Gauge invariance of the $\beta$-function in nonrelativistic quantum electrodynamics

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Abstract

Recently there has been much interest in gauge theories applied to condensed matter physics. I show that for a system of nonrelativistic electrons coupled to a U(1) gauge field in the presence of a Fermi surface, the $\beta$-function to one-loop order is, for a particular family of gauge-choices, independent of the gauge-choice.

1 Introduction

That gauge fields successfully describe the forces between elementary particles is certainly among the greatest discoveries in physics. But gauge theories are not restricted to high energy physics. In condensed matter physics one is often interested in theories with constraints. In a sense a constraint is just a (infinitely strong) force and it is quite natural that these theories also can be formulated as gauge theories.

The gauge field description is in terms of gauge potentials, which are redundant variables. The theory is therefore invariant with respect to a change of gauge. The most important consequence of this gauge symmetry is that it dictates the form of the interaction vertices between the gauge fields and the other fields in the theory. Other than that; the gauge-invariance is just a reminder that we are working with redundant variables. This redundancy can be very annoying, in particular when using path-integrals, and it is often necessary to choose a specific gauge. This choice of gauge cannot affect any physical quantities calculated in the theory.

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In this paper we will consider a theory of $U(1)$ gauge bosons coupled to nonrelativistic fermions. This theory is motivated by the composite fermion approach to the half-filled Landau level state in the Quantum Hall Effect.

Some years ago it was observed that the $\nu = 1/2$ Quantum Hall state exhibits metal-like features [1]. Using the idea of Jain [2], in which two flux quanta are "attached" to each electron; making a composite fermion, Halperin, Lee and Read [3] formulated the theory of the $\nu = 1/2$ state as a theory of composite fermions interacting with a fictitious gauge field, in zero external magnetic field. In the mean field approximation, in which the electrons are uniformly spread out, this theory is just that of free (composite) fermions. Therefore, in the mean field approximation, the $\nu = 1/2$ state is certainly metallic. Striking evidence of the existence of a Fermi surface was subsequently observed experimentally [4],[5],[6].

In order to make a consistent theory of the $\nu = 1/2$ state it is necessary to also include the effects of gauge-fluctuations. However, when fluctuations of the gauge field are taken into account, it seems at first sight that the mean field picture is destroyed. As shown in [3] the self-energy correction due to the transverse gauge fluctuations dominates over the linear in frequency Fermi liquid term at low frequencies.

In an attempt to address the role of the gauge field fluctuations Nayak and Wilczek [7] showed, using a renormalization-group approach, that for electron-electron interactions of shorter range than $1/r$, the fluctuations lead to a strong coupling fixed point in the infra-red, different from the Fermi liquid fixed point. Their result was based on the calculation of the single-particle Green function and gauge-invariance through the Ward identity. A somewhat similar approach was taken by Chakravarty et al [8]. They studied the same system using the $\epsilon$-expansion around $d = 3$, and obtained essentially the same result; a strong coupling fixed point.

Stern and Halperin [9] constructed a Boltzman transport equation for the $\nu = 1/2$ state in the same way as Landau did for the Fermi liquid. This construction is based on the pole structure of the single-particle Green function.

As emphasized in [10] the single-particle Green function is not gauge-invariant, and so any conclusions based on it might not be physical. This leads naturally to the question whether or not the existence of the strong coupling fixed point found in [7],[8], or the pole structure used in [9] is dependent on the choice of gauge?

We will in this paper show that although the single-particle Green function is dependent on the choice of gauge, the $\beta$-function is not, i.e. neither the existence of the strong coupling fixed point found by Nayak and Wilczek nor
the pole structure used by Stern and Halperin is dependent on the choice of gauge.

2 Calculations in $d=2$

The Lagrangian density considered in this paper is

\[
\mathcal{L} = \psi^\dagger (\partial_0 - \mu) \psi - \frac{1}{2m} \psi^\dagger (\partial_i - ig a_i)^2 \psi \\
+ \frac{1}{2} \int d^d y \nabla \times \vec{a} V(|\vec{x} - \vec{y}|) \nabla \times \vec{a} + \frac{1}{2\alpha} (\nabla \cdot \vec{a})^2.
\]  

This Lagrangian is motivated by the composite fermion description of the $\nu = 1/2$ Quantum Hall state, where two flux tubes are attached to each electron forming composite fermions. Since the flux quanta are attached to the electrons, fluctuations in electron-density lead to fluctuations in the gauge field. This is taken care of by the third term in which the electron-electron interaction $V(|\vec{x} - \vec{y}|)$ controls the gauge fluctuations. Here we will consider electron-electron interactions of the form

\[
V(|\vec{x} - \vec{y}|) \propto \frac{1}{|\vec{x} - \vec{y}|^\eta},
\]

where $\eta$ is close to 1. The last term is the gauge-fixing term. It is obtained using the Faddeev-Popov Ansatz to restrict $\vec{a}$ to a gauge in which

\[
\nabla \cdot \vec{a} = f,
\]

and then averaging over a Gaussian distribution, with mean value 0 and width $\alpha$, of such functions $f$. Coulomb gauge corresponds to $\alpha \to 0$.

In order to describe the $\nu = 1/2$ state properly, a Chern-Simons term and a time-like gauge field ($a_0$) should also be included. We omit these terms because it is likely, as for Coulomb interaction in metals, that the $a_0$-field is screened and does not contribute to the fluctuations at low frequencies. There are however subtleties associated with the fluctuations of the $a_0$ field [3],[16] which we do not address here.

Before we go on to study the Lagrangian (1), let us understand why the $\beta$-function in QED is gauge-invariant. The QED-Lagrangian density in terms
of the bare coupling and fields is
\[ L = -\frac{1}{4} F^\mu_\nu F^\nu_\mu + \bar{\psi}_0 \left( \phi - i e_0 A_0 - m_0 \right) \psi_0. \] (4)

By redefining fields and the coupling constant
\[ L = -\frac{1}{4} Z_3 F^\mu_\nu F^\nu_\mu + Z \bar{\psi} \left( \phi - \sqrt{\frac{Z_3}{Z_e}} i e A - Z_m m \right) \psi. \] (5)

The form of the interaction between the fermions and the gauge-field is dictated by gauge-invariance. So for gauge-invariance to hold in the renormalized theory:
\[ Z_3 = Z_e. \] (6)

This identity can also be proven using the Ward identity, which is nothing but the statement of gauge-invariance. Now consider the correlation function
\[ \langle F^{\mu\nu}(x)F^{\alpha\beta}(y) \rangle. \] (7)

This correlation function is obviously gauge-independent, and therefore \( Z_3 \) should also be independent of gauge. This and the above Ward identity (6) ensures that the \( \beta \)-function which is obtained by differentiating \( Z_e \) is independent of the choice of gauge.

We see that neither \( Z \) nor \( Z_m \) play any role in this argument. Let us think about computing the above correlation function (7). The exact answer will of course be a very complicated function of \( m_0 = Z_m m \) and \( e_0 = e Z_e^{-1/2} \). This answer should not change as we change the gauge, and so if \( Z_m \) and \( Z_e \) were dependent on gauge, amazing cancellations would have to occur. Conceivably the only cancellations that can occur should be due to the Ward identity. Since the Ward identity has nothing to do with \( Z_m \) it follows that \( Z_m \) cannot be dependent on gauge either. \( Z \) is not constrained by the above as the correlation function (7) can be calculated using the effective action for the gauge fields in which the fermions are integrated out. \( Z \) will be absorbed into the measure, and there is nothing preventing \( Z \) from being gauge-dependent.

For non-abelian gauge theories the situation is somewhat different from QED. There the gauge-field renormalization constant is dependent on gauge, but in spite of that, the \( \beta \)-function is gauge-independent there also.

Let us now return to the Lagrangian (1). Written in Fourier space the Euclidean action in terms of renormalized couplings and fields is
\[
S = T \sum_n \int \frac{d^d p}{(2\pi)^d} Z \psi^\dagger(p) \left(-i\omega_n + Z_m \left(\frac{p^2}{2m} - \mu\right)\right) \psi(p)
+ T \sum_m \int \frac{d^d q}{(2\pi)^d} Z_3 \frac{1}{2} a^i(q) \left(\delta^{ij} - \frac{q^i q^j}{q^2}\right) + Z_\alpha \frac{1}{\alpha} q^i q^j a^j(-q) \tag{8}
\]
\[
- ZZ_m \sqrt{Z_3 Z_g} \frac{g^2}{2m} T \sum_n \int \frac{d^d p}{(2\pi)^d} T \sum_m \int \frac{d^d q}{(2\pi)^d} \left(2p^i + q^i\right) a_i(q)\psi^\dagger(q+p)\psi(p)
+ ZZ_m \frac{Z_3}{Z_g} \frac{g^2}{2m} T \sum_n \int \frac{d^d p}{(2\pi)^d} T \sum_m \int \frac{d^d q_1}{(2\pi)^d} T \sum_m \int \frac{d^d q_2}{(2\pi)^d} \\
\times \psi^\dagger(p + q_1 + q_2)\psi(p) a^i(q_1) a^i(q_2).
\]

We will assume a circular Fermi surface and eventually take the zero-temperature limit such that \(\mu = p_f^2/2m\). It is of course possible to linearize the energy dispersion around the Fermi surface, in that case \(Z_m\) is naturally named \(Z_{vf}\). This linearization cannot be essential for any important physics and will not be used in this paper. Note that there is no renormalization constant associated with \(p_f\). \(p_f\) measures the total number of particles which does not change. The renormalization of \(V(q) = v_B q^{n-2}\) is taken care of by \(Z_3\).

By simple dimensional analysis one can verify that the relevant coupling between the gauge field and the fermions is \(g^2 v_f p_f^{d-1-n}/v_B\) and not \(g^2\) as in QED. Therefore the relevant \(\beta\)-function for this problem doesn’t involve just \(Z_g\), but \(Z_g, Z_m\) and \(Z_3\). As in QED, gauge invariance requires \(Z_3 = Z_g\). This can be seen explicitly by writing down the Ward identity. Our Lagrangian is not gauge-invariant with respect to gauge-transformations which mix space and time. We must therefore consider the spatial vertex separately in the Ward identity, setting the external frequency to zero. As \(Z_3\) is obviously gauge-invariant, we need only to find if \(Z_m\) depends on the choice of gauge, to make statements about the \(\beta\)-function.

In the mean field approximation the gauge field vanishes. Since the mean field theory describes a metal it is reasonable to proceed as in the theory of metals where one uses the RPA-corrected form of the interaction. The bare photon propagator extracted from the Lagrangian is

\[
D^{ij}_0(q, \epsilon_m) = \frac{1}{V(q)q^2} \left(\delta^{ij} - \frac{q^i q^j}{q^2}\right) + \alpha \frac{q^i q^j}{(q^2)^2} \tag{9}
\]

The RPA-approximation of the propagator is obtained by summing an infinite series of bubble diagrams. The polarization bubble is shown in figure 1, and its expression is

\[
\Pi^{ij}(q, \epsilon_m) = A(q, \epsilon_m) \left(\delta^{ij} - \frac{q^i q^j}{q^2}\right) + B(q, \epsilon_m) \frac{q^i q^j}{q^2} \tag{10}
\]
Fig. 1. One-loop diagrams used in the RPA resummation.

where

\[
A(q, \epsilon_m) = -\left(\frac{g}{2m}\right)^2 \frac{4}{d-1} T \sum_n \int \frac{d^d p}{(2\pi)^d} (p \times q)^2 G_0(p, \omega_n) G_0(p + \bar{q}, \omega_n + \epsilon_m) \\
+ \frac{g^2}{m} T \sum_n \int \frac{d^d p}{(2\pi)^d} G_0(p, \omega_n),
\]

\[
B(q, \epsilon_m) = -\left(\frac{g}{2m}\right)^2 T \sum_n \int \frac{d^d p}{(2\pi)^d} (2p \cdot \bar{q} + q^2)^2 G_0(p, \omega_n) G_0(p + \bar{q}, \omega_n + \epsilon_m) \\
+ \frac{g^2}{m} T \sum_n \int \frac{d^d p}{(2\pi)^d} G_0(p, \omega_n),
\]

and the bare single-particle propagator is

\[
G_0(p, \omega_n) = -\frac{1}{i \omega_n - \left(\frac{\bar{p}^2}{2m} - \mu\right)}.
\]

Summing the infinite series of bubble diagrams we obtain

\[
D^{ij}(q, \epsilon_m) = \frac{1}{V(q)q^2 - A(q, \epsilon_m)} \left( \delta^{ij} - \frac{q^i q^j}{q^2} \right) + \frac{\alpha}{q^2 - \alpha B(q, \epsilon_m)} \frac{q^i q^j}{q^2}.
\]

The transverse part is independent of the gauge-fixing parameter \(\alpha\), and the longitudinal part of the gauge interaction does not include any information about the electron-electron interaction.

The calculations of \(A(q, \epsilon_m)\) and \(B(q, \epsilon_m)\) are straight-forward at \(T = 0\). In the zero-temperature limit of the imaginary time formalism we get in \(d = 2\), for \(|\epsilon| \ll v_f q\) and \(q \ll p_f\):

\[
A(q, \epsilon) = -\frac{g^2}{12\pi} \frac{q^2}{2m} \frac{\epsilon}{\pi} \frac{p_f^2}{v_f q},
\]

\[
B(q, \epsilon) = -\frac{g^2}{\pi} \frac{p_f^2}{2m} \left(\frac{\epsilon}{v_f q}\right)^2.
\]

Having calculated the gauge boson propagator we can now calculate the self-energy diagrams.
The one-loop diagrams contributing to the self-energy are drawn in figure 2. The expression is

\[ \Sigma(p, \omega_n) = \left( \frac{g}{2m} \right)^2 T \sum_m \int \frac{d^d q}{(2\pi)^d} \left( 2p^i + q^i \right) D^{ij}(q, \epsilon_m) \left( 2p^j + q^j \right) G_0(p+q, \omega_n+\epsilon_m) \]

\[ - \frac{g^2}{2m} T \sum_m \int \frac{d^d q}{(2\pi)^d} D^{ii}(q, \epsilon_m). \]  

(17)

For convenience we will split the contributions to the self-energy into two pieces,

\[ \Sigma(p, \omega_n) = \Sigma_t(p, \omega_n) + \Sigma_l(p, \omega_n). \]  

(18)

The first part is the self-energy obtained from the diagrams in figure 2 using the transverse part of the gauge propagator, while the second part is obtained using the longitudinal propagator in the same diagrams. It is obvious from the form of the gauge-propagator (14) that only \( \Sigma_t \) will depend on gauge. Let us first show that the logarithmic singularity necessary for the strong coupling fixed point is obtained by calculating \( \Sigma_t \). Performing the angular integral in \( d = 2 \), the transverse contribution to the one-loop self-energy can be written

\[ \Sigma_t(p, \omega_n) = -\frac{g^2}{2\pi} T \sum_m \int_0^\Lambda dq \frac{q^{-1}}{V(q)q^2 - A(q, \epsilon_m)} \left[ i\omega_n - \frac{p^2}{2m} + \mu \right] \]

\[ - \text{isgn} (\omega_n + \epsilon_m) \left( \frac{p^2q^2}{m^2} - \left( i\omega_n - \frac{p^2}{2m} + \mu + i\epsilon_m - \frac{q^2}{2m} \right)^2 \right)^{1/2}. \]  

(19)

We have inserted an arbitrary cutoff \( \Lambda \) in the \( q \)-integration and used the fact that \( A(q, -i\epsilon_m) = A(q, i\epsilon_m) \). Expanding the square root the self-energy takes the form

\[ \Sigma_t(p, \omega_n) = -\frac{g^2}{2\pi} T \sum_m \int_0^\Lambda dq \frac{1}{q V(q)q^2 - A(q, \epsilon_m)} \]

\[ \times \left[ \left( i\omega_n - \frac{p^2}{2m} + \mu \right) \left( 1 + \text{isgn}(\omega_n + \epsilon_m) \frac{m}{pq} (i\epsilon_m - \frac{q^2}{2m}) + \cdots \right) \]

\[ - \text{isgn}(\omega_n + \epsilon_m) \frac{pq}{m} \left( 1 - \frac{1}{2} \left( \frac{m}{pq} \right)^2 (i\epsilon_m - \frac{q^2}{2m})^2 + \cdots \right) \]
\[+O \left( i\omega_n - \frac{p^2}{2m} + \mu \right)^2 \]. \tag{20}\]

The ellipses indicate higher order terms in the expansion of the square root. These terms are not as singular as the leading term, so they can safely be neglected. Let us now expand the renormalization constants \(Z\) and \(Z_m\) in the coupling constant,

\[
Z = 1 + Z^{(1)} + \cdots, \tag{21}
\]

\[
Z_m = 1 + Z_m^{(1)} + \cdots. \tag{22}
\]

Using this expansion we have

\[
Z \left( i\omega_n - Z_m \left( \frac{p^2}{2m} - \mu \right) \right) = \left( 1 + Z^{(1)} + Z_m^{(1)} \right) \left( i\omega_n - \left( \frac{p^2}{2m} - \mu \right) \right) - Z_m^{(1)} i\omega_n + \cdots. \tag{23}\]

Contributions to \(Z_m^{(1)}\) are distinguished by the fact that they renormalize \(i\omega_n\) differently from \(\left( \frac{p^2}{2m} - \mu \right)\). So it is the second term in the above expression (20) which will contribute to \(Z_m^{(1)}\). The second term is

\[
\Sigma_{Z_m}^{n}(\vec{p}, \omega_n) = i \frac{g^2}{2\pi m} \sum_{m}^{\Lambda} \int_{0}^{\infty} dq \frac{\text{sgn}(\omega_n + \epsilon_m)}{V(q)q^2 - A(q, \epsilon_m)}
\]

\[
\times \left( 1 - \frac{1}{2}(m_{pq})^2(i\epsilon_m - \frac{q^2}{2m})^2 + \cdots \right). \tag{24}\]

When \(\omega \rightarrow 0\) the first term in the last parenthesis will give the leading behaviour. Setting \(T = 0\), \(p = p_f\), \(V(q) = v_Bq^{\eta-2}\) and using the result (15) for \(A(q, \epsilon)\) this leading behaviour is

\[
\Sigma_{Z_m}^{n}(p_f, \omega) = i \frac{g^2}{2\pi v_f} \int_{-\infty}^{\infty} d\epsilon \int_{0}^{\Lambda} dq \frac{\text{sgn}(\omega + \epsilon)}{V_B q^{\eta} + \frac{g^2}{12\pi 2m} + \frac{g^2 p_f^2}{\pi 2m_{pq}} |\epsilon|}
\]

\[
\approx -i \omega \frac{g^2 v_f A^{1-\eta} \left( \frac{g^2 v_f A^{1-\eta} \pi |\omega|}{2\pi^2 v_B} \right)^{\frac{1-\eta}{1+\eta}}}{2\pi^2 v_B} \tag{25}\]

In the special case \(\eta = 1\), corresponding to \(1/r\)-interactions between the fermions, we find in the low-frequency limit:
\[
\Sigma^Z_{\lambda}(p_f, \omega) = -i\omega g^2 v_f \frac{\ln \left( \frac{g^2 v_f \pi |\omega|}{2\pi^2 v_B v_f \Lambda} \right)^2}{4\pi^2 v_B}.
\] (26)

In this case the correction to the \(i\omega\)-behaviour is logarithmic, as observed by Nayak and Wilczek. This case is the marginal case which sets the borderline between the Fermi liquid theory and a strong coupling theory. For \(\eta < 1\), corresponding to interactions more long-range than Coulomb interactions, the self-energy vanishes faster than \(\omega\) and no deviations from Fermi liquid are seen. But for \(\eta > 1\), short-range electron-electron interactions, the self-energy becomes the dominant term at small \(\omega\) and the system is not in the Fermi liquid universality class.

The question is now whether these conclusions depend on the choice of gauge. At first sight they do not, since the transverse gauge-propagator does not depend on \(\alpha\). However, one could imagine that the longitudinal gauge-propagator, which is dependent on \(\alpha\), could give additional competing singularities. We will now show, by computing the contribution to the self-energy using the longitudinal gauge-propagator, that there are no \(\alpha\)-dependent competing singularities.

Performing the angular integration in the expression for the self-energy contribution coming from the longitudinal gauge propagator we get

\[
\Sigma_{\lambda}(p, \omega_n) = \frac{g^2}{2\pi} T \sum_{m}^{\Lambda} \int_{0}^{\infty} dq \frac{\alpha q^{-1}}{q^2 - \alpha B(q, \epsilon_m)} \left[ i\omega_n - \frac{p^2}{2m} + \mu \right. \\
+ \frac{\text{sgn} (\omega_n + \epsilon_m) \left( i\omega_n - \frac{p^2}{2m} + \mu + i\epsilon_m \right)^2}{\sqrt{\left( \frac{\epsilon_m}{m} \right)^2 - \left( i\omega_n - \frac{p^2}{2m} + \mu + i\epsilon_m - \frac{q^2}{2m} \right)^2}} \left. \right].
\] (27)

The gauge boson propagator is invariant with respect to \(\epsilon_m \rightarrow -\epsilon_m\). Therefore no gap in the fermion spectrum is generated by the self-energy, which of course is expected on general grounds. In the same way as for the transverse contribution we expand the square root.

\[
\Sigma_{\lambda}(p, \omega_n) = \frac{g^2}{2\pi} T \sum_{m}^{\Lambda} \int_{0}^{\infty} dq \frac{\alpha q^{-1}}{q^2 - \alpha B(q, \epsilon_m)} \left[ i\omega_n - \frac{p^2}{2m} + \mu \right] \\
\times \left( 1 + 2i\text{sgn} (\omega_n + \epsilon_m) \frac{m}{p} i\epsilon_m + \cdots \right) \\
+ i\text{sgn} (\omega_n + \epsilon_m) (i\epsilon_m)^2 \frac{m}{p} \left( 1 + \frac{1}{2} \frac{m}{p} \left( \frac{\epsilon_m - \frac{q^2}{2m}}{2m} \right)^2 + \cdots \right)
\]
The contribution to $Z_m$ comes from the second term and is

$$
\Sigma^Z_{m}(\vec{p}, \omega_n) = \frac{g^2 T}{2\pi} \sum_m \int_{0}^{\Lambda} d\alpha \frac{\alpha^{-1} \text{isgn}(\omega_n + \epsilon_m)}{q^2 - \alpha B(q, \epsilon_m)} (i\epsilon_m)^2 \frac{m}{pq} (1 + \cdots). \quad (29)
$$

This quantity does not vary fast with $p$ so we set $p = p_f$. Taking the zero-temperature limit, using the result (16) for $B(q, \epsilon)$ and keeping just the dominant term as $\omega \to 0$ we get

$$
\Sigma^Z_{m}(p_f, \omega) = \frac{g^2}{4\pi^2} \int_{-\infty}^{\infty} d\epsilon \int_{0}^{\Lambda} d\alpha \frac{\alpha^{-1} \text{isgn}(\omega + \epsilon)}{q^2 + \alpha^2 \frac{p_f^2}{2m} \frac{\epsilon^2}{v_f^2}} \left( \alpha g^2 v_f \right)^{1/4} \left( \frac{\epsilon}{v_f p_f} \right)^{1/2}

\approx -i \omega \frac{\sqrt{2}}{6} \left( \frac{\alpha g^2 v_f}{2\pi p_f} \right)^{1/4} \left( \frac{\omega}{v_f p_f} \right)^{1/2}

\sim \omega^{3/2},
$$

which vanishes faster than $\omega$ and therefore does not contribute to the singularities of $Z_m$. This shows that the singularities of $Z_m$ are not dependent on $\alpha$ to this order in perturbation theory.
3 Calculations in d=3

In [8] the authors used the $\epsilon$-expansion around $d = 3$, and the free gauge-propagator

$$D^{ij} = \frac{1}{v_{Bq}^2} \left( \delta^{ij} - \frac{q^i q^j}{q^2} \right), \quad (31)$$

to analyze the same theory. We will in this chapter show that the $\beta$-function is independent of gauge also in this case. The computation in three dimensions follows along the same lines as the computations in $d = 2$. The polarization diagrams at $T = 0$ give for $|\epsilon| \ll v_f q$ and $q \ll p_f$,

$$A(q, \epsilon) = -\frac{g^2}{4\pi^2 p_f^2} \frac{p_f^2}{2m} \left( \frac{q}{p_f} \right)^2 - \frac{g^2}{4\pi^2 p_f^2} \frac{p_f^2}{2m v_f q} |\epsilon|, \quad (32)$$

$$B(q, \epsilon) = -\frac{g^2}{\pi^2} \frac{p_f^2}{2m} \left( \frac{\epsilon}{v_f q} \right)^2. \quad (33)$$

The contribution from the transverse gauge boson to the self-energy takes the following form when the angular integrations are performed

$$\Sigma_t(\vec{p}, \omega_n) = -\frac{g^2}{4\pi^2} T \sum_{m} \int_{0}^{\Lambda} dq \frac{1}{V(q)q^2 - A(q, \epsilon_m)} \left[ 2 \left( i \omega_n - \frac{p^2}{2m} + \mu + i \epsilon_m + \frac{q^2}{2m} \right) \right. $$

$$\left. + m pq \left( \left( i \omega_n - \frac{p^2}{2m} + \mu + i \epsilon_m - \frac{q^2}{2m} \right)^2 - \frac{p^2 q^2}{m^2} \right) \right. $$

$$\times \ln \left[ \frac{p^2 m - (i \omega_n - \frac{p^2}{2m} + \mu + i \epsilon_m - \frac{q^2}{2m})}{-\frac{p^2 m}{pq} - (i \omega_n - \frac{p^2}{2m} + \mu + i \epsilon_m - \frac{q^2}{2m})} \right]. \quad (34)$$

For small $\omega_n$, $p^2/2m - \mu$ and $\epsilon_m \ll v_f q$, $q \ll p_f$ the logarithm can be expanded;

$$\ln \left[ \frac{p^2 m - (i \omega_n - \frac{p^2}{2m} + \mu + i \epsilon_m - \frac{q^2}{2m})}{-\frac{p^2 m}{pq} - (i \omega_n - \frac{p^2}{2m} + \mu + i \epsilon_m - \frac{q^2}{2m})} \right] = i \pi \text{sgn}(\omega_n + \epsilon_m)$$

$$- \frac{2m}{pq} \left( i \omega_n - \frac{p^2}{2m} + \mu + i \epsilon_m - \frac{q^2}{2m} \right) + \cdots. \quad (35)$$

When using this expansion in $\Sigma_t$ we can write
\[ \Sigma_t(\vec{p}, \omega_n) = -\frac{g^2}{4\pi^2} T \sum_n \int_0^\Lambda dq \frac{1}{q^2 V(q) - A(q, \epsilon_m)} \left[ (i\omega_n - \frac{p^2}{2m} + \mu) \right. \\
\quad \times \left. \left( 4 + 2\frac{m}{pq}(i\epsilon_m - \frac{q^2}{2m})i\pi \text{sgn}(\omega_n + \epsilon_m) + \cdots \right) \right. \\
\quad + \left. \frac{pq}{m} i\pi \text{sgn}(\omega_n + \epsilon_m) + 4i\epsilon_m + \cdots \right] \\
\quad + O \left( \frac{i\omega_n - \frac{p^2}{2m} + \mu}{\mu} \right)^2. \quad (36) \]

Using the expression (32) for \( A(q, \epsilon_m) \) all terms odd in \( \epsilon_m \) will vanish, and so no mass term is generated. The most singular term contributing to \( Z_m \) is

\[ \Sigma Z_m^t(\vec{p}, \omega_n) = \frac{g^2}{4\pi^2} p T \sum_n \int_0^\Lambda dq \frac{q i\pi \text{sgn}(\omega_n + \epsilon_m)}{q^2 V(q) - A(q, \epsilon_m)}. \quad (37) \]

Taking the zero-temperature limit and setting \( p = p_f \) we get for the dominant term in \( \omega \) as \( \omega \to 0 \):

\[ \Sigma Z_m^t(p_f, \omega_n) = \frac{g^2 v_f}{4\pi^2} i\omega \int_0^\Lambda dq \frac{q i\pi \text{sgn}(\omega_n + \epsilon_m)}{q^2 V(q) + \frac{g^2}{4\pi^2} i\omega + \frac{p_f^2}{2m} (\frac{q}{p_f})^2 + \frac{g^2}{4\pi^2} \frac{p_f^4}{2m v_B q^2}}. \quad (38) \]

where we have used the expression (32) for \( A(q, \epsilon) \). For the free gauge-propagator (31), \( V(q) = v_B \), and the leading behaviour as \( \omega \to 0 \) is

\[ \Sigma Z_m^t(p_f, \omega_n) = -i\omega \frac{g^2 v_f}{12\pi^2 v_B} \ln \left( \frac{g^2 v_f}{8\pi v_B} \frac{p_f^3}{A^3 v_f p_f} |\omega| \right). \quad (39) \]

This logarithmic behaviour leads to a logarithmic correction to the specific heat of the electron gas [17]. This result is not dependent on the choice of gauge as can be seen by again computing the contribution to the self-energy using the longitudinal part of the gauge-propagator,

\[ \Sigma_t(\vec{p}, \omega_n) = \frac{g^2}{4\pi^2} T \sum_n \int_0^\Lambda dq \frac{\alpha}{q^2 - \alpha B(q, \epsilon_m)} \left[ i\omega_n - \frac{p^2}{2m} + \mu + i\epsilon_m \right. \\
\quad + \left. m \frac{pq}{i\omega_n - \frac{p^2}{2m} + \mu + i\epsilon_m} \ln \left[ \frac{-1 + \frac{m}{pq} \left( i\omega_n - \frac{p^2}{2m} + \mu + i\epsilon_m - \frac{q^2}{2m} \right)}{1 + \frac{m}{pq} \left( i\omega_n - \frac{p^2}{2m} + \mu + i\epsilon_m - \frac{q^2}{2m} \right)} \right] \right]. \quad (40) \]

Expanding the logarithm we get
\[
\Sigma_{l}(\vec{p},\omega_n) = \frac{g^2}{4\pi^2} T \sum_{m} \frac{\alpha}{\Lambda} \int_0^\Lambda dq \frac{\alpha}{q^2 - \alpha B(q, \epsilon_m)} \left[ \left( i\omega_n - \frac{p^2}{2m} + \mu \right) \right.
\]
\[
\times \left( 1 + i\frac{m}{pq} \frac{\alpha B(q, \epsilon_m)}{q^2 - \alpha B(q, \epsilon_m)} \right)
\]
\[
+ i\epsilon_m + (i\epsilon_m)^2 \frac{m}{pq} \frac{i\pi \text{sgn}(\omega_n + \epsilon_m)}{q^2 - \alpha B(q, \epsilon_m)} + \cdots
\]
\[
+ O \left( i\omega_n - \frac{p^2}{2m} + \mu \right)^2 \right].
\] (41)

The most singular term contributing to \( Z_m \) is
\[
\Sigma_{l}^Z(p,\omega_n) = \frac{g^2}{4\pi^2} mT \sum_{m} \frac{\alpha q^{-1}(i\epsilon_m)^2 i\pi \text{sgn}(\omega_n + \epsilon_m)}{q^2 - \alpha B(q, \epsilon_m)}. \] (42)

Taking the zero-temperature limit and using the expression (33) for \( B(q, \epsilon) \), we get in the limit \( \omega \to 0 \)
\[
\Sigma_{l}^Z(p_f, \omega) = -i\omega \frac{\alpha g^2 v_f }{16} \left( \frac{v^2}{2\pi^2} \right)^{1/2} \frac{|\omega|}{v_f p_f}.
\] (43)

This vanishes faster than \( \omega \) as \( \omega \to 0 \), so there are no gauge-dependent corrections to the singularities of \( Z_m \) in this case either.

4 Conclusion

There have been many attempts [7]-[15] to go beyond the mean field approximation for the \( \nu = 1/2 \) Quantum Hall state. Some of these attempts [7],[8],[9] are based on the behaviour of the singe-particle Green function. One criticism put forward [10] against arguments based on the single particle Green function is that conclusions based on it might not be physical since the single-particle Green function is not gauge-invariant.

In this paper we have shown that the dominant singular behaviour determining the pole-structure of the single-particle Green function is not affected by the choice of gauge, when the gauge choices are restricted to a particular family of gauges. This is true both in \( d = 2 \) and in \( d = 3 \).

This implies in particular that the \( \beta \)-functions obtained in [7],[8] are independent of gauge, and that the pole-structure used to construct the (singular)
Fermi liquid description in [9] is also gauge-invariant.

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