On the nonKoszulity of \((2p + 1)\)-ary partially associative Operads

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Abstract

We want to present here the part of the work in common with Martin Markl [11] which concerns quadratic operads for \(n\)-ary algebras and their dual for \(n\) odd. We will focus on the ternary case (i.e. \(n = 3\)). The aim is to underline the problem of computing the dual operad and the fact that this last is in general defined in the graded differential operad framework.

We prove that the operad associated to \((2p + 1)\)-ary partially associative algebra is not Koszul. Recall that, in the even case, this operad is Koszul.

1 Introduction

We are interested in the operads associated to \(n\)-ary algebras whose multiplication \(\mu\) satisfies the following relation

\[
\sum_{i=1}^{n} (-1)^{(i-1)(n-1)} \mu(X_1, \ldots, X_{i-1}, \mu(X_i, \ldots, X_{i+n-1}), X_{i+n}, \ldots, X_{2n-1}) = 0.
\]

Such a multiplication is called \(n\)-ary partially associative.

These operads have already been studied several times for instance in the thesis of Gnedbaye [4] and more recently by M. Markl - E. Remm (in term of operads) [11], N. Goze - E. Remm (in term of algebras) [9] and also H. Ataguema - A. Makhlouf [1] and E. Hoffbeck [10].

When computing the free algebras associated to \((2p + 1)\)-ary partially associative algebras, using different arguments in [11] and [9], we saw that the odd and even cases behave in a completely different way. This approach is a little bit different to [4], [1] and [5] where odd and even cases have been studied in
the same way and then contain some misunderstandings in the odd case. In fact, contrary to what we find in the previous papers, the operads associated to 
\((2p+1)\)-ary partially associative algebras are non Koszul so there is no operadic cohomology associated to these algebras (we call it operadic because it uses the dual operad to define a cohomology). It is this result that we are going to expose in this work.

Let us remark that in the first versions of [10], the author thought that the degrees didn’t matter and that the degrees of the operations appear only in the calculation of the dual and even there, if the operations are concentrated in a fixed arity, it wouldn’t change the relations in the dual. Then he concluded that the operad of partially associative \(n\)-ary algebras was Koszul for any \(n\). The author has changed his point of view after that we indicated him the problem. Let us also note that in his habilitation thesis and contrary to his previous works Gnedbaye precises the necessity of distinguishing the even and odd cases assigning the problem on the definition of the generating series what is is insufficient to find the odd case as we are going to see here.

To study Koszulity we need to define correctly the dual operad of a \((2p+1)\)-ary partially associative operad which implies to understand correctly the definition of Ginzburg-Kapranov [3] developed in the case of binary operations in order to extend it to \(n\)-ary operations. When we compute the dual of a graded or nongraded operad, we get graded objects which take some suspension in account. In particular if we consider an \(n\)-ary multiplication of degree 0 and its associated operad \(\mathcal{P}\), an algebra on its dual operad \(\mathcal{P}^!\) corresponds to a \(n\)-ary multiplication of degree \(m \equiv n \mod 2\). In our case, as \(n\) is even, the multiplication of an algebra on the dual operad has to be considered of degree 1. If we forgot this degree, all multiplications related to the dual operad are considered of degree 0. We get something that we could call the nongraded “dual” operad which is the operad of \((2p+1)\)-ary totally associative algebras with operation in degree 0. But this operad is not the dual operad in the Ginzburg-Kapranov’s sense when \(n\) is odd.

As it was not possible to find an operadic cohomology for \((2p+1)\)-ary partially associative algebras, we have developed in [9] a cohomology for \((2p+1)\)-ary partially associative algebras restricting the space of cochains that we developp for the ternary case. We give explicitly the free algebra. In [11] we have a different approach. We consider a graded version of the \((2p+1)\)-ary partially associative algebras with operation in degree 1. In this case the dual operad is the operad of \((2p+1)\)-ary totally associative algebras with operation in degree 0 and both are Koszul. Then we obtain an operadic cohomology.

In the following we consider \(\mathbb{K}\) a field of characteristic 0 and the operads that we consider are generally \(\mathbb{K}\)-linear operads. All definitions and concepts used refers to [3] and [12].
The operad $3\text{Ass}$

This section deals with the operad of 3-ary partially associative algebras that is
algebras defined by a multiplication

\[ \mu : A^{\otimes 3} \to A \]

satisfying the relations

\[ \mu \circ (\mu \otimes I_2) + \mu \circ (I_1 \otimes \mu \otimes I_1) + \mu \circ (I_2 \otimes \mu) = 0 \]

where $I_j : A^{\otimes j} \to A^{\otimes j}$ is the identity map. We have a classical example of
such an algebra when we consider the Hochschild cohomology of an associative
algebra. In fact if $C^k(V, V)$ denotes the space of $k$-cochains of the Hochschild
cohomology of the associative algebra $V$, the Gerstenhaber product $\circ_{n,m}$ is a
linear map

\[ \circ_{n,m} : C^n(V, V) \times C^m(V, V) \to C^{n+m-1}(V, V) \]

given by

\[ f \circ_{n,m} g(X_1 \otimes \cdots \otimes X_{n+m-1}) = \sum_{i=1}^n (-1)^{i-1}(m-1) f(X_1 \otimes \cdots \otimes g(X_i \otimes \cdots \otimes X_{i+m-1}) \otimes \cdots X_{n+m-1}) \]

if $f \in C^n(V)$ and $g \in C^m(V)$. A 3-ary partially associative product is a 3-
cochain $\mu$ satisfying $\mu \circ_{3,3} \mu = 0$.

The notion of quadratic operad is clearly defined in [G.K]. We refer to this
paper. An operad $P$ is a collection \{\(P(n), n \geq 1\)\} of $K$-vector spaces satisfying
the following properties: each $P(n)$ is a $\Sigma_n$-module where $\Sigma_n$ is the symmetric
group of degree $n$, there is an element $1 \in P(1)$ called the unit and linear maps

\[ \circ_i : P(n) \times P(m) \to P(n + m - 1) \]

called comp-$i$ operations satisfying associative conditions: if $\lambda \in P(l), \mu \in P(m), \nu \in P(n)$ then

\[ (\lambda \circ_i \mu) \circ_j \nu = \begin{cases} (\lambda \circ_j \nu) \circ_{i+n-1} \mu & \text{if } 1 \leq j \leq i-1 \\ \lambda \circ_i (\mu \circ_{j+i-1} \nu) & \text{if } i \leq j \leq m+1-1 \\ (\lambda \circ_j \nu) \circ_i \mu & \text{if } i + m \leq j \end{cases} \]

which are compatible with the action of the symmetric group.

Recall that a $P$-algebra is a $K$ vector space $V$ equipped with a morphism of
operad $f : P \to \mathcal{E}_V$ where $\mathcal{E}_V$ is the operad of endomorphisms of $V$. Given a
structure of $P$-algebra on $V$ is the same as giving a collection of linear maps

\[ f_n : P(n) \otimes V^{\otimes n} \to V \]

satisfying natural associativity, equivariance and unit conditions.
If $E$ is a right-$\mathbb{K}[\Sigma_2]$-module, we can define an operad, denoted by $\mathcal{F}(E)$ and called the free operad generated by $E$ which is solution of the following universal problem: for any operad $Q = \{Q(n)\}$ and any $\mathbb{K}[\Sigma_2]$-linear morphism $f : E \to Q(2)$, there exits a unique operad morphism $\hat{f} : \mathcal{F}(E) \to Q$ which coincide with $f$ on $E = \mathcal{F}(E)(2)$. We have for example $\mathcal{F}(E)(3) = (E \otimes E) \otimes \Sigma_2 \mathbb{K}[\Sigma_3]$. If $R$ is a $\mathbb{K}[\Sigma_3]$-submodule of $\mathcal{F}(E)(3)$, it generates an ideal $(R)$ of $\mathcal{F}(E)$. The quadratic operad generated by $E$ with relations $R$ is the operad $\mathcal{P}(E, R) = \{\mathcal{P}(E, R)(n), n \geq 1\}$ with

$$\mathcal{P}(E, R)(n) = \mathcal{F}(E)(n)/R(n).$$

This notion of quadratic operad is related to binary algebras. In [Gn] this notion is adapted to $n$-ary algebras. In this case we consider a generating multiplication $\mu$ which is an $n$-ary multiplication that is $E = \langle \mu \rangle$ a $\mathbb{K}[\Sigma_n]$-module. We define the free operad generated by $E$ in the same sense that in the binary case. Here we get that $\mathcal{F}(E)(n) = 0$ if $n \neq p(n - 1) + 1$ and $\mathcal{F}(E)(p(n - 1) + 1)$ consists as a vector space to ”parenthesized products” of $\mu(n - 1) + 1$ variables indexed by $\{1, \cdots, n\}$. For instance a basis of $\mathcal{F}(E)(n)$ is generated as $\mathbb{K}[\sigma_n]$-module by $(x_1 \cdots x_n)$ and a basis of $\mathcal{F}(E)(2(n - 1) + 1)$ is given by $((x_1 \cdots x_n)x_{n+1} \cdots x_{2n-1}), (x_1 \cdots x_{n+1})x_{n+2} \cdots x_{2n-1}), \cdots, (x_1 \cdots x_{n-1}(x_n \cdots x_{2n-1})$ and all their permutations. The relations that we consider will be quadratic in the sense that we compose two $n$-ary multiplications so $R$ is a $\mathbb{K}[\Sigma_{2n-1}]$-submodule of $\mathcal{F}(E)(2n - 1)$. The operad $\mathcal{R} = (R)$ is the ideal generated by $R$ so in particular $\mathcal{R}(k) = 0$ for $k = 1$ and $k \neq p(n - 1) + 1$. In other words $\mathcal{R}(p(n - 1) + 1)$ consists in all relations in $\mathcal{F}(E)(p(n - 1) + 1)$ induced by the relations $R$.

We recall here the notion of quadratic operad of ternary algebras.

**Definition 1** Let $E$ be a $\mathbb{K}[\Sigma_3]$-module and $\mathcal{F}(E)$ the free operad over $E$. If $R$ is a $\mathbb{K}[\Sigma_3]$-submodule of $\mathcal{F}(E)(5)$ and if $(R)$ is the ideal generated by $R$, then the ternary quadratic operad $\mathcal{P}(E, R)$ generated by $E$ and $R$ is the quotient

$$\mathcal{F}(E)/(R).$$

Note that $\mathcal{F}(E)(n) = 0$ when $n$ is even. If $E \simeq \mathbb{K}[\Sigma_3]$ and $R$ is the $\mathbb{K}[\Sigma_3]$-submodule of $\mathcal{F}(E)(5)$ generated by the vectors

$$(x_1x_2x_3)x_4x_5 + x_1(x_2x_3x_4)x_5 + x_1x_2(x_3x_4x_5)$$

then the corresponding quadratic operad is the ternary quadratic operad

$$3\text{Ass} = \mathcal{F}(E)/(R).$$

We know that for any operad $\mathcal{P}$, the spaces $\mathcal{P}(n)$ are related to the free $\mathcal{P}$-algebras. In [G.R] we have studied the free partially associative algebra of order $3$ $\mathcal{L}_{3,\text{Ass}}(V) = \oplus \mathcal{L}^{2p+1}(V)$ on a vector space $V$. We have computed the dimensions of its homogeneous components, found a basis and a systematic method to write this basis. In particular we have, if $\dim V = 1$,

$$\dim \mathcal{L}^3(V) = 1, \quad \dim \mathcal{L}^5(V) = 2, \quad \dim \mathcal{L}^7(V) = 4$$

4
$$\dim \mathcal{L}^9(V) = 5, \ \dim \mathcal{L}^{11}(V) = 6, \ \dim \mathcal{L}^{13}(V) = 7.$$  

We deduce that

$$\dim((3\text{Ass})(3)) = 6, \ \dim((3\text{Ass})(5)) = 2 \dim K[\Sigma_5] = 240$$

and more generally

$$\dim((3\text{Ass})(2k+1)) = (k+1) \dim K[\Sigma_{2k+1}].$$

The Poincaré serie of \((3\text{Ass})\) called also the generating function is written

$$g_{3\text{Ass}}(x) = \sum_{n=1}^{\infty} \dim((3\text{Ass})(n)) \frac{x^n}{n!}$$

with the convention \(\dim(3\text{Ass})(1) = 1\). Then

$$g_{3\text{Ass}}(x) = x + x^3 + 2x^5 + 4x^7 + 5x^9 + 6x^{11} + 7x^{13} + \cdots$$

Recall that a quadratic operad \(\mathcal{P}\) is Koszul if the homology of any free \(\mathcal{P}\)-algebra \(F_\mathcal{P}(V)\) is trivial except in degree 0. If \(\mathcal{P}\) is a Koszul operad, then its Poincaré serie \(g_\mathcal{P}(x)\) satisfies

$$g_\mathcal{P}(-g_\mathcal{P}'(-x)) = x$$

where \(\mathcal{P}'\) is the dual operad. In the following section we define the dual operad of \((3\text{Ass})\). The previous identity shows, if \(\mathcal{P}\) is of Koszul, that there is a formal serie

$$s(x) = \sum a_n x^n$$

such that \(g_\mathcal{P}(-s(-x)) = x\). Now consider a serie \(s(x) = \sum a_n x^n\) and if we solve the equation \(g_{3\text{Ass}}(-s(-x)) = x\) we find:

$$s(x) = x - x^3 + x^5 - 19x^{11} + O[x]^{12}.$$  

Such a serie cannot be a Poincaré serie of a quadratic operad corresponding to a multiplication of degree 0. In the next section we compute the graded dual operad and we will see that its generating function is a polynomial of degree 5. This will permit to conclude to the non Koszulity of \((3\text{Ass})\).

**Remark.** If we consider the operad \(\mathcal{P} = 3t\text{Ass}\) associated to 3-ary totally associative algebras that is satisfying the relations

$$\mu \circ (\mu \otimes I_2) = \mu \circ (I_1 \otimes \mu \otimes I_1) = \mu \circ (I_2 \otimes \mu) = 0$$

we get the generating function

$$g_{3t\text{Ass}}(x) = x + x^3 + x^5 + x^7 + x^9 + \cdots + x^{2k+1} + \cdots$$

If we suppose this operad to be Koszul we would get that the dual operad \(\mathcal{P}'\) has as generating serie \(g_\mathcal{P}'\) satisfying \(g_\mathcal{P}(-g_\mathcal{P}'(-x)) = x\). But if we compute \(g_\mathcal{P}'\) from these equation we obtain

$$g_\mathcal{P}'(x) = \sum_{i \geq 1} a_n x^n$$

but \(|a_n|\) do not correspond to the dimensions of the operad \(3\text{Ass}\).
3 The dual operad

To compute the dual operad of an operad associated to \( n \)-ary algebras we need some differential graded operad. Recall that if \( \mathcal{C} \) is a monoidal category, a \( \Sigma \)-module \( A \) is represented by a sequence of objects \( \{ A(n) \}_{n \geq 1} \) in \( \mathcal{C} \) with a right-\( \Sigma_n \)-action on \( A(n) \). Then an operad in a (strict) symmetric monoidal category \( \mathcal{C} \) is a \( \Sigma \)-module \( \mathcal{P} \) together with a family of structural morphisms satisfying some associativity, equivariance and unit conditions.

We use the fact that \( dgVect \) the category of differential graded vector spaces over the base field \( \mathbb{K} \) (an object of \( dgVect \) is a graded vector space together with a linear map \( d \) (differential) of degree 1 such that \( d^2 = 0 \); morphisms are linear maps preserving gradings and differentials) is a symmetric monoidal category and we can define an operad in this category \( dgVect \).

A differential graded operad (or dg operad) is a differential graded \( \Sigma \)-module with an operad structure for which the operad structure maps are differential graded morphisms.

A (non graded) operad can be seen as a differential graded operad considering trivial differentials and nongraded objects are objects trivially graded.

Since a dg operad \( \mathcal{P} \) is itself a monoid in a symmetric monoidal category, the bar construction applies to \( \mathcal{P} \), producing a dg operad \( \mathcal{B}(\mathcal{P}) \). The linear dual of \( \mathcal{B}(\mathcal{P}) \) is a dg operad denoted by \( \mathcal{C}(\mathcal{P}) \) and called the cobar complex suitably regarded. Quadratic operad are defined as having a presentation with generators and relations and for them the dual operad will also be quadratic. For quadratic operads there is a natural transformation of functor from \( \mathcal{D}(\mathcal{P}) \) to \( \mathcal{P}^! \) which is a quasi-isomorphism if \( \mathcal{P} \) is Koszul. This concept is similar to the concept of quadratic dual and Koszulness for associative algebras.

Any quadratic operad \( \mathcal{P} \) associated to binary multiplications admits a dual operad which is also quadratic and denoted \( \mathcal{P}^! \). To define it we need to recall some definitions and notations. Let \( E \) be a \( \Sigma \)-module. The dual \( \Sigma \)-module \( E^\# = \{ E^\#(n) \}_{n \geq 1} \) is defined by

\[
E^\#(n) = Hom_{\mathbb{K}}(E(n), \mathbb{K})
\]

and the \( \Sigma_n \) representation on \( E(n) \) determines a dual representation on \( E^\#(n) \) by

\[
(\lambda \cdot \sigma, \mu) := (\lambda, \mu \cdot \sigma^{-1})
\]

for \( \mu \in E(n), \lambda \in E^\#(n) \) and \( \sigma \in \Sigma_n \). The Czech dual is the \( \Sigma \)-module \( E^\vee = \{ E^\vee(n) \}_{n \geq 1} \) with

\[
E^\vee(n) = E^\#(n) \otimes sgn_n.
\]

Then the quadratic dual operad is defined as a quotient of the free operad \( \mathcal{F}(E^\vee) \) by relations orthogonal (in some sense) to the relations defining the original quadratic operad \( \mathcal{P} \). So if \( Q = \mathcal{P}(E, R) \) (i.e \( E \) corresponds to the generators and \( R \) to the relations) the dual operad \( Q^! \) is defined by \( Q^! = \mathcal{P}(E^\#, R^\perp) \) where \( R^\perp \subseteq \mathcal{F}(E^\#)(3) \) is the annihilator with respect to some paring of the relations.
\( R \subseteq \mathcal{F}(E)(3) \) defining \( Q \). But notice that in the general definition of quadratic operad contains a suspension. If we consider a quadratic operad generated by an operation \( E = \langle \mu \rangle \in \mathbb{K}[\Sigma_2] \) where \( \mu \) is of degree 0, the dual operad is still a quadratic operad generated by an operation of degree 0.

Now if we consider \( n \)-ary algebras, we have seen that we can still define the notion of quadratic operad that is we consider a generating multiplication which is a \( n \)-ary multiplication \( \mu \) that is \( E = \langle \mu \rangle \in \mathbb{K}[\Sigma_n] \) and relations which are quadratic that is \( R \) is a \( \mathbb{K}[\Sigma_{2n-1}] \)-submodule. We can also still define a ”scalar product” as in the case of a binary operation but now it is a map \( \langle , \rangle : \mathcal{F}(E)(n) \otimes \mathcal{F}(E')(n) \to \mathbb{K} \). Then we get \( R^\perp \) but the dual operad \( \mathcal{P}^! \) is
\[
\mathcal{P}^!(n) = \mathcal{F}(E')(n)/(R^\perp)(n)
\]
when the generating operation is of degree even. But if the degree odd, the dual operad is a quadratic operad with a generating operation of degree odd and other relations are induced by \( R^\perp \).

This can also be seen as follows: if the operations are not of degree 0, then a nontrivial sign of the composition in the free operad which is introduced.

Now let us come back to the determination to the dual operad of \( 3\text{-Ass} \).

Recall that a totally associative 3-ary algebra is given by a 3-ary product \( \mu \) satisfying
\[
\mu(\mu(x_1, x_2, x_3), x_4, x_5) = \mu(x_1, \mu(x_2, x_3, x_4), x_5) = \mu(x_1, x_2, \mu(x_3, x_4, x_5)).
\]
We denote by \( 3\text{p}\text{-Ass} \) the corresponding quadratic operad.

**Theorem 2** [9] The dual operad of the \( 3\text{-Ass} \) operad is isomorphic to the operad of totally associative algebras with operation of degree 1.

Hence as a vector space a basis of \( 3\text{Ass}^!(n) \) is given by \( (x_{i_1} x_{i_2} \cdots x_{i_n}) \), a basis of \( 3\text{Ass}^!(2n-1) \) is given by \( (x_{i_1} \cdots x_{i_{n-1}}(x_{i_n} \cdots x_{i_{2n-1}})) \). We have also
\[
3\text{Ass}(n)^!(k(n-1)+1) = \{0\} \quad \forall k > 2.
\]
So
\[
dim(3\text{Ass}^!(n)) = dim(3\text{Ass}^!(2n-1)) = 1
\]
and
\[
dim(3\text{Ass}^!(k(n-1)+1)) = 0
\]
for \( k > 2 \).

**Theorem 3** [9] The generating function of the dual operad of \( 3\text{-Ass} \) is
\[
g_{3\text{Ass}^!}(x) = x - x^3 + x^5.
\]
But we have seen in the previous section that any formal serie satisfying \( g_{3\text{Ass}}(-s(-x)) = x \) is of the form
\[
s(x) = -x + x^3 - x^5 + 19x^{11} + O|x|^{12}.
\]
Then it doesnot correspond to the dual of \( 3\text{-Ass} \).

**Corollary 4** The quadratic operad \( 3\text{Ass} \) is not Koszul.
4 The quadratic operad $\widetilde{3A}_{ss}$

In [8] we have defined, given a quadratic operad $\mathcal{P}$, a quadratic operad $\tilde{\mathcal{P}}$ by the following characteristic property:

*For every $\mathcal{P}$-algebra $A$ and every $\tilde{\mathcal{P}}$-algebra $B$, the tensor product $A \otimes B$ is a $\mathcal{P}$-algebra.*

**Proposition 5** We have $\widetilde{3A}_{ss} = 3tA_{ss}$.

Here the product is considered to be of degree 0. In many classical cases, we have seen that $\mathcal{P}^! = \tilde{\mathcal{P}}$. Some examples where this equality is not realized are constructed considering binary non-associative algebras.

**Remark : The operad of Jordan Triple systems.** A Jordan Triple system on a vector space is a 3-ary product $\mu$ satisfying the commutativity condition

$$\mu(x_1, x_2, x_3) = \mu(x_3, x_2, x_1)$$

and

$$\mu(x_1, x_2, \mu(x_3, x_4, x_5)) + \mu(x_3, \mu(x_2, x_1, x_4), x_5) = \mu(\mu(x_1, x_2, x_3), x_4, x_5) + \mu(x_3, \mu(x_1, x_2, x_5)).$$

We denote by $Jord_3$ the operad of algebras defined by a Jordan triple system. In [5] one proves that this quadratic operad is of Koszul computing its dual. But the definition of the dual is based of the nongraded version of [4]. Then they prove that the dual corresponds to $3tA_{ss}$ without degree and we are not sure that the koszulity is satisfied. For instance it is not proved. This calculus using product of degree 1 is making by Nicolas Goze and myself. But the delay imposed to present this paper to publication of the actes of the Algebra, Geometry, and Mathematical Physics Workshop prevents to end this calculus.

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**References**

[1] H. Ataguema, A. Makhlouf. Deformations of ternary algebras. J. Gen. Lie Theory Appl. 1 (2007), no. 1, 41–55.

[2] M. Gerstenhaber. On the deformation of rings and algebras. Ann. of Math. (2) 79 1964 59–103.

[3] V. Ginzburg, M. Kapranov. Koszul duality for operads. Duke Math. J. 76 (1994), no. 1, 203–272.
[4] A. V. Gnedbaye Opérades des algèbres $k+1$-aires. Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995), 83–113, Contemp. Math., 202, Amer. Math. Soc., Providence, RI, 1997.

[5] A.V. Gnedbaye, M. Wambst. Jordan triples and operads. J. Algebra 231 (2000), no. 2, 744–757.

[6] Michel Goze, Elisabeth Remm. Lie-admissible algebras and operads. J. Algebra 273 (2004), no. 1, 129–152.

[7] Michel Goze, Elisabeth Remm. Lie admissible coalgebras. J. Gen. Lie Theory Appl. 1 (2007), no. 1, 19–28.

[8] Elisabeth Remm, Michel Goze. On the algebras obtained by tensor product arXiv:math/0606105.

[9] Nicolas Goze, Elisabeth Remm. $n$-ary associative algebras, cohomology, free algebras and coalgebras. submitted to JGLTA (2008).

[10] Eric Hoffbeck. A Poincaré-Birkhoff-Witt criterion for Koszul operads arXiv:0709.2286v2 and v3.

[11] M. Markl, E. Remm. Déformations des opérades non Koszul. Préprint Mulhouse 2008.

[12] M. Markl, S. Shnider, J. Stasheff. Operads in algebra, topology and physics. Mathematical Surveys and Monographs, 96. American Mathematical Society, Providence, RI, 2002.