CENTRAL VALUE OF AUTOMORPHIC $L$–FUNCTIONS

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Abstract. We prove a generalization to the totally real field case of the Waldspurger’s formula relating the Fourier coefficient of a half integral weight form and the central value of the $L$-function of an integral weight form. Our proof is based on a new interpretation of Waldspurger’s formula in terms of equality between global distributions. As applications we generalize the Kohnen-Zagier formula for holomorphic forms and prove the equivalence of the Ramanujan conjecture for half integral weight forms and a case of the Lindelof hypothesis for integral weight forms. We also study the Kohnen space in the adelic setting.

1. Introduction

In this paper, we generalize Waldspurger’s formula to the totally real field case, and study some of its applications. We use here the term Waldspurger’s formula to mean an identity relating the Fourier coefficients of a half integral weight form and the central twisted $L$–values of an integral weight form. A key point in this paper is a new interpretation of Waldspurger’s formula. We study the formula in the adelic setting. This setting allows us to interpret Waldspurger’s formula as an equality between two global (adelic) distributions. Roughly, we define two global distributions $I$ and $J$, and obtain the following factorization into products as distributions over local fields:

$$I = c_1 \prod_v I_v, \quad J = c_2 \prod_v J_v.$$ 

Here $c_1, c_2$ are two global constants which can be interpreted respectively as the central $L$-value of an integral weight form and as the square of the Fourier coefficient of a half integral weight form. For the more precise formulas, see Propostions 6.2 and 6.4. Waldspurger’s formula is then just the identity $c_1 = c_2$, (or more precisely, a family of such identities). Interpreted this way, Waldspurger’s formula follows immediately from the comparison between the global distributions $I$ and $J$ and the comparison between the local distributions $I_v$ and $J_v$. The formula fits into a more general family of formulas that should result from the comparison of the global distributions. Works on the theory of relative trace formula have proved or conjectured many such comparisons of global distributions, see [Gu], [J2], [M], [M-R].

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etc. The resulting identities from such comparisons in all these cases should also have arithmetic interest as in the case of Waldspurger’s formula.

We look at some applications of the generalization of Waldspurger’s formula, among them the equivalence of the Ramanujan conjecture for half-integral weight forms and a special case of the Lindelöf hypothesis, (this result is useful in the work of Cogdell-Piatetski-Shapiro-Sarnak on ternary forms [C-PS-S]). Our result is stated in the general setting of automorphic forms. As many of its applications are in terms of holomorphic modular forms over \( \mathbb{Q} \), we also translate our result into this language. In the process of the translation, we generalize the well known Kohnen-Zagier formula, with the restrictions on the quadratic twist dropped (Theorem 10.1).

With our interpretation of the Waldspurger’s formula, it is clear how the proof of the identity will proceed. The equality of the global distributions \( I \) and \( J \) follows from the global theory of relative trace formula ([J1]), while the equalities of the local distributions follow from the corresponding local theory, [B-M1], [B-M2]. The subtle part is to show that we can fit the identities \( c_1 = c_2 \) into a family of identities. There we need to use the beautiful results of Waldspurger on theta correspondence.

Below we describe the content of the paper in more detail.

1.1. An explicit version of Waldspurger’s formula. The Shimura correspondence associates a cusp form \( f(z) \) with integral weight \( 2k \) to a half integral weight cusp form \( g(z) \) with weight \( k + 1/2 \). Waldspurger is the first to described a relation between the twisted central \( L \)-values \( L(f, D, k) \) and the Fourier coefficients of \( g(z) \), [W2]. There are many later versions of Waldspurger’s formula. See for example the papers of Shimura [Sh1], [Sh2], Niwa [N], Katok-Sarnak [Ka-S], Kohnen-Zagier [KZ], [K1], Gross [Gr]. The result of Kohnen and Zagier is probably the easiest to describe, it states: [K1].

Let \( f(z) \) be a new form of weight \( 2k \), square free and odd level \( N \), of trivial character. There is a unique (up to a scalar multiple) weight \( k + 1/2 \) form \( g(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi i nz} \) corresponding to \( f \) and lying in the Kohnen Space ([K2]), such that when \( D \) is a fundamental discriminant with \((-1)^k D > 0\) and \( (D) = w_l \) for all prime divisors \( l \) of \( N \),

(1.1) \[
\frac{|c(D)|^2}{<g,g>} = \frac{(k-1)!}{\pi^k} \left|\frac{D}{k-1/2} \right|2^{\nu(N)} \frac{L(f, D, k)}{<f,f>}
\]

In the above statement, we have adopted the notations in [K1], and \( \nu(N) \) is the number of prime divisors of \( N \), \( w_l \) is the eigenvalue of the Atkin-Lehner involution acting on \( f(z) \).

For the many applications of an equation of type (1.1), see for examples [Iw1], [KZ], [O-Sk], [Lu-Ra].

1.2. Waldspurger’s formula: adelic version. In this paper, we will work with the more general notaion of automorphic representations over a totally
real number field $F$. The relationship between the modular forms and the automorphic representations is as follows. A half-integral weight modular form is a vector in the space of an automorphic representation $\pi$ on $\tilde{SL}_2$, the double cover of $SL_2$. An integral weight modular form with trivial character can be considered as a vector in the space of an automorphic representation $\pi$ on $PGL_2$. The representation theoretic version of the Shimura correspondence is a theta correspondence relating $\pi$ on $PGL_2$ to some $\tilde{\pi}$ on $\tilde{SL}_2$, $[W1]$.

Our first task is to define the constants associated to $\tilde{\pi}$ and $\pi$ that are the analogues of the Fourier coefficients of the modular forms. Such constants are defined in §2. For $D \neq 0 \in F^*$, we define the $D$-th “Fourier coefficients” of $\pi$ and $\tilde{\pi}$ to be $d_{\pi}(S, \psi^D)$ and $d_{\tilde{\pi}}(S, \psi^D)$, (see §2 for notations). Our version of the Waldspurger’s formula is (Theorem 4.1): for a $\pi$ and $D$, there is a corresponding representation $\tilde{\pi}(\pi, D)$ of $\tilde{SL}_2$, such that

\begin{equation}
|d_{\pi}(S, \psi^D)|^2 L^S(\pi, 1/2) = |d_{\tilde{\pi}}(S, \psi^D)|^2.
\end{equation}

Here $L^S(\pi, 1/2)$ is the central (partial) $L$–value.

1.3. The role of Waldspurger’s result on theta correspondence. A clear difference between our formula and (1.1) is that the twisted $L$–value does not appear in our formula. To introduce the twisted $L$–value, we apply the formula to $\pi \otimes \chi_D$ where $\chi_D$ is a quadratic character associated to $D \in F^*$. The problem of course is that $\tilde{\pi}(\pi \otimes \chi_D, D)$ is not necessarily the same for all $D$.

The results of Waldspurger on theta correspondence ([W3]) gives a partition of the set $F^*$ into a finite collection of subsets, such that the representation $\tilde{\pi}(\pi \otimes \chi_D, D)$ remains the same for $D$ in a given subset. The equation (1.2) then gives a formula for the twisted $L$–value $L(\pi \otimes \chi_D, 1/2)$ in terms of the $D$–th Fourier coefficients of $\tilde{\pi}$, as long as $D$ lies in this particular subset.

1.4. Generalization of the Kohnen-Zagier formula. We compare our result with the Kohnen-Zagier formula (1.1). We now understand the conditions on $D$ in the Kohnen-Zagier formula. The condition is to ensure that $D$ lies in a given subset of $Q^*$, so that the half-integral weight form appear in (1.1) remains the same. With this understanding, we see that for $D$ in other subsets of $Q^*$, there should also be another version of the Kohnen-Zagier formula, involving a different half integral weight form.

In our generalization, we associate to $f(z)$ a partition of $Q^*$, and to each subset $X$ of the partition a half integral weight forms $g_X(z)$. We get an equation in the form of (1.1) for each $g_X(z)$, which holds for all fundamental discriminants $D$ that lie in the subset $X$.

In the process of deriving the generalized Kohnen-Zagier formula from our adelic version, we give the adelic interpretation of the concept of Kohnen space. To a half-integral weight eigenform of level $4N$ ($N$ odd) associates
a vector \( \tilde{\varphi} \) in the space of an automorphic representation \( \tilde{\varpi} = \otimes \tilde{\pi}_v \) of \( \widetilde{SL}_2 \),
where \( \tilde{\varphi} = \tilde{\varphi}_\infty \otimes \tilde{\varphi}_2 \otimes \tilde{\varphi}_3 \otimes \ldots \) with \( \tilde{\varphi}_v \), being vectors in the space \( V_{\tilde{\pi}_v} \) of \( \tilde{\pi}_v \).
The vector \( \tilde{\varphi}_2 \) could lie in a two dimensional subspace of \( V_{\tilde{\pi}_2} \). The Kohnen space is a subspace of half-integral weight forms. We show that it is exactly the subspace generated by \( \tilde{\varphi}'s \) whose local component \( \tilde{\varphi}_2 \) lies in a particular one dimensional subspace of \( V_{\tilde{\pi}_2} \).

**Structure of the paper:**
The paper is organized as follows: In § 2, we define the constants \( d_\pi(S, \psi) \) and \( d_{\tilde{\pi}}(S, \psi) \). We recall Waldspurger’s result on theta correspondence in § 3. In § 4 we state the main theorems. The relative trace formula of [J1] is reviewed in § 5. We describe the local theory of the relative trace formula in § 6. The proof of the main theorems are given in § 7. In § 8, we compute some examples of local constants appearing in the identity for \( L(\pi, 1/2) \). The computations are just easy exercises, and the results are used in the translation from adelic language to modular form language of our formula, as well as in a proof in § 7. In § 9, we give a dictionary between the language of representation theory and modular forms. We also give a discussion on the Kohnen space. In § 10, we prove the Kohnen-Zagier formula without the conditions on the fundamental discriminant \( D \).

**Notations and background:**
Let \( F \) be a totally real number field, \( A \) its adele ring. We will use \( v \) to denote places of \( F \). When \( v \) is nonarchimedean, let \( \mathcal{O}_v \) be the ring of integers in \( F_v \), \( P_v \) (or \( P \)) be its prime ideal, \( \varpi \) its uniformizer, and \( q_v \) (or \( q \)) the size of the residue field \( \mathcal{O}_v/P \). We will use \( ||_v \) to denote the metric on \( F_v \).

Let \( G = GL_2 \), \( \tilde{G} = GL_2 \) and \( G' = SL_2 \). We will use \( e \) to denote the identity elements of the groups \( G \), \( G' \) and \( \tilde{G} \). Let \( Z \) be the center of \( G \). Then \( PGL_2 = G/Z \). Let \( B \) be the subgroup of \( GL_2 \) consisting of upper triangular matrices, \( \tilde{B} \) be its lifting to \( \tilde{G} \).

We will use \((g, \pm1)\) to denote an element in \( \tilde{G} \). Let \([*, *] \) be the Hilbert symbol. The multiplication in \( \tilde{G} \) takes the form:

\[
(g_1, 1) \cdot (g_2, 1) = (g_1 g_2, [\frac{x(g_1 g_2)}{x(g_1)}, \frac{x(g_1 g_2)}{x(g_2)} \det g_1])
\]

where for \( g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, x(g) = c \) if \( c \neq 0 \) or \( d \) if \( c = 0 \).

Let \( n(x) = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}, \tilde{n}(x) = (n(x), 1) \). Let \( w = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \tilde{w} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \).

Let \( a = \begin{pmatrix} a \\ 1 \end{pmatrix}, \tilde{a} = \begin{pmatrix} a & a^{-1} \\ 1 & 1 \end{pmatrix} \).

We fix a nontrivial additive character \( \psi \) of \( A/F \). Then \( \psi = \prod_v \psi_v \). For \( D \in F^* \), let \( \psi^D(x) = \psi(Dx) \); let \( \chi_D \) be the quadratic character of \( A^*/F^* \) associated to the field extension \( F(\sqrt{D}) \). At a local place \( v \), for \( D \in F_v^* \), we let \( \chi_D \) be the quadratic character of \( F_v^* \) associated to the extension \( F_v(\sqrt{D}) \).
We will fix measures as follows. The choice of additive $dx$ on $F_v$ measure does not matter for the statement of our theorem. We will however fix it to be self-dual for the character $\psi_v$. The multiplicative measure is $d^*a = (1 - q^{-1})^{-1} \frac{da}{|a_v|}$, where $q$ is the size of the residue field when $v$ is $p$-adic, and $q = \infty$ when $v = \infty$. We fix the measures for $Z \backslash GL_2$ and $\tilde{S}L_2$ as in [B-M1] and [B-M2]. Write $g = z(c)n(x)\varepsilon_a n(y)$ for $g \in G(F_v) - B(F_v)$, then $dg = |a_v|d^* a d^* adx dy$ is the measure on $G(F_v)$. The measure on $Z(F_v)$ is $dz(c) = d^*c$, and we use the resulting quotient measure on $Z(F_v) \backslash G(F_v)$.

For $g \in G'(F_v) - B(F_v) \cap G'(F_v)$ with $g = \tilde{n}(x)\tilde{\omega} \tilde{n}(y)$, we define $dg = |a_v|^2 d^* a d x dy$ to be the measure on $G'(F_v)$.

Define the Weil constant $\gamma(a, \psi_v^D)$ over $F_v$ to satisfy:

$$\int \hat{\Phi}(x)\psi_v^D(ax^2)dx = |a_v|^{-1/2}\gamma(a, \psi_v^D) \int \Phi(x)\psi_v(-a^{-1}x^2)dx$$

where

$$\hat{\Phi}(x) = \int \Phi(y)\psi_v^D(-2xy)dy.$$ 

We let $\tilde{\gamma}(a, \psi_v^D) = \frac{\gamma(a, \psi_v^D)}{|(1, \psi_v^D)|} [-1, a]$.

We use $\pi$ to denote an irreducible cuspidal representation of $G(\mathbf{A})$ with trivial central character, and use $\tilde{\pi}$ to denote an irreducible cuspidal representation of $G'(\mathbf{A})$. $\pi$ can be considered as a representation of $PGL_2(\mathbf{A})$. We have $\pi = \otimes \pi_v$ and $\tilde{\pi} = \tilde{\otimes} \tilde{\pi}_v$ as the restricted tensor products of representations over local fields. We will use $V_\pi$, $V_{\tilde{\pi}}$, $V_{\pi, v}$ and $V_{\tilde{\pi}, v}$ to denote the spaces where the representations $\pi, \tilde{\pi}, \pi_v$ and $\tilde{\pi}_v$ act on respectively.

When $\mu$ is a character of $F_v^*$, we will let $\pi(\mu, \mu^{-1})$ denote the principal series representation of $G(F_v)$ induced from $\mu$. It acts by right translation on the space of functions $\phi$ on $G(F_v)$ that satisfies:

$$\phi(n(a)gaz) = \mu(a)|a_v|^{1/2}\phi(g)$$

We use $\tilde{\pi}(\mu, \psi_v)$ to denote the principal series representation of $G'(F_v)$ that acts on the space of functions $\phi$ of $G'(F_v)$ satisfying:

$$\phi(\tilde{n}(a)gaz) = \mu(a)\tilde{\gamma}(a, \psi_v)|a_v|\phi(g)$$

These representations are unramified if $\mu$ and $\psi_v$ are unramified.

The $L$–function $L(\pi, s)$ is defined in [J-L]. So is the factor $\epsilon(\pi, s) = \prod \epsilon(\pi_v, s, \psi_v)$. We note that $\epsilon(\pi_v, 1/2) = \epsilon(\pi_v, 1/2, \psi_v)$ is independent of the choice of $\psi_v$.

By the well known result on the Shimura-Waldspurger (theta) correspondence ([R-Sc], [Sh1], [W3]), for the given $\psi^D$ there associates a unique irreducible cuspidal representation $\tilde{\pi}(\pi, D) = \Theta(\pi, \psi^D)$ of $G'(\mathbf{A})$ ([W1]). Here $\Theta$ denotes the theta correspondence. Similarly given $\tilde{\pi}$ on $G'$, there is a unique irreducible cuspidal representation $\Theta(\tilde{\pi}, \psi^D)$ on $PGL_2$ ([W1]). We note that the space of $\Theta(\pi, \psi^D)$ and $\Theta(\tilde{\pi}, \psi^D)$ could be zero dimensional. The theta correspondence is also defined locally, which we again denote by $\Theta$. 
We will use $S$ to denote a finite set of local places containing all $v$ which is archimedean or has even residue characteristic. For $v \notin S$, the covering $G'(F_v)$ splits over $SL_2(O_v)$. With this splitting, we consider $SL_2(O_v)$ a subgroup of $G'(F_v)$. Explicitly the embedding of $SL_2(O_v)$ in $G'$ is given by $g \mapsto (g, \kappa(g))$, where $\kappa\left(\begin{array}{cc}a & b \\ c & d \end{array}\right) = [c, d]$ if $|c|_v < 1$ and $c \neq 0$, $\kappa(g)$ equals 1 when $|c|_v = 1$ or $c = 0$.

We will use $\|\varphi\|$ to denote the norm of a vector $\varphi$: if $(\ast, \ast)$ is a Hermitian form on a space $V$, for $\varphi \in V$, let $\|\varphi\| = (\varphi, \varphi)^{1/2}$. We use $\{\delta_i\}$ to denote a set of representatives for the square classes in $F^*_v/(F^*_v)^2$, with $\delta_1 = 1$.

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2. Definition of Two Constants

2.1. A constant associated to $\pi$ on $G$. We define a constant $d_\pi(S, \psi)$ which can be considered as the Fourier coefficient of $\pi$. The constant depends only on the character $\psi$, the choice of the finite set of places $S$ and the choice of Haar measures. Note that the choice of Haar measures and $\psi$ are fixed in the introduction.

2.1.1. Whittaker model on $G$. Let $\pi$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character. Let $V_\pi \subset L^2(Z(\mathbb{A})G(F)\backslash G(\mathbb{A}))$ be the space that $\pi$ acts on. For $\varphi \in V_{\pi}$, let

\begin{equation}
W_{\varphi}(g) = \int_{A/F} \varphi(n(u)g)\psi(-u)du.
\end{equation}

Then the space $\{W_{\varphi}|\varphi \in V_{\pi}\}$ gives the global Whittaker model of $\pi$.

Remark 2.1. When an integral weight form $f$ is considered as a vector $\varphi$ in the space of $V_{\pi}$, its Fourier coefficients are roughly the values of $W_{\varphi}(e)$ for various choices of $\psi$, (see § 9).

For any admissible representation $\pi_v$ of $G(F_v)$, the space of its $\psi_v$-Whittaker functional $L_v : V_{\pi_v} \rightarrow \mathbb{C}$ satisfying:

\begin{equation}
L_v(\pi_v(n(u))v) = \psi_v(u)L_v(v), v \in V_{\pi_v}
\end{equation}

is at most one dimensional. For the $\pi_v$’s appear as local components of $\pi$, such a space is one dimensional. We will fix the linear form $L_v$ for any given $\pi_v$.

Let $S$ be a finite set of places as in the introduction, and contain all places $v$ where $\pi_v$ is not unramified. For $v \notin S$, $\pi_v$ is an unramified representation.
of $G(F_v)$; let $\varphi_{0,v} \in V_{\pi,v}$ be the unique vector fixed under the action of $G(O_v)$ such that $L_v(\varphi_{0,v}) = 1$.

We note that

$$L : \varphi \to W_\varphi(e)$$

is a linear form on $V_\pi$ satisfying (2.2). From the uniqueness of the local Whittaker functionals, $L$ can be expressed as a product of local linear forms $L_v$. There is a constant $c_1(\pi, S, \psi, \{L_v\})$, such that whenever $\varphi = \otimes_{v \in S} \varphi_v \otimes_{v \notin S} \varphi_{0,v}$, (here we fix an identification between $V_\pi$ and the restricted tensor product $\otimes V_{\pi,v}$ where $V_{\pi,v}$ is the space of the local component $\pi_v$)

$$(2.3) \quad W_\varphi(e) = c_1(\pi, S, \psi, \{L_v\}) \prod_{v \in S} L_v(\varphi_v).$$

### 2.1.2. Hermitian forms on $G$

Define a Hermitian form on $V_\pi$ by:

$$(\varphi, \varphi') = \int_{Z(A)G(F) \backslash G(A)} \varphi(g) \overline{\varphi'(g)} dg$$

Over a local place $v$, for a unitary representation $\pi_v$ with a nontrivial Whittaker functional $L_v$, we can define a $G_v$-invariant Hermitian form on $V_{\pi,v}$ by:

$$(v, v') = \int_{F^*} L_v(\pi_v(a)v) \overline{L_v(\pi_v(a)v') \frac{da}{|a_v|}}$$

(see [Go]). From the uniqueness of $G_v$-invariant Hermitian forms, we get: there is a constant $c_2(\pi, S, \psi, \{L_v\}) > 0$, such that whenever $\varphi = \otimes_{v \in S} \varphi_v \otimes_{v \notin S} \varphi_{0,v}$,

$$(2.6) \quad ||\varphi|| = c_2(\pi, S, \psi, \{L_v\}) \prod_{v \in S} ||\varphi_v||.$$

### 2.1.3. The constant $d_\pi(S, \psi)$

**Lemma 2.2.** The constant $d_\pi(S, \psi)$ defined by

$$d_\pi(S, \psi) = |c_1(\pi, S, \psi, \{L_v\})|/c_2(\pi, S, \psi, \{L_v\})$$

is independent of the choice of the linear forms $L_v$.

**Proof.** From the uniqueness of local Whittaker functionals, any other choice of linear forms $L'_v$ must have the form $L'_v = a_v L_v$ with $a_v$ some nonzero complex constants. From the definition, we get

$$c_2(\pi, S, \psi, \{L'_v\}) = \prod_{v \in S} |a_v|^{-1} c_2(\pi, S, \psi, \{L_v\}),$$

$$c_1(\pi, S, \psi, \{L'_v\}) = \prod_{v \in S} a_v^{-1} c_2(\pi, S, \psi, \{L_v\}).$$

Thus the constant $d_\pi(S, \psi)$ is independent of the choice of $\{L_v\}$. \qed
This is the “Fourier coefficient” we associate to $\pi$. The constant $d_\pi(S, \psi)$ is well defined as we fixed the choice of $\psi$ and the measure on $G$, (it is easy to check that the constant is independent of the choice of additive measure). Explicitly for any vector $\varphi = \otimes_{v \in S} \varphi_v \otimes_{v \notin S} \varphi_{0,v}$ with $L_v(\varphi_v) \neq 0$ for $v \in S$,

\begin{equation}
(2.7) \quad d_\pi(S, \psi) = \frac{|W_\varphi(e)|}{||\varphi||} \prod_{v \in S} \frac{|\varphi_v|}{|L_v(\varphi_v)|}.
\end{equation}

We make an observation on the dependence of $d_\pi(S, \psi)$ on $\psi$.

**Lemma 2.3.** Let $D \in F^*$. If $S$ is large enough such that $|D|_v = 1$ for all $v \notin S$, then

$$d_\pi(S, \psi) = d_\pi(S, \psi^D).$$

**Proof.** Take a vector $\varphi = \otimes_{v \in S} \varphi_v \otimes_{v \notin S} \varphi_{0,v}$ in the space of $\pi$. We will let $L_v^D(\varphi_v) = L_v(\pi_v(D)\varphi_v)$. Then $L_v^D$ is a nontrivial $\psi_v^D$-Whittaker functional on $\pi_v$. Let $||\varphi_v||_D$ be the norm of $\varphi_v$ defined by (2.5) with $L_v$ replaced by $L_v^D$. When $v \notin S$, clearly $L_v(\pi_v(D)\varphi_{0,v}) = L_v(\varphi_{0,v}) = 1$. Thus using the above explicit form,

$$d_\pi(S, \psi^D) = \frac{|W_\varphi^D(e)|}{||\varphi||} \prod_{v \in S} \frac{|\varphi_v|_D}{|L_v^D(\varphi_v)|}.$$

From a simple change of variable we get $W_\varphi^D(e) = W_{\pi(D)}^D(e)$. By the uniqueness of local Whittaker functional,

$$\frac{|W_\varphi^D(e)|}{\prod_{v \in S} |L_v^D(\varphi_v)|} = \frac{|W_{\pi(D)}^D(e)|}{\prod_{v \in S} |L_v^D(\pi_v(D)\varphi_v)|} = \frac{|W_\varphi(e)|}{\prod_{v \in S} |L_v(\varphi_v)|}.$$

From (2.5), $||\varphi_v||_D = ||\varphi_v||$. Thus we get the equality in the Lemma. \qed

### 2.2. A constant associated to $\tilde{\pi}$ on $G^*$

Let $\tilde{\pi}$ be an irreducible cuspidal automorphic representation of $G^*$. We associate a constant $d_{\tilde{\pi}}(S, \psi^D)$ to $\tilde{\pi}$ in a similar fashion. Let $V_{\tilde{\pi}}$ be the space of automorphic forms that $\tilde{\pi}$ acts on. For $\tilde{\varphi} \in V_{\tilde{\pi}}$, let

$$\tilde{W}_\tilde{\varphi}^D(g) = \int_{A/F} \tilde{\varphi}(\tilde{n}(x) \cdot g) \psi^D(-x) dx.$$

Then the Fourier coefficients of half integral weight form can be interpreted as some $\tilde{W}_\tilde{\varphi}^D(e)$, (see equation (9.2)).

We will assume $\tilde{\pi}$ has a nontrivial $\psi^D$-Whittaker model, namely $\tilde{W}_\tilde{\varphi}^D(g) \neq 0$ for some $\tilde{\varphi} \in V_{\tilde{\pi}}$. Then locally, $\tilde{\pi}_v$ has a nontrivial $\psi_v^D$-Whittaker model, unique up to a scalar multiple. We will fix the corresponding linear forms $\tilde{L}_v^D$ satisfying for all $\tilde{\varphi}_v v \in V_{\tilde{\pi}_v}$,

$$\tilde{L}_v^D(\tilde{\pi}_v(\tilde{n}(x))\tilde{\varphi}_v) = \tilde{L}_v^D(\tilde{\varphi}_v)\psi_v^D(x).$$

Let $S$ be a finite set of places as in the introduction, and contain all places $v$ where $\tilde{\pi}_v$ is not unramified. When $v \notin S$, $\tilde{\pi}_v$ is unramified and possess a
nontrivial $\psi_v^D$-Whittaker model $\tilde{L}_v^D$; there is a unique vector in $\varphi_{0,v}$ that is fixed under $SL_2(O_v)$ and satisfying $\tilde{L}_v^D(\varphi_{0,v}) = 1$.

The space $V_{\tilde{\pi}}$ has the Hermitian form

$$ (\tilde{\varphi}, \tilde{\varphi}') = \int_{SL_2(F) \backslash G'(A)} \tilde{\varphi}(g) \overline{\tilde{\varphi}'(g)} dg. $$

Over the space $V_{\tilde{\pi},v}$, one can define a Hermitian form similar to (2.5), though the definition is more complicated. Let $\{\delta_i\}$ be a set of representatives of $F_v^* / (F_v^*)^2$, with $\delta_1 = 1$. From [B-M1],[B-M2], we see there is a choice of $\psi_v^{D\delta_i}$-Whittaker models $\tilde{L}_v^{D\delta_i}$ (could be trivial) on $V_{\tilde{\pi},v}$, such that $\tilde{L}_v^{D\delta_i} = \tilde{L}_v^D$ and

$$ (\tilde{\varphi}_v, \tilde{\varphi}'_v) = \sum_{\delta_i} \frac{|2|_v}{2} \int \tilde{L}_v^{D\delta_i}(\tilde{\pi}(a)\tilde{\varphi}_v) \overline{\tilde{L}_v^{D\delta_i}(\tilde{\pi}(a)\tilde{\varphi}'_v)} \frac{da}{|a|_v} $$

is a $G'_v$-invariant Hermitian form. (We used the factor $\frac{|2|_v}{2}$ to be consistent with [B-M1] and [B-M2]. There we defined the Hermitian form on $GL_2$ first and restricted the form to $SL_2$. See section (9.7) of [B-M1]). For some explicit constructions of this form, see § 8.

We can now define the constant $d_{\tilde{\pi}}(S, \psi^D)$. From the uniqueness of Hermitian forms and Whittaker models, there exist constants $\tilde{c}_1(\tilde{\pi}, S, \psi^D, \{\tilde{L}_v^D\})$ and $\tilde{c}_2(\tilde{\pi}, S, \psi^D, \{\tilde{L}_v^D\})$ such that whenever $\tilde{\varphi} = \otimes_{v \in S} \tilde{\varphi}_v \otimes \varphi_{0,v}$ (under an identification between $V_{\tilde{\pi}}$ and the restricted tensor product $\otimes V_{\tilde{\pi},v}$):

$$ W_{\tilde{\varphi}}^D(e) = \tilde{c}_1(\tilde{\pi}, S, \psi^D, \{\tilde{L}_v^D\}) \prod_{v \in S} \tilde{L}_v^D(\varphi_v), $$

$$ ||\tilde{\varphi}|| = \tilde{c}_2(\tilde{\pi}, S, \psi^D, \{\tilde{L}_v^D\}) \prod_{v \in S} ||\varphi_v||. $$

As in the case of Lemma 2.2, we have:

**Lemma 2.4.** The constant

$$ d_{\tilde{\pi}}(S, \psi^D) = |\tilde{c}_1(\tilde{\pi}, S, \psi^D, \{\tilde{L}_v^D\}) / \tilde{c}_2(\tilde{\pi}, S, \psi^D, \{\tilde{L}_v^D\})| $$

is independent of the choice of the linear forms $\tilde{L}_v^D$.

When $\tilde{\pi}$ does not have a nontrivial $\psi^D$-Whittaker model, we will set $d_{\tilde{\pi}}(S, \psi^D) = 0$. The constant $d_{\tilde{\pi}}(S, \psi^D)$ is well defined with our fixed choice of $\psi^D$ and the measure on $G'$, (and again it is independent of the choice of additive measure). Explicitly for any vector $\tilde{\varphi} = \otimes_{v \in S} \tilde{\varphi}_v \otimes \varphi_{0,v}$ with $\tilde{L}_v^D(\tilde{\varphi}_v) \neq 0$ for $v \in S$,

$$ d_{\tilde{\pi}}(S, \psi^D) = \frac{|W_{\tilde{\varphi}}^D(e)|}{||\tilde{\varphi}||} \prod_{v \in S} \frac{||\varphi_v||}{|L_v^D(\varphi_v)|}. $$

3. Results on the theta correspondence

As stated in the introduction, the generalization of the Shimura correspondence between the half integral weight modular forms and integral weight modular forms is the theta correspondence between automorphic representations of $PGL_2(\mathbb{A})$ and $\widetilde{SL}_2(\mathbb{A})$. The theta correspondence is dependent on the choice of the additive character $\psi$. With a fixed $\psi$, we denote by $\Theta(\pi, \psi)$ the representation of $\widetilde{SL}_2(\mathbb{A})$ associated to $\pi$ of $PGL_2(\mathbb{A})$, and $\Theta(\pi_v, \psi_v)$ the representation of $\widetilde{SL}_2(F_v)$ associated to $\pi_v$ of $PGL_2(F_v)$ under the theta correspondence. Conversely, the theta correspondence associates to $\pi$ of $\widetilde{SL}_2(\mathbb{A})$ and $\pi_v$ of $\widetilde{SL}_2(F_v)$ representations $\Theta(\pi, \psi)$ of $PGL_2(\mathbb{A})$ and $\pi_v = \Theta(\pi_v, \psi_v)$ of $PGL_2(F_v)$.

In this paper, of particular importance are the representation $\tilde{\pi}^D = \Theta(\pi \otimes \chi_D, \psi^D)$ and its local counterpart $\tilde{\pi}_v^D = \Theta(\pi_v \otimes \chi_D, \psi_v^D)$. In this section, we recall Waldspurger’s beautiful results on these representations. The works in [W1], [W3] tell us that the set $\{\tilde{\pi}^D\}$ (or $\{\tilde{\pi}_v^D\}$) is finite, moreover the dependence of $\tilde{\pi}^D$ (or $\tilde{\pi}_v^D$) on $D$ is also given.

We first recall Waldspurger’s local theory. Fix a place $v$ of $F$. Let $P_{0,v}$ be the set of special or supercuspidal representations (or discrete series representations when $F_v = \mathbb{R}$) of $PGL_2(F_v)$. For $D \in F_v^*$, define $(\frac{D}{\pi_v}) \in \pm 1$ by:

$$(\frac{D}{\pi_v}) = \chi_D(-1)\epsilon(\pi_v, 1/2)/\epsilon(\pi_v \otimes \chi_D, 1/2).$$

We then get a partition of $F_v^* = F_v^+ \cup F_v^-$ where

$$F_v^\pm(\pi_v) = \{D \in F_v^* | (\frac{D}{\pi_v}) = \pm 1\}.$$

**Theorem 3.1.** [W3]

When $\pi_v \notin P_{0,v}$, $F_v^+ = F_v^*$ and $\tilde{\pi}_v^D = \Theta(\pi_v, \psi_v)$.

When $\pi_v \in P_{0,v}$, there are two representations $\tilde{\pi}_v^+$ and $\tilde{\pi}_v^-$ of $G_v'$, such that $\tilde{\pi}_v^D = \tilde{\pi}_v^+$ when $D \in F_v^+$, and $\tilde{\pi}_v^D = \tilde{\pi}_v^-$ when $D \in F_v^-$.

Moreover $\tilde{\pi}_v^D = \Theta(\pi_v, \psi_v)$ if and only if $\Theta(\pi_v, \psi_v)$ has a nontrivial $\psi_v^D$-Whittaker model.

We now state the global counterpart of this Theorem. Let $\tilde{A}_{00}$ be the space of cuspidal automorphic forms on $G'(\mathbb{A})$ that are orthogonal to the theta series generated by quadratic forms of one variable. Let $A_{0,1}$ be the subspace of cuspidal automorphic forms on $PGL_2(\mathbb{A})$, such that for any $\pi$ subrepresentation of $A_{0,1}$, there is $D \in F^*$, with $L(\pi \otimes \chi_D, 1/2) \neq 0$.

For $\tilde{\pi}_1, \tilde{\pi}_2$ irreducible subrepresentations of $\tilde{A}_{00}$, we will say $\tilde{\pi}_1 \sim \tilde{\pi}_2$ if they are near equivalent, that is at almost all places $v$, $\tilde{\pi}_{1,v} \cong \tilde{\pi}_{2,v}$. Denote by $\tilde{A}_{00}$ the quotient of $\tilde{A}_{00}$ by this relation.
Let $\Sigma = \Sigma(\pi)$ be the set of places $v$ where $\pi_v \in P_{0,v}$. Given $D \in F^*$, let $\epsilon(D, \pi) = (\frac{D_v}{\pi_v})_{v \in \Sigma}$. Then $\epsilon(D, \pi) \in \{\pm 1\}^{[\Sigma]}$. We have

\begin{equation}
\epsilon(\pi \otimes \chi_D, 1/2) = \epsilon(\pi, 1/2) \prod_{v \in \Sigma} (\frac{D_v}{\pi_v}).
\end{equation}

We will use $\epsilon = (\epsilon_v)_{v \in \Sigma}$ to denote an element in $\{\pm 1\}^{[\Sigma]}$, with $\epsilon_v \in \{\pm 1\}$. Given such an $\epsilon$, we will let $F^*(\pi)$ to be the set of $D \in F^*$ with $\epsilon(D, \pi) = \epsilon$. Then we get a partition $F^* = \bigcup_{\epsilon \in \{\pm 1\}^{[\Sigma]}} F^*(\pi)$.

**Theorem 3.2.** [W3]

1. (Relation with local correspondence) When $\Theta(\tilde{\pi}, \psi) \neq 0$, $\Theta(\tilde{\pi}, \psi) \cong \otimes_v \Theta(\tilde{\pi}_v, \psi_v)$. When $\Theta(\pi, \psi) \neq 0$, $\Theta(\pi, \psi) \cong \otimes_v \Theta(\pi_v, \psi_v)$.

2. (Nonvanishing of the correspondence) $\Theta(\pi, \psi) \neq 0$ if and only if $L(\pi, 1/2) \neq 0$. $\Theta(\tilde{\pi}, \psi) \neq 0$ if and only if $\tilde{\pi}$ has a nontrivial $\psi$-Whittaker model.

3. (Correspondence as a bijection) For $\tilde{\pi}$ an irreducible subrepresentation of $A_{00}$, there is a unique $\pi$ associated to $\tilde{\pi}$, such that whenever $\Theta(\tilde{\pi}, \psi^D) \neq 0$, $\Theta(\tilde{\pi}, \psi^D) \otimes \chi_D = \pi$. Denote this association by $\pi = S_\psi(\tilde{\pi})$. This association defines a bijection between $A_{00}$ and $A_{0,i}$.

4. (Description of near equivalent class). If $\pi = S_\psi(\tilde{\pi})$, the near equivalence class of $\tilde{\pi}$ consists of all the nonzero $\pi^D$'s.

5. (Dependence of $\pi^D$ on $D$). Let $\epsilon \in \{\pm 1\}^{[\Sigma]}$. If $\prod_{v \in \Sigma} \epsilon_v \neq \epsilon(\pi, 1/2)$, then $\pi^D = 0$ for all $D \in F^*(\pi)$. If $\prod_{v \in \Sigma} \epsilon_v = \epsilon(\pi, 1/2)$, then there is a unique $\tilde{\pi}^\epsilon$ such that for $D \in F^*(\pi)$, $\tilde{\pi}^D = \tilde{\pi}^\epsilon$ when $L(\pi \otimes \chi_D, 1/2) \neq 0$ and $\tilde{\pi}^D = 0$ otherwise.

For convenience, if $\prod_{v \in \Sigma} \epsilon_v \neq \epsilon(\pi, 1/2)$, we set $\tilde{\pi}^\epsilon = 0$.

**4. Statement of the main results**

4.1. **The formula for** $L(\pi, 1/2)$. The definition of the $L$–function $L(\pi, s)$ and the local $L$–functions $L(\pi_v, s)$ can be found in [J-L]. Fix a finite set of places $S$, we use $L^S(\pi, s)$ to denote the partial $L$–function $\prod_{v \in S} L(\pi_v, s)$.

**Theorem 4.1.** For an irreducible cuspidal automorphic representations $\pi$ of $GL_2(A)$ with trivial central character and $L(\pi, 1/2) \neq 0$, for $D \in F^*$, let $\pi^D = \Theta(\pi, \psi^D)$. Let $S$ be a finite set of places as in the introduction and moreover containing all places of $v$ where $\psi$ or $\psi^D$ is not unramified, and all places where $\pi_v$ or $\pi^D$ is not unramified. Then

\begin{equation}
|d_{\pi}(S, \psi)|^2 L^S(\pi, 1/2) = |d_{\pi^D}(S, \psi^D)|^2
\end{equation}

We state a more explicit formula using (2.7) and (2.10). Take any vectors $\varphi = \otimes_{v \in S} \varphi_v \otimes_{v \in S} \varphi_{0,v}$ and $\tilde{\varphi} = \otimes_{v \in S} \tilde{\varphi}_v \otimes_{v \in S} \tilde{\varphi}_{0,v}$ in $V_\pi$ and $V_{\tilde{\pi}}$ such that
$L_v(\varphi_v) \neq 0$ and $\tilde{L}_v(\tilde{\varphi}_v) \neq 0$. Define:

\begin{align*}
e(\varphi_v, \psi_v) &= \frac{||\varphi_v||^2}{|L_v(\varphi_v)|^2} \\
e(\tilde{\varphi}_v, \psi_v^D) &= \frac{||\tilde{\varphi}_v||^2}{|\tilde{L}_v^D(\tilde{\varphi}_v)|^2}
\end{align*}

Then as in the proof of the Lemma 2.2, these constants are independent of our choice of $L_v$ and $\tilde{L}_v^D$ and are well defined. From (2.7) and (2.10), we see the identity (4.1) can be stated as follows:

\begin{equation}(4.4)\end{equation}

4.2. **The adelic version of Waldspurger’s Theorem.** The statement in Theorem 4.1 is a direct result of our interpretation of Waldspurger’s formula. To see the relation with the previous versions of Waldspurger’s formula, we apply the theorem to the case with $\pi$ replaced by $\pi \otimes \chi_D$. Then $\tilde{\pi}_D = \tilde{\pi}^D = \Theta(\pi \otimes \chi_D, \psi^D)$ and we can apply the results in §3. The results are stated in the following two Theorems.

In the rest of the section, we will assume $D$ satisfies that for all odd nonarchimedean places $v$, $|D|_v = 1$ or $|D|_v = q_v^{-1}$, and for all even nonarchimedean places $v$, $1 \leq |D|_v \leq q_v^{-2}$. With a bit of abuse of terminology, we call this $D$ a square free integer in $F^*$.

In [W2], Waldspurger described the Fourier coefficient of a half integral weight form using the data from the corresponding integral weight form. We generalize his result here. For $\tilde{\pi}$ an irreducible subrepresentation of $\tilde{A}_{00}$ and $D \in F^*$, we describe $d_{\tilde{\pi}}(S, \psi^D)$ in terms of the data of $\pi = S_{\psi}(\tilde{\pi})$.

Let $\Sigma = \Sigma(\pi)$ be as in Theorem 3.2. From Theorem 3.2, $\tilde{\pi} = \tilde{\pi}^{\epsilon_0}$ for some $\epsilon_0 \in \{\pm 1\}^{[\Sigma]}$.

**Theorem 4.2.** Let $S$ be as in the introduction and contain the places $v$ where $\tilde{\pi}_v$ or $\psi$ is not unramified. Let $D$ be a square free integer in $F^*$. Let $\pi = S_{\psi}(\tilde{\pi})$ and $\epsilon_0$ be as above. If $D \in F^{\epsilon_0}(\pi)$ then

\begin{equation}(4.5)\end{equation}

If $D \notin F^{\epsilon_0}(\pi)$, $d_{\tilde{\pi}}(S, \psi^D) = 0$.

4.3. **Adelic version of the Kohnen-Zagier formula.** The Kohnen-Zagier formula quoted in the introduction, as opposed to the Waldspurger’s theorem in [W2], describes the twisted $L$–value of $f(z)$ in terms of data of a half integral weight form. We now state its generalization.

Let $\pi \in A_{01}$. Let $\Sigma = \Sigma(\pi)$ be as in Theorem 3.2. For $\epsilon \in \{\pm 1\}^{[\Sigma]}$, we define $\tilde{\pi}^{\epsilon}$ as in Theorem 3.2.
Theorem 4.3. Let $S$ be as in the introduction and contain the places where \( \psi \) or $\pi_v$ is not unramified. Then for $\epsilon \in \{\pm 1\}^{[\Sigma]}$, for $D \in F^\epsilon(\pi)$ a square free integer,

\[
|d_\pi(\psi^D)|^2 = |d_\pi(\psi)|^2 L^S(\pi \otimes \chi_D, 1/2) \prod_{v \in S} |D|_v^{-1}.
\]

Remark 4.4. 1. The equation (4.6) is in fact a finite set of equations, corresponding to the finite partition of $F^\epsilon$ by $F^\epsilon(\pi)$. The conditions on $D$ in (1.1) are precisely the condition $D \in D^{\epsilon_0}(\pi)$ for a given $\epsilon_0$. Thus (1.1) is only one in a set of equations. See § 10 for the whole set of equations.

2. From Theorem 4.2, $|d_\pi(\psi^D)| = 0$ when $D \notin F^\epsilon(\pi)$. Thus for $\pi \in A_{0,i}$, for all $D$ square free integers,

\[
\sum_\epsilon |d_\pi(\psi^D)|^2 = |d_\pi(\psi)|^2 L^S(\pi \otimes \chi_D, 1/2) \prod_{v \in S} |D|_v^{-1}.
\]

3. If $\pi \notin A_{0,i}$, then clearly $L(\pi \otimes \chi_D, 1/2) = 0$ for all $D \in F^\epsilon$.

4.4. Some other consequences. Since $L(\pi_v, 1/2) > 0$ when $\pi_v$ is unitary, from the Theorem 4.1 we have

Corollary 4.5. For all irreducible cuspidal automorphic representations $\pi$ of $PGL_2(A)$, $L(\pi, 1/2) \geq 0$.

This result is first shown in [Gu].

The next corollary can be considered as the adelic version of corollary 2 in [W2]. It follows immediately from Theorem 4.3.

Corollary 4.6. Let $\pi \in A_{0,i}$ and $S$, $\Sigma = \Sigma(\pi)$ be as in Theorem 4.3. Fix $\epsilon \in \{\pm 1\}^{[\Sigma]}$. For two square free integers $D_1, D_2 \in F^\epsilon(\pi)$,

\[
|d_\pi(\psi^{D_1})|^2 L(\pi \otimes \chi_{D_2}, 1/2) = |d_\pi(\psi^{D_2})|^2 L(\pi \otimes \chi_{D_1}, 1/2) \prod_{v \in S} \frac{D_2}{D_1} |D_1|_v.
\]

We also state a result concerning the relation of the $D-$th and $D\Delta^2$-th Fourier coefficients of a half-integral weight form. Note the similarity with Lemma 2.3.

Corollary 4.7. Let $S$ be as in Theorem 4.3. If $D' = D\Delta^2$ for $\Delta \in F^\times$. Let $S_{D,D'}$ be a finite set of places with $|D|_v = |D'|_v = 1$ for all $v \notin S_{D,D'}$. Then

\[
|d_\pi(S_{D,D'} \cup S, \psi^D)| = |d_\pi(S_{D,D'} \cup S, \psi^{D'})|.
\]

4.5. Ramanujan conjecture and Lindelöf Hypothesis. The Ramanujan conjecture for the half integral weight cusp form is as follows:

Let $g(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$ be a cusp form of weight $k+1/2$, $k \in \mathbb{Z}$, such that $g(z)$ is orthogonal to the space generated by the theta series associated to quadratic forms of one variable,
(i.e. \( g(z) \) is a vector in \( \tilde{A}_{00} \)). Then when \( n \) is square free, as \( n \to \infty \),

\[
|c(n)| \ll_{g,\alpha} n^{k/2 - 1/4 + \alpha}
\]

for all \( \alpha > 0 \). The implied constant depends on \( g(z) \) and \( \alpha \) only.

To state the generalization of this conjecture to the case of cusp forms over totally real fields, we find it most natural to use the notion \( d_{\tilde{\pi}}(\tilde{\varphi},S,\psi^D) \) again.

Let \( S \) be a finite set of places as in the introduction and contain all places where \( \tilde{\pi}_v \) is not unramified. For \( S_0 \subset S \), for \( \tilde{\varphi} \) a vector in the space of \( \tilde{\pi} \), we define

\[
d_{\tilde{\pi}}(\tilde{\varphi},S_0,\psi^D) = \frac{\tilde{W}_{\tilde{\varphi}}(e)}{||\tilde{\varphi}||} \prod_{v \in S_0} \frac{||\tilde{\varphi}_v||}{|\tilde{L}_v^D(\tilde{\varphi}_v)|} = d_{\tilde{\pi}}(S,\psi^D) \prod_{v \in S - S_0} \frac{|\tilde{L}_v^D(\tilde{\varphi}_v)|}{||\tilde{\varphi}_v||}.
\]

As before this constant is well defined and independent of the choice of \( \{\tilde{L}_v^D\} \).

Let \( S_\infty \) be the collection of archimedean places of \( F \). For \( D \in F^* \), define

\[
|D|_\infty = \prod_{v \in S_\infty} |D|_v.
\]

**Conjecture 4.8.** *(Ramanujan conjecture).* Let \( \tilde{\pi} \) be an irreducible sub-representation of \( \tilde{A}_{00} \). Let \( \tilde{\varphi} \in V_{\tilde{\pi}} \). For \( D \) a square free integer in \( F^* \), as \( |D| \to \infty \), for all \( \alpha > 0 \)

\[
|d_{\tilde{\pi}}(\tilde{\varphi},S_\infty,\psi^D)| \ll_{\tilde{\pi},\tilde{\varphi},\alpha} |D|_{S_\infty}^{\alpha - 1/2}
\]

where the implied constant depends only on \( \tilde{\pi}, \tilde{\varphi} \) and \( \alpha \).

We will show in \( \S \) 9 that the above conjecture implies inequality (4.10).

The Lindelöf hypothesis is a conjecture on the bound of central value of \( L \)-functions. We state only a special case.

**Conjecture 4.9.** *(Lindelöf hypothesis)* Let \( \pi \) be an irreducible cuspidal automorphic representation of \( G(A) \) with trivial central character, then for \( D \) square free integer, as \( |D| \to \infty \), for all \( \beta > 0 \)

\[
|L^{S_\infty}(\pi \otimes \chi_D, 1/2)| \ll_{\pi,\beta} |D|_{S_\infty}^\beta
\]

where the implied constant depends only on \( \pi \) and \( \beta \).

**Theorem 4.10.** The inequality (4.12) holds for some \( \alpha > 0 \) (and for all \( \tilde{\pi} \) and \( \tilde{\varphi} \) as in Conjecture 4.8) if and only if the inequality (4.13) holds for \( \beta = 2\alpha > 0 \) (for all \( \pi \) as in Conjecture 4.9). In particular, the Conjecture 4.8 is equivalent to the Conjecture 4.9.
5. A RELATIVE TRACE FORMULA

In [J1], Jacquet proved some of Waldspurger’s results on theta correspondence using a relative trace formula. Our result is based on a local analysis of his trace formula and its variation. We recall some of the results on the trace formula, in the process fix some notations. The main result here is Theorem 5.5.

5.1. Definition of the global distributions $I(f, \psi)$ and $J(f', \psi^D)$. Let $f(g) \in C_c^\infty(Z(\mathbb{A}) \backslash G(\mathbb{A}))$ the space of smooth compactly supported functions. Define a kernel function

$$K_f(x, y) = \sum_{\xi \in PGL_2(F)} f(x^{-1} \xi y)$$

Define a distribution $I(f, \psi)$ to be:

$$\int_{A^*/F^*} \int_{A/F} K_f(g, n(u)) \psi(u) dud^* a$$

Let $f'(g) \in C_c^\infty(G'(\mathbb{A}))$, (we use this notation to denote the space of genuine smooth compactly supported functions). Define a kernel function

$$K_{f'}(x, y) = \sum_{\xi \in SL_2(F)} f'(x^{-1} \cdot \xi \cdot y)$$

Here we note that $SL_2(F)$ embeds into $G'(\mathbb{A})$. Define a distribution $J(f', \psi^D)$ to be:

$$\int_{A/F} \int_{A/F} K_{f'}(\tilde{n}(x), \tilde{n}(y)) \psi^D(-x + y) dxdy$$

The relative trace formula is an identity between the distributions $I(f, \psi)$ and $J(f', \psi^D)$. We will state the result on the distributions $I(f, \psi)$, $J(f', \psi^D)$. The computations are available in [J1] and will not be included.

5.2. Comparison of orbital integrals.

Proposition 5.1. [J1] If $f = \otimes f_v$ and $f' = \otimes f'_v$, then

$$I(f, \psi) = \prod I^+_\psi(f_v) + \prod I^-_\psi(f_v) + \sum_{a \in F^*} \prod \mathcal{O}_{f_v}^{\psi_v}(n(a)w)$$

$$J(f', \psi^D) = \prod J^+_{\psi^D}(f'_v) + \prod J^-_{\psi^D}(f'_v) + \sum_{a \in F^*} \prod \mathcal{O}_{f'_v}^{\psi^D}(\tilde{w} \cdot a)$$

In the above equations, $I^\pm_\psi(f_v)$ and $J^\pm_{\psi^D}(f'_v)$ are the so called singular orbital integrals of $f_v$ and $f'_v$, whose precise forms are not important for us, while

$$\mathcal{O}_{f_v}^{\psi_v}(g) = \int f_v(agn(x)) \psi_v(x) dxd^* a$$

$$\mathcal{O}_{f'_v}^{\psi^D}(g) = \int f'_v(\tilde{n}(x) \cdot g \cdot \tilde{n}(y)) \psi^D_v(x + y) dxdy$$
whenever

\[ (5.4) \]

\[
O_f^\psi (n(\frac{a}{4D})w) = O_f^\psi (\bar{\psi} \cdot g) \psi_v(-\frac{2D}{a}|a|^{1/2}\gamma(a^{-1}, \psi^D))^{-1}
\]

and \( I^\pm (f) = J^\pm (f') \). Conversely, given \( f' \), we can find a \( f \) satisfying the equations.

We say the two functions \( f \) and \( f' \) match if the relations in the proposition are satisfied.

Now let \( v \) be a non-Archimedean place with odd residue characteristic, and where \( \psi_v, \psi_v^D \) have order 0. Recall that the Hecke algebra \( \mathcal{H}(G(F_v)/Z(F_v)) \) is the algebra of compactly supported functions on \( G(F_v)/Z(F_v) \) biinvariant under the maximal compact group \( G(F_v) \). The Hecke algebra \( \mathcal{H}(G'(F_v)) \) is similarly defined, except that the functions are genuine, and biinvariant under \( SL_2(O_v) \) embedded in \( G'(F_v) \).

**Proposition 5.3.** [J1] There is an algebra isomorphism \( \eta_v : \mathcal{H}(G(F_v)/Z(F_v)) \to \mathcal{H}(G'(F_v)) \), such that \( f \) and \( \eta_v(f) \) match.

From Propositions 5.1, 5.2 and 5.3, we get

**Theorem 5.4.** Fix any finite set of places \( S \) as in the introduction and containing places where \( \psi \) and \( D \) are not of order 0. For each place \( v \in S \), there is a map \( \rho_v : C_\infty^c(G(F_v)/Z(F_v)) \to C_\infty^c(G'(F_v)) \), such that

\[
I(\otimes_{v \in S} f_v \otimes_{v \in S} \psi_v) = J(\otimes_{v \in S} \rho_v(f_v) \otimes_{v \in S} \eta_v(f_v), \psi^D),
\]

whenever \( f_v \in \mathcal{H}(G(F_v)/Z(F_v)) \) for all \( v \in S \).

### 5.3. Relation with Shimura-Waldspurger correspondence

For \( \pi \) an irreducible cuspidal automorphic representation of \( G(A) \) with trivial central character, define

\[
(5.4) \quad I_\pi (f, \psi) = \sum_{\varphi_i} Z(\pi(f)\varphi_i) \overline{W_{\varphi_i}(\epsilon)}
\]

with \( \varphi_i \) an orthonormal basis of \( V_\pi \); here for \( \varphi \in V_\pi \)

\[
\pi(f)\varphi = \int_{Z(A)\backslash G(A)} f(g)\pi(g)\varphi dg,
\]

\[
Z(\varphi) = \int_{\mathbb{A}^* / \mathbb{F}^*} \varphi(a) d^* a.
\]

For \( \tilde{\pi} \) an irreducible cuspidal representation of \( G'(A) \), define

\[
(5.5) \quad J_{\tilde{\pi}} (f', \psi^D) = \sum_{\varphi'_i} \overline{W_{\tilde{\pi}(f')}_{\varphi'_i}(\epsilon)} \overline{W_{\varphi'_i}^D(\epsilon)}
\]

with \( \varphi'_i \) an orthonormal basis of \( V_{\tilde{\pi}} \); here for \( \tilde{\varphi} \in V_{\tilde{\pi}} \)

\[
\tilde{\pi}(f')\tilde{\varphi} = \int_{G'(A)} f'(g)\tilde{\pi}(g)\tilde{\varphi} dg.
\]
The distributions $I_\pi(f, \psi)$ and $J_\tilde{\pi}(f', \psi^D)$ are the contributions from $\pi$ and $\tilde{\pi}$ to $I(f, \psi)$ and $J(f', \psi^D)$ respectively.

Recall that if $\pi_v$ is unramified, there exists a vector $\varphi_{0,v}$ that is fixed under the action of $G(O_v)$. For $f_v$ a Hecke function on $G(F_v)/Z(F_v)$, there is a constant $\hat{f}_v(\pi_v)$ with

$$\pi_v(f_v)\varphi_{0,v} = \hat{f}_v(\pi_v)\varphi_{0,v}. \quad (5.6)$$

Similarly, if $\tilde{\pi}_v$ be unramified, let $\tilde{\varphi}_{0,v}$ be a vector that is fixed under $SL_2(O_v)$, then for $f'_v$ in the Hecke algebra of $G'(F_v)$, there is a constant $\hat{f}'_v(\tilde{\pi}_v)$ with

$$\tilde{\pi}_v(f'_v)\tilde{\varphi}_{0,v} = \hat{f}'_v(\tilde{\pi}_v)\tilde{\varphi}_{0,v}. \quad (5.7)$$

It is standard to derive from the Theorem 5.4 the following:

**Theorem 5.5.** [J1] For any cuspidal representation $\pi$ of $G$ with trivial central character such that $I_\pi(f, \psi)$ is nontrivial, there is a unique cuspidal representation $\tilde{\pi}$ of $G'$, such that if $f$ and $f'$ match

$$I_\pi(f, \psi) = J_{\tilde{\pi}}(f', \psi^D) \quad (5.8)$$

Moreover, if $S$ satisfies the condition in Theorem 5.4 and contains all places where $\pi_v$ or $\tilde{\pi}_v$ is not unramified, for $v \not\in S$, if $f_v$ is a Hecke function and $f'_v = \eta_v(f_v)$, then

$$\hat{f}_v(\pi_v) = \hat{f}'_v(\tilde{\pi}_v). \quad (5.9)$$

**Remark 5.6.** From the definition of $Z(\varphi)$ and the integral representation of $L-$function $L(\pi, s)$, it is clear that $I_\pi(f, \psi)$ is nontrivial if and only if $L(\pi, 1/2) \neq 0$.

**Proposition 5.7.** In the above theorem, $\tilde{\pi} = \Theta(\pi, \psi^D)$.

**Proof.** From the description of the map $\eta_v$ of Hecke algebras in [J1] and the equation (5.9), we get $\tilde{\pi}$ is in the same near equivalence class as $\Theta(\pi, \psi^D)$. As $J_\tilde{\pi}(f', \psi^D) \neq 0$, $\tilde{\pi}$ has $\psi^D-$Whittaker model from (5.5). Let $\Sigma = \Sigma(\pi)$ be as in Theorem 3.2. Then for $v \in \Sigma$, $\tilde{\pi}_v = \Theta(\pi_v, \psi^D_v)$ by Theorem 3.1. Thus $\tilde{\pi}$ must be the same representation as $\Theta(\pi, \psi^D)$.

---

6. The local distributions

Let $\pi$ and $\tilde{\pi}$ be the cuspidal representations that correspond to each other by Theorem 5.5. Then $\tilde{\pi} = \Theta(\pi, \psi^D)$. Let $S$ be as in Theorem 5.5. Assume $f = \otimes f_v, f' = \otimes f'_v$, where $f$ and $f'$ match, and $f_v, f'_v$ are matching Hecke functions when $v \not\in S$. We write $I_\pi(f, \psi)$ and $J_{\tilde{\pi}}(f', \psi^D)$ as products of local distributions over the places in $S$. We then state the identity between the local distributions, which coupled with Theorem 5.5 gives Theorem 4.1.
6.1. **The distribution** \( I_{\pi,v}(f_v, \psi_v) \). We fix a choice of local Whittaker functionals \( L_v \) on \( \pi_v \), and define the Hermitian form on \( V_{\pi,v} \) using this choice of \( L_v \). For \( v \in S \), we fix for \( \pi_v \) as above an orthonormal basis of \( V_{\pi,v} \), denote it by \( \{ \varphi_{i,v} \} \). For \( v \not\in S \), let \( \varphi_{0,v} \) be the vector given in § 2. For \( \pi = \otimes \pi_v \), from (2.6), the set

\[
\{ \varphi_I \} = \{ c_2(\pi, S, \psi, \{ L_v \})^{-1} \otimes_{v \in S} \varphi_{i,v} \otimes_{v \not\in S} \varphi_{0,v} \}
\]

can be extended to an orthogonal basis of \( V_{\pi} \). Let \( V(\pi, S) \) be the space of vectors generated by the set of \( \{ \varphi_I \} \). With our choice of \( f \), if is clear that if \( \varphi \in V_{\pi} \) is perpendicular to the space \( V(\pi, S) \), then \( \pi(f) \varphi = 0 \). Thus the expression (5.4) for \( I_{\pi}(f, \psi) \) takes the form:

\[
(6.2) \quad \sum_{\varphi_I} Z(\pi(f) \varphi_I) W_{\varphi_I(e)}.
\]

Using the Hecke theory for \( GL_2 \), we show:

**Lemma 6.1.** When \( \varphi = \otimes_{v \in S} \varphi_v \otimes_{v \not\in S} \varphi_{0,v} \),

\[
(6.3) \quad Z(\varphi) = c_1(\pi, S, \psi, \{ L_v \}) L(\pi, 1/2) \prod_{v \in S} \lambda_v(\varphi_v)
\]

where

\[
\lambda_v(\varphi_v) = \left. \frac{\int_{F_v} L_v(\pi_v(a) \varphi_v) |a|^{s-1/2} d^*a}{L(\pi, s)} \right|_{s=1/2}.
\]

**Proof.** Since \( \varphi(a) = \sum_{\delta \in F_v} W_\varphi(\delta a) \), we get:

\[
Z(\varphi) = \int_{A^*} W_\varphi(a) |a|^{s-1/2} d^*a|_{s=1/2},
\]

which by (2.3) equals

\[
L(\pi, 1/2) c_1(\pi, S, \psi, \{ L_v \}) \prod_{v} \left. \int_{F_v} L_v(\pi_v(a) \varphi_v) |a|^{s-1/2} d^*a \right|_{s=1/2}.
\]

When \( v \not\in S \), it is well known that the above local factor equals 1, ([Go]). Thus the Lemma. 

**Proposition 6.2.** Let \( S \) be as in Theorem 5.5. When \( f = \otimes f_v \) where \( f_v \) is a Hecke function if \( v \not\in S \):

\[
(6.4) \quad I_{\pi}(f, \psi) = L(\pi, 1/2) |d_{\pi}(S, \psi)|^2 \prod_{v \in S} I_{\pi,v}(f_v, \psi_v) \prod_{v \not\in S} \hat{f}_v(\pi_v)
\]

where

\[
(6.5) \quad I_{\pi,v}(f_v, \psi_v) = \sum_{\varphi_{i,v}} \lambda_v(\pi_v(f_v) \varphi_{i,v}) L_v(\varphi_{i,v}).
\]

where the sum is taken over the orthonormal basis of \( V_{\pi,v} \).
Remark 6.3. The expression $J_v(\tilde{\pi}, \psi) = c_2(\pi, S, \psi, \{L_v\})^{-1} \otimes_{v \in S} \varphi_v \otimes_{v \notin S} \varphi_{0,v}$ be an element in the orthonormal set (6.1). From (2.3), we get

$$W_{\varphi}(\epsilon) = \frac{c_1(\pi, S, \psi, \{L_v\})}{c_2(\pi, S, \psi, \{L_v\})} \prod_{v \in S} L_v(\varphi_v).$$

From (5.6),

$$\pi(f)\varphi = c_2(\pi, S, \psi, \{L_v\})^{-1} \prod_{v \notin S} \hat{f}_v(\pi_v) \otimes_{v \in S} \pi_v(f_v) \varphi_v \otimes_{v \notin S} \varphi_{0,v}.$$ 

From the above Lemma:

$$Z(\pi(f)\varphi) = |\frac{c_1(\pi, S, \psi, \{L_v\})}{c_2(\pi, S, \psi, \{L_v\})}|^2 L(\pi, 1/2) \prod_{v \notin S} \hat{f}_v(\pi_v) \prod_{v \in S} \lambda_v(\pi_v(f_v) \varphi_v) \overline{L_v(\varphi_v)}.$$ 

The proposition follows from (6.2) and the definition of $d_\pi(S, \psi).$ \hfill \Box

**Remark 6.3.** The expression $I_{\pi,v}(f_v, \psi_v)$ is well defined and independent of the linear form $L_v$ we choose, as a change in $L_v$ will result in a change in the Hermitian form, thus the orthonormal basis of $V_{\pi,v}$, leaving $I_{\pi,v}(f_v, \psi_v)$ unchanged.

6.2. The distribution $J_{\tilde{\pi},v}(f'_v, \psi^D_v).$ We can apply the above argument also to $J_{\tilde{\pi}}(f'_v, \psi^D_v).$ Similarly we have:

**Proposition 6.4.** Let $S$ be as in Theorem 5.5. When $f' = \otimes f'_v$ where $f'_v$ is a Hecke function if $v \notin S$:

$$J_{\tilde{\pi}}(f', \psi^D) = |d_{\tilde{\pi}}(S, \psi^D)|^2 \prod_{v \in S} J_{\tilde{\pi},v}(f'_v, \psi^D_v) \prod_{v \notin S} \hat{f}'_v(\tilde{\pi}_v)$$

where

$$J_{\tilde{\pi},v}(f'_v, \psi^D_v) = \sum_{\tilde{\varphi}_{j,v}} \tilde{L}^D_v(\tilde{\varphi}_v(f'_v) \tilde{\varphi}_{j,v}) \overline{\tilde{L}^D_v'(\tilde{\varphi}_{j,v})}$$

where the sum is taken over the orthonormal basis $\{\tilde{\varphi}_{j,v}\}$ of $V_{\tilde{\pi},v}.$

**Remark 6.5.** Again one can show that the expression $J_{\tilde{\pi},v}(f'_v, \psi^D_v)$ is well defined and independent of the linear form $\tilde{L}^D_v$ we choose.

6.3. Statement of the local identity.

**Theorem 6.6.** Fix a place $v$, when $f_v, f'_v$ match, when $\pi_v$ is a local component of an irreducible cuspidal automorphic representation $\pi$ of $PGL_2(A)$ with $L(\pi, 1/2) \neq 0$, let $\tilde{\pi}_v = \Theta(\pi_v, \psi^D_v),$ then

$$J_{\tilde{\pi},v}(f'_v, \psi^D) = |2D_v| c(\pi_v, 1/2) L(\pi_v, 1/2) I_{\pi,v}(f_v, \psi_v)$$

The proof of this Theorem is quite technical. It is done in [B-M1],[B-M2]. In [B-M1] we established the identity when $v$ is nonarchimedean. In [B-M2], the identity is proved when $v = R.$ In this case, the identity follows from the identities between classical Bessel functions. We established the Theorem in a bit more generality, as we do not assume $\pi_v$ is a local component of an
From Theorem 6.6, we get:

Combine the above two equations, we get:

\[ L(\pi, 1/2)|d_\pi(S, \psi)|^2 \prod_{v \in S} I_{\pi,v}(f_v, \psi_v) = |d_{\tilde{\pi}}(S, \psi^D)|^2 \prod_{v \in S} J_{\tilde{\pi},v}(f'_v, \psi'_v). \]

From Theorem 6.6, we get:

\[ \prod_{v \in S} J_{\tilde{\pi},v}(f'_v, \psi'_v) = \prod_{v \in S} (2D_v|\epsilon(\pi_v, 1/2)L(\pi_v, 1/2)I_{\pi,v}(f_v, \psi_v)). \]

Combine the above two equations, we get:

\[ L(\pi, 1/2)|d_\pi(S, \psi)|^2 = |d_{\tilde{\pi}}(S, \psi^D)|^2 \prod_{v \in S} (2D_v|\epsilon(\pi_v, 1/2)L(\pi_v, 1/2). \]

As \(|2D_v| = 1 \) for \( v \notin S \), we get \( \prod_{v \in S} |2D_v| = 1 \). As \( \epsilon(\pi, 1/2) = 1 \) and for \( v \notin S, \epsilon(\pi_v, 1/2) = 1 \), we get \( \prod_{v \in S} \epsilon(\pi_v, 1/2) = 1 \). Thus we get the identity (4.1). □

**Proof of Theorem 4.2:** First assume \( D \in F^e(\pi) \). Let \( S_D \) be a finite set of places, such that \(|D_v| = 1 \) when \( v \notin S_D \). Let \( S_1 = S \cup S_D, S_2 = S_1 - S \). Let \( \pi = S_2(\tilde{\pi}) \) be as in the Theorem and \( \tilde{\pi}^D = \Theta(\pi \otimes \chi_D, \psi^D) \). Then by Theorem 3.2, we have \( \tilde{\pi}^D = \tilde{\pi} \) or 0.

When \( \tilde{\pi}^D = 0 \), \( L(\pi \otimes \chi_D, 1/2) = 0 \) from Theorem 3.2. Meanwhile from Proposition 30 of [W1], \( \tilde{\pi} \) does not have a \( \psi^D \)-Whittaker model and \( d_{\tilde{\pi}}(S, \psi^D) = 0 \) by definition. The Theorem holds in this case.

Now assume \( \tilde{\pi}^D = \tilde{\pi} \). From Theorem 3.1, for \( v \notin S \), \( \tilde{\pi}_v = \Theta(\pi_v, \psi_v) \) is unramified, thus \( \pi_v \) is unramified (see Proposition 4 in [W3]). As \( \pi_v \otimes \chi_D \) is unramified for \( v \notin S \), we can apply Theorem 4.1 to get:

\[ |d_{\pi \otimes \chi_D}(S_1, \psi)|^2 L_{S_1}(\pi \otimes \chi_D, 1/2) = |d_{\tilde{\pi}}(S_1, \psi^D)|^2. \]

We will take a vector \( \tilde{\varphi} \) in the space of \( \tilde{\pi} \) so that \( \tilde{\varphi} = \otimes \tilde{\varphi}_v \) with \( \tilde{\varphi}_v = \tilde{\varphi}_{0,v} \) when \( v \notin S \). Using the equation (2.10), we get:

\[ d_{\tilde{\pi}}(S_1, \psi^D) = \frac{|\tilde{\varphi}(\epsilon)|}{||\tilde{\varphi}||} \prod_{v \in S_1} \frac{||\tilde{\varphi}_v||}{|\tilde{L}_{\varphi}(\tilde{\varphi}_v)|} = d_{\tilde{\pi}}(S, \psi^D) \prod_{v \in S_2} \frac{||\tilde{\varphi}_{0,v}||}{|\tilde{L}_{\varphi}(\tilde{\varphi}_{0,v})|}. \]
Since D-dimensional, and by definition ∏D S F ψ acting through right translation. Then Lϕ : ϕv → ϕv(e) is a Whittaker functional. The space of Ψ0,ν,S,ψ consists of functions ϕ is such that ϕ0,ν,S,ψ ∈ Vπ,ν,ψ with ϕ0,ν,S,ψ being unramified vector such that Lϕ(ϕ0,ν,S,ψ) = 1. Then ϕχD = ϕ0,S,ψ,χD ⊗ ϕ0,ν,S,ψ,χD is in Vπ⊗χD with ϕ0,ν,S,ψ,χD being unramified vector such that Lϕ(ϕ0,ν,S,ψ,χD) = 1. Clearly Lϕ(ϕv) = Lϕ(ϕv,χD) and ||ϕv|| = ||ϕv,χD|| for all v ∈ S1. From (2.1) and (2.4), we see Wϕ(ϕv) = Wϕ(ϕv,χD) and ||ϕv|| = ||ϕv,χD||. Thus from the explicit formula (2.7), we get the identity (7.5). □

As πν is unramified for ν ∉ S, as in (7.4) we have

(7.6)   |dπ(S, ψ)|2 = |dπ(S, ψ)|2 ∏v∈S2 e(ϕ0,v,ψv).

From equations (7.3), (7.4), (7.6) and the Lemma, we get:

(7.7)   |dπ(S, ψ)|2 L(S1(π ⊗ χD, 1/2) = |dπ(S, ψ)|2 ∏v∈S2 e(ϕ0,v,ψv) e(ϕ0,v,ψv).  

For v ∈ S2, πν is unramified and unitary, πv = Θ(πv, ψv), the quotient e(ϕ0,v,ψv)/e(ϕ0,v,ψv) is given in Propositions 8.1 and 8.2; it equals (|Dv|L(πv ⊗ χD, 1/2))−1. (This is the only place we use the fact that D is a square free integer). As S1 = S ∪ S2, we get from (7.7):

|dπ(S, ψ)|2 L(S1(π ⊗ χD, 1/2) = |dπ(S, ψ)|2 ∏v∈S2 |Dv|−1.

Since D is in F* and |Dv| = 1 when D ∉ S1, ∏v∈S1 |Dv| = 1; thus ∏v∈S2 |Dv|−1 = ∏v∈S |Dv|. We get (4.5).

Now assume D is such that D ∉ Fν0(π). Then for some v ∈ Σ, (Dv/πv) ≠ ϵ0,v; thus Θ(πv ⊗ χD, ψv) ≠ πv, and by Theorem 3.1, πv does not have a nontrivial ψv-Whittaker functional. Therefore π does not have a nontrivial ψv-Whittaker model, and dπ(S, ψ) = 0. □

Proof of Theorem 4.3: When ∏v∈Σ ϵv ≠ ϵ(π, 1/2), we get for D ∈ Fν(π), ϵ(π ⊗ χD) = −1, thus L(π ⊗ χD, 1/2) = 0. In this case, πν is zero dimensional, and by definition dπν(S, ψ) = 0. When ∏v∈Σ ϵv = ϵ(π, 1/2),
by Theorem 3.2, \( \pi = S_\psi(\tilde{\pi}^\epsilon) \). Thus the Theorem follows from equation (4.5).

\[ \square \]

**Proof of Corollary 4.7:** Let \( \pi = S_\psi(\tilde{\pi}) \) as in Theorem 3.2. We note that \( \varepsilon(D, \pi) = \varepsilon(D', \tilde{\pi}) \). Assume \( \tilde{\pi} = \tilde{\pi}^\epsilon_0 \) for some \( \epsilon_0 \). When \( \varepsilon(D, \pi) \neq \epsilon_0 \), from Theorem 4.2, \( d_{\tilde{\pi}}(S \cup S_{D,D'}, \psi^D) = d_{\tilde{\pi}}(S \cup S_{D,D'}, \psi^{D'}) = 0 \). When \( \varepsilon(D, \pi) = \epsilon_0 \), we follow the first part of the proof of Theorem 4.2, replacing \( S \) by \( S \cup S_{D,D'} \) in the argument. We get (see (7.3)):

\[
|d_{\pi \otimes \chi_D}(S \cup S_{D,D'}, \psi)|^2 L^{S \cup S_{D,D'}}(\pi \otimes \chi_D, 1/2) = |d_{\tilde{\pi}}(S \cup S_{D,D'}, \psi^D)|^2,
\]

\[
|d_{\pi \otimes \chi_{D'}}(S \cup S_{D,D'}, \psi)|^2 L^{S \cup S_{D,D'}}(\pi \otimes \chi_{D'}, 1/2) = |d_{\tilde{\pi}}(S \cup S_{D,D'}, \psi^{D'})|^2.
\]

The equation (4.9) follows from the fact that \( \chi_{D'} = \chi_D \) and Lemma 7.1. (The equation can also be established directly as in the proof of Lemma 2.3).

\[ \square \]

**Proof of Theorem 4.10:** First note that for \( D \) a square free integer, for any fixed finite set of places \( S \), the value of \( \prod_{v \in S - S_\infty} |D_v| \) lies in a finite set of positive numbers. Thus we can disregard this quantity in our computation below.

Assume (4.12) holds for some \( \alpha > 0 \). Given any \( \pi \) irreducible cuspidal representation of \( PGL_2(\mathbf{A}) \), we prove (4.13). Let \( D \) be a square free integer, by Theorem 4.3, there is a finite set \( S \) of places, such that equation (4.6) holds for some \( \tilde{\pi}^\epsilon \). We may as well assume that \( L(\pi \otimes \chi_D, 1/2) \neq 0 \). Then \( d_{\pi^*}(S, \psi^D) \neq 0 \), and we can find \( \tilde{\varphi} = \otimes_{v \in S} \tilde{\varphi}_v \otimes_{v \notin S} \tilde{\varphi}_0 \in V_{\tilde{\varphi}} \), with \( W^D_D(e) \neq 0 \). Recall

\[
(7.8) \quad |d_{\tilde{\pi}^*}(S, \psi^D)| = |d_{\tilde{\pi}^*}(\tilde{\varphi}, S_\infty, \psi^D)| \prod_{v \in S - S_\infty} \frac{|\tilde{\varphi}_v|}{|L^D_v(\tilde{\varphi}_v)|}.
\]

For a given \( v \), the value of \( \frac{|\tilde{\varphi}_v|}{|L^D_v(\tilde{\varphi}_v)|} \) does depend on \( D \) (as it depends on \( \psi^D_v \)). We put the dependence on \( D \) in the notation and denote the value as \( \frac{|\tilde{\varphi}_v|}{|L^D_v(\tilde{\varphi}_v)|} \).

**Lemma 7.2.** For a fixed \( v \in S - S_\infty \) and fixed \( \tilde{\varphi}_v \), there are only finitely many possible values of \( \frac{|\tilde{\varphi}_v|}{|L^D_v(\tilde{\varphi}_v)|} \).

**Proof.** As \( |D|_v = 1 \) or \( |D|_v = q^{-1} \), the value of \( D \) lies in finitely many cosets of \( (\mathcal{O}_v^*)^2 \). Write \( D = D_0 \alpha^2 \) with \( \alpha \in \mathcal{O}_v^* \), then we can let \( \tilde{L}^D_v(\tilde{\varphi}_v) = L^D_v(\tilde{\varphi}_v) \equiv L^{D_0 \delta_i(\tilde{\varphi}_v)}(\tilde{\varphi}_v) \) where \( \delta_i \) are representatives of square classes of \( F_v^* \). Then from (2.9), we get \( |\tilde{\varphi}_v|_D = |\tilde{\varphi}_v|_{D_0} \). Meanwhile \( \tilde{L}^D_v(\tilde{\varphi}_v) = L^{D_0}(\tilde{\varphi}_v) \). As \( \tilde{\varphi}_v \) is admissible, the set of \( \{ \tilde{\varphi}_v(\alpha) \}_{\alpha \in \mathcal{O}_v^*} \) is finite. There are only finitely many possible values of \( \tilde{L}^D_v(\tilde{\varphi}_v) \) when \( D = D_0 \alpha^2 \). Thus only finitely many possible values of the quotient \( \frac{|\tilde{\varphi}_v|_D}{|L^D_v(\tilde{\varphi}_v)|} \).

\[ \square \]
From the Lemma, there is a positive constant $c(\tilde{\varphi})$ depending only on $\tilde{\varphi}$, with
\[
\prod_{v \in S - S_\infty} \frac{|\tilde{\varphi}_v|}{|L_D(\tilde{\varphi}_v)|} < c(\tilde{\varphi}).
\]
From (7.8) and (4.12), we get:
\[
|d_{\tilde{\pi}^*}(S, \psi^D)| << \tilde{\pi}, \varphi, \alpha |D|_{S_\infty}^{\alpha - 1/2}.
\]
Using (4.6), we get:
\[
L(\pi \otimes \chi_D, 1/2) << \tilde{\pi}, \varphi, \alpha |D|_{S_\infty}^{2\alpha}.
\]
As the set of $\tilde{\pi}^r$ is finite and determined by $\pi$, we get the inequality (4.13) for $\beta = 2\alpha$.

Conversely, assume the inequality (4.13) holds for some $\beta = 2\alpha > 0$, take any $\tilde{\pi} \in \tilde{\pi}_0$ and $\tilde{\varphi} \in V_{\tilde{\pi}}$, we prove (4.12). We may as well assume $\tilde{\varphi} = \otimes_{v \in S} \tilde{\varphi}_v \otimes_{v \not\in S} \tilde{\varphi}_0$, where $S$ is a large enough finite set of places. Let $\pi = S_{\psi}(\tilde{\pi})$. From (4.7) and our assumption, we get
\[
|d_{\tilde{\pi}^*}(S, \psi^D)| << \tilde{\pi}, \varphi, \alpha |D|_{S_\infty}^{\alpha - 1/2}.
\]
Using the equation (7.8) we get:
\[
|d_{\tilde{\pi}^*}(\tilde{\varphi}, S_\infty, \psi^D)| \prod_{v \in S - S_\infty} \frac{|\tilde{\varphi}_v| |D|}{|L_D(\tilde{\varphi}_v)|} << \tilde{\pi}, \alpha |D|_{S_\infty}^{\alpha - 1/2}.
\]
From Lemma 7.2, we see there is a constant $c'(\tilde{\varphi}) > 0$ such that
\[
\prod_{v \in S - S_\infty} \frac{|\tilde{\varphi}_v| |D|}{|L_D(\tilde{\varphi}_v)|} > c'(\tilde{\varphi})
\]
for all $D$. We get
\[
|d_{\tilde{\pi}^*}(\tilde{\varphi}, S_\infty, \psi^D)| << \tilde{\pi}, \alpha |D|_{S_\infty}^{\alpha - 1/2}.
\]
As $\pi$ is determined by $\tilde{\pi}$, the implied constant only depends on $\alpha$ and $\tilde{\pi}$. □

8. Local factors: some examples

In this section, we compute the local factors $e(\varphi_v, \psi)$ and $e(\tilde{\varphi}_v, \psi^D)$ in equation (4.4) for some specific choices of the vectors $\varphi_v$ and $\tilde{\varphi}_v$. The computation here is standard and fairly easy. The result has already been used in the proof of Theorem 4.2. It is also used when we translate our formula into more explicit results about cusp forms, (see the proof of Theorem 10.1).

In subsections 8.1–8.3, we assume $v$ is a nonarchimedean place, with odd residue characteristic. For simplicity, we will assume $\psi_v$ has order 0, (and denote it simply by $\psi$), and $D$ is either a unit or generates the prime ideal in $O_v$ at nonarchimedean places $v$.

The cases we consider are the following:
1. When \( \pi_v \) is an unramified unitary representation of \( G(F_v) \) where \( v \) is an odd non-archimedean place. Then \( \tilde{\pi}_v^D = \Theta(\pi_v, \psi) \) for all \( D \in F_v^* \) and is an unramified unitary representation of \( G'(F_v) \). We take \( \varphi_v \) and \( \tilde{\varphi}_v \) to be the unramified vectors in \( V_{\pi,v} \) and \( V_{\tilde{\pi},v} \) respectively.

2. When \( \pi_v \) is a holomorphic discrete series representation of \( G(\mathbb{R}) \). Then by Theorem 3.1 \( \tilde{\pi}_v^D \) is either a holomorphic discrete series or an antiholomorphic discrete series representation of \( G'(\mathbb{R}) \). We will only consider the case when \( \tilde{\pi}_v \) is the corresponding holomorphic discrete series. We take \( \varphi_v \) and \( \tilde{\varphi}_v \) to be the minimal weight vectors in \( V_{\pi,v} \) and \( V_{\tilde{\pi},v} \) respectively.

3. When \( \pi_v \) is a special representation of \( G(F_v) \) where \( v \) is a non-archimedean place. Then by Theorem 3.1 \( \tilde{\pi}_v^D \) could be either \( \tilde{\pi}_v^+ \) or \( \tilde{\pi}_v^- \). \( \tilde{\pi}_v^+ \) is a special representation of \( G'(F_v) \) while \( \tilde{\pi}_v^- \) is a supercuspidal representation. We consider both cases. The vectors \( \varphi_v \) and \( \tilde{\varphi}_v \) will be described in subsection 8.3.

When we look at the cuspidal representations corresponding to the integral weight forms of level \( N \) and half integral weight forms of level \( 4N \) with \( N \) being square free, the local components at infinite places and odd non-archimedean places are of the form \( \pi_v \) and \( \tilde{\pi}_v \) considered above. The situation at the even place is more subtle and will be considered in the next section.

8.1. Some principal series at nonarchimedean places. Let \( \pi_v = \pi(\mu, \mu^{-1}) \) be a unitary representation with \( \mu(x) = |x|^s, \mu \in \mathbb{R} \). Let \( \tilde{\pi}_v = \Theta(\pi_v, \psi) \), it is \( \tilde{\pi}(\mu, \psi) \) by Proposition 4 of [W3]. Note that \( \tilde{\pi}(\mu, \psi) \) is unramified. We will let \( \tilde{\varphi}_v \) be \( \tilde{\varphi}_{0,v} \) the unramified vector in \( V_{\tilde{\pi},v} \), and let \( \varphi_v \) be \( \varphi_{0,v} \) the unramified vector in \( V_{\pi,v} \). Then \( \tilde{\varphi}_v \) and \( \varphi_v \) are respectively \( SL_2(\mathcal{O}_v) \) and \( G(\mathcal{O}_v) \) invariant functions in \( \tilde{\pi}(\mu, \psi) \) and \( \pi(\mu, \mu^{-1}) \).

**Proposition 8.1.** With above choices,

\[
e(\varphi_v, \psi) = \frac{1 + q^{-1}}{1 - q^{-2s-1} |D|_v^2},
\]

\[
e(\tilde{\varphi}_v, \psi^D) = \left\{ \begin{array}{ll}
|D|_v^{-1} & \text{when } |D|_v = 1, \\
|D|_v^{-1} \left[ 1 + q^{-1} \frac{1 + q^{-1}}{1 - q^{-2s-1} |D|_v^2} \right] & \text{when } |D|_v = q^{-1}.
\end{array} \right.
\]

Therefore \( \frac{e(\varphi_v, \psi)}{e(\tilde{\varphi}_v, \psi^D)} = |D|_v L(\varphi_v \otimes \chi_D, 1/2) \).

**Proof.** We will use the Whittaker functional on \( V_{\pi,v} \):

\[
L_v(\phi_v) = \int \phi_v(wn(x))\psi(-x)dx, \ \phi_v \in V_{\pi,v}
\]

The formula for spherical Whittaker function is well known, see [Ca-Sl]. The formula for our case is available in [Go]. We have \( L_v(\pi_v(a)\varphi_v) = 0 \) if \( |a|_v > 1 \); it equals

\[
q^{-m/2} \mu^{-1}(a) \frac{(1 - q^{-2s-1})(1 - q^{-2(m+1)s})}{1 - q^{-2s}} \varphi_v(e)
\]
if \( |a|_v = q^{-m} \) with \( m \geq 0 \). It is easy to show from (2.5) that \( ||\varphi_v||^2 \) equals (8.2)
\[
1 - q^{-2s-1} \left| 1 - q^{-2s-1} \right|^2 |\varphi_v(e)|^2 \sum_{m=0}^{\infty} q^{-m} \left( 1 - q^{-1} \right) \left| 1 - q^{-2(m+1)s} \right|^2 = (1 + q^{-1}) |\varphi_v(e)|^2,
\]
and thus the result on \( e(\varphi_v, \psi) \).

To compute \( e(\tilde{\varphi}_v, \psi^D) \), we use the \( \psi^D \)-Whittaker functional
\[
(8.3) \quad \tilde{L}_v^D(\phi) = \int \phi(\tilde{w} \cdot \tilde{n}(x)) \psi^D(-x) \, dx
\]
The formula for spherical Whittaker functions on the metaplectic groups is given in [Bu-F-H]. In our case it is an easy exercise to show when \( |D|_v = 1 \), \( \tilde{L}_v^D(\tilde{\varphi}_{0,v}) = \tilde{\varphi}_{0,v}(e)(1 + q^{-1/2-s} \chi_D(\overline{w})) \); when \( |D|_v = q^{-1} \), it equals \( \tilde{\varphi}_{0,v}(e)(1 - q^{-1-2s}) \).

The Hermitian form on \( V_{\tilde{\pi}, v} \) takes the form ([B-M1] (9.19)):
\[
(8.4) \quad (\phi, \phi') = \sum_{\delta \in F^+_v / (F^*_v)^2} 1/2 \int \hat{L}_v^{D\delta_i}(\hat{\pi}(a)\phi) \overline{\hat{L}_v^{D\delta_i}(\hat{\pi}(a)\phi')} \, da,\quad \frac{\delta i}{|a|_v}
\]
The above form in turn equals: ([B-M1] (9.18))
\[
(8.5) \quad \int |D|_v^{-1} |\phi(\tilde{w} \cdot \tilde{n}(x))\phi'(\tilde{w} \cdot \tilde{n}(x))| \, dx.
\]
Use the second formula for Hermitian form to compute \( ||\tilde{\varphi}_{0,v}||^2 \). From Iwasawa decomposition we get
\[
(8.6) \quad ||\tilde{\varphi}_{0,v}||^2 = |D|_v^{-1} \int_{|x|_v \leq 1} |\tilde{\varphi}_{0,v}(e)|^2 \, dx + \int_{|x|_v > 1} |x|_v^{-2} |\tilde{\varphi}_{0,v}(e)|^2 \, dx
\]
which gives
\[
||\tilde{\varphi}_{0,v}||^2 = |D|_v^{-1} (1 + q^{-1}) |\tilde{\varphi}_{0,v}(e)|^2
\]
This gives the result on \( e(\tilde{\varphi}_{0,v}, \psi^D) \). The result on the quotient \( e(\varphi_v, \psi) / e(\tilde{\varphi}_{0,v}, \psi^D) \) follows from the table on \( L \)-functions in [Go].

8.2. Complementary series at nonarchimedean places. Let \( \pi_v \) be as in subsection 8.1, except that now \( \mu(x) = |x|^s \chi_T(x) \) with \( \tau \) a unit in \( O_v \), and \( |s| < 1/2, s \in \mathbb{R} \). Let \( \tilde{\pi}_v = \Theta(\pi_v, \psi) \). Then as before \( \tilde{\pi}_v = \tilde{\pi}(\mu, \psi) = \tilde{\pi}(\pi_v^s, \psi^t) \). We will choose the vectors \( \varphi_v \) and \( \tilde{\varphi}_v \) as in subsection 8.1.

Proposition 8.2. With above choices,
\[
e(\varphi_v, \psi) = \begin{cases} 
1 + q^{-1} & \text{when } |D|_v = 1, \\
\frac{|D|_v^{-1}}{(1 + q^{-1-2s} \chi_D(\overline{w})(1 + q^{-1-2s} \chi_D(\overline{w}))} & \text{when } |D|_v = q^{-1}.
\end{cases}
\]
\[
e(\tilde{\varphi}_v, \psi^D) = \begin{cases} 
|D|_v^{-1} & \text{when } |D|_v = 1, \\
|D|_v^{-1} \frac{1 + q^{-1}}{(1 + q^{-1-2s} \chi_D(\overline{w})(1 + q^{-1-2s} \chi_D(\overline{w}))} & \text{when } |D|_v = q^{-1}.
\end{cases}
\]
Therefore \( \frac{e(\tilde{\varphi}_v, \psi^D)}{e(\varphi_v, \psi)} = |D|_v L(\pi_v \otimes \chi_D, 1/2) \).
Thus we have the formula for $e(\varphi_v, \psi)$.

The formula for $\bar{L}_v^D(\tilde{\varphi}_{0,v})$ in the proof of Proposition 8.1 remains valid. The Hermitian form however takes a more complicated form. If $z = \Delta^2 \delta$, let

$$
\lambda(z) = |\Delta|^{-2s-2}(1 - q^{-2s})^{-1} |1 - q^{-1}|^{1 - q^{-1}/2} \chi_{\tau}(w), \quad \text{if } |\delta|_v = 1;
$$

$$
\lambda(z) = |z|^{-s-1} \Delta(\psi, \tau, v)(z) \quad \text{where } \Delta(\psi, \tau, v)(z) \text{ is defined in [B-M1] Proposition 9.8.}
$$

From equation (9.22) of [B-M1], the Hermitian form is:

$$(8.7) \quad (\phi, \phi') = \sum_{\delta \in F_v/(F_v^*)^2} 1/2 \int F_v^D(\tilde{\pi}_v(a) \phi) \bar{L}_v^D(\tilde{\pi}_v(a) \phi') \lambda(z) \frac{da}{\lambda(D)|a|_v}.$$  

From equation (9.21) of [B-M1], we see this form can also be written as:

$$(8.8) \quad (\phi, \phi') = \lambda(D)^{-1} \int A\phi(\tilde{w} \cdot \tilde{n}(x)) \bar{\phi}'(\tilde{w} \cdot \tilde{n}(x)) dx.$$ 

where

$$
A\phi(g) = \int \phi(\tilde{w} \cdot \tilde{n}(y) \cdot g) dy
$$

Then $A\tilde{\varphi}_{0,v}$ is the unique vector in the space of $\tilde{\pi}(\mu^{-1} \chi_{-1}, \psi)$ fixed under $SL_2(O_v)$, with $A\tilde{\varphi}_{0,v}(e) = \frac{1 - q^{-1 - 2s}}{1 - q^{2s}} \tilde{\varphi}_{0,v}(e)$. Using the Iwasawa decomposition, we see

$$||\tilde{\varphi}_{0,v}||^2 = (1 + q^{-1}) \lambda(D)^{-1} \frac{1 - q^{-1 - 2s}}{1 - q^{-2s}} |\tilde{\varphi}_{0,v}(e)|^2.$$ 

This gives the formula for $e(\tilde{\varphi}_{0,v}, \psi^D)$. The result on the quotient $\frac{e(\varphi_v, \psi^D)}{e(\tilde{\varphi}_{0,v}, \psi^D)}$ follows from the table on $L-$functions in [Go].

### 8.3. Special representations.

8.3.1. *Description of $\tilde{\pi}_v^+$ and $\tilde{\pi}_v^-$.*

Let $\mu_{\tau}(x) = |x|_{v}^{1/2} \chi_{\tau}(x)$, where $\tau$ is in $F_v^*$. Let $\sigma^\tau = \sigma(\mu_{\tau}, \mu_{\tau}^{-1})$ be the special representation associated to the character $\mu_{\tau}$. We will only consider the case when $|\tau|_v = 1$. The space of $\sigma^\tau$ is the subspace of $\pi(\mu_{\tau}, \mu_{\tau}^{-1})$ consisting of functions $\phi$ with

$$
(8.9) \quad \int \phi(wn(x)) dx = 0
$$

From Theorem 3.1, the set $\{\Theta(\sigma_v^\tau \otimes \chi_D, \psi^D)\}$ consists of two elements. These two elements are described in [W3]. When $D\tau$ is not a square, $\Theta(\sigma_v^\tau \otimes \chi_D, \psi^D) = \tilde{\pi}^+$ is the special representation $\tilde{\sigma}^\tau(\psi)$. The space of
this representation is the subspace of $\pi(\mu, \psi)$ consisting of functions $\phi$ satisfying

$$(8.10) \quad \tilde{L}_\tau \Delta^2(\phi) = \int \phi(\tilde{w} \cdot \tilde{n}(x)) \psi(-\tau \Delta^2 x) dx = 0, \text{ for all } \Delta.$$ 

On the other hand, when $D\tau$ is a square, $\Theta(\sigma_v^\tau \otimes \chi_D, \psi^{D}) = \pi^-$ is a supercuspidal representation of $G'(F_v)$. More precisely, it is the odd component of the Weil representation, denoted $r_{\psi^\tau}^-$. The space of $r_{\psi^\tau}^-$ is the subspace of odd functions of $C_c^\infty(F_v)$, with the action being: for $\Phi(z) \in C_c^\infty(F_v)$, $\Phi(z) = -\Phi(-z)$,

$$(8.11) \quad r_{\psi^\tau}^-(n(x))\Phi(z) = \psi^\tau(xz^2)\Phi(z),$$

$$(8.12) \quad r_{\psi^\tau}(\tilde{w})\Phi(z) = |a|_v^{1/2}\gamma(1, \psi^\tau)/\gamma(a, \psi^\tau)\Phi(az),$$

$$(8.13) \quad r_{\psi^\tau}^-(\tilde{w})\Phi(z) = \gamma(1, \psi^\tau)^2/\gamma(-1, \psi^\tau) \int \Phi(y)\psi^\tau(-2yz)dy.$$

8.3.2. The case of $\pi_v^\pm$. When $\tau D$ is not a square, $\Theta(\sigma_v^\tau \otimes \chi_D, \psi^{D})$ is the special representation $\pi_v^\pm = \tilde{\sigma}^\tau(\psi)$.

Define the Iwahori subgroup $K_0 \subset G(O_v)$ as the group consisting of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $|a|_v = |d|_v = 1, |c|_v < 1, |b|_v \leq 1$. Recall in the introduction we defined an embedding of $SL_2(O_v)$ in $G'$ given by $g \mapsto (g, \kappa(g))$. Let $K'_0$ be the image in $G'$ of the restriction of the splitting to $K_0 \cap SL_2$.

Denote by $\text{char}(G(O_v))$ and Denote by $\text{char}(K_0)$ the characteristic functions if $G(O_v)$ and $K_0$ respectively. Denote by $\text{char}(G'(O_v))$ a function on $G'$ with $\text{char}(G'(O_v))((g, \xi))$ equals 0 if $g \notin SL_2(O_v)$, and equals $\xi\kappa(g)$ otherwise. Let $\text{char}(K'_0)(g, \xi)$ be the genuine function on $G'$ that is 0 if $g \notin K_0$, and equals $\text{char}(G'(O_v))((g, \xi)$ if $g \in K_0$.

**Lemma 8.3.** Let $\varphi_v$ be a function in $\pi(\mu, \mu^{-1})$ such that it equals $\text{char}(G(O_v)) - (q+1)\text{char}(K_0)$ over $G(O_v)$, then $\varphi_v$ is in $V_{\sigma^\tau, v}$ and is fixed under $K_0$.

Let $\tilde{\varphi}_v$ be a function in $\tilde{\pi}(\mu, \psi)$ such that it equals $\text{char}(G'(O_v)) - (q+1)\text{char}(K'_0)$ over $G'(O_v)$, then $\tilde{\varphi}_v$ lies in the space of $\tilde{\sigma}^\tau(\psi)$ and is fixed under $K'_0$.

The spaces of $K_0$ fixed vectors in $\sigma^\tau$ and $K'_0$ fixed vectors in $\tilde{\sigma}^\tau(\psi)$ are one dimensional.

**Proof.** We can consider the vectors in $\pi(\mu, \mu^{-1})$ and $\tilde{\pi}(\mu, \psi)$ as functions on $G(O_v)$ and $SL_2(O_v)$ respectively. Since $B \setminus G/K_0$ and $B \cap G'/K'_0$ both have two elements, the space of vectors in $\pi(\mu, \mu^{-1})$ fixed by $K_0$ is two dimensional, with basis $\{\text{char}(G(O_v)), \text{char}(K_0)\}$; the space of vectors in $\tilde{\pi}(\mu, \psi)$ fixed by $K'_0$ is two dimensional, with basis $\{\text{char}(G'(O_v)), \text{char}(K'_0)\}$.

When $\phi_1$ is the vector corresponding to $\text{char}(G(O_v))$, the integral $\int \phi_1(wn(x))dx$ equals $1 + q^{-1}$. When $\phi_2$ is the vector corresponding to $\text{char}(G(K_0))$, the
integral \( \int \phi_2(u n(x)) dx \) equals \( q^{-1} \). Thus \( \varphi_v \) satisfies the condition (8.9) and generates the one dimensional space fixed by \( K_0 \) in \( \sigma^\tau \).

Next we compute \( \tilde{L}^z_v(\tilde{\varphi}_v) \) with \( z \in \mathbb{P}_v^* \). Let \( \phi'_1 \) be the vector corresponding to \( \text{char}(G'(O_v)) \), then from (8.3) and the Iwasawa decomposition, we get

\[
\tilde{L}^z_v(\phi'_1) = \int_{|x|_v \leq 1} \psi(-z x) dx + \sum_{r=1}^{\infty} \int_{|x|_v = q^r} \tilde{\gamma}(x, \psi)|x|^{\tau} \chi_r(x)[-1, x] \psi(-z x) dx.
\]

When \( |x|_v = q^r \) with \( r \) even, with our assumption on \( \tau \) being a unit,

\[
\tilde{\gamma}(x, \psi) \chi_r(x)[-1, x] = 1.
\]

When \( |x|_v = q^r \) with \( r \) odd,

\[
\tilde{\gamma}(x, \psi) \chi_r(x)[-1, x] = \gamma(x, \psi)[\tau, \varpi].
\]

It is a simple calculation to get the following result: write \( z = \delta \Delta^2 \) with \( |\delta|_v = 1 \) or \( q^{-1} \),

\[
\tilde{L}^{\delta \Delta^2}_v(\phi'_1) = \begin{cases} 0 & \text{when } |\Delta|_v > 1, \\ 1 + q^{-1} + |\Delta|_v(q^{-1}|\tau \delta, \varpi| - q^{-1}) & \text{when } |\delta|_v = 1, |\Delta|_v \leq 1, \\ q^{-1} - |\Delta|_v(q^{-1} + q^{-2}) & \text{when } |\delta|_v = q^{-1}, |\Delta|_v \leq 1. \end{cases}
\]

Let \( \phi'_2 \) be the vector corresponding to \( \text{char}(K'_0) \). Then

\[
\tilde{L}^z_v(\phi'_1 - \phi'_2) = \int_{|x|_v \leq 1} \psi(-z x) dx.
\]

We get

\[
\tilde{L}^{\delta \Delta^2}_v(\phi'_1) = \begin{cases} 0 & \text{when } |\Delta|_v > 1, \\ q^{-1} + |\Delta|_v(q^{-1}|\tau \delta, \varpi| - q^{-1}) & \text{when } |\delta|_v = 1, |\Delta|_v \leq 1, \\ q^{-1} - |\Delta|_v(q^{-1} + q^{-2}) & \text{when } |\delta|_v = q^{-1}, |\Delta|_v \leq 1. \end{cases}
\]

The formula for \( \tilde{L}^z_v(\tilde{\varphi}_v) \) is (8.14)

\[
\tilde{L}^{\delta \Delta^2}_v(\tilde{\varphi}_v) = \begin{cases} 0 & \text{when } |\Delta|_v > 1, \\ 2|\Delta|_v & \text{when } |\delta|_v = 1, |\Delta|_v \leq 1, \delta \tau \text{ is not a square}, \\ 0 & \text{when } |\delta|_v = 1, |\Delta|_v \leq 1, \delta \tau \text{ is a square}, \\ |\Delta|_v(q^{-1} + 1) & \text{when } |\delta|_v = q^{-1}, |\Delta|_v \leq 1. \end{cases}
\]

It is now clear that \( \tilde{L}^z_v(\tilde{\varphi}_v) \) satisfies the condition (8.10). The vector \( \tilde{\varphi}_v \) generates the space of \( K'_0 \) fixed vectors in \( \tilde{\sigma}^\tau(\psi) \) which from the above formulas is clearly one dimensional.

**Proposition 8.4.** Assume \( D \tau \) is not a square. Let \( \varphi_v \) and \( \tilde{\varphi}_v \) be the vectors in Lemma 8.3. Then

\[
e(\varphi_v, \psi) = \frac{1}{1 + q^{-1}},
\]

\[
e(\tilde{\varphi}_v, \psi^D) = \begin{cases} 1/2 & \text{when } |D|_v = 1, \\ q/(1 + q^{-1}) & \text{when } |D|_v = q^{-1}. \end{cases}
\]
Therefore

\[
\begin{align*}
\frac{e(\varphi_v, \psi)}{e(\tilde{\varphi}_v, \psi^D)} = \begin{cases} 
2L(\pi_v \otimes \chi_D, 1/2)|D|_v & \text{when } |D|_v = 1, \\
L(\pi_v \otimes \chi_D, 1/2)|D|_v & \text{when } |D|_v = q^{-1}.
\end{cases}
\end{align*}
\]

**Proof.** We can use the Iwasawa decomposition to compute \( L_v(\sigma^\tau(\tilde{a})\varphi_v) \) where \( L_v \) is defined as in (8.1). We will skip the details. One gets

\[ L_v(\sigma^\tau(\tilde{a})\varphi_v) = \begin{cases} 
0 & \text{when } |a|_v > 1, \\
(1 + q^{-1})|a|_v \chi_\tau(a) & \text{when } |a|_v \leq 1.
\end{cases} \]

Thus \( ||\varphi_v||^2 = (1 + q^{-1}) \) from (2.5) and we get the value of \( e(\varphi_v, \psi) \).

Assume \( |D|_v = 1 \). From (8.14) \( \tilde{L}^D_v(\tilde{\varphi}_v) = 2 \). To find \( ||\tilde{\varphi}_v|| \), we use the Hermitian form (9.23) in [B-M1]:

(8.15) \( (\phi, \phi') = \sum b = D, D\pi, \tau \omega \ c_b/2 \int \tilde{L}^b_v(\tilde{\sigma}(\tilde{a})\phi)\tilde{L}^b_v(\tilde{\sigma}(\tilde{a})\phi') \frac{da}{|a|_v} \)

where \( \tilde{\sigma} = \tilde{\sigma}^\tau(\psi) \) and \( c_b = 1 \) when \( b = D \), and \( c_b = 2q^{-1}/(1 + q^{-1}) \) when \( b = D\pi, \tau \omega \). Using this formula, the formula (8.14) for \( \tilde{L}^b_v(\tilde{\varphi}) \) and the fact that

\[ |\tilde{L}^b_v(\tilde{\sigma}(\tilde{a})\tilde{\varphi})| = |a|_v^{1/2} |\tilde{L}^b_v(\tilde{\varphi})|, \]

we get

\[ ||\tilde{\varphi}_v||^2 = \int_{|a|_v \leq 1} 2|a|_v^3 \frac{da}{|a|_v} + \frac{2q^{-1}}{1 + q^{-1}} \int_{|a|_v \leq 1} (1 + q^{-1})^2 |a|_v^3 \frac{da}{|a|_v} = 2. \]

Thus we get the formula for \( e(\tilde{\varphi}_v, \psi^D) \) when \( |D|_v = 1 \).

When \( |D|_v = q^{-1} \), from (8.14) \( \tilde{L}^D_v(\tilde{\varphi}_v) = 1 + q^{-1} \). The computation of \( ||\tilde{\varphi}_v|| \) goes (2.9) the Hermitian form changes from (8.15) to

\[ (\phi, \phi') = \sum b = D, D\pi, \tau \delta \ c'_b/2 \int \tilde{L}^b_v(\tilde{\sigma}(\tilde{a})\phi)\tilde{L}^b_v(\tilde{\sigma}(\tilde{a})\phi') \frac{da}{|a|_v} \]

where \( \delta \) is a unit in \( \mathcal{O}_v \) such that \( \delta \tau \) is not a square. Here \( c'_D = c'_D\tau = 1 \) and \( c'_\delta = \frac{1 + q^{-1}}{2q^{-1}} \). We get \( ||\tilde{\varphi}_v|| = (1 + q^{-1})/q^{-1} \) and the formula for \( e(\tilde{\varphi}_v, \psi^D) \).

The claim on the quotient follows from the following formulas for \( L \)-values (see [Go]). When \( |D|_v = 1 \) with \( \tau \) not a unit, \( L(\pi_v \otimes \chi_D, 1/2) = L(\sigma^\tau D, 1/2) = (1 + q^{-1})^{-1} \). When \( |D|_v = q^{-1} \), \( L(\pi_v \otimes \chi_D, 1/2) = L(\sigma^\tau D, 1/2) = 1 \).

8.3.3. The case of \( \tilde{\pi}_v \). When \( D \) is a square, \( \Theta(\sigma^\tau_v \otimes \chi_D, \psi^D) \) is the supercuspidal representation \( \tilde{\pi}_v = r_{\psi^\tau} \). Recall by our assumption \( \tau \) and \( D \) are both units in \( \mathcal{O}_v \). We will let \( \varphi_v \) to be the vector defined in Lemma 8.3. Next we describe a vector \( \tilde{\varphi}_v \) in the space of \( r_{\psi^\tau} \).

Let \( K_0 \) be the subgroup of \( K_0 \) consisting of matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) with \( |a|_v = |d|_v = 1, |c|_v < q^{-1}, |b|_v \leq 1 \). Let \( K_0' \) be the image of \( K_0 \cap SL_2 \) embedded in \( G' \). Then \( K_0' = \{(\sigma, \kappa(\sigma)) | \sigma \in K_0\} \). Let \( \chi \) be any odd
character of $\mathcal{O}_v^*$ that is trivial on $1 + P$, $(\chi(-1) = -1)$, then $\chi$ defines a character on $K'_{00}$ by:

$$
\chi(\sigma, \kappa(\sigma)) \mapsto \chi(d), \sigma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in K_{00}.
$$

Let $\text{char}(X)$ denote the characteristic function of a subset $X$ in $F_v$.

**Proposition 8.5.** The space of vectors $\Phi$ in $r_{\psi^r}^-$ satisfying $r_{\psi^r}^-(k)\Phi = \chi(k)\Phi$, $k \in K'_{00}$ is one dimensional. It is generated by the element

$$
\Phi_{\chi}(z) = \sum_{b \in O_v^*/1+P} \chi^{-1}(b) \text{char}(b + P)(z).
$$

**Proof.** Let $\Phi$ be a vector satisfying the relation in the Proposition. With our assumptions on $\psi, \tau$ and the place $v$, the equation (8.12) gives

$$
r_{\psi^r}^-(\tilde{w})\Phi(1) = \Phi(z) = \chi(z^{-1})\Phi(1), \ t \in O_v^*.
$$

Thus $\Phi_0 = \Phi - \Phi(1)\Phi_{\chi}$ vanishes over the set $O_v^*$. Next if $z \notin O_v$, from (8.11), for $x \in O_v$:

$$
r_{\psi^r}^-(\tilde{n}(x))\Phi(z) = \psi(\tau z x^2)\Phi(z) = \Phi(z),
$$

thus $\Phi(z) = 0$. We get $\Phi_0$ is supported on $P$.

We now show $\Phi_{\chi}$ satisfies the relation in the Proposition.

**Lemma 8.6.** For $z \in F_v$, $x \in O_v$, (8.16)

$$
r_{\psi^r}^-(\left( \begin{array}{cc} 1 \\ x x^2 \\ 1 \end{array} \right), 1) \text{char}(z + P) = \text{char}(z + P).
$$

**Proof.** We use the fact that

$$
\left( \begin{array}{cc} 1 \\ x x^2 \\ 1 \end{array} \right), 1) = \tilde{w} \cdot (-e, 1) \cdot \tilde{n}(-x x^2) \cdot \tilde{w}.
$$

From (8.11), (8.12) and (8.13) the left hand side of (8.16) is

$$
r_{\psi^r}^-(\left( \begin{array}{cc} 1 \\ x x^2 \\ 1 \end{array} \right), 1) \text{char}(z + P)(a) = \int \int \text{char}(z + P)(u) \psi(\tau x z x^2 + 2uy - 2ay) dudy.
$$

The integral over $u$ is nonzero only when $y \in P^{-1}$, in which case $\psi(\tau x y^2 x^2) = 1$ and from the Fourier inversion formula, the above integral just equals $\text{char}(z + P)(a)$.

It is easy to check using (8.11) and (8.12) that $r_{\psi^r}^-(k)\Phi_{\chi} = \chi(k)\Phi_{\chi}$ when $k \in K'_{00} \cap \tilde{B}$. From the above Lemma we get for $x \in O_v$,

$$
r_{\psi^r}^-(\left( \begin{array}{cc} 1 \\ x x^2 \\ 1 \end{array} \right), 1)\Phi_{\chi} = \Phi_{\chi}.
$$

Since these group $K'_{00}$ is generated by the elements in $K'_{00} \cap \tilde{B}$ and $\{(( \begin{array}{cc} 1 \\ x x^2 \\ 1 \end{array} ), 1)|x \in O_v\}$, we see $\Phi_{\chi}$ satisfies the relation in the Proposition.
Thus $\Phi_0 = \Phi - \Phi(1)\Phi_\chi$ satisfies the relation in the Proposition and is supported over $P$. To finish the proof, we need to show such a function is identically 0. From the proof of Lemma 8.6, we get:

$$\int_{x \in O_v} \int \Phi_0(u) \psi^\tau(xy^2 y^2 + 2uy - 2ay) dudydx.$$ 

For the integration over $x$ to be nonzero, $y \in P^{-1}$, in which case $\Phi_0(u) \psi^\tau(2uy) = \Phi_0(u)$. Thus the above integral equals

$$\int_{x \in O_v} \int_{y \in P^{-1}} \Phi_0(u) \psi^\tau(-2ay) dudydx$$

which equals a constant times $\text{char}(P)$. Thus for $a \in P$, $\Phi_0(a) = \Phi_0(0)$. Since $\Phi_0$ is an odd function, $\Phi_0$ vanishes over $P$, thus vanishes identically.

The representation $r^\tau_{\psi^\tau}$ is a distinguished representation, in the sense that it only has nontrivial Whittaker functionals for $\psi^\delta$ with $\delta$ in the same square class as $\tau$. Assume $D = \tau\alpha^2$, we can define $\tilde{L}^D_v$ by setting

$$\tilde{L}^D_v(\Phi) = \Phi(\alpha).$$

Then the Hermitian form is just:

$$(\Phi_1, \Phi_2) = 1/2 \int \tilde{L}^D_v(\tilde{r}^\tau_{\psi^\tau}(a)\Phi_1) \bar{\tilde{L}^D_v(\tilde{r}^\tau_{\psi^\tau}(a)\Phi_1)} da_{|_v}.$$ 

Clearly $\tilde{L}^D_v(\Phi_\chi) = \chi(\alpha^{-1})$ and by (8.12),

$$|\tilde{L}^D_v(\tilde{r}^\tau_{\psi^\delta}(a)\Phi_\chi)| = |a|^{1/2} |\Phi_\chi(a\alpha)|$$

which equals 1 when $|a|_v = 1$ and 0 otherwise. Thus

$$||\Phi_\chi||^2 = 1/2 \int_{|a|_v = 1} \frac{da}{|a|_v} = (1 - q^{-1})/2.$$ 

Thus $e(\Phi_\chi, \psi^D) = (1 - q^{-1})/2$. Note that from [Go]

$$L(\pi_v \otimes \chi_D, 1/2) = L(\sigma^\tau, 1/2) = L(\sigma^1, 1/2) = (1 - q^{-1})^{-1}.$$ 

We have

**Proposition 8.7.** Let $\varphi_v$ be a vector in $\pi_v = \sigma^\tau$ given by Lemma 8.3. Let $\tilde{\varphi}_v$ be $\Phi_\chi$ as in Proposition 8.5. Then $e(\varphi_v, \psi) = (1 + q^{-1})^{-1}$, $e(\tilde{\varphi}_v, \psi^D) = (1 - q^{-1})/2$ and $e(\tilde{\varphi}_v, \psi^D) = 2(1 + q^{-1})^{-1} L(\pi_v \otimes \chi_D, 1/2)$. 

8.4. **Holomorphic discrete series.** Let $F_v = \mathbb{R}$. Let $\pi_v$ be the discrete series $\sigma(\mu, \mu^{-1})$ ([W3]) where $\mu(x) = |x|^{s/2}(\text{sgn } x)^{(s+1)/2}$, $k = \frac{s+1}{2}$ being a positive integer. Then according to Theorem 3.1, $\Theta(\pi \otimes \chi_D, \psi^D)$ can be one of the following two representations: $\tilde{\pi}_v = \tilde{\pi}_v^+$ is $\Theta(\pi_v, \psi)$ and $\tilde{\pi}_v^- = \Theta(\pi_v \otimes \text{sgn}, \psi^{-1})$. We note in this case $\pi_v \cong \pi_v \otimes \text{sgn}$.

We now assume $\psi(x) = e^{2\pi i n x}$ with $n$ a positive integer. Then $\tilde{\pi}_v$ is a holomorphic discrete series $\tilde{\sigma}(\mu)$ while $\tilde{\pi}_v^- = \tilde{\sigma}(\mu \text{ sgn})$ is an antiholomorphic discrete series, ([W3]).

Let $\varphi_v$ and $\tilde{\varphi}_v$ be a vector of minimal weight in $V_{\pi,v}$ and $V_{\tilde{\pi},v}$ respectively. These vectors are determined up to a scalar. We have:

**Proposition 8.8.** Let $D$ be a positive integer. With the above notations:

$$e(\varphi_v, \psi) = e^{4\pi n(1 + k)}(2k),$$

$$e(\tilde{\varphi}_v, \psi^D) = 2e^{4\pi n D(1 - k)}(1 - k + k) \Gamma(1/2 + k).$$

Therefore $e(\varphi_v, \psi) = \frac{1}{2}e^{4\pi n (1 + D)} D^{1/2 + k} \pi^{-k}(k - 1)!$.

**Proof.** From [Go], we see the Whittaker model for $\varphi_v$ has the form:

$$L_v(\pi_v(a)\varphi_v) = \begin{cases} \alpha a^k e^{-2\pi n a} & a > 0 \\ 0 & a < 0 \end{cases}$$

where $\alpha$ is some nonzero constant which we may as well fix to be 1. With this model, we get from (2.5)

$$||\varphi_v||^2 = \int_{a > 0} a^{2k} e^{-4\pi n a} d^* a$$

which equals $(4\pi n)^{-(2k)} \Gamma(2k)$. This gives the result for $e(\varphi_v, \psi)$.

From [W1] p.24, we see the Whittaker model for $\tilde{\varphi}_v$ with respect to $\psi^D$ has the form:

$$\tilde{L}_v^D(\tilde{\pi}_v(a)\tilde{\varphi}_v) = \alpha \omega(\text{sgn}(a))|a|^{1/2 + k} e^{-2\pi n \beta a^2}$$

where $\omega$ is the central character of $\tilde{\sigma}(\mu)$. We will again let $a = 1$. Since in this case, $\tilde{\pi}_v$ is distinguished, i.e., the Whittaker functional for $\psi^D$ is always trivial when $z < 0$, the Hermitian form in (2.9) simplifies to:

$$(\phi, \phi') = \int \tilde{L}_v^D(\tilde{\pi}_v(a)\phi) \tilde{L}_v^D(\tilde{\pi}_v(a)\phi') d^* a$$

Apply the formula to compute $||\tilde{\varphi}_v||$, we find that

$$||\tilde{\varphi}_v||^2 = 2(4\pi n D)^{-1/2 + k} \Gamma(1/2 + k).$$

The result for $e(\tilde{\varphi}_v, \psi^D)$ follows. The assertion on the quotient follows from the formula $(k + 1)!$.

$$\Gamma(2k) = \pi^{-1/2} 2^{2k-1} \Gamma(k) \Gamma(k+1), \quad \Gamma(k) = (k-1)!. \quad \square$$
We will apply the results in § 4 to the case of holomorphic cusp forms over \( \mathbb{Q} \). Fix the additive character \( \psi \) as follows: if \( x \in \mathbb{R} \), then \( \psi(x) = e^{2\pi ix} \), at a rational prime \( p \), if \( x \in \mathbb{Q}_p \), choose \( \hat{x} \in \mathbb{Q} \) so that \( |x - \hat{x}|_p \leq 1 \), and set \( \psi(x) = e^{-2\pi i\hat{x}} \). Denote by \( \hat{\gamma}(x) \) the number \( \hat{\gamma}(x, \psi) \). We denote by \( |D|_v \) the absolute value of \( D \) which equals \( |D|_\infty \).

We first recall the correspondence between the cusp forms and automorphic representations. Our reference is [W2] section III. The main result in this section is the choice of a one dimensional subspace in a two dimensional subspace of \( V_{\pi_2} \). This choice is closely related to the definition of the Coleman space of half-integral weight forms.

9.1. The dictionary: integral weight form. Let \( \Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) | c \equiv 0(N) \} \). Let \( S_{2k}(N) \) be the space of cusp form of weight \( 2k \) on \( \Gamma_0(N) \) (of level \( N \)), and with trivial character. Assume from now on that \( N \) is odd and square free. Let \( f \in S_{2k}(N) \) be a newform. Then \( f \) determines a vector in the space of automorphic forms on \( GL_2(\mathbb{A}_\mathbb{Q}) \) by \( f \mapsto \varphi = s(f) \).

The map \( s(f) \) is defined as follows. For \( g_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R}) \), let

\[
 f|_{g_\infty}(z) = f\left(\frac{az + b}{cz + d}\right)(cz + d)^{-2k}
\]

Consider \( g_\infty \) as an element \( (g_\infty, e, e, \ldots) \) in \( GL_2(\mathbb{A}_\mathbb{Q}) \), then \( \varphi(g_\infty) = f|_{g_\infty}(i) \), and \( \varphi(\gamma g k) = \varphi(g) \) whenever \( \gamma \in GL_2(\mathbb{Q})Z(\mathbb{A}_\mathbb{Q}) \), and \( k \in \prod_{v | N} GL_2(\mathbb{Q}_v) \prod_{v \nmid N} K_{0,p} \).

Then \( \varphi \) is a vector in the space of an irreducible cuspidal representation \( \pi \) of \( GL_2(\mathbb{A}_\mathbb{Q}) \), with trivial central character. The representation \( \pi = \otimes \pi_v \), and \( \varphi = \otimes \varphi_v \) can be described as follows:

(9.1.1). When \( v = \infty \), \( \pi_v \) is the discrete series \( \sigma(\mu_\infty, \mu_v^{-1}) \) as in subsection 8.4, with \( \mu_\infty(x) = |x|_v^{k-1/2}(\text{sgn } x) \). The vector \( \varphi_\infty \) is a minimal weight vector.

(9.1.2). When \( v \) is \( p \)-adic, \( p \) not dividing \( N \), then \( \pi_v = \pi(\mu_v, \mu_v^{-1}) \) with \( \mu_v \) an unramified character, and \( \varphi_v \) is an unramified vector.

(9.1.3). When \( v \) is \( p \)-adic, \( p | N \), then \( \pi_v \) is a special representation \( \sigma^{\tau} \) as in subsection 8.3, where \( \tau_v \) is a unit in \( \mathbb{Z}_p \). Then \( \varphi_v \) is the vector described in Lemma 8.3.

Conversely, given an irreducible cuspidal automorphic representation \( \pi = \otimes \pi_v \) with local components as described in (9.1.1)–(9.1.3), pick \( \varphi \) as above (which is unique up to scalar multiple), then \( \varphi \) is a scalar multiple of \( s(f) \) for some newform \( f \) in \( S_{2k}(N) \).

If \( f(z) = \sum_{n=1}^\infty a(n)e^{2\pi inz} \), then \( a(1) = e^{2\pi W_\varphi(e)} \). As we assume \( a(1) = 1 \), for \( \varphi = s(f) \), \( W_\varphi(e) = e^{-2\pi} \).

9.2. Dictionary: half integral weight form. Assume now that \( k \) is a nonnegative integer. Let \( N \) be a positive odd integer. Let \( \chi \) be a Dirichlet
character mod $4N$ such that $\chi(-1) = 1$. Assume $4N = \prod_{p|4N} p^{v(p)}$, then 
$(\mathbb{Z}/4N)^* \simeq \prod_{p|4N}(\mathbb{Z}/p^{v(p)})^*$, and $\chi$ can be decomposed into a product of 
characters $\chi(p)$ of $(\mathbb{Z}/p^{v(p)})^*$ under this isomorphism. We can trivially extend 
$\chi(p)$ to a character of $\mathbb{Z}_p^*$.

Let $S'_{k+1/2}(4N, \chi)$ be the space of holomorphic cusp forms of weight $k + 1/2$, level $4N$ and character $\chi$. The functions in the space satisfies: [W2]

$$
(9.1) \quad g(\frac{az + b}{cz + d}) = j(\sigma, z)^{2k+1} \chi(d)g(z), \quad \sigma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}), \ 4N|c.
$$

Here

$$
\sum_{n=-\infty}^{\infty} e^{2\pi in^2z}.
$$

Let $\tilde{A}'_{k+1/2}(4N, \chi)$ be the space generated by vectors $\tilde{\varphi} = \otimes_v \tilde{\varphi}_v$ in the space 
of cuspidal automorphic forms on $SL_2(\mathbb{A})$ satisfying:

$$(9.2.1) \text{ When } v = \infty, \tilde{\varphi}_v \text{ is a minimal weight vector in the space of a holomorphic discrete series } \tilde{\sigma}(\mu_{\infty}) \text{ where } \mu_{\infty}(x) = |x|^{k-1/2}(\text{sgn } x)^k.$$

$$(9.2.2) \text{ When } v \text{ is } p-\text{adic, } p \text{ not dividing } 2N, \text{ then } \tilde{\varphi}_p \text{ is the unramified vector in the space of } \tilde{\pi}(\mu_v, \psi) \text{ where } \mu \text{ is an unramified character.}
$$

$$(9.2.3) \text{ When } v \text{ is } p-\text{adic, } p \neq 2, \text{ and } p|N, \text{ then } \tilde{\varphi}_p \text{ is a vector in the space of some } \tilde{\pi}_p \text{ such that } \tilde{\pi}_p(\sigma, \kappa(\sigma))\tilde{\varphi}_p = \chi_p(d)\tilde{\varphi}_v \text{ whenever } \sigma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}_p) \text{ with } |c|_p \leq |N|_p.
$$

$$(9.2.4) \text{ When } v \text{ is } p-\text{adic with } p = 2, \tilde{\varphi}_2 \text{ is a vector in the space of some } \tilde{\pi}_2 \text{ such that } \tilde{\pi}_2(\sigma, 1)\tilde{\varphi}_2 = \tilde{\epsilon}_2(\sigma) \chi_2(d)\tilde{\varphi}_2 \text{ whenever } \sigma = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}_2) \text{ with } |c|_2 \leq |4N|_2.
$$

The Proposition 3 of [W2] establishes a bijection from $S'_{k+1/2}(4N, \chi)$ to 
$\tilde{A}'_{k+1/2}(4N, \chi)$. The bijection is given by $g(z) \mapsto \tilde{\varphi} = t(g)$, where $t(g)$ is 
the unique function on $SL_2(\mathbb{A})$ that is continuous and left invariant under 
$SL_2(\mathbb{Q})$ and satisfies:

$$
t(g)\left( \begin{array}{cc} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{array} \right) \left( \begin{array}{cc} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{array} \right), 1 \right) = y^{k/2+1/4}e^{i(k+1/2)\theta}g(x + yi),
$$

where $y > 0, x \in \mathbb{R}$ and $-\pi < \theta \leq \pi$.

The relation between the Whittaker functionals of $t(g)$ and the Fourier 
coefficients of $g(z)$ is given by follows: From Lemma 3 of [W2], we get when 
$g(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$,

$$
c(n) = e^{2\pi n \tilde{W}_{t(g)}(e)}.
$$
Remark on Petersson norm: If \( f \) is a cusp form of weight \( k \in \frac{1}{2} \mathbb{Z} \) on one subgroup \( \Gamma \) of finite index in \( \Gamma_1 = SL_2(\mathbb{Z}) \), we define as usual the norm of \( f \) to be

\[
<f, f> = \frac{1}{|\Gamma(1) : \Gamma|} \int_{\Gamma \backslash \mathcal{H}} |f(z)|^2 y^{k-2} dx dy
\]

where \( z = x + iy \) and \( \mathcal{H} \) is the upper half plane. Then

**Lemma 9.1.** For \( \varphi = s(f) \) and \( \tilde{\varphi} = t(g) \) as above:

\[
(9.3) \quad \frac{||\varphi||^2}{<f, f>} = \frac{||\tilde{\varphi}||^2}{<g, g>}
\]

**Proof.** It is well known that \( ||\varphi||^2 = <f, f> \) and \( ||\tilde{\varphi}||^2 = <g, g> \) when we use the following Haar measures \( d' \) on \( GL_2 \) and \( SL_2 \) instead of the one given in the introduction. When \( v \) is a nonarchimedean place, choose the measure \( d' \) on \( GL_2 \) so that \( G(\mathcal{O}_v) \) has volume 1; choose the measure \( d' \) on \( SL_2 \) so that \( SL_2(\mathcal{O}_v) \) has volume 1. When \( v \) is the infinite place, let \( k(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \). From the Iwasawa decomposition any \( g \in GL_2^+(\mathbb{R}) \) (the subgroup with positive determinant) can be written uniquely as \( g = z(c)n(x) \begin{pmatrix} a & \cdot \\ \cdot & a^{-1} \end{pmatrix} k(\theta) \) with \( c \in \mathbb{R}^*, x \in \mathbb{R}, a > 0 \) and \( 0 \leq \theta < \pi \). Let \( d'g = \frac{1}{2\pi} |a|^{-2} d^* c d x d\theta \) be the measure on \( GL_2^+(\mathbb{R}) \) and thus on \( GL_2(\mathbb{R}) \). Similarly any \( g \in SL_2(\mathbb{R}) \) can be written uniquely as \( g = n(x) \begin{pmatrix} a & \cdot \\ \cdot & a^{-1} \end{pmatrix} k(\theta) \) with \( x \in \mathbb{R}, a > 0 \) and \( 0 \leq \theta < 2\pi \). Let \( d'g = \frac{1}{2\pi} |a|^{-2} dx d\theta \) be the measure on \( SL_2(\mathbb{R}) \).

We now compare the measures \( dg \) and \( d'g \) on \( GL_2 \) and \( SL_2 \) respectively. When \( v \) is a nonarchimedean place for the rational prime \( p \), we have \( dg = (1 + p^{-1}) d'g \) in both case \( GL_2 \) and \( SL_2 \). When \( v \) is archimedean, we note the measures \( d' \) defined on \( GL_2 \) and \( SL_2 \) induce the same quotient measure on \( Z(\mathbb{R}) \backslash GL_2(\mathbb{R}) \cong Z \backslash SL_2(\mathbb{R}) \backslash SL_2(\mathbb{R}) \), and the measures \( d \) defined in the introduction also induce the same quotient measure. Thus our change of the measures is consistent and the equation (9.3) still holds. \( \square \)

### 9.3. Ramanujan conjecture.

We show here that the conjecture (4.12) implies the conjecture (4.10).

Let \( g(z) \) be as in (4.10). Then \( \tilde{\varphi} = t(g) \) is a linear combination of vectors in the space \( \tilde{A}_{00} \cap \tilde{A}'(4N, \chi) \) for some \( N \) and \( \chi \). We may as well assume \( g(z) \) correspond to a vector \( \tilde{\varphi} = t(g) \) in a subrepresentation \( \tilde{\pi} \) of \( \tilde{A}_{00} \). From (9.2), \( c(n) = e^{2\pi n \tilde{W}_{\tilde{\varphi}}}(e) \). From the definition,

\[
|d_{\tilde{\pi}}(\tilde{\varphi}, S_{\infty}, \psi^n)| = \frac{|\tilde{W}_{\tilde{\varphi}}(e)|}{||\tilde{\varphi}||} c(\tilde{\varphi}_{\infty}, \psi^n)^{1/2}
\]
From Proposition 8.8
\[ |d_\pi(\tilde{\varphi}, S_\infty, \psi^n)| = e^{-2\pi n}|c(n)| \left[ \frac{1}{2} e^{4\pi n (4\pi n)^{-1/2}} \Gamma(1/2 + k) \right]^{1/2} / ||\tilde{\varphi}||. \]

Thus \( |d_\pi(\tilde{\varphi}, S_\infty, \psi^n)| = \delta(\tilde{\varphi})|c(n)|n^{-1/4-k/2} \) where \( \delta(\tilde{\varphi}) \) is a positive constant depending only on \( \tilde{\varphi} \). From (4.12),
\[ |d_\pi(\tilde{\varphi}, S_\infty, \psi^n)| < < \tilde{\varphi}, \alpha n^{a-1/2}. \]
Thus we get \( |c(n)| < < \tilde{\varphi}, \alpha n^{k/2-1/4+\alpha} \) and (4.10).

9.4. **Choice of \( \tilde{\varphi}_2 \).** Let \( \tilde{\pi} \) be a cuspidal representation such that the space of \( \tilde{\pi} \) has nontrivial intersection with \( \tilde{A}_{k+1/2}(4N, \chi) \). The condition (9.2.4) puts a restriction on \( \tilde{\pi}_2 \) (the component at place \( v = 2 \) of \( \tilde{\pi} \)). The representation \( \tilde{\pi}_2 \) must be a subrepresentation of \( \tilde{\pi}(\mu \chi_{-1}, \psi) \) where \( \mu \) is an unramified character of \( \mathbb{Z}_2 \), and \( k' = k \) if \( \chi(2)(-1) = 1 \) and \( k' = k + 1 \) if \( \chi(2)(-1) = -1 \). The space of vectors in \( \tilde{\pi}(\mu \chi_{k+1}, \psi) \) satisfying (9.2.4) is then two dimensional. It is spanned by two vectors \( F[2, 1] \) and \( F[2, 2^2] \) ([W2] Proposition 12). We recall their definitions ([W2] p.415, 427). They are the unique functions in the space of \( \tilde{\pi}(\mu \chi_{k+1}, \psi) \) satisfying:
\[ F[2, 1](\tilde{w}) = 1, \quad F[2, 1]\left(\begin{array}{c} 1 \\ c \\ 1 \end{array}\right), 1 = 0, \quad c \in 2\mathbb{Z}_2. \]
\[ F[2, 2^2](\tilde{w}) = 0, \quad F[2, 2^2]\left(\begin{array}{c} 1 \\ c \\ 1 \end{array}\right), 1 = \text{char}(\mathbb{Z}_2)(2^{-2}c). \]

We make a choice of a vector \( \tilde{\varphi}_2 \) in the above two dimensional space. Define the linear combination
\[ \tilde{\varphi}_2 = \mu(2^2) \frac{1 + (-1)^{k'i}}{4} F[2, 1] + F[2, 2^2]. \]

The reason for this choice is explained by the following Proposition. Recall the definition of the Whittaker functional \( \tilde{L}_2^z \) by equation (8.3).

**Proposition 9.2.** When \( (-1)^{k'} z \equiv 2, 3 \ mod \ 4 \), \( \tilde{L}_2^z(\tilde{\varphi}_2) = 0 \).

This is a direct consequence of the following computation of \( \tilde{L}_2^z(F[2, 1]) \) and \( \tilde{L}_2^z(F[2, 2^2]) \).

**Lemma 9.3.** With above definitions, \( \tilde{L}_2^z(F[2, 1]) = \text{char}(\mathbb{Z}_2)(z) \) and
\[ \tilde{L}_2^z(F[2, 2^2]) = \begin{cases} 0 & |z| > 1, \\ \left( \mu(2^2) + \sqrt{7} \mu(2^3) \right) \frac{1 + (-1)^{k'i}}{4} & z \in (-1)^{k'} + P^2, \\ -\mu(2^2) \frac{1 + (-1)^{k'i}}{4} & z \in ((-1)^{k'+1} + P^2) \cup (2 + P^2), \\ \left( \mu(2^2) - \mu(2^4) \right) \frac{1 + (-1)^{k'i}}{4} & (-1)^{k'/2} \in (2 + P^2) \cup (-1 + P^2). \end{cases} \]
Proof. The claim for $F[2, 1]$ is easy to verify. For $F[2, 2^2]$, using the Iwasawa decomposition, we see:

$$\hat{L}_2^*(F[2, 2^2]) = \int_{|x|_2 \geq 2} \mu(x^{-1})|x|_2^{-1}\hat{\gamma}(x)\chi_{-1}(x)^{k'+1}e^{2\pi i x} dx$$

Consider the integral

$$T(z, i) = \int_{|x|_2 = 2^i} \mu(x^{-1})|x|_2^{-1}\hat{\gamma}(x)\chi_{-1}(x)^{k'+1}e^{2\pi i x} dx$$

Then

(9.5) $$\hat{L}_2^*(F[2, 2^2]) = \sum_{i=2}^{\infty} T(z, i).$$

If $l = 2m$ is even, then a change of variable $x \mapsto x^{2^{-l}}$ gives $T(z, l) = \mu(2^l)T(2^{-l}z, 0)$. Over $|x|_2 = 1$, we have ([W2], p. 382)

$$\hat{\gamma}(x) = 1/2(1 - i + (1 + i)\chi_{-1}(x))$$

Define $\eta(\nu, t)$ to be the Gauss sum: ([W2], p.382)

$$\int_{|u|_2 = 1} \nu(x) e^{-2\pi i t u} du$$

Then

$$T(2^{-l}z, 0) = (1 - 2^{-1})^{-1}[\frac{1 - i}{2} \eta(x^{k'+1}, -2^{-l}z) + \frac{1 + i}{2} \eta(x^{k'+1}, -2^{-l}z)].$$

Thus

(9.6) $$T(z, 2m) = 2\mu(2^{2m})[\frac{1 - i}{2} \eta(x^{k'+1}, -2^{-2m}z) + \frac{1 + i}{2} \eta(x^{k'+1}, -2^{-2m}z)].$$

If $l = 2m + 1$ is odd, then using the formula $\hat{\gamma}(2^{-1}x) = \chi_2(x)\hat{\gamma}(x)$ and make a change of variable $x \mapsto 2^{-1}x$, we get

$$T(z, 2m + 1) = \mu(2) \int_{|x|_2 = 2^{2m}} \mu(x^{-1})|x|_2^{-1}\hat{\gamma}(x)\chi_2(x)\chi_{-1}(x)^{k'+1}e^{2\pi i x} dx$$

which by above argument becomes:

(9.7) $$T(z, 2m + 1) = 2\mu(2^{2m+1})[\frac{1 - i}{2} \eta(x^{k'+1}\chi_2, -2^{-2m-1}z) + \frac{1 + i}{2} \eta(x^{k'+1}\chi_2, -2^{-2m-1}z)].$$

Note that Gauss sum $\eta(\nu, t)$ vanish if the conductor of $\nu$ is nonzero and not equal to $-\nu(t)$, or if $\nu$ is unramified and $|t|_2 > 2$. Observe that $\chi_{-1}$ is of conductor 2, and $\chi_2$ is of conductor 3. Thus

$$\hat{L}_2^*(F[2, 2^2]) = \begin{cases} 0 & |z|_2 > 1, \\ T(z, 2) + T(z, 3) & |z|_2 = 1, \\ T(z, 2) & |z|_2 = 2^{-1}, \\ T(z, 2) + T(z, 4) + T(z, 5) & |z|_2 = 2^{-2}, \\ T(z, 2) + T(z, 4) & |z|_2 = 2^{-3}. \end{cases}$$
We can use the following formulas for $\eta$ ([W2], p.383) to finish the computation: $\eta(\chi_2, 2^{-3}) = \frac{1}{\sqrt{2}}$, $\eta(\chi_{-2}, 2^{-3}) = \frac{-i}{\sqrt{2}}$, and $\eta(\chi_{-1}, 2^{-2}) = -i$ (there is a typo in [W2] for this value). Note also that $\chi_{-1}(\pm 1 + P^2) = \pm 1$ and $\eta(\nu, tt') = \eta(\nu, t)\nu^{-1}(t')$ when $|t'|_2 = 1$. Our assertion follows the formulas (9.6) and (9.7).

9.5. The Kohnen space. Kohnen introduced a subspace $S'_{k+1/2}(4N, \chi)$ in $S_{k+1/2}'(4N, \chi)$ in [K2], (we note the notation in [K2] is different from ours). It consists of $g(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$ with the Fourier coefficients $c(n)$ satisfying:

\begin{equation}
(9.8)\quad c(n) = 0, \text{ when } \chi(2)(-1)(-1)^k n \equiv 2, 3 \text{ mod } 4.
\end{equation}

With our definition of $\tilde{\varphi}_2$, the Kohnen space has a natural interpretation in the representation language. Let $\tilde{\varphi}_2$ be a positive integer such that $g(z) = \sum_{n=1}^{\infty} c(n)e^{2\pi inz}$ with the Fourier coefficients $c(n)$ satisfying:

\begin{equation}
(9.8)\quad c(n) = 0, \text{ when } \chi(2)(-1)(-1)^k n \equiv 2, 3 \text{ mod } 4.
\end{equation}

The operator has two different eigenvalues on this space and $\chi(2)(-1)(-1)^k n \equiv 2, 3 \text{ mod } 4$. Since $(\varphi)^{-1} = \chi(2)(-1)(-1)^k$, at $v = 2$, $\tilde{L}_2^0(\varphi_2) = 0$. From the uniqueness of the local Whittaker functionals, $\tilde{W}_v^0(\varphi)$ vanishes when $\tilde{L}_v^0(\tilde{\varphi}_v)$ vanish for any place $v$. We get $\tilde{W}_v^0(\varphi) = 0$ for such $n$. Thus $g(z) = t^{-1}(\varphi)$ lies in the Kohnen space, and $\tilde{A}_{k+1/2}(4N, \chi) \subset t(S'_{k+1/2}(4N, \chi)).$

In Proposition 1 of [K2], Kohnen defined an operator $Q$ on $S'_{k+1/2}(4N, \chi).$ The operator has two different eigenvalues on this space and $S'_{k+1/2}(4N, \chi)$ is the eigenspace of one eigenvalue (denoted $\alpha$). The operator $Q$ induces an operator $Q'$ on the spaces $V_\varphi \cap A'_{k+1/2}(4N, \chi)$. We have a factorization $V_\varphi \cap A'_{k+1/2}(4N, \chi) = \otimes V'_v$ with $V'_{\varphi,2}$ a two dimensional space. Then $Q' = \otimes Q'_v$. In fact $Q'_v$ are all trivial actions for $v \neq 2$. Clearly $\tilde{\varphi}_2$ in (9.4) is the eigenvector of $Q'_2$ with eigenvalue $\alpha$ as the vector $\tilde{\varphi} = \otimes \tilde{\varphi}_v$ with local component $\tilde{\varphi}_2$ lies in $A'_{k+1/2}(4N, \chi)$. Let $\tilde{A}_{k+1/2}^-(4N, \chi)$ be the subspace of $\tilde{A}_{k+1/2}^+(4N, \chi)$ generated by $\bar{\varphi}' = \otimes \bar{\varphi}'_v$ where $\bar{\varphi}'_2$ is the eigenvector for the other eigenvalue. Then $\tilde{A}_{k+1/2}^-(4N, \chi) = \tilde{A}_{k+1/2}^+(4N, \chi) \oplus \tilde{A}_{k+1/2}^-(4N, \chi)$. As $\tilde{A}_{k+1/2}^-(4N, \chi) \cap t(S'_{k+1/2}(4N, \chi)) = 0$, we get the corollary.

9.6. Local computation at the place 2. We compute the local factor as in § 8. The vector $\varphi_2$ is the unramified vector chosen as in subsection 8.1.
Proposition 9.5. Assume \( \mu_2(x) = |x|^{ir} \), with \( r \in \mathbb{R} \). Then
\[
e(\varphi_2, \psi) = 3/2|1 - q^{-1-2ir}|^{-2},
\]
\[
e(\hat{\varphi}_2, \psi)[D] = \begin{cases} 
3/4|1 + 2^{-1/2 - ir}|^{-2} & D \in 1 + P^2, \\
3/4|1 - 2^{-1-2ir}|^{-2}|D|^{-1} & \frac{P}{4} \in (1 + P^2) \cup (-1 + P^2).
\end{cases}
\]
Therefore, when \( D \in 1 + P^2 \),
\[
\frac{e(\varphi_2, \psi)}{e(\hat{\varphi}_2, \psi)[D]} = \begin{cases} 
2|D_2|L(\pi_2, 1/2) & D \in 1 + P^2, \\
2|D| & \frac{P}{4} \in (1 + P^2) \cup (-1 + P^2).
\end{cases}
\]

Proof. The formula for \( e(\varphi_2, \psi) \) is given in Proposition 8.1. One can use (8.5) to compute ||\( \hat{\varphi}_2 || \) when \( \mu_2(x) = |x|^{ir} \) where \( r \in \mathbb{R} \). Since \( F[2,1] \) and \( F[2,2^2] \) are perpendicular, we get
\[
||\hat{\varphi}_2 ||^2 = 1/8||F[2,1]||^2 + ||F[2,2^2]||^2
\]
Use the Iwasawa decomposition it is easy to show
\[
||\hat{\varphi}_2 ||^2 = (1/4 + 1/8)|D|^{-1} = 3/8|D|^{-1}.
\]
From Lemma 9.3, we get the claim for \( e(\hat{\varphi}_2, \psi)[D] \). From the formula on local \( L \)-factor in \([Go]\), we get the last statement of the Proposition. \( \square \)

10. A generalization of the Kohnen-Zagier formula

10.1. Statement of the Theorem. Let \( f(z) \) be a cusp form as in (1.1) with square free level \( N \) (odd) and weight \( 2k \). Let \( S_N \) be the set of primes \( p|N \). Let \( S \) be a (possibly empty) subset of \( S_0 \). Define \( D_S \) to be the set of fundamental discriminants \( D \) such that \( \frac{D}{p} = w_0 \) if and only if \( p \in S \). Then the set of fundamental discriminants is the disjoint union \( \cup_{S \subset S_N} D_S \). For \( D \) a fundamental discriminant, let \( T(D) \) be the set of \( p|N \) that also divides \( D \). Let \( \text{sgn}(D) = D/|D| \). The character \( \psi \) is defined as in \( \S \).

Theorem 10.1. Let \( S \subset S_N \) and \( s \) be the size of \( S \). Let \( N' = \prod_{p \in S} p \), let \( \chi = \prod_{p|2N} \chi(p) \) be any Dirichlet character of \((\mathbb{Z}/4NN')^\ast \) such that \( \chi(p) \equiv 1 \) when \( p|N', \chi(p)(-1) = -1 \) when \( p|N' \) and \( \chi(-1) = 1 \). There exists a unique (up to scalar multiple) cusp form \( g_S(z) \) in \( S_{k+1/2}(4NN', \chi) \), such that the following is true:

1. \( g_S(z) \) is a Shimura lift of \( f(z) \).
2. \( g_S(z) \) lies in the Kohnen space, i.e. if \( g_S(z) = \sum_{n=1}^\infty c(n)e^{2\pi inz} \), then \( c(n) = 0 \) when \((-1)^{s+k} n \equiv 2, 3 \mod 4 \).
3. \( c(D) = 0 \) if \( D > 0 \) is a fundamental discriminant with \((-1)^{s+k} D \notin D_S \).

Moreover for this \( g_S(z) \) and for \( D \in D_S \), if \((-1)^{s+k} \neq \text{sgn}(D) \), then \( L(f, D, k) = 0 \); if \((-1)^{s+k} = \text{sgn}(D) \), then
\[
|c(D)|^2 < g_S, g_S > = L(f, D, k) |D|^{k-1/2} \frac{(k-1)!}{\pi^k} 2^{\nu(N)-t} \prod_{p \in S} p^{t} p + 1.
\]
Here \( t \) is the size of \( T(D) \).
From subsection 9.1, the new form \( f(z) \) determines an irreducible cuspidal representation \( \pi \) of \( GL_2(\mathbb{A}_\mathbb{Q}) \) with trivial central character. Here we say \( g(z) \) is a Shimura lift of \( f(z) \) if \( \hat{\varphi} = t(g) \) lies in a space \( V_{\pi} \) where \( \pi \) satisfies \( \pi = S_v(\hat{\pi}) \) or \( \pi = S_{v^{-1}}(\hat{\pi}) \) (see Theorem 3.2 for the notion \( S_v(\cdot) \)).

The proof of the Theorem involves translating Theorem 4.3 into the language of cusp forms. We will set up the translation in subsections (10.2)–(10.5).

10.2. A Lemma on Atkin-Lehner involution. Let \( \hat{\varphi} = s(\hat{f}) \). For lack of reference, we give a proof of the following well known result.

**Lemma 10.2.** With the above definition, then \( \pi_p = \sigma(\mu, \mu^{-1}) \) with \( \mu(x) = |x|^{1/2} \chi_\tau(x) \), \( \tau \) a unit in \( \mathbb{Z}_p \). We have \( \hat{f} = w_p f \), with \( w_p = 1 \) when \( \tau \) is not a square, and \( w_p = -1 \) when \( \tau \) is a square. Moreover \( \epsilon(\pi_p, 1/2) = w_p \).

**Proof.** The first claim is in [G]. Since \( \hat{\varphi}(g_\infty) = \varphi(\hat{w}_p g_\infty) \), from left \( GL_2(\mathbb{Q})Z(A_\mathbb{Q}) \) invariance,

\[
\hat{\varphi}(g_\infty) = \varphi(g_\infty \prod_{v \neq \infty} \hat{w}_{p,v}^{-1}) = \pi(\prod_{v \neq \infty} \hat{w}_{p,v}^{-1}) \varphi(g_\infty)
\]

As \( \varphi = \otimes_{v} \varphi_v \), we get \( \hat{\varphi} = \varphi_\infty \otimes_{v \neq \infty} \pi_v(\hat{w}_{p,v}^{-1}) \varphi_v \). When \( v \) does not divide \( N \), \( \hat{w}_{p,v}^{-1} \in GL_2(\mathbb{Z}_v) \), thus fixes \( \varphi_v \). When \( v | N \) but \( v \neq p \), \( \hat{w}_{p,v}^{-1} \in K_{0,v} \), thus fixes \( \varphi_v \). When \( v = p \), \( \hat{w}_{p,v}(p a b) = \pi_v(\hat{w}_{p,v}^{-1}) \varphi_v \). As \( \hat{w}_p K_{0,p}(p w) \) lies in \( K_{0,p} \), the vector \( \pi_v(w_p)^{-1} \varphi_v \) is again fixed by \( K_{0,p} \), thus is a scalar multiple of \( \varphi_v \). Denote this multiple by \( w_p \). Then \( \varphi_v = w_p \varphi_v \) which clearly equals \( p \chi_\tau(p) \varphi_v \). Since \( \varphi_v(e) = -p \), we get \( w_p = -\chi_\tau(p) \varphi_v \) which gives the claim in the Lemma. The computation of \( \epsilon(\pi_p, 1/2) \) is given in [Go].

10.3. Definition of \( \epsilon(S) \). Let \( f \) and \( \pi \) be as before. Since \( \pi_v \) is unramified for all places \( v \) where \( v \neq \infty \) and \( |N|_v = 1 \), we can let \( \Sigma \) in Theorem 3.2 to be the set \( \{\infty\} \cup \Sigma \). Let \( S \) be a set as in the Theorem. Then it determines an \( \epsilon(S) = \{\pm 1\}^{\Sigma} \), where the component \( \epsilon(S)_p \) at \( p \in S \) is \( -w_p \), at \( p | N \) and \( p \not\in S \) is \( w_p \), and at \( \infty \) is \( (-1)^{s+k} \).

**Lemma 10.3.** We have \( \epsilon(\pi, 1/2) = \prod_{v \in \Sigma} \epsilon(S)_v \).

**Proof.** The product on the right is \( (-1)^k \prod_{v | N} w_p \). Since \( w_p = \epsilon(\pi_p, 1/2) \) by Lemma 10.2 and \( (-1)^k = \epsilon(\pi_{\infty}, 1/2) \) from [Go], we get the claim. 

From Theorem 3.2, associated to \( \pi \) and the character \( \psi_S(x) = \psi((-1)^{k+s} x) \) is the Shimura lift \( \pi^{\psi(S)} \). Here \( \pi^{\psi(S)} = \Theta(\pi \otimes \chi_D, \psi^D_S) \) for some \( D \in \mathbb{Q}^{\psi(S)}(\pi) \).
10.4. Relation between $D_S$ and $Q^e(S)(\pi)$. Given $D$ a fundamental discriminant, let $\epsilon_v(D) = (\frac{D}{\pi_v})$ for $v \in \Sigma$.

**Lemma 10.4.** When $v = \infty$, $\pi_\infty$ is a discrete series, $\epsilon_\infty(D) = sgn(D)$.

When $p | N$, $\pi_p$ is a special representation $\sigma^\tau_p$ as in subsection 8.3, where $\tau_v$ is a unit in $\mathbb{Z}_p$. Then $\epsilon_p(D) = w_p$ if $p | D$; $\epsilon_p(D) = (\frac{D}{p})$ when $D$ is a unit in $\mathbb{Z}_p$.

**Proof.** At $v = \infty$, $\pi_\infty \cong \pi_\infty \chi_D$, thus $\epsilon_\infty(D) = \chi_D(-1) = sgn(D)$. When $p | N$, as $\Theta(\pi_p, \psi)$ is either a special representation (when $\tau_v$ is not a square) or an odd Weil representation (when $\tau_v$ is a square). Note that $w_p = 1$ if and only if $\tau$ is not a square (see Lemma 10.2). In the case $\tau_v$ is not a square, $\Theta(\pi_p, \psi)$ has a nontrivial $\psi^D$-Whittaker model when $D$ is not a nonsquare unit. In the case $\tau_v$ is a square, then only when $D$ is a square does $\Theta(\pi_p, \psi)$ has a nontrivial $\psi^D$-Whittaker model. From Theorem 3.1, we get the result. \[\square\]

**Lemma 10.5.** When $D \in D_S$, $\epsilon_p(D) = \epsilon(S)_p$ for all $p | N$.

**Proof.** When $p \in S$, then $D$ is a unit in $\mathbb{Z}_p$, and $\epsilon_p(D) = (\frac{D}{p}) = -w_p = \epsilon(S)_p$. When $p \in S_N - S$, then either $p | D$ in which case $\epsilon_p(D) = w_p = \epsilon(S)_p$ or $D$ is a unit in $\mathbb{Z}_p$, in which case $\epsilon_p(D) = (\frac{D}{p}) = w_p = \epsilon(S)_p$. \[\square\]

As the set $Q^e(S)(\pi)$ consists of $D$ with $\epsilon_v(D) = \epsilon(S)_v$ for $v \in \Sigma$, from the above lemma and the formula for $\epsilon_\infty(D)$, we get

**Corollary 10.6.** A fundamental discriminant $D$ lies in $Q^e(S)(\pi)$ if and only if $D \in D_S$ and $\epsilon(D) = (\epsilon(S))^k = sgn(D)$.

10.5. Description of $g_S$. The cusp forms $g_S$ in the Theorem is taken to be the inverse image $t^{-1}((\tilde{\varphi}_S))$ of some vector $\tilde{\varphi}_S$ in the space of $\tilde{\pi}^e(S)$. We describe the choice of $\tilde{\varphi}_S = \otimes \tilde{\varphi}_v$.

Using the explicit description of theta correspondence in [W3], we get the following description on the local components of $\tilde{\pi}^e(S) = \otimes \tilde{\pi}^e(v)$ of $\varphi$ such that $\varphi = s(f)$. Below is the description of $\pi_v, \tilde{\pi}^e(v)$ and the choice of the vectors $\tilde{\varphi}_v$.

10.5.1 When $v = \infty$, $\pi_v$ is the discrete series $\sigma(\mu'_\infty; \mu^{-1}_\infty)$, with $\mu_\infty(x) = |x|^{k-1/2}(\text{sgn}\ x)^k$. When $\epsilon_\infty(D) = (\epsilon(S))_\infty$, we get $sgn(D) = (-1)^{s+k}$ thus $\psi^D_S = \psi^{|D|}$. Thus $\tilde{\pi}^e(\infty) = \Theta(\pi_\infty \otimes \chi_D, \psi^{|D|})$. As $\pi_\infty \otimes \chi_D \cong \pi_\infty \otimes \chi_D$ and $|D|$ and $1$ are in the same square class, we get $\tilde{\pi}^e(\infty) = \Theta(\pi_\infty, \psi) = \delta_\infty(\mu_\infty)$, ([W3]). We take $\tilde{\varphi}_\infty$ to be the vector with minimal weight.

10.5.2 At $p \notin \Sigma$, $\pi_p = \pi(\mu_p, \mu^{-1}_p)$ with $\mu_p$ an unramified character. From a well known result of Deligne, the unramified characters $\mu_p$ has the form $\mu_p(x) = |s|^{ir}$ with $r \in \mathbb{R}$, (this is the Ramanujian conjecture for the integral
weight forms). Then  \( \tilde{\pi}^{(S)}_p = \Theta(\pi_p, \psi_S) = \tilde{\pi}(\mu_p \chi_{-1}^{*+k}, \psi) \). We take  \( \tilde{\varphi}_p \) to be the unramified vector in this unramified representation when  \( p \neq 2 \). We let  \( \tilde{\varphi}_2 \) be the vector defined by (9.4) with  \( k' = k + s \).

(10.5.3) At  \( p \in S_N - S \),  \( \pi_p = \sigma^\tau \) with  \( \tau \in \mathbb{Z}_p^* \). Let  \( D \) be a unit in  \( \mathbb{Z}_p \)

such that  \( \tau D \) is not a square, then  \( \epsilon_p(D) = w_p = \epsilon(S)_p \). Thus  \( \tilde{\pi}^{(S)}_p = \Theta(\sigma^\tau, \psi_S^D) = \sigma^\delta(\psi_S^D) \); here  \( \delta \) is any nonsquare unit. We take  \( \tilde{\varphi}_p \) to be the vector in Lemma 8.3.

(10.5.4) When  \( p \in S \), again  \( \pi_p = \sigma^\tau \) with  \( \tau \in \mathbb{Z}_p^* \). Let  \( D \) be a unit in  \( \mathbb{Z}_p \)

such that  \( \tau D \) is a square, then  \( \epsilon_p(D) = -w_p = \epsilon(S)_p \). Thus  \( \tilde{\pi}^{(S)}_p = \Theta(\sigma^1, \psi_S^D) \) which is  \( r^\sigma_g(D) \) from subsection 8.3. Let  \( \chi_{(p)} \) be the character on  \( \mathbb{Z}_p^* \) defined in the Theorem, we let  \( \tilde{\varphi}_p = \Phi_{\chi_{(p)}} \) where  \( \Phi_{\chi_{(p)}} \) is defined in Lemma 8.5.

Each of the choices of  \( \tilde{\varphi}_p \) is determined unique up to a scalar multiple. Let  \( \tilde{\varphi} = \otimes \tilde{\varphi}_v \). We define the cusp form  \( g_S(z) \) to be  \( t^{-1}(\tilde{\varphi}) \).

10.6. **Proof of the Theorem.**

*Proof.* As  \( S_{\psi_S}(\tilde{\pi}^{(S)}_S) = \pi \), we see  \( g_S(z) \) is a Shimura lift of  \( f \). We can check  \( \tilde{\varphi} = \otimes \tilde{\varphi}_v \in A'_{k+1/2}(4NN', \chi) \). Thus  \( g_S(z) \in S'_{k+1/2}(4NN', \chi) \). It lies in the Kohnen space because of our choice of  \( \tilde{\varphi}_2 \). If  \( D \) is a fundamental discriminant with  \( \pm D \not\in D_S \), then  \( \pm D \not\in \mathbb{Q}^{(S)}(\pi) \). By Theorem 4.2, we get  \( d_{\tilde{\pi}^{(S)}(\Sigma \cup \{2\}, \psi_S^D) = 0. \) As  \( \psi^D \) is one of  \( \psi_S^D \), we get  \( |D| \) from the consideration in §9.

Next we show the uniqueness. If  \( g(z) \neq 0 \) is a Shimura lift of  \( f \) such that  \( t(g) \) lies in the space of  \( \tilde{\pi} \), we show  \( \tilde{\pi} = \tilde{\pi}^{(S)} \). As  \( \tilde{\pi}_\infty \) is a holomorphic discrete series, by Theorem 3.2  \( \tilde{\pi} = \Theta(\pi \otimes \chi_{D_1}, \psi_{[D_1]}) \) for some  \( D_1 \). As  \( g(z) \neq 0 \),  \( c((-1)^{s+k}D_2) \neq 0 \) for some  \( D_2 \in D_S \). The condition implies that  \( D_2 \in \mathbb{Q}^{(S)}(\pi) \) and  \( \tilde{\pi}_v \) has nontrivial  \( \psi_{[D_2]} \)-Whittaker model at all places  \( v \). From Theorem 3.1, we see  \( \tilde{\pi} = \Theta(\pi \otimes \chi_{D_2}, \psi_{[D_2]}) \), where  \( \alpha = |D_1D_2| \) is  \( \pm 1 \). Examine the component  \( \tilde{\pi}_2 \). As  \( \chi_{(2)} (-1) = (-1)^s \) under our assumptions, we get  \( \tilde{\pi}_2 = \Theta(\pi \otimes \chi_{-1}^{*+k}, \psi) \), thus we see  \( \tilde{\pi} = \Theta(\pi \otimes \chi_{D_2}, \psi_{[D_2]}) = \tilde{\pi}^{(S)} \). From Corollary 9.4 and the fact that  \( \tilde{A}^{+}_{k+1/2}(4N, \chi) \) is one dimensional, we get the uniqueness of  \( g_S(z) \).

We now prove the identity (10.1). Let  \( D \in D_S \). If  \( (-1)^{s+k} = \text{sgn}(D) \), then from equation (3.1), Lemma 10.3 and 10.5, we get:

\[
\epsilon(\pi \otimes \chi_D, 1/2) = \epsilon(\pi, 1/2) \prod_{v \in \Sigma} \epsilon_v(D) = -\epsilon(\pi, 1/2) \prod_{v \in \Sigma} \epsilon(S)_v = -1.
\]

Thus  \( L(\pi \otimes \chi_D, 1/2) = 0 \), i.e.  \( L(f, D, k) = 0 \).

When  \( (-1)^{s+k} = \text{sgn}(D) \), then  \( D \in \mathbb{Q}^{(S)}(\pi) \), thus we can apply Theorem 4.3 to get:

\[
(10.2) \quad L^{\Sigma \cup \{2\}}(\pi \otimes \chi_D, 1/2) = \frac{|d_{\tilde{\pi}(S)}(\Sigma \cup \{2\}, \psi_S^D)|^2}{|d_{\pi}(\Sigma \cup \{2\}, \psi_S)|^2} \prod_{v \in \Sigma \cup \{2\}} |D_v|.
\]
From Lemma 2.3, \( d_\pi(\Sigma \cup \{2\}, \psi_S) = d_\pi(\Sigma \cup \{2\}, \psi) \). Observe also \( \psi_S^D = \psi^{|D|} \). From the explicit description of \( d_{\pi(s)}(\Sigma \cup \{2\}, \psi^{|D|}) \) and \( d_\pi(\Sigma \cup \{2\}, \psi) \), we get:

\[
(10.3) \quad \frac{|d_{\pi(s)}(\Sigma \cup \{2\}, \psi^{|D|})|^2}{|d_\pi(\Sigma \cup \{2\}, \psi)|^2} = \frac{|\tilde{W}^{|D|}_\varphi(e)|^2 ||\varphi||^2}{|W_\varphi(e)|^2 ||\varphi_S||^2} \prod_{\nu \in \Sigma \cup \{2\}} \frac{\epsilon(\varphi, \psi^{|D|})}{\epsilon(\varphi, \psi)}.
\]

Recall the results on the local factors from Propositions 8.8, 8.4, 8.7 and 9.5:

\[
(10.4) \quad \frac{\epsilon(\varphi, \psi)}{\epsilon(\varphi, \psi^{|D|})} = \begin{cases} 
\frac{1}{\pi^4(1-|D|)|D|^{1/2+k}\pi^{-k}(k-1)!} & p = \infty, \\
2L(\pi_p \otimes \chi_D, 1/2)|D|_p & p \in S_N - S, p \not\mid N, \\
L(\pi_p \otimes \chi_D, 1/2)|D|_p & p \in S_N - S, p \not\mid N, \\
2(1+q^{-1})^{-1}L(\pi_v \otimes \chi_D, 1/2) & p = 2, D \in 1 + P^2, \\
2|D|_2 L(\pi_2, 1/2) & p = 2, D \not\in (2 + P^2) \cup (-1 + P^2).
\end{cases}
\]

From subsection 9.1, \( W_\varphi(e) = e^{-2\pi} \), and from (9.2), \( \tilde{W}^{|D|}_\varphi(e) = e^{-2\pi|D|}c(|D|) \). Thus we get

\[
(10.5) \quad L^{\infty \cup \{2\}}(\pi \otimes \chi_D, 1/2)_2 = (1/2)^{\nu(N)}|D|^{-k+1/2} \frac{\pi^k}{(k-1)!} \frac{|c(|D|)|^2 ||\varphi||^2}{||\varphi_S||^2} \prod_{p \in S}(1+p^{-1}).
\]

Here we set \( l_2 \) to be \( L(\pi_2, 1/2) \) when \( D \equiv 1 \mod 4 \) and to be 1 when \( D \equiv 0 \mod 4 \).

**Remark 10.7.** We now notice some differences between \( L(\pi \otimes \chi_D, s) \) and \( L(f, D, s') \). First \( L(f, D, s') \) does not have factor at \( \infty \). Secondly, because \( \chi_D(2) \) over \( v = 2 \) is not the same as \( (\frac{D}{2}) \), we need to correct the factor at the place \( v = 2 \). We have actually

\[
L(f, D, k) = L^{\infty \cup \{2\}}(\pi \otimes \chi_D, 1/2)_2.
\]

From the above remark, Lemma 9.1 and (10.5), we get:

\[
(10.6) \quad L(f, D, k) = (1/2)^{\nu(N)-t}|D|^{-k+1/2} \frac{\pi^k}{(k-1)!} \frac{|c(|D|)|^2 < f, f >}{< g, g >}.
\]

Therefore we get (10.1). \( \square \)

### 10.7. Some examples.

**Example 1:** When \( S \) is empty, \( g_S(z) \) is the \( g(z) \) in (1.1). If \( T(D) \) is also empty, we recover (1.1). If \( T(D) \) is nonempty, (1.1) should be revised to:

\[
\frac{|c(|D|)|^2}{< g, g >} = |D|^{k-1/2} \frac{(k-1)!}{\pi^k} 2^{\nu(N)-t} L(f, D, k).
\]

**Example 2:** Look at the case when \( f(z) \) is the new form of weight 2 and level 11. Such a form exists and is unique, with \( w_{11} = -1 \). The theorem says there is a cusp form \( g_{(11)}(z) \) in \( S'_{3/2}(484, \chi) \), where \( \chi \) satisfies the condition
in the Theorem, (for example $\chi(x) = (-1)^{(x-1)/2}(\frac{x}{p})$), such that $c(n) = 0$ whenever $n \equiv 2, 3 \mod 4$, and when the fundamental discriminant $D > 0$ is such that $\left(\frac{D}{p}\right) = 1$,

$$\frac{|c(D)|^2}{<g_{\{11\}}, g_{\{11\}}>^2} = \frac{|D|^{1/2}}{6\pi} \frac{11 L(f, D, k)}{<f, f>}.$$  

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