Anyonic excitations in fast rotating Bose gases

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The role of anyonic excitations in fast rotating harmonically trapped Bose gases in a fractional quantum Hall state is examined. Standard Chern-Simons anyons as well as “nonstandard” anyons obtained from a statistical interaction having Maxwell-Chern-Simons dynamics and suitable nonminimal coupling to matter are considered. Their respective ability to stabilize attractive Bose gases under fast rotation in the thermodynamical limit is studied. Stability can be obtained for standard anyons while for nonstandard anyons, stability requires that the range of the corresponding statistical interaction does not exceed the typical wavelength of the atoms.

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Consider an interacting Bose gas at zero temperature in a rotating harmonic trap with strong confinement in the direction of the rotation axis so that the system is actually two dimensional. The Hamiltonian in the rotating frame [14] and the corresponding action quoted here for further convenience are [15]

\[ H = \sum_{A=1}^{N} \frac{1}{2m}[\mathbf{p}_A - \mathbf{A}(\mathbf{x}_A)]^2 + W + V + \cdots, \]

\[ S_0 = \int x \left( \sum_{A=1}^{N} \frac{m}{2} \dot{x}_A^2 - W - V \right) + \int x \mathbf{A(x)} \cdot \mathbf{J(x,t)} + \cdots, \]

where \( \mathbf{J(x,t)} = \sum_{a} \dot{x}_a(t) \delta[\mathbf{x} - \mathbf{x}_a(t)] \), the external gauge potential \( \mathbf{A}_{1}(\mathbf{x}_a) = m \omega_1 \epsilon_{a1} \mathbf{x}_a \) yields the Coriolis force, \( W = \sum_{A=1}^{N} (m/2)(\omega_1^2 - \omega_2^2) |\mathbf{x}_A|^2 \) \((\omega_1/2\pi)\) is the rotation (resp trapped) frequency, \( V = \sum_{A,B} V(\mathbf{x}_A, \mathbf{V}(\mathbf{x}_A)) \) is the two-body potential felt by the bosons in the plane. It can be well approximated by \( V(\mathbf{x}_A) = g_2 \delta(\mathbf{x}_A) \) since the scattering between ultracold bosons is dominated by the s wave [14]. Here, \( g_2 \) can be viewed as an effective coupling constant encoding basically the effects of the harmonic axial confinement (with trapping frequency \( \omega_2/2\pi \) and localization length \( l_2 = \sqrt{1/m_2} \)) along the rotation axis on the initial three dimensional (3D) scattering properties of the atoms [16]. This will be discussed later on. In (1) and (2), the ellipses denote possible multibody interaction terms. We now consider the “limit” \( \omega = \omega_3 \) for which the trapping and centrifugal potential (nearly) balance each other.

The standard CS description of anyons [10] obtained from (1) and (2) is achieved by minimally coupling to \( S_0 \) a statistical CS gauge potential \( a_\mu : S_0 \rightarrow S_1 = S_0 + \int_{x}(\eta/4) \epsilon_{\mu\nu\rho} \partial^\alpha f^\nu f^\rho + A_\mu J_\mu \) where the CS parameter \( \eta \) is dimensionless, \( J_\mu = (\rho, J) \) with \( \rho = \rho(\mathbf{x}, t) = \sum_1 g_2 \delta(\mathbf{x} - \mathbf{x}_i(t)) \) and \( J_\mu \) is given above \( f_\mu = (\epsilon_{\mu\nu\rho} f^\nu f^\rho, f_\mu = \partial_\mu a_\nu - \partial_\nu a_\mu, f_0 = f_t) \) the statistical magnetic (resp electric) field. The equations of motion for \( a_\mu \) take the field-current identity form \( f_\mu = -(1/\eta)f_\mu \), ensuring that particle-statistical magnetic flux composite anyonic objects are formed [10]. Particle-flux coupling leads to Aharonov-Bohm type interactions. The anyonic character of the wave functions for the quasiparticles is determined by

The experimental realization of Bose-Einstein condensation (BEC) of atomic gases [1] has given rise to a rich variety of phenomena and motivated numerous investigations focused on ultracold atomic Bose gases in rotating harmonic traps. BEC confined to two dimensions have been created [2] and their response under rotation has been studied. Basically, the rotation of a BEC produces vortices in the condensate [3,4]. When the rotation frequency increases, the BEC state is destroyed and for sufficiently high frequency, a state corresponding to fractional quantum Hall effect (FQHE) [5] is expected to possibly occur [6]. In particular, when the rotation frequency is tuned to the characteristic frequency of the harmonic confining potential in the radial plane, FQHE states have been predicted to become possible ground states for the system [7]. This observation has been followed by studies focused on the FQHE for bosons with short range interactions [8,9]. FQHE states involve anyonic excitations [10]. Starting from the standard Chern-Simons (CS) realization for anyons [10], it has been shown in [11] that in the thermodynamical limit, a two-dimensional (2D) harmonically trapped rotating Bose gas with attractive interactions can be stabilized (in a FQHE state) thanks to its anyonic (quasiparticle) excitations [12].

The above mentioned description of anyons is not unique. Another realization has been investigated in [13] where the minimal coupling of the statistical gauge potential to matter, which is constrained to have Maxwell-Chern-Simons (MCS) dynamics, is supplemented by a Pauli-type coupling, as recalled below. Differences do exist between CS and MCS anyons. The former stem from a statistical interaction having finite range (whereas CS interaction has zero range) and have an additional attractive mutual interaction (absent for CS anyons). These differences cannot be observed in electronic quantum Hall systems [5] but may show up in fast rotating attractive Bose gases. We point out that the existence of this attractive mutual interaction may have an impact on the ability of MCS anyonic excitations to stabilize an attractive Bose gas in a FQHE state. We find that stability requires that the range of the MCS statistical interaction does not exceed the typical wavelength for the atoms. The description of Quantum Hall Fluids within the MCS Landau-Ginzburg theory stemming from the MCS description of anyons is also discussed.
the Aharonov-Bohm phase [10] \( \exp(i\mathcal{F} \cdot \mathbf{a} \cdot d\mathbf{x}) = \exp(i\eta) \) (C is some closed curve) that is induced when one quasiparticle moves adiabatically around another one, equivalent to a double interchange of identical quasiparticles in the wave function. Then, their statistics are controlled by \( \eta \); they have Fermi (Bose) statistics when \( \eta = 1/[(2k + 1)2\pi] \), \( k \in \mathbb{Z} \). The statistics are anyonic otherwise. This leads to the CSLG action [17] underlying the analysis presented in [11]. Since the statistical interaction is mediated by a CS gauge potential, its range is zero.

Another realization for anyons has been proposed and discussed in [13]. It is obtained by coupling a MCS statistical gauge potential  to \( S_0 \) through minimal and nonminimial (Pauli-type) coupling term, the strength of this latter being fixed to a specific value [13], namely: 

\[
\mathcal{S}_0 = \mathcal{S}_0 + i\mathcal{F} \cdot \mathbf{a} \cdot d\mathbf{x},
\]

\[-\frac{1}{e^2}e_{\mu \rho} \partial^{\mu} f^{\rho} + \frac{\eta}{4e^2} e_{\mu \rho \tau} f^{\rho} a^{\tau} - (\eta^2) f^{\rho} a^{\rho} + \frac{\eta}{4} \mathcal{F} \mathcal{F}^\dagger \]

\( (\mathbf{e}^2 \) has mass dimension 1), where the coupling constant of the Pauli term (last term in \( S_2 \)) has been already fixed to the specific value. Then, the equations of motion for \( a_\mu \) can be written as 

\[-\frac{1}{e^2}e_{\mu \rho} \partial^{\mu} f^{\rho} + (\frac{\eta}{4e^2} e_{\mu \rho \tau} f^{\rho} a^{\tau}) + \frac{\eta}{4} \mathcal{F} \mathcal{F}^\dagger = 0 \]

which is solved by (i). This latter observation has been used as a starting point to construct an effective theory reproducing the usual anyonic behavior [13]. The Aharonov-Bohm phase defining the statistics of the resulting quasiparticles still verifies \( \exp(i\mathcal{F} \cdot \mathbf{a} \cdot d\mathbf{x}) = \exp(i\eta) \) so that \( \eta \) again controls the statistics. However, the resulting MCS anyons have an additional attractive contact mutual interaction [13]. Combining \( S_2 \) with the second quantization machinery, one obtains the corresponding MCSLG action

\[
\mathcal{S} = \int_x \mathcal{L} \mathcal{D}_\phi \mathcal{D} \phi - \frac{1}{2m} \mathcal{D}_\phi \mathcal{D} \phi^2 - g_2 (\mathcal{F} \cdot \mathcal{F})^2 - U(\phi),
\]

\[
-\frac{1}{e^2}e_{\mu \rho} \partial^{\mu} f^{\rho} + \frac{\eta}{4} \mathcal{F} \mathcal{F}^\dagger = 0,
\]

\[
H = \int_x \left( \frac{1}{2m} \mathcal{D}_\phi \mathcal{D} \phi^2 + \frac{1}{2\eta^2} \mathcal{F} \mathcal{F}^\dagger + (g_2 - g_0) \right) + \nu(\gamma).
\]

In (3), \( \mathcal{F} \) is the order parameter, \( \mathcal{D}_\phi \mathcal{D} \phi = 0 \) denotes the Landau-Ginzburg (LG) potential for multibody interactions, \( U(\phi) \) is a typical interactions potential for multibody in-

\[
\mathcal{L} = \frac{1}{2m} \mathcal{D}_\phi \mathcal{D} \phi - \frac{1}{2\eta^2} \mathcal{F} \mathcal{F}^\dagger + U(\phi),
\]

\[
-\frac{1}{e^2}e_{\mu \rho} \partial^{\mu} f^{\rho} + \frac{\eta}{4} \mathcal{F} \mathcal{F}^\dagger = 0,
\]

\[
\mathcal{D}_\phi = \partial_\phi - i(\mathbf{a} \phi) + \frac{if_0}{\nu e^2},
\]

\[
\mathcal{D}_\phi = \partial_\phi - i(\mathbf{a} \phi) + \frac{if_2}{\nu e^2}.
\]

In (7), \( \omega > 0 \), the positive constants \( C_k \)'s are given by

\[
C_k = \frac{\Gamma(k - \frac{1}{2})/(2k \sqrt{\pi})}{1 - \frac{\omega}{2m\eta}},
\]

\[
g_0 = -\frac{\hbar^2}{2m\eta} \left( 1 - \frac{\Lambda_{z}}{\Lambda} \right) = -\frac{\hbar^2}{2m\eta} \left( 1 - \frac{\Lambda_{z}}{\Lambda} \right)
\]

(\( \hbar \) and \( c \) reinstated), where the rightmost relation stems from \( \hbar \omega \ll m c^2 \) which holds for current experimental values for \( \omega \) [taking \( \omega/2\pi \sim O(10^{-9}) \sim O(10^9) \) Hz as a benchmark]. In the same way, the term \( -\omega C_k \) in (7) can also be neglected [since \( \hbar \omega \ll \eta \mu c^2 \) in view of \( \Lambda_{z} \sim O(\Lambda) \)].
The quartic interaction terms in (7) can be eliminated provided $g_2$ is chosen to be
\[ g_2 = g_0. \]  
(9)

Then, neglecting for the moment (7) the small terms $-\mathcal{O}(\rho^3)$ as in [11], the ground state of (7) is obtained for those configurations satisfying $[\partial_i \gamma - e_i (\partial \gamma / \Theta^{1/2})] = 0$, $i=1,2$. This, combined with (5a), (8), and (9), further expanding the various contributions depending on $\Theta$ in powers of $\rho/m \eta^2 \epsilon^2$ and using the fact that $\rho/m \eta^2 \epsilon^2 \ll 1$ yields $\gamma^2 - (\partial \gamma / \gamma)^2 = \gamma^2 = \gamma^2 + 2 m \omega$ where, in the RHS, $\omega \eta^2 \epsilon^2 \ll 1$ still holds so that this latter equation can be accurately approximated by
\[ \Delta \gamma - \frac{(\partial \gamma)^2}{\gamma} = - \gamma \left( \frac{\gamma^2}{\eta} + 2 m \omega \right). \]  
(10)

When $\omega$ goes to zero, the solutions of (10) reduces smoothly [22] to the solutions of the Liouville equation
\[ \Delta \ln \rho = - \frac{2}{\eta} \rho. \]  
(11)

These latter can be parametrized as $\rho(z) = 4 \eta / h(z)^2 (1 + h(z)^2)^2 \cos(z_1 + i x_1)$ for any holomorphic function $h(z)$. The particular choice $h(z) = (z_1/z)^n \sin(z_1/z)$, $n \in \mathbb{N}$, gives rise to radially symmetric vortex-type solutions $\rho(r) = 4 \eta \gamma^2 / (r_0 r)^2 (r_0 / r)^2 + (r / r_0)^2$ where $r_0$ is some arbitrary length scale and $\phi = \gamma \rho^{\epsilon \eta^2 \circ \eta} \circ \eta$.

From (7)–(9), one realizes that the initial two-body coupling term $-\gamma_2$ can be compensated by the statistical interaction. In view of (7), for fixed $\eta > 0$, the particular value $g_0$ (8) represents the limiting value for $g_2$ below which the Bose gas cannot be stabilized by the statistical interaction. In the CS case, $\Lambda_\mu = 0$ and $g_0$ is negative. Then, when $g_2 = g_0$, corresponding to an initial attractive Bose gas, the system behaves as a free anyon gas whose ground state (neglecting the small $|\phi|^6$ terms) is exactly described by (10) giving rise to nonsingular finite energy matter distribution [22]. Then, one would conclude as in [11] that an attractive Bose gas may be stabilized against collapse in the thermodynamical limit by anyonic excitations stemming from a CS (zero range) statistical interaction. This conclusion is somehow altered for “nonstandard” MCS anyons. In this case, $g_0$ receives an additional positive contribution from the (statistical) magnetic energy and the magnetic coupling in (4a) and the sign of $g_0$ depends now on the relative magnitude of $\Lambda$ and $\Lambda_\mu$ [see (8)]. When $g_2 = g_0$, the system behaves again as a free MCS anyon gas whose ground state (neglecting again small $\gamma^6$ terms) is now accurately described by (10). However, this situation can be reached by an initial attractive Bose gas only if $\Lambda_\mu < 0$ corresponding to negative values for $g_0$. A MCS statistical interaction with range exceeding the typical wavelength $\Lambda$ for the atoms could not protect an attractive Bose gas from collapse.

Now, let us assume that $\Lambda_\mu < 0$ and discuss (9) from a physical viewpoint. (In most of) current experiments, 2D atomic systems are obtained by tightening the axial confinement applied to the initial (harmonically trapped) 3D system, which restricts the dynamics of the atoms along the axial direction to zero point oscillations [16]. Kinematically, the system becomes 2D while the effective coupling constant $g_2$ for the interparticle interaction depends closely on the motion of the atoms in the axial direction [16]. As shown in [16],
\[ g_2 = \frac{4 \pi h^2}{m} \frac{1}{\sqrt{2 \pi \left( \frac{1}{\Lambda_\mu} + \ln \left( \frac{B \omega_{\omega_0}}{\eta} \right) \right)}} \]  
(12)

$[B = 0.915, \epsilon$ is the typical energy for the relative motion of the atoms and $a_\mu$ is the (3D) $s$-wave scattering length] where $\epsilon \ll \omega_{\omega_0}$ must hold. This regime (called quasi-2D regime in [16]), relevant here, seems to be accessible to experiments [2,16]. Then, (9) provides $a_\mu$ is tuned to the negative value
\[ a_\mu = - \sqrt{2} n \frac{8 \pi \eta}{\Lambda} + \ln \left( \frac{B \omega_{\omega_0}}{\epsilon} \right) \]  
(13)

Typical values obtained for $\omega_{\omega_0}/\epsilon \sim O(10^{-2} - 10^3)$ are $|a_\mu|/\Lambda_\mu \sim O(10^{-1})$ [resp $|a_\mu|/\Lambda_\mu \sim O(10^{-1} - 10^{-2})$] for a CS (resp MCS) statistical interaction (assuming possibly reachable current quantum Hall states and $\Lambda_\mu \lesssim \frac{1}{2} \Lambda$). Let us discuss the possible effect of the inclusion of the next to leading order interaction terms $-|\phi|^6$ on the stability of the system [23], assuming that (9) holds. In the CS case, stability requires $g_3 > 0$, as indicated in [11]. In the MCS case, the $|\phi|^6$ term in $U(\phi)$ receives an additional positive (repulsive) contribution coming from the infinite sum in (7). Then, stability in the MCS case is obtained provided $g_3 > 1/(8 n^2 \eta^2 \epsilon^2) > 0$. The actual computation of $g_3$, which here must correspond to the quasi-2D regime, is still lacking and would need to extend the analysis reported in [16] to the case of higher order interactions within the quasi-2D regime.

QHF, usually described in the low energy regime by a CSCLG theory [17], can also be described by a MCSCLG theory as defined in (3) and (4). To compare to the CS case [17], it is convenient to add a chemical potential term to (3), $S' = S + \int d^2 \mu \phi \mu (\mu > 0)$ and set $U(\phi) = 0$ in all. Equation (5a) becomes $i D_0 \phi + (1/2m) D_0 \phi - 2 g_2 (\phi \phi) \phi = \mu \phi$ while the relevant anyonic configurations are still obtained when $\int_\mu = (-1/\eta) \int_\mu$ solving (5b). The equations of motion admit the uniform (constant) density solution minimizing the energy $H' = H - \int d^2 \mu \rho$, given by
\[ \phi = \int n_0, \]  
(12a)
\[ a + A = 0, \]  
(12b)
\[ a_0 = 0, \]  
(12c)
\[ n_0 = \frac{\mu}{2 g_2}, \]  
(12d)

where $g_2 = g_2 = 1/(2 n^2 \epsilon^2)$, similar to the usual uniform density solution supported by the CSCLG action for QHF [17]. Equation (12b) implies that the external magnetic field is screened by the statistical magnetic field. This combined with $f_0 = -(1/\eta) \rho$, yields $\nu = 2 \pi \eta$ where $\nu$ is the filling factor. When statistical transmutation occurs as expected in Hall systems,
n = 1/2k + 1, k ≥ 0. Introducing polar coordinates (r, θ) and setting φ = √(pe^ιθ), a = e_d(r)/r, a_r = a_r(0) [e_p = (sin θ, -cos θ), n ∈ Z is the winding number], one finds that H' admits static finite energy vortex solutions satisfying the boundary conditions ρ ~ n_0, a(r) ~ mα^2 for r → ∞, ρ → 0, α(r) → n for r → 0. The vortex effective magnetic flux Φ = ∫_S (f_0 + F_0) is then Φ = -2m(F_0 = ε_r^d A^d). Using n_0 = η_0, F_0 = ±p^2, Φ is related formally to the “vortex charge” Q = ∫_S (ρ - n_0) through Q = -γ_Φ = 2πmp. When statistical transmutation [10] occurs, Q = n/2k + 1 so that the vortices can be interpreted as (the analog of) the fractionally charged Laughlin quasiparticles, as it is the case in the CS description of QHF [17]. By expanding S' around (12) up to the quadratic order in the fields fluctuations, fixing the gauge freedom, integrating out the fluctuations of a_μ, one obtains the low energy effective action for the matter. It yields the following dispersion relation with a gap

ω_0^2(p) = \left( \frac{p^4}{4m^2} + p^2\frac{λ^2}{m^2} + \frac{1}{m^2}\right)(1 - 2θ_0^2) + p^2(θ_0 + 2θ_0^2), \tag{13}

θ_0 = n_0/m^2 e^2 [ω_0^2(p) = p^4/4m^2 + p^2(μ/m) + n_0^2/m^2 e^2 since θ_0 ≈ 1], so that S' describes an incompressible fluid.

Although CS and MCS frameworks both permit one to describe the expected low energy properties of QHF, MCS anyons must be considered as anyonic objects distinct from the standard CS anyons, since in particular the former feel an additional contact attraction. This can be understood from the mechanism [13] responsible for the formation of each type of these particle-flux anyonic quasiparticles. This reflects itself in each case into the origin of the field-current identity (i), which must hold in order to generate composites with anyonic statistics. Recall that (i) expresses the fact that, for a distribution of point-like particles, only localized (statistical) fields can appear. This is automatically fulfilled in the CS case (see S) since the CS statistical interaction has zero range. In the MCS case for which the interaction has finite range, (i) results from an exact cancellation between electric and Pauli-type coupling effects [13], which amounts to fix the Pauli coupling constant to a specific value (see [24]). This leads to the appearance of an additional attractive contact interaction among the resulting MCS anyons which, apart from this, behave essentially as CS anyons. The question is to examine if some effects specific to the MCS framework, in particular those related to the above contact attraction, may be experimentally observed. Clearly, the “electronic” quantum Hall systems are excluded: the fact that the repulsive Coulomb interaction among electrons cannot be manipulated combined with the specific features of the experimentally accessible observables make these systems only sensitive to the properties both shared by CS and MCS anyons in the long wavelength limit. In fast rotating Bose gases, the interaction among atoms can be manipulated, offering a way to study the possible stabilization of initially attractive Bose gases in a FQHE state through their anyonic excitations. As we have shown, stability may be somehow conditioned by the actual nature of the statistical interaction. Basically, anyons of MCS origin loose their ability to stabilize the system as the range of the MCS interaction grows. An interesting proposal to measure the statistical phase of anyons has been presented in [25]. This, combined with an experimental implementation of the analysis presented here may provide a deeper insight into the physical features of the anyonic composite quasiparticles.

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Our conventions are $\hbar = c = 1$ when not explicitly written; indices $A, B = 1, \ldots, N$ label bosons; $i, j = 1, 2$ label coordinates in the plane, $x = (x_1, x_2)$, $\epsilon_{i2} = 1$, $x_{AB} = x_A - x_B$; space-time metric is $g_{\mu\nu} = \text{diag}(+, −, −, −)$, $\mu, \nu, \ldots = 0, 1, 2$, $\epsilon_{012} = 1$, $\partial^{\mu} = \partial / \partial x^\mu$, $\partial_t = \partial / \partial t$, $\int_x = \int d^2 x$, $\int_t = \int dt$.

It can be realized that the general (static) solutions of (5b) such that $f_\mu = -(1 / \eta) J_\mu$ have infinite energy.