Gravity localization in a string-cigar braneworld

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Abstract
We proposed a six-dimensional string-like braneworld built from a warped product between a 3-brane and the Hamilton cigar soliton space, the string-cigar braneworld. This transverse manifold is a well-known steady solution of the Ricci flow equation that describes the evolution of a manifold. The resulting bulk is an interior and exterior metric for a thick string. Furthermore, the source satisfies the dominant energy condition. It is possible to realize the geometric flow as a result of variations of the matter content of the brane, actually, as its tensions. Furthermore, the Ricci flow defines a family of string-like branes and we studied the effects that the evolution of the transverse space have on the geometric and physical quantities. The geometric flow makes the cosmological constant and the relationship between the Planck masses evolves. The gravitational massless mode remains trapped to the brane and the width of the mode depends on the evolution parameter. For the Kaluza–Klein modes, the asymptotic spectrum of mass is the same as that for the thin string-like brane and the analogue Schrödinger potential also changes according to the flow.

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1. Introduction
In recent years, much attention has been paid to the possibilities and consequences of the extra dimensions. In a five-dimensional manifold, the Randall–Sundrum (RS) models [1, 2] not only explained the hierarchy between the Planck mass $M_P$ and the electroweak mass $M_{EW}$ but also predict a small correction to the Newtonian potential. The cornerstone of these models is the warp product between the brane and the transverse manifold to the brane.

In six dimensions, these warped models are even more rich. Indeed, the two-dimensional transverse space has its own geometry, curvature and symmetries. For instance, a cylindrical symmetric space exterior to a string is conical with angular deficit proportional to the tension of the string [3–8]. On this matter, many authors contributed in recent years. Cohen and Kaplan proposed a warped solution between a flat brane and a cylinder produced by a global string [9]. This solution has a singularity far from the brane, what produces an effective compactification.
Gregory [10] has found a regular and stable solution allowing a time dependence on the bulk geometry and adding a negative cosmological constant. Olasagasti and Vilenkin [3] studied a large class of exterior solutions produced by a global defect for all values of bulk and brane cosmological constant. On the other hand, Oda [11] has extended the warped product between a \((p - 1)\)-brane and a hypersphere, whereas Carlos–Moreno proposed a warped solution which converges asymptotically to a cylinder, the so-called cigar-like universe [12].

In Gherghetta–Shaposhnikov (GS) model [8], an infinite thin string-like brane is embedded in a six-dimensional static and axisymmetric bulk. In opposite to RS models, where the brane cosmological constant must be adjusted to the bulk cosmological constant, the hierarchy problem is solved using only the ratio between the bulk cosmological constant and the string tension. Moreover, the massless and massive graviton modes are localized in that thin brane and it is possible to resolve the boundary singularity by string methods and apply the AdS–CFT correspondence [13]. However, as pointed out by Tinyakov and Zuleta [14], the thin limit of the string-like brane yields a source breaking the dominant energy condition.

Some authors proposed different solutions that do not possess this issue. Giovannini–Meyer–Shaposhnikov (GMS) proposed a string-like braneworld built from an Abelian vortex. They founded by numerical means a static, regular and complete (interior and exterior) solution satisfying the dominant energy condition [15]. Bostok et al [17] and Kofinas [18, 19] proposed string-like defects with higher order corrections. Kanno and Soda [20], Navaro et al [21, 22], Papantonopoulos [23, 24], Cline et al [25] and Vinet and Cline [26] proposed different thick string-like models and they studied mainly the cosmological consequences of these scenarios.

The models described above have two common features: the brane has an azimuthal symmetry about to the transverse space and the brane is generated by fields with fixed physical parameters. In six dimensions, the bulk is built from a warped product between the brane and a disc. Once the exterior geometry reflects the physical properties of the string source, we argue that a geometric flow in the transverse space can be understood as a result of variation of physical parameters of the string.

In this spirit, we propose a complete, regular and static string-like solution. Instead of the disc, we have taken the warped product between the brane and the Hamilton cigar soliton, and we called the resulting solution as a string cigar. The Hamilton cigar is a two-dimensional regular manifold conformal to the disc and asymptotically flat [27]. This space is called a cigar due its cylindrical symmetry whose radius vanishes at the origin and it converges asymptotically to a (cigar-shaped) cylinder [28]. The name Hamilton soliton is due to this manifold being a self-similar solution of the Ricci flow equation, first discovered by Hamilton [29]. In another context, the Hamilton cigar soliton was also studied by Witten as a target space in string theory [30].

The Ricci flow is a geometric flow of a manifold, where its metric evolves with some dimensionless parameter, according to a diffusion partial differential equation [27–29, 31–37]. This equation is used to study the final manifold that some initial manifold can reach evolving under this equation. Ricci flow was first proposed by Hamilton [31] in order to prove some the Poincaré conjecture [33, 36, 37].

The Ricci flow has also applications in physics [38]. In sigma models, the flow represents a first-order approximation to the renormalization flow in the target space [39–42]. In Euclidean gravity, the Ricci flow can be used to prove the existence of black hole solutions [43]. Moreover, since the Arnowitt–Deser–Misner (ADM) mass is invariant under Ricci flow, this flux can be used to prove some statements about the asymptotic behavior of the curved space-times [44]. In the three-dimensional topological gravity, the Ricci flow is useful to study the decay of massive gravitons [45].
The choice of the Hamilton cigar as the transverse space yields an interesting extension of the GS model. On one hand, the string-cigar solution is defined inside and outside the string defect. This enables us to study correction of GS model due to the effects near the brane and inside the core of the string. For instance, this thick string solution, unlike the GS model, satisfies the dominant energy condition. Besides, none of the string tensions is zero inside the core and the relationships between them, which define the Tolman mass and the angular deficit, depend on the string width.

On the other hand, since the cigar soliton depends on an evolution parameter, we have a parameterized solution reflecting the changes of some physical parameters. This allows us to study the effects that variations of the sources has on the geometric and physical quantities. In five-dimensional braneworlds, parameter-dependent domain walls are obtained by means of the deformed potentials [46, 47]. In six dimensions, we proposed an exterior string solution evolving under a resolution flow, where the parameter comes from the 2-cycle of the resolved conifold as the transverse space, a well-known smoothed orbifold of string theory [48–53].

Furthermore, the massless gravitational mode is normalizable and the KK modes differ from the GS model only near the brane. Moreover, the relationship between the Planck masses also evolves upon the geometric flow. This could result in some interesting particle physics effects. In fact, the hierarchy between the weak and the Planck scale would depend on the geometrical changes of the extra dimensions.

The task of find complete string solutions is hard since it requires to solve a system of coupled differential equations for the interior and the exterior region and then to match them [7, 9, 4, 6]. The known solutions have been performed numerically [15, 10, 12]. Therefore, the use of a Ricci flow solution as a transverse space can also be a useful tool to obtain new solutions (analytically or numerically) and to study their stability.

The analysis of the effects on the braneworlds due to non-standard transverse manifolds has already been addressed in the literature. Indeed, Randjbar-Daemi and Shaposhnikov have assumed the transverse manifold as a Ricci-flat or a homogeneous space and they obtained trapped massless gravitational modes and chiral fermions as well [54]. Kehagias proposed a conical tear-drop whose conical singularity drains the vacuum energy to the transverse space, explaining the small value of the cosmological constant [55]. Furthermore, Garriga and Porrati have shown the absence of self-tuning in a compact six-dimensional braneworld taking a football-shaped transverse space [56]. Gogberashvili et al have achieved three-generation for fermions on a 3-brane whose transverse space has the shape of an apple [57]. In the analysis of fermions, Duan et al proposed the torus as a transverse manifold [58]. Some authors have also studied the field behavior for a transverse manifold as the smoothed versions of the conifold, the resolved [53] and deformed [59–61].

Another non-standard solution is the so-called cigar-like universe where the transverse space has a cylindrical shape whose radius shrinks as we move toward the brane [12]. The cigar-like universe interpolates between an interior solution (with conical behavior) and an exterior solution (AdS$_5 \times S^1$). The string-cigar solution is also defined inside the core of the string-like brane, but it converges to an AdS$_6$ space far from the brane.

The rest of the paper is organized as follows. In section 2, we have made a review of the definition, basic properties, important solution and physical application of the Ricci flow. Furthermore, we defined the Hamilton cigar and sketched its basic properties. In section 3, we built the bulk geometry and we studied the main geometrical properties. Furthermore, we analyze the Einstein equation, the corresponding stress–energy–momentum components and string tension, and we studied how the hierarchy between mass scales changes with $k$. In section 4, we studied the behavior of the massless and massive gravitational modes upon this flow. In section 5, some conclusions, final remarks and perspectives are outlined.
2. The Ricci flow and the Hamilton cigar soliton

In this section, we discuss about the definition and the main geometrical and physical properties of the so-called Ricci flow and of one of its solution, i.e. the Hamilton cigar soliton.

Let \((M_D, g_\lambda)\) be a \(D\)-dimensional manifold with a Riemannian metric \(g_\lambda\), \(\lambda \in \mathbb{R}\). For each \(\lambda\), the manifold \(M_D\) has a distinct local geometry. Suppose it could pass continuously from one configuration to another through the partial differential equation:

\[
\frac{\partial g_{ab}(\lambda)}{\partial \lambda} = -2R_{ab}(\lambda). 
\]

This geometric flow is called the Ricci flow due to the Ricci tensor being responsible for driving the ‘evolution’ of metric tensor.

The self-similarity is a key property of Ricci flow. Indeed, consider a scale transformation on the metric tensor [28]

\[
\tilde{g}(x, \lambda) = \xi g\left(x, \frac{\lambda}{\xi}\right).
\]

Since \(\frac{\partial \tilde{g}}{\partial \lambda} = \frac{\partial g}{\partial \lambda}\) and \(\tilde{R}_{ij} = R_{ij}\), the Ricci flow equation is unaltered through scale transformation. Scale invariance provides a relation between Ricci flow and RG flow [39, 33, 36, 28, 38].

Another important feature of the Ricci flow is that it tends to shrink manifolds of positive scalar curvature and stretches those of negative curvature. Indeed, for a sphere,

\[
R_{ij} = \left(\frac{n}{2} - 1\right)g_{ij}. 
\]

Under the Ricci flow, the metric evolves as [27, 28]

\[
g(\lambda) = (1 - 2(n - 1)\lambda)g_0(S^n). 
\]

Since for \(\lambda_f = \frac{1}{2(n-1)} \Rightarrow g(\lambda_f) = 0\), the sphere is contracted to a point in a finite time. For the hyperbolic space \(H^n\), an Einstein space of negative curvature whose Ricci tensor is given by \(R_{ij} = -(n-1)g_{ij}\), the space expands infinitely.

An extension of Einstein spaces are the so-called gradient Ricci solitons that satisfy the equation [27, 28, 34, 36, 37]

\[
R_{ij} + \nabla_i \nabla_j f = \xi g_{ij}, 
\]

where the function \(f\) is called the potential of Ricci flow.

There are three different kinds of gradient Ricci solitons. For \(\xi = 0\), the solutions are called steady; if \(\xi < 0\), we have an expanding Ricci soliton, and for \(\xi > 0\), we have a shrinking soliton. In this paper, we focus on a special steady solution called the Hamilton cigar soliton [27, 28].

The steady solitons have a remarkable property of being extremes of the Perelman energy. Indeed, the Perelman energy functional is defined as [27, 28, 36, 37]

\[
\mathcal{F}(g, f) = \int_M \left( R + |\nabla f|^2 \right) e^{-f} \sqrt{g} \, d^nx. 
\]

Note that the Perelman energy functional is the Euclidean version of the low-energy supergravity action for the gravitational and dilaton fields [50, 36].

The Hamilton cigar \(C_2\) is a two-dimensional steady solution of the Ricci flow equation (5), where

\[
ds_\lambda^2 = \frac{1}{(e^{2\lambda} + r^2)}(dr^2 + r^2 d\theta^2), 
\]

and \(r \in [0, \infty), \theta \in [0, 2\pi], \lambda \in (-\infty, \infty)\) [27–29, 35].
The metric (7) above defines a family of conformal metrics to the disc. Using a new variable defined by $r = e^{2\lambda} \sinh \rho$, the metric (7) yields [27, 35]
\[ ds^2 = d\rho^2 + \tanh^2 \rho \ d\theta^2. \] (8)

The metric (8) has been studied by Witten as a target space metric on sigma models [30].

The scalar curvature of cigar soliton is
\[ R = 4 \text{sech}^2 \rho = \frac{e^{4\lambda}}{r^2 + e^{4\lambda}}. \] (9)

Therefore, that manifold has an everywhere non-negative scalar curvature and it converges to a cylinder asymptotically, and that is why it is called a cigar.

It is worthwhile to mention that the metric (7) is dimensionless. In order to leave it with its right dimension, \[ g \] = $L^2$, we introduced the constants $R_0$ and $c$, with the dimension \[ R_0 = c = L \], yielding to
\[ ds^2 = \frac{R_0^2}{(c^2 e^{4\lambda} + r^2)} (dr^2 + r^2 d\theta^2). \] (10)

Now, let us make the replacement $c e^{2\lambda} = a$. (11)

Since $\lambda \in [-\infty, \infty] \Rightarrow a \in [c, \infty)$, and hence, $a$ has the dimension \[ a = L \]. As $c$ can be any non-zero real number, the parameter $a$ is defined in the range (0, $\infty$). The parameter $a$ defines a family of two-dimensional manifolds and the variation of $a$ defines a flux from one to another.

In order to leave the metric (10) in a simpler form, let us make the change of variable
\[ r = a \sinh k \rho, \] (12)
where $k = \frac{1}{R_0}$. Since the flow has only one free parameter, we must relate $a$ with $k$. Thus, let us choose $k = \frac{1}{a} \Rightarrow R_0 = a$. Hereinafter, we shall call $k$ as the evolution parameter.

The change (12) leaves the metric (10) to the neatly form
\[ ds^2 = \frac{1}{k^2} \tanh^2 k \rho \ d\theta^2. \] (13)

Henceforward, we use (13) as the cigar soliton metric.

3. Bulk geometry

Once described the geometry of the Hamilton cigar which we will use as a transverse space, let us build a six-dimensional bulk $\mathcal{M}_6$ of the form $\mathcal{M}_6 = \mathcal{M}_4 \times C_2$, where $C_2$ is the Hamilton cigar described in the last section and the $\mathcal{M}_4$ is a 3-brane embedded in $\mathcal{M}_6$.

The action for the gravitational field minimally coupled with some matter source is
\[ S_g = \int_{\mathcal{M}_6} \left( \frac{1}{2K_6} R - \Lambda + \mathcal{L}_m \right) \sqrt{-g} \ d^6x, \] (14)
where $K_6 = \frac{8\pi}{M_6^4}$ and $M_6$ is the six-dimensional bulk Planck mass. Note that in this convention, the bulk cosmological constant $\Lambda$ has the dimension \[ \Lambda = L^{-6} = M_6^4. \]

Now, let us propose the following warped metric between the 3-brane and the Hamilton cigar, namely [8, 15, 11, 3, 12]
\[ ds^2_6 = \sigma(\rho, k) \hat{g}_{\mu\nu}(x^i) \ dx^\mu \ dx^\nu + d\rho^2 + \gamma(\rho, k) \ d\theta^2, \] (15)
where
\[ \sigma(\rho, k) = e^{-(k \rho + \tanh k \rho)}. \] (16)
and

\[ \gamma(\rho, k) = \frac{1}{k^2} \left( \tanh k\rho \right)^2 \sigma(\rho, k). \]  

This ansatz has a cylindrical symmetry about the 3-brane that lies at the point \( r = 0 \). Furthermore, this metric represents a space-time inside and outside a string-like defect. Indeed, the warp factors satisfy the usual string-like conditions for regularity at the origin \([8, 15, 14, 6]\), namely

\[ \sigma(0) = 1, \sigma'(0) = 0, \]  
\[ \gamma(0) = 0, (\sqrt{\gamma})'(0) = 1, \]  

where the prime (') denotes the derivative according to \( \rho \) variable. On the other hand, since \( \lim_{\rho \to \infty} \tanh \rho = 1 \), the ansatz asymptotically goes to the string-like exterior solution \([3, 8, 10, 12, 14, 15]\).

The advantage of the metric ansatz proposed in (16), and (17) is that it provides geometric information for points inside the core of the string defect, near and far from the brane. For core of the string, we understand the region close to the string where the stress–energy–momentum tensor is more intense while the far region is the vacuum. Since the stress–energy–momentum tensor decays slowly, there is an intermediary region, called near the string, where the stress–energy–momentum tensor interpolate between those values. As pointed out by many authors \([4–7]\), the geometrical and physical properties are quite different on each region.

Close to the origin (in the core), the warp factor and the angular metric component have a \( Z_2 \) symmetry. It is worthwhile to say that the angular component has a conical behavior near the brane and decays exponentially far from the brane.

The warp factor has the same behavior of another axial complete solution called cigar-like universe \([12]\) (see figure 1). However, the angular metric components agree only in the core of the string, where \( \gamma(\rho) \approx \rho \) (see figure 2). Far from the brane, the cigar-like geometry approaches to a cylinder whereas the string-cigar angular component tends to zero.

The scalar curvature for the metric ansatz (15) is

\[
R = \frac{\hat{R}}{\sigma} - \left[ 4 \left( \frac{\sigma'}{\sigma} \right)' + 5 \left( \frac{\sigma'}{\sigma} \right)^2 + \left( \frac{\gamma'}{\gamma} \right)' + \frac{1}{2} \left( \frac{\gamma'}{\gamma} \right)^2 + 2 \frac{\sigma'}{\sigma} \frac{\gamma'}{\gamma} \right] \\
= \frac{\hat{R}}{\sigma} - k^2 \left[ 15 \tanh^4 k\rho - 16 \tanh k\rho \text{sech}^2(k\rho) - 4\text{sech}^2(k\rho) \right].
\]  

Figure 1. Warp factor. For the string-cigar (thick line) and for the cigar-like universe (dashed line).
The scalar curvature is sketched in figure 3 for $\hat{R} = 0$. Even though the angular metric factor shrinks the radius of the transverse circle, the space-time has no conical singularity at the origin. Inside the core of the string, the space-time has a positive scalar curvature that increases until it reaches a value and then decreases. Near the brane, the curvature turns to be negative and still decreasing. In the far region, the scalar curvature reaches a constant and negative value what signals that the space-time is asymptotically an $\text{AdS}_6$ manifold. Besides, as larger the value of parameter $k$, the larger the value of the curvature near and far from the brane. The dependence of the scalar curvature on the parameter $k$ is related with the $k$-dependence on the cosmological constant, as we will see in the next section.

The geometric properties of the string-cigar solution is similar to the vortex solution found in [15]. Indeed, the metric components, scalar curvature and Ricci tensor have the same behavior.
3.1. Einstein equations

In this section, we study some physical aspects of string-cigar geometry, e.g., the components of stress–energy–momentum tensor, the value of the cosmological constant and the string tensions through the Einstein equation.

First of all, let us assume an axial symmetry ansatz for the stress–energy–momentum tensor [8–11, 14–16]

\[ T^\mu_{\nu} = t_0(r) \delta^\mu_{\nu}, \]  
(21)

\[ T^r_r = t_r(r), \]  
(22)

\[ T^\theta_{\theta} = t_\theta(r), \]  
(23)

where

\[ T_{ab} = \frac{2}{\sqrt{-g}} \frac{\partial L_m}{\partial g^{ab}}. \]  
(24)

Although we shall not propose a specific Lagrangian for the source, it is noteworthy to say that the ansatz of the stress–energy–momentum tensor above

\[ T_{ab} = T_a^c g_{cb}, \]  
(25)

comprises the class of Lagrangians for fields minimally coupled to gravity [16]. Besides, the stress–energy–momentum form for a string-like defect satisfies this ansatz [3, 7–10, 14, 15]

The Einstein equations are

\[ R_{ab} g_{ab} - \frac{1}{2} R g_{ab} = -K_6 (\Lambda_1 g_{ab} + T_{ab}), \]  
(26)

which for the metric ansatz in equation (15) yield the system of coupled differential equations [8, 15, 14]

\[ \frac{3}{2} \left( \frac{\sigma'}{\sigma} \right)' + \frac{3}{2} \left( \frac{\sigma'}{\sigma} \right)^2 + \frac{3}{4} \frac{\sigma'}{\sigma} \frac{\gamma'}{\gamma} + \frac{1}{4} \left( \frac{\gamma'}{\gamma} \right)^2 + \frac{1}{2} \left( \frac{\gamma'}{\gamma} \right)' = -K_6 (\Lambda + t_0(\rho)) + \frac{K_4 \Lambda_4}{\sigma}, \]  
(27)

\[ \frac{3}{2} \left( \frac{\sigma'}{\sigma} \right)^2 + \frac{\sigma'}{\sigma} \frac{\gamma'}{\gamma} = -K_6 (\Lambda + t_0(\rho)) + \frac{2K_4 \Lambda_4}{\sigma}, \]  
(28)

\[ 2 \left( \frac{\sigma'}{\sigma} \right)' + \frac{5}{2} \left( \frac{\sigma'}{\sigma} \right)^2 = -K_6 (\Lambda + t_0(\rho)) + \frac{2K_4 \Lambda_4}{\sigma}, \]  
(29)

where we have used the prime (‘) to represent the derivative \( \frac{d}{d\rho} \). Furthermore, the four-cosmological constant on the 3-brane satisfies

\[ \hat{\mathcal{R}}_{\mu\nu} - \frac{\hat{R}_{\mu\nu}}{2} = -K_4 \Lambda_4 \hat{g}_{\mu\nu}, \]  
(30)

where \( K_4 = \frac{8\pi}{M_4^4} \). However, we have chosen a metric ansatz, where \( \lim_{r \to \infty} \sigma (\rho) = 0 \), and then, the Einstein equations will blow up at infinity. Some authors solved this question by allowing a width to the brane and modifying the warp functions [20–24, 26] or adding higher order terms [17–19]. Here, following the flat-brane configurations [8, 15, 14, 13], henceforward, we set \( \Lambda_4 = 0 \).
Using the metric ansatz (15), we have found the components of the stress–energy–
momentum tensor, namely

\[ t_0(\rho, k) = \frac{k^2}{K_6} \left( 7 \text{sech}^2 k \rho + \frac{13}{2} \text{sech}^2 k \rho \tanh k \rho - \frac{5}{2} \text{sech}^4 k \rho \right), \]  

(31)

\[ t_\rho(\rho, k) = \frac{k^2}{K_6} \left( 5 \text{sech}^2 k \rho + 2 \text{sech}^2 k \rho \tanh k \rho - \frac{5}{2} \text{sech}^4 k \rho \right), \]  

(32)

\[ t_\theta(\rho, k) = \frac{k^2}{K_6} \left( 5 \text{sech}^2 k \rho + 4 \text{sech}^2 k \rho \tanh k \rho - \frac{5}{2} \text{sech}^4 k \rho \right). \]  

(33)

These functions were plotted in figure 4. It is worthwhile to say that all the components have compact support near the origin where the 3-brane is placed. Furthermore, the components satisfy the dominant, strong and weak energy conditions that turn this geometry into an extension of the GS model.

Besides, from the graphic, we can define the core, near and far zones. Indeed, for \( \rho > 3 \), the components of energy tensor are almost zero. Then, we can define this region as the vacuum (far zone). For \( \rho \approx 1.5 \), all the components reach half of their maximum value, and then, for \( 0 \leq \rho \leq 1.5 \), we define a zone as the core of the string defect with width \( \epsilon \approx 1.5 \). The intermediary region is the so-called near zone. Note that for \( \rho \approx 1.5 \), the angular metric component, the scalar curvature has a change in its behavior.

It is worthwhile to mention that the behavior of the energy density \( t_0 \) in the core of the string-cigar solution is similar to the one found in [15] for a vortex solution with the winding number \( n = 2 \). Indeed, this component grows from the origin, reaches a maximum and then it decays. On the other hand, the radial and angular components, unlike the solution in [15], behave as the energy density.

The difference between the string-cigar solution and the vortex model [15] can have many origins, among them: the string-cigar can be a solution for an Abelian Einstein–Maxwell–Higgs model with a higher winding number or a different symmetry breaking potential; furthermore, we argue that as in the cigar-universe solution, it can be a solution for a supersymmetric theory [12].
Since the components of energy–momentum tensor depend on $k$, the maximum value of these functions also depends on the evolution parameter. Then, the geometric flow alters the width of the string core $\epsilon$.

Another consequence of this geometry is that the bulk has a negative and parameter-dependent cosmological constant. Indeed, constant $k$ is related with the cosmological constant of the bulk by the well-known relation [8, 11]

$$k^2 = -\frac{2K_0}{5}\Lambda \Rightarrow \Lambda = \Lambda(k).$$

Therefore, the geometric flow represents a variation of the bulk cosmological constant. This is an expected result since the asymptotic value of scalar curvature depends on the evolution parameter $k$ and $R = 3\Lambda$.

### 3.2. String tensions

In this subsection, we studied the effects of the geometric flow has on the string tensions. We also have shown that these quantities depend on the string width.

We defined the 4-tension per unit of volume of the 3-brane as [8, 11, 14, 15]

$$\mu_k(k) = \int_0^{\infty} t_i(\rho, k)\sqrt{\gamma(\rho, k)}d\rho$$

$$= \frac{1}{k} \int_0^{\infty} t_i(\rho, k) e^{-\frac{1}{2}(k+\tanh k)}\tanh k\rho d\rho.$$

Note the string tensions are all finite and smooth and have compact support around the brane, as expected. Moreover, the higher the value of the $k$, the more located are the tensions. This indicates that the string width $\epsilon$ varies with $k$.

It is worthwhile to mention here that the geometric flow is related to a variation of physical quantities, as the string tensions. Therefore, let us see how the string-cigar solution alters the Tolman mass and the string angular deficit computed in the GS model. Once the stress–energy–momentum tensor vanishes smoothly, we shall consider the integration range $[0, \epsilon_0]$, where $\epsilon_0$ is the width of the core. From the Einstein equations, we obtain

$$\left(\sigma \sqrt{\gamma}\right)'_0 = -\frac{1}{M_0^4} \left( \frac{1}{2} (\mu_\rho + \mu_\phi) + \frac{\Lambda}{2} \int_0^{\epsilon_0} \sigma^2 \sqrt{\gamma} d\rho - 4\Lambda_4 \int_0^{\epsilon_0} \sigma \sqrt{\gamma} d\rho \right)$$

and

$$\sigma^2 (\sqrt{\gamma})'_0 = -\frac{1}{M_0^4} \left( \left( \mu_\rho + \frac{1}{4} \mu_\phi - 3\Lambda_4 \right) + \Lambda \int_0^{\epsilon_0} \sigma^2 \sqrt{\gamma} d\rho - \Lambda_4 \int_0^{\epsilon_0} \sigma \sqrt{\gamma} d\rho \right).$$

The term $\mu_\rho + \mu_\phi$ is known in the literature as the Tolman mass of the string [4, 8, 14]. Thus, equation (36) tell us that the Tolman mass evolves with the evolution parameter $k$ and it depends on the bulk and string cosmological constant. Since equation (37) depends on the derivative of the angular metric component, it provides a measure of the angular deficit of the exterior space-time surrounding the string [4, 8, 14]. Then, the angular deficit also depends on the cosmological constant terms.

From equations (36) and (37), we obtain the relation

$$\mu_0 - \mu_\phi = M_0^4 \tan^3 (k\epsilon) \sigma^2 (\epsilon).$$

The relationship (38) is analogous to that in the GS model [8]. Nevertheless, in string-cigar solution, the relation depends on the cosmological constant ($k$) and on the string width $\epsilon$.

These consequences are due the nonzero value of the core width and they are valid even for the flat-brane $\Lambda_4 = 0$. Indeed, for the GS model, i.e. for the thin string limit $\epsilon \rightarrow 0$, the geometric flow alters the width of the string core $\epsilon$. Another consequence of this geometry is that the bulk has a negative and parameter-dependent cosmological constant. Indeed, constant $k$ is related with the cosmological constant of the bulk by the well-known relation [8, 11]
the terms of cosmological constant drop out of equations (36) and (37). In this limit, we do not need to tune the brane cosmological constant to the bulk cosmological constant, as in RS models [1, 2], although, as shown in [14], the thin string limit contradicts the dominant energy condition.

3.3. Mass hierarchy

In this subsection, we have seen that string tensions, including the string mass, depend on the evolution parameter $k$. In this section, we have analyzed how the geometric evolution alters the relation between the bulk and the brane mass scale.

In this geometry, the relationship between the four-dimensional Planck mass ($M_4$) and the bulk Planck mass ($M_6$) is given by

$$M_4^2 = \frac{2}{\pi} M_6^4 \int_0^\infty \sigma(\rho, k) \sqrt{\gamma(\rho, k)} d\rho.$$  \hfill (39)

Since all the metric components are limited functions,

$$\int_0^\infty e^{-\frac{1}{2}(k \rho - \tanh k \rho)} \tanh k \rho d\rho \approx \frac{1}{k}.$$  \hfill (40)

Then, $M_4^2 \approx \frac{2}{\pi} M_6^4 \frac{1}{k}$. Hence, this geometry can be used to tune the ratio between the Planck masses, explaining the hierarchy between them.

Note, however, that the relationship between the Planck masses depends on the evolution parameter $k$. Hence, a evolution of the bulk geometry could alter the hierarchy between the fundamental forces. For large value of $k$, the brane Planck mass is smaller than the bulk Planck mass. In order to obtain $M_4 \gg M_6$, we must do $k \rightarrow 0$. This is an extension of the GS tuning of the Planck masses [8, 11] for points inside and near the core. Indeed, for points outside the string core, i.e. for $\rho \rightarrow \infty$, the hyperbolic function $\tanh k \rho \approx 1$, and we obtain the GS mass tuning [8].

4. Gravity localization

Now let us study the localization of small perturbations of the background metric around a flat brane in the geometry analyzed so far. The perturbation is such that

$$ds_6^2 = \sigma(\rho, k)(\eta_{\mu\nu} + h_{\mu\nu}(x^\zeta, \rho, \theta, k)) dx^\mu dx^\nu + d\rho^2 + \gamma(\rho, k) d\theta^2.$$  \hfill (41)

Using the traceless transverse gauge

$$h^\mu_\nu = \nabla_\mu h^{\mu\nu} = 0.$$  \hfill (42)

the linearization of the Einstein equations (26) with the source given by equation (21) yields a decoupled equation for the gravitational perturbation, namely [9, 8, 15, 16]

$$\Box h_{\mu\nu} = \partial_\alpha(\sqrt{-g} g^{\alpha\beta} \partial_\beta h_{\mu\nu}) = 0.$$  \hfill (43)

Let us assume that the symmetric tensorial field $h_{\mu\nu}$ is a product of a four-component field with the Poincaré symmetry on the 3-brane $\hat{h}_{\mu\nu}$ and another scalar field living only in the transverse space, the well-known process called Kaluza–Klein (KK) decomposition [8–11, 15]

$$h_{\mu\nu}(x^\xi, \rho, \theta) = \hat{h}_{\mu\nu}(x^\xi) \hat{\phi}(\rho, \theta).$$  \hfill (44)
From the Poincaré symmetry, the tensorial field on the 3-brane $\hat{h}_{\mu\nu}$ must satisfy the mass condition
\[
\Box_4 \hat{h}_{\mu\nu}(x^i) = -m^2 \hat{h}_{\mu\nu}(x^i). \tag{45}
\]
Since $0 \leq \theta \leq 2\pi$, let us assume that $\hat{\phi}(\rho, \theta)$ can be expanded in the Fourier series as
\[
\hat{\phi}(\rho, \theta) = \chi(\rho) \sum_{l=0}^{\infty} e^{il\theta}. \tag{46}
\]
Using the ansatz (46), equation (43) yields the differential equation for the transverse component
\[
(\sigma^2 \sqrt{\beta} \chi'(r))' + \sigma^2 \sqrt{\beta} \left( m^2 - \frac{l^2}{\beta} \right) \chi(\rho) = 0, \tag{47}
\]
where $\beta(\rho, k) = \tanh^2 k \rho/k^2$.
Equation (47) is a Sturm–Liouville-like equation. Furthermore, let us looking for solutions that satisfy the boundary conditions [8]
\[
\chi'(0) = \lim_{\rho \to \infty} \chi'(\rho) = 0. \tag{48}
\]
Given two solutions of equation (47), namely $\chi_i(\rho)$ and $\chi_j(\rho)$, the orthogonality relation between them is given by
\[
\int_0^\infty \sigma(\rho, a) \frac{3}{2} \sqrt{\beta(\rho, k)} \chi_i^* \chi_j d\rho = \delta_{ij}. \tag{49}
\]
We can rewrite equation (47) as
\[
\chi''(\rho) + \left( \frac{5}{2} \sigma' + \frac{1}{2} \frac{\beta'}{\beta} \right) \chi'(\rho) + \frac{1}{\sigma} \left( m^2 - \frac{l^2}{\beta} \right) \chi(\rho) = 0. \tag{50}
\]
Note that equation (50) is similar to that found in string-like geometries [8, 11], regardless the cigar term $\beta(\rho, k)$. Besides, we could define an effective angular number $l_{\text{eff}} = l^2/\beta(\rho, k)$ which would depends on the distance from the brane and on the resolution parameter.

4.1. Massless mode
For $m = 0$, a constant function is a solution of equation (50). Thus, from orthogonality relation (49), we can define the zero mode as
\[
\chi_0(\rho, k) = N \sigma(\rho, k)^{\frac{3}{2}} \left( \frac{\tanh k \rho}{k} \right)^{\frac{1}{2}}, \tag{51}
\]
where
\[
N^2 = \int_0^\infty \sigma(\rho, k)^{\frac{3}{2}} \left( \frac{\tanh k \rho}{k} \right)^{\frac{1}{2}} d\rho. \tag{52}
\]
Since the massless mode has compact support, as can be seen in figure 5, it is normalizable, and then, we claim that the gravitational field is trapped around the string-like 3-brane.
4.2. Massive modes

Now let us study the KK modes of equation (50). Using the expressions for the metric factor yields

\[ \chi'' + \left( -\frac{5k}{2} + k \text{sech}^2(k\rho) \left( \frac{5}{2} + \frac{2}{\tanh k\rho} \right) \right) \chi' + e^{(k\rho - \tanh k\rho)} \left( m^2 - \frac{l^2 k^2}{\tanh^2 k\rho} \right) \chi = 0. \] (53)

Equation (53) together with the boundary conditions (48) is a complex problem to tackle directly and analytically. Hence, we shall concern ourselves to study this equation in the near and far from the brane regimes.

In the limit \( \rho \to \infty \) (vacuum), equation (53) turns out to be

\[ \chi'' - \frac{5k}{2} \chi' + e^{(k\rho - 1)} (m^2 - l^2) \chi = 0. \] (54)

Equation (54) is analogous to the massive modes equation of GS model [8] with a mass term rescaled as \( m \to e^{\frac{m}{2}} m \). The solution of equation (54) can be written in terms of the Bessel function as

\[ \chi(\rho) = e^{\frac{m'\rho}{k}} \left[ C_1 J_\frac{m'}{k} \left( \frac{2m'}{k} e^{\rho} \right) + C_2 Y_\frac{m'}{k} \left( \frac{2m'}{k} e^{\rho} \right) \right]. \] (55)

where \( m' = \frac{m}{e^{\frac{m}{2}}} \). Therefore, we argue that, for asymptotic points, the KK spectrum is similar to the GS model [8].

In the limit \( \rho \to 0 \), i.e. for points close to the brane, let us make the change of variable \( x : [0, \infty) \to [0, 1] \) given by

\[ x = \tanh k\rho, \]

\[ y(x) = \chi(\rho). \] (57)
This transformation allows that equation (53) can be expressed as

\[ y''(x) + \frac{4x^2 - 5x^3 - 8x^2 + 4}{2x(1 - x^2)} y'(x) = - \frac{e^{(-x + \tanh^{-1} x)}}{(1 - x^2)^2} \mu^2 y(x). \]  

(58)

Let us look for the solution of equation (58) in a Taylor series form

\[ y(x) = y(0) + y'(0)x + \frac{y''(0)}{2} x^2 + \frac{y'''(0)}{3!} x^3 + \frac{y^{(4)}(0)}{4!} x^4 + \frac{y^{(5)}(0)}{5!} x^5 + \mathcal{O}(x^6). \]  

(59)

From the boundary conditions expressed in equation (48), we obtain \( y'(0) = 0 \). Substituting the resulting series into the differential equation (58) and retaining only terms up to \( \mathcal{O}(x^4) \), we obtain the approximated solution

\[ y(x) = y(0) \left( 1 - \frac{\mu^2}{6} x^2 + \frac{\mu^2(\mu^2 - 12)}{120} x^4 - \frac{7\mu^2}{180} x^5 \right). \]  

(60)

Hence, the massive mode is smooth in the core and near the brane.

4.3. Analogue quantum potential

Another way to study the massive modes lies in putting equation (50) into a Schrödinger-like equation and study its analogue quantum potential [9].

First, let us perform the change of variable \( z = z(\rho) \), namely

\[ z = z(\rho) = \int^\rho \sigma^{-1/2} d\rho'. \]  

(61)

Furthermore, let us write \( \chi(z) \) in the form

\[ \chi(z) = u(z) \Psi(z). \]  

(62)

Making

\[ \frac{\dot{u}}{u} = -\frac{1}{2} \left( \frac{\dot{\sigma}}{\sigma} + \frac{1}{2} \frac{\dot{\beta}}{\beta} \right), \]  

(63)

the \( \Psi(z) \) function must obey

\[ -\ddot{\Psi}(z) + U(z) \Psi(z) = m^2 \Psi(z), \]  

(64)

where

\[ U(z) = \frac{\dot{\sigma}}{\sigma} + \frac{1}{2} \frac{\dot{\beta}}{\beta} - \frac{3}{16} \left( \frac{\dot{\beta}}{\beta} \right)^2 + \frac{1}{4} \frac{\beta''}{\beta} + \frac{l^2}{\beta}. \]  

(65)

Equation (64) is a time-independent Schrödinger-like equation. We can study the localization of the scalar field by analyzing the behavior of the potential around a potential well. Returning to the \( \rho \) coordinate, the potential can be written as

\[ U(\rho, k, l) = \frac{\sigma''}{\sigma} + \frac{1}{2} \left( \frac{\sigma'}{\sigma} \right)^2 + \frac{\sigma'}{\sigma} \left( \frac{5}{8} \frac{\beta'}{\beta} \right)^2 - \frac{3}{16} \left( \frac{\beta'}{\beta} \right)^2 + \frac{1}{4} \frac{\beta''}{\beta} + \frac{l^2}{\beta}. \]  

(66)

\[ = k^2 e^{-k(\rho - \tanh k \rho)} \left( \frac{3}{2} \tanh^2 k \rho - \frac{9}{4} \text{sech}^2 k \rho \tanh k \rho - \frac{1}{4} \text{sech}^4 k \rho - \text{sech}^2 k \rho \right) \]  

\[ + (kl)^2 \frac{1}{\tanh^2 k \rho}. \]  

(67)

The Schrödinger potential is plotted in figure 6 for \( l = 0 \). It is worthwhile to mention that there is a potential well and a barrier around the origin where the 3-brane is present. Therefore, there are massive modes trapped in the brane.
5. Conclusions and perspectives

In this work, we have found an interior and an exterior gravitational solution for a string-like brane in six dimensions. The metric was build from a warped product between a 3-brane and a two-dimensional manifold called the Hamilton cigar soliton. Since the cigar soliton is a parameter-dependent manifold with axial symmetry, the solution enable us to study the effects of variations of string-like brane configurations have on the physical quantities of the braneworld scenario, as the gravitational field, the relationship between the masses scales and so on, without breaking its own string-like axial symmetry. Furthermore, since the solution is valid for both interior and exterior regions to the string-like brane, it provides an improvement on the analysis of the behavior of the fields inside and near the brane.

The bulk possesses a cosmological constant depending on the cigar parameter, also called the evolution parameter. Thus, the bulk converges asymptotically to a constant curvature manifold whose scalar curvature depends on the evolution parameter. Besides, the stress–energy–momentum tensor also depends on the evolution parameter, satisfying the weak, strong and dominant energy condition for any value of $k$.

Another physical consequence is the parametrization of the relationship between the mass scales of bulk and brane. In order to take $M_4 \gg M_6$, $k$ must be taken quite small. However, allowing $k$ to vary keeping $M_6$ fixed, the brane mass scale would vary which leads to both particle physics and cosmological consequences.

The differences between the string-cigar solution and the GS model arise due to the former being a thick brane solution, while the latter a thin solution. As a result, the relationships between the string tensions, which define the Tolman mass and the angular deficit, depend on the bulk cosmological constant and on the width of the core.

Due to the resemblance to the GMS solution, we argue that the string cigar should be a solution of some vortex model, e.g., a non-Abelian EMH model. Another possibility is that this geometry could be generated by a non-minimal coupling between fields in a modified
gravity theory, as currently generated in cosmology. The deduction of this geometry from a matter Lagrangian should be a future step in order to complete the model.

We have also studied the gravitational perturbations on this scenario. Since the volume of the transverse manifold is finite, the massless mode is trapped in the 3-brane for any value of the evolution parameter. In the interior and exterior regions but close to the brane, the higher the value of the evolution parameter, the higher the amplitude of the massless mode. For the KK modes, the eigenfunction converges to the usual vacuum solution of the GS model written in terms of the Bessel functions, and then, it is trapped to a brane up to a cutoff. The KK spectrum also depends on the evolution parameter since the KK mass depends on the cosmological constant. Nevertheless, the KK spectrum shifts for a same amount since it depends only on the cosmological constant. The localization of the massive modes can also be inferred from the shape of the analogous Schrödinger-like potential. Indeed, there is a potential well and a potential barrier around the brane. Hence, there are massive modes trapped to the brane due the potential barrier. As a perspective, the spectrum of mass can be achieved numerically from this potential by means of the resonances method.

This paper opens new perspectives to develop. We showed that steady geometric flows on the transverse manifold yields a variation of the relation between the bulk and the brane masses. We argue that, whether this scenario be created by an Einstein–Maxwell–Higgs vortex, this mass flow can be linked with the Higgs and vector masses, since these are the mass content of the vortex. Therefore, for the transverse space evolving under a general Ricci flow, it would be interesting to investigate what physical quantities of the brane would change. For other fields living in that geometry, we could study the effects that the evolution parameter has on their properties, as well the behavior of these fields on the region inside and near the string-like brane. Indeed, we could find for fermions, for instance, some geometrical Yukawa potential or some different fermion generations. Since the brane has parameter-dependent tensions, another possibility would be of studying the cosmological consequences of the geometric flow on the brane. Besides, the string cigar extends the cigar-like universe due to the presence of the cosmological constant. Then, a useful study could deduce the string cigar from a Kähler potential, as for the cigar universe. Another interesting feature to be addressed is the gravitational modes behavior of sources non-minimally coupled to gravity, as in the braiding models [62], where the source interferes with the perturbation.

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