Wave Function of Evaporating Black Holes

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ABSTRACT

We study some quantum mechanical aspects of dynamical black holes where the Vaidya metric is used as a model representing evaporating black holes. It is shown that in this model the Wheeler-DeWitt equation is solvable in whole region of spacetime, provided that one considers the ingoing (or outgoing) Vaidya metric and selects a suitable coordinate frame. This wave function has curious features in that near the curvature singularity it oscillates violently owing to large quantum effects while in the other regions the wave function exhibits a rather benign and completely regular behavior. The general formula concerning the black hole radiation, which reduces to the Hawking’s semiclassical result when \( r = 2M \) is chosen, is derived by means of purely quantum mechanical approach. The present formulation can be applied essentially to any system with a spherically symmetric black hole in an arbitrary spacetime dimension.

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Hawking’s discovery that quantum black holes can evaporate by emitting thermal radiation [1] has stimulated not only active researches on quantum field theory in curved spacetime and a theory of quantum gravity but also a great deal of speculation about a grand synthesis of general relativity, thermodynamics, and quantum theory [2]. Unfortunately, however, in spite of much impressive effort it might be fair to say that in four dimensions little progress has been made in generalizing Hawking’s “semiclassical” analysis to mathematically self-consistent, quantum mechanically correct approach.

Recently, we have developed a new approach which is purely quantum mechanical in that not only the matter fields but also the gravitational field are treated as quantum fields, in order to understand the black hole radiation [3, 4], and subsequently applied fruitfully to various problems relevant to quantum black holes, for example, the mass inflation in the Reissner-Nordstrom charged black hole [5], the three dimensional de Sitter black hole [6], the weak cosmic censorship [7], and the two dimensional dilaton gravity [8]. The key observation in this approach is that we can carry out the canonical quantization for a spherically symmetric system with the Vaidya metric representing the dynamical black holes if we limit our consideration to only the region near the apparent horizon and at the same time choose an appropriate coordinate frame.

However, this approach gives us some frustrations. This is because within the approach it is impossible to gain useful informations about quantum structure of spacetime in whole region through the wave function and it is also quite unclear what approximation method considering the specific region of spacetime corresponds to.

The main motivation in this paper is to remove these disadvantages and put our approach on a more sound foundation. As a bonus, we will see that the wave function obtained in this way shows very interesting, physically reasonable features such that it fluctuates violently near the curvature singularity while it
behaves mildly in the other regions. In addition, we can derive the more general formula with respect to the mass loss rate of an evaporating black hole compared to the semiclassical approach.

The action which we consider in this paper is of form

\[
S = \int d^4x \sqrt{-\text{gr}^4} \left( \frac{1}{16\pi G} \text{gr}^4 R - \frac{1}{8\pi} \text{gr}^4 g^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi \right),
\]

(1)

where \( \Phi \) is a real scalar field. To show the four dimensional character explicitly we put the suffix (4) in front of the metric tensor and the curvature scalar. We follow the conventions adopted in the MTW textbook [9] and use the natural units \( G = \hbar = c = 1 \). The Greek indices \( \mu, \nu, ... \) take the values 0, 1, 2, and 3, while the Latin indices \( a, b, ... \) take the values 0 and 1. Of course, the inclusion of other matter fields, the cosmological constants and the surface terms in this formalism is straightforward even if we confine ourselves to the action (1) for the sake of simplicity.

After a general spherically symmetric reduction [10]

\[
ds^2 = (4) g_{\mu\nu} dx^\mu dx^\nu,
\]

(2)

\[
= g_{ab}(x^c) dx^a dx^b + \phi(x^c)^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
\]

the action (1) can be rewritten as

\[
S = \frac{1}{2} \int d^2x \sqrt{-g} \left( 1 + g^{ab} \partial_a \phi \partial_b \phi + \frac{1}{2} R \phi^2 \right)
- \frac{1}{2} \int d^2x \sqrt{-g} \phi^2 g^{ab} \partial_a \Phi \partial_b \Phi.
\]

(3)

Moreover, introducing the ADM parametrization

\[
g_{ab} = \begin{pmatrix}
-\alpha^2 + \beta^2 & \beta \\
\beta & \beta \\
\end{pmatrix},
\]

(4)
and \( n^a \) which is unit vector normal to the foliations \( x^0 = \text{const} \)

\[
n^a = \left( \frac{1}{\alpha}, -\frac{\beta}{\alpha \gamma} \right),
\]

the action (3) becomes

\[
S = \int d^2 x \left[ \frac{1}{2} \alpha \sqrt{\gamma} \left\{ 1 - (n^a \partial_a \phi)^2 + \frac{1}{\gamma} (\phi')^2 - Kn^a \partial_a (\phi^2) \right\} + \frac{\alpha'}{\alpha \gamma} \partial_1 (\phi^2) \right] + \frac{1}{2} \alpha \sqrt{\gamma} \phi^2 \left\{ (n^a \partial_a \Phi)^2 - \frac{1}{\gamma} (\Phi')^2 \right\},
\]

where \( K \) is the trace of the extrinsic curvature defined as \( \frac{1}{\sqrt{-g}} \partial_a (\sqrt{-g} n^a) \) and \( \frac{\partial}{\partial x^0} = \partial_0 \) and \( \frac{\partial}{\partial x^1} = \partial_1 \) are also denoted by an overdot and a prime, respectively.

From (6), the canonical conjugate momenta can be read off

\[
p_\Phi = \sqrt{\gamma} \phi^2 n^a \partial_a \Phi,
\]

\[
p_\phi = -\sqrt{\gamma} n^a \partial_a \phi - \sqrt{\gamma} K \phi,
\]

\[
p_\gamma = -\frac{1}{4 \sqrt{\gamma}} n^a \partial_a (\phi^2).
\]

Then the Hamiltonian can be found to be a linear combination of the Hamiltonian and the momentum constraints as follows:

\[
H = \int dx^1 \left( \alpha H_0 + \beta H_1 \right),
\]

where the constraints are explicitly given by

\[
H_0 = \frac{1}{2 \sqrt{\gamma} \phi^2} p_\Phi^2 - \frac{\sqrt{\gamma}}{2} \frac{\phi'}{\phi} - \frac{\phi'}{2 \sqrt{\gamma}} \partial_1 (\phi^2)
\]

\[
+ \frac{\phi^2}{2 \sqrt{\gamma}} (\Phi')^2 - \frac{2 \sqrt{\gamma}}{\phi} p_\phi p_\gamma + \frac{2 \gamma \sqrt{\gamma}}{\phi^2} p_\gamma^2,
\]

\[
H_1 = \frac{1}{\gamma} p_\Phi \Phi' + \frac{1}{\gamma} p_\phi \phi' - 2 p_\gamma - \frac{1}{\gamma} p_\gamma'.
\]

The canonical formalism explained so far gives a basis to construct a quantum
theory of a system with a spherically symmetric black hole in the below. As a first crucial step toward the canonical quantization, we shall introduce the two dimensional coordinate $x^a$ by

$$x^a = (x^0, x^1) = (v - r, r), \quad (11)$$

where $v$ is the advanced time coordinate. Next let us fix the gauge symmetries by the gauge conditions

$$g_{ab} = \begin{pmatrix} -\alpha^2 + \frac{\beta^2}{\gamma} & \beta \\ \beta & \gamma \end{pmatrix},$$

$$= \begin{pmatrix} -(1 - \frac{2M}{r}) & \frac{2M}{r} \\ \frac{2M}{r} & 1 + \frac{2M}{r} \end{pmatrix}, \quad (12)$$

where the black hole mass $M$ is in general the function of the two dimensional coordinate $x^a$. From (11) and (12) the two dimensional line element takes a form

$$ds^2 = g_{ab} dx^a dx^b,$$

$$= -(1 - \frac{2M}{r}) dv^2 + 2 dv dr. \quad (13)$$

In previous works [3-8], the model which we have described till now and its variants have been widely used to perform the canonical quantization of dynamical black holes near the apparent horizon. The main new achievement in this article is to accomplish the quantization not only near the apparent horizon but over the whole spacetime region. This will enable the wave function to be explored in

1 This choice of the coordinate system is very critical to reach the desired result. For instance, instead of (11) if we choose $x^a = (v, r)$ which reduces to the Eddington-Finkelstein coordinate in vacuum, later analysis would reveal that the wave function is independent of the matter field and static.
considerable detail. To do so, since we are interested in the ingoing Vaidya metric [11], we impose the reasonable assumption on the dynamical fields

\[ \Phi = \Phi(v), \ M = M(v), \ \phi = r, \]  

(14)

which is obviously consistent with the fields equations stemming from (1). Under the assumption (14), the canonical conjugate momenta (7) and the constraints (9) and (10) reduce to

\[ p_{\Phi} = \phi^{2} \partial_{v}\Phi, \]
\[ p_{\phi} = \frac{1}{\gamma} \partial_{v}M + \frac{2M^{2}}{r^{2}\gamma}, \]
\[ p_{\gamma} = \frac{M}{\gamma}, \]  

(15)

\[ \sqrt{\gamma}H_{0} = \gamma H_{1}, \]
\[ = \frac{1}{\phi^{2}} p_{\Phi}^{2} - \gamma p_{\phi} + \frac{2M^{2}}{r^{2}}, \]  

(16)

where \( \gamma = 1 + \frac{2M}{r} \) from (12). Here some comments are in order. First of all, the reason why the two constraints are proportional to each other in (16) is that there remains only one residual symmetry which generates the translation \( v \rightarrow v + \varepsilon \) owing to the assumption (14) whose situation bears some resemblance to the case of the lightcone string field theory [12]. This corrects the wrong reasoning stated in previous our work [6]. Secondly, the dynamical degrees of freedom \( \gamma \) corresponding to “graviton” are frozen because \( p_{\gamma} \) in (15) does not have the term including a differentiation with respect to \( v \). This is nothing but a manifestation of the Birkoff’s theorem [9]. Finally, in (16), \( p_{\phi} \) appears in the linear form so that this equation has the same structure as the Schrödinger equation if we regard a suitable function of \( \phi \) as the time in the superspace.
Substituting $p_\Phi = -i \frac{\partial}{\partial \Phi}$ and $p_\phi = -i \frac{\partial}{\partial \phi}$ into the constraint (16) yields the Wheeler-DeWitt equation.

\[
\left( -\frac{1}{\phi^2} \frac{\partial^2}{\partial \Phi^2} + i \gamma \frac{\partial}{\partial \phi} + \frac{2M^2}{r^2} \right) \Psi = 0.
\]

(17)

It is straightforward to obtain a special solution of (17) by the method of separation of variables. The result is

\[
\Psi = (Be^{\sqrt{A}\Phi(v)} + Ce^{-\sqrt{A}\Phi(v)}) e^{i\frac{A-2M^2}{2M} \log \gamma},
\]

(18)

where $A$, $B$, and $C$ are integration constants. Let us examine the physical implications of this wave function in detail. At the spatial infinity $r \to \infty$, $\gamma \to 1$ so that $\Psi$ consists of only the ingoing matter field in the asymptotically flat spacetime as expected. And at the apparent horizon $r = 2M(v)$, $\gamma = 2$ so that $\Psi$ is composed of the part of the ingoing matter field plus the part affected by the gravitational field there. Finally in the vicinity of the curvature singularity $r \to 0$, $\gamma \to \infty$ thus $\Psi$ oscillates violently. It is important to remember that the field $\gamma$ is exactly the dynamical degrees of the freedom of “graviton”. Although classically these degrees of the freedom are killed owing to a spherically symmetric ansatz, it is of interest that they reappear and play an important role in quantum theory through the wave function. In other words, we can regard that strong quantum effects associated with the gravitational degrees of freedom cause the wave function to be singular at the singularity. Note that $\Psi$ is completely regular over the whole region of spacetime except at the curvature singularity. Also notice that in deriving the Wheeler-DeWitt equation (17) and its solution (18) we have never appealed to any approximation method, thus they are essentially in the nonperturbative regime.

Now under a general definition of the expectation value, the change rate of the
mass function of a black hole can be evaluated by means of (16) and (18)

\[ < \partial_v M > = - \frac{A}{r^2}. \]  \hspace{1cm} (19)

If we set \( r \) to be equal to the black hole radius \( 2M \), (19) reproduces the Hawking’s semiclassical result up to a numerical constant [1]. But (19) is derived from the purely quantum mechanical arguments and provides the more general result than the semiclassical one although I have no idea how to determine the constant \( A \) within the present framework. In order to fix the numerical coefficient in this formula, maybe we would need to impose the boundary conditions on the wave function. Furthermore, this formula strongly suggests that the particles emitted by an evaporating black hole can be traced all the way back to not the surface of the horizon but the singularity. Incidentally, when we consider the outgoing Vaidya metric

\[ ds^2 = -(1 - \frac{2M(u)}{r})du^2 - 2dudr \]  \hspace{1cm} (20)

instead of the ingoing one (13), we can repeat the previous arguments and it then turns out that we can reach similar results to the case of the ingoing Vaidya metric except a minor change where the coordinate \( v \) must be replaced by \( u \). One weakness in the present model is the lack of outgoing radiation escaping to the future null infinity. It is valuable to mention that the loss of the black hole mass in (19) is controlled by sending in negative energy flux from the past null infinity, rather than by having positive energy flux radiated to the future null infinity. In future, it would be interesting to construct a model which takes account of both the energy fluxes at the same time.

It is of interest to inquire whether the wave function has also a singular behavior near the the curvature singularity \( r = 0 \) when we use a different kind of hypersurfaces, for example, \( x^1 = r = const \) which would be more appropriate inside the apparent horizon. Using the canonical formalism constructed previously [4] and
similar thoughts to the above, it is straightforward to derive the Wheeler-DeWitt equation
\[
\left[ -\frac{1}{\phi^2} \frac{\partial^2}{\partial \Phi^2} - i\gamma \frac{\partial}{\partial \phi} - \gamma (1 - \frac{M}{r}) \right] \Psi = 0, \tag{21}
\]
where this time \(\gamma\) is given by \(\gamma = -(1 - \frac{2M}{r}) > 0\) inside the apparent horizon. The wave function satisfying (21) is
\[
\Psi = (Be^{\sqrt{A} \Phi(v)} + Ce^{-\sqrt{A} \Phi(v)}) \ e^{i(r-M \log r) - \frac{rA}{2M} \log \gamma}. \tag{22}
\]
It is obvious that near the singularity the wave function (22) oscillates violently owing to the factors \(\log r\) as well as \(\gamma\) which would be again rooted to strong quantum effects of the gravitational degrees of freedom. It is valuable to comment two points here. One is that we have previously obtained the wave function with similar features to this in a different context where the radial quantization was carried out for “static” black hole models without matter field [13]. Their conformity in the behavior of the wave function suggests that the singular behavior of the wave function at the singularity has a connection with quantum effects of the gravitational field. Of course, the quantum effects associated with the matter field would make the singular behavior there stronger. This point could be also understood from the fact that the change rate of the mass function (19) diverges at the singularity. The other is that we cannot connect smoothly the two wave functions (18) and (22) just on the apparent horizon. This is simply because we have used the different formulations.

In conclusion, we have investigated some quantum aspects of a dynamical black hole corresponding to the Schwarzschild geometry in four dimensions. However, the present analysis is easily generalized to all spherically symmetric black holes in any spacetime dimension. One intriguing question in future is how the present analysis takes over if we use the Kruskal coordinate which enables us to slice the spacetime by the hypersurfaces extending from the left spatial infinity to the right.
spatial infinity across the horizon and to find the wave function holding over the whole Kruskal extention of the Schwarzschild geometry. In this respect, Kuchar’s work might be helpful [14].

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