AN ENERGY-CONSERVING ULTRA-WEAK DISCONTINUOUS GALERKIN METHOD FOR THE GENERALIZED KORTEWEG-DE VRIES EQUATION

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Abstract. We propose an energy-conserving ultra-weak discontinuous Galerkin (DG) method for the generalized Korteweg-De Vries (KdV) equation in one dimension. Optimal a priori error estimate of order \( k + 1 \) is obtained for the semi-discrete scheme for the KdV equation without the convection term on general nonuniform meshes when polynomials of degree \( k \geq 2 \) are used. We also numerically observe optimal convergence of the method for the KdV equation with linear or nonlinear convection terms.

It is numerically observed for the new method to have a superior performance for long-time simulations over existing DG methods.

1. Introduction

In this paper, we present an energy-conserving discontinuous Galerkin (DG) method for the generalized Korteweg-De Vries (KdV) equation:

\[
\frac{\partial u}{\partial t} + f(u)_x + \epsilon u_{xxx} = 0, \quad (x, t) \in (0, 1) \times (0, T),
\]

\[
u(x, 0) = u_0(x),
\]

with periodic boundary conditions, where \( f \) is a (nonlinear) function of \( u \), and \( \epsilon \neq 0 \) is a constant parameter.

A vast amount of literature can be found on the numerical approximation of the above equation, c.f. references cited in [1]. All types of numerical methods, including finite difference, finite element, finite volume and spectral methods have their proponents. Here, we will confine our attention in finite element methods, in particular, DG methods. The DG methods, c.f. [5], belong to a class of finite element methods using discontinuous piecewise polynomial spaces for both the numerical solution and the test functions. They allow arbitrarily unstructured meshes, and have compact stencils. Moreover, they easily accommodate arbitrary \( h-p \) adaptivity.

Various DG methods can be applied to solve (1.1), including the LDG methods [3,10,11], the direct DG method [12], and the ultra-weak DG method [1,4]. Among these cited references, the methods [1,3,12] are energy conserving, while the methods [4,10,11] are energy dissipative. Energy conserving DG methods were numerically shown [1,12] to outperform their dissipative counterparts for long time simulations. However, the above cited energy-conserving DG methods are numerically observed to be suboptimally convergent for odd polynomial degrees, while

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optimal error estimates have only been proven for even polynomial degrees on uniform meshes [13] for the equation (1.1) without the convection term, i.e. \( f(u) = 0 \). To the best of our knowledge, no optimally convergent, energy-conserving DG methods using odd polynomial degrees have been available for the equation (1.1), even numerically, in the literature.

We fill this gap by presenting a new energy conserving DG method that is proven to be optimally convergent for all polynomial degrees \( k \geq 2 \) for the equation (1.1) without the convection term \( (f(u) = 0) \) on general nonuniform meshes, and is numerically shown to be optimally convergent for the equation (1.1) with linear or nonlinear convection terms. Our method is based on the ultra-weak DG formulation [4], and on the idea of doubling the unknowns in our recent work on optimal energy conserving DG methods for linear symmetric hyperbolic systems [6]. We point out that the choice of ultra-weak DG formulation is not essential, one can equivalently apply the idea to the LDG method [10], and obtain an optimal energy conserving LDG method for the equation (1.1). We specifically note that the polynomial degree \( k \geq 2 \) is required for the third order equation (1.1) in the ultra-weak DG formulation, while the LDG method [10] do not have such order restriction. The ultra-weak DG method is computationally more efficient over the LDG method due to its compactness, c.f. [4].

The rest of the paper is organized as follows. In Section 2, we present and analyze the semi-discrete energy-conserving DG method for the equation (1.1). Numerical results are reported in Section 3. Finally, we conclude in Section 4.

2. The numerical scheme

In this section, we present the energy-conserving DG method for the generalized KdV equation (1.1). We first present and analyze the scheme for (1.1) without the convection term \( (f = 0) \). An optimal error estimate is obtained for this case. We then treat the nonlinear convection term using the square entropy-conserving flux [8].

2.1. Notation and definitions in the one-dimensional case. In this subsection, we shall first introduce some notation and definitions in the one-dimensional case, which will be used throughout this section.

2.1.1. The meshes. Let us denote by \( \mathcal{I}_h \) a tessellation of the computational interval \( I = [0, 1] \), consisting of cells \( I_j = (x_{j-rac{1}{2}}, x_{j+rac{1}{2}}) \) with \( 1 \leq j \leq N \), where

\[
0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N+\frac{1}{2}} = 1.
\]

The following standard notation of DG methods will be used. Denote \( x_j = (x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}})/2 \), \( h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \), and \( \rho = \min_j h_j \). The mesh is assumed to be regular in the sense that \( h/\rho \) is always bounded during mesh refinements, namely, there exists a positive constant \( \gamma \) such that \( \gamma h \leq \rho \leq h \). We denote by \( p_j^+ \) and \( p_j^- \) the values of \( p \) at the discontinuity point \( x_{j+\frac{1}{2}} \) from the left cell, \( I_j \), and from the right cell, \( I_{j+1} \), respectively. In what follows, we employ \( [p] = p^+ - p^- \) and \( \{p\} = \frac{1}{2}(p^+ + p^-) \) to represent the jump and the mean value of \( p \) at each element boundary point. The following discontinuous piecewise polynomials space is chosen as the finite element space:

\[
V_h \equiv V_h^k = \{ v \in L^2(I) : v|_{I_j} \in P^k(I_j), \ j = 1, \ldots, N \}, \quad (2.1)
\]
where \( P^k(I_j) \) denotes the set of polynomials of degree up to \( k \geq 2 \) defined on the cell \( I_j \).

\[ 2.1.2. \text{ Function spaces and norms.} \] Denote \( H^s(I) \) as the space of \( L^2 \) functions on \( I \) whose \( s \)-th derivative is also an \( L^2 \) function. Denote \( \| \cdot \|_{I_j} \) the standard \( L^2 \)-norm on the cell \( I_j \), and \( \| \cdot \|_I \) the \( L^2 \)-norm on the whole interval.

\[ 2.2. \text{ Case } f=0. \] In this subsection, we consider the KdV equation without convection effect \((f=0)\):

\[
  u_t + \epsilon u_{xxx} = 0, \quad (x, t) \in I \times (0, T],
\]

with a smooth periodic initial condition \( u(x, 0) = u_0(x) \) for \( x \in I \), and a periodic boundary condition.

To derive the optimal energy-conserving DG method for this equation, we shall following the idea in our recent work [6] on optimal energy-conserving DG methods for linear symmetric hyperbolic systems by doubling the unknowns with the introduction of an auxiliary zero function \( \phi(x, t) = 0 \). Hence, we consider the following \( 2 \times 2 \) system:

\[
  u_t + \epsilon u_{xxx} = 0, \quad (x, t) \in I \times (0, T],
\]

\[
  \phi_t - \epsilon \phi_{xxx} = 0, \quad (x, t) \in I \times (0, T],
\]

with initial condition \( u(x, 0) = u_0(x) \) and \( \phi(x, 0) = 0 \).

We use the ultra-weak DG method of Cheng and Shu [4] to discretize the above equations. The semi-discrete DG method for (2.3) is as follows. Find, for any time \( t \in (0, T] \), the unique function \((u_h, \phi_h) = (u_h(t), \phi_h(t)) \in V_h^k \times V_h^k \) such that

\[
  \int_{I_j} (u_h)_t v_h dx + D_{u,j}((u_h, \phi_h), v_h) = 0, \quad \quad (2.4a)
\]

\[
  \int_{I_j} (\phi_h)_t \psi_h dx - D_{\phi,j}((u_h, \phi_h), \psi_h) = 0, \quad \quad (2.4b)
\]

holds for all \((v_h, \psi_h) \in V_h^k \times V_h^k \) and all \( j = 1, \ldots, N \), where the bilinear forms

\[
  D_{u,j} = -\epsilon \int_{I_j} u_h v_h x dx \quad (2.4c)
\]

\[
  + \epsilon \hat{u}_h(v_h)^- |_{j+1/2} - \epsilon \hat{u}_h(x)(v_h)^- |_{j+1/2} + \epsilon \hat{u}_h(x)(v_h)^+ |_{j+1/2}
\]

\[
  + \epsilon \hat{u}_h(v_h)^+ |_{j-1/2} - \epsilon \hat{u}_h(x)(v_h)^+ |_{j-1/2} + \epsilon \hat{u}_h(x)(v_h)^- |_{j-1/2}
\]

\[
  D_{\phi,j} = -\epsilon \int_{I_j} \phi_h \psi_h x dx \quad (2.4d)
\]

\[
  + \epsilon \hat{\phi}_h(\psi_h)^- |_{j+1/2} - \epsilon \hat{\phi}_h(x)(\psi_h)^- |_{j+1/2} + \epsilon \hat{\phi}_h(x)(\psi_h)^+ |_{j+1/2}
\]

\[
  + \epsilon \hat{\phi}_h(\psi_h)^+ |_{j-1/2} - \epsilon \hat{\phi}_h(x)(\psi_h)^+ |_{j-1/2} + \epsilon \hat{\phi}_h(x)(\psi_h)^- |_{j-1/2}
\]
and the numerical fluxes are chosen as follows, c.f. [6],
\[ \hat{u}_h = \{ u_h \} + \frac{1}{2} \{ \phi_h \}, \quad \hat{\phi}_h = \{ \phi_h \} + \frac{1}{2} \{ u_h \}, \quad (2.4f) \]
\[ \hat{u}_{h,x} = \{ (u_h)_x \} + \frac{1}{2} \{ (\phi_h)_x \}, \quad \hat{\phi}_{h,x} = \{ (\phi_h)_x \} + \frac{1}{2} \{ (u_h)_x \}, \quad (2.4g) \]
\[ \hat{u}_{h,xx} = \{ (u_h)_{xx} \} + \frac{1}{2} \{ (\phi_h)_{xx} \}, \quad \hat{\phi}_{h,xx} = \{ (\phi_h)_{xx} \} + \frac{1}{2} \{ (u_h)_{xx} \}. \quad (2.4h) \]

The energy conservation property of this scheme is presented in the following theorem.

**Theorem 2.1.** The energy
\[ E_h(t) = \int_I ((u_h)^2 + (\phi_h^2)) dx \]
is conserved by the semi-discrete scheme (2.4) for all time.

**Remark 2.2 (Modified energy).** We specifically remark here that it is the total energy
\[ E_h(t) = \int_I ((u_h)^2 + (\phi_h^2)) dx \]
that is conserved, not the quantity \( \int_I (u_h)^2 dx \). The quantity \( \phi_h \) is an approximation to the zero function.

**Proof.** By repeatedly integrating by parts and using the periodic boundary condition, we have
\[ \sum_{j=1}^{N} \left( - \int_{I_j} u_h (u_h)_{xxx} dx \right) \]
\[ = \sum_{j=1}^{N} (\{ u_h \}\{ (u_h)_x \} - \{ (u_h)_x \}\{ u_h \} + \{ (u_h)_x \}\{ u_h \}) |_{j-1/2}. \]

Taking \( v_h = u_h \) and \( \psi_h = \phi_h \) in the scheme (2.4), and summing over all elements, we get
\[ \frac{1}{2} \int_I (u_h)^2 dx + \Theta = 0, \]
\[ \frac{1}{2} \int_I (\phi_h)^2 dx - \Theta = 0, \]
where
\[ \Theta = \frac{1}{2} \sum_{j=1}^{N} (-\{ \phi_h \}\{ (u_h)_{xx} \} + \{ (\phi_h)_{xx} \}\{ u_h \} - \{ (\phi_h)_{xx} \}\{ u_h \}) |_{j-1/2}. \]

We get the desired result by adding the above equations, and integrating over \( t \). \( \square \)

Now, we turn to the error estimates of the scheme (2.4). We start by introducing a set of projections, similar to the ones used in [4]. We shall use the following left
and right generalized Gauss-Radau projections $P_h^\pm \in P^k(I_j)$ for $k \geq 2$.

\[
\int_{I_j} P_h^\pm u(x) v_h dx = \int_{I_j} u(x) v_h dx \quad \forall v_h \in P^{k-3}(I_j),
\]

(2.5a)

\[
(P_h^\pm u)^\pm = u^\pm \quad \text{at } x_j + \frac{1}{2},
\]

(2.5b)

\[
(P_h^\pm u)^+_x = u^+_x \quad \text{at } x_j + \frac{1}{2},
\]

(2.5c)

\[
(P_h^\pm u)^-_{xx} = u^-_{xx} \quad \text{at } x_j + \frac{1}{2},
\]

(2.5d)

where the first set of equations (2.5a) vanishes if $k = 2$. The following approximation properties of $P_h^\pm$ is well-known, c.f. [4],

\[
\|P_h^\pm u - u\|_{I_j} \leq Ch^{k+1}.
\]

(2.5e)

We shall also use the following coupled projection specifically designed for the DG scheme (2.4). For any function $u, \phi \in H^2(I)$, we introduce the following coupled auxiliary projection $(P_h^{1,*}, P_h^{2,*}) \in [V_k]^2$:

\[
\int_{I_j} P_h^{1,*} u(x) v_h dx = \int_{I_j} u(x) v_h dx \quad \forall v_h \in P^{k-3}(I_j),
\]

(2.6a)

\[
\int_{I_j} P_h^{2,*} \phi(x) v_h dx = \int_{I_j} \phi(x) v_h dx \quad \forall v_h \in P^{k-3}(I_j),
\]

(2.6b)

\[
\left( \|P_h^{1,*} u_h\| + \frac{1}{2} \|P_h^{2,*} \phi_h\| \right)_{j-\frac{1}{2}} = u(x_{j-\frac{1}{2}}),
\]

(2.6c)

\[
\left( \|P_h^{2,*} \phi_h\| + \frac{1}{2} \|P_h^{1,*} u_h\| \right)_{j-\frac{1}{2}} = \phi(x_{j-\frac{1}{2}}),
\]

(2.6d)

\[
\left( \|P_h^{1,*} u_h\|_x + \frac{1}{2} \|(P_h^{2,*} \phi_h)_x\| \right)_{j-\frac{1}{2}} = u_x(x_{j-\frac{1}{2}}),
\]

(2.6e)

\[
\left( \|P_h^{2,*} \phi_h\|_x + \frac{1}{2} \|(P_h^{1,*} u_h)_x\| \right)_{j-\frac{1}{2}} = \phi_x(x_{j-\frac{1}{2}}),
\]

(2.6f)

\[
\left( \|P_h^{1,*} u_h\|_{xx} + \frac{1}{2} \|(P_h^{2,*} \phi_h)_{xx}\| \right)_{j-\frac{1}{2}} = u_{xx}(x_{j-\frac{1}{2}}),
\]

(2.6g)

\[
\left( \|P_h^{2,*} \phi_h\|_{xx} + \frac{1}{2} \|(P_h^{1,*} u_h)_{xx}\| \right)_{j-\frac{1}{2}} = \phi_{xx}(x_{j-\frac{1}{2}}),
\]

(2.6h)

for all $j$.

At a first glance, the projection (2.6) seems to be globally coupled. The following Lemma shows that it is actually an optimal local projection.

**Lemma 2.3.** Let $k \geq 2$. The projection (2.6) is well-defined, and it satisfies

\[
P_h^{1,*} u = \frac{1}{2} (P_h^+(u + \phi) + P_h^-(u - \phi)),
\]

(2.7a)

\[
P_h^{2,*} \phi = \frac{1}{2} (P_h^+(u + \phi) - P_h^-(u - \phi)).
\]

(2.7b)

In particular, it satisfies

\[
\|P_h^{1,*} u - u\|_{I_j} \leq Ch^{k+1}, \quad \text{and} \quad \|P_h^{2,*} \phi - \phi\|_{I_j} \leq Ch^{k+1}.
\]

(2.7c)

**Proof.** The proof is identical to [6, Lemma 2.6], and is omitted for simplicity. We specifically mention that the local structure of the projection is due to the careful choice of the boundary terms (numerical fluxes) in (2.6).
Now, we are ready to state our main result on the error estimate.

**Theorem 2.4.** Assume that the exact solution \( u \) of (2.2) is sufficiently smooth. Let \( u_h \) be the numerical solution of the semi-discrete DG scheme (2.4) for \( k \geq 2 \). Then for \( T > 0 \) there holds the following error estimate

\[
\| u(T) - u_h(T) \|_{L^2(I)} + \| \phi_h(T) \|_{L^2(I)} \leq C(1 + T)h^{k+1},
\]

where \( C \) is independent of \( h \) and \( T \).

**Proof.** The proof is a standard energy argument, again, we skip the details. In particular, we get the following energy identity by the specific choice of the projection (2.6):

\[
\frac{1}{2} \int_I (u_h - P_h^1 \cdot u_h)^2 + (\phi_h - P_h^1 \cdot \phi_h)^2 dx = \int_I (u - P_h^1 \cdot u_h)_t (u_h - P_h^1 \cdot u_h) + (-P_h^1 \cdot \phi_h)_t (\phi_h - P_h^1 \cdot \phi_h) dx.
\]

\[\square\]

**Remark 2.5 (\( \phi_h \) approximates zero).** Note that \( \phi_h \) is an order \( k + 1 \) approximation to the zero function.

**Remark 2.6 (Other numerical fluxes).** There are two other choices of numerical fluxes available in the literature \([1, 4]\), with both methods devised for the original equation (2.2). The flux in \([4]\) is given by

\[
\hat{u}_h = (u_h)^-, \quad \hat{u}_{h,x} = (u_h)^+_x, \quad \hat{u}_{h,xx} = (u_h)^{++}_{xx}.
\]

(2.9)

The resulting method is optimally convergent, but not energy conserving. The flux in \([1]\) is given by

\[
\hat{u}_h = (u_h)^-, \quad \hat{u}_{h,x} = \langle (u_h)_x \rangle, \quad \hat{u}_{h,xx} = (u_h)^{+}_{xx}.
\]

(2.10)

The resulting method is energy conserving but suboptimal on general non-uniform mesh.

### 2.3. Case \( f \neq 0 \)

Now, we consider the generalized KdV equation (1.1). Again, we add a zero auxiliary function \( \phi(x, t) \) and consider the following \( 2 \times 2 \) system

\[
u_t + f(u)_x + \epsilon u_{xxx} = 0, \quad (x, t) \in I \times (0, T],
\]

(2.11a)

\[
\phi_t - \epsilon \phi_{xxx} = 0, \quad (x, t) \in I \times (0, T].
\]

(2.11b)

To properly treat the nonlinear convection term, we shall first recall the unique square entropy-conserving flux, c.f. \([2,9]\), for the scalar conservation law

\[
u_t + f(u)_x = 0,
\]

which reads:

\[
fs(u_L, u_R) = \begin{cases} 
\Psi(u_R) - \Psi(u_L) & \text{if } u_L \neq u_R \\
\frac{\Psi(u_L) - f(u_L)}{u_R - u_L} & \text{if } u_L = u_R,
\end{cases}
\]

(2.12)

where \( \Psi(u) := uf(u) - \int^u f(u) \) is the so-called potential flux function for the square entropy \( U(u) = u^2/2 \). Note that the above choice of the nonlinear flux was used in \([1]\).
We denote the following operator
\[
F_{u,j}(u_h, \phi_h, v_h) = \int_{I_j} f(u_h)(v_h)_x dx - \hat{\phi}_{u,j}(v_h)_{-} |_{j+1/2} + \hat{\phi}_{u,j}(v_h)_{+} |_{j-1/2},
\]
(2.13)
with the numerical flux \(\hat{\phi}_{u,j}(u_h, u_h^+)\) being the square entropy conserving flux given in (2.12). Then, the semi-discrete scheme for equations (2.1) reads: Find, for any time \(t \in [0, T]\), the unique function \((u_h, \phi_h) = (u_h(t), \phi_h(t)) \in V_h \times V_h^k\) such that
\[
\int_{I_j} (u_h)_t v_h dx + F_{u,j}(u_h, \phi_h, v_h) + D_{u,j}(u_h, \phi_h, v_h) = 0, \tag{2.14a}
\]
\[
\int_{I_j} (\phi_h)_t \psi_h dx - D_{\phi,j}(u_h, \phi_h, \psi_h) = 0, \tag{2.14b}
\]
holds for all \((v_h, \psi_h) \in V_h^k \times V_h^k\) and all \(j = 1, \ldots, N\).

It is easy to show that this method is energy-conserving in the sense of Theorem 2.1. An error analysis using the same energy argument as in Theorem 2.4 leads to a suboptimal convergence rate of order \(h^k\). However, numerical results presented in the next section suggest that the method is optimally convergent.

Remark 2.7 (Convective flux). When \(f(u) = u\), it is clear that the flux (2.12) is nothing but the central flux \(f_S(u_L, u_R) = \frac{1}{2}(u_L + u_R)\). The DG method for linear scalar conservation law using a central flux is suboptimal in general, and only optimal for even polynomial degrees on uniform meshes, see [6]. Numerically results for the DG method using the flux (2.12), not shown in this paper, for the Burgers equation \((u_t + uu_x = 0)\) also indicates a similar convergence behavior, namely, suboptimal order \(k\) for odd polynomial degrees, and optimal order \(k+1\) for even polynomial degrees on uniform meshes.

However, the method (2.14) is numerically shown to be optimally convergent on general nonuniform meshes, probably due to the interplay between the convection term and dispersion term.

Remark 2.8 (Alternative auxiliary function). We also tried to add the nonlinear convective term into the auxiliary function
\[
\phi_t - f(\phi)_x - c \phi_{xxx} = 0,
\]
and discretize the resulting convective terms with the following operators
\[
F_{u,j}(u_h, \phi_h, v_h) = \int_{I_j} f(u_h)(v_h)_x dx - \hat{f}_{u,h}(v_h)_{-} |_{j+1/2} + \hat{f}_{u,h}(v_h)_{+} |_{j-1/2},
\]
\[
F_{\phi,j}(u_h, \phi_h, \psi_h) = \int_{I_j} f(\phi_h)(\psi_h)_x dx - \hat{f}_{\phi,h}(\psi_h)_{-} |_{j+1/2} + \hat{f}_{\phi,h}(\psi_h)_{+} |_{j-1/2},
\]
with the numerical fluxes given by
\[
\hat{f}_{u,h} |_{j-1/2} = f_S(u_h^-, u_h^+) + \alpha_{j-1/2}[\phi_h],
\]
\[
\hat{f}_{\phi,h} |_{j-1/2} = f_S(\phi_h^-, \phi_h^+) + \alpha_{j-1/2}[u_h],
\]
where
\[
\alpha_{j-1/2} = \max_{\min\{u_h^+, \phi_h^+\} \leq v \leq \max\{u_h^+, \phi_h^+\}} \text{sign}(\|u_h\|)|f'(v)|.
\]
This choice of fluxes mimic the linear scalar case in [6], and is energy-conserving. While the resulting method is also numerically shown to be optimally convergent, we do observe a larger phase error of the method for long time simulations comparing with the method (2.14). For this reason, we will stick with the formulation in (2.14).

2.4. Time discretization. In this paper, we will simply use the SSP-RK3 [7] time discretization. For the method of lines ODE

\[(u_h)_t = L(u_h),\]

the SSP-RK3 method is given by

\[
\begin{align*}
    u^{(1)}_h &= u^n_h + \Delta t L(u^n_h), \\
    u^{(2)}_h &= \frac{3}{4} u^n_h + \frac{1}{4} u^{(1)}_h + \frac{1}{4} \Delta t L(u^{(1)}_h), \\
    u^{n+1}_h &= \frac{1}{3} u^n_h + \frac{2}{3} u^{(2)}_h + \frac{2}{3} \Delta t L(u^{(2)}_h).
\end{align*}
\] (2.15)

This time discretization is explicit and dissipative. For PDEs containing high order spatial derivatives as (1.1), explicit time discretization is subject to severe time step restriction for stability. We may also use the conservative, implicit Runge-Kutta collocation type methods discussed in [1, Section 4]. We do not pursue this issue since the focus of this paper is on the DG spatial discretization.

3. Numerical results

We present numerical results to assess the performance of the proposed energy-conserving DG method. We also compare results with the (dissipative) DG method of Cheng and Shu [4] denoted as (U), and the (conservative) DG method of Bona et al. [1], denoted as (C). We denote the new method as (A). The RK3 time discretization (2.15) is used, with the time step size \( \Delta t = CFL h^3 \), for a sufficiently small CFL number to ensure stability. All numerical simulation are performed using MATLAB.

**Example 4.1: with no convection.** We consider the following equation

\[ u_t + \frac{1}{4\pi^2} u_{xxx} = 0 \] (3.1)

on a unit interval \( I = [0, 1] \) with initial condition \( u(x, 0) = \sin(2\pi x) \), and a periodic boundary condition. The exact solution is

\[ u(x, t) = \sin(2\pi(x + t)). \]

Table 3.1 lists the numerical errors and their orders for the three DG methods at \( T = 10\Delta t_0 \), where \( \Delta t_0 \) is the time step size on the coarsest mesh. We use \( P^k \) polynomials with \( 2 \leq k \leq 4 \) on a nonuniform mesh which is a 10% random perturbation of the uniform mesh.

From the table we conclude that, one can always observe optimal \((k+1)\)th order of accuracy for both the variable \( u_h \) (which approximate the solution \( u \)) and \( \phi_h \) (which approximate the zero function) for the new energy-conserving DG method (2.4). This validates our convergence result in Theorem 2.4. Moreover, the absolute value of the error is slightly smaller than the optimal-convergent DG method (U) for all polynomial degrees. We also observe suboptimal convergence for the method (C) for all polynomial degrees. We specifically point out that while optimal convergence for the method (C) can be shown for even polynomial degrees
on uniform meshes [1], Table 3.1 shows that such optimality no longer holds on nonuniform meshes, regardless of the polynomial degree.

Table 3.1. The $L^2$-errors and orders for Example 4.1 for the three DG methods on a random mesh of $N$ cells.

| $N$ | $(U)$ $\|u - u_h\|$ Order | $(C)$ $\|u - u_h\|$ Order | $(A)$ $\|u - u_h\|$ Order | $\|\phi_h\|$ Order |
|-----|-------------------------|-------------------------|-------------------------|-------------------------|
| 10  | 6.57e-03 0.00           | 6.67e-03 0.00           | 8.73e-04 0.00           | 3.79e-03 0.00           |
| 20  | 2.21e-03 1.57           | 1.52e-03 2.13           | 1.77e-03 -1.02          | 8.70e-04 2.12           |
| 40  | 2.90e-04 2.93           | 2.00e-04 2.93           | 1.63e-04 3.44           | 1.10e-04 2.99           |
| 80  | 3.61e-05 3.01           | 3.23e-05 2.63           | 1.93e-05 3.08           | 1.28e-05 3.10           |
| $(P^2)$ | 10  | 1.94e-04 0.00           | 2.22e-04 0.00           | 7.99e-05 0.00           | 6.31e-05 0.00           |
|     | 20  | 2.04e-05 3.25           | 6.56e-05 1.76           | 2.19e-05 1.87           | 1.72e-05 1.88           |
|     | 40  | 1.29e-06 3.99           | 2.45e-05 1.42           | 1.06e-06 4.37           | 9.63e-07 4.16           |
|     | 80  | 7.99e-08 4.01           | 2.72e-06 3.17           | 5.93e-08 4.16           | 5.47e-08 4.14           |
| $(P^3)$ | 10  | 4.65e-06 0.00           | 4.71e-06 0.00           | 1.42e-06 0.00           | 2.25e-06 0.00           |
|     | 20  | 2.13e-07 4.45           | 1.44e-07 5.03           | 1.67e-07 3.08           | 1.66e-07 3.76           |
|     | 40  | 5.88e-09 5.18           | 5.78e-09 4.64           | 4.17e-09 5.33           | 4.76e-09 5.12           |
|     | 80  | 1.82e-10 5.01           | 3.47e-10 4.06           | 1.46e-10 4.84           | 1.56e-10 4.93           |
| $(P^4)$ | 10  | 4.65e-06 0.00           | 4.71e-06 0.00           | 1.42e-06 0.00           | 2.25e-06 0.00           |
|     | 20  | 2.13e-07 4.45           | 1.44e-07 5.03           | 1.67e-07 3.08           | 1.66e-07 3.76           |
|     | 40  | 5.88e-09 5.18           | 5.78e-09 4.64           | 4.17e-09 5.33           | 4.76e-09 5.12           |
|     | 80  | 1.82e-10 5.01           | 3.47e-10 4.06           | 1.46e-10 4.84           | 1.56e-10 4.93           |

Example 4.2: with linear convection. We consider the following equation

$$u_t - u_x + \frac{1}{4\pi^2}u_{xxx} = 0 \quad (3.2)$$

on a unit interval $I = [0, 1]$ with initial condition $u(x, 0) = \sin(2\pi x)$, and a periodic boundary condition. The exact solution is $u(x, t) = \sin(2\pi(x + 2t))$.

Table 3.2 lists the numerical errors and their orders for the three DG methods at $T = 10\Delta t_0$, where $\Delta t_0$ is the time step size on the coarsest mesh. We use $P^k$ polynomials with $2 \leq k \leq 4$ on a nonuniform mesh which is a 10% random perturbation of the uniform mesh.

The convergence behavior of all three methods are similar to Example 4.1. However, we do not have a proof of optimality of the methods $(U)$ and $(A)$.

Example 4.3: nonlinear convection: cnoidal wave. We consider the following equation

$$u_t + \frac{1}{2}(u^2)_x + cu_{xxx} = 0 \quad (3.3)$$

on a unit interval $I = [0, 1]$ with periodic boundary conditions. Following [1], we take $\epsilon = 1/24^2$ and choose the exact solution to be the following cnoidal-wave solution:

$$u(x, t) = a \cn^2(4K(x - vt - x_0)),$$

where $\cn(z) = \cn(z : m)$ is the Jacobi elliptic function with modulus $m \in (0, 1)$ and the parameters have the values $a = 192m K(m)^2$ and $v = 64c(2m - 1)K(m)^2$, $x_0 = 0.5$. The function $K = K(m)$ is the complete elliptic integral of the first kind. We use the value $m = 0.9$. 


Table 3.2. The $L^2$-errors and orders for Example 4.2 for the three DG methods on a random mesh of $N$ cells.

|   | (U) | (C) | (A) |
|---|-----|-----|-----|
|   | $N$ | $\|u - u_h\|$ | Order | $\|u - u_h\|$ | Order | $\|u - u_h\|$ | Order | $\|\phi_h\|$ | Order |
| $P^2$ | 10 | 3.57e-03 | 0.00 | 4.59e-03 | 0.00 | 1.27e-03 | 0.00 | 3.10e-03 | 0.00 |
|   | 20 | 1.93e-03 | 0.88 | 1.13e-03 | 2.03 | 1.21e-03 | 0.07 | 1.08e-03 | 1.52 |
|   | 40 | 2.89e-04 | 2.74 | 1.82e-04 | 2.63 | 1.63e-04 | 2.90 | 1.15e-04 | 3.22 |
|   | 80 | 3.61e-05 | 3.00 | 2.67e-05 | 2.77 | 1.94e-05 | 3.07 | 1.30e-05 | 3.16 |
| $P^3$ | 10 | 1.93e-04 | 0.00 | 2.22e-04 | 0.00 | 7.97e-05 | 0.00 | 6.30e-05 | 0.00 |
|   | 20 | 2.05e-05 | 3.24 | 6.66e-05 | 1.74 | 2.18e-05 | 1.87 | 1.72e-05 | 1.87 |
|   | 40 | 1.29e-06 | 3.99 | 2.45e-05 | 1.44 | 1.07e-06 | 4.35 | 9.65e-07 | 4.16 |
|   | 80 | 7.99e-08 | 4.01 | 2.71e-06 | 3.18 | 5.99e-08 | 4.16 | 5.54e-08 | 4.12 |
| $P^4$ | 10 | 4.65e-06 | 0.00 | 4.72e-06 | 0.00 | 1.42e-06 | 0.00 | 2.25e-06 | 0.00 |
|   | 20 | 2.13e-07 | 4.45 | 1.37e-07 | 5.11 | 1.70e-07 | 3.06 | 1.69e-07 | 3.73 |
|   | 40 | 1.29e-08 | 4.45 | 2.45e-05 | 1.44 | 1.07e-06 | 4.35 | 9.65e-07 | 4.16 |
|   | 80 | 7.99e-10 | 4.01 | 3.47e-06 | 4.05 | 1.44e-08 | 4.85 | 1.55e-08 | 4.94 |

Table 3.3 lists the numerical errors and their orders for the three DG methods at $T = 10\Delta t_0$, where $\Delta t_0$ is the time step size on the coarsest mesh. We use $P^k$ polynomials with $2 \leq k \leq 4$ on a nonuniform mesh which is a 10% random perturbation of the uniform mesh.

For Table 3.3, similar conclusion as in Example 4.2 can be drawn.

Table 3.3. The $L^2$-errors and orders for Example 4.3 for the three DG methods on a random mesh of $N$ cells.

|   | (U) | (C) | (A) |
|---|-----|-----|-----|
|   | $N$ | $\|u - u_h\|$ | Order | $\|u - u_h\|$ | Order | $\|u - u_h\|$ | Order | $\|\phi_h\|$ | Order |
| $P^2$ | 10 | 6.57e-03 | 0.00 | 6.67e-03 | 0.00 | 8.73e-03 | 0.00 | 3.79e-03 | 0.00 |
|   | 20 | 2.21e-03 | 1.57 | 1.37e-03 | 5.11 | 1.70e-07 | 3.06 | 1.69e-07 | 3.73 |
|   | 40 | 5.88e-06 | 5.18 | 5.77e-09 | 4.57 | 1.07e-06 | 4.35 | 9.65e-07 | 4.16 |
|   | 80 | 1.82e-10 | 5.01 | 3.47e-10 | 4.05 | 1.44e-08 | 4.85 | 1.55e-08 | 4.94 |
| $P^3$ | 10 | 5.51e-03 | 0.00 | 7.69e-03 | 0.00 | 6.30e-03 | 0.00 | 4.10e-03 | 0.00 |
|   | 20 | 9.99e-05 | 3.85 | 4.59e-04 | 2.84 | 9.48e-05 | 4.14 | 9.39e-05 | 4.13 |
|   | 80 | 6.17e-06 | 4.02 | 5.04e-05 | 3.19 | 5.37e-06 | 4.14 | 5.28e-06 | 4.15 |
| $P^4$ | 10 | 2.20e-03 | 0.00 | 3.08e-03 | 0.00 | 1.33e-03 | 0.00 | 9.42e-04 | 0.00 |
|   | 20 | 7.28e-05 | 4.92 | 2.72e-04 | 3.50 | 1.40e-04 | 3.25 | 1.16e-04 | 3.02 |
|   | 40 | 2.71e-06 | 4.75 | 3.65e-06 | 6.22 | 3.68e-06 | 5.25 | 3.91e-06 | 4.90 |
|   | 80 | 8.47e-08 | 5.00 | 5.55e-07 | 2.72 | 1.28e-07 | 4.85 | 1.11e-07 | 5.14 |

Example 4.4: long time simulation: cnoidal wave. We consider the cnoidal wave problem in Example 4.3, and run the simulation with a much larger final time. We use both DG-$P^2$ space with 20 uniform cells, and DG-$P^3$ space with 10 uniform cells for the simulation. The numerical results at time $T = 5$ of the three DG methods are shown in Figure 1. From this figure, we observe large dissipation.
and dispersion error for the method (U). We also observe poor resolution for the method (C), indicating that the mesh might be too coarse. Finally, we observe relatively the best result for the method (A) for both polynomial degrees.

Figure 1. Numerical solution at $T = 5$ for Example 4.4. Top: method (U). Middle: method (C). Bottom: method (A). Left: DG-$P^2$ space, 20 cells. Right: DG-$P^3$ space, 10 cells. Solid line: numerical solution. Dashed line: exact solution.

4. Concluding remarks

In this paper, we have proposed an energy conserving DG method for the generalized KdV equation in one dimension. The method is proven to be optimal convergent in the absence of the convection term, and is numerically shown to be optimally convergent on general non-uniform meshes.
Numerical comparison of the new method with the DG methods \cite{1,4} for long
time simulations is also presented. The new method is found to be better than
both methods \cite{1,4} in terms of the dissipation and dispersion errors.

Extension of the method to higher order odd-degree PDEs as those considered
in \cite{4,10} poses no particular difficulty. Extensions to multi-dimensions are also
straight-forward. Both will be pursued in our future work.

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