A Precise Analytic Delayed Coincidence Efficiency and Accidental Coincidence Rate Calculation

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Abstract

In a delayed coincidence experiment, for example, the recent reactor neutrino oscillation experiments, a precise analytic determination of the delayed coincidence signal efficiency and the accidental coincidence background rate is important for the high accuracy measurement of the oscillation parameters and to understand systematic uncertainties associated with fluctuations in muon rate and random background rate. In this work, a data model is proposed to describe the full time sequence of all possible events on the live time axis. The acceptance of delayed coincidence signals, the rate of accidental backgrounds and other coincidence possibilities are calculated by assuming that all of the ‘net muons’ are uniformly distributed on the live time axis. The intrinsic relative uncertainties in the event rates are at the $10^{-5}$ level for all combinations. The model and predictions are verified with a high statistics Monte Carlo study with a set of realistic parameters.

Keywords: delayed coincidence; accidental coincidence; inverse beta decay

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I. INTRODUCTION

The delayed coincidence method is used very broadly in nuclear and high energy physics experiments. The existence of a delayed signal greatly mitigates the critical requirement on low random background levels. The experimental observation of neutrino oscillation is one of the important breakthroughs in studying the Standard Model physics in the past half century. Recently three neutrino experiments, Daya Bay, RENO and Double CHOOZ \cite{1-3} adopted this technique again to measure the neutrino mixing angle $\theta_{13}$ precisely. Those anti-electron neutrinos from reactors are distinguished by detecting the positron and neutron from an inverse beta decay (IBD): $p + \bar{\nu}_e \rightarrow e^+ + n$. The whole signal is composed of a prompt positron signal and a delayed neutron capture signal, where the prompt positron has a detectable energy between 1 to 10 MeV and the delayed neutron spends about 29 $\mu$s in thermalization to be captured in the gadolinium (Gd) - loaded scintillator region where several photons of total energy 8 MeV are released.

In the past year, the Daya Bay Experiment \cite{1} and the RENO Experiment \cite{2} have reported their first $\theta_{13}$ measurements with delayed neutron capture signals on Gd. So far the accidental background is one of the dominant backgrounds. They both quoted the uncertainty for this background is at the 1% level. Their calculation methods are the same and can be found in their public presentations and papers, e.g. \cite{1, 2}. The formula for accidental background number, $N_{\text{accidental}}$, is repeated here for discussion:

$$N_{\text{accidental}} = N_{n-\text{like}} \times [1 - \exp(R_{e^+ - \text{like}} \times T_{\text{coincidence}})].$$

In a low random background level environment, i.e. the number of neutron-like signal ($N_{n-\text{like}}$) and the rate of positron-like signal ($R_{e^+ - \text{like}}$) are both low, or with a very small coincidence window ($T_{\text{coincidence}}$), all the single signals will be very well separated in time on average, for which this simple formula is valid. But when the random background rate is high, it will become more likely to have other cases, like three-fold coincidence events. In this special physics context, the visible energy ranges of the prompt positron and the delayed neutron overlap, giving this formula an intrinsic bias, i.e. $R_{e^+ - \text{like}}$ will be higher than the real $R_{e^+ - \text{like}}$ by including $n-\text{like}$ signals.

The Double CHOOZ \cite{3} experiment estimated the accidental background by shifting the coincidence window by 1000 $\mu$s, which gave a larger statistical uncertainty of about 10%. There is also an additional systematic uncertainty in this off-window method: because of the spallation products produced by muons, the accidental background rate is lower when further in time after the previous muon, i.e. in the off-window.
For these three experiments, the Gd-loaded liquid scintillator region is enclosed by a pure liquid scintillator region which has an almost equivalent mass as the Gd region. There are proposals to use the hydrogen neutron capture signal to further enhance the neutrino signal statistics, and to even aim at the comparable precision level as the Gd capture result. The effect mentioned above may only have marginal effect on the published $\theta_{13}$ measurement since the accidental background itself is small. The neutron capture on hydrogen will release a 2.2 MeV photon and the average capture time is around 200 $\mu$s. Apparently the delayed neutron signal energy is significantly lower than that in Gd-loaded scintillator which would allow more neutron-like singles. In addition, the coincident window chosen will be much longer. Correspondingly the accidental background will be much higher. This necessitates a precise analytic delayed coincident signal efficiency and accidental coincident background rate calculation.

Another interesting issue is that the full data-taking time is chopped into slices by cosmic muons. Some complex muon veto cuts are used in these analyses, i.e. IBD events are rejected if they tightly follow a muon. The length of veto time relies on the amount of the muon energy deposited or the location of the muon hits. It is imaginable that while the muon rate is high, there is no acceptance for a delayed coincident signal or an accidental coincident background. But how exactly the muon rate is going to affect the signal selection etc. is not transparent.

In this paper, we propose a data model to depict the full time sequence of all kinds of possible events in Sec. II and with a possible signal selection scheme, the acceptance of delayed coincidence signal, the rate of accidental background and other coincidence possibilities are calculated analytically in Sec. III In Sec. IV a high statistics simulation is done to verify the model and predictions.

II. DATA MODEL AND SIMPLIFICATION

A. Data model

A model is built up to demonstrate different types of events. Although the context used is closer to the physics of the reactor neutrino experiments, there is no loss of generality. In the following we use $e$ and $n$ to represent the positron and neutron from an IBD reaction respectively, $s$ for random background (or singles) and $\mu$ for muon. The features of the event types are described below:
1) The most interesting event is the delayed coincident event including a prompt signal, $e$, and a delayed signal, $n$. The event rate of IBDs is assumed to be $R_{IBD}$. The time between the prompt and the delay signal can be complex. For the three neutrino experiments mentioned above, the neutron capture cross section varies with its momentum, so that an exponential distribution is only a pool approximation for the neutron capture time. Here an abstract form $P_{\text{delay}}(t)$ is used to represent the delayed signals’ time distribution.

2) Random signals, or singles, include those decays from residual radioactive nuclei in the detector or other non-correlated detector noise. Random signals occur with a uniform distribution, or equivalently the time interval between two random signals follows an exponential distribution with an average value of $1/R_s$, where $R_s$ is the singles rate.

3) Cosmic Ray muons are always present. They themselves are usually easy to identify because of their high energy deposit. The spallation products of muons, however, can cause a brief increase of random signal rate, which causes a trouble for the delayed coincidence event searching. Therefore a passive veto window is applied for each muon. The chance to produce spallation products depends on how far a muon track is from the sensitive region and the energy deposit. Based on this knowledge, different muon veto window lengths can be chosen to maximize the live time and at the same time have a good control of spallation background.

These different types of signals in the full data-taking time axis are shown in Fig. 1. First we discuss two muon events, A and B, on the full time axis. Muon A is supposed to be closer to the sensitive region of the detector than muon B, and so a longer veto window is applied to muon A. A pair of delayed coincidence signals (a positron and a neutron) occurs between muon A and B, where $T_c$ is the coincidence window used to search for delayed coincidence events and is usually at the scale of the average arrival time of the delayed signal. To ensure the spallation background of muons will not contaminate the search for the prompt signal as well as the delayed signal, the time between a prompt signal and the next muon event cannot be smaller than $T_c$. The total dead time introduced by a muon is the veto time plus $T_c$. After muon B, a single signal occurs, since no delayed signal is found within $T_c$ after the single signal. Finally we show a case where two muons, C and D, are very close to each other in time such that their dead times overlap. In this example all coincident signals and singles are all well separated. We will discuss situations of overlap in the following sections.
B. Simplification

The first attempt at simplification is to project all signals from the full time axis to the live time axis as illustrated in Fig. 1. A muon with its dead time is abstracted as a virtual point on the live time axis and is counted as one net muon, like muons A and B in the plot. Muons C and D have overlapping dead times, so they together are abstracted as one point on the live time axis and only counted as one net muon. The other signals are all transparently moved to the live time axis. Since every muon dead time with a buffer window $T_c$ is already removed from the live time axis, the full length of the live time axis is the total effective time of the prompt signal searching.

The important assumption making this analytic calculation possible is to postulate that the net muons are uniformly distributed on the live time axis and the time interval follows an exponential distribution with an average value of $1/R_\mu$, where $R_\mu$ is the net muon rate on the live time axis.

In the calculation below, we always have $R_{IBD} \ll R_s$. For the three reactor neutrino experiments we took as examples, $R_s$ is about 4 orders of magnitude higher than $R_{IBD}$.

C. Searching for delayed coincidence events and other backgrounds

In this discussion we will use the simplest method to search for delayed coincidence pairs: as depicted in Fig. 1 for the positron and neutron, whenever a signal appears and is not within a muon dead time, a window will be opened for a duration of $T_c$ during which a second signal will be searched for. Two random signals can form an accidental coincidence background if their
separation is less than $T_c$.

Except to predict the event rate of delayed coincidence events and accidental background, other types of combinations are also interesting to check. The following cases will be calculated:

1-fold coincident events: $s, e, n$.
2-fold coincident events: $ss, se, sn, en, es, ns$.
3-fold coincident events: $sss, sse, ses, sn, ssn, sen, ens, esn, ess, nss$.

III. THE METHOD OF CALCULATION

A. Two Steps of Calculation

The calculation of these event rates is divided into two steps: a) determine the probability of a type of signal to start a coincidence searching window; b) determine the probability that there is a second or third signal in the searching window for 2-fold or 3-fold coincidences or there is no other signal for 1-fold events. In the following sections the starting probabilities of a random background signal ($P_{s\text{-start}}$), positron ($P_{e\text{-start}}$) and neutron ($P_{n\text{-start}}$) will be calculated first and then followed by all kinds of event rates.

B. Starting probability

On the live time axis, each signal except a muon can start a coincident searching window, as long as it is not in the previous coincidence searching window. Note that on the live time axis, $R_{IBD}$ and $R_s$ are exactly the same as on the full time axis. First let us consider how a single event may start a searching window in different situations. In the formulas below, $t_\mu$ is the time from a signal event to its previous net muon event.

Case a) As shown in the top panel of Fig. 2, when $t_\mu < T_c$, if there is no other signal between the single event and the muon, a searching window will be started by the single event. The probability of this situation is:

$$P_a = \int_0^{T_c} P(0| R_s t_\mu) \cdot P(t_\mu) \cdot dt_\mu$$

$$= \int_0^{T_c} \frac{(R_s t_\mu)^k}{k!} e^{-R_s t_\mu} |_{k=0} \cdot R_\mu e^{-R_\mu t_\mu} \cdot dt_\mu$$

$$= \frac{R_\mu}{R_s + R_\mu} [1 - e^{-(R_s + R_\mu)T_c}], \quad (1)$$
where $P(0|R_s t_\mu)$ is the probability that there is no other single event in between, which is calculated as a Poisson distribution with a mean of $R_s t_\mu$ and count $k = 0$, and the second term $P(t_\mu)$ gives the probability of finding a muon at $t_\mu$ before the random background being considered, which is calculated according to an exponential distribution with a rate of $R_\mu$.

Case b) While $t_\mu \geq T_c$, if there is no other signal in a $T_c$ window before the single event being considered, a searching window will be started, just as the panel b of Fig. 2 shows. The starting probability of this situation is:

$$P_b = P(0|R_s T_c) \cdot \int_{T_c}^{\infty} P(t_\mu) \cdot dt_\mu$$

$$= \frac{(R_s T_c)^k}{k!} e^{-R_s T_c} \bigg|_{k=0} \cdot \int_{T_c}^{\infty} R_\mu e^{-R_\mu t_\mu} \cdot dt_\mu$$

$$= e^{-(R_s+R_\mu)T_c},$$

where the first term corresponds to no signals in $T_c$ and the second term is to ensure $t_\mu \geq T_c$.

Case c) In case b) when $t_\mu > T_c$, if there is another scenario that the target single can start a searching window as depicted in the panel c of Fig. 2. There is a random signal, $s1$, in front of the target single event; however it is occupied by a previous searching window started by $s2$. The probability of this situation is:
\[ P_c = \int_0^{T_c} dt R_s e^{-R_s t} \cdot [1 - P(0|R_s t)] \cdot \int_{T_c+t}^{\infty} P(t_\mu) \cdot dt_\mu \]
\[ = \int_0^{T_c} dt R_s e^{-R_s t} \cdot \left[ 1 - \frac{(R_s t)^k}{k!} e^{-R_s t}_{|k=0} \right] \cdot \int_{T_c+t}^{\infty} R_\mu e^{-R_\mu t_\mu} \cdot dt_\mu \]
\[ = \frac{R_s}{R_s + R_\mu} e^{-R_s T_c} \left[ 1 - e^{-(R_s + R_\mu)T_c} \right] - \frac{R_s}{2R_s + R_\mu} e^{-R_\mu T_c} \left[ 1 - e^{-(2R_s + R_\mu)T_c} \right], \quad (3) \]

where the first integral gives the probability to find the first random background signal \( s_1 \) at time \( t \) in front of the target single event, the second term calculates the probability of having at least one random background \( s_2 \) in the early \( t \) window, that is \( T_c \) away from the target single event, and the last integral just gives the probability of finding a muon at some time larger than \( T_c + t \).

Finally the starting probability is,

\[ P_{s\text{-start}} = P_a + P_b + P_c. \quad (4) \]

There should be higher order corrections after \( P_c \), but, with the example parameters used in the simulation presented later, they are estimated to be five orders of magnitude smaller than these leading terms.

The starting rate is

\[ R_{s\text{-start}} = R_s \cdot P_{s\text{-start}}. \quad (5) \]

As for IBD signals, it is

\[ P_{e\text{-start}} = P_{s\text{-start}} \cdot \varepsilon_e, \quad (6) \]

where \( R_{IBD} \ll R_s \) and there is a detection efficiency \( \varepsilon_e \) for positrons. Then the rate of prompt signals, i.e. positrons starting a time window, is similar to the rate of random backgrounds starting a time window:

\[ R_{e\text{-start}} = R_{IBD} \cdot P_{s\text{-start}} \cdot \varepsilon_e. \quad (7) \]

The situation of a neutron starting a searching window is a bit complicated, because by design it is always possible that there is a prompt positron signal ahead of it. It is only possible when the positron is failed to be detected. However the method to calculate \( P_{n\text{-start}} \) is the same as above and the final result is listed in the appendix. Then the event rate with neutrons as a start is:

\[ R_{n\text{-start}} = R_{IBD} \cdot P_{n\text{-start}}. \quad (8) \]
C. Construct an Event

After the starting probability or starting rate of a type of signal is known, the rate of accidental background, detectable IBD pairs and other cases can be calculated.

For example, the accidental background rate, i.e. the single-single combination in one $T_c$, is just the rate of one single event to start a searching window multiplied by the probability of a second single event appearing in the same window:

$$R(\text{ss}) = R_{\text{s-start}} \cdot P(1|R_sT_c),$$  \hspace{1cm} (9)

where $R(\text{ss})$ is used to describe the event rate of \text{ss}-type backgrounds and $P(1|R_sT_c)$ is the Poisson probability of one count with a mean of $R_sT_c$. IBD events are not explicitly required to be vetoed from this with $R_{\text{IBD}} \ll R_s$.

The IBD event rate can be obtained in a similar fashion:

$$R(\text{en}) = R_{\text{e-start}} \cdot \varepsilon_{n|e} \int_0^{T_c} p_{\text{delay}}(t) dt \cdot P(0|R_sT_c)$$  \hspace{1cm} (10)

$$= R_{\text{IBD}} \cdot P_{s-start} \cdot \varepsilon_{e} \cdot \varepsilon_{n|e} \cdot \int_0^{T_c} p_{\text{delay}}(t) dt \cdot P(0|R_sT_c),$$  \hspace{1cm} (11)

where $\varepsilon_{n|e}$ is the efficiency of neutron detection after a positron is detected, and the random background is explicitly required to be outside this searching window. The complete IBD event detection efficiency can be expressed as:

$$\varepsilon_{\text{IBD}} = \frac{R(\text{en})}{R_{\text{IBD}}}$$  \hspace{1cm} (12)

The results of other combinations are listed in the appendix.

IV. VERIFICATION WITH MONTE CARLO

Since the muon veto cut is quite complicated, a high statistics Monte Carlo simulation (1E10 events) is done to verify the assumption and the predictions. Three types of events are produced: IBD, random background, muon. They are generated on the full time axis according to three uniform distributions. Their event rates on the full time axis are chosen to be very similar to those of recent reactor neutrino experiments. Usually a veto time of $400\mu s$ is applied for each muon, but 0.05% of them can be shower muons or closer to the sensitive region, for which a one-second long veto is applied. The neutron capture time distribution $P_{\text{delay}}(t)$ is replaced with a simple exponential distribution with rate $= \lambda$. The detection efficiencies of the prompt and delayed
signals are also included. All the parameters used are summarized in Tab. [I] The measurements with the simulated sample are compared to the predictions and are listed in Tab. [II] where the discrepancies are all within a 3 $\sigma$ range.

One interesting discovery is that the net muon rate on the live time axis and the real muon rate on the full time axis are the same within statistical uncertainty. With the analytic formulas all errors on singles rate and muon rate can be easily propagated to the final result.

$$R_s \quad \text{Total muon rate} \quad \text{veto} \quad \text{Shower fraction} \quad \text{Shower veto} \quad R_{IBD} \quad \epsilon_e \quad \epsilon_{\mu}[e] \quad T_c \quad 1/\lambda$$

| Rate [Hz] | $R(s)$ | $R(ss)$ | $R(se)$ | $R(sn)$ | $R(sss)$ | $R(ens)$ | $R(ses)$ | $R(ssn)$ |
|-----------|--------|---------|---------|---------|---------|---------|---------|---------|
| Mea.      | 48.0833 | 0.96165 | 0.0010460 | 0.0001108 | 0.009626 | 0.001342 | 2.10E-5 | 2.27E-6 |
| Sta. Err. | 0.0015 | 0.00022 | 7.2E-6 | 2.3E-6 | 2.2E-5 | 1.1E-5 | 1.4E-6 | 4.7E-7 |
| Pred.     | 48.0856 | 0.96171 | 0.0010499 | 0.0001093 | 0.009617 | 0.001330 | 2.10E-5 | 2.19E-6 |

| Rate [Hz] | $R(e)$ | $R(en)$ | $R(es)$ | $R(sen)$ | $R(ess)$ | $R(n)$ | $R(ns)$ | $R(nss)$ |
|-----------|--------|---------|---------|---------|---------|--------|---------|---------|
| Mea.      | 0.029657 | 0.066447 | 0.0005991 | 0.0008715 | 6.91E-6 | 0.012719 | 0.0002616 | 3.11E-6 |
| Sta. Err. | 0.000038 | 0.000057 | 5.4E-6 | 6.6E-6 | 5.8E-7 | 2.5E-5 | 3.6E-6 | 3.9E-7 |
| Pred.     | 0.029647 | 0.066525 | 0.0005930 | 0.0008735 | 5.93E-6 | 0.012770 | 0.0002554 | 2.55E-6 |

TABLE I: Monte Carlo simulation parameters, including singles rate, muon rate, IBD rate, IBD delayed signal capture time constant and signal selection cuts and efficiencies.

TABLE II: Measured event rates of simulation sample and their predictions. For each type of combination, Mea. gives the measured result, Sta. Err. is its statistical error, and Pred. is the predicted value. $sse$ and $ses$ are not distinguished, as well as $sns$ and $ssn$.

V. CONCLUSION

We brought up a complete mathematical model for the signals distribution on the full time axis for delayed coincidence experiments, and further projected this complicated model onto the live time axis. An analytic calculation was done by assuming all the net muons are uniformly distributed on the live time axis. The intrinsic relative uncertainties for all combinations’ rates
are at the $10^{-5}$ level. The predictions were verified with a high statistics simulation with a set of realistic parameters from the Daya Bay, RENO and Double CHOOZ experiments. The method proposed here is precise and can be used in these reactor neutrino experiments and other related experiments. In the appendix the event rate formulas for all the cases are listed.

VI. ACKNOWLEDGEMENT

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[1] F. P. An et al. (Daya Bay Collaboration), Phys. Rev. Lett. 108, 171803 (2012); F. P. An et al. (Daya Bay Collaboration), Chinese Physics C37, 011001 (2013).

[2] J. K. Ahn et al. (RENO Collaboration), Phys. Rev. Lett. 108, 191802 (2012).

[3] Y. Abe et al. (Double CHOOZ Collaboration), Phys. Rev. Lett. 108, 131801 (2012).
APPENDIX

All 1-fold, 2-fold and 3-fold coincident event rate calculation formulas are listed in Tab. III, where the first column indicates the combination concerned and the second column shows the expression for calculation. $P(k|m)$ is used to describe a Poisson distribution with a mean of $m$ and a count of $k$. The delayed signal follows a simple exponential distribution with time constant $\lambda$. 
\[ R_{s\text{-start}} \]
\[ R_s \cdot \{ \frac{R_0}{R_s+R_\mu} [1 - e^{-(R_s+R_\mu) T_c}] + e^{-(R_s+R_\mu) T_c} + \]
\[ \frac{R_s}{R_s+R_\mu} e^{-R_\mu T_c} [1 - e^{-(R_s+R_\mu) T_c}] - \frac{R_s}{2R_s+R_\mu} e^{-R_\mu T_c} [1 - e^{-(2R_s+R_\mu) T_c}] \} \]

\[ R_{e\text{-start}} \]
\[ R_{IBD} \cdot \epsilon \cdot \{ \frac{R_0}{R_s+R_\mu} [1 - e^{-(R_s+R_\mu) T_c}] + e^{-(R_s+R_\mu) T_c} + \]
\[ \frac{R_s}{R_s+R_\mu} e^{-R_\mu T_c} [1 - e^{-(R_s+R_\mu) T_c}] - \frac{R_s}{2R_s+R_\mu} e^{-R_\mu T_c} [1 - e^{-(2R_s+R_\mu) T_c}] \} \]

\[ R_{n\text{-start}} \]
\[ R_{IBD} \{ \epsilon_n R_\mu [(1 - \epsilon_n) \frac{1 - e^{-(R_s+R_\mu) T_c}}{R_s+R_\mu} + \epsilon_n \frac{1 - e^{-(R_s+R_\mu+\lambda) T_c}}{R_s+R_\mu+\lambda}] \]
\[ + \epsilon_n e^{-(R_\mu+R_s) T_c} (1 - \epsilon_n) + \epsilon_n e^{-(R_\mu+R_s) T_c} \} \]
\[ + \epsilon_n R_s e^{-R_\mu T_c} [1 - e^{-(R_s+R_\mu+\lambda) T_c}] \]
\[ + \epsilon_n e^{-(R_\mu+R_s) T_c} \}
\[ + \epsilon_n e^{-(R_\mu+R_s) T_c} \frac{1 - e^{-(R_s+R_\mu+\lambda) T_c}}{R_s+R_\mu+\lambda} \]
\[ + \epsilon_n e^{-(R_\mu+R_s) T_c} \]
\[ + \epsilon_n e^{-(R_\mu+R_s) T_c} \]

\[ R(s) \]
\[ R_{s\text{-start}} \cdot P(0|R_s T_c) \]

\[ R(e) \]
\[ R_{e\text{-start}} \cdot P(0|R_s T_c) \cdot [1 - \epsilon_n \epsilon (1 - e^{-\lambda T_c})] \]

\[ R(n) \]
\[ R_{n\text{-start}} \cdot P(0|R_s T_c) \]

\[ R(en) \]
\[ R_{e\text{-start}} \cdot (1 - e^{-\lambda T_c}) \cdot P(0|R_s T_c) \cdot \epsilon_n \epsilon \]

\[ R(ss) \]
\[ R_{s\text{-start}} \cdot P(1|R_s T_c) \]

\[ R(se) \]
\[ R_{s\text{-start}} \cdot P(0|R_s T_c) \cdot \epsilon \{ (1 - \epsilon_n \epsilon) (1 - e^{-R_{IBD} T_c}) + \frac{\epsilon_n R_{IBD}}{e^{-R_{IBD} T_c} - e^{-\lambda T_c}} \} \]

\[ R(es) \]
\[ R_{e\text{-start}} \cdot P(1|R_s T_c) \cdot [1 - \epsilon_n \epsilon (1 - e^{-\lambda T_c})] \]

\[ R(ns) \]
\[ R_{n\text{-start}} \cdot P(1|R_s T_c) \]

\[ R(sn) \]
\[ R_s e^{-R_\mu T_c} \cdot \epsilon_n \frac{R_{IBD}}{R_s+R_\mu+\lambda} [1 - e^{-(R_{IBD}+\lambda) T_c}] \]
\[ \{ \frac{R_0}{R_s+R_\mu} (1 - \epsilon_n \epsilon) [1 - e^{-(R_\mu+R_s) T_c}] + \frac{R_0}{R_s+R_\mu+\lambda} \cdot \epsilon_n \epsilon [1 - e^{-(R_\mu+R_s+\lambda) T_c}] + \]
\[ e^{-(R_\mu+R_s) T_c} \cdot (1 - \epsilon_n \epsilon + \epsilon_n \epsilon \cdot e^{-\lambda T_c}) + \epsilon_n \epsilon \frac{R_s}{R_\mu+2R_s} \cdot e^{-R_\mu T_c} \]
\[ [1 - e^{-\lambda T_c} - \frac{\lambda}{R_\mu+2R_\mu+\lambda} (1 - e^{-(R_\mu+2R_\mu+\lambda) T_c})] \}

\[ R(sss) \]
\[ R_{s\text{-start}} \cdot P(2|R_s T_c) \]

\[ R(ess) \]
\[ R_{e\text{-start}} \cdot P(2|R_s T_c) \cdot [1 - \epsilon_n \epsilon (1 - e^{-\lambda T_c})] \]

\[ R(nss) \]
\[ R_{n\text{-start}} \cdot P(2|R_s T_c) \]

\[ R(ses) + R(sse) \]
\[ R(se) \cdot R_s T_c \]

\[ R(sns) + R(snn) \]
\[ R(sn) \cdot R_s T_c \]

\[ R(sen) \]
\[ R_{s\text{-start}} \cdot P(0|R_s T_c) \cdot \epsilon \epsilon_n \epsilon [1 - e^{-R_{IBD} T_c} - \frac{R_{IBD}}{R_s+R_\mu+\lambda} (e^{-R_{IBD} T_c} - e^{-\lambda T_c})] \]

\[ R(esn) + R(ens) \]
\[ R(en) \cdot R_s T_c \]

**TABLE III:** Analytic event rate expressions for all 1-fold, 2-fold and 3-fold coincident events.