Boundary states in boundary logarithmic CFT

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Abstract

There exist logarithmic CFTs (LCFTs) such as the $c_{p,1}$ models. It is also well known that it generally contains Jordan cell structure. In this paper, we obtain the boundary Ishibashi state for a rank-2 Jordan cell structure and, with these states in $c = -2$ rational LCFT, we derive boundary states in the closed string picture, which correspond to boundary conditions in the open string picture. We also discuss the Verlinde formula for LCFT and possible applications to string theory.

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1 Introduction

It is clear that 2d conformal field theory (CFT)\cite{1} is an essential mathematical background to explore string theories which are thought to be candidates of the long-awaited ultimate theory of everything. Also, CFTs provide underlying theories or theoretical interpretations of 2-dimensional statistical physics.

It was first revealed by Rozansky and Saleur that some 4-point functions of CFT have unavoidable logarithmic singularities\cite{2}, and later, Gurarie showed that, with such logarithms, logarithmic fields appear in the theory, which was named, LCFT\cite{3}. The main feature of LCFT is that there is a pair, or maybe more, of primary operators which are not independent and which form a reducible but indecomposable representation of the $L_0$ operator, rank-2 – or even higher rank – Jordan cell structure, where one primary is logarithmic and the other is a state of zero norm\cite{3}. In fact, some minimal models of CFT can have such fields in principle although, in most cases, they are non-unitary or central charges of them are, somehow, irregular. Nevertheless, it is worth seeing the extent to which we can investigate them for new physics based on them, since we can ignore their non-unitary nature by having them as subsystems. Thus far, many studies have been devoted to this subject and have found the same sort of logarithmic behaviour in various models. For example, the gravitationally dressed CFT and WZNW models at different levels or on different groups\cite{2,5-7}, $c_{p,1}$ and non-minimal $c_{p,q}$ models, as mentioned above, $c_{2,1} = -2$ model\cite{3,8-15}, $c = 0$ models, describing critical polymers and percolation\cite{11,15,16,17}, quantum Hall effect, quenched disorder and localisation in planar systems\cite{18}, 2D-magnetohydrodynamic and ordinary turbulence\cite{19}. In string theory, D-brane recoil, target-space symmetries and AdS/CFT correspondence have been studied and discussed with respect to LCFTs in the literature\cite{20,21}.

On boundary CFT, which is CFT with one or more boundaries and boundary conditions, it was shown by Cardy that, with the boundary conditions, a lot of the tools developed in ordinary CFT can be used and hence n-point functions become manageable\cite{22}. These types of theories are essential in both particle physics and condensed matter physics, when some direction is required to be finite or to have one or two ends. For instance, in string theory, theories of open strings are defined on an infinite strip with two boundaries. A periodicity along its boundaries induces dual pictures on it and modular invariance leads to a one-to-one correspondence between the boundary conditions of the open string picture and the boundary states of the closed string picture, which is, on the other hand, quantised on an annulus. It was found in \cite{23} that these boundary states are spanned by boundary Ishibashi states, by which the Verlinde formula is proven to hold for unitary minimal models of boundary CFT\cite{22,24}.
In spite of much progress in both areas, little has been mentioned on LCFT with boundaries and the effects of the presence of boundary, because there is a problem of reducible but indecomposable representations which cannot be applied to boundary CFT in a straightforward way. The first systematic attempt to formulate boundary LCFT was made by Kogan and Wheater in [25], where several important problems were discussed, including the structure of boundary states in LCFT, using the \( c = -2 \) theory as an example, and the Verlinde formula. The arguments on boundary states are based on the conjectured forms of the Ishibashi states and therefore the conjecture remains to be proven. Otherwise, it should be confirmed that we can derive explicit forms of Ishibashi states without relying on it, and whether they reproduce the same result.

In this paper, we briefly review boundary CFT, LCFT and the definition of Jordan cell structure. Thereafter, we prove the existence of the boundary Ishibashi state for the rank-2 Jordan cell structure and show explicit forms of them. We also propose a conjecture of Ishibashi states for all LCFTs which contain rank-2 Jordan cell structure. After introducing the \( c = -2 \) LCFT, we show how these states prescribe the boundary states in the closed string picture, which correspond to boundary conditions in the open string picture. In consequence, we show some typical results, which potentially include the one given in [25], and take the different original result as a conclusion. Finally, we will also discuss the Verlinde formula for LCFT and possible applications to string theory.

## 2 Preliminaries

### 2.1 Boundary CFT

To begin with, consider an infinite strip of width \( L \) on which theories of open strings can lie. By a conformal map, \( w = \pi \ln z \), a theory on a \( z \) upper half-plane is mapped onto a \( w \) infinite strip, where time \( t \) goes along two parallel edges. A pair of conformally invariant boundary conditions is put onto these two edges respectively, labeled by \( \alpha, \beta \), and the Hamiltonian of this system is given by \( (\pi/L)H_{\alpha\beta} \) with a generator of \( t \)-translations, \( H_{\alpha\beta} \).

The eigenstates of \( H_{\alpha\beta} \) fall into irreducible representations of the chiral algebra and the partition function becomes a linear combination of the functions of these representations. By imposing a periodicity \( T \) along \( t \), the partition function of the open string picture reads

\[
Z_{\alpha\beta}^{\text{open}}(q) = \text{Tr} q^{H_{\alpha\beta}} = \sum_i n_{\alpha\beta}^i \chi_i(q),
\]

where \( q \equiv e^{2\pi i \tau}, \tau \equiv iT/2L \) and \( n_{\alpha\beta}^i \) is the number of times which a representation \( i \) occurs in the presence of boundary conditions \( (\alpha\beta) \). \( \chi_i(q) \) denotes a character function.
of the representation $i$.

At this point, a dual picture appears. The periodicity wraps the strip to a cylinder and the dual description of the theory is given by the change of $t$ direction to one across the strip. The boundary conditions turn to the boundary states on the initial and final ends of the cylinder and the partition function of the dual picture is constructed from the theory of closed strings. This cylindrical geometry, an annulus on $\zeta$-plane, is obtained from the strip by a map, $w = iT_2 + \ln \zeta$, and the conformal invariance of the boundary conditions amounts to the following conditions of the boundary states, $\{ | B \rangle \}$:

$$
(W_n - (-1)^s \overline{W}_n) | B \rangle = 0,
$$

where $W_n(\overline{W}_n)$ denotes a $n$-th mode of the (anti-)holomorphic sector of the chiral algebra, and $s$ is a dimension of the operator. Among solutions of eq. (2), Ishibashi states are known to form a basis of boundary-state space and express the partition function of the closed string picture in a more convenient way as below.

$$
Z_{\alpha\beta}^{\text{closed}}(\tilde{q}) = \langle \tilde{\alpha} | q^\frac{1}{2} (L_0 + \overline{L}_0 - \frac{c}{12}) | \tilde{\beta} \rangle = \sum_i \langle \tilde{\alpha} | i \rangle \langle i | \tilde{\beta} \rangle \chi_i(\tilde{q}),
$$

where $\tilde{q} \equiv e^{2\pi i \tilde{\tau}}, \tilde{\tau} \equiv -1/\tau$. Note that $\{ | \tilde{i} \rangle \}$ denote Ishibashi states and the diagonality of them, that is, of the representations is used. The same central charge is assigned to both chiral sectors, i.e. $\tau = c$.

As a consequence, the equivalence of both quantisation schemes ends up with

$$
Z_{\alpha\beta}^{\text{open}}(q) = Z_{\alpha\beta}^{\text{closed}}(\tilde{q}),
$$

where both sides of the equation have the same set of characters but of different variables. Since $\tilde{\tau} = -1/\tau$, modular properties of the characters lead to the relations between boundary states and $n_{\alpha\beta}^i$, by which the forms of the boundary states, in terms of Ishibashi states, and the values of $n_{\alpha\beta}^i$ are equated. Actually, modular properties of characters completely determine the above quantities in unitary minimal models and lead to the Verlinde formula of boundary CFT, provided that $n_{\alpha\beta}^i = \delta^i_\beta$ for some $\alpha$. The condition is satisfied when the theory has a ‘vacuum’ and $n_{\alpha\beta}^i$ is identical to the fusion rule coefficients $N_{\alpha\beta}^i$ of the theory. Since the identification is precisely what the formula means, this should be taken as the self-consistency condition of the formula, which is not sufficient.

In addition, the diagonality of Ishibashi states is essential in this construction, and this seems to be absent in LCFT, since LCFT possesses reducible but indecomposable representations which are obviously not diagonal. Nonetheless, there might be a possibility that Ishibashi states of LCFT allow the similar construction and hence the Verlinde formula. This is worth being carefully examined.
Before we turn to the Ishibashi states, it is better to see what LCFT is like and how initial states of Jordan cell structure can be defined. Being based on them, detailed proof and examinations of Ishibashi states will be given in the next section.

2.2 Jordan cell & LCFT

In unitary minimal models, the theory is characterised by a central charge \( c_{p,q} = 1 - \frac{6(p-q)^2}{pq} \) and conformal dimensions of the fields \( \phi_{r,s}(z) \), \( h_{r,s} = \{ (rp - sq)^2 - (p - q)^2 \}/4pq \), where \( p \geq 2, q = p + 1 \) are integers and the set of integers \((r, s)\) is restricted to a rectangular region, \( 1 \leq r < q, 1 \leq s < p \).

If we remove the constraint on \((p, q)\), the \( c_{p,1} \) models appear as non-unitary theories, where the above rectangular regions vanish and so do the restrictions on \((r, s)\). Instead, due to the relations \( h_{r,s} = h_{-r,-s} = h_{r+1,s+p} \), the region for \((r, s)\) is stretched to a semi-infinite rectangular region \( 1 \leq r, 1 \leq s \leq p \). Remarkably, fusion rules of them result in distinguishable states of the same conformal dimension and hence degenerate theories. They might be simply degenerate and diagonalisable, but in fact, in \( c_{2,1} = -2 \) model, a logarithmic field \( D(z) \) emerges in the fusion rule of the primary \( \mu(z) \equiv \phi_{2,1}(z) \) and gives a logarithmic singularity in its 4-point function. This, together with a normal primary field \( C(z) \) of the same dimension, forms a reducible but indecomposable representation. The general form of the pair of such fields can be written down as

\[
T(z)C(w) \sim \frac{h C(w)}{(z-w)^2} + \frac{\partial_w C(w)}{z-w},
\]

\[
T(z)D(w) \sim \frac{h D(w) + C(w)}{(z-w)^2} + \frac{\partial_w D(w)}{z-w},
\]

where \( h \) is a conformal dimension of both fields and their correlation functions are given by

\[
\langle C(z)C(w) \rangle \sim 0, \quad \langle C(z)D(w) \rangle \sim \frac{\alpha}{(z-w)^{2h}},
\]

\[
\langle D(z)D(w) \rangle \sim \frac{1}{(z-w)^{2h}} (-2\alpha \ln(z-w) + \alpha').
\]

Accordingly, a pair of initial states \( |C\rangle \) and \( |D\rangle \) forms a rank-2 Jordan cell,

\[
L_0 |C\rangle = h |C\rangle, \quad L_0 |D\rangle = h |D\rangle + |C\rangle.
\]

Verma modules of them are obtained from the above states by acting with the chiral algebra successively on them.
2.3 Jordan cell structure

On the way to consistent boundary LCFT, we fix the notations of Jordan cell structure, most of which has been introduced by Rohsiepe.\[12\]

Let $U$ be the universal enveloping algebra of Virasoro algebra $L$. Setting $L^\pm \equiv <L_n \geq 0>$, $L^0 \equiv <L_0, C>$, we can introduce $U^\pm, U^0 \subset U$, the enveloping algebras of them. Note that this can be naturally extended to any chiral algebras that are graded in the same way.

Jordan lowest weight module (JLWM) is defined by Rohsiepe as a $L$-module, $V$, satisfying

$$
\begin{align*}
(0) & \quad Cv^{(i)} = cv^{(i)}, \\
(1) & \quad L_0v^{(i)} = hv^{(i)} + v^{(i-1)}, \quad L_0v^{(0)} = hv^{(0)}, \quad (h, c \in \mathbb{C}) \\
(2) & \quad v^{(i)} \in V_0 \quad (0 \leq i \leq k - 1), \\
(3) & \quad V = U.v^{(k-1)},
\end{align*}
$$

where $V_0 \equiv \{ v \in V|^{\forall}v' \in U^+v = 0, v \neq U^-v' \}$, $h$ is lowest weight, $\{v^{(i)}\}$ are linearly independent lowest weight vectors (JLWV). The integer $k$ is called rank of JLWM. For rank-2 case, $v^{(0)}$ is called upper JLWV and $v^{(1)}$ is lower JLWV.

One may define a representation of $L$ on its dual $V^\ast$ by setting

$$
\begin{align*}
(L_i^\dagger w)^{\dagger} & = L_{-i_1}^{n_1} \cdots L_{-i_p}^{n_p}, \\
\langle L^\phi \rangle (w) & = \phi(L^\dagger w) \text{ for } \phi \in V^\ast, \ w \in V.
\end{align*}
$$

$V^\ast$ appears as a JLWM with lowest weight vectors, $v^{(i)^\ast}$, which satisfy eq.\[(8)\]. Therefore, the dual JLWM, $V^\dagger \subset V^\ast$, is naturally induced by a map, $V = \{u.v\} \rightarrow V^\dagger = \{u.v^\ast\}$, where $u \in U$, $v, v^\ast$ are lowest weight vectors on each side, respectively.

The Shapovalov bilinear form, $\langle \ | \ \rangle$, can be defined as\[2\]

$$
\forall v^{(i)} \in V, \ \exists v^{(j)} \in V^\dagger; \quad \langle v^{(j)}|v^{(i)} \rangle = \delta_{ij}
$$

which, of course, satisfies $\langle v_i|L_i^\dagger v_j \rangle = \langle v_i|L_{-i}v_j \rangle$ and, in fact, this condition prescribes the relation between $*$ and $\dagger$. Namely, the $*$ transformation is an isomorphism which acts on $V$ as

$$
* : v^{(i)} \rightarrow v^{(i)^*} = v^{(k-1-i)^\dagger},
$$

therefore, $|v^{(i)}\rangle \rightarrow \langle v^{(k-1-i)}|$, $\langle v^{(i)}| \rightarrow |v^{(k-1-i)}\rangle$.\[3\]

\[2\]You may change this orthogonality condition for some cases.

\[3\] Note that, even if you have $\langle v^{(0)}|v^{(1)} \rangle \neq 0$ for rank-2 case, we have the same result.
On $\mathcal{V}$, Virasoro generators can be divided into two parts, namely, $L_n = L^d_n + L^n_n$, such that,

\begin{align}
L^d_n : & \quad |v^{(i)}, N\rangle \rightarrow |v^{(i)}, N - n\rangle, \\
L^n_n : & \quad |v^{(i)}, N\rangle \rightarrow |v^{(i-1)}, N - n\rangle \quad \text{for } i \neq 0, \\
|v^0, N\rangle & \rightarrow 0,
\end{align}

where, for simplicity, $|v^{(i)}, N\rangle$ denotes orthogonal basis of $|v^{(i)}\rangle$-descendants at level $N$. We will use this convention from now on.

### 3 Boundary Ishibashi states

It is natural to assume that a solution $|B\rangle$ of eq. (2) takes the form,

$$
|B\rangle \equiv \sum_{\{N\}} |\alpha, N\rangle \otimes |\beta, N\rangle,
$$

because initial(final) states are in the tensor product of Hilbert spaces of both chiral sectors, $\mathcal{H} \otimes \overline{\mathcal{H}}$. Similarly, eq. (2) is equivalent to

$$
\langle j, N_1 | \otimes \langle k, N_2 | (W_n - (-1)^n \overline{W}_n) | B\rangle = 0,
$$

where $j, k, N_1, N_2$ are arbitrary.

Extracting Virasoro parts of the above conditions, let us solve the equation

$$
\langle j, N_1 | \otimes \langle k, N_2 | (L_n - \overline{L}_{-n}) | B\rangle = 0,
$$

where the left hand side can be decomposed and simplified into two parts, according to the decomposition of Virasoro algebra in the previous section.

\begin{align}
\text{lhs} & = \sum_N \left\{ \langle j, N_1 | (L^d_n + L^n_n) | \alpha, N\rangle \langle k, N_2 | \overline{\beta}, N \rangle \\
& \quad - \langle j, N_1 | \alpha, N\rangle \langle k, N_2 | (L^d_n + L^n_n) | \overline{\beta}, N \rangle \right\} \\
& = \text{(diagonal part)} + \text{(non–diagonal part)},
\end{align}

(\text{diagonal part}) \equiv \delta_{N_1, N_2 - n} \delta_{k, \beta} \delta_{j, \alpha} \left\{ \langle \alpha, N_1 | L^d_n | \alpha, N_1 + n\rangle - \langle \beta^*, N_1 | L^n_n | \beta^*, N_1 + n\rangle \right\},

(\text{non–diagonal part}) \equiv \delta_{N_1, N_2 - n} (\delta_{j, \alpha} \delta_{\beta^*, k^*} + \delta_{j, \alpha} \delta_{\beta^*, k^*} + \delta_{j, \alpha} \delta_{\beta^*, k^*}) \\
\times \langle j, N_1 | \langle L_n | \alpha, N_2 | \beta^*, N_2 \rangle - | \alpha, N_1 \rangle \langle \beta^*, N_1 | L_n \rangle | k^*, N_2 \rangle.

where we introduce $\delta_{f>i}$ such that, if $f = v^{(a)}$, $i = v^{(b)}$ and $a > b$, then $\delta_{f>i} = 1$, others vanish. Here we assume that the above bilinear form is a Shapovalov form but not
necessarily simple under $\ast$, and $\langle j, N_1 | L_n | \alpha, N_2 \rangle \neq 0$ if and only if $h_j = h_\alpha$. The diagonal part vanishes if $| \alpha, N \rangle$ and $| \beta^*, N \rangle$ have the same conformal structure, i.e. the same conformal dimensions and null vectors at the same level, etc. For the case of Sugawara construction, this is compensated by setting two states to be in the same multiplet as $| \alpha, 0 \rangle = | 0; l, m \rangle$ and $| \beta^*, 0 \rangle = | 0; l, m' \rangle$. However, for $c_{p,1}$ models, there is a possibility for $| \alpha \rangle$ and $| \beta \rangle$ to be not in the same representation but in the same Jordan cell.

Given that we have just two primary states in a rank-2 Jordan cell, namely, the upper JLWV $| C \rangle = | v^{(0)} \rangle$ and the lower JLWV $| D \rangle = | v^{(1)} \rangle$, while generators of Virasoro algebra merely generate their descendants, i.e. both submodules, $\mathcal{V}_C$ and $\mathcal{V}_D$, of JLWM do not contain any other submodule. By forcing $\alpha, \beta$ to be in this cell, vanishing diagonal part is assured and corresponding conditions for boundary Ishibashi states reduce to

\[
\text{(non-diag part) = 0 \quad \text{for arbitrary } j, k, n, N_1, N_2,}
\]

\[
\text{lhs} = \delta_{N_1, N_2 - \eta} \left[ \delta_{j, \alpha, C} \delta_{\beta, D} \left( \langle \alpha, N_1 | L_n | \alpha, N_2 \rangle \langle C, N_2 | - \langle C, N_1 | L_n | D, N_2 \rangle \right) + \delta_{j, \alpha, D} \delta_{\beta, C} \left( \langle C, N_1 | L_n | D, N_2 \rangle - | D, N_1 \rangle \langle \beta^*, N_1 | L_n | \beta^*, N_2 \rangle \right) + \delta_{j, \alpha, \beta, D} \langle C, N_1 | L_n | D, N_2 \rangle \cdot \langle C, N_2 | D, N_2 \rangle - \langle C, N_1 | D, N_1 \rangle \right].
\] (17)

Then finally we get the following conditions,

\[
\delta_{\alpha, D} = \delta_{\beta, D} = 0. \quad \text{(18)}
\]

Hence, the only allowed Ishibashi state in the Jordan cell is, as expected,

\[
| B \rangle = \sum_{\{N\}} | C, N \rangle \otimes | C, N \rangle.
\] (19)

This new result is valid for all rank-2 indecomposable representations of this type, as long as both $\mathcal{V}_C$ and $\mathcal{V}_D$ have the same conformal structure and spectrum. Unless the assumption is violated, we can extend the chiral algebra as far as possible. Unfortunately, this is not the case of the $c = -2$ rational LCFT, because the conformal tower of the logarithmic state $| \omega \rangle$ contains a subrepresentation. However, it is still a rigorous proof of boundary states in Jordan cell structures and it could be extended to a generic case.

Namely, we state a conjecture that, in general, for all rank = 2 indecomposable representations, only one boundary Ishibashi state is allowed in each representation.

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4 $| \omega \rangle \equiv | D, h = 0 \rangle, | \Omega \rangle \equiv | C, h = 0 \rangle$. For instance, $| \phi \rangle \equiv L_{-1} | \omega \rangle$ should be interpreted as in a subrepresentation.
If all LCFTs contain Jordan cell structure, this result and conjecture become a powerful fundamental tool to tackle to boundary LCFT. Also, it should be noted that this conjecture includes $W$-algebraic cases, which are thought to give rational series of $c_{p,1}$ LCFTs.

4 \( c = -2 \) boundary LCFT

At this stage, we can investigate boundary LCFTs which only contain rank-2 Jordan cell structures and irreducible representations. A useful point in these LCFTs is that the number of independent boundary states coincides with the number of Jordan cells plus the number of irreducible representations$^5$.

In this section, we will show how we can obtain boundary states in a particular LCFT, the $c = -2$ theory, as a first example. In order to do so, we briefly review the $c = -2$ theory in [9, 10, 25] from the point of view of modular properties. Then we examine the boundary states of the theory.

4.1 The $c = -2$ again

The Jordan cell in eq.(7) sits on the vacuum representation, and two Verma modules of the cell are of the same sort. In other words, the characters of them take the same form and become indistinguishable. It may be possible to interpret them as two coincident characters of different representations. When the chiral algebra of the system is Virasoro algebra, characters of the theory are given straightforwardly as Virasoro characters. The rest of our construction seems to be rather easy, but it is to be taken carefully. In fact, the chiral algebra should include $W$-algebra.

In boundary CFTs on a wrapped strip, the partition function as a sum of characters has to be transformed into a sum of the same set of characters under modular transformations. Therefore, at least, we need some sort of rationality which plays a crucial role in unitary rational models. Since above mentioned Virasoro characters in ‘normal’, i.e. without $W$-symmetry, $c = -2$ model are not modular-transformed into the same set but generate more characters, the rationality is missing. However, there is another way to recover the rationality, that is, with $W$-symmetry. Precisely, the number of representations is reduced to be finite and a set of linear combinations of them may possess well-defined modular properties$^5$. This is why we are about to take it in our theory. It should be noted

\[5 \text{ e.g. if there are one Jordan cell and one irreducible representation, there are two independent boundary states because of two independent Ishibashi states.} \]
that the loss of rationality always happens in normal $c_{p,1}$ models.

In $c = -2$, $W(2,3,3,3)$ algebra plays this role, representations of which are summarised as $h = \{-1/8, 0, 3/8, 1\}$. Their $W$-characters are given by

$$\chi_{V_0}(q) = \frac{1}{2\eta(q)} (\Theta_{1,2}(q) + \partial\Theta_{1,2}(q)),
\chi_{V_1}(q) = \frac{1}{2\eta(q)} (\Theta_{1,2}(q) - \partial\Theta_{1,2}(q)),
\chi_{V_{-1/8}}(q) = \frac{1}{\eta(q)} \Theta_{0,2}(q),
\chi_{V_{3/8}}(q) = \frac{1}{\eta(q)} \Theta_{2,2}(q),$$

where $q = e^{2\pi i \tau}$, $\Theta_{l,k}(q) \equiv \sum_{n \in \mathbb{Z}} q^{(2kn+l)^2/4k}$ is a Riemann theta function. and $\partial\Theta_{l,k}(q) \equiv \sum_{n \in \mathbb{Z}} (2kn+l) q^{(2kn+l)^2/4k}$. The first character is of the vacuum representation and of the Jordan cell. Characters in eq.(20) do not close under modular transformations but generate the new function $\Delta\Theta_{1,2}/\eta \equiv i\tau\partial\Theta_{1,2}/\eta$. In fact, linear combinations of those five functions can form a modular invariant set.

One way is to introduce the notion of generalised highest weight representations(hwrep), $R_0$ and $R_1$, and define $\chi_{R_0} = \chi_{R_1} = 2(\chi_{V_0} + \chi_{V_1})$ so that, with $\chi_{V_{-1/8}}, \chi_{V_{3/8}}$, they form a modular invariant set. This was first proposed in [9], based on the analysis of fusion rules, and its $S$-matrices are given in [9, 12].

Another way is to draw a general set of linear combinations and determine the coefficients so that the modular invariant partition function is given by them. Separately, this was given in [10], showing that there are three cases. $S$-matrices of them are also listed.

The way which has been taken in [25] is to define the logarithmic pair by two non-logarithmic primaries in a particular limit and infer their characters should be those in [10]. Selecting four characters, the $S$-transformation is expressed by two different matrices, $S$ and $Q$, the latter of which is for the logarithmic nature. They imposed invariance under $S^2$-transformation and derived the result.

In what follows, we will start from the first way, and then, case(I) in [10] and check the last one. Note that the first two ways are in the scope of LCFT without boundaries and one of necessary conditions is $S^4 = 1$, which should be $S^2 = 1$ in boundary cases.

### 4.2 Boundary states and ‘fusion rule’ coefficients

#### 4.2.1 First approach: on Gaberdiel and Kausch’s construction

In the first approach of the $c = -2$ LCFT, there are two generalised hwrep, $R_0$, $R_1$, and two normal lwrep, $V_{-1/8}, V_{3/8}$, where two different logarithmic pairs reside in each
generalised one. Therefore, we have four linearly independent boundary Ishibashi states, two of which are constructed from upper JLWVs as in the previous section. Fortunately, they are orthogonal to each other and span the space of boundary states.

The main aim of this section is to construct the set of boundary states from them, which correspond to boundary conditions in the open string picture. We then discuss the Verlinde formula for the LCFT.

Let us begin with the closed string picture, that is, LCFT on a cylinder. Extracting upper JLWVs, \( \Omega \) and \( \phi \) out of \( R_{0,1} \), any initial boundary state is expressed as

\[
|\text{initial state}\rangle = a \ |\Omega\rangle + b \ |\phi\rangle + c \ \left\langle \frac{1}{8}\right\rangle + d \ \left\langle \frac{3}{8}\right\rangle,
\]

and final state is done similarly, where each bra(ket) in r.h.s. denotes an Ishibashi state. As there are supposed to be four boundary conditions for the open string picture, the corresponding boundary states may be labeled by \( \tilde{\alpha} = \{\tilde{R}_0, \tilde{R}_1, -\frac{1}{8}, \frac{3}{8}\} \) and are given by

\[
|\tilde{\alpha}\rangle \equiv \sqrt{2} \ a_\alpha \ |\Omega\rangle + \sqrt{2} \ b_\alpha \ |\phi\rangle + \bar{c}_\alpha \ \left\langle \frac{1}{8}\right\rangle + \bar{d}_\alpha \ \left\langle \frac{3}{8}\right\rangle,
\]

\[
\langle \tilde{\alpha} | \equiv \sqrt{2} \ a_\alpha \langle \Omega | + \sqrt{2} \ b_\alpha \langle \phi | + c_\alpha \ \left\langle \frac{1}{8}\right\rangle + d_\alpha \ \left\langle \frac{3}{8}\right\rangle,
\]

with a factor \( \sqrt{2} \) added for later convenience.

With the definition of Ishibashi states, eq.(13), the partition function becomes simple in terms of the characters:

\[
Z_{\alpha\beta}(\tilde{q}) = \langle \tilde{\alpha} | q^{\frac{1}{2}}(L_0 + \tilde{L}_0 - c/12) |\tilde{\beta}\rangle
\]

\[
= 2 \left( a_\alpha \bar{a}_\beta \chi_{\tilde{V}_0}(\tilde{q}) + b_\alpha \bar{b}_\beta \chi_{\tilde{V}_1}(\tilde{q}) \right) + c_\alpha \bar{c}_\beta \chi_{\tilde{V}_{-1/8}}(\tilde{q}) + d_\alpha \bar{d}_\beta \chi_{\tilde{V}_{3/8}}(\tilde{q})
\]

\[
= a_\alpha \bar{a}_\beta \{\gamma \chi_{R_0}(\tilde{q}) + (1 - \gamma) \chi_{R_1}(\tilde{q})\} + c_\alpha \bar{c}_\beta \chi_{\tilde{V}_{-1/8}}(\tilde{q}) + d_\alpha \bar{d}_\beta \chi_{\tilde{V}_{3/8}}(\tilde{q})
\]

\[
= (M_\alpha)_j^\beta \chi_j(\tilde{q}),
\]

where \( \tilde{q} \equiv e^{2\pi i \tilde{\tau}} = e^{-2\pi i/\tau} \), \( j \) is contracted and summed over \( R_0, R_1, V_{-1/8}, \) and \( V_{3/8}. \)

In the second line of eq.(22), the first two coefficients are naturally combined into one with a condition \( a_\alpha \bar{a}_\beta = b_\alpha \bar{b}_\beta \), due to the modular invariance of the theory. In addition, due to \( \chi_{R_0} = \chi_{R_0} \), lack of difference between \( \chi_{R_{0,1}} \), an extra factor \( \gamma \) is introduced to redistribute the combined term to two characters, which emerge in the open string picture. Finally, the form of the partition function is simplified with a matrix

\[ M_\alpha = \begin{pmatrix} a_\alpha \bar{a}_\beta \gamma, a_\alpha \bar{a}_\beta (1 - \gamma), c_\alpha \bar{c}_\beta, d_\alpha \bar{d}_\beta \end{pmatrix} \]

The upper JLWV \( \phi_\alpha \) in \( R_1 \) is a doublet under the \( \mathcal{W} \)-algebra, but we suppress the suffix for simplicity.
By substituting (22) into the equivalence of both partition functions, eq.(4), and rewriting the partition function of the open string picture with $S$-matrix, we obtain

$$(M_\alpha)_\beta^j = (n_\alpha)_\beta^i S_i^j. (23)$$

Fusion rule coefficients and $S$ matrix have been given in [3, 12, 10] and $S$ undertakes two solutions, one of which is given from the other by flipping the role of $R_{0,1}$. With the identification of $n_\alpha$ with $N_\alpha$, the fusion rule matrix, (23) dramatically reduces the potential 25 parameters of $M_\alpha$ to four:

$$\gamma = 1/\epsilon, \quad a_{0,1} = \bar{a}_{0,1} = 0,$$
$$c_0 = 2c_{-\frac{1}{8}} = 2c_{\frac{1}{8}} = \frac{4}{c_0} = \frac{2}{\bar{c}_{-\frac{1}{8}}} = \frac{2}{\bar{c}_{\frac{1}{8}}} \neq 0,$$
$$d_0 = -2d_{-\frac{1}{8}} = -2d_{\frac{1}{8}} = -\frac{4}{d_0} \neq \frac{2}{d_{\frac{1}{8}}} \neq 0,$$
$$a_{-\frac{1}{8}} = -a_{\frac{1}{8}} = \frac{i\epsilon}{2a_{-\frac{1}{8}}} = -\frac{i\epsilon}{2a_{\frac{1}{8}}}, (24)$$

where the infinitesimal parameter $\epsilon$ is introduced to describe the solutions. A general solution of boundary states can be easily obtained by substituting (24) into (21) while the other solution of $S$ merely changes the sign of $\epsilon$ in the last line. The factor $\gamma$ may be regarded as a regulator, taking a limit of $\epsilon \to 0$, and values of $a_\alpha, \bar{a}_\alpha$ are restricted by the limit, while other non-zero free parameters remain intact. In other words, explicit form of solutions totally depends on how we take the limit in $a_\alpha(\bar{a}_\alpha)$ space.

It is easy to see that there are three kinds of limits, and, according to them, there arise three distinct families of solutions. In all cases, in both bra and ket state spaces, boundary conditions $\tilde{R}_0, \tilde{R}_1$ can neither be distinguished nor be excluded by the other and, therefore, we label them by $\tilde{R}_c$ in both state spaces. Setting $c_0 = d_0 = 2$ and $a_\alpha = \bar{a}_\alpha$ for simplicity, it follows that

(i) \[\begin{align*}
\langle \tilde{R}_c \rangle &= 2 \langle -\frac{1}{8} \rangle + 2 \langle \frac{3}{8} \rangle, \\
\langle \tilde{R}_c \rangle &= 2 \langle -\frac{1}{8} \rangle - 2 \langle \frac{3}{8} \rangle \\
\langle \tilde{R}_c \rangle &= 2 \langle -\frac{1}{8} \rangle + 2 \langle \frac{3}{8} \rangle, \\
\langle \tilde{R}_c \rangle &= 2 \langle -\frac{1}{8} \rangle - 2 \langle \frac{3}{8} \rangle \\
\langle \tilde{R}_c \rangle &= 2 \langle -\frac{1}{8} \rangle + 2 \langle \frac{3}{8} \rangle, \\
\langle \tilde{R}_c \rangle &= 2 \langle -\frac{1}{8} \rangle - 2 \langle \frac{3}{8} \rangle \\
\langle \tilde{R}_c \rangle &= 2 \langle -\frac{1}{8} \rangle + 2 \langle \frac{3}{8} \rangle, \\
\langle \tilde{R}_c \rangle &= 2 \langle -\frac{1}{8} \rangle - 2 \langle \frac{3}{8} \rangle \\
\langle \tilde{R}_c \rangle &= 2 \langle -\frac{1}{8} \rangle + 2 \langle \frac{3}{8} \rangle, \\
\langle \tilde{R}_c \rangle &= 2 \langle -\frac{1}{8} \rangle - 2 \langle \frac{3}{8} \rangle
\end{align*}\]
where \(|R\rangle \equiv \sqrt{2}(|\Omega\rangle + |\phi\rangle)\) and \(\tilde{R}_c\) is called the combined logarithmic boundary condition. Note that, with respect to the coefficients in front of \('R' states in case (i) and (ii), only the relative sign of them is important, they can take any value in \(\mathbb{C}\), unless we introduce another criteria to constrain them.

Most notably, these solutions indicate that we have either three states for one end of the closed string tube and two for the other (case (i) and (ii)), or two for each end respectively (case (iii)).

4.2.2 First approach revisited

Despite these interesting results, they should be discarded since final states defined in (21) cannot be constructed by the same way as initial states. A bra state \(\langle C, 0 |\) is not an upper JLWV, but it generates \(\langle D, 0 | \) as \(\langle C, 0 | L_0 = h \langle C, 0 | + \langle D, 0 |\). Thus, by the use of \(\langle C^*, 0 | = \langle D, 0 |\), final states of this case must be replaced by

\[
\langle \tilde{\alpha} \rangle \equiv \sqrt{2}a_{\alpha} \langle \omega \rangle + \sqrt{2}b_{\alpha} \langle \phi^* \rangle + c_{\alpha} \left\langle \frac{1}{8} \right\rangle + d_{\alpha} \left\langle \frac{3}{8} \right\rangle,
\]

where \(\langle \phi^* \rangle\) is not the dual Ishibashi state of \(|\phi\rangle\) but of the lower JLWV of this cell. In the bulk, these Ishibashi states of Jordan cell structures do not propagate from initial boundary to final one and thus disappear from the partition function in the closed string picture. It follows that \((M_{\alpha})^{-1/8}_\beta = c_{\alpha}\bar{c}_\beta\), \((M_{\alpha})^{3/8}_\beta = d_{\alpha}\bar{d}_\beta\), others vanishing. This is valid in any case of \(c = -2\) LCFT which has been proposed so far.

In our first approach, fusion rules and this matrix cause a contradiction in (23). Thus, we conclude that the above \(n_{i\alpha\beta}\) is not the fusion rules given in [9]. Leaving \(n_{i\alpha\beta}\) to be unknown, a part of (23) shows \(n_{0\alpha\beta} = n_{1\alpha\beta}\) and that the non-diagonal part of the \(S^2\)-matrix doesn’t change the partition function. Thus, the theory remains invariant under this transformation. This means \(S^2\) becomes effectively an unit matrix.

Now, (23) reduces to

\[
c_{\alpha}\bar{c}_\beta = 2n_{0\alpha\beta} + n_{-1/8\alpha\beta}, \quad d_{\alpha}\bar{d}_\beta = -2n_{0\alpha\beta} + n_{-1/8\alpha\beta}, \quad n_{0\alpha\beta} = n_{1\alpha\beta}, \quad n_{-1/8\alpha\beta} = n_{3/8\alpha\beta}.
\]

Provided that \(n_{i\alpha\beta}\) is a positive integer and \(\bar{c}_\alpha = c^*_\alpha, \bar{d}_\alpha = d^*_\alpha\), then the form of (27) already prescribes the solutions in three ways. First, it prescribes the phase of coefficients, that is, if \(c_{\alpha} \neq 0\) for some \(\alpha\), \(\bar{c}_\beta\) has the opposite phase for an arbitrary \(\beta\), otherwise it vanishes. So, we can eliminate the phases without loss of generality and set them to be real.  

\(^7\) By setting \(\beta = \alpha\), this means that \(c_{\alpha}\) is a square root of integer, thus, either integer or irrational.
if \( \bar{\alpha}; c_\alpha \in \mathbb{Z} \) then \( \forall \beta; c_\beta \in \mathbb{Z} \), and equivalently, if \( \gamma \) is irrational and \( \bar{\alpha}; c_\alpha \in \gamma \mathbb{Z} \) then \( \forall \beta; c_\beta \in \gamma^{-1} \mathbb{Z} \). In other words, every coefficient must be integer or irrational of the same sort. Lastly, rewriting eq. (27), we draw another attention on the coefficients as

\[
c_\alpha c_\beta + d_\alpha d_\beta = 2n_{\alpha\beta}^{-1/8} \geq 0, \quad c_\alpha c_\beta - d_\alpha d_\beta = 4n_{\alpha\beta}^0 \geq 0, \quad n_{\alpha\alpha}^{-1/8} \geq 2n_{\alpha\alpha}^0,
\]

(28)

where the first prescription is used. Note that there is no restrictions on \( a_\alpha, b_\alpha \) and every possible boundary state has two additional degrees of freedom, since they give no contribution to naive inner products of boundary states. This point will be discussed later.

Collecting the above prescriptions, it becomes a simple task to pick up explicit solutions and several of the simplest ones are shown as below with a condition \( n_{\alpha\beta}^j \leq 4 \). One may treat them as representatives.

\[
(i) \quad \begin{align*}
\hat{0} &= a_0 \mid \Omega \rangle + b_0 \mid \phi \rangle \\
\hat{1} &= a_1 \mid \Omega \rangle + b_1 \mid \phi \rangle \\
\hat{2} &= 2 \mid -\frac{3}{8} \rangle \\
\hat{3} &= 2 \mid -\frac{1}{8} \rangle \pm 2 \mid \frac{3}{8} \rangle 
\end{align*}
\]

\[
\begin{align*}
\left( n_{22}^0, n_{22}^{-1/8} \right) &= (1, 2) \\
\left( n_{33}^0, n_{33}^{-1/8} \right) &= (0, 4) \\
\left( n_{23}^0, n_{23}^{-1/8} \right) &= (1, 2) \\
\text{other } n_{\alpha\beta}^0 \text{ and } n_{\alpha\beta}^{-1/8} \text{ vanish}
\end{align*}
\]

\[
(ii) \quad \begin{align*}
\hat{0} &= a_0 \mid \Omega \rangle + b_0 \mid \phi \rangle \\
\hat{1} &= a_1 \mid \Omega \rangle + b_1 \mid \phi \rangle \\
\hat{2} &= 2 \mid -\frac{1}{8} \rangle + 2 \mid \frac{3}{8} \rangle \\
\hat{3} &= 2 \mid -\frac{4}{8} \rangle - 2 \mid \frac{3}{8} \rangle 
\end{align*}
\]

\[
\begin{align*}
\left( n_{22}^0, n_{22}^{-1/8} \right) &= (0, 4) \\
\left( n_{33}^0, n_{33}^{-1/8} \right) &= (0, 4) \\
\left( n_{23}^0, n_{23}^{-1/8} \right) &= (2, 0) \\
\text{other } n_{\alpha\beta}^0 \text{ and } n_{\alpha\beta}^{-1/8} \text{ vanish}
\end{align*}
\]

\[
(i_a) \quad \begin{align*}
\hat{0} &= a_0 \mid \Omega \rangle + b_0 \mid \phi \rangle \\
\hat{1} &= a_1 \mid \Omega \rangle + b_1 \mid \phi \rangle \\
\hat{2} &= 2\sqrt{2} \mid -\frac{1}{8} \rangle \\
\hat{3} &= \sqrt{2} \mid -\frac{1}{8} \rangle \pm \sqrt{2} \mid \frac{3}{8} \rangle 
\end{align*}
\]

\[
\begin{align*}
\left( n_{22}^0, n_{22}^{-1/8} \right) &= (2, 4) \\
\left( n_{33}^0, n_{33}^{-1/8} \right) &= (0, 2) \\
\left( n_{23}^0, n_{23}^{-1/8} \right) &= (1, 2) \\
\text{other } n_{\alpha\beta}^0 \text{ and } n_{\alpha\beta}^{-1/8} \text{ vanish}
\end{align*}
\]

\[
(ii_a) \quad \begin{align*}
\hat{0} &= a_0 \mid \Omega \rangle + b_0 \mid \phi \rangle \\
\hat{1} &= a_1 \mid \Omega \rangle + b_1 \mid \phi \rangle \\
\hat{2} &= \sqrt{2} \mid -\frac{1}{8} \rangle + \sqrt{2} \mid \frac{3}{8} \rangle \\
\hat{3} &= \sqrt{2} \mid -\frac{1}{8} \rangle - \sqrt{2} \mid \frac{3}{8} \rangle 
\end{align*}
\]

\[
\begin{align*}
\left( n_{22}^0, n_{22}^{-1/8} \right) &= (0, 2) \\
\left( n_{33}^0, n_{33}^{-1/8} \right) &= (0, 2) \\
\left( n_{23}^0, n_{23}^{-1/8} \right) &= (1, 0) \\
\text{other } n_{\alpha\beta}^0 \text{ and } n_{\alpha\beta}^{-1/8} \text{ vanish}
\end{align*}
\]

where two Ishibashi states of Jordan cells in \( \hat{2} \) and \( \hat{3} \) are suppressed for simplicity and final boundary states are given by the complex conjugation \( * \), not by the dual \( \dagger \). Only representatives of nontrivial \( n_{\alpha\beta}^j \) are listed. The solutions, \((i_a)\) and \((ii_a)\) can be obtained.
from the solutions, (i) and (ii), and vice versa, via redefinitions of boundary states, which
would be determined by the detailed analysis of some experiment, if it exists. Note that it is not necessarily two in the solutions that have \( |\frac{-1}{8}\rangle \) and \( |\frac{3}{8}\rangle \), since there is at least one state of a Jordan cell with which we can make another independent state. This will be discussed later.

It is possible to reduce the number of solutions by imposing another condition, which is orthogonality of boundary states. For this purpose, we must introduce the inner product which is defined by Cardy in his paper \[22\],

\[
(\alpha|\beta) \equiv \lim_{q \to 1} \frac{\langle\alpha|q^{L_0}\beta\rangle}{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle^{1/2}}, \tag{30}
\]
as a bilinear form on boundary states. Although the states of Jordan cells become undefined with themselves under this product, the orthogonality of states given below is still valid with arbitrary additions of those Ishibashi states. Namely, it turns out that, among the solutions in eq.(29), only (ii) and (ii\textsubscript{a}) satisfy the orthogonality as

\[
(\tilde{2}|\tilde{3}) = \lim_{q \to 1} \frac{\langle\tilde{2}|q^{L_0}\tilde{3}\rangle}{\langle\tilde{2}|\tilde{2}\rangle^{1/2}} = \lim_{q \to 1} \frac{\langle-\frac{1}{8}|q^{L_0}|-\frac{1}{8}\rangle - \langle\frac{3}{8}|q^{L_0}|\frac{3}{8}\rangle}{\langle-\frac{1}{8}|\frac{1}{8}\rangle + \langle\frac{3}{8}|\frac{3}{8}\rangle} = 0. \tag{31}
\]
The other orthogonality conditions hold trivially. Hence, we may conclude that the simplest and most acceptable solution is (ii\textsubscript{a}). It is remarkable that this is the only solution which satisfies \( n_{\alpha\beta}^i < 4 \).

Simply following these prescriptions, the other approaches can be derived similarly.

4.2.3 Second approach: on Flohr’s construction

In \[10\], it was shown that there is a set of functions which can provide a modular invariant partition function and an explicit form of the \( S \)-matrix. Subsequently, he generalised the set of functions and proposed three different sets of functions and corresponding \( S \)-matrices, as case I, II and III. The case I is determined by requiring integer valued fusion rules, while the case II and case III are by requiring symmetric \( S \)-matrices and matching the characters calculated by the spectrum, that is, keeping the original four \( W \)-characters unchanged.

Let us examine the case I. \( S \)-matrix of this case satisfies \( S^2 = 1 \) and is parameterised by two complex numbers, \( x \) and \( y \), which are related to each other. By setting \( x = 0, y = -1 \), and relabeling the characters as \( \{\tilde{\chi}_{1,2}, \chi_{1,2}, \chi_{-1,2}, \chi_{0,2}, \chi_{2,2}\} = \{\chi_{-1}, \chi_0, \chi_1, \chi_2, \chi_3\} \), we obtain similar equations to (27) from (23) with \( M_\alpha = (0, 0, 0, c_\alpha \bar{c}_\beta, d_\alpha \bar{d}_\beta) \), namely,

\[
c_\alpha \bar{c}_\beta = n_{\alpha\beta}^0 + n_{\alpha\beta}^2, \quad d_\alpha \bar{d}_\beta = -n_{\alpha\beta}^0 + n_{\alpha\beta}^2, \quad n_{\alpha\beta}^{-1} = 0, \quad n_{\alpha\beta}^1 = n_{\alpha\beta}^1, \quad n_{\alpha\beta}^2 = n_{\alpha\beta}^3, \tag{32}
\]
and as the third prescription,
\[ c_\alpha c_\beta + d_\alpha d_\beta = 2n_{\alpha\beta}^2 \geq 0, \quad c_\alpha c_\beta - d_\alpha d_\beta = 2n_{\alpha\beta}^0 \geq 0, \quad n_{\alpha\alpha}^2 \geq n_{\alpha\alpha}^0. \] (33)

Because fusion rules in \([10]\) have negative integer values, it seems necessary to remove the assumption of positive \(n_{\alpha\beta}^i\). However, those fusion rules contradict this \((32)\) and, thus, there is no need to expect the identification \(n_\alpha = \mathcal{N}_\alpha\). We may keep the assumption.

It is easy to follow the same procedure and, if we require \(n_{\alpha\beta}^i < 4\), we find three solutions, one of which is given by multiplying the coefficients and \(n_{\alpha\alpha}^i\) by a factor of \(\sqrt{2}\) and 2, respectively. Two of them are

\[
\begin{align*}
(i) & \quad \left\{ \begin{array}{l}
\langle 0 \rangle = a_0 \left| \Omega \right\rangle + b_0 \left| \phi \right\rangle \\
\langle 1 \rangle = a_1 \left| \Omega \right\rangle + b_1 \left| \phi \right\rangle \\
\langle 2 \rangle = 2 \left| -\frac{1}{8} \right\rangle \\
\langle 3 \rangle = -\left| \frac{1}{8} \right\rangle + \left| \frac{3}{8} \right\rangle
\end{array} \right\}, \\
(ii) & \quad \left\{ \begin{array}{l}
\langle 0 \rangle = a_0 \left| \Omega \right\rangle + b_0 \left| \phi \right\rangle \\
\langle 1 \rangle = a_1 \left| \Omega \right\rangle + b_1 \left| \phi \right\rangle \\
\langle 2 \rangle = -\left| \frac{1}{8} \right\rangle + \left| \frac{3}{8} \right\rangle \\
\langle 3 \rangle = -\left| \frac{1}{8} \right\rangle - \left| \frac{3}{8} \right\rangle
\end{array} \right\},
\end{align*}
\]

where only the second solution satisfies the orthogonality of the states. The above results are valid for any \(x\) and \(y\) unless \((x + y)^2 + 2(xy - 1)^2 = 0\). A point which should be mentioned is that these characters are not based on analysis of primary fields and states, so it is not clear whether there is a Jordan cell other than at \(h = 0\). In the first approach, an introduced generalised hwrep contains a Jordan cell structure at \(h = 1\) and, even if we assume them in this approach, nothing seems to be changed. So, we add two linear combinations of states of Jordan cells as above, in order to show those possibilities.

### 4.2.4 Third approach: on Kogan and Wheater’s construction

This approach is on the basis of the first results which describe LCFT in the presence of a boundary \([25]\). Various 2-point functions have been calculated and shown logarithmic singularities in boundary LCFT. In the \(c = -2\) case, since the discussion only deals with one Jordan cell which appears in the calculated 2-point function, it may be reasonable to assume that there is only one Jordan cell in the theory. We will take this assumption for this approach \([8]\). Since our conjecture is that only one state survive in the cell, 

\[8\] This does not totally exclude the possibility of a Jordan cell at \(h = 1\) and it should be confirmed by the direct calculation of the 4-point function, \(\langle \phi_1,2\phi_2,2\phi_1,2\phi_2,2 \rangle\), without boundaries.
their construction, using both fields in the cell, contradicts ours. However, it would be interesting to take their set of characters in the open string picture and examine how they change the equations. In addition, it is known that there is a representation of conformal dimension one, and that its character is \( \chi_{V_1} \), we then have to admit the states, \(|\phi\rangle\) and \(|\phi^*\rangle = (\phi |\). But, our prescriptions are still applicable, because they never appear in the boundary states due to the fact that \( \chi_{V_1} \) generates \( \Delta\Theta_{1,2}/\eta \) under a modular transformation which is absent in this open string picture.

The eq.(23) and \( M_{\alpha} \) lead to

\[
c_\alpha \bar{c}_\beta = \frac{1}{2} n^0_{\alpha\beta} + n^2_{\alpha\beta}, \quad d_\alpha \bar{d}_\beta = -\frac{1}{2} n^0_{\alpha\beta} + n^2_{\alpha\beta}, \quad n^0_{\alpha\beta} = n_{\alpha\beta}^1, \quad n^2_{\alpha\beta} = n_{\alpha\beta}^3,
\]

and the prescription is

\[
c_\alpha c_\beta + d_\alpha d_\beta = 2n^2_{\alpha\beta} \geq 0, \quad c_\alpha c_\beta - d_\alpha d_\beta = n^0_{\alpha\beta} \geq 0, \quad 2n^2_{\alpha\beta} \geq n^0_{\alpha\beta}.
\]

Note that the above eq.(23) coincides with eq.(44) in [23]. With a trick of defining \( c'_\alpha = \sqrt{2}c_\alpha, \quad d'_\alpha = \sqrt{2}d_\alpha \), solutions which satisfy \( n^i_{\alpha\beta} < 4 \) are easily found as

\[
(i) \quad \begin{cases} |\bar{1}\rangle = a_1 |\Omega\rangle \\ |\bar{2}\rangle = \sqrt{2} |\frac{1}{8}\rangle \\ |\bar{3}\rangle = \sqrt{2} |\text{left\,of\,1}\rangle \pm \sqrt{2} |\frac{3}{8}\rangle \end{cases}, \quad \begin{cases} (n^2_{22}, n^2_{22}) = (2, 1) \\ (n^0_{33}, n^2_{33}) = (0, 2) \\ (n^0_{23}, n^2_{23}) = (2, 1) \\ \text{other } n^0_{\alpha\beta} \text{ and } n^2_{\alpha\beta} \text{ vanish}\end{cases},
\]

\[
(ii) \quad \begin{cases} |\bar{1}\rangle = a_1 |\Omega\rangle \\ |\bar{2}\rangle = \frac{1}{8} |\frac{3}{8}\rangle \\ |\bar{3}\rangle = \frac{1}{8} |\frac{3}{8}\rangle - \frac{3}{8} |\frac{3}{8}\rangle \end{cases}, \quad \begin{cases} (n^2_{22}, n^2_{22}) = (0, 1) \\ (n^0_{33}, n^2_{33}) = (0, 1) \\ (n^0_{23}, n^2_{23}) = (2, 0) \\ \text{other } n^0_{\alpha\beta} \text{ and } n^2_{\alpha\beta} \text{ vanish}\end{cases},
\]

where, again, only the second solution satisfies the orthogonality. Notably, these solutions are different from what is shown in [23], due to our assumption and prescriptions. Precisely speaking, the second prescription directly prohibits us from having those solutions with different types of irrational coefficients. All the prescriptions are derived from the assumption that \( n^i_{\alpha\beta} \) is positive integer and \( \bar{c}_\alpha = c_\alpha, \quad \bar{d}_\alpha = d_\alpha \), so that the second prescription eliminates such solutions. Thus, we may recover their solution by changing the assumption of \( n^i_{\alpha\beta} \). Their solution was derived with a condition, \( n^2_{\alpha\alpha} \leq 1 \), in order to look for the first simplest example. It was also one of the differences from ours.

4.2.5 Conclusions & Remarks on approaches

To summarise, we make some general remarks on all the solutions shown in this section, setting \( \chi_2 \equiv \chi_{-\frac{1}{8}} \) and \( \chi_3 \equiv \chi_{\frac{3}{8}} \) in the first approach.
First, all the solutions have
\[ n^0_{\alpha\beta} = n^1_{\alpha\beta} \text{ and } n^2_{\alpha\beta} = n^3_{\alpha\beta} \] (38)
in common as a part of solutions. Secondly, it must be mentioned that it is possible to have three boundary states whose coefficients of \( \left| -\frac{1}{8} \right\rangle \) or \( \left| \frac{3}{8} \right\rangle \) take non-zero values simultaneously. For instance, in solutions (i), if we split \(|\tilde{3}\rangle\) to \(|\tilde{3}_+\rangle\) and \(|\tilde{3}_-\rangle\) with positive and negative signs in its expressions, we can have such states as \( |\tilde{2}\rangle\), \( |\tilde{3}_+\rangle\) and \( |\tilde{3}_-\rangle\).

Thirdly, amongst all the solutions, we conclude that the most acceptable solutions are illustrated with (ii) (or (ii_a)) because of the orthogonality. Most strikingly, all approaches have the following orthogonal solution,

\[
\begin{align*}
|\tilde{0}\rangle &= a_0 |\Omega\rangle + b_0 |\phi\rangle \\
|\tilde{1}\rangle &= a_1 |\Omega\rangle + b_1 |\phi\rangle \\
|\tilde{2}\rangle &= \sqrt{2} \left| -\frac{1}{8} \right\rangle + \sqrt{2} \left| \frac{3}{8} \right\rangle \\
|\tilde{3}\rangle &= \sqrt{2} \left| -\frac{1}{8} \right\rangle - \sqrt{2} \left| \frac{3}{8} \right\rangle,
\end{align*}
\]

where, of course, whether there are both \(|\tilde{0}\rangle\) and \(|\tilde{1}\rangle\) depends on how many Jordan cells exist in the theory. In this solution, only \( n^0_{22} = n^1_{22} = n \in \mathbb{Z}^+ \) has a dependency on approaches and it is caused by the difference of definitions of characters, \( \chi_0 \) and \( \chi_1 \).

In the first and second approaches, although fusion rules are defined in the original constructions, they cannot be identified with \( n^i_{\alpha\beta} \) and an interpretation of \( n^i_{\alpha\beta} \) as fusion rules is excluded at this point. Besides, when we look at the solution (39), it is obvious that we miss an appropriate \( n^i_{\alpha\beta} \) s.t. \( n^i_{\alpha\beta} = \delta^i_{\beta} \) and lose such a way to the Verlinde formula as in boundary unitary CFTs.

In the closed string picture, whatever boundary states we have, those of Jordan cell cannot travel from one end to the other while those which have nontrivial coefficients of \( | -\frac{1}{8} \rangle \) and \( | \frac{3}{8} \rangle \) can contribute to the partition function. With the definition of inner product of boundary states, Ishibashi states of the cell become null, hence one might think that those states of the cell can be regarded as ‘ghosts’ in this picture.

In the open string picture, in spite of the vanishing physical degrees of freedom of the cell in the other picture, there is non-zero \( n^0_{\alpha\beta} = n^1_{\alpha\beta} \in \mathbb{Z}^+ \). It means that those indecomposable representations can travel along the edges of the strip and appear in the partition function. It would be interesting to check this result in the context of condensed matter physics.

It should be noted that, if the theory on the annulus is the non-chiral theory in \([14]\), our results become inapplicable, because logarithmic fields of the theory are expected to
satisfy the condition (2) by definition. However, this doesn’t mean our construction is totally invalid for all non-chiral theories, though (14) should be modified in some proper way. For instance, by imposing an appropriate restriction on bra states in (14).

5 Summary and Discussions

On the basis of the mathematical definition of JLWM, we have proven that there exists the Ishibashi state in a rank-2 Jordan cell structure and only one is allowed in the structure. We have also shown the explicit form of it in terms of primary states in the Hilbert space of the theory. The result is that descendants of the normal primary state can be in the expression of Ishibashi state, but those of the logarithmic state are excluded. We conjecture that this holds for all LCFTs which contain, at least, one rank-2 JLWM as a submodule. It is prominently useful in a sense that, while Ishibashi states of diagonal representations are given conventionally, the whole set of such states, including those of Jordan cells, provides a general expression of boundary states. Therefore, it leads to relations between boundary states and \( n_{\alpha\beta} \) and, hopefully, the Verlinde formula.

In the previous section, we have focused on the \( c = -2 \) theory as an example. Introducing the \( c = -2 \) theory, its characters and \( S \)-matrices, we have constructed partition functions of the boundary LCFT, and derived boundary states and the relations of \( n_{\alpha\beta} \). On the way to construct the partition function, there are two choices of final boundary states and we have discussed both and excluded the former choice because of the condition (2) and the definition of Jordan cell structure. Both choices finally turned out not to formulate any conventional expression of the Verlinde formula.

Still, it is interesting to interpret the results in the D-brane context, since boundary states are the initial and final states of closed strings, and some of them would correspond to those on the D-branes. Here, we should recall that we do not know the clear microscopic interpretation of them in such a trivial manner as the free and (anti-)symmetric boundary conditions in Ising model. So, the interpretation is still obscure.

As for the Verlinde formula, in spite of the construction in [11] that some of \( S \)-matrices satisfy it and give integer values of fusion coefficients, our results indicate that they never give the conventional formulas as in boundary unitary CFT. The results also imply that it is impossible to have such a formula or, at least, necessary to modify its expression. On the other hand, with characters in [9, 12], it was shown that they do not satisfy the conventional one but lead to a block-diagonal form, which no longer expresses fusion matrices only with \( S \)-matrix. Even this case is not applicable to our results because our \( n_{\alpha\beta} \) do not match the fusion rules. After all, we lost the complete identification of \( n_{\alpha} \) and

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and it suggests that another criteria should be introduced to describe the theory in the open string picture. Apart from it, questions still remain, what else can $n_{\alpha}$ be and whether fusion rules of the open string picture is different from those of ordinary CFT. It would be interesting to answer these questions.

Recently, in [26], another attempt has been done in order to construct boundary states of $c = -2$ rational LCFT from the $(\xi, \eta)$-ghost system given in [8, 11]. By setting $n^{\bar{i}}_{0\bar{i}} = \delta^{\bar{i}}_{i}$, they suspect that it is possible to make Ishibashi states from the ghost system, and show that it is impossible to derive the Verlinde formula in a conventional way. This also supports our observation of incompatibility between Ishibashi states and the fusion rules. So, it is also interesting to confirm whether we can construct our Ishibashi states from the $(\xi, \eta)$-ghost or symplectic fermion system[8, 15], or from any other field representation[3, 7] with our $n^{i}_{\alpha \beta}$.

After this paper was completed, one reference was added to the end of [19], which also deals with boundary LCFT.

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