MEDIUM-SCALE CURVATURE FOR CAYLEY GRAPHS

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Abstract. We introduce a notion of Ricci curvature for Cayley graphs that can be called medium-scale because it is neither infinitesimal nor asymptotic, but based on a chosen finite radius parameter. For this definition, abelian groups are identically flat, and in the other direction we show that $\kappa \equiv 0$ implies the group is virtually abelian. In right-angled Artin groups, the curvature is zero on central elements and negative otherwise. On the other hand, we find nilpotent, CAT(0), and hyperbolic groups with points of positive curvature. We study dependence on generators and behavior under embeddings.

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1. Introduction

The emergence of a theory of large-scale negative curvature, or $\delta$-hyperbolicity, has provided an enormous breakthrough and a powerful tool for the theory of infinite groups. The existence of any $\delta \geq 0$ for which a Cayley graph satisfies this family of metric conditions implies a host of consequences for the geometry of the graph and the spaces acted on by the group. However, for large finite graphs, this theory is not applicable—every bounded-diameter metric space is $\delta$-hyperbolic for $\delta \geq \text{diam}$, so hyperbolicity provides no information. In the analysis of large but finite graphs or networks, we need another tool.

In the 2000s, Yann Ollivier defined a notion of metric Ricci curvature—at finite scales, not asymptotic—suitable for graphs and other non-manifold geometries [7, 8, 9, 10]. To do this, he offered a geometric interpretation of classical Ricci curvature as follows: curvature measures the extent to which corresponding points on spheres are closer together or farther apart than the centers of the spheres. (Negative Ricci curvature occurs when corresponding points are farther and...
positive curvature when they are closer.) The ability to define corresponding points on spheres in the manifold setting relies on parallel transport, so Ollivier’s generalized definition replaces the comparison distance with transportation distance, $L^1$ Wasserstein distance to be precise. With this definition, he can show that in the manifold setting his metric Ricci curvature agrees with classical Ricci curvature on small scales, to first order.

This paper is motivated by the observation that Cayley graphs of finitely generated groups, because of the edge labelings, provide a setting in which transportation distance is not needed, because any two points’ neighbors can be put in natural correspondence. Thus we arrive at a new definition of Ricci curvature of Cayley graphs, which parallels manifold Ricci curvature even more closely than the Ollivier adaptation. We explore its properties below.

1.1. Definitions. We will write $(G, S)$ for a group $G$ with generating set $S$, and $e \in G$ for the identity element. We will assume throughout that $|S| < \infty$, $S = S^{-1}$, and $e \notin S$. We will use $\Gamma$ for graphs, in particular for the defining graph of a right-angled Artin group $A_\Gamma$ and for the Cayley graph $\Gamma(G, S)$ associated to $(G, S)$. For $x \in G$, let $|x|$ denote its length in the $(G, S)$ word metric. We write $B_r(x) = \{ g \in G : |x^{-1}g| \leq r \}$ and $S_r(x) = \{ g \in G : |x^{-1}g| = r \}$ for the ball and sphere of radius $r$ centered at $x$, respectively.

Let $T_r(x, y)$ be the $L^1$ transportation distance for the uniform measure on $B_r(x)$ and $B_r(y)$: it is the “earth mover’s distance,” or the infimal distance for any plan of moving mass from one probability distribution to the other. (See [8].) Then Ollivier defines what we will call the transportation curvature $\kappa^T_r(x, y) = \frac{d(x, y) - T_r(x, y)}{d(x, y)}$.

In a Cayley graph $\Gamma(G, S)$, note that there is a natural correspondence between neighbors of $x$ and neighbors of $y$ given by matching $xa$ with $ya$ for $a \in S$; more generally we can compare $xw$ with $yw$ for any word $w \in G$. Thus let $S_r(x, y)$ be the radius-$r$ spherical comparison distance in the Cayley graph $\Gamma(G, S)$:

$$S_r(x, y) := \frac{1}{|S_r|} \sum_{w \in S_r} d(xw, yw),$$

or similarly $B_r(x, y)$ on balls rather than spheres. Thus we can define a new Ricci curvature on groups via $\kappa^B_r(x, y) = \frac{d(x, y) - B_r(x, y)}{d(x, y)}$ or $\kappa^S_r(x, y) = \frac{d(x, y) - S_r(x, y)}{d(x, y)}$. We call this comparison curvature to distinguish it from Ollivier’s transportation curvature.

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1 We note that Ollivier’s was one of several proposed discretizations of Ricci curvature formulated in the same time period; other quite different definitions can be found in sources such as [4, 5].
For comparison curvature, note that \( \kappa^n_R(x, y) = \kappa^n_R(e, x^{-1} y) \), so to study the curvature in a group it suffices to fix one of the points at the identity. Thus we will write \( \kappa^n_R(g) \) to mean \( \kappa^n_R(e, g) \), and similarly for the spherical comparison curvature \( S \). For all three definitions, when the parameter \( r \) is omitted, we are considering the case \( r = 1 \). In this note we will focus on the spherical comparison curvature with \( r = 1 \), so we will simplify the notation even further, writing
\[
\kappa(g) = \kappa^1_S(e, g).
\]

Indeed, note that \( S_1(g) \) is the the average word length of a conjugate of \( x \) by a generator, so we will introduce the notation \( \text{GenCon}(g) = \text{GenCon}(G, S, g) := \frac{1}{|S|} \sum_{a \in S} |a^{-1} g| = S_1(e, g) \). In terms of conjugation, then, we will study the notion of Ricci curvature on groups given by
\[
\kappa(g) = \frac{|g| - \text{GenCon}(g)}{|g|}.
\]

Negative Ricci curvature (with respect to spheres of radius one) occurs at a group element \( g \) when its conjugates by generators are, on average, longer in word length than \( g \) itself. Respectively, positive curvature occurs when conjugation shortens the length, and zero curvature when it is unchanged.

1.2. Properties and examples.

**Proposition 1** (Basic properties of comparison curvature).

1. The curvature is zero at central elements, because \( g \in Z(G) \implies \text{GenCon}(g) = |g| \). Indeed, all comparison curvatures are identically zero on the center, for balls or spheres of any radius. This is because \( d(xa, ya) = d(ax, ay) = d(x, y) \), so corresponding points have the same distance as the centers, giving \( S = B = d \) and so \( \kappa^B_R = \kappa^S_R = 0 \) for all \( r \). In particular, abelian groups with any generating sets have identically zero curvature.

2. For finite groups, we can take all elements as generators \((S = G \setminus \{e\})\) to make \( \kappa = 0 \), because the Cayley graph is a complete graph so all distances are 1.

3. For \( r = 1 \), the value \( B \) is obtained as a weighted average of the value \( S \) with \( d \), the distance of the centers \( x \) and \( y \). This means that the sign of the comparison curvature is the same whether one considers balls or spheres of radius 1.

4. Since \( T \) is an inf, and correspondence defines one of the competing transportation plans, we have \( T \leq B \), so \( \kappa_T \geq \kappa_B \) in general. This means that for abelian groups, the transportation curvature is non-negative.

5. \( |a^{-1} xa| \) is integer-valued and equals zero only for \( x = e \), so \( \text{GenCon}(e) = 0 \) and \( \text{GenCon}(x) \geq 1 \) for non-identity elements. Generators \( a \in S \) satisfy \( \kappa(a) = 1 - \text{GenCon}(a) \), so \( \kappa(a) \leq 0 \).

6. Suppose that \( \{e, w, w^2, w^3, \ldots\} \) is geodesic in a group \( G \), meaning that \( |w^k| = k|w| \) for all \( k \). It follows that \( \kappa(w) \leq 0 \). Otherwise, there exists some generator \( a \in S \) for which \( |a^{-1} wa| = d(a, wa) \leq |w| - 1 \). In that case we would have \( |w^k| \leq |a^{-1} w^k a| + 2 \leq k|w| - k + 2 \), which for \( k \geq 3 \) contradicts the assumption.

7. For any \( x, y \) and any \( w \in B_r \), we have \( d(x, y) - 2r \leq d(xw, yw) \leq d(x, y) + 2r \), and so the average is also in that range, which means that the correspondence curvature is bounded: \(-2r/d \leq \kappa^B_R \), \( \kappa^S_R \leq 2r/d \) for \( d = d(x, y) \). In particular if you fix \( r \) and make \( d \) large, the curvature becomes pinched near zero. When \( |w| \leq 2 \), we get \( \kappa(w) \leq 2/|w| \leq 1 \). Putting these together, we get \( \kappa \leq 1 \) for every group element.

**Proposition 2** (Damping out). If \( G \) is an infinite group and \( S \) is any generating set, then the average curvature over the ball of radius \( n \) tends to zero as \( n \to \infty \).

**Proof.** The average curvature in the group \( G \) is the limit
\[
\lim_{n \to \infty} \frac{\sum_{g \in B_n} \kappa(g)}{|B_n|}.
\]
Since $|\kappa(g)| \leq \frac{2}{|g|}$ we see that for any $k < n$ we have

$$\lim_{n \to \infty} \frac{\sum_{g \in B_n} |\kappa(g)|}{|B_n|} \leq \lim_{n \to \infty} \frac{|B_k| + \frac{2}{k} |B_n \setminus B_k|}{|B_n|} \leq \frac{2}{k}.$$  

Since this holds for arbitrary $k$, the average curvature tends to 0.

**Example 3** (Dependence on generators in the symmetric group). We will illustrate a simple observation about the dependence on choice of generators by considering the case of the symmetric group $\text{Symm}(n)$. Recall from Proposition 1(2) that with respect to $S = G \setminus \{e\}$, we get $\kappa = 0$. If we focus, for example, on the group element $\sigma = (1 \ldots n) \in \text{Symm}(n)$, then we will be able to manipulate the sign of $\kappa(\sigma)$ by building an appropriate generating set.

**Proposition 4** (Manipulating the sign). Let $\sigma = (1 \ldots n)$ be the basic $n$-cycle in the symmetric group $\text{Symm}(n)$. Given any $\epsilon > 0$, for sufficiently large $n$ there exist two generating sets $S_{\text{pos}}$ and $S_{\text{neg}}$ for $\text{Symm}(n)$ such that

- with respect to $S_{\text{pos}}$, we have $|\sigma| = 2$, $\text{GenCon}(\sigma) < 1 + \epsilon$, and $\kappa(\sigma) > \frac{1-\epsilon}{2}$; and
- with respect to $S_{\text{neg}}$, we have $|\sigma| = 1$, $\text{GenCon}(\sigma) > 3 - \epsilon$, and $\kappa(\sigma) < -2 + \epsilon$.

**Proof.** Let $S_{\text{pos}}(n) = \text{Symm}(n) \setminus \{\epsilon, \sigma\}^{\pm}$, so that $|S_{\text{pos}}| = n! - 3$. Consider the action of $\text{Symm}(n)$ on itself by conjugation. We must consider the length $s^{-1}\sigma s$ for $s \in S$. If $s^{-1}\sigma s \in \{\sigma, \sigma^{-1}\}$, then we can see $|s^{-1}\sigma s| = 2$. This happens if $s$ commutes with $\sigma$ or conjugates it to $\sigma^{-1}$ in the first case there are $n - 3$ elements of $S$ with this property. In the second case there are $n$ elements of $S$. In all other cases $s^{-1}\sigma s \in S$ and so has length 1. Thus we have

$$\text{GenCon}(\sigma) = \frac{1}{n! - 3}((n! - 3 - n - (n - 3)) + 2(n + n - 3)) = \frac{n! + 2n - 6}{n! - 3} \to 1.$$  

Let $S_{\text{neg}}(n) = \{(12), (23), \ldots (n-1 \ n), \sigma^{\pm}\}$, so that $|S_{\text{neg}}| = n + 1$. Consider $|s^{-1}\sigma s|$ for $s \in S_{\text{neg}}$. If $s = \sigma^{\pm}$, then $|s^{-1}\sigma s| = 1$. Since $\sigma$ is not fixed by any transposition we see that $|s^{-1}\sigma s| = 3$ otherwise. Thus,

$$\text{GenCon}(\sigma) = \frac{2 + 3(n - 1)}{n + 1} \to 3.$$  

**Example 5** (Free groups with standard generators). Let $F_n$ be the free group with $n$ standard generators, and let $w = a_1 \cdots a_m$ be a freely reduced word of length $m$, with $a_i$ not necessarily distinct. If this spelling is not cyclically reduced, i.e., $a_1 = a_m^{-1} = x$, then $\text{GenCon}(w) = \frac{(2n-1)(m+2) + (m-2)}{2(m+2)n}$ where the first term corresponds to conjugation by all generators other than $x$, and the last term corresponds to conjugation by $x$. If $w$ is cyclically reduced, then $\text{GenCon}(w) = \frac{(2n-2)(m+2) + m + m}{2n}$.

We have established that $\text{GenCon}(w)$ and $\kappa(w)$ both depend only on $|w|$ and the rank of the free group. We get negative curvature at all non-identity points in free groups of rank at least two.

**Example 6** ($F_2 \times \mathbb{Z}$). If $G = F_2 \times \mathbb{Z} = \langle a, b, z : [a,z] = [b,z] = e \rangle$, with $S = \{a, b, z\}^{\pm}$, then a straightforward calculation shows that $\text{GenCon}(a) = \frac{3}{2}$, so $\kappa(a) = \kappa(b) = -\frac{2}{3}$, as follows. Conjugating $a$ by $a^\pm$ or $z^\pm$ doesn’t change its length, while conjugates by $b^\pm$ have length 3. Thus $\text{GenCon}(a) = \frac{1 + 1 + 1 + 3 + 3}{5} = \frac{5}{3}$. The case of $b$ is similar. That means that $\kappa = -2/3$ on the sphere, or $-3/7$ on the ball. Thus $F_2 \times \mathbb{Z}$ has some points of zero curvature and some points of negative curvature (and no points of positive curvature) in its standard generators.

**Example 7** (General RAAGs). Right-angled Artin groups, or RAAGs, are groups in which the only relations are commuting relations between certain generators. This is encoded in a simple undirected graph $\Gamma$ for which the vertices represent generators and the edges appear between
commuting pairs. For instance the free group $F_n$ is a RAAG defined by a graph with no edges, and the previous example $F_2 \times \mathbb{Z}$ is given by three vertices arranged in a path.

We will use the fact that RAAGs have a natural normal form: a string is geodesic iff it does not contain a substring of the form $x^{-1}(y_1y_2 \ldots y_n)x$ where $x, y_i$ are generators and $x, y_i$ commute for all $i$.

**Proposition 8** (Curvature dichotomy for RAAGs). Let $A_\Gamma$ be the RAAG on the graph $\Gamma$ with the standard generating set $S = V(\Gamma)^{\pm 1}$. Then $\kappa(g) = 0$ iff $g$ is central; else, $\kappa(g) < 0$.

**Proof.** In light of Proposition 1, it suffices to show that if $g$ is not central then $\kappa(g) < 0$. Fix such a $g$. We will show that for all generators $t \in S$, $|t^{-1}gt| + |tgt^{-1}| \geq 2|g|$, and there exists a generator making this inequality strict.

Let $\text{Pref}(g) = \{v \in S \mid g = vh$ is a geodesic spelling of $g\}$ and $\text{Suff}(g) = \{v \in S \mid g = hv$ is a geodesic spelling of $g\}$. These are the letters that can be commuted to the front or the back of a geodesic spelling for $g$, respectively. By the normal forms for RAAGs, these sets are well defined, and neither $\text{Pref}(g)$ nor $\text{Suff}(g)$ can contain both $v$ and $v^{-1}$. Note that all the letters in $\text{Pref}(g)$ (respectively $\text{Suff}(g)$) commute with one another, so the corresponding vertices form a clique in $\Gamma$.

These sets will prove useful in calculating word length.

For $t \in S$, note that $|gt| = |g| \pm 1$, with $|gt| = |g| - 1$ if and only if $t^{-1} \in \text{Suff}(g)$. Similarly, $|t^{-1}g| = |g| + 1 \iff t \notin \text{Pref}(g)$. Considering all the possibilities for whether $t^\pm$ are in $\text{Pref}(g)$ or $\text{Suff}(g)$ one sees that $|t^{-1}gt| + |tgt^{-1}| \geq 2|g|$. Finally, since $g$ is not central, we can find a letter $x$ such that $\{x\} \cup \text{Pref}(g)$ is not a clique, and that means that both $xg$ and $x^{-1}g$ are geodesic. Since both $x$ and $x^{-1}$ can’t be in $\text{Suff}(g)$, this gives $|x^{-1}gx| + |xgx^{-1}| > 2|g|$. \qed

### 1.3. A Laplacian rephrasing.

**Definition 9.** We define the automorphic Cayley graph $\Gamma_A(G, U)$ for a group $G$ and set of automorphisms $U \subseteq \text{Aut}(G)$ as follows: it has vertices for group elements $g \in G$ and (directed) edges $g \sim h$ if there exists $\phi \in U$ such that $\phi(g) = h$.

Note that $\Gamma_A(G, U)$ is often not connected. Note also that averaging over conjugates by generators is the same thing as averaging over conjugates by neighbors:

$$\frac{1}{|S|} \sum_{y \sim x} |y^{-1}xy| = \frac{1}{|S|} \sum_{a \in S} |(xa)^{-1}x(xa)| = \frac{1}{|S|} \sum_{a \in S} |a^{-1}xa| = \text{GenCon}(x).$$

The automorphic Cayley graph of a group is related to $\kappa$ in the following manner:

**Proposition 10** (Curvature and the Laplacian). Let $G$ be a group generated by $S$, and let $S$ act on $G$ by conjugation. Fix $x \in G$ and set $f_x(y) := |y^{-1}xy|$. Then

$$\kappa(x) = \frac{1}{f_x(x)} (\nabla f_x)(x),$$

where $\nabla f$ is the graph Laplacian on $\Gamma_A(G, S)$ pulled back by $f$. In particular, $\kappa \equiv 0$ on $G$ if and only if $f_x$ is harmonic.

### 2. Virtually abelian groups

We have seen that no group can have positive curvature at all nonidentity points, since Proposition 1 observes that $\kappa(a) \leq 0$ for $a \in S$. Now we will consider the consequences of identically zero curvature.

**Theorem 11.** Suppose $\kappa(a) = 0$ for all $a \in S$. Then $G$ is virtually abelian.

**Proof.** Since $\kappa(a) = 0$ we have that $\text{GenCon}(a) = 1$ for all $a \in S$. In particular, this means that the action of $G$ by conjugation stabilizes $S$ as a set. Consequently, since $S$ is finite, it follows that the stabilizer of a point $a \in S$ is finite index in $G$. Since $G$ is acting by conjugation, $\text{Fix}(a) = \text{Z}(a)$. 

Thus, for every \( a \in S \), \( Z(a) \leq G \). Since \( S \) generates \( G \), it follows that \( Z(G) = \bigcap_{a \in S} Z(a) \), and this group is finite index in \( G \) and abelian.

As a corollary, we get that groups of constant zero curvature are virtually abelian.

One might hope that there is a converse, namely that given a virtually abelian group we can find a generating set where every point has \( \kappa = 0 \). We will see soon that this is not possible; however, we do have the following.

**Theorem 12.** Let \( G \) be virtually abelian, so that \( H \leq G, H \cong \mathbb{Z}^n \) is a finite-index normal free abelian subgroup of \( G \). Then there exists a generating set \( S \) of \( G \) such that \( \kappa(h) = 0 \) \( \forall h \in H \).

**Proof.** There exists a finite group \( F \) such that \( G \) fits into the sequence

\[ H \hookrightarrow G \twoheadrightarrow F. \]

Since \( G \) acts on \( H \) by conjugation, there is a well defined map \( \phi: F \to Out(H) = GL_n(\mathbb{Z}) \). Let \( U \) be a set containing a lift of each non-identity element of \( F \) and close it under inversion. Let \( T' \) be a finite generating set for \( H \). Let \( V = \{ uuv | u, v \in U, uvw \in H \} \cup \{ uv | t \in V, u \in (U) \} \) be the set of conjugates of elements of \( T \) by elements of \( \langle U \rangle \). This is a finite set since the orbits of \( U \) in \( H \) are finite. Let \( S = U \cup T' \); by construction \( S \) generates \( G \). We will show that conjugating by \( s \in S \) does not change the word length of any \( h \in H \).

Let \( x_1 \ldots x_l \) be a spelling for \( h \) that is geodesic with respect to \( S = U \cup T' \). Since \( T' \) is preserved by the conjugation action of \( U \) we can assume that all but the first \( k \) letters are in \( T' \). We will show that \( k = 0 \).

Consider the word \( x_1 \ldots x_k \), note that since this represents an element of \( H \) we know that \( k \neq 1 \). First we can see that no subword of the form \( x_i x_{i+1} \) gives an element of \( H \) as we could replace this with a generator from \( T' \). Looking at the final subword \( x_{k-1} x_k \) we can see there is an element \( y \) such that \( yx_{k-1} x_k \) is in \( H \). We can then replace \( x_{k-1} x_k \) by \( y^{-1} t \) for the element \( t = yx_{k-1} x_k \in T' \). This allows us to reduce \( k \) by 1 without changing the length of our word. We can thus keep reducing until we arrive at the case that \( k = 1 \) giving a contradiction.

Since \( H \) is a normal subgroup, \( T' \) is contained in \( H \), which is abelian. Therefore generators in \( T' \) commute with \( h \), so conjugation by \( T' \) generators does not change the word length.

Next consider conjugation by \( u \in U \). Let \( x_i = u^{-1} x_i u \), and note that this is in \( T' \) so its word length is 1 with respect to \( S \) generators. We have a spelling for the group element \( u^{-1} hu \), namely \( x_1 \ldots x_i \), so \( |u^{-1} hu| \leq |h| \). Conversely, if \( y_1 \ldots y_m \) is a word representing \( u^{-1} hu \), then \( y_1 u^{-1} \ldots y_m \) is a word representing \( h \), so \( |h| \leq |u^{-1} hu| \). Thus conjugation fixes the length of every word in \( H \) and the curvature \( \kappa(h) = 0 \) for all \( h \in H \).

In the case where the group is virtually \( \mathbb{Z} \) we can remove the dependence on generating set; however, in general, we must choose the generators carefully. This is established in the next two results.

**Proposition 13.** Let \( G \) be a group with generating set \( S \). Let \( N \) be a normal infinite cyclic subgroup. Then \( \kappa(g) = 0 \) for all \( g \in N \).

**Proof.** Take an element \( g \in Z \) and any generator \( s \in S \). Since \( Z \) is normal we see that conjugation by \( s \) gives an automorphism of \( Z \) which is either the identity or inversion. In either case this preserves the length of the element \( g \).

**Proposition 14.** There is a virtually abelian group \( G \) with a generating set \( S \) such that every finite index free abelian subgroup has a point of positive curvature and a point of negative curvature.

**Proof.** Let \( G = \langle a, b, t | [a, b], t^{-1} at = ab, t^{-1} bt = a^{-1}, t^6 \rangle \) with generating set \( \{ a, b, t \}^{\pm 1} \). This group is isomorphic to \( \mathbb{Z}^2 \rtimes \mathbb{Z}/6\mathbb{Z} \). The subgroup \( \langle a, b \rangle = \mathbb{Z}^2 \) is a normal abelian subgroup and any finite index free abelian subgroup is a subgroup of this.
We will show that the element \((ab)^n\) has positive curvature and the element \(a^n\) has negative curvature for all \(n \geq 1\).

The element \(w = (ab)^n\) can be written as a geodesic using the spelling \(t^{-1}a^n t = w\). So this word has length \(n + 2\). Since \(w = (ab)^n \in \mathbb{Z}^2\), conjugating by the generators \(a, b\) does not change the length. We must now consider the conjugates by \(t\) and \(t^{-1}\). Firstly, \(twt^{-1} = t^{-1}a^n t^{-1} = a^n\) and has length \(n\). Finally, \(t^{-1}wt = t^{-2}a^n t^2 = t^{-1}(ab)^n t = b^n\) which also has length \(n\). We conclude that \(\text{GenCon}(w) = n + \frac{1}{3} < n + 2\), thus \(\kappa(w) > 0\).

The element \(v = a^n\) is a geodesic and so we work with this spelling. Conjugating by elements of \(\langle a, b \rangle\) will not change the length. Conjugating by \(t\) we get \(t^{-1}vt\) is a geodesic and has length \(n + 2\). Conjugating by \(t^{-1}\) we get \(tvt^{-1} = b^{-n}\), which is a geodesic and has length \(n\). This gives \(\text{GenCon}(v) = n + \frac{1}{3} > n\), thus \(\kappa(v) < 0\).

Finally, to see that it is not possible to have a complete converse to Theorem \([\text{11}]\) we will look at the infinite dihedral group.

**Proposition 15.** Let \(D_{\infty} = \mathbb{Z}_2 \ast \mathbb{Z}_2\), which is virtually \(\mathbb{Z}\). This group has no finite generating set giving it constant zero curvature.

**Proof.** First consider \(D_{\infty} = \mathbb{Z}_2 \ast \mathbb{Z}_2\) with the standard presentation \(\langle a, b \mid a^2, b^2 \rangle\). Every element of this group can be represented uniquely and geodesically as a positive word in \(a\) and \(b\) with no occurrence of \(a^2\) or \(b^2\). (The cyclic group generated by \(ab\) has index 2, with \(D_{\infty} = \langle ab \rangle \sqcup a(ab)\).)

If \(D_{\infty}\) had a finite generating set \(S\) with zero curvature everywhere, then \(S\) would be preserved by the conjugation action on itself, i.e., \(s^{-1}ts \in S\) for all \(s, t \in S\). (As in the proof of Theorem \([\text{11}]\) this is because conjugation can’t carry a generator to the identity, so it can only lengthen the generators, but if any of them strictly increases in length, then GenCon will go up and curvature will be negative.)

If every element of \(S\) is a word which starts and ends in different letters, then \(S\) is contained in the subgroup \(\langle ab \rangle\), because \(ba = (ab)^{-1} \in \langle ab \rangle\). Since \(\langle ab \rangle\) is a proper subgroup, we conclude that there is an element \(s \in S\) which starts and ends with the same letter, say \(a\). There are two cases: either \(s = a\) or \(s = ava\) for some nonempty \(w\). We will deal with the case that \(s = a\); the other case works the same way.

There is at least one more element \(t \neq s\) of \(S\) since \(D_{\infty}\) has rank 2. Up to replacing \(t\) by \(t^{-1}\), there are four cases to check: freely reduced spellings \(t = b, t = bvb, t = bva\) or \(t = ava\) for strings \(v \in S^*\).

Firstly, if \(t = b\), then \(t^{-1}st = bab \in S\) and \(s^{-1}(t^{-1}st)s = abab \in S\). Repeatedly conjugating by \(t\) and then \(s\) we find more elements that must belong to \(S\), contradicting finiteness. The proof for \(t = bvb\) is exactly the same.

If \(t = bva\), then we can look at \(t^{-1}st^i = (av^{-1}b)^i a(bva)^i\) which is reduced and must be in \(S\) for all \(i\), showing again that \(S\) would be infinite.

Finally, if \(t = ava\), then we can start by conjugating by \(a\) to see that \(v \in S\). Since \(t\) was a reduced word, we see that \(v\) starts and ends with \(b\), which is a case that has already been treated. \(\Box\)

## 3. Nilpotent Groups

First, note that every nilpotent group has a nontrivial center (containing at least the last group in its lower central series), which means that it has some points with \(\kappa = 0\). In this section we will establish that nilpotent groups can also have infinitely many points of positive and negative curvature, by finding all three cases \((\kappa = 0, > 0, < 0)\) with positive density in the Heisenberg group.

Let \(H(\mathbb{Z}) = \langle a, b : [a, b] \rangle\) be the standard presentation of the Heisenberg group (the free nilpotent group of step two and rank two). Blachère computed the exact formula for the word metric in this distance in \([\text{11}]\) in terms of matrix-entry coordinates. We will instead use Mal’cev coordinates: every group element can be uniquely written in the form \(a^A b^B c^C\), where \(c = [a, b]\), for
integer exponents $A, B, C$. When it is convenient we can use $\mathbb{Z}^3$ coordinates $(A, B, C)$ to denote $a^A b^B c^C$. Then the word metric obeys the following formula: if $A > B > 0$ and $C > 0$, then
\[
(a^A b^B c^C) = \begin{cases}
2[C/A] + A + B, & C \leq A^2 - AB; \\
2[2\sqrt{C + AB}] - A - B, & C \geq A^2 - AB.
\end{cases}
\]

Let us call the first case ($C \leq A^2 - AB$) the low-height case; then the high-height case is $C \geq A^2 - AB$.

**Theorem 16.** The Heisenberg group $H(\mathbb{Z})$ with standard generators $S = \{a, b\}$ has a positive proportion of points with positive, negative and zero curvature, respectively: there exists $\epsilon > 0$ such that
\[
\epsilon < \frac{\#\{g \in B_n : \kappa(g) = 0\}}{\#B_n} < 1 - \epsilon,
\]
and the same is true for $\kappa > 0$ and $\kappa < 0$.

**Proof.** First let us describe the ball $B_n$ in terms of the coordinates $(A, B, C)$ of its elements. For each $(A, B)$ satisfying $|A| + |B| \leq n$, the point $(A, B, C)$ belongs to $B_n$ iff $C$ satisfies a certain quadratic polynomial inequality in $A, B$—this can be read off of the word-length formula with a little bit of work (because of the square root appearing in the high-height case and the quadratic bound for the low-height case), or is described directly in [3].

Now let $w = a^A b^B c^C \in B_n$ and note what happens when we conjugate by generators $a, b$. We have
\[
a^{-1}(a^A b^B c^C) = a^A b^B c^{C-B}; \quad a(a^A b^B c^C) a^{-1} = a^A b^B c^{C+B};
\]
\[
b^{-1}(a^A b^B c^C) = a^A b^B c^{C+A}; \quad b(a^A b^B c^C) b^{-1} = a^A b^B c^{C-A}.
\]

At low height, for $A, B$ fixed, we will show that the sign of $\kappa$ only depends on $C$ mod A. To see this, note that conjugating by $b^\pm$ changes $C$ by adding $\pm A$. By [4], each of these two conjugates changes $|w|$ by exactly 2 in opposite directions, which cancels out in the averaging. Thus only conjugates by $a^\pm$ contribute to the sign of $\kappa$.

In particular if $C$ is a multiple of $A$ then $C/A$ is a whole number, so $[C/A]$ is preserved by $C \mapsto C - A$ and increased by $C \mapsto C + A$, which means that the curvature is negative overall. On the other hand, if $C = kA + 1$, then $[C/A]$ is preserved by $C \mapsto C + B$ and decreased by $C \mapsto C - B$, for positive curvature overall. Generally the sign of $\kappa$ in this case (low height; $A, B$ fixed; $C$ varying) only depends on $C$ mod $A$. To be precise, suppose $C = kA + r$. Then
\[
[(C + B)/A] = \begin{cases}
k + 2, & r > A - B \\
k + 1, & r \leq A - B
\end{cases}
\]
and
\[
[(C - B)/A] = \begin{cases}
k + 1, & r > B \\
k, & r \leq B.
\end{cases}
\]

Within the conditions $A > B > 0, C > 0$, let us further restrict to the sector $\frac{1}{5}A \leq B < \frac{2}{5}A$; clearly a positive proportion of the points in the ball satisfy these inequalities. Then of the possible remainders $r$ for $C$, at least 1/5 satisfy $1 \leq r \leq B$ (Case 1), at least 1/5 satisfy $A - B < r \leq A - 1$ (Case 2), and at least 1/5 satisfy $B < r \leq A - B$ (Case 3). In all three cases, $[C/A] = k + 1$. In Case 1, $[(C \pm B)/A] = k + 1, k$, so $\kappa > 0$. In Case 2, $[(C \pm B)/A] = k + 2, k + 1$, so $\kappa < 0$. And in Case 3, $[(C \pm B)/A] = k + 1, k + 1$, so $\kappa = 0$.

We have established that the sign of the curvature repeats periodically (mod $A$) at low height. To complete the proof, we must be sure that for most of the $(A, B)$ in the ball of radius $n$, the low-height values obtained by $C$ complete at least one period with respect to $A$, and that the low-height case has positive measure. First we check that $A^2 - AB \geq A$ (the low-height inequality is satisfied for a full period): this is true for $A \geq 5$ because $\frac{1}{5}A \leq B < \frac{2}{5}A$. Finally, note that
\[
\int_0^n \int_0^{n-A} (A^2 - AB) \, dA \, dB = \frac{n^4}{24},
\]
which has positive measure since $|B_n|$ is bounded above and below by multiples of $n^4$. \qed
Remark 17. At high height, for $A, B$ fixed, there is a pattern of positive and negative curvature points appearing in an interval of fixed length surrounding each value of $C$ for which $C + AB = k^2$ is a perfect square. Outside of these intervals, $κ = 0$.

This means that in a fixed fiber $(A, B, *), almost every point has zero curvature; on the other hand, we’ve seen that over the ball of radius $n$, a positive proportion of points have $κ \neq 0$.

We note that this proof will carry over to the Heisenberg group with any generating set. We sketch the general proof here. There will still be a low-height/high-height distinction in arbitrary generators, with a positive proportion of the ball in low height. (See \[3\], where the low-height case is called unstable and the proportion is the positive rational number $V_{\text{im}}/V$.) Each generating set has a defining polygon $Q$ in the plane given by the convex hull of the projection of its generating set. (For the standard generators, $Q$ is the diamond $|A| + |B| \leq n$.) It is still the case that conjugation by generators induces a linear change in the $C$ coordinate. We can restrict to a sector of the defining polygon bounded away from its vertices, as we did here, so that there exists $h > 0$ for which $(A, B) ∈ nQ$ and $C ∈ [0, h n^2]$ implies $(A, B, C) ∈ B_n$. This guarantees that the fibers in this sector over each $(A, B)$ will see a full period with respect to $A$ and $B$. The word length will still be given in $(A, B, C)$ by a linear function with rounding in the low-height case. Thus we will get linear inequalities determining whether $κ > 0, κ < 0$, or $κ = 0$, with each one satisfied for a positive proportion of points in $B_n$.

Remark 18. In the 1976 paper \[6\], John Milnor shows that for any left-invariant Riemannian metric on a nonabelian nilpotent Lie group, there are directions of strictly positive Ricci curvature and directions of strictly negative Ricci curvature. In particular, the central direction is positively curved, and non-central directions orthogonal to the commutator subalgebra are negatively curved. Note that the asymptotic cone of the word metric $(H(\mathbb{Z}), \{a, b\}^{\infty})$ is the Lie group $H(\mathbb{R})$ with the (left-invariant) $L^1$ Carnot-Carathéodory metric, which is not Riemannian but only sub-Finsler. The work here shows that this metric on $H(\mathbb{R})$ behaves quite differently from the Riemannian ones studied by Milnor: the $x$-direction is still negatively curved, as in the Riemannian case (because the conjugate of $a^n$ by $b^\pm$ has length $n + 2$ for $n ≥ 1$), but now the $z$-direction (the center) has zero Ricci curvature.

4. Positively curved directions in CAT(0) and hyperbolic groups

4.1. A CAT(0) example. We will build a family of CAT(0) groups which have points of positive curvature. First we recall the following Proposition from \[2\].

Proposition 19 (Bridson-Haefliger pg. 353, Prop 11.13). Let $X$ and $A$ be non-positively curved spaces. If $A$ is compact and $ϕ, ψ: A → X$ are local isometries, then the quotient of $X \sqcup A$ by the equivalence relation generated by $[(a, 0) \sim ϕ(a)$ and $(a, 1) \sim ψ(a)]$ is non-positively curved.

We can now prove the following.

Theorem 20. For $n ≥ 3, k ≥ 2$, the group

$$G_{n,k} = \langle x, y, a_1, \ldots, a_k | a_i x^n a_i^{-1} = y^{n-2}, a_i^{-1} x^n a_i = y^{n-2} \rangle$$

is CAT(0). In each such group, the string $x^n$ is geodesic (i.e., $|x^n| = n$), and $κ(x^n) > 0$.

Proof. Take a rose $R$ with 2 petals one of length $n$ and one of length $n - 2$. Given a rose $S$ with two petals both of length $n(n - 2)$ there are two local isometries $ϕ, ψ: S → R$ wrapping the loops around the loops in $R$ the appropriate number of times. Doing this $k$ times will give a non-positively curved quotient by the above proposition. A simple application of Seifert-van Kampen shows that the fundamental group of this space is $G_{n,k}$.

Suppose $x^n$ is not geodesic. Then there is a geodesic $w$ such that $w = x^n$ in $G$ and $|w| < n$. Using HNN normal forms, $w$ has normal form $x^n$ and so there must be a “pinch,” i.e., $w$ decomposes as
We have \(|x^n| = n\), and by construction, for all \(i\), \(|a_i^{-1}x^n a_i| = |y^n| = n - 2\), and similarly for \(a_i^{-1}\).

Additionally, \(|x^{-1}x^n| = n\), \(|x^n x^{-1}| = n\), and by the triangle inequality, \(|y^{-1}x^n y| \leq n + 2\), and similarly for \(|y^n y^{-1}| \leq n + 2\). All together, we get

\[
\text{GenCon}(x^n) \leq \frac{1}{2k+4} (2k(n-2) + 2n + 2(n+2)) \leq n.
\]

This gives \(\kappa(x^n) = 1 - \frac{1}{n} (\text{GenCon}(x^n)) > 0\), as desired. \(\square\)

### 4.2. A hyperbolic example.

Before we look for a hyperbolic group with such a property we start with the following useful computation for products. Let us adopt the notation that if \(S_1, S_2\) are generating sets for \(G_1, G_2\), respectively, then \(S_1 \bowtie S_2\) will denote the split generating set \(S = (S_1 \times \{e\}) \cup (\{e\} \times S_2)\) for the product \(G_1 \times G_2\).

**Proposition 21 (Product formula).** Let \(G_1\) and \(G_2\) be groups with generating sets \(S_1, S_2\) respectively. Let \(S = S_1 \bowtie S_2\) be the split generating set for \(G_1 \times G_2\). Then

\[
\text{GenCon}((x, y)) = \frac{|S_1| \text{GenCon}(x) + |S_1||y| + |S_2||x| + |S_2| \text{GenCon}(y)}{|S_1| + |S_2|}.
\]

**Proof.** We will compute \((|S_1| + |S_2|) \text{GenCon}((x, y))\). This is just the sum of the lengths of all conjugates of \((x, y)\).

\[
\sum_{s \in S} |s^{-1}(x, y)s| = \sum_{s_1 \in S_1} |(s_1^{-1}, e)(x, y)(s_1, e)| + \sum_{s_2 \in S_2} |(e, s_2^{-1})(x, y)(e, s_2)|
\]

\[
= \sum_{s_1 \in S_1} |(s_1^{-1}xs_1, y)| + \sum_{s_2 \in S_2} |(x, s_2^{-1}ys_2)|
\]

\[
= \sum_{s_1 \in S_1} |(s_1^{-1}xs_1) + |y| + \sum_{s_2 \in S_2} |(s_2^{-1}ys_2) + |x|
\]

\[
= |S_1||y| + |S_2||x| + \sum_{s_1 \in S_1} |s_1^{-1}xs_1| + \sum_{s_2 \in S_2} |s_2^{-1}ys_2|
\]

\[
= |S_1| \text{GenCon}(x) + |S_1||y| + |S_2||x| + |S_2| \text{GenCon}(y). \quad \square
\]

**Theorem 22.** There is a non-elementary hyperbolic group and a sequence of elements \(w_i\) such that \(\kappa(w_i) > 0\).

**Proof.** Consider two random reduced spellings \(x, y \in F(a, b) = \langle a, b \mid \emptyset \rangle\) of length \(l\) and let \(w_1 = axa\) and \(w_2 = y\). Then for any \(\epsilon > 0\), the following properties hold with positive probability as \(l \to \infty\):

- They are cyclically and freely reduced,
- \(\langle a, b \mid w_1, w_2 \rangle\) is a \(C'(\epsilon)\) small cancellation group.

Fix some such \(w_1, w_2\).

Consider the group \(G = \langle a, b, t \mid t^{-1}w_1 t = w_2 \rangle\), and let \(S = \{a, b, t\}^\pm\) denote its generators. By the above hypothesis, we can fix \(\epsilon = 1/6\) and choose \(w_1, w_2\) as relators presenting a \(C'(1/6)\) small cancellation group, which is necessarily a non-elementary hyperbolic group. By small cancellation any trivial word contains at least half a relation, therefore \(w_1\) is a geodesic spelling in the group \(G\).

We have \(|w_1| = l + 2\) and \(|w_2| = l\).

Consider the lengths of the 6 conjugates of \(w_1\), recalling that \(w_1\) starts and ends with \(a\).

\[
|a^{-1}w_1a| = l + 2, \quad |r^{-1}w_1 r| = l, \quad |b^{-1}w_1b| \leq l + 4,
\]

\[
|aw_1 a^{-1}| = l + 2, \quad |tw_1 t^{-1}| \leq |w_1| + 2 = l + 4, \quad |bw_1 b^{-1}| \leq l + 4.
\]

Thus we have \(\text{GenCon}(w_1) \leq \frac{6l+16}{6} = |w_1| + \frac{2}{3}\).
The hyperbolic group we are constructing to prove the theorem is the group \( G \times \text{Symm}(4) \) with generating set \( S \square S_{\text{pos}} \), where \( S_{\text{pos}} \) is the generating set from Proposition \( \text{[4]} \). Note that \( |\sigma| = 2 \) in these generators. From the proof of Proposition \( \text{[4]} \) we know that \( \text{GenCon}(\sigma) = \frac{26}{27} \). We can now compute the curvature \( \kappa((w_1, \sigma)) \) using the formula from Proposition \( \text{[21]} \) obtaining
\[
\text{GenCon}((w_1, \sigma)) \leq \frac{6l + 16 + 12 + 21(l + 2) + 26}{27} = \frac{27l + 96}{27} = l + \frac{32}{9} < l + 4.
\]
This gives us positive curvature: \( \kappa((w_1, \sigma)) = 1 - \frac{1}{l+4} \text{GenCon}((w_1, \sigma)) > 0 \).
To get an infinite sequence of words, let \( v_n = w_1t^{-1}w_2tw_1 \). This is a geodesic word of length \( nl + 2 + 2(l + 2) \) in \((G, S)\). Conjugating \( v_n \) by \( a \) or \( a^{-1} \) preserves length, conjugating by \( t \) will shorten it by 2 and the other three conjugations (by \( t^{-1}, b, b^{-1} \)) will lengthen it by at most 2. Thus, we get \( \text{GenCon}(v_n) \leq |v_n| + \frac{2}{3} \). On the other hand, so we get
\[
\text{GenCon}((v_n, \sigma)) \leq |v_n| + \frac{38}{27} < |v_n| + 2.
\]
Thus we can see that we have \( \kappa((v_n, \sigma)) > 0 \).

5. Behavior under isometric embeddings

Recall the notation \( S_1 \square S_2 = (S_1 \times \{e\}) \cup (\{e\} \times S_2) \)

**Theorem 23** (Positively curved embedding). Let \( G \) be any group with generating set \( S \). For sufficiently large \( n \), there is an isometric embedding \( i: G \rightarrow G \times \text{Symm}(n) \) such that \( \kappa(i(g)) > 0 \) for all \( g \in G \) with respect to a generating set \( S \square S_{\text{pos}} \). Moreover, \( i \) is boundedly far from a homomorphism: \( d(i(gh), i(g)i(h)) \leq 2 \).

**Proof.** Let \( \sigma = (1 \ldots n) \in \text{Symm}(n) \) and let \( S_{\text{pos}} = \text{Symm}(n) \setminus \{e, \sigma, \sigma^\pm\} \) be the generating set for \( \text{Symm}(n) \) designed to make \( \kappa(\sigma) > 0 \), as in Proposition \( \text{[4]} \).

Consider the map \( i: G \rightarrow G \times \text{Symm}(n) \) defined by \( i(g) = (g, \sigma) \). Then by Proposition \( \text{[21]} \) we can write
\[
\text{GenCon}((g, \sigma)) = \frac{|S| + |S_{\text{pos}}||g, \sigma| + |S|(\text{GenCon}(g) - |g|) + |S_{\text{pos}}||\text{GenCon}(\sigma) - |\sigma|)}{|S| + |S_{\text{pos}}|}.
\]
To have positive curvature at the point \((g, \sigma)\) we require \( |S|(\text{GenCon}(g) - |g|) + |S_{\text{pos}}|(\text{GenCon}(\sigma) - |\sigma|) \) to be negative. Since \( S \) is fixed and \( \text{GenCon}(g) - |g| \leq 2 \), we must pick \( n \) large enough so that \( |S_{\text{pos}}|(\text{GenCon}(\sigma) - |\sigma|) < 2|S| \). From Proposition \( \text{[4]} \) \( |S_{\text{pos}}|(\text{GenCon}(\sigma) - |\sigma|) = n! + 2n - 6 - 2(n! - 3) = -n! + 2n \). Taking \( n \) large we can ensure \(-n! + 2n + 2|S| < 0 \). Thus for sufficiently large \( n \) we have \( \text{GenCon}((g, \sigma)) < |(g, \sigma)| \) and \( \kappa((g, \sigma)) > 0 \).

In an exactly similar fashion, we get a corresponding statement for negative curvature.

**Theorem 24** (Negatively curved embedding). Let \( G \) be a group with generating set \( S \). For sufficiently large \( n \), there is an isometric embedding \( i: G \rightarrow G \times \text{Symm}(n) \) such that \( \kappa(i(g)) < 0 \) for all \( g \in G \) with respect to a generating set \( S \square S_{\text{neg}} \). Moreover, \( i \) is boundedly far from a homomorphism: \( d(i(gh), i(g)i(h)) \leq 1 \).

**Proof.** Let \( S_{\text{neg}}(n) = \{(12), (23), \ldots, (n-1 \ n), \sigma^\pm\} \) be the generating set for \( \text{Symm}(n) \) designed to make \( \kappa(\sigma) < 0 \), as in Proposition \( \text{[4]} \). Take the embedding \( i: G \rightarrow G \times \text{Symm}(n) \) defined by \( i(g) = (g, \sigma) \), as above. To have negative curvature at the point \((g, \sigma)\) we require \( |S|(\text{GenCon}(g) - |g|) + |S_{\text{neg}}|(\text{GenCon}(\sigma) - |\sigma|) \) to be greater than 0. Since \( S \) is fixed and \( \text{GenCon}(g) - |g| \geq -2 \), we must pick \( n \) large enough so that \( |S_{\text{neg}}|(\text{GenCon}(\sigma) - |\sigma|) > -2|S| \). From Proposition \( \text{[4]} \) \( |S_{\text{neg}}|(\text{GenCon}(\sigma) - |\sigma|) = 2n - 2 \). Taking \( n > |S| \) we get that \( 2n - 2 - 2|S| > 0 \). Thus for sufficiently large \( n \) \( \text{GenCon}((g, \sigma)) > |(g, \sigma)| \) and \( \kappa((g, \sigma)) < 0 \).
In both examples above the embedding is quasi-dense. If we relax this, we can make the negative curvature embedding a homomorphism.

**Theorem 25** (Negatively curved homomorphic embedding). Let $G$ be a group with generating set $T$. There exists a group $H$ and a homomorphism $\phi: G \to H$ such that:

1. $\phi$ is an isometry.
2. $\kappa(\phi(g)) < 0$ for all $g \in G \setminus \{e\}$.

In fact, $\kappa(h) < 0$ for all $h \in H \setminus \{e\}$.

**Proof.** Let $F_n$ be the free group of rank $n$ with the standard generating set $U$. We defer the choice of $n$ until later. Let $H = G \ast F_n$ with the generating set $S = T \cup U$. Then $\phi(g) = g$ defines a homomorphism which is an isometry onto its image.

We must now compute GenCon$(g)$ for $g \in G$. Let $u$ be a generator of $F_n$. The word $u^{-1}gu$ is a geodesic in the group $G \ast F_n$ and so this element has length $|g| + 2$. Thus,

$$\text{GenCon}(H, S, g) = \frac{|T| \text{GenCon}(G, T, g) + 2n(|g| + 2)}{|T| + 2n}.$$

Since GenCon$(G, T, g) \geq |g| - 2$, taking $n > \frac{|T|}{2}$, we have GenCon$(H, S, g) > |g|$. Thus, $\kappa(g) < 0$.

Let $h$ be an element of $H$. If one geodesic representative starts with an element of $G$, then all geodesic representatives will start with a letter of $G$. Similarly, if any geodesic representative ends with a letter of $G$, then all geodesic representatives end with a letter of $G$.

We now split into cases. Firstly, consider the case that both the first and last letters of $h$ are in $G$. In this case we see that every element of $F_n$ lengthens this word. Thus taking $n$ large enough we can ensure such elements have negative curvature.

Secondly, for an element that has a geodesic representative where both the first and last letter are in $F_n$, we see that all elements of $T$ will lengthen this element. From the calculation in Example 5 we see that for any $n \geq 1$ these elements will have negative curvature.

Finally consider the case that a geodesic representative starts with a letter of $T$ and ends with a letter of $U$. Conjugating by elements of $T$ will either preserve the length or make the word longer. Conjugating by $U$ will have a similar effect. Taking $n \geq 2$ we see that there is an element of $U$ which lengthens the word and thus we get negative curvature.

This completes the proof that the group has negative curvature everywhere. □

Recall the well-known fact that no hyperbolic group has a subgroup isomorphic to $\mathbb{Z}^2$. So using the above construction with $G = F_n \ast \mathbb{Z}^2$, we’ve constructed a non-hyperbolic group with every non-identity element negatively curved.

6. **Curvature outside a ball**

Since generators are always nonpositively curved (see Prop. 15), no nontrivial group can have positive curvature at all points. What happens when we assume that $\kappa(g) > 0$ for all $g$ outside a ball of finite radius?

**Theorem 26.** Let $G$ be a group generated by $S$ and suppose for some $R \in \mathbb{N}$ we have $\kappa(g) > 0$ for all $g \notin B_R$. Then $G$ is a finite group.

**Proof.** Define $A_n = S_{2n} \cup S_{2n+1}$ to be an annulus in the Cayley graph. For all $g \in G$, $s \in S$, let $w_{g,s} = |g| - |s^{-1}gs|$ measure the amount that $g$ is shortened by conjugation by $s$. Let $c_n = \sum_{S_n \times S} w_{g,s}$.

Noting that for $g \in S_n$ we have $|g| = n$, the following chain of equalities relating $c_n$ to $\kappa(g)$. 

Thus for $n > R$ we have $c_n > 0$.

Define the following sets:

$$K_n = \{(g, s) \in A_n \times S \mid s^{-1}gs \in A_{n-1}\}; \quad L_n = \{(g, s) \in S_n \times S \mid w_{g,s} = 1\}; \quad U_n = \{(g, s) \in S_n \times S \mid w_{g,s} = -1\}.$$ 

Let $k_n = \sum_{K_n} w_{g,s}$. The generating set $S$ is closed under inversion so, $\sum_{L_{n+1}} w_{g,s} = -\sum_{U_n} w_{g,s}$ and $\sum_{J_n} w_{g,s} = -\sum_{K_{n+1}} w_{g,s} = -k_{n+1}$.

For $n \gg R$ we have

$$0 < c_{2n} + c_{2n+1} = \sum_{(g, s) \in S_{2n} \times S} w_{g,s} + \sum_{(g, s) \in S_{2n+1} \times S} w_{g,s}$$

$$= \sum_{(g, s) \in K_n} w_{g,s} + \sum_{(g, s) \in L_{n+1}} w_{g,s} + \sum_{(g, s) \in U_2} w_{g,s} + \sum_{(g, s) \in J_n} w_{g,s}$$

$$= \sum_{(g, s) \in K_n} w_{g,s} + \sum_{(g, s) \in J_n} w_{g,s}$$

$$= \sum_{(g, s) \in K_n} w_{g,s} - \sum_{(g, s) \in K_{n+1}} w_{g,s}$$

$$= k_n - k_{n+1}.$$ 

We conclude that for indices bigger than $R$, the sequence $(k_n)$ is strictly decreasing.

Positive curvature ensures that every element of $S_{2n}$ has at least one generator which conjugates it into $A_{n-1}$, so we have

$$|S_{2n}| \leq k_n \leq 2|K_n| \leq 2|S| \cdot |A_n|.$$ 

We are now ready to assemble these facts. If $G$ has linear growth it is virtually $\mathbb{Z}$ and we can appeal to Proposition 13. Thus assume $G$ is a group with superlinear growth. Since $|S_{2n}| \to \infty$, we see that for $m \gg n \gg R$ we get

$$|S_{2m}| \leq k_m < k_n \leq 2|S| \cdot |A_n| \leq 4|S||S_{2n+1}|,$$

giving a contradiction.

\[\square\]

7. Questions for future study

We have seen that only finite groups have positive curvature outside a ball. It seems natural to conjecture that negative curvature outside a ball implies exponential growth, and to hope for a similar argument.

Finally, one reasonable critique of this definition of curvature is that it is somehow too local, since we based our comparisons on spheres of radius $1$. Some of the anomalous behavior (such as positive curvature in hyperbolic groups) might be better controlled by evaluating $\kappa_r^B(x)$ where $r$ is “tuned” as a function of $|x|$. For instance it would be interesting to study the $r = |x|$ case or the $r = \sqrt{|x|}$ case, both for our comparison curvature and for Ollivier’s transportation curvature, where these variants could give a promising way to evaluate the curvature of large but finite graphs.
References

[1] Sébastien Blachère, *Word distance on the discrete Heisenberg group*. Colloq. Math. 95 (2003), no. 1, 21–36.

[2] Martin R. Bridson and André Haefliger *Metric spaces of nonpositive curvature*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 319. Springer-Verlag, Berlin, 1999.

[3] Moon Duchin and Christopher Mooney, *Fine asymptotic geometry in the Heisenberg group*. Indiana Univ. Math. J. 63 (2014), no. 3, 885–916.

[4] Robin Forman, *Bochner’s method for cell complexes and combinatorial Ricci curvature*. Discrete Comput. Geom., 29(3), 323–374, 2003.

[5] Yong Lin and Shing-Tung Yau, *Ricci curvature and eigenvalue estimate on locally finite graphs*. Math. Res. Lett., 17(2): 343–356, 2010.

[6] John Milnor, *Curvatures of left invariant metrics on Lie groups*. Advances in Math. 21 (1976), no. 3, 293–329.

[7] Yann Ollivier, *Ricci curvature of metric spaces*. C. R. Math. Acad. Sci. Paris 345 (2007), no. 11, 643–646.

[8] Yann Ollivier, *Ricci curvature of Markov chains on metric spaces*. J. Funct. Anal. 256 (2009), no. 3, 810–864.

[9] Yann Ollivier, *A survey of Ricci curvature for metric spaces and Markov chains*. Probabilistic approach to geometry, 343–381, Adv. Stud. Pure Math., 57, Math. Soc. Japan, Tokyo, 2010.

[10] Yann Ollivier, *A visual introduction to Riemannian curvatures and some discrete generalizations*. Analysis and geometry of metric measure spaces, 197–220, CRM Proc. Lecture Notes, 56, Amer. Math. Soc., Providence, RI, 2013.