Comparing tautological relations from the equivariant Gromov-Witten theory of projective spaces and spin structures

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Abstract

Pandharipande-Pixton-Zvonkine’s proof of Pixton’s generalized Faber-Zagier relations in the tautological ring of $\overline{M}_{g,n}$ has started the study of tautological relations from semisimple cohomological field theories. In this article we compare the relations obtained in the examples of the equivariant Gromov-Witten theory of projective spaces and of spin structures. We prove an equivalence between the $\mathbb{P}^1$- and 3-spin relations, and more generally between restricted $\mathbb{P}^m$-relations and similarly restricted $(m+2)$-spin relations. We also show that the general $\mathbb{P}^m$-relations imply the $(m+2)$-spin relations.

1 Introduction

The study of the Chow ring of the moduli space of curves was initiated Mumford in [1]. Because the whole Chow ring is in general very complicated he introduced the tautological subrings of classes reflecting the geometry of the objects parametrized by the moduli space. The tautological ring $R^*(\overline{M}_{g,n})$ is compactly described [2] as the smallest system $R^*(\overline{M}_{g,n}) \subseteq A^*(\overline{M}_{g,n})$ of subrings compatible with push-forward under the tautological maps, i.e. the maps obtained from forgetting marked points or gluing curves along common markings.

There is a canonical set of generators parametrized by decorated graphs [3]. The formal vector space $S_{g,n}$ generated by them, the strata algebra, therefore admits a surjective map to $R^*(\overline{M}_{g,n})$ and the structure of the tautological ring is determined by the kernel of this surjection. Elements of the kernel are called tautological relations.

In [4] A. Pixton proposed a set of (at the time conjectural) relations generalizing the relations by Faber-Zagier in $R^*(M_g)$. Furthermore, he conjectured that these generate all tautological relations. The first proof [5] of the fact that the conjectural relations are actual relations (in cohomology) brought cohomological field theories (CohFTs) into the picture.
A CohFT on a free module $V$ of finite rank over a base ring $A$ is a system of classes $\Omega_{g,n}$ behaving nicely under pull-back via the tautological maps. A CohFT can also be used to give $V$ the structure of a Frobenius algebra. The CohFT is called semisimple if, after possible base extension, the algebra $V$ has a basis of orthogonal idempotents.

For semisimple CohFTs there is a conjecture by Givental [6] proven in some cases by himself and in full generality in cohomology by Teleman [7], giving a reconstruction of the CohFT from its genus 0, codimension 0 part and the data of a power series $R(z)$ of endomorphisms of $V$. The formula naturally lifts to the strata algebra.

To get relations from a semisimple cohomological field theory one can use that the reconstructed CohFT of elements in the strata algebra is in general only defined over an extension $B \leftarrow A$. However since one has started out with a CohFT over $A$, this implies that certain linear combinations of elements in the strata algebra have to vanish under the projection to the tautological ring.

This procedure was essentially used in the proof [5] in the special example of the CohFT defined from Witten’s 3-spin class. There the base ring is a polynomial ring in one variable but the reconstructed CohFT seems to have poles.

In [8] (in preparation) the authors construct tautological relations using Witten’s $r$-spin class for any $r \geq 3$. Given a list of integers $a_1, \ldots, a_n \in \{0, \ldots, r-2\}$, Witten’s class $W_{g,n}(a_1, \ldots, a_n)$ is a cohomology class on $\overline{M}_{g,n}$ of pure degree

$$D_{g,n}(a_1, \ldots, a_n) = \frac{(r-2)(g-1) + \sum_{i=1}^n a_i}{r}.$$ 

Witten’s class can be “shifted” by any vector in the vector space $\langle e_0, \ldots, e_{r-2} \rangle$ to obtain a semisimple CohFT. In practice, the authors use two particular shifts for which the answer can be explicitly computed. Shifted Witten’s class is of mixed degree: more precisely, the degrees of its components go from 0 to $D_{g,n}(a_1, \ldots, a_n)$. On the other hand, the Givental-Teleman classification of semisimple CohFTs gives an expression of the shifted Witten class in terms of tautological classes. The authors conclude that the components of this expression beyond degree $D_{g,n}(a_1, \ldots, a_n)$ are tautological relations.

This article studies how relations from spin structures are related to the relations obtained from the CohFT defined from the equivariant Gromov-Witten theory of projective spaces. The following two theorems are our main results.

**Theorem 1** (rough version). The relations obtained from the equivariant Gromov-Witten theory of $\mathbb{P}^m$ imply the $(m+2)$-spin relations.

**Theorem 2** (rough version). A special restricted set of relations from equivariant $\mathbb{P}^m$ is equivalent to a corresponding restricted set of $(m+2)$-spin relations. For $\mathbb{P}^1$ and 3-spin no restriction is necessary.

Since for equivariant $\mathbb{P}^m$ the reconstruction holds in Chow, Theorem 1 implies that the higher spin relations also hold in Chow.
We will give strong evidence that the method of proof for Theorem 2 cannot be extended to an equivalence between the full \( P^m \)- and \((m + 2)\)-spin relations for \( m > 2 \). Possibly, there are more \( P^m \)- than \((m + 2)\)-spin relations.

Any of the theorems gives another proof of the fact that Pixton’s relations hold in Chow. In fact, the proof of Theorem 1 in the case \( m = 1 \) is essentially a simplified version of the author’s previous proof in [9].

This article does not give a comparison between relations from CohFTs of different dimensions, nor does it consider all relations from equivariant \( P^m \). On the other hand, if indeed Pixton’s relations are all tautological relations, the 3-spin relations have to imply the relations from any other semisimple CohFT. Yet, for example it is still open whether they imply the 4-spin relations.

The article is structured as follows. In Section 2, we give definitions of CohFTs, discuss the \( R \)-matrix action on CohFTs and the reconstruction result. We then in Section 2.5 turn to the two examples of equivariant \( P^m \) and the CohFT from the \( A_{m+1} \)-singularity. In Section 2.6, we describe the general procedure of obtaining relations from semisimple CohFTs and general methods of proving that the relations from one CohFT imply the relations from another. We then state precise versions of Theorem 1 and 2. Section 3 discusses explicit expression of the \( R \)-matrices in both theories in terms of asymptotics of oscillating integrals. The constraints following from these expressions will be used in the next sections. We also note a connection to Airy functions. Section 4 and Section 5 give proofs of Theorem 1 and 2. Finally, Section 6 gives evidence why, with the methods used in the proofs of the theorems, an equivalence between \( P^m \)- and \((m + 2)\)-spin relations cannot be established. Since the reconstruction result of Givental we use to get relations in Chow has never appeared explicitly in the literature, we recall its proof in Appendix A.

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2 Cohomological field theories

2.1 Definitions

Cohomological field theories were first introduced by Kontsevich and Manin in [10] to formalize the structure of classes from GW-theory. Let \( A \)
be an integral, commutative $\mathbb{Q}$-algebra, $V$ a free $A$-module of finite rank
and $\eta$ a non-degenerate bilinear form on $V$.

**Definition 1.** A cohomological field theory (CohFT) $\Omega$ on $(V, \eta)$ is a system
$$\Omega_{g,n} \in A^*(\overline{M}_{g,n}) \otimes \mathbb{Q} (V^*)^\otimes n$$
of multilinear forms with values in the Chow ring of $\overline{M}_{g,n}$ satisfying the following properties:

**Symmetry** $\Omega_{g,n}$ is symmetric in its $n$ arguments

**Gluing** The pull-back of $\Omega_{g,n}$ via the gluing map
$$\overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \rightarrow \overline{M}_{g,n}$$
is given by the direct product of $\Omega_{g_1,n_1+1}$ and $\Omega_{g_2,n_2+1}$ with the bivector $\eta^{-1}$ inserted at the two gluing points. Similarly for the gluing map $\overline{M}_{g-1,n+2} \rightarrow \overline{M}_{g,n}$ the pull-back of $\Omega_{g,n}$ is given by $\Omega_{g-1,n+2}$ with $\eta^{-1}$ inserted at the two gluing points.

**Unit** There is a special element $1 \in V$ called the unit such that
$$\Omega_{g,n+1}(v_1, \ldots, v_n, 1)$$is the pull-back of $\Omega_{g,n}(v_1, \ldots, v_n)$ under the forgetful map and
$$\Omega_{0,3}(v, w, 1) = \eta(v, w).$$

**Definition 2.** The quantum product $(u,v) \mapsto uv$ on $V$ with unit $1$ is defined by the condition
$$\eta(uv, w) = \Omega_{0,3}(u, v, w).$$

**Definition 3.** A CohFT is called semisimple if there is a base extension $A \rightarrow B$ such that the algebra $V \otimes_A B$ is semisimple.

### 2.2 First Examples

**Example 1.** For each Frobenius algebra there is the trivial CohFT (also called topological field theory) $\Omega_{g,n}$ characterized by \[1\] and that
$$\Omega_{g,n} \in A^0(\overline{M}_{g,n}) \otimes (V^*)^\otimes n.$$Let us record an explicit formula for Appendix A In the case that $\epsilon_i$ is a basis of orthogonal idempotents of $V$ and that
$$\tilde{\epsilon}_i = \frac{\epsilon_i}{\sqrt{\Delta_i}},$$where $\Delta_i^{-1} = \eta(\epsilon_i, \epsilon_i)$, is the corresponding orthonormal basis of normalized idempotents, we have
$$\Omega_{g,n}(\tilde{\epsilon}_{i_1} \otimes \cdots \otimes \tilde{\epsilon}_{i_n}) = \begin{cases} 
\sum_j \Delta_{i_1}^{j-1}, & \text{if } n = 0, \\
\frac{\Delta_i \delta_{i_1 \cdots i_n}}{\Delta_{i_1} \cdots \Delta_{i_n}}, & \text{if } i_1 = \cdots = i_n, \\
0, & \text{else.}
\end{cases}$$
Example 2. The Chern polynomial \( c_t(E) \) of the Hodge bundle \( E \) gives a 1-dimensional CohFT over \( \mathbb{Q}[t] \).

Example 3. Let \( X \) be a smooth, projective variety such that the cycle class map gives an isomorphism between Chow and cohomology rings. Let \( A = \mathbb{Q}[q^β] \) be its Novikov ring. Then the Gromov-Witten theory of \( X \) defines a CohFT based on the \( A \)-module \( A^*(X) \otimes A \) by the definition

\[
\Omega_{g,n}(v_1, \ldots, v_n) = \sum_β \pi_* \left( \prod_{i=1}^n \text{ev}_i^*(v_i) \cap [\overline{M}_{g,n}(X, \beta)]^{vir} \right) q^β,
\]

where the sum ranges over effective, integral curve classes, \( \text{ev}_i \) is the \( i \)-th evaluation map and \( \pi \) is the forgetful map \( \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n} \). The gluing property follows from the splitting axiom of virtual fundamental classes. The fundamental class of \( X \) is the unit of the CohFT and the unit axioms follow from the identity axiom in GW-theory.

For a torus action on \( X \), this example can be enhanced to give a CohFT from the equivariant GW-theory of \( X \).

2.3 The \( R \)-matrix action

Definition 4. The (upper part of the) symplectic loop group is defined as the subgroup of the group of endomorphism valued power series \( R = 1 + O(z) \) in \( z \) satisfying the symplectic condition

\[ \eta(R(z)v, R(-z)w) = \eta(v, w) \]

for all vectors \( v \) and \( w \).

An action of this group on the space of CohFTs makes it interesting for us. In its definition the endomorphism valued power series \( R \) is evaluated at cotangent line classes and applied to vectors.

Given a CohFT \( \Omega_{g,n} \) the new CohFT \( R\Omega_{g,n} \) takes the form of a sum over dual graphs \( \Gamma \)

\[
R\Omega_{g,n}(v_1, \ldots, v_n) = \frac{1}{\text{Aut}(\Gamma)} \xi_* \left( \prod_{v} \sum_{k=0}^{\infty} \frac{1}{k!} \varepsilon_* \Omega_{g_v, n_v+k}(\ldots) \right),
\]

where \( \xi : \prod_v \overline{M}_{g_v, n_v} \to \overline{M}_{g,n} \) is the gluing map of curves of topological type \( \Gamma \) from their irreducible components, \( \varepsilon : \overline{M}_{g_v, n_v+k} \to \overline{M}_{g_v, n_v} \) forgets the last \( k \) markings and we still need to specify what is put into the arguments of \( \prod_v \Omega_{g_v, n_v+k_v} \).

- Into each argument corresponding to a marking of the curve, put \( R^{-1}(\psi) \) applied to the corresponding vector.
- Into each pair of arguments corresponding to an edge put the bivector

\[
\frac{R^{-1}(\psi_1)\eta^{-1}R^{-1}(\psi_2)^t - \eta^{-1}}{-\psi_1 - \psi_2} \in \text{Hom}(V^*, V)[\psi_1, \psi_2] \cong V^* \otimes V[\psi_1, \psi_2],
\]

where one has to substitute the \( \psi \)-classes at each side of the normalization of the node for \( \psi_1 \) and \( \psi_2 \). By the symplectic condition this is well-defined.
• At each of the additional arguments for each vertex put

\[ T(\psi) := \psi(1 - R^{-1}(\psi))1, \]

where \( \psi \) is the cotangent line class corresponding to that vertex. Since \( T(z) = O(z^2) \) the above \( k \)-sum is finite.

**Reconstruction Conjecture** (Givental). The \( R \)-matrix action is free and transitive on the space of semisimple CohFTs based on a given Frobenius algebra.

**Theorem 3** (Givental\([6]\)). Reconstruction for the equivariant GW-theory of toric targets holds in Chow.

**Theorem 4** (Teleman\([7]\)). Reconstruction holds in cohomology.

**Remark 1.** Givental’s original conjecture was only stated in terms of the underlying Frobenius manifold and there is no explicit proof of Theorem 3 in the literature. Therefore in Appendix A we recall the well-known lift of Givental’s proof to CohFTs.

**Example 4.** By Mumford’s Grothendieck-Riemann-Roch calculation \([1]\) the single entry of the \( R \)-matrix taking the trivial one-dimensional CohFT to the CohFT from Example 2 is given by

\[ \exp \left( \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)} (tz)^{2i-1} \right), \]

where \( B_{2i} \) are the Bernoulli numbers, defined by

\[ \sum_{i=0}^{\infty} B_i \frac{x^i}{i!} = \frac{x}{e^x - 1}. \]

**2.4 Frobenius manifolds and the quantum differential equation**

There is a natural way to deform a CohFT \( \Omega_{g,n} \) on \( V \) over \( A \) to a CohFT over \( A[V] \). For a basis \( \{e_\mu\} \) of \( V \) let

\[ p = \sum t^\mu e_\mu \]

be a formal point on \( V \). Then the deformed CohFT is given by

\[ \Omega^p_{g,n}(v_1, \ldots, v_n) = \sum_{k=0}^{\infty} \frac{1}{k!} \pi_* \Omega_{g,n+k}(v_1, \ldots, v_n, p, \ldots, p). \]

Notice that the deformation is constant in the direction of the unit.

The quantum product on the deformed CohFT gives \( V \) the structure of a (formal) Frobenius manifold \([11]\). The \( e_\mu \) induce flat vector fields on \( V \) corresponding to the flat coordinates \( t^\mu \). Greek indices will stand for flat coordinates with an exception stated in Section 2.5.

A Frobenius manifold is called conformal if it admits an Euler vector field, i.e. a vector field \( E \) of the form

\[ \sum_{\mu} (\alpha_\mu t^\mu + \beta_\mu) \frac{\partial}{\partial t^\mu}, \]
such that the quantum product, the unit and the metric are eigenfunctions of the Lie derivative $L_E$ with eigenvalues $1$, $-1$ and $2-\delta$ respectively. Here $\delta$ is a rational number called conformal dimension. Assuming that $A$ itself is the ring of (formal) functions of a variety $X$ we say that the Frobenius manifold is quasi-conformal if there is vector field $E$ on $X \times V$ satisfying the axioms of an Euler vector field.

A CohFT $\Omega_{g,n}$ is called homogeneous (quasi-homogeneous) if its Frobenius manifold is conformal (quasi-conformal) and the extended CohFT is an eigenvector of $L_E$ of eigenvalue $(g-1)\delta + n$. As the name suggests CohFTs are homogeneous if they carry a grading such that all natural structures are homogeneous with respect to the grading.

We say that the Frobenius manifold $V$ is semisimple if there is a basis of idempotent vector fields $\epsilon_i$ defined after possible base extension of $A$. The idempotents can be formally integrated to canonical coordinates $u_i$. We will use roman indices for them. Let $u$ be the diagonal matrix with entries $u_i$ and $\Psi$ be the transition matrix from the basis of normalized idempotents corresponding to the $u_i$ to the flat basis $e_i$.

The $R$-matrix from the trivial theory to $\Omega^p$ satisfies a differential equation which is related to the quantum differential equation

$$z \frac{\partial}{\partial \alpha} S_j = e_\alpha \star S_j$$

for vectors $S_j$. We assemble the $S_j$ into a matrix $S$.

**Proposition 1** (see [12]). If $V$ is semisimple and after a choice of canonical coordinates $u_i$, there exists a fundamental solution $S$ to the quantum differential equation of the form

$$S = \Psi R e^{u/2},$$

such that $R$ satisfies the symplectic condition $R(z)R^T(-z) = 1$. The matrix $R$ is unique up to right multiplication by a diagonal matrix of the form

$$\exp(a_1z + a_3z^3 + a_5z^5 + \cdots)$$

for constant diagonal matrices $a_i$.

In the case that there exists an Euler vector field $E$, there is a unique matrix $R$ defined from a fundamental solution $S$ by (2) satisfying the homogeneity

$$z \frac{d}{dz} R + L_E R = 0.$$

Such an $R$ automatically satisfies the symplectic condition.

**Remark 2.** The matrix $R$ should be thought as the matrix representation of an endomorphism in the basis of normalized idempotents. The symplectic condition in Proposition 1 is then the same as in Definition 4.

**Remark 3.** The exponential in (2) has to be thought as a formal expression. All the quantities in Proposition 1 are only defined after base change of $A$ necessary to define the canonical coordinates.

**Remark 4.** The quantum differential equation is equivalent to the differential equation

$$[R, du] + z\Psi d(\Psi^{-1} R) = 0$$

for $R$. 7
In the conformal case Teleman showed that the uniquely determined homogeneous $R$-matrix of Proposition 1 is the one appearing in the reconstruction, taking the trivial theory to the given one.

Equivariant projective spaces $\mathbb{P}^n$ only give a quasi-conformal Frobenius manifold. However Givental showed, and we will recall the proof in Appendix A, that in this case in the reconstruction one should take $R$ such that in the classical limit $q \to 0$ it assumes the diagonal form

$$R|_{q=0} = \exp(\text{diag}(b_0, \ldots, b_{k-1})), \quad (4)$$

where, using the notation from Section 2.5

$$b_j = \sum_{i=1}^{\infty} \frac{B_{2i}}{2i(2i-1)} \frac{1}{z^{2i-1}} \sum_{l \neq j} \left(\frac{\lambda_i - \lambda_j}{\lambda_i - \lambda_j}\right)^i.$$ 

The $R$-matrix is uniquely determined by this additional property and the homogeneity property.

### 2.5 The two CohFTs

The cohomological field theory corresponding to the $A_{m+1}$-singularity $f(X) = X^{m+2}/(m+2)$ is defined using Witten’s $(m+2)$-spin class on the moduli of curves with $(m+2)$-spin structures. See [5] for a discussion of different constructions of Witten’s class. In comparison to [5] we use a different normalization for Witten’s class and a different basis for the free module in order to have a more direct comparison to the $\mathbb{P}^m$-theory.

The CohFT is based on the rank $(m+1)$ free module of versal deformations

$$f_t(X) = \frac{X^{m+2}}{m+2} + t^m X^m + \cdots + t^1 X + t^0$$

of $f$. In this article, using the deformation from Section 2.4 we will view the CohFT as being based on

$$k_{A_{m+1}} = \mathbb{Q}[t^1, \ldots, t^m],$$

the space of regular functions on the Frobenius manifold where the $t^0$-coordinate vanishes. Because of dimension constraints we do not need to look at formal functions, and because the CohFT stays constant along the $t^0$ direction we can restrict to the $(t^0 = 0)$-subspace.

The algebra structure is given by $k_{A_{m+1}}[X]/(f_t^\prime)$, where $X^\mu$ corresponds to $\frac{\partial}{\partial t^\mu}$. The metric is given by the residue pairing

$$\eta(a, b) = \frac{1}{2\pi i} \oint f_t^\prime(X) dB(X).$$

Written as a matrix in the basis $1, \ldots, X^m$, the metric $\eta$ has therefore zeros above the antidiagonal, ones at the antidiagonal and again zeros in the first antidiagonal below it. Notice also that $\eta$ has no dependence on $t^1$. Therefore, while the $t^m$ do not give a basis of flat vector fields on the Frobenius manifold, there is a triangular matrix independent of $t^1$, sending the $1, \ldots, X^m$ to a basis of flat vector fields such that $X$ is mapped to
itself. With this we can pretend that the $t^\mu$ were flat coordinates if we consider in the quantum differential equation only differentiation by $t^1$.

For $(\mathbb{C}^*)^{m+1}$-equivariant $\mathbb{P}^m$ the CohFT is based on the equivariant Chow ring

$$A^*_{(\mathbb{C}^*)^{m+1}}(\mathbb{P}^m)[q] \cong k_{\mathbb{P}^m}[H]/\prod_{i=0}^m (H - \lambda_i),$$

of $\mathbb{P}^m$, an $(m+1)$-dimensional free module over

$$k_{\mathbb{P}^m} = \mathbb{Q}[\lambda_0, \ldots, \lambda_m][q],$$

and depends on the Novikov variable $q$ and the torus parameters $\lambda_i$. We will not consider the deformation from Section 2.4. The algebra structure is given by the small quantum equivariant Chow ring

$$QA^*_{(\mathbb{C}^*)^{m+1}}(\mathbb{P}^m) \cong k_{\mathbb{P}^m}[H]/\left(\prod_{i=0}^m (H - \lambda_i) - q\right),$$

and the pairing is the Poincaré pairing

$$\eta(a, b) = \frac{1}{2\pi i} \oint \frac{ab}{\prod_{i=0}^m (H - \lambda_i)} dH$$

in the equivariant Chow ring.

To match up this data we set

$$X = H - \bar{\lambda},$$

$$X^{m+1} + \sum_{\mu=0}^{m-1} (\mu + 1)\mu^{\mu + 1} X^\mu = \prod_{i=0}^m (X + \bar{\lambda} - \lambda_i) - q,$$

where

$$\bar{\lambda} = \sum_{i=0}^m \frac{\lambda_i}{m + 1}.$$ 

So in particular

$$t^1 = -q + \prod_{i=0}^m (\bar{\lambda} - \lambda_i) =: -q - \lambda$$

and we have described a map

$$\Phi : k_{A_{m+1}}[\lambda] \to k_{\mathbb{P}^m},$$

whose image are the polynomials, symmetric in the torus parameters and vanishing if all torus parameters coincide. Therefore, after base extension the Frobenius algebras from the $A_{m+1}$-singularity and equivariant $\mathbb{P}^m$ match completely up.

On the $\mathbb{P}^m$-side, let $Q_i$ be the power series solution to

$$\prod_{i=0}^m (Y + \bar{\lambda} - \lambda_i) = q$$
with limit $\lambda_i - \bar{\lambda}$ as $q \to 0$. In particular, the $Q_i$ are solutions to

$$Y^{m+1} + \sum_{\mu=0}^{m-1} (\mu + 1)t^{\mu+1}Y^\mu.$$ 

On the $A_{m+1}$-side, let the $Q_i$ be the solutions to this equation in any order. On both sides we can then define

$$\Delta_i = \prod_{j \neq i} (Q_i - Q_j) = (m+1)Q_i^m - \sum_{\mu=1}^{m-1} (\mu + 1)\mu t^{\mu+1}Q_i^{\mu-1}$$

and the discriminant

$$\text{disc} = \prod_i \Delta_i \in k_{A_{m+1}}.$$ 

The choice of the $Q_i$ gives a bijection between the idempotents

$$\epsilon_i = \frac{\prod_{j \neq i} (X - Q_j)}{\Delta_i}.$$ 

We will also need to make a choice of square roots of the $\Delta_i$ to be able to define the normalized idempotents

$$\tilde{\epsilon}_i = \frac{\prod_{j \neq i} (X - Q_j)}{\sqrt{\Delta_i}}.$$ 

The $A_{m+1}$-theory is conformal with Euler vector field

$$E = \sum_{i=1}^{m} \frac{m+2 - i}{m+2} \frac{\partial}{\partial t^i},$$

while the equivariant $P^m$-theory is semi-conformal with Euler vector field

$$E = (m+1)\frac{\partial}{\partial q} + \sum_{i=0}^{m} \lambda_i \frac{\partial}{\partial \lambda_i}. $$

### 2.6 Relations from CohFTs

Let $\Omega$ be a semisimple CohFT defined on $V$ over $A$. Formal properties of the reconstruction theorem will imply tautological relations. The main point is that the $R$-matrix from the trivial theory written in flat coordinates lives only in

$$\text{End}(V \otimes_A B)[z],$$

for some $Q$-algebra extension $B$ of $A$. We obtain an exact sequence of $A$-modules

$$0 \to A \to B \xrightarrow{\rho} C \to 0.$$  \hspace{1cm} (5)

The reconstruction gives elements

$$\prod_{g,n} \in S_{g,n} \widehat{\otimes} (V^*)^\otimes n \widehat{\otimes} B,$$

\footnote{In our examples $B = A[\text{disc}^{-1}]$.}
where \( \hat{\otimes} \) is a completed tensor product with respect to the dimension grading of the strata algebra. However since we have started out with a CohFT defined over \( A \), we know that the projection of

\[
p(\Omega_{g,n}) \in S_{g,n} \hat{\otimes} (V^*)^n \hat{\otimes} C
\]

to \( R^*(\Omega_{g,n}) \otimes (V^*)^n \otimes C \) has to vanish. Since \( C \) is a \( \mathbb{Q} \)-vector space, we obtain a system of vector spaces \( T_{g,n} \) of relations. The complete system \( \bar{T}_{g,n} \) of tautological relations obtained from the CohFT \( \Omega \) is the vector space generated by

\[
\xi_* (\pi_* (T_{g_1,n_1+m} P) \times S_{g_2,n_2} \times \cdots \times S_{g_k,n_k}),
\]

where \( P \) is the vector space of polynomials in \( \psi \)-classes, and \( \xi_* \) and \( \pi_* \) are the formal analogues of the push-forwards along gluing and forgetful maps.

We say that a vector space of tautological relations \( T_{g,n} \) implies another \( T'_{g,n} \), if the vector space, obtained from \( T_{g,n} \) by the completion process as described right above, is contained in \( T'_{g,n} \). Using this definition we can also define an equivalence relation between vector spaces of tautological relations.

Let us describe two relation preserving actions on the space of all CohFTs on \( V \) over \( A \). The first is an action of the multiplicative monoid of \( A \). The action of \( \varphi \in A \) is given by multiplication by \( \varphi^d \) in codimension \( d \). This replaces the \( R \)-matrix \( R(z) \) of the theory by \( R(\varphi z) \). Since multiplication by \( \varphi \) is well-defined in \( C \), relations are preserved. The second action is the action of an \( R \)-matrix defined over \( A \).

The second action automatically proves equivalence of relations since \( R \)-matrices are always invertible. Similarly, the first action proves equivalence if \( \varphi \) is invertible.

**Extending scalars** also preserves relations. By this we mean tensoring \( \Omega \) with \( A \to A' \) under the condition that this preserves the exactness of the sequence \([4]\). We call the special case when \( A' = A/I \) for some ideal \( I \) of \( A \) a limit. If \( C \to C \otimes_A A' \) is injective, extending scalars proves an equivalence of relations.

Let us again state our now well-defined results.

**Theorem 1.** The relations from the equivariant Gromov-Witten theory of \( F^m \) imply the \((m+2)\)-spin relations, both CohFTs as defined in Section 2.5.

The main statement necessary to be proven here is that the \( R \)-matrix for \( F^m \) after replacing \( z \mapsto z\lambda^{-1} \) admits the limit \( \lambda^{-1} \to 0 \) and that this limit is the \( R \)-matrix for the \( A_{m+1} \)-theory. In order for this to make sense, one uses the matchup from Section 2.5 and views both as being defined over

\[
\mathbb{Q}[\lambda_0, \ldots, \lambda_m, q][\lambda^{-1}]
\]

In Section 3 we will see that for both original theories to define the \( R \)-matrix it is enough to localize by disc. So the extension of scalars does not lose relations.

Motivated from Section 4.1 let us call the limit \( t^2, \ldots, t^m = 0 \) the Airy limit. For \( F^m \) the Airy limit concretely means, assuming the sum of all

\footnote{Assuming that reconstruction holds in this case.}
torus weights is zero, that we restrict ourselves to the case that up to a factor the torus weights are the \((m+1)\)-th roots of unity.

**Theorem 2.** In the Airy limit the \(P_m\)- and \((m+2)\)-spin relations are equivalent.

The main point for the proof is to show there is a series

\[ \varphi \in \lambda \mathbb{Q}[[t^1 \lambda^{-1}]], \]

and an \(R\)-matrix \(R\) without poles in disc such that the Airy limit \(P_m\)-\(R\)-matrix is obtained from the Airy limit \(A_{m+1}\)-\(R\)-matrix by applying the transformation \(z \mapsto z \varphi\), followed by the action of \(R\). We will show in the proof that there is only one possible choice for \(\varphi\). For Theorem 2 both theories can be viewed as living over

\[ \mathbb{Q}[[\lambda_0, \ldots, \lambda_m, q, t^1 \lambda^{-1}], \lambda^{-1}]/(t^1 + q + \lambda, t^2, \ldots, t^m). \]

In Section 6 we will give evidence that the method of proof of Theorem 2 does not work outside the Airy limit. What we will show is that assuming a procedure as in the proof of Theorem 2 exists and is well-defined in the Airy limit, the information that \(\varphi\) was unique in the limit implies that the \(R\)-matrix in the \(R\)-matrix action cannot be defined over the base ring.

**Relations from degree vanishing**

The classical way of [5] and [8] to obtain tautological relations works by considering cohomological degrees: Assume that \(\Omega\) is in addition quasi-homogenous for an Euler vector field \(E\) and that all \(\beta\) vanish and all \(\alpha_i\) are positive. Then the quasi-homogeneity implies that the cohomological degree of \(\Omega_{g,n}(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_m})\) is bounded by

\[ (g-1)\delta + n - \sum_j \alpha_{ij}. \]

However the reconstructed theory might also contain terms of higher cohomological degree. So these have to vanish, giving tautological relations.

Notice that these relations coming from degree considerations are implied from the relations we have described previously: With respect to the grading on \(B\) induced by the Euler vector field, no element of \(A\) has negative degree. Therefore the negative degree parts of \(B\) and \(C\) are isomorphic. Thus, the homogeneity of the CohFT implies that the dimension vanishing relations are obtained from the previous relations by restricting to the negative degree part of \(C\).

The way of obtaining tautological relations by looking at poles in the discriminant has first been studied by D. Zvonkine.

### 3 Oscillating integrals

#### 3.1 For the \(A_{m+1}\)-singularity

We want to describe the \(A_{m+1}\)-\(R\)-matrix in terms of asymptotics of oscillating integrals.
The quantum differential equation with one index lowered says that

\[ z \frac{\partial}{\partial t} S_{\mu k} = \pm S_{\mu+1 k}, \quad \text{for } \mu < m, \]

\[ z \frac{\partial}{\partial t} S_{mk} = -\sum_{\mu=0}^{m-1} (\mu + 1)^{\mu+1} S_{\mu k}, \]

where the Greek indices stand for components in the basis of the \( X^\mu \). It is not difficult to see that the oscillating integrals

\[ \frac{1}{\sqrt{-2\pi z}} \int_{\Gamma_k} x^\mu \exp(\frac{f_t(x)}{z}) dx, \]

where \( f_t \) as before is the deformed singularity, for varying cycles \( \Gamma_k \), provide solutions to this system of differential equations, and also satisfy homogeneity with respect to the Euler vector field.

To each critical point \( Q_k \) there corresponds a cycle \( \Gamma_k \) through that critical point in the direction of steepest descent, avoiding all other critical points. By moving to the critical point and scaling coordinates we obtain

\[ S_{\mu k} = \frac{e^{u_k/z}}{\sqrt{2\pi \Delta_k}} \int_{-\infty}^{\infty} \left( \frac{x(-z)^{1/2}}{\sqrt{\Delta_k}} + Q_k \right)^\mu \exp \left( -\sum_{l=2}^{m+2} \frac{x^l(-z)^{(l-2)/2}}{l!} \frac{f_t^{(l)}(Q_k)}{\Delta_k^{l/2}} \right) dx, \]

where \( u_k = f_t(Q_k) \). Since the \((l = 2)\)-term in the sum is \(-x^2/2\), we can use the formula for the moments of the Gaussian distribution to write the \( \sqrt{\Delta_k} e^{-u_k/z} S_{\mu k} \) as power series in \( z \) with values in \( k_{A_{m+1}}[Q_k, \Delta_k^{-1}] \).

The entries of the \( R \)-matrix are then given by

\[ R_{ik} \propto \frac{1}{\sqrt{\Delta_i}} e^{-u_k/z} \prod_{j \neq i} \left( \frac{\partial}{\partial t} \frac{\partial}{\partial t} - Q_j \right) S_{0k}. \]

Noticing that the change of basis from normalized idempotents to the basis \( 1, X, \ldots, X^m \) can be defined over \( k_{A_{m+1}}[Q_k, \Delta_k^{-1}] \), recalling that \( \text{disc} = \prod \Delta_i \), and applying Galois theory, we see that the endomorphism \( R \) is defined over \( k_{A_{m+1}}[\text{disc}^{-1}] \).

In the Airy limit \( t^2, \ldots, t^m \to 0 \) the quantum differential equation becomes the slightly modified higher Airy differential equation \[13\]

\[ \left( z \frac{\partial}{\partial t} \right)^{m+1} S_{0k} = -t^2 S_{0k}. \]

The entries of the \( R \)-matrix in this case are therefore related to the asymptotic expansions of the higher Airy functions and their derivatives when their complex argument approaches \( \infty \).

In the case of the \( A_2 \)-singularity we do not need to take any limit and discover the hypergeometric series \( A \) and \( B \) of Faber-Zagier in the
expansions of the (slightly modified) usual Airy function

\[
\frac{\exp\left(\frac{x^3}{3} + t^1 x^{1/2}/z\right)}{\sqrt{-2\pi z}} \int_{\Gamma_k} e^\frac{2}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2} - \frac{x^3}{3} \Delta \frac{z}{\sqrt{2\pi z}}\right) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-x^2}{2} - \frac{x^3}{3} \Delta \frac{z}{\sqrt{2\pi z}}\right) dx
\]

\[
\gamma \left(\frac{-z}{(9\Delta^2)}\right)^i = \frac{1}{\sqrt{\Delta}} \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \left(\frac{-z}{72\Delta^3}\right)^i
\]

and a derivative of it

\[
\frac{d\exp\left(\frac{x^3}{3} + t^1 x^{1/2}/z\right)}{d\Gamma_k} \int_{\Gamma_k} e^\frac{2}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2} - \frac{x^3}{3} \Delta \frac{z}{\sqrt{2\pi z}}\right) dx \approx \frac{1}{\sqrt{-t}} \sum_{i=0}^{\infty} \frac{(6i)!}{(3i)!(2i)!} \left(1 + 6\frac{i}{(2i)}\frac{-z}{72\Delta^3}\right)^i.
\]

Here \(\Delta = 2\sqrt{-t^1}\). The cycle \(\Gamma_k\) determines which square-root of \((-t^1)\) we take.

### 3.2 For equivariant \(\mathbb{P}^m\)

Givental [6] has given explicit solutions to the quantum differential equation for projective spaces in the form of complex oscillating integrals. Let us recall their definition and see how they behave in the match up with the \((m + 2)\)-spin theory.

Using the divisor axiom of Gromov-Witten invariants, the quantum differential equation implies the differential equations

\[
(D + \lambda_i)S_i = H \star S_i
\]

for the fundamental solutions \(S_i\) at the origin. Here we have written \(D = zq\frac{\partial}{\partial q}\). Equivalently, the equation says

\[
D(S_i e^{\ln(q)\lambda_i/z}) = H \star S_i e^{\ln(q)\lambda_i/z}.
\]

Therefore the entries of \(S\) with one index lowered satisfy

\[
(D - \bar{\lambda})(S_{\mu i} e^{\ln(q)\lambda_i/z}) = S_{(\mu + 1)i} e^{\ln(q)\lambda_i/z},
\]

where \(S_{(m+1)i}\) is defined such that

\[
\prod_{j=0}^m (D - \lambda_j)(S_{0i} e^{\ln(q)\lambda_i/z}) = q S_{0i} e^{\ln(q)\lambda_i/z}.
\]

The Greek indices stand for the basis of flat vector fields corresponding to \(1, H - \bar{\lambda}, \ldots, (H - \bar{\lambda})^m\).

Givental’s oscillating integral solutions for \(S_{0i}\) are stationary phase expansions of the integrals

\[
S_{0i} e^{\ln(q)\lambda_i/z} = (-2\pi z)^{-m/2} \int_{\Gamma_i C (\sum T_j = \ln q)} e^{F_i(T)/z} \omega
\]
along \( m \)-cycles \( \Gamma_i \) through a specific critical point of \( F_i(T) \) inside a \( m \)-dimensional \( \mathbb{C} \)-subspace of \( \mathbb{C}^m \), where

\[
F_i(T) = \sum_{j=0}^{m} (e^{T_j} + \lambda_j T_j).
\]

The form \( \omega \) is the restriction of \( dT_0 \wedge \cdots \wedge dT_m \). To see that the integrals are actual solutions, notice that applying \( D - \lambda_i \) to the integral has the same effect as multiplying the integrand by \( e^{T_j} \).

There are \( m+1 \) possible critical points at which one can do a stationary phase expansion of \( S_{0i} \). Let us write \( P_i = Q_i + \bar{\lambda} \) for the solution to

\[
\prod_{i=0}^{m} (X - \lambda_i) = q
\]

with limit \( \lambda_i \) as \( q \to 0 \). For each \( i \) we need to choose the critical point \( e^{T_j} = P_i - \lambda_j \) in order for the factor

\[
\exp(u_i/z) := \exp \left( \sum_{j=0}^{m} (P_i - \lambda_j + \lambda_j \ln(P_i - \lambda_j)) - \lambda_i \ln(q) \right) / z
\]

of \( S_{0i} \) to be well-defined in the limit \( q \to 0 \). Shifting the integral to the critical point and scaling coordinates by \( \sqrt{-1/z} \) we find

\[
S_{0i} = e^{u_i/z} \int \exp \left( - \sum_{j} (Q_i - \bar{\lambda}_j) \sum_{k=3}^{\infty} T_k^k (-z)^{(k-2)/2} / k! \right) d\mu_i
\]

for the conditional Gaussian distribution

\[
d\mu_i = (2\pi)^{-m/2} \exp \left( - \sum_{j} (Q_i - \bar{\lambda}_j) T_j^2 / 2 \right) \omega.
\]

The covariance matrices are given by

\[
\sigma_i(T_k, T_l) = \frac{1}{\Delta_i} \begin{cases} 
- \prod_{j \notin \{k,l\}} (Q_i - \bar{\lambda}_j), & \text{for } k \neq l, \\
\sum_{m \neq k} \prod_{j \notin \{k,m\}} (Q_i - \bar{\lambda}_j), & \text{for } k = l.
\end{cases}
\]

From here we can see that the integral is symmetric in the \( \bar{\lambda}_j \) and therefore we can write it completely in terms of data from \( A_{m+1} \). Since odd moments of Gaussian distributions vanish we find that \( e^{-u_i/z} S_{0i} \) is a power series in \( z \) with values in \( \Delta_i^{-1/2} k_{A_{m+1}} [Q_i, \Delta_i^{-1}, \lambda] \).

So the entries of the \( R \)-matrix in the basis of normalized idempotents are given by

\[
\Delta_k^{-1/2} \prod_{j \neq k} (D + \lambda_i - P_j) \left( e^{-u_i/z} S_{0i} \right).
\]

Since \( \frac{dP_i}{dq} = \frac{1}{\Delta_i} \) these entries are in

\[
k_{A_{m+1}} [Q_0, \ldots, Q_m, \Delta_0^{-1/2}, \ldots, \Delta_m^{-1/2}, \lambda].
\]

So, with the arguments from Section 3.1, the endomorphism \( R \) can be defined over \( k_{Pm} [\text{disc}^{-1}] \).
We need to check that the $R$-matrix given in terms of oscillating integrals behaves correctly in the limit $q \to 0$. By definition, in this limit $P_{1} \to \lambda_{i}$. By symmetry it is enough to consider the 0-th column. Set $x_{t} = e^{\bar{t}}$. Then

$$
\lim_{q \to 0} R_{j0} \propto \lim_{q \to 0} e^{-u_{0}/z} \Delta_{0}^{-1/2} \prod_{k \neq j} (q \frac{d}{dq} + \lambda_{j} - \lambda_{k}) \, S_{00}
$$

$$
= \lim_{q \to 0} \frac{e^{-u_{0}/z}}{\sqrt{\Delta_{0}(-2\pi z)^{m/2}}} \int e^{(\sum_{k \neq j} (e^{T_{k} + (\lambda_{0} - \lambda_{k})T_{k})/z + \sum_{k \neq j} T_{k} \omega)}
$$

$$
= \lim_{q \to 0} \frac{e^{-u_{0}/z}}{\sqrt{\Delta_{0}(-2\pi z)^{m/2}}} \int e^{\left(\sum_{k \neq j} (x_{k} - (\lambda_{0} - \lambda_{k})T_{k}) + \frac{z}{\prod_{k \neq j} T_{k}}\right)/z} \prod_{k \neq j} x_{j} \prod_{k=1}^{m} dT_{k}.
$$

In the last step we have moved to the chart

$$
x_{0} = \frac{q}{\prod_{j \neq 0} x_{j}}.
$$

Since in this chart $\lim_{q \to 0} x_{0} = 0$, we have that $R_{j0}$ vanishes unless $j = 0$. On the other hand in the limit $q \to 0$ the integral for $R_{00}$ splits into one-dimensional integrals

$$
\lim_{q \to 0} R_{00} \propto \lim_{q \to 0} \frac{e^{-u_{0}/z}}{\sqrt{\Delta_{0}(-2\pi z)^{m/2}}} \prod_{k \neq 0} \int_{0}^{\infty} e^{(x - (\lambda_{0} - \lambda_{k}) \ln(x))/z} dx.
$$

Let us temporarily set $z_{k} = -z/(\lambda_{0} - \lambda_{k})$. The prefactors also split into pieces in the limit and we calculate the factor corresponding to $k$ to be

$$
\frac{e^{(1-\ln(\lambda_{0} - \lambda_{k}))}/z_{k}}{\sqrt{-2\pi z(\lambda_{0} - \lambda_{k})}} \int_{0}^{\infty} e^{(x - (\lambda_{0} - \lambda_{k}) \ln(x))/z} dx = \frac{e^{(1-\ln(1/z_{k}))}/z_{k}}{\sqrt{2\pi z_{k}}},
$$

$$
= \frac{\Gamma\left(\frac{1}{2k}\right)}{\sqrt{2\pi z_{k}}} \geq \exp\left(\sum_{l=1}^{\infty} \frac{B_{2l}}{2l(2l-1)} \left(\frac{z}{\lambda_{k} - \lambda_{0}}\right)^{2l-1}\right),
$$

using Stirling’s approximation of the gamma function in the last step. So the product of the factors gives the expected limit $[1]$ of $R_{00}$ for $q \to 0$. This calculation gives a proof for the results [4] of Ionel on the main generating function used in [9] and [8] without having to use Harer stability.

4 $\mathbb{P}^{m}$ relations imply $(m+2)$-spin relations

We prove Theorem [1] in this section. As already mentioned, for this it is enough to show that, after the change $z \mapsto z \lambda^{-1}$, the $\mathbb{P}^{m}-R$-matrix converges to the $A_{m+1}-R$-matrix in the limit $\lambda \to \infty$. For this we have to compare the differential equations satisfied by the $R$-matrices.

Inserting the vector field corresponding to the hyperplane into (3) and using the divisor equation as in Section 3.2 gives the equation

$$
[R_{\mathbb{P}^{m}}, \xi] + zq \frac{dR_{\mathbb{P}^{m}}}{dq} + zq \Psi^{-1} \frac{dR_{\mathbb{P}^{m}}}{dq} = 0,
$$

where $\xi$ denotes the diagonal matrix of quantum multiplication by $H - \lambda$.  

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Lemma 1. $R_{p^m}(z/\lambda)$ admits a limit $R$ for $\lambda \to \infty$. The matrix $R$ satisfies

$$[R, \xi] + z \frac{dR}{dt^1} + z\Psi \frac{d\Psi^{-1}}{dt^1} R = 0$$

$$\frac{dR}{dz} + \sum_{\mu=1}^{m} \frac{m+2-\mu}{m+2} t^\mu \frac{dR}{dt^\mu} = 0$$

Proof. The $P^m$-matrix satisfies the homogeneity property

$$z \frac{dR_{p^m}}{dz} + (m+1)q \frac{dR_{p^m}}{dq} + \sum_{i=0}^{m} \lambda_i \frac{dR_{p^m}}{d\lambda_i} = 0.$$ 

So $R'(z) := R_{p^m}(z/\lambda)$ written with the $A_{m+1}$-variables satisfies

$$(m+2)z \frac{dR'}{dz} + (m+1)\lambda \frac{dR'}{d\lambda} + \sum_{\mu=1}^{m} (m+2-\mu) t^\mu \frac{dR'}{dt^\mu} = 0.$$ 

The main differential equation satisfied by $R'$ is

$$[R', \xi] + z \left(1 + t^1 \lambda \right) \frac{dR'}{dt^1} + z \left(1 + t^1 \lambda \right) \Psi \frac{d\Psi^{-1}}{dt^1} R' = 0.$$ 

From the expression of $R_{p^m}$ in terms of oscillating integrals we know that the entries of the $z^i$-part $R'_i$ of $R'$ live in

$$\lambda^{-i} k_{A_{m+1}} [Q_0, \ldots, Q_m, \Delta_0^{-1/2}, \ldots, \Delta_m^{-1/2}, \lambda].$$

To show that the limit exists we need to show that $\lambda$ occurs in no positive power. We will show this by induction by $i$. It certainly holds for $R'_0 = 1$. Since $\xi$ is diagonal with pairwise distinct entries $Q_j$, the $z^i$-part of the differential equation determines the off-diagonal coefficients of $R'_i$ in terms of $R'_{i-1}$. Because $\Psi \frac{d\Psi^{-1}}{dt^1}$ does not depend on $\lambda$, the off-diagonal coefficients of $R'_i$ will admit the limit $\lambda \to \infty$. Since $\Psi \frac{d\Psi^{-1}}{dt^1}$ in general vanishes on the diagonal the diagonal coefficient of the $z^{i+1}$-part of the differential equation determines the diagonal of $\frac{dR'_i}{dt^i}$ from the off-diagonal entries of $R'_i$. Apart from a possible term constant in $t^1$ we therefore know that also the diagonal entries of $R'_i$ admit the limit.

Let us consider such a possible ambiguity $a_i$. Since all products of $\Delta_j$ have dependence in $t^1$, the “denominator” of $a_i$ can only be a power of $\lambda$ less than $i$. However then $a_i$ cannot possibly satisfy the homogeneity. By induction therefore the limit $R$ exists. The properties of $R$ easily follow from the corresponding ones of $R'$.

By inserting the vector field $\frac{\partial}{\partial t^1}$ into (3) and similar arguments as in the proof of Lemma we one can show the following lemma.

Lemma 2. The $A_{m+1}$-matrix is uniquely determined from the differential equation

$$[R_{A_{m+1}}, \xi] + z \frac{dR_{A_{m+1}}}{dt^1} + z\Psi \frac{d\Psi^{-1}}{dt^1} R_{A_{m+1}} = 0.$$ 

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the homogeneity

\[ z \frac{d R_{A_{m+1}}}{dz} + \sum_{\mu=1}^{m} \frac{m+2-\mu}{m+2} \mu^\mu \frac{d R_{A_{m+1}}}{d\mu} = 0 \]

and that the entries of the z-series coefficients of \( R_{A_{m+1}} \) should lie in

\[ k_{A_{m+1}}(Q_0, \ldots, Q_m, \Delta_0^{-1/2}, \ldots, \Delta_m^{-1/2}). \]

The lemmas imply that the modified \( \mathbb{P}^m \)-R-matrix contains only non-positive powers of \( \lambda \) and the part constant in \( \lambda \) equals the \( A_{m+1} \)-matrix. Therefore the \( A_{m+1} \)-relations are contained in the modified \( \mathbb{P}^m \)-relations as the \( \lambda^0 \)-part, and we have completed the proof of Theorem 1.

5 Equivalence of relations

We want to give a proof of Theorem 2 in this section. So we will consider the CohFTs in the Airy limit, i.e. with all \( t^\mu \) but \( t := t^1 \) set to zero. In this limit the metric becomes \( \eta(X^i, X^j) = \delta_{i+j,m} \), the quantum product stays semisimple and the Euler vector field for the \( A_{m+1} \)-singularity

\[ E = m + 1 \frac{\partial}{\partial t} \]

is a multiple of \( X \).

Rewriting (6) for the \( \tilde{R}_{\mathbb{P}^m} \)-matrix \( \tilde{R}_{\mathbb{P}^m} = \Psi^{-1} R_{\mathbb{P}^m} \Psi \) written in flat coordinates gives

\[ [\tilde{R}_{\mathbb{P}^m}, \xi] - z q L_E \tilde{R}_{\mathbb{P}^m} + z q \tilde{R}_{\mathbb{P}^m} \mu = 0, \]

where \( \xi \) is multiplication by \( E \) in flat coordinates and \( \mu = -(L_E \Psi^{-1}) \).

We need to find a series \( \varphi \) in \( t \) and an \( R \)-matrix \( R \) sending the modified \( A_{m+1} \)-theory to equivariant \( \mathbb{P}^m \):

\[ \tilde{R}_{\mathbb{P}^m}(z) = R(z) \tilde{R}_{A_{m+1}}(z \varphi). \]

We know that \( \tilde{R}_{A_{m+1}} \) satisfies

\[ [\tilde{R}_{A_{m+1}}, \xi] + z L_E \tilde{R}_{A_{m+1}} - z \tilde{R}_{A_{m+1}} \mu = 0 \]

and the weighted homogeneity condition

\[ \left( z \frac{d}{dz} + L_E \right) \tilde{R}_{A_{m+1}} + [\mu, \tilde{R}_{A_{m+1}}] = 0. \]

Putting these together we find that \( R \) must satisfy

\[ 0 = [R, \xi] - z q L_E R + z q \frac{L_E \varphi}{\varphi} R \mu \]

\[ + \frac{1}{\varphi} \left( q + \varphi - q \frac{L_E \varphi}{\varphi} \right) R([\tilde{R}_{A_{m+1}}(z \varphi), \xi] \tilde{R}_{A_{m+1}}^{-1}(z \varphi)). \]

Lemma 3. The series \( \tilde{R}_{A_{m+1}}(z \varphi) \) is not a polynomial in \( z \).
Because of the lemma and the homogeneity of $\tilde{R}_{A_{m+1}}$ we see that in order for $R$ to exist in the limit $\text{disc} \to 0$ the function $\varphi$ has to satisfy

$$q + \varphi - q \frac{L_E \varphi}{\varphi} = 0$$

or equivalently

$$-q^{-1} = \varphi^{-1} + L_E \varphi^{-1}.$$ 

There is a unique solution $\varphi^{-1}$ to this differential equation. Concretely, we have

$$\varphi^{-1} = \lambda^{-1} \sum_{i=0}^{\infty} \frac{m + 2}{m + 2 + i(m + 1)} \left( -\frac{t}{\lambda} \right)^i.$$ 

Since it is not necessary for the proof of Theorem 2, we will prove Lemma 3 in Section 6.

Let us from now on assume that $\varphi$ is this solution. Then the differential equation for $R$ spells

$$[R, \xi] - zq L_E R + zq \frac{L_E \varphi}{\varphi} R_{\mu} = 0. \quad (7)$$

The following lemma implies that the matrix $\tilde{R}_{A_{m+1}}(z) \tilde{R}_{A_{m+1}}^{-1}(z \varphi)$ does not have any poles in $t$ and this concludes the proof of Theorem 2.

**Lemma 4.** For any solution $R(z)$ of (7) of the form

$$R(z) = \sum_{i=0}^{\infty} (R^i_{jk}) z^i = 1 + O(z),$$

for Laurent series $R^i_{jk}$ in $t$, actually all the $R^i_{jk}$ have to be polynomials.

**Proof.** The matrices $\xi$ and $\mu$ can be explicitly calculated

$$\xi_{jk} = t \frac{m + 1}{m + 2} \delta_{j,k+1} (-t)^{\delta_{0,j}}, \quad \mu_{jk} = \frac{2j - m}{2(m + 2)} \delta_{j,k},$$

where all indices are understood modulo $(m + 1)$.

Assume that we have already constructed $R^{i-1}$ and its entries have no negative powers in $t$. Looking at the $z^i$-part of (7) gives expressions for $R^i_{(j+1)k} + \xi_{(j+1)k} - \xi_{(j-1)k} R^i_{(j-1)k}$ as power series with no poles in $t$. From here we see that if we can determine the $R^i_{j0}$ as power series with no poles, then the other entries are given by

$$R^i_{jk} \equiv (-t)^{\delta_{k,j+1}} R^i_{(j-k)0}$$

modulo terms with no poles in $t$, determined from $R^{i-1}$. The exponent $\delta_{k,j+1}$ is 1 for $k > j$ and 0 otherwise.

From the $z^{i+1}$-part of (7) we then get expressions with no poles in $t$ for

$$(m + 1) t \frac{dR^i_{j0}}{dt} + jR^i_{j0},$$

thus determining all $R^i_{j0}$ but $R^i_{00}$ up to a constant. Therefore all the $R^i_{jk}$ are polynomials in $t$.

**Remark 5.** The derivation in this section would have worked the same if $q$ was any other invertible power series in $t$. 

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6 Higher dimensions

We would like to show that for $m > 1$ there is no pair of function $\varphi$ and matrix power series $R(z)$, both well-defined in the limit disc $\to 0$, such that

$$\tilde{R}_m(z) = R(z)\tilde{R}_{m+1}(z\varphi),$$

where again $\tilde{R}_m = \Psi R_* \Psi^{-1}$. We will need to assume that that $\varphi$ is well-defined in the Airy limit. Then we can use the discussion from Section 5 to derive that $\varphi$ is of the form

$$\varphi = \lambda + c_0\lambda^0 + c_{-1}\lambda^{-1} + \cdots,$$

where the $c_i$ are independent of $\lambda$ and $c_{-1}$ in the Airy limit becomes a constant multiple of $(t^1)^2$. For the uniqueness of $\varphi$ we needed Lemma 3.

**Proof of Lemma 3.** Recall that we have to show that $P := \tilde{R}_{m+1}\tilde{R}_{m+1}^{-1}$ is not a polynomial in $z$. From the differential equation for $\tilde{R}_{m+1}$ we obtain a differential equation for $P$.

$$[P, \xi] = z\frac{dP}{dz} - z[P, \mu]$$

By definition we also have the initial condition $P|_{z=0} = \xi$. Write $P = \xi + zP_1 + z^2P_2 + \cdots$. The homogeneity condition for $\tilde{R}_{m+1}$ implies that the only nonzero entries of $P_i$ are at the $(i-1)$-th diagonal, where by this we mean the entries on $j$-th row, $k$-th column such that $k - j \equiv i - 1 \pmod{m + 1}$.

Assume we have shown that $P_i \neq 0$ has a nonzero entry on the $(i-1)$-th diagonal row. Recalling the proof of Lemma 4 we see that essentially the differences of two subsequent entries in the $i$-th diagonal of $P_{i+1}$ are a multiple of an entry on the $(i-1)$-th diagonal of $P_i$. Since the absolute value of any entry of $\mu$ is less than $\frac{1}{2}$, all of these multiples are nonzero. Therefore it is impossible for all entries on the $i$-th diagonal of $P_{i+1}$ to be zero. The lemma follows by induction.

To show that there is no suitable intermediate $R$-matrix $R$ it will be enough to consider the $z^1$-term of $P$. It says

$$\tilde{r}_m = r + \varphi \tilde{r}_{m+1},$$

where $r_*$ stands for the $z^1$-term of $R_*$. Since $\tilde{r}_m$ has no negative powers in $\lambda$, the $\lambda^{-1}$-terms on the right hand side have to cancel. However the bottom-left coefficient of $\tilde{r}_{m+1}$ has a pole in the discriminant. Since for $m > 2$ the coefficient $c_{-1}$ cannot be a multiple of the discriminant for degree reasons, in this case $r$ has to have a pole in the discriminant. Contradiction.

It remains to look at the case $m = 2$. Here it is similarly enough to show that there is one coefficient in the $R$-matrix with a second order pole in the discriminant in order to derive a contradiction. We look at
We need to calculate the $z^1$-coefficient of the asymptotic expansion of

$$\sum Q_1 \sqrt{\frac{2}{\pi}} \Delta \int_{-\infty}^{\infty} \exp \left( -\frac{x^2}{2} - x^3 \sqrt{-2} \frac{Q}{\Delta^{3/2}} - \frac{x^4}{4} (\frac{1}{\Delta}) \right) dx,$$

where we sum over roots $Q$ of the polynomial defining the singularity and here $\Delta = 3Q^2 + 2t^2$. Expanding the Gaussian integral we find the coefficient to be equal to

$$-\sum Q \frac{15}{2} \frac{Q^2}{\Delta^4} + \sum Q \frac{3}{\Delta^3}.$$

It is straightforward to check that the first summand equals

$$\frac{15}{2} \frac{-2(2t^2)^3 + 27(t^1)^2}{(-4(2t^2)^3 - 27(t^1)^2)^2},$$

whereas the second term has only a first order pole in the discriminant.

A Givental’s localization calculation

We want to recall Givental’s localization calculation \[12\] that proves that the CohFT from equivariant $\mathbb{P}^m$ can be obtained from the trivial theory via a specific $R$-matrix action. We first recall localization in the space of stable maps to $\mathbb{P}^m$ in Section A.1 and look at the general procedure in Section A.2 before going into the details of the calculation.

A.1 Localization in the space of stable maps

Let $T = (\mathbb{C}^*)^{m+1}$ act diagonally on $\mathbb{P}^m$. Then the equivariant Chow ring of a point and $\mathbb{P}^m$ are given by

$$A_T^*(\text{pt}) = \mathbb{Q}[\lambda_0, \ldots, \lambda_m]$$

$$A_T^*(\mathbb{P}^m) = \mathbb{Q}[H, \lambda_0, \ldots, \lambda_m] / \prod_{\mu=0}^{m} (H - \lambda_\mu),$$

where $H$ is a lift of the hyperplane class. Let

$$[\mu] = \prod_{\nu \neq \mu} (H - \lambda_\nu)$$

for $\mu \in \{0, \ldots, m\}$ be the equivariant fundamental classes of the fixed points $p_\mu$ of the torus action on $\mathbb{P}^m$. We have

$$[\mu]^2 = [\mu] \prod_{\nu \neq \mu} (\lambda_\mu - \lambda_\nu) =: [\mu] \epsilon_\mu,$$

so that $\phi_\mu := \epsilon^{-1}_\mu [\mu]$ define orthogonal idempotents for the multiplication in the (classical) equivariant Chow ring. Here $\epsilon_\mu$ is also the equivariant Euler class of the tangent bundle of $\mathbb{P}^m$ at $\mu$. 

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The virtual localization formula \([16]\) says that the virtual fundamental class can be written as a sum over fixed loci of local contributions

\[
[M_{g,n}(\mathbb{P}^m, d)]^{vir}_{T} = \sum_{X} \frac{[X]^{vir}_{T}}{e_{T}(N_{X,T}^{vir})},
\]

where \(N_{X,T}^{vir}\) denotes the virtual normal bundle of \(X\) in \(\overline{M}_{g,n}(\mathbb{P}^m, d)\) and \(e_{T}\) the equivariant Euler class. Because of the denominator the fixed point contributions are only defined after localizing by the elements \(\lambda_0, \ldots, \lambda_m\). Using the deformation theory of the moduli space of stable maps \(e_{T}(N_{X,T}^{vir})\) can be calculated explicitly.

The fixed loci can be labeled by certain decorated graphs. They consist of

- a graph \((V, E)\),
- an assignment \(\zeta : V \to \{p_0, \ldots, p_m\}\) of fixed points,
- a genus assignment \(g : V \to \mathbb{Z}_{\geq 0}\),
- a degree assignment \(d : E \to \mathbb{Z}_{> 0}\),
- an assignment \(p : \{1, \ldots, n\} \to V\) of marked points,

such that the graph is connected and contains no self-edges, two adjacent vertices are not assigned to the same fixed point and we have

\[
g = h^1(\Gamma) + \sum_{v \in V} g(v), \quad d = \sum_{e \in E} d(e).
\]

A vertex \(v \in V\) is called stable if \(2g(v) - 3 + n(v) > 0\), where \(n(v)\) is the number of outgoing edges at \(v\).

The fixed locus corresponding to a graph is characterized by the condition that stable vertices \(v \in V\) of the graph correspond to contracted genus \(g(v)\) components of the domain curve, and that edges \(e \in E\) correspond to multiple covers of degree \(d(e)\) of the torus fixed line between two fixed points. Such a fixed locus is isomorphic to a product of moduli spaces of curves

\[
\prod_{v \in V} \overline{M}_{g(v), n(v)}
\]

modulo the group of automorphisms of the decorated graph.

For a fixed locus \(X\) corresponding to a given graph the Euler class \(e_{T}(N_{X,T}^{vir})\) is a product of factors corresponding to the geometry of the graph

\[
e_{T}(N_{X,T}^{vir}) = \prod_{v, \text{ stable}} \frac{e(E^* \otimes T_{\mathbb{P}^m, \zeta(v)})}{e_{\zeta(v)}} \prod_{nodes} \frac{1}{-\psi_1 - \psi_2} \prod_{g(v)=0}^{n(v)=1} (-\psi_v) \prod e \text{Contr}_e.
\]

In the first product \(E^*\) denotes the dual of the Hodge bundle, \(T_{\mathbb{P}^m, \zeta(v)}\) is the tangent space of \(\mathbb{P}^m\) at \(\zeta(v)\), and all bundles and Euler classes should be considered equivariantly. The second product is over nodes forced
onto the domain curve by the graph. They correspond to stable vertices together with an outgoing edge, or vertices of genus 0 and valuation 2. With \( \psi_1 \) and \( \psi_2 \) we denote the (equivariant) cotangent line classes at the two sides of the node. For example, the cotangent line class \( \psi \) at a fixed point \( p_i \) on a line mapped with degree \( d \) to a fixed line should be interpreted as

\[-\psi = \frac{\lambda_j - \lambda_i}{d},\]

where \( p_j \) is the other fixed point on the fixed line. The explicit expressions for the edge contributions \( \text{Contr} \) will play no role here; we only need to know that they are pulled-back from (the localization of) the equivariant Chow ring of a point.

### A.2 General procedure

For \( v_1, \ldots, v_n \in A^*_T(\mathbb{P}^m) \) the (full) CohFT \( \Omega_{g,n} \) from equivariant \( \mathbb{P}^m \) is given by

\[
\Omega_{g,n}^p(v_1, \ldots, v_n) = \sum_{d=0}^{\infty} \sum_{k=0}^{\infty} \frac{q^d}{k!} \pi^* \left( \prod_{i=1}^{v} \ev_i^*(v_1) \varepsilon_\ast \prod_{i=n+1}^{v+k} \ev_i^*(p) \cap \left[ \mathcal{M}_{g,n+k}(\mathbb{P}^m,d) \right]_{\text{vir}} \right),
\]

where

\[
p = \sum_{\mu=0}^{m} y^\mu \phi_\mu
\]

is a point on the formal Frobenius manifold, \( \varepsilon \) forgets the last \( k \) markings and \( \pi \) forgets the map. We want to calculate the push-forward via virtual localization. In the end we will arrive at the formula of the \( R \)-matrix action as described in Section 2.3.

Let us make the formula explicit in the case that all vectors put into \( \Omega \) are flat basis vectors \( \phi_\alpha \). For the topological field theory we use the basis of normalized idempotents. Recall the calculation of the TQFT in Example 1. Since the it vanishes unless one puts the same idempotent element into each argument, the reconstruction formula takes the form of a sum over dual graphs \( \Gamma \) with a coloring \( \zeta \) of the vertices according to which normalized idempotent has been chosen at each vertex. So we want to show

\[
\Omega_{g,n}(\phi_{\mu_1}, \ldots, \phi_{\mu_n}) = \sum_{\Gamma, \zeta} \frac{1}{\text{Aut}(\Gamma)} \xi^\zeta \left( \prod_{v} V^\zeta_{v_1} \zeta_{v_2} (-\psi_1, -\psi_2) \prod_{i=1}^{n} P^\zeta_{v_i} (-\psi_i) \prod_{v} \Delta_{v_i}^{n_e - 1} \sum_{k=0}^{\infty} \frac{1}{k!} \prod_{i=n_0 + 1}^{n_e + k} T^\zeta_{v_i} (-\psi_i) \right),
\]

where the first product is over edges of \( \Gamma \) and \( \psi_i \) should be interpreted as \( \psi \)-classes at each side of the node, the second product is over marked points and the third product is over the vertices of the graph. The series
$T$, $V$ and $P$ are defined in terms of the $R$-matrix and metric by

$$1 + zT(z) = \sqrt{\Delta_1(R^1)^1_1(z)},$$

$$V^{ij}(z, w) = \sqrt{\Delta_i \Delta_j} \sum_{\alpha, \beta} (R^1)^\alpha_i(z) \eta^{\alpha \beta} (R^1)^\beta_j(w) - \eta^{ij} z + w,$$

$$P^i_\mu(z) = \sqrt{\Delta_i} (R^1)^i_\mu(z),$$

where the $\Delta^{-1}_i$ are the norms of the idempotents. Roman indices will always denote coefficients in the normalized idempotent basis whereas Greek indices stand for components in the flat basis. A lower index of 1 stands applying the map to the identity.

The localization formula for the push-forward splits into contributions corresponding to localization graphs. We group contributions according to which dual graph $\Gamma$ in the moduli space of curves they push-forward to. Let us consider the fates $C'$ of domain curves $C$ under the stabilization map.

- For each vertex of $\Gamma$: There is a corresponding component of $C$ together with a collection of trees of rational curves which have to be contracted.
- For each edge: There is a tree of rational curves connecting the corresponding components of $C$ and this tree is contracted to a node of $C'$.
- For each marked leg: There is one component of $C$ the containing the corresponding marked point connected by a (possibly empty) tree of rational curves to the component of $C$ corresponding to the vertex the leg is attached to.

Therefore from the localization formula one immediately obtains an expression of the form of (9) where the coloring of the vertices of $\Gamma$ corresponds to which fixed points of the target the corresponding component is mapped to. However the vertex, edge and marking contributions are not as clearly separated as in (9). For each vertex of $\Gamma$ one still needs to calculate push-forwards of the form

$$\sum_{k=0}^{\infty} \frac{1}{k!} \varepsilon^k \left( \frac{e(E^* \otimes T^m \zeta)}{e(\zeta)} \prod_{i=1}^{n} \frac{1}{x_i - \psi_i} \prod_{i=n+1}^{n+k} Q(-\psi_i) \right),$$

(10)

where $\varepsilon : \overline{M}_{g,n+k} \to \overline{M}_{g,n}$ forgets the $k$ indistinguishable nodes at which a tree of rational curves is attached. For the variables $(-x_i)$ later the cotangent line classes at the other side of the node have to be substituted. The series $Q$ contains the generating series contribution of a single tree to the localization formula.

The factors $(x_i - \psi_i)^{-1}$ bring dependence on the adjacent edges and legs into the vertex contribution (10). In Section A.3 we will study some integrals on $\overline{M}_{0,n}$ which will later be used to move out these factors while slightly modifying the integrand. In Section A.4 we will see how the modified leg, edge and vertex contributions can be interpreted in terms of a fundamental solution $S^i$ to the quantum differential equation. Later in Section A.5 we see how to interpret the occurring Chern classes of Hodge bundles and finish the proof of (9).
A.3 String and Dilaton Flow

We will need some identities on intersection numbers in $\overline{M}_{0,n}$.

Let $Q = Q_0 + zQ_1 + z^2Q_2 + \cdots$ be a formal series viewed as a point on an infinite dimensional manifold $M$. The string and dilaton vector fields $L, D$ are defined by

$$L = \frac{\partial}{\partial Q_0} + Q_1 \frac{\partial}{\partial Q_0} + Q_2 \frac{\partial}{\partial Q_1} + \cdots,$$

$$D = - \frac{\partial}{\partial Q_1} - Q_0 \frac{\partial}{\partial Q_0} - Q_1 \frac{\partial}{\partial Q_1} - Q_2 \frac{\partial}{\partial Q_2} - \cdots.$$

Notice that the two vector fields commute. Define the formal functions

$$u = \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\overline{M}_{0,n+2}} \prod_{i=1}^{n} Q(-\psi_i),$$

$$\Delta = \frac{\partial u}{\partial Q_0} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \int_{\overline{M}_{0,n+2}} \prod_{i=1}^{n} Q(-\psi_i),$$

and notice that the string and dilaton equations imply that $Lu = 1$ and $D\Delta = \Delta$. Since $u(Q) = 0$ if $Q^0 = 0$, for any function $F$ on $M$ which is an eigenvector to $L$ with eigenvalue $a$, we have

$$F(Q) = e^{au(Q)} F(Q')$$

where $Q'$ is the point on $M$ obtained by following the integral curve of $L$ to a point with vanishing $z^0$-coordinate. If $F$ is also an eigenvector to $D$ with eigenvalue $b$, we can similarly move along integral curves of the dilaton flow to a point $Q''$ with vanishing $z^0$- and $z^1$-coordinates. Since $\Delta$ is constant along integral curves of $L$ and is equal to 1 when both $z^0$- and $z^1$-coordinates are zero, we obtain

$$F(Q) = \Delta(Q)^{b/2} e^{au(Q)} F(Q'').$$

We immediately obtain the identities

$$1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\overline{M}_{0,n+2}} \prod_{i=1}^{n} Q(-\psi_i) = e^{u/2}$$

$$\frac{1}{z + w} + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\overline{M}_{0,n+2}} \prod_{i=1}^{n} Q(-\psi_i) = e^{u/z + u/w}$$

since the left hand sides of (13) and (14) are eigenvectors of $L$ with eigenvalues $1/z$ and $1/z + 1/w$ respectively, and the integrals vanish for $Q^0 = 0$.

Similarly, we obtain the following identity for the push-forward under $\varepsilon : \overline{M}_{g,n+k} \to \overline{M}_{g,n}$:

$$\sum_{k=0}^{n+k} \frac{1}{k!} \varepsilon^* \left( \prod_{i=1}^{n} \frac{1}{x_i - \psi_i} \prod_{i=n+1}^{n+k} Q(-\psi_i) \right)$$

$$= \Delta^{2(z-2+n)} \prod_{i=1}^{n} x_i - \psi_i \sum_{k=0}^{n+k} \frac{1}{k!} \varepsilon^* \prod_{i=n+1}^{n+k} Q''(-\psi_i)$$

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To see this notice that the left hand side is an eigenvector to \( L \) with eigenvalue \( x_1^{-1} + \cdots + x_n^{-1} \) and to \( D \) with eigenvalue \( 2g - 2 + n \). At the point \( Q'' \) (or already at \( Q' \)) the factors \( (x_i - \psi_i)^{-1} \) can be pulled out the push-forward.

### A.4 Expressing localization series in terms of Frobenius structures

For \( n \geq 3 \) and functions \( f_1, \ldots, f_k \) in \( n + 1 \) variables we introduce a short hand notation for genus 0 Gromov-Witten integrals

\[
(f_1(\phi, \psi), \ldots, f_k(\phi, \psi)) := \sum_{d=0}^{\infty} \sum_{n=0}^{\infty} \frac{q^d}{n!} \int \prod_{i=1}^{k} f_i(\text{ev}_i^*(\phi_i), \psi_i) \prod_{i=k+1}^{k+n} \text{ev}_i^*(p).
\]

Then define

\[
S^t_{i}(z) := \sqrt{\epsilon_i} \left( \frac{\phi_i}{z - \psi} \right) := \sqrt{\epsilon_i} \left( \frac{\phi_i}{z - \psi}, 1 \right)
\]

viewed as a power series in \( z^{-1} \). The second equation is motivated by the string equation. Notice that by genus 0 topological recursion

\[
z \left( \phi_\alpha, \frac{\phi_i}{z - \psi}, \phi_\beta \right) = \sum_{\gamma, \delta} \langle \phi_\alpha, \phi_\beta, \phi_\gamma \rangle \eta_{\gamma \delta} \left( \frac{\phi_i}{z - \psi}, 1 \right),
\]

and \((S^t)\), therefore satisfies the quantum differential equation

\[
z \frac{\partial}{\partial y^\beta} (S^t)_i = \phi_\beta \ast (S^t)_i.
\]

### A.4.1 Legs

The matrix \( S \) can be calculated using the localization formula in genus 0. We will obtain the formula

\[
e^{-1/2} S_a(z) = \delta_a^\epsilon \frac{e^{\epsilon u^j/z}}{\epsilon_i} + \frac{e^{u^j/z}}{\epsilon_i} (P_\alpha^j(z) - \delta_i^a) = \frac{e^{u^j/z}}{\epsilon_i} P_\alpha^j(z).
\]

whose terms will be explained in its following derivation.

There are two kinds of localization graphs depending on whether the \( \phi_\alpha \)- and \( \phi_i \)-insertions are at the same component of the domain curve or not. These correspond to the two summands in (17).

In the first case an integral as in (13) needs to be calculated, where \( Q \) is the series mentioned in Section A.2 recording the localization contribution of a tree of rational curves. There are actually \( m + 1 \) series \( Q' \) depending on the fixed point the attachment point of the tree is mapped to. Then \( u^\alpha \) is defined as in (11). In (17) the \( u^\alpha \) are shown to give canonical coordinates on the Frobenius manifold.

In the case that the insertions are at different components of the domain curve, let us call the components of the domain curve with \( \phi_\alpha \)- or
\(\phi_i\)-insertions \(\alpha\)- or \(i\)-component. The contribution in this case is strongly related to the generating series \(P^i_\alpha\) recording the leg contribution in the localization formula. The series \(P^i_\alpha\) is defined as \(\delta^i_\alpha\), corresponding to the case that the components corresponding to vertex and leg coincide, plus the sum of localization contributions of the trees of rational curves corresponding to a marking at \(\alpha\) starting from a component \(i\) but with

\[
\frac{e^{u/\psi_i}}{z-\psi_i}
\]

put at the vertex \(i\). This term is motivated by (15).

We use (14) for \(Q = Q^i\), and \(-w\) being the cotangent line class at the other side of the node connecting the \(i\)-component to the tree to the \(\alpha\)-component. The summands on the left hand side in (14) correspond to the case that there are no or \(n \geq 1\) trees at the \(i\)-component, respectively. A part of the right hand side of (14) is put into the definition of the series \(P^i_\alpha\).

We have determined the series \(P^i_\alpha\) in terms of \(S^i_\alpha\):

\[
e_i^{-1/2} P^i_\alpha(z) = e^{-u/\psi_i} S^i_\alpha(z) =: (R^i)^{ij}_\alpha(z) \tag{18}
\]

### A.4.2 Edges

The WDVV equation and the unit axiom in GW-theory imply

\[
\left\langle \frac{\phi_i}{z-\psi}, \frac{\phi_j}{w-\psi} \right\rangle := \frac{zw}{z+w} \left( \frac{\phi_i}{z-\psi} \frac{\phi_j}{w-\psi} - 1 \right) = \frac{(e_i e_j)^{-1/2}}{z+w} \sum_{\alpha,\beta} S^i_\alpha(z) \eta^{\alpha\beta} S^j_\beta(w).
\]

In particular

\[
\sum_{\alpha,\beta} S^i_\alpha(z) \eta^{\alpha\beta} S^j_\beta(-z) = \eta^{ij}.
\]

Using genus 0 localization we find the formula

\[
\left\langle \frac{\phi_i}{z-\psi}, \frac{\phi_j}{w-\psi} \right\rangle = \delta_{ij} \frac{e^{u/\psi_i + u/\psi_j}}{(z+w)e_i} + \frac{e^{u/\psi_i + u/\psi_j}}{e_i e_j} V^{ij}(z,w).
\]

The first summand corresponds to the case that the \(\phi_i\) and \(\phi_j\) insertions are at the same component. Its expression follows from (14).

The second summand corresponds to the case that the insertions are at different components. The generating series \(V^{ij}\) records the contribution of the genus 0 localization formula for a tree connecting components mapped to \(i\) and \(j\) respectively but with

\[
\frac{e^{u/\psi_i}}{z-\psi_i} \quad \text{and} \quad \frac{e^{u/\psi_j}}{w-\psi_j}
\]

put at the ends of the tree. Using (14) for both special components we verify the claimed expression.
Solving for $V^{ij}$ gives

$$(e_i e_j)^{-1/2} V^{ij}(z, w) = \frac{\sum_{\alpha, \beta} e^{-u^i / z} S^i_\alpha(z) \eta^{\alpha \beta} e^{-u^j / w} S^j_\beta(w) - \eta^{ij}}{z + w} = \frac{\sum_{\alpha, \beta} (R^i)_\alpha(z) \eta^{\alpha \beta} (R^j)_\beta(w) - \eta^{ij}}{z + w}.$$ 

**A.4.3 Vertices**

Finally we calculate

$$\left\langle \frac{\phi_i}{z - \psi} \right\rangle := z \left\langle \frac{\phi_i}{z - \psi}, 1 \right\rangle = e_i^{-1/2} z S^i_1(z)$$

via localization. The result is

$$\left\langle \frac{\phi_i}{z - \psi} \right\rangle = \frac{z + Q'(z)}{e_i} + \frac{1}{e_i} \sum_{n=2}^{\infty} \frac{1}{n!} \int_{M_0, n+1} \prod_{j=1}^n Q'(-\psi_j) \frac{1}{z - \psi_{n+1}}. \quad (19)$$

The first summand corresponds to the degenerate degree 0 contribution. The second and third summand correspond to the cases where there are 1 or $n \geq 2$ trees at the $i$-component.

Since the right hand side of (19) is an eigenvector of $\mathcal{L}$ and $\mathcal{D}$ with eigenvalue $1/z$ and $-1$ respectively, we can move along string and dilaton flows to obtain

$$\left\langle \frac{\phi_i}{z - \psi} \right\rangle = \frac{z + Q''(z)}{\sqrt{e_i} \sqrt{\Delta_i}},$$

where $\Delta_i$ is defined as $e_i$ times $\Delta$ as defined in (12). In (17) it is shown that these $\Delta_i$ are the same as in Section A.2. Finally, we obtain

$$Q''(z) = z \left( \sqrt{\Delta_i} e^{-u^i / z} S^i_1(z) - 1 \right) = z \left( (R^i)_1(z) - 1 \right).$$

**A.5 Hodge bundle**

Going back to higher genus push-forward we need to calculate in order to derive (9), notice that (15) can be used to bring the push-forward into the right form, as a sum over colored dual graphs of gluing map push-forwards of products vertex, edge and leg series. In the previous sections we have identified the different contributions.

If the Hodge bundle was trivial and therefore $e(\mathcal{E}^* \otimes T_{\mathcal{V} = \xi}) = e^\xi$, the various appearances of powers of $e_i$ would cancel out and we would arrive (9) with $R$-matrix defined from $S^t$ as defined in (16).

Because of

$$\frac{e(\mathcal{E}^* \otimes T_{\mathcal{V} = \xi})}{e^\xi} = \prod_{j \neq \xi} c_{\xi \to \lambda_j} (\mathcal{E}^*) = \prod_{j \neq \xi} c_{\gamma_j \to \xi} (\mathcal{E})$$

the actual CohFT is obtained by first applying $R$-matrix of a direct sum of CohFTs as in Example 2 to the trivial CohFT and then the $R$-matrix obtained from asymptotic expansion of $S^t$. 

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The $R$-matrix from (18) in the limit $q \to 0$ becomes the identity matrix, as can be seen from the definition of the leg series $P$ and the fact that the limit of the quantum product is the usual equivariant intersection product. Therefore the correct $R$-matrix satisfies the expected behavior in the limit $q \to 0$.

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