Fragmentation of

\textit{SU}(2)-invariant spin ladders

\textit{Giuseppe Albertini}\textsuperscript{*}

Dipartimento di Fisica, Universita’ di Milano
via Celoria 16, 20133 Milano (Italy)
(20.2.2001)

A two-parameter family of quantum spin ladders with local bilinear and bi-quadratic interactions is shown to be solvable by a mapping onto fragments of integrable spin 1 chains. The phase diagram, consisting of four phases, and the ground state properties are discussed. In one novel phase, the ground state is made up of plaquette singlets and rung singlets, alternating with a three-rung periodicity.

PACS numbers: 75.10.Jm

IFUM-681-FT

I. INTRODUCTION

Integrable models have provided extremely valuable insight into the physics of one dimensional quantum spin chains. There is a substantial lack of exact results for analogous two dimensional systems and the solution of quantum spin ladders might provide a first step in that direction. Besides, spin ladders are currently experimentally accessible, interesting in their own and possibly related to high $T_c$ superconductivity \cite{1}.

It has been established that the basic $n$-leg Heisenberg ladders, with bilinear exchange interactions along rungs and legs, are gapful (spin-liquid state) for $n$ even and gapless for $n$ odd \cite{1}. These ladders do not seem to be integrable. On the other hand, examples of integrable ladders, containing additional biquadratic interactions, have been found and solved by some form of Bethe-ansatz (BA) \cite{2,3}. Luckily, biquadratic interactions do arise in physically realizable systems, and a large class of these generalized, but still $SU(2)$-invariant ladders, have been proven to have a matrix-product (MP) ground state \cite{6}. Still, the MP approach determines the ground state but, with the exception of few lucky cases, the whole set of excitations remains unknown.

In this paper, the following two-parameter family of $SU(2)$-invariant ladder hamiltonians $H = \sum_{k=1}^{N} H_{k,k+1}$ will be studied

\textsuperscript{*}E-mail Giuseppe.Albertini@mi.infn.it
and the eigenvalue problem solved for any $\tilde{\mu}_1, \tilde{\mu}_2$. Here $s_k$ and $t_k$ are spin 1/2 matrices sitting on the first and the second leg, respectively. Hamiltonian (1) is actually a special point in a wider three-parameter class for which the relative strength of the first two terms is left arbitrary, (see (11)). All points in this manifold have the property of being reducible, through the introduction of the composite rung spin

$$S_k = s_k + t_k$$

(2)
to the general spin 1 chain

$$H_0(\theta) = \sum_{k=1}^{N} \left( \cos \theta S_k \cdot S_{k+1} + \sin \theta (S_k \cdot S_{k+1})^2 \right)$$

(3)

In particular (1) corresponds to the BA solvable purely biquadratic spin 1 chain, $\theta = -\pi/2$. Selecting $\theta = 0$, the Heisenberg spin 1 chain, would remove the second brackets in (1) [4]. This seems physically more appealing but prevents the detailed analysis allowed by BA, since such chain is not integrable. Details about definitions and exact diagonalization are the subject of sections II and III.

The system defined by (1) has four phases, connected by first order transition lines, namely discontinuities in some first order derivative of the ground state energy per site. At least one phase is novel because its ground state is a threefold degenerate global singlet where dimerization takes place alternatively along legs and rungs with a three-step periodicity. A second phase has a huge ($\sim 3N/2$) ground state degeneracy. All this is discussed in section IV.

In section V, exact elementary excitations are examined in the phase whose ground state is, effectively, that of the biquadratic chain, also a global singlet but made up of rung triplets. The simplest ones are created by introducing one rung singlet and leaving the remaining $N - 1$ triplets locked into the ground state of the biquadratic chain with free ends. If $N - 1$ is odd, such ground state is presumably an $SU(2)$ triplet containing one dynamical kink and one finds a band of singlet-triplet excitations, depending on two degrees of freedom, whose energy is degenerate in the singlet position. Such excitations had been found in a related ladder [3], but their dependence on two degrees of freedom had not been discussed. On the other hand, similar ladders including biquadratic interactions are known to have two-parameter singlet-triplet excitations [7,6], but their nature is different, being a pair of kink-antikink over a dimerized ground state, which is not what happens here. In section VI, some general results are discussed for the wider three-parameter family of spin ladder hamiltonians.
II. DEFINITION OF BASIC INVARIANTS

Each elementary plaquette involves four spins, say \((s_1, t_1, s_2, t_2)\). If only \(SU(2)\) invariance is required, the most general hermitean plaquette hamiltonian \(H_{1,2}\) is a linear combination with real coefficients of 14 hermitean invariants. If, beside \(SU(2)\) invariance, one requires symmetry under the exchange of the two legs and symmetry under the exchange of the two rungs, symmetries implemented by operators \(C\) and \(P\)

\[
CS_iC = t_i \quad (i = 1, 2) \quad \quad C^2 = 1
\]

\[
Ps_1P = s_2 \quad \quad Pt_1P = t_2 \quad \quad P^2 = 1
\]

then \(H_{1,2}\) is a linear combination with real coefficients of 8 hermitean invariants.

The proof goes as follows. The local plaquette Hilbert space is \(H_{1,2} \cong V_1^{(s)} \otimes V_1^{(t)} \otimes V_2^{(s)} \otimes V_2^{(t)} \cong C^{16}\). It breaks into the orthogonal sum of \(SU(2)\) multiplets: one quintuplet (dim=5), three triplets (dim=9) and two singlets (dim=2). Within each multiplet, all states are obtained by application of the lowering operator \(S_{1,2}^{-}\) to the highest weight vector (h.w.v.) \(v^{(s)}\), where

\[
S_{1,2} = s_1 + s_2 + t_1 + t_2 \quad \quad S^\pm_{1,2} = S^\pm_{1,2} \pm iS^0_{1,2}
\]

\[
S_{1,2}^+v^{(s)} = 0 \quad \quad S_{1,2}^2v^{(s)} = s(s + 1)v^{(s)} \quad \quad v^{(s)} \in H_{1,2}
\]

\(H_{1,2}\) is \(SU(2)\)-invariant, i.e. \([H_{1,2}, S_{1,2}] = 0\), therefore it maps h.w.v. into h.w.v. of the same spin. So, spin-2 h.w.v. must be mapped into itself, each spin-1 h.w.v. will in general be mapped into a linear combination of all three spin-1 h.w.v. and the same happens for the two spin-0 h.w.v.. Altogether, imposing hermiticity, 6 real parameters plus 4 complex (hence 8 real) ones. The total spin \(S_{1,2}\) commutes with \(C\) and \(P\), so h.w.v. can be labelled by \(C\) and \(P\) eigenvalues. Define, with self-explanatory notation

\[
e_0 = \left| \uparrow\uparrow \right>_s \quad f_0 = \left| \uparrow\uparrow \right>_t \quad \tau_0 = \left| \downarrow\downarrow \right>_s \quad \mathbf{F}_0 = \left| \downarrow\downarrow \right>_t
\]

\[
e_1 = \left| \downarrow\uparrow \right>_s \quad f_1 = \left| \downarrow\uparrow \right>_t \quad e_2 = \left| \uparrow\uparrow \right>_s \quad f_2 = \left| \uparrow\downarrow \right>_t
\]

The 6 h.w.v. \(v^{(s)}\) will be chosen to be

\[
v^{(2)} = e_0f_0 \quad \quad (2, 1, 1)
\]

\[
v^{(1)}_1 = e_0f_1 - e_0f_2 + e_1f_0 - e_2f_0 \quad \quad (1, 1, -1)
\]

\[
v^{(1)}_2 = e_0f_1 + e_0f_2 - e_1f_0 - e_2f_0 \quad \quad (1, -1, 1)
\]

\[
v^{(1)}_3 = e_0f_1 - e_0f_2 - e_1f_0 + e_2f_0 \quad \quad (1, -1, -1)
\]
\[ v_1^{(0)} = e_1 f_1 + e_2 f_2 - e_1 f_2 - e_2 f_1 \quad (0, 1, 1) \]
\[ v_2^{(0)} = e_1 f_1 + e_1 f_2 + e_2 f_1 + e_2 f_2 - 2 e_0 f_0 - 2 \overline{e_0} f_0 \quad (0, 1, 1) \]

The three numbers on the right column are eigenvalues of \( S_1, C, P \). If \( C \) and \( P \) symmetries are imposed, \( H_{1,2} \) maps \( v^{(2)} \) into itself and each \( v_i^{(1)} \) \((i = 1, 2, 3)\) into itself, while its action on the span of \( \{ v_i^{(0)}; i = 1, 2 \} \) is determined by 4 real numbers

\[
H_{1,2} v^{(2)} = m^{(2)}(2) v^{(2)} \quad H_{1,2} v_i^{(1)} = m_i^{(1)}(1)^{v_i(1)} \quad i = 1, 2, 3
\]
\[
H_{1,2} v_i^{(0)} = \sum_{j=1}^{2} m_j^{(0)}(0) v_j^{(0)} \quad i = 1, 2
\]

This proves that the general \( H_{1,2} \) is a linear combination with real coefficients of 8 linearly independent invariants (notice incidentally that h.w.v. \( v_i^{(0)} \) are orthogonal but unnormalized, \(|v_2^{(0)}|^2 = 3|v_1^{(0)}|^2\), so \( m_{12}^{(0)} = 3 m_{21}^{(0)} \)). One can choose 7 of them to be three bilinear (Heisenberg) terms

\[
I_{1,2}^{(1)} = s_1 \cdot s_2 + t_1 \cdot t_2 \quad (4)
\]
\[
I_{1,2}^{(2)} = s_1 \cdot t_1 + s_2 \cdot t_2 \quad (5)
\]
\[
I_{1,2}^{(3)} = s_1 \cdot t_2 + s_2 \cdot t_1 \quad (6)
\]

three biquadratic (plaquette) terms

\[
I_{1,2}^{(4)} = (s_1 \cdot s_2)(t_1 \cdot t_2) \quad (7)
\]
\[
I_{1,2}^{(5)} = (s_1 \cdot t_1)(s_2 \cdot t_2) \quad (8)
\]
\[
I_{1,2}^{(6)} = (s_1 \cdot t_2)(s_2 \cdot t_1) \quad (9)
\]

and the identity \( I \). An eighth one is necessary to have a complete set, but it involves more complicated combinations of the basic spins and it will not be needed in the following. To show that the six in (4)-(9) plus the identity are indeed linearly independent, write

\[ H_{1,2} = c_0 I + \sum_{k=1}^{6} c_k I^{(k)}_{1,2} \]

It is a matter of easy algebra to find the action of the basic invariants (4)-(9) on the h.w.v., resulting in

\[
m^{(2)} = c_0 + \frac{c_1 + c_2 + c_3}{2} + \frac{c_4 + c_5 + c_6}{16}
\]
\[
m^{(1)}_1 = c_0 + \frac{-c_1 + c_2 - c_3}{2} + \frac{-3c_4 + c_5 - 3c_6}{16}
\]
\[
m^{(1)}_2 = c_0 + \frac{c_1 - c_2 - c_3}{2} + \frac{c_4 - 3c_5 - 3c_6}{16}
\]
\[
m^{(1)}_3 = c_0 + \frac{-c_1 - c_2 + c_3}{2} + \frac{-3c_4 - 3c_5 + c_6}{16} \quad (10)
\]
\[ m_{11}^{(0)} = c_0 + \frac{-3c_1}{2} + \frac{9c_4 + 3c_5 + 3c_6}{16} \]
\[ m_{22}^{(0)} = c_0 + \frac{c_1 - 2c_2 - 2c_3}{2} + \frac{c_4 + 7c_5 + 7c_6}{16} \]
\[ m_{21}^{(0)} = \frac{c_2 - c_3}{2} + \frac{-c_5 + c_6}{8} \]

and \( m_{12}^{(0)} = 3m_{21}^{(0)} \). The vanishing of all \( \{m_i^{(s)}\} \) implies the vanishing of all \( \{c_i\} \), proving linear independence of the invariants chosen.

Scalar products of the rung spin will now be expressed in terms of these invariants. Clearly

\[ S_1 \cdot S_2 = I_{1,2}^{(1)} + I_{1,2}^{(3)} \]

while the biquadratic term \( (S_1 \cdot S_2)^2 \) seems to involve more complicated invariants. But since it is invariant under \( C \) and \( P \), it must be a linear combination of the 8 basic ones. Actually, the six \((4)-(9)\) and the identity are sufficient: from \((10)\)

\[
(S_1 \cdot S_2)^2 = \frac{3I}{4} - \frac{I_{1,2}^{(1)}}{2} + \frac{I_{1,2}^{(2)}}{2} - \frac{I_{1,2}^{(3)}}{2} + 2I_{1,2}^{(4)} + 2I_{1,2}^{(6)}
\]

Consider now the one-parameter spin ladder Hamiltonian \((3)\) where each spin \( S \) is the composite object defined in \((2)\). Two parameters can be added after noticing that the charges

\[ Q_1 = \sum_{k=1}^{N} S_k^2 \quad \quad Q_2 = \sum_{k=1}^{N} S_k^2 S_{k+1}^2 \]

commute with \( H_0(\theta) \) for any \( \theta \). This can easily be checked by a direct calculation. Finally, the Hamiltonian to be studied is

\[
H(\theta, \mu_1, \mu_2) = H_0(\theta) + \mu_1 Q_1 + \mu_2 Q_2 = \\
\sum_{k=1}^{N} \left[ (\cos \theta - \frac{\sin \theta}{2})I_{1,2}^{(1)} + I_{1,2}^{(3)} + 2 \sin \theta (I_{1,2}^{(4)} + I_{1,2}^{(6)}) + (\sin \theta + \mu_1 + 3\mu_2)I_{1,2}^{(2)} \\
+ 4\mu_2 I_{1,2}^{(5)} \right] + N \left[ \frac{3 \sin \theta}{4} + \frac{3\mu_1}{2} + \frac{9\mu_2}{4} \right]
\]

(11)

Eq. (11) is (11), up to a constant shift, if one takes \( \theta = -\pi/2 \) and

\[
\tilde{\mu}_1 = -1 + \mu_1 + 3\mu_2 \quad \quad \tilde{\mu}_2 = 4\mu_2
\]

(12)

In the following, parameters \((\mu_1, \mu_2)\) will be adopted to describe the model. It is always possible, through \((12)\), to revert to the original ones \((\tilde{\mu}_1, \tilde{\mu}_2)\).

One further remark. Since each rung space \( V_k^{(s)} \otimes V_k^{(t)} \) carries a spin \( 1 \oplus 0 \) representation, not just spin 1, \((S_k \cdot S_{k+1})^3 \) is linearly independent from \( S_k \cdot S_{k+1} \) and \((S_k \cdot S_{k+1})^2 \). But, from \((11)\), it can be seen that \((S_k \cdot S_{k+1})^3 \) is a linear combination of the first two powers, \( I \), \( I_{1,2}^{(2)} \) and \( I_{1,2}^{(5)} \), that is why adding a cubic term to \( H_0(\theta) \) does not generalize \((11)\).
The composite rung spin has, of course, been introduced in several previous works. It lies behind the physical idea that even-legged and odd-legged Heisenberg ladders should be in different phases [1]. More closely to this work, it has been used [6] to map two-leg ladders into the AKLT chain [13] at \( \tan \theta = 1/3 \). Earlier, Xian and then Kitakani and Oguchi [4] had studied a state at \( \theta = 0 \), but without the rung-rung interaction \( I_{k,k+1} \) which is actually responsible for the rise of two new phases.

III. FRAGMENTATION AND BETHE-ANSATZ

Charges \( Q_1 \) and \( Q_2 \) have a simple physical meaning. In each four dimensional rung space \( V_k = V_k^{(s)} \otimes V_k^{(t)} \simeq C^4 \) one introduces the singlet-triplet basis \( \{ |s >_k; |t >_k, t = -1, 0, 1 \} \) consisting of the singlet and the triplet of the rung spin [2], i.e.

\[
S^2_k |s >_k = 0 \quad S^2_k |t >_k = 2 |t >_k \quad S^z_k |t >_k = t |t >_k
\]

Then \( Q_1 \) counts the number of triplets and \( Q_2 \) counts the number of pairs of neighboring triplets.

Now consider the \( 3^N \) dimensional subspace (of the total \( 4^N \)-dimensional Hilbert space of the ladder) spanned by \( |t_1, t_2, \ldots, t_N > \), \( (t_k = -1, 0, 1) \). In this sector \( Q_1 \) and \( Q_2 \) are constant, at \( 2N \) and \( 4N \) respectively, and \( H_0 \) acts effectively as a spin-1 chain with periodic boundary conditions (p.b.c.). Such chain is Bethe-ansatz solvable in three cases: (a) \( \theta = \frac{\pi}{4} \), (b) \( \theta = -\frac{\pi}{4} \) and (c) \( \theta = -\frac{\pi}{2} \). In case (a) it is the \( SU(3) \)-invariant Sutherland-Uimin chain [11], in case (b) the Babujian-Takhtajian chain [12] and in case (c) the purely biquadratic chain [15].

Next, consider states containing one singlet and \( N - 1 \) triplets. Eigenvalues of the two conserved charges are fixed at

\[
Q_1 = 2(N - 1) \quad Q_2 = 4(N - 2)
\]

regardless of triplet’s position. If the singlet is, say, on the \( N^{th} \) rung (due to p.b.c. the singlet’s position is actually immaterial), \( H_0 \) acts on these vectors like

\[
H_0 \simeq \sum_{k=1}^{N-2} \left( \cos \theta S_k \cdot S_{k+1} + \sin \theta (S_k \cdot S_{k+1})^2 \right)
\]

that is exactly like a spin-1 chain of length \( N - 1 \) and free boundary conditions (f.b.c.). The singlet can be positioned anywhere, it just opens a fracture in a ring of spins, consequently each eigenvalue of \( (13) \) appears with an \( N \)-fold degeneracy in the ladder spectrum. More importantly, the spectrum of \( (13) \) can be found exactly by Bethe-ansatz, at least in cases (b) and (c) which have been more or less extensively studied for f.b.c. \[14,15\]. The opposite situation arises when all rungs are singlets. There is only one such state and its eigenvalue is trivially zero.
Between these extreme cases, and orthogonal to them, there is all possible intermediate situations, characterized by alternating fragments of triplets and singlets. Sectors are labeled by a sequence of positive integers, the lengths of fragments, \( \{ N^{(0)}_1, N^{(1)}_1, N^{(0)}_2, N^{(1)}_2, \ldots, N^{(0)}_n, N^{(1)}_n \} \), with \( \sum_{j=1}^{n} N^{(0)}_j = N^{(0)} \), \( \sum_{j=1}^{n} N^{(1)}_j = N^{(1)} \) and \( N^{(0)} + N^{(1)} = N \). Each sector is spanned by the 3\( N^{(1)} \) vectors

\[
|\phi> = |s, s, \ldots, s, t_1^{(1)}, t_2^{(1)}, \ldots, t_{N^{(1)}_1}; s, s, \ldots, s, t_1^{(2)}, t_2^{(2)}, \ldots, t_{N^{(1)}_2}; \ldots, t_{N^{(1)}_n}; s, s, \ldots, s>.
\]

On all these basis vectors

\[
H_0|\phi> = \sum_{j=1}^{n} H_0^{(j)}|\phi>
\]

\[
H_0^{(j)}|\phi> = |s, s, \ldots, s >_{N^{(0)}_j} \otimes |t_1^{(1)}; t_2^{(1)}, \ldots, t_{N^{(1)}_1}; s, s, \ldots, s; t_1^{(2)}, t_2^{(2)}, \ldots, t_{N^{(1)}_2}; \ldots, t_{N^{(1)}_j}; s, s, \ldots, s>
\]

where \( H_0^{(j)} \) acts on the string of triplets like a spin-1 chain of length \( N^{(1)}_j \) and f.b.c.. Call \( |\psi >_{N^{(1)}_j} \) any of its eigenvectors and \( E_j(N^{(1)}_j) \) the relevant eigenvalue. Then

\[
|\psi> = |s, s, \ldots, s >_{N^{(0)}_1} \otimes |t_1^{(1)}; s, s, \ldots, s; t_1^{(2)}, t_2^{(2)}, \ldots, t_{N^{(1)}_2}; \ldots, t_{N^{(1)}_n}; s, s, \ldots, s>
\]

is an eigenvector of \( H_0 \) with eigenvalue

\[
E = \sum_{j=1}^{n} E_j(N^{(1)}_j)
\]

Vectors in (14) provide a complete set for the whole ladder to the extent the vectors \( |\psi_j >_{N_j} \) provide a complete set of eigenvectors for the spin-1 chain of length \( N_j \), so diagonalization of the spin 1 chain with p.b.c. and f.b.c. provides a complete solution to the diagonalization problem of the spin ladder (11). In general this is only a partial simplification. Instead, in cases (a), (b) and (c) eigenvalues can in principle be found, for all fragments, by the suitable Bethe-ansatz. What happens here is very similar, actually almost identical, to what happens in the mixed Heisenberg chains studied by Niggemann et al. [5]. There, composite rung spins alternate with single spins and, very much like it happens here, fragmentation takes place when one rung spin is in a singlet state. In the following I will partly adopt notation and methods introduced in their work.

### IV. THE GROUND STATE PROBLEM

To identify the ladder ground state one must minimize (15). An easy first step is to choose, in (15), the lowest eigenvalue for each fragment. To fix the notation, denote with \( E_0^{(p)}(N; 1) \) the
lowest eigenvalue of the $N$-site spin 1 chain with p.b.c. and with $E_0^{(f)}(N'; 1)$ the lowest one for the $N'$-site chain with f.b.c.. Then, in a sector $\{N^{(0)}_j, N^{(1)}_j\}_{j=1}^n$ the lowest eigenvalue is

$$E = \begin{cases} \sum_{j=1}^n \left( E_0^{(f)}(N_j^{(1)}); 1) + (2\mu_1 + 4\mu_2)N_j^{(1)} - 4\mu_2 \right) & N^{(1)} \neq 0, N \\ E_0^{(p)} + (2\mu_1 + 4\mu_2)N & N^{(1)} = N \\ 0 & N^{(1)} = 0 \end{cases}$$

(16)

Following [5], a spin 0 rung is joined to, say, the right of each triplets fragment, bringing its length to $j + 1$. Set $n_j$ to be the number of triplet fragments of length $j + 1$ (i.e. $j$ triplets plus one singlet at the edge). Consider, for the time being, only fragmented configurations, i.e. $N^{(0)} > 0$, specified by $\{n_0, n_1, n_2, \ldots\}$ which must satisfy

$$\sum_{j=0}^{N-1} (j + 1)n_j = N$$

Their energy is 0 for $n_0 = N$, otherwise

$$E(n_0, n_1, \ldots, n_N) = \sum_{j=1}^{N-1} \left( E_0^{(f)}(j; 1) + j(2\mu_1 + 4\mu_2) - 4\mu_2 \right)n_j$$

where $E_0^{(f)}(1; 1) = 0$ by definition. In the limit $N \to \infty$ the densities $w_j = n_j/N$ must be chosen to minimize

$$\lim_{N \to \infty} \frac{E}{N} = \epsilon(w_0, w_1, w_2, \ldots) = \sum_{j=0}^{\infty} \left( E_0^{(f)}(j; 1) + j(2\mu_1 + 4\mu_2) - 4\mu_2 \right)w_j$$

(17)

$$\sum_{j=0}^{\infty} (j + 1)w_j = 1$$

(18)

As observed in [5], the problem (17), (18) has arisen in classical statistical mechanics of systems with competing interactions [20]. Owing to the linearity of (17) in the parameters $\{w_j\}$, the extremum occurs when one $w_j \neq 0$ and all others are zero. Then

$$w_j = \frac{1}{j + 1} \quad w_k = 0 \quad (k \neq j)$$

and the ground state energy per site is

$$\epsilon = \epsilon(j) = \frac{E_0^{(f)}(j; 1) + j(2\mu_1 + 4\mu_2) - 4\mu_2}{j + 1}$$

(19)

The relevant phase is denoted $\langle j \rangle$. One is left with the task of determining the minimum in the sequence of real quantities (19), $j \geq 0$ where it is understood that $\epsilon(0) = 0$. The minimum might be reached at $j \to \infty$, or $\epsilon(\infty) = e_0 + 2\mu_1 + 4\mu_2$ where $e_0$ is the ground state energy per site of the spin chain, independent from boundary conditions. The thermodynamical analysis outlined above does not allow to distinguish between the periodically closed sector $N^{(1)} = N$ and the open
sector where, for instance, one singlet is introduced. In appendix B it is shown in detail that, when 
\[ \inf_{j \geq 0} \epsilon(j) = \lim_{j \to \infty} \epsilon(j) \], then the ground state indeed belongs to the sector \( N^{(1)} = N \).

All said so far is true for any value of \( \theta \) in (11). Yet, a comparison of \( \epsilon(j) \) requires the knowledge of \( E^{(f)}(j; 1) \), in principle for any \( j \). The BA solvable cases (a), (b), (c) are perhaps not the most physically relevant, but they allow an efficient computation of \( E^{(f)}(j; 1) \). In the following I will mostly concentrate on case (c), the purely biquadratic spin 1 chain. The information available for this chain is summarized in Appendix A. The relevant results on finite size ground state energies are gathered in Table I. Some considerations on models with arbitrary \( \theta \) are postponed to section VI.

It can be shown that the \((\mu_1, \mu_2)\) plane is divided into four regions, corresponding to four different ground state energies per site, hence four different phases. Since \( \mu_1 \) and \( \mu_2 \) often appear in the combination \( 2\mu_1 + 4\mu_2 \) define

\[ \mu_3 = 2\mu_1 + 4\mu_2 \]

First, seek the \((\mu_2, \mu_3)\) values for which \( \epsilon(0) \) is lowest, that is \( 0 < \epsilon(j), j \geq 1 \). It is convenient to rewrite this condition adding and subtracting a term containing \( e_0 \) (whose numerical value is reported in (A1))

\[ 4\mu_2 < s_0(j) + j(\epsilon_0 + \mu_3) \quad j \geq 1 \]

\[ s_0(j) \overset{\text{def}}{=} E^{(f)}_0(j; 1) - je_0 \]

The sequence \( s_0(j) \) is bounded, so a first necessary condition is \( \epsilon_0 + \mu_3 > 0 \). Furthermore, from Table 2, \( s_0(j) \) is increasing in the range \( 2 \leq j \leq 51 \), with a minimum

\[ s_0(2) = E^{(f)}_0(2; 1) - 2e_0 = -4 - 2e_0 \approx 1.5937 \]

What can be said for \( j > 51 \)? From Appendix A, \( s_0(j) \) has different limits at \( j \to \infty \) for \( j \) even \( (s^{(e,+)}_0) \) and \( j \) odd \( (s^{(e,-)}_0) \) but they are both larger than \(-4 - 2e_0\). It seems, from Table I, that \( s_0(j) \) at \( j \approx 51 \) is already rather close to the asymptotic regime, so one can safely, although not completely rigorously, conclude that \( s_0(j), j \geq 2 \), is minimal at \( j = 2 \) and that, since \( \epsilon_0 + \mu_3 > 0 \)

\[ \inf_{j \geq 2} \left( s_0(j) + j(\epsilon_0 + \mu_3) \right) = s_0(2) + 2(\epsilon_0 + \mu_3) = -4 + 2\mu_3 \]

Hence, the three necessary and sufficient conditions for being in phase \( < 0 > \) are

\[
\begin{cases}
\epsilon_0 + \mu_3 > 0 \\
2\mu_3 - 4\mu_2 > 4 \\
\mu_3 - 4\mu_2 > 0
\end{cases}
\quad \text{(phase } < 0 > \) (21)
\]

The third condition comes from \( \epsilon(0) < \epsilon(1) \), considered separately.

Phase \( < 1 > \). The conditions for its existence, \( \epsilon(1) < \epsilon(0) \) and \( \epsilon(1) < \epsilon(j), j \geq 2 \) are rewritten, from (11).
\[
\mu_3 - 4\mu_2 < 0 \quad \mu_3 + 4\mu_2 > s_1(j)
\]

\[
s_1(j) \overset{\text{def}}{=} -2E^{(f)}(j;1)/(j-1) \quad (j \geq 2)
\]

where \(s_1(j)\) is listed in Table [II] up to \(j = 51\). It is decreasing in this range and it certainly is at large \(j\). In fact, from the known numerical values (A1), (A8) and (A9), \(2e_0 + 2e^{(s,\pm)} < 0\); moreover

\[
s_1(j) = \frac{-2e_0j - 2e^{(s,\pm)}}{j-1} + o(1/j) = -2e_0 - \frac{2e_0 + 2e^{(s,\pm)}}{j} + o(1/j) \quad (j \to \infty)
\]

Hence one can conclude that \(\sup_{j \geq 2} s_1(j) = s_1(2) = 8\) and phase \(<1>\) appears when

\[
\mu_3 - 4\mu_2 < 0 \quad \mu_3 + 4\mu_2 > 8 \quad \text{(phase} <1>)
\]

Phase \(<2>\). The conditions for its existence are \(\epsilon(2) < \epsilon(0), \epsilon(2) < \epsilon(1)\) and \(\epsilon(2) < \epsilon(j), j \geq 3,\) or, remembering that \(E^{(f)}_0(2;1) = -4\)

\[
\mu_3 - 2\mu_2 < 2 \quad \mu_3 + 4\mu_2 < 8 \quad \mu_3 + 4\mu_2 > s_2(j) \quad (j \geq 3)
\]

\[
s_2(j) \overset{\text{def}}{=} \frac{3E^{(f)}_0(j;1) + 4(j+1)}{-j+2}/(-j+2) \quad (j \geq 3)
\]

The sequence \(s_2(j)\) is listed in Table [II] and it is increasing up to \(j = 51\). It certainly is for large \(j\) where

\[
s_2(j) = \frac{3e_0j + 3e^{(s,\pm)} + 4(j+1)}{-j+2} + o(1/j) = -3e_0 - 4 + \frac{-6e_0 - 3e^{(s,\pm)} - 12}{j} + o(1/j)
\]

because \(-6e_0 - 3e^{(s,\pm)} - 12 < 0\). This time, it is not so straightforward to guess the upper limit of \(s_2(j)\). If \(s_2(j)\) keeps increasing for \(j > 51\) up to the asymptotic regime where one can rely on the previous equation, then \(\sup_{j \geq 3} s_2(j) = \lim_{j \to +\infty} s_2(j) = -3e_0 - 4 \simeq 4.3906\) and phase \(<2>\) appears for

\[
\mu_3 - 2\mu_2 < 2 \quad -3e_0 - 4 < \mu_3 + 4\mu_2 < 8 \quad \text{(phase} <2>)
\]

On the other hand, one cannot rule out the possibility that \(\sup_{j \geq 3} s_2(j)\) lie slightly above and this would leave a tiny window for an additional phase. However, the existence of phase \(<2>\) is unquestionable.

Finally, phase \(<\infty>\), whose ground state is that of the periodic spin 1 chain, shows up when, for each \(j, \epsilon(j) \overset{\text{def}}{=} \lim_{j\to\infty} \epsilon(j).\) Since \(\epsilon(\infty) = e_0 + \mu_3,\) this means

\[
e_0 + \mu_3 < 0 \quad e_0 + \mu_3 < (\mu_3 - 4\mu_2)/2
\]

\[
e_0 + \mu_3 < s_0(j) - 4\mu_2 \quad (j \geq 2)
\]

We already know that \(s_0(j)\) is minimal at \(j = 2,\) so the third condition amounts to \(e_0 + \mu_3 + 4\mu_2 < s_0(2) = -4 - 2e_0\) which, by itself, implies the second inequality. Hence
are the necessary and sufficient conditions that guarantee the appearance of phase \(< \infty \)\. Appendix B completes the proof: the ground state is that of the periodic spin 1 chain because, under conditions \((24)\), local variations around the sector \(N^{(1)} = N\) only lead to an increase in energy.

Under the foregoing assumption on the upper limit of sequence \(s_2(j)\) there is no room for other phases because inequalities \((21)-(24)\), with their separating lines, fill the whole \((\mu_2, \mu_3)\) plane. The ground state energy per site is \(\epsilon(0) = 0\ (\text{phase } < 0)\), \(\epsilon(1) = (\mu_3 - 4\mu_2)/2\ (\text{phase } < 1)\), \(\epsilon(2) = (-4 + 2\mu_3 - 4\mu_2)/3\ (\text{phase } < 2)\) and \(\epsilon(\infty) = \epsilon_0 + \mu_3\ (\text{phase } < \infty)\). It shows first derivative discontinuities at the boundaries.

Here are some general features of the four phases. Define the total spin

\[
S_{\text{tot}} = \sum_{k=1}^{N} S_k \quad \quad S_{\text{tot}}^2 = S_{\text{tot}}(S_{\text{tot}} + 1)
\]

In phase \(< \infty \) the ground state is simply that of the periodic biquadratic spin 1 chain. Supposing \(N\) even, it is a global singlet and, it was conjectured in [15], its degeneracy should be 2. Phase \(< 0 \) has certainly a unique ground state and of course \(S_{\text{tot}} = 0\). As to phase \(< 2 \), the ground state is, again, a tensor product of local singlets, hence a global singlet, \(S_{\text{tot}} = 0\). In fact, it is represented by the sequence \(|\ldots,s,t,t,s,t,t,s,t,t,s,s\ldots>\) where neighboring triplets \(|t_k,t_{k+1}>\) are locked into the singlet ground state of \((-S_k \cdot S_{k+1})^2\). Supposing \(N = 0 \pmod{3}\) to avoid problems with AFM seams, the ground state is threefold degenerate: rung singlets sit on rungs labelled \(k \pmod{3}\), with the freedom of choice \(k = 0,1,2\). It clearly spontaneously breaks translational invariance. Finally, phase \(< 1 \), described by the sequence \(|\ldots,t,s,t,s,t,s,s\ldots>\) is rather different because nothing fixes the state of each isolated triplet resulting in a huge degeneracy, \(2 \cdot 3^{N/2}\). Not only is translational invariance spontaneously broken, but one is dealing with \(N/2\) effectively noninteracting triplets that can combine in a “ferrimagnetic” state \(S_{\text{tot}} = N/2\), or combine in local pairs yielding \(S_{\text{tot}} = 0\) (these are only two of the many possibilities). In other words, the ground state is not necessarily an eigenstate of \(S_{\text{tot}}\).

\[ e_0 + \mu_3 < 0 \quad \mu_3 + 4\mu_2 < -4 - 3e_0 \quad (\text{phase } < \infty) \quad (24) \]

\[ \mu_3 + 4\mu_2 < -4 - 3e_0 \quad (\text{phase } < \infty) \]

\[ \mu_3 + 4\mu_2 < -4 - 3e_0 \quad (\text{phase } < \infty) \]

V. EXCITATIONS

The four phases display a rich variety of excitations. Most of them, especially in the fragmented phases \(< 0 \), \(< 1 \) and \(< 2 \) can easily be determined resorting to \((17)\), therefore only phase \(< \infty \) will be extensively treated in the following. Appendix B gives a complete discussion of how, in phase \(< \infty \), “local” perturbations of the biquadratic ground state can only lead to an increase of energy.

A first kind of excitations, within the sector \(N^{(1)} = N\), are those of the biquadratic chain
itself. Supposing \( N \) even, they exist in even number and have a gap (see Appendix A for a more exhaustive discussion)

\[
\Delta E_{\text{gap}} \simeq 0.173179
\]

A second kind of excitation, called TSST (triplet-to-singlet spin flip) in [9], is obtained by introducing one singlet, sector \( N^{(1)} = N - 1 \). There is \( N \) ways to do so and the eigenvalue is \( N \)-fold degenerate. This implies that linear superpositions of such states produce eigenstates of the shift operator, all with the same energy. As a matter of fact, dispersionless excitations have already been found in akin ladders [8]. The energy difference is, see (16)

\[
\Delta E(\alpha, n) = E^{(f)}_{0}(N - 1; 1) - E^{(p)}_{0}(N; 1) - \mu_3 - 4\mu_2
\]

From conditions (24) it follows that \( \Delta E \approx 2e_0 + 4 + e^{(s, \pi)} \simeq 0.1479 \) confirming that the energy can only be increased by fragmentation. At the same time, that numerical value shows that, at least near the boundary lines, the energy scale is comparable, actually slightly smaller, than \( \Delta E_{\text{gap}} \). Therefore fragmentation may be relevant to the low energy physics of the ladder. Despite what might seem, such excitations depend on two degrees of freedom. In fact, as discussed in Appendix A, the ground state of the fragment of triplets, i.e. the ground state of the open biquadratic chain with \( N - 1 \) (odd) sites, is only the lowest in a continuous band. This adds a second degree of freedom to the singlet’s position. Therefore, if \( n \) is the singlet position

\[
\Delta E(\alpha, n) = e^{(\alpha)} - e^{(\pi)} + e^{(s, \pi)} - e_0 - \mu_3 - 4\mu_2 \quad \alpha \in (0, \pi)
\]

Here \( e^{(\alpha)} - e^{(\pi)} \), always positive, measures the excitation energy of the single kink in the open spin chain and vanishes at \( \alpha \to \pi \), see (A6). The \( n \)-dependence is, of course, trivial. Nonetheless, it is at least plausible that under a small variation of coupling constants in (4), for instance a different relative weight between \( I^{(1)} \) and \( I^{(3)} \), or \( I^{(4)} \) and \( I^{(6)} \) interactions, \( \Delta E \) might acquire a genuine two-parameter form. Since the triplets fragment is presumably locked into a spin 1 state (see Appendix A), \( S_{\text{tot}} = 1 \) for such states. Triplet-singlet excitations have been found in similar ladders, but they arise through a different mechanism [7,8].

When two rung singlets are introduced \( (N^{(1)} = N - 2) \), their energy becomes position dependent. If they sit on rungs \( n_1, n_2 \) and \( d = n_2 - n_1 \geq 1 \), then from (17)

\[
\Delta E(N; n_1, n_2) = E^{(f)}_{0}(d - 1; 1) + E^{(f)}_{0}(N - d - 1; 1) - E^{(p)}_{0}(N; 1) - 2\mu_3 - 8\mu_2 + 4\mu_2\delta_{d,1}
\]

If \( N \to \infty \), keeping \( n_1, n_2 \) fixed \( (p(d) \) is the parity of \( d \))

\[
\Delta E(d) = \begin{cases} 
  s_0(d - 1) - 2e_0 + e^{(s, \pi(d-1))} - 2\mu_3 - 8\mu_2 & (d > 2) \\
  -3e_0 + e^{(s, \pi)} - 2\mu_3 - 8\mu_2 & (d = 2) \\
  -2e_0 + e^{(s, \pi)} - 2\mu_3 - 4\mu_2 & (d = 1)
\end{cases}
\]

(25)
Notice that all these excitation energies are exact. It is easy to see that \( \Delta E(d) > 0 \) always. It represents a sort of “static interaction energy” between singlets, whose distance dependence is governed by \( s_0(d) \). The open chain surface energy is different for even and odd lengths, so \( s_0(d) \) has a uniform \( d \)-dependence only if the parity of \( d \) is fixed. When \( d \) is even, \( \Delta E(d) \) uniformly decreases (see Table I) from a maximum \( \Delta E(2) \) to a minimum \( \lim_{d \to \infty} \Delta E(d) = -2\epsilon_0 + 2\epsilon^{(s,-)} - 2\mu_3 - 8\mu_2 \). When \( d \) is odd, the behavior is certainly uniform for \( d \geq 3 \), because \( \Delta E(d) \) increases from \( \Delta E(3) \) to a maximum \( \lim_{d \to \infty} \Delta E(d) = 2\epsilon^{(s,+)} - 2\epsilon_0 - 2\mu_3 - 8\mu_2 \). Still, \( \Delta E(1) \) is not necessarily lower or higher than \( \Delta E(3) \), since

\[
\Delta E(3) - \Delta E(1) = -4 - 2\epsilon_0 - 4\mu_2
\]

whose sign is left undetermined from conditions (24). The short range behavior can be either repulsive or attractive.

A third possible kind of excitations, named \( n \)-TSSF in [4], is produced by the insertion of a block of \( m \) singlets, represented by \( | \ldots, t, t, t, s, s, s, s, t, t, t, \ldots > \). Now, from (16), supposing \( N \) even as always

\[
\Delta E(N; m) = E_0^{(f)}(N - m; 1) - m\mu_3 - 4\mu_2 - E_0^{(p)}(N; 1)
\]

\[
\Delta E(m) = \lim_{N \to \infty} \Delta E(N; m) = -(m - 1)(\epsilon_0 + \mu_3) + \epsilon^{(s,p(m))} - \epsilon_0 - \mu_3 - 4\mu_2
\]

which is positive under conditions (24). At \( \epsilon_0 + \mu_3 = 0 \) these perturbations determine the instability versus phase \( < 0 > \).

VI. OTHER VALUES OF \( \theta \)

It is easy to see that phases \( < 0 > \), \( < 1 > \) and \( < \infty > \) will be present no matter what value \( \theta \) takes in (11). In this general situation, (16) will formally be the same but now with \( E_0^{(f)}(N_j^{(1)}; 1; \theta) \) and \( E_0^{(p)}(N; 1; \theta) \), the energies of the spin 1 chain (3) with arbitrary \( \theta \). The proof of section 4 goes through unchanged. One has to find the minimum of

\[
\epsilon(j; \theta) = \frac{E_0^{(f)}(j; 1; \theta) + j\mu_3 - 4\mu_2}{j + 1} \qquad \epsilon(0; \theta) = 0
\]

The conditions for phase \( < 0 > \) amount to

\[
4\mu_2 < s_0(j) + j(\epsilon_0(\theta) + \mu_3) \quad (j \geq 1) \quad s_0(j) \overset{\text{def}}{=} E_0^{(f)}(j; 1; \theta) - \epsilon_0(\theta)j
\]

Notice that \( s_0(j) \) is bounded. Set

\[
s_0(j_1) = \inf_{j \geq 1} s_0(j)
\]
Of course $j_1$ might be $+\infty$. In this case, since $\lim_{j \to \infty} s_0(j) = e^{e^{s_1}}$, the two possible surface energies, one has $s_0(j_1) = \min\{e^{s_1}, e^{s_2}\}$. At any rate, $s_0(j_1)$ is finite. A necessary condition required by (28) is $e_0(\theta) + \mu_3 > 0$, but since one does not know the exact value $j_1$ one can only give sufficient conditions for the existence of $<0>$, namely

$$e_0(\theta) + \mu_3 > 0 \quad \text{(phase } < 0 \text{)}$$

$$4\mu_2 < \inf_{j \geq 1} s_0(j) + \inf_{j \geq 1} j(\epsilon(\theta_0) + \mu_3) = s_0(j_1) + e_0(\theta) + \mu_3$$

which can always be simultaneously satisfied by suitable values of $(\mu_2, \mu_3)$.

Likewise, for phase $< \infty>$, the condition $\epsilon(j) > e_0(\theta) + \mu_3$, $\epsilon(j) > 0$, translates into $e_0(\theta) + \mu_3 < 0$ and $e_0(\theta) + \mu_3 + 4\mu_2 < s_0(j)$, $(j \geq 1)$, which can both be fulfilled taking

$$e_0(\theta) + \mu_3 < 0 \quad e_0(\theta) + \mu_3 + 4\mu_2 < s_0(j_1) \quad \text{(phase } < \infty \text{)}$$

These conditions are necessary and sufficient, and are compatible.

Finally, the inequalities for $< 1>$ are rewritten

$$\mu_3 - 4\mu_2 < 0 \quad \mu_3 + 4\mu_2 > -\frac{2E(\theta_0)(j; 1; \theta)}{j - 1} \equiv s_1(j) \quad j \geq 2$$

Sequence $s_1(j)$ is also bounded. Set $s_1(j_2) = \sup_{j \geq 2} s_1(j)$. The ensuing necessary and sufficient conditions are

$$\mu_3 - 4\mu_2 < 0 \quad \mu_3 + 4\mu_2 > s_1(j_2) \quad \text{(phase } < 1 \text{)}$$

On the contrary, it is in general impossible to draw conclusions on the existence of other phases. For example, phase $< 2>$ would be present if it were possible to fulfill simultaneously

$$4\mu_2 - 2\mu_3 > E_0^{(f)}(2; 1; \theta) \quad \mu_3 + 4\mu_2 < -2E_0^{(f)}(2; 1; \theta) \quad \mu_3 + 4\mu_2 > s_2(j) \quad (27)$$

$$s_2(j) \equiv \left( (j + 1)E_0^{(f)}(2; 1; \theta) - 3E_0^{(f)}(j; 1; \theta) \right) / (j - 2) \quad (j \geq 3)$$

Notice that $s_2(j)$ is also bounded. If we set $s_2(j_3) = \sup_{j \geq 3} s_2(j)$, the second and the third inequality in (27) are compatible if and only if

$$s_2(j_2) < -2E_0^{(f)}(2; 1; \theta) \quad (28)$$

At least for $\theta = 0$, $e_0(\theta)$ has been determined to a great accuracy and one can attempt to estimate whether (28) holds. It is known that (22)

$$e_0(\theta = 0) \simeq -1.401484$$

On the other hand, not many data seem to have been published for $E_0^{(f)}(j; 1; 0)$ even at small $j$. Diagonalization of (3), with free boundaries, can easily be carried out numerically up to 7 sites (as
a test, I diagonalized the periodic chain as well and compared the results with those published in [23]. It is sufficient to examine the sector $S_{\text{tot}}^z = 0$ because, owing to $SU(2)$-invariance, all possible eigenvalues are bound to appear there. Now, $E_0^{(f)}(2; 1; 0) = -2$, while the sequence $\{ s_2(j) \}_{j=3}^7$ is found to have the approximate values $\{ 1, 1.9606, 1.8302, 2.0277, 1.9807 \}$. Furthermore

$$\lim_{j \to \infty} s_2(j) = -2 - 3e_0 \simeq 2.20446$$

It is of course impossible to draw a rigorous conclusion from these results, but it looks extremely likely that inequality [28] is true and therefore phase $< 2 >$ is present also for $\theta = 0$.

**VII. DISCUSSION**

It is useful to compare the ladders considered here with those studied in previous papers. To shorten notation, I will denote by $| \psi >_{<x>}$ the ground state of phase $< x >$, or one of them if it is degenerate. In the work of Niggeman et al. similar methods were employed but for different systems, namely the mixed Heisenberg chains. Perhaps the most closely related ladder is that of Xian [4], which is (11) for $\theta = 0$ and $\mu_2 = 0$. Perturbations of this ladder where considered in [8, 9]. Xian finds only two phases, $< 0 >$ and $< \infty >$ in the present notation, but, since $\theta = 0$, his ground state $| \psi >_{\infty}$ is that of the spin 1 Heisenberg chain. The absence of phases $< 1 >$ and $< 2 >$ can be understood when one notices that even for the case considered in this paper, $\theta = -\pi/2$, phases $< 1 >$ and $< 2 >$ would be missing without the rung-rung interaction $I_{k,k+1}^{(5)}$, as can be seen by setting $\mu_2 = 0$ in (22) and (23).

The ground state of a wide class of $SU(2)$-invariant ladders has the matrix-product (MP) form [6]. Such states can be translationally invariant or dimerized

$$| \psi_0^{(\text{inv})}(u) > = \text{Tr} \prod_{n=1}^N g_n(u) \quad | \psi_0^{(\text{dim})}(u_1, u_2) > = \text{Tr} \prod_{n=1}^{N/2} g_{2n-1}(u_1)g_{2n}(u_2)$$

$$g_n(u) = \begin{bmatrix} u|s >_n + |0 >_n \quad -\sqrt{2}|1 >_n \\ \sqrt{2}|1 >_n - |0 >_n \end{bmatrix}$$

For models with translationally invariant MP ground states, $u = 0$ and $u = \infty$ are the only points which have a significant overlapping with the ladders presented here. At $u = 0$ the MP state is made up of rung triplets and is, effectively, the ground state of the AKLT spin 1 chain. That is what one gets for (11) by setting $\tan \theta = 1/3$ and keeping $\mu_1, \mu_2$ small enough. Now, $| \psi >_{\infty}$ is also made up of triplets, but it is that of the biquadratic spin 1 chain, hence a different kind. Furthermore, $| \psi >_{<0>}, | \psi >_{<1>}$ and $| \psi >_{<2>}$ all contain rung singlets and are certainly not $| \psi_0^{(\text{inv})}(0) >$. Conversely, after suitable normalization, the limit $| \psi_0^{(\text{inv})}(\infty) >$ is made up entirely of rung singlets. There is only one such state so it must coincide with $| \psi >_{<0>}$. Of course excitations may differ according to the detailed form of the hamiltonian. At any rate, $| \psi >_{<1>}$ and $| \psi >_{<2>}$
break translational invariance so they can never be of the form $|\psi_0^{(\text{inv})}(u)\rangle$ for any $u$. For the same reason, $|0\rangle_{<2>}$, which is invariant under a three-rung translation, can never be of the form $|\psi_0^{(\text{dim})}(u_1,u_2)\rangle$ either, since the latter is invariant under a two-rung translation. But, within the large lowest energy eigenspace of phase $<1>$, at least one vector is in MP form. To show it, notice that at finite, non-vanishing values of $u_1$, $u_2$, $|\psi_0^{(\text{dim})}(u_1,u_2)\rangle$ is a linear combination necessarily containing kets with all triplets and the ket with all singlets. The only way to reproduce $|0\rangle_{<1>}$ is to set $u_1 = 0$ and to take $u_2 \to \infty$ after proper normalization of $g_{2n}(u_2)$. The outcome is a linear combination of kets which have singlets on even rungs and triplets on odd rungs, exactly like $|0\rangle_{<1>}$. It should be observed though the the MP approach only in few cases allow to find the whole spectrum, as it is instead possible from BA for hamiltonian $[1]$. 

ACKNOWLEDGEMENTS

I am grateful to M.Raciti and F.Riva for useful discussions.

APPENDIX A: THE BQUADRATIC SPIN 1 CHAIN

The purely biquadratic spin 1 chain is integrable for periodic [15] and free [16,18] boundary conditions. The definition of the relevant elliptic parameters differs in the two cases, as treated in the literature. For the sake of readability, a Landen transformation [21] has been applied to the results of [15], in order to make notations compatible.

Define an elliptic modulus from the relation

$$
\frac{K'(k)}{K(k)} = \frac{1}{\pi} \ln(1/q) \quad q = \frac{3 - \sqrt{5}}{2}
$$

where $K(k)$ and $K'(k)$ are the elliptic integrals of first and second kind. The ground state energy, which is the same in the two cases, is then

$$
e_0 = -1 - \frac{\sqrt{5}}{2} \left( 1 + 4 \sum_{n=1}^{+\infty} \frac{q^{2n}}{1 + q^{2n}} \right) \approx -2.796863... \quad (A1)
$$

For the periodic case, energy and momentum of the excitations are given, in the limit $N \to \infty$, by

$$
\Delta E = \sum_{i=1}^{2\nu} \tilde{\epsilon}(p_i) \quad \Delta P = \sum_{i=1}^{2\nu} p_i
$$

$$
\tilde{\epsilon}(p) = \frac{\sqrt{5}}{\pi} K(k) \sqrt{k'^2 + k^2 \sin p} \quad k'^2 + k^2 = 1 \quad (A2)
$$

There is a gap in the spectrum

$$
\Delta E_{\text{gap}} = 2\tilde{\epsilon}(0) = 2 \frac{\sqrt{5}}{\pi} K(k) k' = \sqrt{5} \prod_{n=1}^{+\infty} \left( \frac{1 - q^n}{1 + q^n} \right)^2 \approx 0.173178... \quad (A3)
$$

16
The free boundary case has been solved by noticing that the spectrum is the same, up to degeneracies, as that of the $XXZ$ spin $1/2$ chain with a boundary field \[ H_{XXZ} = -\frac{1}{2} \sum_{k=1}^{N-1} (\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y - \cosh \gamma \sigma_k^z \sigma_{k+1}^z) + \frac{\sinh \gamma}{2} (\sigma_1^z - \sigma_N^z) \] (A4)

\[ E_{bQ} = E_{XXZ} - \frac{7}{4} (N - 1) \quad q = e^{-\gamma} \]

The mapping, valid for any $N$, works because both hamiltonians can be written as sums over generators of the same Temperley-Lieb algebra \[16\]. Consequently, ground state energies of (A4) are for any $N$ identical, up to a shift, to ground state energies of the for biquadratic chain with free boundaries. In turn, (A4) has been solved by coordinate Bethe-ansatz \[17\], so eigenvalues are found from solutions of a set of coupled transcendental equations

\[ \frac{1}{\pi} \Theta(\alpha_j; \frac{\gamma}{2}) - \frac{1}{2\pi N} \sum_{k=1, k \neq j}^{n} \left( \Theta(\alpha_j - \alpha_k; \gamma) + \Theta(\alpha_j + \alpha_k; \gamma) \right) = \frac{I_j}{N} \quad j = 1 \ldots n \]

\[ \Theta(\alpha; x) \overset{df}{=} -i \ln \left[ \frac{\sinh(x + ix)}{\sinh(x - ix)} \right] = 2 \arctan(\tan \frac{\alpha}{2} \coth x) \] (A5)

\[ E = \frac{1}{2} (N - 1) \cosh \gamma - 2 \sinh \gamma \sum_{j=1}^{n} \Theta'(\alpha_j; \frac{\gamma}{2}) \]

The function $\Theta(\alpha; x)$ is defined to be continuous for real $\alpha$. The $\{I_j\}$ in (A5) are positive integers \[17,18\]. The ground state belongs to the sector $n = N/2$ ($N$ even) or $n = (N - 1)/2$ ($N$ odd). In both cases, in (A5), $I_j = j$, $j = 1, 2 \ldots n$. Data in Table I have been obtained by numerically solving (A5) with this choice of integers.

Excitations are, in general, gapful (no dispersion relation can be written here because linear momentum is not conserved).

\[ \Delta E = \sum_{i=1}^{2\nu} \epsilon(\alpha_i) \quad \epsilon(\alpha) = 2 \sinh \gamma \frac{K(k)}{\pi} \text{dn}(\frac{K(k)\alpha}{\pi}; k) \quad \alpha \in (0, \pi) \]

\[ \Delta E_{\text{gap}} = 2\epsilon(\pi) = \sqrt{5} \prod_{n=1}^{\infty} \left( 1 - q^n \right)^2 \]

as in (A3). When $N$ is odd, though, the ground state is at the bottom of a one-parameneter, continuous, gapless (in the $N \to \infty$ limit) band of eigenvalues \[19\]. Within this band

\[ \Delta E(\alpha) = \epsilon(\alpha) - \epsilon(\pi) \quad \alpha \in (0, \pi) \] (A6)

This may be interpreted by saying that the ground state at $N$ odd is not the “vacuum”, but it contains one particle (kink) which can take up a whole band of dynamical states.

Surface energies for open antiferromagnetic chains differ, in general, when the limit $N \to \infty$ is taken for $N$ even ($e^{(s,+)}$) or $N$ odd ($e^{(s,-)}$). They are defined by
and, for the case at hand, they can be computed exactly from (A3). For the biquadratic chain, one finds (A8, A9)

\[
e^{(s,+)} = 1 + 2\sqrt{5} \sum_{n=1}^{+\infty} \frac{q^{2n} - q^{4n}}{1 + q^{4n}} \approx 1.6550092
\]

\[
e^{(s,-)} = 1 + \frac{\sqrt{5}}{2} \left( 1 + 4 \sum_{n=1}^{+\infty} \frac{q^{2n} - q^{4n}}{1 + q^{4n}} - \sum_{n=0}^{+\infty} \frac{4q^{2n+1}}{1 + q^{4n+2}} \right) \approx 1.7415986
\]

**APPENDIX B: GROUND STATE PROOF FOR PHASE \(< \infty >\)**

It has to be proven that, within the boundaries of region (24), the ladder ground state indeed belongs to the sector \(N^{(1)} = N\), therefore it is just the ground state of (3) at \(\theta = -\pi/2\). Here it will be shown that it is the lowest energy state within the class which encompasses the sectors \(\{N^{(1)}_j\}^n_{j=1}\) where \(\{N^{(1)}_j\}^n_{j=1}\) and \(\{N^{(1)}_j\}^n_{j=1}\) are arbitrarily large but kept finite as \(N \to \infty\). In other words, only one triplet fragment, \(N^{(1)}\), has a diverging size in the thermodynamic limit, as dictated by the analysis of section 4. It will be assumed that \(\inf_{j \geq 2} s_0(j) = s_0(2)\).

From (10)

\[
\Delta E(N) = \sum_{j=1}^{n-1} E^f(0)(N_j^{(1)}; 1) + E^f(0)(N^{(1)}_n; 1) + \mu_3 N^{(1)} - 4n\mu_2 - (E^p(0)(N; 1) + \mu_3 N^{(1)})
\]

When \(N \to \infty\), setting \(\Delta E = \lim_{N \to \infty} \Delta E(N)\)

\[
\Delta E \geq \sum_{j=1}^{n-1} \left( E^f(0)(N^{(1)}_j; 1) - e_0 N_j^{(1)} \right) + e^{(s,p(N^{(1)}_n))} - \mu_3 N^{(0)} - 4n\mu_2 - e_0 N^{(1)}
\]

Since \(N^{(1)}_n = N - N^{(0)} - \sum_{j=1}^{n-1} N^{(1)}_j\) and \(e^{(s)} \geq e^{(s,+)}\)

\[
\Delta E \geq \sum_{j=1}^{n-1} \left( E^f(0)(N^{(1)}_j; 1) - e_0 N_j^{(1)} \right) - (e_0 + \mu_3) N^{(0)} + e^{(s,+)} - 4n\mu_2
\]

(B1)

The bracketed term is \(s_0(N^{(1)}_j)\). Actually \(s_0(j)\) has been defined for \(j \geq 2\), whereas in (B1) \(N^{(1)}_j\) can be 1 for some \(j\). The definition of \(s_0(j)\), though, makes sense for \(j = 1\), too. In that case \(s_0(1) = -e_0 > s_0(2)\), so it is also true that \(\inf_{j \geq 1} s_0(j) = s_0(2)\). Then

\[
\Delta E \geq (n - 1)s_0(2) + e^{(s,+)} - (e_0 + \mu_3) N^{(0)} - 4n\mu_2 > ns_0(2) - (e_0 + \mu_3) N^{(0)} - 4n\mu_2
\]

because \(e^{(s,+)} > s_0(2)\). Next, notice that

\[
N^{(0)} \geq n
\]
because to create a new fragment at least one singlet must be added. So, setting $N^{(0)} = n + \Delta N^{(0)}$, 

$$\Delta E > -(e_0 + \mu_3)\Delta N^{(0)} + n(s_0(2) - e_0 - \mu_3 - 4\mu_2)$$

which is positive due to inequalities \[24\].
[18] M.T. Batchelor and C.J. Hamer, J.Phys. A 23, 761 (1990).

[19] G. Albertini, cond-mat/0012439.

[20] M.E. Fisher and W. Selke, Phys. Rev. Lett. 44, 1502 (1980); V.L. Pokrovsky and G.V. Uimin, J.Phys. C 11, 3535 (1978).

[21] R.J. Baxter, Exactly Solved Models in Statistical Mechanics (Academic, New York, 1982).

[22] S.R. White and D.H. Huse, Phys. Rev. B 48, 3844 (1993).

[23] H.W. Blöte, Physica 93 B, 93 (1978); O. Golinelli, Th. Jolicœur and R. Lacaze, Phys. Rev. B 50, 3037 (1994).
| $N$ | $E_0^{(1)}(N;1)$ |
|-----|-----------------|
| 1   | 0               |
| 2   | -4              |
| 3   | -6              |
| 4   | -9.56155281     |
| 5   | -11.76372382    |
| 6   | -15.14325669    |
| 7   | -17.45402100    |
| 8   | -20.73101075    |
| 9   | -23.11096357    |
| 10  | -26.32134617    |
| 11  | -28.74967412    |
| 12  | -31.91290040    |
| 13  | -34.37723072    |
| 14  | -37.50520573    |
| 15  | -39.99742180    |
| 16  | -43.09794702    |
| 17  | -45.61247178    |
| 18  | -48.69096768    |
| 19  | -51.22377887    |
| 20  | -54.28417520    |
| 21  | -56.83226912    |
| 22  | -59.87751203    |
| 23  | -62.43858209    |
| 24  | -65.47094086    |
| 25  | -68.04317437    |
| 26  | -71.06443665    |
| 27  | -73.64638147    |
| 28  | -76.65798212    |
| 29  | -79.24845576    |
| 30  | -82.25156507    |
| 31  | -84.84950992    |
| 32  | -87.84517669    |
| 33  | -90.4493829     |
| 34  | -93.43881050    |
| 35  | -96.04961790    |
| 36  | -99.03246169    |
| 37  | -101.64872625   |
| 38  | -104.62612662   |
| 39  | -107.24734191   |
| 40  | -110.21980250   |
| 41  | -112.84552974   |
| 42  | -115.81348717   |
| 43  | -118.44334259   |
| 44  | -121.40717895   |
| 45  | -124.04082624   |
| 46  | -127.00087653   |
| 47  | -129.63801837   |
| 48  | -132.59457884   |
| 49  | -135.23495129   |
| 50  | -138.18828505   |
| 51  | -140.83165264   |
| n | $s_0(n)$  | $s_1(n)$ | $s_2(n)$ |
|---|---|---|---|
| 1 | n.d. | n.d. | n.d. |
| 2 | 1.59372686 | 8 | n.d. |
| 3 | 2.39059029 | 6 | 2 |
| 4 | 2.62500991 | 6.3746854 | 4.3423292 |
| 5 | 2.22059333 | 5.8818619 | 3.7637238 |
| 6 | 1.63792389 | 6.0573026 | 4.3574425 |
| 7 | 2.12402301 | 5.8180070 | 4.0724126 |
| 8 | 1.64389669 | 5.9231459 | 4.3655053 |
| 9 | 2.06080730 | 5.7777408 | 4.1904129 |
| 10 | 1.64731963 | 5.8491810 | 4.3704930 |
| 11 | 2.01582361 | 5.7499348 | 4.2498137 |
| 12 | 1.6494076 | 5.8023455 | 4.3738701 |
| 13 | 1.98199387 | 5.7295384 | 4.2846992 |
| 14 | 1.65082229 | 5.7700361 | 4.3763014 |
| 15 | 1.95532965 | 5.7139174 | 4.3070973 |
| 16 | 1.65186786 | 5.7469294 | 4.3781315 |
| 17 | 1.93420653 | 5.7015589 | 4.3224943 |
| 18 | 1.65257406 | 5.7283491 | 4.3795644 |
| 19 | 1.91662630 | 5.6915309 | 4.3368084 |
| 20 | 1.65309340 | 5.7141237 | 4.3806958 |
| 21 | 1.90186291 | 5.6832269 | 4.3419372 |
| 22 | 1.65348343 | 5.7026201 | 4.3816268 |
| 23 | 1.88927680 | 5.6762347 | 4.3483687 |
| 24 | 1.65378146 | 5.6931259 | 4.3824010 |
| 25 | 1.87841138 | 5.6702645 | 4.3534575 |
| 26 | 1.65401253 | 5.6851549 | 4.3830545 |
| 27 | 1.86893114 | 5.6651062 | 4.3575657 |
| 28 | 1.65419392 | 5.6783905 | 4.3836132 |
| 29 | 1.86058371 | 5.6600398 | 4.3609395 |
| 30 | 1.65433783 | 5.6725217 | 4.3840962 |
| 31 | 1.85317541 | 5.6563939 | 4.3637508 |
| 32 | 1.65445307 | 5.6674305 | 4.3845176 |
| 33 | 1.84655490 | 5.6531211 | 4.3661236 |
| 34 | 1.65454612 | 5.6629582 | 4.3848848 |
| 35 | 1.84060215 | 5.6499775 | 4.3681470 |
| 36 | 1.65462179 | 5.6589978 | 4.3852172 |
| 37 | 1.83522066 | 5.6471514 | 4.3698908 |
| 38 | 1.65468372 | 5.6554663 | 4.3851055 |
| 39 | 1.83033186 | 5.6445969 | 4.3714061 |
| 40 | 1.65473470 | 5.6522975 | 4.3857738 |
| 41 | 1.82587089 | 5.6422764 | 4.3723306 |
| 42 | 1.65477689 | 5.6494384 | 4.3860115 |
| 43 | 1.82178490 | 5.6401591 | 4.3739031 |
| 44 | 1.65481197 | 5.6468455 | 4.3862270 |
| 45 | 1.81802811 | 5.6382193 | 4.3749137 |
| 46 | 1.65484125 | 5.6444834 | 4.3862340 |
| 47 | 1.81456284 | 5.6364355 | 4.3758678 |
| 48 | 1.65486580 | 5.6423225 | 4.3860297 |
| 49 | 1.81135678 | 5.6347896 | 4.3766902 |
| 50 | 1.65488645 | 5.6403817 | 4.3867678 |
| 51 | 1.80838229 | 5.6332661 | 4.3774481 |