Einstein-Hilbert Graviton Modes modified by the Lorentz-violating Bumblebee Field

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In this work, we investigate the consequences of the spontaneous breaking of Lorentz symmetry, triggered by the bumblebee vector field, on the Einstein-Hilbert usual theory. Specifically, we consider the Einstein-Hilbert action modified by the bumblebee dynamic field, and evaluate the graviton propagator using an extended basis of Barnes-Rivers tensor projectors, involving the Lorentz-violating vector. Once the propagator is carried out, we proceed discussing the consistency of the model, writing the dispersion relations, analyzing causality and unitarity. We verify that graviton physical propagating modes are causal and unitary.

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I. INTRODUCTION

Theories with Lorentz symmetry breaking have been under intensive investigation since the proposal of the Standard Model Extension (SME)¹ ² as a broader version of the usual standard model incorporating tensor terms generated by spontaneous Lorentz violation. The Lorentz-violating (LV) terms, generated as vacuum expectation values of tensors defined in a high energy scale, are coupled to the physical fields yielding coordinate invariance and violation of Lorentz symmetry in the particle frames ¹ ³. This theoretical framework has inspired a large number of investigations in the latest years, encompassing fermion systems ¹ ⁴, CPT-probing experiments ⁵, the electromagnetic CPT- and Lorentz-odd term ⁶ ⁷ ⁸, the cP-violating and Lorentz-odd gauge sector ⁹ ¹⁰. Recent investigations involving higher dimensional operators, nonminimal couplings ¹¹, and possible connections with LV theories have also been reported ¹² ¹³.

The interest in an extension of the SME embracing gravity comes from the fact that Lorentz violation may be a key ingredient of a quantum theory for gravitation. Indeed, Lorentz-violating effects are expected to be significant in regions or situations were the curvature or torsion are large, as in the vicinity of black holes. Furthermore, these effects may also play relevant role in cosmological scenarios described by dark energy or dark matter, or the ones where anisotropy factors can be inserted in the Friedman-Robertson-Walker solutions. Lorentz violation in the gravitational sector may be theoretically investigated in connection with tests sensitive to the inverse square law, the deflection of light, geodesic precession, between others. A consistent formalism to include LV terms in gravity requires a framework compatible with nonnull vacuum expectation values that break local Lorentz symmetry but keeps the general coordinate invariance. The Riemann-Cartan geometry, endowed with dynamic curvature and torsion, was used for such a purpose in Ref. ¹⁴, where the LV coupling terms were constructed using vierbein and spin connections. In Ref. ¹⁵, the connection between Nambu-Goldstone modes and the spontaneous violation of local Lorentz and diffeomorphism symmetries were investigated in the Riemann-Cartan spacetime using the vierbein and spin connection formalism previously developed. In Ref. ¹⁶, signals for Lorentz violation in post-Newtonian gravity were scrutinized in the case of a Riemann spacetime (null torsion) by considering the linearized Einstein equations modified by 20 independent dynamical LV coefficients generated by spontaneous symmetry breaking. New developments were performed in Refs. ¹⁷, ¹⁸. Also, alternative approaches for Lorentz violation in curved space, focused on a more geometric point of view, have been discussed in Refs. ¹⁹ ²¹. Investigations about Lorentz-violating linearized gravitation ²² and high order gravity models modified by Lorentz-violating terms were also reported ²³ ²⁴.

In accordance with these studies the extension of the gravitational sector including Lorentz-violating terms is given by the action

\[ S = S_{EH} + S_{LV} + S_{\text{matter}}, \]

where \( S_{EH} \) represents the usual Einstein-Hilbert action,

\[ S_{EH} = \int d^4x \sqrt{-g} \left( R - 2\Lambda \right), \]

where \( R \) is the curvature scalar, \( \Lambda \) is the cosmological constant. Moreover, the action \( S_{LV} \) accounts for Lorentz-violating leading terms, written as

\[ S_{LV} = \int d^4x \sqrt{-g} \left( uR + s^\mu{}_{\nu} R_{\mu\nu} + t^{\mu\nu\alpha\beta} R_{\mu\nu\alpha\beta} \right), \]

with \( u, s^\mu{}_{\nu} \) and \( t^{\mu\nu\alpha\beta} \) being tensors which enclose the Lorentz-violating coefficients. The dimensionless tensors...
The bumblebee model is a simple example of gravity model where a vector field $B^μ$ acquires a nonzero vacuum expectation value inducing Lorentz and diffeomorphism violations. This model was first considered in the context of string theories $^2$, with the spontaneous Lorentz symmetry breaking being triggered by the potential $V(B^μ) = 1 / 2 B^μ B_μ + b^2 / 2$. In accordance with the literature $^{14, 15}$, the vector bumblebee model can be represented as stated in action (3), whenever $\xi^{μν}=0$ and

$$u = \frac{1}{4}B^μB_μ, \quad s^{μν} = \frac{1}{4}g^{μρ}g^{νσ}B_ρB_σ,$$ \hspace{1cm} (4)

where the trace of $s^{μν}$ is absorbed by the scalar $u$. With such definitions, the action responsible for the dynamical fields is written as

$$S_{LV} = \int d^4x\sqrt{-g} \left[ \frac{1}{2} \left( R + \xi B^μ B_μ - \frac{1}{4} B^{μν}B_μB_ν \right) - V(B^μ) \right],$$ \hspace{1cm} (5)

with $\kappa^2 = 32\pi G$ being the gravitational coupling, and $\xi$ being the $m^{-2}$ dimension constant that establishes the nonminimal coupling between the bumblebee field and the curvature tensor. Moreover,

$$B_μν = \partial_μB_ν - \partial_νB_μ,$$ \hspace{1cm} (6)

is the field strength for the bumblebee vector. Here, the potential that triggers the spontaneous breakdown of diffeomorphism symmetry, $V(B^μ)$, assumes the smooth quadratic form

$$V = \frac{1}{2} (B^μB_μ + b^2)^2,$$ \hspace{1cm} (7)

and $\lambda$ is a dimensionless, $b^2$ is a positive constant that stands for the nonzero vacuum expectation value of this field.

In Ref. $^{16}$, the effects of the linearized version of the bumblebee model on the Einstein-Hilbert gravity were analyzed. This model was also addressed in Refs. $^{16, 18}$. Some additional implications of this model on the Newtonian gravitational potential have been recently addressed in Ref. $^{23}$, where the weak-field formalism of gravity was used to calculate the bumblebee corrections induced on the gravitational potential. It was then shown that the coupling of this field with the curvature tensor, as stated in action (4), implies an anisotropic potential correction proportional to $b_μ b_ν x^μ x^ν$, which breaks the spatial isotropy of the ordinary gravitational potential. Besides, it was also reported an additional correction similar to the well known electric Darwin term, $\frac{1}{2} \partial_μ \partial_ν A_μ \sim \delta^{(3)}(\vec{x})$, giving rise to a very weak and short-ranged contribution to the gravitational interaction.

In spite of the fact that the results of Ref. $^{23}$ are consistent with the literature $^{16, 18}$, it is based on a preliminary form of the Einstein-Hilbert graviton modified by the linearized bumblebee field, evaluated as a perturbative insertion on the usual case. This approach, however, does not provide an exact result, being not suitable to analyze the vacuum structure of this model and the properties of the physical excitations around it. It is known that the spontaneous Lorentz breaking is always accompanied by diffeomorphism violation $^{13–17}$, so the graviton spectrum may undergo nontrivial modifications, as the generation of massive modes, the appearance of nonphysical modes concerning causality (tachyons) and unitarity aspects (ghosts).

In this work, we intend to investigate the graviton spectrum in the context of the linearized Einstein-Hilbert gravity endowed with the spontaneous violation of Lorentz symmetry induced by the bumblebee field, as studied in Ref. $^{26}$. In this sense, we exactly carry out the graviton propagator (in tree-level approximation) applying a general method based on the Barnes-Rivers spin operators $^{26, 27}$ and recently extended in Ref. $^{24}$ for the case of gravity theories with Lorentz-breaking terms. Once the graviton Feynman propagator is evaluated, one also analyzes the consistency (stability, causality, unitarity) of this theory starting from the dispersion relations stemming from the poles of the propagator. In the present work we use the spacetime signature $(+,−,−,−)$ and adopt the following definition for the Ricci tensor: $R_{μν} = \partial_σ\Gamma^σ_μ_ν − \partial_σ\Gamma^μ_σ_ν + \Gamma^σ_μ_λ \Gamma^λ_σ_μ − \Gamma^λ_σ_μ \Gamma^σ_λ_μ$, where $\Gamma^λ_μ_ν = \frac{1}{2} g^{λσ}(\partial_μ g_ν_σ + \partial_ν g_μ_σ − \partial_σ g_μ_ν)$. All quantities are expressed in natural units ($h = c = \epsilon_0 = 1$), including the gravitational constant is $G = 6.707 \times 10^{-57}$ eV$^{-2}$. Moreover, tensors are symmetrized with unit weight, i.e., $(A_{μν}) = \frac{1}{2}(A_{μν} + A_{νμ})$.

The structure of the paper is as follows. In Sec. $^{III}$ we present the linearized bilinear gravity action for which we evaluate the Feynman propagator, using a extended basis of Barnes-Rivers projectors. In Sec. $^{III}$, we present the dispersion relations coming from the poles of the propagator, and discuss the stability and causality issues. The unitarity analysis is investigated in Sec. $^{IV}$, while our concluding comments are presented in Sec. $^{VI}$.

II. THEORETICAL MODEL AND THE GRAVITON PROPAGATOR

In order to determine the influence of the gravity-bumblebee coupling on the graviton dynamics, we consider the actions (3) and (4), following the route described in Refs. $^{16, 18, 23}$. For assessing the linearized version, we split the dynamic fields into the vacuum ex-
pectation values and the nearby quantum fluctuations:

\[ g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \]
\[ B_\mu = b_\mu + \tilde{B}_\mu, \]
\[ B^\mu = b^\mu + \tilde{B}^\mu - \kappa b_\mu h^{\mu\nu}, \]

where \( h_{\mu\nu} \) and \( \tilde{B}_\mu \) represent small perturbations around the Minkowski background and a constant vacuum value \( b_\mu \), respectively. The quantity \( b^\mu = (b_0, \mathbf{b}) \) represents the fixed background responsible for the violation of Lorentz and CPT symmetries in the local frame of particles [14].

Following the procedure outlined in Ref. [16], the solution for the linearized bumblebee equation of motion can be written in the momentum space as

\[
\tilde{B}^\mu = \frac{\kappa p^\mu b_\alpha b_\beta h^{\alpha\beta}}{2 (b \cdot p)} + \frac{2 \sigma b_\alpha R^{\alpha\mu}}{p^2} - \frac{2 \sigma p^\mu b_\alpha b_\beta R^{\alpha\beta}}{p^2 (b \cdot p)} \\
+ \frac{\sigma p^\mu R}{4 \lambda (b \cdot p)} \frac{\sigma b^\mu R}{p^2} + \frac{\sigma p^\mu b^2 R}{p^2 (b \cdot p)},
\]

where \( h = h^{\alpha\alpha} \). As expected, it is possible to verify that the Lagrangian (10) is not invariant under the gauge transformations \( h_{\mu\nu} \to h_{\mu\nu} + i k_\mu \zeta_\nu + i k_\nu \zeta_\mu \), for any arbitrary \( \zeta_\mu \). Furthermore, it is worth noting that there are second-order corrections \( O(\xi^2) \) which introduce higher derivative terms, and are background independent.

Another observation concerns the existing connection between the terms involving the bumblebee field in the squared linearized Lagrangian (10) and the ones stemming from higher order Lagrangian terms, as \( L_{ho} = \beta R_{\mu\nu} R^{\mu\nu} + \gamma R^2 \), as depicted in Refs. [23, 24, 29]. We can show that the linearized terms associated with \( R^2 \), \( R_{\mu\nu} R^{\mu\nu} \), namely, \( h^2 \), \( h\partial_\mu h^{\alpha\beta} \), can be also found in Lagrangian (10). This shows that the bumblebee field also plays the role of inducing high order gravity terms on the Einstein-Hilbert action.

Following the purpose of analyzing the effects of the bumblebee field on the Einstein-Hilbert action, we should add the Lorentz-violating terms of Eq. (10) to the bilinear terms of the linearized Einstein-Hilbert Lagrangian,

\[
\mathcal{L}_{EH} = p_\mu p_\nu h^{\mu\nu} h^{\alpha\beta} - p_\mu p_\nu h^{\mu\nu} h^{\alpha\beta}
\]

\[ + \frac{1}{2} p^2 h^{\mu\nu} h^{\lambda\sigma} - \frac{1}{2} p^2 h^{\mu\nu} h^{\lambda\sigma}. \]

without introducing any gauge fixing-term. Our interest is the kinetic Lagrangian,

\[
\mathcal{L}_{kin} = \mathcal{L}_{EH} + \mathcal{L}_{LN}.
\]

To find the corresponding Feynman propagator for \( \mathcal{L}_{kin} \), we first rewrite the resulting Lagrangian \( \mathcal{L}_{kin} \) into the bilinear form

\[
\mathcal{L}_{kin} = -\frac{1}{2} h^{\mu\nu} \hat{O}_{\mu\nu,\alpha\beta} h^{\alpha\beta},
\]

where the operator \( \hat{O}_{\mu\nu,\alpha\beta} \) is symmetric in the indices \((\mu\nu), (\alpha\beta)\), and under the interchange of the pairs \((\mu\nu)\) and \((\alpha\beta)\). Following the notations and conventions of
Ref. [24], the graviton propagator is defined as

\[ \langle 0 | T [ h_{\mu\nu}(x) h_{\alpha\beta}(y) ] | 0 \rangle = D_{\mu\nu,\alpha\beta}(x - y), \] (14)

where \( D_{\mu\nu,\alpha\beta} \) is the operator that satisfies the Green’s equation, given as

\[ \hat{O}^{\mu\nu}_{\lambda\sigma} D^{\lambda\sigma,\alpha\beta}(x - y) = i \hat{T}^{\mu\nu,\alpha\beta}(x - y), \] (15)

with \( \hat{T}^{\mu\nu,\alpha\beta} = \frac{1}{4} (\eta^{\mu\alpha} \eta^{\nu\beta} + \eta^{\mu\beta} \eta^{\nu\alpha}) \) playing the role of the identity operator. Thus, the problem of determining the propagator is reduced to the inversion of the operator \( \hat{O} \), given in Eq. (13). Once found a closed operator algebra composed of a set of appropriated tensor projectors with the operator \( \hat{O} \) can be expanded, the inversion of the operator becomes a tedious but straightforward task.

As it is well known, a convenient method for obtaining the inverse of symmetric rank-two tensors is based on the spin projector operators found by Barnes and Rivers [24, 27, 28] which constitute a complete orthonormal basis of operators for Lorentz-invariant models in four dimensions. This basis is shown in Eq. (A11). To accommodate the emerging terms containing the LV background vector \( b^\mu \) in gravity theories, an extended basis of the Barnes-Rivers projectors was devised in Ref. [24]. All tools needed to invert our operator \( \hat{O} \) are outlined in the Appendix.

Using the spin-projection operators and the identities as given in the Appendix A we are able to put the operator \( \hat{O} \) in the form (where for simplicity we adopt the notation \( AB \) in place of \( A^{\mu\nu,\rho\sigma} B^{\rho\sigma,\alpha\beta} \) to the contractions)

\[ \hat{O} = a_1 P^{(1)} + a_2 P^{(2)} + a_3 P^{(0-\theta)} + a_4 P^{(0-\omega)} + a_5 \Pi^{(1-\Sigma)} + a_6 \Pi^{(2)} + a_7 \Pi^{(\theta\Sigma)} + a_8 \Pi^{(\theta\Lambda)}, \] (16)

with the scalar coefficients \( a_i \) being functions of the momentum and the background vector, \( b^\mu \), given explicitly by

\[ \begin{align*}
a_1 &= \frac{-4 \xi^2 (b \cdot p)^2}{\kappa^2} + \xi (b \cdot p)^2, \\
a_2 &= \xi (b \cdot p)^2 + p^2, \\
n_3 &= \frac{4 \xi^2 \Box (p)}{\kappa^2} - \frac{6 \xi^2 p^4}{\kappa^2 \lambda} - 2 \xi (b \cdot p)^2 - 2 p^2, \\
a_4 &= \frac{8 \sqrt{3} \xi^2 (b \cdot p)^2}{\kappa^2} - \sqrt{3} \xi (b \cdot p)^2, \\
a_5 &= \frac{4 \xi^2 (b \cdot p)}{\kappa^2}, \\
a_6 &= \frac{4 \xi^2 p^2}{\kappa^2} - \xi p^2, \\
a_7 &= -\frac{8 \sqrt{3} \xi^2 (b \cdot p)}{\kappa^2}, \\
a_8 &= \frac{8 \sqrt{3} \xi^2 p^2}{\kappa^2} - \sqrt{3} \xi p^2.
\end{align*} \] (17)

Then, we write the operator \( \hat{O}^{-1} \) in terms of the whole set of projectors, as

\[ \hat{O}^{-1} = \begin{align*}
b_1 P^{(1)} + b_2 P^{(2)} + b_3 P^{(0-\theta)} + b_4 P^{(0-\omega)} + b_5 P^{(0-\theta\omega)} + b_6 \Pi^{(1-\Sigma)} + b_7 \Pi^{(2)} + b_8 \Pi^{(\theta\Sigma)} + b_9 \Pi^{(\theta\Lambda)} + b_{10} \Pi^{(\theta\Sigma)} + b_{11} \Pi^{(\omega\Lambda-a)} + b_{12} \Pi^{(\omega\Lambda-b)} + b_{13} \Pi^{(\omega\Sigma-a)} + b_{14} \Pi^{(\omega\Sigma-b)},
\end{align*} \] (18)

by using the identity, \( \hat{O} \hat{O}^{-1} = I \) it is established a set of algebraic equations for \( b_i \) coefficients. After a lengthy computation, the coefficients \( b_i \) are

\[ \begin{align*}
b_1 &= \frac{N_1}{\kappa^2 \xi^2 (b \cdot p) \Box \Xi + \kappa^2 p^4}, \\
b_2 &= \frac{1}{\Xi}, \\
b_3 &= -\frac{1}{2 \Xi}, \\
b_4 &= \frac{N_4}{2 \lambda \kappa^2 \xi^2 (b \cdot p)^4 \Box \Xi + \kappa^2 p^4}, \\
b_5 &= \frac{N_5}{2 \xi (b \cdot p)^2 \Box \Xi}, \\
b_6 &= \frac{\xi (b \cdot p)}{p^2 \Xi}, \\
b_7 &= \frac{p^2}{\Xi}, \\
b_8 &= \frac{N_8}{4 \xi (b \cdot p)^2 \Box \Xi}, \\
b_9 &= \frac{-\sqrt{3} \xi p^2}{2 \Xi}, \\
b_{10} &= \frac{N_{11}}{2 \boxtimes \Xi}, \\
b_{11} &= \frac{N_{12}}{8 \xi \kappa^2 (b \cdot p)^2 \Box \Xi}, \\
b_{12} &= \frac{N_{13}}{4 \xi \kappa^2 \xi^2 (b \cdot p)^3 \Box \Xi}, \\
b_{14} &= \frac{N_{14}}{4 \xi (b \cdot p) \Box \Xi},
\end{align*} \] (19)

where one has used \( \boxtimes = \Xi (p) \) and \( \Box = \Box (p) \), defined by the following expressions

\[ \begin{align*}
\Xi (p) &= p^2 + \xi (b \cdot p)^2, \\
\Box (p) &= (b \cdot p)^2 - b^2 p^2,
\end{align*} \] (20, 21)

and for numerators \( N_i \) follow

\[ \begin{align*}
N_1 &= \xi (4 \xi + \kappa^2) \Box \Xi + \kappa^2 p^4, \\
N_4 &= \xi \xi \Box \Xi \left[ p^2 F_1 (p) + \lambda \kappa^2 (b \cdot p)^4 \right] + 4 \lambda \xi \kappa^2 (b \cdot p)^4 \Box \Xi + \xi^3 (b \cdot p)^2 \Box \Xi F_1 (p) - \lambda \kappa^2 p^4 \left[ b^2 p^4 - 4 (b \cdot p)^4 + 2 b^2 p^2 (b \cdot p)^2 \right], \\
N_5 &= \sqrt{3} \left[ -\xi (b \cdot p)^2 \Box \Xi + b^2 p^4 \right], \\
N_8 &= \sqrt{3} (\xi (b \cdot p)^2 - p^2), \\
N_{11} &= p^2 (p^2 - \xi (b \cdot p)^2), \\
N_{12} &= p^4 (2 b^2 p^2 - 3 (b \cdot p)^2) - 2 \xi (b \cdot p)^2 \Box \Xi, \\
N_{13} &= F_2 (p) \xi \xi \Box \Xi - 16 \xi^3 (b \cdot p)^2 \Box \Xi + \xi \kappa^2 b^4 \left[ 2 (b \cdot p)^2 - b^2 p^2 \right], \\
N_{14} &= p^2 \left[ p^2 - \xi (b \cdot p)^2 \right],
\end{align*} \] (22-29)
where
\[ F_1 (p) = 16\lambda (b \cdot p)^2 + 16\lambda b^2 p^2 + p^4, \]
\[ F_2 (p) = \kappa^2 (b \cdot p)^2 [(b \cdot p)^2 + b^2 p^2] \]
\[ + 16p^2 [b^2 p^2 - (b \cdot p)^2]. \]

The Feynman propagator is
\[
D_{\mu\nu,\alpha\beta}(p) = \frac{i}{\Xi(p)} \left\{ \frac{N_1}{\kappa^2 \xi^2 (b \cdot p)^2} P^{(1)}_{\mu\nu,\alpha\beta} + P^{(2)}_{\mu\nu,\alpha\beta} \right. \\
- \frac{1}{2} P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{N_4}{2\kappa^2 \xi^2 (b \cdot p)^2} P^{(0-\omega)}_{\mu\nu,\alpha\beta} - \frac{p^2}{\xi(b \cdot p)} \Pi^{(1)}_{\mu\nu,\alpha\beta} \\
+ \frac{N_8}{4\xi^2 (b \cdot p)^2} \Pi^{(\theta \Sigma)}_{\mu\nu,\alpha\beta} - \frac{\sqrt{3}p^2}{2\xi} \Pi^{(\theta \Lambda)}_{\mu\nu,\alpha\beta} + \frac{p^4}{2\xi} \Pi^{(2\Lambda)}_{\mu\nu,\alpha\beta} \\
+ \frac{N_{11}}{8\kappa^2 \xi^2 (b \cdot p)^2} \Pi^{(\omega \Lambda - a)}_{\mu\nu,\alpha\beta} + \frac{N_{12}}{2\kappa^2 (b \cdot p)^2} \Pi^{(\omega \Lambda - b)}_{\mu\nu,\alpha\beta} \\
+ \frac{N_{13}}{4\kappa^2 \xi^2 (b \cdot p)^2} \Pi^{(\omega \Sigma - a)}_{\mu\nu,\alpha\beta} + \frac{N_{14}}{4\kappa^2 (b \cdot p)^2} \Pi^{(\omega \Sigma - a)}_{\mu\nu,\alpha\beta} \right\}.
\]

The next step is to read off the graviton dispersion relations from the poles of the propagator, aiming at verifying the consistency of the theory concerning causality and unitarity respects.

### III. DISPERSION RELATIONS

In this section, we analyze the dispersion relations stemming from the poles of the graviton propagator, which provide information about the stability and causality of the modes.

We begin considering the pole \( \Xi(p) \), which implies
\[ p^2 + \xi(b \cdot p)^2 = 0. \] (33)

For the timelike configuration, \( b^\mu = (b_0, 0) \), the corresponding dispersion relation is
\[ p_0 = \pm \frac{|p|}{\sqrt{1 + \xi b_0^2}}, \] (34)

which is an energy stable mode and yields the group velocity
\[ u_g = \frac{1}{\sqrt{1 + \xi b_0^2}}, \] (35)

that is smaller than 1 for \( \xi > 0 \), implying causality assurance for \( \xi > 0 \).

For the spacelike configuration, \( b^\mu = (0, b) \), the dispersion relation is
\[ p_0 = \pm |p| \sqrt{1 - \xi |b|^2 \cos^2 \theta}, \] (36)

where \( (b \cdot p) = |b| |p| \cos \theta \). This is a stable mode, related with the group velocity
\[ u_g = \sqrt{1 - \xi |b|^2 \cos^2 \theta}, \] (37)

which becomes smaller than 1 for \( \xi > 0 \) and \( \xi |b|^2 < 1 \). So, this mode is causal for \( \xi > 0 \) for both configurations.

For the pole \( \Xi(p) \), the associated dispersion relation is given by the roots of
\[ (b \cdot p)^2 - b_0^2 p^2 = 0. \] (38)

In a general background, \( b^\mu = (b_0, b) \), the dispersion relation is
\[ p_0 = \frac{|p|}{|b|} \left[ b_0 \cos \theta \pm \sqrt{\left(|b|^2 - b_0^2\right) \sin^2 \theta} \right], \] (39)

It becomes clear that the conditions \( b \neq 0 \) and \( |b|^2 > b_0^2 \) ensure the existence of real roots. So a timelike \( b^\mu \) is an excluded background. We also see from Eq. (39) that the positive and negative energy solutions imply \( E_+ \neq E_- \), which is compatible with charge conjugation (C) violation. These different energies for the particle and antiparticle can be problematic to the locality of the quantum theory, as discussed in Refs. [23, 28].

We should also mention that a dispersion relation much similar to the one of Eqs. (38, 39) was attained in Ref. [23] in the context of gravity with higher derivative terms. Hence, this relation seems to be associated with the bumblebee field contributions yielding the high order gravity terms.

A glance at Eq. (39) reveals two particular cases of possible interest. The first one is \( b^\mu = (0, b) \), for which
\[ p_0 = \pm |p| \sin \theta. \] (40)

Despite being a causal pole, \( u_g = \sin \theta \leq 1 \), the energy of the particle presents a strong dependence on the direction of propagation, standing for an nonphysical behavior. The second one corresponds to a general background configuration, \( b^\mu = (b_0, b) \), with the relation (39) specialized to the case in which \( |p||b| \), yielding
\[ p_0 = b_0 \frac{|p|}{|b|}. \] (41)

In principle, this pole can be compatible with causality requirements, \( u_g = b_0 / |b| \leq 1 \), and could represent a physical excitation. However, as it will be shown in the sequel, the calculation of the residue on this pole ensures that it yields a nonphysical mode.

### IV. TREE-LEVEL UNITARITY

The tree-level unitarity analysis of this model is performed through the saturation of the Feynman propagator with external currents. This method is usually applied in quantum field theory [30], being implemented by means of the saturated propagator (SP),
\[ SP = J_{\mu}^* \text{Res}(\Delta_{\mu\nu}) J^\nu, \] 
a scalar quantity given by the contraction of the external currents \((J^\mu)\) with \(\text{Res}(\Delta_{\mu\nu})\) - the residue of the propagator evaluated at each pole. The conserved current, \(\partial_\mu J^\mu = 0\), implies \(p_\mu J^\mu = 0\). This method was already used to analyze unitarity in the gauge sector of the SME [31].

For the rank-two graviton field, this method can also be equally applied. In this case the saturated residue of propagator is written as

\[ SP = J_{\mu}^* \text{Res}(D_{\mu\nu,\kappa\lambda}) J^{\kappa\lambda}, \tag{42} \]

where \(\text{Res}(D_{\mu\nu,\kappa\lambda})\) is the residue evaluated at each pole of the propagator, and \(J_{\mu}^*\) is a symmetric tensor describing an external conserved current \((\partial_\mu J^\mu = 0)\), which in momentum space reads as \(p_\mu J^\mu = 0\). In accordance with this method, the unitarity analysis is assured whenever the imaginary part of the saturation \(SP\) (at the poles of the propagator) is positive.

Due to the conservation law, \(p_\mu J_{\mu\nu} = 0\), all the proctor terms involving \(u_{\mu\nu}\) and \(\Sigma_{\mu\nu}\) yield null saturation. Hence, nonnull contribution for saturation stems only from the following terms:

\[ J_{\mu\nu}^* p_{\mu\nu,\kappa\lambda} J^{\kappa\lambda} = J_{\kappa\lambda} J^{\kappa\lambda} - \frac{1}{3} (J^\kappa)_\kappa^2, \]
\[ J_{\mu\nu}^* p_{(0-\theta)\kappa\lambda} J^{\kappa\lambda} = \frac{1}{3} (J^\kappa)_\kappa^2, \]
\[ J_{\mu\nu}^* \Pi_{(2)\kappa\lambda} J^{\kappa\lambda} = 2 b_\mu b_\nu J^\kappa b_\kappa J^\lambda, \tag{43} \]
\[ J_{\mu\nu}^* \Pi_{(\theta\kappa\lambda)} J^{\kappa\lambda} = \frac{2}{\sqrt{3}} J_{\kappa\lambda} b_\mu b_\nu J_{\mu\nu}, \]

\[ J_{\mu\nu}^* \Pi_{(\Lambda)\kappa\lambda} J^{\kappa\lambda} = J_{\mu\nu} b_\mu b_\nu b_\kappa b_\lambda J^{\kappa\lambda}. \]

Using the Feynman propagator [32] and the current conservation, the propagator saturated by conserved currents reads

\[ SP = \frac{J_{\kappa\lambda} J^{\kappa\lambda} - \frac{1}{2} (J^\kappa)_\kappa^2}{\square} + \frac{2p^2 b_\mu J^\kappa b_\nu J^\lambda}{\square} \tag{44} \]
\[ - \frac{p^2 J_{\kappa\lambda} b_\mu b_\nu J_{\mu\nu}}{\square} + \frac{p^4 J_{\mu\nu} b_\mu b_\nu b_\kappa b_\lambda J^{\kappa\lambda}}{\square \Box}. \]

Next, we compute the residues in the poles of the propagator, whose corresponding dispersion relations where studied in Sec. III

\[ \text{A. The first pole } \square = p^2 + \xi (b \cdot p)^2 \]

This pole implies the dispersion relation [43]. The corresponding residue obtained from [44] yields the following expression:

\[ \text{Res} (S) \bigg|_{\square = 0} = \frac{J_{\kappa\lambda} J^{\kappa\lambda} - \frac{1}{2} (J^\kappa)_\kappa^2}{\square} - \frac{2 \xi (b_\mu J^\kappa b_\nu)}{1 + \xi b^2} \tag{45} \]
\[ + \frac{\xi (J^\kappa)_\kappa (b_\mu b_\nu J_{\nu\mu})}{1 + \xi b^2} \tag{46} + \frac{\xi^2 (b_\mu b_\nu J_{\nu\mu})^2}{2(1 + \xi b^2)^2}, \]

where \(b^2 = b^\mu b_\mu\).

In the following calculations we will use of the relations

\[ J_{00} = \frac{p_0}{p^2} J_{ca}, \quad J_{0a} = \frac{p}{p^2} J_{ca}, \tag{46} \]

obtained from the current conservation condition.

We should now specialize our analysis for two cases: a timelike background, \(b^\mu = (b_0, 0)\), and a spacelike background, \(b^\mu = (0, b)\).

For the timelike background, \(b^\mu = (b_0, 0)\), the dispersion relation is written as Eq. [44], and the residue [45] becomes

\[ \text{Res} (S) \bigg|_{b=(b_0,0)} = \frac{1}{2} \left[ \frac{p_0 p_c J_{ca}}{p^2} + (J_{dd})^2 \right]^2 \tag{47} \]
\[ + (J_{ab})^2 - (J_{dd})^2 - 2 \left( \frac{p_c J_{ca}}{p^2} \right)^2. \]

It is exactly equal to the one stemming from the usual graviton mode \(p^2 = 0\) of the Einstein-Hilbert’s gravity which preserves unitarity. Therefore, the residue [47] is positive-definite and the pole \(p^2 + \xi (b \cdot p)^2 = 0\), is unitary for all values of \(b_0\).

For the spacelike background, \(b^\mu = (0, b)\), the dispersion relation is given by Eq. [46], or

\[ \langle p_0 \rangle^2 = p^2 - \xi (b \cdot p)^2. \tag{48} \]

The residue [46] for this background configuration is evaluated by using the following set of orthogonal vectors

\[ u_1 = u_3 \times u_b, \quad u_2 = u_3 \times u_1, \quad u_3 = p / |p|, \tag{49} \]

where \(u_b = b / |b|\) has the direction of the LV background. So the residue [46] results to be

\[ \text{Res} (S) \bigg|_{b=(0, b)} = \frac{1}{1 - \xi b^2} \left[ \frac{2(J_1^2)}{1 - \xi (b_3)^2} \right] + \frac{(S_2)^2}{2(1 - \xi b^2)}, \tag{50} \]

with the terms

\[ S_1 = \left[ 1 - \xi (b_3)^2 \right] J_{12}^u - \xi b_2 b_3 J_{13}^u, \tag{51} \]
\[ S_2 = \frac{(\xi b_2 b_3)^2 J_{33}^u}{1 - \xi (b_3)^2} + \frac{J_{22}^u}{1 - \xi (b_3)^2} \left[ 1 - \xi (b_3)^2 \right] \tag{52} \]
\[ - J_{11}^u \left( 1 - \xi b^2 \right) - 2 \xi b_2 b_3 J_{23}^u. \]

Here, we have used the definitions \(b = b_0 u_0 + b_2 u_2\), and \(J_{ij}^u = [J_{ij}]^u \cdot (J_{ij})\), with \([J_{ij}]^u\) being \(J_{ij}\) the spatial components of the tensor \(J_{\mu\nu}\). The residue [46] is positive whenever

\[ \xi b^2 < 1. \tag{53} \]

As the magnitude of the LV background should be small, the unitarity of this mode, for \(b^\mu = (0, b)\), is assured.
B. The second pole $\square = (b \cdot p)^2 - b^2 p^2$

It is a double pole implying the dispersion relation $b^2 \nu \kappa b_\nu J^{\kappa \nu} = 0$, which is physically senseful for or $b \parallel p$ or $b_0 = 0$. Its residue, computed from the saturated propagator, is

$$\text{Res} \left( S \right)_{\square = 0} = R_1 + R_2,$$

with

$$R_1 = - \frac{2 b_\nu J^\nu b_\kappa J^{\kappa \nu} - J^\nu b_\nu b_\kappa J^{\kappa \nu}}{b^2 (1 + \xi b^2)},$$

$$R_2 = \frac{(J^{\mu \nu} b_\mu b_\nu)^2 (1 + 2 \xi b^2)}{2 b^4 (1 + \xi b^2)}. \quad (55)$$

Using the identities $\nu \kappa$, the quantities $J^{\mu \nu} b_\mu b_\nu$, $b_\nu J^\nu b_\kappa J^{\kappa \nu}$, and $J^\kappa \nu$ become

$$J^{\mu \nu} b_\mu b_\nu = \frac{(b_0)^2 (p \cdot J p)}{p_0^2} - 2 \frac{b_0 (b \cdot J p)}{p_0} + b \cdot J b, \quad (57)$$

$$b_\nu J^\nu b_\kappa J^{\kappa \nu} = \frac{(b_0)^2 (p \cdot J p)^2}{(p_0)^2} - 2 \frac{b_0 (p \cdot J p) (b \cdot J p)}{(p_0)^2} + \frac{(b \cdot J p)^2}{p_0} + 2 \frac{b_0 (b \cdot J^2 p)}{p_0} - b \cdot J^2 b,$$

$$J^\kappa \nu = J_{00} - J_{aa} = \frac{(p \cdot J p)}{p_0^2} - J_{aa}. \quad (59)$$

The case $b \parallel p$: it holds the relation $p = p_0 b / b_0$, and one obtains $J^{\mu \nu} b_\mu b_\nu = 0$ and $b_\nu J^\nu b_\kappa J^{\kappa \nu} = 0$ implying $R_1 = 0 = R_2$, so the residue is null

$$\text{Res} \left( S \right)_{p_0 = b_0, \square = 0} = 0, \quad (60)$$

providing a null saturation $(SP = 0)$. It means that this double pole represents excitation that do not contribute to the scattering amplitude or cross section.

The case $b_0 = 0$: the dispersion relation is

$$p_0 = \pm \frac{|p \times b|}{|b|},$$

and the expressions $J_{00} - J_{aa}$ can be written as

$$b^2 (1 - \xi b^2) R_1 = \frac{2 (p \cdot J b)^2}{(p_0)^2} - 2 (b \cdot J^2 b)$$

$$- \frac{(b \cdot J b) (p \cdot J p)}{(p_0)^2} + (b \cdot J b) J_{aa}, \quad (61)$$

$$b^2 (1 - \xi b^2) R_2 = \frac{(b \cdot J b) (p \cdot J p) (1 - 2 \xi b^2)}{2 b^2 (1 - \xi b^2)}. \quad (62)$$

To simplify the above expressions, we use the orthonormal basis

$$u_1 = \frac{p \times b}{|p \times b|} = \frac{p \times u_3}{p_0}, \quad u_2 = u_3 \times u_1, \quad u_3 = \frac{b}{|b|}, \quad (63)$$

so we have the following expressions:

$$p = p_2 u_2 + p_3 u_3, \quad b = |b| u_3, \quad p \times b = |b| p_2 u_1, \quad J^\alpha = u_3 (J u_1),$$

with $J = [J_{ij}]$ being $J_{ij}$ the spatial components of the tensor $J^{\mu \nu}$. After some algebra we obtain the residue

$$\text{Res} \left( S \right)_{\square = 0} = \frac{J_{11}^n J_{33}^n - 2 (J_{13}^n)^2}{1 - \xi b^2} \quad (64)$$

$$+ \frac{2 p_2 p_3 J_{13}^u J_{33}^u + (p_3)^2 (J_{33}^u)^2}{(p_2)^2 (1 - \xi b^2)} - \frac{(J_{13}^u)^2}{2 (1 - \xi b^2)} = 0,$$

which is not definite-positive, meaning, in general, a non unitary excitation.

It means that this double pole represents excitations that do not contribute to the scattering amplitude or cross section or represent ghost excitations. Henceforth, Eq. (55) is a non physical dispersion relation.

Thus, we conclude that this graviton model provides physical modes related only to the dispersion relation $\mu \nu$, which preserve tree level unitarity. Such situation also happens for the pole $p^2 = 0$ of the graviton propagator of Einstein-Hilbert’s theory.

V. DEGREES OF FREEDOM

Now, we show that the gravitational field modified by the vacuum expectation value of bumblebee field is still represented for a massless spin 2 symmetric tensor field with only two physical degrees of freedom. The equation of motion obtained from the effective Lagrangian [12] is

$$\hat{\mathcal{O}}_{\mu \nu, \alpha \beta} h^{\alpha \beta} = 0,$$

with the operator $\hat{\mathcal{O}}$ defined by Eq. (16). By saturating Eq. (65) with $p^\mu p^\nu$, we obtain the following constraint

$$p_\mu p_\nu h^{\mu \nu} = p^2 h,$$

where we have assumed $\xi \neq 0$ and $(b \cdot p) \neq 0$. As we can easily observe, due to the presence of the background field $b_\mu$, we can still saturate Eq. (65) with the combinations $b^\mu p^\nu$ and $b^\mu b^\nu$. In addition we also perform the saturation of the equation of motion (65) with the metric $\eta_{\mu \nu}$.

Such procedures imply in the following three equations,

$$0 = (b \cdot p)^2 h - 2 (b \cdot b) b_{(\mu} p_{\nu)} h^{\mu \nu} + p^2 b_\mu b_\nu h^{\mu \nu},$$

$$0 = [p^2 (1 - 3 \xi b^2) + 2 \xi (b \cdot p)^2] b_\mu b_\nu h^{\mu \nu} + (b \cdot p)^2 (1 - \xi b^2) h - 2 (b \cdot p) (1 - \xi b^2) b_{(\mu} p_{\nu)} h^{\mu \nu},$$

$$0 = -2 (b \cdot p) (16 \xi - \kappa^2) b_{(\mu} p_{\nu)} h^{\mu \nu} + p^2 (16 \xi - 5 \kappa^2) b_\mu b_\nu h^{\mu \nu} + (b \cdot p)^2 (16 \xi - 3 \kappa^2) h,$$
where we have used the condition (60). Thus, it is straightforward to see that these three equations imply the following constraint relations which $h_{\mu\nu}$ and $b_\mu$ must satisfy:

$$b_\mu b_\nu h^{\mu\nu} = 0, \quad b_{(\mu} p_{\nu)} h^{\mu\nu} = 0, \quad h = 0. \quad (70)$$

Immediately, from (66) it follows that

$$p_\mu p_\nu h^{\mu\nu} = 0. \quad (73)$$

We can achieve even more restrictions when we perform the contraction of Eq. (65) with $p_\mu$ or $b_\mu$ separately. By following a similar procedure as above, we obtain the following constraints

$$p_\mu h^{\mu\nu} = 0, \quad (74)$$

$$b_\mu h^{\mu\nu} = 0, \quad (75)$$

representing a total of 8 additional conditions to be satisfied by the fields. Thus, the set of Eqs. (60)–(65) provides a total of 12 constraints which can be used to reduce the 14 initial degrees of freedom contained in the graviton and bumblebee fields. Consequently, we are left with only two physical degrees of freedom such as it happens for the Einstein-Hilbert’s graviton.

Finally, by applying Eqs. (70)–(75) to Eq. (65), it becomes

$$[p^2 + \xi (b \cdot p)^2] h_{\mu\nu} = 0, \quad (76)$$

providing the correct energy-momentum dispersion relation (33) associated to the physical pole as previously determined. Therefore, we can conclude that the mechanism of spontaneous Lorentz symmetry breaking triggered by bumblebee field did not generate mass for the graviton but only modifies the Einstein-Hilbert dispersion relation: $p^2 = 0 \rightarrow p^2 + \xi (b \cdot p)^2 = 0$.

An interesting feature of the bumblebee models is to break three Lorentz transformations and one diffeomorphism, in such a way that there arise four Nambu-Goldstone (NG) modes associated with the fluctuations about the vacuum

$$\delta B_\mu = \delta B = (B_\mu - b_\mu), \quad (77)$$

which are explicitly written in the form (3). Note that the NG modes can be eliminated by the constraint equations (70)–(75), which is compatible with the existence of only two degrees of freedom.

VI. REMARKS AND CONCLUSIONS

In this work, we have investigated the Einstein-Hilbert action in the presence of the Lorentz-violating bumblebee field. We initially have considered the linearized version of this theory, including the bumblebee solution. The bilinear bumblebee action in the linearized graviton field $h_{\mu\nu}$ is added to the bilinear linearized Einstein-Hilbert action, implying the action of our interest. The corresponding Feynman propagator is exactly carried out using an extended basis of spin projectors based on the Barnes-Rivers usual ones. The graviton dispersions relations are extracted from the poles of the propagator and used to study the physical consistency of the model. It is verified that this theory has two physical propagating modes, modified by the bumblebee field. These physical modes are causal (for a positive coupling constant, $\xi > 0$) and unitary (for any values of $b_0$ and small values of the background $|b|$). Hence, one verifies that the gravity theory modified by the bumblebee field preserves the causality and unitarity aspects of the Einstein-Hilbert graviton modes.

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Appendix A: Barnes-Rivers operators and Lorentz-symmetry breaking

In the calculation of the inverse kinetic operator, necessary to determine the graviton propagator, we employ an algorithm based on the Barnes–Rivers rank-2 spin-projectors [24, 25, 26], given by:

$$P^{(1)}_{\mu\nu,\kappa\lambda} = \frac{1}{2} \left( \theta_{\mu\nu} \omega_{\kappa\lambda} + \theta_{\mu\kappa} \omega_{\nu\lambda} + \theta_{\nu\kappa} \omega_{\mu\lambda} + \theta_{\nu\lambda} \omega_{\mu\kappa} \right),$$

$$P^{(2)}_{\mu\nu,\kappa\lambda} = \frac{1}{2} \left( \theta_{\mu\nu} \theta_{\kappa\lambda} + \theta_{\mu\kappa} \theta_{\nu\lambda} + \theta_{\nu\kappa} \theta_{\mu\lambda} \right) - \frac{1}{3} \theta_{\mu\nu} \theta_{\kappa\lambda},$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda} = \frac{1}{3} \theta_{\mu\nu} \theta_{\kappa\lambda},$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda} = \frac{1}{\sqrt{3}} \left( \theta_{\mu\nu} \omega_{\kappa\lambda} + \theta_{\kappa\lambda} \omega_{\mu\nu} \right), \quad (A1)$$

where $\theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}$, $\omega_{\mu\nu} = k_{\mu} k_{\nu}/k^2$ are the transverse and longitudinal projectors, respectively. The usual spin operators for the subspace of symmetric rank-2 tensors satisfy the following tensorial completeness relation:

$$[p^{(1)} + p^{(2)} + p^{(0-\theta)} + p^{(0-\omega)}]_{\mu\nu,\kappa\lambda} = \frac{\eta_{\mu\kappa} \eta_{\nu\lambda} + \eta_{\mu\lambda} \eta_{\nu\kappa}}{2}. \quad (A2)$$

The Barnes–Rivers basis was extended to gravity theories involving Lorentz-symmetry violation in Ref. [24]. In such extension, the spin operators induced by the Lorentz-violating background, $b_\mu$, yields the whole set of structures listed below:

$$\Sigma_{\mu\nu} = b_\mu k_\nu + b_\nu k_\mu, \quad \Lambda_{\mu\nu} = b_\mu b_\nu, \quad (A3)$$
\[\Pi^{(1-\Sigma)}_{\mu\nu,\kappa\lambda} = \frac{\theta_{\mu\kappa}\Sigma_{\nu\lambda} + \theta_{\mu\lambda}\Sigma_{\nu\kappa} + \theta_{\nu\kappa}\Sigma_{\mu\lambda} + \theta_{\nu\lambda}\Sigma_{\mu\kappa}}{2},\]
\[\Pi^{(2)}_{\mu\nu,\kappa\lambda} = \frac{\theta_{\mu\kappa}\Lambda_{\nu\lambda} + \theta_{\mu\lambda}\Lambda_{\nu\kappa} + \theta_{\nu\kappa}\Lambda_{\mu\lambda} + \theta_{\nu\lambda}\Lambda_{\mu\kappa}}{2},\]
\[\Pi^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda} = \frac{\theta_{\mu\kappa}\Sigma_{\nu\lambda} + \theta_{\nu\kappa}\Sigma_{\mu\lambda}}{\sqrt{3}},\]
\[\Pi^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda} = \frac{\theta_{\mu\kappa}\Lambda_{\nu\lambda} + \theta_{\nu\kappa}\Lambda_{\mu\lambda}}{\sqrt{3}},\]
\[\Pi^{(\Lambda)}_{\mu\nu,\kappa\lambda} = \Lambda_{\mu\kappa}\Lambda_{\nu\lambda}, \quad \Pi^{(\omega\Lambda-\alpha)}_{\mu\nu,\kappa\lambda} = \Sigma_{\mu\kappa}\Sigma_{\nu\lambda},\]
\[\Pi^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda} = \omega_{\mu\kappa}\Lambda_{\nu\lambda} + \omega_{\nu\kappa}\Lambda_{\mu\lambda},\]
\[\Pi^{(\omega\Sigma-a)}_{\mu\nu,\kappa\lambda} = \omega_{\mu\kappa}\Sigma_{\nu\lambda} + \omega_{\nu\kappa}\Sigma_{\mu\lambda},\]
\[\Pi^{(\Lambda\Sigma-a)}_{\mu\nu,\kappa\lambda} = \Lambda_{\mu\kappa}\Sigma_{\nu\lambda} + \Lambda_{\nu\kappa}\Sigma_{\mu\lambda}.\]

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