A Riemannian Corollary of Helly’s Theorem

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Abstract

We introduce a notion of halfspace for Hadamard manifolds that is natural in the context of convex optimization. For this notion of halfspace, we generalize a classic result of Grünbaum, which itself is a corollary of Helly’s theorem. Namely, given a probability distribution on the manifold, there is a point for which all halfspaces based at this point have at least $\frac{1}{n+1}$ of the mass. As an application, the gradient oracle complexity of convex optimization is polynomial in the parameters defining the problem.

1 Overview

1.1 Introduction

The extrema of functions are of fundamental importance in mathematics and its applications. Much of numerical optimization studies this topic. Most of the theory focuses on convex functions, as it has proven hard to find other classes that are both useful and tractable. The motivation for this paper comes from the desire to expand the boundaries of this class of tractable functions.

Rigorous study of convergence rates was initiated in [8] for first order methods for convex functions on Hadamard manifolds. That is, gradient descent methods for simply connected manifolds of nonpositive sectional curvature. Such manifolds are diffeomorphic to $\mathbb{R}^n$ and exhibit natural convex functions; so in a sense they give new classes of functions for which optimization is tractable.

Still, up to this point as far as the author is aware, all known algorithms for general convex optimization on Riemannian manifolds have iteration complexity depending polynomially on $\epsilon^{-1}$. To achieve better convergence rates, further conditions are added such as strong convexity, dominated gradients, or recently robust second-order [8], [7], [9]. One major unresolved question for interesting Hadamard manifolds like $SL_n/\text{SO}_n$ is, does convexity enable algorithms whose time complexity depends polynomially on $\log(\epsilon^{-1})$?

For Euclidean optimization, cutting plane methods are the standard, general approach to get $\log(\epsilon^{-1})$ complexity. They are based on reducing the feasible set by halfspace cuts indicated by the gradient of the convex function. An important feature of this approach is
that Euclidean convex sets have what are commonly termed centerpoints; roughly speaking, all hyperplanes based at a centerpoint are approximately balanced. Ellipsoid methods explicitly maintain a radially symmetric set, so hyperplanes are exactly balanced. More generally, Grünbaum’s result in [3] shows that by computing the gradient at a centerpoint, we may reduce the volume of the feasible set by a $\frac{1}{n+1}$ factor.

Here, we replicate the result of Grünbaum in the more general setting of Hadamard manifolds in the hope that others find the result encouraging, useful, or intrinsically interesting. The main result is from Section 2.

**Theorem 1.** Suppose a subset $S$ of Hadamard manifold $M$ is convex and compact, and $p$ is a probability distribution on $S$ that is absolutely continuous with respect to the Riemannian volume measure. Then there exists a $\frac{1}{n+1}$-centerpoint $c \in S$ for the measure $p$.

This leads to a bound on the number of gradient calls needed to optimize a function.

**Theorem 2.** Suppose a subset $S$ of Hadamard manifold $M$ is convex and compact and also a convex $L$-Lipschitz function $f : S \to \mathbb{R}$. Additionally assume the optimum $x_*$ is in the $\epsilon$-interior of $S$, in that $b_{x_*}(\epsilon) \subset S$. Then it is possible to find a point $x \in S$ such that $f(x) - f(x_*) \leq \epsilon$ using $O(n^2 \log(nLm(S)\epsilon^{-1}))$ gradient calls.

### 1.2 Definitions and Notation

In this section we create the definitions needed to frame the problem and results. Only basic notions of Riemannian geometry are needed in this paper; these are surveyed in Appendix A, with the present section mainly providing non-standard or less common definitions.

For the remainder of this paper, we are generally interested in triples $(M, g, p)$. $M$ is always an $n$-dimensional Hadamard manifold with Riemannian metric $g$, whose inner product at $x \in M$ we will denote by $\langle \cdot, \cdot \rangle_x$. We additionally work with a probability measure $p$ defined on $M$. Usually it is the Riemannian volume measure with support restricted to a subset. The metric produces geodesics, which are locally distance minimizing paths, and are the analog of straight lines in Euclidean space. The exponential map at $x$ follows geodesics starting at $x$. As explained in the Riemannian geometry overview, $\exp_x(\cdot)$ is a diffeomorphism from $T_xM$ to $M$ when $M$ is a Hadamard manifold.

**Definition 3.** $f$ is convex on its convex domain $D$ if its restrictions to geodesics $f_\gamma(t) : \mathbb{R} \to \mathbb{R}$ are convex.

As explained in Appendix A, for differentiable functions, this is equivalent to for all $x, y \in D$

$$f(y) \geq f(x) + t \langle \nabla f(x), \hat{v} \rangle_x$$

where $y = \exp_x(t\hat{v})$ for the unit length tangent $\hat{v} \in T_xM$. We have adopted the notation $\nabla f$ for the Riemannian gradient of $f$.

Our primary object of interest is a halfspace.

**Definition 4.** An open halfspace based at $x$ is formed by applying $\exp_x(\cdot)$ to a halfspace of $T_xM$,

$$H_x(\hat{v}) = \{\exp_x(v) | \langle v, \hat{v} \rangle_x < 0\}$$
Although such halfspaces are not convex sets in the general setting of Hadamard manifolds, they arise naturally as “cutting planes” for convex functions. This notion of cutting is justified by the following.

**Lemma 5.** Consider convex function $f : S \to \mathbb{R}$ where $S$ is a convex subset of Hadamard manifold $M$. Then for any $x \in S$, the optimum of $f$ within $S$ is either obtained at $x$ or within $H_x(\nabla f(x))$.

**Proof.** If $y \notin H_x(\nabla f(x))$, the corresponding $v = \exp^{-1}_x(y)$ satisfies

$$\langle v, \nabla f(x) \rangle_x \geq 0$$

and we have

$$f(y) \geq f(c) + \langle \nabla f(x), v \rangle_x \geq f(x)$$

Cutting plane methods need to find a point for which no halfspace based at that point has too much volume. This can be captured through the notion of a centerpoint.

**Definition 6.** A $\beta$-centerpoint of the probability measure $p$ on a manifold $M$ is a point $c$ such that

$$p(H_c(\tilde{v})) \leq 1 - \beta$$

for all $\tilde{v} \in T_c M$.

Finally, a few notations that recur.

- $b_x(r)$ is the open ball of Riemannian radius $r$, based at $x$
- $m(\cdot)$ is used for the Riemannian volume

### 1.3 Overview and Conclusion

The remainder of this paper is organized as follows:

- Section 2 analyzes the existence of centerpoints on Hadamard manifolds.
- Section 3 presents the brief application of the above to upper bound gradient oracle complexity.
- Appendix A recalls the relevant notions of Riemannian geometry, providing references.

To be clear, the problem of constructing an efficient optimization procedure is far from resolved. However, the interesting consequence of Helly’s theorem does carry over to the more general setting of Hadamard manifolds, showing that in some sense there is not an information theoretic obstacle to developing a Hadamard manifold analog to cutting plane methods.

We hope our main result is of interest and encourages others to study centerpoints in the manifold setting. Targeting optimization procedures, we believe focusing on the spaces $SL_n/\text{SO}_n$ would be of greatest interest, both for theory and applications. Speaking informally, computing a centerpoint from a discrete point set would be a notable advancement. It would also be useful to be able to sample from the Riemannian volume restricted to a convex subset.
2 Existence of Centerpoints

Helly’s Theorem is not classically part of cutting plane methods in $\mathbb{R}^n$, because the centroid is an approximately optimal point at which to call the separation oracle. However, all proofs we are aware of critically use the Brunn-Minkowski Theorem, whose manifold analogs do not seem suitable for this application. However, Helly’s Theorem is somewhat the opposite, working in situations in which the distance function is convex. \cite{5} and \cite{4} have proofs that amount to:

**Theorem 7.** Let $M$ be an $n$-dimensional Riemannian manifold of nonpositive sectional curvature. Suppose that there exists a convex compact set $C$ with closed convex set family $C_\alpha \subset C$. Then if for any $n + 1$ sets, $C_1 \cap \ldots \cap C_{n+1} \neq \emptyset$, it follows that $\cap_\alpha C_\alpha \neq \emptyset$.\cite{5}

The paper \cite{4} actually proves this result for $\text{Cat}(0)$ geodesic spaces.

That the halfspace notion of Definition \cite{4} is convex in few situations limits the applicability of this generalization of Helly’s Theorem. The remainder of this section proves a result that could be a considered a Riemannian variant of a well-known corollary of Helly’s theorem, whose classical proof relies upon the convexity of halfspaces. The result can be found in \cite{3}, which we summarize as

**Lemma 8.** A $\frac{1}{n+1}$-centerpoint exists for any compactly supported probability measure on $\mathbb{R}^n$, endowed with the usual Borel $\sigma$-algebra.

To generalize this result, we rely on a few simple regularity properties of sets of Euclidean centerpoints, which we now collect. In the following lemma, the halfspaces are Euclidean halfspaces, and $D$ is the Hausdorff distance.

**Lemma 9.** Let $p_x(\cdot)$ be a family of compactly supported probability measures on $\mathbb{R}^n$ that are absolutely continuous with respect to Lebesgue measure. Assume the index set is compact, and the measures vary continuously with respect to total variation distance. Define the Euclidean centrality function

$$G(x,y) := \sup_{\hat{v} \in S^{n-1}} p_x(H_y(\hat{v}))$$

Then $G$ is continuous and $G(x,\cdot)$ is a quasi-convex function for a fixed $x$. Also define

$$U_x := \left\{ y \in \mathbb{R}^n \mid G(x, y) \in \left(0, \frac{1}{n+1}\right) \right\}$$

Then as $x_i \to x$, $D(U_{x_i}, U_x) \to 0$.

**Proof.** Each $\{ y \mid p_x(H_y(\hat{v})) < a \}$ is a halfspace. Indeed, there is a unique halfspace with normal $\hat{v}$ that contains $a$ of the mass of $p$, and the previous set is precisely the points contained in this halfspace. Therefore the intersection over all $\hat{v}$ is a convex set. This shows that preimages under $G(x,\cdot)$ of sets $(-\infty,a)$ are convex, which is the definition of quasi-convex.

To prove continuity, we may assume the domain of $G$ is a compact set $K$. It is easy to see that $g(x,y,\hat{v}):(x,y,\hat{v}) \to p(H_x(\hat{v}))$ is a continuous function on its domain $K \times S^{n-1}$. Therefore, in particular, choose $\delta$ so that when $|x-x'|<\delta$ and $|y-y'|<\delta$, then

$$|G(x,y) - G(x',y')| < \epsilon$$
By compactness in the last argument, we may let $G(x,y) = g(x, y, \hat{v}_{x,y})$, and therefore

$G(x,y) - G(x', y') = g(x, y, \hat{v}_{x,y}) - g(x', y', \hat{v}_{x', y'}) > g(x, y, \hat{v}_{x,y}) - (g(x, y, \hat{v}_{x', y'}) + \epsilon) > -\epsilon$

Switching roles gives the reverse inequality, $G(x,y) - G(x', y') < \epsilon$, which proves continuity.

Recall the Hausdorff distance is the maximum distance from one set to the other. Therefore for the final observation, the alternative is that there exists a sequence of points $(x_i, y_i) \in U_x$ with $y_i$ that are bound away from $U_x$. Compactness implies an accumulation point $(x, y)$. Continuity of $G$ requires $y \in U_x$, because each $G(x_i, y_i) \in (0, \frac{1}{n+1}]$. This contradicts the premise that $y_i$ are bound away from $U_x$.

We are now ready for our main result.

**Proposition 10.** Let $p$ be a probability measure on a convex and compact subset $S$ of a Hadamard manifold $M$. Further assume $S$ is absolutely continuous with respect to the Riemannian volume measure. Then there exists a $\frac{1}{n+1}$-centerpoint for $p$ contained in $S$.

Before going into the proof details, here is conceptual overview of the proof. We will define a continuous function $f$ from $S$ to itself, and an application of Brouwer’s theorem will show there is a fixed point. We design $f$ so that the fixed point is a $\frac{1}{n+1}$-centerpoint. To do this, we adopt normal coordinates at $x$ and pull back the measure $p$ from $M$ (i.e. the measure of $U \subset \mathbb{R}^n$ is $p(\exp_x(U))$). In these coordinates, there is a Euclidean-convex set of Euclidean centerpoints $U_x$ provided by the previous lemma, for the pulled-back measure. We select the closest of these centerpoints to $x$ and denote this point by $u_x$. $f(x)$ is then defined by projecting $u_x$ onto $S$. As stated precisely in the appendix, it is the Hadamard assumption that implies a strictly convex distance function, making this projection possible.

The technical part of the proof mostly involves showing continuity of $f(x)$, as it is not hard to show that fixed points are $\frac{1}{n+1}$-centerpoints. Lemma 9 provides the needed tools. The main obstacle is to show that $u_x$ varies continuously. To establish this, we note that the pulled back measures vary continuously with respect to total variation. Then the lemma shows that the Euclidean centerpoint sets $U_x, U_{x'}$ are close in Hausdorff distance, provided $x, x'$ are close. Combining this with convexity of the centerpoint sets, we are able to make $|u_x - u_{x'}|$ small.

We now provide the details.

**Proof.** Fix an orthonormal frame $V = (e_1, \ldots e_n)$ on $S$, so as to determine normal coordinate charts at each $x \in S$, defined by $\psi_x : y \rightarrow \exp_x(y' e_i(x))$. Because it is absolutely continuous with respect to the Riemannian volume measure, the measure $p$ pulls back under these coordinate charts to measures we denote by $p_x(y)dy$.

Identifying $S$ with a subset of $\mathbb{R}^n$ through a chart $\psi_x$, Lemma 9 shows that there is a nonempty closed convex set $U_x \subset \mathbb{R}^n$ of Euclidean $\frac{1}{n+1}$-centerpoints. There is a unique point $u_x \in U_x$ that is closest to $x$. However, it is not necessarily the case that $u_x$ is inside $\psi^{-1}_x(S)$, because the latter is not convex. To work around this, project $u_x$ onto $S$. That is,

$$f(x) := \pi(u_x) := \arg \min_{s \in S} d(s, \psi_x(u_x))$$
That the projection is well-defined and continuous can be found in Corollary 5.6 of [1]. Therefore $f$ is well-defined. Moreover, the exponential map provides a homeomorphism between a compact convex set with non-empty interior, and the closed $n$-ball. A proof of this is in the subsequent Lemma [11]. So by using Brouwer’s fixed point theorem for continuous functions on a closed topological ball, it is sufficient to verify the following properties

- If $f(x) = x$, then $x$ is a $\frac{1}{n+1}$-centerpoint
- $f(x)$ is continuous

We first show that fixed points are centerpoints. One of the key properties of normal coordinates at $x$, is that $\psi_x^{-1}(x) = 0$ and geodesics through $x$ appear as lines. As a consequence, $\psi_x^{-1}(H_e(v^i e_i))$ is the Euclidean halfspace $H_0(v^1, \ldots, v^n)$. Therefore if $x$ is a $\frac{1}{n+1}$-centerpoint for $p$, then $u_x = 0$ and $f(x) = x$. Conversely, suppose $x$ is not a $\frac{1}{n+1}$-centerpoint but is also fixed. The only way this could happen is if $\psi_x(u_x) \neq x$, and also $\psi_x(H_0(-u_x)) \cap S$ is nonempty. Taking $s$ in the latter intersection, the geodesic between $x, s$ is contained in $S$. Triangle inequalities (or as an alternative on Hadamard manifolds, convexity of the distance function) show that, initially, moving from $x$ to $s$ along the geodesic decreases the distance to $u_x$. This means it is not the case that $\pi(\psi_x(u_x)) = x$.

Next we consider the continuity claim. Once we show $u_x \in \mathbb{R}^n$ is continuous, then $f(x)$ is as well, because the projection is continuous. As a first step, we remark that the pull-back probability densities $p_x(y)$ vary continuously with respect $x$, because they are defined by smoothly varying functions (the frame and exponential map). Since $S$ is compact, there is $\delta$ so that $|x - x'| < \delta$ implies $|p_x(y) - p_{x'}(y)| < \epsilon$. This establishes continuity for the family of measures $p_x(y)dy$, with respect to total variation distance. We can now make use of the regularity properties provided by Lemma 9.

From the lemma’s last part, by requiring $d(x, x') < \delta$, one can ensure $D(U_x, U_{x'}) < \epsilon$. Let $h_x \in U_x$ be the point closest to $u_x$; this ensures $|u_x - h_x| < \epsilon$. It is also easy to see that $|u_x| - |u_{x'}| < \epsilon$. Therefore $|u_x| - |h_x| < 2\epsilon$. Critically, the Euclidean distance to the origin is strongly convex and $u_x$ minimizes it on the Euclidean convex set $U_x$, which also includes $h_x$. Therefore, qualitatively, since $|h_x|$ and $|u_x|$ are close, we know that $|h_x - u_x|$ is small. Making this quantitative through the Euclidean law of cosines,

$$|u_x - h_x|^2 \leq |h_x|^2 - |u_x|^2 = (|h_x| - |u_x|)(|h_x| + |u_x|) < \epsilon R$$

where a sufficiently large $R$ can be taken to be twice the diameter of $S$. We conclude

$$|u_x - u_{x'}| \leq |u_x - h_x| + |h_x - u_{x'}| < \sqrt{R} + \epsilon$$

\[ \square \]

**Lemma 11.** If $S$ is compact, convex, and has non-empty interior, then there is a homeomorphism from $S$ to the closed $n$-ball, $D^n$.

**Proof.** We will first define the map, and then prove it is a homeomorphism. Fix a point $x_0$ in the interior of $S$. The function $t : \mathbb{R}\setminus \{0\} \to \mathbb{R}$ is to be given by

$$t(v) = \sup \{ t : \exp_{x_0}(tv) \in S \}$$

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And then the homeomorphism is \( f : D^n \to S \), given by
\[
f(v) = \exp_{x_0}(t(v)|v|v)\]

By compactness and convexity of \( S \), it is clear that \( t \) is well-defined and bounded from above and below. Therefore \( f \) is well defined and bijective. Because the domain and codomain are compact and Hausdorff respectively, it remains to show that \( f \) is continuous. This follows once we show that \( t \) is continuous, which we will now establish.

For a given sequence \( v_n \to v \), we must also show that \( t(v_n) \to t(v) \). Consider the supporting halfspace \( H \) to \( S \) based at \( \exp_{x_0}(t(v)|v|v) \). If \( \limsup t(v_n) > t(v) \), then eventually the sequence \( \exp_{x_0}(t(v_n)v_n) \) would lie in \( H \) and therefore outside \( S \), contradicting the definition of \( t \).

For contradiction, we assume that \( \limsup t(v_n) < t(v) \). That way, we have a subsequence \( v_n \) for which \( \exp_{x_0}(t(v_n)v_n) \) approaches \( \exp_{x_0}(t(v) - \epsilon)v \) for some \( \epsilon > 0 \). However, \( x_0 \) is an interior point, and therefore has a small open ball \( b \subset S \) containing it of radius \( \delta \). The hull formed by geodesics between points in \( b \) and \( \exp_{x_0}(t(v)v) \) is full dimensional, contains \( \exp_{x_0}(t(v) - \epsilon)v \), and its only boundary point is \( \exp_{x_0}(t(v)v) \). It therefore also contains the later points in the sequence \( \exp_{x_0}(t(v_n)v_n) \). This contradicts that these points are contained in \( \partial S \). We conclude that \( \lim t(v_n) = t(v) \) completing the proof.

3 Upper Bound on Needed Gradient Calls

The proof of Theorem 2 is now a rather straightforward consequence.

**Lemma 12.** Suppose \( f \) is convex on convex \( S \subset M \) and \( L\)-Lipschitz, and \( M \) is Hadamard. Additionally assume the optimum is in the \( \epsilon \)-interior of \( S \). Now suppose a sequence of cutting planes \( H_{c_i}(\tilde{v}_i) \) with \( c_i \in S \) are used, leaving
\[
m(S' := S \cap_i H_{c_i}(\tilde{v}_i)) < \frac{(\epsilon/L)^n}{n^n}
\]

Then one of the \( c_i \) satisfies \( f(c_i) - f(x_*) \leq \epsilon \).

And now to complete the proof of the application,

**Proof.** The main fact to be established is that \( m(b(\frac{\epsilon}{L})) > m(S') \). By comparison methods, this geodesic ball has as much volume as a geodesic ball of the same radius in Euclidean space. A reference justifying this is included in Appendix A. Using a rough estimate for the volume of a Euclidean ball, we deduce
\[
m(b(\frac{\epsilon}{L})) > \frac{(\epsilon/L)^n}{n^n} > m(S')
\]

It follows that \( S' \) does not contain some \( x' \in b_{x_*(\frac{\epsilon}{L})} \); let \( x' \notin H_{c_i}(\tilde{v}_i) \). From Lemma 5, \( f(c_i) \leq f(x') \). The Lipschitz bound on \( f \) then gives
\[
f(c_i) - f(x_*) \leq f(x') - f(x_*) \leq \epsilon
\]
Proof for Theorem \[\text{Lemma 12 shows that one of the origins of the cuts is } \epsilon \text{ from optimal for the function } f \text{ as soon as the remaining set } S' \text{ has volume } O\left(\frac{\epsilon^n}{n L^m} \right).\]

Prop 10 applied to the Riemannian density on \(S'\) shows that we may choose the cut centers to be \(O\left(\frac{1}{n} \right)\) centerpoints \(x \in S\) for the remaining set \(S' \subset S\), so that the volume is reduced by a factor \(O\left(1 - \frac{1}{n} \right)\) each cut. This means the number of iterations needed is \(O\left(n^2 \log \left(n L m(S) \epsilon^{-1} \right)\right)\). \[\square\]

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### A Riemannian Overview

We will be working in the setting of Riemannian geometry, but will not use much machinery. We provide an informal overview. The definitions we introduce here are generally standard and formalized in introductory texts, one such being [6].

An \(n\)-dimensional (smooth) manifold \(M\) can be understood as a space that is locally diffeomorphic to \(\mathbb{R}^n\), so we identify these subsets of \(M\) with coordinates \((x_1, \ldots, x_n)\). This allows us to define smooth curves \(\gamma : \mathbb{R} \rightarrow M\) by requiring its coordinate representation \((x_1(t), \ldots, x_n(t))\) to be smooth. We may define velocities \(\gamma'(t)\) by associating them with \((x'_1(t), \ldots, x'_n(t))\), leading to the notion of the tangent spaces \(T_pM \cong \mathbb{R}^n\) and their union, the tangent bundle \(TM\).

Riemannian manifolds additionally specify a metric for measuring the size of these velocities, by defining an inner product \(\langle \cdot, \cdot \rangle_p\) on each tangent space. This immediately enables the definition of curve length, as \(\int |\gamma'(t)| \gamma(t) dt\). It also gives a method of measuring volume; if \(g_{ij}\) is the bilinear form for the metric in a local coordinate choice, then \(\sqrt{|g|} dx\) can be used to find Riemannian volumes. For smooth \(f : M \rightarrow \mathbb{R}\), although usually \(\nabla f\) means the covariant derivative of \(f\), which coincides with the differential (or pushforward) \(df\), we use it to mean the gradient of \(f\). The gradient is defined by duality using the metric, \(\nabla f \in TM\) satisfies \(\langle \nabla f, \cdot \rangle = df(\cdot)\).

It also turns out to be helpful to compute directional derivatives for vector fields (or acceleration along curves). Requiring a few natural conditions leads to a unique Riemannian connection \(\nabla : T_pM \times T_pM \rightarrow T_pM\) determined by the metric. It is known as the Levi-Civita connection. In the coordinates of a local frame \(E = (e_1, \ldots, e_n)\) for \(TU \subset TM\),

\[
\nabla_{e_i} e_j = \Gamma^k_{ij} e_k
\]

where \(\Gamma^k_{ij}\) are the Christoffel symbols. When the acceleration of a curve is 0, i.e. \(\nabla \gamma'(t) \gamma'(t) = 0\), we say that curve is a geodesic. This is a second order nonlinear ODE system for \(\gamma(t) = (x_1(t), \ldots, x_n(t))\),

\[
\ddot{x}^k(t) + \dot{x}^i \dot{x}^j \Gamma^k_{ij}(x(t)) = 0
\]
A unique solution will exist locally provided we specify the initial position and velocity.

**Definition 13.** We say that $S \subset M$ is convex if points $x, y \in S$ are joined by a unique geodesic contained within $S$, which is also distance minimizing.

Because of the geodesic equation, $\gamma(t)$ is determined by its initial position $p$ and velocity $v$. Then the exponential map $\exp : \xi \subset TM \to M$ can be defined as $\exp_p(v) = \gamma(1)$, its domain being the $p, v$ for which the ODE solution exists for a unit time. We will actually further restrict to Hadamard manifolds, which will be defined soon; the exponential map is globally defined for these manifolds.

One critical consequence of the metric is that Riemannian manifolds are not locally equivalent to $\mathbb{R}^n$ with the Euclidean metric. The Riemann curvature endomorphism is introduced to provide a local characterization and measure the deviation from $\mathbb{R}^n$. It takes as inputs 3 vector fields and outputs a vector field,

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where $[X,Y]$ is the Lie bracket for vector fields. Although the curvature endomorphism has intuitive geometric meaning, it is often more helpful to derive certain quantities from it. The sectional curvature assigns a scalar value to the 2-plane spanned by orthonormal $v, w \in T_p M$,

$$K(v, w) = \langle R(v, w)w, v \rangle_p$$

More concretely, this quantity is the Gaussian curvature (product of the two principle curvatures) of the surface generated by the 2-plane. Sectional curvature lower and upper bounds enable generalizations of Euclidean tools like ball volume and triangle trigonometry estimates. These methods introduce notions like Jacobi fields and shape operator to quantitatively characterize the effect of curvature. One important result along these lines, which we use, is the Bishop-Gromov volume comparison theorem. Although usually stated for its volume upper bound by assuming just a lower bound on curvature, it is understood that the proof also provides a lower volume bound. A quite direct path to the result can be found in the lecture notes [2].

**Theorem 14.** Bishop-Gromov Let $M$ be complete, and $r$ small enough so that $b_p(r)$ does not contain any of the cut points of $p$. Then provided the sectional curvatures are bound between $\kappa_1, \kappa_2$,

$$H(r) \leq m(b_p(r)) \leq S(r)$$

where $H(r), S(r)$ are the volumes of balls in the model spaces of constant $\kappa_1, \kappa_2$ curvature respectively.

Hadamard manifolds are simply connected manifolds of nonpositive sectional curvature. They have been extensively studied in mathematical literature. We collect two commonly used facts which we made use of or provide intuition. For Hadamard manifolds,

- The exponential maps $\exp_x(\cdot)$ are diffeomorphisms from $T_x M$ to $M$ (Cartan-Hadamard theorem)
- The square of the distance to a point, $d^2(x, \cdot)$, is strictly convex
• Geodesics between points are unique and distance minimizing
• Projection onto closed, convex sets is well defined and continuous

All of these can be found in [1]. In particular, their Corollary 5.6 proves the last.

One additional fact, mentioned in the introduction, is the generalization of the idea that convexity along lines implies supporting planes to the graph of the function. We provide a short justification for this simple fact, in case it helps to work through the definitions.

Lemma 15. Suppose \( f(x) \) is convex along geodesics, on a convex domain. Then
\[
f(y) \geq f(x) + \langle \nabla f(x), \exp_x^{-1}(y) \rangle_x
\]

Proof. Let \( y = \exp_x(t_0v) \). That \( f \) is convex on geodesics means \( f(\exp_x(tv)) \) is convex in \( t \), so
\[
f(y) \geq f(x) + t_0f(\exp_x(tv))' (0)
\]
But using the chain rule and that \( d(\exp_x)|_0 = I \) (see [6]),
\[
f(\exp_x(tv))' (0) = df(d \exp_x|_0(tv)) = df(v) = \langle \nabla f(x), v \rangle_x
\]

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