An Asymptotically Optimal Bound for Covering Arrays of Higher Index

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Abstract

A covering array is an $N \times k$ array ($N$ rows, $k$ columns) with each entry from a $v$-ary alphabet, and for every $N \times t$ subarray, all $v^t$ tuples of size $t$ appear at least $\lambda$ times. The covering array number is the smallest number $N$ for which such an array exists. For $\lambda = 1$, the covering array number is asymptotically logarithmic in $k$, when $v,t$ are fixed. Godbole, Skipper, and Sunley proved a bound of the form $\log k + \lambda \log \log k$ for the covering array number for arbitrary $\lambda$ and $v,t$ constant. The author proved a similar bound via a different technique, and conjectured that the $\log \log k$ term can be removed. In this short note we answer the conjecture in the affirmative with an asymptotically tight upper bound. In particular, we employ the probabilistic method in conjunction with the Lambert $W$ function.

1 Introduction

Let $N,t,k,v,\lambda$ be positive integers. A covering array is an $N \times k$ array $A$, where each entry is picked from a $v$-ary alphabet, such that for every set of $t$ columns $S = \{s_1, \ldots, s_t\}$, the restriction of the columns of $A$ to $S$ has that each of the $v^t$ tuples appears in these columns at least $\lambda$ times. We say each of these $t$-tuples is covered if they appear at least $\lambda$ times in this way, and an interaction to be a set of column/value tuples of size $t$. We use the notation $\text{CA}_\lambda(N; t, k, v)$ for this object. Much research has been devoted to determine the smallest $N$ for which a covering array exists \[3\]; we define $\text{CAN}_\lambda(t, k, v)$ = the minimum $N$ for which a $\text{CA}_\lambda(N; t, k, v)$ exists. In this short note we determine an asymptotically tight upper bound on $\text{CAN}_\lambda(t, k, v)$ when $v, t$ are fixed.

For $\lambda = 1$, previous work has shown that $\text{CAN}_1(t, k, v) = \Theta_{v,t}(\log k)$ \[3\], where the hidden constant depends on $v, t$; subsequent work has attempted at improving the constant for $\log k$. When not all $t$-sets of columns need to have this coverage property, some families of so-called “variable-strength covering arrays” have been shown to exhibit sub-logarithmic growth \[12\].

For the case of general $\lambda$, it is evident that $\text{CAN}_\lambda(t, k, v) = O_{v,t}(\lambda \log k)$: one can vertically juxtapose a $\text{CA}_1 \lambda$ times; since every interaction of values in the $\text{CA}_1$ is covered at least once, the vertical duplication shows that each interaction is now covered at least $\lambda$ times.
Godbole, Skipper, and Sunley [9] proved a better upper bound, in that $\text{CAN}_\lambda(t, k, v) = O(v^t \log k + v^t \lambda \log \log k)$ using the probabilistic method [1]. More recently, the author [5] showed a similar upper bound for perfect hash families, namely a bound of the form $C_1 \log k + C_2 \lambda \log \log k + o(\lambda)$, when $v, t$ are constants (and $C_1, C_2$ are constants only depending on $v, t$).

Like Godbole, Skipper, and Sunley, this proof also used the probabilistic method, but also employed a different technique within, which we expand upon in this note. Additionally, the author conjectured that the $o(\lambda)$ term can be removed; here we show not only is this possible, but the $\log \log k$ multiplication with $\lambda$ can also be removed.

As we prove in Corollary 2 below, this is asymptotically optimal. A sketch of the asymptotics we derive in this note appears in [8]; this note provides all of the details needed for such a proof, as well as an explicit upper bound (whereas [8] does not have such a bound).

Nearly every computational technique for constructing covering arrays of index $\lambda = 1$ can be applied to the higher index setting; [6] uses a different approach, based on the sum derived in Theorem 1, and is thus also asymptotically optimal.

2 The Lambert $W$ Function

The Lambert $W$ function is defined to be the inverse of $f(W) = W \exp(W)$, and $W(x)$ is a real number if $x \geq -1/e$. If $-1/e < x < 0$, then $W(x)$ has two real solutions, as shown in Figure 1. As our setup will operate in this regime of $x$, call the larger of the two solutions $W_0(x)$, and the smaller of the two $W_{-1}(x)$.

**Lemma 1.** Let $t, k, v, \lambda$ be positive integers such that $k \geq t \geq 2$, $v \geq 2$, $p = 1/v^t \leq 1/4$, and $a = \sqrt{(1-p)^{2\lambda - p^{2\lambda}}}/(1-2p)$. Denote $x = \frac{\log(1-p)}{e((1/v^t)^{v^t a(1-p)})^{1/\lambda}}$. Then $-1/e < x < 0$.

**Proof.** $x$ is negative because $\log(1 - p) < 0$, and all other parameters are positive. The inequality $-1/e < x$ is equivalent to:

$$
(1 - p)^{1 - 1/\lambda} \left( \frac{1}{v^t a} \right)^{1/\lambda} \log \frac{1}{1 - p} < 1.
$$

Simple analysis of each of the terms in the above expression yields the lemma statement.

**Lemma 2.** Let $t, k, v, \lambda$ be positive integers such that $k \geq t \geq 2$, $v \geq 2$, $p = 1/v^t \leq 1/4$, and $a = \sqrt{(1-p)^{2\lambda - p^{2\lambda}}}/(1-2p)$. Then

$$
W_0 \left( \frac{\log(1-p)}{e((1/v^t)^{v^t a(1-p)})^{1/\lambda}} \right) < v^t.
$$

**Proof.** Note that $\log(1 - p) < 0$, and thus the argument to $W_0(\cdot)$ is negative. The argument to $W_0(\cdot)$ is strictly between $-1/e$ and 0 by Lemma 1 and so guarantees that $W_0(x)$ is a real (negative) number. $W_0(-1/e) = -1$ and $-1/e$ is the only real number for which $W_0$ achieves this value. Since $1/\log(1 - x) > 1/x$ for all $0 < x < 1$, the lemma statement can be verified with routine algebra.
We now state a simple fact about $W$ that can be verified by routine calculation, and applying the definition of $W$:

**Lemma 3.** If $ab^nc = d$, then $n = \frac{1}{\log b}W\left(\frac{1}{c}\left(\frac{d}{a}\right)^{1/c}\log b\right)$.

### 3 A Stein-Lovász-Johnson Bound

The methods of Stein [15], Lovász [11], and Johnson [10] have been applied to covering arrays, and a more modern proof appears in the work of Sarkar and Colbourn [13]. We generalize their proof to provide an asymptotically tight upper bound on $\text{CAN}_\lambda(t, k, v)$ for any positive integers $t, k, v, \lambda$.

**Theorem 1.** Let $t, k, v, \lambda$ be positive integers such that $k \geq t \geq 2$. Denote $p = 1/v^t$, and $a = \sqrt{(1-p)^2 - p^2 \lambda^{-1}}$. Then

$$\text{CAN}_\lambda(t, k, v) \leq 1 + \frac{e^{-\lambda}}{\log(1 - p)W_{-1}\left(\frac{\log(1-p)}{e((\binom{k}{t})v^t a(1-p))^{1/\lambda}}\right)}.$$

**Proof.** Let $N$ be an integer to be determined later, and let $A$ be an $N \times k$ array in which each entry is uniformly and independently selected from a $v$-ary alphabet. The probability that a given interaction $T$ is not $\lambda$-covered in $A$ is $\sum_{i=0}^{\lambda-1} \binom{N}{i} p^i (1-p)^{N-i}$. The expected number of non-$\lambda$-covered interactions in $A$, therefore, is $\binom{k}{t} v^t \sum_{i=0}^{\lambda-1} \binom{N}{i} p^i (1-p)^{N-i}$. Since for any fixed array the number of interactions not $\lambda$-covered is always an integer, if

$$\binom{k}{t} \sum_{i=0}^{\lambda-1} \binom{N}{i} p^i (1-p)^{N-i} < 1,$$

(1)
then \( A \) has positive probability of being a \( \text{CA}_\lambda \), thus proving that \( \text{CAN}_\lambda(t, k, v) \leq N \). We repeatedly find upper bounds on the left-hand side of Equation (1). We first use the Cauchy-Schwarz inequality to give an upper bound on the summation:

\[
\sum_{i=0}^{\lambda-1} \binom{N}{i} p^i (1-p)^{N-i} \leq \sqrt{\left( \sum_{i=0}^{\lambda-1} p^{2i} (1-p)^{2N-2i} \right) \left( \sum_{i=0}^{\lambda-1} \binom{N}{i}^2 \right)}.
\]

The quantity \( \sqrt{\sum_{i=0}^{\lambda-1} p^{2i} (1-p)^{2N-2i}} \) can be routinely verified to be equal to \( (1-p)^{N-\lambda+1} \sqrt{\frac{(1-p)^{2\lambda}-p^{2\lambda}}{1-2p}} \).

Since \( \sqrt{x^2 + y^2} \leq x + y \) for all \( x, y \geq 0 \), it follows that

\[
\sqrt{\sum_{i=0}^{\lambda-1} \binom{N}{i}^2} \leq \sum_{i=0}^{\lambda-1} \binom{N}{i} \leq \left( \frac{eN}{\lambda} \right)^\lambda,
\]

where the last inequality can be proven via induction on \( \lambda \). Thus, it follows that to obtain an upper bound on \( \text{CAN}_\lambda \), one needs to solve the following equation for \( N \) and add 1:

\[
\binom{k}{t} v^t (1-p)^{N-\lambda+1} \left( \frac{eN}{\lambda} \right)^\lambda = 1.
\]

By using the Lambert \( W \) function \( W(x) \), we apply Lemma 3 to obtain the following upper bound on \( N \):

\[
N \leq 1 + \frac{\lambda}{\log(1-p)} \log \left( \frac{\log(1-p)}{e(k) v^t a(1-p)^{1/\lambda}} \right).
\]

The argument to \( W(\cdot) \) is negative since the numerator is negative and the denominator is positive; additionally, it is larger than \( -1/e \), by Lemma 1. Therefore, there are two solutions \( y_0, y_{-1} \) to \( y_i = W(\cdot) \), where \( y_{-1} \leq y_0 < 0 \), as is shown in Figure 1. By Lemma 2, we must choose \( y_{-1} \), since if \( y_0 \) is chosen, then \( N < \lambda v^t \), a contradiction.

Note that one can improve both instances of \( \lambda \) to \( \lambda - 1 \) in the right-hand side of the inequality in Equation (2), which will slightly improve the constants in the derived upper bound.

**Corollary 1.** Let \( t, k, v, \lambda \) be positive integers such that \( k \geq t \geq 2 \). Denote \( p = 1/v^t \), and \( a = \sqrt{(1-p)^{2\lambda}-p^{2\lambda}} \). Then

\[
\text{CAN}_\lambda(t, k, v) \leq 1 + \frac{\lambda e}{(e-1) \log(1/(1-p))} \left( 1 + \log \left( 1 + \frac{(k) v^t a(1-p)^{1/\lambda}}{\log(1/(1-p))} \right) \right),
\]


Proof. Alzahrani and Salem [2] show that $W_{-1}(-e^{-z-1}) > -\alpha(z + 1)$, where $\alpha = e/(e - 1)$, and $z \geq 0$. Solving for $z$ using the argument from Equation (3) in the proof of Theorem 1 yields the following equality:

$$z = \log \left( \frac{(k_t) v^t a(1 - p)^{1/\lambda}}{\log(1/(1 - p))} \right).$$

Substitution of the inequality and $z$ into Equation (3) yields the corollary statement. □

**Corollary 2.** \(\text{CAN}_\lambda(t, k, v) = \Theta_v(t, k, v) (\log k + \lambda).\)

**Proof.** The upper bound is a result of Corollary 1. For the lower bound, we can assume without loss of generality that any covering array does not have two identical columns. Then it must be that a CA\(_1\) must have at least $\Omega_v(t, k, v)$ rows. To complete this array into a CA\(_\lambda\), at least $\lambda - 1$ more rows are required, showing that $\text{CAN}_\lambda(t, k, v) = \Omega_v(t, k, v) (\log k + \lambda)$. □

An analysis of Theorem 1 shows that the found upper bound is approximately equal to

$$\lambda v^t + v^t \log \left( \binom{k}{t} \right) + t \log v.$$

Since $v^t$ is much larger than $\log v^t$, we have that $\text{CAN}_\lambda(t, k, v) = O(v^t \log \binom{k}{t} + \lambda v^t)$.

## 4 Differentiation

In specific circumstances, we can improve the constants over Theorem 1. Here we analyze the situation where $\lambda = 2$, and $t, k, v$ are variable.

**Theorem 2.** Let $t, k, v$ be positive integers such that $k > t \geq 2$, and $(k, t) \neq (3, 2)$. Then

$$\text{CAN}_2(t, k, v) \leq \frac{1}{\log(1 - 1/v^t)} \left( W_{-1} \left( \frac{-e(1 - 1/v^t)v^t}{2^{(k)}_t} \right) - (v^t - 1) \log(1 - 1/v^t) - 1 \right).$$

**Proof.** The proof structure is very similar to Theorem 1. Let $A$ be an array with $N$ rows, with entries chosen uniformly at random, independently. The expected number of uncovered interactions in $A$ is $(k_t)v^t((1 - p)^N + Np(1 - p)^{N-1})$. To complete the remaining interactions, one can add $\lambda = 2$ rows for each uncovered interaction, yielding a covering array of index $\lambda$ with $N + 2(k_t)v^t((1 - p)^N + Np(1 - p)^{N-1})$ rows. The minimum occurs when the derivative of this expression with respect to $N$ is equal to 0. Solving this equation for $N$ yields the equation in the theorem statement, apart from the usage of $W_{-1}$.

Note that the argument to $W(\cdot)$ is negative. If $(k_t) > e^2/2 \approx 3.69$, this argument is strictly larger than $-1/e$, and thus by Lemmas 1 and 2, we must choose the negative branch for $W$, as was done in Theorem 1. Since $k \geq t \geq 2$, this inequality fails only for $k = t$ for any $k, t$, and for $k = 3, t = 2$. □
When we use the same lower bounds for $W_{-1}$ as done in the proof of Theorem 1, we obtain the following bound on CAN$_2$:

**Corollary 3.** Let $t, k, v$ be positive integers such that $k > t \geq 2$, and $(k, t) \neq (3, 2)$. Then

$$
\text{CAN}_2(t, k, v) \leq \frac{e^t}{e-1} \left( \log \frac{k}{t} + \log \frac{v}{v^t - 1} \right) + 1 - v^t.
$$

Routine verification shows that Corollary 3 improves upon Theorem 1 when $\lambda = 2$. Additionally, the bound for standard covering arrays obtained by Sarkar and Colbourn [14] for $\lambda = 1$ is very similar in size to that of Corollary 3 here, only a relatively small number of additional rows are needed to have every interaction guarantee to be covered (at least) twice.

5 Future Work

We plan on expanding this work in several directions. First, we have found improvements in the bounds derived above using the Lovász Local Lemma; see [12] for its definition and related usage. Second, the “differentiation” construction of Theorem 2 has been extended to arbitrary $\lambda$, but requires additional bounds not given here since the equation for $\lambda \geq 3$ does not appear to be directly analytically solvable, whereas for $\lambda = 2$ it was in terms of $W$. And third, we have applied the same techniques to covering perfect hash families and related objects (see [4] for the former’s definition, and [7] for a related object). The bounds achieved for the covering arrays resulting from these hash families are not as asymptotically strong as the results we have obtained above, and will investigate why that is the case.

A future direction that is worth pursuing is to find (1) sharper lower bounds on the Lambert $W$ function for the negative branch, (2) sharper bounds for sums of binomial coefficients, (3) tighter upper bounds for the Cauchy-Schwarz inequality, or (4) a proof technique that avoids the $W$ function entirely. The first and the fourth seem possible, whereas the second and third do not.

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