Variational principles for involutive systems of vector fields

Giuseppe Gaeta*
Dipartimento di Matematica, Università di Milano
via Saldini 50, I–20133 Milano (Italy)

Paola Morando†
Dipartimento di Matematica, Politecnico di Torino
Corso Duca degli Abruzzi 24, I–10129 Torino (Italy)

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Summary. In many relevant cases – e.g., in hamiltonian dynamics – a given vector field can be characterized by means of a variational principle based on a one-form. We discuss how a vector field on a manifold can also be characterized in a similar way by means of an higher order variational principle, and how this extends to involutive systems of vector fields.

Introduction

The paradigm of a vector field identified by a variational principle comes from Mechanics, and takes the form of the Euler-Lagrange or Hamilton equations, depending on the formulation of the theory.

In these cases, a vector field on a manifold $P$ (in the Lagrangian formulation, $P = TV$ with $V$ the configuration space; in the Hamiltonian one, $P = T^*V$), termed the phase space, is identified in terms of a variational principle defined by a one-form on a fiber bundle having the extended phase space $M = P \times \mathbb{R}$ as total space, with base the factor $\mathbb{R}$ corresponding to physical time.

More generally, consider a $n$-dimensional bundle $(M, \pi, B)$ on a $k$-dimensional manifold $B$; denote by $\Gamma(\pi)$ the set of smooth sections of this bundle. If $\vartheta$ is a $k$-form on $M$ satisfying certain non-degeneration conditions (depending on the fibration $\pi$), and $D$ any given domain in $B$, we consider for any $\phi \in \Gamma(\pi)$ the integral

$$I[\phi] := \int_D \phi^*(\vartheta);$$

*e-mail: giuseppe.gaeta@mat.unimi.it ; g.gaeta@tiscali.it
†e-mail: paola.morando@polito.it
thus $I$ identifies a smooth real function $I : \Gamma(\pi) \to \mathbb{R}$. The request that for a variation of $\varphi$ of order $\varepsilon$, vanishing at $\partial D$, the variation of $I$ is of order $o(\varepsilon)$, defines a variational problem (see below for more precise statements, concerning this and other concepts mentioned in this introduction).

Consider first the case where $B = \mathbb{R}$. If the equations expressing the condition $\delta I / \delta \varphi = 0$ identifies a section $\varphi$ which is the integral line of a vector field $X$, we say that $X$ is identified (up to normalization) by the variational principle given by $\vartheta$.

For $B$ higher dimensional, we would obtain field equation, and the critical sections would be $k$-dimensional submanifolds of $M$. It was shown in [7] that for $k = n - 2$ – i.e. for the higher possible degree of $\vartheta$, see below – these manifolds are actually integral manifolds of a one-dimensional module of vector fields; that is, in this case as well the variational principle identifies (up to normalization) a vector field $X$.

**Remark 1.** It is important to stress that a variational principle by itself will always identify a module (over $C^\infty(M)$) of vector fields rather than a single one; in order to single out a specific vector field from this module, one needs an additional requirement; usually this is simply a normalization condition.

The purpose of this note is twofold: on the one hand we want to illustrate how a vector field can also be identified by a maximal order ($k = n - 2$ in our present notation) variational principle, as proven in recent work [6, 7]; on the other hand we want to discuss if, under suitable conditions, a variational characterization is also possible for systems of vector fields in involution (rather than a single vector field); we will answer this question in the positive and identify this with $N(\text{d} \vartheta)$, i.e. with the characteristic distribution of the variational ideal $\mathcal{J}(\vartheta, \pi)$, see below. We will not try to give a general discussion, but just study a special class of forms $\vartheta$: those for which $\text{d} \vartheta$ is a decomposable form satisfying certain nondegeneracy conditions.

The point raised in remark 1 will also be relevant here: that is, the variational principle by itself will identify a module of vector fields rather than a finite dimensional set; we can reduce to the latter only by additional conditions.

It turns out that the convenient language to discuss this problem is provided by the theory of Cartan ideals; we will actually to a large extent make use of the framework laid down in [7], adapting it to our present purposes.

Sections 1-3 will be devoted to illustrate this framework as well as (the parts we need of) classical Cartan ideals theory. In section 4 we discuss the relation between the theory developed in previous section and reduction (in the sense of proposition 4, see sect.1), i.e. how a variational principle based on a $k$-form (with $k > 1$) on a $n$-dimensional manifold, which of course produces a system of PDEs and provides critical sections $\sigma$ corresponding to $k$-dimensional submanifolds of $M$, can also identify a ($q = n - k - 1$ dimensional) module of vector fields. The determination of critical sections $\sigma$ can then be reduced to determining a $(k - q)$-dimensional manifold $\sigma_0$ which is in away the quotient of $\sigma$ by the action of the vector fields. Section 5 will recall the results that are obtained in the “maximal degree” case $k = n - 2$, where the module is one-dimensional...
The geometry of PDEs is naturally discussed using Cartan ideals, see e.g. [1, 2, 3, 4, 11, 14].

In section 6 we briefly discuss, for the sake of completeness, the case of non-proper variational principles. In the last three sections we provide completely explicit examples, dealing respectively with the “maximally characteristic” and the “non maximally characteristic” cases (see sect.6), and with a “non proper” variational principle.

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1 Cartan ideals

In this section we will recall some basic notions from the theory of Cartan ideals, i.e. ideals of differential forms. The reader is referred to [4] for further detail, and [2, 3, 14] for modern expositions and further developments.\footnote{See e.g. [2, 8] for the use of Cartan’s ideals in the study of PDEs and in analytical mechanics, including standard variational formulation of the latter. The relation between Cartan ideals and variational problem is studied in great detail, for B one dimensional, in [9]. The geometry of PDEs is naturally discussed using Cartan ideals, see e.g. [1, 2, 3, 4, 11, 14].}

From now on \(M\) will be a smooth \(n\)-dimensional manifold; we will denote by \(i\) the canonical inclusion, so that a submanifold \(S \subset M\) will also be denoted by \(i : S \rightarrow M\).

**Definition 1.** We say that \(\mathcal{J} \subset \Lambda(M)\) is a Cartan ideal iff: (i) it is an ideal in \(\Lambda(M)\) under exterior product; (ii) \(\mathcal{J}_k := \mathcal{J} \cap \Lambda^k(M)\) is a module over \(\Lambda^0(M)\) for all \(k = 0, ..., n\). These are also rephrased as follows: (i) for all \(\eta \in \mathcal{J}, \psi \in \Lambda(M), \eta \wedge \psi \in \mathcal{J}\); and (ii) for all \(\beta_i \in \mathcal{J}_k, f_i \in \Lambda^0(M) (i = 1, 2), f_1\beta_1 + f_2\beta_2 \in \mathcal{J}_k\) (for all \(k = 0, ..., n\)).

**Definition 2.** Let \(i : S \rightarrow M\) be a smooth submanifold of \(M\); \(S\) is said to be an integral manifold of the Cartan ideal \(\mathcal{J}\) iff \(i^*(\eta) = 0\) for all \(\eta \in \mathcal{J}\). In other words, \(S \subset M\) is an integral manifold of \(\mathcal{J}\) iff all \(\eta \in \mathcal{J}\) vanish on \(S\).

The Cartan ideal \(\mathcal{J}\) is said to be generated by the forms \(\{\eta^{(\alpha)}, \alpha = 1, ..., r\}\) (with \(\eta^{(\alpha)} \in \mathcal{J}\)) if each \(\zeta \in \mathcal{J}\) can be written as \(\zeta = \sum \rho(\alpha) \wedge \eta^{(\alpha)}\) for a suitable choice of \(\rho(\alpha) \in \Lambda(\mathcal{M}), \alpha = 1, ..., r\).

**Proposition 1.** If \(\mathcal{J}\) is generated by \(\{\eta^{(\alpha)}, \alpha = 1, ..., r\}\), then \(i : S \rightarrow M\) is an integral manifold for \(\mathcal{J}\) iff \(i^*(\eta^{(\alpha)}) = 0\) for all \(\alpha = 1, ..., r\).

The Cartan ideal \(\mathcal{J}\) is said to be closed if it is closed under exterior differentiation, i.e. if \(d\eta \in \mathcal{J}\) for all \(\eta \in \mathcal{J}\). In this case one also says that \(\mathcal{J}\) is a differential ideal.

If the Cartan ideal \(\mathcal{J}\) is generated by \(\{\eta^{(\alpha)}, \alpha = 1, ..., r\}\), it can always be completed to a differential ideal by adding the \(d\eta^{(\alpha)} \notin \mathcal{J}\) to the system of generators. We denote by \(\hat{\mathcal{J}}\) the completion of the ideal \(\mathcal{J}\) obtained in this way; obviously \(\mathcal{J} \subseteq \hat{\mathcal{J}}\), the equality corresponding to the case where \(\mathcal{J}\) is closed.
Note that if \( \eta \) vanishes on \( S \), the same is true of \( d\eta \); thus, integral manifolds of \( \mathcal{J} \) are also integral manifolds of \( \hat{\mathcal{J}} \).\(^2\) We will always assume that \( \mathcal{J} \) does not include 0-forms; by the previous remark, this is not actually a limitation (but simplifies discussions).

Note that if \( \eta = d\alpha \) and \( i : S \to M \), then \( i^*(\eta) = 0 \) means that \( \alpha \) is constant on \( S \). In particular, if we deal with a set of equations \( F_a = 0 \) on \( M \), we can pass to the system \( \eta_a := dF_a = 0 \); integral manifolds for the (closed) Cartan ideal generated by the \( \eta_a \) will be manifolds on which the \( F_a \) are constant; the solution to the original problem will be provided by the manifold on which they are constant and all equal to zero.

Given a Cartan ideal \( \mathcal{J} \), we associate to any point \( x \in M \) the subspace \( D_x(\mathcal{J}) \subset T_xM \) defined by

\[
D_x(\mathcal{J}) := \{ \xi \in T_xM : \xi \downarrow \mathcal{J}_x \subset \mathcal{J}_x \}.
\]

If \( D_x(\mathcal{J}) \) has constant dimension, the Cartan ideal \( \mathcal{J} \) is said to be non singular, and the distribution \( D(\mathcal{J}) = \{ D_x(\mathcal{J}), x \in M \} \) is its characteristic distribution; any vector field \( X \in D(\mathcal{J}) \) (by this we mean that \( X(x) \in D_x(\mathcal{J}) \) at all points \( x \in M \)) is said to be a characteristic field for \( \mathcal{J} \).

**Remark 2.** Note that if all the generators \( \eta^{(\alpha)} \) of \( \mathcal{J} \) are of the same degree \( k \), then all forms in \( \mathcal{J} \) are of degree not smaller than \( k \), and \( \mathcal{J}_m = \{ 0 \} \) for \( m < k \). If \( \mathcal{J}_m = \{ 0 \} \) for \( m < k \), then \( X \in D(\mathcal{J}) \) satisfies \( X \downarrow \mathcal{J} = 0 \) for all \( \xi \in \mathcal{J}_k \), and in particular \( X \in D(\mathcal{J}) \) iff \( X \downarrow \eta^{(\alpha)} = 0 \). Indeed by definition any \( \xi \in \mathcal{J} \) is written as \( \xi = \rho(\alpha) \wedge \eta^{(\alpha)} \), and \( X \downarrow \mathcal{J} = \sigma(\alpha) \wedge \eta^{(\alpha)} \) with \( \sigma(\alpha) = X \downarrow \rho(\alpha) \).

**Definition 3.** An integral manifold for a distribution \( D \) on \( M \) is a submanifold \( i : N \to M \) such that \( i_* (T_xN) \subset D_{i(x)} \) for all \( x \in N \). In other words, any vector field tangent to \( N \) is in \( D \) (the converse is in general not true).

It should be stressed that integral manifolds of \( D(\mathcal{J}) \) are always integral manifolds of \( \mathcal{J} \), but the converse is in general not true.

**Definition 4.** The \( p \)-dimensional distribution \( D \) on \( M \) is said to be completely integrable if through each point \( x \in M \) passes a \( p \)-dimensional integral manifold of \( D \). In this case, the \( p \)-dimensional integral submanifolds are also said to be the Cauchy characteristics for \( D \).

**Proposition 2.** If \( \mathcal{J} \) is a closed nonsingular differential Cartan ideal, then \( D(\mathcal{J}) \) is completely integrable.

It should be stressed that the Cauchy characteristics of an integrable \( p \)-dimensional distribution \( D \) provide a foliation of \( M \) by \( p \)-dimensional submanifolds [13]. Thus if \( \mathcal{J} \) is a closed nonsingular Cartan ideal with \( p \)-dimensional characteristic distribution \( D(\mathcal{J}) \), then \( \mathcal{J} \) always has \( p \)-dimensional integral manifolds, and \( M \) is foliated by these (see below the notion of complete ideal).

\(^2\)In Cartan’s words, “La recherche des solutions d’un système différentiel peut toujours être ramenée à la recherche des solutions d’un système différentiel fermé” (see [4], p. 52).
The following theorem (proposition 3) is most useful in performing computations with Cartan ideals; it appears in different forms in [3, 4, 14]. Proposition 4 is an immediate consequence of it; see section 45 of [4].

**Proposition 3.** Let $\mathcal{J}$ be a nonsingular differential Cartan ideal, and let its characteristic distribution $\mathcal{D}(\mathcal{J})$ be $p$-dimensional. Then in a neighbourhood of any point $x \in M$ we can choose local coordinates $(x^1, \ldots, x^p; y^1, \ldots, y^{n-p})$ such that $\mathcal{J}$ admits a system of generators $\{\theta_1, \ldots, \theta_r\}$ with the property that, locally around $x$, the $\theta$ and $d\theta$ do not involve the variables $x^j$ nor the forms $dx^j$. ♦

The local coordinates whose existence is guaranteed by this theorem will be called Cartan canonical coordinates; if we consider locally a fibration of $M$ over $\mathbb{R}^p$ for which the $x^i$ are horizontal and the $y_j$ are vertical coordinates, $\mathcal{D}(\mathcal{J})$ spans horizontal planes identified as $y^j = \text{const}$, $j = 1, \ldots, n-p$.

**Proposition 4.** Let $\mathcal{J}$ be a nonsingular differential Cartan ideal, and let $\mathcal{G}$ be the $p$-dimensional characteristic distribution for $\mathcal{J}$; let $i: S \to M$ be a $q$-dimensional integral manifold of $\mathcal{J}$. Assume that $\mathcal{G}$ is nowhere tangent to $i(S)$, and denote by $\mathcal{G}(x)$ the local integral manifold for $\mathcal{G}$ through a point $x$. The $(p+q)$-dimensional local manifold $\Phi : \mathcal{G}(S) \to M$ defined by the union of the $G(x)$ through points in $S$ is a local integral manifold of $\mathcal{J}$. ♦

Finally, let us consider the useful notion of the complete ideal (sometimes also called characteristic ideal), see [15], related to a Cartan ideal. Consider the ideal $\mathcal{J}$ and its characteristic distribution $\mathcal{D}(\mathcal{J})$. The complete ideal $\overline{\mathcal{J}}$ is the set of forms $\omega \in \Lambda^1(M)$ which are annihilated by all vectors in $\mathcal{D}(\mathcal{J})$, i.e.

$$\overline{\mathcal{J}} := \{ \omega : X \cdot \omega = 0 \ \forall X \in D(\mathcal{J}) \};$$

note that this can and in general (i.e. unless $\mathcal{J}$ is generated by a set of one-forms) will include forms of degree lower than those in $\mathcal{J}$.

The integrability of $\mathcal{D}(\mathcal{J})$ can be studied by means of the forms $\alpha_i$ generating $\overline{\mathcal{J}}$ (this is just another version of Frobenius theorem, see [15]).

The complete ideal $\overline{\mathcal{J}}$ can always be generated – as a Cartan ideal – by a set of one-forms $\alpha_i \in \Lambda^1(M)$. In the case of interest here, i.e. for a non-singular ideal $\mathcal{J}$, these are easily built as follows: if $\{X_1, \ldots, X_p\}$ are vector fields spanning $\mathcal{D}(\mathcal{J})$ as a module, complete the set by any set of vectors $\{Y_1, \ldots, Y_{n-p}\}$ such that the $\{X_i; Y_j\}$ together span $TM$, and choose these so that $(X_i \cdot Y_j) = 0$ for all $i, j$. Then the $\alpha_i$ are the one-forms dual to the $Y_i$.

## 2 Variational principles and variational modules

In this section we recall the construction of variational modules given in [7] (see there for further detail), and its relation to standard notions in the calculus of variations.

Let $\pi : M \to B$ be a smooth bundle; we assume that $M$ is $n$-dimensional, and $B$ is a smooth manifold of dimension $k$, with $1 \leq k < n$.  

We denote, as customary, by $\Gamma(\pi)$ the set of smooth sections of the bundle $\pi : M \to B$, and by $\mathcal{V}(\pi)$ the set of vector fields in $M$ which are vertical for this fibration. For $D$ a domain in $B$, we denote by $\mathcal{V}_D(\pi) \subset \mathcal{V}(\pi)$ the set of vertical vector fields which vanish on all of $\pi^{-1}(\partial D)$. We will use such notations for all bundles.

Consider a form $\vartheta \in \Lambda^k(M)$ (not basic for the fibration $\pi$); then to any domain $D \subset B$ we associate a functional $I_D : \Gamma(\pi) \to \mathbb{R}$ by

$$I_D(\varphi) := \int_D \varphi^*(\vartheta).$$

Let $V \in \mathcal{V}(\pi)$ and $\gamma \in \Gamma(\pi)$; denote by $\psi_s$ the flow of $V$ on $M$. This induces a flow in $\Gamma$, and the flow of $\gamma$ is the one-parameter family of local sections $\tilde{\psi}_s(\gamma) := \psi_s \circ \gamma$. The variation under $V$ of $I_D$ at $\varphi \in \Gamma(\pi)$ is defined as

$$(\delta_V I_D)(\varphi) := \frac{d}{ds} \left[ \int_D (\tilde{\psi}_s(\varphi))^*(\vartheta) \right]_{s=0}. \quad (2)$$

The requirement that $(\delta_V I_D)(\varphi) = 0$ for all $V \in \mathcal{V}_D(\pi)$ [we write $\delta I_D(\varphi)$ for short] is the variational principle on $\pi : M \to B$ defined by $\vartheta$. With reference to the degree of $\vartheta$ (equal to the dimension of $B$), we say this is a variational principle of degree $k$. If $d\vartheta = 0$, then the variational principle defined by $\vartheta' = \vartheta + \alpha$ is equivalent to the one defined by $\vartheta$.

We want to exclude the possibility that the variation of $\vartheta$ be identically zero along some vertical direction; this leads us to introduce the notion of proper variational principle.

**Definition 5.** The variational principle on $\pi : M \to B$ defined by $\vartheta$ is proper if $d\vartheta$ is nowhere zero and there is no vertical field along which the variation is zero for all sections, i.e. there is no vertical field $V \in \mathcal{V}(\pi)$ such that $V \lrcorner d\vartheta = 0$. \(\Diamond\)

A section $\varphi \in \Gamma(\pi)$ is critical for $I_D$ if and only if $(\delta_V I_D)(\varphi) = 0$ whenever $V \in \mathcal{V}_D(\pi)$. A well known criterion for a section to be critical is as follows (see e.g. [10]).

**Proposition 5.** A section $\varphi \in \Gamma(\pi)$ is critical for $I_D$ if and only if $\varphi^*(V \lrcorner d\vartheta) = 0$ for all $V \in \mathcal{V}_D(\pi)$. \(\Diamond\)

We introduce now the concept of variational module, and provide an equivalent criterion for $\varphi$ to be critical in terms of this [7].

Consider a basis $\{V_1, ..., V_r\}$ (here and below, $r = n - k$) of vertical vector fields, generating $\mathcal{V}(\pi)$ as a module. Then any $V \in \mathcal{V}(\pi)$ can be written as $V = \sum_{i=1}^r f^i(x)V_i$, and $V \in \mathcal{V}_D(\pi) \subset \mathcal{V}(\pi)$ if and only if $f^i(x) = 0$ for all $x \in \pi^{-1}(\partial D)$ and for all $i = 1, ..., r$.

Define the forms $\Psi_j \in \Lambda^k(M)$ as $\Psi_j := V_j \lrcorner d\vartheta$ (for $j = 1, ..., r$). The module $\mathcal{W}(\pi, \vartheta)$ generated by $\{\Psi_1, ..., \Psi_r\}$ is the variational module associated to the variational principle over $\pi : M \to B$ defined by $\vartheta$. Note that $\mathcal{W}(\pi, \vartheta)$ does not depend on the choice of the basis $\{V_j\}$; moreover, the variational modules for $\vartheta$
and for \( \vartheta' = \vartheta + \beta \) with \( \beta \) closed, are equivalent. If \( \mathcal{W}(\vartheta, \pi) \) is \( r \)-dimensional, we say it is nondegenerate.

We can then rephrase proposition 5 as follows [7] (note that this condition is manifestly independent of \( D \)):

**Proposition 6.** A section \( \varphi \in \Gamma(\pi) \) is critical for \( I_D \) if and only if \( \varphi^*(\mathcal{W}) = 0 \), i.e. iff \( \varphi^*(\Psi) = 0 \) for all \( \Psi \in \mathcal{W}(\pi, \vartheta) \).

In studying the variational principle defined by \( \vartheta \), a central role is played by the annihilator of \( d\vartheta \). Let us hence consider the annihilator \( \mathcal{N}(\eta) \) of a form \( \eta \in \Lambda^{k+1}(M) \), i.e. the module of vector fields \( Y \) on \( M \) such that \( Y \lrcorner \eta = 0 \). It is shown in [7] that if \( \{ X; V_1, ..., V_r \} \) are \( n - k + 1 \) independent and nonzero vector fields on \( M \) \((r = n - k)\), then \( V_j \lrcorner (X \lrcorner \eta) = 0 \) for all \( j = 1, ..., r \) is equivalent to \( X \lrcorner \eta = 0 \).

By specializing to \( \eta = d\vartheta \), this implies that if \( \vartheta \in \Lambda^k(M) \) is non closed (and non basic for \( \pi : M \to B \)), then a vector field \( X \notin \mathcal{V}(\pi) \) satisfies \( X \lrcorner \mathcal{W}(\vartheta, \pi) = 0 \) iff \( X \in \mathcal{N}(d\vartheta) \).

In other words, the set of vector fields which are transversal to the fibers of \( \pi \) and annihilate \( \mathcal{W}(\vartheta, \pi) \) corresponds to the set of vector fields in \( \mathcal{N}(d\vartheta) \) which are not vertical.

Note that if \( \vartheta \) defines a proper variational principle in \( \pi : M \to B \), then (by definition 5) \( \mathcal{N}(d\vartheta) \) will not contain any vector field which is vertical for \( \pi \). We thus have the

**Lemma 1.** Let \( \vartheta \) define a proper variational principle in \((M, \pi, B)\). Then the characteristic distribution \( \mathcal{D} = D[\mathcal{J}(\vartheta, \pi)] \) of the Cartan variational ideal \( \mathcal{J}(\vartheta, \pi) \) coincides with the distribution \( \mathcal{N}(d\vartheta) \).

**Remark 3.** For a generic nonzero \( \eta \), we are not guaranteed that \( \mathcal{N}(\eta) \neq \{0\} \), nor that there are nonzero independent vectors \( \{ X; V_1, ..., V_r \} \) as above. Moreover, the rank of \( \mathcal{N}_x(\eta) := \{ \xi \in T_xM : \eta_x(\xi) = 0 \} \) could be different at different points \( x \in M \).

3 Cartan ideals and variational principles

We will now consider the Cartan ideal \( \mathcal{J} \) generated by \( \mathcal{W}(\vartheta, \pi) \); by this we mean the ideal generated by a set of generators of \( \mathcal{W}(\vartheta, \pi) \), which corresponds to a set of generators \( V_j \) for \( \mathcal{V}(\pi) \). Note that, as remarked above, this does not depend on the choice of the \( V_j \), and is invariant under adding to \( \vartheta \) a closed form.

**Definition 6.** The Cartan ideal \( \mathcal{J}(\vartheta, \pi) \) generated by \( \mathcal{W}(\vartheta, \pi) \) is the “Cartan ideal associated to the variational principle on \( \pi \) defined by \( \vartheta \)”. We will refer to it, for short, as the variational ideal.

Note that if \( (d\vartheta)_{x_0} = 0 \) at some point \( x_0 \in M \), then \( \Psi_j = \partial_j \lrcorner (d\vartheta) \) also vanish at that point, and \( D_{x_0}(\mathcal{J}) = T_{x_0}M \). Thus in order to have a nonsingular \( \mathcal{J}(\vartheta, \pi) \), we have to require that \( d\vartheta \) is nowhere zero (if the variational principle is proper, this is automatically true).
We can characterize critical sections of the variational principle on \( \pi: M \to B \) defined by \( \vartheta \) by noting that the critical sections of the variational principle on \( \pi: M \to B \) based on \( \vartheta \) are integral manifolds of the Cartan ideal \( \mathcal{J}(\vartheta, \pi) \). We can therefore rephrase proposition 6 (which was a restatement of proposition 5) in terms of Cartan ideals.

**Proposition 7.** A section \( \varphi \in \Gamma(\pi) \) is critical for the proper variational principle on \( \pi: M \to B \) defined by \( \vartheta \) if and only if \( \varphi \) is an integral manifold of the Cartan variational ideal \( \mathcal{J}(\vartheta, \pi) \).

This proposition justifies calling \( \mathcal{J}(\vartheta, \pi) \) the Cartan ideal associated to the variational principle \( \delta I_D = 0 \): indeed, it implies that in order to study (critical sections for) the variational principle \( (\delta I_D)(\varphi) = 0 \), we can just study (integral manifolds of) the Cartan ideal \( \mathcal{J}(\vartheta, \pi) \).

We stress that, more precisely, we have to study integral manifolds of \( \mathcal{J}(\vartheta, \pi) \) that are sections of \( \pi: M \to B \); this means in particular that they are of dimension \( k \) and everywhere transversal to fibers of the bundle \( \pi: M \to B \).

We have thus completely characterized critical sections \( \varphi \) for a variational principle as sections which are integral manifolds for the associated Cartan variational ideal.

In the previous section, we considered the (necessarily non-vertical, if the variational principle is proper) vector fields \( X \in \mathcal{N}(d\vartheta) \). These are, by construction, characteristic for the variational ideal \( \mathcal{J}(\pi, \vartheta) \) and will therefore be tangent to its integral manifold, i.e. – see proposition 7 – to the critical section for the variational principle.

### 4 Variational principles and reduction

Consider the variational principle on \( \pi: M \to B \) defined by \( \vartheta \in \Lambda^k(M) \); assume \( \mathcal{J} := \mathcal{J}(\vartheta, \pi) \) is nonsingular, and \( \mathcal{D} := D[\mathcal{J}(\vartheta, \pi)] \) is an integrable \( q \)-dimensional distribution.

**Remark 4.** We stress that both these assumptions are non generic; needless to say, the discussion of this section will apply only under these hypotheses.

The result of proposition 4 can be applied to reduce the problem of determining critical sections of a variational principle, i.e. \( k \)-dimensional integral manifolds of \( \mathcal{J}(\vartheta, \pi) \) transversal to the fibers of \( \pi: M \to B \), down to that of determining \( (k-q) \)-dimensional ones satisfying suitable transversality conditions with respect to the fibration \( \pi: M \to B \) and also to the foliation provided by \( \mathcal{D} \) (these conditions are automatically satisfied if \( \vartheta \) defines a proper variational principle in \( \pi: M \to B \)).

We say that the submanifold \( M_0 \subset M \) is non characteristic for \( \mathcal{J} \) if \( T_x M_0 \cap [D(\mathcal{J})]_x = \{0\} \) for all \( x \in M_0 \), i.e. if it is everywhere transversal to the characteristic distribution of the ideal. Then a local integral manifold for \( \mathcal{J} \) is specified by assigning a manifold \( M_0 \) which is integral and non characteristic for \( \mathcal{J} \), and “pulling” it along the characteristic distribution \( \mathcal{D} \).
In a less pictorial way, we build – as described in proposition 4 – a local integral manifold for $\mathcal{J}$ as a local bundle over $M_0$, with fibers corresponding to integral manifolds for $\mathcal{D}$ (see proposition 2 and the remark after it); note this only uses the Frobenius integrability of $\mathcal{D}$ [2, 4].

It should be stressed that, of course, such a general reduction is not always possible; actually when the fibers of $\pi : M \to B$ have dimension greater than two it is generally impossible to perform it (the case of two-dimensional fibers presents several peculiarities also in this respect, see section 5 below), as we now briefly discuss.

When looking for integral manifolds of $\mathcal{J}$ which are sections of $\pi : M \to B$, this reduction would require to consider the subset $\mathcal{D}_\pi \subseteq \mathcal{D}$ which is transversal to fibers of $\pi : M \to B$, and extend integral manifolds of $\mathcal{J}$ over a submanifold $B_0 \subset B$ of codimension equal to the dimension of $\mathcal{D}_\pi$ to a local critical section.

Note that several additional conditions are required for the reduction procedure to be viable: the dimension of $\mathcal{D}_\pi$ can vary even if that of $\mathcal{D}$ is constant; moreover, the involutivity of $\mathcal{D}$ does not imply, in general, involutivity and hence integrability of $\mathcal{D}_\pi$. In practice, this means that this approach can be applied to the construction of critical sections, i.e. integral manifolds of $\mathcal{J}$ which are sections of the bundle $\pi : M \to B$, only if $D[J(\vartheta, \pi)]$ is transversal to the fibers of $\pi : M \to B$; that is, there are nondegeneracy conditions which must be satisfied by $\vartheta$ or equivalently by $W(\vartheta, \pi)$. These are automatically satisfied when $\mathcal{J}$ is the variational ideal for a proper variational principle in $\pi : M \to B$.

An even more substantial obstacle is that $\mathcal{N}(d\vartheta)$ (and thus the “useful” part of $D(\mathcal{J})$, see section 2) is in general empty when $\vartheta$ does not have degree $k = n - 2$ (see [7] for a discussion of the special features of the latter case). In this case, of course, we miss the main ingredient of the reduction procedure.

Remark 5. The above discussion can be better reinterpreted in terms of the Cartan canonical coordinates (see proposition 3). We work in $\pi : M \to B$ and look for integral manifolds of a Cartan ideal $\mathcal{J}$ which are sections for $\pi$.

The Cartan coordinates define a (local) natural fibration $\kappa : M \to L$ over a $p$-dimensional manifold $L$, spanned (in the notation of proposition 3) by the coordinates $x^1, \ldots, x^p$.

Thus we have two local fibrations in $M$, i.e. $\pi : M \to B$ and $\kappa : M \to L$. The latter is such that $D(\mathcal{J})$ is transversal to fibers $\kappa^{-1}(\ell)$ for all $\ell \in L$, but in order to apply the reduction procedure to integral manifolds of $\mathcal{J}$ which are sections of $\pi : M \to B$, we need that $D(\mathcal{J})$ be transversal to fibers $\pi^{-1}(b)$ for all $b \in B$. This condition is in general not satisfied, but it is automatically met when $\mathcal{J} = J(\vartheta, \pi)$ is the variational ideal for a proper variational principle in $(M, \pi, B)$. $\Box$
5 The maximal degree case

In the maximal degree case\(^3\), i.e. for \(\vartheta \in \Lambda^k(M)\) with \(k = n - 2\), the non-degenerate form \(\eta := d\vartheta\) is of degree \(n - 1\); it is well known that in this case \(N(d\vartheta)\) is necessarily a one-dimensional module. This implies that our general construction applies here, as we discuss in this section.

5.1 Abstract results

In this case our general discussion concretizes into the following results, see [7].

**Proposition 8.** Let \(\pi : M \to B\) be a smooth fiber bundle of dimension \(n\) with base manifold \(B\) of dimension \(k = n - 2\); let \(\vartheta \in \Lambda^k(M)\) be non basic for this fibration, and such that \(\eta := d\vartheta\) is nowhere zero on \(M\). Then the Cartan ideal \(\mathcal{J}(\vartheta, \pi)\) is nonsingular and admits a one-dimensional characteristic distribution \(\mathcal{D}[\mathcal{J}(\vartheta, \pi)]\); this coincides with \(N(d\vartheta)\).

Note that vector fields in \(\mathcal{D}[\mathcal{J}(\vartheta, \pi)]\) differ only by a nonzero function, and can thus be uniquely determined by a normalization prescription (see remark 1 above). Thus a maximal degree proper variational principle over \(\pi : M \to B\) together with a normalization condition determine a unique vector field in \(M\).

In this case one can also apply the reduction procedure discussed above:

**Proposition 9.** Let \(B_0 \subset B\) be a smooth submanifold of codimension one in \(B\), and \(\pi_0 : \pi^{-1}(B_0) \to B_0\) the associated subbundle\(^4\) of \(\pi : M \to B\). Let \(\varphi_0 \in \Gamma(\pi_0)\), seen as a submanifold of \(M\), be an integral manifold for the Cartan ideal \(\mathcal{J}(\vartheta, \pi)\), nowhere tangent to integral manifolds of \(\mathcal{D}[\mathcal{J}(\vartheta, \pi)]\). Then the critical local sections for the maximal degree variational principle on \(\pi\) defined by \(\vartheta\) can be built by pulling \(\varphi_0\) along integral curves of \(\mathcal{D}[\mathcal{J}(\vartheta, \pi)]\).

Finally, let us also consider the inverse problem: given a vector field \(X\) on \(M\), characterize it, up to normalization, in terms of a maximal degree variational principle (see below for the case of Liouville vector fields).

**Proposition 10.** Let \(M\) be a smooth \(n\)-dimensional manifold, and \(X\) a vector field on \(M\). Assume there is an exact form \(\eta = d\vartheta \in \Lambda^{n-1}(M)\) such that: (i) \(X \in N(\eta)\) (ii) \(\eta\) is nowhere vanishing, (iii) \(\eta\) is not basic for the fibration \(\pi : M \to B\) over a \((n-2)\)-dimensional manifold \(B \subset M\). Then \(X\) generates the characteristic distribution of the Cartan ideal associated to the (maximal degree) variational principle on \(\pi : M \to B\) defined by \(\vartheta\).

5.2 Coordinate approach

It is worth discussing – also in view of later extensions – how the above abstract results are embodied in concrete computations using local coordinates

\(^3\)If \(k = n - 1\), we have \(d\vartheta \in \Lambda^n(M)\), so it is either degenerate or a volume form; in both cases it does not define a proper variational principle.

\(^4\)This is defined by \(M_0 = \pi^{-1}(B_0) \subset M\), with \(\pi_0\) the restriction of \(\pi\) to \(M_0\); sections of this subbundle will be denoted as \(\Gamma(\pi_0)\).
We will take coordinates \( \{x^1, \ldots, x^k \} \) on \( B \), and \( \{y^1, \ldots, y^p \} \) on the fiber. As \( k = n - 2 \), we have \( p = 2 \); we will write \( z \equiv y^1 \), \( w \equiv y^2 \) to avoid a plethora of indices.

We write \( \omega = dz^1 \wedge \ldots \wedge dz^k \) for the reference volume form in \( B \); the reference volume form in \( M \) will of course be \( \pi^* (\omega) \wedge dw \wedge d\nu \); in the following we will write, with a slight abuse of notation, \( \omega \) for \( \pi^*(\omega) \).

One should focus on \( \eta := d\nu \in \Lambda^{n-1}(M) \); we can always write any \( \eta \in \Lambda^{n-1}(M) \) in the form

\[
\eta = \sum_{\mu=1}^k A^\mu [\omega_{(\mu)} \wedge dz \wedge dw] + (-1)^k f [\omega \wedge dw] + (-1)^{k+1} g [\omega \wedge dz] ,
\]

with \( \mu = 1, 2 \), \( A^\mu, f, g \) smooth functions of \( (x, z, w) \), and \( \omega_{(\mu)} := \partial_{\mu} \omega \).

In the following, we will assume that the vector \( A = (A^1, \ldots, A^k) \) is not identically zero (if this was the case, the variational principle would not be proper).

We choose \( \partial_z \) and \( \partial_w \) as generators of \( \mathcal{V}(\pi) \), i.e. \( \Psi_1 = \partial_z \mathcal{W} \eta \), \( \Psi_2 = \partial_w \mathcal{W} \eta \). With \( \varphi \in \Gamma(\pi) \), we have

\[
\Psi_1 = (-1)^{k-1} [A^\mu (\omega_{(\mu)} \wedge dw) + (-1)^k g \omega] ;
\]

\[
\Psi_2 = (-1)^k [A^\mu (\omega_{(\mu)} \wedge dz) + (-1)^k f \omega] .
\]

\[
\varphi^* (\Psi_1) = \varphi^* [A^\mu (\partial w / \partial x^\mu) - g] \omega ;
\]

\[
\varphi^* (\Psi_2) = - \varphi^* [A^\mu (\partial z / \partial x^\mu) - f] \omega .
\]

Requiring the vanishing of both \( \varphi^*(\Psi_j) \) for \( j = 1, 2 \) means looking for solutions of two quasilinear first order PDEs; writing \( Y = A^\mu \partial_{x^\mu} \), and with \( \mathcal{L}_Y \) the Lie derivative, these are

\[
\varphi^* [\mathcal{L}_Y (z) - f] = 0 ; \quad \varphi^* [\mathcal{L}_Y (w) - g] = 0 . \tag{3}
\]

The relevant property is that the equations can be written in terms of the action of the same (nonzero) vector field \( Y \), or more precisely [2] in terms of the (non vertical, as \( Y \neq 0 \)) vector field \( W = Y + f \partial_z + g \partial_w \) on \( M \), i.e.

\[
W = \sum_{\mu=1}^{n-2} A^\mu (x; z, w) \frac{\partial}{\partial x^\mu} + f(x; z, w) \frac{\partial}{\partial z} + g(x; z, w) \frac{\partial}{\partial w} . \tag{4}
\]

Indeed, see e.g. [2], the \( \mathbb{R}^2 \)-valued function \( u(x, t) = (z(x, t), w(x, t)) \) is a solution to the system of quasilinear PDEs (3) if and only if its graph is an integral manifold for the associated characteristic system

\[
dx^\mu / ds = A^\mu \ , \ dz / ds = f \ , \ dw / ds = g .
\]

This is just the \( W \) given above, and it is thus entirely natural to call \( W \) the characteristic vector field for the maximal degree variational principle on \( \pi : M \to B \) defined by \( \vartheta \).
Note that we have a one-dimensional module of characteristic vector fields (all differing by multiplication by a nowhere zero smooth function); these define a unique direction field on $M$ [2].

Summarizing, with the above discussion we have proved that:

**Proposition 11.** The section $\varphi \in \Gamma(\pi)$ is critical for the maximal degree proper variational principle defined by $\vartheta$ if and only if it is an invariant manifold of the characteristic vector field $W$, i.e. is foliated by integral lines of $W$. ♦

If one of the $A^{\mu}$, say $A^1$, is nowhere zero, we can divide this out from $W$, and obtain a vector field of the form $Z := \partial_1 + X$, with $X \text{ } dx^1 = 0$ and hence satisfying the normalization condition $\partial_t Z = 1$.

5.3 The distribution $N(\text{d}\vartheta)$.

We will discuss in some detail, due to its relevance in our general reduction procedure, the geometry of the (one-dimensional) distribution $N(\text{d}\vartheta)$ in this case. This is generated by $W$, as seen above, so we are actually discussing properties of $W$.

We know that $W$ is tangent to sections $\varphi \in \Gamma(\pi)$ such that $\varphi^*(\Psi_1) = 0 = \varphi^*(\Psi_2)$, see above and [2]; on the other hand, $\varphi^*(\Psi_i) = 0$ means that $\Psi_i$ vanish on vector fields tangent to $\varphi$, hence vanish if evaluated on $W$. This shows that the characteristic vector field satisfies

$W \cdot \Psi_i \equiv W \cdot V_i \cdot \text{d}\vartheta = 0 \quad (i = 1, 2)$.

With local coordinates $(x, z, w)$ as before (so $V_1 = \partial_z, V_2 = \partial_w$), consider a vector field $X$ which is nonzero and non vertical; hence $V_i \cdot (X \cdot \text{d}\vartheta) = 0$ for $j = 1, 2$ means that $\chi := X \cdot \text{d}\vartheta$ does not contain $dz$ or $dw$ factors. However, this is impossible unless $X \cdot \text{d}\vartheta = 0$: The condition $V \cdot (X \cdot \text{d}\vartheta) = 0$ for all $V \in \mathcal{V}(\pi)$ implies – and is thus equivalent to – $-X \cdot \text{d}\vartheta = 0$.

Indeed, $\chi \in \Lambda^k(M)$, hence we should have $\chi = F(x, z, w)dx^1 \wedge ... \wedge dx^k$; this cannot be obtained by $\chi = X \cdot \text{d}\vartheta$ if $X$ is not vertical, i.e. $X = X_0 + \beta_1 \partial_z + \beta_2 \partial_w$ with $X_0 = \alpha_\theta \partial / \partial x^1$ nonzero.

In fact, this would mean that either $d\theta = \eta_0 \wedge dz$ or $d\theta = \eta_0 \wedge dw$, with $\eta_0$ semibasic for the fibration $\pi : M \to B$; but in this case the variational principle would not be proper.

Note that this applies to $W$ provided this is non vertical, i.e. provided the $A^{\mu}$ identifying $d\theta$ – see above – are not all identically vanishing. This is excluded by the assumption the variational principle is proper. We have thus proven that

**Proposition 12.** The characteristic vector field $W$ for the variational principle defined by $\vartheta$ satisfies $W \cdot \text{d}\vartheta = 0$, i.e. $W \in N(\text{d}\vartheta)$. ♦

Recalling that $N(\text{d}\vartheta)$ is one dimensional (see also remark 1 for its explicit description) we have in fact shown that $N(\text{d}\vartheta)$ coincides with the one-dimensional module generated by the characteristic vector field $W$ for the variational principle identified by $\vartheta$. 12
We can summarize our discussion by introducing a suitable definition:

**Definition 7.** A vector field \( W \) on \( M \) satisfying \( W \lrcorner d\vartheta = 0 \), i.e. \( W \in N(d\vartheta) \), is a **characteristic vector field** for the maximal degree proper variational principle on \( \pi : M \to B \) defined by \( \vartheta \).

### 5.4 Liouville dynamics

A vector field in the phase space \( P \) is said to be **Liouville** – or to define a Liouville dynamics – if it preserves a volume in phase space. The geometry of Liouville vector fields has been discussed by several authors in parallel with the geometry of Hamilton vector field, see e.g. [12]. Here we show how our discussion for maximal degree variational principles applies to Liouville dynamics; see [5, 6, 7] for further detail.

Note in this case there is a preferred independent variable, i.e. time. We will thus write \( B = \mathbb{R} \times Q \) and \( M = \mathbb{R} \times P \). We assume \( P \) is a connected orientable manifold, and denote by \( \Omega \) the reference volume form on it. We will choose a form \( \sigma \) such that \( d\sigma = \Omega \).

The vector field \( X \) on the phase space \( P \) is (globally) Liouville with respect to \( \Omega \) if there is a form \( \gamma \) such that

\[
X \lrcorner \Omega = d\gamma .
\]  

(5)

To \( X \) we associate a vector field \( Z = \partial_t + X \) on the extended phase space \( M = \mathbb{R} \times P \). One can then prove the following result.

**Proposition 13.** Let \( X \) be a Liouville vector field on \( P \), \( X \lrcorner \Omega = d\gamma \) and \( Z = \partial_t + X \) be the associated vector field on \( M = \mathbb{R} \times P \). Then \( Z \) is the unique characteristic vector field for the maximal degree variational principle on \( \pi : M \to B \) defined by

\[
\vartheta := \sigma + (-1)^s \gamma \wedge dt \in \Lambda^{n-2}(M)
\]

(6)

(\( s = \pm 1 \) depending on orientation) satisfying \( Z \lrcorner dt = 1 \). 

It may be worth mentioning that \( \vartheta \) can be determined via Hodge duality [6]. Denote as usual by \( *\alpha \) the Hodge dual to the form \( \alpha \); and by \( (\tilde{Z}) \) the one-form in \( M \) dual to the vector field \( Z \): if \( Z = \partial_t + f^i \partial_i \), this will be \( dt + g_{ij}f^jdx^i \), with \( g \) the metric in \( P \). Then we have:

**Proposition 14.** The form \( \vartheta \) defining the variational principle associated to the Liouville vector field \( X \) satisfies \( d\vartheta = \sqrt{|g^{-1}|} \ast \vartheta \). 

Note that this condition completely determines the variational principle: indeed it identifies \( \vartheta \) up to a closed form, which has no role in the variation of \( I(\varphi) = \int_B \varphi^*(\vartheta) \).
6 The decomposable case

As discussed in section 4, the reduction procedure based on proposition 4 is in general not viable, as \( N(d\vartheta) \) fails to exist. There are, however, cases in which the reduction procedure discussed above can be performed.

We will deal with the simplest occurrence of the case, i.e. that where \( d\vartheta \) is a decomposable form. This is enough to show the main ingredient of the procedure discussed in sect.4 at work, and to describe a multidimensional module of vector fields in terms of a variational principle. A more general discussion will be given elsewhere.

6.1 The decomposable case in general

In particular, let us consider the case where \( \eta = d\vartheta \) is a decomposable form, i.e. there are 1-forms \( \alpha_i \) \((i = 1, ..., k+1)\) such that

\[
\eta := d\vartheta = \alpha_1 \wedge ... \wedge \alpha_{k+1}.
\]

(7)

In this case we will say that \( \vartheta \) is \( d \)-decomposable.

We will denote by \( N_\pi(\eta) \) the subset of vector fields in \( N(\eta) \) which are transversal to fibers of \( \pi : M \rightarrow B \).

Definition 8. The decomposable form \( \eta \in \Lambda^{k+1}(M) \) is nondegenerate if the forms \( \{\alpha_i\} \) are independent at all points \( x \in M \). It is compatible with the fibration \( \pi \) if the dimension of \( N_\pi(\eta) \) is constant, and adapted to \( \pi : M \rightarrow B \) if \( N(\eta) \cap V(\pi) = \emptyset \), i.e. if \( N_\pi(\eta) = N(\eta) \).

Lemma 2. Let \( M \) be a \( n \)-dimensional manifold, and \( \eta \in \Lambda^{k+1}(M) \) a nondegenerate decomposable form. Then \( \mathcal{N}(\eta) \) is a \( q = (n - k - 1) \) dimensional module over \( \Lambda^0(M) \).

Proof. As \( \eta \) is decomposable and nondegenerate, we have

\[
\mathcal{N}(\eta) = N(\alpha_1) \cap ... \cap N(\alpha_{k+1}) ;
\]

note that each \( N(\alpha_i) \) spans a distribution of codimension one, hence \( \mathcal{N}(\eta) \) has codimension \( k + 1 \), i.e. dimension \( n - k - 1 \).

Equivalently, denote by \( Y_i \) the vector field dual to the one-form \( \alpha_i \); the nondegeneration of \( d\eta \) implies that the \( \{Y_1, ..., Y_{k+1}\} \) span at each point \( x \in M \) a \((k + 1)\)-dimensional subspace \( Y_x \) of \( T_x M \). The vector fields \( X \in \mathcal{N}(d\vartheta) \) are then vector fields which are in the orthogonal complement to \( Y_x \) at each point \( x \in M \), hence they span a \((n - k - 1)\)-dimensional module.

In this case, not only \( \mathcal{N}(d\vartheta) \) is not empty (see sect.4), but has dimension \( q = (n - k - 1) \). Note that \( \vartheta \) is of degree \( k \), and \( B \) of dimension \( k \); this means that critical sections will be submanifolds of \( M \) also of dimension \( k \). Thus, in order to have a proper variational principle based on a form \( \vartheta \) such that \( \eta := d\vartheta \) is decomposable and nondegenerate, a necessary (but not sufficient) condition is that \( n - k - 1 \leq k \), i.e.

\[
n \leq 2k + 1 .
\]

(8)
We will refer to the case $n = 2k + 1$ as the **maximally characteristic** case.

We also recall that for the d-decomposable form $\vartheta$ to define a proper variational principle in $(M, \pi, B)$, it is necessary that none of the vectors in $\mathcal{N}(d\vartheta)$ is vertical, i.e. that $d\vartheta$ is adapted to $\pi : M \to B$.

**Lemma 3.** Let $\vartheta \in \Lambda^k(M)$ be a d-decomposable form, such that it defines a proper variational principle in $(M, \pi, B)$ and $d\vartheta = \alpha_1 \wedge ... \wedge \alpha_{k+1}$. Then the complete ideal $\mathcal{J}(\vartheta, \pi)$ associated to the Cartan variational ideal is generated by $\{\alpha_1, ..., \alpha_{k+1}\}$.

**Proof.** As the $\alpha_i$ are independent, a vector field $Y$ in $M$ can satisfy $Y \cdot \eta = 0$ if and only if $Y \cdot \alpha_i = 0$ (this, of course, is again the remark that $\mathcal{N}(\eta) = \mathcal{N}(\alpha_1) \cap ... \cap \mathcal{N}(\alpha_{k+1})$). From this and the definition of $\mathcal{J}(\vartheta, \pi)$ the statement is immediate.

**Lemma 4.** If $\vartheta$ is such that $d\vartheta$ is decomposable, nondegenerate and compatible with the fibration $\pi$, then $\mathcal{D} = D[\mathcal{J}(\pi, \vartheta)]$ is integrable.

**Proof.** The properties assumed on $d\vartheta = \eta$ imply that the complete ideal $\mathcal{J}(\vartheta, \pi)$ is a differential ideal. In fact, $\mathcal{J}(\vartheta, \pi)$ is generated by the $\alpha_i$, and $d\eta = 0$ guarantees that $d\alpha_i \in \mathcal{J}(\vartheta, \pi)$ as well. The characteristic distribution $\mathcal{D}$ of $\mathcal{J}(\vartheta, \pi)$ is also the characteristic distribution of $\mathcal{J}(\vartheta, \pi)$, by definition, and proposition 2 implies this is integrable.

It will be convenient to state some simple general results, also to establish a convenient notation for our later discussion. We stress that here we assume the condition (8) is satisfied; see sect. 7 for the opposite case.

We work in a local chart, i.e. in $\mathbb{R}^n$ (recall that we deal with a local variational principle), and we will deal with the case of euclidean metric.\footnote{The modifications needed to take a more general metric into account are rather obvious, but dealing with this case will keep notation simpler.}

We introduce local orthogonal coordinates $\{x^1, ..., x^k\}$ in $B$, and $\{z^1, ..., z^p\}$ (with $p = n - k \leq k + 1$ for the variational principle to be proper; and necessarily $n \geq k + 2$, see footnote 3) on the fiber $F \simeq \pi^{-1}(x)$. We write $h = 2k + 1 - n$, hence $p = k + 1 - h$ and (8) implies $0 \leq h \leq k - 1$.

It is convenient to write the forms $\alpha_i$ as

$$\alpha_i = M_{ij} dx^j + L_{ia} dz^a$$

where $i = 1, ..., k + 1$ and the dummy indices $j$ and $a$ run, respectively, from 1 to $k$ and from 1 to $p$. The $(k+1) \times k$ dimensional matrix $M$ and the $(k+1) \times p$ dimensional matrix $L$ are of course functions of $(x, z)$.

As we assumed $\eta$ to be nondegenerate and adapted to the fibration $\pi$, the rank of $L$ is constant and equal to $p < k + 1$. We can thus, with a point-dependent change of coordinates (or considering linear combinations of the $\alpha_i$), take a square submatrix $L_0$ of $L$ – say the one given by its first $p$ rows – to diagonal form and set to zero the remaining rows; with a rescaling of the (new)
$z$ coordinates, $L_0$ can be assumed to be the identity. In this way, we can limit to deal with
\[
\begin{cases}
\alpha_j = dz^j + B_{jm} dx^m & \text{for } j = 1, \ldots, p; \\
\alpha_{p+i} = C_{im} dx^m & \text{for } i = 1, \ldots, h
\end{cases}
\]
(no confusion should be possible between the matrix $B$ and the base manifold of $\pi : M \to B$). Note now that
\[
\frac{\partial}{\partial z^a} \alpha_j = \begin{cases}
\delta_{aj} & \text{for } j \leq p, \\
0 & \text{for } j > p.
\end{cases}
\]
We introduce now the decomposable forms $\chi_s \in \Lambda^k(M)$ obtained by the wedge product of all the $\alpha_i$ but $\alpha_s$, with a factor $(-1)^{s-1}$. That is,
\[
\chi_s := (-1)^{s-1} \alpha_1 \wedge \ldots \wedge \alpha_{s-1} \wedge \alpha_{s+1} \wedge \ldots \wedge \alpha_{k+1}.
\]

**Lemma 5.** With $\alpha_i$ as in (9) and $\eta$ given by (7), we have
\[
\Psi_a := \frac{\partial}{\partial z^a} \eta = \chi_a
\]

**Proof.** This follows immediately from (10) and (11). \(\triangle\)

**Theorem 1.** Let $\vartheta$ define a proper variational principle in $(M, \pi, B)$. Then, with the notation introduced above where $\eta := d\vartheta$, the equations $\Delta_a = 0$ identifying critical sections $\varphi \in \Gamma(\pi)$ for the variational principle defined by $\vartheta$ are given by
\[
\varphi^*(\Psi_a) = \varphi^*(\chi_a) = 0.
\]

**Proof.** This is a restatement of our discussion, and follows immediately from Lemma 5. \(\triangle\)

The forms $\varphi^*(\Psi_a) \in \Lambda^k(B)$ can necessarily be written as $\Delta_a \omega$, where $\omega := dx^1 \wedge \ldots \wedge dx^k$ is the volume form in $B$. Thus (13) can also be written as
\[
\Delta_a = 0;
\]
these are first order nonlinear PDEs for the dependent variables $z^a$ in terms of the independent variables $x^i$.

We will now provide a compact way of writing the $\Delta_a$. Given a matrix $P$ of dimension $(k+1) \times k$, we will denote by $P_i$ the $k \times k$ matrix obtained suppressing the $i$-th row, and by $|P_i|$ its determinant with a factor $(-1)^{i+1}$.

We recall that if we have $k$ one-forms $\beta_i = A_{ij} dx^j \in \Lambda^1(\mathbb{R}^k)$, and write $\omega = dx^1 \wedge \ldots \wedge dx^k$, then
\[
\beta_1 \wedge \ldots \wedge \beta_k = \|A\| \omega.
\]
Theorem 2. Consider a variational principle on \((M, \pi, B)\) identified by \(\vartheta\) such that \(d\vartheta\) is decomposable, nondegenerate and compatible with the fibration \(\pi\). Write \(\eta = d\vartheta\) as in (7) and in general use the notations introduced in this section. Then the equations (1) identifying critical sections are written as

\[
||\hat{P}_a|| = 0 \quad \text{for} \quad a = 1, \ldots, p^6,
\]

with the \((k + 1) \times k\) matrix \(P\) given by

\[
P_{ij} := \begin{cases}
B_{ij} + \frac{\partial z^i}{\partial x^j} & \text{for} \quad i = 1, \ldots, p, \\
C_{i-p,j} & \text{for} \quad i = p + 1, \ldots, h.
\end{cases}
\]

Proof. With \(\alpha_i\) in the form (9), we have

\[
\begin{align*}
\varphi^*(\alpha_i) &= [B_{im} + (\partial z^i/\partial x^m)] dx^m & \text{for} \quad i \leq p, \\
\varphi^*(\alpha_{p+j}) &= C_{jm} dx^m & \text{for} \quad j \leq h.
\end{align*}
\]

Note that, with an abuse of notation, we write \(f\) for \(\varphi^*(f)\) when dealing with functions.

It follows from the definition (11) of \(\chi_s\), together with (12), (15) and standard properties of the pullback operation, that

\[
\varphi^*(\Psi_a) = \left[ \frac{1}{p!} \epsilon_{a_1 \ldots a_p} P_{i_1 j_1} \ldots P_{i_p j_p} C_{i_{1} j_{p+1}} \ldots C_{i_{h} j_{k}} dx^{j_1} \wedge \ldots \wedge dx^{j_k} = \right.
\]

\[
\left. = \frac{1}{k! p!} \epsilon_{i_1 \ldots i_k} \epsilon_{a_1 \ldots a_p} P_{i_1 j_1} \ldots P_{i_p j_p} C_{i_{1} j_{p+1}} \ldots C_{i_{h} j_{k}} \omega := \Delta_{\omega}
\right]
\]

with \(P\) given indeed by (16). It suffices now to note that

\[
\epsilon_{i_1 \ldots i_k} \epsilon_{a_1 \ldots a_p} P_{i_1 j_1} \ldots P_{i_p j_p} C_{i_{1} j_{p+1}} \ldots C_{i_{h} j_{k}}
\]

coincides with \(||\hat{P}_j||\).

\(\Box\)

Remark 6. For the maximally characteristic case \(n = 2k + 1\), we have \(h = 0\) and one should understand \(C = 0\) in the above discussion and formulas; see next subsection.

\(\Box\)

The expression (17) also provides a way of writing the equations \(\Delta_a = 0\) in terms of a certain set of vector fields in \(M\). Indeed, introducing the vector fields

\[
X_i := \frac{\partial}{\partial x^i} + B_{ai} \frac{\partial}{\partial z^a} \quad (i = 1, \ldots, k),
\]

we rewrite (16) as

\[
P_{ij} := \begin{cases}
X_j(z^i) & \text{for} \quad i = 1, \ldots, p, \\
C_{i-p,j} & \text{for} \quad i = p + 1, \ldots, h.
\end{cases}
\]

This makes clear that the equations \(\Delta_a = 0\) will be written in the form (we omit a combinatorial factor \([k!(p-1)!]^{-1}\))

\[
\Delta_a := \epsilon_{m_1 \ldots m_k} \epsilon_{a_1 \ldots a_{p-1}} \left[ X_{m_1}(z^{b_1}) \ldots X_{m_{p-1}}(z^{b_{p-1}}) \right] \Theta_{m_p \ldots m_k} = 0,
\]

where \(\Theta\) depends only on \(x\) and \(z\), not on the derivatives \(\partial_i z^j\).

\(\uparrow\)Note no condition is set on \(||\hat{P}_j||\) for \(j > p\).
6.2 The maximally characteristic case

Let us now consider the case \( n = 2k + 1 \); as we have seen before, in this case \( \text{dim}[\mathcal{N}(d\vartheta)] = k \). If \( \mathcal{N}(d\vartheta) \) is nowhere vertical, i.e. if \( d\vartheta \) is adapted to the fibration \( \pi : M \to B \) and hence \( \vartheta \) defines a proper variational principle, then the critical sections will be spanned by the distribution \( \mathcal{N}(d\vartheta) \).

Thus the (partial differential) equations issued by the variational principle should be equivalent to a set of (ordinary differential) equations, each of them defining a vector field on \( M \); these vector fields in turn generate the module \( \mathcal{N}(d\vartheta) \). This is indeed what happens. We will make it precise in the following statement.

**Theorem 3.** Let \( \pi : M \to B \) be a smooth fiber bundle of dimension \( n = 2k + 1 \) with base manifold \( B \) of dimension \( k \); let \( \vartheta \in \Lambda^k(M) \) such that \( d\vartheta \) is decomposable, nondegenerate and adapted to the fibration \( \pi \). Then \( \vartheta \) defines a proper variational principle in \( (M, \pi, B) \), and the variational Cartan ideal \( J(\vartheta, \pi) \) is nonsingular and admits a \( k \)-dimensional characteristic distribution \( D = D[J(\vartheta, \pi)] \); this coincides with \( \mathcal{N}(d\vartheta) \). Moreover \( D \) is completely integrable and integral manifolds of \( D \) coincide with critical sections for the variational problem defined by \( \vartheta \). \( \Diamond \)

**Proof.** First of all we note that now \( q = n - (k - 1) = k \), hence the dimension of \( D \) given in the statement agrees with our general results.

We have seen that a section \( \varphi \in \Gamma(\pi) \) is critical for the variational problem defined by \( \vartheta \) if and only if \( \varphi \) is an integral manifold for the variational ideal \( J(\vartheta, \pi) \) (see proposition 7). Moreover, in the decomposable case, the characteristic distribution \( D[J(\vartheta, \pi)] \equiv D \) of the variational ideal \( J(\vartheta, \pi) \) is a completely integrable \( k \)-dimensional distribution, see lemma 4.

Then, recalling that integral manifolds of \( D \) are also integral manifolds of \( J(\vartheta, \pi) \), we can conclude, using also the compatibility condition between \( \vartheta \) and the fibration \( \pi : M \to B \), that critical sections for the variational principle can be identified with integral manifolds of the distribution \( D[J(\vartheta, \pi)] \). \( \triangle \)

**Remark 7.** Note that the condition \( ||\hat{P}_i|| = 0 \) for all \( i = 1, ..., p \) means that \( \text{rank}(P) < k \). \( \bigcirc \)

In order to help comparison with later results, we give a corollary which is essentially a restatement of theorem 3:

**Corollary 1.** In the maximally characteristic case, a variational principle in the bundle \( (M, \pi, B) \) based on \( \vartheta \in \Lambda^k(M) \) satisfying the hypotheses of theorem 3, uniquely identifies the \( k \)-dimensional integrable distribution \( D = D[J(\vartheta, \pi)] \); this coincides with the module \( \mathcal{N}(d\vartheta) \) of vector fields which are tangent to all the critical sections, and conversely critical sections are the manifolds for which all tangent vectors are in \( \mathcal{N}(d\vartheta) \). \( \Diamond \)

Note that this remark is not interesting for the maximal degree case \( n - k = 2 \); indeed \( k + 2 = n = 2k + 1 \) enforces \( k = 1 \) (and \( n = 3 \)), i.e. we would be in a case where the variational principle is defined by a one-form, and is thus obvious it produces a vector field – or more precisely a direction field, see remark 1 above.
6.3 The non maximally characteristic case

In the more general case \( n < 2k + 1 \), we will have a situation similar to that described by Proposition 9. We will reformulate the latter for the case at hand.

Recall preliminarily that \( \mathcal{N}(d\vartheta) \) is \( q \)-dimensional, with \( q = n - k - 1 \) (see lemma 2), and \( n = 2k + 1 - h \), with \( 0 \leq h \leq k - 1 \); thus we also have \( q = k - h \). The extremal cases \( h = k - 1 \) and \( h = 0 \) correspond, respectively, to the case of maximal degree variational principles and to the maximally characteristic case. Here we consider the case \( 0 < h < k - 1 \).

**Theorem 4.** Let the \( d \)-decomposable form \( \vartheta \) define a proper variational principle in \((M, \pi, B)\). Let \( B_0 \subset B \) be a smooth \( h \)-dimensional submanifold of \( B \), and \( \pi_0 : \pi^{-1}(B_0) \to B_0 \) the associated subbundle of \( \pi : M \to B \). Let the section \( \varphi_0 \in \Gamma(\pi_0) \) be an integral manifold for the Cartan ideal \( \mathcal{J}(\vartheta, \pi) \), nowhere tangent to integral manifolds of \( D[\mathcal{J}(\vartheta, \pi)] \). Then the critical local sections for the variational principle defined by \( \vartheta \) can be built by pulling \( \varphi_0 \) along local integral manifolds of the \( q = k - h \) dimensional distribution \( D[\mathcal{J}(\vartheta, \pi)] \).

\[ \text{Proof.} \]

From lemma 4 we know that the characteristic distribution \( D[\mathcal{J}(\vartheta, \pi)] \) of the variational ideal is completely integrable and \( q \)-dimensional, with \( q = k - h \). Moreover, by proposition 4, we can build local integral manifolds of the differential ideal \( \mathcal{J}(\vartheta, \pi) \) by pulling a lower dimensional local integral manifold along the local integral manifold of the characteristic distribution \( D[\mathcal{J}(\vartheta, \pi)] \) (as all vector in \( D[\mathcal{J}(\vartheta, \pi)] \) are tangent to integral manifold of \( \mathcal{J}(\vartheta, \pi) \)).

Then, let us start from the submanifold \( \varphi_0 \subset M \): this is an integral manifold for the Cartan ideal \( \mathcal{J}(\vartheta, \pi) \), nowhere tangent to integral manifolds of \( D[\mathcal{J}(\vartheta, \pi)] \). Hence we can obtain a local integral manifolds \( \Phi \) for \( \mathcal{J}(\vartheta, \pi) \), such that \( \varphi_0 \subset \Phi \), pulling along local integral manifolds of \( D \), see proposition 4.

Recalling that \( \vartheta \) is nondegenerate and adapted to the fibration, and proposition 7, we conclude that the \( k \)-dimensional submanifolds obtained in this way are also critical sections for the variational principle defined by \( \vartheta \).

In this case we will also say that vector fields in \( \mathcal{N}(d\vartheta) \) are characteristic vector fields for the variational principle identified by \( \vartheta \), see definition 6.

**Corollary 2.** In the non maximally characteristic case, a (proper) variational principle in the \( n \)-dimensional fiber bundle \((M, \pi, B)\) based on \( \vartheta \in \Lambda^k(M) \) (with \( k + 2 < n < 2k + 1 \)) such that \( \vartheta \) is \( d \)-decomposable, nondegenerate and adapted to the fibration \( \pi \), uniquely identifies the \( q \)-dimensional \( (q = n - k - 1 < k) \) integrable distribution \( D = D[\mathcal{J}(\vartheta, \pi)] \); this coincides with the module \( \mathcal{N}(d\vartheta) \) of vector fields which are tangent to all the critical sections.

\[ \text{\Box} \]

7 Non proper variational principles

In this section we want to discuss the case where \( d\vartheta \) is decomposable, nondegenerate and compatible with the fibration \( \pi : M \to B \), but not adapted to it. That is, \( \mathcal{N}(d\vartheta) \) will include some vertical vector field.
Note that as $\mathrm{d}\vartheta$ is nondegenerate, $\mathcal{D} = \mathcal{D}[\mathcal{J}(\vartheta, \pi)]$ is a distribution, i.e. $\mathcal{N}(\mathcal{D}[\vartheta])$ has constant rank. This, in combination with the assumption $\mathrm{d}\vartheta$ is compatible with $\pi$ (recall this means that $\mathcal{D}[\vartheta]$ has constant dimension) implies that the module $\mathcal{N}(\mathcal{D}[\vartheta]) \cap \mathcal{V}(\pi)$ has also constant dimension, as it is the complementary to $\mathcal{D}[\vartheta]$ in $\mathcal{N}(\mathcal{D}[\vartheta])$.

In this case we introduce local orthogonal coordinates $\{x^1, ..., x^k\}$ in $B$, and $\{y^1, ..., y^p\}$ (with $p = n - k \geq k + 1$) on the fiber $F \simeq \pi^{-1}(x)$.

It will be convenient to separate the vertical coordinates in two subsets, i.e. a set of $k + 1$ ones, which we denote as $\{z^1, ..., z^{k+1}\}$, and a residual set of $s = p - (k + 1) > 0$ ones which we denote as $\{w^1, ..., w^s\}$. The reason for this splitting is the following.

We can write, in full generality, the forms $\alpha_i$ as

$$\alpha_i = L_{ia} \mathrm{d}y^a + M_{ij} \mathrm{d}x^j;$$

here $L$ is a $(k + 1) \times p$ dimensional matrix, and $M$ is a $(k + 1) \times k$ dimensional one (both of these are a function of the point $(x, y)$, of course). Assuming that $\eta$ is nondegenerate and compatible with the fibration $\pi$, the rank of the matrix $L$ is constant and equal to $k + 1 < p$; thus there exists a change of coordinates (depending on $\xi$) in which a $(k + 1)$-dimensional square submatrix of $L$ is diagonal (see the discussion in sect.6). The $z^a$ will be the corresponding coordinates. Thus we write

$$\alpha_i = \mathrm{d}z^i + B_{ij} \mathrm{d}x^j + G_{im} \mathrm{d}w^m. \quad (18)$$

We introduce now the decomposable forms $\chi_s \in \Lambda^k(M)$ defined as in sect.6, i.e. obtained by the wedge product of all the $\alpha_i$ but $\alpha_s$, with a factor $(-1)^{s-1}$.

**Lemma 6.** With $\eta$ given by (7), and $\alpha_i$ as in (18), we have

$$\frac{\partial}{\partial z^a} \bigwedge \eta = \chi_a \quad , \quad \frac{\partial}{\partial w^m} \bigwedge \eta = G_{jm} \chi_j. \quad (19)$$

**Proof.** The first equation is just the definition of $\chi_s$, due to (18). The second follows immediately from the expression of the $\alpha_i$ and of $\eta$. \triangle

Consider the variational principle on $(M, \pi, B)$, defined by $\vartheta$, and the associated variational ideal $\mathcal{J}(\pi, \vartheta)$. This is generated by $\{\Psi_j\}$ with $j = 1, ..., p$. We can decide in full generality that

$$\Psi_a = \begin{cases} (\partial/\partial z^a) \bigwedge \mathrm{d}\vartheta & \text{for } a = 1, ..., k + 1 \\ (\partial/\partial w^m) \bigwedge \mathrm{d}\vartheta & \text{for } a = k + 1 + m, \ m = 1, ..., s \end{cases}. \quad (20)$$

**Lemma 7.** If $\vartheta$ is such that $\mathrm{d}\vartheta$ is decomposable, nondegenerate and compatible with the fibration $\pi$, then with the choice (20), $\mathcal{J}(\pi, \vartheta)$ is generated by $\{\Psi_1, ..., \Psi_{k+1}\}$. \blacklozenge

20
Proof. It follows from (19) and (20) that
\[
\Psi_{k+1+m} = \sum_{a=1}^{k+1} G_{ma}^T \Psi_a .
\] (21)
The lemma is an immediate consequence of (21).

The equations identifying critical sections \( \varphi \) given in coordinates by \( y^a = y^a(x) \) will be obtained simply from the requirement
\[
\Delta_a := \varphi^*(\Psi_a) = 0 \quad \text{for } a = 1, ..., k+1 .
\] (22)
Indeed, if these are satisfied, the ones for \( a = k + 1 + m, \ m = 1, ..., s \) are also satisfied, see (21). Needless to say, this is just another way of seeing lemma 7.

We will now provide a compact way of writing the equations \( \Delta_a = 0 \ (a = 1, ..., k+1) \) identifying critical sections, similarly to what was done in sect.6 and using the same notation. That is, given a matrix \( P \) of dimension \((k+1) \times k\), we will denote by \( ||\hat{P}_i|| \) the determinant of the \( k \times k \) matrix obtained suppressing the \( i \)-th row (with a sign \((-1)^{i-1}\)).

Lemma 8. Consider a variational principle on \((M, \pi, B)\) identified by \( \vartheta \) such that \( d\vartheta \) is decomposable, nondegenerate and compatible with the fibration \( \pi \). Write \( \eta = d\vartheta \) as in (7), with \( \alpha_i \) as in (18). Then the equations (22) identifying critical sections are written as \( ||\hat{P}_a|| = 0 \) (for all \( a = 1, ..., k \)) for \( P \) the \((k+1) \times k\) matrix given by
\[
P_{ij} = B_{ij} + \partial \frac{\partial z^i}{\partial x^j} + G_{im} \frac{\partial w^m}{\partial x^j} ,
\] (23)
where \( B \) and \( G \) are the matrices appearing in (18).

Proof. With \( \alpha_i \) given by (18), we have
\[
\varphi^*(\alpha_i) = \left[ B_{ij} + \frac{\partial z^i}{\partial x^j} + G_{im} \frac{\partial w^m}{\partial x^j} \right] dx^j := P_{ij} dx^j .
\] (24)
Proceeding as in the proof of theorem 2, we identify the condition of vanishing of \( \varphi^*(\Psi_a) \) for \( a = 1, ..., p \) with the condition that \( ||\hat{P}_a|| = 0 \), or equivalently \( \text{rank}(P) < k \). △

We can also rewrite \( P \) in terms of the action of vector fields, similarly to the case of proper variational principles. We define
\[
X_i := \frac{\partial}{\partial x^i} + B_{ji} \frac{\partial}{\partial z^j}
\] (25)
and with these we have
\[
P_{ij} = X_j(z^i) + G_{im} X_j(w^m) .
\]
Let us now come to the reduction theorem in this case, i.e. the analogue of theorems 3 and 4. We preliminarily note that the first part of these theorems, ensuring the variational principles identifies an integrable distribution – which is just the module $\mathcal{N}(d\vartheta)$ – of vector fields, has a counterpart in this case:

**Theorem 5.** Let $\pi : M \to B$ be a smooth fiber bundle of dimension $n = 2k+1+s$ ($s > 0$) with base manifold $B$ of dimension $k$; let $\vartheta \in \Lambda^k(M)$ such that $d\vartheta$ is decomposable, nondegenerate and compatible with the fibration $\pi$. Then $\vartheta$ defines a (necessarily non proper) variational principle in $(M, \pi, B)$; the variational Cartan ideal $\mathcal{J}(\vartheta, \pi)$ is nonsingular and admits a $(k+s)$-dimensional characteristic distribution $D = D[\mathcal{J}(\vartheta, \pi)]$; this coincides with $\mathcal{N}(d\vartheta)$. Moreover $D$ is completely integrable.

**Proof.** First of all we note that now $q = n - (k-1) = k+s$, hence the dimension of $D$ given in the statement agrees with our general results. Proceeding as in the proof of theorem 3, we recall that in the decomposable case, the characteristic distribution $D$ of the variational ideal $\mathcal{J}(\vartheta, \pi)$ is a completely integrable $(k+s)$-dimensional distribution, see lemma 4.

Let us now discuss how $D = \mathcal{N}(d\vartheta)$ can be used for reduction in the spirit of section 4 in this case. From proposition 2 we have at once that integral manifolds of $D$ are also integral manifolds for the variational ideal $\mathcal{J} = \mathcal{J}(\vartheta, \pi)$. Proposition 4 would also allow to build integral manifolds of $\mathcal{J}$ starting from non-characteristic lower dimensional integral manifolds and pulling them along integral manifolds of $D$.

Note however that we are not interested in generic integral manifolds for the variational ideal $\mathcal{J}(\vartheta, \pi)$, but only in those which are also sections for the bundle $\pi : M \to B$ (i.e. critical sections). Thus, roughly speaking, we should use only the part of $D$ which is transversal to fibers $\pi^{-1}(b)$, i.e. $D_\pi$, for pulling lower dimensional integral manifolds of $\mathcal{J}$. Note that $D_\pi$ has constant dimension $r$ since $\vartheta$ is compatible with the fibration $\pi$; we assume that $r > 0$.

The problem with using $D_\pi$ to generate higher dimensional integral manifolds lies in that proposition 4 relies on the fact that $D$ is integrable; but integrability of $D$ does not imply integrability of $D_\pi$, as the commutator of transversal vector fields could fail to be transversal. Hence we are not guaranteed $D_\pi$ is an integrable distribution and in general we can not just use this for our reduction procedure (see the example in sect.10 for an illustration of this).

In the very special case where $D_\pi$ is integrable, we can state a very close analogue of theorems 3 and 4:

**Lemma 9.** Let the hypotheses of theorem 5 be verified. Assume moreover that $D_\pi \subset D$ is an integrable distribution. If $D_\pi$ has dimension $k$, then critical sections for the variational problem defined by $\vartheta$ coincide with integral manifolds for $D_\pi$.

**Proof.** Follow the proof of theorem 3, using $D_\pi$ rather than $D$.

**Lemma 10.** Let the hypotheses of theorem 5 be verified. Assume moreover that $D_\pi \subset D$ is an integrable distribution, of dimension $r = k-h$ ($0 < h < k$). Let
$B_0 \subset B$ be a smooth $h$-dimensional submanifold of $B$, $\pi_0 : \pi^{-1}(B_0) \to B_0$ the associated subbundle of $\pi : M \to B$, and $\varphi_0 \in \Gamma(\pi_0)$ be an integral manifold for the Cartan ideal $\mathcal{J}(\vartheta, \pi)$, nowhere tangent to integral manifolds of $\mathcal{D}_\pi$. Then the critical local sections for the variational principle defined by $\vartheta$ can be built by pulling $\varphi_0$ along integral manifolds of the $r = k - h$ dimensional distribution $\mathcal{D}_\pi$.

\textbf{Proof.} Follow the proof of theorem 4, using $\mathcal{D}_\pi$ rather than $\mathcal{D}$. \qed

Let us now consider the general case, i.e. the one where we are not guaranteed that the $r$-dimensional distribution $\mathcal{D}_\pi$ is integrable. We write again $r = k - h$ with $0 \leq h < k$; for $h = 0$, the role of $B_0$ and $\varphi_0$ in the theorem below is played by any point $m_0 \in \pi^{-1}(b_0)$.

\textbf{Theorem 6.} Let the hypotheses of theorem 5 hold, and let $\mathcal{D}_\pi$ be of dimension $k$. Then critical sections for the variational problem defined by $\vartheta$ are submanifolds of integral manifolds of $\mathcal{D}$, and their tangent vector fields belong to $\mathcal{D}_\pi$. \qed

\textbf{Proof.} Integral manifolds of $\mathcal{D}$ are also integral manifolds of $\mathcal{J}(\vartheta, \pi)$; thus submanifolds of the former are submanifolds of the latter. This applies in particular to submanifolds of integral manifolds of $\mathcal{D}$ whose tangent vectors are in $\mathcal{D}_\pi$; such submanifolds are $k$-dimensional and everywhere transversal to fibers of $\pi : M \to B$ and are thus sections of this bundle. This also implies they are critical sections for the variational principle defined by $\vartheta$. Conversely, consider a critical section $\varphi$ through a point $m$: this necessarily belongs to the integral submanifold through $m$ of the integrable distribution $\mathcal{D}$, and tangent vectors to $\varphi \subset M$ are necessarily in $\mathcal{D}_\pi \subset \mathcal{D}$.

\textbf{Theorem 7.} Let the hypotheses of theorem 5 hold, and let $\mathcal{D}_\pi$ be of dimension $r = k - h$, $0 < h < k$. Let $B_0 \subset B$ be a smooth $h$-dimensional submanifold of $B$, and $\pi_0 : \pi^{-1}(B_0) \to B_0$ the associated subbundle of $\pi : M \to B$. Let the section $\varphi_0 \in \Gamma(\pi_0)$ be an integral manifold for the Cartan ideal $\mathcal{J}(\vartheta, \pi)$, nowhere tangent to integral manifolds of $\mathcal{D} = \mathcal{D}(\mathcal{J}(\vartheta, \pi))$. Then the local critical sections $\varphi$ for the variational principle defined by $\vartheta$ are submanifolds of the manifolds built by pulling $\varphi_0$ along integral manifolds of the $q = k + s$ dimensional distribution $\mathcal{D}$; the tangent vector fields to $\varphi$ are in $\mathcal{D}_\pi$. \qed

\textbf{Proof.} The distribution $\mathcal{D}$ is integrable (proposition 2). Thus, by proposition 4, we can build a local integral manifold $\Phi$ of the differential ideal $\mathcal{J}(\vartheta, \pi)$ by pulling $\varphi_0 \subset M$ along the local integral manifold of $\mathcal{D}$. Note that as $\mathcal{D}_\pi$ has dimension $r = k - h$ and $\varphi$ has dimension $h$, for any point $m \in \Phi \subset M$ there is a subspace of $T_m\Phi$ which is transversal to fibers of $\pi$ and of dimension $k$. This means that there are $k$-dimensional submanifolds $\varphi \subset \Phi$ which are transversal to fibers of $\pi$, i.e. which are sections for $\pi : M \to B$. As they are also integral manifolds for $\mathcal{J}(\vartheta, \pi)$, they are indeed critical sections for the variational principle defined by $\vartheta$. The tangent vector fields to these are by construction in $\mathcal{D}$, and transversality ensures they are actually in $\mathcal{D}_\pi$. Conversely, consider a critical section $\hat{\varphi}$ such that $\varphi_0 \subset \hat{\varphi}$: necessarily this is a submanifold of $\Phi$ considered above, $\hat{\varphi} \subset \Phi$, and by unicity we conclude that actually $\hat{\varphi} = \varphi$. \qed
Corollary 3. The (necessarily non proper) variational principle in the n-dimensional fiber bundle \( (M, \pi, B) \) based on \( \vartheta \in \Lambda^k(M) \) (with \( n > 2k + 1 \)) such that \( \vartheta \) is \( d \)-decomposable, nondegenerate and compatible with the fibration \( \pi \), uniquely identifies the integrable distribution \( \mathcal{D} = D[J(\vartheta, \pi)] \), which coincides with the module \( \mathcal{N}(d\vartheta) \). If the transverse part \( \mathcal{D}_\pi \) of \( \mathcal{D} \) have positive dimension, then vector fields in \( \mathcal{D}_\pi \) – i.e. non vertical vector fields in \( \mathcal{N}(d\vartheta) \) – are tangent to all critical sections for the variational principle.

8 Example 1: maximally characteristic case

Let us consider the space \( M = \mathbb{R}^n \), seen as a fiber bundle \( (M, \pi, B) \) on the space \( B = \mathbb{R}^k \), and a \( d \)-decomposable form \( \vartheta \in \Lambda^k(M) \) defining a proper variational principle in \( (M, \pi, B) \); this implies that \( \eta = d\vartheta \) is adapted to \( \pi : M \to B \).

The simplest occurrence of the mechanism described in abstract terms in section 6.2 is for \( M = \mathbb{R}^5 \) with euclidean metric and \( B = \mathbb{R}^2 \), i.e. \( n = 5 \) and \( k = 2^8 \). We will analyze this case in full detail. We will take coordinates \((x^1, x^2, z^1, z^2, z^3)\) in \( \mathbb{R}^5 \); the space \( B \) will correspond to the \((x^1, x^2)\) (i.e. \( B \subset M \) is given by \( z^1 = z^2 = z^3 = 0 \)), so that the \( z \) represent coordinates along the fiber, i.e. vertical ones.

We write \( d\vartheta = \eta \in \Lambda^3(M) \) in the form \( \eta = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \). We choose (see the discussion in sect.6)

\[ \alpha_a = dz^a + B_{aj} dx^j. \]

Thus the explicit expression for \( \eta \) is:

\[ \eta = dz^1 \wedge dz^2 \wedge dz^3 + B_{31} dz^1 \wedge dx^1 + B_{32} dz^2 \wedge dx^1 + B_{33} dz^3 \wedge dx^1 + B_{21} dz^2 \wedge dz^1 \wedge dx^2 + B_{22} dz^2 \wedge dx^2 \wedge dx^1 + B_{11} dz^3 \wedge dz^1 \wedge dx^2 + B_{12} dz^3 \wedge dz^2 \wedge dx^1 + B_{13} dz^3 \wedge dx^1 \wedge dx^2 + (B_{21} B_{32} - B_{22} B_{31}) dx^1 \wedge dx^2 \wedge dx^1 + (B_{12} B_{31} - B_{11} B_{32}) dx^1 \wedge dx^2 \wedge dz^1 + (B_{11} B_{22} - B_{12} B_{21}) dx^1 \wedge dx^2 \wedge dz^2 . \]

In this case, the variational ideal \( J(\vartheta, \pi) \) is generated by the three 2-forms

\[ \psi_a := (\partial/\partial z^a) \wedge \eta. \]

They are given explicitly by

\[ \psi_1 = dz^2 \wedge dz^3 + B_{31} dz^1 \wedge dx^1 + B_{32} dz^2 \wedge dx^1 + B_{33} dz^3 \wedge dx^1 + B_{22} dz^3 \wedge dz^1 \wedge dx^2 + (B_{21} B_{32} - B_{22} B_{31}) dx^1 \wedge dx^2 , \]

\[ \psi_2 = dz^3 \wedge dz^1 + B_{11} dz^3 \wedge dx^1 + B_{12} dz^3 \wedge dx^2 - B_{31} dz^1 \wedge dx^2 + (B_{12} B_{31} - B_{11} B_{32}) dx^1 \wedge dx^2 , \]

\[ \psi_3 = dz^1 \wedge dz^2 + B_{21} dz^1 \wedge dx^1 + B_{22} dz^1 \wedge dx^2 - B_{11} dz^2 \wedge dx^1 + B_{12} dz^2 \wedge dx^2 + (B_{11} B_{22} - B_{12} B_{21}) dx^1 \wedge dx^2 . \]

*We need \( 1 < k < n - 2 \): in the case \( k = 1 \) the variational principle identifies a vector field in a standard way, and \( k = n - 2 \) gives a maximal degree variational principle.
Let us now consider a section $\varphi$, described in coordinates by $z^a = z^a(x^1, x^2)$. We write $\varphi^*(\psi_a) := \Delta_a dx^1 \wedge dx^2 = \Delta_a \omega$, and $\varphi$ is critical if and only if the $z^a(x^1, x^2)$ satisfy the equations $\Delta_a = 0$ for $a = 1, 2, 3$.

By explicit computations, and writing $\partial_i := \partial/\partial x^i$, we obtain that these equations are:

$$
\Delta_1 := (\partial_1 z^2) (\partial_2 z^3) - (\partial_2 z^2) (\partial_1 z^3) - B_{31} (\partial_2 z^2) + B_{32} (\partial_1 z^2) + B_{21} (\partial_2 z^3) - B_{22} (\partial_1 z^3) + B_{21} B_{32} - B_{22} B_{31} = 0 ,
$$

$$
\Delta_2 := (\partial_2 z^1) (\partial_1 z^3) - (\partial_1 z^1) (\partial_2 z^3) + B_{31} (\partial_2 z^1) - B_{32} (\partial_1 z^1) + B_{12} (\partial_1 z^3) - B_{11} (\partial_2 z^3) + B_{12} B_{31} - B_{11} B_{32} = 0 ,
$$

$$
\Delta_3 := (\partial_1 z^1) (\partial_2 z^2) - (\partial_2 z^1) (\partial_1 z^2) - B_{21} (\partial_2 z^1) + B_{22} (\partial_1 z^1) + B_{11} (\partial_2 z^2) - B_{12} (\partial_1 z^2) + B_{11} B_{22} - B_{12} B_{21} = 0 .
$$

These equations can also be obtained (see theorem 2 and remark 7) by requiring that rank($P$) $< 2$, with $P$ the matrix given by

$$
P = \begin{pmatrix}
B_{11} + (\partial_1 z^1) & B_{12} + (\partial_2 z^1) \\
B_{21} + (\partial_1 z^2) & B_{22} + (\partial_2 z^2) \\
B_{31} + (\partial_1 z^3) & B_{32} + (\partial_2 z^3)
\end{pmatrix}
$$

Indeed, $\Delta_a$ is the determinant of the matrix $\hat{P}_a$ obtained from $P$ by elimination of its $a$-th row.

Note also that introducing the vector fields

$$
X_i = \frac{\partial}{\partial x^i} + B_{ai} \frac{\partial}{\partial z^a}
$$

the matrix $P$ is rewritten as

$$
P = \begin{pmatrix}
X_1(z^1) & X_2(z^1) \\
X_1(z^2) & X_2(z^2) \\
X_1(z^3) & X_2(z^3)
\end{pmatrix}
$$

and the condition rank($P$) $< 2$ reads

$$
e_{ijk} X_1(z^j) X_2(z^k) = 0 \quad i = 1, 2, 3 .
$$

The characteristic distribution $\mathcal{D} = D[\mathcal{J}(\vartheta, \pi)]$ associated to the variational ideal $\mathcal{J}(\vartheta, \pi)$ is given by the vector field $Y$ on $M$ such that $Y \cdot \psi_a = 0$ for $a = 1, 2, 3$; by lemma 1, $\mathcal{D}$ coincides with $N(\vartheta)$. We have by explicit computation that $N(\eta)$ is a 2-dimensional integrable distribution generated by

$$
Y_1 = (\partial/\partial x^1) - [B_{11} (\partial/\partial z^1) + B_{21} (\partial/\partial z^2) + B_{31} (\partial/\partial z^3)] ,
$$

$$
Y_2 = (\partial/\partial x^2) - [B_{12} (\partial/\partial z^1) + B_{22} (\partial/\partial z^2) + B_{32} (\partial/\partial z^3)] .
$$

Then, critical sections for the variational principle associated to $\vartheta$ can be obtained as sections of $\pi : M \to B$ that are integral manifold (in the sense of definition 3) for the characteristic distribution $\mathcal{D} = D[\mathcal{J}(\vartheta, \pi)]$ generated by $Y_1$ and $Y_2$. 

25
In particular, let us consider a section $\varphi$ of $\pi : M \to B$ given by $z^a = \varphi^a(x^1, x^2)$, for $a = 1, 2, 3$. It is immediate to check that a vector field

$$X = f^1 \frac{\partial}{\partial z^1} + f^2 \frac{\partial}{\partial z^2} + F^1 \frac{\partial}{\partial z^1} + F^2 \frac{\partial}{\partial z^2} + F^3 \frac{\partial}{\partial z^3} \quad (26)$$

is tangent to the section $\varphi$ if and only if

$$F^a = f^1 \frac{\partial \varphi^a}{\partial x^1} + f^2 \frac{\partial \varphi^a}{\partial x^2}, \quad a = 1, 2, 3.$$

Looking for sections of $\pi$ which are integral manifolds of the characteristic distribution $D$ is equivalent to requiring that the vector field $X$ defined in (26) belongs to $D$. By (26), $X = f^1 Y_1 + f^2 Y_2$ if and only if

$$(\partial \varphi^1 / \partial x^1)f^1 + (\partial \varphi^1 / \partial x^2)f^2 = -f^1 B_{11} - f^2 B_{12},$$

$$(\partial \varphi^2 / \partial x^1)f^1 + (\partial \varphi^2 / \partial x^2)f^2 = -f^1 B_{21} - f^2 B_{22}$$

$$(\partial \varphi^3 / \partial x^1)f^1 + (\partial \varphi^3 / \partial x^2)f^2 = -f^1 B_{31} - f^2 B_{32}$$

It is a trivial computation to prove that these equations are equivalent to $\Delta_a = 0$, just eliminating the variables $f^i$.

Then, critical section for the variational principle defined by $\vartheta$ can be obtained as integral manifold of the characteristic distribution $D = D[\mathcal{J}(\vartheta, \pi)]$ generated by $Y_1$ and $Y_2$.

9 Example 2: non maximally characteristic case

Let us now consider a non maximally characteristic case, i.e. a case with $h \neq 0$ (see sect.6.3). The simplest such case is obtained for $n = 6$ and $k = 3$, with $h = 2k + 1 - n = 1$.

Thus we consider as $M$ the euclidean $\mathbb{R}^6$ space, fibered over $B = \mathbb{R}^3$. We denote by $(x^1, x^2, x^3)$ coordinates on $B$, and by $(z^1, z^2, z^3)$ coordinates in the fibers $\pi^{-1}(b)$. Proceeding according to our general discussion, we write

$$\begin{align*}
\alpha_a &= dz^a + B_{ak} dx^k \quad (a = 1, 2, 3), \\
\alpha_4 &= C_k dx^k,
\end{align*}$$

and $\eta = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_4$. Note that $(\partial/\partial z^a) \cdot \alpha_m = \delta_{am}$. Hence

$$\begin{align*}
\Psi_1 &= \alpha_2 \wedge \alpha_3 \wedge \alpha_4 = \chi_1, \\
\Psi_2 &= -\alpha_1 \wedge \alpha_3 \wedge \alpha_4 = \chi_2, \\
\Psi_3 &= \alpha_1 \wedge \alpha_2 \wedge \alpha_4 = \chi_3.
\end{align*}$$

In considering the pullbacks $\varphi^*(\Psi_a)$, it is convenient to introduce $\omega = dx^1 \wedge dx^2 \wedge dx^3$ and write $\varphi^*(\Psi_a) = \Delta_a \cdot \omega$. Note that

$$\begin{align*}
\varphi^*(\alpha_a) &= (\partial z^a / \partial x^k + B_{ak}) dx^k := F_{ak} dx^k \quad \text{for } a = 1, 2, 3, \\
\varphi^*(\alpha_4) &= C_k dx^k.
\end{align*}$$
Therefore, with standard algebra,

\[ \varphi^*(\Psi_a) = (1/2) \epsilon_{abc} F_{by} F_{cz} C_\sigma \, dx^b \wedge dx^c \wedge dx^\sigma \]

and hence, omitting a constant (1/12) factor,

\[ \Delta_a = \epsilon_{abc} \epsilon_{\mu
u\sigma} F_{by} F_{cz} C_\sigma . \]

The equations \( \Delta_a = 0 \) for \( a = 1, 2, 3 \) can also be written in terms of the matrix

\[
P = \begin{pmatrix}
B_{11} + \partial_1 z_1^1 & B_{12} + \partial_2 z_1^1 & B_{13} + \partial_3 z_1^1 \\
B_{21} + \partial_1 z_2^1 & B_{22} + \partial_2 z_2^1 & B_{23} + \partial_3 z_2^1 \\
B_{31} + \partial_1 z_3^1 & B_{32} + \partial_2 z_3^1 & B_{33} + \partial_3 z_3^1 \\
C_1 & C_2 & C_3
\end{pmatrix}
\]

as the requirement that all the three-dimensional submatrices \( \tilde{P}_a \) obtained deleting from \( P \) the \( a \)-th row, for \( a = 1, 2, 3 \) have zero determinant. Note this does not set any requirement on \( \tilde{P}_4 \).

We can rewrite the matrix \( P \) in terms of three vector fields, transversal to the fibers of \( \pi : M \to B \) and defined as

\[ X_i := \frac{\partial}{\partial x^i} + B_{ai} \frac{\partial}{\partial z^a} \quad (i = 1, 2, 3) . \]

With these, \( P \) is rewritten as

\[
P = \begin{pmatrix}
X_1(z^1) & X_1(z^2) & X_1(z^3) \\
X_2(z^1) & X_2(z^2) & X_2(z^3) \\
X_3(z^1) & X_3(z^2) & X_3(z^3) \\
C_1 & C_2 & C_3
\end{pmatrix} .
\]

The equations \( \Delta_a = 0 \) are then written as

\[
\Delta_1 := [X_2(z^2)X_3(z^3) - X_2(z^3)X_3(z^2)] C_1 + [X_2(z^3)X_3(z^1) - X_2(z^1)X_3(z^3)] C_2 + [X_2(z^1)X_3(z^2) - X_2(z^2)X_3(z^1)] C_3 ;
\]

\[
\Delta_2 := [X_1(z^2)X_3(z^3) - X_1(z^3)X_3(z^2)] C_1 + [X_1(z^3)X_3(z^1) - X_1(z^1)X_3(z^3)] C_2 + [X_1(z^1)X_3(z^2) - X_1(z^2)X_3(z^1)] C_3 ;
\]

\[
\Delta_3 := [X_1(z^2)X_2(z^3) - X_1(z^3)X_2(z^2)] C_1 + [X_1(z^3)X_2(z^1) - X_1(z^1)X_2(z^3)] C_2 + [X_1(z^1)X_2(z^2) - X_1(z^2)X_2(z^1)] C_3 .
\]

Let us now consider \( \mathcal{N}(\eta) \). Writing generic vector fields in the form

\[ Y = f^i \frac{\partial}{\partial x^i} + F^a \frac{\partial}{\partial z^a} , \]

these are in \( \mathcal{N}(\eta) \) if the coefficients satisfy the relations

\[ F^a = -B_{ai} f^i , \quad C_k f^k = 0 . \]
The vector fields satisfying these conditions form a two dimensional module; we can take as generators of $N(\eta)$ e.g. the vector fields

\[
Y_1 = C_3(\partial/\partial x^1) - C_4(\partial/\partial x^2) + [B_{a3}C_1 - B_{a1}C_3] (\partial/\partial x^a), \\
Y_2 = C_3(\partial/\partial x^2) - C_4(\partial/\partial x^3) + [B_{a3}C_2 - B_{a2}C_3] (\partial/\partial x^a).
\]

10 Example 3: non proper variational principle

As a third and final example we will consider a case where $d\theta$ admits an annihilating vertical vector field, i.e. where the variational principle defined by $\theta$ is non proper, and $\eta$ not adapted to the fibration $\pi : M \to B$. We will of course require that $\eta$ is compatible with $\pi : M \to B$.

The simplest such case of interest in the present context is obtained for $k = 2$ and $n = 6$; note here $n > 2k + 1$.

We will take coordinates $(x^1, x^2, z^1, z^2, z^3, w)$ in euclidean $\mathbb{R}^6$; the $(x^1, x^2)$ will be coordinates in the space $B$, and the $(z^a, w)$ represent coordinates along the fibers, i.e. vertical ones.

Consider a form $\vartheta \in \Lambda^2(M)$ such that $\eta = d\vartheta$ is nondegenerate and decomposable; we write it as $\eta = \alpha_1 \wedge \alpha_2 \wedge \alpha_3$, and choose (see sect.7)

\[
\alpha_1 = dz^1 + B_{1k}dx^1 + B_{12}dx^2 + C_1dw, \\
\alpha_2 = dz^2 + B_{21}dx^1 + B_{22}dx^2 + C_2dw, \\
\alpha_3 = dz^3 + B_{31}dx^1 + B_{32}dx^2 + C_3dw.
\]

Then we have the following explicit expression for $\eta$:

\[
\eta = dz^1 \wedge dz^2 \wedge dz^3 + B_{31}dz^1 \wedge dz^2 \wedge dx^1 + B_{32}dz^1 \wedge dz^2 \wedge dx^2 \\
+ C_3dz^1 \wedge dz^2 \wedge dw + B_{21}dz^2 \wedge dz^1 \wedge dx^1 + B_{22}dz^3 \wedge dz^1 \wedge dx^2 \\
+ C_2dz^1 \wedge dz^2 \wedge dw + B_{11}dz^2 \wedge dz^3 \wedge dx^1 + B_{12}dz^2 \wedge dz^3 \wedge dx^2 \\
+ C_1dz^2 \wedge dz^3 \wedge dw + (B_{21}B_{32} - B_{22}B_{31})dx^1 \wedge dx^2 \wedge dz^1 \\
+ (B_{12}B_{31} - B_{11}B_{32})dx^1 \wedge dx^2 \wedge dz^2 + (B_{11}B_{22} - B_{12}B_{21})dx^1 \wedge dz^2 \wedge dz^3 \\
+(C_1B_{21} - C_2B_{11})dx^1 \wedge dz^3 \wedge dw + (C_1B_{22} - C_2B_{12})dx^2 \wedge dz^3 \wedge dw \\
+(C_2B_{32} - C_3B_{22})dx^3 \wedge dz^1 \wedge dw + (C_3B_{12} - C_1B_{32})dx^2 \wedge dz^2 \wedge dw \\
+(C_1B_{31} - C_3B_{11})dx^3 \wedge dz^1 \wedge dw + (C_3B_{11} - C_1B_{31})dx^3 \wedge dz^2 \wedge dw \\
+ (C_1B_{32}B_{21} + C_3B_{11}B_{22} + C_2B_{31}B_{12} - C_1B_{22}B_{31} \\
- C_2B_{32}B_{11} - C_3B_{12}B_{31})dx^1 \wedge dx^2 \wedge dw.
\]

In this case, the variational ideal $J(\vartheta, \pi)$ is generated by the three 2-forms $\psi_4 := (\partial/\partial z^a) \wedge \eta$, as $\psi_4$ will be a linear combination of these, and more precisely $\psi_4 = C_\alpha \psi^\alpha$. 

28
We have indeed

\[
\psi_1 = dz^2 \wedge dz^3 + B_{31} dz^2 \wedge dx^1 + B_{32} dz^2 \wedge dx^2 + C_3 dz^2 \wedge dw
\]

\[
- B_{21} dz^3 \wedge dx^1 - B_{22} dz^3 \wedge dx^2 - C_2 dz^3 \wedge dw
\]

\[
+(B_{21} B_{32} - B_{22} B_{31}) dx^1 \wedge dx^2
\]

\[
-(C_2 B_{32} - C_3 B_{22}) dx^2 \wedge dw - (C_2 B_{31} - C_3 B_{21}) dx^1 \wedge dw
\]

\[
\psi_2 = dz^2 \wedge dz^3 + B_{11} dz^3 \wedge dx^1 + B_{12} dz^3 \wedge dx^2 + C_1 dz^3 \wedge dw
\]

\[
- B_{11} dz^1 \wedge dx^1 - B_{12} dz^1 \wedge dx^2 - C_1 dz^1 \wedge dw
\]

\[
+(B_{11} B_{32} - B_{12} B_{31}) dx^1 \wedge dx^2
\]

\[
-(C_1 B_{32} - C_1 B_{32}) dx^2 \wedge dw - (C_1 B_{11} - C_1 B_{31}) dx^1 \wedge dw
\]

\[
\psi_3 = dz^1 \wedge dx^1 + B_{21} dz^1 \wedge dx^1 + B_{22} dz^1 \wedge dx^2 + C_2 dz^1 \wedge dw
\]

\[
- B_{11} dz^2 \wedge dx^1 - B_{12} dz^2 \wedge dx^2 - C_1 dz^2 \wedge dw
\]

\[
+(B_{11} B_{32} - B_{12} B_{31}) dx^1 \wedge dx^2
\]

\[
-(C_1 B_{21} - C_2 B_{11}) dx^1 \wedge dx^2 - (C_1 B_{22} - C_2 B_{12}) dx^1 \wedge dw
\]

while \( \psi_4 \) is given by

\[
\psi_4 = C_3 dz^1 \wedge dz^2 + C_2 dz^3 \wedge dz^2 + C_1 dz^2 \wedge dz^3
\]

\[
+(C_1 B_{21} - C_2 B_{31}) dx^1 \wedge dz^1 + (C_2 B_{11} - C_2 B_{12}) dx^1 \wedge dz^2
\]

\[
+(C_2 B_{32} - C_2 B_{32}) dx^2 \wedge dz^1 + (C_2 B_{31} - C_3 B_{21}) dx^1 \wedge dz^1
\]

\[
+(C_2 B_{11} - C_1 B_{32}) dx^2 \wedge dz^2 + (C_3 B_{11} - C_3 B_{31}) dx^1 \wedge dz^2
\]

\[
+[C_1 (B_{21} B_{32} - B_{31} B_{21}) + C_2 (B_{31} B_{32} - B_{31} B_{32}) + C_3 (B_{11} B_{21} - B_{11} B_{22})]
\]

The equations \( \varphi^*(\psi_a) := \Delta_2 \omega = 0 \) can be written in terms of the matrix

\[
P = \begin{pmatrix}
B_{11} + \partial_1 z^1 + C_3 \partial_1 w & B_{12} + \partial_2 z^1 + C_1 \partial_2 w \\
B_{21} + \partial_1 z^2 + C_2 \partial_1 w & B_{22} + \partial_2 z^2 + C_2 \partial_2 w \\
B_{31} + \partial_1 z^3 + C_3 \partial_1 w & B_{32} + \partial_2 z^3 + C_3 \partial_2 w
\end{pmatrix}
\]

as the requirement that all the two-dimensional submatrices \( \hat{P}_a \) obtained deleting from \( P \) the \( a \)-th row, for \( a = 1, 2, 3 \) have zero determinant.

Let us now consider \( \mathcal{N}(d\vartheta) \). This is generated by the vector fields

\[
Y_1 = (\partial/\partial x^1) - [B_{11}(\partial/\partial z^1) + B_{21}(\partial/\partial z^2) + B_{31}(\partial/\partial z^3)],
\]

\[
Y_2 = (\partial/\partial x^2) - [B_{12}(\partial/\partial z^1) + B_{22}(\partial/\partial z^2) + B_{32}(\partial/\partial z^3)],
\]

\[
Y_3 = (\partial/\partial w) - [C_1(\partial/\partial z^1) + C_2(\partial/\partial z^2) + C_3(\partial/\partial z^3)],
\]

note that \( Y_3 \) is vertical for \( \pi \).

The integral manifolds of \( \mathcal{D} = \mathcal{N}(d\vartheta) \) will be three-dimensional. We are actually interested in integral manifolds for the variational ideal \( J \) which are sections for the bundle \( (M, \pi, B) \) (we call these critical sections for short); this means in particular that they are two dimensional and transversal to fibers of \( \pi \). Note this means that they are not maximal integral manifolds for \( \mathcal{D} \), at difference with the cases considered before.

When we try to determine critical sections making use of our knowledge of \( \mathcal{N}(d\vartheta) \), we should consider general sections \( \varphi \in \Gamma(\pi) \), i.e. manifolds \( \{x, z, w\} \) identified by \( z^a = \varphi^a(x^1, x^2) \) and \( w = \varphi^4(x^1, x^2) \), and require that vector fields
which are tangent to $\varphi$ are in the distribution $D = \mathcal{N}(d\vartheta)$. If this is the case, the section $\varphi$ is indeed a critical section.

Proceeding in this way, we write a general $X$ in the form

$$X = f^i \frac{\partial}{\partial x^i} + F^a \frac{\partial}{\partial z^a} + F^4 \frac{\partial}{\partial w}; \quad (27)$$

this is tangent to the section $\varphi$ if and only if

$$F^a = \frac{\partial \varphi^a}{\partial x^i} f^i. \quad (28)$$

A vector field $X$ in the form (27) and satisfying (28) is in $D$ if

$$(\partial \varphi^1 / \partial x^1) f^1 + (\partial \varphi^1 / \partial x^2) f^2 = -B_{11} f^1 - B_{12} f^2 - C_1 f^1 (\partial \varphi^4 / \partial x^1)$$

$$-C_1 f^2 (\partial \varphi^4 / \partial x^2),$$

$$(\partial \varphi^2 / \partial x^1) f^1 + (\partial \varphi^2 / \partial x^2) f^2 = -B_{21} f^1 - B_{22} f^2 - C_2 f^1 (\partial \varphi^4 / \partial x^1)$$

$$-C_2 f^2 (\partial \varphi^4 / \partial x^2),$$

$$(\partial \varphi^3 / \partial x^1) f^1 + (\partial \varphi^3 / \partial x^2) f^2 = -B_{31} f^1 - B_{32} f^2 - C_3 f^1 (\partial \varphi^4 / \partial x^1)$$

$$-C_3 f^2 (\partial \varphi^4 / \partial x^2).$$

It is a simple matter to check that, eliminating the variables $f^i$ from this system, we recover the equations $\Delta_a = 0 \ (a = 1, 2, 3)$. 

30
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