Polarization effects in the reactions
\[ p + ^3 He \rightarrow \pi^+ + ^4 He, \quad \pi^+ + ^4 He \rightarrow p + ^3 He \]
and quantum character of spin correlations
in the final \((p, ^3 He)\) system

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Abstract. The general consequences of \(T\) invariance for the direct and inverse binary reactions \(a + b \rightarrow c + d, \quad c + d \rightarrow a + b\) with spin-1/2 particles \(a, b\) and unpolarized particles \(c, d\) are considered. Using the formalism of helicity amplitudes, the polarization effects are studied in the reaction \(p + ^3 He \rightarrow \pi^+ + ^4 He\) and in the inverse process \(\pi^+ + ^4 He \rightarrow p + ^3 He\). It is shown that in the reaction \(\pi^+ + ^4 He \rightarrow p + ^3 He\) the spins of the final proton and \(^3 He\) nucleus are strongly correlated. A structural expression through helicity amplitudes, corresponding to arbitrary emission angles, is obtained for the correlation tensor. It is established that in the reaction \(\pi^+ + ^4 He \rightarrow p + ^3 He\) one of the “classical” incoherence inequalities of the Bell type for diagonal components of the correlation tensor is necessarily violated and, thus, the spin correlations of the final particles have the strongly pronounced quantum character.

1. Consequences of \(T\) invariance for binary reactions

Let us consider the reaction \(a + b \rightarrow c + d\), where \(a, b\) are the spin-1/2 particles and the particles \(c, d\) have arbitrary spins. The effective cross-section \(\sigma_{a+b\rightarrow c+d}\) in the c.m. frame of the particles \(a\) and \(b\), summed over spin projections of the final particles \(c\) and \(d\), has the structure \([1]\):

\[ \sigma_{a+b\rightarrow c+d}(k_a, P^{(a)}, P^{(b)}; k_c) = \sigma_0(E, \theta) L(k_a, P^{(a)}, P^{(b)}; k_c), \]

where \(\sigma_0(E, \theta)\) is the respective cross-section for unpolarized particles \(a, b\) and \(L\) is the linear function of the polarization vectors \(P^{(a)}\) and \(P^{(b)}\):

\[ L(k_a, P^{(a)}, P^{(b)}; k_c) = 1 + A(E, \theta)(P^{(a)}n) + B(E, \theta)(P^{(b)}n) + C(E, \theta)(P^{(a)}P^{(b)}) + +D(E, \theta)(P^{(a)}1)(P^{(b)}1) + F(E, \theta)(P^{(a)}m)(P^{(b)}m) + G(E, \theta)(P^{(a)}1)(P^{(b)}m) + H(E, \theta)(P^{(a)}m)(P^{(b)}1). \]

Here \(1, m, n\) are the mutually orthogonal unit vectors, defined as:

\[ l = k_a/k_c; \quad m = \frac{Y-1(Y1)}{\sin \theta}; \quad n = \frac{1 \times Y}{\sin \theta} \]

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( \mathbf{P} = \mathbf{k}_c / k_c ); k_a, k_c are the respective momenta of the particles a and c, E is the total energy in the c.m. frame and \( \theta = \arccos (1 \mathbf{P}) \) is the emission angle.

Meanwhile, for the inverse reaction \( c + d \rightarrow a + b \) with the unpolarized particles \( c, d \) and the fixed polarization vectors of the final particles \( \mathbf{c}^{(a)}, \mathbf{c}^{(b)} \) the effective cross-section takes, due to \( T \) invariance and the principle of detailed balance, the following form [1]:

\[
\sigma_{c+d \rightarrow a+b}(\mathbf{k}_c; k_c, \mathbf{c}^{(a)}, \mathbf{c}^{(b)}) = \frac{1}{4} \tilde{\sigma}_0(E, \theta) L(-k_a, -\mathbf{c}^{(a)}; -\mathbf{c}^{(b)}; -k_c),
\]

where

\[
\tilde{\sigma}_0(E, \theta) = \frac{4k^2}{k^2(2j_c + 1)(2j_d + 1)} \sigma_0(E, \theta)
\]
is the cross-section of the inverse reaction, summed over the spin projections of the final particles \( a, b \). The two-particle spin density matrix \( \hat{\rho}^{(a,b)} \) for the final particles \( a, b \) can also be expressed through the same function \( L \), replacing the polarization vectors by the vector Pauli operators:

\[
\hat{\rho}^{(a,b)} = \frac{1}{4} \left[ \hat{1}^{(a)} \otimes \hat{1}^{(b)} + (\mathbf{P}^{(a)}(E, \theta) \hat{\sigma}^{(a)}) \otimes \hat{1}^{(b)} + \hat{1}^{(a)} \otimes (\mathbf{P}^{(b)}(E, \theta) \hat{\sigma}^{(b)}) + \right.
\]

\[
\left. + \sum_{i=1}^{3} \sum_{k=1}^{3} T_{ik}(E, \theta) \hat{\delta}^{(a)}_k \otimes \hat{\delta}^{(b)}_k \right] = \frac{1}{4} \hat{L}(-k_a, -\hat{\sigma}^{(a)}; -\hat{\sigma}^{(b)}; -k_c).
\]

In Eq. (6) \( \hat{a}^{(a)}, \hat{1}^{(b)} \) are two-row unit matrices, \( \mathbf{P}^{(a)}(E, \theta) = A(E, \theta) \mathbf{n}, \mathbf{P}^{(b)}(E, \theta) = -B(E, \theta) \mathbf{n} \) are the polarization vectors of the particles \( a \) and \( b \),

\[
T_{ik}(E, \theta) = C(E, \theta) \delta_{ik} + D(E, \theta) l_k l_i + F(E, \theta) m_i m_k + G(E, \theta) l_k m_i + H(E, \theta) m_i l_k
\]
are components of the correlation tensor describing spin correlations in the final two-particle system \( (a, b) \). In doing so, all the functions \( A, B, C, D, F, G, H, \sigma_0 \) and the unit vectors \( \mathbf{l}, \mathbf{m}, \mathbf{n} \) in Eqs. (1)–(7) are the same as for the direct reaction \( a+b \rightarrow c+d \).

Thus, due to \( T \) invariance, the dependence of the effective cross-section of the direct reaction \( a+b \rightarrow c+d \) upon the polarizations of initial particles completely determines the polarization vectors and spin correlations for the same particles \( a, b \) produced in the inverse reaction \( c+d \rightarrow a+b \) with unpolarized primary particles.

2. Polarization effects in the reaction \( p + ^3\text{He} \rightarrow \pi^+ + ^4\text{He} \)

This reaction belongs to the type \( 1/2 + 1/2 \rightarrow 0 + 0 \) (the proton and \( ^3\text{He} \) nucleus have spin \( 1/2 \), \( \pi^+ \) and \( ^4\text{He} \) have zero spin). Thus, on account of the negative internal parity of the \( \pi^+ \) meson, this reaction can proceed only from triplet states of the system \( (p, ^3\text{He}) \) [2,3] (as follows from the parity and angular momentum conservation).

Let us choose the axis of the total spin quantization \( z \) along the vector \( \mathbf{l} = k_p / k_p \). There exist three possible triplet states of the \( (p, ^3\text{He}) \) system with the spin projections \( +1, -1 \) and 0 onto \( z \):

\[
|+1, 1\rangle = |+1/2, 1\rangle^{(p)} \otimes |+1/2, 1\rangle^{(^3\text{He})}, \quad |-1, 1\rangle = |-1/2, 1\rangle^{(p)} \otimes |-1/2, 1\rangle^{(^3\text{He})},
\]

\[
|0, 1\rangle = \frac{1}{\sqrt{2}} \left( |+1/2, 1\rangle^{(p)} \otimes |-1/2, 1\rangle^{(^3\text{He})} + |-1/2, 1\rangle^{(p)} \otimes |+1/2, 1\rangle^{(^3\text{He})} \right).
\]

The two-particle spin density matrix for the \( (p, ^3\text{He}) \) system is:

\[
\hat{\rho}^{(p, ^3\text{He})} = \frac{1}{4} \left( \hat{1}^{(p)} + \mathbf{P}^{(p)} \sigma^{(p)} \right) \otimes \left( \hat{1}^{(^3\text{He})} + \mathbf{P}^{(^3\text{He})} \sigma^{(^3\text{He})} \right)
\]

(9)
and it is symmetric under the interchange of spin quantum numbers of the proton and $^3H\epsilon$.

$|\psi\rangle = \sum_{\lambda} R_0^\lambda(E, \theta)|\lambda, 1\rangle$.

(10)

Finally, using Eq. (11) and the formula (9) for the spin density matrix, we find that the cross-section $\sigma_{p^+ H^- \rightarrow p + ^3H\epsilon}$ (10) is described by the general structural formula for $\sigma_{a \rightarrow c + d}$ (1)–(2), where the functions $\sigma_0$, $A$, $B$, $C$, $D$, $F$, $G$, $H$ are bilinear combinations of the helicity amplitudes $R_1$, $R_0$ [1]:

$$\sigma_0(E, \theta) = \frac{1}{4}|\psi\rangle |\psi\rangle = \frac{1}{4}(|R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2),$$  

$$A(E, \theta) = B(E, \theta) = \frac{1}{\sqrt{2}}\text{Im}(R_1(E, \theta)R_0^\dagger(E, \theta)), $$  

$$C(E, \theta) = 1, \quad D(E, \theta) = \frac{|R_0(E, \theta)|^2}{2\sigma_0(E, \theta)}, \quad F(E, \theta) = -\frac{|R_1(E, \theta)|^2}{2\sigma_0(E, \theta)}, $$  

$$G(E, \theta) = H(E, \theta) = \frac{1}{\sqrt{2}\sigma_0(E, \theta)}\text{Re}(R_1(E, \theta)R_0^\dagger(E, \theta)).$$

For the particular cases $\theta = 0$ and $\theta = \pi$, when $R_1(E, \theta) = 0$, the expression for cross-section $\sigma_{p^+ H^- \rightarrow p + ^3H\epsilon}$ takes a considerably simpler form:

$$\sigma_{p^+ H^- \rightarrow p + ^3H\epsilon} = \frac{1}{4}|R_0|^2 \left(1 + |P(p)|P(H\epsilon) - 2(P(p)P(H\epsilon)|P(p)|P(H\epsilon))\right).$$  

$$3. \text{ Spin effects in the inverse reaction } \pi^+ + ^4H\epsilon \rightarrow p + ^3H\epsilon$$

In the reaction $\pi^+ + ^4H\epsilon \rightarrow p + ^3H\epsilon$ the $(p, ^3H\epsilon)$ system is produced in the triplet state only. This state, normalized to unity, is as follows [1]:

$$|\tilde{\psi}\rangle = \frac{1}{(|R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2)^{1/2}}$$  

$$\times \left[R_1(E, \theta) \left(|1 + 1/2, 1\rangle (p) \otimes | + 1/2, 1\rangle (H\epsilon) - | - 1/2, 1\rangle (p) \otimes | - 1/2, 1\rangle (H\epsilon)\right) + \right]  

+ \frac{1}{\sqrt{2}}R_0(E, \theta) \left(|1 + 1/2, 1\rangle (p) \otimes | - 1/2, 1\rangle (H\epsilon) + | - 1/2, 1\rangle (p) \otimes | + 1/2, 1\rangle (H\epsilon)\right),$$  

and it is symmetric under the interchange of spin quantum numbers of the proton and $^3H\epsilon$. 

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Basing on the $T$ invariance, we obtain for the effective cross-section of the reaction $\pi^+ + ^4He \rightarrow p + ^3He$ in the c.m. frame, summed over the spin projections in the final state, for the polarization vectors of the proton and $^3He$ in the final system and for the correlation tensor of the $(p, ^3He)$ system [1] (all notations in Eqs. (17)-(20) are the same as in Section 2):

$$\bar{\sigma}_{0}(E, \theta) = (k_p/k_x)^2 |R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2;$$

$$P^{(p)}(E, \theta) = \langle \bar{\psi} | \bar{\sigma}^{(p)} | \bar{\psi} \rangle = P^{(He)}(E, \theta) = \langle \bar{\psi} | \bar{\sigma}^{(He)} | \bar{\psi} \rangle =$$

$$= -A(E, \theta)n = -2\sqrt{2} \frac{\text{Im}(R_1(E, \theta)R_0^*(E, \theta))}{|R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2} n;$$

$$T_{ik}(E, \theta) = \langle \bar{\psi} | \bar{\sigma}_k^{(He)} | \bar{\psi} \rangle = \delta_{ik} - \frac{2}{|R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2} \times$$

$$\times \left[ |R_0(E, \theta)|^2 l_i l_k + 2|R_1(E, \theta)|^2 m_i m_k - \sqrt{2} \text{Re}(R_1(E, \theta)R_0^*(E, \theta))(l_i m_k + m_i l_k) \right].$$

In accordance with Eqs. (19),(20), the spins of the proton and the $^3He$ nucleus in the reaction $\pi^+ + ^4He \rightarrow p + ^3He$ must be tightly correlated (see [1] for more details).

4. Violation of the incoherence inequalities for the correlation tensor

As it was established in the paper [4], in the case of incoherent mixtures of factorizable two-particle states of spin-1/2 fermions the following inequalities for the diagonal components of the correlation tensor should be satisfied:

$$|T_{11} + T_{22} + T_{33}| \leq 1; \quad |T_{11} + T_{22}| \leq 1; \quad |T_{11} + T_{33}| \leq 1; \quad |T_{22} + T_{33}| \leq 1. \quad (21)$$

However, for non-factorizable quantum mechanical superpositions these inequalities may be violated. The triplet state $|\psi\rangle$ (17) of the final system in the reaction $\pi^+ + ^4He \rightarrow p + ^3He$ represents a characteristic example of such non-factorizable spin states (it is well seen that the state $|\psi\rangle$ cannot be reduced to the product of one-particle spin states).

Let us calculate the components of the correlation tensor $T_{ik}$ (20) for the system $(p, ^3He)$ in the coordinate frame with $z \parallel l$, $x \parallel m$, $y \parallel n$. Finally, we obtain the following expressions:

$$T_{11} = \frac{|R_0(E, \theta)|^2 - 2|R_1(E, \theta)|^2}{|R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2}; \quad T_{22} = 1; \quad T_{33} = \frac{2|R_1(E, \theta)|^2 - |R_0(E, \theta)|^2}{|R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2} = -T_{11}; \quad (22)$$

$$T_{13} = T_{31} = \frac{2\sqrt{2}}{|R_0(E, \theta)|^2 + 2|R_1(E, \theta)|^2} \text{Re}(R_1R_0^*); \quad T_{12} = T_{21} = T_{13} = T_{32} = 0 \quad (23)$$

(indexes: $1 \rightarrow x$, $2 \rightarrow y$, $3 \rightarrow z$); in doing so, $tr(T) = 1$).

Thus, as follows from Eq. (22), in the reaction $\pi^+ + ^4He \rightarrow p + ^3He$ one of the incoherence inequalities (21) for the diagonal components of the correlation tensor is necessarily violated, irrespective of the concrete mechanism of generation of the system $(p, ^3He)$. Indeed, if $|R_0|^2 > 2|R_1|^2$, we obtain that $|T_{11} + T_{22}| > 1$; if, on the contrary, $|R_0|^2 < 2|R_1|^2$, then $|T_{22} + T_{33}| > 1$. Meantime, in both the cases the other three incoherence inequalities (21) are satisfied.

References

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