Study of compound nucleus formation via bremsstrahlung emission in proton α-particle scattering

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In this paper a role of many-nucleon dynamics in formation of the compound $^5$Li nucleus in the scattering of protons off α-particles at the proton incident energies up to 20 MeV is investigated. We propose a bremsstrahlung model allowing to extract information about probabilities of formation of such nucleus on the basis of analysis of experimental cross-sections of the bremsstrahlung photons. In order to realize this approach, the model includes elements of microscopic theory and also probabilities of formation of the short-lived compound nucleus. Results of calculations of the bremsstrahlung spectra are in good agreement with the experimental cross-sections.

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I. INTRODUCTION

Understanding of nuclear interactions between nucleons is hot topic of physics. In such a line, bremsstrahlung photons emitted during nuclear reactions provide important information about dynamics of nucleons of composed nuclear system, which is formed on the basis of interactions. The bremsstrahlung of the emitted photons in nuclear reactions where nuclei are described on the microscopic level, has been studied for a long time [1–13]. In such frameworks, bremsstrahlung photons are successfully applied for deeper understanding of nucleon-nucleon interactions. The nucleon-nucleon scattering data (for example, Nijmegen data set [14]) and properties of the deuteron are used for controlling parameters of the microscopical cluster models.

A crucial point in description of scattering properties is accurate (precise) determination of wave functions for scattering states of the nuclear system. In their determination, scattering theory and theory of nuclear reactions have traditional way, where nuclei from lightest up to super-heavy are included into consideration. In such a frameworks, interactions between the scattering objects are described often in potential approach, which parameters are extracted from existed experimental data. Classical objects for study here are effects of collective motion, strong quantum phenomena, processes of fusion and breakup inside the nuclear system, etc.

Results of such a work are used as a basis for construction of other bremsstrahlung models in nuclear physics. Our bremsstrahlung model is developed along such a line [15–27]. According to its logic, determination of the wave functions for the scattering states is quantum mechanical task with interacting potentials, that allows to keep maximally accurate quantum effect (that can be interesting in cases where they are strong, for example, see [28]). As attractive point of such a line, inverse scattering theory [29] can be naturally added, to achieve exact coincidence between experimental and calculated cross-sections (that provides information about the corresponding potentials). Our approach was successfully tested on set of tasks where experimental data on the emission of the bremsstrahlung photons in nuclear reactions is essentially larger. These are experimental data for α-decay [17, 18, 30, 34] (see also some calculations [35–40]), scattering of protons off nuclei [50, 51] (see also reviews on situation in such a research [60, 61]), spontaneous fission [62–68], etc. (see also predicted spectra for emission of protons from nuclei [23, 25, 26, 69], ternary fission [24]).

So, all existed variety of experimental data of emission of the bremsstrahlung photons in nuclear reactions can be

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separated into two following groups. To the first group one can include the data \cite{70,73}, analyzed in the microscopic cluster models developed formalism of resonating group method \cite{74,76} and generator coordinate method \cite{77}. Such a data are obtained in the coplanar geometry, where maxima in the bremsstrahlung cross-sections are observed at some energies (i.e. resonances). Usually, such maxima are explained by resonant states of the compound nuclear system. In frameworks of such a formalism, kinematic relations between energy of the emitted photon and energy of relative motion of two nuclei or nucleus and nucleon are applied.

The experimental data of the second group are obtained relatively energy of the emitted photons (here, in calculations usually kinematic relations between energy of the emitted photons and energy of relative motion between two nuclei or nucleus and nucleon are not imposed). Such cross-sections are obtained in the large regions of energy, they have smooth continuous shapes where any sharp resonant peaks are not observed. However, potentials of interactions usually do not include explicitly such resonant states of the compound nuclear system at definite energies. Absence of the fixed dependence between the energy of the emitted photons and the energy of the relative motion between two nuclei or nucleus and nucleon is justified by that such experimental bremsstrahlung cross-sections for different types of nuclear decays and the scattering of protons off nuclei at fixed incident proton energy are continuous, usually have no resonant peaks and are presented inside enough large region of the emitted photon energy. In description of such an experimental data our approach proves oneself enough accurate (successfully). From such a point of view, it could be interesting for us to investigate the experimental data of the first group on the basis of our formalism. We see that understanding of the nuclear interactions on the basis of many nucleon forces, is one of the most important tasks in nuclear physics. Therefore, in order to base possibility to study the interactions between nucleons in our formalism, we include some elements of the microscopic theory into it. These ideas represent a main aim of this paper.

Another attractive advance of our approach is possibility to include into operator of emission the additional terms, connected spinor properties of nucleons with their momenta (i.e. possibility to study influence of dynamics and spins of nucleons (their interference) on the emission of photons). Such corrections are appeared in result of many-nucleon generalization of Dirac equation and the corresponding Hamiltonian of the nuclear system (where the first approximation is many-nucleon Pauli equation, see \cite{25} for details). So, they have another origin than appearance of the spin states in the resonating group method and the generator coordinate method. In particular, in frameworks of such a model \cite{26}, we explain the hump-shaped plateau in the intermediate and high energy regions of the experimental bremsstrahlung data \cite{59} by essential presence of the incoherent emission (formed by interactions between the scattering proton and nucleus with the internal many-nucleon structure) for the scattering of $p + ^{208}$Pb at the proton incident energies of 140 and 145 MeV, and the scattering of $p + ^{12}$C, $p + ^{58}$Ni, $p + ^{107}$Ag and $p + ^{197}$Au at the proton incident energy of 190 MeV, while at low energies the coherent emission (formed by interaction between the scattering proton and nucleus as a whole without internal many-nucleon structure) predominates which produces the logarithmic shape spectrum. Such an approach allows to develop theory for description of the bremsstrahlung photons emitted in the nuclear reactions in the relativistic region of energies.

Else one useful advance of our model is description of quantum effects shown during fusion and break-up processes inside the compound nuclear system. According to \cite{28}, quantum effects participating in capture of the $\alpha$ particle by the $^{40}$Ca and $^{44}$Ca nuclei, are not small, and inclusion of their description into the model allows to essentially improve agreement between the calculated spectra and experimental data (this method found new parametrization of the $\alpha$–nucleus potential and fusion probabilities and decreased the error by 41.72 times for $\alpha + ^{40}$Ca and 34.06 times for $\alpha + ^{44}$Ca in a description of experimental data \cite{28} in comparison with previously existing results). From this point of view, it could be interesting to include ideas of such a formalism of the processes of fusion and breakup into the bremsstrahlung model. But, in order to include formalism of many-nucleon interactions, we draw attention on one of light systems, which were studied previously and for which the experimental data exists. This is the scattering of protons off the $\alpha$-particles, for which the bremsstrahlung cross-sections were measured in the coplanar geometry \cite{70}. Here, suitable object for investigations of processes of nuclear formation and breakup is the system from five nucleons in form of the short-lived $^5$Li nucleus. Analysis of such a task is also included into our research in this paper.

In Sec. III our improved bremsstrahlung model applied for the scattering of protons off alpha-particles is presented. Here, after formulating of operator of emission of bremsstrahlung photons for such a nuclear system, emphasis is made on construction of formalism for the matrix elements in the microscopic approach, with next application of the multipolar expansion for wave function of photons and obtaining the bremsstrahlung cross-sections. In Sec. III we perform theoretical analysis of the bremsstrahlung emission in this reaction. Here, after analysis of phase shifts of the scattering wave functions obtained in our approach and comparing them with empirical data \cite{29}, we calculate contributions of the emitted photons from different transitions between states, obtain the full bremsstrahlung spectrum comparing it with experimental data \cite{70} of Arndt, Roper, and Shotwell, and calculations of Liu, Tang and Kanada \cite{4}, and Dohet-Eraly \cite{12}. Concluding remarks are presented in Sec. IV Calculations of one-nucleon space matrix elements over bound states of nucleus are added in Appendixes A and B for convenience.
II. MODEL

A. Operator of emission and wave function of the proton - nucleus system

We shall start from the leading form Eq. (7) of the photon emission operator $\hat{H}_s$ in [26, 27], generalizing it for the system of the scattering proton and nucleus composed of $A$ nucleons in the laboratory system. Using presentation for the vector potential of the electromagnetic field in form (5) in [27], we obtain

$$\hat{H}_s = -e\sqrt{\frac{2\pi}{w_{ph}}} \sum_{\alpha=1,2} e^{(\alpha),*} \left\{ \frac{z_p}{m_p} e^{-ikr_p} p_p + \sum_{j=1}^{A} \frac{z_j}{m_j} e^{-ikr_j} p_j \right\}. \quad (1)$$

Here, star denotes complex conjugation, $z_j$ and $m_j$ are the electromagnetic charge and mass of the nucleon with number $j$, $m_p$ is mass of proton, $p_j = -i\hbar \frac{d}{dr_j}$ is the momentum operator for the nucleon with number $j$ (we number nucleons of the nucleus by index $j$). $e^{(\alpha)}$ are unit vectors of the polarization of the photon emitted [$e^{(\alpha),*} = e^{(\alpha)}$], $k$ is the wave vector of the photon and $w_{ph} = kc = |k| c$. Vectors $e^{(\alpha)}$ are perpendicular to $k$ in the Coulomb gauge. We have two independent polarizations $e^{(1)}$ and $e^{(2)}$ for the photon with momentum $k$ ($\alpha = 1, 2$). In this paper we shall use the system of units where $\hbar = 1$ and $c = 1$.

Now we turn to the center-of-mass frame. We define coordinate of centers of masses for the nucleus as $R$ having form $R_A = \sum_{j=1}^{A} m_j r_{Aj}/m_A$, $R = (m_A R_A + m_p r_p)/(m_A + m_p)$, where $m_A$ and $m_A$ are masses of the scattering proton and nucleus. Introducing new relative coordinates $\rho_{Aj}$ and $r$, as $r_j = R_A + \rho_{Aj}$, $r = r_p - R_A$, we find the corresponding momenta $p_j = P_A + \tilde{P}_{Aj}$, $p = p_p - P_A$, where $p_p = -i\hbar \frac{d}{dr_p}$, $P_A = -i\hbar \frac{d}{d\rho_{Aj}}$, $\tilde{P}_{Aj} = -i\hbar \frac{d}{d\rho_{Aj}}$. Using these formulas, we obtain

$$R_A = R - c_p r, \quad r_p = R + c_A r, \quad r_j = R - c_p r + \rho_{Aj}, \quad (2)$$

where we introduced $c_A = \frac{m_A}{m_A + m_p}$ and $c_p = \frac{m_p}{m_A + m_p}$. Substituting these expressions to eq. (1), we find

$$\hat{H}_s = -e\sqrt{\frac{2\pi}{w_{ph}}} \sum_{\alpha=1,2} e^{(\alpha),*} e^{-ikr} \left\{ \frac{z_p}{m_p} + \sum_{j=1}^{A} \frac{z_j}{m_j} e^{-ik\rho_{Aj}} \right\} P +$$

$$+ \left[ c_A e^{-ikr} \frac{z_p}{m_p} - c_p \sum_{j=1}^{A} \frac{z_j}{m_j} e^{-ik\rho_{Aj}} \right] P + \sum_{j=1}^{A} \frac{z_j}{m_j} e^{-ik\rho_{Aj}} \tilde{P}_{Aj} \quad (3).$$

We define the wave function of the full nuclear system as

$$\Psi(1, 2, \ldots A + 1) = \mathcal{A} [\psi_{\lambda_1}(1), \psi_{\lambda_2}(2) \ldots \psi_{\lambda_{A+1}}(A + 1)], \quad (4)$$

with $\mathcal{A}$ being an antisymmetrization operator. One-nucleon functions $\psi_{\lambda_s}(s)$ represent the multiplication of space and spin-isospin functions as

$$\psi_{\lambda_s}(s) = \varphi_{n_s}(r_s) |\sigma^{(s)} r^{(s)}\rangle,$$  

where $\varphi_{n_s}$ is space function of the nucleon with number $s$, $n_s$ is number of state of the space function of the nucleon with number $s$, $|\sigma^{(s)} r^{(s)}\rangle$ is spin-isospin function of the nucleon with number $s$.

B. Matrix element of emission

We shall assume $\Phi_{\delta}(\mathbf{R}) = e^{-i\mathbf{K}_s \cdot \mathbf{R}}$ where $\delta = i$ or $f$ (indexes $i$ and $f$ denote the initial state, i.e. the state before emission of photon, and the final state, i.e. the state after emission of photon), $\mathbf{K}_s$ is momentum of the total system [80]. Suggesting $\mathbf{K}_i = 0$, we calculate the matrix element:

$$\langle \Psi_f | \hat{H}_s | \Psi_i \rangle = - \frac{e}{m_p} \sqrt{\frac{2\pi}{w_{ph}}} \sum_{\alpha=1,2} e^{(\alpha),*} \left\{ M_1 + M_2 + M_3 \right\}, \quad (6)$$
where

\[ M_1 = \left\langle \Psi_f \left| e^{i (K_f - k) \cdot r} e^{i c_v kr} \left[ e^{-i k r \cdot z_p} + \sum_{j=1}^A z_j \frac{m_p}{m_j} e^{-i k \rho_{A_j}} \right] \right| \Psi_i \right\rangle, \]

\[ M_2 = \left\langle \Psi_f \left| e^{i (K_f - k) \cdot r} e^{i c_v kr} \left[ e^{-i k r \cdot c_A z_p} - c_v \sum_{j=1}^A z_j \frac{m_p}{m_j} e^{-i k \rho_{A_j}} \right] \right| \Psi_i \right\rangle, \]

\[ M_3 = \left\langle \Psi_f \left| e^{i (K_f - k) \cdot r} e^{i c_v kr} \sum_{j=1}^A z_j \frac{m_p}{m_j} e^{-i k \rho_{A_j}} \tilde{P}_{A_j} \right| \Psi_i \right\rangle. \]

(7)

We will not use the first term \( M_1 \) (as we shall study decay in the center-of-mass system and neglect by possible response), and the third term \( M_3 \) (as we shall not study contribution of photon emission caused by the deformation of the daughter nucleus during decay). So, we shall calculate the second matrix element. Substituting representation for the wave function, we obtain:

\[ M_2 = \delta(K_f - k) \left\{ c_A \left\langle \psi_f(1 \cdots A) \left| e^{i c_v kr} e^{-i k r \cdot p} \right| \psi_i(1 \cdots A) \right\rangle - \right. \]

\[ - c_p \sum_{j=1}^A \left\langle \psi_f(1 \cdots A) \left| e^{i c_v kr} f_{A_j}(\rho_{A_j}) \left| \psi_i(1 \cdots A) \right\rangle \right. \right\}, \]

(8)

where

\[ f_{A_j}(\rho_{A_j}) = z_j \frac{m_p}{m_j} e^{-i k \rho_{A_j}}. \]

(9)

Matrix element from operator, dependent on two nucleons with numbers \( i \) and \( j \), can be written in form of linear combination of two-nucleon matrix elements as

\[ \left\langle \psi_f(1 \cdots A) \left| \tilde{V}(r_i, r_j) \right| \psi_i(1 \cdots A) \right\rangle = \]

\[ = \frac{1}{A (A - 1)} \sum_{k=1}^A \sum_{m=1}^A \left\{ \left( \psi_k(i) \psi_m(j) \right) \tilde{V}(r_i, r_j) \left| \psi_k(i) \psi_m(j) \right\rangle - \left( \psi_k(i) \psi_m(j) \right) \tilde{V}(r_i, r_j) \left| \psi_k(i) \psi_m(j) \right\rangle \right\}. \]

(10)

Here, summation is performed over all states of nucleons. But, if operator is not dependent on relative distances between nucleons and it does not act on spin and isospin states of nucleons, then it needs to calculate all terms of matrix element separately. In result, we obtain:

\[ M_2 = \delta(K_f - k) \left\{ M_{21} - M_{22} \right\}, \]

(11)

where

\[ M_{21} = \frac{c_A}{A + 1} \sum_{k=1}^{A+1} \left\langle \psi_k(i) \left| e^{i c_v kr} f_p(\rho_p) \right| \psi_k(i) \right\rangle, \]

\[ M_{22} = c_p \sum_{j=1}^Z \frac{1}{A (A + 1)} \sum_{k=1}^{A+1} \sum_{m=1, m \neq k}^{A+1} \left\{ \left( \psi_k(i) \right) \left| e^{i c_v kr} \left| \psi_k(i) \right\rangle \left( \psi_m(j) \right) f_j(\rho_{A_j}) \right| \psi_m(j) \right\} - \]

\[ - \left( \psi_k(i) \right) \left| e^{i c_v kr} \left| \psi_k(i) \right\rangle \left( \psi_m(j) \right) f_j(\rho_{A_j}) \right| \psi_k(j) \right\} \right\}. \]

(12)

C. Matrix element in multipole expansion of wave function of photons

After summation over spin-isospin states (see Appendix A for details), in a case of the \( \alpha \)-particle as the target nucleus, we obtain:

\[ M_{21} = \frac{c_A}{5} \left\{ 2 i (c_p - 1) k e^{-(c_p - 1)^2(a^2 k_x^2 + b^2 k_y^2 + c^2 k_z^2)/4} + J_1(k) \right\}, \]

\[ M_{22} = \frac{c_p}{10} \left\{ 2 J_2(k) e^{-(a^2 k_x^2 + b^2 k_y^2 + c^2 k_z^2)/4} + i c_p k J_3(k) e^{-(a^2 k_x^2 + b^2 k_y^2 + c^2 k_z^2)/4} \right\}. \]

(13)
where we introduced the integrals

\[
J_1(k) = \left\langle \varphi_k(\rho) \bigg| e^{i(c_0-1)k\rho} p \bigg| \varphi_k(\rho) \right\rangle_{k=2(\text{scat.state})},
\]

\[
J_2(k) = \left\langle \varphi_k(\rho) \bigg| e^{i(c_0)k\rho} p \bigg| \varphi_k(\rho) \right\rangle_{k=2(\text{scat.state})},
\]

\[
J_3(k) = \left\langle \varphi_k(\rho) \bigg| e^{-i(k\rho)} p \bigg| \varphi_k(\rho) \right\rangle_{k=2(\text{scat.state})}.
\]

(14)

According to results of variational analysis for the \(\alpha\)-particle \([81]\), it has spherically symmetric shape in the ground state (we have \(a = b = c = 1.02 \text{ fm}\) and eqs. \([13]\) are simplified as

\[
M_{21} = \frac{c_A}{5} \left\{ 2i(c_0 - 1)k e^{-(c_0-1)^2a^2k^2/4} + J_1(k) \right\},
\]

\[
M_{22} = \frac{c_p}{10} \left\{ 2 J_2(k) e^{-a^2k^2/4} + i c_p k J_3(k) e^{-c_p^2a^2k^2/4} \right\},
\]

(15)

where \(k^2 = k_x^2 + k_y^2 + k_z^2\). Now we calculate multiplications of these functions on vectors of polarization of photons. Taking into account that vectors \(e_{1,2}^\alpha\) and \(k\) are perpendicular, we have property:

\[
\sum_{\alpha=1,2} e_\alpha^{(\alpha),*} M_{21} = \frac{c_A}{5} \sum_{\alpha=1,2} e_\alpha^{(\alpha),*} J_1(k),
\]

\[
\sum_{\alpha=1,2} e_\alpha^{(\alpha),*} M_{22} = \frac{c_p}{5} \sum_{\alpha=1,2} e_\alpha^{(\alpha),*} J_2(k) e^{-a^2k^2/4}
\]

(16)

and the full matrix element \([6]\) is equal to

\[
\left\langle \Psi_f \bigg| \hat{H}_s (1) \bigg| \Psi_i \right\rangle = -\frac{e}{m_p} \sqrt{\frac{2\pi}{\omega_{ph}}} p_{fi} \delta(K_f - k),
\]

(17)

where

\[
p_{fi} = \frac{1}{5} \left\{ c_A \sum_{\alpha=1,2} e_\alpha^{(\alpha),*} J_1(k) - c_p \sum_{\alpha=1,2} e_\alpha^{(\alpha),*} J_2(k) \right\}.
\]

(18)

For calculation of these integrals, we apply multipole expansion of wave function of photons. According to formalism in \([7]\), at quantum numbers \(l_f = 0, \mu_f = 1\) and \(l_{ph} = 1\) we have the following formula:

\[
\sum_{\alpha=1,2} e_\alpha^{(\alpha)} \left\langle k_f \bigg| e^{-i(kr)} \nabla \bigg| k_i \right\rangle_r = c_M \cdot J(1,1) + c_{E1} \cdot J(1,0) + c_{E2} \cdot J(1,2)
\]

(19)

where

\[
J(l_f, n) = \int_0^{+\infty} \frac{dR_z(r)}{dr} R_f^t(l, r) j_{\mu}(kr) r^2 dr,
\]

(20)

and

\[
c_M = \sqrt{\frac{3\pi}{2}} \sum_{\mu=\pm1} h\mu I(1,1,\mu), \quad c_{E1} = \sqrt{\pi} \sum_{\mu=\pm1} h\mu I(1,0,\mu), \quad c_{E2} = -\sqrt{\pi} \sum_{\mu=\pm1} h\mu I(1,2,\mu).
\]

(21)

Here, \(I(l_f, l_{ph}, n, \mu)\) are angular integrals defined in \([23, 24]\). In the approximation of the leading integrals (in calculations, \(J(1,0)\) is the largest almost always, about on 10 times than other integrals), we obtain:

\[
\sum_{\alpha=1,2} e_\alpha^{(\alpha)} \left\langle k_f \bigg| e^{-i(kr)} \nabla \bigg| k_i \right\rangle_r = c_{E1} \cdot J(1,0).
\]

(22)
We define the cross-section of the emitted photons on the basis of matrix element \( \langle 17 \rangle \) in frameworks of formalism given in \([23, 26]\) and we do not repeat it in this paper. In result, we obtain the bremsstrahlung cross-section as

\[
\frac{d^2 \sigma(\theta_f)}{d\omega_{ph} d\cos \theta_f} = \frac{e^2}{2\pi c^3} w_{ph} E_i \left\{ p_{fi} \frac{d p_{fi}^\prime(\theta_f)}{d \cos \theta_f} + \text{c.c.} \right\},
\]

where c. c. is complex conjugation, \( p_{fi} \) is proportional to the electrical component \( p_{el} \) in Eqs. (10) in \([25]\) and \( d p_{fi}^\prime(\theta_f)/d \cos \theta_f \) is defined by the same way as \( d p / (k_i, k_f, \theta_f)/d \cos \theta_f \) in Ref. \([25]\).

Experimental data of Wölfl, Hall and Müller \([71]\) provides information about the emitted photons in the coplanar geometry. So, in order to compare our approach and calculations with those data, we have to apply kinematic relations given by Eqs. (2)–(3) in Ref. \([3]\) (see also Eqs. (15) and (16) in \([3]\), also \([2]\)):

\[
E_{ph} = E_p \left\{ 1 - \frac{4 \sin^2 \theta_\alpha + \sin^2 \theta_p}{4 \sin^2(\theta_\alpha) + \theta_p} \right\}, \quad E_f = E_i - E_{ph},
\]

where \( E_p \) is incident proton energy in the laboratory frame, \( E_{ph} \) is photon energy, \( E_i \) and \( E_f \) are relative energies in the system-of-mass frame of the proton – \( \alpha \)-particle system in the states before emission of photon (i.e. initial state) and after this emission (i.e. final state). For the energy of the system in the initial state (i.e. before the emission of photon) in the center-of-mass frame we have \( E_i = m_{\alpha i}/(m_{\alpha i} + m_{\alpha}) E_\alpha \). In the experimental setup \([71]\) the coplanar arrangement of detectors was used. Here, protons are measured at the laboratory angle of 70° concerning the beam axis (i.e. \( \theta_p = 70^\circ \)), while the \( \alpha \)-particles are measured at opposite angle of 30° (\( \theta_\alpha = 30^\circ \)).

For the scattering states, we use the space wave function \( \varphi_s(\rho) \) in the spherically symmetric approximation (i.e. where state of the full system is dependent only on relative distance \( \rho = |\rho| \) between the scattered proton and center-of-mass-of the \( \alpha \)-particle). We have

\[
\varphi(\rho) = \varphi(\rho, \theta, \phi) = \frac{R_l(\rho)}{\rho} Y_{lm}(\theta, \phi),
\]

where \( R_l(\rho) \) and \( Y_{lm}(\theta, \phi) \) are radial and angular wave functions, \( l \) and \( m \) are quantum numbers. General solution of the wave function we find as linear combination of two independent partial solutions \( c_{l1}(\rho) \) and \( c_{2l}(\rho) \) as

\[
R_l(\rho) = A_l c_{l1}(\rho) + B_l c_{2l}(\rho),
\]

where \( A_l \) and \( B_l \) are unknown amplitudes. In the asymptotic limit, where we have only action of Coulomb forces, we apply

\[
c_{l1}(\rho) \rightarrow G_l(\rho), \quad c_{2l}(\rho) \rightarrow F_l(\rho),
\]

where \( G_l(\rho) \) and \( F_l(\rho) \) are Coulomb wave functions. Also quantum mechanics requires to apply condition of finite value of wave function \( R_l(\rho) \) at zero \( \rho = 0 \). These conditions and normalization of the full wave function (for the scattering states in the continuous energy spectrum) allow us to determine the amplitudes \( A_l \) and \( B_l \). Using determination of our wave functions above, we can find phase shifts \( \delta_l \), analyzed in details in \([4]\). For this, we use definition (9) in \([3]\), and obtain:

\[
\tan \delta_l = \frac{A_l}{B_l},
\]

The radial wave functions \( c_{l1} \) and \( c_{2l} \) we calculate numerically. For description of interactions between proton and nucleus we extrapolate potential with parameters from Ref. \([82]\), which was intensively studied and tested in wide region of nuclei for long time\(^1\). In order to provide a numerical basis for analysis of experimental data via computer,
we shall use the following functions of errors \[28\]:

\[
\begin{align*}
\varepsilon_1 &= \frac{1}{N} \sum_{k=1}^{N} \left| \frac{\sigma^{(\text{theor})}(E_k) - \sigma^{(\text{exp})}(E_k)}{\sigma^{(\text{exp})}(E_N)} \right|, \\
\varepsilon_2 &= \frac{1}{N} \sum_{k=1}^{N} \left| \frac{\sigma^{(\text{theor})}(E_k) - \sigma^{(\text{exp})}(E_k)}{\sigma^{(\text{exp})}(E_k)} \right|, \\
\varepsilon_3 &= \frac{1}{N} \sum_{k=1}^{N} \left| \frac{\ln(\sigma^{(\text{theor})}(E_k)) - \ln(\sigma^{(\text{exp})}(E_k))}{\ln(\sigma^{(\text{exp})}(E_k))} \right|,
\end{align*}
\]  

(29)

where \(\sigma^{(\text{theor})}(E_k)\) and \(\sigma^{(\text{exp})}(E_k)\) are theoretical and experimental bremsstrahlung cross-sections at the proton incident energy \(E_k\), the summation is performed over experimental data set, and \(N\) is number of experimental data points (\(N = 6\) for data \[70\])².

### III. CALCULATIONS, ANALYSIS

We applied the method above to calculate the spectrum of the bremsstrahlung photons emitted during the scattering of protons off \(\alpha\)-particles at the incident proton energies up to 20 MeV. Previous results given in Ref. \[4\] for the proton incident energies up to 25 MeV are based on the calculations of the corresponding wave functions for the scattering states, and maxima in the bremsstrahlung spectra are explained by phase shift of these wave functions. By such a reason, we started our calculations from analysis of the phase shifts of the wave functions for the scattering states, which our approach provides. Results of such calculations for the first some states are presented in Fig. 1. Comparing

![Phase shifts for the scattering of protons off \(\alpha\)-particles calculated by Eq. (28) in frameworks of our model inside the incident proton energies till 20.0 MeV. Here, blue solid line is the calculated shift for \(s_{1/2}\) state, red dashed line is the calculated shift for \(p_{3/2}\) state, green dash-dotted line is the calculated shift for \(p_{1/2}\) state, blue squared points, red triangular points, and green circular points are empirical data for the \(s_{1/2}\), \(p_{3/2}\) and \(p_{1/2}\) states taken from Table III in \[79\]. One can see that these calculations qualitatively corresponds to data calculated by Liu, Tang and Kanada \[4\] with potential set III of \[83\] and empirical data of Arndt, Roper, and Shotwell \[79\] (see Fig. 1 in \[4\], for details). This result can be considered as some good indication of logical coincidence of the wave functions for the scattering states in our model and the corresponding wave functions in the model \[4\]. As a next step, one can obtain essentially higher coincidence between our phase shifts and empirical data \[79\] that is subject of inverse theory (i.e., can be resolved completely, in principle) and, so, we should like to do not consider this technical task in this paper.

\[\text{FIG. 1: (Color online) Phase shifts for the scattering of protons off } \alpha\text{-particles calculated by Eq. (28) in frameworks of our model inside the incident proton energies till 20.0 MeV. Here, blue solid line is the calculated shift for } s_{1/2} \text{ state, red dashed line is the calculated shift for } p_{3/2} \text{ state, green dash-dotted line is the calculated shift for } p_{1/2} \text{ state, blue squared points, red triangular points, and green circular points are empirical data for the } s_{1/2}, p_{3/2} \text{ and } p_{1/2} \text{ states taken from Table III in } [79]. \text{ One can see that these calculations qualitatively corresponds to data calculated by Liu, Tang and Kanada } [4] \text{ with potential set III of } [83] \text{ and empirical data of Arndt, Roper, and Shotwell } [79] \text{ (see Fig. 1 in } [4], \text{ for details). This result can be considered as some good indication of logical coincidence of the wave functions for the scattering states in our model and the corresponding wave functions in the model } [4]. \text{ As a next step, one can obtain essentially higher coincidence between our phase shifts and empirical data } [79] \text{ that is subject of inverse theory (i.e., can be resolved completely, in principle) and, so, we should like to do not consider this technical task in this paper.} \]

² We introduced three different functions of errors in Eqs. (28) in order to analyze the spectra more carefully in the different energy regions, as they usually have exponential forms (for example, see Fig. 2(b)).
these calculation with data calculated by Liu, Tang and Kanada \[4\] with potential set III of \[83\] and empirical data of Arndt, Roper, and Slottweli \[79\], we conclude that our calculated wave functions (for any states) have the same behavior, and are in some not bad agreement (also see Fig. 1 in \[4\] for comparison). Supposing, that better agreement between our calculations of the phase shifts and others results can be achieved using inverse scattering theory and is independent more technical task, in this paper we should like to focus on the physical insight of what our approach would give in analysis of the bremsstrahlung experimental data.

As a next step, we calculated contributions of the emitted bremsstrahlung photons from some first transitions to the full spectrum in frameworks of our model. Such calculations compared with experimental data of Wöllfl, Hall and Müller \[70\] (we take them from Table 1 in that paper) are presented in Fig. 2. Here, one can see that the spectra are increased monotonously with increasing of the energy of proton for almost all studied transitions. But we observe a visible maximum at the proton energy of $E_p = 6\text{ MeV}$ in the spectrum for the transition between states $p_{i,1/2}$ $(L_i = 1$ and $S_i = -1/2$) and $p_{j,3/2}$ $(L_f = 1$ and $S_f = +1/2$). Also there are little variations in the spectra for the transitions between states with $L_i = 1$ and $L_f = 2$ at low proton energies (which can be related with limits in accurate calculations for the smaller cross-sections).

The full bremsstrahlung spectrum is obtained at summation of the different contributions. Different ratios between contributions give different shapes of the resulting spectrum. In particular, one can find that hump in the experimental cross-sections \[71\] can be described if contribution formed in transition $p_{i,1/2} \rightarrow p_{j,3/2}$ is leading. Moreover, changing ratios between contributions, one can displace this hump along the proton incident energy axis. For example, in Fig. 2a) we demonstrate shift of such hump of the summarized spectrum, by changing ratio (defined by factor $f$) between contribution for transition $p_{i,1/2} \rightarrow p_{j,3/2}$ and contribution for transition $s_{i,1/2} \rightarrow p_{j,1/2}$.

Now we suppose that the wave function of the scattering state of proton off the $\alpha$-particle should include also possibility to form a combined system of $^5\text{Li}$ which lives for some short time. In formation of such a combined system the fusion and opposite disintegration processes play important role. For example, a clear picture of importance to study fusion in forming composed systems the $\alpha$-decay provide us, which can be analyzed in three different stages: (1) formation of a quantum object ($\alpha$- cluster having purely wave nature, behavior and quantum description) from some nucleons inside a space region of a parent nucleus, (2) internal oscillations of this cluster before its further escaping outside, and (3) tunneling transition of this cluster from nuclear region outside (related with tunneling through the barrier). It is impossible to calculate correctly $\alpha$-decay half-lives for any nuclei without two first stages. But, at the
So, we see that such coefficients characterize intensity of formation (fusion) and breakup (disintegration).

\[ \alpha \text{ of protons off the} \]

(such data are obtained on the basis of minimization of \( \varepsilon_2 \), we obtain \( \varepsilon_2 = 0.141 \)) brown dash-double dotted line (calc. 2) is our calculated spectrum based on transitions given in Fig. 2(a) only, green dash-dotted line is the spectrum calculated by Dohet-Eraly with used non-Siegert operator in \([12]\) (see Fig. 3(a) in that paper). Our calculated curves for full spectrum have oscillatory behavior, but they can be and some straighten (and agreement with experimental data can be improved) after using complex values for the amplitudes \( f \) and inclusion of the other scattering states.

same time, calculations of the bremsstrahlung spectra in \( \alpha \)-decay, based only on inclusion of the third stage into the model, are very successful in description of experimental data (see \([16, 19, 30, 37, 39, 49]\), and reference therein).

Here, we note our previous progress in quantum study of fusion processes in \( \alpha \)-capture by the \( ^{40} \text{Ca} \) and \( ^{44} \text{Ca} \) nuclei \([28]\). As we demonstrated in that paper, quantum effects in fusion play important role (i.e. they can be not small), their inclusion into the model and calculations allows to essentially improve agreement with experimental data, and quantum description of such processes can be performed via additional amplitudes (characterizing probabilities of presence of these processes in the different states). Along with a logic in \([28]\), instead of formula \([22]\) we introduce a new one as (here, we add numbers \( i_l, s_i, s_f \) to designation of integral)

\[
\sum_{\alpha=1,2} e^{(\alpha)} \left( k_f \left| e^{-ikr} \right| k_i \right)_r = c E_1 \sum_{l_i, l_f, s_i, s_f} f_{l_i, s_i, l_f, s_f} f(l_i, s_i, l_f, s_f, n = l_f - 1),
\]

where \( f_{l_i, s_i, l_f, s_f} \) are new real amplitudes characterized a possibility of formation of the compound \(^5\text{Li}\) nucleus and its breakup (disintegration) during the transition between the initial and final states (further in the paper, amplitudes of the compound nucleus formation, we have \( 0 \leq f_{l_i, s_i, l_f, s_f} \leq 1 \)). Case of \( f_{l_i=0, s_i=+1/2, l_f=1, s_f=\pm 1/2} = 1 \) and \( f_{l_i, s_i, l_f, s_f} = 0 \) at other quantum numbers transforms formula \([30]\) to the old Eq. \([22]\), that corresponds to complete absence of the formation of the compound nucleus in the scattering (i.e. the scattering takes place without any appearance of \(^5\text{Li}\)). So, we see that such coefficients characterize intensity of formation (fusion) and breakup (disintegration).

In order to clarify, if such processes are negligibly small or strong, we shall look for values of the amplitudes \( f_{l_i, s_i, l_f, s_f} \), at which agreement between calculations and experimental data is the best. In Fig. 3(b) we present results of such calculations in comparison with experimental data, where the found non-zero amplitudes are

\[
f_{0, +1/2, 1, -1/2} = f(s_{1/2} \rightarrow p_{1/2}) = 0.45263, \quad f_{1, -1/2, 2, -1/2} = f(p_{1/2} \rightarrow d_{3/2}) = 0.655.
\]

\[\text{FIG. 3: (Color online) The coplanar calculated cross sections of the bremsstrahlung photons emitted during the scattering of protons off the } \alpha \text{ particles, and experimental data (squared points) of Wöllli, Hall and Müller [73]. (a) In this picture we demonstrate how one can displace hump of the full spectrum along the proton incident energy axis, by changing ratio (defined by factor } f \text{) between contribution for transition } p_{1/2} \rightarrow p_{3/2} \text{ and contribution for transition } s_{1/2} \rightarrow p_{1/2}. \text{ (b) Here, blue solid line (calc. 1) is our calculated full spectrum based on all transitions given in Fig. 2(a, b) with amplitudes given in Eq. } \text{ (30)} \text{ (such data are obtained on the basis of minimization of } \varepsilon_2 \text{, we obtain } \varepsilon_2 = 0.141 \text{) brown dash-double dotted line (calc. 2) is our calculated spectrum based on transitions given in Fig. 2(a) only, green dash-dotted line is the spectrum calculated by Dohet-Eraly with used non-Siegert operator in } \text{[12]} \text{ (see Fig. 3(a) in that paper). Our calculated curves for full spectrum have oscillatory behavior, but they can be and some straighten (and agreement with experimental data can be improved) after using complex values for the amplitudes } f \text{ and inclusion of the other scattering states.} \]
From analysis of the obtained calculations we conclude that (1) inclusion of the amplitudes of the compound nucleus formation allows to essentially improve agreement between the full bremsstrahlung spectrum and the experimental data, (2) transition $p_{1/2} \rightarrow p_{3/2}$ with a visible maximum at the proton energy of $E_p = 6$ MeV in the spectrum (see brown dashed line in Fig. 2(a)) does not play a role in description of hump in the experimental bremsstrahlung data.

IV. CONCLUSIONS

In this paper we develop microscopic formalism of our bremsstrahlung model, which was previously successfully tested in description of experimental data in alpha-decay, proton emission from nuclei, spontaneous fission, ternary fission, scattering of protons (at fixed incident energies) off nuclei in region of the emitted photons from lowest up to intermediate. We focus on the scattering of protons off the $\alpha$-particles. In description of the scattering states we implement our quantum formalism for calculations of wave functions based on the scattering theory. In result, we obtain enough good agreement between the spectra calculated in frameworks of our approach and experimental data \cite{70} (see Fig. 3(b)). We formulate conclusions of application of this model in analysis of experimental data for this reaction.

1. In the model connection between the bound states of nucleons inside the $\alpha$-particle, the scattering states and parameters of the emitted photon is obtained (see Eq. (13)). But, influence of the parameters of the one-nucleon wave function on the bremsstrahlung spectrum is very small (less than 1 percent).

2. Analyzing bremsstrahlung experimental data, we observe compound nucleus $^5$Li formed in scattering of protons off the $\alpha$-particles. Main transitions responsible for creation of such a nucleus are $s_{1/2} \rightarrow p_{1/2}$ and $p_{1/2} \rightarrow d_{3/2}$.

3. In order to provide qualitative description of formation of the compound $^5$Li nucleus, we introduce the new amplitudes characterizing probabilities of formation of this nucleus and its breakup (disintegration) at definite transitions. Analyzing experimental data, we obtain non-zero values $f(s_{1/2} \rightarrow p_{1/2}) = 0.45263$ and $f(p_{1/2} \rightarrow d_{3/2}) = 0.655$.

Note that only one experimental data point in \cite{70} at the emitted photon energy at 6.9 MeV gives a decreasing tendency of the spectrum at decreasing of energy of photons. This is not enough for a proper basis to conclude about such a behavior of the spectrum at the smaller photon energy. If to compare the experimental data \cite{70} with the existed experimental data of the bremsstrahlung photons at the scattering of protons off nuclei \cite{50, 51, 52} (see also reviews on situation in such a research \cite{61, 62}), $\alpha$-decay \cite{17, 18, 30, 34}, heavy-ion collisions \cite{84}, spontaneous fission \cite{62, 68}, neutron-induced fission \cite{85}, one can find that experimental information from \cite{70} is essentially not so rich. In such regards, we see a sense in re-measuring of the bremsstrahlung emission of photons at the scattering of protons off the $\alpha$-particles inside the studied region of the emitted photon energy, and we propose for experimental people to organize such experiments.

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Appendix A: Summation over spin-isospin states

At first, we shall find summations in the term $M_{22}$ over spin and isospin states. We shall study a case where nucleus is composed from even number $Z$ of protons and even number $N$ of neutrons. For simplicity, we shall analyze a case of the scattering proton with spin -1/2 in this paper. Write summations over proton and neutrons states separately as

$$
\sum_{k=1}^{A+1} \sum_{m=1, m \neq k}^{A+1} S_{km} = \sum_{k=1}^{Z+1} \sum_{m=1, m \neq k}^{Z+1} \delta_{\tau_k, 1/2} \delta_{\tau_m, 1/2} S_{km} + \sum_{k=1}^{Z+1} \sum_{m=1, m \neq k}^{N} \delta_{\tau_k, 1/2} \delta_{\tau_m, -1/2} S_{km} + \\
+ \sum_{k=1}^{N} \sum_{m=1, m \neq k}^{Z+1} \delta_{\tau_k, -1/2} \delta_{\tau_m, 1/2} S_{km} + \sum_{k=1}^{N} \sum_{m=1, m \neq k}^{N} \delta_{\tau_k, -1/2} \delta_{\tau_m, -1/2} S_{km}.
$$

(A1)
Here, we consider the first term where we select summations over spin states:

\[
\begin{align*}
\sum_{k=1}^{Z+1} \sum_{m=1,m \neq k}^{Z+1} \delta_{\tau_k,1/2} \delta_{\tau_m,1/2} S_{km} &= \\
&= \sum_{k=1}^{Z/2+1} \sum_{m=1,m \neq k}^{Z/2+1} \delta_{\tau_k,1/2} \delta_{\tau_m,1/2} \delta_{\sigma_k,-1/2} \delta_{\sigma_m,1/2} S_{km} + \sum_{k=1}^{Z/2} \sum_{m=1,m \neq k}^{Z/2} \delta_{\tau_k,1/2} \delta_{\tau_m,1/2} \delta_{\sigma_k,-1/2} \delta_{\sigma_m,1/2} S_{km} + \\
&+ \sum_{k=1}^{Z/2} \sum_{m=1,m \neq k}^{Z/2} \delta_{\tau_k,1/2} \delta_{\tau_m,1/2} \delta_{\sigma_k,1/2} \delta_{\sigma_m,1/2} S_{km}.
\end{align*}
\]

(A2)

Now we take into account that operator in (12) does not act on spin and isospin states. So, we use properties of orthogonality of the wave functions in spin and isospin states. For spin states we have

\[
\langle \downarrow_j | \downarrow_j \rangle = \langle \uparrow_i | \uparrow_j \rangle = \delta_{ij}, \quad \langle \downarrow_i | \uparrow_j \rangle = \langle \uparrow_i | \downarrow_j \rangle = 0.
\]

(A3)

Analogical formulas we have for isospin states. At first, we calculate terms for proton states in matrix element \( M_{22} \):

\[
\begin{align*}
\sum_{k=1}^{Z+1} \sum_{m=1,m \neq k}^{Z+1} \left\{ \langle \psi_k(i_p) | e^{i c_p k \mathbf{r}} \mathbf{p} | \psi_k(i_p) \rangle \langle \psi_m(j_A) | f_j(\rho_{Aj}) | \psi_m(j_A) \rangle - \\
- \langle \psi_k(i_p) | e^{i c_p k \mathbf{r}} \mathbf{p} | \psi_m(i_p) \rangle \langle \psi_m(j_A) | f_j(\rho_{Aj}) | \psi_k(j_A) \rangle \right\} &= \\
&= 4 \sum_{k=1}^{Z/2} \langle \varphi_k(\mathbf{r}) | e^{i c_p k \mathbf{r}} \mathbf{p} | \varphi_k(\mathbf{r}) \rangle \sum_{m=1,m \neq k}^{Z/2} \langle \varphi_m(\rho_{Aj}) | f_j(\rho_{Aj}) | \varphi_m(\rho_{Aj}) \rangle + \\
&+ 2 \langle \varphi_k(\mathbf{r}) | e^{i c_p k \mathbf{r}} \mathbf{p} | \varphi_k(\mathbf{r}) \rangle_{k \neq Z/2+1} \sum_{m=1}^{Z/2} \langle \varphi_m(\rho_{Aj}) | f_j(\rho_{Aj}) | \varphi_m(\rho_{Aj}) \rangle + \\
&+ 2 \sum_{k=1}^{Z/2} \langle \varphi_k(\mathbf{r}) | e^{i c_p k \mathbf{r}} \mathbf{p} | \varphi_k(\mathbf{r}) \rangle \langle \varphi_m(\rho_{Aj}) | f_j(\rho_{Aj}) | \varphi_m(\rho_{Aj}) \rangle_{m \neq Z/2+1}.
\end{align*}
\]

(A4)

Taking into account orthogonality conditions between isospin states for the one-nucleon wave functions, and that operator \( f_j(\rho_{Aj}) \) gives zero in acting on the one-nucleon wave function at neutron states, we find other terms in matrix elements (11) to be equal to zero. Summarizing, we find the term \( M_{22} \):

\[
M_{22} = c_p \sum_{j=1}^{Z_A} \frac{1}{A(A+1)} \left\{ 4 \sum_{k=1}^{Z/2} \langle \varphi_k(\mathbf{r}) | e^{i c_p k \mathbf{r}} \mathbf{p} | \varphi_k(\mathbf{r}) \rangle \sum_{m=1,m \neq k}^{Z/2} \langle \varphi_m(\rho_{Aj}) | f_j(\rho_{Aj}) | \varphi_m(\rho_{Aj}) \rangle + \\
+ 2 \langle \varphi_k(\mathbf{r}) | e^{i c_p k \mathbf{r}} \mathbf{p} | \varphi_k(\mathbf{r}) \rangle_{k \neq Z/2+1} \sum_{m=1}^{Z/2} \langle \varphi_m(\rho_{Aj}) | f_j(\rho_{Aj}) | \varphi_m(\rho_{Aj}) \rangle + \\
+ 2 \sum_{k=1}^{Z/2} \langle \varphi_k(\mathbf{r}) | e^{i c_p k \mathbf{r}} \mathbf{p} | \varphi_k(\mathbf{r}) \rangle \langle \varphi_m(\rho_{Aj}) | f_j(\rho_{Aj}) | \varphi_m(\rho_{Aj}) \rangle_{m \neq Z/2+1} \right\}.
\]

(A5)

Now we shall calculate the term \( M_{21} \). Let us consider such a summation where we write separately terms for different spin and isospin states

\[
\begin{align*}
\sum_{k=1}^{A+1} S_{km} &= \sum_{k=1}^{Z/2+1} \delta_{\tau_k,1/2} \delta_{\sigma_k,-1/2} S_{km} + \sum_{k=1}^{Z/2} \delta_{\tau_k,1/2} \delta_{\sigma_k,1/2} S_{km} + \\
&+ \sum_{k=1}^{Z/2} \delta_{\tau_k,-1/2} \delta_{\sigma_k,-1/2} S_{km} + \sum_{k=1}^{Z/2} \delta_{\tau_k,-1/2} \delta_{\sigma_k,1/2} S_{km}.
\end{align*}
\]

(A6)
Taking into account orthogonality conditions between isospin states for the one-nucleon wave functions, and that operator in matrix element $M_{21}$ gives zero in acting on the one-nucleon wave function at neutron states, we obtain:

$$M_{21} = \frac{c_A}{A + 1} \left\{ 2 \sum_{k=1}^{\frac{Z}{2}} \langle \varphi_k(r) | e^{i c_p k r} e^{-i k r} f_p(\rho_p) | \varphi_k(r) \rangle + 2 \sum_{k=1}^{\frac{N}{2}} \langle \varphi_k(r) | e^{i c_p k r} e^{-i k r} f_p(\rho_p) | \varphi_k(r) \rangle \right\} + \left( \langle \varphi_k(r) | e^{i c_p k r} e^{-i k r} f_p(\rho_p) | \varphi_k(r) \rangle \right)_{k=\frac{Z}{2}+1} \right\}$$

where

$$\langle \varphi_000(\rho) | e^{-i k \rho} | \varphi_000(\rho) \rangle = e^{-\left( a^2 k_x^2 + b^2 k_y^2 + c^2 k_z^2 \right) / 4},$$

$$\langle \varphi_000(\rho) | e^{i c_p k \rho} p | \varphi_000(\rho) \rangle = \frac{i c_p}{2} e^{-c_p^2 \left( a^2 k_x^2 + b^2 k_y^2 + c^2 k_z^2 \right) / 4},$$

$$\langle \varphi_000(\rho) | e^{i c_p k \rho} e^{-i k \rho} p | \varphi_000(\rho) \rangle = \frac{1}{2} (c_p - 1) k e^{-\left( c_p - 1 \right)^2 \left( a^2 k_x^2 + b^2 k_y^2 + c^2 k_z^2 \right) / 4}. (A7)$$

In a case of the $\alpha$-particle (in the ground state) as the target-nucleus, we calculate such integrals (see Appendix B):

$$\langle \varphi_{k_{\text{bound}}}(r) | e^{i c_p k r} p | \varphi_{k_{\text{bound}}}(r) \rangle, \langle \varphi_{k_{\text{bound}}}(\rho) | f_j(\rho_A) | \varphi_{k_{\text{bound}}}(\rho_A) \rangle. (B1)$$

For the scattering state (which we label by index $k_{\text{scat}} = \frac{Z}{2} + 1$) we have

$$\langle \varphi_{k_{\text{scat}}}(r) | e^{i c_p k r} p | \varphi_{k_{\text{scat}}}(r) \rangle, \langle \varphi_{k_{\text{scat}}}(\rho) | f_j(\rho_A) | \varphi_{k_{\text{scat}}}(\rho_A) \rangle. (B2)$$

For the first term of the matrix element we have:

$$\langle \varphi_{k_{\text{bound}}}(r) | e^{i c_p k r} e^{-i k r} f_p(\rho_p) p | \varphi_{k_{\text{bound}}}(r) \rangle, \langle \varphi_{k_{\text{scat}}}(r) | e^{i c_p k r} e^{-i k r} f_p(\rho_p) p | \varphi_{k_{\text{scat}}}(r) \rangle, (B3)$$

where $f_p(\rho_p) = 1$ for proton.

### 1. One-nucleon space wave function and matrix element

We shall choose the space wave function of one nucleon in the gaussian form

$$\varphi_{n_x, n_y, n_z}(r) = N_x N_y N_z \cdot \exp \left\{ -\frac{1}{2} \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) \right\} \cdot H_{n_x} \left( \frac{x}{a} \right) H_{n_y} \left( \frac{y}{b} \right) H_{n_z} \left( \frac{z}{c} \right), (B4)$$

where $H_{n_x}$, $H_{n_y}$ and $H_{n_z}$ are the Hermitian polynomials, $N_x$, $N_y$, $N_z$ are the normalized coefficients. The unknown normalized coefficients are calculated from the normalization condition:

$$\int N_x \exp \left\{ -\frac{1}{2a^2} \right\} H_{n_x} \left( \frac{x}{a} \right) dx = 1, \ \int N_y \exp \left\{ -\frac{1}{2b^2} \right\} H_{n_y} \left( \frac{y}{b} \right) dy = 1, \ \int N_z \exp \left\{ -\frac{1}{2c^2} \right\} H_{n_z} \left( \frac{z}{c} \right) dz = 1. (B5)$$

Taking the properties of the Hermitian polynomials into account (see [?], p. 749):

$$\int e^{-\frac{x^2}{2a^2}} H_{n_x}(x) dx = 2^n n! \sqrt{\pi}, \ H_0 = 1, \ H_1 = 2x, \ H_2 = 4x^2 - 2, (B6)$$

we obtain:

$$N_x = \frac{1}{\pi^{1/4} \sqrt{a} 2^{n_x} n_x!}, \ N_y = \frac{1}{\pi^{1/4} \sqrt{b} 2^{n_y} n_y!}, \ N_z = \frac{1}{\pi^{1/4} \sqrt{c} 2^{n_z} n_z!}. (B7)$$
We shall calculate the second matrix element in eqs. (B11) for protons in form
\[
\left\langle \varphi_{\text{bound}}(\rho) \right| f_{AB}(\rho) \left| \varphi_{\text{bound}}(\rho) \right\rangle = \left\langle \varphi_{\text{bound}}(\rho) \right| e^{-ik\rho} \left| \varphi_{\text{bound}}(\rho) \right\rangle,
\] (B8)
where we take into account \( z_p = 1 \) for protons. Substituting form of the wave function into Eq. (B14), we calculate the matrix element for the \( \alpha \) particle:
\[
\left\langle \varphi_{n_x,n_y,n_z}(\rho) \right| e^{-ik\rho} \left| \varphi_{n_x,n_y,n_z}(\rho) \right\rangle = \int \varphi_{n_x,n_y,n_z}(\rho) e^{-ik\rho} \rho = I_x(n_x) I_y(n_y) I_z(n_z),
\] (B9)
where
\[
I_x = N_{\alpha,x}^2 \exp \left[ -a^2 k_x^2 / 4 \right] \int \exp \left[ -\frac{(x + i a^2 k_x / 2)^2}{a^2} \right] H_n^2 \left( \frac{x}{a} \right) dx. \tag{B10}
\]
and solutions for \( I_y(n_y) \) and \( I_z(n_z) \) are obtained after change of indexes \( x \to y \) and \( x \to z \).

2. Case of the scattering of the proton off \( \alpha \)-particle

Now let us consider case when the \( \alpha \) particle is in the ground state \( (n_x = n_y = n_z = 0) \). We have \( H_{n_x=0} = 1, H_{n_y=0} = 1, H_{n_z=0} = 1 \). In approximation, integral in eq. (B10) over complex variable \( \tilde{x} = x + i a^2 k_x / 2 \) has solution:
\[
\int \exp \left[ -\frac{(x + i a^2 k_x / 2)^2}{a^2} \right] dx_i = N_{\alpha,x}^{-2} \tag{B11}
\]
and we obtain:
\[
I_{\alpha,x}(n_x = 0) = \exp \left[ -a^2 k_x^2 / 4 \right]. \tag{B12}
\]
Now we calculate the matrix element (B9):
\[
\left\langle \varphi_{000}(\rho) \right| e^{-ik\rho} \left| \varphi_{000}(\rho) \right\rangle = e^{- (a^2 k_x^2 + b^2 k_y^2 + c^2 k_z^2) / 4}. \tag{B13}
\]
We shall calculate the first matrix element (B11) for protons in form
\[
\left\langle \varphi_{\text{bound}}(\rho) \right| e^{i\epsilon_p k\rho} \rho \left| \varphi_{\text{bound}}(\rho) \right\rangle = -i\hbar \left\langle \varphi_{\text{bound}}(\rho) \right| e^{i\epsilon_p k\rho} \frac{d}{d\rho} \left| \varphi_{\text{bound}}(\rho) \right\rangle. \tag{B14}
\]
Consider a case of the \( \alpha \)-particle in the ground state. We have:
\[
\left\langle \varphi_{000}(\rho) \right| e^{i\epsilon_p k\rho} \frac{d}{d\rho} \left| \varphi_{000}(\rho) \right\rangle = I_{2,x}(n_x = 0) I_{2,y}(n_y = 0) I_{2,z}(n_z = 0), \tag{B15}
\]
where
\[
I_{2,x}(n_x, a) = N_{\alpha,x}^2 \int e^{- a^2 x^2 / 2} \frac{d}{dx} e^{- a^2 x^2} e^{- i\epsilon_p k_x x} dx \tag{B16}
\]
and solutions for \( I_y(n_y) \) and \( I_z(n_z) \) are obtained after change of indexes \( x \to y \) and \( x \to z \). Calculate this integral:
\[
I_{2,x}(n_x, a) = - N_{\alpha,x}^2 \int x e^{- a^2 x^2} e^{- i\epsilon_p k_x x} dx. \tag{B17}
\]
We simplify this integral and obtain:
\[
I_{2,x}(n_x, a) = - N_{\alpha,x}^2 \exp \left[ -c_{\epsilon_p}^2 a^2 k_x^2 / 4 \right] \int (x + i\epsilon_p a^2 k_x / 2) \exp \left[ -\frac{(x + i\epsilon_p a^2 k_x / 2)^2}{a^2} \right] dx +
+ N_{\alpha,x}^2 \exp \left[ -c_{\epsilon_p}^2 a^2 k_x^2 / 4 \right] (i\epsilon_p a^2 k_x / 2) \int \exp \left[ -\frac{(x + i\epsilon_p a^2 k_x / 2)^2}{a^2} \right] dx. \tag{B18}
\]

For next integration of the first integral in this expression, we use property:

\[
\frac{d}{dx} \exp \left[ - \frac{(x + ic_p a^2 k_x/2)^2}{a^2} \right] = - \frac{2}{a^2} \left( x + ic_p a^2 k_x/2 \right) \exp \left[ - \frac{(x + ic_p a^2 k_x/2)^2}{a^2} \right].
\]

and we write

\[
(x + ic_p a^2 k_x/2) \exp \left[ - \frac{(x + ic_p a^2 k_x/2)^2}{a^2} \right] = - \frac{a^2}{2} \frac{d}{dx} \exp \left[ - \frac{(x + ic_p a^2 k_x/2)^2}{a^2} \right].
\]

Taking this equation into account, we find the first term in eq. (B18) as

\[
- \frac{N^2_{s,a}}{a^2} \exp \left[ - c_p^2 a^2 k_x^2/4 \right] \int (x + ic_p a^2 k_x/2) \exp \left[ - \frac{(x + ic_p a^2 k_x/2)^2}{a^2} \right] dx = 0,
\]

where we suppose

\[
\exp \left[ - \frac{(x + ic_p a^2 k_x/2)^2}{a^2} \right] \bigg|_{x=-\infty}^{x=\infty} = 0.
\]

The second term in eq. (B18) can be expressed via the previous found one:

\[
\frac{N^2_{s,a}}{a^2} \exp \left[ - c_p^2 a^2 k_x^2/4 \right] (ic_p a^2 k_x/2) \int \exp \left[ - \frac{(x + ic_p a^2 k_x/2)^2}{a^2} \right] dx = \frac{ic_p k_x}{2} \exp \left[ - c_p^2 a^2 k_x^2/4 \right].
\]

So, we find solution for integral:

\[
I_{s,a}(n_x,a) = \frac{ic_p k_x}{2} \exp \left[ - c_p^2 a^2 k_x^2/4 \right]
\]

and we calculate the matrix element (B2) and (B3). Now we rewrite the all obtained results:

\[
\langle \varphi_{000}(\rho) | e^{-ik\rho} \varphi_{000}(\rho) \rangle = e^{- (a^2 k_z^2 + b^2 k_y^2 + c^2 k_x^2) / 4},
\]

\[
\langle \varphi_{000}(\rho) | e^{i\rho \cdot k \rho} \varphi_{000}(\rho) \rangle = \frac{ic_p k}{2} \left( c^2 (a^2 k_z^2 + b^2 k_y^2 + c^2 k_x^2) / 4, \right.
\]

\[
\langle \varphi_{000}(\rho) | e^{i\rho \cdot k \rho} e^{-ik\rho} \varphi_{000}(\rho) \rangle = \frac{ic_p}{2} (\rho - 1) \left( c^2 (a^2 k_z^2 + b^2 k_y^2 + c^2 k_x^2) / 4, \right.
\]

\[
\bigg. \bigg. e^{- (c_p - 1)^2 (a^2 k_z^2 + b^2 k_y^2 + c^2 k_x^2) / 4}.
\]

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