

Supersymmetric $AdS_6$ black holes from matter coupled $F(4)$ gauged supergravity



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ABSTRACT: In matter coupled $F(4)$ gauged supergravity in six dimensions, we study supersymmetric $AdS_6$ black holes with various horizon geometries. We find new $AdS_2 \times \Sigma_{g_1} \times \Sigma_{g_2}$ horizons with $g_1 > 1$ and $g_2 > 1$, and present the black hole solution numerically. The full black hole is an interpolating geometry between the asymptotically $AdS_6$ boundary and the $AdS_2 \times \Sigma_{g_1} \times \Sigma_{g_2}$ horizon. We also find black holes with horizons of Kähler four-cycles in Calabi-Yau fourfolds and Cayley four-cycles in Spin(7) manifolds.

KEYWORDS: AdS-CFT Correspondence, Supergravity Models

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1 Introduction and conclusions

In $F(4)$ gauged supergravity in six dimensions [1], there is a unique supersymmetric fixed point which is dual to 5d superconformal USp(2$N$) gauge theory [2, 3]. As it was shown in [4], $F(4)$ gauged supergravity is a consistent truncation of massive type IIA supergravity [5]. The fixed point uplifts to $AdS_6 \times S^4$ near-horizon geometry of the D4-D8 brane system [6, 7]. In the spirit of [8], supergravity solutions of wrapped D4-branes on various supersymmetric cycles were studied in $F(4)$ gauged supergravity. D4-branes wrapped on two- and three-cycles were studied in [9]. They found $AdS_4$ and $AdS_3$ fixed point solutions. See [10] for more recent results.

Recently, by considering D4-branes wrapped on supersymmetric four-cycles, we found supersymmetric $AdS_6$ black holes of $F(4)$ gauged supergravity in [11]. To be specific, we found the full black hole solutions which is an interpolating geometry between the asymptotically $AdS_6$ boundary and the $AdS_2 \times H^2 \times H^2$ horizon. Via the AdS/CFT

\footnote{Previously, D4-branes wrapped on supersymmetric four-cycles were studied in [12], but the equations and solutions were incorrect.}
correspondence, [13], analogous to the $AdS_4$ black hole cases in [14–16], the Bekenstein-Hawking entropy of the black holes nicely matched with the topologically twisted index of 5d USp(2N) gauge theory on $\Sigma_{g_1} \times \Sigma_{g_2} \times S^1$ in the large $N$ limit [17, 18]. We also considered black hole horizons of Kähler four-cycles in Calabi-Yau fourfolds and Cayley four-cycles in Spin(7) manifolds.

Pure $F(4)$ gauged supergravity is a consistent truncation of massive type IIA supergravity [4] and type IIB supergravity [19–21] on a four-hemisphere. Although it is not known whether it is also a consistent truncation of ten-dimensional supergravity, one can couple vector multiplets to pure $F(4)$ gauged supergravity [22]. In this theory, new fixed points and holographic RG flows were studied in [23–25]. See [26–28] also for other studies in this theory.

In this paper, in matter coupled $F(4)$ gauged supergravity, we continue our study on supersymmetric $AdS_6$ black holes. We consider $F(4)$ gauged supergravity coupled to three vector multiplets, and its $U(1) \times U(1)$-invariant truncation first considered in [25]. We consider black hole solutions with a horizon which is a product of two Riemann surfaces, $AdS_2 \times \Sigma_{g_1} \times \Sigma_{g_2}$. We derive supersymmetry equations and obtain new $AdS_2$ solutions. The $AdS_2$ horizon exists only for the $H^2 \times H^2$ background, and not for the $H^2 \times S^2$ or $S^2 \times S^2$ backgrounds. We present the full black hole solutions numerically.

We also consider black holes with horizons of Kähler four-cycles in Calabi-Yau fourfolds and Cayley four-cycles in Spin(7) manifolds. For Cayley four-cycles in Spin(7) manifolds, we consider the $SU(2)_{\text{diag}}$-invariant truncation of $F(4)$ gauged supergravity coupled to three vector multiplets. We find new $AdS_2$ horizons. It will be interesting to have a field theory interpretation of this $AdS_2$ solution.

In section 2, we review matter coupled $F(4)$ gauged supergravity in six dimensions. In section 3, we consider $F(4)$ gauged supergravity coupled to three vector multiplets, and its $U(1) \times U(1)$-invariant truncation. We consider supersymmetric black hole solutions with a horizon which is a product of two Riemann surfaces. In section 4, we consider supersymmetric black hole solutions with horizons of Kähler four-cycles in Calabi-Yau fourfolds and Cayley four-cycles in Spin(7) manifolds. In appendix A, we present the equations of motion for the $U(1) \times U(1)$-invariant truncation.

Note added. In the final stage of this work, we became aware of [29] which has some overlap with the results presented. The solutions presented in sections 3 and 4.1 are identical to the solutions found in sections 5 and 6 of [29], respectively.

2 Matter coupled $F(4)$ gauged supergravity

We review matter coupled $F(4)$ gauged supergravity in six dimensions [22]. The gravity multiplet consists of

\[
\left( e^\alpha_\mu , \psi^A_\mu , A^\alpha_\mu , B_\mu , \chi^A , \sigma \right),
\]

where they denote the graviton, gravitinon, four vector fields, a two-form gauge potential, dilatini, and a real scalar field, respectively. The vector fields, $A^\alpha_\mu$, $\alpha = 0, 1, 2, 3$, can be
used to gauge the SU(2)_R \times U(1) gauge symmetry. The vector multiplet consists of
\begin{equation}
(A_{\mu}, \lambda, \varphi^\alpha)^I, \quad (2.2)
\end{equation}
where they denote a vector field, gaugini, and four real scalar fields, respectively, and \( I = 1, \ldots, n \) labels the vector multiplets. The fermionic fields are eight-dimensional pseudo-Majorana spinors and transform in the fundamental representation of the SU(2) \_R \simeq USp(2)_R \text{ R-symmetry denoted by indices, } A, B = 1, 2. \text{ We denote the coupling constants of gauge fields from the gravity and vector multiplets by } g_1 \text{ and } g_2, \text{ respectively, and the mass parameter of the two-form gauge potential by } m. \text{ When there is no vector multiplet, the theory reduces to pure } F(4) \text{ gauged supergravity } [1]. \text{ In pure } F(4) \text{ gauged supergravity, there are five inequivalent theories: } \mathcal{N} = 4^+ \text{ (} g_1 > 0, \text{ } m > 0), \mathcal{N} = 4^- \text{ (} g_1 < 0, \text{ } m > 0), \mathcal{N} = 4^g \text{ (} g_1 > 0, \text{ } m = 0), \mathcal{N} = 4^m \text{ (} g_1 = 0, \text{ } m > 0), \mathcal{N} = 4^0 \text{ (} g_1 = 0, \text{ } m = 0). \text{ The } \mathcal{N} = 4^+ \text{ theory admits a supersymmetric AdS}_6 \text{ fixed point when } g_1 = 3m. \text{ At the supersymmetric AdS}_6 \text{ fixed point, all the fields are vanishing except the AdS}_6 \text{ metric.} \text{ The scalar fields from the gravity and vector multiplets parametrize each factor of the coset manifold,}
\begin{equation}
\mathcal{M} = SO(1,1) \times \frac{SO(4,n)}{SO(4) \times SO(n)}, \quad (2.3)
\end{equation}
respectively. The coset representative of the second factor is given by
\begin{equation}
P^I_\alpha = (P^I_0, P^I_r) = (\Omega^I_0, \Omega^I_r), \quad (2.4)
\end{equation}
where we define
\begin{equation}
\Omega^\Lambda \Sigma = (L^{-1})^\Lambda \Pi \nabla L^\Pi \Sigma, \quad \nabla L^\Lambda \Sigma = dL^\Lambda \Sigma - f^\Lambda_{\Gamma \Pi} A^\Gamma L^\Pi \Sigma, \quad (2.5)
\end{equation}
and \( \Lambda = (\alpha, I) \) or \( \Lambda = (0, r, I) \) with \( r = 1, 2, 3. \text{ The indices, } \Lambda, \Sigma, \ldots, \text{ are raised and lowered by} \)
\begin{equation}
\eta_{\Lambda \Sigma} = \text{diag}(1, 1, 1, 1, -1, \ldots, -1). \quad (2.6)
\end{equation}
The bosonic Lagrangian is given by
\begin{align}
e^{-1}L = & -\frac{1}{4} R + \partial_\mu \sigma \partial^\mu \sigma - \frac{1}{4} P_{\alpha \mu} P^{\alpha \mu} - V - \frac{1}{8} e^{-2\sigma} N_{\Lambda \Sigma} F^\Lambda_{\mu \nu} F^{\Sigma \mu \nu} + \frac{3}{64} e^{4\sigma} H_{\mu \nu \lambda} H^{\mu \nu \lambda} \\
& - \frac{1}{64} e^{2\sigma} B_{\mu \nu} \left( \eta_{\Lambda \Sigma} F^\Lambda_{\rho \sigma} F^{\Sigma}_{\tau \kappa} + m B_{\rho \sigma} F^{\rho 0}_{\tau \kappa} + \frac{1}{3} m^2 B_{\rho \sigma} B_{\tau \kappa} \right), \quad (2.7)
\end{align}
and
\begin{align}
V = & -e^{2\sigma} \left( \frac{1}{36} A^2 + \frac{1}{4} B^4 B_t + \frac{1}{4} C^I_i C_{It} + D^I_i D_{It} \right) \\
& + m^2 e^{-6\sigma} N_{00} - me^{-2\sigma} \left( \frac{2}{3} A L_{00} - 2 B^I L_{0I} \right), \quad (2.8)
\end{align}
where we define
\begin{equation}
N_{\Lambda \Sigma} = L^0_\Lambda (L^{-1})_{0 \Sigma} + L^{r}_{\Lambda} (L^{-1})_{r \Sigma} - L^{I}_{\Lambda} (L^{-1})_{I \Sigma}, \quad (2.9)
\end{equation}
\[ A = \epsilon^{rst} K_{rst}, \quad B^I = \epsilon^{trs} K_{rso}, \quad C_I^t = \epsilon^{trs} K_{rls}, \quad D_{II} = K_{0II}. \] (2.10)

The field strengths are collectively defined by
\[ F^\Lambda_{\mu\nu} = \partial_{[\mu} A^\Lambda_{\nu]} - m \delta_0^\Lambda B_{\mu\nu}, \quad H_{\mu\nu\lambda} = \partial_{[\mu} B_{\nu\lambda]}\]. (2.11)

The supersymmetry variations of the fermionic fields are given by
\[ \delta \psi_{\mu A} = D_{\mu} \epsilon_A + \frac{i}{24} \left[ A e^\sigma + 6 m e^{-3\sigma} (L^{-1})_{00} \right] \gamma_\mu \epsilon_A - \frac{i}{8} \left[ B t e^\sigma - 2 m e^{-3\sigma} (L^{-1})_{00} \right] \gamma^7 \gamma_\mu \sigma_{AB}^I \epsilon^B + \frac{1}{16} e^{-\sigma} \left[ (L^{-1})_{00} F^\Lambda_{\mu\nu} - m B_{\nu\lambda} \delta_0^\Lambda \right] \gamma_\gamma \epsilon_{AB} - \left( L^{-1} \right)_{rA} F^\Lambda_{\mu\nu} \sigma^I_{AB} \left( \gamma_\mu \nu_\lambda - 6 \delta_0^{\mu_\nu} \gamma_\lambda \right) \epsilon^B + \frac{i}{32} e^{2\sigma} H_{\mu\nu\lambda} \gamma_\gamma \gamma_\mu \lambda \gamma_\mu \epsilon_A, \] (2.12)
\[ \delta \chi_A = \frac{i}{2} \gamma_\mu \partial_{\nu} \epsilon_A + \frac{1}{24} [ A e^\sigma - 18 m e^{-3\sigma} (L^{-1})_{00} ] \epsilon_A + \frac{1}{8} [ B t e^\sigma + 6 m e^{-3\sigma} (L^{-1})_{00} ] \gamma^7 \sigma_{AB}^I \epsilon^B + \frac{1}{16} e^{-\sigma} \left[ (L^{-1})_{00} F^\Lambda_{\mu\nu} - m B_{\nu\lambda} \delta_0^\Lambda \right] \gamma_7 \epsilon_{AB} + \left( L^{-1} \right)_{rA} F^\Lambda_{\mu\nu} \sigma^I_{AB} \gamma_\mu \nu \epsilon^B + \frac{1}{32} e^{2\sigma} H_{\mu\nu\lambda} \gamma_\gamma \gamma_\mu \lambda \gamma_\mu \epsilon_A, \] (2.13)
\[ \delta \lambda^I_{\mu} = i D^I_{\mu} \gamma_\mu \sigma_{AB}^r \epsilon^B - i D^I_{\mu} \gamma^7 \gamma_\mu \epsilon_A - \sigma [ \sigma^I_{AB} \epsilon^B - 2 m e^{-3\sigma} (L^{-1})_{00} \gamma_7 \epsilon_A ] + \frac{i}{2} e^{-\sigma} \left( L^{-1} \right)_{00} \gamma_\mu \sigma_{AB}^r \epsilon^B, \] (2.14)

where we define
\[ K_{rst} = g_1 \epsilon_{lmn} L^I_r (L^{-1})_{s}^m L^n_t + g_2 C_{1JK} L^I_j (L^{-1})_{r}^s J L^K_t, \]
\[ K_{rst} = g_1 \epsilon_{lmn} L^I_r (L^{-1})_{s}^m L^n_t + g_2 C_{1JK} L^I_j (L^{-1})_{r}^s J L^K_t, \]
\[ K_{rIT} = g_1 \epsilon_{lmn} L^I_r (L^{-1})_{l}^m L^n_t + g_2 C_{1JK} L^I_j (L^{-1})_{r}^s J L^K_t, \]
\[ K_{0IT} = g_1 \epsilon_{lmn} L^I_0 (L^{-1})_{l}^m L^n_t + g_2 C_{1JK} L^I_j (L^{-1})_{r}^s J L^K_t, \] (2.15)

and
\[ D_{\mu} = \partial_{\mu} \epsilon_A + \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} \epsilon_A + \frac{i}{2} \left( \frac{1}{2} g_1 \epsilon^{rst} \Omega_{rst} + i \gamma_7 \Omega_{0rt} \right) \sigma_{rAB}^r \epsilon^B. \] (2.16)

The Pauli matrices, \( \sigma^{IAB} \), satisfy the relations,
\[ \sigma_{AB}^I = \sigma^{IC} B_C^A, \] (2.17)

and \( \sigma_{AB}^I = \sigma_{(AB)}^I \). We also define the chirality matrix by
\[ \gamma = i \gamma_0 \gamma_1 \gamma_2 \gamma_3 \gamma_4 \gamma_5, \] (2.18)

with \( \gamma_5^2 = -1 \) and \( \gamma_5^7 = -\gamma_7 \). We employ the mostly minus signature, \((+-+-+-+)\).

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2 The numerical factors are due to the unusual convention of form fields, e.g., \( \omega = \omega_{\mu} dx^{\nu} dx^{\nu} \), in [22].

3 There is a missing \( \gamma_7 \) in (4.32) and also in the \( m \) dependent term in (4.35) of [22].
2.1 \textit{F}(4) gauged supergravity coupled to three vector multiplets

In \textit{F}(4) gauged supergravity coupled to three vector multiplets, we have a scalar field from the gravity multiplet and four scalar fields from each vector multiplet: total thirteen scalar fields. The scalar fields from the gravity multiplet and the vector multiplets parametrize each factor of the coset manifold, respectively,
\[
\mathcal{M} = \text{SO}(1,1) \times \frac{\text{SO}(4,3)}{\text{SO}(4) \times \text{SO}(3)}.
\] (2.19)

There are three vector fields from the gravity multiplet and three vector fields from the three vector multiplets. Two sets of three vector fields can be used to gauge SU(2)_R \times SU(2) gauge group. We denote the coupling constants of two SU(2) factors by \(g_1\) and \(g_2\). The structure constant in (2.5) splits into
\[
j_{rst} = g_1 \epsilon_{rst}, \quad j_{IJK} = g_2 C_{IJK} = g_2 \epsilon_{IJK}.
\] (2.20)

The generators of SO(4), SU(2)_R, SU(2), and the non-compact SO(4;3) could be represented by, respectively [25],
\[
J^{\alpha\beta} = \epsilon^{\beta\alpha} - \epsilon^{\alpha\beta}, \quad J^r_s = \epsilon^{s,r} - \epsilon^{r,s},
\]
\[
J^I_J = \epsilon^{J+3,I+3} - \epsilon^{I+3,J+3}, \quad Y_{\alpha I} = \epsilon^{\alpha,I+3} + \epsilon^{I+3,\alpha},
\] (2.21)

where we define
\[
(\epsilon^\Lambda) = \delta_{\Lambda}^\Sigma, \delta_{\Sigma}^\Pi.
\] (2.22)

3 Black holes with \textit{AdS}_2 × \Sigma_{g_1} × \Sigma_{g_2} horizon

3.1 The U(1)×U(1)-invariant truncation

We truncate the theory to the U(1)×U(1)-invariant sector, which was first considered in section 3.1 of [25]. The U(1)×U(1) are generated by \(J^{12}_1\) and \(J^{12}_2\). We find two non-compact SO(4,3) generators, \(Y_{03}^3\) and \(Y_{33}^3\), which are invariant under the action of \(J^{12}_1\) and \(J^{12}_2\). We exponentiate the non-compact generators and obtain the coset representative,
\[
L = e^{\epsilon_1 Y_{03}^3}e^{\epsilon_2 Y_{33}^3}.
\] (3.1)

We also have two U(1) gauge fields, \(A^3\) and \(A^6\), and a two-form gauge potential, \(B_{\mu
u}\), in the U(1)×U(1)-invariant truncation. The Lagrangian of the truncation is given by
\[
e^{-1}L = \frac{1}{4} R + \partial_\mu \sigma \partial^\mu \sigma + \frac{1}{4} \cosh^2 \varphi_2 \partial_\mu \varphi_1 \partial^\mu \varphi_1 + \frac{1}{4} \partial_\mu \varphi_2 \partial^\mu \varphi_2 - V
\]
\[
- \frac{1}{8} e^{-2\sigma} \cosh(2\varphi_2) F_{\mu\nu}^3 F_{3\mu\nu}^3 - \frac{1}{8} e^{-2\sigma} (\cosh^2 \varphi_1 \cosh(2\varphi_2) + \sinh^2 \varphi_1) F_{\mu\nu}^6 F_{6\mu\nu}^6
\]
\[
+ \frac{1}{4} e^{-2\sigma} \cosh \varphi_1 \sinh(2\varphi_2) F_{\mu\nu}^3 F_{6\mu\nu}^6 - \frac{1}{8} m^2 e^{-2\sigma} B_{\mu\nu} B^{\mu\nu} + \frac{3}{64} e^{4\sigma} H_{\mu\nu\rho} H^{\mu\nu\rho}
\]
\[
- \frac{1}{64} e^{\mu\rho\sigma\tau\kappa} B_{\mu\nu} \left( F_{\rho\sigma}^3 F_{\tau\kappa}^3 - F_{\rho\sigma}^6 F_{\tau\kappa}^6 + \frac{1}{3} m^2 B_{\rho\sigma} B_{\tau\kappa} \right),
\] (3.2)

where the scalar potential is
\[
V = -g_1^2 e^{2\sigma} - 4g_1 m e^{-2\sigma} \cosh \varphi_1 \cosh \varphi_2 + m^2 e^{-6\sigma} (\cosh^2 \varphi_1 + \sinh^2 \varphi_1 \cosh(2\varphi_2)).
\] (3.3)
\[ F^\gamma e^{-\gamma} \epsilon_A + \frac{i}{2} \left( g_1 e^\sigma \cos \varphi_2 + me^{-3\sigma} \cos \varphi_1 \right) \epsilon_A - \frac{i}{2} me^{-3\sigma} \sinh \varphi_1 \sinh \varphi_2 \gamma_\sigma \sigma_{AB}^3 B^B \]

\[ + \frac{1}{4} e^{-\sigma} \sinh \varphi_1 \left( b_1 e^{-2G_1} \gamma_1 \dot{\varphi}_0 + b_2 e^{-2G_2} \gamma_2 \dot{\varphi}_2 \right) \gamma_\gamma \epsilon_A \]

\[ - \frac{1}{4} e^{-\sigma} \cosh \varphi_2 \left( a_1 e^{-2G_1} \gamma_1 \dot{\varphi}_0 + a_2 e^{-2G_2} \gamma_2 \dot{\varphi}_2 \right) \sigma_{AB}^3 B^B \]

\[ + \frac{1}{4} e^{-\sigma} \sinh \varphi_1 \sinh \varphi_2 \left( b_1 e^{-2G_1} \gamma_1 \dot{\varphi}_0 + b_2 e^{-2G_2} \gamma_2 \dot{\varphi}_2 \right) \sigma_{AB}^3 B^B \]

\[ + \frac{3}{8m} (a_1 a_2 - b_1 b_2) e^{-2G_1 - 2G_2} \gamma_{AB} \gamma \gamma \epsilon_A = 0, \]

\[ G^\gamma e^{F} \gamma_{AB} \epsilon_A + \frac{i}{2} \left( g_1 e^\sigma \cos \varphi_2 + me^{-3\sigma} \cos \varphi_1 \right) \epsilon_A - \frac{i}{2} me^{-3\sigma} \sinh \varphi_1 \sinh \varphi_2 \gamma_\sigma \sigma_{AB}^3 B^B \]

\[ - \frac{1}{4} e^{-\sigma} \sinh \varphi_1 \left( 3b_1 e^{-2G_1} \gamma_1 \dot{\varphi}_0 - b_2 e^{-2G_2} \gamma_2 \dot{\varphi}_2 \right) \gamma_\gamma \epsilon_A \]

\[ + \frac{1}{4} e^{-\sigma} \cosh \varphi_2 \left( 3a_1 e^{-2G_1} \gamma_1 \dot{\varphi}_0 - a_2 e^{-2G_2} \gamma_2 \dot{\varphi}_2 \right) \sigma_{AB}^3 B^B \]

\[ - \frac{1}{4} e^{-\sigma} \sinh \varphi_1 \sinh \varphi_2 \left( 3b_1 e^{-2G_1} \gamma_1 \dot{\varphi}_0 - b_2 e^{-2G_2} \gamma_2 \dot{\varphi}_2 \right) \sigma_{AB}^3 B^B \]

\[ - \frac{1}{8m} (a_1 a_2 - b_1 b_2) e^{-2G_1 - 2G_2} \gamma_{AB} \gamma \gamma \epsilon_A = 0, \]
\[
G^e_{\mu} e^{-F^{\gamma^\dagger}} \epsilon_A + \frac{i}{2} \left( g_1 e^{\sigma} \cosh \varphi_2 + m e^{-3\sigma} \cosh \varphi_1 \right) \epsilon_A - \frac{i}{2} m e^{-3\sigma} \sinh \varphi_1 \sinh \varphi_2 \gamma \sigma^3_{AB} \epsilon^B \\
- \frac{1}{4} e^{-\sigma} \sinh \varphi_1 \left( 3b_2 e^{-2G_{2\gamma}} \dot{\theta}_1 + b_1 e^{-2G_{2\gamma}} \dot{\theta}_2 \right) \gamma \gamma \epsilon_A \\
+ \frac{1}{4} e^{-\sigma} \cosh \varphi_2 \left( a_1 e^{-2G_{2\gamma}} \dot{\theta}_1 - a_2 e^{-2G_{2\gamma}} \dot{\theta}_2 \right) \sigma^3_{AB} \epsilon^B \\
- \frac{1}{4} e^{-\sigma} \cosh \varphi_1 \sinh \varphi_2 \left( b_2 e^{-2G_{2\gamma}} \dot{\theta}_1 - b_1 e^{-2G_{2\gamma}} \dot{\theta}_2 \right) \sigma^3_{AB} \epsilon^B \\
- \frac{1}{8m} (a_1 a_2 - b_1 b_2) e^{\sigma} - 2G_{2\gamma} \gamma \gamma \epsilon_A = 0 ,
\]

(3.11)

\[
\sigma^e_{\mu} e^{-F^{\gamma^\dagger}} \epsilon_A - \frac{i}{2} \left( g_1 e^{\sigma} \cosh \varphi_2 - 3m e^{-3\sigma} \cosh \varphi_1 \right) \epsilon_A - \frac{3i}{2} m e^{-3\sigma} \sinh \varphi_1 \sinh \varphi_2 \gamma \sigma^3_{AB} \epsilon^B \\
- \frac{1}{4} e^{-\sigma} \sinh \varphi_1 \left( b_1 e^{-2G_{2\gamma}} \dot{\theta}_1 + b_2 e^{-2G_{2\gamma}} \dot{\theta}_2 \right) \gamma \gamma \epsilon_A \\
+ \frac{1}{4} e^{-\sigma} \cosh \varphi_2 \left( a_1 e^{-2G_{2\gamma}} \dot{\theta}_1 + a_2 e^{-2G_{2\gamma}} \dot{\theta}_2 \right) \sigma^3_{AB} \epsilon^B \\
- \frac{1}{4} e^{-\sigma} \cosh \varphi_1 \sinh \varphi_2 \left( b_1 e^{-2G_{2\gamma}} \dot{\theta}_1 + b_2 e^{-2G_{2\gamma}} \dot{\theta}_2 \right) \sigma^3_{AB} \epsilon^B \\
+ \frac{1}{8m} (a_1 a_2 - b_1 b_2) e^{\sigma} - 2G_{2\gamma} \gamma \gamma \epsilon_A = 0 ,
\]

(3.12)

\[
\varphi_{\mu}^e e^{-F^{\gamma^\dagger}} \epsilon_A - \varphi_{\mu}^e e^{-F} \cosh \varphi_2 \gamma \gamma \dot{\theta}_1 \sigma^3_{AB} \epsilon^B - 2i g_1 e^{\sigma} \sinh \varphi_2 \epsilon_A - 2i m e^{-3\sigma} \sinh \varphi_1 \cosh \varphi_2 \gamma \sigma^3_{AB} \epsilon^B \\
- e^{-\sigma} \sinh \varphi_2 \left( a_1 e^{-2G_{2\gamma}} \dot{\theta}_1 + a_2 e^{-2G_{2\gamma}} \dot{\theta}_2 \right) \sigma^3_{AB} \epsilon^B \\
+ e^{-\sigma} \cosh \varphi_1 \cosh \varphi_2 \left( b_1 e^{-2G_{2\gamma}} \dot{\theta}_1 + b_2 e^{-2G_{2\gamma}} \dot{\theta}_2 \right) \sigma^3_{AB} \epsilon^B = 0 ,
\]

(3.13)

where the hatted indices are the flat indices. The \( t_-, \theta_1-, \) and \( \theta_2-\)components of the gravitino variations give (3.9), (3.10), (3.11), the dilatino variation gives (3.12), and the gaugino variation gives (3.13). The \( \phi_1-, \phi_2-\)components of the gravitino variations are identical to the \( \theta_1-, \) and \( \theta_2-\)components beside few more terms,

\[
\epsilon_A = -2i g_1 a_1 \gamma \dot{\theta}_1 \sigma^3_{AB} \epsilon^B , \quad \epsilon_A = -2i g_1 a_2 \gamma \dot{\theta}_2 \sigma^3_{AB} \epsilon^B .
\]

(3.14)

We employ the projection conditions,

\[
\gamma^\dagger \epsilon_A = i \epsilon_A , \quad \gamma \dot{\theta}_1 \sigma^3_{AB} \epsilon^B = -i \lambda \epsilon_A , \quad \gamma \dot{\theta}_2 \sigma^3_{AB} \epsilon^B = -i \lambda \epsilon_A ,
\]

(3.15)

where \( \lambda = \pm 1 \). Solutions with the projection conditions preserve 1/8 of the supersymmetries. As we check later, in order to have consistent supersymmetry equations with the equations of motion, it is required to have

\[
\varphi_1 = 0 .
\]

(3.16)

Therefore, from now on, we set \( \varphi_1 \) to vanish.
We present the complete supersymmetry equations,

\[
F' e^{-F} = -\frac{1}{2} \left( g_1 e^\sigma \cosh \varphi_2 + me^{-3\sigma} \right) - \frac{3}{8m} (a_1 a_2 - b_1 b_2) e^{\sigma - 2G_1 - 2G_2} - \frac{\lambda}{4} e^{-\sigma} \cosh \varphi_2 \left( a_1 e^{-2G_1} + a_2 e^{-2G_2} \right) + \frac{\lambda}{4} e^{-\sigma} \sinh \varphi_2 \left( b_1 e^{-2G_1} + b_2 e^{-2G_2} \right),
\]

\[
G_1' e^{-F} = -\frac{1}{2} \left( g_1 e^\sigma \cosh \varphi_2 + me^{-3\sigma} \right) + \frac{1}{8m} (a_1 a_2 - b_1 b_2) e^{\sigma - 2G_1 - 2G_2} + \frac{\lambda}{4} e^{-\sigma} \cosh \varphi_2 \left( 3a_1 e^{-2G_1} - a_2 e^{-2G_2} \right) - \frac{\lambda}{4} e^{-\sigma} \sinh \varphi_2 \left( 3b_1 e^{-2G_1} - b_2 e^{-2G_2} \right),
\]

\[
G_2' e^{-F} = -\frac{1}{2} \left( g_1 e^\sigma \cosh \varphi_2 + me^{-3\sigma} \right) + \frac{1}{8m} (a_1 a_2 - b_1 b_2) e^{\sigma - 2G_1 - 2G_2} + \frac{\lambda}{4} e^{-\sigma} \cosh \varphi_2 \left( 3a_2 e^{-2G_2} - a_1 e^{-2G_1} \right) - \frac{\lambda}{4} e^{-\sigma} \sinh \varphi_2 \left( 3b_2 e^{-2G_2} - b_1 e^{-2G_1} \right),
\]

\[
\sigma' e^{-F} = +\frac{1}{2} \left( g_1 e^\sigma \cosh \varphi_2 - 3me^{-3\sigma} \right) + \frac{1}{8m} (a_1 a_2 - b_1 b_2) e^{\sigma - 2G_1 - 2G_2} + \frac{\lambda}{4} e^{-\sigma} \cosh \varphi_2 \left( a_1 e^{-2G_1} + a_2 e^{-2G_2} \right) - \frac{\lambda}{4} e^{-\sigma} \sinh \varphi_2 \left( b_1 e^{-2G_1} + b_2 e^{-2G_2} \right),
\]

\[
\varphi_2' e^{-F} = +2g_1 e^{\sigma} \sinh \varphi_2 - \lambda e^{-\sigma} \sinh \varphi_2 \left( a_1 e^{-2G_1} + a_2 e^{-2G_2} \right) + \lambda e^{-\sigma} \cosh \varphi_2 \left( b_1 e^{-2G_1} + b_2 e^{-2G_2} \right).
\]  

(3.17)

We also obtain twist conditions on the magnetic charges from (3.14),

\[
a_1 = -\frac{k}{2\lambda g_1}, \quad a_2 = -\frac{k}{2\lambda g_1}.
\]  

(3.18)

where \( k = +1 \) for the \( S^2 \times S^2 \) background and \( k = -1 \) for the \( H^2 \times H^2 \) background.\(^4\)

There is no condition on \( b_1 \) and \( b_2 \). The supersymmetry equations are consistent with the equations of motion. We present the equations of motion in appendix \( \text{A} \).

### 3.3 The \( \text{AdS}_2 \) solutions

In this section, we find \( \text{AdS}_2 \) solutions of the supersymmetry equations. The solutions describe the \( \text{AdS}_2 \times \Sigma_{g_1} \times \Sigma_{g_2} \) horizon of six-dimensional black holes.

Now we will consider the \( \mathcal{N} = 4^+ \) theory, \( g_1 > 0, m > 0 \). When \( b_1 = b_2 = 0 \), we find an \( \text{AdS}_2 \) fixed point solution for the \( H^2 \times H^2 \) background with \( k = -1 \),

\[
e^F = \frac{1}{2^{5/4} g_1^{3/4} m^{1/4}}, \quad e^{G_1} = e^{G_2} = \frac{1}{2^{3/4} g_1^{3/4} m^{1/4}}, \quad e^\sigma = \frac{2^{1/4} m^{1/4}}{g_1^{1/4}}, \quad e^{\varphi_2} = 1,
\]

(3.19)

which is the \( \text{AdS}_2 \) solution first found in [11].\(^5\) When we consider the \( S^2 \times S^2 \) background with \( k = +1 \), \( \text{AdS}_2 \) fixed point does not exist.

\(^4\)It is possible to have geometries like \( S^2 \times H^2 \) for \( k_1 = +1 \) and \( k_2 = -1 \), or vice versa. One can easily generalize our supersymmetry equations and the twist conditions to that case.

\(^5\)In order to compare with [11], we have to reparametrize our parameters by

\[
\sigma \to \frac{1}{\sqrt{2}} \phi, \quad g_1, m \to \frac{1}{2\sqrt{2}} g, \frac{1}{2\sqrt{2}} m, \quad a_1, a_2 \to \sqrt{2} a_1, \sqrt{2} a_2.
\]

(3.20)
When we consider for non-zero $b_1$ and $b_2$, we obtain new $AdS_2$ solutions,

$$e^F = \frac{e^{-\sigma}}{2g_1 \cosh \varphi_2 \rho},$$

$$e^{2G_1} = \frac{\lambda}{2m} e^{2\alpha} (a_1 \cosh \varphi_2 - b_1 \sinh \varphi_2),$$

$$e^{2G_2} = \frac{\lambda}{2m} e^{2\alpha} (a_2 \cosh \varphi_2 - b_2 \sinh \varphi_2),$$

$$e^{4\alpha} = \frac{m}{2} \frac{2}{g_1} \cosh 2\varphi_2 (a_1 \cosh \varphi_2 - b_1 \sinh \varphi_2) (a_2 \cosh \varphi_2 - b_2 \sinh \varphi_2),$$

$$e^{2\varphi_2} = \frac{1}{(a_1 - b_1)(a_2 - b_2)} \left[ (a_1 a_2 + a_1 b_2 + a_2 b_1 - 3b_1 b_2)^2 - 2(\Phi/2)^{1/3} \right]$$

$$- \frac{2}{(\Phi/2)^{1/3}} \left( a_1 a_2^2 b_1 + a_2^2 a_2 b_2 - a_1 a_2 b_1 b_2 - 2a_2 b_1^2 b_2 - 2a_1 b_1^2 b_2 + 3b_1^2 b_2^2 \right),$$

where we define

$$\Phi = -(a_1 a_2 - 2b_1 b_2) (a_1^2 b_1^2 + a_2^2 b_2^2) - (a_1 b_2 + a_2 b_1) (a_1^2 a_2^2 + 10b_1^2 b_2^2 - 7a_1 a_2 b_1 b_2)$$

$$- 2b_1 b_2 (a_1^2 a_2^2 - 5b_1^2 b_2^2 + a_1 a_2 b_1 b_2)$$

$$+ \sqrt{(a_1 - b_1)^2 (a_2 - b_2)^2 (a_1 a_2 - 2b_1 b_2) (a_1^2 b_1^2 + a_1 a_2 b_1 b_2 - 2b_1^2 b_2) (a_2^2 b_1 + a_1 a_2 b_2 - 2b_1 b_2^2).}$$

All the fields are parametrized by the magnetic charges, $(a_1, a_2, b_1, b_2)$. As $(a_1, a_2)$ are fixed by the twist condition in (3.18), there are two free parameters left, $(b_1, b_2)$.

In order to have $AdS_2$ solutions, we should choose $(b_1, b_2)$ which makes

$$e^F > 0, \quad e^{2G_1} > 0, \quad e^{2G_2} > 0, \quad e^{4\alpha} > 0, \quad e^{2\varphi_2} > 0.$$

We plot the range of $(b_1, b_2)$ which satisfies the positivity conditions for $H^2 \times H^2$, $H^2 \times S^2$, $S^2 \times H^2$ and $S^2 \times S^2$, respectively. The positivity ranges are depicted in figure 1. We set $m = 1/2$ and $g_1 = 3m$ to have a unit radius for the $AdS_5$ boundary. From the plots, we conjecture that only the $H^2 \times H^2$ background gives the $AdS_2$ solutions.

Even in the large region in the graph for the $H^2 \times H^2$ background, only a small part near origin yields $AdS_2$ solutions.

### 3.4 Numerical black hole solutions

Now we present the full black hole solution numerically. The full black hole solution is an interpolating geometry between the asymptotically $AdS_6$ boundary and the $AdS_2 \times H^2 \times H^2$ horizon. We introduce a new radial coordinate,

$$\rho = F + \sigma.$$

---

We note that, when $G_1 \leftrightarrow G_2$, the solutions are invariant under $a_1 \leftrightarrow a_2$ and $b_1 \leftrightarrow b_2$. However, as $a_1 = a_2$ from (3.18), they are invariant under $b_1 \leftrightarrow b_2$. When we plot the positivity range for $H^2 \times S^2$ and $S^2 \times H^2$, there are small and irregular distributions of points. Most of the points do not respect the invariance under $b_1 \leftrightarrow b_2$. Even for the points invariant under $b_1 \leftrightarrow b_2$ seem not to give $AdS_2$ solutions. We presume that the appearance of these irregular distribution in the positivity range would be due to the complexity of the conditions.
This kind of coordinate was introduced in [30]. Employing the supersymmetry equations, we obtain

\[ \frac{\partial \rho}{\partial r} = F' + \sigma' = -e^F D, \]  

(3.25)

where we define

\[ D = 2me^{-3\sigma} + \frac{1}{2m} (a_1a_2 - b_1b_2)e^{\sigma-2G_1-2G_2}. \]  

(3.26)

Then, the supersymmetry equations are

\[
-D\frac{\partial F}{\partial \rho} = -\frac{1}{2} \left( g_1 e^\sigma \cosh \varphi_2 + me^{-3\sigma} \right) - \frac{3}{8m} (a_1a_2 - b_1b_2) e^{\sigma-2G_1-2G_2} \\
- \frac{\lambda}{4} e^{-\sigma} \cosh \varphi_2 \left( a_1e^{-2G_1} + a_2e^{-2G_2} \right) + \frac{\lambda}{4} e^{-\sigma} \sinh \varphi_2 \left( b_1e^{-2G_1} + b_2e^{-2G_2} \right),
\]

\[
-D\frac{\partial G_1}{\partial \rho} = -\frac{1}{2} \left( g_1 e^\sigma \cosh \varphi_2 + me^{-3\sigma} \right) + \frac{1}{8m} (a_1a_2 - b_1b_2) e^{\sigma-2G_1-2G_2} \\
+ \frac{\lambda}{4} e^{-\sigma} \cosh \varphi_2 \left( 3a_1e^{-2G_1} - a_2e^{-2G_2} \right) - \frac{\lambda}{4} e^{-\sigma} \sinh \varphi_2 \left( 3b_1e^{-2G_1} - b_2e^{-2G_2} \right),
\]

\[
-D\frac{\partial G_2}{\partial \rho} = -\frac{1}{2} \left( g_1 e^\sigma \cosh \varphi_2 + me^{-3\sigma} \right) + \frac{1}{8m} (a_1a_2 - b_1b_2) e^{\sigma-2G_1-2G_2} \\
+ \frac{\lambda}{4} e^{-\sigma} \cosh \varphi_2 \left( 3a_2e^{-2G_2} - a_1e^{-2G_1} \right) - \frac{\lambda}{4} e^{-\sigma} \sinh \varphi_2 \left( 3b_2e^{-2G_2} - b_1e^{-2G_1} \right),
\]
Figure 2. Numerical black hole solutions with $m = 1/2$ and $g_1 = 3m$. For the magnetic charges, $(b_1, b_2)$, we have $(-0.1, -0.1)$, purple, $(-0.01, -0.01)$, red, and $(-0.001, -0.001)$, orange.

\[-D \frac{\partial \sigma}{\partial \rho} = \frac{1}{2} \left( g_1 e^\sigma \cosh \varphi_2 - 3m e^{-3\sigma} \right) - \frac{1}{8m} \left( a_1 a_2 - b_1 b_2 \right) e^{\sigma - 2G_1 - 2G_2} \]

\[+ \frac{\lambda}{4} e^{-\sigma} \cosh \varphi_2 \left( a_1 e^{-2G_1} + a_2 e^{-2G_2} \right) - \frac{\lambda}{4} e^{-\sigma} \sinh \varphi_2 \left( b_1 e^{-2G_1} + b_2 e^{-2G_2} \right),\]

\[-D \frac{\partial \varphi_2}{\partial \rho} = 2 g_1 e^\sigma \sinh \varphi_2 - \lambda e^{-\sigma} \sinh \varphi_2 \left( a_1 e^{-2G_1} + a_2 e^{-2G_2} \right) \]

\[+ \lambda e^{-\sigma} \cosh \varphi_2 \left( b_1 e^{-2G_1} + b_2 e^{-2G_2} \right). \tag{3.27}\]

In the $r$-coordinate, the UV or asymptotically $AdS_6$ boundary is at $r = 0$, and the IR or $AdS_2 \times H^2 \times H^2$ horizon is at $r = \infty$. In this $\rho$-coordinate, the UV is at $\rho = +\infty$, and the IR is at $\rho = -\infty$. We present some representative plots of the full black hole solutions in figure 2.

4 Black holes with other horizons

In this section, we obtain more black hole solutions with other horizon geometries by considering D4-branes wrapped on Kähler four-cycles in Calabi-Yau fourfolds and on Cayley four-cycles in Spin(7) manifolds. We believe these are all possible four-cycles on which D4-branes can wrap in $F(4)$ gauged supergravity. D4-branes on two Riemann surfaces in the previous section fall into a special case of D4-branes on Kähler four-cycles in Calabi-Yau fourfolds. The analogous solutions of M5-branes wrapped on supersymmetric four-cycles were studied in [31, 32].

4.1 Kähler four-cycles in Calabi-Yau fourfolds

We consider the $U(1) \times U(1)$-invariant truncation presented in section 3.1. We consider the metric,

\[ds^2 = e^{2F(r)} \left( dt^2 - dr^2 \right) - e^{2G(r)} ds_{M_4}^2, \tag{4.1}\]
where \( M_4 \) is a Kähler four-cycle in Calabi-Yau fourfolds. The curved coordinates on the Kähler four-cycles will be denoted by \( \{ x_1, x_2, x_3, x_4 \} \), and the hatted ones are the flat coordinates. For Kähler four-cycles in Calabi-Yau fourfolds, there are four directions transverse to D4-branes in the fourfolds. The normal bundle of the four-cycle has \( U(2) \subset SO(4) \) structure group. We identify \( U(1) \) part of the structure group with \( U(1) \) gauge field from the non-Abelian SU(2) gauge group, \([31, 33]\). The only non-vanishing components of the field strength of SU(2) gauge field, \( A_\mu^A, \Lambda = 0, 1, \ldots, 6 \), are given by

\[
F_{\hat{x}_1 \hat{x}_2}^A = a_1 e^{-2G}, \quad F_{\hat{x}_3 \hat{x}_4}^A = a_2 e^{-2G},
F_{\hat{x}_1 \hat{x}_2}^B = b_1 e^{-2G}, \quad F_{\hat{x}_3 \hat{x}_4}^B = b_2 e^{-2G},
\]

where the magnetic charges, \( a_1, a_2, b_1, b_2 \), are constant. The only non-vanishing component of the two-form gauge potential is

\[
B_{tr} = -\frac{1}{2m^2} (a_1 a_2 - b_1 b_2) e^{2\sigma + 2F - 4G}.
\]

We employ the projection conditions,

\[
\gamma^j \epsilon_A = i \epsilon_A, \quad \gamma^{\hat{x}_1 \hat{x}_2} \sigma_{AB}^3 \epsilon^B = -i \lambda \epsilon_A, \quad \gamma^{\hat{x}_3 \hat{x}_4} \sigma_{AB}^3 \epsilon^B = -i \lambda \epsilon_A,
\]

where \( \lambda = \pm 1 \). Solutions with the projection conditions preserve 1/8 of the supersymmetries. By employing the projection conditions, we obtain the complete supersymmetry equations,

\[
F' e^{-F} = -\frac{1}{2} (g_1 e^\sigma \cosh \varphi_2 + me^{-3\sigma}) - \frac{\lambda}{2} e^{-\sigma - 2G} (a \cosh \varphi_2 - b \sinh \varphi_2)
- \frac{3}{8m} (a^2 - b^2) e^{\sigma - 4G}
\]

\[
G' e^{-F} = -\frac{1}{2} (g_1 e^\sigma \cosh \varphi_2 + me^{-3\sigma}) + \frac{\lambda}{2} e^{-\sigma - 2G} (a \cosh \varphi_2 - b \sinh \varphi_2)
+ \frac{1}{8m} (a^2 - b^2) e^{\sigma - 4G}
\]

\[
\sigma' e^{-F} = +\frac{1}{2} (g_1 e^\sigma \cosh \varphi_2 - 3me^{-3\sigma}) + \frac{\lambda}{2} e^{-\sigma - 2G} (a \cosh \varphi_2 - b \sinh \varphi_2)
- \frac{1}{8m} (a^2 - b^2) e^{\sigma - 4G}
\]

\[
\varphi_2' e^{-F} = +2g_1 e^\sigma \sinh \varphi_2 - 2\lambda e^{-\sigma - 2G} (a \sinh \varphi_2 - b \cosh \varphi_2).
\]

with the twist conditions,

\[
a \equiv a_1 = a_2 = -\frac{k}{2\lambda g_1}, \quad b \equiv b_1 = b_2.
\]

where \( k \) determines the curvature of the Kähler four-cycles in Calabi-Yau fourfolds. There is no condition on \( b_1 \) and \( b_2 \).

The product of two Riemann surfaces considered in the previous section is a special case of Kähler four-cycles in Calabi-Yau fourfolds. When we identify \( G \equiv G_1 = G_2 \) in the supersymmetry equations for D4-branes wrapped on two Riemann surfaces, \((3.17)\), we obtain the supersymmetry equations here, \((4.5)\). By solving the supersymmetry equations, we find the \( AdS_2 \) fixed point solutions which are identical to the ones obtained in the previous section.
4.2 Cayley four-cycles in Spin(7) manifolds

4.2.1 The SU(2)-invariant truncation

We considered matter coupled \( F(4) \) gauged supergravity coupled to three vector multiplets which has \( SU(2)_R \times SU(2) \) gauge symmetry. In this section we truncate the theory to the \( SU(2)_{\text{diag}} \subset SU(2)_R \times SU(2) \) invariant sector, which was first considered in [23] and again in section 4.1 of [25]. There is one singlet under \( SU(2)_{\text{diag}} \) which corresponds to \( Y_{11} + Y_{22} + Y_{33} \) by the non-compact generators defined in (2.21). We exponentiate the non-compact generators and obtain the coset representative,

\[
L = e^\varphi (Y_{11} + Y_{22} + Y_{33}) .
\]  

(4.7)

We also have a non-Abelian \( SU(2) \) gauge field and a two-form gauge potential in the \( SU(2)_{\text{diag}} \)-invariant truncation. The Lagrangian of the truncation is given by

\[
e^{-1}L = -\frac{1}{4} \bar{R} + \partial_\mu \sigma \partial^\mu \sigma + \frac{3}{4} \partial_\mu \varphi \partial^\mu \varphi - V
\]

\[
+ \frac{1}{8} \sinh^2(2\varphi) \left[ (g_1 A_{\mu}^1 - g_2 A_{\mu}^4)^2 + (g_1 A_{\mu}^2 - g_2 A_{\mu}^5)^2 + (g_1 A_{\mu}^3 - g_2 A_{\mu}^6)^2 \right]
\]

\[
- \frac{1}{8} e^{-2\varphi} \cosh(2\varphi) \left( F_{\mu\nu}^1 F^{1\mu\nu} + F_{\mu\nu}^2 F^{2\mu\nu} + F_{\mu\nu}^3 F^{3\mu\nu} + F_{\mu\nu}^4 F^{4\mu\nu} + F_{\mu\nu}^5 F^{5\mu\nu} + F_{\mu\nu}^6 F^{6\mu\nu} \right)
\]

\[
+ \frac{1}{4} e^{-2\varphi} \sinh(2\varphi) \left( F_{\mu\nu}^1 F^{4\mu\nu} + F_{\mu\nu}^2 F^{5\mu\nu} + F_{\mu\nu}^3 F^{6\mu\nu} \right) - \frac{1}{8} m^2 e^{-2\sigma} B_{\mu\nu} B^{\mu\nu} + \frac{3}{64} e^{4\sigma} H_{\mu\nu\rho} H^{\mu\nu\rho}
\]

\[
- \frac{1}{64} e^{\mu\rho\sigma\tau\kappa} B_{\mu\nu} \left( F_{\rho\sigma}^1 F_{\tau\kappa}^1 + F_{\rho\sigma}^2 F_{\tau\kappa}^2 + F_{\rho\sigma}^3 F_{\tau\kappa}^3 - F_{\rho\sigma}^4 F_{\tau\kappa}^4 - F_{\rho\sigma}^5 F_{\tau\kappa}^5 - F_{\rho\sigma}^6 F_{\tau\kappa}^6 + \frac{1}{3} m^2 B_{\rho\sigma} B_{\tau\kappa} \right),
\]

(4.8)

where the scalar potential is

\[
V = \frac{1}{16} g_1^2 e^{2\varphi} (\cosh(6\varphi) - 9 \cosh(2\varphi) - 8) + \frac{1}{16} g_2^2 e^{2\varphi} (\cosh(6\varphi) - 9 \cosh(2\varphi) + 8)
\]

\[
- \frac{1}{2} g_1 g_2 e^{2\varphi} \sinh^3(2\varphi) - 4 g_1 m e^{-2\sigma} \cosh^3 \varphi + 4 g_2 m e^{-2\sigma} \sinh^3 \varphi + m^2 e^{-6\sigma} .
\]

(4.9)

4.2.2 The supersymmetry equations

We consider the metric,

\[
ds^2 = e^{2F(r)} \left( dt^2 - dr^2 \right) - e^{2G(r)} ds_{M_4}^2 ,
\]

(4.10)

where \( M_4 \) is a Cayley four-cycle in manifolds with Spin(7) holonomy. The curved coordinates on the Cayley four-cycles will be denoted by \( \{ x_1, x_2, x_3, x_4 \} \), and the hatted ones are the flat coordinates. In order to preserve supersymmetry for D4-branes wrapped on Cayley four-cycles in Spin(7) manifolds, we identify self-dual \( SU(2)_+ \) subgroup of the SO(4) isometry of the four-cycle,

\[
SO(4) \rightarrow SU(2)_+ \times SU(2)_- ,
\]

(4.11)
with the non-Abelian SU(2) gauge group, [31, 33]. The self-duality is defined by
\[ \gamma_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho} \gamma^{\rho}, \]  
and we denoted the self-duality and anti-self-duality by + and −, respectively. For the self-dual part, components are identified by
\[ \gamma^{\hat{x}_1 \hat{x}_2} = \gamma^{\hat{x}_3 \hat{x}_4}, \quad \gamma^{\hat{x}_1 \hat{x}_3} = \gamma^{\hat{x}_4 \hat{x}_2}, \quad \gamma^{\hat{x}_1 \hat{x}_4} = \gamma^{\hat{x}_2 \hat{x}_3}. \]  
The only non-vanishing components of the field strength of the SU(2) gauge field, \( A^A_{\mu}, \Lambda = 0, 1, \ldots, 6 \), are given by
\[ F^{1}_{\hat{x}_1 \hat{x}_2} = F^{1}_{\hat{x}_3 \hat{x}_4} = a_1 e^{-2G}, \quad F^{4}_{\hat{x}_1 \hat{x}_2} = F^{1}_{\hat{x}_3 \hat{x}_4} = b_1 e^{-2G}, \]
\[ F^{2}_{\hat{x}_1 \hat{x}_3} = F^{2}_{\hat{x}_4 \hat{x}_5} = a_2 e^{-2G}, \quad F^{5}_{\hat{x}_2 \hat{x}_3} = F^{5}_{\hat{x}_2 \hat{x}_5} = b_2 e^{-2G}, \]
\[ F^{3}_{\hat{x}_1 \hat{x}_4} = F^{3}_{\hat{x}_3 \hat{x}_5} = a_3 e^{-2G}, \quad F^{6}_{\hat{x}_2 \hat{x}_3} = F^{6}_{\hat{x}_2 \hat{x}_5} = b_3 e^{-2G}, \]  
where the magnetic charges, \( a_1, a_2, a_3, b_1, b_2, b_3, \) are constant. As we have one SU(2)\(_{\text{diag}}\) ⊂ SU(2)\(_R\) × SU(2) non-Abelian gauge field, they are related by
\[ g_1 a_1 = g_2 b_1, \quad g_1 a_2 = g_2 b_2, \quad g_1 a_3 = g_2 b_3, \]  
where \( g_1 \) and \( g_2 \) are the gauge coupling constants of SU(2)\(_R\) and SU(2), respectively. The only non-vanishing component of the two-form gauge potential is
\[ B_{tr} = -\frac{1}{2m^2} (a_1^2 + a_2^2 + a_3^2 - b_1^2 - b_2^2 - b_3^2) e^{2\sigma + 2F - 4G}. \]  
We employ the projection conditions,
\[ \gamma^{\hat{x}_1 \hat{x}_2} \sigma_{AB} = i \epsilon_A, \]
\[ \gamma^{\hat{x}_1 \hat{x}_2} \sigma_{AB}^1 = \gamma^{\hat{x}_3 \hat{x}_4} \sigma_{AB}^1 = -i \lambda \epsilon_A, \]
\[ \gamma^{\hat{x}_1 \hat{x}_3} \sigma_{AB}^2 = \gamma^{\hat{x}_4 \hat{x}_5} \sigma_{AB}^2 = -i \lambda \epsilon_A, \]
\[ \gamma^{\hat{x}_1 \hat{x}_4} \sigma_{AB}^3 = \gamma^{\hat{x}_2 \hat{x}_3} \sigma_{AB}^3 = -i \lambda \epsilon_A, \]
where \( \lambda = \pm 1 \). Solutions with the projection conditions preserve 1/16 of the supersymmetries.

Employing the projection conditions, from \( \delta \lambda^I_A = 0 \), we obtain for \( I = 1, 2, 3 \), respectively,
\[ \varphi' e^{-F} = + (g_1 e^\sigma \cosh \varphi - g_2 e^\sigma \sinh \varphi \sinh(2\varphi) - 2\lambda e^{-\sigma - 2G} (a_1 \sinh \varphi - b_1 \cosh \varphi)), \]
\[ \varphi' e^{-F} = + (g_1 e^\sigma \cosh \varphi - g_2 e^\sigma \sinh \varphi \sinh(2\varphi) - 2\lambda e^{-\sigma - 2G} (a_2 \sinh \varphi - b_2 \cosh \varphi)), \]
\[ \varphi' e^{-F} = + (g_1 e^\sigma \cosh \varphi - g_2 e^\sigma \sinh \varphi \sinh(2\varphi) - 2\lambda e^{-\sigma - 2G} (a_3 \sinh \varphi - b_3 \cosh \varphi)). \]  
Therefore, we conclude that the magnetic charges are
\[ a = a_1 = a_2 = a_3, \quad b = b_1 = b_2 = b_3. \]
We present the complete supersymmetry equations,
\[F' e^{-F} = - \frac{1}{2} \left( g_1 e^\varphi \cosh^3 \varphi - g_2 e^\varphi \sinh^3 \varphi + m e^{-3\sigma} \right) - \frac{3\lambda}{2} e^{-\sigma-2G} \left( a \cosh \varphi - b \sinh \varphi \right) \]
\[- \frac{9}{8m} \left( a^2 - b^2 \right) e^{\sigma-4G}, \]
\[G' e^{-F} = - \frac{1}{2} \left( g_1 e^\varphi \cosh^3 \varphi - g_2 e^\varphi \sinh^3 \varphi + m e^{-3\sigma} \right) + \frac{3\lambda}{2} e^{-\sigma-2G} \left( a \cosh \varphi - b \sinh \varphi \right) \]
\[+ \frac{3}{8m} \left( a^2 - b^2 \right) e^{\sigma-4G}, \]
\[\sigma' e^{-F} = + \frac{1}{2} \left( g_1 e^\varphi \cosh^3 \varphi - g_2 e^\varphi \sinh^3 \varphi - 3m e^{-3\sigma} \right) + \frac{3\lambda}{2} e^{-\sigma-2G} \left( a \cosh \varphi - b \sinh \varphi \right) \]
\[- \frac{3}{8m} \left( a^2 - b^2 \right) e^{\sigma-4G}, \]
\[\varphi' e^{-F} = + \left( g_1 e^\varphi \cosh \varphi - g_2 e^\varphi \sinh \varphi \right) \sinh(2\varphi) - 2\lambda e^{-\sigma-2G} \left( a \cosh \varphi - b \cosh \varphi \right). \]

We also obtain twist conditions on the magnetic charges,
\[a = - \frac{k}{6\lambda g_1}, \quad b = - \frac{k}{6\lambda g_2}, \quad (4.21)\]
where \(k\) determines the curvature of the Cayley four-cycles in Spin(7) manifolds. The twist condition on \(b\) comes from the SU(2)\(_{\text{diag}}\) condition, (4.15).

### 4.2.3 The AdS\(_2\) solutions

Now we will consider the \(\mathcal{N} = 4^+\) theory, \(g_1 > 0, m > 0\). When \(b = 0\), we find an AdS\(_2\) fixed point solution for \(k = -1\),
\[e^F = \frac{3^{1/4}}{2^{3/2}g_1^{3/4}m^{1/4}} r, \quad e^G = \frac{1}{2^{1/2}3^{1/4}g_1^{3/4}m^{1/4}}, \quad e^{\sigma} = \frac{2^{1/2}m^{1/4}}{3^{1/4}g_1^{1/4}}, \quad e^{\varphi} = 1. \quad (4.22)\]

After the reparametrization of the parameters given in (3.20), this is the AdS\(_2\) solution first found in [11].

When we consider for non-zero \(b\), we find new AdS\(_2\) solutions in terms of the scalar field, \(\varphi\),
\[e^F = \frac{e^{-\sigma}}{2(g_1 \cosh^3 \varphi - g_2 \sinh^3 \varphi) r}, \quad e^{2G} = \frac{3\lambda}{2m} e^{2\sigma} \left( a \cosh \varphi - b \sinh \varphi \right), \]
\[e^{4\sigma} = \frac{m}{6} \left( 5(a^2 - b^2) + 3(a^2 + b^2) \cosh(2\varphi) - 6ab \sinh(2\varphi) \right) \]
\[\left( g_1 \cosh^3 \varphi - g_2 \sinh^3 \varphi \right) \left( a \cosh \varphi - b \sinh \varphi \right)^2. \quad (4.23)\]

Then, the scalar field, \(\varphi\), should be expressed in terms of the magnetic charges, \(a\) and \(b\), but the expression is very unwieldy. Alternatively, we present the magnetic charge, \(b\), in terms of the scalar field, \(\varphi\),
\[b = - a \left[ g_1 \left( 1 + e^{2\varphi} \right)^3 \left( 1 - 6e^{2\varphi} + e^{4\varphi} \right) + g_2 \left( 1 - e^{2\varphi} \right)^3 \left( 1 + 6e^{2\varphi} + e^{4\varphi} \right) \right. \]
\[+ 4e^{2\varphi} \sqrt{g_2^2 \left( 1 + e^{2\varphi} \right)^4 \left( 5 - 6e^{2\varphi} + 5e^{4\varphi} \right) + g_2^2 \left( 1 - e^{2\varphi} \right)^4 \left( 5 + 6e^{2\varphi} + 5e^{4\varphi} \right) + 10g_1 g_2 \left( 1 - e^{4\varphi} \right)^3} \]
\[\left. / \left( (1 - e^{4\varphi}) \left( g_1 \left( 1 - 13e^{2\varphi} - 13e^{4\varphi} + e^{6\varphi} \right) + g_2 \left( 1 - 7e^{2\varphi} + 7e^{4\varphi} - e^{6\varphi} \right) \right) \right) \right]. \quad (4.24)\]
The solutions are parametrized by two magnetic charges, $a$ and $b$. It will be interesting to have a field theory interpretation of this $AdS_2$ fixed point solution.

Unlike the black holes with a horizon of two Riemann surfaces, for this case, we could not device a way to determine the positivity range for $AdS_2$ solutions. However, as we see in the next subsection, we obtained a number of $AdS_2$ solutions with negative curvature horizon, $k = -1$, numerically. On the other hand, we could not find any solutions with positive curvature horizon, $k = +1$. Thus, we will concentrate on solutions with negative curvature horizon, $k = -1$.

4.2.4 Numerical black hole solutions

Now we present the full black hole solution numerically. The full black hole solution is an interpolating geometry between the asymptotically $AdS_6$ boundary and the $AdS_2 \times Cayley_4$ horizon. As we explained at the end of the last subsection, we will concentrate on solutions with negative curvature horizon, $k = -1$. We introduce a new radial coordinate,

$$\rho = F + \sigma.$$  \hspace{1cm} (4.25)

This kind of coordinate was introduced in [30]. Employing the supersymmetry equations, we obtain

$$\frac{\partial \rho}{\partial r} = F' + \sigma' = -e^F D,$$  \hspace{1cm} (4.26)

where we define

$$D = 2m e^{-3\sigma} + \frac{3}{2m} (a^2 - b^2) e^{-4G}.$$  \hspace{1cm} (4.27)

Then, the supersymmetry equations are

$$-D \frac{\partial F}{\partial \rho} = -\frac{1}{2} \left( g_1 e^\sigma \cosh^3 \varphi - g_2 e^\sigma \sinh^3 \varphi + me^{-3\sigma} \right) - \frac{3\lambda}{2} e^{-\sigma - 2G} (a \cosh \varphi - b \sinh \varphi) - \frac{9}{8m} (a^2 - b^2) e^{\sigma - 4G},$$

$$-D \frac{\partial G}{\partial \rho} = -\frac{1}{2} \left( g_1 e^\sigma \cosh^3 \varphi - g_2 e^\sigma \sinh^3 \varphi + me^{-3\sigma} \right) + \frac{3\lambda}{2} e^{-\sigma - 2G} (a \cosh \varphi - b \sinh \varphi) + \frac{3}{8m} (a^2 - b^2) e^{\sigma - 4G},$$

$$-D \frac{\partial \sigma}{\partial \rho} = + \frac{1}{2} \left( g_1 e^\sigma \cosh^3 \varphi - g_2 e^\sigma \sinh^3 \varphi - 3me^{-3\sigma} \right) + \frac{3\lambda}{2} e^{-\sigma - 2G} (a \cosh \varphi - b \sinh \varphi) - \frac{3}{8m} (a^2 - b^2) e^{\sigma - 4G},$$

$$-D \frac{\partial \varphi_2}{\partial \rho} = + (g_1 e^\sigma \cosh \varphi - g_2 e^\sigma \sinh \varphi) \sinh(2\varphi) - 2\lambda e^{-\sigma - 2G} (a \sinh \varphi - b \cosh \varphi).$$  \hspace{1cm} (4.28)

In the $r$-coordinate, the UV or asymptotically $AdS_6$ boundary is at $r = 0$, and the IR or $AdS_2 \times Cayley_4$ horizon is at $r = \infty$. In this $\rho$-coordinate, the UV is at $\rho = +\infty$, and the IR is at $\rho = -\infty$. We set $m = 1/2$ and $g_1 = 3m$ to have a unit radius for the $AdS_6$ boundary. Then, $a$ is determined by (4.21), and there is one free parameter left, $b$. Employing (4.23) and (4.24) to determine boundary conditions, we solve the supersymmetry
Figure 3. Numerical black hole solutions with \( m = 1/2 \) and \( g = 3m \). We have \( b = -0.1 \) for purple, \( b = -0.3 \) for red, and \( b = -0.5 \) for orange.

equations numerically. With some choices of \( b \), we present representative plots of the full black hole solutions in figure 3.

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A The equations of motion

In this appendix, we present the equations of motion for the U(1) \( \times U(1) \)-invariant truncation in (4.8) with \( \varphi_1 = 0 \) as in (3.16),

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 2 \left[ V g_{\mu\nu} + 2 \left( \partial_\mu \sigma \partial_\nu \sigma - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \sigma \partial_\sigma \right) 
+ \frac{1}{2} \left( \partial_\mu \varphi_2 \partial_\nu \varphi_2 - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \varphi_2 \partial_\sigma \varphi_2 \right) 
- \frac{1}{2} e^{-2\sigma} \cosh(2\varphi_2) \left( g^{\rho\sigma} F_{\mu\rho}^3 F_{\nu\sigma}^3 - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}^3 F_{3\rho\sigma}^3 \right) 
- \frac{1}{2} e^{-2\sigma} \cosh(2\varphi_2) \left( g^{\rho\sigma} F_{\mu\rho}^6 F_{\nu\sigma}^6 - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}^6 F_{6\rho\sigma}^6 \right) 
+ e^{-2\sigma} \sinh(2\varphi_2) \left( g^{\rho\sigma} F_{\mu\rho}^3 F_{\nu\sigma}^6 - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma}^3 F_{3\rho\sigma}^6 \right) 
- \frac{1}{2} m^2 e^{-2\sigma} \left( g^{\rho\sigma} B_{\mu\rho} B_{\nu\sigma} - \frac{1}{4} g_{\mu\nu} B_{\rho\sigma} B^{\rho\sigma} \right) \right],
\]  

(A.1)
\[
\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu \nu} \partial_{\nu} \sigma \right) + \frac{1}{2} \frac{\partial V}{\partial \sigma} - \frac{1}{8} e^{-2\sigma} \cosh(2\varphi_2) F_{\mu \nu}^3 F^{3\mu \nu} \right) - \frac{1}{8} e^{-2\sigma} \cosh(2\varphi_2) F_{\mu \nu}^6 F^{6\mu \nu} \\
+ \frac{1}{4} e^{-2\sigma} \cosh(2\varphi_2) F_{\mu \nu}^3 F^{6\mu \nu} - \frac{1}{8} m^2 e^{-2\sigma} B_{\mu \nu} B^{\mu \nu} = 0,
\]
(A.2)

\[
\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu \nu} \partial_{\nu} \varphi_2 \right) + \frac{1}{2} \frac{\partial V}{\partial \varphi_2} + \frac{1}{2} e^{-2\sigma} \cosh(2\varphi_2) F_{\mu \nu}^3 F^{3\mu \nu} + \frac{1}{2} e^{-2\sigma} \cosh(2\varphi_2) F_{\mu \nu}^6 F^{6\mu \nu} \\
- e^{-2\sigma} \cosh(2\varphi_2) F_{\mu \nu}^3 F^{6\mu \nu} = 0.
\]
(A.3)

\[
\mathcal{D}_\mu \left( e^{-2\sigma} F^{\Lambda \mu \nu} \right) = \frac{1}{24} e^\mu e^{\rho \sigma \tau \kappa} F_{\nu \rho} H_{\sigma \tau \kappa},
\]
(A.4)

\[
\mathcal{D}_\mu \left( e^{4\sigma} H^{\rho \mu \nu} \right) = -\frac{1}{16} e^{\mu \rho \sigma \tau \kappa} \eta_{\Lambda \Sigma} F_{\nu \rho}^\Lambda F_{\tau \kappa}^\Sigma - m e^{-2\sigma} \delta^0 \Lambda F^{\Lambda \mu \nu}.
\]
(A.5)

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