THE MIRROR CONJECTURE FOR MINUSCULE FLAG VARIETIES

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Abstract. We prove Rietsch’s mirror conjecture that the Dubrovin quantum connection for minuscule flag varieties is isomorphic to the character $D$-module of the Berenstein-Kazhdan geometric crystal. The idea is to recognize the quantum connection as Galois and the geometric crystal as automorphic. We reveal surprising relations with the works of Frenkel-Gross, Heinloth–Ngô–Yun and Zhu on Kloosterman sheaves. The isomorphism comes from global rigidity results where Hecke eigensheaves are determined by their local ramification. As corollaries we obtain combinatorial identities for counts of rational curves and the Peterson variety presentation of the small quantum cohomology ring.

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1. Introduction

Let $G$ be a complex semisimple algebraic group, $B \subset G$ be a Borel subgroup, and $P \subset G$ a parabolic subgroup containing $B$. Let $P^\vee \subset P^\vee \subset G^\vee$ denote the Langlands duals. In the case that $P^\vee$ is a minuscule maximal parabolic subgroup, we prove the mirror theorem that the quantum connection of the partial flag variety $G^\vee/P^\vee$ is isomorphic to the character $D$-module of the geometric crystal associated to $(G, P)$. This isomorphism is the top row of the following diagram of $D$-modules, where the bottom row is an instance of the geometric Langlands program.
We now discuss this diagram in detail.

1.1. **Quantum cohomology and mirror symmetry for flag varieties.** The study of the topology of flag varieties $G^\vee/B^\vee$ has a storied history. Borel [14] computed the cohomology rings $H^*(G^\vee/B^\vee, \mathbb{C})$ to be isomorphic to the coinvariant algebras of the Weyl group $W$ acting on the natural reflection representations. This work is continued by the works of Chevalley, Bernstein–Gelfand–Gelfand, Demazure, Lascoux–Schützenberger, and many others on the Schubert calculus of flag varieties.

Much progress was made on the quantum cohomology of flag varieties in the last two decades. Givental and Kim [61] and Ciocan-Fontanine [33] (for $G^\vee$ of type $A$), and Kim [90] (for general $G^\vee$) identified the quantum cohomology rings $QH^*(G^\vee/B^\vee, \mathbb{C})$ with the ring of regular functions on the nilpotent leaf of the Toda lattice of $G$. Subsequently, Givental [60] formulated a mirror conjecture that oscillatory integrals over the mirror manifold should be solutions to the quantum $D$-module, and established this result for $G^\vee$ of type $A$ (see also [41]). This result was extended to general $G^\vee$ by Rietsch [117]. These oscillatory integrals gave new integral formulae for Whittaker functions.

By contrast, our understanding of mirror symmetry for partial flag varieties $G^\vee/P^\vee$ is much more limited. Peterson [111] discovered a uniform geometric description of the quantum cohomology rings $QH^*(G^\vee/P^\vee, \mathbb{C})$, but this work remains unpublished (see however [96, 116, 117]). The quantum $D$-modules of $G^\vee/P^\vee$ have remained largely unstudied in full generality. Batyrev–Ciocan-Fontanine–Kim–van Straten [4] proposed a mirror conjecture for GL($n$)/$P^\vee$, and Rietsch formulated a mirror conjecture for arbitrary $G^\vee/P^\vee$, in the style of Givental.

One of the main aims of this work is to establish Rietsch’s mirror conjecture in the case that $P^\vee$ is minuscule. This class of spaces includes projective spaces, Grassmannians, and orthogonal Grassmannians (see Figure 2 for the full list). Even for the case of Grassmannians, whose quantum cohomology rings are well studied [11, 123, 133] and a large part of the mirror conjecture established in [104], our results are new.

1.2. **Small quantum $D$-module.** We now let $P^\vee$ be a minuscule parabolic. The small quantum cohomology ring $QH^*(G^\vee/P^\vee)$ is isomorphic to $\mathbb{C}[q, q^{-1}] \otimes H^*(G^\vee/P^\vee)$ as a vector space, with quantum multiplication denoted by $*_q$.

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1In this paper, cohomologies and quantum cohomologies are all taken with $\mathbb{C}$ coefficients.
Let $\mathbb{C}_q^\times = \text{Spec}(\mathbb{C}[q,q^{-1}])$ be the one-dimensional torus with coordinate $q$. The small quantum $D$-module (at $\hbar = 1$) [40] is the connection on the trivial $H^*(G^\vee/P^\vee)$–bundle over $\mathbb{C}_q^\times$ given by

$$Q^{G^\vee/P^\vee} := d + (\sigma q) \frac{dq}{q}$$

where $\sigma \in H^*(G^\vee/P^\vee, \mathbb{Z})$ is the effective divisor class, and we consider

$$\sigma q \in \text{End}(H^*(G^\vee/P^\vee)) \otimes \mathbb{C}[q,q^{-1}].$$

In [31], Chevalley gave a combinatorial formula for the cup-product in $H^*(G^\vee/P^\vee)$ with the divisor class $\sigma$, i.e. for (1.2.2) at $q = 0$. A quantum Chevalley formula (see Theorem 1.3) evaluating (1.2.2) in terms of Schubert classes for general flag varieties was stated by Peterson [111] and proved by Fulton and Woodward [50]. This formula has been extended to the equivariant case by Mihalcea [105] and to the cotangent bundle of partial flag varieties by Peterson [111] and proved by Fulton and Woodward [50]. For recent developments in the minuscule case, see [21].

In the sequel, we also let $Q^{G^\vee/P^\vee}$ denote the corresponding algebraic $D$-module, where $D = D_{\mathbb{C}_q^\times} = \mathbb{C}[q,q^{-1}][\partial_q]$ is the ring of differential operators on $\mathbb{C}_q^\times$, and $\partial_q := \frac{d}{dq}$.

1.3. The character $D$-module of a geometric crystal. Berenstein–Kazhdan [7], based on previous works by Lusztig and Berenstein–Zelevinsky [9], have constructed geometric crystals which are certain complex algebraic varieties equipped with rational maps. The motivation of the construction was the birational lifting of the combinatorics of Lusztig’s canonical bases [100] and Kashiwara’s crystal bases [86].

Fix opposite Borel subgroups $B$ and $B_-$ of $G$ with unipotent subgroups $U$ and $U_-$, and let $T = B \cap B_-$. Let $R$ denote the root system, and $R^\pm$ denote the subsets of positive and negative roots. Let $\psi : U \to \mathbb{G}_a$ be a non-degenerate additive character.

For a parabolic subgroup $P \subset G$, let $W_P \subset W$ be the Weyl group of the Levi subgroup $L_P$, and let $I_P \subset I$ be the corresponding subset of the Dynkin diagram. There is a unique set $W^P \subset W$ of minimal length coset representatives for the quotient $W/W_P$. Define $w^{-1}_P \in W$ to be the longest element in $W^P$. The (parabolic) geometric crystal $X = X_{(G,P)}$ is the subvariety

$$X = UZ(L_P)\bar{w}_PU \cap B_- \subset G$$

where $Z(L_P)$ denotes the center of the Levi subgroup $L_P$, and $\bar{w}_P \in G$ is a representative of $w_P \in W$, equipped with geometric crystal actions $e_i : \mathbb{G}_m \times X \to X$ and three maps of importance to us:

- $f : X \to \mathbb{A}^1$, $u_1t\bar{w}_Pu_2 \mapsto \psi(u_1) + \psi(u_2)$ called the decoration function,
- $\gamma : X \to T$, $x \mapsto x \mod U_- \subset B_-/U_- \cong T$ called the weight function,
- $\pi : X \to Z(L_P)$, $u_1t\bar{w}_Pu_2 \mapsto t$ called the highest weight function.

The fiber $X_t := \pi^{-1}(t)$ for $t \in Z(L_P)$ is called the geometric crystal with highest weight $t$. For any $t \in Z(L_P)$, $X_t$ is a log Calabi-Yau variety isomorphic to the open projected Richardson variety $G/P \subset G/P$ [91], the complement in $G/P$ of a particular anticanonical divisor $\partial_{G/P}$.

The affine variety $G/P$ has a distinguished holomorphic volume form $\omega$, with logarithmic singularities along the boundary divisor $\partial_{G/P}$.
The (formal) geometric character of \( X \) is the integral function

\[
\psi_\lambda(t) = \int_{\Gamma_t \subset X_t} \lambda(\gamma(x)) \exp(-f(x))\omega
\]

where \( t \in Z(L_P) \), and \( \lambda : T \to \mathbb{G}_m \) is a character of \( T \). The integral can be made convergent by choosing the cycle \( \Gamma_t \) to be the totally positive part of \( X_t \). Just as the geometric crystal \( X \) tropicalizes to Kashiwara’s combinatorial crystals, as explained in Lam [95] and Chhaibi [32], formally tropicalizing (1.3.1) one obtains a (usual) irreducible character of \( G^\vee \).

On \( \mathbb{A}^1 \) we consider the cyclic \( D \)-module \( E := D_{\mathbb{A}^1}/D_{\mathbb{A}^1}(\partial_x - 1) \) with generator the exponential function, where \( D_{\mathbb{A}^1} = \mathbb{C}[x](\partial_x) \). The pullback \( f^*E \) is a \( D \)-module on \( X \). We define the character \( D \)-module of the geometric crystal \( X \) by

\[
Cr_{(G,P)} := R\pi_1 f^*E,
\]

which is a \( D \)-module on \( Z(L_P) \). A priori \( Cr_{(G,P)} \) is a complex of \( D \)-modules, but we show that it is just a \( D \)-module. Our proof is via the left-hand side of Figure 11 which enables us to recognize this statement as the Ramanujan property, in the context of the geometric Langlands program, for a certain cuspidal automorphic \( D \)-module \( A_\psi \), see below and [76].

The integral function \( \psi_\lambda(t) \) is formally the solution of \( Cr_{(G,P)} \). More generally, we shall define in (1.4.1) a character \( D \)-module \( Cr_{(G,P)}(\lambda) \) with solution \( \psi_\lambda(t) \).

This article seems to be the first time the properties of the character \( D \)-module \( Cr_{(G,P)} \) are studied. There are other geometric crystals, and as we shall see below, other families of Landau–Ginzburg models that one could apply this construction to. We also note that automorphic \( D \)-modules with wild ramification, and geometric analogues of Arthur conjectures, which both play an important role in our study, are themes which been largely unexplored at the present time.

1.4. Rietsch’s Landau–Ginzburg model. In [117], Rietsch constructed conjectural Landau–Ginzburg mirror partners of all partial flag varieties \( G/P \). Her construction was motivated by earlier works of Givental [58], Joe–Kim [85], and Batyrev–Ciocan-Fontanine–Kim–van Straten [4] for type \( A \) flag varieties, and also by the Peterson presentation of \( QH^*(G^\vee/P^\vee) \).

Rietsch’s mirror construction are families of varieties fibered over \( q \in \text{Spec}(\mathbb{C}[q^{\pm 1} \mid I \notin I_P]) \), equipped with holomorphic superpotentials \( f_q \), and holomorphic volume forms \( \omega_q \).

It was observed by Lam [95], and Chhaibi [32] that Rietsch’s mirror construction could be obtained from the group geometry of geometric crystals. Thus, after identifying \( \text{Spec}(\mathbb{C}[q^{\pm 1} \mid I \notin I_P]) \) with \( Z(L_P) \), Rietsch’s mirror family is \( \pi : X \to Z(L_P) \), equipped with the superpotential \( f_t := f|_{X_t} : X_t \to \mathbb{A}^1 \); henceforth we will use \( f_q \) or \( f_t \) interchangeably (\( q \) being a point in \( \text{Spec}(\mathbb{C}[q^{\pm 1} \mid I \notin I_P]) \) and \( t \) a point in \( Z(L_P) \)).

This candidate mirror Landau–Ginzburg model is a partial compactification of the Hori-Vafa mirror which is a Laurent polynomial obtained by degeneration of \( G/P \) to a toric variety, see [80,109,110]. In the literature this distinction also appears in the form of “strong mirror” versus “weak mirror”.

Stated informally our main goal in this paper is to show:

If \( P^\vee \) is minuscule then \( G^\vee/P^\vee \) and \( (G^\vee, f_q) \) form a Fano type mirror pair.

On the \( A \)-model side \( G^\vee/P^\vee \) is a projective Fano variety, and on the \( B \)-model side \( G^\vee/P \) is a log Calabi-Yau variety; see [89] for general expectations for Fano type mirror pairs. We show that some of the mirror symmetry expectations hold.
1.5. The mirror isomorphism. The following is a central result of this paper and establishes the top row of Figure 1.

**Theorem 1.6 (Theorem 8.3).** Suppose $P^\lor$ is minuscule. The geometric crystal $D$-module $Cr_{(G,P)}$ and the quantum cohomology $D$-module $Q^{G^\lor/P^\lor}$ for $G^\lor/P^\lor$ are isomorphic.

For $G^\lor/P^\lor$ a projective space $\mathbb{P}^n$, the result is well-known [57,87]. The homological mirror symmetry version is established in [44]. Our approach gives an original perspective in terms of hyper-Kloosterman sheaves studied in SGA4 1/2 [37].

For $G^\lor/P^\lor$ a Grassmanian $Gr(k,n)$, the result is already new. Partial results are obtained by Marsh-Rietsch [104], notably a canonical injection of $Q^{G^\lor/P^\lor}$ into $Cr_{(G,P)}$, who establish as a consequence a conjecture of Batyrev–Ciocan-Fontanine–Kim–VanStraten [3, Conj. 5.2.3].

Our Theorem 1.6 is stronger, indeed it establishes the conjecture of [104, §3] that the canonical injection is bijective, and thereby also another conjecture of Batyrev–Ciocan-Fontanine–Kim–VanStraten [4, Conj. 5.1.1].

For $G^\lor/P^\lor$ an even-dimensional quadric, the injection of $Q^{G^\lor/P^\lor}$ into $Cr_{(G,P)}$ is obtained by Pech-Rietsch-Williams [109], and our Theorem 1.6 establishes a conjecture of [109, §4].

Although both sides of Theorem 1.6 are described explicitly, this does not lead to a way of establishing the isomorphism. Indeed our proof will follow a lengthy path, where the isomorphism will eventually arise from Langlands reciprocity for the automorphic form $A_G$ over the rational function field $\mathbb{C}(t)$.

Givental and Rietsch’s original mirror conjectures were formulated in terms of oscillatory integrals. The following Corollary establishes [117, Conj. 8.2] in the minuscule case.

**Corollary 1.7 (Corollary 12.14).** Suppose $P^\lor$ is minuscule. A full set of solutions to the quantum differential equation of $Q^{G^\lor/P^\lor}$ is given by integrals

$$I_{\Gamma}(t) = \int_{\Gamma_t} e^{f_t \omega},$$

where $\Gamma_t$ is a horizontal section of the $Z(L_P)$-local system of middle-dimensional rapid decay cycles on $G/P$ relative to $f_t$.

1.8. Kloosterman sums, Kloosterman sheaves, and Kloosterman $D$-modules. For a prime $p$ and a finite field $\mathbb{F}_q$, $q = p^n$, define the two maps

$$f : (\mathbb{F}_q^n)^n \to \mathbb{F}_q \quad (x_1, x_2, \ldots, x_n) \mapsto x_1 + x_2 + \cdots + x_n$$

$$\pi : (\mathbb{F}_q^n)^n \to \mathbb{F}_q^n \quad (x_1, x_2, \ldots, x_n) \mapsto x_1 x_2 \cdots x_n.$$

The (hyper)Kloosterman sum in $(n - 1)$-variables is

$$Kl_n(a) := (-1)^{n-1} \sum_{x \in \pi^{-1}(a)} \exp\left(\frac{2\pi i}{p} Tr_{\mathbb{F}_q/F_p} f(x)\right),$$

where $a \in \mathbb{F}_q^\times$. Deligne [37] defines the (hyper)Kloosterman sheaf to be the $\ell$-adic sheaf on $\mathbb{F}_p$ given by

$$Kl_n^\ell := R\pi^*_f AS_\psi[n-1]$$

where $AS_\psi$ is the Artin-Schreier sheaf on $\mathbb{A}^1$ corresponding to a nontrivial character $\psi : \mathbb{F}_p \to \mathbb{C}$. For an appropriate embedding $\iota : \mathbb{C}_\ell \to \mathbb{C}$, the Kloosterman sum (1.8.1) is identified as the Frobenius trace of the Kloosterman sheaf (1.8.2): $Kl_n(a) = \iota(Tr(Frob_n, Kl_n))$. 
The *Kloosterman D-module* is defined \cite{87} by replacing the Artin-Schreier sheaf with the exponential D-module:

\[(1.8.3) \quad \text{Kl}_n := R\pi_! f^* E.\]

The pair \((\pi : (\mathbb{F}_q^\times)^n \rightarrow \mathbb{F}_q^\times, f)\) and (1.8.2) should be compared with the geometric crystal mirror family \((\pi : X \rightarrow Z(L_P), f)\) and (1.3.2).

Heinloth–Ngô–Yun \cite{76} generalize Kloosterman sheaves and D-modules to reductive groups. More precisely, for a representation \(V\) of \(G^\vee\), they define a generalized Kloosterman D-module \(\text{Kl}_{(G^\vee, V)}\) on \(\mathbb{C}^\times\). Their construction uses the moduli stack \(\text{Bun}_G\) of \(G\)-bundles on \(\mathbb{P}^1\), where \(G\) is a particular nonconstant group scheme over \(\mathbb{P}^1\) (see \S7); equivalently, \(\text{Bun}_G\) classifies \(G\)-bundles with specified ramification behavior. Heinloth–Ngô–Yun construct an automorphic Hecke eigen-D-module \(A_G\) on the Hecke stack over \(\text{Bun}_G\). The generalized Kloosterman D-module \(\text{Kl}_{(G^\vee, V)}\) is defined to be the Hecke eigenvalue of \(A_G\). The projection and superpotential maps \(\pi\) and \(f\) are replaced in this setting by the projection maps of the Hecke moduli stack.

A remarkable feature of the automorphic D-module \(A_G\) is that it is rigid: it can be characterised uniquely by its local components. Indeed, the existence of the rigid local systems constructed by Heinloth–Ngô–Yun was predicted by work of Gross \cite{70}, who constructed \(A_G\) over finite fields via the stable trace formula. We refer to \cite{137} for a comprehensive survey of rigid automorphic forms. Rigid local systems have been introduced and systematically studied by Katz \cite{88}.

The following result gives an automorphic interpretation of geometric crystals.

**Theorem 1.9** (Theorem 7.12). Let \(P \subset G\) be a cominuscule parabolic and \(V\) the corresponding minuscule representation of \(G^\vee\). The character D-module \(\text{Cr}_{(G^\vee, P)}\) is isomorphic to the Kloosterman D-module \(\text{Kl}_{(G^\vee, V)}\) defined as the \(V\)-Hecke eigenvalue of the automorphic D-module \(A_G\).

The proof of Theorem 1.9 is by a direct comparison of the geometry of the Hecke moduli stack and that of parabolic geometric crystals.

The above discussion suggests a striking parallel between exponential sums over finite fields and Landau–Ginzburg models. The same construction applied to other mirror families produces interesting \(\ell\)-adic sheaves which we believe could be studied analogously \cite{87}. Although we do not pursue this direction in the present paper, we observe for example the precise compatibility between the recent conjecture of Katzarkov–Kontsevich–Pantev \cite{89} (3.1.5)], specialized to \(G^\vee/P^\vee = \mathbb{P}^n\), and the classical theorem of Dwork and Sperber \cite{126} on the Newton polygon of hyper-Kloosterman sums. See also \cite{132} \S3.

### 1.10. Frenkel–Gross’s rigid connection.

In \cite{48}, Frenkel–Gross study a rigid irregular connection on the trivial \(G^\vee\)-bundle on \(\mathbb{P}^1\) given by the formula

\[(1.10.1) \quad \nabla^{G^\vee} := d + f dq + x_\theta dq,\]

where \(f \in g^\vee = \text{Lie}(G^\vee)\) is a principal nilpotent, and \(x_\theta \in g^\vee_\theta\) lives in the highest root space. For any \(G^\vee\)-representation \(V\), we have an associated connection \(\nabla^{(G^\vee, V)}\).

When \(V\) is the minuscule representation of \(G^\vee\) corresponding to parabolic \(P^\vee\), we have a natural isomorphism \(L : H^*(G^\vee/P^\vee) \cong V\).
Theorem 1.11. Under the isomorphism \( L : H^*(G^\vee/P^\vee) \cong V \), the quantum connection \( Q^{G^\vee/P^\vee} \) is isomorphic to the connection \( \nabla^{(G^\vee,V)} \).

The isomorphism \( L \) sends the Schubert basis of \( H^*(G^\vee/P^\vee) \) to the canonical basis of \( V \). The proof of Theorem 1.11 is via a direct comparison of the Frenkel–Gross operator in the canonical basis with the quantum Chevalley formula.

1.12. Zhu’s theorem. In their seminal work, Beilinson–Drinfeld have introduced a class of connections called opers, extending earlier work of Drinfeld and Sokolov. They use opers to construct (part of) the Galois to automorphic direction of the geometric Langlands correspondence. Frenkel–Gross have observed that (1.10.1) can be put into oper form after a gauge transformation.

Zhu has extended Beilinson–Drinfeld’s construction to allow certain nonconstant group schemes, or equivalently to allow specified ramifications. He thereby confirms the following conjecture of Heinloth–Ngô–Yun.

Theorem 1.13. The Kloosterman D-module \( K_1(G^\vee,V) \) is isomorphic to the Frenkel–Gross connection \( \nabla^{(G^\vee,V)} \).

Theorem 1.13 is obtained by composing the isomorphisms of Theorems 1.6, 1.11 and 1.13.

1.14. Equivariant generalization. Figure 1 has an equivariant generalization. We briefly discuss the new features.

We may replace the quantum cohomology ring \( QH^*(G^\vee/P^\vee) \) by the \( T^\vee \)-equivariant quantum cohomology ring \( QH^*_{T^\vee}(G^\vee/P^\vee) \). The corresponding equivariant quantum connection \( Q^{G^\vee/P^\vee}(S) \) is a connection on the trivial \( H^*_{T^\vee}(G^\vee/P^\vee) \)-bundle over \( \mathbb{C}^*_q \), where instead of \( \sigma_* \) in (1.2.1), we have the operator \( c_1^T(O(1)) \ast_{q}^\gamma \) of equivariant quantum multiplication in \( QH^*_{T^\vee}(G^\vee/P^\vee) \) (see [1]). Identifying \( S = H^*_T(pt) \) with \( \text{Sym}(t) \cong \mathbb{C}[t^*] \), where \( t = \text{Lie}(T) \), we may equivalently consider the family of connections \( Q^{G^\vee/P^\vee}(h) := Q^{G^\vee/P^\vee}(S) \otimes_S \mathbb{C} \) indexed by \( h \in t^* \). These are again connections on the trivial \( H^*(G^\vee/P^\vee) \)-bundle over \( \mathbb{C}^*_q \).

Let us now discuss the character \( D \)-module of the geometric crystal \( X \). Let \( \ell : T \to t \) be the logarithm (multi-valued). The character \( D \)-module (1.3.2) parametrized by \( h \in t^* \) is defined to be

\[
Cr_{(G,P)}(h) := R\pi_![f + (\ell(\gamma), h)]^*E.
\]

When \( h \in t^*_G \) is integral, it can be identified with a character \( \lambda : T \to \mathbb{G}_m \), and the geometric character \( \psi_\lambda(t) \) of (1.3.1) is a solution to \( Cr_{(G,P)}(h) \).

The automorphic analogue of the Kloosterman \( D \)-module was constructed in [76]. The automorphic \( D \)-module \( A_G \) can be generalized to an automorphic \( D \)-module \( A_G(h) \) which further depends on the choice of a character of \( T \). In (1.8.2), one then replaces the Artin-Schreier sheaf by the tensor product of an Artin-Schreier sheaf and a Kummer sheaf.

An equivariant analogue of the Frenkel–Gross connection has not appeared in the literature as far as we know. We define it to be

\[
\nabla^G(h) := d + (f + h)\frac{dq}{q} + xq dq.
\]

With these modifications, Figure 1 and Theorems 1.6, 1.9 and 1.11 all hold with their equivariant counterparts. With some mild variation, we also extend Zhu’s theorem to the equivariant setting.
1.15. **Peterson isomorphism.** Given a regular function on an algebraic variety, one can consider the sheaf of Jacobian ideals generated by all the first-order derivatives. Its quotient ring defines a subscheme, possibly non-reduced, of critical points of the function. Since $G/P$ is affine, applying this construction to the equivariant potential, we obtain a (relative) Jacobian ring $\text{Jac}(G/P, f_q + \ell(\gamma))$, which has the structure of a $\mathbb{C}[t^*, q, q^{-1}]$-algebra.

**Theorem 1.16** (Homological mirror isomorphism – Theorem 11.16). If $P^\vee$ is minuscule, then we have an isomorphism of $\mathbb{C}[t^*, q, q^{-1}]$-algebras $\mathcal{Q}H^*_T(G^\vee/P^\vee) \cong \text{Jac}(G/P, f_q + \ell(\gamma))$.

Specializing to non-equivariant cohomology, we obtain the mirror isomorphism of $\mathbb{C}[q, q^{-1}]$-algebras $\mathcal{Q}H^*(G^\vee/P^\vee) \cong \text{Jac}(G/P, f_q)$. The same isomorphism is expected to hold for every Fano mirror dual pair. It is established for toric Fano varieties in [35, 49, 65, 108].

The equivariant Peterson variety $\mathcal{Y}$ is the closed subvariety of $G/B_\times t^*$ defined by

$$\mathcal{Y} := \{(gB_-h) \in (G/B_-) \times t^* \mid g^{-1} \cdot (f - h) \text{ vanishes on } [u^\vee, u^\vee] \}$$

where $f \in g^* = g^\vee$ is a principal nilpotent and $u^\vee := \text{Lie}(U^\vee)$. It contains an open subscheme

$$\mathcal{Y}^* := \mathcal{Y} \cap B_-w_0B_-/B_-$$

obtained by intersecting with the open Schubert cell $B_-w_0B_-/B_-$, where $w_0$ denotes the longest element of $W$. The intersection of $\mathcal{Y}^*$ with the opposite Schubert stratification $\{BwB_-/B_-\}$ gives the $2^{|\mathfrak{g}|}$ strata

$$(1.16.1) \quad \mathcal{Y}_P^* := \mathcal{Y}^* \cap Bw_0^PB_-/B_-$$

where $w_0^P$ is the longest element of $W_P \subset W$ and the intersections are to be taken scheme-theoretically. In [11], Peterson announced the isomorphism $\mathcal{Y}_P^* \cong \text{Spec}(\mathcal{Q}H^*_T(G^\vee/P^\vee))$.

Rietsch [117] has proved that $\text{Jac}(G/P, f_q + \ell(\gamma))$ is isomorphic to $\mathbb{C}[\mathcal{Y}_P^*]$ as $\mathbb{C}[t^*, q, q^{-1}]$-algebras. We thus obtain:

**Corollary 1.17** (Equivariant Peterson isomorphism – Corollary 11.17). If $P^\vee$ is minuscule, then we have an isomorphism of $\mathbb{C}[t^*, q, q^{-1}]$-algebras $\mathcal{Q}H^*_T(G^\vee/P^\vee) \cong \mathcal{C}[\mathcal{Y}_P^*]$.

The Peterson isomorphism has been established directly for Grassmannians by Rietsch [115], for quadrics by Pech–Rietsch–Williams [109], and for Lagrangian and Orthogonal Grassmannians by Cheong [30], all in the non-equivariant case (that is, specializing $h \in t^*$ to 0). In the equivariant case, the results of [115] and [104] can be combined to also obtain Corollary 11.17 for Grassmannians, see [104, §5]. For some other works on the spectrum of classical equivariant cohomology rings, which correspond to the specialization $q = 0$, see [66, 67].

The main idea of the proof of Theorem 11.16 is to pass to the semiclassical limit in the equivariant mirror isomorphism $\mathcal{Q}G^\vee/P^\vee(h) \cong \mathcal{C}r_{(G,P)}(h)$. Thus we introduce back the parameter $\hbar$, rescale the exponential $D$-module $E$ by $1/\hbar$, and consider the limit $\hbar \to 0$. A framework to rigorously justify this limit is to extend the mirror theorem to an isomorphism of $D_\hbar$-modules, where $D_\hbar := \mathbb{C}[q, q^{-1}, \hbar]/(\hbar d_q)$. This is done by exploiting the grading of the quantum product on one side, and the homogeneity of the potential $f_q$ on the other side.

1.18. **Mirror pairs of Fano type and towards mirror symmetry for Richardson varieties.** In our mirror theorem, the A-model $G^\vee/P^\vee$ and the B-model $(X_t, f_t)$ plays distinctly different roles. On the other hand, the geometry of $G/P$ features prominently in the construction of $X_t$. This suggests a more symmetric mirror conjecture should exist.
One such setting could be the mirror pairs of compactified Landau–Ginzburg models studied in [89], and one might speculate on the mirror symmetry of the pair of compactified Landau–Ginzburg models

\[(G/P, g, \omega_{G/P}, f_{G/P}) \text{ and } (G^\vee/P^\vee, g^\vee, \omega_{G^\vee/P^\vee}, f_{G^\vee/P^\vee}).\]

where \(g\) is a Kähler form, \(\omega_{G/P}\) denotes the volume form of [91], and \(f_{G/P}\) denotes the potential function on \(G/P\) discussed above. If such a mirror theorem holds, we would expect a matching of the cohomologies of the log Calabi-Yau manifolds \(\hat{G}/P\) and \(\hat{G}^\vee/P^\vee\).

Namely, we identify the Weyl group of \(G\) and of \(G^\vee\), and denote it by \(W\). For \(v, w \in W\) with \(v \leq w\), the open Richardson variety \(\mathcal{R}_w^\vee \subset G/B\) is the intersection of the Schubert cell \(BvB/B\) with the opposite Schubert cell \(BwB/B\). We denote by \(\mathcal{R}_v^\vee \subset G^\vee/B^\vee\) the Richardson variety attached to \(G^\vee\). Then we have the equality \(H^*(\mathcal{R}_v^\vee) \cong H^*(\mathcal{R}_v^\vee)\) (Proposition 13.4). We are thus led to the question: can our mirror theorems be generalized to Richardson varieties?

Let us also comment that the open Richardson varieties \(\mathcal{R}_v^\vee\) are expected to be cluster varieties [98]. We refer to [64, 71, 72] for recent results on canonical bases on log Calabi–Yau varieties assuming the existence of a cluster structure. For a discussion of the cluster structure of \(G/P\), see [6] §2.

1.19. Other related works. Witten [39, 134], based on previous works by Hausel–Thaddeus [75], Kapustin–Witten and Gukov–Witten, has related Langlands reciprocity for connections with possibly irregular singularities and mirror symmetry of Hitchin moduli spaces of Higgs bundles. The present work may perhaps be seen as an instance of this relation in the case of rigid connections, although we are considering rather the moduli spaces of holomorphic bundles. Another important difference is that automorphic \(D\)-modules appear in [134] as the \(A\)-side, as opposed to the \(B\)-side in the present work.

Recently, the rigid connections of [18, 76] have been generalized by Yun and Chen [29, 136] to parahoric structures and Yun [138] considered rigid automorphic forms ramified at three points. See also [13, 18] for recent advances on wild character varieties and [107, 121] for the Betti side.

Quantum multiplication by divisor classes in the equivariant quantum cohomology ring \(QH^*_{T^*G/P^\vee}(T^*G^\vee/P^\vee)\) of the cotangent bundle has been recently computed for any partial flag variety by Su [129], extending work of Braverman–Maulik–Okounkov [17] for the cotangent bundle \(T^*G^\vee/B^\vee\) of the full flag variety. Specializing the \(\mathbb{C}^\times\)-equivariant parameter to zero, it recovers the \(T\)-equivariant quantum Chevalley formula for \(Q^{G^\vee/P^\vee}\) considered in this paper. It would be interesting to investigate generalizations of Rietsch’s mirror conjecture to this setting. See [130] for work in this direction.

A different approach to mirror phenomena for partial flag varieties is the study of period integrals of hypersurfaces in \(G/P\) by Lian–Song–Yau [99]. Their “tautological system” is further studied in [82], where geometry such as the open projected Richardson \(G/P\) also makes an appearance.

Since \(QH^*(G^\vee/P^\vee)\) is known [27] to be semisimple for minuscule \(P^\vee\), the Dubrovin conjecture concerning full exceptional collections of vector bundles on \(G^\vee/P^\vee\) and the Stokes matrix of \(Q^{G^\vee/P^\vee}\) at \(q = \infty\) is expected to hold. It has been established for projective spaces by Dubrovin and Guzzetti, and more generally for Grassmannians by Ueda [131].
Recent works on exceptional collections for projective homogeneous spaces include [94] for $G^\vee$ classical and [43] for $G^\vee$ of type $E_6$.

Related to the Dubrovin conjecture are the Gamma conjectures [52]. The relation with mirror symmetry is discussed in [33, 89]. The conjectures are currently known for Grassmannians [52], for certain toric varieties [33], and is compatible with taking hyperplane sections [33]. Also [62] establish the Gamma conjecture I for Fano 3-folds with Picard rank one, exploiting notably the modularity of the quantum differential equation which holds for 15 of the 17 families from the Iskovskikh classification.

2. Preliminaries

2.1. Notation for root systems and Weyl groups. Let $G$ denote a complex almost simple algebraic group, $T \subset G$ a maximal torus, and $B, B_-$ opposed Borel subgroups. The Lie algebras are denoted $\mathfrak{g}, \mathfrak{t}, \mathfrak{b}, \mathfrak{b}_-$ respectively. Let $R$ denote the root system of $G$ and $I$ denote the vertex set of the Dynkin diagram. The simple roots are denoted $\alpha_i \in \mathfrak{t}^*$, and the simple coroots are denoted $\alpha_i^\vee \in \mathfrak{t}$. The pairing between $\mathfrak{t}$ and $\mathfrak{t}^*$ is denoted by $\langle ., . \rangle$. Thus $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$ are the entries of the Cartan matrix. Let $R^+, R^- \subset R$ denote the subsets of positive and negative roots. Let $\theta \in R^+$ denote the highest root and $\rho := \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ be the half sum of positive roots.

We let $W$ denote the Weyl group, and $s_i, i \in I$ denote the simple generators. For a root $\alpha \in R$, we let $s_{\alpha} \in W$ denote the corresponding reflection. The length of $w$ is denoted $\ell(w)$. For $w \in W$, we let $\text{Inv}(w) := \{ \alpha \in R^+ \mid \alpha w \in R^- \}$ denote the inversion set of $w$. Thus $|\text{Inv}(w)| = \ell(w)$. Let $\leq$ denote the Bruhat order on $W$ and $< \subset$ denote a cover relation (that is, $w < v$ if $w < v$ and $\ell(w) = \ell(v) - 1$).

Let $P \subset G$ denote the standard parabolic subgroup associated to a subset $I_P \subset I$. Let $W_P \subset W$ be the subgroup generated by $s_i, i \in I_P$. Let $W^P$ be the set of minimal length coset representatives for $W/W_P$. Let $\pi_P : W \to W^P$ denote the composition of the natural map $W \to W/W_P$ with the bijection $W/W_P \cong W^P$. Let $R_P \subset R$ denote the root system of the Levi subgroup of $P$. Let $\rho_P := \frac{1}{2} \sum_{\alpha \in R^+_P} \alpha$.

For a weight $\lambda$ of $\mathfrak{g}$, we let $V_\lambda$ denote the irreducible highest-weight representation of $\mathfrak{g}$ with highest weight $\lambda$.

2.2. Root vectors and Weyl group representatives. We pick a generator $x_\alpha$ for the weight space $\mathfrak{g}_\alpha$ for each root $\alpha$. Write $y_j$ for $x_{-\alpha_j}$ and $x_j$ for $x_{\alpha_j}$. Define

$$\dot{s}_j := \exp(-x_j) \exp(y_j) \exp(-x_j) \in G.$$ 

Then if $w = s_{i_1} \cdots s_{i_l}$ is a reduced expression, the group element $\dot{w} = \dot{s}_{i_1} \cdots \dot{s}_{i_l}$ does not depend on the choice of reduced expression.

We assume that the root vectors $x_\alpha$ have been chosen to satisfy:

1. $\dot{w} \cdot x_\alpha = \pm x_{w\alpha}$,
2. $[x_\alpha, y_\alpha] = \alpha^\vee$,

where $w \cdot x_\alpha$ denotes the adjoint action. See [127, Chapter 3]. In (3.9.1), we will make a choice of sign for $x_\theta$.

2.3. Quantum roots. If $G$ is simply-laced, we consider all roots to be long roots. Otherwise, we have both long and short roots. Let $\tilde{R}^+ \subset R^+$ be the subset of positive roots defined by

$$\tilde{R}^+ = \{ \beta \in R^+ \mid \ell(s_\beta) = \langle 2\rho, \beta^\vee \rangle - 1 \}.$$
A root $\beta \in \tilde{R}^+$ is called a *quantum root* (terminology to be explained in §4.2). If $G$ is simply-laced, then $\tilde{R}^+ = R^+$. Otherwise, it is a proper subset. A root $\alpha \in \tilde{R}^+$ belongs to $\tilde{R}^+$ if one of the following is satisfied (see [17]):

1. $\alpha$ is a long root, or
2. no long simple roots $\alpha_i$ appear in the expansion of $\alpha$ in terms of simple roots.

If $G$ is of type $B_n$, then $\tilde{R}^+$ is the union of the long positive roots with the short simple root $\alpha_n$. If $G$ is of type $C_n$, then $\tilde{R}^+$ is the union of the long positive roots, with the short roots of the form $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ where $1 \leq i \leq j \leq n - 1$.

### 2.4. Minuscule weights

Let $i \in I$ and $\varpi_i$ denote the corresponding fundamental weight. We call $i$, or $\varpi_i$, *minuscule*, if the weights of $V = V_{\varpi_i}$ is exactly the set $W \cdot \lambda$. Equivalently, $i \in I$ is minuscule if the coefficient of $\alpha_i^\vee$ in every coroot $\alpha_i^\vee$ is $\leq 1$.

Let $P = P_i \subset G$ be the parabolic subgroup associated to $I_P = I \setminus \{i\}$. Then $W_P$ is the stabilizer of $\varpi_i$. We have natural bijections between $W^P$, $W/W_P$, and $W : \varpi_i$. We have that $\alpha \in R_P$ if the simple root $\alpha_i$ does not occur in $\alpha$.

The minuscule nodes for each irreducible root system are listed in Figure 2. Our conventions follow the Bourbaki numbering, see [16, Chap. VIII, §7.4, Prop. 8].

If $G$ is simply-laced then a minuscule node is also cominuscule. Thus the coefficient of $\alpha_i$ in every root $\alpha \in R^+$ is $\leq 1$. This means that the nilradical of $P$ is abelian, hence by Borel–deSiebenthal theory, $G/P$ is a compact Hermitian symmetric space, see e.g. [69].
2.5. A remarkable quantum root. Fix a minuscule node $i$ and corresponding parabolic $P = P_i$. Define the long root $\gamma = \gamma(i) \in R$ by:

$$\gamma := \begin{cases} \alpha_i & \text{if } G \text{ is simply-laced,} \\ \alpha_{n-1} + 2\alpha_n & \text{if } G \text{ is of type } B_n \text{ (and thus } i = n), \\ 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n = \theta & \text{if } G \text{ is of type } C_n \text{ (and thus } i = 1). \end{cases}$$

Since $\gamma$ is a long root, it is also a quantum root.

Let $I_Q = \{ j \in I_P \mid \langle \alpha_j, \gamma^\vee \rangle = 0 \} = \{ j \in I_P \mid \langle \gamma, \alpha_j^\vee \rangle = 0 \}$. Then $\alpha \in R_Q$ if no simple root $\alpha_j$ with $j \in I_Q$ occurs in $\alpha$. If $G$ is simply-laced, then $I_Q$ is the set of nodes in $I$ not adjacent to $i$. If $G$ is of type $B_n$, then $I_Q = \{1, 2, \ldots, n-3, n-1\}$. If $G$ is of type $C_n$, then $I_Q = \{2, 3, \ldots, n\} = I_P$.

**Lemma 2.6.** The root $\gamma$ has the following properties:

1. $\langle \varpi_i, \gamma^\vee \rangle = 1$, and
2. $\langle \alpha, \gamma^\vee \rangle = -1$ for $\alpha \in R^+_P \setminus R^+_Q$.

**Proof.** Direct check. \qed

It turns out that the root $\gamma$ can be characterized in a number of ways.

**Proposition 2.7.** Let $\beta \in R^+_P \setminus R^+_P$. Then the following are equivalent:

1. $\beta = \gamma$;
2. $\beta \in \check{R}^+$ and $\langle \alpha, \beta^\vee \rangle \in \{-1, 0\}$ for all $\alpha \in R^+_P$;
3. there exists $w \in W^P$ such that $\beta = -w^{-1}(\theta)$.

Define $W(\gamma) := \{ w \in W^P \mid w\gamma = -\theta \}$. Let $w_{P/Q} \in W_P$ be the longest element that is a minimal length coset representative in $w_{P/Q}W_Q$. Note that $\text{Inv}(w_{P/Q}) = R^+_P \setminus R^+_Q$ (see Lemma 14.3). Denote $s'_\gamma := s_\gamma w_{P/Q}^{-1}$.

**Proposition 2.8.** Suppose $w \in W(\gamma)$. Then:

1. $\ell(ws\gamma) = \ell(w) - \ell(s_\gamma)$,
2. $\ell(ws'\gamma) = \ell(w) - \ell(s_\gamma) - \ell(w_{P/Q}) = \ell(w) - \ell(s'_\gamma)$,
3. $ws'_\gamma = \pi_P(ws\gamma)$,
4. there is a unique length-additive factorization $w = uw'$, where $u \in W_J$ and $w' \in W(\gamma)$ is the minimal length element in the double coset $W_JwW_P$. Here, $W_J$ is a standard parabolic subgroup all of whose generators stabilize $\theta$.

Conversely, suppose $w \in W^P$ satisfies (1) and (2). Then $w \in W(\gamma)$.

Proofs of Propositions 2.7 and 2.8 are given in §14.

3. Frenkel-Gross connection

We caution the reader that the roles of $G$ and $G^\vee$ are reversed in §3–§5 compared to the rest of the paper.
3.1. Principal $\mathfrak{sl}_2$. Let $f := \sum_{i \in I} y_i$ which is a principal nilpotent in $\mathfrak{b}_-$. Let $2\rho^\vee = \sum_{a \in R^+} a^\vee$ viewed as an element of $\mathfrak{t}$. We have

$$2\rho^\vee = 2 \sum_{i \in I} \omega_i^\vee = \sum_{i \in I} c_i \alpha_i^\vee,$$

where the $c_i$ are positive integers. Let $e := \sum_{i \in I} c_i x_i \in \mathfrak{b}$. Then $(e, 2\rho^\vee, f)$ is a principal $\mathfrak{sl}_2$-triple, see [68] and [16] Chap. VIII, §11, n° 4).

Let $\mathfrak{z}(f)$ be the centralizer of $f$ which is an abelian subalgebra of dimension equal to the rank of $\mathfrak{g}$. The adjoint action of $2\rho^\vee$ preserves $\mathfrak{z}(f)$ and the eigenvalues are non-negative even integers. We denote the eigenspaces by $\mathfrak{z}(f)_m$ with $m \geq 0$. Thus $\mathfrak{z}(f)_0 = \mathfrak{z}(\mathfrak{g})$. The integers $m \geq 1$ counted with multiplicity $\dim \mathfrak{z}(f)_m$ coincide with the exponents $m_1 \leq \cdots \leq m_r$ of the root system $R$. Kostant has shown that $\mathfrak{g} = \oplus_{m \geq 0} \text{Sym}^{2m}(\mathbb{C}^2) \otimes \mathfrak{z}(f)_m$ as a representation of the principal $\mathfrak{sl}_2$. It implies that twice the sum of exponents is equal to the number of roots $|\mathcal{R}|$.

The first exponent is $m_1 = 1$ since $\mathfrak{z}(f)_2$ contains $f$. The last exponent is $m_r = c - 1$ which is the height of the highest root $\theta$ because $x_{-\theta} \in \mathfrak{z}(f)$. In fact, $m_i + m_{r-i} = c$ for any $i$.

3.2. Rigid irregular connection. Frenkel and Gross [18] construct a meromorphic connection $\nabla^G$ on the trivial $G$-bundle on $\mathbb{P}^1$ by the formula:

$$(3.2.1) \quad \nabla^G := d + \frac{dq}{q} + x_{\theta} dq.$$

Here $d$ is the trivial connection and $f \frac{dq}{q} + x_{\theta} dq$ is the $\mathfrak{g}$-valued connection 1-form attached to the trivialization $G \times \mathbb{P}^1$. For any finite-dimensional $G$-module $V$, it induces a meromorphic flat connection $\nabla^{(G,V)}$ on the trivial vector bundle $V \times \mathbb{P}^1$. If $V_\lambda$ is an irreducible highest module, we also write $\nabla^{(G,\lambda)}$ for $\nabla^{(G,V_\lambda)}$.

The formula (3.2.1) is in oper form because $\nabla^G$ is everywhere transversal to the trivial $B$-bundle $B \times \mathbb{P}^1$ inside $G \times \mathbb{P}^1$. The connection $\nabla^G$ has a regular singularity at the point 0 with monodromy generated by the principal unipotent $\exp(2i\pi f)$. It has an irregular singularity at the point $\infty$, and it is shown in [18] that the slope is $1/c$ where $c$ is the Coxeter number of $G$. One of the main results of [18] is that the connection is rigid in the sense of the vanishing of the cohomology of the intermediate extension to $\mathbb{P}^1$ of $\nabla^{(G,\text{Ad})}$, viewed as an holonomic $D$-module on $\text{Spec}[q, q^{-1}] \cong \mathbb{C}_q^\times \subset \mathbb{P}^1$. Here $\text{Ad}$ is the adjoint representation of $G$ on $\mathfrak{g}$.

3.3. Outer automorphisms. In certain cases, the connection $\nabla^G$ admits a reduction of the structure group. This is related to outer automorphisms of $G$, and thus to automorphisms of the Dynkin diagram. If $G$ is of type $A_{2n-1}$ then $\nabla^G$ can be reduced to type $C_n$. If $G$ is of type $E_6$ then $\nabla^G$ can be reduced to type $F_4$. If $G$ is of type $D_{n+1}$ with $n \geq 4$ then $\nabla^G$ can be reduced to type $B_n$. In particular, there is a reduction from type $D_4$ to type $B_3$. In fact, by using the full group $S_3$ of automorphisms of the Dynkin diagram, if $G$ is of type $D_4$, then $\nabla^G$ can be reduced to type $G_2$. As a consequence, there is also a reduction of $\nabla^G$ from type $B_3$ to type $G_2$ even though $B_3$ has no outer automorphism. It follows from [18] §6 and §13] who determine the monodromy group of $\nabla^G$ for every $G$, that the above is a complete list of possible reductions.
3.4. **Homogeneity.** We make the observation that the connection $\nabla^G$ is compatible with the natural grading on $\mathfrak{g}$ induced by the adjoint action of $\rho^\vee$. Precisely we have a $\mathbb{G}_m$-action on $\mathfrak{g}$ induced by $\zeta \mapsto \text{Ad}(\rho^\vee(\zeta))$ where $\zeta \in \mathbb{G}_m$. Consider also the $\mathbb{G}_m$-action on $\mathbb{C}_q^\times$ given by $\zeta \cdot q = \zeta^c q$, where we recall that $c$ is the Coxeter number of $\mathfrak{g}$. It induces a natural $\mathbb{G}_m$-equivariant linear action on $T^*\mathbb{C}_q^\times$ and also on the bundle $\mathfrak{g} \otimes T^*\mathbb{C}_q^\times$.

**Lemma 3.5.** The connection 1-form $f dq^\vee q + x_\theta dq$ in $\Omega^1(\mathbb{C}_q^\times, \mathfrak{g})$ is homogeneous of degree one under the above $\mathbb{G}_m$-equivariant action.

**Proof.** We have seen in §3.1 that $f$ has degree 1 and $x_\theta$ has degree $1 - c$, which immediately implies the assertion. \qed

3.6. **Frenkel–Gross operator acting on the minuscule representation.** Let $i$ be a minuscule node and $V = V_{\varpi_i}$ denote the minuscule representation. In this section, we explicitly compute $\nabla^{(G,V)}$. We shall use the canonical basis of $V$, constructed in [54, §3.1, 10].

There is a basis $\{v_w \mid w \in W^P\}$ of $V$ characterized by the properties:

$$x_j(v_w) = \begin{cases} v_{s_iw} & \text{if } \langle w\varpi_i, \alpha_j^\vee \rangle = -1 \\ 0 & \text{otherwise.} \end{cases}$$

and the condition that $v_w$ has weight $w\varpi_i$. Note that in the formulae above, $s_iw$ always lies in $W^P$.

The following result follows from [54, Lemma 3.1] and the discussion after [54, Lemma 3.3]. We caution that our $\hat{s}_j$ is equal to Geck’s $n_j(-1)$.

**Lemma 3.7.**

1. For $w \in W^P$, we have $\hat{w}v_w = v_w$. For $u \in W$ and $w \in W^P$, we have $iuw = \pm v_{\pi_P(uw)}$.
2. For $\alpha \in R^+$ and $w \in W^P$, we have

$$x_\alpha(v_w) = \begin{cases} \pm v_{s_\alpha w} & \text{if } \langle w\varpi_i, \alpha^\vee \rangle = -1 \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 3.8.** Let $j \in I$ and $w \in W^P$. Then

$$y_j v_w = \begin{cases} v_{w_{s_\beta}} & \text{if } \beta = \pi^{-1}(\alpha_j) \in R^+ \setminus R^+_P \\ 0 & \text{otherwise.} \end{cases}$$

In the first case, we automatically have $ws_\beta > w$ and $ws_\beta \in W^P$.

**Proof.** Let $\beta = \pi^{-1}(\alpha_j)$. The condition that $\beta > 0$ is equivalent to $s_jw > w$. In this case, $\text{Inv}(s_jw) = \text{Inv}(w) \cup \{\beta\}$, so the condition that $s_jw \in W^P$ is equivalent to $\beta \notin R^+_P$. The condition $\langle w\varpi_i, \alpha_j^\vee \rangle = 1$ is thus equivalent to $\beta \in R^+ \setminus R^+_P$. \qed

Recall that we have defined a distinguished root $\gamma = \gamma(i) \in R^+$ and a subset $W(\gamma) \subset W$ in §2.5.

**Lemma 3.9.** There is a sign $\varepsilon \in \{+1, -1\}$, not depending on $w \in W^P$, such that

$$\varepsilon x_\theta v_w = \begin{cases} v_{\pi_P(ws_\beta)} & \text{if } w \in W(\gamma) \\ 0 & \text{otherwise.} \end{cases}$$
Proof. Let \( \beta = -w^{-1}(\theta) \). By Lemma 3.9(2), \( x_\theta v_w \neq 0 \) if and only if \( \langle w w_i, \theta^\vee \rangle = -1 \). By a similar argument to the proof of Lemma 3.8, this holds if and only if \( \beta \in R^+ \setminus R^+_P \). By Proposition 2.7, we have \( x_\theta v_w \neq 0 \) if and only if \( \beta = \gamma \), that is, \( w \in W(\gamma) \).

Suppose \( w \in W(\gamma) \). By Proposition 2.8(4), we have \( w = uw' \) where \( u \in W_J \) is an element of a standard parabolic subgroup stabilizing \( \theta \), and the product \( uw' \) is length-additive. Then we have

\[
x_\theta v_w = x_\theta w w_1 \epsilon' \epsilon'' w_{-\gamma}(w')^{-1} v_{w'} = v_{w w',\epsilon' \epsilon''} = v_{w w'},
\]

where \( \epsilon', \epsilon \) are signs not depending on \( w \). For the first equality we have used Lemma 3.7(1). For the second equality, we used that \( \hat{u} \) is a product of elements \( s_j \), where \( s_j \theta = \theta \) and thus \( s_j \) commutes with \( x_\theta \). For the third inequality, we used \( x_{-\gamma} v_e = v_{w,\epsilon} \) which follows from Lemma 3.7(2) and Lemma 2.6(1). In the last two equalities, we used Proposition 2.8(2) applied to \( w, w' \in W(\gamma) \). In the last inequality we also used that \( \ell(w w') = \ell(u) + \ell(w' s_{\gamma}) \). \( \square \)

From now on we make the assumption that

\[
(3.9.1) \quad x_\theta \in g_\theta \text{ is chosen so that } \epsilon = 1 \text{ in Lemma 3.9}
\]

4. Quantum cohomology connection

4.1. Quantum cohomology of partial flag varieties. Let \( P \subset G \) be an arbitrary standard parabolic subgroup. Let \( QH^*(G/P) \) denote the small quantum cohomology ring of \( G/P \). It is an algebra over \( \mathbb{C}[q_i \mid i \notin I_{\mathcal{P}}] \), where we write \( q_i \) for \( q_{\alpha_i^\vee} \). For \( w \in W_P \), let \( \sigma_w \in QH^*(G/P) \) denote the quantum Schubert class. For each \( i \in I \) let \( \sigma_i := \sigma_{w,\alpha_i} \). Then we have

\[
QH^*(G/P) \cong \bigoplus_{w \in W_P} \mathbb{C}[q_i \mid i \notin I_{\mathcal{P}}] \cdot \sigma_w.
\]

4.2. Quantum Chevalley formula. The quantum Chevalley formula for a general \( G/P \) is due to Fulton–Woodward [50] and Peterson [111]. Let \( \eta_P : Q^\vee \to Q^\vee/Q_P^\vee \) be the quotient map, where \( Q^\vee = \oplus_i \mathbb{Z} \alpha_i^\vee \) (resp. \( Q_P^\vee = \oplus_{i \in I_P} \mathbb{Z} \alpha_i^\vee \)) is the coroot lattice. Recall that \( \rho_P = \frac{1}{2} \sum_{\alpha \in R^+_P} \alpha \). The following version of the quantum Chevalley rule is from [96, Theorem 10.14 and Lemma 10.18].

Theorem 4.3. For \( w \in W_P \), we have

\[
\sigma_i * q \sigma_w = \sum_{\beta} \langle \varpi_i, \beta^\vee \rangle \sigma_{w w,\beta} + \sum_{\delta} \langle \varpi_i, \delta^\vee \rangle q_{\eta_P(\delta^\vee)} \sigma_{\pi_P(w w_\delta)}
\]

where the first summation is over \( \beta \in R^+ \setminus R^+_P \) such that \( w w_\beta \succ w \) and \( w w_\beta \in W_P \), and the second summation is over \( \delta \in R^+ \setminus R^+_P \) such that

\[
(4.3.1) \quad \ell(w w_\delta) = \ell(w) - \ell(s_\delta), \quad \text{and}
\]

\[
(4.3.2) \quad \ell(\pi_P(w w_\delta)) = \ell(w) + 1 - \langle 2(\rho - \rho_P), \delta^\vee \rangle.
\]

4.4. Degrees. The quantum cohomology ring \( QH^*(G/P) \) is a graded ring. The degree of \( \sigma_w \) is equal to \( 2 \ell(w) \). The degrees of the quantum parameters \( q_i = q_{\alpha_i^\vee} \) for \( i \in I \setminus I_{\mathcal{P}} \) are given by

\[
\deg q_i = 2 \int_{G/P} c_1(T_{G/P}) \cdot \sigma_{w_0 s_i} = \langle 4(\rho - \rho_P), \alpha_i^\vee \rangle.
\]
The second equality is [30] Lemma 3.5. Indeed, the first Chern class of \( G/P \) satisfies

\[
(4.4.1) \quad c_1(T_{G/P}) = \sum_{i \in I \setminus I_P} \langle 2(\rho - \rho_P), \alpha_i^\vee \rangle \sigma_i.
\]

We verify that the quantum multiplication \( \sigma_i \ast_q \) is homogeneous of degree 2 directly from Theorem 4.3. Indeed, \( \sigma_{ws_i} \) has degree \( 2\ell(w) + 2 \), and

\[
de\eta_{\rho_P(\delta^\vee)} + 2\ell(\pi_P(ws_i)) = \langle 4(\rho - \rho_P), \eta_P(\delta^\vee) \rangle + 2\ell(w) + 2 - 2\langle 2(\rho - \rho_P), \delta^\vee \rangle = 2\ell(w) + 2,
\]

where the second equality follows because \( \rho - \rho_P \) is orthogonal to \( Q_P^\perp \).

4.5. Quantum connection and quantum \( D \)-module. We let \( C_q := \text{Spec } \mathbb{C}[q_i \mid i \notin I_P] \) and \( C_q^\times := \text{Spec } \mathbb{C}[q_i^\pm \mid i \notin I_P] \). We can attach a quantum connection \( \mathcal{Q}_{G/P}^q \) on the trivial bundle \( C_q^\times \times H^*(G/P) \) over \( C_q^\times \) as follows. For each \( i \in I \setminus I_P \), the connection \( \mathcal{Q}_{q,\delta_i}^G \) in the direction of \( q_i \) is given by

\[
q_i \frac{\partial}{\partial q_i} + \sigma_i \ast_q
\]

where \( \ast_q \) is quantum multiplication with quantum parameter \( q \). The connection is integrable, which is equivalent to the associativity of the quantum product. The associated connection 1-form is

\[
(4.5.1) \quad \sum_{i \in I \setminus I_P} (\sigma_i \ast_q) \frac{dq_i}{q_i} \in \Omega^1(C_q^\times, \text{End}(H^*(G/P))).
\]

Define a \( \mathbb{G}_m \)-action on \( H^*(G/P) \) by \( \zeta \cdot \sigma = \zeta^i \sigma \) for \( \zeta \in \mathbb{G}_m \) and \( \sigma \in H^{2i}(G/P^\vee) \). Also define a \( \mathbb{G}_m \)-action on \( C_q^\times \) by \( \zeta \cdot q_i = \zeta^{\deg(q_i)/2} q_i \) for \( i \notin I_P \). Then it is clear that the connection 1-form \( 4.5.1 \) is homogeneous of degree one for the action of \( \mathbb{G}_m \).

Remark 4.6. We may identify the universal cover of \( C_q^\times \) with \( H^2(G/P) \) and define a flat connection on \( H^2(G/P) \) instead, which would correspond to the general framework of Frobenius manifolds (see [40, 77, 103]). Viewing \( \{ \sigma_i \mid i \notin I_P \} \) as a basis of \( H^2(G/P) \), the link is the change of parameters given by \( q \mapsto \sum_{i \in I \setminus I_P} \log(q_i) \sigma_i \). See e.g. [83, §2.2]. Intrinsically \( C_q^\times \) is identified with the quotient \( H^2(G/P)/2\pi H^2(G/P, \mathbb{Z}) \), see also Lemma 8.2 below.

4.7. Minuscule case. For minuscule \( G/P \), with \( I_P = I \setminus \{ i \} \), where \( i \) is a minuscule node, we shall simplify Theorem 4.3. The Schubert divisor class \( \sigma_i \in H^2(G/P) \) is a generator of \( \text{Pic}(G/P) \). It defines a minimal homogeneous embedding \( G/P \subset \mathbb{P}(V) \), and \( G/P \) is realized as the closed orbit of the highest weight vector \( v \in V \). The hyperplane class of \( \mathbb{P}(V) \) restricts to \( \sigma \). The following is established in [25, 124]:

Lemma 4.8. If \( P = P_i \) is a minuscule parabolic, then \( \langle 2(\rho - \rho_P), \alpha_i^\vee \rangle = c \), the Coxeter number of \( G \).

It then follows from \( 4.4.1 \) that the first Chern class \( c_1(T_{G/P}) \) is equal to \( c \sigma \). There is only one quantum parameter \( q = q_i = q_\alpha^\vee_i \) which has degree \( 2c \).

Proposition 4.9. Let \( \gamma = \gamma(i) \) be the long root of \( \gamma(i) \). Then for \( w \in W_P \), we have

\[
\sigma_i \ast_q \sigma_w = \sum_{\beta} \sigma_{ws_{\beta}} + \chi(w) q \sigma_{P(ws_{\gamma})}
\]
where the first summation is over \( \beta \in R^+ \setminus R^+_P \) such that \( ws_\beta \succ w \) and \( ws_\beta \in WP \), and \( \chi(w) \) equals 1 or 0 depending on whether \( w \in W(\gamma) \) or not.

Proof. For \( \beta^\vee \in R^+ \), the coefficient \( \langle w_\beta, \beta^\vee \rangle \) is either 0 or 1, and it is equal to 1 if \( \beta^\vee \in R^+ \setminus R^+_P \). This explains the first summation.

Suppose that \( \delta \in R^+ \setminus R^+_P \) and we have \((4.3.1)\) and \((4.3.2)\). Define \( I_{Q'} \) by \( I_{Q'} = \{ j \in I_P \mid \langle \alpha_j, \delta^\vee \rangle = 0 \} = \{ j \in I_P \mid \langle \gamma_j, \delta^\vee \rangle = 0 \} \).

We have \( \text{Inv}(ws_\delta) \cap R^+_P = \emptyset \) and thus \( \text{Inv}(ws_\delta) \cap R^+_P \subseteq (R^+_P \setminus R_{Q'}) \). But \((4.3.1)\) implies that \( s_{\gamma} \in WP \) and therefore \( \langle \alpha, \delta^\vee \rangle < 0 \) for \( \alpha \in R^+_P \setminus R^+_Q \), so

\[ |R^+_P \setminus R^+_Q| \leq - \sum_{\alpha \in R^+_P \setminus R^+_Q} \langle \alpha, \delta^\vee \rangle = - \sum_{\alpha \in R^+_P} \langle \alpha, \delta^\vee \rangle = -\langle 2\rho_P, \delta^\vee \rangle. \]

Condition \((4.3.2)\) guarantees that we have equality and hence that \( \langle \alpha, \delta^\vee \rangle = -1 \) for \( \alpha \in R^+_P \setminus R^+_Q \). By Proposition \(2.7\), we conclude that \( \delta = \gamma \). It follows from the last sentence of Proposition \(2.8\) that \( w \in W(\gamma) \).

Example 4.10. Suppose that \( G/P = \text{Gr}(n - 1, n) = \mathbb{CP}^{n-1} \). The minimal representative permutations \( w \in WP \) are determined by the value \( w(n) \in [1, n] \), or equivalently by a Young diagram which is a single column of length \( w(n) - 1 \). Denote the Schubert classes by \( \sigma_0 = 1, \sigma_1 = \sigma_{s_0}, \sigma_2, \ldots, \sigma_{n-1} \). Then \( \sigma_1 \cdots \sigma_j = \sigma_j \) for \( 1 \leq j \leq n-1 \) and \( \sigma_1 \ast q \sigma_{n-1} = q \). The quantum cohomology ring has presentation \( \mathbb{C}[\sigma_1, q]/(\sigma_1^2 - q) \).

Chaput–Manivel–Perrin \([25,27]\) study the quantum cohomology of minuscule and cominuscule flag varieties. In particular they obtain a combinatorial description in terms of certain quivers \([25, \text{Prop. 24}]\), which may be compared with Proposition \(4.9\) above.

4.11. Minuscule representation. Define the linear isomorphism

\begin{equation}
L : H^*(G/P) \to V \quad \sigma_w \mapsto v_w \text{ for } w \in WP.
\end{equation}

Recall the principal \( sl_2\)-triple \((e, 2\rho^\vee, f)\), where \( f = \sum_{i \in I} y_i \).

Proposition 4.12 (Gross \(69\)). The isomorphism \( L \) intertwines the action of the Lefschetz \( sl_2 \) on \( H^*(G/P) \) and the action of the principal \( sl_2 \) on \( V \).

Proof. If the term \( \sigma_{ws_j} \) occurs in \( \sigma_1 \ast \sigma_w \) then \( \sigma^\beta = \sigma_j \) for some \( j \) (see \([128]\)). It then follows from Lemma \(3.8\) that \( L(\sigma_1 \ast \sigma_w) = f v_w = f \circ L(\sigma_w) \). On the other hand, we have \( \dim(G/P) = \langle w, 2\rho^\vee \rangle \) and \( \ell(w) = \langle w, \rho^\vee \rangle - \langle w, \rho^\vee \rangle \), see \([69, \text{§6}]\). Since \( L(\sigma_w) = v_w \) has weight \( w \omega_i \) (see \([3.6]\), for every \( d \in [0, 2 \dim(G/P)] \)), the image \( L(H^d(G/P)) \) is equal to the \( 2\rho^\vee\)-eigenspace of \( V \) of eigenvalue \( \dim(G/P) - d \). \[ \square \]

Consider quantum multiplication \( \sigma_1 \ast q \) as an operator on \( H^*(G/P) \) with coefficients in \( \mathbb{C}[q] \).

Proposition 4.13. We have \( L \circ \sigma_1 \ast q = (f + qx_\theta) \circ L \).

Proof. Let \( \sigma_1 \ast q = D_1 + D_2 \), where \( D_1 \) and \( D_2 \) correspond to the two terms of Proposition \(4.9\). We have seen in the proof of Proposition \(4.12\) that \( L \circ D_1 = f \circ L \). Assumption \((3.9.1)\) and Proposition \(4.9\) show that \( L \circ D_2 = qx_\theta \circ L \). \[ \square \]
Golyshov and Manivel [63] study “quantum corrections” to the geometric Satake correspondence. Their main result is closely related to our Proposition [4.13] for the simply-laced cases. Recall from [4.13] that the quantum connection on $\mathbb{C}^\times$ is given by

\begin{equation}
Q^{G/P} = d + \sigma_1 *_q dq.
\end{equation}

**Theorem 4.14.** If $P \subset G$ is minuscule with minuscule representation $V$, then under the isomorphism $L : H^*(G/P) \rightarrow V$, the quantum connection $Q^{G/P}$ is isomorphic to the rigid connection $\nabla^{(G,V)}$. Moreover, the isomorphism is graded with respect to the gradings in [3.3] and [4.3].

4.15. **Automorphism groups.** The connected automorphism group of a projective homogeneous space $H/P$ is equal to $H$ except in the following three exceptional cases, see [2, §3.3]:

- If $H = \text{Sp}(2n)$ is of type $C_n$, $n \geq 2$, and $i = 1$ is the unique minuscule node, then $H/P_1$ is isomorphic to projective space $\mathbb{P}^{2n-1}$. Thus it is homogeneous under the bigger automorphism group $G = \text{GL}(2n)$.
- If $H = \text{SO}(2n + 1)$ is of type $B_n$, $n \geq 2$, and $i = n$ is the unique minuscule node, then the odd orthogonal Grassmannian $SO(2n + 1)/P_n$ is isomorphic to the even orthogonal Grassmannian $SO(2n + 2)/P_{n+1}$.
- If $H$ is of type $G_2$ and $i = 1$, then $H/P_1$ is a five-dimensional quadric which is also isomorphic to $SO(7)/P_1$. In this case $i$ corresponds to the unique short root, which is therefore also the shortest highest root, and thus $H/P_1$ is quasi-minuscule and coadjoint, but it is neither minuscule nor cominuscule. On the other hand the five-dimensional quadric is cominuscule as an homogeneous space under $G = \text{SO}(7)$.

In each of the above cases the quantum cohomology rings coincide, hence the quantum connections also coincide. In the first two cases we can apply Theorem 4.14 to deduce that the corresponding rigid connections associated to a minuscule representation $V$ coincide. In view of [3.3] we conclude that if there is a minuscule Grassmannian $H/P$ whose connected automorphism group is $G$, then $\nabla^G$ can be reduced to $\nabla^H$.

4.16. **Quantum period solution.** The connection $Q^{G/P}$ has regular singularities at $q = 0$. Let $S(q)$ be the flat section of the dual connection that is asymptotic to $\sigma_{w_0w_0}^\alpha$ as $q \rightarrow 0$. Here, $w_0w_0^P$ (resp. $w_0$, and $w_0^P$) is the longest element of $W^P$ (resp. $W$, and $W_P$). The quantum period of $G/P$ is $\langle S(q), 1 \rangle$. Here, $\langle \cdot , \cdot \rangle$ denotes the intersection pairing on $H^*(G/P)$, so $\langle S(q), 1 \rangle$ is equal to the coefficient of $\sigma_{w_0w_0}^\gamma$ in the Schubert expansion of $S(q)$. The quantum period $\langle S(q), 1 \rangle$ has a power series expansion in $q$ with non-negative coefficients, which one can determine using the Frobenius method. We determine the first term in the $q$-expansion in the following.

**Lemma 4.17.** As $q \rightarrow 0$,

$$\langle S(q), 1 \rangle = 1 + q \int_{G/P} \sigma_1 \sigma_{\pi_P(w_0w_0^P s_\gamma)} + O(q^2).$$

The integral above is the number of paths in Bruhat order inside $W^P$ from $\pi_P(w_0w_0^P s_\gamma)$ to $w_0w_0^P$. It is a positive integer.

**Proof.** We write $S(q) = \sigma_{w_0w_0}^\gamma + q v + O(q^2)$, where $v \in H^*(G/P)$. Since $S$ is a flat section of the connection dual to $Q^{G/P}$, we have $\frac{dS}{dq} = \sigma_1 * q S(q)$. Using the quantum Chevalley formula
in Proposition 4.9 this implies
\[ v = \sigma_1 \ast_0 v + \sigma_{\pi_P(w_0 w^P_0 s_{\gamma})}. \]
Since \(\sigma_1 \ast_0\) is nilpotent, the equation uniquely determines \(v\).

We have \(\ell(w_0 w^P_0) = \dim(G/P)\), and
\[ \ell(\pi_P(w_0 w^P_0 s_{\gamma})) = \dim(G/P) + 1 - c, \]
where \(c = \langle 2(\rho - \rho_P), \gamma' \rangle\) is the Coxeter number of \(G\) by Lemma 4.8. Hence we find that \(\langle v, 1 \rangle\) is as stated in the lemma.

The interpretation as counting paths in Bruhat order follows from the classical Chevalley formula for the cup product with \(\sigma_1\). It is a general fact that the Bruhat order of any \(W^P\) is a directed poset with maximal element \(w_0 w^P_0\). In particular, there exists always a path in Bruhat order from any element to the top, and the count is positive. \(\square\)

5. Examples

5.1. Grassmannians. Let \(G = GL_n\). Then \(G/P\) is the Grassmannian \(Gr(k,n)\) for \(1 \leq k \leq n - 1\). The Weyl group \(W = S_n\) and the simple root \(\alpha_i = \gamma\) corresponds to the transposition of \(k\) and \(k+1\). We have \(\langle 2(\rho - \rho_P), \gamma' \rangle = n\). The maximal parabolic subgroup is \(W_P = S_k \times S_{n-k}\). The minimal representatives \(w \in W_P\) are the permutations such that \(w(1) < \cdots < w(k)\) and \(w(k+1) < \cdots < w(n)\). Any such permutation can be identified with a Young diagram that fits inside a \(k \times (n-k)\) rectangle, and \(\ell(w)\) is the number of boxes in the diagram. The projection \(\pi_P : W \rightarrow W_P\) consists in reordering the values \(w(1), \ldots, w(k)\) in increasing order and similarly for \(w(k+1), \ldots, w(n)\).

In the quantum Chevalley formula of Proposition 4.9 the condition that \(w s_{\beta} \gg w\) means that \(\beta \in \mathbb{R}^+ \setminus \mathbb{R}^+_P\) is the transposition of \(l \in [1,k]\) and \(m \in (n-k,n]\) with \(w(m) = w(l) + 1\). Equivalently the Young diagram of \(w s_{\beta}\) has one additional box on the \(k-l+1\)th row. In the second term of the quantum Chevalley formula, the condition \(\ell(\pi_P(w s_{\gamma})) = \ell(w) + 1 - n\) is equivalent to \(w \gamma = -\theta\), which is in turn equivalent to \(w(k) = n\) and \(w(k+1) = 1\). This can also be seen from the fact that the element \(\pi_P(w s_{\gamma})\) has Young diagram obtained by deleting the rim of the diagram of \(w\), see [12]. A presentation for the quantum cohomology ring of Grassmannians is given in [20][22][123].

The first term in the \(q\)-expansion in Lemma 4.17 is \((\frac{n-2}{2})_k\). Indeed, \(\pi_P(w_0 w^P_0 s_{\gamma})\) has Young diagram the \((k-1) \times (n-k-1)\) rectangle. The number of paths in Bruhat order is equal to the number of ways to sequentially add boxes to form the \(k \times n\) rectangle which corresponds to the maximal element \(w_0 w^P_0\) of \(W^P\). This is consistent with the \(q\)-expansion of the quantum period in terms of binomial coefficients in [4] Thm. 5.1.6 and [104] Cor. 4.7.

The fundamental representation \(V = V_{w_{\gamma}}\) is the exterior product \(\Lambda^k \mathbb{C}^n\). The highest weight vector is \(v = e_1 \wedge \cdots \wedge e_k\). For every \(w \in W^P\), the basis vector is \(v_w = w \cdot v = e_{w(1)} \wedge \cdots \wedge e_{w(k)}\). The Schubert class \(\sigma_w\) is the \(B\)-orbit closure of \(\text{span}(e_{w(1)}, \ldots, e_{w(k)})\) inside \(Gr(k,n)\).

Example 5.2. Assume that \(k = 2\) and \(n = 4\). Denote the Schubert classes by \(\sigma_0 = 1, \sigma_1 = \sigma_{s_1}, \sigma_{11}, \sigma_2, \sigma_{21}\) and \(\sigma_{22}\). The quantum Chevalley formula gives the identities \(\sigma_1 \ast_q \sigma_1 = \sigma_{11} + \sigma_2, \sigma_1 \ast_q \sigma_{11} = \sigma_{11} \ast_q \sigma_2 = \sigma_{21}, \sigma_1 \ast_q \sigma_{21} = \sigma_{22} + q\) and \(\sigma_1 \ast_q \sigma_{22} = q \sigma_{21}\).

5.3. Type D. If \(G = SO(2n)\) is of type \(D_n\), \(n \geq 4\), and \(i = 1\), then \(G/P_i\) is a quadric of dimension \(2n - 2\) in \(\mathbb{P}^{2n}\). The quantum cohomology ring is described in [26][109].
The two minuscule nodes \(i = n\) and \(i = n - 1\) are equivalent, and then \(G/P_n\) is the orthogonal Grassmannian of maximal isotropic subspaces in \(\mathbb{C}^{2n}\). A presentation for the quantum cohomology ring is given in [93].

5.4. **Exceptional cases.** A presentation of the quantum cohomology ring of the Cayley plane \(E_6/P_6\) (resp. the Freudenthal variety \(E_7/P_7\)) is given in [25, Thm. 31] (resp. [25, Thm. 34]). The quantum corrections in the quantum Chevalley formula are also described in terms of the respective Hasse diagram. There are 6 (resp. 12) corrections terms for \(E_6/P_6\) (resp. \(E_7/P_7\)).

5.5. **Six-dimensional quadric, triality of \(D_4\).** A case of special interest is \(G = SO(8)\) of type \(D_4\) where all minuscule nodes \(1, 3, 4\) are equivalent. The homogeneous space \(G/P_1\) is a six-dimensional quadric. It also coincides with the Grassmannian \(SO(7)/P_3\) of isotropic spaces of dimension 3 inside \(\mathbb{C}^7\).

The quadric is minuscule both as a \(SO(8)\)-homogeneous space and as a \(SO(7)\)-homogeneous space. Theorem 4.14 applies in both cases so that \(Q^{SO(8)/P_1} \simeq Q^{SO(7)/P_3}\) is isomorphic to the Frenkel–Gross connection \(\nabla^{(G,V)}\) for both \(G = SO(8)\) and \(G = SO(7)\). Here, the representation \(V\) is either the standard representation of \(SO(8)\), or its restriction to \(SO(7)\) which remains irreducible and is isomorphic to the spinor representation.

**Proposition 5.6.**

(i) The quantum connection \(Q^{SO(8)/P_1}\) of the six-dimensional quadric is the direct sum of two irreducible constituents of dimensions one and seven respectively.

(ii) The monodromy group is \(G_2\).

*Proof.* We have seen in §3.3 that \(\nabla^G\) for \(G\) of type \(D_4\) reduces to \(\nabla^G\) for \(G\) of type \(G_2\). Thus it suffices to observe that the above defining representation \(V\) of \(SO(8)\) when restricted to \(G_2\) decomposes into the trivial representation plus the representation of dimension seven. \(\square\)

A presentation of the quantum cohomology ring of the homogeneous space \(G/P_1\) is given in [26]. It is also given in [93] as a particular case of Grassmannian of isotropic spaces and in [109] as a particular case of even-dimensional quadrics. From either of these presentations or from the quantum Chevalley formula, we find the quantum multiplication by \(\sigma\) in the Schubert basis, thus

\[
Q^{G/P_1} = q \frac{d}{dq} + \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & q \\
0 & 0 & 0 & 0 & 0 & q & 0 \\
0 & 0 & 0 & 0 & q & 0 & 0 \\
0 & 0 & 0 & q & 0 & 0 & 0 \\
0 & 0 & q & 0 & 0 & 0 & 0 \\
0 & q & 0 & 0 & 0 & 0 & 0 \\
q & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The middle cohomology \(H^6(G/P_1)\) is two-dimensional, spanned by \(\{\sigma_3^+, \sigma_3^-\}\). Since \(\sigma_q \sigma_3^+ = \sigma_q \sigma_3^-\), the subspace \(\mathbb{C}(\sigma_3^+ - \sigma_3^-)\) is in the kernel of \(\sigma\) and in particular is a stable one-dimensional subspace of the connection. The other stable subspace, denoted \(H^#(G/P_1)\) following [73], has dimension seven and is spanned by \(\sigma_3^+ + \sigma_3^-\) and all the cohomology in the remaining degrees. This is consistent with Proposition 5.6(i).

The rank seven subspace \(H^#(G/P_1)\) is generated as an algebra by \(H^2(G/P_1)\), and moreover the vector 1 is cyclic for the multiplication by \(\sigma\). The quantum \(D\)-module \(Q^{G/P_1}\) is then given in scalar form as \(D/DL\) where

\[
L := \left(q \frac{d}{dq}\right)^7 + 4q^2 \frac{d}{dq} + 2q.
\]
The differential Galois group of $L$ on $\mathbb{P}^1_{\{0,\infty\}}$ has monodromy $G_2$ according to Proposition 5.6 (ii). Recall from [85] that ultimately the reason for the monodromy to be $G_2$ is the triality of $D_4$ and the invariance of the Frenkel–Gross connection $\nabla^{SO(8)}$ under outer automorphisms which reduces it to $\nabla^{G_2}$.

After rescaling $L$ by $q \mapsto -q/4$, the $D$-module $D/ DL$ becomes isomorphic to the hypergeometric $D$-module $_1F_0\left(\frac{1}{2}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1}, \frac{1}{1} \right)$ studied in [87] with the notation $H(0, 0, 0, 0, 0, 0, 0; 1/2)$. Katz proved in [87, Thm 4.1.5] that the monodromy group is $G_2$ which is consistent with Proposition 5.6 (ii).

Our work gives a new interpretation of $D/ DL$ studied by Katz and Frenkel–Gross as the quantum connection $Q^{G/P}$. Hence the following

**Question 5.7.** Is it possible to see “a priori” that the differential Galois group of the quantum connection of the six-dimensional quadric is $G_2$?

The question seems subtle because for example the quantum connection of the 5-dimensional quadric, which is homogeneous under $G_2$, has rank 6 (see §5.8 below), and thus its monodromy group is unrelated to the group $G_2$.

### 5.8. Odd-dimensional quadrics.

More generally, let $G = \text{SO}(2n + 1)$ be of type $B_n$ with $n \geq 3$. Then $G/P_1$ is a $(2n - 1)$-dimensional quadric and is cominuscule. The cohomology has total dimension $2n$. There is one Schubert class $\sigma_k$ in each even degree $2k \leq 4m - 2$. The quantum product is determined in [26, §4.1.2]. In particular $\sigma_1 \ast_q \sigma_{k-1} = \sigma_k$ for $1 \leq k \leq n - 1$ and $n + 1 \leq k \leq 2n - 2$, $\sigma_1 \ast_q \sigma_{n-1} = 2\sigma_n$, $\sigma_1 \ast_q \sigma_{2n-2} = \sigma_{2n-1} + q$ and $\sigma_1 \ast_q \sigma_{2n-1} = q\sigma_1$, which also follows from the quantum Chevalley formula. The relation between the quantum connection and hypergeometric $D$-modules is studied in detail by Pech–Rietsch–Williams [109].

After a rescaling of coordinate $q \mapsto q/2$ the connection $Q^{G/P_1}$ becomes isomorphic to the connection $\nabla^{(H,V)}$ for $H = \text{SO}(2n)$ of type $D_n$. By [3, §3] $\nabla^H$ reduces to $\nabla^{H'}$ with $H'$ of type $B_{n-1}$.

The space $G_2/P_1$ is a five-dimensional quadric. Its connected automorphism group is $\text{SO}(7)$ by [4, §5]. It is coadjoint as a $G_2$-homogeneous space and cominuscule as a $\text{SO}(7)$-homogeneous space. The cohomology has total dimension 6. A presentation of the quantum cohomology ring is $\mathbb{C}[\sigma, q]/(\sigma^6 - 4q)$, see [28, §5.1].

### 6. Character $D$-module of a geometric crystal

In this section, we introduce the character $D$-module for the geometric crystal of Berenstein and Kazhdan [7]. The roles of $G$ and $G'$ are interchanged relative to §3.5.

#### 6.1. Double Bruhat cells.

Let $U \subset B$ and $U_+ \subset B_+$ be opposite maximal unipotent subgroups. For each $w \in W$, define

$$B_w := U \dot{w} U \cap B_+$$

$$U^w := U \cap B_+ \dot{w} B_-$$

**Lemma 6.2.** Let $U(w) := U \cap \dot{w} U_+ \dot{w}^{-1}$. For $u \in U^w$, there is a unique $\eta(u) \in B_w$ and a unique $\tau(u) \in U(w)$ such that

$$\eta(u) = \tau(u) \dot{w} u.$$  

The twist map $\eta : U^w \to B_w$ is a biregular isomorphism and $\tau : U^w \hookrightarrow U(w)$ is an injection.
Proof. This is \cite{HLM} Prop. 5.1 and 5.2; see also \cite{BHM} thm. 4.7 and \cite{BL} Claim 3.25. Since our conventions differ from those in \cite{BHM,LH} slightly, we provide a proof.

If we define the subgroup \( U'(w) := U_- \cap \hat{w}U_-\hat{w}^{-1} \), then the multiplication maps \( U(w) \times U'(w) \to \hat{w}U_-\hat{w}^{-1} \) and \( U'(w) \times U(w) \to \hat{w}U_-\hat{w}^{-1} \) are bijective. In particular, \( B_-\hat{w}B_-\hat{w}^{-1} = B_-\hat{w}U_-\hat{w}^{-1} = B_-U'U(w) = B_-U(w) \).

We have \( u^{-1} \in U^w \). Thus \( u^{-1}\hat{w}^{-1} \in B_-U(w) \subseteq B_-U \). Since \( B_- \cap U = 1 \), the factorization \( u^{-1}\hat{w}^{-1} = \eta(u)^{-1}\tau(u) \) with \( \eta(u) \in B_- \) and \( \tau(u) \in U(w) \) is unique. Moreover \( \eta(u) \in B^w \).

Since \( \tau(w)w \in B_-u^{-1} \), it follows similarly that \( u \mapsto \tau(u) \) is injective.

Conversely, \( \hat{w}^{-1}U\hat{w}U = (\hat{w}^{-1}U \cap U_-)U = \hat{w}^{-1}U(w)\hat{w}U \). Hence, given \( x \in B^w \) we have \( \hat{w}^{-1}x \in \hat{w}^{-1}U(w)\hat{w}U \), which provides by factorization an inverse element \( \eta^{-1}(x) \in U^w \). □

Lemma 6.3. For \( t \in T \) let \( s := \hat{w}t\hat{w}^{-1} \). Each of \( U^w \) and \( U(w) \) is Ad(\( T \))-stable, and for \( u \in U^w \),

\[
\tau(tu^{-1}) = s\tau(u)s^{-1}, \quad \eta(tu^{-1}) = sn(u)t^{-1}.
\]

Proof. Since \( \hat{w}t = \hat{w}t \), we have \( sn(u)t^{-1} = s\tau(u)s^{-1} \hat{w}tu^{-1} \), hence the assertion follows. □

6.4. Geometric crystals. Fix an arbitrary standard parabolic subgroup \( P \subseteq G \). Let \( w_0 \in W \) be the longest element of \( W \) and \( w_0^P \subseteq W_P \) be the longest element of \( W_P \). Define \( w_P := w_0^P w_0 \) so that \( w_P^P \) is the longest element in \( W_P \). In this case, the subgroup \( U_P := U(w_P) \) is the unipotent radical of \( P \). The parabolic geometric crystal associated to \( (G,P) \) is

\[
X := UZ(L_P)\hat{w}_P U \cap B_- = Z(L_P)B^w_P.
\]

We now define three maps \( \pi, \gamma, f \) on \( X \), called the highest weight map, the weight map, and the decoration or superpotential.

The highest weight map is given by

\[
\pi : X \to Z(L_P) \quad x = u_1t\hat{w}_Pu_2 \mapsto t.
\]

Let \( X_t = \pi^{-1}(t) = \{u_1t\hat{w}_Pu_2 \in B_- \} \) be the fiber of \( X \) over \( t \). We call \( X_t \) the geometric crystal with highest weight \( t \). Since the product map \( Z(L_P) \times B^w_P \to X \) is an isomorphism, we have a natural isomorphism \( X_t \cong B^w_P \). Geometrically we think of \( X \) as a family of open Calabi–Yau manifolds fibered over \( Z(L_P) \).

The weight map is given by

\[
\gamma : X \to T \quad x \mapsto x \mod U_- \in B_-/U_- \cong T.
\]

For \( i \in I \), let \( \chi_i : U \to \mathbb{A}^1 \) be the additive character uniquely determined by

\[
\chi_i(\exp(tx_j)) = \delta_{ij}t.
\]

Let \( \psi = \sum_{i \in I} \chi_i \). The decoration, or superpotential is given by

\[
f : X \to \mathbb{A}^1 \quad x = u_1t\hat{w}_Pu_2 \mapsto \psi(u_1) + \psi(u_2).
\]

It follows from \cite{HLM} Lemma 5.2 that \( f \) agrees with Rietsh’s superpotential.

Set \( \psi_t(u) := \psi(tu^{-1}) \) for \( t \in T \) and \( u \in U \). For \( t \in Z(L_P) \), the potential can be expressed as a function of \( u \in U^{w_P} \) as follows:

\[
f_t(u) := f(t\eta(u)) = \psi_t(\tau(u)) + \psi(u).
\]

Equivalently, the potential is expressed on \( B^{w_P} = U\hat{w}_P U \cap B_- \) by

\[
f_t(u_1\hat{w}_Pu_2) = \psi_t(u_1) + \psi(u_2).
\]
Example 6.5. Let $G = \text{SL}(2)$ and $P = B$. With the parametrizations

$$u_1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \quad t = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}, \quad \hat{w}_P = s_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 & t^2/a \\ 0 & 1 \end{pmatrix}\) 

the geometric crystal $X$ is the set of matrices

$$X = \left\{ \begin{pmatrix} a/t & 0 \\ 1/t & t/a \end{pmatrix} \mid a,t \in \mathbb{C}^\times \right\} \subset \text{SL}(2)$$

equipped with the functions

$$f(x) = a + t^2/a, \quad \pi(x) = \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix}, \quad \text{and} \ \gamma(x) = \begin{pmatrix} a/t & 0 \\ 0 & t/a \end{pmatrix}.$$

6.6. Open projected Richardson varieties. For $v,w \in W$ with $v \leq w$, the open Richardson variety $R^w_v \subset G/B$ is the intersection of the Schubert cell $B \cdot vB/B$ with the opposite Schubert cell $BwB/B$. The map $u \mapsto uw_0 \pmod{B}$ induces an isomorphism $U^w \cong R^w_{w_0}$. For every $t \in \mathbb{Z}(L_P)$, we have a sequence of isomorphisms

$$X_t \cong B^p_r \cong U^w_r \cong R^w_{u_0}$$

given by $x = tu_1\hat{w}_pu_2 \mapsto u_1\hat{w}_pu_2 \mapsto u_2^{-1} \mapsto u_2^{-1}\hat{w}_0B$, where in the factorization we assume $u_1 \in U(w_P)$. We describe directly the composition of these isomorphisms as follows.

Lemma 6.7. For every $x = u_1\hat{w}_pu_2 \in X_t$ with $u_1 \in U_P = \text{U}(w_P)$, we have $x^{-1}\hat{w}_0^pB = u_2^{-1}\hat{w}_0B$.

Proof. We have $x^{-1}\hat{w}_0^pB = u_2^{-1}\hat{w}_0^{-1}u_1\hat{w}_pu_0B$. It suffices to observe that $\hat{w}_0\hat{w}_p^1u_1\hat{w}_pu_0 \in U$ since $u_1 \in U(w_P)$. \hfill \Box

The projection $p : G/B \to G/P$ induces an isomorphism of $R^w_{u_0}$ onto its image $G/P$, the open projected Richardson variety of $G/P$. The complement of $G/P$ in $G/B$ is an anticanonical divisor $\partial_{G/P}$ in $G/P$ \cite{91}. The divisor $\partial_{G/P}$ is the multiplicity-free union of the divisors $D^i, i \in I$ and $D_i, i \not\in I_P$, where

$$D^i := p(R^w_{u_0} \circ s_i) \quad \text{and} \quad D_i := p(R^w_{u_0} \circ t_i).$$

There is, up to scalar, a unique holomorphic anticanonical section $1/\omega$ on $G/P$ which has simple zeroes exactly along $\partial_{G/P}$. The meromorphic form $\omega$ has no zeroes, and simple poles along $\partial_{G/P}$.

By construction, the potential $f_i$ can be identified with a rational function on $G/P$. We now show that the polar divisor of $f_i$ is equal to the anticanonical divisor $\partial_{G/P} \subset G/P$. In the following, we shall assume that $G$ is simply-connected. Since the partial flag variety $G/P$ only depends on the type of $G$, we lose no generality.

For a fundamental weight $\varpi_i$ and elements $u,w \in W$, there is a generalized minor $\Delta_{u\varpi_i,w\varpi_i} : G \to A^1$, defined in \cite{15}. This function is equal to the matrix coefficient $g \mapsto \langle g \cdot v_{u\varpi_i}, v_{w\varpi_i} \rangle$ of $G$ acting on the irreducible representation $V_{w\varpi_i}$, with respect to extremal weight vectors $v_{u\varpi_i} := \hat{w}_i v_{\varpi_i}$ and $v_{w\varpi_i} := \hat{w}_i v_{\varpi_i}$ with weights $w\varpi_i$ and $w\varpi_i$ respectively. Here $v_{\varpi_i}$ denotes a fixed highest weight vector with weight $\varpi_i$.

Proposition 6.8. The potential $f_i$, viewed as a rational function on $G/P$ has polar divisor $\partial_{G/P}$. It can thus be written as the ratio $(1/\eta_i)/(1/\omega)$ of two holomorphic anticanonical
sections on $G/P$, where $1/\omega$ is the unique (up to scalar) holomorphic anticanonical section with simple zeroes along $\partial G/P$.

Proof. Let $x = tu_1\dot{w}_0u_2 \in X_I$, where $u_1 \in U(w_P)$. Under \eqref{6.6.1}, we have that $x$ is sent to $zB \in R_{w_0^P}$ where $z := u_2^{-1}\dot{w}_0$. The image of $x$ in $G/P$ is then $p(zB) = u_2^{-1}\dot{w}_0P$.

Given $zB \in G/B$, and $i \in I$, we have

\begin{equation}
\Delta_{u_0,\omega_i,\omega_i}(z) = 0 \iff z \in B\dot{w}_0s_iB/B
\end{equation}

and for $zP \in G/P$ and $i \notin I_P$, we have

\begin{equation}
\Delta_{\omega_i,\omega_i,\omega_i}(z) = 0 \iff z \in B\dot{w}_0P/P.
\end{equation}

Indeed \eqref{6.8.1} is clear from \cite{45} Prop. 2.4, and for \eqref{6.8.2} if we assume that $z = b_-\dot{v}\dot{w}_0^P$ for some $b_- \in B_-$ and some $v \in W_P$, then $\Delta_{\omega_i,\omega_i,\omega_i}(z) = 0$ if and only if $\Delta_{\omega_i,\omega_i,\omega_i}(\dot{v}) = 0$, from which the result follows. See also \cite{117} §4.

We have by \cite{45} Prop. 2.6 and \cite{7} (1.8)

\[ \chi_i(u_2) = -\chi_i(u_2^{-1}) = -\langle u_2^{-1}v_{u_0,\omega_i}, v_{u_0s_i,\omega_i} \rangle = -\frac{\Delta_{u_0s_i,\omega_i,\omega_i}(u_2^{-1})}{\Delta_{u_0,\omega_i,\omega_i}(u_2^{-1})}. \]

Since $u_2^{-1} = z\dot{w}_0$, we find

\[ \chi_i(u_2) = \frac{\Delta_{u_0s_i,\omega_i,\omega_i}(z)}{\Delta_{u_0,\omega_i,\omega_i}(z)} = \frac{\Delta_{u_0s_i,\omega_i,\omega_i}(zB)}{\Delta_{u_0,\omega_i,\omega_i}(zB)} \]

where the denominator is necessary for the expression to be well defined on $zB$. By \eqref{6.8.1}, the rational function $\chi_i(u_2)$, viewed as a rational function on $G/P$, has a simple pole along $D_i$.

Since $u_1 = xz\dot{w}_0P^{-1}$, we have

\[ \chi_i(u_1) = \langle u_1v_{\omega_i}, v_{s_i,\omega_i} \rangle = \frac{\Delta_{s_i,\omega_i,\omega_i}(u_1)}{\Delta_{\omega_i,\omega_i}(u_1)} = \frac{\Delta_{s_i,\omega_i,\omega_i}(xz\dot{w}_0P)}{\Delta_{\omega_i,\omega_i}(xz\dot{w}_0P)} = \frac{\Delta_{s_i,\omega_i,\omega_i,\omega_i}(xzP)}{\Delta_{\omega_i,\omega_i,\omega_i,\omega_i}(xzP)}. \]

Now $\Delta_{s_i,\omega_i,\omega_i,\omega_i}(xzP) = \Delta_{s_i,\omega_i,\omega_i,\omega_i}(zP)$ so by \eqref{6.8.2}, $\chi_i(u_1)$ has a simple pole along $D_i$. Note that $\Delta_{s_i,\omega_i,\omega_i,\omega_i}(xzP)$ does depend on $t$!

Since $u_1 \in U(w_P)$, then $f(x) = \sum_{i \in I_P} \chi_i(u_1) + \sum_{i \notin I} \chi_i(u_2)$. Since all the divisors $D_i$ and $D_i$ are distinct, the rational function $f_t$ has polar divisor exactly $\partial G/P$. \hfill $\Box$

Remark 6.9. Proposition 6.8 is one manifestation of mirror symmetry of Fano manifolds. For example the potentials of mirrors of toric Fano varieties are constructed in \cite{59} and the same property can be seen to hold. In general, it is explained in Katzarkov–Kontsevich–Pantev \cite{89} Remark, 2.5 (ii) by the fact that the cup product by $c_1(K_{G/\nu}/P^\nu)$ on the cohomology of the mirror manifold $G^\nu/P^\nu$ is a nilpotent endomorphism.

Remark 6.10. The zero divisor of $1/\omega$ and the zero divisor of $1/\eta_i$ may intersect, so $f_t$ has points of indeterminacy. Indeed, this happens in the example of $\mathbb{P}^2$, see also \cite{89} Remark. 2.5 (i)].
6.11. Explicit formula for superpotential. Given $g = b_\nu v$ where $b_\nu \in B_\nu$ and $v \in U$, we set $\pi_+(g) = v$. Also let $g \mapsto g^T$ denote the transpose antiautomorphism of $G$ (see for example [45]). Let $g \mapsto g^{-T}$ denote the composition of the inverse and transpose antiautomorphisms (which commute). There is an involution $\ast : I \to I$ determined by $w_0 \cdot \alpha_i = -\alpha_{i\ast}$. We let $P^\ast$ be the standard parabolic subgroup determined by $I_{P^\ast} = (I_P)^\ast$.

**Lemma 6.12.** For $u \in \mathcal{U}^{w_{P}}$, we have

$$\pi_+((\dot{w}_0)^{-1}u^T\dot{w}_0^{P^\ast}) = (\dot{w}_0)^{-1}\tau(u)^{-T}\dot{w}_0.$$  

**Proof.** Let $v = \tau(u)$. Then $x = \tau(v)w_Pu \in B_\nu$ and $u = (\dot{w}_P)^{-1}v^{-1}x$. Noting that $(\dot{w})^T = (\dot{w})^{-1}$, we have

$$(\dot{w}_0)^{-1}u^T = (\dot{w}_0)^{-1}x^Tv^{-T}\dot{w}_P = [(\dot{w}_0)^{-1}x^Tv_0][[(\dot{w}_0)^{-1}v^{-T}\dot{w}_0][[(\dot{w}_0)^{-1}]^{-1}\dot{w}_P].$$

We note that $[(\dot{w}_0)^{-1}x^Tv_0] \in B_\nu$ and $[(\dot{w}_0)^{-1}v^{-T}\dot{w}_0] \in U$, so the claim follows from the equality

$$(\dot{w}_0)^{-1}\dot{w}_P = (\dot{w}_0^{P^\ast})^{-1}.$$  

We first argue that $(\dot{w}_0)^{-1}\dot{s}_i\dot{w}_0 = \dot{s}_{i\ast}$. Write $\alpha_i^\nu(t)$ for the cocharacter $\mathbb{G}_m \to T$. Then $\alpha^\nu_i(-1) = (\dot{s}_i)^2 \in T$ and $\alpha^\nu_i(-1)^2 = 1$. Let $w' = s_iw_0 = w_0s_i$ and compute

$$(\dot{w}_0)^{-1}\dot{s}_i\dot{w}_0 = (\dot{w}_0)^{-1}\alpha^\nu_i(-1)\dot{w}' = (\dot{w}_0)^{-1}\alpha^\nu_i(-1)\dot{w}_0\alpha^\nu_i(-1)\dot{s}_{i\ast} = \dot{s}_{i\ast}$$

where we have used $(\dot{w}_0)^{-1}\alpha^\nu_i(t)\dot{w}_0 = w_0 \cdot \alpha_i(t) = \alpha_{i\ast}(t^{-1})$. It follows that

$$(\dot{w}_0)^{-1}\dot{w}_P = (\dot{w}_0)^{-1}\dot{w}_P\dot{w}_0(\dot{w}_0)^{-1} = \dot{w}_{P^\ast}(\dot{w}_0)^{-1} = (\dot{w}_0^{P^\ast})^{-1}$$

as required. \hfill \Box

For the following result, we assume $G$ to be simply-connected.

**Lemma 6.13.** For $u \in \mathcal{U}^{w_{P}}$, we have

$$\psi(\tau(u)) = \sum_{i \in I_{P^\ast}} \frac{\Delta_{w_0^{P^\ast}w_{1(\nu)i},w_0w_1(\nu)(u)}}{\Delta_{w_0^{P^\ast}w_1(\nu),w_0w_1(\nu)}(u)}.$$  

Thus

$$f_i(u) = \psi(u) + \sum_{i \in I_{P^\ast}} \frac{\Delta_{w_0^{P^\ast}w_{1(\nu)i},w_0w_1(\nu)}(u)}{\Delta_{w_0^{P^\ast}w_1(\nu),w_0w_1(\nu)}(u)}.$$  

**Proof.** First note that $\chi_i(\pi_+(g)) = \frac{\Delta_{w_{i(\nu)i},w_{1(\nu)i}}(g)}{\Delta_{w_1(\nu),w_{1(\nu)}}(g)}$. We have $\psi(\tau(u)) = \sum_{i \notin P^\ast} \chi_i(\tau(u))$. Since $(\dot{w}_0)^{-1}\exp(ty_i)\dot{w}_0 = \exp(-t\alpha_i^\nu)$, we have $\chi_i(\tau(u)) = \chi_i((\dot{w}_0)^{-1}\tau(u)^{-T}\dot{w}_0)$. By Lemma 6.12, we have for $i^\ast \notin I_P$, the equalities $\chi_{i^\ast}(\tau(u)) = \frac{\Delta_{w_1(i^\ast),w_1(\nu)}(u)}{\Delta_{w_0(i^\ast),w_0w_1(\nu)}(u)}$.

For the last formula, we note that for $t \in Z(L_P)$, we have $\chi_i(t\tau(u)t^{-1}) = \alpha_i(t)\chi_i(\tau(u))$.

Fix a reduced word $i = i_1i_2\cdots i_t$ of $w_P^{-1}$. We have the Lusztig rational parametrization $\mathbb{G}_m^{(w_P)} \to \mathcal{U}^{w_{P}}$ given by

$$a = (a_1,a_2,\ldots,a_{\ell}) \longmapsto x_i(a) = x_{i_1}(a_1)x_{i_2}(a_2)\cdots x_{i_t}(a_{\ell})$$

where $x_i(t) := \exp(tx_i)$ denotes a one-parameter subgroup of $G$.  

Corollary 6.14. In the Lusztig parametrization, the superpotential $f_t : G_m^{(w_P)} \to \mathbb{A}^1$ is given by the function

$$f_t(a_1, a_2, \ldots, a_\ell) = a_1 + a_2 + \cdots + a_\ell + \sum_{i \in I \setminus J_P} \alpha_i(t) P_i$$

where $P_i$ is a Laurent polynomial in $a_1, a_2, \ldots, a_\ell$ with positive coefficients.

Proof. We may assume that $G$ is simply-connected and apply Lemma 6.13. We have $\psi(x_1(a)) = a_1 + a_2 + \cdots + a_\ell$. Now, for any $i$, the generalized minor $\Delta_{w_0^{P^*,s_i,\omega_i}}(x_1(a))$ is a polynomial in $a_1, a_2, \ldots, a_\ell$ with positive coefficients by [9] Theorem 5.8 and $\Delta_{w_0^{P^*,\omega_i}}(x_1(a))$ is a monomial in $a_1, a_2, \ldots, a_\ell$ by [9] Corollary 9.5. □

Corollary 6.14 generalizes [32, Theorem 5.6] to the parabolic setting.

Example 6.15. Let us pick $G = SL(5)$ and $i = 2$. A reduced word for $w_P^{-1}$ is 234123. Using the reversed reduced word for $w_P^{-1}$, we obtain the parametrization

$$(a_1, a_2, a_3, a_4, a_5, a_6) \mapsto u = x_i(a) = \begin{pmatrix} 1 & a_3 & a_3 a_6 & 0 & 0 \\ 0 & 1 & a_2 + a_6 & a_2 a_5 & 0 \\ 0 & 0 & 1 & a_1 + a_5 & a_1 a_4 \\ 0 & 0 & 0 & 1 & a_4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\tau(u) = \begin{pmatrix} 1 & 0 & \frac{1}{a_1 a_2 a_3} & \frac{1}{a_1 a_2} & \frac{1}{a_1} \\ 0 & 1 & \frac{a_1 a_2 a_3 a_4 a_5 a_6}{a_1 a_2 a_3 a_4 a_5 a_6} & \frac{1}{a_1 a_2 a_3 a_4 a_5 a_6} & \frac{1}{a_1 a_2 a_3 a_4 a_5 a_6} \\ 0 & 0 & 1 & \frac{a_1 a_2 a_3 a_4 a_5 a_6}{a_1 a_2 a_3 a_4 a_5 a_6} & \frac{1}{a_1 a_2 a_3 a_4 a_5 a_6} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Thus $\psi(\tau(u)) = (a_1 a_2 + a_1 a_5 + a_5 a_6)/a_1 a_2 a_3 a_4 a_5 a_6$. This is equal to the ratio $\Delta_{235,345}(u)/\Delta_{123,345}(u)$, agreeing with Lemma 6.13. Here, $\Delta_{I,J}$ denotes the minor using rows $I$ and columns $J$. Thus the superpotential is given by

$$f_t = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + \frac{a_1 a_2 + a_1 a_6 + a_5 a_6}{a_1 a_2 a_3 a_4 a_5 a_6},$$

where $q = \alpha_i(t)$.

6.16. The character $D$-module of a geometric crystal. Let $E := D_{\mathbb{A}^1}/D_{\mathbb{A}^1}(\partial_x - 1)$ be the exponential $D$-module on $\mathbb{A}^1$. Let $E^f = f^* E$ be the pullback $D$-module on $X$. Finally, define the character $D$-module of the geometric crystal $X$ by

$$(6.16.1) \quad \text{Cr}_{(G,P)} := R\pi_! E^f \text{ on } Z(L_P).$$

A priori $\text{Cr}_{(G,P)}$ lies in the derived category of $D$-modules on $Z(L_P)$. But in Theorem 7.12 we shall see that $R^i \pi_! E^f = 0$ for $i \neq 0$, and thus $\text{Cr}_{(G,P)}$ is just a $D$-module.

6.17. Homogeneity. Recall that $\rho_P = \frac{1}{2} \sum_{\alpha \in R_P^+} \alpha$ and that $w_P = w_0 P$. We compute $w_P(\rho) = w_0 P(\rho) = w_0 P(\rho) = w_0 P(\rho) = w_0 P(\rho) = - (\rho - \rho_P) + \rho_P = - \rho + 2 \rho_P$. □

Lemma 6.18. We have $w_P(\rho) = - \rho + 2 \rho_P$.

Proof. The element $w_0 P$ sends $R_P^+$ to $R_P^-$ and permutes the elements of $R^+ \setminus R_P^+$. We compute

$$w_P(\rho) = w_0 P(\rho) = w_0 P(\rho) = w_0 P(\rho) = w_0 P(\rho) = - (\rho - \rho_P) + \rho_P = - \rho + 2 \rho_P. \quad \square$$
We view $2\rho^\vee$ as a cocharacter $G_m \to T$. Similarly, we view $2\rho^\vee - 2\rho_P^\vee$ as a cocharacter $G_m \to Z(L_P)$.

**Lemma 6.19.** For any $u \in U^{w_P}$, $t \in Z(L_P)$ and $\zeta \in G_m$,

$$f_{(\rho^\vee - \rho_P^\vee)(\zeta^2)t}(\text{Ad}(\rho^\vee(\zeta))(u)) = \zeta f_t(u).$$

**Proof.** We have $\psi_{\rho^\vee(\zeta)}(u) = \zeta \psi(u)$ for any $u \in U$ and $\zeta \in G_m$. Thus in view of (6.4.2), we have

$$f_{(\rho^\vee - \rho_P^\vee)(\zeta^2)t}(\text{Ad}(\rho^\vee(\zeta))(u)) = \psi_{(\rho^\vee - \rho_P^\vee)(\zeta^2)t}(\tau(\text{Ad}(\rho^\vee(\zeta))(u))) - \zeta \psi(u),$$

and we are now reduced to treating the first term. It follows from Lemma 6.3 that

$$\tau(\text{Ad}(\rho^\vee(\zeta))(u)) = \text{Ad}(w_P(\rho^\vee))(\zeta)(\tau(u))$$

and therefore the first term above is equal to

$$\psi_{(w_P(\rho^\vee)))(\zeta)(\rho^\vee - \rho_P^\vee)(\zeta^2)t}(\tau(u)) = \psi_{(w_P(\rho^\vee) + 2\rho^\vee - 2\rho_P^\vee)(\zeta^2)t}(\tau(u)) = \psi_{\rho_P^\vee(\zeta^2)t}(\tau(u)) = \zeta \psi_t(\tau(u)).$$

In the second line we have used Lemma 6.18, but with $\rho^\vee$ and $\rho_P^\vee$ instead of $\rho$ and $\rho_P$. \qed

We define the following $G_m$-actions on $X$, $Z(L_P)$, and $A^1$. For $\zeta \in G_m$, we have

$$\begin{align*}
\zeta \cdot x &= \rho^\vee(\zeta)x\rho^\vee(\zeta)^{-1} \\
\zeta \cdot t &= (2\rho^\vee - 2\rho_P^\vee)(\zeta)t \\
\zeta \cdot a &= \zeta a
\end{align*}$$

for $x \in X$, $t \in Z(L_P)$, and $a \in A^1$.

Also equip $T$ with the trivial $G_m$-action.

**Proposition 6.20.** The maps $\pi : X \to Z(L_P)$, $f : X \to A^1$, and $\gamma : X \to T$ are $G_m$-equivariant.

**Proof.** We have $\zeta \cdot x = (2\rho^\vee - 2\rho_P^\vee)(\zeta)w_P(\rho^\vee)(\zeta)x\rho^\vee(\zeta)^{-1}$. We verify using Lemma 6.3 that $x \mapsto w_P(\rho^\vee)(\zeta)x\rho^\vee(\zeta)^{-1}$ is an automorphism of $B_{\rho^\vee}$. This shows that $\pi(\zeta \cdot x) = \zeta \cdot \pi(x)$. The second claim follows from Lemma 6.19. The last claim is immediate from the definitions. \qed

**Corollary 6.21.** For any $h \in C^\times$, we have

$$\pi_tE/h \cong [q \mapsto (2\rho^\vee - 2\rho_P^\vee)(h)q]_\text{Cr}(G,P).$$

**Proof.** By definition the left-hand side is equal to $\pi_tf^*[a \mapsto a/h]^*E$. In view of Proposition 6.20 it is isomorphic to

$$\pi_1(x \mapsto h \cdot x)f^*E = [q \mapsto (2\rho^\vee - 2\rho_P^\vee)(h)q]_1 \pi_1f^*E = [q \mapsto (2\rho^\vee - 2\rho_P^\vee)(h)q]_1 \text{Cr}(G,P),$$

which concludes the proof. \qed

We record the following lemma which will be needed in (12) in the context of rapid decay cycles.

**Lemma 6.22.** The meromorphic form $\omega$ on $G/P$ with simple poles along the anticanonical divisor $\partial_{G/P}$ is preserved under the $T$-action.
Proof. First we observe that each irreducible component of the divisor \( \partial_{G/P} \) is \( T \)-invariant and thus the sections cutting them out are \( T \)-weight vectors. Now for \( P = B \), the sections cutting out the \( 2|I| \) divisor components \( D_i \) and \( D^i \) have \( T \) weights \( \omega_1, \ldots, \omega_n \) and \( w_0 \omega_1, \ldots, w_0 \omega_n \). The sum of these weights is 0, so the form \( \omega_{G/B} \) must be \( T \)-invariant. Each open Richardson variety \( R^w \) has its own canonical form \( \omega_{R^w} \) which is obtained from \( \omega_{G/B} \) by taking residues, so again these forms are \( T \)-invariant. Finally, for each parabolic \( P \), the projection map \( p : G/B \to G/P \) induces an isomorphism of \( R^w_{\omega_P} \) onto its image \( G/P \). Since \( p \) is \( T \)-equivariant, the result follows. \( \square \)

6.23. Convention for affine Weyl groups. Let \( w \tau^\lambda \in W_{af} = W \ltimes Q^\vee \) denote an element of the affine Weyl group, and let \( \delta \) denote the null root of the affine root system. Then for \( \mu \in P \),

\[
(6.23.1) \quad w \tau^\lambda \cdot (\mu + n\delta) = w\mu + (n - \langle \mu, \lambda \rangle)\delta.
\]

6.24. Cominuscule case. We now assume that \( G \) is simple and of adjoint type. Fix a cominuscule node \( i \) of \( G \), which is also a minuscule node of \( G^\vee \). Let \( P = P_i \) be the corresponding (maximal) parabolic, and identify \( Z(L_P) \) with \( \mathbb{G}_m \cong \mathbb{P}^1_{\{0,\infty\}} \) via the simple root \( \alpha_i \).

Lemma 6.25. If \( P \) is a cominuscule parabolic, then the composition of \( (2\rho^\vee - 2\rho_P^\vee) : \mathbb{G}_m \to Z(L_P) \) with \( \alpha : Z(L_P) \cong \mathbb{G}_m \), is the character \( q \mapsto q^c \), where \( c \) is the Coxeter number of \( G \).

Proof. We have \( \rho_G^\vee - \rho_P^\vee = \rho_c^\vee - \rho_P^\vee \) for the dual minuscule parabolic group \( P^\vee \) of \( G^\vee \). Since \( \alpha_i \) is a simple coroot of \( G^\vee \), it follows from Lemma 4.8 that \( \langle 2(\rho_G^\vee - \rho_P^\vee), \alpha_i \rangle = c \) where \( c \) is the Coxeter number of \( G \). \( \square \)

Let \( \Omega \) be the quotient of the coweight lattice of \( G \) by the coroot lattice. Thus \( \Omega \) is isomorphic to the center of \( G^\vee \). Let \( \gamma \in \Omega \) be the element corresponding to the cominuscule node \( i \). Namely, \( \gamma \equiv \varpi_i^\vee \) under this identification (see [96, Section 11.2]). For a coweight \( \lambda \) of \( G \), we abuse notation by letting \( \tau^\lambda \in G((\tau)) \) denote the corresponding element, which is a lift of the translation element \( \tau^\lambda \in W_{af} \). Let \( \hat{\gamma} = \hat{w}_P \tau^\gamma \). Then \( \hat{\gamma} \in G((\tau)) \) is a lift of \( \gamma \) to the loop group. Note that \( \hat{\gamma}|_{\tau^{-1}} = \hat{w}_P^{-1} \).

Let \( G[\tau^{-1}]_1 := \ker(G[\tau^{-1}]_{ev_{\infty}} \to G) \), where \( ev_{\infty} \) is given by \( \tau^{-1} = 0 \).

Lemma 6.26.

(a) For \( u \in U_P \), we have \( \hat{\gamma}^{-1} u \hat{\gamma} \in G[\tau^{-1}]_1 \).
(b) We have \( w_P^{-1} \alpha_i = -\theta \).

Proof. We have \( \text{Inv}(w_P^{-1}) = R^+ \setminus R_P^+ \). Thus \( w_P^{-1} \) acts as a bijection from \( R^+ \setminus R_P^+ \) to \( -(R^+ \setminus R_P^+) \). In particular, \( w_P^{-1} \alpha = -(R^+ \setminus R_P^+) \) for \( \alpha \in R^+ \setminus R_P^+ \). We compute by (6.23.1)

\[
\gamma^{-1} \cdot \alpha = \tau^{-\varpi_i^\vee} \cdot w_P^{-1} \cdot \alpha = \tau^{-\varpi_i^\vee} \cdot w_P^{-1}(\alpha) = w_P^{-1} \alpha - \langle w_P^{-1} \alpha, -\varpi_i^\vee \rangle \delta \in R^- - \delta,
\]

where \( \delta \), the null root of the affine root system, is the weight of \( \tau \). We have used that \( i \) cominuscule implies \( \langle \beta, \varpi_i^\vee \rangle = 1 \) for all \( \beta \in R^+ \setminus R_P^+ \). This proves (a).

We prove (b). Since \( i \) is cominuscule, every root \( \beta \in R^+ \setminus R_P^+ \) is of the form \( \beta = \alpha_i \mod \sum_{j \in I_P} \mathbb{Z}_{\geq 0} \alpha_j \). Since \( w_P^{-1} \alpha_j > 0 \) for all \( j \in I_P \), we deduce that we must have \( w_P^{-1} \alpha_i = -\theta \). \( \square \)
Thus we obtain an inclusion
\begin{equation}
\iota_t : X_t \longrightarrow G[\tau^{-1}]_1
\end{equation}
\begin{equation}
x = u_1 t \dot{w}_P u_2 \longmapsto \dot{\gamma}^{-1} t^{-1} u_1 t \dot{\gamma} \in G[\tau^{-1}]_1
\end{equation}
where \(u_1 \in U_P\) and \(u_2 \in U_{\eta^{-1}}\).

6.27. Embedding the geometric crystal into the affine Grassmannian. We interpret the inclusion \(\iota_t\) via the affine Grassmannian.

Let \(\text{Gr} = G((\tau))/G[[\tau]]\) denote the affine Grassmannian of \(G\). The connected components \(\text{Gr}^\gamma\) of \(\text{Gr}\) are indexed by \(\gamma \in \Omega\). For a dominant weight \(\lambda\), let \(\text{Gr}_\lambda := G[[\tau]]^\lambda \subseteq \text{Gr}\) denote the \(G[[\tau]]\)-orbit. For \(\lambda = \varpi_i^\gamma\), we have that \(\text{Gr}^\varpi_i^\gamma \cong G/P\) is closed in \(\text{Gr}\). Indeed, the map \(G \rightarrow \text{Gr}\) given by \(\tau^\varpi_i^\gamma \mapsto g \tau^\varpi_i^\gamma\) mod \(G[[\tau]]\) has stabilizer \(P\), giving a closed embedding \(G/P \cong \text{Gr}^\varpi_i^\gamma \hookrightarrow \text{Gr}^\gamma\).

Since \(\dot{w}_P^{-1} U_P \dot{w}_P \cap P = \{e\}\), we have an inclusion \(X_t \hookrightarrow G/P\) given by \(x = u_1 t \dot{w}_P u_2 \mapsto t^{-1} u_1^{-1} \dot{w}_P\) mod \(P\), where \(u_1 \in U_P\) and \(u_2 \in U_{\eta^{-1}}\).

Composed with the isomorphism \(G/P \cong \text{Gr}^\varpi_i^\gamma\), we obtain an inclusion
\begin{equation}
X_t \longrightarrow \text{Gr}^\varpi_i^\gamma
\end{equation}
\begin{equation}
x = u_1 t \dot{w}_P u_2 \mapsto t^{-1} u_1 t \dot{w}_P \tau^\varpi_i^\gamma = t^{-1} u_1 t \cdot \dot{\gamma} = \dot{\gamma} \iota_t(x).
\end{equation}

7. Heinloth–Ngô–Yun’s Kloosterman \(D\)-module

While [76] works over a finite field we will work over the complex numbers \(\mathbb{C}\) (cf. [76, Section 2.5]). In this section, we assume that \(G\) is simple and of adjoint type.

7.1. A group scheme over \(\mathbb{P}^1\). Take \(t\) to be the coordinate on \(\mathbb{P}^1\), and set \(s = t^{-1}\). Let
\begin{align*}
I(0) &= I_\infty(0) := \{g \in G[[s]] \mid g(0) \in B\} \\
I(1) &= I_\infty(1) := \{g \in G[[s]] \mid g(0) \in U\}
\end{align*}
Similarly define
\begin{align*}
I_{0}^{\text{opp}}(0) &= \{g \in G[[t]] \mid g(0) \in B_-\} \\
I_{0}^{\text{opp}}(1) &= \{g \in G[[t]] \mid g(0) \in U_-\}
\end{align*}
Also let \(I(2) = [I(1), I(1)]\) so that \(I(1)/I(2) \cong \bigoplus_{\lambda \in I_{\text{aff}}} \mathbb{A}^1\).

Denote by \(\mathcal{G} = \mathcal{G}(1, 2)\) the group scheme over \(\mathbb{P}^1\) in [76], satisfying
\begin{align*}
\mathcal{G}(1, 2)|_{\mathbb{C}_m} &\cong G \times \mathbb{G}_m \\
\mathcal{G}(1, 2)(\mathcal{O}_0) &= I_{0}^{\text{opp}}(1) \\
\mathcal{G}(1, 2)(\mathcal{O}_\infty) &= I_\infty(2).
\end{align*}
The group scheme \(\mathcal{G}\) is constructed as follows. First, the group scheme \(\mathcal{G}(1, 1)\) is obtained from the dilatation of the constant group scheme \(G \times \mathbb{P}^1\) along \(U_- \times \{0\} \subset G \times \{0\}\) and along \(U \times \{\infty\} \subset G \times \{\infty\}\). Then \(\mathcal{G}(1, 2)\) is the dilatation of \(\mathcal{G}(1, 1)\), which is an isomorphism away from \(\infty\) and at \(\infty\) it induces \(\mathcal{G}(1, 2)(\mathcal{O}_\infty) = I(2) \subset I(1) = \mathcal{G}(1, 1)(\mathcal{O}_\infty)\). We refer the reader to [15] for the general theory of dilatations, and to [135] for a concise introduction.
7.2. Hecke modifications. Let $\text{Bun}_G$ denote the moduli stack $\text{Bun}_{G(1,2)}$ of $G(1,2)$-bundles on $\mathbb{P}^1$ defined in [76]. Let $\text{Bun}_G^\gamma$ denote the $\gamma$-component, for $\gamma \in \Omega$, and let $\ast_\gamma$ denote the basepoint of $\text{Bun}_G^\gamma$. (Recall that $\Omega$ is the quotient of the coweight lattice of $G$ by the coroot lattice.) Under the isomorphism $\text{Bun}_G^0 \cong \text{Bun}_G^\gamma$, the basepoint $\ast_\gamma$ is the image of the point corresponding to the trivial bundle $\ast$.

The stack of Hecke modifications is the stack which for a $\mathbb{C}$-scheme $S$ takes value the groupoid

$$\text{Hecke}_G(S) := \left\{ (\mathcal{E}_1, \mathcal{E}_2, x, \phi) \mid \mathcal{E}_i \in \text{Bun}_G(S), x : S \to \mathbb{P}^1 \setminus \{0, \infty\}, \phi : \mathcal{E}_1|_{(\mathbb{P}^1 - x) \times S} \xrightarrow{\cong} \mathcal{E}_2|_{(\mathbb{P}^1 - x) \times S} \right\}.$$ 

It has two natural forgetful maps

$$\begin{array}{ccc}
\text{Bun}_G & \xrightarrow{\text{pr}_1} & \text{Hecke}_G \\
\downarrow & & \downarrow \text{pr}_2 \\
\text{Bun}_G \times \mathbb{P}^1 \setminus \{0, \infty\} & & \\
\end{array}$$

The geometric fibers of $\text{pr}_2$ over $\text{Bun}_G \times \mathbb{P}^1 \setminus \{0, \infty\}$ are isomorphic to the affine Grassmannian $\text{Gr}_G$. Locally in the smooth topology on $\text{Bun}_G \times \mathbb{P}^1 \setminus \{0, \infty\}$, the projection $\text{pr}_2$ is a locally trivial fibration ([76] Remark 4.1]). The $G[[\tau]]$-orbits $\text{Gr}_\lambda$ (and closures on $\overline{\text{Gr}}_\lambda$) $\text{Gr}_G$ define substacks $\text{Hecke}_\lambda \subset \text{Hecke}_G$ (and $\overline{\text{Hecke}}_\lambda$).

7.3. Gross’ rigid automorphic form. Let $\phi : I(1)/I(2) \to \mathbb{A}^1$ be the standard additive character. Precisely, the exponential map identifies $I(1)/I(2)$ with the span $\bigoplus_{i \in I_{af}} g_{\alpha_i}$. We fix root vectors $x_i = x_{\alpha_i} \in g_{\alpha_i}$ and define $\phi$ by insisting that $\phi(\exp(tx_{\alpha_i})) = -t$ for all $i$. The choice of $x_i$ for $i \in I$ is already fixed in [16]. Since $g_{\alpha_0}$ can be identified with $s g_{-\theta}$, the choice of $x_0 \in g_{\alpha_0}$ is equivalent to a choice of $x_{-\theta} \in g_{-\theta}$. The choice of $x_{-\theta}$ satisfying the conventions of [12] is equivalent to a choice of a sign, which will be fixed in [76]. More generally, the constructions of this section work for any generic affine character $\chi : I(1)/I(2) \to \mathbb{A}^1$, that is, any additive character taking nonzero values on each $x_i, i \in I_{af}$.

Let $j_\gamma : T \times I(1)/I(2) \hookrightarrow \text{Bun}_G^\gamma$ denote the inclusion of the big cell into the $\gamma$-component of $\text{Bun}_G^\gamma$. Recall that $E = D_{\mathbb{A}^1}/(\partial_t - 1)$ denotes the exponential $D$-module on $\mathbb{A}^1$. Define the Artin-Schreier $D$-module, or exponential $D$-module $E^\phi$ on $I(1)/I(2)$ by

$$E^\phi := \phi^*(E).$$

Abusing notation, also use $E^\phi$ to denote the similarly defined $D$-module on $T \times I(1)/I(2)$ that is constant along $T$.

**Lemma 7.4.** We have $j_{\gamma!*}E^\phi = j_{\gamma!*}E^\phi$.

**Proof.** This is the $D$-module version of [76, Lemma 2.3].

Define $A_G$ to be the $D$-module on $\text{Bun}_G$, which is given by $j_{\gamma!*}E^\phi = j_{\gamma!*}E^\phi$ on each connected component $\text{Bun}_G^\gamma$. It is $(I(1)/I(2), E^\phi)$-equivariant at $\infty \in \mathbb{P}^1$ and $T$-invariant at $0 \in \mathbb{P}^1$. The existence and unicity of $A_G$ was found by Gross [70]. The above construction is due to Heinloth–Ngô–Yun [76].
7.5. Kloosterman $D$-module. Let $\lambda$ be an (integral) coweight for $G$, and $\gamma \in \Omega$ be such that $\text{Gr}_\lambda$ lies in the $\gamma$-component of $\text{Gr}_G$. Restrict (7.2.1) to a diagram

\[
\begin{array}{ccc}
\text{Bun}_G^0 & \xrightarrow{\text{pr}_2} & \ast_\gamma \times \mathbb{P}^1_{\{0,\infty\}} \\
\text{pr}_1 & & \\
\text{Hk}_\lambda & \xrightarrow{\text{pr}_2} & \\
\end{array}
\]

(7.5.1)

where $\text{Hk}_\lambda$ is the restriction of $\text{Hecke}_\lambda$ to $\ast_\gamma \times \mathbb{P}^1_{\{0,\infty\}}$.

Let $\mathcal{O}_\lambda$ denote the structure sheaf of $\text{Gr}_\lambda$, considered a $D(\text{Gr}_\lambda)$-module. Denote the minimal extension of $\mathcal{O}_\lambda$ under the inclusion $j : \text{Gr}_\lambda \hookrightarrow \overline{\text{Gr}}_\lambda$ by $D_\lambda$. Abusing notation, also denote by $D_\lambda$ the corresponding holonomic $D$-module on $\text{Hk}_\lambda$. Define the Kloosterman $D$-module $\text{Kl}_{(G^\vee,\lambda)}$ on $\mathbb{P}^1_{\{0,\infty\}}$ by

\[
\text{Kl}_{(G^\vee,\lambda)} := R\text{pr}_{2,!}(\text{pr}_1^*A_G \otimes_{\mathcal{O}_\lambda} D_\lambda).
\]

The following lemma implies that $\text{Kl}_{(G^\vee,\lambda)}$ is a $D$-module, rather than a complex of $D$-modules.

**Lemma 7.6.** We have $\text{pr}_{2,!}(\text{pr}_1^*A_G \otimes D_\lambda) = \text{pr}_{2,*}(\text{pr}_1^*A_G \otimes D_\lambda)$ and $R^i\text{pr}_{2,*}(\text{pr}_1^*A_G \otimes D_\lambda) = 0$ for $i > 0$.

**Proof.** This is the $D$-module version of [76, §4.1] and [76, §4.2].

Over number fields, the analogous construction for $\text{GL}(2)$ goes back to Poincaré (1912) and H. Petersson (1930’s), and Bump–Friedberg–Goldfeld [23] for $\text{GL}(n)$; other developments, such as [19] for metaplectic groups, feature connections with crystal bases, which could be related with the present work.

7.7. Parametrization. Assume now that $(G, P)$ are as in §6.24. In particular, $G$ is simple and of adjoint type, so $Z(L_P)$ is one-dimensional. We now fix the isomorphism $\alpha_t : Z(L_P) \cong \mathbb{P}^1_{\{0,\infty\}}$.

\[
\alpha_t : Z(L_P) \cong \mathbb{P}^1_{\{0,\infty\}} \quad z \mapsto \alpha_t(z).
\]

Via (7.7.1), we may use “$t$” as both a coordinate on $\mathbb{P}^1$ and a coordinate on $Z(L_P)$.

We follow [76, Section 5.2] in the following. Let $\text{Hk}$ be the restriction of the stack of Hecke modifications to $\ast_\gamma \times \mathbb{P}^1_{\{0,\infty\}} \subset \text{Bun}_G \times \mathbb{P}^1_{\{0,\infty\}}$ and let $\text{Hk}_q$ be the restriction to $\ast_\gamma \times \{q\}$ for $q \in \mathbb{P}^1_{\{0,\infty\}}$. Let $\text{Hk}^0 \subset \text{Hk}$ denote the inverse image of the big cell $T \times I(1)/I(2) \subset \text{Bun}_G^0$ under $\text{pr}_1$, and similarly define $\text{Hk}^0_q$ and $\text{Hk}^0_\lambda$. Denote the map $\text{Hk}^0 \to T \times I(1)/I(2) \cong T \times U_{-\theta} \times U/[U, U]$ by

\[
(f_T, f_0, f_+) : \text{Hk}^0 \longrightarrow T \times U_{-\theta} \times U/[U, U].
\]

Our immediate aim is to parametrize $\text{Hk}^0$ and compute $f_T, f_0, f_+$. Let $\mathcal{E}_0 = G \times \mathbb{P}^1$ be the trivial bundle and let $\mathcal{E}_\gamma$ be the $G$-bundle corresponding to the basepoint $\ast_\gamma \in \text{Bun}_G^0$. The bundle $\mathcal{E}_\gamma$ is obtained by gluing the trivial bundle on $\mathbb{P}^1_{\{0,\infty\}}$ with the trivial bundle on the formal disk around $\infty$ via the transition function $\gamma(t^{-1}) = \hat{w}_pt^\infty t^\infty$.

We use the local parameter $\tau = 1 - t/q$ at $q$. Thus $\tau = 0, 1, \infty$ (or $\tau^{-1} = \infty, 1, 0$) corresponds to $t = q, 0, \infty$ respectively. Let

\[
\gamma(t^{-1}) = \hat{\gamma} = \hat{w}_pt^\infty t^\infty \in G[\tau, \tau^{-1}].
\]
We view $\gamma(\tau^{-1})$ as an isomorphism

$$
\gamma(\tau^{-1}) : \mathcal{E}_0|_{\mathbb{P}^1_{\infty}} \to \mathcal{E}_\gamma|_{\mathbb{P}^1_{\infty}}
$$

using trivializations of $\mathcal{E}_0$ and $\mathcal{E}_\gamma$ over $\mathbb{P}^1_{\infty}$. Since

$$(7.7.2) \quad \tau^{-1} = -qt^{-1} + O((t^{-1})^2),$$

the Laurent expansions of $\gamma(\tau^{-1})$ and $\gamma(t^{-1})$ in $t^{-1}$ differ by an element of $G[[t^{-1}]]$. Thus $\gamma(\tau^{-1})$ extends to an isomorphism

$$(7.7.3) \quad \gamma(\tau^{-1}) : \mathcal{E}_0|_{\mathbb{P}^1_{\gamma}} \to \mathcal{E}_\gamma|_{\mathbb{P}^1_{\gamma}}.$$

Any point in $H_k^0$ can be obtained by precomposing $\gamma(\tau^{-1})$ by an element of $\text{Aut}(\mathcal{E}_0|_{\mathbb{P}^1_{\gamma}}) \cong G[\tau^{-1}]$. Let $g(\tau^{-1})\gamma(\tau^{-1})$ be such a point. The isomorphism $(7.7.3)$ preserves the level structure at $\infty$, and since $\gamma$ has the trivial level structure at $\infty$, we deduce that the level structure of $g(\tau^{-1})\gamma(\tau^{-1})$ at $\infty$ is given by $ev_{\infty}[g(\tau^{-1})^{-1}] = g(0)^{-1} \in G$. Similarly, we may take the level structure at $0$ to be given by $ev_{t=0}[g(\tau^{-1})\gamma(\tau^{-1})^{-1}] = \dot{w}_p^{-1}g(1)^{-1} \in G$. The condition that $g(\tau^{-1})\gamma(\tau^{-1})$ projects under $pr_1$ to the big cell $T \times I(1)/I(2)$ is thus given by

$$
g(0) \in U \iff g(0) \in U$$

$$
g(1)^{-1}\dot{w}_p^{-1} \in B_- \iff \dot{w}_pg(1) \in B_-.$$

We have a natural evaluation map $ev_q : H_k^0 \to Gr_q$ given by considering $g(\tau^{-1})\gamma(\tau^{-1})$ as an element of $G((\tau))/G[[\tau]] \cong Gr_q$. The image $ev_q(H_k^0)$ is denoted $Gr_q^0$. We may further rigidify the moduli problem by precomposing with an element of $\text{Aut}(\mathcal{E}_0) = \text{Aut}(G \times \mathbb{P}^1) = G$ to obtain an isomorphism $\gamma(\tau^{-1})h(\tau^{-1}) = \gamma(\tau^{-1})g(\tau^{-1})g(0)^{-1} : \mathcal{E}_0|_{\mathbb{P}^1_{\gamma}} \to \mathcal{E}_\gamma|_{\mathbb{P}^1_{\gamma}}$ which is the identity at $\infty$. This gives the parametrization

$$
H_k^0 \cong \{ h(\tau^{-1}) \in G[\tau^{-1}]_1 \mid h(1) \in \dot{w}_p^{-1}B_-U \}.
$$

and varying $q$,

$$(7.7.4) \quad H_k^0 \cong \{ h(\tau^{-1}) \in G[\tau^{-1}]_1 \mid h(1) \in \dot{w}_p^{-1}B_-U \} \times \mathbb{P}^1_{\{0,\infty\}}.
$$

Under this parametrization, the image of $h(\tau^{-1})$ in $Gr_q \cong G((\tau))/G[[\tau]]$ is equal to $\dot{\gamma}h(\tau^{-1})$.

**Lemma 7.8.** Under the parametrization $(7.7.4)$, write $\dot{w}_p^{-1}h(1) = b_-u$ for $u \in U$ and $b_- \in B_-$. Then we have

$$
f_T(h, q) = b_-^{-1} \mod U_- \in B_-/U_- \cong T$$

$$
f_+(h, q) = u \mod [U, U] \in U/[U, U]$$

$$
f_0(h, q) = qa_0(g) \in U_0 \cong \mathfrak{g}_0$$

where $a_0 : G[\tau^{-1}] \to \mathfrak{g}_0$ sends $h$ to the $\mathfrak{g}_0$-part of the tangent vector $\frac{dh(\tau^{-1})}{d(\tau^{-1})} \bigg|_{\tau^{-1}=0} \in \mathfrak{g}$.

**Proof.** The formulae for $f_T$ and $f_+$ follow from the parametrization $(7.7.4)$. The function $f_0(h, q)$ is obtained by expanding $h(\tau^{-1})^{-1}$ at $t = \infty$ using the local parameter $t^{-1}$. By $(7.7.2)$, we have

$$
\frac{dh(\tau^{-1})^{-1}}{d(t^{-1})} \bigg|_{t^{-1}=0} = \frac{d\tau^{-1}}{d(t^{-1})} \frac{dh(\tau^{-1})^{-1}}{d(\tau^{-1})} \bigg|_{\tau^{-1}=0} = -q \frac{dh(\tau^{-1})^{-1}}{d(\tau^{-1})} \bigg|_{\tau^{-1}=0} = q \frac{dh(\tau^{-1})}{d(\tau^{-1})} \bigg|_{\tau^{-1}=0} \in \mathfrak{g}.$$
where for the last equality we have used the condition $h(0) = 1 \in G$: if $h(\tau^{-1}) = 1 + h_1 \tau^{-1} + O(\tau^{-2})$ then $h(\tau^{-1}) = 1 - h_1 \tau^{-1} + O(\tau^{-2})$.

7.9. **Comparison.** The inclusion $\iota_t : X_t \to G[\tau^{-1}]_1$ of (6.26.1) can be extended to an inclusion

\[
\tilde{\iota} = (\iota, \pi) : X \longrightarrow G[\tau^{-1}] \times \mathbb{P}^1_{\{0, \infty\}}
\]

\[
x = u_1 tw_p u_2 \longmapsto (\iota(x) = \tau^{-1} t^{-1} u_1 t \tilde{\gamma}, q = t)
\]

where $u_1 \in U_P$ and $u_2 \in U_{w_p}$. 

**Lemma 7.10.** Under the identification (7.7.4), we have an isomorphism $\tilde{\iota} : X \cong \text{Hk}^\circ_{\varphi}$. 

**Proof.** Recall our convention that $h(\tau^{-1}) \in \text{Hk}^\circ_{\varphi}$ if and only if $h(\tau^{-1}) \tilde{\gamma} \in \text{Gr}^\circ_{\varphi} = \text{Gr}^{\circ} \cap \text{Gr}_{\varphi}^{\circ}$. The condition $\iota(x) \tilde{\gamma} \subset \text{Gr}^\circ_{\varphi}$ follows from (6.27.1). The inclusion $\iota(x) \subset \text{Hk}^{\circ}$ follows from

\[
(7.10.1) \quad \dot{w}_p (\tau^{-1} t^{-1} u_1 t \tilde{\gamma}) |_{\tau^{-1} = 1} = t^{-1} u_1 t \dot{w}_p = (t^{-1} x)(u_2^{-1})
\]

where $x = u_1 t \dot{w}_p u_2$. Suppose that $h(\tau^{-1}) = \tau^{-1} g \tilde{\gamma}$ where $g \in G$ is constant. Then the condition that $h(\tau^{-1}) \subset G[\tau^{-1}]_1$ is equivalent to $g \in U_P$ (see Lemma 6.26(a)). The condition that $h(1) \in \dot{w}_p^{-1} B_- U$ is then equivalent to $h(\tau^{-1}) \subset \iota_t(X_t)$, for any $t \in Z(L_P)$. 

Recall from (6.3) the choice of the standard additive character $\psi : U \to \mathbb{A}^1$. 

**Proposition 7.11.** There is a choice of root vector $x_{-\theta} \in g_{-\theta}$ such that

\[
\begin{align*}
 f_T(\tilde{\iota}(x)) &= t \gamma(x)^{-1} \in B_- / U_- \cong T \\
 f_+(\tilde{\iota}(x)) &= u_2^{-1} \mod [U, U] \in U / [U, U] \\
 f_0(\tilde{\iota}(x)) &= -\psi(u_1) x_{-\theta} \in U_{w_p} \cong g_{-\theta}.
\end{align*}
\]

**Proof.** Let $\iota(x) = h(\tau^{-1})$. Then by (7.10.1), we have

\[
\dot{w}_p h(1) = (t^{-1} x)(u_2^{-1}).
\]

By Lemma 7.8 we have $f_T(\tilde{\iota}(x)) = x^{-1} t$ mod $U_- \cong \tau(x)^{-1} \in T$ and $f_+(\tilde{\iota}(x)) = u_2^{-1}$ mod $[U, U]$. 

It remains to compute $f_0(\tilde{\iota}(x))$. Let $u_P = \text{Lie}(U_P)$. Then $u_P = \bigoplus_{\alpha \in R^+ \setminus \Pi^+} g_\alpha$. Since $\tilde{\iota}$ is cominuscule, $\alpha_i$ occurs in every $\alpha \in R^+ \setminus R^+_P$ with coefficient one. It follows that $\alpha + \beta$ is never a root for $\alpha, \beta \in R^+ \setminus R^+_P$. In particular, $u_P$ is an abelian Lie algebra and $U_P$ is an abelian algebraic group, and so is $\mathfrak{g}_{\gamma^{-1} u P \tilde{\gamma}}$. Now, $\mathfrak{g}_{\gamma^{-1} \alpha \gamma}$ can be identified with the root space $\mathfrak{g}_{\gamma^{-1} \alpha}$, and $\alpha$ is now an affine root. By Lemma 6.26 $\dot{w}_p^{-1} \alpha_i = -\theta$, so

\[
\gamma^{-1} \cdot \alpha_i = \dot{w}_p^{-1} \cdot \tau^\varphi \cdot \alpha_i = \dot{w}_p^{-1} \cdot (\alpha_i - \delta) = -\theta - \delta,
\]

where $\delta$ denotes the null root.

Let $\text{exp} : u_P \to U_P$ be the exponential map, which is an isomorphism. If $h(\tau^{-1}) = \dot{\gamma}^{-1} u \tilde{\gamma}$ for $u \in U_P$, it follows that

\[
a_{-\theta}(h) = \dot{w}_p^{-1} \cdot \text{exp}^{-1}(u) \alpha_i = -\psi(u) x_{-\theta}
\]

for any appropriate choice of the root vector $x_{-\theta}$.
By Lemma 7.8 and (7.7.1), we thus have
\[ f_0(i(x)) = -\alpha_1(t)\psi(t^{-1}u_1t)x_{-\theta} = -\alpha_1(t)\alpha_1(t^{-1})\psi(u_1)x_{-\theta} = -\psi(u_1)x_{-\theta} \in g_{-\theta}, \]
as claimed.

Henceforth we assume that
\[ (7.11.1) \quad x_{-\theta} \in g_{-\theta} \]
has been chosen to satisfy Proposition 7.11.

Note that the choice (7.11.1) is independent of (3.9.1) because $g$ here is $g^\vee$ in (3).

We can now prove the main result of this section. It then follows from [76, Thm 1.3] that $Cr_{(G,P)}$ is a flat connection (smooth and concentrated in one degree).

**Theorem 7.12.** The character $D$-module $Cr_{(G,P)}$ is isomorphic to the Kloosterman $D$-module $Kl_{(G',\bar{\omega})}$.

**Proof.** Recall that by definition
\[ Kl_{(G',\bar{\omega})} = \text{pr}_{2,!}(\text{pr}_1^* A_{G} \otimes D_{\bar{\omega}}) \]
where $\text{pr}_1^* A_{G} \otimes D_{\bar{\omega}}$ is a $D$-module on $H_{K\bar{\omega}}$. Since $Gr_\omega = \text{Gr}^\omega$, we have $D_{\bar{\omega}} \cong \mathcal{O}_{H_{K\bar{\omega}}}$. Thus $Kl_{(G',\bar{\omega})} = R\text{pr}_{2,!}(\text{pr}_1^* (A_G)) = R\text{pr}_{2,!}(\text{pr}_1^* (A_G))$, the latter equality by Lemma 7.6.

By definition (7.3), the $D$-module $A_G$ on $Bun_G$ is supported on the big cell $T \times I(1)/I(2)$ of $Bun_G$. Thus $\text{pr}_1^*(A_G)$ is supported on $H_{K\bar{\omega}}$. The restriction of $\text{pr}_1^*(A_G)$ to $H_{K\bar{\omega}}$ is equal to $(f_+,f_0)^*(E^\phi)$. Thus we may restrict to considering the diagram

\[ (7.12.1) \quad I(1)/I(2) \cong U/[U,U] \times A^1 \]
\[ \xymatrix{ \text{HK}_{K\bar{\omega}} \ar[rr]^{(f_+,f_0)} & & \ast \gamma \times \mathbb{P}^1_{(0,\infty)} } \]

where $g_{-\theta}$ is identified with $A^1$ via the root vector $x_{-\theta}$. We then have $Kl_{(G',\bar{\omega})} = R\text{pr}_{2,!}((f_+,f_0)^*(E^\phi))$. By Lemma 7.10 and Proposition 7.11 diagram (7.12.1) is isomorphic to the diagram

\[ (7.12.2) \quad X \]
\[ \xymatrix{ \theta: u_1 \mapsto \psi(u_1)
\xymatrix{ U/[U,U] \times A^1 \ar[rr]^-{\pi} & & Z(L_P) } \]

We note that with our choices $\phi(u,a) = -\psi(u) - a$ for $(u,a) \in U/[U,U] \times A^1$. The definition of the character $D$-module can then be written as
\[ Cr_{(G,P)} = R\pi_1(E^{\bar{\phi}}) = R\pi_1(\theta^*(E^{-\phi})) = R\text{pr}_{2,!}((f_+,f_0)^*(E^\phi)) = Kl_{(G',\bar{\omega})} \]
where $\theta$ is the left arrow in (7.12.2). It also follows from this calculation that $Cr_{(G,P)}$ is a $D$-module, rather than a complex of $D$-modules. \qed

**Remark 7.13.** Similarly, over a finite field $\mathbb{F}_q$ equipped with a non-degenerate additive character $\psi: \mathbb{F}_q \to \mathbb{Q}_l^\times$, we can define the Artin-Schreier $\ell$-adic sheaf $\mathcal{L}_{\psi(f)} := f^* \mathcal{L}_\psi$ on $X$ and a geometric crystal $\ell$-adic sheaf $\pi_1^* \mathcal{L}_{\psi(f)}$ on $Z(L_P)$. The comparison with generalized Kloosterman $\ell$-adic sheaves is the same.
7.14. **Homogeneity.** In [139] Section 2.6.4, a $\mathbb{G}_m$-action is defined on $H^k$. Under the parametrization $(7.7.4)$, $\zeta \in \mathbb{G}_m$ acts by conjugation by $\rho^\vee(\zeta)$ on the first factor $G^{[m^{-1}]_1}$ and by $q \mapsto \zeta^c q$ on the second factor $\mathbb{P}_{1\{0,\infty\}}$, where $c$ is the Coxeter number. The map $(f_+, f_0) : H^k \to I(1)/I(2)$ is $\mathbb{G}_m$-equivariant where $\mathbb{G}_m$ acts on $I(1)/I(2)$ by scalar multiplication in every affine simple root space.

The $\mathbb{G}_m$-action on $H^k$ preserves $H^k$, and under the isomorphism of the diagrams $(7.12.1)$ and $(7.12.2)$, this $\mathbb{G}_m$-action is identified with the one in $(6.17)$.

8. **The mirror isomorphism for minuscule flag varieties**

8.1. **D-module mirror theorem.** Assume as before that $G$ is of adjoint type and $G^\vee$ is simply-connected. Let $P \subset G$ be a parabolic subgroup and $P^\vee \subset G^\vee$ be the corresponding parabolic of the dual group.

**Lemma 8.2.** There is a canonical exact sequence

$$2i\pi H^2(G^\vee / P^\vee, \mathbb{Z}) \to H^2(G^\vee / P^\vee) \to Z(L_P).$$

**Proof.** By Borel’s theorem there is a canonical isomorphism $H^2(G^\vee / P^\vee) \cong t^{WP}$. We have $Z(L_P) = T^{WP}$ and thus it only remains to apply the exponential map.

Recall that the character $D$-module $\text{Cr}_{(G,P)}$ attached to the Berenstein–Kazhdan parabolic geometric crystal has been constructed in §6 and that the quantum connection $\mathcal{Q}^{G^\vee/P^\vee}$ for the projective homogeneous space $G^\vee / P^\vee$ has been described in §4 in terms of the quantum Chevalley formula. The base of the geometric crystal $D$-module is $Z(L_P)$, and the base of the quantum $D$-module is $\mathbb{C}_q^\times \cong H^2(G^\vee / P^\vee) / 2i\pi H^2(G^\vee / P^\vee, \mathbb{Z})$. By the above lemma the two tori are canonically isomorphic.

**Theorem 8.3.** Suppose that $P$ is a minuscule parabolic subgroup of $G$ and let $P^\vee$ be the dual minuscule parabolic subgroup of $G^\vee$. The geometric crystal $D$-module $\text{Cr}_{(G,P)}$ and the quantum cohomology $D$-module $\mathcal{Q}^{G^\vee/P^\vee}$ for $G^\vee / P^\vee$ are isomorphic.

**Proof.** Let $i$ be the minuscule node corresponding to $P$. We decompose the canonical isomorphism $Z(L_P) \cong H^2(G^\vee / P^\vee) / 2i\pi H^2(G^\vee / P^\vee, \mathbb{Z})$ of Lemma 8.2 into the composition

$$Z(L_P) \xrightarrow{\alpha_i} \mathbb{P}_{1\{0,\infty\}} = \mathbb{C}_q^\times \xrightarrow{\log} \mathbb{C} / 2i\pi \mathbb{Z} \xrightarrow{\sigma_i} H^2(G^\vee / P^\vee) / 2i\pi H^2(G^\vee / P^\vee, \mathbb{Z}).$$

Indeed, the Schubert class $\sigma_i \in H^2(G^\vee / P^\vee, \mathbb{C})$ corresponds to the fundamental coweight $\varpi_i^\vee \in t^{WP}$ under Borel’s isomorphism. Thus composing with the exponential map, we see that the isomorphism $\mathbb{C}_q^\times \to T^{WP}$ is given by the cocharacter $q \mapsto \varpi_i^\vee(q)$. Composing with $\alpha_i$ the claim follows from $\langle \alpha_i, \varpi_i^\vee \rangle = 1$.

The proof of the theorem follows by combining the following three results:

- Theorem 7.12 says that $\text{Cr}_{(G,P)}$ is isomorphic to the Kloosterman $D$-module $\text{Kl}_{(G^\vee, \varpi_i^\vee)}$ if we identify $Z(L_P)$ with $\mathbb{P}_{1\{0,\infty\}}$ via $(7.7.1)$;
- Zhu proved [140] that $\text{Kl}_{(G^\vee, \varpi_i^\vee)}$ is isomorphic to the Frenkel–Gross connection $\nabla^{(G^\vee, \varpi_i^\vee)}$;
- Theorem 4.14 says that $\nabla^{(G^\vee, \varpi_i^\vee)}$ is isomorphic to $\mathcal{Q}^{G^\vee/P^\vee}$, if we identify the bases via $\mathbb{P}_{1\{0,\infty\}} = \mathbb{C}_q^\times$.

In Zhu’s isomorphism (see Theorem 9.7) the choice of affine generic character $\phi$ in the definition of $\text{Kl}_{(G^\vee, \varpi_i^\vee)}$ matches with a particular choice of highest root vector in the definition...
of $\nabla^{(G^\vee, \omega^\vee)}$. It is clear that all our sign choices leads to a single overall sign, which is equivalent to an isomorphism $q \mapsto \pm q$ of the curve $\mathbb{P}^1_{\mathbb{C}[\{0, \infty\}]}$.

To determine this sign and conclude that $\text{Cr}_{(G,P)}$ is isomorphic to $Q^{G^\vee/P^\vee}$, we consider the quantum period solution $\langle S(q), 1 \rangle$ of $Q^{G^\vee/P^\vee}$. From Lemma 3.17, we know that the first term in the $q$-expansion is positive. On the other hand, the corresponding solution of $\text{Cr}_{(G,P)}$ is

$$\int e^{f_t(x)} \omega = \int e^{a_1 + \cdots + a_\ell + \alpha_i(t) P_i} \frac{da_1}{a_1} \cdots \frac{da_\ell}{a_\ell},$$

where we use the expression of the superpotential from Corollary 6.14. Since $P_i$ is a Laurent polynomial with positive coefficients, and $\alpha_i(t) = q$, we deduce from Cauchy’s residue theorem that the first term in the $q$-expansion of the above integral is also positive. □

If $G$ is of type $A_n$ this proves a conjecture of Marsh-Rietsch [104, §3], and if $G$ is of type $D_n$ a conjecture of Pech-Rietsch-Williams [109, §4]. They construct in both cases a $D$-module homomorphism $Q^{G^\vee/P^\vee} \to \text{Cr}_{(G,P)}$ and show that it is injective. The conjecture was whether it is an isomorphism, or equivalently whether the dimension of $H^\ast(G^\vee/P^\vee)$ is equal to the rank of $\text{Cr}_{(G,P)}$. This follows from Theorem 8.3. It follows from Proposition 4.12 below that our $D$-module isomorphism coincides with that of [104,109].

9. Equivariant case

We extend the mirror isomorphism of Theorem 8.3 to the equivariant case.

9.1. Equivariant Frenkel–Gross connection. We use the notation from §3 except that $G$ and $G^\vee$ are swapped. For an element $h \in \mathfrak{t}^\ast$, define the equivariant Frenkel–Gross connection evaluated at $h$ by

$$\nabla^{G^\vee}(h) := d + (f + h) \frac{dq}{q} + x_\theta dq.$$

Thus $\nabla^{G^\vee}(0)$ is the connection considered in §3. As before, the equivariant Frenkel–Gross connection depends on a choice of basis vector $x_\theta$, but this choice is suppressed in the notation. If the choice of $x_\theta$ is not mentioned, by default we will use (3.9.1). As before we also have the associated bundles $\nabla^{(G^\vee, V)}$ or $\nabla^{(G^\vee, X)}$.

9.2. Equivariant quantum connection. Let $S = \text{Sym}(\mathfrak{t}) = H_{T^\vee}(\text{pt})$. Let $QH_{T^\vee}(G^\vee/P^\vee)$ denote the torus-equivariant small quantum cohomology ring of $G^\vee/P^\vee$. It is an algebra over $\mathbb{C}[q_i \mid i \notin I_P] \otimes S$. For $w \in W^P$, we abuse notation by also writing $\sigma_w \in QH_{T^\vee}(G^\vee/P^\vee)$ for the equivariant quantum Schubert class. The following equivariant quantum Chevalley formula for a general $G^\vee/P^\vee$ is due to Mihalcea [105].

**Theorem 9.3.** For $w \in W^P$, we have in $QH_{T^\vee}(G^\vee/P^\vee)$

$$\sigma_i \ast_q \sigma_w = (\varpi_i^\vee - w \cdot \varpi_i^\vee) \sigma_w + \sum_{\beta^\vee} \langle \varpi_i^\vee, \beta \rangle \sigma_{w \beta} + \sum_{\gamma^\vee} \langle \varpi_i^\vee, \gamma \rangle q_{\beta \gamma} \sigma_{\pi P(w \beta)}$$

where $\varpi_i^\vee \in \mathfrak{k}$ denotes a fundamental weight of $\mathfrak{g}^\vee$, and $\beta^\vee, \gamma^\vee$ denote roots of $\mathfrak{g}^\vee$. The last two summations are as in Theorem 4.3.
We have a canonical map $QH^*_T(G^\vee/P^\vee) \to \text{Spec}(\text{Sym}(t))$. For $h \in t^\ast$, we write $QH^*_h(G^\vee/P^\vee)$ for the fiber of this map over $h \in t^\ast \cong \text{Spec}(S)$. The ring $QH^*_h(G^\vee/P^\vee)$ is again a free $\mathbb{C}[q]$-module with Schubert basis $\{\sigma_w \mid w \in W^P\}$.

Now assume that $P^\vee \subset G^\vee$ is minuscule. Let $O(1)$ be the line bundle on $G^\vee/P^\vee$ arising from the natural embedding $G^\vee/P^\vee \hookrightarrow \mathbb{P}(V_{\omega^\vee})$. We define the equivariant quantum connection $Q^{G^\vee/P^\vee}(h)$ on the trivial $H^*(G/P)$-bundle on $\mathbb{C}_q^\times$ by

$$Q^{G^\vee/P^\vee}(h) := d + c_1^T(O(1)) \ast_d q \frac{dq}{q}$$

where $c_1^T(O(1))$ denotes the equivariant Chern class of $O(1)$, and $\ast_d q$ denotes equivariant quantum multiplication evaluated at $h \in t$. We have that $c_1^T(O(1)) = \sigma_i - \omega_i \sigma_1$ in $QH^*_T(G^\vee/P^\vee)$, so by Theorem 9.3

$$c_1^T(O(1)) \ast_d \sigma_w = -w \cdot \omega_i \sigma_w + \sum_{\beta^\vee} \langle \omega_i^\vee, \beta \rangle \sigma_{w \beta} + \sum_{\gamma^\vee} \langle \omega_i^\vee, \gamma \rangle q_{\eta_p(\gamma)} \sigma_{p(w, \gamma)}.$$

Theorem 9.4 has the following equivariant generalization.

**Theorem 9.4.** Suppose $h \in t^\ast$. If $P^\vee \subset G^\vee$ is minuscule and with corresponding minuscule representation $V_{\omega^\vee}$, then under the isomorphism $L : H^*(G^\vee/P^\vee) \to V_{\omega^\vee}$ of (4.11.1), the equivariant quantum connection $Q^{G^\vee/P^\vee}(h)$ is isomorphic to the connection $\nabla^{G^\vee, \omega^\vee}(\cdot - h)$.

**Proof.** The extra term in $c_1^T(O(1)) \ast_d \sigma_w$ not present in the non-equivariant case is $-w \cdot \omega_i \sigma_w$. Evaluating at $h$, we get the term $-\langle w \cdot \omega_i^\vee, h \rangle \sigma_w$. This agrees with the calculation $-h \cdot v_w = -\langle w \cdot \omega_i^\vee, h \rangle v_w$ for $g^\vee$ acting on $v_w \in V$. The result then follows from the calculation in Theorem 9.14. \hfill \Box

9.5. Kloosterman D-modules associated to an additive character and a multiplicative character. Define the $D_T$-module $M^h$ on $T$ as the free $O_T$-module with basis element $\langle x^h \rangle$, with the action of $D_T$ given by

$$\partial_\lambda \cdot x^h := \langle \lambda, h \rangle x^h$$

for $\lambda \in t \subset S$. Noting being as in (37) for a generic affine character $\chi : I(1)/I(2) \to \mathbb{A}^1$, consider the $D$-module $M^h \otimes E^\chi$ on $T \times I(1)/I(2)$. On each connected component $\text{Bun}_h^\vee$, define

$$A_G(\chi, h) := j_{\gamma!(M^h \otimes E^\chi)} = j_{\gamma!*}(M^h \otimes E^\chi).$$

$A_G(\chi, h)$ is an automorphic Hecke eigen-$D$-module which is $(T, M^h)$-equivariant at $0 \in \mathbb{P}^1$ and $(I(1)/I(2), E^\chi)$-equivariant at $\infty \in \mathbb{P}^1$. Let $Kl_{G^\vee}(\chi, h)$ denote the corresponding Hecke eigenvalue $D$-module on $\mathbb{P}^1_{\{0, \infty\}}$. As before, we have the associated $D$-modules $Kl_{G^\vee, V}(\chi, h)$ and $Kl_{G^\vee, \omega^\vee}(\chi, h)$. We set $Kl_{G^\vee}(h) := Kl_{G^\vee}(\phi, h)$ and similarly for $Kl_{G^\vee, V}(h)$ and $Kl_{G^\vee, \omega^\vee}(h)$.

9.6. Equivariant reciprocity theorem.

**Theorem 9.7.** For any generic affine character $\chi$, and $h \in t^\ast$, there is a choice of basis vector $x_\theta \in g_0$ such that we have an isomorphism

$$Kl_{G^\vee}(\chi, h) \cong \nabla^{G^\vee}(h).$$

We will normalize Theorem 9.7 by matching $Kl_{G^\vee}(\phi, h)$ with the choice of $x_\theta$ from (3.9.1). When $h = 0 \in t$, i.e. $\chi$ is the trivial character, Theorem 9.7 reduces to the main result of [140]. In [10] we explain how a minor modification of Zhu’s results leads to Theorem 9.7.
9.8. **Equivariant character $D$-module of a geometric crystal.** For $h \in t^*$, define the equivariant character $D$-module of the geometric crystal $X$ by

\[
Cr_{(G,P)}(h) := R\pi_!(E^f \otimes \gamma^*M^h)
\]
on $Z(L_P)$, where we recall that $\gamma : X \to T$ is the weight map (6.4.1).

Theorem (7.12) has the following equivariant generalization.

**Theorem 9.9.** Suppose $P = P_\delta$ is cominuscule. Then the character $D$-module $Cr_{(G,P)}(h)$ is isomorphic to the Kloosterman $D$-module $Kl_{(G\vee, x_0^\vee)}(-h)$.

**Proof.** The proof is the same as that of Theorem (7.12) so we sketch the main differences. According to Proposition (7.11), we have $t^{-1}f_T(i(x)) = \gamma(x)^{-1}$. Thus adding $f_T$ to the diagram (7.12.2) we can write

\[
Cr_{(G,P)}(h) = R\pi_!(\theta^*(E^{-\phi}) \otimes f_T^*M^h \otimes \pi^*M^{-1}) = R\pi_!((f_+, f_0)^*(E^\phi) \otimes f_T^*M^h) \otimes M^{-1} = Kl_{G\vee}(\phi, h) \otimes M^{-1}
\]

where we have used the projection formula ([81, Corollary 1.7.5]) for the second equality. Since $M^{-1}$ is isomorphic to $O_{\mathfrak{z}'_{(0, \infty)}}$ as $D$-modules, the conclusion follows.

9.10. **The equivariant mirror theorem.** Combining Theorems 9.4, 9.7 and 9.9 we obtain the following equivariant analogue of Theorem 8.3.

**Theorem 9.11.** Suppose that $P$ is a cominuscule parabolic subgroup of $G$ and let $P^{\vee}$ be the dual minuscule parabolic subgroup of $G^{\vee}$. For any $h \in t^*$, we have the isomorphism

\[
Cr_{(G,P)}(h) \cong Q^{G^{\vee}/P^{\vee}}(h).
\]

In the case that $G^{\vee}/P^{\vee}$ is a Grassmannian, an injection from $Q^{G^{\vee}/P^{\vee}}(h)$ into $Cr_{(G,P)}(h)$ is constructed by Marsch-Rietsch [104, Theorem 5.5].

10. **Equivariant generalization of Zhu’s theorem**

The aim of this section is to explain how Zhu’s results in [140] establishes Theorem 9.7. For a point $x \in \mathbb{P}^1$, we let $O_x$ denote the completed local ring at $x$ and $F_x = \text{Frac}(O_x)$ denote its fraction field. Denote by $D_x = \text{Spec} O_x$ and $D_x^\times = \text{Spec} F_x$ the formal disk and formal punctured disk at $x$. We write $\omega_{O_x}$ for the $O_x$-module $O_x \cdot dt$ (after choosing a local coordinate $t$).

In this section, we typically write $\text{Fun} S$ to denote the commutative algebra of regular functions on a space $S$.

10.1. **Classical Hitchin map.** We use notation from §7.1. We let $V = I_\infty(1)/I_\infty(2)$ and identify $V$ with its Lie algebra via the exponential map.

**Lemma 10.2.** The stack $\text{Bun}_G$ is good in the sense of [51 §1.1.1]; that is, we have $\dim T^*\text{Bun}_G = 2 \dim \text{Bun}_G$.

**Proof.** Let $G(0,1)$ be the group scheme over $\mathbb{P}^1$ obtained from the dilatation of the constant group scheme $G \times \mathbb{P}^1$ along $U \times \{\infty\} \subset G \times \{\infty\}$. The proof is identical to [140, Lemma 17], after noting that $\text{Bun}_G$ is a principal bundle over $\text{Bun}_{G(0,1)}$ under the group $I_0^{opp}(0)/I_0^{opp}(1) \times I_\infty(1)/I_\infty(2) \cong T \times V$. 

\[\square\]
Let \( c^* := \text{Spec } k[\mathfrak{g}^* G] \cong \mathfrak{t}^* / W \). We have a canonical \( G_m \)-action on \( c^* \) giving rise to a decomposition \( c^* = \bigoplus c_i^* \) (see [140, §3.1]) into one-dimensional subspaces, where the integers \( d_1, d_2, \ldots, d_r \) are the degrees of \( W \).

We next consider the Hitchin map

\[
h^\text{cl} : T^* \text{Bun}_G \to \text{Hitch}(\mathbb{P}^1)_G.
\]

The Hitchin space \( \text{Hitch}(\mathbb{P}^1)_G \) is defined to be the image of the usual Hitchin map \( h^\text{cl} : T^* \text{Bun}_G \to \text{Hitch}(G_m) \). Here \( \text{Hitch}(G_m) := \Gamma(G_m, c^* \times \mathbb{C}^* \omega_{G_m}) \) (we temporarily write \( \mathbb{C}^* \) instead of \( G_m \) to distinguish the group acting on \( c^* \) and \( \omega_{G_m} \) from the open curve \( G_m \subset \mathbb{P}^1 \)). Given \( E \in \text{Bun}_G \), we have \( E' = E|_{G_m} \in \text{Bun}_{G \times G_m} \). The cotangent space \( T^*_G \text{Bun}_G \) maps to \( \Gamma(G_m, \mathfrak{g}_c^* \otimes \omega_{G_m}) \) where \( \mathfrak{g}_c^* \) is the bundle on \( G_m \) associated to \( E' \) and the representation \( \mathfrak{g}^* \) of \( G \). The \( G \)-invariant map \( \mathfrak{g}^* \to c^* \) gives rise, as \( E \) varies, to the map \( h^\text{cl} : T^* \text{Bun}_G \to \text{Hitch}(G_m) \).

Applying the results of [140, §4], we deduce that \( \text{Hitch}(\mathbb{P}^1)_G \) is isomorphic to

\[
(10.2.1) \quad \text{Hitch}(\mathbb{P}^1)_G \cong \bigoplus_{1 \leq i < r} \Gamma(\mathbb{P}^1, \omega^d_i(d_i(0d_i+r_1(0)+\cdots) \otimes c_i^* \bigoplus \Gamma(X, \omega^d_i(d_i(0d_i+r_1(0)+\cdots)) \otimes c_i^*.
\]

Note that in our case the integer \( m \) in [140, §4] is given by \( m = d_r \), which is equal to the Coxeter number \( c \) of \( \mathfrak{g} \). The space \( \text{Hitch}(\mathbb{P}^1)_G \) is thus isomorphic to \( \mathbb{A}^r \times \mathbb{A}^1 \).

Let \( \mu : T^* \text{Bun}_G \to \mathfrak{t}^* \times \mathfrak{v}^* \) be the moment map for the action of \( T \times V \) on \( \text{Bun}_G \).

**Proposition 10.3.** We have the following commutative diagram, with all maps surjective, and the bottom map is an isomorphism and the top map is flat.

\[
\begin{array}{ccc}
T^* \text{Bun}_G & \xrightarrow{\mu} & \mathfrak{t}^* \times \mathfrak{v}^* \\
| & | & | \\
\text{Hitch}(\mathbb{P}^1)_G & \cong & c^* \times \mathfrak{v}^* / T
\end{array}
\]

**Proof.** The global Hitchin map embeds into the product of the local Hitchin maps at 0 and \( \infty \).

For \( i = 0, 1, 2 \), let \( \mathfrak{p}_\infty(i) \subset \mathfrak{g}_\infty \) denote the Lie algebra of \( I_\infty(i) \). Similarly define \( \mathfrak{p}_0(i) \subset \mathfrak{g}_0 \) using \( I_0^{\text{opp}}(i) \). For a \( \mathcal{O} \)-lattice \( \mathfrak{p} \subset \mathfrak{g} \otimes \mathcal{O} \), we define \( \mathfrak{p}^\perp := \mathfrak{p}^\perp \otimes \omega_\mathcal{O} \), where \( \mathfrak{p}^\perp \subset \mathfrak{g}^\perp \otimes \mathcal{O} \) is the \( \mathcal{O} \)-dual of \( \mathfrak{p} \). The two local Hitchin maps give the following two commutative diagrams ([140, Remark 4.4] and [140, Proposition 14]):

\[
\begin{array}{ccc}
\mathfrak{p}_0(1)^\perp & \cong & \mathfrak{p}_0(1)^\perp / \mathfrak{p}_0(0)^\perp \\
| & | & | \\
\text{Hitch}(D_0)_{\text{RS}} & \to & c^* \\
| & | & | \\
\mathfrak{p}_\infty(2)^\perp & \cong & \mathfrak{p}_\infty(2)^\perp / \mathfrak{p}_\infty(1)^\perp \\
| & | & | \\
\text{Hitch}(D_\infty)_{1/c} & \to & \mathfrak{v}^* / T
\end{array}
\]
where the local Hitchin spaces are defined by

$$\text{Hitch}(D_0)_{\text{RS}} = \bigoplus_{i} \omega_{\mathcal{O}_0}^{d_i}(d_i) \otimes c_{d_i}^*$$

$$\text{Hitch}(D_{\infty})_{1/c} = \bigoplus_{i} \omega_{\mathcal{O}_0}^{d_i}(d_i) \otimes c_{d_i}^* \bigoplus \omega_{\mathcal{O}_0}^{d_r}(d_r + 1) \otimes c_{d_r}^*.$$

The bottom map of the left diagram is obtained by taking the residue at 0 \([140, \S 2]\). The bottom map of the right diagram is explained in \([140, (4.11)]\).

This establishes the commutativity of \((10.3.1)\). The explicit description \((10.2.1)\) of Hitch\((\mathbb{P}^1)_{\mathcal{G}}\) establishes the isomorphism of the bottom map (see \([140, (4.9)\) and Proof of Lemma 19]). The left map of \((10.3.1)\) is surjective by definition. The right map of \((10.3.1)\) is a quotient map and thus surjective. The top map of \((10.3.1)\) is surjective because Bun\(_G\) is a principal \(T \times V\)-bundle.

The proof of the last claim is identical to \([140, \text{Lemma } 18]\), which we repeat. The Hamiltonian reduction \(\mu^{-1}(0)/(T \times V)\) is naturally identified with \(T^*\text{Bun}_{\mathcal{G}(0,1)}\). Since Bun\(_{\mathcal{G}(0,1)}\) is good, we have that \(T^*\text{Bun}_{\mathcal{G}(0,1)}\) has dimension zero from \([140, \text{Proof of Lemma } 17]\). This implies that \(\dim \mu^{-1}(0) = \dim (T \times V)\). Let \(S \subset T^*\text{Bun}_G\) be the largest open subset such that the fibers of \(\mu|_S\) have dimension \(\dim (T \times V)\). Then \(S\) is \(G_m\)-invariant, and since \(\mu^{-1}(0) \subset S\), we have \(S = T^*\text{Bun}_G\), so all fibers of \(\mu : T^*\text{Bun}_G \to t^* \times V^*\) have dimension \(\dim (T \times V)\). Since \(t^* \times V^*\) is smooth and \(T^*\text{Bun}_G\) is locally a complete intersection (\([5, \S 1.1.1]\)), we conclude that \(\mu\) is flat.

**10.4. Quantization.** We first recall the descriptions of certain spaces of \(\mathfrak{g}^\vee\)-opers from \([5, \S 1.1.1, 140]\). (The Lie algebra \(\mathfrak{g}^\vee\) will be suppressed from the notation). Fix an \(\mathfrak{sl}_2\)-triple \(\{e, h, f\}\) in \(\mathfrak{g}^\vee\), where \(f\) is the principal nilpotent. The space \(\text{Op}(D_x^\vee)\) of opers on the formal punctured disk centered at \(x\) can be identified with the space of operators

$$\{\nabla = d + (f + (\mathfrak{g}^\vee)^e \otimes F_x)dz\}$$

where \(z\) is a local coordinate at \(x\). The space \(\text{Op}(D_0)_{\text{RS}}\) of opers with regular singularities at 0 can be identified with the space of operators

$$\left\{\nabla = d + \left(\frac{f}{t} + \frac{1}{t^2} \mathfrak{b}^\vee \otimes \mathcal{O}_0 + \frac{1}{t^2} \mathfrak{g}^\vee \otimes \mathcal{O}_0\right)dt\right\}/\text{U}^\vee(\mathcal{O}_\infty)$$

where \(t = 1/q\). The spaces \(\text{Op}(D_0)_{\text{RS}}\) and \(\text{Op}(D_{\infty})_{1/c}\) are subschemes of \(\text{Op}_D\) and \(\text{Op}_{D_{\infty}}\) respectively. In \([140, \S 2]\), a subscheme of opers \(\text{Op}(\mathbb{P}^1)_{\mathcal{G}} \subset \text{Op}(G_m)\) is defined, and according to \([140, \text{Lemma } 5]\), we have

\begin{equation}
\text{Op}(\mathbb{P}^1)_{\mathcal{G}} \cong \text{Op}(D_0)_{\text{RS}} \times_{\text{Op}(\mathfrak{d}_{\mathcal{G}})} \text{Op}(G_m) \times_{\text{Op}(D_{\infty})} \text{Op}(D_{\infty})_{1/c}.
\end{equation}

The description \((10.4.1)\) is a quantization of \((10.2.1)\), and \(\text{Fun Op}(\mathbb{P}^1)_{\mathcal{G}}\) has a filtration such that \(\text{gr(Fun Op}(\mathbb{P}^1)_{\mathcal{G}}) \cong \text{Fun Hitch}(\mathbb{P}^1)_{\mathcal{G}}\).

Let \(U(t)\) and \(U(V)\) denote the universal enveloping algebras of \(t\) and \(V\). Thus \(\text{gr}(U(t)) \cong \text{Fun } t^*\) and \(\text{gr}(U(V)) \cong \text{Fun } V^*\). Let \(D'\) be the sheaf of algebras on the smooth site \((\text{Bun}_G)_{sm}\).
defined in [5], see [140, §3.2]. Then \( \Gamma(\text{Bun}_G, D') \) is a filtered commutative algebra such that \( \text{gr}(\Gamma(\text{Bun}_G, D')) \cong \text{Fun} T^* \text{Bun}_G \).

The following commutative diagram is the quantization of Proposition 10.3.

**Proposition 10.5.** We have the following commutative diagram, where the top map is an isomorphism and the bottom map is flat.

\[
\begin{array}{ccc}
U(t)^W \otimes U(V)^T & \cong & \text{Fun}(\text{Op}(\mathbb{P}^1)_G) \\
\downarrow & & \downarrow h_v \\
U(t) \otimes U(V) & \to & \Gamma(\text{Bun}_G, D')
\end{array}
\]

**Proof.** The commutativity follows from commutative diagrams analogous to those in the proof of Proposition 10.3, see [140, Proposition 15]. The remaining statements follow by taking the associate graded of the corresponding statements in Proposition 10.3. \( \Box \)

**10.6. Proof of Theorem 9.7.** Let \( \eta : \text{Op}(D_0)_{RS} \to c^* \) be the residue map. Let \( h \in t^* \) and denote by \( \varpi(h) \) the image of \( h \) in \( c^* \). We now compute the space \( \text{Op}(\mathbb{P}^1)_G \cap \eta^{-1}(\varpi(h)) \). The space of opers on \( G_m \) is the space of operators of the form

\[
\nabla = d + f \frac{dq}{q} + vdq
\]

where \( v(q) \in (\mathfrak{g}^\vee)^e[q, q^{-1}] \). The condition of regular singularities at 0 implies that \( v \in q^{-1}(\mathfrak{g}^\vee)^e[q] \). Write \( v = a/q + v' \) for \( v' \in (\mathfrak{g}^\vee)^e[q] \). The residue of \( \nabla \) at 0 is \( \varpi(h) \in c^* \). By Kostant’s theorem [92], \( f+(\mathfrak{g}^\vee)^e \) maps isomorphically to \( c^* \) under the map \( \mathfrak{g}^\vee \to \mathfrak{g}^\vee / G^\vee \to c^* \). Thus the element \( a = a_h \in (\mathfrak{g}^\vee)^e \) is uniquely determined.

Writing \( t = 1/q \), the operator becomes

\[
\nabla = d + (f + a_h) \frac{dt}{t} + v'(1/t) \frac{dt}{t^2}.
\]

The condition at \( \infty \) implies that \( v' \in \mathfrak{g}^\vee_0 \) must be constant. Thus the space of opers \( \text{Op}(\mathbb{P}^1)_G \cap \eta^{-1}(\varpi(h)) \) is the space of operators of the form

\[
(10.6.1) \quad \nabla = \nabla_\alpha = d + (f + a_h) \frac{dq}{q} + \alpha x dq
\]

for \( \alpha \in \mathbb{C} \). Thus \( \text{Op}(\mathbb{P}^1)_G \cap \eta^{-1}(\varpi(h)) \cong \mathbb{A}^1 \cong \text{Spec}(U(V)^T) \). Let \( u \in U(V)^T \) correspond to the function \( \alpha \in \text{Fun}(\text{Op}(\mathbb{P}^1)_G \cap \eta^{-1}(\varpi(h))) \) under this isomorphism.

Recall the equivariant Frenkel-Gross connection

\[
(10.6.2) \quad \nabla^G(h) = d + (f + h) \frac{dq}{q} + x dq.
\]

By construction, the two elements \( f + a_h \) and \( f + h \) in \( \mathfrak{g}^\vee \) have the same image in \( \mathfrak{g}^\vee / G^\vee \cong t^*/W \) and are therefore conjugate by a group element \( g \in G^\vee \). Again by Kostant’s theorem, \( U^\vee \) acts freely on \( f + b^\vee \) via the adjoint action and the quotient is isomorphic to \( f + (\mathfrak{g}^\vee)^e \). Thus \( f + a_h \) and \( f + h \) are conjugate by an element in the unipotent subgroup \( U^\vee \subset G^\vee \). It follows that \( \text{Ad}_g(x_\varpi) = x_\varpi \) so that the two connections (10.6.1) and (10.6.2) are gauge equivalent via a constant gauge transformation.
To complete the proof of Theorem 9.7, it remains to show that for each generic affine character \( \chi \), the Kloosterman \( D \)-module \( \text{Kl}_{G^\vee}(\chi, \hbar) \) is isomorphic to the \( M \)-module of \( \text{Aut}(\chi, \hbar) \) of [76, Remark 6.1]. The tensor product is defined using the bottom map of Proposition 10.3. The Hecke eigen-\( D \)-module \( \text{Aut}(\chi, \hbar) \) is holonomic since \( \mu^{-1}(0) \) is Lagrangian, and by [140, Corollary 9] and the flatness of the bottom map in Proposition 10.3, it has the connection \( \nabla_{\varphi(a)} \) as its Hecke eigenvalue.

It remains to argue that \( \text{Aut}(\chi, \hbar) \) is isomorphic to the automorphic \( D \)-module of [76]. Let \( \text{Bun}_G \subset \text{Bun}_G \) be the open substack mapping to the (open) basepoint \(* \subset \text{Bun}_G(0,1)\) corresponding to trivializable \( G \)-modules. Then \( \text{Bun}_G \) is isomorphic to \( T \times V \). By [140, Remark 6.1], \( \omega_{\text{Bun}}^{-1/2} \) is canonically trivialized on \( \text{Bun}_G \). It follows that the restriction of \( \text{Aut}(\chi, \hbar) \) to \( \text{Bun}_G \cong T \times V \) is isomorphic to \( M^h \otimes E^\chi \). Furthermore, \( \text{Aut}(\chi, \hbar) \) is a \((T \times V, M^h \otimes E^\chi)\)-equivariant \( D \)-module on \( \text{Bun}_G \). By [76, Remark 2.5], \( \text{Aut}(\chi, \hbar) \) is automatically (intermediate) clean extension of \( \text{Aut}(\chi, \hbar) \) on \( \text{Bun}_G \). Thus \( \text{Aut}(\chi, \hbar) \) is isomorphic to the automorphic Hecke eigen-\( D \)-module \( A(\chi, \hbar) \) of [76] for which \( \text{Kl}_{G^\vee}(\chi, \hbar) \) is an eigenvalue. This shows that \( \text{Kl}_{G^\vee}(\chi, \hbar) \) is isomorphic to \( \nabla_{\varphi(a)} \) and thus to the equivariant Frenkel–Gross connection, completing the proof.

11. The Peterson isomorphism

This section has two parts. We begin by establishing Theorem 11.13 which is a stronger version of Theorem 9.11. Let \( S = \text{Sym}(t) = H^*_{\vee}(\text{pt}) \) so that \( \text{Spec}(S) = t^* \). Instead of considering the \( D \)-modules \( Q_{G^\vee/P^\vee}(h), \nabla_{G^\vee,\varphi(a)}(h), \text{Kl}_{G^\vee,\varphi(a)}(h), \text{Cr}(h) \) for each \( h \in t^* \) separately, we shall work with \( D \)-modules with an action of \( S \). Furthermore, we introduce an additional parameter \( \hbar \) and work with \( D_h \otimes S \)-modules. In the second part, we deduce the Peterson isomorphism by specializing \( \hbar = 0 \) (Theorem 11.16).

11.1. \( D_h \otimes S \)-modules. The definition of the sheaf \( D_{h,X} \) of \( h \)-differential operators on a scheme \( X \) equipped with a \( \mathbb{G}_m \)-action is recalled in [15]. An \( S \)-structure on a \( D_{h,X} \)-module \( \mathcal{M} \) is an action of \( S \) on \( \mathcal{M} \) that commutes with the \( D_{h,X} \)-action. Equivalently, \( \mathcal{M} \) is a module for the sheaf \( D_{h,X} \otimes S \), where elements of \( S \) are considered “scalars”. For any \( D_{h,X} \)-module \( \mathcal{M} \), the sheaf \( \mathcal{M} \otimes S \) is a \( D_{h,X} \otimes S \)-module.

Our basic example is the multiplicative \( D_{h,T} \otimes S \)-module \( M^{S,h} \) on \( T \), defined as follows. Let \( \ell : T \to \text{Fun}(t^*) = S \) denote the multi-valued function

\[(11.1.1) \quad \ell(t)(h) = \langle \log(t), h \rangle, \quad \text{where } h \in t^*.
\]

Define the \( D_{h,T} \otimes S \)-module \( M^{S,h} \) as the free \( O_T \otimes S \)-module with basis element \( e^{\ell/h} \), with the action of \( \xi_\lambda \in D_{h,T} \) given by

\[\xi_\lambda \cdot e^{\ell/h} := (h \mapsto \langle \lambda, h \rangle) \cdot e^{\ell/h} = \lambda e^{\ell/h}\]
for $\lambda \in \mathfrak{t} \subset S$. Here $\xi_\lambda$ should be thought of as "$\hbar \partial_\lambda$". We give $T$ the trivial $\mathbb{G}_m$-action and furthermore declare that $\lambda \in \mathfrak{t} \subset S$ has degree one. This gives $M^{S,h}$ the structure of a graded $D_{h,T}$-module.

Recall that an $\hbar$-connection on a bundle $E$ over $X$ is a $\mathbb{C}$-linear operator $\nabla : \Gamma(X,E) \to \Gamma(X,E) \otimes \Omega_X$ such that $\nabla(fs) = f\nabla(s) + hs \otimes df$ where $f \in \mathcal{O}_X$ and $s \in \Gamma(X,E)$ are sections. A $\hbar$-connection for $\hbar = 1$ is simply a connection in the usual sense. An alternative description of $M^{S,h}$ is as follows: take the trivial $S[h]$-bundle on $T$ and equip it with the $\hbar$-connection $\hbar d - \lambda$ where $\lambda \in \mathfrak{t} \subset S$.

Suppose we have $\mathbb{G}_m$-actions on $E$ and $X$ such that the projection $E \to X$ is $\mathbb{G}_m$-equivariant. We then say that the $\hbar$-connection $\nabla$ is graded if $\hbar^{-1}\nabla$ is $\mathbb{G}_m$-equivariant, where $\hbar$ is taken to be degree one for the $\mathbb{G}_m$-action. Equivalently, if $\nabla = \hbar d + \eta$, we require that $\eta$ has degree one for the $\mathbb{G}_m$-action.

11.2. **Equivariant Frenkel-Gross connection revisited.** Let $V$ be a finite-dimensional $G^\vee$-module and let $\mu : V \times t^* \to V$ denote the action map of $t^*$. Let $\mu^* : V \to V \otimes S$ denote the map defined by $\mu^*(v) = v \otimes \lambda$ if $v \in V$ has weight $\lambda \in \mathfrak{t}$. By extending scalars, we obtain a map $\mu^* : V \otimes S \to V \otimes S$. For a $G^\vee$-module $V$, define the equivariant Frenkel-Gross $\hbar$-connection to be

$$\nabla^{(G^\vee,V)}(S,h) = \hbar d + (f + \mu^*) \frac{dq}{q} + x_d dq$$

acting on the trivial $V \otimes S[h]$-bundle on $\mathbb{C}^X_q$. Thus for $h \in \text{Spec}(S) \cong t^*$ and $\hbar = 1$, we have $\nabla^{(G^\vee,V)}(S,1) \otimes \mathbb{C} \cong \nabla^{(G^\vee,V)}(h)$ reduces to $\nabla^{(G^\vee)}$.

Declaring that $\lambda \in \mathfrak{t} \subset S$ sits in degree one, the $\mathbb{G}_m$-action of $\nabla^{(G^\vee)}$ extends to the equivariant setting, so that the 1-form $(f + \mu^*) \frac{dq}{q} + x_d dq$ has degree one.

11.3. **Equivariant quantum connection revisited.** Define the equivariant $\hbar$-quantum connection

$$Q^{G^\vee/P^\vee}(S,h) := \hbar d + c^T(O(1))_q \frac{dq}{q}.$$

acting on the trivial bundle over $\mathbb{C}^X_q = \text{Spec} \mathbb{C}[q,q^{-1}]$ with fiber the equivariant cohomology $H^*_T(G^\vee/P^\vee) \otimes \mathbb{C}[h]$. Here $c^T(O(1))_q$ denotes the equivariant quantum cohomology action. Define $\nabla^{G^\vee/P^\vee}(h) := Q^{G^\vee/P^\vee}(S,1) \otimes \mathbb{C}$. Then $\nabla^{G^\vee/P^\vee}(h) \text{ from } \left[4.2\right]$ is equal to $\nabla^{G^\vee/P^\vee}(h,1)$.

As in $\left[4.3\right]$ we define a $\mathbb{G}_m$-action on $QH^*_T(G^\vee/P^\vee)$ by using half the topological degree. As before, $\lambda \in \mathfrak{t} \subset S$ sits in degree one. The connection 1-form $(\sigma_1 \ast q,h - \omega^\vee_i) \frac{dq}{q}$ is then homogeneous of degree one for the $\mathbb{G}_m$-action.

We then have the following variation of Theorem $\left[4.3\right]$.

**Theorem 11.4.** If $P^\vee \subset G^\vee$ is minuscule and with corresponding minuscule representation $\mathcal{V}_{\omega^\vee_i}$, then under the isomorphism $L : H^*_T(G^\vee/P^\vee) \to \mathcal{V}_{\omega^\vee_i}$ of $\left[4.1\right]$, the equivariant quantum connection $Q^{G^\vee/P^\vee}(S,h)$ is identified with the equivariant Frenkel–Gross connection $\nabla^{(G^\vee,V)}(-S,h)$. This is an isomorphism of graded $h$-connections.

Here "$-S$" means that the $S$-module structure is negated.

**Proposition 11.5.** For any $h \in t^*$ and $h \in \mathbb{C}^X_q$, there is an isomorphism

$$Q^{G^\vee/P^\vee}(h,h) \cong [q \mapsto q/hc^T] \ast Q^{G^\vee/P^\vee}(\frac{h}{h},1).$$
Proof. Recall that $QH^*_C(G^\vee/P^\vee)$ is a graded ring with the topological degree $\deg \sigma_w = 2\ell(w)$ and that it follows from Lemma 4.8 that $\deg q = 2c$. The gauge transformation $\sigma_w \mapsto h^{\ell(w)}\sigma_w$ then gives the desired isomorphism between the two connections. \hfill \Box

11.6. Kloosterman $D$-modules associated to an additive character and a multiplicative character revisited. Define the exponential $D_{h,\lambda^1}$-module by

$$E^x/h := D_{h,\lambda^1}/D_{h,\lambda^1}(h\partial_x - 1).$$

For a generic affine character $\chi : I(1)/I(2) \to A^1$, we let $E^x/h := \chi^*(E^x/h)$ denote the pullback. Let $A_G(\chi, S, h)$ denote the $D_h \otimes S$-module on $\text{Bun}_{G(1,2)}$ given by taking the $D_h \otimes S$-module $M^{S,h} \boxtimes E^x/h$ on $T \times I(1)/I(2)$ and pushing it forward to $\text{Bun}_{G(1,2)}$. Using the formalism of Hecke functors, we may define a Kloosterman $D_h \otimes S$-module $\text{Kl}_{G^\vee}(\chi, S, h)$ on $\mathbb{P}^1_{\{0,\infty\}}$. As before, we have associated $D_h \otimes S$-modules $\text{Kl}_{G^\vee,V}(\chi, S, h)$ and $\text{Kl}_{G^\vee,\omega^\vee}(\chi, S, h)$. We set $\text{Kl}_{G^\vee}(S, h) := \text{Kl}_{G^\vee}(\phi, S, h)$ and similarly for $\text{Kl}_{G^\vee,V}(S, h)$ and $\text{Kl}_{G^\vee,\omega^\vee}(S, h)$. The $\mathbb{G}_m$-action of 11.4 gives the structure of a graded $D_h \otimes S$-module.

11.7. Equivariant reciprocity theorem revisited.

**Theorem 11.8.** Let $\omega^\vee$ be a minuscule fundamental weight of $G^\vee$. For any generic affine character $\chi$, there is a choice of basis element $x_0 \in \mathfrak{g}_0^\vee$ such that we have an isomorphism of graded $D_{h,\mathbb{G}_m} \otimes S$-modules

$$\text{Kl}_{G^\vee,\omega^\vee}(\chi, S, h) \cong \nabla^{G^\vee,\omega^\vee}(S, h).$$

As before, we normalize conventions so that $\text{Kl}_{G^\vee,\omega^\vee}(\phi, S, h) = \text{Kl}_{G^\vee,\omega^\vee}(S, h)$ matches with the choice of $x_0$ from 3.9.1.

**Proof.** The proof is a variation of the proof of Theorem 9.7. We start by arguing that $\text{Kl}_{G^\vee}(\chi, S, h) \cong \nabla^{G^\vee}(S, h)$ holds with $h = 1$. Notationwise, the convention is that omitting $h$ from the notation of a $D_h$-module gives the corresponding $D$-module at $h = 1$. Let $\iota : S^W \to S$ denote the natural inclusion. Consider the automorphic sheaf

$$\text{Aut}(\chi, S) = \omega^{-1/2}_{\text{Bun}_g} \otimes (D' \otimes S^W \otimes U(V), \iota \otimes \phi)(S \otimes \mathbb{C}).$$

defined using Proposition 10.5 and the natural isomorphism $U(t) \cong S$. The same argument as in the proof of Theorem 9.7 gives that $\text{Aut}(\chi, S)$ is a holonomic $D' \otimes S$-module. The technology of 3.140 shows that $\text{Aut}(\chi, S)$ is a Hecke-eigensheaf on $\text{Bun}_g$. Let $E$ denote its Hecke-eigenvalue and for a finite-dimensional $G$-module $V$, let $E^V$ denote its associated bundle. Then $E^V$ is a $D_{G_m} \otimes S$-module isomorphic to $\nabla^{G^\vee,V}$.

On the other hand, as in the proof of Theorem 9.7 $\text{Aut}(\chi, S)$ restricted to $\text{Bun}_g \cong T \times V$ is isomorphic to $\text{M}^S \boxtimes E^V$. Furthermore, $\text{Aut}(\chi, S)$ is a $(T \times V, \text{M}^S \boxtimes E^V)$-equivariant $D$-module on $\text{Bun}_g$. It follows that $\text{Aut}(\chi, S) \cong A_{\chi,S}$. Thus $\text{Kl}_{G^\vee,V}(\chi, S, h) \cong \nabla^{G^\vee,V}(S)$ for any $V$, or equivalently, $\text{Kl}_{G^\vee}(\chi, S) \cong \nabla^{G^\vee}(S)$.

We note that the $\mathbb{G}_m$-actions of 3.3 and 7.14 are in agreement: they are both induced by the trivial $\mathbb{G}_m$-action on $T$, the dilation action on $V \cong I(1)/I(2)$, and the action $\zeta = \zeta^c q$ of the curve $\mathbb{C}^\times_q$ (noting that the Coxeter numbers of $G$ and $G^\vee$ coincide). Thus $\text{Kl}_{G^\vee}(\chi, S) \cong \nabla^{G^\vee}(S) \cong D_{\mathbb{G}_m} \otimes S$-modules, where the filtration is induced by the $\mathbb{G}_m$-action on $\mathbb{C}^\times_q$ as explained in 15.1.

Now we consider the associated $D$-modules with $V_\omega$ a minuscule fundamental representation. By definition, $\nabla^{G^\vee,\omega^\vee}(S, h)$ is a free $\mathbb{C}[h]$-module isomorphic to $\nabla^{G^\vee,\omega^\vee}(S) \otimes_{\mathbb{C}} \mathbb{C}[h]$.\hfill \Box
with the obvious action of $D_{h,S_m}$ arising from the action of $\nabla^{(G^\vee,\omega^\vee)}(S)$. It thus suffices to show that $Kl_{(G^\vee,\omega^\vee)}(\chi,S,h)$ is $\hbar$ torsion-free, for then it will be isomorphic to $Kl_{(G^\vee,\omega^\vee)}(\chi,S) \otimes \mathbb{C}[h]$, which is in turn isomorphic to $\nabla^{(G^\vee,\omega^\vee)}(S) \otimes \mathbb{C}[h]$. This follows from Proposition \[15.12\] and Theorem \[11.1\].

11.9. Equivariant geometric crystal $D$-module revisited. We use notation similar to \cite{6}. Let $Cr_{(G,P)}(S,h) := R\pi_!(\gamma^*M^{S,h} \otimes E/f^h)$ be the pushforward $D_{h,Z(L_P)} \otimes S$-module on $Z(L_P) \cong \mathbb{G}_m$. According to Proposition \[6.20\], we have that $\pi : X \to Z(L_P)$, $f : X \to \mathbb{A}^1$ and $\gamma : X \to T$ are $\mathbb{G}_m$-equivariant. Thus $Cr_{(G,P)}(S,h)$ acquires a natural structure of a graded $D_{h,Z(L_P)} \otimes S$-module. In Proposition \[15.12\], we show that $Cr_{(G,P)}$ is $\hbar$-torsion free.

**Proposition 11.10.** (i) For any $h \in \mathfrak{t}^*$ and $h \in \mathbb{C}^\times$, there is an isomorphism of $D_{Z(L_P)}$-modules
\[ Cr_{(G,P)}(h,h) \cong [q : q/\hbar^c]Cr_{(G,P)}\left(\frac{h}{\hbar}, 1\right), \]
where $c$ is the Coxeter number of $G$.

(ii) There is an isomorphism of $D_{h,Z(L_P)} \otimes S$-modules
\[ Cr_{(G,P)}(S,h) \cong Cr_{(G,P)}(S,1) \otimes \mathbb{C}[h]. \]

**Proof.** Assertion (i) follows from the homogeneity of the potential $f$ established in \cite{6.17} combined with Corollary \[6.21\] and Lemma \[6.25\]. Note that equivariant part $\gamma^*M^{S,h}$ is multiplicative and thus invariant under any Kummer pullback.

From (i) we deduce that $Cr_{(G,P)}(S,h)$ and $Cr_{(G,P)}(S,1) \otimes \mathbb{C}[h]$ are isomorphic after localizing $D_{h,S_m}$ at $(\hbar)$. Proposition \[15.12\] says that $Cr_{(G,P)}(S,h)$ is $\hbar$-torsion free, and $Cr_{(G,P)}(S,h)$ is also $\hbar$-torsion free by construction, hence the isomorphism extends to $D_{h,S_m}$.

The following result has an identical proof to Theorem \[9.9\].

**Theorem 11.11.** Suppose $P = P_1$ is cominuscule. Then the graded character $D_{h,Z(L_P)} \otimes S$-module $Cr_{(G,P)}(S,h)$ is isomorphic to the graded Kloosterman $D_{h,S_m} \otimes S$-module $Kl_{(G^\vee,\omega^\vee)}(S,h)$.

11.12. The $D_{h} \otimes S$ mirror theorem. Combining Theorems \[11.4\] \[11.8\] and \[11.11\] we obtain the following result.

**Theorem 11.13.** We have isomorphisms of graded $D_{h,S_m} \otimes S$-modules
\[ Cr_{(G,P)}(S,h) \cong Q^{G^\vee/P^\vee}(S,h). \]

11.14. The Gauss-Manin model. In this subsection, we describe the $D_{G_m} \otimes S[h]$-module $Cr_{(G,P)}(S,h)$ more explicitly. In the following, we write $f_S : X \to S$ for the equivariant multivalued potential, that is $f_S := f + \ell \circ \gamma$, where $\ell$ is defined in \[11.11\].

By \[176\], we have $Cr_{(G,P)}(S,h) := \pi_!E^{f^h/S} \cong \pi_!E^{f^h/S}$. By \[81\] Proposition 1.5.28(i)], we may compute $\pi_!E^{f^h/S}$ by computing the sheaf pushforward $GM^\bullet$ along $\pi$ of the deRham complex $DR^\bullet(E^{f^h/S})$. Since $X \cong \mathcal{R} \times Z(L_P)$ where $\mathcal{R}$ and $Z(L_P)$ are both affine (and thus also $D$-affine), it suffices to work with the modules of global sections. The complex $GM^\bullet$ is the sequence
\[ \Omega^0(X/Z(L_P)) \otimes \mathbb{C}[X] S[X] \to \cdots \to \Omega^{d-1}(X/Z(L_P)) \otimes \mathbb{C}[X] S[X] \to \Omega^d(X/Z(L_P)) \otimes \mathbb{C}[X] S[X] \]
where $d = \dim \mathcal{R}$, and $\Omega^k(X/Z(L_P))$ is the module of relative global rational differentials. Here the space of global sections of the rank one $D_{X,h} \otimes S$-module $E^{f^h/S}$ have been identified with $S[X] = \mathbb{C}[X] \otimes S$.
By Proposition 15.12, we know that \( R\pi_*(E^{f/h}) \) vanishes except in one degree, so the only nonzero cohomology group of \( \text{GM}^\bullet \) is
\[
\text{GM}_h := \text{cokernel}(\Omega^{d-1}(X/Z(L_P)) \otimes S \to \Omega^d(X/Z(L_P)) \otimes S),
\]
and the differential is given by \( hd + df_S \). Here the differential \( d \) and the form \( df_S \) are both relative: no differentiation is made in the \( q \) or \( h \) directions.

Now, \( X \cong \mathcal{R} \times Z(L_P) \) is an open subset of affine space: specifically, \( \mathcal{R} \) is an open subset of a Schubert cell in \( G/P \). Let \( x_1, x_2, \ldots, x_d \) be coordinates for this Schubert cell. Let \( A = S[h, q, q^{-1}] \). Then \( \mathbb{C}[\mathcal{R}] \) is a localization of \( \mathbb{C}[x_1, \ldots, x_d] \), and we have isomorphisms of \( A[\mathcal{R}] \)-modules
\[
\Omega^d(X/Z(L_P)) \otimes S \cong A[\mathcal{R}] \cdot \omega
\]
\[
\Omega^{d-1}(X/Z(L_P)) \otimes S \cong \sum_i A[\mathcal{R}] \cdot \omega_i
\]
where \( \omega = \prod_{j=1}^d dx_j \) and \( \omega_i = \prod_{j \neq i} dx_j \). Thus the Gauss-Manin module \( \text{GM}_h \) can be written explicitly in terms of coordinates by computing the partial derivatives \( \partial f_S / \partial x_j \).

The Gauss-Manin module \( \text{GM}_h \) is a \( A(\hbar \partial_q) \)-module where \( \hbar \partial_q \) acts via the operator \( \hbar \partial_q + \frac{\partial f_S}{\partial q} \).

### 11.15. Peterson isomorphism

Let
\[
\text{Jac}(X/Z(L_P), f_S) := S[q^{\pm 1}][\mathcal{R}]/\langle \frac{\partial f_S}{\partial x_1}, \ldots, \frac{\partial f_S}{\partial x_d} \rangle
\]
denote the Jacobian ring of \( f_S \). It is independent of the choice of coordinates because it can be identified with the cokernel of the wedge map with \( df_S \) from \( \Omega^{d-1}(X/Z(L_P)) \) to \( \Omega^d(X/Z(L_P)) \).

### Theorem 11.16

We have an isomorphism of \( S[q^{\pm 1}] \)-algebras
\[
\text{Jac}(X/Z(L_P), f_S) \cong QH_{T^\vee}^*(G^\vee/P^\vee).
\]
Moreover, multiplication by \( q \frac{\partial f_S}{\partial q} \) on the left-hand side corresponds to quantum multiplication by \( c_1^T(O(1)) = \sigma_i - \varpi_i^\vee \) on the right-hand side.

**Proof.** By Theorem 11.13 we have an isomorphism of \( A(\hbar \partial_q) \)-modules between \( \text{GM}_h \) and the equivariant quantum connection \( Q^G/P^\vee(S, h) \), which is the \( A(\hbar \partial_q) \)-module \( QH_{T^\vee}^*(G^\vee/P^\vee) \otimes \mathbb{C}[\hbar] \) with the action of \( \hbar q \) \( \partial_q \) given by \( \hbar q \partial_q + (\sigma_i \ast q) - \varpi_i \).

At \( \hbar = 0 \), the map is given by wedging with the relative differential \( df_S \), so we have \( \text{GM}_0 \cong \text{Jac}(X/Z(L_P), f_S) \) as an \( S[q^{\pm 1}][\hbar \partial_q] \)-module with the action of \( q \hbar \partial_q \) given by multiplication by \( \frac{\partial f_S}{\partial q} \) in the right-hand side which we denote by \( \text{Jac}(f_S) \) for short.

Under the above isomorphism
\[
\gamma : QH_{T^\vee}^*(G^\vee/P^\vee) \cong \text{Jac}(f_S)
\]
of \( S[q^{\pm 1}] \)-modules, quantum multiplication by \( \sigma_i - \varpi_i \) corresponds to multiplication by \( q \frac{\partial f_S}{\partial q} \) in \( \text{Jac}(f_S) \).

Since \( QH_{T^\vee}^*(G^\vee/P^\vee) \) is a free \( S[q^{\pm 1}] \)-module we deduce that \( \text{Jac}(f_S) \) is also free. Let \( \gamma(1_H) \) be the image of the identity \( 1_H \) of the ring \( H_{T^\vee}^*(G^\vee/P^\vee) \), and let \( 1_J \in \text{Jac}(f_S) \) denote the identity of the ring \( \text{Jac}(f_S) \). It also follows that there exists \( \zeta \in \text{Jac}(f_S) \otimes \mathbb{C}(t^\vee) \) so that \( \gamma(1_H) \cdot \zeta = 1_J \). Let \( \zeta \gamma : H_{T^\vee}^*(G^\vee/P^\vee) \cong \text{Jac}(f_S) \) denote the composition of the \( S[q^{\pm 1}] \)-module isomorphism \( \gamma \) with left multiplication by \( \zeta \). Then \( \zeta \gamma(1_H) = 1_J \) and \( \zeta \gamma \) sends quantum multiplication by \( \sigma_i \) to multiplication by \( q \frac{\partial f_S}{\partial q} \).
Recall that $S = \mathbb{C}[t^*]$, so the fraction field is $\mathbb{C}(t^*)$. By \cite[Cor. 6.5]{105} and \cite[Lemma 4.1.3]{34}, $QH^*_T(G^\vee/P^\vee) \otimes \text{Frac}(S)$ generated over $\text{Frac}(S)[q^\pm 1]$ by $\sigma_i$, and thus also by $c^T_i(O(1)) = \sigma_i - x_i^\vee$. We deduce that $\zeta\gamma$ induces a $\text{Frac}(S)[q^\pm 1]$-algebra isomorphism after localization. Since the $S[q^\pm 1]$-algebras $QH^*_T(G^\vee/P^\vee)$ and $J_f,S$ algebras are already free as $S$-modules, if follows that $\zeta\gamma$ is an isomorphism of $S[q^\pm 1]$-algebras. \hfill $\square$

Recall from \cite[1.16.1]{116} the definition of the Peterson stratum $\mathcal{Y}_p^\ast$. Rietsch \cite{117} has proved that $\text{Jac}(X/Z(L_P), f_S)$ is isomorphic to $\mathbb{C}[\mathcal{Y}_p^\ast]$. We thus obtain the following corollary.

**Corollary 11.17.** If $P^\vee$ is minuscule, then we have an isomorphism of $\mathbb{C}[t^*, q, q^{-1}]$-algebras

$$QH^*_T(G^\vee/P^\vee) \cong \mathbb{C}[\mathcal{Y}_p^\ast].$$

11.18. **Example.** Consider the case $G^\vee/P^\vee = \text{Gr}(1, n + 1) = \mathbb{P}^n$. For $h \neq 0$, we have that the equivariant quantum $A(\partial_q)$-module $Q^{\mathbb{P}^n}(h, h)$ is given by the connection

$$q \frac{d}{dq} + \frac{1}{h} \begin{pmatrix} h_1 & h_2 & q & \cdots & h_{n+1} \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix},$$

where $\sum_{i=1}^{n+1} h_i = 0$, and we identify $h = (h_1, h_2, \ldots, h_{n+1}) \in t^*$ in the usual way. Its dual is isomorphic to $A(\partial_q)/A(\partial_q)L$, where

$$L := \prod_{i=1}^{n+1} (hq q \frac{d}{dq} - h_i) - q.$$

This is a hypergeometric differential operator of type $0F_n$. In the notation of \cite[§3]{87}, we see that $Q^{\mathbb{P}^n}(h, h)$ is the hypergeometric $D$-module $\mathcal{H}_h(L_{\mathbb{P}^n}/\emptyset)$. On the other hand the character $A(\partial_q)$-module $\text{Cr}(h, h)$ is given by the $\pi_1$-pushforward of $\gamma^* M^{h/h} \otimes E^{h/h}$, that is

$$\int_{x_1 \cdots x_{n+1} = q} x_1^{h_1/h} \cdots x_{n+1}^{h_{n+1}/h} \frac{dx_1 \cdots dx_{n+1}}{x_1 \cdots x_{n+1}}.$$

The mirror isomorphism $Q^{\mathbb{P}^n}(h, h) \cong \text{Cr}(h, h)$ of Theorem 11.13 follows in this case from a result of Katz on convolution of hypergeometric $D$-modules \cite[Thm.5.3.1]{87}. In the semiclassical limit $h \to 0$, we recover the equivariant quantum cohomology algebra

$$QH^*_T(\mathbb{P}^n) = \mathbb{C}[x, q, q^{-1}, t^*]/\left( \prod_{i=1}^{n+1} (x - h_i) = q \right),$$

from the quantum connection $Q^{\mathbb{P}^n}$ on the one hand. And on the other hand, from the potential function $f_S$, and in view of

$$x_i \frac{\partial f_S}{\partial x_i} = x_i + h_i - \frac{q}{x_1 \cdots x_n},$$

we recover the Jacobi ring $\text{Jac}(f_S)$. By letting $x := x_i + h_i$, which is independent of $i$, we see that $\frac{\partial f_S}{\partial x_i} = 0$ is equivalent to $\prod_{i=1}^{n+1} (x - h_i) = q$, in agreement with Theorem 11.16.
12. PARABOLIC BESSEL FUNCTIONS

The Givental integral formulae [58] for Whittaker functions (see also [55] and more generally [1.3.1]) arise in the present context as solutions to $\text{Cr}_{(G,P)}$ via a natural pairing with homology groups. Equivalently these are special functions that are solutions of the quantum differential equation. The final [12.21] treats the case of the classical $I_0$ and $K_0$-Bessel functions as an illustration of the main concepts.

12.1. Solution of the geometric crystal $D$-module. We allow $P$ to be arbitrary until §12.6. The solution complex of $\text{Cr}_{(G,P)}(h) := \text{Cr}_{(G,P)}(h = 0, h)$ is defined [81] §4.2 to be

$$\text{Sol}_{(G,P)}(h) := \text{RHom}_D(\text{Cr}^{\text{an}}_{(G,P)}, \mathcal{O}^{\text{an}}_{L(P)}).$$

Recall that $\text{Cr}_{(G,P)}(h) = \pi_\ast \mathcal{E}^{f/h}$. By [81] Thm. 4.2.5 we can interpret the stalks of $\text{Sol}_{(G,P)}(h)$ as dual to the algebraic de Rham cohomology $H^i_{\text{dR}}(G/P, \mathcal{E}^{f/h})$. Concretely, $\text{Sol}_{(G,P)}(h)$ is the local system of holomorphic flat sections of the connection dual to $\text{Cr}_{(G,P)}(h)$.

If $P$ is cominuscule, then by Theorem [8.3] $\text{Cr}_{(G,P)}(h)$ is a coherent $D$-module, hence $\text{Sol}_{(G,P)}(h)$ is a local system on $Z(L_P)$. For every $h \in \mathbb{C}^\times$ and $t \in Z(L_P)$, we deduce that $H^i_{\text{dR}}(G/P, \mathcal{E}^{f/h})$ is zero unless $i = d = \dim(G/P)$, and that $\dim H^i_{\text{dR}}(G/P, \mathcal{E}^{f/h})$ is constant equal to $|W_P|$.

12.2. Rapid decay homology. We want to define oscillatory integrals with parameters $h$ and $t$,

$$I_\Gamma(h, t) := \int_{\Gamma_t} e^{f/h} \omega_t,$$

and interpret them as horizontal sections of $\text{Sol}_{(G,P)}(h)$. Here we view $\omega_t$ as an element of $H^i_{\text{dR}}(G/P, \mathcal{E}^{f/h})$. In general $I_\Gamma$ will be multi-valued since $Z(L_P)$ is not simply-connected, and we could lift the parameter to the universal cover $H^2(G^\vee/P^\vee) \to Z(L_P)$ (see Lemma 8.2), but we shall proceed in the framework of local systems on $Z(L_P)$.

For each $h \in \mathbb{C}^\times$ and $t \in Z(L_P)$, consider the exponential $D$-module $\mathcal{E}^{f/h}$ on $G/P$. Hironaka’s theorem on elimination of points of indeterminacy applies to the potential $f_t$ on $G/P$, viewed as a rational function $f_t : G/P \to \mathbb{P}^1$. It implies the existence of a resolution of singularities of the anticanonical divisor $\partial_{G/P}$ (the complement of $G/P$ in $G/P$) such that $f_t$ can be lifted to a regular function to $\mathbb{P}^1$. Then Bloch–Esnault and Hien–Roucairol [79] have defined a space of rapid decay homology cycles $H^i_{\text{rd}}(G/P, \mathcal{E}^{-f/h})$, and showed that it is independent of the choice of resolution of singularities. We think of $\Gamma_t$ as a rapid decay cycle in this sense. Moreover, by [79] Thm. 2.4 the oscillatory integral (12.2.1) induces, for every $i$, a perfect pairing

$$H^i_{\text{rd}}(G^\vee/P, \mathcal{E}^{f/h}) \times H^i_{\text{rd}}(G^\vee/P, \mathcal{E}^{-f/h}) \to \mathbb{C}.$$

As $t \in Z(L_P)$ varies, it is expected that $H^i_{\text{rd}}(G^\vee/P, \mathcal{E}^{-f/h})$ is a constructible sheaf, see [79] for the case where $Z(L_P)$ is one-dimensional. Assuming this, the above pairing can be interpreted as a canonical isomorphism in $D^b(\mathbb{C}_{Z(L_P)})$ between the sheaf of rapid decay cycles on $G/P$ relative to $f_t/h$ and the solution sheaf $\text{Sol}_{(G,P)}(h)$.

Remark 12.3. We have focused on the non-equivariant case for simplicity. The constructions work in the equivariant case with the following modifications. For the equivariant
D-module $M^{h,h} \otimes E^{f_t/h}$ on $G/P$, which appears in the construction of $Cr_{(G,P)}(h,h)$ and is more general than $E^{f_t/h}$, there is a generalization due to T. Mochizuki and K. Kedlaya of the elimination of points of indeterminacy. The generalized rapid decay cycles and duality pairing are developed by Hien [78].

12.4. Compact cycles. The space of rapid decay cycles $H^\bullet_{rd}(G/P)$ contains the usual homology group $H^\bullet(G/P)$ of compact cycles. The following proposition holds for any open Richardson variety so we state and prove it in that generality. For $u \leq w$ in $W$, recall that $R^u_w$ denotes the open Richardson variety, defined to be the intersection of $B_\lambda \check{B}/B$ with $B\check{w}B/B$.

**Proposition 12.5.** $H_{\text{middle}}(R^u_w)$ is one-dimensional.

**Proof.** By Poincaré duality it is equivalent to treat the cohomology with compact support $H^c_{\text{middle}}(R^u_w)$. By [114, Prop.4.2.1] there is a canonical isomorphism

$$H^\bullet(R^u_w) \cong \text{Ext}^{\bullet+\ell(w)-\ell(u)}(M_w, M_u)$$

where $M_w$ and $M_u$ denote the Verma modules in the principal block. Since $R^u_w$ has real dimension $2(\ell(w) - \ell(u))$ we have $H^{\text{middle}}_{c}(R^u_w) \cong \text{Hom}(M_w, M_u)$. This space is one-dimensional as follows from [10].

To construct a middle dimension cycle generating $H_{\text{middle}}(R^u_w)$, we use that $R^u_w$ contains many tori. (In fact by Leclerc [98], $\mathbb{C}[R^u_w]$ contains a cluster algebra, and is conjectured to be equal to one.) We choose any cluster torus $(\mathbb{C}^\times)^{\ell(w)-\ell(u)} \subset R^u_w$ and consider the middle dimension cycle given by a compact torus $(S^1)^{\ell(w)-\ell(u)}$. We denote integration along this cycle by $\oint$. We can normalize the form $\omega$ from [91] which has simple poles along the boundary of $R^u_w$ such that

$$\oint \frac{\omega}{(2i\pi)^{\ell(w)-\ell(u)}} = 1.$$

In view of Proposition 12.5, the cycle is well-defined and independent of the choice of tori.

Recall from §6.6 that $G/P \cong R^u_w_{w_0Pw_0}$ which can be identified with the open projected Richardson variety in $G/P$. Thus we have shown that the space $H_{\text{middle}}(G/P)$ is one-dimensional and generated by the above compact cycle. For the case of full flag varieties $G/B$ a related construction appears in [118, §7.1], and for the Grassmannian in [104, Thm. 4.2].

12.6. Cominuscule case. We now assume that $P$ is cominuscule and come back to studying $\text{Sol}_{(G,P)}(h)$ from §12.1 above.

**Proposition 12.7.** For every $h \in \mathbb{C}^\times$ and $t \in Z(L_P)$, $H^\bullet_{rd}(G/P, E^{-f_t/h})$ is zero unless $i = d$, and

$$\dim H^\bullet_{rd}(G/P, E^{-f_t/h}) = |W^P| = \sum_{i=0}^{d} \dim H^\bullet_{i}(G/P).$$

Assume that the ramification set denoted $\Sigma_2$ in [79, Prop. 3.5] can be chosen to be empty, i.e. that the middle-dimensional rapid decay cycles on $G/P$ relative to $f_t/h$ form a local system on $Z(L_P)$. The oscillatory integral (12.2.1) induces an isomorphism with the local system $\text{Sol}_{(G,P)}(h)$.

We shall denote horizontal sections by $\Gamma_t$, viewed as half-dimensional cycles inside $G/P$. 
Proof. It follows from Theorem 8.3 that $\text{Sol}_{(G,P)}(h)$ is a local system of rank $|W^P|$. In particular the ramification set denoted $\Sigma_1$ in [79, Prop. 3.3], can be chosen to be empty. Also [79, Thm. 2.4] implies the first assertion. The second assertion follows from [79, Thm. 3.7]. □

12.8. Poincaré duality. For $w \in W^P$, we define $\text{PD}(w) := w_0 w w_0^P$, which is still an element of $W^P$. This is an involution and we have $\ell(\text{PD}(w)) = \dim(G^\vee/P^\vee) - \ell(w)$. Moreover the Schubert class $\sigma_w \in H^{2(\ell(w))}(G^\vee/P^\vee)$ is Poincaré dual to $H^{2(\ell(\text{PD}(w)))(G^\vee/P^\vee)}$. Since $G^\vee/P^\vee$ is minuscule, a reduced expression for $w \in W^P$ is unique up to commutation relations. It is always [25, §2.4] a subexpression in any reduced expression for the longest element $w_f^P = w_0 w_0^P = \text{PD}(1)$ of $W^P$.

12.9. Givental fundamental solution. Givental has introduced solutions $S_w(h,q)$ of the quantum connection $Q^{G^\vee/P^\vee}(h)$ in terms of a generating series of gravitational descendants of Gromov–Witten invariants, see [56, §4.1], [36, §10], [72, §5] and [83, §2.3] for details. The functions $S_w$, for $w \in W^P$, form a fundamental solution of $Q^{G^\vee/P^\vee}$ near the regular singular point $q = 0$, see [52, §2].

The Givental $J$-function is defined by

$$J^{G^\vee/P^\vee}(h,q) := \sum_{w \in W^P} \langle S_w(h,q), 1 \rangle \sigma_{\text{PD}(w)}.$$ 

It gives rise to a multivalued holomorphic section

$$J^{G^\vee/P^\vee} : \mathbb{C}_q^\times \times \mathbb{C}_q^\times \rightarrow H^*(G/P),$$

which becomes single-valued when factored through the universal cover $H^2(G^\vee/P^\vee) \rightarrow \mathbb{C}_q^\times$. Using the notation of [36, Lemma 10.3.3],

$$J^{G^\vee/P^\vee}(h,q) = \exp \left( \frac{\log q}{\hbar} \sigma_1 \right) \left( 1 + \sum_{d=1}^{\infty} \sum_{w \in W^P} q^d \left( \frac{\sigma_w}{\hbar - \epsilon} \right) \sigma_{\text{PD}(w)} \right).$$

Intrinsically the $J$-function is the solution to the dual connection to $Q^{G^\vee/P^\vee}$ that is asymptotic to 1 as $q$ approaches the regular singular point 0, see [53].

Example 12.10. For $\mathbb{P}^n$, we have [36, §10]

$$(12.10.1) \quad J^{\mathbb{P}^n}(h,q) = \exp \left( \frac{\log q}{\hbar} \sigma_1 \right) \sum_{d=0}^{\infty} q^d \prod_{j=1}^{d} \frac{1}{(\sigma_i + j\hbar)^{n+1}}.$$

The case of quadrics is treated in [109, §5].

Of particular importance is the component $\langle J^{G^\vee/P^\vee}(h,q), \sigma_{\text{PD}(1)} \rangle = \langle S_{\text{PD}(1)}(h,q), 1 \rangle$ which is a power series in $h^{-1}, q$. In [4,16] we used the notation $S(q)$ for $S_{\text{PD}(1)}(1,q)$. The single-valuedness follows from considering the kernel of the monodromy operator which is the usual cup product with $\sigma_i$. Precisely,

$$(12.10.2) \quad \langle S_{\text{PD}(1)}(h,q), 1 \rangle = 1 + \sum_{d=1}^{\infty} q^d \left( \frac{\sigma_{\text{PD}(1)}}{\hbar - \epsilon}, 1 \right)_{0,d}.$$

It is called the hypergeometric series of $G^\vee/P^\vee$ in [3,11] and called the quantum period in [51,53]. The sum can be simplified further by expanding $(h - \epsilon)^{-1}$ in power series of $h^{-1}$, see [109, §5.2] who also consider more generally $\langle S_{\text{PD}(1)}, \sigma_w \rangle$ for any $w \in W^P$. 
12.11. Degrees and irregular Hodge filtration. The isomorphism \( \mathcal{Q}^{G^\vee/P^\vee}(h) \cong \mathcal{C}_{r(G,P)}(h) \) from Theorem 11.13 induces for every \( h \in \mathbb{C}^\times \) and \( t \in Z(L_P) \) an isomorphism

\[
\bigoplus_{i=0}^{d} H^{2i}(G^\vee/P^\vee) \cong H^d_{\text{dr}}(G/P, E^{f_t/h}).
\]

In this isomorphism, the left-hand side visibly carries a gradation by degree, which can be transported to the right-hand side. We want to spell this out precisely and derive an important corollary.

It is easy to see that the filtration associated to the Jordan decomposition of the linear endomorphism given by the cup-product by \( \sigma_i \) coincides with the filtration by degree on \( H^*(G^\vee/P^\vee) \). The cup-product by \( \sigma_i \) is the monodromy at \( q = 0 \) of the connection \( \mathcal{Q}^{G^\vee/P^\vee}(h) \). Thus we conclude from the mirror isomorphism \( \mathcal{Q}^{G^\vee/P^\vee}(h) \cong \mathcal{C}_{r(G,P)}(h) \) that the filtration by degree is transported on \( H^d_{\text{dr}}(G/P, E^{f_t/h}) \) to the monodromy filtration of \( \mathcal{C}_{r(G,P)}(h) \).

Proposition 12.12. In the isomorphism (12.11.1), the line spanned by the top class \( \mathbb{C} \cdot \sigma_{PD(1)} \) corresponds to the line spanned by the cohomology class of the volume form \( \omega \) from §6.6. In particular this cohomology class is nonzero.

This was previously established for Grassmannians by Marsh–Rietsch [104] and for quadrics by Pech–Rietsch–Williams [109].

Proof. In view of the above discussion we only need to analyse the monodromy filtration on \( H^d_{\text{dr}}(G/P, E^{f_t/h}) \) near \( \alpha_i(t) = 0 \) or equivalently near \( h = \infty \). A convenient way to do so is via the Kontsevich complex \( \Omega^\bullet_{f_t} \) of \( f_t \)-adapted log-forms, which again involves a resolution \( \widetilde{G}/P \) of the singularities of \((G/P,f_t)\). It is established in [42, Cor. 1.4.8] that

\[
H^d_{\text{dr}}(G/P, E^{f_t/h}) \cong \bigoplus_{p+q=d} H^q \left( \widetilde{G}/P, \Omega^p_{f_t} \right).
\]

The right-hand side is independent of \( h \) which makes it possible to write down the monodromy operator. It is possible to verify that the decreasing monodromy filtration corresponds to the gradation by \( p - q \). We omit the details which are discussed in [12] [74] [89].

Then by the above \( H^{2d}(G^\vee/P^\vee) \) corresponds under the isomorphism (12.11.1) to \( H^0(\widetilde{G}/P, \Omega^d_{f_t}) \) where by the definition of \( \Omega^d_{f_t} \), this coincides with the space \( H^0(\widetilde{G}/P, \Omega^d(\log)) \) of log differential holomorphic top forms. It is known from [91] that \( H^0 \) is one-dimensional and spanned by the form \( \omega \).

Remark 12.13. We observe that in the case of \( \mathbb{P}^n \), the above essentially amounts to a remarkable theorem of Dwork and Sperber [126] on the slopes of hyper-Kloosterman sums.
Thus we are led to conjecture that the slopes of the minuscule Kloosterman sums $K_l(G, \omega_i)$ can be read from the cohomology of $G^\vee/P^\vee$. This would follow from a suitable $p$-adic comparison isomorphism for differential equations of exponential type between Dwork $p$-adic cohomology and complex Hodge theory, which does not seem to be available in the literature yet. Interestingly the same Hodge numbers appear in the $(\mathfrak{g}, K)$-cohomology of a certain $L$-packet of discrete series [69].

The linear map $\Gamma \mapsto I_\Gamma(h, t)$ induces an isomorphism between the local system of relative rapid decay homology of half-dimension $H_{rd}^d(G/P, E_{f/t}/h)$ and the local system of holomorphic flat sections of the dual of the quantum connection $Q_{G^\vee/P^\vee}(h)$. Specializing $\omega_t$ to the class of $\omega$, we then obtain the following which is close to the original formulation of the mirror conjecture in the work of Givental [118, §7.3] and Rietsch [117, Conj. 8.2].

**Corollary 12.14.** Assume that $P^{\vee}$ is a minuscule parabolic subgroup of $G^\vee$. Then for any $h \in \mathbb{C}^\times$, a full set of solutions to the quantum differential equation of $G^\vee/P^{\vee}$ is given by integrals

$$I_\Gamma(h, t) = \int_{\Gamma_t} e^{f_t/h} \omega,$$

where $\Gamma_t$ is an horizontal section of the $Z(L_{P^{\vee}})$-local system of middle-dimensional rapid decay cycles on $G/P$ relative to $f_t/h$.

**Proof.** We combine Proposition [12.7] and Proposition [12.12]. Note that the volume form $\omega$ in [117, Prop. 7.2] coincides with the volume form $\omega$ from Proposition [12.12] constructed from [91].

**12.15. Enumerative formula.** We can deduce from the above mirror theorem an integral representation for the hypergeometric series and combinatorial formulas for certain Gromov–Witten invariants.

**Theorem 12.16.** (i) The hypergeometric series (12.10.2) of $G^\vee/P^{\vee}$ is equal to the integral of the potential on the middle-dimensional compact cycle of $G/P$,

$$I_{cpt}(h, q) := \oint e^{f_t/h} \frac{\omega}{(2i\pi)^{\dim(G/P)}}.$$

(ii) For every integer $d \geq 0$, the genus 0 and degree $d$ Gromov–Witten correlator $\langle \tau_{cd-2\sigma_{PD(1)}} \rangle_{0,d}$ of $G^\vee/P^{\vee}$ is equal to the constant term of $f_{1}^{cd}$ in any cluster chart of $G/P$, divided by $(cd)!$.

Property (ii) is referred to as weak Landau–Ginzburg model in [112]. For quadrics the theorem can be established directly by computing both sides as shown in [109, §5.3].

**Proof.** Assertion (ii) follows from (i) by taking residues. Recall that $c$ is the Coxeter number of $G$. To establish (i) we observe as consequence of the mirror Theorem 8.3 that $I_{cpt}(h, q)$ is solution of the quantum connection $Q_{G^\vee/P^\vee}(h)$. It is a power series in $q$ by Cauchy’s residue formula. The same holds for the fundamental solution $S_{PD(1)}(h, q)$. We can then deduce the desired equality of the two solutions up to scalar from the Frobenius method at the regular singularity $q = 0$. More precisely, we need to consider the equivariant connection $Q_{G^\vee/P^\vee}(h, h)$ and equivariant Gromov-Witten correlators. For generic $h \in \mathfrak{t}^*$, the monodromy at $q = 0$ is regular semisimple. We then specialize the equivariant parameter to $h = 0$. 
To conclude the proof of (i) we need to specialize the solution \( S_{PD(1)}(h, q), 1 \) to the specific component in (12.10.2). It is a power series in \( q \) with constant term 1. Similarly we evaluate \( I_{cpt}(h, q) \) against the form \( \omega \) in (12.2.1) which implies the identity in view of Proposition [12.12] □

The hypergeometric series typically has infinitely many zeros. As explained by Deligne [38, p. 128] this implies that the Hodge filtration on \( H^d_d(G/P, E^f/h) \) does not come from a Hodge structure.

Remark 12.17. More generally, the work of Marsh–Rietsch [104] for Grassmannians and Pech–Williams–Rietsch [109] for quadrics suggests the more general formula that \( S_{PD(1)}, \sigma_w \) should be equal to the residue integral \( \oint p_w e^{f_q/h} \frac{\omega}{(2i\pi)^{dim(G/P)}} \), with the Plücker coordinate \( p_w \) added. This would be compatible with the Gamma conjecture and central charges discussed in [53, 83].

Remark 12.18. In a series of works, see e.g. [56], Gerasimov–Lebedev–Oblezin study the Givental integral from various viewpoints, motivated by archimedean \( L \)-functions, integrable systems of Toda type, and Whittaker functions.

12.19. Projective spaces. For \( \mathbb{P}^n = Gr(1, n+1) \), the Coxeter number is \( c = n + 1 \). We deduce from (12.10.1) that the hypergeometric series \( S_{PD(1)}(h, q), 1 \) is equal to

\[
\sum_{d=0}^{\infty} \frac{1}{(d!)^{n+1}} \left( \frac{q}{h^{n+1}} \right)^d = \frac{1}{\Gamma_n} \left( 1 + \frac{q}{h^{n+1}} \right).
\]

On the other hand,

\[
I_{cpt}(h, q) = \oint e^{\frac{x_1}{2} + \cdots + x_n + \frac{q}{x_1 \cdots x_n}} \frac{dx_1 \cdots dx_n}{(2i\pi)^n x_1 \cdots x_n}.
\]

Hence Theorem [12.16] reduces to Erdélyi’s integral representation.

Remark 12.20. The hypergeometric series for a general minuscule homogeneous space \( G^\vee/P^\vee \) are related to the Bessel functions of matrix argument introduced by C. Herz, see [102, 122].

12.21. Classical Bessel functions. For \( \mathbb{P}^1 = Gr(1, 2) \), we have \( f_q(x) = x + \frac{q}{x} \) for \( x \in \mathbb{G}_m = \mathbb{P}^1, \omega = \frac{dx}{x} \), and

\[
H^1_d(G_m, E^{f_q/h}) = \begin{cases} 
0 & \text{if } i = 0, \\
\mathbb{C} \omega \oplus \mathbb{C} x \omega & \text{if } i = 1.
\end{cases}
\]

Deligne defines a Hodge filtration and shows in [38, p.127] that \( F^1 H^1_d(G_m, E^{f_q/h}) = \mathbb{C} \omega \), which corresponds to Theorem [12.12] above.

The space \( H^1_d(G_m, E^{-f_q/h}) \) is generated by the two cycles \( \oint \) and \( \int_0^\infty \), denoted by \( e_1, -e_2 \) in [38]. Note that the cycle \( \int_0^\infty \) depends on \( q \) and \( h \) and approaches 0 and \( \infty \) in the direction of rapid decay of the exponential.

The hypergeometric series is \( \frac{1}{\Gamma_n} \frac{1}{\Gamma_n} \left( 1 + \frac{q}{h^{n+1}} \right) \) is equal to \( I_0(2\sqrt{q}/h) = \oint e^{f_q/h} \frac{\omega}{2i\pi} \). The other integrals are expressed as follows: \( \int_0^\infty e^{f_q/h} \omega = 2K_0(2\sqrt{q}/h); \oint e^{f_q/h} x \omega = 2\sqrt{q}I_1(2\sqrt{q}/h); \int_0^\infty e^{f_q/h} x \omega = -2\sqrt{q}K_1(2\sqrt{q}/h). \) Note that \( I_0 = I_1 \) and \( K_0 = -K_1. \)

\[2\text{Also } \omega \text{ is } T\text{-invariant by Lemma [6.22] in particular invariant by } \rho^\vee. \text{ This implies the identity } I_{cpt}(h, q) = I_{cpt}(1, q/h^c), \text{ which is also satisfied by the hypergeometric series.} \]
The determinant of periods
\[
\begin{vmatrix}
\int_0^\infty e^{f_n/h_x} & \int_0^\infty e^{f_n/h_x} \\
\int_0^\infty e^{f_n/h} & \int_0^\infty e^{f_n/h} \\
\end{vmatrix} = -2i\pi h,
\]
established in the last paragraph of \cite{38} corresponds to the Wronskian formula
\[
I_\nu(y) K_{\nu+1}(y) + I_{\nu+1}(y) K_\nu(y) = 1/y
\]
for all \(y \in \mathbb{R}_{>0}\) and \(\nu \in \mathbb{C}\).

More generally, we consider the equivariant version. Let \(h \in \mathbb{C}\) and \(h\alpha \in t^*\), where \(\alpha\) denotes the positive simple root. We consider the integral solutions to \(Cr(hh\alpha, h)\),
\[
\oint x^2 h q e^{f_n/h} \omega = q^h \sum_{k=0}^\infty \frac{h^{-2k-2}q^k}{k!\Gamma(k+2h+1)} = \frac{(q/h^2)^h}{\Gamma(1+2h)} F_1 \left(1+2h; \frac{q}{h^2}\right) = I_{2h}(2\sqrt{q}/h),
\]
where compared to Example \cite{6,5} we have \(q = t^2\) and the factor \(x^2 h q^{-1}\) is equal to \((h\alpha)(\gamma(x))\). Similarly the integral from 0 to \(\infty\) is equal to \(K_{2h}(2\sqrt{q}/h)\).

On the quantum connection side, let \(\{1, \sigma\}\) be the Schubert basis of \(H^*(\mathbb{P}^1)\). Then the equivariant quantum Chevalley formula is
\[
\sigma \ast q, \sigma = q.1 + (\omega - s \cdot \omega).\sigma = q.1 + \alpha^\vee .\sigma.
\]
Here, \(s\) denotes the unique simple reflection. Thus
\[
\sigma \ast_{q, h} \sigma = q.1 + 2h.\sigma
\]
since \(\langle \alpha^\vee, \alpha \rangle = 2\). The equivariant quantum connection \(Q^{\mathbb{P}^1}(h\alpha)\) is
\[
h q \frac{d}{dq} + \begin{pmatrix} -hh & q \\ 1 & hh \end{pmatrix}.
\]
This is equivalent to the second order differential operator
\[
(h q \frac{d}{dq})^2 - (q + h^2 h^2),
\]
which has solutions the modified Bessel functions \(I_{2h}(2\sqrt{q}/h)\) and \(K_{2h}(2\sqrt{q}/h)\). This agrees with Theorem \cite{9,11}.

13. Compactified Fano and log Calabi-Yau mirror pairs

Our Theorems \cite{8,3} and \cite{11,13} verify two specific mirror symmetry predictions. In this section the goal is to briefly recast the mirror symmetry of flag varieties in view of recent advances, and provide some evidence for potential generalizations.

\footnote{The minus sign compared to \cite{38} is because we chose the cycle \(\int_0^\infty \) which is \(-e_2\).}
13.1. Mirror pairs of Fano type. The notion of mirror pairs of Fano type is explained in [89] §2.1 and [80]. In the context of Rietsch’s conjecture that we study in this paper, we have a family of mirror pairs indexed on one side by $H$ with canonical volume form $[91]$. We denote by $\Gamma$ the A and B-sides now play a symmetric role. Rietsch’s mirror conjecture corresponds to $Y$ varieties $\tilde{Y}$, namely a quasi-projective Calabi-Yau manifold $X$ by taking the complement of the anticanonical divisor. One obtains triples $(X,g,\omega,f)$ consisting of a projective Fano variety $X$, a complexified Kähler form $g$ and an anticanonical section $1/\omega$. In the context of Rietsch’s conjecture the variety is $X = G^\vee/P^\vee$, the Kähler class is varying in $H^2(G^\vee/P^\vee)$ and the anticanonical section is the one constructed in [91], see also [117].

The B-model is another triple $((Y,f),\eta,\omega_Y)$ consisting of a Landau–Ginzburg model, namely a quasi-projective Calabi-Yau manifold $Y$ with a regular function $f$ (Landau–Ginzburg potential), a Kähler form $\eta$ and a non-vanishing canonical section $\omega_Y$ (holomorphic volume form). In the context of Rietsch’s conjecture, the Landau–Ginzburg model is given by the Berenstein–Kazhdan geometric crystal $Y = X_{(G,P)}$. The underlying variety is the open Richardson $G/P$ in $G/P$, the Landau-Ginzburg potential is the decoration function $f_t$ of Berenstein–Kazhdan, which depends on the parameter $t \in Z(L_P)$. The volume form $\omega_Y$ is again the one constructed in [91].

13.2. Mirror pairs of compactified Landau–Ginzburg models. Following [89] §3.2.4, one may also consider quadruples $(X,g,\omega,f)$ consisting of a projective Fano variety $X$, a complexified Kähler form $g$, a canonical section $\omega$, and a potential function $f$.

It is natural to conjecture, for appropriate choices of Kähler forms, the mirror symmetry between $(G/P,g,\omega_{G/P},f_{G/P})$ and $(G^\vee/P^\vee,\omega^\vee_{G^\vee/P^\vee},f_{G^\vee/P^\vee})$. The A and B-sides now play a symmetric role. Rietsch’s mirror conjecture corresponds to omitting some of the data on both sides. The full mirror conjecture between these compactified mirror pairs involves the matching of a variety of homological data on both sides.

For example a Fano type mirror pair gives rise to a pair of open Calabi–Yau manifolds by taking the complement of the anticanonical divisor. One obtains triples $(X,g,\omega)$ of a log Calabi-Yau manifold, a Kähler form and a volume form. In our setting, the log Calabi-Yau varieties are $G^\vee/P^\vee$ and $G/P$ respectively. The volume form is as before. Thus from the general mirror predictions [89] Table 2], one expects a matching of cohomology of the open projected Richardson variety $H^*(G/P)$ and the cohomology of the Langlands dual open projected Richardson variety $H^*(G^\vee/P^\vee)$. As we observe in the next subsection, this matching holds more generally for arbitrary Richardson varieties.

13.3. Open Richardson varieties. Recall the Richardson varieties $\mathcal{R}_u^w \subset G/B$, where $u, w \in W$ with $u \leq w$, and the special case $\mathcal{R}_{u_0}^{w_0} \cong G/P$. They are log Calabi-Yau variety with canonical volume form [91]. We denote by $\mathcal{R}_u^w \subset G^\vee/B^\vee$ the Richardson varieties inside the flag variety of the dual group.

Proposition 13.4. For any $i \geq 0$ and $u, w \in W$ with $u \leq w$, there is an isomorphism $H^i(\mathcal{R}_{u}^{w}) \cong H^i(\mathcal{R}_{u}^{w})$.

Proof. As in the proof of Proposition [125] the statement is equivalent to the isomorphism $\text{Ext}^i(M_w,M_u) \cong \text{Ext}^i(M^\vee_w,M^\vee_u)$ where $M_w$ (resp. $M^\vee_w$) denotes a Verma module in the principal block of category $\mathcal{O}$ for $g$ (resp. $g^\vee$). By the work of Soergel [125], the principal
blocks of category $\mathcal{O}$ for $\mathfrak{g}$ and $\mathfrak{g}^\vee$ are equivalent, and the isomorphism of Ext-groups follows.

\[ \square \]

Question 13.5. Can this isomorphism be an indication of mirror symmetry between open Richardson varieties $\mathcal{R}_u^w \subset G/B$ and $\mathcal{R}_u^w \subset G^\vee/B^\vee$?

14. Proofs from Section 2.5

14.1. Proof of Proposition 2.7. (1) $\implies$ (2) and (1) $\implies$ (3) are easy to check directly.

We show that (2) $\implies$ (1). Suppose $G$ is simply-laced. The condition $\langle \alpha, \beta' \rangle \in \{-1, 0\}$ for all $\alpha \in R^+_P$ is equivalent to $s_\beta \in W^P$. Since $\alpha_i$ appears in the expansion of $\beta$ in terms of simple roots, by [128, Corollary 3.3] we have a reduced expression of the form $s_\beta = u_1^{-1}$. Furthermore, since $\alpha_i$ appears in $\beta$ with coefficient 1, the simple generator $s_i$ cannot occur in $u$. It follows that $s_\beta \in W^P$ only if $u = 1$, or equivalently, $\beta = \alpha_i$.

For $G$ not simply-laced, the result follows from a direct computation. (Use the explicit description of $\tilde{R}$ from [2.3])

We show that (3) $\implies$ (1). Suppose $G$ is simply-laced. Suppose $\beta = -w^{-1}(\theta) \in R^+ \setminus R^+_P$, but $\beta \neq \alpha_i$. Then $i$ is also cominuscule so $\beta = \alpha_i + \beta'$ where $\beta'$ is a nonzero linear combination of $\alpha_j$ for $j \neq i$. Since $w \in W^P$, we have $w \alpha_j \in R^+$ for $j \neq i$. Thus $w^2 - w \alpha_i \in \mathbb{Z}_{\geq 0} R^+ \setminus \{0\}$. Since $w \alpha_i$ is a root, it would be impossible for $w \beta = -\theta$.

Suppose $G$ is of type $B_n$. Choosing coordinates for $R$, we have $\theta = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_n = \epsilon_1 + \epsilon_2$. We may identify $W$ with the group of signed permutations on $\{1, 2, \ldots, n\}$, and $W^P$ is identified with signed permutations that are increasing, under the order $1 < 2 < \cdots < n < -n < -(n - 1) < \cdots < -1$. We have $|W^P| = 2^n$. For example, $w = (2, 4, 5, -3, -1) \in W^P$ and $w^{-1}(\epsilon_1 + \epsilon_2) = -\epsilon_5 + \epsilon_1$. It follows by inspection that $-w^{-1}(\theta) \in R^+ \setminus R^+_P$ implies that $-w^{-1}(\theta) = \epsilon_4 + \epsilon_5 = \alpha_{n-1} + 2\alpha_n$.

Suppose $G$ is of type $C_n$. We have $\theta = 2\alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{n-1} + \alpha_n$. The elements of $W^P$ are

$$1, s_1, s_2 s_1, s_3 s_2 s_1, \ldots, s_n s_{n-1} \cdots s_1, s_{n-1} s_n s_{n-1} \cdots s_2 s_1, \ldots, s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2 s_1,$$

and we have $|W^P| = 2n$. We have $-w^{-1}(\theta) \in R^+ \setminus R^+_P$ if and only if $w = s_1 s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2 s_1$, and the statement follows.

14.2. Proof of Proposition 2.8. We first note the following properties of $w_{P/Q}$.

Lemma 14.3.

1. $\text{Inv}(w_{P/Q}) = R^+_P \setminus R^+_Q$.
2. $\ell(w_{P/Q}) = -2\rho_P, \gamma^\vee$.
3. $\ell(w_{P/Q} s_\gamma) = \ell(w_{P/Q}) + \ell(s_\gamma) = (2(\rho - \rho_P), \gamma^\vee) - 1$.

Proof. Let $w_P$ (resp. $w_Q$) be the maximal element of $W_P$ (resp. $W_Q$). Then $w_{P/Q}w_Q = w_P$ is length-additive, so $\text{Inv}(w_{P/Q}) = w_Q(\text{Inv}(w_P) \setminus \text{Inv}(w_Q)) = R^+_P \setminus R^+_Q$, proving (1). Formula (2) follows from Lemma 2.6(2). Since $\text{Inv}(s_\gamma) \cap R^+_P = \emptyset$, it follows that the product $w_{P/Q} s_\gamma$ is length-additive. (3) follows from (2) and $\gamma \in \tilde{R}$. \[ \square \]

\footnote{In the simply-laced case, [128] considers roots to be short.}
14.3.1. **Proof of (1) in Proposition 2.8.** It is equivalent to show that $\text{Inv}(w) \supset \text{Inv}(s_{\gamma})$. Suppose $\alpha \in \text{Inv}(s_{\gamma})$. Then $\alpha - \langle \alpha, \gamma \rangle \gamma = s_{\gamma} \alpha < 0$, where $a = \langle \alpha, \gamma \rangle > 0$. Thus

$$-a\theta = aw(\gamma) = w(\alpha) - w(s_{\gamma} \alpha),$$

and it follows that $w\alpha < 0$ because $w(s_{\gamma} \alpha)$ is a root.

14.3.2. **Proof of (2) in Proposition 2.8.** After Lemma 14.3(1,3), it is equivalent to show that $\text{Inv}(ws_{\gamma}) \supset R_{P}^{+} \setminus R_{Q}^{+}$. Let $\alpha \in R_{P}^{+} \setminus R_{Q}^{+}$. Then $s_{\gamma} \alpha = \gamma + \alpha$ by Lemma 2.6. Thus $ws_{\gamma} \alpha = w\alpha + w\beta = w\alpha - \theta$. Since $\theta$ is the highest root, we deduce that $\alpha \in \text{Inv}(ws_{\gamma})$.

14.3.3. **Proof of (3) in Proposition 2.8.** Since $w_{P/Q}^{-1} \in W_{P}$, it suffices to show that $ws_{\gamma} \in W_{P}$. It suffices to check that $\text{Inv}(ws_{\gamma}) \cap R_{Q}^{+} = \emptyset$. But $s_{\gamma}$ fixes every element in $R_{Q}^{+}$, and $\text{Inv}(w) \cap R_{Q}^{+} = \emptyset$ since $w \in W_{P}$. The claim follows.

14.3.4. **Proof of (4) in Proposition 2.8.** The standard parabolic subgroup $J$ is given as follows: for type $A_n$, we have $J = \{2, 3, \ldots, n - 1\}$; for type $D_n$ or $E_n$, we have $J = [n] \setminus \{2\}$; for type $E_7$, we have $J = [n] \setminus \{1\}$; for type $B_n$, we have $J = [n] \setminus \{1, 2\}$; for type $C_n$, we have $J = \emptyset$. In all cases, it is clear that $W_{J}$ stabilizes $\theta$.

If $w, v \in W(\gamma)$ then clearly $wv^{-1}$ belongs to the stabilizer of $\theta$. In the simply-laced types, this stabilizer is exactly the group $W_{J}$. In type $B_n$, the stabilizer of $\theta$ is $W_{[n] \setminus \{1\}}$, but from the description in the proof of Proposition 2.7, it is clear that $wv^{-1} \in W_{J}$. In type $C_n$, as noted previously we have $W(\gamma) = \{s_{\gamma}\}$ consists of a single element.

The double coset $W_{J}wW_{P}$ contains a unique minimal element $w'$, and since $w \in W_{P}$, we have a length-additive factorization $w = uu'$, where $u \in W_{J}$. Since $w' \in W_{P}$ and $(w')^{-1}(\theta) = w^{-1}(\theta) = -\gamma$, we have $w' \in W(\gamma)$.

14.3.5. **Proof of final sentence in Proposition 2.8.** We assume that $w \in W_{P}$ satisfies $\text{Inv}(w) \supset \text{Inv}(s_{\gamma})$ and $\text{Inv}(ws_{\gamma}) \supset R_{P}^{+} \setminus R_{Q}^{+}$. Suppose first that $G$ is simply-laced, so that $\gamma = \alpha_{i}$. Suppose that $-w^{-1}(\theta) = \alpha \neq \alpha_{i}$. Let $w\alpha_{i} = -\eta < 0$. Since $w \in W_{P}$, we have $\alpha \notin R_{P}^{+}$. On the other hand, we have $w(\alpha - \alpha_{i}) = -\theta + \eta < 0$. Again because $w \in W_{P}$, this shows that $\alpha \notin R_{P}^{+}$. Thus $\alpha \in R_{-}$.

Let $\delta = -\alpha \in R_{P}^{+}$. Since $w\delta = \theta$ and $w \in W_{P}$, we have that $\delta + \lambda$ cannot be a root whenever $0 \neq \lambda \in \sum_{j \in P_{Z}} \mathbb{Z}_{\geq 0} \alpha_{j}$. If $\delta \in R_{P}^{+}$, it follows that $\delta \in R_{P}^{+} \setminus R_{Q}^{+}$. But then $s_{\delta} \delta = \delta + \gamma$ implies that $(ws_{\gamma}) \delta = w\delta + w\gamma > 0$ contradicts the assumption that $\text{Inv}(ws_{\gamma}) \supset R_{P}^{+} \setminus R_{Q}^{+}$.

Thus $\delta \in R_{P}^{+} \setminus R_{Q}^{+}$, and again since $w \in W_{P}$, we may assume that $\delta = \theta$. Thus $w\theta = \theta$, so $w$ lies in the stabilizer $W' \subset W$ of $\theta$. In types $E_6, E_7$, or $D_n$, $n \geq 4$, it is easy to see that $\text{Inv}(ws_{\gamma})$ for $w \in W'$ cannot contain $R_{P}^{+} \setminus R_{Q}^{+}$ since $W'$ is a parabolic subgroup that contains the minuscule node $i$, but does not contain the adjoint node (node 2 in types $D_n$ or $E_6$ and node 1 in type $E_7$). In type $A_n$, the whole claim is easy to check directly, and we conclude that $w \in W'(\gamma)$.

Suppose that $G$ is of type $B_n$. We use notation from the proof of Proposition 2.7. We have $s_{\gamma} = s_{\gamma-1} s_{\gamma} s_{\gamma-1}$ and for a signed permutation $w = w_{1} w_{2} \cdots w_{n} \in W_{P}$, we have $ws_{\gamma} = w_{1} w_{2} \cdots (-w_{n}) (-w_{n-1})$. Thus the first condition $\text{Inv}(w) \supset \text{Inv}(s_{\gamma})$ is equivalent to $w_{n-1}, w_{n} < 0$. The second condition $\text{Inv}(ws_{\gamma}) \supset R_{P}^{+} \setminus R_{Q}^{+}$ is equivalent to the condition that $\{w_{1}, w_{2}, \ldots, w_{n-2}\}$ are all bigger than $-w_{n}$ and $-w_{n-1}$ under the order $1 < 2 < \cdots < n < -(n-1) < \cdots < -1$. It follows that $w_{n-1} = -2$ and $w_{n} = -1$, so that $w\gamma = -\theta$.

Suppose that $G$ is of type $C_{n}$. Then $\ell(s_{\gamma}) \geq \ell(w)$ for $w \in W_{P}$ with equality if and only if $w = s_{\gamma} = s_{1} s_{2} \cdots s_{n-1} s_{n} s_{n-1} \cdots s_{2} s_{1}$. The claim follows easily.
15. Background on $D_h$-modules

15.1. Filtered and graded categories. Let $X$ be a smooth irreducible affine algebraic variety equipped with a $\mathbb{G}_m$-action. Its structure sheaf $O_X$ is naturally graded by $\mathbb{G}_m$-homogeneous sections. Denote by $p : T^*X \to X$ the cotangent bundle of $X$. Denote by $D_X$ the sheaf of differential operators on $X$. This is a sheaf of noncommutative rings. It is equipped with a natural filtration

$$\ldots \subset D_{X,-1} \subset D_{X,0} \subset D_{X,1} \subset \ldots$$

induced by the gradation of $O_{T^*X}$ plus the order of the differential operator. The filtration is admissible in the sense of [24]. Examples of admissible filtrations on $D_X$ include the Bernstein filtration, the filtration by the order of the differential operator which corresponds to a trivial $\mathbb{G}_m$-action, and the V-filtration of Kashiwara-Malgrange.

Let $MF(D_X,\bullet)$ denote the category of filtered left $D_X,\bullet$-modules that are quasi-coherent as $O_X$-modules. An object $M_\bullet \in MF(D_X,\bullet)$ is equipped with a filtration $\ldots \subset M_1 \subset M_0 \subset \ldots$ satisfying $D_{X,j}M_i \subset M_{i+j}$. The category $MF(D_X,\bullet)$ is an additive category but not an abelian category; it can be made into an exact category by declaring a sequence $0 \to M'_i \to M_i \to M''_i \to 0$ to be exact if $0 \to M'_i \to M_i \to M''_i \to 0$ is exact for all $i$. (This is stronger than asking for the sequence of underlying unfiltered $D$-modules to be exact.) As shown in [97], one can define the derived category of $MF(D_X,\bullet)$; we let $D^bF(D_X,\bullet)$ denote the bounded derived category of $MF(D_X,\bullet)$. There are natural forgetful functors $MF(D_X,\bullet) \to M(D_X)$ and $D^bF(D_X,\bullet) \to D^b(D_X)$ sending a filtered module $M_\bullet$ to the underlying $D_X$-module $M$, and a complex $M_\bullet$ of filtered modules to the underlying complex $M$.

The associated graded of $D_X,\bullet$ is the sheaf $\text{gr}D_X,\bullet = p_*O_{T^*X}$ of graded commutative rings on $X$, where the grading comes from the grading of $O_X$ together with the declaration that vector fields have degree one. Since $p$ is affine, we have equivalences of categories

$$M(O_{T^*X}) \cong M(p_*O_{T^*X}), \quad \text{and} \quad D^b(O_{T^*X}) \cong D^b(p_*O_{T^*X})$$

between the corresponding categories of quasi-coherent $O_{T^*X}$-modules and quasi-coherent $p_*O_{T^*X}$-modules, and bounded derived categories. We have an associate graded functor, and derived functor

$$\text{gr} : MF(D_X,\bullet) \to M(O_{T^*X}), \quad \text{and} \quad \text{gr} : D^bF(D_X,\bullet) \to D^b(O_{T^*X}).$$

Let $D_{h,X}$ denote the sheaf of graded noncommutative rings with a central section $h$, locally generated by $f \in O_X$ and sections $\xi \in \Theta_X$ of the tangent sheaf with the relations $[f, \xi] = h(\xi \cdot f)$ and $\xi \eta - \eta \xi = h[\xi, \eta]$. The grading is given by the assignment $\text{deg}(h) = 1$ and the homogeneous degrees of $f$ and $\xi$ induced by the $\mathbb{G}_m$-action. The sheaf $D_{h,X}/h$ is isomorphic to the sheaf $p_*O_{T^*X}$, while the localization $D_{h,X}$ at $h$ is isomorphic to $D_X[h^{\pm 1}]$. Let $MG(D_{h,X})$ denote the category of sheaves of graded left $D_{h,X}$-modules that are quasi-coherent as graded $O_X$-modules. To an object $M_\bullet \in MF(D_X,\bullet)$ we associate an object $M_\bullet \otimes \mathbb{C}[h] = M^h = \bigoplus M_i^h \in MG(D_{h,X})$ by defining $M_i^h = M_i$. The section $h$ acts by the identity, thought of as a map from $M_i^h$ to $M_{i+1}^h$. It is clear that $\otimes \mathbb{C}[h] : MF(D_X,\bullet) \to MG(D_{h,X})$ is an exact functor.

For the following result see [97, Section 7] [119, Section 4].

Proposition 15.2. The functor

$$\otimes \mathbb{C}[h] : MF(D_X,\bullet) \to MG(D_{h,X}), \quad M_\bullet \mapsto M^h_\bullet = M_\bullet \otimes \mathbb{C}[h]$$
is an equivalence between $\text{MF}(D_{X, \bullet})$ and the full subcategory of $h$-torsion free $D_{h, X}$-modules. It induces a derived functor $\otimes \mathbb{C}[h]$ giving an equivalence of categories

$$\otimes \mathbb{C}[h] : D^b F(D_{X, \bullet}) \cong D^b(D_{h, X}).$$

We also have a functor $\otimes \mathbb{C}[h] : \text{MG}(D_{h, X}) \to \text{M}(O_{T^* X})$ and a left derived functor $\otimes \mathbb{C}[h] C : D^b(D_{h, X}) \to D^b(O_{T^* X})$, setting $h = 0$.

**Proposition 15.3.** We have commutative diagrams

$$\begin{array}{ccc}
\text{MF}(D_{X, \bullet}) & \xrightarrow{\otimes \mathbb{C}[h]} & \text{MG}(D_{h, X}) \\
\text{gr} & \downarrow & \text{gr} \\
\mathbb{M}(O_{T^* X}) & \xrightarrow{\otimes \mathbb{C}[h]} & D^b(O_{T^* X}) \\
\end{array}$$

\[ D^b F(D_{X, \bullet}) \xrightarrow{\otimes \mathbb{C}[h]} D^b(D_{h, X}) \]

\[ D^b(D_{h, X}) \xrightarrow{\otimes \mathbb{C}[h]} D^b(O_{T^* X}) \]

**Example 15.4.** Consider $X = \mathbb{G}_m^n \times \mathbb{G}_m$, with coordinates $(x_1, \ldots, x_n, q)$, and equipped with the $\mathbb{G}_m$-action

$$\zeta \cdot (x_1, \ldots, x_n, q) = (\zeta x_1, \ldots, \zeta x_n, \zeta^{n+1} q).$$

The ring $\mathbb{C}[X]$ of Laurent polynomials has a corresponding gradation by homogeneous polynomials. The potential $f = x_1 + \cdots + x_n = x_1 \cdots x_n$ has degree one. The ring of differential operators $D_X$ is filtered by the subspaces $D_{X, i}$, which for each $i \in \mathbb{Z}$, are the linear span of the operators

$$x_1^{\alpha_1} \frac{\partial^{\beta_1}}{\partial x_1^{\beta_1}} \cdots x_n^{\alpha_n} \frac{\partial^{\beta_n}}{\partial x_n^{\beta_n}} q^i \frac{\partial^\delta}{\partial q^\delta}, \quad \alpha_1 + \alpha_2 + \cdots + \alpha_n + (n+1) \gamma - n \delta \leq i.$$

The $D_X$-module $D_X/D_X(d - df \wedge)$ that we denote $E^f$ is equipped with a natural filtration, and becomes an object of $\text{MF}(D_{X, \bullet})$. The ring $D_{h, X}$ is the graded noncommutative ring generated by functions and differential operators $\zeta_{x_k}, \zeta_q$ with notably the relations

$$[\zeta_{x_k}, x_k] = h, \quad [\zeta_q, q] = h.$$

The degrees are given by $\deg(x_k) = 1$, $\deg(\zeta_k) = 0$, $\deg(q) = n + 1$, $\deg(\zeta_q) = -n$, $\deg(h) = 1$. One can think $\zeta_{x_k}$ as representing $\frac{h \partial}{\partial x_k}$, and $\zeta_q$ as representing $\frac{h \partial}{\partial q}$. Applying the functor of Proposition 15.2 we have that $E^f \otimes \mathbb{C}[h]$ becomes the $D_{h, X}$-module $E^{f/h}$ which we can describe as follows. We define $E^{f/h}$ as the quotient of $D_{h, X}$ by the left ideal generated by the operators $\zeta_{x_k} - \frac{\partial f}{\partial x_k}$ and $\zeta_q - \frac{\partial f}{\partial q}$. The operators are all homogeneous, hence $E^{f/h}$ is an element of $\text{MG}(D_{h, X})$, and moreover it is $h$-torsion free.

**15.5. Pushforward functors.** In this and the next subsection only we write $\int_\pi$ to denote the pushforward functor for $D$-modules, and reserve $\pi_*$ for the pushforward functor of quasi-coherent sheaves. Let $\pi : X \to Y$ be a $\mathbb{G}_m$-equivariant morphism between complex irreducible smooth varieties $X$ and $Y$ equipped with $\mathbb{G}_m$-actions. We recall results concerning the pushforward functors of $D_X$, $D_{h, X}$, and $O_{T^* X}$-modules under $\pi$. Though we shall not need it, the functors of Proposition 15.3 are also compatible with pullbacks under $\pi$.

Let $\omega_X$ (resp. $\omega_Y$) denote the canonical line bundles of $X$ (resp. $Y$). The sheaf $\omega_X$ acquires a grading from the $\mathbb{G}_m$-action so that it becomes a filtered right $D_{X, \bullet}$-module. Define

$$D_{Y \leftarrow X} := \pi^{-1}(D_Y \otimes_{O_Y} \omega_Y^{-1}) \otimes_{\pi^{-1}O_Y} \omega_X$$
which is a \((\pi^{-1}D_Y, D_X)\)-bimodule on \(X\). The module \(D_{Y\leftarrow X}\) inherits a filtration from the filtrations of \(D_Y\), \(\omega_Y\), and \(\omega_X\). We obtain a filtered \((\pi^{-1}D_Y, D_X)\)-bimodule \(D_{Y\leftarrow X}\) on \(X\), satisfying \(\pi^{-1}D_{Y,j} \cdot D_{Y\leftarrow X,i} \cdot D_{X,k} \subset D_{Y\leftarrow X,i+j+k}\). We define the direct image functor by

\[
\int_{\pi} M := R\pi_* (D_{Y\leftarrow X} \otimes_{D_X} M)
\]

where \(M \in D^bF(D_X)\). Similarly define \(\int_{\pi} : D^b(D_X) \to D^b(D_Y)\) by forgetting filtrations.

**Proposition 15.6 ([97, (5.6.1.1)])**. The following diagram commutes:

\[
\begin{array}{ccc}
D^bF(D_X) & \xrightarrow{f_{\pi}} & D^bF(D_Y) \\
\downarrow & & \downarrow \\
D^b(D_X) & \xrightarrow{f_{\pi}} & D^b(D_Y)
\end{array}
\]

where the vertical arrows are the natural forgetful functors.

Let \(T^*Y \times_Y X\) be the pullback of the cotangent bundle \(T^*Y\) to \(X\), fitting into the commutative diagram [97 (5.0.1)]

\[
\begin{array}{ccc}
T^*X & \xrightarrow{\Pi} & T^*Y \times_Y X \\
\downarrow & & \downarrow \pi \\
T^*Y & \xrightarrow{p_Y} & Y
\end{array}
\]

Moreover, we have

\[\text{gr}D_{Y\leftarrow X} = \pi^* O_{T^*Y} \otimes_{O_X} \omega_{X/Y}\]

which has a natural structure of a graded \((\pi^* O_{T^*Y}, O_{T^*X})\)-bimodule. We now define a functor \(f_{\pi} : D^b(O_{T^*X}) \to D^b(O_{T^*Y})\) by

\[
(15.6.1) \quad \int_{\pi} M_0 := (R\pi_* \circ \Pi^! \langle d \rangle)(M_0)
\]

where \(d = \dim X - \dim Y\) and \(\Pi^! : D^b(O_{T^*X}) \to D^b(O_{T^*Y \times_Y X})\) denotes the upper-shriek functor on derived categories of quasi-coherent sheaves.

We will only use (15.6.1) when the map \(\pi : X \to Y\) is smooth, in which case, we have

\[
(15.6.2) \quad \Pi^! \langle d \rangle(-) = L\Pi^*(-) \otimes_{O_{T^*Y \times_Y X}} p_Y^* \omega_{X/Y}.
\]

where \(L\Pi^* : D^b(O_{T^*X}) \to D^b(O_{T^*Y \times_Y X})\) is the left derived functor of the usual pullback functor \(\Pi^*\) of quasi-coherent sheaves.

We have the following compatibility result of pushforwards.
Proposition 15.7 ([97 (5.6.1.2)]). The following diagram commutes:

\[
\begin{array}{c}
D^b F(D_{X,\bullet}) \xrightarrow{f_*} D^b F(D_{Y,\bullet}) \\
\downarrow \text{gr} \quad \downarrow \text{gr} \\
D^b (O_{T^* X}) \xrightarrow{f_*} D^b (O_{T^* Y})
\end{array}
\]

Finally, we describe the pushforward functor for \( D_{X,h} \)-modules. We define \( D_{Y,\leftarrow X,h} := D_{Y,\leftarrow X,\bullet} \otimes \mathbb{C}[h] \), which is a graded \( (\pi^{-1} D_{Y,h}, D_{X,h}) \)-bimodule. We define the direct image functor \( \int_{\pi} : D^b (D_{X,h}) \to D^b (D_{Y,h}) \) by

\[
\int_{\pi} \mathcal{M} := \mathcal{R} \pi_* (D_{Y,\leftarrow X,h} \otimes_{D_{X,h}} \mathcal{M}).
\]

Proposition 15.8. The following diagram commutes:

\[
\begin{array}{c}
D^b F(D_{X,\bullet}) \xrightarrow{f_*} D^b F(D_{Y,\bullet}) \\
\otimes \mathbb{C}[h] \quad \otimes \mathbb{C}[h] \\
D^b (D_{X,h}) \xrightarrow{f_*} D^b (D_{Y,h})
\end{array}
\]

Proof. A direct comparison shows that

\[
(D_{Y,\leftarrow X,h} \otimes_{D_{X,h}} \mathcal{M}) \otimes \mathbb{C}[h] = D_{Y,\leftarrow X,h} \otimes_{D_{X,h}} (\mathcal{M} \otimes \mathbb{C}[h])
\]

as graded \( \pi^{-1}(D_{Y,h}) \)-modules. Similarly, \( \otimes \mathbb{C}[h] \) is an exact functor, so it commutes with \( \mathcal{R} \pi_* \). □

Proposition 15.9. The following diagram commutes:

\[
\begin{array}{c}
D^b (D_{X,h}) \xrightarrow{f_*} D^b (D_{Y,h}) \\
\otimes_{\mathbb{C}[h]} \otimes_{\mathbb{C}[h]} \\
D^b (O_{T^* X}) \xrightarrow{f_*} D^b (O_{T^* Y})
\end{array}
\]

where the vertical arrows are the natural forgetful functors.

Proof. Combine Proposition 15.3 with Propositions 15.7 and 15.8. □

Example 15.10. Consider \( Y = \mathbb{G}_m \), graded by \( \text{deg}(q) = n + 1 \). The ring \( D_{h,Y} = \mathbb{C}[\hat{q}^\pm, \hat{h}] \langle \xi_q \rangle \) satisfies the relation \( [\xi_q, q] = \hat{h} \), and the gradation is given by \( \text{deg}(\xi_q) = -n \), \( \text{deg}(\hat{h}) = 1 \). The quantum differential operator \( (q \xi_q)^{n+1} - q \) is homogeneous of degree \( n + 1 \). It shall follow from the next subsection that it is isomorphic to the pushforward \( \pi_* E^{1/h} \).
15.11. **Application to the character $D_h$-module.** Let $\pi : X \to Z(L_P)$ denote the geometric crystal and $f : X \to \mathbb{A}^1$ denote the superpotential. Recall that we defined $\mathbb{G}_m$-actions on $X$ and $Z(L_P)$ in \[6.17\].

**Proposition 15.12.** The character $D_{h,Z(L_P)} \otimes S$-module $C_{r(G,P)}(S,h) \in D^b(D_{h,Z(L_P)} \otimes S)$ is a $h$-torsion free $D_{h,Z(L_P)} \otimes S$-module concentrated in a single degree.

**Proof.** To simplify the notation we will prove the proposition without $S$. Thus let

$$M^h = D_{h,X}/(\xi - (\xi \cdot f))$$

denote the cyclic $D_{h,X}$-module generated by a single section $e^{f/h}$. Here $\xi \in \Gamma(X, \Theta_X)$ denotes a vector field on $X$. We shall show that $N^h := \int_\pi M^h \in D^b(D_{h,Z(L_P)})$ is isomorphic to a $h$-torsion free $D_{Z(L_P),h}$-module concentrated in a single degree. The condition that $N^h$ is $h$-torsion free and concentrated in one cohomological degree is equivalent to the condition that the object $N_0 = N^h L \mathbb{C}[\hbar] \subset D^b(O_{T^*Z(L_P)})$ (see \[15.1\]) is concentrated in a single cohomological degree.

Let $M_0 = M^h \otimes_{\mathbb{C}[\hbar]} \mathbb{C} \in M(O_{T^*X})$. Then $M_0$ isomorphic to $O_V$, where $V \subset T^*X$ is cut out by the equations $\xi - (\xi \cdot f)$. By Proposition \[15.9\] we have $N_0 = \int_\pi M_0$. Denote $T^*Z(L_P) \times_{Z(L_P)} X$ by $W$. By \[15.6.1\] and \[15.6.2\], we have

$$\int_\pi M_0 = R\pi'_* (LF^*(M_0) \otimes_{O_W} \tilde{\pi}^* \omega_{X/Z(L_P)})$$

where $\tilde{\pi} : W \to X$ and $\pi' : W \to T^*Z(L_P)$ are the two projections and $F : W \to T^*X$ is the natural inclusion. We first show that $LF^*(M_0) \in D^b(O_W)$ is concentrated in a single cohomological degree. This is equivalent to the condition that $\text{Tor}^i_{O_{T^*X}}(O_W, O_V) = 0$ for $i > 0$. It is easy to see that both $V$ and $W$ are smooth subvarieties of $T^*X$, and hence Cohen-Macaulay.

The fiber of $W \cap V$ under $W \to T^*Z(L_P) \to Z(L_P)$ over a point $q \in Z(L_P)$ can be identified with the critical point set of $f|_{\pi^{-1}(q)}$. Rietsch \[117\] showed that this critical point set is 0-dimensional, and it follows that $W \cap V$ is pure of dimension 1. Since $\dim V = \dim X$ and $\dim W = \dim X + 1$, it follows that the intersection $W \cap V$ is proper.

If $\text{Tor}^i_{O_{T^*X}}(O_W, O_V)$ is nonzero then it is nonzero after localizing to some irreducible component of $C$ of $W \cap V$. Applying \[120\] V.6, Corollary] we obtain $\text{Tor}^i_{O_{T^*X}}(O_{W,C}, O_{V,C}) = 0$ for all $i > 0$, where $O_{T^*X,C}$ (resp. $O_{W,C}, O_{V,C}$) denotes the localization. Thus $\text{Tor}^i_{O_{T^*X}}(O_W, O_V) = 0$ for $i > 0$ and we deduce that $LF^*(M_0) = 0$ for $i > 0$. Since $\pi : X \to Z(L_P)$ is affine, the map $\pi'$ is also affine, so $R\pi'_*(F^*(M_0))$ is concentrated in a single degree. \hfill $\square$

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