FUZZY PATH HOMOTOPY IN FUZZY PEANO SPACE

1K. Sugapriya,  2Dr. B. Amudhambigai

Department of Mathematics, Sri Sarada College for Women, Salem-636016 ,Tamil Nadu, India

E-mail: 1nirmalapriya97@gmail.com,  2rbamudha@yahoo.co.in

Abstract. The concepts of fuzzy path connected homotopy in fuzzy Peano Space, Contiguous position of fuzzy path connected homotopy and fuzzy path connected homotopy equivalence in fuzzy Peano space are introduced in this paper. Also several interesting properties of fuzzy path connected homotopy in fuzzy Peano space are discussed.

1. INTRODUCTION

The famous Mathematician, Computer scientist, Electrical engineer and Artificial intelligence researcher Zadeh [7] is the founder of fuzzy mathematics, fuzzy set theory, and fuzzy logic. In 1968, Chang [1] proposed the idea of fuzzy topological space. The concept of classical homotopy theory was introduced by Massey[6]. Fuzzy homotopy theory was defined and developed by Cuvalcioglu and Citil[3]. The concept of fuzzy Peano space was introduced by Sugapriya and Amudhambigai[4]. The concepts of fuzzy path homotopy in fuzzy Peano space, Contiguous position of fuzzy path homotopy and fuzzy path homotopy equivalence in fuzzy Peano Space are introduced in this paper. In this connection several interesting properties of fuzzy path homotopy in fuzzy Peano space are discussed.

2. PRELIMINARIES

In this section the basic and necessary definitions required for this paper are discussed.

Definition 2.1[7] Let $X$ be a non-empty set and $I$ be the unit interval $[0, 1]$. A fuzzy set in $X$ is an element of the set $/g^{13,5,0,25}$ of all function from $X$ to $I$.

Definition 2.2 [1] A fuzzy topology is a family $/g^{20,23}$ of fuzzy sets in $X$ which satisfies the following conditions :

(i) $/g^{0,25}, /g^{26,0,25} \in /g^{20,23}$,

(ii) If $/g^{20,19}, /g^{20,20} \in /g^{20,23}$ then $/g^{20,19} \wedge /g^{20,20} \in /g^{20,23}$,

(iii) If $/g^{20,19} /g^{4,20,19} \in /g^{20,23}$ for each $i \in J$ then $\vee /g^{20,19} /g^{4,20,19} \in /g^{20,23}$.

$/g^{20,23}$ is called a fuzzy topology for $X$, and the pair $(X, T)$ is a fuzzy topological space, or fts for short.

Definition 2.3[6] A fuzzy point in $X$ is a special fuzzy set with membership function defined by

$$P(x) = \begin{cases} \lambda & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

where $0 < \lambda \leq 1$. $P$ is said to have support $y$, value $\lambda$ and is denoted by $P(y, \lambda)$.

Definition 2.4[6] A fuzzy point $P$ in $X$ is a special fuzzy set with membership function defined by

$$P(x) = \begin{cases} \lambda & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

where $0 < \lambda \leq 1$. $P$ is said to have support $y$, value $\lambda$ and is denoted by $P(y, \lambda)$.

Definition 2.5[5] Let $(X, T)$ be a fuzzy topological space. Then
is a fuzzy topology on $X$, called the fuzzy topology on $X$ introduced by $T$ and $(X, \tilde{T})$ is called the fuzzy topological space introduced by $(X, T)$.

**Definition 2.6** \[2\] Let $(X, T), (Y, S)$ be any two fuzzy topological spaces and $f, g \in FC(X, Y)$. If there exist a fuzzy continuous function $F : (X, T) \times (I, \varepsilon_I) \rightarrow (Y, S)$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$, then $f$ and $g$ are fuzzy homotopic.

**Definition 2.7** \[2\] Let $(X, T), (Y, S)$ be any two fuzzy topological spaces and $f \in FC(X, Y), g \in FC(Y, X)$. If $f \sim g \sim 1_X$ and $f \sim g 1_X$, then $(X, T)$ and $(Y, S)$ are called fuzzy homotopic equivalent or having the same homotopy type. Also $f$ and $g$ are called fuzzy equivalences.

**Notation 2.1** \[4\] Let $(X, T)$ be a Fuzzy Peano Space. The collection of all fuzzy path connected points in $(X, T)$ is denoted by $\mathcal{FP}(X)$. Also $f$ and $g$ are called fuzzy equivalences.

**3. FUZZY PATH HOMOTOPY**

In this section the concepts of fuzzy path connected homotopy in fuzzy Peano space, Contiguous position of fuzzy path connected homotopy and fuzzy path connected homotopy equivalence in fuzzy Peano space are discussed.

**Notation 3.1** Let $\varepsilon_i$ and $\zeta_j$ denote the Euclidean subspace fuzzy topologies on $I$ and $(I, \varepsilon_i')$, $(I, \zeta_j')$ denote the fuzzy Peano spaces introduced by the fuzzy topological spaces $(I, \varepsilon_I)$ and $(I, \zeta_j)$ respectively, where $I = [0, 1]$.

**Theorem 3.1** Let $(I, \varepsilon_i')$ and $(I, \zeta_j')$ be any two fuzzy Peano spaces. The collection of all fuzzy path connected points in $(I, \varepsilon_i')$ and $(I, \zeta_j')$ is denoted by $\mathcal{FP}(\varepsilon_i')$ and $\mathcal{FP}(\zeta_j')$ respectively.

**Proposition 3.1** Let $(X, T)$ be a fuzzy Peano space. Then every fuzzy Path connected homotopy relation $\cong_{fp}$ is an equivalence relation.

**Proof** (i) Reflexivity of the relation $\cong_{fp}$ is obvious.

(ii) Suppose $p_1 \cong_{fp} p_2$. Then by Definition 3.1, there exists a fuzzy continuous function $P : (I, \varepsilon_i') \times (I, \zeta_j') \rightarrow \mathcal{FP}(X)$ such that $P(0, \zeta_j) = x_t, P(1, \zeta_j) = y_t$ for all $\zeta_j \in \mathcal{FP}(\zeta_j')$ and $P(\varepsilon_i, 0) = p_1(x_t), P(\varepsilon_i, 1) = p_2(y_t)$ for all $\varepsilon_i \in \mathcal{FP}(\varepsilon_i')$. Let us consider a fuzzy continuous function $Q : (I, \varepsilon_i') \times (I, \zeta_j') \rightarrow \mathcal{FP}(X)$ which is defined by $Q(\varepsilon_i, \zeta_j) = P(\varepsilon_i, 1 - \varepsilon_i) \cdot P(\varepsilon_i, \zeta_j)$ for all $\varepsilon_i \in \mathcal{FP}(\varepsilon_i')$ and $\zeta_j \in \mathcal{FP}(\zeta_j')$. Now $Q(0, \zeta_j) = P(0, \zeta_j, 1) = y_t$ and $Q(1, \zeta_j) = P(1, \zeta_j, 0) = p_2(x_t)$. Hence $p_1 \equiv_{fp} p_2$.

(iii) Let $x_t, y_t, z_t \in \mathcal{FP}(X)$ and let $p_1 : [0, 1] \rightarrow \mathcal{FP}(X)$ be a fuzzy Path from $x_t$ to $y_t$ such that $p_1(0) = x_t, p_1(1) = y_t$. Let $p_2 : [0, 1] \rightarrow \mathcal{FP}(X)$ be a fuzzy Path from $y_t$ to $z_t$ such that $p_2(0) = y_t$ and $p_2(1) = z_t$. Suppose $p_2 \cong_{fp} p_3$ and $p_2 \cong_{fp} p_3$. Let $P : (I, \varepsilon_i') \times (I, \zeta_j') \rightarrow \mathcal{FP}(X)$ be the fuzzy Path connected homotopy between $p_2$ and $p_3$. Define $P_2 : (I, \varepsilon_i') \times (I, \zeta_j') \rightarrow \mathcal{FP}(X)$ as follows:

$$P_2(\varepsilon_i, \zeta_j) = \begin{cases} P(\varepsilon_i, 2\zeta_j), & 0 \leq \zeta_j \leq \frac{1}{2} \\ P(\varepsilon_i, 2\zeta_j - 1), & \frac{1}{2} \leq \zeta_j \leq 1 \end{cases}$$

Now $P_2(\varepsilon_i, 0) = P(\varepsilon_i, 0)$ and $P_2(\varepsilon_i, 1) = P(\varepsilon_i, 1)$. Therefore from Definition 3.1, $p_1 \equiv_{fp} p_3$. 

\[2\]
\textbf{Proposition 3.2} Let \((I, \varepsilon'_1)\) and \((I, \xi'_2)\) be any two fuzzy Peano spaces introduced by the Euclidean subspace fuzzy topologies \(\varepsilon_1\) and \(\xi_2\) respectively. Let \((X, T)\) be a fuzzy Peano Space. Suppose \(f : (I, \varepsilon'_1) \rightarrow (I, \xi'_2)\) is a fuzzy continuous function defined as \(f(0) = 0\) and \(f(1) = 1\). Then for any fuzzy Path \(p_1 : [0, 1] \rightarrow \mathcal{FPCP}(X)\), the fuzzy Path connected homotopy will be \(p_1 \equiv_p p_2 \circ f\).

**Proof** Define a fuzzy continuous function \(P : (I, \varepsilon'_1) \times (I, \xi'_2) \rightarrow \mathcal{FPCP}(X)\) by \(P(\varepsilon'_1, \xi'_2) = p_1(\varepsilon'_1) \circ f(\xi'_2) + (1 - \varepsilon'_1)K_1\) where \(\varepsilon'_1 \in \mathcal{FPCP}(\varepsilon'_1)\) and \(\xi'_2 \in \mathcal{FPCP}(\xi'_2)\). Therefore, \(P(0, \xi'_2) = p_1(\xi'_2)\) and \(P(1, \xi'_2) = f(\xi'_2) = p_1(0) \circ f(\xi'_2)\). Similarly, \(P(\varepsilon'_1, 0) = p_2(0)\) and \(P(\varepsilon'_1, 1) = p_1(1)\). Hence the fuzzy Path connected ending points are fixed. Therefore, \(P_1 \equiv_p P_2 \circ f\).

\textbf{Definition 3.2} Let \((I, \varepsilon'_1)\) and \((I, \xi'_2)\) be any two fuzzy Peano spaces introduced by the Euclidean subspace fuzzy topologies \(\varepsilon_1\) and \(\xi_2\) respectively. Let \((X, T)\) be a fuzzy Peano Space. Suppose \(p_1, p_2 : [0, 1] \rightarrow \mathcal{FPCP}(X)\) be any two fuzzy paths from the fuzzy path connected point \(x_t\) to \(y_t\) where \(x_t, y_t \in \mathcal{FPCP}(X)\). Assume that the fuzzy path connected ending point of \(p_1 : [0, 1] \rightarrow \mathcal{FPCP}(X)\) is the fuzzy path connected starting point of \(p_2 : [0, 1] \rightarrow \mathcal{FPCP}(X)\) such that \(p_1(1) = p_2(0)\). Then the fuzzy paths \(p_1\) and \(p_2\) can become a contiguous to produce a fuzzy path from \(p_1(0)\) to \(p_2(1)\). This contiguous is called the contiguous position of \(p_1\) and \(p_2\). The contiguous position of \(p_1\) and \(p_2\) is denoted by \(p_1 * c p_2\) and it is defined by

\[
(p_1 * c p_2)(t') = \begin{cases} 
  p_1(4t') & 0 \leq t' \leq \frac{1}{4} \\
  p_2(4t' - 3) & \frac{1}{4} \leq t' \leq 1
\end{cases}
\]

\textbf{Proposition 3.3} Let \((I, \varepsilon'_1)\) and \((I, \xi'_2)\) be any two fuzzy Peano spaces introduced by the Euclidean subspace fuzzy topologies \(\varepsilon_1\) and \(\xi_2\) respectively. Let \((X, T)\) be a fuzzy Peano Space. Let \(p_1, p_2 : [0, 1] \rightarrow \mathcal{FPCP}(X)\) be any two fuzzy paths from the fuzzy path connected point \(x_t\) to \(y_t\) where \(x_t, y_t \in \mathcal{FPCP}(X)\) which are fuzzy path connected homotopic, starting at \(p_0(1)\) then \(p_0 * c p_1 \equiv_p p_0 * c p_2\).

**Proof** Let \(P : (I, \varepsilon'_1) \times (I, \xi'_2) \rightarrow \mathcal{FPCP}(X)\) be a fuzzy path connected homotopy between \(p_1\) and \(p_2\). Therefore by Definition 3.1, \(P(0, \xi'_2) = x_t = p_1(0)\) and \(P(1, \xi'_2) = y_t = p_1(1) = p_2(0)\) for all \(\xi'_2 \in \mathcal{FPCP}(\xi'_2)\). Similarly, \(P(\varepsilon'_1, 0) = p_0(x_t)\) and \(P(\varepsilon'_1, 1) = p_1(y_t)\) for all \(\varepsilon'_1 \in \mathcal{FPCP}(\varepsilon'_1)\). Define the fuzzy path connected homotopy \(Q(\varepsilon'_1, \xi'_2)\) as

\[
Q(\varepsilon'_1, \xi'_2) = \begin{cases} 
  p_0(4\xi'_2) & 0 \leq \xi'_2 \leq \frac{3}{4} \\
  p_1(4\xi'_2 - 3) & \frac{3}{4} \leq \xi'_2 \leq 1
\end{cases}
\]

Therefore \(Q(0, \xi'_2) = p_0 * c p_1\) and \(Q(1, \xi'_2) = p_0 * c p_2\). Hence the proof.

\textbf{Proposition 3.4} Let \((X, T)\) be a fuzzy Peano Space. Suppose \(p_1 : [0, 1] \rightarrow \mathcal{FPCP}(X)\) be a fuzzy Path from \(x_t\) to \(y_t\) such that \(p_1(0) = x_t, p_1(1) = y_t, p_2 : [0, 1] \rightarrow \mathcal{FPCP}(X)\) be a fuzzy Path from \(y_t\) to \(z_t\) such that \(p_2(0) = y_t\) and \(p_2(1) = z_t\) and \(p_3 : [0, 1] \rightarrow \mathcal{FPCP}(X)\) be a fuzzy Path from \(x_t\) to \(z_t\) such that \(p_3(0) = x_t\) and \(p_3(1) = z_t\). If \(p_1(1) = p_2(0)\); \(p_2(1) = p_3(0)\) then \(p_1 * c p_2 * c p_3 \equiv_p p_1 * c (p_2 * c p_3)\).

**Proof** By the direct calculation we get the Proof:

\[
(p_1 * c p_2) * c p_3 = \begin{cases} 
  p_3(y_t) & 0 \leq y_t \leq \frac{1}{8} \\
  p_2 & \frac{(y_t - 1)}{8} \leq y_t \leq \frac{3}{4} \\
  p_3(4y_t - 3) & \frac{3}{4} \leq y_t \leq 1
\end{cases}
\]

On the other hand,

\[
p_1 * c (p_2 * c p_3) = \begin{cases} 
  p_1(4y_t) & 0 \leq y_t \leq \frac{1}{4} \\
  p_2(4y_t - 2) & \frac{1}{4} \leq y_t \leq \frac{3}{8} \\
  p_3 & \frac{(y_t - 5)}{8} \leq y_t \leq 1
\end{cases}
\]

Let \((I, \varepsilon'_1)\) be any fuzzy Peano space introduced by the Euclidean subspace fuzzy topology \(\varepsilon_1\). Now for all \(t' \in (I, \varepsilon'_1)\) define \(f : (I, \varepsilon'_1) \rightarrow (I, \varepsilon'_1)\) by
Proposition 3.5 Let \((I, \varepsilon'_i)\) and \((I, \varepsilon'_j)\) be any two fuzzy Peano spaces introduced by the Euclidean subspace fuzzy topologies \(\varepsilon_i\) and \(\varepsilon_j\) respectively. Let \((X, T), (Y, S)\) and \((Z, R)\) be any three fuzzy Peano Spaces. Let \(f_1, f_2, g_1, g_2: \mathcal{FPCP}(X) \to \mathcal{FPCP}(Y)\) and \(g_1, g_2: \mathcal{FPCP}(Y) \to \mathcal{FPCP}(Z)\) be any two fuzzy continuous functions, and let \(g_1 \circ f_1, f_2 \circ f_1\) and \(g_1 \circ g_2, g_2 \circ g_2\) respectively. Define a fuzzy continuous function \(R : \mathcal{FPCP}(X) \times (I, \varepsilon'_i) \to \mathcal{FPCP}(Y)\) and \(Q : \mathcal{FPCP}(Y) \times (I, \varepsilon'_i) \to \mathcal{FPCP}(Z)\) be any two fuzzy path connected homotopy equivalence. If \(f_1 \cong f_1 \circ f_2\) and \(g_1 \cong g_1 \circ g_2\), then \(g_1 \circ g_2 \cong g_2 \circ g_2\).

Proof Let \(P : \mathcal{FPCP}(X) \times (I, \varepsilon'_i) \to \mathcal{FPCP}(Y)\) and \(Q : \mathcal{FPCP}(Y) \times (I, \varepsilon'_i) \to \mathcal{FPCP}(Z)\) be any two fuzzy path connected homotopy equivalence. Define a fuzzy continuous function \(R : \mathcal{FPCP}(X) \times (I, \varepsilon'_i) \to \mathcal{FPCP}(Z)\) by \(R(x_1, \varepsilon'_i) = Q(P(x_1, \varepsilon'_i), \varepsilon'_i)\) for all \(x_1 \in \mathcal{FPCP}(X)\) and \(\varepsilon'_i \in \mathcal{FPCP}(\varepsilon'_i)\). Therefore \(R(x_1, 0) = Q(P(x_1, 0), 0) = Q(f_1(x_1), 0) = g_1(f_1(x_1))\) and \(R(x_1, 1) = Q(P(x_1, 1), 1) = Q(f_2(x_1), 1) = g_2(f_2(x_1))\). Hence \(g_1 \circ g_2 \cong g_2 \circ g_2\).

Proposition 3.6 The fuzzy path connected homotopy equivalence \(\cong_{fp}\) is an equivalence relation.

Proof (i) Reflexivity: Since the identity function \(id_X : (X, T) \to (X, T)\) is a fuzzy homeomorphism, Reflexivity is obvious. (ii) Symmetry: Suppose \(g : (X, T) \to (Y, S)\) is a fuzzy path connected homotopy equivalence with fuzzy path connected homotopy inverse \(h : (Y, S) \to (X, T)\). Then \(h : (Y, S) \to (X, T)\) is a fuzzy path connected homotopy equivalence with fuzzy path connected homotopy equivalence \(g \circ h : (X, T) \to (Y, S)\) and \(h \circ g : (Y, S) \to (X, T)\) is a fuzzy path connected homotopy equivalence with fuzzy path connected homotopy equivalence \(h \circ g : (Y, S) \to (X, T)\) and \(id_X : (X, T) \to (X, T)\) is a fuzzy path connected homotopy equivalence with fuzzy path connected homotopy equivalence \(id_X : (X, T) \to (X, T)\). (iii) Transitivity: Suppose \(g : (X, T) \to (Y, S)\) is a fuzzy path connected homotopy equivalence with fuzzy path connected homotopy inverse \(h : (Y, S) \to (X, T)\) and \(j : (Y, S) \to (Z, R)\) is a fuzzy path connected homotopy equivalence with fuzzy path connected homotopy inverse \(j : (Y, S) \to (Z, R)\). Then by Proposition 3.4, \(j \circ g : (X, T) \to (Z, R)\) is a fuzzy path connected homotopy equivalence with fuzzy path connected homotopy equivalence \(j \circ g : (X, T) \to (Z, R)\).

Proposition 3.7 Let \(f : (X, T) \to (Y, S)\) be a fuzzy path connected homotopy equivalence. If \((X, T)\) is a fuzzy path connected space, then so is \((Y, S)\).

Proof Since the image of a function \(f\) is a fuzzy path connected space, it is enough to show that there is a fuzzy path between any fuzzy path connected point \(y_1 \in \mathcal{FPCP}(Y)\) to a fuzzy path connected point of \(x_1 \in f(X)\). Let \(g : (Y, S) \to (X, T)\) be any function such that \(f \circ g\) is fuzzy path connected homotopic to \(id_Y\), through the fuzzy path connected homotopy \(h : (Y, S) \times (I, \varepsilon'_j) \to (Y, S)\). Let \(y_1 \in \mathcal{FPCP}(Y)\) and \(y'_1 = f(g(y_1)) \in f(X)\) and for \(\gamma(t) \in (Y, S)\), \(\gamma(t) = h(y(t), t)\) is a fuzzy path from \(y'_1\) to \(y_1\). Hence the proof.

Proposition 3.8 (X, T) and (Y, S) be any two fuzzy Peano spaces and let \(f : (X, T) \to (Y, S)\) be a fuzzy path connected homotopy equivalence. Then \((X, T)\) is a fuzzy path connected space if and only if \(f(X)\) is a fuzzy path connected space.

Proof If \(f(X)\) is a fuzzy path connected space, then by Proposition 3.6, \((Y, T)\) is a fuzzy path connected space. Hence \((X, T)\) is a fuzzy path connected space.

Conclusion In this paper the concepts of fuzzy path connected homotopy in fuzzy Peano Space, Contiguous position of fuzzy path connected homotopy and fuzzy path connected homotopy equivalence in fuzzy Peano space are discussed. Also several interesting properties of fuzzy path connected homotopy in fuzzy Peano space are established.
Acknowledgment
The authors would like to extend their gratitude to the referees for their valuable and constructive suggestions which have improved this paper.

References

[1] C.L. Chang, Fuzzy topological Spaces, J. of Math Anal. and Appl., 24(1968), 182-190.
[2] David Ran, "An Introduction to the fundamental group"
[3] G. Cuvalcioglu and M. Citil, On fuzzy homotopy theory, Adv. Stud. Contemp. Math., 12(2006), 163-166.
[4] K. Sugapriya and B. Amudhambigai, "A view on fuzzy Peano space and its application on rail connectivity network" (Communicated).
[5] M. N. Mukherjee and S. P. Sinha, “On some near-fuzzy continuous functions between fuzzy topological spaces,” Fuzzy Sets and Systems, vol. 46, pp. 316-328, 1984.
[6] W. S. Massey, Algebraic Topology - An Introduction, Harcourt, Brace and World, New York (1967).
[7] Zadeh, L. A., 1965, "Fuzzy Sets," Information and Control, 8, pp. 338-353