On the local pressure of the Navier-Stokes equations and related systems

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Abstract

In the study of local regularity of weak solutions to systems related to incompressible viscous fluids local energy estimates serve as important ingredients. However, this requires certain informations on the pressure. This fact has been used by V. Scheffer in the notion of a suitable weak to the Navier-Stokes equation, and in the proof of the partial regularity due to Caffarelli. Kohn and Nirenberg. In general domains, or in case of complex viscous fluid models a global pressure doesn’t necessarily exist. To overcome this problem, in the present paper we construct a local pressure distribution by showing that every distribution $\partial_t u + F$, which vanishes on the set of smooth solenoidal vector fields can be represented by a distribution $\partial_t \nabla p_h + \nabla p_0$, where $\nabla p_h \sim u$ and $\nabla p_0 \sim F$.

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1 Introduction

Let $\Omega \subset \mathbb{R}^n (n \geq 2)$ be an open set. By $W^{1,q}_0(\Omega)^n (1 < q < +\infty)$ we denote the
 closure of $C_c^\infty(\Omega)^n$ under the usual Sobolev norm. By $W^{-1,q}(\Omega)^n$ we denote the dual of
$W^{1,q}_0(\Omega)^n$. For $p \in L^q(\Omega)$ by $\nabla_q p$ we mean the functional in $W^{-1,q}(\Omega)^n$ determined by

$$\langle \nabla_q p, \psi \rangle = - \int_{\Omega} p \nabla \cdot \psi \, dx, \quad \psi \in W^{1,q}_0(\Omega)^n. \tag{1.1}$$

In the present paper we are interested in the existence of a projection $E_q : W^{1,q}_0(\Omega)^n \to
W^{1,q}_0(\Omega)^n$ such that

$$E_q(\psi) = 0 \quad \forall \psi \in C_c^\infty(\Omega)^n \text{ with } \nabla \cdot \psi = 0, \tag{1.2}$$
$$E_q(\nabla \phi) = \nabla \phi \quad \forall \phi \in C_c^\infty(\Omega). \tag{1.3}$$

As we will see below, the existence of such projection is ensured if $\Omega$ is sufficiently
 regular. However, (1.2) and (1.3) do not guarantee the uniqueness of $E_q$, so we may
 replace condition (1.3) by a more restrictive one. Moreover, due to (1.2) the dual
 projection $E_q^* : W^{-1,q}(\Omega)^n \to W^{-1,q}(\Omega)^n$ enjoys the property

$$E_q^*(\nabla_q p) = \nabla_q p \quad \forall p \in L^q(\Omega), \tag{1.4}$$

so that $E_q^*$ appears to be a useful tool for constructing the pressure of weak solutions
to the equations modelling the motion of an incompressible fluid. In three dimensions,
this systems consist of four equations, formed by the conservation of momentum and
the conservation of volume including four unknowns, the pressure $p$ and the velocity
field $\mathbf{u} = (u^1, u^2, u^3)$. Among this fluid models perhaps the Navier-Stokes system is
one of the most popular. Since the pioneering work by J. Leray [13] the theory of the

1) Here $q' = \frac{q}{q-1}$ if $1 < q < +\infty$, $q' = +\infty$ if $q = 1$ and $q' = 1$ if $q = +\infty$.

2) Here $\nabla \cdot \mathbf{v}$ stands for the divergence $\sum_{i=1}^n \partial_i v^i$ of the field $\mathbf{v} = (v^1, \ldots, v^n)$. 

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Navier-Stokes equations has been widely developed, where fundamental problems such as existence of weak solutions, conditions for global and local regularity or asymptotical behaviour have been solved. However, despite strong efforts one of the most important question, the existence of a unique global regular solution for general smooth data is still open. Partial answers to this fundamental question have been given, such as sufficient conditions for global or local regularity, which have been relaxed step by step in recent years. Concerning the local regularity, a first result goes back to V. Scheffer [15], who introduced the notion of a suitable weak solutions to the Navier-Stokes equations, fulfilling a local energy inequality. In 1982, based on Scheffer’s notion, L. Caffarelli, R. Kohn and L. Nirenberg [3] obtained an optimal result of partial regularity of suitable weak solutions, by showing that the one-dimensional parabolic Hausdorff measure of the singular set is zero. Recently, this result has been improved logarithmically by Choe and Lewis in [5]. For alternative proofs of Caffarelli-Kohn-Nirenberg theorem we refer to [14, 12, 18, 20].

The proof of the Caffarelli-Kohn-Nirenberg theorem rests on decay estimates, derived from the local energy inequality, which holds for suitable weak solutions (cf. [15, 3]). Unlike weak Leray-Hopf solutions, which are constructed by using Galerkin approximation, suitable weak solutions have to be constructed differently. It is still unclear, whether a Leray-Hopf solution is suitable or not. Furthermore, since the existence of a suitable weak solution depends on the existence of a global pressure (cf. [17, 16] and [7] for general uniform $C^2$-domains) this method requires that $\Omega$ is sufficiently regular. Note, that the same problem occurs in other fluid models, such as non-Newtonian fluids and fluids with variable viscosity. In the recent paper [22] a local pressure projection has been introduced to obtain the Caffarelli-Kohn-Nirenberg theorem for local suitable weak solutions to the Navier-Stokes equations in arbitrary cylindrical domain domains $Q = \Omega \times [0, T]$, for any open set $\Omega \subset \mathbb{R}^3$. In fact, this method suggests to even work with distributional weak solutions to the Navier-Stokes equations with variable viscosity, and related systems. However, one has to be careful as the following example of a potential-like solution shows (cf. [9, footnote p.79]).

Example. Let $\eta : ]0, T[ \to \mathbb{R}$ be any function and let $\phi : \Omega \to \mathbb{R}^3$ be harmonic. Then it is not difficult to check that $u(x, t) = \nabla \phi(x) \eta(t)$ solves the Navier-Stokes equations

\[
\nabla \cdot u = 0 \quad \text{in} \quad \Omega \times ]0, T[,
\]
\[
\partial_t u + (u \cdot \nabla) u - \Delta u = -\nabla p \quad \text{in} \quad \Omega \times ]0, T[.
\]

in the sense of distribution, where the pressure is given by the following distribution

\[
p = -\phi \eta' - \frac{1}{2} |\nabla \phi|^2 \eta^2.
\]

Note that in this example we have not imposed any boundary condition on $u$. In fact, in case of no slop boundary condition and if $\Omega$ is sufficiently smooth $u$ becomes trivial.

As the above example shows, the pressure might not be a Lebesgue function and it is unlikely to improve the time regularity for a distributional solution $u$. On the other hand, the pressure has the following form

\[
p = \partial_t p_h + p_0, \quad p_h = -\phi \eta, \quad p_0 = -\frac{1}{2} |\nabla \phi|^2 \eta^2,
\]

(1.5)
where \( p_h(t) \) is harmonic for a.e. \( t \in ]0, T[ \), which suggests to introduce the local pressure taken as in (1.5) on suitable subdomains. This will be done with help of a projection \( E_q^* \) fulfilling (1.2) and (1.3). In fact, such method of pressure representation on subdomains has introduced first in [19] and later used in [20] to construct a weak solution to the equations of non-Newtonian fluids in general domains. This method has played also an significant role to achieve further results concerning existence and regularity of weak solutions to models related to incompressible viscous fluids (cf. [2], [20], [21]).

In the present paper we wish to generalize the method introduced above such that it can be used for any given distributional solution to the Navier-Stokes equation or related systems in a cylindrical domain \( Q = \Omega \times ]0, T[ \) \( (0 < t < +\infty) \) governing incompressible viscous fluids. Our main result will be the characterization of distributions of the form \( \partial_t u + \nabla \cdot A \) in \( Q \) vanishing on the space of all smooth solenoidal fields with compact support in \( Q \), by a distribution involving gradient fields only. More precisely, for every \( C^1 \) subdomain \( G \subset \Omega \) ³ there are pressure functions \( p_{h,G}(t) \) and \( p_{0,G}(t) \) with \( \nabla p_{h,G}(t) \sim u(t) \) and \( p_{0,G}(t) \sim A(t) \) (for a.e. \( t \in ]0, T[ \)) satisfying \( \partial_t u + \nabla \cdot A = -\partial_t \nabla p_{h,G} - \nabla p_{0,G} \) in the sense of distributions, i.e.

\[
\int_0^T \int_G -\nabla \cdot \nabla \varphi - A : \nabla \varphi dxdt = \int_0^T \int_G -p_{h,G} \cdot \partial_t \varphi - p_{0,G} \nabla \cdot \varphi dxdt \tag{1.6}
\]

for all \( \varphi \in C_c^\infty(Q)^n \). Here \( u = (u^1, \ldots, u^n) \) stands for the velocity field of the fluid and \( A = \{A_{ij}\} \) for an \( n \times n \) tensor modelling the fluid system. For instance, the Navier-Stokes equation is modelled by

\[
A = u \otimes u - \nu \nabla u, \quad \nabla \cdot u = 0 \quad \text{in} \quad Q,
\]

where \( \nu = \text{const} > 0 \) denotes the viscosity of the fluid.

The paper is organized as follows. In Section 2 we provide some notations and function spaces used throughout the paper. In Section 3 we introduce the space \( G^{-1,q}(\Omega)^n \) containing all functionals \( u^* \in W^{-1,q}(\Omega)^n \) which vanish on the space of solenoidal vector fields. As we will see below the space \( G^{-1,q}(\Omega)^n \) contains distributions of the form \( \nabla p \). Then we are interested in domains \( \Omega \) for which there exists a projection \( E_q^* \) from \( W^{-1,q}(\Omega)^n \) onto \( G^{-1,q}(\Omega)^n \) being the dual of a projection \( E_{q'} \) of \( W_0^{1,q'}(\Omega)^n \) onto a closed subspace, fulfilling (1.2) and (1.3). Such domains will be called \( \nabla_q \)-regular, and they will be used for the construction of the pressure representation \( \nabla p = \partial_t \nabla p_h + \nabla p_0 \). Section 4 deals with the existence of unique \( q \)-weak solutions to the Stokes-like system, which appears to be a sufficient criterion for \( \nabla_q \)-regularity. This will be verified for the following cases, (i) \( \Omega = \mathbb{R}^n \), (ii) \( \Omega \) is a bounded \( C^1 \) domain and (iii) \( \Omega \) is a exterior \( C^1 \)-domain. In Section 5 we first introduce a sufficient criterion for \( \nabla_q \)-regularity based on the existence and uniqueness of \( q \)-weak solutions to the Stokes-like system. Next, in Section 6 we present our first main result concerning the local pressure decomposition

\[\text{Here, for two sets } A, B \subset \mathbb{R}^n, \text{ the notation } A \Subset B \text{ means } \overline{A} \subset B, \text{ and } \overline{A} \text{ compact.} \]

\[\text{For matrices } A, B \in \mathbb{R}^{n \times n} \text{ by } A : B \text{ we denote the scalar product } \sum_{i,j=1}^n A_{ij} B_{ij}. \]
for time-dependent distribution. Then, in Section 7 for the case $q = 2$ we provide a
global pressure representation for a general domain by using orthogonal projections due
to the Hilbert space structure. We complete our discussion by applying the result of
Section 7 to the case of the generalized Navier-Stokes equations. At the end of the pa-
ter we have added Appendix A, recalling some well-known properties of vector-valued
function in suitable Bochner spaces and Appendix B, where we discuss the continuity
of time dependent potentials of spatial gradients being continuous in space and time,
which is related to the continuity of the harmonic pressure.

2 Notations and function spaces

Let $\Omega \subset \mathbb{R}^n$ ($n \in \mathbb{N}, n \geq 2$) denote a domain. If necessary the properties of $\Omega$ will be
specified. By $W^{k,q}(\Omega), W^{k,q}_0(\Omega)$ ($1 < q < +\infty; k \in \mathbb{N}$) we denote the usual Sobolev
spaces. If $\Omega$ is bounded, the space $W^{k,q}_0(\Omega)$ will be equipped with the norm
\[
\|u\|_{W^{k,q}_0(\Omega)} = \left( \sum_{|\alpha| = k} \|D^\alpha u\|_{L^q}^q \right)^{1/q},
\]
otherwise with the usual Sobolev norm. For $1 < q < +\infty$, the dual of $W^{k,q}_0(\Omega)$ will be denoted by $W^{-k,q}(\Omega)$. Throughout, without any reference vector valued or
tensor-valued functions will be denoted by boldface letters.

Next, by $C_{c,\text{div}}^\infty(\Omega)^n$ we denote the space of all smooth solenoidal vector fields $\psi : \Omega \to \mathbb{R}^n$ having its support in $\Omega$. Then by $W^{1,q}_{0,\text{div}}(\Omega)^n$ we denote the closure of $C_{c,\text{div}}^\infty(\Omega)^n$
with respect to the norm in $W^{1,q}_{0,\text{div}}(\Omega)^n$ ($1 \leq q < +\infty$). Similarly, by $L^q_{\text{div}}(\Omega)^n$ we denote the
closure of $C_{c,\text{div}}^\infty(\Omega)^n$ with respect to the $L^q$-norm ($1 \leq q < +\infty$). Furthermore,
in case meas $\Omega < +\infty$ by $L^q_{\text{div}}(\Omega)^n$ we denote the subspace of all $p \in L^q(\Omega)$ such that
\[
\int_\Omega p \, dx = 0.
\]
Let $J_q = J_q,\Omega : W^{1,q}_0(\Omega)^n \to W^{-1,q}(\Omega)^n$ be defined by
\[
\begin{align*}
\langle J_q u, v \rangle &= \int_\Omega \nabla u : \nabla v dx \quad \text{if } \Omega \text{ is bounded} \\
\langle J_q u, v \rangle &= \int_\Omega \nabla u : \nabla v + u \cdot v dx \quad \text{if } \Omega \text{ is unbounded} \\
\end{align*}
\]
\[u \in W^{1,q}_0(\Omega)^n, v \in W^{1,q}_0(\Omega)^n \quad (1 < q < +\infty).\]

Note that $J_q$ defines an isomorphism in the cases (i) $\Omega = \mathbb{R}^n$, (ii) $\Omega = \mathbb{R}^n_+$ or (iii) $\Omega$
is a $C^1$-domain with compact boundary. Note that if $\Omega \subset \mathbb{R}^n$ is a bounded Lipschitz
domain then there exists $3 < q_1 < +\infty$ if $n \geq 3$ or $4 < q_0 < +\infty$ if $n = 2$ such that
$J_q$ is an isomorphism for all $q_0' < q < q_0$, which is in some sense sharp (cf. [11]). In
the special case $q = 2$, $J_2$ defines isomorphism for any domain, which is due to the
Hilbert space structure. In fact, here $J_2$ coincides with the duality map for $W^{1,2}_0(\Omega)^n$,
while $J_2^{-1} u^* \in W^{1,2}_0(\Omega)^n$ appears to be the Riesz representation of the functional
$u^* \in W^{-1,2}(\Omega)^n$. 

Let \( u \in L^1_{\text{loc}}(\Omega)^n \). We say \( u \in W^{-1,q}(\Omega)^n \) for \( 1 < q < +\infty \) if there exists \( c = \text{const} > 0 \) such that

\[
\langle u, \psi \rangle := \int_\Omega u \cdot \psi dx \leq c\|\psi\|_{W^{1,q}} \quad \forall \psi \in C^\infty_c(\Omega)^n.
\]

Hence, there exists a unique \( f \in W^{-1,q}(\Omega)^n \) such that \( \langle u, \psi \rangle = \langle f, \psi \rangle \) for all \( \psi \in C^\infty_c(\Omega)^n \). In this case we may identify \( u \) with \( f \), which justifies the above notation.

For our discussion below we use the notation \( \nabla_q = \nabla_{q, \Omega} \) for the gradient operator, mapping from \( L^q(\Omega) \) into \( W^{-1,q}(\Omega)^n \) defined by (1.1). From this definition we immediately derive that the dual \( \nabla^*_q \) equals the divergence operator \( -\nabla \cdot \) mapping from \( W^{1,q}_0(\Omega)^n \) into \( L^q(\Omega) \), i.e. \( \nabla^*_q v = -\nabla \cdot v \) for \( v \in W^{1,q}_0(\Omega)^n \).

Let \( X \) be a Banach space with norm \( \| \cdot \|_X \). Let \( -\infty \leq a < b \leq +\infty \). By \( L^s(a,b; X) \) \((1 \leq s \leq +\infty)\) we denote the space of all Bochner measurable functions \( f : ]a,b[ \to X \) such that

\[
\int_a^b \| f(t) \|_X^s dt < \infty \quad \text{if } 1 \leq s < \infty; \quad \text{ess sup}_{t \in (a,b)} \| f(t) \|_X < \infty \quad \text{if } q = \infty.
\]

## 3 The space \( G^{-1,q}(\Omega)^n \)

The present section deals with properties of functionals \( u^* \in W^{-1,q}(\Omega)^n \) vanishing on \( C^\infty_{c, \text{div}}(\Omega)^n \). To this end, we introduce the notion of \( W^{-1,q} \)-potential.

**Definition 3.1.** Let \( 1 < q < +\infty \). A function \( p \in L^q_{\text{loc}}(\Omega) \) is called a \( W^{-1,q} \)-potential if there exists \( u^* \in W^{-1,q}(\Omega)^n \) such that

\[
(3.1) \quad \langle u^*, \psi \rangle = -\int_{\Omega} p \nabla \cdot \psi dx \quad \forall \psi \in C^\infty_c(\Omega)^n.
\]

The set of all \( W^{-1,q} \)-potentials will be denoted by \( L^q_{\text{pot}}(\Omega) \).

If \( p \in L^q_{\text{pot}}(\Omega) \) and \( u^* \in W^{-1,q}(\Omega)^n \) fulfilling (3.1) we use the brief notation \( u^* = \nabla_q p \). Then we define

\[
(3.2) \quad G^{-1,q}(\Omega)^n := \{ \nabla_q p \in W^{-1,q}(\Omega)^n \mid p \in L^q_{\text{pot}}(\Omega) \}.
\]

**Remark 3.2.** If \( p \in L^q(\Omega) \) we have \( -\int_{\Omega} p \nabla \cdot \psi dx \leq n\|p\|_{L^q} \|\psi\|_{W^{1,q}} \) for all \( \psi \in C^\infty_c(\Omega)^n \), which shows that \( p \in L^q_{\text{pot}}(\Omega) \). Thus, there exists a unique \( u^* \in W^{-1,q}(\Omega)^n \) such that \( u^* = \nabla_q p \). This implies

\[
(3.3) \quad L^q(\Omega) \subset L^q_{\text{pot}}(\Omega) \subset L^q_{\text{loc}}(\Omega).
\]

The next lemma provides a well-known characterization of \( G^{-1,q}(\Omega)^n \) (see also Cor.III.5.2).
Lemma 3.3. For $u^* \in W^{-1,q}(\Omega)^n$ the following statements are equivalent

1° $\langle u^*, \psi \rangle = 0$ for all $\psi \in W^{1,q}_{0,\text{div}}(\Omega)^n$;

2° $u^* \in G^{-1,q}(\Omega)^n$.

Proof: 1. The implication $2 \Rightarrow 1$ holds, since $C^\infty_{c,\text{div}}(\Omega)^n$ is dense in $W^{1,q}_{0,\text{div}}(\Omega)^n$.

2. To prove $1 \Rightarrow 2$ we choose a sequence of balls $B_i \in \Omega (i \in \mathbb{N})$ such that $\Omega_j := \bigcup_{i=1}^j B_i$ is connected for all $j \in \mathbb{N}$, and $\bigcup_{j=1}^\infty \Omega_j = \Omega$. Since $W^{1,q}_{0,\text{div}}(\Omega_j)^n \hookrightarrow W^{1,q}_{0,\text{div}}(\Omega)^n$ there holds

$$\langle u^*, v \rangle = 0 \quad \forall v \in W^{1,q}_{0,\text{div}}(\Omega_j)^n.$$ 

By the aid of [3] Cor.III.5.1] we get a unique $p_j \in L^q(\Omega_j)$ with $(p_j)_{B_i} = 0$ and

$$\langle u^*, v \rangle = \int_{\Omega_j} p_j \nabla \cdot v \, dx \quad \forall v \in W^{1,q}_{0,\text{div}}(\Omega_j)^n \quad (j \in \mathbb{N}).$$

As $W^{1,q}_{0,\text{div}}(\Omega_j)^n \hookrightarrow W^{1,q}_{0,\text{div}}(\Omega_{j+1})^n$ we see that $p_{j+1} - p_j = \text{const a.e. in } \Omega_j$, which must vanish since $(p_{j+1} - p_j)_{B_i} = 0$. Thus, $p_j = p_{j+1}|_{\Omega_j}$. Hence, there exists a unique $p \in L^q_{\text{loc}}(\Omega)$ such that $p_{B_i} = 0$ and $p|_{\Omega_j} = p_j$. Thus, setting $p = p_j$ a.e. in $\Omega_j (j \in \mathbb{N})$ it follows $p \in L^q_{\text{pot}}(\Omega)$ satisfying $\nabla q p = u^*$. Whence, $u^* \in G^{-1,q}(\Omega)^n$. \[\square\]

Remark 3.4. Lemma 3.3 implies that $G^{-1,q}(\Omega)^n = \left( W^{1,q}_{0,\text{div}}(\Omega)^n \right)^{\circ}$ is a closed subspace of $W^{-1,q}(\Omega)^n$.

Next, corresponding to a given functional $u^* \in G^{-1,q}(\Omega)^n$ there exists a unique $[p] \in L^q_{\text{pot}}(\Omega)/\mathbb{R}$ such that

$$\nabla q p = u^* \quad \forall p \in [p]. \quad (3.4)$$

The element $[p]$ will be denoted by $\overrightarrow{\mathcal{P}}_{q,\Omega}(u^*)$, which defines a bijective linear mapping from $G^{-1,q}(\Omega)^n \rightarrow L^q_{\text{pot}}(\Omega)/\mathbb{R}$.

Remark 3.5. Let $1 < q_1, q_2 < +\infty$. From the above definition it is immediately clear that

$$\overrightarrow{\mathcal{P}}_{q_1,\Omega}(u^*) = \overrightarrow{\mathcal{P}}_{q_2,\Omega}(u^*) \quad \forall u^* \in G^{-1,q_1} \cap G^{-1,q_2}(\Omega)^n. \quad (3.5)$$

If no confusion can arise, we may omit both subscript $q$ and $\Omega$ and write $\overrightarrow{\mathcal{P}}$ in place of $\overrightarrow{\mathcal{P}}_{q,\Omega}$.

Let $u^* \in W^{-1,q}(\Omega)^n$. By $\nabla \cdot u^*$ we denote the functional

$$v \mapsto -\langle u^*, \nabla v \rangle, \quad v \in W^{2,q}_{0}\Omega.$$ 

which belongs to $W^{-2,q}(\Omega)$. Then we have the following

---

\[\text{6)}\] Let $X$ be a Banach space. For a subset $M \subset X$ we define the annihilator $M^\circ$ which contains of all functionals $x^* \in X^*$ such that $\langle x^*, x \rangle = 0$ for all $x \in M$. Note, that $M^\circ$ is always a closed subspace of $X^*$. 

---
Lemma 3.6. Let \( u^* \in G^{-1,q}(\Omega)^n \). Suppose, \( \nabla \cdot u^* = 0 \). Then, every \( p \in \overline{\mathcal{P}(u^*)} \) is harmonic.

Proof: Note, that \( p \in \overline{\mathcal{P}(u^*)} \) is equivalent to \( p \in L^q_{\text{pot}}(\Omega) \) with \( u^* = \nabla q \). Thus, in view of (3.1) we have

\[
\int_{\Omega} p \Delta \phi dx = -\langle \nabla q, \nabla \phi \rangle = -\langle u^*, \nabla \phi \rangle = \langle \nabla \cdot u^*, \phi \rangle = 0
\]

for all \( \phi \in C_0^\infty(\Omega) \). Hence, by Weyl’s lemma \( p \) is harmonic.

Next, we derive some interesting properties of functions \( p \in L^q_{\text{pot}}(\Omega) \). We begin with the following definition.

Definition 3.7. A subdomain \( U \subset \Omega \) is called \( q \)–suitable if \( p|_U \in L^q(U) \) for all \( p \in L^q_{\text{pot}}(\Omega) \).

Remark 3.8. 1. If \( U \subset \Omega \) is \( q \)–suitable then \( \text{meas } U < +\infty \), since \( 1 \in L^q_{\text{pot}}(\Omega) \).
2. Every \( U \subset \Omega \) is \( q \)–suitable.
3. If \( L^q_{\text{pot}}(\Omega) = L^q(\Omega) \) then every subdomain \( U \subset \Omega \) is \( q \)–suitable.

We have the following weak Poincaré-type inequality.

Lemma 3.9 (Weak Poincaré-type inequality). Let \( U \subset \Omega \) be a \( q \)–suitable subdomain. Then there exists a constant \( c > 0 \) such that

\[
\|p - p_U\|_{L^q(U)} \leq c\|\nabla q\|_{W^{-1,q} \Omega} \quad \forall \ p \in L^q_{\text{pot}}(\Omega).
\]

Proof: In view of Remark 3.8 there holds \( \text{meas } U < +\infty \). On \( G^{-1,q}(\Omega)^n \) we introduce the following equivalence relation. Let \( u^*, v^* \in G^{-1,q}(\Omega)^n \). We say \( u^* \sim_U v^* \) if \( u^*|_{W^{1,q}(U)} = v^*|_{W^{1,q}(U)} \).

We define the linear mapping \( \Phi : G^{-1,q}(\Omega)^n/ \sim_U \rightarrow L^q_{0}(U) \) by setting \( \Phi([u^*]) := p|_U - p_U \), where \( p \in \overline{\mathcal{P}(u^*)} \). By the assumption of the lemma, \( \Phi \) is surjective. On the other hand, if \( \Phi([u^*]) = 0 \) there exists \( p \in L^q_{\text{pot}}(\Omega) \) vanishing on \( U \) such that \( u^* = \nabla q \). Hence, \( \langle u^*, u \rangle = 0 \) for all \( u \in W^{1,q}_0(U)^n \), which shows that \( \Phi \) also is injective and hence bijective. Its inverse is bounded, which follows from

\[
\|\Phi^{-1} p\|_{W^{-1,q} \Omega/ \sim U} = \inf \left\{ \|u^*\|_{W^{-1,q}} \mid u^* \in \Phi^{-1} p \right\}
\leq \sup_{u \in W^{1,q}_0(U)^n} \int_U p \nabla \cdot u dx \leq \|p\|_{L^q(U)} \|\nabla \cdot u\|_{L^q}
\leq n\|p\|_{L^q(U)}
\]
for all $p \in L^q_0(U)$. By the closed range theorem we deduce that $\Phi$ also is bounded. This implies for $p \in L^q_\text{pot}(\Omega)$ and $u^* = \nabla_q(p - p_U)$
\[
\|p - p_U\|_{L^q(U)} = \|\Phi([u^*])\|_{L^q(U)} \\
\leq c\|[u^*]\|_{W^{-1,q}(\Omega)/L^q(U)} \\
\leq c\|\nabla_q(p - p|U)\|_{W^{-1,q}} = c\|\nabla_qp\|_{W^{-1,q}}.
\]
This proves (3.6).  

In what follows, let $U \subset \Omega$ be a fixed $q$-suitable domain. Then, for $u^* \in G^{-1,q}(\Omega)^n$ by $\mathcal{P}(U)(u^*)$ we denote the unique pressure $p \in \mathcal{P}(u^*)$ fulfilling $p_U = 0$. If $L^q_\text{pot}(\Omega) = L^q(\Omega)$ we might take $U = \Omega$. In this case, we shortly write $\mathcal{P}$ in place of $\mathcal{P}(\Omega)$.

From Lemma 3.9 we easily derive the following

Lemma 3.10. For every $q$-suitable subdomain $G \subset \Omega$ there exists a constant $c_G > 0$ such that
\begin{equation}
\|\mathcal{P}(U)(u^*)\|_{L^q(G)} \leq c_G\|u^*\|_{W^{-1,q}(\Omega)} \quad \forall u^* \in G^{-1,q}(\Omega)^n.
\end{equation}

Proof: Firstly, assume that $G \cap U \neq \emptyset$. Let $u^* \in G^{-1,q}(\Omega)^n$, and set $p := \mathcal{P}(U)(u^*)$. Clearly, as $p_U = 0$ we easily find
\[
\frac{1}{\text{meas}(G)} \int_G |p|^q \, dx \\
\leq \frac{1}{\text{meas}(G)} \int_G |p - p_G|^q + 4^{q-1}|p_G - p_{G \cap U}|^q + 4^{q-1}|p_{G \cap U} - p_U|^q.
\]
(3.8) Estimating the second term on the right of (3.8) by means of
\[
|p_G - p_{G \cap U}|^q \leq \frac{2^q}{\text{meas}(G \cap U)} \int_G |p - p_G|^q \, dx,
\]
and the third term by a similar one, we are led to
\[
\int_G |p|^q \, dx \leq c \int_G |p - p_G|^q + c \int_U |p - p_U|^q \, dx.
\]
Recalling that both $U$ and $G$ are $q$-suitable, we are in a position to apply Lemma 3.9 to both terms on the right-hand side of the above estimate. This implies (3.7).

Secondly, if $G \cap U = \emptyset$, we take $G \subset G_0 \subset \Omega$ such that $G_0$ is $q$-suitable and $G_0 \cap U \neq \emptyset$. Then (3.7) immediately follows from the first case.  

Remark 3.11. 1. Lemma 3.10 says that for every $q$-suitable $G \subset \Omega$, the mapping $u^* \mapsto \mathcal{P}(U)(u^*)|_G$ is a bounded linear operator from $G^{-1,q}(\Omega)^n$ into $L^q(G)$.

2. If $L^q_\text{pot}(\Omega) = L^q(\Omega)$ then $\mathcal{P} : G^{-1,q}(\Omega)^n \to L^q(\Omega)$ is bounded.
4 Projections onto $G^{-1,q}(\Omega)^n$

As we have seen in the previous section functionals in $W^{-1,q}(\Omega)^n$ vanishig on $C_{c,\text{div}}^\infty(\Omega)^n$ equal to functionals of the form $\nabla q p$ with potential $p$. Furthermore, in many applications such functionals are expressed by a sum, namely $\nabla q p = u_1^* + u_2^*$ such that $u_i^* \in W^{-1,q_i}(\Omega)$ (1 < $q_i$ < $+\infty$) ($i = 1, 2$). If there is an operator $E^*$ which simultaneously projects $u_1^*$ into $G^{-1,q_1}(\Omega)^n$ and $u_2^*$ into $G^{-1,q_2}(\Omega)^n$ we are able to write

\[ \nabla_q p = E^*(\nabla q p) = \nabla_q p_1 + \nabla_q p_2, \quad p_1 \in L^q_{\text{pot}}(\Omega), \quad p_2 \in L^{q_2}_{\text{pot}}(\Omega). \]

Unfortunately, the existence of an operator $E^*$ which implies (4.1) isn’t necessarily guaranteed, unless the domain $\Omega$ enjoys certain regularity properties. On the other hand, if such projection $E^*$ exists, there are infinite projections leading to (4.1). Nevertheless, among all such projections there is a canonical one which is related to the existence and uniqueness of weak solutions to the Stokes system (or a Stokes-like system for unbounded domains).

Unlike the case $q \neq 2$, the case $q = 2$ appears to be special, due to the Hilbert space structure we are in a position to define $E^*_2$ as the orthogonal projection of $W^{-1,2}(\Omega)^n$ onto the closed subspace $G^{-1,2}(\Omega)^n$, where the scalar product in $W^{-1,2}(\Omega)^n$ is given by

\[ \langle (u^*, v^*) \rangle = \langle u^*, J_2^{-1} v^* \rangle = \langle (J_2^{-1} u^*, J_2^{-1} v^*) \rangle, \quad u^*, v^* \in W^{-1,2}(\Omega)^n. \]

Here $(\cdot, \cdot)$ denotes the usual scalar product in $W_0^{1,2}(\Omega)^n$. Then, by $E_2$ we denote the dual of $E^*_2$ which appears to be a projection from $W_0^{1,2}(\Omega)^n$ onto a closed subspace. Then then (1.2) and (1.3) are fulfilled. In fact, (1.2) follows by the aid of the closed range theorem together with Lemma (3.3) as

\[ \text{im} \ E^*_2 = G^{-1,2}(\Omega)^n = (W_0^{1,2}(\Omega)^n)^\circ = (\ker E_2)^\circ. \]

In addition, there holds

\[ E^*_2 J_2 = J_2 E_2 \] \[ \text{(1.2)} \]

Noting that $J_2(\nabla \phi) \in G^{-1,2}(\Omega)^n$ for all $\phi \in C_c^\infty(\Omega)$, the property (1.3) furnishes (1.2), i.e.

\[ E_2(\nabla \phi) = \nabla \phi \quad \forall \phi \in C_c^\infty(\Omega). \]

Bearing in mind (1.3), we give the following definition of the projection $E_{q'}$ for $1 < q < +\infty$.

**Definition 4.1.** Let $1 < q < +\infty$. A domain $\Omega \subset \mathbb{R}^n$ is called $\nabla_q$-regular if there exists a projection $E_{q'}: W_0^{1,q'}(\Omega)^n \rightarrow W_0^{1,q'}(\Omega)^n$ fulfilling

\[ \text{ker} \ E_{q'} = W_0^{1,q'}(\Omega)^n, \]

\[ E_{q'}(\psi) = E_2(\psi) \quad \forall \psi \in C_c^\infty(\Omega)^n. \]

\[ \text{(4.5)} \]

\[ \text{(4.6)} \]

\[ \text{This can be readily seen by} \ \langle E^*_2 J_2 u, v \rangle = \langle (E^*_2 J_2 u, J_2 v) \rangle = \langle (J_2 u, E_2 J_2 v) \rangle = \langle J_2 v, E_2 u \rangle = \langle J_2 E_2 u, v \rangle \text{ for all} \ v \in W_0^{1,2}(\Omega)^n. \]
Remark 4.2. 1. As \( C_c^\infty(\Omega)^n \) is dense in \( W_0^{1,q}(\Omega)^n \) the projection \( E_q^* \) is uniquely defined by (4.6).

2. From (4.6), by using (4.4) we immediately get

\[
E_q^*(\nabla \phi) = \nabla \phi \quad \forall \phi \in C_c^\infty(\Omega).
\] (4.7)

3. If \( W_0^{1,q}(\Omega)^n = W_0^{1,2}(\Omega)^n \cap W_0^{1,q}(\Omega)^n \) then (4.5) follows from (4.6) since (4.6) implies \( \psi - E_q^* \psi = \psi - E_2 \psi \in W_0^{1,2}(\Omega)^n \cap W_0^{1,q}(\Omega)^n \) for all \( \psi \in C_c^\infty(\Omega)^n \).

4. The dual operator \( E_q^* = (E_q')' : W^{-1,q}(\Omega)^n \to W^{-1,q}(\Omega)^n \) defines a projection onto \( G^{-1,q}(\Omega)^n \). By the aid of the closed range theorem we infer

\[
W^{-1,q}(\Omega)^n = G^{-1,q}(\Omega)^n \oplus \ker E_q^*.
\] (4.8)

\[
W_0^{1,q}(\Omega)^n = \text{im} E_q^* + W_0^{1,q}(\Omega)^n.
\] (4.9)

If \( J_q \) is an isomorphism then

\[
\text{im} E_q^* = \{ v \in W_0^{1,q}(\Omega)^n \mid J_q v \in G^{-1,q}(\Omega)^n \},
\] (4.10)

\[
\ker E_q^* = \{ J_q v \mid v \in W_0^{1,q}(\Omega)^n \}.
\] (4.11)

Remark 4.3. Let \( 1 < q_1, q_2 < +\infty \). Suppose \( \Omega \) is \( \nabla_{q_1} \)-regular and also \( \nabla_{q_2} \)-regular. As \( C_c^\infty(\Omega)^n \) is dense in both \( W_0^{1,q_1}(\Omega)^n \) and \( W_0^{1,q_2}(\Omega)^n \), (4.6) implies \( E_{q_1}^* u^* = E_{q_2}^* u^* \) for all \( u^* \in W^{-1,q_1} \cap W^{-1,q_2}(\Omega)^n \). Thus, we may write shortly \( E^* \) in place of \( E_{q_1}^* \) or \( E_{q_2}^* \). In particular, \( E^* \) is a projection in \( W^{-1,q_1} \cap W^{-1,q_2}(\Omega)^n \).

On the other hand, let \( u^* = u_1^* + u_2^* \in (W^{-1,q_1} + W^{-1,q_2})(\Omega)^n \), then \( E^* u^* = E_{q_1}^* u_1^* + E_{q_2}^* u_2^* \in (W^{-1,q_1} + W^{-1,q_2})(\Omega)^n \). Elementary,

\[
\langle E^* u^*, v \rangle \leq c \max\{\|u_1^*\|_{W^{-1,q_1}}, \|u_2^*\|_{W^{-1,q_2}}\} (\|v\|_{W^{1,q_1}} + \|v\|_{W^{1,q_2}})
\]

\[\forall v \in W^{1,q_1} \cap W^{1,q_2}(\Omega)^n.\]

Hence, \( E^* \) is a projection in \( (W^{-1,q_1} + W^{-1,q_2})(\Omega)^n \) which fulfills (4.4). The dual of \( \tilde{E} \) of \( E^* \) is a projection from \( W_0^{1,q_1} \cap W_0^{1,q_2}(\Omega)^n \) into itself satisfying \( \tilde{E} \psi = E_2 \psi \) for all \( \psi \in C_c^\infty(\Omega)^n \). However, we don’t know whether \( \tilde{E} v = E_{q_1} v = E_{q_2} v \) for all \( v \in W_0^{1,q_1} \cap W_0^{1,q_2}(\Omega)^n \). Eventually, this property holds if \( C_c^\infty(\Omega)^n \) is dense in \( W_0^{1,q_1} \cap W_0^{1,q_2}(\Omega)^n \) which is true for uniform Lipschitz domains.

The condition (4.6) implies the following properties of \( E_q^* \).

Lemma 4.4. Let \( 1 < q < +\infty \), and let \( \Omega \) be \( \nabla_q \)-regular. Then the following statements are true.

1. There holds

\[
J_q E_q^*(\psi) = E_q^* J_q(\psi) \quad \forall \psi \in C_c^\infty(\Omega)^n.
\] (4.12)

2. Let \( u^* \in W^{-1,q}(\Omega)^n \) such that \( \langle u^*, \nabla \phi \rangle = 0 \) for all \( \phi \in C_c^\infty(\Omega) \). Then, every potential \( p \in L^0_{\text{pot}}(\Omega) \) of \( E_q^* u^* \) is harmonic.
In particular, there holds
\[ \langle E_q^* J_q(\psi), \phi \rangle = \langle J_q \psi, E_q(\phi) \rangle = \langle \psi, J_2 E_2(\phi) \rangle \]
\[ = \langle \psi, E_2^* J_2(\phi) \rangle \langle J_2 E_2(\psi), \phi \rangle = \langle J_q E_q(\psi), \phi \rangle. \]
Whence, (4.12).

2. Let \( p \in L^q_{\text{pot}}(\Omega) \) such that \( \nabla q p = E_q^* u^* \). In light of (4.7) we calculate
\[ \int_{\Omega} p \Delta \phi dx = -\langle E_q^* u^*, \nabla \phi \rangle = -\langle u^*, \nabla \phi \rangle = 0 \]
for all \( \phi \in C_c^\infty(\Omega) \). Hence, by Weyl’s lemma \( p \) is harmonic.

Next, we turn to the decomposition of the pressure by using the projection \( E^* \).

**Theorem 4.5.** Let \( 1 < q_i < +\infty \quad (i = 1, \ldots, N) \). Suppose \( \Omega \) is \( \nabla q_i \)-regular for all \( i = 1, \ldots, N \). Let \( u_i^* \in W^{-1, q_i}(\Omega)^n \quad (i = 1, \ldots, N) \) such that
\[ \sum_{i=1}^{N} \langle u_i^*, \psi \rangle = 0 \quad \forall \psi \in \bigcap_{i=1}^{N} W^{1, q_i}_{0, \text{div}}(\Omega)^n. \]
Then, for every \( U \Subset \Omega \) there holds
\[ \sum_{i=1}^{N} \langle u_i^*, \psi \rangle = -\sum_{i=1}^{N} \int_{\Omega} p_i \nabla \cdot \psi dx \quad \forall \psi \in C_c^\infty(\Omega)^n, \]
where \( p_i = \mathcal{P}^{(U)}(E_q^* u_i^*) \quad (i = 1, \ldots, N) \).

**Proof:** Let \( \psi \in C_c^\infty(\Omega)^n \) be arbitrarily chosen. Observing (4.6), we get \( \psi - E_2(\psi) = \psi - E_q(\psi) \in W^{-1, q_i}(\Omega)^n \) for all \( i = 1, \ldots, N \), in view of (4.13) we infer
\[ \sum_{i=1}^{N} \langle E_q^* u_i^*, \psi \rangle = \sum_{i=1}^{N} \langle u_i^*, E_q(\psi) \rangle = \sum_{i=1}^{N} \langle u_i^*, \psi \rangle. \]
Let \( U \Subset \Omega \). According to Remark 3.8, \( U \) is \( q \)-suitable. Setting \( p_i = \mathcal{P}^{(U)}(E_q^* u_i^*) \quad (i = 1, \ldots, N) \), as \( u_i^* = \nabla q_i p_i \), the identity (4.14) follows from the latter identity. This completes the proof of the theorem.

**Remark 4.6.** 1. Let \( \Omega \subset \mathbb{R}^n \) be a \( \nabla q \)-regular domain \((1 < q < +\infty)\), and let \( U \Subset \Omega \) be \( q \)-suitable. Then for every \( u^* \in W^{-1, q}(\Omega)^n \) there exists a unique associate pressure \( p \in L^q_{\text{pot}}(\Omega) \) with \( p_U = 0 \), defined by \( p := \mathcal{P}^{(U)}(E_q^* u^*) \). This pressure will be denoted shortly by \( \mathcal{P}^{(U)}(u^*) \). In fact, \( \mathcal{P}^{(U)} \) defines a linear map from \( W^{-1, q}(\Omega)^n \) onto \( L^q_{\text{pot}}(\Omega) \). In particular, there holds
\[ \langle E_q u^*, \psi \rangle = -\int_{\Omega} \mathcal{P}^{(U)}(u^*) \nabla \cdot \psi dx \quad \forall \psi \in C_c^\infty(\Omega)^n. \]

On the other hand for every subdomain \( G \subset \Omega \) being \( q \)-suitable, the mapping \( u^* \mapsto \mathcal{P}^{(U)}(u^*)|_G \) is a bounded linear operator from \( W^{-1, q}(\Omega)^n \) onto \( L^q(G) \).

Respectively, if \( L^q_{\text{pot}}(\Omega) = L^q(\Omega) \), we define \( \mathcal{P}(u^*) := \mathcal{P}(E_q^* u^*) \) such that \( \mathcal{P} \) is a bounded linear operator from \( W^{-1, q}(\Omega)^n \) onto \( L^q(\Omega) \) (see Remark 3.11 for the definition of \( \mathcal{P}(E_q^* u^*) \)).
5 Sufficient condition for $\nabla_q$-regularity

In this section we wish to present a sufficient condition for $\Omega$ being $\nabla_q$-regular, based on the existence and uniqueness of $q$-weak solutions to the system

\begin{align}
-\Delta u + \delta u &= u^* - \nabla p, \quad \nabla \cdot u = 0 \quad \text{in } \Omega,
\end{align}

\begin{align}
u &= 0 \quad \text{on } \partial \Omega,
\end{align}

with $\delta = 0$ or $\delta = 1$. If $\delta = 0$, (5.1), (5.2) forms the Stokes system. In case $\delta = 1$ we call (5.1), (5.2) the Stokes-like system. Concerning weak solutions to the Stokes or the Stokes-like system we give the following definition.

**Definition 5.1.** Let $1 < q < +\infty$. Let $u^* \in W_{-1, q}(\Omega)^n$. Then $u \in W_{1, q^0}(\Omega)$ is called a $q$-weak solution to (5.1), (5.2) if $\nabla \cdot u = 0$ a.e. in $\Omega$, and there holds

\begin{align}
\int_{\Omega} \nabla u : \nabla v + \delta u \cdot v dx = \langle u^*, v \rangle \quad \forall v \in W_{0, \text{div}}(\Omega)^n.
\end{align}

**Remark 5.2.**
1. Using the canonical embedding $L^q(\Omega)^n \hookrightarrow W_{-1, q}(\Omega)^n$, and employing Lemma 3.3, we see that $u \in W_{1, q^0}(\Omega)$ is a $q$-weak solution to (5.1), (5.2) iff there exists $p \in L^q_{\text{pot}}(\Omega)$ such that

\begin{align}
\int_{\Omega} \nabla u : \nabla v + \delta u \cdot v dx = \langle u^*, v \rangle \quad \forall v \in W_{1, q^0}(\Omega)^n.
\end{align}

2. Clearly, (5.3) can be interpreted as an operator equation

\begin{align}
T_q u = u^* |_{W_{0, \text{div}}(\Omega)},
\end{align}

where $T_q u$ stands for the restriction of $-\Delta u + \delta u$ to $W_{0, \text{div}}(\Omega)^n$ which appears to be a bounded linear operator from $W_{1, q^0}(\Omega)^n$ into $(W_{0, \text{div}}(\Omega)^n)'$. By using a routine functional analytic argument, we infer that the existence and uniqueness of $q$-weak solutions to the system (5.1), (5.2) is equivalent to $T_q$ being an isomorphism. Furthermore, having

\begin{align}
(T_q)' = T_{q'} \quad \forall 1 < q < +\infty,
\end{align}

the existence and uniqueness of $q$-weak solutions to (5.1), (5.2) is equivalent to the existence and uniqueness of $q'$-weak solutions to (5.1), (5.2).

Next we shall introduce a sufficient condition for $\Omega$ being $\nabla_{q'}$-regular.

**Lemma 5.3.** Let $\Omega \subset \mathbb{R}^n$ be a domain. Let $1 < q < +\infty$. Suppose that

(a) For every $u^* \in W^{-1, q}(\Omega)^n$ there exists a unique $q$-weak solution $u \in W_{0, \text{div}}(\Omega)^n$ to (5.1), (5.2) ($\delta = 0$ if $\Omega$ is bounded or $\delta = 1$ otherwise).
(b) For every \( \psi \in C^\infty_c(\Omega)^n \) the \( q \)-weak solution to (5.1), (5.2) with \( u^* = -\Delta \psi + \delta \psi \) belongs to \( W^{1,2}_{0,\text{div}}(\Omega)^n \).

Then \( \Omega \) is \( \nabla_q \)-regular.

**Proof:** 1. Let \( v \in W^{1,q'}_0(\Omega)^n \). Note that, according to Remark 5.2/2., (a) continuous to hold after replacing \( q \) by \( q' \) therein. Thus, there exists a unique \( q' \)-weak solution \( u \in W^{1,q'}_0(\Omega)^n \) to (5.1), (5.2) with right-hand side \( u^* = -\Delta_q v + \delta v = J_q v \). Setting \( E_q v = v - u \), and having \( \|E_q v\|_{W^{1,q'}} \leq c\|v\|_{W^{1,q'}} \), we see that \( E_q \) is a linear bounded operator from \( W^{1,q'}_0(\Omega)^n \) into itself. By the definition of \( E_q \), it is readily seen that \( \ker E_q = W^{1,q'}_0(\Omega)^n \). In particular, owing to \( v - E_q v \in W^{1,q'}_0(\Omega)^n \) it follows \( E_q(v - E_q v) = 0 \) for all \( v \in W^{1,q'}_0(\Omega)^n \). Consequently, \( E_q^2 = E_q \). This, shows that \( E_q \) is a projection enjoying (4.5).

2. Now, it remains to verify the condition (4.6). To see this, let \( \psi \in C^\infty_c(\Omega)^n \) be arbitrarily chosen. Let \( u \in W^{1,q'}_0(\Omega) \) denote the unique \( q' \)-weak solution to (5.1), (5.2) with right-hand side \( u^* = -\Delta \psi + \delta \psi \). Due to (b) we have \( u \in W^{1,2}_{0,\text{div}}(\Omega)^n \). Recalling the definition of \( E_2 \), we get \( E_2 \psi = \psi - u = E_q \psi \). Whence, (4.6). This shows that \( \Omega \) is \( \nabla_q \)-regular.

**Remark 5.4.** 1. Recalling the definition of \( E_2 \), we see that \( I - E_2 \) becomes the orthogonal projection onto \( W^{1,2}_{0,\text{div}}(\Omega)^n \). Let \( v \in W^{1,2}_0(\Omega)^n \). Define, \( u = v - E_2 v \) in view of (4.3) we see that

\[
J_2 u = -\Delta_2 u + \delta u = J_2 v - E_2^* J_2 v.
\]

Thus, \( v - E_2 v \) is the unique \( 2 \)-weak solution to (5.1), (5.2) with right-hand side \( u^* = J_2 v \). On the other hand for any \( u^* \), the function \( u = J_2^{-1} u^* - E_2^* u^* \) becomes the unique \( 2 \)-weak solution to (5.1), (5.2).

2. If \( \Omega \subset \mathbb{R}^n \) is bounded the condition (a) implies that \( \Omega \) is \( \nabla_q \)-regular and \( \nabla_{q'} \)-regular as well.

Indeed, for \( 1 < q < 2 \) verifying \( W^{1,q'}_0(\Omega)^n \subset W^{1,2}_{0,\text{div}}(\Omega)^n \) we immediately get \( u = \psi - E_q \psi \in W^{1,2}_{0,\text{div}}(\Omega)^n \), and thus (b) holds. If \( 2 \leq q < +\infty \) we argue as follows.

Let \( \psi \in C^\infty_c(\Omega)^n \). As we have mentioned above in 1., \( u = \psi - E_2 \psi \in W^{1,2}_{0,\text{div}}(\Omega)^n \) is the unique \( 2 \)-weak solution to (5.1), (5.2) with \( u^* = -\Delta \psi \). Since \( W^{1,2}_{0,\text{div}}(\Omega)^n \subset W^{1,q'}_0(\Omega)^n \), from (a) it follows that \( u \) is the \( q' \)-weak solution to the corresponding system. This shows that (b) holds. According to Lemma 5.3 \( \Omega \) is \( \nabla_{q'} \)-regular, and in view of Remark 5.2 \( \Omega \) is \( \nabla_q \)-regular.

Next, we shall present examples of \( \nabla_q \)-domains, which occur in various applications. However, the list of domains below is not complete, and it will be left to the reader to find more relevant examples used in further applications.

### 5.1 The whole space \( \mathbb{R}^n \)

By employing the well-known Calderón-Zygmund inequality along with the classical regularity theory and Sobolev’s inequalities, one verifies that the operator \( -\Delta + I : \)
$W^{2,q}(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is an isomorphism for all $1 < q < +\infty$. By using an interpolation argument, the above statement implies that $J_q = -\Delta_q + I : W^{1,q}(\mathbb{R}^n)^n \to W^{-1,q}(\mathbb{R}^n)^n$ becomes an isomorphism for all $1 < q < +\infty$. This yields the following

**Theorem 5.5.** Let $1 < q < +\infty$. For every $u^* \in W^{-1,q}(\mathbb{R}^n)^n$ there exists a unique $q$-weak solution $u \in W^{1,q}(\mathbb{R}^n)^n$ to (5.1), (5.2) for $\delta = 1$.

**Proof:** First, note that $W_{0,\text{div}}^{-1,q}(\mathbb{R}^n)^n$ equals the space of all $u \in W^{1,q}(\mathbb{R}^n)^n$ with $\nabla \cdot u = 0$. This easily follows from

$$W^{1,q}(\mathbb{R}^n)^n = W_{0,\text{div}}^{1,q}(\mathbb{R}^n)^n \oplus \{\nabla f \mid f \in W^{2,q}(\mathbb{R}^n)\}$$

which can be proved by the aid of Calderón-Zygmund’s inequality along with a duality argument. Now, let $P_q$ denote the usual Helmholtz projection defined by

$$P_q u = u - \nabla \Delta^{-1} \nabla \cdot u, \quad u \in W^{1,q}(\mathbb{R}^n)^n.$$

Again using Calderón-Zygmund’s inequality we see that $P_q$ is a projection operator from $W^{1,q}(\mathbb{R}^n)^n$ onto $W_{0,\text{div}}^{1,q}(\mathbb{R}^n)^n$.

Now, let $u^* \in W^{-1,q}(\mathbb{R}^n)^n$ be arbitrarily chosen. Set $u = J_q^{-1} P_q u^*$. Verifying $J_q^{-1} P_q = P_q J_q^{-1}$, we see that $u \in W_{0,\text{div}}^{1,q}(\mathbb{R}^n)^n$ is a $q'$-weak solution to (5.1), (5.2). This solution also is unique, since $J_q w \in G^{q',1}(\Omega)^n$ implies $P_q J_q w = J_q P_q w = J_q w = 0$, and thus $w = 0$. Whence, the assertion of the theorem is proved.

As Theorem 5.5 shows, condition (a) of Lemma 5.3 is satisfied. Furthermore, as $P_q \psi = P_2 \psi$ for all $\psi \in C_c^\infty(\Omega)^n$, condition (b) also is fulfilled. Thus, by Lemma 5.3 we immediately get

**Corollary 5.6.** $\mathbb{R}^n$ is $\nabla_q$-regular for all $1 < q < +\infty$.

### 5.2 Bounded domains

To apply Lemma 5.3 for bounded domains it will be sufficient to recall the existence and uniqueness of weak solutions to the Stokes system, which has been proved first by Cattabriga in [4] for a bounded three dimensional $C^2$ domain. For the general case we quote from [10] the following

**Theorem 5.7 (Galdi, Simader, Sohr).** Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^1$-domain. Let $1 < q < +\infty$. For every $u^* \in W^{-1,q}(\Omega)^n$ and $g \in L^q_0(\Omega)$ there exits a unique pair $(u,p) \in W_0^{1,q}(\Omega)^n \times L^q(\Omega)$ such that

\[
\begin{align*}
\nabla \cdot u &= g \quad \text{a.e. in} \quad \Omega \\
-\Delta q u &= u^* - \nabla_q p \quad \text{in} \quad W^{-1,q}(\Omega)^n.
\end{align*}
\]

In addition there holds

\[
\|\nabla u\|_{L^q(\Omega)} + \|p\|_{L^q(\Omega)} \leq c(q,n,\Omega) \left(\|u^*\|_{W^{-1,q}(\Omega)} + \|g\|_{L^q(\Omega)}\right).
\]

In particular, if $g = 0$ we have $u \in W_{0,\text{div}}^{1,q}(\Omega)^n$. 

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Theorem 5.7 shows the existence and uniqueness of weak $q$-solutions to Stokes system (5.1), (5.2) holds for every bounded $C^1$ domain. Thus, according to Remark 5.4/2 we get the following

**Corollary 5.8.** Every bounded $C^1$-domain is $\nabla_q$-regular.

**Remark 5.9.**
1. Let $1 < q < +\infty$. The statement in Theorem 5.7 continues to hold if $\Omega$ is a Lipschitz domain with sufficiently small Lipschitz constant (cf. [10]). In this case $\Omega$ is $\nabla_q$-regular.
2. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. As it has been proved in [1] there exists a number $3 < q_0 < +\infty$ such that the existence and uniqueness of weak $q$-solutions holds for every $q_0 < q < q_0$. Thus, $\Omega$ is $\nabla_q$-regular for all $q^0 < q < q_0$.

As a consequence of Theorem 5.7 we get the existence and uniqueness of $q$-weak solutions to (5.1), (5.2) for the case $\delta = 1$ too.

**Corollary 5.10.** Let $\Omega \subset \mathbb{R}^n$ be a bounded $C^1$-domain. Let $1 < q < +\infty$. Then for every $u^* \in W^{-1,q}(\Omega)^n$ there exists a unique $q$-weak solution $u \in W^{1,q}_0(\Omega)^n$ to (5.1), (5.2) with $\delta = 1$ and a unique $p \in L^q(\Omega)$ satisfying

$$ -\Delta_q u + u = u^* - \nabla_q p \quad \text{in} \quad W^{-1,q}(\Omega)^n. $$

In addition, there holds

$$ \|u\|_{W^{1,q}(\Omega)} + \|p\|_{L^q(\Omega)} \leq c(q,n,\Omega)\|u^*\|_{W^{-1,q}(\Omega)}. $$

**Proof:** In view of Remark 5.2/2. we may restrict ourselves to the case $2 \leq q < +\infty$. In this case, since every $q$-weak solution is a $2$-weak solution to (5.1), (5.2) the uniqueness is obvious.

Let $u^* \in W^{-1,q}(\Omega)^n$ be arbitrarily chosen. Clearly, as $u^* \in W^{-1,2}(\Omega)^n$ there exists a $2$-weak solution to (5.1), (5.2). This solution also is a $2$-weak solution to the Stokes system with right-hand side $u^* - u$. Consulting Theorem 5.7 replacing $u^*$ by $u^* - u$ therein and applying a bootstrapping argument we obtain $u \in W^{1,q}_0(\Omega)^n$. Whence, $u$ is a $q$-weak solution to (5.1), (5.2). The estimate (5.10) follows from the closed mapping theorem.

**Remark 5.11.** Let $1 < q < +\infty$. If $\Omega$ is a bounded Lipschitz domain, then for every $f \in L^q(\Omega)$ the equation $\nabla \cdot u = f$ has a solution in $W^{1,q}_0(\Omega)^n$ (cf. [8, Thm.III.3.1]). Hence, by the closed range theorem we see that $\text{im} \nabla_q = (W^{1,q}_0(\Omega)^n)^\circ = G^{-1,q}(\Omega)^n$. If $\Omega$ is $\nabla_q$-regular, then for every $u^* \in W^{-1,q}(\Omega)^n$ we may define the associate pressure as $p := \mathcal{P}(E_q u^*)$, which belongs to $L^q(\Omega)$. In addition, we have the estimate

$$ \|p\|_{L^q} \leq c\|u^*\|_{W^{-1,q}}, $$

where the constant $c > 0$ depends on $q, n$, and the geometric properties of $\Omega$ only.

At the end of this subsection we examine the scaling properties of the projection $E_q$. In particular, we will see that if $\Omega = B_R(x_0)$, the constant $c$ in (5.11) is independent of $R > 0$. Without loss of generality we may assume that $x_0 = 0$, since all estimates...
which will be used are invariant under translations. Instead of \( B_R(0) \) we shortly write \( B_R \). Define,

\[
\Phi_q(u)(y) = R^{n/q - 1}u(Ry), \quad y \in B_1, \quad u \in W^{1,q}_0(B_R)^n.
\]

Clearly, according to \( \|\Phi_q(u)\|_{W^{1,q}_0(B_R)} = \|u\|_{W^{1,q}(B_R)} \), the map \( \Phi_q \) defines an isometry between \( W^{1,q}_0(B_R)^n \) and \( W^{1,q}(B_1)^n \). Accordingly, its dual \( \Phi_q^* \) is an isometry between \( W^{-1,q}(B_1)^n \) and \( W^{-1,q}(B_R)^n \). On the other hand, it is readily seen that

\[
\Phi_q(W^{1,q}_0(B_R)^n) = W^{1,q}(B_1)^n, \quad \Phi_q^*(G^{-1,q}(B_1)^n) = G^{-1,q}(B_R)^n.
\]

It can be easily checked,

\[
E_{q,B_1} = \Phi_q E_{q,B_R}^* \Phi_q^{-1}. \tag{5.12}
\]

From (5.12) we infer

\[
E_{q,B_R}^* = \Phi_q^* E_{q,B_1}^* (\Phi_q^*)^{-1}. \tag{5.13}
\]

In order to understand the relation between \( \mathcal{P}_{B_R} \) and \( \mathcal{P}_{B_1} \), we define

\[
\Psi_q(f)(y) = R^{n/s}f(Ry), \quad y \in B_1, \quad f \in L^q(B_R).
\]

Clearly, \( \Psi_q \) defines an isometry between \( L^q(B_R) \) and \( L^q(B_1) \). Furthermore, we have \( \Psi_q(L^q_0(B_R)) = L^q_0(B_1) \). Let \( p \in L^q(B_1) \). By an elementary calculus we obtain

\[
\langle \Phi_q^* \nabla q p, u \rangle = - \int_{B_1} p \nabla \cdot \Phi_q u \, dy = -R^{-n/s} \int_{B_R} p(x/R) \nabla \cdot u(x) \, dx
\]

for all \( u \in W^{1,q}_0(B_R)^n \). Consequently,

\[
\Phi_q^* \nabla q = \nabla q \Psi_q^{-1}, \quad \mathcal{P}_{B_R} E_{q,B_R}^* = \Psi_q^{-1} \mathcal{P}_{B_1} E_{q,B_1}^* (\Phi_q^*)^{-1}. \tag{5.14}
\]

Thus, we have the following

**Corollary 5.12.** There exists a constant \( c = c(q,n) \) such that for all \( 0 < R < +\infty \) the following is true:

\[
\|p\|_{L^q(B_R)} \leq c\|u^*\|_{W^{-1,q}(B_R)} \quad \forall u^* \in W^{-1,q}(B_R)^n,
\]

where \( p = \mathcal{P}_{B_R}(E_{q,B_R}^* u^*) \).

---

\(^8\) Obviously, the operator on the right is a projection with kernel \( W^{1,q}_0(B_1)^n \). For \( \psi \in C_c^\infty(B_1)^n \) we have \( \Phi_q^{-1}(\psi) \in C_c^\infty(B_1)^n \) and thus \( E_{q,B_1}(\psi) = \Phi_2 E_{2,B_R} \Phi_2^{-1}(\psi) \). Since \( \Phi_2 \) is an isometry the operator on the right is self-adjointed with range \( (W^{1,q}_0(B_1)^n)^\perp \). This shows that \( \Phi_2 E_{2,B_R} \Phi_2^{-1}(\psi) = E_{2,B_1}(\psi) \) and thus (5.6).
5.3 Exterior domains

Next, let us investigate the case $\Omega$ being an exterior domain. For the existence and uniqueness of weak solutions to the Stokes-like system we argue similar as in \cite{8} Chap. V. with slight modification. Accordingly, we have the following

**Theorem 5.13.** Let $\Omega$ be an exterior $C^1$-domain. Let $1 < q < +\infty$. Then for every $u^* \in W^{-1,q}(\Omega)^n$ there exists a unique $q$-weak solution to (5.1), (5.2) ($\delta = 1$). In addition, there holds

\[(5.16) \quad \|u\|_{W^{1,q}} \leq c \|u^*\|_{W^{-1,q}}.\]

Now, Theorem 5.13 yields

**Corollary 5.14.** Every exterior $C^1$-domain $\Omega$ is $\nabla_q$-regular for all $1 < q < +\infty$.

**Proof:** By virtue of Theorem 5.13 we only need to show, that condition (b) in Lemma 5.3 is fulfilled. Then the assertion will immediately follow from Lemma 5.3.

**Proof of (b):** Let $\psi \in C^\infty(\Omega)^n$. Then $u^* = -\Delta \psi + \psi \in C^\infty(\Omega)^n$, which is embedded into $W^{-1,q}(\Omega)^n$. According to Theorem 5.13 there is a unique solution $u \in W^{1,q}_{0,\text{div}}(\Omega)^n$ of (5.1), (5.2) with $\delta = 1$. By a standard regularity argument making use of Theorem 5.5 and Corollary 5.10 we deduce that $u \in W^{1,2}_{0,\text{div}}(\Omega)^n$. Whence (b), and the assertion of the corollary is completely proved.

6 The associate pressure for time dependent distributions

Here we consider time distributions of the form $u' + F$ in an interval $]a, b[ (-\infty < a < b < +\infty)$ (for the distributional time derivative cf. appendix below). Let $\Omega \subset \mathbb{R}^n$ be a domain. By $Q$ we denote the cylindrical domain $\Omega \times ]a, b[$.

In the discussion below by $L^s(a, b; L^q_{\text{loc}}(\Omega))$ we mean the linear space of all $f : ]a, b[ \to L^q_{\text{loc}}(\Omega)$ such that

$$f|_{G \times ]a, b[} \in L^s(a, b; L^q(G)) \quad \forall G \in \Omega \quad (1 \leq s, q < +\infty).$$

Our first main result is the following

**Theorem 6.1.** Let $1 < q < +\infty$. Let $\Omega \subset \mathbb{R}^n$ be a $\nabla_q$-regular domain together with the projection $E^*_{q} : W^{-1,q}(\Omega)^n \to G^{-1,q}(\Omega)^n$. Let $U \subset \Omega$ be a $q$-suitable subdomain.

1. Let $g, f \in L^1(a, b; W^{-1,q}(\Omega)^n)$ such that

\[(6.1) \quad \int_a^b \langle g(t), \psi \rangle \eta(t) - \langle f(t), \psi \rangle \eta'(t) dt = 0 \quad \forall \psi \in C^\infty_{\text{c,div}}(\Omega)^n, \ \eta \in C^\infty_{\text{c}}(]a, b[).\]
Then,

\[
(6.2) \left\{ \begin{array}{l}
\int_a^b \langle g(t), \varphi(t) \rangle \eta(t) - \left\langle f(t), \frac{d\varphi}{dt}(t) \right\rangle \eta'(t) dt = \int_Q (-p_h \partial_t \varphi + p_0 \nabla \varphi) dx dt \\
\forall \varphi \in C_c^{\infty}(Q)^n,
\end{array} \right.
\]

where \( p_h(t) = -\mathcal{P}_\Omega^{(U)}(f(t)) \) and \( p_0(t) = -\mathcal{P}_\Omega^{(U)}(g(t)) \) for a. e. \( t \in ]a, b[ \) (cf. Remark 4.6).

2. If \( \nabla \cdot f = 0 \) in the sense of distribution, then \( p_h(t) \) is harmonic for a. e. \( t \in ]a, b[ \). If, in addition \( f \in C_w([a, b]; W^{-1, q}(\Omega)^n) \), then \( p_h \) is continuous in \( \Omega \times ]a, b[ \).

**Proof:** 1. Since \( C_{c, \text{div}}(\Omega)^n \) is dense in \( W_{0, \text{div}}^{1, q}(\Omega)^n \) (6.1) remains true for all \( \psi \in W_{0, \text{div}}^{1, q}(\Omega)^n = \ker E_q^* \) (cf. 4.5). Thus, we are in a position to apply Theorem A.4 for \( X = W_0^{1, q}(\Omega)^n, Y = X^* = W_0^{-1, q}(\Omega)^n, E = E_q^*, f = f, g^* = g \). This implies that \( f - E_q^*f \) admits a distributive time derivative \( (f - E_q^*f)' = E_q^*g - g \) which satisfies (A.11), i.e.

\[
(6.3) \int_a^b \langle g(t), \varphi(t) \rangle - \left\langle f(t), \frac{d\varphi}{dt}(t) \right\rangle dt = \int_a^b -\left\langle E_q^*(f(t)), \frac{d\varphi}{dt}(t) \right\rangle + \langle E_q^*(g(t)), \varphi(t) \rangle dt
\]

for all \( \varphi \in C_c^{\infty}(Q)^n \). Define,

\[
p_h(t) = -\mathcal{P}_\Omega^{(U)}(g(t)), \quad p_0(t) = -\mathcal{P}_\Omega^{(U)}(f(t)), \quad \text{for a. e. } t \in ]a, b[.
\]

Clearly, both \( p_h \) and \( p_0 \) belong to \( L^1(a, b; L^q_{\text{loc}}(\Omega)) \). Thus, (6.2) follows from (6.3) by using (4.13).

2. If \( \nabla \cdot f = 0 \) in the sense of distributions, then Lemma A.4 implies that \( p_h(t) \) is harmonic for a. e. \( t \in ]a, b[ \).

In addition, if \( f \in C_w([a, b]; W^{-1, q}(\Omega)^n) \), from the first part of the theorem we see that

\[
(f + \nabla_q p_h)' = (g + \nabla_q p_0) \quad \text{in} \quad L^1(a, b; W^{-1, q}(G)^n).
\]

Thus, appealing to Lemma A.1 with \( X = Y = W^{-1, q}(\Omega)^n \), we get \( f + \nabla_q p_h \in C([a, b]; W^{-1, q}(\Omega)^n) \). In particular, as \( f \in C_w([a, b]; W^{-1, q}(\Omega)^n) \) we see that for every \( \zeta \in C_c^{\infty}(\Omega)^n \) the function

\[
t \mapsto \int_\Omega \nabla p_h(t) \cdot \zeta dx = \int_\Omega \langle f(t), E_q^* \zeta \rangle dx
\]

9) Let \( f \in L^1(a, b; W^{-1, q}(\Omega)^n) \). Observing (3.7) we see that for every ball \( B, \mathcal{P}_\Omega^{(U)}(f)|_B \in L^1(a, b; L^q(B)) \), which shows that \( \mathcal{P}_\Omega^{(U)}(f) \in L^1(a, b; L^q_{\text{loc}}(\Omega)) \).
is continuous on $[a, b]$. Taking $\zeta$ to be radial symmetric, recalling the mean value formula of harmonic functions it follows that

\[
\nabla p_h(x_0, t) \to \nabla p(x_0, t_0) \quad \text{as} \quad t \to t_0 \quad \forall (x_0, t_0) \in \Omega \times [a, b].
\]

Let $(x_0, t_0) \in Q$ be fixed. Let $B_R = B_R(x_0) \subset \Omega$. Recall that $\mathcal{P}_{\Omega}^{(\nu)}|_{B_R}$ is a bounded linear operator from $W^{-1, q}(\Omega)$ into $L^q(B_R)$ (cf. Remark 4.3 and (3.7)), which gives

\[
\|p_h(t)\|_{L^q(B_R)} \leq C\|f(t)\|_{W^{-1, q}(\Omega)} \leq C\|f\|_{L^\infty(a, b; W^{-1, q}(\Omega))} \quad \forall t \in [a, b].
\]

By using the properties of harmonic functions we find

\[
\max_{x \in B_{R/2}} |\nabla^2 p_h(x, t)| \leq CR^{n-2}\|f\|_{L^\infty(a, b; W^{-1, q}(\Omega))} = CR^{n-2} \quad \forall t \in [a, b]
\]

with a constant $C$ independent on $(x, t)$ and $R$. Using Newton-Leibniz formula we infer

\[
|\nabla p_h(x, t) - \nabla p_h(x_0, t)| \leq CR^{n-2}|x - x_0| \quad \forall x \in B_{R/2}(x_0), \; t \in [a, b].
\]

Then by the aid triangle inequality we get

\[
|\nabla p_h(x, t) - \nabla p_h(x_0, t_0)| \leq CR^{n-2}|x - x_0| + |\nabla p_h(x_0, t) - \nabla p_h(x_0, t_0)|.
\]

Thus, taking into account (6.4) we deduce that $\nabla p_h$ is continuous in $\Omega \times [a, b]$.

As $p_h = p_h - (p_h)_\nu$ we are in a position to apply Lemma 3.2 which completes the proof of the theorem.

**Remark 6.2.** 1. Let $\Omega \subset \mathbb{R}^n$ be any domain. According to Definition 4.1, $\Omega$ is $\nabla_2$-regular with the orthogonal projection $E_\ast^2$. Thus, the statement of Theorem 6.1 holds for $q = 2$ without any restriction on $\Omega$.

Next, we wish to introduce the local pressure projection associated to a bounded $C^{1}$-subdomain $G \subset \Omega$. To this end, we recall the definition of $\mathcal{P}_{\nu, G} : W^{-1, q}(G)^n \to L^q_0(G)$,

\[
\mathcal{P}_{\nu, G}(u^*) = \mathcal{P}_{\nu, G}(E_{\nu, G}^* u^*), \quad u^* \in W^{-1, q}(G)^n.
\]

Let $1 < q_1, q_2 < +\infty$. According to Remark 3.1 and Remark 4.3 we see that

\[
\mathcal{P}_{\nu, G}(u^*) = \mathcal{P}_{\nu, G}(u^*) \quad \forall u^* \in W^{-1, q_1} \cap W^{-1, q_2}(G)^n.
\]

Hence, if no confusion can arise we omit the subscript $q$ and write $\mathcal{P}_G$ in place of $\mathcal{P}_{\nu, G}$. Correspondently, we write $E_\ast^G$ in place of $E_{\nu, G}^*$. Is readily seen that

\[
\nabla_q \mathcal{P}_G = E_\ast^G, \quad \mathcal{P}_G \nabla_q \mathcal{P}_G = \mathcal{P}_G.
\]

From consulting [10] we get the following

**Lemma 6.3.** Let $G \subset \mathbb{R}^n$ be a bounded $C^k$-domain $(k \in \mathbb{N})$. Let $1 < q < +\infty$. Then $\mathcal{P}_G(f) \in W^{k-1, q}(G)^n$ for every $f \in W^{k-2, q}(G)^n$. In addition, there holds

\[
\|\mathcal{P}_G(f)\|_{W^{k-1, q}(G)} \leq c(q, k, n, G)\|f\|_{W^{k-2, q}(G)}.
\]

Here $W^{0, q}(G)$ equals $L^q(G)$. 

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Our second main result is the construction of the following local pressure representation.

**Theorem 6.4.** Let \( \Omega \subset \mathbb{R}^n \) be an open set. Let \( u \in L^1(a, b; L^1_{\text{loc}}(\Omega)) \) and \( F \in L^1(a, b; W^{-1,q'}(\Omega)) \) (1 < q < +\( \infty \)) such that

\[
(6.8) \quad \int_a^b \langle F(t), \psi \rangle \eta(t) dt - \int_Q u(t) \cdot \psi' \eta(t) dx dt = 0 \quad \forall \psi \in C^\infty_{c,\text{div}}(\Omega), \ \eta \in C^\infty([a, b]).
\]

Then, for every bounded subdomain \( G \subseteq \Omega \) with \( C^1 \)-boundary we have

\[
(6.9) \quad \begin{cases}
\int_a^b \langle F(t), \varphi(t) \rangle dt - \int_Q u(t) \cdot \partial_t \varphi dt = \int_Q (-p_{h,G} \partial_t \nabla \cdot \varphi + p_{0,G} \nabla \cdot \varphi) dt \\
\forall \varphi \in C^\infty_c(\Omega), \ n \in [a, b]^n,
\end{cases}
\]

where \( p_{h,G}(t) = -\mathcal{P}_G(u(t)|_G) \) and \( p_{0,G}(t) = -\mathcal{P}_G(F(t)|_G) \) for a.e. \( t \in [a, b]. \)

**Proof:** Let \( G \subseteq \Omega \) be a bounded \( C^1 \)-domain. Fix, \( 1 < s < \min \left\{ q, \frac{n}{n-1} \right\} \). As \( C^\infty_{c,\text{div}}(G)^n \) is dense in \( W^{0,1}_s(G)^n \), (6.8) yields

\[
(6.10) \quad \int_a^b \langle F(t), \psi(t) \rangle \eta(t) dt - \int_a^b \int_G u(t) \cdot \psi dy dy' (t) dt = 0
\]

for all \( \psi \in W^{0,1}_s(G)^n = \ker(E_{s,G}) \) and \( \eta \in C^\infty([a, b]) \). Thus, (6.10) allows to apply Theorem[A.4] with \( X = W^{0,1}_s(G)^n \), \( Y = L^1(G)^n \), and the projection \( E = E_{s,G} \). Indeed, by means of Sobolev’s embedding theorem we have \( Y \hookrightarrow X^* \) in the following sense

\[
\langle \psi, \eta \rangle = \int_G \psi \cdot \eta dx \leq \| \psi \|_{L^1(G)} \max_G | \eta | \leq c \| \psi \|_{L^1(G)} \| \nabla \eta \|_{L^{1'}(G)}
\]

for all \( \psi \in L^1(G)^n \) and \( \eta \in W^{0,1}_s(G)^n \). Hence, the assumption (A.9) of Theorem[A.4] is fulfilled for \( f(t) = u(t) \) and \( g^*(t) = F(t)|_G \). Consequently,

\[
(u - E_{s,G}^* u)' + F - E_{s,G}^* F = 0 \quad \text{in} \quad G \times [a, b]
\]

in the sense of distributions. Thus, setting \( p_{h,G}(t) = -\mathcal{P}_G(u(t)|_G) \) and \( p_{0,G}(t) = \mathcal{P}_G(F(t)|_G) \), from (A.11) we immediately get

\[
\int_a^b \left\langle u(t) + \nabla_q p_{h,G}(t), \frac{d\varphi}{dt}(t) \right\rangle + \int_a^b \left\langle F(t) + \nabla_q p_{0,G}(t), \varphi(t) \right\rangle dt = 0
\]

for all \( \varphi \in C^1_c(a, b; W^{0,1}_s(G)^n) \). Whence, (6.9). \( \blacksquare \)

As consequence of Theorem[6.4] we have
Corollary 6.5. Suppose all assumptions of Theorem 6.4 are fulfilled. Then the following statements are true.

1. Suppose \( \nabla \cdot \mathbf{u} = 0 \) in the sense of distributions. Then for every bounded \( C^1 \)-domain \( G \subseteq \Omega \) the pressure \( p_{h,G}(t) \) is harmonic for a.e. \( t \in [a,b] \), and there holds

\[
\begin{aligned}
\int_a^b (F(t), \varphi(t)) dt - \int_Q (\mathbf{u} + \nabla p_{h,G}) \cdot \partial_t \varphi dx dt &= \int_Q p_0(G) \nabla \cdot \varphi dx dt \\
\forall \varphi \in C_c^\infty(G \times [a,b]^n).
\end{aligned}
\]

2. If \( \mathbf{u} \in L^1([a,b]; L^q_{\text{loc}}(\Omega)) \) for some \( 1 < q < +\infty \) then for every bounded \( C^1 \)-domain \( G \subseteq \Omega \) there holds \( p_{h,G} \in L^1(a,b; W_{\text{loc}}^{1,q}(G)) \) fulfilling (6.9). If, in addition, \( G \) is a \( C^2 \)-domain we have \( p_{h,G} \in L^1(a,b; W^{1,q}(G)) \) together with the estimate

\[
\|\nabla p_{h,G}(t)\|_{L^q(G)} \leq c(n,q,G)\|\mathbf{u}(t)\|_{L^q(G)} \text{ for a.e. } t \in [a,b].
\]

3. If \( F = F_1 + \ldots + F_N \), for \( F_i \in L^1(a,b; W^{1,q}_{\text{loc}}(\Omega)^n) \) \( (1 < q_i < +\infty; i = 1, \ldots, N) \) then for every bounded \( C^1 \)-domain \( G \subseteq \Omega \) we have

\[
p_{0,G} = p_{0,G}^1 + \ldots + p_{0,G}^N, \quad \text{where } p_{0,G}^i = \mathcal{P}_G(F_i|G) \quad (i = 1, \ldots, N).
\]

Furthermore, there holds

\[
\|p_{h,G}^i(t)\|_{L^{q_i}(G)} \leq c(n,q_i,G)\|F_i(t)\|_{W^{-1,q_i}(G)} \text{ for a.e. } t \in [a,b].
\]

4. Assume that \( \mathbf{u} \in C_w([a,b]; L^1_{\text{loc}}(\Omega)^n) \) with \( \nabla \cdot \mathbf{u} \) in the sense of distributions, then for every bounded \( C^1 \)-domain \( G \subseteq \Omega \) the harmonic pressure \( p_{h,G} \) is continuous in \( G \times [a,b] \).

**Proof:**

1. As \( \nabla \cdot \mathbf{u}(t) = 0 \) in the sense of distributions for a.e. \( t \in [a,b] \) we may apply Lemma 6.6 for \( \mathbf{u}(t)|_G \), which implies \( p_{h,G} \in \mathcal{P}(F_{q,G}^* \mathbf{u}(t)|_G) \) is harmonic for a.e. \( t \in [a,b] \).

2. This first statement follows immediately from the local regularity of weak solutions to the Stokes equation, while the second is an immediate consequence of (6.7) taking \( k = 2 \) therein (cf. Lemma 6.3).

3. Set \( s = \min\{q_1, \ldots, q_N\} \). Owing to

\[
p_{0,G}(t) = \mathcal{P}_G(F(t)|_G) = \mathcal{P}_G(F_1(t)|_G) + \ldots + \mathcal{P}_G(F_N(t)|_G),
\]

the assertion is easily obtained by applying (6.7) \( (k = 1) \) to each of \( p_{0,G}^i \quad (i = 1, \ldots, N) \).

4. Let \( G \subseteq \Omega \) be any bounded \( C^1 \)-domain. According to the first statement of the corollary \( p_{h,G}(t) \) is harmonic for every \( t \in [a,b] \), and by virtue of Theorem 6.4 we have

\[
(\mathbf{u} + \nabla_q p_{h,G})' = -F|_G - \nabla_q p_{0,G} = 0 \quad \text{in} \quad L^1(a,b; W^{-1,q}(G)^n).
\]

Appealing to Theorem 6.1, we immediately see that \( p_{h,G} = p_{h,G} - (p_{h,G})_G \) is continuous in \( G \times [a,b] \).
7 An application to distributional solutions to the generalized Navier-Stokes equation

We consider distributional solutions of the following generalized Navier-Stokes equations

\( \nabla \cdot u = 0 \text{ in } Q, \)  
\( \partial_t u + u \cdot \nabla u - \nabla \cdot (a(x,t)D(u)) = \nabla \cdot f - \nabla p \text{ in } Q, \)

where \( u = (u^1, \ldots, u^n) \) denotes the velocity field \((n = 2 \text{ or } n = 3), p \) the pressure, \( a > 0 \) the viscosity and \( \nabla \cdot f \) the external force. Here \( D(u) \) stands for the matrix of the symmetric gradient given by

\[ D_{ij}(u) = \frac{1}{2}(\partial_i u^j - \partial_j u^i), \quad i, j = 1, \ldots, n. \]

Regarding distributional solution to (7.1), (7.2) we give the following definition

**Definition 7.1.** Let \( a \in L^\infty(Q) \) and \( f \in L^2(Q)^n \). Then \( u \) is called a distributional solution with bounded energy if

\( u \in C_w([a, b]; L^2(\Omega)^n) \cap L^2(a, b; W^{1,2}(\Omega)^n), \)
\( \nabla \cdot u = 0 \text{ a.e. in } Q, \)

and the following identity holds for all \( \varphi \in C_c^\infty(Q)^n \) with \( \nabla \cdot \varphi = 0 \)

\[ \int_Q u \cdot \partial_t \varphi - u \otimes u : \nabla \varphi + aD(u) : D(\varphi) dx dt = \int_Q f : \nabla \varphi dx dt. \]  

The following result is a direct application of Theorem 6.1 and the results of Section 6.

**Theorem 7.2.** Given \( a \in L^\infty(Q) \) and \( f \in L^2(Q)^n \), let \( u \) be a distributional solution to (7.1), (7.2) with bounded energy. Let \( U \Subset \Omega \) be 2-suitable. Then there holds

\[ \int_Q (u + \nabla p_h) \cdot \partial_t \varphi - u \otimes u : \nabla \varphi + aD(u) : D(\varphi) dx dt = \int_Q f : \nabla \varphi dx dt + \int_Q p_0 \nabla \cdot \varphi dx dt \]

for all \( \varphi \in C_c^\infty(Q)^n \), where

\[ p_h(t) = -\mathcal{D}^{(U)}(E_2 u(t)) \]
\[ p_0(t) = \mathcal{D}^{(U)} \left[ E^*(\nabla \cdot (aD(u(t)) - u(t) \otimes u(t) + f(t))) \right] \]
for a.e. $t \in [a, b]$. In particular, $p_h$ is harmonic with respect to $x \in \Omega$ and continuous in $\Omega \times [a, b]$.

If, in addition, $\Omega$ is a bounded $C^1$ domain then $p_0 = p_0^1 + p_0^2 + p_0^3$, where

\begin{align*}
p_0^1(t) &= \mathcal{P}(\nabla \cdot (aD(u(t)))), \\
p_0^2(t) &= -\mathcal{P}(\nabla \cdot (u(t) \otimes u(t))), \\
p_0^3(t) &= \mathcal{P}(\nabla \cdot f(t)),
\end{align*}

while the harmonic pressure is given by $p_h(t) = -\mathcal{P}(u(t))$ ($t \in [a, b]$).

**Proof:** Let $u$ be a distributional solution to (7.1), (7.2). From the above definition it follows that $u \in L^\infty(a, b; L^2(\Omega)^n)$.

By a standard interpolation argument along with Sobolev’s embedding theorem we infer $u \in L^{8/n}(a, b; L^4(\Omega)^n)$. Thus, $aD(u) - u \otimes u + f \in L^{4/n}(a, b; L^2(\Omega)^n)$. Define, $F(t) \in W^{-1,2}(\Omega)^n$ by

$$
\langle F(t), v \rangle = \int_\Omega (aD(u(t)) - u(t) \otimes u(t) + f(t)) : \nabla v dx, \quad v \in W^{1,2}(\Omega)^n.
$$

for a.e. $t \in ]a, b[$. As one can easily check $F \in L^{4/n}(a, b; W^{-1,2}(\Omega)^n)$. Now, applying Theorem 6.1 together with Corollary 6.5, from (7.5) we get (7.6). The second statement immediately follows from Corollary 6.5. \hfill \blacksquare

## A Vector valued functions

Let $X$ be a Banach space with norm $\| \cdot \|_X$. Let $-\infty < a < b \leq +\infty$. By $L^s(a, b; X)$ ($1 \leq s \leq +\infty$) we denote the space of all Bochner measurable functions $f :]a, b[ \rightarrow X$ such that

$$
\int_a^b \| f(t) \|^s dt < \infty \quad \text{if } 1 \leq s < \infty; \quad \text{ess sup}_{t \in [a, b]} \| f(t) \|_X < \infty \quad \text{if } s = \infty.
$$

**The Steklov mean** Let $f \in L^1(a, b; X)$. We extend $f$ outside $]a, b]$ by zero, and denote this extension again by $f$. For $\lambda \in \mathbb{R} \setminus \{0\}$ we define the Steklov mean $f_\lambda : [a, b] \rightarrow X$, by

$$
f_\lambda(t) = \frac{1}{\lambda} \int_t^{t+\lambda} f(\tau) d\tau, \quad t \in [a, b]^{10}.
$$

The following properties of the Steklov mean are well-known and can be found in the standard literature:

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10) Here, for $\lambda < 0$ we set $\int_t^{t+\lambda} f(\tau) d\tau := -\int_{t+\lambda}^t f(\tau) d\tau$. 

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(i) \( f_\lambda \in C([a, b]; X) \) for all \( \lambda \in \mathbb{R} \setminus \{0\} \);

(ii) If \( f \in L^s(a, b; X) \) \((1 \leq s < +\infty)\) then
\[
\| f_\lambda \|_{L^s(a, b; X)} \leq \| f \|_{L^s(a, b; X)} \quad \forall \lambda \in \mathbb{R} \setminus \{0\},
\]
and \( f_\lambda \to f \) in \( L^s(a, b; X) \) as \( \lambda \to 0 \);

(iii) Let \( f \in L^s(a, b; X) \) and \( g \in L^{s'}(a, b; X) \) \((1 \leq s \leq +\infty)\), then
\[
\int_a^b f_\lambda g \, dt = \int_a^b f g_\lambda \, dt \quad \forall \lambda \in \mathbb{R} \setminus \{0\}.
\]

Let \( Y \) be a further Banach space, its norm being denoted by \( \| \cdot \|_Y \). Let \( T \in \mathcal{L}(X, Y) \), i.e. \( T : X \to Y \) is linear and bounded. Then \( T \) forms a linear and bounded operator from \( L^s(a, b; X) \) into \( L^s(a, b; Y) \) \((1 \leq s \leq +\infty)\), which again will be denoted by \( T \) such that
\[
T f(t) = T(f(t)) \text{ for a.e. } t \in [a, b].
\]

In particular, we have
\[
\int_a^b \| T f(t) \|_Y^s \, dt \leq \| T \|_s \int_a^b \| f(t) \|_X^s \, dt, \quad \forall f \in L^s(a, b; X).
\]

**Distributive time derivative** Let \( X, Y \) are Banach spaces such that \( X \) is continuously, and densely embedded into \( Y \). Let \( f \in L^1(a, b; X) \). A Bochner function \( g \in L^1(a, b; Y) \) is called a **distributive time derivative** of \( f \) if
\[
\int_a^b g(t) \eta(t) \, dt = -\int_a^b f(t) \eta'(t) \, dt \quad \forall \eta \in C^\infty_c([a, b]).
\]

Clearly, the distributive time derivative is unique and will be denoted by \( f' \). The following important properties are well known and can be found in the standard literature.

**Lemma A.1.** Let \( f \in L^1(a, b; X) \) with distributive time derivative \( f' \in L^1(a, b; Y) \).

Then, eventually redefining \( f \) on a subset of \([a, b]\) of measure zero we have \( f \in C([a, b]; Y) \), and there holds
\[
f(t) = f(s) + \int_s^t f'(\tau) \, d\tau \quad \text{in } Y \quad \forall a \leq s \leq t \leq b.
\]

In addition, we have
\[
\int_a^b \langle v^*(t), f'(t) \rangle \, dt = -\int_a^b \left\langle \frac{dv^*}{dt}(t), f(t) \right\rangle \, dt \quad \forall v^* \in C^1_c([a, b]; Y^*).
\]
Lemma A.2. Assume X to be reflexive. Let \( f \in L^\infty(a,b;X) \) with distributive time derivative \( f' \in L^1(a,b;Y) \). Then \( f \in C_\text{w}([a,b];X) \).

**Proof:** Thanks to Lemma A.1 there holds \( f \in C([a,b];Y) \). Let \( t \in ]a,b[ \). Then \( (f_\lambda(t)) \) is bounded in \( X \). Since \( X \) is reflexive there exists a sequence \( \lambda_j \to 0 \) and \( \xi \in X \) such that \( f_\lambda(t) \to \xi \) in \( X \) as \( j \to +\infty \). According to \( Y^* \hookrightarrow X^* \) and \( f_\lambda(t) \to f(t) \) in \( Y \) as \( j \to +\infty \) we get

\[
\langle v^*, f_\lambda(t) \rangle \to \langle v^*, f(t) \rangle \quad \forall v^* \in Y^* \quad \text{as} \quad j \to +\infty.
\]

Consequently, \( \xi = f(t) \). In particular, \( f(t) \in X \) for all \( t \in [a,b] \), and by the lower semi continuity of the norm we have

\[
(A.7) \quad \| f(t) \|_X \leq \| f \|_{L^\infty(a,b,X)} \quad \forall t \in [a,b].
\]

Next, let \( t_k \to t \) in \( [a,b] \). Since \( (f(t_k)) \) is bounded in \( X \) (cf. (A.7)), as \( X \) is reflexive there exists a subsequence \( (t_{k_j}) \) and \( \xi \in X \) such that \( f(t_{k_j}) \to \xi \) as \( j \to +\infty \). As above, \( f \in C([a,b];Y) \) yields \( \xi = f(t) \). Since the limit is unique we get the convergence of the whole sequence, which proves the lemma. \( \blacksquare \)

Lemma A.3. Let \( X,Y,Z \) are Banach spaces. As above we assume \( X \hookrightarrow Y \), densely.

Let \( T : Y \to Z \) be linear and bounded. If \( f \in L^1(a,b;X) \) with distributive time derivative \( f' \in L^1(a,b;Y) \), then \( Tf \in L^1(a,b;Z) \) admits a distributive time derivative \( (Tf)' \in L^1(a,b;Z) \) given by

\[
(A.8) \quad (Tf)'(t) = Tf'(t) \quad \text{in} \ Z \quad \text{for a. e.} \ t \in ]a,b[.
\]

**Proof:** Clearly, with help of (A.3) we get \( Tf' \in L^1(a,b;Z) \). Let \( \eta \in C_c^\infty([a,b]) \). Recalling the definition of the distributive time derivative, we see that

\[
\int_a^b f'(t)\eta(t)dt = - \int_a^b f(t)\eta'(t)dt \quad \text{in} \ Y.
\]

Then applying the operator \( T \) to both sides of the above identity, we obtain

\[
\int_a^b Tf'(t)\eta(t)dt = - \int_a^b Tf(t)\eta'(t)dt \quad \text{in} \ Z.
\]

This, shows that \( (Tf)' = Tf' \), which completes the proof of the assertion. \( \blacksquare \)

**Projections** Let \( X,Y \) are Banach spaces such that \( X \hookrightarrow Y \hookrightarrow X^* \), each of the embeddings are dense \(^{11}\). The following theorem has been used in the proof of our main result Theorem 6.1

\(^{11}\) In the literature \{\( X,Y,X^* \)\} usual is called a *Gelfand triple*. 

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Theorem A.4. Let $E : X \to X$ be a projection together with the direct sum $X = \text{im } E \oplus \ker E$. Let $f \in L^1(a,b,Y)$ and $g^* \in L^1(a,b,X^*)$ such that

$$
(A.9) \quad - \int_a^b \langle f(t), \psi \rangle \eta'(t) dt = \int_a^b \langle g^*(t), \psi \rangle \eta(t) dt
$$

for all $\psi \in \ker E$ and $\eta \in C^\infty_c([a,b])$. Then, $f - E^*f$ admits a distributive time derivative $(f - E^*f)' \in L^1(a,b;X^*)$ such that

$$
(A.10) \quad (f - E^*f)'(t) = g^*(t) - E^*g^*(t) \quad \text{in} \quad X^* \quad \text{for a.e. } t \in ]a,b[
$$

In particular, there holds

$$
(A.11) \quad - \int_a^b \left\langle f(t) - E^*f(t), \frac{d \varphi}{dt}(t) \right\rangle dt = \int_a^b \langle g^* - E^*g^*(t), \varphi(t) \rangle dt
$$

for all $\varphi \in C^1_c(a,b;X)$. This shows that $f - E^*f$ is continuous.

Proof: From the assumption of the theorem it follows that $I - E$ is a projection from $X$ onto $\ker E$. From (A.9) we deduce

$$
- \int_a^b \langle f(t), (I - E)\psi \rangle \eta'(t) dt = \int_a^b \langle g^*(t), (I - E)\psi \rangle \eta(t) dt
$$

for all $\psi \in X$ and $\eta \in C^\infty_c([a,b])$. This shows that

$$
- \int_a^b \langle f(t) - E^*f(t), \eta'(t) \rangle dt = \int_a^b \langle (I - E^*)g^*(t), \eta(t) \rangle dt \quad \text{in} \quad X^* \quad \forall \eta \in C^\infty_c([a,b]).
$$

Thus, $(f - E^*f)' \in L^1(a,b;X^*)$, and there holds (A.10). Finally, (A.11) can be obtained with help of (A.6) taking into account the canonical embedding $X \hookrightarrow X^{**}$.

B Continuity of potentials of continuous gradient fields

In this appendix we discuss the question of continuity of time dependent gradient fields being continuous in a subcylinder. This result has been used in the proof of the continuity of the harmonic pressure (cf. Theorem 6.1, Corollary 6.5).

Lemma B.1. Let $\Omega \subset \mathbb{R}^n$ be a domain. Let $-\infty < a < b < +\infty$. Let $p : \Omega \times [a,b] \to \mathbb{R}$ such that $p(\cdot,t) \in C^1(\Omega)$ for all $t \in [a,b]$. Furthermore, suppose that $\nabla p$ is continuous in $\Omega \times [a,b]$, and there exists $x_0 \in \Omega$ such that $p(x_0, \cdot) \in C([a,b])$. Then $p$ is continuous in $\Omega \times [a,b]$.

12) Here $E^* : X^* \to X^*$ stands for the dual of $E$. 27
Proof: Firstly, we claim that $p(x, \cdot) \in C([a, b])$ for all $x \in \Omega$. To prove this, let us denote by $\Omega_c$ the set of all points $x \in \Omega$ such that $p(x, \cdot) \in C([a, b])$. Since $x_0 \in \Omega_c$, this set is nonempty. Thus, in order to prove that $\Omega_c = \Omega$ we only need to show that $\Omega_c$ is open and relatively closed.

(i) $\Omega_c$ is open. Let $x \in \Omega_c$. Fix a ball $B \Subset \Omega$ centered in $x$. Then by applying the Newton-Leibniz formula for all $y \in B$ we find

$$p(y, t) = p(y, t) - p(x, t) + p(x, t) = \int_0^1 \nabla p(x + \tau(y - x), t) \cdot (y - x)d\tau + p(x, t).$$

Due to our assumptions both functions on the right-hand side belong to $C([a, b])$, and thus $x \in \Omega_c$.

(ii) $\Omega_c$ is relatively closed Let $x \in \overline{\Omega_c}^\text{rel} \subset \Omega$. Let $B \Subset \Omega$ be a ball having its center in $x$. Clearly, there exists $y \in B \cap \Omega_c$. As above we see that

$$p(x, t) = p(x, t) - p(y, t) + p(y, t) = \int_0^1 \nabla p(y + \tau(x - y), t) \cdot (x - y)d\tau + p(y, t).$$

Hence, as the term on the right-hand side belongs to $C([a, b])$ we deduce that $x \in \Omega_c$. Whence, $\Omega_c = \Omega$.

Secondly, let $(x, t) \in \Omega \times [a, b]$. Let $B \Subset \Omega$ be a ball with center $x$. By using the triangle inequality, and Newton-Leibniz formula, we get for all $(y, s) \in B \times [a, b]$

$$|p(x, t) - p(y, s)| \leq |p(x, t) - p(x, s)| + \int_0^t \nabla p(y + \tau(x - y), s) \cdot (x - y)d\tau.$$

From this inequality we infer that $p$ is continuous in $(x, t)$, since the first term tends to zero as $s \to t$ according to the first part of the proof, while the second term tends to 0 as $y \to x$, since $\nabla p(\cdot, s)$ is bounded on $B$.

**Lemma B.2.** Let $\Omega \subset \mathbb{R}^n$ be a domain. Let $-\infty < a < b < +\infty$. Let $p : \Omega \times [a, b] \to \mathbb{R}$ such that $p(\cdot, t) \in C^1(\Omega)$ for all $t \in [a, b]$. Furthermore, suppose that $\nabla p$ is continuous in $\Omega \times [a, b]$. Then for every subdomain $U \subset \Omega$ with meas $U < +\infty$ and $(p - p_U)|_U \in L^\infty(a, b; L^q(U))$ ($1 < q < +\infty$) the function $p - p_U$ is continuous in $\Omega \times [a, b]$.

**Proof:** According to Lemma [B.1] it will be sufficient to prove the existence of $x_0 \in \Omega$ such that $p(x_0, \cdot) - p(\cdot)U \in C([a, b])$. If $U \Subset \Omega$ is a ball with center $x_0$, then by using Newton-Leibniz formula it follows

$$p(x_0, \cdot) - p(\cdot)U = \frac{1}{\text{meas } U} \int_U \int_0^1 \nabla p(x - \tau(x_0 - x), \cdot) \cdot (x - x_0)d\tau dx.$$
Since $\nabla p$ is continuous on $\overline{U} \times [a,b]$ we infer that $p(x_0, \cdot) - p(\cdot)_U \in C([a,b])$. Consequently, $p - p_U$ is continuous in $\Omega \times [a,b]$.

Next, suppose $U \Subset \Omega$. Let $B \Subset \Omega$ be a ball with center $x_0$. Then

$$p(x_0, \cdot) - p(\cdot)_U = (p(x_0, \cdot) - p(\cdot)_B) + (p(\cdot)_B - p(\cdot)_U).$$

The function which occurs in first parenthesis belongs to $C([a,b])$, which has been shown above, while the second one can be evaluated as follows

$$p(\cdot)_B - p(\cdot)_U = \frac{1}{\text{meas } U} \int_U p(\cdot)_B - p(\cdot)_U \, dy.$$ 

As the integrant belongs to $C(\overline{U} \times [a,b])$ the function on the left-hand side is continuous on $[a,b]$. Thus, Lemma B.1 implies the assertion.

Finally, if $U \subset \Omega$ with $\text{meas } U < +\infty$ and $(p - p_U)|_U \in L^\infty(a,b; L^q(U))$ we may choose a sequence $U_m \Subset U_{m+1} \Subset U$ of increasing domains such that $\cup_{m=1}^\infty U_m = U$. According to the second step we have $p - p_{U_m} = \pi - \pi_{U_m}$ is continuous in $\Omega \times [a,b]$, where $\pi = p - p_U$. Let $x_0 \in \Omega$. As $\pi_U = 0$ in order to prove that $p(x_0, \cdot) - p(\cdot)_U = \pi(x_0, \cdot)$ is continuous on $[a,b]$ it will be sufficient to verify that $\pi_{U_m} \to 0$ uniformly on $[a,b]$. In fact, this is true since

$$\pi(t)_{U_m} = -\frac{1}{\text{meas } U_m} \int_{U \setminus U_m} \pi(y,t) \, dy \leq \frac{\text{meas}(U \setminus U_m)^{1/g}}{\text{meas } U_1} \|\pi\|_{L^\infty(a,b; L^q(U))}.$$ 

Hence, Lemma B.1 gives the desired continuity of $p - p_U$. $lacksquare$

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