Nucleon localization and formation of cluster structures characterize not only light α-conjugate nuclei [1–6], but also heavier nuclear systems [7]. Several microscopic models, for instance the anti-symmetrized molecular dynamics [1,2], have very successfully been applied to a description of cluster states in relatively light nuclei. A more general approach based on energy density functionals (EDFs) [8–12] has to be employed in order to study the occurrence and structure of nucleon clusters in medium-heavy and heavy nuclei. Studies based on the latter method have related the conditions for nucleon localization and formation of clusters to the underlying single-nucleon dynamics, geometric shape transitions and surface effects.

The EDF framework enables a systematic analysis of nucleon localization as a precondition for cluster formation. The Wigner parameter [13] can be used to describe the transition between the nuclear quantum liquid phase and a hybrid phase of cluster states in terms of spatial localization. Such a localization parameter can be microscopically calculated as the ratio of the dispersion of a single-nucleon wave function to the average inter-nucleon distance [9,14]. When

\[ \alpha_{loc} \sim \frac{b}{r_0} = \frac{\sqrt{\hbar A^{1/6}}}{(2m V_0 \rho_0^2)^{1/4}} \] (1)

where \( b = \sqrt{\hbar/m_0} \) fm is the oscillator length, and \( r_0 \approx 1.25 \) fm is the typical inter-nucleon distance determined by nuclear saturation density (\( \rho \approx 0.16 \) fm\(^{-3} \)). The resulting expression includes the nucleon number \( A \), the mass of the nucleon \( m_0 \), and the depth of the confining potential \( V_0 \). As shown in Ref. [14], the oscillator length can be related to the spatial dispersion \( \Delta r = \sqrt{\langle r^2 \rangle - \langle r \rangle^2} \): \( b \approx 2 \Delta r \) for the first s, p, and d HO wave functions. When the dispersion of the single-nucleon wave function is of the same size as the inter-nucleon distance, \( \alpha_{loc} \) is of the order of 1 and this facilitates the formation of \( \alpha \)-clusters. The dependence of the localization parameter on \( A^{1/6} \) means that cluster states are preferably formed in lighter nuclei, and the transition from coexisting cluster and mean-field states to a Fermi liquid state should occur for nuclei with \( A \approx 20–30 \), in qualitative agreement with experiment. In infinite nuclei the spatial dispersion and, therefore the localization parameter, will explicitly depend on the quantum numbers of specific single-nucleon orbitals. In this work we generalize expression (1) and derive an explicit dependence of the localization parameter on single-nucleon quantum numbers. We also compare the spatial dispersions of the HO wave functions with those obtained in a fully self-consistent microscopic calculation of nuclear ground states, and perform a systematic microscopic calculation of single-nucleon dispersions in axially symmetric nuclei over the entire nuclear chart.

In the first step we perform a systematic microscopic calculation, based on the EDF framework, of dispersions...
The localization and cluster effects in nuclear systems [19].

approach provides a realistic approximation for studies of
classical three-dimensional harmonic oscillator. The HO ap-
proximation of the single-nucleon wave function, for the case
when the nuclear potential is approximated by a spher-
oidal momenta and radial quantum numbers arise because
between points that correspond to the same orbital angu-
lar momentum, just as in the case of a fully
spherical structure. By considering a heavy spher-
oidal nucleus with many occupied levels we can analyze
the dependence of the corresponding dispersions on the
radial quantum number and orbital angular momentum. A particu-
larly interesting result is that for single-neutron states
weakly on the orbital angular momentum. A particu-
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depending on the orbital angular momentum.

The depth of the self-consistent neutron potential is
\( V_0 = 78.6 \text{ MeV} \), and the dispersions \( \Delta r \) are
plotted as functions of the single-particle radial quantum
state. The corresponding dispersions, of the size of the average
inter-nucleon distance. We note that the small splittings
between points that correspond to the same orbital angular
momenta and radial quantum numbers arise because of
deformation: the self-consistent mean-field solution is
not fully spherical symmetric (the quadrupole deformation
parameter is \( \beta_2 = 0.07 \)).

Next we derive an analytic expression for the disper-
sion of the single-nucleon wave function, for the case
when the nuclear potential is approximated by a spher-
oidal three-dimensional harmonic oscillator. The HO ap-
proach provides a realistic approximation for studies of
localization and cluster effects in nuclear systems [19]. The \(< r^2 >\) term is easy to evaluate and reads:

\[
< r^2 > = b^2 \left( N + \frac{3}{2} \right) = b^2 \left( 2n' + l + \frac{3}{2} \right) \tag{2}
\]

where \( N = 2(n-1) + l \) is the principal quantum number,
and \( n' \equiv n - 1 \). The \(< r >\) term is considerably more
complicated. Using the HO wave functions, it can be
expressed in the following form

\[
\frac{< r >}{b} = \sum_{q=0}^{n'} \frac{(-1)^q(l + q + 1)!\Gamma(n' - q - \frac{1}{2})}{q!(n' - q)\Gamma(l + q + \frac{3}{2})\Gamma(-q - \frac{1}{2})} \tag{3}
\]

where \( \Gamma \) is the Euler function. To compare with the
microscopic results shown in Fig. 1, the corresponding
dispersions for the single-particle wave functions of the
harmonic oscillator potential of \(^{288}\text{Cf}\) are evaluated
numerically using Eqs. 2 and 3, and plotted in Fig. 2.
The dispersion, of course, increases with the number of
radial nodes, but shows very little dependence on the or-
bital angular momentum, just as in the case of a fully
microscopic calculation. It should be noted that the mi-
icroscopic dispersion (cf. Fig. 1) is typically 1.2 times
larger than the corresponding one in the HO approxima-
tion, because the actual self-consistent nuclear potential
is more diffuse. Indeed, a Woods-Saxon potential can be
approximated by a HO with a length of about 1.2b, thus
explaining this ratio.

Therefore, if only \( n = 1 \) states are occupied in a nu-
cleus, all nucleons have similar and minimal spatial dis-
ersion, of the order of 1 fm. The pronounced localization
will favor formation of \( \alpha \)-like clusters, whereas the occu-
pation of \( n > 1 \) states breaks the coherence of spatial
localization. Of course, nuclei in which only levels orig-
inating from the \( n = 1 \) spherical states are occupied are
the light ones up to about silicon \(( Z = 14, 1s, 1p, 1d \) lev-
els occupied). These are indeed nuclear systems in which
cluster structures are empirically most pronounced [20].
To derive a generalization of the expression for the localization parameter in the HO approximation Eq. (1), but now taking explicitly into account the quantum numbers of occupied states, we simplify the $n$ and $l$ dependence in Eq. (3). For $l = 0$ one obtains

$$<\frac{r}{b}> = \frac{2}{\sqrt{\pi}} \frac{(2n'+1)!!}{(2n)!!} \simeq \frac{2}{\sqrt{\pi}} \left( \frac{5n'}{4} + 1 \right)^{1/2}$$

where the rhs is an accurate approximation with a $<1\%$ error for $n' = 20$. Thus, using Eqs. (2) and (4), the $l = 0$ dispersion reads:

$$\left( \frac{\Delta r}{b} \right)^2 \simeq \left( 2 - \frac{5}{\pi} \right) n' + \left( \frac{3}{2} - \frac{4}{\pi} \right) \simeq 0.4n' + 0.23$$

Let us now consider the case of large angular momenta $l$ in Eq. (6). In this limit [21]:

$$\frac{(l+q+1)!}{\Gamma(l+q+\frac{3}{2})} \simeq \sqrt{l} + \frac{1}{\sqrt{l}} \left( \frac{q}{2} + \frac{5}{8} \right),$$

and the expression Eq. (6) reduces to:

$$<\frac{r}{b}> \simeq \sqrt{l} + \frac{1}{\sqrt{l}} \left( \frac{5}{8} + \frac{3n'}{4} \right).$$

The corresponding dispersion for large $l$ values reads:

$$\left( \frac{\Delta r}{b} \right)^2 \simeq \frac{n'}{2} + \frac{1}{4}.$$

The close agreement of the expressions for $l = 0$ (Eq. (6)) and in the large $l$ limit (Eq. (8)), reflects the weak dependence of the HO dispersion on orbital angular momentum. The corresponding dispersions of occupied states of $^{288}$Cf: minimal values corresponding to Eq. (5) and maximal values computed using Eq. (8), are indicated by open symbols in Fig. 2. Both expressions, of course, yield very similar dispersions. Eq. (5) implies, as also shown in Fig. 2, that the occupation of an $n = 2$ state leads to a dispersion that is by a factor $\sqrt{3} \sim 1.7$ larger than the one of $n = 1$ states. This corresponds to the case of medium-heavy nuclei, typically above silicon, in which there is no clear evidence of cluster states at low energy and angular momenta.

From Eqs. (1) and (8), we finally derive the approximate expression for the HO localization parameter:

$$\alpha_{loc} \simeq \frac{b}{r_0} \sqrt{2n-1} = \frac{\sqrt{h(2n-1)}}{(2mV_0r_0^2)^{1/4}} A^{1/6}. $$

In nuclei the depth of the confining potential is rather constant, as well as the average inter-nucleon distance, hence the two key parameters that determine localization are $A$ and the radial quantum number $n$. For relatively light nuclei, with $A \leq 30$ and $n = 1$ states occupied, $\alpha_{loc} \lesssim 1$ and this favors the formation of $\alpha$-like clusters.

In heavier nuclei levels that originate from $n > 1$ spherical states are largely delocalized and this explains the predominant liquid drop nature of these systems.

An interesting possibility however, is the formation of individual $\alpha$-like clusters from valence nucleons in heavy nuclei. We have performed a systematic, fully self-consistent Relativistic Hartree-Bogoliubov (RHB) [22] calculation of single-nucleon dispersions in axially symmetric nuclei over the entire nuclear chart, using the functional DD-ME2. Pairing correlations have been taken into account by employing an interaction that is separable in momentum space, and is completely determined by two parameters adjusted to reproduce the empirical bell-shaped pairing gap in symmetric nuclear matter [23]. The Dirac-Hartree-Bogoliubov equations are solved by expanding the nucleon spinors in a large axially-symmetric HO basis. The microscopic values of the dispersion $\Delta r$ have been calculated for each single-particle state. Figure 3 indicates, on the table of nuclides in the $N - Z$ plane, those nuclei in the RHB calculation for which both the neutron and proton valence states (having an occupation probability larger than 0.1) exhibit a significantly small dispersion, of the order of 1 fm. For deformed nuclei it can be shown that these Nilsson levels do originate from $n = 1$ spherical states with the degeneracy raised by deformation. One notices that pronounced localization, as precondition for the formation of cluster structures, is present in light nuclei but also occurs among valence nucleons in medium-heavy and heavy nuclei, in agreement with empirically known $\alpha$- and cluster-radioactive nuclei. For instance, favorable condition for clustering is predicted for $^{212}$Po, in accordance with experimental evidence [10]. The EDF-based approach used in this work provides a global interpretation of the occurrence of cluster structures by means of spatial dispersion of single-nucleon wave functions.

The role of deformation, which is known to favor cluster formation [10, 23, 25], is illustrated in Fig. 4 where we show the self-consistent mean-field results for $^{20}$Ne calculated using the relativistic density functional DD-ME2 in the RMF approach. Pairing does not play an important role for the effect that we consider in this particular nucleus, and it has not been included in the RMF calculation restricted to axial symmetry. Figure 4 displays the occupied single-nucleon levels as functions of the axial deformation parameter, the dispersion of the wave function corresponding to the highest level occupied by the two valence neutrons, and the partial intrinsic densities of the valence neutrons for values of deformation that correspond to the peaks and minima of dispersion. In general, the spatial dispersion increases with deformation until a level crossing occurs for the last occupied state. The largest and sharpest increase of the spatial dispersion takes place at the deformation at which a $1/2^+$ state (originating from the $2s_{1/2}$ spherical state) becomes occupied. It is remarkable that at this point the dispersion increases by the factor $\sim 1.7$, which we encountered above when discussing the filling of the $n = 2$ HO state (cf. Eq.
FIG. 3: (Color online) Microscopic axially-symmetric RHB prediction of nuclei that have small radial dispersion of the single-particle states of valence nucleons (red circles), plotted on the background of empirically known nuclides in the $N-Z$ plane. The single-nucleon dispersions have been calculated using the functional DD-ME2 and separable pairing, and assuming axial symmetry.

The present study can also be related to the discussion of (multi)clustering in super-deformed and hyper-deformed states [24, 26]. These specific states (ratio of deformed HO frequencies of 2 and 3, respectively) can be described as irreducible representations of SU(3). The magic numbers of super (hyper) deformed states are obtained from the sum of two (three) magic numbers of the spherical system. These relations involve small values of radial quantum numbers and, through Eq. (9), this can be linked to more localized states. The present approach however, as illustrated in Fig. 4, establishes a connection between spatial dispersion and clustering for all deformations, rather than only for specific super- and hyper-deformed states.

In summary, we have used the self-consistent mean-field framework based on nuclear energy density functionals, and the spherical harmonic oscillator approximation for the nuclear potential, to analyze the radial dispersion of single-nucleon wave functions. It has been shown that the dispersion exhibits a pronounced dependence on the radial quantum number, but essentially does not depend on the orbital angular momentum. In particular, for single-neutron states with $n=1$ the dispersion is of the size of the average inter-nucleon distance, and the correspondingly small value of the localization parameter $\alpha_{loc}$ indicates a transition between the nuclear quantum liquid phase and a hybrid phase of cluster states that coexists with mean-field states. Based on the HO approximation, we have derived an analytic expression for the localization parameter that, in addition to the dependence on the depth of the nuclear potential and the nucleon number, explicitly takes into account the radial quantum number of occupied single-nucleon states. For $A \leq 30$ and $n=1$ states occupied, $\alpha_{loc} \lesssim 1$ and the formation of $\alpha$-clusters is favored. Although in heavier nuclei levels that originate from $n>1$ spherical states are largely delocalized and these systems exhibit the quantum liquid phase of nucleonic matter, individual $\alpha$-like clusters can be formed from valence nucleons filling Nilsson levels that can be traced back to the $n=1$ spherical mean-field states. The role of deformation in the evolution of spatial dispersion of single-nucleon levels has

FIG. 4: (Color online) The occupied single-neutron levels $\Omega^+$ of $^{20}$Ne in the RMF approach as functions of the axial deformation parameter (solid curves), the dispersion of the wave function corresponding to the highest level occupied by the two valence neutrons (blue dotted curve), and the partial intrinsic densities of the valence neutrons for values of deformation that correspond to the peaks and minima of dispersion.
been microscopically analyzed in the example of $^{20}$Ne, showing the robustness of the present analysis and conclusions. This study provides a general basis for understanding the conditions for cluster formation in light and heavy nuclei.

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