USING CONSTANT LEARNING RATE OF TWO TIME-SCALE UPDATE RULE FOR TRAINING GENERATIVE ADVERSARIAL NETWORKS

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Abstract. Previous numerical results have shown that a two time-scale update rule (TTUR) using constant learning rates is practically useful for training generative adversarial networks (GANs). Meanwhile, a theoretical analysis of TTUR to find a stationary local Nash equilibrium of a Nash equilibrium problem with two players, a discriminator and a generator, has been given using decaying learning rates. In this paper, we give a theoretical analysis of TTUR using constant learning rates to bridge the gap between theory and practice. In particular, we show that, for TTUR using constant learning rates, the number of steps needed to find a stationary local Nash equilibrium decreases as the batch size increases. We also provide numerical results to support our theoretical analyzes.

1. Introduction
1.1. Background. Generative adversarial networks (GANs) have attracted attention (see, e.g., [12, 23, 24, 27]) with the development of real-world applications [1, 2, 9, 25]. The generator network in GANs constructs synthetic data from random variables, and the discriminator network separates the synthetic data from the real-world data. One way of training GANs is to solve a Nash equilibrium problem [15, 16] with two players, a discriminator that minimizes the synthetic-real discrimination error and a generator that maximizes this error.

Many optimizers have been presented for training GANs (see, e.g., [3, 8, 10, 14, 20, 24]). In this paper, we focus on a two time-scale update rule (TTUR) [10] for finding a stationary local Nash equilibrium of the Nash equilibrium problem. TTUR uses sequences generated by each of the generator and the discriminator.

For example, let us consider a TTUR based on stochastic gradient descent (SGD) [4, 7, 21]; let $\theta_n$ be the point generated by the generator at iteration $n$, and $w_n$ the point generated by the discriminator at iteration $n$. The generator uses SGD with a learning rate $\alpha_n^G$ and a mini-batch stochastic gradient at $\theta_n$ using a batch of size $b$ for minimizing its loss function $L_G(\cdot, w_n)$. Meanwhile, the discriminator uses SGD with a learning rate $\alpha_n^D$ and a mini-batch stochastic gradient at $w_n$ using a batch of size $b$ for minimizing its loss function $L_D(\theta_n, \cdot)$. If $\alpha_n^G$ and $\alpha_n^D$ are decaying learning rates, then TTUR based on SGD converges almost surely to a stationary local Nash equilibrium [10, Theorem 1].

TTUR based on adaptive methods can be defined by replacing SGD with adaptive methods for training deep neural networks. For example, TTUR based on adaptive moment estimation (Adam) [13] with decaying learning rates $\alpha_n^G$ and $\alpha_n^D$ converges almost surely to a stationary local Nash equilibrium [10, Theorem 2].
1.2. Motivation. The above subsection described that TTUR using decaying learning rates can in theory be applied to local Nash equilibrium problems in GANs. Meanwhile, the numerical results in [10,28] showed that TTUR and deep learning optimizers using constant learning rates perform well. The motivation behind this work is thus to identify whether TTUR with constant learning rates can in theory be applied to local Nash equilibrium problems in GANs. In so doing, we may bridge the gap between theory and practice for TTUR.

Previous numerical evaluations [10, Figure 5] indicated that increasing the batch size used in TTUR tends to decrease the Fréchet inception distance (FID) [6, (4)], which is a performance measure of optimizers for training GANs. This implies that using large batch sizes is desirable for training GANs. We are interested in examining the theoretical relationship between the performance of TTUR using constant learning rates and the batch size.

1.3. Contribution. The main contribution of this paper is a theoretical analysis of TTUR using constant learning rates. We examine TTURs based on several adaptive methods (see Algorithm 1 and Table 2), such as Adam [13], adaptive mean square gradient (AMSGGrad) [19], and AdaBelief (short for adapting step sizes by the belief in observed gradients) [28]. Let \( ((\theta_n, w_n))_{n \in \mathbb{N}} \subset \mathbb{R}^\Theta \times \mathbb{R}^W \) be the sequence generated by TTUR using constant learning rates \( \alpha^G \) and \( \alpha^D \). We show that, under certain assumptions, there exist accumulation points \((\theta^*, w^*)\) and \((\theta_\ast, w_\ast)\) of \((\theta_n, w_n))_{n \in \mathbb{N}}\) such that

\[
E [\|\nabla_\theta L_G(\theta^*, w^*)\|^2] \leq \frac{\alpha^G (\sigma^2_G b^{-1} + M_G^2)}{2\beta^G \tau^G G^2 h_{0,\ast}^G} + \frac{\Theta \text{Dist}(\tilde{\theta}) }{b} \left( \frac{\sigma^2_G }{M_G^2} \right) \frac{\beta^G }{\beta^G },
\]

\[
E [\|\nabla_w L_D(\theta_\ast, w_\ast)\|^2] \leq \frac{\alpha^D (\sigma^2_D b^{-1} + M_D^2)}{2\beta^D \tau^D D^2 h_{0,\ast}^D} + \frac{\text{WDist}(\tilde{\omega}) }{b} \left( \frac{\sigma^2_D }{M_D^2} \right) \frac{\beta^D }{\beta^D },
\]

where \( E[X] \) denotes the expectation of a random variable \( X \), \( \|x\| \) denotes the norm of a vector \( x \), \( \nabla_\theta L_G(\cdot, w) \) is the gradient of \( L_G(\cdot, w) \) \( (w \in \mathbb{R}^W) \), \( \nabla_w L_D(\theta, \cdot) \) is the gradient of \( L_D(\theta, \cdot) \) \( (\theta \in \mathbb{R}^\Theta) \), \( \sigma^2_G, \sigma^2_D \geq 0, M_G, M_D, \text{Dist}(\tilde{\theta}), \text{Dist}(\tilde{\omega}) > 0, \beta^G, \beta^D, \tau^G, \tau^D \in [0, 1], \beta = 1 - \beta, \tau = 1 - \gamma, \) and \( h_{0,\ast}^G, h_{0,\ast}^D > 0 \) (see Section 3.1 and Theorem 3.1 for details).

This result implies that subsequences of the whole sequence \((\theta_n, w_n))_{n \in \mathbb{N}}\) approximate a stationary local Nash equilibrium \((\theta^*, w^*)\) satisfying \( \nabla_\theta L_G(\theta^*, w^*) = 0 \) and \( \nabla_w L_D(\theta^*, w^*) = 0 \) (see Problem 2.1). Hence, TTUR using constant learning rates \( \alpha^G \) and \( \alpha^D \) should converge unstably. Figure 1 plots TTUR based on Adam with constant learning rates defined by \( \alpha^G = 5 \times 10^{-5} \) and \( \alpha^D = 10^{-5} \) [10, Table 1], and a fixed batch size \( b = 2^4 \) for training a deep convolutional GAN (DCGAN) [18] on the CIFAR-10 dataset. Although the values of the loss functions of the generator and discriminator fluctuate, we can see that there are subsequences of the whole sequence which converge to some point. Accordingly, our theoretical result (Theorem 3.1) would be adequate from a practical viewpoint.

This result also implies that, for TTUR with constant learning rates, using a large batch size \( b \) would be useful for training GANs since the upper bounds of
\[ \mathbb{E}[\| \nabla_\theta L_G(\theta^*, w^*) \|^2] \text{ and } \mathbb{E}[\| \nabla_w L_D(\theta^*, w^*) \|^2] \]
become small when large batch sizes are used.

**Figure 1.** Training losses of generator and discriminator for TTUR based on Adam with constant learning rates \( \alpha^G = 5 \times 10^{-4} \) and \( \alpha^D = 10^{-5} \) and batch size \( b = 2^6 \) versus number of steps for training DCGAN on the CIFAR-10 dataset.

Previous results \cite{22, 26} on training deep neural networks in practical tasks showed that, for each deep learning optimizer, the number of steps to train a deep neural network halves for each doubling of the batch size and there are diminishing returns beyond a critical batch size. Motivated by the results in \cite{22, 26}, we examine the relationship between batch size \( b \) and the number of steps \( N \) needed for training GAN, i.e., for finding a stationary local Nash equilibrium in the Nash equilibrium problem in GANs. Here, suppose that the generator runs in at most \( N_G \) steps, defined for batch size \( b \) by

\[
N_G(b) := \frac{A_G b}{\epsilon_G^2 - C_G b - B_G},
\]
and the discriminator runs in at most \( N_D \) steps, defined for batch size \( b \) by

\[
N_D(b) := \frac{A_D b}{\epsilon_D^2 - C_D b - B_D},
\]

where \( \epsilon_G \) and \( \epsilon_D \) are precisions for the Nash equilibrium problem and \( A_G, A_D, B_G, B_D, C_G, \) and \( C_D \) are nonnegative constants depending on parameters used in TTUR (see Section 3.2 for details). Then, we show that TTUR approximates a stationary local Nash equilibrium and \( N_G \) and \( N_D \) are monotone decreasing and convex functions for the batch size (Theorem 3.2). This result implies that large batch sizes are desirable in the sense of minimizing the number of steps for training GANs.
A particularly interesting concern is how large the batch size $b$ should be. Here, we consider the stochastic first-order oracle (SFO) complexities defined for $N_G$ in (1.1), $N_D$ in (1.2), and $b$ by $N_G(b) b$ and $N_D(b) b$. We show that $N_G(b) b$ and $N_D(b) b$ are convex functions of $b$ and that there exist global minimizers $b^*_G$ and $b^*_D$ of $N_G(b) b$ and $N_D(b) b$ (Theorem 3.3). It would be desirable to use batch sizes $b^*_G$ and $b^*_D$ since the SFO complexity, which is the stochastic gradient computation cost, can be minimized by doing so.

We also provide numerical results to support our theoretical results (Section 4). In particular, we numerically show that increasing the batch size decreases the number of steps needed to train the GANs and a batch size that minimizes the SFO complexity exists.

2. Mathematical Preliminaries

2.1. Assumptions and main problem. The notation used in this paper is summarized in Table 1.

We assume the following standard conditions:

Assumption 2.1.

(S1) $L_G^{(i)}(\cdot, w): \mathbb{R}^\Theta \rightarrow \mathbb{R}$ and $L_D^{(i)}(\theta, \cdot): \mathbb{R}^W \rightarrow \mathbb{R}$ are continuously differentiable.

(S2) Let $((\theta_n, w_n))_{n \in \mathbb{N}} \subset \mathbb{R}^\Theta \times \mathbb{R}^W$ be the sequence generated by an optimizer.

(i) For each iteration $n$,

$$
\mathbb{E}_{\xi_{n}^{G}} \left[ G_{\xi_{n}^{G}}(\theta_n) \right] = \nabla_{\theta} L_G(\theta_n, w_n) \quad \text{and} \quad \mathbb{E}_{\xi_{n}^{D}} \left[ G_{\xi_{n}^{D}}(w_n) \right] = \nabla_{w} L_D(\theta_n, w_n).
$$

(ii) There exist nonnegative constants $\sigma_G^2$ and $\sigma_D^2$ such that

$$
\mathbb{E}_{\xi_{n}^{G}} \left[ \|G_{\xi_{n}^{G}}(\theta_n) - \nabla_{\theta} L_G(\theta_n, w_n)\|^2 \right] \leq \sigma_G^2 \quad \text{and} \quad \mathbb{E}_{\xi_{n}^{D}} \left[ \|G_{\xi_{n}^{D}}(w_n) - \nabla_{w} L_D(\theta_n, w_n)\|^2 \right] \leq \sigma_D^2.
$$

(S3) For each iteration $n$, the optimizer samples mini-batches $S_n \subset \mathcal{S}$ and $\mathcal{R}_n \subset \mathcal{R}$ and estimates the full gradients $\nabla L_G$ and $\nabla L_D$ as

$$
\nabla L_{G,S_n}(\theta_n) := \frac{1}{b} \sum_{i \in [b]} G_{\xi_{n,i}^{G}}(\theta_n) = \frac{1}{b} \sum_{\{i : z^{(i)} \in \mathcal{S}_n\}} \nabla_{\theta} L_G^{(i)}(\theta_n, w_n) \quad \text{and} \quad \nabla L_{D,R_n}(w_n) := \frac{1}{b} \sum_{i \in [b]} G_{\xi_{n,i}^{D}}(w_n) = \frac{1}{b} \sum_{\{i : x^{(i)} \in \mathcal{R}_n\}} \nabla_{w} L_D^{(i)}(\theta_n, w_n).
$$

This paper considers the following local Nash equilibrium problem with two players, a generator and a discriminator [10]:

Problem 2.1. Under Assumption 2.1 find a stationary local Nash equilibrium $(\theta^*, w^*) \in \mathbb{R}^\Theta \times \mathbb{R}^W$ satisfying

$$
\nabla_{\theta} L_G(\theta^*, w^*) = 0 \quad \text{and} \quad \nabla_{w} L_D(\theta^*, w^*) = 0.
$$

TTURs [10] (1), (2) were presented on the basis of SGD and Adam for solving Problem 2.1 and it is guaranteed that the optimizers using decaying learning rates converge to stationary local Nash equilibria [10 Theorems 1 and 2].
The expectation with respect to $A$

The mini-batch stochastic gradient of $\xi$

The set of $R$

Mini-batch of $R$

The history of process $G$

The stochastic gradient of $A$

A random variable generated from the $G$

A total loss function of the generator $G$

A set of real-world samples $S$

A loss function of the generator $G$

A set of synthetic samples $\xi$

The set of $N$

The number of elements of a set $A$

A random variable supported on $\Xi$

Assumption 2.2. described by Algorithm 1, for solving Problem 2.1.

To analyze Algorithm 1, we assume the following conditions:

**Assumption 2.2.**

(A1) $H_n^G = \text{diag}(h_{n,i}^G)$ depends on $\xi_{\theta_{n,i}}^G$ and $H_n^D = \text{diag}(h_{n,i}^D)$ depends on $\xi_{\theta_{n,i}}^D$. Moreover, $h_{n+1,i}^G \geq h_{n,i}^G$ and $h_{n+1,j}^D \geq h_{n,j}^D$ hold for all $n \in \mathbb{N}$, all $i \in [\Theta]$, and all $j \in [W]$. 

| Notation | Description |
|----------|-------------|
| $\mathbb{N}$ | The set of all nonnegative integers |
| $[N]$ | $[N] := \{1, 2, \ldots, N\}$ (for $N \in \mathbb{N} \setminus \{0\}$) |
| $|A|$ | The number of elements of a set $A$ |
| $\mathbb{R}^d$ | A $d$-dimensional Euclidean space with inner product $\langle \cdot, \cdot \rangle$, which induces the norm $\| \cdot \|_d$ |
| $\mathbb{R}_d^+$ | $\mathbb{R}_d^+ := \{ x \in \mathbb{R}_d : x_i \geq 0 \ (i \in [d]) \}$ |
| $\mathbb{R}_d^{++}$ | $\mathbb{R}_d^{++} := \{ x \in \mathbb{R}_d : x_i > 0 \ (i \in [d]) \}$ |
| $\mathbb{S}_d^{++}$ | The set of $d \times d$ symmetric positive-definite matrices |
| $\mathbb{D}_d$ | The set of $d \times d$ diagonal matrices, i.e., $\mathbb{D}_d = \{ M \in \mathbb{R}_d^{d \times d} : M = \text{diag}(x_i), \ x_i \in \mathbb{R} \ (i \in [d]) \}$ |
| $E[\xi]$ | The expectation with respect to $\xi$ of a random variable $X$ |
| $S$ | A set of synthetic samples $\xi^{(i)}$ |
| $\mathcal{R}$ | A set of real-world samples $x^{(i)}$ |
| $S_n$ | Mini-batch of $b$ synthetic samples $\xi^{(i)}$ at time $n$ |
| $\mathcal{R}_n$ | Mini-batch of $b$ real-world samples $x^{(i)}$ at time $n$ |
| $L_G^G(\cdot, w)$ | A loss function of the generator for $w \in \mathbb{R}^W$ and $\xi^{(i)}$ |
| $L_G(\cdot, w)$ | A total loss function of the generator for $w \in \mathbb{R}^W$, i.e., $L_G(\cdot, w) := |S|^{-1} \sum_{\xi^{(i)} \in S} L_G^G(\cdot, w)$ |
| $\xi^G$ | A random variable supported on $\Xi^G$ that does not depend on $w \in \mathbb{R}^W$ and $\theta \in \Theta$ |
| $\xi_n^G$ | $\xi_0^G, \xi_1^G, \ldots$ are independent samples and $\xi_n^G$ is independent of $(\theta_k)_{k=0}^n \subset \Theta$ and $w \in \mathbb{R}^W$ |
| $\xi_{n,i}^{G,\theta}$ | A random variable generated from the $i$-th sampling at time $n$ |
| $\xi_{[n]}^G$ | The history of process $\xi_0^G, \xi_1^G, \ldots$ to time step $n$, i.e., $\xi_{[n]}^G := (\xi_0^G, \xi_1^G, \ldots, \xi_n^G)$ |
| $G_{\xi^G}(\theta)$ | The stochastic gradient of $L_G(\cdot, w)$ at $\theta \in \Theta$ |
| $\nabla L_{G,S_n}(\theta_n)$ | The mini-batch stochastic gradient of $L_G(\theta_n, w_n)$ for $S_n$, i.e., $\nabla L_{G,S_n}(\theta_n) := b^{-1} \sum_{i \in [b]} G_{\xi_n^G,i}(\theta_n)$ |
We assume the following conditions:

1. Convergence analysis of Algorithm 1 using constant learning rates.
2. By referring to the results in [5, 11, 28], we can check that \(H_n^G \in S_{++}^+ \cap D^\Theta\) and \(H_n^D \in S_{++}^W \cap D^W\) (steps 6 and 13) in Algorithm 1 (\(\eta^G, \eta^D, \sigma^G, \sigma^D \in [0, 1]\))

### Examples of \(H_n^G\) and \(H_n^D\)

| Algorithm | \(H_n^G\) | \(H_n^D\) |
|-----------|------------|------------|
| SGD \((\beta = \gamma = 0)\) | \(H_n^G\) is the identity matrix. | \(H_n^D\) is the identity matrix. |
| Momentum \((\gamma = 0)\) | \(H_n^G\) is the identity matrix. | \(H_n^D\) is the identity matrix. |
| Adam \((v_{n,i} \leq v_{n+1,i})\) | \(p_n^G = \nabla L_{G,Sn}(\theta_n) \odot \nabla L_{G,Sn}(\theta_n)\) | \(p_n^G = \nabla L_{D,Rn}(w_n) \odot \nabla L_{D,Rn}(w_n)\) |
| AMSGrad \((\gamma = 0)\) | \(H_n^G = \text{diag}(\sqrt{v_{n,i}})\) | \(H_n^D = \text{diag}(\sqrt{v_{n,i}})\) |
| AdaBelief \((s_{n,i}^G \leq s_{n+1,i}^G, s_{n,i}^D \leq s_{n+1,i}^D)\) | \(H_n^G = \text{diag}(\sqrt{s_{n,i}})\) | \(H_n^D = \text{diag}(\sqrt{s_{n,i}})\) |

### Examples of \(H_n^G\) and \(H_n^D\)

| Algorithm | \(H_n^G\) | \(H_n^D\) |
|-----------|------------|------------|
| SGD \((\beta = \gamma = 0)\) | \(H_n^G\) is the identity matrix. | \(H_n^D\) is the identity matrix. |
| Momentum \((\gamma = 0)\) | \(H_n^G\) is the identity matrix. | \(H_n^D\) is the identity matrix. |
| Adam \((v_{n,i} \leq v_{n+1,i})\) | \(p_n^G = \nabla L_{G,Sn}(\theta_n) \odot \nabla L_{G,Sn}(\theta_n)\) | \(p_n^G = \nabla L_{D,Rn}(w_n) \odot \nabla L_{D,Rn}(w_n)\) |
| AMSGrad \((\gamma = 0)\) | \(H_n^G = \text{diag}(\sqrt{v_{n,i}})\) | \(H_n^D = \text{diag}(\sqrt{v_{n,i}})\) |
| AdaBelief \((s_{n,i}^G \leq s_{n+1,i}^G, s_{n,i}^D \leq s_{n+1,i}^D)\) | \(H_n^G = \text{diag}(\sqrt{s_{n,i}})\) | \(H_n^D = \text{diag}(\sqrt{s_{n,i}})\) |

(A2) For all \(i \in [\Theta]\), there exists a positive number \(H_i^G\) such that \(\sup_{n \in \mathbb{N}} \mathbb{E}[h_{n,i}^G] \leq H_i^G\). For all \(j \in [W]\), there exists a positive number \(H_j^D\) such that \(\sup_{n \in \mathbb{N}} \mathbb{E}[h_{n,j}^D] \leq H_j^D\).

Examples of \(H_n^G \in S_{++}^+ \cap D^\Theta\) and \(H_n^D \in S_{++}^W \cap D^W\) satisfying Assumption 2.2 are listed in Table 2. By referring to the results in [5, 11, 28], we can check that \(H_n^G\) and \(H_n^D\) in Table 2 satisfy (A1) and (A2).

### 3. Main Results

#### 3.1. Convergence analysis of Algorithm 1 using constant learning rates.

We assume the following conditions:

**Assumption 3.1.**

1. \(\alpha_n^G := \alpha^G\) and \(\alpha_n^D := \alpha^D\) for all \(n \in \mathbb{N}\).
2. There exist positive numbers \(M_G\) and \(M_D\) such that \(\mathbb{E}[\|\nabla_{\theta} L_G(\theta_n, w_n)\|^2] \leq M_G^2\) and \(\mathbb{E}[\|\nabla_{w} L_D(\theta_n, w_n)\|^2] \leq M_D^2\).
3. For all \(\theta = (\theta_i) \in \Theta^\Theta\) and all \(w = (w_i) \in \mathbb{R}^W\), there exist positive numbers \(\text{Dist}(\theta)\) and \(\text{Dist}(\theta)\) such that \(\max_{i \in [\Theta]} \sup\{\|\theta_{n,i} - \theta_i\|^2: n \in \mathbb{N}\} \leq \text{Dist}(\theta)\) and \(\max_{i \in [W]} \sup\{\|w_{n,i} - w_i\|^2: n \in \mathbb{N}\} \leq \text{Dist}(\theta)\).
The previous study in [10] used a decaying learning rate $O(n^{-\tau})$, where $\tau \in (0,1]$, for TTUR based on Adam to train GANs. This paper investigates the performance of TTUR based on adaptive methods using constant learning rates defined by (C1). Condition (C2) is used here to provide upper bounds on the performance measures defined by the expectations of the gradients of $L_G$ and $L_D$ (see Theorem 3.1 for details). (C2) was also used to analyze adaptive methods for training deep neural networks (see, e.g., [5,28]). Condition (C3) was used to provide upper bounds on the performance measures and for analyzing both convex and nonconvex optimization in deep neural networks (see, e.g., [13,19,28]).

The following is a convergence analysis of Algorithm 1 (The proof of Theorem 3.1 is in the Appendix).

**Theorem 3.1.** Suppose that Assumptions 2.1, 2.2, and 3.1 hold and consider the sequence $((\theta_n, w_n))_{n \in \mathbb{N}}$ generated by Algorithm 1. Then, the following hold:

(i) For all $\theta \in \mathbb{R}^\Theta$ and all $w \in \mathbb{R}^W$,

$$\liminf_{n \to +\infty} \mathbb{E} \left[ \langle \theta_n - \theta, \nabla_{\theta} L_G(\theta_n, w_n) \rangle \right] \leq \frac{\alpha^G(\sigma^2_G b^{-1} + M^2_G)}{2\beta^G \gamma^2_G k^G_0} + \sqrt{\Theta \text{Dist}(\theta) \left( \frac{\sigma^2_G}{b_G} + M^2_G \right) \beta^G},$$

Theorem 3.1. Suppose that Assumptions 2.1, 2.2, and 3.1 hold and consider the sequence $((\theta_n, w_n))_{n \in \mathbb{N}}$ generated by Algorithm 1. Then, the following hold:

(i) For all $\theta \in \mathbb{R}^\Theta$ and all $w \in \mathbb{R}^W$,

$$\liminf_{n \to +\infty} \mathbb{E} \left[ \langle \theta_n - \theta, \nabla_{\theta} L_G(\theta_n, w_n) \rangle \right] \leq \frac{\alpha^G(\sigma^2_G b^{-1} + M^2_G)}{2\beta^G \gamma^2_G k^G_0} + \sqrt{\Theta \text{Dist}(\theta) \left( \frac{\sigma^2_G}{b_G} + M^2_G \right) \beta^G},$$
Here, we would like to check how increasing $b$ affects the number of steps $N$ needed to train GANs. Indeed, the evidence that using sufficiently small $\alpha$ to use small learning rates is, the smaller the upper bounds in Theorem 3.1 depend on $\alpha$ and $\alpha^D$. Meanwhile, (ii) indicates that there is no evidence that using sufficiently small $\alpha^G$ and $\alpha^D$ is good for training GANs since the upper bounds in Theorem 3.1 depend on $\alpha^G$, $\alpha^D$, $1/\alpha^G$, and $1/\alpha^D$. Indeed, the results reported in [10] used small $\alpha^G$, $\alpha^D = 10^{-4}, 10^{-5}$.

3.2. Relationship between batch size and steps for Algorithm 1. Fix $\alpha^G$, $\alpha^D$, $\beta^G$, and $\beta^D$. Theorem 3.1 (i) implies that, if the batch size $b$ is sufficiently large, then the upper bounds of $\mathbb{E}[\|\nabla_{\theta} L_G(\theta^*, w^*)\|^2]$ and $\mathbb{E}[\|\nabla_w L_D(\theta^*, w^*)\|^2]$ are small. Hence, it would be useful for Algorithm 1 to use a sufficiently large $b$ to train GANs. Here, we would like to check how increasing $b$ affects the number of steps $N$ needed.

\[
\lim_{n \to +\infty} \mathbb{E} \left[ (w_n - w, \nabla_w L_D(\theta_n, w_n)) \right] \\
\leq \frac{\alpha^D (\sigma^D_\beta b^{-1} + M^2_D)}{2\beta^D \gamma^D h^D_{0,*}} + \sqrt{W \text{Dist}(w) \left( \frac{\sigma^2_\beta}{b} + M^2_D \right) \frac{\beta^D}{\beta^G}},
\]

where $\beta^G := 1 - \beta^G$, $\beta^D := 1 - \beta^D$, $\gamma^G := 1 - \gamma^G$, $\gamma^D := 1 - \gamma^D$, $h^G_{0,*} := \min_{i \in [\Theta]} h^G_{0,i}$, and $h^D_{0,*} := \min_{j \in [W]} h^D_{0,j}$. Furthermore, there exist accumulation points $(\theta^*, w^*)$ and $(\theta_n, w_n)$ of $((\theta_n, w_n))_{n \in \mathbb{N}}$ such that

\[
\mathbb{E} \left[ \|\nabla_{\theta} L_G(\theta^*, w^*)\|^2 \right] \leq \frac{\alpha^G (\sigma^2_G b^{-1} + M^2_G)}{2\beta^G \gamma^G h^G_{0,*}} + \sqrt{\Theta \text{Dist}(\bar{\theta}) \left( \frac{\sigma^2_G}{b} + M^2_G \right) \frac{\beta^G}{\beta^G}},
\]

\[
\mathbb{E} \left[ \|\nabla_w L_D(\theta_n, w_n)\|^2 \right] \leq \frac{\alpha^D (\sigma^2_D b^{-1} + M^2_D)}{2\beta^D \gamma^D h^D_{0,*}} + \sqrt{W \text{Dist}(\bar{w}) \left( \frac{\sigma^2_D}{b} + M^2_D \right) \frac{\beta^D}{\beta^D}},
\]

where $\bar{\theta} := \theta^* - \nabla_{\theta} L_G(\theta^*, w^*)$ and $\bar{w} := w^* - \nabla_w L_D(\theta^*, w^*)$.

(ii) For all $\theta \in \mathbb{R}^\Theta$, all $w \in \mathbb{R}^W$, and all $N \geq 1$,

\[
\frac{1}{N} \sum_{n \in [N]} \mathbb{E} \left[ (\theta - \theta_n, \nabla_{\theta} L_G(\theta_n, w_n)) \right] \\
\leq \frac{\Theta \text{Dist}(\theta) H^G}{2\alpha^G \beta^G N} + \frac{\alpha^G}{2\beta^G \gamma^G h^G_{0,*}} \left( \frac{\sigma^2_G}{b} + M^2_G \right) + \frac{\beta^G}{\beta^G} \sqrt{\Theta \text{Dist}(\theta) \left( \frac{\sigma^2_G}{b} + M^2_G \right)},
\]

\[
\frac{1}{N} \sum_{n \in [N]} \mathbb{E} \left[ (w - w_n, \nabla_w L_D(\theta_n, w_n)) \right] \\
\leq \frac{W \text{Dist}(w) H^D}{2\alpha^D \beta^D N} + \frac{\alpha^D}{2\beta^D \gamma^D h^D_{0,*}} \left( \frac{\sigma^2_D}{b} + M^2_D \right) + \frac{\beta^D}{\beta^D} \sqrt{W \text{Dist}(w) \left( \frac{\sigma^2_D}{b} + M^2_D \right)},
\]

where $H^G := \max_{i \in [\Theta]} h^G_i$ and $H^D := \max_{j \in [W]} h^D_j$.
for solving Problem 2.1 i.e., the steps $N$ satisfying that
\[
\frac{1}{N} \sum_{n \in [N]} \mathbb{E} [(\theta - \theta_n, \nabla_{\theta} L_G(\theta_n, w_n))] \leq \epsilon_G^2,
\]
(3.1)
\[
\frac{1}{N} \sum_{n \in [N]} \mathbb{E} [(w - w_n, \nabla_{w} L_D(\theta_n, w_n))] \leq \epsilon_D^2,
\]
where $\epsilon_G$ and $\epsilon_D$ are precisions, $\theta \in \mathbb{R}^G$, and $w \in \mathbb{R}^W$. Theorem 3.1(ii) ensures that it is sufficient for (3.1) to consider
\[
\Theta_{\text{Dist}}(\theta) H^G \left( \frac{1}{2} \alpha G \right) \left( \frac{1}{N} \right) + \frac{2 \beta G \alpha G \sigma^2}{b} + \frac{M_G^2 \alpha G}{b} + \frac{\beta G}{\alpha G} \sqrt{\Theta_{\text{Dist}}(\theta)} (\sigma^2_M + M_G^2) \leq \epsilon_G^2,
\]
and
\[
\text{WDist}(w) H^D \left( \frac{1}{2} \alpha D \right) \left( \frac{1}{N} \right) + \frac{2 \beta D \alpha D \sigma^2}{b} + \frac{M_D^2 \alpha D}{b} + \frac{\beta D}{\alpha D} \sqrt{\text{WDist}(w)} (\sigma^2_M + M_D^2) \leq \epsilon_D^2.
\]

The relationship between $b$ and the number of steps $N$ satisfying (3.2) is as follows:

**Theorem 3.2.** Suppose that Assumptions 2.1, 2.2, and 3.1 hold and consider Algorithm 1. Then, the following $N_G$ and $N_D$ hold and satisfy (3.1):
\[
N_G(b) := \frac{A_G b}{(\epsilon_G^2 - C_G) b - B_G} \leq N \text{ for } b > \frac{B_G}{\epsilon_G^2 - C_G},
\]
and
\[
N_D(b) := \frac{A_D b}{(\epsilon_D^2 - C_D) b - B_D} \leq N \text{ for } b > \frac{B_D}{\epsilon_D^2 - C_D},
\]
where $A_G$, $B_G$, $C_G$, $A_D$, $B_D$, and $C_D$ are defined as in (3.2), and $\epsilon_G^2 - C_G > 0$ and $\epsilon_D^2 - C_D > 0$ are assumed. Moreover, the lower bounds $N_G$ and $N_D$ of $N$ defined by (3.3) are monotone decreasing and convex for $b > B_G/(\epsilon_G^2 - C_G)$ and $b > B_D/(\epsilon_D^2 - C_D)$.

Theorem 3.2 indicates that $N_G$ and $N_D$ of $N$ defined by (3.3) decrease as the batch size increases. Accordingly, it is useful to choose a sufficiently large $b$ in the sense of minimizing the number of steps $N$ needed for (3.1).

A particularly interesting concern is how large $b$ should be. To this end, we consider the SFO complexity defined for the lower bounds on the number of steps needed for (3.1) and the batch size by
\[
N_G(b) = \frac{A_G b^2}{(\epsilon_G^2 - C_G) b - B_G} \text{ and } N_D(b) = \frac{A_D b^2}{(\epsilon_D^2 - C_D) b - B_D}.
\]

The following theorem guarantees the existence of global minimizers of $N_G(b)$ and $N_D(b)$ defined by (3.4).
Theorem 3.3. Suppose that Assumptions 2.1, 2.2 and 3.1 hold and consider Algorithm 1. Then there exist

\[ b_G^* := \frac{2B_G}{\epsilon_G^2 - C_G} \quad \text{and} \quad b_D^* := \frac{2B_D}{\epsilon_D^2 - C_D} \]

such that \( b_G^* \) minimizes a convex function \( N_G(b)b \) \((b > B_G/(\epsilon_G^2 - C_G))\) and \( b_D^* \) minimizes a convex function \( N_D(b)b \) \((b > B_D/(\epsilon_D^2 - C_D))\).

Previous results [22, 26] on training deep neural networks in practical tasks showed that, for each deep learning optimizer, the number of steps for training the network halves for each doubling of the batch size and diminishing returns exist beyond a critical batch size. Theorem 3.2 indicates that the number of steps needed for \( (3.1) \) decreases as the batch size increases, as is also seen in training deep neural networks. Theorem 3.3 indicates that critical batch sizes exist in the sense of minimizing the SFO complexities \( N_G(b)b \) and \( N_D(b)b \). We are interested in verifying whether or not a critical batch size exists for training GANs. Hence, the next section numerically examines the relationship between the batch size \( b \) and not only the number of steps \( N \) but also the SFO complexity \( Nb \) and verifies whether a critical batch \( b^* \) exists such that the SFO complexity \( N(b)b \) is minimized at \( b^* \).

4. Numerical Examples

We evaluated the performance of TTUR based on SGD, Momentum, and Adam in training a deep convolutional GAN (DCGAN) on the CIFAR-10 dataset. The experimental environment consisted of two Intel(R) Xeon(R) Gold 6148 2.4-GHz CPUs with 20 cores each, a 16-GB NVIDIA Tesla V100 900-Gbps GPU, and Red Hat Enterprise Linux 7.6 OS. The code was written in Python 3.8.2 using the NumPy 1.17.3 and PyTorch 1.3.0 packages. TTUR used the identity matrices \( H_G \) and \( H_D \) and \( \beta^G = \beta^D = \gamma^G = \gamma^D = 0 \), and TTUR based on Adam used the identity matrices \( H_G \) and \( H_D \) with \( \beta^G = \beta^D = 0.9 \), and \( \gamma^G = \gamma^D = 0 \). Adam used \( H_G \) and \( H_D \) defined in Table 2, \( \beta^G = \beta^D = \gamma^G = \gamma^D = 0.9 \), and \( \eta^G = \eta^D = \zeta^G = \zeta^D = 0.999 \).

First, we evaluated TTUR based on Adam with different learning rates and a fixed batch size \( b = 2^6 \). Figure 2 plots the training losses of the generator and the discriminator for TTUR based on Adam with constant learning rates \( \alpha^G = 5 \times 10^{-4} \) and \( \alpha^D = 10^{-5} \) [10, Table 1]. The results show that there are subsequences of the whole sequences \( (L_G(\theta_n, w_n))_{n \in \mathbb{N}} \) and \( (L_D(\theta_n, w_n))_{n \in \mathbb{N}} \) which converge to some point. This result supports Theorem 3.1 indicating that using small constant learning rates \( \alpha^G \) and \( \alpha^D \) can approximate a stationary local Nash equilibrium.

Figure 3 plots the training losses of the generator and the discriminator for TTUR based on Adam with constant learning rates \( \alpha^G = 1 \) and \( \alpha^D = 1 \). The results show that the convergent point of TTUR with \( \alpha^G = 1 \) and \( \alpha^D = 1 \) is different from the one of TTUR with \( \alpha^G = 5 \times 10^{-4} \) and \( \alpha^D = 10^{-5} \) (Figure 2). To verify whether TTUR with \( \alpha^G = 1 \) and \( \alpha^D = 1 \) converged to an appropriate point, we checked the Fréchet inception distance (FID) [6, (4)] of which a small value is desirable for training GANs. FID for TTUR with \( \alpha^G = 1 \) and \( \alpha^D = 1 \) was between 300 and 330, while FID for TTUR with \( \alpha^G = 5 \times 10^{-4} \) and \( \alpha^D = 10^{-5} \) was 120. Hence, we can see that TTUR with \( \alpha^G = 1 \) and \( \alpha^D = 1 \) is not good for training the DCGAN.
Figure 4 plots the training losses of the generator and the discriminator for TTUR based on Adam with constant learning rates $\alpha_G = 10^{-10}$ and $\alpha_D = 10^{-10}$. The results show that the behaviors of TTUR with $\alpha_G = 10^{-10}$ and $\alpha_D = 10^{-10}$ are different from those of TTUR with $\alpha_G = 5 \times 10^{-4}$ and $\alpha_D = 10^{-5}$. We also found that FID for TTUR with $\alpha_G = 10^{-10}$ and $\alpha_D = 10^{-10}$ was between 290 and 330, which implies that TTUR with $\alpha_G = 5 \times 10^{-4}$ and $\alpha_D = 10^{-5}$ performed better than TTUR with $\alpha_G = 10^{-10}$ and $\alpha_D = 10^{-10}$.

Figure 5 plots the training losses of the generator and the discriminator for TTUR based on Adam with decaying learning rates $\alpha_G = 1/\sqrt{n}$ and $\alpha_D = 1/\sqrt{n}$. The results show that TTUR using decaying learning rates is not good from a practical viewpoint, while in theory it can be used to train GANs [10, Theorems 1 and 2]. Overall, the above results indicate that it is desirable to use small constant learning rates such as $\alpha_G = 5 \times 10^{-4}$ and $\alpha_D = 10^{-5}$ [10, Table 1] to train DCGANs.

Next, we evaluated the TTUR with different batch sizes and fixed constant learning rates $\alpha_G = 5 \times 10^{-4}$ and $\alpha_D = 10^{-5}$ [10, Table 1]. The performance measures were the number of steps $N$ and the SFO complexity $Nb$ satisfying that FID is less than or equal to 200. Figure 6 plots the number of steps $N$ versus the batch size $b$. The figure indicates that $N$ for TTUR based on Adam was monotone decreasing for $b \geq 2^1$, as in Theorem 3.2. TTUR based on Momentum also decreased $N$ except for $b = 2^7$. Meanwhile, $N$ for TTUR based on SGD was monotone decreasing for $b \geq 2^4$.

Figure 7 plots the SFO complexity $Nb$ versus batch size $b$. Here, the SFO complexity $Nb$ for TTUR based on each of Momentum and Adam was a convex function of $b$, as in Theorem 3.3. Moreover, the SFO complexity $Nb$ for TTUR based on SGD was convex for $b \geq 2^4$. This figure also shows that the SFO complexity of TTUR based on Momentum was minimized at $b = 2^3$ and the SFO complexity of TTUR based on Adam was minimized at $b = 2^6$. This result supports Theorem 3.3 indicating that there exists a batch size minimizing the SFO complexity.

5. Conclusion

This paper considered a local Nash equilibrium problem in a GAN and performed a theoretical analysis of TTUR using constant learning rates to find a stationary local Nash equilibrium of the problem. We evaluated the upper bound of the expectation of the gradient of the loss function of the discriminator and the generator and showed that the upper bound is small when small constant learning rates and a large batch size are used. Next, we examined the relationship between the number of steps needed for solving the problem and batch size and showed that the number of steps decreases as the batch size increase. Moreover, we evaluated the SFO complexity of TTUR to check how large the batch size should be and showed that there is a critical batch size minimizing the SFO complexity, which is a convex function of the batch size. Finally, we provided numerical examples to support our theoretical analyzes. In particular, the numerical results showed that TTUR using small constant learning rates can be used to train a DCGAN, the number of steps needed to train DCGAN is monotone decreasing for batch size, and a critical batch size that it minimizes the SFO complexity exists.
Figure 2. Training losses of the generator and the discriminator for TTUR based on Adam with constant learning rates $\alpha^G = 5 \times 10^{-4}$ and $\alpha^D = 10^{-5}$ and batch size $b = 2^6$ versus number of steps for training DCGAN on the CIFAR-10 dataset (FID was 120).

Figure 3. Training losses of the generator and the discriminator for TTUR based on Adam with constant learning rates $\alpha^G = 1$ and $\alpha^D = 1$ and batch size $b = 2^6$ versus number of steps for training DCGAN on the CIFAR-10 dataset (FID fluctuated between 300 and 330).

Figure 4. Training losses of the generator and the discriminator for TTUR based on Adam with constant learning rates $\alpha^G = 10^{-10}$ and $\alpha^D = 10^{-10}$ and batch size $b = 2^6$ versus number of steps for training DCGAN on the CIFAR-10 dataset (FID fluctuated between 290 and 330).

Figure 5. Training losses of the generator and the discriminator for TTUR based on Adam with decaying learning rates $\alpha^G_n = 1/\sqrt{n}$ and $\alpha^D_n = 1/\sqrt{n}$ and batch size $b = 2^6$ versus number of steps for training DCGAN on the CIFAR-10 dataset (FID fluctuated between 325 and 330).
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Figure 6. Number of steps for TTUR based on SGD, Momentum, and Adam versus batch size needed to train DCGAN on CIFAR-10. The number of steps $N$ for TTUR based on Adam is a monotone decreasing function of $b$.

Figure 7. SFO complexities for TTUR based on SGD, Momentum, and Adam versus batch size needed to train DCGAN on CIFAR-10. The SFO complexity $Nb$ for TTUR based on each of Momentum and Adam is a convex function of $b$.

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A.1. Lemmas.
Lemma A.1. Suppose that (S1), (S2)(i), and (S3) hold and consider Algorithm 1. Then, for all \( \theta \in \mathbb{R}^2 \) and all \( n \in \mathbb{N} \),

\[
\mathbb{E} \left[ \| \theta_{n+1} - \theta \|_{H_\beta}^2 \right] = \mathbb{E} \left[ \| \theta_n - \theta \|_{H_\beta}^2 \right] + \alpha_n^G \mathbb{E} \left[ \| d_n^G \|_{H_\beta}^2 \right] \\
+ 2\alpha_n^G \left\{ \frac{\beta^G}{\gamma_n^G} \mathbb{E} \left[ (\theta - \theta_n, m_{n-1}^G) \right] + \frac{\beta^G}{\gamma_n^G} \mathbb{E} \left[ (\theta - \theta_n, \nabla L_G(\theta_n, w_n)) \right] \right\},
\]

where \( \tilde{\beta}^G := 1 - \beta^G \) and \( \tilde{\gamma}_n^G := 1 - \gamma_n^{G+1} \).

Proof. Let \( \theta \in \mathbb{R}^2 \) and \( n \in \mathbb{N} \). The definition of \( \theta_{n+1} \) implies that

\[
\| \theta_{n+1} - \theta \|_{H_\beta}^2 = \| \theta_n - \theta \|_{H_\beta}^2 + 2\alpha_n^G (\theta_n - \theta, d_n^G)_{H_\beta} + \alpha_n^G \| d_n^G \|_{H_\beta}^2.
\]

Moreover, the definitions of \( d_k, m_k \), and \( \tilde{m}_k \) ensure that

\[
\langle \theta_n - \theta, d_n^G \rangle_{H_\beta} = \frac{1}{\tilde{\gamma}_n^G} \langle \theta - \theta_n, m_n^G \rangle \\
= \frac{\beta^G}{\tilde{\gamma}_n^G} \langle \theta - \theta_n, m_{n-1}^G \rangle + \frac{\beta^G}{\gamma_n^G} \langle \theta - \theta_n, \nabla L_G, S_n(\theta_n) \rangle.
\]

Hence,

\[
\| \theta_{n+1} - \theta \|_{H_\beta}^2 = \| \theta_n - \theta \|_{H_\beta}^2 + \alpha_n^G \| d_n^G \|_{H_\beta}^2 \\
+ 2\alpha_n^G \left\{ \frac{\beta^G}{\gamma_n^G} \langle \theta - \theta_n, m_{n-1}^G \rangle + \frac{\beta^G}{\gamma_n^G} \langle \theta - \theta_n, \nabla L_G, S_n(\theta_n) \rangle \right\}.
\]

(A.1)

Conditions (S2)(i) and (S3) guarantee that

\[
\mathbb{E} \left[ \mathbb{E} \left[ (\theta - \theta_n, \nabla L_G, S_n(\theta_n)) \bigg| \theta_n \right] \right] = \mathbb{E} \left[ \langle \theta - \theta_n, \nabla L_G, S_n(\theta_n) \bigg| \theta_n \right] \rangle \\
= \mathbb{E} \left[ (\theta - \theta_n, \nabla L_G, S_n(\theta_n), w_n) \right].
\]

Therefore, the lemma follows by taking the expectation with respect to \( \xi_n^G \) on both side of (A.1).

A discussion similar to the one for proving Lemma A.1 leads to the following lemma.

Lemma A.2. Suppose that (S1), (S2)(i), and (S3) hold and consider Algorithm 1. Then, for all \( w \in \mathbb{R}^W \) and all \( n \in \mathbb{N} \),

\[
\mathbb{E} \left[ \| w_{n+1} - w \|_{H_\beta}^2 \right] = \mathbb{E} \left[ \| w_n - w \|_{H_\beta}^2 \right] + \alpha_n^D \mathbb{E} \left[ \| d_n^D \|_{H_\beta}^2 \right] \\
+ 2\alpha_n^D \left\{ \frac{\beta^D}{\gamma_n^D} \mathbb{E} \left[ (w - w_n, m_{n-1}^D) \right] + \frac{\beta^D}{\gamma_n^D} \mathbb{E} \left[ (w - w_n, \nabla w L_D(\theta_n, w_n)) \right] \right\},
\]

where \( \tilde{\beta}^D := 1 - \beta^D \) and \( \tilde{\gamma}_n^D := 1 - \gamma_n^{D+1} \).

Lemma A.3. Algorithm 1 satisfies that, under (S2)(i), (ii) and (C2), for all \( n \in \mathbb{N} \),

\[
\mathbb{E} \left[ \| m_n^G \|^2 \right] \leq \frac{\sigma_n^G}{\tilde{\beta}^G} + M_n^G.
\]
Under (A1) and (C2), for all $k \in \mathbb{N}$,

$$
E \left[ \|d_n^G\|_{H_n^G}^2 \right] \leq \frac{1}{(1 - \gamma^G)^2 h_{0,s}^G} \left( \frac{\sigma_G^2}{b} + M_G^2 \right),
$$

where $h_{0,s}^G := \min_{i \in \Theta} h_{0,i}^G$.

**Proof.** Let $n \in \mathbb{N}$. From (S2)(i), we have that

$$
E \left[ \|\nabla_{G,S_n}(\theta_n)\|^2 | \theta_n \right] = E \left[ \|\nabla_{G,S_n}(\theta_n) - \nabla_{G}(\theta_n, w_n) + \nabla_{G}(\theta_n, w_n)\|^2 | \theta_n \right]
$$

$$
= E \left[ \|\nabla_{G,S_n}(\theta_n) - \nabla_{G}(\theta_n, w_n)\|^2 | \theta_n \right] + E \left[ \|\nabla_{G}(\theta_n, w_n)\|^2 | \theta_n \right]
$$

$$
+ 2E \left[ \langle \nabla_{G,S_n}(\theta_n) - \nabla_{G}(\theta_n, w_n), \nabla_{G}(\theta_n, w_n) \rangle | \theta_n \right]
$$

$$
= E \left[ \|\nabla_{G,S_n}(\theta_n) - \nabla_{G}(\theta_n, w_n)\|^2 | \theta_n \right] + \|\nabla_{G}(\theta_n, w_n)\|^2,
$$

which, together with (S2)(ii) and (C2), implies that

(A.2) \quad $E \left[ \|\nabla_{G,S_n}(\theta_n)\|^2 \right] \leq \frac{\sigma_G^2}{b} + M_G^2$.

The convexity of $\| \cdot \|^2$, together with the definition of $m_n^G$ and (A.2), guarantees that, for all $n \in \mathbb{N}$,

$$
E \left[ \|m_n^G\|^2 \right] \leq \beta^G E \left[ \|m_{n-1}^G\|^2 \right] + (1 - \beta^G) E \left[ \|\nabla_{G,S_n}(\theta_n)\|^2 \right]
$$

$$
\leq \beta^G E \left[ \|m_{n-1}^G\|^2 \right] + (1 - \beta^G) \left( \frac{\sigma_G^2}{b} + M_G^2 \right).
$$

Induction thus ensures that, for all $n \in \mathbb{N}$,

(A.3) \quad $E \left[ \|m_n^G\|^2 \right] \leq \max \left\{ \|m_{n-1}^G\|^2, \frac{\sigma_G^2}{b} + M_G^2 \right\} = \frac{\sigma_G^2}{b} + M_G^2$, where $m_{n-1}^G = 0$ is used. For $n \in \mathbb{N}$, $H_n^G \in S_{++}^{g}$ guarantees the existence of a unique matrix $H_n^G \in S_{++}^{g}$ such that $H_n^G = \hat{H}_n^G \hat{H}_n^{-1}$ [?, Theorem 7.2.6]. We have that, for all $x \in \mathbb{R}^{\Theta}, \|x\|_{H_n^G} = \|\hat{H}_n^G x\|^2$. Accordingly, the definitions of $d_n^G$ and $m_n^G$ imply that, for all $n \in \mathbb{N}$,

$$
E \left[ \|d_n^G\|_{H_n^G}^2 \right] = E \left[ \left\| H_n^{-1} H_n^G d_n^G \right\|^2 \right] \leq \frac{1}{\tilde{\gamma}_n^G} E \left[ \left\| H_n^{-1} m_n^G \right\|^2 \right]
$$

$$
\leq \frac{1}{(1 - \gamma^G)^2} \frac{1}{\tilde{\gamma}_n^G} E \left[ \left\| H_n^{-1} \right\|^2 \left\| m_n^G \right\|^2 \right],
$$

where

$$
\left\| H_n^{-1} \right\| = \left\| \text{diag} \left( h_{n,i}^{-\frac{1}{2}} \right) \right\| = \max_{i \in \Theta} h_{n,i}^{-\frac{1}{2}}
$$

and $\tilde{\gamma}_n^G := 1 - \gamma^G \geq 1 - \gamma^G$. Moreover, (A1) ensures that, for all $n \in \mathbb{N}$,

$$
h_{n,i}^G \geq h_{0,i}^G \geq h_{0,s}^G := \min_{i \in \Theta} h_{0,i}^G.$$

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Hence, (A.3) implies that, for all \( k \in \mathbb{N} \),
\[
E \left[ \left\| d^G_n \right\|_{H^G_n}^2 \right] \leq \frac{1}{(1 - \gamma^G)^2 h^G_{0,*}} \left( \frac{\sigma^2_G}{b} + M^2_G \right),
\]
completing the proof. \( \square \)

A discussion similar to the one for proving Lemma A.3 leads to the following lemma.

**Lemma A.4.** Algorithm 1 satisfies that, under (S2)(i), (ii) and (C2), for all \( n \in \mathbb{N} \),
\[
E \left[ \left\| m^D_n \right\|^2 \right] \leq \sigma^2_D b + M^2_D.
\]
Under (A1) and (C2), for all \( k \in \mathbb{N} \),
\[
E \left[ \left\| d^D_n \right\|_{H^D_n}^2 \right] \leq \frac{1}{(1 - \gamma^D)^2 h^D_{0,*}} \left( \frac{\sigma^2_D}{b} + M^2_D \right),
\]
where \( h^D_{0,*} := \min_{i \in [\Theta]} h^D_{0,i} \).

Lemmas A.1 and A.3 lead to the following:

**Lemma A.5.** Suppose that (S1)–(S3), (A1), and (C2)–(C3) hold and define
\[
X^G_n(\theta) := E \left[ \left\| \theta_n - \theta \right\|_{H^G_n}^2 \right] \text{ for all } n \in \mathbb{N} \text{ and all } \theta \in \mathbb{R}^\Theta. \text{ Then, for all } \theta \in \mathbb{R}^\Theta \text{ and all } n \in \mathbb{N},
\]
\[
X^G_{n+1}(\theta) \leq X^G_n(\theta) + \text{Dist}(\theta) E \left[ \sum_{i \in [\Theta]} (h^G_{n+1,i} - h^G_{n,i}) \right] + \frac{\alpha^G_n \sigma^2_G}{(1 - \gamma^G)^2 h^G_{0,*}} \left( \frac{\sigma^2_G}{b} + M^2_G \right)
\]
\[
+ 2 \alpha^G_n \left( \frac{\beta^G}{\tilde{\gamma}^G_n} \sqrt{\Theta \text{Dist}(\theta)} \left( \frac{\sigma^2_G}{b} + M^2_G \right) + \frac{\tilde{\beta}^G}{\tilde{\gamma}^G_n} E \left[ \langle \theta - \theta_n, \nabla_{\theta} L_G(\theta_n, w_n) \rangle \right] \right).
\]

**Proof.** Lemma A.3 and Jensen’s inequality guarantee that
\[
E \left[ \left\| m^G_n \right\| \right] \leq \sqrt{\frac{\sigma^2_G}{b} + M^2_G}.
\]
Condition (C3) implies that, for all \( \theta \in \mathbb{R}^\Theta \),
\[
\left\| \theta_n - \theta \right\|^2 = \sum_{i \in [\Theta]} (\theta_{n,i} - \theta_i)^2 \leq \Theta \text{Dist}(\theta).
\]
The Cauchy-Schwarz inequality thus ensures that
\[
(\text{A.4}) \quad E \left[ \left\langle \theta - \theta_n, m^G_{n-1} \right\rangle \right] \leq E \left[ \left\| \theta - \theta_n \right\| \left\| m^G_{n-1} \right\| \right] \leq \sqrt{\Theta \text{Dist}(\theta)} \left( \frac{\sigma^2_G}{b} + M^2_G \right).
\]
We define \( X^G_n(\theta) := E \left[ \left\| \theta_n - \theta \right\|_{H^G_n}^2 \right] \) for all \( n \in \mathbb{N} \) and all \( \theta \in \mathbb{R}^\Theta \). Then, we have
\[
X^G_{n+1}(\theta) - E \left[ \left\| \theta_{n+1} - \theta \right\|_{H^G_n}^2 \right] = E \left[ \sum_{i \in [\Theta]} (h^G_{n+1,i} - h^G_{n,i})(\theta_{n+1,i} - \theta_i)^2 \right],
\]
which, together with (C3), implies that

\[ X_{n+1}^G(\theta) - \mathbb{E} \left[ \|\theta_{n+1} - \theta\|^2_{H_0^2} \right] \leq \text{Dist}(\theta) \mathbb{E} \left[ \sum_{i \in [\Theta]} (h_{n+1,i}^G - h_{n,i}^G) \right]. \]

Hence, Lemmas A.1 and A.3 lead to the assertion in Lemma A.5. \( \square \)

A discussion similar to the one for proving Lemma A.5, together with Lemmas A.2 and A.4 leads to the following lemma.

**Lemma A.6.** Suppose that (S1)–(S3), (A1), and (C2)–(C3) hold and define

\[ X_n^D(w) := \mathbb{E} \|w_n - w\|^2_{H_0^2} \] for all \( n \in \mathbb{N} \) and all \( w \in \mathbb{R}^W \). Then, for all \( w \in \mathbb{R}^W \) and all \( n \in \mathbb{N} \),

\[ X_{n+1}^D(w) \leq X_n^D(w) + \text{Dist}(w) \mathbb{E} \left[ \sum_{i \in [W]} (h_{n+1,i}^D - h_{n,i}^D) \right] + \frac{\alpha_{n+1}^D}{(1 - \gamma_n^D)2h_{0,*}^D} \left( \frac{\sigma_n^2}{b} + M^2_D \right) \]

\[ + 2\alpha_n^D \left\{ \frac{\beta_n^D}{\gamma_n^D} \sqrt{W \text{Dist}(w) \left( \frac{\sigma_n^2}{b} + M^2_D \right)} + \frac{\tilde{\beta}_n^D}{\gamma_n^D} \mathbb{E} \left[ (w - w_n, \nabla_w L_D(\theta_n, w_n)) \right] \right\}. \]

**A.2. Proof of Theorem 3.1(i).**

**Proof of Theorem 3.1(i).** Let us assume (C1), i.e., \( \alpha_n^G := \alpha^G \) for all \( n \in \mathbb{N} \). Then, Lemma A.5 ensures that, for all \( \theta \in \mathbb{R}^{\Theta} \) and all \( n \in \mathbb{N} \),

\[ X_{n+1}^G(\theta) \leq X_n^G(\theta) + \text{Dist}(\theta) \mathbb{E} \left[ \sum_{i \in [\Theta]} (h_{n+1,i}^G - h_{n,i}^G) \right] + \frac{\alpha_{n+1}^G}{(1 - \gamma_n^G)2h_{0,*}^G} \left( \frac{\sigma_n^2}{b} + M^2_G \right) \]

\[ + 2\alpha_n^G \left\{ \frac{\beta_n^G}{\gamma_n^G} \sqrt{\Theta \text{Dist}(\theta) \left( \frac{\sigma_n^2}{b} + M^2_G \right)} + \frac{\tilde{\beta}_n^G}{\gamma_n^G} \mathbb{E} \left[ (\theta - \theta_n, \nabla_\theta L_G(\theta_n, w_n)) \right] \right\}. \]

Since we have that \( \gamma_n^G = 1 - \gamma_n^{G_n+1} \leq 1 \), \( \gamma_n^{G_n+1} (X_{n+1}^G(\theta) - X_n^G(\theta)) \leq \gamma_n^{G_n+1} X_{n+1}^G(\theta) \), and \( h_{n+1,i}^G \geq h_{n,i}^G \) (by (A1)) for all \( n \in \mathbb{N} \), we also have that

\[ X_{n+1}^G(\theta) \leq X_n^G(\theta) + \gamma_n^{G_n+1} X_{n+1}^G(\theta) + \text{Dist}(\theta) \mathbb{E} \left[ \sum_{i \in [\Theta]} (h_{n+1,i}^G - h_{n,i}^G) \right] \]

\[ + \frac{\alpha_{n+1}^G}{(1 - \gamma_n^G)2h_{0,*}^G} \left( \frac{\sigma_n^2}{b} + M^2_G \right) \]

\[ + 2\alpha_n^G \left\{ \frac{\beta_n^G}{\gamma_n^G} \sqrt{\Theta \text{Dist}(\theta) \left( \frac{\sigma_n^2}{b} + M^2_G \right)} + \frac{\tilde{\beta}_n^G}{\gamma_n^G} \mathbb{E} \left[ (\theta - \theta_n, \nabla_\theta L_G(\theta_n, w_n)) \right] \right\}. \]
Let us show that, for all $\theta \in \mathbb{R}^\Theta$ and all $\epsilon > 0$,

$$\begin{align*}
\liminf_{n \to +\infty} E \left[ (\theta_n - \theta, \nabla_\theta L_G(\theta_n, w_n)) \right] \\
(A.6) & \leq \frac{\alpha^G}{2\beta^G (1 - \gamma^G)^2 h_{0,*}^G} \left( \frac{\sigma^2_G}{b} + M^2_G \right) + \sqrt{\Theta \text{Dist}(\theta)} \left( \frac{\sigma^2_G}{b} + M^2_G \right) \frac{\beta^G}{\beta^G} + \epsilon.
\end{align*}$$

If (A.6) does not hold, then there exist $\hat{\theta} \in \mathbb{R}^\Theta$ and $\epsilon_0 > 0$ such that

$$\begin{align*}
\liminf_{n \to +\infty} E \left[ (\theta_n - \hat{\theta}, \nabla_\theta L_G(\theta_n, w_n)) \right] \\
& > \frac{\alpha^G}{2\beta^G (1 - \gamma^G)^2 h_{0,*}^G} \left( \frac{\sigma^2_G}{b} + M^2_G \right) + \sqrt{\Theta \text{Dist}(\hat{\theta})} \left( \frac{\sigma^2_G}{b} + M^2_G \right) \frac{\beta^G}{\beta^G} + \epsilon_0.
\end{align*}$$

Then, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\begin{align*}
\mathbb{E} \left[ (\theta_n - \hat{\theta}, \nabla_\theta L_G(\theta_n, w_n)) \right] \\
& > \frac{\alpha^G}{2\beta^G (1 - \gamma^G)^2 h_{0,*}^G} \left( \frac{\sigma^2_G}{b} + M^2_G \right) + \sqrt{\Theta \text{Dist}(\hat{\theta})} \left( \frac{\sigma^2_G}{b} + M^2_G \right) \frac{\beta^G}{\beta^G} + \epsilon_0.
\end{align*}$$

Meanwhile, the condition $\gamma^G \in [0, 1)$, (C3), and (A1)–(A2) guarantee that there exists $n_1 \in \mathbb{N}$ such that, for all $n \geq n_1$,

$$\gamma^{G+1} X^G_{n+1}(\hat{\theta}) + \text{Dist}(\hat{\theta}) \mathbb{E} \left[ \sum_{i \in \Theta} (h^G_{n+1,i} - h^G_{n,i}) \right] \leq \frac{\alpha^G \beta^G \epsilon_0}{2}.$$ 

Accordingly, from (A.5), for all $n \geq n_2 := \max\{n_0, n_1\}$,

$$\begin{align*}
X^G_{n+1}(\hat{\theta}) & < X^G_n(\hat{\theta}) + \frac{\alpha^G \beta^G \epsilon_0}{2} + \frac{\alpha^G}{(1 - \gamma^G)^2 h_{0,*}^G} \left( \frac{\sigma^2_G}{b} + M^2_G \right) \\
& + 2\alpha^G \beta^G \sqrt{\Theta \text{Dist}(\hat{\theta})} \left( \frac{\sigma^2_G}{b} + M^2_G \right) - 2\alpha^G \beta^G \left\{ \frac{\alpha^G}{2\beta^G (1 - \gamma^G)^2 h_{0,*}^G} \left( \frac{\sigma^2_G}{b} + M^2_G \right) \right\} \\
& = X^G_n(\hat{\theta}) - \frac{\alpha^G \beta^G \epsilon_0}{2} \\
& < X^G_{n_2}(\hat{\theta}) - \frac{\alpha^G \beta^G \epsilon_0}{2} (n + 1 - n_2).
\end{align*}$$

Note that the right-hand side of the above inequality approaches minus infinity as $n$ approaches positive infinity, producing a contradiction. Therefore, (A.6) holds,
which implies that, for all $\theta \in \mathbb{R}^\Theta$,

$$
\liminf_{n \to +\infty} \mathbb{E} \left[ \langle \theta_n - \theta, \nabla_\theta L_G(\theta_n, w_n) \rangle \right]
$$

(A.7)

$$
\leq \frac{\alpha^G}{2\beta^G(1 - \gamma^G)^2 h_{0,*}^G} \left( \frac{\sigma^2_G}{b} + M^2_G \right) + \sqrt{\Theta \text{Dist}(\theta)} \left( \frac{\sigma^2_G}{b} + M^2_G \right) \frac{\beta^G}{\beta^G}.
$$

A discussion similar to the one for showing (A.7), together with Lemma A.6 leads to the finding that, for all $w \in \mathbb{R}^W$,

$$
\liminf_{n \to +\infty} \mathbb{E} \left[ \langle w_n - w, \nabla_w L_D(\theta_n, w_n) \rangle \right]
$$

(A.8)

$$
\leq \frac{\alpha^D}{2\beta^D(1 - \gamma^D)^2 h_{0,*}^D} \left( \frac{\sigma^2_D}{b} + M^2_D \right) + \sqrt{\Theta \text{Dist}(w)} \left( \frac{\sigma^2_D}{b} + M^2_D \right) \frac{\beta^D}{\beta^D}.
$$

Let $\theta \in \mathbb{R}^\Theta$. From (A.7), there exists a subsequence $((\theta_{n_i}, w_{n_i}))_{i \in \mathbb{N}}$ of $((\theta_n, w_n))_{n \in \mathbb{N}}$ such that

$$
\lim_{i \to +\infty} \mathbb{E} \left[ \langle \theta_{n_i} - \theta, \nabla_\theta L_G(\theta_{n_i}, w_{n_i}) \rangle \right]
$$

$$
\leq \frac{\alpha^G}{2\beta^G(1 - \gamma^G)^2 h_{0,*}^G} \left( \frac{\sigma^2_G}{b} + M^2_G \right) + \sqrt{\Theta \text{Dist}(\theta)} \left( \frac{\sigma^2_G}{b} + M^2_G \right) \frac{\beta^G}{\beta^G}.
$$

Conditions (S1) and (C3) guarantee that there exists $((\theta_{n_{i_j}}, w_{n_{i_j}}))_{j \in \mathbb{N}}$ of $((\theta_n, w_n))_{n \in \mathbb{N}}$ such that $((\theta_{n_{i_j}}, w_{n_{i_j}}))_{j \in \mathbb{N}}$ converges almost surely to $(\theta^*, w^*) \in \mathbb{R}^\Theta \times \mathbb{R}^W$. Therefore, for all $\theta \in \mathbb{R}^\Theta$,

$$
\mathbb{E} \left[ \langle \theta^* - \theta, \nabla_\theta L_G(\theta^*, w^*) \rangle \right]
$$

$$
\leq \frac{\alpha^G}{2\beta^G(1 - \gamma^G)^2 h_{0,*}^G} \left( \frac{\sigma^2_G}{b} + M^2_G \right) + \sqrt{\Theta \text{Dist}(\theta^*)} \left( \frac{\sigma^2_G}{b} + M^2_G \right) \frac{\beta^G}{\beta^G}.
$$

Hence, letting $\theta = \tilde{\theta} := \theta^* - \nabla_\theta L_G(\theta^*, w^*)$ implies that

$$
\mathbb{E} \left[ \langle \nabla_\theta L_G(\theta^*, w^*) \rangle^2 \right]
$$

(A.9)

$$
\leq \frac{\alpha^G}{2\beta^G(1 - \gamma^G)^2 h_{0,*}^G} \left( \frac{\sigma^2_G}{b} + M^2_G \right) + \sqrt{\Theta \text{Dist}(\tilde{\theta})} \left( \frac{\sigma^2_G}{b} + M^2_G \right) \frac{\beta^G}{\beta^G}.
$$

A discussion similar to the one showing (A.9), together with (A.8), implies that there exists $(\theta_*, w_*) \in \mathbb{R}^\Theta \times \mathbb{R}^W$ such that

$$
\mathbb{E} \left[ \langle \nabla_w L_D(\theta_*, w_*) \rangle^2 \right]
$$

$$
\leq \frac{\alpha^D}{2\beta^D(1 - \gamma^D)^2 h_{0,*}^D} \left( \frac{\sigma^2_D}{b} + M^2_D \right) + \sqrt{\Theta \text{Dist}(\tilde{w})} \left( \frac{\sigma^2_D}{b} + M^2_D \right) \frac{\beta^D}{\beta^D},
$$

where $\tilde{w} = w_* - \nabla_w L_D(\theta_*, w_*)$. This completes the proof. □
A.3. **Proof of Theorem 3.1(ii).** Lemmas A.1 and A.3 lead to the following lemma:

**Lemma A.7.** Suppose that (S1)–(S3), (A1)–(A2), and (C2)–(C3) hold and consider Algorithm 1 where \((\alpha_n^G)_{n \in \mathbb{N}}\) is monotone decreasing. Then, for all \(\theta \in \mathbb{R}^\Theta\) and all \(N \geq 1\),

\[
\frac{1}{N} \sum_{n \in [N]} \mathbb{E}[(\theta - \theta_n, \nabla_\theta L_G(\theta_n, w_n))]
\]

\[
\leq \frac{\Theta_{\text{Dist}}(\theta) H_G}{2\alpha_N^G \beta_G N} + \frac{1}{N} \sum_{n \in [N]} \frac{\alpha_n^G}{2\beta_G (1 - \gamma^G)^2 h_{0,*}^G} \left( \frac{\sigma_n^2}{b} + M_G^2 \right) + \frac{\beta_G}{\beta_G} \sqrt{\Theta_{\text{Dist}} \left( \frac{\sigma_G^2}{b} + M_G^2 \right)},
\]

where \(H_G := \max_{i \in [\Theta]} H_i^G\).

**Proof.** Lemma A.1 implies that, for all \(\theta \in \mathbb{R}^\Theta\) and all \(n \in \mathbb{N}\),

\[
\mathbb{E}[(\theta_n - \theta, \nabla_\theta L_G(\theta_n, w_n))] = \frac{\gamma_n^G}{2\alpha_n^G \beta_G} \left\{ \mathbb{E} \left[ \|\theta_n - \theta\|_{H_n^G}^2 \right] - \mathbb{E} \left[ \|\theta_n - \theta\|_{H_n^G}^2 \right] \right\} + \frac{\alpha_n^G z_n^G}{2\beta_G} \mathbb{E} \left[ \|d_n^G\|_{H_n^G}^2 \right] + \frac{\beta_G}{\beta_G} \mathbb{E} \left[ (\theta - \theta_n, m_{n-1}^G) \right],
\]

which implies that, for all \(\theta \in \mathbb{R}^\Theta\) and all \(N \geq 1\),

\[
\frac{1}{N} \sum_{n \in [N]} \mathbb{E}[(\theta_n - \theta, \nabla_\theta L_G(\theta_n, w_n))]
\]

\[
= \frac{1}{N} \sum_{n \in [N]} \frac{\gamma_n^G}{2\alpha_n^G \beta_G} \left\{ \mathbb{E} \left[ \|\theta_n - \theta\|_{H_n^G}^2 \right] - \mathbb{E} \left[ \|\theta_n - \theta\|_{H_n^G}^2 \right] \right\} + \frac{1}{N} \sum_{n \in [N]} \frac{\alpha_n^G z_n^G}{2\beta_G} \mathbb{E} \left[ \|d_n^G\|_{H_n^G}^2 \right] + \frac{1}{N} \sum_{n \in [N]} \frac{\beta_G}{\beta_G} \mathbb{E} \left[ (\theta - \theta_n, m_{n-1}^G) \right].
\]

Let \(\delta_n^G := \frac{\gamma_n^G}{2\alpha_n^G \beta_G}\) for all \(n \in \mathbb{N}\). Then, we have that

\[
\Theta_N := \delta_1^G \mathbb{E} \left[ \|\theta_1 - \theta\|_{H_1^G}^2 \right] + \sum_{n=2}^{N} \left\{ \delta_n^G \mathbb{E} \left[ \|\theta_n - \theta\|_{H_n^G}^2 \right] - \delta_{n-1}^G \mathbb{E} \left[ \|\theta_n - \theta\|_{H_{n-1}^G}^2 \right] \right\} - \delta_{N}^G \mathbb{E} \left[ \|\theta_{N+1} - \theta\|_{H_{N}^G}^2 \right] .
\]

Since \(H_n^G \in \mathcal{S}_{\Theta, +}\) exists such that \(H_n^G = H_n^G\), we have \(\|x\|_{H_n^G}^2 = \|H_n^G x\|^2\) for all \(x \in \mathbb{R}^\Theta\). Accordingly, we have

\[
\hat{\Theta}_N = \mathbb{E} \left[ \sum_{n=2}^{N} \left\{ \delta_n^G \left\| H_n^G (\theta_n - \theta) \right\|^2 - \delta_{n-1}^G \left\| H_{n-1}^G (\theta_n - \theta) \right\|^2 \right\} \right].
\]
Hence, for all $N \geq 2$,

\[
\Theta_N = \mathbb{E} \left[ \sum_{n=2}^{N} \sum_{i=1}^{\Theta} \left( \delta_{n}^{G} h_{n,i}^{G} - \delta_{n-1}^{G} h_{n-1,i}^{G} \right) (\theta_{n,i} - \theta_{i})^2 \right].
\]

(A.10)

Since $(\alpha_{n}^{G})_{n \in \mathbb{N}}$ is monotone decreasing, we have that $\delta_{n+1}^{G} \geq \delta_{n}^{G}$ ($n \in \mathbb{N}$). Hence, from (A1), we have that, for all $n \geq 1$ and all $i \in [\Theta]$,

\[
\delta_{n}^{G} h_{n,i}^{G} - \delta_{n-1}^{G} h_{n-1,i}^{G} \geq 0.
\]

Moreover, from (C3), $\max_{i \in [\Theta]} \sup_{n \in \mathbb{N}} (\theta_{n,i} - \theta_{i})^2 \leq \Dist(\theta)$. Accordingly, for all $N \geq 2$,

\[
\Theta_N \leq \Dist(\theta) \mathbb{E} \left[ \sum_{n=2}^{N} \sum_{i=1}^{\Theta} \left( \delta_{n}^{G} h_{n,i}^{G} - \delta_{n-1}^{G} h_{n-1,i}^{G} \right) \right] = \Dist(\theta) \mathbb{E} \left[ \sum_{i=1}^{\Theta} \left( \delta_{2}^{G} h_{2,i}^{G} - \delta_{1}^{G} h_{1,i}^{G} \right) \right].
\]

Therefore, $\delta_{1}^{G} \mathbb{E}[\|\theta - \theta\|_{H_{1}^{G}}^2] \leq \Dist(\theta) \delta_{1}^{G} \mathbb{E}[\sum_{i=1}^{\Theta} h_{1,i}^{G}]$, and (A2) imply, for all $N \geq 1$,

\[
\Theta_N \leq \delta_{1}^{G} \Dist(\theta) \mathbb{E} \left[ \sum_{i=1}^{\Theta} h_{1,i}^{G} \right] + \Dist(\theta) \mathbb{E} \left[ \sum_{i=1}^{\Theta} \left( \delta_{N}^{G} h_{N,i}^{G} - \delta_{1}^{G} h_{1,i}^{G} \right) \right]
\[
\leq \delta_{N}^{G} \Dist(\theta) \sum_{i=1}^{\Theta} H_{i}^{G}
\[
\leq \delta_{N}^{G} \Theta \Dist(\theta) H^{G},
\]

where $H^{G} = \max_{i \in [\Theta]} H_{i}^{G}$. From $\delta_{n}^{G} := \delta_{n}^{G}/(2\alpha_{n}^{G} \beta_{n}^{G})$ and $\delta_{n}^{G} = 1 - \gamma_{n+1}^{G} \leq 1$, we have

\[
\Theta_N \leq \frac{\Theta \Dist(\theta) H^{G}}{2\alpha_{N}^{G} \beta_{N}^{G}}.
\]

(Lemma A.3) implies that, for all $N \geq 1$,

\[
A_N := \sum_{n \in [N]} \frac{\alpha_{n}^{G} \delta_{n}^{G}}{2\beta_{n}^{G}} \left[ \mathbb{E}[d_{n}^{G}]^2 \right] \leq \sum_{n \in [N]} \frac{\alpha_{n}^{G}}{2\beta_{n}^{G}(1 - \gamma_{n}^{G})^2 h_{0,n}^{G}} \left( \frac{\sigma_{n}^2}{b} + M_{n}^{G} \right).
\]

From (A.4), we have

\[
B_N := \sum_{n \in [N]} \frac{\beta_{n}^{G}}{2\beta_{n}^{G}} \mathbb{E}[\|\theta - \theta_{n} - m_{n-1}^{G}\|_{H_{n}^{G}}] \leq \frac{\beta_{N}^{G} N \Dist(\theta) \left( \frac{\sigma_{N}^2}{b} + M_{N}^{G} \right)}{\beta_{N}^{G}}.
\]

Therefore, (A.11), (A.12), and (A.13) lead to the assertion in Lemma A.7. This completes the proof. \qed
A discussion similar to the one for showing Lemma A.7 together with Lemmas A.2 and A.4 leads to the following lemma:

**Lemma A.8.** Suppose that (S1)–(S3), (A1)–(A2), and (C2)–(C3) hold and consider Algorithm [1] where \((\alpha_n^D)_{n \in \mathbb{N}}\) is monotone decreasing. Then, for all \(w \in \mathbb{R}^W\) and all \(N \geq 1\),

\[
\frac{1}{N} \sum_{n \in [N]} \mathbb{E}[(w - w_n, \nabla_w L_D(\theta_n, w_n))]
\leq \frac{W \text{Dist}(w) H^D}{2\alpha_n^D \beta_n^D N} + \frac{1}{N} \sum_{n \in [N]} \frac{\alpha_n^D}{2\beta_n^D(1 - \gamma^D)^2 h_{\theta_n}^D} \left( \frac{\sigma_n^2}{b} + M_n^D \right) + \frac{\beta_n^D}{\beta^D \gamma^D} \sqrt{W \text{Dist}(w) \left( \frac{\sigma_n^2}{b} + M_n^D \right)},
\]

where \(H^D := \max_{j \in [W]} H_j^D\).

**Proof of Theorem 3.1(ii).** Let \(\alpha_n^G := \alpha^G\) and \(\alpha_n^D := \alpha^D\) for all \(n \in \mathbb{N}\). Lemmas A.7 and A.8 thus guarantee that, for all \(\theta \in \mathbb{R}^\Theta\) and all \(N \geq 1\),

\[
\frac{1}{N} \sum_{n \in [N]} \mathbb{E}[(\theta - \theta_n, \nabla_\theta L_G(\theta_n, w_n))]
\leq \frac{\Theta \text{Dist}(\theta) H^G}{2\alpha_n^G \beta_n^G N} + \frac{\alpha_n^G}{2\beta_n^G(1 - \gamma^G)^2 h_{\theta_n}^G} \left( \frac{\sigma_n^2}{b} + M_n^G \right) + \frac{\beta_n^G}{\beta^G} \sqrt{\Theta \text{Dist}(\theta) \left( \frac{\sigma_n^2}{b} + M_n^G \right)},
\]

\[
\frac{1}{N} \sum_{n \in [N]} \mathbb{E}[(w - w_n, \nabla_w L_D(\theta_n, w_n))]
\leq \frac{W \text{Dist}(w) H^D}{2\alpha_n^D \beta_n^D N} + \frac{\alpha_n^D}{2\beta_n^D(1 - \gamma^D)^2 h_{\theta_n}^D} \left( \frac{\sigma_n^2}{b} + M_n^D \right) + \frac{\beta_n^D}{\beta^D} \sqrt{W \text{Dist}(w) \left( \frac{\sigma_n^2}{b} + M_n^D \right)},
\]

which completes the proof. \(\square\)

### A.4. Proofs of Theorems 3.2 and 3.3

**Proof of Theorem 3.2.** Condition (3.2) leads to (3.3). We have that, for \(b > B_G/(\epsilon_G^2 - C_G)\),

\[
\frac{d N_G(b)}{db} = \frac{-A_G B_G}{((\epsilon_G^2 - C_G)b - B_G)^2} \leq 0,
\]

\[
\frac{d^2 K(b)}{db^2} = \frac{2 A_G B_G (\epsilon_G^2 - C_G)}{((\epsilon_G^2 - C_G)b - B_G)^3} \geq 0.
\]

Hence, \(N_G\) is monotone decreasing and convex for \(b > B_G/(\epsilon_G^2 - C_G)\). We also have that \(N_D\) is monotone decreasing and convex for \(b > B_D/(\epsilon_D^2 - C_D)\). This completes the proof. \(\square\)

**Proof of Theorem 3.3.** We have that, for \(b > B_G/(\epsilon_G^2 - C_G)\),

\[
N_G(b) b = \frac{A_G b^2}{(\epsilon_G^2 - C_G)b - B_G}.
\]
Hence,

\[
\frac{d N_G(b)b}{db} = \frac{A_G(b)\{c_G^2 - C_Gb - 2B_G\}}{\{(c_G^2 - C_G)b - B_G\}^2},
\]

\[
\frac{d^2 N_G(b)b}{db^2} = \frac{2A_GB_G^2}{\{(c_G^2 - C_G)b - B_G\}^3} \geq 0,
\]

which implies that \(N_G(b)b\) is convex for \(b > B_G/(c_G^2 - C_G)\) and

\[
\frac{d N_G(b)b}{db} \begin{cases} < 0 & \text{if } b < b_G^*, \\ = 0 & \text{if } b = b_G^* = \frac{2B_G}{c_G^2 - C_G}, \\ > 0 & \text{if } b > b_G^*. \end{cases}
\]

The point \(b_G^*\) attains the minimum value \(N_G(b_G^*)b_G^*\) of \(N_G(b)b\). A discussion similar to the one for showing the results for \(N_G\) leads to the results for \(N_D\). This completes the proof. \(\square\)