Analysis of Iterative Waterfilling Algorithm for Multiuser Power Control in Digital Subscriber Lines

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Abstract

In modern digital subscriber line (DSL) systems where multiple users must coexist in the same frequency band, multiuser power control is an effective technique for reducing crosstalk interference and improving total system throughput. The popular distributed iterative waterfilling algorithm (IWFA) for DSL power control lets each user locally measure the total noise plus interference powers in all frequencies and optimally allocate its power across frequency tones according to a “waterfilling” procedure to maximize its own achievable rate. Though effective and empirically convergent in extensive simulations, IWFA has only been shown to be convergent in limited cases under restrictive interference assumptions. In this paper we present an equivalent linear complementarity problem (LCP) formulation of the noncooperative Nash game resulting from the DSL power control problem. Based on this LCP reformulation, we establish the linear convergence of IWFA for arbitrary symmetric interference environment and for certain asymmetric channel conditions with any number of users. In the case of symmetric interference crosstalk coefficients, we show that the users of IWFA in fact, unknowingly but willingly, cooperate to minimize a common quadratic cost function whose gradient measures the received signal power from all users. This is surprising since the DSL users in the IWFA have no intention to cooperate as they each maximize its own rate to reach a Nash equilibrium. In the case of asymmetric coefficients, the convergence of the IWFA is due to a contraction property of the iterates. In addition, the LCP reformulation enables us to solve the DSL power control problem under no restrictions on the interference coefficients using existing LCP algorithms, e.g., Lemke’s method. Indeed, we use the latter method to benchmark the empirical performance of IWFA in the presence of strong crosstalk interference.

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1 Introduction

In modern DSL systems, all users share the same frequency band and crosstalk is known to be the dominant source of interference. Since the conventional interference cancellation schemes require access to all users’ signals from different vendors in a bundled cable, they are difficult to implement in an unbundled service environment. An alternative strategy for reducing crosstalk interference and increasing system throughput is power control whereby interference is controlled (rather than cancelled) through the judicious choice of power allocations across frequency. This strategy does not require vendor collaboration and can be easily implemented to mitigate the effect of crosstalk interference and maximize total throughput.

A typical measure of system throughput is the sum of all users’ rates. Unfortunately the problem of maximizing the sum rate subject to individual power constraints turns out to be nonconvex with many local maxima [11]. To obtain a global optimal power allocation solution, a simulated annealing method was proposed in [8]; however this method suffers from slow convergence and lacks a rigorous analysis. More recently, a dual decomposition approach [2] was developed to solve the nonconvex rate maximization problem, whose complexity was claimed by the authors to be linear in terms of the number of frequency tones but exponential in the number of users. Notice that all of these approaches require a centralized implementation whereby a spectrum management center collects all the channel and noise information, calculates rate-maximizing power spectra vectors and send them to individual users for implementation. In a departure from this centralized framework, Yu, Ginis and Cioffi [13] proposed a distributed game-theoretic approach for the power control problem. The key observation is that each DSL user’s data rate is a concave function of its own power spectra vector when the interfering users’ power vectors are fixed. Letting each user locally measure the interference plus noise levels and greedily allocate its power to maximize its own rate gives rise to a noncooperative Nash game (called DSL game hereafter) [13, 7]. The resulting distributed power control scheme is known as the iterative waterfilling algorithm (IWFA) and has become a popular candidate for the dynamic spectrum management standard for future DSL systems.

Despite its popularity and its apparent convergent behavior in extensive computer simulations, IWFA has only been theoretically shown to converge in limited cases where the crosstalk interferences are weak [6] and/or the number of users is two [13]. The goal of this paper is to present a convergence analysis of IWFA in more realistic channel settings and for arbitrary number of users. Our approach is based on a key new result that establishes a simple reformulation of the noncooperative Nash game (resulting from the distributed power control problem) as a linear complementarity problem (LCP) of the “copositive-plus” type [3]. Based on this equivalent LCP reformulation, we establish the linear convergence of IWFA for arbitrary symmetric interference environment as well as for diagonally dominant asymmetric channel conditions with any number of users. Moreover, in the case of symmetric interference crosstalk coefficients, we show a surprising result that the users of IWFA in fact, unknowingly but willingly, cooperate to minimize a common quadratic cost function whose gradient measures the total received signal power from all users, subject to the constraints that each user must allocate all of its budgeted power across the frequency tones. This “virtual collaborating behavior” is unexpected since the DSL users in IWFA never have any intention nor incentives to cooperate as they each simply maximize its own rate to reach a Nash equilibrium. Another major advantage of this LCP reformulation is that it opens up the possibility to solve the DSL power control problem using the existing well-developed algorithms for LCP, e.g., Lemke’s method [3, 9]. The latter method requires no restriction on the interference coefficients.
and therefore can be used to benchmark the performance of IWFA, especially in the presence of strong crosstalk interference which leads to multiple Nash equilibrium solutions.

Our current work was partly inspired by the recent work of [12] which presented a nonlinear complementarity problem (NCP) formulation of the DSL game using the Karush-Kuhn-Tucker (KKT) optimality condition for each user’s own rate maximization problem. Such an NCP approach can be implemented in a distributed manner despite the need for some small amount of coordination among the DSL users through a spectrum management center. It was shown [12] that the resulting NCP belongs to the $P_0$ class under certain conditions on the crosstalk interference coefficients among the users relative to the various frequency tones. It was further shown that, under the same conditions, the solution to the NCP is “B-regular” [5]; as a consequence, the NCP can be solved in this case by a host of Newton-type methods as described in the Chapter 9 of the latter monograph. In contrast to [12], our present work shows that the DSL game is basically a linear problem. This simple result has important consequences as we will see.

The rest of this paper is organized as follows. In Section 2, we present the Nash game formulation of the DSL power control problem and develop an equivalent mixed LCP formulation, based on which we obtain a new uniqueness result of the Nash equilibrium solution to the game. In Section 3, we convert the mixed LCP formulation of the DSL game into a standard LCP and show that the well-known Lemke method will successfully compute a Nash equilibrium of the DSL game, under essentially no conditions on the interference and noise coefficients. Section 4 is devoted to the convergence analysis of the IWFA where we apply an existing convergence theory for a symmetric LCP and the contraction principle in the asymmetric case to show the linear convergence of IWFA under two sets of channel conditions. These convergence results significantly enhance those of [6, 13] by allowing arbitrary number of users and more realistic channel conditions. Section 5 reports simulation results of Lemke’s algorithm and IWFA. It is observed that the IWFA delivers robust convergent behavior under all simulated channel conditions and achieves superior sum rate performance. The final Section 6 gives some concluding remarks and suggestions for future work.

## 2 LCP Formulation

Let there be $m$ DSL users who wish to communicate with a central office in an uplink multi-access channel. Let $n$ denote the total number of frequency tones available to the DSL users. Each user $i$ has its own power budget described by the feasible set

$$
\mathcal{P}^i = \left\{ p^i \in \mathbb{R}^n \mid 0 \leq p^i_k \leq \text{CAP}_k^i, \quad \forall k = 1, \ldots, n, \quad \sum_{k=1}^n p^i_k \leq P^i_{\text{max}} \right\}
$$

for some positive constants $\text{CAP}_k^i$ and $P^i_{\text{max}}$, where $p^i = (p^i_1, p^i_2, \ldots, p^i_n)$ denotes the power spectra vector of user $i$ with $p^i_k$ signifying the power allocated to frequency tone $k$. In this model, we allow $\text{CAP}_k^i \leq \infty$. To avoid triviality, we assume throughout the paper that

$$
P^i_{\text{max}} < \sum_{k=1}^n \text{CAP}_k^i,
$$

which ensures that the budget constraint $\sum_{k=1}^n p^i_k \leq P^i_{\text{max}}$ is not redundant.
Taking $p_j^i$ for $j \neq i$ as fixed, IWFA lets user $i$ solve the following concave maximization problem in the variables $p_k^i$ for $k = 1, \ldots, n$,

$$\text{maximize} \quad f_i(p^1, \ldots, p^m) \equiv \sum_{k=1}^{n} \log \left( \frac{1 + \frac{p_k^i}{\sigma_k^i + \sum_{j \neq i}^{m} \alpha_{kj}^i p_j^i}}{\sigma_k^i + \sum_{j \neq i}^{m} \alpha_{kj}^i p_j^i} \right)$$  \hspace{1cm} (2.2)

subject to $p^i \in P^i$,

where $\sigma_k^i$ are positive scalars and $\alpha_{kj}^i$ are nonnegative scalars for all $i \neq j$ and all $k$ representing noise power spectra and channel crosstalk coefficients, respectively. A Nash equilibrium of the DSL game is a tuple of strategies $p^* \equiv (p^*_i)^m_{i=1}$ such that, for every $i = 1, \ldots, m$, $p^*_i \in P^i$ and

$$f_i(p^*_1, \ldots, p^*_{i-1}, p^*_i, p^*_{i+1}, \ldots, p^*_m) \geq f_i(p^*_1, \ldots, p^*_{i-1}, p^*_i, p^*_{i+1}, \ldots, p^*_m), \quad \forall p^i \in P^i.$$  \hspace{1cm} (2.2)

The existence of such an equilibrium power vector $p^*$ is well known. Subsequently, we will give some new sufficient conditions for $p^*$ to be unique; see Proposition 2. Our main goal in the paper pertains the computation of $p^*$. Throughout the paper, we let $\alpha_{ii}^i = 1$ for all $i$ and $k$.

Letting $u_i$ be the multiplier of the inequality $\sum_{k=1}^{n} p_k^i \leq P_{i\text{max}}^i$, and $\gamma_k^i$ be the multiplier of the upper bound constraint: $p_k^i \leq \text{CAP}_k^i$, we can write down the KKT conditions for user $i$'s problem (2.2) as follows (where $a \perp b$ means that the two scalars (or vectors) $a$ and $b$ are orthogonal):

$$0 \leq p_k^i \perp \frac{1}{\sigma_k^i + \sum_{j=1}^{m} \alpha_{kj}^i p_j^i} + u_i + \gamma_k^i \geq 0, \quad \forall k = 1, \ldots, n$$

$$0 \leq u_i \perp P_{i\text{max}}^i - \sum_{k=1}^{n} p_k^i \geq 0 \hspace{1cm} (2.3)$$

$$0 \leq \gamma_k^i \perp \text{CAP}_k^i - p_k^i \geq 0, \quad \forall k = 1, \ldots, n.$$  \hspace{1cm} (2.3)

Although the above KKT system is nonlinear, Proposition 1 shows that, under the assumption (2.1), the system is equivalent a mixed linear complementarity system.

**Proposition 1** Suppose that (2.1) holds. The system (2.3) is equivalent to:

$$0 \leq p_k^i \perp \sigma_k^i + \sum_{j=1}^{m} \alpha_{kj}^i p_j^i + v_i + \varphi_k^i \geq 0, \quad \forall k = 1, \ldots, n$$

$v_i$ free, $P_{i\text{max}}^i - \sum_{k=1}^{n} p_k^i = 0$ \hspace{1cm} (2.4)

$$0 \leq \varphi_k^i \perp \text{CAP}_k^i - p_k^i \geq 0, \quad \forall k = 1, \ldots, n.$$  \hspace{1cm} (2.4)

**Proof.** Let $(p_k^i, u_i, \gamma_k^i)$ satisfy (2.3). We must have

$$\sigma_k^i + \sum_{j=1}^{m} \alpha_{kj}^i p_j^i > 0, \quad \forall k = 1, \ldots, n.$$
We claim that \( u_i > 0 \). Indeed, if \( u_i = 0 \), then
\[
\gamma_k^i \geq \frac{1}{\sigma_k^i + \sum_{j=1}^{m} \alpha_{k j}^i p_k^j} > 0, \quad \forall k = 1, \ldots, n,
\]
which implies \( p_k^i = \text{CAP}_k^i \) for all \( k = 1, \ldots, n \). Thus
\[
P_{\text{max}} \geq \sum_{k=1}^{n} p_k^i = \sum_{k=1}^{n} \text{CAP}_k^i,
\]
which contradicts (2.1). Hence to get a solution to (2.4), it suffices to define
\[
v_i \equiv -\frac{1}{u_i} \quad \text{and} \quad \varphi_k^i \equiv \frac{\gamma_k^i \left( \sigma_k^i + \sum_{j=1}^{m} \alpha_{k j}^i p_k^j \right)}{u_i}.
\]
Conversely, suppose \((p_k^i, v_i, \varphi_k^i)\) satisfies (2.4). We must have \( v_i < 0 \); otherwise, complementarity yields \( p_k^i = 0 \) for all \( k = 1, \ldots, n \), which contradicts the equality constraint. Consequently, letting
\[
u_i \equiv -\frac{1}{v_i} \quad \text{and} \quad \gamma_k^i \equiv \frac{\varphi_k^i}{v_i} \left( \sigma_k^i + \sum_{j=1}^{m} \alpha_{k j}^i p_k^j \right),
\]
we easily see that (2.3) holds. \( \square \)

In turn, the mixed LCP (2.4) is the KKT condition of the affine variational inequality (AVI) defined by the affine mapping: \( p \equiv (p^i)_{i=1}^{m} \in \mathbb{R}^{mn} \rightarrow \sigma + Mp \in \mathbb{R}^{mn} \) and the polyhedron \( X \equiv \prod_{i=1}^{m} \hat{P}^i \), where \( \sigma \equiv (\sigma^i)_{i=1}^{m} \) with \( \sigma^i \) being the \( n \)-dimensional noise power vector \( (\sigma_k^i)_{k=1}^{n} \) for user \( i \), \( M \) is the block partitioned matrix \( (M^{ij})_{i,j=1}^{m} \) with each \( M^{ij} \equiv \text{Diag} \left( \alpha_{k j}^i \right)_{k=1}^{n} \) being the \( n \times n \) diagonal matrix of power interferences (note: \( M^{ii} \) is an identity matrix), and
\[
\hat{P}^i \equiv \left\{ p^i \in \mathbb{R}^{n} \mid 0 \leq p_k^i \leq \text{CAP}_k^i, \quad \forall k = 1, \ldots, n, \quad \sum_{k=1}^{n} p_k^i = P_{\text{max}}^i \right\}.
\]
Consequently, the tuple \( p \) is a Nash equilibrium to the DSL game if and only if \( p \in X \) and
\[
(p' - p)^T \left( \sigma + Mp \right) \geq 0, \quad \forall p' \in X.
\]
We denote this AVI by the triple \((X, \sigma, M)\) and refer the reader to [5] for a comprehensive study of the finite-dimensional variational inequality. Among its consequences, the above AVI reformulation of the DSL game enables us to obtain some new sufficient conditions for the uniqueness of a Nash equilibrium solution. To present these conditions, we define the \( m \times m \) matrix \( B = [b_{ij}] \) by
\[
b_{ij} \equiv \max_{1 \leq k \leq n} \alpha_{k j}^i, \quad \forall i, j = 1, \ldots, m.
\]
Note that $b_{ii} = 1$. In what follows, we review some background results in matrix theory, which can be found in [3].

Let $B_{\text{dia}}$, $B_{\text{low}}$ and $B_{\text{upp}}$ be the diagonal, strictly lower, and strictly upper triangular parts of $B$, respectively. Since $\alpha_{ij}^k$ are all nonnegative, $B$ is a nonnegative matrix. Hence $B_{\text{dia}} - B_{\text{low}}$ is a “Z-matrix”; i.e., all its off-diagonal entries are nonpositive. Since all principal minors of $B_{\text{dia}} - B_{\text{low}}$ are equal to one, $B_{\text{dia}} - B_{\text{low}}$ is a “P-matrix”, and thus a “Minkowski matrix” (also known as an “M-matrix”). It follows that $(B_{\text{dia}} - B_{\text{low}})^{-1}$ exists and is a nonnegative matrix. Therefore, so is the matrix $\bar{B} \equiv (B_{\text{dia}} - B_{\text{low}})^{-1}B_{\text{upp}}$. Let $\rho(\bar{B})$ denote the spectral radius of $\bar{B}$, which is equal to its largest eigenvalue, by the well-known Perron-Frobenius theorem for nonnegative matrices. The matrix $\bar{B}$ is called an $H$-matrix if $\bar{B}$ is also a P-matrix. There are many characterizations for the latter condition to hold; we mention two of these: (a) $\rho(\bar{B}) < 1$, and (b) for every nonzero vector $x \in \mathbb{R}^m$, there exists an index $i$ such that $x_i(\bar{B}x)_i > 0$.

For each $k = 1, \ldots, n$, we call the $m \times m$ matrix $M_k$, where

$$( M_k )_{ij} \equiv \alpha_{ij}^k, \quad \forall i, j = 1, \ldots, m,$$

a tone matrix. Notice that the matrix $M$ in the AVI $(X, \sigma, M)$ is a principal rearrangement of the block diagonal matrix with $M_k$ as its diagonal blocks, for $k = 1, \ldots, n$. This rearrangement simply amounts to the alternative grouping of the tuple $p$ by tones, instead of users as done above.

**Proposition 2** Suppose that

$$\max_{1 \leq i \leq m} \sum_{k=1}^n \sum_{j=1}^m \alpha_{ij}^k p_{ki} p_{kj} > 0, \quad \forall p \equiv (p^i)_{i=1}^m \neq 0.$$  \hfill (2.5)

There exists a unique Nash equilibrium to the DSL game. In particular, this holds if either one of the following two conditions is satisfied:

(a) for every $k = 1, \ldots, n$, the tone matrix $M_k$ is positive definite;

(b) $\rho(\bar{B}) < 1$.

**Proof.** As $X$ is the Cartesian product of the sets $\hat{P}^i$, it follows that the AVI $(X, \sigma, M)$ has a unique solution if $M$ has the “uniform P property” relative to the Cartesian structure of $X$; see [5]. This property says that for any nonzero tuple $p \equiv (p^i)_{i=1}^m$,

$$\max_{1 \leq i \leq m} (p^i)^T \sum_{j=1}^m M^{ij} p^j > 0.$$  

Since $M^{ij} = \text{Diag} (\alpha_{ij}^k)_{k=1}^n$, the above condition is precisely (2.5). Under condition (a), the matrix $M$ is positive definite because it is a principal rearrangement of $\text{Diag} (M_k)_{k=1}^n$. It is easy to verify that

$$p^T M p = \sum_{i=1}^m \sum_{k=1}^n \sum_{j=1}^m \alpha_{ij}^k p_{ki} p_{kj}.$$
Hence condition (a) implies (2.5). To show that condition (b) also implies (2.5), write
\[
\sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_{k}^{ij} p_k p_j^i = \sum_{k=1}^{n} (p_k^i)^2 + \sum_{j \neq i}^{n} \alpha_{k}^{ij} p_k p_j^i \\
\geq \sum_{k=1}^{n} (p_k^i)^2 - \sum_{j \neq i}^{n} \alpha_{k}^{ij} |p_k^i||p_j^j| \\
\geq \sum_{k=1}^{n} (p_k^i)^2 - \sum_{j \neq i}^{n} \max_{1 \leq k \leq n} \alpha_{k}^{ij} \left( \sum_{k=1}^{n} (p_k^i)^2 \right)^{1/2} \left( \sum_{k=1}^{n} (p_k^j)^2 \right)^{1/2} \\
\geq \sum_{k=1}^{n} (p_k^i)^2 - \sum_{j \neq i}^{n} \max_{1 \leq k \leq n} \alpha_{k}^{ij} \left( \sum_{k=1}^{n} (p_k^i)^2 \right)^{1/2} \left( \sum_{k=1}^{n} (p_k^j)^2 \right)^{1/2} \\
= \left( \sum_{k=1}^{n} (p_k^i)^2 \right)^{1/2} \sum_{j=1}^{m} b_{ij} \left( \sum_{k=1}^{n} (p_k^j)^2 \right)^{1/2},
\]
where the first and third inequality are obvious and the second is due to the Cauchy-Schwarz inequality. Hence letting
\[
q_i \equiv \left( \sum_{k=1}^{n} (p_k^i)^2 \right)^{1/2},
\]
we have
\[
\sum_{j=1}^{m} \sum_{k=1}^{n} \alpha_{k}^{ij} p_k p_j^i \geq q_i \sum_{j=1}^{m} b_{ij} q_j = q_i (\bar{B}q)_i, \quad \forall i = 1, \ldots, m.
\]
By what has been mentioned above, condition (b) implies
\[
\max_{1 \leq i \leq m} q_i (\bar{B}q)_i > 0,
\]
because \( q \) is obviously a nonzero vector; thus (2.5) holds. \( \square \)

Proposition 2 significantly extends the current existence and uniqueness result of [6, 7, 13] which required \( 0 \leq \alpha_{k}^{ij} \leq 1/n \), for all \( i \neq j \) and all \( k \). Under the latter condition, it can be shown that the symmetric part of each tone matrix \( M_k, \frac{1}{2}(M_k + M_k^T) \), is strictly diagonally dominant; hence each \( M_k \) is positive definite. The condition \( \rho(\overline{Y}) < 1 \) is quite broad; for instance, it includes the case where each matrix \( M_k \) is “strictly quasi-diagonally dominant”, i.e., where for each \( k \), there exist positive scalars \( d_k^i \) such that
\[
d_k^i > \sum_{j=1}^{m} \alpha_{k}^{ij} d_k^j, \quad \forall i = 1, \ldots, m.
\]
In Section 4, we will see that the condition \( \rho(\overline{Y}) < 1 \) is responsible for the convergence of the IWFA with asymmetric interference coefficients.

As another application of the AVI formulation of the DSL game, we show that if each tone matrix \( M_k \) is positive semidefinite (but not definite), it is still possible to say something about the uniqueness of certain quantities.
Proposition 3 Suppose that the tone matrices $M_k$ for $k = 1, \ldots, n$ are all positive semidefinite. Then the set of DSL Nash equilibria is a convex polyhedron; moreover, the quantities

$$\sum_{j=1}^{m} (\alpha_{k}^{ij} + \alpha_{k}^{ji})p_{k}^{j}, \quad \forall i = 1, \ldots, m; \ k = 1, \ldots, n,$$

are constants among all Nash equilibria.

Proof. Under the given assumption, the matrix $M$ is positive semidefinite. As such, the polyhedrality of the set of Nash equilibria follows from well-known monotone AVI theory [5]. Furthermore, in this case, the vector $(M + M^T)p$ is a constant among all such equilibria $p$. By unwrapping the structure of the matrix $M$, the desired constancy of the displayed quantities follows readily. □

We can interpret $(\alpha_{k}^{ij} + \alpha_{k}^{ji})/2$ as the “average interference coefficient” between user $i$ and user $j$ at frequency $k$. In this way, the invariant quantity $\frac{1}{2} \sum_{j=1}^{m} (\alpha_{k}^{ij} + \alpha_{k}^{ji})p_{k}^{j}$ represents the average of signal and interference power received and caused by user $i$ across all frequency tones.

3 Solution by Lemke’s Method

We next discuss the solution of the mixed LCP (2.4) by the well-known Lemke method [3]. The Lemke solution can used as a benchmark to evaluate the empirical performance of IWFA; see Section 5. For convenience, let us first convert this problem into a standard LCP. Let

$$w_{k}^{i} \equiv \sigma_{k}^{i} + \sum_{j=1}^{m} \alpha_{k}^{ij} p_{k}^{j} + v_{i} + \varphi_{k}^{i}, \quad \forall k = 1, \ldots, n,$$

from which we obtain, considering $k = 1$ and substituting $p_{1}^{j} = P_{\max}^{j} - \sum_{k=2}^{n} p_{k}^{j}$ for all $j = 1, \ldots, m$,

$$v_{i} = -\sigma_{1}^{i} + w_{1}^{i} - \sum_{j=1}^{m} \alpha_{1}^{ij} p_{1}^{j} - \varphi_{1}^{i}$$

$$= -\sigma_{1}^{i} + w_{1}^{i} - \sum_{j=1}^{m} \alpha_{1}^{ij} \left( P_{\max}^{j} - \sum_{k=2}^{n} p_{k}^{j} \right) + \varphi_{1}^{i}$$

$$= -\sigma_{1}^{i} - \sum_{j=1}^{m} \alpha_{1}^{ij} P_{\max}^{j} + w_{1}^{i} + \sum_{j=1}^{m} \sum_{k=2}^{n} \alpha_{1}^{ij} p_{k}^{j} - \varphi_{1}^{i}.$$ 

Substituting this into the expression of $w_{k}^{i}$ for $k \geq 2$, we deduce

$$w_{k}^{i} \equiv \sigma_{k}^{i} - \sigma_{1}^{i} - \sum_{j=1}^{m} \alpha_{1}^{ij} P_{\max}^{j} + w_{1}^{i} + \sum_{j=1}^{m} \alpha_{k}^{ij} p_{k}^{j} + \sum_{j=1}^{m} \sum_{\ell=2}^{n} \alpha_{1}^{ij} p_{\ell}^{j} + \varphi_{k}^{i} - \varphi_{1}^{i}$$

$$= \hat{\sigma}_{i}^{1} + \sum_{j=1}^{m} \sum_{\ell=2}^{n} (\alpha_{1}^{ij} + \alpha_{1}^{ij} \delta_{k\ell}) p_{\ell}^{j} + \varphi_{k}^{i} - \varphi_{1}^{i},$$
where \( \delta_{k\ell} \) is kronecker delta, i.e.,
\[
\delta_{k\ell} \equiv \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{otherwise} \end{cases}
\]
and
\[
\hat{\sigma}_k^i \equiv \sigma_k^i - \sum_{j=1}^{m} \alpha_{ij}^i P_{\max}^j, \quad \forall k = 2, \ldots, n.
\]
Consequently, the concatenation of the system (2.4) for all \( i = 1, \ldots, m \) is equivalent to: for all \( i = 1, \ldots, m \) and all \( k = 2, \ldots, n \),
\[
0 \leq p_k^i \perp w_k^i = \hat{\sigma}_k^i + \sum_{j=1}^{m} \sum_{\ell=2}^{n} (\alpha_{ij}^i + \alpha_{ij}^\ell \delta_{k\ell}) p_{\ell}^j + w_1^i + \varphi_k^i - \varphi_1^i \geq 0
\]
\[
0 \leq w_1^i \perp p_1^i = P_{\max}^i - \sum_{k=2}^{n} p_k^i \geq 0
\]
\[
0 \leq \varphi_k^i \perp \text{CAP}_k^i - p_k^i \geq 0
\]
\[
0 \leq \varphi_1^i \perp \text{CAP}_1^i - P_{\max}^i + \sum_{k=2}^{n} p_k^i \geq 0.
\]
The above is an LCP of the standard type:
\[
0 \leq z \perp q + Mz \geq 0,
\] (3.6)
where the constant vector \( q \) is given by
\[
q \equiv \begin{pmatrix}
\hat{\sigma}_k^i & i = 1, \ldots, m; \ k = 2, \ldots, n \\
P_{\max}^i & i = 1, \ldots, m \\
\text{CAP}_k^i & i = 1, \ldots, m; \ k = 2, \ldots, n \\
\text{CAP}_1^i - P_{\max}^i & i = 1, \ldots, m
\end{pmatrix},
\] (3.7)
z is the vector of variables:
\[
z \equiv \begin{pmatrix}
p_k^i & i = 1, \ldots, m; \ k = 2, \ldots, n \\
w_1^i & i = 1, \ldots, m \\
\varphi_k^i & i = 1, \ldots, m; \ k = 2, \ldots, n \\
\varphi_1^i & i = 1, \ldots, m
\end{pmatrix},
\] (3.8)
and the matrix \( M \), partitioned in accordance with the vectors \( q \) and \( z \), is of the form:
\[
M \equiv \begin{bmatrix}
\hat{M} & N & I & -N \\
-N^T & 0 & 0 & 0 \\
-I & 0 & 0 & 0 \\
N^T & 0 & 0 & 0
\end{bmatrix},
\] (3.9)
where the leading principal submatrix \( \hat{M} \) is a nonnegative (albeit asymmetric) matrix with positive diagonals, and \( N \) is a special nonnegative matrix. (The details of the matrices \( \hat{M} \) and \( N \) are not important, except for the distinctive features mentioned here.) Based on (3.9), it follows that the matrix \( M \) is copositive plus (i.e., \( z^T M z \geq 0 \) for all \( z \geq 0 \), and \( [z \geq 0, z^T M z = 0] \) implies \((M + M^T)z = 0\)). Consequently, Lemke’s algorithm can successfully compute a solution to the LCP (3.6), provided that this LCP is feasible; see [3]. But the latter feasibility condition trivially holds by the nonemptiness of the sets \( \hat{P}^i \) for \( i = 1, \ldots, m \), which is a blanket assumption that we have made. Summarizing this discussion, we obtain the following result.

\[ \text{Theorem 4} \quad \text{Suppose that (2.1) holds and that } \hat{P}^i \neq \emptyset \text{ for all } i = 1, \ldots, m. \text{ For all nonnegative coefficients } \alpha^{ij}, i \neq j \text{ and all positive } \sigma^i_k, \text{ there exists a Nash equilibrium solution which can be obtained by Lemke’s algorithm applied to the LCP (3.6) with } q \text{ and } M \text{ given by (3.7) and (3.9), respectively.} \]

This existence result extends that of [13] which required the condition that \( \max_k \{\alpha^{21}_k \alpha^{12}_k\} < 1 \) and was only for the two user case.

4 Convergence Analysis of the IWFA

The LCP formulation (3.6) of the DSL game, where each user’s variables associated with tone 1 are eliminated, facilitates the computation of a Nash equilibrium by Lemke’s method (see Section 5 for numerical results). Nevertheless, for the convergence analysis of the IWFA, it would be convenient to return to the AVI \((X, q, M)\), where all variables are left in the formulation. It is well known [5] that the latter AVI is equivalent to the fixed-point equations: for all \( i = 1, \ldots, m \),

\[ p^i = \left[ p^i - \sigma^i - \sum_{j=1}^{m} M^{ij} p^j \right]_{\hat{P}_i} = \left[ -\sigma^i - \sum_{j \neq i} M^{ij} p^j \right]_{\hat{P}_i}, \]

where \( [\cdot]_{\hat{P}_i} \) denotes the Euclidean projection operator onto \( \hat{P}_i \), i.e.,

\[ [x]_{\hat{P}_i} = \arg\min_{p^i \in \hat{P}_i} \|x - p^i\|. \]

As briefly described in Section 2, the IWFA [6, 7, 13] is a distributed algorithm for solving the DSL game; it has the attractive feature of not requiring the coordination of the DSL users. In fact, each DSL user \( i \) simply maximizes its rate \( f_i(p^1, \ldots, p^m) \) on the feasible set \( P^i \) by adjusting its own power vector \( p^i \) while assuming other users’ powers are fixed but unknown. In so doing, user \( i \) measures the aggregated interference powers

\[ \sum_{j \neq i} (M^{ij} p^j)_k = \sum_{j \neq i} \alpha^{ij}_k p^j_k, \quad \forall k, \]

locally without the specific knowledge of other users’ power allocations \( p^j \) or crosstalk coefficients \( \alpha^{ij}_k, j \neq i \). Such aggregated interference powers are sufficient for user \( i \) to carry out its own rate maximization (2.2).
Specifically, the iterative waterfilling method can be described as follows: at each iteration, user $i$ measures the aggregated interferences and updates the new iterate by

$$
(p_i)^{\text{new}} = \left[ -\sigma^i - \left( \sum_{j=1}^{i-1} M^{ij} (p_j)^{\text{new}} + \sum_{j=i+1}^{m} M^{ij} (p_j)^{\text{old}} \right) \right]_{\tilde{p}_i}. \tag{4.10}
$$

In other words, user $i$ simply projects the negative of the aggregated interferences plus the noise power vector onto the polyhedral set $\tilde{P}_i$. This simple geometric interpretation of the IWFA is key to its convergence analysis, which we separate into two cases: symmetric and nonsymmetric interferences.

**Symmetric interferences**

When the DSL users are symmetrically located, the corresponding interference coefficients are symmetric: $\alpha_{jk} = \alpha_{kj}$ for all $i, j, k$. In this case, it follows that $M^{ij} = M^{ji}$ for all $i, j$. Hence the matrix $M$ is symmetric. Consequently, the mixed LCP (2.4) is precisely the KKT condition for the following quadratic program (QP):

$$
\begin{aligned}
&\text{minimize } g(p) \equiv \frac{1}{2} p^T M p + \sum_{i=1}^{m} (\sigma^i)^T p^i \\
&\text{subject to } p = (p_i)_{i=1}^{m} \in \prod_{i=1}^{m} \tilde{P}_i.
\end{aligned} \tag{4.11}
$$

Notice that the gradient of $g(p)$ measures precisely the total received signal power by every user at each frequency. Moreover, the set of Nash equilibrium points for the noncooperative rate maximization game (2.2) correspond exactly to the set of stationary points of the quadratic minimization problem (4.11), which is not necessarily convex because the matrix $M$ is not positive semidefinite in general. More importantly, the IWFA (4.10) can be viewed as a block Gauss-Seidel coordinate descent iteration to solve the QP (4.11). As such, its convergence behavior can be established by appealing to the following general convergence result for the Gauss-Seidel algorithm [10, Proposition 3.4].

**Proposition 5** Consider the following quadratic minimization problem

$$
\begin{aligned}
&\text{minimize } \theta(x_1, x_2, \ldots, x_n) \\
&\text{subject to } x_i \in X_i, \ \forall i = 1, 2, \ldots, n,
\end{aligned} \tag{4.12}
$$

with each $X_i$ a given polyhedral set. Suppose that $X = X_1 \times X_2 \times \cdots \times X_n$ is nonempty and that $\theta$ is strongly convex in each variable $x_i$. Let $\bar{X}$ denote the set of stationary points of (4.12) and let $x^0$, $x^1$, $x^2$, $\ldots$ be a sequence of iterates generated by the following fixed-point iteration:

$$
x_i^{r+1} = \left[ x_i^r - \nabla x_i \theta(x_1^{r+1}, x_2^{r+1}, \ldots, x_i^{r+1}, x_{i+1}^{r+1}, \ldots, x_n^{r+1}) \right]_{X_i}. \tag{4.13}
$$

Then $\{x^r\}$ converges linearly to an element of $\bar{X}$ and $\{\theta(x^r)\}$ converges linearly and monotonically.

□
Under the following identifications
\[ x_i \equiv p^i, \quad X_i \equiv \tilde{P}^i, \quad \theta(x) \equiv g(p), \]
iteration (4.10) is precisely the equation (4.13). Since \( M^{ii} \) is the identity matrix for each \( i \), it follows that the quadratic function \( g(p) \) is strongly convex in each variable \( p^i \). Thus, we can invoke Proposition 5 to conclude the following.

**Corollary 6** If the interference coefficients are symmetric, i.e., \( \alpha_{ij} = \alpha_{ji} \) for all \( i, j, k \), then the iterates \( \{p^\nu \equiv (p^{\nu,i})_{i=1}^m\} \) generated by the IWFA converges linearly to a Nash equilibrium point of the noncooperative DSL game. Moreover, \( \{g(p^\nu)\} \) converges linearly and monotonically. \( \square \)

Notice that in the original IWFA, each user acts greedily to maximize its own rate without coordination. What is surprising is that this seemingly totally distributed algorithm can in fact be viewed equivalently as a coordinate descent algorithm for the minimization of a single quadratic function. In other words, the users actually collaborate, implicitly and willingly, to minimize a common quadratic objective function \( g(p) \) whose gradient corresponds to precisely the total received signal power by every user at each frequency. This important insight is the key to the convergence of the IWFA in the symmetric case.

**Asymmetric interferences**

In general, the DSL users may not be symmetrically located. In this case, the interference matrix \( M \) is not symmetric and the aggregated interference power vectors can not be viewed as the gradient of a scalar function. Thus, Proposition 5 is no longer applicable. More importantly, there is now a lack of an obvious objective function which serves as a monitor for the progress of the IWFA, making the convergence analysis of this algorithm less straightforward. Nevertheless, it is still possible to establish the convergence of the IWFA by imposing the spectral radius condition \( \rho(\Upsilon) < 1 \) introduced in Proposition 2.

**Theorem 7** Suppose that \( \rho(\Upsilon) < 1 \). Then the iterates \( \{p^\nu \equiv (p^{\nu,i})_{i=1}^m\} \) generated by the IWFA converge linearly to the unique Nash equilibrium of the DSL game.

**Proof.** Our proof is by a vector contraction argument; see [3]. Let \( p^* \equiv (p^{*,i})_{i=1}^m \) be the unique Nash equilibrium solution, which satisfies
\[
p^{*,i} = \left[ p^{*,i} - \sigma^i - \sum_{j=1}^m M^{ij} p^{*,j} \right] \tilde{p}_i = \left[ -\sigma^i - \sum_{j \neq i} M^{ij} p^{*,j} \right] \tilde{p}_i, \quad \forall i = 1, \ldots, m.
\]
For each \( \nu \), we have
\[
p^{\nu+1,i} = \left[ -\sigma^i - \left( \sum_{j=1}^{i-1} M^{ij} p^{\nu+1,j} + \sum_{j=i+1}^m M^{ij} p^{\nu,j} \right) \right] \tilde{p}_i, \quad \forall i = 1, \ldots, m.
\]
Let $\| \cdot \|$ denote the Euclidean norm in $\mathbb{R}^m$. By the nonexpansiveness property of projection operator (i.e., $\| x_{\hat{P}_i} - y_{\hat{P}_i} \| \leq \| x - y \|$ for all $x, y$), we have, for all $i = 1, \ldots, m$,

$$\| p^{\nu+1,i} - p^*,i \|$$

$$= \left\| -\sigma^i - \left( \sum_{j=1}^{i-1} M^{i,j} p^{\nu+1,j} + \sum_{j=i+1}^{m} M^{i,j} p^{\nu,j} \right) \right\|_{\hat{P}_i} - \left\| -\sigma^i - \left( \sum_{j=1}^{i-1} M^{i,j} p^*,j + \sum_{j=i+1}^{m} M^{i,j} p^*,j \right) \right\|_{\hat{P}_i}$$

$$\leq \left\| \sum_{j=1}^{i-1} M^{i,j} (p^{\nu+1,j} - p^*,j) + \sum_{j=i+1}^{m} M^{i,j} (p^{\nu,j} - p^*,j) \right\|$$

$$\leq \sum_{j=1}^{i-1} \| M^{i,j} (p^{\nu+1,j} - p^*,j) \| + \sum_{j=i+1}^{m} \| M^{i,j} (p^{\nu,j} - p^*,j) \|$$

$$\leq \sum_{j=1}^{i-1} b_{ij} \| p^{\nu+1,j} - p^*,j \| + \sum_{j=i+1}^{m} b_{ij} \| p^{\nu,j} - p^*,j \|.$$

Hence,

$$\sum_{j=1}^{i} b_{ij} \| p^{\nu+1,j} - p^*,j \| \leq \sum_{j=i+1}^{m} b_{ij} \| p^{\nu+1,j} - p^*,j \|.$$

Letting $e^\nu \equiv (e_i^\nu)_{i=1}^m$ with $e_i^\nu \equiv \| p^{\nu,j} - p^*,j \|$ and concatenating the above inequalities for all $i = 1, \ldots, m$, we deduce

$$(B_{\text{dia}} - B_{\text{low}}) e^\nu \leq B^{\text{upp}} e^\nu,$$

which implies

$$0 \leq e^{\nu+1} \leq (B_{\text{dia}} - B_{\text{low}})^{-1} B^{\text{upp}} e^\nu = \Upsilon e^\nu, \quad \forall \nu,$$

where we have used the fact that $(B_{\text{dia}} - B_{\text{low}})^{-1}$ is nonnegative entry-wise; see the discussion preceding Proposition 2. Since $\rho(\Upsilon) < 1$, the above inequality implies that the sequence of error vectors $\{e^\nu\}$ contract according to a certain norm. Consequently, the sequence converges to zero, implying that the sequence of waterfilling iterates $\{p^\nu\}$ converges linearly to the unique solution $p^*$ of the DSL game. \hfill $\Box$

Theorem 7 strengthens the existing convergence results [6, 13]. Specifically, the condition required for convergence is weaker. In particular, it can be seen that the strong diagonal dominance condition ($\alpha_k^{ij} \leq 1/(m - 1)$) required in [6] and the respective condition for two user case [13] both imply the condition $\rho(\Upsilon) < 1$. Thus, Theorem 7 covers the convergence for a broader class of DSL problems.

5 Numeric Simulations

In this section, we present some computer simulation results comparing the convergence behavior of IWFA with Lemke’s algorithm under various channel conditions and system load (i.e., number of users). In all simulated cases, the channel background noise levels $\sigma_k^i$ are chosen randomly from the
Table 1: Average sum rate: two user case

| $n$  | $\alpha_{12}^k$, $\alpha_{21}^k \in (0, 1)$ | $\alpha_{12}^k$, $\alpha_{21}^k \in (0, 1.5)$ |
|------|------------------------------------------|--------------------------------------------|
|      | Lemke | IWFA | Lemke | IWFA |
| 256  | 704   | 698  | 829.73 | 826.5787 |
| 512  | $1.402 \times 10^4$ | $1.398 \times 10^4$ | $1.6555 \times 10^3$ | $1.6333 \times 10^3$ |
| 1024 | $2.786 \times 10^3$ | $2.811 \times 10^3$ | $3.3028 \times 10^3$ | $3.2968 \times 10^3$ |

Table 2: Average sum rate: $m = 10$ user case

| $n$  | $\alpha_{i,j}^k \in (0, 1/(m-1))$ |
|------|-----------------------------------|
|      | Lemke | IWFA |
| 256  | $2.8216 \times 10^4$ | $2.824 \times 10^4$ |
| 512  | $5.6464 \times 10^3$ | $5.6457 \times 10^3$ |
| 1024 | $1.1284 \times 10^4$ | $1.1296 \times 10^4$ |

interval $(0, 0.1/(m-1))$ and the total power budgets $P^i_{\text{max}}$ are chosen uniformly from the interval $(n/2, n)$. All sum rates are averaged over 100 independent runs. The IWFA and Lemke’s method are both implemented on a Pentium 4 (1.6GHz) PC using Matlab 6.5 running under Windows XP. For IWFA we set a maximum of 400 iteration cycles (among all users), while the maximum pivoting steps for Lemke’s method is set to be $\min(1000, 25mn)$.

Table 1 reports the achieved sum rates (averaged over 100 independent runs) of Lemke’s method and IWFA for 2 users and various numbers $n$ of frequency tones. In this case we have chosen crosstalk coefficients $\{\alpha_{i,j}^k\}$ from the intervals $(0, 1)$ and $(0, 1.5)$ respectively, for all $k$, and all $i, j$. This represents strong cross talk interference scenarios. According to the table, the average rates achieved by both algorithms are comparable (within 2%), suggesting that the IWFA is capable of computing approximate Nash solutions with high sum rates. Moreover, the results show that stronger interference actually lead to Nash solutions with higher overall sum rates. This seems to indicate that the well-known Braess paradox [1] exist in DSL games. [In fact, using the QP characterization of Nash game (cf. Section 4), it is possible to construct simple examples whereby more transmission power for individual users do not necessarily lead to Nash solutions with higher sum rate.]

For the case with more ($m = 10$) users, the situation is similar, as shown in Table 2. Indeed, when $\alpha_{i,j}^k \in (0, 1/(m-1))$, the condition for the uniqueness of Nash solution is satisfied and the two methods have identical performance. The results in both tables show that IWFA solutions are comparable in quality to the respective solutions generated by the Lemke method.

6 Conclusions

In this paper we reformulate the DSL Nash game (resulting from the distributed implementation of IWFA) as an equivalent LCP, and apply the rich theory for LCP to analyze the convergence
behavior of IWFA. Our analysis not only significantly strengthens the existing convergence results, but also yields surprising insight on IWFA. In particular, in the case of symmetric interference, the users of IWFA in fact collaborate unknowingly to minimize a common quadratic cost, even though their original intention is to maximize their individual rates. Moreover, the LCP reformulation makes it possible to solve the DSL game with existing LCP solvers, such as Lemke’s method. With the latter as a benchmark, we show via computer simulations that IWFA tends to converge to good Nash solutions with high sum rates. Our theoretical and simulation work affirms the potential of IWFA as a promising candidate for the dynamic power spectra management in DSL environment.

As a future work, we are interested in further analyzing the behavior of IWFA under no assumptions on the crosstalk coefficients. Perhaps the compactness of the feasible set and the nonnegativity of the crosstalk coefficients already ensure the convergence of IWFA, or at least eliminate the possibility of finite limit cycles. These issues and the design of an efficient optimal power allocation algorithm for the nonconvex sum rate maximization problem are interesting topics for future research.

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