How to count the number of zeros that a polynomial has on the unit circle?

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Abstract

The classical problem of counting the number of real zeros of a real polynomial was solved a long time ago by Sturm. The analogous problem of counting the number of zeros that a polynomial has on the unit circle is, however, still an open problem. In this paper, we show that the second problem can be reduced to the first one through the use of a suitable pair of Möbius transformations — often called Cayley transformations — that have the property of mapping the unit circle to the real line and vice versa. Although the method applies to arbitrary complex polynomials, we discuss in detail several classes of polynomials with symmetric zeros as, for instance, the cases where the polynomial is self-conjugate, self-adjoint, self-inversive, self-reciprocal or skew-reciprocal. We show that faster algorithms can be implemented in these cases. Finally, an application of this method to Salem polynomials and to polynomials with small Mahler measure is also discussed.

Keywords: Self-inversive polynomials, Self-reciprocal polynomials, Salem polynomials, Sturm theorem, Möbius transformations, Cayley transformations.

1. Methods for counting the number of real zeros of real polynomials

The first exact method for counting the number of real zeros of a given real polynomial without knowing its zeros explicitly was presented by Sturm in 1829 [1]. In its simplest form, Sturm method works as follows: let \( p(z) \) be a polynomial of degree \( n \) with real coefficients; further, let \( a < b \) be two real numbers which are not a multiple zero of \( p(z) \). Then, construct the so-called Sturm sequence

\[
S(z) = \{S_0(z), S_1(z), S_2(z), \ldots, S_m(z)\},
\]

(1.1)

where

\[
S_0(z) = p(z), \quad S_1(z) = p'(z),
\]

(1.2)

and

\[
S_k(z) = -\text{rem}[S_{k-2}(z), S_{k-1}(z)], \quad 2 \leq k \leq m,
\]

(1.3)

where \( m \) is determined by the condition that \( S_m(z) \) has degree zero. Notice that each \( S_k(z) \) for \( 2 \leq k \leq m \) corresponds to the opposite of the polynomial remainder obtained in the Euclidean division of \( S_{k-2}(z) \) by \( S_{k-1}(z) \), so that \( S_m(z) \) is proportional to the greatest common divisor of \( p(z) \) and \( p'(z) \). Furthermore, let \( \text{var}[S(z)] \) denote the number of sign variations in the sequence \( S(z) \) for \( z = \zeta \). Then, Sturm theorem states that the number \( N \) of distinct zeros of \( p(z) \) in the half-open interval \( I = (a, b) \) is

\[
N = \text{var}[S(b)] - \text{var}[S(a)].
\]

(1.4)

The proof is based on the fact that, as we vary \( z \) from \( a \) to \( b \) on the real line, the sequence \( S(z) \) suffers a sign variation when, and only when, \( z \) passes through a zero of \( p(z) \). Thus, the number of sign variations of \( S(z) \) from \( a \) to \( b \) counts the exact number of distinct real zeros of \( p(z) \) in this interval — for the proof, see [2, 3].

We remark that Sturm’s method requires \( p(z) \) a real polynomial with no multiple zeros at the endpoints \( a \) and \( b \). Besides, it excludes from the counting the eventual zero of \( p(z) \) at \( z = a \) but includes the zero at \( z = b \); it is, however, an easy matter to verify if \( p(z) \) has or not a zero at \( z = a \), so that we can also count the number of zeros of \( p(z) \) in any closed interval \([a, b]\). Notice moreover that Sturm method described above counts only the number of distinct real zeros of \( p(z) \). Nonetheless, Sturm had shown in a subsequent paper [4] that the number of non-real zeros of \( p(z) \) in the interval \([a, b]\) can be determined from his method as well and Thomas in [5] showed that a generalization of Sturm algorithm can also get account for the multiplicity of the zeros.

It is worth to mention that Sturm derived this theorem during his researches on qualitative aspects of differential equations, which gave rise to the so-called Sturm-Liouville theory [6]. In fact, in an interval of weeks Sturm published similar theorems regarding the distribution of zeros of orthogonal functions, which are solutions of Sturm-Liouville differential equation [7].

Sturm was influenced by the works of Fourier and, as a matter of a fact, his method can be thought of as a refinement of Fourier’s previous result [8] that establishes an upper bound for the number of real zeros of \( p(z) \) in a
given half-open interval \((a, b]\) of the real line through the number of sign variations in the Fourier sequence
\[
F(z) = \left\{ p(z), p'(z), \ldots, p^{(n)}(z) \right\},
\]
for \(z\) running from \(a\) to \(b\) over the real line. Thus, we can say that Sturm’s method makes Fourier’s exact.

Other methods for counting or isolating the real zeros of a given real polynomial were formulated since Sturm’s fundamental papers. In 1834, Vincent published a paper [9] (republished two years later, with few additions, in [10]), in which a method based on successive replacements in terms of continued fractions was proposed. His method was based on a previous work of Budan [11], who established a theorem equivalent to that of Fourier [8], although in a different form. Unfortunately, Vincent’s work was almost forgotten thenceforward and, in fact, it was only rescued from oblivion in 1976 by Collins and Akritas, who formulated a powerful bisection method based on Vincent’s theorem for isolating the zeros of a given real polynomial [12]. Two years later, Akritas [13, 14] gave a fundamental contribution to this method by replacing the uniform substitutions that take place in Vincent’s algorithm by non-uniform ones based on previously calculated bounds for the zeros of the testing polynomial. With that modification, Akritas was able to reduce the complexity of Vincent’s method from exponential to polynomial type. Further improvements of these methods gave rise to some of the fastest algorithms known up to date for counting or isolating the real zeros of real polynomials — see [3, 15–19] and references therein.

2. The Cayley transformations and polynomials

The methods described above determine the exact number of zeros of a real polynomial on the real line. The correspondent problem of determining the exact number of zeros of a given polynomial on the unit circle is still unsolved. In fact, this is an old question whose first works remount to the end of XIX century: we can cite, for instance, the pioneer works of Eneström [20, 21], Kakeya [22], Schur [23], Kempner [24–26] and Cohn [27].

In the recent years, a great interest on this problem has emerged, usually in connection with the theory of the so-called self-inversive polynomials, which are polynomials whose zeros are all symmetric with respect to the unit circle [28, 29]. Self-inversive polynomials are important in both pure and applied mathematics: they appear in connection with the theory of numbers, algebraic curves, knots theory, error-correcting codes, cryptography and also in some topics of physics as in quantum and statistical mechanics — see [30–35]. There is a countless number of papers that presents conditions for all, some, or no zero of a self-inversive polynomial to lie on the unit circle — see [36] and references therein.

In this paper, we present a method that reduces the problem of counting the number of zeros that an arbitrary complex polynomial has on the unit circle to the problem of counting the number of zeros of a real polynomial on the real line. Because the second problem is completely addressed by Sturm (or Akritas) method, our approach also solves the first problem completely. The method is based on the use of the following pair of Möbius transformations:
\[
\mu(z) = \frac{z - i}{z + i}, \quad \text{and} \quad \omega(z) = -i \left( \frac{z + 1}{z - 1} \right),
\]
which are often called Cayley transformations [37]. Together with the relations
\[
\mu(\infty) = 1, \quad \mu(-i) = \infty, \quad \omega(1) = \infty, \quad \omega(\infty) = -i,
\]
these two transformations become the inverse of each other in the extended complex plane \(\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}\). It can be easily verified that \(\mu(z)\) maps the real line onto the complex unit circle \(S = \{z \in \mathbb{C} : |z| = 1\}\), while \(\omega(z)\) maps the unit circle onto the real line\(^1\). Besides, \(\mu(z)\) sends any point in the upper-half (lower-half) plane to the interior (exterior) of \(S\), so that \(\omega(z)\) sends any point in the inside (outside) of \(S\) to the upper-half (lower-half) plane.

Given a complex polynomial \(p(z)\) of degree \(n\), we define the Cayley-transformed polynomials \(q_\mu(z)\) and \(q_\omega(z)\) by the formulas:
\[
q_\mu(z) = (z + i)^n p(\mu(z)), \tag{2.3}
\]
and
\[
q_\omega(z) = \left( \frac{1}{i} \right)^n (z - 1)^n p(\omega(z)). \tag{2.4}
\]
The factor \((\frac{1}{i})^n\) in front of the second formula is to make the two mappings (2.3) and (2.4) the inverse each of the other.

The following theorems discuss some properties of these Cayley-transformed polynomials and their zeros.

Theorem 1. Let \(p(z)\) be a complex polynomial of degree \(n\). If \(p(z)\) has a zero of multiplicity \(m\) at the point \(z = 1\), then \(q_\mu(z)\) defined by (2.3) will be a polynomial of degree \(n - m\). Similarly, if \(p(z)\) has a zero of multiplicity \(m\) at the point \(z = -i\), then \(q_\omega(z)\) defined by (2.4) will be a polynomial of degree \(n - m\).

Proof. Suppose that \(p(z)\) has a zero at \(z = 1\) of multiplicity \(m\), where \(0 \leq m \leq n\). Write, \(p(z) = (z - 1)^m r(z)\), where \(r(z)\) is a polynomial of degree \(n - m\) with no zeros at \(z = 1\). From (2.3) we get that \(q_\mu(z) = (-2i)^m s(z)\), where \(s(z) = (z + i)^{n-m} r(\mu(z))\). Now, expanding \(s(z)\) in powers of \(z\) we can verify that its leading coefficient equals \(r(1)\); because \(r(1) \neq 0\) we conclude that \(s(z)\) is a polynomial of degree \(n - m\) and so it is \(q_\mu(z)\). By the same argument, if \(p(z)\) has a zero of multiplicity \(m\) at the point \(z = -i\), then \(q_\omega(z)\) as given by (2.4) will be a polynomial of degree \(n - m\).

\[\square\]

\(^1\)We remark that the transformations (2.1) are not the only pair of Möbius transformations that maps the unit circle onto the real line and vice versa: they are, however, the most adequate ones for our purposes.
Thus, the inverting of (2.3), we get that,
\[ p(\zeta_k) = (\frac{1}{\mu})^n q_\mu(\omega(\zeta_k)) = 0, \quad 1 \leq k \leq n, \] (2.7)
but \( \zeta_k \neq 1 \), from which follows that \( \zeta_k = \omega(\zeta_k) \) is a zero of \( q_\mu(z) \). Similarly, inverting (2.4) we get that
\[ p(\zeta_k) = (\zeta_k + i)^n q_\mu(\mu(\zeta_k)) = 0, \quad 1 \leq k \leq n, \] (2.8)
and the condition \( \zeta_k \neq 1 \) implies that \( \eta_k = \mu(\zeta_k) \) is a zero of \( q_\mu(z) \).

Theorem 2 shows us that whenever a polynomial is transformed through a Cayley transformation, its zeros are accordingly transformed through the inverse transformation. Besides, from relations (2.2), we see that if \( p(z) \) has a zero at the point \( z = 1 \) (respectively, \( z = -i \)), then the transformed polynomial \( q_\mu(z) \) (respectively, \( q_\mu(z) \)) will have a zero at infinity, which confirms again that the transformed polynomial cannot have the same degree as \( p(z) \) in this case.

The previous results imply the following theorem, which is a keystone in the what follows:

**Theorem 3.** Let \( p(z) \) be a complex polynomial of degree \( n \) that has \( m \) zeros on the unit circle, counted with multiplicity, and such that \( p(1) \neq 0 \). Then the transformed polynomial \( q_\mu(z) \) will have exactly \( m \) real zeros, also counted with multiplicity. Similarly, if \( p(z) \) is a complex polynomial of degree \( n \) that has \( m \) zeros on the real line, counted with multiplicity, and such that \( p(-i) \neq 0 \), then the transformed polynomial \( q_\mu(z) \) will have \( m \) zeros on the unit circle, also counted with multiplicity.

**Proof.** These statements follow directly from the theorems proved above and from the fact that the Cayley transformations \( \mu(z) \) and \( \omega(z) \) map the real line onto the unit circle and vice versa, respectively. \( \square \)

3. The number of zeros on the unit circle of a general complex polynomial

From Theorem 3 becomes clear how we can count the number of zeros that a polynomial has on the unit circle: all we need to do is to compute the Cayley-transformed polynomial \( q_\mu(z) = (z + i)^n p(\mu(z)) \) and to count the number of zeros on the real line of \( q_\mu(z) \). Thus, we would be done if was not for one detail: Sturm (and Akritas) method requires a real polynomial to work with, while the transformed polynomial \( q_\mu(z) \) usually have non-real coefficients\(^2\) (of course, if some root-counting-method worked with general complex polynomial this would not be a problem). To work around this issue we can proceed in two ways. The first way consists of multiplying the transformed polynomial \( q_\mu(z) \) by its complex conjugate\(^3\), \( q_\mu^\star(z) \), so that a polynomial of degree \( 2n \) is obtained in place:

\[ Q(z) = q_\mu(z)q_\mu^\star(z). \] (3.1)

It is clear that the zeros of \( q_\mu^\star(z) \) are the complex-conjugate of the zeros of \( q_\mu(z) \), from which it follows that \( Q(z) \) has the same number of real zeros than \( q_\mu(z) \), counted without multiplicity. Sturm (or Akritas) procedure can now be used to the number of real zeros of \( Q(z) \), which, according to Theorem 3, will correspond to the number of zeros that the original polynomial \( p(z) \) has on the unit circle, provided \( p(1) \neq 0 \) (if \( p(1) = 0 \) then all we need to do is to add 1 to the final result). This is described in Algorithm 1, where \( \text{RRC} [Q(z), \alpha, \beta] \) means any real-root-counting procedure that gives the exact number of distinct zeros that a given real polynomial \( Q(z) \) has on the interval \( (\alpha, \beta) \) of the real line.

The second way consists of writing the transformed polynomial in the form \( q_\mu(z) = r(z) + is(z) \), where \( r(z) = \frac{1}{n} \left[ q_\mu(z) + q_\mu^\star(z) \right] \) and \( s(z) = \frac{1}{2n} \left[ q_\mu(z) - q_\mu^\star(z) \right] \), so that both \( r(z) \) and \( s(z) \) are both real polynomials (this alternative was suggested already in [38]). From this we may realize that any common zero of \( r(z) \) and \( s(z) \) is also a zero of \( q_\mu(z) \). Conversely, if \( \zeta \) is a zero of \( q_\mu(z) \), then either \( \zeta \) is a common zero of \( r(z) \) and \( s(z) \) or \( r(\zeta)/s(\zeta) = -i \). The last condition, however, can not be satisfied whenever \( \zeta \) is real, which means that any real zero of \( q_\mu(z) \) is a necessary a common root of \( r(z) \) and \( s(z) \). Thus we can compute the greatest common divisor of \( r(z) \) and \( s(z) \) and define

\[ Q(z) = \gcd [r(z), s(z)]. \] (3.2)

Thereby \( Q(z) \) has, again, the same number of real zeros than \( q_\mu(z) \), whence Sturm (or Akritas) procedure can be used to count the number of real zeros of \( Q(z) \), which gives indirectly the number of zeros of \( p(z) \) on the unit circle.

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\(^2\)In Theorem 5 we show that the the transformed polynomial \( q_\mu(z) = (z + i)^n p(\mu(z)) \) will be a real polynomial only if the original polynomial \( p(z) \) is self-adjoint.

\(^3\)The star means complex conjugation so that, if \( p(z) = p_0 + p_1 z + \cdots + p_{n-1} z^{n-1} + p_n z^n \), then,
\[ p^\star(z) = p_0 + p_1 z^* + \cdots + p_{n-1} z^{*n-1} + p_n z^*, \]
\[ p^\star^\star(z) = p_0 + p_1 z + \cdots + p_{n-1} z^{n-1} + p_n z^*, \]
\[ p^\star^\star^\star(z) = p_0 + p_1 z^* + \cdots + p_{n-1} z^{*n-1} + p_n z^*. \]
circle. This is described in Algorithm 2.

We remark that both the alternatives described above have their caveats: in the first case \( Q(z) \) has degree 2\( n \), twice the degree of \( p(z) \), while in the second case, although \( Q(z) \) has degree \( n \), it is necessary to compute the \( \gcd \) of two polynomials to obtain it. Besides, notice that both algorithms count only the number of distinct zeros of \( p(z) \). If we are interested in the number of zeros counted with multiplicity then we need to employ a suitable real-root-counting method that takes this into account — for example, Thomas algorithm [5].

Finally, we highlight that we can also count the number of zeros of \( p(z) \) in a given arc of the unit circle. Let \( \mathcal{J} = (e^{\alpha_1}, e^{\alpha_2}) \) be the referred arc of the unit circle. In the simplest case, we assume that \( 0 \leq \alpha < \beta \leq 2\pi \), so that the interval \( \mathcal{J} \) is mapped to the interval \( \mathcal{I} = (a, b) \) on the real line, where \( a = \omega(e^{\alpha_1}) \) and \( b = \omega(e^{\beta_2}) \) [with the following conventions: \( \lim_{\theta \to 0} \omega(e^{i\theta}) = \infty \) and \( \lim_{\theta \to 2\pi} \omega(e^{i\theta}) = \infty \)]. The number of zeros of \( p(z) \) on the arc \( \mathcal{J} \) can thereby be found by counting the number of real zeros that the polynomial \( Q(z) \) — as given by (3.1) or (3.2) —, has on the interval \( \mathcal{I} \) of the real line. In the case where \( \alpha > \beta \) (which corresponds to an open interval on the unit circle that contains the point \( z = 1 \)), we need to split the algorithm into two parts because, in this case, the interval \( \mathcal{I} \) on the real line will be composed by two disjoint intervals — namely, we have \( \mathcal{I}(a, \beta] = (-\infty, a) \cup (a, \infty) \). Thus, the procedure \( \text{RRC} [Q(z), a, b] \) must be replaced by \( \text{RRC} [Q(z), -\infty, b] + \text{RRC} [Q(z), a, \infty] \) in this case. Finally, if the point \( z = 1 \) belongs to the interval \( \mathcal{J} \) and \( p(1) = 0 \), then we should add 1 to the final result. This is described in Algorithm 3 (for sake of simplicity, we have defined the polynomial \( Q(z) \) through (3.1) there).

4. The number of zeros of self-conjugate and self-inversive polynomials on the unit circle

The algorithms presented above apply to any complex polynomial. In the most important cases, however, the coefficients of the test polynomial enjoy certain symmetries which allow us to implement faster algorithms. In the what follows, we shall specialize into two classes of polynomials whose zeros are symmetric with respect to either the real line or the unit circle.

Let us suppose first that all the zeros of \( p(z) \) are symmetric with respect to the real line. This means that, for any zero \( \zeta \) of \( p(z) \), the complex conjugate number \( \zeta^* \) is also a zero of it. Of course, any real polynomial has this property, but there can be non-real polynomials with this property as well. This suggests us to call any complex polynomial \( p(z) \) whose zeros are all symmetric with respect to the real line as a self-conjugate polynomial. The necessary and sufficient condition for a complex polynomial \( p(z) = p_0 z^n + \cdots + p_0 \) of degree \( n \) to be self-conjugate is that there exists a fixed complex number \( \epsilon \) of modulus 1 such that,

\[
p(z) = \epsilon p^*(z^*).
\] (4.1)

In fact, if this condition is satisfied, then \( p(z) \) is clearly self-conjugate because, for any zero \( \zeta \) of \( p(z) \), the complex conjugate \( \zeta^* \) will also be a zero of it. Now let \( p(z) \) be a self-conjugate polynomial of degree \( n \). In this case, we can write \( p(z) = p_n r(z) \), where \( p_n \) is the leading coefficient of \( p(z) \) (which can be any non-null complex number), and \( r(z) \) is a monic real polynomial of degree \( n \). Evaluating \( p(z) \) at \( z^* \) and taking the complex conjugate, we get that
Algorithm 3: The number of zeros that a complex polynomial \( p(z) \) of degree \( n \) has on the arc \( \mathcal{J} = (e^{i\alpha}, e^{i\beta}) \) of the unit circle.

**input**: A complex polynomial \( p(z) \) of degree \( n \) and two real numbers \( \alpha \) and \( \beta \) such that \( 0 \leq \alpha < 2\pi \) and \( 0 < \beta \leq 2\pi \).

**output**: The number of zeros of \( p(z) \) on the arc \( \mathcal{J} = (e^{i\alpha}, e^{i\beta}) \) of the unit circle.

1 begin
2    if \( \alpha = 0 \) then
3        \( a = -\infty \);
4    else
5        \( a := -i \left( \frac{e^{i\alpha} + 1}{e^{i\alpha} - 1} \right) \);
6    end
7    if \( \beta = 2\pi \) then
8        \( b = \infty \);
9    else
10       \( b := -i \left( \frac{e^{i\beta} + 1}{e^{i\beta} - 1} \right) \);
11    end
12    \( n := \text{degree}(p(z)) \);
13    \( q(z) := (z + i)^n p \left( \frac{z - i}{z + i} \right) \);
14    \( Q(z) := q(z)q^*(z^*) \);
15    if \( \alpha > \beta \) then
16        \( N := \text{RRC}[Q(z), -\infty, b] + \text{RRC}[Q(z), a, \infty] \);
17        if \( p(1) = 0 \) then
18            \( N := N + 1 \);
19        end
20        return \( N \).
21    end
22    \( N := \text{RRC}[Q(z), a, b] \);
23    if \( p(1) = 0 \land a + b = 2\pi \) then
24        \( N := N + 1 \);
25    end
26    return \( N \).
27 end

\( p^*(z^*) = p_n^* r(z) \), so that (4.1) follows after we define \( \epsilon = p_n/p_n^* \). Notice that the coefficients of any self-conjugate polynomial \( p(z) \) of degree \( n \) satisfy the properties:

\[
p_k = c p_k^*, \quad 0 \leq k \leq n.
\] (4.2)

Notice that for a self-conjugate polynomial \( p(z) \) of degree \( n \), the polynomial \( Q(z) \) given by (3.1) will actually be a polynomial of degree \( 2n \) in the variable \( z^2 \). This is the content of the following:

**Theorem 4.** Let \( p(z) \) be a self-conjugate polynomial of degree \( n \) such that \( p(1) \neq 0 \). Then the polynomial \( Q(z) \) as defined by (3.1) will be a real polynomial of degree \( 2n \) in the variable \( z^2 \).

**Proof.** According to (2.3) and (3.1), we have that

\[
Q(z) = (z^2 + 1)p(\mu(z))p^*(\mu(z^*)),
\] (4.3)

which is clearly a real polynomial. If, moreover, \( p(z) \) is self-conjugate, then we get that

\[
Q(z) = \epsilon(z^2 + 1)p(\mu(z))p^*(\mu(z^*)).
\] (4.4)

But \( \mu^*(z^*) = 1/\mu(z) \), so that we obtain:

\[
Q(z) = \epsilon(z^2 + 1)p(\mu(z))p \left( \frac{1}{\mu(z)} \right).
\] (4.5)

Now, replacing \( z \) by \( -z \) and using the fact that \( \mu(-z) = 1/\mu(z) \), we see that \( Q(-z) = Q(z) \), from which we conclude that \( Q(z) \) has only even powers of \( z \).

From this property, we can see that, in the case where \( p(z) \) is self-conjugate, the \( 2n \) degree polynomial \( Q(z) \) can be transformed into an \( n \) degree polynomial by the replacement \( z \leftrightarrow \sqrt{z} \). The number of positive zeros of \( Q(\sqrt{z}) \) will, therefore, equal the half of the number of real zeros of \( Q(z) \). Thus, for counting the number of zeros on the unit circle of a given self-conjugate polynomial \( p(z) \) of degree \( n \) we can proceed as in Algorithm 1, except that we can replace \( Q(z) \) by \( Q(\sqrt{z}) \) and the procedure \( \text{RRC}[Q(z), -\infty, \infty] \) by \( 2\text{RRC}[Q(\sqrt{z}), 0, \infty] \) [we should also notice that Sturm’s procedure will not take into account the eventual zero of \( Q(z) \) at \( z = 0 \), which should be added to the counting if it is the case; the condition \( Q(0) = 0 \) is equivalent to the condition \( p(-1) = 0 \)]. This is exemplified in Algorithm 4.

Finally, we remark that, if we are interested in the number of zeros of a self-conjugate polynomial \( p(z) \) of degree \( n \) in a given interval \( \mathcal{J} = (e^{i\alpha}, e^{i\beta}) \) of the unit circle, then the change of variable \( z \leftrightarrow \sqrt{z} \) is not adequate because this map is not one-to-one. In fact, in this case we cannot ensure anymore that the number of real zeros of \( Q(z) \) on this interval corresponds to twice the number of zeros of \( Q(\sqrt{z}) \) in the respective positive interval of the real line. In this case it is better to use Algorithm 3 instead.

Now, let us consider the case of a complex polynomial \( p(z) \) whose zeros are all symmetric with respect to the
Algorithm 4: The number of zeros that a self-conjugate polynomial $p(z)$ of degree $n$ has on the unit circle.

**input**: A self-conjugate polynomial $p(z)$ of degree $n$.

**output**: The number of zeros of $p(z)$ on the unit circle.

1. \( n \leftarrow \text{degree}(p(z)); \)
2. \( q(z) \leftarrow \frac{(z + i)^n}{z + i}; \)
3. \( Q(z) \leftarrow q(z)q^*(z^*); \)
4. \( Q(z) \leftarrow Q(\sqrt{z}); \)
5. \( N \leftarrow 2 \text{RRC} \{Q(z), 0, \infty\}; \)
6. if $p(1) = 0$ then
   7. \( N \leftarrow N + 1; \)
7. if $p(-1) = 0$ then
   8. \( N \leftarrow N + 1; \)
9. end
10. return $N$. 

unit circle. This means that, for any zero $\zeta$ of $p(z)$, the complex number $1/\zeta^*$ is also a zero of it. Any polynomial of this kind is called a self-inversive polynomial \[^{[29]}\] and the necessary and sufficient condition for a polynomial $p(z) = p_n z^n + \cdots + p_0$ of degree $n$ to be self-inversive is that $p_n p_0 \neq 0$ and that there exists a complex number $\epsilon$ with modulus 1 such that,

\[
 p(z) = \epsilon z^n p^* \left( \frac{1}{z^*} \right). \tag{4.6}
\]

Indeed, if the condition above is satisfied, then $p(z)$ is clearly self-inversive. Then, suppose that all the zeros of $p(z)$ are symmetric with respect to the unit circle. In this case we can write $p(z) = p_n s(z)$, where $p_n$ is the leading coefficient of $p(z)$ and $s(z)$ is a monic polynomial. On the other hand, evaluating $p(z)$ at $1/z^*$, taking the complex conjugate and multiplying by $z^n$ we shall obtain the polynomial,

\[
 z^n p^* \left( \frac{1}{z^*} \right) = p_n^* z^n + \cdots + p_0^* = p_0^* s(z), \tag{4.7}
\]

whose zeros are the same as before. From this, we promptly see that (4.6) will be satisfied provided we define $\epsilon = p_n/p_0^*$. Besides, $\epsilon$ must have modulus 1 as the product of the zeros of any polynomial whose zeros are all symmetric with respect to the unit circle has modulus 1, so that we get $|p_0/p_n| = 1$. Notice that the coefficients of any self-inversive polynomial $p(z)$ of degree $n$ satisfy the properties:

\[
 p_{n-k} = \epsilon p_k^*, \quad 0 \leq k \leq n. \tag{4.8}
\]

If a given polynomial is self-inversive with $\epsilon = 1$ we shall call it a self-adjoint polynomial\[^{[4]}\]. Any self-adjoint polynomial $p(z)$ of degree $n$ satisfies, therefore, the property,

\[
 p(z) = z^n p^* \left( \frac{1}{z^*} \right), \tag{4.9}
\]

and its coefficients satisfy the relations:

\[
 p_{n-k} = p_k^*, \quad 0 \leq k \leq n. \tag{4.10}
\]

The following theorem shows that there is a one-to-one correspondence between the sets of self-inversive and self-conjugate polynomials, as well as between the sets of self-adjoint and real polynomials.

**Theorem 5.** Let $p(z)$ be a self-inversive polynomial. Then, the transformed polynomial $q_\mu(z)$ defined by (2.3) will be a self-conjugate polynomial. Moreover, if $p(z)$ is a self-adjoint polynomial, then the transformed polynomial $q_\mu(z)$ will be a real polynomial. Similarly, let $p(z)$ be a self-conjugate polynomial. Then the polynomial $q_\omega(z)$ defined by (2.4) will be a self-inversive polynomial and if $p(z)$ is a real polynomial, then $q_\omega(z)$ will be a self-adjoint polynomial.

**Proof.** Let $p(z)$ be self-inversive. Then, the Cayley-transformed polynomial $q_\mu(z)$ is given by:

\[
 q_\mu(z) = (z + i)^n p(\mu(z)) = \epsilon (z + i)^n \mu(z)^n p^* \left( \frac{1}{\mu^*(z)} \right) = \epsilon (z - i)^n p^* \left( \frac{1}{\mu^*(z)} \right). \tag{4.11}
\]

But we have the identity $1/\mu^*(z) = \mu(z^*)$, from which we get,

\[
 q_\mu(z) = \epsilon (z - i)^n p^* \left( \mu(z^*) \right) = cq_n^*(z^*), \tag{4.12}
\]

which proves that $q_\mu(z)$ is self-conjugate. Notice that if $\epsilon = 1$, so that $p(z)$ is self-adjoint polynomial, then we get that $q_\mu(z)$ will be a real polynomial because the value of $\epsilon$ is preserved during this transformation.

Now, suppose that $p(z)$ is a self-conjugate polynomial. Then the Cayley-transformed polynomial $q_\omega(z)$ is given by
by:

\[ q_{\omega}(z) = \left( \frac{1}{z} \right)^n (z - 1)^n p(\omega(z)) = \epsilon (z - 1)^n p^*(\omega^*(z)). \]  

(4.13)

But we have the identity \( \omega^*(z) = \omega(1/z^*) \), from which it follows that,

\[ q_{\omega}(z) = \epsilon \left( \frac{1}{z} \right)^n (z - 1)^n p^* \left( \frac{1}{z^*} \right) = \epsilon z^n q_{\omega^*} \left( \frac{1}{z^*} \right), \]

(4.14)

which proves that \( p(z) \) is self-inversive. In the case where \( \epsilon = 1 \), so that \( p(z) \) is a real polynomial, we get that \( q(z) \) will be a self-adjoint polynomial. \( \square \)

Now, let us see how we can count the number of zeros that a self-adjoint and self-inversive polynomial has on the unit circle. Let us first consider the case where \( p(z) \) is self-adjoint. In this case, Theorem 5 ensures that \( q_{\omega}(z) \) is already a real polynomial, so that there is no need of defining the polynomial \( Q(z) \) given by (3.1). This results in the Algorithm 5, which is a faster version of Algorithm 1. The number of zeros of \( p(z) \) in a given arc of the unit circle can also be found by a simplified version of Algorithm 3.

**Algorithm 5:** THE NUMBER OF ZEROS THAT A SELF-ADJOINT POLYNOMIAL \( p(z) \) OF DEGREE \( n \) HAS ON THE UNIT CIRCLE.

```
Algorithm 5: \( p(z) \) of degree \( n \) with \( \epsilon \neq 1 \).

input : A self-adjoint polynomial \( p(z) \) of degree \( n \).
output: The number of zeros of \( p(z) \) on the unit circle.

1 begin
2 \( n \leftarrow \text{degree}(p(z)); \)
3 \( q(z) \leftarrow (z + i)^n p \left( \frac{z - i}{z + i} \right); \)
4 \( N \leftarrow \text{RRC} [q(z), -\infty, \infty]; \)
5 if \( p(1) = 0 \) then
6 \( N \leftarrow N + 1; \)
7 end
8 return \( N. \)
9 end
```

Now, let us suppose \( p(z) \) self-inversive with \( \epsilon \neq 1 \). In this case, the Cayley-transformed polynomial \( q_{\omega}(z) \) is not real. Of course, this issue can be overcome through algorithms 1 or 2, but we shall show in the what follows that for self-inversive polynomials there is another way of approaching this problem, namely, that a self-adjoint polynomial \( s(z) \), with the same degree as that of \( p(z) \), can always be found by a simple change of variable. This is the content of the following:

**Theorem 6.** Let \( p(z) \) be a self-inversive polynomial of degree \( n \) such that \( \epsilon \neq 1 \). Then, there exist \( n \) values for the real variable \( \phi \) in the interval \( 0 < \phi \leq 2\pi \) for which the composition \( s(z) = p(e^{i\phi} z) \) will provide a self-adjoint polynomial of degree \( n \). The possible values of \( \phi \) are related with \( \epsilon \) through the formula \( \phi = (i \log \epsilon) / n - 2\pi (k/n) \), for \( 1 \leq k \leq n \), such that \( \epsilon = e^{-in\phi} \) for any admissible value of \( \phi \). Conversely, if \( s(z) \) is a self-adjoint polynomial of degree \( n \), then \( p(z) = s(e^{-i\phi} z) \) will provide a self-inversive polynomial of degree \( n \) such that \( \epsilon = e^{in\phi} \).

Proof. Let \( p(z) = p_0 z^n + p_{n-1} z^{n-1} + \cdots + p_1 z + p_0 \) be a self-inversive polynomial of degree \( n \). Making the change of variable \( z \leftarrow e^{i\phi} z \), we get that,

\[ p(e^{i\phi} z) = p_0 e^{in\phi} z^n + p_{n-1} e^{i(n-1)\phi} z^{n-1} + \cdots + p_1 e^{i\phi} z + p_0. \]

(4.15)

This is polynomial of degree \( n \) in the variable \( z \) which we may call \( s(z) \):

\[ s(z) = \sum_{k=0}^n s_k z^k, \quad s_k = p_k e^{ik\phi}, \quad 0 \leq k \leq n. \]

(4.16)

Now, using the fact that the coefficients of \( p(z) \) should satisfy the properties (4.8), we get that \( s(z) \) will be a self-adjoint polynomial provided \( \epsilon \) be related to \( \phi \) through the formula:

\[ \epsilon = e^{-in\phi}. \]

(4.17)

Inverting this relation, we get that \( \phi = \phi_k \), where,

\[ \phi_k = \frac{i \log \epsilon}{n} - \frac{2\pi k}{n}, \quad k \in \mathbb{Z}. \]

(4.18)

This shows us that there exist \( n \) possible values for \( \phi \) in the interval \( 0 < \phi \leq 2\pi \) for which \( s(z) \) will be self-adjoint. Thus, for each admissible value of \( \phi \), a given self-adjoint polynomial is obtained and we can write:

\[ s^{(k)}(z) = p(e^{i\phi_k} z), \quad 1 \leq k \leq n. \]

(4.19)

Furthermore, in terms of \( \epsilon \) we also have that,

\[ s^{(k)}(z) = p \left( \frac{z}{e^{ik\phi_k} z^{1/n}} \right), \quad \phi_k^* = 2\pi i \left( \frac{k}{n} \right), \quad 1 \leq k \leq n. \]

(4.19)

Finally, given a self-adjoint polynomial \( s(z) \) of degree \( n \), then it is clear that \( p(z) = s(e^{-i\phi} z) \) will be a self-inversive polynomial with \( \epsilon = e^{in\phi} \) for any admissible value of \( \phi \). \( \square \)

We highlight that Theorem 6 means that any self-inversive polynomial can be thought as a rotated self-adjoint polynomial. In fact, if \( \sigma_1, \ldots, \sigma_n \) denote the zeros of a self-inversive polynomial \( p(z) \) of degree \( n \), and \( \sigma^{(j)} \), \( \ldots, \sigma^{(j)} \) the correspondent zeros of the self-adjoint polynomials \( s^{(j)} = p(e^{i\phi_j}), 1 \leq j \leq n \), as provided by Theorem 6, then it is straightforward matter to show that

\[ \sigma_k^{(j)} = e^{-i\phi_j} \sigma_k = \frac{\sigma_k}{\phi_k^{1/n}}, \quad \phi_k^* = 2\pi i \left( \frac{k}{n} \right), \]

(4.20)

for any \( j \) and \( k \) running from 1 to \( n \).
Therefore, the zeros of \( s^{(j)}(z) \) are rotated with respect to the zeros of \( p(z) \) by an angle equal to \( e^{-1/n} \) divided by \( \sqrt[n]{n} \) — the \( j \)th root of unity of degree \( n \) — in the clockwise direction. Theorem 6 also shows us that if we rotate the zeros of a given polynomial \( p(z) \) of degree \( n \) by an angle equal to any multiple of a root of unity of degree \( n \), then we shall obtain another self-inversive polynomial with the same \( \epsilon \). Therefore, there are exactly \( n \) self-inversive polynomials conjugated in this way.

Now, from Theorem 6, we can implement a faster algorithm to count the number of zeros on the unit circle of any self-inversive polynomial. This is described in Algorithm 6.

**Algorithm 6: The Number of Zeros That a Self-Inversive Polynomial \( p(z) \) Has on the Unit Circle.**

```plaintext
input : A self-inversive polynomial \( p(z) \) of degree \( n \).
output : The number of zeros of \( p(z) \) on the unit circle.

1 begin
2     \( n := \text{degree}(p(z)) \);
3     \( \epsilon := \frac{p_n}{p_0} \);
4     if \( \epsilon \neq 1 \) then
5         \( p(z) \leftarrow p \left( \frac{z}{\epsilon^{1/n}} \right) \);
6     end
7     \( q(z) := (z + i)^n p \left( \frac{z - i}{z + i} \right) \);
8     \( N := \text{RRC} [q(x), -\infty, \infty] \);
9     if \( p(1) = 0 \) then
10        \( N \leftarrow N + 1 \);
11     end
12     return \( N \).
13 end
```

Moreover, we can also count the number of zeros that \( p(z) \) has in a given arc \( J = (e^{i\alpha}, e^{i\beta}] \) by making a few modifications in Algorithm 3. These modifications consist of declaring the polynomial \( Q(z) \) through (3.1), we use Theorem 6 to transform the self-inversive polynomial \( p(z) \), if \( \epsilon \neq 1 \), into the self-adjoint polynomial \( s(z) = p(z/\epsilon^{1/n}) = p(e^{i\phi}z) \), where \( \phi \) is related to \( \epsilon \) by formula (4.17). Then, we need to rotate as well the endpoints \( e^{i\alpha} \) and \( e^{i\beta} \) of the interval \( J \) by the angle \( \phi \) in the clockwise direction, so that the new endpoints on the unit circle become,

\[
e^{iA} = e^{i(\alpha - \phi)}, \quad \text{and} \quad e^{iB} = e^{i(\beta - \phi)}.
\]

Finally, the interval endpoints \( a \) and \( b \) used in the Sturm (or Akritas) procedure are found from the Cayley transformation \( \omega(z) \) applied to the rotated endpoints \( A \) and \( B \) on the unit circle, that is, \( a = \omega(e^{iA}) \) and \( b = \omega(e^{iB}) \), and from this point forward we can proceed as before.

5. The number of zeros of self-reciprocal and skew-reciprocal polynomials on the unit circle. An application to Salem polynomials

As the last case to be discussed in this work, let us suppose the possibility of a complex polynomial \( p(z) \) of degree \( n \) which is, at the same time, self-conjugate and self-inversive. From the properties (4.1) and (4.6), it follows therefore that such polynomials \( p(z) \) should satisfy the property:

\[
p(z) = \epsilon z^n p \left( \frac{1}{z} \right).
\]

Contrary to the previous cases, however, \( \epsilon = p_n/p_0 \) can assume only the values 1 or \(-1\). In fact, replacing \( z \) by \( 1/z \) in the formula above we immediately realize that \( \epsilon^2 = 1 \). Hence, any polynomial which is simultaneously self-conjugate and self-inversive is actually a real polynomial.

In the first case where \( \epsilon = 1 \), we say that \( p(z) \) is a self-reciprocal polynomial, while in the second case where \( \epsilon = -1 \), \( p(z) \) is often called a skew-reciprocal polynomial. The coefficients of any self-reciprocal or skew-reciprocal polynomial satisfy, respectively, the relations:

\[
p_{n-k} = p_k, \quad \text{and} \quad p_{n-k} = -p_k, \quad 0 \leq k \leq n.
\]

The results of the previous sections are dramatically simplified when \( p(z) \) is a self-reciprocal or skew-reciprocal polynomial:

**Theorem 7.** Let \( p(z) \) be a self-reciprocal polynomial of even degree, say \( n = 2m \). Then, \( q_n(z) \) as defined by (2.3) will be a real polynomial of degree \( m \) in the variable \( z^2 \). Moreover, if \( p(z) \) is a self-reciprocal polynomial of odd degree, say \( n = 2m + 1 \), then \( q_n(z) \) will be a real polynomial of degree \( m \) in the variable \( z^2 \), multiplied by \( z \). Similarly, let \( p(z) \) be a skew-reciprocal polynomial of degree \( n \). Then, \( q_n(z) \) as defined by (2.3) will be a pure imaginary polynomial of degree \( m \) in the variable \( z^2 \).

**Proof.** Let us suppose first that \( p(z) \) is a self-reciprocal polynomial of even degree, say, \( n = 2m \). Because the coefficients of any self-reciprocal polynomial satisfy the first set of relations in (5.2), it follows that \( p(z) \) can be written as,

\[
p(z) = p_m z^m + \sum_{k=0}^{m-1} p_k \left( z^{2m-k} + z^k \right)
\]

\[
= z^m \left[ p_m + \sum_{k=0}^{m-1} p_k \left( z^{m-k} + z^{k-m} \right) \right].
\]
On the other hand, the transformed polynomial \( q_\mu(z) \) defined in (2.3) becomes,

\[
q_\mu(z) = \left((z^2 + 1)^m\right)^m p_m + \sum_{k=0}^{m-1} p_k \left((z^2 + 1)^k \left[(z + i)^{2m-2k} + (z - i)^{2m-2k}\right]\right),
\]

(5.4)
after a simplification. Therefore, we can plainly see that \( q_\mu(z) \) is an even function of \( z \), which means that \( q(z) \) is in fact a polynomial of degree \( m \) on the variable \( z^2 \). Furthermore, \( q_\mu(z) \) is also a real polynomial because all the imaginary terms inside the brackets will cancel after we expand the binomials.

Now, let us suppose \( p(z) \) a self-reciprocal polynomial of odd degree, say, \( n = 2m + 1 \). In this case, \( p(z) \) always has a zero at \( z = -1 \), so that we can write \( p(z) = (z + 1)r(z) \), where \( r(z) \) is a self-reciprocal polynomial of degree \( n - 1 = 2m \). Thus, the transformed polynomial \( q_\mu(z) \) will be, in this case,

\[
q_\mu(z) = (z + i)^{2m+1} p(\mu(z)) = (z + i)^{2m+1} (\mu(z) + 1) r(\mu(z)) = 2z(z + i)^{2m} r(\mu(z)).
\]

(5.5)

If \( s(z) \) denotes the transformed polynomial associated with \( r(z) \), that is, \( s(z) = (z + i)^{2m} r(\mu(z)) \), then we see that \( q_\mu(z) = 2zs(z) \). But \( r(z) \) is a self-reciprocal polynomial of even degree, so that we conclude that \( s(z) \) will be a real polynomial in the variable \( z^2 \) and that \( q_\mu(z) \) will be a real polynomial in the variable \( z^2 \), multiplied by \( z \).

Finally, let us suppose \( p(z) \) is a skew-reciprocal polynomial of degree \( n \). If \( p(z) \) is of even degree, say, \( n = 2m \), then relations (5.2) imply that \( p_{n/2} = p_m = 0 \). Therefore, we can write:

\[
p(z) = \sum_{k=0}^{m-1} p_k \left((z^{2m-k} - z^k)^m \sum_{k=0}^{m-1} p_k \left(z^{m-k} - z^{k-m}\right)\right).
\]

(5.6)
The transformed polynomial \( q_\mu(z) \) becomes, now,

\[
q_\mu(z) = \sum_{k=0}^{m-1} p_k \left((z^2 + 1)^k \left[(z - i)^{2m-2k} - (z + i)^{2m-2k}\right]\right).
\]

(5.7)

After expanding the binomials we conclude that \( q_\mu(z) \) is a pure imaginary polynomial in the variable \( z^2 \). If, on the other hand, \( p(z) \) has odd degree, say, \( n = 2m + 1 \), then \( p(z) \) necessarily has a zero at \( z = 1 \). Thus, \( p(z) = (z - 1)r(z) \), where \( r(z) \) is a self-reciprocal polynomial of degree \( n - 1 = 2m \). The transformed polynomial is, in this case,

\[
q_\mu(z) = (z + i)^{2m+1} p(\mu(z)) = (z + i)^{2m+1} (\mu(z) - 1) r(\mu(z)) = -2i(z + i)^{2m} r(\mu(z)),
\]

(5.8)

from which we conclude that \( q_\mu(z) \) is also a pure imaginary polynomial in the variable \( z^2 \), although its degree is \( n - 1 = 2m \) instead of \( n = 2m + 1 \) (this, of course, is due to the fact that the transformation \( \omega(z) \) defined in (2.1) maps the point \( z = 1 \) to the infinity).

Theorem 7 provides a powerful way of counting the number of zeros of self-reciprocal (or skew-reciprocal) polynomials on the unit circle. In fact, we can make the change of variable \( z \leftarrow z^2 \) in order to cut by a half the degree of the testing polynomial used in Sturm (or Akritas) procedure. This is exemplified in Algorithm 7.

We remark again that, if we are interested in counting the number of zeros that a given self-reciprocal or skew-reciprocal polynomial has on some arc of the unit circle, then the change of variable \( z \leftarrow \sqrt{z} \) is not adequate, as we discussed before. In this case it is better to use the respective algorithm for self-inversive polynomials.

**Algorithm 7: The number of zeros that a self-reciprocal or skew-reciprocal polynomial \( p(z) \) of degree \( n \) has on the unit circle.**

**input**: A self-reciprocal or skew-reciprocal polynomial \( p(z) \) of degree \( n \).

**output**: The number of zeros of \( p(z) \) on the unit circle.

1. \textbf{begin}
2. \hspace{1em} \( n := \text{degree}(p(z)); \)
3. \hspace{1em} \( \epsilon := \frac{p_k}{p_0}; \)
4. \hspace{1em} \textbf{if} \( \epsilon = -1 \) \textbf{then}
5. \hspace{2em} \( q(z) := i(z + i)^{n} p\left(\frac{z - i}{z + i}\right); \)
6. \hspace{1em} \textbf{else}
7. \hspace{2em} \( q(z) := (z + i)^{n} p\left(\frac{z - i}{z + i}\right); \)
8. \hspace{1em} \textbf{end}
9. \( q(z) \leftarrow q(\sqrt{z}); \)
10. \( N := \text{RRC}[q(z), -\infty, \infty]; \)
11. \textbf{if} \( p(1) = 0 \) \textbf{then}
12. \hspace{1em} \( N \leftarrow N + 1; \)
13. \hspace{1em} \textbf{end}
14. \textbf{return} \( N \).
15. \textbf{end}

Finally, notice that from Algorithm 7 we can easily test if a given polynomial is a Salem polynomial or not, without knowing explicitly its zeros. Remember that a Salem polynomial \( p(z) \) is a monic self-reciprocal polynomial of degree \( n \geq 4 \) with integer coefficients whose all zeros but two lies on the unit circle [39–41]. The two zeros not lying on the unit circle are necessarily real and positive — say \( \zeta \) and
1/ζ. The greatest real zero of a Salem polynomial is usually called the Salem number associated with this polynomial. A Salem number is called “small” [42, 43] if it is less than \( \rho \approx 1.324 \) — the lowest Pisot number [44], also known as the plastic number, which corresponds to the value of the unique real zero of the polynomial \( p(z) = z^3 - z - 1 \).

Up to date there was found only 47 small Salem numbers, and the lowest one has the value \( \lambda \approx 1.176 \), which is the greatest real zero of the so-called Lehmer polynomial [45],

\[
L(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.
\]

It is still an open problem to know if Lehmer’s number \( \lambda \) is the lowest Salem number, or even if there exists a lowest Salem number after all. We highlight that Algorithm 7 provides a powerful tool to look for polynomials with small Salem numbers [42, 43] and, in a more general way, to find polynomials with small Mahler measure [46, 47]. In fact, we report that from a slightly improved algorithm, running in a simple desktop computer, we were able to reproduce all small Salem numbers known to date with this method. A detailed analysis of such researches will be communicated in a forthcoming paper.

6. Conclusions

In this work, we presented a simple method for counting the exact number of zeros that a complex polynomial \( p(z) \) has on the unit circle. The method makes use of a special Möbius transformation that maps the unit circle onto the real line in a one-to-one way, so that its action over a complex polynomial \( p(z) \) provides a transformed polynomial \( q(z) \) whose number of real zeros equals the number of unimodular zeros of \( p(z) \). In general, the transformed polynomial \( q(z) \) will have non-real coefficients, nonetheless a real polynomial can be easily obtained from it in such a way that its number of real zeros remains unchanged. Thereby, any real-root-counting method, as for instance Sturm or Akritas methods, can be used to count the number of real zeros of the real transformed polynomial, which indirectly gives the number of the original complex polynomial on the unit circle.

We have discussed in details the cases where the original polynomial \( p(z) \) has symmetric zeros with respect to the unit circle or to the real line, besides the case were \( p(z) \) is an arbitrary complex polynomial. Polynomials with symmetric zeros include the cases of self-conjugate, self-inversive, self-adjoint, self-reciprocal and skew-reciprocal polynomials and for each case we presented specific algorithms. A very powerful algorithm is obtained for the case where \( p(z) \) is either self-reciprocal or skew-reciprocal polynomial. In fact, in these cases we showed that the degree of the transformed polynomial can be reduced by a half through the change of variable \( z \leftarrow \sqrt{z} \).

Our approach can also be used to find the number of zeros of a complex polynomial in a given arc of the unit circle and to isolate the intervals on the unit circle containing exactly one zero of the polynomial. It can also be adapted to take into account the multiplicity of the zeros by employing a real-counting-method that takes into account the multiplicity of the zeros (e.g., Thomas generalization of Sturm’s method). Of course, we can also count the number of zeros that a complex polynomial has in any circle or straight line of the complex plane by replacing the transformations given in (2.1) by another pair of Möbius transformations of the form,

\[
m(z) = \frac{ax + b}{cx + d}, \quad w(z) = -\left( \frac{dx - b}{cx - a} \right), \quad ad - bc \neq 0,
\]

so that \( m(z) \) and \( w(z) \) map that given circle or straight line of the complex plane onto the real line and vice versa. The method can also be adapted to count the number of zeros that a given complex function has on the unit circle, provided that a suitable method for counting the number of real zeros of such functions is available.

Finally, we mention that the idea of using Möbius transformations to study the distribution of the zeros of a polynomial is not new, although this topic seems not to be explored in detail before. In fact, as far as we known, the use of such transformations to test if a given polynomial has some or all zeros on the unit circle was mentioned only in an old reference due to Kempner [24–26] and, more recently, in an expository note due to Conrad [38] (who credited F. Rodriguez Villegas for this idea). Kempner, however, considered only a real polynomial \( p(z) \) and defined the transformed polynomial \( q(z) \) by the formula,

\[
q(z) = (z^2 + 1) \left( \frac{z - i}{z + i} \right)^p \left( \frac{z + i}{z - i} \right).
\]

so that \( q(z) \) become a real polynomial as well; this essentially corresponds to the case discussed by us in Algorithm 4. Conrad, on the other hand, considered non-real polynomials in some examples, but made no reference to Sturm algorithm or any other real-root-counting method (the zeros of the transformed polynomials were found by numeric methods only). We believe that our detailed treatment of the problem can be of interest in future applications, for instance in the search of polynomials with small Salem numbers and small Mahler measure.

This work was supported by Coordination for the Improvement of Higher Education Personnel (CAPES).

References

[1] C. F. Sturm, Analyse d’un mémoire sur la résolution des équations numériques, in: Collected Works of Charles François Sturm, Springer, 2009, pp. 323–326. doi:10.1007/978-3-7643-7990-2_24.

[2] B. L. Van Der Waerden, Algebra, Vol. 1, Springer, 2003.

[3] A. G. Akritas, Elements of computer algebra with applications, Vol. 3, Wiley New York, 1989.

[4] C. F. Sturm, Mémoire sur la résolution des équations numériques, in: Collected Works of Charles
