The $R$- and $L$-orders of the Thompson-Higman monoid $M_{k,1}$ and their complexity

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Abstract

We study the monoid generalization $M_{k,1}$ of the Thompson-Higman groups, and we characterize the $R$- and the $L$-preorder of $M_{k,1}$. Although $M_{k,1}$ has only one non-zero $J$-class and $k-1$ non-zero $D$-classes, the $R$- and the $L$-preorder are complicated; in particular, $<_R$ is dense (even within an $L$-class), and $<_L$ is dense (even within an $R$-class).

We study the computational complexity of the $R$- and the $L$-preorder. When inputs are given by words over a finite generating set of $M_{k,1}$, the $R$- and the $L$-preorder decision problems are in $P$. The main result of the paper is that over a “circuit-like” generating set, the $R$-preorder decision problem of $M_{k,1}$ is $\Pi^P_2$-complete, whereas the $L$-preorder decision problem is $\text{coNP}$-complete. We also prove related results about circuits: For combinational circuits, the surjectiveness problem is $\Pi^P_2$-complete, whereas the injectiveness problem is $\text{coNP}$-complete.

1 Introduction

The groups of Richard J. Thompson are well known and have been extensively studied since their introduction in the 1960s [18, 13, 19]. They were generalized by Graham Higman [12]. A classical survey is [7]. Here we will follow the notation of [6] (which is similar to [17]).

The Thompson-Higman group $G_{k,1}$ (for any integer $k \geq 2$) is defined by taking all maximally extended right-ideal isomorphisms between essential right ideals of a free monoid $A^*$ (where $A$ is an alphabet with $k$ elements). This can be directly generalized to a monoid $M_{k,1}$ consisting of all right-ideal homomorphisms between right ideals of the free monoid $A^*$ [2]. Subsection 1.1 gives more detailed definitions.

The monoid $M_{k,1}$ (and some of its subgroups) have been, first, its remarkable collection of properties (e.g., $G_{k,1}$ is finitely presented, it is simple, it contains all finite groups and all countable free groups), and second, its magical appearance in many contexts (see e.g. the many references in [6, 2]). The monoid $M_{k,1}$ also has remarkable properties: it is finitely generated, congruence-simple, $J$-0-simple, it has exactly $k-1$ $D$-classes, and its word problem is in $P$ [2]. Moreover, there are strong connections between $M_{k,1}$ and circuit complexity [4, 3]. Indeed, although $M_{k,1}$ is finitely generated, it is also interesting to use generating sets of the form $\Gamma \cup \tau$ where $\Gamma$ is any finite generating set of $M_{k,1}$ and $\tau$ consists of all position transpositions in words in $A^*$ (the exact definition of $\tau$ is given at the beginning of Section 4). Then every combinational circuit $C$ (of size $|C|$) can be represented by a word of $M_{k,1}$ over $\Gamma \cup \tau$ of length $O(|C|)$; conversely, if a function $A^m \to A^n$ can be described by a word $w$ over $\Gamma \cup \tau$ of length $|w|$ then this function has a circuit of size $O(|w|^2)$ (Prop. 2.4 and Theorem 2.9 in [3]). We call generating sets of $M_{k,1}$ of the form $\Gamma \cup \tau$ circuit-like generating sets.

The monoid $M_{k,1}$ (and the Thompson-Higman group $G_{k,1}$ too) has a hybrid nature with respect to finiteness and infinity: On the one hand, each element of $M_{k,1}$ has a finite description (by a finite function between finite prefix codes). However, $M_{k,1}$ is countably infinite, and its elements have a partial action on $A^*$; moreover, $M_{k,1}$ acts faithfully on the Cantor space $A^\omega$, which is uncountable.
The subject of this paper is to characterize the \( R \)- and the \( L \)-preorder of \( M_{k,1} \). The results reflect the hybrid nature of \( M_{k,1} \): The characterization of the \( R \)- and \( L \)-preorders is quite simple in terms of the action on the uncountable space \( A^\ast \), but looks a little bit more complicated when formulated in terms of the pre-action on the countable set \( A^\ast \). The \( R \)- and \( L \)-preorders are dense, i.e., whenever \( \psi \prec \varphi \) in \( M_{k,1} \) then \( \psi \prec \chi \prec \varphi \) for some \( \chi \in M_{k,1} \). We also study the computational complexity of the \( R \)- and the \( L \)-preorder of \( M_{k,1} \). We prove that over a finite generating set the \( R \)- and the \( L \)-preorder decision problems of \( M_{k,1} \) are in \( P \). The main result of the paper is that over a “circuit-like” generating set, however, the \( R \)-preorder decision problem of \( M_{k,1} \) is \( \Pi^P_2 \)-complete, whereas the \( L \)-preorder decision problem is \( \text{coNP} \)-complete. We also prove related results about circuits: For combinational circuits, the surjectiveness problem is \( \Pi^P_2 \)-complete, and the injectiveness problem is \( \text{coNP} \)-complete. We also prove that when \( \psi \prec \varphi \) over a circuit-like generating set then the length of right-multipliers cannot be polynomially bounded, unless the polynomial hierarchy collapses. Also, for surjective elements of \( M_{k,1} \) polynomially bounded size, unless the polynomial hierarchy collapses.

This paper is a continuation of [2], where the monoid generalizations \( M_{k,1} \) of the Thompson-Higman groups \( G_{k,1} \) was introduced. The present paper focuses on the \( R \)- and \( L \)-orders of \( M_{k,1} \), and the complexity of problems associated with the \( R \)- and \( L \)-orders of \( M_{k,1} \). The \( J \)-order and the \( D \)-relation of \( M_{k,1} \), and their complexity, will be studied in [1].

1.1 Definition of the Thompson-Higman groups and monoids

We first review some of the material from [2]. Let \( A \) be a finite alphabet of cardinality \( k \), and let \( A^\ast \) denote the free monoid over \( A \) (consisting of all finite sequences of elements of \( A \)); elements of \( A^\ast \) are called words. The empty word is denoted by \( \varepsilon \). The length of \( w \in A^\ast \) is denoted by \( |w| \). For \( u,v \in A^\ast \), the concatenation is denoted by \( uv \) or by \( u \cdot v \); more generally, the concatenation of \( B,C \subseteq A^\ast \) is \( BC = \{w : u \in B, v \in C\} \).

We say that two sets \( X \) and \( Y \) intersect iff \( X \cap Y \neq \emptyset \). A right ideal of \( A^\ast \) is a subset \( R \subseteq A^\ast \) such that \( RA^\ast \subseteq R \). A right ideal \( R \) is said to be essential iff \( R \) intersects every right ideal of \( A^\ast \). For right ideals \( R' \subseteq R \subseteq A^\ast \) we say that \( R' \) is essential in \( R \) (or that \( R' \) is an essentially equal right subideal of \( R \)) iff \( R' \) intersects all right subideals of \( R \).

We say that a right ideal is generated by a set \( C \) iff \( R \) is the intersection of all right ideals that contain \( C \); equivalently, \( R = CA^\ast \). We say that \( u \in A^\ast \) is a prefix of \( v \in A^\ast \) iff \( uz = v \) for some \( z \in A^\ast \), and we write \( u \text{ pref } v \). A prefix code is a subset \( C \subseteq A^\ast \) such that no element of \( C \) is a prefix of another element of \( C \). A prefix code is maximal iff it is not a strict subset of another prefix code. It is easy to prove that a right ideal \( R \) has a unique minimal (under inclusion) generating set, and that this minimal generating set is a prefix code; this prefix code is maximal iff \( R \) is an essential right ideal.

It is often helpful to picture \( A^\ast \) as the infinite \( k \)-ary tree, with vertex set \( A^\ast \) and edge set \( \{(v,va) : v \in A^\ast, a \in A\} \). This is the right Cayley graph of the monoid \( A^\ast \) with generating set \( A \), and we will call it the tree of \( A^\ast \). We turn this into a rooted tree by choosing the empty word \( \varepsilon \) as the root. A path in this tree will always be taken to be directed away from the root. In general, a rooted tree is called a \( k \)-ary tree iff every vertex has \( \leq k \) children. A \( k \)-ary tree is called saturated iff every non-leaf vertex has exactly \( k \) children.

A word \( v \) is a prefix of a word \( w \) iff \( v \) is an ancestor of \( w \) in the tree of \( A^\ast \). A set \( P \) is a prefix code iff no two elements of \( P \) are on the same path. A set \( R \) is a right ideal iff any path that starts in \( R \) has all its vertices in \( R \). A finitely generated right ideal \( R \) is essential iff every infinite path of the tree reaches \( R \) (and then stays in it from there on). Similarly, for two finitely generated right ideals \( R' \subset R \), \( R' \) is essential in \( R \) iff any infinite path starting in \( R \) also intersects \( R' \) (and then stays in \( R' \) from there on). A finite prefix code \( P \) is maximal iff any infinite path starting at the root intersects \( P \).
A right ideal homomorphism of $A^*$ is a function $\varphi : R_1 \to A^*$ with domain $R_1$ such that $R_1$ is a right ideal of $A^*$, and such that for all $x_1 \in R_1$ and all $w \in A^*$: $\varphi(x_1 w) = \varphi(x_1) w$. It is easy to prove that the image set of a right ideal homomorphism is a right ideal, which is finitely generated (as a right ideal) if the domain $R_1$ is finitely generated. Let $f : A^* \to A^*$ be any partial function; then $\text{Dom}(f)$ denotes the domain and $\text{Im}(f)$ denotes the image (range) of $f$.

For any set $X \subseteq A^*$ we denote the partial identity of $X$ by $\text{id}_X$, i.e., $\text{id}_X(x) = x$ for all $x \in X$, and $\text{id}_X(x)$ is undefined when $x \not\in X$. For any sets $X, Y \subseteq A^*$ we have $\text{id}_X \circ \text{id}_Y = \text{id}_Y \circ \text{id}_X = \text{id}_{X \cap Y}$.

Note that we write the action of functions on the left of the argument and thus, functions are read and composed from right to left.

Let $\varphi : R_1 \to R_2$ be a right ideal homomorphism with $R_1 = \text{Dom}(\varphi)$ and $R_2 = \text{Im}(\varphi)$. Then $\varphi$ can be described by a total surjective function $P_1 \to S_2$, where $P_1$ is the prefix code (not necessarily maximal) that generates $R_1$ as a right ideal, and $S_2$ is a set (not necessarily a prefix code) that generates $R_2$ as a right ideal. This function $P_1 \to S_2$, which is just the restriction of $\varphi$ to $P_1$, is called the table of $\varphi$. The prefix code $P_1$ is called the domain code of $\varphi$ and we write $P_1 = \text{domC}(\varphi)$. When $S_2$ is a prefix code we call $S_2$ the image code of $\varphi$ and we write $S_2 = \text{imC}(\varphi)$.

We say that right ideal homomorphism $\Phi : R'_1 \to A^*$ is an essentially equal restriction of a right ideal homomorphism $\varphi : R_1 \to A^*$ (or, equivalently, that $\varphi$ is an essentially equal extension of $\Phi$) iff $R'_1$ is essential in $R_1$, and for all $x'_1 \in R'_1$: $\varphi(x'_1) = \Phi(x'_1)$. In the earlier papers [6, 5, 4, 2, 3] we used the terms “essential restriction (extension)” instead of “essentially equal restriction (extension)”; the new terminology is more precise and will help prevent mix-ups with other concepts that will be introduced later in this paper.

The following is a crucial fact: Every homomorphism $\varphi$ between finitely generated right ideals of $A^*$ has a unique maximal essentially equal extension, denoted $\text{max}(\varphi)$ (Prop. 1.2(2) in [2]). Another interesting fact (remark after Prop. 1.2 in [2]): Every right ideal homomorphism $\varphi$ has an essentially equal restriction $\varphi'$ whose table $P' \to Q'$ is such that both $P'$ and $Q'$ are prefix codes.

We can now define the Higman-Thompson monoid $M_{k,1}$: As a set, $M_{k,1}$ consists of all homomorphisms (between finitely generated right ideals of $A^*$) that have been maximally essentially equally extended. In other words, as a set,

$$M_{k,1} = \{\text{max}(\varphi) : \varphi \text{ is a homomorphism between finitely generated right ideals of } A^*\}.$$ 

The multiplication is composition followed by maximal essentially equal extension. This multiplication is associative (Prop. 1.4 in [2]). Similarly, we define the uncountable monoid $M_{k,1}$ exactly like $M_{k,1}$ except that we allow all homomorphisms between (not necessarily finitely generated) right ideals of $A^*$.

An important motivation for considering $M_{k,1}$ is that its elements are closely related to combinational circuits, as mentioned before. Another possible motivation is the connection with $C^*$-algebras; indeed, $M_{k,1}$ is a submonoid of the multiplicative part of the Cuntz algebra $O_k$ [2].

In [2] it was proved that $M_{k,1}$ is finitely generated, and that its word problem is decidable in deterministic polynomial time; moreover, $M_{k,1}$ is congruence-simple, i.e., the only congruences on $M_{k,1}$ are the trivial congruence (in which all of $M_{k,1}$ is one class), and the discrete congruence (in which each element constitutes a class). It was also proved that the Higman-Thompson group $G_{k,1}$ is the group of units of $M_{k,1}$ (i.e., the set of invertible elements).

To describe the structure of a semigroup, the Green relations play an important role; they determine the left, right, and two-sided ideal structure of the semigroup. We consider the following Green relations: $\leq_J$ (the $J$-preorder), $\leq_L$ (the $L$-preorder), $\leq_R$ (the $R$-preorder), and $\equiv_D$ (the $D$-equivalence relation). From $\leq_J$, $\leq_L$, and $\leq_R$ one also defines $\equiv_J$ (meaning “$\leq_J$ AND $\geq_J$”), and similarly, $\equiv_L$, and $\equiv_R$. In this paper we will not consider the Green relations $\leq_H$ and $\equiv_H$. See e.g. [8, 9] for more information on the Green relations. For any $u, v \in M$ (where $M$ is a monoid) we have, by definition, $u \leq_J v$ iff every ideal of $M$ containing $v$ also contains $u$; equivalently, $u \leq_J v$ iff there
exist \( x, y \in M \) such that \( u = xvy \). Similarly, \( u \leq_L v \) iff any left ideal of \( M \) containing \( v \) also contains \( u \); equivalently, there exists \( x \in M \) such that \( u = xv \); the definition of \( \leq_R \) is similar. By definition, \( u \equiv_D v \) iff there exists \( s \in M \) such that \( u \equiv_R s \equiv_L v \); this is equivalent to the existence of \( t \in M \) such that \( u \equiv_L t \equiv_R v \).

In \([2]\) it was proved that \( M_{k,1} \) has only one non-zero \( \equiv_J \)-class, and that \( M_{k,1} \) has exactly \( k - 1 \) non-zero \( \equiv_P \)-classes. These results mean that \( M_{k,1} \) has almost no structure as far as congruences, \( \leq_J, \equiv_J \) and \( \equiv_D \) are concerned. However, we will see in this paper that \( \leq_L, \leq_R \) have a complicated structure.

### 1.2 Cantor space and topological aspects of the Thompson-Higman monoids

For the study of \( \leq_L \) and \( \leq_R \) in \( M_{k,1} \) it will be useful to consider the action of \( M_{k,1} \) on infinite words, i.e., on the Cantor space \( A^\omega \); this is somewhat similar to Thompson’s \([19]\) original definition of the group \( G_{2,1} \). We also call the elements of \( A^\omega \) the **ends** of the tree of \( A^* \), or the ends of \( A^\omega \). For an end \( z = (z_n : n \geq 1) \) (with \( z_n \in A \) for all \( n \)), we call \( \{z_1, \ldots, z_n : n \geq 1\} \cup \{\varepsilon\} \) the set of prefixes of \( z \). For notational convenience, we will define \( \omega \) to consist of the positive integers (starting at 1).

Let \( R \) be a finitely generated right ideal of \( A^* \) and let \( z = (z_n : n \in \omega) \in A^\omega \). We say that the **end** \( z \) belongs to the right ideal \( R \) iff \( z_1 \ldots z_m \in R \) for some \( m \geq 1 \); since \( R \) is a right ideal, this is equivalent to the existence of \( m \geq 0 \) such that for all \( n \geq m \): \( z_1 \ldots z_n \in R \). The set of all ends in a finitely generated right ideal \( R \subseteq A^* \) is denoted by \( \text{ends}(R) \).

The set of ends \( A^\omega \) can be given the **Cantor space topology**, defined by taking \( \{vA^\omega : v \in A^\omega\} \) as a base of open sets. Then a set is **open** iff it is of the form \( PA^\omega \), where \( P \) is any subset of \( A^* \); moreover, \( P \) can be taken to be a prefix code. Hence, for every right ideal \( R \) the set \( \text{ends}(R) \) is open. Every subset \( S \) of \( A^\omega \) can be written as \( S = PA^\omega \cap E \), with \( E \cap PA^\omega = \emptyset \), where \( P \subseteq A^* \) is countable, and \( PA^\omega \) is the interior of \( S \); moreover, here \( P \) can be chosen to be a prefix code; the set \( E \subseteq A^\omega \) is the (countable) set of isolated elements of \( S \). By definition, \( w \in S \subseteq A^\omega \) is an isolated point of \( S \) if \( w \) has a neighborhood \( N_w \) such that \( S \cap N_w = \{w\} \). A set \( S \subseteq A^\omega \) is **closed** (or, equivalently, compact) iff \( S \) can be written as \( S = PA^\omega \cup E \), with \( P \) and \( E \) as above, with the additional condition that \( P \) and \( E \) are finite. Finally, a set \( S \subseteq A^\omega \) is **clopen** iff \( S = PA^\omega \) for some finite set \( P \subseteq A^* \); moreover, here \( P \) can be taken to be a finite prefix code (see e.g. \([15]\)). Hence for any right ideal \( R \) we have: \( R \) is finitely generated iff \( \text{ends}(R) \) is clopen in the Cantor space \( A^\omega \). For a right ideal \( R \), \( \text{ends}(R) \) is always open; hence, since \( A^\omega \) is compact, we also have for any right ideal \( R \): \( R \) is finitely generated iff \( \text{ends}(R) \) is compact.

This yields a topological characterization of the Higman-Thompson monoid \( M_{k,1} \) within \( \mathcal{M}_{k,1} \). For every \( \varphi \in \mathcal{M}_{k,1} \) the following are equivalent: (1) \( \varphi \in \mathcal{M}_{k,1} \), (2) \( \text{ends}(\text{Dom}(\varphi)) \) is compact, (3) \( \text{ends}(\text{Im}(\varphi)) \) is compact.

For any set \( S \subseteq A^\omega \) we denote the **closure** of \( S \) by \( \text{cl}(S) \). By definition of closure we have: \( w \in \text{cl}(S) \) iff for every prefix \( p \) of \( w \), \( pA^\omega \) intersects \( S \). Indeed, \( \{pA^\omega : p \in A^\omega\} \) \( \text{is} \) a neighborhood base of \( w \). In particular we also have for any right ideal \( R \subseteq A^* \) and any \( w \in A^\omega \): \( w \in \text{cl}(\text{ends}(R)) \) iff there exists a prefix \( p \) of \( w \) such that for all \( s \in A^* \), \( ps \in R \).

Any element of \( \varphi \in \mathcal{M}_{k,1} \), represented by a homomorphism between right ideals of \( A^* \), can be extended to a partial function \( \Phi \) on \( A^* \cup A^\omega \); indeed, if \( \varphi(u) = v \) for \( u, v \in A^* \) then we define \( \Phi(\omega) = \varepsilon \). The extension \( \Phi \) can now be restricted to a partial function \( \Phi' \) on \( A^\omega \); \( \Phi' \) is well defined as a partial function, and \( \Phi' \) uniquely determines \( \Phi \) and \( \varphi \). Indeed, if \( \varphi \) is a right ideal homomorphism of the form \( PA^* \rightarrow QA^* \), where \( P, Q \subseteq A^* \) are prefix codes, then \( \Phi' \) is a total function \( PA^\omega \rightarrow QA^\omega \); and vice-versa. Hence, \( \mathcal{M}_{k,1} \) acts faithfully on \( A^\omega \), and when the elements of \( \mathcal{M}_{k,1} \) are represented by partial functions on \( A^\omega \), the multiplication is just composition (without any further maximal extension). In this action on \( A^\omega \), the elements of \( \mathcal{M}_{k,1} \) are **continuous** partial functions, with open domains and ranges.
1.3 Three descriptions of the Thompson-Higman groups and monoids

The Thompson-Higman groups $G_{k,1}$ and monoids $M_{k,1}$ can be defined in equivalent ways, by finite, countable, or uncountable structures. Each description has advantages and drawbacks.

In the finite description, every element of $G_{k,1}$ or $M_{k,1}$ is given by a function between finite prefix codes over the alphabet $A$. Such a function can be represented concretely by a finite table. However, the representation is not unique: “extensions” or “restrictions” of a table do not change the element of $M_{k,1}$ being represented. In a restriction, a table entry $(x,y)$ is replaced by the $k$ entries $(xa_1,ya_1),\ldots,(xa_k,ya_k)$, where $A = \{a_1,\ldots,a_k\}$; for $G_{k,1}$, an extension is the inverse of a restriction. The notion of restriction of tables looks a little contrived, but it has a natural interpretation in the countable description below, and in the uncountable description, restrictions and extensions are not used at all. For background, see Thompson’s Def. 1.1 in [19], Higman’s pages 24–25 in [12], and Lemma 2.2 in [6]; the functions between the leaves of finite trees, in [7], are very similar to the description by tables; for $M_{k,1}$, see Prop. 1.2 in [2]. Nevertheless, every element of $M_{k,1}$ has a unique maximally extended table. Multiplication is somewhat complicated in the finite description: first the tables have to be restricted so as to become composable (i.e., so that the image code of the first equals the domain code of the second), then they are composed, and finally the resulting table has to be extended. Multiplication is somewhat simpler in the countable description below, and in the uncountable description, restrictions and extensions are not used at all. For background, see Thompson’s Def. 1.1 in [19], Higman’s pages 24–25 in [12], and Lemma 2.2 in [6]; the functions between the leaves of finite trees, in [7], are very similar to the description by tables; for $M_{k,1}$, see Prop. 1.2 in [2]. Nevertheless, every element of $M_{k,1}$ has a unique maximally extended table. Multiplication is somewhat complicated in the finite description: first the tables have to be restricted so as to become composable (i.e., so that the image code of the first equals the domain code of the second), then they are composed, and finally the resulting table has to be maximally extended (if the unique representation is desired). Associativity is not obvious. The $\mathcal{L}$- and $\mathcal{R}$-preorders of $M_{k,1}$ are complicated to characterize in the finite description.

In the countable description, every element of $G_{k,1}$ or $M_{k,1}$ is given by right ideal homomorphisms between finitely generated right ideals of $A^*$. Again, the representation is not unique: extensions or restrictions of a right ideal homomorphism do not change the element of $M_{k,1}$ being represented. However, the concept of extension or restriction is a special case of the usual one (for partial functions), and not ad hoc (as it was for tables). Every element of $M_{k,1}$ can be represented by a unique maximally extended right ideal homomorphism. Multiplication is a little simpler than for tables: it is just the usual function composition, followed by maximal extension (if the unique representation is desired). Associativity is not obvious. The characterization of the $\mathcal{L}$- and $\mathcal{R}$-preorders of $M_{k,1}$ is manageable but somewhat complicated.

In the uncountable description, every element of $G_{k,1}$ or $M_{k,1}$ is given by a permutation, respectively a partial function on the Cantor space $A^\omega$. Multiplication is simply composition of permutations or of partial functions. However, to define which permutations or partial functions on $A^\omega$ belong to $G_{k,1}$ or $M_{k,1}$, the countable description needs to be referred to (at least indirectly). The characterization of the $\mathcal{L}$- and $\mathcal{R}$-preorders of $M_{k,1}$ is easy to state.

1.4 Overview

In Section 2 the $\mathcal{R}$-order of $M_{k,1}$ is characterized (in terms of image sets), and in Section 3 the $\mathcal{L}$-order is characterized (in terms of right congruences). In Section 4 the $\mathcal{R}$- and $\mathcal{L}$-orders are embedded into the idempotent order, and various density properties of the $\mathcal{R}$- and $\mathcal{L}$-orders are shown. In Section 5 it is proved that the $\leq_\mathcal{R}$-decision problem of $M_{k,1}$ is in \textsf{P} when a finite generating set is used, and that it is $\Pi^P_2$-complete with circuit-like generating sets; the surjectiveness problem for combinational circuits is also proved to be $\Pi^P_2$-complete. In Section 6 it is proved that the $\leq_\mathcal{L}$-decision problem of $M_{k,1}$ is in \textsf{P} when a finite generating set is used, and that it is \textsf{coNP}-complete with circuit-like generating sets; the injectiveness problem for combinational circuits is also proved to be \textsf{coNP}-complete.

2 The $\mathcal{R}$-order of $M_{k,1}$

The monoid $M_{k,1}$ has some similarities with the monoid $\text{PF}_X$ of all partial functions on a set $X$. In both cases, the $\mathcal{R}$-order between functions is related to the inclusion order of the image sets. However, for $M_{k,1}$ the notion of “inclusion” has several different characterizations, and depends on whether we
use the finite, countable, or uncountable description of $M_{k,1}$. We show that the $\mathcal{R}$-order of $M_{k,1}$ is dense.

**Notation.** For a finite set $S \subset A^*$, the length of a longest word in $S$ will be denoted by $\ell(S)$.

Recall that a tree is called $k$-ary iff every vertex has $\leq k$ children, and at least one vertex has exactly $k$ children. A $k$-ary tree is said to be *saturated* iff every non-leaf vertex has exactly $k$ children.

### 2.1 Characterization of the $\mathcal{R}$-order of $M_{k,1}$

**Theorem 2.1 (R-order of $M_{k,1}$).** For all $\psi, \varphi \in M_{k,1}$ the following are equivalent:

1. $\psi(.) \preceq \mathcal{R} \varphi(.)$;
2. $\text{ends}(\text{Im}(\psi)) \subseteq \text{ends}(\text{Im}(\varphi))$;
3. every right ideal of $A^*$ that intersects $\text{Im}(\psi)$ also intersects $\text{Im}(\varphi)$;
4. every monogenic right ideal of $A^*$ that intersects $\text{Im}(\psi)$ also intersects $\text{Im}(\varphi)$;
5. every path in the tree of $A^*$, starting at $\varepsilon$, of length $\max\{\ell(\text{Im}C(\varphi)), \ell(\text{Im}C(\psi))\}$, that intersects $\text{Im}C(\psi)$, also intersects $\text{Im}C(\varphi)$;
6. for every $y \in \text{Im}C(\psi)$ we have:
   - either $\text{Im}C(\varphi)$ contains a prefix of $y$,
   - or the subtree of the tree of $A^*$ with root $y$ and leaf-set $yA^* \cap \text{Im}C(\varphi)$ is saturated.

The characterizations above reflect the various representations of $M_{k,1}$, (2) the uncountable representation, (3) and (4) the countable infinite representation, (5) and (6) the finite representation. We first prove intermediate results, and then the Theorem.

**Lemma 2.2.**

For any two right ideals $R_1, R_2 \subseteq A^*$ the following are equivalent:

(a) $\text{clos}(\text{ends}(R_1)) \subseteq \text{clos}(\text{ends}(R_2))$.
(b) Every right ideal of $A^*$ that intersects $R_1$ also intersects $R_2$.
(c) Every monogenic right ideal of $A^*$ that intersects $R_1$ also intersects $R_2$.

If $R_2$ is finitely generated then (a) is equivalent to the following:

(d) $\text{ends}(R_1) \subseteq \text{ends}(R_2)$.

If $R_1$ and $R_2$ are generated by the prefix code $P_1$, respectively $P_2$, then (d) is equivalent to the following (where we allow $\max\{\ell(P_1), \ell(P_2)\}$ to be infinite):

(e) Every path in the tree of $A^*$, starting at $\varepsilon$, of length $\max\{\ell(P_1), \ell(P_2)\}$, that intersects $P_1$, also intersects $P_2$.

If $R_1$ and $R_2$ are generated by the prefix code $P_1$, respectively $P_2$, and if $P_2$ is finite, then (d) is equivalent to the following:

(f) For every $y \in P_1$ we have:
   - either $P_2$ contains a prefix of $y$,
   - or the subtree of the tree of $A^*$ with root $y$ and leaf-set $yA^* \cap P_2$ is saturated.

**Proof.** [(a) ⇒ (b)] Let $R$ be any right ideal that intersects $R_1$, and let $w \in R_1 \cap R$. Then there is an end in $\text{ends}(R_1 \cap R)$ with $w$ as a prefix (in fact, all ends with prefix $w$ are in $\text{ends}(R_1 \cap R)$). By (a), this end is also in $\text{clos}(\text{ends}(R_2))$. Hence, some finite word of the form $wx$ belongs to $R_2$. Since $w \in R$ and since $R$ is a right ideal, $wx \in R$. So, $wx \in R \cap R_2$, hence $R$ and $R_2$ intersect.

[(b) ⇒ (c)] This is trivial.
Lemma 2.4

Let \( z = z_1 \ldots z_m \ldots \in \operatorname{clos}(\text{ends}(R_1)) \). Hence, \( z_1 \ldots z_m \in R_1 \) for some \( m \geq 0 \). Hence, for all \( n > m : z_1 \ldots z_m \ldots z_n \in R_1 \), so \( z_1 \ldots z_m \ldots z_n A^* \) intersects \( R_1 \). By (c) this implies that \( z_1 \ldots z_m \ldots z_n A^* \) intersects \( R_2 \), for all \( n > m \). So for each \( n > m \) there exists \( u_n \in A^* \) such that \( z_1 \ldots z_m \ldots z_n u_n \in R_2 \).

Let \( P_2 \) be a finite generating set of \( R_2 \), i.e., \( R_2 = P_2 A^* \). So for each \( n > m \) there exists \( u_n \in A^* \) and there exist \( p_n \in P_2 \) and \( v_n \in A^* \) such that \( z_1 \ldots z_m \ldots z_n u_n = p_n v_n \). When \( n \) is longer than the longest word in \( P_2 \) (such an \( n \) exists since \( P_2 \) is finite), \( p_n \) will be a prefix of \( z_1 \ldots z_m \ldots z_n \). Then some element of \( P_2 (\subseteq R_2) \) is a prefix of the end \( z \). It follows that the end \( z \) is in \( R_2 \).

(a) \( \Rightarrow \) (d) when \( R_2 \) is finitely generated Trivially, (d) always implies (a). Suppose now that (a) holds. When \( R_2 \) is finitely generated, \( \text{ends}(R_2) \) is compact, so \( \text{ends}(R_2) = \operatorname{clos}(\text{ends}(R_2)) \). Then \( \text{ends}(R_1) \subseteq \operatorname{clos}(\text{ends}(R_1)) \subseteq \operatorname{clos}(\text{ends}(R_2)) = \text{ends}(R_2) \), so (d) follows.

(b) \( \Leftrightarrow \) (e) This is fairly obvious.

c) \( \Rightarrow \) (f), when \( R_2 \) is finitely generated Let \( p \) be a long-enough path, intersecting \( P_1 \) at some element \( y \). By (f), if some element \( x \) of \( P_2 \) is a prefix of \( y \) then \( x \) is also on the path \( p \), hence \( p \) also intersects \( P_2 \) (at \( x \)). If, on the other hand, the subtree of \( A^* \) with root \( y \) and leaf-set \( yA^* \cap P_2 \) is saturated, then every way to continue the path \( p \) beyond \( y \) will lead to an intersection with \( P_2 \).

d) \( \Rightarrow \) (f), when \( R_2 \) is finitely generated Consider any \( y \in P_1 \) and consider any long path \( p \) through \( y \). By (e), \( p \) intersects \( P_2 \) at some point, say \( x \in P_2 \). If \( x \) is shorter than \( y \) then \( x \) is between \( y \) and the root \( \varepsilon \), hence \( x \) is a prefix of \( y \), and no other path through \( y \) intersects \( P_2 \) (indeed, there is only one path from the root to, and that path intersects \( P_2 \)).

If \( x \) is longer than \( y \) then the intersection of the paths through \( y \) and \( P_2 \) is \( yA^* \cap P_2 \). If the tree with root \( y \) and leaves \( yA^* \cap P_2 \) were not saturated then some path through \( y \) (away from the root) could “escape” without intersecting \( P_2 \).

Remark 2.2 Part (d) of Lemma 2.2 is not true if we omit the assumption that \( R_2 \) is finitely generated. For example, let \( A = \{a, b\}, R_1 = a\{a, b\}^*, R_2 = \{a^m b : m \geq 0\} \{a, b\}^* = \{a^* b \{a, b\}^*\} \). Then every right ideal that intersects \( R_1 \) also intersects \( R_2 \). However, the end \( a^\omega = (a^n : n \geq 0) \) is in \( R_1 \) but not in \( R_2 \).

Definition 2.3 For any two finitely generated right ideals \( R_1, R_2 \subseteq A^* \) we say that \( R_1 \) is \textbf{end-included} in \( R_2 \) iff \( R_1, R_2 \) satisfy the equivalent properties of Lemma 2.2. We denote this by \( R_1 \subseteq_{\text{end}} R_2 \).

Two finitely generated right ideals \( R_1, R_2 \subseteq A^* \) are called \textbf{essentially equal} (or \textbf{end-equality}) iff \( R_1 \subseteq_{\text{end}} R_2 \) and \( R_2 \subseteq_{\text{end}} R_1 \). We denote this by \( R_1 =_{\text{ess}} R_2 \).

The relation \( \subseteq_{\text{end}} \) is a pre-order (reflexive and transitive) on the set of finitely generated right ideals of \( A^* \). Moreover, \( \subseteq_{\text{end}} \) is a lattice pre-order, i.e., the order on the set of \( \equiv_{\text{ess}}\)-equivalence classes is a lattice order. We have \( R_1 \subseteq_{\text{end}} R_2 \) iff \( R_1 =_{\text{ess}} R_1 \cap R_2 \) iff \( R_1 \cup R_2 =_{\text{ess}} R_2 \). Clearly, \( R_1 \subseteq_{\text{end}} R_2 \) does not imply \( R_1 \subseteq R_2 \), and essential equality does not imply equality; e.g., all finitely generated \textit{essential} right ideals of \( A^* \) (as defined in Subsection 1.1) are essentially equal.

Lemma 2.4 Let \( R_1 = P_1 A^* \) and \( R_2 = P_2 A^* \) be right ideals, where \( P_1 \) and \( P_2 \) are finite prefix codes. Then \( R_1 =_{\text{ess}} R_2 \) iff \( P_2 \) can be transformed into \( P_1 \) by a finite sequence of replacements steps of the following form:

\begin{enumerate}
\item[(r1)] For a finite prefix code \( C \) and for \( c \in C \), replace \( C \) by \( (C - \{ c \}) \cup cA \).
\item[(r2)] For a finite prefix code \( C' \) such that \( cA \subseteq C' \) for some word \( c \), replace \( C' \) by \( (C' - cA) \cup \{ c \} \).
\end{enumerate}

Proof. This is straightforward. \( \square \)
Lemma 2.5 (Lemma 3.3 of [6]). Let $P, Q, R \subseteq A^*$ be such that $PA^* \cap QA^* = RA^*$, and $R$ is a prefix code. Then $R \subseteq P \cup Q$.

As a consequence, the intersection of two finitely generated right ideals is finitely generated. □

Definition 2.6. By $\text{riHom}(A^*)$ we denote the set of right ideal homomorphisms between finitely generated right ideals of $A^*$. This is a monoid under function composition.

Recall (Definition 1.3 in [2]) that for $\psi, \psi_0 \in \text{riHom}(A^*)$ we say that $\psi_0$ is an essentially equal restriction of $\psi$ iff $\psi_0$ is a restriction of $\psi$ and $\text{Dom}(\psi_0) = \text{ess Dom}(\psi)$.

Lemma 2.7 Assume $\psi, \psi_0 \in \text{riHom}(A^*)$ are such that $\psi_0$ is an essentially equal restriction of $\psi$. Then $\text{Im}(\psi) = \text{ess Im}(\psi_0)$ and $\text{Dom}(\psi) = \text{ess Dom}(\psi_0)$.

Proof. This follows directly from the definition of essentially equal restriction. □

Proposition 2.8 If $\psi, \varphi \in \text{riHom}(A^*)$ represent the same element of $M_{k,1}$ then $\text{Im}(\psi) = \text{ess Im}(\varphi)$ and $\text{Dom}(\psi) = \text{ess Dom}(\varphi)$.

Proof. $\varphi$ and $\psi$ represent the same element of $M_{k,1}$ iff $\text{max}(\psi) = \text{max}(\varphi)$. By Lemma 2.7, $\text{Im}(\psi) = \text{ess Im}(\text{max}(\psi)) = \text{ess Im}(\varphi)$, and similarly for Dom. □

Lemma 2.9 Let $\varphi, \psi \in \text{riHom}(A^*)$ and assume that $\text{Im}(\psi) \subseteq \text{Im}(\varphi)$. Then there exist $\varphi_0, \psi_0 \in \text{riHom}(A^*)$ such that

- $\varphi_0$ is an essentially equal restriction of $\varphi$, and $\psi_0$ is an essentially equal restriction of $\psi$,
- $\text{imC}(\psi_0)$ and $\text{imC}(\varphi_0)$ are prefix codes,
- we have the inclusion (in the ordinary sense): $\text{imC}(\psi_0) \subseteq \text{imC}(\varphi_0)$.

Proof. By essentially restricting $\varphi$ and $\psi$, if necessary, we can assume that $\text{imC}(\varphi)$ and $\text{imC}(\psi)$ are prefix codes. Let $P = \text{imC}(\varphi)$ and $Q = \text{imC}(\psi)$, so $\text{Im}(\psi) = QA^*$ and $\text{Im}(\varphi) = PA^*$. By Lemma 2.5 (Lemma 3.3 of [6]), there exists a prefix code $Q_0$ such that $PA^* \cap QA^* = Q_0A^*$ (hence, obviously, $Q_0A^* \subseteq QA^*$), and $Q_0 \subseteq P \cup Q$. Moreover, since $\subseteq \text{end}$ is a lattice pre-order, $\text{Im}(\psi) \subseteq \text{Im}(\varphi)$ implies that $\text{Im}(\psi) \cap \text{Im}(\varphi) = \text{ess Im}(\varphi)$, so $Q_0A^* = \text{ess QA}^*$.

We now restrict $\psi$ (whose image is $\text{Im}(\psi) = QA^*$) so as to obtain $\psi_0$ with $\text{Im}(\psi_0) = Q_0A^* (\subseteq QA^*)$. Since $Q_0A^* = \text{ess} Q_0A^* \subseteq QA^*$, $\psi_0$ is an essentially equal restriction of $\psi$.

Next, we partition $P$ into $P_1 = \{ p \in P : p$ is a prefix of an element of $Q_0 \}$, and $P_2 = P - P_1$. Then we define $P_0 = Q_0 \cup P_2$. Since $Q_0A^* \subseteq PA^*$, every element $p$ of $P$ is either a prefix of an element of $Q_0$ or prefix-incomparable with all of $Q_0A^*$. It follows that $P_0A^* \subseteq PA^*$ and $P_0A^* = \text{ess} PA^*$.

Finally, we restrict $\varphi$ (whose image is $\text{Im}(\varphi) = PA^*$) so as to obtain $\varphi_0$ with $\text{Im}(\varphi_0) = P_0A^* (\subseteq PA^*)$. Since $P_0A^* = \text{ess} PA^*$, $\varphi_0$ is an essentially equal restriction of $\varphi$.

Now we have, $\text{ImC}(\psi_0) = Q_0 \subseteq Q_0 \cup P_2 = P_0 = \text{ImC}(\varphi_0)$.

Proof of Theorem 2.1

Because of Lemma 2.2 we only need to prove that $\psi \leq_R \varphi$ iff $\text{Im}(\psi) \subseteq \text{Im}(\varphi)$.

[⇒] Suppose $\psi(\cdot) = \text{max}(\varphi \circ \alpha(\cdot))$, for some $\alpha \in M_{k,1}$. It is easy to see that $\text{Im}(\varphi \circ \alpha(\cdot)) \subseteq \text{Im}(\varphi(\cdot))$.

It follows now from Lemma 2.7 that $\text{Im}(\varphi \circ \alpha(\cdot)) \subseteq \text{Im}(\text{max}(\varphi \circ \alpha(\cdot)))$.

[⇐] If $\text{Im}(\psi) \subseteq \text{Im}(\varphi)$ we can apply Lemma 2.9 and choose representations $\psi_0, \varphi_0$ of $\psi$, respectively $\varphi$, such that $\text{imC}(\psi_0) \subseteq \text{imC}(\varphi_0)$. We will now define $\alpha \in \text{riHom}(A^*)$ so that $\varphi_0 \circ \alpha(\cdot) = \psi_0(\cdot)$:

- We pick $\text{domC}(\alpha) = \text{domC}(\psi_0)$;
- for every $y \in \text{imC}(\psi_0)$ we choose an element $\overline{y} \in \varphi_0^{-1}(y)$;
- for each $x \in \text{domC}(\psi_0)$ we define $\alpha(x) = \overline{\psi_0(x)}$ (if $\overline{\psi_0(x)}$).

Now for every $x \in \text{domC}(\psi_0)$ we have: $\varphi_0 \circ \alpha(x) = \varphi_0(\overline{\psi_0(x)}) \in \varphi_0(\varphi_0^{-1}(\psi_0(x))) = \psi_0(x)$. So, $\varphi_0 \circ \alpha(\cdot) = \psi_0(\cdot)$.

□
3 The \( \mathcal{L} \)-order of \( M_{k,1} \)

Just as for the \( \mathcal{R} \)-order, the \( \mathcal{L} \)-order of the monoid \( M_{k,1} \) has some similarities with the \( \mathcal{L} \)-order of the monoid \( PF_X \) of all partial functions on a set \( X \). In both cases, the \( \mathcal{L} \)-order between functions is related to the refinement order of the partitions on the domains of the functions. However, for \( M_{k,1} \) we need more complicated notions of partition and of refinement.

In the following subsections we first define right congruences in \( A^* \) and essential equality of right congruences. We associate a right congruence with every element of \( M_{k,1} \). We define the refinement order of right congruences, and finally we use that to characterize the \( \mathcal{L} \)-order of \( M_{k,1} \).

3.1 Right congruences, prefix code partitions, and essential equivalence

A right congruence on a right ideal \( R \subseteq A^* \) is an equivalence relation \( \simeq \) on \( R \) such that for all \( x,y \in R \) and all \( w \in A^* \): \( x \simeq y \) implies \( xw \simeq yw \); moreover, \( x \simeq y \) is undefined if \( x \) or \( y \) are not both in \( R \). The right ideal \( R \) is called the domain of \( \simeq \) and we denote it by \( \text{Dom}(\simeq) \). We will only consider the case when \( R \) is finitely generated as a right ideal. The equivalence class containing an element \( x \) will be denoted by \([x]\).

Let \( P \subseteq A^* \) be a finite prefix code, and let \( \equiv =_P \) be an equivalence relation on \( P \). Then \( \equiv =_P \) determines a right congruence \( \simeq =_P \) on the right ideal \( PA^* \), as follows: If \( p_1,p_2 \in P \) and \( p_1 \equiv =_P p_2 \) then \( p_1w \equiv =_P p_2w \) for all \( w \in A^* \); and if \( p_1x,p_2y \in PA^* \) are such that \( p_1 \neq p_2 \) or \( x \neq y \), then \( p_1x \neq p_2y \). Thus, if the set of equivalence classes of \( \equiv =_P \) is \( \{P_1,\ldots,P_n\} \), then the set of congruence classes of \( \simeq =_P \) is \( \{P_jw : w \in A^*, 1 \leq j \leq n\} \). Hence \( \simeq =_P \) is the coarsest right congruence on \( PA^* \) that agrees with \( \equiv =_P \) on \( P \). We also say that \( \simeq =_P \) is the right congruence generated by the equivalence relation \( \equiv =_P \).

Definition 3.1 Let \( P \subseteq A^* \) be a finite prefix code, let \( \simeq \) be a right congruence on the right ideal \( PA^* \), and let \( \equiv =_P \) be the restriction of \( \simeq \) to \( P \). We call \( \simeq \) a prefix code congruence iff \( \simeq \) is equal to the right congruence generated by its restriction \( \equiv =_P \). In that case we call \( P \) the domain code of \( \simeq \), and denote it by \( \text{domC}(\simeq) \).

Not every right congruence on \( PA^* \) is a prefix code congruence. For example, \( \simeq \) is not determined by its restriction to \( P \) if \( pu \simeq pvw \) (for some \( p \in P \) and some \( u,v \in A^* \) with \( v \) non-empty), or if \( p_1x \simeq p_2y \) (for some \( p_1 \neq p_2 \in P \) and some \( x,y \in A^* \)). In this paper we will only consider prefix code congruences.

A prefix code congruence \( \simeq \) can be extended to the ends of \( \text{Dom}(\simeq) \). For \( w_1,w_2 \in \text{ends}(\text{Dom}(\simeq)) \), we say that \( w_1 \simeq w_2 \) iff there exist \( p_1,p_2 \in \text{Dom}(\simeq) \) and \( v \in A^w \) such that
\[
\quad w_1 = p_1v, \quad w_2 = p_2v, \quad \text{and} \quad p_1 \simeq p_2.
\]
Hence, the set of right congruence classes of \( \simeq \) in \( \text{ends}(\text{Dom}(\simeq)) \) is \( \{[p]v : p \in \text{Dom}(\simeq), v \in A^w\} \).

Notational Remark: Although \( \simeq \) can be extended to a partition of \( \text{ends}(\text{Dom}(\simeq)) \), our notation \( \text{Dom}(\simeq) \) will continue to refer to the right ideal of finite words on which \( \simeq \) is defined; i.e., in our notation we still have \( \text{Dom}(\simeq) \subseteq A^*. \)

For a prefix code congruence \( \simeq \), the prefix code \( \text{domC}(\simeq) \) is finite, by definition. It follows that for a prefix code congruence we have: Every \( \simeq \)-class in \( \text{ends}(\text{Dom}(\simeq)) \) or in \( \text{Dom}(\simeq) \) is finite, with cardinality uniformly bounded from above by \( |\text{domC}(\simeq)| \).

Definition 3.2 Let \( \simeq_1 \) and \( \simeq_2 \) be two prefix code congruences. We say that \( \simeq_1 \) is an essentially equal extension of \( \simeq_2 \) (and that \( \simeq_2 \) is an essentially equal restriction of \( \simeq_1 \)) iff the following three conditions hold:

(1) \( \text{Dom}(\simeq_2) =_{\text{ess}} \text{Dom}(\simeq_1) \), and

(2) \( \text{Dom}(\simeq_2) \subseteq \text{Dom}(\simeq_1) \), and

(3) \( \simeq_2 \) agrees with \( \simeq_1 \) on \( \text{Dom}(\simeq_2) \); i.e., for all \( x,y \in \text{Dom}(\simeq_2) \) : \( x \simeq_2 y \iff x \simeq_1 y \).
Conditions (1) and (2) are equivalent to saying that every element of \( \text{Dom}(\simeq_1) \) is the prefix of some element of \( \text{Dom}(\simeq_2) \), and that element of \( \text{Dom}(\simeq_2) \) has a prefix in \( \text{Dom}(\simeq_1) \).

Conditions (2) and (3) are equivalent to saying that every \( \simeq_2 \)-class is also a \( \simeq_1 \)-class; (2) and (3) are also equivalent to saying that \( \text{Dom}(\simeq_2) \subseteq \text{Dom}(\simeq_1) \) and for every \( x \in \text{Dom}(\simeq_2) \), \( [x]_1 = [x]_2 \). Here, \([x]_1\) and \([x]_2\) denote the equivalence class of \( x \) for \( \simeq_1 \), respectively \( \simeq_2 \).

Essentially equal restrictions and extensions of prefix code congruences can be determined by a replacement (or rewriting) process, based on the following replacement rules (where \( A = \{a_1, \ldots , a_k\} \)):

\[
\begin{align*}
(3.3) & \quad \text{Replace the class } C \text{ in the domain code } P \text{ by the sets of classes } \{Ca_1, \ldots , Ca_k\}. \\
(3.4) & \quad \text{Replace the set of classes } \{Ca_1, \ldots , Ca_k\} \text{ of the domain code } P \text{ by the new class } C.
\end{align*}
\]

When rule (3.3) is applied, the domain code \( P \) is replaced by \((P - C) \cup CA\); as a result, an essentially equal restriction of the prefix code congruence is obtained. Similarly, when rule (3.4) is applied the domain code \( P \) is replaced by \((P - CA) \cup C\); as a result, an essentially equal extension of the prefix code congruence is obtained. The replacement steps (3.3) and (3.4) can be iterated. It turns out that all essentially equal restrictions and extensions can be obtained in the above way:

**Proposition 3.5** Let \( \simeq_1 \) and \( \simeq_2 \) be two prefix code congruences. Then \( \simeq_2 \) is an essentially equal restriction of \( \simeq_1 \) iff \( \simeq_2 \) can be obtained from \( \simeq_1 \) by a finite sequence of replacements of the form (3.3). And \( \simeq_1 \) is an essentially equal extension of \( \simeq_2 \) iff \( \simeq_1 \) can be obtained from \( \simeq_2 \) by a finite sequence of replacements of the form (3.4).

**Proof.** The proof is similar to the proof of Prop. 1.4 in [2] (as well as the proof of Lemma 2.2 in [4], going back to Thompson). The direction \([\Leftarrow]\) is easy to see.

Conversely, suppose that \( \simeq_2 \) is an essentially equal restriction of \( \simeq_1 \). Let \( P_1 = \text{dom}C(\simeq_1) \) and \( P_2 = \text{dom}C(\simeq_2) \). Since \( \text{Dom}(\simeq_1) \) and \( \text{Dom}(\simeq_2) \) are essentially equal and \( \text{Dom}(\simeq_2) \subseteq \text{Dom}(\simeq_1) \), every path (in the tree of \( A^* \)) starting in \( P_1 \) reaches \( P_2 \). Hence, the set difference \( \text{Dom}(\simeq_1) - \text{Dom}(\simeq_2) \) is finite. Also, since \( \simeq_2 \) agrees with \( \simeq_1 \) on \( \text{Dom}(\simeq_2) \) it follows that \( \text{Dom}(\simeq_1) - \text{Dom}(\simeq_2) \) consists of \( \simeq_1 \)-equivalence classes.

Let \( \pi \) be a \( \simeq_1 \)-equivalence class that lies in \( \text{dom}C(\simeq_1) - \text{dom}C(\simeq_2) \); the latter set is not empty, otherwise \( \simeq_1 \) and \( \simeq_2 \) would be equal. Removing \( \pi \) from \( \simeq_1 \) yields a new prefix code congruence \( \simeq_1' \) with domain \( \text{Dom}(\simeq_1') = \text{Dom}(\simeq_1) - \pi \). In the tree of \( A^* \), the children of the elements of \( \pi \) are \( \bigcup_{i=1}^k \pi a_i \). So, \( \text{dom}C(\simeq_1') = (P_1 - \pi) \cup \bigcup_{i=1}^k \pi a_i \). This amounts to applying rule of type (3.3) to \( \simeq_1 \).

Since \( |\text{Dom}(\simeq_1') - \text{Dom}(\simeq_2)| < |\text{Dom}(\simeq_1) - \text{Dom}(\simeq_2)| \), we conclude by induction that \( \simeq_2 \) can be obtained from \( \simeq_1' \) by rules of type (3.3). Hence, the essentially equal restrictions from \( \simeq_1 \) to \( \simeq_1' \) and from there to \( \simeq_2 \) can be carried out by applying rules of type (3.3).

In a similar way one proves that essentially equal extensions can be carried out by rules of type (3.4). \( \Box \)

**Definition 3.6** Two prefix code congruences \( \simeq_1 \) and \( \simeq_2 \) are essentially equal iff \( \simeq_1 \) and \( \simeq_2 \) can be obtained from each other by a finite sequence of essentially equal extensions and essentially equal restrictions. We denote this by \( \simeq_1 =_{\text{ess}} \simeq_2 \).

By Proposition 3.5 this could be defined equivalently by: \( \simeq_1 \) and \( \simeq_2 \) are essentially equal iff \( \simeq_1 \) and \( \simeq_2 \) can be obtained from each other by a finite sequence of replacement steps of form (3.3) and (3.4). We also have the following characterization.

**Proposition 3.7** Two prefix code congruences \( \simeq_1 \) and \( \simeq_2 \) are essentially equal iff:

(1) \( \text{Dom}(\simeq_2) =_{\text{ess}} \text{Dom}(\simeq_1) \), and

(2) \( \simeq_2 \) agrees with \( \simeq_1 \) on \( \text{Dom}(\simeq_1) \cap \text{Dom}(\simeq_2) \).
Proof. By Definition 3.2 if $\simeq_1$ is an essentially equal extension of $\simeq_2$ then $\simeq_1$ and $\simeq_2$ agree on $\text{Dom}(\simeq_1) \cap \text{Dom}(\simeq_2)$. For finite sequence $\simeq_1, \ldots, \simeq_i, \ldots, \simeq_n$ of prefix code congruences, successively obtained from each other by essentially equal extensions and restrictions, $\simeq_1$ and $\simeq_n$ will agree on $\bigcap_{i=1}^n \text{Dom}(\simeq_i)$. But then, if we extend (by applying replacement steps (3.4)), $\simeq_1$ and $\simeq_n$ will agree on $\text{Dom}(\simeq_1) \cap \text{Dom}(\simeq_2)$.

Conversely, suppose $\simeq_1$ and $\simeq_2$ agree on $\text{Dom}(\simeq_1) \cap \text{Dom}(\simeq_2)$. Then the restriction $\simeq$ of $\simeq_1$ to $\text{Dom}(\simeq_1) \cap \text{Dom}(\simeq_2)$ is an essentially equal restriction of $\simeq_1$. Moreover, the extension of $\simeq$ from $\text{Dom}(\simeq_1) \cap \text{Dom}(\simeq_2)$ to $\text{Dom}(\simeq_2)$ is an essentially equal extension of $\simeq$. So $\simeq_1$ and $\simeq_2$ are essentially equal. □

The replacement system consisting of the rules of type (3.4) is terminating (since the number of classes in the finite prefix code decreases at each step) and confluent (there are no overlaps between left-sides of rules). Hence, there is a unique result for the iterated replacement in the direction (3.4). So we proved:

**Proposition 3.8** Every prefix code congruence has a unique maximal essentially equal extension. □

This maximal essentially equal extension of $\simeq$ is denoted by $\text{max}(\simeq)$.

**Proposition 3.9** For prefix code congruences $\simeq_1$ and $\simeq_2$ the following are equivalent:

1. $\simeq_1 =_{\text{ess}} \simeq_2$;
2. $\text{max}(\simeq_1) = \text{max}(\simeq_2)$;
3. $\text{ends}(\text{Dom}(\simeq_1)) = \text{ends}(\text{Dom}(\simeq_2))$, and $\simeq_2$ agrees with $\simeq_1$ on ends(\text{Dom}(\simeq_1)).

**Proof.** [(1) $\Rightarrow$ (2)] If $\simeq_1$ and $\simeq_2$ can be obtained from each other by rewriting according to (3.3) and (3.4), both can be rewritten to $\text{max}(\simeq_1)$, as well as to $\text{max}(\simeq_2)$. By uniqueness of the maximal essentially equal extension, we obtain (2).

[(2) $\Leftrightarrow$ (1)] If $\simeq_1$ and $\simeq_2$ have the same maximal essentially equal extension, we can rewrite $\simeq_1$ to $\simeq_2$ via this common maximal essentially equal extension. So, $\simeq_1 =_{\text{ess}} \simeq_2$.

[(2) $\Leftrightarrow$ (3)] The set of ends $\text{ends}(\text{Dom}(\simeq_1))$ uniquely determines the prefix code $\text{domC}(\text{max}(\simeq_1))$. Namely, we take the set shortest prefixes of ends in $\text{ends}(\text{Dom}(\simeq_1))$ that are not prefixes of ends that are not in $\text{ends}(\text{Dom}(\simeq_1))$. Thus, we can write each end $w$ in $\text{ends}(\text{Dom}(\simeq_1))$ uniquely as $w = pv$ with $p \in \text{domC}(\text{max}(\simeq_1))$, and $v \in A^*$. The partition $\simeq_1$ on ends(\text{Dom}(\simeq_1)) then uniquely determines $\text{max}(\simeq_1)$ on $\text{domC}(\text{max}(\simeq_1))$.

Conversely, $\text{max}(\simeq_1)$ on $\text{domC}(\text{max}(\simeq_1))$ determines the partition $\simeq_1$ on ends(\text{Dom}(\simeq_1)). Since $\text{max}(\simeq_1)$ determines $\simeq_1$ on ends(\text{Dom}(\simeq_1)), and vice versa, it follows that (2) is equivalent to (3). □

### 3.2 The prefix code congruence of a right ideal homomorphism

It is well known that with any partial function $f : X \rightarrow Y$ one can associate an equivalence relation $\equiv_f$ on $X$, defined by $x_1 \equiv_f x_2$ iff $f(x_1) = f(x_2)$; the set of equivalence classes is $\{f^{-1}(y) : y \in \text{Im}(f)\}$. When $X$ and $Y$ have a structure and $f$ is a homomorphism for that structure, then $\equiv_f$ is a congruence for that structure.

**Definition 3.10 (The partition $\text{part}(\varphi)$).** For $\varphi \in \text{riHom}(A^*)$ we consider the right congruence on the right ideal $\text{Dom}(\varphi)$, defined by $x_1 \equiv x_2$ iff $\varphi(x_1) = \varphi(x_2)$. This right congruence is called $\text{part}(\varphi)$.
As the following example shows, part(ϕ) is not always a prefix code congruence (according to Definition 3.11). Let A = {a, b}, and let ϕ be given by the table
\[
\begin{bmatrix}
a & b \\
\text{aa} & a \\
\end{bmatrix}
\]
which has an essentially equal restriction ϕ' with table
\[
\begin{bmatrix}
a & ba & bb \\
\text{aa} & aa & ab \\
\end{bmatrix}.
\]
The set of classes of part(ϕ) is \{\{b\}\} ∪ \{av, bab : v ∈ \{a, b\}^*\} ∪ \{bbw : w ∈ \{a, b\}^*\}. The presence of the class \{b\} prevents part(ϕ) from being a prefix code congruence; indeed, for the class \{b\} we see that \{b\} a is a strict subset of the class \{a, ba\}. On the other hand, part(ϕ') is a prefix code congruence.

Interestingly, this is related to an issue that was mentioned when \(M_{k,1}\) was first defined in \(\mathbb{F}_2\), namely the fact that \(ϕ(\text{dom}(ϕ))\) is not necessarily a prefix code. In the above example, \(ϕ(\text{dom}(ϕ)) = \{aa, a\}\). The connection between these issues is given by the following.

**Proposition 3.11** For any \(ϕ \in \text{riHom}(A^*)\) and its congruence part(ϕ) we have:

\[
\text{part}(ϕ) \text{ is a prefix code congruence} \iff \text{ϕ(dom(C(ϕ))) is a prefix code.}
\]

*Proof.* [⇒] We assume that \(ϕ(\text{dom}(ϕ)) = \text{im}(ϕ)\) is a prefix code. Let \(x_1, x_2 \in \text{dom}(ϕ)\) and \(u, v \in A^*\) be such that \(ϕ(x_1u) = ϕ(x_2v)\). Then \(ϕ(x_1), ϕ(x_2) \in \text{im}(ϕ)\), and since this is a prefix code it follows that \(ϕ(x_1)\) and \(ϕ(x_2)\) are either prefix-incomparable or equal. Since \(ϕ(x_1u) = ϕ(x_2v)\) it follows that \(ϕ(x_1) = ϕ(x_2)\) and that \(u = v\). So, \(x_1u, x_2v \in ϕ^{-1}(y) = ϕ^{-1}(y) u\) for some \(y \in \text{im}(ϕ)\) (where \(y = ϕ(x_1) = ϕ(x_2)\)). In other words, every class of part(ϕ) is of the form \(ϕ^{-1}(y) u\) for some \(y \in \text{part}(ϕ)\), and \(u \in A^*\).

Moreover, \(ϕ^{-1}(y) \subseteq \text{dom}(ϕ)\) for every \(y \in \text{im}(ϕ) = \text{ϕ(dom(C(ϕ)))}\). Indeed, if \(ϕ(xu) = y \in \text{im}(ϕ)\) for some \(x \in \text{dom}(ϕ)\) and \(u \in A^*\), then \(ϕ(x) \in \text{im}(ϕ) = \text{ϕ(dom(C(ϕ)))}\); hence \(u = ε\), since \(\text{im}(ϕ)\) is a prefix code, so \(xu = x \in \text{dom}(ϕ)\). Hence, part(ϕ) is a prefix code congruence, with domain code domC(ϕ).

[⇒] We assume that \(ϕ(\text{dom}(ϕ))\) is not a prefix code. So, there exist \(y, yu \in ϕ(\text{dom}(ϕ))\) with \(u \in A^*\) and \(u \neq ε\). Let \(x_1, x_2 \in \text{dom}(ϕ)\) be such that \(ϕ(x_1) = y\) and \(ϕ(x_2) = yu\). Since \(y \neq yu\), \([x_1] \cap [x_2] = ∅\). On the other hand, \(ϕ(x_1u) = y\), so \(x_1u \in [x_1] u [x_2] \neq ∅\), and \([x_2] \neq [x_1] u\) (since \([x_1] u \subseteq \text{dom}(ϕ)\) would imply that \(x_1\) and \(x_1u\) both belong to the prefix code \(\text{dom}(ϕ)\), which is impossible since \(u \neq ε\)). Hence, \([x_1] u\) is not a class of part(ϕ), so part(ϕ) is not a prefix code congruence.

This motivates the following.

**Definition 3.12** Within the monoid \(\text{riHom}(A^*)\) we consider the submonoid
\[
\text{riHom}_{\text{pc}}(A^*) = \{ϕ \in \text{riHom}(A^*) : ϕ(\text{dom}(ϕ)) \text{ is a prefix code}\}.
\]
The elements of \(\text{riHom}_{\text{pc}}(A^*)\) are called prefix code preserving.

The subscript “pc” stands for “prefix code”. It is easy to check that \(\text{riHom}_{\text{pc}}(A^*)\) is indeed a monoid. The reason for calling the elements of \(\text{riHom}_{\text{pc}}(A^*)\) “prefix code preserving” is the following.

**Proposition 3.13** For all \(ϕ \in \text{riHom}(A^*)\) we have: \(ϕ(\text{dom}(ϕ))\) is a prefix code iff for every prefix code \(P \subseteq A^*, ϕ(P)\) is a prefix code.

*Proof.* The right-to-left implication is trivial. We prove the left-to-right implication by contraposition. Let \(x_1, x_2 \in A^*\) prefix incomparable, but assume by contradiction that \(ϕ(x_2) = ϕ(x_1) w\), for some non-empty \(w \in A^*\). Assume also that \(x_1, x_2 \in \text{Dom}(ϕ)\), so there are \(p_1, p_2 \in \text{dom}(ϕ)\) such that \(x_1 = p_1 u_1, x_2 = p_2 u_2\) (for some \(u_1, u_2 \in A^*\)). Then \(ϕ(x_2) = ϕ(x_1) w\) implies \(ϕ(p_2) = ϕ(p_1) u_1 w\), which implies that \(ϕ(p_2)\) and \(ϕ(p_1)\) are prefix comparable. □

The following further demonstrates the importance of the monoid \(\text{riHom}_{\text{pc}}(A^*)\).
Proposition 3.14  Every $\varphi \in \text{riHom}(A^*)$ has an essential restriction to some element of $\text{riHom}_{pc}(A^*)$.

Proof. It is easy to restrict $\varphi$ (to some element $\Phi$) such that the image code becomes $\text{imC}(\Phi) = A^\ell$, where $\ell$ is the length of a longest word in $\varphi(\text{domC}(\varphi))$. Obviously, $A^\ell$ is a prefix code.  \hfill \Box

Henceforth, when we use $\text{part}(\varphi)$ we will always assume that $\varphi$ is a prefix code congruence; equivalently, we assume that $\varphi(\text{domC}(\varphi)) = \text{imC}(\varphi)$ is a prefix code.

A related issue is the fact that essentially equal restrictions and extensions of prefix code congruences are defined in a more limited way than restrictions and extensions of elements of $\text{riHom}(A^*)$: In an essentially equal restriction of a prefix code congruence, an entire class $C$ is replaced by the set of classes $\{Ca_1, \ldots, Ca_k\}$. On the other hand, in an essentially equal restriction of $\varphi \in \text{riHom}(A^*)$, a single element $x \in \text{domC}(\varphi)$ is replaced by $\{xa_1, \ldots, xa_k\}$ (with accompanying replacements of the image $\varphi(x)$ by $\{\varphi(x)a_1, \ldots, \varphi(x)a_k\}$). So, besides the general essentially equal restrictions of $\varphi \in \text{riHom}(A^*)$ we will consider the following important special case:

Definition 3.15  Let $\varphi, \Phi \in \text{riHom}(A^*)$. Then $\Phi$ is an essentially equal class-wise restriction of $\varphi$ (and $\varphi$ is an essentially equal class-wise extension of $\Phi$) iff $\Phi$ is an essentially equal restriction of $\varphi$ such that $\text{Dom}(\varphi) - \text{Dom}(\Phi)$ is a union of classes of $\text{part}(\varphi)$.

The best way to understand class-wise restrictions (or extensions) is to think of them as restrictions (or extensions) from the point of view of $\text{imC}(\varphi)$. More precisely, to create such a restriction we take $y \in \text{imC}(\varphi)$, replace $\text{imC}(\varphi)$ by $(\text{imC}(\varphi) - \{y\}) \cup \{ya_1, \ldots, ya_k\}$, and then do the corresponding replacement in $\text{domC}(\varphi)$ (as described in Definition 3.13).

It is easy to see that $\Phi$ is an essentially equal class-wise restriction of $\varphi$ iff $\Phi$ can be obtained from $\varphi$ by a finite number of the following type of replacement steps. Below, $\chi \in \text{riHom}(A^*)$ is any intermediate element obtained.

(3.16)  Replace $\{(x, \chi(C)) : x \in C\}$ in the table by $\{(xa_i, \chi(C)a_i) : x \in C, \ 1 \leq i \leq k\}$,

where $C$ is a class of $\text{part}(\chi)$ in $\text{domC}(\chi)$. An essentially equal class-wise extension is obtained by repeated replacements in the opposite direction, i.e., of the form

(3.17)  Replace $\{(xa_i, \chi(C)a_i) : x \in C, \ 1 \leq i \leq k\}$ in the table by $\{(x, \chi(C)) : x \in C\}$,

where $Ca_1, \ldots, Ca_k$ are classes of $\text{part}(\chi)$ in $\text{domC}(\chi)$.

The significance of these replacement rules is demonstrated by the following.

Proposition 3.18  For any $\varphi \in \text{riHom}_{pc}(A^*)$ we have: An essentially equal restriction or extension of $\varphi$ leads again to an element of $\text{riHom}_{pc}(A^*)$ iff this restriction or extension is an essentially equal class-wise restriction or extension.

Proof. Consider an essentially equal restriction of $\varphi$ in which a class $C$ of $\text{part}(\varphi)$ (contained in $\text{domC}(\varphi)$) is replaced in part. In other words, there are $x_1, x_2 \in C$ such that $(x_1, y)$ is left unchanged (where $\varphi(C) = y \in \text{imC}(\varphi)$), and $(x_2, y)$ is replaced by $\{(x_2a_1, ya_1), \ldots, (x_2a_k, ya_k)\}$. Then for the resulting element $\Phi \in \text{riHom}(A^*)$ obtained from $\varphi$ we have $\ y, ya_1, \ldots, ya_k \in \Phi(\text{domC}(\Phi)), \text{ hence } \Phi(\text{domC}(\Phi))$ is not a prefix code.

On the other hand, assume only entire classes are replaced; e.g., for a class $C$ of $\text{part}(\varphi)$ in $\text{domC}(\varphi)$, we replace all of $\{(x, y) : x \in C\}$ by $\{(xa_i, ya_i) : x \in C, \ 1 \leq i \leq k\}$, where $y = \varphi(C)$. Then $\text{imC}(\varphi)$ is replaced by $(\text{imC}(\varphi) - \{y\}) \cup yA$; this is a prefix code if $\text{imC}(\varphi)$ is a prefix code.

For essentially equal extensions, the proof is similar.  \hfill \Box
By Prop. 3.18 \( \text{riHom}_\text{pc}(A^*) \) is closed under essential class-wise restriction and extension, as introduced in Definition 3.15.

The replacement rules (3.16) and (3.17). Are easily seen to form a confluent and terminating rewriting system, in the direction (3.17). Hence, for each \( \varphi \in \text{riHom}(A^*) \) there exists a unique maximal element \( \max_{\text{pc}}(\varphi) \) for the rules (3.17).

**Proposition 3.19.**

1. For all \( \varphi_1, \varphi_2 \in \text{riHom}_\text{pc}(A^*) \) we have: \( \varphi_1 = \varphi_2 \) in \( M_{k,1} \) iff \( \varphi_1 \) and \( \varphi_2 \) can be obtained from each other by a finite number of applications of the replacement rules (3.16) and (3.17).
2. For all \( \varphi_1, \varphi_2 \in \text{riHom}_\text{pc}(A^*) \) we have: \( \varphi_1 = \varphi_2 \) in \( M_{k,1} \) iff \( \max_{\text{pc}}(\varphi_1) = \max_{\text{pc}}(\varphi_2) \).
3. For all \( \varphi \in \text{riHom}_\text{pc}(A^*) \), \( \max_{\text{pc}}(\varphi) \) is the maximum class-wise extension of \( \varphi \).

**Proof.** For (1), the implication \( \Leftarrow \) is obvious. For \( \Rightarrow \) we consider a common essentially equal class-wise restriction \( \varphi_0 \) of both \( \varphi_1 \) and \( \varphi_2 \) (which exists since \( \varphi_1 \) and \( \varphi_2 \) are equal as elements of \( M_{k,1} \)). Next, we can extend \( \varphi_0 \) to \( \varphi_1 \) and to \( \varphi_2 \) by essentially equal class-wise extension steps. The proofs of (2) and (3) are straightforward. \( \square \)

**Proposition 3.20** If \( \varphi_1, \varphi_2 \in \text{riHom}_\text{pc}(A^*) \) represent the same element of \( M_{k,1} \) then \( \text{part}(\varphi_1) =_{\text{ess}} \text{part}(\varphi_2) \).

**Proof.** By Propositions 3.5 and by Prop. 3.8 it is enough to prove that if \( \varphi_2 \) is obtained from \( \varphi_1 \) by one essential congruence extension (or restriction) step, then \( \text{part}(\varphi_2) \) is obtained from \( \text{part}(\varphi_1) \) by essential extension (or restriction) steps. We only consider the case of an extension step, the case of a restriction step being similar. Suppose \( C_a, \ldots, C_k \) are classes of \( \text{part}(\varphi_1) \) in \( \text{domC}(\varphi_1) \), and suppose \( \varphi_1(C_a) = y_1, \ldots, \varphi_1(C_k) = y_k \) for some \( y \in A^* \). Let \( \varphi_2 \) be obtained by extending \( \varphi_1 \) by \( \varphi_2(C) = y \). Then \( \varphi_2^{-1}(y_a) = \varphi_2^{-1}(y) = \varphi_1^{-1}(y) \in \text{part}(\varphi_1) \), for \( i = 1, \ldots, k \). Rule (3.14) can be applied to this situation; this leads to a new prefix code congruence, obtained by adding \( \varphi_2^{-1}(y) = C \) to \( \text{part}(\varphi_1) \). But this new prefix code congruence is precisely \( \text{part}(\varphi_2) \), since \( \varphi_2 \) is obtained from \( \varphi_1 \) by adding \( \varphi_2(C) = y \). So, \( \text{part}(\varphi_2) \) is obtained from \( \text{part}(\varphi_1) \) by one extension step. \( \square \)

The converse of Proposition 3.20 is obviously not true. E.g., for every \( \varphi \in G_{k,1} \) the essential congruence \( \max(\text{part}(\varphi)) \) is the same, namely the congruence given by the prefix code partition \( \{\{\varepsilon\}\} \) (consisting of a single class, where \( \varepsilon \) is the empty word); the prefix code congruence that corresponds to this is the discrete partition of \( A^* \) (with singletons as classes). This example also gives an instance where \( \varphi \) is maximally extended, whereas \( \text{part}(\varphi) \) is not maximally extended (neither class-wise nor in the general sense).

We will show in Prop. 3.24 that every prefix code congruence is the prefix code congruence of some right-ideal homomorphism. First we will need a characterization of the idempotents of \( \text{riHom}(A^*) \) and \( M_{k,1} \).

**Lemma 3.21.**

1. An element \( \eta \in \text{riHom}(A^*) \) is an idempotent (for the operation of composition) iff for every \( y \in \text{Im}(\eta) \) : \( y \in \eta^{-1}(y) \). The latter is equivalent to \( \eta(y) = y \) for all \( y \in \text{Im}(\eta) \).
2. If \( \eta \in \text{riHom}(A^*) \) is an idempotent (for the operation of composition) then all essentially equal extensions and restrictions of \( \eta \) are also idempotents of \( \text{riHom}(A^*) \).
3. If \( \eta \in \text{riHom}(A^*) \) is an idempotent then \( \eta \in \text{riHom}_\text{pc}(A^*) \).

**Proof.** Statement (1) is a basic fact about composition of partial functions.

Proof of (2): If \( \eta(y) = y \) for some \( y \in \text{Im}(\eta) \) then \( \eta(yw) = yw \) for all \( w \in A^* \). Hence, essentially equal restrictions of \( \eta \) are also idempotents.
If \( y_{a_1}, \ldots, y_{a_k} \in \text{Im}(\eta) \) and \( \eta(y_{a_1}) = y_{a_1}, \ldots, \eta(y_{a_k}) = y_{a_k} \) then an extension of \( \eta \) will be a function \( \eta' \) with the additional mapping \( \eta'(y) = y \). This preserves the condition for an idempotent.

Proof of (3): Since \( \eta \) is an idempotent, it follows from (1) that \( \text{Im}(\eta) \subseteq \text{Dom}(\eta) \). Let \( y_1, y_2 \in \text{imC}(\eta) \subseteq \text{Dom}(\eta) \). Then there exist \( p_1, p_2 \in \text{domC}(\eta) \) and \( u_1, u_2 \in A^* \) such that \( y_1 = p_1u_1 \) and \( y_2 = p_2u_2 \). If \( y_2 \) and \( y_1 \) were prefix-comparable then \( p_1 \) and \( p_2 \) would also be prefix-comparable, contradicting the fact that \( \text{domC}(\eta) \) is a prefix code. \( \square \)

**Proposition 3.22** An element \( \eta \in \text{riHom}(A^*) \) represents an idempotent of \( M_{k,1} \) iff \( \eta \) is an idempotent of \( \text{riHom}(A^*) \) (for the operation of composition).

An element \( \varphi \in M_{k,1} \) is an idempotent iff at least one representative of \( \varphi \) in \( \text{riHom}(A^*) \) is an idempotent, iff all representatives of \( \varphi \) in \( \text{riHom}(A^*) \) are idempotents. In other words, the inverse of the function \( \varphi \in \text{riHom}(A^*) \mapsto \text{max}(\varphi) \in M_{k,1} \) preserves idempotents.

**Proof.** We first prove the following.

**Claim.** If for \( \eta \in \text{riHom}(A^*) \) we have \( \eta = \text{max}(\eta) \) and \( \text{max}(\eta \circ \eta) = \eta \), then \( \eta \circ \eta = \eta \).

**Proof of Claim.** For any \( x_i \in \text{domC}(\eta) \) and any \( w \in A^* \) we have \( \eta(x_i;w) = y_iw \) for some \( y_i \in \text{imC}(\eta) \). We also have \( \eta \circ \eta(x_i;w) = \eta(y_iw) \) if \( w \) is long enough so that \( y_iw \in \text{Dom}(\eta) \). Since \( \text{max}(\eta \circ \eta) = \eta \), we then have \( \eta(y_iw) = y_iw \). Hence, for all \( y_i \in \text{imC}(\eta) \) and all long enough \( w \in A^* \) we have: \( \eta(y_iw) = y_iw \). Since \( \eta \) was assumed to be maximally essentially extended it follows that for all \( y_i \in \text{imC}(\eta) \): \( \eta(y_i) = y_i \). Therefore, for all \( x_i \in \text{domC}(\eta) \): \( \eta \circ \eta(x_i) = \eta(y_i) = y_i \), so \( \eta \circ \eta = \eta \). This proves the Claim.

We complete the proof of the Proposition. If \( \eta \in \text{riHom}(A^*) \) is an idempotent then it represents an idempotent of \( M_{k,1} \), since \( M_{k,1} \) is a homomorphic image of \( \text{riHom}(A^*) \).

If \( \varphi \in M_{k,1} \) is an idempotent then \( \varphi \) can be represented by \( \eta \in \text{riHom}(A^*) \) such that \( \eta = \text{max}(\eta) \), and \( \text{max}(\eta \circ \eta) = \eta \). By the Claim, \( \eta \) is an idempotent of \( \text{riHom}(A^*) \).

Moreover, if \( \eta = \text{max}(\eta) \) is an idempotent of \( \text{riHom}(A^*) \) then by Lemma 3.21 all its essentially equal restrictions (i.e., all representatives of \( \varphi \)) are idempotents of \( \text{riHom}(A^*) \). \( \square \)

**Definition 3.23** With a prefix code congruence \( \simeq \) we associate the following two right-ideal homomorphisms, \( \text{func}_0(\simeq) \), \( \text{func}_1(\simeq) \) \( \in \text{riHom}(A^*) \). Both have domain \( \text{Dom}(\simeq) \), and they are defined by

\[
\begin{align*}
\text{func}_0(\simeq) : & \quad x \in \text{Dom}(\simeq) \mapsto \text{min}_{\text{dict}}([x]) \in [x], \\
\text{func}_1(\simeq) : & \quad x \in \text{Dom}(\simeq) \mapsto \text{max}_{\text{dict}}([x]) \in [x],
\end{align*}
\]

where \([x]\) denotes the \( \simeq \)-class of \( x \), and \( \text{min}_{\text{dict}}([x]) \) or \( \text{max}_{\text{dict}}([x]) \) denotes the minimum, respectively maximum, element in \([x]\) according to the dictionary order on \( A^* \).

It follows from Lemma 3.21 and Prop. 3.22 that \( \text{func}_0(\simeq) \) and \( \text{func}_1(\simeq) \) are idempotents, both as elements of \( \text{riHom}(A^*) \) and of \( M_{k,1} \).

**Proposition 3.24** The operations \( \text{part} \) and \( \text{func} \) are inverses of each other, in the following sense:

1. For any prefix code congruence \( \simeq \) and for \( j = 0,1 \), we have:
   \[ \text{part}(\text{func}_j(\simeq)) = \simeq. \]

2. For any right-ideal homomorphism \( \varphi \) and for \( j = 0,1 \), we have:
   \[ \text{func}_j(\text{part}(\varphi)) \equiv_{\text{L}} \varphi, \]

where \( \equiv_{\text{L}} \) is the \( \text{L} \)-equivalence of \( \text{riHom}(A^*) \).

Hence (by Prop. 3.22), we also have \( \text{func}_j(\text{part}(\varphi)) \equiv_{\text{L}} \varphi \) for the \( \equiv_{\text{L}} \)-relation of \( M_{k,1} \).
Proof. Part (1) follows immediately from the definitions of part and \( \func_j \).

For part (2) let \( \varphi : P \rightarrow QA \) be a right-ideal homomorphism where \( P \) and \( Q \) are finite prefix codes. Let \( \part(\varphi) = \{P_1, \ldots, P_m\} \), where \( m = |Q| \), and let \( Q = \{q_1, \ldots, q_m\} \), where \( q_i = \varphi(P_i) \).

We give the proof for \( \func_1 \); for \( \func_0 \) the proof is the same. Let \( f_1 \) be a short-hand for \( \func_1(\part(\varphi)) \). We want to show that \( f_1 \geq_r \varphi \) and \( \varphi \geq_f f_1 \).

For all \( p \in P_i \) (\( i = 1, \ldots, m \)) we have \( \varphi \circ f_1(p) = \varphi(\max_{\dict}(P_i)) = \varphi(p) \), since \( p \) and \( \max_{\dict}(P_i) \) belong to \( P_i \). So, \( f_1 \geq_r \varphi \).

Let \( \psi \in \riHom(A^*) \) be defined by \( \psi(q_i) = \max_{\dict}(P_i) \) for \( i = 1, \ldots, m \); so, \( \domC(\psi) = Q \). Then for all \( p \in P_i \) (\( i = 1, \ldots, m \)) we have \( \psi \circ \varphi(p) = \psi(q_i) = \max_{\dict}(P_i) = f_1(p) \). Hence, \( \varphi \geq_f f_1 \). \( \square \)

It follows from Prop. 3.24(1) that every prefix code congruence is the partition of some right-ideal homomorphism.

**Lemma 3.25** Let \( \simeq \) be a prefix code congruence. Then \( \simeq \) is maximally extended iff \( \func_j(\simeq) \) is maximally extended. It follows that (for \( j = 0, 1 \)),

\[
\func_j(\max(\simeq)) = \max(\func_j(\simeq)) = \max(\func_j(\max(\simeq))).
\]

**Proof.** An extension of \( \simeq \) is possible iff \( \simeq \) contains the classes \( Ca_1, \ldots, Ca_k \), but not \( C \). This is equivalent to having \( \func_1(\simeq) \) mapping \( Ca_i \) to \( \max_{\dict}(Ca_i) = \max_{\dict}(C) \) \( a_i \) for \( i = 1, \ldots, k \). So \( \simeq \) is extendable iff \( \func_1(\simeq) \) is extendable. The proof for \( \func_0(\simeq) \) is the same. \( \square \)

**Proposition 3.26** For prefix code congruences \( \simeq_1 \) and \( \simeq_2 \) the following are equivalent:

- \( \simeq_1 \) and \( \simeq_2 \) are essentially equal,
- \( \func_0(\simeq_1) = \func_0(\simeq_2) \) in \( M_{k,1} \),
- \( \func_1(\simeq_1) = \func_1(\simeq_2) \) in \( M_{k,1} \).

**Proof.** We prove this only for \( \func_1 \); for \( \func_0 \) the proof works in the same way. If \( \simeq_1 = \ess \simeq_2 \) then \( \max(\simeq_1) = \max(\simeq_2) \), by Proposition 3.2(2). Hence by Lemma 3.24 \( \max(\func_1(\simeq_1)) = \max(\func_1(\simeq_2)) \), hence, \( \func_1(\simeq_1) = \func_1(\simeq_2) \) in \( M_{k,1} \).

Conversely, if \( \func_1(\simeq_1) = \func_1(\simeq_2) \) in \( M_{k,1} \) then \( \part(\func_1(\simeq_1)) = \ess \part(\func_1(\simeq_2)) \), by Prop. 3.20 By Prop. 3.21(1), \( \part(\func_1(\simeq_1)) = \simeq_1 \) and, similarly for \( \simeq_2 \). Hence, \( \simeq_1 = \ess \simeq_2 \). \( \square \)

### 3.3 Refinements of prefix code congruences

**Definition 3.27** Let \( \simeq_1 \) and \( \simeq_2 \) be prefix code congruences. We say that \( \simeq_1 \) is an *end refinement* of \( \simeq_2 \) iff there exist essentially right congruences \( \simeq_1' \) and \( \simeq_2' \) such that:

- \( \simeq_1' \) is an essentially equal restriction of \( \simeq_1 \) (for \( i = 1, 2 \)),
- \( \domC(\simeq_2') \subseteq \domC(\simeq_1') \), and
- every class of \( \simeq_2' \) is a union of classes of \( \simeq_1' \).

**Notation:** If \( \simeq_1 \) is an end refinement of \( \simeq_2 \) we denote this by \( \simeq_2 \leq_{\text{end}} \simeq_1 \).

**Lemma 3.28** Let \( \simeq_2' \) and \( \simeq_1' \) be prefix code congruences.

1. If \( \simeq_2' \leq_{\text{end}} \simeq_1' \) then \( \Dom(\simeq_2') \subseteq_{\text{end}} \Dom(\simeq_1') \).
2. Assume that every class of \( \simeq_2' \) is a union of classes of \( \simeq_1' \), and that \( \domC(\simeq_2') \subseteq \domC(\simeq_1) \). And assume \( \simeq_2' \) is essentially extendable, in one replacement step \( \{i\} \), to \( \simeq_2 \). Then \( \simeq_1' \) is essentially extendable to a prefix code congruence \( \simeq_1 \) such that every \( \simeq_2 \)-class is a union of \( \simeq_1 \)-classes, and \( \domC(\simeq_2) \subseteq \domC(\simeq_1) \).
3. Assume that every \( \simeq_2 \)-class is a union of \( \simeq_1 \)-classes, and that \( \domC(\simeq_2') \subseteq \domC(\simeq_1') \). And assume that \( \simeq_2' \) can be essentially restricted, in one replacement step \( \{i\} \), to \( \simeq_2'' \). Then \( \simeq_1' \) can be
essentially restricted to a prefix code congruence \( \simeq''_1 \) such that every \( \simeq''_2 \)-class is a union of \( \simeq''_1 \)-classes and \( \text{dom} C(\simeq''_2) \subseteq \text{dom} C(\simeq''_1) \).

**Proof.** (1) Let \( \simeq'_1, \simeq'_2 \) be as in Definition \ref{def:equivalent}. Then \( \text{dom} C(\simeq'_2) \subseteq \text{dom} C(\simeq'_1) \) implies \( \text{Dom}(\simeq'_2) \subseteq \text{Dom}(\simeq'_1) \). Therefore, \( \text{Dom}(\simeq_2) =_{\text{ess}} \text{Dom}(\simeq'_2) \subseteq \text{Dom}(\simeq'_1) \), since \( \simeq'_1 \) and \( \simeq'_2 \) are essentially equal restrictions of \( \simeq_1 \), respectively \( \simeq_2 \). Hence, \( \text{Dom}(\simeq_2) \subseteq_{\text{end}} \text{Dom}(\simeq_1) \).

(2) Let \( A = \{a_1, \ldots, a_k\} \). If \( \simeq'_2 \) is extendable, it has a subset of classes of the form \( Ca_1, \ldots, Ca_k \), with \( Ca_1 \cup \ldots \cup Ca_k \subseteq \text{dom} C(\simeq'_2) \). Since every class of \( \simeq'_2 \) is a union of \( \simeq'_1 \)-classes, we have for each \( i = 1, \ldots, k \) : \( Ca_i = \bigcup_j Q_{i,j} \), where each \( Q_{i,j} \) is a \( \simeq'_1 \)-class, and \( Q_{i,j} \subseteq \text{dom} C(\simeq'_1) \). It follows that \( Q_{i,j} \) has the form \( P_ja_i \) for all \( i, j \), and that \( \bigcup_j P_j = C \). Hence \( \simeq'_1 \) contains the classes \( P_ja_1, \ldots, P_ja_k \). So, \( \simeq'_1 \) can be essentially extended to a prefix code congruence \( \simeq_1 \) by adding the classes \( P_j \) to \( \simeq'_1 \) for all \( j \). The domain code of \( \simeq_1 \) is obtained from \( \text{dom} C(\simeq'_2) \) by replacing the set \( \bigcup_j P_ja_i \) by \( \bigcup_j P_j \). Since \( \bigcup_j P_j = C \), it follows that every \( \simeq_2 \)-class is a union of \( \simeq_1 \)-classes, and that \( \text{dom} C(\simeq'_2) \subseteq \text{dom} C(\simeq'_1) \).

(3) The proof is similar to the proof of (2). \( \qed \)

The following generalizes Definition 5.2 from [5].

**Definition 3.29** Let \( Q, P \subset A^* \) be finite prefix codes such that \( PA^* \subset QA^* \). A complement of \( P \) in \( QA^* \) is any finite prefix code \( C \subset A^* \) such that \( CA^* \cap PA^* = \emptyset \) and \( CA^* \cup PA^* =_{\text{ess}} QA^* \).

For the ends this means: \( \text{end}(CA^*) \cap \text{end}(PA^*) = \emptyset \), and \( \text{end}(CA^*) \cup \text{end}(PA^*) = \text{end}(QA^*) \).

**Lemma 3.30** Let \( Q, P \subset A^* \) be finite prefix codes such that \( PA^* \subset QA^* \). Then there exists a complement of \( P \) in \( QA^* \).

**Proof.** Let \( \ell = \max\{|p| : p \in P\} \), i.e., \( \ell \) is the length of the longest word in \( P \). We pick \( C = \{x \in QA^* - PA^* : |x| = \ell\} \).

Since all elements of \( C \) have the same length, \( C \) is a finite prefix code. Also, the definition of \( C \) immediately implies that \( CA^* \cap PA^* = \emptyset \).

Let us prove that \( CA^* \cup PA^* =_{\text{ess}} QA^* \). It is enough to show that every end \( w \in \text{ends}(QA^*) \) that does not pass through \( P \) passes through \( C \). The latter is true, since the prefix \( x \) of \( w \) of length \( \ell \) belongs to \( C \), by the definition of \( C \). \( \qed \)

**Proposition 3.31 (Characterizations of \( \leq_{\text{end}} \)).** Let \( \simeq_2 \) and \( \simeq_1 \) be prefix code congruences. The following are equivalent:

1. \( \simeq_2 \leq_{\text{end}} \simeq_1 \);
2. there is an essentially equal restriction \( \simeq'_2 \) of \( \simeq_2 \) such that every \( \simeq'_2 \)-class is a union of \( \simeq_1 \)-classes;
3. there is an essentially equal extension \( \simeq''_1 \) of \( \simeq_1 \) such that every \( \simeq_2 \)-class is a union of \( \simeq''_1 \)-classes;
4. every \( \simeq_2 \)-class is a union of \( \text{max}(\simeq_1) \)-classes;
5. every \( \text{max}(\simeq_2) \)-class is a union of \( \text{max}(\simeq_1) \)-classes;
6. \( \text{ends}(\text{Dom}(\simeq_2)) \subseteq \text{ends}(\text{Dom}(\simeq_1)) \), and every \( \simeq_2 \)-class of \( \text{ends}(\text{Dom}(\simeq_2)) \) is a union of \( \simeq_1 \)-classes of ends(\( \text{Dom}(\simeq_1) \)).

**Proof.** [(1) \( \Rightarrow \) (2)] Let \( \simeq'_2 \) and \( \simeq'_1 \) be as in Definition \ref{def:equivalent} so every \( \simeq'_2 \)-class is a union of \( \simeq'_1 \)-classes. Moreover, every \( \simeq'_2 \)-class is also a \( \simeq_1 \)-classes, since \( \simeq'_1 \) is an essentially equal restriction of \( \simeq_1 \). Thus, every \( \simeq'_2 \)-class is a union of classes of \( \simeq_1 \).
[(2) $\Rightarrow$ (3)] Let $\simeq_2$, $\simeq_1$, and $\simeq_1$ be as in (2). Now we apply Lemma [3.28](2) to $\simeq_2$ and $\simeq_1$, repeatedly, until $\simeq_2'$ has been rewritten to its essentially equal extension $\simeq_1$. In this process, $\simeq_1$ is rewritten to some essentially equal extension $\simeq_1'$ such that (3) holds.

[(3) $\Rightarrow$ (4)] If every $\simeq_2$-class is a union of $\simeq_1^*$-classes, then $\simeq_2$-class is also a union of $\max(\simeq_1)$-classes, since $\max(\simeq_1)$ is an end refinement of $\simeq_1^*$.

[(4) $\Rightarrow$ (5)] We repeatedly apply Lemma [3.28](2) to $\simeq_2$ and $\max(\simeq_1)$, until $\simeq_2$ has been rewritten to $\max(\simeq_2)$. In the process, $\max(\simeq_1)$ does not change (being already maximally extended). As a result, every $\max(\simeq_2)$-class is a union of $\max(\simeq_1)$-classes.

[(5) $\Rightarrow$ (6)] It follows immediately from (5) that every element of a $\simeq_2$-class is also in a $\simeq_1$-class; thus, $\Dom(\max(\simeq_2)) \subseteq \Dom(\max(\simeq_1))$. Hence, $\Ends(\Dom(\simeq_2)) \subseteq \Ends(\Dom(\simeq_1))$.

Let $w_1, w_2 \in \Ends(\Dom(\simeq_2))$ be such that $w_1 \simeq_1 w_2$; we want to show that $w_1 \simeq_2 w_2$. It follows from $w_1 \simeq_1 w_2$ that $w_1 = q_1 v$ and $w_2 = q_2 v$ with $q_1 \simeq_1 q_2$, for some $q_1, q_2 \in \Dom(\simeq_1)$, $v \in A^\omega$. Moreover, $q_1 \simeq_1 q_2$ implies that $q_1 = p_1 x$ and $q_2 = p_2 x$ with $p_1 \max(\simeq_1) p_2$, for some $p_1, p_2 \in \Dom(\max(\simeq_1))$, $x \in A^*$. By (5), this implies $p_1 \max(\simeq_2) p_2$, and hence $q_1 \max(\simeq_2) q_2$ (since this is a right congruence). Hence, $q_1 \simeq_2 q_2$ (since $q_1, q_2 \in \Dom(\simeq_1)$). Therefore, $w_1 \simeq_2 w_2$ (by the definition of $\simeq_2$ on ends).

[(6) $\Rightarrow$ (1)] Since $\Dom(\simeq_1)$ and $\Dom(\simeq_2)$ are finitely generated right ideals, there intersection is also a finitely generated right ideal (by Lemma [3.26]). So, $\Dom(\simeq_2) \cap \Dom(\simeq_1) = P_2 A^*$ for some finite prefix code $P_2$. By (6), $\Ends(\Dom(\simeq_2)) \subseteq \Ends(\Dom(\simeq_1))$; hence

$$\Ends(P_2 A^*) = \Ends(\Dom(\simeq_2)) \cap \Ends(\Dom(\simeq_1)) = \Ends(\Dom(\simeq_2)).$$

In other words, $P_2 A^* = \ess \Dom(\simeq_2)$. Let $\simeq_2'$ be the essentially equal restriction of $\simeq_2$ to $P_2 A^*$.

By Lemma [3.30] there exists a complementary prefix code (let’s call it $Q_1$) of $P_2$ within $\Dom(\simeq_1)$. Since $Q_1$ is a complementary prefix code, $(Q_1 \cup P_2) A^* = \ess \Dom(\simeq_1)$. Let $\simeq_1'$ be the essentially equal restriction of $\simeq_1$ to $(Q_1 \cup P_2) A^*$.

Then $\simeq_2$ and $\simeq_1'$ satisfy the conditions of Definition [3.31], so (1) holds. □

For prefix codes congruences $\simeq_1$ and $\simeq_2$ we have

$$\simeq_1 = \ess \simeq_2 \iff \simeq_1 \leq_{\text{end}} \simeq_2 \text{ and } \simeq_1 \geq_{\text{end}} \simeq_2.$$

This follows immediately from Propositions [3.39](2) and [3.31](5). Recall that for prefix codes congruences, $=\ess$ was defined in Def. [3.6] and $\leq_{\text{end}}$ was defined in Def. [3.27].

The relation $\leq_{\text{end}}$ is a lattice pre-order on the set of all prefix code congruences of $A^\omega$. For two prefix code congruences $\simeq_1$ and $\simeq_2$ we consider the prefix code congruence $\simeq_1 \wedge \simeq_2$, called the meet or wedge. Its domain is $\Dom(\simeq_1 \wedge \simeq_2) = \Dom(\simeq_1) \cap \Dom(\simeq_2)$, and for all $u, v \in \Dom(\simeq_1 \wedge \simeq_2)$ we have: $u (\simeq_1 \wedge \simeq_2) v$ iff $u \simeq_1 v$ and $u \simeq_2 v$. Similarly, the join $\simeq_1 \vee \simeq_2$ has domain $\Dom(\simeq_1 \vee \simeq_2) = \Dom(\simeq_1) \cup \Dom(\simeq_2)$, and is defined by the transitive closure of the relation $\simeq_1 \cup \simeq_2 = \{(u, v) \in A^* \times A^*: u \simeq_1 v \text{ or } u \simeq_2 v\}$. Equivalently, we start with all the classes that belong to $\simeq_1$ or $\simeq_2$, and we iteratively replace classes that intersect by their union, until no two classes intersect.

### 3.4 Characterization of the $L$-order of $M_k,1$

**Theorem 3.32 (L-order of $M_k,1$).** For any $\varphi, \psi \in M_{k,1}$:

$$\psi(.) \leq_L \varphi(.) \iff \Part(\psi) \leq_{\text{end}} \Part(\varphi)$$

**Proof.** $\Rightarrow$ If $\psi \leq_L \varphi$ then there exists $\alpha \in M_{k,1}$ such that $\psi$ and $\alpha \circ \varphi$ represent the same element of $M_{k,1}$. Hence (by Lemma [2.8] and Prop. [3.20]), $\Dom(\psi) = \ess \Dom(\alpha \circ \varphi)$, $\Im(\psi) = \ess \Im(\alpha \circ \varphi)$,
Lemma 4.1

\[ \text{Proof.} \]

Let \( u, v \in \text{Dom}(\alpha \circ \varphi) \). Then \( u \) and \( v \) are related by \( \text{part}(\varphi) \) iff \( \varphi(u) = \varphi(v) \), which implies \( \alpha \circ \varphi(u) = \varphi(v) \), hence \( u \) and \( v \) are in the same \( \text{part}(\alpha \circ \varphi) \)-class. It follows that \( \text{part}(\varphi) \) is a refinement of \( \text{part}(\alpha \circ \varphi) \). Since \( \text{part}(\psi) =_{\text{ess}} \text{part}(\alpha \circ \varphi) \), it follows that \( \text{part}(\varphi) \) is an end refinement of \( \text{part}(\psi) \).

\[ \left[ \leq \right] \] If \( \text{part}(\varphi) \) is an end refinement of \( \text{part}(\psi) \), then (by Definition 3.27) there exists an essentially equal restriction \( \simeq_2^{\psi} \) of \( \text{part}(\psi) \), and an essentially equal restriction \( \simeq_1^{\psi} \) of \( \text{part}(\varphi) \), such that every \( \simeq_2^{\psi} \)-class is a union of \( \simeq_1^{\psi} \)-classes. Let \( P_2 = \text{domC}(\simeq_2^{\psi}) \) and let \( P_1 = \text{domC}(\simeq_1^{\psi}) \). Let \( \psi_0 \) be the restriction of \( \psi \) to \( P_2A^* \), and let \( \varphi_0 \) be the restriction of \( \varphi \) to \( P_1A^* \). Since \( \simeq_2^{\psi} \) is an essentially equal restriction, \( \psi_0 \) and \( \varphi_0 \) represent the same element of \( M_{k,1} \); similarly, \( \varphi_0 \) and \( \varphi \) represent the same element of \( M_{k,1} \).

We define a right ideal homomorphism \( \alpha \) with domain \( \text{Dom}(\alpha) = \text{Im}(\varphi_0) \) and image \( \text{Im}(\alpha) = \text{Im}(\psi_0) \), as follows:

\[ \alpha : \ z \in \text{Im}(\varphi_0) \mapsto \psi_0(\varphi_0^{-1}(z)) \in \text{Im}(\psi_0). \]

Then \( \alpha \) is a function. Indeed, \( z \in \text{Im}(\varphi_0) \) can be written as \( z = \varphi_0(x) \) for any \( x \in \varphi_0^{-1}(z) \). Then \( \alpha(z) = \psi_0(\varphi_0^{-1}(\varphi_0^{-1}(z))) \); and since every \( \simeq_2^{\psi} \)-class is a union of \( \simeq_1^{\psi} \)-classes, the latter is a subset of \( \psi_0(\varphi_0^{-1}(\varphi_0^{-1}(z))) = \psi_0(z) \), independently of the choice of \( x \).

It follows also from the definition of \( \alpha \) that for all \( z \in \text{Im}(\varphi_0) : \alpha \circ \varphi_0(z) = \psi_0 \circ \varphi_0^{-1} \circ \varphi_0(z) \), and since every \( \simeq_2^{\psi} \)-class is a union of \( \simeq_1^{\psi} \)-classes, the latter is a subset of \( \psi_0 \circ \varphi_0^{-1} \circ \psi_0(z) = \psi_0(z) \). So, \( \psi_0 = \alpha \circ \varphi_0 \leq_{L} \varphi_0 \). Since \( \psi_0 \) represents the same element as \( \psi \) in \( M_{k,1} \), and \( \varphi_0 \) represents the same element as \( \varphi \) in \( M_{k,1} \), we obtain \( \psi \leq_{L} \varphi \). \qed

4 Infinite \( L \)- and \( R \)-chains, and density

4.1 Embedding of the \( L \)- and \( R \)-orders into the idempotent order

Lemma 4.1

\[ \text{If } \varphi \in M_{k,1} \text{ is represented by } \varphi : P_1A^* \rightarrow P_2A^* \text{ in riHom}(A^*) \text{ where } P_1 = \text{domC}(\varphi) \text{ and } P_2 = \text{imC}(\varphi) \text{ are prefix codes, then we have: } \varphi \equiv_R \text{id}_{P_2A^*}. \]

\[ \text{Proof.} \]

\[ \left[ \geq_R \right] \text{ Obviously, } \text{id}_{P_2A^*} \circ \varphi(\cdot) = \varphi(\cdot), \text{ so } \text{id}_{P_2A^*} \geq_R \varphi. \]

\[ \left[ \leq_R \right] \text{ We want to define } \alpha : P_1A^* \rightarrow P_2A^* \text{ so that } \varphi \circ \alpha(\cdot) = \text{id}_{P_2A^*}. \text{ For every } y \in P_2 \text{ we choose an element } \gamma \in \varphi^{-1}(y) \leq P_1; \text{ next, for every } y \in P_2 \text{ define } \alpha(y) = \gamma. \]

\[ \text{Then, for every } y \in P_2 \text{ we have: } \varphi(\alpha(y)) = \varphi(\gamma) \in \varphi(\varphi^{-1}(y)) = y. \] \qed

The following provides an embedding of the \( R \)-order on the set of \( R \)-classes of \( M_{k,1} \) into the idempotent order of \( M_{k,1} \).

Proposition 4.2 (embedding of the \( R \)-order into the idempotent order).

\[ \text{Let } X, Y \subseteq A^* \text{ be finitely generated right ideals. We have:} \]

\[ \text{(1) } Y =_{\text{ess}} X \text{ iff } \text{id}_Y = \text{id}_X \text{ in } M_{k,1}. \]

\[ \text{(2) } Y \subseteq_{\text{end}} X \text{ iff } \text{id}_Y \leq \text{id}_X \text{ for the idempotent order of } M_{k,1}. \]

\[ \text{(3) The } R \text{-order \{on } R \text{-classes} \} \text{ of } M_{k,1} \text{ is embedded in the idempotent-order order of partial identity elements of } M_{k,1}. \]

\[ \text{Proof.} \]

\[ \left[ \Rightarrow \right] \text{ Y } =_{\text{ess}} X \text{ implies that } X \text{ and } Y \text{ have the same ends, hence } X \cap Y \text{ also has the same ends as } X \text{ and } Y. \text{ Hence, the restrictions of both } \text{id}_{Y} \text{ and } \text{id}_{X} \text{ to } \text{id}_{X \cap Y} \text{ are essentially equal restrictions, hence } \text{id}_{Y} = \text{id}_{X \cap Y} = \text{id}_{X} \text{ in } M_{k,1}. \]

\[ \left[ \Leftarrow \right] \text{ If } \text{id}_{Y} = \text{id}_{X} \text{ in } M_{k,1} \text{ then } \text{id}_{Y} \text{ and } \text{id}_{X} \text{ agree on some common essential subideal } Z \text{ of } X \text{ and } Y. \text{ Then both } X \text{ and } Y \text{ are essentially equal to } Z, \text{ hence } Y =_{\text{ess}} X. \]
(2) Since $\subseteq_{\text{end}}$ is a lattice pre-order we have $Y \subseteq_{\text{end}} X$ iff $Y =_{\text{ess}} X \cap Y$. By (1) the latter is equivalent to $\text{id}_Y = \text{id}_{X \cap Y}$, and the latter is equal to $\text{id}_X \circ \text{id}_Y = \text{id}_Y \circ \text{id}_X$. Moreover, $\text{id}_Y = \text{id}_V \circ \text{id}_Y = \text{id}_Y \circ \text{id}_X$ is equivalent to $\text{id}_Y \leq \text{id}_X$ for the idempotent order.

(3) This follows now from (2) and Lemma 4.1.

Proposition 4.3 (embedding of the $\mathbf{L}$-order into the idempotent order). The $\mathbf{L}$-order on the set of $\mathbf{L}$-classes of $M_{k,1}$ is embeddable into the idempotent order of $M_{k,1}$ as follows. The function $\func_j(\part(\cdot))$, defined on the set of $\mathbf{L}$-classes of $M_{k,1}$, is injective (for $j = 0, 1$), and for any $\psi, \varphi \in \riHom(A^*)$ and for $j = 0, 1$, we have:

$$\psi \leq \mathbf{L} \varphi \iff \func_j(\part(\psi)) \leq \func_j(\part(\varphi))$$

where $\leq$ is the idempotent order of $M_{k,1}$.

Proof. The injectiveness of $\func_j(\cdot)$, as a from the set of $\mathbf{L}$-classes of $M_{k,1}$ into the set of idempotents of $M_{k,1}$, follows from Theorem 3.32. So we have an embedding, but we still need to show that this is order-preserving.

$[\Leftarrow]$ If $\func_j(\part(\psi)) \leq \func_j(\part(\varphi))$ then, since $\func_j(\part(\chi)) \equiv_{\mathbf{L}} \chi$ for all $\chi$ (by Prop. 3.24(2)), we obtain $\psi \leq_{\mathbf{L}} \varphi$.

$[\Rightarrow]$ If $\psi \leq_{\mathbf{L}} \varphi$ then (by Prop. 3.24(2)), $\func_j(\part(\psi)) \leq_{\mathbf{L}} \func_j(\part(\varphi))$. We also need to show this order relation for $\leq_{\mathbf{R}}$, i.e., we need to show $\text{Im}(\func_j(\part(\psi))) \subseteq_{\text{end}} \text{Im}(\func_j(\part(\varphi)))$. We prove the result only for $\func_0$; for $\func_1$ the proof is similar.

Let the tables for $\psi$ and $\varphi$ be $\psi : S \to T$ and $\varphi : P \to Q$, where $S,T,P,Q$ are finite prefix codes. After an essential class-wise restriction (if necessary) we have $S \subseteq P$, and every class $S_i$ of $\part(\psi)$ in $S$ is a union of classes $P_1, \ldots, P_{n_i}$ of $\part(\psi)$ in $S (\subseteq P)$; this follows from $\psi \leq_{\mathbf{L}} \varphi$. Then $\func_0(\part(\psi))$ maps all of $S_i$ to $\min_{\text{dict}}(S_i) \in S_i = \bigcup_{j=1}^{n_i} P_j$. So there is $j$ with $\min_{\text{dict}}(S_i) \in P_j$; hence, $\min_{\text{dict}}(S_i) = \min_{\text{dict}}(P_j)$. Therefore, $\func_0(\part(\psi))$ maps all of $S_i$ to an element of $\text{Im}(\func_0(\part(\psi)))$. Therefore, $\text{Im}(\func_j(\part(\psi))) \subseteq \text{Im}(\func_j(\part(\varphi)))$. □

4.2 Density

Proposition 4.4 (Density of the $\mathbf{R}$-order). For any two elements $\varphi, \psi \in M_{k,1}$ such that $\varphi >_{\mathbf{R}} \psi$ there exists $\chi \in M_{k,1}$ such that $\varphi >_{\mathbf{R}} \chi >_{\mathbf{R}} \psi$.

Proof. Let $P = \text{imC}(\varphi)$ and $Q = \text{imC}(\psi)$. After applying essentially equal reductions, if necessary, we can assume that $P$ and $Q$ are prefix codes. Since $\varphi >_{\mathbf{R}} \psi$ we have $PA^* \neq_{\text{ess}} QA^*$, by Theorem 2.1. Hence, $PA^*$ contains some end $\eta$ which does not belong to $QA^*$. Let $w \in PA^*$ be a prefix of this end that is strictly longer than any element in $P$ or $Q$, and let $p$ be the prefix of $w$ that belongs to $P$. Consider the right ideal $DA^* = (Q \cup \{w\})A^*$. Then $QA^* \subseteq DA^* \subset PA^*$. Moreover, $DA^* \neq_{\text{ess}} QA^*$ since $DA^*$ contains the end $\eta$. And $DA^* \neq_{\text{ess}} PA^*$ since $PA^*$ contains all ends with prefix $p$, while $DA^*$ doesn’t ($p$ being strictly shorter than $w$). Hence (by Lemma 1.1 and Theorem 2.1), we have $\varphi >_{\mathbf{R}} \text{id}_{DA^*} >_{\mathbf{R}} \psi$. □

Proposition 4.5 (Density of the $\mathbf{L}$-order). For any two elements $\varphi, \psi \in M_{k,1}$ such that $\varphi >_{\mathbf{L}} \psi$ there exists $\chi \in M_{k,1}$ such that $\varphi >_{\mathbf{L}} \chi >_{\mathbf{L}} \psi$.

Proof. By Theorem 3.32 $\text{Dom}(\psi) \subseteq_{\text{end}} \text{Dom}(\varphi)$ and $\part(\varphi)$ is an end-refinement of $\part(\psi)$. To construct $\chi$ we distinguish two cases: Case (1), when $\text{Dom}(\psi) \neq_{\text{end}} \text{Dom}(\varphi)$; case (2), when $\text{Dom}(\psi) =_{\text{ess}} \text{Dom}(\varphi)$, and $\part(\varphi)$ is a strict end-refinement of $\part(\psi)$. By Prop. 3.31 we can essentially restrict $\varphi$ and $\psi$ so that $\text{domC}(\psi) \subseteq \text{domC}(\varphi)$ and $\part(\varphi)$ is a finer partition of $\text{domC}(\psi)$ than $\part(\psi)$. So, after essentially equal restrictions, case (1) becomes $\text{domC}(\psi) \subseteq \text{domC}(\varphi)$; case (2) becomes $\text{domC}(\psi) = \text{domC}(\varphi)$ and $\part(\varphi)$ is strictly finer than $\part(\psi)$. □
Case (1): $\text{domC}(\psi) \subsetneq \text{domC}(\varphi)$.

Consider the prefix code $\{u_1, \ldots, u_n\} = \text{domC}(\varphi) - \text{domC}(\psi)$. We take the essentially equal restriction of $\varphi$ to $u_1 A \cup \{u_2, \ldots, u_n\} \cup \text{domC}(\psi)$. Then we define $\chi$ to be the (non-essential) restriction of $\varphi$ to $u_1 A \cup \{u_2, \ldots, u_n\} \cup \text{domC}(\psi)$. This implies $\text{Dom}(\psi) \subsetneq \text{Dom}(\chi) \subsetneq \text{Dom}(\varphi)$.

Case (2): $\text{domC}(\psi) = \text{domC}(\varphi)$ and $\text{part}(\varphi)$ is strictly finer than $\text{part}(\psi)$.

In this case we will actually do use the fact that $\text{domC}(\psi) = \text{domC}(\varphi)$ in order to construct $\chi$.

To simplify the notation we only prove case (2) when $k = 2$ and $A = \{a, b\}$, but the same method works in general.

Let $\{Q_1, \ldots, Q_n\}$ be the classes of $\text{part}(\psi)$ on $\text{domC}(\psi)$, and let $\{P_1, \ldots, P_m\}$ be the classes of $\text{part}(\varphi)$ on $\text{domC}(\varphi)$. So, $\text{imC}(\psi) = \{y_1, \ldots, y_n\}$, where $y_i = \psi(Q_i)$ for $1 \leq i \leq n$; this is one word since $Q_1$ is a class of $\text{part}(\psi)$. By renaming the classes of $\text{part}(\psi)$, if necessary, we can assume that $Q_1$ is the union of at least two classes of $\text{part}(\varphi)$: $Q_1 = P_1 \cup \cdots \cup P_s$; for some $s$ with $2 \leq s \leq m$. We take an essentially equal restriction $\psi'$ of $\psi$ such that $\text{part}(\chi')$ on $\text{domC}(\chi')$ is $\{Q_1a, Q_1b, Q_2, \ldots, Q_n\}$.

Similarly, we essentially restrict $\varphi$ to $\varphi'$ so that $\text{part}(\varphi')$ on $\text{domC}(\varphi')$ is

$\{P_1a, P_2a \} \cup \{P_1b, P_2b \} \cup \{P_{s+1}, \ldots, P_m\}$.

So, $\text{domC}(\chi') = \text{domC}(\varphi')$, $P_1a \cup \cdots \cup P_s a = Q_1a$, and $P_1b \cup \cdots \cup P_s b = Q_1b$. We now define $\chi$ as follows, where $\{c_1, \ldots, c_s\}$ is any prefix code of cardinality $s$:

- $\text{domC}(\chi) = \text{domC}(\psi')$,
- $\text{part}(\chi)$ on $\text{domC}(\chi)$ is $\{P_1a, P_2a \} \cup \{Q_1, Q_2, \ldots, Q_n\}$,
- the values of $\chi$ on $\text{domC}(\chi)$ are

\[
\begin{cases}
\chi(P_1a) = y_1ac_j & \text{for } j = 1, \ldots, s, \\
\chi(Q_1b) = y_1b, \\
\chi(Q_j) = y_j & \text{for } 2 \leq j \leq n.
\end{cases}
\]

Recall that $y_i = \psi(Q_i)$, $1 \leq i \leq n$.

Then on $\text{domC}(\chi) (= \text{domC}(\psi') = \text{domC}(\varphi'))$ we have: $\text{part}(\varphi')$ refines $\text{part}(\chi)$, and $\text{part}(\chi)$ refines $\text{part}(\psi')$. Hence Theorem 3.32 and Prop. 3.31 imply $\psi <_L \chi <_L \varphi$. □

**Proposition 4.6 (Density of the $\mathcal{L}$-order within an $\mathcal{R}$-class).**

1. For any two elements $\varphi, \psi \in M_{k,1}$ with $\varphi \equiv \mathcal{R} \psi$ and $\varphi >_\mathcal{L} \psi$, there exists $\chi \in M_{k,1}$ such that $\varphi \equiv \mathcal{R} \chi \equiv \mathcal{R} \psi$ and $\varphi >_\mathcal{L} \chi >_\mathcal{L} \psi$.
2. For every $\varphi \in M_{k,1} - \{\emptyset\}$ there exists $\chi \in M_{k,1}$ such that $\chi \equiv \mathcal{R} \varphi$ and $\varphi >_\mathcal{L} \chi$.

And for every $\psi \in M_{k,1}$ such that $\psi$ is not $\mathcal{L}$-maximal, there exists $\chi \in M_{k,1}$ such that $\chi \equiv \mathcal{R} \psi$ and $\chi >_\mathcal{L} \psi$.

**Proof.** (1). As in the proof of Prop. 4.15 we consider two cases, and we take essentially equal class-wise restrictions. Then we have: Case (1), $\text{domC}(\psi) \subsetneq \text{domC}(\varphi)$; case (2), $\text{domC}(\psi) = \text{domC}(\varphi)$ and $\text{part}(\varphi)$ is strictly finer than $\text{part}(\psi)$.

To simplify the notation we only give the proof when $k = 2$ and $A = \{a, b\}$, but the same method works in general.

Case (1): $\text{domC}(\psi) \subsetneq \text{domC}(\varphi)$.

Consider the prefix code $\{u_1, \ldots, u_n\} = \text{domC}(\varphi) - \text{domC}(\psi)$. We take the essentially equal restriction $\varphi'$ of $\varphi$ to $\{u_1a, u_1b, u_2, \ldots, u_n\} \cup \text{domC}(\psi)$. Then we define $\chi$ on $\text{domC}(\chi) = \text{domC}(\psi) \cup \{u_1a\}$ by

\[
\begin{cases}
\chi(x) = \psi(x) & \text{if } x \in \text{domC}(\psi), \\
\chi(u_1a) = \varphi(u_1)a.
\end{cases}
\]

Then $\text{im}(\psi) \subseteq \text{im}(\chi) \subseteq \text{im}(\varphi')$, hence $\chi \equiv \mathcal{R} \psi$ (since $\text{im}(\psi) = \text{ess im}(\varphi')$).
Also, $\text{Dom}(\psi) \not\preceq_{\text{end}} \text{Dom}(\chi)$ (since $u_1a$ and its ends are missing from $\text{Dom}(\psi)$), and $\text{Dom}(\chi) \not\preceq_{\text{end}} \text{Dom}(\varphi)$ (since $u_1b$ and its ends are missing from $\text{Dom}(\chi)$). So, $\psi \vartriangleleft \chi \vartriangleleft \varphi$.

Case (2): $\text{domC}(\psi) = \text{domC}(\varphi)$ and $\text{part}(\varphi)$ is strictly finer than $\text{part}(\psi)$.

In this case we will actually not use the fact that $\text{domC}(\psi) = \text{domC}(\varphi)$ in order to construct $\chi$.

Let $\{Q_1, \ldots, Q_n\}$ be the classes of $\text{part}(\psi)$ on $\text{domC}(\psi)$, and let $\{P_1, \ldots, P_m\}$ be the classes of $\text{part}(\varphi)$ on $\text{domC}(\varphi)$. So, $\text{imC}(\psi) = \{y_1, \ldots, y_n\}$, where $y_i = \psi(Q_i)$ for $1 \leq i \leq n$; this is one word since $Q_i$ is a class of $\text{part}(\psi)$. By renaming the classes of $\text{part}(\psi)$, if necessary, we can assume that $Q_1$ is the union of at least two classes of $\text{part}(\varphi)$: $Q_1 = P_1 \cup \ldots \cup P_s$, for some $s$ with $2 \leq s \leq m$. We take the essentially equal class-wise restriction $\psi'$ of $\psi$ so that $\text{part}(\psi') = \{Q_1a, Q_1b, Q_2, \ldots, Q_n\}$, and we define $\chi$ on $\text{domC}(\chi) = \text{domC}(\psi')$ by

$$
\begin{align*}
\chi(Q_1) &= y_i \text{ for } 2 \leq i \leq n, \\
\chi(Q_1b) &= y_1b, \\
\chi(P_1a) &= y_1aa, \\
\chi(Q_1a - P_1a) &= y_1ab.
\end{align*}
$$

Then $\text{imC}(\chi) = \{y_1aa, y_1ab, y_1b, y_2, \ldots, y_n\}$, hence, since $\{aa, ab, b\}$ is a maximal prefix code, we have $\text{Im}(\chi) =_{\text{ess}} \text{Im}(\psi)$; so $\chi \equiv_R \psi$.

And $\text{part}(\chi)$ is finer than $\text{part}(\psi')$ since $Q_1a$ is partitioned into $P_1a$ and $Q_1a - P_1a$; and $\text{part}(\chi)$ is finer than $\text{part}(\chi)$ since $Q_1b$ is partitioned into $P_1b$, $P_2b$, $\ldots$, $P_nb$.

(2) The proof is just a simpler version of the proof of (1). From $\varphi$ we can construct $\chi$ with an essentially smaller domain or an essentially coarser partition, while leaving the image unchanged. And from $\psi$ we can construct $\chi$ with an essentially larger domain (if $\text{domC}(\psi)$ is not a maximal prefix code) or with an essentially finer partition (if $\psi$ is not injective), while leaving the image unchanged. \(\Box\)

We observe that the proof of Prop. 4.6 also shows the following:

If $\varphi \equiv_R \psi$ and $\varphi \vartriangleright_R \psi$, and in addition, $\text{Dom}(\psi) =_{\text{ess}} \text{Dom}(\varphi)$, then there exists $\chi$ such that $\varphi \equiv_R \chi \equiv_R \psi$, $\psi \vartriangleleft_R \chi \vartriangleleft_R \varphi$, and in addition, $\text{Dom}(\chi) =_{\text{ess}} \text{Dom}(\psi)$.

The next proposition shows that for idempotents it is not possible to have all these properties at the same time.

**Proposition 4.7** If $\eta_0, \eta_1 \in M_{k, 1}$ are idempotents such that $\text{Dom}(\eta_0) =_{\text{ess}} \text{Dom}(\eta_1)$ and $\text{Im}(\eta_0) =_{\text{ess}} \text{Im}(\eta_1)$, then either $\eta_0 = \eta_1$ or $\eta_0$ and $\eta_1$ are incomparable in the $\mathcal{L}$-order.

**Proof.** Since $\text{Dom}(\eta_0) =_{\text{ess}} \text{Dom}(\eta_1)$ we can take essentially equal restrictions so that $\text{domC}(\eta_0) = \text{domC}(\eta_1)$. By Lemma [4.2][3], all idempotents of $\text{riHom}(A^*)$ belong to $\text{riHom}_{\mathcal{E}}(A^*)$; so all essentially equal restrictions of idempotents are essentially equal class-wise restrictions.

If $\text{part}(\eta_0)$ and $\text{part}(\eta_1)$ are not comparable for refinement then $\eta_0$ and $\eta_1$ are not $\mathcal{L}$-comparable; so, let us assume now that $\text{part}(\eta_0)$ refines $\text{part}(\eta_1)$. If $\text{part}(\eta_0) = \text{part}(\eta_1)$ then $\eta_0 \equiv_\mathcal{L} \eta_1$, which in combination with $\text{Im}(\eta_0) =_{\text{ess}} \text{Im}(\eta_1)$ implies that $\eta_0 = \eta_1$. So we now assume that $\text{part}(\eta_0)$ strictly refines $\text{part}(\eta_1)$. Let $P = \text{domC}(\eta_0) = \text{domC}(\eta_1)$. Let $Q_1$ be a class of $\text{part}(\eta_1)$ in $P$, which is the union of at least two classes $P_1, \ldots, P_s$ of $\text{part}(\eta_0)$ in $P$. Then $\eta_1(Q_1) = p_i \in P_i \subset Q_1$ for some $i = 1, \ldots, n$. So all ends in $\eta_1(Q_1A^*)$ have $p_i$ as a prefix, whereas ends in $\eta_0(Q_1A^*)$ can have prefixes in $P_j$ with $j \neq i$. This implies that $\text{Im}(\eta_0) \neq_{\text{ess}} \text{Im}(\eta_1)$, contrary to the assumptions. So $\text{part}(\eta_0)$ cannot strictly refine $\text{part}(\eta_1)$.

**Proposition 4.8** (Density of the $\mathcal{R}$-order within an $\mathcal{L}$-class).

1. For any two elements $\varphi, \psi \in M_{k, 1}$ with $\varphi \equiv_\mathcal{L} \psi$ and $\varphi \vartriangleright_\mathcal{R} \psi$, there exists $\chi \in M_{k, 1}$ such that $\chi \equiv_\mathcal{L} \varphi \equiv_\mathcal{L} \psi$ and $\varphi \vartriangleright_\mathcal{R} \chi \vartriangleright_\mathcal{R} \psi$.

2. For every $\varphi \in M_{k, 1} - \{\emptyset\}$ there exists $\chi \in M_{k, 1}$ such that $\chi \equiv_\mathcal{L} \varphi$ and $\varphi \vartriangleright_\mathcal{R} \chi$.

And for every $\psi \in M_{k, 1}$ such that $\psi$ is not $\mathcal{R}$-maximal, there exists $\chi \in M_{k, 1}$ such that $\chi \equiv_\mathcal{L} \psi$ and $\chi \vartriangleright_\mathcal{R} \psi$. 22
Proof. (1). Since \( \varphi \equiv_{\mathcal{L}} \psi \) we have, after an essentially equal restriction if necessary: \( \text{domC}(\varphi) = \text{domC}(\psi) \), and the \( \text{part}(\varphi) \) on \( \text{domC}(\varphi) \) is equal to \( \text{part}(\psi) \) on this domain code; let us denote this partition by \( \{P_1, \ldots, P_m\} \). Let \( v_i = \psi(P_i) \) for \( 1 \leq i \leq m \).

We essentially restrict \( \psi \) to \( \psi' \) such that \( \text{part}(\psi') \) on \( \text{domC}(\psi') \) is \( \{P_1a, P_1b, P_2, \ldots, P_m\} \). Since \( \varphi \geq_{\mathcal{R}} \psi \), there are ends in \( \text{Im}(\varphi) \) that do not belong to \( \text{Im}(\psi) \). Let \( q \in A^* \) be a prefix of such an end; we can choose \( q \) to be longer than any element of \( \text{imC}(\varphi) \cup \text{imC}(\psi') \).

We define \( \chi \) such that \( \text{domC}(\chi) = \text{domC}(\psi') \), and \( \text{part}(\chi) \) on \( \text{domC}(\chi) \) is equal to \( \text{part}(\psi') \), i.e.,

\[
\text{domC}(\chi) = \{P_1a, P_1b, P_2, \ldots, P_m\}.
\]

Hence \( \chi \equiv_{\mathcal{L}} \psi' \). The values of \( \chi \) are defined by

\[
\begin{align*}
\chi(P_i) &= v_i \text{ for } 2 \leq i \leq m, \\
\chi(P_1b) &= v_1, \\
\chi(P_1a) &= q.
\end{align*}
\]

Then \( \psi' \leq_{\mathcal{R}} \chi \) since \( \text{Im}(\psi) \) does not have any end with prefix \( q \). And \( \chi \leq_{\mathcal{R}} \varphi' \), since \( q \) is longer than any element of \( \text{imC}(\varphi') \), hence \( \text{Im}(\varphi') \) has ends that are missing in \( \text{imC}(\chi) \).

(2) The proof is just a simpler version of the proof of (1); compare with [14, 23]. \( \square \)

5 Complexity of the \( \mathcal{R} \)-order

We are interested in the computational difficulty of deciding whether \( \psi \leq_{\mathcal{R}} \varphi \) or \( \psi \equiv_{\mathcal{R}} \varphi \). We assume at first that \( \psi, \varphi \in M_{k,1} \) are given either by explicit tables, or by words over a finite generating set of \( M_{k,1} \).

Then we consider circuit-like generating sets of \( M_{k,1} \); they have the form \( \Gamma \cup \tau \), where \( \Gamma \) is any finite subset of \( M_{k,1} \), and \( \tau = \{\tau_{i,i+1} : i \geq 1\} \). The permutation \( \tau_{i,i+1} \in G_{k,1} \) is the letter transposition which swaps positions \( i \) and \( i + 1 \) in any word over \( A \). More precisely, \( \text{domC}(\tau_{i,i+1}) = \text{imC}(\tau_{i,i+1}) = A^{-1} \), and

\[
\tau_{i,i+1} : u\ell_i\ell_{i+1} \mapsto u\ell_{i+1}\ell_i
\]

for all \( u \in A^{-1} \) and \( \ell_i, \ell_{i+1} \in A \). Including \( \tau \) into the generating set makes word-length polynomially equivalent to circuit-size (see [5, 8, 23] for details, especially Prop. 2.4 and Theorem 2.9 in [3]). The word problem of \( M_{k,1} \) over \( \Gamma \) is in \( \text{P} \), but over \( \Gamma \cup \tau \) it is \( \text{coNP} \)-complete [2].

The word-length of \( \varphi \in M_{k,1} \) over \( \Gamma \) or \( \Gamma \cup \tau \) is the length of a shortest word over \( \Gamma \), respectively \( \Gamma \cup \tau \), that represents \( \varphi \); for this, the length of an element of \( \Gamma \) is counted as 1, and the length of \( \tau_{i,i+1} \in \tau \) is counted as \( i + 1 \). We denote these word-lengths of \( \varphi \) by \( |\varphi|_\Gamma \), respectively \( |\varphi|_{\Gamma \cup \tau} \).

General references on combinational circuits and circuit complexity are [21, 16], and chapters 14 and 2 in [20].

5.1 Deciding \( \leq_{\mathcal{R}} \) over a finite generating set \( \Gamma \)

The first problems whose complexity we would like to know are as follows.

- Input: \( \psi, \varphi \in M_{k,1} \), given by tables, or by words over a finite generating set \( \Gamma \) of \( M_{k,1} \).
- Question (the \( \mathcal{R} \)-order decision problem): Does \( \psi \leq_{\mathcal{R}} \varphi \) hold?
- Search (the right-multiplier search problem): If \( \psi \leq_{\mathcal{R}} \varphi \), find some \( \alpha \in M_{k,1} \) such that \( \psi(.) = \varphi \alpha(.) \); the multiplier \( \alpha \) should be expressed by a word over \( \Gamma \).

Theorem 5.1 The \( \mathcal{R} \)-order decision problem of \( M_{k,1} \) is decidable in deterministic polynomial time, if inputs are given by tables or by words over a finite generating set.

Proof. By Corollary 4.11 in [6], if \( \psi \) and \( \varphi \) are given by words over a finite generating set \( \Gamma \) of \( M_{k,1} \) then \( \text{imC}(\psi) \) and \( \text{imC}(\varphi) \) can be computed (as explicit lists of words over \( A \)) in deterministic
polynomial time (where time is taken as a function of the word-lengths of $\psi$ and $\varphi$ over $\Gamma$). If $\psi$ and $\varphi$ are given by tables, then $\im C(\psi)$ and $\im C(\varphi)$ can be immediately read from the tables. By the characterization of $\leq_R$ of $M_{k,1}$ (Theorem 2.1) it is now sufficient to solve the following problem in deterministic polynomial time.

The end inclusion problem for finite prefix codes:

- **Input:** Two finite prefix codes $P, Q \subset A^*$, given explicitly as lists of words.
- **Question:** $\text{ends}(QA^*) \subseteq \text{ends}(PA^*)$?

By Lemma 2.2 this question has several equivalent formulations. Recall part (f) of Lemma 2.2, which says that $\text{ends}(QA^*) \subseteq \text{ends}(PA^*)$ iff for all $y \in Q$ we have:
- either $P$ contains a prefix of $y$,
- or the tree $T(y, P)$ is saturated (where $T(y, P)$ is the subtree of $A^*$ with root $y$ and leaf-set $yA^* \cap P$).

This yields the following algorithm.

\begin{align*}
\text{EndInclusion}(Q, P) &\quad \text{|| Decide whether } \text{ends}(QA^*) \subseteq \text{ends}(PA^*). \\
\text{for } y \in Q &\quad \text{if } P \text{ does not contain a prefix of } y \\
&\quad \text{then}
L := \text{Leaves}(y, P) \\
N := \text{NonLeaves}(y, P) \\
&\quad \text{if not saturated}(L, N) \\
&\quad \quad \text{then return false and halt}
\text{return true and halt}.
\end{align*}

This algorithm uses the following three sub-routines.

\begin{align*}
\text{Leaves}(y, P) &\quad \text{|| Find the set of leaves of } T(y, P), \text{ i.e., the set } yA^* \cap P. \\
&\quad \text{return the set of all words in } P \text{ that have } y \text{ as a prefix.}
\end{align*}

\begin{align*}
\text{NonLeaves}(y, P) &\quad \text{|| Find the set of non-leaf vertices of } T(y, P). \\
&\quad \text{return the set of all strict prefixes of words in } P \text{ that have } y \text{ as a prefix.}
\end{align*}

\begin{align*}
\text{saturated}(L, N) &\quad \text{|| Is the tree with leaf set } L \text{ and non-leaf set } N \text{ saturated?} \\
\text{for } x \in N &\quad \text{if } (xa_1 \notin N \cup L \text{ or } \ldots \text{ or } xa_k \notin N \cup L) \\
&\quad \text{then return false and halt}
\text{return true and halt.}
\end{align*}

Since the prefix code $Q$ is given explicitly, the main loop “for $y \in Q$” repeats a linear number of times. Checking prefix relations between words takes linear or quadratic time (depending on the model of computation). Since $y$ and the prefix code $P$ is given explicitly, the functions $\text{Leaves}(y, P)$, $\text{NonLeaves}(y, P)$, and $\text{saturated}(L, N)$ execute in polynomial time. Hence, each iteration of the main loop “for $y \in Q$” takes polynomial time. Note that since every vertex of $T(y, P)$ is a prefix of some element of $P$, the number of vertices of $T$ at most $\sum_{p \in P} |p|$. \(\square\)

Before we show that the right-multiplier search problem is also solvable in polynomial time, we need more definitions and lemmas.

By definition, if $\{(u_1, v_1), \ldots, (u_n, v_n)\}$ is a table for an element of $M_{k,1}$, the total table size of that table is $\sum_{i=1}^n |u_i| + |v_i|$. 

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Lemma 5.2 (From tables to generators). Let \( \Gamma \) be a finite generating set of \( M_{k,1} \). The following search problem is solvable in deterministic polynomial time.

Input: \( \varphi \in M_{k,1} \), described by a table of total size \( n \).

Output: A word over \( \Gamma \) that represents \( \varphi \).

It follows that \( |\varphi|_\Gamma \) has a polynomial upper bound in terms of the total table size of \( \varphi \).

Proof outline. First, if \( \varphi \in G_{k,1} \) we apply the constructions from \([6]\): By Prop. 3.9 in \([6]\) we can factor \( \varphi \) as \( \varphi = \beta \varphi \pi \varphi \alpha \varphi \) where \( \pi \varphi \) is a permutation of a finite maximal prefix code, and \( \beta \varphi, \alpha \varphi \in F \); \( F \) is the Thompson group consisting of the elements of \( G_{2,1} \) that preserve the dictionary order of \( A^* \). Then we factor elements of \( F \) into generators (Prop. 3.10 in \([6]\), and \([7]\)). And we factor \( \pi \varphi \), first into word transpositions, then into generators (Lemma 3.11 in \([6]\)). The process takes polynomial time. In \([6]\) the construction was done for \( G_{2,1} \), but the same method applies to \( G_{k,1} \) for all \( k \geq 2 \).

Second, if \( \varphi \in \text{Inv}_{k,1} \) (the monoid of injective elements of \( M_{k,1} \)), we use the proof of Theorem 3.4 and Lemma 3.3 in \([2]\), which is constructive and provides a polynomial-time algorithm.

Finally, for a general element \( \varphi \in M_{k,1} \) we use the proof of Theorem 3.5 (especially the Claim) in \([2]\); again, the proof is constructive, and provides a polynomial-time algorithm. \( \square \)

Remark. The converse of Lemma 5.2 is not true: By Prop. 4.1 in \([2]\), \( M_{k,1} \) has infinitely many elements whose total table size is exponentially larger than their word-length.

When \( \varphi \in M_{k,1} \) is by a word over a finite generating set \( \Gamma \) of \( M_{k,1} \) we are interested in finding an inverse of \( \varphi \), i.e., an element \( \chi \in M_{k,1} \) such that \( \varphi \chi \varphi = \varphi \) and \( \chi \varphi \chi = \chi \).

Lemma 5.3 (Find an inverse). Let \( \Gamma \) be a finite generating set of \( M_{k,1} \). For every \( \varphi \in M_{k,1} \) there exists an inverse \( \chi \in M_{k,1} \) which satisfies

(a) \( \varphi \chi(.) = \text{id}_{\text{im}(\varphi)} \) and \( \chi \varphi(.) = \eta_{\varphi} \), where \( \eta_{\varphi} \) is an idempotent such that \( \text{Dom}(\varphi) = \text{Dom}(\eta_{\varphi}) \), \( \text{part}(\eta_{\varphi}) = \text{part}(\varphi) \);

(b) the number of entries in the table of \( \chi \) is \( ||\chi|| = |\text{imC}(\varphi)| \);

(c) the length \( \ell(\chi) \) of a longest word in the table of \( \chi \) satisfies \( \ell(\chi) \leq O(|\varphi|_\Gamma) \).

Moreover the inverse search problem, specified as follows, is solvable in deterministic polynomial time.

Input: \( \varphi \in M_{k,1} \), given by a word over \( \Gamma \).

Output: An inverse \( \chi \) of \( \varphi \) with properties (a), (b), (c); \( \chi \) should be described by a word over \( \Gamma \).

Proof. By Corollary 4.11 in \([2]\) we can compute \( \text{imC}(\varphi) \) as an explicit list of words, in polynomial time. For each \( y \in \text{imC}(\varphi) \), to define \( \chi(y) \) we choose one element in \( \varphi^{-1}(y) \). Any way of doing this will make \( \chi \) satisfy conditions (a), (b), and (c); for (b) and (c) this holds by Corollary 4.7 in \([2]\).

By Corollary 4.15 in \([2]\), we can construct a finite automaton \( B_y \) in polynomial time, such that \( B_y \) accepts \( \varphi^{-1}(y) \); the construction takes as input the given word over \( \Gamma \) that represents \( \varphi \), and an automaton \( A_y \) accepting just \( \{y\} \); \( A_y \) can trivially constructed from \( y \); \( A_y \) has \( |y| + 1 \) states. The number of states of \( B_y \) is \( \leq |y| + 1 + O(n) \), where \( n \) is the length of the word over \( \Gamma \) that represents \( \varphi \).

Then from \( B_y \) we can quickly pick a word \( x \) accepted (i.e., \( x \in \varphi^{-1}(y) \)), e.g. by depth-first search in the transition graph of \( B_y \). Then \( |x| \leq |y| + 1 + O(n) \). Thus, \( \chi(y) = x \in \varphi^{-1}(y) \) can be chosen so that the inverse search problem is solved in polynomial time.

It follows that the table of \( \chi \) contains \( \leq O(|\varphi|_\Gamma) \) words of length \( \leq O(|\varphi|_\Gamma) \). By Lemma 5.2 it follows that \( |\chi|_\Gamma \leq O(|\varphi|_\Gamma) \). \( \square \)

As a consequence of Lemma 5.3 the elements of \( M_{k,1} \), when described by words over a finite generating set, are not one-way functions (for any reasonable definition of “one-way function”).
Proposition 5.4  The right-multiplier search problem of \( M_{k,1} \) is solvable in deterministic polynomial time, if inputs are given by tables or by words over a finite generating set.

Proof. Again, by Corollary 4.11 in [6], if \( \psi \) and \( \varphi \) are given by words over a finite generating set \( \Gamma \) of \( M_{k,1} \) then \( Q = \text{imC}(\psi) \) and \( P = \text{imC}(\varphi) \) can be computed (as explicit lists of words over \( A \)) in deterministic polynomial time.

By Lemma 4.11 above, we have \( \varphi \equiv_R \text{id}_{P_A} \) and \( \psi \equiv_R \text{id}_{Q_A} \). The latter is equivalent to \( \psi = \text{id}_{Q_A} \circ \psi \). Moreover, \( \psi \leq R \varphi \) is equivalent to \( \text{id}_{Q_A} = \text{id}_{P_A} \circ \text{id}_{Q_A} \). Thus, after finding \( \beta \in M_{k,1} \) such that \( \text{id}_{P_A} = \varphi \beta \) (by using Lemma 5.3), we will have \( \psi = \varphi \circ \beta \circ \text{id}_{Q_A} \circ \psi \). Next, we express \( \beta \) and \( \text{id}_{Q_A} \) as words over \( \Gamma \) (by using Lemma 5.2). For \( \varphi \) we already have a word over \( \Gamma \) as part of the input. This yields a word for \( \varphi \circ \beta \circ \text{id}_{Q_A} \circ \psi \) in polynomial time, and this is a right-multiplier (since \( \psi = \varphi \circ \beta \circ \text{id}_{Q_A} \circ \psi \)). \( \square \)

5.2 Deciding \( \leq_R \) over a circuit-like generating set \( \Gamma \cup \tau \)

Let \( \Gamma \) be any finite generating set of \( M_{k,1} \) and let \( \tau = \{ \tau_{i,i+1} : i \geq 1 \} \) be the set of transpositions of neighboring letters. We saw in [5] that the set \( \tau \) plays an important role in the representation of combinational circuits by elements of \( M_{k,1} \) (in such a way that circuit size is polynomially related to word-length). For words over the generating set \( \Gamma \cup \tau \) we define the length by \(|\gamma| = 1 \) for \( \gamma \in \Gamma \), and \(|\tau_{i,i+1}| = i + 1 \). For \( \varphi = \gamma_m \ldots \gamma_1 \) we define \(|\varphi| = \sum_{j=1}^{m} |\gamma_j|\).

We saw in [2] that the word problem of \( M_{k,1} \) is \( \text{coNP} \)-complete when the generating set \( \Gamma \cup \tau \) is used to write words (and this is even true for \( G_{k,1} \) over \( \Gamma_G \cup \tau \), where \( \Gamma_G \) is any finite generating set of \( G_{k,1} \), [5]). At first impression one might think that the \( \leq_R \) decision problem of \( M_{k,1} \) over \( \Gamma \cup \tau \) might be in \( \Sigma^P_2 \), since \( \psi \leq_R \varphi \) holds iff \( \exists \forall \exists \{ \psi(x) = \varphi \alpha(x) \} \); however, there is no guarantee that \( \alpha \) has polynomial word-length. But, surprisingly at first, the problem turns out to be \( \Pi^P_2 \)-complete; the proof uses the characterization of the \( \text{R-order} \) (Theorem 2.1). The connection with \( \Sigma^P_2 \) reappears in Subsection 5.3.

The complexity class \( \Pi^P_2 \) consists of all decision problems that can be decided by alternating polynomial-time Turing machines of type \( \forall \exists \), i.e., nondeterministic polynomial-time Turing machines whose computations first visit universal states and then existential states; see e.g. [10, 11, 20, 14] for details. The class \( \Pi^P_2 \) contains interesting complete problems, e.g., the problem \( \text{\( \forall \exists \)-QBF} \) (also denoted by \( \text{\( \forall \exists \text{Sat} \)} \)), which asks whether a given \( \forall \exists \)-quantified boolean formula is true; see e.g. [10] pp. 84-89, [11] pp. 270-274. More precisely, \( \forall \exists \)-QBF consists of all fully quantified boolean formulas of the form \( \forall x \exists y (\beta(y,x)) \), where \( x \) and \( y \) are finite sequences of boolean variables (each boolean variable ranging over \{0,1\}), and \( \beta(y,x) \) is a boolean formula (in the usual sense).

We also consider two special versions of the \( \text{R-order decision problem} \), called the “lower-bound \( \text{R-order decision problem} \)” and the “upper-bound \( \text{R-order decision problem} \)” for \( M_{k,1} \) over the generating set \( \Gamma \cup \tau \). First we choose an element \( \alpha \in M_{k,1} \), called the bound-parameter of the problem. The problems are then specified as follows.

- **Input**: \( \varphi \in M_{k,1} \), given by a word over \( \Gamma \cup \tau \).
- **Question (lower-bound \( \text{R-order decision problem for a fixed} \ \alpha \))**: \( \varphi \geq_R \alpha \) ?
- **Question (upper-bound \( \text{R-order decision problem for a fixed} \ \alpha \))**: \( \varphi \leq_R \alpha \) ?

We consider also the following problems for \( M_{k,1} \) over the generating set \( \Gamma \cup \tau \).

- **Input**: \( \varphi, \psi \in M_{k,1} \), given by words over \( \Gamma \cup \tau \).
- **Question (\( \equiv_R \text{-decision problem} \))**: Does \( \varphi \equiv_R \psi \) hold?
- **Input**: \( \varphi \in M_{k,1} \), given by a word over \( \Gamma \cup \tau \).
- **Question (\( \equiv_R 1 \text{-decision problem} \))**: Does \( \varphi \equiv_R 1 \) hold?
And we consider the membership problems of the domain, the domain code, and the image of an element $\varphi \in M_{k,1}$ (over $\Gamma \cup \tau$), specified as follows.

- **Input:** $\varphi \in M_{k,1}$, given by a word over $\Gamma \cup \tau$, and a word $z \in A^*$.
- **Question (domain membership problem):** $z \in \text{Dom}(\varphi)$?
- **Question (domain code membership problem):** $z \in \text{domC}(\varphi)$?
- **Question (image membership problem):** $z \in \text{Im}(\varphi)$?

We also consider the special version of the image problem, namely the **image membership problem for a fixed test string**. We first choose a word $z_0 \in A^*$, called the test string or the test parameter of the problem; for a fixed $z_0$ we consider:

- **Input:** $\varphi \in M_{k,1}$, given by a word over $\Gamma \cup \tau$.
- **Question:** $z_0 \in \text{im}(\varphi)$?

**Notation:** For $x_2, x_1 \in A^*$ we write $x_1 \text{ pref } x_2$ iff $x_1$ is a prefix of $x_2$. We also write $x_1 \text{ spref } x_2$ iff $x_1$ is a prefix of $x_2$ and $x_1 \neq x_2$ (i.e., $x_1$ is a strict prefix of $x_2$).

**Proposition 5.5.**

1. The domain membership problem, and the domain code membership problem for $M_{k,1}$ over $\Gamma \cup \tau$ are in $\text{P}$.
2. The image membership problem for $M_{k,1}$ over $\Gamma \cup \tau$ is $\text{NP}$-complete. The special image membership problem (with a fixed test string) is always in $\text{NP}$, and it is $\text{NP}$-complete for certain test strings.

**Proof.** (1a) The domain membership problem:

Let $\gamma_N \ldots \gamma_1$ be the generator sequence that represents $\varphi$, with $\gamma_j \in \Gamma \cup \tau$ ($N \geq j \geq 1$). We simply apply this generator sequence to $z$, and check if the result is always defined. By Theorem 4.5(2) in [2], applied to $\gamma_n \ldots \gamma_1 \cdot \text{id}_z$, we have for all $r = 1, \ldots, n$:

$$|\gamma_r \ldots \gamma_1 \cdot \text{id}_z| \leq |z| + \sum_{j=1}^r \ell(\gamma_j),$$

Here, $\ell(\gamma_j)$ is the length of a longest word in the table of $\gamma_j$. For every $\gamma_j \in \Gamma$, $\ell(\gamma_j) \leq c$ for some constant $c$. For $\gamma_j = \pi_i, i+1$, $\ell(\gamma_j) = i + 1$. It follows that $\sum_{j=1}^r \ell(\gamma_i) \leq c |\varphi|_{|\gamma_r \ldots \gamma_1 \cdot \text{id}_z|}$. Hence, all the intermediate words (in $A^*$) obtained as all the generators $\gamma_j$ are applied, have length $\leq O(|z| + |\varphi|_{|\gamma_r \ldots \gamma_1 \cdot \text{id}_z|})$, i.e., the length is linearly bounded in terms of the size of the input $z, \varphi$. Also, applying a generator to a word of linearly bounded length takes polynomial time.

(1b) The domain code membership problem:

As in (1a) we first check whether $z \in \text{Dom}(\varphi)$; if the answer is “no” then, obviously, $z \not\in \text{domC}(\varphi)$. If “yes”, we check whether the prefix $s$ of $z$, obtained by removing the right-most letter, is in $\text{Dom}(\varphi)$; if $s \in \text{Dom}(\varphi)$ then $z \not\in \text{domC}(\varphi)$, otherwise $z \in \text{domC}(\varphi)$.

(2) The image membership problem, in general, and with fixed test string:

The problems are in $\text{NP}$. Indeed, to check whether $z \in \text{Im}(\varphi)$ we can guess $x \in A^*$ and check whether $\varphi(x) = z$. By the reasoning in (1) above, the length of $x$ is linearly bounded from above by the input length $|z| + |\varphi|_{|\gamma_r \ldots \gamma_1 \cdot \text{id}_z|}$; moreover, $\varphi(x)$ can be computed in polynomial time by successive application of the generators $\gamma_j$.

To show $\text{NP}$-hardness we reduce the satisfiability problem for boolean formulas to the image membership problem with fixed test string (namely the test string $1 \in \{0,1\}^*$). Here we identify $\{0,1\}$ with $\{a_1, a_2\} \subseteq \{a_1, \ldots, a_k\} = A$. Let $\beta(x)$ be a boolean formula with list of boolean variables $x = (x_1, \ldots, x_m)$. So, $\beta$ also represents a function $\{0,1\}^m \rightarrow \{0,1\}$. Then $\beta(x)$ is satisfiable iff $1 \in \text{Im}(\beta)$. A boolean formula can be viewed as an element of $M_{k,1}$, with word length over $\Gamma \cup \tau$ linearly related to the formula size (see [3], Proposition 2.4). Hence the satisfiability problem for boolean formulas reduces to the special image membership problem. \qed
It is interesting that, complexity-wise, the membership problems of the domain and the image are quite different. This is ultimately the cause of the discrepancy between the complexities of the \( \mathcal{R} \)- and \( \mathcal{L} \)-orders.

**Lemma 5.6** The \( \mathcal{R} \)-order decision problem of \( M_{k,1} \) over \( \Gamma \cup \tau \) is in \( \Pi^P_2 \).

**Proof.** Let \( \psi, \varphi \in M_{k,1} \) be given by words over \( \Gamma \cup \tau \), and let \( n = \max \{|\psi|_\Gamma, |\varphi|_\Gamma\} \). By Theorem 4.5(2) in \( [2] \), the length of a longest word in \( \text{im}C(\psi) \) or \( \text{im}C(\varphi) \) satisfies \( \ell(\text{im}C(\psi)) \leq O(n) \), and \( \ell(\text{im}C(\varphi)) \leq O(n) \). Also, \( \ell(\text{dom}C(\psi)) \leq O(n) \) and \( \ell(\text{dom}C(\varphi)) \leq O(n) \). Indeed, if \( \varphi = \gamma_1 \ldots \gamma_n \) (where \( \gamma_i \in \Gamma \cup \tau \) for \( i = 1, \ldots, N \)), then \( \ell(\varphi) = \sum_{i=1}^N \ell(\gamma_i) \) and this is \( \leq O(|\varphi|_\Gamma) \); note that \( \ell(\tau_{i-1,i}) = |i| \in \text{dom}C \).

Recall Theorem 2.1(6) which says: \( \psi \leq \mathcal{R} \varphi \) iff for every \( y \in \text{im}C(\psi) \), either \( \text{im}C(\varphi) \) contains a prefix of \( y \), or the subtree \( T_{y,\text{im}C(\varphi)} \) is saturated. Here, \( T_{y,\text{im}C(\varphi)} \) is the subtree of the tree of \( A^* \) with root \( y \) and leaf-set \( yA^* \cap \text{im}C(\varphi) \). So the set of leaves \( \text{L}(T_{y,\text{im}C(\varphi)}) \) of \( T_{y,\text{im}C(\varphi)} \) is characterized by: \( z \in \text{L}(T_{y,\text{im}C(\varphi)}) \) iff \( z \in \text{im}C(\varphi) \) and \( y \) is a prefix of \( z \). Hence,

\[
z \in \text{L}(T_{y,\text{im}C(\varphi)}) \iff (\exists x \in \text{dom}C(\varphi)) \ [z = \varphi(x) \land y \text{ pref } \varphi(x)].
\]

The vertex set of the subtree \( T_{y,\text{im}C(\varphi)} \) consists of the words that are a prefix of a word in \( \text{L}(T_{y,\text{im}C(\varphi)}) \) and that have \( y \) as a prefix, i.e., \( z \in \text{V}(T_{y,\text{im}C(\varphi)}) \) iff \( (\exists s \in \text{im}C(\varphi)) \ [y \text{ pref } z \text{ pref } s] \). Hence,

\[
z \in \text{V}(T_{y,\text{im}C(\varphi)}) \iff (\exists x \in \text{dom}C(\varphi)) \ [y \text{ pref } z \text{ pref } \varphi(x)].
\]

Similarly, the set of non-leaf vertices of \( T_{y,\text{im}C(\varphi)} \) is characterized by

\[
z \in \text{V}(T_{y,\text{im}C(\varphi)}) - \text{L}(T_{y,\text{im}C(\varphi)}) \iff (\exists m \in \text{dom}C(\varphi)) \ [y \text{ pref } z \text{ sPref } \varphi(x)].
\]

The subtree \( T_{y,\text{im}C(\varphi)} \) is saturated iff \( (\forall z \in \text{V}(T_{y,\text{im}C(\varphi)}) - \text{L}(T_{y,\text{im}C(\varphi)})) \ [zA \subseteq \text{V}(T_{y,\text{im}C(\varphi)})] \), since \( zA \) is the set of children of \( z \) in the tree of \( A^* \). Thus \( \psi \leq \mathcal{R} \varphi \) is equivalent to

\[
(\forall y \in \text{im}C(\varphi)) \ [ (\exists t \in \text{im}C(\varphi))[t \text{ pref } y] \lor (\forall z \in (\text{V}(T_{y,\text{im}C(\varphi)}) - \text{L}(T_{y,\text{im}C(\varphi)}))) \ [zA \subseteq \text{V}(T_{y,\text{im}C(\varphi)})]].
\]

Since \( y \in \text{im}C(\psi) \) iff \( (\exists x \in \text{dom}C(\psi)) \ [y = \psi(x)] \), the above formula is equivalent to

\[
(\forall x \in \text{dom}C(\psi)) \ [ (\exists t \in \text{im}C(\varphi))[t \text{ pref } \psi(x)] \lor (\forall z \in (\text{V}(T_{\psi(x),\text{im}C(\varphi)}) - \text{L}(T_{\psi(x),\text{im}C(\varphi)}))) \ [zA \subseteq \text{V}(T_{\psi(x),\text{im}C(\varphi)})]].
\]

We saw that \( z \in \text{V}(T_{\psi(x),\text{im}C(\varphi)}) - \text{L}(T_{\psi(x),\text{im}C(\varphi)}) \iff (\exists r \in \text{dom}C(\varphi)) \ [\psi(x) \text{ pref } z \text{ sPref } \varphi(x)] \). Also, every \( z \in \text{V}(T_{\psi(x),\text{im}C(\varphi)}) - \text{L}(T_{\psi(x),\text{im}C(\varphi)}) \) has length \( |z| < \ell(\text{im}C(\varphi)) \leq c\ell(\text{dom}C(\varphi)) \), where \( c \geq 1 \) is a constant (depending on the choice of the finite set \( \Gamma \)). Thus, the above formula is equivalent to

\[
(\forall x \in \text{dom}C(\psi)) \ [ (\exists s \in \text{dom}C(\varphi)) \ [\varphi(s) \text{ pref } \psi(x)] \lor (\forall z \in A^{\leq c\ell(\text{dom}C(\varphi))}) (\exists r \in \text{dom}C(\varphi)) \ [\psi(x) \text{ pref } z \text{ sPref } \varphi(r) \Rightarrow zA \subseteq \text{V}(T_{\psi(x),\text{im}C(\varphi)})]].
\]

Obviously, \( X \Rightarrow Y \iff \text{not}{X \lor Y} \). Also, \( zA \subseteq \text{V}(T_{\psi(x),\text{im}C(\varphi)}) \iff \bigwedge_{i=1}^k (za_i \in \text{V}(T_{\psi(x),\text{im}C(\varphi)})). \) Moreover, \( za_i \in \text{V}(T_{\psi(x),\text{im}C(\varphi)}) \iff (\exists x_i \in \text{dom}C(\varphi)) \ [\psi(x) \text{ pref } za_i \text{ sPref } \varphi(x)] \). Hence, the above formula is equivalent to

\[
(\forall x \in \text{dom}C(\psi)) \ [ (\exists s \in \text{dom}C(\varphi)) \ [\varphi(s) \text{ pref } \psi(x)] \lor (\forall z \in A^{\leq c\ell(\text{dom}C(\varphi))}) (\exists r \in \text{dom}C(\varphi)) \ [\text{not}{\psi(x) \text{ pref } z \text{ sPref } \varphi(r)} \lor (\exists x_1, \ldots, x_k \in \text{dom}C(\varphi)) \ [\psi(x) \text{ pref } za_i \text{ sPref } \varphi(x_i)])].
\]

We transform this to a \( \forall \exists \)-formula by using the following facts, where \( C \) is any formula that does not contain the variable \( v \). First, we apply \( C \lor (\forall v \in S)B(v) \equiv (\forall v \in S)[C \lor B(v)] \). Then, three times we apply \( C \lor (\exists v \in S)B(v) \equiv (\exists v \in S)[C \lor B(v)] \). We then obtain the following \( \forall \exists \)-formula that characterizes the \( \mathcal{R} \)-order of \( M_{k,1} \):

\[
28
\]
∀ formula that satisfies \( \beta \) obviously, prefix relations can be checked in polynomial time. Moreover, it is a \( \Pi_2 \) polynomial time; indeed (by Prop. 5.5), membership in essential right ideal iff the formula \((\forall \text{ codes in Subsections 1.2, 1.3, and 2.1)}\).\( \ell \) and \( \psi \) equivalence (2) \( \Leftrightarrow \) and a right ideal is essential iff its generating prefix code is maximal. By Theorem 2.1(1)(2), the \( R \) to the The following gives the connection between the surjectiveness problem for \( M_{k,1} \), namely the surjectiveness problem for \( M_{k,1} \) over \( \Gamma \cup \tau \). It is specified as follows:

- **Input:** \( \varphi \in M_{k,1} \), given by a word over \( \Gamma \cup \tau \).
- **Question:** Is \( \varphi \) surjective (on \( A^\omega \))?

It is easy to see that in \( M_{k,1} \), the surjective elements are the same thing as the epimorphisms of \( M_{k,1} \), i.e., elements \( \varphi \in M_{k,1} \) such that for all \( \psi_1, \psi_2 \in M_{k,1} : \psi_1 \varphi(.) = \psi_2 \varphi(.) \Rightarrow \psi_1 = \psi_2 \).

We will also use the surjectiveness problem for combinational circuits, specified as follows:

- **Input:** A combinational circuit \( C \).
- **Question:** Is the input-output function of \( C \) surjective?

The following gives the connection between the surjectiveness problem for \( M_{k,1} \) and the \( \equiv_R \)-decision problem.

**Lemma 5.7** The following are equivalent for any \( \varphi \in M_{k,1} \):

1. \( \varphi \) is surjective (on \( A^\omega \)), or equivalently, \( \varphi \) is an epimorphism of \( M_{k,1} \).
2. \( \text{im}(\varphi) \) is a maximal prefix code.
3. \( \varphi \equiv_R 1 \) in \( M_{k,1} \).
4. \( (\forall y \in A^N)(\exists x \in A^{\leq N}) [y \text{ is a prefix of } \varphi(x)] \), where \( N = c \cdot |\varphi|_{\max} \) and \( c \) is the constant \( c = \max\{\ell(\gamma) : \gamma \in \Gamma \} \).

**Proof.** The equivalence (1) \( \Leftrightarrow \) (2) is straightforward (see the discussion of ends, ideals, and prefix codes in Subsections 1.2, 1.3, and 2.1).

We have of course \( \text{im}(1) = A^* \). Moreover, \( A^* \) is essentially equal to every essential right ideal, and a right ideal is essential iff its generating prefix code is maximal. By Theorem 2.1(1)(2), the equivalence (2) \( \Leftrightarrow \) (3) then follows.

Let us prove the equivalence (4) \( \Leftrightarrow \) (2). By Theorem 4.5(2) in [2], all words in \( \text{im}(\varphi) \cup \text{dom}(\varphi) \) have lengths \( \leq N = c \cdot |\varphi|_{\max} \), where \( c = \max\{\ell(\gamma) : \gamma \in \Gamma \} \). Thus, \( A^N A^* \subseteq \text{im}(\varphi) \). Hence, \( \text{im}(\varphi) \) is an essential right ideal iff every word in \( A^N \) has a prefix in \( \text{im}(\varphi) \). This holds iff the formula \((\forall y \in A^N)(\exists z \in \text{im}(\varphi))[y \text{ pref } z] \) is true. Since all words in \( \text{dom}(\varphi) \) have lengths \( \leq N \), the statement \((\exists z \in \text{im}(\varphi))[y \text{ pref } z] \) is equivalent to \((\exists x \in A^{\leq N})[y \text{ pref } \varphi(x)] \). Thus, \( \text{im}(\varphi) \) is an essential right ideal iff the formula \((\forall y \in A^N)(\exists x \in A^{\leq N})[y \text{ pref } \varphi(x)] \) is true. □

We mentioned already that the problem \( \forall \exists \text{-QBF} \) is \( \Pi_2 \)-complete. It will be useful to prove \( \Pi_2 \)-completeness for a slightly special form of \( \forall \exists \text{-QBF} \), where the input consists of formulas of the form \( \forall y \exists x \beta_1(x, y) \), where \( x \in \{0,1\}^M \) and \( y \in \{0,1\}^N \) (for some \( M, N \)), and where \( \beta_1(x, y) \) is a boolean formula that satisfies \( \beta_1(1^{M+N}) = 1 \). We call this problem \( \forall \exists \text{-QBF}_1 \).
Lemma 5.8  The problem $\forall \exists -$QBF$_1$ is $\Pi^P_2$-complete.

Proof. The problem $\forall \exists -$QBF$_1$ is obviously in $\Pi^P_2$ since $\forall \exists -$QBF is. To prove hardness we map any formula $\forall x_2 \exists x_1 B(x_1, x_2)$ (where $x_1 \in \{0,1\}^m$ and $x_2 \in \{0,1\}^n$) to a formula $\forall b \forall x_2 \exists x_1 \beta(x_1, x_2, b)$, where $b \in \{0,1\}$ and where $\beta$ is defined for all $x_1, x_2$ by

$$
\beta(x_1, x_2, 0) = B(x_1, x_2)
$$

$$
\beta(x_1, x_2, 1) = 1
$$

Then $\beta^{(m+n+1)} = 1$. Moreover, $\forall b \forall x_2 \exists x_1 \beta(x_1, x_2, b)$ is true iff $\forall x_2 \exists x_1 B(x_1, x_2)$ is true. Indeed, $\forall b \forall x_2 \exists x_1 \beta(x_1, x_2, b) \iff \forall x_2 \exists x_1 \beta(x_1, x_2, 0) \land \forall x_2 \exists x_1 \beta(x_1, x_2, 1) \iff \forall x_2 \exists x_1 B(x_1, x_2) \land 1$.

Theorem 5.9.

(1) The surjectiveness problem for combinational circuits is $\Pi^P_2$-complete.

(2) As a consequence, the following problem for $M_{2,1}$ over the generating set $\Gamma_{2,1} \cup \tau$ are $\Pi^P_2$-hard (where $\Gamma_{2,1}$ is any finite generating set of $M_{2,1}$): The surjectiveness problem for elements of $M_{2,1}$, the $\equiv_R$ 1 decision problem, the $\equiv_R$ decision problem, and the $\leq_R$ decision problem.

Proof. (1). It is easy to see that the surjectiveness problem for combinational circuits is in $\Pi^P_2$. Indeed, $C$ is surjective iff $\forall y \exists x [C(x) = y]$; and for a given $(x,y)$, one can check in deterministic polynomial time whether $C(x) = y$.

To prove hardness we will reduce $\forall \exists -$QBF$_1$ to the the surjectiveness problem for combinational circuits. Let $\beta(x,y)$ be a boolean formula where $x$ is a sequence of $m$ boolean variables, and $y$ is a sequence of $n$ boolean variables. We map the formula $\beta$ to the combinational circuit $C_{\beta,m,n}$ with input-output function defined by

$$(x,y) \mapsto C_{\beta,m,n}(x,y) = \begin{cases} y & \text{if } \beta(x,y) = 1, \\ 1^n & \text{if } \beta(x,y) = 0. \end{cases}$$

From the formula $\beta(x,y)$ one can easily construct a combinational circuit or a word over $\Gamma \cup \tau$ for $C_{\beta,m,n}$. Moreover, $\text{Im}(C_{\beta,m,n}) = \{1^n\} \cup \{y : \exists x \beta(x,y)\} = \{y : \exists x \beta(x,y)\}$; the latter equality comes from the fact that $\beta^{(m+n)} = 1$. Hence, $\forall y \exists x \beta(x,y)$ is true iff $\text{Im}(C_{\beta,m,n}) = \{0,1\}^n$, i.e., iff $C_{\beta,m,n}$ is surjective.

(2) By Lemma 5.7 it follows that the $\equiv_R$ 1 decision problem is $\Pi^P_2$-hard. Since the latter is a special case of the $R$-equivalence problem and of the $\leq_R$ decision problem, these are also $\Pi^P_2$-hard.

Theorem 5.10  The $\equiv_R$ 1 decision problem, the $\equiv_R$ decision problem, and the $\leq_R$ decision problem of $M_{k,1}$ are $\Pi^P_2$-complete, if inputs are given by words over $\Gamma \cup \tau$.

The lower-bound $R$-order decision problem is always in $\Pi^P_2$, and it is $\Pi^P_2$-complete for certain choices of the lower-bound parameter (if inputs are given by words over $\Gamma \cup \tau$, where $\Gamma$ is any finite generating set of $M_{k,1}$).

Proof. By Lemma 5.7 (3), and by Lemma 5.6 the problems are in $\Pi^P_2$. By Theorem 5.9 the $\equiv_R$ 1, $\equiv_R$, and $\leq_R$ decision problems are $\Pi^P_2$-hard for $M_{k,1}$ over $\Gamma \cup \tau$.

Since $\equiv_R 1$ is equivalent to $\geq_R 1$, the lower-bound $R$-order decision problem is $\Pi^P_2$-hard when 1 is chosen as the bound-parameter.

The lower-bound $R$-order decision problem is also $\Pi^P_2$-hard for $M_{k,1}$ for each $k \geq 2$, by the same reasoning as at the end of the proof of Proposition 5.5.

After seeing that the lower-bound decision problem “$\varphi \geq_R 1$?” is $\Pi^P_2$-complete (when $\varphi$ is given by a word over $\Gamma \cup \tau$), we wonder what might be the complexity of upper-bound problems “$\varphi \leq_R \alpha$?” for a fixed $\alpha$. Surprisingly we have:
Proposition 5.11 The upper-bound \(R\)-decision problem of \(M_{k,1}\) over \(\Gamma \cup \tau\) is always in coNP, and it is coNP-complete for certain choices of the bound parameter.

Proof. The problem is in coNP:

Since coNP is \(\Pi^p_1\), it is enough to find a \(\Pi^p_1\)-formula that, for a fixed element \(\alpha \in M_{k,1}\) and an input \(\varphi \in M_{k,1}\), expresses that \(\varphi \leq_R \alpha\). Here, \(\varphi\) is given by a generator sequence \(\gamma_N \cdot \ldots \cdot \gamma_1\) with \(\gamma_j \in \Gamma \cup \tau\), \(n \geq j \geq 1\). The length of the longest element in \(imC(\varphi)\) or in \(domC(\varphi)\) satisfies \(\ell(domC(\varphi))\).

\(\ell(imC(\varphi)) \leq c \cdot |\varphi|_{\Gamma \cup \tau}\), as we saw in part (1a) of the Proof of Prop. 5.5. By Theorem 2.1, \(\varphi \leq_R \alpha\) iff \(\Im(\varphi) \subseteq_{ends} \Im(\alpha)\). Also, \(\Im(\alpha) = imC(\alpha) \cdot A^*\), where \(imC(\alpha)\) is a fixed finite set, and \(imC(\alpha) \cdot A^*\) is accepted by a finite automaton. We will use the following claims.

Claim 1: If \(x \in Dom(\varphi)\) and \(|x| \geq \ell(domC(\varphi)) + m\) then \(|\varphi(x)| \geq m\).

Indeed, if \(\varphi\) is applied to \(x\), at most the \(\ell(domC(\varphi))\) left-most letters of \(x\) will be changed; in particular, if there is a length decrease from \(|x|\) to \(|\varphi(x)|\), the amount of decrease will be at most \(\ell(domC(\varphi))\).

Claim 2: Let \(N = \ell(domC(\varphi)) + \ell(imC(\varphi))\). Then,

\[\Im(\varphi) \subseteq_{ends} \Im(\alpha)\] iff \((\forall x \in Dom(\varphi) \cap A^{\geq N}) [\varphi(x) \in \Im(\alpha)]\).

Indeed, if \(\Im(\varphi) \subseteq_{ends} \Im(\alpha)\) then every word \(\varphi(x)\) of length \(\geq \ell(imC(\alpha))\) is in \(\Im(\alpha) = imC(\alpha) \cdot A^*\). Also, if \(x \in Dom(\varphi)\) and \(|x| \geq N\) then (by Claim 1), \(|\varphi(x)| \geq \ell(imC(\alpha))\). Hence, \(\varphi(x) \in \Im(\alpha)\) for every \(x \in Dom(\varphi) \cap A^{\geq N}\).

Conversely, if \(\varphi(Dom(\varphi) \cap A^{\geq N}) \subseteq \Im(\alpha)\), then \(\Im(\varphi) \subseteq_{ends} \Im(\alpha)\), since the right ideal \(Dom(\varphi) \cap A^{\geq N}\) is essential in \(Dom(\varphi)\). This proves Claim 2.

The ideal \(Dom(\varphi) \cap A^{\geq N}\) is generated by the finite prefix code \(Dom(\varphi) \cap A^N\). Hence, by Claim 2, the upper-bound relation \(\varphi \leq_R \alpha\) is characterized by

\(\varphi \leq_R \alpha\) iff \((\forall x \in A^N) [x \notin Dom(\varphi) \lor \varphi(x) \in imC(\alpha) \cdot A^*]\).

This is a \(\Pi^p_1\)-formula, since the word-length of the quantified variable \(x\) is linearly bounded; indeed, we saw that \(N \leq c \cdot |\varphi|_{\Gamma \cup \tau} + c_\alpha\), where \(c_\alpha = \ell(imC(\alpha))\), which is a constant. Moreover, the predicates in the formula can be decided in deterministic polynomial time. Indeed, the membership problem of \(Dom(\varphi)\) is in P, and \(imC(\alpha)\) is a fixed finite set so \(imC(\alpha) \cdot A^*\) is a fixed regular language (decided by a finite automaton). Moreover, \(\varphi(x)\) can be computed in deterministic polynomial time, as we saw in part (1a) of the Proof of Prop. 5.5.

Proof of coNP-hardness:

We consider the tautology problem for boolean formulas, i.e., the question whether \(\forall x B(x)\) is true; here, \(B(x)\) is any boolean formula with some list of boolean variable \(x = (x_1, \ldots, x_m)\). The tautology problem is a well known coNP-complete problem. The formula \(B(x)\) determines a function \(\{0,1\}^m \rightarrow \{0,1\}\), which we also denote by \(B(.\)\), and this function determines an element of \(M_{2,1}\), which we will denote by \(\beta\). By Prop. 2.4 in [3], the word-length of \(\beta\) over \(\Gamma \cup \tau\) is linearly bounded by the formula size of \(B(x)\), and an expression of \(\beta\) as a word over \(\Gamma \cup \tau\) can be found in polynomial time from the formula \(B(x)\).

We have: \(\forall x B(x)\) iff \(\Im(B(.)) = \{1\}\) iff \(\Im(\beta) \subseteq 1 \{0,1\}^*\). Since \(B(.)\) is a total function (everywhere defined on \(\{0,1\}^m\), the latter is equivalent to \(\Im(\beta) \subseteq_{ess} 1 \{0,1\}^*\). By the characterization of \(\leq_R\) (Theorem 2.1), this is equivalent to \(\beta \leq_R const_{m,1}\), where \(const_{m,1}\) is the function \(xw \mapsto 1w\) (for all \(x \in \{0,1\}^m\) and all \(w \in \{0,1\}^*\)). So the upper-bound \(R\)-order decision problem with bound parameter \(const_{m,1}\) is coNP-hard for \(M_{2,1}\). The upper-bound problem is also coNP-complete for \(M_{k,1}\) for each \(k \geq 2\), by the same reasoning as at the end of the proof of Proposition 5.5. \(\square\)

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5.3 The right-multiplier search problem for \( M_{k,1} \) over \( \Gamma \cup \tau \)

Theorem 5.12.

(1) The lengths of right-multipliers for \( \leq_R \) are not polynomially bounded, unless the polynomial hierarchy \( \text{PH} \) collapses.

More precisely, suppose there is a polynomial \( p(.) \) such that for all \( \psi, \varphi \in M_{k,1} \) we have the following: \( \psi \leq_R \varphi \) implies that there is a right-multiplier \( \alpha \in M_{k,1} \) such that \( \psi = \varphi \alpha \), and such that \( |\alpha|_{\Gamma \cup \tau} \leq p(|\psi|_{\Gamma \cup \tau} + |\varphi|_{\Gamma \cup \tau}) \). Then \( \Pi_2^P = \Sigma_2^P = \text{PH} \).

(2) The lengths of right-inverses of surjective elements of \( M_{k,1} \) are not polynomially bounded, unless the polynomial hierarchy \( \text{PH} \) collapses.

More precisely, suppose there is a polynomial \( p(.) \) such that we have: For every \( \varphi \in M_{k,1} \) that is surjective (on \( A^\omega \)) there exists a right-inverse \( \alpha \in M_{k,1} \) (i.e., \( \varphi \alpha(.) = 1 \)) such that \( |\alpha|_{\Gamma \cup \tau} \leq p(|\varphi|_{\Gamma \cup \tau}) \). Then \( \Pi_2^P = \Sigma_2^P = \text{PH} \).

(3) The circuit sizes of right-inverses of surjective boolean functions are not polynomially bounded, unless \( \Pi_2^P = \Sigma_2^P = \text{PH} \).

Proof. (1) follows from (2), since by Lemma 5.7 \( \varphi \) is surjective iff \( 1 \leq_R \varphi \).

Proof of (2): Recall that \( \varphi \) is surjective iff \( 1 \equiv_R \varphi \), iff \( \varphi \) has a right-inverse. Let us assume for a contradiction that for all \( \varphi \in M_{k,1} \) with \( 1 \equiv_R \varphi \) there exists a right-inverse \( \alpha \in M_{k,1} \) with \( |\alpha|_{\Gamma \cup \tau} \leq p(|\varphi|_{\Gamma \cup \tau}) \). This would imply that the \( \equiv_R 1 \) decision problem is in \( \Sigma_2^P \). Indeed, \( 1 \leq_R \varphi \) is equivalent to \( (\exists \alpha)(\forall x)[\varphi \alpha(x) = x \lor \varphi \alpha(x) = \emptyset] \). Here, the lengths of the quantified variables are polynomially bounded (as a function of \( n = |\varphi|_{\Gamma \cup \tau} \)); indeed, \( |\alpha|_{\Gamma \cup \tau} \leq p(n) \) by assumption, and \( x \in \text{domC}(\varphi \alpha) \subseteq A^{\leq N} \) for some \( N \leq c \cdot (p(n) + n) \) by Theorem 4.5 in [2]. And \( \varphi \alpha(x) \) can be computed in polynomial time when \( x, \varphi, \) and \( \alpha \) are given, by part (1a) of the Proof of Prop. 5.5.

Since we also saw in Theorem 5.10 that the question \( "1 \equiv_R \varphi?" \) is an \( \Pi_2^P \)-complete problem, it follows that \( \Pi_2^P = \Sigma_2^P \).

The proof of (3) is similar to the proof of (2). For circuits, as for elements of \( M_{2,1} \) in general, we have: \( C \) is surjective iff there exists a circuit \( \alpha \) such that \( C \circ \alpha(.) = \text{id} \). Indeed, if the circuit \( C \) is surjective then \( C \) is also surjective as an element of \( M_{2,1} \), hence (by Lemma 5.7), there exists \( \alpha \in M_{2,1} \) such that \( C \circ \alpha(.) = \text{id} \). Let \( C : \{0,1\}^m \rightarrow \{0,1\}^n \) and \( \text{id} : \{0,1\}^n \rightarrow \{0,1\}^n \), for some \( n, m \). Then we have \( \alpha : \{0,1\}^n \rightarrow \{0,1\}^m \). Since both \( C \) and \( \text{id} \) are total (i.e., defined on all inputs in \( \{0,1\}^m \), respectively \( \{0,1\}^n \)), it follows that \( \alpha \) is also total, so \( \alpha \) belongs to the monoid \( \text{lep} M_{2,1} \) (studied in [3]), i.e., \( \alpha \) is the input-output function of a combinational circuit. By Prop. 2.4 and Theorem 2.9 in [3], the circuit-size of \( \alpha \) and its word-length over \( \Gamma \cup \tau \) (as an element of \( M_{2,1} \)) are polynomially related. Hence, if \( \alpha \) always had polynomially bounded circuit-size then the surjectiveness problem for combinational circuits would be in \( \Sigma_2^P \). We saw that the surjectiveness problem for combinational circuits is \( \Pi_2^P \)-complete (Theorem 5.9), hence \( \Pi_2^P = \Sigma_2^P \). □
Theorem 6.1.
(1) The $L$-order decision problem of $M_{k,1}$ is decidable in deterministic polynomial time, if inputs are given by tables or by words over a finite generating set.

(2) The left-multiplier search problem of $M_{k,1}$ is solvable in deterministic polynomial time, if inputs are given by tables or by words over a finite generating set.

Proof. (1) Given $\varphi$, we can find an inverse $\varphi'$ in deterministic polynomial time, satisfying $\varphi' \varphi(.) = \eta \varphi$, with $\eta \varphi$ as in Lemma 5.3. In particular, $\eta \varphi$ is an idempotent; and from $\varphi' \varphi(.) = \eta \varphi$ it follows that $\varphi \equiv \varphi \eta \varphi$. Also, by Coroll. 5.3, $|\varphi'| \leq O(|\varphi|)$, hence (since $\eta \varphi = \varphi' \varphi$) we also have $|\eta | \leq O(|\varphi|)$. Similarly, for $\psi$ we have an inverse $\psi'$ with all the properties of Lemma 5.3 so $\psi' \psi = \eta \psi$, $\eta \psi$ is an idempotent, $\psi \equiv \eta \psi$, $|\eta \psi| \leq O(|\psi|)$.

So, $\psi \equiv \varphi \eta \psi$, and the latter holds iff $\eta \psi = \eta \psi \eta \varphi$ (since they are idempotents). The question “$\eta \psi = \eta \psi \eta \varphi$?” is an instance of the word problem of $M_{k,1}$ over $\Gamma$. Since the word problem of $M_{k,1}$ over $\Gamma$ is in $\mathcal{P}$, as was proved in [2], the $\leq \mathcal{L}$-decision problem is in $\mathcal{P}$.

(2) By Lemma 5.3 and the proof of (1) above, we have $\psi = \psi \eta \psi = \psi \eta \psi \eta \varphi = \psi \eta \psi \varphi \eta \varphi = \psi \varphi' \varphi = \psi \varphi' \varphi$. So, $\varphi \varphi'$ serves as a left multiplier, and by Lemma 5.3, $\varphi \varphi'$ can be found (as a word over $\Gamma$) in deterministic polynomial time. \qed

6.2 Deciding $\leq \mathcal{L}$ over a circuit-like generating set $\Gamma \cup \tau$

We saw that the $\leq \mathcal{R}$-decision problem is $\mathcal{P}^{\mathcal{L}}$-complete. The characterization of $\leq \mathcal{L}$ (Theorem 3.32) is more complicated than the characterization of $\leq \mathcal{R}$. Nevertheless we will see that the $\leq \mathcal{L}$-decision problem is easier than the $\leq \mathcal{R}$-decision problem over $\Gamma \cup \tau$ (assuming that $\mathcal{NP}$ is not equal to $\text{coNP}$). Before we deal with the $\leq \mathcal{L}$-decision problem we consider related problems that are of independent interest.

Let 0 denote the zero element of $M_{k,1}$ (represented by the empty function). We will consider a special case of the word problem of $M_{k,1}$, called the 0 word problem.

• Input: $\varphi \in M_{k,1}$, given by a word over the generating set $\Gamma \cup \tau$,

• Question: Is $\varphi = 0$ in $M_{k,1}$?

Proposition 6.2 The 0 word problem of $M_{k,1}$ over $\Gamma \cup \tau$ is $\text{coNP}$-complete.

Proof. We reduce the tautology problem for boolean formulas to the 0 word problem. Let $B$ be any boolean formula, with corresponding boolean function $\{0,1\}^m \rightarrow \{0,1\}$. We identify $\{0,1\}$ with $\{a_1, a_2\} \subseteq \{a_1, \ldots, a_k\} = A$. The function $B$ can be viewed as an element $\beta \in M_{k,1}$, represented by a word over $\Gamma \cup \tau$. The length of that word is linearly bounded by the size of the formula $B$ (by Prop. 2.4 in [3]). In $M_{k,1}$ we consider the element $id_{0,A^*}$ (i.e., the identity function restricted to $0A^*$), and we assume that some fixed representation of $id_{0,A^*}$ by a word over $\Gamma$ has been chosen. We have:

$id_{0,A^*} \circ \beta(.) = 0$ iff $\text{Im}(\beta) \subseteq 1A^*$.

The latter holds iff $B$ is a tautology. Thus we reduced the tautology problem for $B$ to the special word problem $id_{0,A^*} \beta = 0$. Note that $id_{0,A^*}$ is fixed, and independent of $B$.

It follows that the 0 word problem of $M_{k,1}$ over $\Gamma \cup \tau$ is $\text{coNP}$-hard for all $k \geq 2$. Moreover, since the word problem of $M_{k,1}$ over $\Gamma \cup \tau$ is in $\text{coNP}$ (by [2]), it follows that the 0 word problem is $\text{coNP}$-complete. \qed

Lemma 6.3 The $\equiv \mathcal{L}$ 0 decision problem for $M_{k,1}$ over $\Gamma \cup \tau$ is $\text{coNP}$-hard. Hence, the $\mathcal{L}$ upper-bound decision problem is $\text{coNP}$-hard for certain choices of the upper-bound.

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Proof. This follows from the coNP-hardness of the 0 word problem of $M_{k,1}$ over $\Gamma \cup \tau$ (Prop. 6.2), since $\varphi \equiv \xi \ 0$ if $\varphi = 0$ in $M_{k,1}$. Moreover, this is an $\xi$ upper-bound decision problem since $\varphi = 0$ if $\varphi \leq \xi \ 0$. □

Lemma 6.4 An element $\varphi \in M_{k,1}$ is an injective total function (on the ends space $A^\omega$) iff $\varphi \equiv \xi \ 1$.

Proof. Suppose $\varphi \equiv \xi \ 1$, and let $\alpha \varphi = 1$. If $\varphi(w)$ were not defined (for some $w \in A^\omega$), then $\alpha \varphi(w)$ would also be undefined; but $\alpha \varphi(w) = 1(w) = w$, which is defined. So $\varphi$ is total. If $\varphi(w_1) = \varphi(w_2)$ for some $w_1, w_2 \in A^\omega$ then $w_1 = w_2$ (by application of $\alpha$ on the left). So $\varphi$ is injective.

Conversely, if $\varphi$ is total and injective on $A^\omega$ then $\text{Dom}(\varphi)$ is an essential right ideal, and the partition of $\varphi$ is the identity. Thus, $\varphi$ has the same partition as 1 so, by Theorem 3.32, $\varphi \equiv \xi \ 1$. □.

It is easy to see the total injective elements of $M_{k,1}$ are the same thing as the monomorphisms of $M_{k,1}$, i.e., the elements $\xi \varphi \in M_{k,1}$ such that for all $\varphi_1, \varphi_2 \in M_{k,1}$: $\varphi_1(\cdot) = \varphi_2(\cdot) \Rightarrow \varphi_1 = \varphi_2$. By Lemma 6.4, the $\xi$-class of 1 is exactly the set of monomorphisms of $M_{k,1}$.

As a consequence of Lemma 6.4 the following problems about acyclic circuits are relevant for the complexity of the $\xi$-order.

• Input: An acyclic boolean circuit $B$.

• Question (injectiveness problem): Is the input-output function of $B$ injective?

• Question (identity problem): Is the input-output function of $B$ the identity function?

Proposition 6.5 The injectiveness problem and the identity problem for acyclic boolean circuits are coNP-complete.

Proof. It is easy to see that the injectiveness problem and the identity problem are in coNP. To show hardness we reduce the tautology problem for boolean formulas to the injectiveness problem (and to the identity problem) for acyclic circuits. Let $B$ be any boolean formula with $n$ variables; the formula defines a function $\{0,1\}^n \rightarrow \{0,1\}$, which we also call $B$. From $B$ we define a new boolean function $F : \{0,1\}^{n+1} \rightarrow \{0,1\}^{n+1}$ by

$$F(x_1, \ldots, x_n, x_{n+1}) = \begin{cases} (x_1, \ldots, x_n, x_{n+1}) & \text{if } B(x_1, \ldots, x_n) = 1 \text{ or } x_{n+1} = 0, \\ (0, \ldots, 0, 1) & \text{otherwise.} \end{cases}$$

Let us check that $B$ is a tautology iff $F$ is injective, iff $F$ is the identity function on $\{0,1\}^{n+1}$.

When $B(x_1, \ldots, x_n) = 1$ then $F(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n, x_{n+1})$. So, if $B$ is a tautology, i.e., $B(x_1, \ldots, x_n) = 1$ always holds, then $F$ is the identity function (which is injective) on $\{0,1\}^{n+1}$.

If $B$ is a not a tautology then $B(c_1, \ldots, c_n) = 0$ for some $(c_1, \ldots, c_n) \in \{0,1\}^n$, hence $F(c_1, \ldots, c_n, 1) = (0, \ldots, 0, 1)$. But we also have $F(0, \ldots, 0, 0) = (0, \ldots, 0, 1)$. Hence, $F$ is not injective (and hence not the identity function). □.

Lemma 6.6 The $\equiv \xi \ 1$ decision problem for $M_{k,1}$ over $\Gamma \cup \tau$ is coNP-hard. Hence, the $\xi$ lower-bound decision problem is coNP-hard for certain choices of the lower-bound.

Proof. By Lemma 6.4 for $\xi \varphi \in M_{k,1}$ we have $\varphi \equiv \xi \ 1$ if $\varphi$ is total and injective on $A^\omega$, or equivalently, if $\varphi$ is total and injective on $A^n$ for some $n$. Combinational circuits are a special case of elements of $M_{k,1}$ given over a generating set $\Gamma \cup \tau$, and circuit-size is polynomially related to word-length in $M_{k,1}$ over $\Gamma \cup \tau$ (by Prop. 2.4 and Theorem 2.9 in [3]). The input-output functions of combinational circuits are total. Hence there is a reduction from the injectiveness problem of combinational circuits to the $\equiv \xi \ 1$ decision problem for $M_{k,1}$ over $\Gamma \cup \tau$. Since the injectiveness problem for circuits is coNP-complete, the Lemma follows. □.
Theorem 6.7  The $\leq_L$, the $\equiv_L$, and the $\equiv_L 1$ (i.e., the monomorphism) decision problems of $M_{k,1}$ are coNP-complete, if inputs are given by words over $\Gamma \cup \tau$.

The lower-bound and upper-bound $L$-order decision problem are always in coNP, and they are coNP-complete for certain choices of the bound-parameters (if inputs are given by words over $\Gamma \cup \tau$).

Proof. (1) Hardness of the problems follows immediately from Lemmas 6.3 and 6.6.

(2) To show that the $L$-order decision problem is in coNP we use the characterization of the $L$-order from Theorem 3.32 for all $\varphi, \psi \in M_{k,1}$:

$\psi(.) \leq_L \varphi(.)$ if $\text{part}(\psi) \leq_{\text{end}} \text{part}(\varphi)$.

Let $N = \max\{\ell(\text{domC}(\psi)), \ell(\text{domC}(\varphi))\}$. By the Prop. 5.5(1) (based on Theorem 4.5(2) in [2]), we have $N \leq O(|\psi|_{\Gamma \cup \tau} + |\varphi|_{\Gamma \cup \tau})$. Let $\Psi$ and $\Phi$ be the restrictions of $\psi$, respectively $\varphi$, to $A^N A^*$. Then we have

$\text{part}(\psi) \leq_{\text{end}} \text{part}(\varphi)$

iff

$\text{domC}(\Psi) \subseteq \text{domC}(\Phi)$ (\subseteq $A^N$), and

$\text{part}(\Psi) \leq \text{part}(\Phi)$ (where “$\leq$” means “is a coarser partition than”)

iff

$$(\forall x \in \text{domC}(\Psi)) [x \in \text{domC}(\Phi)] \land \quad (\forall x_1, x_2 \in \text{domC}(\Psi)) [\Phi(x_1) = \Phi(x_2) \implies \Psi(x_1) = \Psi(x_2)].$$

This is a $\forall$-formula. Moreover, all arguments have linearly bounded length $N \leq O(|\psi|_{\Gamma \cup \tau} + |\varphi|_{\Gamma \cup \tau})$. Membership in $\text{domC}(\Psi)$ or $\text{domC}(\Phi)$ can be tested in deterministic polynomial time, by Prop. 5.5(1). Also, on input $\varphi, \psi$ (as words over $\Gamma \cup \tau$), and $x_1, x_2 \in A^N$ we can compute $\Phi(x_i) (= \varphi(x_i))$ and $\Psi(x_i) (= \psi(x_i)) (i = 1, 2)$ in deterministic polynomial time (by Prop. 5.5(1)). So, the above is a $\Pi_P^1$-formula, i.e., the problem is in coNP. \(\square\)

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