On a Boltzmann-type price formation model

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In this paper, we present a Boltzmann-type price formation model, which is motivated by a parabolic free boundary model for the evolution of price presented by Lasry and Lions in 2007. We discuss the mathematical analysis of the Boltzmann-type model and show that its solutions converge to solutions of the model by Lasry and Lions as the transaction rate tends to infinity. Furthermore, we analyse the behaviour of the initial layer on the fast time scale and illustrate the price dynamics with various numerical experiments.

1. A Boltzmann-type model for price formation

Market microstructure analysis studies the trading mechanisms of assets in (financial) markets. According to O’Hara [1], markets have two principal functions: they provide liquidity and facilitate the price. The evolution of the price is influenced by the trading system and the nature of the players, and is hence an emergent phenomenon from microscopic interaction that calls for advanced mathematical modelling and analysis.

While there has been a lot of research in economics and statistics related to either the micro- or macro-structure of markets, few investigations have dealt with systematically analysing the micro-macro transition. In 2007, Lasry & Lions [2] revived mathematical interest by introducing a price formation model that describes the evolution of price by a system of
parabolic equations for trader densities (as functions of the bid–ask price), with the agreed price entering as a free boundary. The authors motivated the model using mean field game theory, but the detailed microscopic origin remained unclear.

In this paper, we provide a simple agent-based trade model with standard stochastic price fluctuations together with discrete trading events. By modelling trading events between vendors and buyers as kinetic collisions, we obtain a Boltzmann-type model for the densities. Then, we prove rigorously that, in the limit of large trading frequencies, the proposed Boltzmann model converges to the Lasry and Lions free boundary problem (FBP). We also analyse other asymptotics beyond the scales that the free boundary model can describe. Hence, we provide a basis for deriving macroscopic limits from the microscopic structure of trading events, which allows for various generalizations to make the model more realistic.

To set up the model, we consider the price formation process of a certain good which is traded between two groups, namely a group of buyers and a group of vendors. The groups are described by the two positive densities \( f_\sigma \) and \( g_\sigma \) between two groups, namely a group of buyers and a group of vendors, respectively, and \( \sigma \) for all times \( t \geq 0 \). The volume of transactions at a price \( x \) is given by

\[
\mu^k(x) = k f^k(x,t) g^k(x,t).
\]

Note that we have conservation of the number of buyers and vendors, i.e.

\[
\int_{\mathbb{R}} f^k(x,t) \, dx = \int_{\mathbb{R}} f_\sigma(x) \, dx \quad \text{and} \quad \int_{\mathbb{R}} g^k(x,t) \, dx = \int_{\mathbb{R}} g_\sigma(x) \, dx,
\]

for all times \( t > 0 \).

The mathematical modelling of (1.1) was inspired by a price formation model presented by Lasry & Lions [2]. They considered the same situation described above, but proposed a parabolic FBP to model the evolution of the price. The model presented by Lasry and Lions reads

\[
f_\sigma(x,t) = \frac{\sigma^2}{2} f_{xx}(x,t) + \lambda(t) \delta(x - p(t) + a) \quad \text{for} \quad x < p(t) \quad \text{and} \quad f(x,t) = 0 \quad \text{for} \quad x > p(t) \tag{1.3a}
\]

and

\[
g_\sigma(x,t) = \frac{\sigma^2}{2} g_{xx}(x,t) + \lambda(t) \delta(x - p(t) - a) \quad \text{for} \quad x > p(t) \quad \text{and} \quad g(x,t) = 0 \quad \text{for} \quad x < p(t), \tag{1.3b}
\]
where the free boundary $p = p(t)$ denotes the agreed price of the trading good at time $t$ and $\lambda(t) = -f_x(p(t), t) = g_x(p(t), t)$. Note that the difference in buyer and vendor densities $v = f - g$ satisfies the following equation:

$$v_t(x, t) = v_{xx}(x, t) + \lambda(t)(\delta(x - p(t)) + \delta(x - p(t) - a)).$$

(1.4)

The function $\lambda(t)$ is the transaction rate at time $t$, which corresponds to the flux of buyers and vendors. The Dirac deltas correspond to trading events that take place at the agreed price $p = p(t)$, shifted by the transaction cost $a$. The analysis of system (1.3) was studied in a number of papers; see Markowich et al. [3], Chayes et al. [4] and Caffarelli et al. [5,6].

The main difference between system (1.1) and system (1.3) is that the agreed price $p = p(t)$ enters as a free boundary in (1.3). Furthermore, the density of buyers $f$ and vendors $g$ is zero, if the price is greater or smaller than the agreed price. Therefore, agents trade only at the agreed price $p = p(t)$ and not at all prices as in the Boltzmann-type model. In (1.1), the mean, median or maximum of $\mu^k$ gives an estimate for the price. In the following, we shall show that solutions of the Boltzmann-type equation (1.1) converge to solutions of (1.3) as the transaction rate tends to infinity, i.e. $k \to \infty$, thus giving a mathematical justification for (1.3) as a pure formation model.

This limit is motivated by the agent-based interpretation of (1.1), because the number of trading events increases with the number of interacting agents.

Another interesting question is the behaviour of an initial layer on the fast time scale, i.e. the limiting functions $f^\infty$ and $g^\infty$ at time $t = 0$ (obtained in the limit $k \to \infty$) differ from the initial data $(f_1, g_1)$ and $(f_t, g_t)$. This problem is motivated by our previous work on initial layers in the price formation model by Lasry and Lions (1.3). In Caffarelli et al. [5], we show that a fat-free boundary, i.e. an interval $[p_{\text{min}}, p_{\text{max}}]$, where $f(p(t), t) = 0$ for all $p \in [p_{\text{min}}, p_{\text{max}}]$, cannot occur. This question can be translated for the Boltzmann-type model (1.1): what happens with an initial layer as $\tau \to \infty$ and $k \to \infty$. We present sufficient conditions on the initial data $f_1$ and $g_1$, which allows us to identify the fast scale limits and discuss whether the limits $\tau \to \infty$ and $k \to \infty$ commute.

In many markets, goods are traded at very high frequencies and little or no transaction costs. This motivates the study of (1.1) in the limit $k \to \infty$, $a \to 0$ and $ka = c > 0$. Furthermore, we compare these results with the limiting behaviour of the Lasry and Lions model (1.3) as $a \to 0$.

Finally, we present numerical simulations of the Boltzmann-type model (1.1) and the model by Lasry and Lions (1.3) to illustrate the different behaviours in various situations. We focus particularly on the behaviour of (1.1) for the different limiting cases and discuss the simulations with respect to the analytic results.

This paper is organized as follows: we start with a detailed presentation of the mathematical modelling of (1.1) in §2. In §3, we show that solutions of (1.1) converge to solutions of (1.3) as $k \to \infty$. The initial layer problem for (1.1) is discussed in §4, the scaling limit $k \to \infty, a \to 0$ and $ka = c > 0$ in §5. This limit corresponds to high-frequency trading, where computers trade goods on a rapid basis with little or no transaction costs involved. The behaviour of both models is illustrated by numerical simulations in §6.

2. Modelling: from agent behaviour to Boltzmann-type equations

In order to describe price formation at a reasonably simple agent-based level, we consider a set-up of a large number, say $N$, of buyers and a large number, $M$, of vendors. With the exception of discrete events, the price changes are subject to random fluctuations, which we model as Brownian motions with diffusivity $\sigma$. The trading events can be modelled as kinetic collisions. After the usual kinetic limit $N \to \infty, M \to \infty$, we associate the density $f = f(x, t)$ with the group of buyers at time $t$, and the density $g = g(x, t)$ with the group of vendors at time $t$. Here, $f(x, t)$ and $g(x, t)$ denote the fraction of the buyers and vendors willing to trade in an infinitesimal interval around $x$ at time $t$. If a buyer with state $x$ meets a vendor with state $y$, then they will trade with a certain probability depending on their state. After collision, i.e. trade, a buyer becomes a vendor and vice versa. Owing to conservation and indistinguishability, we do not need to take
into account the change of roles, but just model a standard collision \((x, y) \rightarrow (x', y')\) with collision kernel \(K = K(x, y, x', y')\). More precisely, \(K(x, y, x', y')\) counts the number of trading events per unit time of buyers willing to buy at price \(x\) and reselling after the trading event at price \(x'\), and the number of vendors willing to sell at price \(y\) and re-buying after the trading event at price \(y'\). Using the notations for densities as above, we arrive (with appropriate time scaling) at

\[
f_t(x, t) - \frac{\sigma^2}{2} f_{xx}(x, t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} K(x', y', x, y) f(x', t) g(y', t) \, dy \, dx' \, dy'
- \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} K(x, y, x', y') f(x, t) g(y, t) \, dy \, dx' \, dy'
\]  

(2.1a) and

\[
g_t(x, t) - \frac{\sigma^2}{2} g_{xx}(x, t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} K(x', y', y, x) f(y', t) g(x', t) \, dx \, dx' \, dy'
- \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} K(y, x, x', y') f(y, t) g(x, t) \, dx \, dx' \, dy'
\]  

(2.1b)

The peculiar aspect in the modelling of the collisions related to the Lasry–Lions approach is introduced by the transaction costs. Right after selling at a certain price, it only makes sense to re-buy at a price that is at least lower than the previous execution price minus the transaction costs, and similar reasoning applies to the buyers. Thus, if we assume symmetric transaction costs \(a\) and denote by \(r(x, y)\) the price at which a buyer with state \(x\) and a vendor with state \(y\) trade, then

\[x' = r(x, y) - a, \quad y' = r(x, y) + a,\]  

(2.2) and hence \(K\) is of the form

\[K(x, y, x', y') = K_0(x, y) \delta(x' - r(x, y) + a) \delta(y' - r(x, y) - a),\]  

(2.3)

where \(\delta\) denotes the Dirac \(\delta\)-distribution. The detailed properties of \(K_0\) and \(r\) now depend on the modelling of the limit order book (see [7–11]). Here, we make a simple assumption that \(x\) and \(y\) can be interpreted as the bids of buyers and vendors, which are tried to be matched exactly. Hence, \(K_0\) is centred around \(x = y\), and the simplest choice is to assume

\[K_0(x, y) = k \delta(x - y),\]  

(2.4)

where the constant \(k\) is the trading frequency. We need to specify only

\[r(x, x) = x.\]

This leads to the collision kernel

\[K(x, y, x', y') = k \delta(x - y) \delta(x' - r(x, y) + a) \delta(y' - r(x, y) - a).\]

For smooth test functions \(\varphi = \varphi(x, y, x', y')\), we have

\[\langle K, \varphi \rangle = \int_{\mathbb{R}} \varphi(x, x, r(x, x) - a, r(x, x) + a) \, dx\]

\[= \int_{\mathbb{R}} \varphi(x, x, x - a, x + a) \, dx.\]

Applying this to (2.1) leads to (1.1) (using the appropriate notation \(f^k\) and \(g^k\) instead of \(f\) and \(g\)).

For large numbers \(N\) and \(M\), it appears natural to consider the case of large \(k\), because a high potential for transactions is available. One observes that the volume of transactions in (1.1) is given by \(\mu^k = k^4 g^k\); hence, the scaled density \(\rho^k = c f^k g^k\) with \(c = 1/\int_{\mathbb{R}} f^k g^k \, dx\) yields a density of currently traded prices. If \(\rho^k\) is well centred, then the mean, median or maximum \(\rho^k\) will give an
estimate of the price \( p(t) \), which we will also exploit numerically in comparison with the Lasry–Lions model. For large \( k \), we will see that \( \rho^k \) concentrates to a Dirac \( \delta \) distribution centred at \( p(t) \). Note that the mean supply and demand price satisfy

\[
\frac{d}{dt} \int_{\mathbb{R}} x f^k(x, t) \, dx = -a \int_{\mathbb{R}} k f^k(x, t) g^k(x, t) \, dx \leq 0
\]

and

\[
\frac{d}{dt} \int_{\mathbb{R}} x g^k(x, t) \, dx = a \int_{\mathbb{R}} k f^k(x, t) g^k(x, t) \, dx \geq 0.
\]

This means that the average demand price is decreasing, whereas the average supply price increases as long as \( f^k g^k \) does not vanish. This shift is clearly inherent in the collision model for the trades, because each collision decreases the demand price of an agent and increases the supply price of another one.

We finally mention that the above modelling is reasonably simple for studying basic effects, but the agent-based interpretation easily allows various generalizations that can make it more realistic. An obvious first example is replacing the Brownian motion by more general stochastic differential equations, which simply changes the differential operator to some other parabolic equation, but leaves the collisions unchanged. Even herding effects could be modelled this way, for example, by making the drift of a buyer dependent on the empirical density \( f^k \), such as \( \int_{\mathbb{R}} x f^k(x, t) \, dx \). The extension to more complicated trading models is inherent in our model by specifying \( K_0 \) and \( r \). A simple alternative model is

\[
K_0(x, y) = k \mathbf{1}_{[y \leq x]}, \quad r(x, y) = \frac{x + y}{2}, \quad (2.5)
\]

which corresponds to a self-organized trading using the mean value of the prices when \( y \leq x \).

### 3. Limiting equations as \( k \to \infty \)

In this section, we show that the difference \( f^k - g^k \) of solutions of the system (1.1) converges to the solution of (1.4) as the transaction rate \( k \) tends to infinity and that, as \( k \to \infty \), \( f = (f^k - g^k)^+ \) and \( g = (f^k - g^k)^- \). Without loss of generality, we set \( \sigma^2/2 = 1 \) in the rest of the paper. Note that the existence and uniqueness of non-negative solutions of (1.1) for \( k \geq 0 \) follow trivially from the estimates stated in this section. For the following, we assume that \( f_1 \) and \( g_1 \) are independent of \( k \) and satisfy

(A) Let \( f_1, g_1 \geq 0 \) on \( \mathbb{R} \) and \( f_1, g_1 \in \mathcal{S}(\mathbb{R}) \).

Note that assumption (A) can be weakened, i.e.

\[
f_1, g_1 \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap H^1(\mathbb{R}) \text{ with sufficiently fast decay as } |x| \to \infty,
\]

using parabolic regularity estimates, standard, in particular, for the one-dimensional heat equation.

**Proposition 3.1.** Let assumption (A) be satisfied. Then, system (1.1) has unique positive solutions \( f^k, g^k \in L^\infty(0, \infty; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})) \) bounded uniformly as \( k \to \infty \). Furthermore

\[
f^k g^k \to 0 \quad \text{in } D'(\mathbb{R} \times [0, \infty]). \quad (3.1)
\]

**Proof.** Note that the function \( u(x, t) = f^k(x, t) + g^k(x + a, t) \) satisfies the initial value problem (IVP) for the heat equation

\[
\begin{cases}
u_t(x, t) = \nu_{xx}(x, t) \\ u(x, 0) = f_1(x) + g_1(x + a) \end{cases}
\]

and

\[
u_t(x, t) = a \int_{\mathbb{R}} k f^k(x, t) g^k(x, t) \, dx \geq 0.
\]
The heat equation has a unique positive solution $u = u(x, t) \in L^\infty(0, \infty; L^\infty(\mathbb{R}))$ for initial data $f_1, g_1 \in \mathcal{S}(\mathbb{R})$, $g_1(x) \geq 0, f_1(x) \geq 0$ independent of $k$. Hence, we deduce that the weak limits $f^\infty, g^\infty$ satisfy

$$ f^\infty(t, x) \in L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R}) \quad \text{and} \quad g^\infty(t, x) \in L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R}) \quad \text{uniformly in } t > 0. $$

To show the weak convergence of $f^k$ to zero, we rewrite the first equation in (1.1) as

$$ \frac{1}{k} f^k_t(x, t) - \frac{1}{k} f^k(x, t) = -f^k(x, t) g^k(x, t) + f^k(x + a, t) g^k(x + a, t). $$

We define $\xi^k(x, t) = f^k(x, t) g^k(x, t)$, $\xi^k \in L^\infty(0, \infty; L^1_+(\mathbb{R}) \cap L^\infty(\mathbb{R}))$. Then, in the limit $k \to \infty$, we deduce that

$$ -\xi^k(x, t) + \xi^k(x + a, t) = o(1) \in \mathcal{D}'(\mathbb{R} \times [0, \infty)) \Rightarrow \xi^\infty(x, t) = \xi^\infty(x + a, t). $$

The only periodic function in $L^1(\mathbb{R})$ is the zero-function, which concludes the proof.

Next, we introduce, as in Caffarelli et al. [5,6], the functions

$$ F^k(x, t) = \sum_{l=0}^{\infty} f^k(x + al, t) \quad \text{and} \quad G^k(x, t) = \sum_{l=0}^{\infty} g^k(x - al, t). \quad (3.3) $$

**Proposition 3.2.** Let assumption (A) be satisfied. The functions $F^k, G^k$ defined by the infinite series (3.3) satisfy $F^k, G^k \in L^\infty(0, \infty; L^\infty(\mathbb{R}))$. The series converges locally uniform in $x$ and $t$ as $k \to \infty$.

**Proof.** Because $f^k, g^k \in \mathcal{S}(\mathbb{R})$, we can estimate the series $F^k$ by the following integral:

$$ I(x) = \int_0^{\infty} \frac{c}{(1 + |x + y|)^2} \ dy = \int_x^{\infty} \frac{c}{(1 + |z|)^2} \ dz \quad \text{for some } c > 0. $$

For $x > 0$, the integral $I(x) = (1/(q - 1))(1/(1 + x))$ tends to $0$ as $x \to \infty$. In the case $x < 0$, we obtain the following estimate for the integral:

$$ I(x) \leq c_1 + \frac{c_2}{(1 + |x|)}, $$

where $c_1, c_2$ are independent of $x$.

Note that $F^k$ and $G^k$ satisfy

$$ F^k_t(x, t) = -k f^k(x, t) g^k(x, t) + F^k_{xx}(x, t) \quad (3.4a) $$

and

$$ G^k_t(x, t) = -k f^k(x, t) g^k(x, t) + G^k_{xx}(x, t). \quad (3.4b) $$

Then, the difference $\Phi := F^k - G^k$ satisfies the heat equation $\Phi_t = \Phi_{xx}$. Note that $\Phi$ is independent of $k$ for $f_1, g_1$ independent of $k$. From system (3.4), we deduce that $\mu_k = k f^k g^k$ converges as $k \to \infty$ to a locally bounded positive measure on $\mathbb{R} \times [0, \infty)$. Then, the limiting functions $f^\infty$ and $g^\infty$ satisfy the following system:

$$ f^\infty_t(x, t) = -\mu^\infty(x, t) + \mu^\infty(x - a, t) + f^\infty_{xx}(x, t) \quad (3.5a) $$

and

$$ g^\infty_t(x, t) = -\mu^\infty(x, t) + \mu^\infty(x - a, t) + g^\infty_{xx}(x, t). \quad (3.5b) $$

Note that the difference $v = f^\infty - g^\infty$ satisfies a parabolic partial differential equation with a similar structure to the price formation model (1.4), i.e.

$$ v_t(x, t) = v_{xx}(x, t) + \mu^\infty(x + a, t) - \mu^\infty(x - a, t). \quad (3.6) $$
(a) *A priori estimates*

We derive various *a priori* estimates for the solutions $f^k$ and $g^k$ of system (1.1), which will be used for the identification of the limiting system as $k \to \infty$.

**Proposition 3.3.** The solution of (1.1) satisfies the following *a priori* estimate:

$$\int_0^T \int_{\mathbb{R}} \left( f^k(x,t) + g^k(x,t) \right)^2 \, dx \, dt \leq \text{const.},$$

(3.7)

uniformly as $k \to \infty$.

**Proof.** In the following calculations, we neglect the dependence of the functions $f^k$, $g^k$, $u$, $F^k$ on the variables $x$ and $t$, and only state their arguments if necessary. We multiply system (1.1) by $f^k$ and $g^k$, respectively, and obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (f^k)^2 \, dx = -k \int_{\mathbb{R}} (f^k)^2 g^k \, dx + k \int_{\mathbb{R}} f^k(x+a,t) f^k(x,t) g^k(x+a,t) \, dx$$

$$- \int_{\mathbb{R}} (f^k_x)^2 \, dx$$

and

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} (g^k)^2 \, dx = -k \int_{\mathbb{R}} (g^k)^2 f^k \, dx + k \int_{\mathbb{R}} f^k(x-a,t) g^k(x,t) g^k(x-a,t) \, dx$$

$$- \int_{\mathbb{R}} (g^k_x)^2 \, dx.$$

With changes of variables in the second integral on the right-hand side of both equations, we deduce for the sum of both equations that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} ((f^k)^2 + (g^k)^2) \, dx = -k \int_{\mathbb{R}} \left[ f^k g^k (f^k + g^k) - f^k g^k (f^k(x-a,t) + g^k(x+a,t)) \right] \, dx$$

$$- \int_{\mathbb{R}} (g^k_x + f^k_x)^2 \, dx.$$

Next, we use $g^k(x+a,t) = u(x,t) - f^k(x,t)$ and $f^k(x-a,t) = u(x-a,t) - g^k(x,t)$ to obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}} ((f^k)^2 + (g^k)^2) \, dx = -k \int_{\mathbb{R}} \left[ f^k g^k (f^k + g^k) + (f^k)^2 g^k + f^k (g^k)^2 \right] \, dx$$

$$+ \int_{\mathbb{R}} k f^k g^k (u(x,t) - u(x-a,t)) \, dx - \int_{\mathbb{R}} (g^k_x + f^k_x)^2 \, dx.$$

(3.8)

Multiplication of (3.4) by $u = u(x,t)$ and, respectively, by $u(x+a,t)$ gives

$$\int_{\mathbb{R}} (F^k u)_t \, dx - \int_{\mathbb{R}} F^k u_t \, dx = \int_{\mathbb{R}} F^k u_{xx} \, dx - k \int_{\mathbb{R}} f^k g^k u \, dx.$$

Note that all integrals are well defined, because $u = u(x,t)$ decays algebraically as $|x| \to \infty$. Hence, we find after integration over the time interval $(0, T)$ that

$$\int_0^T \int_{\mathbb{R}} k f^k(x,t) g^k(x,t) u(x,t) + u(x+a,t) \, dx \, dt \leq \text{const.}$$

(3.9)

uniformly as $k \to \infty$. Using (3.9) in (3.8) concludes the proof. \[\blacksquare\]
For a function generalized version of the Aubin–Lions lemma (see Lions [12]), given by:

\[ \lim_{k \to \infty} f_k = f, \quad \lim_{k \to \infty} g_k = g \]

we obtain

\[ \int_0^T \int_{\Omega} k^2 f_k \varphi \, dx \, dt \leq \int_0^T \int_{\Omega} F^k \varphi_i \, dx \, dt + \int_0^T \int_{\Omega} F^k \varphi \, dx \, dt + \int_0^T \int_{\Omega} F^k \varphi_{xx} \, dx \, dt \leq \text{const.} \]

If \( \varphi(\cdot, t) \) has compact support in \((-3R, 3R)\) and \( \varphi \equiv 1 \) in \((-2R, 2R)\), then we deduce that

\[ \int_0^T \int_{|x|<R} k^2 f_k(x,t) \varphi(x,t) \, dx \, dt \leq \text{const.} \]

uniformly in \( k \). Then, the functions \( f_k \) and \( g_k \) satisfy

\[ f_k, g_k \in L^1_{\text{loc}}(\mathbb{R} \times [0, \infty)) + L^2_{\text{loc}}(0, \infty; H^{-1}(\mathbb{R})) \] (3.10) and

\[ f_k, g_k \in L^2_{\text{loc}}(0, \infty; H^1_{\text{loc}}(\mathbb{R})) \] (3.11)

uniformly in \( k \).

**b) Strong convergence**

Let \( \Omega \) be an interval on the real line \( \Omega = (-R, R) \). To show strong convergence of (3.1), we use a generalized version of the Aubin–Lions lemma (see Lions [12]), given by:

**Theorem 3.4 (generalized version of the Aubin–Lions lemma).** Let \( B_0 \) be a normed linear space embedded compactly into another normed linear space \( B \) which is continuously embedded into a Hausdorff locally convex space \( B_1 \), and \( 1 \leq p < +\infty \). If \( v, v_i \in L^p(0, T; C_0; B_0) \), \( i \in \mathbb{N} \), then the sequence \( \{v_i\}_{i \in \mathbb{N}} \) converges weakly to \( v \) in \( L^p(0, T; C_0; B_0) \), and \( \{\partial v_i / \partial t\}_{i \in \mathbb{N}} \) is bounded in \( L^1(0, T; C_0; B_1) \), then \( v_i \) converges strongly in \( L^p(0, T; C_0; B) \).

We define the following spaces:

\[ H^1(\Omega) \subset C(\Omega) \subset H^{-1}(\Omega). \] (3.12)

For a function \( v \in L^1(0, T; H^{-1}(\Omega)) \), we estimate the corresponding norm by

\[ \int_0^T \sup_{\varphi \in H_0^1(\Omega)} \int_{\Omega} \frac{|u \varphi|}{\|\varphi\|_{H_0^1(\Omega)}} \, dx \, dt \leq \int_0^T \sup_{\varphi \in H_0^1(\Omega)} \int_{\Omega} \frac{|u|}{\|\varphi\|_{H_0^1(\Omega)}} \, dx \, dt \leq c \int_0^T \int_{\Omega} |u(x, t)| \, dx \, dt. \]

Owing to (3.10) and the previous estimate, we deduce that \( f_k, g_k \in L^1(0, T; H^{-1}(\Omega)) \). Because \( f_k \) and \( g_k \) converge weakly to \( f^\infty \) and \( g^\infty \) in \( L^2(0, T; H^1(\mathbb{R})) \), we use the general version of the Aubin–Lions lemma (see Lions [12]) to obtain

\[ \lim_{k \to \infty} (f_k, g_k) = (f^\infty, g^\infty) \quad \text{in} \quad (L^2(0, T; L^2(\Omega)))^2, \] (3.13)

for every bounded interval \( \Omega \). Hence, we conclude strong convergence of the product in \( L^1_{\text{loc}}(0, \infty; \mathbb{R}) \), i.e.

\[ \lim_{k \to \infty} f_k, g_k = f^\infty, g^\infty = 0. \] (3.14)

**c) The limiting equations as \( k \to \infty \)**

Now, we are able to identify the price formation model given by Lasry and Lions (1.3) in the limiting case \( k \to \infty \). We have already shown that the Boltzmann-type price formation model has a similar structure in the limit \( k \to \infty \); see (3.6). Therefore, it remains to be shown that the measures on the right-hand side of (3.6) are indeed the transaction rates defined by Lasry and Lions. We shall do this for the initial data \( f_1 \) and \( g_1 \), which satisfy the compatibility conditions necessary for (1.3), that is,
We start with an additional regularity result.

We find by applying the Lebesgue dominated convergence theorem that
\[
F^\infty(x,t) = \sum_{l=0}^{\infty} f^\infty(x + al, t) \quad \text{and} \quad G^\infty(x,t) = \sum_{l=0}^{\infty} g^\infty(x - al, t).
\]

Then, the difference \( \Phi(x,t) = F^\infty(x,t) - G^\infty(x,t) \) solves the IVP for the heat equation
\[
\Phi_t(x,t) = \Delta \Phi(x,t) \tag{3.15a}
\]
and
\[
\Phi(x,0) = \Phi_1(x) := \sum_{l=0}^{\infty} f_1(x + al) - \sum_{l=0}^{\infty} g_1(x - al). \tag{3.15b}
\]

We start with an additional regularity result.

**Proposition 3.5.** The solutions \( f^k \) and \( g^k \) of (1.1) satisfy \( f^k, g^k \in L^1(0,\infty; C_b(\mathbb{R})) \) for almost every time \( t > 0 \). This implies the continuity of \( F^k \) and \( G^k \) for almost every time \( t > 0 \).

**Proof.** The proposition follows from the compact embedding of \( H^1(\mathbb{R}) \) into \( C_b(\mathbb{R}) \); therefore, 
\[
f^k, g^k \in L^1(0,\infty; C_b(\mathbb{R})) \quad \text{for almost every time } t.
\]

We now prove the main result of this section.

**Theorem 3.6.** Let \( f_1 \) and \( g_1 \) satisfy assumption (A).

(i) Then, the limiting functions \( f^\infty \) and \( g^\infty \) are given by
\[
f^\infty(x,t) = \Phi^+(x,t) - \Phi^+(x+a,t) \quad \text{and} \quad g^\infty(x,t) = \Phi^-(x,t) - \Phi^-(x-a,t).
\]

(ii) Additionally, let (B) hold. Then, \( f^\infty \) and \( g^\infty \) satisfy system (1.3).

The main ingredient of the proof is:

**Lemma 3.7.** Let \( f_1 \) and \( g_1 \) satisfy (A). Then

(i) \( F^\infty = (F^\infty - G^\infty)^+ \) and \( G^\infty = (F^\infty - G^\infty)^- \).

(ii) Additionally, let (B) hold. Then, there exists a unique globally defined continuous function \( p = p(t) \) (the price), which satisfies
\[
F^\infty(p(t),t) = G^\infty(p(t),t) = 0.
\]

**Proof.** (i) Take \((x,t) \in \mathbb{R} \times (0,\infty)\) such that \( F^\infty(\cdot,t), G^\infty(\cdot,t) \) are continuous and \( F^\infty(x,t) > 0 \). Then, there exists an \( l \) such that \( f^\infty(x + al, t) > 0 \) and therefore \( g^\infty(x + al, t) = 0 \), owing to (3.14). The function \( u(x,t) = f^k(x,t) + g^k(x+a,t) \) satisfies the heat equation (3.2) and is positive; hence, we can deduce that \( f^\infty(x - a(l-1), t) > 0 \) and subsequently that \( g^\infty(x + a(l-1), t) = 0 \). By repeating this argument, we can show that \( g^\infty(x - al, t) = 0 \) for \( l = 0,1,2,\ldots \), which implies that \( G^\infty(x,t) = 0 \). Analogously, we prove \( G^\infty(x,t) > 0 \) implies \( F^\infty(x,t) = 0 \).

(ii) From the results in the work by Caffarelli et al. [5,6], we deduce that there exists a unique globally defined continuous function \( p = p(t) \) (the price) which satisfies
\[
\Phi(p(t),t) = 0 = F^\infty(p(t),t) - G^\infty(p(t),t). \tag{3.16}
\]
Therefore, \( F^\infty(p(t),t) = G^\infty(p(t),t) \). Assuming that \( F^\infty(p(t),t) > 0 \) leads to a contradiction by proceeding similarly to the proof of (i).

This allows us to identify the limiting functions \( f^\infty \) and \( g^\infty \) in case (ii) of theorem 3.6 and conclude, as in Caffarelli et al. [5,6], that \( f^\infty \) and \( g^\infty \) satisfy system (1.3).
We remark that the theory developed in Caffarelli et al. [5,6] applies, strictly speaking, only if \( \sup(f_{1}(x) > 0) = \inf(g_{1}(x) > 0) \); however, an extension for \( \sup(f_{1}(x) > 0) \leq \inf(g_{1}(x) > 0) \) is straightforward.

**Remark 3.8.** In case (ii), the transaction volume in the limit \( k \to \infty \) is

\[
\mu^{\infty}(x, t) = \lambda(t) \delta(x - p(t)),
\]

and the limited density of traded prices is \( \rho^{\infty}(x, t) = \delta(x - p(t)) \). To see this, we compare (3.6) with (1.4) and obtain

\[
\mu^{\infty}(x + a, t) - \mu^{\infty}(x - a, t) = \lambda(t)(\delta(x - p(t) + a) - \delta(x - p(t) - a)).
\]

This implies that \( \mu^{\infty}(x, t) = \lambda(t)\delta(x - p(t)) + A(t, x) \), where \( A \) is a non-negative and 2\( a \)-periodic function. Now, we expand \( A \) into its Fourier series and consider a single harmonic term in the series, given by \( a_{l}(t) \exp(\imath \pi l x / a), l \in \mathbb{Z} \). When taking this term as an inhomogeneity in the heat equation

\[
z_{t} = z_{xx} - a_{l}(t) \exp(\imath \pi l x / a)
\]

and

\[
z(x, t = 0) = 0,
\]

we easily compute the solution

\[
z(x, t) = -\exp(\imath \pi l x / a) \int_{0}^{t} e^{-\pi l^{2} / a}(t-s) a_{l}(s) \, ds.
\]

Passing to the limit \( k \to \infty \) in (3.4a) gives the heat equation for \( F^{\infty} \) with \( -\mu^{\infty} \) as the inhomogeneity. From the proof of proposition 3.2, it becomes clear that \( F^{\infty} \) tends to 0 for \( x \to \infty \) (for every fixed \( t \)) and thus \( F^{\infty} \) does not admit \( x \)-periodic modes. We conclude (3.17).

If assumption (B) does not hold, then it can be shown that the local transaction rate satisfies in the limit \( k \to \infty \)

\[
\mu^{\infty}(x, t) = \sum_{j \in I(t)} |\Phi(\cdot, p_{j}(t), t)\delta(x - p_{j}(t))|
\]

where \( I(t) \) denotes the index set (finite or countably finite) such that \( \{p_{j}(\cdot, t) \mid j \in I(t)\} \) is the set of zeros of \( \Phi(\cdot, \cdot) \).

### 4. The initial layer problem: preparation of the initial data

In this section, we discuss the behaviour of the initial layer on the fast time scale. This initial layer occurs if \( (f^{\infty}(x, t = 0), g^{\infty}(x, t = 0)) \), as computed in theorem 3.6 (i), differs from \( (f_{1}, g_{1}) \).

Let \( \varepsilon = 1/k \), then system (1.1) for the functions \( f^{\varepsilon} \) and \( g^{\varepsilon} \) (instead of \( f^{k} \) and \( g^{k} \)) reads

\[
e_{f}^{\varepsilon}(x, t) = -f^{\varepsilon}(x, t)g^{\varepsilon}(x, t) + f^{\varepsilon}(x + a, t)g^{\varepsilon}(x + a, t) + e_{x}^{\varepsilon}(x, t)
\]

and

\[
e_{g}^{\varepsilon}(x, t) = -g^{\varepsilon}(x, t)g^{\varepsilon}(x, t) + f^{\varepsilon}(x - a, t)g^{\varepsilon}(x - a, t) + e_{x}^{\varepsilon}(x, t).
\]

Let \( \tau = t/\varepsilon \) denote the fast time scale, and the corresponding fast-scale-dependent variables are denoted by

\[
\alpha^{\varepsilon}(x, \tau) := f^{\varepsilon}(x, t) \quad \text{and} \quad \beta^{\varepsilon}(x, \tau) := g^{\varepsilon}(x, t).
\]

Then, \( \alpha^{\varepsilon} = \alpha^{\varepsilon}(x, \tau) \) and \( \beta^{\varepsilon} = \beta^{\varepsilon}(x, \tau) \) satisfy

\[
\alpha^{\varepsilon}_{\tau}(x, \tau) = -\alpha^{\varepsilon}(x, \tau) \beta^{\varepsilon}(x, \tau) + \beta^{\varepsilon}(x + a, \tau) \alpha^{\varepsilon}(x + a, \tau) + \varepsilon \alpha^{\varepsilon}_{xx}(x, \tau)
\]

and

\[
\beta^{\varepsilon}_{\tau}(x, \tau) = -\beta^{\varepsilon}(x, \tau) \alpha^{\varepsilon}(x, \tau) + \beta^{\varepsilon}(x - a, \tau) \alpha^{\varepsilon}(x - a, \tau) + \varepsilon \beta^{\varepsilon}_{xx}(x, \tau),
\]

with initial data

\[
\alpha^{\varepsilon}(x, 0) = f_{1}(x) \quad \text{and} \quad \beta^{\varepsilon}(x, 0) = g_{1}(x).
\]
Proposition 4.1. In the limit $\varepsilon \to 0$, the fast-scale variables $\alpha^\varepsilon = \alpha^\varepsilon(x, \tau)$ and $\beta^\varepsilon = \beta^\varepsilon(x, \tau)$ converge to
\[
\alpha^0 \to \alpha^0, \beta^0 \to \beta^0 \quad \text{in } L^2_{\text{loc}}(0, \infty; L^2_{\text{loc}}(\mathbb{R})).
\]
The limits $\alpha^0 = \alpha^0(x, \tau)$ and $\beta^0 = \beta^0(x, \tau)$ satisfy the ODE system
\[
\alpha^0_t(x, \tau) = -\alpha^0(x, \tau)\beta^0(x, \tau) + \alpha^0(x + a, \tau)\beta^0(x + a, \tau)
\]
and
\[
\beta^0_t(x, \tau) = -\alpha^0(x, \tau)\beta^0(x, \tau) + \alpha^0(x - a, \tau)\beta^0(x - a, \tau).
\]
Furthermore, the system (4.2) with initial conditions (4.1c) has a unique continuous space–time solution.

Proof. Note that $v^\varepsilon(x, \tau) = \alpha^\varepsilon(x, \tau) + \beta^\varepsilon(x + a, \tau)$ satisfies the heat equation
\[
v^\varepsilon_t(x, \tau) = \varepsilon v^\varepsilon_{xx}(x, \tau)
\]
and
\[
v^\varepsilon(x, 0) = f_1(x) + g_1(x + a).
\]
Solutions of (4.3a) with initial datum $f_1(x), g_1(x) \in S(\mathbb{R})$ decay algebraically fast and are smooth, and we deduce that $\alpha^\varepsilon, \beta^\varepsilon \in L^\infty(0, \infty; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}))$ uniformly as $\varepsilon \to 0$. Next, we differentiate equation (4.1a) with respect to $x$, multiply with $\alpha^\varepsilon_x$ and integrate over $\mathbb{R}$ to obtain
\[
\frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}} (\alpha^\varepsilon_x)^2 \, dx \leq K \int_{\mathbb{R}} (\alpha^\varepsilon_x)^2 \, dx + K \int_{\mathbb{R}} (\beta^\varepsilon_x)^2 \, dx - \varepsilon \int_{\mathbb{R}} (\alpha^\varepsilon_{xx})^2 \, dx.
\]
The same holds for (4.1b), i.e.
\[
\frac{1}{2} \frac{d}{d\tau} \int_{\mathbb{R}} (\beta^\varepsilon_x)^2 \, dx \leq K \int_{\mathbb{R}} (\alpha^\varepsilon_x)^2 \, dx + K \int_{\mathbb{R}} (\beta^\varepsilon_x)^2 \, dx - \varepsilon \int_{\mathbb{R}} (\beta^\varepsilon_{xx})^2 \, dx.
\]
Integration over $(0, T)$ gives, for arbitrary $T > 0$,
\[
\int_{\mathbb{R}} [(\alpha^\varepsilon_x)^2(x, \tau) + (\beta^\varepsilon_x)^2(x, \tau)] \, dx \leq K_1(T),
\]
for $\tau \in (0, T)$. Similar calculations, i.e. differentiating equations (4.1a) and (4.1b) with respect to $\tau$ and multiplication by $\alpha^\varepsilon_t$ and $\beta^\varepsilon_t$, respectively, lead to
\[
\int_{\mathbb{R}} [(\alpha^\varepsilon_t)^2(x, \tau) + (\beta^\varepsilon_t)^2(x, \tau)] \leq K_2(T),
\]
for $\tau \in (0, T)$. Therefore, $\alpha^\varepsilon$ and $\beta^\varepsilon$ are bounded in $H^1_{\text{loc}}(\mathbb{R} \times (0, \infty))$ uniformly as $\varepsilon \to 0$. Owing to the compact embedding of $H^1_{\text{loc}}$ into $L^2_{\text{loc}}$, we can deduce that, after extraction of a subsequence,
\[
\alpha^\varepsilon \to \alpha^0, \beta^\varepsilon \to \beta^0 \quad \text{in } L^2_{\text{loc}}(0, \infty; L^2_{\text{loc}}(\mathbb{R})).
\]
The existence of a unique continuous space–time solution of (4.2), given by $(\alpha^0, \beta^0)$, is immediate.

Next, we study the behaviour of $\alpha^0 = \alpha^0(x, \tau)$ and $\beta^0 = \beta^0(x, \tau)$ as $\tau \to \infty$, which corresponds to the ‘end of the initial layer’.

Proposition 4.2.

(a) Solutions $\alpha^0 = \alpha^0(x, \tau)$ and $\beta^0 = \beta^0(x, \tau)$ of (4.2) converge in the limit $\tau \to \infty$
\[
\lim_{\tau \to \infty} \alpha^0(x, \tau) = \alpha^\infty(x), \quad \lim_{\tau \to \infty} \beta^0(x, \tau) = \beta^\infty(x) \quad \text{and} \quad \alpha^\infty(x)\beta^\infty(x) \equiv 0 \quad \text{on } \mathbb{R}.
\]

(b) If $f_1(x)g_1(x) \equiv 0$ on $\mathbb{R}$, then $\alpha^0(x, t) \equiv f_1(x)$ and $\beta^0(x, t) \equiv g_1(x)$ on $\mathbb{R} \times [0, \infty)$. 
Then, \( \alpha \) and \( \beta \) are such that
\[
\frac{d}{dt} \left[ \alpha(x, t) + \beta(x, t) \right] = w(x, t) = w(x - a, t) - [\alpha(x, t) + \beta(x, t)] w(x, t) + \beta(x, t) w(x + a, t).
\]
and
\[
w(0, 0) = \alpha(0, 0) \beta(0, 0) = f_1(x) g_1(x).
\]
If \( f_1(x) g_1(x) \equiv 0 \) on \( \mathbb{R} \), then \( w(0, t) = \alpha(0, t) \beta(0, t) \equiv 0 \). Hence, (4.2) gives \( \alpha(0, t) = f_1(x) \) and \( \beta(0, t) = g_1(x) \) on \( \mathbb{R} \times [0, \infty) \).

We observe that
\[
\alpha(0, t) + \beta(0, t) + \beta(0, t) \equiv f_1(x) + g_1(x + a)
\]
and
\[
\sum_{l=0}^{\infty} \alpha(x + al, t) - \sum_{l=0}^{\infty} \beta(x - al, t) = \sum_{l=0}^{\infty} f_1(x + al, t) - \sum_{l=0}^{\infty} g_1(x - al, t).
\]

Note that \( \alpha(0, t), \beta(0, t) \in \mathcal{S}^+(\mathbb{R}) \), thus both sequences \( A_0^0 \) and \( B_0^0 \) converge uniformly locally in \( x \). Moreover, both sequences are decreasing in time, because
\[
A_0^0 = -w^0 \leq 0 \quad \text{and} \quad B_0^0 = -w^0 \leq 0.
\]
Therefore, for every \( x \in \mathbb{R} \), we have
\[
A_0^0(x, t) \downarrow A^\infty(x) \quad \text{and} \quad B_0^0(x, t) \downarrow B^\infty(x) \quad \text{as} \quad t \to \infty.
\]
Integration of (4.4) in time gives
\[
A_0^0(x, t) - A_0^0(x, 0) = \int_0^t A_0^0(x, s) \, ds = - \int_0^t w^0(x, s) \, ds.
\]
Therefore, we conclude from the existence of the limits \( A^\infty(x) \) and \( B^\infty(x) \) for \( t \to \infty \) that
\[
w^0(\cdot, t) \in L^1_+(0, \infty) \quad \text{for every} \quad x \in \mathbb{R}.
\]
Then, \( \alpha_0^0(x, t) = -w^0(x, t) + w^0(x + a, t) \) and integration over the interval \([0, t]\) gives
\[
\alpha_0^0(x, t) - f_1(x) = - \int_0^t w^0(x, s) \, ds + \int_0^t w^0(x + a, s) \, ds.
\]
We know that \( w^0(\cdot, t) \in L^1_+(0, \infty) \) and therefore we conclude for \( t \to \infty \) that
\[
\lim_{t \to \infty} \alpha_0^0(x, t) = \alpha^\infty(x), \quad \lim_{t \to \infty} \beta_0^0(x, t) = \beta^\infty(x) \quad \text{and} \quad \alpha^\infty(x) \beta^\infty(x) = w^\infty(x).
\]
Thus, \( \alpha_0^0(x, t) \downarrow -w^\infty(x) + w^\infty(x + a) = 0 \), and we conclude that \( \alpha^\infty(x) \beta^\infty(x) \equiv 0 \) on \( \mathbb{R} \). \( \blacksquare \)

The properties of \( \alpha^\infty \) and \( \beta^\infty \) can be summarized as follows:

\[
\begin{align*}
(P1) \quad & \alpha^\infty(x) + \beta^\infty(x + a) = f_1(x) + g_1(x + a), \\
(P2) \quad & \sum_{l=0}^{\infty} \alpha^\infty(x + al) - \sum_{l=0}^{\infty} \beta^\infty(x - al) = \sum_{l=0}^{\infty} f_1(x + al) - \sum_{l=0}^{\infty} g_1(x - al).
\end{align*}
\]
\[
\begin{align*}
(P3) \quad & \alpha^\infty(x) \beta^\infty(x) = 0, \\
(P4) \quad & \alpha^\infty(x) \geq 0, \beta^\infty(x) \geq 0 \quad \text{on} \quad \mathbb{R}.
\end{align*}
\]

Furthermore, we set \( C^\infty(x) = A^\infty(x) - B^\infty(x) \).

**Theorem 4.3.** Let \( f_1, g_1 \) satisfy (A) and \( h(x) := f_1(x) + g_1(x + a), x \in \mathbb{R} \) satisfy

\( i \) If for some \( x_1 \in \mathbb{R} : h(x_1) > 0 \) and if there is an \( l_1 \in \mathbb{N} \) with \( h_1(x_1 - al_1) = 0 \), then \( h_1(x_1 - a(l_1 + 1)) = 0 \).
(ii) If for some \( x_2 \in \mathbb{R} : h(x_2) > 0 \) and if there is an \( l_2 \in \mathbb{N} \) with \( h_1(x_2 + a l_2) = 0 \), then \( h_1(x_2 + a(l_2 + 1)) = 0 \).

Set \( \Phi_1(x) := \sum_{i=0}^{\infty} f_1(x + al) - \sum_{i=0}^{\infty} g_1(x - al) \). Then, the limiting functions \( \alpha^\infty(x) = \lim_{\tau \to \infty} \alpha^0(x, \tau) \) and \( \beta^\infty(x) = \lim_{\tau \to \infty} \beta^0(x, \tau) \) can be identified as

\[
\alpha^\infty(x) = \Phi_1^+(x) - \Phi_1^+(x + a) \quad \text{and} \quad \beta^\infty(x) = \Phi_1^-(x) - \Phi_1^-(x - a). \tag{4.5}
\]

**Proof.** The proof proceeds in analogy to the proof of proposition 3.7 and is split into two parts.

First, we show that the positivity of either \( \alpha^\infty \) or \( \beta^\infty \) at \( x \in \mathbb{R} \) implies that the other variable is zero at \( x \) and \( x \pm al \) as well. In the second step, we identify the limiting functions.

(a) Let \( x_0 \in \mathbb{R} \) be such that \( \alpha^\infty(x_0) > 0 \). Then, \( \beta^\infty(x_0) = 0 \) for all \( l \in \mathbb{N} \), because of property (P3). Next, we show by induction that \( \beta^\infty(x_0 - al) = 0 \), \( \forall l \in \mathbb{N} \). Therefore, let for \( l_1 \in \mathbb{N} \) hold \( \beta^\infty(x_0 - al) = 0 \) for all \( l \in \{0, 1, 2, \ldots, l_1 - 1\} \). Then, we have

\[
\alpha^\infty(x_0 - al_1) + \beta^\infty(x_0 - a(l_1 - 1)) = h_1(x_0 - al_1) \tag{4.6}
\]

and

\[
\alpha^\infty(x_0) + \beta^\infty(x_0 + a) = h_1(x_0) > 0.
\]

We distinguish between two cases:

Case 1: If \( h_1(x_0 - al_1) > 0 \), then \( \alpha^\infty(x_0 - al_1) > 0 \) (because of (4.6)). Hence, we deduce, using property (P3), that \( \beta^\infty(x_0 - al_1) = 0 \).

Case 2: If \( h_1(x_0 - al_1) = 0 \), then by assumption (i) of the theorem, we have \( h(x_0 - a(l_1 + 1)) = 0 \). Property (P1) gives \( \alpha^\infty(x_0 - a(l_1 + 1)) + \beta^\infty(x_0 - al_1) = h_1(x_0 - a(l_1 + 1)) = 0 \) and we conclude that \( \beta^\infty(x_0 - al_1) = 0 \).

(b) Let \( x_0 \in \mathbb{R} \) with \( \beta^\infty(x_0) > 0 \). From property (P3), we conclude that \( \alpha^\infty(x_0) = 0 \). Again, we use an induction argument to show that \( \alpha^\infty(x_0 + al) = 0 \) for all \( l \in \mathbb{N} \). Therefore, for \( l_2 \in \mathbb{N} \), let \( \alpha^\infty(x_0 + al) = 0 \) for all \( l \in \{0, 1, \ldots, l_2 - 1\} \). In this case

\[
\alpha^\infty(x_0 + a(l_2 - 1)) + \beta^\infty(x_0 + al_2) = h_1(x_0 + a(l_2 - 1)) \tag{4.7}
\]

and

\[
\alpha^\infty(x_0 - a) + \beta^\infty(x_0) = h_1(x_0 - a) > 0.
\]

Again, we consider the two different cases:

Case 1: If \( h_1(x + a(l_2 - 1)) > 0 \), then \( \beta^\infty(x_0 + al_2) > 0 \) (because of (4.7)) and therefore \( \alpha^\infty(x_0 + al_2) = 0 \) follows.

Case 2: If \( h_1(x + a(l_2 - 1)) = 0 \), then, by assumption (ii) of the theorem, we know that \( h(x_0 + al_2) = 0 \) and therefore \( \alpha^\infty(x_0 + al_2) + \beta^\infty(x_0 + a(l_2 + 1)) = h_1(x + al_2) = 0 \). Thus, \( \alpha^\infty(x_0 + al_2) = 0 \).

Now, we identify the limiting functions \( \alpha^\infty = \alpha^\infty(x) \) and \( \beta^\infty = \beta^\infty(x) \). We choose an \( x_0 \in \mathbb{R} \) such that \( A^\infty(x_0) > 0 \). Then, there exists \( l_2 \in \mathbb{N} \cup \{0\} \), such that \( \alpha^\infty(x_0 + al_2) > 0 \). From (a), we deduce \( \beta^\infty((x_0 + al_2) - al) = \beta^\infty(x_0 - a(l - l_2)) = 0 \) for all \( l \in \mathbb{N} \). Therefore, \( B^\infty(x_0) = 0 \). Analogously, \( B^\infty(x_1) > 0 \) implies \( A^\infty(x_1) = 0 \). Therefore, \( A^\infty(x) = C^\infty(x)^+ \) and \( B^\infty(x) = C^\infty(x)^- \), i.e.

\[
A^\infty(x) = \Phi_1^+(x) \quad \text{and} \quad B^\infty(x) = \Phi_1^-(x).
\]

We conclude that

\[
\alpha^\infty(x) = \Phi_1^+(x) - \Phi_1^+(x + a) \quad \text{and} \quad \beta^\infty(x) = \Phi_1^-(x) - \Phi_1^-(x - a). \]

\[\blacksquare\]
Under the assumption of theorem 4.3 on \( f_1 \) and \( g_1 \), we computed the unique solution candidate satisfying properties (P1)–(P4). Because existence of a solution of (P1)–(P4) already follows from the \( \tau \to \infty \) argument, we conclude that we have indeed calculated \( \alpha^\infty, \beta^\infty \).

**Remark 4.4.** Note that the function \( \Phi \), which serves to identify the (slow-time-scale) limits \( f^\infty \) and \( g^\infty \) in theorem 3.6, is continuous in \( x \in \mathbb{R} \) and \( t \in [0, \infty) \). Thus, the fast-time-scale problem (4.2) describes the temporal initial layer correctly iff (4.5) holds. The conditions (i) and (ii) of theorem 4.3 are certainly sufficient for this but not necessary.

Let, for example, \( f_1 \) be supported in \([-2a, -a] \) and \( g_1 \) in \([a, 2a] \). Then, clearly condition (i) in theorem 4.3 is violated; however, (4.5) holds because of proposition 4.2(b). In addition, it is easy to see that \( \alpha^\infty, \beta^\infty \) are not always well prepared according to assumption (B). To construct an example for this, take \( f_1 \) to be supported in \([a/8, 3a/8] \) and \( g_1(x) = f_1(x + a/2) \). Then, \( \alpha^\infty(x) = f_1(x), \beta^\infty(x) = g_1(x) \) (again by proposition 4.2(b) and (4.5) is satisfied).

We remark that \( u^\varepsilon(x, \tau) = \alpha^\varepsilon(x, \tau) + \beta^\varepsilon(x + a, \tau) \) satisfies the heat equation (4.3), thus

\[
\lim_{\tau \to \infty} (\alpha^\varepsilon, \beta^\varepsilon) = 0 \quad \text{in } L^\infty(\mathbb{R}),
\]

and, actually,

\[
0 \leq \alpha^\varepsilon(x, \tau) + \beta^\varepsilon(x, \tau) \leq \frac{1}{\sqrt{4\pi \varepsilon \tau}} ||h_1||_{L^1(\mathbb{R})} \quad \forall x \in \mathbb{R}, \forall \tau > 0.
\]

Therefore, the limits \( \tau \to \infty \) and \( \varepsilon \to 0 \) do not commute. Note that the initial layer problem (4.1) does in all generality not provide well prepared initial data satisfying assumption (B). We conjecture that small diffusion helps in this process, i.e. that \( \alpha^\varepsilon, \beta^\varepsilon \) are close to well prepared on large time scales \( \tau = \tau(\varepsilon) \) with \( \varepsilon \tau(\varepsilon) \) small (see example 6.3 in §6).

### 5. Motivation by fast trading with small transaction rates

In this section, we consider the scaling limit \( k \to \infty, a \to 0 \) with \( ka = c = \text{const} \). We rewrite system (1.1) as

\[
f_{1}^{k}(x, t) = c \frac{f_{1}^{k}(x + a, t) - f_{1}^{k}(x, t)}{a} + f_{xx}^{k}(x, t) \quad (5.1a)
\]

and

\[
g_{1}^{k}(x, t) = c \frac{g_{1}^{k}(x - a, t) - g_{1}^{k}(x, t)}{a} + g_{xx}^{k}(x, t). \quad (5.1b)
\]

Formally, it is clear that the limiting system with solutions \( f^{0} = f^{0}(x, t) \) and \( g^{0} = g^{0}(x, t) \) as \( k \to \infty \) and \( a \to 0 \) is

\[
f_{1}^{0}(x, t) = c(f^{0}g^{0})_{x}(x, t) + f_{xx}^{0}(x, t) \quad (5.2a)
\]

and

\[
g_{1}^{0}(x, t) = -c(f^{0}g^{0})_{x}(x, t) + g_{xx}^{0}(x, t). \quad (5.2b)
\]

Note that \( u^{0}(x, t) = f^{0}(x, t) + g^{0}(x, t) \) satisfies again the heat equation \( u^{0}_{t}(x, t) = u^{0}_{xx}(x, t) \) with initial datum \( u^{0}(x, 0) = f_{1}(x) + g_{1}(x) \). The rigorous statement for this limit is given in theorem 5.1.

**Theorem 5.1.** Let \( k > 0, a > 0, ka = c \) and \( f_{1}, g_{1} \in S^{+}(\mathbb{R}) \). Then, the weak limits \( f^{0}, g^{0} \) satisfy system (5.2) subject to \( f^{0}(x, 0) = f_{1}(x) \) and \( g^{0}(x, 0) = g_{1}(x) \).
Proof. We reiterate that \( u(x,t) = f^k(x,t) + g^k(x+a,t) \) satisfies the heat equation (3.2); therefore, we can deduce that
\[
\|u\|_{L^\infty(0,\infty;\mathbb{R})} \leq K \text{ uniformly as } k \to \infty, a \to 0 \quad \text{with } ka = c.
\]

Multiplication of system (5.1) with \( f^k \) and \( g^k \), respectively, gives
\[
\frac{1}{2} \frac{d}{dt} \int_\mathbb{R} (f^k)^2 \, dx = c \int_\mathbb{R} f^k(x+a,t)g^k(x+a,t) - (f^k)^2(x,t)g^k(x,t) \, dx - \int_\mathbb{R} |f^k_x|^2 \, dx
\]
and
\[
\frac{1}{2} \frac{d}{dt} \int_\mathbb{R} (g^k)^2 \, dx = c \int_\mathbb{R} f^k(x+a,t)g^k(x+a,t) - f^k(x,t)(g^k)^2(x,t) \, dx - \int_\mathbb{R} |g^k_x|^2 \, dx.
\]

Now, we look at the first term on the right-hand side of the equation for \( f^k \) and deduce
\[
\int_\mathbb{R} \frac{f^k(x+a,t)g^k(x+a,t)f^k(x,t) - (f^k)^2(x,t)g^k(x,t)}{a} \, dx
\]
\[
= \int_\mathbb{R} \left[ f^k(x+a,t)g^k(x+a,t) - \frac{g^k(x+a,t)}{a} \right] \left[ f^k(x,t)g^k(x,t) - \frac{f^k(x,a)}{a} \right] \, dx
\]
\[
\leq \|f^k\|_{L^\infty} \int_\mathbb{R} |f^k(x,t)| \left| \frac{g^k(x+a,t) - g^k(x,t)}{a} \right| \, dx + \|g^k\|_{L^\infty} \int_\mathbb{R} |f^k(x,t)| \left| \frac{f^k(x+a,t) - f^k(x,t)}{a} \right| \, dx
\]
\[
\leq \frac{1}{2} \|f^k\|_{L^\infty} \frac{1}{\mu} \int_\mathbb{R} |f^k|^2(x,t) \, dx + \frac{1}{2} \|g^k\|_{L^\infty} \frac{1}{\mu} \int_\mathbb{R} (f^k)^2(x,t) \, dx
\]
\[
+ \frac{\mu}{2} \|f^k\|_{L^\infty} \int_\mathbb{R} \left| \frac{g^k(x+a,t) - g^k(x,t)}{a} \right|^2 \, dx + \frac{\mu}{2} \|g^k\|_{L^\infty} \int_\mathbb{R} \left| \frac{f^k(x+a,t) - f^k(x,t)}{a} \right|^2 \, dx.
\]

The difference quotient for \( f^k \) and \( g^k \) can be estimated using the Fourier transform, i.e.
\[
\int_\mathbb{R} \left| \frac{f^k(x+a,t) - f^k(x,t)}{a} \right|^2 \, dx = \int_\mathbb{R} \left| \hat{f}^k(\xi) \right|^2 \frac{e^{2\pi i a \xi} - 1}{a} \, d\xi.
\]

Because \(|(a^{2\pi i a \xi} - 1)/a|\) is bounded by \(|\xi|^2\), we obtain that
\[
\int_\mathbb{R} \left| \frac{f^k(x+a,t) - f^k(x,t)}{a} \right|^2 \, dx \leq L \int_\mathbb{R} \left| \hat{f}^k(\xi, t) \right|^2 \, d\xi.
\]

Similar computations hold for the second equation in \( g \). Now, we choose \( \mu^k = 1/(2(\|f^k\|_{L^\infty} + \|g^k\|_{L^\infty})L) \), then there exists a \( \sigma > 0 \) such that
\[
\frac{1}{2} \frac{d}{dt} \int_\mathbb{R} (f^k(x,t))^2 + (g^k(x,t))^2 \, dx + \sigma \int_\mathbb{R} ((f^k_x(x,t))^2 + (g^k_x(x,t))^2) \, dx \leq M.
\]

Thus
\[
f^k, g^k \in L^2_{\text{loc}}(0,\infty; H^1(\mathbb{R})) \text{ uniformly for } a \to 0.
\]

By the Aubin–Lions lemma (see [13]), we conclude that \( f^k \to f^0 \) and \( g^k \to g^0 \) as \( a \to 0 \). Thus, we can pass to the limit in system (5.1), making the formal limit rigorous.

Remark 5.2. Consider the price formation FBP (1.4) subject to the initial condition \( \nu(x,0) = f_1(x) - g_1(x) \), where \( f_1 \) and \( g_1 \) satisfy (A) and (B). Being interested in the limit as the transaction fee \( a \) tends to zero, we denote \( \nu = v^a, f = f^0 \) and \( g = g^a \). The initial data \( f_1 \) and \( g_1 \) are assumed to
be independent of $a$. It is an easy exercise to show that the weak limits $v^a \xrightarrow{a \to 0} v^0, f^a \xrightarrow{a \to 0} f^0$ and $g^a \xrightarrow{a \to 0} g^0$ exist and that
\[
v^0 = -\frac{d}{dx}|F^0|, \quad f^0 = (v^0)^+ = -\frac{d}{dx}(F^0)^+, \quad g^0 = (v^0)^- = \frac{d}{dx}(F^0)^-
\]
holds, where $F^0$ satisfies the IVP for the heat equation
\[
F_t^0 = F_{xx}^0, \quad x \in \mathbb{R}, \; t > 0
\]
and
\[
F^0(x, 0) = \int_0^\infty f_1(x + y) \, dy - \int_0^\infty g_1(x - y) \, dy.
\]
Using the methods of Caffarelli et al. [5,6], we conclude the existence of a unique continuous function $p^0 = p^0(t)$ (the price of the limiting system) such that
\[
F^0(p^0(t), t) = 0.
\]
The parabolic Hopf lemma implies $(\partial/\partial x)F^0(p^0(t), t) < 0$. Therefore, the function $v^0(\cdot, t)$ has a jump discontinuity at $x = p^0(t)$, with
\[
\lim_{x \to p^0(t)^+} v^0(x, t) = -\lim_{x \to p^0(t)^-} v^0(x, t) < 0.
\]
We conclude that $v^a(x, t)$ has an internal layer of rapid transition at $p = p^a(t)$ for a small. Moreover, it follows that the limits $k \to \infty$ (from the Boltzmann-type model to the Lasry–Lions FBP) and $a \to 0$ (from the Lasry–Lions FBP to the limit as discussed in the remark) consecutively do not give the same result as $k \to \infty, a \to 0$ and $ka = c > 0$ in the Boltzmann-type model.

We conclude that, for markets with many transactions with small fees, the Boltzmann price formation model is very sensitive to the relative sizes of the transaction frequencies and transaction fees. We refer to §6 for numerical illustrations.

6. Numerical simulations

In this section, we illustrate the behaviour of the Boltzmann-type price formation model (1.1) on a bounded domain $\Omega$. We supplement system (1.1) with homogeneous Neumann boundary conditions; hence, the total mass of buyers and vendors is conserved over time. We simulate the behaviour of solutions of (1.1) for different values of $k$ and compare them with the solutions of (1.3).

Furthermore, we perform numerical simulations in the case of not well prepared initial data and study the evolution of the initial layer on the fast time scale as discussed in §§4 and 5.

Simulations are performed on the interval $\Omega = (0, p_{\text{max}})$, where $p_{\text{max}}$ corresponds to the maximum price of the traded good, usually scaled to 1. The domain $\Omega$ is split into $N$ intervals of size $\Delta t$. We denote the discrete grid points by $x_i$ and the time steps by $\Delta t$. Then, the discrete solutions $f^n_j$ and $g^n_j$ correspond to the functions $f$ and $g$ at time $t^n = n\Delta t$ and $x_j = j\Delta t$.

The simulations of the FBP (1.3) are based on its formulation as an IVP for the heat equation, i.e.
\[
V_t(x, t) = V_{xx}(x, t) \quad \text{for all } x \in \Omega
\]
and
\[
V_x(0, t) = V_x(a, t) \quad \text{and} \quad V_x(1, t) = V_x(1 - a, t),
\]
with $v = v(x, t)$ being a solution to (1.4) and $V$ being given by
\[
V(x, t) = \begin{cases}
\sum_{n=0}^{\infty} v^+(x + at, t) \\
-\sum_{n=0}^{\infty} v^-(x + at, t)
\end{cases}
\]
as in Caffarelli et al. [5]. Note that (6.1b) corresponds to the transformed homogeneous Neumann boundary conditions. System (6.1) is solved using an implicit in time finite-difference method. The solutions \( f \) and \( g \) to (1.3) can be calculated from

\[
v(x,t) = V(x,t) - V^+(x+a,t) + V^-(x-a,t).
\]

The price \( p = p(t) \) corresponds to the zero-level set of \( V = V(x,t) \).

System (1.1) is solved using a semi-implicit in time discretization, i.e. the diffusion is discretized implicitly in time, the ‘collision’ terms explicitly. Note that we omit the superscript \( k \) in the description of the numerical discretization of (1.1) to enhance readability. The resulting scheme reads as

\[
\frac{f^n_j - f^{n-1}_j}{\Delta t} = -k f^{n-1}_j s^{n-1}_j + \frac{1}{h^2} (f^n_{j+1} - 2f^n_j + f^n_{j-1})
\]

(6.2a)

and

\[
\frac{g^n_j - g^{n-1}_j}{\Delta t} = -k g^{n-1}_j s^{n-1}_j + \frac{1}{h^2} (g^n_{j+1} - 2g^n_j + g^n_{j-1}).
\]

(6.2b)

We set \( a = \tilde{a}h, \tilde{a} \in \mathbb{N} \); therefore, the terms \( f^{n-1}_j \) and \( g^{n-1}_j \) correspond to the evaluation of the function \( f \) and \( g \) at the discrete points \( x_{i\pm a} \). In all examples, the stationary price \( p \) remained inside the interval \([h, p_{\text{max}} - h]\); hence, no further precautions in the evaluation of \( f^{n-1}_{j\pm a} \) and \( g^{n-1}_{j\pm a} \) had to be taken into account. The solution of (6.2) corresponds to solving the linear system

\[
(I + \Delta t A)\begin{pmatrix} f^n \cr g^n \end{pmatrix} = (I + \Delta t B)\begin{pmatrix} f^{n-1} \cr g^{n-1} \end{pmatrix},
\]

where the matrices \( I, A \) and \( B \) are the identity, the discrete Laplacian and the discrete ‘collision’ term, respectively.

**Example 6.1.** Comparison of the models for large trading rates \( k \).

We compare the behaviour of solutions of (1.1) with those for (1.3) using the same initial data \( f_1 = f_1(x) \) and \( g_1 = g_1(x) \). Therefore, we choose an initial datum, which satisfies the compatibility condition \( \lambda(t) = -(\partial f/\partial x)(p(0), 0) = (\partial g/\partial x)(p(0), 0) \), namely

\[
f_1(x) = \begin{cases} 
1, & \text{if } x \leq 0.5 \\
-10x + 6, & \text{if } 0.5 < x < 0.6 \\
0, & \text{otherwise},
\end{cases} \quad g_1(x) = \begin{cases} 
10x - 6, & \text{if } x > 0.6 \\
0, & \text{otherwise}.
\end{cases}
\]

Because (1.1) converges to (1.3) as \( k \to \infty \), we set \( k = 10^6 \). The other parameters are set to: \( a = 10h, \sigma = 1, \Delta t = 1/k = 10^{-6} \) and \( h = 0.002 \). Figure 1 shows the evolution of the price and the distribution of buyers and vendors at time \( t = 1 \) for both models. We observe that the buyer–vendor distribution as well as the evolution of the price agree very well.

**Example 6.2.** Behaviour of solutions of (1.1) for different trading rates.

Next, we illustrate how solutions of (1.1) change for different values of \( k \). The initial datum is set to

\[
f_1(x) = \begin{cases} 
15(x - 0.3)(0.5 - x), & \text{if } 0.3 \leq x \leq 0.5 \\
0, & \text{otherwise}
\end{cases} \quad g_1(x) = \begin{cases} 
15(0.55 - x)(x - 0.8), & \text{if } 0.55 \leq x \leq 0.8 \\
0, & \text{otherwise}.
\end{cases}
\]

The simulations were performed with the following parameters: \( h = 10^{-3}, \tau = 1/k, k = 10^5 \) and \( 10^6, a = 10h \). The final distribution of buyers and vendors as well as the evolution of the price is depicted in figure 2. We observe that the value \( c = f(x, 0.5) = f(x, 0.5) \) decreases if \( k \)
increases; also, the evolution of the price towards its stationary value is slower. The support of the trading distribution (1.2) decreases for larger $k$ and converges to a Dirac $\delta$ as in (1.3).

**Example 6.3.** Not well prepared initial datum on the fast time scale.

In this example, we study the behaviour of solutions in the case of not well prepared (in the sense of assumption (B)) initial data $f_1$ and $g_1$ on the fast time scale. Hence, we simulate system (4.1) with an initial datum, where the densities of vendors and buyers are switched, i.e. the initial data are not well prepared in the sense of assumption (B). We set

$$f_1(x) = \begin{cases} 15(x - 0.65)(0.95 - x), & \text{if } 0.65 \leq x \leq 0.95 \\ 0, & \text{otherwise} \end{cases}$$

and

$$g_1(x) = \begin{cases} 12(x - 0.25)(0.5 - x), & \text{if } 0.25 \leq x \leq 0.5 \\ 0, & \text{otherwise}. \end{cases}$$

We choose a spatial resolution of $h = 2 \times 10^{-3}$, the time steps are $\Delta t = 2 \times 10^{-4}$, the transaction rate $k = 5 \times 10^2$ and the transaction cost $a = 10h$. The evolution of $\sup f(x) > h$ and $\inf g(x) > h$ as well as the final distribution of buyers and vendors are depicted in figure 3. We observe that even
though the initial data are not well prepared $\alpha^\varepsilon$ and $\beta^\varepsilon$ converge to a well-prepared solution in the sense of assumption (B).

**Example 6.4.** Scaling limit.

Finally, we consider the scaling limit discussed in §5. We choose a large domain $\Omega = [0, 20]$ and an initial datum of the form

\[
    f_1(x) = \begin{cases} 
        1, & \text{if } 9 \leq x \leq 9.5 \\
        -2x + 20, & \text{if } 9.5 < x < 10 \\
        0, & \text{otherwise,}
    \end{cases}
\]

\[
    g_1(x) = \begin{cases} 
        2x - 20, & \text{if } 10 \leq x > 11 \\
        0, & \text{otherwise,}
    \end{cases}
\]

to avoid any interference with boundary conditions. We observe the evolution of the price for the following set of parameters for (1.1):

\[
    a = h = 2 \times 10^{-5}, \quad k = 5 \times 10^{4}, \quad \sigma = 1 \quad \text{and} \quad \tau = 2 \times 10^{-5}.
\]

This parameter set corresponds to the case where $ak = O(1)$, i.e. the scaling limit in §5. Figure 4 shows the solutions of (1.1) and (1.3) at time $t = 1$ and the evolution of the price $p = p(t)$. We observe that the consecutive limits $k \to \infty$ and $a \to 0$ in Lasry & Lions [2] do not give the same
result as the scaling limit discussed in §5. In addition, the layer of fast transition in the Lasry–Lions [2] solutions $f$ and $g$ is clearly visible in figure 4a, as mentioned in remark 5.2 of §5.

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