A BROWNIAN MOTION ON $\text{Diff}(S^1)$

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Abstract. Let $\text{Diff}(S^1)$ be the group of orientation preserving $C^\infty$ diffeomorphisms of $S^1$. In [13] P. Malliavin and then in [5] S. Fang constructed a canonical Brownian motion associated with the $H^{3/2}$ metric on the Lie algebra $\text{diff}(S^1)$. The canonical Brownian motion they constructed lives in the group Homeo$(S^1)$ of Hölderian homeomorphisms of $S^1$, which is larger than the group $\text{Diff}(S^1)$. In this paper, we present another way to construct a Brownian motion that lives in the group $\text{Diff}(S^1)$, rather than in the larger group Homeo$(S^1)$.

1. Introduction

Let $\text{Diff}(S^1)$ be the group of orientation preserving $C^\infty$-diffeomorphisms of $S^1$, and let $\text{diff}(S^1)$ be the space of $C^\infty$-vector fields on $S^1$. The space $\text{diff}(S^1)$ can be identified with the space of $C^\infty$-functions on $S^1$. Therefore, $\text{diff}(S^1)$ carries a natural Fréchet space structure. In addition $\text{diff}(S^1)$ is an infinite dimensional Lie algebra: for any $f, g \in \text{diff}(S^1)$, the Lie bracket is given by $[f, g] = f'g - fg'$. Thus the group $\text{Diff}(S^1)$ associated with the Lie algebra $\text{diff}(S^1)$ becomes an infinite dimensional Fréchet Lie group [14]. Our goal in this paper is to construct a Brownian motion in the group $\text{Diff}(S^1)$.

In general, to construct a Brownian motion in a Lie group, one might solve a Stratonovich stochastic differential equation (SDE) on such a group. The method is best illustrated for a finite dimensional compact Lie group.

Let $G$ be a finite dimensional compact Lie group. Denote by $\mathfrak{g}$ the Lie algebra of $G$ identified with the tangent space $T_eG$ to the group $G$ at the identity element $e \in G$. Let $L_g : G \to G$ be the left translation of $G$ by an element $g \in G$, and let $(L_g)_* : \mathfrak{g} \to T_gG$ be the differential of $L_g$. If we choose a metric on $\mathfrak{g}$ and let $W_t$ be the standard Brownian motion on $\mathfrak{g}$ corresponding to this metric, we can develop the Brownian motion $W_t$ onto $G$ by solving a Stratonovich stochastic differential equation

$$\delta \tilde{X}_t = (L_{\tilde{X}_t})_* \delta W_t$$

where $\delta$ stands for the Stratonovich differential. The solution $\tilde{X}_t$ is a Markov process on $G$ with the generator being the Laplace operator on $G$. We call $\tilde{X}_t$ the Brownian motion on the group $G$ [11, 12].

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In case when $G$ is an infinite dimensional Hilbert Lie group, one can solve Equation (1.1) by using the theory of stochastic differential equations in Hilbert spaces as developed by G. DaPrato and J. Zabczyk in [4]. Using this method, M. Gordina [6, 7, 8] and M. Wu [10] constructed a Brownian motion in several Hilbert-Schmidt groups. The construction relied on the fact that these Hilbert-Schmidt groups are Hilbert Lie groups.

In the present case, we would like to replace $G$ by Diff $(S^1)$ and $g$ by diff $(S^1)$ and solve Equation (1.1) correspondingly. But because the group Diff $(S^1)$ is a Fréchet Lie group, which is not a Hilbert Lie group, Equation (1.1) does not even make sense as it stands. First, we need to interpret the Brownian motion $W_t$ in the Fréchet space diff $(S^1)$ appropriately. Second, we are lacking a well developed stochastic differential equation theory in Fréchet spaces to make sense of Equation (1.1).

In 1999, P. Malliavin [13] first constructed a canonical Brownian motion on Homeo $(S^1)$, the group of Hölderian homeomorphisms of $S^1$. In 2002, S. Fang [5] gave a detailed construction of this canonical Brownian motion on the group Homeo $(S^1)$. Their constructions were essentially by interpreting and solving the same Equation (1.1) on the group Diff $(S^1)$.

To define the Brownian motion $W_t$ in Equation (1.1), P. Malliavin and S. Fang chose the $H^{3/2}$ metric of the Lie algebra diff $(S^1)$. Basically, this metric uses the set

$$\{n^{-3/2} \cos(n\theta), m^{-3/2} \sin(m\theta)|m, n = 1, 2, 3, \cdots\}, \quad (1.2)$$

which is a subset of the Lie algebra diff $(S^1)$, as an orthonormal basis to form a Hilbert space $H^{3/2}$. Then they defined $W_t$ to be the cylindrical Brownian motion in $H^{3/2}$ with the covariance operator being the identity operator on $H^{3/2}$. But since the coefficients $n^{-3/2}$ and $m^{-3/2}$ do not decrease rapidly enough, the Hilbert space $H^{3/2}$ is not contained in the Lie algebra diff $(S^1)$. Therefore, the Brownian motion $W_t$ they defined on $H^{3/2}$ does not live in diff $(S^1)$ either. This is the essential reason why the canonical Brownian motion they constructed lives in a larger group Homeo $(S^1)$, but not in the group Diff $(S^1)$.

To interpret and solve Equation (1.1), S. Fang treated it as a family of stochastic differential equations on $S^1$: for each $\theta \in S^1$, S. Fang considered the equation

$$\delta \tilde{X}_{\theta,t} = (L_{\tilde{X}_{\theta,t}}) \cdot \delta W_{\theta,t}, \quad (1.3)$$

which is a stochastic differential equation on $S^1$. By solving Equation (1.3) for each $\theta \in S^1$, S. Fang obtained a family of solutions $\tilde{X}_{\theta,t}$ parameterized by $\theta$. Then he used a Kolmogorov type argument to show that the family $\tilde{X}_{\theta,t}$ is Hölderian continuous in the variable $\theta$. Using this method, he proved that for each $t \geq 0$, $\tilde{X}_{\theta,t}$ is a Hölderian homeomorphism of $S^1$. Thus, he constructed the canonical Brownian motion on the group Homeo $(S^1)$. But this Kolmogorov type argument cannot be pushed further to show that $\tilde{X}_{\theta,t}$ is differentiable in $\theta$. Therefore, S. Fang’s method does not seem to be suitable to construct a Brownian motion that lives in the group Diff $(S^1)$, rather than in Homeo $(S^1)$.
In the current paper, our goal is to construct a Brownian motion that lives in the group $\text{Diff}(S^1)$. To achieve this, we need another way to interpret and solve Equation (1.1).

First, instead of the $H^{3/2}$ metric that P. Malliavin and S. Fang used, we choose a very “strong” metric on the Lie algebra $\text{diff}(S^1)$: let $\{\lambda(n)\}_{n=1}^{\infty}$ be a sequence of rapidly decreasing positive numbers. We use the set

$$\{\lambda(n) \cos(n\theta), \lambda(m) \sin(m\theta)|m, n = 1, 2, 3, \cdots\},$$

which is a subset of the Lie algebra $\text{diff}(S^1)$, as an orthonormal basis to form a Hilbert space $H_{\lambda}$. Then we define the Brownian motion $W_t$ to be the cylindrical Brownian motion in $H_{\lambda}$ with the covariance operator being the identity operator on $H_{\lambda}$. Because the coefficients $\lambda(n)$ are rapidly decreasing, the Hilbert space $H_{\lambda}$ is a subspace of the Lie algebra $\text{diff}(S^1)$. Therefore, the Brownian motion $W_t$ lives in the Lie algebra $\text{diff}(S^1)$, and the solution of Equation (1.1) has a better chance to live in the group $\text{Diff}(S^1)$.

Second, in contrast to Fang’s method of interpreting Equation (1.1) “pointwise” as a family of stochastic differential equations on $S^1$, we interpreted it as a sequence of stochastic differential equations on a sequence of “Hilbert” spaces. To do this, we embed the group $\text{Diff}(S^1)$ into an affine space $\widetilde{\text{diff}}(S^1)$ that is isomorphic to the Lie algebra $\text{diff}(S^1)$. Let $H^k$ be the $k$th Sobolev space over $S^1$. It is a separable Hilbert space. Let $\widetilde{H}^k$ be the corresponding affine space that is isomorphic to $H^k$. For the precise definition of the space $\text{diff}(S^1)$ and $\widetilde{H}^k$, see Section 2. It is well known that the space $\text{diff}(S^1)$ is the intersection of the Sobolev spaces $H^k$. Similarly, $\widetilde{\text{diff}}(S^1)$ is the intersection of the affine spaces $\widetilde{H}^k$. Now we have the embedding

$$\text{Diff}(S^1) \subseteq \widetilde{\text{diff}}(S^1) \subseteq \widetilde{H}^k, \quad k = 1, 2, 3, \cdots$$

Thus, we can interpret Equation (1.1) as a sequence of stochastic differential equations on the sequence of affine spaces $\{\widetilde{H}^k\}_{k=1}^{\infty}$ each of which is isomorphic to the Hilbert space $H^k$. These stochastic differential equations can be solved by DaPrato and Zabczyk’s method [4].

In accordance with the notations used by DaPrato and Zabczyk in [4], in the rest of this paper, we will denote the operator $(L_{\widetilde{X}})_*$ in Equation (1.1) by $\widetilde{\Phi}(\widetilde{X}_t)$. The operator $\widetilde{\Phi}$ will be discussed in detail in Section 2. After adding the initial condition, we can now re-write Equation (1.1) as

$$\delta \widetilde{X}_t = \widetilde{\Phi}(\widetilde{X}_t) \delta W_t, \quad \widetilde{X}_0 = id$$

where $id$ is the identity element in $\text{Diff}(S^1)$.

Equation (1.6) is interpreted as a stochastic differential equation in each “Hilbert” space $\widetilde{H}^k$. To use DaPrato and Zabczyk’s method to solve this equation, we need to establish the Lipschitz condition of the operator $\widetilde{\Phi}$. In Section 2, it turns out that the operator $\widetilde{\Phi}$ is locally Lipschitz. So the explosion time of the solution needs to be discussed.

After solving Equation (1.6) in $\widetilde{H}^k$ for each $k$, it is relatively easy to prove that the solution lives in the affine space $\text{diff}(S^1)$ (Proposition 3.17). By the embedding
In general, to prove a process lives in a group rather than in an ambient space, one needs to construct an inverse process. To construct the inverse process, usually one needs to solve another stochastic differential equation – the SDE for the inverse process \[6, 10\]. In our case, we have derived the SDE for the inverse process:

\[
\delta \tilde{Y}_t = \tilde{\Psi}(\tilde{Y}_t) \delta W_t
\] (1.7)

where \(\tilde{\Psi}\) is an operator such that for \(\tilde{g} \in \text{Diff}(S^1)\) and \(f \in \text{diff}(S^1)\), \(\tilde{\Psi}(\tilde{g})f = D\tilde{g} \cdot f\), where \(D = d/d\theta\) and \(\cdot\) is the pointwise multiplication of two functions. Because the operator \(D\) causes loss of one degree of smoothness, we cannot interpret Equation (1.7) in \(\widetilde{H}^k\) as we did for Equation (1.6), and we were forced to give up this method.

But we managed to get around this problem. We first observed that an element \(\tilde{f} \in \text{diff}(S^1)\) belongs to Diff\((S^1)\) if and only if \(\tilde{f}'(\theta) > 0\) for all \(\theta \in S^1\). Based on this observation, we showed that the solution is contained in the group Diff\((S^1)\) up to a stopping time. Then we can “concatenate” this small piece of solution with another small piece of solution to make a new solution up to a longer stopping time. The key idea is Proposition (3.14) and the following remark (Remark 3.15). Finally, we were able to prove the following theorem (Theorem 3.19):

**Theorem 1.1.** There is a unique \(\widetilde{H}^k\)-valued solution with continuous sample paths to Equation (1.6) for all \(k = 0, 1, 2, \cdots\). Furthermore, the solution is non-explosive and lives in the group Diff\((S^1)\).

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**2. An interpretation of Equation (1.6)**

**2.1. The group Diff\((S^1)\) and the Lie algebra diff\((S^1)\).** Let Diff\((S^1)\) be the group of orientation preserving \(C^\infty\) diffeomorphisms of \(S^1\), and diff\((S^1)\) be the space of \(C^\infty\) vector fields on \(S^1\). We have the following identifications for the space diff\((S^1)\):

\[
diff(S^1) \cong \{ f : S^1 \to \mathbb{R} : f \in C^\infty \}
\]

\[
\cong \{ f : \mathbb{R} \to \mathbb{R} : f \in C^\infty, f(x) = f(x + 2\pi), \text{ for all } x \in \mathbb{R} \}
\] (2.1)

Using this identification, we see that the space diff\((S^1)\) has a Fréchet space structure. In addition, this space has a Lie algebra structure, namely, for \(f, g \in \text{diff}(S^1)\) the Lie bracket is given by

\[
[f, g] = f'g - fg',
\] (2.2)

where \(f'\) and \(g'\) are derivatives with respect to the variable \(\theta \in S^1\). Therefore, the group Diff\((S^1)\) is a Fréchet Lie group as defined in [13].
Notation 2.1. Using the above identification, we also have an identification for $\text{Diff}(S^1)$

$$\text{Diff}(S^1) \cong \{ \hat{f} : \mathbb{R} \to \mathbb{R} : \hat{f} = \text{id} + f, f \in \text{diff}(S^1), \hat{f}' > 0\}, \quad (2.3)$$

where $\text{id}$ is the identity function from $\mathbb{R}$ to $\mathbb{R}$. We note that the set on the right hand side of the above identification is a group with the group multiplication being composition of functions. We require that for $\hat{f}, \hat{g} \in \text{Diff}(S^1)$, $f\hat{g} = \hat{g} \circ \hat{f}$. Under this identification, the left translation of $\text{Diff}(S^1)$ is given by $L_\hat{g} \hat{f} = \hat{g} \circ \hat{f}$.

Denote

$$\tilde{\text{diff}}(S^1) = \{ \hat{f} : \mathbb{R} \to \mathbb{R} | \hat{f} = \text{id} + f, f \in \text{diff}(S^1)\} \quad (2.4)$$

The space $\tilde{\text{diff}}(S^1)$ is an affine space which is isomorphic to the vector space $\text{diff}(S^1)$. We denote the isomorphism by $\sim$, that is, $\sim : \text{diff}(S^1) \to \tilde{\text{diff}}(S^1)$, $f \mapsto \hat{f} = \text{id} + f$. Comparing (2.3) and (2.4), we have the embedding

$$\text{Diff}(S^1) \subseteq \tilde{\text{diff}}(S^1). \quad (2.5)$$

With this embedding, the differential of a left translation $L_\hat{g}$ becomes $(L_\hat{g})_* : \text{diff}(S^1) \to \tilde{\text{diff}}(S^1)$, and is given by $(L_\hat{g})_* f = f \circ \hat{g}$ for $f \in \text{diff}(S^1)$.

The following proposition is an immediate observation from the identification (2.3) and definition of $\text{diff}(S^1)$ given by (2.4). Yet, it plays a key role in proving the main theorem Theorem 1.1.

Proposition 2.2. An element $\hat{f} \in \tilde{\text{diff}}(S^1)$ belongs to $\text{Diff}(S^1)$ if and only if $\hat{f}' > 0$, or equivalently $f' > -1$.

2.2. The Hilbert space $H_\lambda$ and the Brownian motion $W_t$. To define the Brownian motion $W_t$ in Equation (1.4), we need to choose a metric on the Lie algebra $\text{diff}(S^1)$. Comparing with the $H^{3/2}$ metric that P. Malliavin and S. Fang chose, the metric we choose here is a very “strong” metric.

Definition 2.3. Let $S$ be the set of even functions $\lambda : \mathbb{Z} \to (0, \infty)$ such that $\lim_{n \to \infty} |n|^k \lambda(n) = 0$ for all $k \in \mathbb{N}$. For $\lambda \in S$, let $e_n^\lambda = e_n^\lambda(\lambda) \in \text{diff}(S^1)$ be defined by

$$e_n^\lambda(\theta) = \begin{cases} \lambda(n) \cos(n \theta), & n \geq 0 \\ \lambda(n) \sin(n \theta), & n < 0 \end{cases} \quad (2.6)$$

Let $H_\lambda$ be the Hilbert space with the set $\{e_n^\lambda\}_{n \in \mathbb{Z}}$ as an orthonormal basis.

Note that the function $\lambda$ is rapidly decreasing, therefore the Hilbert space $H_\lambda$ defined above is a proper subspace of $\text{diff}(S^1)$. We also remark that $\text{diff}(S^1) = \bigcup_{\lambda \in S} H_\lambda$.

Let $\alpha, \lambda \in S$ be defined by $\lambda(n) = |n| \alpha(n)$, and let $H_\alpha$ and $H_\lambda$ be the corresponding Hilbert subspaces of $\text{diff}(S^1)$. Then we have $H_\alpha \subset H_\lambda$, and the inclusion map $\iota : H_\alpha \hookrightarrow H_\lambda$ that sends $e_n^{\alpha(\lambda)}$ to $e_n^{\alpha} = \frac{1}{|n|} e_n^\alpha$ is a Hilbert-Schmidt operator. The adjoint operator $\iota^* : H_\lambda \to H_\alpha$ that sends $e_n^{\lambda}$ to $\frac{1}{|n|} e_n^{\alpha}$ is also a Hilbert-Schmidt operator. The operator $Q_\lambda = \iota^* : H_\lambda \to H_\lambda$ is a trace class operator on $H_\lambda$, and $H_\alpha = Q_\lambda^{1/2} H_\lambda$. 
Definition 2.4. Let \( W_t \) be a Brownian motion defined by

\[
W_t = \sum_{n \in \mathbb{Z}} B_{t}^{(n)} \hat{e}_n = \sum_{n \in \mathbb{Z}} \frac{1}{|n|} B_{t}^{(n)} \hat{e}_n
\] (2.7)

where \( \{B_{t}^{(n)}\}_n \) are mutually independent standard \( \mathbb{R} \)-valued Brownian motions.

We see that \( W_t \) is a cylindrical Brownian motion on \( H_\alpha \) with the covariance operator being the identity operator on \( H_\alpha \). Also, \( W_t \) is a Brownian motion on \( H_\lambda \) with the covariance operator being the operator \( Q_\lambda \).

2.3. The Sobolev space \( H^k \) and the affine space \( \tilde{H}^k \). Now we turn to the Sobolev spaces over \( S^1 \). Let us first recall some basic properties of the Sobolev spaces over \( S^1 \) found for example in [1].

Let \( k \) be a non-negative integer. Denote by \( C^k \) the space of \( k \)-times continuously differentiable real-valued functions on \( S^1 \), and denote by \( H^k \) the \( k \)th Sobolev space on \( S^1 \). Recall that \( H^k \) consists of functions \( f : S^1 \to \mathbb{R} \) such that \( f^{(k)} \in L^2 \), where \( f^{(k)} \) is the \( k \)th derivative of \( f \) in distributional sense. The Sobolev space \( H^k \) has a norm given by

\[
\|f\|_{H^k}^2 = \|f\|_{L^2}^2 + \|f^{(k)}\|_{L^2}^2
\] (2.8)

The Sobolev space \( H^k \) is a separable Hilbert space, and \( C^k \) is a dense subspace of \( H^k \). We will make use of the following standard properties of the spaces \( H^k \).

Theorem 2.5 ([1]). Let \( m, k \) be two non-negative integers.

1. If \( m \leq k \) and \( f \in H^k \), then \( \|f\|_{H^m} \leq \|f\|_{H^k} \).
2. If \( m < k \) and \( f \in H^k \), then there exists a constant \( c_k \) such that \( \|f^{(m)}\|_{L^\infty} \leq c_k \|f\|_{H^k} \).
3. \( H^{k+1} \subseteq H^k \) for all \( k = 0, 1, 2, \ldots \), and \( \text{diff}(S^1) = \bigcap_{k=0}^\infty H^k \).

An element \( f \in H^k \) can be identified with a \( 2\pi \)-periodic function from \( \mathbb{R} \) to \( \mathbb{R} \). Define

\[
\tilde{H}^k = \{ \tilde{f} : \mathbb{R} \to \mathbb{R} : \tilde{f} = \text{id} + f, f \in H^k \}
\] (2.9)

Then \( \tilde{H}^k \) is an affine space that is isomorphic to the Sobolev space \( H^k \). We denote the isomorphism by \( \sim \), that is, \( \sim : H^k \to \tilde{H}^k, f \mapsto \tilde{f} = \text{id} + f \). The image of \( C^k \) under the isomorphism, denoted by \( \tilde{C}^k \), is a dense subspace of the affine space \( \tilde{H}^k \). An element \( \tilde{f} \in \tilde{H}^k \) can be identified as a function from \( S^1 \) to \( S^1 \). By item (3) in Theorem 2.5 we have \( \tilde{H}^{k+1} \subseteq \tilde{H}^k \) and \( \text{diff}(S^1) = \bigcap_{k=0}^\infty \tilde{H}^k \).

Now we have the following embeddings:

\[
\text{Diff}(S^1) \subseteq \text{diff}(S^1) \subseteq \cdots \subseteq \tilde{H}^3 \subseteq \tilde{H}^2 \subseteq \tilde{H}^1,
\] (2.10)

and we can interpret Equation (1.6) as a sequence of stochastic differential equations on the sequence of affine spaces \( \{\tilde{H}^k\}_{k=1}^\infty \).
2.4. The operator $\tilde{\Phi}$ and $\Phi$. For $\tilde{g} \in \text{Diff}(S^1)$, let $(L_{\tilde{g}})_*$ be the differential of the left translation. In accordance with the notation used by DaPrato and Zabczyk in [11], we denote $(L_{\tilde{g}})_*$ by $\tilde{\Phi}(\tilde{g})$.

Initially, $\tilde{\Phi} : \text{Diff}(S^1) \to (\text{Diff}(S^1) \to \text{Diff}(S^1))$, which means $\tilde{\Phi}$ takes an element $\tilde{g} \in \text{Diff}(S^1)$ and becomes a linear transformation $\tilde{\Phi}(\tilde{g})$ from $\text{Diff}(S^1)$ to $\text{Diff}(S^1)$ (see subsection 2.1). Because we want to interpret Equation (1.6) as an SDE on $H$ and use DaPrato and Zabczyk's theory [4], we need the operator $\Phi$ to be extended as $\tilde{\Phi} : \tilde{H}^k \to (H_{\lambda} \to \tilde{H}^k)$, which means $\tilde{\Phi}$ takes an element $\tilde{g} \in \tilde{H}^k$ and becomes a linear transformation $\tilde{\Phi}(\tilde{g})$ from $H_{\lambda}$ to $\tilde{H}^k$ [4].

Let $L(H_{\lambda}, H^k)$ be the space of linear transformations from $H_{\lambda}$ to $H^k$. Define a mapping

$$\tilde{\Phi} : \tilde{C}^k \to L(H_{\lambda}, H^k)$$

(2.11)

such that if $\tilde{f} \in \tilde{C}^k$, $g \in H_{\lambda}$, then $\tilde{\Phi}(\tilde{f})(g) = g \circ \tilde{f}$. The mapping $\tilde{\Phi}$ is easily seen to be well defined. Sometimes, it is easier to work with the vector space $C^k$. So we similarly define a mapping

$$\Phi : C^k \to L(H_{\lambda}, H^k)$$

(2.12)

such that if $f \in C^k$, $g \in H_{\lambda}$, then $\Phi(f)(g) = g \circ \tilde{f}$, where $\tilde{f} = id + f$ is the image of $f$ under the isomorphism $\sim$.

Let $L^2(H_{\lambda}, H^k)$ denote the space of Hilbert-Schmidt operators from $H_{\lambda}$ to $H^k$. The space $L^2(H_{\lambda}, H^k)$ is a separable Hilbert space. For $T \in L^2(H_{\lambda}, H^k)$, the norm of $T$ is given by

$$\|T\|_{L^2(H_{\lambda}, H^k)} = \sum_{n \in \mathbb{Z}} \|T e_n^{(\lambda)}\|_{H^k}^2$$

where $e_n^{(\lambda)}$ is defined in Definition (2.3).

To use DaPrato and Zabczyk’s theory [4], we need $\Phi$ to be $\tilde{\Phi} : \tilde{H}^k \to L^2(H_{\lambda}, H^k)$ or equivalently, we need $\Phi$ to be $\Phi : H^k \to L^2(H_{\lambda}, H^k)$. We will also need some Lipschitz condition of $\tilde{\Phi}$ and $\Phi$. These are proved in proposition (2.7) and (2.8). Both propositions need the Faà di Bruno’s formula for higher derivatives of a composition function.

**Theorem 2.6** (Faà di Bruno’s formula [11]).

$$f(g(x))^{(n)} = \sum_{k=0}^{n} f^{(k)}(g(x)) B_{n,k}(g'(x), g''(x), \ldots, g^{(n-k+1)}(x)), \quad (2.13)$$

where $B_{n,k}$ is the Bell polynomial

$$B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum_{j_1! \cdots j_{n-k+1}!} \frac{n!}{j_1! \cdots j_{n-k+1}!} \left(\frac{x_{j_1}}{1!}\right)^{j_1} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}},$$

and the summation is taken over all sequences of $\{j_1, \ldots, j_{n-k+1}\}$ of nonnegative integers such that $j_1 + \cdots + j_{n-k+1} = k$ and $j_1 + 2j_2 + \cdots + (n-k+1)j_{n-k+1} = n$.

We remark that after expanding expression (2.13), $f(g(x))^{(n)}$ can be viewed as a summation of several terms, each of which has the form

$$f^{(j)}(g(x))m(g', g'', \ldots, g^{(n)}).$$
where \( j \leq n \) and \( m(g', g'', \cdots, g^{(n)}) \) is a monomial in \( g', g'', \cdots, g^{(n)} \). Also observe that, the only term that involves the highest derivative of \( g \) is \( f'(g(x))g^{(n)}(x) \).

**Proposition 2.7.** For any \( f \in C^k, k = 0, 1, 2, \cdots \), \( \Phi(f) \in L^2(H, H^k) \).

**Proof.**

\[
\|\Phi(f)\|_{L^2(H, H^k)}^2 = \sum_{n \in \mathbb{Z}} \|\Phi(f)(\hat{e}_n)\|_{H^k}^2 \\
= \sum_{n \in \mathbb{Z}} \|\hat{e}_n(id + f)\|_{L^2}^2 + \|\hat{e}_n(id + f)^{(k)}\|_{L^2}^2,
\]

where \( \hat{e}_n \) is defined in Definition 2.3 and we have suppressed the index \( \lambda \) here. \( \hat{e}_n(id + f) \) denotes the function \( \hat{e}_n \) composed with \( id + f \), and \( \hat{e}_n(id + f)^{(k)} \) is the \( k \)th derivative of \( \hat{e}_n(id + f) \).

First, we have

\[
\|\hat{e}_n(id + f)\|_{L^2}^2 \leq \lambda(n)^2.
\]

We apply Faà di Bruno’s formula (2.13) to \( \hat{e}_n(id + f)^{(k)} \), and then expand it to a summation of several terms. We are going to deal with the terms with and without \( f^{(k)} \), the highest derivative of \( f \), separately. So we write the summation as

\[
\hat{e}_n(id + f)^{(k)} = \ldots \text{ terms without } f^{(k)}\ldots + \hat{e}_n(id + f)f^{(k)},
\]

where each term without \( f^{(k)} \) has the form

\[
\hat{e}_n^{(j)}(id + f)m(f', f'', \cdots, f^{(k-1)})
\]

with \( j \leq k \) and \( m(f', f'', \cdots, f^{(k-1)}) \) a monomial in \( f', f'', \cdots, f^{(k-1)} \). Let \( d \) be the degree of the monomial \( m(f', f'', \cdots, f^{(k-1)}) \). Then from Faà di Bruno’s formula we see that \( d \leq k \) for all monomials.

By Definition 2.3 of \( \hat{e}_n \) and using item 4 in Theorem 2.5 we have

\[
\|\hat{e}_n^{(j)}(id + f)m(f', f'', \cdots, f^{(k-1)})\|_{L^2} \\
\leq \|\hat{e}_n^{(j)}(id + f)\|_{L^\infty}\|m(f', f'', \cdots, f^{(k-1)})\|_{L^\infty} \\
\leq \lambda(n)|n|^k c_k^m \|f\|_{H^k}^k.
\]

For the last term in expression (2.14), we have

\[
\|\hat{e}_n'(id + f)f^{(k)}\|_{L^2} \leq \|\hat{e}_n'(id + f)\|_{L^\infty}\|f^{(k)}\|_{L^2} \\
\leq \lambda(n)|n|\|f\|_{H^k} \leq \lambda(n)|n|^k c_k^m \|f\|_{H^k}^m.
\]

By (2.15) and (2.16), we have

\[
\|\hat{e}_n(id + f)^{(k)}\|_{L^2}^2 \leq K\lambda(n)^2|n|^{2k}c_k^{2k} \|f\|_{H^k}^{2k},
\]

where \( K \) is the number of terms in expression (2.14), which depends on \( k \) but does not depend on \( n \). Therefore,

\[
\|\Phi(f)\|_{L^2(H, H^k)}^2 \leq \sum_{n \in \mathbb{Z}} (\lambda(n)^2 + K\lambda(n)^2|n|^{2k}c_k^{2k} \|f\|_{H^k}^{2k})
\]
Because $\lambda(n)$ is rapidly decreasing (Definition 2.3), $\sum_{n \in \mathbb{Z}} \lambda(n)^2|n|^{2k} < \infty$. Therefore, we have

$$\|\Phi(f)\|_{L^2(H_\lambda, H^k)}^2 < \infty$$

Now $\Phi$ can be viewed as a mapping $\Phi : C^k \to L^2(H_\lambda, H^k)$. Similarly, $\tilde{\Phi}$ can be viewed as a mapping $\tilde{\Phi} : \tilde{C}^k \to L^2(H_\lambda, H^k)$. To use DaPrato and Zabczyk’s theory [4], we will need the Lipschitz condition of $\Phi$ and $\tilde{\Phi}$. It turns out that they are locally Lipschitz. Let us recall the concept of local Lipschitzness: Let $A$ and $B$ be two normed linear spaces with norm $\| \cdot \|_A$ and $\| \cdot \|_B$ respectively. A mapping $f : A \to B$ is said to be locally Lipschitz if for $R > 0$, and $x, y \in A$ such that $\|x\|, \|y\| \leq R$, we have

$$\|f(x) - f(y)\|_B \leq C_R \|x - y\|_A,$$

where $C_R$ is a constant which in general depends on $N$.

**Proposition 2.8.** For any $k = 0, 1, 2, \cdots$, $\Phi : C^k \to L^2(H_\lambda, H^k)$ is locally Lipschitz.

**Proof.** Let $R > 0$, and $f, g \in C^k$ be such that $\|f\|_{H^k}, \|g\|_{H^k} \leq R$. We have

$$\|\Phi(f) - \Phi(g)\|_{L^2(H_\lambda, H^k)}^2 = \sum_{n \in \mathbb{Z}} \|\Phi(f) - \Phi(g)\|^2_{H_\lambda} = \sum_{n \in \mathbb{Z}} \|\hat{e}_n(id + f) - \hat{e}_n(id + g)\|^2_{H_\lambda}$$

$$= \sum_{n \in \mathbb{Z}} \|\hat{e}_n(id + f) - \hat{e}_n(id + g)\|^2_{L^2} + \|\hat{e}_n(id + f)(k) - \hat{e}_n(id + g)(k)\|^2_{L^2},$$

where $\hat{e}_n$ is defined in Definition 2.3 and we have suppressed the index $\lambda$ here. $\hat{e}_n(id + f)$ and $\hat{e}_n(id + g)$ denote the function $\hat{e}_n$ composed with $id + f$ and $id + g$ respectively. $\hat{e}_n(id + f)(k)$ and $\hat{e}_n(id + g)(k)$ are the $k$th derivatives of $\hat{e}_n(id + f)$ and $\hat{e}_n(id + g)$ respectively.

First, by the mean value theorem we have

$$\|\hat{e}_n(id + f) - \hat{e}_n(id + g)\|_{L^2} = \|\hat{e}'_n(id + \xi)(f - g)\|_{L^2} \leq \lambda(n)|\xi| \|f - g\|_{H^k}$$

(2.17)

We apply Faà di Bruno’s formula (2.18) to $\hat{e}_n(id + f)(k)$, and then expand it to a summation of several terms. We are going to deal with the terms with and without $f^{(k)}$, the highest derivative of $f$, separately. So we write the summation as

$$\hat{e}_n(id + f)(k) = \cdots \text{ terms without } f^{(k)} \cdots + \hat{e}_n(id + f)f^{(k)},$$

(2.19)

where each term without $f^{(k)}$ has the form

$$\hat{e}^{(j)}_n(id + f)m(f', f'', \cdots, f^{(k-1)})$$

with $j \leq k$ and $m(f', f'', \cdots, f^{(k-1)})$ a monomial in $f', f'', \cdots, f^{(k-1)}$. Let $d$ be the degree of the monomial $m(f', f'', \cdots, f^{(k-1)})$. Then from Faà di Bruno’s formula we see that $d \leq k$ for all monomials. By replacing $f$ with $g$ in (2.19), we obtain

$$\hat{e}_n(id + g)(k) = \cdots \text{ terms without } g^{(k)} \cdots + \hat{e}_n(id + g)g^{(k)}$$

(2.20)
Next, we need a simple observation: suppose $A_1A_2A_3\ldots$ and $B_1B_2B_3\ldots$ are two monomials with the same number of factors. By telescoping, we can put $A_1A_2A_3\ldots - B_1B_2B_3\ldots$ into the form

$$(A_1 - B_1)A_2A_3\ldots + B_1(A_2 - B_2)A_3\ldots + B_1B_2(A_3 - B_3)\ldots + \cdots$$

Using this observation, we can put $\hat{e}_n(id + f)^{(k)} - \hat{e}_n(id + g)^{(k)}$ into the form

$$\hat{e}_n(id + f)^{(k)} - \hat{e}_n(id + g)^{(k)} = \ldots \text{terms without } f^{(k)} \text{ and } g^{(k)} \ldots$$ (2.21)

$$+ (\hat{e}'_n(id + f) - \hat{e}'_n(id + g)) f^{(k)} + \hat{e}'_n(id + g) \left(f^{(k)} - g^{(k)}\right)$$

In expression (2.21), there are two types of terms without $f^{(k)}$ and $g^{(k)}$. One type has the form

$$\left(\hat{e}'_n(id + f) - \hat{e}'_n(id + g)\right) m_A(f', \ldots, f^{(k-1)}, g', \ldots, g^{(k-1)}), \quad (2.22)$$

where $j \leq k$ and $m_A$ is a monomial in $f', \ldots, f^{(k-1)}, g', \ldots, g^{(k-1)}$. We denote such a term by $A$. Another type has the form

$$\hat{e}'_n(id + g) \left(f^{(j)} - g^{(j)}\right) m_B(f', \ldots, f^{(k-1)}, g', \ldots, g^{(k-1)}) \quad (2.23)$$

where $i, j \leq k$ and $m_B$ is a monomial in $f', \ldots, f^{(k-1)}, g', \ldots, g^{(k-1)}$. We denote such a term by $B$.

Now we want to find an $L^2$ bound of each term in (2.21). For the term $A$, by the mean value theorem we have

$$[\hat{e}'_n(id + f) - \hat{e}'_n(id + g)] = \hat{e}'_n(id + \xi)(f - g).$$

By Definition 2.3 of $\hat{e}_n$, and using Item 1 and 2 in Theorem 2.4 we have

$$\|A\|_{L^2} \leq \|\hat{e}'_n(id + \xi)\|_{L^\infty} \|m_A\|_{L^\infty} \|f - g\|_{L^2} \leq \lambda(n)|n|^{k+1} |c_k| \|f - g\|_{H^k}. \quad (2.24)$$

For the term $B$, we have

$$\|B\|_{L^2} \leq \|\hat{e}'_n(id + g)\|_{L^\infty} \|m_B\|_{L^\infty} \|f^{(j)} - g^{(j)}\|_{L^2} \leq \lambda(n)|n|^{k+1} |c_k| \|f - g\|_{H^k}. \quad (2.25)$$

For the last two terms in expression (2.21), using Item 1 and 2 in Theorem 2.5 again, we have

$$\|\hat{e}'_n(id + f) - \hat{e}'_n(id + g)f^{(k)}\|_{L^2} = \|\hat{e}'_n(id + \xi)(f - g)f^{(k)}\|_{L^2} \leq \|\hat{e}'_n(id + \xi)\|_{L^\infty} \|f - g\|_{L^\infty} \|f^{(k)}\|_{L^2} \leq \lambda(n)|n|^k |c_k| \|f - g\|_{H^k} \quad (2.26)$$

and

$$\|\hat{e}'_n(id + g)[f^{(k)} - g^{(k)}]\|_{L^2} \leq \lambda(n)|n|\|f - g\|_{H^k}. \quad (2.27)$$

By (2.24) - (2.27), we see that $\lambda(n)|n|^{k+1} |c_k| \|f - g\|_{H^k}$ is a common $L^2$ bound for all terms in (2.21). So,

$$\|\hat{e}_n(id + f)^{(k)} - \hat{e}_n(id + g)^{(k)}\|_{L^2} \leq K\lambda(n)|n|^{k+1} |c_k| \|f - g\|_{H^k} \quad (2.28)$$
where $K$ is the number of terms in expression (2.21), which depends on $k$ but does not depend on $n$.

Finally,

$$
\|\Phi(f) - \Phi(g)\|_{L^2(H_\lambda, H^k)}^2 \\
\leq \sum_{n \in \mathbb{Z}} \lambda(n)^2 |n|^2 \|f - g\|^2_{H^k} + K^2 \lambda(n)^2 |n|^{2k+2} c_k^2 R^{2k} \|f - g\|_{H^k}^2
$$

$$
\leq Kc_k^2 R^k \|f - g\|_{H^k} \left( \sum_{n \in \mathbb{Z}} \lambda(n)^2 |n|^{2k+2} \right)^{1/2}
$$

Let

$$
C_R = \left( \sum_{n \in \mathbb{Z}} \lambda(n)^2 |n|^2 + K^2 \lambda(n)^2 |n|^{2k+2} c_k^2 R^{2k} \right)^{1/2},
$$

Because $\lambda(n)$ is rapidly decreasing (Definition 2.9), $\sum_{n \in \mathbb{Z}} \lambda(n)^2 |n|^{2k} < \infty$. So $C_R$ is a finite number that depends on $R$ and $k$. Therefore,

$$
\|\Phi(f) - \Phi(g)\|_{L^2(H_\lambda, H^k)} \leq C_R \|f - g\|_{H^k}. \tag{2.29}
$$

By the above proposition, $\Phi : C^k \to L^2(H_\lambda, H^k)$ is locally Lipschitz. So $\Phi$ is uniformly continuous on $C^k$. But $C^k$ is a dense subspace of $H^k$ (see subsection 2.3). Therefore, we can extend the domain of $\Phi$ from $C^k$ to $H^k$, and obtain a mapping $\Phi : H^k \to L^2(H_\lambda, H^k)$. Similarly, we can also extend the domain of $\bar{\Phi}$ from $\tilde{C}^k$ to $\tilde{H}^k$, and obtain a mapping $\bar{\Phi} : \tilde{H}^k \to L^2(H_\lambda, H^k)$. After extension, $\Phi$ and $\bar{\Phi}$ are still locally Lipschitz.

**Definition 2.9.** Define $\tilde{\Phi} : \tilde{H}^k \to L^2(H_\lambda, H^k)$ to be the extension of $\bar{\Phi} : \tilde{C}^k \to \tilde{H}^k$ to $\tilde{H}^k$, and $\tilde{\Phi} : H^k \to L^2(H_\lambda, H^k)$ to be the extension of $\Phi : C^k \to L^2(H_\lambda, H^k)$ from $C^k$ to $H^k$. By the remark in the previous paragraph, $\Phi$ and $\tilde{\Phi}$ are still locally Lipschitz.

### 3. The main result

In this section, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $\mathcal{F}_t = \{\mathcal{F}_t, t \geq 0\}$ that is right continuous and such that each $\mathcal{F}_t$ is complete with respect to $\mathbb{P}$.

Equation (1.6) is now interpreted as a Stratonovich stochastic differential equation on $\tilde{H}^k$ for each $k = 0, 1, 2, \ldots$. Let us fix such a $k$.

#### 3.1. Changing Equation (1.6) into the Itô form

To solve Equation (1.6), we first need to change it into the Itô form. Here we follow the treatment of S. Fang in [5]. In Definition 2.21, $W_t = \sum_{n \in \mathbb{Z}} B^{(n)}_t e^{(n)}_n$, where $\alpha$ is a rapidly decreasing function as described in Definition 2.9. Using the definition of $\tilde{\Phi}, W_t$, and $e^{(n)}_n$, we can write Equation (1.6) as

$$
\delta \tilde{X}_t = \alpha(0) + \sum_{n=1}^\infty \alpha(n) \cos(n \tilde{X}_t) \delta B^{(n)}_t + \sum_{m=1}^\infty \alpha(m) \sin(m \tilde{X}_t) \delta B^{(m)}_t. \tag{3.1}
$$
Using the stochastic contraction of \( dB^{(n)}_t \cdot dB^{(m)}_t = \delta_{mn} dt \), we have:

\[
\alpha(n) d \cos(n \tilde{X}_t) \cdot dB^{(n)}_t = -\alpha(n)^2 \sin(n \tilde{X}_t) \cos(n \tilde{X}_t) dt \\
\alpha(n) d \sin(m \tilde{X}_t) \cdot dB^{(m)}_t = \alpha(m)^2 \sin(m \tilde{X}_t) \cos(m \tilde{X}_t) dt
\]

So the stochastic contraction of the right hand side of (3.1) is zero. Therefore Equation (3.1) can be written in the following Itô form:

\[
d\tilde{X}_t = \alpha(0) + \sum_{n=1}^{\infty} \alpha(n) \cos(n \tilde{X}_t) dB^{(n)}_t + \sum_{m=1}^{\infty} \alpha(m) \sin(m \tilde{X}_t) dB^{(m)}_t \tag{3.2}
\]

Using the definition of \( W_t \) and \( \tilde{\Phi} \) again, Equation (3.2) becomes

\[
d\tilde{X}_t = \tilde{\Phi}(\tilde{X}_t) dW_t \tag{3.3}
\]

Therefore, Equation (3.3) is equivalent to the following Itô stochastic differential equation

\[
d\tilde{X}_t = \tilde{\Phi}(\tilde{X}_t) dW_t, \quad \tilde{X}_0 = 0 \tag{3.4}
\]

This equation is considered in the affine space \( \tilde{H}^k \).

If we write \( \tilde{X}_t = id + X_t \) with \( X_t \) a process with values in the Sobolev space \( H^k \) and use the definition of \( \Phi \) (see subsection 2.4), Equation (3.4) is equivalent to the following equation

\[
dX_t = \Phi(X_t) dW_t, \quad X_0 = 0 \tag{3.5}
\]

This equation is considered in the Sobolev space \( H^k \).

### 3.2. Truncated stochastic differential equation

By Proposition (2.3) the operator \( \Phi \) is locally Lipschitz. To use G. DaPrato and J. Zabczyk’s theory [4], we need to “truncate” the operator \( \Phi \): Let \( R > 0 \). Let \( \Phi_R : H^k \rightarrow L^2(H_\alpha, H^k) \) be defined by

\[
\Phi_R(x) = \begin{cases} 
\Phi(x), & \|x\|_{H^k} \leq R \\
\Phi(Rx/\|x\|_{H^k}), & \|x\|_{H^k} > R
\end{cases} \tag{3.6}
\]

Then \( \Phi_R \) is globally Lipschitz. Let us consider the following “truncated” stochastic differential equation

\[
dX_t = \Phi_R(X_t) dW_t, \quad X_0 = 0 \tag{3.7}
\]

in the Sobolev space \( H^k \). The following definition is in accordance with G. DaPrato and J. Zabczyk’s treatments (p.182 in [4]).

**Definition 3.1.** Let \( T > 0 \). An \( \mathcal{F}_t \)-adapted \( H^k \)-valued process \( X_t \) with continuous sample paths is said to be a mild solution to Equation (3.7) up to time \( T \) if

\[
\int_0^T \|X_s\|_{H^k}^2 ds < \infty, \quad \mathbb{P}\text{-a.s.}
\]

and for all \( t \in [0, T] \), we have

\[
X_t = X_0 + \int_0^t \Phi_R(X_s) dW_s, \quad \mathbb{P}\text{-a.s.}
\]

For Equation (3.7), a strong solution is the same as a mild solution. The solution \( X_t \) is said to be unique up to time \( T \) if for any other solution \( Y_t \), the two processes
X_t and Y_t are equivalent up to time T, that is, the stopped processes X_{t\wedge T} and Y_{t\wedge T} are equivalent.

**Remark 3.2.** In the above definition, we require a solution to have continuous sample paths.

**Proposition 3.3.** For each $T > 0$, there is a unique solution $X^{(T)}$ to Equation (3.7) up to time $T$.

**Proof.** The proof is a simple application of Theorem 7.4, p.186 from [4]. We need to check the conditions to use Theorem 7.4 from [4]. By definition of $\Phi_R$, we see that $\Phi_R$ satisfies the following growth condition:

$$\|\Phi_R(x)\|_{L^2(H_n, H_k)}^2 \leq C(1 + \|x\|_{H_k}^2), \quad x \in H_k$$

for some constant $C$. All other conditions to use Theorem 7.4 from [4] are easily verified. Therefore, we have the conclusion.

Let us choose a sequence \{T_n\}_{n=1}^{\infty} such that $T_n \uparrow \infty$, and let each $X^{(T_n)}$ be the unique solution to Equation (3.7) up to time $T_n$. By the uniqueness of the solution, and by the continuity of sample paths, for $1 \leq i < j$, the sample paths of $X^{(T_i)}$ coincide with the sample paths of $X^{(T_j)}$ up to time $T_i$ almost surely. To be precise, we have, for almost all $\omega \in \Omega$,

$$X^{(T_i)}(t, \omega) = X^{(T_j)}(t, \omega), \quad \text{for all } t \in [0, T_i]$$

Therefore, we can extend the sample paths to obtain a process $X^R$: For almost all $\omega \in \Omega$, let

$$X^R(t, \omega) = \lim_{n \to \infty} X^{(T_n)}(t, \omega) \quad \text{for all } t \in [0, \infty)$$

Then the process $X^R$ is a unique solution with continuous sample paths to Equation (3.7) up to time $T$ for all $T > 0$.

**Remark 3.4.** The above construction of the process $X^R$ is independent of the choice of the sequence \{T_n\}_{n=1}^{\infty}. Let \{S_n\}_{n=1}^{\infty} be another sequence such that $S_n \uparrow \infty$. Let $Y^R$ be the process constructed as above but using the sequence \{S_n\}_{n=1}^{\infty}. Then $X^R$ and $Y^R$ are equivalent up to $T$ for all $T > 0$. Therefore, they are equivalent.

**Definition 3.5.** For every $R > 0$, we define $X^R$ to be the $H^k$-valued process with continuous sample paths as constructed above. Define

$$\tau_R = \inf\{t : \|X^R(t)\|_{H^k} \geq R\} \quad (3.8)$$

### 3.3. Solutions up to stopping times

Let us consider Equation (3.5) in the Sobolev space $H^k$. The following definition is in accordance with E. Hsu’s treatments in [11].

**Definition 3.6.** Let $\tau$ be an $\mathcal{F}_\tau$-stopping time. An $\mathcal{F}_\tau$-adapted process $X_t$ with continuous sample paths is said to be a solution to Equation (3.5) up to time $\tau$ if for all $t \geq 0$

$$X_{t\wedge \tau} = X_0 + \int_0^{t\wedge \tau} \Phi(X_s)dW_s$$
The solution $X_t$ is said to be unique up to $\tau$ if for any other solution $Y_t$, the two processes $X_t$ and $Y_t$ are equivalent up to $\tau$, that is, the stopped processes $X_{t\wedge\tau}$ and $Y_{t\wedge\tau}$ are equivalent.

**Remark 3.7.** We can similarly define an $\mathcal{H}^k$-valued process being the unique solution to Equation (3.5) up to a stopping time $\tau$. Clearly, we have the following: If $X_t$ is the solution to Equation (3.5) up to a stopping time $\tau$, then the $\mathcal{H}^k$-valued process $\widetilde{X}_t = id + X_t$ is the solution to Equation (3.5) up to time $\tau$ and vice versa.

**Remark 3.8.** If $X_t$ is a solution to Equation (3.5) up to $\tau$, then it is also a solution up to $\sigma$ for any $\mathcal{F}_s$-stopping time $\sigma$ such that $\sigma \leq \tau$ a.s.

**Proposition 3.9.** Let $R > 0$. Let $X^R$ and $\tau_R$ be defined as in Definition (3.5). Then $X^R$ is the unique solution to Equation (3.5) up to $\tau_R$.

**Proof.** Because $X^R$ is the unique solution to Equation (3.7) up to $T$ for all $T > 0$, we have

$$X^R_t = \int_0^t \Phi_R(X^R_s)dW_s$$

for all $t \geq 0$. By the definition of $\Phi_R$, we have $\Phi_R(X^R_s) = \Phi(X^R_s)$ for $s \leq \tau_R$. So,

$$X^R_{t\wedge\tau_R} = \int_0^{t\wedge\tau_R} \Phi_R(X^R_s)dW_s = \int_0^{t\wedge\tau_R} \Phi(X^R_s)dW_s$$

Therefore, $X^R$ is a solution to Equation (3.5) up to $\tau_R$.

Suppose $Y_t$ is another solution to Equation (3.5) up to $\tau_R$. Then $Y_t$ is also a solution to Equation (3.7) up to $\tau_R$. But $X^R_t$ is the unique solution to Equation (3.7) up to $T$ for all $T > 0$. Therefore, $Y_t$ and $X^R_t$ are equivalent up to $\tau_R$. $\square$

Let us choose a sequence $\{R_n\}_{n=1}^{\infty}$ such that $R_n \uparrow \infty$, and let $X^{R_n}$ and $\tau_{R_n}$ be defined as in Definition (3.5). For $1 \leq i < j$, we have $\Phi_{R_i}(x) = \Phi_{R_j}(x)$ for $\|x\|_{H^k} \leq R_i$. Thus, $X^{R_j}$ is also a solution to Equation (3.7) up to $\tau_R$. Therefore, by the uniqueness of solution and by the continuity of sample paths of solution, the sample paths of $X^{R_j}$ coincide with the sample paths of $X^{R_i}$ almost surely. To be precise, we have, for almost all $\omega \in \Omega$,

$$X^{R_i}(t, \omega) = X^{R_j}(t, \omega), \quad \text{for all } t \in [0, \tau_{R_j}(\omega)]$$

Consequently, $\{\tau_{R_n}\}_{n=1}^{\infty}$ is an increasing sequence of stopping times. Let

$$\tau_{\infty} = \lim_{n \to \infty} \tau_{R_n} \quad (3.9)$$

Now we can extend the sample paths of $X^{R_n}$ to obtain a process $X^{\infty}$: For almost all $\omega \in \Omega$, let

$$X^{\infty}(t, \omega) = \lim_{n \to \infty} X^{R_n}(t, \omega) \quad \text{for all } 0 \leq t < \tau_\infty(\omega)$$

Then the process $X^{\infty}$ is a unique solution with continuous sample paths to Equation (3.5) up to time $\tau_R$ for all $R > 0$. Also, the stopping time $\tau_R$ defined in Definition (3.5) is realized by the process $X^{\infty}$:

$$\tau_R = \inf\{t : \|X^{\infty}(t)\|_{H^k} \geq R\}$$
Remark 3.10. The above constructions of the process $X^\infty$ and the stopping time $\tau_\infty$ are independent of the choice of the sequence $\{R_n\}_{n=1}^\infty$: Let $\{S_n\}_{n=1}^\infty$ be another sequence such that $S_n \uparrow \infty$. Let $\sigma_\infty$ be the stopping time and $Y^\infty$ be the process constructed as above but using the sequence $\{S_n\}_{n=1}^\infty$. First, we can combine the two sequences $\{R_n\}_{n=1}^\infty$ and $\{S_n\}_{n=1}^\infty$ to form a new sequence $\{K_n\}_{n=1}^\infty$ such that $K_n \uparrow \infty$. Let $\gamma_\infty$ be the stopping time constructed as above but using the sequence $\{K_n\}_{n=1}^\infty$. Then $\tau_\infty = \sigma_\infty = \gamma_\infty$. Also, $X^\infty$ and $Y^\infty$ are equivalent up to $\tau_{R_n}$ and $\tau_{S_n}$ for all $n = 1, 2, \cdots$. Therefore, they are equivalent up to $\tau_\infty$.

Definition 3.11. We define $X^\infty$ to be the $H^k$-valued process and $\tau_\infty$ to be the stopping time as constructed above. We call $\tau_\infty$ the explosion time of the process $X^\infty$. We also define the $\tilde{H}^k$-valued process $\tilde{X}^\infty$ to be $\tilde{X}^\infty = id + X^\infty$.

We can slightly extend Definition (3.10) and make the following definition:

Definition 3.12. Let $\tau$ be an $\mathcal{F}_\tau$-stopping time. An $\mathcal{F}_\tau$-adapted process $X_t$ with continuous sample paths is said to be a solution to Equation (3.15) up to time $\tau$ if there is an increasing sequence of $\mathcal{F}_\tau$-stopping time $\{\tau_n\}_{n=1}^\infty$ such that $\tau_n \uparrow \tau$ and $X_t$ is a solution to Equation (3.15) up to time $\tau_n$ in the sense of Definition (3.11) for all $n = 1, 2, \cdots$. The solution $X_t$ is said to be unique up to $\tau$ if it is unique up to $\tau_n$ for all $n = 1, 2, \cdots$.

We have proved the following proposition:

Proposition 3.13. Let $k$ be a non-negative integer. The process $X^\infty$ as defined in Definition (3.11) is the unique solution with continuous sample paths to Equation (3.11) up to the explosion time $\tau_\infty$.

3.4. The main result. In this subsection, we will prove that the explosion time $\tau_\infty$ defined in Definition (3.11) is infinity almost surely. We will also prove that the process $\tilde{X}^\infty$ defined in Definition (3.11) lives in the group Diff($S^1$). The key idea to both proofs is the following proposition:

Proposition 3.14. Let $\tilde{X}_t$ be an $\mathcal{F}_t$-adapted $\tilde{H}^k$-valued process with continuous sample paths and $\tau$ an $\mathcal{F}_\tau$-stopping time. If $\tilde{X}_t$ is a solution to
\[
d\tilde{X}_t = \tilde{\Phi}(\tilde{X}_t)dW_t, \quad \tilde{X}_0 = id
\]
up to $\tau$, then $\tilde{X}_t \circ \xi$ is a solution to
\[
d\tilde{X}_t = \tilde{\Phi}(\tilde{X}_t)dW_t, \quad \tilde{X}_0 = \xi
\]
up to $\tau$, where $\xi$ is a bounded $\tilde{H}^k$-valued random variable and "$\circ$" is the composition of two functions.

Proof. By assumption
\[
\tilde{X}_{t\wedge \tau} = id + \int_0^{t\wedge \tau} \tilde{\Phi}(\tilde{X}_s)dW_s
\]
By definition of the operator $\tilde{\Phi}$ (see subsection 2.4), this can be written as
\[
\tilde{X}_{t\wedge \tau} = id + \int_0^{t\wedge \tau} dW_s \circ \tilde{X}_s
\]
So
\[ \tilde{X}_{t \land \tau} \circ \tilde{\xi} = \tilde{\xi} + \int_0^{t \land \tau} dW_s \circ \tilde{X}_s \circ \tilde{\xi} \]
that is
\[ \tilde{X}_{t \land \tau} \circ \tilde{\xi} = \tilde{\xi} + \int_0^{t \land \tau} \tilde{\Phi}(\tilde{X}_s \circ \tilde{\xi}) dW_s \]
Therefore, \( \tilde{X} \circ \tilde{\xi} \) is a solution to
\[ d\tilde{X}_t = \tilde{\Phi}(\tilde{X}_t) dW_t, \quad \tilde{X}_0 = \tilde{\xi} \]
up to \( \tau \).

\[ \square \]

Remark 3.15. (Concatenating procedure.) Let \( R > 0 \). Let \( \tilde{\xi} = \tilde{X}_\infty(\tau_R) \). Then \( \tilde{\xi} \) is an \( \tilde{H}^k \)-valued bounded random variable. Let \( W'_t = W_{t+t_R} - W_{t_R} \). By the strong Markov property of the Brownian motion \( W_t \), we have \( W'_t = W_t \) in distribution for all \( t \geq 0 \). Therefore, similar to the construction of \( X_\infty \) and \( \tilde{X}_\infty \), we can construct \( Y_\infty \) and \( \tilde{Y}_\infty \) with \( \tilde{Y}_\infty \) a solution to the following equation
\[ d\tilde{X}_t = \tilde{\Phi}(\tilde{X}_t) dW'_t, \quad \tilde{X}_0 = \tilde{\xi} \]
up to stopping time
\[ \tau'_R = \inf\{ t : ||Y_\infty(t)||_{H^k} \geq R \} \]
Using the strong Markov property of the Brownian motion \( W_t \) again, we see that \( \tau_R = \tau'_R \) in distribution, and they are independent with each other. By Proposition (3.14), \( Y_\infty \circ \tilde{\xi} \) is the solution up to time \( \tau'_R \) to the following equation
\[ d\tilde{X}_t = \tilde{\Phi}(\tilde{X}_t) dW'_t, \quad \tilde{X}_0 = \tilde{\xi} \]
Because \( \tilde{\xi} = \tilde{X}_\infty(\tau_R) \), we can concatenate the two processes \( X_\infty \) and \( \tilde{Y}_\infty \) to form a new process \( \tilde{Z}_\infty \) as follows:
\[ \tilde{Z}_t = \begin{cases} \tilde{X}_t, & \text{for } t \leq \tau_R \\ \tilde{Y}_t \circ \tilde{\xi}, & \text{for } t > \tau_R \end{cases} \quad (3.10) \]
By the choice of \( W'_t \), we see that the process \( \tilde{Z}_\infty \) is a solution to Equation (3.4) up to time \( \tau_R + \tau'_R \). By the uniqueness of solution, \( \tilde{Z}_\infty \) is equivalent to \( \tilde{X}_\infty \) up to time \( \tau_R + \tau'_R \).

We can carry out this “concatenating” procedure over and over again. Thus, for any \( n \in \mathbb{N} \), we can construct a process \( \tilde{Z}_\infty \) which is a solution to Equation (3.4) and is equivalent to \( \tilde{X}_\infty \) up to time \( \tau_R + \tau'_R + \cdots + \tau_R^{(n)} \) with \( \tau_R, \tau'_R, \cdots \) being identical in distribution and mutually independent with each other.

Proposition 3.16. Let \( \tau_\infty \) be the explosion time of the process \( X_\infty \) defined as in Definition (3.11). Then \( \tau_\infty = \infty \) almost surely.

Proof. We can carry out the above “concatenating” procedure as many times as we want. Thus, for any \( n \in \mathbb{N} \), we can construct a process \( \tilde{Z}_\infty \) which is a solution to Equation (3.4) and is equivalent to \( \tilde{X}_\infty \) up to time \( \tau_R + \tau'_R + \cdots + \tau_R^{(n)} \).
By the triangle inequality in $H^k$, we have
\[ \tau_R + \tau'_R + \cdots + \tau^{(n)}_R \leq \tau_{Rn} \leq \tau_\infty, \]
On the other hand, because $\tau_R, \tau'_R, \cdots$ have the same distributions and are mutually independent with each other,
\[ \lim_{n \to \infty} \tau_R + \tau'_R + \cdots + \tau^{(n)}_R = \infty \ a.s. \]
Therefore, the explosion time $\tau_\infty = \infty$ almost surely.

**Proposition 3.17.** Let $X^\infty$ be the $H^k$-valued process defined in Definition (3.11). Then $X^\infty$ actually lives in the space $\text{diff}(S^1)$.

**Proof.** The construction of $X^\infty$ in subsection 3.3 is for a fixed $k$. But the method is valid for all $k = 0, 1, 2, \cdots$. Let us denote by $X^{k, \infty}$ the $H^k$-valued process as constructed in subsection 3.3. Because Equation (3.5) takes the same form in each space $H^k$, $k = 0, 1, 2, \cdots$, also, $H^{k+1} \subseteq H^k$, we see that the $H^{k+1}$-valued process $X^{k+1, \infty}$ is also a solution to Equation (3.5) in the space $H^k$. By uniqueness of the solution, $X^{k+1, \infty}$ is equivalent to $X^{k, \infty}$. Therefore, we can also say the solution $X^{k, \infty}$ to Equation (3.5) in the space $H^k$ is also the solution to Equation (3.5) in the space $H^{k+1}$. By induction, the solution $X^{k, \infty}$ actually lives in $H^{k+i}$ for all $i = 0, 1, 2, \cdots$. Therefore it lives in $\bigcap_{i=0}^{\infty} H^{k+i} = \text{diff}(S^1)$.

By the above proposition, the $\tilde{H}^k$-valued process $\tilde{X}^\infty$ lives in the affine space $\text{diff}(S^1)$. In the next proposition we will prove that $\tilde{X}^\infty$ actually lives in the group $\text{Diff}(S^1)$. The key to the proof is Proposition (2.2) together with the “concatenating” procedure (remark 3.13).

**Proposition 3.18.** The process $\tilde{X}^\infty$ defined in Definition (3.11) lives in the group $\text{Diff}(S^1)$.

**Proof.** Let us fix a $k \geq 2$. Suppose $\tilde{f} \in \tilde{H}^k$. By item (2) in Theorem 2.5, $\|f'\|_{L^\infty} \leq c_k \|f\|_{H^k}$. Thus, by controlling the $H^k$-norm of $f$ we can control the $L^\infty$-norm of $f'$. When $\|f'\|_{L^\infty} < 1$, we have $f' > -1$, or equivalently, $\tilde{f}' > 0$. If we also know that $\tilde{f}$ is $C^\infty$, then by Proposition (2.2), we can conclude that $\tilde{f}$ is actually a diffeomorphism of $S^1$.

The process $X^\infty$ has values in the $R$-ball
\[ B(0, R) = \{ x \in H^k : \|x\|_{H^k} \leq R \} \]
up to time $\tau_R$. Let us choose $R$ so that $f \in B(0, R)$ implies $\|f'\|_{L^\infty} < 1$. Then up to $\tau_R$, the first derivative $\|X^\infty(t, \omega)(1)\|_{L^\infty} < 1$ almost surely. So up to $\tau_R$, $X^\infty(t, \omega)(1) > -1$, or equivalently $\tilde{X}^\infty(t, \omega)(1) > 0$ almost surely. Also by Proposition (3.17), $\tilde{X}^\infty$ lives in the affine space $\text{diff}(S^1)$, which means: every element $\tilde{X}^\infty(t, \omega)$ is $C^\infty$. Therefore, by Proposition (2.2), $\tilde{X}^\infty$ lives in the group $\text{Diff}(S^1)$ up to time $\tau_R$.

In the “concatenating” procedure (remark 3.13), the process $\tilde{Y}^\infty$ lives in the group $\text{Diff}(S^1)$ up to time $\tau'_R$ for the same reason. Because $\xi = \tilde{X}^\infty(\tau_R)$, it is
now a \( \text{Diff}(S^1) \)-valued random variable. So we have \( \tilde{Y}^\infty \circ \tilde{\xi} \) lives in \( \text{Diff}(S^1) \) up to time \( \tau_R' \). By concatenation, the process \( \tilde{Z}^\infty \) lives in \( \text{Diff}(S^1) \) up to time \( \tau_R + \tau_R' \). Because \( \tilde{X}^\infty \) is equivalent to \( \tilde{Z}^\infty \) up to time \( \tau_R + \tau_R' \), we have the process \( \tilde{X}^\infty \) lives in \( \text{Diff}(S^1) \) up to time \( \tau_R + \tau_R' \). We can carry out this “concatenating” procedure over and over again. Therefore, the process \( \tilde{X}^\infty \) lives in \( \text{Diff}(S^1) \) up to the explosion time \( \tau_\infty \) which is infinity by Proposition (3.16).

\[ \square \]

Putting together Propositions (3.13), (3.16) and (3.18), we have proved the main result of the paper:

**Theorem 3.19.** There is a unique \( \tilde{H}^k \)-valued solution with continuous sample paths to Equation (3.4) for all \( k = 0, 1, 2, \ldots \). Furthermore, the solution is non-explosive and lives in the group \( \text{Diff}(S^1) \).

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