Magnetic field generation in first order phase transition bubble collisions

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Abstract

We consider the formation of a ring-like magnetic field in collisions of bubbles of broken phase in an abelian Higgs model. Particular attention is paid on multiple collisions. The small collision velocity limit, appropriate to the electroweak phase transition, is discussed. We argue that after the completion of the electroweak phase transition, when averaged over nucleation center distances, there exists a mean magnetic field \(B \simeq 2.0 \times 10^{20}\) G with a coherence length \(9.1 \times 10^3\) GeV\(^{-1}\) (for \(m_H = 68\) GeV). Because of the ring-like nature of \(B\), the volume average behaves as \(B \sim 1/L\). Taking into account the turbulent enhancement of the field by inverse cascade, we estimate that colliding electroweak bubbles would give rise to a mean field \(B_{rms} \simeq 10^{-21}\) G at 10 Mpc comoving scale today.

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1 Introduction

Cosmological first order phase transitions may give rise to primordial magnetic fields, which then could act as the seed field required for the dynamo explanation of the observed galactic magnetic fields [1]. A first order phase transition proceeds by nucleation of bubbles of broken phase in the background of unbroken phase. In the case of a first order electroweak (EW) phase transition, the Higgs field inside a given bubble has an arbitrary phase. The bubbles expand and eventually collide, while new bubbles are continuously formed, until the phase transition is completed. This also involves the equilibration of the phases of the complex Higgs fields, the gradients of which act as a source for gauge fields, thus making the generations of magnetic fields possible. The growth of the bubble is much affected by the non-linear interplay between the field configuration constituting the bubble and the background plasma. In the EW case hydrodynamical studies show [2, 3] that the expanding bubble is preceded by a shock front, which heats up the universe back to the critical temperature. The transition is then completed by the merging of the slowly expanding bubbles.

There are two different, but not mutually exclusive, theoretical scenarios for generating magnetic fields in first order phase transitions. One employs directly gradients of the complex Higgs field in collisions between bubbles of broken phase, as discussed in [4]. The other is based on the spontaneous appearance of electrical currents and turbulent flow near the bubble walls and has been applied both to QCD [5] and EW [6, 7] phase transitions. In the EW phase transition separation of electric charges occurs at the phase boundary because of baryon number gradients. These give rise to a net current and hence magnetic fields, the fate of which is dependent on the hydrodynamical details of bubble dynamics. The various dynamical features have been studied carefully in [7], where it was argued that field strengths of the order of $10^{-29}$ G on a 10 Mpc comoving scale could be achieved in EW phase transition by this mechanism (and in the case of QCD, even larger fields). As discussed in [8], after the phase transition hydromagnetic turbulence is likely to enhance the seed field by several orders of magnitude, thus making the primordial field a plausible candidate for the seed field.

In the present paper we will focus on magnetic fields created in bubble collisions, following the treatment of Kibble and Vilenkin [4], who showed that in a collision of two bubbles a ring-like magnetic field is formed. (Bubble collisions have also been treated in [9], but mainly with an eye on the defect formation). The starting point is an abelian Higgs model, the properties of which are likely to reflect the properties of the full EW $SU(2) \times U(1)$ model. In Section 2 we discuss the collision of the bubbles, and in particular the multiple collisions, and show that in all cases the resulting magnetic field looks qualitatively the same. In Section 3 we introduce diffusion and consider very slow collision velocities, which are typical to the EW case. We find out the average
field by folding in the spectrum of separation of nucleation centers and performing the volume average over the randomly inclined ring-like magnetic field configurations. In Section 4 we present our conclusions.

2 Bubble collisions

2.1 Basic features

Let us begin by recapitulating some of the features of colliding bubbles in abelian Higgs model [4]. We shall begin by first ignoring diffusion. The starting point is the U(1)-symmetric lagrangian

\[ L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi (D^\mu \Phi)^\dagger + V(|\Phi|), \]

(1)

where \( \Phi \equiv \frac{1}{\sqrt{2}} \rho e^{i\Theta} \) is the complex Higgs field and the potential \( V \) is assumed to have minima at \( \rho = 0 \) and \( \rho = \eta/\sqrt{2} \). The equations of motion for \( \Theta \) and \( \rho \) read

\[ \partial_\mu \partial^\mu \rho - (\partial_\nu \Theta - e A_\nu)^2 \rho + 2 \frac{\partial V}{\partial |\Phi|^2} \rho = 0, \]

\[ \partial_\mu \partial^\mu \Theta + e \partial_\mu A^\mu + 2 (\partial^\mu \Theta - e A^\mu) \partial_\mu \rho \frac{1}{\rho} = 0. \]

(2)

A gauge invariant phase difference can be defined in terms of an integral over the gradient \( D_\rho \Theta \) [4]. Before the collision, the phase angle within each bubble may be taken constant. Following Kibble and Vilenkin [4] we assume that inside the bubble the radial mode \( \rho \) settles rapidly to its equilibrium value \( \eta \) and can thus be treated as a constant. It then follows that a Klein-Gordon equation holds for \( A \) and \( \Theta \) separately:

\[ (\partial_\mu \partial^\mu + e^2 \eta^2) X = 0, \]

(3)

where \( X = A_\mu \) or \( \Theta \). These are not independent but are related by virtue of the Maxwell equations:

\[ \partial_\mu j^\mu = 0, \]

\[ \partial_\mu F^{\mu\nu} = j^\nu, \]

(4)

where \( j^\mu = -e \rho^2 (\partial^\mu \Theta + e A^\mu) \). The simplest case is that two bubbles nucleate, one at \((x, y, z, t) = (0, 0, z_0, 0)\) and the other at \((x, y, z, t) = (0, 0, -z_0, t_0)\), and keep expanding with velocity \( v \) even after colliding. Their radii at the collision are denoted by \( R_1 \) and \( R_2 \). It is easy to find out the intersection volume of the bubbles:

\[ x^2 + y^2 = R_2^2(t) - (z + z_0)^2; \quad z > 0, \]

\[ x^2 + y^2 = R_1^2(t) - (z - z_0)^2; \quad z \leq 0. \]

(5)
Denoting $R_1(t) = vt$ and $R_2(t) = v(t - t_0)$ one may solve for the time of intersection:

$$t = t_I \equiv \frac{t_0}{2} + \frac{z_0^2(v^2t_0^2 - 4z_0^2 - 4(x^2 + y^2))}{v^2(v^2t_0^2 - 4z_0^2)}. \quad (6)$$

In this Section we will always take $v = 1$ and $t_0 = 0$ for simplicity, so that $R_1(t_I) = R_2(t_I) \equiv R$. As we will discuss later, this is not true for EW phase transition where $v \ll 1$, but nevertheless the assumption serves a useful illustrative purpose.

Because of the symmetry in the bubble collision, one can now assume \[4\] that the phase angle $\Theta$ is actually a function of $z$ and $\tau = \sqrt{t^2 - x^2 - y^2}$. Also, the symmetry of the problem dictates that in this case the electromagnetic potential has the form $A^\mu = x^\mu f(\tau, z)$, where $f$ is a function to be determined later. One then finds that in the intersection volume the electric and magnetic fields read (in cylindrical coordinates)

$$E = -2\frac{r}{\tau} \frac{\partial}{\partial \tau} f(\tau, z) e_r - \frac{\partial}{\partial z} f(\tau, z) e_z,$$

$$B = \frac{r}{\tau} \frac{\partial}{\partial z} f(\tau, z) e_{\phi}, \quad (7)$$

where $r \equiv \sqrt{x^2 + y^2}$.

The solutions for $\Theta$ and $f$, in the gauge $A^z = 0$ and with $\rho =$constant, are obtained with the initial conditions \[4\]

$$\Theta|_{\tau = R} = \Theta_0 \epsilon(z - z_1), \quad \partial_\tau \Theta|_{\tau = R} = 0,$$

$$f|_{\tau = R} = 0, \quad \partial_\tau f|_{\tau = R} = \frac{\Theta_0 \epsilon R^2}{\tau} \epsilon(z - z_1), \quad (8)$$

and read

$$\Theta = \frac{\Theta_0 R}{\pi \tau} \int_{-\infty}^{\infty} \frac{dk}{k} \sin k(z - z_1) \{ \cos \omega(\tau - R) + \frac{1}{\omega R} \sin \omega(\tau - R) \},$$

$$f = \frac{\Theta_0 R \epsilon \eta^2}{\pi \tau^3} \int_{-\infty}^{\infty} \frac{dk}{k} \sin k(z - z_1) \{ \frac{R - \tau}{\omega^2 R} \cos \omega(\tau - R)$$

$$+ \left( \frac{\tau}{\omega} + \frac{1}{\omega^3 R} \right) \sin \omega(\tau - R) \}, \quad (9)$$

where $\omega \equiv \sqrt{k^2 + e^2 \eta^2}$ and $R$ is the radius of the bubbles at the collision and $z_1$ the point of first collision on the $z$-axis.

In Fig. 1 and 2 we display the time evolution of the absolute value of the magnetic field and $\Theta$. One can see that $B$ spreads out in the whole intersection region (which in Fig. 1 and 2 correspond to the range $50 - t \lesssim z \lesssim 50 + t$) and oscillates rapidly with increasing frequency as one moves from the center of the regime to the edge. As time goes on, the amplitude of $B$ oscillation decreases as $1/\tau^2$ while the frequency increases. Therefore the mean field at sufficiently large scales, of the order of $10/(e \eta)$, is zero everywhere else except in the middle of the collision regime. The energy density $\int_{-\infty}^{\infty} B^2 dV/V$ can be seen to scale as $1/t$. 

Figure 1: The time evolution of the phase angle $\Theta$ along the radius $r = 1$ for $t=10, 20, 30$ and $40$ after the initial collision. The radius of the bubbles at the collision has been chosen here $R = 10$, and the collision point on the $z$-axis is $z_1 = 50$. (The units are such that $e \eta = 1$).

2.2 Full-empty collisions

After the bubbles have intersected, they will be subject to new collisions. Indeed, first order phase transitions are locally completed by merging of several bubbles. Let us assume that a sufficiently long time has elapsed since the first collision so that we may approximate the result of the collision by a spherical bubble with radius $R_1$. It contains a magnetic field and a phase angle (we shall call it “full” bubble), which are given by Eq. (9). Strictly speaking, these solutions cannot be valid inside the whole full bubble, but we assume that they are valid in the collision region where the full bubble merges with another bubble.

When a full bubble collides with a recently nucleated bubble with no gauge field inside (which we shall call an “empty” bubble), some of symmetries of the empty-empty collision remain. Because of the reduced symmetry, we will now take $\tau = \sqrt{t^2 - x^2}$.

The electromagnetic potential in the full-empty case can be obtained by approximating the initial conditions by $f|_{r=R_1,y=0,z\geq 0} = f_0(1 - e^{-Cz})(1 - e^{-Cz^2})^{-1}H(z_2 - z)$, where $H$ is the Heaviside theta-function, $f_0$ the asymptotic form of the electromagnetic potential and $\partial_r f|_{r=R_1,y=0,z\geq 0} = 0$. $z_2$ is the point of collision on the $z$-axis, which is chosen as the symmetry axis of the collision. These initial conditions should approximate quite well the average behaviour of the gauge potential in the ‘full’ bubble, which
subsequently collides with an empty one assuming that the bubbles are big enough so that the vector potential $A$ can be taken to point to a constant direction in the collision region of the full bubble.

The vector potential is zero at $z = 0$ and $A_i = x_i f_0$ in the asymptotic $z \to \infty$ region, where $f_0$ is given by

$$f_0 = f(z \to \infty) = \frac{\Theta_0 \eta R}{\tau^3} \left[ \frac{R-\tau}{e\eta R} \cos(e\eta(\tau - R)) + \left( \tau + \frac{1}{e\eta R} \right) \sin(e\eta(\tau - R)) \right]. \quad (10)$$

The parameter $C$ can be found by matching the $z$-derivatives of $f_0(1 - e^{-Cz})(1 - e^{-Cz})^{-1}$ and Eq. (9) at $z = z_1 = 0$.

This approximates rather well the overall behaviour of $f$, but does not take into account the rapid oscillations involved. However, when diffusion is included (Sect. 3), it is precisely these rapid oscillations which get diffused. An expression for $A$, obeying the approximate initial conditions, can then be found, but it cannot be expressed in a simple form and we do not reproduce it here. A more tractable picture can be obtained by making the assumption that the empty bubble is much smaller than the full bubble. This also seems a natural assumption. We may then write $f = f_0 + \delta f$, where $\delta f$ is a perturbation and $f_0$ is the asymptotic solution obtained from the collision of two empty bubbles, Eq. (10).
Let us choose a new coordinate system so that the electromagnetic potential in the full bubble can be written in the form
\[
A = R_1 f_0 \sin \alpha e_y ,
\]
where \( f_0 \) is the asymptotic form of \( f \), Eq. (10), \( \alpha \) the intersection angle and \( R_1 \) the radius of the large bubble. Also, we take \( \Theta \) to be at its asymptotic value as given by
\[
\Theta_0 = \frac{\Theta_{00} R}{\tau} \left( \cos(\eta(\tau - R)) + \frac{\sin(\eta(\tau - R))}{\eta R} \right).
\]
Here \( \Theta_{00} \) is the original phase difference of the two initial colliding empty bubbles, and \( R \) refers to the radii of these bubbles. The initial conditions can now be written in the form
\[
\Theta_{\tau=R_1} = \Theta_0 \epsilon(z-z_1) + \Theta_1 , \quad \partial_\tau \Theta_{|\tau=R_1} = 0,
\]
\[
f_{\tau=R_1} = a(1-\epsilon(z-z_1)) , \quad \partial_\tau f_{|\tau=R_1} = b(1-\epsilon(z-z_1)),
\]
where \( a = f_0/2 \) and \( b = -\epsilon R^2 (\Theta_0 + 3f_0/2\eta^2)/R_1 \). Here \( \tau \) is once again \( \tau^2 \equiv t^2 - x^2 - y^2 \) and \( \omega^2 = k^2 + \eta^2 \). One then finds that
\[
\Theta = \frac{\Theta_0 R_1}{\pi \tau} \int_{-\infty}^{\infty} \frac{\pi \Theta_1 \delta(l)}{\Theta_0} \cos l(z-z_1) + \sin l(z-z_1)] \left( \cos \sqrt{\epsilon^2 \eta^2 + l^2 (\tau - R_1)} \right)\,dl ,
\]
and
\[
f = \int_{-\infty}^{\infty} -\frac{1}{\omega^3 \tau^2} \left[ \cos(\omega \tau) \left( 3a \omega^2 R_1 + b \omega^2 R_1^2 \right) + \sin(\omega R_1) \left( a \omega^3 R_1^2 - 3a \omega - b \omega R_1 \right) \right]
\times \left[ \sin(\omega \tau) + \frac{\cos(\omega \tau)}{\omega \tau} \right] + \frac{R_1}{2 (\sin(\omega R_1) - \omega R_1 \cos(\omega R_1))} \left( (a \omega^4 R_1^2 - \cos(2\omega R_1) \right)
\times \left[ 6a \omega^2 + 2b \omega^2 R_1 - a \omega^4 R_1^2 \right] + \sin(2\omega R_1) \left[ b \omega + \frac{3a \omega}{R_1} - 4a \omega^3 R_1 - b \omega^3 R_1^2 \right] \right)
\times \left( -\cos(\omega \tau) + \frac{\sin(\omega \tau)}{\omega \tau} \right) \left[ \frac{1}{\pi k} \right] - \delta(k) \cos k(z-z_1) \right] \,dk .
\]
From the complicated expression Eq. (13) one then finds that approximately
\[
B \simeq R_1 \sin \alpha (\partial_z f) e_x .
\]
To take into account the angle \( \alpha \) between \( B \) in the full bubble and the line of collision, one should substitute \( y \to y \cos \alpha - z \sin \alpha \) and \( z \to z \cos \alpha + y \sin \alpha \). Qualitatively, the situation is however equivalent to the case of empty-empty collision. As can be seen from Eq. (16), the resulting magnetic field is again a ring, although it is slanted by the angle \( \alpha \) with respect to the z-axis. Its time evolution is also similar to what is presented in Fig. 1. Because almost every collision that takes place between empty and full bubbles can be treated in the approximation above, the structure of magnetic fields generated will all be essentially equal.
2.3 Subsequent collisions

As time goes on, more and more bubbles may have collided at least once, and the subsequent collisions take place between already “full” bubbles, i.e. bubbles which contain magnetic fields. In such collisions there are no symmetries left, and the resulting equations of motion are much more complicated compared with the previous cases. However, in the case where one of the bubbles is much larger than the other, it is again possible to approximate the small bubble as a small perturbation on a bubble with infinite radius. We consider a collision where the magnetic fields inside the bubbles point to arbitrary directions. The bubbles may again be taken to collide along the $z$-axis with the gauge potentials almost constant in the colliding region. Because all symmetries are lost, one can not assume that $A_i = x_i f$. In general, however, these collisions can be very well approximated with the initial conditions given by Eq. (13) for $\Theta$ and $A_i$. Therefore, qualitatively the full-full collision does not differ from the empty-empty collision discussed in Sect. 2.1.

Because both $\Theta$ and $A_i$ obey the Klein-Gordon equation, the solutions are simple waves. That is why in the case of two colliding bubbles the qualitative behaviour of the generated magnetic field is found not to depend on whether the bubbles had already collided or not. A more involved situation arises when one considers a simultaneous collision of more than two bubbles. The waves can then interfere and produce more complicated patterns.

To demonstrate this, let us assume that 3 (empty) bubbles move along the $z$-axis and collide simultaneously. We can then use symmetry and again define $A_i = x_i f$. The initial conditions read as in Sect. 2.1, but now for the three bubbles. We have solved the equation of motion numerically, and the results are presented in Fig. 3 and Fig. 4, where we show the time evolution of $\Theta$ and $B$.

A simple way to discuss the collision of several bubble is to assume that one of the bubbles is much larger than the others, as was done in Sect. 2.2. When $\tau \to \infty$, the equation of motion is approximately the wave equation, and thus the small bubbles only create small wave-like perturbations which subsequently interfere with each other. Our conclusion is that, qualitatively, all possible collisions between the bubbles look like collisions of empty bubbles.

3 Colliding electroweak bubbles

3.1 Flux spreading and diffusion

The magnetic field generated in bubble collisions will be imprinted on the background plasma. In the early universe electrical conductivity is high but not infinite. When
Figure 3: The time evolution of $\Theta$ in symmetric collision of three bubbles each having radius $R = 10$ at the collision. Here $r = 1$ and time refers to the time elapsed after the initial collision of the bubbles. The points of initial collision of the bubbles on the $z$-axis are 40 and 60 (and units are $e\eta = 1$).

$1 \text{ GeV} \ll T \lesssim m_W$ it is given by $[10]$ 

$$\sigma \simeq 6.7T$$ \hspace{1cm} (17)

When $T \gg m_W$, and above the electroweak phase transition, one should also account for the $W$, $Z$ and the higgs, but their presence will change $\sigma$ only slightly (note that quarks, including the top, contribute very little $[10]$ to $\sigma$). Therefore we will adopt Eq. (17) as appropriate for electrical conductivity both in the broken and in the unbroken phase. Finite conductivity gives rise to a diffusion term in the MHD equation 

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) - \frac{1}{\sigma} \nabla \times (\nabla \times B)$$ \hspace{1cm} (18)

so that diffusion begins with small scales, and diffusion time $t_d \simeq \sigma L^2$.

In real cosmological phase transitions the velocity of the bubble wall is definitely less than $c$. In such a situation the magnetic flux will escape the region of intersection and penetrate the interior of the bubbles, and eventually the false vacuum outside $[4]$. This issue can be addressed by taking the bubble radius to be $R(t) = vt$ and assuming $r \equiv x^2 + y^2 \ll 1$ so that we may still take $\tau \equiv \sqrt{v^2 t^2 - r^2}$. (Here we assume that the nucleated bubbles have the same radius.) To include dissipation in the equation
Figure 4: The time evolution of $B$ in a symmetric collision of three bubbles each having radius $R = 10$ at the collision. Here $r = 1$ and time refers to the time elapsed after the initial collision of the bubbles. The points of initial collision of the bubbles on the $z$-axis are 40 and 60 (and units are $e\eta = 1$).

Figure 5: $B$ with $v = 1$, $r = 1$, $\sigma = 7$, $R = 10$ and $t = 10$, 20, 30 and 40 after the initial collision. The point of initial collision on the $z$-axis is $z_1 = 50$ (and units are $e\eta = 1$).
of motion properly, we add a conduction current term $j_c^\mu$ to the Maxwell’s equations \cite{footnote} and obtain the following equation of motion for $f(\tau, z)$:

\[
- \left( \frac{r^2 - v^2(\tau^2 + r^2)}{\tau^2} \right) \frac{\partial^2 f}{\partial \tau^2} + \left( \frac{2}{\tau} - \frac{1}{\tau^3} \{r^2(1 + v^2) - 2(\tau^2 + r^2) \} \right) + \sigma \frac{(1 + v^2)\sqrt{\tau^2 + r^2}}{2v\tau} \frac{\partial f}{\partial \tau} - \frac{\partial^2 f}{\partial z^2} + e^2\eta^2 f = 0.
\]

(19)

At the time of intersection, $\tau = R$, with $R \gg 1/T$, so that $1/\sigma \tau \simeq 1/\sigma R \ll 1$. Neglecting higher order terms results in

\[
\left( \frac{v^2\partial^2}{\partial \tau^2} + \frac{\sigma(1 + v^2)}{2v} \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2} + e^2\eta^2 \right) f = 0.
\]

(20)

The consequence of the diffusion term in Eq. (20) is to smooth out the rapid oscillations of $B$, as is demonstrated in Fig. 5, where we display the numerical solution of Eq. (20) for the case $v = 1$. Fig. 5 should be compared with Fig. 2, which does not include diffusion.

As will be discussed below, in electroweak phase transition the bubbles will in fact intersect with non-relativistic velocities. In Fig. 6 we show the velocity effect on the created magnetic field. As can be seen from Fig. 6 (and comparing with Fig. 5), the smaller the velocity, the smaller the magnetic field.

Fig. 6 does not yet take fully into account the MHD equation, which should hold also outside the intersection region. Taking Eq. (18) in conjunction with Eq. (19) results in the escape of the magnetic flux, which is most easily seen in the case of low conductivity ($\sigma = 0$) and is demonstrated in Fig. 7. In Fig. 8 magnetic diffusion is again switched on. Again the magnetic field escapes from the intersection region and moves outwards with the speed of light. Here our conclusion is different from \cite{footnote}, where it was argued that with bubble wall velocities in the range $v \simeq 0.1 - 1$ no
magnetic field should be present outside the collision region. For very low velocities \(v \leq 0.01\) with \(\sigma = 7\) the magnetic field looks however just like the case with \(\sigma = 0\). This is due to the fact that for small bubble wall velocity the magnetic field outside the bubbles has enough time to decay. Note, however, that decay time is dependent on the scale of magnetic structure, i.e. the radii of the bubbles, which in these examples are very small compared to the realistic EW case (the reason being that bubbles with very large radii are not easily tractable by numerical analysis and that in the region of interest, \(\tau \simeq R\), the structure of the magnetic field does not depend essentially on \(R\)).

### 3.2 Electroweak phase transition

The real-time history of a first order electroweak phase transition depends in an essential way on the hydrodynamical, non-equilibrium dynamics. Recent lattice simulations \([11]\) have now provided reliably the parameters pertaining the phase transition in the case \(m_H \lesssim m_W\), and a real-time simulation of the bubble growth and collision has been performed in \([3]\). In the following we shall make use of these results.

When bubbles of broken symmetry are first nucleated, they usually grow rapidly.
Figure 8: $B$ along $r = 1$ with $R = 10$, $\sigma = 7$ and $t = 30$ after collision with $v = 0.8$ and 0.1. For the cases $v = 0.01$ and 0.001 the field looks like in Fig. 6. The point of initial collision on the z-axis is $z_1 = 50$, and the outer edge of the intersection region is at $\simeq 50 \pm vt$. The outer edge of the magnetic field is at $\simeq 50 \pm t$ (and units are $e\eta = 1$).

In the electroweak case the initial growth of the bubble wall is by subsonic deflagration, with velocities of the order of $0.05c$, depending on the assumed friction strength. The wall is preceded by a shock front, which may collide with other bubbles. This results in reheating, oscillations of the bubble radii, but eventually a phase equilibrium is attained. The ensuing bubble growth is very slow and takes place because of the expansion of the universe [3]. Note that because the universe has been reheated back to $T_c$, no new bubbles are formed during the slow growth phase.

As an example, let us choose as the reference point the set of numbers related to the case of a small friction coefficient [3]:

$$r_{ave} = 9.5 \times 10^{-8} t_H \; ; \; \; v = 1.2 \times 10^{-4} \; ; \; \; \delta = 0.03 \; , \; \; (21)$$

where $r_{ave}$ is the average distance of the nucleation centers, $v$ is the velocity of the bubbles after the collision of the shock fronts, and $\delta$ is the fraction of the volume in the broken phase at the onset of the slow bubble expansion (the initial velocity of the wall in this case is $v_{init} = 0.089$). The numbers refer to the case of $m_H = 68$ GeV, a weak first order transition. To be definite, we will adopt $T_c = 100$ GeV and $t_H = 3.55 \times 10^{13} \; \text{GeV}^{-1}$.

In EW phase transition the collision of the bubbles thus takes place with a very slow velocity, and the bubbles are very large. Therefore the discussion in the previous Section is not directly applicable, but we may nevertheless assume that the gross features hold, in particular that the collision between EW bubbles will (for practical purposes) always resemble the collision of two empty bubbles.

In Fig. 6 we displayed the numerical solution to the equation of motion Eq. (20). For large EW bubbles numerical methods are not accurate enough to provide a reliable result. Therefore we need an analytic estimate of Eq. (20). Let us write $f(\tau, z) = \ldots$
b(\tau)c(z) and use the initial conditions \( f(R, z) = 0 \) and \( \partial_\tau f(R, z) = e\eta \Theta_0 \cdot \epsilon(z)/R \), where \( \epsilon(z) \) is the sign-function. Then in the small \( r \equiv \sqrt{x^2 + y^2} \) limit we find that

\[
B = 4\frac{e\eta^2 \Theta_0 v^3 R}{\pi R} e^{\frac{1}{2} \sigma (1 + v^2)(R - \tau)/2v^3} \\
\times \int_{-\infty}^{\infty} \cos(k(z - z_1)) \sinh\left(\frac{1}{2} \sqrt{\sigma^2(1 + v^2)^2 - 16q^2 v^4(\tau - R)/2v^3}\right) \frac{dk}{\sqrt{\sigma^2(1 + v^2)^2 - 16q^2 v^4}}, \tag{22}
\]

where \( z_1 \) is the point of initial collision on the z-axis, \( q^2 \equiv (k^2 + e^2 \eta^2) \) and \( \tau^2 = v^2 t^2 - r^2 \).

From Eq. (22) one sees that \( B \) at extremely small scales (\( L < \sim 4v/\sigma = 7.1 \times 10^{-5}/T \)) decays essentially instantaneously, although strictly speaking one cannot apply MHD at length scales less than the interparticle distance \( \sim 1/T \). At scales \( L > \sim 1/T \) Eq. (22) implies that at a given time \( t_d \approx R/v \) the magnetic field at length scales

\[
L \approx \left(\frac{\sqrt{R^2 + r^2} - R}{v\sigma}\right)^{1/2} \tag{23}
\]

has already decayed. Note that at the outer edges of the intersection region diffusion cannot remove \( B \), unless \( r \approx 0, L \ll R \).

Roughly speaking, at the onset of the slow bubble expansion, the bubbles almost touch. Once they have collided, the phase transition will be completed in a time \( \Delta t \approx O(R/v) \). Although strictly speaking Eq. (22) is valid only for \( r \ll vt \), let us use it to estimate the magnitude of \( B \) at the end of phase transition. From Eq. (24) we can see that the magnetic field has decayed from the centre of the collision region. Eq. (22) tells us that new magnetic field is created only in the region where \( \tau \approx R \). The flux escapes, as discussed in Sect. 3.1, and dissipation outside the bubbles is very slow (at scales much larger than \( L \) in Eq. (23)). Inside the bubbles there is also a region where \( B \) has not yet decayed. The largest field is obtained, however, around \( \tau \approx R \), so that the field looks like a narrow ring around the z-axis at \( z = 0 \) and \( r = \sqrt{v^2 t^2 - R^2} \). From Eq. (22) we then obtain an estimate for the strength of the magnetic field in the ring

\[
B(R) \approx \frac{e\eta^2 v}{R} \sqrt{\gamma^2 + 2\gamma R}, \tag{24}
\]

where we have defined \( \gamma \equiv v\Delta t = O(1)R \). This yields the size and the coherence length \( L_0 \) of the field as

\[
B \approx 4.0\sqrt{\gamma^2 + 2\gamma R/R \text{ GeV}^2 = 2.0 \times 10^{20} \sqrt{\gamma^2 + 2\gamma R/R \text{ G}},
\]

\[
L_0 \approx 6.5 \times 10^5 ((1 + \frac{\gamma^2 + 2\gamma R}{R^2})^{1/2} - 1)^{1/2}/T_c, \tag{25}
\]

where we have assumed \( \Theta_0 = 1, T_c = e\eta = 100 \text{ GeV} \) and used Eq. (21).

Thus we may argue that at the end of EW phase transition there are about \( (t_H/r_{ave})^3 \approx 10^{21} \) magnetic rings within each horizon volume. To obtain a reliable
estimate of the average $B$, we should average over the possible bubble sizes, on which $B$ depends. The spectrum of separation of the adjacent shocked spherical bubbles in a first order phase transition has been estimated in \[12\] and reads

$$
P(R) = \frac{96}{185} \left( \frac{S'}{v_s} \right)^3 R^2 \left[ \exp(-S'R/v_s) \left( \frac{S'R}{2v_s} - \frac{2}{3} \right) + \exp(-2S'R/v_s) \left( \frac{S'R}{4v_s} + \frac{2}{3} \right) \right],$$  \hspace{1cm} (26)

where $v_s = 1/\sqrt{3}$ is the velocity of the shock front, and

$$
S'/v_s \equiv S'(t_f)/v_s = (\pi)^{1/3}/r_{ave} = 4.35 \times 10^{-7} \text{ GeV}
$$  \hspace{1cm} (27)

is the derivative of the tunneling action at nucleation time $t_f$ \[13\], and the number is for the reference values Eq. (21). Assuming that Eq. (26) also gives the distribution of the bubble radii at collision, we thus arrive at the average magnetic field

$$
\langle B \rangle = \int P(R)B(R)dR = 3.1 \times 10^{20} \text{ G},
$$  \hspace{1cm} (28)

where we have taken $\gamma = 2R$ for definiteness.

Thus, on the average, each volume of a radius $r_{ave}$ contains a large ring-like field $\langle B \rangle$, but with the planes of inclination randomly distributed. For large volumes, we should average over all possible inclinations, which corresponds to a random walk on the 2d surface of a sphere. This results in a highly entangled field with a root-mean-square value $\langle B_{rms} \rangle = \sqrt{\sum_{n,k=1}^{N} \langle B_n \rangle \langle B_k \rangle / N^2} = \langle B \rangle L_0/L$. Therefore, the rms-field at a given time and comoving scale $L$ reads

$$
B_{rms}(t, L) = f_T(t) \langle B \rangle \left( \frac{R_{EW}(t)}{R(t)} \right)^2 \frac{L_0(t)}{L} = f_T(t) \langle B \rangle \left( \frac{R_{EW}(t)}{R(t)} \right) \left( \frac{r_{ave}}{L} \right),
$$  \hspace{1cm} (29)

where the factor $f_T(t)$ accounts for the turbulent enhancement of $B$ at large length scales, which persists until the plasma becomes matter dominated and even in the presence of large plasma viscosity (i.e. Silk damping) \[8\]. The reason for such an inverse energy cascade is the non-linear nature of the MHD equations. Effectively, the small magnetic loops will merge to form larger magnetic loops, thereby transferring energy to larger length scales. Numerical simulations, using so-called shell models to simulate the full 3d MHD equations, together with scaling arguments, suggest that $f_T(t) \approx t^p$ with $p \approx 0.25$. The total enhancement may thus be estimated as $f_T \approx 10^6$ (but with somewhat large uncertainty). At the comoving scale of 10 Mpc today we thus find for the set of the reference values Eq. (24) (which implicitly assume that $m_H = 68 \text{ GeV}$)

$$
B_{rms} \approx 10^{-21} \text{ G}.
$$  \hspace{1cm} (30)

(Here we assumed that the transition from radiation dominated to matter dominated era takes place at $t \approx 10^5 \text{ yrs}$). It is encouraging that the magnetic field given by Eq. (30) appears to be of the correct order of magnitude to provide the seed field for the galactic dynamo.
4 Discussion

Although we have completely neglected the non-abelian nature of the electroweak theory, it is unlikely to affect the gross features of true electroweak bubble collisions. They, like in the abelian Higgs model we have considered, are bound to depend mostly on the existence of a phase difference and magnetic diffusion (although the abelian anomaly could be important for magnetic fields [15]). We have not treated the full (and complicated) hydrodynamics of the bubbles either. These may include deformations of the spherical bubbles and kinematic viscosity playing an important part in the dynamical evolution. If the surface tension of the electroweak bubbles is not extremely low, sphericity should be a good approximation even during bubble collisions. The resulting magnetic fields, although not necessarily perfectly ring-like, should be of the same order of magnitude as discussed here. For very low surface tension the situation might be different.

Our treatment is obviously a simplification of the highly complex chain of bubble collisions. However, it serves to emphasize the decisive role the velocity of the colliding bubbles play in magnetic field generation. Velocity itself depends on hydrodynamical details such as the strength of the shock front, and in the final analysis on the parameters of the Higgs potential, in particular on the mass of the higgs. Increasing the higgs mass decreases the velocity of the colliding bubbles so that the magnetic field would also decrease according to Eq. (24). For example, moving from $m_H = 50$ GeV to $m_H = 68$ GeV entails a reduction of $v$ by a factor of ten [3]. At the same time the surface tension decreases by a factor of twenty. It is also known [16] that for high enough higgs mass, the transition is no longer of first order. It therefore seems likely that for large enough higgs mass the present treatment is no longer valid, and no magnetic field is created.

The fact that reasonable assumptions seem to produce primordial magnetic fields which could serve as the seed field for the galactic dynamo nevertheless emphasizes the significance of the electroweak phase transition. There now exist proposals to measure the strength of the intergalactic magnetic field to a very high precision [14]. It is interesting that such measurements could provide information also on the nature of the electroweak phase transition.

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References

[1] P.J.E. Peebles, *The Large Scale Structure of the Universe* (Princeton University Press 1980); I. Wasserman, Astrophys. J. **224** (1978) 337; E. Harrison, *Mon. Not. R. Astr. Soc.* **147** (1970) 279; *ibid.* **165** (1973) 185; for a recent review, see R. Beck, A. Brandenburg, D. Moss, A. Shukurov, and D. Sokoloff, *Ann. Rev. Astron. Astrophys.* **34** (1996) 153.

[2] M. Dine, R. G. Leigh, P. Huet, A. D. Linde, and D. Linde, *Phys. Rev.* **D46** (1992) 550; N. Turok, *Phys. Rev. Lett.* **68** (1992) 1803; B. H. Liu, L. McLerran, and N. Turok, *Phys. Rev.* **D46** (1992) 2668; S. Yu. Khlebnikov, *Phys. Rev.* **D46** (1992) 3223; P. Arnold, *Phys. Rev.* **D48** (1993) 1539; G. Moore and T. Prokopec, *Phys. Rev. Lett.* **75** (1995) 777.

[3] H. Kurki-Suonio and M. Laine, *Phys. Rev. Lett.* **77** (1996) 3951.

[4] T. W. B. Kibble and A. Vilenkin, *Phys. Rev.* **D52** (1995) 679.

[5] B. Cheng and A.V. Olinto, *Phys. Rev.* **D50** (1994) 2421.

[6] G. Baym, D. Bödeker and L. McLerran, *Phys. Rev.* **D53** (1996) 662.

[7] G. Sigl, A.V. Olinto and K. Jedamzik, [astro-ph/9610201](https://arxiv.org/abs/astro-ph/9610201).

[8] A. Brandenburg, K. Enqvist and P. Olesen, *Phys. Rev.* **D54** (1996) 1291; *Phys. Lett.* **B392** (1997) 395.

[9] M. Hindmarsh, A.C. Davis and R. Brandenberg, *Phys. Rev.* **D49** (1994) 1444; A.M. Srivastava, *Phys. Rev.* **D46** (1992) 1353; A. Melfo and L. Perivolaropoulos, *Phys. Rev.* **D52** (1995) 992; A. Ferrera and A. Melfo, *Phys. Rev.* **D53** (1996) 6853.

[10] J.T. Ahonen and K. Enqvist, *Phys. Lett.* **B382** (1996) 40.

[11] K. Kajantie, M. Laine, K. Rummukainen, and M. Shaposhnikov, *Nucl. Phys.* **B466** (1996) 189; M. Gürtler, E.-M. Ilgenfritz, J. Kripfganz, H. Perlt, and A. Schiller, *Nucl. Phys.* **B suppl. 49** (1996) 312; *Nucl. Phys.* **B483** (1997) 383.

[12] B.S. Meyr, C.R. Alcock and G.J. Mathews, *Phys. Rev.* **D43** (1991) 1079.

[13] K. Enqvist, J. Ignatius, K. Kajantie and K. Rummukainen, *Phys. Rev.* **D45** (1992) 3415.

[14] R. Plaga, *Nature* **374** (1995) 430; S. Lee, A. Olinto and G. Sigl, *Astrophys. J. Lett.* **455** (1995) L21.
[15] M. Joyce and M. Shaposnikov, hep-ph/9703005.

[16] K. Kajantie, M. Laine, K. Rummukainen, M. Shaposhnikov, Phys. Rev. Lett. 77 (1996) 2887.