The q-Deformed NJL Model ”Revisited ”

S.S. Avancini, J.R. Marinelli, D.P.Menezes and M.M. Watanabe de Moraes

Depto de Física - CFM - Universidade Federal de Santa Catarina
Florianópolis - SC - CP. 476 - CEP 88.040 - 900 - Brazil

Abstract

In this work we investigate the chiral symmetry breaking in the $q$–deformed version of the NJL model and its consequent mass generation mechanism. We show that the deformation of the NJL model, in the mean field approximation, may take into account correlations that go beyond the mean field and, in a certain limit, approaches the more realistic lattice calculations.

PACS number(s):11.30.Rd: 03.65.Fd: 12.40.-y

Quantum algebras, or simply $q$-deformed algebras, known in the case where just a single deformation parameter is introduced, have received some attention last years in the study of many-body problems. Because they can provide us with a class of symmetries that is richer than the usual Lie algebras, they are appropriate to describe physical systems and (or) models which are not properly described by the last ones. On the other hand, the introduction of the $q$–parameter in the theory can be viewed as a way to take into account correlations in many-body systems. As examples, we mention a set of works [1], [2] related with the deformation of the su(2) algebra (also called su$_q$ (2) algebra) in a time dependent Hartree–Fock (TDHF) approximation to solve the Lipkin model. Following similar reasonings, the quon algebra, which describes particles whose statistics interpolates between the bosonic and fermionic ones [3], was then used to perform a $q$–deformed boson analysis of the random–phase–approximation (RPA) solution for the Lipkin and the two-level pairing models [4]. In that case there is a strong indication that the introduction of correlations through the deformation of the RPA bosons can also be interpreted as a restoration of a broken symmetry caused by using the regular (approximated) RPA solution. This is manifested in the smearing out of the phase transition at the so-called critical point [5].
Recently, the dynamical breaking of chiral symmetry in the Nambu-Jona-Lasinio (NJL) model \[6\] and the corresponding generation of a dynamical mass for quarks, were analyzed at the light of a $q$-deformed version of that model \[7\]. Once $q$-deformation appears as a powerful tool to study symmetries and their generalizations, this seems to be a quite appealing investigation. For that purpose, a particular prescription for the deformation was chosen, based on reference \[8\] and two important conclusions of that analysis were drawn: the phase transition of the NJL model continues to be sharp and occur at the same critical interaction point, when compared with the non–deformed case, which also means that any possible explicit chiral symmetry breaking terms in the Lagrangian cannot be simulated by the deformation procedure used; the main modification caused by the deformation is an enhancement of the condensate, i.e., the quark mass is effectively increased for a given value of the interaction constant.

In the present work, it is our intention to proceed further with the analysis introduced in reference \[7\]. We propose a method to solve the NJL model based on a Hamiltonian density written in terms of the $su(2)$ operators. As we show bellow, this is in fact possible for the NJL model and corresponds to a $J=1$ angular momentum algebra \[9\], once we are restricted to the well-known BCS type ansatz for the variational solution in the Hartree approximation. With that result in hand, it is then straightforward to build an $su_q(2)$ version of the NJL model. This has the advantage that it is then possible to exploit different $q$-deformation schemes \[10\]. Moreover, this procedure allows us to generalize some previously obtained results and to explore all features introduced by $q$-deforming the model.

We start with the Lagrangian density of the NJL model \[3\]

$$L = \overline{\psi} i \partial_\mu \gamma^\mu \psi + G \left( (\overline{\psi} \psi)^2 + (\overline{\psi} i \gamma_5 \tau \psi)^2 \right),$$

(1)

where $\psi$ represents the quark field amplitude, the second term represents the interaction mechanism and $G$ is the coupling constant. From (1), the corresponding Hamiltonian density can be obtained and reads:

$$\mathcal{H} = -i \overline{\psi} \gamma \cdot \nabla \psi - G \left( (\overline{\psi} \psi)^2 - (\overline{\psi} i \gamma_5 \tau \psi)^2 \right).$$

(2)

As it is well known, a solution for the above problem can be found in the Hartree approximation, using a BSC-like variational state \[11\] of the form:

$$|NJL> = \prod_{ps} \left( \cos \theta(p) - s \sin \theta(p) b^\dagger(p, s) d^\dagger(-p, s) \right) |0>,$$

(3)
where \( \mathbf{p} \) is the vector momentum, \( s = \pm 1 \) are the helicity state eigenvalues and the particle and anti–particle operators are defined such that

\[
b(\mathbf{p}, s) |0\rangle = 0, \quad d(\mathbf{p}, s) |0\rangle = 0.
\]

In terms of the creation and annihilation operators, we expand the field operators at \( t=0 \), yielding

\[
\psi(x, 0) = \sum_s \int \frac{d^3p}{(2\pi)^3} \left( b(\mathbf{p}, s) u(\mathbf{p}, s) e^{i\mathbf{p} \cdot \mathbf{x}} + d(\mathbf{p}, s) v(\mathbf{p}, s) e^{-i\mathbf{p} \cdot \mathbf{x}} \right),
\]

where \( u(\mathbf{p}, s) \) and \( v(\mathbf{p}, s) \) are the normalized spinor eigenfunctions for particles and anti–particles with momentum \( \mathbf{p} \) and helicity \( s \).

At this point, we introduce the following angular momentum operators:

\[
J_{+, \mathbf{p}} = \sqrt{2} A^\dagger_{\mathbf{p}}, \quad J_{-, \mathbf{p}} = \sqrt{2} A_{\mathbf{p}}, \quad J_{0, \mathbf{p}} = \frac{N_p}{2} - 1,
\]

where

\[
A^\dagger_{\mathbf{p}} = \frac{1}{\sqrt{2}} [b^\dagger(\mathbf{p}, +) d^\dagger(-\mathbf{p}, +) - b^\dagger(\mathbf{p}, -) d^\dagger(-\mathbf{p}, -)],
\]

\[
N_p = \sum_s [b^\dagger(\mathbf{p}, s) b(\mathbf{p}, s) + d^\dagger(\mathbf{p}, s) d(\mathbf{p}, s)],
\]

the operators \( J_{+, \mathbf{p}}, J_{-, \mathbf{p}} \) and \( J_{0, \mathbf{p}} \) obey the usual \( \text{su}(2) \) commutation relations and

\[
(J_{+, \mathbf{p}})^n |0\rangle = 0, \quad \text{if } n > 2, \quad J_{-, \mathbf{p}} |0\rangle = 0, \quad J_{0, \mathbf{p}} |0\rangle = -|0\rangle.
\]

The variational ansatz defined in equation (8) can then be written as:

\[
|NJL\rangle = \mathcal{N} e^{-\frac{1}{2} \sum_p \xi_p A_p^\dagger} |0\rangle,
\]

with

\[
\mathcal{N} = \frac{1}{\prod_p (1 + |\xi_p|^2/2)},
\]

where \( \xi_p = \sqrt{2} \tan \theta(\mathbf{p}) \). In terms of those operators, the hamiltonian (2) (in the Hartree approximation) can be put in the form:

\[
H = \int d^3x \mathcal{H} = \int \frac{d^3p}{(2\pi)^3} \left\{ p^2 J_{0, \mathbf{p}} + M (J_{+, \mathbf{p}} + J_{-, \mathbf{p}}) \right\},
\]

where \( M \) represents the mass of the arising condensate.
We now minimize the mean value of the Hamiltonian with the above NJL state, using $\theta(p)$ as the variational function and end up with the well known NJL solution for the condensate, i.e., $\langle \bar{\psi} \psi \rangle = -M/(2G)$. At this point we introduce the deformation in the model. Once the algebraic $su(2)$ structure above underlies the model, there exists a direct procedure to extend the solution to the $su_q(2)$ formalism \[10\]. The $q$-deformed angular momentum operators obey the following commutation relations:

$$[J_{0,p}, J_{\pm,p}] = \pm J_{\pm,p} \quad [J_{+,p}, J_{-,p}] = (qr)^{J-J_0,p} [2J_{0,p}],$$

(11)

where $J = \sqrt{J_{+,p}^2 + J_{-,p}^2 + J_{0,p}^2}$ and $[X]$ represents the deformed version of the operator (or c-number) $X$. Its definition is not unique. Throughout this paper we use the following definition \[12\]:

$$[X] = q^X - r^{-X} \quad q - r^{-1},$$

(12)

and only cases where just one deformation parameter is introduced is considered, namely, $r = q$ and $r = 1$, respectively. Note that in the first case ($r = q$), $q$ can be either a real or a complex number ($q = e^{i\tau}, \tau$ being real), and in the second one ($r = 1$) it must be real. To proceed with the deformed solution we introduce a $q$-deformed variational ansatz, which is:

$$|NJL>_q = \mathcal{N}_q e_q^{\sum \frac{\xi_p}{2} J_{+,p}} |0>,$$

(13)

where

$$\mathcal{N}_q = \frac{1}{\prod_p \left(1 + \frac{[\xi_p]^2}{2} + (\frac{[\xi_p]^2}{2})^2\right)}$$

(14)

and the $q$-exponential is defined as $e_q(ax) = \sum_{n=0}^{\infty} \frac{a^n}{[n]!} x^n$, with $[n]! = [n][n-1].....[2][1]$.

Again, we minimize the mean value of the Hamiltonian \[11\], using the variational state defined in \[13\] and bearing in mind that the angular momentum operators obey the deformed commutation rules. After a straightforward calculation we obtain the modified gap equation:

$$M = 4G[2] \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{\tan \theta(p)(1 + \tan^2 \theta(p))}{(1 + [2] \tan^2 \theta(p) + \tan^4 \theta(p))} \right\},$$

(15)

where $\theta(p)$ is a solution of the equation

$$\tan^6 \theta(p) + \frac{2p}{M} \tan^5 \theta(p) + (3 - [2]) \tan^4 \theta(p) + \frac{8p}{[2]M} \tan^3 \theta(p)$$
\[-(3 - [2]) \tan^2 \theta(p) + \frac{2p}{M} \tan \theta(p) - 1 = 0.\]

At this point it is useful to make some comments about the above equations. Our choice for a \(q\)-deformed exponential in the NJL ansatz is somewhat arbitrary, in the sense that we could have introduced the deformation just by modifying the \(\text{su}(2)\) commutation rules. If we had chosen to write the variational state \([13]\) in terms of a regular exponential function, our results would reduce to:

\[
M^* = 4G^* \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{M^*}{\sqrt{p^2 + (M^*)^2}} \right\},
\]

(16)

where \(M^* = \sqrt{\frac{p^2}{2}} M\) and \(G^* = G[2]\). As \([2] \to 2\) when \(q \to 1\), this last result also makes clear that we reproduce the right limit for the gap equation \([11]\). We have verified numerically that the difference between the results obtained from the gap equations (15) and (16) is negligible, unless we investigate the behavior of the dynamical mass for \(q\) values very far from 1, which is not the case in the present study. Thus, the introduction of the deformed exponential, instead of the regular one, in the variational ansatz, does not affect the results discussed next.

In figures 1a and 1b we show the condensate as a function of the inverse coupling constant for the two kinds of deformation considered here, i.e., \(r = q\) and \(r = 1\) respectively. In the first case, \(q\) can acquire any complex values, but in the second one it must be real. We have chosen two values in each case in order to show the main modifications introduced by the deformation. In figure 1a, for \(r = q\), we see that for \(q\) real we obtain an increase of the condensate, in agreement to what was found in \([7]\), and for a complex \(q\) the condensate decreases, in relation to the condensate obtained for \(q = 1\). This is not surprising, once according to \([15]\) and \([16]\), complex \(q\)-values mimic a repulsive interaction while real ones play the role of an attractive interaction. A similar effect can be seen when \(r = 1\), for which only real \(q\)-values are allowed. The results are displayed in figure 1b. In that case, however, for \(q < 1\) a repulsive effect is simulated, while for \(q > 1\) an additional attraction appears. In all cases we notice that the critical value for the coupling constant depends on the deformation parameter. It can be shown that the phase-transition point is given by \(G_c^* = \frac{3G_c}{[2]}\), with \(G_c = \frac{s^2}{\Lambda^2}\) where \(\Lambda\) is the well-known NJL cutoff. Throughout this paper we have used \(\Lambda = 600\text{MeV}\). This is a feature not present in the result obtained in \([5]\) and we return to this point latter.

It is also worth mentioning that there is a certain degree of ambiguity in the way a
physical system can be $q$-deformed. This problem has already been extensively discussed in the literature [13, 14]. In this way, equation (10) can be substituted by

$$H = \int d^3x \mathcal{H} = \int \frac{d^3p}{(2\pi)^3} \{p [2 J_{0,p}] + M (J_{+,p} + J_{-,p}) \}.$$  

(17)

before the minimization procedure is performed within the deformation scheme. As the only modification refers to the way we write the kinetic term, i.e., we replace $2J_0$ by $[2J_0]$, the corresponding results are called the deformed kinetic (DK) results and the previous results obtained by minimization of the Hamiltonian (10) are called non-deformed kinetic results (NDK). A comparison between the two approaches is shown in figures 2a and 2b, for $r = q$ and $r = 1$ respectively. In figure 2a, we can see that the DK results simply turn off the $q$ dependence in $G_c$, yielding the same behavior obtained in reference [7]. However, for $r = 1$ the DK results can lead to strong quantitative modifications, specially in what concerns the phase transition point, as is made clear for $q = 2$ in figure 2b.

In order to get some insight on the physical background embedded in the above results, we have decided to make a qualitative comparison with more elaborate calculations in the NJL model [17]. In that work, a lattice Monte Carlo simulation and a Schwinger-Dyson (S-D) calculation are applied to obtain the condensate and compared with the usual gap equation solution. The first important result that emerges is the fact that the value of the critical coupling constant ($G_c$) depends on the approximation used. The better the approximation, the bigger the critical coupling constant. Moreover, both the Monte Carlo simulation and the S-D method gives a smaller value for the condensate for any given value of the coupling constant, compared with the gap equation result, as can be seen in figure 3 of [17]. These two features can be simultaneously reproduced by our $q$-deformed NJL calculation for a complex $q$-value ($r = q$) or for $q < 1$ ($r = 1$). In figures 3a and 3b we plot the condensate as a function of $G/G_c$ for a better comparison with the results in [17].

In summary, we have introduced deformation in the NJL model by means of the angular momentum operators, which are the generators of the $su_q(2)$ algebra. The condensate was then obtained by minimizing the deformed Hamiltonian with the help of a BCS ansatz for the $q$ variational state. The results show that the phase transition is never suppressed, but the behavior of the condensate depends on the definition of the deformed quantity and on the numerical value of $q$. Two different prescriptions to deform the Hamiltonian and two different ways of introducing the $q$-deformed variational state were used. A qualitative comparison was made with calculations that go beyond the familiar Hartree solution.
We can obtain the same behavior as these sophisticated calculations for any $q$-value which simulates a decrease in the residual attractive interaction. Our calculations give rise to a $q$ dependence in the critical strength $G_c$. This dependence is consistent with the repulsive character that can be mimiced by the deformation and goes to the right direction as compared with realistic calculations for the condensate.

Acknowledgments

This work was partially supported by CNPq - Brazil.

References

[1] S. S. Avancini, D. P. Menezes, M. M. W. de Moraes and F. F. de Souza Cruz, J. Phys. A27 (1994) 831.

[2] L. Brito, C. Providência, J. Providência, S.S. Avancini, F.F. de Souza Cruz, D.P. Menezes and M.M. Watanabe de Moraes, Phys. Rev. A52 (1995) 92.

[3] S.S. Avancini, F.F. de Souza Cruz, J.R. Marinelli and D.P. Menezes, Phys. Lett. A 267 (2000) 109.

[4] S. S. Avancini, F. F. de Souza Cruz, J. R. Marinelli and D. P. Menezes, Phys. Rev. C62 (2000) 024312.

[5] P. Ring and P Schuck, The Nuclear Many-Body Problem, Springer-Verlag(1980) New York.

[6] Y. Nambu and G. Jona-Lasinio, Phy. Rev. 122 (1961) 345.

[7] V. S. Timóteo and C. L. Lima, Phys. Lett. B448 (1999) 1.

[8] M. Ubriaco, Phys. Lett. A219 (1996) 205.

[9] T. Hatsuda and T. Kunihiro, Phys. Rep. 247 (1994) 221.

[10] D. Bonatsos and C. Daskaloyannis, Prog. in Part. and Nucl. Phys. 43 (1999) 537.

[11] S.P. Klewansky, Rev. Mod. Phys. 64 (1992) 2947.
[12] M.R. Kibler, Introduction to Quantum Algebras, Second International School on Theoretical Physics, Poland (1992), World Scientific.

[13] D. Bonatsos, S.S. Avancini, D.P. Menezes and C. Providência, Phys. Lett. A 192 (1994) 192; F.W. Fávero, L.O.E. dos Santos and D.P. Menezes, Int. J. Mod. Phys. E 3 (1995) 547.

[14] E.G. Floratos, J. Phys. A 24 (1991) 4739.

[15] D. Bonatsos, L. Brito, D.P. Menezes, C. Providência and J. Providência, J. Phys. A 26 (1993) 895; Corrigendum J. Phys. A 26 (1993) 5185.

[16] S.S. Sharma and N.K. Sharma, Phys. Rev. C 50 (1994) 2323.

[17] A. Ali Khan, M. Göckeler, R. Horsley, P.E. L. Rakow, G. Schierholz and H. Stüben, Phys. Rev. D 51 (1995) 3751.
Fig. 1a. The condensate as a function of $G_c/G$, for $r=q$ in the deformed quantity definition.
**fig 1b.** Same as fig. 1a for $r=1$. 

Condensate ($10^{-3}$ GeV$^3$) vs $G_c/G$. Lines represent different values of $q$: 
- **solid line**: $q=1.0$
- **dashed line**: $q=0.5$
- **dotted line**: $q=1.5$
fig2a. Same as fig. 1a for the deformed kinetic (KD) operator and for the non deformed kinetic (NDK) operator.
Condensate ($10^{-3}$ GeV$^3$) vs. $G_c/G$.

- $q=1.0$
- $q=0.8$ DK
- $q=0.8$ NDK
- $q=2.0$ NDK
- $q=2.0$ DK

**fig. 2b.** Same as fig. 2a with $r=1$. 
\( q = 1.0 \)

\( \tau = 0.7 \)

\( q = 2.0 \)

**fig3a.** The condensate as a function of \( G/G_c \) for \( r=q \).
fig. 3b. Same as fig. 3a with $r=1$. 

Condensate ($x10^{-3}$ GeV$^3$)

$q=1.0$
$q=1.5$
$q=0.5$