Exponential bases on multi-rectangles of $\mathbb{R}^d$

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Abstract. We produce explicit exponential bases on finite union of disjoint rectangles in $\mathbb{R}^d$ with rational vertices. The proof of our main result relies on the semigroup properties and precise norm estimates of a remarkable family of linear operators on $\ell^2(\mathbb{Z}^d)$ that generalize the discrete Hilbert transform.

1. Introduction

The main purpose of this paper is to construct explicit Riesz bases made of exponential functions (or: exponential bases) on multi-rectangles in $\mathbb{R}^d$ with rational vertices. By multi-rectangle we mean a finite union of disjoint rectangles in the form of $\prod_{j=1}^d [a_j, b_j)$, with $-\infty < a_j < b_j < \infty$. In this paper we assume $a_j, b_j \in \mathbb{Q}$. We may denote with multi-interval a finite union of disjoint intervals in $\mathbb{R}$, i.e., a multi-rectangle in dimension $d = 1$. We have stated the definition of Riesz basis and other preliminary results in Section 2.

Exponential bases are known to exist on any multi-interval of $\mathbb{R}$ and on any multi-rectangle in $\mathbb{R}^d$. See the recent [15] and [16], and see also [25] and [6]. In these papers no explicit exponential bases are produced.

Since we consider only multi-rectangles with vertices in $\mathbb{Q}^d$, after perhaps a translation and dilation of coordinates (see Corollary 5.2) we can restrict our attention to multi-rectangles with vertices in $(\frac{1}{2} + \mathbb{Z})^d$, the lattice of points in $\mathbb{R}^d$ with half-integer coordinates. Such multi-rectangles can be written as disjoint union of translated of the unit cube $Q_0 = [-\frac{1}{2}, \frac{1}{2})^d$. We let

\begin{equation}
Q = Q(\vec{M}_1, ..., \vec{M}_N) = \bigcup_{p=1}^N Q_0 + \vec{M}_p,
\end{equation}

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where we assume \( \vec{M}_p \in \mathbb{Z}^d \) and \( \vec{M}_p \neq \vec{M}_q \) if \( p \neq q \). We find exponential bases of \( L^2(Q) \) in the form of

\[
(1.2) \quad \mathcal{B} = \mathcal{B}(\vec{\delta}_1, ..., \vec{\delta}_N) = \bigcup_{j=1}^{N} \{ e^{2\pi i \langle \vec{n} + \vec{\delta}_j, x \rangle} \}_{\vec{n} \in \mathbb{Z}^d}
\]

where \( \vec{\delta}_p = (\delta_{p,1}, ..., \delta_{p,d}) \in \mathbb{R}^d \). Each set \( \{ e^{2\pi i \langle \vec{n} + \vec{\delta}_j, x \rangle} \}_{\vec{n} \in \mathbb{Z}^d} \) is an orthonormal exponential basis on any unit cube of \( Q \). The idea of combining exponential bases on individual rectangles to form a basis on their disjoint union is not new. See e.g. the introduction of [15], and [16] and [24, Sect. 4]. Our main result is the following

**Theorem 1.1.** \( \mathcal{B} \) is a Riesz basis of \( L^2(Q) \) if and only if the matrix

\[
(1.3) \quad \Gamma = \Gamma((\vec{M}_p, \vec{\delta}_j)) = \{ e^{2\pi i \langle \vec{\delta}_j, \vec{M}_p \rangle} \}_{1 \leq j, p \leq N}
\]

is nonsingular. The optimal frame constants of \( \mathcal{B} \) are the maximum and minimum singular value of \( \Gamma \).

We recall that the singular values of a matrix \( M \) are the eigenvalues of \( M^* M \), where \( M^* \) is the conjugate transpose of \( M \).

The optimal frame constants of \( \mathcal{B} \) can be explicitly evaluated when \( N = 2 \) (see Corollary 5.4); the case \( N > 2 \) is discussed in Section 6.

Exponential bases or frames on multi-intervals of the real line have been investigated by several authors. In addition to [15], see also [2], and [24, Sect. 4] and the references cited in these papers. The case \( d = 1 \) of Theorem 1.1 extends the main theorem and Remark 4 in [2].

In general, it is not true that an exponential frame contains a Riesz basis or that an exponential Riesz sequences can be completed to a Riesz basis (see e.g. [30]). But when \( Q = Q(\vec{M}_1, ..., \vec{M}_N) \) is as in (1.1), sets of exponentials \( \mathcal{B} = \mathcal{B}(\vec{\delta}_1, ..., \vec{\delta}_N) \) as in (1.2) enjoy the following remarkable properties.

**Theorem 1.2.** The following are equivalent:
- \( \mathcal{B} \) is a Riesz sequence in \( L^2(Q) \)
- \( \mathcal{B} \) is a frame
- \( \mathcal{B} \) is a Riesz basis.

**Theorem 1.3.** The set

\[
\mathcal{L} = \{ (\vec{\delta}_1, ..., \vec{\delta}_N) \in (\mathbb{R}^d)^N : \mathcal{B}(\vec{\delta}_1, ..., \vec{\delta}_N) \text{ is not a Riesz basis of } L^2(Q) \}
\]

has Lebesgue measure zero.

The following is a special and significant case of Theorem 1.1.

**Theorem 1.4.** Let \( \vec{\delta} \in \mathbb{R}^d \) be fixed; the set

\[
(1.4) \quad \mathcal{S}(\vec{\delta}) = \bigcup_{j=1}^{N} \{ e^{2\pi i \langle \vec{n} + (j-1)\vec{\delta}, \vec{x} \rangle} \}_{\vec{n} \in \mathbb{Z}^d}.
\]
is a Riesz basis of $L^2(Q)$ if and only if
\begin{equation}
\langle \tilde{M}_p - \tilde{M}_q, \tilde{\delta} \rangle \notin \mathbb{Z}, \quad \text{for every } 1 \leq p \neq q \leq N.
\end{equation}

The frame constants of $S(\tilde{\delta})$ are the maximum and minimum eigenvalue of the matrix $\tilde{\mathbf{B}} = \{ \tilde{\beta}_{p,q} \}_{1 \leq p,q \leq N}$, where
\begin{equation}
\tilde{\beta}_{p,q} = \begin{cases} 
\frac{\sin(\pi N \langle \tilde{\delta}, \tilde{M}_q - \tilde{M}_p \rangle)}{\sin(\pi \langle \tilde{\delta}, \tilde{M}_q - \tilde{M}_p \rangle)} & \text{if } p \neq q, \\
N & \text{if } p = q 
\end{cases}
\end{equation}

The proof of Theorem 1.1 relies on the semigroup properties and precise norm estimates of a remarkable family of linear operators on $\ell^2(\mathbb{Z}^d)$ that we discuss in section 3.

The rest of the paper is organized as follows. In Section 4 we prove our main results. We have collected a number of corollaries and examples in Section 5. In Section 6 we estimate the frame constants of $B$ and $S(\tilde{\delta})$.

2. Preliminaries

2.1. Bases and frames. We have used [20] for standard linear algebra results and the excellent textbook [11] for definitions and properties of bases and frames in Hilbert spaces. See also [4] and [32].

A sequence of vectors $V = \{ v_j \}_{j \in \mathbb{N}}$ in a separable Hilbert space $H$ is a frame if there exist constants $A, B > 0$ such that for every $w \in H$,
\begin{equation}
A \|w\|^2 \leq \sum_{j=1}^{\infty} |\langle w, v_j \rangle|^2 \leq B \|w\|^2.
\end{equation}
Here, $\langle \cdot, \cdot \rangle$ and $\| \cdot \| = \sqrt{\langle \cdot, \cdot \rangle}$ are the inner product and the norm in $H$. The sequence $V$ is a tight frame if $A = B$, it is a Parseval frame if $A = B = 1$, and a Riesz sequence if the following inequality is satisfied for all finite sequences $\{a_j\}_{j \in J} \subset \mathbb{C}$.
\begin{equation}
A \sum_{j \in J} |a_j|^2 \leq \left\| \sum_{j \in J} a_j v_j \right\|^2 \leq B \sum_{j \in J} |a_j|^2.
\end{equation}
A Riesz basis is a frame and a Riesz sequence, i.e., a set of vectors that satisfies (2.1) and (2.2).

If $H = L^2(D)$, with $D \subset \mathbb{R}^d$ of finite Lebesgue measure $|D|$, Riesz bases (or frames) made of exponential functions are especially relevant in the applications. An exponential basis of $L^2(D)$ is a Riesz basis in the form of $E(\Lambda) = \{ e^{2\pi i \langle \lambda, x \rangle} \}_{\lambda \in \Lambda}$, where $\Lambda$ is a discrete set of $\mathbb{R}^d$. Exponential bases are important to provide unique and stable representation of functions in $L^2(D)$ in terms of the functions in $E(\Lambda)$, with coefficients that are easy to calculate. Unfortunately, our understanding of exponential bases is still very incomplete. There are very few examples of domains in which it is known
how to construct exponential bases, and no example of domain for which
exponential bases are known not to exist. See [9], [14] and the references
cited there.

Because frames are not necessarily linearly independent, they are often
more easily constructible than bases. For example, when \( D \subset Q_0 = [-\frac{1}{2}, \frac{1}{2}]^d \)
we can easily verify that \( E(Z^d) \) is a Parseval frame on \( D \). See also [6, Prop.
2.1]. The construction of exponential frames on unbounded sets of finite
measure is a difficult problem that has been recently solved in [27].

2.2. Exponential bases on \( L^2(Q_0) \). It is proved in [23] that \( E(\Lambda) \) is an
orthonormal basis on \( L^2(Q_0) \) if and only if the sets \( \{Q_0 + \lambda\}_{\lambda \in \Lambda} \) tiles \( \mathbb{R}^d \),
that is, if \( \bigcup_{\lambda \in \Lambda} (Q_0 + \lambda) = \mathbb{R}^d \) and \( |(Q_0 + \lambda_j) \cap (Q_0 + \lambda_i)| = 0 \) whenever \( \lambda_i \neq \lambda_j \).

A bounded domain \( D \subset \mathbb{R}^d \) is called spectral if \( L^2(D) \) is has an orthogo-
nal exponential basis. The connection between tiling and spectral properties
of domains of \( \mathbb{R}^d \) is deep and fascinating and has spur intense investigation
since when B. Fuglede formulated his famous tiling \( \iff \) spectral conjecture
in [7]. See also [13] and the references cited there.

Non-orthogonal exponential bases in \( L^2(-\frac{1}{2}, \frac{1}{2}) \) were first investigated
by Paley and Wiener [28] and Levinson [22] and extensively studied by sev-
eral other authors. A complete characterization of exponential bases on
\( L^2(-\frac{1}{2}, \frac{1}{2}) \) was given by Pavlov in [29]. It is proved in [31, Lemma 2.1] that
if \( \Lambda = \{(\lambda_{n_1}, \ldots, \lambda_{n_d})\}_{(n_1,\ldots,n_d) \in \mathbb{Z}^d} \), the set \( E(\Lambda) \) is an exponential basis on
\( Q_0 = [-\frac{1}{2}, \frac{1}{2}]^d \) if and only if the sets \( \{e^{2\pi i \lambda_n x}\}_{n \in \mathbb{Z}} \) are exponential bases
on \( [-\frac{1}{2}, \frac{1}{2}]^d \). To the best of our knowledge, no complete characterization of
exponential bases on \( L^2(Q_0) \) exists in the literature.

2.3. Stability of Riesz bases. Riesz bases are stable, in the sense that
a small perturbation of a Riesz basis produces a Riesz basis. The celebrated
Kadec stability theorem states that any set \( \{e^{2\pi i \lambda_n x}\}_{n \in \mathbb{Z}} \) is Riesz basis of
\( L^2(-\frac{1}{2}, \frac{1}{2}) \) if \( |\lambda_n - n| \leq L < \frac{1}{4} \) whenever \( n \in \mathbb{Z} \). In [22] it is shown that the constant \( \frac{1}{4} \)
cannot be replaced by any larger constant. See [12] or [32] for a proof of Kadec theorem.

The following multi-dimensional generalization of Kadec theorem is in
[31].

\textbf{Theorem 2.1. Let} \( \Lambda = \{\lambda_{\vec{n}} = (\lambda_{\vec{n},1}, \ldots, \lambda_{\vec{n},d}) \in \mathbb{R}^d \}_{\vec{n} \in \mathbb{Z}^d} \) \textbf{be a sequence}
in \( \mathbb{R}^d \) for which \( ||\vec{\lambda}_{\vec{n}} - \vec{n}||_\infty = \sup_{1 \leq j \leq d} |\lambda_{\vec{n},j} - n_j| \leq L < \frac{1}{4} \) whenever \( \vec{n} \in \mathbb{Z}^d \).

\textbf{Then,} \( E(\Lambda) \) \textbf{is an exponential basis of} \( L^2(Q_0) \) \textbf{and the constant} \( \frac{1}{4} \) \textbf{cannot be replaced by any larger constant.}

2.4. Three useful lemmas.

\textbf{Lemma 2.2. Let} \( \mathcal{V} \) \textbf{be a Riesz basis in} \( H \); \textbf{the optimal constants} \( A \) \textbf{and}
\( B \) \textbf{in the inequalities} (2.1) \textbf{and} (2.2) \textbf{are the same.}
The following is Lemma 3 in [16].

**Lemma 2.3.** \( E(\Lambda) \subset E(\mathbb{Z}^d) \) is a frame on a domain \( D \subset Q_0 \) if and only if \( E(\mathbb{Z}^d - \Lambda) \) is a Riesz sequence on \( Q_0 - D \).

The following is Proposition 3.2.8 in [4]

**Lemma 2.4.** A sequence of unit vectors \( \mathcal{V} \subset H \) is a Parseval frame if and only if it is an orthonormal Riesz basis.

From Lemma 2.4 follows that an exponential basis of \( L^2(D) \) is orthogonal if and only if it is a tight frame of \( L^2(D) \) with frame constant \(|D|\).

### 3. Families of isometries in \( \ell^2(\mathbb{Z}^d) \)

The proof of Theorem 1.1 in dimension \( d = 1 \) lead G. Shaikh Samad and the author of this paper to the investigation of a one-parameter family of operators \( \{T_t\}_{t \in \mathbb{R}} \) defined in \( \ell^2 = \ell^2(\mathbb{Z}) \) as follows:

\[
(T_t(\tilde{a}))_m = \begin{cases} \frac{\sin(\pi t)}{\pi} \sum_{n \in \mathbb{Z}} \frac{a_n}{m - n + t} & \text{if } t \notin \mathbb{Z} \\ (-1)^t a_{m+t} & \text{if } t \in \mathbb{Z}. \end{cases}
\]

See [5]. When \( t \) is not an integer (and in particular when \( t \in (-1, 1) \)), these operators can be viewed as discrete versions of the Hilbert transform in \( L^2(\mathbb{R}) \) (see [19] and the reference therein). The main result in [5] is the following:

**Theorem 3.1.** The family \( \{T_t\}_{t \in \mathbb{R}} \) defined in (3.1) is a strongly continuous Abelian group of isometries in \( \ell^2 \); its infinitesimal generator is \( \pi H \), where \( (H(\tilde{a}))_m = \frac{1}{\pi} \sum_{n \neq m} \frac{a_n}{m - n} \) is the discrete Hilbert transform.

That is, we proved that \( T_s \circ T_t = T_{s+t} \); that for every \( \tilde{a} \in \ell^2 \), the application \( t \to T_t(\tilde{a}) \) is continuous in \( \mathbb{R} \); that \( ||T_t(\tilde{a})||_{\ell^2} = ||\tilde{a}||_{\ell^2} \); and finally that, for every \( \tilde{a} \in \ell^2 \), \( \lim_{t \to 0} \frac{T_t(\tilde{a}) - \tilde{a}}{t} = \pi H(\tilde{a}) \), where the limit is in \( \ell^2 \).

Furthermore, \( T_t^* = T_{-t} = T_t^{-1} \) (see Lemma 4.4 in [5]).

In this paper we define multi-variable versions of the operators \( T_t \). We let

\[
\ell^2(\mathbb{Z}^d) = \{ \tilde{a} = (a_{\tilde{n}})_{\tilde{n} \in \mathbb{Z}^d} : ||\tilde{a}||_{\ell^2(\mathbb{Z}^d)}^2 = \sum_{\tilde{n} \in \mathbb{Z}^d} |a_{\tilde{n}}|^2 < \infty \}.
\]

We denote with \( \tilde{e}_j \) the vector in \( \mathbb{R}^d \) whose components are all \( 0 \), with the exception of the \( j \)-th component which is \( 1 \).

Let \( j \leq d \) and \( s \in \mathbb{R} \); we define the operator \( T_{s\tilde{e}_j} : \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d) \) as follows:

\[
(T_{s\tilde{e}_j}(\tilde{a}))_{\tilde{n}} = \begin{cases} \frac{\sin(\pi s)}{\pi} \sum_{m \in \mathbb{Z}} \frac{a_{(n_1, \ldots, n_{j-1}, m, n_{j+1}, \ldots, n_d)}}{n_j - m + s} & \text{if } s \notin \mathbb{Z} \\ (-1)^s a_{(n_1, \ldots, n_j + s, \ldots, n_d)} & \text{if } s \in \mathbb{Z}. \end{cases}
\]

**Proof.** Follows from [4, Proposition 3.5.5]. \( \square \)
For a given \( \vec{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d \), we let \( T_{\vec{t}} = T_{t_1 \vec{e}_1} \circ \cdots \circ T_{t_d \vec{e}_d} \).

For example, when \( \vec{t} \in (0,1)^d \) we can easily verify that \( (T_{\vec{t}}(\vec{a}))_{\vec{m}} = \prod_{j=1}^d \sin(\pi t_j) \sum_{\vec{n} \in \mathbb{Z}^d} \frac{a_{\vec{n}}}{\prod_{j=1}^d (m_j - n_j + t_j)} \).

The following multi-dimensional version of Theorem 3.1 and Lemma 4.4 in [5] are easy to prove.

**Theorem 3.2.** For every \( \vec{a}, \vec{t} \in \mathbb{R}^d \) and every \( \vec{a} \in \ell^2(\mathbb{Z}^d) \),
\[
T_{\vec{t}} \circ T_{\vec{a}}(\vec{a}) = T_{\vec{t}} \circ T_{\vec{a}}(\vec{a}) = T_{\vec{t} + \vec{a}}(\vec{a});
\]

furthermore
\[
T_{\vec{a}}^{-1}(\vec{a}) = T_{-\vec{a}}(\vec{a}),
\]

and
\[
\|T_{\vec{a}}(\vec{a})\|_{\ell^2(\mathbb{Z}^d)} = \|\vec{a}\|_{\ell^2(\mathbb{Z}^d)}.
\]

b) Let \( T_{\vec{t}}^* \) be the adjoint of \( T_{\vec{t}} \). Then, for every \( \vec{t} \in \mathbb{R}^d \) and every \( \vec{a}, \vec{b} \in \ell^2(\mathbb{Z}^d) \), we have that
\[
T_{\vec{t}}^*(\vec{a}) = T_{-\vec{t}}(\vec{a})
\]

and
\[
\langle T_{\vec{t}}(\vec{a}), T_{\vec{t}}^*(\vec{b}) \rangle = \langle \vec{a}, \vec{b} \rangle.
\]

4. Proofs

Let \( Q = Q_1 \cup \cdots \cup Q_N \) be as in (1.1). By (2.1) and (2.2), the set
\[
B = \bigcup_{p=1}^N \{ e^{2\pi i \vec{s}(\vec{a} + \vec{b})} \mid \vec{n} \in \mathbb{Z}^d \}
\]
is a Riesz basis in \( L^2(Q) \) if and only if there exists \( A, B > 0 \) for which the following hold:
\[
A\|f\|^2_{L^2(Q)} \leq \sum_{j=1}^N \sum_{\vec{n} \in \mathbb{Z}^d} \langle f, e^{2\pi i (\vec{n} + \vec{b}, \vec{x})} \rangle^2_{L^2(Q)} \leq B\|f\|^2_{L^2(Q)}
\]
whenever \( f \in L^2(Q) \), and
\[
A \sum_{j=1}^N \sum_{\vec{n} \in \mathbb{Z}^d} |a_{j,\vec{n}}|^2 \leq \left\| \sum_{j=1}^N \sum_{\vec{n} \in \mathbb{Z}^d} a_{j,\vec{n}} e^{2\pi i (\vec{n} + \vec{b}, \vec{x})} \right\|^2_{L^2(Q)} \leq B \sum_{j=1}^N \sum_{\vec{n} \in \mathbb{Z}^d} |a_{j,\vec{n}}|^2
\]
for every finite set of coefficients \( \{ a_{j,\vec{n}} \} \).
\[ \mathbf{\alpha} = \begin{pmatrix} -1 \end{pmatrix}^{n_1+\ldots+n_d} e^{2\pi i (\bar{n}, \bar{M})} a_{\bar{n}} \bar{n} \bar{\mathbf{\alpha}} = \begin{pmatrix} -1 \end{pmatrix}^{m_1+\ldots+m_d} e^{2\pi i (\bar{n}, \bar{M})} b_m \bar{m} \bar{\mathbf{\alpha}}. \]

**Proof.** Assume \( d = 1 \) since the proof for \( d \geq 1 \) is similar. Assume first \( s - t \notin \mathbb{Z} \); We have

\[ \langle \sum_{n \in \mathbb{Z}} a_n e^{2\pi i (n+s)x}, \sum_{m \in \mathbb{Z}} b_m e^{2\pi i (m+t)x} \rangle_{L^2(M^{-1}, M+1)} = \sum_{n,m \in \mathbb{Z}} a_n \bar{b}_m \int_{M^{1/2}}^{M+1/2} e^{2\pi i (n-m-s-t)x} dx \]

\[ = e^{2\pi i (s-t)M} \sum_{n,m \in \mathbb{Z}} e^{2\pi i (n-m)M} a_n \bar{b}_m \frac{\sin(\pi(n-m+s-t))}{\pi(n-m+s-t)} \]

\[ = e^{2\pi i (s-t)M} \langle \mathbf{T}_{s-t}(\mathbf{\alpha}), \mathbf{\beta} \rangle_{L^2(\mathbb{Z})}. \]

The identity (3.3) in Theorem 3.2 yields (4.3).

When \( s - t \in \mathbb{Z} \), the integral in (4.4) equals 1 if \( m = n + s - t \) and is \( = 0 \) in all other cases. We can write

\[ (4.4) = \sum_{n \in \mathbb{Z}} a_n \bar{b}_{n+s-t} = (-1)^{s-t} e^{2\pi i (s-t)M} \sum_{n \in \mathbb{Z}} \alpha_n \bar{\beta}_{n+s-t} \]

\[ = (-1)^{s-t} e^{2\pi i (s-t)M} \langle \mathbf{T}_{s-t}(-\mathbf{\alpha}), \mathbf{T}_{s-t}(\mathbf{\beta}) \rangle. \]

as required. \( \sqcup \)

### 4.1. Proof of Theorem 1.1

Let \( \mathbf{\Gamma} = \mathbf{\Gamma}((\bar{\mathbf{M}}_p, \bar{\mathbf{\delta}}_j)) \) be as in (1.3). We show that (4.1) holds if and only if \( \det \mathbf{\Gamma} \neq 0 \). Let \( f \in L^2(Q) \). For every \( j \leq N \), we obtain

\[ \sum_{\bar{n} \in \mathbb{Z}^d} \langle f, e^{2\pi i (\bar{n}+\bar{\delta}_j, x)} \rangle^2_{L^2(Q)} = \sum_{\bar{n} \in \mathbb{Z}^d} \sum_{p=1}^{N} \langle f, e^{2\pi i (\bar{n}+\bar{\delta}_j, x)} \rangle_{L^2(Q_0+\bar{\mathbf{M}}_p)}^2 \]

\[ = \sum_{\bar{n} \in \mathbb{Z}^d} \sum_{p,q=1}^{N} \langle f, e^{2\pi i (\bar{n}+\bar{\delta}_j, x)} \rangle_{L^2(Q_0+\bar{\mathbf{M}}_p)} \langle f, e^{2\pi i (\bar{n}+\bar{\delta}_j, x)} \rangle_{L^2(Q_0+\bar{\mathbf{M}}_q)} . \]

Observe that

\[ \langle f, e^{2\pi i (\bar{n}+\bar{\delta}_j, x)} \rangle_{L^2(Q_0+\bar{\mathbf{M}}_p)} = \int_{Q_0+\bar{\mathbf{M}}_p} f(x) e^{-2\pi i (\bar{n}+\bar{\delta}_j, x)} dx \]

\[ = \int_{Q_0} f(y+\bar{\mathbf{M}}_p) e^{-2\pi i (\bar{n}+\bar{\delta}_j, y+\bar{\mathbf{M}}_p)} dy \]
Λ = \max_x \gamma \in \mathbb{B}
\text{the unit sphere of (4.8)}

and from (4.5) and (4.7) follow that
\( \Lambda = \Gamma \) (4.9)
where we have let

\( \sum_{n \in \mathbb{Z}^d} \)

Similarly, we prove that
\[ \sum_{j=1}^{N} \left| \langle f, e^{2\pi i (n+\delta_j, x)} \rangle_{L^2(Q_0)} \right|^2 = \sum_{p,q=1}^{N} \beta_{p,q} \langle \tau_{\tilde{M}_p} f, \tau_{\tilde{M}_q} f \rangle_{L^2(Q_0)} \]

where we have let
\[ \beta_{p,q} = \sum_{j=1}^{N} e^{2\pi i (\delta_j, \tilde{M}_q - \tilde{M}_p)}, \quad 1 \leq p, q \leq N. \]

Let \( B \) be the \( N \times N \) matrix whose elements are the \( \beta_{p,q} \). We can let \( \gamma_{j,p} = e^{2\pi i (\delta_j, \tilde{M}_p)} \) and write \( \beta_{p,q} = \sum_{j=1}^{N} \gamma_{j,p} \gamma_{j,q} \); thus, \( B = \Gamma \Gamma^* \) where \( \Gamma = \{ \gamma_{j,p} \}_{1 \leq j, p \leq N} \) is as in (1.3) and \( \Gamma^* \) is the conjugate transpose of \( \Gamma \).

The maximum (minimum) value of the Hermitian form \( \langle B \tilde{v}, \tilde{v} \rangle \) on \( \mathbb{C}^N \), the unit sphere of \( \mathbb{C}^N \), equal the maximum (minimum) eigenvalue of \( B \). Let \( \Lambda = \max_{x \in \mathbb{S}^{N-1}} \langle B x, x \rangle \), and \( \lambda = \min_{x \in \mathbb{S}^{N-1}} \langle B x, x \rangle \). We have
\[ \sum_{p,q=1}^{N} \beta_{p,q} \langle \tau_{\tilde{M}_q} f, \tau_{\tilde{M}_p} f \rangle_{L^2(Q_0)} = \int_{Q_0} \sum_{p,q=1}^{N} \beta_{p,q} f(x + \tilde{M}_p) f(x + \tilde{M}_q) dx \]
\[ \leq \Lambda \int_{Q_0} \sum_{p=1}^{N} |f(x + \tilde{M}_p)|^2 dx = \Lambda \sum_{p=1}^{N} \int_{Q_0 + \tilde{M}_p} |f(x)|^2 dx = \Lambda ||f||^2_{L^2(Q)}. \]

Similarly, we prove that
\[ \sum_{p,q=1}^{N} \beta_{p,q} \langle \tau_{\tilde{M}_q} f, \tau_{\tilde{M}_p} f \rangle_{L^2(Q_0)} \geq |\lambda||f||^2_{L^2(Q)} \]
from (4.8) and the inequalities above follows (4.1), with \( A = \lambda \) and \( B = \Lambda \).

We prove that \( A = \lambda \) and \( B = \Lambda \) are the optimal frame bounds of \( B \).

Let \( \tilde{w} = (w_1, ..., w_d) \) be an eigenvector of \( B \) with eigenvalue \( \Lambda \); let \( g(x) = \sum_{q=1}^{N} w_q \chi_{Q_0 - \tilde{M}_q}(x) \), where \( \chi \) is the characteristic function. The identity
(4.8) yields
\[
\sum_{j=1}^{N} \sum_{\tilde{n} \in \mathbb{Z}^d} |\langle g, e^{2\pi i (\tilde{n} + \tilde{\delta}_j, x) \rangle \rangle_{L^2(Q)}|^2 = \int_{Q_0} \sum_{p,q=1}^{N} \beta_{p,q} \tau_{\tilde{M}_p} g(x) \tau_{\tilde{M}_q} g(x) \, dx
\]
\[
= \sum_{p,q=1}^{N} \sum_{h,k=1}^{N} \beta_{p,q} \omega_k \tilde{\omega}_k \int_{Q_0} \chi_{Q_0-\tilde{M}_h} (x + \tilde{M}_p) \chi_{Q_0-\tilde{M}_k} (x + \tilde{M}_q) \, dx
\]
\[
= \sum_{p,q=1}^{N} \beta_{p,q} \omega_p \tilde{\omega}_q = \Lambda \sum_{p=1}^{N} |w_p|^2 = \Lambda \|g\|_{L^2(Q)}^2.
\]

A similar argument shows that \( A = \lambda \) is the optimal lower frame bound of \( B \).

Let us prove that (4.2) holds if and only if \( \det \Gamma \neq 0 \). Let \( \tilde{a}_1, \ldots, \tilde{a}_N \in \ell^2(\mathbb{Z}^d) \) be such that \( a_{j,\tilde{n}} = 0 \) for all but a finite set of \( \tilde{n} \in \mathbb{Z}^d \). Let \( S_j = \sum_{\tilde{n} \in \mathbb{Z}^d} a_{j,\tilde{n}} e^{2\pi i (\tilde{n} + \tilde{\delta}_j, x)} \). The central sum in (4.2) equals \( \|S_1 + \ldots + S_N\|_{L^2(Q)}^2 \).

By Lemma 4.1
\[
\|S_1 + \ldots + S_N\|_{L^2(Q)}^2 = \sum_{1 \leq i,j \leq N} \langle S_i, S_j \rangle_{L^2(Q)}
\]
\[
= \sum_{1 \leq i,j \leq N} \sum_{p=1}^{N} \langle S_i, S_j \rangle_{L^2(Q_0 + \tilde{M}_p)}
\]
\[
= \alpha_{i,j} \langle T_{\tilde{\delta}_i} (\tilde{a}_i), T_{\tilde{\delta}_j} (\tilde{a}_j) \rangle_{L^2(\mathbb{Z}^d)}
\]
(4.10)

where we have let
\[
\alpha_{i,j} = \sum_{p=1}^{N} e^{2\pi i (\tilde{\delta}_i - \tilde{\delta}_j, \tilde{M}_p)}
\]

Let \( A \) be the \( N \times N \) matrix whose elements are the \( \alpha_{i,j} \). It is easy to verify that \( A = \Gamma \Gamma^* \), where \( \Gamma \) is as in (1.3). The matrices \( A = \Gamma \Gamma^* \) and \( B = \Gamma^* \Gamma \) have the same eigenvalues (see Lemma 4.2 below) and for every \( \tilde{v} \in \mathbb{C}^N \),

\[
\lambda \|\tilde{v}\|^2 \leq \sum_{i,j=1}^{N} \alpha_{i,j} v_i \overline{v}_j \leq \Lambda \|\tilde{v}\|^2
\]
(4.12)

where \( \Lambda \) and \( \lambda \) are as in the first part of the proof.

Let \( T_{\tilde{\delta}_k} (\tilde{a}_k) = \tilde{b}_k \). In view of (4.10),

\[
\|S_1 + \ldots + S_N\|_{L^2(Q)}^2 = \sum_{i,j=1}^{N} \alpha_{i,j} \langle \tilde{b}_i, \tilde{b}_j \rangle_{\ell^2(\mathbb{Z}^d)} = \sum_{\tilde{n} \in \mathbb{Z}^d} \sum_{i,j=1}^{N} \alpha_{i,j} b_{i,\tilde{n}} \overline{b}_{j,\tilde{n}}.
\]
(4.13)
Fix \( \vec{n} \in \mathbb{Z}^d \); by (4.12),

\[
\lambda (|b_{1,\vec{n}}|^2 + \ldots + |b_{N,\vec{n}}|^2) \leq \sum_{i,j=1}^{N} \alpha_{i,j} b_{i,\vec{n}} b_{j,\vec{n}} \leq \Lambda (|b_{1,\vec{n}}|^2 + \ldots + |b_{N,\vec{n}}|^2)
\]

and when we sum with respect to \( \vec{n} \) we obtain

\[
(4.14) \quad \lambda \sum_{j=1}^{N} ||\vec{b}_j||^2_{\ell^2(\mathbb{Z}^d)} \leq \sum_{\vec{n} \in \mathbb{Z}^d} \sum_{i,j=1}^{N} \alpha_{i,j} b_{i,\vec{n}} b_{j,\vec{n}} \leq \Lambda \sum_{j=1}^{N} ||\vec{b}_j||^2_{\ell^2(\mathbb{Z}^d)}.
\]

Recalling that \( \vec{b}_j = T_{\vec{\delta}_j} (\vec{a}_j) \) and that the \( T_{\vec{\delta}_j} \) are invertible isometries, we obtain \( ||\vec{b}_j||_{\ell^2(\mathbb{Z}^d)} = ||\vec{a}_j||_{\ell^2(\mathbb{Z}^d)} \). From (4.13) and (4.14) follows that

\[
\lambda \sum_{j=1}^{N} ||\vec{a}_j||^2_{\ell^2(\mathbb{Z}^d)} \leq ||S1 + \ldots + S_N||^2_{L^2(Q)} \leq \Lambda \sum_{j=1}^{N} ||\vec{a}_j||^2_{\ell^2(\mathbb{Z}^d)},
\]

and (4.2) is proved with \( B = \Lambda \) and \( A = \lambda \). By Lemma 2.2, \( \Lambda \) and \( \lambda \) are the optimal constants in (4.2). The proof of Theorem 1.1 is completed. \( \square \)

**Remark.** The proof of Theorem 1.1 shows that \( B \) is a frame \( \iff \det B \neq 0 \iff \det \Gamma \neq 0 \iff \det A \neq 0 \iff B \) is a Riesz sequence.

This observation proves Theorem 1.2.

For the sake of completeness, we prove the following easy lemma.

**Lemma 4.2.** Let \( M \) be a square matrix. The matrices \( MM^* \) and \( M^*M \) have the same eigenvalues.

**Proof.** Let \( \lambda \) be an eigenvalue of \( MM^* \) with eigenvector \( \vec{v} \). Thus, \( MM^* \vec{v} = \lambda \vec{v} \), and if we multiply this identity to the right by \( M^* \), we obtain \( (M^*M)M^* \vec{v} = \lambda M^* \vec{v} \). Thus, \( \lambda \) is an eigenvalue of \( M^*M \) with eigenvector \( M^* \vec{v} \).

**Proof of Theorem 1.3.** For a given the multi-rectangle \( Q = Q(M_1, \ldots, M_N) \) as in (1.1), we let

\[
Z = \{ (\vec{\delta}_1, \ldots, \vec{\delta}_N) \in (\mathbb{R}^d)^N : \det \Gamma((M_{\vec{\delta}_1}, \ldots, M_{\vec{\delta}_N})) = 0 \}.
\]

By Theorem 1.1, \( B(\vec{\delta}_1, \ldots, \vec{\delta}_N) \) is a Riesz basis of \( L^2(Q) \) if and only if \( (\vec{\delta}_1, \ldots, \vec{\delta}_N) \notin Z \). Consider the function \( \psi : (\mathbb{C}^d)^N \to \mathbb{C}, \psi(\vec{\eta}_1, \ldots, \vec{\eta}_N) = \det \Gamma((M_{\vec{\eta}_1}, \ldots, M_{\vec{\eta}_N})). \) Since \( \psi \) is holomorphic, by [10, Corollary 10], the set where \( \psi \equiv 0 \) has zero Lebesgue measure in \( (\mathbb{C}^d)^N \). Thus, \( Z \) has zero Lebesgue measure in \( \mathbb{R}^d \).

\( \square \)
4.2. Proof of Theorem 1.4. Let \( S(\delta) \) be as in (1.4). By Theorem 1.1, \( S(\delta) \) is a Riesz basis of \( L^2(Q) \) if and only if the matrix \( \Gamma = \{ \gamma_{j,p} \}_{1 \leq j, p \leq n} \), with \( \gamma_{j,p} = e^{2\pi i (\bar{M}_p, (j-1)\delta)} \) is non-singular. Since \( \gamma_{j,p} = (e^{2\pi i (\bar{M}_p, \delta)})^{J-1} \), \( \Gamma \) is a Vandermonde matrix, i.e., a matrix with the terms of a geometric progression in each row. We have

\[
\det \Gamma = \prod_{p < q} \left( e^{2\pi i (\bar{M}_p, \delta)} - e^{2\pi i (\bar{M}_q, \delta)} \right),
\]

and so \( \Gamma \) is non-singular if and only if \( e^{2\pi i (\bar{M}_p, \delta)} - e^{2\pi i (\bar{M}_q, \delta)} \neq 0 \) whenever \( p \neq q \). Equivalently, \( \Gamma \) is non-singular if and only if \( \langle \bar{M}_q - \bar{M}_p, \delta \rangle \notin \mathbb{Z} \) whenever \( p \neq q \).

By Theorem 1.1 and (4.9), the frame constants of \( \Gamma^* \Gamma = B \) = \( \{ \beta_{p,q} \}_{1 \leq p,q \leq N} \), where

\[
\beta_{p,q} = \sum_{j=1}^{N} e^{2\pi i (j-1)(\delta, \bar{M}_q - \bar{M}_p)} = \frac{1 - e^{2\pi i N (\delta, \bar{M}_q - \bar{M}_p)}}{1 - e^{2\pi i (\delta, \bar{M}_q - \bar{M}_p)}} = e^{\pi i (N-1)(\delta, \bar{M}_q - \bar{M}_p)} \frac{\sin(\pi N \langle \bar{M}_q - \bar{M}_p \rangle)}{\sin(\pi \langle \bar{M}_q - \bar{M}_p \rangle)}.
\]

Let \( \tilde{B} \) be the matrix whose elements are \( \tilde{\beta}_{p,q} = \frac{\sin(\pi N (\delta, \bar{M}_q - \bar{M}_p))}{\sin(\pi (\delta, \bar{M}_q - \bar{M}_p))} \) when \( p \neq q \) and \( \tilde{\beta}_{p,q} = N \) when \( p = q \); we have \( \langle B \vec{v}, \vec{v} \rangle = \langle \tilde{B} (\vec{v} \circ \vec{w}), (\vec{v} \circ \vec{w}) \rangle \), where \( \vec{w} = (e^{\pi i (N-1)(\delta, \bar{M}_1)}, ..., e^{\pi i (N-1)(\delta, \bar{M}_N)}) \) and \( \circ \) denotes the Hadamard (componentwise) product.

Since \( ||\vec{w}|| = ||\vec{v}|| \), the Hermitian forms \( \vec{v} \rightarrow \langle B \vec{v}, \vec{v} \rangle \) and \( \vec{v} \rightarrow \langle \tilde{B} \vec{w}, \vec{v} \rangle \) have the same maximum and minimum on \( S_{\mathbb{C}}^{N-1} \); consequently, the optimal frame bounds of \( S(\delta) \) are the maximum and minimum eigenvalues of \( \tilde{B} \), as required. \( \square \)

Remark. It is interesting to observe that

\[
|\det \Gamma|^2 = \det B = \prod_{p < q} \left| e^{2\pi i (\bar{M}_p, \delta)} - e^{2\pi i (\bar{M}_q, \delta)} \right|^2 = 2^{N(N-1)/2} \prod_{p < q} (1 - \cos(2\pi \langle \bar{M}_p - \bar{M}_q, \delta \rangle)) = 2^{N(N-1)} \prod_{p < q} \sin^2(\pi \langle \bar{M}_p - \bar{M}_q, \delta \rangle).
\]

5. Corollaries and examples

In this section we prove a number of corollaries of Theorems 1.1 and 1.4. We use the following notation: we denote with \( \bar{a} \odot \bar{b} = a_1b_1 + ... + a_db_d \) the
of (4.11), the optimal frame constant of $B$ We can easily verify that $A$ eigenvalue of the matrix $\prod$ and only if the matrix $\Gamma \cup \delta \in \mathbb{Z}$. The optimal frame constants of $B$ are the maximum and minimum eigenvalues of the matrix $\tilde{A} = \{\tilde{a}_{i,j}\}_{1 \leq i,j \leq N}$, where

$$\tilde{a}_{i,j} = \begin{cases} \sin(\pi(N(i-j))) & \text{if } i \neq j, \\ N & \text{if } i = j. \end{cases}$$

**Proof.** Without loss of generality, we can let $I = \left[-\frac{1}{2}, N - \frac{1}{2}\right) = \bigcup_{p=0}^{N-1}\left[p - \frac{1}{2}, p + \frac{1}{2}\right)$; by Theorem 1.1, $B(\delta_1, \ldots, \delta_N)$ is a Riesz basis of $L^2(I)$ if and only if the matrix $\Gamma = \{e^{2\pi ip}\}_{0 \leq j, p \leq N-1}$ is nonsingular. But $e^{2\pi ip} = (e^{2\pi i\delta})^p$, and so $\Gamma$ is a Vandermonde matrix whose determinant is $\det \Gamma = \prod_{0 \leq i < j \leq N-1}(e^{2\pi i\delta_i} - e^{2\pi i\delta_j})$. Therefore, $\det \Gamma \neq 0 \iff \delta_i - \delta_j \notin \mathbb{Z}$. In view of (4.11), the optimal frame constant of $B$ are the maximum and minimum eigenvalue of the matrix $A = \{\alpha_{i,j}\}_{1 \leq i,j \leq N}$, with $\alpha_{i,j} = \sum_{p=0}^{N-1} e^{2\pi i(\delta_i - \delta_j)p}$.

We can easily verify that $\alpha_{i,j} = e^{i\pi(N-1)(\delta_i - \delta_j)}\frac{\sin(\pi(N(\delta_i - \delta_j)))}{\sin(\pi(\delta_i - \delta_j))}$ when $i \neq j$, and $\alpha_{i,j} = N$ when $i = j$; if we argue as in the second part of the proof of Theorem 1.4, we can conclude that the frame constants of $B$ are the maximum and minimum eigenvalue of the matrix $\tilde{A}$ defined above.

**Remark.** In view of [31, Lemma 2.1] (see Section 2.2) it is not too difficult to prove a multi-dimensional version of Corollary 5.1. We leave the details to the interested reader.

### 5.1. Multi-rectangles with vertices in $\mathbb{Q}^d$ and a stability theorem

Let $R$ be a multi-rectangle with vertices in $\mathbb{Q}^d$. Let $l_1, \ldots, l_d$ be the smallest positive integers for which $R = R \cap (l_1, \ldots, l_d)$ has vertices in $\mathbb{Z}^d$. Let $\tilde{l} = (l_1, \ldots, l_d)$ and $L = \prod_{j=1}^{N} l_j$. The multi-rectangle $\tilde{R}$ is a union of $N = L|R|$ disjoint unit cubes with vertices in $\mathbb{Z}^d$. Let $\tilde{\delta}_1, \ldots, \tilde{\delta}_N \in \mathbb{R}^d$ and $B(\tilde{\delta}_1, \ldots, \tilde{\delta}_N)$ as in (1.2). A simple scaling in (4.1) and (4.2) proves the following

**Corollary 5.2.** $B(\tilde{\delta}_1, \ldots, \tilde{\delta}_N)$ is a Riesz basis of $L^2(\tilde{R})$ with constant $A$ and $B$ if and only if $\tilde{B} = \bigcup_{j=1}^{N}\{e^{2\pi i(\tilde{\delta} + \tilde{\delta}_j) \circ \tilde{x}}\}_{n \in \mathbb{Z}^d}$ is a Riesz basis of $L^2(R)$ with constants $\frac{A}{2}$ and $\frac{B}{2}$.

Our next result is a stability theorem for the basis $E(\mathbb{Z})$ on $L^2(0,1)$:
Corollary 5.3. Let $E = \{e_j\}_{j \in \mathbb{Z}} \subset \mathbb{R}$; assume that, for some integer $N \geq 2$, we have that for which

(5.1) \quad \epsilon_j = \epsilon_{N+j}, \quad j \in \mathbb{Z}

The set $U = \{2\pi i (n+\epsilon_j)x\}_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(0,1)$ if and only if

(5.2) \quad \frac{\epsilon_i - \epsilon_j + i-j}{N} \notin \mathbb{Z} \quad \text{whenever} \quad i \neq j.

In particular, $U$ is a Riesz basis of $L^2(0,1)$ whenever $\epsilon_i - \epsilon_j \notin \mathbb{Z}$.

Kadec theorem implies that a set $E(\Lambda) = \{e^{2\pi i \lambda_n x}\}_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(0,1)$ whenever $\ell = \sup_{j \in \mathbb{Z}} |j - \lambda_j| < \frac{1}{4}$. When $\ell \geq \frac{1}{4}$, $E(\Lambda)$ may not be a Riesz basis of $L^2(-\frac{1}{2}, \frac{1}{2})$, but that is the case if the $\epsilon_j = \lambda_j - j$ satisfy (5.2) and (5.1).

We prove first the prove the following

**Proof of Corollary 5.3.** By (5.1), $\epsilon_k = \epsilon_m$ whenever $m = nN + k$ for some $n \in \mathbb{Z}$. Thus,

$$U = \{2\pi i (m+\epsilon_m)x\}_{m \in \mathbb{Z}} = \bigcup_{j=0}^{N-1} \{2\pi i (Nn+j+\epsilon_j)x\}_{n \in \mathbb{Z}} = \bigcup_{j=0}^{N-1} \{2\pi i N(n+j\epsilon_j)x\}_{n \in \mathbb{Z}}.$$

By Corollary 5.2, $U$ is a Riesz basis of $L^2(0,1)$ if and only if the set $\hat{U} = \bigcup_{j=0}^{N-1} \{e^{2\pi i (n+\frac{j+\epsilon_j}{N})x}\}_{n \in \mathbb{Z}}$ is a Riesz basis of $L^2(0,N)$. We can apply Corollary 5.1, with $\delta_j = \frac{j+\epsilon_j}{N}$, and conclude that $\delta_j - \delta_j \notin \mathbb{Z}$ is equivalent to (5.2). □

**Example 1.** Fix $s \in (0,1)$, and define $U = \{e^{2\pi i (m+\mu_m)x}\}_{m \in \mathbb{Z}}$ where $\mu_m = s$ when $m$ is even and $\mu_m = -s$ when $m$ is odd. By (5.2), $U$ is a Riesz basis of $L^2(0,1)$ if and only if $\frac{2m-1}{N} \notin \mathbb{Z}$, or $0 < s < \frac{1}{2}$.

It is interesting to compare Example 1 with a famous example by Ingham. In [17, p. 378] it is proved that $U$ is not a Riesz basis of $L^2(0,1)$ when $\mu_m = \frac{1}{2}$ if $m > 0$, $\mu_m = -\frac{1}{2}$ if $m < 0$ and $\mu_0 = 0$. Ingham’s example shows that the constant $\frac{1}{2}$ in Kadec’s theorem cannot be replaced by any larger constant. See also [32].

5.2. **Two cubes in $\mathbb{R}^d$.** Let $\vec{M}_1 \neq \vec{M}_2 \in \mathbb{Z}^d$ and let $Q = \tau_{\vec{M}_1} Q_0 \cup \tau_{\vec{M}_2} Q_0$. Let $\vec{\delta}_1, \vec{\delta}_2 \in \mathbb{R}^d$. We prove the following

**Corollary 5.4.** a) The set $B = B(\delta_1, \delta_2)$ is a Riesz basis of $L^2(Q)$ if and only if $(\vec{M}_1 - \vec{M}_2, \vec{\delta}_1 - \vec{\delta}_2) \notin \mathbb{Z}$. The optimal frame constants of $B$ are $A = 2(1 - |\cos(\pi \langle \vec{M}_1 - \vec{M}_2, \vec{\delta}_1 - \vec{\delta}_2 \rangle)|)$, $B = 2(1 + |\cos(\pi \langle \vec{M}_1 - \vec{M}_2, \vec{\delta}_1 - \vec{\delta}_2 \rangle)|)$.

In particular, $B$ is an orthogonal Riesz basis of $L^2(Q)$ if and only if

$$2\langle \vec{M}_1 - \vec{M}_2, \vec{\delta}_1 - \vec{\delta}_2 \rangle \in \mathbb{Z} \quad \text{and} \quad (\vec{M}_1 - \vec{M}_2, \vec{\delta}_1 - \vec{\delta}_2) \notin \mathbb{Z}.$$
Proof. After perhaps a translation, we can let $Q = Q_0 \cup +\tau \bar{M} Q_0$, with \( \bar{M} = \bar{M}_2 - \bar{M}_1 \). By Theorem 1.1, $B$ is a Riesz basis if and only if the matrix

$$A = \begin{pmatrix}
2, & e^{2\pi i \langle \bar{M}, \delta_1 - \delta_2 \rangle} \\
1 + e^{-2\pi i \langle \bar{M}, \delta_1 - \delta_2 \rangle}, & 2
\end{pmatrix}$$

is nonsingular. The eigenvalues of $A$ are the zeros of the characteristic polynomial,

$$\det(A - sI) = (2 - s)^2 - \left| 1 + e^{2\pi i \langle \bar{M}, \delta_1 - \delta_2 \rangle} \right|^2$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We can easily verify that

$$\det(A - sI) = s^2 - 4s + 4 \sin^2(\pi \langle \bar{M}, \delta_1 - \delta_2 \rangle) = 0 \iff s = 2(1 \pm |\cos(\pi \langle \bar{M}, \delta_1 - \delta_2 \rangle)|).$$

Thus, $\lambda = 2(1 - |\cos(\pi \langle \bar{M}, \delta_1 - \delta_2 \rangle)|)$ and $\Lambda = 2(1 + |\cos(\pi \langle \bar{M}, \delta_1 - \delta_2 \rangle)|)$ are the optimal frame constants of $B$.

When $\cos(\pi \langle \bar{M}, \delta_1 - \delta_2 \rangle) = 0$, i.e. when $\langle \bar{M}, \delta_1 - \delta_2 \rangle$ is an odd multiple of $\frac{\pi}{2}$, $B$ is a tight frame with constants $\lambda = \Lambda = 2$. Since the functions in $B$ have norm $= 2$ on $L^2(Q)$, Lemma 2.4 implies that $B$ is orthogonal.

\[ \square \]

Remark. Let $Q$ be the union of two disjoint unit cubes with vertices in $\mathbb{Z}^d$. We can verify that $Q$ tiles $\mathbb{R}^d$ by translation; by Corollary 5.4, we can always find an orthogonal basis on $L^2(Q)$ and so $Q$ is a spectral domain of $\mathbb{R}^d$. It is proved in [18] that the union of two disjoint intervals of nonzero length is spectral if and only if it tiles $\mathbb{R}$ by translation. To the best of our knowledge, the analog of the main theorem in [18] has not been proved (or disproved) for unions of two disjoint rectangles in $\mathbb{R}^d$.

5.3. Spectral domains in $\mathbb{R}^d$. In this section we show examples of spectral multi-rectangles in $\mathbb{R}^d$. We let $Q = Q(M_1, ..., M_N) \subset \mathbb{R}^d$ be as in (1.1) and $S(\delta) = \bigcup_{j=1}^{N-1} \{ e^{2\pi i (\bar{r} + (j-1)\delta, x)} \}_{\bar{r} \in \mathbb{Z}^d}$ be as in (1.4). The following Corollary can be viewed as a generalization of Corollary 5.4.

Corollary 5.5. The set $S(\delta)$ is an orthogonal basis of $L^2(Q)$ if and only if, for every $p \neq q$,

\begin{equation}
(\delta) \notin \mathbb{Z} \quad \text{and} \quad N(\bar{M}_p - \bar{M}_q, \delta) \in \mathbb{Z}.
\end{equation}

Proof. Assume that (5.3) holds. By Theorem 1.4, the first condition in (5.3) yields that $S(\delta)$ is a Riesz basis of $L^2(Q)$; the second condition implies the matrix $\tilde{B}$ as in (1.6) is diagonal. Thus, $S(\delta)$ is a tight frame with frame constant $N = |Q|$, and by Lemma 2.4, it is an orthogonal basis of $L^2(Q)$.

Conversely, assume that $S(\delta)$ is orthogonal. Thus, the frame constant of $S(\delta)$ equal $A = B = N$ and by Theorem 1.4, the maximum and minimum eigenvalues of the matrix $\tilde{B}$ equal $N$ as well. Since $\mathbb{C}^N$ has a basis of
eigenvectors of $\tilde{\mathbf{B}}$, we can infer that $\tilde{\mathbf{B}}\vec{v} = N\vec{v}$ for every $\vec{v} \in \mathbb{C}^N$ and that $\tilde{\mathbf{B}}$ is diagonal. Recalling that the elements of $\tilde{\mathbf{B}}$ are as in (1.6), we deduce (5.3).

Example 2. Let $Q = Q(\tilde{M}_1, ..., \tilde{M}_N)$, with $N \leq d$. Assume that $Q_N = Q_0$ (so that $\tilde{M}_N = (0, ..., 0)$) and that the $\tilde{M}_1, ..., \tilde{M}_{N-1}$ are linearly independent. We show that $Q$ is spectral.

Let $\mathbf{M}$ be the matrix whose rows are $\tilde{M}_1, ..., \tilde{M}_{N-1}$. By assumption, $\mathbf{M}$ has rank $N - 1$, and so we can find $\vec{\sigma} \in \mathbb{R}^d$ that satisfies $\langle \tilde{M}_j, \vec{\sigma} \rangle = \frac{1}{N}$ for every $j = 1, ..., N - 1$. By Corollary 5.5, $\mathcal{S}(\vec{\sigma})$ is an orthogonal basis of $L^2(Q)$.

5.4. Extracting Riesz bases from frames. Let $Q = Q(\tilde{M}_1, ..., \tilde{M}_N)$ be as in (1.1). Without loss of generality, we can assume $Q \subset [-\frac{1}{2}, T - \frac{1}{2})^d$ for some $T > 0$. From [16, Theorem 2] follows that a basis of $L^2(Q)$ can be extracted from $E((\frac{T}{2}\mathbb{Z})^d)$, which is an orthogonal basis of $[0,T)^d$ and an exponential frame of $L^2(Q)$. When $T$ is an integer, it is easy to verify that

$$E((\frac{T}{2}\mathbb{Z})^d) \supset \mathcal{S}(\frac{1}{T})$$

where $\mathcal{S}(\frac{1}{T})$ as in (1.4), and $\vec{a} = (a, ..., a)$ when $a \in \mathbb{R}$. Indeed

$$E((\frac{T}{2}\mathbb{Z})^d) = \{e^{2\pi i((m_1 + \frac{1}{T})x_1 + \ldots + (m_d + \frac{1}{T})x_d)}\}_{(m_1, ..., m_d) \in \mathbb{Z}^d}$$

$$= \bigcup_{j=0}^{T-1} \{e^{2\pi i((m_1 + \frac{1}{T})x_1 + \ldots + (m_d + \frac{1}{T})x_d)}\}_{(m_1, ..., m_d) \in \mathbb{Z}^d}$$

$$= \mathcal{S}(\frac{1}{T}).$$

Let $\bar{T} = \sup_{1 \leq p \neq q \leq N} \|\tilde{M}_p - \tilde{M}_q\|_{\infty}$ be the smallest positive integer for which $Q \subset [0, \bar{T})^d$. If (1.5) in Theorem 1.4 is satisfied, then $\mathcal{S}(\frac{1}{T})$ is an exponential basis of $L^2(Q)$. For that we need $\langle \tilde{M}_p - \tilde{M}_q, \vec{1} \rangle \not\in \mathbb{Z}\bar{T}$ for every $1 \leq p \neq q \leq N$. Otherwise, we can let $L > \bar{T}$ be the smallest positive integer for which $\frac{1}{L}(\tilde{M}_p - \tilde{M}_q, \vec{1}) \not\in \mathbb{Z}$ and conclude that $\mathcal{S}(\frac{1}{T})$ is an exponential basis of $L^2(Q)$ extracted from the exponential frame $E((\frac{1}{T}\mathbb{Z})^d)$. We have proved the following

**Corollary 5.6.** Let $Q$ be defined as above. We can find an integer $L > 0$ for which $Q \subset [-\frac{1}{2}, L - \frac{1}{2})^d$ and $\mathcal{S}(\frac{1}{T})$ is a Riesz basis of $L^2(Q)$.

From Lemma 2.3 and Theorem 1.2 we have the following.
Corollary 5.7. Under the assumptions of Corollary 5.6, the set $S(\bar{\delta})$ is a Riesz basis of $L^2(Q)$ if and only if $E((\frac{1}{2}\mathbb{Z})^d) - S(\bar{\delta})$ is a Riesz basis on $L^2([-\frac{1}{2}, L - \frac{1}{2}]^d - Q)$.

6. Estimating the frame constants

Let $Q = Q(M_1, ..., M_N)$ be as in (1.1), and let $B = B(\delta_1, ..., \delta_N)$ as in (1.2) be a a Riesz basis on $L^2(Q)$ with optimal frame constants $0 < \lambda < \Lambda$. By Theorem 1.1 and (4.11) and (4.9), $\Lambda$ and $\lambda$ are the maximum and minimum eigenvalues of the matrices $A = \{\alpha_{i,j}\}_{1 \leq i, j \leq N}$ and $B = \{\beta_{p,q}\}_{1 \leq p, q \leq N}$, with

$$
\alpha_{i,j} = \sum_{p=1}^{N} e^{2\pi i (\delta_i - \delta_j, M_p)}, \quad \beta_{p,q} = \sum_{j=1}^{N} e^{2\pi i (M_{p} - \bar{M}_q, \delta_j)}.
$$

When $B = S(\bar{\delta})$ is as in (1.4), the frame constant of $B$ are the maximum and minimum eigenvalues of the matrix $\tilde{B} = \{\tilde{\beta}_{p,q}\}_{1 \leq p, q \leq N}$ defined in (1.6).

Gershgorin theorem provides a powerful tool for estimating the eigenvalues of complex-valued matrices. It states that each eigenvalue of a square matrix $M = \{m_{i,j}\}_{1 \leq i, j \leq n}$ is in at least one of the disks $D_j = \{z \in \mathbb{C} : |z - m_{j,i}| \leq R_j\}$, and in at least one of the disks $D'_j = \{z \in \mathbb{C} : |z - m_{j,i}| \leq C_j\}$, where $R_j$ (resp. $C_j$) are the sum of the off-diagonal elements of the $j$–th row (column) of $M$, i.e.

$$
R_j = \sum_{i=1}^{n} |m_{j,i}|, \quad C_j = \sum_{i=1}^{n} |m_{i,j}|.
$$

See [8], and also [26, pg. 146] and [3].

Observe that if $|m_{j,i}| > \max\{R_j, C_j\}$ for every $j$, (i.e., if $M$ is diagonally dominant), then $M$ is nonsingular.

The following refinement of Gershgorin theorem is in [1].

Theorem 6.1. Let $M$ be an Hermitian matrix with eigenvalues $\lambda_1, ..., \lambda_n$. Let $R_j = C_j$ be as in (6.2). We have

$$
\max_{1 \leq j \leq n} \{m_{j,j} - R_j\} \leq \max_{1 \leq j \leq n} \lambda_j \leq \max_{1 \leq j \leq n} \{m_{j,j} + R_j\}
$$

(6.3) \hspace{1cm}

$$
\min_{1 \leq j \leq n} \{m_{j,j} - R_j\} \leq \min_{1 \leq j \leq n} \lambda_j \leq \min_{1 \leq j \leq n} \{m_{j,j} + R_j\}.
$$

We can use Theorem 6.1 to estimate the optimal frame constants of $B$ and $S(\bar{\delta})$.

Theorem 6.2. Let $B$ be a Riesz basis of $L^2(Q)$. Let

$$
r_i = \sum_{1 \leq j \leq N} \left(1 - \frac{4}{N^2} \sum_{p,q=1, p \neq q}^{N} \sin^2(\pi (\bar{\delta}_i - \delta_j, \bar{M}_p - \bar{M}_q))\right)^{\frac{1}{2}},
$$

where $\bar{M}_p$ and $\bar{M}_q$ are the maximum and minimum eigenvalues of $\tilde{B}$. Then

$$
r_i \leq R(i),
$$

where $R(i)$ is the maximum of the sum of the off-diagonal elements of the $i$–th row (column) of $M$. 


\[
\rho_p = \sum_{1 \leq q \leq N \atop q \neq p} \left( 1 - \frac{4}{N^2} \sum_{i,j=1 \atop i < j}^N \sin^2(\pi (\vec{\delta}_i - \vec{\delta}_j, \vec{M}_p - \vec{M}_q)) \right)^{\frac{1}{2}}.
\]

The optimal frame constants of \(B\) satisfy
\[
N(1 - \min_{1 \leq p \leq N} \{r_i, \rho_p\}) \leq \lambda \leq \Lambda \leq N(1 + \max_{1 \leq p \leq N} \{r_i, \rho_p\})
\]

b) If \(B = S(\vec{\delta})\), we have
\[
N(1 - \min_{1 \leq p \leq N} s_p) \leq \lambda \leq \Lambda \leq N(1 + \max_{1 \leq p \leq N} s_p)
\]

where \(s_p = \frac{\left| \sin(\pi N(\vec{\delta}, \vec{M}_q - \vec{M}_p))) \right|}{N \sin(\pi (\vec{\delta}, \vec{M}_q - \vec{M}_p))} \).

**Proof.** a) The optimal frame constants of \(B\) are the maximum and minimum eigenvalues of the matrices \(A\) and \(B\). In view of (6.1), it is easy to verify that
\[
\sum_{1 \leq i \leq N \atop i \neq j} |\alpha_{i,j}| = \sum_{1 \leq i \leq N \atop i \neq j} \left( N + 2 \sum_{p,q=1 \atop p < q}^N \cos(2\pi (\vec{\delta}_i - \vec{\delta}_j, \vec{M}_p - \vec{M}_q)) \right)^{\frac{1}{2}}
\]
\[
= \sum_{1 \leq i \leq N \atop i \neq j} \left( N + 2 \sum_{p,q=1 \atop p < q}^N (1 - 2\sin^2(\pi (\vec{\delta}_i - \vec{\delta}_j, \vec{M}_p - \vec{M}_q))) \right)^{\frac{1}{2}}
\]
\[
= \sum_{1 \leq i \leq N \atop i \neq j} \left( N^2 - 4 \sum_{p,q=1 \atop p < q}^N \sin^2(\pi (\vec{\delta}_i - \vec{\delta}_j, \vec{M}_p - \vec{M}_q)) \right)^{\frac{1}{2}}
\]
\[
= Nr_i.
\]

We can prove that \(\sum_{1 \leq q \leq N \atop q \neq p} |\beta_{p,q}| = N \rho_p\) in a similar manner.

Since \(A\) and \(B\) have the same eigenvalues, we can apply Theorem 6.1 with \(m_{i,j} = N\) and \(R_i = Nr_i\) or \(R_i = N\rho_i\), and (6.5) follows.

b) When \(B = S(\vec{\delta})\) we can apply Theorem 6.1 to the matrix \(\tilde{B}\); the inequality (6.6) follows from (1.6).

**Corollary 6.3.** a) Let \(B\), \(\lambda\) and \(\Lambda\) be as in in Theorem 6.2. If
\[
\min_{1 \leq p,q \leq N \atop p < q} \sin^2(\pi (\vec{\delta}_i - \vec{\delta}_j, \vec{M}_p - \vec{M}_q)) \geq \frac{N}{2(N-1)} \left( 1 - \left( \frac{1-a}{N-1} \right)^2 \right)
\]

for some \(0 < a < 1\), we have
\[aN \leq \lambda \leq \Lambda \leq (2-a)N.\]
b) Let $B = S(\vec{\delta})$, with $\min_{1 \leq p, q \leq N} |\sin(\pi \langle \vec{\delta}_i - \vec{\delta}_j, \vec{M}_q - \vec{M}_p \rangle)| \geq \frac{N-1}{N(1-a)}$. Then,

$$Na \leq \lambda \leq \Lambda \leq N(2 + a).$$

**Proof.** a) Let $s = \min_{1 \leq p, q \leq N} |\sin(\pi \langle \vec{\delta}_i - \vec{\delta}_j, \vec{M}_q - \vec{M}_p \rangle)|$. By (6.4) and (6.7), $r_j \leq (N - 1) \left(1 - \frac{2(N-1)s^2}{N} \right)^{\frac{1}{2}} \leq 1 - a$. A similar inequality holds for $\rho_p$. By Theorem 6.2, part a) of the corollary is proved.

For part b), we let $s = \min_{1 \leq p, q \leq N} |\sin(\langle \vec{\delta}, \vec{M}_q - \vec{M}_p \rangle)|$. With the notation of Theorem 6.2 b),

$$s_p \leq \sum_{1 \leq q \leq N, q \neq p} \frac{1}{N} \frac{1}{\sin(\pi \langle \vec{\delta}_i - \vec{\delta}_j, \vec{M}_q - \vec{M}_p \rangle)} \leq \frac{N - 1}{Ns}.$$

By (6.6), the proof of the corollary is concluded

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