Establishment and Analysis of the Spectral Relationship Between Range-Rate and Gravity Potential Based on Energy Conservation

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Abstract  The spectral relationship between range-rate measurements and the gravity potential for low-low satellite-to-satellite tracking mission was established based on the energy conservation theory. Then the performances of satellite separation, the orbital altitude, and the accuracy of range-rate measurements in recovering the earth’s gravity field were simulated and analyzed by this method. Finally, the cumulative geoid errors of the reference mode were obtained by using the configuration parameters of the GRACE mission. By comparing the cumulative geoid errors of the reference mode and GGM02S and EIGEN-GRACE02S models, it basically reflected the performance of GRACE and proved the feasibility of this method.

Keywords  Earth’s gravity field; energy conservation; spectrum; GRACE

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Introduction

At the beginning of the 21st century, the research on Earth’s gravity field with low-low satellite-to-satellite tracking (SST-II) techniques has been in a notable development period due to the successful launch of Gravity Recovery and Climate Experiment (GRACE) mission. The dedicated gravity satellite mission not only provides the long- and medium-wave length part of static gravity field with high accuracy and high resolution, but also provides measurements of time-variable gravity field parameters. It will greatly improve our knowledge of the Earth’s gravity field, and extend the application fields of the Earth’s gravity field in related earth science disciplines. Due to the development prospect of the gravity satellites and their scientific significance, it is imperative to develop our gravity exploring satellites\(^{[1,2]}\). In recent years, the performances of the technique indexes of SST in recovering the Earth’s gravity field have been studied and simulated by many domestic scholars\(^{[2,7]}\), and these achievements provide references for the design and demonstration of our future gravity exploring satellite mission. In this paper, the spectral relationship between range-rate measurements and the
gravity potential for SST-II mission was established on the basis of the energy conservation and error propagation law. Through some simulation experiments, the performances of satellite separation, the orbital altitude, and the accuracy of range-rate measurements in recovering the Earth’s gravity field were analyzed, and the feasibility of this method was also verified. Finally, some useful conclusions were obtained, and the favorable conclusion can be as a reference for the preliminary design of our future SST-II mission.

1 Spectral relationship between range-rate and gravity potential

If the non-gravitational forces are ignored, then according to the energy conservation theory, the gravity satellite’s kinetic energy plus the potential energy will be conserved. Let us suppose the satellite is unit mass; then the energy conservation equation can be written as:

\[ \frac{1}{2} v^2 - U = E \]  

(1)

where \( E \) is the total energy, \( U \) is the potential, and \( v \) is the satellite velocity. The potential \( U \) and the satellite’s velocity \( v \) can be decomposed as:

\[
\begin{align*}
U &= U_0 + T' \\
v &= v_0 + \Delta v
\end{align*}
\]  

(2)

in which \( U_0 \) is the reference gravity potential representing the gravity at the reference ellipsoid, \( v_0 \) is the reference velocity corresponding to the reference gravity potential \( U_0 \), \( T' \) is disturbing potential representing the difference between the actual gravity and the reference gravity, \( \Delta v \) denotes the velocity variation due to disturbing gravity potentials.

The reference velocity \( v_0 \) is related to the reference gravity potential \( U_0 \) as the following energy equation:

\[ \frac{1}{2} v_0^2 - U_0 = E \]  

(3)

Substituting Eq.(2) and Eq.(3) into Eq.(1), and ignoring the higher order term of the velocity variation, the relationship between the velocity variation and the disturbing potential for one satellite at the satellite altitude can be obtained:

\[ \Delta v = \frac{1}{v_0} T' \]  

(4)

From Eq.(4), the velocity difference for each of the two satellites is directly proportional to the geopotential difference at their respective locations:

\[ \Delta v_2 - \Delta v_1 = \frac{T'_2}{v_{02}} - \frac{T'_1}{v_{01}} \]  

(5)

The subscripts 1 and 2 represent the location of the two satellites traveling at the same altitude and on the same orbit plane. Eqs.(4) and (5) show that if the velocity variation is perfectly measured, then it can recover the Earth’s gravity field with high accuracy. However, there always exists measurement noise which prevents the accurate recovery of the Earth’s gravity field. Our work is to figure out how the measurement noise propagates to the geopotential estimate, so the determination of the actual disturbing potential is not of interest. For this kind of work, it is better to manipulate in spectral domain rather than in space domain, and the relationship between the space domain and the spectral domain needs to be derived.

The velocity variation can be expressed in the spectral domain using the spherical harmonics as [8]:

\[ \Delta v(\phi, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \bar{P}_{mn}(\sin \phi)(\Delta \bar{V}_{mn}^c \cos m\lambda + \Delta \bar{V}_{mn}^s \sin m\lambda) \]  

(6)

where \( \phi, \lambda \) are geodetic latitude and longitude, respectively; \( \Delta \bar{V}_{mn}^c \), \( \Delta \bar{V}_{mn}^s \) are the harmonic coefficient and \( \bar{P}_{mn}(\sin \phi) \) are the normalized associated Legendre function. The disturbing potential at the satellite altitude can be expressed in a similar way [9]:

\[ T(r, \phi, \lambda) = \frac{GM}{R_e} \sum_{n=2}^{\infty} \sum_{m=0}^{n-1} \left( \frac{R_e}{r} \right)^{n+1} \bar{P}_{nm}(\sin \phi)(\bar{C}_{nm} \cos m\lambda + \bar{S}_{nm} \sin m\lambda) \]  

(7)

in which \( R_e \) is the Earth’s average equatorial radius, \( r \) is the geocentric distance, \( GM \) is the gravitational coefficient, and the coefficients \( \bar{C}_{nm} \), \( \bar{S}_{nm} \) are the normalized spherical harmonic coefficients.

Defining \( T \) as the disturbing potential on the Earth’s surface, and \( T' \) as the disturbing gravity potential at the satellite altitude. The error variance of \( T \) may be defined as an average of the expectation of the squared error over the unit sphere. Due to the orthogonality, the error variance of \( T' \) has a similar relationship with
the degree error variance of $T$ as follows:

$$\sigma^2(\delta T') = \sum_{n=2}^{\infty} \left( \frac{R}{r} \right)^{2n+2} \sigma^2_n(\delta T)$$  

(8)

where the degree error variance of $T$ is given by:

$$\sigma^2_n(\delta T) = \sum_{n=0}^{\infty} (\delta C^2_{mn} + \delta S^2_{mn})$$  

(9)

in which $\delta C^2_{mn}$, $\delta S^2_{mn}$ are the degree error variance of potential coefficients.

The error variance of the velocity variation is obtained in the same way. After substituting Eq.(6) and Eq.(7) into Eq.(4), the relationship between the degree error variance of velocity and gravity is given by:

$$\sigma^2_n(\delta v) = \frac{GM}{R_e} \left( \frac{R}{r} \right)^{2n+2} \sigma^2_n(\delta T)$$  

(10)

For the SST-II, the range-rate observable is the velocity difference component along the line-of-sight. For small fixed separation angle $\theta$ and the circular reference orbit, this observable can be approximated as the velocity difference between the two satellites:

$$\hat{\rho} = (v_1 - v_2) \cdot \hat{e}_{12} = (|v_1| - |v_2|) \cos \frac{\theta}{2}$$  

(11)

where $\hat{e}_{12}$ is unit vector of the line-of-sight. The range-rate $\hat{\rho}$ is not only a function of the separation distance but also a function of direction. However, it is difficult to retain the directional property through spectral mapping, so an average value is used instead. The root-mean-square of range-rate over all possible directions is defined as:

$$\overline{\rho} \approx \frac{1}{4\pi} \int_0^{2\pi} (\Delta v_1 - \Delta v_2)^2 \, d\alpha$$  

(12)

The variance of this root-mean-square value over the unit sphere $\Omega$ is given by:

$$\sigma^2(\overline{\rho}) = \frac{1}{4\pi} \int_0^{2\pi} \overline{\rho}^2 \, d\Omega$$  

$$= \sigma^2(\Delta v) - \text{cov}(\Delta v_1, \Delta v_2)$$  

(13)

Assuming the velocity variation $\Delta v$ is isotropic and homogeneous on the sphere. The covariance function can be expanded as:

$$\text{cov}(\Delta v_1, \Delta v_2) = \sum_{n=2}^{\infty} \sigma^2_n(\delta v) P_n(\cos \theta)$$  

(14)

where $P_n(\cos \theta)$ is the Legendre function, $\theta$ is the separation angle between two satellites. The relationship between $\theta$ and the satellites separation $d$ can be given by:

$$\cos \theta = 1 - d^2 / 2r^2$$  

(15)

According to Eqs.(13) and (14), the degree variance of the range-rate is then related to that of the velocity variance as:

$$\sigma^2_n(\delta \phi) = \sigma^2_n(\delta v) [1 - P_n(\cos \theta)]$$  

(16)

Since the factor, $1 - P_n(\cos \theta)$ is less than 1 in most degree, the range-rate error is lower than the velocity error. This is one of the advantages of using the inter-satellite range-rate measurements instead of the velocity measurements.

Substituting Eq.(16) into Eq.(10) yields the spectral relationship between the range-rate error and the gravity potential error as:

$$\sigma^2(\delta r) = \frac{1}{1 - P_n(\cos \theta) \frac{GM}{R_e}} \left( \frac{r}{R_e} \right)^{2n+2} \sigma^2(\delta \phi)$$  

(17)

Assuming that the measurement error of the range-rate is white noise, the variance of which is $\sigma^2_{\rho}$, then the measurement noise spectrum is:

$$\sigma^2_{\rho}(\delta r) = \sigma^2_{\rho} / N_{\text{max}}$$  

(18)

where $N_{\text{max}}$ is the maximum degree of the Earth’s gravity model, which was obtained according to the Nyquist criterion. The approximate formula is:

$$N_{\text{max}} = \frac{\pi}{\delta t} \sqrt{\frac{a^3}{GM}}$$  

(19)

where $a$ is the semimajor axis of satellite orbit and $\delta t$ is time sampling interval.

Substituting Eq.(18) into Eq.(17) yields the relationship between the measurement error variance of the range-rate and the degree variance of disturbing potential can be obtained:

$$\sigma^2(\delta r) = \frac{1}{1 - P_n(\cos \theta) \frac{GM}{R_e}} \left( \frac{r}{R_e} \right)^{2n+2} \sigma^2_{\rho} N_{\text{max}}$$  

(20)

If the error variance of the range-rate is available, the accuracy of recovery of Earth’s gravity can be obtained from the above equation.

## 2 Simulation and analysis

### 2.1 Separation

According to Eq.(20), when the orbital altitude and
the accuracy of range-rate measurements are given, the performance of SST is mainly dependent on the term $1/(1-P_n(cos\theta))$ theoretically. This is a weighting function, which determines how much the SST noise is transformed into the gravity error. The smaller this weighting function is, the higher the accuracy of the gravity field recovery is.

Fig.1 shows the variation of the weighting function for four separations. Higher value means higher error in gravity solution for the same level of noise. In the low degrees, the larger the separation distances, the lower the values of the weighting function are. But in the high degrees, the weighting function may have a higher value than the others when the separation distance increase, which mainly manifests as a fluctuation change with the separation. Obviously, it is very necessary to select a suitable separation for the SST-II mission.

Fig.2 shows the effect of the separation distance on the gravity recovery, four numerical simulations were performed with different separations. Fig.2 shows that the geoid height errors vary with different separations of the satellite pair (450 km of satellite altitude, $1.0 \times 10^{-6}$ m/s accuracy of range-rate measurements, and 5s of sampling interval). From Fig.2, we can see that: (1) A longer separation is better for the low degree gravity recovery, but the accuracy of gravity recovery cannot be equally improved by the separation which is equally increased. The largest difference is found to occur between the separation of 100 km and 300 km, while the smallest difference corresponds to the separation between 500 km and 700 km. When the separation increases from 500 km to 700 km, the effect for improving the gravity field recovery accuracy is not obvious. When the separation is more than 500 km, it is no longer helpful for improving the accuracy of geopotential. (2) According to Kaula’s rule, the maximum degrees of the potential coefficients corresponding to the separations of 100 km, 300 km, 500 km and 700 km obtained in gravity field recovery were 108, 112, 110 and 111, respectively. It is within a variation of 4 degrees, which means that the variation of separation has little influence on spatial resolution.

2.2 Orbital altitude

Due to the attenuation effect of the gravitational field with increasing altitude, a lower altitude is preferred for gravity estimation in general. Fig.3 shows the geoid height errors vary with different orbital altitude (220 km of separation distance, $1.0 \times 10^{-6}$ m/s accuracy of range-rate measurements, and 5s of sampling interval). Fig.3 also shows that a lower satellite altitude is better for the high degree gravity recovery. According to Kaula’s rule, the maximum degrees of the potential coefficients corresponding to the altitudes of 300 km, 400 km and 500 km obtained in gravity field
recovery were 104, 125 and 155, respectively. The maximum degrees increased more than 20 degrees with the altitude decrease per 100km. It means that a lower satellite altitude is beneficial to improve the spatial resolution of gravity field recovery.

2.3 Range-rate measurements

When the orbital altitude and the separation are given, the performance of SST is mainly dependent on the accuracy of range-rate measurements. Fig.4 shows that the geoid height errors vary with different accuracy of range-rate measurements (220 km of separation distance, 450 km of satellite altitude, and 5s of sampling interval), and higher accuracy of range-rate measurements is helpful to improve both the accuracy of potential coefficient and the spatial resolution of gravity recovery. According to Kaula's rule, the maximum degrees of the potential coefficients corresponding to the accuracies of range-rate measurements of $1.0 \times 10^{-5}$ m/s, $1.0 \times 10^{-6}$ m/s, and $1.0 \times 10^{-7}$ m/s obtained in gravity field recovery are 85, 113 and 142, respectively. The maximum degrees increased about 28 degrees as the accuracy of range-rate measurements increased one order of magnitude.

![Fig.4 Geoid height errors vary with different accuracy of range-rate measurements](image)

2.4 Comparison and validation

Taking the index parameters of the GRACE mission for example, the performance of a reference mode (220 km of separation distance, 450 km of satellite altitude, $1.0 \times 10^{-6}$ m/s accuracy of range-rate measurements, and 5s of sampling interval) was simulated. Table 1 gives the geoid height error generated from the reference mode. Fig.5 shows the comparison of cumulative geoid height errors derived from the reference mode and the Earth's gravity field models of EGM96, GGM02S and EIGEN-GRACE02S.

| Harmonic degree | Geoid height error by degree | Cumulative geoid height error |
|-----------------|-----------------------------|-----------------------------|
| $n=2$           | $<0.001$ 0                  | /                           |
| $3<=n<=10$      | $<0.000$ 7                  | $<0.001$ 8                  |
| $10<=n<=70$     | $<0.004$ 3                  | $<0.013$ 3                  |
| $70<=n<=100$    | $<0.027$ 7                  | $<0.080$ 1                  |
| $100<=n<=120$   | $<0.105$ 7                  | $<0.297$ 8                  |

Table 1 shows that the geoid error of the reference mode cumulated up to degree 100 reached about 8cm, which is better than the accuracy computed in reference [11] (geoid height error cumulated up to degree 100 was about 42cm, only 30 days of GRACE range-rate measurements were used). Fig.5 shows that the error curve of the reference mode is similar to that of GGM02S, but the expected gravity recovery accuracy of the reference mode is slightly lower than the models of GGM02S and EIGEN-GRACE02S. After analyzing the causes, we think it is related to the observation data that is used in gravity recovery. The observation data used in the reference mode was only the range-rate measurements, while the GGM02S and EIGEN-GRACE02S models were computed from the combined observation data, such as range and range-rate measurements, and the SST measurements from GPS satellites as well. Therefore, the more the observation data are used, the higher the accuracy of gravity field recovery can be.

![Fig.5 Cumulative geoid height errors of reference mode and comparison models](image)
3 Conclusion

The expected gravity recovery accuracy simulated in this paper was slightly lower than those models obtained from GRACE data, but it basically reflects the performance of satellite separation, orbital altitude, and accuracy of range-rate measurements in gravity field recovery. The simulation analysis models were established based on some reasonable approximation and simplification, for example, the circular polar orbit and white noise in range-rate measurements were assumed, and the non-gravitational forces were ignored. For the successful implementation of our gravity exploring satellite mission, the analysis models must be improved and perfected in future work.

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