Hyperspherical Trigonometry and Corresponding Elliptic Functions

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We develop the basic formulae of hyperspherical trigonometry in multidimensional Euclidean space, using multidimensional vector products, and their conversion to identities for elliptic functions. We show that the basic addition formulae for functions on the 3-sphere embedded in 4-dimensional space lead to addition formulae for elliptic functions, associated with algebraic curves, which have two distinct moduli. We give an application of these formulae to the cases of a multidimensional Euler top, using them to provide a link to the Double Elliptic model.
I. INTRODUCTION

Spherical trigonometry, a branch of geometry dealing with the goniometry of the angles of triangles confined to the surface of a 2-sphere, is an area of mathematics that has existed since the ancient Greeks, with its foundations laid by Menelaus and Hipparchus. It is of vital importance for many calculations in astronomy, navigation and cartography. Further advances made in the Islamic world, in order to help calculate its Holy Days based on the phases of the moon, resulted in giving us the basis of spherical trigonometry in its modern form. In the western world in the 17th century spherical trigonometry was first considered as a separate mathematical discipline, independent of astronomy. One of the protagonists of this era, John Napier, in his work of 1614, treated spherical trigonometry alongside his work introducing logarithms. Other protagonists include Delambre, Euler, Cagnoli and l’Huillier.

For obvious reasons the trigonometry of hyperspheres in higher dimensions was not studied so intently. McMahon produced a number of formulae as generalisations of some of those from spherical trigonometry. More recently, Sato considered the relationship between the dihedral angles of spherical simplices and those of their polars. There also exists a significant amount of work looking into the ‘Law of Sines’, generalisations of the sine rule from the spherical case. Various formulae have also arisen in the the work of Derevnin, Mednykh and Pashkevich in their work on spherical volumes. Recently, it has been shown by Petrera and Suris the the cosine rule for spherical triangles and tetrahedra define integrable systems.

It is well known that elliptic functions are related to spherical trigonometry through their addition formulae. These functions are commonly defined through the inversion of integrals, and they consequently obey differential equations associated with certain algebraic curves. A great deal of research took place in this area in the 19th century, comprising works by many of the great mathematicians, including Euler, Jacobi, Legendre and Frobenius. As for higher dimensional hyperspherical trigonometry no such connection is known. It may be expected that this link would be through higher genus elliptic curves, for example Abelian functions. However, this is not in fact the case. We show that instead the link is through the ‘Generalised Jacobi Elliptic Functions’ explicitly defined by Pawellek, and building upon Jacobi’s work in this area, as an elliptic covering, dependent on two distinct moduli.

In this paper we develop a complete set of formulae for hyperspherical trigonometry,
and explore their link with elliptic functions. We establish a novel connection between the
generalised Jacobi elliptic functions and the formulae of hyperspherical trigonometry. We
show that through this connection the basic addition formulae of hyperspherical trigonometry
lead to addition formulae for these generalised Jacobi elliptic functions. We apply this
connection to an example, the four-dimensional Euler top. We then show that this system
is equivalent to the one-particle Double Elliptic model.

Our interest in this subject area arises in relation to the tetrahedron equation, a higher
dimensional generalisation of the Yang-Baxter equation.\(^3\) Zamolodchikov presented an in-
tuitive solution to the latter dependent on spherical trigonometry, cf. also\(^6\). It is therefore
natural for higher generalisations to be solved in terms of higher dimensional hyperspherical
trigonometry.

The outline of this paper is as follows. In section II we review multidimensional vector
products as a higher dimensional analogue of the standard cross-product of vectors, which
we need to obtain the formulae of hyperspherical trigonometry. In section III we provide
a summary of the well-known formulae of spherical trigonometry. We deduce analogous
formulae for the 4 dimensional hyperspherical case, and using the same principles, do the
same for the general \(n\)-dimensional case. In section IV we review the link between spheri-
cal trigonometry and the Jacobi elliptic functions, and generalise this to the link between
four-dimensional hyperspherical trigonometry and elliptic functions. Section V is a brief
introduction to the generalised Jacobi elliptic functions,\(^2\) complete with a link between the
formulae of hyperspherical trigonometry and these functions. We conclude with two ex-
amples, the first being the four-dimensional Euler top, and the second the double elliptic
(DELL) model, providing a connection between the two.

**Note:** This paper summarises some collaborative results which appeared in the PhD thesis
of one of the authors\(^1\).

II. MULTIDIMENSIONAL VECTOR PRODUCTS

A. Definition and Relations

The formulae involved in spherical trigonometry are dependent on the cross product
between vectors. The vector product \(\mathbf{a} \times \mathbf{b}\) in 3D Euclidean space is a binary operation
defined by

\[(a \times b)_i = \text{det}(a, b, e_i), \quad (i = 1, 2, 3), \quad (1)\]

with \(e_1, \ e_2, \ e_3\) the standard unit vectors in the orthogonal basis. This product obeys the rules:

- **Anti-commutative:** \(a \times b = -b \times a\).

- **Vector Triple Product:** \((a \times b) \times c = (a \cdot c)b - (b \cdot c)a = -\begin{vmatrix} a & b \\ a \cdot c & b \cdot c \end{vmatrix}\), from which it follows \((a \times b) \times c - (a \times c) \times b = a \times (b \times c)\).

- **Area of a Parallelogram:** The modulus of the vector product, \(|a \times b|\), is equivalent to the area of the parallelogram defined by these vectors, \(|a \times b| = \sin \theta\), with \(\theta\) the obtuse angle between them \((0 \leq \theta \leq \pi)\).

For higher dimensional spherical trigonometry an \(n\)-ary operation between \(n\) vectors is required. A natural vector product in four dimensions will therefore be a ternary vector product, \(a \times b \times c\), of three vectors in 4D Euclidean space \(E_4\), defined in a similar manner, by

\[(a \times b \times c) \cdot d = \text{det}(a, b, c, d), \quad \text{for all vectors } d \in E_4. \quad (2)\]

More generally, this could be extended to an \(n\)-dimensional vector product as an \(n\)-ary operation of \(n-1\) vectors in \(E_n\), defined by the expression \(^{14}\) \(^{18}\)

\[(a_1 \times a_2 \times \cdots \times a_{n-1}) \cdot a_n = \text{det}(a_1, a_2, \ldots, a_{n-1}, a_n). \quad (3)\]

It clearly follows from this definition that the multiple vector products are antisymmetric with respect to the interchangement of their constituent vectors. Furthermore, these multiple vector products are perpendicular to any one of their constituent vectors. In fact, the multidimensional vector product follows directly from the wedge product of the exterior algebra \(^{18}\).

The following nested product identity involving five vectors holds for the triple vector product in \(E_4 = \mathbb{R}^4\),

\[(a \times b \times c) \times d \times e = -\begin{vmatrix} a & b & c \\ a \cdot d & b \cdot d & (c \cdot d) \\ a \cdot e & b \cdot e & (c \cdot e) \end{vmatrix}. \quad (4)\]
This follows from the more general higher-dimensional analogue.

**Proposition 1.** (Nested Vector Product Identity). For 2\(n\)-1 vectors, \(a_i \in \mathbb{R}^{n+1}, i = 1, \ldots, 2n - 1\), we have the identity

\[
(a_1 \times a_2 \times \cdots \times a_n) \times a_{n+1} \times \cdots \times a_{2n-1} = - \begin{vmatrix}
  a_1 & a_2 & \cdots & a_n \\
  (a_1 \cdot a_{n+1}) & (a_2 \cdot a_{n+1}) & \cdots & (a_n \cdot a_{n+1}) \\
  \vdots & \vdots & \ddots & \vdots \\
  (a_1 \cdot a_{2n-1}) & (a_2 \cdot a_{2n-1}) & \cdots & (a_n \cdot a_{2n-1})
\end{vmatrix}.
\tag{5}
\]

**Proof.** Consider the determinantal expression

\[
\det(a_1, \ldots, a_{n+1}) = \sum_{i_1} \cdots \sum_{i_{n+1}} \varepsilon_{i_1 \cdots i_{n+1}} (a_1)_{i_1} \cdots (a_{n+1})_{i_{n+1}},
\tag{6}
\]

where \(\varepsilon_{i_1 \cdots i_{n+1}}\) is the \((n+1)\)-dimensional Levi-Civita symbol. From this it follows that

\[
[(a_1 \times a_2 \times \cdots \times a_n) \times a_{n+1} \times \cdots \times a_{2n-1}]_{j_n}
= \sum_{i_1} \cdots \sum_{i_{n+1}} \sum_{j_1} \cdots \sum_{j_{n+1}} \varepsilon_{j_1 i_1 \cdots i_{n+1}} \varepsilon_{j_1 \cdots j_{n+1}} (a_1)_{i_1} \cdots (a_n)_{i_n} (a_{n+1})_{j_1} \cdots (a_{2n-1})_{j_n}.
\tag{7}
\]

Noting that the Levi-Civita symbol satisfies the following product rule

\[
\sum_{j_1} \varepsilon_{j_1 i_1 \cdots i_n} \varepsilon_{j_1 \cdots j_{n+1}} = \begin{vmatrix}
  \delta_{i_1 j_2} & \cdots & \delta_{i_1 j_{n+1}} \\
  \vdots & \ddots & \vdots \\
  \delta_{i_n j_2} & \cdots & \delta_{i_n j_{n+1}}
\end{vmatrix},
\tag{8}
\]

the result follows. \(\square\)

Vectorial addition identities follow from the Plücker relations in projective geometry. The Plücker relations are identities involving minors of non-square matrices which are the Plücker coordinates of corresponding Grassmannians.

**Proposition 2** (Plücker Relations). For \((2n - 2)\) vectors \(a_1, a_2, \ldots, a_{2n-2} \in \mathbb{R}^{n-1},\)

\[
(a_1, a_{n+1}, \ldots, a_{2n-2})(a_2, \ldots, a_n) - (a_2, a_{n+1}, \ldots, a_{2n-2})(a_1, a_3, \ldots, a_n) + \cdots + (-1)^{n-1}(a_n, a_{n+1}, \ldots, a_{2n-2})(a_1, \ldots, a_{n-1}) = 0.
\tag{9}
\]

**Proof.** Consider \(n\) vectors, \(a_1, a_2, \ldots, a_n\) in \(\mathbb{R}^{n-1}\). As there are \(n\) vectors in \((n-1)\) dimensional space they must be linearly dependent, hence,

\[
\begin{vmatrix}
  (a_1) & (a_2) & \cdots & (a_n) \\
  a_1 & a_2 & \cdots & a_n
\end{vmatrix} = 0,
\tag{10}
\]
where \((a_i)\) represents the column vector of the components of \(a_i, i \in \{1,2,\ldots,n\}\). This implies
\[
\mathbf{a}_1(\mathbf{a}_2, \ldots, \mathbf{a}_n) - \mathbf{a}_2(\mathbf{a}_1, \mathbf{a}_3, \ldots, \mathbf{a}_n) + \cdots + (-1)^{n-1}\mathbf{a}_n(\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}) = 0, \tag{11}
\]
which, in turn gives
\[
(\mathbf{a}_1, \mathbf{a}_{n+1}, \ldots, \mathbf{a}_{2n-2})(\mathbf{a}_2, \ldots, \mathbf{a}_n) - (\mathbf{a}_2, \mathbf{a}_{n+1}, \ldots, \mathbf{a}_{2n-2})(\mathbf{a}_1, \mathbf{a}_3, \ldots, \mathbf{a}_n)
+ \cdots + (-1)^{n-1}(\mathbf{a}_n, \mathbf{a}_{n+1}, \ldots, \mathbf{a}_{2n-2})(\mathbf{a}_1, \ldots, \mathbf{a}_{n-1}) = 0, \tag{12}
\]
for some arbitrary \(\mathbf{a}_{n+1}, \ldots, \mathbf{a}_{2n-2} \in \mathbb{R}^{n-1}\), as required. \(\square\)

**Corollary 1.** For vectors \(\mathbf{a}_1, \ldots, \mathbf{a}_{2n-2} \in \mathbb{R}^{n-1}\),
\[
(\mathbf{a}_1 \times \cdots \times \mathbf{a}_n) \times \mathbf{a}_{n+1} \times \cdots \times \mathbf{a}_{2n-1}
+ \mathbf{a}_n \times (\mathbf{a}_1 \times \cdots \times \mathbf{a}_{n-1} \times \mathbf{a}_{n+1}) \times \mathbf{a}_{n+2} \times \cdots \times \mathbf{a}_{2n-1} + \cdots
+ \mathbf{a}_n \times \cdots \times \mathbf{a}_{2n-2} \times (\mathbf{a}_1 \times \cdots \times \mathbf{a}_{n-1} \times \mathbf{a}_{2n-1})
= \mathbf{a}_1 \times \cdots \times \mathbf{a}_{n-1} \times (\mathbf{a}_n \times \cdots \times \mathbf{a}_{2n-1}). \tag{13}
\]

**Proof.** From the Plücker relations, \([9]\), set \(\mathbf{a}_{2n-2} = \mathbf{a}_n = e_i\) and sum over \(i\). This implies
\[
(\mathbf{a}_1 \times \mathbf{a}_{n+1} \times \cdots \times \mathbf{a}_{2n-3}) \cdot (\mathbf{a}_2 \times \cdots \times \mathbf{a}_{n-1})
- (\mathbf{a}_2 \times \mathbf{a}_{n+1} \times \cdots \times \mathbf{a}_{2n-3}) \cdot (\mathbf{a}_1 \times \mathbf{a}_3 \cdots \times \mathbf{a}_{n-1})
+ \cdots + (-1)^n(\mathbf{a}_{n-1} \times \mathbf{a}_{n+1} \times \cdots \times \mathbf{a}_{2n-3}) \cdot (\mathbf{a}_1 \times \cdots \times \mathbf{a}_{n-2})
= 0, \tag{14}
\]
which, using \((\mathbf{a}_1 \times \mathbf{a}_2 \times \ldots \mathbf{a}_{n-2}) \cdot \mathbf{a}_{n-1} = -(\mathbf{a}_2 \times \cdots \times \mathbf{a}_{n-1}) \cdot \mathbf{a}_1\), implies
\[
\mathbf{a}_1 \times \mathbf{a}_{n+1} \times \cdots \times \mathbf{a}_{2n-4} \times (\mathbf{a}_2 \times \cdots \times \mathbf{a}_{n-1})
- \mathbf{a}_2 \times \mathbf{a}_{n+1} \times \cdots \times \mathbf{a}_{2n-4} \times (\mathbf{a}_1 \times \mathbf{a}_3 \cdots \times \mathbf{a}_{n-1})
+ \cdots + (-1)^n\mathbf{a}_{n-1} \times \mathbf{a}_{n+1} \times \cdots \times \mathbf{a}_{2n-4} \times (\mathbf{a}_1 \times \cdots \times \mathbf{a}_{n-2})
= 0. \tag{15}
\]
Summing over \((n - 2)\) copies, the result follows. \(\square\)

**Corollary 2.** More specifically, for \(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e} \in \mathbb{R}^4\),
\[
(\mathbf{a} \times \mathbf{b} \times \mathbf{c}) \times \mathbf{d} \times \mathbf{e} + (\mathbf{a} \times \mathbf{b} \times \mathbf{d}) \times \mathbf{e} \times \mathbf{c} + (\mathbf{a} \times \mathbf{b} \times \mathbf{e}) \times \mathbf{c} \times \mathbf{d} = \mathbf{a} \times \mathbf{b} \times (\mathbf{c} \times \mathbf{d} \times \mathbf{e}). \tag{16}
\]

This identity will be particularly important in proving the various hyperspherical identities in the four dimensional case.
III. HYPERSPHHERICAL TRIGONOMETRY

In this section we review the formulae for spherical trigonometry as a preparation for developing similar formulae in higher dimensions. The approach we take is by exploiting the higher dimensional vector product introduced in section II.

A. Spherical Trigonometry

We review the derivation of the basic formulae of spherical trigonometry, cf. also\cite{37}. Consider a big spherical triangle on the surface of a 2-sphere of unit radius embedded in three dimensional Euclidean space, that is a triangle bounded by three big circles on $S^2$, restricted such that each side is less than a semicircle, and hence, each angle is less than $\pi$. Take the centre of the sphere to be the origin in $E_3 = \mathbb{R}^3$ and denote the position vectors of the three vertices of the spherical triangle by $\mathbf{n}_1, \mathbf{n}_2$ and $\mathbf{n}_3$, with the angles, $\theta_{ij}$, between them, corresponding to the edges, being defined by

$$\mathbf{n}_i \cdot \mathbf{n}_j \equiv \cos \theta_{ij}, \quad i, j = 1, 2, 3. \quad (17)$$

Introduce the vectors

$$\mathbf{u}_{ij} \equiv \frac{\mathbf{n}_i \times \mathbf{n}_j}{|\mathbf{n}_i \times \mathbf{n}_j|}, \quad (18)$$

and define the spherical angles, $\alpha_j$, between them by

$$\mathbf{u}_{ij} \cdot \mathbf{u}_{jk} \equiv -\cos \alpha_j. \quad (19)$$

**Theorem 1** (Spherical Excess). The area, $A$, of a spherical triangle is given by its Spherical Excess,

$$A = \alpha_i + \alpha_j + \alpha_k - \pi. \quad (20)$$

*Proof.* Schlafli’s Differential Volume formula states that the area of a spherical triangle must satisfy

$$\mathrm{d}A = \sum_{i=1}^{3} \mathrm{d}\alpha_i. \quad (21)$$

Integrating this gives

$$A = \alpha_i + \alpha_j + \alpha_k + c, \quad (22)$$
where \( c = -\pi \) is the integration constant determined by considering, say, the area of a spherical triangle with planar angles \( \alpha_i = \alpha_j = \alpha_k = \frac{\pi}{2} \) and comparing this to the known area of \( 1/8 \) of a sphere.

This theorem may also be proven by considering the areas of intersecting lunars.

**Definition 1** (Polar Triangle). *The spherical triangle defined by the vectors \( u_{ij}, u_{ik}, u_{jk} \) is called the polar of the spherical triangle defined by the vectors \( n_i, n_j, n_k \).*

By considering various relations for the scalar and vector products between the polar vectors \( u_{ij} \), we can define a number of important relations between the angles of a spherical triangle.

**Proposition 3** (Cosine Rule).

\[
\cos \alpha_j = \frac{\cos \theta_{ik} - \cos \theta_{ij} \cos \theta_{jk}}{\sin \theta_{ij} \sin \theta_{jk}}, \quad (23)
\]

for all \( i, j, k = 1, 2, 3 \).
Proof. Consider the scalar product,
\[ u_{ij} \cdot u_{jk} = \frac{(n_i \times n_j) \cdot (n_j \times n_k)}{|n_i \times n_j| |n_j \times n_k|}, \]
\[ = \frac{\cos \theta_{ij} \cos \theta_{jk} - \cos \theta_{ik}}{\sin \theta_{ij} \sin \theta_{jk}}. \]  
(24)

Hence, the Cosine Rule follows. \( \square \)

Definition 2. The \textit{generalised sine function of three variables}, \( \sin(\theta_{ij}, \theta_{jk}, \theta_{ik}) \), is defined by
\[ \sin(\theta_{ij}, \theta_{jk}, \theta_{ik}) = \sqrt{1 - \cos^2 \theta_{ij} - \cos^2 \theta_{jk} - \cos^2 \theta_{ik} + 2 \cos \theta_{ij} \cos \theta_{jk} \cos \theta_{ik}}, \]
\[ = \begin{vmatrix} 1 & \cos \theta_{ij} & \cos \theta_{ik} \\ \cos \theta_{ij} & 1 & \cos \theta_{jk} \\ \cos \theta_{ik} & \cos \theta_{jk} & 1 \end{vmatrix}^{\frac{1}{2}}, \quad (0 \leq \theta_{ij}, \theta_{ik}, \theta_{jk} \leq \pi). \]  
(25)

Note that the restrictions for a spherical triangle, that each side is less than a semi-circle and each angle less than \( \pi \), ensure that the generalised sine function is always real and positive.

This sine function is equivalent to the volume of a parallelepiped with edges \( n_i, n_j \) and \( n_k \), given by
\[ \text{Volume} = |n_i \cdot (n_j \times n_k)| = \sin(\theta_{ij}, \theta_{ik}, \theta_{jk}). \]  
(26)

This identity can be viewed of as a consequence of the triple product when we embed the 3D vectors \( n_i, n_j \) and \( n_k \) in \( E_4 \) by
\[ \tilde{n}_i = \begin{pmatrix} n_i \\ 0 \end{pmatrix}, \quad \tilde{n}_j = \begin{pmatrix} n_j \\ 0 \end{pmatrix}, \quad \tilde{n}_k = \begin{pmatrix} n_k \\ 0 \end{pmatrix}, \]
(27)

such that
\[ \tilde{n}_i \times \tilde{n}_j \times \tilde{n}_k = \begin{pmatrix} 0 \\ 0 \\ \det(n_i, n_j, n_k) \end{pmatrix}, \]  
(28)

where the last entry equals \( |n_i \cdot (n_j \times n_k)| \).

Proposition 4 (Sine Rule).
\[ \frac{\sin \alpha_i}{\sin \theta_{ij}} = \frac{\sin \alpha_j}{\sin \theta_{jk}} = \frac{\sin \alpha_k}{\sin \theta_{ik}} = \frac{\sin(\theta_{ij}, \theta_{jk}, \theta_{ik})}{\sin \theta_{ij} \sin \theta_{jk} \sin \theta_{ik}} = k, \]
(29)

where \( k \) is a constant.
Proof. Consider the ratio
\[
\frac{|\mathbf{u}_{ij} \times \mathbf{u}_{jk}|}{|\mathbf{n}_i \times \mathbf{n}_k|} = \frac{\sin \alpha_j}{\sin \theta_{ik}}. \tag{30}
\]
Now, note the vector product
\[
\mathbf{u}_{ij} \times \mathbf{u}_{jk} = \left(\frac{\mathbf{n}_i \times \mathbf{n}_j}{|\mathbf{n}_i \times \mathbf{n}_j|}\right) \left(\frac{\mathbf{n}_j \times \mathbf{n}_k}{|\mathbf{n}_j \times \mathbf{n}_k|}\right), \tag{31}
\]
This implies
\[
|\mathbf{u}_{ij} \times \mathbf{u}_{jk}| = \frac{|\mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k)|}{\sin \theta_{ij} \sin \theta_{jk}} = \frac{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})}{\sin \theta_{ij} \sin \theta_{jk}}, \tag{32}
\]
and hence, the sine rule follows.

**Proposition 5** (Polar Cosine Rule).

\[
\cos \theta_{jk} = \frac{\cos \alpha_j \cos \alpha_k + \cos \alpha_i}{\sin \alpha_j \sin \alpha_k}, \tag{33}
\]
for all \(i, j, k = 1, 2, 3\).

Proof. Consider the product
\[
\frac{(\mathbf{u}_{ij} \times \mathbf{u}_{jk}) \cdot (\mathbf{u}_{jk} \times \mathbf{u}_{ik})}{|\mathbf{u}_{ij} \times \mathbf{u}_{jk}| |\mathbf{u}_{jk} \times \mathbf{u}_{ik}|}. \tag{34}
\]
This product can be calculated in two ways. First,
\[
\frac{(\mathbf{u}_{ij} \times \mathbf{u}_{jk}) \cdot (\mathbf{u}_{jk} \times \mathbf{u}_{ik})}{|\mathbf{u}_{ij} \times \mathbf{u}_{jk}| |\mathbf{u}_{jk} \times \mathbf{u}_{ik}|} = -\frac{\cos \alpha_j \cos \alpha_k - \cos \alpha_i}{\sin \alpha_j \sin \alpha_k}, \tag{35}
\]
and second, using (31),
\[
\frac{(\mathbf{u}_{ij} \times \mathbf{u}_{jk}) \cdot (\mathbf{u}_{jk} \times \mathbf{u}_{ik})}{|\mathbf{u}_{ij} \times \mathbf{u}_{jk}| |\mathbf{u}_{jk} \times \mathbf{u}_{ik}|} = \frac{1}{\sin \alpha_j \sin \alpha_k} \left(\frac{\mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k)}{\sin \theta_{ij} \sin \theta_{jk}} \mathbf{n}_j \right) \cdot \left(\frac{\mathbf{n}_i \cdot (\mathbf{n}_k \times \mathbf{n}_j)}{\sin \theta_{ik} \sin \theta_{jk}} \mathbf{n}_k \right), \tag{36}
\]
Reducing this using the Sine Rule and equating with the previous result gives the Polar Cosine Rule. \qed

Note that, by now considering
\[
k^2 = \frac{\sin^2 \alpha_i}{\sin^2 \theta_{jk}} = \frac{\sin^2 \alpha_i}{1 - \cos^2 \theta_{jk}}, \tag{37}
\]
and substituting in the polar cosine rule,
\[
k^2 = \frac{\sin^2 \alpha_i}{1 - \left(\frac{\cos \alpha_i + \cos \alpha_j \cos \alpha_k}{\sin \alpha_j \sin \alpha_k}\right)^2} = \frac{\sin^2 \alpha_i \sin^2 \alpha_j \sin^2 \alpha_k}{1 - \cos^2 \alpha_i - \cos^2 \alpha_j - \cos^2 \alpha_k - 2 \cos \alpha_i \cos \alpha_j \cos \alpha_k}, \tag{38}
\]
the constant $k$ may also be written in terms of the spherical angles, so that the sine rule now becomes

\[
\frac{\sin \alpha_i}{\sin \theta_{jk}} = \frac{\sin \alpha_j}{\sin \theta_{ik}} = \frac{\sin \alpha_k}{\sin \theta_{ij}} = \sqrt{1 - \cos^2 \alpha_i - \cos^2 \alpha_j - \cos^2 \alpha_k - 2 \cos \alpha_i \cos \alpha_j \cos \alpha_k}. \tag{39}
\]

We will use both forms of the sine rule later when discussing the link between spherical trigonometry and the Jacobi elliptic functions.

B. Hyperspherical Trigonometry in 4-Dimensional Euclidean Space

We now extend these principles to the four dimensional hyperspherical case. Consider a 3-sphere embedded in four dimensional Euclidean space, $E_4 = \mathbb{R}^4$. In this case, we have four unit vectors, $\mathbf{n}_i, i = 1, \ldots, 4$ pointing to the four vertices of a hyperspherical tetrahedron.

FIG. 2. A hyperspherical tetrahedron

We now extend these principles to the four dimensional hyperspherical case. Consider a 3-sphere embedded in four dimensional Euclidean space, $E_4 = \mathbb{R}^4$. In this case, we have four unit vectors, $\mathbf{n}_i, i = 1, \ldots, 4$ pointing to the four vertices of a hyperspherical tetrahedron.
The angles between these unit vectors, \( \theta_{ij} \), are defined, in the same way as for the spherical case, by
\[
n_i \cdot n_j \equiv \cos \theta_{ij}, \tag{40}
\]
whereas now we must also define orthogonal vectors \( u_{ijk} \) to each hyperplane \([i, j, k]\) (defined by \( n_i, n_j, n_k \)). This is done using the ternary cross product,
\[
u_{ijk} \equiv \frac{n_i \times n_j \times n_k}{|n_i \times n_j \times n_k|}. \tag{41}\]
The four vectors \( u_{123}, u_{124}, u_{134}, u_{234} \) define the polar of the hyperspherical tetrahedron, between which we define dihedral angles \( \phi_{jk} \) by
\[
u_{ijk} \cdot u_{jkl} \equiv -\cos \phi_{jk}. \tag{42}\]
We also have
\[
|n_i \times n_j \times n_k| = \sin(\theta_{ij}, \theta_{ik}, \theta_{ik}), \tag{43}\]
and together with
\[
|u_{ijk} \times u_{jkl} \times u_{kli}| = \sin(\phi_{jk}, \phi_{ik}, \phi_{kl}). \tag{44}\]

The four faces of the hyperspherical tetrahedron are spherical triangles, with the various sine and cosine rules for the spherical case still holding true for these. For ease of notation, we now label the angles in the faces of the spherical triangles \( \alpha_{i}^{(ijk)} \) to indicate which triangle is being considered.

In addition to the spherical relations, there are now various relations involving the dihedral angles, \( \phi_{ij} \), which connect the various faces of the hyperspherical tetrahedron. By taking various relations between the scalar and vector products for the vectors \( u_{ijk} \), we can again derive a number of relations.

**Proposition 6 (Cosine Rule).**
\[
\cos \phi_{jk} = \frac{\cos \theta_{ij} \cos \theta_{ik} \cos \theta_{il}}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk}) \sin(\theta_{jk}, \theta_{jl}, \theta_{kl})}, \tag{45}\]
for all \( i, j, k, l = 1, 2, 3, 4 \).
Proof. From the computation

\[(n_i \times n_j \times n_k) \cdot (n_j \times n_k \times n_l) = -(n_j \times n_k \times (n_j \times n_k \times n_l))n_i,\]

\[= \begin{vmatrix} n_j & n_k & n_l \\ n_j \cdot n_j & n_j \cdot n_k & n_j \cdot n_l \\ n_k \cdot n_j & n_k \cdot n_k & n_k \cdot n_l \end{vmatrix} \cdot n_i, \quad (46)\]

\[= \begin{vmatrix} n_i \cdot n_j & n_i \cdot n_k & n_i \cdot n_l \\ n_j \cdot n_j & n_j \cdot n_k & n_j \cdot n_l \\ n_k \cdot n_j & n_k \cdot n_k & n_k \cdot n_l \end{vmatrix},\]

we find that for the following inner product (for all \(i, j, k, l = 1, 2, 3, 4\)) we have

\[u_{ijk} \cdot u_{jkl} = \frac{(n_i \times n_j \times n_k) \cdot (n_j \times n_k \times n_l)}{|n_i \times n_j \times n_k| |n_j \times n_k \times n_l|}, \quad (47)\]

\[= \frac{\sin(\theta_{ij}, \theta_{ik}, \theta_{il}) \sin(\theta_{jk}, \theta_{jl}, \theta_{kl})}{\sin(\theta_{ij}, \theta_{ik}, \theta_{il}) \sin(\theta_{jk}, \theta_{jl}, \theta_{kl})}.\]

Hence, the Cosine Rule follows. \(\square\)

We now need to extend to definition of the triple sine function for the four-dimensional case, to give a further generalisation of the sine function dependent on six variables.

**Definition 3.** The generalised sine function of six variables, \(\sin(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl})\), is the four dimensional analogue of the triple sine function, and is given by

\[\sin(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl}) = \frac{1}{\sin(\theta_{ij}, \theta_{ik}, \theta_{il}) \sin(\theta_{jk}, \theta_{jl}, \theta_{kl})} \begin{vmatrix} 1 & \cos \theta_{ij} & \cos \theta_{ik} & \cos \theta_{il} \\ \cos \theta_{ij} & 1 & \cos \theta_{jk} & \cos \theta_{jl} \\ \cos \theta_{ik} & \cos \theta_{jk} & 1 & \cos \theta_{kl} \\ \cos \theta_{il} & \cos \theta_{jl} & \cos \theta_{kl} & 1 \end{vmatrix}, \quad (48)\]

\[0 \leq \theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl} \leq \pi. \quad (49)\]

Note that the restrictions on being a spherical tetrahedron ensure that the generalised sine function is always real and positive.
This sine function is equivalent to the volume of a 4D parallelootope with edges \( \mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k \) and \( \mathbf{n}_l \), given by

\[
\text{Volume} = |\mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l)| = \sin(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jkl}). \tag{50}
\]

This relation follows from embedding 4D vectors \( \mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k \) and \( \mathbf{n}_l \) into \( E_4 \) in a similar manner to the three-dimensional case.

**Proposition 7** (Sine Rule).

\[
\frac{\sin(\phi_{ij}, \phi_{ik}, \phi_{il})}{\sin(\theta_{jk}, \theta_{jl}, \theta_{kl})} = \frac{\sin(\phi_{ij}, \phi_{jk}, \phi_{jl})}{\sin(\theta_{ik}, \theta_{il}, \theta_{kl})} = \frac{\sin(\phi_{ik}, \phi_{jk}, \phi_{kl})}{\sin(\theta_{ij}, \theta_{il}, \theta_{kl})},
\]

\[
= \frac{\sin^2(\theta_{ij}, \theta_{ik}, \theta_{jl}, \theta_{kl})}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk}, \theta_{jl}, \theta_{kl})} = k_H,
\]

where \( k_H \) is constant.

**Proof.** Consider the triple product

\[
\mathbf{u}_{ijk} \times \mathbf{u}_{jkl} \times \mathbf{u}_{kli} = \frac{(\mathbf{n}_i \times \mathbf{n}_j \times \mathbf{n}_k) \times (\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l) \times (\mathbf{n}_k \times \mathbf{n}_l \times \mathbf{n}_i)}{|\mathbf{n}_i \times \mathbf{n}_j \times \mathbf{n}_k||\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l||\mathbf{n}_k \times \mathbf{n}_l \times \mathbf{n}_i|},
\]

\[
= \mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l) \quad 0 \quad 0
\]

\[
- \quad \mathbf{n}_j \cdot (\mathbf{n}_k \times \mathbf{n}_l \times \mathbf{n}_i) \quad 0 \quad 0
\]

\[
= \sin(\theta_{ij}, \theta_{ik}, \theta_{jk}, \theta_{jl}, \theta_{kl}) \sin(\theta_{jl}, \theta_{kl}, \theta_{li}) \sin(\theta_{kl}, \theta_{li}, \theta_{ij}) \sin(\theta_{ij}, \theta_{il}, \theta_{il}),
\]

\[
= \frac{\sin^2(\theta_{ij}, \theta_{ik}, \theta_{jl}, \theta_{kl})}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk}, \theta_{jl}, \theta_{kl})} \mathbf{n}_i.
\]

This implies

\[
|\mathbf{u}_{ijk} \times \mathbf{u}_{jkl} \times \mathbf{u}_{kli}| = \frac{(\mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l))^2}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk}, \theta_{jl}, \theta_{kl}) \sin(\theta_{jl}, \theta_{kl}, \theta_{li}) \sin(\theta_{kl}, \theta_{li}, \theta_{ij}) \sin(\theta_{ij}, \theta_{il}, \theta_{il})},
\]

\[
= \frac{\sin^2(\theta_{ij}, \theta_{ik}, \theta_{jl}, \theta_{kl})}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk}, \theta_{jl}, \theta_{kl})} \mathbf{n}_i.
\]

From this we have that the ratio

\[
\frac{|\mathbf{u}_{ijk} \times \mathbf{u}_{jkl} \times \mathbf{u}_{kli}|}{|\mathbf{n}_i \times \mathbf{n}_j \times \mathbf{n}_l|} = \frac{\sin(\phi_{ik}, \phi_{jk}, \phi_{kl})}{\sin(\theta_{ij}, \theta_{il}, \theta_{jl})},
\]

is symmetric with respect to the interchange of the labels \( i, j, k, l \). Thus, it follows that

\[
\frac{\sin(\phi_{ik}, \phi_{jk}, \phi_{kl})}{\sin(\theta_{ij}, \theta_{il}, \theta_{jl})} = \frac{\sin^2(\theta_{ij}, \theta_{ik}, \theta_{jl}, \theta_{kl})}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk}, \theta_{jl}, \theta_{kl}) \sin(\theta_{jl}, \theta_{kl}, \theta_{li}) \sin(\theta_{kl}, \theta_{li}, \theta_{ij}) \sin(\theta_{ij}, \theta_{il}, \theta_{il})},
\]

and hence, the hyperspherical sine rule follows. \( \square \)
There also exists a simpler sine relationship between the standard sine functions of the central angles and the sine of the dihedral angles.

**Proposition 8.**

\[
\sin \phi_{kl} = \frac{\sin(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl})}{\sin(\theta_{ik}, \theta_{il}, \theta_{kl}) \sin(\theta_{jk}, \theta_{jl}, \theta_{kl})} \sin \theta_{kl}. \tag{56}
\]

In order to prove this proposition, we require a determinantal identity, attributed to Desnanot for the $n \leq 6$ case\textsuperscript{12}, and to Jacobi in the general case.\textsuperscript{17}

**Theorem 2** (Desnanot-Jacobi Identity). Let $M$ be a $n \times n$ square matrix, and denote by $M_i^p$ the matrix obtained by removing both the $i$-th row and $p$-th column. Similarly, let $M_{ij}^{p,q}$ denote the matrix obtained by deleting the $i$-th and $j$-th rows, and the $p$-th and $q$-th columns, respectively, with $1 \leq i, j, p, q \leq n$. Then,

\[
\det(M) \det(M_{i,n}^{1,n}) = \det(M_1^1) \det(M_n^n) - \det(M_i^1) \det(M_i^n). \tag{57}
\]

Diagrammatically, this is

\[
\begin{pmatrix}
\times & \vline & \times & \vline & \times
\end{pmatrix}
\begin{pmatrix}
\| & \vline & 1 & \vline & 1 \\
1 & \vline & \cos \theta_{ij} & \vline & \cos \theta_{il} \\
\cos \theta_{ij} & \vline & 1 & \vline & \cos \theta_{jl} \\
\cos \theta_{il} & \vline & \cos \theta_{jl} & \vline & 1 \\
\end{pmatrix}
\begin{pmatrix}
\| & \vline & \cos \theta_{ik} & \vline & \cos \theta_{kl} \\
\cos \theta_{ik} & \vline & 1 & \vline & \cos \theta_{jk} \\
\cos \theta_{jl} & \vline & \cos \theta_{kl} & \vline & 1 \\
\cos \theta_{jl} & \vline & \cos \theta_{kl} & \vline & 1 \\
\end{pmatrix}.
\tag{58}
\]

**Proof.** The result follows directly from the Desnanot-Jacobi determinant identity

\[
\begin{pmatrix}
1 & \cos \theta_{ij} & \cos \theta_{ik} & \cos \theta_{jk} \\
\cos \theta_{ij} & 1 & \cos \theta_{jl} & \cos \theta_{jl} \\
\cos \theta_{ik} & \cos \theta_{ik} & 1 & \cos \theta_{kl} \\
\cos \theta_{il} & \cos \theta_{jl} & \cos \theta_{kl} & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & \cos \theta_{ij} & \cos \theta_{ik} & \cos \theta_{jk} \\
\cos \theta_{ij} & 1 & \cos \theta_{jl} & \cos \theta_{jl} \\
\cos \theta_{ik} & \cos \theta_{ik} & 1 & \cos \theta_{kl} \\
\cos \theta_{il} & \cos \theta_{jl} & \cos \theta_{kl} & 1 \\
\end{pmatrix}
\begin{pmatrix}
1 & \cos \theta_{ik} & \cos \theta_{jk} \\
\cos \theta_{ik} & 1 & \cos \theta_{kl} \\
\cos \theta_{kl} & \cos \theta_{kl} & 1 \\
\end{pmatrix}.
\tag{59}
\]

The result follows directly. \qed
From this proposition, it hence follows that the hyperspherical sine rule may be rewritten as
\[
\frac{\sin \phi_{ij} \sin \phi_{kl}}{\sin \theta_{ij} \sin \theta_{kl}} = \frac{\sin \phi_{ik} \sin \phi_{jl}}{\sin \theta_{ik} \sin \theta_{jl}} = \frac{\sin \phi_{il} \sin \phi_{jk}}{\sin \theta_{il} \sin \theta_{jk}} = k_H. \tag{60}
\]
The constant \(k_H\) may also be neatly expressed in terms of cosines. For this we need the Polar Cosine rule.

**Proposition 9** (Polar Cosine Rule).

\[
\cos \theta_{kl} = \frac{-\cos \phi_{jk} \cos \phi_{ik} - \cos \phi_{ij}}{\sin(\phi_{ik}, \phi_{jk}, \phi_{kl}) \sin(\phi_{il}, \phi_{jl}, \phi_{kl})}, \tag{61}
\]
for all \(i, j, k, l = 1, 2, 3, 4\).

**Proof.** Consider the product
\[
\frac{(u_{ijk} \times u_{jkl} \times u_{kli}) \cdot (u_{jkl} \times u_{kli} \times u_{lji})}{|u_{ijk} \times u_{jkl} \times u_{kli}| |u_{jkl} \times u_{kli} \times u_{lji}|} = \frac{|u_{ijk} \cdot u_{jkl} \cdot u_{kli} \cdot u_{lji}|}{\sin(\phi_{ik}, \phi_{jk}, \phi_{kl}) \sin(\phi_{il}, \phi_{jl}, \phi_{kl})}, \tag{62}
\]
On the other hand, using \((52)\), we have
\[
u_{ijk} \times u_{jkl} \times u_{kli} = \frac{(n_i \cdot (n_j \times n_k \times n_l))^2}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk}) \sin(\theta_{jk}, \theta_{jl}, \theta_{kl}) \sin(\theta_{kl}, \theta_{ki}, \theta_{li})} n_k. \tag{63}
\]
Reducing this using the Sine Rule and equating the two results gives the hyperspherical polar cosine rule. \[\square\]

**Proposition 10.**
\[
\frac{\cos \phi_{ij} \cos \phi_{kl} - \cos \phi_{ik} \cos \phi_{jl}}{\cos \theta_{ij} \cos \theta_{kl} - \cos \theta_{ik} \cos \theta_{jl}} = \frac{\cos \phi_{ik} \cos \phi_{jl} - \cos \phi_{il} \cos \phi_{jk}}{\cos \theta_{ik} \cos \theta_{jl} - \cos \theta_{il} \cos \theta_{jk}}, \tag{64}
\]
\[
= \frac{\cos \phi_{il} \cos \phi_{jk} - \cos \phi_{ij} \cos \phi_{kl}}{\cos \theta_{il} \cos \theta_{jk} - \cos \theta_{ij} \cos \theta_{kl}},
\]
\[
= k_H.
\]
Proof. Consider the numerator

\[ \cos \phi_{ij} \cos \phi_{kl} - \cos \phi_{ik} \cos \phi_{jl}, \]

and rewrite this in terms of the central angles using the cosine rule,

\[ \cos \phi_{ij} \cos \phi_{kl} - \cos \phi_{ik} \cos \phi_{jl} = \]

\[ \frac{\cos \theta_{ik} \cos \theta_{jk} \cos \theta_{kl}}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})} - \frac{\cos \theta_{ij} \cos \theta_{il} \cos \theta_{jl}}{\sin(\theta_{ij}, \theta_{il}, \theta_{jl})} \]

Multiplying this out and factorising, this reduces to

\[ \cos \phi_{ij} \cos \phi_{kl} - \cos \phi_{ik} \cos \phi_{jl} = (\cos \theta_{ij} \cos \theta_{kl} - \cos \theta_{ik} \cos \theta_{jl}) k_H, \]

from which the result follows.

Furthermore, in 4D there are also relations involving the spherical angles \( \alpha_j^{(ij\cdot)} \) at a common vertex, \( n_j \), of the tetrahedron involving the different spherical triangles.

**Proposition 11** (Vertex Cosine Rule).

\[ \cos \phi_{jk} = \frac{\cos \alpha_j^{(ij\cdot)} \cos \alpha_j^{(jkl)} - \cos \alpha_j^{(ijl)}}{\sin \alpha_j^{(ijk)} \sin \alpha_j^{(jkl)}}. \]  

Proof. Expanding out the cosine rule \((45)\) and applying the sine rule \((51)\) to the denominator gives

\[ \cos \phi_{jk} = (\cos \theta_{ij} \cos \theta_{jl} + \cos \theta_{ik} \cos \theta_{kl} - \cos \theta_{ij} \cos \theta_{jk} \cos \theta_{kl} - \cos \theta_{ik} \cos \theta_{jl} \cos \theta_{il} + \cos \theta_{il} \cos^2 \theta_{jk}) \]

\[ /\sin \alpha_j^{(ijk)} \sin \alpha_j^{(jkl)} \sin \theta_{ij} \sin^2 \theta_{jk} \sin \theta_{jl}. \]

This can be factorised as

\[ \cos \phi_{jk} = \frac{(\cos \theta_{ik} - \cos \theta_{ij} \cos \theta_{jk})}{\sin \theta_{ij} \sin \theta_{jk}} \frac{(\cos \theta_{ik} - \cos \theta_{il} \cos \theta_{jl})}{\sin \theta_{ij} \sin \theta_{jl}} - \frac{(\cos \theta_{il} - \cos \theta_{ij} \cos \theta_{kl})}{\sin \theta_{ij} \sin \theta_{jl}} \frac{(\cos \theta_{il} - \cos \theta_{il} \cos \theta_{jl})}{\sin \theta_{ij} \sin \theta_{jl}}. \]

Applying \((28)\) the result follows. \(\square\)
This set of relations imply, in turn, corresponding vertex polar cosine relations, expressing
the spherical angles at a vertex in terms of the dihedral angles as follows,

$$\cos \alpha_{ij}^{(ijl)} = \frac{\cos \phi_{jk} + \cos \phi_{ij} \cos \phi_{jl}}{\sin \phi_{ij} \sin \phi_{jl}},$$

(71)

together with a corresponding sine rule of the form

$$\frac{\sin \phi_{jk}}{\sin \alpha_{ij}^{(ijl)}} = \frac{\sin \phi_{jl}}{\sin \alpha_{ij}^{(jkl)}} = \frac{\sin \phi_{ij}}{\sin \alpha_{ij}^{(ijl)}} = \frac{\sin \left(\alpha_{ij}^{(ijk)}, \alpha_{ij}^{(ijl)}, \alpha_{ij}^{(jkl)}\right)}{\sin \alpha_{ij}^{(ijl)} \sin \alpha_{ij}^{(jkl)}},$$

(72)

for $i, j, k, l = 1, 2, 3, 4$. Note the similarities between these formulae and those of the spherical

case.

We also present a, to our knowledge, new formula which again follows from the Desnanot-
Jacobi identity, which proves particularly useful when providing the link between hyperspherical
trigonometry and elliptic functions given later.

**Proposition 12.** The following identity holds between the various tetrahedral angles:

$$\frac{\sin^2(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl}) \cos \alpha_{ij}^{(jkl)}}{\sin^2(\theta_{jk}, \theta_{jl}, \theta_{kl})} \sin \theta_{jl} \sin \theta_{kl} = \sin(\theta_{ij}, \theta_{il}, \theta_{jl}) \sin(\theta_{ik}, \theta_{il}, \theta_{kl}) \left(\cos \phi_{jl} \cos \phi_{kl} - \cos \phi_{il}\right)$$

(73)

**Proof.** Consider the Desnanot-Jacobi determinantal identity applied to $\sin^4(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl})$. This yields:

$$\begin{vmatrix}
1 & \cos \theta_{ij} & \cos \theta_{ik} & \cos \theta_{il} \\
\cos \theta_{jk} & 1 & \cos \theta_{jk} & \cos \theta_{jl} \\
\cos \theta_{ik} & \cos \theta_{jk} & 1 & \cos \theta_{kl} \\
\cos \theta_{jl} & \cos \theta_{jk} & \cos \theta_{kl} & 1
\end{vmatrix} = \begin{vmatrix}
1 & \cos \theta_{jk} & \cos \theta_{jl} \\
\cos \theta_{ik} & 1 & \cos \theta_{ik} \\
\cos \theta_{kl} & \cos \theta_{kl} & 1
\end{vmatrix} \begin{vmatrix}
1 & \cos \theta_{ij} & \cos \theta_{il} \\
\cos \theta_{jl} & 1 & \cos \theta_{jl} \\
\cos \theta_{kl} & \cos \theta_{kl} & 1
\end{vmatrix}$$

(74)

from which the result follows. \qed
In Appendix A, we consider the expansion of one of the radial vectors, \( n_i \), in terms of the other radial vectors. We compare this with the expansion in terms of an orthogonal frame, obtained by Gram-Schmidt orthonormalisation to obtain the four-parts formula. We follow a similar procedure for the orthogonal vectors, \( u_{ijk} \), to obtain the five-parts formula. We believe these expansions will be of particular use in considering a model associated with quadrilateral nets introduced by Sergeev in \(^{34}\).

C. \( m \)-Dimensional Hyperspherical Trigonometry

Some of the results in this subsection were derived in \(^{33}\) from a different perspective. Consider an \((m-1)\)-dimensional hyperspherical simplex on the surface of an \(m\)-dimensional hypersphere, an \((m-1)\)-sphere. Let the \(m\) vectors, \( n_1, n_2, \ldots, n_m \) be the position vectors of the \(m\) vertices of this simplex, such that the edges of this simplex are given by \( \theta_{ij}^{[1]} \), with

\[
\mathbf{n}_i \cdot \mathbf{n}_k \equiv \cos \theta_{ij}^{[1]},
\]  

for \( i_j, i_k \in \{1, 2, \ldots, m\} \). Now define orthogonal vectors \( u_{i_1i_2\ldots i_{m-1}} \) using the \((m-1)\)-ary vector product,

\[
\mathbf{u}_{i_1i_2\ldots i_{m-1}} \equiv \frac{\mathbf{n}_{i_1} \times \mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_{m-1}}}{|\mathbf{n}_{i_1} \times \mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_{m-1}}|}.
\]  

Define the angles between these orthogonal vectors by

\[
\mathbf{u}_{i_1i_2\ldots i_{m-1}} \cdot \mathbf{u}_{i_2i_3\ldots i_m} \equiv -\cos \theta_{i_2i_3\ldots i_{m-1}}^{[m-1](i_1\ldots i_m)},
\]  

Note that the superscript \((i_1\ldots i_m)\) denotes the simplex that is being considered. In the highest dimensional case this may be omitted. We also have

\[
|\mathbf{n}_{i_1} \times \mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_{m-1}}| = \sin \left( \Theta_{i_1\ldots i_{m-1}}^{[1]} \right),
\]  

where \( \sin \left( \Theta_{i_1\ldots i_{m-1}}^{[1]} \right) \) is the generalised sine function of \(m(m-1)/2\) variables,

\[
\sin \left( \Theta_{i_1\ldots i_{m-1}}^{[1]} \right) = \sin \left( \left\{ \theta_{ij}^{[1]} \left| i_j < i_k, i_j, i_k = 1, \ldots, m-1 \right. \right\} \right),
\]  

together with

\[
|\mathbf{u}_{i_1\ldots i_{m-1}} \times \mathbf{u}_{i_2\ldots i_m} \times \mathbf{u}_{i_3\ldots i_{m-1}i_1} \times \cdots \times \mathbf{u}_{i_{m-1}i_mi_1\ldots i_{m-3}}| = \sin \left( \Theta_{i_1\ldots i_m}^{[m-1]} \right),
\]  

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where similarly $\sin (\Theta_{i_1\ldots i_m}^{[m-1]})$ is the multiple sine of all of the angles between each pair of orthogonal vectors. There follow a number of identities, similar to those for the lower dimensional cases.

**Proposition 13** (Cosine Rule).

$$
\cos \theta_{i_2\ldots i_{m-1}}^{[m-1]} = \frac{\cos \theta_{i_{1i_2}}^{[1]} \ldots \cos \theta_{i_{1i_m}}^{[1]}}{\sin (\Theta_{i_1\ldots i_{m-1}}^{[1]}) \sin (\Theta_{i_2\ldots i_m}^{[1]})}.
$$  \hspace{1cm} (81)

**Proof.** Consider the scalar product

$$
\mathbf{u}_{i_1\ldots i_{m-1}} \cdot \mathbf{u}_{i_2\ldots i_m} = \frac{\mathbf{n}_{i_1} \times \cdots \times \mathbf{n}_{i_{m-1}} \cdot (\mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_m})}{|\mathbf{n}_{i_1} \times \cdots \times \mathbf{n}_{i_{m-1}}| |\mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_m}|},
$$

$$
= \frac{\mathbf{n}_{i_1} \cdot \mathbf{n}_{i_2} \ldots \mathbf{n}_{i_1} \cdot \mathbf{n}_{i_m}}{\sin (\Theta_{i_1\ldots i_{m-1}}^{[1]}) \sin (\Theta_{i_2\ldots i_m}^{[1]})},
$$

$$
= \frac{\cos \theta_{i_{1i_2}}^{[1]} \ldots \cos \theta_{i_{1i_m}}^{[1]}}{\sin (\Theta_{i_1\ldots i_{m-1}}^{[1]}) \sin (\Theta_{i_2\ldots i_m}^{[1]})}.
$$  \hspace{1cm} (82)

Hence, the cosine rule follows. \qed

**Proposition 14** (Sine Rule).

$$
\frac{\sin (\Theta_{i_1\ldots i_m}^{[1](i_{m-1})})}{\sin (\Theta_{i_{m1}\ldots i_{m-2}}^{[m-1]})} = \frac{\sin^{m-2} (\Theta_{i_1\ldots i_m}^{[1]})}{\sin (\Theta_{i_{m1}\ldots i_{m-2}}^{[m-1]}) \ldots \sin (\Theta_{i_{m1}i_{m1}'\ldots i_{m-3}}^{[m-1]})} = k, \text{ a constant},
$$  \hspace{1cm} (83)

where

$$
\sin (\Theta_{i_1\ldots i_m}^{[1](i_{m-1})}) = \sin \left( \left\{ \theta_{j_1\ldots j_{i_{m-1}}}^{[1]} \mid i_{m-1} = j_k \text{ for some } k \in \{1, \ldots, m-1\} \right\} \right).
$$  \hspace{1cm} (84)

**Proof.** Consider the ratio

$$
\frac{|\mathbf{u}_{i_1\ldots i_{m-1}} \times \mathbf{u}_{i_2\ldots i_m} \times \cdots \times \mathbf{u}_{i_{m-1}i_{m1}'\ldots i_{m-3}}|}{|\mathbf{u}_{i_{m1}'\ldots i_{m-2}}|} = \frac{\sin (\Theta_{i_1\ldots i_m}^{[1](i_{m-1})})}{\sin (\Theta_{i_{m1}\ldots i_{m-2}}^{[m-1]})}.
$$  \hspace{1cm} (85)
However,

\[
\begin{align*}
(u_{1\ldots i_{m-1}} \times u_{i_2 \ldots i_m} \times \cdots \times u_{i_{m-1}i_1\ldots i_{m-3}}) &= (n_{i_1} \times \cdots \times n_{i_{m-1}}) \times \cdots \times (n_{i_{m-1}} \times n_{i_m} \times n_{i_{m-1}} \times \cdots \times n_{i_{m-3}}) \\
&= \frac{n_{i_1} \times \cdots \times n_{i_{m-1}} \cdot n_{i_1} \times \cdots \times n_{i_{m-1}} \cdot n_{i_1} \times \cdots \times n_{i_{m-1}}}{n_{i_1} \times \cdots \times n_{i_{m-1}} \times n_{i_m} \times n_{i_{m-1}} \times \cdots \times n_{i_{m-3}}} \\
&= \frac{n_{i_1} \cdot (n_{i_2} \times \cdots \times n_{i_m}) \times \cdots \times n_{i_{m-1}} \cdot (n_{i_2} \times \cdots \times n_{i_m})}{\sin (\Theta_{i_1\ldots i_{m-1}}) \cdots \sin (\Theta_{i_{m-1}i_{1\ldots i_{m-3}}})} \\
&= \frac{(n_{i_1} \cdot (n_{i_2} \times \cdots \times n_{i_m}))(m-2)n_{i_{m-1}}}{\sin (\Theta_{i_1\ldots i_{m-1}}) \cdots \sin (\Theta_{i_{m-1}i_{1\ldots i_{m-3}}})},
\end{align*}
\]

Hence,

\[
|u_{1\ldots i_{m-1}} \times u_{i_2 \ldots i_m} \times \cdots \times u_{i_{m-1}i_1\ldots i_{m-3}}| = \frac{(n_{i_1} \cdot (n_{i_2} \times \cdots \times n_{i_m}))(m-2)}{\sin (\Theta_{i_1\ldots i_{m-1}}) \cdots \sin (\Theta_{i_{m-1}i_{1\ldots i_{m-3}}})} \sin^{m-2} (\Theta_{i_{1\ldots i_{m-1}}} \cdots \Theta_{i_{m-1}i_{1\ldots i_{m-3}}}),
\]

and so, the sine rule follows. \(\square\)

**Proposition 15 (Polar Cosine Rule).**

\[
\cos \theta_{i_{m-1}i_1}^{[1]} = \det(X),
\]

where

\[
(X)_{jk} = \begin{cases} 
-1; & j = k - 1, \\
-\cos \theta_{i_1\ldots i_{j-1}i_{j+1}\ldots i_{k-2}i_k\ldots i_m}; & j \neq k - 1, \ m \text{ odd}, \\
(-1)^{j+k+1} \cos \theta_{i_1\ldots i_{j-1}i_{j+1}\ldots i_{k-2}i_k\ldots i_m}; & j \neq -1, \ m \text{ even}.
\end{cases}
\]

**Proof.** Consider the product

\[
(u_{1\ldots i_{m-1}} \times u_{i_2 \ldots i_m} \times u_{i_3 \ldots i_1} \times \cdots \times u_{i_{m-1}i_1\ldots i_{m-3}}) \cdot (u_{i_2 \ldots i_m} \times \cdots \times u_{i_{m-1}i_1\ldots i_{m-2}})
\]

\[
= \begin{vmatrix} 
\mathbf{u}_{i_1\ldots i_{m-1}} & \mathbf{u}_{i_2 \ldots i_m} & \cdots & \mathbf{u}_{i_1\ldots i_{m-1}} \\
\vdots & \ddots & \ddots & \vdots \\
\mathbf{u}_{i_{m-1}i_1\ldots i_{m-3}} & \mathbf{u}_{i_2 \ldots i_m} & \cdots & \mathbf{u}_{i_{m-1}i_1\ldots i_{m-3}} \\
\end{vmatrix}
= \det(X),
\]
where \( X \) is as defined previously. Alternatively, recalling (86) gives

\[
\begin{align*}
(u_{i_1 \ldots i_{m-1}} \times u_{i_2 \ldots i_m} \times u_{i_3 \ldots i_m} \times \cdots \times u_{i_m \ldots i_m}) \cdot (u_{i_2 \ldots i_m} \times \cdots \times u_{i_m \ldots i_m}) &= \left( \frac{\mathbf{n}_{i_1} \cdot (\mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_m})}{\sin(\Theta_{i_1 \ldots i_{m-1}})} \right) \sin(\Theta_{i_{m-1} i_m}) \left( \frac{\mathbf{n}_{i_2} \cdot (\mathbf{n}_{i_3} \times \cdots \times \mathbf{n}_{i_m} \times \mathbf{n}_{i_{m-1}})}{\sin(\Theta_{i_{m-1} i_m})} \right) \sin(\Theta_{i_{m-1} i_{m-2}}) \\
&= \frac{\left( \mathbf{n}_{i_1} \cdot (\mathbf{n}_{i_2} \times \cdots \times \mathbf{n}_{i_m}) \right)^{2n-4} \cos^2 \Phi_{i_{m-1} i_m}}{\sin(\Theta_{i_1 \ldots i_{m-1}}) \sin(\Theta_{i_m \ldots i_{m-1}}) \sin^2(\Theta_{i_{m-1} i_m}) \sin^2(\Theta_{i_{m-1} i_{m-2}})}.
\end{align*}
\]

(91)

By applying the sine rule, the Polar Cosine rule then follows.

Note that the facets of the \((m-1)\)-dimensional simplex are \((m-2)\)-dimensional simplices with the various cosine and sine rules still holding true in these facets. The same applies to these \((m-2)\)-dimensional facets, and so on, forming a complete hierarchy of cosine and sine rules between the angles for the \(j\)-dimensional facets, and those for the \((j-1)\)-dimensional facets. So, in terms of the cosine rule we have

\[
\cos \theta_{i_2 \ldots i_j} = \begin{vmatrix}
\cos \theta_{i_2 \ldots i_{j-1}}^{j-1}(i_1 \ldots i_{j-1}) & \cos \theta_{i_2 \ldots i_{j-1}}^{j-1}(i_1 \ldots i_{j-2} i_j) \\
\cos \theta_{i_2 \ldots i_{j-1}}^{j-1}(i_2 \ldots i_j) & 1 \\
\sin \theta_{i_2 \ldots i_{j-1}}^{j-1}(i_1 \ldots i_{j-1}) & \sin \theta_{i_2 \ldots i_{j-1}}^{j-1}(i_2 \ldots i_j)
\end{vmatrix}.
\]

(92)

This can be extended so that the angles of any facet can be expressed in terms of those for any other dimensional facets to give a cosine rule of the form

\[
\cos \theta_{i_2 \ldots i_j}^{j} = \begin{vmatrix}
\cos \theta_{i_2 \ldots i_j}^{k}(i_1 \ldots i_k) & \cos \theta_{i_2 \ldots i_j}^{k}(i_1 \ldots i_{k+2}) & \cdots & \cos \theta_{i_2 \ldots i_j}^{k}(i_1 \ldots i_{k+i_j}) \\
\cos \theta_{i_2 \ldots i_j}^{k}(i_2 \ldots i_{k+1}) & \cos \theta_{i_2 \ldots i_j}^{k}(i_2 \ldots i_{k+2}) & \cdots & \cos \theta_{i_2 \ldots i_j}^{k}(i_2 \ldots i_{k+i_j}) \\
\cos \theta_{i_2 \ldots i_j}^{k}(i_3 \ldots i_{k+1}) & \cos \theta_{i_2 \ldots i_j}^{k}(i_3 \ldots i_{k+2}) & \cdots & \cos \theta_{i_2 \ldots i_j}^{k}(i_3 \ldots i_{k+i_j}) \\
\vdots & \vdots & \ddots & \vdots \\
\cos \theta_{i_2 \ldots i_j}^{k}(i_{j-1} \ldots i_{k+1}) & \cos \theta_{i_2 \ldots i_j}^{k}(i_{j-1} \ldots i_{k+2}) & \cdots & \cos \theta_{i_2 \ldots i_j}^{k}(i_{j-1} \ldots i_{k+i_j}) \\
\sin \theta_{i_2 \ldots i_j}^{k}(i_1 \ldots i_k) & \sin \theta_{i_2 \ldots i_j}^{k}(i_2 \ldots i_k) & \cdots & \sin \theta_{i_2 \ldots i_j}^{k}(i_{j-1} \ldots i_k)
\end{vmatrix}.
\]

(93)

From these cosine rules, there arise corresponding sine rules. Specifically, note that

\[
\sin \theta_{i_2 \ldots i_j}^{j} = k_{j-1} \sin \theta_{i_2 \ldots i_{j-1}}^{j-1},
\]

(94)
where
\[ k_{j-1} = \frac{\sin\left(\theta_{i_1...i_{j-1}+1}(\theta_{i_2...i_{j-1}}, \theta_{i_1...i_{j-1}}, \theta_{i_2...i_{j-1}})ight)}{\sin\left(\theta_{i_2...i_{j-1}}(\theta_{i_1...i_{j-1}})ight) \sin\left(\theta_{i_2...i_{j-1}}(\theta_{i_1...i_{j-1}}+1)\right) \sin\left(\theta_{i_2...i_{j-1}}(\theta_{i_1...i_{j-1}})ight)}, \]

(95)

with \( k_{j-1} \) being constant. Hence, from this, it follows that a full hierarchy of intertwined sine rules exists between the angles of any two different dimensional facets.

**IV. LINK BETWEEN HYPERSPHERICAL TRIGONOMETRY AND ELLIPTIC FUNCTIONS**

**A. Link between Spherical Trigonometry and Jacobi Elliptic Functions**

There exists a well-known link between the formulae of spherical trigonometry and the Jacobi elliptic functions through their addition formulae\(^{20,21}\). Here, we follow a derivation given by Irwin\(^{15}\), but with a slight modification, taking as a starting point the sine rule for spherical trigonometry,

\[ \frac{\sin \alpha_i}{\sin \theta_{jk}} = \frac{\sin \alpha_j}{\sin \theta_{ik}} = \frac{\sin \alpha_k}{\sin \theta_{ij}} = k, \]

(96)

where

\[ k = \frac{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})}{\sin \theta_{ij} \sin \theta_{ik} \sin \theta_{jk}}, \]

(97)

and considering the expression

\[ W = k^2 \sin^2 \theta_{ij} \sin^2 \theta_{ik} \sin^2 \theta_{jk} - \sin^2(\theta_{ij}, \theta_{ik}, \theta_{jk}). \]

(98)

It follows that the derivative of \( W \) with respect to one of the angles, \( \theta_{ij} \), is

\[ \frac{\partial W}{\partial \theta_{ij}} = -2 \sin \theta_{ij} \left( \cos \theta_{ij} - \cos \theta_{ik} \cos \theta_{jk} - k^2 \cos \theta_{ij} \sin^2 \theta_{ik} \sin^2 \theta_{jk} \right), \]

(99)

\[ = 2 \sin \theta_{ij} \sin \theta_{ik} \sin \theta_{jk} \cos \alpha_i \cos \alpha_j. \]

If the radial angles \( \theta_{ij}, \theta_{ik} \) and \( \theta_{jk} \) all vary in such a way as to keep \( k \) constant, then since this is equivalent to \( W = 0 \), this implies

\[ \cos \alpha_i \cos \alpha_j d \theta_{ij} + \cos \alpha_i \cos \alpha_k d \theta_{ik} + \cos \alpha_j \cos \alpha_k d \theta_{jk} = 0, \]

(100)

which is of course equivalent to

\[ \frac{d \theta_{ij}}{\cos \alpha_k} + \frac{d \theta_{jk}}{\cos \alpha_i} + \frac{d \theta_{ik}}{\cos \alpha_j} = 0. \]

(101)
However, the sine rule gives

\[ \cos \alpha_i = \pm \sqrt{1 - k^2 \sin^2 \theta_{jk}}, \text{ etc.,} \]  

(102)

and so, as the signs of \( \theta_{ij}, \theta_{ik} \) and \( \theta_{jk} \) can be chosen arbitrarily, without loss of generality they can all be chosen to be positive, giving

\[ \frac{d\theta_{ij}}{\sqrt{1 - k^2 \sin^2 \theta_{ij}}} + \frac{d\theta_{ik}}{\sqrt{1 - k^2 \sin^2 \theta_{ik}}} + \frac{d\theta_{jk}}{\sqrt{1 - k^2 \sin^2 \theta_{jk}}} = 0 \]  

(103)

Writing \( \theta_{ij} = \text{am}(a_k) \), with

\[ a_k = \int_0^{\text{am}(a_k)} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}, \]  

(104)

and similarly, \( \theta_{ik} = \text{am}(a_j) \) and \( \theta_{jk} = \text{am}(a_i) \), the integral of the differential relation, (103), implies that

\[ a_i + a_j + a_k = \delta, \text{ constant,} \]  

(105)

together with

\[ \sin \theta_{ij} = \sin(\text{am}(a_k)) = \text{sn}(a_k), \]
\[ \sin \theta_{ik} = \sin(\text{am}(a_j)) = \text{sn}(a_j), \]  

(106)
\[ \sin \theta_{jk} = \sin(\text{am}(a_i)) = \text{sn}(a_i). \]

Therefore, having introduced uniformising variables \( a_i, i = 1, 2, 3, \) associated with the three spherical angles \( \theta_{jk} \), the various spherical trigonometric functions can be identified with the Jacobi elliptic functions by using the identifications

\[ \sin(\theta_{jk}) = \text{sn}(a_i; k) \iff \sin \alpha_i \equiv k \text{sn}(a_i; k), \]  

(107)

in which \( k \) is the modulus of the elliptic function given by (97). These identifications, through the usual relations between the three Jacobi elliptic functions \( \text{sn}, \text{cn} \) and \( \text{dn} \),

\[ \text{cn}^2(u; k) + \text{sn}^2(u; k) = 1, \quad \text{dn}^2(u; k) + k^2 \text{sn}^2(u; k) = 1, \]  

(108)

lead to

\[ \cos \theta_{jk} = \text{cn}(a_i; k), \text{ and } \cos \alpha_i = \text{dn}(a_i; k), \]  

(109)

for \( i, j, k \) cyclic. Addition formulae follow readily from the various spherical trigonometric relations. In particular, the cosine and polar cosine rules yield the relations

\[ \text{cn}(a_i) = \text{cn}(a_j)\text{cn}(a_k) + \text{sn}(a_j)\text{sn}(a_k)\text{dn}(a_i), \]
\[ \text{dn}(a_i) = -\text{dn}(a_j)\text{dn}(a_k) + k^2\text{sn}(a_j)\text{sn}(a_k)\text{cn}(a_i), \]  

(110)
respectively. Solving these relations as functions of \( a_i \) gives

\[
\begin{align*}
\text{cn}(a_i) &= \frac{\text{dn}(a_j) \text{dn}(a_k) \text{sn}(a_j) \text{sn}(a_k) - \text{cn}(a_j) \text{cn}(a_k)}{1 - k^2 \text{sn}^2(a_j) \text{sn}^2(a_k)}, \\
\text{dn}(a_i) &= \frac{\text{dn}(a_j) \text{dn}(a_k) - k^2 \text{cn}(a_j) \text{cn}(a_k) \text{sn}(a_j) \text{sn}(a_k)}{1 - k^2 \text{sn}^2(a_j) \text{sn}^2(a_k)},
\end{align*}
\]

(111)

the addition formulae for the Jacobi elliptic functions with

\[
\begin{align*}
\text{cn}(a_i) &= -\text{cn}(a_j + a_k), \\
\text{dn}(a_i) &= \text{dn}(a_j + a_k).
\end{align*}
\]

(113)

The Jacobi Elliptic functions are periodic in \( K(k) \) and \( K'(k) \) as

\[
\begin{align*}
\text{cn}(a_j + 2mK + 2niK'; k) &= (-1)^m \text{cn}(a_j; k), \\
\text{dn}(a_j + 2mK + 2niK'; k) &= (-1)^n \text{dn}(a_j; k),
\end{align*}
\]

(114)

for all \( a_j \), and so we must restrict \( \delta \) such that \( a_i + a_j + a_k = 2K + 2iK' \). The addition formula for \( \text{sn} \) follows as a consequence. From these addition formulae a number of intertwined addition relations follow

\[
\begin{align*}
\text{cn}(a_j) \text{sn}(a_i + a_j) &= \text{sn}(a_i) \text{dn}(a_j) + \text{dn}(a_i) \text{sn}(a_j) \text{cn}(a_i + a_j), \\
\text{dn}(a_i) \text{sn}(a_i + a_j) &= \text{cn}(a_i) \text{sn}(a_j) + \text{sn}(a_i) \text{cn}(a_j) \text{dn}(a_i + a_j), \\
\text{sn}(a_i) \text{cn}(a_i + a_j) + \text{sn}(a_j) \text{dn}(a_i + a_j) &= \text{cn}(a_i) \text{dn}(a_j) \text{sn}(a_i + a_j).
\end{align*}
\]

(115) \( 116 \) \( 117 \)

Denoting

\[
\begin{align*}
w_1(a_j) &= \frac{\rho}{\text{sn}(a_j)}, \quad w_2(a_j) = \frac{\rho \text{cn}(a_j)}{\text{sn}(a_j)}, \quad w_3(a_j) = \frac{\rho \text{dn}(a_j)}{\text{sn}(a_j)},
\end{align*}
\]

(118)

these addition formulae may be rewritten as

\[
\begin{align*}
w_i(a_j)w_j(a_i) + w_j(a_k)w_k(a_j) + w_k(a_i)w_i(a_k) = 0, \quad i, j, k = 1, 2, 3,
\end{align*}
\]

(119)

with \( a_1 + a_2 + a_3 = 2K + 2iK' \). Note that this relation is the functional Yang-Baxter relation.35

Now, by considering the reciprocal of \( k \),

\[
\frac{\sin \theta_{jk}}{\sin \alpha_i} = \frac{\sin \theta_{ik}}{\sin \alpha_j} = \frac{\sin \theta_{ij}}{\sin \alpha_k} = \frac{1}{k},
\]

(120)

following Irwin’s method, we can derive a similar, yet slightly simpler, relationship. First, recall

\[
k = \frac{\sin \alpha_k}{\sin \theta_{ij}} = \frac{\sin \alpha_i \sin \alpha_j \sin \alpha_k}{\sqrt{1 - \cos^2 \alpha_i - \cos^2 \alpha_j - \cos^2 \alpha_k - 2 \cos \alpha_i \cos \alpha_j \cos \alpha_k}},
\]

(121)
and let
\[ W = \frac{1}{k^2} \sin^2 \alpha_i \sin^2 \alpha_j \sin^2 \alpha_k - (1 - \cos^2 \alpha_i - \cos^2 \alpha_j - \cos^2 \alpha_k - 2 \cos \alpha_i \cos \alpha_j \cos \alpha_k). \] (122)

Now, consider the derivative \( \partial W / \partial \alpha_i \),
\[ \partial W / \partial \alpha_i = -2 \cos \theta_{ij} \cos \theta_{ik} \sin \alpha_i \sin \alpha_j \sin \alpha_k. \] (123)

Varying \( \alpha_i, \alpha_j \) and \( \alpha_k \) as to keep \( k \) constant, it follows that
\[ \frac{d\alpha_i}{\sqrt{1 - \frac{1}{k^2} \sin^2 \alpha_i}} + \frac{d\alpha_j}{\sqrt{1 - \frac{1}{k^2} \sin^2 \alpha_i}} + \frac{d\alpha_k}{\sqrt{1 - \frac{1}{k^2} \sin^2 \alpha_i}} = 0. \] (124)

Writing \( \alpha_i = \text{am}(b_i) \), with
\[ b_i = \int_0^{\text{am}(b_i)} \frac{dt}{\sqrt{1 - \frac{1}{k^2} \sin^2 t}}, \] (125)
and similarly, \( \alpha_j = \text{am}(b_j) \) and \( \alpha_k = \text{am}(b_k) \), the integral of (124) implies
\[ b_i + b_j + b_k = \gamma, \text{ constant}, \] (126)

\[ \sin \alpha_i = \sin(\text{am}(b_i)) = \text{sn}(b_i), \]
\[ \sin \alpha_j = \sin(\text{am}(b_j)) = \text{sn}(b_j), \]
\[ \sin \alpha_k = \sin(\text{am}(b_k)) = \text{sn}(b_k). \] (127)

Therefore, having introduced spherical angles \( b_i, i = 1, 2, 3 \), this time associated with the three spherical angles \( \alpha_i \), the various spherical trigonometric functions can be identified with the Jacobi elliptic functions via the identifications
\[ \sin \alpha_i = \text{sn} \left( b_i; \frac{1}{k} \right) \iff \sin \theta_{jk} = \frac{1}{k} \text{sn} \left( b_i; \frac{1}{k} \right), \] (128)
in which \( 1/k \) is the modulus of the elliptic function. These identifications lead to
\[ \cos \alpha_i = \text{cn} \left( b_i; \frac{1}{k} \right), \text{ and } \cos \theta_{jk} = \text{dn} \left( b_i; \frac{1}{k} \right), \] (129)

In this case, the Jacobi elliptic function addition formulae are satisfied providing we have
\[ \text{cn}(b_i) = \text{cn}(b_j + b_k), \]
\[ \text{dn}(b_i) = -\text{dn}(b_j + b_k). \] (130)

The periodicity ensures \( \gamma = 2K \),
\[ b_i + b_j + b_k = 2K. \] (131)

Note that making the identification in this way results in a slightly simpler restriction, dependent only on \( K \), and not also \( K' \).
B. Link between Hyperspherical Trigonometry and Elliptic Functions

We now look to provide a similar link for the 4 dimensional hyperspherical case and elliptic functions following a similar procedure as before. We take as a starting point the hyperspherical trigonometry relation,

\[
\frac{\sin(\phi_{ik}, \phi_{jk}, \phi_{kl})}{\sin(\theta_{ij}, \theta_{il}, \theta_{jl})} = \frac{\sin^2(\theta_{ij}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl})}{\sin(\theta_{ij}, \theta_{il}, \theta_{jk}, \theta_{jl}), \sin(\theta_{ik}, \theta_{iil}, \theta_{jkl})} = k, \tag{132}
\]

from which we introduce

\[W = \sin^4(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl}) - k^2 \sin^2(\theta_{ij}, \theta_{ik}, \theta_{kl}) \sin^2(\theta_{ij}, \theta_{il}, \theta_{kl}) \sin^2(\theta_{kl}, \theta_{jl}, \theta_{kl}). \tag{133}\]

Taking \(W\)’s derivative with respect to \(\theta_{ij}\) gives

\[
\frac{\partial W}{\partial \theta_{ij}} = 2 \sin^2(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl}) \left( \frac{\partial \sin^2(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl})}{\partial \theta_{ij}} \right)
- k^2 \left( \frac{\partial \sin^2(\theta_{ij}, \theta_{ik}, \theta_{kl})}{\partial \theta_{ij}} \right) \sin^2(\theta_{ij}, \theta_{il}, \theta_{jl}) \sin^2(\theta_{ik}, \theta_{il}, \theta_{kl}) \sin^2(\theta_{jk}, \theta_{jl}, \theta_{kl}) \tag{134}\]

\[
- k^2 \sin^2(\theta_{ij}, \theta_{ik}, \theta_{jk}) \left( \frac{\partial \sin^2(\theta_{ij}, \theta_{il}, \theta_{jl})}{\partial \theta_{ij}} \right) \sin^2(\theta_{ik}, \theta_{il}, \theta_{kl}) \sin^2(\theta_{jk}, \theta_{jl}, \theta_{kl}).
\]

Considering the derivatives of the squares of the triple sine functions, we have

\[
\frac{\partial \sin^2(\theta_{ij}, \theta_{ik}, \theta_{jk})}{\partial \theta_{ij}} = 2 \cos \theta_{ij} \sin \theta_{ij} - 2 \sin \theta_{ij} \cos \theta_{ik} \cos \theta_{jk}
= 2 \cos \alpha_k^{(ijk)} \sin \theta_{ij} \sin \theta_{ik} \sin \theta_{jk}, \tag{135}
\]

and similarly,

\[
\frac{\partial \sin^2(\theta_{ij}, \theta_{il}, \theta_{jl})}{\partial \theta_{ij}} = 2 \cos \alpha_l^{(ijl)} \sin \theta_{ij} \sin \theta_{il} \sin \theta_{jl}. \tag{136}
\]

We also have, for the six term sine function,

\[
\frac{\partial \sin^2(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl})}{\partial \theta_{ij}} = -2 \sin \theta_{ij} \cos \phi_{kl} \sin(\theta_{ik}, \theta_{iil}, \theta_{kl}) \sin(\theta_{jk}, \theta_{jkl}, \theta_{kl}). \tag{137}
\]

Substituting these back into (134) gives

\[
\frac{\partial W}{\partial \theta_{ij}} = -2 \sin \theta_{ij} \sin^2(\theta_{ij}, \theta_{il}, \theta_{kl}) \sin(\theta_{jk}, \theta_{kl}) \sin(\theta_{jk}, \theta_{kl})
\times \left( 2 \cos \phi_{kl} \sin(\theta_{ik}, \theta_{iil}, \theta_{kl}) \sin(\theta_{jk}, \theta_{jkl}, \theta_{kl})
+ \sin^2(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jl}, \theta_{ij}, \theta_{kl}) \left( \frac{\cos \alpha_k^{(ijk)} \sin \theta_{ik} \sin \theta_{jk}}{\sin^2(\theta_{ij}, \theta_{ik}, \theta_{jk})} + \frac{\cos \alpha_l^{(ijl)} \sin \theta_{il} \sin \theta_{jl}}{\sin^2(\theta_{ij}, \theta_{il}, \theta_{jl})} \right) \right). \tag{138}
\]
In order to simplify this derivative, we now consider the Desnanot-Jacobi determinantal identity,

$$
\begin{vmatrix}
\cos \theta_{jk} & \cos \theta_{jl} & 1 \\
\cos \theta_{ik} & \cos \theta_{jk} & \cos \theta_{jl} \\
\cos \theta_{ik} & \cos \theta_{jk} & \cos \theta_{jl} \\
\cos \theta_{il} & \cos \theta_{jl} & \cos \theta_{kl} & 1
\end{vmatrix}
$$

which, reduces neatly to

$$
\frac{\sin^2(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl}) \cos \alpha_l^{(jkl)}}{\sin^2(\theta_{jk}, \theta_{jl}, \theta_{kl})} \sin \theta_{lj} \sin \theta_{kl} = \sin(\theta_{ij}, \theta_{il}, \theta_{jl}) \sin(\theta_{ik}, \theta_{il}, \theta_{kl}) \cos(\phi_{jl} \cos \phi_{kl} - \cos \phi_{il}).
$$

Substituting this expression into \((138)\), the derivative reduces to

$$
\frac{\partial W}{\partial \theta_{ij}} = -2 \sin \theta_{ij} \sin^2(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl}) \sin(\theta_{ik}, \theta_{il}, \theta_{kl}) \sin(\theta_{jk}, \theta_{jl}, \theta_{kl}) \sin(\theta_{ij}, \theta_{il}, \theta_{jl}) \sin(\theta_{ij}, \theta_{il}, \theta_{jl}) \sin(\theta_{ij}, \theta_{ik}, \theta_{jk}) \sin(\theta_{ij}, \theta_{ik}, \theta_{jk})
$$

which, using \((56)\), may be rewritten as

$$
\frac{\partial W}{\partial \theta_{ij}} = -2 \sin \theta_{ij} \sin(\theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl}) \sin(\theta_{jk}, \theta_{jl}, \theta_{kl}) \sin(\theta_{ij}, \theta_{il}, \theta_{jl}) \sin(\theta_{ij}, \theta_{ik}, \theta_{jk}) \sin(\theta_{ij}, \theta_{ik}, \theta_{jk})
$$

$$
\cdot \sin \phi_{ij} (\cos \phi_{jl} \cos \phi_{kl} + \cos \phi_{il} \cos \phi_{jk}).
$$

If we vary the six ‘\(\theta\)’s such that \(k\) remains constant, then since this is equivalent to \(W = 0\), setting \(dW = 0\) implies

$$
\sum_{\text{perm}} \sin \phi_{ij} (\cos \phi_{il} \cos \phi_{jl} + \cos \phi_{ik} \cos \phi_{jk}) d\theta_{ij} = 0,
$$

\((143)\)

where \(\text{perm}\) denotes all six index pairs \(ij, ik, il, jk, jl, kl\). In spite of the similarity between \((143)\) and the analogous formula \((100)\) in the spherical case, the former cannot be simplified along the same line as in the spherical case. A way out, albeit not quite satisfactory, is to focus on a special symmetric case, in which the opposite dihedral angles are equivalent, i.e.

$$
\phi_{ij} = \phi_{kl}, \quad \phi_{ik} = \phi_{jl}, \quad \phi_{il} = \phi_{jk},
$$

28
which we will refer to as the symmetric hyperspherical tetrahedron.

Restricting ourselves to this symmetric case, the differential relation (143) reduces to

$$\sum_{\text{perm}} \sin \phi_{ij} \cos \phi_{ik} \cos \phi_{jk} \, d\theta_{ij} = 0,$$

or equivalently

$$\sum_{\text{perm}} \frac{\sin \phi_{ij}}{\cos \phi_{ij}} \, d\theta_{ij} = 0. \quad (145)$$

The restrictions on the tetrahedron being symmetric also simplify the hyperspherical sine rule (51), reducing it to

$$\sin^2 \phi_{ij} = k_{H} \sin^2 \theta_{ij}, \quad (146)$$

and substituting this into (145) we obtain

$$\sum_{\text{perm}} \frac{\sqrt{k_{H}} \sin \theta_{ij}}{\sqrt{1 - k_{H} \sin^2 \theta_{ij}}} \, d\theta_{ij} = 0, \quad (147)$$

which yields a sum of elliptic integrals of the form

$$\int \frac{\sqrt{k_{H}} u}{\sqrt{(1 - k_{H} u^2)(1 - u^2)}} \, du.$$

Thus, for the case of a symmetric hyperspherical tetrahedron, the hyperspherical trigonometric functions which in a sense uniformise the hyperspherical sine rule (132) are again associated with elliptic functions, suggesting that for the general case this remains true. However, in that general case the connection becomes a bit more entangled and require the introduction of a generalised class of elliptic functions, introduced by Pawellek.25

V. GENERALISED JACOBI ELLIPTIC FUNCTIONS AND THEIR LINK TO HYPERSPHERICAL TRIGONOMETRY

In this section we review a class of ‘generalised’ Jacobi elliptic functions as introduced by Pawellek25, and provide a link between these functions and the formulae of hyperspherical trigonometry.

A. Generalised Jacobi Elliptic Functions

In25, Pawellek introduced the generalised Jacobi elliptic functions $s(u, k_1, k_2)$, $c(u, k_1, k_2)$, $d_1(u, k_1, k_2)$ and $d_2(u, k_1, k_2)$. These functions are algebraic curves with two distinct moduli,
and are based upon Jacobi’s elliptic functions. After assuming without loss of generality that \(1 > k_1 > k_2 > 0\) as moduli parameters, they are defined as the inversion of the hyperelliptic integrals

\[
u(x, k_1, k_2) = \int_0^{x=s(u)} \frac{dt}{\sqrt{(1-t^2)(1-k_1^2t^2)(1-k_2^2t^2)}}, \quad (148a)
\]

\[
u(x, k_1, k_2) = \int_0^{1} \frac{dt}{\sqrt{(1-t^2)(k_1^2t^2+k_2^2t^2)(k_2^2+k_1^2t^2)}}, \quad (148b)
\]

\[
u(x, k_1, k_2) = k_1 \int_0^{1} \frac{dt}{\sqrt{(1-t^2)(t^2-k_1^2t^2)(k_2^2+k_1^2t^2)}} \quad (148c)
\]

\[
u(x, k_1, k_2) = k_2 \int_0^{1} \frac{dt}{\sqrt{(1-t^2)(t^2-k_2^2t^2)(k_2^2+k_1^2t^2)}} \quad (148d)
\]

respectively, with \(k'_i = \sqrt{1-k_i^2}\). These generalised Jacobi functions are associated with an algebraic curve of the form

\[C : y^2 = (1-x^2)(1-k_1^2x^2)(1-k_2^2x^2), \quad (149)\]

which can be modeled as a Riemann surface of genus \(2\), but is, in fact, a double cover of an elliptic curve \(E\),

\[E : w^2 = z(1-z)(1-k_1^2z)(1-k_2^2z), \quad (150)\]

via the cover map \(C \xrightarrow{\pi} E\),

\[(w, z) = \pi(y, x) = (xy, x^2). \quad (151)\]

The functions satisfy a number of identities,

\[c^2(u) = 1 - s^2(u), \quad d_1^2(u) = 1 - k_1^2s^2(u), \quad d_2^2(u) = 1 - k_2^2s^2(u), \quad (152)\]

\[d_i^2(u) - k_i^2c^2(u) = 1 - k_i^2, \quad i = 1, 2; \quad k_1^2d_2^2(u) - k_2^2d_1^2(u) = k_1^2 - k_2^2, \quad (152)\]

and are related to the Jacobi elliptic functions by

\[
s(u, k_1, k_2) = \frac{\text{sn}(k_2^2u, \kappa)}{\sqrt{k_2^2 + k_1^2\text{sn}^2(k_2^2u, \kappa)}}, \quad (153a)\]

\[
c(u, k_1, k_2) = \frac{k_2^2\text{cn}(k_2^2u, \kappa)}{\sqrt{1-k_2^2\text{cn}^2(k_2^2u, \kappa)}}, \quad (153b)\]

\[
d_1(u, k_1, k_2) = \frac{k_1^2 - k_2^2}{\sqrt{k_1^2 - k_2^2\text{dn}^2(k_2^2u, \kappa)}}, \quad (153b)\]

\[
d_2(u, k_1, k_2) = \frac{k_2^2}{\sqrt{k_1^2 - k_2^2\text{dn}^2(k_2^2u, \kappa)}}, \quad (153b)\]

with

\[\kappa = \frac{k_1^2 - k_2^2}{1-k_2^2} \quad (154)\]
The first derivatives of these functions are given by
\[ s'(u) = c(u)d_1(u)d_2(u), \quad c'(u) = -s(u)d_1(u)d_2(u), \]
\[ d_1'(u) = -k_1^2 s(u)c(u)d_2(u), \quad d_2'(u) = -k_2^2 s(u)c(u)d_1(u). \] (155)

These generalised Jacobi functions with moduli \( k_1 \) and \( k_2 \) satisfy the following addition formulae:
\[ s(u \pm v) = \frac{s(u)d_2(u)c(v)d_1(v) \pm s(v)d_2(v)c(u)d_1(u)}{\sqrt{[d_2^2(u)d_2^2(v) - \kappa^4 k_2^4 s^2(u)s^2(v)]^2 + k_2^2 [s(u)d_2(u)c(v)d_1(v) \pm s(v)d_2(v)c(u)d_1(u)]}}, \] (157a)
\[ c(u \pm v) = \frac{c(u)d_2(u)c(v)d_2(v) \mp k_2^2 s(u)d_1(u)s(v)d_1(v)}{\sqrt{[d_2^2(u)d_2^2(v) - \kappa^4 k_2^4 s^2(u)s^2(v)]^2 + k_2^2 [s(u)d_2(u)c(v)d_1(v) \pm s(v)d_2(v)c(u)d_1(u)]}}, \] (157b)
\[ d_1(u \pm v) = \frac{d_1(u)d_2(u)d_1(v)d_2(v) \mp k_2^2 k_1^2 s(u)c(u)s(v)c(v)}{\sqrt{[d_2^2(u)d_2^2(v) - \kappa^4 k_2^4 s^2(u)s^2(v)]^2 + k_2^2 [s(u)d_2(u)c(v)d_1(v) \pm s(v)d_2(v)c(u)d_1(u)]}}, \] (157c)
\[ d_2(u \pm v) = \frac{d_2^2(u)d_2^2(v) - \kappa^4 k_2^4 s^2(u)s^2(v)}{\sqrt{[d_2^2(u)d_2^2(v) - \kappa^4 k_2^4 s^2(u)s^2(v)]^2 + k_2^2 [s(u)d_2(u)c(v)d_1(v) \pm s(v)d_2(v)c(u)d_1(u)]}}. \] (157d)

These formulae follow from the addition formula for the standard Jacobi elliptic functions.

B. Link between Hyperspherical Trigonometry and the Generalised Jacobi Elliptic Functions

We are also able to show a connection between the generalised Jacobi elliptic functions and hyperspherical trigonometry. As the generalised Jacobi elliptic functions are dependent on two moduli, then so must the hyperspherical trigonometry be. It is the interplay of these moduli that govern the connection. Recall, that for a hyperspherical tetrahedron, we have
\[ \sin \alpha^{(ijk)}_i = k_1 \sin \theta_{jk}, \] (158)
and
\[ \sin \phi_{il} = k_2 \sin \alpha^{(ijk)}_i, \] (159)
for all \( i, j, k, l = 1, 2, 3, 4 \), where
\[ k_1 = \frac{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})}{\sin \theta_{ij} \sin \theta_{ik} \sin \theta_{jk}}, \] (160)
and

\[ k_2 = \frac{\sin(\alpha_i^{(ijk)}, \alpha_i^{(ijl)}, \alpha_i^{(ikl)})}{\sin \alpha_i^{(ijk)} \sin \alpha_i^{(ijl)} \sin \alpha_i^{(ikl)}}, \]

respectively. Separately, these each have the same link to the Jacobi elliptic functions as the functions of a spherical triangle. They imply that

\[ \sin \phi_{il} = k_1 k_2 \sin \theta_{jk}. \]  

Introducing uniformising variables \( a_{jk}, j, k = 1, 2, 3, 4 \) with \( k > j \), associated with the six \( \theta_{jk} \), such that

\[ a_{jk} = \int_0^{\theta_{jk}} \frac{dt}{\sqrt{(1 - k_1^2 \sin^2 t)(1 - k_1^2 k_2^2 \sin^2 t)}}, \]

with \( \theta_{jk} = \text{am}(a_{jk}) \), the various hyperspherical trigonometric functions can be identified with Pallewek's generalised Jacobi elliptic functions via the identifications

\[ s(a_{jk}) \equiv \sin \theta_{jk} \iff k_1 s(a_{jk}) = \sin \alpha_i^{(ijk)} \iff k_1 k_2 s(a_{jk}) = \sin \phi_{il}, \]

in which, \( k_1 \) and \( k_1 k_2 \) are the two moduli of the functions, and are as given earlier. From these and the identities listed previously it follows that

\[ c(a_{jk}) = \cos \theta_{jk}, \quad d_1(a_{jk}) = \cos \alpha_i^{(ijk)}, \quad d_2(a_{jk}) = \cos \phi_{il}. \]

Note that under these identifications the identities [152] for the generalised Jacobi functions are still obeyed.

The modulus \( k_1 \) governs the relations between the spherical trigonometry of each of the faces, and the elliptic functions. The second modulus, \( k_1 k_2 \), acts as an overall modulus for the spherical tetrahedron. Note its dependence on the first modulus, \( k_1 \). These moduli remain constant, so that if one of the vertices of the spherical tetrahedron were moved, the others must be adjusted to compensate. These movements result in a change to the faces of the tetrahedron, but not to \( k_1 \), the facial modulus.

VI. EXAMPLES

As an example of where these hyperspherical formulae and their connection with the generalised Jacobi elliptic functions may be used, we consider a multidimensional generalisation of the Euler top in relation to Nambu mechanics. We provide a link between this example and the double elliptic model.
A. Multidimensional Euler Top

The Euler top describes a free top, moving in the absence of any external torque. In their paper on Nambu Quantum mechanics\textsuperscript{24}, Minic and Tze reformulated the mechanics of this top in terms of a Nambu system of order three, which they then solved in terms of the Jacobi elliptic functions. We consider a higher dimensional analogue of their example, formulating the top’s mechanics in terms of a Nambu system of order four, and then solving this system in terms of the Generalised Jacobi elliptic functions.

The Euler top admits two conserved quantities, the total energy,

\[ H_1 = \frac{1}{2} \sum_{i=1}^{3} \frac{1}{I_i} M_i^2, \]

(166)

and the square of the angular momentum,

\[ H_2 = \frac{1}{2} \sum_{i=1}^{3} M_i^2, \]

(167)

where \( I_i \) denote the principal moments of inertia. Minic and Tze take these both to be Hamiltonians for the system, with the action of the top given by

\[ S = \int M_1 dM_2 \wedge dM_3 - H_1 dH_2 \wedge dt. \]

(168)

The equations of motion,

\[ \frac{dM_i}{dt} = \{ H_1, H_2, M_i \}, \quad i = 1, 2, 3, \]

(169)

follow from \( \delta S = 0 \), where \( \{ H_1, H_2, M_i \} \) is the Nambu 3-bracket, a ternary bracket, defined by Nambu\textsuperscript{21}, satisfying the axioms for Nambu brackets given by Takhtajan\textsuperscript{36}.

- Skew-symmetry

\[ \{ A_1, A_2, A_3 \} = (-1)^{\epsilon(p)} \{ A_{p(1)}, A_{p(2)}, A_{p(3)} \}, \]

(170)

where \( p(i) \) is the permutation of indices and \( \epsilon(p) \) is the parity of the permutation.

- Derivation (the Leibniz rule)

\[ \{ A_1 A_2, A_3, A_4 \} = A_1 \{ A_2, A_3, A_4 \} + \{ A_1, A_3, A_4 \} A_2. \]

(171)
• Fundamental Identity (analogue of Jacobi identity)

\[
\{\{A_1, A_2, A_3\}, A_4, A_5\} + \{A_3, \{A_1, A_2, A_4\}, A_5\} + \{A_3, A_4, \{A_1, A_2, A_5\}\} = \{A_1, A_2, \{A_3, A_4, A_5\}\}.
\]

(172)

Note the similarity between the Fundamental Identity and the vectorial identity (16). Without loss of generality, taking \(I_3 > I_2 > I_1\), this gives the equations of motions as

\[
\frac{dM_1}{dt} = \left(\frac{1}{I_3} - \frac{1}{I_2}\right) M_2 M_3, \quad (173a)
\]

\[
\frac{dM_2}{dt} = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) M_1 M_3, \quad (173b)
\]

\[
\frac{dM_3}{dt} = \left(\frac{1}{I_2} - \frac{1}{I_1}\right) M_1 M_2, \quad (173c)
\]

which have solutions of the form

\[
M_1(t) = K_1 \text{sn}(c(t - t_0); k), \quad (174a)
\]

\[
M_2(t) = K_2 \text{cn}(c(t - t_0); k), \quad (174b)
\]

\[
M_3(t) = K_3 \text{dn}(c(t - t_0); k), \quad (174c)
\]

with

\[
A_1^2 = -\frac{K^2 k^2 I_1^2 I_2 I_3}{(I_2 - I_1)(I_3 - I_1)}, \quad A_2^2 = \frac{K^2 k^2 I_1 I_2^2 I_3}{(I_2 - I_1)(I_3 - I_2)}, \quad A_3^2 = \frac{K^2 k^2 I_1 I_2 I_3^2}{(I_3 - I_1)(I_3 - I_2)},
\]

with \(K\) constant, and where the modulus \(k\) of the Jacobi elliptic functions is given by

\[
k = \frac{I_2 - I_1}{I_3 - I_2} \frac{2H_1 I_3 - H_2}{2H_1 I_1 - 2H_2}, \quad (175)
\]

Using Irwin’s correspondence\textsuperscript{15}, these solutions may be reparameterised in terms of spherical trigonometry. Note that in this case \(\theta_{ij} = \alpha_k\) for all \(i, j, k = 1, 2, 3\).

A generalisation of this method can be made in higher dimensions, and in the case of a four dimensional space, we formulate the mechanics of the top in terms of a Nambu mechanical system of order four, which we then solve using Pawellek’s generalised Jacobi elliptic functions discussed in the previous section. Consider a spinning top given by the action

\[
S = \int M_1 dM_2 \wedge dM_3 \wedge dM_4 - H_1 dH_2 \wedge dH_3 \wedge dt, \quad (176)
\]
where the cross products are the triple cross products defined earlier, and $H_1$, $H_2$ and $H_3$ are Hamiltonians, given by

$$H_1 = \frac{1}{2} \sum_{i=1}^{4} M_i^2, \quad H_2 = \frac{1}{2} \sum_{i=1}^{4} \alpha_i M_i^2, \quad H_3 = \frac{1}{2} \sum_{i=1}^{4} \beta_i M_i^2,$$

with $\alpha_i$ and $\beta_i$ constants. The top’s equations of motion are given by the Nambu-Poisson brackets

$$\frac{dF}{dt} = \{F, H_1, H_2, H_3\},$$

where

$$\{F, G, H, I\} = \epsilon^{ijkl} \partial_M F \partial_{M_j} G \partial_{M_k} H \partial_{M_l} I.$$

This gives the intertwined differential system of four variables

$$\dot{M}_1 = (\alpha_2 \beta_3 - \alpha_3 \beta_2 + \alpha_3 \beta_4 - \alpha_4 \beta_3 + \alpha_4 \beta_2 - \alpha_2 \beta_4) M_2 M_3 M_4,$$

$$\dot{M}_2 = (\alpha_1 \beta_4 - \alpha_4 \beta_1 + \alpha_3 \beta_1 - \alpha_1 \beta_3 + \alpha_4 \beta_3 - \alpha_3 \beta_4) M_1 M_3 M_4,$$

$$\dot{M}_3 = (\alpha_1 \beta_2 - \alpha_2 \beta_1 + \alpha_2 \beta_4 - \alpha_4 \beta_2 + \alpha_4 \beta_1 - \alpha_1 \beta_4) M_1 M_2 M_4,$$

$$\dot{M}_4 = (\alpha_1 \beta_3 - \alpha_3 \beta_1 + \alpha_2 \beta_1 - \alpha_1 \beta_2 + \alpha_3 \beta_2 - \alpha_2 \beta_3) M_1 M_2 M_3.$$

These equations can be integrated in closed form in terms of the generalised Jacobi elliptic functions $s$, $c$, $d_1$ and $d_2$, giving

$$M_1(t) = A_1 s(K(t - t_0); k_1, k_2),$$

$$M_2(t) = A_2 c(K(t - t_0); k_1, k_2),$$

$$M_3(t) = A_3 d_1(K(t - t_0); k_1, k_2),$$

$$M_4(t) = A_4 d_2(K(t - t_0); k_1, k_2),$$

with $A_i$, $K$ and $t_0$ all constants satisfying

$$\frac{A_1 K}{A_2 A_3 A_4} = \alpha_2 \beta_3 - \alpha_3 \beta_2 + \alpha_3 \beta_4 - \alpha_4 \beta_3 + \alpha_4 \beta_2 - \alpha_2 \beta_4,$$

$$\frac{A_2 K}{A_1 A_3 A_4} = \alpha_1 \beta_4 - \alpha_4 \beta_1 + \alpha_3 \beta_1 - \alpha_1 \beta_3 + \alpha_4 \beta_3 - \alpha_3 \beta_4,$$

$$\frac{k_1^2 A_3 K}{A_1 A_2 A_4} = \alpha_1 \beta_2 - \alpha_2 \beta_1 + \alpha_2 \beta_4 - \alpha_4 \beta_2 + \alpha_4 \beta_1 - \alpha_1 \beta_4,$$

$$\frac{k_2^2 A_4 K}{A_1 A_2 A_3} = \alpha_1 \beta_3 - \alpha_3 \beta_1 + \alpha_2 \beta_1 - \alpha_1 \beta_2 + \alpha_3 \beta_2 - \alpha_2 \beta_3.$$
Hence, the motion of the top may be entirely parameterised in terms of the generalised Jacobi elliptic functions, and therefore, through the link with hyperspherical trigonometry, the angles of hyperspherical tetrahedra. Note again that this identification ensures that \( \theta_{ij} = \alpha_k = \phi_{kl} \), for all \( i, j, k, l = 1, 2, 3, 4 \).

**B. Double Elliptic Systems (DELL)**

The so-called DELL, or double-elliptic, model is a conjectured generalisation of the Calogero-Moser and Ruijsenaars-Schneider models (integrable many-body systems), which is elliptic in both the momentum and position variables. So far, only the 2-particle model (reducing to one degree of freedom) has been explicitly constructed\(^7,8\). A possible Hamiltonian for the three-particle model was later suggested in terms of Riemann theta functions\(^3,27\), supported by numerical evidence in\(^26\) although it remained to be proven.

The Dell Hamiltonian for this 2-particle model,

\[
H(P, Q) = \alpha(Q) \frac{c(P; k_1, k_2)}{d_2(P; k_1, k_2)},
\]

(182)

with

\[
\alpha^2(Q) = \alpha_{rat}^2(Q) = 1 - \frac{2g^2}{Q^2},
\]

(183)

can easily be parameterised in terms of Pawellek’s generalised Jacobi elliptic functions using the identity

\[
cn(k'_u, \kappa) = \frac{c(u; k_1, k_2)}{d_2(u; k_1, k_2)}, \quad \kappa^2 = \frac{k_1^2 - k_2^2}{k_2^2},
\]

(184)
given by Pawellek. From this, it follows that the Hamiltonian takes the much neater form

\[
H(P, Q) = \alpha(Q) \frac{c(P; k_1, k_2)}{d_2(P; k_1, k_2)},
\]

(185)

with

\[
k_1 = k, \\
k_2 = k \sqrt{(1 - \alpha^2(Q))}.
\]

(186)

The Hamilton equations of motion read

\[
\dot{P} = -\frac{\partial H}{\partial Q} = -\alpha'_{ell}(Q; \k) \frac{c(P; k_1, k_2)}{d_2(P; k_1, k_2)},
\]

(187a)

\[
\dot{Q} = \frac{\partial H}{\partial P} = -k_2^2 \alpha'_{ell}(Q; \k) \frac{s(P; k_1, k_2)d_1(P; k_1, k_2)}{d_2^2(P; k_1, k_2)},
\]

(187b)
where
\[ \alpha'_{\text{ell}}(Q; \tilde{k}) = \frac{d\alpha_{\text{ell}}(Q; \tilde{k})}{dQ} = 4g^2 \text{sn}(Q; \tilde{k}) \text{cn}(Q; \tilde{k}) \text{dn}(Q; \tilde{k}). \]

Therefore, from this it makes sense to take a closer look at the model in terms of the generalised Jacobi elliptic functions.

The following system of four quadrics in \( \mathbb{C}^6 \) provides a phase space for this two-body double elliptic system:

\[
\begin{align*}
Q_1 & : \quad x_1^2 - x_2^2 = 1, \\
Q_2 & : \quad x_3^2 - x_4^2 = k^2, \\
Q_3 & : \quad -g^2 x_1^2 + x_4^2 + x_5^2 = 1, \\
Q_4 & : \quad -g^2 x_1^2 + x_4^2 + \tilde{k}^{-2} x_6^2 = \tilde{k}^{-2},
\end{align*}
\]

(188)

where \( g \) is a coupling constant, and \( x_i, i = 1, 2, 3, 4 \) are affine coordinates. The first pair of equations provides the embedding of an elliptic curve, while the second pair provides a second elliptic curve which is locally fibred over the first. Note that when \( g = 0 \), the system simply becomes two copies of elliptic curves embedded in \( \mathbb{C}^3 \times \mathbb{C}^3 \). The Poisson brackets are given by

\[
\{x_i, x_j\} = \varepsilon_{ijk_1 \cdots k_4} \frac{\partial Q_1}{\partial x_{k_1}} \frac{\partial Q_2}{\partial x_{k_2}} \frac{\partial Q_3}{\partial x_{k_3}} \frac{\partial Q_4}{\partial x_{k_4}},
\]

(189)

with the polynomials \( Q_i \) yielding the Casimirs of the algebra. The relevant Poisson brackets for this system of quadrics are then:

\[
\begin{align*}
\dot{x}_1 & = \{x_1, x_5\} = x_2 x_3 x_4 x_6, & \dot{x}_2 & = \{x_2, x_5\} = x_1 x_3 x_4 x_6, \\
\dot{x}_3 & = \{x_3, x_5\} = x_1 x_2 x_4 x_6, & \dot{x}_4 & = \{x_4, x_5\} = x_1 x_2 x_3 x_6, \\
\dot{x}_5 & = \{x_5, x_5\} = 0, & \dot{x}_6 & = \{x_6, x_6\} = 0.
\end{align*}
\]

(190)

These equations of motion can solved in terms of Jacobi elliptic functions, where the \( x_4, x_5, x_6 \) depend on complicated expressions for the moduli (as in (182)), cf.\( \text{K} \), (we omit the formulae), but they are more readily solved in terms of the generalised Jacobi elliptic functions, namely by the following expressions:

\[
\begin{align*}
x_1 & = A_1 s(K(t - t_0); k_1, k_2), & x_2 & = A_2 c(K(t - t_0); k_1, k_2), \\
x_3 & = A_3 d_1(K(t - t_0); k_1, k_2), & x_4 & = A_4 d_2(K(t - t_0); k_1, k_2), \\
x_5 & = E, \text{ the energy}, & x_6 & = K, \text{ constant},
\end{align*}
\]

(191)
with
\[ A_1^2 = \frac{k_1 k_2}{\sqrt{-g^2}}, \quad A_2^2 = \frac{-k_1 k_2}{\sqrt{-g^2}}, \]
\[ A_3^2 = \frac{-k_2}{k_1 \sqrt{-g^2}}, \quad A_4^2 = \frac{k_1 \sqrt{-g^2}}{k_2}. \tag{192} \]
respectively, identifying \( k_1 = 1/\tilde{k} \) and \( k_2 = \tilde{k} \sqrt{-2g^2} \). Note that \( K \) is related to the energy, \( E \), through the final two quadrics
\[ \tilde{k}^2 (1 - E^2) = 1 - K^2. \tag{193} \]

This connection through the quadrics suggests that the generalised Jacobi functions form the natural parametrisation for the 2-particle DELL model. Note that these equations of motion also coincide with those for the four-dimensional Euler top example of the previous subsection. Hence, these two systems must be equivalent up to scaling.

\section*{VII. CONCLUSION}

We have investigated a novel connection between the formulae governing 4-dimensional hyperspherical geometry and elliptic functions. This connection is via the generalised Jacobi elliptic functions, defined through Abelian integrals associated with a doubly elliptic cover of an elliptic curve. These generalised Jacobi elliptic functions have two distinct moduli, \( k_1 \) and \( k_2 \), and are expressible in terms of the usual Jacobi elliptic functions, . As such these functions may be of interest for the study of certain elliptic integrable models where solutions in terms of elliptic functions with different moduli appear, for example the Q4 lattice equation of the ABS classification, and the DELL model. The link between addition formulae and hyperspherical geometry suggests that discrete integrable models, such as discretisation of the DELL model, may be derived exploiting this connection. This is a link that has been exploited in the spherical case by Petrera and Suris\cite{PetreraSuris} in producing an integrable map, in the sense of multidimensional consistency, based upon the cosine and polar cosine rules. They have shown this map to be related to the Hirota-Kimura discretisation of the Euler top. Whilst this touches slightly on our work, we are more concerned with the higher dimensional hyperspherical trigonometry, and its links to elliptic functions. The Dell model, which arises in our case, although equivalent to a higher dimensional Euler top, seems to be different from Petrera and Suris’ model.
We have also generalised the nested structure of hyperspherical trigonometry here to any
dimension using higher-dimensional vector products, although the inter-relations between
the formulae become increasingly complicated.

In recent years there has been substantial progress in obtaining formulas for hyperspher-
ical tetrahedra in 4D in terms of arc lengths and dihedral angles, cf. These generalise
results for symmetric tetrahedra obtained earlier, refining the formula for the general
case found in. The expressions involve Schäfli type symbols, and can be expressed in terms
of dilogarithms (Lobachevsky functions in the hyperbolic case). The occurrence of such
functions in Lagrangians of certain discrete integrable systems, seems to point to suggest
a connection between the two types objects from a geometrical perspective.

Finally, the occurrence of the Pawellek functions, which involve elliptic functions of two
distinct moduli, is reminiscent of the bi-elliptic addition formulae that occur naturally in
the seed and soliton solutions of the important Q4 lattice equation, discovered by Adler.

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Appendix A: Four and Five Parts Formulae

We write a general position vector on the surface of a sphere in terms of its basis vectors,
and derive the four and five parts formulae.

Given 2 vectors in three dimensions, we can express any other vector on the sphere in
terms of these by

$$\mathbf{n}_i = A\mathbf{n}_j + B\mathbf{n}_k + C\mathbf{n}_j \times \mathbf{n}_k, \quad (A1)$$

with

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & \cos \theta_{jk} \\ \cos \theta_{jk} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \cos \theta_{ij} \\ \cos \theta_{ik} \end{pmatrix} = \frac{1}{\sin^2 \theta_{jk}} \begin{pmatrix} \sin \theta_{ik} \sin \theta_{jk} \cos \alpha_k \\ \sin \theta_{ij} \sin \theta_{jk} \cos \alpha_j \end{pmatrix}, \quad (A2)$$
and
\[ C = \frac{\mathbf{n}_i \cdot (\mathbf{n}_j \times \mathbf{n}_k)}{\sin^2 \theta_{jk}} = \frac{\sin \alpha_i \sin \theta_{ik} \sin \theta_{ij}}{\sin^2 \theta_{jk}}, \] \hspace{1cm} (A3)
giving
\[ \mathbf{n}_i = \frac{\sin \theta_{ik} \cos \alpha_k}{\sin \theta_{jk}} \mathbf{n}_j + \frac{\sin \theta_{ij} \cos \alpha_j}{\sin \theta_{jk}} \mathbf{n}_k + \frac{\sin \alpha_i \sin \theta_{ik} \sin \theta_{ij}}{\sin^2 \theta_{jk}} \mathbf{n}_j \times \mathbf{n}_k. \] \hspace{1cm} (A4)

Using the spherical sine rule this reduces to
\[ \mathbf{n}_i = \frac{\sin \alpha_j \cos \alpha_k}{\sin \alpha_i} \mathbf{n}_j + \frac{\sin \alpha_k \cos \alpha_j}{\sin \alpha_i} \mathbf{n}_k + \frac{\sin \alpha_i \sin \alpha_k \sin \alpha_j}{\sin \alpha_i} \mathbf{n}_j \times \mathbf{n}_k. \] \hspace{1cm} (A5)

Using the Gram-Schmidt Process \( \mathbf{n}_i \) may be written in terms of an orthonormal basis,
\[ \mathbf{n}_i = A' \mathbf{n}_j + B' \mathbf{n}'_k + C' \mathbf{n}_j \times \mathbf{n}_k, \] \hspace{1cm} (A6)
where \( \mathbf{n}_j \) is as is before, and \( \mathbf{n}'_k \) is given by
\[ \mathbf{n}'_k = \frac{\mathbf{n}_k - (\mathbf{n}_j \cdot \mathbf{n}_k) \mathbf{n}_j}{|\mathbf{n}_k - (\mathbf{n}_j \cdot \mathbf{n}_k) \mathbf{n}_j|} = \frac{\mathbf{n}_k - (\mathbf{n}_j \cdot \mathbf{n}_k) \mathbf{n}_j}{\sin \theta_{jk}}. \] \hspace{1cm} (A7)

This implies
\[ C' = (\sin \theta_{jk}) C, \quad B' = (\sin \theta_{jk}) B, \quad A' = A + \cot \theta_{jk} B', \] \hspace{1cm} (A8)
from which it follows that
\[ \mathbf{n}_i = \frac{\sin \theta_{ik} \cos \alpha_k + \cos \theta_{jk} \sin \theta_{ij} \cos \alpha_j}{\sin \theta_{jk}} \mathbf{n}_j + \sin \theta_{ij} \cos \alpha_j \mathbf{n}'_k + \sin \theta_{ij} \sin \alpha_j \mathbf{n}_j \times \mathbf{n}'_k. \] \hspace{1cm} (A9)

Equating the two expression we have for \( \mathbf{n}_i \) gives the four-parts formula
\[ \cot \theta_{ij} \sin \theta_{jk} = \cot \alpha_k \sin \alpha_j + \cos \theta_{jk} \cos \alpha_j. \] \hspace{1cm} (A10)

Similarly, for
\[ \mathbf{u}_{ij} = P \mathbf{u}_{ik} + Q \mathbf{u}_{kj} + R \mathbf{u}_{ik} \times \mathbf{u}_{kj} \] \hspace{1cm} (A11)
it follows that
\[ \mathbf{u}_{ij} = \frac{\sin \alpha_j}{\sin \alpha_k} \cos \theta_{jk} \mathbf{u}_{ik} + \frac{\sin \alpha_i}{\sin \alpha_k} \cos \theta_{ik} \mathbf{u}_{kj} - \frac{\sin \theta_{ik} \sin \theta_{kj}}{\sin \theta_{ij}} \mathbf{u}_{ik} \times \mathbf{u}_{kj}, \] \hspace{1cm} (A12)
or alternatively, using the spherical sine rule,
\[ \mathbf{u}_{ij} = \frac{\sin \theta_{ik} \cos \theta_{jk}}{\sin \theta_{ij}} \mathbf{u}_{ik} + \frac{\sin \theta_{jk} \cos \theta_{ik}}{\sin \theta_{ij}} \mathbf{u}_{kj} - \frac{\sin \theta_{ik} \sin \theta_{kj}}{\sin \theta_{ij}} \mathbf{u}_{ik} \times \mathbf{u}_{kj}. \] \hspace{1cm} (A13)
Similarly, using the Gram-Schmidt Process,
\[
\mathbf{u}_{ij} = \frac{\sin \alpha_j \cos \theta_{jk} - \cos \alpha_k \sin \alpha_i \cos \theta_{ik}}{\sin \alpha_k} \mathbf{u}_{ik} + \sin \alpha_i \cos \theta_{ik} \mathbf{u}_{k'} + \sin \theta_{kj} \sin \alpha_j \mathbf{u}_i \times \mathbf{u}_{k'}. \tag{A14}
\]

Equating the two expressions we have for \( u_{ij} \) gives the five-parts formula
\[
\cot \theta_{jk} \sin \alpha_j = \cos \alpha_i \sin \alpha_k + \cos \theta_{ik} \sin \alpha_i \cos \alpha_k. \tag{A15}
\]

In 4-dimensional Euclidean space, for the vectors \( \mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k, \mathbf{n}_l \in E_4 \) of a hyperspherical tetrahedron we have the expansions:
\[
\mathbf{n}_i = \frac{\sin(\theta_{ik}, \theta_{il}, \theta_{kl}) \cos \phi_{kl}}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})} \mathbf{n}_j + \frac{\sin(\theta_{ij}, \theta_{il}, \theta_{jl}) \cos \phi_{jl}}{\sin(\theta_{jk}, \theta_{il}, \theta_{kl})} \mathbf{n}_k + \frac{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk}) \cos \phi_{jk}}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})} \mathbf{n}_l \tag{A16}
\]
\[+ \frac{\sqrt{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk}) \sin(\theta_{ij}, \theta_{il}, \theta_{jl}) \sin(\theta_{jk}, \theta_{il}, \theta_{kl}) \sin(\phi_{ij}, \phi_{ik}, \phi_{jl})}}{\sin(\theta_{jk}, \theta_{il}, \theta_{kl})^2} \mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l \tag{A17}
\]
as well as
\[
\mathbf{n}_i = \cos \theta_{ij} \mathbf{n}_j + \sin \theta_{ij} \cos \alpha^{(ijk)}_j \mathbf{n}_k + \sin \theta_{ij} \sin \alpha^{(ijk)}_j \cos \phi_{jk} \mathbf{n}_l \tag{A18}
\]
\[+ \sin \theta_{ij} \sin \alpha^{(ijk)}_j \sin \phi_{jk} \mathbf{n}_j \times \mathbf{n}_k \times \mathbf{n}_l, \tag{A19}
\]
in a (Gram-Schmidt) orthogonal basis \( \mathbf{n}_j, \mathbf{n}_k, \mathbf{n}_l \). Comparing the results provides the following set of equations:
\[
\frac{\sin(\theta_{ik}, \theta_{il}, \theta_{kl})}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})} \cos \phi_{jk} = \cos \theta_{il} \left( 1 - \tan \theta_{il} \cot \theta_{ik} \cos \phi_{ij} \cos \alpha^{(ijk)}_j \sin \alpha^{(ijl)}_{il} \right. \\
- \tan \theta_{il} \cot \theta_{ij} \cos \alpha^{(ijl)}_{il} \left( 1 - \tan \alpha^{(ijl)}_{il} \cot \alpha^{(ijk)}_{il} \cos \phi_{ij} \right), \tag{A20}
\]
\[
\frac{\sin(\theta_{ik}, \theta_{il}, \theta_{kl})}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})} \cos \phi_{ik} = \frac{\sin(\theta_{il} \cos \alpha^{(ijl)}_{il})}{\sin(\theta_{ij})} \left( 1 - \tan \alpha^{(ijl)}_{il} \cot \alpha^{(ijk)}_{il} \cos \phi_{ij} \right)
\]
which form the hyperspherical analogue of the four-parts formula.

Similarly, for the expansions of the polar vectors \( \mathbf{u}_{ijk}, \mathbf{u}_{ijl}, \mathbf{u}_{ikl}, \mathbf{u}_{jkl} \) we have
\[
\mathbf{u}_{ijk} = \frac{\sin(\theta_{ij}, \theta_{il}, \theta_{jl})}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})} \cos \theta_{kl} \mathbf{u}_{ijl} - \frac{\sin(\theta_{jk}, \theta_{jl}, \theta_{kl})}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})} \cos \theta_{il} \mathbf{u}_{ijk} + \frac{\sin(\theta_{ik}, \theta_{il}, \theta_{kl})}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})} \cos \theta_{jl} \mathbf{u}_{ikl} \\
+ \frac{\sin(\theta_{ij}, \theta_{il}, \theta_{jl}) \sin(\theta_{jk}, \theta_{jl}, \theta_{kl}) \sin(\theta_{ik}, \theta_{il}, \theta_{kl})}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk}, \theta_{jl}, \theta_{kl})} \mathbf{u}_{ijl} \times \mathbf{u}_{jkl} \times \mathbf{u}_{ikl} \tag{A21}
\]
and in terms of an orthonormal basis,

\[
\mathbf{u}_{ijk} = \left( \frac{\sin(\theta_{ij}, \theta_{il}, \theta_{jl}) \cos \theta_{kl}}{\sin(\theta_{ij}, \theta_{ik}, \theta_{jk})} - \left( \frac{\cos \phi_{jk} - \cos \phi_{jl} \cos \phi_{ij}}{\sin \phi_{jl}} \right) \cos \phi_{jl} \right) \cos \phi_{jl} \\
+ \frac{\cos \theta_{jl} \cos \phi_{il} \sin \phi_{jl} \sin(\phi_{ij}, \phi_{jk}, \phi_{jl})}{\sin(\phi_{il}, \phi_{jl}, \phi_{kl})} \mathbf{u}_{ijl} \\
+ \frac{\cos \phi_{jk} - \cos \phi_{ij} \cos \phi_{jl}}{\sin \phi_{jl}} \mathbf{u}'_{jlk} + \frac{\cos \theta_{jl} \sin(\phi_{ij}, \phi_{jk}, \phi_{jl})}{\sin \phi_{ij}} \mathbf{u}'_{lki} \\
+ \sin(\theta_{ij}, \theta_{ik}, \theta_{jl}, \theta_{kl}) \mathbf{u}_{ijkl} \times \mathbf{u}'_{jlk} \times \mathbf{u}'_{kli}. \tag{A22}
\]

Comparing the results provides

\[
\cos \phi_{jk} \cos \phi_{jl} \sin(\theta_{ij}, \theta_{ik}, \theta_{jk}) = \cos \phi_{ij} \sin \phi_{jl} \sin(\theta_{ij}, \theta_{ik}, \theta_{jk}) \\
+ \cos \theta_{kl} \sin \phi_{jl} \sin(\theta_{ij}, \theta_{il}, \theta_{jl}) \\
+ \cos^2 \phi_{jl} \cos \phi_{ij} \sin(\theta_{ij}, \theta_{ik}, \theta_{jk}) \\
+ \cos \theta_{jl} \cos \phi_{il} \sin^2 \phi_{jl} \sin(\theta_{ik}, \theta_{il}, \theta_{kl}), \tag{A23}
\]

the hyperspherical analogue of the five-parts formula.

**Appendix B: Angle Addition Formulas**

We derive the cosine addition formulae by collapsing a spherical triangle. We then extend this to see what happens when we collapse a hyperspherical tetrahedron. Recall that the generalised sine function of three variables gives the volume of a three dimensional parallelepiped defined by vectors \( \mathbf{n}_i, \mathbf{n}_j \) and \( \mathbf{n}_k \) embedded in four dimensional Euclidean space,

\[
\sin(\theta_{ij}, \theta_{ik}, \theta_{jk}) = \begin{vmatrix}
1 & \cos \theta_{ij} & \cos \theta_{ik} \\
\cos \theta_{ij} & 1 & \cos \theta_{jk} \\
\cos \theta_{ik} & \cos \theta_{jk} & 1
\end{vmatrix}^{\frac{1}{2}}. \tag{B1}
\]

When the three vectors \( \mathbf{n}_i, \mathbf{n}_j \) and \( \mathbf{n}_k \) become coplanar, the volume of the parallelepiped collapses to zero, and hence, so does the generalised sine function. When this occurs

\[
\sin^2(\theta_{ij}, \theta_{ik}, \theta_{jk}) = 0. \tag{B2}
\]

By expanding this out and completing the square in terms of \( \cos \theta_{ij} \), this reduces to

\[
(\cos \theta_{ij} - \cos \theta_{ik} \cos \theta_{jk})^2 = \sin^2 \theta_{ik} \sin^2 \theta_{jk}. \tag{B3}
\]
Solving this gives
\[
\cos \theta_{ij} = \cos \theta_{ik} \cos \theta_{jk} \pm \sin \theta_{ik} \sin \theta_{jk}.
\] (B4)

This occurs when \( \theta_{ij} = \theta_{ik} \mp \theta_{jk} \), hence giving us the standard addition formula for cosine,
\[
\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B.
\] (B5)

Similarly, the generalised sine function of six variables gives the volume of a four-dimensional parallelootope defined by vectors \( \mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k \) and \( \mathbf{n}_l \) embedded in five dimensional Euclidean space,
\[
\sin(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl}) = \left| \begin{array}{cccccc}
1 & \cos \theta_{ij} & \cos \theta_{ik} & \cos \theta_{il} \\
\cos \theta_{ij} & 1 & \cos \theta_{jk} & \cos \theta_{jl} \\
\cos \theta_{ik} & \cos \theta_{jk} & 1 & \cos \theta_{kl} \\
\cos \theta_{il} & \cos \theta_{jl} & \cos \theta_{kl} & 1
\end{array} \right|^{1/2}.
\] (B6)

When the four vectors \( \mathbf{n}_i, \mathbf{n}_j, \mathbf{n}_k \) and \( \mathbf{n}_l \) become linearly dependent, the volume of the 4-parallelepiped collapses to zero, and hence, so does the generalised sine function. When this occurs, recalling the hyperspherical sine rule,
\[
\sin^2(\theta_{ij}, \theta_{ik}, \theta_{il}, \theta_{jk}, \theta_{jl}, \theta_{kl}) = 0 \iff k_H = 0.
\] (B7)

This gives
\[
\begin{align*}
\sin(\phi_{ij}, \phi_{ik}, \phi_{il}) &= 0 \\
\sin(\phi_{ij}, \phi_{jk}, \phi_{jl}) &= 0 \\
\sin(\phi_{ik}, \phi_{jk}, \phi_{kl}) &= 0 \\
\sin(\phi_{il}, \phi_{jl}, \phi_{kl}) &= 0
\end{align*}
\] (B8)

which, in turn from the spherical case implies
\[
\begin{align*}
\phi_{ij} + \phi_{ik} + \phi_{il} &= 0 \\
\phi_{ij} + \phi_{jk} + \phi_{jl} &= 0 \\
\phi_{ik} + \phi_{jk} + \phi_{kl} &= 0 \\
\phi_{il} + \phi_{jl} + \phi_{kl} &= 0
\end{align*}
\] (B9)

and hence,
\[
\phi_{ij} + \phi_{ik} + \phi_{il} + \phi_{jk} + \phi_{jl} + \phi_{kl} = 0,
\] (B10)

the sum of the dihedral angles is zero.
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