PONCELET AND THE ARQUIMEDEAN TWINS

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Abstract. We give a sharp construction for twins in arbelos, based on polar reciprocity. In the process, new circles displaying arquimedean affinities came into scene.

1. Introduction

Arbelos is the greek for 'shoemaker’s knife’, the shape bounded by three pairwise tangent semicircles with diameters lying on the same line. As a geometric object, it was first studied by Arquimedes in his Book of Lemmas, hence dates back more than 2200 years ago. If we draw the common (internal) tangent to the arbelos circles then the two circles that tangent this line, the arbelos outer circle and one of the arbelos’ circles are called "the twins"; see figure 5. Arquimedes had already spotted them and proved that their radius is half the harmonic mean of the two arbelos i-circles. As a tribute, circles in arbelos, congruent with the twins are called arquimedean; chasing arquimedean circles in arbelos was is a constant theme ever-since.

Perhaps the most humble arquimedean circle is those whose diameter is the parallel through $M$ at the bases of the (rectangular) trapeze of basis $R_1, R_2$ and altitude $R_1 + R_2$; see figure 2; in fact the proof of this fact uses elementary proprieties in trapezius. Longer, yet straightforward computation confirms that the tangents from $O_1$ and $O_2$ to this circle meet precisely at the center of the arbelos i-circle. This circle was spotted in [B] (somehow backwardly that presented here), but we presume Arquimedes already knew about it.

Main results. Rather than chasing arquimedean circles, here we are interested in explain why the twins are identical and how to draw them. We foreseen the (classic) twins as solutions of two degenerated Apollonius’ problem, and we find their centers as intersection of special conics.

Keywords: arbelos, Apollonius’s problem, circle inversion, poles, polar duals.

Figure 1. Twins in arbelos are solutions of two distinct Apollonius’ problems: but why are they arquimedean?
Figure 2. An arquimedean circle (bordeaux) canonically attached to an arbelos; two other circles of same radius (solid yellow) can be drawn tangent to the arbelos' circles (grey) and to their common tangent; but in this case, why do they also the arbelos' external circle?

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To perform their intersection, we use polar reciprocity, the method tailored by Poncelet ([P]) in order to proof his Porism. The proof itself led to a sharp geometric construction for their centers, which is our main result; see theorem 1. The method also explains why the radii of the twins is the same. We then attach another pair of circles, which has twice the radius of the initial one, and study a new tern of related circles. One of the circles of the tern is arquimedean, and was first spotted by Scotch in [S]; the other two are new and verify an arquimede type relation which we prove in theorem 4.

Related work. The literature on arbelos is so rich and the references are so abundant, that we limit ourselves to a small sample that show that interest for this problem is still vivid. In [B] a famous construction for the i-circle in arbelos, much simpler than the one from Archimedes’ proof, due to the existence of a third arquimedean circle; [W] also provides a simple construction of the i-circle in (classic) arbelos; in [D-L] the authors study inversions mapping that switch two given circles and apply it to arbelos; [O] studies a generalization of arbelos; [O-W] study twin circles in skewed arbelos. [We] is a handy source that collect many of these facts and related bibliography.

Notations. We shall note by (O) a circle centered in O. By reflection in (O) we mean the symmetry (or inversion) with respect to circle (O). We freely use "dual curve", "polar dual" or "reciprocal curve" as synonymous.

2. Twins’ centers and the Apolonius’ problem
Twins in arbelos are the two circles that tangent the arbelos’ nested circles and their common (internal) tangent line; thus, each twin is a solution of an Apollonius
problem; hence, as in [G], their centers can be obtained by intersecting some special conics.

Let a circle and a tangent line to it be given, as in figure 3.

**Lemma 1.** The locus of the centers of the circles that tangent both the circle and the line is a parabola focused at circle’s center and whose vertex is at the tangency point of the circle and the line.

Let \((O)\) and \((O_1)\) be two internally tangent circles, as in figure 3.

**Lemma 2.** The locus of the centers of the circles that tangent two nested (internally) tangent circles is an ellipse focused at the centers of the two nested circles and passing through their common tangency point.

Both lemma 1 and 2 have elementary proofs that we omit. When we specialize to arbelos, we get a sharper result. Refer to figure 3.

**Proposition 1.** The center of each twin is the intersection between an ellipse and a parabola.

Each ellipse has one vertex at the common tangency point of each internal arbelos circle with the external one, (points \(A_1\) and \(A_2\) respectively), has one focus in \(O\) and the other focus in \(O_1\) and \(O_2\), respectively. The vertex of the parabolas is the common tangency point of the internal arbelos’ circles \((O_1), (O_2)\) (point \(M\)) and their focus is in \(O_1\) and \(O_2\), respectively.

At this point, any drawing software is able to perform the intersection of these two conics. Nevertheless, we are "not done", since the task is to perform a geometric construction: a construction with a straight-line and a compass only. In a geometric construction, one cannot intercept "continuous curves", other then circles and lines.

In fact, while two arbitrary conics cannot be (geometrically) intersected, the conics that aroused here, have a special feature: a common focus. And here is where polar reciprocity comes into scene. The reader not acquainted with this topic, may see Appendix.

**Lemma 3.** The intersection of two conics that have a common focus are the poles of the common tangents of their duals, w.r. to an inversion circle centered at their (common) focus.

For more details on poles, polars and polar reciprocity, see [A], [S], [GSO].

Now we may specify who these duals are. Refer to figure 4 and choose as the inversion circle \((O_1)\).

**Lemma 4.** The polar duals (w.r. to \((O_1)\)) of the ellipses \(E_1\) and the parabola \(P_1\) in figure 4, are two circles that:

(i) are tangent to \((O_1)\) at \(A_1\) and \(M\) respectively;
(ii) the diameter of the parabola’s dual is \([O_1M] = R_1\) and the diameter of the ellipse’s dual is \([A_1O_2'] = \frac{2R_1^2 + R_1R_2}{R_1 + R_2}\) where \(O_2'\) is the reflection of \(O_2\) in \((O_1)\);
(iii) the similitude centers of these two circles is \(O_2\).

**Proof.** The proof uses known facts on the polar of a conic, that we collect in Appendix. Both the parabola and the ellipse in figure 4 have one common focus in \(O_1\); therefore, their duals w.r. to \((O_1)\) are circles. The dual of a parabola w.r. to an inversion circle centered at its focus is a circle, whose diameter is \([O_1M]\), where \(M\) is the reflection of the parabola’s vertex and \(O_1\) is the center of inversion. Since the vertex \(M\) is located on the inversion circle, its is invariant by reflection; thus, the dual of the parabola is a circle with diameter \([O_1M] = R_1\).
Figure 3. I) (left) The locus of the centers of the circles that tangents two internally tangent circles is an ellipse (orange) which has the foci at the centers of the two circles and one vertice at their common tangency point. II) (right) The locus of the centers of the circles that tangents externally one circle (solid grey) and a line (which tangent the circle) is a parabola (purple), which has the focus into the center of the circle and one vertice at the common tangency point of the circle and the line.

Figure 4. I) The polar dual w.r. to $(O_1)$ of the ellipse focused in $O_1$ and $O$ and passing through $A_1$ is a circle (dotted orange) tangent at $A_1$ to $(O)$. Its diameter is $[A_1O_2]$ where $O_2$ is the reflection of $O_2$ in $(O)$. II) The polar dual w.r. to $(O_2)$ of the parabola focused in $O_1$ and vertex $M$ is the circle of diameter $[O_2M]$. III) The similitude center of these two circles is $O_2$, the center of the second arbelos’ circle. IV) The two (real) intersection points of the parabola and the ellipse are the poles (w.r. to $(O_2)$) of their common tangents : shown is point $D_1$, the center of one of the arbelos’ twins.
The ellipse focused in $O_1$ and $O$, and passing through $A_1$ has its second vertice at $O_2$. Therefore, its polar dual w.r. to $(O_1)$ is a circle whose diameter is $[A_1O'_2]$, where $O'_2$ is the reflection of $O_2$ in $(O_1)$; the vertice $A_1$ is invariant, since it is a point of the inversion circle. Thus, the diameter of the dual is

$$(1) \quad A_1O'_2 = A_1O_1 + O_1O'_2 = R_1 + \frac{R_1^2}{R_1 + R_2} = 2R_1^2 + R_1R_2.$$

Let $o_1$ and $S_1$ be the centers of these dual circles; their radius are, respectively

$$(2) \quad r_1 = \frac{A_1M}{2} = \frac{2R_1^2 + R_1R_2}{2(R_1 + R_2)} \quad \text{and} \quad r_m = \frac{R_1}{2}.$$

In order to prove that the similitude center is $O_2$, we prove that

$$(3) \quad \frac{o_1O_2}{S_1O_2} = \frac{r_1}{r_m}.$$

In fact,

$$S_1O_2 = \frac{R_1}{2} + R_2 = \frac{R_1 + 2R_2}{2}$$

and

$$o_1O_2 = A_1O_2 - A_1o_1 = 2R_1 - \frac{R_1(2R_1 + R_2)}{2(R_1 + R_2)} = \frac{(2R_1 + R_2)(R_1 + 2R_2)}{2(R_1 + R_2)};$$

hence

$$\frac{o_1O_2}{S_1O_2} = \frac{(2R_1 + R_2)}{(R_1 + R_2)}.$$
\[ \frac{r_1}{r_m} = \frac{R_1(2R_1 + R_2)}{2(R_1 + R_2)} \quad \frac{2}{R_1} = \frac{(2R_1 + R_2)}{(R_1 + R_2)} \]

equation 3 is verified.

We may attach to any arbelos a pair of tangent circles, "the siblings:" these are two circles, mutually tangent at the common tangency points of the arbelos circles, and of half radius each.

The results proved above justifies the following new and sharp construction of the centers of the twins; refer to figure 5.

**Theorem 1.** The centers of the twins are the poles of the tangents drawn from the center of a arbelos’ circle, to the opposite sibling.

3. Twins’ radii

At this point, we draw the twins’ centers using polar reciprocity. This does not guarantees that the circles are arquimedean. Fortunately, the method is proper for computation purposes, as well. Refer to figure 5.

**Lemma 5.** Let \((O_1)\) and \((O_2)\) the (internal) arbelos circles and let \((S_1)\) and \((S_2)\), be their siblings: two circles whose diameters are \([O_1M]\) and \([O_2M]\). Let \(O_1T_2\) the tangent from \(O_1\) to circle \((S_2)\); let \(D_2\) be its pole w.r. to \((O_2)\); construct similarly \(D_1\).

Then

\[ O_2D_2 - R_2 = O_1D_1 - R_1. \]

**Proof.** Let \(r_1, r_2\) be half of the radius \(R_1, R_2\). By construction,

\[ \triangle O_1O_2P_2 \sim \triangle O_1S_2T_2, \]

hence

\[ \frac{r_2}{O_2P_2} = \frac{2r_1 + r_2}{2r_1 + 2r_2}, \quad \text{or} \quad O_2P_2 = \frac{2r_2(r_1 + r_2)}{2r_1 + r_2}. \]

Since \(D_2\), the pole of the line \(O_1T_2\) is the reflection of \(P_2\), the projection of \(O_2\) to the line \(O_1T_2\), with respect to \((O_2)\).

\[ O_2P_2 \cdot O_2D_2 = 4r_2^2, \]

hence

\[ O_2D_2 = \frac{4r_2^2}{O_2P_2} = \frac{4r_2^2(2r_1 + r_2)}{2r_2(r_1 + r_2)} = \frac{2r_2(2r_1 + r_2)}{r_1 + r_2}. \]

This gives

\[ O_2D_2 - R_2 = O_2D_2 - 2r_2 = \frac{2r_2(2r_1 + r_2)}{r_1 + r_2} - 2r_2 = 2r_2 \left\{ \frac{(2r_1 + r_2)}{r_1 + r_2} - 1 \right\}, \]

hence

\[ (4) \quad r = O_2D_2 - R_2 = \frac{2r_1 \cdot r_2}{r_1 + r_2}. \]

Formula 4 is symmetric with respect to \(r_1\) and \(r_2\); an identical computation, obtained by switching the indices 1 and 2, ends the proof.

Lemma 3 and Lemma 5, prove the result on the twins; refer to figure 5.
Figure 6. The center of the arbelos’ i-circle as intersection of two ellipses: one focused in $O$ and $O_1$ and passing through $O_2$, (green) and the other focused in $O$ and $O_2$ and passing through $O_1$. Their intersection is the pole w.r. to $(O)$ of the common tangent (blue) to their reciprocal circles (orange and green circles).

**Theorem 2.** The circles centered in $D_1, D_2$ and of radius $r$ are arquimedean twins in arbelos: they are tangent to both $(O_1)$ $(O_2)$ and $(O)$, and also are tangents to the line $l$ and their (common) radius $r$ is

$$\frac{1}{r} = \left[ \frac{1}{R_1} + \frac{1}{R_2} \right],$$

where $R_1, R_2$ denotes the radii of the arbelos’ circles.

**Proof.** The fact that the points $D_1$ and $D_2$ are the centers of the twins is guaranteed by Lemma 3. Thus, the circle centered in $D_2$ and whose radius is $D_2O_2 - R_2$ tangents the line $l$, as well, hence these are the twins. The relation 4 ends the proof. \qed

**Arbelos i-circle.** For the sake of completeness, and because is effortless, we show how may draw the arbelos’ i-circle $I$, the circle that tangents the three arbelos’ circles.

**Theorem 3.** Refer to figure 6.

(i) The center of $I$ is obtainable as the intersection of two ellipses: one focused in $O$, and $O_1$, and axis $[O_2A_1]$ and the other focused in $O$ and $O_2$ and axis $[O_1A_2]$.

(ii) The intersection points of these ellipses are the poles of the common tangents to their reciprocal circles, w.r. to $(O)$.

(iii) The radius of the arbelos i-circle is

$$R = \frac{R_1R_2(R_1 + R_2)}{R_1^2 + R_1R_2 + R_2^2}.$$
Figure 7. The cousin i-circle (golden) tangents internally \((O)\) and externally \((A_1)\) and \((A_2)\). Its center obtains as intersection of an ellipse (violet) and a hyperbola, focused in \(A_2, O\) and \(A_1, A_2\), respectively. \(S\) is the pole of the common tangent at their reciprocal circles, w.r. to \((A_2)\); its radius is the same as those of the classic twins: 
\[
R = \sqrt{R_1 + R_2 + R_1 R_2 R_2}. 
\]

Proof. The two ellipses have a common focus in \(O\) and tangent the inversion circle \((O)\) at their vertices \(A_1\) and \(A_2\), respectively. Hence their duals are two circles of diameters 
\[
[A_1 O_2'] \text{ and } [A_2 O_1'] \text{ respectively, where } O_1' \text{ is the reflection of } O_1 \text{ in } (O) \text{ and } O_2' \text{ is the reflection of } O_2 \text{ in } (O). 
\]

Let \(H\) be the similitude center of the reciprocal circles. Then
\[
\triangle HC_1 P_1 \sim \triangle HOP \sim \triangle HC_2 P_2. 
\]
Hence
\[
OP = \frac{(R_1 + R_2)(R_1^2 + R_1 R_2 + R_2^2)}{R_1^2 + R_2^2}. 
\]
Since \(D\) is the pole of \(HP\),
\[
OD \cdot OP = (R_1 + R_2)^2. 
\]
The radius of the i-circle is
\[
R = (R_1 + R_2) - OD = (R_1 + R_2) - \frac{(R_1 + R_2)^2}{OP} 
\]
which gives
\[
R = \frac{R_1 R_2 (R_1 + R_2)}{R_1^2 + R_1 R_2 + R_2^2}. 
\]

4. Arquimedean circles in doubling arbelos

We now associate to a classic arbelos, two other circles, passing through the common tangency points \(M\) and centered at each of the end-point of arbelos’s diameter; we call it a doubling arbelos; see figure 7 and 9.

By i-circle in a doubling arbelos, we mean the circle that tangents externally the two new-added circles and internally the arbelos diametral circle. Such circle was first spotted by Scotch, in [S], who proved that it is arquimedean; above, we indicate
another proof for this fact, and also a construction for the center of this circle, both based on polar reciprocity.

**Theorem 4.** The i-circle in a doubling arbelos is arquimedean: \( \frac{1}{s} = \left[ \frac{1}{R_1} + \frac{1}{R_2} \right] \).

**Proof.** Refer to figure 7. We foresee the i-circle associated to a doubling arbelos as a solution of an Apollonius’ problem; therefore, its center \( S \), can be obtained as an intersection between an ellipses focused in \( O \) and \( A_2 \), passing through \( O_1 \) and a hyperbola focused in \( A_1 \) and \( A_2 \), and passing through \( M \). Since these two conics have a common focus in \( A_2 \), a polar dual w.r. to \( (A_2) \) maps them into a pair of circles, whose diameters are the the reflection of their (ellipses and hyperbola’s) vertices. \( S \), the center of the i-circle in a doubling arbelos is the pole of the common tangent to their reciprocal circles. Straightforward computations, similar to those in the proof of theorem 2 provide an give proof to the Scotch’s result.

These method embeds the geometric construction of \( S \), as a pole of the common tangents to these reciprocal circles.

Now we consider two new circles associated to a doubling arbelos: the twin-cousins are the circles that tangents externally the two arbelo’s circles of diameters \( [A_1,A_2] \) and \( [A_1,M] \), as well as internally the arbelos external circle and one of the (new) circles centered at \( A_1 \) and of radii \( A_1M \) or \( A_2 \) and of radii \( A_2M \); see figure 9 (bottom side).

The twin-cousins (solid blue circles) are not congruent; nevertheless they verify an arquimedean-type metric relation.

**Theorem 5.** Let \( s_1, s_2 \) be the radius of the twin-cousins; then

\[
\frac{1}{s_1} + \frac{1}{s_2} = 3\left[ \frac{1}{R_1} + \frac{1}{R_2} \right]
\]

**Proof.** Refer to figure 8. The center of twin cousin \( S_1 \), obtains as intersection of two ellipses: \( \mathcal{E}_1 \) the ellipses focused in \( O \) and \( O_1 \), and passing through \( A_1 \), and another ellipses, \( \mathcal{E}' \), focused in \( O_1 \) and \( A_1 \), and passing through \( M \). Perform a dual transform w.r. to \( (O_1) \); since \( O_1 \) is the common focus of the ellipses \( \mathcal{E}_1 \) and \( \mathcal{E}' \), their duals are two circles. The dual of \( \mathcal{E}_1 \) is the circle \( C_1' \) of diameter \( [A_1O_2] \), where \( O_2' \) is the reflection of \( O_2 \) in \( (O_1) \); denote by \( o_2' \) and \( R_2' \) be its center and radius. Similarly, the dual of \( \mathcal{E}' \) is the circle \( C_2' \) of diameter \( [B_1'M] \), where \( B_1' \) is the reflection of the ellipse’s vertice \( B_1 \) in \( (O_1) \); denote by \( o_1' \) and \( R_1' \) be its center and radius. The common tangent, \( p \) of these dual circles intercept the line of centers in \( \Omega_1 \), their similitude center. Let \( P_1, P_1', P_2' \) denote respectively the projections of \( O_1, o_1', o_2' \) on the tangent \( p \).

Since \( \mathcal{E}' \) the ellipses focused in \( O_1 \) and \( A_1 \), and passing through \( M \), and since \( O_1B_1 = 2 \cdot R_1 \),

\[
O_1B_1 \cdot O_1B_1' = R_1^2, \quad O_1B_1' = \frac{R_1}{2};
\]

therefore,

\[
R_1' = \frac{3R_1}{4}, \quad O_1O_2' = \frac{R_1}{4}.
\]  

Now we find the segment \( O_1O_2 \). The diameter \( A_1A_1' \) of the dual circle of the ellipses \( \mathcal{E}' \) is

\[
A_1A_1' = O_1A_1 + O_1A_1' = R_1 + \frac{R_2^2}{R_1 + R_2} = \frac{2R_1^2 + R_1R_2}{R_1 + R_2};
\]
Figure 8. \( S_1 \), the center of a twin-cousin, obtained as intersection of two ellipses, as well as a pole of the common tangent at their dual circles (dotted, centered at \( o'_1 \) and \( o'_2 \), respectively).

hence

\[
R'_1 = O_1 A'_1 = \frac{R_1^2}{R_1 + R_2}
\]

Finally,

\[
O_1 O'_2 = \frac{A_1 A'_1}{2} - O_1 A'_1 = \frac{2R_1^2 + R_1 R_2}{2(R_1 + R_2)} - \frac{R_1^2}{(R_1 + R_2)} = \frac{R_2^2}{2(R_1 + R_2)}.
\]

Thus,

\[
O_1 O'_2 = O_1 A_1 - O'_2 A_1 = R_1 - \frac{A_1 A'_1}{2} = R_1 - \frac{2R_1^2 + R_1 R_2}{2(R_1 + R_2)}.
\]

In other words,

\[
O_1 O'_1 = O_1 M - O'_1 M = R_1 - R'_1 = \frac{R_1 R_2}{2(R_1 + R_2)}.
\]

and

\[
O'_1 O'_2 = O_1 O'_2 + O_1 O'_1 = \frac{R_1 R_2}{2(R_1 + R_2)} + \frac{R_1}{4} = \frac{R_1^2 + 3R_1 R_2}{4(R_1 + R_2)}.
\]

Now let \( x = \Omega_1 O'_1 \) and \( p = O_1 P_1 \); by hypothesis,

\[
\triangle \Omega O'_1 P'_1 \sim \triangle \Omega O_1 P_1 \sim \triangle \Omega O'_2 P'_2.
\]

Using the relations above, we obtain

\[
x = \frac{3R_1(R_1 + 3R_2)}{4(R_1 - R_2)}, \quad p = \frac{R_1(R_1 + 2R_2)}{(R_1 + 3R_2)}.
\]
Figure 9. Metric coincidences in arbelos: I) The (classic) twins (solid golden) and the cousin i-circle (blue) are congruent and arquimedean: \( s = r_1 = r_2 \) and \( \frac{1}{s} = \frac{1}{r_1} + \frac{1}{r_2} \). II) The two cousin-circles (solid blue) verifies \( \frac{3}{s} = \left[ \frac{1}{s_1} + \frac{1}{s_2} \right] \) hence \( \frac{1}{s_1} + \frac{1}{s_2} = 3 \left[ \frac{1}{R_1} + \frac{1}{R_2} \right] \).

Hence
\[
O_1'O_2' = \frac{R_1^2 + 3R_1R_2}{4(R_1 + R_2)}, \quad R_1' = \frac{3R_1}{4}, \quad R_2' = \frac{2R_2^2 + R_1R_2}{2(R_1 + R_2)}.
\]

With these ingredients in place, a straightforward computations led to the radius of the cousin-twin centered at \( S_1 \)
\[
s_1 = \frac{R_1R_2}{R_1 + 2R_2}.
\]

If we interchange the indices 1 and 2, we obtain
\[
s_2 = \frac{R_1R_2}{R_2 + 2R_1}.
\]

Thus
\[
\frac{1}{s_1} + \frac{1}{s_2} = 3\left[ \frac{1}{R_1} + \frac{1}{R_2} \right].
\]

\[\square\]

**Appendix. Brief recall on polar reciprocity**

We include here a brief recall on polar reciprocity.

Let us fix \( C(\Omega, R) \) a circle centered in \( \Omega \) and of radius \( R \), which we shall call inversion circle.

**Definition 1.** If \( p_0 \) is a line that does not pass through \( \Omega \), its pole is the inverse of the projection of the center \( \Omega \), on the line \( p_0 \).

**Definition 2.** If \( P_0 \) is a point \( (P_0 \neq \Omega) \), the polar of \( P_0 \) is the perpendicular line on \( \Omega P_0 \), that pass through \( P_1 \), the inverse of \( P_0 \).
Definition 3. The polar dual (or a reciprocal curve) of a regular curve (w.r. to an inversion circle) defines as the curve whose points are the poles of the tangents of the original curve.

When we perform the dual of a circle, w.r. to an inversion circle, we obtain conics.

A 1. (see e.g. [S], art. 306 and 309) The dual of a circle $\gamma = C(O, r)$, w.r. to an inversion circle $C(\Omega, R)$, is a conic, $\Gamma$; if $d$ denotes the distance between the centres of the reciprocated and inversion circles, $d = \Omega O$, then:

i) $\Gamma$ is an ellipse, if $r < d$;
ii) $\Gamma$ is a parabola, if $r = d$;
iii) $\Gamma$ is a hyperbola, if $r > d$.

Moreover, (one of) the the focus of the dual conic $\Gamma$ is precisely $\Omega$, the center of the inversion circle; its directrix is the polar of $O$, the center of the reciprocated circle and the eccentricity is $e = \frac{r}{d}$.

This theorem has a very useful counterpart.

A 2. The dual of a conic $\Gamma$, w.r. to an inversion circle centered into its focus, is a circle, $\gamma$. The symmetric of the vertices of the conic $\Gamma$, are a pair of diametrically opposite points of the dual circle, $\gamma$. The pole of the directrix of $\Gamma$, is the center of the circle $\gamma$.

All the geometrical elements of the dual conic, can be drawn with straight-line and compass, since all the steps involves drawing the symmetric of a point and the pole of a line.

A final useful fact regards intersections of (regular) curves.

A 3. The intersection of two curves are the poles of their common tangents.

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