Magnus approximation in neutrino oscillations

Mario A. Acero, Alexis A. Aguilar-Arevalo and J. C. D’Olivo
Instituto de Ciencias Nucleares, Departamento de Física de Altas Energías, Universidad Nacional Autónoma de México (ICN-UNAM), Apdo. Postal 70-543, México, D.F. 04510, México.
E-mail: mario.acero@nucleares.unam.mx

Abstract. Oscillations between active and sterile neutrinos remain as an open possibility to explain some anomalous experimental observations. In a four-neutrino (three active plus one sterile) mixing scheme, we use the Magnus expansion of the evolution operator to study the evolution of neutrino flavor amplitudes within the Earth. We apply this formalism to calculate the transition probabilities from active to sterile neutrinos with energies of the order of a few GeV, taking into account the matter effect for a varying terrestrial density.

1. Introduction
Experimental results from solar, atmospheric, reactor and accelerator neutrino experiments, have proved the existence of neutrino mixing, showing also that neutrinos are massive particles. The dominant behavior of neutrino oscillations is well described by a three–neutrino picture with two mass splitting parameters and three mixing angles, plus a CP–violating phase.

Analytical treatments of the neutrino oscillation phenomenon have been implemented, in the framework of a two– or three–neutrino system, considering the matter potential felt by neutrinos traveling through the Earth [1, 2, 3, 4, 5, 6, 7]. It has been proved that the Magnus expansion [8] is a useful tool to obtain an approximate analytical description of matter effects on neutrino oscillations considering two or three neutrinos [9, 10, 11, 12].

On the other hand, some anomalous experimental results from the Gallium radioactive source experiments, LSND, MiniBooNE and MINOS collaborations, can be interpreted as an indication of the existence of additional (sterile) neutrinos (see for example Refs. [13, 14, 15, 16, 17, 18, 19, 20]).

Here we apply the Magnus expansion formalism to describe the propagation of neutrinos through the Earth, in the case of four neutrinos: the three standard active neutrinos plus and additional sterile neutrino.

2. Evolution of the four–neutrino system
The flavor amplitudes of the four–neutrino system at time \( t \) can be written as

\[
\begin{pmatrix}
\psi_e(t) \\
\psi_\mu (t) \\
\psi_\tau (t) \\
\psi_s(t)
\end{pmatrix}
= \mathcal{U}(t,t_0)
\begin{pmatrix}
\psi_e(t_0) \\
\psi_\mu (t_0) \\
\psi_\tau (t_0) \\
\psi_s(t_0)
\end{pmatrix}.
\]
where the evolution operator \( \mathcal{U}(t, t_0) \) satisfies the Schrödinger-like equation (\( \hbar = c = 1 \))

\[
\frac{d\mathcal{U}}{dt}(t, t_0) = H(t)\mathcal{U}(t, t_0),
\]

with the initial condition \( \mathcal{U}(t_0, t_0) = I \). In the flavor basis \( \{ |\nu_\alpha\rangle, \alpha = e, \mu, \tau, s \} \), the Hamiltonian \( H(t) \) takes the form

\[
H(t) = U H_0 U^\dagger + V(t),
\]

where \( V(t) = \text{diag}(V_{CC}(t), 0, 0, -V_{\text{NC}}(t)) \) is the matter potential, \( H_0 = \text{diag}(0, 0, \Delta_{31}, \Delta_{41}) \) and \( U \), is the mixing matrix that relates the neutrino flavor states to the states with definite masses \( m_i \), \( i = 1, \ldots, 4 \); \( |\nu_\alpha\rangle = \sum_i U^\alpha_i |\nu_i\rangle \). Here, \( \Delta_{ij} = \Delta m^2_{ij}/2E = (m_i^2 - m_j^2)/2E \), and \( V_{CC} \) and \( V_{\text{NC}} \) are the charge current and neutral current contributions to the matter effective potential for the active neutrinos. Since we are interested in neutrinos with an energy \( E \) of a few GeV and higher, the effect of the \( \Delta m^2_{21} \) mass splitting can be safely discarded and we put \( \Delta m^2_{21} = 0 \). Also we subtracted a common term \( V_{\text{NC}}(t) \), which has no observable effect on the oscillations.

Hereafter, we assume that neutrinos are of Dirac type and parametrize the mixing matrix as follows:

\[
U = R_{23} \Gamma_3 R_{34} \Gamma_3^* R_{14} \Gamma_2 R_{24} \Gamma_2^* \Gamma_1 R_{13} \Gamma_1^* R_{12},
\]

which helps to simplify the calculations. In this formula, \( R_{ij} \) are \( 4 \times 4 \) rotation matrices expressed in terms of the vacuum mixing angles \( \theta_{ij} \) and the diagonal matrices \( \Gamma_i \) include the three CP-violating phases \( \delta_i \) \( (i = 1, 2, 3) \) that appear in the present scheme. For \( \theta_{14} = 0 \), \( U \) reduces to the standard 3–neutrino mixing matrix [21], with \( \delta_1 \) identified to the corresponding CP-violating phase for this case.

It is interesting to see the important effect that the neutral current potential \( V_{\text{NC}} \) has on the four–neutrino system. If there were four active neutrinos (assuming that the sterile neutrino interacts as the \( \nu_\mu \) and \( \nu_\tau \)), one could expect a behavior similar to the three–neutrino system, given that the contribution of \( V_{\text{NC}} \) could be subtracted of from Eq. (3). This can be seen in Fig. 1, where the (numerical) evolution of the eigenvalues for a system composed by four active neutrinos is shown. In this case (where for completeness we used \( \Delta m^2_{21} \neq 0 \)), there three resonance regions which are associated with the mixing angles \( \theta_{12}, \theta_{13}, \theta_{14} \). After each of these regions, one of the neutrinos decouples from the rest.

On the other hand, a more complex evolution is observed when one of the four neutrinos is sterile (see Fig. 2). In this case, \( V_{\text{NC}} \) is not a common factor for all the neutrinos and the potential matrix \( V(t) \) in the Hamiltonian is the one given after Eq. (3). The effect of the neutral current potential is evident. The three resonances corresponding to those of Fig. 1 are also present here, but now the neutrinos do not decouple after each of them. On the contrary, as a direct effect of \( V_{\text{NC}} \) there appear other resonance regions related to the other mixing angles involved in the neutrino system.

For our analytical treatment, it is convenient to work within a new basis, by introducing the new vector \( \Psi \equiv \bar{U}^{\dagger} \Phi \), with \( U = R_{23} \Gamma_3 R_{34} \Gamma_3^* R_{14} \). Its evolution is driven by the Hamiltonian \( \hat{H} = \hat{U}^\dagger \hat{H} \hat{U} \) which, under the assumption \( \Delta m^2_{41} \gg \Delta m^2_{31}, V_{CC}, V_{\text{NC}} \), becomes

\[
\hat{H}(t) \cong \begin{pmatrix}
D_1 & 0 & Z_1^* & 0 \\
0 & D_3 & 0 & Z_2^* \\
Z_1 & 0 & D_3 & 0 \\
0 & Z_2 & 0 & D_4
\end{pmatrix},
\]

with

\[
\begin{align*}
D_1 &= \Delta_{31} s^2_{13} + V_{CC} c^2_{14} - V_{\text{NC}} s^2_{14} c^2_{34}, \\
D_3 &= \Delta_{41} s^2_{24}, \\
D_4 &= \Delta_{41} s^2_{24} + V_{CC} c^2_{14} - V_{\text{NC}} s^2_{14} c^2_{34}, \\
D_3 &= \Delta_{31} c^2_{13} - V_{\text{NC}} s^2_{14}, \\
Z_1 &= \Delta_{31} c^2_{13} s^2_{13} e^{i\theta_{13}} - V_{\text{NC}} s_{14} c_{34} s_{34} e^{i\theta_{23}}, \\
Z_2 &= \Delta_{41} c^2_{24} s_{24} e^{i\theta_{23}}.
\end{align*}
\]
For brevity we have defined $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$.

The matrix in Eq. (5) can be diagonalized by a time-dependent unitary transformation

$$U_m = \Gamma_2 R_{24}(\theta_{24}^m) \Gamma_2^* \Gamma_1 R_{13}(\theta_{13}^m) \Gamma_1^* R_{12}(\pi/2),$$

such that $U_m \hat{H} U_m^\dagger = H_D = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, with the energy eigenvalues in matter $\lambda_k(t)$ ordered in such a way that $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$. The mixing angles in matter are given by formulas similar to the familiar ones for two neutrinos:

$$\cos 2\theta_{13}^m = \frac{D_3 - D_1}{\sqrt{(D_3 - D_1)^2 + 4|Z_1|^2}}$$

$$\cos 2\theta_{24}^m = \frac{D_4 - D_2}{\sqrt{(D_4 - D_2)^2 + 4|Z_2|^2}}$$

Figure 3 shows the mixing angles in matter and the energy eigenvalues $\lambda_i$ as functions of the neutrino energy. We see that $\lambda_2$ and $\lambda_3$ are almost degenerate at around the same energy at which the angle $\theta_{13}^m$ exhibits the characteristic resonant behavior for the terrestrial density. Notice that, for the energy range under consideration, the largest eigenvalue $\lambda_4$ decreases as $E^{-1}$, while the smallest eigenvalue $\lambda_1$ remains constant as a function of the neutrino energy.

3. Solution of the evolution operator

The evolution operator in the flavor basis can be expressed as $\mathcal{U}(t, t_0) = O_m(t) \mathcal{U}_A(t, t_0) O_m^\dagger(t_0)$, where $O_m(t) = \hat{U} U_m(t)$ is the mixing matrix that relates the flavor states with the set $\{\nu_k^\nu(t)\}, k = 1, \ldots, 4$ of the (instantaneous) energy eigenstates in matter. The latter define the adiabatic basis and in it the evolution operator $\mathcal{U}_A(t, t_0)$ obeys Eq. (2) with the Hamiltonian

$$H_A(t) = H_D(t) - iU_m^\dagger(t) \hat{A} U_m(t),$$

where dot means differentiation with respect to time.

Discarding the second term in Eq. (10) corresponds to solving the problem in the adiabatic approximation. In any case, the time dependence generated by $H_D$ can be integrated exactly.
by a change of the representation accomplished by means of the unitary transformation \( P(t) = \exp[-\int_{t_0}^t dt' H_D(t')] \). For the remaining part, in general it is not possible to find an analytical solution in a close form and one has to rest on some approximation to determine it. As mentioned in the introduction, an appropriate procedure, which preserves unitarity, is based in the exponential expansion of the evolution operator. Accordingly, next we write \( \mathcal{U}_A(t, t_0) = P(t) \exp \Omega(t, t_0) \) and evaluate \( \Omega \) in terms of the first two terms of its Magnus expansion: \( \Omega \cong \Omega_1 + \Omega_2 \).

Proceeding in this manner, after some algebraic manipulations we arrive at:

\[
\mathcal{U}_A(t_f, t_0) = \mathcal{I}_{32} e^{-i\alpha_4} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & (c_\xi - is_\xi \xi(2)) e^{i\phi_{t \rightarrow t_f}} & 0 & 0 \\
0 & ie^{i \delta_1} s_\xi \xi(1) & (c_\xi + is_\xi \xi(2)) e^{-i\phi_{t \rightarrow t_f}} & 0 \\
0 & 0 & 0 & \mathcal{I}_{32} e^{-i\alpha_4}
\end{pmatrix}
\]

where \( \alpha_i = \int_{t_0}^{t_f} dt' \lambda_i(t') \) and

\[
\mathcal{I}_{32} = \exp \left\{ -\frac{i}{2} [\alpha_3(t) + \alpha_2(t)] \right\},
\]

\[
\phi_{t \rightarrow t_f} = \int_{t}^{t_f} dt' [\lambda_3(t') - \lambda_2(t')],
\]
and we have introduced the notations \( c_\xi = \cos \xi \), \( s_\xi = \sin \xi \), and \( \xi = \sqrt{\xi_{(1)}^2 + \xi_{(2)}^2} \). The quantities

\[
\begin{align*}
\xi_{(1)} &= 2 \int_0^t dt' \hat{\theta}^{\nu_{13}}_{13}(t') \sin \phi_{\nu \rightarrow \nu'}, \\
\xi_{(2)} &= \int_0^t dt' \int_0^{t'} dt'' \hat{\theta}^{\nu_{13}}_{13}(t') \hat{\theta}^{m}_{13}(t'') \sin \phi_{\nu' \rightarrow \nu''},
\end{align*}
\]

appear in the explicit calculation of \( \Omega_1(t_f, t_0) \) and \( \Omega_2(t_f, t_0) \), respectively [11], taking into account the fact that the potential is symmetric with respect to the middle point of the neutrino trajectory \( \tilde{t} = (t_f + t_0)/2 \).

4. Transition Probabilities

We are interested in the transition probabilities of muon neutrinos coming from the atmosphere and traversing the Earth. With the results of the previous section, we are able to compute the oscillation probabilities through the relation

\[
P_{\nu_\mu \rightarrow \nu_\alpha}(t) = |\Psi_\alpha(t)|^2 = |\mathcal{U}_A(t_f, t_0)\Psi_\mu(t_0)|^2 = |\hat{U}_m U_m(t)U_{\nu}(t_f, t_0)\Psi_\mu(t_0)|^2,
\]

which depends on the vacuum–mixing parameters through \( \hat{U} \) and on the matter potential by means of \( U_m \) and \( U(t_f, t_0) \).

In Figures 5 and 6, we show the behavior of the transition probabilities as a function of the neutrino energy for different values of the parameters involved in the analysis. We consider a simplified model for the electron density inside the Earth, the so-called mantle-core-mantle [22] (Fig. 4). One can see that the Magnus approximation agrees very well with the numerical calculation for \( E \gtrsim 7 \text{ GeV} \). As the plots show, the effect of the inclusion of the sterile neutrino generates noticeable changes on the \( \nu_\mu \rightarrow \nu_\alpha \) (\( \alpha = e, s \)) transition probabilities. Moreover, the \( \nu_\mu \rightarrow \nu_s \) transition remains as an open possibility. These effects increase for larger values of the three additional mixing angles \( \theta_{i4} \) (\( i = 1, 2, 3 \)). This is particularly relevant for \( \theta_{24} \) and \( \theta_{34} \), on which no stringent limits have been placed directly, contrary to the case of \( \theta_{14} \) that has been constrained by the Bugey experiment [23].

Finally, taking into account that neutrinos produced in the atmosphere reach the Earth’s surface with different zenith angle \( \Theta \) (see Fig. 4), in Figures 7 and 8 we plot contours of the oscillation probability \( P(\nu_\mu \rightarrow \nu_{e,s}) \) in vacuum and in the terrestrial matter, using our result of Eq. 16. It is important to note that we consider the initial neutrino state to be a muon neutrino, at the moment it arrives to the Earth surface, i.e., vacuum oscillations are assumed not to occurs in the atmosphere.

It is clear from the figures that Earth matter potential enhances the probability that muon neutrinos oscillate to electron neutrinos, through the MSW resonant effect [24, 25]. This effect is notorious in the high probability (\( > 40\% \)) regions shown in Fig. 8 at neutrino energies around \( 3 \text{ GeV} \) and \( 6 \text{ GeV} \). Being aware of some subtle differences, our results are in good agreement with those presented, for instance, in Ref. [26].

5. Conclusions

We used the Magnus expansion to calculate the flavor transition of a system composed by the three active neutrinos plus a sterile neutrino under the effect of the terrestrial potential. This approximation coincides very well with the numerical computations for high neutrino energies (\( E \gtrsim 7 \text{ GeV} \)). The inclusion of the fourth neutrino has a significative effect on the evolution of

5
Figure 5. $P(\nu_\mu \rightarrow \nu_e, s)$ as a function of the energy for a neutrino crossing the center of the Earth. Parameters: $\Delta m_{31}^2 = 2.5 \times 10^{-3}$ eV$^2$, $\Delta m_{41}^2 = 1.0$ eV$^2$, $\theta_{12} = 34^\circ$, $\theta_{13} = 7^\circ$, $\theta_{23} = 45^\circ$, $\theta_{14} = 5^\circ$.

Figure 6. $P(\nu_\mu \rightarrow \nu_e, s)$ as a function of the energy for a neutrino crossing the center of the Earth. The oscillation parameters are same as in Fig. 5, changing $\theta_{i4}$ to $\theta_{i4} = 5^\circ$, $\theta_{24} = \theta_{34} = 10^\circ$.

matter eigenstates of the system as well as on $P(\nu_\mu \rightarrow \nu_\alpha)$ which depends strongly on $\theta_{i4}$ and exhibits a rapid oscillatory behavior superimposed to the standard oscillations for the active neutrinos.

Acknowledgments
This work has been partially supported by PAPIIT-UNAM through grant IN117210 and CONACYT through grant 83534 and Red FAE. M. A. A. acknowledges support by a postdoctoral grant from the UNAM; he also thanks the organizers for the support to participate in MSPF2010.

References
[1] E. Lisi and D. Montanino, Phys. Rev. D 56, 1792 (1997).
Figure 7. $P(\nu_\mu \to \nu_e)$ in vacuum as a function of the neutrino energy and the zenith angle $\Theta$. Oscillation parameters are $\theta_{12} = 34^\circ$, $\theta_{13} = 11.5^\circ$, $\theta_{23} = 45^\circ$, $\theta_{14} = 0^\circ$, $\theta_{24} = \theta_{34} = 10^\circ$, $\Delta m^2_{31} = 2.5 \times 10^{-3}$ eV$^2$, $\Delta m^2_{41} = 0.1$ eV$^2$

Figure 8. $P(\nu_\mu \to \nu_e)$ in matter as a function of the neutrino energy and the zenith angle $\Theta$. Oscillation parameters are $\theta_{12} = 34^\circ$, $\theta_{13} = 11.5^\circ$, $\theta_{23} = 45^\circ$, $\theta_{14} = 0^\circ$, $\theta_{24} = \theta_{34} = 10^\circ$, $\Delta m^2_{31} = 2.5 \times 10^{-3}$ eV$^2$, $\Delta m^2_{41} = 0.1$ eV$^2$

[2] M. Freund, Phys. Rev. D 64, 053003 (2001).
[3] E. K. Akhmedov, M. A. Tortola and J. W. F. Valle, JHEP 0405, 057 (2004).
[4] M. Blennow and T. Ohlsson, J. Math. Phys. 45, 4053 (2004).
[5] P. C. de Holanda, W. Liao and A. Y. Smirnov, Nucl. Phys. B 702, 307 (2004).
[6] A. N. Ioannisian and A. Y. Smirnov, Phys. Rev. Lett. 93, 241801 (2004).
[7] A. N. Ioannisian, N. A. Kazarian, A. Y. Smirnov and D. Wyler, Phys. Rev. D 71, 033006 (2005).
[8] W. Magnus, Commun. Pure Appl. Math., 7, 649 (1954).
[9] J. C. D’Olivo, Phys. Rev. D 45, 924 (1992).
[10] J. C. D’Olivo and J. A. Oteo, Phys. Rev. D 54, 1187 (1996).
[11] A.D. Supanitsky, J.C. D’Olivo and G. Medina-Tanco, Phys. Rev. D 78, 0455024 (2008)
[12] A. Ioannisian and A.Yu. Smirnov, Nucl. Phys. B916, 94 (2009).
[13] M. Maltoni and T. Schwetz, Phys. Rev. D 76, 093005 (2007).
[14] M. A. Acero, C. Giunti and M. Laveder, Phys. Rev. D 78, 073009 (2008).
[15] Y. Farzan, T. Schwetz and A. Y. Smirnov, JHEP 0807, 067 (2008).
[16] G. Karagiorgi, Z. Djuricic, J. M. Conrad, M. H. Shaevitz and M. Sorel, Phys. Rev. D 80, 073001 (2009) [Erratum-ibid. D 81, 039902 (2010)].
[17] P. Adamson et al. [The MINOS Collaboration], Phys. Rev. D 82, 051102 (2010).
[18] P. Adamson et al. [The MINOS Collaboration], Phys. Rev. D 81, 052004 (2010).
[19] C. Giunti and M. Laveder, Phys. Rev. D 82, 053005 (2010).
[20] C. Giunti and M. Laveder, arXiv:1012.0267 [hep-ph].
[21] C. Amsler et al. [Particle Data Group], Phys. Lett. B 667, 1 (2008).
[22] Stacey, F. D. 1977, Physics of the earth, John Wiley & Sons, New York, NY (USA) 414 p.
[23] Y. Declais et al., Nucl. Phys. B 434, 503 (1995).
[24] L. Wolfenstein, Phys. Rev. D 17, 2369 (1978).
[25] S. P. Mikheyev, A. Yu. Smirnov, Yad. Fiz. 42, 1441 (1985); Sov. H. Nucl. Phys. 42.
[26] J. Hosaka et al. [Super-Kamiokande Collaboration], Phys. Rev. D 74, 032002 (2006).