Existence of flipped orthogonal conjugate symmetric Jordan canonical bases for real $H$-selfadjoint matrices

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ABSTRACT
For real matrices selfadjoint in an indefinite inner product there are two special canonical Jordan forms, that is (i) flipped orthogonal (FO) and (ii) $\gamma$-conjugate symmetric (CS). These are the classical Jordan forms with certain additional properties induced by the fact that they are $H$-selfadjoint. In this paper, we prove that for any real $H$-selfadjoint matrix, there is a $\gamma$-FOCS Jordan form that is simultaneously flipped orthogonal and $\gamma$-conjugate symmetric.

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1. Introduction

1.1. $H$-selfadjoint matrices and the affiliation relation

Let $\cdot, \cdot$ denote the indefinite inner product, that is $\cdot, \cdot$ satisfies all the axioms of the usual inner product except for positivity.

For example, for every $n \times n$ invertible hermitian matrix $H$

$$[x, y]_H = y^* H x, \text{ for } x, y \in \mathbb{C}^n$$

(1)

determines an indefinite inner product. Conversely, for any indefinite inner product, we can find such $H$ that (1) holds true. If $H$ is positive-definite then $[\cdot, \cdot]_H$ is a classical inner product, but in what follows we assume that $H$ is just Hermitian.

$H$-selfadjoint matrices. A matrix $A$ is called $H$-selfadjoint if

$$A = H^{-1} A^* H (\text{or } [Ax, y]_H = [x, Ay]_H).$$

(2)

These matrices have many applications and have been studied by many authors (e.g. see [1] and many references therein).

It follows from (2) that $H$-selfadjoint matrices have eigenvalues symmetric about the real axis. Moreover, the pairs of conjugate eigenvalues have the Jordan blocks of the same size.
**Affiliation relation.** Let us consider the change of basis

\[ x \mapsto u = T^{-1} x, \quad y \mapsto v = T^{-1} y. \]

It follows that

\[ [x, y]_H = y^* H x = (Tv)^* H (Tu) = v^* (T^* HT) u = [u, v]_G. \]

Therefore, if \( T \) is a change of basis matrix, then in the new basis the same inner product is given by a new congruent matrix \( G = T^* HT \).

Two pairs \((A, H)\) and \((B, G)\) are called affiliated (or unitarily similar as in [1]) if

\[ T^{-1} A T = B \quad \text{and} \quad T^* HT = G. \]

The relation \((A, H) \overset{T}{\mapsto} (B, G)\) is called the affiliation relation. That is, if we change the basis, the first matrix in a new basis is similar (as in the general case), but the second is congruent as described above.

Note that the affiliation relation preserves selfadjointness in the indefinite inner product, that is if \( A \) is \( H \)-selfadjoint, then \( B \) is \( G \)-selfadjoint.

As for canonical forms, in the \( H \)-selfadjoint case, we are looking for the affiliation \((A, H) \mapsto (J, P)\) where not only \( J \) is the Jordan canonical form but also \( P \) has a certain simple form. One example of such a form is considered next.

### 1.2. Flipped orthogonal bases

Let us consider the following example

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & -2 & 0 \\
\end{bmatrix} = TJT^{-1}, \quad H = \begin{bmatrix}
0 & 2 & 0 & 1 \\
2 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix},
\]

where \( A = H^{-1} A^* H \) and

\[
J = \begin{bmatrix}
i & 1 & 0 & 0 \\
0 & i & 0 & 0 \\
0 & 0 & -i & 1 \\
0 & 0 & 0 & -i \\
\end{bmatrix}, \quad P = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{bmatrix},
\]

and

\[
T = \frac{1}{2} \begin{bmatrix}
-i & 0 & i & 1 \\
1 & -i & 1 & 0 \\
i & 2 & -i & 1 \\
-1 & 3i & -1 & -2i \\
\end{bmatrix}.
\]

Moreover, \( J = P^{-1} J^* P \) and \( P = T^* HT \), so that

\[(A, H) \overset{T}{\mapsto} (J, P).\]

This is a very special affiliation relation, since \((J, P)\) is of a particular form with \( J \) being the Jordan canonical form of \( A \), and \( P \) was called the sip matrix in [1, Section 5.5], which is uniquely defined up to the perturbation of the Jordan blocks in \( J \).
Thus, \((J, P)\) is a canonical form of the matrix pair \((A, H)\). It first appeared in the works by Weierstrass [2, 3] (see also [1, Chapter 5] and [4, Chapter 3]). It can be seen that the columns of \(T = [t_j]\) satisfy
\[
[t_i, t_j]H = [e_i, e_j]P = \delta_{i,4-j},
\]
where \(\delta_{i,j}\) is the Kronecker symbol. Therefore, we suggest to call the Jordan basis \([t_1, \ldots, t_n]\) flipped orthogonal in agreement with its usage in [5].

### 1.3. \(\gamma\)-Conjugate symmetric bases

While the above definitions were given for the general complex case, in this paper we focus on the case, where \(A\) and \(H\) are real matrices.

*Are there any canonical forms for the real case?*

One immediate example is given next. Here the matrices \(A\) and \(H\) are the same as in (3).

Let us consider another Jordan basis for \(A\).

\[
R = \frac{1}{2} \begin{bmatrix}
-i & 2 & i & 2 \\
1 & i & 1 & -i \\
i & 0 & -i & 0 \\
-1 & i & -1 & -i \\
\end{bmatrix}.
\] (6)

Note that the columns of the matrix \(R = [r_1, r_2, r_3, r_4]\) are conjugate of each other:

\[r_1 = \overline{r_3} \text{ and } r_2 = \overline{r_4}.\]

Hence, we say that \(R\) captures a conjugate symmetric basis of \(A\) from (3) and \((A, H) \mapsto (J, G)\), where

\[
G = R^*HR = \begin{bmatrix}0 & 0 & 0 & 1 \\
0 & 0 & 1 & -3i \\
0 & 1 & 0 & 0 \\
1 & 3i & 0 & 0 \end{bmatrix}
\]

is not antidiagonal as \(P\) in (5). That is, \(R\) is CS but not FO. Also, if we look at the FO basis in \(T\) from (4), it is not CS.

### 1.4. FOCS bases

Now that we have introduced the FO and \(\gamma\)-CS canonical bases we might ask: *Is there a Jordan basis for real \(H\)-selfadjoint matrices that are simultaneously flipped orthogonal and \(\gamma\)-conjugate symmetric?*

Consider the following matrix \(M\) whose columns also capture a Jordan basis of the same matrix \(A\) as in Section 1.2.

\[
M = \frac{1}{4} \begin{bmatrix}
-2i & 1 & 2i & 1 \\
2 & -i & 2 & i \\
2i & 3 & -2i & 3 \\
-2 & 5i & -2 & -5i \end{bmatrix}.
\]

In fact, \((A, H) \mapsto (J, P)\). Moreover, \(M\) is CS.
We summarize all the above examples in Figure 1. In this case, we say that \( M \) captures the \textbf{flipped orthogonal conjugate symmetric basis}. This is only an example. We show the existence of such bases in the next section.

1.5. Main results

In this paper, we consider the case of real matrices \( A \) and \( H \). For the real case, we prove the existence of a ‘refined’ FO basis which we call the \( i \)-FOCS basis (i.e. flipped orthogonal and also \( i \)-conjugate symmetric). Namely, the Jordan chains of \( A \) corresponding to \( \lambda \) and \( \overline{\lambda} \) are just scaled conjugates of each other, still enjoying the flipped orthogonal property.

2. Three canonical bases

We have already presented an example of an FO basis. The next subsection shows the existence of such a basis for any \( H \)-selfadjoint matrix.

2.1. FO bases

The next proposition establishes the existence of FO bases for any (potentially non-real) pair \((A, H)\).

\textbf{Proposition 2.1:} (Theorem 5.5.1 in [1]). Let \( A, H \in \mathbb{C}^{n \times n} \) be given with \( H = H^\ast \) and \( A \) being the \( H \)-selfadjoint matrix. Let \( J \) be its Jordan form

\[ J = J(\lambda_1) \oplus \cdots J(\lambda_\alpha) \oplus \widehat{J}(\lambda_{\alpha+1}) \oplus \cdots \oplus \widehat{J}(\lambda_\beta) \]  

where \( \lambda_1, \ldots, \lambda_\alpha \in \mathbb{R} \), \( \lambda_{\alpha+1}, \ldots, \lambda_\beta \notin \mathbb{R} \), \( J(\lambda_i) \) is a Jordan block for real eigenvalues \( \lambda_1, \ldots, \lambda_\alpha \) and

\[ \widehat{J}(\lambda_k) = \begin{bmatrix} J(\lambda_k) & 0 \\ 0 & \overline{J(\lambda_k)} \end{bmatrix} \]  

is the direct sum of two Jordan blocks of the same size corresponding to \( \lambda_k \) and \( \overline{\lambda_k} \). Then there exists an invertible matrix \( T \) such that

\[ (A, H) \leftrightarrow (J, P) \]
where
\[ P = P_1 \oplus \cdots \oplus P_\alpha \oplus P_{\alpha+1} \oplus \cdots \oplus P_\beta \]  \hspace{1cm} (9)

where \( P_k \) is a sign matrix \( \epsilon_k \tilde{I}_k \) (i.e. 
\[
\begin{bmatrix}
0 & \cdots & 0 & \epsilon_k \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & \epsilon_k \\
\epsilon_k & 0 & \cdots & 0
\end{bmatrix}
\]
) of the same size as \( J(\lambda_k) \) and \( \epsilon_k = \pm 1 \) for \( k = 1, \ldots, \alpha \) and a sign matrix \( \tilde{I}_k \) (i.e. 
\[
\begin{bmatrix}
0 & \cdots & 0 & 1 \\
\vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & 1 \\
1 & 0 & \cdots & 0
\end{bmatrix}
\]
) of the same size as \( \tilde{J}(\lambda_k) \) for \( k = \alpha + 1, \ldots, \beta \).

Why flipped orthogonal?
If \( t_i, t_j \) are the \( i \)th, \( j \)th columns of \( T \) respectively, then
\[
[t_i, t_j]^H = [e_i, e_j] P.
\]
In other words, if we partition \( T \) in the following way
\[
T = [T_1 \ldots T_\alpha | T_{\alpha+1} \ldots T_\beta],
\]
for \( k = 1, \ldots, \alpha \) the columns of
\[
T_k = [g_{0,k}, \ldots, g_{p_k-1,k}]
\]
form the Jordan chain of \( A \) corresponding to real \( \lambda_k \). Similarly, the columns of
\[
\begin{array}{c}
T_k = [g_{0,k}, \ldots, g_{p_k-1,k}] \\
\lambda_k
\end{array}
\begin{array}{c}
\overset{h_{0,k}, \ldots, h_{p_k-1,k}}{\lambda_k}
\end{array}
\begin{array}{c}
\tilde{\lambda}_k
\end{array}
\]
for \( k = \alpha + 1, \ldots, \beta \) form two Jordan chains, corresponding to \( \lambda_k \) and \( \tilde{\lambda}_k \) respectively.

Then the structure of the signature matrix \( P \) implies the following orthogonality relation,
\[
g_{ik}^* H_{gjm} = 0, \quad h_{ik}^* H_{gjm} = 0, \quad h_{ik}^* H_{hjm} = 0, \quad \text{for } k \neq m.
\]

Further, we have that
\[
g_{i,k}^* H_{g_{j,k}} = \begin{cases} 
\epsilon_k, & j = p_k - 1 - i \\
0 & \text{otherwise}
\end{cases} \quad \text{for } k = 1, \ldots, \alpha
\]
\[
g_{i,k}^* H_{g_{j,k}} = 0, \quad h_{i,k}^* H_{h_{j,k}} = 0, \quad h_{i,k}^* H_{g_{j,k}} = \begin{cases} 
1, & j = p_k - 1 - i \\
0 & \text{otherwise}
\end{cases} \quad \text{for } k = \alpha + 1, \ldots, \beta.
\]

In terms of our example, it means that
\[
\begin{bmatrix}
[t_1, t_1]^H & [t_1, t_2]^H & [t_1, t_3]^H & [t_1, t_4]^H \\
[t_2, t_1]^H & [t_2, t_2]^H & [t_2, t_3]^H & [t_2, t_4]^H \\
[t_3, t_1]^H & [t_3, t_2]^H & [t_3, t_3]^H & [t_3, t_4]^H \\
[t_4, t_1]^H & [t_4, t_2]^H & [t_4, t_3]^H & [t_4, t_4]^H
\end{bmatrix}
\]
\[
\begin{bmatrix}
[e_1, e_1]_P & [e_1, e_2]_P & [e_1, e_3]_P & [e_1, e_4]_P \\
[e_2, e_1]_P & [e_2, e_2]_P & [e_2, e_3]_P & [e_2, e_4]_P \\
[e_3, e_1]_P & [e_3, e_2]_P & [e_3, e_3]_P & [e_3, e_4]_P \\
[e_4, e_1]_P & [e_4, e_2]_P & [e_4, e_3]_P & [e_4, e_4]_P
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix} = P.
\]

As we showed by our example in the introduction – not every Jordan basis is necessarily flipped orthogonal. However, the above proposition yields that the flipped orthogonal basis always exists for \( H \)-selfadjoint matrices. In fact, in view of the structure of \( P \), it is captured by the columns of \( T \).

To show that \( \gamma \)-FOCS bases exist, we need to introduce yet another canonical form.

### 2.2. \( \gamma \)-CS bases

Looking back at (16), one question arises.

*Why do the columns of \( R \) from our example in the introduction capture the Jordan basis?*

For any real matrix \( A \), if \( \lambda \) is its non-real eigenvalue, then so is \( \overline{\lambda} \). Moreover, if \( g_k \to g_{k-1} \to \ldots \to g_0 \to 0 \) is a Jordan chain corresponding to \( \lambda \) then

\[
\gamma q_{k-1} = \gamma (A - \lambda I)q_k = (A - \overline{\lambda}I)\gamma q_k = (A - \overline{\lambda}I)q_k,
\]

i.e. \( \gamma g_k \to \gamma g_{k-1} \to \ldots \to \gamma g_0 \to 0 \) is also a Jordan chain of \( A \) but corresponds to \( \overline{\lambda} \) for any non-zero \( \gamma \in \mathbb{C} \).

This observation leads to the following definition.

Suppose \( A \in \mathbb{R}^{n \times n} \) and that there exists an invertible matrix \( N \) such that \( A = NJN^{-1} \), where

\[
J = J(\lambda_1) \oplus \cdots \oplus J(\lambda_\alpha) \oplus \widehat{J}(\lambda_{\alpha+1}) \oplus \cdots \oplus \widehat{J}(\lambda_\beta)
\]

is the Jordan canonical form and \( \widehat{J}(\lambda_k) \) defined in (8). The matrix

\[
N = [N_1 | \ldots | N_\alpha | N_{\alpha+1} | \ldots | N_\beta]
\]

(10)

can be chosen that

\[
N_k = [Q_k | \gamma \overline{Q}_k]
\]

(11)

for any \( \gamma \neq 0 \) and \( k = \alpha + 1, \ldots, \beta \). The columns of \( N \) capture the Jordan basis of \( A \), and because of (11) we call it the \( \gamma \)-conjugate symmetric basis.

In this paper, we show that for any real \( H \)-selfadjoint matrix \( A \), there exists a Jordan basis that is simultaneously FO and \( i \)-CS. For this, we need to consider the real canonical form.

### 2.3. Real canonical form

The following well-known result (see [1, Theorem 6.1.5]) describes a purely real relation

\[
(A, H) \overset{R}{\mapsto} (J_R, P)
\]

(12)

where all five matrices are real.
Proposition 2.2: Let $A, H \in \mathbb{R}^{n \times n}$ be given with $H = H^\top$ and $A$ is an $H$-selfadjoint matrix. Let $J_R$ be its real Jordan form

$$J_R = J(\lambda_1) \oplus \cdots \oplus J(\lambda_\alpha) \oplus \tilde{J}_R(\lambda_{\alpha+1}) \oplus \cdots \oplus \tilde{J}_R(\lambda_\beta),$$

(13)

where $\lambda_1, \ldots, \lambda_\alpha \in \mathbb{R}$, $\lambda_{\alpha+1}, \ldots, \lambda_\beta \notin \mathbb{R}$, $J(\lambda_i)$ is a Jordan block for real eigenvalues $\lambda_1, \ldots, \lambda_\alpha$ and

$$\tilde{J}_R(\lambda_k) = \begin{bmatrix}
\sigma_k & \tau_k & 1 & 0 & 0 & \cdots & 0 \\
-\tau_k & \sigma_k & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & \sigma_k & \tau_k & 1 & \cdots & 0 \\
0 & 0 & -\tau_k & \sigma_k & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & 0 & \sigma_k & \tau_k \\
0 & 0 & \cdots & \cdots & -\tau_k & \sigma_k & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & \cdots & \cdots & 0 & 0 & \sigma_k & \tau_k \\
\end{bmatrix},$$

where $\lambda_k = \sigma_k + i\tau_k$. Then there exists an invertible real matrix $R$ such that $(A, H) \xrightarrow{R} (J_R, P)$ where $P$ is a sip matrix of the form

$$P = P_1 \oplus \cdots \oplus P_\alpha \oplus P_{\alpha+1} \oplus \cdots \oplus P_\beta,$$

where $P_k$ is a sip matrix $\epsilon_k I$ of the same size as $J(\lambda_k)$ for $k = 1, \ldots, \alpha$ and a sip matrix $\tilde{I}$ of the same size as $\tilde{J}_R(\lambda_k)$ for $k = \alpha + 1, \ldots, \beta$ and $\epsilon_k = \pm 1$ for $k = 1, \ldots, \alpha$.

Suppose that $A \in \mathbb{R}^{n \times n}$ and $J_R$ is the real Jordan form in (13). We say that columns of matrix $R$ in (12) form a real canonical basis of $A$.

Finally, we are ready to prove the main result.

3. Existence of $i$-FOCS bases

The FO basis was defined for pairs of matrices $(A, H)$, where $A$ and $H$ are not necessarily real. The $\gamma$-CS basis is defined for the case of $A$ being a real matrix. So what about real $H$-selfadjoint matrices?

Proposition 2.1 implies the existence of the FO basis for $(A, H)$, and the latter, generally, does not have the CS property.

Theorem 3.1: (Existence of an $i$-FOCS basis) Let $A$ be a real $H$-selfadjoint matrix where $H$ is real, invertible, and symmetric. Then $N$ in $(A, H) \xrightarrow{N} (J, P)$ can be chosen $i$-conjugate symmetric and flipped orthogonal at the same time.
To establish this result, let us discuss the relation between RC and i-FOCS bases. As we will see, for an arbitrary real pair \((A, H)\), the relation between i-FOCS and RC bases is carried over with the help of the fixed invertible matrix \(S\) in Lemma 3.2.

**Lemma 3.2:** Let \(A, H \in \mathbb{R}^{n \times n}\), where \(H = H^\top\), and \(A\) is \(H\)-selfadjoint. If the columns of \(R\)
\[
R = [N_1, \ldots, N_\alpha |K_{\alpha+1}, \ldots, K_\beta]
\]  
(14)
capture the RC basis of \(A\), then for \(S = \text{diag}(I_1, \ldots, I_\alpha |S_{\alpha+1}, \ldots, S_\beta)\), where the explicit formula for \(S_j\) is:
\[
S_j = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & 0 & \cdots & 0 & i & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & i & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & \ddots & \ddots & \ddots & i \\
0 & \cdots & 0 & i & 0 & \cdots & 0 & 1
\end{bmatrix},
\]
the columns of \(N = RS\) capture the i-FOCS basis of \(A\).

The next result is the converse to the one in Lemma 3.2

**Lemma 3.3:** If the columns of
\[
N = [N_1, \ldots, N_\alpha |N_{\alpha+1}, \ldots, N_\beta] \quad \text{and} \quad N_k = [Q_k |i\overline{Q}_k], \quad k = \alpha + 1, \ldots, \beta
\]  
(15)
capture the i-FOCS basis of \((A, H)\).

Further, note that \(S^{-1} = \text{diag}(I_1, \ldots, I_\alpha |S_{\alpha+1}^{-1}, \ldots, S_\beta^{-1})\), where
\[
S_j^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix}
1 & -i & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & -i & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & -i & 0 & 0 \\
0 & \cdots & 0 & 0 & 1 & -i & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & 1 & 0 & 0 & -i \\
0 & \cdots & 0 & 0 & -i & 0 & 1
\end{bmatrix},
\]
then the matrix
\[
R = NS^{-1}
\]  
(16)
is real and its columns capture the RC basis of \(A\). Moreover, from (14) we get \(K_j = N_j S_j^{-1}\) for \(j = \alpha + 1, \ldots, \beta\).
**Proof of Lemma 3.2.** First, let us show that the columns of matrix $N$ in (16) capture a basis of $A$, then that this basis is $i$-CS. Without loss of generality, we consider the case of a single real Jordan block, assuming that $\sigma(A) = \{\lambda, \bar{\lambda}\}$. The relation $S^{-1}J_R S = f$ follows from

$$J_R S = \frac{1}{\sqrt{2}} \begin{bmatrix} \sigma + i\tau & 1 & 0 & \ldots & 0 & i\sigma + \tau & i & 0 & \ldots & 0 \\ -\tau + i\sigma & i & 0 & \ldots & 0 & -i\tau + \sigma & 1 & 0 & \ldots & 0 \\ 0 & \sigma + i\tau & 1 & \ldots & 0 & 0 & i\sigma + \tau & i & 0 & \ldots & 0 \\ 0 & -\tau + i\sigma & i & \ldots & 0 & 0 & -i\tau + \sigma & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & i & 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & \sigma + i\tau & 0 & 0 & 0 & \ldots & i\sigma + \tau \\ 0 & 0 & 0 & \ldots & -\tau + i\sigma & 0 & 0 & 0 & \ldots & -i\tau + \sigma \end{bmatrix} = SJ,$$

where $\lambda = \sigma + i\tau$. Therefore,

$$A = RJ_R R^{-1} = (RS)(S^{-1}J_R S)(RS)^{-1} = NJN^{-1}.$$

That is the columns of $N$ indeed capture a basis of $A$. For a pair of vectors $a$ and $b$

$$i \cdot a + ib = i(a - ib) = ia + b.$$

This is exactly what multiplication by $S$ on the left does to real columns of $R$. Hence, we get the $i$-conjugate symmetry for $N$.

Next, we want to show that the columns of $N$ form an FO basis. The relation $S^* PS = P$ in (17) follows from

$$S^* PS = P \text{ being equivalent to } PS^* PS = PP = I.$$

Thus, we just need to show that $PS^* PS = I$.

$$PS^* PS = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 & \ldots & -i & 1 \\ 0 & 0 & 0 & 0 & 1 & \ldots & i \\ 0 & 0 & 0 & 1 & 0 & 0 & \ldots & i \\ 0 & 0 & 1 & 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & i & 0 & 0 & 0 & 1 & 0 & \ldots & i \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & i \\ i & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & \ldots & 0 \end{bmatrix} = I.$$

Since $(A, H) \overset{R}{\mapsto} (J_R, P)$, the above relation implies that

$$(J_R, P) \overset{S}{\mapsto} (J, P). \tag{17}$$

Hence, $(A, H) \overset{N}{\mapsto} (J, P)$ and the basis in question is FO as well as $i$-CS, i.e. it is $i$-FOCS. \[\blacksquare\]

The converse statement follows from the relation $(J, P) \overset{S^{-1}}{\mapsto} (J_R, P)$, using a similar argument.

Combining Proposition 2.2 and Lemma 3.2, we get the result of Theorem 3.1.

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