STANLEY DEPTH AND SYMBOLIC POWERS OF MONOMIAL IDEALS

S. A. SEYED FAKHARI

ABSTRACT. The aim of this paper is to study the Stanley depth of symbolic powers of a squarefree monomial ideal. We prove that for every squarefree monomial ideal $I$ and every pair of integers $k, s \geq 1$, the inequalities $s\text{depth}(S/I(ks)) \leq s\text{depth}(S/I(s))$ and $s\text{depth}(I(ks)) \leq s\text{depth}(I(s))$ hold. If moreover $I$ is unmixed of height $d$, then we show that for every integer $k \geq 1$, $s\text{depth}(I(k+d)) \leq s\text{depth}(I(k))$ and $s\text{depth}(S/I(k+d)) \leq s\text{depth}(S/I(k))$. Finally, we consider the limit behavior of the Stanley depth of symbolic powers of a squarefree monomial ideal. We also introduce a method for comparing the Stanley depth of factors of monomial ideals.

1. Introduction

Let $K$ be a field and $S = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over the field $K$. Let $M$ be a nonzero finitely generated $\mathbb{Z}^n$-graded $S$-module. Let $u \in M$ be a homogeneous element and $Z \subseteq \{x_1, \ldots, x_n\}$. The $K$-subspace $uK[Z]$ generated by all elements $uv$ with $v \in K[Z]$ is called a Stanley space of dimension $|Z|$, if it is a free $K[Z]$-module. Here, as usual, $|Z|$ denotes the number of elements of $Z$. A decomposition $\mathcal{D}$ of $M$ as a finite direct sum of Stanley spaces is called a Stanley decomposition of $M$. The minimum dimension of a Stanley space in $\mathcal{D}$ is called Stanley depth of $\mathcal{D}$ and is denoted by $s\text{depth}(\mathcal{D})$. The quantity

\[ s\text{depth}(M) := \max \{s\text{depth}(\mathcal{D}) \mid \mathcal{D} \text{ is a Stanley decomposition of } M \} \]

is called Stanley depth of $M$. Stanley [9] conjectured that

\[ \text{depth}(M) \leq s\text{depth}(M) \]

for all $\mathbb{Z}^n$-graded $S$-modules $M$. As a convention, we set $s\text{depth}(M) = 0$, when $M$ is the zero module. For a reader friendly introduction to Stanley depth, we refer the reader to [7].

In this paper, we introduce a method for comparing the Stanley depth of factors of monomial ideals (see Theorem 2.1). We show that our method implies the known results regarding the Stanley depth of radical, integral closure and colon of monomial ideals (see Propositions 2.2, 2.3, 2.4, 2.5).

In Section 3, we apply our method for studying the Stanley depth of symbolic powers of monomial ideals. We show that for every pair of integers $k, s \geq 1$ the
Stanley depth of the $k$th symbolic power of a squarefree monomial ideal $I$ is an upper bound for the Stanley depth of the $(ks)$th symbolic power of $I$ (see Theorem 3.2). If moreover $I$ is unmixed of height $d$, then we show that for every integer $k \geq 1$, the Stanley depth of the $k$th symbolic power of $I$ is an upper bound for the Stanley depth of the $(k+d)$th symbolic power of $I$ (see Theorem 3.7). Finally, in Theorem 3.10 we show that the limit behavior of the Stanley depth of unmixed squarefree monomial ideals can be very interesting.

2. A comparison tool for the Stanley depth

The following theorem is the main result of this section. Using this result, we deduce some known results regarding the Stanley depth of radical, integral closure and colon of monomial ideals. We should mention that in the following theorem by $\operatorname{Mon}(S)$, we mean the set of all monomials in the polynomial ring $S$.

**Theorem 2.1.** Let $I_2 \subseteq I_1$ and $J_2 \subseteq J_1$ be monomial ideals in $S$. Assume that there exists a function $\phi : \operatorname{Mon}(S) \to \operatorname{Mon}(S)$, such that the following conditions are satisfied.

(i) For every monomial $u \in \operatorname{Mon}(S)$, $u \in I_1$ if and only if $\phi(u) \in J_1$.

(ii) For every monomial $u \in \operatorname{Mon}(S)$, $u \in I_2$ if and only if $\phi(u) \in J_2$.

(iii) For every Stanley space $u\mathbb{K}[Z]$ and every monomial $v \in \operatorname{Mon}(S)$, $v \in u\mathbb{K}[Z]$ if and only if $\phi(v) \in \phi(u)\mathbb{K}[Z]$.

Then

$$\operatorname{sdepth}(I_1/I_2) \geq \operatorname{sdepth}(J_1/J_2).$$

**Proof.** Consider a Stanley decomposition

$$D : J_1/J_2 = \bigoplus_{i=1}^{m} t_i \mathbb{K}[Z_i]$$

of $J_1/J_2$, such that $\operatorname{sdepth}(D) = \operatorname{sdepth}(J_1/J_2)$. By assumptions, for every monomial $u \in I_1 \setminus I_2$, we have

$$\phi(u) \in J_1 \setminus J_2.$$ 

Thus for each monomial $u \in I_1 \setminus I_2$, we define $Z_u := Z_i$ and $t_u := t_i$, where $i \in \{1, \ldots, m\}$ is the uniquely determined index, such that $\phi(u) \in t_i \mathbb{K}[Z_i]$. It is clear that

$$I_1 \setminus I_2 \subseteq \sum u\mathbb{K}[Z_u],$$

where the sum is taken over all monomials $u \in I_1 \setminus I_2$. For the converse inclusion note that for every $u \in I_1 \setminus I_2$ and every monomial $h \in \mathbb{K}[Z_u]$, clearly we have $uh \in I_1$. By the choice of $t_u$ and $Z_u$, we conclude $\phi(u) \in t_u \mathbb{K}[Z_u]$ and therefore by (iii),

$$\phi(uh) \in \phi(u)\mathbb{K}[Z_u] \subseteq t_u \mathbb{K}[Z_u].$$

This implies that $\phi(uh) \notin J_2$ and it follows from (ii) that $uh \notin I_2$. Thus

$$I_1/I_2 = \sum u\mathbb{K}[Z_u],$$
where the sum is taken over all monomials \( u \in I_1 \setminus I_2 \).

Now for every \( 1 \leq i \leq m \), let
\[
U_i = \{ u \in I_1 \setminus I_2 : Z_u = Z_i \text{ and } t_u = t_i \}.
\]
Without lose of generality we may assume that \( U_i \neq \emptyset \) for every \( 1 \leq i \leq l \) and \( U_i = \emptyset \) for every \( l + 1 \leq i \leq m \). Note that
\[
I_1/I_2 = \sum_{i=1}^{l} \sum u \mathbb{K}[Z_i],
\]
where the second sum is taken over all monomials \( u \in U_i \). For every \( 1 \leq i \leq l \), let \( u_i \) be the greatest common divisor of elements of \( U_i \). We claim that for every \( 1 \leq i \leq l \), \( u_i \in U_i \).

**Proof of claim.** It is enough to show that \( \phi(u_i) \in t_i \mathbb{K}[Z_i] \). This, together with (i) and (ii) implies that \( u_i \in I_1 \setminus I_2 \) and \( Z_u = Z_i \) and \( t_u = t_i \) and hence \( u_i \in U_i \).

So assume that \( t_i \) does not divide \( \phi(u_i) \). Then there exists \( 1 \leq j \leq n \), such that \( \deg_{x_j}(\phi(u_i)) < \deg_{x_j}(t_i) \), where for every monomial \( v \in S \), \( \deg_{x_j}(v) \) denotes the degree of \( v \) with respect to the variable \( x_j \). Also by the choice of \( u_i \), there exists a monomial \( u \in U_i \), such that \( \deg_{x_j}(u) = \deg_{x_j}(u_i) \). We conclude that
\[
\phi(u) \in \phi(u_i) \mathbb{K}[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n],
\]
and hence by (iii)
\[
\deg_{x_j}(\phi(u)) = \deg_{x_j}(\phi(u_i)) < \deg_{x_j}(t_i).
\]

It follows that \( t_i \) does not divide \( \phi(u) \), which is a contradiction, since \( \phi(u) \in t_i \mathbb{K}[Z_i] \). Hence \( t_i \) divides \( \phi(u_i) \). On the other hand, since \( u_i \) divides every monomial \( u \in U_i \), (iii) implies that for every monomial \( u \in U_i \), \( \phi(u_i) \) divides \( \phi(u) \). Note that by the definition of \( U_i \), for every for every monomial \( u \in U_i \), \( \phi(u) \in t_i \mathbb{K}[Z_i] \). It follows that
\[
\phi(u_i) \in t_i \mathbb{K}[Z_i]
\]
and this completes the proof of our claim.

Our claim implies that for every \( 1 \leq i \leq l \), we have
\[
u_i \mathbb{K}[Z_i] \subseteq \sum_{u \in U_i} u \mathbb{K}[Z_i].
\]
On the other hand (iii) implies that, for every monomial \( u \in U_i \), \( \phi(u_i) \) divides \( \phi(u) \). Since
\[
\phi(u_i) \in t_i \mathbb{K}[Z_i] \quad \text{and} \quad \phi(u) \in t_i \mathbb{K}[Z_i],
\]
we conclude that
\[
\phi(u) \in \phi(u_i) \mathbb{K}[Z_i]
\]
and it follows from (iii) that
\[
u \in u_i \mathbb{K}[Z_i]\]
and thus

\[ u_i \mathbb{K}[Z_i] = \sum_{u \in U_i} u \mathbb{K}[Z_i]. \]

Therefore

\[ I_1/I_2 = \sum_{i=1}^{l} u_i \mathbb{K}[Z_i]. \]

Next we prove that for every \( 1 \leq i, j \leq l \) with \( i \neq j \), the summands \( u_i \mathbb{K}[Z_i] \) and \( u_j \mathbb{K}[Z_j] \) intersect trivially. By contradiction, let \( v \) be a monomial in \( u_i \mathbb{K}[Z_i] \cap u_j \mathbb{K}[Z_j] \). Then there exist \( h_i \in \mathbb{K}[Z_i] \) and \( h_j \in \mathbb{K}[Z_j] \), such that \( u_i h_i = v = u_j h_j \). Therefore \( \phi(u_i h_i) = \phi(v) = \phi(u_j h_j) \). But \( u_i \in U_i \) and hence \( \phi(u_i) \in t_i \mathbb{K}[Z_i] \), which by (iii) implies that

\[ \phi(u_i h_i) \subseteq \phi(u_i) \mathbb{K}[Z_i] \subseteq t_i \mathbb{K}[Z_i]. \]

Similarly \( \phi(u_j h_j) \subseteq t_j \mathbb{K}[Z_j] \). Thus

\[ \phi(v) \in t_i \mathbb{K}[Z_i] \cap t_j \mathbb{K}[Z_j], \]

which is a contradiction, because \( \bigoplus_{i=1}^{m} t_i \mathbb{K}[Z_i] \) is a Stanley decomposition of \( J_1/J_2 \). Therefore

\[ I_1/I_2 = \bigoplus_{i=1}^{l} u_i \mathbb{K}[Z_i] \]

is a Stanley decomposition of \( I_1/I_2 \) which proves that

\[ \text{sdepth}(I_1/I_2) \geq \min_{i=1}^{l} |Z_i| \geq \text{sdepth}(J_1/J_2). \]

\( \Box \)

Using Theorem 2.1 we are able to deduce many known results regarding the Stanley depth of factors of monomial ideals. For example, it is known that the Stanley depth of the radical of a monomial ideal \( I \) is an upper bound for the Stanley depth of \( I \). In the following proposition we show that this result follows from Theorem 2.1.

**Proposition 2.2.** (See [15]) Let \( J \subseteq I \) be monomial ideals in \( S \). Then

\[ \text{sdepth}(I/J) \leq \text{sdepth}(\sqrt{I}/\sqrt{J}). \]

**Proof.** Let \( G(\sqrt{I}) = \{u_1, \ldots, u_s\} \) be the minimal set of monomial generators of \( \sqrt{I} \). For every \( 1 \leq i \leq s \), there exists an integer \( k_i \geq 1 \) such that \( u_i^{k_i} \in I \). Let \( k_I = \text{lcm}(k_1, \ldots, k_s) \) be the least common multiple of \( k_1, \ldots, k_s \). Now for every \( 1 \leq i \leq s \), we have \( u_i^{k_I} \in I \) and this implies that \( u^{k_I} \in I \), for every monomial \( u \in \sqrt{I} \). It follows that for every monomial \( u \in S \), we have \( u \in \sqrt{I} \) if and only if \( u^{k_I} \in I \). Similarly there exists an integer \( k_J \), such that for every monomial \( u \in S \), \( u \in \sqrt{J} \) if and only if \( u^{k_J} \in J \). Let \( k = \text{lcm}(k_I, k_J) \) be the least common multiple of \( k_I \) and \( k_J \). For every monomial \( u \in S \), we define \( \phi(u) = u^k \). It is clear that \( \phi \) satisfies the hypothesis of Theorem 2.1. Hence it follows from that theorem that

\[ \text{sdepth}(I/J) \leq \text{sdepth}(\sqrt{I}/\sqrt{J}). \]
Let $I \subseteq S$ be an arbitrary ideal. An element $f \in S$ is integral over $I$, if there exists an equation
\[ f^k + c_1 f^{k-1} + \ldots + c_{k-1} f + c_k = 0 \quad \text{with} \quad c_i \in I^i. \]
The set of elements $\mathcal{T}$ in $S$ which are integral over $I$ is the integral closure of $I$. It is known that the integral closure of a monomial ideal $I \subseteq S$ is a monomial ideal generated by all monomials $u \in S$ for which there exists an integer $k$ such that $u^k \in I^k$ (see [8, Theorem 1.4.2]).

Let $I$ be a monomial ideal in $S$ and let $k \geq 1$ be a fixed integer. Then for every monomial $u \in S$, we have $u \in \mathcal{T}$ if and only if $u^s \in I^s$, for some $s \geq 1$ if and only if $u^{k s'} \in I^{k s'}$, for some $s' \geq 1$ if and only if $u^k \in \mathcal{T}$. This shows that by setting $\phi(u) = u^k$ in Theorem 2.1 we obtain the following result from [8]. We should mention that the method which is used in the proof of Theorem 2.1 is essentially a generalization of one which is used in [8].

**Proposition 2.3.** ([8, Theorem 2.1]) Let $J \subseteq I$ be two monomial ideals in $S$. Then for every integer $k \geq 1$
\[ \operatorname{sdepth}(\overline{I}/\overline{J}) \leq \operatorname{sdepth}(\overline{I}/\overline{J}). \]

Similarly, using Theorem 2.1 we can deduce the following result from [8].

**Proposition 2.4.** ([8, Theorem 2.8]) Let $I_2 \subseteq I_1$ be two monomial ideals in $S$. Then there exists an integer $k \geq 1$, such that for every $s \geq 1$
\[ \operatorname{sdepth}(I_1^{s k}/I_2^{s k}) \leq \operatorname{sdepth}(\overline{I_1}/\overline{I_2}). \]

**Proof.** Note that by Remark 8, there exist integers $k_1, k_2 \geq 1$, such that for every monomial $u \in S$, we have $u^{k_1} \in I_1^{k_1}$ (resp. $u^{k_2} \in I_2^{k_2}$) if and only if $u \in \overline{I_1}$ (resp. $u \in \overline{I_2}$). Let $k = \text{lcm}(k_1, k_2)$ be the least common multiple of $k_1$ and $k_2$. Then for every monomial $u \in S$, we have $u^k \in I_1^k$ (resp. $u^k \in I_2^k$) if and only if $u \in \overline{I_1}$ (resp. $u \in \overline{I_2}$). Hence for every monomial $u \in S$ and every $s \geq 1$, we have $u^{s k} \in I_1^{s k}$ (resp. $u^{s k} \in I_2^{s k}$) if and only if $u \in \overline{I_1}$ (resp. $u \in \overline{I_2}$). Set $\phi(u) = u^{s k}$, for every monomial $u \in S$ and every $s \geq 1$. Now the assertion follows from Theorem 2.1.

Let $I$ be a monomial ideal in $S$ and $v \in S$ be a monomial. It can be easily seen that $(I : v)$ is a monomial ideal. Popescu [9] proves that $\operatorname{sdepth}(I : v) \geq \operatorname{sdepth}(I)$. On the other hand, Cimpoeas [2] proves that $\operatorname{sdepth}(S/(I : v)) \geq \operatorname{sdepth}(S/I)$. Using Theorem 2.1 we prove a generalization of these results.

**Proposition 2.5.** Let $J \subseteq I$ be monomial ideals in $S$ and $v \in S$ be a monomial. Then
\[ \operatorname{sdepth}(I/J) \leq \operatorname{sdepth}((I : v)/(J : v)). \]

**Proof.** It is just enough to use Theorem 2.1 by setting $\phi(u) = vu$, for every monomial $u \in S$.

3. STANLEY DEPTH OF SYMBOLIC POWERS

Let $I$ be a squarefree monomial ideal in $S$ and suppose that $I$ has the irredundant primary decomposition

$$I = p_1 \cap \ldots \cap p_r,$$

where every $p_i$ is an ideal of $S$ generated by a subset of the variables of $S$. Let $k$ be a positive integer. The $k$th symbolic power of $I$, denoted by $I^{(k)}$, is defined to be

$$I^{(k)} = p_1^k \cap \ldots \cap p_r^k.$$

As a convention, we define the $k$th symbolic power of $S$ to be equal to $S$, for every $k \geq 1$.

We now use Theorem 2.1 to prove a new result. Indeed, we use Theorem 2.1 to compare the Stanley depth of symbolic powers of squarefree monomial ideals.

**Theorem 3.1.** Let $J \subseteq I$ be squarefree monomial ideals in $S$. Then for every pair of integers $k, s \geq 1$

$$\text{sdepth}(I^{(ks)}/J^{(ks)}) \leq \text{sdepth}(I^{(s)}/J^{(s)}).$$

**Proof.** Suppose that $I = \cap_{i=1}^r p_i$ is the irredundant primary decomposition of $I$ and let $u \in S$ be a monomial. Then $u \in I^{(s)}$ if and only if for every $1 \leq i \leq r$

$$\sum_{x_j \in P_i} \deg x_j u \geq s$$

if and only if

$$\sum_{x_j \in P_i} \deg x_j u^k \geq sk$$

if and only if $u^k \in I^{(sk)}$. By a similar argument, $u \in J^{(s)}$ if and only if $u^k \in J^{(sk)}$. Thus for proving our assertion, it is enough to use Theorem 2.1 by setting $\phi(u) = u^k$, for every monomial $u \in S$. $\square$

The following corollary is an immediate consequence of Theorem 3.1.

**Corollary 3.2.** Let $I$ be a squarefree monomial ideals in $S$. Then for every pair of integers $k, s \geq 1$, the inequalities

$$\text{sdepth}(S/I^{(ks)}) \leq \text{sdepth}(S/I^{(s)})$$

and

$$\text{sdepth}(I^{(ks)}) \leq \text{sdepth}(I^{(s)})$$

hold.

**Remark 3.3.** Let $t \geq 1$ be a fixed integer. Also let $I$ be a squarefree monomial ideal in $S$ and suppose that $I = \cap_{i=1}^r p_i$ is the irredundant primary decomposition of $I$. Assume that $A \subseteq \{x_1, \ldots, x_n\}$ is a subset of variables of $S$, such that

$$|p_i \cap A| = t,$$
for every $1 \leq i \leq r$. We set $v = \prod_{x_i \in A} x_i$. It is clear that for every integer $k \geq 1$ and every integer $1 \leq i \leq r$, a monomial $u \in \text{Mon}(S)$ belongs to $p_i^k$ if and only if $uv$ belongs to $p_i^{k+t}$. This implies that for every integer $k \geq 1$, a monomial $u \in \text{Mon}(S)$ belongs to $I^{(k)}$ if and only if $uv$ belongs to $I^{(k+t)}$. This shows

$$(I^{(k+t)} : v) = I^{(k)}$$

and thus Proposition 2.5 implies that

$$\text{sdepth}(I^{(k+t)}) = \text{sdepth}(I^{(k)})$$

and

$$\text{sdepth}(S/I^{(k+t)}) = \text{sdepth}(S/I^{(k)}).$$

In particular, we conclude the following result.

**Proposition 3.4.** Let $I$ be a squarefree monomial ideal in $S$ and suppose there exists a subset $A \subseteq \{x_1, \ldots, x_n\}$ of variables of $S$, such that for every prime ideal $p \in \text{Ass}(S/I)$,

$$|p \cap A| = 1.$$ 

Then for every integer $k \geq 1$, the inequalities

$$\text{sdepth}(I^{(k+1)}) \leq \text{sdepth}(I^{(k)})$$

and

$$\text{sdepth}(S/I^{(k+1)}) \leq \text{sdepth}(S/I^{(k)})$$

hold.

As an example of ideals which satisfy the assumptions of Proposition 3.4, we consider the cover ideal of bipartite graphs. Let $G$ be a graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. A subset $C \subseteq V(G)$ is a minimal vertex cover of $G$ if, first, every edge of $G$ is incident with a vertex in $C$ and, second, there is no proper subset of $C$ with the first property. For a graph $G$ the cover ideal of $G$ is defined by

$$J_G = \bigcap_{\{v_i, v_j\} \in E(G)} \langle x_i, x_j \rangle.$$ 

For instance, unmixed squarefree monomial ideals of height two are just cover ideals of graphs. The name cover ideal comes from the fact that $J_G$ is generated by squarefree monomials $x_{i_1}, \ldots, x_{i_r}$ with $\{v_{i_1}, \ldots, v_{i_r}\}$ is a minimal vertex cover of $G$. A graph $G$ is bipartite if there exists a partition $V(G) = U \cup W$ with $U \cap W = \emptyset$ such that each edge of $G$ is of the form $\{v_i, v_j\}$ with $v_i \in U$ and $v_j \in W$.

**Corollary 3.5.** Let $G$ be a bipartite graph and $J_G$ be the cover ideal of $G$. Then for every integer $k \geq 1$, the inequalities

$$\text{sdepth}(J_G^{(k+1)}) \leq \text{sdepth}(J_G^{(k)})$$

and

$$\text{sdepth}(S/J_G^{(k+1)}) \leq \text{sdepth}(S/J_G^{(k)})$$

hold.
Proof. Let $V(G) = U \cup W$ be a partition for the vertex set of $G$. Note that
\[ \text{Ass}(S/J_G) = \{ \langle x_i, x_j \rangle : \{v_i, v_j\} \in E(G) \}. \]
Thus for every $p \in \text{Ass}(S/J_G)$, we have $|p \cap A| = 1$, where
\[ A = \{ x_i : v_i \in U \}. \]
Now Proposition 3.4 completes the proof of the assertion. \qed

It is known [4, Theorem 5.1] that for a bipartite graph $G$ with cover ideal $J_G$, we have $J_G^{(k)} = J_G^{(k)}$, for every integer $k \geq 1$. Therefore we conclude the following result from Corollary 3.5.

**Corollary 3.6.** Let $G$ be a bipartite graph and $J_G$ be the cover ideal of $G$. Then for every integer $k \geq 1$, the inequalities
\[ \text{sdepth}(J_G^{(k+1)}) \leq \text{sdepth}(J_G^{(k)}) \]
and
\[ \text{sdepth}(S/J_G^{(k+1)}) \leq \text{sdepth}(S/J_G^{(k)}) \]
hold.

Let $G$ be a non-bipartite graph and let $J_G$ be its cover ideal. We do not know whether the inequalities
\[ \text{sdepth}(J_G^{(k+1)}) \leq \text{sdepth}(J_G^{(k)}) \]
and
\[ \text{sdepth}(S/J_G^{(k+1)}) \leq \text{sdepth}(S/J_G^{(k)}) \]
hold for every integer $k \geq 1$. However, we will see in Corollary 3.8 that we always have the following inequalities.
\[ \text{sdepth}(J_G^{(k+2)}) \leq \text{sdepth}(J_G^{(k)}) \quad \text{sdepth}(S/J_G^{(k+2)}) \leq \text{sdepth}(S/J_G^{(k)}) \]
In fact, we can prove something stronger as follows.

**Theorem 3.7.** Let $I$ be an unmixed squarefree monomial ideal and assume that $\text{ht}(I) = d$. Then for every integer $k \geq 1$ the inequalities
\[ \text{sdepth}(I^{(k+d)}) \leq \text{sdepth}(I^{(k)}) \]
and
\[ \text{sdepth}(S/I^{(k+d)}) \leq \text{sdepth}(S/I^{(k)}) \]
hold.

Proof. Let $A = \{x_1, \ldots, x_n\}$ be the whole set of variables. Then for every prime ideal $p \in \text{Ass}(S/I)$, we have $|p \cap A| = d$. Hence the assertion follows from Remark 3.3. \qed

Since the cover ideal of every graph $G$ is unmixed of height two, we conclude the following result.
Corollary 3.8. Let $G$ be an arbitrary graph and $J_G$ be the cover ideal of $G$. Then for every integer $k \geq 1$, the inequalities

$$\text{sdepth}(J_G^{(k+2)}) \leq \text{sdepth}(J_G^{(k)})$$

and

$$\text{sdepth}(S/J_G^{(k+2)}) \leq \text{sdepth}(S/J_G^{(k)})$$

hold.

Corollary 3.9. Let $I$ be an unmixed squarefree monomial ideal and assume that $\text{ht}(I) = d$. Then for every integer $1 \leq \ell \leq d$ the sequences

$$\left\{ \text{sdepth}(S/I^{(kd+\ell)}) \right\}_{k \in \mathbb{Z}_{\geq 0}} \quad \text{and} \quad \left\{ \text{sdepth}(I^{(kd+\ell)}) \right\}_{k \in \mathbb{Z}_{\geq 0}}$$

converge.

Proof. Note that by Theorem 3.7, the sequences

$$\left\{ \text{sdepth}(S/I^{(kd+\ell)}) \right\}_{k \in \mathbb{Z}_{\geq 0}} \quad \text{and} \quad \left\{ \text{sdepth}(I^{(kd+\ell)}) \right\}_{k \in \mathbb{Z}_{\geq 0}}$$

are both nonincreasing and so convergent. □

We do not know whether the Stanley depth of symbolic powers of a squarefree monomial ideal stabilizes. However, Corollary 3.9 shows that one can expect a nice limit behavior for the Stanley depth of symbolic powers of squarefree monomial ideals. Indeed it shows that for unmixed squarefree monomial ideals of height $d$, there exist two sets $L_1, L_2$ of cardinality $d$, such that

$$\text{sdepth}(S/I^{(k)}) \in L_1 \quad \text{and} \quad \text{sdepth}(I^{(k)}) \in L_2,$$

for every $k \gg 0$. The following theorem shows that the situation is even better. Indeed we can even choose the sets $L_1$ and $L_2$ of smaller cardinality.

Theorem 3.10. Let $I$ be an unmixed squarefree monomial ideal and assume that $\text{ht}(I) = d$. Suppose that $t$ is the number of positive divisors of $d$. Then

(i) There exists a set $L_1$ of cardinality $t$, such that $\text{sdepth}(S/I^{(k)}) \in L_1$, for every $k \gg 0$.

(ii) There exists a set $L_2$ of cardinality $t$, such that $\text{sdepth}(I^{(k)}) \in L_2$, for every $k \gg 0$.

Proof. (i) Based on Corollary 3.9 it is enough to prove that for every couple of integers $1 \leq \ell_1, \ell_2 \leq d$, with $\gcd(d, \ell_1) = \ell_2$, we have

$$\lim_{k \to \infty} \text{sdepth}(S/I^{(kd+\ell_1)}) = \lim_{k \to \infty} \text{sdepth}(S/I^{(kd+\ell_2)}).$$

Set $m = \frac{\ell_1}{\ell_2}$. Then by Corollary 3.2

$$\lim_{k \to \infty} \text{sdepth}(S/I^{(kd+\ell_2)}) \geq \lim_{k \to \infty} \text{sdepth}(S/I^{(mkd+m\ell_2)}) = \text{sdepth}(S/I^{(mkd)}) = \text{sdepth}(S/I^{(kd)}) = \text{sdepth}(S/I^{(kd+\ell_2)}).$$
\[
\lim_{k \to \infty} \text{sdepth}(S/I^{(mkd+\ell_1)}) = \lim_{k \to \infty} \text{sdepth}(S/I^{(kd+\ell_1)}),
\]
where the last equality holds, because the sequence

\[\left\{ \text{sdepth}(S/I^{(mkd+\ell_1)}) \right\}_{k \in \mathbb{Z}_0}\]

is a subsequence of the convergent sequence

\[\left\{ \text{sdepth}(S/I^{(kd+\ell_1)}) \right\}_{k \in \mathbb{Z}_0}.
\]

On the other hand, since \(\gcd(d, \ell_1) = \ell_2\), there exists an integer \(m' \geq 1\), such that \(m'\ell_1\) is congruence \(\ell_2\) modulo \(d\). Now by a similar argument as above, we have

\[
\lim_{k \to \infty} \text{sdepth}(S/I^{(kd+\ell_1)}) \geq \lim_{k \to \infty} \text{sdepth}(S/I^{(m'kd+m'\ell_1)}) = \lim_{k \to \infty} \text{sdepth}(S/I^{(kd+\ell_2)}),
\]

and hence

\[
\lim_{k \to \infty} \text{sdepth}(S/I^{(kd+\ell_1)}) = \lim_{k \to \infty} \text{sdepth}(S/I^{(kd+\ell_2)}).
\]

(ii) The proof is similar to the proof of (i). \(\square\)

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