Hypomonotonicity of the normal cone and proximal smoothness

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Definition

A set $A \subset X$ is said to be proximally smooth with constant $R$ if the distance function $x \to \rho(x, A)$ is continuously differentiable on set $U(R, A) = \{x \in X : 0 < \rho(x, A) < R\}$.

We denote by $\Omega_{PS}(R)$ the set of all closed proximally smooth sets with constant $R$ in $X$. 
Proposition 1.

Let $A$ be a closed set in a Hilbert space $H$ and $R > 0$. The following conditions are equivalent

1. the set $A \in \Omega_{PS}(R)$;
2. for any vectors $x_1, x_2 \in A$, $p_1 \in N(x_1, A)$, $p_2 \in N(x_2, A)$ such that $\|p_1\| = \|p_2\| = 1$, the following inequality holds

$$\langle p_2 - p_1, x_2 - x_1 \rangle \geq -\frac{\|x_2 - x_1\|^2}{R}.$$ 

where

$$N(a_0, A) = \{p \in X^* : \forall \varepsilon > 0 \exists \delta > 0 : \forall a \in A \cap \mathcal{B}_\delta(a_0) \langle p, a - a_0 \rangle \leq \varepsilon \|a - a_0\|\}.$$
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Question

Are the conditions 1) and 2) of Proposition 1 equivalent in an arbitrary Banach space?
Definition

The function $\delta_X(\cdot) : [0, 2] \to [0, 1]$ is referred to as the modulus of convexity

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : x, y \in \mathcal{B}_1(o), \|x - y\| \geq \varepsilon \right\}.$$
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Normed space $X$ is called uniformly convex, if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. 
**SPECIAL BANACH SPACES**

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**Definition**

The function $\rho_X : [0, +\infty) \rightarrow \mathbb{R}$ is referred to as the modulus of smoothness

$$\rho_X(\tau) = \sup \left\{ \frac{\|x+y\|+\|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}.$$
**Definition**

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Normed space \( X \) is called **uniformly smooth**, if \( \lim_{\tau \to +0} \frac{\rho_X(\tau)}{\tau} = 0 \).
Proposition 2.

Let $X$ be a uniformly convex and uniformly smooth Banach space. Let $\rho_X(\tau) \asymp \tau^2$ as $\tau \to 0$. Then the proximally smooth set $A \subset X$ with constant $r > 0$ satisfies condition 2) of Proposition 1 for some constant $R > 0$.

Proposition 3.

Let the convexity and smoothness moduli be of power order at zero in the Banach space $X$. Let $\delta_X(\varepsilon) \asymp \varepsilon^2$ as $\varepsilon \to 0$. Then, if the set $A$ satisfies condition 2) of Proposition 1, it is proximally smooth with some constant $r > 0$.

Let $f$ and $g$ be two non-negative functions, each one defined on a segment $[0, \varepsilon]$. We shall consider $f$ and $g$ as equivalent at zero, denoted by $f(t) \asymp g(t)$ as $t \to 0$, if there exist positive constants $a, b, c, d, e$ such that $af(bt) \leq g(t) \leq cf(dt)$ for $t \in [0, \varepsilon]$. 
Definition

Let a function $\psi : [0, +\infty) \to [0, +\infty)$ be given. The set $A \subset X$ satisfies the $\psi$-hypomonotonicity condition with constant $R > 0$ if for some $\varepsilon > 0$ and for any $x_1, x_2 \in A$, $p_1 \in N(x_1, A)$, $p_2 \in N(x_2, A)$, $\|p_1\| = \|p_2\| = 1$ such that $\|x_1 - x_2\| \leq \varepsilon$, the inequality

$$\langle p_2 - p_1, x_2 - x_1 \rangle \geq -R\psi\left(\frac{\|x_2 - x_1\|}{R}\right)$$

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Through $\Omega^\psi_N(R)$ we denote the class of all closed sets $A \subset X$ that satisfy the $\psi$-hypomonotony condition with constant $R > 0$. 
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Through $\Omega_N^\psi(R)$ we denote the class of all closed sets $A \subset X$ that satisfy the $\psi$-hypomonotony condition with constant $R > 0$.

**Definition**

Through $\mathcal{M}$ denote the class of convex functions $\psi : [0, +\infty) \to [0, +\infty)$ such that $\psi(0) = 0$. 
Theorem 1.

In a uniformly convex and uniformly smooth Banach space $X$ the following statements are equivalent for the function $\psi \in \mathcal{M}$:

1. there exists $k_1 > 0$ such that $\Omega_{PS}(R) \subset \Omega_{N}^{k_1\psi}(R)$ for any $R > 0$;
2. $\rho_{X}(\tau) = O(\psi(\tau))$ as $\tau \to 0$.

We shall say that function $x(\tau)$ is big-$O$ of function $y(\tau)$, and write $x(\tau) = O(y(\tau))$ as $\tau \to 0$, if the following inequality holds

$$|x(\tau)| \leq A|y(\tau)| \quad \forall \tau \in [0, \varepsilon] \quad \text{(for some } \varepsilon > 0, A > 0).$$
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Lemma 1.

In a uniformly smooth and uniformly convex Banach space $X$ the inclusion

$$(X \setminus \text{int } \mathcal{B}_1(0)) \in \Omega_{N}^{\frac{1}{17} \rho_{X}(\cdot)}(1)$$

holds.
Definition

We say that the function $N : [0, +\infty) \to [0, +\infty)$ such that $N(0) = 0$, satisfies the Figiel condition if there exists a constant $K$ such that the function $N(\cdot)$ on some interval $(0, \varepsilon)$ satisfies the condition

$$\frac{N(s)}{s^2} \leq K \frac{N(t)}{t^2} \quad \forall 0 < t \leq s < \varepsilon.$$
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Remark 1.

The modulus of smoothness of an arbitrary Banach space satisfies the Figiel condition

Definition

Through $\mathcal{M}_2$ denote the class of functions from $\mathcal{M}$ that satisfy the Figiel condition.
Theorem 2.

In a uniformly convex and uniformly smooth Banach space $X$ the following statements are equivalent for the function $\psi \in \mathcal{M}_2$:

1. there exists $k_2 > 0$ such that $\Omega_{N}^{k_2\psi}(R) \subset \Omega_{PS}(R)$ for any $R > 0$;
2. $\psi(\varepsilon) = O(\delta_X(\varepsilon))$ as $\varepsilon \to 0$. 
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In a uniformly convex and uniformly smooth Banach space $X$ the following statements are equivalent for the function $\psi \in \mathcal{M}_2$:

1. there exists $k_2 > 0$ such that $\Omega_{N}^{k_2 \psi}(R) \subset \Omega_{PS}(R)$ for any $R > 0$;
2. $\psi(\varepsilon) = O(\delta_X(\varepsilon))$ as $\varepsilon \to 0$.

Theorem 3.

Suppose in a Banach space $X$ for some function $\psi \in \mathcal{M}$ there exist $k_1 > 0$, $k_2 > 0$ such that the inclusions

$$\Omega_{N}^{k_1 \psi}(R) \subset \Omega_{PS}(R) \subset \Omega_{N}^{k_2 \psi}(R)$$

hold. Then $\delta_X(\varepsilon) \asymp \rho_X(\varepsilon) \asymp \varepsilon^2$ as $\varepsilon \to 0$, and, therefore, the space $X$ is isomorphic to a Hilbert space.
Hypothesis 1.

The equality $\Omega_{PS}(R) = \Omega_{N}^{\psi}(R)$ holds only in a Hilbert space provided that $\psi(t) = t^2$. 
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The equality $\Omega_{PS}(R) = \Omega_{\psi}^N(R)$ holds only in a Hilbert space provided that $\psi(t) = t^2$.

Hypothesis 2.

If in a uniformly convex and uniformly smooth Banach space $X$ the set $A \subset X$ belongs to the class $\Omega_{PS}(R)$ and to the class $\Omega_{\psi}^N(r)$ for some function $\psi \in \mathcal{M} \setminus \mathcal{M}_2$ and constants $R > 0, r > 0$, then it belongs to the class $\Omega_{\psi_1}^N(r)$, where $\psi_1(t) = ct^2$ for some $c \geq 0$. 
