FREE INVOLUTIONS ON $S^2 \times S^3$

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Abstract. In this paper, we classify smooth 5-manifolds with fundamental group isomorphic to $\mathbb{Z}/2$ and universal cover diffeomorphic to $S^2 \times S^3$. As a consequence, a classification of smooth free involutions on $S^2 \times S^3$ up to conjugation is obtained.

1. Introduction

The study of group actions on manifolds is an important topic of topology. Free involutions on manifolds have been studied by several authors. Free involutions on the spheres $S^n$ were studied by Wall, Browder, Liversay, Lopez de Medrano and others. In [4], Hambleton studied the existence of free involutions on simply-connected 5-manifolds with trivial second Stiefel-Whitney class. A classification of free involutions on simply-connected 5-manifolds with torsion free second homology and trivial action on $H_2$ is obtained by Hambleton and Su in [6].

The classification of free involutions on a manifold up to conjugation is equivalent to the classification of the quotient spaces. In the paper we study the classification of smooth 5-manifolds with fundamental group $\mathbb{Z}/2$ and universal cover $S^2 \times S^3$, and henceforth obtain a classification of smooth free involutions on $S^2 \times S^3$.

Before stating the main theorem, we give some examples of smooth free involutions on $S^2 \times S^3$. First of all, there are some obvious examples. Let $T_i = (-1, \tau_i) : S^2 \times S^3 \rightarrow S^2 \times S^3$ be a free involution, where $-1$ denotes the antipodal map on $S^2$ and $\tau_i$ is the restriction of a linear map with $i \ (-1)$-eigenvalues and $(4 - i)$ 1-eigenvalues, $i = 0, 1, 2, 3, 4$. We denote the corresponding quotient manifold by $Y_i$. Let $\eta_2$ be the canonical line bundle over $\mathbb{R}P^2$, then $Y_i$ is the sphere bundle of the vector bundle $i\eta_2 \oplus (4 - i)\mathbb{R}$. Similarly, let $T'_i = (\tau'_i, -1) : S^2 \times S^3 \rightarrow S^2 \times S^3$ be a free involution, where $-1$ denotes the antipodal map on $S^3$ and $\tau'_i$ is the restriction of a linear map with $i \ (-1)$-eigenvalues and $(3 - i)$ 1-eigenvalues, $i = 0, 1, 2, 3$. We denote the corresponding quotient manifold by $Z_j$. Let $\eta_3$ be the canonical line bundle over $\mathbb{R}P^3$, then $Z_j$ is the sphere bundle of the vector bundle $j\eta_3 \oplus (3 - j)\mathbb{R}$.

Other classes of examples are given by the following constructions. $S^2 \times S^3$ can be realized as the link $\Sigma^5_q$ of a Brieskorn singularity of type $A_q$, $q = 0, 2, 4, 6, 8$. There is a smooth free involution $T$ on $\Sigma^5_q$ induced by an involution of the ambient space. We denote the quotient manifold by $\Sigma^5_q/T$. On the other hand, by the $S^1$-connected sum operation on 5-dimensional fake projective spaces, we obtain manifolds $X^5(q)$ ($q = 0, 2, 4, 6, 8$) with $\pi_1(X^5(q)) \cong \mathbb{Z}/2$ and universal cover $S^2 \times S^3$. Detailed description of these manifolds will

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be given in Section 2. These manifolds correspond to nonlinear involutions on $S^2 \times S^3$ and appear in the main theorem.

Let $T: S^2 \times S^3 \to S^2 \times S^3$ be a smooth free involution, $M^5 = S^2 \times S^3 / T$ be the quotient space. Then according to the action of $T$ on $\pi_2(S^2 \times S^3)$, as a $\mathbb{Z}[\mathbb{Z}/2]$-module, $\pi_2(M)$ is isomorphic to $\mathbb{Z}_+$ (the trivial module) or $\mathbb{Z}_-$ (the nontrivial module). The second space of the Postnikov tower of $M$, $P_2(M)$, is a fibration over $\mathbb{R}P^\infty$ with fiber $\mathbb{C}P^\infty$. By [1], there are two such fibrations in the $\mathbb{Z}_-$-case, one admits a section and the other one doesn’t. We denote the former by $P$ and the latter by $Q$. More precise constructions of $P$ and $Q$ will be given in Section 2.

**Theorem 1.1.** Let $M^5$ be a smooth 5-manifold with $\pi_1(M) \cong \mathbb{Z}/2$ and universal cover $\tilde{M} \cong S^2 \times S^3$.

1. If $\pi_2(M) \cong \mathbb{Z}_+$, then $M$ is orientable, and the classification of $M$ up to diffeomorphism is given in the following table

| $w_2(M) = 0$ | $w_2(M) \neq 0$ |
|---------------|------------------|
| $S^2 \times \mathbb{R}P^3$ | $X^5(q), \ q = 0, 2, 4, 6, 8$ |

2. If $\pi_2(M) \cong \mathbb{Z}_-$ and $M$ is orientable, then $P_2(M) = Q$ and the classification of $M$ up to diffeomorphism is given in the following table

| $w_2(M) = 0$ | $w_2(M) \neq 0$ |
|---------------|------------------|
| $Y_1$ | $\Sigma^5_q/T, \ q = 0, 2, 4, 6, 8$ |

For (1) and (2), in the $w_2(M) \neq 0$ cases the manifolds are classified by the $\text{Pin}^+$-bordism class of the characteristic submanifold $N$ of $M$, $[N] \in \Omega^\text{Pin}^+ / \pm = \{0, 1, \ldots, 8\}$.

3. If $\pi_2(M) \cong \mathbb{Z}_-$ and $M$ is nonorientable, then the classification of $M$ up to diffeomorphism is given in the following table

| $w_2(M) = 0$ | $w_2(M) \neq 0$ |
|---------------|------------------|
| $P_2(M) = P$ | $Z_1$ |
| $P_2(M) = Q$ | $Y_2$ | $S^3 \times \mathbb{R}P^2$ |

As a corollary, we have a classification of smooth free involutions on $S^2 \times S^3$.

**Corollary 1.2.** There are exactly 12 orientation preserving smooth involutions on $S^2 \times S^3$ and 3 orientation reversing ones. The quotient spaces are given in the above tables.

**Remark 1.3.** It will be interesting to give explicit description of the involutions with quotient space $X^5(q)$.

**Remark 1.4.** There are topological free involutions on $S^2 \times S^3$ which are not conjugate to smooth ones. In [6], a 5-manifold $*(S^2 \times \mathbb{R}P^3)$ is constructed. This manifold is homotopy
equivalent to $S^2 \times \mathbb{R}P^3$, but doesn’t admit any smooth structure. Therefore the deck-transformation on the universal cover is a non-smoothable free involution on $S^2 \times S^3$. The method of this paper is also valid for the classification in the topological category.

The outline of this paper is as follows: in Section 2 description and properties of the second space of the Postnikov tower of $M_k$ such a fibration is determined by its involution on $S^3$ in [8].

Theorem 1.1 is proved in Section 3, using the method of modified surgery developed in [9].

2. Preliminaries

§2A. Second space of the Postnikov tower. Let $M^5$ be the quotient space of a free involution on $S^2 \times S^3$, then $\pi_1(M) \cong \mathbb{Z}/2$ and $\pi_2(M) \cong \mathbb{Z}$. According to [11], the second space of the Postnikov tower of $M$, $P_2(M)$, is a fibration over $\mathbb{R}P^\infty$ with fiber $\mathbb{C}P^\infty$, and such a fibration is determined by its $k$-invariant $k \in H^3(\mathbb{R}P^\infty; \pi_2(\mathbb{C}P^\infty))$, which is the obstruction for a section.

If $\pi_2(M) \cong \mathbb{Z}_+$ as a $\mathbb{Z}[\mathbb{Z}/2]$-module, since $H^3(\mathbb{R}P^\infty; \pi_2(\mathbb{C}P^\infty)) = 0$, we have

$$P_2(M) = \mathbb{R}P^\infty \times \mathbb{C}P^\infty.$$

If $\pi_2(M) \cong \mathbb{Z}_-$, since $H^3(\mathbb{R}P^\infty; \pi_2(\mathbb{C}P^\infty)) \cong \mathbb{Z}/2$, there are two such fibrations up to fiber homotopy equivalence. These can be described as follows:

Let $c$ denote the complex conjugation on $\mathbb{C}P^\infty$, $(-1)$ denote the antipodal map on $S^\infty$. Let $P = (\mathbb{C}P^\infty \times S^\infty)/(c, -1)$, then there is a fibration $\mathbb{C}P^\infty \to P \to \mathbb{R}P^\infty$ with a section $\sigma: \mathbb{R}P^\infty \to P$, $x \mapsto [x, \tilde{x}]$, where $\tilde{x}$ is any preimage of $x$ in $S^\infty$.

There is a free involution $\tau$ on $\mathbb{C}P^\infty$, with $\tau_\ast = (-1)$ on $H_2(\mathbb{C}P^\infty)$. (Under homogeneous coordinates, $\tau$ maps $[z_0, z_1, z_2, z_3, \cdots]$ to $[-z_1, z_0, -z_3, z_2, \cdots]$.) Let $Q = (\mathbb{C}P^\infty \times S^\infty)/(\tau, -1)$, then there is a fibration $\mathbb{C}P^\infty \to Q \to \mathbb{R}P^\infty$ which corresponds to the nontrivial $k$-invariant. For details see [10] pp. 44]

The homology groups of $Q$ were calculated by Olbermann in [10]

**Lemma 2.1.** [10] pp. 48-49

1. $H^*(Q; \mathbb{Z}/2) \cong \mathbb{Z}/2[t, q]/t^3$, where $|t| = 1$, $|q| = 4$.
2. the integral homology of $Q$ up to dimension 6 is given by

|       | 1  | 2  | 3  | 4  | 5  | 6  |
|-------|----|----|----|----|----|----|
| $H_*(Q; \mathbb{Z})$ | $\mathbb{Z}/2$ | 0  | 0  | $\mathbb{Z}/2$ | $\mathbb{Z}/2$ | 0  |

3. $H_4(Q; \mathbb{Z}) \cong \mathbb{Z}$, $H_5(Q; \mathbb{Z}) = 0$, $H_6(Q; \mathbb{Z}) \cong \mathbb{Z}$.

Analogously, we compute the homology groups of $P$:

**Lemma 2.2.**

1. $H^*(P; \mathbb{Z}/2) \cong \mathbb{Z}/2[t, x]$, where $t$ is the pull-back of the nontrivial element of $H^1(\mathbb{R}P^\infty; \mathbb{Z}/2)$ and $x$ restricts to the nontrivial element of $H^2(\mathbb{C}P^\infty; \mathbb{Z}/2)$.
2. the integral homology of $P$ up to dimension 6 is given by
(3) $H_5(P; \mathbb{Z}_-)$ is mapped injectively into $H_5(P; \mathbb{Z}/2)$, whose image is dual to $t^3x$; $H_6(P; \mathbb{Z}_-)$ is isomorphic to $\mathbb{Z} \oplus (\mathbb{Z}/2)^2$.

**Proof.** We apply the Serre spectral sequence for $H^*(- ; \mathbb{Z}/2)$ to the fibration $\mathbb{CP}^\infty \to P \to \mathbb{RP}^\infty$. Since there is a section $\sigma: \mathbb{RP}^\infty \to P$, all $E^2_{p,q}$-terms survive to infinity. By the multiplicative structure of the spectral sequence, this implies that $H^*(M; \mathbb{Z}/2) \cong \mathbb{Z}/2[t, x]$. We apply the Serre spectral sequence for $H_*(M; \mathbb{Z})$ and the universal coefficient theorem one computes $H_*(M; \mathbb{Z})$ in low dimensions. Using the Bockstein sequence associated to the short exact sequences $\mathbb{Z}_+ \to \mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z}_-$ and $\mathbb{Z}_- \to \mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z}_-$, one computes $H_*(M; \mathbb{Z}_-)$ in low dimensions.

The long exact sequence associated to the coefficient sequence $\mathbb{Z}_- \xrightarrow{2} \mathbb{Z}_- \to \mathbb{Z}/2$ shows that $H_5(P; \mathbb{Z}_-) \to H_5(P; \mathbb{Z}/2)$ is injective. Let $Z_1$ be the 5-manifold given in Section 1, we have a commutative diagram

$$
\begin{array}{ccc}
S^2 & \xrightarrow{f_2} & \mathbb{CP}^\infty \\
\downarrow & & \downarrow f_2 \\
Z_1 & \xrightarrow{f_2} & \mathbb{RP}^3 \\
\end{array}
$$

where $f_2: Z_1 \to P$ is the second stage Postnikov map. By comparison of the Serre spectral sequences of the two fibrations, we see that $f_{2*}: H_5(Z_1; \mathbb{Z}/2) \to H_5(P; \mathbb{Z}/2)$ is injective and the image is dual to $t^3x$. Therefore from the commutative diagram

$$
\begin{array}{ccc}
H_5(Z_1; \mathbb{Z}_-) & \xrightarrow{f_{2*}} & H_5(Z_1; \mathbb{Z}/2) \\
\downarrow & & \downarrow f_{2*} \\
H_5(P; \mathbb{Z}_-) & \xrightarrow{f_{2*}} & H_5(P; \mathbb{Z}/2) \\
\end{array}
$$

we conclude that the image of $H_5(P; \mathbb{Z}_-) \to H_5(P; \mathbb{Z}/2)$ is dual to $t^3x$.

If $M$ has $\pi_2(M) \cong \mathbb{Z}_-$, then there is an exact sequence (cf. [2])

$$
H_3(\mathbb{Z}/2) \to \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/2]} \mathbb{Z}_- \to H_2(M) \to H_2(\mathbb{Z}/2).
$$

Thus $H_2(M)$ is either trivial or isomorphic to $\mathbb{Z}/2$. We see from Lemma 2.1 and Lemma 2.2 that these two cases correspond to the different Postnikov towers.

**Corollary 2.3.** Let $P_2(M)$ be the second space of the Postnikov tower of $M$. Then $P_2(M) = Z$ if and only if $H_2(M) \cong \mathbb{Z}/2$; $P_2(M) = Q$ if and only if $H_2(M) = 0$.

**Lemma 2.4.** Let $M^5$ be the quotient space of a free involution on $S^2 \times S^3$ with $\pi_2(M) \cong \mathbb{Z}_+$. Then $M$ is orientable.

**Proof.** Assume that $M$ is nonorientable, then as a $\mathbb{Z}[\mathbb{Z}/2]$-module, $H_3(S^2 \times S^3) \cong \mathbb{Z}_-$. An easy calculation with the homology Serre spectral sequence with $\mathbb{Z}$-coefficients for the fibration $\tilde{M} \to M \to \mathbb{RP}^\infty$ shows that $H_5(M; \mathbb{Z}) \neq 0$, a contradiction.
Lemma 2.5. Let $M^5$ be the quotient space of an orientation preserving free involution on $S^2 \times S^3$ with $\pi_2(M) \cong \mathbb{Z}_2$. Then $H_2(M; \mathbb{Z}) = 0$. Therefore by Corollary 2.3, $P_2(M) = Q$.

Proof. Assume that $H_2(M) \neq 0$, then $H_2(M) \cong \mathbb{Z}/2$. Let $x \in H_2(M)$ be the nontrivial element, then the non-singular linking form $b$ on $\text{tors} H_2(M)$ must have value $b(x, x) = 1/2 \in \mathbb{Q}/\mathbb{Z}$. Note that $\widetilde{M} = S^2 \times S^3$ is spin, thus $w_2(M)$ is zero on $H_2(M)$ (cf. \[6\] Lemma 2.1). Also we have a surjection $\pi_2(M) \to H_2(M)$. Therefore we can do surgery on an embedded $S^2$ representing $x$ to kill $x$. Using \[13\] Lemma 3 it is seen that there is no new homology class created. Thus the result manifold $M'$ has $H_2(M') = 0$. This means $M$ can be obtained by doing surgery on an embedded $S^2$ in $M'$. Since $H_2(M') = 0$, we have $H_2(M) \cong \mathbb{Z}$, which is a contradiction. \hfill $\square$

Lemma 2.6. Let $M^5$ be the quotient space of a free involution on $S^2 \times S^3$ with $P_2(M) = P$. Then $M$ is nonorientable and $w_2(M) = 0$.

Proof. That $M$ is nonorientable is a consequence of the previous lemma. To show that $w_2(M) = 0$, consider the cohomology Serre spectral sequence with $\mathbb{Z}/2$-coefficients for the fibration $\widetilde{M} \to M \to \mathbb{RP}^\infty$. $H^2(M; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^2$ implies that the differentials $d_2: E^3_{p, 2} \to E^3_{p+3, 0}$ are trivial. Therefore $H^3(M; \mathbb{Z}/2) = \mathbb{Z}/2 < t^2, x >$ and $H^3(M; \mathbb{Z}/2) = \mathbb{Z}/2 < t^3, tx >$, where $t$ is the pull-back of the nontrivial element of $H^1(\mathbb{RP}^\infty; \mathbb{Z}/2)$ and $x$ restricts to the nontrivial element of $H^2(\widetilde{M}; \mathbb{Z}/2)$. By Poincaré duality, $H^5(M; \mathbb{Z}/2)$ is generated by $t^3x$.

Let $v = v_1 + v_2$ denote the Wu class of $M$. Then $v_1 = w_1(M) = t$, and $w_2(M) = (Sq^2)v_2 = v_1^2 + v_2 = t^2 + v_2$, where $v_2$ is determined by $Sq^2: H^3(M; \mathbb{Z}/2) \to H^3(M; \mathbb{Z}/2)$. We have $Sq^2t^3 = t^5 = 0$ and $Sq^2tx = t^2Sq^1x$.

Let $f_2: M \to P = P_2(M)$ be the Postnikov map, there is a commutative diagram

$$
\begin{array}{ccc}
H^2(P; \mathbb{Z}/2) & \xrightarrow{f_2^*} & H^2(M; \mathbb{Z}/2) \\
\downarrow{Sq^3} & & \downarrow{Sq^3} \\
H^3(P; \mathbb{Z}/2) & \xrightarrow{f_2^*} & H^3(M; \mathbb{Z}/2)
\end{array}
$$

by Lemma 2.2 and the structure of $H^3(M; \mathbb{Z}/2)$, both $f_2^*$ are isomorphisms. Therefore we only need to determine $Sq^1x$ for a special $M$. Consider the manifold $Z_1 = S(Q \oplus 3\mathbb{R})$. It is easy to compute that $H_2(Z_1) = \mathbb{Z}/2$ and $w_2(Z_1) = 0$. This implies that $Sq^1x = tx$ and $v_2 = t^2$. Hence $w_2(M) = 0$. \hfill $\square$

§2B. Characteristic submanifolds and $\text{Pin}^\pm$-structures. Recall that for a manifold $M^n$ with fundamental group $\mathbb{Z}/2$, a characteristic submanifold $N^{n-1} \subset M$ is defined in the following way (see \[3\], §5): there is a decomposition $\widetilde{M} = A \cup TA$ such that $\partial A = \partial TA = \tilde{N}$, where $T$ is the deck-transformation. Then $N := \tilde{N}/T$ is called the characteristic submanifold of $M$. For example, if $M = \mathbb{RP}^n$, then $N = \mathbb{RP}^{n-1}$. In general, let $f: M \to \mathbb{RP}^k$ ($k$ large) be the classifying map of the universal cover, transverse to $\mathbb{RP}^{k-1}$, then $N$ can be taken as $f^{-1}(\mathbb{RP}^{k-1})$. By equivariant surgery we may assume that $\pi_1(N) \cong \mathbb{Z}/2$ and that the inclusion $i: N \subset M$ induces an isomorphism on $\pi_1$. 
Recall that there are central extensions of $O(n)$ by $\mathbb{Z}/2$ (see \cite{7} §1 and \cite{5} §2):

\[ 1 \to \mathbb{Z}/2 \to \text{Pin}^\pm(n) \to O(n) \to 1, \]

Let $\dagger \in \{+,-\}$. A Pin$^\dagger$-structure on a vector bundle $\xi$ over a space $X$ is the fiber homotopy class of lifts of the classifying map $c_X: X \to BO$.

**Lemma 2.7.** \cite{5} Lemma 1]

(1) A vector bundle $E$ over $X$ admits a Pin$^\dagger$-structure if and only if

\[ w_2(E) = 0 \quad \text{for } \dagger = +, \]

\[ w_2(E) = w_1(E)^2 \quad \text{for } \dagger = -, \]

(2) Pin$^\pm$-structures are in bijection with $H^1(X; \mathbb{Z}/2)$.

**Lemma 2.8.** \cite{3} Lemma 9] Let $M$ be a smooth, orientable 5-manifold with $\pi_1(M) \cong \mathbb{Z}/2$. Let $N \subset M$ be the characteristic submanifold (with $\pi_1(N) \cong \mathbb{Z}/2$). Then $TN$ admits a Pin$^\dagger$-structure, where

\[ \dagger = \begin{cases} - & \text{if } w_2(M) = 0 \\ + & \text{if } w_2(M) \neq 0 \text{ and } w_2(\tilde{M}) = 0 \end{cases} \]

$N$ admits a pair of Pin$^\dagger$-structures. The Pin$^\dagger$-bordism class of the pair of structures is independent of the choice of characteristic submanifold $N$. The Pin$^\dagger$-bordism groups in low dimensions are computed by Kirby and Taylor in \cite{7}, we have $\Omega^{\text{Pin}^+}_4 \cong \mathbb{Z}/16$ generated by $\pm \mathbb{R}P^4$ and $\Omega^{\text{Pin}^-}_4 = 0$.

**§2C. Construction of manifolds.** In this subsection we give detailed description of the manifolds in Theorem \cite{11}.

The manifolds $\Sigma^5_q/T$ are described in \cite{3} §1: Let $V^6_q$ be the complex hypersurface in $\mathbb{C}^4$ given by the equation

\[ z_0^q + z_1^2 + z_2^2 + z_3^2 = 0, \]

where $q = 0, 1, \ldots, 8$. The origin is an isolated singularity of $V^6_q$, and the link $\Sigma^5_q$ of this singularity, i.e., the intersection of $V^6_q$ with the unit sphere $S^7 \subset \mathbb{C}^4$, is shown to be diffeomorphic to $S^5$ for $q$ odd and diffeomorphic to $S^2 \times S^3$ for $q$ even. The involution $T: \Sigma^5_q \to \Sigma^5_q$ given by

\[ T(z_0, z_1, z_2, z_3) = (z_0, -z_1, -z_2, -z_3) \]

is free on $\Sigma^5_q$.

Thus for $q = 0, 2, 4, 6, 8$, we have an orientation preserving free involution on $S^2 \times S^3$, we denote the quotient space by $\Sigma^5_q/T$. It was shown in \cite{3} Proposition 6, Lemma 11] that $H_2(\Sigma^5_q/T) = 0$, $w_2(\Sigma^5_q/T) \neq 0$ and the Pin$^+$-bordism class of the characteristic submanifold of $\Sigma^5_q/T$ is $\pm q \in \Omega^{\text{Pin}^+}_4 \cong \mathbb{Z}/16$. By Corollary \cite{2.3} $P_2(\Sigma^5_q/T) = Q$.

For $q = 1, 3, 5, 7$, the above construction gives a free involution on $S^5$, the quotient space $\Sigma^5_q/T$ is a fake $\mathbb{R}P^5$ — a smooth manifold homotopy equivalent to $\mathbb{R}P^5$ (see \cite{14} §14D). We denote it by $H\mathbb{R}P^5_q$. Furthermore, it is shown that these $H\mathbb{R}P^5_q$ are all the possible
fake $\mathbb{RP}^5$. The $\text{Pin}^+$-bordism class of the corresponding characteristic submanifold is $\pm q \in \Omega_4^\text{Pin}^+$.

The manifolds $X^5(q)$ are constructed by the “connected-sum along $S^1$” operation on the fake projective spaces $H\mathbb{RP}_q^5$ (see \cite{F} §3). Denote the trivially oriented 4-dimensional real disc bundle over $S^1$ by $E$. Choose embeddings of $E$ into $H\mathbb{RP}_q^5$ and $H\mathbb{RP}_q^5$, representing a generator of $\pi_1$, such that the first embedding preserves the orientation and the second reverses it. Then we define

$$H\mathbb{RP}_q^5 \#_{S^1} H\mathbb{RP}_q^5 := (H\mathbb{RP}_q^5 - E) \cup_{\partial} (H\mathbb{RP}_q^5 - E),$$

Note that $H\mathbb{RP}_q^5$ admits orientation reversing automorphisms, thus the construction doesn’t depend on the orientations. (The fact that $H\mathbb{RP}_q^5$ admits orientation reversing automorphisms follows from that $\mathbb{RP}_q^5$ admits orientation reversing automorphisms and that the action of $\text{Aut}(\mathbb{RP}_q^5)$ on the structure set $\mathcal{S}(\mathbb{RP}_q^5)$ is trivial.)

The Seifert-van Kampen theorem implies that $\pi_1(H\mathbb{RP}_q^5 \#_{S^1} H\mathbb{RP}_q^5) \cong \mathbb{Z}/2$. The Mayer-Vietoris exact sequence implies that

$$H_2(H\mathbb{RP}_q^5 \#_{S^1} H\mathbb{RP}_q^5) \cong \mathbb{Z}, \quad H_2(H\mathbb{RP}_q^5 \#_{S^1} H\mathbb{RP}_q^5; \mathbb{Z}/2) \cong \mathbb{Z}.$$

Since $\pi_1 SO(4) \cong \mathbb{Z}/2$, there are actually two possibilities to form $H\mathbb{RP}_q^5 \#_{S^1} H\mathbb{RP}_q^5$. However, note that the characteristic submanifold of $H\mathbb{RP}_q^5 \#_{S^1} H\mathbb{RP}_q^5$ is $N_1 \#_{S^1} N_2$, where $N_1$ (resp. $N_2$) is the characteristic submanifold of $H\mathbb{RP}_q^5$ (resp. $H\mathbb{RP}_q^5$). (See \cite{F} p.651 for the definition of $\#_{S^1}$ for nonorientable 4-manifolds with fundamental group $\mathbb{Z}/2$). Therefore if we fix $\text{Pin}^+$-structures on each of the characteristic submanifolds, then the manifold $H\mathbb{RP}_q^5 \#_{S^1} H\mathbb{RP}_q^5$ is well-defined.

This construction allows us to construct manifolds with given bordism class of characteristic submanifold. Note that $N_1 \#_{S^1} N_2$ corresponds to the addition in the bordism group $\Omega_4^\text{Pin}$. Now for $q = 0, 2, 4, 6, 8$, choose $l, l' \in \{1, 3, 5, 7\}$ and appropriate $\text{Pin}^+$-structures on the characteristic submanifolds of $H\mathbb{RP}_l^5$ and $H\mathbb{RP}_{l'}^5$, we can form a manifold $H\mathbb{RP}_l^5 \#_{S^1} H\mathbb{RP}_{l'}^5$ such that the $\text{Pin}^+$-bordism class of the characteristic submanifold is $\pm q \in \Omega_4^\text{Pin}$. We denote this manifold by $X^5(q)$. For example, we can form $X^5(0) = H\mathbb{RP}_1^5 \#_{S^1} H\mathbb{RP}_1^5$ and $X^5(2) = H\mathbb{RP}_2^5 \#_{S^1} H\mathbb{RP}_1^5$ with different glueing maps.

The manifold $X^5(q)$ has fundamental group $\mathbb{Z}/2$ and $w_2(X^5(q)) \neq 0$. The universal cover $\tilde{X^5(q)}$ is a simply-connected 5-manifold with trivial second Stiefel-Whitney class and $H_2(\tilde{X^5(q)}) \cong \mathbb{Z}$. Therefore, by the classification theorem of Smale \cite{S}, $\tilde{X^5(q)} \cong S^2 \times S^3$. The action of $\pi_1(\tilde{X^5(q)})$ on $\pi_2(\tilde{X^5(q)})$ is trivial (cf. \cite{F} pp. 6-7).

The manifolds $Y_1$ and $Z_1$ are sphere bundles over projective spaces. Their homology groups and characteristic classes can be computed by standard methods. Especially we have $H_2(Y_1) = 0, w_1(Y_1) = w_2(Y_1) = 0; H_2(Y_2) = 0, w_1(Y_2) \neq 0, w_2(Y_2) = 0; H_2(Z_1) \cong \mathbb{Z}/2, w_1(Z_1) \neq 0, w_2(Z_1) = 0$.

3. Bordism and Surgery

§3A. Modified surgery. The classification of free involutions on a manifold is equivalent to the classification of the quotient manifolds. For involutions on the spheres $S^n$,
the quotient spaces are all homotopy equivalent to $\mathbb{R}P^n$. Therefore the classical surgery program can be applied to the classification problem of the quotient spaces ([14, §14D]).

For free involutions on $S^2 \times S^3$, the homotopy type of the quotient spaces is not constant. However, the lower Postnikov system and the normal data of the quotient spaces are relatively simple. This is appropriate for the settings of the modified surgery program in [3]. In this subsection we briefly describe the modified surgery program. For the sake of simplicity, we only consider 5-dimensional manifolds here.

Let $p: B \to BO$ be a fibration. A normal $B$-structure of $M$ is a lift $\bar{\nu}: M^5 \to B$ of the normal Gauss map $\nu: M \to BO$. $\bar{\nu}$ is called a normal 2-smoothing if it is a 3-equivalence. Manifolds with normal $B$-structures form a bordism theory. Suppose $(M^5, \bar{\nu}_i)$ ($i = 1, 2$) are two normal 2-smoothings in $B$, $(W^6, \bar{\nu})$ is a $B$-bordism between $(M^5, \bar{\nu}_1)$ and $(M^5, \bar{\nu}_2)$. Then $W$ is bordant rel. boundary to an s-cobordism (implying that $M_1$ and $M_2$ are diffeomorphic) if and only if an obstruction $\theta(W, \bar{\nu}) \in L_6(\pi_1(B), w_1(B))$ is zero [3, p.730].

The obstruction group $L_6(\pi_1(B), w_1(B))$ is related to the ordinary Wall’s $L$-group in the following exact sequence

$$0 \to L_6^s(\pi_1(B), w_1(B)) \to L_6(\pi_1(B), w_1(B)) \to \text{Wh}(\pi_1(B)),$$

where $L_6^s(\pi_1(B), w_1(B))$ is the Wall’s $L$-group and $\text{Wh}(\pi_1(B))$ is the Whitehead group of $\pi_1(B)$. For $\pi_1(B) = \mathbb{Z}/2$, $\text{Wh}(\mathbb{Z}/2) = 0$ ([2]), therefore we have an isomorphism $L_6^s(\pi_1(B), w_1(B)) \cong L_6(\pi_1(B), w_1(B))$. According to [14], the elements in $L_6^s(\mathbb{Z}/2^\pm)$ are detected by the Kervaire-Arf invariant. And if $\theta(W, \bar{\nu})$ is nonzero, then one can do surgery on $(W, \bar{\nu})$ such that the result manifold $(W', \bar{\nu}')$ has trivial surgery obstruction. Therefore we have the following

**Proposition 3.1.** Two smooth 5-manifolds $M_1$ and $M_2$ with fundamental group $\mathbb{Z}/2$ are diffeomorphic if they have bordant normal 2-smoothings in some fibration $B$.

The fibration $B$ is called the normal 2-type of $M$ if $p$ is 3-coconnected. This is an invariant of $M$. By this proposition, the solution to the classification problem consists of two steps: first we need to determine the normal 2-types $B$ for the 5-manifolds under consideration. The second step is to compute the corresponding bordism groups $\Omega_5^{(B,p)}$.

**§3B. Normal 2-types.** In this subsection we determine the normal 2-types of the quotient manifolds of smooth free involutions on $S^2 \times S^3$.

Let $p_2: B \text{Spin} \to BO$ be the canonical projection, $\oplus: BO \times BO \to BO$ be the $H$-space structure on $BO$ induced by the Whitney sum of vector bundles. Let $\eta$ be the canonical real line bundle over $\mathbb{R}P^\infty$. Let $\pi: P_2(M) \to \mathbb{R}P^\infty$ be the projection in the Postnikov tower, $L = \pi^*\eta$ be the pull-back line bundle and $kL$ be the Whitney sum of $k$ copies of $L$. Let $f_2: M \to P_2(M)$ be the second stage Postnikov map.

Consider the fibration

$$p: B = P_2(M) \times B \text{Spin} \xrightarrow{p_1 \times p_2} BO \times BO \xrightarrow{\oplus} BO,$$
where \( p_1: P_2(M) \to BO \) is the classifying map of \( kL \) with

\[
k = \begin{cases} 
0 & \text{if } w_1(M) = w_2(M) = 0 \\
1 & \text{if } w_1(M) \neq 0, w_2(M) \neq 0 \\
2 & \text{if } w_1(M) = 0, w_2(M) \neq 0 \\
3 & \text{if } w_1(M) \neq 0, w_2(M) = 0 
\end{cases}
\]

A straightforward calculation shows that \( w_1(\nu M - f_2^*kL) = w_2(\nu M - f_2^*kL) = 0 \). This implies that \( \nu M - f_2^*kL \) admits a Spin-structure. Such a structure induces a map \( M \to B\text{Spin} \). Together with \( f_2 \) we have a lift \( \tilde{\nu} \) of \( \nu \). It is easy to see that \((B, p)\) is the normal 2-type of \( M \) and \( \tilde{\nu} \) is a normal 2-smoothing. (Compare [6, §5A])

§3C. Computation of \( \Omega^{(B, p)}_5 \). To apply Proposition 3.1 according to the normal 2-types given above and Lemma 2.4, 2.5, 2.6 we need to compute the following bordism groups \( \Omega^{Spin}_5(B, p) \):

- \( \Omega^{Spin}_5(\mathbb{RP}^\infty \times \mathbb{CP}^\infty, kL), k = 0, 2; \)
- \( \Omega^{Spin}_5(Q; kL), k = 0, 1, 2, 3 \) and \( \Omega^{Spin}_5(P; 3L) \).

Computation of \( \Omega^{Spin}_5(Q; kL) \):

We apply the Atiyah-Hirzebruch spectral sequence and the Adams spectral sequence to compute the bordism groups \( \Omega^{Spin}_5(Q; kL) \cong \tilde{\Omega}^{Spin}_{5+k}(\text{Th}(kL)) \).

For \( k = 1, 3 \), the Atiyah-Hirzebruch spectral sequence has \( E^2_{p,q} = H^p(Q; \Omega^{Spin}_q) \) (here we use the Thom isomorphism to identify \( \tilde{H}^{p+k}(\text{Th}(kL); \Omega^{Spin}_q) \) and \( H^p(Q; \Omega^{Spin}_q) \)). Using Lemma 2.1 the relevant terms and differentials in the spectral sequence are depicted as follows:

```
\begin{center}
\begin{tikzpicture}
\draw[very thin, gray!30] (-1,-1) grid (7,7);
\fill[black] (0,0) circle (1.8pt);
\fill[black] (1,0) circle (1.8pt);
\fill[black] (2,0) circle (1.8pt);
\fill[black] (3,0) circle (1.8pt);
\fill[black] (4,0) circle (1.8pt);
\fill[black] (5,0) circle (1.8pt);
\fill[black] (6,0) circle (1.8pt);
\fill[black] (0,1) circle (1.8pt);
\fill[black] (1,1) circle (1.8pt);
\fill[black] (2,1) circle (1.8pt);
\fill[black] (3,1) circle (1.8pt);
\fill[black] (4,1) circle (1.8pt);
\fill[black] (5,1) circle (1.8pt);
\fill[black] (6,1) circle (1.8pt);
\fill[black] (0,2) circle (1.8pt);
\fill[black] (1,2) circle (1.8pt);
\fill[black] (2,2) circle (1.8pt);
\fill[black] (3,2) circle (1.8pt);
\fill[black] (4,2) circle (1.8pt);
\fill[black] (5,2) circle (1.8pt);
\fill[black] (6,2) circle (1.8pt);
\fill[black] (0,3) circle (1.8pt);
\fill[black] (1,3) circle (1.8pt);
\fill[black] (2,3) circle (1.8pt);
\fill[black] (3,3) circle (1.8pt);
\fill[black] (4,3) circle (1.8pt);
\fill[black] (5,3) circle (1.8pt);
\fill[black] (6,3) circle (1.8pt);
\fill[black] (0,4) circle (1.8pt);
\fill[black] (1,4) circle (1.8pt);
\fill[black] (2,4) circle (1.8pt);
\fill[black] (3,4) circle (1.8pt);
\fill[black] (4,4) circle (1.8pt);
\fill[black] (5,4) circle (1.8pt);
\fill[black] (6,4) circle (1.8pt);
\fill[black] (0,5) circle (1.8pt);
\fill[black] (1,5) circle (1.8pt);
\fill[black] (2,5) circle (1.8pt);
\fill[black] (3,5) circle (1.8pt);
\fill[black] (4,5) circle (1.8pt);
\fill[black] (5,5) circle (1.8pt);
\fill[black] (6,5) circle (1.8pt);
\fill[black] (0,6) circle (1.8pt);
\fill[black] (1,6) circle (1.8pt);
\fill[black] (2,6) circle (1.8pt);
\fill[black] (3,6) circle (1.8pt);
\fill[black] (4,6) circle (1.8pt);
\fill[black] (5,6) circle (1.8pt);
\fill[black] (6,6) circle (1.8pt);
\end{tikzpicture}
\end{center}
```

Each black dot denotes a copy of \( \mathbb{Z}/2 \) and the circle denotes a copy of \( \mathbb{Z} \).

(1) For \( k = 1 \), the differential \( d_2: E^2_{4,1} \to E^2_{2,2} \) is dual to \( Sq^2 + t \cdot Sq^1 \), \( d_2: E^2_{6,0} \to E^2_{4,1} \) is reduction mod 2 composed with the dual of \( Sq^2 + t \cdot Sq^1 \) (cf. [12]). From Lemma 2.1 both are trivial. Hence \( \Omega^{Spin}_5(Q; L) \) is either trivial or isomorphic to \( \mathbb{Z}/2 \). By
comparison with the Adams spectral sequence for $\Omega^\text{Spin}_5(Q; L) = \pi_5(TL \wedge M\text{Spin})$, Olbermann showed in [10] that $\Omega^\text{Spin}_5(Q; L) = 0$.

(2) For $k = 3$, the differential $d_2: E^2_{4,0} \to E^2_{2,1}$ is reduction mod 2 composed with the dual of $Sq^2 + t \cdot Sq^1 + t^2 \cdot$. It is shown in [10] that $Sq^1 q = Sq^2 q = 0$ (using the fact the $Q \simeq \mathbb{C}P^\infty/\tau$ and there is a fiber bundle $\mathbb{R}P^2 \to \mathbb{C}P^\infty/\tau \to \mathbb{H}P^\infty$). Therefore $Sq^2 q + t \cdot Sq^1 q + t^2 \cdot q = t^2 q$ is the nontrivial element in $H^6(Q; \mathbb{Z}/2)$, and hence $d_2$ is surjective. Therefore $\Omega^\text{Spin}_5(Q; 3L) = 0$.

For $k = 0, 2$, the Atiyah-Hirzebruch spectral sequence has $E^2_{p,q} = H^p(Q; \Omega^\text{Spin}_q)$. Using Lemma 2.1 the relevant terms and differentials of the spectral sequence are depicted as follows:

![Spectral Sequence Diagram](image)

Each black dot denotes a copy of $\mathbb{Z}/2$. $d_2: E^2_{4,1} \to E^2_{2,2}$ is dual to $Sq^2$ for $k = 0$ and to $Sq^2 + t^2 \cdot$ for $k = 2$. In both cases $d_2$ is trivial.

(1) For $k = 2$, by Lemma 2.8 there is a homomorphism (see also [6, pp. 18])

$$\phi: \Omega^\text{Spin}_5(Q; 2L) \to \Omega^\text{Pin}_4$$

sending a singular manifold $[X, f]$ to the bordism class of its characteristic submanifold. We know that $\Omega^\text{Pin}_4$ is isomorphism to $\mathbb{Z}/16$ and the characteristic submanifold of $\Sigma^8_q/T$ ($q = 0, 2, 4, 6, 8$) corresponds to $\pm q \in \Omega^\text{Pin}_4$. It is seen from the spectral sequence that $\Omega^\text{Spin}_5(Q; 2L)$ has at most 8 elements. Therefore $\phi$ is injective and $\Omega^\text{Spin}_5(Q; 2L) \cong \mathbb{Z}/8$, determined by the characteristic submanifold.

(2) For $k = 0$, we apply the Adams spectral sequence. It suffices to concentrate on the prime 2. The $E^2$-terms are $\text{Ext}^s_A(H^*(Q \wedge M\text{Spin}; \mathbb{Z}/2), \mathbb{Z}/2)$, and $E^\infty_{s,t}$ is the graded group of a filtration of $\tilde{\Omega}^\text{Spin}_t(Q) = \pi_{t-s}(Q \wedge M\text{Spin})$. The spectral sequence in low degrees is described as follows (compare [10, pp. 57]):
As usual, the horizontal coordinate is $t-s$ and the vertical coordinate is $s$. The differentials $d_2$ are both trivial or nontrivial. They are trivial if and only if the edge homomorphism $\Omega_5^{\text{Spin}}(Q) \to H_5(Q; \mathbb{Z})$ is nontrivial. Note that $H_5(Q; \mathbb{Z}) \cong H_5(Q; \mathbb{Z}/2)$ and the latter group is detected by $tq \in H^5(Q; \mathbb{Z}/2)$. For $[X, f] \in \Omega_5^{\text{Spin}}(Q)$, we may assume that $f_*$ is an isomorphism on $\pi_1$. Then $f$ induces a map between the fibrations $\tilde{X} \to X \to \mathbb{R}P^\infty$ and $\mathbb{C}P^\infty \to Q \to \mathbb{R}P^\infty$. By comparison of the cohomology Serre spectral sequences with $\mathbb{Z}/2$-coefficients for these fibrations, it’s easy to see that $f^*(q) = 0$. Then the edge homomorphism is trivial and the two differentials in the Adams spectral sequence are nontrivial. Hence $\Omega_5^{\text{Spin}}(Q) \cong \mathbb{Z}/2$.

It’s easy to see that the map $Q \cong \mathbb{C}P^\infty/\tau \to \mathbb{H}P^\infty$ induces an isomorphism $\Omega_5^{\text{Spin}}(Q) \xrightarrow{\cong} \Omega_5^{\text{Spin}}(\mathbb{H}P^\infty) = \Omega_5^{\text{Spin}}(S^4)$. Let $s_0 \in S^4$ be a base point, for a map $f : X^5 \to S^4$ transversal to $s_0$, $f^{-1}(s_0)$ is a framed 1-manifold. The spin structure on $X$ induces a spin structure on $f^{-1}(s_0)$, thus we obtained an element in $\Omega_1^{\text{Spin}} \cong \mathbb{Z}/2$. This is an isomorphism between $\Omega_5^{\text{Spin}}(S^4)$ and $\Omega_1^{\text{Spin}}$. If $\pi_1(X) \cong \mathbb{Z}/2$, then changing the spin structure on $X$ will change the induced spin structure on $f^{-1}(s_0)$, thus different spin structures on $X$ correspond to different elements in $\Omega_5^{\text{Spin}}(Q)$.

Let $f_2 : Y_1 \to Q$ be the second stage Postnikov map, then the two elements in $\Omega_5^{\text{Spin}}(Q)$ are represented by $[(Y_1, \sigma_1), f_2]$, where $\sigma_1, \sigma_2$ are the two spin structures on $Y_1$.

**Computation of $\Omega_5^{\text{Spin}}(P; 3L)$:**

The $E^2$-terms of the Atiyah-Hirzebruch spectral sequence are $E^2_{p,q} = H_p(P; \Omega_5^{\text{Spin}})$, the differential $d_2 : E^2_{p,1} \to E^2_{p-2,2}$ is dual to $Sq^2 + t \cdot Sq^1 + t^2 \cdot$, $d_2 : E^2_{p,0} \to E^2_{p-2,1}$ is reduction mod 2 composed with the dual of $Sq^2 + t \cdot Sq^1 + t^2 \cdot$. The image of the mod 2 reduction $H_6(P; \mathbb{Z}/2) \to H_6(P; \mathbb{Z}/2)$ contains an element $a$ such that $\langle t^0, a \rangle = 1$ and an element $b$ such that $\langle t^2 x^2, b \rangle = 1$. (Using similar method in the proof of Lemma 2.2) $a$ is obtain from
the section \( σ : \mathbb{R}P^∞ \to P \) and \( b \) is obtained from the commutative diagram of fibrations

\[
\begin{array}{ccc}
\mathbb{C}P^2 & \longrightarrow & V^6 & \longrightarrow & \mathbb{R}P^2 \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{C}P^∞ & \longrightarrow & P & \longrightarrow & \mathbb{R}P^∞
\end{array}
\]

where \( V^6 = (\mathbb{C}P^2 \times S^2)/(c, -1) \).

Having this information, a standard calculation shows that all terms in the line \( p + q = 5 \) are killed except for \( E^2_{5,0} \cong H_5(P; \mathbb{Z}_-) \). Therefore the edge homomorphism \( \Omega^\text{Spin}_5(P; 3L) \to H_5(P; \mathbb{Z}_-) \) is an isomorphism, \( \Omega^\text{Spin}_5(P; 3L) \cong \mathbb{Z}/2 \), and a bordism class \([X, f]\) is determined by the number \( \langle t^3 x, f_*[X]_{\mathbb{Z}/2} \rangle \).

The groups \( \Omega^\text{Spin}_5(\mathbb{R}P^∞ \times \mathbb{C}P^∞; kL) \), \( k = 0, 2 \) were computed in [6, Proposition 5.3, Proposition 5.6].

We summarize the above computations in:

**Proposition 3.2.**

1. \( \Omega^\text{Spin}_5(\mathbb{R}P^∞ \times \mathbb{C}P^∞) \cong \mathbb{Z}/4 \). A generating bordism class \([X^5, f]\) is detected by the invariant \( \langle t^3 x, f_*[X] \rangle \in \mathbb{Z}/2 \), the two generating bordism classes are interchanged if we compose \( f \) with the involution \( τ \) on \( \mathbb{C}P^∞ \).

   There is a short exact sequence
   
   \[
   0 \to \mathbb{Z}/4 \to Ω^\text{Spin}_5(\mathbb{R}P^∞ \times \mathbb{C}P^∞; 2L) \to Ω^\text{Pin+}_4 \to 0,
   \]
   
   the bordism classes \([X, f]\) \( \in \{±1\} \subset \mathbb{Z}/4 \) are detected by the invariant \( \langle t^3 x, f_*[X] \rangle \in \mathbb{Z}/2 \), and they are interchanged if we compose \( f \) with the involution \( τ \) on \( \mathbb{C}P^∞ \).

2. \( \Omega^\text{Spin}_5(Q) \cong \mathbb{Z}/2 \), the two elements are represented by \((Y_1, f_2)\) with the two spin structures on \( Y_1 \); \( Ω^\text{Spin}_5(Q; L) = Ω^\text{Spin}_5(Q; 3L) = 0 \); \( Ω^\text{Spin}_5(Q; 2L) \cong \mathbb{Z}/8 \) is determined by the characteristic submanifold.

3. \( Ω^\text{Spin}_5(P; 3L) \cong \mathbb{Z}/2 \), a bordism class \([X, f]\) is determined by the bordism number \( \langle t^3 x, f_*[X]_{\mathbb{Z}/2} \rangle \).

The proof of Theorem 3.1 is a direct consequence of Proposition 3.1 and 3.2.

**Proof of Theorem 3.1.** The \( \pi_2(M) \cong \mathbb{Z}_+ \) case was shown in [6, Theorem 3.1, Theorem 3.6]. Here we only need to consider the \( \pi_2(M) \cong \mathbb{Z}_- \) cases.

For \( w_1(M) \neq 0 \) and \( P_2(M) = Q \), since the corresponding bordism group \( Ω^{(B,p)}_5 \) is 0, there is only one diffeomorphism type, which is given in the table: \( Y_2 \) and \( S^3 \times \mathbb{R}P^2 \) respectively.

For \( w_1(M) \neq 0 \) and \( P_2(M) = P \), the bordism group is \( Ω^{(B,p)}_5 = Ω^\text{Spin}_5(P; 3L) \cong \mathbb{Z}/2 \). For a normal 2-smoothing \([M, \tilde{v}]\), the bordism number \( \langle t^3 x, \tilde{v}_*[M] \rangle \) must be 1. Therefore there is only one diffeomorphism type, which is represented by \( Z_1 \).
For \( w_1(M) = 0 \) (and therefore \( P_2(M) = Q \)), if \( w_2(M) = 0 \), then \([M, \bar{\nu}_M] = [Y_1, f_2] \in \Omega_5^{\text{Spin}}(Q)\). Thus \( M \) is diffeomorphic to \( Y_1 \). If \( w_2(M) \neq 0 \), the corresponding bordism group \( \Omega_5^{(B,p)} = \Omega_5^{\text{Spin}}(Q; 2L) \cong \mathbb{Z}/8 \). Each bordism class is represented by a normal 2-smoothing \([\Sigma_5^q/T, \bar{\nu}]\). Therefore each \( M \) in this class is diffeomorphic to a \( \Sigma_5^q/T \).

\[ \square \]

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