ACCELERATION AND COLLIMATION OF RELATIVISTIC MAGNETOHYDRODYNAMIC DISK WINDS

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Received 2009 October 16; accepted 2009 December 16; published 2010 January 11

ABSTRACT

We perform axisymmetric relativistic magnetohydrodynamic simulations to investigate the acceleration and collimation of jets and outflows from disks around compact objects. Newtonian gravity is added to the relativistic treatment in order to establish the physical boundary condition of an underlying accretion disk in centrifugal and pressure equilibrium. The fiducial disk surface (respectively a slow disk wind) is prescribed as boundary condition for the outflow. We apply this technique for the first time in the context of relativistic jets. The strength of this approach is that it allows us to run a parameter study in order to investigate how the accretion disk conditions govern the outflow formation. Substantial effort has been made to implement a current-free, numerical outflow boundary condition in order to avoid artificial collimation present in the standard outflow conditions. Our simulations using the PLUTO code run for 500 inner disk rotations and on a physical grid size of 100 × 200 inner disk radii. The simulations evolve from an initial state in hydrostatic equilibrium and an initially force-free magnetic field configuration. Two options for the initial field geometries are applied—an hourglass-shaped potential magnetic field and a split monopole field. Most of our parameter runs evolve into a steady state solution which can be further analyzed concerning the physical mechanism at work. In general, we obtain collimated beams of mildly relativistic speed with Lorentz factors up to 6 and mass-weighted half-opening angles of 3–7 deg. The split-monopole initial setup usually results in less collimated outflows. The light surface of the outflow magnetosphere tends to align vertically—implying three relativistically distinct regimes in the flow—an inner subrelativistic domain close to the jet axis, a (rather narrow) relativistic jet and a surrounding subrelativistic outflow launched from the outer disk surface—similar to the spine-sheath structure currently discussed for asymptotic jet propagation and stability. The outer subrelativistic disk-wind is a promising candidate for the X-ray absorption winds that are observed in many radio-quiet active galactic nuclei. The hot winds under investigation acquire only low Lorentz factors due to the rather high plasma-$\beta$ we have applied in order to provide an initial force-balance in the disk corona. When we increase the outflow Poynting flux by injecting an additional disk toroidal field into the outflow, the jet velocities achieved are higher. These flows gain super-magnetosonic speed and remain Poynting flux dominated.

Key words: accretion, accretion disks – galaxies: active – galaxies: jets – ISM: jets and outflows – magnetohydrodynamics (MHD) – relativistic processes

1. INTRODUCTION

Astrophysical jets emanate from sources spanning a huge range in energy output or length scale—among them young stellar objects (YSOs), stellar mass compact objects such as X-ray binaries or $\mu$-quasars, or the powerhouses of some active galactic nuclei (AGNs) which host a super-massive black hole. In particular for radio-loud quasars, for which synchrotron emission dominates the radio spectrum, relativistic jets are a generic feature. Due to the omnipresent angular momentum conservation, mass accretion to all of these objects features a disk structure around the central mass. It is commonly believed that jets are launched as disk winds, which are further accelerated and collimated by magnetic forces (see Blandford & Payne 1982; Pudritz & Norman 1983; Camenzind 1986b; Beskin 1997; Heyvaerts & Norman 2003; Pudritz et al. 2007). Relativistic jets may gain further energy by interaction with the black hole magnetosphere (Blandford & Znajek 1977; Ghosh & Abramowicz 1997; Komissarov 2005).

The magnetohydrodynamic (MHD) self-collimation of non-relativistic jets has been proven in general by time-dependent simulations (Üstüugova et al. 1995; Ouyed & Pudritz 1997) and have been investigated in further detail considering additional physical effects as magnetic diffusivity by Fendt & Čemeljić (2002), a variation in Ouyed & Pudritz (1999), non-axisymmetric instabilities in the launching region (Ouyed et al. 2003), or a variation in the mass flow profile or the magnetic field geometries (Fendt 2006; Pudritz et al. 2006), or the influence of a central magnetic field (Fendt 2009; Matsakos et al. 2008).

In the case of relativistic jets the efficiency of MHD self-collimation is under debate. The main reason is the existence of electric fields which are negligible for non-relativistic MHD and which are commonly thought to have a net de-collimating effect on the jet. Essentially, Chiueh et al. (1991) have demonstrated that the current carrying relativistic jet can be highly collimated. However, the actual structure of these jets still remains unclear—mainly due to the need for simplifying assumptions to solve the corresponding set of MHD equations.

So far, a variety of theoretical models have been developed for the case of self-similar jets (Li et al. 1992; Contopoulos 1994, 1995; Vlahakis & Königl 2003; Meliani et al. 2006), although it seems clear that relativity does not obey self-similarity. Fully 2.5-dimensional theoretical solutions for the internal magnetic jet structure could be obtained by neglecting matter inertia (Fendt 1997a; Fendt & Memola 2001). These force-free solutions for the field structure can in principle be coupled to the dynamical wind solution along the field lines (Fendt & Camenzind 1996; Fendt & Greiner 2001; Fendt & Ouyed 2004). Fendt (1997a) obtained solutions for the internal jet force balance in Kerr metric with an asymptotically cylindrical jet emerging from...
a disk-like structure around the central rotating black hole. The shape of the collimating jet boundary was obtained as a result of the internal force equilibrium, in particular considering the regularity condition along the jet outer light surface.

Time-dependent simulations of relativistic MHD jet formation have been performed considering a general relativistic metric, including also the evolution of the underlying accretion disk. Early—seminal—simulations did last for a few inner disk rotations only (Koide et al. 1998, 1999), which is sufficient time to demonstrate the launching of an outflow, but hardly sufficient in order to investigate the long-term dynamical evolution of the emerging jet.

More recent simulations were able to follow several 100 disk rotations and show the formation of a so-called funnel flow origin in the shear layer between the horizon and the inner disk radius (De Villiers et al. 2005; McKinney & Narayan 2007; Tchekhovskoy et al. 2008; McKinney & Blandford 2009). These simulations indicate highly time-variable mass ejections of rather low degree of collimation. However, the funnel flow achieves Lorentz factors of up to 50. General relativistic MHD simulations are also able to determine the interrelation between jet formation and the Blandford–Znajek mechanism (McKinney 2005; Komissarov & McKinney 2007).

Ultra-relativistic MHD simulations of accelerating and collimating jets have been presented by Komissarov et al. (2007), spanning over a huge range of length scale and providing jets of large Lorentz factor $\Gamma \sim 10$. Their simulations, however, did not start from the very base of the jet—the accretion disk, but at some fiducial boundary above the equatorial plane. Since the jet has been launched already with super-escape speed, gravity has not been considered. The jet flow has been confined within a rigid wall of predefined shape which naturally affects the opening angle of the MHD jet nozzle and thus jet collimation and acceleration.

The focus of our present paper is (1) to concentrate on the formation and acceleration of a relativistic MHD jet right from the launching area of the accretion disk surface, (2) to investigate the (self-) collimation of relativistic MHD jets under the influence of de-collimating electric forces and an “open” boundary condition for the outflow, (3) to consider gravity as an essential gradient to provide a realistic disk boundary condition in equilibrium, (4) to run long-term simulations lasting more than 1000 inner disk rotations until the jet reaches steady state, and (5) to concentrate on MHD disk jets as disks are the natural origin for the mass load for AGN jets.

The outline of this paper is as follows. In Section 2, we discuss the concepts of ideal special relativistic MHDs in the perspective of jet formation. Section 3 is devoted to the initial- and boundary conditions of the numerical simulations, whose results are shown in Section 4. We conclude in Section 5.

2. CONCEPTS OF RELATIVISTIC MHD JETS

It is well known that relativistic jets must be strongly magnetized (Michel 1969; Camenzind 1986b; Li 1993). This simply reflects the fact that the lower the mass flux, the more electromagnetic energy (Poynting flux) can be transferred into high kinetic energy per unit mass. Relativistic MHD is also limited for very strong magnetization as then the MHD assumption can be violated since a sufficiently large amount of electric charges is lacking which are needed to drive the electric current system. Such a situation might arise in the ultra-relativistic regime of pulsar winds but is unlikely for the disk winds investigated here.

This paper deals with the time-dependent formation of relativistic MHD jets by using the special relativistic MHD module of the PLUTO code provided by Mignone et al. (2007) and applying a Newtonian description of gravity.

2.1. Relativistic MHD Equations

The relativistic MHD module of PLUTO solves the system of special relativistic conservation laws. In a covariant formulation, the equations follow naturally as a set of hyperbolic equations. It is solved for energy and momentum conservation

$$\partial_\alpha T^{\alpha\beta} = 0$$

of an ideal magnetized fluid

$$T^{\alpha\beta} = (\rho h + b^2)u^\alpha u^\beta + \left( p + \frac{1}{2}b^2 \right) g^{\alpha\beta} - b^\alpha b^\beta$$

with the specific plasma enthalpy

$$h = \frac{\gamma p}{\gamma - 1} + 1,$$

the isotropic gas pressure $p$, density $\rho$ (both in the local rest-frame), four-velocity $(u^\alpha) = (\Gamma, \Gamma \beta)^T$, velocity $\beta = v/c$, Lorentz factor $\Gamma = 1 - \beta^2)^{-\frac{1}{2}}$, and the magnetic field pseudo vector

$$b^\alpha = -\frac{1}{2} e^{\alpha\gamma\delta} u_\beta F_{\gamma\delta}.$$

Assuming infinite conductivity, thus vanishing electric fields in the rest frame of the plasma $F^{\alpha\beta} u_\beta = 0$, the homogenous Maxwell equation

$$\partial_\alpha *F^{\alpha\beta} = 0$$

can be written solely in terms of the magnetic four-vector $b^\alpha$

$$*F^{\alpha\beta} \equiv \frac{1}{2} e^{\alpha\gamma\delta} F_{\gamma\delta} = b^\alpha u_\beta - b_\beta u^\alpha$$

and for the field vector components it follows

$$B^i = *F^{i0} = b^i u^0 - b^0 u^i$$

$$E^i = \epsilon^{ijk} b^j u^k.$$

Equation (8) represents the ideal MHD condition $E = -\beta \times B$ and is the reason why all electric fields can be eliminated from the equations. The magnetic four-vector turns out as $b^0 = B^i u^i$; $b^i = (B^i + b^0 u^i)/u^0$. The conservation of the Faraday tensor given by Equation (6) results in the non-relativistic (ideal) induction equation and the solenoidal condition $\nabla \cdot B = 0$. Mass conservation is guaranteed by the continuity equation

$$\partial_\alpha (\rho u^\alpha) = 0.$$

We apply a polytropic equation of state for the gas with the polytropic index $\gamma = 5/3$.

Following the convention that Greek indices run from 0 to 4 whereas Latin indices go from 1 to 3.
2.2. Gravity in Special Relativity

The outcome of MHD simulations is mainly determined by its boundary conditions and therefore requires great care in describing the proper physical state of interest. Since in our simulations the jet is considered to be launched as a wind from a rotating disk, it is essential to take into account a proper disk model as boundary condition. For the disk boundary we choose a (sub-) Keplerian rotation profile and a hydrostatic pressure model as boundary condition. For the disk boundary we choose a rotating disk, it is essential to take into account a proper disk simulations the jet is considered to be launched as a wind from describing the proper physical state of interest. Since in our its boundary conditions and therefore requires great care in

\[ a(R) = \frac{GM}{(R + r_s)^2 R} \]  

(10)

with a softening length of \( r_s = 1/3 \) that may be related to the Schwarzschild radius of a non-rotating black hole. The corresponding acceleration reads

\[ a = -\nabla \phi = -\frac{GM}{(R + r_s)^2 R} r \]  

(11)

and hence instead of solving Equation (1), we solve

\[ \partial_t T^{\alpha \beta} = f^{\beta} \]  

(12)

with the four force density \( (f^\beta) = \Gamma \rho (a \cdot v, a)^T \) as a local source term on the right-hand side. This is incorporated in PLUTO as a “body force” \((\Gamma a)\) using the infrastructure of the code.

Omission of softening would lead to numerical errors (due to the unresolved steep gradients in the potential close to the origin), piling up to produce artificial acceleration along the spine of the jet close to the axis. Softening is clearly a compromise avoiding the singularity (by limiting the required resolution) on little cost of realism. Another choice could be the well-known pseudo-potential by Paczynsky & Wiita (1980) which has just the negative softening \( \phi_{PW} = -GM/(R - r_s) \). For the cylindrical geometry of our choice, the softening would become even more problematic, complicating the setup a great deal.

2.3. Relations in Axisymmetric MHD

The region of jet formation may be fairly well approximated in axisymmetry. In fact, non-axisymmetric distortions may actually hinder the formation of powerful jets as probably demonstrated by the existence of a variety of strongly magnetized, rapidly rotating accretion disk systems which, however, do not exhibit jets (e.g., cataclysmic variables or most pulsars).

Under the assumed symmetry in a cylindrical coordinate system, the magnetic field vector can be written as

\[ B = B_\phi + B_\phi e_\phi, \]  

(13)

where \( B_\phi \) can now be an arbitrary function of \( r \) and \( z \), as the solenoidal condition translates to \( \nabla \cdot B_\phi = 0 \). The stream function \( \Psi(r, z) = (1/2\pi) \int dS \cdot B_\phi = r A_\theta \) measures the magnetic flux through the surface area \( S \) and follows from the toroidal component of the vector potential,

\[ B_\phi = \nabla \times A_\phi = \nabla \times \frac{\Psi}{r} = \frac{1}{r} \nabla \Psi \times e_\phi. \]  

(14)

For the electric field, the ideal MHD condition \( E = B \times \beta \) gives

\[ E = \frac{r}{c} \frac{\Omega F}{r_L} B_\phi n = \frac{r}{r_L} B_\phi n \]  

(15)

in terms of the so-called angular velocity of the field line \( \Omega F = (\Omega - v_\phi B_\phi/B_p)/r \), or the so-called light cylinder radius of a field line \( r_L \equiv c/\Omega F \). The direction of the electric field is given by \( n = B_\phi/B_p \times e_\phi \) and is perpendicular to the magnetic flux surface \( \Psi(r, z) \). The poloidal Poynting flux \( S = (c/4\pi)E \times B_\phi \) simplifies to

\[ S = -\frac{c}{4\pi r_L} B_\phi B_p = -r \Omega F \frac{B_\phi B_p}{4\pi}. \]  

(16)

2.3.1. Perpendicular and Parallel Force-balance

The processes leading to flow collimation can be identified directly from the (steady-state) trans-field force-balance equation (Chiueh et al. 1991; Appl & Camenzind 1993). Here, we adopt the notation of the latter paper when investigating the collimation behavior in the quasi steady-state time domain of our simulations. The curvature \( \kappa \equiv n \cdot (B_\phi \cdot V) B_p/B_\phi^2 \) of a flux surface \( \Psi(r, z) \) results from the summation of perpendicular forces,

\[ \kappa \frac{B_\phi^2}{4\pi} \left( 1 - \frac{M^2 - r^2 \Omega F^2}{c^2} \right) = \]  

\[ + \left( 1 - \frac{r^2 \Omega F^2}{c^2} \right) \nabla_{\perp} \left( \frac{B_\phi^2}{8\pi} + \frac{\nabla \cdot B_\phi}{8\pi} + \nabla \cdot \rho \Omega F \right) + \frac{B_\phi^2}{4\pi} \left( \frac{r \rho u_\phi^2}{r} \right) \nabla_{\perp} r - \frac{B_\phi^2 \Omega F}{4\pi c^2} \nabla \perp \left( r^2 \Omega F^2 \right) + \nabla \perp \Psi, \]  

(17)

where we have added the collimating component of the gravitational force. For the ease of use in Section 4.2.2, we label the terms as \( (F_{\text{curv}}, F_{\text{ph}}, F_{\text{phi}}, F_p, F_{\text{punch}}, F_{\text{elf}}, F_{\text{grav}}) \) in the order of their appearance in Equation (17). (The poloidal) Alfvén Mach number \( M \) is relativistically defined as

\[ M^2 = \frac{4\pi \rho u_\phi^2}{B_\phi^2}. \]  

(18)

The gradient \( \nabla_{\perp} \equiv n \cdot \nabla \) is projected perpendicular to the magnetic flux surfaces \( \Psi \), and thus along the (inward pointing) electric field. The light surface of a magnetosphere is located where \( r_L(\Psi) = r_L(r, z) \equiv c/\Omega F(\Psi) \), hence it depends on the flux-geometry as well as on the rotation profile \( \Omega F \). Each flux
surface/magnetic field line crosses the light surface at most once (see also the discussion in Fendt 1997b) Some field lines $\Psi(r, z)$ never cross the light surface, indicating an asymptotic radius $r_{\infty}(\Psi) < r_L, \Psi$. For these field lines relativistic effects due to rotation (electric fields) are less important. For others, the asymptotic radius is $r_{\infty}(\Psi) > r_L, \Psi$. The light surface constitutes a critical point of the stationary axisymmetric wind equation only in the mass-less limit in which it is identical to the modified poloidal Alfvén surface $M^2_A = 1 - (r_A/r_L)^2$ (Camenzind 1986a, 1986b). However, it is essential to note that at the light surface the dynamical behavior of the poloidal magnetic pressure term changes—the force changes sign. This leads to the existence of three dynamically different regimes in the asymptotic (collimated) region of a relativistic jet (see Figure 1). In region $I$, for all field lines $r_{\infty}(\Psi) < r_L, \Psi$ corresponding to $(1 - (r/r_L)^2 > 0$, and, thus, a de-collimating magnetic pressure term. Field lines in region $II$ do cross their light cylinder, and, since $r > r_L$, the magnetic pressure term acts as collimating for $r > r_L$. Field lines in region $III$ never reach their light cylinder, and here the magnetic pressure term is de-collimating again. The slope of the outer part of the light surface critically depends on the dynamics and the magnetic field structure of the outflow in the very inner part.

Similarly, the forces due to the electric field $E = r/r_L B_p$ (second last term in Equation (17)) scale with the relative position to the light surface, hence they are important in region $II$ of the jet formation region only.

Equation (17) together with Figure 1 once more demonstrates the need to resolve the whole acceleration and collimation region of a jet in radial and vertical directions. Only when the light surface is taken into account self-consistently, the proper force-balance is applied along and across the flow.

Similarly one can derive the parallel-field force equation, it becomes

$$B_p^2/4\pi \nabla_r M^2 = \kappa_B^2/4\pi (1 - M^2) - \nabla_r \left( \rho + B_p^2/8\pi + B_\phi^2/8\pi \right)$$

$$- \left( B_p^2/4\pi r^2 - \rho u^2 \right) \nabla_r r - \Gamma \rho \nabla_r \phi$$

(19)

with the necessary definitions $\nabla_r \equiv B_p/B_p \nabla$ and $\kappa_B^2 \equiv B_p^2/B_p (B_p \cdot \nabla) B_p$. We see how the change in the Mach-number is mediated by the interplay of tension-, pressure-, pinch-, centrifugal- and gravitational-acceleration. Electric fields (pointing in the perpendicular direction) do not contribute and the equation reduces to the Newtonian case. In steady state, there is no electric acceleration!

A number of self-similar approaches to the relativistic jet formation have been published (e.g., Vlahakis & Königl 2003). While the self-similar ansatz is a powerful and highly successful tool to solve the non-relativistic MHD problem (starting with the Blandford–Payne solution), we believe that using self-similarity for relativistic MHD jets is problematic.

We note that neither the light surface nor the relativistic Alfvén surface obeys a self-similar structure. It is well known that forcing self-similarity into the relativistic MHD equations constrains the rotation law for the magnetosphere $\Omega^F(r) \propto r^{-1}$ (see also the discussion in Li et al. 1992; Li 1993). This is a major difference from the non-relativistic self-similar approach.

We further note that also the scaling for the electric field depends on the radial position of the light surface (see Equation (15)). This is, however, of uttermost importance for the structure of relativistic magnetospheres as the electric field forces play a leading role in the trans-field force-balance (Equation (17)). Similar arguments hold for the inner light surfaces around Kerr black holes or the geometry of the black hole ergosphere.

We therefore believe that a steady-state self-similar relativistic MHD approach is intrinsically inconsistent with the relativistic characteristic of the flow.

2.3.2. Field Line Constants

Stationary axisymmetric MHD flows conserve the following five quantities along the magnetic flux-function $\Psi$. From the isotropization law together with the ideal MHD condition follows the rest-mass energy flux per magnetic induction,

$$k = k(\Psi) \equiv \rho u_p/B_p$$

(20)

and the isorotation parameter

$$\Omega^F = \Omega^F(\Psi) = 1/r \left( v_\phi - v_p B_\phi/B_p \right)$$

(21)

(often interpreted as angular velocity of the field lines). In the absence of shocks the (pseudo-) entropy

$$Q = \rho \rho^\gamma = Q(\Psi)$$

(22)

is conserved as well as the angular momentum flux

$$l = - l_{2\pi k c} + r u_\phi = l(\Psi)$$

(23)

and the flux ratio of total energy to rest-mass energy,

$$\mu = \frac{S + \kappa + M + T + G}{M} \equiv \mu(\Psi),$$

(24)

where we identify the individual terms as (purely) kinetic energy flux $K \equiv (\Gamma - 1) \rho u_p$, rest-mass energy flux $M \equiv \rho u_p$, thermal energy flux $T \equiv T_{\gamma - 1} \rho u_p$, and gravitational energy flux $G \equiv \rho \phi u_p$, respectively. The cold, asymptotic limit of Equation (24) is particularly of interest, it reads

$$\mu = \Gamma (\gamma + 1),$$

(25)

where $\gamma = S/(K + M)$ is the customarily defined magnetization parameter—the ratio of Poynting to kinetic flux. This simple
relation provides a theoretical maximum for the Lorentz factor \(\Gamma^* = \mu\), when the entire electromagnetic energy is converted into kinetic energy.

The essential point in the quest for relativistic jets is to find a highly energetic disk solution with values of \(\mu\) beyond the anticipated Lorentz factor. Previous studies obtaining highly relativistic jets by Komissarov et al. (2007) do not start from an anticipated Lorentz factor. Previous studies obtaining highly energetic disk solutions with values of \(\mu < \mu_{\text{max}}\) in the present study, we are aiming to improve on the problems just mentioned by applying a physical boundary condition as a Keplerian disk corona in equilibrium.

2.4. Accretion Disk Coronae

It is our ambition to connect the wind solutions to the ambiance of a realistic accretion disk. In our simulations, the flow originates in the high entropy atmosphere called a corona. Optically thin coronae are an integral part in models of the X-ray features of AGNs (e.g., Mushotzky et al. 1993) and \(\mu\)-Quasars (e.g., Nowak et al. 2002; Markoff et al. 2003).

While Compton cooling can provide the observed spectra, the heating mechanism is not easily found. Just as in the case of the Sun, the coronal heat cannot directly be transferred from the colder photosphere/accretion disk (according to the second law of thermodynamics) and the nature of vertical energy transport is an active field of research. External irradiation of flared disks by the central object (or central disk) is certainly present in a multitude of objects (see Czerny et al. 2008, for a review) but might not be the primary energy source. Among the most promising mechanisms we should highlight magnetic reconnection heating as proposed by Haardt & Maraschi (1991).

Between the mid-plane and the coronal point of injection in our simulations, ideal MHD cannot provide a realistic picture. In order for an accretion disk to work, a torque of viscous or magnetic origin has to be exerted onto the material. Additionally, the flux-freezing constraint of ideal MHD must be relaxed since it would lead to an accumulating magnetic pressure that ultimately stops the accretion process. Studies modeling both the accretion motion and the super Alfvénic jet based on a stationary isorotation is kept constant in time and follows a Keplerian rotation law \(\Omega^0\) owing to the diffusion of magnetic field. For these reasons \(\Omega^0\) should closely follow the expected disk rotational angular velocity of the material \(\omega(r)\). Relaxation of the infinite conductivity constraint would, however, lead to an inequality \(\Omega^F \leq \omega(r)\) owing to the diffusion of magnetic field.

3. MODEL SETUP FOR THE MHD SIMULATIONS

With the aforementioned considerations we choose the following model for our investigation. A global-scale poloidal field favorable of wind acceleration is adopted. Whether it is advected by the accretion flow or created by an underlying dynamo is not of our concern. The jet base resembles a corona in the sense that it is hot (electron temperature \(\sim 10^6\)K), has no mechanism of cooling, is non-turbulent (no viscosity), and highly ionized (infinite conductivity). We choose a Keplerian rotation profile for the field lines. The flow starts with subsescape velocity and we investigate submagnetosonic injection where mass loading is determined by the internal dynamics as well as mass fluxes imposed by the boundary condition. We perform axisymmetric special relativistic MHD simulations of jet formation for a set of different magnetic field geometries and field strengths. In the following, we discuss the numerical realization of our problem.

3.1. Boundary Conditions

Given the 2.5-dimensional nature of the problem, three geometrical boundaries have to be prescribed. These are the inlet boundary along \(z = 0\) from which material is injected into the domain (inflow) and the two outer boundaries at \(r = r_{\text{end}}\) and \(z = z_{\text{end}}\) where we expect material to leave the computational domain (outflow). The boundary condition along \(r = 0\) \((R_{\text{beg}})\) follows from cylindrical symmetry. Figure 2 gives an overview of the different regions.

3.1.1. Injection Boundary \((Z_{\text{beg}})\)

Pursuing the aim to follow the acceleration of a disk wind from as close to the accretion disk as possible, we start with a subslow magnetosonic wind. We are hence free to choose four constraining boundary conditions without overdetermining the system (see Bogovalov 1997 and Appendix B for more details).

Our choice is to fix the toroidal electric field component \(E_\phi = 0\). This suppresses the evolution of the bounding poloidal magnetic field and is realized by requiring \(\mathbf{v}_\phi \parallel \mathbf{B}_p\). The field line isorotation is kept constant in time and follows a Keplerian rotation law \(\Omega^0 \propto r^{-1.5}\). Unless specified otherwise, the boundary condition starts initially in a force-free state with zero toroidal field \(\Omega^F = v_\phi(r)/r\) corresponding to a disk in hydrodynamic equilibrium. This is an essential ingredient as—within stationary ideal MHD—\(\Omega^F\) just equals the mid-plane angular velocity of the material \(\omega(r)\). Relaxation of the infinite conductivity constraint would, however, lead to an inequality \(\Omega^F \leq \omega(r)\) owing to the diffusion of magnetic field. For these reasons \(\Omega^F\) should closely follow the expected disk rotational
profile and should be limited by the maximal velocity in the mid-plane, typically at the inner edge of the disk located at \( r = 1 \).

A radial force-equilibrium along the whole boundary is enforced by balancing the centrifugal and pressure support against gravity via the sub-Keplerianity of the rotation \( \sqrt{\chi} = v_\phi(r = 1)/v_K \) that also determines the inlet density, where \( v_K \) is the circular velocity that alone sustains against gravity at \( r = 1 \). A more convenient parameterization is in terms of the relative temperature \( \epsilon \equiv c_s^2/v_\phi^2 = (\gamma - 1)(1 - \chi)/\chi \). We investigate two cases—a hot corona with \( \epsilon = 2/3 \) and a version with \( \epsilon = 1/6 \) (\( \chi = 0.5 \) and \( \chi = 0.8 \)).

If we interpret \( r = 1 \) as the innermost stable circular orbit (ISCO) around a black hole, \( v_K \) is a measure of the black hole spin. In the case of a Schwarzschild black hole it is \( v_K \approx 0.6c \) while we choose the scaling velocity \( v_\phi(r = 1) = 0.5c \) for convenience.

The inner disk edge is numerically difficult to model because of the transition to the inflow of the disk wind and the steep gradients in gravity. Within \( r < 1 \), the so-called plunged region, a physical solution would allow for (radial and vertical) accretion onto the central object. Since the dynamics in this area would then require a general relativistic treatment which we cannot provide in this context, we simply minimize the dynamical effect of this region by freezing the hydrostatic solution initially and apply the condition \( \partial_z B_\phi = \partial_z B_z = 0 \) (while \( B_z \) then follows from \( \nabla \cdot \mathbf{B} = 0 \)).

By letting the jet solution alone determine Poynting and mass flux, we lose control over the energy flux parameter \( \mu \) and the limiting asymptotic Lorentz factor \( \Gamma^* \). It will rather be a consequence of the MHD under the constraints we have given, while we have used our freedom to provide a boundary most closely resembling a realistic hot disk corona. A graphical summary of the disk-wind boundary conditions is shown in Figure 3.

In order to extend the parameter space toward higher \( \mu \), we also investigate cases where we have overdetermined the boundary conditions by specifying the mass flux through

\[ v_z(r, 0) = v_{\text{inj}} v_\phi(r, 0) \]

as we apply the condition \( \partial_z B_\phi = \partial_z B_z = 0 \) (while \( B_z \) then follows from \( \nabla \cdot \mathbf{B} = 0 \)).

Density given by Equation (26) and the coronal pressure \( p \) constitute the third and fourth fixed in time conditions.

We emphasize that it is not possible to specify both injection velocity and density profile and thus the mass flux for sub-(magneto)sonic flows, as this is determined by the sonic point. Therefore, we match the vertical velocity \( v_z \) to the domain via \( \partial_z v_z = 0 \), while the radial component follows from the \( E_\phi = 0 \) condition. We limit the injection speed by the local slow magnetosonic speed in the case when the velocity just above the boundary becomes trans-sonic. This provides the fifth constraint needed in that case.

With the induction of a toroidal magnetic field component in the jet, the rotational velocity needs to be adjusted in order to satisfy Equation (21),

\[ v_\phi = r \Omega F^\epsilon + \frac{v_p}{\mu} B_\phi \]

Figure 2. Sketch of the different regimes of our grid and boundary. In both directions we set 20 equidistant cells in [0, 1]. Then follows a stretched grid until we add five equidistant cells one unit radius before the outflow boundary. For \( r \in [0, 1] \), the hydrostatic corona is fixed to minimize the influence of the central region on the disk wind.

![Figure 2](image_url)

Figure 3. Profiles of the fixed in time variables for the inlet in hydrodynamic equilibrium. Here we give constraints on \( \Omega F^\epsilon, p, E_\phi \) (\( E_\phi = 0 \) not shown). Parameters are \( v_K = 0.5, \epsilon = 2/3 \). The thin dotted line is the Fermi step function used to smooth those variables experiencing a sharp transition at the inner disk radius \( r = 1 \).

![Figure 3](image_url)
In the case of subfast-magnetosonic outflows, this strategy is unfortunately insufficient as the flow inside of the domain will depend on the flow beyond the boundary via the incoming characteristics. Just as for the inlet boundary, the now missing information has to be supplied by constraints that describe best the physical conditions downstream of the boundary. In the case of an outflow, the conditions leading to an untampered flow are however impossible to know a priori. A way to circumvent this unphysical feedback is to avoid any causal contact by moving the boundary far away such that the characteristics will not enter the domain of interest within the simulated time.

When considering a boundary outside of causal contact “very far away,” we estimate for Alfvén waves to travel over $10^3$ scale radii within the anticipated simulation time. The computational effort of such huge grids does not allow a large parameter study at the current time and we must leave this option for future endeavors.

In the absence of a substantially better solution, zero-gradients are used for the primitive variables except for magnetic fields for which this simple approach leads to artificial electric currents implying an inward-pointed Lorentz force. Especially for low plasma-$\beta$ this may result in a devastating artificial collimation—preventing any steady state to establish and artificially collimating the outflow increasingly thin with time.

Ustyugova et al. (1999) have performed a systematic study comparing different approaches for outflow conditions including a (toroidal) force-free condition $j_p\parallel B_p = 0$ and a more sophisticated version including an additional numerical factor that needs to be determined a posteriori. For the outflow conditions in our simulations we instead recover the magnetic field components by imposing constraints on the poloidal ($j_r = -\partial_z B_\phi$, $j_z = r^{-1}\partial_r r B_\phi$) and toroidal ($j_\theta = \partial_z B_r - \partial_r B_z$) electric currents. For the toroidal magnetic field (poloidal electric current) we radially extrapolate the expected $1/r$ law of a marginal $j_z$ at the radial end ($R_{\text{crit}}$). Using $\partial_z B_\phi = 0$ allows to specify $j_\theta$ at the upper end of the domain ($Z_{\text{crit}}$). Concerning the poloidal magnetic field components we implement a current-free boundary condition by enforcing $j_\phi = 0$. This is a novel approach designed to minimize spurious effects of collimation. We convinced ourselves that boundary effects have only a marginal effect on the solution by varying the grid-size and geometry. For a detailed discussion and comparison of various outflow conditions we refer to Appendix A.

We note that, as a further complication, a fully relativistic version for a force-free or force-balance boundary conditions would also need to take into account electric forces. We have estimated the impact of such an upgrade and found that due to the geometry of our outflow (in particular the location of the light surface) it would play a minor role and is thus not worth the effort to implement.

### 3.2. Initial Conditions

As initial state we prescribe a force-free coronal magnetic field, $F^\alpha_j = 0$, together with a gas distribution in hydrostatic equilibrium. Both is essential in order to avoid artificial relaxation processes caused by a non-equilibrium initial condition. We apply a polytropic equation of state $p = K \rho^\gamma$ with a “classical” polytropic index of $\gamma = 5/3$ since our flows are always cold when compared to the rest-mass. To further strengthen this choice, we performed a comparison simulation with the Taub (1948) equation of state as described by Mignone et al. (2005) which produced an identical jet once the hot shock has passed through. The constant $K$ is determined by the radial force-balance of the inlet.

For the initial magnetic field configuration we apply two different geometries. Field configuration A is a potential field of hourglass shape as applied by Ouyed & Pudritz (1997) and Fendt & Čemeljić (2002) with the magnetic field components

$$B_r = \frac{1}{r} \left[ 1 - \frac{z + z_d}{(r^2 + (z + z_d)^2)^{1/2}} \right]$$  \hspace{1cm} (30)

$$B_z = \frac{1}{(r^2 + (z + z_d)^2)^{1/2}},$$  \hspace{1cm} (31)

where $z_d$ is a (toroidal) force-free condition ($z_d > 0$) electric current implying an inward-pointed Lorentz force. For the toroidal magnetic field (poloidal electric current) we implement a current-free boundary condition

$$B_r = \frac{r}{(r^2 + (z + z_d)^2)^{1/2}},$$  \hspace{1cm} (33)

$$B_z = \frac{z + z_d}{(r^2 + (z + z_d)^2)^{1/2}}.$$  \hspace{1cm} (34)

In cylindrical coordinates with $B_r = -\partial_z A_\phi$ and $B_z = r^{-1}\partial_r r A_\phi$. The dimensionless disk thickness $z_d$ with $(z_d + z) > 0$ is introduced to avoid kinks in the field distribution for $z < 0$ (the ghost zones) and we choose $z_d = 1$ for convenience.

Our other option for the initial magnetic field (configuration B) is the “split monopole” (Sakurai 1987) with the magnetic field components

$$A_\phi = \frac{1}{r} \left[ \sqrt{r^2 + (z + z_d)^2} - (z + z_d) \right]$$  \hspace{1cm} (32)

$$A_\phi = \frac{1}{r} \left[ \sqrt{r^2 + (z + z_d)^2} - (z + z_d) \right].$$  \hspace{1cm} (35)

The fields are scaled to satisfy the choice of the plasma-$\beta$

$$\beta \equiv \frac{B_p^2}{8\pi p} \bigg|_{r=1,z=0}$$  \hspace{1cm} (36)

at the inner disk radius. It should be kept in mind that plasma-$\beta$ largely varies along the disk boundary. In configuration A, the profile $\beta(r)$ monotonically decreases until for large radii it is $\beta(r) \propto r^{-0.5}$ leading to a magnetically dominated outer corona. In the split-monopole, $\beta(r)$ decreases first to a minimum value (at $r^*(\theta = 77^\circ) \approx 5$ and $r^*(\theta = 85^\circ) \approx 15$) and increases for large radii according to $\beta(r) \propto r^{1.5}$ leading to thermal dominance.

In summary, for our injection boundary condition we are left with the following five dynamical parameters,

$$(v_K, \beta, \epsilon, v_{inj}, \eta),$$  \hspace{1cm} (37)

where strictly speaking we are only allowed to choose the first three when launching subslow. An overview of the simulations performed in this parameterization is shown in Table 2.
3.3. Numerical Grid and Physical Scaling

We use a numerical grid of 512 × 1024 cells applying cylindrical coordinates. Onward from the inner region (r < 1, z < 1), which is resolved with 20 × 20 equidistant cells, we apply a stretched grid with the element size increasing by a factor of \( \sim 1.005 \). This leads to a domain size of \((r \times z) = (100 \times 200) r_i\) corresponding to \((300 \times 600) r_i\) if \( r_i = 3 r_s\) (see the sketch in Figure 2). Staggered magnetic fields treated via constrained transport (Balsara & Spicer 1999) are used to ensure \( \nabla \cdot \mathbf{B} = 0 \).

Because of the constraints imposed on the cell aspect ratio by the zero-current boundary (Appendix A), we set the last five grid cells to be equally spaced with maximal aspect ratios < 3/1. The dimensionless nature of our simulations allows for various astrophysical interpretations. We provide a physical scaling of simulation variables (marked with a prime) in the following paragraph.

Since velocities are given in terms of the speed of light \((c' = 1)\), relativistic simulations are in need of only two additional scales. The simulation variables are connected to their physical counterparts via

\[
v = v' c; \quad l = l' l_0; \quad t = t' t_0 = t' l_0 / v_0; \quad \rho = \rho' \rho_0 \quad (38)
\]

\[
p = p' \rho_0 = p' \rho_0 c'^2; \quad B = B' B_0 = B' \sqrt{4 \pi \rho_0 c^2}. \quad (39)
\]

If we assume a Schwarzschild black hole as central body, we may set the spatial scale \( l_0 = 6 r_s\), equating the inner disk radius with the ISCO. Then it becomes

\[
v_0 = 3 \times 10^{10} \text{ cm s}^{-1} \quad (40)
\]

\[
l_0 = 9 \times 10^5 \text{ cm} \left( \frac{M_*}{M_\odot} \right) \quad (41)
\]

\[
t_0 = 3 \times 10^{-5} \text{ s} \left( \frac{M_*}{M_\odot} \right). \quad (42)
\]

Assuming a physical outflow mass-loss rate in terms of the Eddington limited accretion rate \( \dot{M} = 0.01 M_{\text{edd}} \) we can provide a scale for the density by comparison to the mass-loss rate of the simulation \( \dot{M}' \)

\[
\rho_0 = 6 \times 10^{-7} \frac{1}{\dot{M}'} \left( \frac{M_*}{M_\odot} \right)^{-1} \text{ g cm}^{-3}, \quad (43)
\]

where we applied a radiative efficiency of \( \eta^* = 0.1 \). The scaling of pressure and magnetic fields then follows as

\[
p_0 = 5 \times 10^{14} \frac{1}{\dot{M}'} \left( \frac{M_*}{M_\odot} \right)^{-1} \text{ g cm}^{-1} \text{ s}^{-2} \quad (44)
\]

\[
B_0 = 8 \times 10^7 \dot{M}'^{-0.5} \left( \frac{M_*}{M_\odot} \right)^{-0.5} \text{ Gauss}. \quad (45)
\]

Under these considerations, the only remaining scaling parameter is the mass of the compact object \( M_* \). Neglecting additional physical processes as radiation pressure or radiative cooling leaves us with a scale-free model that can be applied to any disk-wind launched jet around compact objects. Table 1 provides a fiducial scaling for a microquasar with \( M_* = 10 M_\odot \) and for an AGN with \( M_* = 10^8 M_\odot \). The scale-free nature becomes obvious if we recall the rest-frame temperatures for an ideal gas, \( T = \frac{p' c'}{\rho'} (44) \) assuming a physical mass-loss rate of \( \dot{M} = 1 \% \dot{M}_{\text{edd}} \) with an efficiency of \( \eta_{\text{vis}} = 0.1 \).

| Table 1 | Fiducial Scaling |
|---------|------------------|
| \( \frac{M_*}{M_\odot} \) | \( t/t' \) | \( t/t' \) | \( \rho/\rho' \) | \( \rho/\rho' \) | \( B/B' \) | (Gauss) |
| 10^6 | 9 \times 10^{11} | 3 \times 10^{10} | 1.8 \times 10^{-10} | 1.5 \times 10^7 | 1.4 \times 10^7 |
| 10 | 9 \times 10^6 | 3 \times 10^{-4} | 1.8 \times 10^{-9} | 1.5 \times 10^{12} | 4.4 \times 10^6 |

Note. Scaling for simulation WA05 (\( \dot{M}' = 32.67 \)) assuming a physical mass-loss rate of \( \dot{M} = 1 \% \dot{M}_{\text{edd}} \) with an efficiency of \( \eta_{\text{vis}} = 0.1 \).

4. RESULTS AND DISCUSSION

We now present the results of our numerical simulations considering the formation of relativistic MHD jets from accretion disks. Each simulation consumed approximately 48 hr on 16 processors. The overall goal is to test whether the paradigm of MHD self-collimation of non-relativistic jets established from numerical simulations (Ustyugova et al. 1995; Ouyed & Pudritz 1997; Krasnopolsky et al. 1999; Fennd & Čemeljić 2002) also holds in the relativistic case.

4.1. Overall Evolution of the Outflow

The initial evolution of the disk corona is governed by the propagation of toroidal Alfvén waves launched due to the rotation of the field line foot points. The initial force-free magnetic field structure is adapted to a new dynamic equilibrium according to a rotating wind magnetosphere.

A wind is launched from the disk boundary and is continuously accelerated driving a shock front through the initial hydrostatic corona and sweeping this material out of the computational domain (Figure 4). The disk wind evolves into a collimated outflow of super-magnetosonic speed. Along the symmetry axis the hydrostatic initial condition is very well preserved. Once the bow shock has passed through the domain, the jet mass flux declines to a value which is solely governed by the internal outflow dynamics and the injection boundary conditions. Similarly, the post-shock magnetic field distribution follows as well from the initial outflow dynamics and has in principle little in common with the initial setup. Certain combinations of boundary conditions for mass flux and magnetic field will result in a quasi-stationary state of the outflow evolution (see the next section). From this point onward we can start our investigations of collimation and acceleration. In this paper, we concentrate on analysis when the flow has reached a quasi-steady-state. We usually terminate our simulations after 500 inner disk rotations \( P \), while a quasi-steady-state is established over most of the domain after about 200 rotations.
Figure 4 shows the time evolution for two exemplary simulations with an initial hourglass-shaped potential field distribution (case A) and a split-monopole field distribution (case B), each for the parameter choice \((\beta, v_K, \epsilon) = (1, 0.5, 2/3)\).

The figure shows the Lorentz factor, the poloidal magnetic field lines, poloidal electric current flow lines, and the critical MHD surfaces. In addition the light surface is drawn.

Phenomenologically, the solutions form a magnetic nozzle with, depending on the disk flux distribution, considerable difference in the width, but comparable final opening angles of the fast component. A broader initial field distribution (case B) also results in broader and faster winds where the material originating from the inner disk is more effectively thinned out. In analogy to hydrodynamic nozzles, the flow reaches the slow-magnetosonic speed directly above the throat. Collimation happens mainly before the fast-magnetosonic surface is reached. Afterward, the opening angle of a given field line is approximately conserved.

Of particular interest is the electric current distribution (shown for the time step \(T/P = 250\)). The electric current distribution is a consequence of the dynamical evolution of the outflow and therefore a direct outcome of the disk boundary magnetic flux profile and the Keplerian field line rotation.

In general, the electric current leaves the outer disk to return within the fast component of the outflow. It is expected to enter the inner disk and then flow radially outward closing with the outgoing current. Such butterfly-shaped circuits are expected in Keplerian disks while the \(j_r\) plays a leading role in the disk-jet feedback (Ferreira 1997). A positive radial electric current in the disk corona supports accretion by braking the disk material due to its magnetic torque \(j_r \times B_z\) similar to a Barlow wheel.\(^6\)

\(^6\) However, this region is not resolved within our numerical domain, as it is located below our injection boundary as part of the underlying non-ideal MHD accretion disk. See Casse & Keppens (2002) and Zanni et al. (2007) for non-relativistic simulations of the disk-jet interaction.
The inclination between the poloidal current vector and the magnetic field line indicates the direction of (de-)collimating magnetic forces acting on the flow. When the inclination becomes less than 90°, the Lorentz force $j_{\|} \times B_{\|}$ changes from collimation to de-collimation. This can be clearly seen in the snapshots at $T/P = 250$ of the case A simulation where actual field lines are indicated in white and initial field lines in red. In the actual field distribution, the field lines are somewhat pushed away from the surface $j_{\|} \perp B_{\|}$ (this is also where magnetic acceleration is most effective). For case B this happens beyond the light surface. As a result, both electric and magnetic forces deflect the flow toward the disk boundary which leads to a highly unstable layer just above the outer disk. We will provide an indepth analysis including all forces acting on the flow in Section 4.2.2.

The locations of the characteristic surfaces are signatures for the MHD flow. Depending on the initial magnetic flux distribution (cases A,B) and the mass flux profile (see also Fendt 2006), this location may vary a great deal. In our case B simulations we generally observe surfaces which leave the domain in radial direction (parallel to the disk surface). For the case A simulations these surfaces tend to “collimate” leaving the domain in vertical direction. The latter implies a two-layered structure of the jet—a central super-fast magnetosonic jet surrounded by a sub-Alfvénic outflow. This is an interesting aspect for observational modeling and for stability analysis of sheath-spine jets (Pushkarev et al. 2005; Mizuno et al. 2007; Kovalev et al. 2007; Hardee 2007; Beskin & Nokhrina 2009). The broad wind launched from the outer regions of the disk has much lower velocities, decreasing continuously with increasing launching-radius. For example, the terminal velocity of the flow originating from $r_{\text{fp}} > 32$ of the case A simulations drops below $0.2c$, consistent with the X-ray absorption features observed in a mounting number of AGNs (Cappi 2006; Turner & Miller 2009).

In principle, our dynamical models can provide basic ingredients (e.g., flow geometries and velocity gradients) for the modeling of spectral line profiles of disk winds (Knaige et al. 1995; Sim et al. 2008).

Following Fendt (2006), we may define an average collimation degree $\xi$ of the outflow measured as the fraction of vertical and radial mass flux through equal-area surfaces at a certain height (here at $z = z_m$),

$$\xi = \frac{\int_{0}^{\infty} r\Gamma v_r \rho|_{zm} \, dr}{\int_{z_m - r_{m}/2}^{z_m} r_m\Gamma v_r \rho|_{zm} \, dz}.$$

The corresponding values for $\xi$ and $r_{\text{jet}}$ derived for $z_m = 200$ at the upper end of the domain and for time $t/P = 500$ are given in Table 2 along with the maximum Lorentz factor $\Gamma_{\text{max}}$, the maximum poloidal velocity $v_{p,\text{max}}$, and the total mass flux $M$. Figure 5 shows the time evolution of these quantities in the top panel. In general, we observe that the collimation degree $\xi$ is the most sensitive tracer for secular trends among the observables mentioned. In the lower panel, we show the evolution of jet-power in the individual energy channels leaving the computational domain (radial and vertical). After the re-configuration of the initial stationary state to the dynamical solution, the partitioning of energies is completed at around...
Figure 6. Logarithmic (rest-frame) density of the stationary flow (simulation run WA04). Shown are poloidal magnetic field lines (solid white), electric current flow lines (solid black), characteristic MHD surfaces (various dot-dashed green), surface of escape velocity (dotted green), light surface (solid green). The arrows in the top plot indicate the velocity field. The bottom figure is an enlarged picture of the central region indicating the three regimes defined by the light surface.

$t/P = 100$. Thermal energy-flux peaks when the hot bow shock passes through the upper boundary. Far away from the central object, gravitational and thermal energy flux are negligible. The integrated energy flux is dominated by rest mass, reflecting the fact that only the inner component reaches significant Lorentz factors. The balance between Poynting and kinetic flux is of particular interest. Figure 5 shows merely the end result of the spatial conversion history with the remaining electromagnetic energy $S$ above the purely kinetic part $K$. More detailed insight into how this is established is provided in the following section using an individual field line.

4.2. Stationary State Analysis

Simulations starting from an initial field distribution $A$ evolve into a quasi-stationary flow solution after about 200 inner disk rotations. Figure 6 shows our reference simulation WA04 at time $t/P = 250$, including an enlarged subgrid of the innermost area of the domain.

Steady-state solutions are helpful to understand the flow structure for a number of reasons. First, by using MHD conservation laws, the conserved quantities (see Section 2.3.2) allow to identify the momentum and energy channels of the flow during acceleration and collimation. Second, by using the force-balance Equations (17) and (19) we may identify the leading forces on the material along the outflow. Third, the cross-check for conserved quantities provides another test for the quality of our setup and the numerical approach. A secondary indicator of stationarity is the alignment of poloidal velocities with the poloidal magnetic field lines, $E_\phi = 0$. Figure 6 shows corresponding velocity vectors confirming this picture.

This is confirmed by checking in detail the complete set of integrals of motion of the MHD-flow $k$, $Q^F$, $Q$, $l$, $\mu$ as defined in Equations (20)–(24). Figure 7 shows the relative deviation of these quantities from their average value along a given field line after $t/P = 500$. The integrals are conserved within 1%-accuracy already right above the injection boundary—clearly demonstrating the quality of the choice of our numerical setup, in particular the injection boundary conditions carefully constructed from an equilibrium of Keplerian rotation and gas pressure.

Due to the differential rotation law, the number of Keplerian rotations $t/P(r)$ scales with radius as $t/P(r) = (t/P)_{r_{1}}^{-3/2}$, implying that at the end of our simulations ($t/P = 500$), we have performed roughly one rotation at $r = 64$ and half a rotation at $r = 100$. Nonetheless the integrals of motion for the field line $r_{fp} = 64$ are conserved within 0.1%.

4.2.1. Energy Conversion

In the simulations where injection is submagnetoslow, the energy flux is not a free parameter, but is consistently determined by the simulation of the disk wind. It is hence of interest how the partitioning and conversion is realized. From the values of $\mu_{max}$ given in Table 2 it is obvious that our disk corona supports only mildly relativistic flows below $\Gamma = 1.5$ (Section 2.3.2).

In Figure 8 (bottom left panel), we show the efficiency $\sigma$ of Poynting flux to kinetic flux conversion along the field line...
with $r_{fp} = 2$ in the fast component of the jet. Here, $\sigma$ is below equipartition already at the inlet and it further decreases as $\Gamma$ approaches $\mu$. In Figure 8 (top left), the toroidal velocity shows that the flow decouples from co-rotation with the magnetic field at the Alfvén point. Beyond the Alfvén point, angular momentum is then carried predominantly by the magnetic field. The poloidal velocity increases from low injection value (sonic velocity) to $\sim 0.5c$. Further acceleration cannot be expected as the bulk of the energy is already in kinetic form. The right panel of Figure 8 shows the individual energy channels compared to the rest-mass flux for the same field line. At the base of the jet, the strong poloidal electric currents (a strong toroidal field) give rise to an outflow with $K < T < -G < S < M$, predominantly transporting energy via rest mass and Poynting flux.

The kinetic energy flux surpasses the thermal flux at the Alfvén point and further overcomes the gravitational binding energy term shortly thereafter. This is not surprising, since the escape surface can be close to the Alfvén surface at least for the inner field lines (see also Figure 6).

Only then, the cold limit $\mu = \Gamma(\sigma + 1)$ is applicable—it is certainly valid in the asymptotical outflow where thermal and gravitational energy fluxes are negligible.

### 4.2.2. Collimating and Accelerating Forces

In this section, we identify the forces responsible for jet acceleration and collimation applying the steady-state parallel and transversal force-equilibrium Equations (17) and (19).

Figure 9 compares these forces for a number of reference simulations (WB01, WA02, WA05) along a field line rooted at $r_{fp} = 2$. As check for consistency, we also show the gradient of the Mach number $a \equiv B_p^2/(4\pi)\nabla_\| M^2$ which just coincides with the summation of the parallel forces, indicating a steady state (see the yellow solid and black dashed lines).

In general, the outflow starts with sonic speed and is first launched by thermal pressure in the hot disk corona, respectively, the centrifugal force in the colder version. Until the Alfvén point, the Lorentz force of the poloidal electric current ($F_{\text{pol}} + F_{\text{pinch}}$) is the main magnetic driver. Ultimately the poloidal tension ($F_{\text{curv}}$) keeps the acceleration up even above the fast surface.

Concerning the transverse force, we reproduce the expected sign change of the curvature (tension) force (first collimating until the Alfvén surface, de-collimating beyond) and the poloidal pressure force (de-collimating until the light cylinder, collimating beyond). For the cross-field balance, we observe the following three regimes.

Just on top of the inlet, the main de-collimating forces besides poloidal magnetic pressure are thermal pressure in the hot case (WB02, WA02) and centrifugal support in the colder case (WA05). Gravity is here the strongest force toward the origin and the situation just reflects the radial force-equilibrium we have applied for the inlet boundary. This is the hydrodynamic regime.

At the Alfvén point, the residual of the pinch- and toroidal pressure-force ($j_p \times B_\phi$) is the main collimator, balanced by the centrifugal term. Thermal pressure quickly loses importance. This is the MHD regime.

In the asymptotic region beyond the light cylinder, de-collimation by electric forces overcomes the centrifugal force and is balanced by the poloidal magnetic pressure that changes its sign at the light cylinder (best seen in WB01). This is the relativistic regime.

To get a global impression on the relative importance of the individual forces we show a radial cut throughout the
asymptotic jet in Figure 10. The strongest forces arise across the inner asymptotic light surface which separates field lines of high angular velocity from those in the non-rotating corona along the axis. Here the electric de-collimation is essential. The $B_\phi(r)$ profile is curled up from the inner disk radius along the outflow—resulting in a magnetic pressure gradient that works in unison with the toroidal field pinch force until at some radius the toroidal field surpasses its maximum and decreases (negative gradient).

The strong gradients in toroidal field and rotation induce a current sheet and give rise to an electric charge. The space charge $\rho_e = (1/4\pi) \nabla \cdot E$ is positive close to the axis and changes its sign at a critical line as defined by Goldreich & Julian (1969).

4.3. Dependence on the Launching Environment

For the simulations described up to now, we have performed in addition several parameter runs in order to investigate how the resulting jet dynamics depends on the (prescribed) launching conditions—the disk corona (see Table 2). We now focus on the impact of the plasma $\beta$ and the disk temperature parameter $\epsilon$. 
In general, in a low $\beta$ (a stronger magnetic field) we find that the outflow tends to collimate more, as indicated by the higher average collimation degree $\xi$ and a lower momentum weighted jet radius $r_{\text{jet}}$. This in principle decreases the MHD acceleration efficiency which critically depends on the divergence of flux surfaces. It is straightforward to define the mass flux-weighted efficiency which critically depends on the divergence of flux field distribution.

The impact of the magnetic field strength on the amount of mass flux is not clearly visible, as the two simulations with $\epsilon = 1/6, 2/3$ show a different trend. As the $\epsilon$-parameter is simply a proxy for the disk corona density, it will affect the collimation in the following manner. A higher inflow density lowers the Alfvén surface toward the disk surface which in turn broadens the current topology and therefore widens the flow. This is also the trend that we observe in the indicators $\xi$ and $r_{\text{jet}}$. Given that the injection speed calculated iteratively from the outflow simulation approaches the slow magnetosonic speed, we expect the mass flux to scale as $\dot{M} \propto \sqrt{\pi \rho \beta}$. In fact, this is approximately realized since we have $M(\epsilon = 1/6)/M(\epsilon = 2/3) = \sqrt{2/3} \approx 1.6$.

The change of the initial split-monopole inclination $\theta$ has little effect on the overall jet collimation angle. We observe an opposite trend as the wider initial field with $\theta = 77^\circ$ ends up slightly more collimated than the one with $\theta = 85^\circ$. Clearly a wider initial field leads to a larger jet radius $r_{\text{jet}}$. Here, we like to stress the point that for the final steady-state solutions in our simulations the initial field structure is important only insofar as it also prescribes the poloidal magnetic field profile along the outflow launching boundary. The field structure is completely changed from the initial steady structure to a new dynamic equilibrium. Thus it makes no sense to compare the collimation of the initial field with the collimation of the outflow field distribution.

Having pointed out the crucial role of the vertical energy flux from the disk surface $\mu$ and the closely related quantity $\sigma = S/(K + M)$, we now study the two most promising handles in increasing $\mu$. That is (1) a decrease in mass flux $\dot{M}$ and (2) an increase in Poynting flux $S$. We first focus on (1) and describe (2) thereafter.

4.3.1. Toward Low Mass Loading

A way to obtain high-speed jets seems to be a lower mass load injected into a similarly strong magnetic flux. According to the well-known Michel-scaling (Michel 1969), the asymptotic outflow velocity depends on the mass flux $u_\infty \propto M^{-1/3}$. We investigate this interrelation running another set of simulations where we change the mass flux by prescribing a low injection speed it approaches some seemingly unphysical offset value. At higher $v_{\text{inj}}$, the mass flux follows the expected linear dependence on the injection parameter (thin solid line).

![Figure 11. Jet collimation and dynamics against injection speed parameter $v_{\text{inj}}$. The horizontal line indicates the mass flux when $v_{\text{inj}}$ is not specified (WA01). For $v_{\text{inj}} < 0.4$, the mass flux approaches an obviously unphysical offset value. At higher $v_{\text{inj}}$, the mass flux follows the expected linear dependence on the injection parameter (thin solid line).](image)

![Figure 10. Trans-field force cut at $z = 200$, inner ($l_c$) and outer ($l_c$) light cylinder. The differentially rotating field lines are fastest at the inner disk radius, resulting in the inner light cylinder—here electric de-collimation is important. The $B_\phi(r)$ profile is curled up from the inner disk radius onward and results in a magnetic pressure gradient that works in unison with the pinch force until the toroidal field surpasses its maximum. Within $r < 3$, we omit the curves for thermal and poloidal pressure. These terms fluctuate around $\pm 10^{-5}$ while balancing each other (run WA02).](image)
injected flow is submagnetosonic and thus overdetermined by simultaneously assigning a profile in $\rho$ and $v_z$. Although this is in principle problematic, it is not necessarily fatal for the investigation of low mass flux outflows. The mass flux is governed by the magneto-slow point and is thus rearranged along the flow. Indeed we find that above the magneto-slow surface the field line constants are very well conserved. This indicates that the flow dynamics transcends at the magneto-slow surface from a seemingly unphysical state into a MHD flow that satisfies the critical conditions at the magnetosonic surfaces.

This technique of self-adjusting the subslow mass inflow, however, turned out to be limited toward lower mass fluxes. The resulting mass fluxes are unfortunately, still too high and do not allow substantially higher magnetization (e.g., $\mu_{\text{max}} = 2.01$ for simulation 1001 compared to $\mu_{\text{max}} = 1.33$ in run WA01). Figure 11 shows the trends concerning collimation $\xi$, jet radius $r_{\text{jet}}$, and integral mass flux $M$ in the asymptotic flow. Increasing the mass flux enhances collimation while $\mu_{\text{max}}$ decreases accordingly. For high injection speed, we observe that the wind originating from the very inner disk evolves into a thin ballistic flow layer that has little in common with the jets we are interested in.

### 4.3.2. Poynting Dominated Flows

Given the limitations mentioned above, we prescribe a priori the limiting energy flux parameter $\mu$ by constraining both the mass flux and the Poynting flux along the injection boundary— with the hope of thus providing a sufficiently energetic disk wind.

To achieve this, we adopted a fixed-in-time toroidal magnetic field distribution, $B_\phi \propto -\eta/r$ which necessarily changes $\Omega^2(r)$. The toroidal field distribution following a $1/r$ profile corresponds to $j_z = 0$ and has a profound physical motivation as it anticipates the radial currents expected in the disk corona.

Table 4 summarizes the simulation runs performed within this setup. These simulations have a considerably stronger toroidal magnetic field at the injection point. We apply up to $B_\phi \sim 4B_p$.

An exemplary process enhancing large-scale toroidal fields in the disk corona could be the MRI-driven dynamo under current investigation by many authors (e.g., Miller & Stone 2000; von Rekowski et al. 2003). The jet eventually evolving from these disks is not propelled by the Blandford & Payne (1982) mechanism, but driven by the toroidal magnetic pressure (Contopoulos 1996; De Villiers et al. 2003; Kato et al. 2004). These so-called Tower jets have initially been proposed by Lynden-Bell (1996) and directly extract Poynting flux from the disk rather than first converting rotational energy into the twisted magnetosphere that is present at the Alfvén surface.\(^7\)

\(^7\) In case of a central black hole causality requires that $r\Omega^2(r) < 0.6$ which corresponds to the ISCO velocity for the Schwarzschild case with $r_{\text{ISCO}} = 1$ in our scaling.

\(^8\) See Kato (2007) for a review. Note also that these jets have successfully been reproduced by Lebedev et al. (2005) in laboratory experiments with purely radial current distributions at the base.

The simulations described here are of a mixed type, since they combine large-scale open field lines with a toroidal field emerging from a radial current. For an example simulation of this type, we show the conversion of energy in Figure 12 similar to Figure 8 for the low-energy case. By design, the injected Poynting flux surpasses the rest-mass flux with $\sigma \simeq 5$ within the fast outflow component.

The outflow, which is initially Poynting flux dominated, does not reach equipartition at the fast magnetosonic surface $r = r_F$, where we merely find $\Gamma \sim \mu^{1/3}$ following Michel (1969) and Beskin et al. (1998). In the asymptotic region $r \gg r_F$, the length scales for additional flow acceleration and collimation would increase exponentially with $\Gamma \propto (\mu \ln r)^{1/3}$ (e.g., Tomimatsu 1994), which is clearly beyond the reach of our numerical method.

Our simulations indicate that, given sufficient Poynting flux, bulk Lorentz factors derived from AGN jet observations can be obtained within several hundred Schwarzschild radii, $\Gamma \sim 6$ for model M08. Since acceleration has proven to be most effective around the Alfvén point which is expected to be very close to the central object ($r_A < r_{\text{ISC}}$), we expect this conclusion to remain valid also for higher $\mu$ and thus higher terminal $\Gamma$ flows. However, we note that when increasing $\sigma$, the energy conversion efficiency decreases, a situation commonly denoted as $\sigma$-problem in pulsar winds (Rees & Gunn 1974; Kennel & Coroniti 1984). Several authors have recently addressed this issue with partly controversial results (Komissarov et al. 2009; Tchekhovskoy et al. 2009; Luystersky 2009), so that after the successful acceleration toward relativistic speeds, Poynting flux could still remain and one is tempted to ask: is there a $\sigma$-problem for AGN jets? The simulations presented here are not fit to answer this question satisfactorily. However, we certainly know that the mildly relativistic disk winds presented earlier do not suffer from this, as they are launched already in subequipartition.

\begin{table}[h]
\centering
\caption{Poynting Dominated Flows}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|}
\hline
ID & Top & $\beta$ & $\epsilon$ & $v_{\text{inj}}$ & $\eta$ & Remarks & $\Gamma_{\text{max}}$ & $\mu_{\text{max}}$ & $\xi$ & $v_{p,\text{max}}$ & $r_{\text{jet}}$ & $M$ \\
\hline
M02 & A & 0.2 & 2/3 & 0.1 & 2 & & 2.18 & 3.29 & 27.55 & 0.86 & 23.71 & 22.48 \\
M04 & A & 0.2 & 2/3 & 0.1 & 4 & & 3.51 & 7.46 & 8.96 & 0.94 & 29.00 & 48.85 \\
M08 & A & 0.2 & 2/3 & 0.1 & 8 & & 6.11 & 25.61 & 6.8377 & 0.97 & 35.89 & 80.40 \\
\hline
\end{tabular}
\end{table}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure12.png}
\caption{As in Figure 8, however calculated for simulation run M04. The injected flow is strongly magnetized and remains Poynting-dominated when leaving the computational domain.}
\end{figure}
5. SUMMARY

We have presented ideal MHD simulations of the formation of special relativistic disk winds using the PLUTO 3.0 code. On the technical side, the key points are as follows.

1. The inclusion of (Newtonian) gravity allows us to specify an astrophysically sensible boundary condition of a hydrodynamically stable disk corona. We can thus consistently follow the acceleration from initially subescape velocity winds.

2. Much dedication has been put in the development and testing of a novel realization for the outflow boundary that enables us to simulate for hundreds of inner disk rotations while minimizing spurious collimation due to artificial boundary currents. Our detailed study of jet collimation is possible only through this effort.

As a general result we obtain well collimated jets with a mass flux weighted half-opening angle of 3°–7° and mildly relativistic velocities depending on the launching conditions for the outflow. The flow collimation happens mainly in the classical (non-relativistic) regime before the light surface. A major result of our simulations is that we—for the first time—can self-consistently calculate the shape of that light surface. The light surface determines the “relativistic” character of the flow. Material which traverses the light surface experiences the full relativistic effects.

We can identify three dynamically distinct regions in terms of flow collimation.

1. In the hydrodynamic regime upstream of the Alfvén surface, gravity balances thermal and magnetic pressure, respectively, the centrifugal force in the colder case.

2. In the MHD regime following the Alfvén surface downstream, the residuals of magnetic pinch and the toroidal magnetic pressure gradient balances the centrifugal force.

3. In the relativistic regime located downstream of the light surface, the poloidal magnetic pressure gradients now impose a collimating force against electric field de-collimation. Electric forces ultimately overcome the classical magneto-centrifugal contribution.

A steep rotation profile of the field line as given by a Keplerian disk results in a light surface geometry which steepens for large radii. Depending on the magnetic field profile, the light surface may even collimate along the flow for large radii. In such a case the relativistic core inside the light surface is naturally confined. The light surface determines the “relativistic” character of the flow. Material which traverses the light surface experiences the full relativistic effects.

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A steep rotation profile of the field line as given by a Keplerian disk results in a light surface geometry which steepens for large radii. Depending on the magnetic field profile, the light surface may even collimate along the flow for large radii. In such a case the relativistic core inside the light surface is naturally confined by a non-relativistic wind. The ability of both the relativistic jets and the non-relativistic disk winds to collimate may provide confining agents for an axial ultra-relativistic funnel which could confine jets and the non-relativistic disk winds to collimate may provide confining agents for an axial ultra-relativistic funnel which could

Figure 13. Construction of the $\nabla \times B_p = 0$ and $\nabla \cdot B = 0$ boundary condition. Shown is the last grid slab of the domain $(i_{\text{end}}, j)$ and a ghost zone of two elements.

We thank Andrea Mignone and the PLUTO team for the possibility to use the PLUTO code and for his support with the relativistic module. Numerical simulations were performed on the PIA cluster of the Max Planck Institute for Astronomy (Heidelberg) located at the Rechen–Zentrum in Garching. O.P. likes to thank Bhargav Vaidya for his commitment in numerous discussions. We are thankful for precise and constructive comments by the anonymous referee. We are pleased to acknowledge clarifying conversations with Max Camenzind.

APPENDIX A

ZERO CURRENT BOUNDARY

One of the major goals of this paper is to investigate the collimation behavior of relativistic outflows. It is therefore essential to exclude any numerical artifacts leading to a spurious flow collimation. We find that the standard zero-gradient outflow boundary conditions may lead to an unphysical Lorentz force in radial direction implying such spurious collimation (or de-collimation).

Thus, we put substantial effort in implementing and testing an enhanced outflow boundary condition to the code.

To get a handle on the Lorentz force $j \times B$, one has to address the toroidal electric currents at the grid boundary. In principle there are (at least) two options. One is the possibility to copy the toroidal electric current across the boundary. While this approach should minimize spurious collimation efficiently, we observed that the overall stability of the simulation was decreased. Thus, we decide to use the following zero-toroidal current outflow boundary conditions in our simulations.

In this case, we take advantage of the staggered grid by enforcing zero toroidal currents while simultaneously satisfying the solenoidal condition $\nabla \cdot B = 0$. In the following, our procedure is described in detail. We consider computational grid cells $(i_{\text{end}}, j)$, adjacent to the domain boundary at $(i_{\text{end}} + 1, j)$, as illustrated in Figure 13. The magnetic field components of the domain, $B_i(i_{\text{end}} + 1/2, j)$, $B_m(i_{\text{end}} + 1/2, j)$, and $B_n(i_{\text{end}} + 1/2, j + 1)$,
together with the transverse field component $B_t(\ell_{\text{end}}+1, j + 1/2)$ of the first ghost zone, constitute a toroidal corner-centered electric current $I_\phi(\ell_{\text{end}}+1/2, j + 1/2)$. Utilizing Stokes theorem, $I_\phi = \int dS \cdot \nabla \times B_p = \int dI \cdot B_p$, we then solve for the unknown field component $B_t(\ell_{\text{end}}+1, j + 1/2)$ under the constraint that $I_\phi = 0$.

$$B_t|_{\ell_{\text{end}}+1,j+1/2} = B_t|_{\ell_{\text{end}},j+1/2} + \frac{\Delta r}{\Delta z} \left[ B_n|_{\ell_{\text{end}}+1/2,j+1} - B_n|_{\ell_{\text{end}}+1/2,j} \right], \quad (A1)$$

where we have assumed an equally spaced grid for clarity of the argument. Once $B_t(\ell_{\text{end}}+1, j + 1/2)$ is known for all $j$, the next layer of normal field components $B_n(\ell_{\text{end}}+3/2, j)$ can be inferred from the $\nabla \cdot B = 0$ constraint in its integral form,

$$B_n|_{\ell_{\text{end}}+3/2,j+1} = \frac{\Delta S_i B_t|_{\ell_{\text{end}}+1/2,j+1} + (\Delta S_i B_t|_{\ell_{\text{end}}+1/2,j+1} - \Delta S_i B_t|_{\ell_{\text{end}}+1/2,j+3/2})}{\Delta S_i|_{\ell_{\text{end}}+3/2,j+1}}. \quad (A2)$$

For the next grid layer, the transverse field components can again be found applying Equation (A1), and the process is repeated for each layer.

Some words of caution. We find that the current-free magnetic field boundary condition can only be realized when the grid cell aspect ratio $\Delta z/\Delta r$ is not too large. An aspect ratio of, e.g., $12/1$ resulted in errors of $100\%$ in $B_n$ at the most critical areas close to the symmetry axis leading to an overall unstable flow evolution. We find that as a rule of thumb, an aspect ratio of $3/1$ should not be exceeded. We also emphasize that it is essential to treat the grid corners consistently. This is because field components in the corner, $B_t(\ell_{\text{end}}+1, j + 1/2)$, and $B_n(\ell_{\text{end}}+1/2, j_{\text{end}}+1)$, are interrelated which would lead to an ambiguity. In order to avoid this ambiguity, we decided to extrapolate the values in question which does provide the information that is missing otherwise.

We demonstrate quality of our approach by showing results of simulations which do not apply the zero current but the zero-gradient or the zero second derivative outflow condition with otherwise the same flow parameters as in simulation WA04 (Figure 14). As it can be seen, for zero-gradient boundary conditions, the effect of collimation by artificial currents is so strong that no steady state can be reached and the flow is continuously squeezed toward the axis. Also in zero second derivative, we observe an artificial alignment with the grid geometry.

Figure 14. Comparison of simulations applying a variation of outflow boundary conditions for the magnetic fields at the time of 100 inner disk rotations. The parameters are equal to those in simulation WA04. The gray scale indicates the Lorentz factor $\log(\Gamma - 1)$ as in Figure 4, the poloidal magnetic field (the poloidal electric current) is shown in thick (thin) white contours. Standard zero gradient, zero second derivative, zero current boundary condition, respectively (from left to right).

We check the geometry dependence of the zero current outflow boundary by several realizations of the fiducial run WA04, each with the same resolution but with a different grid size or shape. Figure 15 (left panel) compares the steady-state flow characteristics for various boxes with ratios $\Delta z/\Delta r \in \{1/1, 2/1, 4/1\}$. While geometries and sizes with $\Delta z/\Delta r \geq 2/1$ are in excellent agreement, the quadratic domains show significantly thinner characteristics. The reason for this discrepancy is the sub-Alfvénic flow that traverses the $z_{\text{end}}$ boundary in “broad” domains. In these underdetermined simulations, current circuits start to uncloase at the sub-Alfvénic part of $Z_{\text{end}}$ which ultimately destroys the butterfly shape in the entire domain. As also noticed and extensively discussed by Krasnopolsky et al. (1999), a sub-Alfvénic (vertical) outflow cannot obtain the proper critical point information and leads to erroneous extensive collimation. This problem can be avoided by taking the position of the critical Alfvén surface into account, hence we choose a ratio of $2/1$ for our science simulations.

Finally, we check convergence by comparison to a half-resolution run with $256 \times 512$ grid elements. The solutions are
in good agreement, indicated by contours of the Alfvén mach number in Figure 15 (right panel). In conclusion, we use a grid of 512 × 1024 cells with a domain size of \((r, z) = (102, 204)\) inner disk radii, ensuring that the presented results depend mostly on the disk corona boundary.

### APPENDIX B

**THE INJECTION BOUNDARY IN MHD-JET SIMULATIONS**

The number of constraints imposed on a given boundary must equal the number of waves allowed through the boundary to the simulation domain (e.g., Bogovalov 1997). In MHD, the number of characteristics equals the number of “variables” minus one—due to the \(\nabla \cdot B = 0\) constraint—to give a total number of seven.

Another way of looking at this is by considering the critical surfaces as internal boundaries. For example, in a subsonic flow, the intersection of the two characteristics \(C_{\pm}\) with respective wave velocities \(v \pm c_s\) determines the hydrodynamical state. With respect to any subsonic boundary condition, only one characteristic is incoming, while the outgoing characteristics originate at the sonic point where its velocity vanishes. The same is true for the supersonic case, only now \(C_-\) transports the information of the sonic point downstream.

Applying a \(x-t\) diagram one may understand why the sonic point constitutes a fixed-in-time boundary condition. That is because here \(C_-\) becomes singular, and hence this part of information (the Riemann-invariant) starts to travel upstream and downstream from the critical point (see also Landau & Lifshitz 1959, Chapter X).

Following these general considerations, we see that the correct number of constraints for a submagnetoSlow boundary is four, since the flow is expected to pass through three characteristics. In other words, there are four outgoing waves: the slow magnetosonic wave, the entropy wave, the Alfvén wave, and the fast magnetosonic wave. Naturally, along a boundary where the flow is super magnetoslow this number equals five. Within this limited freedom, those boundary conditions which best prescribe the astrophysical problem should be used.

To allow the outflow to settle into a steady state requires certain conditions to be met also at the boundary. In our paper, we follow the argument by Krasnopolsky et al. (1999). According to the (axially symmetric) induction equation it is

\[
\partial_t B_z = 1/r \partial_r (r \partial_r E_\phi); \quad \partial_t B_r = -\partial_r E_\phi; \quad \partial_t B_\phi = \partial_r E_r - \partial_z E_\phi. \quad (B1)
\]

Since in steady state \(\partial_t B_z = \partial_t B_r = 0\), the only physical solution is \(E_\phi = 0\) which is satisfied by \(v_B \parallel B_0\).

A number of authors prescribe in addition a fixed-in-time value for \(E_r = \Omega^2 B_z\), however, this is equivalent to keeping the isorotation parameter \(\Omega^2\) constant in time, as the time evolution of \(B_z\) is already suppressed by the choice of \(E_\phi\). Unlike often stated, the certainly proper choice of constraining \((E_\phi, E_r)\) is not dictated by the ideal MHD condition—vanishing electric fields in the co-moving frame—which is respected by design, but by the steady-state considerations given above. In the literature of “disk-as-boundary” jet formation simulations a variety of choices for the injection boundary exist. Table 5 reviews a couple of them in chronological order.

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