Self-consistent multiple complex-kink solutions in Bogoliubov–de Gennes and chiral Gross–Neveu systems

Daisuke A. Takahashi$^{1,2,*}$ and Muneto Nitta$^{2,3}$

$^1$Department of Basic Science, The University of Tokyo, Tokyo 153-8902, Japan
$^2$Research and Education Center for Natural Sciences, Keio University, Hiyoshi 4-1-1, Yokohama, Kanagawa 223-8521, Japan
$^3$Department of Physics, Keio University, Hiyoshi 4-1-1, Yokohama, Kanagawa 223-8521, Japan

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We exhaust all exact self-consistent solutions of complex-valued fermionic condensates in the 1+1 dimensional Bogoliubov–de Gennes and chiral Gross–Neveu systems under uniform boundary conditions. We obtain a complex (twisted) kink, or a grey soliton, with $2n$ parameters corresponding to their positions and phase shifts. Each soliton can be placed at an arbitrary position while the self-consistency requires its phase shift to be quantized by $\pi/N$ for $N$ flavors.

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Introduction.—The search for inhomogeneous self-consistent fermionic condensates including states such as the Fulde–Ferrell (FF) [1] and Larkin–Ovchinnikov (LO) [2] states having phase and amplitude modulations, respectively, in superconductors has attracted considerable attentions for more than half a century since theoretical predictions were made about their existence. While amplitude modulations are well studied in conducting polymers [3–7], the FFLO state is theoretically shown to be a ground state of superconductors under a magnetic field [8]. Recently, the FFLO state has also been discussed in the context of an ultracold atomic Fermi gas [9,10]. In general, inhomogeneous self-consistent fermionic condensates with a gap function and fermionic excitations can be treated simultaneously using the Bogoliubov–de Gennes (BdG) and gap equations [11]. The gap functions are real and complex for conducting polymers [12] and superconductors, respectively. In the quantum field theory, these systems correspond to the Gross–Neveu (GN) model [13] and the Nambu–Jona-Lasinio (or chiral GN) model [14], which were proposed as models of dynamical chiral symmetry breaking in 1+1 or 2+1 dimensions. Therefore, BdG and (chiral) GN systems have been studied and developed together from the viewpoint of both condensed matter physics and high energy physics (see Ref. [15] for a review). For instance, fermion number fractionization is one of the topics that has been studied from this viewpoint [16,17]. Recently, it has been shown that the solutions in 1+1 dimensions can be promoted to 3+1 dimensions [18,19], thereby leading to extensive study of the modulated phases of these systems in terms of quantum chromodynamics (QCD) [20].

Inhomogeneous self-consistent solutions are often studied numerically because analytic solutions are generally difficult to obtain. However, several analytic solutions are available in the case of the real-valued condensates in 1+1 dimensions, which describe the conducting polymers and the real GN model. Under uniform boundary conditions at spatial infinities, a real kink was constructed by Dashen et al. [21] by using the inverse scattering method, and later, it was reconstructed in polyacetylene [22] in the continuum limit of the lattice model [23]. Subsequently, a bound state of a kink and an anti-kink, which is called a polaron, was constructed in polyacetylene [24,25], for which achieving self-consistency in the system requires the distance between the kink and anti-kink to be fixed. Furthermore, three kinks (kink and polaron placed at arbitrary positions) [26,27] and more general solutions [28] were obtained. The attractive interaction between two polarons was also investigated [29]. For a periodic boundary condition, the existence of real kink crystals (the LO state) has been known for a long time [30–7].

On the other hand, when compared with real condensates, only a few self-consistent solutions have thus far been obtained for complex condensates, such as a complex (or twisted) kink or a grey soliton, and their crystals [31,32]. In these complex-valued crystals, both the amplitude and phase are modulated (the FFLO state), and this modulated phase has important applications in both superconductors and QCD, such as in the phase diagram of the chiral GN model [33]. An attempt to construct more general solutions was made [34,35] by using a technique of integrable systems known as the nonlinear Schrödinger or Ablowitz–Kaup–Newell–Segur hierarchy [36].

In this Letter, we exhaust all exact self-consistent solutions of complex condensates under uniform boundary conditions, and we find that they describe multiple twisted kinks. Unlike polarons in real condensates, where the distance between the kink and anti-kink is fixed, the situation is drastically simplified in our multiple twisted-kink solutions; we determine the filling rate of fermions for bound states of each kink, and we find that each kink can be placed at any position and has any phase shift quantized by $\pi/N$ with the number of flavors $N$.

Fundamental equations.—The fundamental equations which we consider in this Letter appear in both condensed matter and high energy physics. In the condensed matter language, they are the one-dimensional BdG system with the Andreev approximation consisting of the BdG equation for...
right movers (BdG$_R$)

\[
\begin{pmatrix}
-i\partial_x & \Delta(x) \\
\Delta(x)^* & i\partial_x
\end{pmatrix}
\begin{pmatrix}
u_R \\
v_R
\end{pmatrix}
= \epsilon
\begin{pmatrix}
u_R \\
v_R
\end{pmatrix},
\]

(1)

the BdG equation for left movers (BdG$_L$)

\[
\begin{pmatrix}
 i\partial_x & \Delta(x) \\
\Delta(x)^* & -i\partial_x
\end{pmatrix}
\begin{pmatrix}
u_L \\
v_L
\end{pmatrix}
= \epsilon
\begin{pmatrix}
u_L \\
v_L
\end{pmatrix},
\]

(2)

and the gap equation as a self-consistent condition

\[
-\frac{\Delta(x)}{g} = \sum_{\text{occupied states}} (\nu_R v_R^* + \nu_L v_L^*).
\]

(3)

For a derivation from the second quantized Hamiltonian, see, e.g., Ref. [8].

In high energy physics, this problem is equivalent to the chiral GN model with $N$ flavors,

\[
\mathcal{L} = \bar{\psi}i\partial_\mu \psi \cdot \frac{g^2}{2N} \left[(\bar{\psi}\sigma^\mu \psi)^2 + (\bar{\psi}i \gamma_5 \psi)^2\right]
\]

(4)

with $\psi(x) = (\psi_1(x), \cdots, \psi_N(x))^T$ [13, 14, 21, 30]. Introducing the auxiliary fields $\sigma(x)$ and $\pi(x)$, this can be rewritten as

\[
\mathcal{L} = \bar{\psi}i\partial_\mu \psi - g\bar{\psi}(\sigma + i\gamma_5)\psi \cdot \frac{N}{2}(\sigma^2 + \pi^2).
\]

(5)

Eliminating $\sigma(x)$ and $\pi(x)$ by their equations of motion, $\sigma = -(g/N)\bar{\psi}\psi$ and $\pi = -(g/N)i\bar{\psi}\gamma_5\psi$, takes us back to (4). Instead, we integrate out $\psi(x)$ to obtain $Z = \int \mathcal{D}\sigma \mathcal{D}\pi \exp(iS_{\text{eff}})$ with

\[
S_{\text{eff}} = N \left[-i \ln \det [i\partial - g(\sigma + i\gamma_5)] - \frac{1}{2}(\sigma^2 + \pi^2)\right].
\]

(6)

Defining $\Delta(x) = \sigma(x) + i\pi(x)$, the gap equation is obtained in the large-$N$ limit as the stationary condition for $\Delta'(x)$

\[
\Delta(x) = -4i \frac{\delta}{\delta\Delta'(x)} \ln \det [i\partial - g(\sigma + i\gamma_5)].
\]

(7)

In the Hartree-Fock formalism, we consider $H_R\psi_R = \epsilon\psi_R$ and $H_L\psi_L = \epsilon\psi_L$ with single-particle Hamiltonians $H_R = -i\gamma_5\partial_x + \gamma_0(\sigma + i\gamma_5)$ and $H_L = +i\gamma_5\partial_x + \gamma_0(\sigma + i\gamma_5)$, reducing to the BdG Eqs. (1) and (2) with $\gamma_0 = \sigma_1$, $\gamma_1 = -i\sigma_2$ and $\gamma_3 = \sigma_3$, while the consistency condition $\Delta = -(g/N)(\bar{\psi}\psi + i\bar{\psi}i\gamma_5\psi)$ reduces to Eq. (3).

Result from the inverse scattering theory.—First, we briefly summarize the mathematical expressions of the $n$-soliton solution and its eigenstates of the self-defocusing Zakharov–Shabat eigenvalue problem [37]

\[
\begin{pmatrix}
-i\partial_x & \Delta(x) \\
\Delta(x)^* & i\partial_x
\end{pmatrix}
\begin{pmatrix}
u \\
v
\end{pmatrix}
= \epsilon
\begin{pmatrix}
u \\
v
\end{pmatrix},
\]

(8)

obtained by the inverse scattering method [38]. The detailed derivation is provided in the Supplemental Material [39].

Let us assume that the gap function obeys the following asymptotically uniform boundary condition:

\[
|\Delta(x)| \to m (> 0), \quad x \to \pm \infty.
\]

(9)
\[ \theta_1, \ldots, \theta_n \text{ are all different from each other and there is no degeneracy.} \]

We further introduce the following notation:

\[ e_j(x) = \sqrt{x} e^{i(x-x_j)}, \quad (j = 1, \ldots, n). \quad (12) \]

Here, the real constant \( x_j \) represents the position of the \( j \)-th soliton up to an additive constant when solitons are well separated from each other, as shown below. Furthermore, we define the functions \( f_1(x), \ldots, f_n(x) \) as solutions of the following linear equation:

\[
\begin{pmatrix}
 f_1 \\
 f_2 \\
 \vdots \\
 f_n
\end{pmatrix} + \frac{\epsilon_1}{m} \begin{pmatrix}
 c_{11} & c_{12} & \cdots & c_{1n} \\
 c_{21} & c_{22} & \cdots & c_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 c_{n1} & c_{n2} & \cdots & c_{nn}
\end{pmatrix} \begin{pmatrix}
 f_1 \\
 f_2 \\
 \vdots \\
 f_n
\end{pmatrix} = 0. \quad (13)
\]

Here, the argument \( x \) is abbreviated.

By using the above notations, the \( n \)-soliton solution can be expressed as

\[ \Delta(x) = m \pm 2i \sum_{j=1}^{n} s_j^{-1} e_j(x) f_j(x). \quad (14) \]

The complex-valued \( n \)-soliton solution has \( 2n \) parameters \( s_1, \ldots, s_n, x_1, \ldots, x_n \), and this number of parameters is exactly twice that of the real-valued soliton solution. This \( \Delta(x) \) has the following asymptotic form:

\[ \Delta(x) \rightarrow \begin{cases} 
 m & \text{as } x \rightarrow -\infty, \\
 me^{-2i(\theta_1+\theta_2+\cdots+\theta_n)} & \text{as } x \rightarrow +\infty.
\end{cases} \quad (15) \]

If the solitons are sufficiently separated from each other, the phase shift brought about by the \( j \)-th soliton is \( s_j^{-2} = e^{-2x_j} \), and the position of the \( j \)-th soliton \( X_j \) is given by

\[ X_j = x_j + \frac{1}{k_j} \sum_{l \neq j \text{ s.t. } x_l < x_j} \log \frac{\sin \theta_l + \theta_j}{\sin \theta_l - \theta_j}. \quad (16) \]

Figure 2 shows an example of the three-soliton solution.

The reduction to the real-valued soliton solution is obtained as follows. When the number of solitons is even \( (n = 2n') \), the relations

\[ s_{2j-1} = -s_{2j}^*, \quad x_{2j-1} = x_{2j} \quad (j = 1, \ldots, n') \quad (17) \]

yield real-valued solutions. When the number of solitons is odd \( (n = 2n' + 1) \), we need to consider the term \( s_{2n'+1} \) in addition to Eq. (17) while \( x_{2n'+1} \) remains arbitrary. By this reduction, we obtain \( f_{2j-1}(x) = f_j(x)^* \) and \( f_{2n'+1}(x) = f_{2n'+1}(x)^* \), and the imaginary part of Eq. (14) vanishes. The bound state with \( s = s_j \) \((\leftrightarrow \epsilon = m \cos \theta_j)\) is given by

\[ \begin{pmatrix}
 u_j(x) \\
 v_j(x)
\end{pmatrix} = \begin{pmatrix}
 f_j(x) \\
 s_j f_j(x)^*
\end{pmatrix} \quad (j = 1, \ldots, n). \quad (18) \]

\[ \begin{array}{c}
 \epsilon = m \quad \cdots \\
 \epsilon = -\epsilon_3 \\
 \epsilon = 0 \\
 \epsilon = \epsilon_1 \\
 \epsilon = \epsilon_2 \\
 \epsilon = \epsilon_3 \\
 \epsilon = -m \quad \cdots
\end{array}
\]

FIG. 3. Diagram of the occupation states considered in this Letter. In this example figure, the number of flavors is \( N = 6 \) and the number of solitons is \( n = 3 \). The filling rates defined by Eq. (23) are given by \( v_1 = -1/6, v_2 = 1/3, \) and \( v_3 = 1/2 \).

We can show that this state is already normalized, \( i.e., \int dx (|u_j|^2 + |v_j|^2) = 1 \) holds.

Finally, let \( s \) be real. Then, the scattering states are given by

\[
\begin{pmatrix}
 u(x, s) \\
 v(x, s)
\end{pmatrix} = e^{i\theta(s)} \begin{pmatrix}
 1 \\
 \frac{2i}{m} \sum_{j=1}^{n} s_j^{-1} \left[ s_j f_j(x)^* \right]
\end{pmatrix}
\]

which are obviously reflectionless as observed from the expression. The amplitudes of these solutions at \( x = \pm \infty \) are

\[ |u(\pm \infty, s)|^2 + |v(\pm \infty, s)|^2 = 1 + s^{-2}. \quad (20) \]

**Occupation states and gap equation.**—From this point onwards, we consider the occupation states of the BdG system with \( N \) internal degrees of freedom, or equivalently, the chiral GN model with \( N \) flavors. We first note that the following relation exists between the solutions of the right and left movers:

\[ (\epsilon, u(x), v(x)) \text{ is a solution of BdGr.} \]

\[ \leftrightarrow (-\epsilon^*, -v(x)^*, u(x)^*) \text{ is a solution of BdGl.} \quad (21) \]

Thus, we can rewrite all quasiparticle wavefunctions of the left movers using those of the right movers. In the light of examining low-energy excited states of condensed matter systems, we consider the configurations in which all the negative-energy scattering states are filled by fermions and positive-energy states are completely vacant. As for bound states, we label the bound states of BdGr as \((u_{j,R}, v_{j,R}) \quad (j = 1, \ldots, n)\), and we also label the corresponding bound states of BdGl with the energy of the opposite sign related by Eq. (21) as \((u_{j,L}, v_{j,L})\). These states are assumed to be filled partially, and
we write the occupation number as $N_{R}$ and $N_{L}$, as schematically shown in Fig. [5]. The gap equation subsequently becomes

$$\Delta(x) = \sum_{\epsilon > 0} N_{R} \nu_{R} + \sum_{\epsilon < 0} N_{L} \nu_{L} + \sum_{\text{b.s.}} N_{j} \nu_{j}^{*} + \sum_{\text{b.s.}} N_{j} \nu_{j}^{*} \epsilon_{j} \Delta(x) = N\left(\sum_{\epsilon > 0} \nu_{R}^{*} - \sum_{\epsilon < 0} \nu_{R}^{*} + \sum_{\text{b.s.}} \nu_{j}^{*} \epsilon_{j} \Delta(x)\right) + \sum_{\text{b.s.}} N_{j} \nu_{j}^{*} \epsilon_{j} \Delta(x).$$

(22)

Here, the notation s.s. (b.s.) denotes the scattering (bound) states, and the relation [21] is used to obtain the second equality in the above equation. Defining the filling rate by

$$\nu_{j} := \frac{N_{R} - N_{L}}{N}, \quad -1 \leq \nu_{j} \leq 1 \quad (j = 1, \ldots, n),$$

(23)

the above equation can be rewritten as follows:

$$\frac{\Delta(x)}{\tilde{g}} = \sum_{\epsilon > 0} \nu_{R}^{*} - \sum_{\epsilon < 0} \nu_{R}^{*} + \sum_{\text{b.s.}} \nu_{j}^{*} \epsilon_{j} \Delta(x)$$

(24)

with $\tilde{g} := N_{R}$. It is to be noted that the sum of positive-energy scattering states in Eq. (24) has a negative sign because of the relation [21], and it is equivalent to the stationary condition of the action in the GN model, as given in Ref. [30]. Thus, we can again confirm the equivalence of the problems between the BdG and GN systems.

Henceforth, we always use the quantities of the BdG system, and we omit the subscript R. Considering the limit $L \to \infty$, where L denotes the system size, we replace the sum of the scattering states of the gap equation [Eq. (24)] by the corresponding integral. After renormalization of the coupling constant

$$\frac{1}{\tilde{g}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{\sqrt{k^{2} + m^{2}}}.$$

(25)

and subtracting the logarithmically divergent terms from both sides, we obtain the following expression:

$$0 = \sum_{\text{b.s.}} \nu_{j}^{*} \mu(x_{j}) \nu_{j} \epsilon_{j} \sin \theta_{j}$$

$$+ \int_{-\infty}^{0} \int_{0}^{\infty} \frac{dk}{2\pi} \frac{\Delta(x)}{\sqrt{k^{2} + m^{2}}} - \frac{\nu_{k}^{*} \nu_{k} \epsilon_{k}^{*} \Delta(x)}{|\nu_{k}^{*}|^{2} + |\nu_{k}|^{2}},$$

(26)

where we have written the scattering states with the wavenumber k as $(\mu_{k}(x), \nu_{k}(x))$, and their amplitudes at infinity as $(\mu_{k,\infty}, \nu_{k,\infty})$. (We note that $(\mu_{k}(x), \nu_{k}(x))$ for a positive energy and that for a negative energy are different from each other, though we use the same notation.) It is convenient to rewrite the above integral in terms of the uniformizing variable s introduced in Eq. (10). Using the relation $\sqrt{k^{2} + m^{2}} = \frac{1}{2}|s|^{1 + s^{2}}$, we obtain

$$0 = \sum_{\text{b.s.}} \nu_{j}^{*} \mu(x_{j}) \nu_{j} \epsilon_{j} \sin \theta_{j}$$

$$+ \frac{1}{2} \int_{-\infty}^{0} \int_{0}^{\infty} \frac{dk}{2\pi} \frac{m(1 + s^{2}) \mu(x, s) \nu(x, s)}{|\nu(x, s)|^{2} + |\nu_{s}(\infty, s)|^{2}} - \frac{\Delta(x)}{2s}.$$

(27)

Here, we have written the scattering states labeled by s as $(\mu(x, s), \nu(x, s))$.

**Self-consistent condition for the n-soliton solution.**—We first present our main result in the following theorem, and then provide the proof.

**Theorem.** Let $\Delta(x)$ be an n-soliton solution given by Eq. (14). The gap equation [Eq. (27)] holds if and only if the filling rate $\nu_{j}$ satisfies

$$\nu_{j} = \frac{2\theta_{j} - \pi}{\pi} \quad (j = 1, \ldots, n).$$

(28)

Here, we remark on certain aspects of this theorem:

1. This theorem provides all self-consistent solutions under the uniform boundary condition [Eq. (29)], because $\Delta(x)$ needs to be a reflectionless potential in order for the gap equation to hold [21, 28, 30], and n-soliton solutions cover all reflectionless potentials with n bound states.

2. The filling rate $\nu_{j}$ for the j-th bound state only depends on the phase shift of the j-th soliton, and it is not affected by other soliton parameters. Thus, the self-consistent condition is decoupled for each bound state (or each soliton).

3. The parameter $x_{j}$ (j = 1, ..., n), which represents the position of the soliton up to an additive constant [Eq. (16)], is arbitrary and is not related to the self-consistency. This contrasts with the case of real-valued condensates. Because they must be real, the distance between two solitons must be fixed to a specific value, such as in the case of the polars in polyacetylene [24–26] and the topologically trivial soliton in the GN model [27].

4. For the N-flavor system, the possible values of the filling rate are given by $\nu = \frac{N-1}{N}, \frac{N-2}{N}, \ldots, \frac{2}{N}$. Correspondingly, the possible phase shift of each soliton is also discretized. For example, only the trivial value $\nu = 0$ is allowed for N = 1, which corresponds to the real kink $\theta = \pi$. When N = 2, the values $\nu = -\frac{1}{2}, 0, \frac{1}{2}$ are allowed, which correspond to $\theta = \frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}$. The cases N = 1 and 2 correspond to s-wave superconductors and polyacetylene, respectively. The cases of N are also obtained as a dimensional reduction of nonrelativistic field theories in 3+1 dimensions, for which N is the number of patches of the Fermi surface [19]. On the other hand, any soliton solution can be self-consistent when $N = \infty$.

**Proof.** Upon substituting Eqs. (14), (19), and (20) into the integrand of the gap equation (27), the terms $1 + s^{2}$ and $(\mu(\pm \infty, s))^{2} + (\nu(\pm \infty, s))^{2}$ in the first term in the bracket undergo cancellation, thereby yielding

$$\frac{m}{2} \mu(x, s) \nu(x, s) - \frac{\Delta(x)}{2s}$$

$$= 2 \sum_{j} s_{j}^{-1} \epsilon_{j} f_{j}^{*} \sin \theta_{j} |s_{j} - s|^{2} - 4i \sum_{m} e_{j} f_{j}^{*} e_{j} f_{j} \sin \theta_{j}$$

$$= - 2 \sum_{j} s_{j}^{-1} f_{j}^{*} \sin \theta_{j} |s_{j} - s|^{2}.$$
Recalling that the bound states are given by Eq. (18), we finally obtain

\[ \int_{-\infty}^{0} - \int_{0}^{\infty} \frac{m}{2} \left( \frac{d}{dx} u(x) v(x, s)^* - \frac{\Delta(x)}{2s} \right) \, dx = - \sum_{j} s_j^{-1} f_j^2 \frac{2\theta_j - \pi}{\pi}. \]  

(30)

Since the functions \( f_1(x)^2, \ldots, f_N(x)^2 \) are linearly independent of each other, the theorem holds. \( \Box \)

**Summary.**—In summary, we have constructed all the exact self-consistent solutions of complex condensates under uniform boundary conditions. Our multiple \( n \)-twisted kink solution contains \( 2n \) parameters, and each kink has one bound state. Each kink can be placed at any position, while the self-consistency of the system requires the phase shift of each kink to be quantized by \( \pi/N \) with the number of flavors \( N \). Our solution describes multiple grey solitons in ultracold atomic fermion gases, and our predictions require experimental verification. The dynamics and scattering of these solitons should be studied as a future research topic. Further research also needs to be conducted on the construction of self-consistent solutions under non-uniform boundary conditions.

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*etakashi@vortex.c.u-tokyo.ac.jp*
Supplemental Material

In this Supplemental Material, we review the inverse scattering theory of the self-defocusing Zakharov–Shabat (ZS) operator under the finite-density boundary condition [37,38]. Subsequently, as a special solution, we derive the $n$-soliton solution and its eigenstates.

I. JOST SOLUTIONS

In this section, we consider the direct problem of the ZS operator. We introduce Jost solutions and check several properties of them and transition coefficients defined by their asymptotic form.

The ZS eigenvalue problem in the self-defocusing case is given by

$$
\mathcal{L} \begin{pmatrix} u \\ v \end{pmatrix} = \epsilon \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} -i\partial_x & \Delta(x) \\ \Delta(x)^* & i\partial_x \end{pmatrix}. \tag{1}
$$

We consider this problem under the finite-density boundary condition:

$$
\Delta(x) \rightarrow \begin{cases} m, & (x \rightarrow -\infty), \\ me^{2i\theta}, & (x \rightarrow +\infty), \quad (m > 0). \tag{2}
\end{cases}
$$

Since the ZS operator is self-adjoint in the self-defocusing case, its eigenvalues $\epsilon$ are always real. More precisely, if $(u, v)^T$ is a bounded function, $\epsilon$ must be real. If we allow exponentially divergent solutions, the solution exists for an arbitrary complex number $\epsilon$.

When the gap function is constant $\Delta = me^{2i\theta}$, the dispersion relation of the plane-wave solution $(u, v) \propto e^{ikx}$ becomes

$$
\epsilon^2 = k^2 + m^2. \tag{3}
$$

Thus, the energy spectrum has a gap $-m < \epsilon < m$. Therefore, if $\Delta(x)$ satisfies the boundary condition \([3] \), the discrete eigenvalues, which give normalizable bound states, appear in $|\epsilon| < m$.

**Wronskian**

Let $f_1 = (u_1, v_1)^T$ and $f_2 = (u_2, v_2)^T$ be two solutions of Eq. (1) for a given $\epsilon$. The Wronskian is defined as

$$
W(f_1, f_2) = u_1v_2 - u_2v_1. \tag{4}
$$

It is easy to show that $dW/dx = 0$, i.e., $W$ is a constant. Furthermore, if $W \neq 0$, these two solutions are linearly independent of each other, and therefore, they form a basis of the solution space.

**Complex conjugate solution**

Taking the complex conjugate of Eq. (1), we obtain the relation

$$
\begin{cases} u \\ v \end{cases} \text{ is a solution for } \epsilon \iff \begin{cases} v^* \\ u^* \end{cases} \text{ is a solution for } \epsilon^*. \tag{5}
$$

**Uniformizing variable**

We parametrize the energy $\epsilon$ and the wavenumber $k$ using the uniformizing variable $s$ defined by

$$
\epsilon(s) = \frac{m}{2} \left( s + s^{-1} \right), \quad k(s) = \frac{m}{2} \left( s - s^{-1} \right). \tag{6}
$$

The uniformizing variable is convenient because we can avoid to introduce the Riemann surface consisting of two sheets, which is necessary to make the square-root function $k = \pm \sqrt{\epsilon^2 - m^2}$ single-valued. We can confirm that the dispersion relation \([5] \) holds for arbitrary complex $s$. The following relations are obvious from the definition:

$$
\begin{align*}
\epsilon(s') &= \epsilon(s)^*, & k(s') &= k(s)^*.
\end{align*} \tag{7}
$$

It is also easy to check

$$
\text{Im } s \geq 0 \iff \text{Im } k(s) \geq 0. \tag{9}
$$

For real $\epsilon$, it holds that $|\epsilon| \geq m \iff s \in \mathbb{R}$ and $|\epsilon| \leq m \iff |s| = 1$. Therefore, eigenstates corresponding to $s$ on the real axis are scattering states, and those corresponding to $s$ on the unit circle are bound states. Since $s$ and $s^*$ on the unit circle correspond to the same bound state, it is sufficient to consider the unit circle in the upper half-plane when we count the number of bound states. The edges of the continuous spectrum are given by $\epsilon = \pm m \iff s = \pm 1$. See Figure 1 of the main article.

**Right Jost solutions**

The basis of the solutions of Eq. (1) with $\epsilon = \epsilon(s)$ under the constant potential $\Delta(x) = me^{2i\theta}$ can be written as

$$
\begin{pmatrix} se^{i\theta} \\ e^{-i\theta} \end{pmatrix} e^{ikx} \text{ and } \begin{pmatrix} e^{i\theta} \\ se^{-i\theta} \end{pmatrix} e^{-ikx}. \tag{10}
$$

Using the solutions \([10] \), we define the right Jost solution $f_s(x, s)$ as a solution which has the following asymptotic forms:

$$
\begin{align*}
\begin{cases} a(s) \frac{s}{1} e^{ikx} + b(s) \frac{1}{s} e^{-ikx} & (x \rightarrow -\infty) \\
\left( \begin{array}{c} se^{i\theta} \\ e^{-i\theta} \end{array} \right) e^{ikx} & (x \rightarrow +\infty).
\end{cases}
\end{align*} \tag{11}
$$
The solution with the above asymptotic form at \(x \to +\infty\) is uniquely determined, and hence the transition coefficients \(a(s)\) and \(b(s)\) are also defined uniquely. Since \(\Delta \to m\) at \(x \to -\infty\), the phase factor \(e^{i\sigma(s)x}\) is unnecessary for the form of \(x \to -\infty\). We do not need to introduce a new symbol for the right Jost solution with wavenumber \(-k\); it can be obtained simply by replacing \(s \to 1/s\), because of Eq. (7):

\[
\frac{a(s^{-1})}{e^{i\sigma(s)x}} + \frac{b(s^{-1})}{e^{i\sigma(s)x}} \left( x \to -\infty \right) \quad \left( x \to +\infty \right).
\] (12)

Here we summarize the fundamental properties of the right Jost solutions and transition coefficients:

(i) The set of solutions \(\{f_+(x, s), sf_+(x, s^{-1})\}\) are linearly independent of each other unless \(s = \pm 1\), and hence they span the solution space.

(ii) The relations \(a(s^{-1}) = a(s)^*\) and \(b(s^{-1}) = b(s)^*\) hold. Specifically, \(a(s^{-1}) = a(s)^*\) and \(b(s^{-1}) = b(s)^*\) hold when \(s\) is real.

(iii) The relation \(a(s)a(s^{-1}) - b(s)b(s^{-1}) = 1\) holds. Specifically, \(a(s)^2 - b(s)^2 = 1\) holds when \(s\) is real.

**Proof.** (i) and the former part of (iii) follow from the evaluation of the Wronskian of these two solutions at \(x = \pm \infty\):

\[
W(+\infty) = s^2 - 1
\]

\[
= W(-\infty) = (s^2 - 1) \left( a(s)a(s^{-1}) - b(s)b(s^{-1}) \right).
\] (13)

By taking the complex conjugate of Eq. (11) and using the relation (5), one can show

\[
\sigma_1 f_+(x, s)^* = s' f_+(x, s^{-1}).
\] (14)

Comparing the asymptotic forms of both sides of this equation, one obtains (ii). The latter part of (iii) follows from (ii). \(\square\)

If we write \(t(s) = 1/a(s)\) and \(r(s) = b(s)/a(s)\), the latter part of (iii) represents the conservation law \(|t(s)|^2 + |r(s)|^2 = 1\), and they are interpreted as the transmission and reflection coefficients.

**Left Jost solutions**

We define the left Jost solution by the following asymptotic form:

\[
f_-(x, s) \to \left( \frac{1}{s} \right) e^{-ik(s)x} \quad (x \to -\infty).
\] (15)

The counterpart with the plus-sign wavenumber \(+k\) can be written as \(sf_-(x, s^{-1})\) by the same discussion with the right Jost solutions.

Using the asymptotic forms (11) and (12), the transformation matrix between the right and left Jost solutions is obtained as

\[
\left( f_+(x, s) \quad sf_+(x, s^{-1}) \right) = \left( sf_-(x, s^{-1}) \quad f_-(x, s) \right) \left( a(s) \quad b(s) \right) \left( a(s^{-1}) \quad b(s^{-1}) \right).
\] (16)

The determinant of the coefficient matrix is unity because of the property (iii). Therefore, the inverse relation is given by

\[
\left( f_-(x, s^{-1}) \quad f_-(x, s) \right) = \left( sf_+(x, s^{-1}) \quad f_+(x, s) \right) \left( a(s^{-1}) \quad b(s^{-1}) \right) \left( a(s) \quad b(s) \right)^{-1}.
\] (17)

By using Eq. (17), we can also write down the asymptotic forms for left Jost solutions.

**Bound states**

The discrete spectrum, which gives normalizable bound states, is given by the zeros of the transition coefficient \(a(s)\). As already mentioned, the zeros exist on the unit circle in the \(s\)-plane. If the system has \(n\) bound states, there are \(n\) zeros in the upper-half plane, and also \(n\) zeros in the lower-half plane. Since two zeros complex conjugate to each other represent the same bound state, it is sufficient to consider the zeros in the upper-half plane only.

Let the zeros in the upper-half plane and corresponding wavenumbers be \(s_j (j = 1, \ldots, n)\) and \(k(s_j) = ik_j\), respectively. Then, Eq. (16) becomes

\[
\frac{f_+(x, s_j)}{b(s_j)} = \left(\frac{1}{s_j}\right)^{\frac{1}{2}} e^{ik_j x} \quad (x \to -\infty).
\] (19)

The following Proposition holds for the normalization constant of bound states.

**Proposition 1.** Let us write one of bound states as \(f_+(x, s_j) = (u_j(x), v_j(x))^T\) and define the normalization constant \(c_j^2\) as follows:

\[
c_j^{-2} := \int_{-\infty}^{\infty} dx \left( |u_j(x)|^2 + |v_j(x)|^2 \right).
\] (20)

By definition \(c_j^2\) is real, positive and finite. The following relation between the transition coefficients and the normalization constant holds:

\[
i\hat{a}(s_j) b(s_j)^* s_j = \frac{m}{2} c_j^{-2}.
\] (21)


Here, the dot represents the differentiation with respect to s, i.e., \( \dot{a}(s) = \frac{da(s)}{ds} \). As a corollary of Eq. (27), it also follows that \( \dot{a}(s, j) \neq 0 \), which implies that all the zeros of \( a(s) \) are simple.

**Proof.** We often omit arguments of functions when it is clear from the context. The dot and the subscript \( x \) denote the differentiation with respect to \( s \) and \( x \), respectively. First, we prove the following relation for real \( \epsilon \):

\[
-i(\dot{u}u^* - \dot{v}v^*)_x = \dot{\epsilon} \left( |u|^2 + |v|^2 \right).
\]  

(22)

It can be shown as follows. Equation (11) and its derivative with respect to \( s \) are

\[
\begin{align*}
-\dot{u}u_s + \Delta u &= \epsilon u, \\
\dot{v}v_s + \Delta^* u &= \epsilon u,
\end{align*}
\]

(23)

\[
\begin{align*}
\dot{u}u_s + \Delta \dot{v} &= \epsilon u + \epsilon \dot{u}, \\
\dot{v}v_s + \Delta^* \dot{u} &= \epsilon v + \epsilon \dot{v}.
\end{align*}
\]

(24)

respectively. From the complex conjugate of (24), we obtain

\[
\Delta = \frac{\epsilon^* v^* + \dot{v}v^*}{u^*}, \quad \Delta^* = \frac{\epsilon^* u^* - \dot{u}u^*}{\dot{v}v^*}.
\]

(25)

Using these, we can eliminate \( \Delta, \Delta^* \) from Eq. (24), yielding

\[
-i(\dot{u}u^* - \dot{v}v^*)_x = \dot{\epsilon} \left( |u|^2 + |v|^2 \right) + (\epsilon - \epsilon^*)(\dot{u}u^* + \dot{v}v^*),
\]

(26)

which reduces to Eq. (22) if \( \epsilon = \epsilon^* \). On the other hand, with recalling that \( a(s, j) = 0 \) and \( k(s, j) = ik_j \), substituting \( s = s_j \) into the differentiation of Eq. (11) with respect to \( s \) yields

\[
\begin{aligned}
\dot{f}_+(x, s_j) &\rightarrow \left\{ \begin{array}{ll}
\dot{a}(s_j) e^{-\xi_s x} & + (b(s_j) - ib(s_j)k(s_j)x) e^{\xi_s x} \\
\left( s_j e^{i\theta} (1 + is_j k(s_j)x) \right) e^{-\xi_s x} & \quad (x \to -\infty)
\end{array} \right. \\
\left( s_j e^{i\theta} k(s_j)x \right) e^{-\xi_s x} & \quad (x \to +\infty).
\end{aligned}
\]

(27)

When \( \epsilon = \epsilon(s, j) \) and \( (u, v)^T = f_+(x, s_j) \), one can show from Eqs. (19) and (27) that

\[
\begin{align*}
\lim_{x \to +\infty} \dot{u}u^* - \dot{v}v^* &= 0, \\
\lim_{x \to -\infty} \dot{u}u^* - \dot{v}v^* &= \dot{\epsilon}(s_j) b(s_j)^*(s_j - s_j^*).
\end{align*}
\]

(28)

Therefore, the integration of both sides of Eq. (22) over \( x \) yields

\[
\dot{\epsilon}(s_j) e^{-2\xi_s} = i \dot{\epsilon}(s_j) b(s_j)^*(s_j - s_j^*).
\]

(29)

Since the relations \( \dot{\epsilon}(s) = k(s)/s \) and \( s_j^* = s_j^{-1} \) hold, we obtain Eq. (21). \( \square \)

**Jost solutions for large \( \epsilon \)**

When \( |\epsilon| \) is sufficiently large, the contribution of \( \Delta(x) \) becomes relatively negligible, and therefore the Jost solution comes close to a simple plane wave. Actually, the following relations hold for large \( |\epsilon| \):

\[
\begin{align*}
f_+(x, s) e^{-i\xi(s)x} &= \left( \frac{xe^{i\theta}}{e^{2\theta}} \right) \hat{O}(s), \\
f_-(x, s) e^{i\xi(s)x} &= \left( \frac{1}{s} \right) \hat{O}(s).
\end{align*}
\]

(30)

(31)

Here \( \hat{O}(s) \) denotes

\[
\hat{O}(s) = \begin{cases} \hat{O}(1) & \text{if } |s| \gg 1 \\ \hat{O}(s) & \text{if } |s| \ll 1. \end{cases}
\]

(32)

These relations are shown by deriving the Volterra integral equation from Eq. (11) and solving it iteratively. The transition coefficient \( a(s) \) can be written as

\[
a(s) = \frac{W(f_+(x, s), f_-(x, s))}{s^2 - 1} \quad \text{by using Eqs. (11) and (17).}
\]

(33)

From Eqs. (30), (31) and (33), we obtain

\[
a(s) = \begin{cases} e^{i\theta} + O(|s|)^{-1} & \text{if } |s| \gg 1 \\ e^{-i\theta} + O(|s|) & \text{if } |s| \ll 1. \end{cases}
\]

(34)

\( a(s) \) expressed in terms of scattering data

Let us define the following function in the region \( \text{Im } s \geq 0 \):

\[
\tilde{a}(s) = e^{i\theta} a(s) \prod_{j=1}^{n} \frac{s - s_j^*}{s - s_j}.
\]

(35)

By definition, \( \tilde{a}(s) \) has no zero and no pole in the upper-half plane, and \( \tilde{a}(s) = 1 + O(|s|)^{-1} \) for \( |s| \gg 1 \). Therefore the function \( \log \tilde{a}(s) \) is analytic and satisfies \( \log \tilde{a}(s) = O(|s|)^{-1} \) for large \( |s| \) in the upper-half plane. From the Cauchy’s integral formula, the relation

\[
\log \tilde{a}(s) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log |\tilde{a}(z)|}{z - s} \quad \text{for } \text{Im } s > 0
\]

(36)

follows. Rewriting this expression in terms of \( a(s) \) by using the relation \( |\tilde{a}(z)|^2 = |a(z)|^2 = |t(z)|^2 = (1 - |r(z)|^2)^{-1} \), we obtain

\[
a(s) = e^{i\theta} \prod_{j=1}^{n} \frac{s - s_j^*}{s - s_j} \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(1 - |r(z)|^2)}{s - z} \right]
\]

(37)

for \( \text{Im } s > 0 \). This relation shows how to determine \( a(s) \) from the scattering data. In the limit \( s \to 0 \), we obtain

\[
e^{2i\theta} = \prod_{j=1}^{n} \frac{s_j^*}{s_j} \exp \left[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\log(1 - |r(z)|^2)}{z} \right]
\]

(38)

with the use of Eq. (34). As a special case, if the reflection coefficient vanishes identically, it reduces to

\[
e^{2i\theta} = \prod_{j=1}^{n} \frac{s_j^*}{s_j},
\]

(39)

which provides the derivation of Eq. (15) of the main article.
II. INVERSE PROBLEM

In this section, we introduce the integral representation of Jost solutions using the integral kernel $K(x, y)$. Subsequently, we derive the Gel’fand–Levitan–Marchenko (GLM) equation, that is, the integral equation which determines $K(x, y)$ from scattering data.

Integral representation of Jost solutions

Let us assume that the left Jost solution can be expressed in terms of a $2 \times 2$ matrix integral kernel $K(x, y)$ as follows:

$$f_-(x, s) = \begin{pmatrix} 1 & e^{-ik(s)x} \\ 1 & e^{-ik(s)y} \end{pmatrix} \int_{-\infty}^\infty dy K(x, y) \begin{pmatrix} 1 & e^{-ik(s)y} \end{pmatrix},$$  

where the integrand should not diverge at $y = -\infty$. If $K(x, y)$ decreases exponentially in the limit $y \to -\infty$, the expression (40) is well defined for $\text{Im} k \geq 0 \leftrightarrow \text{Im} s \geq 0$. By replacement $s \to 1/s$, we obtain the similar expression for the other left Jost solution

$$s f_-(x, s^{-1}) = \begin{pmatrix} 1 & e^{ik(s)x} \\ 1 & e^{ik(s)y} \end{pmatrix} \int_{-\infty}^\infty dy K(x, y) \begin{pmatrix} 1 & e^{ik(s)y} \end{pmatrix}. \tag{41}$$

This expression, on the other hand, is well defined for $\text{Im} s \leq 0$. We note that both expressions in Eqs. (40) and (41) are simultaneously well defined only when $s$ is real.

By the same logic of deriving Eq. (14),

$$\sigma_1 f_-(x, s) = s^* f_-(x, s^{-1}) \tag{42}$$

follows, and we obtain the following relation from this relation and Eq. (40):

$$K(x, y) = \sigma_1 K(x, y)^* \sigma_1. \tag{43}$$

In each component, we obtain $K_{22} = K_{11}$ and $K_{12} = K_{21}$. Thus, only the first column of $K(x, y)$ is independent, and it has the following form:

$$K(x, y) = \begin{pmatrix} K_{11}(x, y) & K_{21}(x, y)^* \\ K_{21}(x, y) & K_{11}(x, y)^* \end{pmatrix}. \tag{44}$$

Equations for $K(x, y)$

Proposition 2. The integral kernel $K(x, y)$ satisfies the following equations:

$$\frac{\partial K(x, y)}{\partial x} + \sigma_3 \left( \frac{\partial K(x, y)}{\partial y} - K(x, y) M \right) \sigma_3 - U(x) K(x, y) = 0. \tag{45}$$

Here, we have introduced the following notations:

$$U(x) := \begin{pmatrix} 0 & -i\Delta(x) \\ i\Delta(x)^* & 0 \end{pmatrix}, \tag{46}$$

$$M := U(-\infty) = m \sigma_2. \tag{47}$$

In particular, Eq. (45) can be used to construct the potential $\Delta(x)$:

$$\Delta(x) = m + 2iK_{21}(x, x)^*. \tag{49}$$

Proof. In order to make descriptions short, we introduce the following temporary notations:

$$F(x, s) = \begin{pmatrix} sf_-(x, s^{-1}) & f_-(x, s) \end{pmatrix}, \tag{50}$$

$$Z(x, s) = \begin{pmatrix} se^{ik(s)x} & e^{-ik(s)x} \\ e^{ik(s)x} & se^{-ik(s)x} \end{pmatrix}. \tag{51}$$

$F(x, s)$ and $Z(x, s)$ satisfy the following ZS eigenvalue problem:

$$\frac{\partial_x F(x, s)}{s} = \left( i\epsilon(s)\sigma_3 + U(x) \right) F(x, s), \tag{52}$$

$$\frac{\partial_x Z(x, s)}{s} = \left( i\epsilon(s)\sigma_3 + M \right) Z(x, s). \tag{53}$$

Equations (40) and (41) can be expressed as

$$F(x, s) = Z(x, s) + \int_{-\infty}^\infty dy K(x, y) Z(y, s). \tag{54}$$

Henceforth, we omit the argument $s$ and simply write them as $\epsilon = \epsilon(s)$, $F(x) = F(x, s)$, and $Z(x) = Z(x, s)$. By differentiating Eq. (54) with respect to $x$ and using Eq. (53), we obtain

$$\frac{\partial_x F(x)}{s} = \left( i\epsilon\sigma_3 + M \right) Z(x) + K(x, x) Z(x) + \int_{-\infty}^\infty dy \frac{\partial K(x, y)}{\partial x} Z(y). \tag{55}$$

Next, let us rewrite the R.H.S. of (52). It follows that

$$i\epsilon\sigma_3 \int_{-\infty}^\infty dy K(x, y) Z(y)$$

$$= \int_{-\infty}^\infty dy \sigma_3 K(x, y) \sigma_3 (i\epsilon\sigma_3 Z(y))$$

$$= \int_{-\infty}^\infty dy \sigma_3 K(x, y) \sigma_3 (\partial_x Z(y) - MZ(y))$$

$$= \sigma_3 K(x, x) \sigma_3 Z(x) + \int_{-\infty}^\infty dy \sigma_3 \left( K(x, y) M - \frac{\partial K(x, y)}{\partial y} \right) \sigma_3 Z(y). \tag{56}$$

Here, we have used $\sigma_3^2 = 1$ in the second line, Eq. (53) in the third line, and $\sigma_3 M = -M \sigma_3$ in the last line. Using Eqs. (54) and (56), the R.H.S. of (52) can be rewritten as

$$(i\epsilon\sigma_3 + U(x)) F(x)$$

$$= (i\epsilon\sigma_3 + U(x) + \sigma_3 K(x, x) \sigma_3) Z(x)$$

$$+ \int_{-\infty}^\infty dy \left[ \sigma_3 \left( K(x, y) M - \frac{\partial K(x, y)}{\partial y} \right) \sigma_3 + U(x) K(x, y) \right] Z(y). \tag{57}$$

From (R.H.S of (52)) = (R.H.S of (57)), we obtain

$$\sigma_3 K(x, x) \sigma_3 - K(x, x) + U(x) - M$$

$$= \int_{-\infty}^\infty dy \left[ \frac{\partial K(x, y)}{\partial x} + \sigma_3 \left( \frac{\partial K(x, y)}{\partial y} - K(x, y) M \right) \right] \sigma_3$$

$$- U(x) K(x, y) \right] Z(y, s) Z(x, s)^{-1}. \tag{58}$$
Here, we again write the \( s \)-dependence of \( Z(x,s) \) explicitly. In Eq. (58), the L.H.S. is a function dependent only on \( x \). On the other hand, the R.H.S. depends on \( x \) and \( s \). Therefore, the L.H.S. vanishes when we differentiate both sides with respect to \( s \). In order for the relation \( \partial_s (\text{R.H.S.}) = 0 \) to hold for any \( x \) and \( s \), the integrand must vanish identically, we thus obtain Eq. (56). As a result, the L.H.S. of (58) also vanishes, and Eq. (45) follows. □

**GLM equation**

**Theorem 3.** Let \( s_j (j = 1, \ldots, n) \) be zeros of \( a(s) \) in the upper-half plane. The equation

\[
K(x,y) \int_0^x + F(x+y) + \int_{-\infty}^x dz K(x,z)F(z+y) = 0
\]

holds for \( y < x \). Here \( F(x) \) is defined by

\[
F(x) := F_r(x) + F_d(x),
\]

\[
F_r(x) := \frac{m}{4\pi} \int_{-\infty}^{+\infty} ds r(s) \left( \frac{s^2}{1} \right) e^{-ik(s)x},
\]

\[
F_d(x) := \sum_{j=1}^n |b(s_j)|^2 e_j \left( \frac{1}{s_j} \right) e^{\kappa_j x},
\]

where \( r(s) = b(s)/a(s) \) is a reflection coefficient, \( \kappa_j = -ik(s_j) \) is a complex wavenumber, and \( e_j^2 \) is the normalization constant introduced in Proposition [1].

Before giving the proof, we remark that this theorem does not contain the case of \( y = x \), since the discussion on the convergence becomes rather sensitive in this case. Indeed, if one tries to include the case \( y = x \) in the theorem, one finds that the proof is not valid in a concrete example of the n-soliton solution discussed in the next section. However, we must use the case of \( y = x \) in order to construct the potential \( \Delta(x) \), as shown in Eq. (49). The safest way to overcome this dilemma is to interpret \( K(x,x) \) as a limiting value, i.e.,

\[
K(x,x) := \lim_{y \to x} K(x,y).
\]

**Proof.** From Eq. (16), the equation

\[
\frac{1}{a(s)} f_r(x,s) = s f_r(x,s^{-1}) + \frac{b(s)}{a(s)} f_r(x,s)
\]

holds. By rewriting this by using Eqs. (40) and (41), we obtain

\[
\frac{1}{a(s)} f_r(x,s) - \left( \frac{1}{s} \right) e^{ik(s)x}
= \int_{-\infty}^{+\infty} dz K(x,z) \left( \frac{1}{s} \right) e^{ik(z)x}
+ \frac{b(s)}{a(s)} \left( \frac{1}{s} \right) e^{-ik(s)x} + \int_{-\infty}^{+\infty} dz K(x,z) \left( \frac{1}{s} \right) e^{-ik(s)x}.
\]

Henceforth, we calculate

\[
\frac{m}{2} \int_{-\infty}^{+\infty} ds \frac{e^{-ik(s)y}}{s} \tag{64}
\]

(\( y < x \)).

First, let us confirm that this integral has a finite value. From Eq. (50), the integrand of L.H.S. of Eq. (65) for large and small \( |s| \) can be estimated as follows:

\[
\begin{align*}
\frac{1}{a(s)} f_r(x,s) - \left( \frac{1}{s} \right) e^{ik(s)x} = O(|s|^{-1}) \times e^{-\frac{\pi}{2} s^{-1} (x-y)} \quad (|s| \gg 1) \\
\frac{1}{a(s)} f_r(x,s) - \left( \frac{1}{s} \right) e^{ik(s)x} = O(1) \times e^{-\frac{\pi}{2} s^{-1} (x-y)} \quad (|s| \ll 1).
\end{align*}
\]

Therefore, the integration for large \( |s| \) converges, unless \( x - y = 0 \). Though we encounter the rapidly oscillating function around \( s = 0 \), the integration gives a finite result.

Let us calculate the L.H.S. of Eq. (65). Since \( e^{ik(s)x-y} \) decreases exponentially in the upper-half plane, we can use the residue theorem. Here we note that all zeros of \( a(s) \) are simple, as mentioned in Proposition [1]. As usual, considering the contour consisting of real axis and semicircle with radius \( R \) in the upper-half plane, and taking the limit \( R \to \infty \), we obtain

\[
\text{L.H.S. of Eq. (65)} = -2\pi \sum_{j=1}^n \frac{1}{a(s_j)} F_r(x,s_j) e^{-ik(s_j)y} = -2\pi \sum_{j=1}^n \left| b(s_j) \right|^2 e_j^2 F_r(x,s_j) e^{\kappa_j y}.
\]

Here, we have used Eqs. (18) and (21) to show the second equality. Furthermore, using the expression (49) and the definition (62), we obtain

\[
\text{L.H.S. of Eq. (65)} = -F_d(x+y) - \int_{-\infty}^{+\infty} dz K(x,z) F_d(z+y).
\]

Next, let us consider the R.H.S. of Eq. (65). Let us note the formulae

\[
\frac{m}{2} \int_{-\infty}^{+\infty} ds e^{ik(s)x} = 2\pi \delta(x),
\]

\[
\frac{m}{2} \int_{-\infty}^{+\infty} ds e^{ik(s)x} = 0
\]

which can be shown by dividing the integral into two regions, \([0, \infty] \) and \([-\infty, 0] \), and substituting \( s' = -s^{-1} \) in the latter integral. By using these formulae, the expression

\[
\text{R.H.S. of Eq. (65)} = K(x,y) \left( \frac{1}{0} \right) + F_r(x+y) + \int_{-\infty}^{+\infty} dz K(x,z) F_d(z+y)
\]

follows by straightforward calculation. From Eqs. (68) and (71), we obtain the theorem. □

**III. MULTI-SOLITON SOLUTION AND EIGENSTATES**

In this section we solve the GLM equation (59) when the reflection coefficient vanishes identically: \( r(s) = 0 \). All the expressions for the n-soliton solution and its eigenstates shown in the main article without proof are provided here.
\textbf{\textit{n}-soliton solution}

We consider the case where the potential has \textit{n} bound states. With letting \( C_j = |b(s_j)|e^{i\epsilon_j} \) (> 0), the form of \( F(x) \) is given by

\[
F(x) = \sum_{j=1}^{n} C_j \left( \frac{1}{s_j} \right) e^{i\epsilon_j s_j}.
\] (72)

Let \( K_1(x, y) \) be the first column of \( K(x, y) \). Then the second column can be written as \( \sigma_1 K_1(x, y)^* \) from Eq. (43). When \( F(x) \) has the form (72), the solution of the GLM equation can be obtained by the following ansatz:

\[
K_1(x, y) = \sum_{j=1}^{n} C_j \left( \frac{f_j(x)}{s_j f_j(x)^*} \right) e^{i\epsilon_j s_j}.
\] (73)

Substituting Eqs. (72) and (73) into Eq. (59) yields

\[
\sum_{j=1}^{n} C_j e^{i\epsilon_j s_j} \left( \frac{f_j(x)}{s_j f_j(x)^*} \right) = 0,
\] (74)

\[
F_j(x) := f_j(x) + C_j e^{i\epsilon_j s_j} + \sum_{j=1}^{n} \left( 1 + s_j^{-1} s_j \right) C_j C_j e^{i\epsilon_j s_j} \frac{e^{i\epsilon_j s_j}}{\kappa_l + \kappa_j} f_j(x).
\] (75)

By taking the complex conjugate in the second component of Eq. (74), we obtain

\[
\sum_{j=1}^{n} C_j \left( \frac{1}{s_j} \right) e^{i\epsilon_j s_j} F_j(x) = 0.
\] (76)

Since \( (e^{i\epsilon_j s_j}, s_j e^{i\epsilon_j s_j})^T \) \(( j = 1, \ldots, n)\) are linearly independent of each other, all \( F_j(x) \)'s must vanish.

The numerical factor appearing in \( F_j(x) \) can be rewritten as

\[
\frac{1 + s_j^{-1} s_j}{\kappa_l + \kappa_j} = -\frac{2i}{m} \frac{1}{s_j^{-1} - s_j}.
\] (77)

In order to simplify descriptions, we also introduce the following symbol:

\[
e_j(x) = C_j e^{i\epsilon_j s_j}, \quad (j = 1, \ldots, n).
\] (78)

If we parametrize \( C_j = \sqrt{k_j} e^{i\epsilon_j s_j} \), it is equivalent to Eq. (12) of the main article. The equation \( F_j(x) = 0 \) \(( j = 1, \ldots, n)\) can be represented as follows:

\[
\begin{pmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_n
\end{pmatrix}
+ \left( \begin{pmatrix}
  \epsilon_1 \\
  \epsilon_2 \\
  \vdots \\
  \epsilon_n
\end{pmatrix}
- \frac{2i}{m} \begin{pmatrix}
  1 & \epsilon_1 & \ldots & \epsilon_n \\
  1 & 1 & \ldots & 1 \\
  \vdots & \vdots & \ddots & \vdots \\
  1 & 1 & \ldots & 1
\end{pmatrix} \begin{pmatrix}
  f_1 \\
  f_2 \\
  \vdots \\
  f_n
\end{pmatrix}
\right) = 0.
\] (79)

Solving this equation, \( K_1(x, y) \) can be obtained as

\[
K_1(x, y) = \sum_{j=1}^{n} \left( \frac{f_j(x)}{s_j f_j(x)^*} \right) e_j(y),
\] (80)

and using this, the potential \( \Delta(x) \) is constructed as

\[
\Delta(x) = m + 2i K_{21}(x, x)^* = m + 2i \sum_{j=1}^{n} s_j^{-1} e_j(x) f_j(x).
\] (81)

\textbf{Eigenstates}

Since \( K_1(x, y) \) is given by Eq. (73), the left Jost solution \(40 \) can be evaluated as follows:

\[
f_{-}(x, s) = e^{-ik(s)x} \left\{ \begin{pmatrix}
  1 \\
  \frac{1}{s_j} \sum_{j=1}^{n} C_j (1 + s_j^{-1} s_j) \left( \frac{f_j(x)}{s_j f_j(x)^*} \right) e^{i\epsilon_j s_j} \end{pmatrix}
\right\} \left( \frac{e^{i\epsilon_j s_j}}{\kappa_j - ik(s)} \right).
\] (82)

By replacement \( s \rightarrow 1/s \) in Eq. (82), we also obtain

\[
f_{-}(x, s^{-1}) = e^{ik(s)x} \left\{ \begin{pmatrix}
  1 \\
  \frac{2i}{m} \sum_{j=1}^{n} \frac{e_j(x)}{s_j - s^{-1}} \left( \frac{f_j(x)}{s_j f_j(x)^*} \right)
\end{pmatrix}
\right\}.
\] (83)

It represents the scattering states if \( s \) is real.

The bound states can be obtained by substitution \( s = s_l (l = 1, \ldots, n) \) in Eq. (82). Because of Proposition 1, the normalization constant of \( f_-(x, s_l) \) is equal to \( C_l = |b(s_l)|e_l \). We thus obtain the normalized bound states

\[
-C_l f_{-}(x, s_l) = -e_l(x) \left\{ \begin{pmatrix}
  1 \\
  \frac{2i}{m} \sum_{j=1}^{n} \frac{e_j(x)}{s_j - s_l^{-1}} \left( \frac{f_j(x)}{s_j f_j(x)^*} \right)
\end{pmatrix}
\right\} = \left\{ \begin{pmatrix}
  f_l(x) \\
  s_l f_l(x)^*
\end{pmatrix}
\right\}.
\] (84)

Here, we have used Eq. (79) to show the second equality. Equations (83) and (84) provide the eigenstates shown in the main article without derivation.

\textbf{Positions of solitons}

Finally, let us derive the approximate expression for positions of solitons when they are well-separated from each other. As in the main article, we use the convention \( e_j(x) = \sqrt{\kappa_j} e^{i\epsilon_j (x - s_j)} \). For convenience, we prepare the notations for 1-soliton solution with phase shift \( e^{i\epsilon_j s_j} = e^{-2i\theta_j} \) located at \( x = 0 \) and its bound state:

\[
\Delta_{1-\text{sol}}(x, s_1) = m \left( e^{i\epsilon_1 x} + e^{-\kappa_1 x} \right)
\] (85)

\[
= me^{-i\theta_1} \left( \cos \theta_1 - i \sin \theta_1 \tanh \kappa_1 x \right),
\] (86)

\[
f_{1-\text{sol}}(x, s_1) = -\frac{\sqrt{\kappa_1}}{2} \frac{1}{\cosh \kappa_1 x}.
\] (87)

Our purpose is to show the following Proposition.
Proposition 4. Let us assume that all \( x_j \)'s are sufficiently separated from each other. In this situation, we can relabel the indices of solitons so that \( x_1 < x_2 < \cdots < x_n \) holds. The approximate position of the \( j \)-th soliton \( X_j \) is given as follows:

\[
X_j = x_j + \frac{1}{\kappa_j} \sum_{i=1}^{j-1} \log \left| \frac{1 - s_i s_j}{s_i - s_j} \right| \quad (j \geq 2).
\]

The approximate expressions of the potential \( \Delta(x) \) and the \( j \)-th bound state near \( x \approx X_j \) are given by

\[
\Delta(x) \approx e^{-2i(\theta_1 + \cdots + \theta_{j-1})}\Delta_{1\text{-sol}}(x - X_j, s_j),
\]

\[
f_j(x) \approx e^{-i(\theta_1 + \cdots + \theta_{j-1})} \prod_{l=1}^{j-1} (\theta_l - \theta_j) f_{1\text{-sol}}(x - X_j, s_j).
\]

Proof. Since the bound states are localized to each of the solitons, all \( f_j(x) \)'s except for \( f_j(x) \) are negligible near the \( j \)-th soliton. Furthermore, \( e(x) \) for \( j \) is also negligible since \( x_j - x_1 \ll 0 \). Thus, Eq. (79) approximately reduces to the following equation near the \( j \)-th soliton:

\[
\begin{pmatrix}
0 \\
\vdots \\
0
\end{pmatrix} + \frac{2i}{m} \begin{pmatrix}
\frac{e}{x_1 - s_1} & \frac{e}{x_1 - s_2} & \cdots & \frac{e}{x_1 - s_j} \\
\frac{e}{x_2 - s_2} & \frac{e}{x_2 - s_3} & \cdots & \frac{e}{x_2 - s_j} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{e}{x_j - s_1} & \frac{e}{x_j - s_2} & \cdots & \frac{e}{x_j - s_j}
\end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_j \end{pmatrix} = 0.
\]

Comparing the coefficient of \( x^{j-1} \), we obtain Eq. (94). When \( l = j \), Eq. (92) is simplified as

\[
f_j(x) = - \frac{1}{m} \prod_{p=1}^{j-1} \frac{s_p^{-1}}{s_p - s_j} \frac{\kappa_j f_j(x)}{\kappa_j + f_j(x)^2}.
\]

Using Eqs. (92) and (94), \( \Delta(x) \) can be obtained as

\[
\Delta(x) = m \sum_{l=1}^{j-1} s_l^{-1} e_l f_l = m \prod_{p=1}^{j-1} s_p^{-1} \frac{\kappa_j + s_j^{-2} I_j}{\kappa_j + I_j^2}.
\]

Equations (96) and (97) with (93) are equivalent to Eqs. (89) and (90), and these expressions imply that \( X_j \) represents the position of the \( j \)-th soliton. □