ON THE STABLE RANK OF ALGEBRAS OF OPERATOR FIELDS OVER METRIC SPACES

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ABSTRACT. Let $\Gamma$ be a finitely generated, torsion-free, two step nilpotent group. Let $C^*(\Gamma)$ be the universal $C^*$-algebra of $\Gamma$. We show that $\text{acsr}(C^*(\Gamma)) = \text{acsr}(C((\hat{\Gamma}))_1)$, where for a unital $C^*$-algebra $A$, $\text{acsr}(A)$ is the absolute connected stable rank of $A$, and where $(\hat{\Gamma})_1$ is the space of one-dimensional representations of $\Gamma$. For the case of stable rank, we have close results. In the process, we give a stable rank estimate for maximal full algebras of operator fields over metric spaces.

1. INTRODUCTION

Rieffel [17] introduced the notion of stable rank for $C^*$-algebras as the noncommutative version of complex dimension of ordinary topological spaces. It turns out that the stable rank of a unital $C^*$-algebra is the same as its Bass stable rank (see [8]).

There has been much work in computing the stable ranks of the universal $C^*$-algebras of various connected Lie groups. The greatest progress has been made in the case of type I solvable Lie groups (see [20], [22] and [23]). Roughly speaking, it has been shown that the stable rank of the universal $C^*$-algebra of a type I solvable Lie group $G$ is controlled by the ordinary topological dimension of the space of one-dimensional representations of $G$.

Recently, the stable ranks of the universal $C^*$-algebras of a class of non-type I solvable Lie groups (which include the Mautner group) have been computed (see [21]).

In this paper, we compute the stable ranks of the universal $C^*$-algebras of a class of non-type I amenable discrete groups. Specifically, our main result is

Theorem 1.1. Let $\Gamma$ be a finitely generated, torsion-free, two step nilpotent group. Let $C^*(\Gamma)$ be the universal $C^*$-algebra of $\Gamma$. Then

1. $\text{acsr}(C^*(\Gamma)) = \text{acsr}(C((\hat{\Gamma}))_1)$,
2. $\text{sr}(C((\hat{\Gamma}))_1) \leq \text{sr}(C^*(\Gamma)) \leq \text{sr}(C((\hat{\Gamma}))_1) + 1$,
3. if the topological dimension $\text{dim}(\hat{\Gamma})_1$ is even, then $\text{sr}(C^*(\Gamma)) = \text{sr}(C((\hat{\Gamma}))_1)$.

Here $(\hat{\Gamma})_1$ is the space of one-dimensional representations of $\Gamma$. Also, for a unital $C^*$-algebra $A$, $\text{sr}(A)$ is the stable rank of $A$, and $\text{acsr}(A)$ is the absolute connected stable rank of $A$.

We note that for a unital $C^*$-algebra $A$, the absolute connected stable rank of $A$ is numerically the same as the stable rank of the tensor product $C[0,1] \otimes A$.

A key step in our proof of Theorem 1.1, is the following stable rank estimate for algebras of operator fields over metric spaces, which is of independent interest:

Theorem 1.2. Suppose that $X$ is a $\sigma$-compact, locally compact, $k$-dimensional metric space. Suppose that $A$ is a maximal full algebra of operator fields over $X$ with fibre algebras, say, $\{A_t\}_{t \in X}$ such that $A_t$ is unital for all $t \in X$. Then the stable rank of $A$ satisfies the inequality

$$\text{sr}(A) \leq \sup_{t \in X} \text{sr}(C([0,1]^k) \otimes A_t).$$

We note that Theorems 1.1 and 1.2 generalize results from [13], where we compute the stable ranks of the universal $C^*$-algebras of the (possibly higher rank) discrete Heisenberg groups.

General references for stable rank are [14] and [17]. General references for full algebras of operator fields are [6], [11] and [24]. General references for the representation theory of finitely generated, two-step nilpotent groups are [2], [15] and [16] (also see [9]).

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In what follows, for a $C^*$-algebra $\mathcal{A}$, “sr($\mathcal{A}$)” and “acsr($\mathcal{A}$)” will denote the stable rank and absolute connected stable rank of $\mathcal{A}$ respectively. If, in addition, $\mathcal{A}$ is unital, then for every positive integer $M > 0$, $Lg_M(\mathcal{A})$ will be the set of all $M$-tuples $(a_1, a_2, ..., a_M)$ in $\mathcal{A}^M$ such that $\sum_{j=1}^{M} (a_j)^*a_j$ is an invertible element of $\mathcal{A}$. Also, for a metric space $X$, for a point $x \in X$ and real number $r > 0$, “$B(x,r)$” will denote the open ball of radius $r$ (with respect to the metric on $X$) about $x$.

2. Main results

In [13], we proved the following result:

**Theorem 2.1.** Suppose that $\mathcal{A}$ is a unital maximal full algebra of operator fields with base space the $k$-cube $[0,1]^k$ and fibre algebras, say, $\{\mathcal{A}_t\}_{t \in [0,1]^k}$. Then the stable rank of $\mathcal{A}$ satisfies the inequality

$$sr(\mathcal{A}) \leq \sup_{t \in [0,1]^k} sr(C([0,1]^k) \otimes \mathcal{A}_t).$$

The key technique within the proof of Theorem 2.1, was the following technical result, which we state as a lemma:

**Lemma 2.2.** Suppose that $\mathcal{A}$ is a unital maximal full algebra of operator fields with base space $[0,1]$ and fibre algebras, say, $\{\mathcal{A}_t\}_{t \in [0,1]}$. Suppose that $M = \sup_{t \in [0,1]} sr(C[0,1] \otimes \mathcal{A}_t)$ is a finite number. Let $q$, $r$ be real numbers, with $0 < q < r < 1$, and let $\mathcal{A}([0,r])$ and $\mathcal{A}([q,1])$ be the restrictions of the operator fields in $\mathcal{A}$ to $[0,r]$ and $[q,1]$ respectively. Now let $\epsilon > 0$ be given and suppose that for $j = 1, 2, ..., M$, $\{a_j(t)\}_{t \in [0,r]}$ is an operator field in $\mathcal{A}([0,r])$ and $\{b_j(t)\}_{t \in [q,1]}$ is an operator field in $\mathcal{A}([q,1])$ such that

(a) $|a_j(t) - b_j(t)| < \epsilon$ for all $t \in [q, r]$ and for $j = 1, 2, ..., M$,

(b) $\sum_{j=1}^{M} (a_j(t))^*a_j(t)$ is an invertible element of $\mathcal{A}_t$ for all $t \in [0,r]$, and

(c) $\sum_{j=1}^{M} (b_j(t))^*b_j(t)$ is an invertible element of $\mathcal{A}_t$ for all $t \in [q,1]$.

Then there are operator fields $\{c_j(t)\}_{t \in [0,1]}$ in $\mathcal{A}$, $j = 1, 2, ..., M$, such that

1. $|c_j(t) - a_j(t)| < \epsilon$ for all $t \in [0,r]$ and for $j = 1, 2, ..., M$,

2. $|c_j(t) - b_j(t)| < \epsilon$ for all $t \in [q,1]$ and for $j = 1, 2, ..., M$, and

3. $\sum_{j=1}^{M} (c_j(t))^*c_j(t)$ is invertible in $\mathcal{A}_t$ for all $t \in [0,1]$.

**Proof of Theorem 1.2.** By [12] IV.7 page 85 last paragraph and [19] Theorem 1.1, we may assume that we have a metric on $X$ such that for every point $x \in X$ and for every real number $r > 0$, the boundary of the open ball $B(x, r)$ (with respect to this new metric) is at most $k - 1$-dimensional. Henceforth, we will be working with this metric.

Suppose that $X$ is noncompact. Then let $X_{\infty}$ be the one-point compactification of $X$, with point at infinity $\infty$. We may view $\mathcal{A}$ as a maximal full algebra of operator fields with base space $X_{\infty}$ and fibre algebras $\{\mathcal{A}_t\}_{t \in X_{\infty}}$, where $\mathcal{A}_t$ is the same as before when $t \neq \infty$, and $X_{\infty} = \{0\}$ the zero $C^*$-algebra. Let $\mathcal{A}^+$ be the unitization of $\mathcal{A}$. By [10] Theorem 1 and Corollary 1, $\mathcal{A}^+$ is a unital maximal full algebra of operator fields with base space $X_{\infty}$ and fibre algebras $\{(\mathcal{A}_t)\}_{t \in X_{\infty}}$ where $(\mathcal{A}_t)_{t \in X_{\infty}} = \mathcal{A}_t$ for $t \neq \infty$ and $(\mathcal{A}_\infty) = \mathbb{C}$ (the complex numbers). A continuity structure $\mathcal{F}$ is the set of all operator fields of the form $a + \alpha t$, where $a$ is in $\mathcal{A}$ and $\alpha$ is a complex number. If $X$ is compact, then $\mathcal{A}$ will automatically be unital, and (in the arguments that follow) we let $X_{\infty} = X$ and $\mathcal{A}^+ = \mathcal{A}$, and we need not consider the point $\infty$ at all.

Now suppose that $M = \sup_{t \in X} sr(C([0,1]^k) \otimes \mathcal{A}_t)$ is a finite number. Let $(a_1, a_2, ..., a_M)$ be an $M$-tuple in $(\mathcal{A}^+)^M$ and let $\epsilon > 0$ be given. By adding a small scalar multiple of the unit if necessary, we may assume that $(a_1, a_2, ..., a_M)$ is nonzero at $\infty$ (noncompact) case. We may also assume that $\epsilon$ is small enough so that for any other $M$-tuple $(c_1, c_2, ..., c_M)$, if $c_j(\infty)$ is within $\epsilon$ of $a_j(\infty)$ for all $j$, then the $M$-tuple $(c_1, c_2, ..., c_M)$ is also not the zero vector at $\infty$.

Now we can choose a sequence of nonempty open balls $\{B(x_i, r_i)\}_{i=1}^\infty$ in $X$ and a sequence of $M$-tuples $\{(f_{i,1}, f_{i,2}, ..., f_{i,M})\}_{i=1}^\infty$ in $(\mathcal{A}^+)^M$ such that

(a) $X$ is covered by the union of all the open balls $B(x_i, r_i)$, $r = 1, 2, ...,$

(b) for every $i, i = 1, 2, 3, ...$ there is a strictly positive number $\delta_i$ such that $\sum_{j=1}^{M} f_{i,j}(t)^*f_{i,j}(t)$ is an invertible element of $\mathcal{A}_t$ for all $t \in B(x_i, r_i + \delta_i)$, and

(c) there is an increasing sequence of integers $\{N_n\}_{n=1}^\infty$ such that $f_{i,j}$ is within $\epsilon/2^n$ of $a_j$ for $i \geq N_n$, $i = 1, 2, 3, ...$ and $1 \leq j \leq M$. 

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Condition (a) uses the $\sigma$-compactness of $X$. Condition (b) requires the use of existence and continuity of operator fields in full algebras of operator fields (see the definition of full algebras of operator fields in [6], [10] or [24]). One also needs to use the fact that in a unital $C^*$-algebra, any element close enough to the unit is invertible. Condition (c) requires the maximality of the full algebra of operator fields (see [10] Proposition 1 and [24] Theorem 1.1) as well as the $\sigma$-compactness of $X$. Henceforth, we let \( (+) \) denote properties (a) - (c) collectively.

By the $\sigma$-compactness of $X$, we may additionally assume that such that for every $n$, if $i \leq N_n$ and $j > N_n$ then $r_j < (1/8)r_i$, for all $i, j$, i.e., the size of the open balls $B(x_i, r_i)$ are, approximately, “decreasing uniformly” with rate $1/8$. We may also assume that for each $i$, $i = 1, 2, 3, \ldots$, the closure of $B(x_i, r_i + \delta_i)$ is a compact subset of $X$.

Our procedure for constructing an $M$-tuple in $Lg_M(A^+)$ which will approximate $(a_1, a_2, \ldots, a_M)$ to within $\epsilon$ is to construct $M$ sequences of operator fields \( \{a_j^n\}_{n=1}^\infty \) $j = 1, 2, \ldots, M$, which satisfy the following conditions:

1. $a_j^n$ is an operator field over $\bigcup_{i=1}^{N_n} B(x_i, r_i)$ for $n = 1, 2, 3, \ldots$ and for $1 \leq j \leq M$,
2. $a_j^n$ is within $\epsilon/2^n$ of $a_j$ over $\bigcup_{i=1}^{N_n} B(x_i, r_i)$,
3. $a_j^n$ is within $\epsilon/2^n$ of $a_j$ over $B(x_{N_n}, r_{N_n})$,
4. $\sum_{j=1}^M (\alpha_j^n(t) - \alpha_j^n(t))$ is invertible in $A_t$ for all $t \in B(x_i, r_i)$ and for $i \leq N_n$, and
5. for $i \leq m \leq n$, $(\alpha_j^n, \alpha_j^{n+1})$ $(\alpha_j^n, \alpha_j^{n+1})$ over the ball $B(x_i, r_i/2)$.

We let \( (*) \) denote conditions (1) - (4) collectively.

For simplicity, let us assume that for every integer $n$, $N_n = n$. We now construct the operator fields \( \{a_j^n\}_{n=1}^\infty \) $1 \leq j \leq M$, recursively on $n$ (for all $j$ at each step $n$). For $n = 1$, just let \( \{\alpha_1, \alpha_2, \ldots, \alpha_M\} = (f_{1,1}, f_{1,2}, \ldots, f_{1,M}) \). Now suppose that \( \{\alpha_1^n, \alpha_2^n, \ldots, \alpha_M^n\} \) has been constructed. To construct \( \{\alpha_1^{n+1}, \alpha_2^{n+1}, \ldots, \alpha_M^{n+1}\} \), we need to “connect” \((f_{n+1,1}, f_{n+1,2}, \ldots, f_{n+1,M})\) with \( \{\alpha_1^n, \alpha_2^n, \ldots, \alpha_M^n\} \) over an appropriate subset of $X$. We may assume that $\bigcup_{i=1}^{N_n} B(x_i, r_i)$ is nonempty (for otherwise, it would be immediate).

Let $d$ be the positive real number which is the minimum of the quantities $\delta_{n+1}$ and $r_{n+1}$. Let $F$ be the set of all points $x$ in $\bigcup_{i=1}^{N_n} B(x_i, r_i)$ whose distance from $x_{n+1}$ is between (and including) $r_{n+1}$ and $r_{n+1} + d$. For $s \in [0, 1]$, let $F_s$ be the set of points in $F$ which have distance $(1-s)r_{n+1} + s(r_{n+1} + d)$ from $x_{n+1}$. Let $A(F)$ be the $C^*$-algebra gotten by taking the restriction of the operator fields in $A$ to $F$. Then $A(F)$ can be realized as a unital maximal full algebra of operator fields with base space $[0, 1]$ and fibre algebras, say, $\{B_s\}_{s \in [0, 1]}$. For each $s \in [0, 1]$, the fibre algebra $B_s$ is the restriction of $A$ to $F_s$, and for each element $a \in A$, its fibre at $s \in [0, 1]$ (with respect to this continuous field representation) is the restriction of $a$ to $F_s$. Continuity and maximality follows from the continuity and maximality of the algebra of operator fields $A$.

Therefore, $C[0, 1] \otimes A(F)$ can be realized as a unital maximal full algebra of operator fields with base space $[0, 1]$ and fibre algebras $\{C[0, 1] \otimes B_s\}_{s \in [0, 1]}$. The continuity structure consists of all operator fields of the form $s \mapsto \sum_{i=1}^N f_i \otimes b_i(s)$, where the $f_i$s are in $C[0, 1]$ and the $b_i$s are continuous operator fields in $A(F)$ (with respect to the continuous field decomposition of $A(F)$ in the previous paragraph). Hence, by Theorem 2.1, the stable rank of $C[0, 1] \otimes A(F)$ satisfies $sr(C[0, 1] \otimes A(F)) \leq sup_{s \in [0, 1]} sr(C[0, 1] \otimes B_s)$.

But for $s \in [0, 1]$, $B_s$ can be realized as a unital maximal full algebra of operator fields with base space $F_s$ (a compact metric space) and fibre algebras $\{A_t\}_{t \in F_s}$. (Since $B_s$ is the restriction of $A$ to $F_s$. Hence, for $s \in [0, 1]$, $C[0, 1] \otimes B_s$ can be realized as a unital maximal full algebra of operator fields with base space $F_s$ and fibre algebras $\{C[0, 1] \otimes A_t\}_{t \in F_s}$. But by our assumption on the metric in the first paragraph of this proof, $F_s$ is a metric space with dimension less than or equal to $k - 1$. Hence, we have, by induction, that $sr(C[0, 1] \otimes B_s) \leq sup_{t \in F_s} sr(C[0, 1] \otimes B_{s \otimes t})$ (the induction is on the dimension of the base space). Note that when $F_s$ is zero-dimensional, the stable rank estimate will be immediate, since we can choose a finite, clopen covering for $F_s$, which satisfies the properties in (+). Hence the base case is immediate). From this and the previous paragraph, $sr(C[0, 1] \otimes A(F)) \leq M$.

By Lemma 2.2, it follows that there is an $M$-tuple of operator fields $\{a_j^{n+1}, a_j^{n+1}, \ldots, a_j^{n+1}\}$ on $B(x_{n+1}, r_{n+1}) \cup \bigcup_{i=1}^M B(x_i, r_i)$ such that $a_j^{n+1} = f_{n+1,j}$ on $B(x_{n+1}, r_{n+1})$, $a_j^{n+1} = a_j^n$ on $\bigcup_{i=1}^M B(x_i, r_i) - F$, and the $a_j^{n+1}$s satisfy (1), (2) and (4) in (*). Condition (5) in (*) is satisfied, since $d$ was chosen to be less than or equal to $r_{n+1}$, and the latter is strictly less than $(1/8)r_n$. Finally, condition (3) in (*) is satisfied since $f_{n+1,j}$ is within $\epsilon/2^{n+1}$ of $a_j$. 
In the general case where \( N_{n+1} \) is not necessarily equal to \( n+1 \), we need to repeat the preceding procedure a finite number of times, in the natural way, in order to go from \((\alpha_1^n, \alpha_2^n, ..., \alpha_M^n)\) to \((\alpha_1^{n+1}, \alpha_2^{n+1}, ..., \alpha_M^{n+1})\).

Also, when \( X \) is compact, the preceding procedure will stop at finitely many steps and the sequences \( \{\alpha_j^{n}\}_{n=1}^{\infty}, 1 \leq j \leq M, \) will all be finite. We leave to the reader the obvious modifications that need to be made.

Now suppose that we have constructed sequences of operator fields \( \{\alpha_j^{n}\}_{n=1}^{\infty}, j = 1, 2, ..., M \) as in (*). We then construct an \( M \)-tuple of continuous operator fields in \((A^+)^M\) as follows: let \( \alpha_j(t) = \alpha_j^n(t) \) for \( t \in B(x_n, r_n/2) \) and (in the noncompact case) let \( \alpha_j(\infty) = a_j(\infty) \). Continuity at the point \( \infty \) is ensured by condition (3) in (*). Then \( \alpha_j \) is within \( \epsilon/2 \) of \( a_j \) for \( j = 1, 2, ..., M \). Moreover, that \( \sum_{j=1}^{M}(\alpha_j(\infty))^*a_j(\infty) \) is invertible follows from the invertibility of \( \sum_{j=1}^{M}(a_j(\infty))^*a_j(\infty) \) and the smallness of the \( \epsilon \), both of which were assumed at the beginning. Hence, \((\alpha_1, \alpha_2, ..., \alpha_M) \in Lg_M(A^+) \) and \( \alpha_j \) is within \( \epsilon \) of \( a_j \) for \( j = 1, 2, ..., M \).

\[ \square \]

The result of the next computation is surely known (See [18], the comments after the proof of Proposition 3.10).

**Lemma 2.3.** If \( A_\Theta \) is a simple noncommutative torus and \( \mathbb{T}^k \) the ordinary \( k \)-torus, then \( sr(C(\mathbb{T}^k) \otimes A_\Theta) = 2 \).

**Proof.** By [3] Theorem 1.5 and [17] the proof of Corollary 7.2, \( sr(C(\mathbb{T}^k) \otimes A_\Theta) \) is a finite number. Hence by [17] Theorem 6.1, let \( l \) be a positive integer such that both \( sr(M_{2l}(C) \otimes C(\mathbb{T}^k) \otimes A_\Theta) \) and \( sr(M_{2l}(C) \otimes (C(\mathbb{T}^k) \otimes A_\Theta) \) are less than or equal to \( 2 \). Let \( A_\Theta = \bigcup_{n=1}^{\infty} A_n \) be the inductive limit decomposition of \( A_\Theta \) given in [3] Corollary 2.10.

Now let a positive real number \( \epsilon > 0 \) and a positive integer \( m > 0 \) be given. Let \( a_1 \) and \( a_2 \) be arbitrary elements of \( C(\mathbb{T}^k) \otimes A_\Theta \). Choose an integer \( n > m \) such that there are \((b_1, b_2) \in L_{2l}(M_{2l}(C) \otimes C(\mathbb{T}^k) \otimes A_\Theta) \) and \((c_1, c_2) \in L_{2l}(M_{2l}(C) \otimes C(\mathbb{T}^k) \otimes A_\Theta) \), with \( b_j \) within \( \epsilon \) of \( a_j \otimes 1_{M_{2l}} \) and \( c_j \) within \( \epsilon \) of \( a_j \otimes 1_{M_{2l}} \), \( j = 1, 2 \). Then it follows from the proof of [3] Corollary 2.10, that we can choose an integer \( N > n \) and choose a finite dimensional subalgebra \( B \subseteq A_N \) such that there exists \((d_1, d_2) \in L_{2l}(C(\mathbb{T}^k) \otimes C^*(A_n, B)) \) with \( d_j \) being within \( \epsilon \) of \( a_j \) for \( j = 1, 2 \). But, \( m \) and \( a_j \) were arbitrary. Hence, \( sr(C(\mathbb{T}^k) \otimes A_\Theta) \leq 2 \).

Now \( K_1(A_\Theta) = \mathbb{Z}^{2^{n-1}} \neq 0 \) where \( p \) is the dimension of the noncommutative torus \( A_\Theta \) (i.e., \( A_\Theta \) is a noncommutative \( p \)-torus). So we can find a positive integer \( n \) such that \( GL_n(A_\Theta) \neq GL_n(A_\Theta_0) \), where \( GL_n(A_\Theta) \) is the group of invertibles in \( M_n(A_\Theta) \) and \( GL_n(A_\Theta) \) is the connected component of the identity in \( GL_n(A_\Theta) \). Therefore, the connected stable rank \( csr(M_n(A_\Theta)) \geq 2 \). But \( csr(M_n(A_\Theta)) \leq sr(M_{n}(C) \otimes C(\mathbb{T}^k) \otimes A_\Theta) \leq sr(C(\mathbb{T}^k) \otimes A_\Theta) \). Hence, \( sr(C(\mathbb{T}^k) \otimes A_\Theta) \geq 2 \).

**Proof of Theorem 1.1.** Since \( C((\mathbb{R})_1) \) is naturally a quotient of \( C^*(\Gamma) \), we must have that \( sr(C^*(\Gamma)) \geq sr((\mathbb{R})_1) \) and \( acsr(C^*(\Gamma)) \geq acsr((\mathbb{R})_1) \).

By [2] page 390 last paragraph and page 391 first paragraph, and by [15] Theorem 1.2, \( C^*(\Gamma) \) can be realized as a unital maximal full algebra of operator fields with base space \( \overline{Z(\Gamma)} \) and fibre algebras, say, \( \{A_\lambda\}_{\lambda \in \hat{Z(\Gamma)}} \). Here, \( Z(\Gamma) \) is the centre of \( \Gamma \), and \( \overline{Z(\Gamma)} \) is the Pontryagin dual of the centre of \( \Gamma \). Moreover, the continuous open surjection corresponding to this continuous field decomposition of \( C^*(\Gamma) \) is the map \( p : Prim(C^*(\Gamma)) \to \overline{Z(\Gamma)} \) which brings a primitive ideal of \( C^*(\Gamma) \) to its restriction to \( Z(\Gamma) \).

Also, by [2] page 390 last paragraph and page 391 first paragraph, by [15] Theorem 1.2, and by [16] Theorem 1, for fixed \( \lambda \in \hat{Z(\Gamma)} \) \( A_\lambda \) (as in the previous paragraph) can in turn be realized as a unital maximal full algebra of operator fields with base space of the form \( T^g \times T \) for some commutative \( g \)-torus \( T^g \) and finite set \( T \). The integer \( g \) is less than or equal to the rank of \( \Gamma/Z(\Gamma) \). The fibre algebras are all isomorphic. Let \( B_\lambda \) be the unique \( C^* \)-algebra which all the fibre algebras are isomorphic to. Then \( B_\lambda \) will be either of the form \( M_n(C) \) (a full matrix algebra) or \( M_n(C) \otimes A_\Theta \) where \( A_\Theta \) is a simple noncommutative torus (the former case will occur if \( p^{-1}(\lambda) \) consists of \( n \)-dimensional representations and the latter will occur if \( p^{-1}(\lambda) \) consists of infinite dimensional representations). We note that \( g, T \) and \( B_\lambda \) will all depend on \( \lambda \).

Now let \( \Gamma^{(2)} \) be the commutator subgroup of \( \Gamma \) (i.e., the subgroup of \( \Gamma \) generated by elements of the form \( xy^{-1}y^{-1}x^{-1} \) where \( x, y \in \Gamma \)). Let \( \Gamma_s^{(2)} \) be the saturation of \( \Gamma^{(2)} \) (i.e., the smallest subgroup \( H \) of \( \Gamma \) containing \( \Gamma^{(2)} \) such that for every \( x \in \Gamma \), if \( x^n \in H \) for some strictly positive integer \( n \) then \( x \in H \). Since \( Z(\Gamma) \) (the
centre of \( \Gamma \) is a saturated subgroup of \( \Gamma \) (i.e., \( x \in \Gamma \) and \( x^n \in Z(\Gamma) \) for some strictly positive integer \( n \) implies that \( x \in Z(\Gamma) \)), \( \Gamma_s^{(2)} \) is a saturated subgroup of \( Z(\Gamma) \). Hence, we have a decomposition \( Z(\Gamma) = \Gamma_s^{(2)} \oplus F \), where \( F \) is a saturated free abelian subgroup of \( Z(\Gamma) \). This in turn gives a decomposition \( \hat{Z}(\Gamma) = \hat{\Gamma_s^{(2)}} \times \hat{F} \).

Now let \( N \) be a positive integer such that for all \( n \geq N \), \( sr(M_n(\mathbb{C}) \otimes C(\mathbb{T}^{h+1})) \leq 2 \), where \( h \) is the rank of \( \Gamma \). Let \( S \) be the set of all \( \lambda \in \hat{Z}(\Gamma) \) such that \( \pi^{-1}(\lambda) \) consists of \( m \)-dimensional representations with \( m \leq N \). With respect to the decomposition of \( \hat{Z}(\Gamma) \) given in the previous paragraph, \( S \) must have the form

\[
\{ \lambda_1, \lambda_2, ..., \lambda_k \} \times \hat{F},
\]

for a finite set of points \( \lambda_i \in \hat{\Gamma_s^{(2)}} \). (Suppose that \( x_1, x_2, ..., x_q \) are elements of \( \Gamma \) so that \( x_1/Z(\Gamma), x_1/Z(\Gamma), x_q/Z(\Gamma) \) give a basis for \( \Gamma/Z(\Gamma) \). Suppose that \( \pi \) is an \( m \)-dimensional representation of \( \Gamma \). Then the scalar values \( \pi(x_i x_j x_i^{-1}(x_j)^{-1}) \), for \( i \leq j \leq q \), must all be rational numbers which can be placed in the form \( r/q \) where \( q \leq m \). These scalar values determine the values of \( \pi \) on \( \Gamma^{(2)} \) and there are only finitely many possibilities for them. And since \( \Gamma^{(2)} \) has finite index in \( \Gamma_s^{(2)} \), there are only finitely many possibilities for the restriction of \( \pi \) to \( \Gamma_s^{(2)} \).

Let \( J \) be the ideal of \( C^*(\Gamma) \) consisting of all operator fields which vanish on \( S \). Then \( J \) is a maximal full algebra of operator fields with base space \( \hat{Z}(\Gamma) - S \). The quotient \( C^*(\Gamma)/J \) is a unital maximal full algebra of operator fields with base space \( S \). Indeed, \( C^*(\Gamma)/J \) is the restriction, to \( S \), of the operator fields in \( C^*(\Gamma) \). Now from the exact sequence \( 0 \rightarrow C[0,1] \otimes J \rightarrow C[0,1] \otimes C^*(\Gamma) \rightarrow C[0,1] \otimes C^*(\Gamma)/J \rightarrow 0 \), and by \([7]\) Corollary 2.22, we get that \( sr(C[0,1] \otimes C^*(\Gamma)) = \max\{sr(C[0,1] \otimes J), sr(C[0,1] \otimes C^*(\Gamma)/J) \} \). By Theorem 1.2, and our definitions of \( N \) and \( S \), \( sr(C[0,1] \otimes J) \leq \sup_{\lambda \in \hat{Z}(\Gamma)-S} sr(C[0,1] \otimes A_{\lambda}) \). But for \( \lambda \in \hat{Z}(\Gamma) - S \), we have, by the definitions of \( N \) and \( S \), by Lemma 2.3, by \([17]\) Proposition 1.7 and Theorem 6.1, and by our discussion of the continuous field decomposition of \( A_{\lambda} \), \( sr(C[0,1] \otimes A_{\lambda}) \leq 2 \). Hence, \( sr(C[0,1] \otimes J) \leq 2 \).

Now by Theorem 1.2, \( sr(C[0,1] \otimes C^*(\Gamma)/J) \leq \sup_{\lambda \in S} sr(C[0,1] \otimes A_{\lambda}) \). But for \( \lambda \in S \), we have that the continuous field decomposition of \( A_{\lambda} \) has base space \( T^g \times T \) where \( g \) is less than the rank of \( \Gamma/Z(\Gamma) \) and \( T \) is a finite set. The fibre algebras are all isomorphic to \( \mathcal{M}_m(\mathbb{C}) \), for some integer \( m \). Hence, by Theorem 1.2, for \( \lambda \in S \), \( sr(C[0,1] \otimes C^*(\Gamma)) \leq sr(C(T^{g+1})) = sr(C[0,1] \otimes C((\hat{\Gamma}_1)) \). From this and the previous paragraph, \( acsr(C^*(\Gamma)) = acsr(C((\hat{\Gamma}_1))) \).

The proofs of the other statements of the theorem now follow from our computation for absolutely connected stable rank.

By \([17]\) Theorem 4.3 and our result for absolutely connected stable rank, we have that \( sr(C^*(\Gamma)) \leq sr(C[0,1] \otimes C^*(\Gamma)) = sr(C[0,1] \otimes C((\hat{\Gamma}_1))) \)

But by \([17]\) Corollary 7.2, \( sr(C[0,1] \otimes C^*(\Gamma)) \leq sr(C^*(\Gamma)) + 1 \). Hence, \( sr(C((\hat{\Gamma}_1))) \leq sr(C^*(\Gamma)) \leq sr(C((\hat{\Gamma}_1))) + 1 \).

Also, by \([17]\) Proposition 1.7, if \( dim((\hat{\Gamma}_1)) \) is even, then \( sr(C[0,1] \otimes C((\hat{\Gamma}_1))) = sr(C((\hat{\Gamma}_1))) \). Hence, if \( dim((\hat{\Gamma}_1)) \) is even, then \( sr(C^*(\Gamma)) = sr(C((\hat{\Gamma}_1))) \).

\[\square\]

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