Averaging of Set Integrodifferential Equations

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Abstract In this article we prove the substantiation of the method of averaging for the set integrodifferential equations with small parameter. Thereby we expand a circle of systems to which it is possible to apply Krylov-Bogolyubov method of averaging.

Keywords Set Integrodifferential Equation, Method of Averaging

1. Introduction

Many important problems of analytical dynamics are described by the nonlinear mathematical models that as a rule are presented by the nonlinear differential or the integrodifferential equations. The absence of exact universal research methods for nonlinear systems has caused the development of numerous approximate analytic and numerically-analytic methods that can be realized in effective computer algorithms.

The averaging methods combined with the asymptotic representations (in Poincare sense) began to be applied as the basic constructive tool for solving the complicated problems of analytical dynamics described by the differential equations. Averaging theory for ordinary differential equations has a rich history, dating to back to the work of N.M. Krylov and N.N. Bogoliubov[1], and has been used extensively in engineering applications[2-6]. Books that cover averaging theory for differential equations and inclusions include[7-10].

In recent years the development of the calculus in metric spaces has attracted some attention[8-14]. Earlier, F.S. de Blasi, F. Iervolino[15] started the investigation of set differential equations (SDEs) in semilinear metric spaces. This has now evolved into the theory of SDEs as an independent discipline: properties of solutions[8,10,33], the impulse equations[45-48] and asymptotic methods[49-53]. On the other hand, SDEs are useful in other areas of mathematics. For example, SDEs are used, as an auxiliary tool, to prove existence results for differential inclusions[8,33,38,42]. Also, one can employ SDEs in the investigation of fuzzy differential equations[9,13,28,30,32,33]. Moreover, SDEs are a natural generalization of the usual ordinary differential equations in finite (or infinite) dimensional Banach spaces.

In this article we prove the substantiation of the method of averaging for the set integrodifferential equations with small parameter. Thereby we expand a circle of systems to which it is possible to apply Krylov-Bogolyubov method of averaging.

2. Preliminaries

Let \( \text{conv}(\mathbb{R}^n) \) be a set of all nonempty (convex) compact subsets from the space \( \mathbb{R}^n \),

\[
h(A,B) = \min_{r \geq 0} \{ S_r(A) \supset B, S_r(B) \supset A \}
\]

be Hausdorff distance between sets \( A \) and \( B \), \( S_r(A) \) is \( r \)-neighborhood of set \( A \).

Let \( A, B, C \) be in \( \text{conv}(\mathbb{R}^n) \). The set \( C \) is the Hukuhara difference of \( A \) and \( B \), if \( B + C = A \), i.e.

\[
C = A - B
\]

From Radstrom’s Cancellation Lemma[54], it follows that if this difference exists, then it is unique.

Definition[55]. A mapping \( X : [0,T] \rightarrow \text{conv}(\mathbb{R}^n) \) is differentiable in the sense of Hukuhara at \( t \in [0,T] \) if for some \( \delta > 0 \) the Hukuhara differences

\[
X(t + \Delta) = X(t) + \Delta^{-1} \left( X(t + \Delta) H X(t) - X(t) H X(t - \Delta) \right)
\]

exists in \( \text{conv}(\mathbb{R}^n) \) for all \( 0 < \Delta < \delta \) and there exists an \( DX(t) \in \text{conv}(\mathbb{R}^n) \) such that

\[
\lim_{\Delta \rightarrow 0_+} h \left( \Delta^{-1} \left( X(t + \Delta) - X(t) \right), DX(t) \right) = 0
\]

and

\[
\lim_{\Delta \rightarrow 0_+} h \left( \Delta^{-1} \left( X(t) - X(t - \Delta) \right), DX(t) \right) = 0
\]

Here \( DX(t) \) is called the Hukuhara derivative of \( X(t) \) at \( t \).
3. The First Scheme of an Average

Consider the Cauchy problem with small parameter

\[ DX = \left[ F(t, X) + \int_0^T \Phi(\tau, s, X(s))ds \right] dt, \quad X(0) = X_0, \tag{1} \]

where \( \varepsilon > 0 \) is a small parameter, \( t, s \in \mathbb{R}_+ \), \( F : \mathbb{R}_+ \times \text{conv}(\mathbb{R}^n) \rightarrow \text{conv}(\mathbb{R}^n) \) is a multivalued mapping, \( \Phi : \mathbb{R}_+ \times \mathbb{R}_+ \times \text{conv}(\mathbb{R}^n) \rightarrow \text{conv}(\mathbb{R}^n) \) is a multivalued mapping. Here the integral is understood in the sense of [55] (the integral exists for example if \( X(\cdot) \) is measurable and the real mapping \( t \rightarrow h(X(t),[0]) \) is integrable on \( I \subset \mathbb{R}_+ \).

**Definition.** A mapping \( X : [0,T] \rightarrow \text{conv}(\mathbb{R}^n) \) is a solution to the problem (1) if and only if it is continuous and satisfies the integral equation

\[ X(t) = X_0 + \varepsilon \int_0^t \left[ F(\tau, X(\tau)) + \int_0^\tau \Phi(\tau, s, X(s))ds \right] d\tau \]

for all \( t \in [0,T] \).

In this section we associate with the equation (1) the following averaged integro-differential equation

\[ DY = \varepsilon \left[ \overline{F}(Y) + \int_0^t \overline{\Phi}(t, Y(s))ds \right], \quad Y(0) = X_0, \tag{2} \]

where

\begin{align*}
\lim_{T \rightarrow \infty} \varepsilon \int_0^T \frac{1}{T} \left[ F(t, X(t), t) dt, \overline{F}(X) \right] = 0, \tag{3} \\
\lim_{T \rightarrow \infty} \varepsilon \int_0^T \frac{1}{T} \left[ \int_0^T \Phi(t, s, X(s))ds dt, \overline{\Phi}(t, X, X) ds dt \right] = 0. \tag{4}
\end{align*}

**Remark.** In this paper we will consider a case when the limits (3),(4) exist.

**Definition.** We say that the limits (3),(4) exist uniformly in \( X \), if for any \( \alpha > 0 \) and any \( X \in G \in \text{conv}(\mathbb{R}^n) \) there exists \( T(\alpha) \) such that

\[ h\left( \frac{1}{T} \int_0^T F(t, X(t), t) dt, \overline{F}(X) \right) < \alpha, \]

\[ h\left( \frac{1}{T} \int_0^T \int_0^T \Phi(t, s, X(s))ds dt, \frac{1}{T} \int_0^T \int_0^T \Phi(t, X, X) ds dt \right) < \alpha \]

for all \( T > T(\alpha) \).

**Theorem.** Let in domain

\[ Q = \left\{ (t, X) \mid t \geq 0, X \in G \in \text{conv}(\mathbb{R}^n) \right\} \]

the following hold:
1) \( F(t, X) \) is continuous in \( (t, X) \in \mathbb{R}_+ \times G \);
2) \( \Phi(t, s, X) \) is continuous in \( (t, s, X) \in \mathbb{R}_+ \times \mathbb{R}_+ \times G \);
3) there exist continuous functions \( K(t), P(t, s) \), and constants \( K_0, P_0 \) such that

\[ h(F(t,X),[0]) \leq K(t), \quad h(\Phi(t,s,X),[0]) \leq P(t,s), \]

\[ \int_{t_1}^t K(\tau) d\tau \leq K_0(t_2-t_1), \quad \int_{t_1}^t P(t,s)ds dt \leq P_0(t_2-t_1) \]

for any \( 0 \leq t_1 \leq t_2 < \infty \);
4) there exist continuous functions \( N(t), M(t,s) \), and constants \( N_0, M_0, M_1 \) such that

\[ h(F(t,X_1),F(t,X_2)) \leq N(t) h(X_1,X_2), \]

\[ h(\Phi(t,s,X_1),\Phi(t,s,X_2)) \leq M(t) h(X_1,X_2), \]

\[ \int_{t_1}^T \int_{t_1}^T M(t,s)ds dt \leq M_0(t_2-t_1), \]

\[ \int_{t_1}^T N(t) ds dt \leq N_0(t_2-t_1), \]

\[ \int_{t_1}^T \int_{t_1}^T M(t,s)(t-s)ds dt \leq M_1(t_2-t_1) \]

for any \( 0 \leq t_1 \leq t_2 < \infty \);
5) there exist continuous function \( V(t) \), and constants \( \lambda, V_0, V_1 \) such that

\[ h(\overline{F}(X_1),\overline{F}(X_2)) \leq \lambda h(X_1,X_2), \]

\[ h(\overline{\Phi}(t,X_1),\overline{\Phi}(t,X_2)) \leq V(t) h(X_1,X_2), \]

\[ \int_{t_1}^T V(t) dt \leq V_0(t_2-t_1), \]

\[ \int_{t_1}^T V(t) dt \leq V_1(t_2-t_1) \]

for any \( 0 \leq t_1 \leq t_2 < \infty \);
6) the limits (3),(4) exist uniformly in \( X \in G \);
7) for any \( X_0 \in G \subset \mathbb{R}^n \) and \( t \geq 0 \) the solution of the equation (2) together with a \( \sigma \)-neighborhood belong to the domain \( G \).

Then for any \( \eta > 0 \) and \( L > 0 \) there exist \( \varepsilon \in (0, \varepsilon_0) \) such that for all \( \varepsilon \in (0, \varepsilon_0) \) and \( t \in [0,Le^{-1}] \) the following statement fulfill:

\[ h(X(t),Y(t)) < \eta, \tag{5} \]

where \( X(t), Y(t) \) are the solutions of the initial and the averaged equations.

**Proof.** Since

\[ X(t) = X_0 + \varepsilon \int_0^t \left[ F(\tau, X(\tau)) + \int_0^\tau \Phi(\tau, s, X(s))ds \right] d\tau, \]

\[ Y(t) = X_0 + \varepsilon \int_0^t \left[ \overline{F}(Y) + \int_0^\tau \overline{\Phi}(\tau, Y(s))ds \right] d\tau \]

we have

\[ h(X(t),Y(t)) \leq \varepsilon h\left( \int_0^t F(\tau,X(\tau))d\tau, \int_0^t \overline{F}(\tau,Y(\tau))d\tau \right) + \]

\[ + \varepsilon h\left( \int_0^t \Phi(\tau,s,X(s))ds d\tau, \int_0^t \overline{\Phi}(\tau,Y(s))ds d\tau \right). \]

Hence

\[ h(X(t),Y(t)) \leq \varepsilon h\left( \int_0^t F(\tau,X(\tau))d\tau, \int_0^t \overline{F}(\tau,X(\tau))d\tau \right) + \]

\[ + \varepsilon h\left( \int_0^t \Phi(\tau,X(s))ds d\tau, \int_0^t \overline{\Phi}(\tau,Y(s))ds d\tau \right) + \]

\[ + \varepsilon h\left( \int_0^t \overline{F}(\tau,X(s))ds d\tau, \int_0^t \overline{\Phi}(\tau,Y(s))ds d\tau \right) \]

\[ \leq \varepsilon h\left( \int_0^t F(\tau,X(\tau))d\tau, \int_0^t \overline{F}(\tau,X(\tau))d\tau \right) + \]

\[ + \varepsilon h\left( \int_0^t \Phi(\tau,s,X(s))ds d\tau, \int_0^t \overline{\Phi}(\tau,X(s))ds d\tau \right) + \]

\[ + \varepsilon h\left( \int_0^t \overline{F}(\tau,s,X(s))ds d\tau, \int_0^t \overline{\Phi}(\tau,Y(s))ds d\tau \right) \]

\[ \leq \varepsilon h\left( \int_0^t F(\tau,X(\tau))d\tau, \int_0^t \overline{F}(\tau,X(\tau))d\tau \right) + \]

\[ + \varepsilon h\left( \int_0^t \Phi(\tau,s,X(s))ds d\tau, \int_0^t \overline{\Phi}(\tau,Y(s))ds d\tau \right) + \]

\[ + \varepsilon h\left( \int_0^t \overline{F}(\tau,s,X(s))ds d\tau, \int_0^t \overline{\Phi}(\tau,Y(s))ds d\tau \right) \]
\[ + \varepsilon \int_{0}^{t} h(\mathcal{F}(X(\tau)), \mathcal{F}(Y(\tau))) \, d\tau + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau \]
\[ + \varepsilon \int_{0}^{t} h(\mathcal{F}(\tau, X(s)), \Phi(\tau, X(s))) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau + \int_{0}^{t} \mathcal{F}(\tau, X(s)) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau + \int_{0}^{t} \mathcal{F}(\tau, X(s)) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau + \int_{0}^{t} \mathcal{F}(\tau, X(s)) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau + \int_{0}^{t} \mathcal{F}(\tau, X(s)) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau + \int_{0}^{t} \mathcal{F}(\tau, X(s)) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau + \int_{0}^{t} \mathcal{F}(\tau, X(s)) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau + \int_{0}^{t} \mathcal{F}(\tau, X(s)) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau + \int_{0}^{t} \mathcal{F}(\tau, X(s)) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau + \int_{0}^{t} \mathcal{F}(\tau, X(s)) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau + \int_{0}^{t} \mathcal{F}(\tau, X(s)) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau + \int_{0}^{t} \mathcal{F}(\tau, X(s)) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau + \int_{0}^{t} \mathcal{F}(\tau, X(s)) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ + \varepsilon \int_{0}^{t} \Phi(\tau, s, X(s)) \, ds \, d\tau + \int_{0}^{t} \mathcal{F}(\tau, X(s)) \, ds \, d\tau \leq \varepsilon \lambda \int_{0}^{t} h(X(\tau), Y(\tau)) \, d\tau + \varepsilon \int_{0}^{t} F(\tau, X(\tau)) \, d\tau + \int_{0}^{t} \mathcal{F}(X(\tau)) \, d\tau \]
\[ \leq \varepsilon \sum_{i=0}^{m-1} \left( \int_t^{t_i} F(r, X(t_i))d\tau + \int_t^{t_i} F(r, X(t_i))d\tau \right) + \varepsilon \sum_{i=0}^{m-1} \left( \int_t^{t_i} F(r, X(t_i))d\tau + \int_t^{t_i} F(r, X(t_i))d\tau \right) + \varepsilon \sum_{i=0}^{m-1} \left( \int_t^{t_i} F(r, X(t_i))d\tau + \int_t^{t_i} F(r, X(t_i))d\tau \right) + \varepsilon \sum_{i=0}^{m-1} \left( \int_t^{t_i} F(r, X(t_i))d\tau + \int_t^{t_i} F(r, X(t_i))d\tau \right) \]

where

\[ \varphi_1(\varepsilon) = \sup_{\tau \in [0, L]} \left( \int \Phi(\tau, s, X(t_i))d\tau \right), \quad \varphi_2(\varepsilon) = \sup_{\tau \in [0, L]} \left( \int \Phi(\tau, s, X(t_i))d\tau \right) \]

Fix \( m \). Then for any \( \eta > 0 \), there exists \( \varepsilon_2 > 0 \) such that the following estimate is true for \( 0 < \varepsilon < \varepsilon_2 \):

\[ \Sigma_2 \leq 2m(\varphi_1(\varepsilon) + \varphi_2(\varepsilon)) \leq \frac{\eta}{2} e^{-L(\lambda + \beta)} . \]  

(9)

Now, we get \( \varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\} \). Combining (6), (8), and (9), we obtain

\[ h(X(t), Y(t)) \leq \leq \varepsilon \int_0^T \lambda h(X(t), Y(t))d\tau + \varepsilon \int_0^T h(X(t), Y(t))d\tau + \varepsilon \leq \int_0^T \lambda h(X(t), Y(t))d\tau + \varepsilon \leq \int_0^T h(X(t), Y(t))d\tau + \varepsilon \leq \int_0^T \lambda h(X(t), Y(t))d\tau + \varepsilon \]

Using Gronwall-Bellman's inequality, we obtain

\[ h(X(t), Y(t)) < \eta . \]

This concludes the proof.

4. The Second Scheme of an Average

In this section we associate with the equation (1) the averaged integrodifferential equation (2), where

\[ \lim_{t \to \infty} \frac{1}{T} \int_0^T F(r, X(t))dt, F(X) = 0 , \]  

(10)

\[ \overline{\Phi}(t, X) = \begin{cases} \frac{1}{t} \Phi(t, X), & t > 0, \\ \Phi(0, X) = \int_0^t \Phi(s, X)ds . \end{cases} \]  

(11)

Remark. In this paper we will consider a case when the limits (10),(11) exist.

Theorem. Let in domain

\[ Q = \{(t, X) | t \geq 0, X \in G \cap \text{conv}(\mathbb{R}^n)\} \]

the following hold:

1) \( F(t, X) \) is continuous in \( (t, X) \in \mathbb{R}_+ \times G \);
2) \( \Phi(t, X) \) is continuous in \( (t, X) \in \mathbb{R}_+ \times \mathbb{R}_+ \times G \);
3) there exist continuous functions \( K(t) \), \( P(t, s) \), and constants \( K_0, P_0 \) such that

\[ h(F(t, X), [0]) \leq K(t), \quad h(\Phi(t, X), [0]) \leq P(t, s) , \]

\[ \int_0^t K(t)dt \leq K_0(t_2 - t_1), \quad \int_0^t P(t, s)ds \leq P_0(t_2 - t_1) \]

for any \( 0 \leq t_1 \leq t_2 < \infty \);
4) there exist continuous functions \( N(t) \), \( M(t, s) \), and constants \( N_0, M_0, M_1 \) such that

\[ h(F(t, X_1), F(t, X_2)) \leq N(t)h(X_1, X_2) , \]

\[ h(\Phi(t, X_1), \Phi(t, X_2)) \leq M(t, s)h(X_1, X_2) . \]

\[ \int_0^t N(t)dt \leq N_0(t_2 - t_1) , \quad \int_0^t M(t, s)ds \leq M_0(t_2 - t_1) . \]

\[ \int_0^t M(t, s)ds \leq M_1(t_2 - t_1) \]

for any \( 0 \leq t_1 \leq t_2 < \infty \);
5) there exist continuous function \( V(t) \), and constants \( \lambda, V_0, V_1 \) such that

\[ h(\overline{F}(X_1), \overline{F}(X_2)) \leq \lambda h(X_1, X_2) , \]

\[ h(\overline{\Phi}(t, X_1), \overline{\Phi}(t, X_2)) \leq V(t)h(X_1, X_2) , \]

\[ \int_0^t V(t)dt \leq V_0(t_2 - t_1) , \quad \int_0^t M(t, s)ds \leq V_1(t_2 - t_1) \]

for any \( 0 \leq t_1 \leq t_2 < \infty \);
6) the limits (3),(4) exist uniformly in \( X \in G \);
7) for any \( X_0 \in G \subset G \) and \( t \geq 0 \) the solution of the equation (2) together with a \( \sigma \)-neighborhood belong to the domain \( G \).

Then for any \( \eta > 0 \) and \( L > 0 \) there exists \( \varepsilon^0(\eta, L) \in (0, \sigma) \) such that for all \( \varepsilon \in (0, \varepsilon^0] \) and \( t \in [0, Le^{-1}] \) the following statement fulfil:

\[ h(X(t), Y(t)) < \eta , \]

where \( X(t), Y(t) \) are the solutions of the initial and the averaged equations.

Proof. Since

\[ X(t) = X_0 + \varepsilon \int_0^t F(r, X(r)) + \int_0^t \Phi(r, s, X(s))ds d\tau , \]

\[ Y(t) = X_0 + \varepsilon \int_0^t F(r, Y(r)) + \int_0^t \overline{\Phi}(r, Y(s))ds d\tau , \]

we have

\[ h(X(t), Y(t)) \leq \]

\[ \leq \varepsilon L h(X(t), Y(t))d\tau + \varepsilon \int_0^t \int_0^t \Phi(r, s, X(s))d\tau ds + \varepsilon \int_0^t \int_0^t \overline{\Phi}(r, Y(s))d\tau ds \]

(12)

Now we will estimate last summands in (12). Divide the interval \([0, Le^{-1}]\) into partial intervals by the points

\[ t_i = \frac{IL}{mc}, \quad i = 0, \ldots, m, \quad m \in \mathbb{N} . \]

Then
\[
e h \left( \int_0^1 F(r, X(r)) \, dr, \int_0^1 F(X(r)) \, dr \right) + \\
e h \left( \int_0^1 \Phi(r, s, X(s)) \, ds \, dr, \int_0^1 \Phi(X(r), s) \, ds \, dr \right) \leq \\
\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5,
\]

where
\[
\Sigma_1 = \varepsilon \sum_{i=0}^{m-1} \int \Phi(r, X(r), r, X(r)) \, dr,
\]
\[
\Sigma_2 = \varepsilon \sum_{i=0}^{m-1} \int \Phi(r, s, X(s), \Phi(r, s, X(s))) \, ds \, dr,
\]
\[
\Sigma_3 = \varepsilon \sum_{i=0}^{m-1} \int \Phi(r, s, X(s), \Phi(r, X(\tau))) \, ds \, dr,
\]
\[
\Sigma_4 = \varepsilon \sum_{i=0}^{m-1} \int \Phi(r, X(\tau)) \, dr,
\]
\[
\Sigma_5 = \varepsilon \sum_{i=0}^{m-1} \int \Phi(r, X(\tau)) \, dr.
\]

By condition 3) of the theorem, we get
\[
h(X(r), X(\tau)) \leq \frac{L}{m} (K_0 + P_0),
\]
for \( \tau \in [t_i, t_{i+1}] \).

From (13), and conditions of the theorem, we have
\[
\Sigma_1 \leq LN_0 \frac{L}{m} (K_0 + P_0), \quad \Sigma_2 \leq (K_0 + P_0) \left( \frac{\varepsilon M_1 + \frac{LM_0}{m}}{2} \right),
\]
\[
\Sigma_3 \leq L (K_0 + P_0) \left( \frac{V_1}{2} + \frac{V_0}{2} \right),
\]
\[
\Sigma_4 \leq \frac{\lambda}{2} (K_0 + P_0) \frac{L^2}{m}.
\]

Hence, for any \( \eta > 0 \), there exist \( m \), and \( \varepsilon_1 > 0 \) such that the following estimate is true for \( 0 < \varepsilon < \varepsilon_1 \):
\[
\sum_{i=0}^{m-1} \Sigma_i \leq LN_0 \frac{L}{m} (K_0 + P_0) + \frac{L (K_0 + P_0) \left( \frac{\varepsilon M_1 + \frac{LM_0}{m}}{2} + \frac{\lambda L}{2} + \frac{V_1}{2} + \frac{LV_0}{m} \right)}{2} \leq \frac{\eta}{2} e^{-L(\lambda + \varepsilon_1)}.
\]

From condition 6) of the theorem, it follows that there exists the increasing function \( \theta(t) \) such that
1) \( \lim_{t \to \infty} \theta(t) = 0 \);
2) \( \int_0^t F(r, X) \, dr, \int_0^t F(X) \, dr \leq t \theta(t) \).

Then
\[
\Sigma_5 \leq \varepsilon \sum_{i=0}^{m-1} \int \Phi(r, X(\tau)) \, dr, \int \Phi(X(\tau)) \, dr \leq \\
\leq \varepsilon \sum_{i=0}^{m-1} \int \Phi(r, X(\tau)) \, dr, \int \Phi(X(\tau)) \, dr + \\
+ \varepsilon \sum_{i=0}^{m-1} \int \Phi(r, X(\tau)) \, dr, \int \Phi(X(\tau)) \, dr \leq \\
\leq 2m \psi(\varepsilon),
\]

where
\[
\psi(\varepsilon) = \sup_{r \in [0, t]} \left[ r \theta \left( \frac{r}{\varepsilon} \right) \right], \quad \tau = dt.
\]

Fix \( m \). Then for any \( \eta > 0 \), there exists \( \varepsilon_2 > 0 \) such that the following estimate is true for \( 0 < \varepsilon < \varepsilon_2 \):
\[
\Sigma_5 \leq 2m \psi(\varepsilon) \leq \frac{\eta}{2} e^{-L(\lambda + \varepsilon_1)}.
\]

By (11), we have
\[
\Sigma_0 = \varepsilon \sum_{i=0}^{m-1} \int \Phi(r, s, X(s), \Phi(r, s, X(s))) \, ds \, dr \leq \\
= \varepsilon \sum_{i=0}^{m-1} \int \Phi(r, s, X(s), \Phi(r, X(\tau))) \, ds \, dr \leq \\
\leq \varepsilon \sum_{i=0}^{m-1} \int \Phi(r, s, X(s), \Phi(r, X(\tau))) \, ds \, dr \leq 0.
\]

Now, we get \( \varepsilon_0 = \min \{ \varepsilon_1, \varepsilon_2 \} \cdot \) Combining (12), (14), and (15), we obtain
\[
h(X(\tau), Y(\tau)) \leq \\
\leq \varepsilon \left( \int_0^\tau h(X(\tau), Y(\tau)) + V(\tau) \int_0^\tau h(X(s), Y(s)) \, ds + \eta \right) e^{-L(\lambda + \varepsilon_1)}.
\]

Using Gronwall-Bellman's inequality, we obtain
\[
h(X(\tau), Y(\tau)) \leq \eta.
\]

This concludes the proof.

5. Conclusions

Here we used the approach of Hukuhara at definition of the derivative which has essential shortages. However the given approach is well investigated by many authors. Also in the literature exist other approaches to definition of the derivative[9,10,16,28,36,43], but also they have the shortages. It is easily possible to show that this outcome will be true for some other cases with little changes.

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