A note about Euler’s inequality and automated reasoning with dynamic geometry

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Abstract. Using implicit loci in GeoGebra Euler’s $R \geq 2r$ inequality can be investigated in a novel way. Some unavoidable side effects of the implicit locus computation introduce unexpected algebraic curves. By using a mixture of symbolic and numerical methods a possible approach is sketched up to investigate the situation.

Keywords: Euler’s inequality, incircle, circumcircle, excircle, GeoGebra, computer algebra, computer aided mathematics education, automated theorem proving

1 GeoGebra: a symbolic tool for obtaining generalizations of geometric statements

GeoGebra [1] is a well known dynamic geometry software package with millions of users worldwide. Its main purpose is to visualize geometric knowledge of mathematical truth. Recently GeoGebra has been supporting investigation of geometric constructions also symbolically by harnessing the strength of the embedded computer algebra system (CAS) Giac [2]. One direct use of the embedded CAS is automated reasoning [3]. In this paper we use in particular the implicit locus derivation feature [4] in GeoGebra, by using the command LocusEquation with a Boolean expression $B$ and the sought mover point $M$.

Obtaining implicit loci is a recent method in GeoGebra to get interesting facts on classic theorems. These facts are closely related to algebraic curves which usually describe generalization of the classic results. Sometimes it is computationally difficult to obtain the curves quickly enough, but some new improvements in Giac’s elimination algorithm opened the road to effectively investigate a large number of geometric constructions [5] including Holfeld’s 35th problem [5,7], a generalization of the Steiner-Lehmus theorem [5,8] or the right triangle altitude theorem [3].

We need to admit that the possibility to generalize well known theorems is a consequence of using unordered geometry [9] p. 97] in the applied tools and
theories. In unordered geometry one cannot designate only one intersection point of a line and a conic (or two conics), so both will be considered at the same time. This results in obtaining a larger set of points for the resulting algebraic curve as expected. The obtained set may be inconvenient in some cases, but can be fruitful to obtain some interesting generalizations.

2 Euler’s inequality

Since GeoGebra’s Automated Reasoning Tools \cite{3} use Gröbner bases in the background, inequalities cannot really be investigated by them automatically. Certain experiments can however be started by fixing the ratio of the studied quantities, here $R$ and $r$. For example, one can start with some concrete experience by comparing $R$ and say $3r$ (see Fig. 1), and then simply change the constant 3 to some different value.

![Fig. 1](image)

**Fig. 1.** An implicit curve (in red) as the output of GeoGebra command `LocusEquation[R==3r,C]`. Here points $A$ and $B$ are fixed in the plane and $C$ is a free point. (In other words: Triangle $ABC$ has fixed vertices $A$ and $B$.) The computation of the sought set of mover point $C$ is a moderately difficult problem: an Intel(R) Core(TM) i7 CPU 860 @ 2.80GHz computer requires 3.7 seconds for the whole computation.

The result seems complicated for the first look. By doing some more experiments, it turns out that the two inner oval parts of the curve show relevant information on the concrete question, but the other parts show something different. That is, by setting $C$ to an arbitrary point of the inner oval parts, the
equality $R = 3r$ will occur. For the other parts we will see later in Sec. 2.2 that the radii $r_a$, $r_b$ and $r_c$ of the excircles will take the role of $r$ over.

After doing further experiments by changing the constant 3 to lower values, when getting close to 2 the inner oval parts seem to disappear even more and more (Fig. 2), and finally for the experiment $R = 2r$ the inner oval parts are not visible any longer (Fig. 3).

![Fig. 2. Result of LocusEquation$[R=2.1r, C]$ and LocusEquation$[R=2.01r, C]$. To properly plot the latter a suitable zoom factor may be required due to possible inaccuracies in the plotting routine in GeoGebra.](image)

The first confusing result is why the points $\left(\frac{1}{2}, \pm \frac{\sqrt{3}}{2}\right)$ are not plotted in this graph—we recall that the equality holds if and only if the triangle is equilateral. Unfortunately, the plotting routine in GeoGebra does not show this isolated point. In fact, other systems (including Wolfram\,Alpha and Desmos) are also unable to automatically plot even the easiest examples of a very similar situation, namely that a curve has an acnode. Such a basic example is the curve $x^3 - x^2 - y^2 = 0$ for which the point $(0,0)$ is not shown in the graph, but is clearly an isolated point of the curve $10$. 
Fig. 3. The inner oval parts disappear when plotting the case $R = 2r$.

The result of the command `LocusEquation[R==2r, C]` is

$$
\begin{align*}
&x^{18} - 225 y^{18} - 1481 x^2 y^{16} - 4004 x^4 y^{14} - 5460 x^6 y^{12} \\
&- 3262 x^8 y^{10} + 770 x^{10} y^8 + 2604 x^{12} y^6 + 1756 x^{14} y^4 \\
&+ 535 x^{16} y^2 - 504 x^{17} + 1256 x y^{16} + 6752 x^3 y^{14} \\
&+ 13632 x^5 y^{12} + 10336 x^7 y^{10} - 4400 x^9 y^8 - 14304 x^{11} y^6 \\
&- 11008 x^{13} y^4 - 3808 x^{15} y^2 + 1764 x^{16} + 1276 y^{16} \\
&+ 1416 x^2 y^{14} - 6936 x^4 y^{12} - 11544 x^6 y^{10} + 8008 x^8 y^8 \\
&+ 33048 x^{10} y^6 + 30104 x^{12} y^4 + 11896 x^{14} y^2 - 3528 x^{15} \\
&- 2888 x y^{14} + 2296 x^3 y^{12} + 6296 x^5 y^{10} - 7000 x^7 y^8 \\
&- 41944 x^9 y^6 - 47080 x^{11} y^4 - 21368 x^{13} y^2 + 4410 x^{14} \\
&- 1094 y^{14} + 2854 x^2 y^{12} - 1262 x^4 y^{10} - 370 x^6 y^8 \\
&+ 31246 x^8 y^6 + 46226 x^{10} y^4 + 24230 x^{12} y^2 - 3528 x^{13} \\
&- 2888 x y^{12} + 592 x^3 y^{10} + 5704 x^5 y^8 - 12704 x^7 y^6 \\
&- 29240 x^9 y^4 - 17840 x^{11} y^2 + 1764 x^{12} + 1276 y^{12} \\
&- 1136 x^2 y^{10} - 5940 x^4 y^8 + 1472 x^6 y^6 + 11604 x^8 y^4 \\
&+ 8368 x^{10} y^2 - 504 x^{11} + 1256 x y^{10} + 2984 x^3 y^8 \\
&+ 912 x^5 y^6 - 2608 x^7 y^4 - 2296 x^9 y^2 + 63 x^{10} - 225 y^{10} \\
&- 581 x^2 y^8 - 330 x^4 y^6 + 246 x^6 y^4 + 283 x^8 y^2 = 0.
\end{align*}
$$

By using GeoGebra’s `Substitute[I, {x=1/2, y=sqrt(3)/2}]` command (here $I$ denotes the obtained implicit curve object) we get $0 = 0$ which shows that the
expected point is indeed an element of the curve. The same result can be seen for the point \( \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right) \).

The obtained polynomial can be factored by using GeoGebra’s \texttt{Factor[LeftSide[I]-RightSide[I]]} command. The factorization is

\[
(x^2 + y^2) \cdot (7x^8 - 28x^7 + 12x^6y^2 + 42x^6 - 36x^5y^2 - 28x^5 - 6x^4y^4 + 34x^4y^2 + 7x^4 + 12x^3y^4 - 8x^3y^2 - 20x^2y^6 - 26x^2y^4 - 2x^2y^2 + 20x^2y^4 + 20x^4y^6 + 46y^6 - 9y^4)
\]

\[
\cdot (9x^8 - 36x^7 + 52x^6y^2 + 54x^6 - 156x^5y^2 - 36x^5 + 102x^4y^4 + 190x^4y^2 + 9x^4 - 204x^3y^4 - 120x^3y^2 + 84x^2y^6 + 186x^2y^4 + 34x^2y^2 - 84x^6y^4 - 84x^4y^4 + 25y^8 - 14y^6 + 25y^4).
\]

Here the first factor \( p_1 = x^2 + y^2 \) clearly corresponds to the point \( A \). (It is not unusual to get non-symmetrical result with respect to the vertices, since both \( R \) and \( r \) were constructed by using two arbitrary angle/perpendicular bisectors for intersection.) The second factor \( p_2 = 7x^8 - \ldots \) shows all real points of the curve \( I \) (without the points \( \left( \frac{1}{2}, \pm \frac{\sqrt{3}}{2} \right) \)), and the third factor \( p_3 = 9x^8 - \ldots \) has seemingly no real points, but after computing its acnodes by solving the inequality system \( p_3 = 0, (p_3)_x' = 0, (p_3)_y' = 0, H(p_3) > 0 \), where \( H \) denotes the Hessian matrix, we may explore symbolically that the polynomial indeed describes the two expected isolated real points as well.

This approach with the Hessian cannot be achieved in GeoGebra. Instead, a numerical way can be tried to visualize the function \( f(x, y) = p_3 \) in 3 dimensions (Fig. 4) to find the real roots, namely \( \left( \frac{1}{2}, \pm \frac{\sqrt{3}}{2} \right), (0, 0) \) and \( (1, 0) \). Also in some other computer algebra systems a contour plot may help (see Fig. 5).

Finally we remark that by using Maple’s \texttt{evala(AFactor(\ldots))} command we can verify that \( p_2 \) and \( p_3 \) are irreducible over \( \mathbb{R} \).

2.1 Summary of the difficulties

The above shows some difficulties in our case. First of all, by using Gröbner bases there is no automatic way to obtain Euler’s inequality. One needs to start some experiments by choosing the ratio between \( R \) and \( r \) randomly. This problem can be resolved by using real geometry and quantifier elimination.

The second problem is that the plotted graph can be inaccurate: the equilateral case for \( R = 2r \) cannot be read off by the user. It would be expected that the output curve should contain the set of points where the equality holds—this does not seem to be the case here because of the failure of the plotting algorithm. The case of failure even for some easy cubic examples show that this problem cannot be easily worked around.

For similar reasons the factorization does not directly help finding the equilateral case, either. Only a 3D plot—actually a numerical approach—gives some hints where to look for the equality.
Fig. 4. A 3D plot of $p_3$ in GeoGebra. Here we used the command

\[ f(x,y) := \text{Element}[\text{Factors}[\text{LeftSide}[I]-\text{RightSide}[I]],3,1] \]

and opened the Graphics 3D View.
2.2 Why the octic?

Similarly to the Steiner-Lehmur generalization in [8] here we silently introduced three other circles as extensions of the incircle. They are the excircles—in unordered geometry one cannot distinguish between internal and external angle bisectors.

After some experimenting it can be concluded that different sections of the octic describe different circles among the three excircles (Fig. 6).

2.3 The inequality does not hold for excircles

Continuing the process that changing the constant 3 to lower values, including less numbers than 2, we learn that the inner oval parts of the curve will not be visible any longer. This is the case e. g. for 1.9: there are no visible inner oval parts (and they do not exist, either, because of Euler’s inequality), but the other parts still do (Fig. 7). This supports the idea that the inequality with respect to \( r \) cannot be transferred to \( r_a \), \( r_b \) or \( r_c \). That is, in this way Euler’s inequality cannot be generalized on excircles.

3 Conclusion

We used a novel method to obtain implicit loci in GeoGebra to investigate Euler’s inequality. This well known statement can also be approached by a mixture
Fig. 6. Various parts of the octic show the sought moving points $C$ for statements $R = 2r_a$ (red), $R = 2r_b$ (green) and $R = 2r_c$ (blue). The figure was produced with GeoGebra by attaching a point to the octic, constructing $r_a$, $r_b$, $r_c$ and $R$ by using the attached point as $C$, and computing which excircle would be connected with the appropriately chosen attached point—then the color of $C$ was dynamically set by using the RGB scheme. Finally tracing and animation was switched on for $C$.

Fig. 7. Result of LocusEquation[$1.9r=R,C$].
of symbolic and numerical observations. Our experiments are clearly not ac-
ceptable as a new way of proof, but steps to claim promising conjectures. For
other investigations of classic or new statements, that is, to generalize geometric
equations or inequalities, this kind of approach may be hopefully fruitful.

Also we highlight that a better approach might be to use real geometry and
quantifier elimination. To find the most efficient way to formalize and prove
Euler’s inequality and present it in an adequate form in a dynamic geometry
software tool is an on-going work of the authors.

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