Labeled and unlabeled Hamiltonian cycles in complete $n$-partite graphs $K_{d,d,...,d}$ having exactly $d$ vertices in each part are enumerated. Recurrence relations for the exact values $b_n^{(d)}$ of such cycles for arbitrary $n$ and $d$ are obtained. Bibliography: 14 titles.

1. Introduction

The problems of enumerating Hamiltonian cycles in different classes of graphs are interesting from both applied and theoretical points of view. Such problems arise in theoretical physics [1], chemistry, biology and bioinformatics [2], as well as in the theory of algorithms. For example, Hamiltonian paths and Hamiltonian cycles can serve in polymer science as excellent simple models for dense packed globular proteins [3]. In physics, Hamiltonian paths find applications for studying magnetic systems with certain symmetry [1]. Finally, the exact enumeration of Hamiltonian cycles might be useful for the development of statistical algorithms that provide unbiased sampling of such cycles.

Determining whether a graph contains a Hamiltonian cycle is a difficult ($NP$-complete) problem. It is even harder to determine how many distinct Hamiltonian cycles a graph has. Because of this difficulty, different methods have been used for enumerating Hamiltonian cycles in different classes of graphs. In some cases, it is possible to obtain an exact solution of this problem, for example, by using the method of transfer matrices [4]. For some other classes of graphs, special enumerative algorithms have been developed [5]. Multiple results regarding upper and lower bounds for the numbers of Hamiltonian cycles are also known [6–10].

Fig. 1. A chord diagram.

The class of problems of enumerating Hamiltonian cycles in graphs belonging to a certain parametrized family and possessing a high degree of symmetry can be distinguished separately. It is known that the graphs of many such families are Hamiltonian and the main issue is to determine the exact number of Hamiltonian cycles. In the present paper, we consider complete $n$-partite graphs $K_{d,d,...,d}$ that have $d$ vertices in each part. For $d = 2$, such graphs...
and Hamiltonian cycles in them turn out to be closely related to so-called chord diagrams (see [11–13]). A chord diagram is a circle with $2n$ points labeled with numbers $1, 2, \ldots, 2n$ in cyclic order; these points are then joined pairwise by chords (Fig. 1). We say that a chord is a loop if it connects two neighboring points (such as the chord $\{1, 2\}$ in Fig. 1). A loopless diagram is a diagram without loops.

The following bijection between Hamiltonian paths in octahedrons and loopless chord diagrams was found in [13]. Take an $n$-dimensional octahedron with distinguished Hamiltonian cycle (Fig. 2a) and draw it so that this cycle forms a circle on the plane (Fig. 2b). Then we remove all of its edges not belonging to the Hamiltonian cycle and add the chords between the vertices not connected in the original graph (Fig. 2c). The resulting object is a chord diagram which is necessarily loopless: traversing a Hamiltonian cycle in $K_{2,2,\ldots,2}$, we cannot visit two vertices of the same part one by one. Clearly, this transformation is invertible.

![Fig. 2. The correspondence between Hamiltonian cycles in octahedrons and chord diagrams.](image)

In the general case of a complete $n$-partite graph $K_{d,d,\ldots,d}$, any Hamiltonian cycle of the graph (Fig. 3a) can be represented by a generalized chord diagram $B_{n}^{(d)}$ (Fig. 3b) constructed on $n \cdot d$ vertices. A generalized chord diagram consists of generalized chords isomorphic to the graphs $K_{d}$ connecting $d$ points of the diagram (see, for example, the generalized chord $\{1, 5, 10\}$ in Fig 3b). A loop in a generalized chord diagram is an edge of a subgraph $K_{d}$ that connects two neighboring points of this diagram. Note that one subgraph of $K_{d}$ can define multiple loops. Similarly to the case $d = 2$, a Hamiltonian cycle in the graph $K_{d,d,\ldots,d}$ corresponds to a generalized chord diagram without loops.

![Fig. 3](image)
In the first part of the present paper, we obtain explicit formula (1) for the number \( b^{(d)}_n \) of loopless chord diagrams \( B^{(d)}_n \) in terms of the numbers \( a^{(d)}_{n,k} \) of so-called generalized linear diagrams with \( k \) loops. In addition, we find a recurrence relations (formulas (2)–(3)) for the numbers \( a^{(d)}_{n,k} \).

The use of the obtained recurrence relations turns out to be computationally inefficient: they represent a system of \( n \cdot (d-1) \) equations for any given \( n \). Using some additional combinatorial arguments, we reduce this system to system (9) of recurrence relations containing \( d \) equations only.

Depending on the chosen notion of isomorphism, two diagrams are said to be isomorphic if one of them can be obtained from the other either by rotations or by a combination of rotations and reflections of the circle. The isomorphism classes of labeled generalized chord diagrams are said to be unlabeled generalized chord diagrams. For enumerating these diagrams, and hence for enumerating unlabeled Hamiltonian cycles in the graphs \( K_{d,d,...,d} \), we obtain a system of recurrence relations. We provide answers for both notions of isomorphism: for rotations only, as well as for rotations and reflections.

2. Enumeration of generalized chord diagrams \( B^{(d)}_n \)

Instead of counting generalized loopless chord diagrams \( B^{(d)}_n \) directly, it is convenient for us to enumerate so-called generalized loopless linear diagrams \( A^{(d)}_n \) first. To transform a chord diagram into a linear diagram, we remove the arc connecting the points 1 and \( n \cdot d \). This means that we no longer consider these points to be neighboring (Fig. 4a). If the chord diagram \( B^{(d)}_n \) is loopless, the corresponding linear diagram \( A^{(d)}_n \) is also loopless.

Fig. 4. A chord diagram and a linear diagram corresponding to it.

Let \( a^{(d)}_{n,k} \) be the number of generalized linear diagrams \( A^{(d)}_{n,k} \) constructed on \( d \cdot n \) points, consisting of \( n \) complete subgraphs \( K_d \), and having \( k \) loops, \( 0 \leq k \leq n(d-1) \). Let us show that the number \( b^{(d)}_n \) of generalized loopless chord diagrams can be expressed through the numbers \( a^{(d)}_{n,k} \) by the formula

\[
b^{(d)}_n = a^{(d)}_{n,0} - \sum_{k=0}^{d-2} \left( \frac{d(n-1)-k-1}{d-2-k} \right) a^{(d)}_{n-1,k}.
\]  

Indeed, among all \( a^{(d)}_{n,0} \) generalized linear diagrams \( A^{(d)}_{n,0} \) without loops, we are interested in only those that have no chord connecting the two end vertices. Assume that after deleting a subgraph \( K_d \) that contains such a chord, we obtain a linear diagram \( A^{(d)}_{n-1,k} \) with \( k \) loops. The number of ways to obtain a generalized linear diagram \( A^{(d)}_{n,0} \) from an arbitrary linear diagram
$A_{n-1,k}^{(d)}$ could be counted as follows. Note that the diagram $A_{n-1,k}^{(d)}$ has $d(n-1) + 1$ positions for placing the vertices of the subgraph $K_d$. Two of these vertices must be placed on the first and the last positions of the diagram and the remaining $d(n-1) - k$ vertices can be placed into the $d(n-1) + 1$ positions in $\binom{d(n-1)-k-1}{d-2-k}^{(d)}$ ways. Summing $\binom{d(n-1)-k-1}{d-2-k}^{(d)} a_{n-1,k}^{(d)}$ over all possible $k$, we obtain the number of diagrams $A_{n,0}^{(d)}$ that have the first and the last points connected.

To find the numbers $b_n^{(d)}$ according to formula (1), we need some recurrence relations for the numbers $a_n^{(d)}$, $k = 0, \ldots, d - 2$. It is easier to start with some recurrence relations for a broader range of possible values of $k$. Namely, let us show that for $k = 0, \ldots, n(d - 1)$, the following relations hold:

$$a_{n,k}^{(d)} = \sum_{t=k-d+1}^{k+d-1} c_{n,k,t}^{(d)} \cdot a_{n-1,t}, \quad n > 0, \quad 0 \leq k \leq n(d - 1),$$

(2)

$$c_{n,k,t}^{(d)} = \sum_{i=0}^{d-1} \binom{d-1}{i} \binom{t}{t+i-k} \binom{d(n-1)-t}{d-2-i-t+k-1},$$

(3)

$$a_{n,0}^{(d)} = 1, \quad a_{n,k}^{(d)} = 0 \text{ for } n < 0, \quad k < 0 \text{ and } k > n(d - 1).$$

The proof of relations (2)–(3) is again based on the procedure of removing the subgraph $K_d$ that contains the rightmost point of the linear diagram $A_{n,k}^{(d)}$. Since the removal of $K_d$ may add or remove at most $d - 1$ loops, this procedure yields a generalized linear diagram $A_{n-1,t}^{(d)}$ with $t$ loops $t \in [k - (d - 1), k + (d - 1)]$.

![Fig. 5. Generalized linear diagram $A_{3,8}^{(d)}$.](image)

As an example, we show in Fig. 5 a generalized linear diagram $A_{n,k}^{(d)}$ for $n = 3$, $d = 6$, and $k = 8$. Removing the subgraph $K_6$ containing the rightmost point 18, yields a generalized linear diagram $A_{2,6}^{(6)}$.

Now assume that we are given a diagram $A_{n-1,t}^{(d)}$. The number of ways to transform it into a diagram $A_{n,k}^{(d)}$ by adding a subgraph $K_d$ containing its rightmost point could be counted as follows. Denote by $i$ the number of loops formed by neighboring vertices of $K_d$ after adding it to $A_{n-1,t}^{(d)}$ ($i = 3$ for the subgraph $K_5$ depicted in Fig. 5). In order for the diagram $A_{n,k}^{(d)}$ to have exactly $k$ loops after adding $K_d$, this subgraph $K_d$ must destroy $t + i - k$ existing loops ($t + i - k = 1$ for the diagram $A_{2,6}^{(6)}$ in Fig 5) by its vertices placed under these loops. Also, we have $d(n - 1) - t$ remaining positions among which we need to distribute $d - i - 1 - (t + i - k)$ remaining vertices of the subgraph $K_d$ (6 positions for the only vertex in the example shown in Fig 5). Counting the total number of ways to perform these combinatorial actions, we arrive at formulas (2)–(3).

We note the following important cases of relations (1)–(3). For $d = 2$, formula (1) can be rewritten as

$$b_n^{(2)} = a_n^{(2)} - a_{n-1}^{(2)}, \quad n \geq 2; \quad b_1^{(2)} = 0,$$

(4)
and formulas (2)–(3) imply

\[
\begin{align*}
a^{(2)}_{n+1,k} &= a^{(2)}_{n,k} - (2n - k)a^{(2)}_{n,k} + (k + 1)a^{(2)}_{n,k+1}, \\
a^{(2)}_{n,k} &= 0 \text{ if } k > n \text{ or } k < 0, \quad a^{(2)}_{0,0} = 1.
\end{align*}
\]

These results agree with formulas (1) and (2) in [13]. For \(d = 3\), the expression for \(b^{(3)}_n\) takes the form

\[
b^{(3)}_n = a_{n,0} - (3n - 4)a^{(3)}_{n-1,0} + a^{(3)}_{n-1,1},
\]

and system (2)–(3) can be rewritten as

\[
\begin{align*}
a^{(3)}_{n,k} &= a^{(3)}_{n-1,k-2} + 2(3(n - 1) - (k - 1))a^{(3)}_{n-1,k-1} + \left(3(n - 1) - k\right)a^{(3)}_{n-1,k} + \left(k + 1\right)(3(n - 1) - (k - 1))a^{(3)}_{n-1,k+1} + \left(k + 2\right)a^{(3)}_{n-1,k+2}. \\
3. A closed system of \(d\) recurrence relations for \(a^{(d)}_{n,k}\)

The recurrence relations (2)–(3) allow us, in principle, to obtain the values of \(a^{(d)}_{n,i}\), \(i = 0, \ldots, d - 2\), that are sufficient for finding \(b^{(d)}_n\). However, from the computational point of view, this approach could be improved; ideally, we need to find a system of recurrences involving only those values of \(a^{(d)}_{n,k}\) that explicitly appear in (1). For \(d = 2\), the approach described in [13] is to rewrite system (2)–(3) as a system of recurrence relations, find the generating function \(w(z,t)\) for \(a_{n,k}\), and then substitute \(z = 0\) into it. The generating function \(\phi(t) = w(z,0)\) obtained as a result of substitution defines the numbers \(a^{(2)}_{n,0} \equiv a^{(2)}_{n}\) sufficient for calculating \(b^{(2)}_n\). However, this approach no longer works in the case \(d > 2\). An alternative approach is to derive the corresponding system by a combinatorial argument. This approach works perfectly for \(d = 2\) (see [12,13]), but even for \(d = 3\) an analogous combinatorial proof becomes quite cumbersome, and for \(d > 3\) the problem becomes practically intractable.

It turns out that there is a combined approach: use combinatorial arguments together with the already obtained system of recurrence relations (2)–(3) for the numbers \(a^{(d)}_{n,k}\). With this approach, we can obtain a closed system for the sequences \(a^{(d)}_{n,k}\), \(k = 0, \ldots, d - 1\), the number of which is only one greater than the number of terms \(a^{(d)}_{n,k}\) in formula (1). Namely, substituting \(k = 0\) into formula (2), we obtain the recurrence relation

\[
a_{n,0}^{(d)} = \sum_{t=0}^{d-1} c_{n,0,t}^{(d)} a_{n-1,t}^{(d)}.
\]

Fig. 6. Generalized linear diagram \(A_{3,2}^{(5)}\).

For \(k\) ranging from 1 to \(d - 1\), relations for \(a_{n,k}^{(d)}\) could be obtained by using combinatorial arguments. Namely, we consider a generalized linear diagram \(A_{n,k}^{(d)}\) that has \(k\) loops distributed
over $l$ subgraphs isomorphic to $K_d$, $1 \leq l \leq k < d$. We begin with the simplest case $l = 1$ when all $k$ loops are formed by a single subgraph $K_d$ (see Fig. 6, where the corresponding subgraph $K_3$ is shown in blue).

Fig. 7. Reduced linear diagram.

Fig. 8. Generalized linear diagram $A_{2,2}^{(5)}$.

Fig. 9. Generalized linear diagram $A_{4,3}^{(5)}$.

For $l > 1$, the argument becomes slightly more complicated. Indeed, let $d - r_i$, $i = 1, \ldots, l$, be the number of loops in a diagram $A_{n,k}^{(d)}$ belonging to the $i$th subgraph $K_d$, and also

$$
\sum_{i=1}^{l} (d - r_i) = k, \quad 1 \leq r_1 \leq r_2 \leq \ldots \leq r_l
$$

(see the diagram $A_{4,3}^{(5)}$ with $l = 2$, $r_1 = 3$, $r_2 = 4$ in Fig. 9).

Contracting each such loop into a point, we obtain a reduced linear diagram with $n - l$ subgraphs $K_d$ and $l$ subgraphs $K_{r_i}$ (Fig. 10).

Assume that after deleting the subgraphs $K_{r_i}$ we obtain a generalized linear diagram $A_{n-1,m}^{(d)}$, $m = 0, \ldots, ld - k$ (see Fig. 11 corresponding to the diagram $A_{2,4}^{(5)}$). Let us see how many
diagrams $A_{n,k}^{(d)}$ with $k$ loops distributed among $l$ subgraphs $K_r$ can be obtained from this diagram $A_{n-l,m_1}^{(d)}$.

Consider a diagram $A_{n-l,m_1}^{(d)}$, $m_1 \equiv m$, and add a subgraph $K_{r_1}$ to it (Fig. 12). Some of the existing loops can be destroyed by the vertices of $K_{r_1}$. At the same time, the new diagram can have additional loops formed by neighboring vertices of the subgraph $K_{r_1}$. Denote by $s_1$ the number of loops destroyed by $K_{r_1}$, and by $j_1$ the number of loops formed by the vertices of $K_{r_1}$ ($s_1 = 2$, $j_1 = 1$ for the diagram shown in Fig 12).

Instead of $K_{r_1}$, we consider other subgraph $K_{t_1}$, $t_1 = r_1 - j_1$, obtained by contracting $j_1$ loops of the subgraph $K_{r_1}$ (Fig. 13). This subgraph $K_{t_1}$ can be placed into the original diagram $A_{n-l,m_1}^{(d)}$ so that $s_1$ vertices of the subgraph $K_{t_1}$ split the loops of the diagram $A_{n-l,m_1}^{(d)}$ and the remaining $t_1 - s_1$ vertices are distributed among $v_1 - m_1$ positions free from the loops, $v_1 := d(n - l) + 1$, in $(m_1)_{(v_1 - m_1)}$ ways. Splitting $t_1$ vertices of the subgraph $K_{t_1}$ into $r_1$ vertices so that the vertices of the obtained subgraph $K_{r_1}$ form $j_1$ additional loops can be done in $\binom{t_1}{j_1}$ ways. The number

$$\binom{m_1}{s_1} \binom{v_1 - m_1}{r_1 - j_1 - s_1} \binom{r_1 - 1}{j_1}$$

of ways obtained in the first step should be multiplied by the number

$$\binom{m_2}{s_2} \binom{v_2 - m_2}{r_2 - j_2 - s_2} \binom{r_2 - 1}{j_2}$$
of ways to add a subgraph $K_{r_2}$ by placing its vertices into $v_2 := v_1 + r_1$ positions in a way that destroys $s_2$ loops of the linear diagram with $m_2 = m_1 + j_1 - s_1$ loops and adds $j_2$ its own loops.

Continuing this process further, we arrive at the final step where one needs to add a subgraph $K_{r_1}$ to a linear diagram. This step is special, because after the adding, there are no loops in the diagram: after adding $K_{r_1}, \ldots, K_{r_l}$ we get a loopless reduced linear diagram (see Fig. 10). Consequently, at this final step one destroys all loops obtained on the previous step (that is, set $m_l = s_l$), and the subgraph $K_{r_l}$ itself has not any loop of its own (that is, $j_l = 0$).

Taking this into account, we obtain the following final formula for the numbers $a_{n,k}^{(d)}$ for $l > 1$:

$$a_{n,k,l>1}^{(d)} = \sum_{R} \frac{\alpha_R}{\beta_1! \cdots \beta_{d-1}!} \sum_{m=0}^{ld-k} p_{n,R,m} \cdot a_{n-l,m}^{(d)}.$$ (9)

Here, $R$ is an ordered multiset $\{r_1, \ldots, r_l\}$ satisfying conditions (8) and the external summation runs over all multisets $R$ such that $p_{n,R,m} := \sum_{j_1=0}^{r_1} \cdots \sum_{j_l=0}^{r_l} \sum_{s_1=0}^{\min\{m_1,r_1-j_1\}} \sum_{s_1=0}^{r_l-1-\min\{m_l,r_l-j_l\}} \sum_{i=1}^{l-1} \prod_{i=1}^{l-1} (m_i - r_i + j_i) (r_i - 1) (v_1 - m_l),$

$$m_{i+1} := m_i + j_i - s_i, \quad v_{i+1} := v_i + r_i, \quad i > 1; \quad m_1 := m, \quad v_1 := d(n-l) + 1,$

$$\alpha_R := \prod_{i=1}^{l} \left( \frac{r_i}{d - r_i} \right) = \prod_{i=1}^{l} \left( \frac{d-1}{r_i - 1} \right).$$

The multiplier $\alpha_R$ in formula (9) describes the number of ways to transform the subgraphs $K_{r_i}$ into $K_d$. The coefficient $1/(\beta_1! \cdots \beta_{d-1}!)$ takes into account the fact that we delete the subgraphs $K_{r_i}$ not simultaneously but one after the other; that is, all the cliques $K_{r_i}$ with the same number of loops are distinct. Consequently, if there are $\beta_u$ instances of a subgraph $K_u$ among all cliques $K_{r_i}$, we need to divide the result by $\beta_u!$.

Finally, we note that for $l > 1$, the numbers $a_{n-l,m}^{(d)}$ with $m \geq d$ appear in formula (9) for $a_{n,k}^{(d)}$. These numbers can always be eliminated by using recurrence relation (2) rewritten as

$$a_{n-1,k,d-1}^{(d)} = \frac{a_{n,k}^{(d)} - \sum_{t=k-d+1}^{k+d-2} c_{n,k,t}^{(d)} \cdot a_{n-1,t}^{(d)}}{c_{n,k,d-1}^{(d)}}.$$ (10)

For instance, substituting $n - 1$ instead of $n$ into (10), we express the numbers $a_{n-2,d}^{(d)}$, $a_{n-2,d+1}^{(d)}$, ..., through the numbers $a_{n-1,k}^{(d)}$ and $a_{n-2,m}^{(d)}$, $0 \leq k, m \leq d - 1$. In a similar way, we can express the numbers $a_{n-3,m}^{(d)}$, $a_{n-4,m}^{(d)}$, ..., up to $a_{n-l,m}^{(d)}$.

Let us illustrate the above approach for special cases $d = 2$ and $d = 3$. Substituting $d = 2$ into formula (6), we obtain the recurrence relation

$$a_{n,0}^{(2)} = (2n - 2)a_{n-1,0}^{(2)} + a_{n-1,1}^{(2)}.$$
One can see that along with \( a_{n,0}^{(2)} \) this equation also contains the numbers \( a_{n,1}^{(2)} \) describing linear diagrams \( A_{n,1}^{(2)} \) with single loop. For these numbers, we can use recurrence relation (7). Substituting the values \( k = 1, \ d = 2 \) into it, we obtain

\[
a_{n,1}^{(2)} = (2n - 1)a_{n-1,0}^{(2)} + a_{n-1,1}^{(2)}.
\]

Expressing the numbers \( a_{n,1}^{(2)} \) from these relations, we get a second-order recurrence relation

\[
a_{n+1,0}^{(2)} = (2n + 1)a_{n,0}^{(2)} + a_{n-1,0}^{(2)}; \quad a_{0,0}^{(2)} = 1, \quad a_{1,0}^{(2)} = 0
\]

for the number of loopless linear diagrams.

Now we consider a more interesting example \( d = 3 \). Substituting \( d = 3 \) into (6), we have

\[
a_{n,0}^{(3)} = \left(\frac{3n - 3}{2}\right) a_{n-1,0}^{(3)} + (3n - 4) a_{n-1,1}^{(3)} + a_{n-1,2}^{(3)}.
\]

The recurrence relation for \( a_{n,1}^{(3)} \) as well as the recurrence relation for \( c_{n,2}^{(3)} \), corresponding to the case of both loops belonging to a single subgraph \( K_3 \), can be obtained from formula (7):

\[
a_{n,1}^{(3)} = 2 \left(\frac{3n - 2}{2}\right) a_{n-1,0}^{(3)} + (3n - 3) a_{n-1,1}^{(3)} + a_{n-1,2}^{(3)},
\]

\[
a_{n,2,t=1}^{(3)} = (3n - 2) a_{n-1,0}^{(3)} + a_{n-1,1}^{(3)}.
\]

However, in contrast to the case \( d = 2 \), it may happen that both loops of a diagram \( A_{n,2}^{(3)} \) belong to two different subgraphs \( K_3 \). To count the number of such diagrams we can use formula (9). In our special case,

\[
l = 2, \quad R = \{ r_1, r_2 \} = \{ 2, 2 \}, \quad \alpha_R = 2 \cdot 2, \quad \beta_2 = 2!, \quad j_1 = j_2 = 1.
\]

Consequently,

\[
a_{n,2,t=2}^{(3)} = \frac{2 \cdot 2}{2!} \sum_{m=0}^{4} \sum_{j=0}^{1} \sum_{s=0}^{\min(m,2-j)} \left(\begin{array}{c} m \\ s \end{array}\right)
\times \left(\begin{array}{c} 3(n - 2) + 1 - m \\ 2 - j - s \end{array}\right) \left(\begin{array}{c} 3(n - 2) + 3 - m - j + s \\ 2 - m - j + s \end{array}\right) a_{n-2,m}^{(3)}.
\]

In its turn, the numbers \( a_{n,2,3}^{(3)} \) and \( a_{n,2,4}^{(3)} \) are expressed through \( a_{n-1,1}^{(3)} \) and \( a_{n-2,1}^{(3)} \) with the help of relation (10):

\[
a_{n-2,3}^{(3)} = \frac{a_{n-1,1}^{(3)} - \sum_{t=0}^{2} c_{n-1,1,t}^{(3)} a_{n-2,t}^{(3)}}{c_{n-1,3,3}^{(3)}}, \quad a_{n-2,4}^{(3)} = \frac{a_{n-1,2}^{(3)} - \sum_{t=0}^{3} c_{n-1,2,t}^{(3)} a_{n-2,t}^{(3)}}{c_{n-1,2,4}^{(3)}}.
\]

The obtained system of recurrence relations for \( d = 3 \) can be simplified and rewritten as

\[
a_{n,0}^{(3)} = \left(\frac{3n - 3}{2}\right) a_{n-1,0}^{(3)} + (3n - 4) a_{n-1,1}^{(3)} + a_{n-1,2}^{(3)}; \quad a_{0,0}^{(3)} = 1, \quad a_{1,0}^{(3)} = 0;
\]

\[
a_{n,1}^{(3)} = 2a_{n,0}^{(3)} + 2(3n - 3)a_{n-1,0}^{(3)} + a_{n-1,1}^{(3)}, \quad a_{0,1}^{(3)} = 0, \quad a_{1,1}^{(3)} = 0;
\]

\[
a_{n,2}^{(3)} = 2a_{n,0}^{(3)} + (9n - 10)a_{n-1,0}^{(3)} + 5a_{n-1,1}^{(3)} + 2a_{n-2,0}^{(3)}, \quad a_{0,2}^{(3)} = 0, \quad a_{1,2}^{(3)} = 1.
\]
4. Enumeration of unlabeled generalized chord diagrams

In this section, we briefly describe the results related to the problem of enumerating Hamiltonian cycles in unlabeled graphs $K_{d,d,...,d}$. More precisely, we solve an equivalent problem of enumerating unlabeled generalized chord diagrams without loops. The number $\hat{b}_n^{(d)}$ of such diagrams can be calculated using the Burnside lemma,

$$\hat{b}_n^{(d)} = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|.$$ (11)

Here, $|\text{Fix}(g)|$ is the number of labeled diagrams fixed by the action of the element $g$ of some group $G$ which defines the isomorphism relation between diagrams. In our case, $G$ is either cyclic group $C_{d \cdot n}$ of rotations or dihedral group $D_{d \cdot n}$ of rotations and reflections.

First, we consider first the case of cyclic group. For this case, denote by $\tilde{b}_n^{(d)}$ the number of unlabeled generalized chord diagrams with $d \cdot n$ points and $n$ chords, and consider the action of the group $C_{d \cdot n}$ on the set of these diagrams. Let $m$ be a divisor of $d \cdot n$ and $\phi(m)$ the Euler function. There are $\phi(m)$ elements of order $m$ in $C_{d \cdot n}$. Any such element fixes the same number $f(d \cdot n, m)$ of diagrams which are said to be $m$-symmetric. Consequently, (11) can be rewritten as

$$\tilde{b}_n^{(d)} = \frac{1}{d \cdot n} \sum_{m \mid d \cdot n} \phi(m) f(d \cdot n, m).$$ (12)

To calculate the values of $f(d \cdot n, m)$, it is convenient to begin with counting so-called generalised $m$-linear diagrams (Fig. 14). Any such diagram with $d \cdot n$ points is obtained by cutting the circle of an $m$-symmetric generalized chord diagram into $m$ sectors between the points $v$ and $v+1, 2v$ and $2v+1, \ldots, m \cdot v$ and 1. Each of the sectors has $v := d \cdot n/m$ points. The cut between points $i$ and $i+1$ means that these points are no longer considered to be neighbors.

We note that 1-linear diagrams are just linear diagrams considered in the previous section.

Denote by $A^{(m,d)}_{v,k}$ the set of generalized $m$-linear diagrams having exactly $k$ loops in each of $m$ its sectors.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig14.png}
\caption{Diagram $A^{(3,4)}_{8,0}$ (a) and Diagram $A^{(6,6)}_{4,0}$ (b).}
\end{figure}
Lemma 4.1. The number $a_{v,k}^{(m,d)}$ of generalized $m$-linear diagrams $A_{v,k}^{(m,d)}$ can be calculated by using the following system of recurrence relations:

$$a_{v,k}^{(m,d)} = \sum_{l|\text{gcd}(m,d)} \sum_{t=k-d/l+1}^{k+d/l-1} c_{v,k,t}^{(m,l,d/l)} \cdot a_{v-d/l,k,t}^{(m,d)},$$

$$a_{v,k} = 0 \quad \text{for} \quad v \leq 0 \quad \text{or} \quad k < 0, \quad \text{except for} \quad a_{0,0}^{(m,d)} = 1,$$

where

$$c_{v,k,t}^{(m,d)} = \sum_{i=0}^{d-1} \binom{d-1}{i} \cdot \left( \binom{t}{t+i-k} \cdot \frac{m^{t+i-k}}{v^i} \cdot \tilde{P}_{\tilde{v}}^{(m,v-d/t+i-k)} \right), \quad \tilde{v} := d - 2i - t + k - 1, \quad (14)$$

and the number $q_{k,t}^{(m,v,l)}$ can be calculated from the following recurrence relation:

$$q_{k,t}^{(m,v,l)} = (2k - 1 + l - t) q_{k-1,t-1}^{(m,v,l)} + (t + 1)(m - 1) q_{k-1,t+1}^{(m,v,l)} + (m(v + k) + l(m - 1) - 2k - 2 - t - t(m - 1)) q_{k-1,t}^{(m,v,l)}, \quad (16)$$

$$q_{0,0}^{(m,v,l)} = 1, \quad q_{0,t}^{(m,v,l)} = 0 \quad \text{for} \quad t \neq 0, \quad q_{k,t}^{(m,v,l)} = 0 \quad \text{for} \quad k < 0 \quad \text{or} \quad t < 0.$$

Theorem 4.1. The number $f(d \cdot n, m)$ can be expressed in terms of the numbers $a_{v,k}^{(m,d)}$ of generalized $m$-linear diagrams $A_{v,k}^{(m,d)}$ by the formula

$$f(d \cdot n, m) = a_{v,0}^{(m,d)} - \sum_{l|\text{gcd}(m,d)} \sum_{k=0}^{l-2} \frac{(m/l)^k}{(d/l - 2 - k)!} \cdot \tilde{P}_{d/l-2-k}^{(m,v-d/l-k,k)} \cdot a_{v-d/l,k}^{(m,d)} \cdot v := \frac{dn}{m}, \quad (17)$$

where

$$\tilde{p}_{k}^{(m,v,l)} = \sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^i \cdot \binom{k}{i} \cdot \frac{k!}{(k-i-j)!} \cdot q_{0,k-i-j}^{(m,v,l)}.$$

The proofs of these statements conceptually do not differ from the proofs presented in the previous sections, but are rather cumbersome. We omit them in this paper.

Next, we proceed with problem of determining the number of unlabeled diagrams considered up to the action of the dihedral group $D_{d \cdot n}$. Denote the number of such diagrams with $d \cdot n$ points and $n$ chords by $\tilde{b}_{n}^{(d)}$. The Burnside lemma for this case can be rewritten as

$$\tilde{b}_{n}^{(d)} = \frac{1}{2dn} \sum_{m|dn} \phi(m) f(dn, m) + \frac{h^{(0)}(n) + 2h^{(1)}(n) + h^{(2)}(n)}{2}, \quad (18)$$

where $h^{(i)}(n)$ denotes the number of chord diagrams symmetric under reflection about the axis passing through $i$ points of the diagram and consisting of $n$ subgraphs $K_d$ (see Fig. 15). The next series of results describing the calculation of the numbers $h^{(i)}(n)$, $i = 0, 1, 2$, is also given without proof.
Theorem 4.2. The number $h^{(0)}(n)$ for the case of odd $d$ coincides with the number $a^{(2,d)}_{dn/2,0}$, and for the case of even $d$ is calculated by formulas:

$$h^{(0)}(n) = a^{(2,d)}_{dn/2,0} - 2 \sum_{k=0}^{d/2-1} \alpha_k^{(1)} \cdot a^{(2,d)}_{d(n-1)/2,k} + \sum_{k=0}^{d-2} \alpha_k^{(2)} \cdot a^{(2,d)}_{d(n-2)/2,k} - \sum_{k=0}^{d-2} \alpha_k^{(3)} \cdot a^{(2,d)}_{d(n-2)/2,k},$$

where

$$\alpha_k^{(1)} := \binom{(n-1)d/2 - 1 - k}{d/2 - 1 - k}, \quad \alpha_k^{(3)} := \binom{(n-1)d/2 - 1 - k}{d/2 - 2 - k},$$

$$\alpha_k^{(2)} := \sum_{j=0}^{d/2-1 \text{ min}(k,d/2-1-j)} \sum_{s=0}^{j} \binom{d/2 - 1}{j} \binom{k}{s} \binom{(n-2)d/2 - k}{d/2 - 1 - j - s} \cdot \binom{(n-1)d/2 - 1 - (k+j-s)}{d/2 - 1 - (k+j-s)}.$$

Theorem 4.3. The number $h^{(1)}(n)$ is expressed in terms of the numbers $a^{(2,d)}_{(n-1)d/2,k}$ by the formula:

$$h^{(1)}(n) = \sum_{k=0}^{(d-1)/2} \binom{(n-1)d/2 - 1 - k}{(d-1)/2 - k} \cdot a^{(2,d)}_{(n-1)d/2,k}.$$

Theorem 4.4. The number $h^{(2)}(n)$ in the case of even $d$ can be calculated by the formula:

$$h^{(2)}(n) = \sum_{k=0}^{d/2-1} \alpha_k^{(1)} \cdot a^{(2,d)}_{(n-1)d/2,k}.$$

For an odd $d$, the number $h^{(2)}(n)$ is nonzero only if $n$ is even; in this case it is equal to

$$h^{(2)}(n) = \sum_{k=0}^{d/2-1 \text{ min}(k,d/2-1-j)} \binom{(d-1)/2}{j} \binom{k}{s} \binom{(n-2)d/2 - k(d-1)/2 - j - s}{(d-1)/2 - (k+j-s)} \cdot a^{(2,d)}_{(n-2)d/2,k}.$$

Conclusion

The final numbers for various classes of linear and chord diagrams as well as the corresponding Hamiltonian cycles can be found in Tables 1–4.
Table 1. Loopless diagrams by the number $n$ of $K_3$. 

| $n$ | Linear, $a_n^{(3)}$ | Chord labeled, $b_n^{(3)}$ | Unlabeled, $\tilde{b}_n^{(3)}$ | Unlabeled, $\overline{b}_n^{(3)}$ |
|-----|---------------------|---------------------------|-------------------|-------------------|
| 1   | 1                   | 1                         | 0                 | 0                 |
| 2   | 242024155203489931  | 35365193760349696135      | 2031103059452895  | 2484801856062817 |
| 3   | 873480201569580280  | 12318355186526416031     | 6357187620745286  | 8081714854841484 |
| 4   | 189565884864690704  | 2163220895025390670       | 1109202680936560116 | 1413207046096804 |
| 5   | 4420321081         | 3980871156                | 1895655889481099  | 21577786374       |
| 6   | 1133879136649      | 1035707510307             | 473263816879517    | 5457187460105824  |
| 7   | 372419001449076    | 343866839138005           | 1273580886689917   | 168701814516501   |
| 8   | 142371487909124891 | 13898635038345978         | 473263816879517    | 5457187460105824  |
| 9   | 457212708009124151 | 4295434200061293456      | 1109202680936560116 | 1413207046096804 |
| 10  | 152466248712342181 | 1413207046096804      | 473263816879517    | 5457187460105824  |
| 11  | 76134462292157828285 | 7163220895025390670     | 1895655889481099  | 21577786374       |
| 12  | 457212708009124151 | 4295434200061293456      | 1109202680936560116 | 1413207046096804 |
| 13  | 32173412855800049 | 293618056206900486501   | 1109202680936560116 | 1413207046096804 |

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| $n$ | Linear, $a_{4n}^{(4)}$ | Chord labeled, $b_{4n}^{(4)}$ | Unlabeled, $b_{n}^{(4)}$ | Unlabeled, $b_{4n}^{(4)}$ |
|-----|----------------------|------------------------|----------------|---------------------|
| 1   | 0                    | 0                      | 0              | 0                   |
| 2   | 1                    | 1                      | 1              | 1                   |
| 3   | 182                  | 134                    | 15             | 13                  |
| 4   | 94376                | 75843                  | 4790           | 2576                |
| 5   | 98371884             | 83002866               | 4151415        | 2081393             |
| 6   | 182502973885         | 158861646466           | 6619291247     | 3309962320          |
| 7   | 551248366509999      | 4902945332924         | 1751651893528  | 875527727334        |
| 8   | 253682368373613858   | 2292204611710892971   | 71631394311300461 | 3581569861383466   |
| 9   | 1690430114210704364659 | 15459367618357013402267 | 429426878302882412435 | 214713439275724149414 |
| 10  | 156690501089429126239232946 | 144663877588996810362218074 | 3616596939726424941979785 | 1808298469877117320495867 |
| $n$ | Linear, $a_n^{(5)}$ | Chord labeled, $b_n^{(5)}$ | Unlabeled, $\tilde{b}_n^{(5)}$ | Unlabeled, $\tilde{b}_n^{(5)}$ |
|-----|-------------------|-------------------|-------------------|-------------------|
| 1   | 0                 | 0                 | 0                 | 0                 |
| 2   | 1                 | 1                 | 1                 | 1                 |
| 3   | 1198              | 866               | 60               | 42               |
| 4   | 5609649           | 4446741           | 222477           | 112418           |
| 5   | 66218360625       | 55279816356       | 2211192688       | 1105696796       |
| 6   | 1681287695542855  | 1450728060971387  | 48357603758012   | 24178822553773   |
| 7   | 81644850343968535401 | 72078730629785795963 | 2059392303708166507 | 1029696155560021174 |
| 8   | 69452221450215084802489929 | 623504815522509308061949 | 1558762038801414144480 | 77938101941693076258854 |
Table 4. Loopless diagrams by the number \( n \) of \( K_6 \).

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