Modified Korteweg–de Vries Hierarchies in Multiple–Times Variables and the Solutions of Modified Boussinesq Equations

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We study solitary–wave and kink–wave solutions of a modified Boussinesq equation through a multiple–time reductive perturbation method. We use appropriated modified Korteweg–de Vries hierarchies to eliminate secular producing terms in each order of the perturbative scheme. We show that the multiple–time variables needed to obtain a regular perturbative series are completely determined by the associated linear theory in the case of a solitary–wave solution, but requires the knowledge of each order of the perturbative series in the case of a kink–wave solution. These appropriate multiple–time variables allow us to show that the solitary–wave as well as the kink–wave solutions of the modified Boussinesq equation are actually respectively a solitary–wave and a kink–wave satisfying all the equations of suitable modified Korteweg–de Vries hierarchies.

I. INTRODUCTION

In this paper we study two model equations of the modified Boussinesq type

\[ u_{tt} - u_{xx} + u_{xxxx} + 2\alpha (u^p)_{xx} = 0, \quad (1) \]

with \( u(x, t) \) a one–dimensional field, \( \alpha = \pm 1 \), \( p = 3 \) and subscripts denoting partial differentiation. These equations approximately describe the propagation of waves in certain nonlinear dispersive systems. For example they appear when we study the continuous limit of a Fermi–Pasta–Ulam dynamical system with cubic nonlinearity (Dodd et al. 1982). They can appear also governing the evolution of long internal waves of moderate amplitude, (Ablowitz et al. 1973) or describing the dynamic of a stretched string (Ablowitz et al. 1973; Mott 1973).

They are considered as intermediate long–wave equations since they represent an intermediate dynamic, in complexity and completeness, situated between the complete dynamic of the full initial equations describing any wave number and any amplitude, and some strong long–waves and small–amplitude limits. Their appropriated long–wave and small–amplitude limit with a further restriction to unidirectional propagation yields the modified Korteweg–de Vries equation (mKdV).

Asymptotic methods are very often employed and very useful to study problems of this type. Hence, to study the evolution of long–waves of equations (1), we will consider a perturbative scheme based on the reductive perturbation method of Taniuti (Taniuti 1974). We will include the classical slow space–variable \( \xi \), but we will modify it by introducing an infinite number of slow time–variables: \( \tau_3, \tau_5, \tau_7, \ldots \). As we are going to see, by using these slow variables together with all the equations of an appropriated modified Korteweg–de Vries hierarchy (Chern & Peng 1979), we can obtain the solitary–wave solution of the modified Boussinesq equation (1) with \( \alpha = 1 \) (for shorteness mBI)

\[ u(x, t) = k \sech \left\{ k \left( x - \sqrt{1-k^2} \right) t \right\}, \quad (2) \]

as well as the kink–wave solution of the modified Boussinesq equation (1) with \( \alpha = -1 \) (mBII)

\[ u(x, t) = k \tanh \left\{ k \left( x - \sqrt{1+2k^2} \right) t \right\}. \quad (3) \]

The solutions (2) and (3), written in the laboratory coordinates \((x, y)\), are actually built up respectively from the solitary–wave solution and the kink–wave solution of the whole set of equations of appropriated mKdV hierarchies written in the slow space–variable \( \xi \) and in each one of the slow time–variables \( \tau_3, \tau_5, \tau_7, \ldots \).

These results follow from the general long–wave perturbation theory and from the requirement of uniformity for large time of the associated perturbative series. This last fact makes the perturbative series truncates for solutions of type (2) or (3), rendering thus exact solutions for the modified Boussinesq equations. Furthermore, we will show that the elimination of the secular producing terms in the perturbative series linked to the solution (2), is completely accounted for only the linear theory associated to (1), which allows us to know ”a priori” all the constants which
define the slow times–variables $\tau^s$ in function of $t$. These properly normalized slow time coordinates automatically give us a perturbative series which is free of secularities (uniform expansion).

On the other hand, to solve the same problem for the solution (3), that is, to eliminate the secular producing terms linked to solution (3), requires the knowledge of each term of the perturbative series.

This paper is organized as follows. In Sec.II the multiple time formalism is introduced for the modified Boussinesq equations (1), the first few evolution equations are obtained and the problems associated with them are exhibited. In Sec.III we discuss how the mKdV hierarchy shows up. In Sec.IV we show how they can be used, in the case $\alpha = 1$, to eliminate the soliton related secularities of the evolution equations for the higher–order terms of the wave fields. In Sec.V by returning from the slow variables to the laboratory coordinates we obtain the above mentioned relation between the solitary–wave of mBI and the corresponding modified Korteweg–de Vries hierarchy. Sec.VI is consecrated to the case of mBII ($\alpha = -1$). In Sec.VII we give a general proof that the symmetries of time derivatives lead to the modified Korteweg–de Vries hierarchy. Finally in Sec.VIII we summarize and discuss the results obtained.

II. THE MULTIPLE TIME FORMALISME: MIXED–SECULAR TERMS

In order to study the far–field dynamics of long–wave solutions of eq.(1), we will need to define slow space and time variables. A small parameter $\epsilon$ giving the long–wave character of the studied solution is introduced via the definition

$$k = \epsilon \kappa,$$

where $k$ is the wave number and $\kappa$ a parameter of order one. Accordingly, we define a slow space variable

$$\xi = \epsilon (x - t),$$

as well as an infinity of slow time coordinates

$$\tau_3 = \epsilon^3 t, \quad \tau_5 = \epsilon^5 t, \quad \tau_7 = \epsilon^7 t, ....$$

Consequently, we have

$$\frac{\partial}{\partial x} = \epsilon \frac{\partial}{\partial \xi},$$

and

$$\frac{\partial}{\partial t} = -\epsilon \frac{\partial}{\partial \xi} + \epsilon^3 \frac{\partial}{\partial \tau_3} + \epsilon^5 \frac{\partial}{\partial \tau_5} + \epsilon^7 \frac{\partial}{\partial \tau_7} + ....$$

Moreover, we consider a small–amplitude solution of (1) and we make the expansion

$$u = \epsilon \hat{u} = \epsilon (u_0 + \epsilon^2 u_2 + \epsilon^4 u_4 + ...).$$

We suppose that $u_{2n} = u_{2n}(\xi, \tau_3, \tau_5, ...)$ for $n = 0, 1, 2, ...$ which corresponds to an extension of the function $u$ in the Sandri’s sense (Sandri 1965). Substituting eqs (5), (6) and (7) into the Boussinesq equations (1), the resulting expression, up to terms of order $\epsilon^4$, is:

\[
-2 \frac{\partial^2}{\partial \xi \partial \tau_3} + \frac{\partial^4}{\partial \xi^4} + \epsilon^2 \left( \frac{\partial^2}{\partial \tau_3^2} - 2 \frac{\partial^2}{\partial \xi \partial \tau_5} \right) + \epsilon^4 \left( -2 \frac{\partial^2}{\partial \xi \partial \tau_7} + 2 \frac{\partial^2}{\partial \tau_3 \partial \tau_5} \right) + ... \hat{u}
\]

\[
2\alpha \frac{\partial^2}{\partial \xi^2} \left[ u_0^3 + \epsilon^2 (3u_0^2u_2) + \epsilon^4 (3u_0^2u_4 + 3u_0u_2^2) + ... \right] = 0,
\]

At order $\epsilon^0$, after an integration in $\xi$, we get

\[
-2 \frac{\partial u_0}{\partial \tau_3} + \frac{\partial^3 u_0}{\partial \xi^3} + 6\alpha u_0 \frac{\partial u_0}{\partial \xi} = 0,
\]

which is the mKdV equation.

At order $\epsilon^2$, eq. (10) yields, using (3) and integrating once in $\xi$
\[-2 \frac{\partial u_2}{\partial \tau_3} + \frac{\partial^3 u_2}{\partial \xi^3} + 6\alpha \frac{\partial (u_0^2 u_2)}{\partial \xi} = \frac{2 \partial u_0}{\partial \tau_5} - \frac{1}{4} \frac{\partial^5 u_0}{\partial \xi^5} - 3\alpha \left( \frac{\partial u_0}{\partial \xi} \right)^3 - 9\alpha u_0 \frac{\partial u_0}{\partial \xi} \frac{\partial^2 u_0}{\partial \xi^2} \]

\[-3\alpha u_0^2 \frac{\partial^3 u_0}{\partial \xi^3} - 9\alpha^2 u_0^4 \frac{\partial u_0}{\partial \xi}.\]

Equation (12) is a linearized inhomogeneous mKdV whose general solution consists of a sum of a general solution to the homogeneous equation and a particular solution to the nonhomogeneous equation. As it stands, (12), presents two problems. First, the inhomogeneity (source term) is unknown because the evolution of $u_0$ in the time $\tau_5$ is not known. The second problem is related to nonuniformity of the expansion for $u$. When we considerer a soliton type solution of eq.(11), case $\alpha = 1$, the term $\frac{\partial^3 u}{\partial \xi^3}$ is proportional to $\frac{\partial u}{\partial \xi}$ which is a solution of the associated homogeneous equation. Hence the general solution $u_2$ of (12) contains a term proportional to $\tau_3 \frac{\partial u_0}{\partial \xi}$ which gives rise to a nonuniformity in the perturbative series (mixed-secular term).

For $mBII$, case $\alpha = -1$, we have kink-type solutions to eq. (11). Solutions of type (2) exist but are complex and we will not consider them. In this case the linear term of eq.(12), and actually some nonlinear ones, produces secularity as well.

In the next sections we will deal with these two problems.

III. THE RISE OF THE MODIFIED KORTEWEG–DE VRIES HIERARCHY

As we have seen, the field $u_0$ satisfies the mKdV equation in the time $\tau_3$. The evolution of the same field $u_0$ in any of the higher order times $\tau_{2n+1}$ can be obtained in the following way (Kraenkel et al. 1995).

First, to have a well ordered perturbative scheme we impose that each one of the equations for $u_{0,\tau_{2n+1}}$ be $\epsilon$–independent when passing from the slow variables ($u_0, \xi$, $\tau_{2n+1}$) to the laboratory coordinates ($u, x, t$). This step selects all possible terms which can appear in $u_{0,\tau_{2n+1}}$. For instance, the evolution of $u_0$ in $\tau_5$ is restricted to the form

$$u_{0,\tau_5} = au_{0,3\xi} + bu_0^2 u_{0,3\xi} + cu_0 u_{0,\xi} u_{0,2\xi} + d(u_0,\xi)^3 + eu^4_0 u_{0,\xi} + fu_0 u_{0,4\xi} + gu_0^2 u_{0,2\xi} + hu_0 \xi u_{0,3\xi} + iv^3_0 + j(u_{0,2\xi})^3,$$

where $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$, $i$ and $j$ are unknown constants. Then, by imposing the natural (in the multiple time formalism) compatibility condition

$$\left(u_{0,\tau_5}\right)_{\tau_{2n+1}} = \left(u_{0,\tau_{2n+1}}\right)_{\tau_3},$$

with $u_0$ satisfying mKdV in $\tau_3$, it is possible to determine any $u_{0,\tau_{2n+1}}$ i.e to determine all the contants appearing in $u_{0,\tau_{2n+1}}$.

As it will be shown in Sec.VI, the resulting equations are those of the mKdV hierarchy. In particular, for $u_{0,\tau_5}$ and $u_{0,\tau_7}$, using the mKdV in its canonical form (for shorteness mKdVI) with $\alpha = 1$

$$u_{0,\tau_5} + u_{0,3\xi} + 6u_0^2 u_{0,\xi} = 0,$$

we obtain

$$u_{0,\tau_5} = \alpha_5 \left( u_{0,5\xi} + 10u_0^2 u_{0,3\xi} + 40u_0 u_{0,\xi} u_{0,2\xi} + 10(u_0,\xi)^3 + 30u_0^4 u_{0,\xi} \right),$$

and

$$u_{0,\tau_7} = -\alpha_7 \left( + u_{0,7\xi} + 84u_0 u_{0,\xi} u_{0,4\xi} + 560u_0^3 u_{0,\xi} u_{0,2\xi} + 14u_0^2 u_{0,5\xi} + 140u_0 u_{0,2\xi} u_{0,3\xi} + 126u_0 \xi (u_0,\xi)^2 + 182u_0 \xi (u_0,2\xi)^2 + 70u_0^4 u_{0,3\xi} + 420(u_0)^2 (u_0,\xi)^3 + 140u_0 \xi (u_0)^6 \right).$$

For the alternative form, mKdVII ($\alpha = -1$)
At first order, this definition of $\tau$ gives the operator (8) now reads

$$u_{0,\tau_5} + u_{0,3\xi} - 6u_0^2u_{0,\xi} = 0,$$

(18)

$u_{0,\tau_5}$ and $u_{0,\tau_7}$ are given by

$$u_{0,\tau_5} = \beta_5\left(u_{0,5\xi} - 10u_0^2u_{0,3\xi} - 40u_0u_{0,\xi}u_{0,2\xi} - 10(u_{0,\xi})^3 + 30u_0^4u_{0,\xi}\right),$$

(19)

$$u_{0,\tau_7} = \beta_7\left(-u_{0,7\xi} + 84u_0u_{0,\xi}u_{0,4\xi} - 560u_0^3u_{0,\xi}u_{0,2\xi} + 14u_0^2u_{0,5\xi} + 140u_0u_{0,2\xi}u_{0,3\xi} + 126u_{0,3\xi}(u_{0,\xi})^2 + 182u_{0,\xi}(u_{0,2\xi})^2 - 70(u_0)^4u_{0,3\xi} - 420(u_0)^2(u_{0,\xi})^3 + 140u_{0,\xi}(u_0)^6\right).$$

(20)

The coefficients $\alpha_5$, $\alpha_7$, $\beta_5$, $\beta_7$, are free parameters which are not determined by the algebraic system originated from eq.(14). These free parameters are related to different possible normalizations of the slow time variables and we will use them to obtain a secular free perturbative solution of (6).

We will see in Sec.V and VI that the genesis of the values of these parameters are rather different for the perturbation theory associated with the two cases $\alpha = 1$ or $\alpha = -1$.

**IV. HIGHER ORDER EVOLUTION EQUATIONS FOR MODIFIED BOUSSINESQ I**

Let us consider the perturbation theory associated with mBI ($\alpha = 1$ in eq.(3)) with $\tau_5$, $\tau_7$, ..., as defined by (11) and $\tau_3$ redefined by

$$\tau_3 = -\frac{\epsilon^3 t}{2}.$$  

(21)

Hence the operator (8) now reads

$$\frac{\partial}{\partial t} = -\epsilon \frac{\partial}{\partial \xi} - \frac{1}{2}\epsilon^3 \frac{\partial}{\partial \tau_3} + \frac{1}{2}\epsilon^5 \frac{\partial}{\partial \tau_5} + \frac{1}{7}\epsilon^7 \frac{\partial}{\partial \tau_7} + \ldots$$

(22)

At first order, this definition of $\tau_3$ gives the mKdVI in canonical form (17). At order $\epsilon^2$ we obtain the following equation:

$$L_I(u_2) = 2u_{0,\tau_5} - \frac{1}{4} \int_{-\infty}^{\xi} u_{0,2\tau_5} d\xi',$$

(23)

where $L_I$ is the linearized mKdVI operator defined by

$$L_I(v) = v_{\tau_5} + v_{3\xi} + 6(u_0^2 v)_{\xi}.$$  

(24)

Substituting $u_{0,\tau_3}$ and $u_{0,\tau_5}$ from (13) and (14) we have

$$L_I(u_2) = 2\left(\alpha_5 - \frac{1}{8}\right)u_{0,5\xi} + (20\alpha_5 - 3)u_0^2u_{0,3\xi} + (80\alpha_5 - 9)u_0u_{0,\xi}u_{0,2\xi} + (20\alpha_5 - 3)(u_{0,\xi})^3 + (60\alpha_5 - 9)u_0^2u_{0,\xi}.$$  

(25)

If we assume the solution of the mKdVI (eq.(13)) to be the solitary–wave solution

$$u_0 = \kappa \text{sech} (\kappa \xi - \kappa^3 \tau_3 + \theta),$$

(26)

where $\theta$ is a phase, it is easy to see that only the term $u_{0,5\xi}$ is a secular–producing term. It can be eliminated if we choose $\alpha_5 = \frac{1}{8}$. In this case, eq.(21) becomes

$$L_I(u_2) = -\frac{1}{2}u_0^2u_{0,3\xi} + u_0u_{0,\xi}u_{0,2\xi} - \frac{1}{2}(u_{0,\xi})^3 - \frac{3}{2}u_0^4u_{0,\xi}.$$  

(27)

Using for $u_0$ the solitary–wave solution (26), we see that the right–hand side of eq.(27) vanishes, leading to
This is a general result that will repeat at any higher order. For respectively by eqs. (4), (5), (6) and (21) and we have found later that

\[ u_2 = 0. \]  

(29)

At order \( \epsilon^4 \), and already assuming that \( u_2 = 0 \), we obtain

\[ \mathcal{L}_I(u_4) = 2u_{0,\tau_7} + \int_{-\infty}^{\tau_7} u_{0,\tau_3,\tau_5,\tau_7} d\xi'. \]  

(30)

Using equations (15) and (16), with \( \alpha_5 = \frac{1}{8} \), to express respectively \( u_{0,\tau_3} \) and \( u_{0,\tau_5} \) we obtain

\[ \mathcal{L}_I(u_4) = 2 \left( u_{0,\tau_7} - \frac{1}{16} u_{0,7\xi} \right) - \frac{65}{4} u_{0,3\xi}(u_{0,\xi})^2 - \frac{35}{2} u_{0} u_{0,2\xi} u_{0,3\xi} - 10 u_{0,0,2\xi} u_{0,4\xi} \]
\[ - 2 u_{0,5\xi} u_{0}^2 - \frac{45}{2} u_{0,5\xi}(u_{0,2\xi})^2 - \frac{105}{2} u_{0,0} u_{0,2\xi}^2 - 75 u_{0,0,0,2\xi} u_{0}^3 - \frac{45}{4} u_{0,3\xi} u_{0}^4 \]
\[ - \frac{45}{4} u_{0,0,0,0} u_{0}^6. \]  

(31)

Now the source term proportional to \( u_{0,7\xi} \) is the only resonant. Then, in the same way we did before, we first use the higher mKdVI, eq.(17), to express \( u_{0,\tau_7} \) and then we choose the free parameter \( \alpha_7 \) in such a way to eliminate the resonant term \( u_{0,7\xi} \). This choice corresponds to \( \alpha_7 = -\frac{1}{16} \), which brings eq.(31) to the form

\[ \mathcal{L}_I(u_4) = \frac{1}{2} u_{0} u_{0,0,0,2\xi} u_{0,4\xi} - 5 u_{0,0,5\xi} u_{0} u_{0,2\xi}^3 - \frac{1}{4} u_{0,5\xi} u_{0}^2 - \frac{1}{2} u_{0,3\xi}(u_{0,\xi})^2 \]
\[ + \frac{1}{4} u_{0,0,0,0} u_{0}^6. \]  

(32)

Substituting again the solitary–wave solution (26) for \( u_0 \), we see that the nonhomogeneous term of eq.(32) vanishes, leading to

\[ \mathcal{L}_I(u_4) = 0. \]  

(33)

Again, we take the trivial solution

\[ u_4 = 0. \]  

(34)

This is a general result that will repeat at any higher order. For \( n \geq 1 \) the evolution equation for \( u_{2n} \), after using the mKdVI hierarchy equations to express \( u_{0,7\tau_n+1} \) and after eliminating the secular producing term coming from the solitary–wave solution (24), is given by an homogeneous linearized mKdVI equation.

Consequently, the solution \( u_{2n} = 0 \), for \( n \geq 1 \), can be assumed for any higher order.

V. BACK TO THE LABORATORY COORDINATES. CONNECTION TO THE DISPERSION RELATION

Actually, to obtain a perturbative scheme free of secular producing terms at any higher order, we assume that \( u_0 \) is the solitary–wave solution of all the equations of the mKdVI hierarchy, each one in a different slow–time variable. Such a solution may be obtained and is given by

\[ u_0(\xi, \tau_3, \tau_5, \tau_7, ...) = \kappa \sech \left[ \kappa \xi - \kappa^3 \tau_3 + \kappa^5 \alpha_5 \tau_5 - \kappa^7 \alpha_7 \tau_7 + ... \right]. \]  

(35)

First, recall that we have expanded \( u \) according to eq.(6). Thereafter, we have found a particular solution in which \( u_{2n} = 0 \) for \( n \geq 1 \). Consequently expansion (8) truncates, leading to an exact solution of the form

\[ u = \epsilon u_0, \]  

(36)

with \( u_0 \) given by eq.(33). Moreover, the slow variables \( (\kappa, \xi, \tau_{2n+1}) \) are related to the laboratory ones \( (k, x, t) \), respectively by eqs. (6), (3), (8) and (21) and we have found later that

\[ \alpha_5 = \frac{1}{8}, \quad \alpha_7 = -\frac{1}{16}, ... \]  

(37)
Then, in the laboratory coordinates, the exact solution (36) is written
\[ u(k, x, t) = k \operatorname{sech} \left[ x - \left( 1 - \frac{1}{2} k^2 - \frac{1}{8} k^4 - \frac{1}{16} k^6 - \ldots \right) t \right]. \]
(38)

Now, the series appearing inside the parenthesis can be summed:
\[ 1 - \frac{1}{2} k^2 - \frac{1}{8} k^4 - \frac{1}{16} k^6 - \ldots = \sqrt{1 - k^2}. \]
(39)

Consequently, we get
\[ u = k \operatorname{sech} k \left( x - \sqrt{1 - k^2} t \right), \]
(40)
which is the solitary-wave solution of mBI.

Let us now take the linear mBI dispersion relation
\[ \omega(k) = k \sqrt{1 + k^2}. \]
(41)

Its long-wave expansion \((k = \epsilon \kappa)\) is given by
\[ \omega(k) = \epsilon \kappa + \frac{1}{2} \epsilon^3 \kappa^3 - \frac{1}{8} \epsilon^5 \kappa^5 + \frac{1}{16} \epsilon^7 \kappa^7 + \ldots. \]
(42)

In passing we notice that the absolute value of the coefficients of the expansion coincide exactly with those found: \(\alpha_3\) to obtain the mKdVI in canonical form and \(\alpha_5, \alpha_7, \ldots\), necessary to eliminate the secular producing term in each order of the perturbative scheme.

With this expansion, the solution of the associated linear mBI reads
\[ u = \exp i \left[ \kappa \epsilon (x - t) - \frac{1}{2} \epsilon^3 \kappa^3 t + \frac{1}{8} \epsilon^5 \kappa^5 t - \frac{1}{16} \epsilon^7 \kappa^7 t + \ldots \right]. \]
(43)
Therefore, if we define from the beginning, as given by this expression, the properly normalized slow time coordinates
\[ \tau_3 = -\frac{1}{2} \epsilon^3 t, \quad \tau_5 = \frac{1}{8} \epsilon^5 t, \quad \tau_7 = -\frac{1}{16} \epsilon^7 t, \quad \ldots, \]
(44)
the mKdVI \((13)\) will be obtained at first order and the resulting perturbative theory will be automatically free of secular producing terms for the soliton solution.

VI. HIGHER ORDER EVOLUTION EQUATION FOR THE MODIFIED BOUSSINESQ II

We consider now the case \(\alpha = -1\) in eq.(1). We assume the same operators \(\partial_x\) and \(\partial_t\) given by (7) and (22), and the same expansion for \(u\). We obtain at order \(\epsilon^0\) and \(\epsilon^2\) the following equations
\[ u_{0,\tau_3} + u_{0,3\xi} - 6u_{0}^2 u_{0,\xi} = 0, \]
(45)
which is mKdVII, and
\[ \mathcal{L}_{u_2}(u_2) = 2u_{0,\tau_5} - \frac{1}{4} \int_{-\infty}^{\xi} u_{0,2\tau_3} d\xi', \]
(46)
where \(\mathcal{L}_{u_2}\) is the linearized mKdVII operator. Using eq.(45) to express \(u_{0,\tau_3}\) and eq.(19) to express \(u_{0,\tau_5}\) in (46) we obtain
\[ \mathcal{L}_{u_2}(u_2) = \left( 2\beta_5 - \frac{1}{4} \right) u_{0,5\xi} - \left( 20\beta_5 - 3 \right) u_{0}^2 u_{0,3\xi} - \left( 80\beta_5 - 9 \right) u_{0} u_{0,\xi} u_{0,2\xi} - \left( 20\beta_5 - 3 \right) (u_{0,\xi})^3 + \left( 60\beta_5 - 9 \right) u_{0,\xi}^4. \]
(47)
In this case the real solution of mKdVII is of kink–wave type. It reads

\[ u_0(x, t) = \kappa \tanh (\kappa \xi + 2\kappa^3 \tau_3 + \theta) \, . \]  

(48)

When we substitute (48) in (47), resonant terms appear. Contrary to the case of mBI, they do not come only from the linear term, but also from the nonlinear ones. They are proportional to

\[ u_{0,\xi} = \kappa \text{sech}^2 (\kappa \xi + 2\kappa^3 \tau_3 + \theta), \]

and we have

\[ \mathcal{L}_{II}(u_2) = -\kappa \text{sech}^2 (\kappa \xi + 2\kappa^3 \tau_3 + \theta) \left( 1 - 12\beta_5 \right). \]

(50)

Hence, the value

\[ \beta_5 = \frac{1}{12} \]  

(51)

eliminates the secular producing term at order \( \epsilon^2 \), and gives \( \mathcal{L}_{II}(u_2) = 0 \). We then assume the trivial solution \( u_2 = 0 \).

At order \( \epsilon^4 \) we have

\[ \mathcal{L}_{II}(u_4) = 2u_{0,\tau_7} + \int_{-\infty}^{\xi} u_{0,\tau_3,\tau_5} d\xi'. \]

(52)

Using (48), (49) and (21) to express respectively \( u_{0,\tau_3} \), \( u_{0,\tau_5} \) and \( u_{0,\tau_7} \) we obtain, (using (48))

\[ \mathcal{L}_{II}(u_4) = \kappa \text{sech}^2 (\kappa \xi + 2\kappa^3 \tau_3 + \theta) \left( 1 + 40\beta_7 \right). \]

(53)

We choose

\[ \beta_7 = -\frac{1}{40} \]

(54)

to eliminate the secular producing term at this order. Again, we take the trivial solution

\[ u_4 = 0. \]

(55)

Actually, as in the previous case, we can assume \( u_{2n} = 0 \) for \( n \geq 1 \) and we have the exact solution

\[ u = \epsilon^2 u_0, \]

(56)

where \( u_0 \) is the kink–wave solution to all equations of the mKdVII hierarchy, which reads

\[ u = \kappa \tanh \left( \kappa \xi + 2\kappa^3 \tau_3 + 6\kappa^5 \beta_5 \tau_5 + 20\kappa^7 \beta_7 \tau_7 + ... \right). \]

(57)

In the laboratory coordinates, the exact solution (57) is written as

\[ u = k \tanh k \left[ x - \left( 1 + k^2 - \frac{1}{2} k^4 + \frac{1}{2} k^6 - ... \right) t \right]. \]

(58)

The series appearing inside the parenthesis can be summed:

\[ 1 + k^2 - \frac{1}{2} k^4 + \frac{1}{2} k^6 - ... = \sqrt{1 + 2k^2}, \]

(59)

and we obtain the kink–wave solution of mBII

\[ u = k \tanh \left( kx - k \sqrt{1 + 2k^2} t \right). \]

(60)

To obtain (60) we used the definitions

\[ \tau_3 = -\frac{1}{2} \epsilon^3 t, \quad \tau_5 = \frac{1}{12} \epsilon^5 t, \quad \tau_7 = -\frac{1}{40} \epsilon^7 t, \quad ... . \]

(61)

In this case the coefficients are not those obtained from the long-wave expansion of the linear dispersion relation of mBII.
VII. COMMUTATIVITY OF TIME DERIVATIVES IMPLIES THE MKDV HIERARCHY: A GENERAL PROOF

In this section we give a general proof that the symmetries of time derivatives, with the scale invariance requirement, leads to the mKdV hierarchy in \(\tau_3, \tau_7, \ldots\), for a field which satisfies mKdV in \(\tau_3\). We do this for mKdVI.

Let \(M_n\) be a polynomial in terms of the form

\[ u^{a_0} u_{\xi}^{a_1} u_{2\xi}^{a_2} \ldots u_{l\xi}^{a_l}, \tag{62} \]

with \(a_i \in N\). We define the Rank \((R)\) of \(M_n\) as

\[ R(M_n) = \sum_{j=0}^{l} (1 + j)a_j. \tag{63} \]

Let us now consider our principle which requires that each one of the evolution equations

\[ u_{\tau_{2n-3}} = M_n, \quad n = 2, 3, \ldots, \tag{64} \]

be \(\epsilon\)-independent when passing from the slow \((u_0, \xi, \tau_{2n-1})\) to the laboratory coordinates \((u, x, t)\). It is easy to see that this is equivalent to require that

\[ u_{\tau_{2n-3}} = M_n, \quad n = 2, 3, \ldots, \tag{65} \]

with

\[ R(M_n) = 2n - 2, \quad n = 2, 3, \ldots. \tag{66} \]

Equation \((65)\) and the condition \((66)\) give us the correct terms in the expression of \(u_{\tau_{2n-3}}\) but they do not determine their coefficients.

Let us recall that the Maurer–Cartan equation associated with the group of \(2 \times 2\) real unimodular matrices (Chern & Peng 1979), leads to the mKdV hierarchy – in variables \(x, t\) – in the form

\[ u_t = u^{-1} R_{n+1, x}, \tag{67} \]

where

\[ R_0 = -1. \tag{68} \]

\(R_n\) for \(n \geq 1\) may be generated from the recursion formula

\[ u^{-1} R_{n+1, x} = \frac{1}{4} \left(u^{-1} R_{n, x}\right)_{2x} + \left(u R_n\right)_x. \tag{69} \]

Let us renormalize the coefficients of highest order derivative term in each one of the equations of the hierarchy to +1 or −1 by defining

\[ A_{n, x} = (-1)^n 4^n - 1 u^{-1} R_{n, x}. \tag{70} \]

Hence, we write the higher–order mKdVI equations

\[ u_t = A_{n-1, x} \quad n = 1, 2, \ldots, \tag{71} \]

with

\[ A_{n+1, x} = -A_{n, 3x} - 4 \left( u \int_{-\infty}^{x} u A_{n, \xi} d\xi \right) + u x \delta_{n, 0}, \quad n = 0, 1, \ldots, \tag{72} \]

where \(\delta_{n, 0}\) is the Kronecker symbol. With the above statements we establish the following theorem

THEOREM

If \((u_3)_{\tau_{2n-3}} = (u_{\tau_{2n-3}})_{\tau_3}\) with the invariance condition \(u_{\tau_{2n-3}} = M_n\) where
\[ M_3 = -u_3 \xi - 6u^2_2 u_\xi, \]  
(73)

and

\[ R(M_n) = 2n - 2, \]  
(74)

then we have

\[ M_n = A_{n-1, \xi}, \quad n = 1, 2, ..., \]  
(75)

where the \( A_n \) are the \( n^{th} \) conserved densities of mKdVI.

Proof:

The commutativity of time derivatives and the invariance condition lead to the linearized mKdVI for \( M_n \)

\[ M_{n, \tau_3} + M_{n, 3\xi} + 6\left(u^2 M_n\right)_{\xi} = 0. \]  
(76)

Thus we will prove that

\[ \left(A_{n-1, \xi}\right)_{\tau_3} + \left(A_{n-1, \xi}\right)_{3\xi} + 6\left(u^2 A_{n-1, \xi}\right)_{\xi} = 0. \]  
(77)

We proceed by induction. For \( n = 3 \), eq.(77) gives mKdVI, and (75) clearly holds if we assume that the arbitrary function of integration in \( \tau_3 \) that appears is zero. This is justified because each \( M_n \) must be a polynomial in terms of (62). Assuming that it holds for \( n - 2 \), we have

\[ \left(A_{n-2, \xi}\right)_{\tau_3} + \left(A_{n-2, \xi}\right)_{3\xi} + 6\left(u^2 A_{n-2, \xi}\right)_{\xi} = 0. \]  
(78)

We use (72), the mKdVI and the inductive hypothesis (78) to show – after some heavy algebra – that

\[ \left(A_{n-1, \xi}\right)_{\tau_3} + \left(A_{n-1, \xi}\right)_{3\xi} + 6\left(u^2 A_{n-1, \xi}\right)_{\xi} = 0. \]  
(79)

This proves the theorem.

VIII. CONCLUSION

We have applied a multiple–time version of the reductive perturbation method to study the solitary–wave type solution and the kink–wave type solution of two families of modified Boussinesq model equations.

In the first case we have eliminated the solitary–wave related secular producing terms through the use of the equations of an appropriated modified Korteweg–de Vries hierarchy. We have shown that the solitary–wave solution of the associated modified Boussinesq equation is given by a solitary–wave satisfying, in the slow variables, all the equations of the modified Korteweg–de Vries hierarchy. Accordingly, while the modified Korteweg–de Vries solitary–wave only depends on one slow variable \( \tau_3 \), the solitary–wave solution of the modified Boussinesq equation can be thought of as depending on the infinite slow time variables.

Hence, solitary–wave solutions of intermediate model equations, like the modified Boussinesq, contain complete information concerning all degrees of long–waves present in the system.

In the second case, we have eliminated the kink–wave related secular producing terms through the use of another appropriated modified Korteweg–de Vries hierarchy. We have shown that the kink–wave solution of the associated modified Boussinesq equation can be built from the kink–wave solution of all equations of the corresponding modified Korteweg–de Vries hierarchy.

The main difference between the solitary–wave case and the kink–wave case is situated in the formulae of tranformation giving the slow–time variables \( \tau^a \) in function of the laboratory time coordinate \( t \). In the first case a formula exists and is given by the long–wave expansion of the linear dispersion relation of the initial equation, independently of the associated perturbation theory.

In the second case such a formula does not exist and we know the \( \tau^a \) in function of \( t \) only "a posteriori", that is, by inspecting each order of the perturbative theory.

In the solitary–wave case the sources of secularities at each order of the associated perturbative series are the linear terms only. If we linearize the perturbative series \( (u_n = 0 \text{ for } n = 1, 2, ...) \) we obtain the equations of the associated
Fourier transform of the extended initial function (Sandri 1965) if, and only if, the definitions of $\tau^n$ like a function of $t$ was done according to the development of the linear dispersion relation for long–waves.

Hence, the requirement of a perturbative scheme free of solitary–wave related secularities and the existence of a compatible linear limit of the theory are completely equivalent.

In the kink–wave case, there are two sources of secularity at each order: the linear terms and also some nonlinear ones. Hence the elimination of the linear terms, or the compatibility with the associated Fourier theory is not sufficient to obtain a regular perturbative series. We must carry out a second renormalization of the kink–wave frequency which gives us the right coefficients transforming $\tau_{2n+1}$ in a function of $t$. In this paper (Sec.VI) we realized these two renormalizations in only one step, but they can be realized separately. In this case the second renormalization of the frequency clearly appears like a reminiscence of the celebrated Stokes’ hypothesis (Whitham 1974) on the frequency amplitude–dependence in water waves.

Let us remember that obstacles to asymptotic integrability can appear in the case of nonintegrable systems (Kodama & Mikhailov 1995). Such obstacles was exhibited in the case of the nonintegrable RLW equation (Kraenkel et al 1996). For integrable systems, like the one consider here, the multiple scale method will be able to handle both, the solitary–wave and the $N$–soliton related secularities since no obstacles to asymptotic integrability will be present.x

Finally in Sec.VII we have shown that the commutativity of time derivatives, together with the scale invariance requirement, lead to the mKdV hierarchy in $\tau_5$, $\tau_7$, ..., for a field which satisfies mKdV in $\tau_3$.

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