, we have

$$K^G(O_A) \cong R^{W[A]}.$$

In the special case that $A = \text{diag}(v)$, the above result implies that

$$K^G(F_v) \cong R^{W[v]}.$$

Statement (a) follows.

The first part of (b) is clear. We now prove the second part of (b). Suppose that $O_A$ is closed. The fiber of the projection $p_1: O_A$ is isomorphic to $P_F/(P_F \cap P_{F'})$. Thus the class $T^*_A \in R$ of the relative cotangent bundle $T^*_{p_1,A}$ at the origin is

$$T^*_A = \sum_{(s,t)} x_s/x_t \sum_{(l,m)} (x_l/x_m + x_lx_m) \sum_{(p,q)} x_px_q,$$

where the sums run over the following ranges.

$$(s,t) \in [A]_{ja} \times [A]_{jb}, \quad 1 \leq a < b \leq N, \quad j \in [1, n];$$

$$(l,m) \in [A]_{n+1,a} \times ([A]_{n+1,b} \cup [A]_{n+1,n+1}), \quad a, b \in [1, n] \quad \text{and} \quad a < b;$$

$$(p,q) \in [A]_{n+1,a} \times [A]_{n+1,a}, \quad p \leq q, \quad 1 \leq a \leq n.$$
By Lefschetz formula, for any sheaf $\mathcal{F} \in K^G(\mathcal{O}_A)$, we have

$$R_{pl,A^*} [\mathcal{F}] = W_{[V_1]}/W_{[A]} (([\mathcal{F}] \otimes (T_A^*)^{-1})$$

where $T_A^* = \sum_i (-1)^i T^i_A$. Statement (b) follows. \hfill \Box

Let $A = \mathbb{C}[q, q^{-1}]$ and $K = \mathbb{C}(q)$. For any vector space $V$ over $\mathbb{C}$, we set $V_A = V \otimes A$ and $V_K = V \otimes K$. For any $A \in \Xi_d$, let $Z_A$ be the conormal bundle of $\mathcal{O}_A$, and $Z_{\leq A} = \bigcup_{B \leq A} Z_B$. Proposition 2.1 implies that $K^{C^* \times G}(Z)$ has a filtration structure indexed by $\Xi_d$ as follows.

$$K^{C^* \times G}(Z_{\leq A}) \star K^{C^* \times G}(Z_{\leq B}) \subseteq K^{C^* \times G}(Z_{\leq A \circ B}).$$

On the other hand, by the cellular fibration lemma [CG Lemma 5.5.1], there is a canonical short exact sequence

$$0 \to K^{C^* \times G}(Z_{\leq A}) \to K^{C^* \times G}(Z_{\leq B}) \to K^{C^* \times G}(Z_A) \to 0,$$

where $Z_{\leq A} = \bigcup_{B \leq A, B \neq A} Z_{\leq B}$. By Proposition 2.2(a) and Thom isomorphism [CG Theorem 5.4.17], we have the following $A$-module isomorphism

$$(6) \quad \oplus_{A \in \Xi_d} K^{C^* \times G}(Z_{\leq A})/K^{C^* \times G}(Z_A) \cong \oplus_{A \in \Xi_d} K^{C^* \times G}(Z_A) \cong \oplus_{A \in \Xi_d} R_A^{W[A]}[c].$$

By Proposition 2.1, for any $A' \preceq A, B' \preceq B$ and $(A', B') \neq (A, B)$, we have

$$Z_{\leq A'} \circ Z_{\leq B'} \subseteq Z_{\leq A \circ B}.$$

By (6), the convolution product on $K^{C^* \times G}(Z)$ induces a well-defined algebra structure on $\oplus_{A \in \Xi_d} R_A^{W[A]}$, denoted by $\star$.

**Example 2.3.** In the case that $N = 3$, let

$$A_1 = \begin{pmatrix} v_1 & 0 & 0 \\ a & v_2 + 2b & a \\ 0 & 0 & v_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} v_1 + a & 0 & 0 \\ b & v_2 & b \\ 0 & 0 & v_1 + a \end{pmatrix},$$

$$A_3 = \begin{pmatrix} v_1 & 0 & 0 \\ b + a & v_2 & b + a \\ 0 & 0 & v_1 \end{pmatrix}.$$  

It is obvious that $A_1 \circ A_2 = A_3$. Set

$$I_1 = [1, v_1],$$

$$I_2 = [v_1 + 1, v_1 + a],$$

$$I_3 = [v_1 + a + 1, v_1 + a + b],$$

$$I_4 = [v_1 + a + b + 1, v_1 + a + b + v_2],$$

$$I_5 = [v_1 + a + b + v_2 + 1, v_1 + v_2 + a + 2b],$$

$$I_6 = [v_1 + v_2 + a + 2b + 1, v_1 + v_2 + 2a + 2b],$$

$$I_7 = [v_1 + v_2 + 2a + 2b + 1, 2d].$$
Thus
\[ W_{[A_1]}^\epsilon = W_{(1,1,1,1,1,1,1,1)} = S_{I_1} \times S_{I_2} \times (\mathbb{Z}_2^{[(I_3 \cup I_4)\epsilon]} \ltimes S_{(I_3 \cup I_4)\epsilon}), \]
\[ W_{[A_2]}^\epsilon = W_{(1,1,1,1,1,1,1,1)} = S_{I_1 \cup I_2} \times S_{I_3} \times (\mathbb{Z}_2^{[I_4\epsilon]} \ltimes S_{I_4}^\epsilon), \]
\[ W_{[A_3]}^\epsilon = W_{(1,1,1,1,1,1,1,1)} = S_{I_1} \times S_{I_2} \times S_{I_3} \times (\mathbb{Z}_2^{[I_4\epsilon]} \ltimes S_{I_4}^\epsilon). \]

By Proposition 2.2(b), we have
\[ \tilde{\pi} : R_{W_{[A_1]}^\epsilon} \otimes R_{W_{[A_2]}^\epsilon} \to R_{W_{[A_1 \circ A_2]}^\epsilon}, \]
\[ f \otimes g \mapsto W_{[A_3]}^\epsilon/(W_{[A_1]}^\epsilon \cap W_{[A_2]}^\epsilon)(fg \prod_{i \in I_2, j \in I_3} \frac{1-q^2 x_j/x_i}{1-x_i/x_j}), \]
where \( W_{[A_3]}^\epsilon/(W_{[A_1]}^\epsilon \cap W_{[A_2]}^\epsilon) = S_{I_2 \cup I_3}/(S_{I_2} \times S_{I_3}). \) For any \( f \in R_{S_{I_2 \cup I_3}}, \) we have
\[ S_{I_2 \cup I_3}/(S_{I_2} \times S_{I_3})(f \prod_{i \in I_2, j \in I_3} \frac{1-q^2 x_j/x_i}{1-x_i/x_j}) = P(q)f, \]
where \( P(q) \) is a polynomial in \( q \) which can be computed explicitly as follows.
\[ P(q) = q^{ab}[a+b]!/[a]![b]!, \quad [k] = k \prod_{i=1}^{k}[i], \quad \text{and} \quad [k] = \frac{q^k-q^{-k}}{q-q^{-1}}. \]

Since \( P(q) \) is invertible in \( \mathbb{K} \), the map \( \tilde{\pi} : R_{W_{[A_1]}^\epsilon} \otimes R_{W_{[A_2]}^\epsilon} \to R_{W_{[A_1 \circ A_2]}^\epsilon} \) is surjective.

**Proposition 2.4.** Let \( A \) be a diagonal matrix. For any \( B \in \Xi_d \) with \( \text{co}(A) = \text{ro}(B) \), we have
\[ f \tilde{\pi} g = fg, \quad \forall f \in R_{W_{[A]}^\epsilon}, \ g \in R_{W_{[B]}^\epsilon}. \]

**Proof.** Since \( A \) is a diagonal matrix, the projection map \( p_{13} : p_{12}^{-1}O_A \cap p_{23}^{-1}O_B \to O_B \) is an isomorphism. By Corollary 3 in [V98], the proposition follows. \( \square \)

Let \( E_{ij} = (a_{kl})_{N \times N} \) such that \( a_{kl} = 1 \) if \( (k,l) = (i,j) \) and 0, otherwise. Define \( E_{ij}^\theta = E_{ij} + E_{N+1-i-N+1-j}^\theta \) and \( E_{ij}^\theta(v,a) = \text{diag}(v) + aE_{ij}^\theta \), where \( v = (v_1, \ldots, v_N) \) such that \( v_i = v_{N+1-i} \) and \( \sum_{i=1}^{N} v_i = 2d - 2a. \)

**Proposition 2.5.** Fix \( v \in \Lambda, A, B \in \Xi_d \) with \( \text{co}(A) = \text{ro}(B) \).

(a) Let \( A = E_{h,h+1}^\theta(v,a) \), and \( l = \max\{i \mid b_{h+1,i} \neq 0\} \). If \( b_{h+1,l} \geq a \), then
\[ A \circ B = B + a(E_{hl}^\theta - E_{h+1,l}^\theta). \]
Moreover, for any \( f \in R_{W_{[A]}^\epsilon} \) and \( g \in R_{W_{[B]}^\epsilon} \), we have
\[ f \tilde{\pi} g = fg \in R_{W_{[A \circ B]}^\epsilon}. \]

(b) Let \( A = E_{h,h-1}^\theta(v,a) \), and \( l = \min\{i \mid b_{h-1,i} \neq 0\} \). If \( b_{h-1,l} \geq a \), then
\[ A \circ B = B + a(E_{hl}^\theta - E_{h-1,l}^\theta), \]
Moreover, for any \( f \in \mathbb{R}_\Delta^{W_{[\Lambda]}} \) and \( g \in \mathbb{R}_\Delta^{W_{[\mathcal{B}^\epsilon]}} \), we have
\[
f \ast g = fg \in \mathbb{R}_\Delta^{W_{[\Lambda \circ \mathcal{B}^\epsilon]}}.
\]

**Proof.** We show (a) first. By Bruhat order or [BKLW14, Lemma 3.9], the first part of (a) follows. We now show the second part of (a). Let
\[
\mathcal{T} = \{(F, F', F'') \in p_{12}^{-1}(\mathcal{O}_A) \cap p_{23}^{-1}(\mathcal{O}_B) \mid (F, F') \in \mathcal{O}_{A \circ B}\}.
\]
Consider the projection map \( p_{13} : \mathcal{T} \to \mathcal{O}_{A \circ B}; (F, F', F'') \mapsto (F, F'') \). If \( b_{hk} = 0, \forall k \geq l \), the fact that the coefficient of leading term is 1 in [BKLW14, Proposition 3.3] implies that \( p_{13} \) is an isomorphism. The second part of (a) follows from Corollary 3 in [V]. Statement (b) can be proved similarly. \( \square \)

Let \( p_1 : \mathcal{O}_{E_{i+1}^N(v_1,1)} \to \mathcal{F}_{v+e+e_{N+1-i}} \) (resp. \( q_1 : \mathcal{O}_{E_{i+1}^N(v_1,1)} \to \mathcal{F}_{v+e+e_{N-i}} \)) be the first projection. Let \( T^*p_1 \) (resp. \( T^*q_1 \)) be the relative cotangent sheaf along the fibers of \( p_1 \) (resp. \( q_1 \)) and \( \operatorname{Det}(T^*p_1) \) (resp. \( \operatorname{Det}(T^*q_1) \)) the determinant bundle on \( \mathcal{O}_{E_{i+1}^N(v_1,1)} \) (resp. \( \mathcal{O}_{E_{i+1}^N(v_1,1)} \)). Let \( \pi_1 : \mathcal{Z}_{E_{i+1}^N(v_1,1)} \to \mathcal{O}_{E_{i+1}^N(v_1,1)} \) (resp. \( \rho_1 : \mathcal{Z}_{E_{i+1}^N(v_1,1)} \to \mathcal{O}_{E_{i+1}^N(v_1,1)} \)) be the first projection. We define
\[
\mathcal{E}_{i,v} = \pi_1(\operatorname{Det}(T^*p_1)), \quad \mathcal{F}_{i,v} = \rho_1(\operatorname{Det}(T^*q_1)).
\]
For \( i = 1, 2, \ldots, n+1 \), let \( \mathcal{H}_{i,v} \) be the sheaf supported on the \( \mathcal{Z}_{\operatorname{diag}(v)} \), which corresponds to \( q^v \) via the isomorphism \( K^{C^\times G}(\mathcal{Z}_{\operatorname{diag}(v)}) \cong \mathbb{R}_\Delta^{W_{i,v}} \). Set
\[
\mathcal{E}_i = \sum_{v \in \Lambda_t} (-q)^{-v} \mathcal{E}_{i,v}, \quad \mathcal{F}_i = \sum_{v \in \Lambda_t} (-q)^{-v_i} \mathcal{F}_{i,v},
\]
\[
\mathcal{H}_i = \sum_{v \in \Lambda_t} \mathcal{H}_{i,v}.
\]

**Theorem 2.6.** The convolution algebra \( (K^{C^\times G}(Z))] \) is generated by \( \mathcal{E}_i, \mathcal{F}_i \) and the sheaves supported on the orbits indexed by a diagonal matrix in \( \Xi_d \).

**Proof.** The proof is similarly as that for [V98, Proposition 10]. For readers’ convenience, we repeat it here. Fix a matrix \( C = (c_{ij}) \in \Xi_d \). We prove it by induction on
\[
l(C) = \sum_{i>j} \binom{|i-j|+1}{2} c_{ij}.
\]
If \( l(C) = 0 \) or \( l(C) = 1 \), then \( C \) itself is a generator of \( K^{C^\times G}(Z_C)_F \). There is nothing to show. We now assume that \( l(C) \geq 1 \). We only need show it for the case that \( h > n \).

In this case \( E_{h+1}^N = E_{N-2-h,N+1-h}^N \). Let
\[
(h,l) = \min\{(i,j) \mid 1 \leq j < i \leq n, c_{ij} \neq 0\},
\]
where
\[
(i,j) \geq (s,t) \iff j > t \quad \text{or} \quad (j = t \quad \text{and} \quad i > s).
\]
Let \( B = C + c_{hl}(E_{h-1,l}^N - E_{h,l}) \) and \( A = E_{h,h-1}^N(v - c_{h,l}e_h - c_{h,l}e_{2n+2-h}, c_{h,l}) \). By Proposition 2.5, we have
\[
f \ast g = fg \in \mathbb{R}_\Delta^{W_{C[\epsilon]}}, \quad \forall \ f \in \mathbb{R}_\Delta^{W_{[\Lambda][\epsilon]}}, \ g \in \mathbb{R}_\Delta^{W_{[\mathcal{B}^\epsilon]}}.
\]
If $(h - 1, l) \neq (n + 1, n + 1)$, by the same argument as that for [V98] Proposition 10, the map $\tilde{x} : R^W[\mathcal{A}]^\ell \otimes R^W[\mathcal{B}]^\ell \to R^W[\mathcal{C}]^\ell$ is surjective.

If $(h - 1, l) = (n + 1, n + 1)$, for any partition $(v_1, v_2, v_1)$ of $2d$, the algebra $R^W[\mathcal{D}]^\ell$ is generated by $R^{Z_{b,d} \times s_d}$ and $R^{S_{v_1}}$. Thus the surjectivity of the map $\tilde{x} : R^W[\mathcal{A}] \otimes R^W[\mathcal{B}] \to R^W[\mathcal{C}]$ follows from the surjectivity of the map

$$R_{\mathcal{A}h,b-1} \otimes R_{\mathcal{A}h}^{s_{1,n+1}} \otimes C \to R_{\mathcal{A}h}^{s_{1,n+1}} \otimes S_{c}^{c+2} \otimes C R^{s_{c+2}}$$

Similar to Example 2.3, any sheaf supported on $Z_{E_{b+1,1},(v,a)}$ can be obtained by the convolution product of the sheaves supported on $Z_{E_{b+1,1}(v+ke_i,ke_{2n+2-i}-1)}$, $k = 0, 1, \ldots, a - 1$. Thus $(K^C \times G(Z))_{\mathcal{A}}$ can be generated by classes of the sheaves supported on the irreducible components $Z_{\mathcal{A}}$ with $\mathcal{A}$ an $\Lambda_d$-diagonal or type $E_{t+1,f}(v,1)$.

By Proposition 2.4 for any $v \in \Lambda_t$, the map

$$\tilde{x} : R^W[v+e_{i+1}+e_{2n+1-i}] \otimes [\mathcal{F}_{i,v}] \otimes R^W[v+e_{i+1}+e_{2n+1-i}] \to R^W[\mathcal{E}_{b+1,1}(v,1)]$$

is surjective. The theorem follows.

2.5. The $K^C \times G(\mathcal{Z})$-module structure on $K^C \times G(\mathcal{M})$. Recall $\mathcal{Z} = \mathcal{M} \times_\mathcal{N} \mathcal{M}$. Convolution product naturally defines an action of $K^C \times G(\mathcal{Z})$ on $K^C \times G(\mathcal{M})$ as follows.

$$\rho : K^C \times G(\mathcal{Z}) \otimes K^C \times G(\mathcal{M}) \to K^C \times G(\mathcal{M}), \quad f \otimes g \to f \ast g.$$

Lemma 2.7. [CG, claim 7.6.7] The action $\rho$ gives a faithful representation of $K^C \times G(\mathcal{Z})$ on $K^C \times G(\mathcal{M})$.

In the rest of this subsection, we shall identify $K^C \times G(\mathcal{M})$ with $\oplus_{v \in \Lambda_t} R^W[v]^\ell$.

2.5.1. Action of $\mathcal{C}$ on $K^C \times G(\mathcal{M})$.

Proposition 2.8. Fix a partition $v \in \Lambda_t$. For any $1 \leq i \leq n$, $f \in K^C \times G(T^s \mathcal{F}_{v+e_{i+1}+e_{2n+1-i}})$, we have

$$\rho(\mathcal{C}_{i,v} \otimes f) = W_{[v+e_{i+1}+e_{2n+1-i}]}/W_{[E_{b,i+1}(v,1)]}(\prod_{\hat{v}_i-1 < t \leq \hat{v}_i} \frac{(q - q^{-1}t/x_{\hat{v}_i+1})}{(1 - t/x_{\hat{v}_i+1})} \cdot f).$$

Proof. Equation 5 implies that $T^s_{p_1} = \sum_{\hat{v}_i-1 < t \leq \hat{v}_i} x_t/x_{\hat{v}_i+1}$. Hence, we further have

$$\det(T^s_{p_1}) = \prod_{\hat{v}_i-1 < t \leq \hat{v}_i} x_t/x_{\hat{v}_i+1},$$

$$[\bigwedge^q T^s_{p_1}] = \prod_{\hat{v}_i-1 < t \leq \hat{v}_i} (1 - q^2 x_{\hat{v}_i+1}/x_t),$$

$$[\bigwedge^p T^s_{p_1}] = \prod_{\hat{v}_i-1 < t \leq \hat{v}_i} (1 - x_t/x_{\hat{v}_i+1}).$$

By Proposition 2.2 and Corollary 4 in [V98], the proposition follows.
2.5.2. Action of \( \mathcal{F}_{i, \nu} \) on \( K^{\ast} \times G(M) \).

**Proposition 2.9.** Let \( \nu = (v_1, \ldots, v_N) \) be a partition of \( 2d - 2 \) such that \( v_i = v_{N+1-i} \).

(a) If \( 1 \leq i < n \), for any \( f \in K^{\ast} \times G(T^* \mathcal{F}_{\nu + e_i + e_{2n+1-i}}) \), we have

\[
\rho(\mathcal{F}_{i, \nu} \otimes f) = W_{[\nu + e_{i+1} + e_{2n+1-i}]} / W_{[\nu + e_{i+1} + e_{2n+1-i}]} \left( \prod_{\bar{e}_{i+1} + 1 < t \leq \bar{e}_{i+1} + 1} \frac{(q - q^{-1}x_{\bar{e}_{i+1}}/x_t)}{(1 - x_{\bar{e}_{i+1}}/x_t)} \cdot f \right).
\]

(b) If \( i = n \), for any \( f \in K^{\ast} \times G(T^* \mathcal{F}_{\nu + e_n + e_{n+2}}) \), we have

\[
\rho(\mathcal{F}_{n, \nu} \otimes f) = W_{[\nu + 2e_{n+1}]} / W_{[\nu + 2e_{n+1}]} \left( \prod_{\bar{e}_{n+1} + 1 < t \leq d} \frac{q - q^{-1}x_t^{-1}x_{\bar{e}_{n+1}} + q - q^{-1}x_t x_{\bar{e}_{n+1}}}{1 - x_t^{-1}x_{\bar{e}_{n+1}} - 1 - x_t x_{\bar{e}_{n+1}}} \cdot f \right).
\]

**Proof.** If \( 1 \leq i < n \), Equation (5) implies that \( T^*_{p_1} = \sum_{\bar{e}_{i+1} + 1 < t \leq \bar{e}_{i+1} + 1} x_{\bar{e}_{i+1}}/x_t \). Hence, we have

\[
\text{Det}(T^*_{p_1}) = \prod_{\bar{e}_{i+1} + 1 < t \leq \bar{e}_{i+1} + 1} x_{\bar{e}_{i+1}}/x_t,
\]

\[
[\Lambda \sum_{q^2} T_{p_1}] = \prod_{\bar{e}_{i+1} + 1 < t \leq \bar{e}_{i+1} + 1} (1 - q^2 x_t/x_{\bar{e}_{i+1}}),
\]

\[
[\Lambda \sum_{q^2} T^*_{p_1}] = \prod_{\bar{e}_{i+1} + 1 < t \leq \bar{e}_{i+1} + 1} (1 - x_{\bar{e}_{i+1}}/x_t).
\]

In the case that \( i = n \), we have

\[
T^*_{p_1} = \sum_{\bar{e}_{n+1} + 1 < t \leq d} (x_t^{-1}x_{\bar{e}_{n+1}} + x_t x_{\bar{e}_{n+1}}) + x_{\bar{e}_{n+1}}^2,
\]

\[
\text{Det}(T^*_{p_1}) = \prod_{\bar{e}_{n+1} + 1 < t \leq d} (x_t^{-1}x_{\bar{e}_{n+1}} \cdot x_t x_{\bar{e}_{n+1}}) \cdot x_{\bar{e}_{n+1}}^2,
\]

\[
[\sum_{q^2} T_{p_1}] = (\prod_{\bar{e}_{n+1} + 1 < t \leq d} (1 - q^2 x_t x_{\bar{e}_{i+1}}^{-1}) (1 - q^2 (x_t^{-1}x_{\bar{e}_{n+1}}^{-1}))) (1 - q^2 x_{\bar{e}_{n+1}}^{-2}),
\]

\[
[\sum_{q^2} T^*_{p_1}] = (\prod_{\bar{e}_{n+1} + 1 < t \leq d} (1 - x_t^{-1}x_{\bar{e}_{n+1}}) (1 - x_t x_{\bar{e}_{n+1}})) (1 - x_{\bar{e}_{n+1}}^2).
\]

By Proposition 2.2 and Corollary 4 in [V98] again, the proposition follows. \(\square\)

3. The coideal subalgebra of \( U_q(\mathfrak{gl}_N) \)

3.1. The algebra \( U^b \). Let us recall the presentation of the algebra \( U^b \) (denoted by \( U^d \) therein) from \( [BKLW14] \).

**Definition 3.1.** The algebra \( U^b \) is the unitary associative \( \mathbb{C}(q) \)-algebra generated by \( e_i, f_i, h_a^1, \quad \forall i \in [1, n], a \in [1, n + 1], \)
subject to the following relations
\[
\begin{align*}
  h_a h_b &= h_b h_a, & h_a h_a^{-1} &= 1, \\
  h_a e_i h_a^{-1} &= q^{\delta_{a,i} - \delta_{a,i+1} - \delta_{2n+2-a,i+1}} e_i, & h_a f_i h_a^{-1} &= q^{-\delta_{a,i} + \delta_{a,i+1} + \delta_{2n+2-a,i+1}} f_i, \\
  e_i f_j - f_j e_i &= \delta_{ij} \frac{h_i h_{i+1}^{-1} - h_i^{-1} h_{i+1}}{q - q^{-1}}, & \text{if } i, j \neq n, \\
  e_i e_j &= e_j e_i, & f_i f_j &= f_j f_i, & \text{if } |i - j| > 1, \\
  e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 &= 0, & \text{if } |i - j| = 1, \\
  f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 &= 0, & \text{if } |i - j| = 1, \\
  e_i n f_n + f_n e_i^2 &= (q + q^{-1})(e_i f_n e_n - e_n (q h_n h_{n+1}^{-1} + q^{-1} h_n^{-1} h_{n+1})), \\
  f_n e_n + e_n f_n &= (q + q^{-1})(f_n e_n f_n - (q h_n h_{n+1}^{-1} + q^{-1} h_n^{-1} h_{n+1}) f_n)
\end{align*}
\]
for \(i, j = 1, 2, \ldots, n\) and \(a, b = 1, 2, \ldots, n + 1\).

The algebra \(U_b\) is a coideal subalgebra of \(U_q(\mathfrak{gl}_N)\) with coideal structure \(\Delta^b : U_b \to U_b \otimes U_q(\mathfrak{gl}_N)\) given by
\[
\begin{align*}
\Delta^b(e_i) &= e_i \otimes H_i + H_i^{-1} e_i, & \Delta^b(f_i) &= f_i \otimes H_i + H_i^{-1} f_i, \\
\Delta^b(h_i) &= h_i \otimes H_i + H_i^{-1} h_i.
\end{align*}
\]

### 3.2. A polynomial representation of \(U_b\)

Recall \(R = \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_d^{\pm 1}]\). Set \(K = \bigoplus_{\nu \in \Lambda} R^{W_{[\nu]}}\). Let
\[
\hat{h}_i \in \bigoplus_{\nu \in \Lambda} \text{Hom}(R^{W_{[\nu]}}_A, R^{W_{[\nu]}}_A),
\]
\[
\hat{e}_i, \hat{f}_i \in \bigoplus_{\nu \in \Lambda} \text{Hom}(R^{W_{[\nu]}}_A, R^{W_{[\nu]}}_A)
\]
be the following operators
\[
\hat{h}_i(f) = q^{v_i} f,
\]
\[
\hat{e}_i(f) = W_{[\nu]}/(W_{[\nu]} \cap W_{[\nu]})(\prod_{\theta_i < t \leq \theta_i} \frac{(q - q^{-1} x_i / x_{\theta_i+1})}{(1 - x_i / x_{\theta_i+1})} f),
\]
\[
\hat{f}_i(f) = \begin{cases} 
W_{[\nu]}/(W_{[\nu]} \cap W_{[\nu]})(\prod_{\theta_i < t \leq \theta_i} \frac{(q - q^{-1} x_i / x_{\theta_i+1})}{(1 - x_i / x_{\theta_i+1})} f) & \text{if } i < n, \\
W_{[\nu]}/(W_{[\nu]} \cap W_{[\nu]})(\prod_{\theta_i < t \leq d} \frac{q - q^{-1} x_{\theta_i+1}}{1 - x_i x_{\theta_i+1}} \frac{q - q^{-1} x_i x_{\theta_i}}{1 - x_i x_{\theta_i}} \frac{q^{-1} x_i^2}{x_{\theta_i}^2} f) & \text{if } i = n,
\end{cases}
\]
where \(f \in R^{W_{[\nu]}}_A, \nu' = \nu + e_{i+1} e_{2n+2-i} - e_{i+1} e_{2n+1-i}, \nu'' = \nu - e_i e_{2n+2-i} + e_{i+1} e_{2n+1-i}\).

### Proposition 3.2

The map
\[
e_i \mapsto \hat{e}_i, \ f_i \mapsto \hat{f}_i, \ h_i \mapsto \hat{h}_i,
\]
defines a representation of \(U_b\) on \(K_\mathbb{C}\).

**Proof.** We shall give a detail proof in appendix. □
3.3. K-theoretic approach to $U^b$. By Propositions 2.8, 2.9 and 3.2, we have the following proposition.

**Proposition 3.3.** There is an algebra homomorphism $U^b \rightarrow \text{End}(K^{C^*\times G}(\mathcal{M})_k)$ sending

$$e_i \mapsto \mathcal{E}_i, \quad f_i \mapsto \mathcal{F}_i, \quad h^\pm_i \mapsto \mathcal{H}_i^\pm.$$

By Lemma 2.7 and Proposition 3.3, we obtain the following main result of this section.

**Theorem 3.4.** The assignment

$$e_i \mapsto \mathcal{E}_i, \quad f_i \mapsto \mathcal{F}_i, \quad h^\pm_i \mapsto \mathcal{H}_i^\pm$$

can be extended to an algebra homomorphism $U^b \rightarrow K^{C^*\times G}(\mathcal{Z})_k$.

3.4. The coideal structure. For any semisimple element $\alpha = (c, s) \in C^* \times G$, let $\epsilon : R(C^* \times G) \rightarrow C$, $f \mapsto f(\alpha)$ be the evaluation map and $C_\alpha$ be the 1-dimensional $R(C^* \times G)$-module with $f$ acting via multiplication by $f(\alpha)$. For any algebraic variety $Y$, denote by $Y^\alpha$ the fixed point subvariety of $Y$ by $\alpha$. Assume that $s$ is a semisimple element in $G$ with $2k - 1$ distinct eigenvalues $\{\lambda_1^{\pm 1}, \ldots, \lambda_k^{\pm 1}, 1\}$ with multiplicity $d_1, \ldots, d_{k-1}, 2d_k$, respectively and set $\alpha = (1, s) \in C^* \times G$. There are natural isomorphisms

$$G^s \cong GL_{d_1} \times GL_{d_2} \times \cdots \times GL_{d_{k-1}} \times Sp_{d_k}, \quad Z^\alpha \cong Z_{d_1}^a \times Z_{d_2}^a \times \cdots \times Z_{d_{k-1}}^a \times Z_{d_k}^c,$$

where $Z_{d_i}^a$ is the Steinberg variety of type A of rank $d_i$ and $Z_{d_k}^c$ is the generalized Steinberg variety of type C of rank $d_k$. Let $N^*_a$ be the conormal bundle of the subvariety $Z^\alpha \hookrightarrow Z$. The $C^* \times G$ action on $Z$ induces a natural $\alpha$-action on $N^*_a$. Set

$$\lambda(Z^\alpha) = \sum_k (-1)^k \text{tr}(\alpha; \wedge^k N^*_a).$$

Let $\mathcal{M}_1, \mathcal{M}_2$ be the $C^* \times G$ variety $\mathcal{M} = T^* \mathcal{F}$ and $j : Z \hookrightarrow \mathcal{M}_1 \times \mathcal{M}_2$ be the imbedding. Define a morphism $r_\alpha : C_\alpha \otimes R(C^* \times G) K^{C^*\times G}(Z) \rightarrow C_\alpha \otimes R((C^* \times G)^\alpha) K^{(C^* \times G)^\alpha}(Z^\alpha)$ by

$$c \otimes \mathfrak{g} \mapsto c \otimes \sum_k (-1)^k \text{Tor}^k_{O_{\mathcal{M}_1^\alpha \times \mathcal{M}_2^\alpha}}(\lambda(M_1^\alpha)^{-1} \times O_{\mathcal{M}_2^\alpha}, j_! \mathfrak{g}).$$

The Künneth formula implies that there is a natural isomorphism

$$\tau : K^{(C^* \times G)^\alpha}(Z) \cong K^{C^* \times GL_{d_1}}(Z_{d_1}^a) \otimes \cdots \otimes K^{C^* \times GL_{d_{k-1}}}(Z_{d_{k-1}}^a) \otimes K^{C^* \times G_{d_k}}(Z_{d_k}).$$

Then by the bivariant fixed-point theorem [GV93 Theorem 5.7] and the Künneth formula, we have the following theorem.

**Theorem 3.5.** Let $\Delta^{k-1}$ be the composition of $r_\alpha$ and $\tau$, then

$$\Delta^{k-1} : C_\alpha \otimes K^{C^* \times G}(Z) \rightarrow C_\alpha \otimes K^{C^* \times GL_{d_1}}(Z_{d_1}^a) \otimes \cdots \otimes K^{C^* \times GL_{d_{k-1}}}(Z_{d_{k-1}}^a) \otimes K^{C^* \times G_{d_k}}(Z_{d_k})$$

is an algebra homomorphism.

By Theorem 8 in [V], there is an algebra homomorphism $U_q(gl_{d_i}) \rightarrow K^{C^* \times GL_{d_i}}(Z_{d_i}^a)$. Hence the above theorem gives a coideal structure of $U^b$. 

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4. The coideal subalgebra $U^c$

4.1. The algebra $U^c$. In [BKLW14, Section 6], authors consider another $\tau$-quantum group, $U^c$ (denoted by $\mathbb{C}U$ therein). By using perverse sheaves on $\mathcal{F}^c \times \mathcal{F}^c$, a geometric approach to $U^c$ is obtained. It has been shown that $U^c \simeq U^b$ in [BKLW14]. In this section, we shall provide a K-theoretic approach to $U^c$. Theorem 4.1. In this subsection, we shall calculate the bilinear form $(\cdot, \cdot)$. For any $q \in \Lambda$, we further set

$$\mathcal{F}_q = \{ F = (0 = V_0 \subset V_1 \cdots \subset V_{2n+1} = V) \mid V_i = V_{2n+1-i}^\perp, \dim(V_i/V_{i-1}) = v_i, \forall i \}.$$ 

The componentwise action of $G_b$ on $\mathcal{F}_q$ is transitive. For any $F \in \mathcal{F}_q$, let $P_v$ be the stabilizer of $F$ in $G_b$. Therefore, we have $G_b/P_v \simeq \mathcal{F}_q$. Let $\mathcal{F}_q^b = \sqcup_{v \in \Lambda_b} \mathcal{F}_q$. Group $G_b$ naturally acts on $\mathcal{F}_q^b$. Similar as Section 2.3, we define $\mathcal{M}_b^b = T^*\mathcal{F}_q$ and $\mathcal{Z}_b^b = \mathcal{M}_b^b \times_{N} \mathcal{M}_b^b$. The later is the generalized Steinberg variety of type B. In this subsection, we shall calculate the convolution algebra $(K^{\mathfrak{C}^* \times G_b}(\mathcal{Z}_q^b), \star)$ similarly as Section 2.

Let $\Pi_d = (a_{ij})_{1 \leq i, j \leq N}$ be the set of all $N \times N$ matrices with nonnegative integer entries such that $a_{ij} = a_{N+1-i,N+1-j}$ for $i, j \leq n+1$ and $\sum_{i,j} a_{ij} = 2d + 1$. Similarly, there is a bijection between $G_b$-orbits in $\mathcal{F}_q^b \times \mathcal{F}_q^b$ and $\Pi_d$. Moreover, we have the following two bijections

$$\varphi : \Lambda_b \to \Lambda_c, \quad v \mapsto v - (\delta_i,n+1)_{i=1,\ldots,N},$$

$$\psi : \Pi_d \to \Xi_d, \quad A \mapsto A - E_{n+1,n+1}.$$ 

Then there is an order “$\preceq$” on $\Pi_d(v,w), \forall v, w \in \Lambda_b$ via the above bijections. Set

$$W_{[v]}^b = W_{[\varphi(v)]}^c, \quad W_{[A]}^b = W_{[\psi(A)]}^c.$$ 

By Proposition 6.7 in [BKLW14], the orbit structure of type B is the same as that of type C under the above bijections. So we can obtain similar results as Section 2.4. Define $\mathcal{E}_{i,v}$ and $\mathcal{F}_{i,v}$ in the same way as $\mathcal{E}_{i,v}$ and $\mathcal{F}_{i,v}$, respectively. Let $\mathcal{H}_{i,v}^\pm$ be the sheaves supported on the $\mathcal{Z}_{\text{diag}(v)}^b$, which corresponds to $q^{\pm(v_i - \delta_{n+1,i})}$. We set

$$\mathcal{E}_i = \sum_{v \in \Lambda_b} (-q)^{-v_i} \mathcal{E}_{i,v}, \quad \mathcal{F}_i = \sum_{v \in \Lambda_b} (-q)^{-v_i} \mathcal{F}_{i,v},$$

$$\mathcal{H}_i = \sum_{v \in \Lambda_b} \mathcal{H}_{i,v}.$$ 

The following is an analogue of Theorem 2.6. We shall skip the proof.

Theorem 4.1. The convolution algebra $(K^{\mathfrak{C}^* \times G_b}(\mathcal{Z}_q^b), \star)$ is generated by $\mathcal{E}_i$, $\mathcal{F}_i$ and the sheaves supported on the orbits indexed by a diagonal matrix in $\Pi_d$. 


In order to calculate the relations of $K^c\times G_b(\mathcal{Z}^b)$ in term of the generators in Theorem 4.1, we still consider the faithful action $\rho_b$ of $K^c\times G_b(\mathcal{Z}^b)$ on $K^c\times G_b(\mathcal{M}^b)$ given by the convolution product

$$\rho_b : K^c\times G_b(\mathcal{Z}^b) \otimes K^c\times G_b(\mathcal{M}^b) \longrightarrow K^c\times G_b(\mathcal{M}^b), \quad f \otimes g \mapsto f \ast g.$$ 

For a partition $\nu \in \Lambda_b$, the action of $\mathcal{E}_{i,\nu}$, $1 \leq i \leq n$ (resp. $\mathcal{F}_{j,\nu}$, $1 \leq j < n$) are given by the same as that of $\mathcal{E}_{i,\varphi(\nu)}$ (resp. $\mathcal{F}_{j,\varphi(\nu)}$). Here we only need to give the action of $F_{n,\nu}$ on $K^c\times G_b(\mathcal{M}^b)$. In this case, we have

$$T_{p_1}^* = \sum_{\emptyset_{n+1} \subset \leq d} x_l^{-1} x_{\emptyset_{n+1}} + x_l x_{\emptyset_{n+1}} + x_{\emptyset_{n+1}},$$

$$\text{Det}(T_{p_1}^*) = \prod_{\emptyset_{n+1} \subset \leq d} x_l^{-1} x_{\emptyset_{n+1}} \cdot x_l x_{\emptyset_{n+1}} \cdot x_{\emptyset_{n+1}},$$

$$\bigwedge T_{p_1}^* = \bigwedge \left( \prod_{\emptyset_{n+1} \subset \leq d} (1 - x_l^{-1} x_{\emptyset_{n+1}})(1 - x_l x_{\emptyset_{n+1}})(1 - x_{\emptyset_{n+1}}) \right),$$

$$\bigwedge q T_{p_1}^* = \bigwedge \left( \prod_{\emptyset_{n+1} \subset \leq d} (1 - q^{-2} x_l^{-1} x_{\emptyset_{n+1}})(1 - q^2 x_l x_{\emptyset_{n+1}})(1 - q^2 x_{\emptyset_{n+1}}) \right).$$

For any $f \in K^c\times G_b(T^*F_{n,e_n+e_{n+1}}^b)$, we have

$$\rho_b(F_{n,\nu} \otimes f) = W_{[\nu+2e_{n+1}]}/W_{[E_{n+1,n}^\emptyset(\nu,1)]} \left( \prod_{\emptyset_{n+1} \subset \leq d} \frac{q - q^{-1} x_l^{-1} x_{\emptyset_{n+1}}}{1 - x_l^{-1} x_{\emptyset_{n+1}}}, \frac{q - q^{-1} x_l x_{\emptyset_{n+1}}}{1 - x_l x_{\emptyset_{n+1}}}, \frac{q - q^{-1} x_{\emptyset_{n+1}}}{1 - x_{\emptyset_{n+1}}} \right) f.$$ 

**Proposition 4.2.** The map

$$e_i \mapsto \mathcal{E}_i, \quad f_i \mapsto \mathcal{F}_i, \quad h_{\pm 1} \mapsto \mathcal{H}_{\pm 1}$$

extends uniquely to a representation of $U^c$ on $K^c\times G_b(\mathcal{M}^b)_k$.

**Proof.** We only need to prove the last two relations in Definition 3.1 for the case $n = 1$. Assume $n = 1$, fix a partition $I = (0, i, 2d + 1 - 2i, i)$, and $Q \in (R(T) \otimes C)^W_{[i]}$. The proof of proposition is almost the same as that of the Proposition 3.2 except the following difference.

$$\omega(\pm l, \pm k) = \frac{q x_l^{\pm 1} - q^{-1} x_k^{\pm 1}}{x_l^{\pm 1} - x_k^{\pm 1}}, \omega(\pm l, \mp k) = \frac{q x_l^{\pm 1} - q^{-1} x_k^{\mp 1}}{x_l^{\pm 1} - x_k^{\mp 1}}, \forall l, k \in [1, d], l \neq k$$

$$\omega(\pm k, \mp k) = \frac{q - q^{-1} x_k^{\mp 1}}{1 - x_k^{\mp 1}}, \forall k \in [1, d].$$
Then we have
\[ h_1 h_2^{-1}(Q \otimes e_i) = q^{3i-2d} Q \otimes e_i, \quad h_1^{-1} h_2(Q \otimes e_i) = q^{2d-3i} Q \otimes e_i, \]
\[ e_1(Q \otimes e_i) = \sum_{l=1}^{1+i} \prod_{s=1, s \neq l}^{i} \omega(l, s)(l, i+1)Q \otimes e_{i+1}, \]
\[ f_1(Q \otimes e_i) = \sum_{k=i}^{d} \prod_{t=1, t \neq k}^{d} \omega(t, k) \omega(-t, k) \omega(-k, k)(k, i)Q \otimes e_{i-1} \]
\[ + \sum_{k=i}^{d} \prod_{t=1, t \neq k}^{d} \omega(t, -k) \omega(-t, -k) \omega(k, -k)[r]_k(k, i)Q \otimes e_{i-1}. \]

Then the remaining calculation is the same as that of Proposition 3.2. We shall skip it.

By the faithful representation and the Proposition 4.2, we obtain the main following theorem.

**Theorem 4.3.** The assignment
\[ e_i \mapsto E_i, \quad f_i \mapsto F_i, \quad h_i ^{\pm 1} \mapsto H_i ^{\pm 1} \]
can be extended to an algebra homomorphism \( U^c \to K^{C^* \times G_b}(Z_b)^b \).

4.3. **Coideal structure of** \( U^c \). Fix a semisimple matrix \( s \in G_b \) with \( 2k-1 \) distinct eigenvalues \( \{ \lambda_1 ^\pm 1, \lambda_2 ^\pm 1, \ldots, \lambda_{k-1} ^\pm 1, 1 \} \) with multiple \( d_1, d_2, \ldots, d_{k-1}, 2d_k + 1 \) and set \( \sigma = (1, s) \in C^* \times G_b \). There are natural isomorphisms
\[ G_b^\sigma \cong GL_{d_1} \times GL_{d_2} \times \cdots \times GL_{d_{k-1}} \times G_{b, d_k}, \]
\[ Z_b^{\sigma, a} \cong Z_{d_1}^a \times Z_{d_2}^a \times \cdots \times Z_{d_{k-1}}^a \times Z_{d_k}^b. \]

Similarly as Section 3.4, we define a morphism \( r_\alpha : C_\alpha \otimes R(C^* \times G_b) K^{C^* \times G_b}(Z_b) \to C_\alpha \otimes R((C^* \times G_b) ^\sigma)(Z_b ^\alpha) \) by
\[ c \otimes \mathcal{F} \mapsto c \otimes \sum (-1)^k \text{Tor}^{k}_{\mathcal{O}_{M^{\alpha}_{b, 1} \times M^{\alpha}_{b, 2}}} (\lambda(M_{b, 1} ^\alpha)^{-1} \times \mathcal{O}_{M^{\alpha}_{b, 2}}, \mathcal{F}). \]

Similarly as Theorem 3.5, we have the following theorem.

**Theorem 4.4.** There is an algebra homomorphism
\[ \Delta_b^{k-1} : C_\alpha \otimes K^{C^* \times G_b}(Z_b) \to C_\alpha \otimes K^{C^* \times GL_{d_1}}(Z_{d_1}^a) \otimes \cdots \otimes K^{C^* \times GL_{d_{k-1}}}(Z_{d_{k-1}}^a) \otimes K^{C^* \times G_{b, d_k}}(Z_{d_k}^b). \]

4.4. **Li’s conjecture.** In [L18], Li defines \( \sigma \)-quiver variety for the Satake diagram without black dot, which is a fixed point subvariety of Nakajima quiver variety corresponding to underlying Dynkin diagram \( \Gamma \). We refer Section 4 in [L18] for detail definitions. Let \( U(g^w) \) be the envelope algebra of the \( \sigma \)-fixed subalgebra of Lie algebra \( g^w \) associated to \( \Gamma \), \( Y_\zeta(w) \) be the Steinberg type \( \sigma \)-quiver variety, and \( H_{top}(Y_\zeta(w)) \) be the top Borel-Moore homology. Li made the following conjecture.

**Conjecture 4.5.** [L18 Conjecture 5.3.4] There is a nontrivial algebra homomorphism \( U(g^w) \to H_{top}(Y_\zeta(w)) \).
Theorem 4.6. In the case with the Satake diagram in (1), Conjecture 4.3 holds.

Proof. It is proved that cotangent bundle of the isotropic flag variety of type B/C is isomorphic to certain special σ-quiver variety \([L18]\) Theorem 6.2.1. From \([L18]\) Lemma 4.2.10, we can know that \(G_0(w)\) is the nilpotent variety of \(\mathfrak{so}_{2d+1}\) or \(\mathfrak{sp}_{2d}\). Thus the Steinberg type σ-quiver variety \(\mathfrak{G}_c(w)\) is isomorphic to the generalized Steinberg variety of type B/C. Hence, \(U(q^c_1) = U^b_{q=1}(\text{resp. } U^c_{q=1})\), where \(U^b_{q=1}\) (resp. \(U^c_{q=1}\)) is the \(\mathbb{C}^*\)-algebras obtained by specializing \(U^b\) (resp. \(U^c\)) to \(q = 1\).

Fix semisimple elements \(\alpha = (t, s) \in \mathbb{C}^* \times G\) and \(\beta = (t_1, s_1) \in \mathbb{C}^* \times G\), the bivariant localization theorem \([CG, Theorem 5.11.10]\) and the bivariant Riemann-Roch theorem \([CG, Theorem 5.11.11]\) imply that there are surjective algebra homomorphisms

\[
K^{\mathbb{C}^* \times G}(\mathcal{Z}^\alpha) \twoheadrightarrow H_*(\mathcal{Z}^\alpha),
\]

\[
K^{\mathbb{C}^* \times G}(\mathcal{Z}^\beta) \twoheadrightarrow H_*(\mathcal{Z}^\beta),
\]

where \(H_*(\mathcal{Z}^\alpha)\) and \(H_*(\mathcal{Z}^\beta)\) are the Borel-Moore homology. For \(\alpha = \beta = (1, 1)\), by Theorem 3.4 and Theorem 4.3, we have two algebra homomorphisms

\[
U^b_{q=1} \to K^{\mathbb{C}^* \times G}(\mathcal{Z})_{q=1} \to H_*(\mathcal{Z}^\alpha),
\]

\[
U^c_{q=1} \to K^{\mathbb{C}^* \times G}(\mathcal{Z})_{q=1} \to H_*(\mathcal{Z}^\alpha).
\]

Thus the theorem follows. \(\square\)

**Appendix A. The proof of Proposition 3.2**

We only need to prove the last two relations in the definition of \(U^b\), the others are straightforward. it is suffice to prove it for the case \(n = 1\). Fix a partition \(I = (0, i, 2d - 2i, i)\), and \(Q \in (R(T) \otimes \mathbb{C})^{W[I]}\). Set

\[
\omega(\pm l, \pm k) = \frac{qx^{\pm 1}_l - q^{-1}x^{\mp 1}_k}{x^{\pm 1}_l - x^{\mp 1}_k}, \omega(\pm l, \mp k) = \frac{qx^{\pm 1}_l - q^{-1}x^{\mp 1}_k}{x^{\pm 1}_l - x^{\mp 1}_k}, \forall l, k \in [1, d].
\]

Then we have

\[
k_1(Q \otimes e_i) = h_1 h_2^{-1}(Q \otimes e_i) = q^{3i - 2d} Q \otimes e_i,
\]

\[
k_1^{-1}(Q \otimes e_i) = h_1^{-1} h_2(Q \otimes e_i) = q^{2d - 3i} Q \otimes e_i,
\]

\[
e_1(Q \otimes e_i) = \sum_{l=1}^{1+i} \left( l, 1 + i \right) \left( \prod_{s=1}^{i} \frac{qx_{l+1} - q^{-1}x_s}{x_{l+1} - x_s} \right) Q \otimes e_{l+1}
\]

\[
= \sum_{l=1}^{1+i} \prod_{s=1,s\neq l}^{i} \omega(l, s)(l, i + 1) Q \otimes e_{l+1},
\]

\[
f_1(Q \otimes e_i) = \sum_{k=1}^{d} \mathbb{Z}_{2k} \times (k, i) [\prod_{t=i+1}^{d} \frac{q_{x_t} - q^{-1}x_t}{x_t - x_i} \frac{q_{x^{-1}_t} - q^{-1}x_i}{x^{-1}_t - x_i} \frac{q_{x^{-1}_i} - q^{-1}x_i}{x^{-1}_i - x_i}] Q \otimes e_{-1}
\]

\[
= \sum_{k=1}^{d} \prod_{t=i, t \neq k}^{d} \omega(t, k) \omega(-t, k) \omega(-k, k)(k, i) Q \otimes e_{-1}
\]
We now show the penultimate identity in Definition 3.1. By a direct calculation, we have

\[
\sum_{k=1}^{d} \prod_{t=i, t \neq k}^{d} \omega(t, -k)\omega(-t, -k)\omega(k, -k)[1]_k(k, i)Q \otimes e_{i-1}.
\]

\[
\frac{e_1^2 f_1(Q \otimes e_i)}{\sum_{l=1}^{i} \prod_{k=i+1}^{d} \prod_{n=1, s=1, s \neq l}^{d} \omega(i + 1, n)\omega(l, s)\omega(t, k)\omega(-t)\omega(-k, k)
\]

\[
\omega(l, k)\omega(-l, -k)(l, k)Q \otimes e_{i+1}
\]

\[
+ \sum_{l=1}^{i} \prod_{k=i+1}^{d} \prod_{n=1, s=1, s \neq l}^{d} \omega(i + 1, n)\omega(l, s)\omega(t, l)\omega(-t, l)\omega(-l, l)Q \otimes e_{i+1}
\]

\[
+ \sum_{l=1}^{i} \prod_{k=i+1}^{d} \prod_{n=1, s=1, s \neq l}^{d} \omega(i + 1, n)\omega(l, s)\omega(t, -k)\omega(-t, -k)\omega(k, -k)
\]

\[
\omega(l, -k)\omega(-l, -k)(l, k)[1]_l Q \otimes e_{i+1}
\]

\[
+ \sum_{l=1}^{i} \prod_{k=i+1}^{d} \prod_{n=1, s=1, s \neq l}^{d} \omega(i + 1, n)\omega(l, s)\omega(t, -l)\omega(-t, -l)\omega(-l, -l)[1]_l Q \otimes e_{i+1}
\]

\[
+ \sum_{m=1}^{i} \sum_{l=1, l \neq m}^{i+1} \prod_{k=i+2}^{i+1} \prod_{n=1, n \neq m}^{i+1} \prod_{s=1, s \neq l}^{i+1, t=i+2} \omega(m, n)\omega(l, s)\omega(t, m)\omega(-t, m)\omega(-m, m)
\]

\[
\omega(l, m)\omega(-l, m)(l, i + 1)Q \otimes e_{i+1}
\]

\[
+ \sum_{m=1}^{i} \sum_{k=i+2}^{i+1} \prod_{n=1, n \neq m}^{i+1} \prod_{s=1, s \neq m}^{i, t=i+1, t \neq k} \omega(m, n)\omega(i + 1, s)\omega(t, k)\omega(-t, k)\omega(-k, k)
\]

\[
\omega(m, k)\omega(-m, k)(k, m)Q \otimes e_{i+1}
\]

\[
+ \sum_{m=1}^{i} \prod_{n=1, n \neq m}^{i+1} \prod_{s=1, s \neq m}^{i+1} \prod_{t=i+1}^{i+1} \omega(m, n)\omega(i + 1, s)\omega(t, m)\omega(-t, m)\omega(-m, m)Q \otimes e_{i+1}
\]

\[
+ \sum_{m=1}^{i} \sum_{t=1, t \neq m}^{i} \prod_{n=1, n \neq m}^{i} \prod_{s=1, s \neq t}^{i+1, t} \omega(m, n)\omega(t, l)\omega(-t, l)\omega(-l, l)
\]

\[
\omega(m, l)\omega(-m, l)(i + 1, m)Q \otimes e_{i+1}
\]
\[\begin{align*}
&+ \sum_{m=1}^{i} \prod_{l=1}^{i+1} \prod_{k=1}^{i} \prod_{t+i+1}^{d} \omega(m, n)\omega(l, s)\omega(t, i + 1)\omega(-t, i + 1)\omega(-i + 1, i + 1) \\
&\quad \omega(m, i + 1)\omega(-m, i + 1)(m, i + 1)Q \otimes e_{i+1} \\
&+ \sum_{m=1}^{i} \prod_{l=1}^{i+1} \prod_{k=1}^{i} \prod_{t+i+1}^{d} \omega(m, n)\omega(l, s)\omega(t, -k)\omega(-t, -k)\omega(k, -k) \\
&\quad \omega(l, -k)\omega(l, -k)\omega(m, -k)\omega(-m, -k)(m, i + 1)(k, l)[1]Q \otimes e_{i+1} \\
&+ \sum_{m=1}^{i} \prod_{l=1}^{i+1} \prod_{k=1}^{i} \prod_{t+i+1}^{d} \omega(m, n)\omega(l, s)\omega(t, -m)\omega(-m, -m) \\
&\quad \omega(l, -m)\omega(-l, -m)(l, i + 1)[1]mQ \otimes e_{i+1} \\
&+ \sum_{m=1}^{i} \prod_{l=1}^{i+1} \prod_{k=1}^{i} \prod_{t+i+1}^{d} \omega(m, n)\omega(l, s)\omega(t, -k)\omega(-t, -k)\omega(k, -k) \\
&\quad \omega(m, -k)\omega(-m, -k)(k, m)[1]mQ \otimes e_{i+1} \\
&+ \sum_{m=1}^{i} \prod_{l=1}^{i+1} \prod_{k=1}^{i} \prod_{t+i+1}^{d} \omega(m, n)\omega(l, s)\omega(t, -m)\omega(-m, -m)[1]mQ \otimes e_{i+1} \\
&+ \sum_{m=1}^{i} \prod_{l=1}^{i+1} \prod_{k=1}^{i} \prod_{t+i+1}^{d} \omega(m, n)\omega(l, s)\omega(t, -l)\omega(-t, -l)\omega(l, -l) \\
&\quad \omega(m, -l)\omega(-m, -l)(i + 1, m)[1]Q \otimes e_{i+1} \\
&+ \sum_{m=1}^{i} \prod_{l=1}^{i+1} \prod_{k=1}^{i} \prod_{t+i+1}^{d} \omega(m, n)\omega(l, s)\omega(t, -(i + 1))\omega(-t, -(i + 1))\omega(i + 1, -(i + 1)) \\
&\quad \omega(m, -(i + 1))\omega(-m, -(i + 1))(m, i + 1)[1]mQ \otimes e_{i+1}
\end{align*}\]

Then we have

\[\frac{e_{i+1}^{2}f_{1}}{q + q^{-1}}(Q \otimes e_{i})\]

\[= \sum_{l=1}^{i} \prod_{k=i+1}^{i+1} \prod_{n=1, n \neq l}^{i} \prod_{s=1, s \neq l}^{i} \prod_{t+i+1}^{d} \omega(i + 1, n)\omega(l, s)\omega(t, k)\omega(-t, -k)\omega(-k, k) \\
\quad \omega(l, k)\omega(-l, k)(l, k)Q \otimes e_{i+1} \\
+ \sum_{l=1}^{i} \prod_{k=i+1}^{i+1} \prod_{n=1, n \neq l}^{i} \prod_{s=1, s \neq l}^{i} \prod_{t+i+1}^{d} \omega(i + 1, n)\omega(l, s)\omega(t, l)\omega(-t, -l)\omega(-l, l)Q \otimes e_{i+1} \\
+ \sum_{l=1}^{i} \prod_{k=i+1}^{i+1} \prod_{n=1, n \neq l}^{i} \prod_{s=1, s \neq l}^{i} \prod_{t+i+1}^{d} \omega(i + 1, n)\omega(l, s)\omega(t, -k)\omega(-t, -k)\omega(k, -k) \\
\quad \omega(l, -k)\omega(-l, -k)(l, k)[1]lQ \otimes e_{i+1}\]
Similarly, we have

\[ \frac{f_1 e_1^2}{q + q^{-1}} (Q \otimes e_i) \]

\[ = \sum_{k=1+2}^{d} \sum_{l=1, l>m}^{d} \prod_{t=i+2, t\neq k}^{d} \prod_{n=1, n\neq l, m \neq i, s}^{i+1} \prod_{m=1, m \neq n, s}^{d} \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(m, n) \omega(l, s) \]

\[ + \sum_{k=1+2}^{d} \sum_{l=1, l>m}^{d} \prod_{t=i+2, t\neq k}^{d} \prod_{n=1, n\neq l, m \neq i, s}^{i+1} \prod_{m=1, m \neq n, s}^{d} \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(m, n) \omega(k, s) \]

\[ + \sum_{k=1+2}^{d} \sum_{l=1, l>m}^{d} \prod_{t=i+2, t\neq k}^{d} \prod_{n=1, n\neq l, m \neq i, s}^{i+1} \prod_{m=1, m \neq n, s}^{d} \omega(t, -k) \omega(-t, -k) \omega(k, -k) \omega(m, n) \omega(l, s) \]

\[ + \sum_{k=1+2}^{d} \sum_{l=1, l>m}^{d} \prod_{t=i+2, t\neq k}^{d} \prod_{n=1, n\neq l, m \neq i, s}^{i+1} \prod_{m=1, m \neq n, s}^{d} \omega(t, -k) \omega(-t, -k) \omega(k, -k) \omega(m, n) \omega(-k, s) \]

Similarly, we have

\[ e_1 f_1 e_1 (Q \otimes e_i) \]

\[ = \sum_{k=1+2}^{d} \sum_{l=1, l>m}^{d} \prod_{t=i+2, t\neq k}^{d} \prod_{n=1, n\neq l, m \neq i, s}^{i+1} \prod_{m=1, m \neq n, s}^{d} \omega(k, l) \omega(l, i+1) (k, m) Q \otimes e_i \]
\[
\begin{align*}
&= \sum_{i} \prod_{d} \prod_{i} \prod_{s=1, s \neq l} \omega(i + 1, n) \omega(t, i + 1) \omega(-t, i + 1) \omega(-i + 1, i + 1) \omega(l, s) \\
&\hspace{1em} \omega(l, i + 1)(l, i + 1) Q \otimes e_{i+1} \\
&+ \sum_{k=i+2}^{d} \sum_{l=1}^{i} \prod_{i} \prod_{d} \prod_{s=1, s \neq l} \omega(i + 1, n) \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(l, s) \\
&\hspace{1em} \omega(l, k)(l, k) Q \otimes e_{i+1} \\
&+ \sum_{k=i+1}^{d} \prod_{i} \prod_{d} \prod_{s=1, s \neq l} \omega(i + 1, n) \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(k, s) Q \otimes e_{i+1} \\
&+ \sum_{m=1}^{i+1} \prod_{d} \prod_{i} \prod_{s=1, s \neq l} \omega(m, n) \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(l, s) \\
&\hspace{1em} \omega(l, k) \omega(m, k) \omega(-m, k) \omega(l, i + 1) (m, i + 1)(l, k) Q \otimes e_{i+1} \\
&+ \sum_{m=1}^{i+1} \prod_{d} \prod_{i} \prod_{s=1, s \neq l} \omega(m, n) \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(i + 1, s) \\
&\hspace{1em} \omega(i + 1, k) \omega(m, k) \omega(-m, k)(m, k) Q \otimes e_{i+1} \\
&+ \sum_{m=1}^{i+1} \prod_{d} \prod_{i} \prod_{s=1, s \neq l} \omega(m, n) \omega(t, m) \omega(-t, m) \omega(-m, m) \omega(l, s) \\
&\hspace{1em} \omega(l, m) \omega(l, i + 1) (l, i + 1) Q \otimes e_{i+1} \\
&+ \sum_{m=1}^{i+1} \prod_{d} \prod_{i} \prod_{s=1, s \neq m} \omega(m, n) \omega(t, m) \omega(-t, m) \omega(-m, m) \omega(i + 1, s) \\
&\hspace{1em} \omega(i + 1, m) Q \otimes e_{i+1} \\
&+ \sum_{m=1}^{i+1} \prod_{d} \prod_{i} \prod_{s=1, s \neq m} \omega(m, n) \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(k, s) \\
&\hspace{1em} \omega(m, k) \omega(-m, k)(k, i + 1) (m, i + 1) Q \otimes e_{i+1} \\
&+ \sum_{m=1}^{i+1} \prod_{d} \prod_{i} \prod_{s=1, s \neq m} \omega(m, n) \omega(t, m) \omega(-t, m) \omega(-m, m) \omega(m, s) \\
&\hspace{1em} \omega(m, i + 1) (m, i + 1) Q \otimes e_{i+1} \\
&+ \sum_{k=i+1}^{d} \sum_{l=1}^{i} \prod_{d} \prod_{i} \prod_{s=1, s \neq l} \omega(i + 1, n) \omega(t, -k) \omega(-t, -k) \omega(k, -k) \omega(l, s) \\
&\hspace{1em} \omega(l, -k)(l, k)[1] Q \otimes e_{i+1}
\end{align*}
\]
Therefore, we have
\[
\sum_{k=i+1}^{d} \prod_{n=1}^{i} \prod_{t=i+1, t \neq k}^{d} \prod_{s=1}^{i} \omega(i + 1, n)\omega(t, -k)\omega(-t, -k)\omega(k, -k)\omega(-k, s)Q \otimes e_{i+1}
\]
\[
+ \sum_{m=1}^{i} \sum_{k=i+2}^{d} \prod_{n=1, n \neq m}^{i} \prod_{t=i+2, t \neq k}^{d} \prod_{s=1, s \neq m}^{i} \omega(m, n)\omega(t, -k)\omega(-t, -k)\omega(k, -k)\omega(l, s)
\]
\[
+ \sum_{m=1}^{i} \sum_{k=i+2}^{d} \prod_{n=1, n \neq m}^{i} \prod_{t=i+2, t \neq k}^{d} \prod_{s=1, s \neq m}^{i} \omega(m, n)\omega(t, -m)\omega(-t, -m)\omega(m, -m)\omega(l, s)
\]
\[
+ \sum_{m=1}^{i} \prod_{i=1}^{i+1} \prod_{d=1}^{i} \prod_{t=i+2}^{d} \prod_{s=1, s \neq m}^{i} \omega(m, n)\omega(t, -m)\omega(-t, -m)\omega(m, -m)\omega(i + 1, s)
\]
\[
+ \sum_{m=1}^{i} \prod_{i=1}^{i+1} \prod_{d=1}^{i} \prod_{t=i+2}^{d} \prod_{s=1, s \neq m}^{i} \omega(m, n)\omega(t, -k)\omega(-t, -k)\omega(k, -k)\omega(-k, s)
\]
\[
+ \sum_{m=1}^{i} \prod_{i=1}^{i+1} \prod_{d=1}^{i} \prod_{t=i+2}^{d} \prod_{s=1, s \neq m}^{i} \omega(m, n)\omega(t, -m)\omega(-t, -m)\omega(m, -m)\omega(-m, s)
\]

Therefore, we have
\[
\frac{e_1^2f_1 + f_1e_1^2}{q + q^{-1}} - e_1f_1e_1
\]
\[
= A \prod_{n=1}^{i} \omega(i + 1, n)Q \otimes e_{i+1} + \sum_{l=1}^{i} \prod_{s=1, s \neq l}^{i+1} \omega(l, s)(l, i + 1)Q \otimes e_{i+1}
\]
\[
+ \sum_{l=1}^{i} \sum_{k=i+2}^{d} C_{kl}(k, l)Q \otimes e_{i+1} + \sum_{m=1}^{i} \sum_{l=1, l < m}^{d} \sum_{k=i+2}^{m} D_{klm}(m, i + 1)(k, l)Q \otimes e_{i+1}
\]
\[
+ \sum_{l=1}^{i} \sum_{k=i+2}^{d} E_{kl}(k, l)[1]Q \otimes e_{i+1} + \sum_{m=1}^{i} \sum_{l=1, l < m}^{d} \sum_{k=i+2}^{m} F_{klm}(m, i + 1)(k, l)Q \otimes e_{i+1}.
\]
Here

\[
A = \sum_{t=1}^{i} \prod_{l=1}^{i} \prod_{s \neq l}^{d} \omega(l, s) \omega(t, l) \omega(-t, l) \omega(-l, l) \omega(-(i+1), l)
\]

\[
+ \prod_{t=i+3}^{d} \prod_{s=1}^{i} \omega(t, i+2) \omega(-t, i+2) \omega(-(i+2), i+2) \omega(i+2, s)
\]

\[
+ \sum_{k=i+3}^{d} \prod_{t=i+2, t \neq k}^{d} \prod_{s=1}^{i} \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(k, s)
\]

\[
+ \prod_{t=i+3}^{d} \prod_{s=1}^{i} \omega(t, -(i+2)) \omega(-t, -(i+2)) \omega(i+2, -(i+2)) \omega(-(i+2), s)
\]

\[
+ \sum_{k=i+3}^{d} \prod_{t=i+2, t \neq k}^{d} \prod_{s=1}^{i} \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(-k, s)
\]

\[
- \sum_{k=i+1}^{d} \prod_{t=i+1, t \neq k}^{d} \prod_{s=1}^{i} \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(k, s)
\]

\[
- \sum_{m=1}^{i} \prod_{t=i+2}^{i} \prod_{s=1}^{d} \omega(m, s) \omega(t, m) \omega(-t, m) \omega(-m, m)
\]

\[
- \sum_{k=i+1}^{d} \prod_{t=i+1, t \neq k}^{d} \prod_{s=1}^{i} \omega(t, -k) \omega(-t, -k) \omega(k, -k) \omega(-k, s)
\]

\[
B_l = \prod_{n=1, n \neq l}^{i} \prod_{t=i+2}^{d} \omega(i+1, n) \omega(t, i+1) \omega(-t, i+1) \omega(-(i+1), i+1) \omega(-l, i+1)
\]

\[
+ \sum_{m=1, m \neq l}^{i+1} \prod_{n=1, n \neq l, m}^{d} \omega(m, n) \omega(t, m) \omega(-t, m) \omega(-m, m) \omega(-l, m)
\]

\[
+ \prod_{t=i+3}^{d} \prod_{n=1, n \neq l}^{i+1} \omega(t, i+2) \omega(-t, i+2) \omega(-(i+2), i+2) \omega(i+2, n)
\]

\[
+ \sum_{k=i+3}^{d} \prod_{t=i+2, t \neq k}^{d} \prod_{n=1, n \neq l}^{i} \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(k, n)
\]

\[
+ \prod_{t=i+3}^{d} \prod_{n=1, n \neq l}^{i} \omega(t, -(i+2)) \omega(-t, -(i+2)) \omega(i+2, -(i+2)) \omega(-(i+2), n)
\]

\[
+ \sum_{k=i+3}^{d} \prod_{t=i+2, t \neq k}^{d} \prod_{n=1, n \neq l}^{i+1} \omega(t, -k) \omega(-t, -k) \omega(k, -k) \omega(-k, n)
\]
\[ C_{kl} = \prod_{n=1}^{i} \prod_{s=1}^{i} \prod_{t=i+1}^{d} \omega(i+1, n) \omega(t, i+1) \omega(-t, i+1) \omega(-(i+1), i+1) \]

\[ \quad + \prod_{n=1}^{i} \prod_{s=1}^{i} \prod_{t=i+1}^{d} \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(-k, i+1, k) \]

\[ \quad - \prod_{n=1}^{i} \prod_{s=1}^{i} \prod_{t=i+1}^{d} \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(-k, i+1, k) \]

\[ \quad - \prod_{n=1}^{i+1} \prod_{s=1}^{i+1} \prod_{t=i+2, t \neq k}^{d} \omega(l, n) \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(i+1, l, k) \omega(-l, k) \]

\[ D_{klm} = \prod_{n=1}^{i+1} \prod_{s=1}^{i+1} \prod_{t=i+2, t \neq k}^{d} \omega(m, n) \omega(l, s) \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(l, k) \omega(-l, k) \]

\[ \quad + \prod_{n=1}^{i+1} \prod_{s=1}^{i+1} \prod_{t=i+2, t \neq k}^{d} \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(l, n) \omega(m, s) \omega(l, k) \omega(m, k) \]

\[ \quad - \prod_{n=1}^{i+1} \prod_{s=1}^{i+1} \prod_{t=i+2, t \neq k}^{d} \omega(m, n) \omega(t, k) \omega(-t, k) \omega(-k, k) \omega(l, s) \omega(l, k) \omega(m, k) \]

\[ \quad \omega(-m, k) \omega(l, i+1) \]

- EQUIVARIANT K-THEORY APPROACH TO \( r \)-QUANTUM GROUPS
By a direct calculation, we have

$$\omega(l, n)\omega(t, k)\omega(-t, k)\omega(-k, l)\omega(m, s)\omega(m, k)\omega(l, k)$$

$$\omega(-l, k)\omega(m, i + 1)$$

$$E_{kl} = \prod_{n=1, n \neq l}^{i+1} \prod_{s=1, s \neq l}^{i} \prod_{t=i+2, t \neq k}^{d} \omega(i+1, n)\omega(l, s)\omega(t, -k)\omega(-t, -k)\omega(k, -k)\omega(l, -k)\omega(-l, -k)$$

$$\omega(l, -k)\omega(i+1, -k)$$

$$- \prod_{n=1, n \neq l}^{i} \prod_{s=1, s \neq l}^{i} \prod_{t=i+1, t \neq k}^{d} \omega(i+1, n)\omega(t, -k)\omega(-t, -k)\omega(k, -k)\omega(l, -k)\omega(l, k)\omega(-l, k)$$

$$\omega(i+1, -k)\omega(l, -k)\omega(-l, -k)$$

$$F_{klm} = \prod_{n=1, n \neq l, m}^{i+1} \prod_{s=1, s \neq l, m}^{i+1} \prod_{t=i+2, t \neq k}^{d} \omega(m, n)\omega(l, s)\omega(t, -k)\omega(-t, -k)\omega(k, -k)\omega(l, -k)$$

$$\omega(-l, -k)\omega(m, -k)\omega(-m, -k)$$

$$+ \prod_{n=1, n \neq l, m}^{i+1} \prod_{s=1, s \neq l, m}^{i+1} \prod_{t=i+2, t \neq k}^{d} \omega(t, -k)\omega(-t, -k)\omega(k, -k)\omega(l, n)\omega(m, s)\omega(l, -k)\omega(m, -k)$$

$$\omega(m, -k)\omega(l, i + 1)$$

$$- \prod_{n=1, n \neq m}^{i+1} \prod_{s=1, s \neq m}^{i} \prod_{t=i+2, t \neq k}^{d} \omega(m, n)\omega(t, -k)\omega(-t, -k)\omega(k, -k)\omega(l, s)\omega(l, -k)\omega(m, -k)$$

$$\omega(-m, -k)\omega(l, i + 1)$$

$$- \prod_{n=1, n \neq l}^{i+1} \prod_{s=1, s \neq l, m}^{i+1} \prod_{t=i+2, t \neq k}^{d} \omega(l, n)\omega(t, -k)\omega(-t, -k)\omega(k, -k)\omega(m, s)\omega(m, -k)\omega(l, -k)$$

$$\omega(-l, -k)\omega(m, i + 1)$$

By a direct calculation, we have

$$A = B_l = -(q^{2d-3i-1} + q^{3i+1-2d}), \quad C_{kl} = D_{klm} = E_{kl} = F_{klm} = 0.$$

That is,

$$e_1^2 f_1 + f_1 e_1^2 = (q + q^{-1})(e_1 f_1 e_1 - e_1(q^{-1}k_1^{-1} + qk_1)).$$
Similarly, we can prove the last identity. Proposition follows.

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