Application of model order reduction to a finite element model of cryogenic turning

Steven Becker¹*, Hendrik Hotz², Benjamin Kirsch², Jan C. Aurich², and Ralf Müller¹

¹ Technische Universität Kaiserslautern, Institute of Applied Mechanics, P.O. Box 3049, 67653 Kaiserslautern
² Technische Universität Kaiserslautern, Institute for Manufacturing Technology and Production Systems, P.O. Box 3049, 67653 Kaiserslautern

Metastable austenitic steels offer the opportunity of a surface layer hardening integrated in the machining process. The hardening effect is achieved by a deformation induced austenite-martensite phase transformation, for which high mechanical loads and low temperatures are necessary, typically below room temperature. These conditions can be accomplished during cryogenic turning, allowing a phase transformation in the surface layer of the workpiece. To study the austenite-martensite transformation behavior, information about the temperatures in the contact zone between tool and workpiece during machining is necessary, which can hardly be measured. Therefore, FE simulations of the process are utilized to calculate the transient process temperatures. In this paper the idea of projection-based model order reduction and its implementation into the finite element program FEAP are described, allowing a solution speed-up while maintaining accuracy of the full-order model.

© 2019 The Authors Proceedings in Applied Mathematics & Mechanics published by Wiley-VCH Verlag GmbH & Co. KGaA Weinheim

1 Motivation and Model

In [1, 2] a model has been presented to calculate the temperatures arising during dry and cryogenic turning of metastable austenitic steel AISI 347. Pre-calculations were necessary to determine the magnitude of boundary heat fluxes first, which are mainly induced by the tool and the cryogenic cooling. This happens in an iterative way in terms of multiple forward solutions in order to correlate measured and simulated temperatures. However, the repeated calculation of such a finite element problem may take a considerable amount of time due to the number of equations to be solved. To counteract this drawback, model order reduction techniques may be considered, which reduce the number of equations and thus solution time, while maintaining accuracy of the solution. In the present case projection-based model order reduction, which is based on the method of snapshots [5], has been implemented into the finite element code of FEAP [6].

The underlying equation is the heat equation, which in the present case considers the convective time derivative as well, resulting in an Eulerian representation, c.f. [2]. The finite element system of equations can then be formulated by $K \dot{T} + M \ddot{T} = \hat{F}$, which may be referred to as the full-order model (FOM) of size $K, M \in \mathbb{R}^{m \times m}$ and $\hat{T}, \hat{F} \in \mathbb{R}^{m}$.

Model order reduction states the existence of an approximation $\hat{T} \approx U_r T$, where the projection matrix $U_r$ is calculated from snapshots of FOM solutions. In more detail, $n$ snapshots are gathered in a snapshot matrix $\in \mathbb{R}^{m \times n}$, which is analyzed using singular value decomposition. The sought projection matrix stems from the left singular vectors $U_r = U[1 : d], d \in [1, n]$. Additionally, further projection of the FOM is needed to overcome an over-determined system of equations. In case of Galerkin projection, the projection matrix $U_r$ is used for this second projection, like it was utilized in the present approach. In case of Petrov-Galerkin projection, a matrix different from $U_r$ performs the second projection. Here the system of equations after Galerkin projection with $U_r$ results in

$$K \hat{T} + M \ddot{T} = \hat{F} \quad \Rightarrow \quad U_r^T KU_r \hat{T} + U_r^T M U_r \ddot{T} = U_r^T \hat{F} \quad \Rightarrow \quad \hat{K} \hat{T} + \hat{M} \ddot{T} = \hat{F}$$

where the reduced-order model (ROM) consists of $\hat{K}, \hat{M} \in \mathbb{R}^{d \times d}$ and $\hat{T}, \hat{F} \in \mathbb{R}^{d}$. The reduced system is typically of size $d \ll mn$.

Contrary to the implementation of the FOM, the element matrices of the ROM do not need to be assembled into global system matrices, c.f. [4]. Instead it is possible to calculate element projection matrices $U_r^T I_c = U_r^T \hat{I}_c$, where the necessary rows of $U_r$ corresponding to the current element nodes are extracted by the operator $I_c$. By design, the element system matrices of the ROM already are of size of the reduced global system matrices and can thus be accumulated.

2 Results

After the implementation of the model order reduction code, dry turning has been considered as a use case. Since the material parameters, e.g. the heat conductivity, were assumed temperature independent, and since the heat transfer coefficient at a specific position was additionally held constant, the system matrices $K, M$ resulted to be constant throughout the simulation as well. Thus, the projections on the left side of the third equation in Eq. 1 could be pre-calculated, rendering the ROM more efficient.
The following results show the accuracy of the ROM by comparing its solutions with solutions of the FOM for different numbers of projection modes $d$. The projection matrix was built from solutions with varying heat fluxes. The ROM is conducted with an intermediate heat flux value, for which its error is calculated relative to the corresponding FOM solution.

![Normalized singular values](image1)

**Fig. 1:** a) Normalized singular values, and b) energy of the singular values resulting from an SVD of FOM snapshots.

The red line in Fig. 1a marks the number of modes $d$ for which the normalized singular values decay below $10^{-6}$. This typical value can be found in literature and marks the number of modes, which build the projection matrix $U_r = U[1 : d]$.

Another criterion to determine the number of modes for the projection matrix is the so-called energy $\epsilon$ of the singular values $\sigma$, $\sum \sigma_i \geq \epsilon$, $\epsilon \in [0, 1]$. The red lines in Fig. 1b mark the number of modes where the energy is just above 95% ($d = 29$) and 99% ($d = 35$), respectively.

| unknowns | FOM | ROM | ROM | ROM | ROM |
|----------|-----|-----|-----|-----|-----|
| sol. time | 646 s | 116 s | 125 s | 136 s | 148 s |
| avg. rel. err. | / | 14.7% | 14.6% | 0.699% | 0.687% |

**Table 1:** Comparison of FOM and ROM in terms of unknowns, solution time and average relative error over time along the tool path.

As can be seen in Tab. 1 the number of modes taken from the singular value criterion results in a low relative average error between FOM and ROM of 0.687%. Since the temperatures along the tool path are most interesting for the analysis of the process, this error is calculated along and averaged over the tool path and time. In the present case, the number of modes corresponding to singular value energies of $\epsilon = 95\%$ and $\epsilon = 99\%$ result in considerably higher errors. Thus, energy bounds $> 99\%$ should be considered instead, when utilizing the energy criterion.

In contrast to the system matrices, the flux term on the right side of Eq. 1 changed per time step, since it represents the moving heat source. In addition, it had to be projected in every time step and could not be pre-computed. Referring to Tab. 1, this fact breaks the scaling between the number of unknowns and the corresponding solution times, when comparing the FOM with a particular ROM. Nevertheless, solution time could be reduced by a factor $> 4$. This term thus represents a contribution of non-linear model order reduction, for which further reduction methods (‘hyper-reduction’) such as DEIM [3] are available and will be investigated in future work.

**Acknowledgements**  Funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) - Projektnummer 172116086 - SFB 926.

**References**

[1] S. Becker, H. Hotz, B. Kirsch, J. C. Aurich, E. v. Harbou, R. Müller, Proc. Appl. Math. Mech. 16, 1 (2016), pp. 303–304.
[2] S. Becker, H. Hotz, B. Kirsch, J. C. Aurich, E. v. Harbou, R. Müller, J. Manuf. Sci. Eng. 140, 10 (2018).
[3] S. Chaturantabut, D. C. Sorensen, SIAM J. Sci. Comp. 32, 5 (2010), pp. 2737–2764.
[4] A. Radermacher, S. Reese, Int. J. Mat. Eng. Inno. 4, 2 (2013), pp. 149–165.
[5] L. Sirovich, Quart. Appl. Math. 45, 3 (1987), pp. 561–571.
[6] R. L. Taylor, FEAP - Finite Element Analysis Program (University of California, Berkeley, 2014).