On the Number of Linear Functions Composing Deep Neural Network: Towards a Refined Definition of Neural Networks Complexity

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Abstract

The classical approach to measure the expressive power of deep neural networks with piecewise linear activations is based on counting their maximum number of linear regions. However, when considering the two different models which are the fully connected and the permutation invariant ones, this measure is unable to distinguish them clearly in term of expressivity. To tackle this, we propose a refined definition of deep neural networks complexity. Instead of counting the number of linear regions directly, we first introduce an equivalence relation between the linear functions composing a DNN and then count those functions relatively to that equivalence relation. We continue with a study of our new complexity measure and verify that it has the good expected properties. It is able to distinguish clearly between the two models mentioned above, it is consistent with the classical measure, and it increases exponentially with depth. That last point confirms the high expressive power of deep networks.

1 Introduction

Deep neural networks with rectified linear units (ReLU) as activation function have been remarkably successful in computer vision, speech recognition, and other domains [Alex et al., 2012], [Goodfellow et al., 2013], [Wan et al., 2013], [Silver et al., 2017]. However, the theoretical understanding to support these experimental progress is still insufficient. Bridging this gap is an important issue that has driven many researchers to solve.

A fundamental theoretical problem is the expressivity of neural networks: which class of functions, and with which performance, can neural network compute, relatively to its architecture (depth, width, layer type, activation function). A well known related topic is the understanding of the empirically observed benefits of depth over width [Pascanu et al., 2013], [Montufar et al., 2014], [Telgarsky, 2016], [Eldan and Shamir, 2016], [Yarotsky, 2017], [Arora et al., 2016], [Serra et al., 2018].

To evaluate the expressive power of a neural network, we need to define a measure of its complexity. In the case of ReLU activation function, one way to describe complexity is to count the number of linear regions. From this perspective, we can theoretically explain the empirically observed benefit of depth over width [Pascanu et al., 2013], [Montufar et al., 2014], [Arora et al., 2016], [Serra et al., 2018]. Nevertheless, these invariants using the number of linear regions, as we will see later, do not adequately reflect the properties of the function. Other possible measures of complexity such as using Betti numbers of the linear regions [Bianchini and Scarselli, 2014], trajectories in the input space [Raghu et al., 2017], or the volumes of the boundaries of linear regions [Hanin and Rolnick, 2019] have been proposed.

In our study, we investigate how the properties of a function affect its complexity. Concretely, we consider the permutation invariant functions and the model introduced by Zaheer et al [Zaheer et al., 2017]. This permutation invariant model is proved to be a universal approximator for the class of permutation invariant continuous functions [Maron et al., 2019], [Zaheer et al., 2017]. Since this model is permutation invariant, its expressive power is strictly lower than for the fully connected model. However, we point out the maximal number of linear regions for both of them are asymptotically similar.

This highlights the fact that the connection between number of linear regions and expressive power may not be so straightforward, because it cannot distinguish between these two models clearly. Thus, we propose a new complexity measure that allows us to make this distinction.

Our main contribution is to introduce such a measure (Definition 2) and to prove that the invariant model and
the fully connected model actually have different values (Theorem 1, Theorem 3). To define our measure, we consider not the number of linear regions but the number of linear functions on them. Our measure counts them relatively to a certain equivalence relation. This relation identifies the linear functions (and their inherent linear region) that can be mapped from one to another thought a certain Euclidean transformation, i.e., isometric affine transformation. For permutation invariant shallow models, the proposed measure of complexity is the same as the number of the orbits of linear regions by permutation action. Technically, we apply theory of hyperplane arrangement which is stable by group action studied in [Kamiya et al., 2012] to count the number of orbits.

2 Preliminaries and background

A (feedforward) neural network of depth \( L + 1 \) is a composition of layers of units which defines a function \( F : \mathbb{R}^{n_0} \to \mathbb{R}^{n_{L+1}} \) of the form

\[
F(x) = f_{L+1} \circ g_L \circ f_L \circ \cdots \circ g_1 \circ f_1(x),
\]

where \( f_i : \mathbb{R}^{n_{i-1}} \to \mathbb{R}^{n_i} \) is an affine map and \( g_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_i} \) is a nonlinear activation function. Throughout this paper, we consider the rectifier linear units (ReLU) as the activation functions \( g_i \), i.e., for \( x = (x_1, \ldots, x_{n_i})^T \in \mathbb{R}^{n_i} \),

\[
\text{ReLU}(x) = (\max\{0, x_1\}, \ldots, \max\{0, x_{n_i}\})^T \in \mathbb{R}^{n_i}.
\]

Let \( n = (n_0, n_1, \ldots, n_{L+1}) \) and \( K \) be a connected compact \( n_0 \)-dimensional subset of \( \mathbb{R}^{n_0} \). Then, we define \( \mathcal{H}^{\text{full}}_{K}(n) = \mathcal{H}^{\text{full}}_{K}(n_0, n_1, \ldots, n_{L+1}) \) as the set of the restriction to \( K \) of the neural networks of the form (2.1). For any \( l = 1, \ldots, L \), we call such a network a ReLU neural network. The affine transformation \( f_1 \) can be written as \( f_1(x) = W_1x + c_1 \) with a weight matrix \( W_1 \in \mathbb{R}^{n_1 \times n_{l-1}} \) and a bias vector \( c_1 \in \mathbb{R}^{n_i} \). We call the feedforward neural network shallow (resp. deeper) if \( L = 1 \) (resp. \( L > 1 \)).

Because ReLU is a continuous piecewise linear function, a function realized by a ReLU neural network is also continuous and piecewise linear. We are interested in the structures of such piecewise linear functions. Any piecewise linear function is encoded as the set of pairs of a linear region and a linear function on it. Here, for a connected compact \( m \)-dimensional subset \( K \subset \mathbb{R}^m \) and a piecewise linear function \( f : K \to \mathbb{R}^n \), a connected region \( D \subset K \) is called a linear region of \( f \) if \( f \) is linear on \( D \) and for any connected region \( D' \subset K \) satisfying \( D \subseteq D' \), \( f \) is not linear on \( D' \). For a piecewise linear function \( f \), \( c^\#(f) \) denotes the number of linear regions of \( f \). For a set of piecewise linear functions \( \mathcal{H} \), we set \( c^\#(\mathcal{H}) = \max\{c^\#(f) \mid f \in \mathcal{H}\} \).

2.1 The number of linear regions for shallow fully connected neural networks

In order to calculate the maximum number of linear regions for shallow ReLU neural network, we use arguments from hyperplane arrangement theory as in [Pascanu et al., 2013]. Let us consider a shallow ReLU neural network \( F \in \mathcal{H}^{\text{full}}_K(n_0, n_1, n_2) \), i.e., a network of the form

\[
F(x) = f_2 \circ g_1 \circ f_1(x),
\]

where \( f_1 : K \to \mathbb{R}^{n_1} \) and \( f_2 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2} \) are two affine maps and \( g_1 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_1} \) is ReLU.

The linear regions of \( F \) depend only on the affine map \( f_1 \). We write \( f_1(x) = Wx + c \) for \( W = (a_{ij}) \in \mathbb{R}^{n_1 \times n_0} \) and \( c = (c_i) \in \mathbb{R}^{n_1} \). Let \( H_i \) be the hyperplane in \( \mathbb{R}^{n_1} \) defined as

\[
a_{i0}x_1 + \cdots + a_{i_{n_0}}x_{n_0} + c_i = 0 \quad \cdots \quad H_i
\]

for \( i = 1, \ldots, n_1 \). Then, the linear regions of \( F \) are exactly the chambers of the hyperplanes arrangement defined by \( A = \{H_1, \ldots, H_{n_1}\} \), i.e., the connected components of the complement \( \mathbb{R}^{n_0} \setminus \bigcup H_i \). Let \( \text{Ch}(A) \) denotes the set of chambers of arrangement \( A \). Then, Schlaffi showed that the cardinality \( |\text{Ch}(A)| \) of \( \text{Ch}(A) \) satisfies

\[
|\text{Ch}(A)| \leq \sum_{i=0}^{n_0} \binom{n_1}{i}
\]

and the equality holds if \( A \) is in general position [Orik and Terao, 2013, Introduction]. Here, we say that the hyperplane arrangement \( A = \{H_1, \ldots, H_{n_1}\} \) is in general position if \( A \) satisfies that for any \( r = 1, \ldots, n_0 \), the codimension of the intersection \( H_{i_1} \cap \cdots \cap H_{i_r} \) is equal to \( r \) if \( r \leq n_1 \) and \( H_{i_1} \cap \cdots \cap H_{i_r} = \emptyset \) if \( r > n_1 \) (see Appendix A.1 for an illustration). For the hyperplane arrangement \( A \) defined by the fully connected shallow neural network above, we remark that it is always possible to make it being in general position by perturbing the weight matrix \( W \) and the bias vector \( c \). Moreover, for any connected compact \( n_0 \)-dimensional subset \( K \subset \mathbb{R}^{n_0} \) and a hyperplane arrangement \( A \), we can take another hyperplane arrangement \( A' = \{H'_i \mid i = 1, \ldots, n_1\} \) such that \( |\text{Ch}(A')| \) is equal to the number of connected components of \( K \setminus \bigcup_{i} (H_i) \) by translating or scaling \( A \) if it is necessary. In particular, the maximal number \( c^\#(\mathcal{H}^{\text{full}}_{K}(n_0, n_1, n_2)) \) of linear regions of the fully connected shallow ReLU neural network having a \( n_0 \)-dimensional input layer and a \( n_1 \)-dimensional hidden layer is \( \sum_{i=0}^{n_0} \binom{n_1}{i} \). For \( n_0 \) such that \( 0 \leq n_0 \leq n_1/2 \), by [Ash, 1965, Section 4.7], the estimate of the sum of
binomial coefficients is
\[
\frac{2^n, H(n_0/n_1)}{\sqrt{S n_0/(1 - n_0/n_1)}} \leq \binom{n_1}{n_0} \leq e^{\#(H_{n, n_1, n_2})}
\]
where \( H(p) \) is the binary entropy function defined as
\[
H(p) = -p \log_2 p - (1 - p) \log_2 (1 - p)
\]
for \( 0 < p < 1 \) and \( H(0) = H(1) = 0 \).

2.2 The number of linear regions for the permutation invariant model

We review the permutation invariant shallow model introduced in [Zaheer et al., 2017] and show that this model can have as many linear regions as fully connected shallow neural network, though this model has a lower expressive power than the fully connected model. We illustrate this calculation on a simple example in Appendix A.2.

We define the permutation action on \((\mathbb{R}^n)^m\) of permutation group \(S_n\) by the following way. For \(\sigma \in S_n\) and \(X = (x_1, \ldots, x_m) \in (\mathbb{R}^n)^m\) where \(x_i = (x_{i1}, \ldots, x_{in})^\top \in \mathbb{R}^n\), we define
\[
\begin{align*}
\sigma \cdot X &= (\sigma \cdot x_1, \ldots, \sigma \cdot x_m), \\
\sigma \cdot x_i &= (x_{i\sigma^{-1}(1)}, \ldots, x_{i\sigma^{-1}(n)})^\top.
\end{align*}
\]
For a subset \(K\) of \(\mathbb{R}^n\), we say that \(K\) is stable by permutation action if for any \(x \in K\), \(\sigma \cdot x \in K\) holds for any \(\sigma \in S_n\).

We consider a permutation invariant shallow network as
\[
F(x) = f_2 \circ g_1 \circ f_1(x)
\]
where \(f_1: \mathbb{R}^n \to (\mathbb{R}^n)^m\) is a permutation equivariant affine map, i.e., \(f_1(\sigma \cdot x) = \sigma \cdot f_1(x)\) for any \(\sigma \in S_n\) and \(x \in \mathbb{R}^n\), and \(f_2: (\mathbb{R}^n)^m \to \mathbb{R}^{m'}\) is a permutation invariant affine map i.e., \(f_2(\sigma \cdot X) = f_2(X)\) for any \(X \in (\mathbb{R}^n)^m\) and \(\sigma \in S_n\), and \(g_1: (\mathbb{R}^n)^m \to (\mathbb{R}^n)^m\) is ReLU. Then, the realized function \(F\) is permutation invariant, i.e., \(F(\sigma \cdot x) = F(x)\) for any \(\sigma \in S_n\). Let \(K\) be a connected compact \(n\)-dimensional subset of \(\mathbb{R}^n\) which is stable by permutation action. Then, we define \(\mathcal{H}_{\text{inv}}^{\text{ReLU}}(n, mn, m')\) as the set of the restrictions to \(K\) of the permutation invariant ReLU neural networks of the form \((2.5)\). By universal approximation theorem [Maron et al., 2019], any permutation invariant continuous function on \(K\) can be approximated by such a neural networks for large enough \(m'\).

The set of linear regions of the above model depends only on the affine map \(f_1\) as in the fully connected case. Using [Zaheer et al., 2017] Lemma 3, by the permutation equivariance of \(f_1\), if we set \(f_1(x) = Wx + c\) for some \(W \in (\mathbb{R}^{n \times n})^m\) and \(c \in (\mathbb{R}^n)^m\), these \(W\) and \(c\) can be written as
\[
W = \begin{pmatrix} a_1 I + b_1 (I - 11^\top) \\ \vdots \\ a_m I + b_m (I - 11^\top) \end{pmatrix}, \quad c = \begin{pmatrix} c_1 1 \\ \vdots \\ c_m 1 \end{pmatrix}
\]
for some \(a_1, \ldots, a_m, b_1, \ldots, b_m, c_1, \ldots, c_m \in \mathbb{R}\). Here, \(I\) is the identity matrix in \(\mathbb{R}^{n \times n}\) and \(1\) is the all one vector in \(\mathbb{R}^n\). Thus, the set linear regions of \(F\) is sequential to the set of chambers of the hyperplanes arrangement \(B_{m,n} = \{H_{11}, \ldots, H_{mn}\}\) defined for \(i = 1, \ldots, m\) by
\[
\begin{align*}
&\quad a_1 x_1 + b_1 x_2 + \cdots + b_i x_n + c_i = 0 \quad \cdots \quad H_{i1} \\
&b_1 x_1 + a_1 x_2 + \cdots + b_i x_n + c_i = 0 \quad \cdots \quad H_{i2} \\
&\quad \vdots \\
&b_1 x_1 + b_1 x_2 + \cdots + a_i x_n + c_i = 0 \quad \cdots \quad H_{in}.
\end{align*}
\]

We calculate the number of chambers of the arrangement \(B_{m,n}\). As in Theorem 2.3, the number of chambers are bounded from above by \(\sum_{i=0}^{\infty} (mn)^i\) and attains this bound if the arrangement \(B_{m,n}\) is in general position. However, this arrangement \(B_{m,n}\) cannot be in general position. Indeed, the hyperplanes in the arrangement \(B_{m,n}\) satisfy
\[
H_{i_1,j_1} \cap H_{i_2,j_2} \cap \cdots \cap H_{i_3,j_3} = \emptyset,
\]
\[
H_{i_1,j_1} \cap H_{i_2,j_2} \cap H_{i_3,j_3} = H_{i_1,j_1} \cap H_{i_2,j_2} \cap H_{i_3,j_3} = H_{i_1,j_1} \cap H_{i_2,j_2} \cap H_{i_3,j_3} = \emptyset.
\]

for \(i_1, i_2, i_3 = 1, \ldots, m\) and \(j_1, j_2, j_3 = 1, \ldots, n\). Nevertheless, we can calculate the number of chambers of the arrangement \(B_{m,n}\) by applying the Deletion-Restriction theorem (Theorem 6 in Appendix B [Orlik and Terao, 2013] Theorem 2.56 and Theorem 2.68) under the assumption \((2.7)\) and \((2.8)\). The detail of the calculation is in Appendix B.

Proposition 1. We assume that \(m > n/2\). Then, the maximum \(b_{m,n}\) of the number of chambers of \(B_{m,n}\) is bounded from below by a function \(g(m, n)\) which is a polynomial with respect to \(m\) of degree \(n\) of which the coefficient of the leading term is bounded from below by \((2^{n/2}/n)^n/(n \sqrt{2})\).

2.3 Comparison of the numbers of linear regions

To equalize the number of hidden units in both models, we consider the fully connected models \((2.2)\) as \(n_0 = n\) and \(n_1 = mn\). Let \(K\) be a connected compact \(n\)-dimensional subset of \(\mathbb{R}^n\) which is stable by permutation action. By a universal approximation theorem
Sonoda and Murata, 2017], if we increase the number of hidden units of the fully connected shallow models, the elements of \( \mathcal{H}_{K}^{\text{full}}(n, mn, m') \) can approximate any continuous maps on a compact set of \( \mathbb{R}^n \). On the other hand, although elements of \( \mathcal{H}_{K}^{\text{inv}}(n, mn, m') \) are also universal approximators for permutation invariant functions [Maron et al., 2019], any function which is not permutation invariant cannot be approximated by the elements of \( \mathcal{H}_{K}^{\text{inv}}(n, mn, m') \). This implies that the expressive power of the permutation invariant shallow models is strictly lower than for the fully connected shallow models.

In keeping with this observation, we compare maximum number of linear regions for the fully connected (2.2) and the permutation invariant shallow models (2.5). By the estimate (2.4) with \( n_0 = n \) and \( n_1 = mn \), we have

\[
c^\#(\mathcal{H}_{K}^{\text{full}})(n, mn, n') \geq \frac{2\sqrt{2}n}{n\sqrt{2}} m^n + O(m^{n-1}).
\]

On the other hand, by Proposition [1] the maximal number of linear regions of permutation invariant shallow models is also bounded from below as

\[
c^\#(\mathcal{H}_{K}^{\text{inv}})(n, mn, n') \geq \frac{(\sqrt{2}/2)^n}{\sqrt{2}/n} m^n + O(m^{n-1}).
\]

In particular, although there is a difference of bases, \( c^\#(\mathcal{H}_{K}^{\text{inv}})(n, mn, n') \) does also increase exponentially with respect to \( n \). This means that the maximum numbers of linear regions cannot represent the difference of expressive powers of these models clearly.

This observation indicates that we should consider some refined measure for complexity and expressive power to be able to distinguish between these two classes of models more clearly.

### 3 Measure of complexity as the numbers of equivalent classes of linear functions

In this section, we introduce a measure of complexity which can distinguish permutation invariant shallow models from fully connected shallow models. Before proposing a refined measure of complexity, we observe the structure of piecewise linear function which is permutation invariant.

Let \( K \) be a connected compact \( n \)-dimensional subset of \( \mathbb{R}^n \) which is stable by permutation action and \( f: K \to \mathbb{R}^n \) be a piecewise linear function which is permutation invariant by the permutation group \( S_n \), and \( \mathcal{F}(f) = \{ (f_\lambda, D_\lambda) \mid \lambda \in \Lambda \} \) the set of pairs of linear regions \( D_\lambda \subset K \) of \( f \) and the linear associated function \( f_\lambda \) on \( D_\lambda \), i.e., \( f_\lambda \) is the restriction \( f|_{D_\lambda} \) of \( f \) on \( D_\lambda \). We call this set \( \mathcal{F}(f) \) the set of linear functions of \( f \). We often abbreviate an element \( (f_\lambda, D_\lambda) \in \mathcal{F}(f) \) to \( f_\lambda \). Then, it is easy to show that for any permutation \( \sigma \in S_n \) and any linear region \( D \) of \( f \), the image \( \sigma(D) \) of \( D \) by \( \sigma \) is also a linear region.

By this fact and the permutation invariance of \( f \), for any \( (f_\lambda, D_\lambda) \) and \( \sigma \in S_n \), there is a \( \lambda' \) such that \( \sigma(D_\lambda) = D_{\lambda'} \) and \( f_{\lambda'} = f_\lambda \circ \sigma|_{D_\lambda} \). Here, we regard the permutation \( \sigma \) as a linear transformation on \( \mathbb{R}^n \). Then, the linear transformation induced by permutation \( \sigma \) is isometric with respect to \( L^2 \)-norm, because the map taking \( L^2 \)-norm \( \|x\|_2 \) is permutation invariant.

Inspired from this observation, we define an equivalence relation \( \sim \) on the set of pairs \( \mathcal{F}(f) \) of linear functions and regions for piecewise linear function \( f: K \to \mathbb{R}^n \) as follows:

**Definition 1.** Let \( f: K \to \mathbb{R}^n \) be a piecewise linear function and let \( \mathcal{F}(f) = \{ (f_\lambda, D_\lambda) \mid \lambda \in \Lambda \} \) be the set of the linear functions of \( f \). Then, we say that \( f_\lambda \) is equivalent to \( f_{\lambda'} \), denoted by \( f_\lambda \sim f_{\lambda'} \), if there is a Euclidean transformation \( \phi: \mathbb{R}^n \to \mathbb{R}^n \) satisfying (1) \( \phi(D_\lambda) = D_{\lambda'} \) and (2) \( f_\lambda = f_{\lambda'} \circ \phi|_{D_\lambda} \). Here, a Euclidean transformation \( \phi \) is an affine map written as \( \phi(x) = A x + b \) for an orthogonal matrix \( A \) and a vector \( b \).

We can characterize the invariant function for a group action as follows (the proof is in Appendix C):

**Proposition 2.** Let \( f \) be a piecewise linear function on \( K \) and \( \mathcal{F}(f) = \{ (f_\lambda, D_\lambda) \mid \lambda \in \Lambda \} \) the set of linear functions of \( f \). We assume that there is a set \( \Phi = \{ \phi_1, \ldots, \phi_n \} \) of Euclidean transformations on \( \mathbb{R}^n \) such that for any \( \phi \in \Phi \) and any linear regions \( D_\lambda \), there is a \( \lambda' \in \Lambda \) such that (1) \( \phi(D_\lambda) = D_{\lambda'} \) and (2) \( f_\lambda = f_{\lambda'} \circ \phi|_{D_\lambda} \). Then, \( f \) is \( \Phi \)-invariant, where \( \hat{\Phi} = \{ \phi_1, \ldots, \phi_n \} \) is the group generated by \( \Phi \).

The relation \( \sim \) is an equivalence relation. Then, we propose the following measure of complexity:

**Definition 2.** We define the measure of complexity \( c^\sim(f) \) of \( f \) by the number of equivalent classes \( \mathcal{F}(f)/\sim \). For a set \( \mathcal{H} \) of piecewise linear functions, we define the measure of complexity \( c^\sim(\mathcal{H}) \) of \( \mathcal{H} \) by the maximum of \( c^\sim(f) \) for any \( f \in \mathcal{H} \).

As a trivial upper bound, \( c^\sim(f) \) is bounded from above by \( |\mathcal{F}(f)| \), i.e., the number of linear regions. In general, the set \( \mathcal{F}(f) \) of linear functions of \( f \) may be infinite. However, if \( f \) is realized by a ReLU neural network of finite width and finite depth, \( \mathcal{F}(f) \) is finite.

We calculate this measure of complexity for the previous two classes of models. We remark that any Euclidean
transformation $\phi$ does not change the volumes of linear regions. Thus, if the volumes of two linear regions $D_1$ and $D_2$ are different, then $f_1$ and $f_2$ cannot be equivalent. We use this observation later to count the number of equivalent classes.

We show examples for 1-dimensional case in Appendix A.4.

### 3.1 Fully connected shallow models

In this subsection, we show that there is a fully connected model as (2.2) satisfying that the proposed complexity is same as $\sum_{i=0}^{n_0} \binom{n_0}{i}$. Let $F$ be a ReLU shallow neural network model as (2.2) and $\mathcal{F}(F) = \{(F, D_i) \mid i = 1, \ldots, N\}$ be the set of linear functions of $F$. As remarked above, by the condition of Definition 1, if the volumes of two linear regions $D_i$ and $D_j$ are different, the corresponding linear functions $F_i$ and $F_j$ cannot be equivalent. Therefore, if all the linear regions $D_i$ have different volumes, all the equivalence class of $F$ are singletons, and its complexity $c^F(F)$ is equal to the number $N$ of linear regions. By perturbing the weight matrix $W$ or the bias vector $c$, we can make $F$ satisfy this condition. Hence, the measure of the complexity $c^\sim(H^{\text{full}}_{k}(n_0, n_1, n_2))$ of fully connected shallow ReLU neural networks is same as $\sum_{i=0}^{n_0} \binom{n_0}{i}$.

**Theorem 1.** The measure of complexity of $H^{\text{full}}_k(n_0, n_1, n_2)$ is equal to $c^\#(H^{\text{full}}_{k}(n_0, n_1, n_2))$. In particular, the following holds:

$$c^\sim(H^{\text{full}}_{k}(n_0, n_1, n_2)) = \frac{2^{n_0}H(n_1/n_0)}{\sqrt{8n_0(1-n_0/n_1)}}.$$ 

### 3.2 Permutation invariant models

Next, we consider the complexity for the permutation invariant models. In this case, the permutation action of permutation group $S_n$ induces equivalence on linear regions. This effect causes the gap between our complexity and the number of linear regions. In particular, for a permutation invariant model $F$, the complexity $c^F(F)$ is equal to the number of orbits of linear regions via permutation action. To calculate the number of orbits of linear regions, We use arguments from group action stable hyperplanes arrangement theory investigated in Kamiya et al [Kamiya et al., 2012]. See Appendix A.3 for an illustration on a simple example.

Let $K$ be a connected compact $n$-dimensional subset of $\mathbb{R}^n$ which is stable by permutation action. As in Section 2.2, the linear regions of the restriction to $K$ of permutation invariant model $F$ defined in (2.5), are the chambers of the hyperplanes arrangement $B_{m,n} = \{H_{ij} \mid i = 1, \ldots, m, j = 1, \ldots, n\}$ defined in (2.6).

Then, $B_{m,n}$ is stable by the permutation action, i.e., for any $\sigma \in S_n$, $\sigma(H_{ij}) = H_{\sigma(i) \sigma(j)}$ holds, where $\sigma(H_{ij}) = \{\sigma \cdot x \mid x \in H_{ij}\}$. Then, the set of chambers $\text{Ch}(B)$ is also stable by permutation action. We remark that the measure of complexity $c^*(H^{\text{perm}}_{n,m,m'})$ is equal to the maximum number of orbits of $\text{Ch}(B)$, because by perturbing weight matrix or bias, we may assume that any two chambers in different orbits have different volumes. We set $A_n$ to be the arrangement $\{W_{ij} \mid 1 \leq i < j \leq n\}$ called the Coxeter arrangement of $S_n$, where $W_{ij}$ is the hyperplane defined by the equation $x_i - x_j = 0$. We may assume that $A_n \cap B_{m,n} = \emptyset$ by perturbing weight matrix or bias vector if we need. Let $C_{m,n} = A_n \cup B_{m,n}$. Then, by [Kamiya et al., 2012] Th. 2.6, the following holds:

**Theorem 2.** The number of orbits of $\text{Ch}(B_{m,n})$ with respect to permutation action is equal to $|\text{Ch}(C_{m,n})|/n!$.

This theorem allows us to reduce the calculation of the number of orbits of chambers of $\text{Ch}(B_{m,n})$ to the calculation of the number $|\text{Ch}(C_{m,n})|$ of chambers of $\text{Ch}(C_{m,n})$. This can be calculated inductively by using the Deletion-Reduction theorem (Theorem 6 in Appendix B). Then, we obtain the following estimate of the complexity of permutation invariant shallow model:

**Theorem 3.** The measure of complexity of $H^{\text{perm}}_{n,m,m'}$ satisfies $c^*(H^{\text{perm}}_{n,m,m'}) \leq (n + \alpha)!/\alpha!n!$. Here, $\alpha = 2^{mH(1/m)}$ and $\gamma$ for positive real number $\gamma$ is the generalized factorial defined by $\gamma! = \prod_{0 \leq k \leq \gamma} (\gamma - k)$.

**Proof.** We set $c^\#_n$ as the numbers of the chambers of the hyperplane arrangement $c^\#_{k,n} = A_k \cup B_{m,n}$ for $A_k = \{W_{ij} \mid 1 \leq i < j \leq k\}$. This $A_k$ can be regarded as the Coxeter arrangement for $S_k$. By using this notation, it is easy to show that $c^\#_n$ satisfies the following recurrence relation:

$$c^\#_n = c^\#_{n-1} + kc^\#_{n-1}.$$ 

Using this relation, we have

$$|\text{Ch}(C_{m,n})| = \sum_{l=0}^{n} \left( \sum_{1 \leq k_1 < \cdots < k_l \leq n} k_1 \cdots k_l \right) c^0_{n-l}.$$ 

If we use the upper bound of $\alpha^{n-l}$, where $\alpha = 2^{mH(1/m)}$ as in (2.4), we have

$$|\text{Ch}(C_{m,n})| \leq \sum_{l=0}^{n} \left( \sum_{1 \leq k_1 < \cdots < k_l \leq n} k_1 \cdots k_l \right) \alpha^{n-l} = \prod_{k=1}^{n} (\alpha + k) = \frac{(n + \alpha)!}{\alpha!}.$$
Hence, by combining this and Theorem 2, the number of orbits of \( \text{Ch}(B_{m,n}) \) is bounded from above as

\[
\left( \text{number of orbits of } \text{Ch}(B_{m,n}) \right) = \frac{|\text{Ch}(B_{m,n})|}{n!} \leq \frac{(n+\alpha)!}{\alpha! n!}. \tag*{\square}
\]

3.3 Comparison of the measures between fully connected and permutation invariant models

We compare these complexities between fully connected shallow model and permutation invariant shallow model. To equalize the number of hidden units in both models, we consider \( n_0 = n \) and \( n_1 = mn \). Let \( K \) be a connected compact \( n \)-dimensional subset \( K \subset \mathbb{R}^n \) which is stable by permutation action. Then, because the maximum number of equivalent classes for fully connected shallow models is bounded from below by \( (n+\alpha)!/\sqrt{8n(1-1/m)} \) as in (2.4), where \( \alpha = 2^{mnH(1/m)} \). This means that the measure of complexity increases exponentially when \( n \) increases. Meanwhile, by Theorem 3, the maximum number of equivalent classes for permutation invariant shallow models is bounded from above by

\[
\frac{(n+\alpha)!}{\alpha! n!} \leq \frac{(n+\alpha)(n+\alpha-1)\cdots(n+\alpha-[\alpha])}{\alpha!}.
\]

In the second inequality, we used the fact that \( n+\alpha-[\alpha]-k \leq n-k+1 \). By this argument, the measure of complexity \( c^-((H_{K}^{\text{inv}}(m, mn, n')) \) of the set of the permutation invariant shallow models is bounded from above by a polynomial with respect to \( n \) of degree \( [\alpha]+1 \). By comparing these measures, we have

\[
c^-((H_{K}^{\text{inv}}(m, mn, n')) \leq \frac{(n+\alpha)!}{\alpha! n!} \leq \frac{\alpha^n}{n^\sqrt{8n(1-1/m)}} \leq c^-((H_{K}^{\text{full}}(m, mn, n')).
\]

In particular, \( c^-((H_{K}^{\text{inv}}(m, mn, n')) \) is strictly smaller than \( c^-((H_{K}^{\text{full}}(m, mn, n')) \). Therefore, the proposed complexity behaves better to evaluate expressive power than simply counting linear regions.

4 Specific deeper models

In this section, we provide a variant of the model which have been introduced by Montúfar et al [Montúfar et al., 2014] and show that this can be used to show that deeper models can have much higher complexity than the shallow models have.

4.1 A variant of the model of Montúfar et al

We here introduce a variant of the model of Montúfar et al [Montúfar et al., 2014]. The original model introduced by Montúfar et al [Montúfar et al., 2014] is a deep neural network defined by some special affine maps designed to cause “folding” efficiently. From the way it is constructed, the hidden layers divide the input space into grid of hypercubes, and the division into linear regions produced by the output layer is copied into each hypercube. We modify this model to be able to control the length of hypercubes’ sides to obtain hypercuboids which have different volumes.

The model is defined as follows: We consider a neural network of depth \( L + 1 \) and width as \( (2.1) \). We assume that \( n \leq n_t \) for any \( l \) and set \( p_l = [n_t/n] \). For \( j \in \{1, 2, \ldots, n\} \), we set \( w_{j1} = (0, 0, 0, 0, 0, 0) \) as the vector \( w_j \in \mathbb{R}^n \) whose \( j \)-th entry is 1 and the others are 0. For \( l = 1, 2, \ldots, L - 1 \), we define \( \tilde{h}^{(l)} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) as follows. We take, for any \( j = 1, 2, \ldots, n \), positive integers \( a_j^{(l)}, \ldots, a_{p_l}^{(l)} \) satisfying \( \sum_{k=1}^{p_l} a_k^{(l)} = 1 \) and set \( b_k^{(l)} = (a_k^{(l)})^{-1} \) and

\[
c_k^{(l)} = \begin{cases} -b_k^{(l)} (a_k^{(l)} + \cdots + a_{k-1}^{(l)}) & \text{if } k \text{ is even,} \\ -b_k^{(l)} (a_k^{(l)} + \cdots + a_{k-1}^{(l)}) & \text{if } k \text{ is odd.} \end{cases}
\]

For \( j = 1, 2, \ldots, n \) and \( k = 1, \ldots, p_l \), we define the function \( h_k^{(l)} : \mathbb{R}^n \rightarrow \mathbb{R} \) as

\[
h_k^{(l)}(x) = \begin{cases} \max\{0, b_1^{(l)} w_j^T x\} & \text{if } k = 1, \\ \max\{0, (b_1^{(l)} + b_2^{(l)}) w_j^T x + \sum_{s=2}^k c_s^{(l)} \} & \text{if } k \geq 2. \end{cases}
\]

Using these \( h_k^{(l)} \), we define the map \( \tilde{h}^{(l)} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) as

\[
\tilde{h}^{(l)}(x) = \sum_{k=1}^{p_l} (-1)^{k-1} h_k^{(l)}(x). \tag*{\square}
\]

Then, as in Figure 1, for \( x \in \mathbb{R}^n \) such that \( a_1^{(l)} + \ldots + a_{i-1}^{(l)} < x_j < a_i^{(l)} + \ldots + a_{i+1}^{(l)} \), \( \tilde{h}^{(l)}(x) \) satisfies

\[
\tilde{h}^{(l)}(x) = (-1)^{i+1} (b_i^{(l)} x_j + c_i^{(l)}).
\]

We remark that although the input space of \( \tilde{h}^{(l)}(x) \) is \( \mathbb{R}^n \), this map depends only on \( j \)-th entry of \( x \). Hence, we can regard this as \( \mathbb{R} \rightarrow \mathbb{R} \). Moreover, this map \( \tilde{h}^{(l)} \) divides the subinterval \([0,1]\) of \( x_j \)-axis into \( p_l \) regions \((-\infty, 0], [0, a_1^{(l)}], [a_1^{(l)}, a_1^{(l)} + a_2^{(l)}], \ldots, [a_{p_l-1}^{(l)}, \infty)\) and the image of each regions by \( \tilde{h}^{(l)} \) is \([0, 1]\). This construction makes a \( p_l \)-fold “folding”.

Figure 1: The graph of $\tilde{h}^{1,(l)}$ for $p = 3$

![Diagram of $\tilde{h}^{1,(l)}$ for $p = 3$]

Figure 2: A decomposition of $[0,1]^2$ into rectangles (2-dim hypercuboids) by $\tilde{h}^{(l)}$ for $n = 2$, $p = 3$

We define $\tilde{h}^{(l)}: \mathbb{R}^n \to \mathbb{R}^n$ by $\tilde{h}^{(l)} = (\tilde{h}_1^{1,(l)}, \ldots, \tilde{h}_n^{n,(l)})^\top$. By the construction, this map $\tilde{h}^{(l)}$ can be realized as a ReLU neural network as

$$\mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}^n; \quad x \mapsto (h_1^{1,(l)}(x), \ldots, h_p^{n,(l)}(x), 0, \ldots, 0)^\top \mapsto (\tilde{h}_1^{1,(l)}(x), \ldots, \tilde{h}_n^{n,(l)}(x))^\top.$$ 

This map $\tilde{h}^{(l)}$ divides $[0,1]^n \subset \mathbb{R}^n$ into $p_i^n$ $n$-dimensional hypercuboids. We remark that the volume of the $(i_1, \ldots, i_n)$-th hypercuboid is $a_i^{1,(l)} a_i^{2,(l)} \cdots a_i^{n,(l)}$ as in Figure 2.

Then, the composition $\tilde{h}^{(L-1)} \circ \cdots \circ \tilde{h}^{(1)}$ defines the deep neural network of depth $L$ and width $n_0, n_1, \ldots, n_L$ and output $\mathbb{R}^n$. This map sends $[0,1]^n \subset \mathbb{R}^n$ to $[0,1]^n \subset \mathbb{R}^n$ and divides $[0,1]^n$ into the $(p_1 p_2 \cdots p_{L-1})^n$ $n$-dimensional hypercuboids with linear regions as in Figure 3. Then, the volume of $(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$-th hypercube is

$$\left( a_{i_1}^{1,(L-1)} \cdots a_{i_1}^{1,(1)} \right) \cdot \left( a_{i_2}^{2,(L-1)} \cdots a_{i_2}^{2,(1)} \right) \cdots \left( a_{i_n}^{n,(L-1)} \cdots a_{i_n}^{n,(1)} \right),$$

where $\tilde{t}_k = (i_{k1}, \ldots, i_{kL-1}) \in \prod_{l=1}^{L-1} \{1, \ldots, p_l\}$.

In particular, by perturbing weights if it is necessary, we may assume that any hypercuboids have different volumes.

Next, we choose a map $F: \mathbb{R}^n \to \mathbb{R}^n$ which gives a hyperplane arrangement whose chambers have different volumes introduced in Section 3.1 and by scaling, we assume that all the intersections of hyperplanes is in the interior of the hypercube $[0,1]^n \subset \mathbb{R}^n$. Finally, we take the composition $\tilde{h}_L \circ \cdots \circ \tilde{h}_1$ with $F$. Then, the hyperplanes arrangement in $[0,1]^n$ defined by $F$ is copied into each hypercuboids as in Figure 3. If we need, by perturbing weights again, we may assume that any linear region has different volume. This implies that the measure of complexity $c^{-}(F \circ \tilde{h}_L \circ \cdots \circ \tilde{h}_1)$ coincides with the maximum of the number of linear regions. In particular, this is equal to

$$\prod_{i=1}^{L-1} \left( \frac{n_i}{n} \right)^n \left( \sum_{k=0}^{n} \binom{n}{k} \right).$$

This shows the following:

**Theorem 4.** The measure of complexity $c^{-}(\mathcal{H}^{\text{full}}_{[0,1]_n}(n_0, n_1, \ldots, n_L, n_{L+1}))$ for the models of above ReLU deep neural networks is bounded from below by $\prod_{i=1}^{L-1} \left( \frac{n_i}{n} \right)^n \left( \sum_{k=0}^{n} \binom{n}{k} \right)$.

As a consequence of the arguments of Section 3.1 and this section, both of the complexities for fully connected models which appear there are same as maximum numbers of linear regions. Hence, by similar argument to [Montufar et al., 2014], the complexity of deeper models is exponentially larger than the shallow models. This also shows a benefit of depth of neural network.

### 4.2 A benefit of depth for deep set models

We here consider a permutation invariant deep model, called deep set model introduced by Zaheer et al [Zaheer et al., 2017]. This model is made by stacking some permutation equivariant maps and one invariant map. Thus, the obtained map is permutation invariant. This model has some common features with the model of Montufar et al [Montufar et al., 2014]. Indeed, the original model of Montufar et al, except for the map from the last hidden layer to the output layer, is equivalent to the deep set model. We shall modify the variant model
introduced in Section 4.1 to be a deep set model and show that deep set models also have a similar benefit of depth.

As mentioned above, deep set model is defined by stacking permutation equivariant affine maps and one invariant map. Precisely speaking, the ReLU deep neural network $f_{L+1} \circ \text{ReLU} \circ f_L \circ \cdots \circ \text{ReLU} \circ f_1$ for affine maps $f_i : (\mathbb{R}^n)^{m_{i-1}} \to (\mathbb{R}^n)^{m_i}$ is called a deep set model if $f_1, \ldots, f_L$ are permutation equivariant and $f_{L+1} : (\mathbb{R}^n)^{m_L} \to (\mathbb{R}^n)^{m_{L+1}}$ is permutation invariant. For $m = (m_1, m_2, \ldots, m_L, m_{L+1})$, let $\mathcal{H}^{mv}_{[0,1]}(n, m)$ be the set of the restrictions to $[0,1]^n$ of the above deep set models.

If we assume that the variant of model which we introduced in Section 4.1 satisfies that $h^{(l)}_{k} = \cdots = h^{(l)}_{k}$ for any $k$ and any $t$, and that $F : (\mathbb{R}^n)^{m_L} \to (\mathbb{R}^n)^{m_{L+1}}$ is permutation invariant, then the obtained neural network $F \circ \tilde{h}(L-1) \circ \cdots \circ \tilde{h}(1)$ is in $\mathcal{H}^{mv}_{[0,1]}(n, m)$. In this case, $a^{(1)}_{k} = \cdots = a^{(n)}_{k}$ holds for any $k$ and $l$. We set $a^{(l)}_{k}$ to be this number. The obtained a neural network providing the $(\prod_{i=1}^{L} p_i)^n$-dimensional hypercuboids. However, the volume of $(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$-th hypercuboid is

$$\left( \prod_{i=1}^{L-1} a^{(1)}_{i_1, \ldots, i_{L-1}} \right) \cdot \left( \prod_{i=1}^{L-1} a^{(1)}_{i_2, \ldots, i_{L-1}} \cdot \ldots \cdot a^{(1)}_{i_n, \ldots, i_{L-1}} \right),$$

where $\tilde{t}_k = (i_{k,1}, \ldots, i_{k,L-1}) \in \prod_{i=1}^{L-1} \{1, \ldots, p_i\}$. We regard the index set $\prod_{i=1}^{L-1} \{1, \ldots, p_i\}$ as an ordered set by the lexicographic order $\leq$. Then, by perturbing the weights or biases if we need, we may assume that any hypercuboid in the set of hypercuboids whose index $(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)$ satisfies $\tilde{t}_1 < \tilde{t}_2 < \cdots < \tilde{t}_n$ have different volumes, and the number of such hypercuboids is $\binom{p_1 \cdots p_{L-1}}{n}$. We choose the affine map from $\mathbb{R}^n$ to output layer $(\mathbb{R}^n)^{m_L}$ to be the one which achieves the measure of complexity of $\mathcal{H}^{mv}_{[0,1]}(n, n m_{L+1})$ as in Section 3.2. Hence, the measure of complexity of $\mathcal{H}^{mv}_{[0,1]}(n, m)$ is bounded from below by $C \cdot (m_1 \cdots m_L)^{n_n}/n!$ for a positive constant $C$. In particular, the following holds:

**Theorem 5.** The following holds: $c^- \cdot \tilde{h}(2) \circ \tilde{h}(1)$ for $n = 2, p_1 = p_2 = 3$ and an image of copies of hyperplane arrangement in $[0,1]^2$ by $F$ into the rectangles

![Figure 3: A grid decomposition of $[0,1]^2$ into rectangles by $h(2) \circ h(1)$ for $n = 2, p_1 = p_2 = 3$ and an image of copies of hyperplane arrangement in $[0,1]^2$ by $F$ into the rectangles](image)

5 Conclusion

In this paper, we defined a new measure of complexity of ReLU neural networks, which is closer to expressive power than the number of linear regions. Specifically, we considered fully connected and permutation invariant models as examples, which are indistinguishable from the conventional measure of linear regions but have different expressive power. The new complexity is introduced as the number of equivalence classes that identify linear regions and linear functions on them with those transferred by a Euclidean transformation. Considering that, we have shown that the values of the measure for the two networks above are actually different. In this sense, the proposed measure of complexity can be considered to represent the expressive power of the function more closely. We also proved that the value of the proposed measure increases exponentially for deeper networks by refining the model of Montufar et al. for both the fully connected model and the deep set model.
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A Illustrations and examples

A.1 Fully connected shallow model

In this section, we illustrate linear regions calculation for simple examples in the plane.

In the two dimensional plan, "general position" means that two lines always intersect and three lines never are concurrent. Let us see an example in the case $n_0 = 2, n_1 = 4$, i.e. four lines in the plan.

\[
\begin{align*}
&\begin{cases}
  x - y + 1 = 0 : H_1 \\
  x - y - 1 = 0 : H_2 \\
  x + y - 2 = 0 : H_3 \\
  x = \frac{1}{2} : H_4,
\end{cases}
\end{align*}
\]

This arrangement is not in general position because $H_1$ and $H_2$ are parallel or $H_1$, $H_3$ and $H_4$ are concurrent. Its number of chambers is 9 (Figure 4).

Let’s modify $H_2$ to make the arrangement being general position. Now we have :

\[
\begin{align*}
&\begin{cases}
  x - y + 1 = 0 : H_1 \\
  y = 1 : H_2 \\
  x + y - 2 = 0 : H_3 \\
  x = \frac{1}{2} : H_4,
\end{cases}
\end{align*}
\]

Now the number of chambers is $11 = \sum_{i=0}^{n_0} \binom{n_1}{i}$. It is maximal for a 4 lines arrangement in the real plan (Figure 5).

![Figure 4: The line arrangement not in general position. The number of chambers is 9.](image)

![Figure 5: The line arrangement in general position. The number of chambers is 11 and is maximal.](image)

A.2 Permutation invariant shallow model

Let us consider an example of permutation invariant shallow model with $m = n = 2$, i.e this model also implement a function from $\mathbb{R}^2$ to $\mathbb{R}^4$. We have the following two pairs of lines (Figure 6).
A.3 Measure of complexity as the number of equivalent classes

Let us consider again the last invariant model example:

\[
\begin{align*}
2x + \frac{1}{2}y - 3 &= 0 : H_{11} \\
\frac{1}{2}x + 2y - 3 &= 0 : H_{12} \\
-x + 6y &= 0 : H_{21} \\
6x - y &= 0 : H_{22}.
\end{align*}
\]

In this case, \( S_2 \) has a single element which is the permutation \( \sigma = (1 \ 2) \). Here, the action of \( \sigma \) on \( \mathbb{R}^2 \) is exactly the action of the reflection symmetry through the line \( x = y \). Then, the corresponding Euclidean transformation \( \phi \) is \( \phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) and the underlying group is \( \hat{\Phi} = \{ I, \phi \} \).

In Figure 7 we identify regions belonging to the same equivalent classes. In this case, a region is identified to its symmetric through the line \( x = y \). Therefore, we count 7 equivalent classes of linear regions: \{R1\}, \{R2,R6\}, \{R3,R7\}, \{R4,R8\}, \{R5,R10\}, \{R9\}, \{R11\}.

A.4 Examples of our measure of complexity in 1-dimensional case

Here, we show some examples in the 1-dimensional case and calculate our measures of complexity. For simplicity, we consider them on the interval \( K = [0, 1] \).

In Section 2, we introduced a measure of complexity for piecewise linear functions. Let \( f \) be a piecewise linear function on \([0, 1]\) and \([0, 1] = \bigcup_{i=1}^{m} D_i \) be the decomposition by linear regions \( D_i \) for \( f \) and \( f_i = f|_{D_i} \). Then, \((f_i, D_i) \sim (f_j, D_j)\) holds only if \(|D_i| = |D_j|\), where \(|D|\) for interval \( D = [p, q] \) is the length \( q - p \). Indeed,
Figure 7: The dashed line is the line of equation $x = 2$. We identify the equivalent regions respectively to the symmetry through the line $x = y$. The number of orbits is 7.

by Definition 1, there is a Euclidean transformation $\phi: \mathbb{R}^1 \rightarrow \mathbb{R}^1; x \mapsto ax + b$ such that $\phi(D_j) = D_i$. As $\phi$ is Euclidean transformation, $a$ is equal to $\pm 1$. If we set $D_i = [p_i, q_i]$, then $\phi(D_j) = D_i$ is equivalent to $[p_j, q_j] = [\phi(p_i), \phi(q_i)] = [ap_i + b, aq_i + b]$. As $a = \pm 1$, this implies that $|D_j| = |D_i|$. Moreover, then, $ap_i + b = p_j$ holds. In particular, $b = p_j - ap_i$ holds. Because $a$ is 1 or $-1$, there are only two choices of the Euclidean transformation $\phi: D_i \rightarrow D_j$.

Furthermore, we set $f_i(x) = \alpha_i x + \beta_i$ for $i = 1, \ldots, m$. Then, $(f_j \circ \phi)|_{D_i} = f_i$ holds. Hence, for $x \in D_i$,

$$\alpha_i x + \beta_i = \alpha_j (ax + b) + \beta_j = a\alpha_i x + \alpha_j b + \beta_j$$

holds. Thus, we have $\alpha_i = a\alpha_j$ and $\beta_i = b\alpha_j + \beta_j$.

By combining these arguments, there are at most two linear functions on $D_j = [p_j, q_j]$ equivalent to $(f_i, D_i)$ where $f_i(x) = \alpha_i x + \beta_i$ and $D_i = [p_i, q_i]$: For $f_i(x) = \alpha_i x + \beta_i$,

$$f_j(x) = \begin{cases} 
\alpha_i x + \beta_i - (p_j - p_i)\alpha_i & \text{if } a = 1, \\
-\alpha_i x + \beta_i + (p_j + p_i)\alpha_i & \text{if } a = -1.
\end{cases}$$

Based on this observation, we show some examples on $[0, 1]$.

**Example 1.** Let $f: [0, 1] \rightarrow \mathbb{R}$ be the piecewise linear function defined by $\{(f_i, D_i) = i = 1, 2, 3, 4\}$, where $D_1 = [0, 1/4], D_2 = [1/4, 1/2], D_3 = [1/2, 3/4], D_4 = [3/4, 1]$ and

$$\begin{align*}
f_1(x) &= ax + \beta & \text{for } x \in D_1, \\
f_2(x) &= a(x - 1/4) + \beta & \text{for } x \in D_2, \\
f_3(x) &= a(x - 1/2) + \beta & \text{for } x \in D_3, \\
f_4(x) &= a(x - 3/4) + \beta & \text{for } x \in D_4,
\end{align*}$$

as Figure 7. By the Euclidean transformation $\phi: [0, 1/4] \rightarrow [1/4, 1/2], x \mapsto x + 1/4$, $(f_1, D_1) \sim (f_2, D_2)$ holds. Similarly, $(f_1, D_1) \sim (f_1, D_1)$ holds for any $i$. Hence, $c^\sim(f) = 1$.

**Example 2.** Let $f: [0, 1] \rightarrow \mathbb{R}$ be the piecewise linear function defined by $\{(f_i, D_i) = i = 1, 2, 3, 4\}$, where $D_1 = [0, 1/4], D_2 = [1/4, 1/2], D_3 = [1/2, 3/4], D_4 = [3/4, 1]$ and

$$\begin{align*}
f_1(x) &= ax + \beta & \text{for } x \in D_1, \\
f_2(x) &= -a(x - 1/4) + \alpha/4 + \beta & \text{for } x \in D_2, \\
f_3(x) &= a(x - 1/2) + \beta & \text{for } x \in D_3, \\
f_4(x) &= -a(x - 3/4) + \alpha/4 + \beta & \text{for } x \in D_4,
\end{align*}$$

where $\alpha = \pm 1$, $\alpha \neq 0$, and $b = \pm 1$. As $\alpha \neq 0$, we identify the equivalent regions respectively to the symmetry through the line $x = y$. The number of orbits is 7.
as Figure 10. By the Euclidean transformation \( \phi: [0, 1/4] \to [1/4, 1/2], x \mapsto -x + 1/2, (f_i, D_i) \sim (f_2, D_2) \) holds. Similarly, \((f_i, D_i) \sim (f_1, D_1)\) holds for any \(i\). Hence, \(c^-(f) = 1\).

**Example 3.** Let \(f: [0, 1] \to \mathbb{R}\) be the piecewise linear function defined by \(\{(f_i, D_i) = i = 1, 2, 3, 4\}\), where \(D_1 = [0, 1/7], D_2 = [1/7, 2/5], D_3 = [2/5, 2/3], D_4 = [2/3, 1]\) as Figure 10. Then, we have \(|D_1| = 1/7, |D_2| = 9/35, |D_3| = 4/15, \text{ and } |D_4| = 1/3\). By the above argument in the beginning of this section, there is no Euclidean transformation \(\phi\) such that \(\phi(D_j) = D_i\) for any \(i \neq j\). Hence, \(c^-(f) = 4\).

**Example 4.** Let \(f: [0, 1] \to \mathbb{R}\) be the piecewise linear function defined by \(\{(f_i, D_i) = i = 1, 2\}\), where \(D_1 = [0, 1/2], D_2 = [1/2, 1]\),

\[
\begin{align*}
  f_1(x) &= \alpha_1 x + \beta_1 & \text{for } x \in D_1, \\
  f_2(x) &= \alpha_2 x + \beta_2 & \text{for } x \in D_2,
\end{align*}
\]

such that \(|\alpha_1| \neq |\alpha_2|\) as Figure 11. Then, there is no Euclidean transformation \(\phi\) such that \(\phi(D_1) = D_2\). Hence, \(c^-(f) = 2\).

**Example 5.** Let \(f: [0, 1] \to \mathbb{R}\) be the piecewise linear function defined by \(\{(f_i, D_i) = i = 1, 2\}\), where \(D_1 = [0, 1/2], D_2 = [1/2, 1]\),

\[
\begin{align*}
  f_1(x) &= \beta_1 & \text{for } x \in D_1, \\
  f_2(x) &= \beta_2 & \text{for } x \in D_2,
\end{align*}
\]

such that \(\beta_1 \neq \beta_2\) as Figure 12. Then, there is no Euclidean transformation \(\phi\) such that \(\phi(D_1) = D_2\). Hence, \(c^-(f) = 2\).

### B Proof of Proposition 1

In this section, we prove Proposition 1. To show this, we use the Deletion-Restriction theorem [Orlik and Terao, 2013, Theorem 2.56 and Theorem 2.68].
Theorem 6 (Brylawsky, Zaslavsky). For a hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^n$ and a fixed hyperplane $X \in \mathcal{A}$, let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be the triple defined as $\mathcal{A}' = \mathcal{A} \setminus \{X\}$ and

$$\mathcal{A}'' = \{H \cap X \mid H \in \mathcal{A} \setminus \{X\}, H \cap X \neq \emptyset\}.$$ 

Then, the following holds:

$$|\text{Ch}(\mathcal{A})| = |\text{Ch}(\mathcal{A}')| + |\text{Ch}(\mathcal{A}'')|.$$ 

By apply Theorem 6 to our hyperplane arrangement, we obtain a recurrence relation and calculate the number of linear regions for permutation invariant models.

Proof of Proposition 1. Let $B_{m,n} = \{H_{ij} \subset \mathbb{R}^n \mid i = 1, \ldots, m, j = 1, \ldots, n\}$ be the hyperplane arrangement defined by (2.6). We recall that hyperplanes of this arrangement $B_{m,n}$ satisfy the following equations:

$$H_{i_1,j_1} \cap H_{i_2,j_2} \cap H_{i_3,j_3} = \emptyset,$$

$$H_{i_1,j_1} \cap H_{i_2,j_2} \cap H_{i_3,j_3} = H_{i_1,j_1} \cap H_{i_2,j_2} \cap H_{i_3,j_3} = H_{i_1,j_1} \cap H_{i_2,j_1} \cap H_{i_2,j_3} = H_{i_1,j_1} \cap H_{i_2,j_1} \cap H_{i_2,j_3} = H_{i_1,j_1} \cap H_{i_2,j_1} \cap H_{i_2,j_3}$$

for $i_1, i_2, i_3 = 1, \ldots, m$ and $j_1, j_1, j_2 = 1, \ldots, n$.

We apply Theorem 6 to $B_{m,n}$ and $H_{m,n} \in B_{m,n}$. Then, we have

$$B'_{m,n} = \{H_{11}, \ldots, H_{1,n}, \ldots, H_{m,1}, \ldots, H_{m,n-1}\},$$

$$B''_{m,n} = \{H_{11} \cap H_{m,n}, \ldots, H_{1,n} \cap H_{m,n}, \ldots, H_{m,1} \cap H_{m,n}, \ldots, H_{m,n-1} \cap H_{m,n}\}$$

and $|B_{m,n}| = |B'_{m,n}| + |B''_{m,n}|$ Here, because $H_{m,n}$ is a hyperplane bijective to $\mathbb{R}^{n-1}$, $H_{ij} \cap H_{m,n}$ can be regarded as a hyperplane in $H_{m,n} = \mathbb{R}^{n-1}$.

Next, we consider deletion and restriction for $B''_{m,n}$ and $H_{m-1,n} \cap H_{m,n}$. Then, we have

$$B'''_{m,n} = \{H_{1,1} \cap H_{m,n}, \ldots, H_{m-2,n} \cap H_{m,n}, H_{m-1,1} \cap H_{m,n}, \ldots, H_{m-1,n-1} \cap H_{m,n}\}.$$ 

Then, in the above $(B''_{m,n})''$, by the relation (2.3), we have

$$H_{i,n} \cap H_{m-1,n} \cap H_{m,n} = \emptyset$$

for any $i = 1, \ldots, m - 2$. Hence, the any hyperplane of the form $H_{i,n} \cap H_{m-1,n} \cap H_{m,n}$ is vanished from $(B''_{m,n})''$. Moreover, by the relation (2.4), for any $j = 1, \ldots, n - 1$,

$$H_{m,j} \cap H_{m-1,n} \cap H_{m,n} = H_{m-1,j} \cap H_{m-1,n} \cap H_{m,n}$$

holds. By this relation, we can unify the hyperplanes of forms of $H_{m,j} \cap H_{m-1,n} \cap H_{m,n}$ and $H_{m-1,j} \cap H_{m-1,n} \cap H_{m,n}$. By these arguments, $(B''_{m,n})'''$ can be written by

$$(B''_{m,n})''' = \{H_{i,j} \cap H_{m-1,n} \cap H_{m,n} \subset \mathbb{R}^{n-2} \mid i = 1, \ldots, m - 1, j = 1, \ldots, n - 1\}.$$ 

Once, we set $\overline{H}_{i,j} = H_{i,j} \cap H_{m-1,n} \cap H_{m,n} \in (B''_{m,n})'''$. Then, it is easy to show that the obtained arrangement $(B''_{m,n})''' = \{\overline{H}_{i,j} \subset \mathbb{R}^{n-2} \mid i = 1, \ldots, m - 1, j = 1, \ldots, n - 1\}$ satisfies the following relations:

$$\overline{H}_{i_1,j_1} \cap \overline{H}_{i_2,j_2} \cap \overline{H}_{i_3,j_3} = \emptyset,$$

$$\overline{H}_{i_1,j_1} \cap \overline{H}_{i_2,j_2} \cap \overline{H}_{i_3,j_3} = \overline{H}_{i_1,j_1} \cap \overline{H}_{i_2,j_2} \cap \overline{H}_{i_3,j_3} = \overline{H}_{i_1,j_1} \cap \overline{H}_{i_2,j_1} \cap \overline{H}_{i_2,j_2} = \overline{H}_{i_1,j_1} \cap \overline{H}_{i_2,j_1} \cap \overline{H}_{i_2,j_2}.$$ 

for $i_1, i_2, i_3 = 1, \ldots, m - 1$ and $j, j_1, j_2 = 1, \ldots, n - 1$. This means that the hyperplane arrangement $(B''_{m,n})'''$ can be regarded as an arrangement “$B_{m-1,n-1}$ in $\mathbb{R}^{n-2}$”. We will justify this argument more precisely later.
Before we do it, we shall observe the deletion and restriction for $B'_{m,n}$ with $H_{m-1,n} \in B'_{m,n}$. Then, we have the following arrangements:

$$ (B'_{m,n})' = \{ H_{1,1}, \ldots, H_{m-2,n}, H_{m-1,1}, \ldots, H_{m-1,n-1}, \ldots, H_{m,1}, \ldots, H_{m,n-1} \}, $$

$$ (B'_{m,n})'' = \{ H_{11} \cap H_{m-1,n}, \ldots, H_{1,n} \cap H_{m-1,n}, \ldots, H_{m-1,n} \cap H_{m-1,n}, \ldots \}. $$

Then, we remark that $(B'_{m,n})''$ is same as $(B''_{m,n})'$ if we exchange $H_{m-1,j}$ and $H_{m,j}$. By these relations, we have the following diagram:

$$
\begin{array}{c}
B_{m,n} \\
\cap H_{m,n} \\
\downarrow H_{m,n} \\
B'_{m,n} \\
\cap H_{m-1,n} \\
\downarrow H_{m-1,n} \\
(B'_{m,n})' \\
\cap (B''_{m,n})' \\
\downarrow (H_{m-1,n} \cap H_{m,n}) \\
(B''_{m,n})'' \\
\end{array}
$$

To extract a recurrence relation from this diagram, we introduce an other notation: Let

$B^n_{m,n} = \{ X_{i,j} \subset \mathbb{R}^\ell \mid i = 1, \ldots, m, j = 1, \ldots, n \}$

be a hyperplane arrangement in $\mathbb{R}^\ell$ satisfying the following relations:

$$ X_{i_1,j_1} \cap X_{i_2,j_2} \cap X_{i_3,j_3} = \emptyset, \quad \text{for } i_1, i_2, i_3 = 1, \ldots, m \text{ and } j_1, j_2, j_3 = 1, \ldots, n. \quad \text{(B.3)} $$

$$ X_{i_1,j_1} \cap X_{i_2,j_2} \cap X_{i_3,j_3} = X_{i_1,j_1} \cap X_{i_2,j_2} \cap X_{i_2,j_3} = X_{i_1,j_1} \cap X_{i_2,j_1} \cap X_{i_2,j_2} \quad \text{(B.4)} $$

for $i_1, i_2, i_3 = 1, \ldots, m$ and $j_1, j_2, j_3 = 1, \ldots, n$. Then, by the above arguments and a simple consideration, we have the following diagram:

$$
\begin{array}{c}
B^n_{m,n} \\
\cap X_{m,n} \\
\downarrow X_{m,n} \\
(B^n_{m,n})' \\
\cap X_{m-1,n} \\
\downarrow X_{m-1,n} \\
(B^n_{m,n})'' \\
\cap (X_{m-1,n} \cap X_{m,n}) \\
\downarrow (X_{m-1,n} \cap X_{m,n}) \\
\vdots \\
\cap X_{2,n} \\
\downarrow X_{2,n} \\
\cap (X_{1,n} \cap X_{2,n}) \\
\downarrow (X_{1,n} \cap X_{2,n}) \\
\vdots \\
\cap X_{1,n} \\
\end{array}
$$

$$ B^n_{m,n-1} $$

Here, $B$ is the hyperplane arrangement in $\mathbb{R}^n$ defined by

$$ B = B^n_{m,n-1} \cup \{ X_{1,n} \}. $$

Let $b^n_{m,n} = |\text{Ch}(B^n_{m,n})|$. Then, by Theorem 5 with the diagram (B.5), we have the recurrence relation

$$ b^n_{m,n} = b^n_{m,n-1} + m b^n_{m-1,n-1} + \frac{m(m-1)}{2} b^{n-2}_{m-1,n-1}. $$
Moreover, by considering recursively, we can show that the following holds for \( \ell, m, n \geq 1 \):

\[
b_{m,n}^\ell = b_{m,n-1}^\ell + m b_{m,n-1}^\ell - 1 + \frac{m(m-1)}{2} b_{m-1,n-1}^\ell.
\]  

(B.6)

Here, \( b_{m,n}^0 = b_{m,0}^0 = b_{0,0}^0 = 1 \) for any \( \ell, m, n \geq 0 \) and we set \( b_{m,n}^\ell = 0 \) for \( \ell < 0 \). Then, for example, by (B.6), we have \( b_{m,n}^1 = mn + 1 \) for any \( m, n \geq 0 \), \( b_{m,1}^\ell = m^2 / 2 + m / 2 + 1 \) for any \( \ell \geq 2 \) and \( m \). In particular, \( b_{m,n}^\ell \) is a polynomial with respect to \( m \).

By this recurrence relation (B.6), we can represent \( b_{m,n}^\ell \) as

\[
b_{m,n}^\ell = \sum_{k=0}^{n/2} \sum_{\ell=0}^{n} d_{\ell,k}(m) b_{m-k,0}^{\ell-2k} = \sum_{k=0}^{n/2} \sum_{\ell=0}^{n} d_{\ell,k}(m),
\]

where \( d_{\ell,k}(m) \) is a non-negative integer. Here, the last equation follows from \( b_{m-k,0}^{\ell-2k} = 1 \) for any \( k, \ell, m \) such that \( m - k \geq 0 \) and \( n - 2k - \ell \geq 0 \). Then, it is easy to show that \( d_{\ell,k}(m) \) is obtained as a sum of multiples of \( k \) times \( m(m-1)/2^\ell \), \( \ell \) times \( m \), and \( n - k - \ell \) times 1. Here, these double quotations mean that these vary in accordance with the order of the operations. Indeed, the iteration relation (B.6) can be represented as a higher-dimensional analogue of Pascal’s triangle as Figure 13. However, because we will calculate only the coefficient of leading term of

\[
b_{m,n}^n, \ 	ext{as a polynomial of variable} \ m, \ 	ext{we may not take care of the orders. Then, the degree of} \ d_{\ell,k}(m) \ 	ext{as a polynomial of variable} \ m \ 	ext{is equal to} \ 2k + \ell. \ 	ext{This means that the leading term of} \ b_{m,n}^0 \ 	ext{as a polynomial of variable} \ m \ 	ext{is equal to the sum of terms} \ d_{\ell,k}(m) \ 	ext{for} \ 2k + \ell = n. \ 	ext{Moreover, by the fact} \ d_{\ell,k}(m) \geq 0, \ 	ext{we have}
\]

\[
b_{m,n}^n = \sum_{k=0}^{n/2} \sum_{\ell=0}^{n} d_{\ell,k}(m) \geq \sum_{k=0}^{n/2} d_{n-2k,k}(m) = \text{(the leading term of} b_{m,n}^n \ 	ext{as a polynomial of variable} \ m).
\]

We calculate a lower bound of the leading term. Then, the leading term of \( d_{n-2k,k}(m) \) as a polynomial of \( m \) can be written as

\[
d_{n-2k,k}(m) = \binom{n}{k, k, n-2k} \frac{1}{2^k} m^n + O(m^{n-1}),
\]

where \( \binom{n}{k_1, \ldots, k_m} \) for positive integers \( k_1, \ldots, k_m \) such that \( n = k_1 + \cdots + k_m \) is the multinomial coefficient defined by

\[
\binom{n}{k_1, \ldots, k_m} = \frac{n!}{k_1! \cdots k_m!} = \left( \begin{array}{c} k_1 \\ k_2 \\ \vdots \\ k_m \end{array} \right).
\]  

(B.7)
Indeed, as mentioned before, \( d_{n-2k,k}(m) \) is obtained as a sum of multiples of \( k \) times of \("m(m - 1)/2", \( n - 2k \) times of \("m", and \( k \) times of \( 1 \). Although the terms in the double quotations varies in accordance with the orders of the operations, the leading term is independent of the orders. Hence, the leading term of \( d_{n-2k,k}(m) \) is the sum of multiples of \( k \) times of \( 1/2, n - 2k \) times of \( 1 \), and \( k \) times of \( 1 \). The number of such multiples in the sum is same as \( (\frac{n}{k,k,n-2k}) \). Hence, we have

\[
d_{n-2k,k}(m) = \left( \frac{n}{k,k,n-2k} \right) \frac{1}{2^k} m^n + O(m^{n-1}).
\]

By the form of RHS of equation (B.7) and the estimate in (2.4), we have

\[
\left( \frac{n}{k,k,n-2k} \right) = \left( \begin{array}{c} k \\ n \\ k \\ n-2k \end{array} \right) \left( \begin{array}{c} k \\ n-2k \\ k \\ n \end{array} \right) = \left( \begin{array}{c} 2k \\ n-2k \end{array} \right) \left( \begin{array}{c} n \\ n-2k \end{array} \right) \geq \frac{2^{k,\Omega(1/2)}}{\sqrt{8k(1-1/2)}} \frac{2^{n,\Omega((n-2k)/n)}}{\sqrt{8k(n-2k)(1-(n-2k)/n)}} \frac{\Omega(2^{2k,\Omega(1/2)})}{\Omega(2^{n,\Omega((n-2k)/n)})} = \frac{\Omega(2^{2k,\Omega(1/2)})}{\Omega(2^{n,\Omega((n-2k)/n)})}.
\]

In the last inequality follows from \( H(1/2) = 1 \).

We evaluate the coefficient of the leading term at \( k = n/4 \). Then, we have

\[
d_{n/2,n/4}^{n}(m) \geq \frac{(2^{5/4})^n}{n\sqrt{2}} m^n + O(m^{n-1}).
\]

In particular, the coefficient of leading term of \( b_{m,n}^{n} \) is bounded from below by \( (2^{5/4})^n / (n\sqrt{2}) \). This concludes the proof.

\[\square\]

**C Proof of Proposition 2**

*Proof of Proposition 2* Let \( x \in \mathbb{R}^n \). Then, for a \( \lambda \in \Lambda \), \( x \in D_\lambda \). For \( \phi \in \Phi \), we assume that \( \phi \) satisfies (1) \( \phi(D_\lambda) = D_{\lambda'} \) and (2) \( f_\lambda = f_\lambda \circ \phi|_{D_\lambda} \). Then, we have

\[
f(\phi(x)) = f_{\lambda'}(\phi(x)) = (f_{\lambda'} \circ \phi|_{D_\lambda})(x) = f_\lambda(x) = f(x).
\]

(C.1)

This equation holds for any \( x \) and any \( \phi \in \Phi \). Because \( \phi \in \Phi \) is a Euclidean transformation, \( \phi \) is isomorphism. In particular, the inverse of \( \phi \) exists. As for any \( y \in \mathbb{R}^n \), there is a \( x \) such that \( y = \phi(x) \), by the equation (C.1), we have

\[
f(\phi^{-1}(y)) = f(x) = f(\phi(x)) = f(y).
\]

(C.2)

Hence, \( f \) is invariant by the action of \( \phi^{-1} \) for any \( \phi \in \Phi \). Now, \( \hat{\Phi} \) was the subgroup of the group of Euclidean transformations generated by \( \Phi \). This means that any element \( \phi \in \hat{\Phi} \) is a composition of finite elements of \( \{\phi_1, \ldots, \phi_t, \phi_1^{-1}, \ldots, \phi_t^{-1}\} \). Hence, by combining this fact and equations (C.1) and (C.2), \( f \) is invariant by the action of the group \( \hat{\Phi} \). \[\square\]