Abstract
Motivated by some recent speculative attempts to model the dark energy, scalar fields with negative kinetic energy coupled to gravity without a cosmological constant are considered. It is shown that in the presence of an ordinary fluid, any solution of the vacuum Einstein equations with cosmological constant is a solution provided \( \rho - P = \frac{\Lambda}{4\pi G} \). The solutions can be interpreted as a steady state in which matter or entropy is being continuously created (or destroyed). The motion of the matter is not determined by the background Einstein spacetime, many different matter flows can be found giving rise to the same metric. Solutions without ordinary matter are also considered. Anti-gravitating multi-solutions and repulsive solutions which can chase ordinary matter or black holes are exhibited. These results may also have applications to gravity theories with higher derivatives.

1 Introduction
Desperate times evidently call for desperate measures. In order to model the observed acceleration of the scale factor \( a(t) \) of our universe Caldwell [1] and others [2] have turned to matter with negative energy density. In particular resort has been made to scalar fields with negative kinetic energies [2]. Such ghost or phantom scalar fields have not been seen in cosmology since the days of the old Steady State theory with its negative kinetic energy creation field \( C(x) \) [3]. The authors of that theory showed that the field equations, essentially gravity plus a pressure free fluid and massless scalar with negative kinetic energy, admit a de-Sitter solution with constant Hubble constant \( H \) with an exponential scale factor \( a(t) = e^{Ht} \) and flat spatial cross sections satisfying the Perfect Cosmological Principle. Of course de-Sitter spacetime satisfies the Einstein equations with cosmological constant \( \Lambda = 3H^2 \). One of the aims of this paper is
to point out that, apparently unknown to those and all subsequent authors, the equations of motion also admit as solution any solution of the Einstein equations with cosmological constant

\[ R_{\alpha\beta} = \Lambda g_{\alpha\beta}. \]  

Moreover, given any solution of (1), there are infinitely many solutions for \( C(x) \), and hence for the distribution of matter. In other words there is a great deal of indeterminateness in the theory. Nevertheless what we know of solutions of (1) that at late times strongly indicates that they will indeed tend locally to a de-Sitter like state, at least inside the cosmological horizon of any observer. This is what is called Cosmic Baldness \[4\]. The necessity of formulating the statement locally, is clearly shown by the example of Taub-NUT-de-Sitter spacetime \[5\]. The distinction between Cosmic Baldness and Cosmic No Hair is made in \[6\]. It is one of asymptotic stability versus uniqueness. As shown in \[6\] the uniqueness property certainly fails in higher dimensions. For further discussion see \[7\] and \[8\].

2 k-Essence

In their attempts to save appearances, a number of cosmologists have resorted to the idea of an as yet undetected and unknown tensile substance called quintessence whose function is to cause the scale factor of the universe to accelerate at late times by violating the strong energy condition. One suggestion is ‘k-essence’, a scalar field \( C \) with Lagrangian, in its most general form, \( L = L(C, y) \), where \( y = g^{\mu\nu}\partial_\mu C\partial_\nu C \). The energy momentum tensor is

\[ T_{\mu\nu} = -2L_y C\partial_\mu C \partial_\nu C + Lg_{\mu\nu}. \]  

From (2) one easily works out the conditions \( L \) such that the weak dominant or strong energy conditions are full filled. An simple example which satisfies the weak and dominant energy conditions but violates the strong energy condition is Sen’s tachyon Lagrangian

\[ L = -V(C)\sqrt{1 + y}. \]  

If \( V(C) = \text{constant} \) we have a Born-Infeld scalar which behaves like a Chaplygin gas with the exotic equation of state \( P = -\frac{\rho}{A}, \quad A > 0 \). Despite its exotic properties this is a causal theory, small disturbances around a background are never super-luminal. In general, the characteristic cones are given by the co-metric \[11\]

\[ (G^{-1})^{\mu\nu} = g^{\mu\nu} - 2L_y g^{\rho\sigma}C\partial_\rho C\partial_\sigma C \]  

with inverse

\[ G_{\mu\nu} = g_{\mu\nu} - \frac{2L_y}{L_y + 2L_{yy}}. \]  

Ones’ confidence is the existence of k-essence is clearly diminished if small disturbances can travel super-luminally. This particular problem does not arise for
a phantom field for which \( L = \frac{1}{2} y \) and in a sense
\[
\left( G^{-1} \right)^{\mu \nu} = -g^{\mu \nu}.
\] (6)

Purely formally one may interpret (6) as saying that phantoms or ghosts feel the opposite spacetime signature from ordinary matter.

A general feature of k-essence models is that the solutions will in general develop shocks, typically associated with caustics at which the equations of motion break down and must be supplemented by additional physical assumptions. Models exhibiting shocks are thus essentially incomplete, just like classical general relativity with its spacetime singularities. Certain k-essence theories however are exceptional in this regard and do not admit shocks. Among them is the Born-Infeld theory, and its cousin Sen’s Lagrangian for the tachyon [9]. Cosmological consequences of caustics have been discussed recently in [10]. Clearly, one’s confidence in k-essence models being realized in nature is increased if they are both causal and shock free. However the later condition may not be essential if one thinks of them as being effective, rather than fundamental theories. Both conditions certainly hold for Sen’s tachyon Lagrangian [9]. Sadly however, it does not seem to be suitable to explain the acceleration of the scale factor of the universe at late times [11].

3 Coupling of phantom fields to a perfect fluid

For the simplest phantom field one has \( L = \frac{1}{2} y \) and the equations of motion which we are trying to solve are:
\[
R_{\mu \nu} - \frac{1}{2} R g_{\mu \nu} = 8 \pi G \left( (\rho + P) U_\mu U_\nu + P g_{\mu \nu} - \partial_\mu C \partial_\nu C + \frac{1}{2} g_{\mu \nu} g^{\alpha \beta} \partial_\alpha C \partial_\beta C \right).
\] (7)

We begin by choosing a Gaussian normal coordinate system
\[
ds^2 = -dt^2 + g_{ij}(t, x) dx^i dx^j.
\] (8)

There are infinitely many ways of doing this for any given spacetime. Now set
\[
U_\mu = \partial_\mu t = \delta^0_\mu
\] (9)

and
\[
C = at + b,
\] (10)

where \( a \) and \( b \) are constants. We now impose (11) with
\[
\frac{\Lambda}{4 \pi G} = \rho - P
\] (11)

\[
\rho + P = \frac{1}{2} a^2.
\] (12)

Some remarks are in order. If one writes the right hand side of (11) as
\[
8 \pi G \left( T_{\mu \nu}^{\text{fluid}} + T_{\mu \nu}^{\text{phantom}} \right),
\] (13)
then the sum $T_{\mu\nu} = T_{\mu\nu}^{\text{fluid}} + T_{\mu\nu}^{\text{phantom}}$ is conserved by the Bianchi identity but the individual summands are not. This was interpreted by the Steady State theorists as creation of matter by the creation field in order to maintain a constant matter density. In fact the construction works for any equation of state but the originators of the theory restricted themselves to the case $P = 0$. In the Lagrangian formulation originally suggested by Pryce, the $C$ field is coupled to point particles of mass $m$ by

$$- \sum m \int d\tau - \sum m \int \partial_\mu C dx^\mu. \quad (14)$$

The second term in (14) does not contribute to the equation of motion of the particles but gives an end-point contribution expressing that the 4-momentum of the point particles comes from the creation field.

More explicitly, the Bianchi identity applied to (7) yields

$$(\rho + P) \left( \dot{U}^\nu + \frac{P^\mu}{\rho + P} \right) = (hU^\mu)_{;\mu} U^\nu - C^\mu \nabla^2 C, \quad (15)$$

where $h = \rho + P$ is the enthalpy density. In our situation the right hand side of (15) vanishes and we find

$$a \nabla^2 C = (hU)^\mu_{;\mu}. \quad (16)$$

Thus the source of the creation field is the rate of production of enthalpy. On might have thought that the divergence of the entropy current should have arisen here. However in the present situation where $\rho$ and $P$ are constant, the entropy density $s$ is a constant multiple of the enthalpy density $h$. In fact one may re-arrange (15) using the first law of thermodynamics to yield

$$(\rho + P) \left( \dot{U}^\nu + h^\mu \frac{P^\mu}{\rho + P} \right) = T (sU^\mu)_{;\mu} U^\nu - C^\mu \nabla^2 C, \quad (17)$$

where $h^{\mu\nu} = g^{\mu\nu} + U^\mu U^\nu$ and $T$ is the temperature.

Physically one can is tempted to regard the solution as an out of equilibrium self-organized system maintaining itself in a steady state through a uniform dissipation rate.

By (9) the coordinates $t, x^i$ are co-moving with respect to the fluid, but as stated above, for a given solution of (1) there are many geodesic vorticity free congruences. However almost all of these coordinate systems will develop caustics or other types of singularities. The region of spacetime covered will thus be geodesically incomplete. The standard example is de-Sitter spacetime $dS_4$. In its Steady State guise as a $k = 0$, FLRW model the coordinates cover just one half of the full de-Sitter hyperboloid embedded in $\mathbb{E}^{4,1}$ to the future of a null hyperplane passing through the origin. By contrast in its $k = 1$ guise, the coordinate system covers, in a time symmetric fashion the entire space. Particles are first destroyed and then created.

As long as the fluid satisfies the dominant energy condition, $0 \leq P \leq \rho$, the cosmological constant is non-negative. The limiting case $\rho = P$, stiff matter,
gives a vanishing cosmological constant. This is because vorticity free stiff mat-
ter is equivalent to an ordinary massless scalar field with positive kinetic energy.
This just cancels the contribution of the phantom or creation field \( C(x) \).

4 Anti-self-gravitating phantoms

We now consider what happens in the absence of the fluid. We shall see that
negative energy scalars coupled to gravity can have some strange properties.
For example there is a static non-singular solution with a wormhole, strictly an
Einstein-Rosen bridge, with vanishing ADM mass \([13]\). It is given by

\[
 ds^2 = -dt^2 + \frac{l^2}{4} \left( 1 + \frac{1}{r^2} \right)^2 (dx^2 + dy^2 + dz^2),
\]

(18)

with \( r^2 = x^2 + y^2 + z^2 \), and

\[
 C = \tan^{-1} \left( \frac{r^2 - 1}{2r} \right). \tag{19}
\]

The involution

\[
 r \rightarrow \frac{1}{r} \tag{20}
\]

interchanges the two sheets of the wormhole. Because the metric is ultra-static,
\( g_{tt} = -1 \) there is no event horizon and one can see through from one side to the
other.

Perhaps even more bizarrely, because the ghost scalar field gives rise to
repulsive rather than attractive forces, one also has static multi-object solutions.
Using the techniques developed in \([13]\) one makes the ansatz

\[
 ds^2 = -\exp(2U) dt^2 + \exp(-2U) \gamma_{ij} dx^i dx^j \tag{21}
\]

and finds that static solutions may be obtained as stationary points of the
dimensionally reduced action

\[
 \int \sqrt{\gamma} \left( \mathcal{R}(\gamma) - 2\gamma^{ij} \partial_i U \partial_j U + 2\kappa^2 \gamma^{ij} \partial_i C \partial_j C \right) \tag{22}
\]

with \( \kappa^2 = 4\pi G \). It is now easy to see that solutions exist of the form

\[
 ds^2 = -\exp e2U(x) dt^2 + \exp^{-2U(x)} dx^2, \tag{23}
\]

with

\[
 \kappa(C - C_\infty) = U = \sum \frac{-M_i}{|x - x_i|}. \tag{24}
\]

Each constant \( M_i \) can take either sign.

The occurrence of solutions with negative ADM mass and the consequent
gravitational repulsions can give rise to runaway accelerating solutions in which
a negative mass particle chases a positive mass particle as described in the
context of general relativity by Bondi \([19]\). The positive mass particle can be a
black hole \([20]\). Thus one might expect if theories with phantoms are correct,
to find black holes being chased around by supernatural beings.
5 Higher spin phantom fields

Once one has lost one’s inhibitions about negative energy, there is no need to stop at a single scalar field. One could consider negative energy tensor fields. The case of vector fields, which would arise for example by dimensional reduction of theories with more than one time, was discussed in [14]. One could consider \( p \)-form gauge fields, and these may possibly be related to the creation of \( p \)-branes. Ghosts are found in higher derivative theories [15] Second rank symmetric tensor ghost are found in higher derivative gravity theories and these have been invoked to provide inflationary solutions (see [16] for a recent discussion). The underlying inflationary mechanism may well be related to that described in this paper.

On an even more unorthodox tack, it is interesting to note that bi-metric theories typically exhibit negative energies. Take as a representative example, Rosen’s theory [17, 18]. This is based in maps \( g_{\alpha\beta}(t,x) \) from Minkowski spacetime \( \mathbb{E}^{3,1} \) equipped with its standard Lorentzian metric \( \eta_{\mu\nu} \) to the space \( N \equiv GL(4,\mathbb{R})/SO(3,1) \) of Lorentzian metrics \( g_{\alpha\beta} \). The Lagrangian density is taken to be

\[
L = \frac{1}{4} \text{Tr} (g^{-1} \partial_{\mu} g^{-1} \partial_{\nu} g) + \frac{1}{8} (\text{Tr} g^{-1} \partial g)^2, \tag{25}
\]

where the contractions on the partial derivatives are taken with respect to the Minkowski metric \( \eta_{\mu\nu} \). This is like a non-linear sigma model except that the field \( g_{\alpha\beta} \) transforms like a symmetric tensor under Lorentz-transformations of the domain

\[
g(x) \rightarrow L^t g(Lx)L, \tag{26}
\]

where \( L \) is a Lorentz transformation.

In fact the ten-dimensional target space \( N \) is endowed with an \( SL(4,\mathbb{R}) \)-invariant metric with signature \((7,3)\) which is moreover the product of a one-dimensional factor times a solution of the Einstein equations. Kinematically at least, we may think of the model in which a 3-brane is immersed into a ten-dimensional spacetime with three times. To see this explicitly we set \( g = \exp \phi U \), where \( \text{det} U = -1 \). The target space metric on \( N \) associated to the action becomes

\[
12d\phi^2 + \text{Tr} (U^{-1} dUU^{-1}dU). \tag{27}
\]

The second term in (27) is the standard invariant metric on the symmetric space \( SL(4,\mathbb{R})/SO(3,1) \) induced from the Killing metric of \( SL(4,\mathbb{R}) \) on the totally geodesic submanifold consisting of metrics invariant under the involution \( g \rightarrow g^t \). The Killing metric on \( SL(4,\mathbb{R}) \) has signature \((9,-6)\), the \(-6\) coming from the maximal compact subgroup \( SO(4) \). The Killing metric on the Lorentz group has signature \((3,3)\), the positive directions being the non-compact boosts and the negative directions the maximal compact subgroup \( SO(3) \). The result is that the metric (27) has signature \((7,3)\), the 3 negative signs coming from the components \( dg_{\alpha\beta} \).

\[1\text{as written this is of course a right action} \]
Thus, taking into account my signature convention (− + + +) Rosen’s Lagrangian \(25\) has 7 negative energy excitations. Because simple solutions exist of a wavelike character, the metric \(g_{\alpha\beta}\) can be an arbitrary function of retarded time \(t - x^3\), where we have chosen the wave to be moving in the \(x^3\) direction, it is clear that both positive and negative energy excitations can propagate freely.

What about antigravity? Using the theory of harmonic maps, it is easy to see that it admits static anti-self-gravitating multi-object solutions of the same form as \(24\). More generally, by making a diagonal static ansatz in \(25\) it is clear that there are anisotropic solutions of the form

\[
g_{\alpha\beta} = \text{diag}(-\exp 2V_0(x), \exp 2V_1(x), \exp V_2(x), \exp 2V_3(x)),
\]

where \(V_0(x), V_1(x), V_2(x), V_3(x)\) are arbitrary harmonic functions on three-dimensional Euclidean space \(\mathbb{E}^3\).

The structure of some variable speed of light theories \(21\) is rather similar to Rosen’s theory and they may therefore be vulnerable to the same problems.

6 Conclusion

Personally, I think it is rather premature to abandon the widespread prejudice against theories with negative energy, but if observations were to confirm the existence of anti-gravitating phantoms one might be inclined to reconsider. That the idea is not completely ridiculous is clear from the example of Lifshitz transitions in condensed matter physics. The coefficients multiplying the terms involving the gradient of an order parameter are temperature dependent and can become negative in a certain temperature range. This is a disaster, it merely signals that configurations in which the order parameter varies spatially are thermodynamically favoured over configurations in which the order parameter is spatially homogeneous. Another exotic phenomenon encountered in low temperature physics is a dispersion relation for rotons in liquid helium allowing momentum and velocity to be anti-parallel. Naively this means accepting negative masses. In both these cases, the fundamental underlying theory has positive energy. It is just that one is dealing with fluctuations around a situation which is not a global equilibrium state. In any event, it is important to understand what one is letting oneself in for, if one does contemplate scalars with negative kinetic energy.

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