Modified Bernstein’s inequality for sums of independent random variables.

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Abstract

We modify the classical Bernstein’s inequality for the sums of independent centered random variables (r.v.) in the terms of relative tails or moments. We built also some examples in order to show the exactness of offered results.

Key words and phrases.

Probability space, random variables (r.v.), Expectation, Variance, Banach spaces of r.v., tail characteristic, Banach rearrangement invariant (r.i.) functional space consisting on random variables, theory of great deviations, Grand Lebesgue Spaces, slowly varying function, Bernstein’s inequality, ordinary and relative tail of distribution, independence, convexity, Young - Fenchel transform, moments, norm, norming sum of random variables (r.v.)

1 Introduction. Definitions. Statement of problem.

Let \((\Omega = \{\omega\}, \mathcal{B}, P)\) be certain non-trivial probability space with expectation \(E\) and variance \(\text{Var}\). Let also \(\{\xi_i\}, i = 1, 2, \ldots\) be a sequence of centered:
\( E \xi_i = 0 \) independent random variables (r.v.). The classical Bernstein’s inequality, see [3], [4], states that if this sequence satisfies the following condition

\[
\exists \kappa, \nu = \text{const} \in (0, \infty), \forall m = 2, 3, 4 \ldots \Rightarrow E|\xi_i|^m \leq \nu m! \kappa^{m-2}/2,
\]

then

\[
P \left\{ \left| \sum_{i=1}^{n} \xi_i \right| > t \right\} \leq 2 \exp \left\{ -\frac{t^2}{2\nu m + 2\kappa t} \right\}, \quad t \geq 0,
\]

and moreover

\[
P \left\{ \max_{j=(1,2,\ldots,n)} \left| \sum_{i=1}^{j} \xi_i \right| > t \right\} \leq 2 \exp \left\{ -\frac{t^2}{2\nu m + 2\kappa t} \right\}, \quad t \geq 0.
\]

There are many generalizations and applications of this result, especially in the theory of great deviations, see e.g. [1], [2], [18], [23], [24], [40], [41], [42] etc.

We offer in this short report a new approach, a new statement of problem for uniform estimate the tail of distribution of the natural normalized sums of the centered independent random variables, in the terms of the so-called relative tails or moments.

We need to introduce several notations in order to formulate our statement of problem. Put \( \sigma_i = \sqrt{\text{Var}(\xi_i)} \), and suppose henceforth that \( 0 < \sigma_i < \infty \), \( i = 1, 2, \ldots \). Denote in turn

\[
S_n = \frac{\sum_{i=1}^{n} \xi_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}},
\]

so that \( \text{Var}(S_n) = 1 \), a natural norming.

Further, let \( X = (X, \| \cdot \| X) \) be some rearrangement invariant (r.i.) Banach functional space consisting on the numerical valued r.v. \( \{\eta\} \) having finite norms \( \|\eta\|_X < \infty \), for instance the classical Lebesgue - Riesz one \( L_p(\Omega, \mathbf{P}) \), \( p \geq 1 \), Orlicz, Grand Lebesgue Space \( G\psi \), Lorentz, Marcinkiewicz ones etc. Define for arbitrary such a space the so-called its tail characteristic \( T^X(t), t \geq 0 \) as follows:

\[
T^X(t) \overset{\text{def}}{=} \sup_{\tau \in X, \|\tau\| X \leq 1} T_\tau(t),
\]

where as ordinary \( T_\tau(t) \) is defined as usually the tail function for the numerical valued r.v. \( \tau \)

\[
T_\tau(t) := P(|\tau| \geq t), \quad t \geq 0.
\]
As an example: tail characteristic for the classical Lebesgue - Riesz spaces 
\( L(p) = L(p, \Omega) \), \( p, t \geq 1 \) for the sufficiently rich probability space.

**Lemma 1.1.**

\[ T^{L(p)}(t) = t^{-p}, \ p \geq 1, t > 1. \]  
\[ (5) \]

**Proof.** The inequality \( T^{L(p)}(t) \leq t^{-p}, p \geq 1, t > 1 \) follows immediately from the classical Tchebychev - Markov inequality. In order to ground the opposite inequality, let us consider the following example. Let \( \nu \) be a non - negative r.v. with the following distribution
\[ P(\nu = 0) = 1 - t^{-p}, P(\nu = t) = t^{-p}, t, p > 1. \]

Then \( E\nu^p = 1 \), but \( P(\nu = t) = t^{-p} \), Q.E.D.

Evidently, if the non - zero r.v. \( \zeta \) belongs to the space \( X : 0 < ||\zeta||X < \infty \), then

\[ T_\zeta(t) \leq T^X(t/||\zeta||X), t \geq 0. \]  
\[ (6) \]

**Definition 1.1.** The r.i. space \( X = (X, \cdot, ||X||X) \) belongs, by definition, to the class \( B_2 \), write \( X \in B_2 \), iff for arbitrary independent centered r.v. \( \eta_1, \eta_2 \) belonging to the space \( X \) there holds

\[ ||\eta_1 + \eta_2||X \leq \sqrt{(||\eta_1||X)^2 + (||\eta_2||X)^2}. \]  
\[ (7) \]

Following, if the r.v. \( \{\eta_k\}, k = 1, 2, \ldots, n \) are centered and commonly independent, and belonging to space \( X \) of the class \( B_2 \), then

\[ ||\sum_{k=1}^n \eta_k||X \leq \sqrt{\sum_{k=1}^n(||\eta_k||X)^2}, n = 3, 4, \ldots. \]  
\[ (8) \]

For instance, the space \( L_2(\Omega, P) \) belongs to the class \( B_2 \).

**Definition 1.2.** The r.i. space \( X = (X, \cdot, ||X||) \) belongs, by definition, to the class \( WB_2 \), (Weak \( B_2 \) space), write \( X \in WB_2 \), iff there exists a finite constant \( K = K(X) \) such that for arbitrary sequence of commonly independent centered random variables \( \eta_1, \eta_2, \ldots \) belonging to the space \( X \) there holds

\[ ||\sum_{k=1}^n \eta_k||X \leq K(X) \cdot \sqrt{\sum_{k=1}^n(||\eta_k||X)^2}, n = 2, 3, 4, \ldots. \]  
\[ (9) \]
For instance, the space $L^p(\Omega, \mathcal{P})$, $p > 2$ belongs to the class $WB_2$. This assertion follows immediately from the classical Rosenthal’s inequality, see [25], [34]. The exact value of the ”constants” $K(L^p)$, $p > 2$ is calculated in [15], [26]:

$$K(L^p) \leq C[R] \cdot \frac{p}{\ln p}, \quad C[R] \approx 1.77638..., \quad (10)$$

$C(R)$ may be named as a ”Rosenthal’s constant”.

**Definition 1.3.** The pair of two r.i. spaces built as before over source probability space, $X = (X, \| \cdot \|_X)$, $Y = (Y, \| \cdot \|_Y)$, is named a weak pair of the $B_2$ spaces, write $(X,Y) \in V\Phi_2$, iff there exists a finite constant $U = U(X,Y)$ such that for arbitrary sequence of commonly independent centered r.v. $\eta_1, \eta_2, \ldots$ belonging to the space $X$ there holds the following estimate

$$\|\sum_{k=1}^{n} \eta_k\|_Y \leq U(X,Y) \cdot \sqrt{\sum_{k=1}^{n}(\|\eta_k\|_X)^2}, \quad n = 2, 3, 4, \ldots \quad (11)$$

Evidently, if the space $X$ belongs to the class $WB_2$, then $(X,X) \in V\Phi_2$ and herewith $U(X,X) = K(X)$. See also several another examples further.

**Statement of problem.**

Suppose that the source sequence of centered independent random variables $\{\xi_i\}$ is such that for some rearrangement invariant Banach functional space $X = (X, \| \cdot \|_X)$ built over $(\Omega, \mathcal{B}, \mathcal{P}$ there holds $\xi_i \in X$:

$$\gamma_i := \|\xi_i/\sigma_i\|_X < \infty, \quad (12)$$

a relative norm estimate.

Our target is to obtain the uniform norm and following tail estimations for natural normed sums of these variables in the described above relative terms.

**2 Main result.**

**Theorem 2.1.** Assume that $X = (X, \| \cdot \|_X)$, $Y = (Y, \| \cdot \|_Y)$, is the weak pair of r.i. spaces built as before over source probability space: $(X,Y) \in V\Phi_2$. Let also $\{\xi_i\}$, $i = 1, 2, 3, \ldots$ be as before the sequence of independent centered non-zero random variables belonging to the space $X$. Moreover, suppose $\sigma_i^2 \in (0, \infty)$ and that
\[ \kappa := \sup_i \|\xi_i\| X / \sigma_i < \infty. \] (13)

Our proposition:

\[ \sup_n \|S_n\| Y \leq \kappa \cdot U(X, Y) \] (14)

with correspondent uniform tail estimate (6)

\[ \sup_n T_{S_n}(t) \leq T^Y \{ t/[\kappa \cdot U(X, Y)] \}, \ t \geq 0. \] (15)

**Proof.** Denote \( \xi_i = \sigma_i \eta_i, \ i = 1, 2, \ldots; \) then \( \|\eta_i\| X \leq \kappa, \)

\[ S_n = \frac{\sum \xi_i}{\sqrt{\sum \sigma_i^2}} = \frac{\sum \sigma_i \eta_i}{\sqrt{\sum \sigma_i^2}}; \]

\[ \sum \overset{\text{def}}{=} \sum_{i=1}^n. \] We have

\[ \|S_n\| Y = \left\| \frac{\sum \sigma_i \eta_i}{\sqrt{\sum \sigma_i^2}} \right\| Y \leq \frac{U(X, Y)}{\sqrt{\sum \sigma_i^2}} \cdot \sqrt{\sum \sigma_i^2 \|\eta_i\|_i X} \leq \frac{U(X, Y)}{\sqrt{\sum \sigma_i^2}} \cdot \sqrt{\kappa^2 \sum \sigma_i^2} = U(X, Y) \cdot \kappa, \]

Q.E.D.

**Remark 2.1.** A particular case. Suppose here that the space \( X = (X, \| \cdot \|_X) \) belongs to the class \( WB_2. \) If again \( \kappa \in (0, \infty), \) then

\[ \sup_n \|S_n\| X \leq \kappa \cdot K(X) \] (16)

with correspondent uniform tail estimate

\[ \sup_n T_{S_n}(t) \leq T^X \{ t/[\kappa \cdot K(X)] \}, \ t \geq 0. \] (17)

**Remark 2.2.** A more particular case. Suppose now that the space \( X = (X, \| \cdot \|_X) \) belongs to the class \( B_2. \) If as before \( \kappa \in (0, \infty), \) then

\[ \sup_n \|S_n\| X \leq \kappa \] (18)

with correspondent uniform tail estimate

\[ \sup_n T_{S_n}(t) \leq T^X \{ t/\kappa \}, \ t \geq 0. \] (19)
3 Examples. Lebesgue - Riesz and Grand Lebesgue Spaces.

A first example: Lebesgue - Riesz spaces.

Let now \( X = L^s(\Omega, P) \), where \( s \in (2, \infty) \). This space belongs to the class 
\( WB_2 \), and we deduce therefore under our notations and assumptions, in particular, 
\( \sigma_i \in (0, \infty) \),

\[
\sup_n ||S_n||_s \leq C_s \sup_i ||\xi_i/\sigma_i||_s, \quad C_s < \infty.
\]  

We bring now as an examples of these spaces for offered estimates the so-called Grand Lebesgue Spaces (GLS).

Let \( \lambda_0 = \text{const} \in (0, \infty] \) and let \( \phi = \phi(\lambda) \) be an even strong convex function 
defined in the segment \( \lambda \in (-\lambda_0, \lambda_0) \) which takes only positive values when \( \lambda \neq 0 \), 
twice continuously differentiable; briefly \( \phi = \phi(\lambda) \) is a Young-Orlicz function, such that

\[
\phi(0) = 0, \quad \phi'(0) = 0, \quad \phi''(0) \in (0, \infty).
\]

We denote the set of all these Young-Orlicz function as \( \Phi : \Phi = \{\phi(\cdot)\} \).

Definition 3.1.

Let \( \phi \in \Phi \). We say that the centered random variable \( \xi \) belongs to the space 
\( B(\phi) \) iff there exists a constant \( \tau \geq 0 \) such that

\[
\forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbb{E} \exp(\pm \lambda \xi) \leq \exp(\phi(\lambda \tau)).
\]

The minimal non-negative value \( \tau \) satisfying (22) for any \( \lambda \in (-\lambda_0, \lambda_0) \) is 
named \( B(\phi) \)-norm of the variable \( \xi \) and we write

\[
||\xi||_{B(\phi)} \overset{\text{def}}{=} \inf\{\tau \geq 0 : \forall \lambda \in (-\lambda_0, \lambda_0) \Rightarrow \mathbb{E} \exp(\pm \lambda \xi) \leq \exp(\phi(\lambda \tau))\}. \tag{23}
\]

For instance if \( \phi(\lambda) = \phi_2(\lambda) := 0.5 \lambda^2 \), \( \lambda \in \mathbb{R} \), the r.v. \( \xi \) is \textit{subgaussian} and in 
this case we denote the space \( B(\phi_2) \) with Sub. Namely we write \( \xi \in \text{Sub} \) and

\[
||\xi||_{\text{Sub}} \overset{\text{def}}{=} ||\xi||_{B(\phi_2)}.
\]

It is proved in particular that \( B(\phi) \), \( \phi \in \Phi \), equipped with the norm (23) and 
under the ordinary algebraic operations, are Banach rearrangement invariant functional spaces, which are equivalent the so-called Grand Lebesgue spaces as well as
to Orlicz exponential spaces. These spaces are very convenient for the investigation of the r.v. having an exponential decreasing tail of distribution; for instance, for investigation of the limit theorem, the exponential bounds of distribution for sums of random variables, non-asymptotical properties, problem of continuous and weak compactness of random fields, study of Central Limit Theorem in the Banach space, etc.

Let \( g : \mathbb{R} \to \mathbb{R} \) be numerical valued measurable function, which can perhaps take the infinite value. Denote by \( \text{Dom}[g] \) the domain of its finiteness:

\[
\text{Dom}[g] := \{ y, \ g(y) \in (-\infty, +\infty) \}.
\] (24)

Recall the definition \( g^*(u) \) of the Young-Fenchel or Legendre transform for the function \( g : \mathbb{R} \to \mathbb{R} \):

\[
g^*(u) \overset{\text{def}}{=} \sup_{y \in \text{Dom}[g]} (yu - g(y)),
\] (25)

but we will use further the value \( u \) to be only non-negative.

In particular, we denote by \( \nu(\cdot) \) the Young-Fenchel or Legendre transform for the function \( \phi \in \Phi \): \( \nu(x) = \nu[\phi](x) \overset{\text{def}}{=} \sup_{\lambda : |\lambda| \leq \lambda_0} (\lambda x - \phi(\lambda)) = \phi^*(x) \)\]. (26)

It is important to note that if the non-zero r.v. \( \xi \) belongs to the space \( B(\phi) \) then

\[
P(\xi > x) \leq \exp \left( -\nu(x/||\xi||_{\phi}) \right).
\] (27)

The inverse conclusion is also true up to a multiplicative constant under suitable conditions, see e.g. [20].

On the other words,

\[
T^{B(\phi)}(t) \leq \exp \left( -\nu(t) \right), \ t \geq 0.
\] (28)

We recall here for reader convenience some known definitions and facts about the so-called Grand Lebesgue Spaces (GLS) using in this article.

Let \( \psi = \psi(p) \), \( p \in [1, b) \) where \( b = \text{const}, \ 1 < b \leq \infty \) be positive measurable numerical valued function, not necessary to be finite in every point, such that \( \inf_{p \in [1, b)} \psi(p) > 0 \). For instance

\[
\psi_m(p) := p^{1/m}, \ m = \text{const} > 0, \ p \in [1, \infty)
\]
or

\[
\psi^{(b; \beta)}(p) := (b - p)^{-\beta}, \ p \in [1, b), \ b = \text{const}, \ 1 < b < \infty; \ \beta = \text{const} \geq 0.
\]
Definition 3.2.

By definition, the (Banach) Grand Lebesgue Space (GLS) $G\psi = G\psi(b)$, consists of all the real (or complex) numerical valued random variable (measurable functions) $f : \Omega \to R$ defined on whole our space $\Omega$ and having a finite norm

$$|| f || = ||f||_{G\psi} \overset{\text{def}}{=} \sup_{p \in [1,b)} \left[ \frac{|f|^p}{\psi(p)} \right].$$

(29)

The function $\psi = \psi(p)$ is named as the generating function for this space. If for instance

$$\psi(p) = \psi^{(r)}(p) = 1, \ p = r; \ \psi^{(r)}(p) = +\infty, \ p \neq r,$$

where $r = \text{const} \in [1,\infty)$, $C/\infty := 0$, $C \in R$, (an extremal case), then the correspondent $G\psi^{(r)}(p)$ space coincides with the classical Lebesgue - Riesz space $L_r = L_r(\Omega, P)$.

These spaces are investigated in many works, e.g. in [9], [11], [12], [16], [17], [19], [22], [27] - [30] etc. They are applied for example in the theory of Partial Differential Equations [11], [12], in the theory of Probability [7], [29] - [30], in Statistics [27], chapter 5, theory of random fields [19], [30], in the Functional Analysis [27], [28], [30] and so one.

These spaces are rearrangement invariant (r.i.) Banach functional spaces; its fundamental function is considered in [30]. They not coincides in general case with the classical spaces: Orlicz, Lorentz, Marcinkiewicz etc., see [22], [28].

The belonging of some r.v. $f : \Omega \to R$ to some $G\psi$ space is closely related with its tail behavior $T_f(t)$ as $t \to \infty$, see [19], [20]. In detail, define for arbitrary generating function $\psi = \psi(p)$

$$h[\psi](p) = h(p) := p \ \ln \psi(p), \ p \in \text{Dom}[\psi].$$

If $0 \neq \xi \in G\psi$ and if we denote $||\xi||_{G\psi} = c \in (0, \infty)$, then

$$T_\xi(t) \leq \exp\left( -h^*(\ln(t/c)) \right), \ t \geq ce;$$

(30)

and the inverse assertion holds true. Namely, it follows for arbitrary r.v. $\xi$ from the relation (30) that $\xi \in G\psi$.

We conclude as before as a consequence

$$T_{G\psi}(t) \leq \exp\left( -h^*(\ln t) \right), \ t \geq e.$$ 

(31)

Bernstein’s inequality for Grand Lebesgue Spaces.
Let us return to the formulated above problem of modified Bernstein’s inequality for the Grand Lebesgue Spaces.

It is known, see [19], [5] that if the (centered!) r.v. $\xi_i$ are independent and subgaussian, then

$$||\sum_{i=1}^{n} \xi_i||_{\text{Sub}} \leq \sqrt{\sum_{i=1}^{n} ||\xi_i||^2_{\text{Sub}}}.$$  \hfill (32)

A more general statement.

**Definition 3.3.** Let us impose the following important condition on the function $\phi(\cdot)$. Indeed, we assume that the function $\lambda \to \phi(\sqrt{\lambda})$, $\lambda \in [0, \lambda_0)$ is convex. Write: $\phi(\cdot) \in \Phi(\text{conv})$.

**Proposition 3.1,** see [19], [20]. Suppose that $\phi(\cdot) \in \Phi(\text{conv})$. Then the space $B(\phi)$ belongs to the class $B_2$.

Another examples of the functions belonging to this class:

$$\phi_{m,L}(\lambda) \overset{\text{def}}{=} m^{-1} \lambda^m L(\lambda), \; \lambda \geq 1, \; m = \text{const} \geq 2,$$ \hfill (33)

where $L = L(\lambda)$, $\lambda \geq 1$ is positive continuous slowly varying at infinity function.

It is known, see e.g. [21], that as $t \to \infty$

$$\phi^*_{m,L}(t) \sim g_{m,L}(t) \overset{\text{def}}{=} \frac{m-1}{m} t^{m/(m-1)} \cdot L^{-1/(m-1)} \left( t^{1/(m-1)} \right).$$ \hfill (34)

The inverse proposition is also true. Indeed, suppose that some r.v. $\eta$ is such that

$$T_\eta(t) \leq c_1 \exp \left( -g_{m,L}(c_2 t) \right), \; t \geq 1,$$

then

$$\eta \in B_{\phi_{m,L}} : ||\eta||_{B_{\phi_{m,L}}} \leq c_3(c_1, c_2, m, L) < \infty.$$

As a consequence:

**Proposition 3.2.** It follows immediately from Remark 2.2 that if $\phi(\cdot) \in \Phi(\text{conv})$ and

$$\sup_i ||\xi_i/\sigma_i||_{B\phi} = \nu \in (0, \infty),$$

then also

$$\sup_n ||S_n||_{B\phi} \leq \nu$$ \hfill (35)
with correspondent uniform tail estimate

$$\sup_n T_{S_n}(t) \leq T^{B\phi} \{t/\nu\}, \ t \geq 0. \quad (36)$$

**The general case of Grand Lebesgue Spaces.**

Suppose now that there exists a $\psi = \psi(p), \ 1 \leq p < b, \ b = \text{const} \in (1, \infty]$—function such that

$$\sup_i ||\xi_i/\sigma_i||_{G\psi} < \infty.$$  

This generating function $\psi(\cdot)$ may be constructively defined as follows

$$\psi(p) \overset{\text{def}}{=} \sup_i ||\xi_i/\sigma_i||_{p},$$

natural way; if of course it is finite at last for certain value $p > 2$.

Put also

$$\tilde{\psi}(p) \overset{\text{def}}{=} CR \frac{p}{\ln p} \psi(p), \ p \in [2, b),$$

where $b = \sup\{p, \ \psi(p) < \infty \}$; the case when $b = \infty$ can not be excluded.

We conclude by virtue of Rosenthal’s inequality:

**Proposition 3.3.**

$$\sup_n ||S_n||_{G\tilde{\psi}} \leq \sup_i ||\xi_i/\sigma_i||_{G\psi}, \quad (37)$$

with correspondent tail estimation.

**4 Non - improvability of our estimations.**

It is no hard to obtain the lower estimates for introduced here tail function $T_{S_n}(t), \ t \geq 1$. Namely, note that when $\sigma_i = 1, \ i = 1, 2, \ldots$; then the our generalized Bernstein’s sum $S_n$ coincides quite with the classical ones

$$S_n = n^{-1/2} \sum_{i=1}^n \xi_i,$$

for which the lower tails estimations are well known, see [7], [8], [19], [27], [40] etc.

Consider for instance the following example. Let $\{\zeta_i\}, \ i = 1, 2, \ldots$ be a sequence if independent centered identical distributed r.v. such that $\sigma_i = 1$ with the following tails of distributions
\[ P(|\zeta_i| > t) = \exp(-tm), \ t \geq 1, \ m = \text{const} > 1. \]

Then there is a "constant" \( C = C(m) \in (0, \infty) \) such that

\[ \sup_n T_{S_n}(t) \leq \exp\left(-C(m) t^{\min(m,2)}\right), \ t \geq 1, \]

(38)

and the power \( \min(m,2) \) is essentially non-improvable. Therefore, it is true also for the Bernstein’s statement of problem.

5 Multivariate generalization.

We consider in this section the case when the centered independent r.v. \( \{\xi_i\} \) taking value in certain separable Banach space, may be finite dimensional \( R^d \). The Bernstein’s inequality in the classical statement of this problem is investigated in many works: [31], [32], [33], [36] - [39] and so one.

We introduce now the variables

\[ \beta_i := ||\xi_i||_X, \ V_n \overset{\text{def}}{=} \frac{\sum \xi_i}{\sqrt{\sum \beta_i^2}}, \]

and impose as above on the spaces \( X, Y \) in addition the following condition: for arbitrary centered independent r.v. \( \xi, \eta \) belonging to the space \( X \)

\[ ||\xi + \eta||_X \leq \sqrt{||\xi||^2 + ||\eta||^2}. \]

(39)

**Proposition 5.1.** We find as before

\[ \sup_n ||V_n||_Y \leq K(X,Y) \sup_i ||\xi_i||_X. \]

(40)

6 Concluding remark.

It is interest in our opinion to deduce the *maximal* version of our estimations, for instance (40): to find the upper estimate for the variable

\[ \tau(m)(t) \overset{\text{def}}{=} P(\max_{n \in [1,2,\ldots,m]} ||Y_n|| > t), \ t > 1, \]

in the spirit in particular of the [35].

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