On Conjectures Concerning the Smallest Part and Missing Parts of Integer Partitions

Damanvir Singh Binner and Amarpreet Rattan

Department of Mathematics
Simon Fraser University
Burnaby, BC V5A 1S6
Canada
dbinner@sfu.ca
rattan@sfu.ca

Abstract

For positive integers $L \geq 3$ and $s$, Berkovich and Uncu (Ann. Comb. 23 (2019) 263–284) conjectured an inequality between the sizes of two closely related sets of partitions whose parts lie in the interval $\{s, \ldots, L+s\}$. Further restrictions are placed on the sets by specifying impermissible parts as well as a minimum part. The authors proved their conjecture for the cases $s = 1$ and $s = 2$. In the present article, we prove their conjecture for general $s$ by proving a stronger theorem. We also prove other related conjectures found in the same paper.

1 Introduction

Let $n$ be a nonnegative integer. A partition $\pi = (\pi_1, \pi_2, \ldots)$ of $n$ is a weakly decreasing list of positive integers whose sum is $n$, and we write $|\pi| = n$ to indicate this. We allow the empty partition as the unique partition of 0. Each $\pi_i$ is known as a part of $\pi$. In the present article, it is more convenient to use the notation that expresses the number of parts of each size in a partition. In this notation, we write $\pi = (1^{f_1}, 2^{f_2}, \ldots)$, where $f_i$ is the frequency of $i$ or the number of times a part $i$ occurs in $\pi$. Thus, each frequency $f_i$ is a nonnegative integer, and when $f_i = 0$ this expresses that $\pi$ has no part of size $i$. When the frequency of a number is 0, it may or may not be omitted in the expression. In the latter notation, it is clear that $|\pi| = \sum_i i \cdot f_i$. Thus $(4, 4, 2, 2, 1), (1^1, 2^2, 3^0, 4^2, 6^0)$ and $(1^1, 2^2, 4^2, 5^0)$ all represent the same partition of 13.

In [BU19], Berkovich and Uncu conjectured some intriguing partition inequalities regarding the relative sizes of certain sets of partitions. We recall their definitions. For positive integers $L$ and $s$,

- $C_{L,s}$ denotes the set of partitions where the smallest part is $s$, all parts are $\leq L + s$, and $L + s - 1$ does not appear as a part;
Conjecture 1 (Berkovich and Uncu (2019)). For positive integers \(L \geq 3\) and \(s\), there exists an \(M\), which only depends on \(s\), such that

\[|\{\pi \in C_{L,s} : |\pi| = N\}| \geq |\{\pi \in D_{L,s} : |\pi| = N\}|,\]

for every \(N \geq M\).

They proved in [BU19, Theorem 1.1, Theorem 3.1] Conjecture 1 for \(s = 1\) (with bound \(M = 1\)) and \(s = 2\) (with bound \(M = 10\)). In both cases, the authors found a suitable injection. Conjecture 1 is therefore a natural generalization of those theorems. Their investigations suggested further conjectures, three of which we give below. To state the first, for positive integers \(L\) and \(s\), if \(L \geq s + 1\),

- \(C_{L,s}\) denotes the set of partitions where the smallest part is \(s\), all parts are \(\leq L + s\), and \(L\) does not appear as a part.

The next conjecture is found in [BU19, Conjecture 3.3].

Conjecture 2 (Berkovich and Uncu (2019)). For positive integers \(L \geq 3\) and \(s\), there exists an \(M\), which only depends on \(s\), such that

\[|\{\pi \in C_{L,s}^* : |\pi| = N\}| \geq |\{\pi \in D_{L,s} : |\pi| = N\}|,\]

for every \(N \geq M\).

In the definition of \(C_{L,s}^*\), we must have \(L \geq s + 1\), so the inclusion of \(L \geq 3\) in the conjecture is to exclude the case \(L = 2\) and \(s = 1\).

Conjectures 1 and 2 are part of a broader body of recent work concerning sets of partitions whose parts come from some interval. See for example [ABR15, BU17, Cha16]. While we further resolve additional related conjectures from [BU19] below, there are a number of other research directions suggested in that article that we do not pursue here.

While Conjectures 1 and 2 motivated our work, we in fact prove a stronger result. For positive integers \(L, s\) and \(k\), with \(s + 1 \leq k \leq L + s\),

- \(I_{L,s,k}\) is the set of partitions where the smallest part is \(s\), all parts are \(\leq L + s\), and \(k\) does not appear as a part.

Whenever a part cannot occur from a range of allowable parts, as with \(k\) in the definition of \(I_{L,s,k}\), we refer to that as an impermissible part. The sets \(C_{L,s}\) and \(C_{L,s}^*\) above are the special cases of \(I_{L,s,k}\) given by \(I_{L,s,L+s-1}\) and \(I_{L,s,L}\), respectively. Thus, the parameter \(k\) allows us to deal with impermissible parts in the set \(\{s + 1, \ldots, L + s\}\) collectively. The next theorem generalizes Conjectures 1 and 2.

\footnote{For recent progress on both Conjectures 1 and 2, see Section 1.1.}
Theorem 3. For positive integers $L$, $s$ and $k$, with $L \geq 3$ and $s + 1 \leq k \leq L + s$, we have

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| > |\{\pi \in D_{L,s} : |\pi| = N\}|,$$

for all $N \geq \Gamma(s)$, where $\Gamma(s)$ is defined in (17).

At this point, the precise value of $\Gamma(s)$ is not important. We have, however, stated Theorem 3 with the constant $\Gamma(s)$ inserted to emphasize that it is explicitly known and only depends on $s$. It also allows us to easily reference this bound when using the partition inequality presented in Theorem 3 to prove other results. We prove Theorem 3 in Section 3.2. While our methods are elementary and involve constructing injective maps between the relevant sets, they entail analyzing many cases.

For the remaining conjectures of Berkovich and Uncu considered in the present article, define

- the $q$-Pochhammer symbol by
  
  $$(a; q)_n := (1 - a)(1 - aq)(1 - aq^2) \cdots (1 - aq^{n-1}),$$

  for an integer $n \geq 1$, with $(a; q)_0 := 1$;

- the series $H_{L,s,k}(q)$ by
  
  $$H_{L,s,k}(q) := \frac{q^s(1 - q^k)}{(q^s; q)_{L+1}} - \left(\frac{1}{(q^{s+1}; q)_L} - 1\right),$$

  for positive integers $L$, $s$ and $k$.

A series $\sum_{n \geq 0} a_n q^n$ is said to be eventually positive if there exists some $l \in \mathbb{N}$ such that $a_n > 0$ for all $n \geq l$. The next conjecture is found in [BU19, Conjecture 7.1].

Conjecture 4 (Berkovich and Uncu (2019)). For positive integers $L$, $s$ and $k$, with $L \geq 3$ and $k \geq s + 1$, the series $H_{L,s,k}(q)$ is eventually positive.

As stated, the bound $l$ guaranteeing the coefficient of $q^N$ in $H_{L,s,k}(q)$ is positive for all $N \geq l$ may depend on $L$, $s$ or $k$ in Conjecture 4. When $s + 1 \leq k \leq L + s$, elementary partition theory gives the coefficient of $q^N$ in $H_{L,s,k}(q)$ as

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| - |\{\pi \in D_{L,s} : |\pi| = N\}|. \quad (1)$$

Hence, Theorem 3 proves Conjecture 4 when $s + 1 \leq k \leq L + s$, and indeed Conjectures 1, 2 and 4 motivated Theorem 3. However, Conjecture 4 is valid for values of $k$ that do not have the combinatorial interpretation specified in (1). We prove a result stronger than Conjecture 4 that also generalizes Theorem 3.

Theorem 5. For positive integers $L$, $s$ and $k$, with $L \geq 3$ and $k \geq s + 1$, the coefficient of $q^N$ in $H_{L,s,k}(q)$ is positive whenever $N \geq \Gamma(s)$, where $\Gamma(s)$ is defined in (17).

Footnote 2

See Footnote 1, Page 2

3
Again, we emphasize that the bound given in Theorem 5 only depends on $s$, is explicitly known, and is the same as the bound in Theorem 3. We use Theorem 3 along with other results to prove Theorem 5 in Section 3.3.

For the last conjecture of Berkovich and Uncu considered here, given a positive integer $L$,

- $G_{L,2}(q)$ is the series

$$G_{L,2}(q) := \sum_{\pi \in \mathfrak{P}, \ s(\pi) = 2, \ l(\pi) - s(\pi) \leq L} q^{s(\pi)} - \sum_{\pi \in \mathfrak{P}, \ s(\pi) \geq 3, \ l(\pi) - s(\pi) \leq L} q^{s(\pi)},$$

where $s(\pi)$ and $l(\pi)$ denote the smallest and largest parts of $\pi$, respectively, and $\mathfrak{P}$ denotes the set of partitions $\pi$ with $|\pi| > 0$.

A series $S = \sum_{n \geq 0} a_n q^n$ is said to be nonnegative if $a_n \geq 0$ for all $n$. The nonnegativity of the series $S$ is denoted by $S \succeq 0$.

We prove the next conjecture, found in [BU19, Conjecture 5.3], in Section 4.

**Conjecture 6** (Berkovich and Uncu (2019)). For $L = 3$,

$$G_{L,2}(q) + q^3 + q^9 + q^{15} \succeq 0;$$

for $L = 4$,

$$G_{L,2}(q) + q^3 + q^9 \succeq 0;$$

and for $L \geq 5$,

$$G_{L,2}(q) + q^3 \succeq 0.$$

**Remark.** Our statement of Conjecture 6 differs slightly from the one given in [BU19], as their statement is not strictly correct. In the case $L = 3$, the conjecture in [BU19] is stated as $G_{3,2}(q) + q^3 + q^9 \succeq 0$. However, it can be checked, either through machine computation or by hand, that the coefficient of $q^{15}$ in $G_{3,2}(q)$ is -1, and hence the discrepancy between our statement and theirs. Subject to this minor modification, their conjecture is as stated above. We further remark that in [BU19] the authors prove a similar nonnegativity result for a series $G_{L,1}(q)$. The definition of $G_{L,1}(q)$ is the same as the one for $G_{L,2}(q)$, except the restriction on $s(\pi)$ is that $s(\pi) = 1$ in the first sum and $s(\pi) \geq 2$ in the second.

Our proofs rely heavily on the following two lemmas, the first of which is a well-known result of Sylvester.

**Lemma 7** (Sylvester (1882)). For natural numbers $a$ and $b$ such that $\gcd(a, b) = 1$, the equation $ax + by = n$ has a solution $(x, y)$, with $x$ and $y$ nonnegative integers, whenever $n \geq (a - 1)(b - 1)$.

The quantity $(a - 1)(b - 1)$ is known as the *Frobenius number* of $a$ and $b$, and it is known to be the smallest number with the property in Lemma 7. Lemma 7 was first known.

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The Frobenius number for two coprime integers $a$ and $b$ is usually defined as the largest integer not expressible as a nonnegative linear combination of $a$ and $b$, and this number is always $(a - 1)(b - 1) - 1$. Here, we have chosen to call the smallest number $N$ such that every $n \geq N$ can be written as a linear combination of $a$ and $b$ the Frobenius number. As indicated, it is known that $N = (a - 1)(b - 1)$. 

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to Sylvester in [Syl82], but we refer the reader to the four proofs in [Alf05] Pages 31-34]. More recently, an elementary proof was given in [Bin20, Corollary 12]. In addition to Sylvester’s theorem, we also require the following simple lemma.

**Lemma 8.** Let $s$ and $n$ be positive integers such that $n \geq s + 1$. Then the equation

$$(s + 1)X_{s+1} + (s + 2)X_{s+2} + \cdots + (2s + 1)X_{2s+1} = n$$

has a solution $(X_{s+1}, X_{s+2}, \ldots, X_{2s+1})$, where $X_i$ is a nonnegative integer for all $i$.

**Proof.** We use the division algorithm to write $n = (s + 1)q + r$ for some $q \geq 1$ and $0 \leq r \leq s$. If $r = 0$, setting $X_{s+1} = q$ with all other $X_i = 0$ gives a suitable solution. Otherwise $1 \leq r \leq s$, and then

$$n = (s + 1)(q - 1) + (s + 1 + r).$$

Note that $s + 2 \leq s + 1 + r \leq 2s + 1$, so setting $X_{s+1} = q - 1$ and $X_{s+1+r} = 1$ with all other $X_i = 0$ gives a solution, completing the proof.

## 1.1 Recent Proofs of the Above Conjectures

Recently, Zang and Zeng [ZZ20] gave proofs of Conjectures 1, 2 and 3. We highlight the specific similarities and differences between their proofs and ours, as well as the strengths of both approaches. In our case, our approach almost always involves constructing an injective map between the relevant sets of partitions, and those maps heavily rely on Lemmas 7 and 8. Furthermore, our approach allows us to give, in all cases, explicit bounds for when the partition inequalities hold. Our methods also allow us to prove Conjecture 6, which was unresolved until now.

In their paper and ours, a crucial step is to prove Conjecture 4 and then use it to prove the other conjectures. Another similarity is that the proofs of Conjecture 4 are separated into two cases: the case with large $L$ and $k$ (compared to $s$) and the case with small $L$ or $k$.

The comparisons between the two approaches to Conjecture 4 are as follows.

- **An important achievement in both papers is to show that there exists an $M$, which depends only on $s$, such that the coefficient of $q^N$ in $H_{L,s,k}(q)$ is positive whenever $N \geq M$. Thus, our work and theirs both achieve this strengthening of Conjecture 4. They prove this strengthening only for $max(s+1,L) \leq k \leq L+s$ ([ZZ20] Theorem 1.1]), while we show this for any $k \geq s + 1$ (Theorem 5).**

- **For large $L$ and $k$, their proofs and ours are different, as are the lower bounds on $L$ and $k$ for when these results hold. The bounds on $N$ guaranteeing the coefficient of $q^N$ in $H_{L,s,k}(q)$ is positive are similar. The lower bounds on $L$ and $k$ are lower in our case ($L \geq s + 2$ and $k \geq 2s + 2$) versus their case ($k \geq L \geq 2s^3 + 5s^2 + 1$). For their results and ours, the coefficient of $q^N$ in the series $H_{L,s,k}(q)$ is positive when $N$ exceeds a lower bound of order $O(s^3)$. However, their restriction on $k$, that $k \geq L$, means that for a given $s$, if $L$ becomes arbitrarily large, their result does not have this lower bound on $N$ for an arbitrarily large set of values of $k$, whereas in our case the bound is valid whenever $k \geq 2s + 2$. This is especially important in the limiting case $L \to \infty$. For**
example, for a given $s$, for any fixed $k \geq 2s + 2$, and any $N > (25s)^5$, our approach shows that the number of partitions of $N$ with smallest part $s$ and no part equal to $k$ is more than the number of partitions of $N$ with smallest part greater than $s$. On the other hand, their approach does not yield any such result since they require $k \geq L$ and here $L \to \infty$.

- For small $L$ or $k$, their approach and ours differ greatly. In [ZZ20], the authors use a celebrated result of Frobenius and Schur (related to the Frobenius coin problem), which states that for a set $A = \{a_1, \ldots, a_m\}$ of positive integers whose greatest common divisor is 1, the number of partitions of a positive integer $n$ whose parts are restricted to $A$ is approximately

$$n^{n-1}/(m-1)!a_1 \cdots a_m.$$ 

This result is asymptotic in $n$, and it is unknown when this approximation is accurate. In fact, even finding for which $n$ onwards there is at least one such partition is a well-known open problem; see [AH05]. Therefore, the result of Zang and Zeng is also asymptotic. In contrast, our methods are combinatorial, and we produce explicit bounds on when $H_{L,s,k}(q)$ is eventually positive. For large $L$ and small $k$ ($L \geq 3s + 3$ and $s + 1 \leq k \leq 2s + 1$), our bounds are $O(s^{10})$, while for small $L$ ($L \leq 3s + 2$), our bounds are of the order $O\left((6s)^{(6s)^{18s}}\right)$.

Some advantages of the proof of Zang and Zeng in this case is that it is short, elegant, and easily understood. Also, their methods show an intriguing connection between the present problems and the above mentioned theorem of Frobenius and Schur. They further show, in [ZZ20, Theorem 1.3], the eventual positivity of a series $H_{L,s,r,k_1,k_2}(q)$ that generalizes the series $H_{L,s,k}(q)$. In this case also, their results are asymptotic. The chief advantage of our methods is that they produce explicit bounds, and they also lead to a proof of Conjecture 6. Indeed, in [ZZ20, Page 12], the authors state that techniques that produce explicit bounds on when $H_{L,s,k}(q)$ is eventually positive may lead to a proof of Conjecture 6. Our methods confirm this.

A final remark about our results is that while we find explicit bounds throughout this paper, we make no claims about the optimality of those bounds. The question of finding the minimal bounds for when these results hold remains open.

# 2 The case when $L$ is large for Theorem 3

In this section, we build to proving Theorem 3 when $L$ is relatively large, by which we mean $L$ is larger than a constant times $s$. In each case, our lower bound on $L$ is explicitly stated. We begin with a case pertaining to Conjecture 11 and then later generalize those arguments to prove the large $L$ case of Theorem 3. The general technique used in this manuscript will be illustrated in this section.
2.1 The case when the impermissible part $k$ is $L + s - 1$, and $L$ is relatively large

In this section, we focus on the case $k = L + s - 1$ in Theorem \[\text{\textup{[3]}}\] with $L \geq s + 3$. That is, we are considering the case $C_{L,s}$ (or equivalently $I_{L,s,L+s-1}$) when $L \geq s + 3$. This corresponds to the case of $L \geq s + 3$ in Conjecture \[\text{\textup{[H]}}\].

For any $s \geq 1$, define the quantities:

- $F(s) = (10s - 2)(15s - 3) + 8s$;
- $\kappa(s) = (12s - 1)((s + 1) + (s + 2) + \cdots (F(s) - 1)) + 1$.

The number $\kappa(s)$ serves as $M$ in our proof of Conjecture \[\text{\textup{[H]}}\] when $L \geq s + 3$.

**Theorem 9.** If $s$ and $L$ are positive integers with $L \geq s + 3$ and $N \geq \kappa(s)$, then

$$|\{\pi \in C_{L,s} : |\pi| = N\}| > |\{\pi \in D_{L,s} : |\pi| = N\}|.$$

**Proof.** We construct an injective map

$$\phi : \{\pi \in D_{L,s} : |\pi| = N\} \rightarrow \{\pi \in C_{L,s} : |\pi| = N\}.$$

To show strict inequality, at the end of the proof we show that there is an element in the codomain of $\phi$ not in its range.

For $\pi \in D_{L,s}$, the image of $\pi$ under $\phi$ is given in cases depending on the frequency of $L + s - 1$ in $\pi$. Hence, for brevity, we set $f = f_{L+s-1}$, so any $\pi \in D_{L,s}$ has the form

$$\pi = ((s + 1)^{f_{s+1}}, \ldots, (L + s - 1)^{f}, (L + s)^{f_{L+s}}).$$

Our definition of the image of $\pi$ under $\phi$ is given in two cases, when $f = 0$ and when $f \geq 1$, and each case contains several subcases. Our strategy for ensuring $\phi$ is injective is to have images of partitions under $\phi$ from different cases have different frequencies of $s$, while ensuring that in each case itself $\phi$ is injective. To make our strategy and arguments on injectivity clear, we summarize in Table \[\text{\textup{[I]}}\] the frequencies of $s$ in partitions in the image of $\phi$ for each case. As the right hand column of Table \[\text{\textup{[I]}}\] contains disjoint sets, the frequency of $s$ in a partition in the image immediately determines from what case its preimage comes. Then we only need to ensure that $\phi$ is injective in each case.

Case 1: Suppose that $f = 0$ in $\pi$. In this case, we obtain $\phi(\pi)$ by inserting some number of parts equal to $s$ into the partition $\pi$; as $f = 0$, we do not need to remove the parts of size $L + s - 1$, but must compensate by altering the other parts of $\pi$. The number of parts equal to $s$ to be inserted into $\pi$ is given by the subcases below.

Case 1(a): Suppose that there exists an $m$ such that $s + 1 \leq m \leq F(s) - 1$ and $f_m \geq 12s$. Let $m_0$ be the least such number. Then define

$$\phi(\pi) = (s^{12m_0}, (s + 1)^{f_{s+1}}, \ldots, m_0^{f_{m_0} - 12s}, \ldots).$$

We can see that $\phi$ is injective in this case because from the frequency of $s$ in $\phi(\pi)$ we can easily determine $m_0$, and from this $\pi$ can be recovered.
Table 1: The frequency of $s$ in the image of a partition under the function $\phi$ in the different cases for Theorem 9

| Case | Possible frequencies of $s$ |
|------|-----------------------------|
| 1(a) | Multiples of 12              |
| 1(b)(i) | 15                           |
| 1(b)(ii) | 20                          |
| 1(b)(iii) | 2, 4, 6, 8                |
| 2(a) | Odd numbers other than 15          |
| 2(b) | 14                           |

Case 1(b): Suppose that the condition of Case 1(a) does not hold. That is, for every $m$ such that $s + 1 \leq m \leq F(s) - 1$, we have $f_m \leq 12s - 1$. Note that if such partitions do not exist, then Case 1(b) does not arise, and there is no need to construct an injection. Since $N \geq \kappa(s)$, we must have $L + s \geq F(s)$, and also there must exist an $h \geq F(s)$ such that $f_h > 0$. Let $l$ be the least such number. Thus, we can write $\pi$ as

$$\pi = ((s + 1)^{f_{s+1}}, \ldots, (F(s) - 1)^{f_{F(s)-1}}, \ldots, l^1, \ldots).$$

We have some further subcases.

Case 1(b)(i): If $f_{5s+1} \geq 1$ and $f_{10s-1} \geq 1$, then define

$$\phi(\pi) = (s^{15}, (s + 1)^{f_{s+1}}, \ldots, (5s + 1)^{f_{5s+1}-1}, \ldots, (10s - 1)^{f_{10s-1}-1}, \ldots).$$

The injectivity of $\phi$ is clear in this case.

Case 1(b)(ii): If $f_{5s+1} = 0$ or $f_{10s-1} = 0$ and $f_{5s+2} \geq 1$ and $f_{15s-2} \geq 1$, then define

$$\phi(\pi) = (s^{20}, (s + 1)^{f_{s+1}}, \ldots, (5s + 2)^{f_{5s+2}-1}, \ldots, (15s - 2)^{f_{15s-2}-1}, \ldots).$$

The injectivity of $\phi$ is also clear in this case.

Case 1(b)(iii): If $f_{5s+1} = 0$ or $f_{10s-1} = 0$ and $f_{5s+2} = 0$ or $f_{15s-2} = 0$. Then at least one of the following statements is true:

- $T_1$: $f_{5s+1} = 0$ and $f_{5s+2} = 0$;
- $T_2$: $f_{5s+1} = 0$ and $f_{15s-2} = 0$;
- $T_3$: $f_{10s-1} = 0$ and $f_{5s+2} = 0$;
- $T_4$: $f_{10s-1} = 0$ and $f_{15s-2} = 0$.

The indices in each of the statements are intentionally chosen to be coprime with each other. For example, let us show that $5s+1$ and $15s-2$ are coprime. If $g = \gcd(5s+1, 15s-2)$, then $g \mid (5s + 1)$ and $g \mid (15s - 2)$. But then $g \mid 3(5s + 1) - (15s - 2) = 5$. Therefore $g = 1$ or $g = 5$, but $g \neq 5$ since $5 \nmid (5s + 1)$. The other pairs can be shown to be coprime with similar ease.
Since $F(s) − 8s = (10s − 2)(15s − 3)$ and the aforementioned indices in each statement are coprime, by Lemma 7, the following equations have nonnegative integer solutions for all $m ≥ F(s)$:

- $(5s + 1)x_m + (5s + 2)y_m = m + 2s$;
- $(5s + 1)z_m + (15s − 2)w_m = m + 4s$;
- $(10s − 1)u_m + (5s + 2)v_m = m + 6s$;
- $(10s − 1)p_m + (15s − 2)q_m = m + 8s$.

That the lower bound on $m$ is sufficient for all the equations to have nonnegative integer solutions follows from the lower bound being sufficient for the last equation to have such solutions; there the lower bound on $m$ is the one specified by Lemma 7. For each $m ≥ F(s)$, fix some values of $x_m, y_m, z_m, w_m, u_m, v_m, p_m$ and $q_m$ that satisfy the equations, and keep these values fixed throughout the proof. Recall that $l$ was defined to be the least number greater than or equal to $F(s)$ that appears with nonzero frequency in the partition $π$. Then we have the following cases:

- if $T_1$ is true, define
  $$φ(π) = (s^2, (s + 1)^{f_{s+1}}, \ldots, (5s + 1)^{x_l}, (5s + 2)^{y_l}, \ldots, (F(s) − 1)^{f_{F(s)−1}}, \ldots, l^{f_l}, \ldots);$$

- if $T_1$ is false and $T_2$ is true, define
  $$φ(π) = (s^4, (s + 1)^{f_{s+1}}, \ldots, (5s + 1)^{x_l}, (15s − 2)^{w_l}, \ldots, (F(s) − 1)^{f_{F(s)−1}}, \ldots, l^{f_l}, \ldots);$$

- if $T_1$ and $T_2$ are false and $T_3$ is true, define
  $$φ(π) = (s^6, (s + 1)^{f_{s+1}}, \ldots, (5s + 2)^{y_l}, \ldots, (10s − 1)^{u_l}, \ldots, (F(s) − 1)^{f_{F(s)−1}}, \ldots, l^{f_l}, \ldots);$$

- if $T_1, T_2$ and $T_3$ are false and $T_4$ is true, define
  $$φ(π) = (s^8, (s + 1)^{f_{s+1}}, \ldots, (10s − 1)^{p_l}, \ldots, (15s − 2)^{q_l}, \ldots, (F(s) − 1)^{f_{F(s)−1}}, \ldots, l^{f_l}, \ldots).$$

The function $φ$ is injective in Case 1(b). To see why, given a partition $\hat{π} = φ(π)$ whose frequency of $s$ is 2, 4, 6 or 8, we can recognize $π$ as coming from this case. Then if, for example, the frequency of $s$ in $\hat{π}$ is 2, then $T_1$ is true, and the frequencies of $5s + 1$ and $5s + 2$ in $\hat{π}$ give the values of $x_l$ and $y_l$, respectively. Then from the defining equation for $x_l$ and $y_l$ given by

$$(5s + 1)x_l + (5s + 2)y_l = l + 2s,$$
we can recover $l$. From there it is easy to find $π$. A similar argument applies if the frequency of $s$ in $\hat{π}$ is 4, 6 or 8. Thus, in all of Case 1(b), $φ$ is injective.

This completes the description of $φ$ for the case $f = 0$. Note, in aggregate, the function $φ$ is injective in Case 1. If $\hat{π} = φ(π)$, then the frequency of $s$ in $\hat{π}$ indicates from which subcase $π$ comes and, as shown above, $π$ is then recoverable.
Case 2: Suppose $f \geq 1$. Hence, to produce the image of $\pi$ under $\phi$ in this case, we must remove all parts of size $L + s - 1$. Recall that $\pi$ has the form

$$\pi = ((s + 1)^{f+1}, \ldots, (L + s - 1)^{f}, (L + s)^{f_{L+s}}).$$

Since $L \geq s + 3$, we have $(L - s - 1)j \geq 1$ for all $j \geq 1$, and therefore

$$(L + s - 1)j - s(2j - 1) \geq s + 1. \tag{2}$$

Then, by Lemma 8 for all $j \geq 1$, the equation

$$(L + s - 1)j = s(2j - 1) + (s + 1)r_{s+1,j} + (s + 2)r_{s+2,j} + \cdots + (2s + 1)r_{2s+1,j} \tag{3}$$

has nonnegative integer solutions $r_{s+1,j}, r_{s+2,j}, \ldots, r_{2s+1,j}$. For each $j \geq 1$, fix a solution $r_{s+1,j}, r_{s+2,j}, \ldots, r_{2s+1,j}$.

Case 2(a): Suppose $f \neq 8$. Since $L \geq s + 3$, we have $L + s - 1 > 2s + 1$. Define

$$\phi(\pi) = (s^{2f-1}, (s + 1)^{f+1+r_{s+1,j}}f, \ldots, (2s + 1)^{f_{s+1}+r_{s+1,j}}, (2s + 2)^{f_{s+2}}, \ldots, (L + s - 2)^{f_{L+s-2}}, (L + s - 1)^{0}, (L + s)^{f_{L+s}}). \tag{4}$$

The case $f = 8$ is dealt with separately below to ensure injectivity of $\phi$ since then $2f - 1 = 15$, and the frequency of 15 for $s$ in partitions in the image of $\phi$ has already been used in Case 1(b)(i).

Case 2(b): Suppose $f = 8$. It follows from $(2)$ that

$$8(L + s - 1) - 15s \geq s + 1,$$

and thus

$$8(L + s - 1) - 14s \geq s + 1.$$ 

Therefore, by Lemma 8 the equation

$$8(L + s - 1) = 14s + (s + 1)t_{s+1} + (s + 2)t_{s+2} + \cdots + (2s + 1)t_{2s+1} \tag{5}$$

has a nonnegative integer solution. Fix a solution $t_{s+1}, t_{s+2}, \ldots, t_{2s+1}$ of this equation. Thus, for

$$\pi = ((s + 1)^{f+1}, \ldots, (L + s - 1)^{8}, (L + s)^{f_{L+s}}),$$

we define

$$\phi(\pi) = (s^{14}, (s + 1)^{f_{s+1}+t_{s+1}}, \ldots, (2s + 1)^{f_{s+1}+t_{s+1}}, (2s + 2)^{f_{s+2}}, \ldots, (L + s - 2)^{f_{L+s-2}}, (L + s - 1)^{0}, (L + s)^{f_{L+s}}).$$

To see why $\phi$ is injective in Case 2, suppose that $\tilde{\pi} = \phi(\pi)$ and that the frequency of $s$ in $\tilde{\pi}$ is either an odd number not equal to 15, or 14. In the former case, from the frequency of $s$, we can determine $f = f_{L+s-1}$ from $(4)$ for its preimage; then from $(3)$, one can determine the constants $r_{s+1,f}, \ldots, r_{2s+1,f}$. From this, it is clear the partition $\pi$ can be reconstructed, so $\phi$ is injective in Case 2(a). In the latter case when the frequency of $s$ is 14, we can apply a similar argument using $(5)$. We conclude $\phi$ is injective in Case 2.
We refer the reader back to Table 1 to note that the map $\phi$ is injective overall. If $\hat{\pi} = \phi(\pi)$, then the frequency of $s$ in $\hat{\pi}$ gives the case from which $\pi$ came, and in each case itself $\phi$ was shown to be injective.

The injection above shows that $|\{\pi \in C_{L,s} : |\pi| = N\}| \geq |\{\pi \in D_{L,s} : |\pi| = N\}|$, for every $N \geq \kappa(s)$. To complete the proof of Theorem 9, we prove that the inequality is in fact strict. To show this, we find a partition of $N$ that is in $C_{L,s}$ but not in the image of $\phi$. Since $N \geq \kappa(s)$ is large enough, by Lemma 7, there exist nonnegative integers $x_0$ and $y_0$ such that

$$N = 10s + (s + 1)x_0 + (s + 2)y_0.$$ 

Consider the partition $\lambda_N = (s^{10}, (s + 1)^{x_0}, (s + 2)^{y_0})$ of $N$. Since $L \geq 3$, we have $L + s - 1 > s + 2$, so $\lambda_N$ is in $C_{L,s}$. However, the partition $\lambda_N$ is not in the image of $\phi$ since the frequency of $s$ in $\lambda_N$ is 10, and 10 does not occur as a frequency of $s$ in a partition in the image of $\phi$ by Table 1.

2.2 Generalizing Theorem 9 if the impermissible part $k$ is large

A careful analysis of the proof of Theorem 9 in the previous section shows that we have not used the fact that the impermissible part is one less than the largest allowable part. Therefore, the proof can be extended for a general impermissible part $k$ under some restrictions. We presented the proof for $k = L + s - 1$ in the previous section first to keep the case analysis simpler and illustrate our techniques for the later proofs. In the proof below, we explain how the proof of Theorem 9 can be easily generalized.

We modify the definitions of $F(s)$ and $\kappa(s)$. For $s \geq 1$, define the quantities:

- $F'(s) = (21s - 2)(35s - 3) + 8s$;
- $\kappa'(s) = (12s - 1)((s + 1) + (s + 2) + \cdots (F'(s) - 1)) + 1$.

**Theorem 10.** Suppose $L, s$ and $k$ are positive integers such that $L \geq s + 2$ and $2s + 2 \leq k \leq L + s$. Then

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| > |\{\pi \in D_{L,s} : |\pi| = N\}|,$$

for any $N \geq \kappa'(s)$.

**Proof.** The proof of Theorem 10 is the same as the proof of Theorem 9 with $L + s - 1$ being replaced with $k$ everywhere, with some minor modifications. Note that we do not replace $L + s$ with $k + 1$; we only change the impermissible part from $L + s - 1$ to $k$. Since $k \geq 2s + 1$, when $L + s - 1$ is replaced by $k$, the crucial equation (2) becomes

$$kj - s(2j - 1) \geq s + 1$$

and holds for all $j \geq 1$. The condition $k \geq 2s + 2$ is required to ensure that the impermissible part $k$ is different from $2s + 1$, which may have been added as a part in the analogue of (3), given by

$$kj = s(2j - 1) + (s + 1)r_{s+1,j} + (s + 2)r_{s+2,j} + \cdots + (2s + 1)r_{2s+1,j}.$$
Finally, we observe that the proof of Theorem 9 requires modification if the impermissible part \(k\) is one of the numbers \(5s + 1, 5s + 2, 10s - 1\) or \(15s - 2\) because, to produce the image of a partition under \(\phi\), these numbers were added as parts in Case 1(b). If \(k\) is one these numbers, then we can repeat the same proof with the numbers \(5s + 1, 5s + 2, 10s - 1\) and \(15s - 2\) replaced with \(7s + 1, 7s + 2, 21s - 1\) and \(35s - 2\), respectively, and it is this modification that necessitates changing the constants \(F(s)\) and \(\kappa(s)\) to \(F'(s)\) and \(\kappa'(s)\), respectively. Note that for \(s = 1\), the numbers \(10s - 1\) and \(7s + 2\) coincide and are equal to 9. Thus, for \(s = 1\) and \(k = 9\), we choose a set of numbers different from \(5s + 1, 5s + 2, 10s - 1\) and \(15s - 2\) (6, 7, 9 and 13 when \(s = 1\)). The choice of 7, 13, 21 and 29 (instead of 6, 7, 9 and 13) solves the problem for \(s = 1\) and \(k = 9\). These are all the necessary modifications needed to obtain a proof of the theorem.

2.3 An analogue of Theorem 9 if the impermissible part \(k\) is small

Theorem 10 requires the condition \(k \geq 2s + 2\). The next theorem focuses on when \(s + 1 \leq k \leq 2s + 1\). For \(s \geq 1\), we modify \(F(s)\) and \(\kappa(s)\) as follows:

- \(F''(s) = (120(s + 1) - 2)(180(s + 1) - 3) + 420s + 1\);
- \(\kappa''(s) = (300(s + 1) - 1)((s + 1) + (s + 2) + \cdots (F''(s) - 1)) + 1\).

Theorem 11. Suppose \(L, s\) and \(k\) are positive integers such that \(L \geq 3s + 3\) and \(s + 1 \leq k \leq 2s + 1\). Then

\[ |\{\pi \in I_{L,s,k} : |\pi| = N\}| > |\{\pi \in D_{L,s} : |\pi| = N\}|, \]

for any \(N \geq \kappa''(s)\).

Proof. Although the proof of Theorem 11 is similar in style and essence to that of Theorem 9 it requires several more substantial modifications, so we explain them in detail.

We again construct an injective map

\[ \phi : \{\pi \in D_{L,s} : |\pi| = N\} \to \{\pi \in I_{L,s,k} : |\pi| = N\}. \]

To show strict inequality, at the end of the proof we show that there is an element in the codomain of \(\phi\) not in its range.

For \(\pi \in D_{L,s}\), the image of \(\pi\) under \(\phi\) is given in cases depending on the frequency of \(k\) in \(\pi\). For brevity, we set \(f = f_k\), so any \(\pi \in D_{L,s}\) has the form

\[ \pi = ((s + 1)^{f_{s+1}}, \ldots, k^f, \ldots, (L + s)^{f_{L+s}}). \]

Our definition of the image of \(\pi\) under \(\phi\) is given in two cases, when \(f \geq 3\) and when \(f \leq 2\), and each case contains several subcases. Our strategy for ensuring \(\phi\) is injective is, like in Theorem 9 to have the image of partitions under \(\phi\) from different cases have different frequencies of \(s\).

Case 1: Suppose \(f \geq 3\). Then there exists a unique \(j(f) \geq 0\) such that

\[ (60(s + 1) - 3)j(f) + 3 \leq f \leq (60(s + 1) - 3)(j(f) + 1) + 2. \]
We regard \( j(\cdot) \) as a function from \( \{3, 4, \ldots\} \) to the set of nonnegative integers. Because \( k \geq s + 1 \), for any \( i \geq 3 \), we have

\[
ki - s(i + 3j(i) - 2) \geq 2s + 2.
\]

Then, by Lemma 8 (applied with \( s \) replaced by \( 2s + 1 \) there), for all \( i \geq 3 \), the equation

\[
ki = s(i + 3j(i) - 2) + (2s + 2)r_{2s+2,i} + (2s + 3)r_{2s+3,i} + \cdots + (4s + 3)r_{4s+3,i}
\]

has nonnegative integer solutions \( r_{2s+2,i}, r_{2s+3,i}, \ldots, r_{4s+3,i} \). For each \( i \geq 3 \), fix a solution \( r_{2s+2,i}, r_{2s+3,i}, \ldots, r_{4s+3,i} \). Since \( L \geq 3s + 3 \), we have \( L + s \geq 4s + 3 \), so \( 2s + 2, \ldots, 4s + 3 \) are valid parts for partitions in \( D_{L,s} \). Define

\[
\phi(\pi) = (s^{f+3j(f)-2}, \ldots, k^0, \ldots, (2s + 2)^{f_{2s+2}+r_{2s+2,f}}, \ldots, (4s + 3)^{f_{4s+3}+r_{4s+3,f}}, \ldots).
\]

Note that because \( s + 1 \leq k \leq 2s + 1 \), the frequency \( k \) in \( \phi(\pi) \) is genuinely 0, as it is not one of the parts whose frequency has increased.

To see why \( \phi \) is injective in Case 1, note that each member of the set

\[
V = \{i + 3j(i) - 2 : i \geq 3\}
\]

is uniquely determined by its defining value of \( i \); if \( i > i' \), then \( j(i) \geq j(i') \), and so \( i + 3j(i) - 2 > i' + 3j(i') - 2 \). Thus, if \( \hat{\pi} = \phi(\pi) \) and the frequency of \( s \) in \( \hat{\pi} \) is in \( V \), we can reverse the process above. From the frequency of \( s \) in \( \hat{\pi} \), we can recover \( f \); from there we can use \( f, j(f) \) and (6) to determine the constants \( r_{2s+2,f}, r_{2s+3,f}, \ldots, r_{4s+3,f} \). From this point, determining \( \pi \) is straightforward. We note here, for showing that \( \phi \) is injective overall later, that the members of \( V \) are congruent to \( 1, 2, 3, 4, \ldots, -3 \) modulo \( 60(s + 1) \). That is, no member of \( V \) is congruent to 0, -1 or -2 modulo \( 60(s + 1) \).

Case 2: Suppose \( f \leq 2 \). As in Case 1, to obtain the image of a partition under \( \phi \), we must remove the parts of size \( k \) (if any) and insert parts of \( s \). To ensure \( |\phi(\pi)| = |\pi| \), we must alter the frequencies of other parts to compensate. The number of parts equal to \( s \) to be inserted into \( \pi \) is given by the subcases below. We describe all subcases of Case 2, and then discuss why \( \phi \) is injective in Case 2.

Case 2(a): Suppose that there exists an \( m \) such that \( s + 1 \leq m \leq F''(s) - 1 \) and \( f_m \geq 300s(s + 1) \). Let \( m_0 \) be the least such number. Notice that \( m_0 \neq k \) because \( f \leq 2 \). Then define

\[
\phi(\pi) = (s^{300s(s+1)m_0-f}, (s + 1)^{f_{s+1}}, \ldots, k^0, \ldots, (s + k)^{f_{s+k}+f}, \ldots, m_0^{f_{m_0}-300s(s+1)}, \ldots).
\]

Case 2(b): Suppose that the condition of Case 2(a) does not hold. That is, for every \( m \) such that \( s + 1 \leq m \leq F''(s) - 1 \), we have \( f_m \leq 300s(s + 1) - 1 \). Note that if such partitions do not exist, then Case 2(b) does not arise, and there is no need to construct an injection. Since \( N \geq k''(s) \), we must have \( L + s \geq F''(s) \), and also there must exist an \( h \geq F''(s) \) such that \( f_h > 0 \). Let \( l \) be the least such number. Thus, we can write \( \pi \) as

\[
\pi = ((s + 1)^{f_{s+1}}, \ldots, k^f, \ldots, (F''(s) - 1)^{f_{F''(s)-1}}, \ldots, l^l, \ldots).
\]
We have some further subcases. To ease notation, we define the following quantities:

- \( \alpha = 60s(s + 1) + 1 \);
- \( \beta = 60s(s + 1) + 2 \);
- \( \gamma = 120s(s + 1) - 1 \);
- \( \delta = 180s(s + 1) - 2 \).

Note that the quantities \( \alpha, \beta, \gamma \) and \( \delta \) are chosen such that they are distinct from \( k \) and

- \( \alpha, \beta, \gamma, \delta < F''(s) \),
- \( \gcd(\alpha, \beta) = \gcd(\alpha, \delta) = 1 \),
- \( \gcd(\gamma, \beta) = \gcd(\gamma, \delta) = 1 \),
- \( \alpha + \gamma = 180s(s + 1) \),
- \( \beta + \delta = 240s(s + 1) \).

Case 2(b)(i): If \( f_\alpha \geq 1 \) and \( f_\gamma \geq 1 \), then define

\[
\phi(\pi) = (s^{180(s+1)-f}, (s+1)^{f_\alpha}, ..., k^0, ..., (s+k)^{f_\alpha + f}, ..., \alpha^{f_\alpha - 1}, ..., \gamma^{f_\gamma - 1}, \ldots).
\]

Case 2(b)(ii): If \( f_\alpha = 0 \) or \( f_\gamma = 0 \) and \( f_\beta \geq 1 \) and \( f_\delta \geq 1 \), then define

\[
\phi(\pi) = (s^{240(s+1)-f}, (s+1)^{f_\beta}, ..., k^0, ..., (s+k)^{f_\beta + f}, ..., \beta^{f_\beta - 1}, ..., \delta^{f_\delta - 1}, \ldots).
\]

Case 2(b)(iii): If \( f_\alpha = 0 \) or \( f_\gamma = 0 \) and \( f_\beta = 0 \) or \( f_\delta = 0 \), then at least one of the following statements is true:

- \( T_1: f_\alpha = 0 \) and \( f_\beta = 0 \);
- \( T_2: f_\alpha = 0 \) and \( f_\delta = 0 \);
- \( T_3: f_\gamma = 0 \) and \( f_\beta = 0 \);
- \( T_4: f_\gamma = 0 \) and \( f_\delta = 0 \).

Since \( F''(s) - 420s(s + 1) = (\gamma - 1)(\delta - 1) \), and since the relevant numbers are coprime, by Lemma 7 the following equations have nonnegative integer solutions for all \( m \geq F''(s) \):

\[
\begin{align*}
\alpha x_m + \beta y_m &= m - 60s(s + 1); \\
\alpha z_m + \delta w_m &= m - 120s(s + 1); \\
\gamma u_m + \beta v_m &= m - 360s(s + 1); \\
\gamma p_m + \delta q_m &= m - 420s(s + 1).
\end{align*}
\]

That the lower bound on \( m \) is sufficient for all the equations to have nonnegative integer solutions follows from the lower bound being sufficient for the last equation to have such solutions; there the lower bound on \( m \) is the one specified by Lemma 7. For each \( m \geq F''(s) \), fix some values of \( x_m, y_m, z_m, w_m, u_m, v_m, p_m \) and \( q_m \) that satisfy the equations, and keep these values fixed throughout the proof. Recall that \( l \) was defined to be the least number greater than or equal to \( F''(s) \) that appears with nonzero frequency in the partition \( \pi \). Then we have the following cases:
\begin{itemize}
  \item if $T_1$ is true, define
    \[
    \phi(\pi) = (s^{60(s+1)-f}, (s+1)f_{s+1}, \ldots, k^0, \ldots, (s+k)f_{s+k}+f, \ldots, \alpha^{x_1}, \beta^{y_1}, \ldots, (F''(s)-1)^{f''(s)-1}, l^{f_{l-1}}, \ldots);
    \]
  \item if $T_1$ is false and $T_2$ is true, define
    \[
    \phi(\pi) = (s^{120(s+1)-f}, (s+1)f_{s+1}, \ldots, k^0, \ldots, (s+k)f_{s+k}+f, \ldots, \alpha^{z_1}, \delta^{w_1}, \ldots, (F''(s)-1)^{f''(s)-1}, l^{f_{l-1}}, \ldots);
    \]
  \item if $T_1$ and $T_2$ are false and $T_3$ is true, define
    \[
    \phi(\pi) = (s^{360(s+1)-f}, (s+1)f_{s+1}, \ldots, k^0, \ldots, (s+k)f_{s+k}+f, \ldots, \alpha^{\nu_1}, \gamma^{u_1}, \ldots, (F''(s)-1)^{f''(s)-1}, l^{f_{l-1}}, \ldots);
    \]
  \item if $T_1$, $T_2$ and $T_3$ are false and $T_4$ is true, define
    \[
    \phi(\pi) = (s^{420(s+1)-f}, (s+1)f_{s+1}, \ldots, k^0, \ldots, (s+k)f_{s+k}+f, \ldots, \alpha^{p_1}, \delta^{v_1}, \ldots, (F''(s)-1)^{f''(s)-1}, l^{f_{l-1}}, \ldots).
    \]
\end{itemize}

Note in all cases $\phi(\pi)$ has at least one part of size $s$, no parts of size $k$, and all parts are \leq L + s, so $\phi(\pi) \in I_{L,s,k}$.

Define the following sets:
\begin{itemize}
  \item $V_1 = \{60(s+1)i : i \geq 1\}$;
  \item $V_2 = \{60(s+1)i - 1 : i \geq 1\}$;
  \item $V_3 = \{60(s+1)i - 2 : i \geq 1\}$.
\end{itemize}

Note that the frequency of $s$ in a partition in the image of $\phi$ in Case 2 lies in $V_1$, $V_2$ or $V_3$ according to whether $f$ is 0, 1 or 2, respectively. To see why $\phi$ is injective in Case 2, suppose that $\hat{\pi} = \phi(\pi)$ and that the frequency of $s$ in $\hat{\pi}$ lies in one of those three sets. This frequency can be a member of one of two sets:

\[
\{300i(s+1) - h : i \geq 2, h = 0, 1, 2\} \quad \text{or} \quad \{60i(s+1) - h : i = 1, 2, 3, 4, 6, 7, h = 0, 1, 2\}. \quad (8)
\]

The former set of values pertains to Case 2(a), while the latter to Case 2(b). These possible values for the frequency of $s$ in $\hat{\pi}$ distinguish which subcase $\pi$ comes from. Say, for example, the frequency of $s$ in $\hat{\pi}$ is $120(s+1) - h$ for some $h = 0, 1, 2$. From this, we recover $f$ as $h$. We also know from the frequency of $s$ in $\hat{\pi}$ that for $\pi$ the condition $T_1$ above is false, but $T_2$ is true. Hence, the frequencies of $\alpha$ and $\delta$ in $\hat{\pi}$ give the values of $z_l$ and $w_l$, respectively. We can then use the second equation in (7) to find the value of $l$. Once the value of $l$ is known, we can reconstruct $\pi$. When the frequency of $s$ in $\hat{\pi}$ is some other value in the sets given in (8), we can similarly reconstruct $\pi$. 

15
Finally, we note that \( \phi \) is injective overall. As discussed in both Cases 1 and 2 separately, \( \phi \) is injective. However, the sets \( V, V_1, V_2 \) and \( V_3 \) are all pairwise disjoint. Indeed, members of \( V_1, V_2 \) and \( V_3 \) are congruent to 0, -1 and -2 modulo \( 60(s+1) \), respectively, whereas members of \( V \) as noted in Case 1, are not congruent to 0, -1 or -2 modulo \( 60(s+1) \). As these are the possible values of the frequency of \( s \) in a partition in the image of \( \phi \), they distinguish the different cases for preimages under \( \phi \), and we conclude that \( \phi \) is injective.

The injection above shows that 
\[
| \{ \pi \in I_{L,s,k} : |\pi| = N \} | \geq | \{ \pi \in D_{L,s} : |\pi| = N \} | ,
\]
for every \( N \geq \kappa''(s) \). To complete the proof of Theorem 9, we prove that the inequality is in fact strict by finding a partition of \( N \) that is in \( I_{L,s,k} \) but not in the image of \( \phi \). Since \( N \geq \kappa''(s) \) is large enough, by Lemma 7 there exist nonnegative integers \( x_0 \) and \( y_0 \) such that
\[
N = 480s(s+1) + (2s+2)x_0 + (2s+3)y_0.
\]
This gives a partition \( \lambda_N = (s^{480(s+1)}, (2s+2)x_0, (2s+3)y_0) \) of \( N \) that is in \( I_{L,s,k} \) (because \( s+1 \leq k \leq 2s+1 \)), but not in the image of \( \phi \) since the frequency of \( s \) in \( \lambda_N \) is \( 480(s+1) \), a number which is different from the frequencies of \( s \) in partitions in the image of \( \phi \).

3 Proofs of Theorems 3 and 5

In Section 3.1 we prove \( H_{L,s,k}(q) \) is eventually positive for \( s+1 \leq k \leq L+s \) (Theorem 12). The bound \( M \) for which \( N \geq M \) guarantees the coefficient of \( q^N \) in \( H_{L,s,k}(q) \) is positive is initially given as depending on \( L \) and \( s \), so we do not immediately obtain a proof of Theorem 3. However, from this result and results in Section 2, we are able to obtain a quick proof of Theorem 3 in Section 3.2. As noted in Section 1, Conjectures 1 and 2 can be obtained as corollaries of Theorem 3. In Section 3.3 we prove Theorem 5.

3.1 The case \( s+1 \leq k \leq L+s \) for \( H_{L,s,k}(q) \)

For positive integers \( L \geq 3 \) and \( s \), define:

- \( P_{L,s} = (s+1)(s+2) \ldots (s+L) \);
- \( \gamma(L,s) = ((s+1) + (s+2) + \cdots + (s+L)) \cdot \left( \frac{p_{L,s-1}^2}{P_{L,s-1}^2} + \frac{P_{L,s}}{P_{L,s}} \right)^L - (L-2) P_{L,s} \).

The number \( \gamma(L,s) \), which only depends on \( L \) and \( s \), serves as our bound \( M \) in the next theorem.

**Theorem 12.** For positive integers \( L, s \) and \( k \), with \( L \geq 3 \) and \( s+1 \leq k \leq L+s \), the inequality
\[
| \{ \pi \in I_{L,s,k} : |\pi| = N \} | > | \{ \pi \in D_{L,s} : |\pi| = N \} |
\]
holds for every \( N \geq \gamma(L,s) \).
Proof. For \( N \geq \gamma(L, s) \), we construct an injective map
\[
\psi : \{ \pi \in D_{L,s} : |\pi| = N \} \longrightarrow \{ \pi \in I_{L,s,k} : |\pi| = N \}.
\]
To show strict inequality, at the end of the proof we show that there is an element in the codomain of \( \psi \) not in its range.

We shall separate our argument into cases and subcases depending on the frequencies of parts of \( \pi = ((s + 1)^{f_{s+1}}, \ldots, k^{f_k}, \ldots, (L + s)^{f_{L+s}}) \) in the domain. The part whose frequency defines the cases is \( k \); hence, to simplify notation, we set \( f = f_k \), so a partition in the domain has the form
\[
\pi = ((s + 1)^{f_{s+1}}, \ldots, k^{f_k}, \ldots, (L + s)^{f_{L+s}}).
\]

Case 1: Suppose \( f = 0 \). Since \( N \geq \gamma(L, s) \) is large enough, there is an \( m \) such that \( s + 1 \leq m \leq L + s \) and \( f_m \geq s \). Let \( m_0 \) be the least such number. Clearly, the number \( m_0 \) cannot be \( k \), since \( f = 0 \). Then define
\[
\psi(\pi) = (s^{m_0}, (s + 1)^{f_{s+1}}, \ldots, m_0^{f_{m_0} - s}, \ldots).
\]
As \( f_k = f = 0 \), and the frequency of \( s > 0 \) in \( \psi(\pi) \), we see that \( \psi(\pi) \in I_{L,s,k} \).

It is clear that \( \psi \) is injective on the domain in this case, and the frequency of \( s \) in a partition in its image is in the set
\[
U_1 = \{ s + 1, \ldots, L + s \}.
\]

Case 2: Suppose that \( f \neq 0 \) in \( \pi \). We have some subcases. As the partitions of \( I_{L,s,k} \) have no part equal to \( k \) but must have a part of size \( s \), to obtain \( \psi(\pi) \) from \( \pi \) we remove the parts of size \( k \), add parts of size \( s \), and compensate in some way so that \( |\psi(\pi)| = |\pi| \). Choose \( \alpha \) and \( \beta \) as follows:

- if \( k \neq s + 1 \) and \( k \neq s + 2 \), then choose \( \alpha = 1 \) and \( \beta = 2 \);
- if \( k = s + 1 \), choose \( \alpha = 2 \) and \( \beta = 3 \);
- if \( k = s + 2 \), choose \( \alpha = 1 \) and \( \beta = 3 \).

Since \( L \geq 3 \), we have \( s + 1 \leq s + \alpha < s + \beta \leq L + s \). Importantly, the numbers \( \alpha \) and \( \beta \) are chosen so that \( k \neq s + \alpha \) and \( k \neq s + \beta \). Furthermore, the numbers \( s + \alpha \) and \( s + \beta \) are coprime except possibly when \( k = s + 2 \); in that case, if \( s \) is odd, the greatest common divisor of \( s + \alpha \) and \( s + \beta \) is 2. The case \( k = s + 2 \) with \( s \) is odd will require special treatment below because of this.

Case 2(a): Suppose \( \pi \) has \( f \geq P_{L,s}^2 \). For any \( j \geq P_{L,s}^2 \), let \( h(j) \) be the integer satisfying \( P_{L,s}^{h(j)} \leq j < P_{L,s}^{h(j)+1} \), and consider the set
\[
U_a = \{ j + (h(j) - 3)P_{L,s} - 2 : j \geq P_{L,s}^2 \}.
\]
Clearly every \( j \geq P_{L,s}^2 \) gives a unique member of \( U_a \). Conversely, from each member of \( U_a \), its defining value of \( j \) can be recovered; if \( j > j' \), then \( h(j) \geq h'(j) \), and so \( j + (h(j) - 3)P_{L,s} - 2 > j' + (h(j') - 3)P_{L,s} - 2 \).
As \( N \geq \gamma(L, s) \), it is easy to verify, for any \( j \geq P_{L,s}^2 \), that
\[
kj \geq s(j + (h(j) - 3)P_{L,s} - 2) + (s + \alpha - 1)(s + \beta - 1).
\]
Therefore, if \( k \neq s + 2 \), or \( k = s + 2 \) and \( s \) is even, by Lemma 7 since \( s + \alpha \) and \( s + \beta \) are coprime, for any \( j \geq P_{L,s}^2 \), the equation
\[
kj = s(j + (h(j) - 3)P_{L,s} - 2) + (s + \alpha)x_j + (s + \beta)y_j
\]
has nonnegative integer solutions \( x_j \) and \( y_j \).

We noted earlier that when \( k = s + 2 \) and \( s \) is odd then \( s + \alpha = s + 1 \) and \( s + \beta = s + 3 \) have a greatest common factor of 2, so we cannot immediately apply Lemma 7 to obtain \( x_j \) and \( y_j \) in (10). We can however overcome this difficulty as follows. Since \( P_{L,s} \) is even, the integer \( kj - s(j + (h(j) - 3)P_{L,s} - 2) \) is also even for all \( j \) and is greater than or equal to \((s + 1)(s + 3)\). Thus, it follows that
\[
Q = \frac{kj - s(j + (h(j) - 3)P_{L,s} - 2)}{2}
\]
is an integer greater than or equal to the Frobenius number of the coprime integers \( \frac{s + 1}{2} \) and \( \frac{s + 3}{2} \). Thus, by Lemma 7 the quantity \( Q \) can be expressed as a nonnegative integer combination of \( \frac{s + 1}{2} \) and \( \frac{s + 3}{2} \), which can be used to find solutions \( x_j \) and \( y_j \) to (10). So, even in the case when \( k = s + 2 \) and \( s \) is odd, (10) has nonnegative integer solutions.

In any case, for each \( j \geq P_{L,s}^2 \), fix a solution \( x_j \) and \( y_j \) to (10) and keep it fixed throughout the entire proof.

Define
\[
\psi(\pi) = (s^j + (h(j) - 3)P_{L,s} - 2, \ldots, \alpha s^j + x_j, \ldots, (s + \beta) s^j + y_j, \ldots, k^0, \ldots),
\]
where it is understood that the part \( k \) is not precisely placed (it may, for example, be the case that \( k < s + \beta \)).

To see that \( \psi \) is injective in this case, suppose that \( \tilde{\pi} = \psi(\pi) \) and that the frequency of \( s \) in \( \tilde{\pi} \) is in \( U_a \). As the defining values of members of \( U_a \) can be recovered, we can recover \( f \) from the frequency of \( s \). Once \( f \) is found, we can use (10) to find \( x_f \) and \( y_f \). From here, we can easily recover \( \pi \).

We repeat the above strategy for the remaining cases. To that end, define the set
\[
U_b = \{P_{L,s}^h + (h - 4)P_{L,s} : h \geq 3\} \cup \{P_{L,s}^h + (h - 4)P_{L,s} + 1 : h \geq 3\},
\]
and note that the union is disjoint. The reader is invited to confirm that 1) each member of \( U_b \) uniquely determines its defining value of \( h \), and 2) the sets \( U_1, U_a \) and \( U_b \) are pairwise disjoint. The image of \( \pi \) under \( \psi \) in each of the remaining cases has its frequency of \( s \) lie in \( U_b \). Hence it suffices to show that when \( \psi \) is restricted to the domain of the remaining cases, it is injective.

The remaining case is when \( f < P_{L,s}^2 \).

Case 2(b): Suppose that \( 0 < f < P_{L,s}^2 \) in \( \pi \). Since \( N \geq \gamma(s) \) is large enough, there exists an \( h \) such that \( 1 \leq h \leq L \) and
\[
f_{s+h} \geq P_{L,s}^{(f-1)\gamma + (h+2)} + ((f-1)\gamma + (h-2))P_{L,s}.
\]
For brevity of notation, for any \(0 < i < P_{L,s}^2\) and \(1 \leq h \leq L\), set
\[
m_{i,h} = P_{L,s}^{(i-1)L+(h+2)} + ((i-1)L + (h-2))P_{L,s}.
\]
We make a few key observations about the numbers \(m_{i,h}\), the first of which is that \(m_{i,h} \in U_b\) for any valid \(i\) and \(h\). Second, each number \(m_{i,h}\) is determined uniquely by its defining value of \((i, h)\). To see why, if \(m_{i,h} = m_{i',h'}\), then, as the exponent in the first term of \(m_{i,h}\) is at least three, we have
\[
(i - 1)L + h + 2 = (i' - 1)L + h' + 2,
\]
and thus \(h - h' = (i' - i)L\). But since \(1 \leq h, h' \leq L\), this implies that \(h = h'\), and so \(i' = i\).

Let \(p\) be the least integer \(1 \leq h \leq L\) for which (11) is satisfied. Notice that the restriction on \(f\) prevents \(s + p\) from being \(k\). By definition,
\[
f_{s+p} \geq m_{f,p}.
\]
Notice that \(m_{f,p}\) is divisible by \(P_{L,s}\), and thus also by \((s + p)\); hence, we can define \(j_{f,p}\) by
\[
(s + p)j_{f,p} = sm_{f,p}.
\]
From (12) and (13), it is easy to verify that
\[
f_{s+p} \geq j_{f,p} + 2s.
\]
For any integer \(u\), we define \(\eta_u\) to be 1 if \(u\) is odd and 0 otherwise. One can easily verify that for any \(1 \leq t \leq P_{L,s}^2\) and any \(1 \leq i \leq L\), we have
\[
2s(s + i) + tk - s(\delta_{k,s+2})\eta_s\eta_t \geq (s + \alpha - 1)(s + \beta - 1),
\]
(14)
We explain the peculiar term \(s(\delta_{k,s+2})\eta_s\eta_t\). As before, when \(k \neq s + 2\), or \(k = s + 2\) and \(s\) is even, the numbers \(s + \alpha\) and \(s + \beta\) are coprime, so by Lemma 7 there exist nonnegative integer solutions \(z_{t,i}\) and \(w_{t,i}\) such that
\[
2s(s + i) + tk = s(\delta_{k,s+2})\eta_s\eta_t \geq (s + \alpha)z_{t,i} + (s + \beta)w_{t,i}.
\]
(15)
When \(k = s + 2\) and \(s\) is odd, recall that \(s + \alpha = s + 1\) and \(s + \beta = s + 3\) are not coprime, but we can remedy this issue as before. The term \(s(\delta_{k,s+2})\eta_s\eta_t\) guarantees that the left hand side of (14) is even. We then apply the same fix as earlier: we divide the left hand side of (14) by 2, and the result is greater than the Frobenius number of the coprime integers \(s + 1\) and \(s + 3\). Then applying Lemma 7 gives us nonnegative integer solutions to (15) in this case as well.

In any case, for each \(1 \leq t \leq P_{L,s}^2\) and \(1 \leq i \leq L\), fix a solution \(z_{t,i}\) and \(w_{t,i}\) to (15).

Let \(n_{f,p}\) be defined as \(m_{f,p} + (\delta_{k,s+2})\eta_s\eta_t\). Notice that \(n_{f,p} \in U_b\). To obtain \(\psi(\pi)\) from \(\pi\), we add to \(\pi\) a part \(s\) with frequency \(n_{f,p}\), reduce the frequency of \(s + p\) by \(j_{f,p} + 2s\), remove the \(f\) parts of \(k\), and add \((s + \alpha)\) and \((s + \beta)\) with frequencies of \(z_{f,p}\) and \(w_{f,p}\), respectively. Thus, if \(p \neq \alpha\) or \(p \neq \beta\), we define
\[
\psi(\pi) = \left(s^{n_{f,p}}, \ldots, (s + \alpha)^{j_{f,p} + z_{f,p}}, \ldots, (s + \beta)^{j_{f,p} + w_{f,p}} \ldots (s + p)^{j_{f,p} - 2s}, \ldots, k^0, \ldots\right).
\]
If \( p = \alpha \), we define
\[
\psi(\pi) = (s^{n_{f,\alpha}} , \ldots , (s + \alpha)^{f + z_{f,\alpha} - j_{f,\alpha} - 2s}, \ldots , (s + \beta)^{f + \beta + w_{f,\alpha}}, \ldots k^0, \ldots).
\]
If \( p = \beta \), we define
\[
\psi(\pi) = (s^{n_{f,\beta}} , \ldots , (s + \alpha)^{f + z_{f,\beta}}, \ldots , (s + \beta)^{f + \beta + w_{f,\beta} - j_{f,\beta} - 2s}, \ldots k^0, \ldots).
\]

We first note that it follows from (13) and (15) that \( |\pi| = |\psi(\pi)| \). As the frequency of \( s \) in \( \psi(\pi) \) is \( n_{f,p} \), it lies in \( U_b \), as noted earlier. To see why \( \psi \) is injective in this case, suppose that \( \hat{\pi} = \psi(\pi) \) and that the frequency of \( s \) in \( \hat{\pi} \) has the form \( n_{f,p} \). Recall that both \( f \) and \( p \) can be recovered from \( m_{f,p} \). However, from \( n_{f,p} \) it is clear that we can recover \( m_{f,p} \); if \( n_{f,p} \) is divisible by \( P_{L,s} \), then \( m_{f,p} = n_{f,p} \), and otherwise \( m_{f,p} = n_{f,p} - 1 \). Thus \( f \) and \( p \) are recoverable from \( n_{f,p} \) as well. From there, using (13), we can determine \( j_{f,p} \). Furthermore, from \( f \) and \( p \), we can use (15) to determine \( z_{f,p} \) and \( w_{f,p} \). From this point, we can easily recover \( \pi \).

We summarize the possible frequencies of \( s \) in a partition in the image of \( \psi \) and the case to which it pertains in Table 2. As discussed earlier, as \( U_1, U_a \) and \( U_b \) are pairwise disjoint, the frequency of \( s \) in a partition in the image of \( \psi \) characterizes the case from which its preimage comes. The injectivity of \( \psi \) over all cases follows from \( \psi \) being injective in each case, which has been demonstrated.

The injection above shows that
\[
|\{\pi \in I_{L,s,k} : |\pi| = N\}| \geq |\{\pi \in D_{L,s} : |\pi| = N\}|
\]
for every \( N \geq \gamma(L,s) \). To complete the proof of Theorem 12 we must prove that the inequality is in fact strict, so we find a partition of \( N \) that is in \( I_{L,s,k} \) but not in the image of \( \psi \). Since \( N \geq \gamma(L,s) \) is large enough, by Lemma 7 there exist nonnegative integers \( x_0 \) and \( y_0 \) such that
\[
N = s(L + s + 1 + (\delta_{k,s+2}) \eta_{s\eta_{N-L}}) + (s + \alpha)x_0 + (s + \beta)y_0.
\]
This gives a partition
\[
\lambda_N = (s^{L+s+1+(\delta_{k,s+2})\eta_{s\eta_{N-L}}}, (s + \alpha)^{x_0}, (s + \beta)^{y_0})
\]
of \( N \). Again, the term \((\delta_{k,s+2})\eta_{s\eta_{N-L}}\) is to ensure that (16) has a solution even in the case \( k = s+2 \) and \( s \) is odd, as before. It is easy to see \( \lambda_N \) has the desired properties. Its frequency of \( s \) is either \( L+s+1 \) or \( L+s+2 \) and its frequency of \( k \) is 0, so it is in \( I_{L,s,k} \). Furthermore, it is easy to see its frequency of \( s \) is not a member of \( U_1, U_a \) or \( U_b \), so \( \lambda_N \) is not in the range of \( \psi \).

\( \square \)
3.2 Proofs of Theorem 3 and Conjectures 1 and 2

As noted earlier, while the bound in Theorem 12 depends on both $L$ and $s$, we can use that theorem in tandem with Theorems 10 and 11 to give a proof of Theorem 3. For that, define

$$\Gamma(s) = \gamma(3s + 2, s),$$

(17)

where $\gamma(\cdot, \cdot)$ is defined as in $\[3\]$.

Proof of Theorem 3. We show that if $N \geq \Gamma(s)$, the inequality

$$|\{\pi \in I_{L,s,k} : |\pi| = N\}| > |\{\pi \in D_{L,s} : |\pi| = N\}|$$

holds.

If $L \geq 3s + 3$ and $2s + 2 \leq k \leq L + s$, then by Theorem 10 the inequality holds for all $N \geq \kappa'(s)$.

However, if $L \geq 3s + 3$ and $s + 1 \leq k \leq 2s + 1$, then by Theorem 11 the inequality holds for all $N \geq \kappa''(s)$. Thus, combining these two results, we see that for $L \geq 3s + 3$ and $s + 1 \leq k \leq L + s$, if $N \geq \max(\kappa'(s), \kappa''(s))$ then the inequality holds.

Finally, if $3 \leq L \leq 3s + 2$ and $s + 1 \leq k \leq L + s$, then by Theorem 12 the inequality holds for all $N \geq \gamma(3s + 2, s) = \Gamma(s)$. It follows that for all $L \geq 3$ and $s + 1 \leq k \leq L + s$, the inequality holds for all $N$ larger than the three constants $\kappa'(s), \kappa''(s)$ and $\Gamma(s)$. A simple comparison reveals that $\Gamma(s)$ is the largest of the three constants, and we give a short proof of this. First, an easy calculation shows that $\kappa'(s) < 10^5(s+1)^5$ and $\kappa''(s) < 10^{11}(s+1)^{10}$. Next, we show that $\Gamma(s)$ is always larger than these numbers. Observe that $P_{3s+2,s} > (s+1)^{3s+2} \geq 32$ and thus $(P^2_{3s+2,s} - 1) > 1000$. Using this, we find

$$\Gamma(s) = \gamma(3s + 2, s) > P^{P_{3s+2,s} - 1}_{3s+2,s}$$

$$> P^{1000}_{3s+2,s}$$

$$> (s + 1)^{1000(3s+2)}$$

$$\geq (s + 1)^{5000}$$

$$\geq 2^{4990}(s + 1)^{10}$$

$$> 10^{11}(s + 1)^{10}$$

$$> \max(\kappa'(s), \kappa''(s)),$$

completing the proof.

Remark. If $s + 2 \leq L \leq 3s + 2$ and $k \geq 2s + 2$, then we can use the bound $\kappa'(s)$ in Theorem 10 instead of the larger bound $\gamma(3s + 2, s)$ suggested by the proof of Theorem 3.

As remarked in Section 1, setting $k = L + s - 1$ and $k = L$ in Theorem 3 proves Conjectures 1 and 2 respectively. We state these separately, however, in the next corollaries so that the bound $M$ for when the partition inequalities hold is explicit.

Corollary 13. If $L \geq 3$ and $s$ are positive integers, then

$$|\{\pi \in C_{L,s} : |\pi| = N\}| \geq |\{\pi \in D_{L,s} : |\pi| = N\}|$$

(Conjecture 1)

and

$$|\{\pi \in C^*_{L,s} : |\pi| = N\}| \geq |\{\pi \in D_{L,s} : |\pi| = N\}|$$

(Conjecture 2)

for all $N \geq \Gamma(s)$, where $\Gamma(s)$ is defined in (17).
3.3 The proof of Theorem 5

In the previous section, we proved Theorem 3 and, as remarked in Section 1, this proves Theorem 5 in the cases \( s + 1 \leq k \leq L + s \). We will, however, be able to use Theorem 3 to prove Theorem 5 in general. We first prove some lemmas.

**Lemma 14.** For positive integers \( L \geq 3 \), \( s \) and \( k \geq s + 1 \), the difference \( H_{L,s,k}(q) - H_{L,s,k}(q) \) is nonnegative.

**Proof.** The lemma is stated in the simplest form that we need. We shall, however, prove a stronger statement: for \( L, s \) and \( k \) as in the lemma, the difference \( H_{L,s,k+i}(q) - H_{L,s,k}(q) \) is nonnegative for all \( s \leq i \leq L + s \).

We have

\[
H_{L,s,k+i}(q) - H_{L,s,k}(q) = \frac{q^{s+k}(1 - q^i)}{(1 - q^s)(1 - q^{s+1}) \cdots (1 - q^{L+s})}.
\]

For \( s \leq i \leq L + s \), the factor \((1 - q^i)\) in the numerator is also present in the denominator, so it cancels. Hence the difference \( H_{L,s,k+i}(q) - H_{L,s,k}(q) \) is nonnegative.

From Theorem 3 and Lemma 14, it follows that \( H_{L,s,k}(q) \) is eventually positive for all \( k \geq s + 1 \) whenever \( L \geq s \). However, we are still left with various cases when \( L < s \). For example, if \( s = 10 \) and \( L = 3 \), then Theorem 3 shows that \( H_{L,s,k}(q) \) is eventually positive whenever \( k \) is 11, 12 or 13. Then, according to Lemma 14, the series \( H_{L,s,k}(q) \) is eventually positive whenever \( k \) is 21, 22 or 23, which leaves the gap \( 14 \leq k \leq 20 \). To complete the proof of Theorem 5, a close analysis of the cases covered by a combination of Theorem 3 and Lemma 14 gives that, when \( L < s \), it suffices to prove that \( H_{L,s,k}(q) \) is eventually positive whenever \( L + s < k \leq 2s \). We do so in Lemma 16, but before that we need another lemma.

**Lemma 15.** For positive integers \( L \geq 3 \) and \( s \), the coefficient of \( q^N \) in the series

\[
\frac{q^s - q^{L+s-1}}{(q^s; q)_{L+1}}
\]

is positive whenever \( N \geq \gamma(s,s) \), where \( \gamma(\cdot, \cdot) \) is defined in (9).

**Proof.** Given a natural number \( N \),

- \( A_N \) is the set of partitions of \( N \) with parts in \( \{ s, \ldots, L + s \} \), and \( s \) appears as a part at least once;
- \( B_N \) is the set of partitions of \( N \) with parts in \( \{ s, \ldots, L + s \} \), and \( L + s - 1 \) appears as a part at least once.

Proving the lemma is equivalent to showing that \( |A_N| \geq |B_N| \) for all \( N \geq \gamma(s,s) \). When \( L \geq s + 1 \), this is easy to show for all \( N \geq 1 \); if \( B_N \) is nonempty and \( \pi \in B_N \), remove a part of size \( L + s - 1 \) from \( \pi \) and insert parts \( s \) and \( t \), where \( t = L - 1 \geq s \). This process clearly defines an injective function from \( B_N \) to \( A_N \). We must therefore deal with the case when \( L \leq s \). For that, given a natural number \( N \),
\begin{itemize}
\item $C_N$ is the set of partitions of $N$ where the smallest part is $s$, all parts are $\leq L + s$, and $L + s - 1$ does not appear as a part;
\item $D_N$ is the set of partitions of $N$ with parts in the set $\{s+1, \ldots, L+s\}$;
\item $E_N$ is the set of partitions of $N$ with parts in the set $\{s, \ldots, L+s\}$, and $L+s-1$ does not appear as a part;
\item $F_N$ is the set of partitions of $N$ with parts in the set $\{s, \ldots, L+s\}$.
\end{itemize}

Notice that $C_N = \{ \pi \in C_{L,s} : |\pi| = N \}$ and $D_N = \{ \pi \in D_{L,s} : |\pi| = N \}$. We could use Corollary 13 here, but since $L \leq s$, we can use Theorem 12 to obtain a stronger bound by setting $k = L + s - 1$ there; the inequality $|C_N| > |D_N|$ then holds for all $N \geq \gamma(s,s)$. Since $C_N \subset E_N$, we also have $|E_N| > |D_N|$ for all $N \geq \gamma(s,s)$. Notice that $B_N = F_N \setminus E_N$ and $A_N = F_N \setminus D_N$. Hence $|A_N| > |B_N|$ for all $N \geq \gamma(s,s)$.

As noted above, the following result completes the proof of Theorem 5.

**Lemma 16.** For positive integers $L$, $s$ and $k$ such that $3 \leq L < s$ and $L + s \leq k \leq 2s$, the coefficient of $q^N$ in $H_{L,s,k}(q)$ is positive whenever $N \geq \Gamma(s)$.

**Proof.** Our proof is by strong induction. The base case ($k = L + s$) has already been proven in Theorem 3. Next, assume that for some $i$ such that $L + s \leq i < 2s$, the coefficient of $q^N$ in $H_{L,s,j}(q)$ is positive whenever $N \geq \Gamma(s)$ for all $L + s \leq j \leq i$. We shall prove that the coefficient of $q^N$ in $H_{L,s,i+1}(q)$ is also positive whenever $N \geq \Gamma(s)$. Consider the difference

$$H_{L,s,i+1}(q) - H_{L,s,i-L+2}(q) = \frac{q^i(q^{i-L+2} - q^{i+1})}{(q^s;q)_{L+1}} = \frac{q^{i-L+2}(q^s - q^{L+s-1})}{(q^s;q)_{L+1}}.$$ 

Thus, from Lemma 13 the coefficient of $q^N$ in the difference $H_{L,s,i+1}(q) - H_{L,s,i-L+2}(q)$ is positive whenever $N \geq \gamma(s,s) + 2s$ (because $i < 2s$). It is easy to verify, for any positive integer $s$, that $\Gamma(s) \geq \gamma(s,s) + 2s$. Thus, the coefficient of $q^N$ in $H_{L,s,i+1}(q) - H_{L,s,i-L+2}(q)$ is positive whenever $N \geq \Gamma(s)$.

The induction hypothesis states that the coefficient of $q^N$ in $H_{L,s,j}(q)$ is positive whenever $N \geq \Gamma(s)$ for all $L + s \leq j \leq i$. Combining this with Theorem 3, the coefficient of $q^N$ in $H_{L,s,j}(q)$ is positive whenever $N \geq \Gamma(s)$ for all $s + 1 \leq j \leq i$. Since $L \geq 3$ and $i \geq L + s$, we have $s + 1 \leq i - L + 2 \leq i$, and thus the coefficient of $q^N$ in $H_{L,s,i-L+2}(q)$ is positive whenever $N \geq \Gamma(s)$.

Thus, we have shown whenever $N \geq \Gamma(s)$ that the coefficient of $q^N$ in both the series $H_{L,s,i+1}(q) - H_{L,s,i-L+2}(q)$ and $H_{L,s,i-L+2}(q)$ is positive. This shows that the coefficient of $q^N$ in $H_{L,s,i+1}(q)$ is positive whenever $N \geq \Gamma(s)$, completing the induction argument.

We collect all of these results together to complete the proof of Theorem 5, which, as noted in Section 1, generalizes Conjecture 4 and Theorem 3.
Proof of Theorem 5. Suppose $N \geq \Gamma(s)$. If $L \geq s$, then Theorem 3 and Lemma 14 immediately prove Theorem 5.

If $L < s$ and $s + 1 \leq k \leq L + s$, then Theorem 12 completes the proof of Theorem 5. If $L < s$ and $L + s < k \leq 2s$, then Lemma 16 completes the proof of Theorem 5. Thus, combining these two cases, the theorem holds for $L < s$ and $s + 1 \leq k \leq 2s$. Since the result holds for $L < s$ and $s + 1 \leq k \leq 2s$, an application of Lemma 14 covers the cases $L < s$ and $k > 2s$. This covers all cases and completes the proof. 

4 Proof of Conjecture 6

In this section, we prove Conjecture 6, which pertains to the series $G_{L,2}(q)$. In [BU19], Berkovich and Uncu found an alternative expression for $G_{L,2}(q)$.

Theorem 17 (Berkovich and Uncu (2019)). For $L \geq 3$,

$$G_{L,2}(q) = \frac{H_{L,2,L}(q)}{1 - q^L}.$$ 

From Theorem 5, we know that $H_{L,2,L}(q)$ is eventually positive, as $H_{L,2,L}(q)$ is the particular case of $s = 2$ and $k = L$ in that theorem, and from there it can be shown that $G_{L,2}(q)$ is also eventually positive. However, to prove Conjecture 6 we need to prove that the required series is not merely eventually positive, but that all its coefficients, with the exception of a few small terms, are nonnegative. We therefore need a method for the particular case $s = 2$ in the series $H_{L,s,L}(q)$ that analyzes the coefficients of $q^n$ for small $n$. As for the previous conjectures, our methods highly depend on Lemma 7 and Lemma 8.

Recall the following notation from Section 1, which we require throughout this section. For a positive integer $L \geq 3$,

- $C^*_{L,2} = I_{L,2,L}$ denotes the set of partitions such that the smallest part is 2, all parts are $\leq L + 2$, and $L$ is not a part;
- $D_{L,2}$ denotes the set of nonempty partitions with parts in the set $\{3, 4, \ldots, L + 2\}$.

From (1), for $N \geq 0$, the coefficient of $q^N$ in $H_{L,2,L}(q)$ is

$$\left|\{\pi \in C^*_{L,2} : |\pi| = N\}\right| - \left|\{\pi \in D_{L,2} : |\pi| = N\}\right|.$$ 

We use this combinatorial interpretation along with Theorem 17 to obtain information about $G_{L,2}(q)$. We prove Conjecture 6 by first proving it for large $L$ (i.e., $L \geq 11$), and then we prove it for smaller values (i.e., for $5 \leq L \leq 10$). The cases $L = 3$ and $L = 4$ will be done separately afterwards.

Theorem 18. For $L \geq 11$,

$$G_{L,2}(q) + q^3 \succeq 0.$$
Proof. For $N > 3$, we construct an injective map
\[
\phi : \{\pi \in D_{L,2} : |\pi| = N\} \to \{\pi \in C^*_L : |\pi| = N\}.
\]

For a partition $\pi$ in the domain, we let $f$ be the frequency of $L$, so $\pi$ has the form $(3^{f_3}, \ldots, L^f, \ldots, (L + 2)^{f_{L+2}})$. Our definition of the image of $\pi$ under $\phi$ is given in two cases, when $f > 0$ and $f = 0$, with the latter case containing several subcases. We describe all the cases first, and show that $\phi$ is injective later. For the reader wishing to look ahead, Table 3 contains a summary of the cases.

Case 1: Suppose $f > 0$. Since $L \geq 11$, for any $i \geq 1$, we have $(L - 8)i \geq 3$, and thus by Lemma 8 (applied with $s = 2$ there), there are nonnegative integers $x_i, y_i$ and $z_i$ such that
\[
Li = 8i + 3x_i + 4y_i + 5z_i.
\]

For each $i \geq 1$, fix some values of $x_i, y_i$ and $z_i$ and keep them fixed throughout the proof. Define
\[
\phi(\pi) = (2^{4f}, 3^{x_f + f_3}, 4^{y_f + f_4}, 5^{z_f + f_5}, 6^{f_6}, \ldots, L^0, \ldots, (L + 2)^{f_{L+2}}).
\]

Case 2: When $f = 0$, to obtain $\phi(\pi)$ from $\pi$, there are no parts of $L$ to remove. To obtain $\phi(\pi)$, we must insert parts of size 2 into $\pi$ and compensate in some way. For this, we must consider several subcases. We denote the smallest part of $\pi$ by $s(\pi)$.

Case 2(A): When $s(\pi) \geq 5$, we define
\[
\phi(\pi) = (2^1, (s(\pi) - 2)^1, (s(\pi))^{f_s(\pi) - 1}, \ldots).
\]

Case 2(B): Suppose $s(\pi) \leq 4$, so $s(\pi)$ is either 3 or 4.

Case 2(B)(i): If $f_4 \geq 1$, we define
\[
\phi(\pi) = (2^{2f_4}, 3^{f_3}, 4^{f_4 - 1}, \ldots).
\]

Case 2(B)(ii): Suppose $f_4 = 0$, so $s(\pi) = 3$. We have further subcases.

Case 2(B)(ii)(a): If $f_3 \geq 2$, we define
\[
\phi(\pi) = (2^3, 3^{f_3 - 2}, 5^{f_5}, \ldots).
\]

Case 2(B)(ii)(b): Suppose $f_3 = 1$. Then $\pi = (3, 5^{f_5}, \ldots)$. Since $N > 3$, there exists an $m \geq 5$ such that $f_m \geq 1$. Let $m_0$ be the least such number. We have further subcases depending on whether $m_0$ is odd or even.

Case 2(B)(ii)(b)(a): If $m_0$ is odd, we define
\[
\phi(\pi) = \left(2^1, 3^{0}, \ldots, \left(\frac{m_0 + 1}{2}\right)^2, \ldots, m_0^{f_{m_0} - 1}, \ldots\right) \quad \text{if } m_0 > 5,
\]
\[
\text{and } \phi(\pi) = (2^1, 3^2, 4^0, 5^{f_5 - 1}, \ldots) \quad \text{if } m_0 = 5.
\]

In both cases, the part 3 and a part $m_0$ were removed from $\pi$, and a part 2 and two parts $\frac{m_0 + 1}{2}$ were inserted into $\pi$ to obtain $\phi(\pi)$. This ensures $|\phi(\pi)| = |\pi|$ regardless of the value of $m_0$. 


Case 2(B)(ii)(b)(β): If \( m_0 \) is even, we define
\[
\phi(\pi) = \left(2^1, 3^0, \ldots, \left(\frac{m_0}{2}\right)^1, \left(\frac{m_0}{2} + 1\right)^1, \ldots, m_0^{m_0-1}, \ldots \right)
\]
if \( m_0 > 6 \),
and
\[
\phi(\pi) = \left(2^1, 3, 4, 5^0, \ldots \right)
\]
if \( m_0 = 6 \).

In both cases, the part 3 and a part \( m_0 \) were removed from \( \pi \), and a part \( \frac{m_0}{2} \) and a part \( \frac{m_0}{2} + 1 \) were inserted into \( \pi \) to obtain \( \phi(\pi) \). This ensures \(|\phi(\pi)| = |\pi|\) regardless of the value of \( m_0 \).

| Case | Description of case | \( f_2 \) in \( \phi(\pi) \) | Next parts |
|------|---------------------|------------------|------------|
| 1    | \( f > 0 \)         |                  | mult. of 4* |
| 2(A) | \( f = 0 \) \( s(\pi) \geq 5 \) | 1                | \( s(\pi) - 2, s(\pi)^\dagger \) |
| 2(B)(i) | \( f = 0 \) \( s(\pi) \leq 4 \) \( f_4 \geq 1 \) | 2                | *          |
| 2(B)(ii)(a) | \( f = 0 \) \( s(\pi) \leq 4 \) \( f_4 = 0 \) \( f_3 \geq 2 \) | 3                | *          |
| 2(B)(ii)(b)(α) | \( f = 0 \) \( s(\pi) \leq 4 \) \( f_4 = 0 \) \( f_3 = 1 \) \( m_0 \) odd | 1                | \( \frac{m_0 + 1}{2} \)² |
| 2(B)(ii)(b)(β) | \( f = 0 \) \( s(\pi) \leq 4 \) \( f_4 = 0 \) \( f_3 = 1 \) \( m_0 \) even | 1                | \( \frac{m_0}{2}, \frac{m_0}{2} + 1 \) |

\( \dagger \) \( s(\pi) \) may appear with frequency 0.

Table 3: The frequency of 2 in the image of a partition under the function \( \phi \) in the different cases for Theorem\[18\]. The quantity \( m_0 \) is defined in Case 2(B)(ii)(b)(\( β \)). The column “Next parts” indicate the second and third smallest parts, which are equal in the Case 2(B)(ii)(b)(\( β \)).

The map \( \phi \) is easily seen to be injective in each case separately. To see it is injective overall, observe that images under \( \phi \) in most of the cases are separated by the frequency of 2; the only cases in which images have the same frequency of 2 are Cases 2(A), 2(B)(ii)(b)(\( α \)) and 2(B)(ii)(b)(\( β \)). The Case 2(B)(ii)(b)(\( α \)) is separated from the other two cases by the frequency, 2, of the second smallest part. Finally, the Cases 2(A) and 2(B)(ii)(b)(\( β \)) are separated by the difference between the second smallest and the third smallest parts; this is 1 for Case 2(B)(ii)(b)(\( β \)) and at least 2 for Case 2(A). These differences are listed in Table 3.

Thus we have shown that for \( N > 3 \),
\[
|\{\pi \in \mathcal{C}_{L,2}^*: |\pi| = N\}| - |\{\pi \in \mathcal{D}_{L,2}: |\pi| = N\}| \geq 0.
\]

However, if \( N = 3 \), this is not true; it is easy to see that \(|\{\pi \in \mathcal{D}_{L,2}: |\pi| = 3\}| = 1\) and \(|\{\pi \in \mathcal{C}_{L,2}^*: |\pi| = 3\}| = 0\). Furthermore, if \( N = 1 \) or 2, we easily find that \(|\{\pi \in \mathcal{C}_{L,2}^*: |\pi| = N\}| \geq |\{\pi \in \mathcal{D}_{L,2}: |\pi| = N\}|\).

Let \( H_{L,2,L}(q) = \sum_{n \geq 0} a_n q^n \). Then the above combinatorial results imply that \( a_n \geq 0 \) for all \( n \neq 3 \) and \( a_n = -1 \) for \( n = 3 \), whence \( H_{L,2,L}(q) + q^3 \geq 0 \). We can in fact make a stronger claim about the coefficients \( a_n \) when \( n \geq 14 \); we have \( a_n \geq 1 \) for all \( n \geq 14 \). To see this, we find a partition of \( n \) in \( \mathcal{C}_{L,2}^* \) not in the image of \( \phi \). Recall \( L \geq 11 \). For \( n = 14 \), the partition \( \pi_{14} = (2^1, 3^1) \) is in \( \mathcal{C}_{L,2}^* \) but not in the image of \( \phi \). For \( n \geq 15 \), we have \( n - 11 \geq 4 \), and thus, by Lemma\[17\] (with \( s = 3 \) there), there are nonnegative integers \( x_n, y_n, z_n \) and \( u_n \) such that
\[
n = 11 + 4x_n + 5y_n + 6z_n + 7u_n.
\]
For each \( n \geq 15 \), fix some choice of \( x_n, y_n, z_n \) and \( u_n \). Thus \( \pi_n = (2^1, 3^3, 4^{x_n}, 5^{y_n}, 6^{z_n}, 7^{u_n}) \) is a partition of \( n \) in \( C_{L,2}^* \) not in the image of \( \phi \).

Let \( G_{L,2}(q) = \sum_{n \geq 0} b_n q^n \). To prove the theorem, we are required to show \( b_n \geq 0 \) whenever \( n \neq 3 \), and \( b_3 \geq -1 \). By Theorem 17, for any \( n \geq 0 \),

\[
b_n = a_n + a_{n-L} + a_{n-2L} + \cdots.
\]

Using the division algorithm, we have \( n = Lq + r \) where \( 0 \leq r < L \), and thus we can rewrite \( b_n \) as

\[
b_n = \sum_{i=0}^{q} a_{Li+r}.
\]  

(18)

If \( r \neq 3 \), then none of the terms on the right hand side of (18) are negative and thus \( b_n \geq 0 \) as required. If \( r = 3 \) and \( q \geq 1 \), then from (18), we have \( b_n \geq a_{L+3} + a_3 \). Since \( L \geq 11 \), we have \( L + 3 \geq 14 \), and thus \( a_{L+3} \geq 1 \), which implies \( b_n \geq 0 \). Finally, if \( r = 3 \) and \( q = 0 \), then \( b_n = b_3 = a_3 = -1 \).

Next, we prove Conjecture 6 for \( 5 \leq L \leq 10 \).

**Theorem 19.** For \( 5 \leq L \leq 10 \),

\[
G_{L,2}(q) + q^3 \succeq 0.
\]

**Proof.** Let \( N_L = \frac{L(L+3)}{2} + 2 \); we give the values of \( N_L \) in Table 4. For \( 5 \leq L \leq 10 \) and \( N \geq N_L \), we construct an injective map

\[
\psi : \{ \pi \in D_{L,2} : |\pi| = N \} \rightarrow \{ \pi \in C_{L,2}^* : |\pi| = N \}.
\]

| \( L \) | 5 | 6 | 7 | 8 | 9 | 10 |
|-----|---|---|---|---|---|----|
| \( N_L \) | 22 | 29 | 37 | 46 | 56 | 67 |

Table 4: The table of values of \( N_L \) versus \( L \) for \( 5 \leq L \leq 10 \).

We again let \( f \) be the frequency of \( L \) in \( \pi \in D_{L,2} \); so, \( \pi = (3^{f_3}, 4^{f_4}, \ldots, L^f, \ldots, (L+2)^{f_{L+2}}) \).

To define the image of \( \pi \) under the map \( \psi \), we consider several cases depending on \( f \). We describe \( \psi \) first and explain why it is injective later.

**Case 1:** Suppose that \( f \) is a positive even number. Then define

\[
\psi(\pi) = \left( 2^{\frac{fL}{2}}, 3^{f_3}, \ldots, L^0, \ldots, (L+2)^{f_{L+2}} \right).
\]

**Case 2:** Suppose that \( f \) is a positive odd number. Then define

\[
\psi(\pi) = \left( 2^L \left( \frac{f+1}{2} \right)^{+1}, 3^{f_3}, \ldots, (L - 2)^{f_{L-2}+1}, \ldots, L^0, \ldots, (L+2)^{f_{L+2}} \right).
\]

**Case 3:** Suppose \( f = 0 \). Since \( N \geq N_L \) is large enough, there exists an \( i \) such that \( 3 \leq i \leq L+2 \) and \( f_i \geq 2 \). Let \( \hat{i}_0 \) be the least such number. Note that \( \hat{i}_0 \neq L \) since \( f = 0 \). We have further subcases depending on whether \( \hat{i}_0 = L+1 \) or not.
Case 3(i): Suppose \( i_0 \neq L + 1 \). Then define
\[
\psi(\pi) = (2^{i_0}, 3^{f_3}, \ldots, i_0^{f_{i_0}-2}, \ldots, L^0, \ldots, (L + 2)^{f_{L+2}}).
\]

Case 3(ii): Suppose \( i_0 = L + 1 \). Then define
\[
\psi(\pi) = (2^2, 3^{f_3}, \ldots, (L - 1)^{f_{L-1}+2}, L^0, (L + 1)^{f_{L+1}+2}, \ldots, (L + 2)^{f_{L+2}}).
\]

It is easy to see that \( \psi \) is injective in each case. To see that \( \psi \) is injective overall, note that the frequency of 2 modulo \( L \) in the image distinguishes the cases with one exception: when \( i_0 = L + 1 \) and \( i_0 = L + 2 \). The frequencies of 2 in the image in those cases are 2 and \( L + 2 \), respectively. However, while those frequencies are the same modulo \( L \), they are different numbers and so distinguish the cases. Hence the map \( \psi \) is injective.

Let \( H_{L,2,L}(q) = \sum_{n \geq 0} a_{L,n}q^n \) and \( G_{L,2}(q) = \sum_{n \geq 0} b_{L,n}q^n \). Then, from Theorem 17, for all \( n \geq 0 \)
\[
b_{L,n} = a_{L,n} + a_{L,n-L} + a_{L,n-2L} + \cdots. \tag{19}
\]
From the injectivity of \( \psi \), we have \( a_{L,N} \geq 0 \) for all \( 5 \leq L \leq 10 \) and \( N \geq N_L \). In fact, we can show \( a_{L,N} \geq 1 \) for all \( 5 \leq L \leq 10 \) and \( N \geq N_L \). To see this, we find a partition in \( C_{L,2}^* \) which is not in the image of \( \psi \). For \( L \geq 5 \), note that \( N_L \geq 2L + 12 \). But from \( N \geq N_L \), we conclude \( N - 2(L + 3) \geq 6 \). Hence, by Lemma 7, there are nonnegative integers \( x_L \) and \( y_L \) such that
\[
N = 2(L + 3) + 3x_L + 4y_L.
\]
For each \( 5 \leq L \leq 10 \), fix some values of \( x_L \) and \( y_L \). Thus, there is a partition \( \lambda_L \) of \( N \) given by
\[
\lambda_L = (2^{L+3}, 3^{x_L}, 4^{y_L}).
\]
Note that \( \lambda_L \) is not in the image of \( \psi \) since the frequency of 2 is \( L + 3 \), which is not possible for any partition in the image of \( \psi \).

We are therefore left with determining the nonnegativity of \( a_{L,N} \) when \( N \leq N_L \). Since \( 5 \leq L \leq 10 \), the numbers \( a_{L,N} \) for \( N \leq N_L \) can easily be calculated by, for example, a short Magma program. The negative values of \( a_{L,N} \) for \( 5 \leq L \leq 10 \) and \( N \leq N_L \) are given in Table 5. In that table, we have also given the value of \( a_{L,N+L} \) when \( a_{L,N} \) is negative.

Using (19) and Table 5 along with the facts that \( a_{5,2} = 1 \) and \( a_{7,2} = 1 \), we conclude that \( b_{L,N} \geq 0 \) for \( 5 \leq L \leq 10 \) if \( N \neq 3 \), and \( b_{L,3} = -1 \). Hence, for all \( 5 \leq L \leq 10 \), we see that \( G_{L,2}(q) + q^3 \geq 0 \).

Finally, we prove Conjecture 6 for the cases \( L = 3 \) and \( L = 4 \) in the next two theorems.

**Theorem 20.** For \( L = 3 \),
\[
G_{L,2}(q) + q^3 + q^9 + q^{15} \geq 0.
\]

**Proof.** For \( N > 43 \), we construct an injective map
\[
\phi : \{ \pi \in D_{3,2} : |\pi| = N \} \to \{ \pi \in C_{3,2}^* : |\pi| = N \}.
\]
Recall that each \( \pi \in D_{3,2} \) has the form \( \pi = (3^{f_3}, 4^{f_4}, 5^{f_5}) \), and each partition in \( C_{3,2}^* \) has 2 as a part, but cannot have 3 as a part. We have cases depending on the frequency of 3 in \( \pi \). We fully describe \( \phi \) and later show it is injective.
Table 5: The values $5 \leq L \leq 10$ and $0 \leq N \leq N_L$ where $a_{L,N}$ is negative. Also given is $a_{L,N+L}$ in those cases.

Case 1: Suppose that $f_3$ is a positive even number. Then define

$$\phi(\pi) = \left(2^{\frac{3f_3}{2}}, 3^0, 4^{f_4}, 5^{f_5}\right).$$

Case 2: Suppose that $f_3 \geq 3$ is an odd number. Then define

$$\phi(\pi) = \left(2^{3\left(\frac{f_3-3}{2}\right)+2}, 3^0, 4^{f_4}, 5^{f_5+1}\right).$$

Case 3: Suppose $f_3 = 1$. Then $\pi = (3^1, 4^{f_4}, 5^{f_5})$. Since $N > 43$, either $f_4 \geq 1$ or $f_5 \geq 1$. We have further subcases based on these conditions.

Case 3(i): Suppose $f_4 \geq 1$. Then define

$$\phi(\pi) = \left(2^{1}, 3^0, 4^{f_4-1}, 5^{f_5+1}\right).$$

Case 3(ii): Suppose $f_4 = 0$ and $f_5 \geq 1$. Then define

$$\phi(\pi) = \left(2^{4}, 3^0, 5^{f_5-1}\right).$$

Case 4: Suppose $f_3 = 0$. Then $\pi = (4^{f_4}, 5^{f_5})$. Since $N > 43$, either $f_4 \geq 8$ or $f_5 \geq 4$. We have further subcases based on these conditions.

Case 4(i): If $f_4 \geq 8$, then define

$$\phi(\pi) = \left(2^{16}, 3^0, 4^{f_4-8}, 5^{f_5}\right).$$

Case 4(ii): If $f_4 \leq 7$ and $f_5 \geq 4$, then define

$$\phi(\pi) = \left(2^{10}, 3^0, 4^{f_4}, 5^{f_5-4}\right).$$

In each case it is clear that we can recover $\pi$ from $\phi(\pi)$. To see $\phi$ is injective overall, the frequency of 2 in partitions in the image distinguishes the different cases; in Cases 1 and 2, the frequency of 2 is congruent to 0 or 2 modulo 3, while in the other cases the frequency of 2 is either 1, 4, 16 or 10. Thus, the map $\phi$ is injective overall whenever $N > 43$.

Let $H_{3,2,3}(q) = \sum_{n \geq 0} a_n q^n$ and $G_{3,2}(q) = \sum_{n \geq 0} b_n q^n$. From Theorem 17

$$b_n = a_n + a_{n-3} + a_{n-6} + \cdots.$$
| n  | 0  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| a_n| 0  | 0  | 1  | −1| 0  | −1| 1  | 0  | 0  | −1| 1  | 0  | 1  | −1| 2  | −1| 2  | 0  | 2  |    |

Table 6: The values $a_n$ for $0 \leq n \leq 18$ in Theorem 20.

The injectivity of $\phi$ shows that $a_n \geq 0$ for all $n > 43$. For $n \leq 43$, $a_n$ can be calculated easily using a computer. It can be verified that $a_n$ is negative only when $n$ is either 3, 5, 9, 13 or 15 and in all of these cases, $a_n = -1$. Table 6 contains the values of $a_n$ for $n \leq 18$. From Table 6 and the fact that $a_n$ is negative only when $n$ is either 3, 5, 9, 13 or 15 (and in all of these cases, $a_n = -1$), we obtain $b_n \geq 0$ whenever $n \neq 3$, $n \neq 9$ and $n \neq 15$. Also apparent is that $b_3 = -1$, $b_9 = -1$ and $b_{15} = -1$. This proves that $G_{3,2}(q) + q^3 + q^9 + q^{15} \geq 0$.

**Theorem 21.** For $L = 4$,

$$G_{L,2}(q) + q^3 + q^9 \geq 0.$$ 

**Proof.** For $N > 20$, we construct an injective map

$$\psi : \{ \pi \in D_{4,2} : |\pi| = N \} \rightarrow \{ \pi \in C_{4,2}^* : |\pi| = N \}.$$ 

Recall that partitions $\pi \in D_{4,2}$ have the form $\pi = (3f_3, 4f_4, 5f_5, 6f_6)$, whereas partitions in $C_{4,2}^*$ must have a part of size 2 and cannot have a part of size 4. We have cases depending on the frequency of 4 in $\pi$. We describe the map $\psi$ fully and then show it is injective later.

Case 1: Suppose $f_4 \geq 1$. Then define

$$\psi(\pi) = (2^{2f_4}, 3f_3, 4^0, 5f_5, 6f_6).$$

Case 2: Suppose $f_4 = 0$. Then $\pi = (3f_3, 5f_5, 6f_6)$. Since $N > 20$, either $f_3 \geq 2$, $f_5 \geq 2$ or $f_6 \geq 3$. We consider subcases based on these conditions.

Case 2(i): Suppose $f_3 \geq 2$. Then define

$$\psi(\pi) = (2, 3f_3-2, 4^0, 5f_5, 6f_6).$$

Case 2(ii): Suppose $f_3 \leq 1$ and $f_5 \geq 2$. Then define

$$\psi(\pi) = (2^5, 3f_3, 4^0, 5f_5-2, 6f_6).$$

Case 2(iii): Suppose $f_3 \leq 1$, $f_5 \leq 1$ and $f_6 \geq 3$. Then define

$$\psi(\pi) = (2^6, 3f_3, 4^0, 5f_5, 6f_6-3).$$

It is easy to see that $\psi$ is injective in each case. To see that $\psi$ is injective overall, notice that the frequency of 2 in partitions in the image distinguishes cases. In Case 1, the frequency of 2 is even, and in the other cases the frequency of 2 is 3, 5 or 9. Thus, $\psi$ is injective.

Let $H_{4,2,4}(q) = \sum_{n \geq 0} a_n q^n$ and $G_{4,2}(q) = \sum_{n \geq 0} b_n q^n$. Then

$$b_n = a_n + a_{n-4} + a_{n-8} + \cdots.$$ 

The injectivity of $\psi$ shows that $a_n \geq 0$ for all $n > 20$. For $n \leq 20$, $a_n$ can be calculated easily using a computer (or by hand). It can be verified that $a_n$ is negative only when $n$ is either 3, 6 or 9, and in all of these cases, $a_n = -1$. Table 7 contains the values of $a_n$ for $n \leq 13$.

From Table 7 and the fact that $a_n$ is negative only when $n$ is either 3, 6 or 9 (and in all of these cases, $a_n = -1$), we get $b_n \geq 0$ whenever $n \neq 3$ and $n \neq 9$. Also, $b_3 = -1$ and $b_9 = -1$. This proves that $G_{4,2}(q) + q^3 + q^9 \geq 0$. 

$$\square$$

30
| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|-----|---|---|---|---|---|---|---|---|---|---|----|----|----|----|
| $a_n$ | 0 | 0 | 1 | -1 | 0 | 0 | -1 | 1 | 1 | -1 | 1 | 1 | 0 | 2 |

Table 7: The values $a_n$ in Theorem 21 for $0 \leq n \leq 13$.

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