Abstract

In min-min optimization or max-min optimization, one has to compute the gradient of a function defined as a minimum. In most cases, the minimum has no closed-form, and an approximation is obtained via an iterative algorithm. There are two usual ways of estimating the gradient of the function: using either an analytic formula obtained by assuming exactness of the approximation, or automatic differentiation through the algorithm. In this paper, we study the asymptotic error made by these estimators as a function of the optimization error. We find that the error of the automatic estimator is close to the square of the error of the analytic estimator, reflecting a super-efficiency phenomenon. The convergence of the automatic estimator greatly depends on the convergence of the Jacobian of the algorithm. We analyze it for gradient descent and stochastic gradient descent and derive convergence rates for the estimators in these cases. Our analysis is backed by numerical experiments on toy problems and on Wasserstein barycenter computation. Finally, we discuss the computational complexity of these estimators and give practical guidelines to chose between them.
1 Introduction

In machine learning, many objective functions are expressed as the minimum of another function: functions $\ell$ defined as

$$\ell(x) = \min_{z \in \mathbb{R}^m} L(z, x),$$

where $L : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$. Such formulation arises for instance in dictionary learning, where $x$ is a dictionary, $z$ a sparse code, and $L$ is the Lasso cost [Mairal et al., 2010]. In this case, $\ell$ measures the ability of the dictionary $x$ to encode the input data. Another example is the computation of the Wasserstein barycenter of distributions in optimal transport [Agueh and Carlier, 2011]: $x$ represents the barycenter, $\ell$ is the sum of distances to $x$, and the distances themselves are defined by minimizing the transport cost. In the field of optimization, formulation (1) is also encountered as a smoothing technique, for instance in reweighted least-squares [Daubechies et al., 2010] where $L$ is smooth but not $\ell$. In game theory, such problems naturally appear in two-players maximin games [von Neumann, 1928], with applications for instance to generative adversarial nets [Goodfellow et al., 2014]. In this setting, $\ell$ should be maximized.

A key point to optimize $\ell$ – either maximize or minimize – is usually to compute the gradient of $\ell$, $g^*(x) \triangleq \nabla_x \ell(x)$. If the minimizer $z^*(x) = \text{arg min}_z L(z, x)$ is available, the first order optimality conditions impose that $\nabla_z L(z^*(x), x) = 0$ and the gradient is given by

$$g^*(x) = \nabla_x L(z^*(x), x).$$

However, in most cases the minimizer $z^*(x)$ of the function is not available in closed-form. It is approximated via an iterative algorithm, which produces a sequence of iterates $z_t(x)$. There are then three ways to estimate $g^*(x)$:

The **analytic** estimator corresponds to plugging the approximation $z_t(x)$ in (2)

$$g^1_t(x) = \nabla_x L(z_t(x), x).$$

The **automatic** estimator is $g^2_t(x) \triangleq \partial_x [L(z_t(x), x)]$, where the derivative is computed with respect to $z_t(x)$ as well. The chain rule gives

$$g^2_t(x) = \nabla_x L(z_t(x), x) + \frac{\partial z_t}{\partial x} \nabla_z L(z_t(x), x).$$

This expression can be computed efficiently using automatic differentiation [Baydin et al., 2018], in most cases at a cost similar to that of computing $z_t(x)$.

If $\nabla_{zz} L(z^*(x), x)$ is invertible, the implicit function theorem gives $\frac{\partial z^*(x)}{\partial x} = -\nabla_{z} L(z^*(x), x) [\nabla_{zz} L(z^*(x), x)]^{-1}$. The **implicit** estimator is

$$g^3_t(x) \triangleq \nabla_x L(z_t(x), x) + \mathcal{J}(z_t(x), x) \nabla_z L(z_t(x), x).$$


This estimator can be more costly to compute than the previous ones, as a $m \times m$ linear system has to be solved.

These estimates have been proposed and used by different communities. The analytic one corresponds to alternate optimization of $L$, where one updates $x$ while considering that $z$ is fixed. It is used for instance in dictionary learning [Olshausen and Field 1997, Mairal et al. 2010] or in optimal transport [Feydy et al., 2019]. The second is common in the deep learning community as a way to differentiate through optimization problems [Gregor and Le Cun, 2010]. Recently, it has been used as a way to accelerate convolutional dictionary learning [Tolooshams et al., 2018]. It has also been used to differentiate through the Sinkhorn algorithm in optimal transport applications [Boursier and Perchet, 2019, Genevay et al. 2015]. It integrates smoothly in a machine learning framework, with dedicated libraries [Abadi et al., 2016, Paszke et al., 2019]. The third one is found in bi-level optimization, for instance for hyperparameter optimization [Bengio, 2000]. It is also the cornerstone of the use of convex optimization as layers in neural networks [Agrawal et al., 2019].

**Contribution** In this article, we want to answer the following question: which one of these estimators is the best? The central result, presented in section 2, is the following convergence speed, when $L$ is differentiable and under mild regularity hypothesis (Proposition 1, 2 and 3):

$$|g^1_t(x) - g^*(x)| = O (|z_t(x) - z^*(x)|) ,$$
$$|g^2_t(x) - g^*(x)| = o (|z_t(x) - z^*(x)|) ,$$
$$|g^3_t(x) - g^*(x)| = O (|z_t(x) - z^*(x)|^2) .$$

This is a super-efficiency phenomenon for the automatic estimator, illustrated in Figure 1 on a toy example. As our analysis reveals, the bound on $g^2$ depends on the convergence speed of the Jacobian of $z_t$, which itself depends on the optimization algorithm used to produce $z_t$. In section 3, we build on the work of Gilbert [1992] and give accurate bounds on the convergence of the Jacobian for gradient descent (Proposition 5) and stochastic gradient descent (Proposition 8 and 9) in the strongly convex case. We then study a simple case of non-strongly convex problem (Proposition 12). To the best of our knowledge, these bounds are novel. This analysis allows us to refine the convergence rates of the gradient estimators. In section 4, we start by recalling and extending the consequence of using wrong gradients in an optimization algorithm (Proposition 13 and 14). Then, since each gradient estimator comes at a different cost, we put the convergence bounds developed in the paper in perspective with a complexity analysis. This leads to practical and principled guidelines about which estimator should be used in which case. Finally, we provide numerical illustrations of the aforementioned results in section 5.

**Notation** The $l^2$ norm of $z \in \mathbb{R}^m$ is $|z| = \sqrt{\sum_{i=1}^m z_i^2}$. The operator norm of $M \in \mathbb{R}^{m \times n}$ is $\|M\| = \sup_{|z|=1} |Mz|$ and the Frobenius norm is $\|M\|_F = \sqrt{\sum_{i=1}^m \sum_{j=1}^n M_{ij}^2}$.
\[ \sum_{i,j} M_{ij}^2. \] The vector of size $n$ full of 1’s is $\mathbb{1}_n$. The Euclidean scalar product is $\langle \cdot, \cdot \rangle$.

The proofs are only sketched in the article, full proofs are deferred to appendix.

2 Convergence speed of gradient estimates

We consider a compact set $K = K_z \times K_x \subset \mathbb{R}^m \times \mathbb{R}^n$. We make the following assumptions on $\mathcal{L}$.

H1: $\mathcal{L}$ is twice differentiable over $K$ with second derivatives $\nabla_{zz} \mathcal{L}$ and $\nabla_{xz} \mathcal{L}$ respectively $L_{zz}$ and $L_{xz}$-Lipschitz.

H2: For all $x \in K_x$, $z \rightarrow \mathcal{L}(z, x)$ has a unique minimizer $z^*(x) \in \text{int}(K_z)$. The mapping $z^*(x)$ is differentiable, with Jacobian $J^*(x) \in \mathbb{R}^{n \times m}$.

H1 implies that $\nabla_{zz} \mathcal{L}$ and $\nabla_{xz} \mathcal{L}$ are Lipschitz, with constants $L_z$ and $L_x$. The Jacobian of $z_t$ at $x \in K_x$ is $J_t \triangleq \frac{\partial z_t(x)}{\partial x} \in \mathbb{R}^{n \times m}$. For the rest of the section, we consider a point $x \in K_x$, and we denote $g^* = g^*(x)$, $z^* = z(x)$ and $z_t = z_t(x)$.

2.1 Analytic estimator $g^1$

The analytic estimator (3) approximates $g^*$ as well as $z_t$ approximates $z^*$ by definition of the $L_x$-smoothness.

**Proposition 1** (Convergence of the analytic estimator). The analytic estimator verifies $|g^1_t - g^*| \leq L_x |z_t - z^*|$.
2.2 Automatic estimator $g^2$

The automatic estimator $g^2$ can be written as

$$g^2 = g^* + R(J_t)(z_t - z^*) + R_xz + J_t R_{xz}$$

where

$$R_xz \triangleq \nabla_x L(z_t, x) - \nabla_x L(z^*, x) - \nabla_{xx} L(z^*, x)(z_t - z^*)$$

$$R_{xz} \triangleq \nabla_z L(z_t, x) - \nabla_{zz} L(z^*, x)(z_t - z^*)$$

are Taylor’s rests and

$$R(J) \triangleq J \nabla_{zz} L(z^*, x) + \nabla_{zz} L(z^*, x).$$

The implicit function theorem states that $R(J^*) = 0$. Importantly, in a non strongly-convex setting where $\nabla_{zz} L(z^*, x)$ is not invertible, it might happen that $R(J_t)$ goes to 0 even though $J_t$ does not converge to $J^*$. $H1$ implies a quadratic bound on the rests

$$|R_xz| \leq \frac{L_{xz}}{2}|z_t - z^*|^2$$

and

$$|R_{xz}| \leq \frac{L_{zz}}{2}|z_t - z^*|^2.$$  (8)

We assume that $J_t$ is bounded $\|J_t\| \leq L_J$. This holds when $J_t$ converges, which is the subject of section 3. The triangle inequality in Equation 6 gives:

**Proposition 2** (Convergence of the automatic estimator). We define

$$L \triangleq L_{xx} + L_J L_{zz}$$

Then $|g^2_t - g^*| \leq \|R(J_t)||z_t - z^*| + \frac{L}{2}|z_t - z^*|^2.$

This proposition shows that the rate of convergence of $g^2$ depends on the speed of convergence of $R(J_t)$. For instance, if $R(J_t)$ goes to 0, we have

$$g^2_t - g^* = o(|z_t - z^*|).$$

Unfortunately, it might happen that, even though $z_t$ goes to $z^*$, $R(J_t)$ does not go to 0 since differentiation is not a continuous operation. In section 3, we refine this convergence rate by analyzing the convergence speed of the Jacobian in different settings.

2.3 Implicit estimator $g^3$

The implicit estimator $g^3$ is well defined provided that $\nabla_{zz} L$ is invertible. We obtain convergence bounds by making a Lipschitz assumption on $J(z, x) = -\nabla_{xx} L(z, x)[\nabla_{zz} L(z, x)]^{-1}$.  

**Proposition 3.** (Convergence of the implicit estimator) Assume that $J$ is $L_J$-Lipschitz with respect to its first argument, and that $\|J_t\| \leq L_J$. Then, for $L$ as defined in (9),

$$|g^3_t - g^*| \leq \left(\frac{L}{2} + L_J L_z\right)|z_t - z^*|^2.$$  (10)
Sketch of proof. The proof is similar to that of Proposition 2 using \( \| R(J(z_t, x)) \| \leq L_z L_J |z_t - z^*| \).

Therefore this estimator converges twice as fast as \( g^1 \), and at least as fast as \( g^2 \). Just like \( g^1 \) this estimator does not need to store the past iterates in memory, since it is a function of \( z_t \) and \( x \). However, it is usually much more costly to compute.

2.4 Link with bi-level optimization

Bi-level optimization appears in a variety of machine-learning problems, such as hyperparameter optimization [Pedregosa, 2016] or supervised dictionary learning [Mairal et al., 2012]. It considers problems of the form

\[
\min_{x \in \mathbb{R}^n} \ell'(x) \triangleq L'(z^*(x), x) \text{ s.t. } z^*(x) \in \arg \min_{z \in \mathbb{R}^m} L(z, x),
\]

where \( L' : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R} \) is another objective function. The setting of our paper is a special instance of bi-level optimization where \( L' = L \). The gradient of \( \ell' \) is

\[
g^* = \nabla_x \ell'(x) = \nabla_x L'(z^*, x) + J^* \nabla_z L'(z^*, x).
\]

When \( z_t(x) \) is a sequence of approximate minimizers of \( L \), gradient estimates can be defined as

\[
\begin{align*}
g'^1 &= \nabla_x L'(z_t(x), x), \\
g'^2 &= \nabla_x L'(z_t(x), x) + J_t \nabla_z L'(z_t(x), x), \\
g'^3 &= \nabla_x L'(z_t(x), x) + J_t(z_t(x), x) \nabla_z L'(z_t(x), x).
\end{align*}
\]

Here, \( \nabla_x L'(z^*, x) \neq 0 \), since \( z^* \) does not minimize \( L' \). Hence, \( g'^1 \) does not estimate \( g'^* \). Moreover, in general,

\[
\nabla_x L'(z^*, x) + J^* \nabla_z L'(z^*, x) \neq 0.
\]

Therefore, there is no cancellation to allow super-efficiency of \( g'^2 \) and \( g'^3 \) and we only obtain linear rates

\[
g'^2 - g^* = O(|z_t - z^*|), \quad g'^3 - g^* = O(|z_t - z^*|).
\]

3 Convergence speed of the Jacobian

In order to get a better understanding of the convergence properties of the gradient estimators – in particular \( g'^2 \) – we analyze it in different settings. A large portion of the analysis is devoted to the convergence of \( R(J_t) \) to 0, since it does not directly follow from the convergence of \( z_t \). In most cases, we show convergence of \( J_t \) to \( J^* \), and use

\[
\| R(J_t) \| \leq L_z \| J_t - J^* \|
\]

in the bound of Proposition 2.
3.1 Contractive setting

When $z_t$ are the iterates of a fixed point iteration with a contractive mapping, we recall the following result due to Gilbert [1992].

**Proposition 4** (Convergence speed of the Jacobian). Assume that $z_t$ is produced by a fixed point iteration

$$z_{t+1} = \Phi (z_t, x),$$

where $\Phi : K_z \times K_x \to K_z$ is differentiable. We suppose that $\Phi$ is contractive: there exists $\kappa < 1$ such that for all $(z, z', x) \in K_z \times K_z \times K_x$, $|\Phi(z, x) - \Phi(z', x)| \leq \kappa |z - z'|$. Under mild regularity conditions on $\Phi$:

- $z_t$ converges to a differentiable function $z^*$ such that $z^* = \Phi(z^*, x)$, with Jacobian $J^*$.
- $|z_t - z^*| = O(\kappa^t)$ and $\|J_t - J^*\| = O(t \kappa^t)$

3.2 Gradient descent in the strongly convex case

We consider the gradient descent iterations produced by the mapping $\Phi(z, x) = z - \rho \nabla_z L(z, x)$, with a step-size $\rho \leq 1/L_z$. We assume that $L$ is $\mu$-strongly convex with respect to $z$, i.e. $\nabla_z L \succeq \mu I$ for all $z \in K_z$, $x \in K_x$. In this setting, $\Phi$ satisfies the hypothesis of Proposition 4, and we obtain precise bounds.

**Proposition 5.** [Convergence speed of the Jacobian of gradient descent in a strongly convex setting] Let $z_t$ produced by the recursion $z_{t+1} = z_t - \rho \nabla_z L(z_t, x)$ with $\rho \leq 1/L_z$ and $\kappa \triangleq 1 - \rho \mu$. We have $|z_t - z^*| \leq \kappa^t |z_0 - z^*|$ and $\|J_t - J^*\| \leq t \kappa^t \rho L |z_0 - z^*|$ where $L$ is defined in [9].

**Sketch of proof** (C.1). We show that $\delta_t = \|J_t - J^*\|$ satisfies the recursive inequality $\delta_{t+1} \leq \kappa \delta_t + \rho L |z_0 - z^*| \kappa^t$.

As a consequence, Prop. 1, 2, 3 together with Eq. (12) give in this case

$$|g^1 - g^*| \leq L_z |z_0 - z^*| \kappa^t,$$
$$|g^2 - g^*| \leq (\rho L_z t + \kappa L_z) |z_0 - z^*|^2 \kappa^{2t-1},$$
$$|g^3 - g^*| \leq (\frac{L}{2} + L_z^2 L_z) |z_0 - z^*|^2 \kappa^t .$$

We get the convergence speed $g^2 - g^* = O(t \kappa^t)$, which is almost twice better than the rate for $g^1$. Importantly, the order of magnitude in Proposition 5 is tight, as it can be seen in Proposition 15.

3.3 Stochastic gradient descent in $z$

We provide an analysis of the convergence of $J_t$ in the stochastic gradient descent setting, assuming once again the $\mu$-strong convexity of $L$. We suppose that $L$ is an expectation

$$L(z, x) = \mathbb{E}_\xi [C(z, x, \xi)] ,$$
where $\xi$ is drawn from a distribution $d$, and $C$ is twice differentiable. Stochastic gradient descent (SGD) with steps $\rho_t$ iterates

$$z_{t+1}(x) = z_t(x) - \rho_t \nabla z C \left(z_t(x), x, \xi_{t+1}\right) \text{ where } \xi_{t+1} \sim d.$$ 

In the stochastic setting, Proposition 2 becomes

**Proposition 6.** Define

$$\delta_t = \mathbb{E} \left[ \| J_t - J^* \|^2_F \right] \text{ and } d_t = \mathbb{E} \left[ \| z_t - z^* \|^2 \right].$$

We have $\mathbb{E} \| g^2 - g^* \| \leq L \sqrt{t} \sqrt{d_t} + \frac{r}{2} d_t$.

**Sketch of proof (C.5).** We use Cauchy-Schwarz and the norm inequality $\| \cdot \| \leq \| \cdot \|_F$ to bound $\mathbb{E} \| R(J_t) \| \| z_t - z^* \|$.

We begin by deriving a recursive inequality on $\delta_t$, inspired by the analysis techniques of $d_t$.

**Proposition 7.** [Bounding inequality for the Jacobian] We assume bounded Hessian noise, in the sense that $\mathbb{E} \| \nabla zz C(z, x, \xi) \|^2_F \leq \sigma^2_{zz}$ and $\mathbb{E} \| \nabla xx C(z, x, \xi) \|^2_F \leq \sigma^2_{xx}$. Let $r = \min(n, m)$, and $B^2 = \sigma^2_{zz} + L^2 \sigma^2_{zz}$. We have

$$\delta_{t+1} \leq (1 - 2\rho_t \mu) \delta_t + 2\rho_t \sqrt{rL} \sqrt{d_t} \sqrt{\delta_t} + \rho_t^2 B^2.$$

**Sketch of proof (C.4).** A standard strong convexity argument gives the bound

$$\delta_{t+1} \leq (1 - 2\rho_t \mu) \delta_t + 2\rho_t \sqrt{rL} \mathbb{E} || J_t - J^* \|_F \| z_t - z_0 \| + \rho_t^2 B^2.$$

The middle term is then bounded using Cauchy-Schwarz inequality.

Therefore, any convergence bound on $d_t$ provides another convergence bound on $\delta_t$ by unrolling Eq. (15). We first analyze the fixed step-size case by using the simple “bounded gradients” hypothesis and bounds on $d_t$ from Moulines and Bach [2011]. In this setting, the iterates converge linearly until they reach a threshold caused by gradient variance.

**Proposition 8.** [SGD with constant step-size] Assume that the gradients have bounded variance $\mathbb{E} \| \nabla z C(z, x, \xi) \|^2 \leq \sigma^2$. Assume $\rho_t = \rho < 1/L_z$, and let $\kappa_2 = \sqrt{1 - 2\rho \mu}$ and $\beta = \sqrt{\frac{\sigma^2}{2 \rho}}$. In this setting

$$\delta_t \leq \left( \kappa_2^t \left( \| J^* \|_F + t \alpha \right) + B^2 \right)^2,$$

where $\alpha = \frac{\rho \sqrt{L}}{\kappa_2} \| z^* - z_0 \|$ and $B^2 = \frac{\rho \sqrt{L} \beta}{\kappa_2 (1 - \kappa_2)} + \frac{\rho B}{1 - \kappa_2}$.

**Sketch of proof (C.4).** Moulines and Bach [2011] give $d_t \leq \kappa_2^t \| z_0 - z^* \| + \beta^2$, which implies $\sqrt{d_t} \leq \kappa_2 \| z_0 - z^* \| + \beta$. A bit of work on Eq. (15) then gives

$$\sqrt{d_{t+1}} \leq \kappa_2 \sqrt{d_t} + \rho \kappa_2^t + (1 - \kappa_2) B_2.$$

Unrolling the recursion gives the proposed bound.
This bound showcases that the familiar “two-stages” behavior also stands for \( \delta_t \): a transient linear decay at rate \( (1 - 2\rho_\mu)t \) in the first iterations, and then convergence to a stationary noise level \( B^2_2 \). This bound is not tight enough to provide a good estimate of the noise level in \( \delta_t \). Let \( B^2_3 \) the limit of the sequence defined by the recursion (15). We find

\[
B^2_3 = (1 - 2\rho_\mu)B^2_3 + 2\rho\sqrt{rL}\beta B_3 + \rho^2 B^2,
\]

which gives by expanding \( \beta \)

\[
B_3 = \sqrt{\rho} \frac{\sqrt{rL}\sigma}{2(\mu)^2} \left( 1 + \sqrt{1 + \frac{4\mu^2 B^2}{rL^2\sigma^2}} \right).
\]

This noise level scales as \( \sqrt{\rho} \), which is observed in practice. In this scenario, Prop. 1, 2 and 3 show that \( g^1 - g^* \) reaches a noise level proportional to \( \sqrt{\rho} \); while both \( g^2 - g^* \) and \( g^3 - g^* \) reach a noise level proportional to \( \rho \). \( g^2 \) and \( g^3 \) can estimate \( g^* \) to an accuracy that can never be reached by \( g^1 \). We now turn to the decreasing step-size case.

**Proposition 9.** [SGD with decreasing step-size] Assume that \( \rho_t = \rho_0 t^{-\alpha} \) with \( \alpha \in (0, 1) \). Assume a bound on \( d_t \) of the form \( d_t \leq d^2 t^{-\alpha} \). Then

\[
\delta_t \leq 4\rho_0 B^2_2 \mu + rL^2 d^2 \mu^2 t^{-\alpha} + o(t^{-\alpha}).
\]

**Sketch of proof (C.6).** We use (15) to obtain a recursion

\[
\delta_{t+1} \leq (1 - \mu \rho_0 t^{-\alpha}) \delta_t + (B^2_2 \rho_0^2 + rL^2 d^2 \rho_0) t^{-2\alpha},
\]

which is then unrolled.

When \( \rho_t \propto t^{-\alpha} \), we have \( d_t = O(t^{-\alpha}) \) so the assumption \( d_t \leq d^2 t^{-\alpha} \) is verified for some \( d \). One could use the precise bounds of [Moulines and Bach, 2011] to obtain non-asymptotic bounds on \( \delta_t \) as well.

Overall, we recover bounds for \( \delta_t \) with the same behavior than the bounds for \( d_t \). Plugging them in Proposition 6 gives the asymptotic behaviors for the gradient estimators.

**Proposition 10** (Convergence speed of the gradient estimators for SGD with decreasing step). Assume that \( \rho_t = Ct^{-\alpha} \) with \( \alpha \in (0, 1) \). Then

\[
\mathbb{E}_\xi[|g^1 - g^*|] = O(\sqrt{t^{-\alpha}}), \quad \mathbb{E}_\xi[|g^2 - g^*|] = O(t^{-\alpha})
\]

\[
\mathbb{E}_\xi[|g^3 - g^*|] = O(t^{-\alpha})
\]

The super-efficiency of \( g^2 \) and \( g^3 \) is once again illustrated, as they converge at the same speed as \( d_t \).
3.4 Beyond strong convexity

All the previous results rely critically on the strong convexity of $\mathcal{L}$. A function $f$ with minimizer $z^*$ is $p$-Łojasiewicz [Attouch and Bolte 2009] when $\mu(f(z) - f(z^*))^{p-1} \leq \|\nabla f(z)\|^p$ for some $\mu > 0$. Any strongly convex function is 2-Łojasiewicz: the set of $p$-Łojasiewicz functions for $p \geq 2$ offers a framework beyond strong-convexity that still provides convergence rates on the iterates. The general study of gradient descent on this class of function is out of scope for this paper. We analyze a simple class of $p$-Łojasiewicz functions, the least mean $p$-th problem, where

$$\mathcal{L}(z, x) \triangleq \frac{1}{p} \sum_{i=1}^{n} (x_i - [Dz]_i)^p$$

for $p$ an even integer and $D$ is overcomplete ($\text{rank}(D) = n$). In this simple case, $\mathcal{L}(\cdot, x)$ is minimized by cancelling $x - Dz$, and $g^* = (x - Dz^*)^{p-1} = 0$.

In the case of least squares ($p = 2$) we can perfectly describe the behavior of gradient descent, which converges linearly.

**Proposition 11.** Let $z_t$ the iterates of gradient descent with step $\rho \leq \frac{1}{2L}$ in (16) with $p = 2$, and $z^* \in \arg \min \mathcal{L}(z, x)$. It holds

$$g^1 = D(z_t - z^*), \quad g^2 = D(z_{2t} - z^*) \quad \text{and} \quad g^3 = 0.$$

**Proof.** The iterates verify $z_t - z^* = (I - D^\top D)^t(z_0 - z^*)$, and we find $J_t \nabla \mathcal{L}(z_t, x) = (I_n - (I_n - D^\top D)^t)(x - Dz_t)$. The result follows. \hfill $\square$

The automatic estimator therefore goes exactly twice as fast as the analytic one to $g^*$, while the implicit estimator is exact. Then, we analyze the case where $p \geq 4$ in a more restrictive setting.

**Proposition 12.** For $p \geq 4$, we assume $DD^\top = I_n$. Let $\alpha \triangleq \frac{p-1}{p-2}$. We have

$$|g^1_t| = O(t^{-\alpha}), \quad |g^2_t| = O(t^{-2\alpha}), \quad g^3_t = 0.$$

**Sketch of proof** [C.7]. We first show that the residuals $r_t = x - Dz_t$ verify $r_t = \left(\frac{1}{p-2} \right)^{\frac{1}{p-2}} (1 + O(\frac{\log(t)}{t}))$, which gives the result for $g^1$. We find $g^2_t = M_t r_t^{p-1}$ where $M_t = I_n - J_t D^\top$ verifies $M_{t+1} = M_t(I_n - (p-1)(p\text{diag}(r_t^{p-2})))$. Using the development of $r_t$ and unrolling the recursion concludes the proof. \hfill $\square$

For this problem, $g^2$ is of the order of magnitude of $g^1$ squared and as $p \to +\infty$, we see that the rate of convergence of $g^1$ goes to $t^{-1}$, while the one of $g^2$ goes to $t^{-2}$.

4 Consequence on optimization

In this section, we study the impact of using the previous inexact estimators for first order optimization. These estimators nicely fit in the framework of inexact oracles introduced by Devolder et al. [2014].
4.1 Inexact oracle

We assume that $\ell$ is $\mu_x$-strongly convex and $L_x$-smooth with minimizer $x^*$. A $(\delta, \mu, L)$-inexact oracle is a couple $(\ell_\delta, g_\delta)$ such that $\ell_\delta : \mathbb{R}^m \to \mathbb{R}$ is the inexact value function, $g_\delta : \mathbb{R}^m \to \mathbb{R}^m$ is the inexact gradient and for all $x, y$

$$\frac{\mu}{2} |x - y|^2 \leq \ell(x) - \ell_\delta(y) - (g_\delta(y) |x - y|) \leq \frac{L}{2} |x - y|^2 + \delta . \quad (17)$$

Devolder et al. [2013] show that if the gradient approximation $g^i$ verifies $|g^*(x) - g^i(x)| \leq \Delta_i$ for all $x$, then $(\ell, g^i)$ is a $(\delta_i, \frac{\mu}{2}, 2L_x)$-inexact oracle, with

$$\delta_i = \frac{1}{2\mu_x} \left( \frac{1}{2L_x} \right) . \quad (18)$$

We consider the optimization of $\ell$ with inexact gradient descent: starting from $x_0 \in \mathbb{R}^n$, it iterates

$$x_{q+1} = x_q - \eta g^i(x_q) , \quad (19)$$

with $\eta = \frac{1}{2L_x}$, a fixed $t$ and $i = 1, 2$ or 3.

Proposition 13. [Devolder et al. 2013, Theorem 4] The iterates $x_q$ with estimate $g^i$ verify

$$\ell(x_q) - \ell(x^*) \leq 2L_x(1 - \frac{\mu}{4L_x})^q |x_0 - x^*|^2 + \delta_i$$

with $\delta_i$ defined in (18).

As $q$ goes to infinity, the error made on $\ell(x^*)$ tends towards $\delta_i = \mathcal{O}(|g^i - g^*|^2)$. Thus, a more precise gradient estimate achieves lower optimization error. This illustrates the importance of using gradients estimates with an error $\Delta_i$ as small as possible.

We now consider stochastic optimization for our problem, with loss $\ell$ defined as

$$\ell(x) = \mathbb{E}_v[h(x, v)] \text{ with } h(x, v) = \min_z H(z, x, v) .$$

Stochastic gradient descent with constant step-size $\eta \leq \frac{1}{2L_x}$ and inexact gradients iterates

$$x_{q+1} = x_q - \eta g^i(x_q, v_{q+1}) ,$$

where $g^i(x_q, v_{q+1})$ is computed by an approximate minimization of $z \to H(z, x_q, v_{q+1})$.

Proposition 14. We assume that $H$ is $\mu_x$-strongly convex, $L_x$-smooth and verifies

$$\mathbb{E}[|\nabla_x h(x, v) - \nabla_x \ell(x)|^2] \leq \sigma^2 .$$

The iterates $x_q$ of SGD with approximate gradient $g^i$ and step-size $\eta$ verify

$$\mathbb{E}[x_q - x^*|^2 \leq (1 - \frac{\eta^2}{2})^q |x_0 - x^*|^2 + \frac{2\eta}{\mu_x} \sigma^2 + \frac{4}{\mu_x} \delta_i$$

with $\delta_i = \frac{1}{2\mu_x} \left( \frac{1}{2L_x} + 2\eta \right)$. 

11
Table 1: Computational cost for a quadratic loss $L$. Here $c \geq 1$ corresponds to the relative added cost of automatic differentiation.

| Gradient estimate | Computational cost |
|-------------------|--------------------|
| $g_1^t$           | $O(mnt)$           |
| $g_2^t$           | $O(cmnt)$          |
| $g_3^t$           | $O(mnt + m^3 + m^2n)$ |

The proof is deferred to Appendix D. In this case, it is pointless to achieve an estimation error on the gradient $\Delta_i$ smaller than some fraction of the gradient variance $\sigma^2$.

As a final note, these results extend without difficulty to the problem of maximizing $\ell$, by considering gradient ascent or stochastic gradient ascent.

4.2 Time and memory complexity

In the following, we put our results in perspective with a computational and memory complexity analysis, allowing us to provide practical guidelines for optimization of $\ell$.

Computational complexity of the estimators The cost of computing the estimators depends on the cost function $L$. We give a complexity analysis in the least squares case [16] which is summarized in Table 1. In this case, computing the gradient $\nabla_z L$ takes $O(mn)$ operations, therefore the cost of computing $z_t$ with gradient descent is $O(mnt)$. Computing $g_1^t$ comes at the same price. The estimator $g_2^t$ requires a reverse pass on the computational graph, which costs a factor $c \geq 1$ of the forward computational cost: the final cost is $O(cmnt)$. Griewank and Walther [2008] showed that typically $c \in [2, 3]$. Finally, computing $g_3^t$ requires a costly $O(m^2n)$ Hessian computation, and a $O(m^3)$ linear system inversion. The final cost is $O(mnt + m^3 + m^2n)$. The linear scaling of $g_1^t$ and $g_2^t$ is highlighted in Figure A.3. In addition, computing $g_2^t$ usually requires to store in memory all intermediate variables, which might be a burden. However, some optimization algorithms are invertible, such as SGD with momentum [Maclaurin et al., 2015]. In this case, no additional memory is needed.

Linear convergence: a case for the analytic estimator In the time it takes to compute $g_2^t$, one can at the same cost compute $g_3^t$. If $z_t$ converges linearly at rate $\kappa^t$, Proposition 1 shows that $g_3^t - g^* = O(\kappa^t)$, while Proposition 2 gives, at best, $g_1^t - g^* = O(\kappa^2)$: $g_1^t$ is a better estimator of $g^*$ than $g_3^t$, provided that $c \geq 2$. In the quadratic case, we even have $g_1^t = g_3^t$. Further, computing $g_2^t$ might requires additional memory: $g_1^t$ should be preferred over $g_3^t$ in this setting. However, our analysis is only asymptotic, and other effects might come into play to tip the balance in favor of $g_2^t$.

As it appears clearly in Table 1, choosing $g_1^t$ over $g_3^t$ depends on $t$: when $mnt \gg m^3 + m^2n$, the additional cost of computing $g_3^t$ is negligible, and it should be preferred since it is more accurate. This is however a rare situation in a large scale setting.
Figure 2: Evolution of $|g_i^t - g^*|$ with the number of iteration $t$ for (a) the Ridge Regression $\mathcal{L}_1$, (b) the Regularized Logistic Regression $\mathcal{L}_2$, (c) the Least Mean $p$-th Norm $\mathcal{L}_3$ in log-scale and (d) the Wasserstein Distance $\mathcal{L}_4$. In all cases, we can see the asymptotic super-efficiency of the $g_2$ estimator compared to $g_1$. The $g_3$ estimator is better in most cases but it is unstable in (d).

**Sublinear convergence**  We have provided two settings where $z_t$ converges sub-linearly. In the stochastic gradient descent case with a fixed step-size, one can benefit from using $g^2$ over $g^1$, since it allows to reach an accuracy that can never be reached by $g^1$. With a decreasing step-size, reaching $|g_1^t - g^*| \leq \varepsilon$ requires $O(\varepsilon^{-2/\alpha})$ iterations, while reaching $|g_2^t - g^*| \leq \varepsilon$ only takes $O(\varepsilon^{-1/\alpha})$ iterations. For $\varepsilon$ small enough, we have $c\varepsilon^{-1/\alpha} < \varepsilon^{-2/\alpha}$: it is always beneficial to use $g^2$ if memory capacity allows it.

The story is similar for the simple non-strongly convex problem studied in subsection 3.4 because of the slow convergence of the algorithms, $g_2^t$ is much closer to $g^*$ than $g_1^t$. Although our analysis was carried in the simple least mean $p$-th problem, we conjecture it could be extended to the more general setting of $p$-Łojasiewicz functions [Attouch and Bolte, 2009].

5 Experiments

All experiments are performed in Python using pytorch [Paszke et al., 2019]. The code to reproduce the figures is available online.\footnote{See Appendix.}

5.1 Considered losses

In our experiments, we considered several losses with different properties. For each experiments, the details on the size of the problems are reported in subsection A.1.

**Regression**  For a design matrix $D \in \mathbb{R}^{n \times m}$ and a regularization parameter
$\lambda > 0$, we define

$$\mathcal{L}_1(z, x) = \frac{1}{2} |x - Dz|^2 + \frac{\lambda}{2} |z|^2,$$

$$\mathcal{L}_2(z, x) = \sum_{i=1}^{n} \log \left(1 + e^{-x_i|Dz|_i}\right) + \frac{\lambda}{2} |z|^2,$$

$$\mathcal{L}_3(z, x) = \frac{1}{p} |x - Dz|^p; \quad p = 4 .$$

$\mathcal{L}_1$ corresponds to Ridge Regression, which is quadratic and strongly convex when $\lambda > 0$. $\mathcal{L}_2$ is the Regularized Logistic Regression. It is strongly convex when $\lambda > 0$. $\mathcal{L}_3$ is studied in subsection 3.4 and defined with $DD^\top = I_n$.

**Regularized Wasserstein Distance** The Wasserstein distance defines a distance between probability distributions. In Cuturi [2013], a regularization of the problem is proposed, which allows to compute it efficiently using the Sinkhorn algorithm, enabling many large scale applications. As we will see, the formulation of the problem fits nicely in our framework. The set of histograms is $\Delta^m_+ = \{ a \in \mathbb{R}^m_+ | \sum_{i=1}^{m} a_i = 1 \}$. Consider two histograms $a \in \Delta^m_{ma}$ and $b \in \Delta^m_{mb}$. The set of couplings is $U(a, b) = \{ P \in \mathbb{R}^{ma \times mb} | P1_{ma} = a, \quad P^\top 1_{mb} = b \}$. The histogram $a$ (resp. $b$) is associated with set of $m_a$ (resp. $m_b$) points in dimension $k$, $(X_1, \ldots, X_{ma}) \in \mathbb{R}^k$ (resp. $(Y_1, \ldots, Y_{mb})$). The cost matrix is $C \in \mathbb{R}^{ma \times mb}$ such that $C_{ij} = |X_i - Y_j|^2$. For $\epsilon > 0$, the entropic regularized Wasserstein distance is $W^2_\epsilon(a, b) = \min_{P \in U(a, b)} (C, P) + \epsilon (\log(P), P)$. The dual formulation of the previous variational formulation is [Peyré and Cuturi 2019, Prop. 4.4.]:

$$W^2_\epsilon(a, b) = \min_{z_a, z_b} \langle (a, z_a) + (b, z_b) + \epsilon (e^{-z_a/\epsilon}, e^{-C/\epsilon} e^{-z_b/\epsilon})\rangle_{\mathcal{L}_4((z_a, z_b), a)} \quad (20)$$

This loss is strongly convex up to a constant shift on $z_a, z_b$. The Sinkhorn algorithm performs alternate minimization of $\mathcal{L}_4$:

$$z_a \leftarrow \epsilon (\log(e^{-C/\epsilon} e^{-z_b/\epsilon}) - \log(a)), $$

$$z_b \leftarrow \epsilon (\log(e^{-C/\epsilon} e^{-z_a/\epsilon}) - \log(b)).$$

This optimization technique is not covered by the results in section 3, but we will see that the same conclusions hold in practice.

**5.2 Examples of super-efficiency**

To illustrate the tightness of our bounds, we evaluate numerically the convergence of the different estimators $g^1, g^2$ and $g^3$ toward $g^*$ for the losses introduced above. For all problems, $g^*$ is computed by estimating $z^*(x)$ with gradient descent for a very large number of iterations and then using (2).

**Gradient Descent** Figure 2 reports the evolution of $|g_t - g^*|$ with $t$ for the losses $\{\mathcal{L}_j\}_{j=1}^4$, where $z_t$ is obtained by gradient descent for $\mathcal{L}_1, \mathcal{L}_2$ and $\mathcal{L}_3$, and
Figure 3: Expected performances of $g^i$ for the SGD; (a) noise level as $t \to +\infty$ for a constant step-size $\rho$; (b) Expected error as a function of the number of iteration for decreasing step-size $\rho_t = Ct^{-\alpha}$. The solid line displays the mean values and the shaded area the first and last decile.

by Sinkhorn iterations for $L_4$. For the strongly convex losses (a),(b), $|g_1^i - g^*|$ converges linearly with the same rate as $|z_i - z^*|$ while $|g_2^i - g^*|$ converges about twice as fast. This confirms the theoretical findings of Proposition 4 and (13). The estimator $g^3$ also converges with the predicted rates in (a),(b), however, it fails in (d) as the Hessian of $L_4$ is ill-conditioned, leading to numerical instabilities. For the non-strongly convex loss $L_3$, Figure 2(c) shows that the rates given in Proposition 12 are correct as $g_1$ converges with a rate $t^{-2}$ while $g_2^i$ converges as $t^{-3}$. Here, we did not include $g_3^i$ as it is equal to 0 due to the particular form of $L_3$.

Stochastic Gradient Descent In Figure 3, we investigate the evolution of expected performances of $g^i$ for the SGD, in order to validate the results of subsection 3.3. We consider $L_2$. The left part (a) displays the asymptotic expected performance $E[|g_1^i - g^*|]$ in the fixed step case, as a function of the step $\rho$, computed by running the SGD with sufficiently many iterations to reach a plateau. As predicted in subsection 3.3, the noise level scales as $\sqrt{\rho}$ for $g_1^i$ while it scales like $\rho$ for $g_2^i$ and $g_3^i$. The right part (b) displays the evolution of $E[|g_1^i - g^*|]$ as a function of $t$, where the step-size is decreasing $\rho_t \propto t^{-\alpha}$. Here again, the asymptotic rates predicted by Proposition 10 is showcased: $g_1^i - g^*$ is $O(\sqrt{t^{-\alpha}})$ while $g_2^i - g^*$ and $g_3^i - g^*$ are $O(t^{-\alpha})$.

5.3 Example on a full training problem

We are now interested in the minimization of $\ell$ with respect to $x$, possibly under constraints. We consider the problem of computing Wasserstein barycenters
using mirror descent, as proposed in [Cuturi and Doucet, 2014]. For a set of histograms \( b_1, \ldots, b_N \in \Delta_n^+ \) and a cost matrix \( C \in \mathbb{R}^{n \times m} \), the entropic regularized Wasserstein barycenter of the \( b_i \)’s is

\[
x \in \arg \min_{x \in \Delta_n^+} \ell(x) = \sum_{i=1}^{N} W_2^*(x, b_i),
\]

where \( W_2^* \) is defined in (20), and we have:

\[
\ell(x) = \min_{z_1^x, \ldots, z_x^x, z_b^1, \ldots, z_b^N} \sum_{i=1}^{N} L_4((z_x^i, z_b^i), x).
\]  \tag{21}

The dual variables \( z_x^i, z_b^i \) are obtained with \( t \) iterations of the Sinkhorn algorithm. In this simple setting, \( \nabla_x L_4((z_x, z_b), x) = z_x \). The cost function is then optimized by mirror descent, with approximate gradient \( g^j: x_{q+1} = P_\Delta(\exp(-\eta g^j) x_q) \), where \( P_\Delta(x) = x/\sum_{i=1}^{n} x_i \) is the projection on \( \Delta_n^+ \). Figure 3 displays the scale of the error \( \delta_i = \ell(x_q) - \ell(x^*) \). We excluded \( g^3 \) here as the computation were unstable – as seen in Figure 2(c) – and too expensive. The error decreases much faster with number of inner iteration \( t \) by using \( g^2 \) compared to \( g^1 \). However, when looking at the time taken to reach the asymptotic error, we can see that \( g^1 \) is a better estimator in this case. This illustrates the fact that while \( g^2 \) is almost twice as good at approximating \( g^* \) as \( g^1 \), it is at least twice as expensive, as discussed in subsection 4.2.

**Conclusion**

In this work, we have described the asymptotic behavior of three classical gradient estimators for a special instance of bi-level estimation. We have highlighted a super-efficiency phenomenon of automatic differentiation. However, our complexity analysis shows that it is faster to use the standard analytic estimator when the optimization algorithm converges linearly, and that the super-efficiency
can be leveraged for algorithms with sub-linear convergence. This conclusion should be taken with caution, as our analysis is only asymptotic. This suggests a new line of research interested in the non-asymptotic behavior of these estimators. Extending our results to a broader class of non-strongly convex functions would be another interesting direction, as we observe empirically that for logistic-regression, $g_2 - g^* \simeq (g_1 - g^*)^2$. However, as the convexity alone does not ensure the convergence of the iterates, it raises interesting questions for the gradient estimation. Finally, it would also be interesting to extend our analysis to non-smooth problems, for instance when $z_t$ is obtained with the proximal gradient descent algorithm as in the case of ISTA for dictionary learning.

Acknowledgement

P.A. and G.P. acknowledge support from the European Research Council (ERC-NORIA). This work was funded in part by the French government under management of Agence Nationale de la Recherche as part of the "Investissements d'avenir" program, reference ANR-19-P3IA-0001 (PRAIRIE 3IA Institute).

References

Martın Abadi, Paul Barham, Jianmin Chen, Zhifeng Chen, Andy Davis, Jeffrey Dean, Matthieu Devin, Sanjay Ghemawat, Geoffrey Irving, Michael Isard, Manjunath Kudlur, Josh Levenberg, Rajat Monga, Sherry Moore, Derek G Murray, Benoit Steiner, Paul Tucker, Vijay Vasudevan, Pete Warden, Martin Wicke, Yuan Yu, and Xiaoqiang Zheng. TensorFlow: A system for large-scale machine learning. In 12th USENIX Symposium on Operating Systems Design and Implementation (OSDI), pages 265–283, 2016.

Akshay Agrawal, Brandon Amos, Shane Barratt, and Stephen Boyd. Differentiable Convex Optimization Layers. In Advances in Neural Information Processing Systems (NeurIPS), pages 9558–9570, Vancouver, BC, Canada, 2019.

Martial Agueh and Guillaume Carlier. Barycenters in the wasserstein space. *SIAM Journal on Mathematical Analysis*, 43(2):904–924, 2011.

Hedy Attouch and Jérôme Bolte. On the convergence of the proximal algorithm for nonsmooth functions involving analytic features. *Mathematical Programming*, 116(1-2):5–16, 2009.

Atılım Gunes Baydin, Barak A Pearlmutter, Alexey Andreyevich Radul, and Jeffrey Mark Siskind. Automatic Differentiation in Machine Learning: A Survey. *Journal of Machine Learning Research (JMLR)*, 18:1–43, 2018.

Yoshua Bengio. Gradient-based optimization of hyperparameters. *Neural computation*, 12(8):1889–1900, 2000.
Etienne Boursier and Vianney Perchet. Utility/Privacy Trade-off through the lens of Optimal Transport. In *International Conference on Artificial Intelligence and Statistics (AISTATS)*, Palermo, Italie, October 2019.

Marco Cuturi. Sinkhorn distances: Lightspeed computation of optimal transport. In *Advances in neural information processing systems*, pages 2292–2300, 2013.

Marco Cuturi and Arnaud Doucet. Fast computation of wasserstein barycenters. *Journal of Machine Learning Research (JMLR)*, 2014.

Ingrid Daubechies, Ronald DeVore, Massimo Fornasier, and C Sinan Güntürk. Iteratively reweighted least squares minimization for sparse recovery. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 63(1):1–38, 2010.

Olivier Devolder, François Glineur, and Yurii Nesterov. First-order methods with inexact oracle: The strongly convex case. CORE Discussion Paper CORE Discussion paper, CORE, 2013.

Olivier Devolder, François Glineur, and Yurii Nesterov. First-order methods of smooth convex optimization with inexact oracle. *Mathematical Programming*, 146(1-2):37–75, August 2014.

Jean Feydy, Thibault Séjourné, François-Xavier Vialard, Shun-ichi Amari, Alain Trouvé, and Gabriel Peyré. Interpolating between Optimal Transport and MMD using Sinkhorn Divergences. In *International Conference on Artificial Intelligence and Statistics (AISTATS)*, pages 2681–2690, Okinawa, Japan, 2019.

Aude Genevay, Gabriel Peyré, and Marco Cuturi. Learning generative models with sinkhorn divergences. In *International Conference on Artificial Intelligence and Statistics*, pages 1608–1617, 2018.

Jean-Charles Gilbert. Automatic differentiation and Iterative Processes. *Optimization Methods and Software*, 1:13–21, 1992.

Ian Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative adversarial nets. In *Advances in neural information processing systems*, pages 2672–2680, 2014.

Karol Gregor and Yann Le Cun. Learning Fast Approximations of Sparse Coding. In *International Conference on Machine Learning (ICML)*, pages 399–406, 2010.

Andreas Griewank and Andrea Walther. *Evaluating derivatives: principles and techniques of algorithmic differentiation*, volume 105. Siam, 2008.

Dougal Maclaurin, David Duvenaud, and Ryan Adams. Gradient-based hyperparameter optimization through reversible learning. In *International Conference on Machine Learning*, pages 2113–2122, 2015.
Julien Mairal, Francis R. Bach, Jean Ponce, and Guillermo Sapiro. Online Learning for Matrix Factorization and Sparse Coding. *Journal of Machine Learning Research (JMLR)*, 11(1):19–60, 2010.

Julien Mairal, Francis R. Bach, and Jean Ponce. Task-driven dictionary learning. *IEEE Transactions on Pattern Analysis and Machine Intelligence (PAMI)*, 34(4):791–804, 2012.

Eric Moulines and Francis R Bach. Non-Asymptotic Analysis of Stochastic Approximation Algorithms for Machine Learning. In *Advances in Neural Information Processing Systems (NeurIPS)*, pages 451–459, Grenada, Spain, 2011.

Bruno A Olshausen and David J Field. Sparse coding with an overcomplete basis set: A strategy employed by v1? *Vision research*, 37(23):3311–3325, 1997.

Adam Paszke, Sam Gross, Francisco Massa, Adam Lerer, James Bradbury, Gregory Chanan, Trevor Killeen, Zeming Lin, Natalia Gimelshein, Luca Antiga, Alban Desmaison, Andreas Kopf, Edward Yang, Zachary DeVito, Martin Raison, Alykhan Tejani, Sasank Chilamkurthy, Benoit Steiner, Lu Fang, Junjie Bai, and Soumith Chintala. PyTorch: An Imperative Style, High-Performance Deep Learning Library. In *Advances in Neural Information Processing Systems (NeurIPS)*, page 12, Vancouver, BC, Canada, 2019.

Fabian Pedregosa. Hyperparameter optimization with approximate gradient. *arXiv preprint arXiv:1602.02355*, 2016.

Gabriel Peyré and Marco Cuturi. *Computational optimal transport*, volume 11. Now Publishers, Inc., 2019.

Bahareh Tolooshams, Sourav Dey, and Demba Ba. Scalable convolutional dictionary learning with constrained recurrent sparse auto-encoders. In *IEEE International Workshop on Machine Learning for Signal Processing (MLSP)*, 2018.

John von Neumann. Zur theorie der gesellschaftsspiele. *Mathematische annalen*, 100(1):295–320, 1928.
A Experiments details and extra experiments

A.1 Experiments details

Super-efficiency of \( g^2 \) for gradient descent

For Figure 2, the problem sizes are:

- **Ridge regression** \( \mathcal{L}_1 \): we use an overcomplete design matrix \( D \) with \( n = 50 \) and \( m = 100 \) with entries \( D_{i,j} \) drawn \( iid \) from a normal distribution \( \mathcal{N}(0, 1) \). The vector \( x \) to evaluate the gradient is sampled also with \( iid \) entries following a normal distribution. We take \( \lambda = \frac{1}{n} \). To compute \( g^*(x) \), we used the gradient descent with step-size \( \frac{1}{Lz} \) for 14,000 iterations.

- **Regularized Logistic regression** \( \mathcal{L}_2 \): we use an overcomplete design matrix \( D \) with \( n = 50 \) and \( m = 100 \) with entries \( D_{i,j} \) drawn \( iid \) from a normal distribution \( \mathcal{N}(0, 1) \). The vector \( x \) to evaluate the gradient is sampled also with \( iid \) entries following a normal distribution. We take \( \lambda = \frac{1}{n} \). To compute \( g^*(x) \), we used the gradient descent with step-size \( \frac{1}{Lz} \) for 40,000 iterations.

- **Least mean \( p \)-th norm** \( \mathcal{L}_3 \): In this setting, the convergence is much slower than in the previous ones. We use an overcomplete design matrix \( D \) with \( n = 5 \) and \( m = 10 \). To meet the condition of Proposition 12, we sample the entries of a matrix \( A \in \mathbb{R}^{n \times m} \) iid with normal distribution \( \mathcal{N}(0, 1) \), take the SVD of \( A = U^\top \Lambda V \) with \( U \) unitary and \( V \) and define \( D = U^\top V \). This ensures that \( DD^\top = I_n \). We choose \( p = 4 \), and use \( g^* = 0 \).

- **Wasserstein Distance** \( \mathcal{L}_4 \): we consider here the problem of computing the Wasserstein distance between two distributions supported on an Euclidean grid in \([0, 1] \) and \( C \) is defined as the \( \ell_2 \) distance between the points of the grid. \( q \) is in \( \Delta^m_+ \) with \( n = 100 \) and \( b \in \Delta^m_+ \) with \( m = 30 \). We sample \( a \in \Delta^m_+ \) by first sampling the \( \hat{a} \) iid from a uniform distribution \( \mathcal{U}(0, 1) \) and then take \( a = \frac{\hat{a}}{\sum_{j=1}^m a_j} \). We used \( \epsilon = 0.1 \) and \( g^*(x) \) is computed by running 2,000 iteration of Sinkhorn.

Super-efficiency of \( g^2 \) for SGD

For Figure 3, we consider for both experiments the penalized logistic loss and an over-complete design matrix \( D \) with \( n = 30 \) and \( m = 50 \) with entries \( D_{i,j} \) drawn \( iid \) from a normal distribution \( \mathcal{N}(0, 1) \). The vector \( x \) to evaluate the gradient is sampled also with \( iid \) entries following a normal distribution. We take \( \lambda = \frac{1}{n} \). To compute \( g^*(x) \), we used the gradient descent with step-size \( \frac{1}{Lz} \) for 1,000 iterations.

In Figure 3(a), we compute the gradient estimates \( g_t^i \) using the output \( z_t \) of the SGD with constant step-size \( \rho \) for 20 values of \( \rho \in [0.001, 0.1] \) in log-scale for 50 realization of the SGD. We report the mean value of \( |g_t^i - g^*| \) computed for \( t \) large enough to have reached the regime where only the noise term is significant in Proposition 8. This corresponds to the value of \( \mathbb{E}[|g_t^i - g^*|] \) estimated by taking
the value at the right end of the curve displayed in Figure A.1 (a) for different values of $\rho$.

In Figure 3 (b), we illustrate the evolution with $t$ of $E[|g^i_t - g^*|]$ for $g^i_t$ computed for SGD with decreasing step-sizes $t^{-\alpha}$ for $\alpha = .8$. The expectation is estimated by averaging 50 realizations of $|g^i_t - g^*|$ and the area around the curve correspond to the first and 9-th deciles.

A.2 Gradient descent with inexact gradients

To evaluate the impact of the different gradient estimators on the optimization of the global function $\ell$, we run mirror descent for the loss $\ell$ defined in (21). We use $n = m = 1,000$ and $N = 20$ for the dimensions of $x \in \Delta_n^+$ and $b_i \in \Delta_m^+$ and we sample following the same procedure as for the Wasserstein Distance. We set $\epsilon = 0.05$ and we used a step-size of $\eta = 0.05$. We compute $x^*$ by running the mirror descent algorithm with analytic gradient estimator $g^1_t$ for $t = 1,000$ and $q = 5,000$. Figure A.3 reports the residual errors $\ell(x_q) - \ell(x^*)$ for $g^1_t$ and $g^2_t$ relatively to the number of iterations $t$ used to compute them (a) as well as to the time taken to compute them (b). We exclude $g^3$ from this analysis as it is much more costly to compute in this case (see subsection A.3) and it can be ill-conditionned – as it is illustrated in Figure 2. Figure A.2 displays for each $t$ used to compute Figure 4 the evolution of the cost function in iteration $q$ and in time. We can see here in both figures that as the number of iteration $t$ to compute the gradient increases, the final optimization error decrease, as predicted in Proposition 13. However, the computational cost for $g^2_t$ scales with a factor $c$ compared to computing $g^1_t$ and $c$ is larger than 2 in this case (see
subsection A.3). As the convergence of \( z_t \) is linear, using \( g^2_t \) is not beneficial for the global optimization as it possible to compute \( g^1_t \) instead which reduces the optimization error compared to \( g^2_t \), as discussed in subsection 4.2.

### A.3 Computation time of the gradient

To evaluate the relative computational cost of the gradient estimates \( g^i_t \), we time the computation of the gradient using the loss \( \ell \) defined in (21). We use \( n = 500 \), \( m = 1,000 \) and \( N = 100 \) for the dimensions of \( x \in \Delta^n \) and \( b_i \in \Delta^m \) and we sample following the same procedure as for the Wasserstein Distance. Then, we time the computational time for the gradient estimators \( g^i_t \) for different values of \( t \), and report in Figure A.3 the median value of this computation time computed on 50 realization as well as the first and last decile values to get an idea of the variation of this value. The results are coherent with the computational complexity analysis in Table 1, \( g^1 \) and \( g^2 \) computation time scales linearly with \( t \), with a constant factor between them which capture the value of \( c \) which is around 3.5 in our case. For this scale of problem, \( g^3 \) requires inverting \( N \) matrices \( n \times n \). This cost dominates the cost of computing \( z_t \), \( g^1_t \) and \( g^2_t \) for small value of \( t \) and it becomes less prohibitive as \( t \) grows.

### B Proof for section 2

**Proposition 3.** [Convergence of the implicit estimator] Assume that \( J \) is \( L_J \)-Lipschitz with respect to its first argument, and that \( \|J_t\| \leq L_J \). Then, for \( L \) as defined in (9),

\[
|g^3_t - g^*| \leq (\frac{L}{2} + L_J L_z)|z_t - z^*|^2 .
\]  

(10)

**Proof.** We define \( J_t = J(z_t, x) \). The implicit gradient \( g^3 \) reads

\[
g^3 = g^* + R(J_t)(z_t - z^*) + R_{zz} + J_t R_{zz} .
\]
Then, we have
\[ R(J_t) = R(J_t) - R(J^*) = (J_t - J^*) \nabla_z \mathcal{L}(z^*, x) \]
We recall that \( J^* = \mathcal{J}(z^*, x) \). It follows that
\[ \| J_t - J^* \| \leq L \| z_t - z^* \|, \]
and
\[ \| R(J_t) \| \leq L \| z_t - z^* \|. \]
The result follows using Equation 8. \( \square \)

C Proof section 3

C.1 Proof of Proposition 5

Proposition 5. /Convergence speed of the Jacobian of gradient descent in a strongly convex setting/ Let \( z_t \) produced by the recursion \( z_{t+1} = z_t - \rho \nabla_z \mathcal{L}(z_t, x) \) with \( \rho \leq 1/L \) and \( \kappa \triangleq 1 - \mu \). We have \( |z_t - z^*| \leq \kappa^t |z_0 - z^*| \) and \( ||J_t - J^*|| \leq t \kappa^{t-1} \rho L |z_0 - z^*| \) where \( L \) is defined in [9].

Proof. Differentiating the gradient descent recursion, we find that \( J_t \) follows the recursion
\[ J_{t+1} = J_t - \rho G_t, \quad (22) \]
where $G_t = J_t \nabla_{zz} L(z_t, x) + \nabla_{xx} L(z_t, x)$. Using $\|I - \rho \nabla_{zz} L(z, x)\| \leq \kappa$, a first crude upper bounding gives

$$\|J_{t+1}\| \leq \kappa \|J_t\| + \alpha,$$

where $\alpha$ is an upper-bound of $\|\nabla_{xx} L\|$. This shows that $\|J_t\|$ is bounded. Next, denoting $\tilde{G}_t = J_t \nabla_{zz} L(z_t, x) + \nabla_{xx} L(z_t, x)$, we find

$$\Delta_t \triangleq G_t - \tilde{G}_t = J_t (\nabla_{zz} L(z_t, x) - \nabla_{zz} L(z^*, x)) + \nabla_{xx} L(z_t, x) - \nabla_{xx} L(z^*, x)$$

Using the third-order differentiability of $L$, the rate of convergence of $z_t$, and that $J_t$ is bounded, we find that there exists $\beta > 0$ such that $\|\Delta_t\| \leq \beta \kappa t$. Eq. [22] finally gives

$$J_{t+1} - J^* = (I_d - \rho \nabla_{zz} L(z^*, x))(J_t - J^*) - \rho \nabla_{z} L(z^*, x) \Delta_t.$$

Taking norms and using the triangular inequality, we find

$$\|J_{t+1} - J^*\| \leq \kappa \|J_t - J^*\| + \gamma \kappa^t,$$

where $\gamma = \rho \mu \beta$. Unrolling the recursion gives, as expected,

$$\|J_t - J^*\| \leq \gamma t \kappa^{t-1}.$$

### C.2 Tightness of the bound in Proposition 5

Importantly, the rate of $O(t \kappa^t)$ given in Proposition 5 is tight. Indeed, it is reached with in the following example.

**Proposition 15.** For $x \in \mathbb{R}^m$, let $\lambda_x = \sum_{i=1}^m x_i$. Consider $L(z, x) = \frac{1}{2} \lambda_x \|z\|^2$. The iterates produced by gradient descent with step $\rho \leq 1/\lambda_x$ verify $z_t = \kappa^t z_0$ with $\kappa = 1 - \rho \lambda_x$, and we have $J_t = -\rho t \kappa^{t-1} n_{z_0}^\top$.

### C.3 Proof of Proposition 6

**Proposition 6.** Define

$$\delta_t = \mathbb{E}[\|J_t - J^*\|_F^2] \text{ and } d_t = \mathbb{E}[\|z_t - z^*\|^2].$$

We have $\mathbb{E}[\|g^2 - g^*\|] \leq L_z \sqrt{d_t} \sqrt{d_t} + \frac{L}{2} d_t$.

**Proof.** Taking expectations in Proposition 2 and using Equation 12 gives

$$\mathbb{E}[\|g^2 - g^*\|] \leq L_z \mathbb{E}[\|J_t - J^*\|_F \|z_t - z^*\|] + \frac{L}{2} \mathbb{E}[\|z_t - z^*\|^2].$$

Cauchy-Schwarz on the first term gives

$$\mathbb{E}[\|J_t - J^*\|_F \|z_t - z^*\|] \leq \sqrt{\mathbb{E}[\|J_t - J^*\|^2]} \sqrt{d_t},$$

and then $\|J_t - J^*\|^2 \leq \|J_t - J^*\|_F^2$ gives the advertised result.  

\[\square\]
C.4 Proof of Proposition 7

Proposition 7. [Bounding inequality for the Jacobian] We assume bounded Hessian noise, in the sense that $\mathbb{E} \left[ \left\| \nabla_z z(z, x, \xi) \right\|_F^2 \right] \leq \sigma_z^2$ and $\mathbb{E} \left[ \left\| \nabla_x x(z, x, \xi) \right\|_F^2 \right] \leq \sigma_x^2$. Let $r = \min(n, m)$, and $B^2 = \sigma_z^2 + L_j^2 \sigma_x^2$. We have

$$\delta_{t+1} \leq (1 - 2\rho_t \mu) \delta_t + 2\rho_t \sqrt{L} \sqrt{d_t} \delta_t + \rho_t^2 B^2. \tag{15}$$

Proof. Let $U_t = \nabla_z z(z_t, x, \xi_{t+1})J_t + \nabla_x x(z_t, x, \xi_{t+1})$. We have $\mathbb{E}_{\xi_{t+1}}[U_t] = \nabla_z z(z^*, x)(J_t - J^*) + \Delta_t$, where $\Delta_t = (\nabla_z z(z_t, x) - \nabla_z z(z^*, x))J_t + \nabla_x x(z_t, x) - \nabla_x x(z^*, x)$. We find

$$\|J_{t+1} - J^*\|^2_F = \|J_t - J^*\|^2_F - 2\rho_t (J_t - J^*, U_t)_F + \rho_t^2 \|U_t\|^2_F.$$

Taking expectations with respect to $\xi_{t+1}$ yields

$$\mathbb{E}_{\xi_{t+1}}[\|J_{t+1} - J^*\|^2_F] = \|J_t - J^*\|^2_F - 2\rho_t (J_t - J^*, \nabla_z z(z^*, x)(J_t - J^*))_F - 2\rho_t (J_t - J^*, \Delta_t)_F + \rho_t^2 \mathbb{E}_{\xi_{t+1}}[\|U_t\|^2_F].$$

Using strong-convexity for the second term and Cauchy-Schwarz for the third term, we find

$$\mathbb{E}_{\xi_{t+1}}[\|J_{t+1} - J^*\|^2_F] \leq (1 - 2\rho_t \mu) \|J_t - J^*\|^2_F + 2\rho_t \|J_t - J^*\|_F \|\Delta_t\|_F + \rho_t^2 \mathbb{E}_{\xi_{t+1}}[\|U_t\|^2_F].$$

Taking expectations over the whole past

$$\delta_{t+1} \leq (1 - 2\rho_t \mu) \delta_t + 2\rho_t \mathbb{E} \left[ \|J_t - J^*\|_F \|\Delta_t\|_F \right] + \rho_t^2 \mathbb{E} \left[ \|U_t\|^2_F \right].$$

To majorize the last term,

$$\|U_t\|^2_F \leq \|\nabla_z z(z, x, \xi_{t+1})J_t\|^2_F + \|\nabla_x x(z, x, \xi_{t+1})\|^2_F \leq \|J_t\|^2 \|\nabla_z z(z, x, \xi_{t+1})\|_F^2$$

Therefore

$$\mathbb{E} \left[ \|U_t\|^2_F \right] \leq L_j \sigma_z^2 + \sigma_x^2$$

Cauchy-Schwarz on the middle term yields

$$\mathbb{E} \left[ \|J_t - J^*\|_F \|\Delta_t\|_F \right] \leq \sqrt{r} \sqrt{\delta_t} \sqrt{\mathbb{E} \left[ \|\Delta_t\|^2 \right]} \leq \sqrt{r} L \sqrt{\delta_t d_t}.$$ 

Combining everything provides the final bound.

\[\square\]
C.5 Proof of Proposition 8

Proposition 8. [SGD with constant step-size] Assume that the gradients have bounded variance \( \mathbb{E}_\xi [\|
abla_z C(z, x, \xi)\|^2] \leq \sigma^2 \). Assume \( \rho_t = \rho < 1/L_z \), and let \( \kappa_2 = \sqrt{1 - 2\rho\mu} \) and \( \beta = \sqrt{\frac{\sigma^2\rho}{2\mu}} \). In this setting

\[
\delta_t \leq \left( \kappa_2^t (\|J^*\|_F + t\alpha) + B_2 \right)^2 ,
\]

where \( \alpha = \frac{\rho\sqrt{L\beta}}{\kappa_2(1-\kappa_2)} \) and \( B_2 = \frac{\rho B}{(1-\kappa_2)} \).

Proof. We start by obtaining a simpler bound than (15) by completing the squares

\[
(1 - 2\rho t\mu)\delta_t + 2\sqrt{rL}\rho_t\sqrt{d_t}\sqrt{\delta_t} \leq \left( \sqrt{1 - 2\rho t\mu}\sqrt{\delta_t} + \frac{\sqrt{rL}\rho_t}{\sqrt{1 - 2\rho t\mu}}\sqrt{d_t} + \rho t B \right)^2 .
\]

And bounding crudely \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \), we obtain a simple recursion on \( \sqrt{\delta_t} \)

\[
\sqrt{\delta_{t+1}} \leq \sqrt{1 - 2\rho t\mu}\sqrt{\delta_t} + \frac{\sqrt{rL}\rho_t}{\sqrt{1 - 2\rho t\mu}}\sqrt{d_t} + \rho t B . \tag{23}
\]

[Moulines and Bach, 2011] give

\[
\mathbb{E}[|z_t - z^*|^2] \leq (1 - 2\rho t\mu)^t|z_0 - z^*|^2 + \beta^2 .
\]

Using \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \) for \( a, b \geq 0 \), we get

\[
\sqrt{d_t} \leq \kappa_2^t|z_0 - z^*| + \beta
\]

Eq. (23) then gives

\[
\sqrt{\delta_{t+1}} \leq \kappa_2\sqrt{\delta_t} + \rho\kappa_2^t + (1-\kappa_2)B_2
\]

Unrolling the recursion and using \( \sum_{i=0}^{t} \kappa_2^i \leq \frac{1}{1-\kappa_2} \) gives the proposed bound on \( \delta_t \).

C.6 Proof of Proposition 9

Proposition 9. [SGD with decreasing step-size] Assume that \( \rho_t = \rho_0 t^{-\alpha} \) with \( \alpha \in (0, 1) \). Assume a bound on \( d_t \) of the form \( d_t \leq d^2 t^{-\alpha} \). Then

\[
\delta_t \leq 4\rho_0 B_2^2 \mu + rL^2 d^2 t^{-\alpha} + o(t^{-\alpha}) .
\]

Proof. Under the assumptions, Eq. (15) becomes

\[
\delta_{t+1} \leq (1 - 2\mu C t^{-\alpha})\delta_t + 2\sqrt{rLdt^{-\alpha}}\sqrt{\delta_t} + B_2^2 C^2 t^{-2\alpha} .
\]
We get rid of the problematic middle term using the inequality, valid for all \( \chi > 0 \),
\[
t^{-\frac{4\alpha}{2}} \sqrt{\delta_t} \leq \frac{1}{2} \left( \frac{\chi \delta_t}{t^\alpha} + \frac{1}{\chi t^{2\alpha}} \right),
\]
which gives
\[
\delta_{t+1} \leq (1 - (2\mu C - \chi \sqrt{rLdC})t^{-\alpha}) \delta_t + (B^2C^2 + \frac{\sqrt{rLdC}}{\chi})t^{-2\alpha}.
\]
We take \( \chi = \frac{\mu}{\sqrt{rLd}} \), so that the first term becomes
\[
(1 - \mu C t^{-\alpha}) \delta_{t+1} + (B^2C^2 + \frac{\sqrt{rLd}}{\mu})t^{-2\alpha}.
\]
We take \( \chi = \frac{\mu}{\sqrt{rLd}} \), so that the first term becomes
\[
1 - \mu C t^{-\alpha}. \text{ Note that it does not give optimal rates, but makes computations much simpler. In [Moulines and
Bach 2011], it is shown that for } a, b > 0, \text{ a recursion satisfying}
\[
\delta_{t+1} \leq (1 - at^{-\alpha}) \delta_t + bt^{-2\alpha}
\]
verifies \( \delta_t \leq 4 \frac{b}{a} t^{-\alpha} + o(t^{-\alpha}) \). The result follows by taking \( a = \mu C \) and \( b = B^2C^2 + \frac{\sqrt{rLd}}{\mu^2} \).

\section*{C.7 Proof of Proposition 12}

\textbf{Proposition 12.} For \( p \geq 4 \), we assume \( DD^\top = I_n \). Let \( \alpha \triangleq \frac{p-1}{p-2} \). We have
\[
|g_1| = O(t^{-\alpha}), \quad |g_2| = O(t^{-2\alpha}), \quad g_3 = 0.
\]

\textbf{Proof.} Gradient descent iterates
\[
z_{t+1} = z_t - \rho D^\top (D z_t - x)^{p-1}.
\]
The residuals \( r_t = x - D z_t \) therefore verify the recursion
\[
r_{t+1} = r_t - \rho D D^\top r_t^{p-1}.
\]
Since we assume \( DD^\top = I_n \), they verify \( r_{t+1} = r_t - \rho r_t^{p-1} \). Each entry of \( r_t \) therefore evolves independently, following the 1-d recursive equation
\[
u_{t+1} = u_t - \rho u_t^{p-1}
\]
Standard analysis techniques show that this gives \( u_t = \left( \frac{1}{\rho(p-2)t} \right)^{\frac{1}{p-2}} (1 + O(\frac{\log(t)}{t})) \), and therefore each coefficient of \( r_t \) satisfies the same asymptotic development. The Jacobian verifies
\[
J_{t+1} = J_t - (p-1) \rho (J_t D^\top - I_n) \text{ Diag}(r_t^{p-2}) D,
\]
and denoting \( M_t = I_n - J_t D^\top \), we find
\[
M_{t+1} = M_t (I_n - (p-1) \rho \text{ Diag}(r_t^{p-2}))
\]
Since the rightmost term is diagonal, we can rewrite this recursion as:

\[ M_{t+1} = M_t \text{Diag}(1_n - (p-1)\rho r_{t}^{p-2}) \]

Unrolling this recursion gives:

\[ M_t = M_0 \text{Diag}(\prod_{j \leq t-1} (1 - (p-1)\rho u_j^{p-2})) \]

We can then majorize each coefficient in the \(\text{Diag}\) by:

\[ u_t^{p-2} = \frac{1}{\rho(p-2)t} + O(\frac{\log(t)}{t^2}) \]

and as a consequence, denoting \(\alpha = \frac{p-1}{p-2}\):

\[ \exp(\sum_{j \leq t-1} -(p-1)\rho u_j^{p-2}) = O(t^{-\alpha}) \]

Overall, we have \(M_t = O(t^{-\alpha})\) and \(r_t^{p-1} = O(t^{-\alpha})\), so \(g_2 = M_t r_t^{p-1} = O(t^{-2\alpha})\).

\[\Box\]

D Proof of Proposition 14

We start by giving the convergence rate of the SGD with \((\delta, L, \mu)\)-inexact oracle for a function \(f\) defined on \(x \in \mathbb{R}^n\) as

\[ f(x) = \mathbb{E}_v[F(x,v)] \]

for \(v\) a random variable distributed with probability \(d_v\). For \(v_0 \sim d_v\), we denote \((F_{\delta}(\cdot, v_0), G_{\delta}(\cdot, v_0))\) a \((\delta, L, \mu)\)-inexact oracle of \(F(\cdot, v_0)\), uniform in \(v_0\).

**Lemma 1.** For a \(\mu\)-strongly convex \(L\)-smooth function \(f\) and a \((\delta, \mu, L)\)-inexact oracle \((F_\delta, G_\delta)\) of \(F\) such that

\[ \mathbb{E}_v[F_\delta(x,v)] = f_\delta(x) , \quad \mathbb{E}_v[G_\delta(x,v)] = g_\delta(x) , \]

and

\[ \mathbb{E}_v[|G_\delta(x,v) - g_\delta(x)|^2] \leq \sigma^2 . \]

Then, the iterates of the stochastic gradient descent with constant step-size \(\eta < \frac{1}{L}\) verify

\[ \mathbb{E}[x_\eta - x^*]^2 \leq (1 - \eta\mu)^q |x_0 - x^*|^2 + \frac{\eta}{\mu} \sigma^2 + \frac{\delta}{\mu} . \]
Proof. Consider the solution estimate \( x_{q+1} \) at iteration \( q \), obtained through stochastic gradient descent i.e. \( x_{q+1} = x_q - \eta \nabla_x G_\delta(x_q, v_{q+1}) \). We denote \( r_{q+1} = \mathbb{E}[|x_{q+1} - x^*|^2] \) and \( \tilde{r}_{q+1} = \mathbb{E}[|x_{q+1} - x^*|^2|v_{q+1}] \). Then

\[
|x_{q+1} - x^*|^2 \leq |x_q - x^*|^2 - 2\eta \langle G_\delta(x_q, v_{q+1}), x_q - x^* \rangle + \eta^2 |G_\delta(x_q, v_{q+1})|^2
\]

We take the expectation relatively to \( v_{q+1} \)

\[
\tilde{r}_{q+1} \leq |x_q - x^*|^2 - 2\eta \langle g_\delta(x_q), x_q - x^* \rangle + \eta^2 |g_\delta(x_q)|^2 + \eta^2 \mathbb{E}_{v_q}[|G_\delta(x_q, v_q) - g_\delta(x_q)|^2]
\]

\[
\leq |x_q - x^*|^2 + 2\eta \langle g_\delta(x_q), x^* - x_q \rangle + \eta^2 |g_\delta(x_q)|^2 + \eta^2 \sigma^2
\]

Using \( \langle g_\delta| x^* - x_q \rangle \leq f(x^*) - f_\delta(x_q) - \frac{\mu}{2} |x_q - x^*|^2 \), we get

\[
\tilde{r}_{q+1}^2 \leq |x_q - x^*|^2 + 2\eta (f(x^*) - f_\delta(x_q) - \frac{\mu}{2} |x_q - x^*|^2) + \eta^2 |g_\delta(x_q)|^2 + \eta^2 \sigma^2
\]

\[
\leq (1 - \eta \mu) |x_q - x^*|^2 + \eta^2 \sigma^2 + 2\eta |f(x^*) - f_\delta(x_q)| + \frac{\eta}{2} |g_\delta(x_q)|^2
\]

We introduce \( \overline{x} = x_q - \eta g_\delta(x_q) \). Using the rhs of the \( \delta \)-inexact oracle, we have

\[
f(\overline{x}) - f_\delta(x_q) - \langle g_\delta(x_q) | \overline{x} - x_q \rangle \leq \frac{L\eta^2}{2} |g_\delta(x_q)|^2 + \delta
\]

i.e.

\[
f(\overline{x}) - f_\delta(x_q) + \frac{\eta}{2} |g_\delta(x_q)|^2 \leq \delta - (1 - \eta L) \frac{\eta}{2} |g_\delta(x_q)|^2.
\]

Using this in the previous equation, we obtain

\[
r_{q+1}^2 \leq (1 - \eta \mu) |x_q - x^*|^2 + \eta^2 \sigma^2 + 2\eta \delta + 2\eta (f(x^*) - f(\overline{x})) - (1 - \eta L) \eta^2 |g_\delta(x_q)|^2 \leq 0,
\]

where the two terms on the second line are non-positive. Indeed, \( \eta \leq \frac{1}{L} \) and \( f(x^*) \leq f(\overline{x}) \). Taking the expectation relatively to \( v_0, \ldots, v_{q-1} \) gives the following recursion relationship

\[
r_{q+1}^2 \leq (1 - \eta \mu) r_q^2 + \eta^2 \sigma^2 + 2\eta \delta.
\]
applying this recursion $q$ times yields

$$r_{q+1}^2 \leq (1 - \eta \mu)^q r_0^2 + (\eta^2 \sigma^2 + 2\eta \delta) \sum_{k=0}^{q} (1 - \eta \mu)^k ,$$

and we obtain the desired results as $\sum_{k=0}^{q} (1 - \eta \mu) < \frac{1}{\eta \mu}$.

**Proposition 14.** We assume that $H$ is $\mu_x$-strongly convex, $L_x$-smooth and verifies

$$\mathbb{E}[|\nabla_x h(x, v) - \nabla_x \ell(x)|^2] \leq \sigma^2.$$ 

The iterates $x_q$ of SGD with approximate gradient $g^i$ and step-size $\eta$ verify

$$\mathbb{E}[x_q - x^*]^2 \leq (1 - \frac{\eta \mu_x}{2})^q |x_0 - x^*| + \frac{2\eta}{\mu_x} \sigma^2 + \frac{4}{\mu_x} \delta_i$$

with $\delta_i = \Delta_i^2(\frac{1}{\mu_x} + \frac{1}{2L_x} + 2\eta)$.

**Proof.** As shown in subsection 4.1, $(\ell, g^i)$ is a $(\tilde{\delta}_i, \frac{\mu_x}{2}, 2L_x)$-inexact oracle with $\tilde{\delta}_i = \Delta_i^2(\frac{1}{\mu_x} + \frac{1}{2L_x})$.

The variance of the stochastic inexact oracles can be bounded as follow

$$\mathbb{E}[|g^i(x, v) - g^i(x)|^2] \leq 2(\mathbb{E}[|g^i(x, v) - \nabla h(x, v)|^2$$

$$+ |\nabla h(x, v) - \nabla \ell(x)|^2$$

$$+ |\nabla \ell(x) - g^i(x)|^2) \leq 2(\sigma^2 + 2\Delta_i^2)$$

Using these two bounds with the previous result yield

$$\mathbb{E}[x_q - x^*]^2 \leq (1 - \frac{\eta \mu}{4})^q |x_0 - x^*| + \frac{4\eta}{\mu} \sigma^2 + \frac{2}{\mu} \Delta_i^2$$

$$\leq (1 - \frac{\eta \mu}{4})^q |x_0 - x^*| + \frac{4\eta}{\mu} \sigma^2 + \frac{2}{\mu} \delta_i$$

with $\delta = \Delta_i^2(\frac{1}{\mu} + \frac{1}{\mu} + 2\eta)$.