Bilinear pseudo-differential operators with exotic symbols, II

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Abstract The boundedness from $H^p \times L^2$ to $L^r$, $1/p + 1/2 = 1/r$, and from $H^p \times L^\infty$ to $L^p$ of bilinear pseudo-differential operators is proved under the assumption that their symbols are in the bilinear Hörmander class $BS^m_{\rho,\rho}$, $0 \leq \rho < 1$, of critical order $m$, where $H^p$ is the Hardy space. This combined with the previous results of the same authors establishes the sharp boundedness from $H^p \times H^q$ to $L^r$, $1/p + 1/q = 1/r$, of those operators in the full range $0 < p, q \leq \infty$, where $L^r$ is replaced by $BMO$ if $r = \infty$.

Keywords Bilinear pseudo-differential operators · Bilinear Hörmander symbol classes · Hardy spaces

Mathematics Subject Classification 42B20 · 42B30 · 47G30

1 Introduction

This paper is a continuation of the paper [8]. We continue the study of the boundedness of bilinear pseudo-differential operators with symbols in the so-called exotic classes.
As for the background of this subject, see Introduction of [8]. Here we begin by recalling necessary definitions.

Let $m \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$. We say that a function $\sigma(x, \xi, \eta) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n)$ belongs to the bilinear Hörmander symbol class $BS_{\rho,\delta}^m = BS_{\rho,\delta}^m(\mathbb{R}^n)$ if for every triple of multi-indices $\alpha, \beta, \gamma \in \mathbb{N}_0^n = \{0, 1, 2, \ldots\}^n$ there exists a constant $C_{\alpha,\beta,\gamma} > 0$ such that

$$|\partial_\alpha^\varepsilon \partial_\beta^\eta \sigma(x, \xi, \eta)| \leq C_{\alpha,\beta,\gamma} (1 + |\xi| + |\eta|)^{m+\delta|\alpha| - \rho(|\beta| + |\gamma|)}.$$ 

In this paper, we consider the class $BS_{\rho,\delta}^m$ with $0 \leq \rho = \delta < 1$.

The bilinear pseudo-differential operator $T_{\sigma}, \sigma \in BS_{\rho,\rho}^m$, is defined by

$$T_{\sigma}(f, g)(x) = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^n \times \mathbb{R}^n} e^{i(x-(\xi+\gamma))} \sigma(x, \xi, \eta) \hat{f}(\xi) \hat{g}(\eta) \, d\xi d\eta, \quad f, g \in \mathcal{S}(\mathbb{R}^n).$$

If $X, Y, Z$ are function spaces on $\mathbb{R}^n$ equipped with quasi-norms $\| \cdot \|_X, \| \cdot \|_Y, \| \cdot \|_Z$ and if there exists a constant $A_{\sigma}$ such that the estimate $\|T_{\sigma}(f, g)\|_Z \leq A_{\sigma} \|f\|_X \|g\|_Y$ holds for all $f \in \mathcal{S}\cap X$ and all $g \in \mathcal{S}\cap Y$, then we shall simply say that $T_{\sigma}$ is bounded from $X \times Y$ to $Z$ and write $T_{\sigma} : X \times Y \rightarrow Z$. For the function spaces $X$ and $Y$, we consider the Hardy spaces $H^p, 0 < p \leq \infty$. For $Z$, we consider the Lebesgue spaces $L^r, 0 < r < \infty$, or $BMO$. Notice that $H^p = L^p$ for $1 < p \leq \infty$. The definitions of $H^p$ and $BMO$ are given in Sect. 2.

For $0 \leq \rho < 1$ and for $0 < p, q, r \leq \infty$ satisfying $1/p + 1/q = 1/r$, we write

$$m_{\rho}(p, q) = (1 - \rho)m_0(p, q),$$

$$m_0(p, q) = -n \left( \max \left\{ \frac{1}{2}, \frac{1}{p}, \frac{1}{q}, 1 - \frac{1}{r}, \frac{1}{r} - \frac{1}{2} \right\} \right).$$

Observe that the region of $(1/p, 1/q), 0 \leq 1/p, 1/q < \infty$, is divided into 5 regions, on each of which $m_{\rho}(p, q)$ is an affine function of $1/p$ and $1/q$ (see [8, Introduction]).

The number $m_{\rho}(p, q)$ is the critical order as the following proposition shows.

**Proposition A** Let $0 \leq \rho < 1, 0 < p, q, r \leq \infty$, and suppose $1/p + 1/q = 1/r$. If $r < \infty$, then

$$m_{\rho}(p, q) = \sup \{ m \in \mathbb{R} : T_{\sigma} : H^p \times H^q \rightarrow L^r \text{ for all } \sigma \in BS_{\rho,\rho}^m \}.$$ 

When $p = q = r = \infty$, the above equality holds if we replace $H^p \times H^q \rightarrow L^r$ by $L^\infty \times L^\infty \rightarrow BMO$.

In fact, this proposition is a conclusion of several previous works: Michalowski–Rule–Staubach [5] (for $(1/p, 1/q)$ in the triangle with vertices $(1/2, 1/2), (1/2, 0), (0, 1/2)$), Bényi–Bernicot–Maldonado–Naibo–Torres [1] (in the range $1/p + 1/q \leq 1$), and Miyachi–Tomita [7,8] (full range $0 < p, q \leq \infty$). For a proof of Proposition A, see [8, Appendix A].
It should be an interesting problem to prove the sharp boundedness, i.e., the boundedness $T_\sigma : H^p \times H^q \to L^r$, $1/r = 1/p + 1/q$, with $L^r$ replaced by $BMO$ if $r = \infty$, for $\sigma \in BS^m_{\rho, \rho}$ with $m = m_\rho(p, q)$.

In the case $\rho = 0$, this sharp boundedness was proved in [7].

Recently, the authors proved the following theorem, which gives the sharp boundedness in the range $1 \leq p, q, r \leq \infty$.

**Theorem B** ([8, Corollary 1.4]) Let $0 \leq \rho < 1$, $1 \leq p, q, r \leq \infty$, $1/p + 1/q = 1/r$, and $m = m_\rho(p, q)$. Then all bilinear pseudo-differential operators with symbols in $BS^m_{\rho, \rho}(\mathbb{R}^n)$ are bounded from $L^p(\mathbb{R}^n) \times L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, where $L^p(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$ should be replaced by $H^1(\mathbb{R}^n)$ if $p = 1$ or $q = 1$ and $L^r(\mathbb{R}^n)$ should be replaced by $BMO(\mathbb{R}^n)$ if $r = \infty$.

Here it should be mentioned that Theorem B with $p = q = r = \infty$ and $0 < \rho < 1/2$ was also proved by Naibo [9].

Now the purpose of the present paper is to prove the sharp boundedness in the remaining cases and establish the sharp boundedness in the full range $0 < p, q, r \leq \infty$. The following is the conclusion of the present paper.

**Theorem 1.1** Let $0 \leq \rho < 1$, $0 < p, q, r \leq \infty$, $1/p + 1/q = 1/r$, and $m = m_\rho(p, q)$. Then all bilinear pseudo-differential operators with symbols in $BS^m_{\rho, \rho}(\mathbb{R}^n)$ are bounded from $H^p(\mathbb{R}^n) \times H^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, where $L^r(\mathbb{R}^n)$ should be replaced by $BMO(\mathbb{R}^n)$ if $p = q = r = \infty$.

The above theorem follows with the aid of complex interpolation and symmetry if the sharp boundedness is proved in the following 5 cases:

(i) $(p, q) = (\infty, \infty)$, $m_\rho(\infty, \infty) = -(1 - \rho)n$;
(ii) $(p, q) = (2, 2)$, $m_\rho(2, 2) = -(1 - \rho)n/2$;
(iii) $(p, q) = (2, \infty)$, $m_\rho(2, \infty) = -(1 - \rho)n/2$;
(iv) $0 < p < 1$, $q = 2$, $m_\rho(p, 2) = -(1 - \rho)n/p$;
(v) $0 < p < 1$, $q = \infty$, $m_\rho(p, \infty) = -(1 - \rho)n/p$.

(For the interpolation argument, see, e.g., [1, Proof of Theorem 2.2] or [7, Proof of the ‘if’ part of Theorem 1.1]). By symbolic calculus of $BS^m_{\rho, \rho}$ as given by Bényi–Maldonado–Naibo–Torres [2] and by duality, the cases (ii) and (iii) are essentially the same (see, e.g., [8, Section 5]). Thus Theorem 1.1 will follow if we prove (i), (ii)=(iii), (iv), and (v). Among these 4 critical cases, (i) and (ii)=(iii) are covered by Theorem B. Thus in order to prove Theorem 1.1 it is sufficient to prove (iv) and (v), which we shall state here as the following two theorems.

**Theorem 1.2** Let $0 \leq \rho < 1$, $0 < p < 1$, $1/p + 1/2 = 1/r$, and $m = -(1 - \rho)n/p$. Then all bilinear pseudo-differential operators with symbols in $BS^m_{\rho, \rho}(\mathbb{R}^n)$ are bounded from $H^p(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$.

**Theorem 1.3** Let $0 \leq \rho < 1$, $0 < p < 1$, and $m = -(1 - \rho)n/p$. Then all bilinear pseudo-differential operators with symbols in $BS^m_{\rho, \rho}(\mathbb{R}^n)$ are bounded from $H^p(\mathbb{R}^n) \times L^\infty(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. 
Here are some comments on the proofs of the theorems. First, although the case (i) (the sharp $L^\infty \times L^\infty \to BMO$ boundedness) was directly proved in [8,9], the argument of the present paper gives an alternate proof. In fact, by virtue of the symbolic calculus of $BS^m_{\rho,\rho}$ and by the duality $(H^1)' = BMO$, the sharp $L^\infty \times L^\infty \to BMO$ boundedness is equivalent to the sharp $H^1 \times L^\infty \to L^1$ boundedness and the latter follows from the cases (iii) and (v) (Theorem 1.3) by interpolation. Secondly, for the proof of the case (iv) (Theorem 1.2), the method of [7] given for $\rho = 0$ does not seem to work for $0 < \rho < 1$. Our proof of Theorem 1.2 is based on a new method, which covers $\rho = 0$ and $0 < \rho < 1$ simultaneously. Finally, the case (v) (Theorem 1.3) is rather easy. In fact, by freezing $g$ of $T_\alpha (f, g)$ we can follow the argument used in the case of linear pseudo-differential operators.

The contents of this paper are as follows. In Sect. 2, we recall some preliminary facts. In Sects. 3 and 4, we prove Theorems 1.2 and 1.3, respectively.

2 Preliminaries

For two nonnegative quantities $A$ and $B$, the notation $A \lesssim B$ means that $A \leq C B$ for some unspecified constant $C > 0$, and $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. We denote by $1_S$ the characteristic function of a set $S$, and by $|S|$ the Lebesgue measure of a measurable set $S$ in $\mathbb{R}^n$.

Let $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$ be the Schwartz space of rapidly decreasing smooth functions and the space of tempered distributions, respectively. We define the Fourier transform $\mathcal{F} f$ and the inverse Fourier transform $\mathcal{F}^{-1} f$ of $f \in S(\mathbb{R}^n)$ by

$$
\mathcal{F} f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx \quad \text{and} \quad \mathcal{F}^{-1} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) \, d\xi.
$$

For $m \in L^\infty(\mathbb{R}^n)$, the linear Fourier multiplier operator $m(D)$ is defined by

$$
m(D) f(x) = \mathcal{F}^{-1} [m \hat{f}](x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} m(\xi) \hat{f}(\xi) \, d\xi, \quad f \in S(\mathbb{R}^n).
$$

We recall the definitions and some properties of Hardy spaces and the space $BMO$ on $\mathbb{R}^n$ (see, e.g., [10, Chapters 3 and 4]). Let $0 < p \leq \infty$, and let $\phi \in S(\mathbb{R}^n)$ be such that $\int_{\mathbb{R}^n} \phi(x) \, dx \neq 0$. Then the Hardy space $H^p(\mathbb{R}^n)$ consists of all $f \in S'(\mathbb{R}^n)$ such that

$$
\| f \|_{H^p} = \| \sup_{0 < t < \infty} |\phi_t * f| \|_{L^p} < \infty,
$$

where $\phi_t(x) = t^{-n} \phi(x/t)$. It is known that $H^p(\mathbb{R}^n)$ does not depend on the choice of the function $\phi$ and $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ for $1 < p \leq \infty$. For $0 < p \leq 1$, a function $a$ on $\mathbb{R}^n$ is called an $H^p$-atom if there exists a cube $Q = Q_\alpha$ such that

$$
\supp a \subset Q, \quad \| a \|_{L^\infty} \leq |Q|^{-1/p}, \quad \int_{\mathbb{R}^n} x^\alpha a(x) \, dx = 0, \quad |\alpha| \leq L - 1,
$$

(2.1)
where $L$ is any fixed integer satisfying $L > n/p - n$ ([10, p. 112]). It is known that every $f \in H^p(\mathbb{R}^n)$ can be written as

$$f = \sum_{i=1}^{\infty} \lambda_i a_i \text{ in } S'(\mathbb{R}^n),$$

where $\{a_i\}$ is a collection of $H^p$-atoms and $\{\lambda_i\}$ is a sequence of complex numbers with $\sum_{i=1}^{\infty} |\lambda_i|^p < \infty$. Moreover,

$$\|f\|_{H^p} \approx \inf \left( \sum_{i=1}^{\infty} |\lambda_i|^p \right)^{1/p},$$

where the infimum is taken over all representations of $f$. The space $BMO(\mathbb{R}^n)$ consists of all locally integrable functions $f$ on $\mathbb{R}^n$ such that

$$\|f\|_{BMO} = \sup_Q \frac{1}{|Q|} \int_Q |f(x) - f_Q| \, dx < \infty,$$

where $f_Q$ is the average of $f$ on $Q$ and the supremum is taken over all cubes $Q$ in $\mathbb{R}^n$. It is known that the dual space of $H^1(\mathbb{R}^n)$ is $BMO(\mathbb{R}^n)$.

### 3 Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2. We assume $0 \leq \rho < 1$, $0 < p < 1$, $1/p + 1/2 = 1/r$, $m = -(1 - \rho)n/p$, and $\sigma \in BS^m_{\rho, \rho}$, and prove the $H^p \times L^2 \to L^r$ boundedness of $T_{\sigma}$.

We first observe that the desired boundedness follows if we prove the following: for an $H^p$-atom $a$ and an $L^2$-function $g$ there exist a function $\tilde{a}$ depending only on $a$ and a function $\tilde{g}$ depending only on $g$ such that

$$|T_{\sigma}(a, g)(x)| \lesssim \tilde{a}(x)\tilde{g}(x), \quad \|\tilde{a}\|_{L^p} \lesssim 1, \quad \|\tilde{g}\|_{L^2} \lesssim \|g\|_{L^2}. \quad (3.1)$$

In fact, if this is proved, we can deduce the $H^p \times L^2 \to L^r$ boundedness of $T_{\sigma}$ as follows. Given $f \in H^p$, we decompose it as

$$f = \sum_{i} \lambda_i a_i, \quad \left( \sum_{i} |\lambda_i|^p \right)^{1/p} \lesssim \|f\|_{H^p},$$

where $a_i, i \geq 1$, are $H^p$-atoms. Then, taking the functions $\tilde{a}_i$ and $\tilde{g}$ satisfying (3.1) for $a = a_i$, we have

$$\|T_{\sigma}(f, g)\|_{L^r} = \left\| \sum_{i} \lambda_i T_{\sigma}(a_i, g) \right\|_{L^r} \lesssim \left\| \left( \sum_{i} |\lambda_i a_i| \right) \tilde{g} \right\|_{L^r}.$$
\[
\leq \left\| \sum_i |\lambda_i| \tilde{a}_i \right\|_{L^p} \| \tilde{g} \|_{L^2} \lesssim \left( \sum_i |\lambda_i|^p \right)^{1/p} \| g \|_{L^2} \lesssim \| f \|_{H^p} \| g \|_{L^2}.
\]

(This argument was already used in [6]. The idea goes back to [4].)

Let \( a \) be an \( H^p \)-atom satisfying (2.1) with \( L > n/p - n \). We denote by \( c_Q \) the center of \( Q \), by \( \ell(Q) \) the side length of \( Q \), and by \( Q^* \) the cube with the same center as \( Q \) but expanded by a factor of \( 2\sqrt{n} \). To obtain (3.1), we shall prove

\[
|T_\sigma(a, g)(x)|I_{\{Q^*\}}(x) \lesssim u(x)v(x), \quad \|u\|_{L^p} \lesssim 1, \quad \|v\|_{L^2} \lesssim \|g\|_{L^2}, \quad (3.2)
\]

\[
|T_\sigma(a, g)(x)|I_Q(x) \lesssim u'(x)v'(x), \quad \|u'\|_{L^p} \lesssim 1, \quad \|v'\|_{L^2} \lesssim \|g\|_{L^2}, \quad (3.3)
\]

where \( u, u' \) depend only on \( a \) and \( v, v' \) depend only on \( g \). Once (3.2) and (3.3) are proved, we can take \( u + u' \) and \( v + v' \) as \( \tilde{\sigma} \) and \( \tilde{g} \) in (3.1).

Let \( \psi_0 \in S(\mathbb{R}^d) \) be such that \( \psi_0 = 1 \) on \( \{ \zeta \in \mathbb{R}^d : |\zeta| \leq 1 \} \) and \( \text{supp} \psi_0 \subset \{ \zeta \in \mathbb{R}^d : |\zeta| \leq 2 \} \), and set \( \psi(\zeta) = \psi_0(\zeta) - \psi_0(2\zeta) \) and \( \psi_j(\zeta) = \psi(\zeta/2^j) \), \( j \geq 1 \). Then

\[
\text{supp} \psi_j \subset \{ \zeta \in \mathbb{R}^d : 2^{j-1} \leq |\zeta| \leq 2^{j+1} \}, \quad j \geq 1, \quad \sum_{j=0}^\infty \psi_j(\zeta) = 1, \quad \zeta \in \mathbb{R}^d. \quad (3.4)
\]

We also use functions \( \widetilde{\psi}_0, \widetilde{\psi} \in S(\mathbb{R}^n) \) satisfying \( \widetilde{\psi}_0 = 1 \) on \( \{ \eta \in \mathbb{R}^n : |\eta| \leq 4 \} \), \( \text{supp} \widetilde{\psi}_0 \subset \{ \eta \in \mathbb{R}^n : |\eta| \leq 8 \} \), \( \widetilde{\psi} = 1 \) on \( \{ \eta \in \mathbb{R}^n : 1/4 \leq |\eta| \leq 4 \} \), and \( \text{supp} \widetilde{\psi} \subset \{ \eta \in \mathbb{R}^n : 1/8 \leq |\eta| \leq 8 \} \), and set \( \widetilde{\psi}_\ell(\eta) = \psi(\eta/2^\ell) \), \( \ell \geq 1 \). In order to obtain (3.2) and (3.3), we decompose \( T_\sigma(a, g) \) as

\[
T_\sigma(a, g)(x) = \sum_{j=0}^\infty \sum_{\ell=0}^\infty T_{\sigma_j,\ell}(a, g)(x) = \sum_{\ell \leq j(1-\rho)+2} T_{\sigma_j,\ell}(a, g_{j,\ell})(x) \quad (3.5)
\]

with

\[
\sigma_j,\ell(x, \xi, \eta) = \sigma(x, \xi, \eta)\Psi_j(\xi, \eta)\psi_\ell(\eta/2^{j\rho})
\]

and

\[
g_{j,\ell}(x) = \widetilde{\psi}_\ell(D/2^{j\rho})g(x),
\]

where \( \Psi_j \) and \( \psi_\ell \) are as in (3.4) with \( d = 2n \) and \( d = n \) respectively, and \( [j\rho] \) is the integer part of \( j\rho \). Here, we used the fact

\[
\sum_{j \geq 0} \Psi_j(\xi, \eta) = \sum_{j \geq 0} \sum_{\ell \geq 0} \Psi_j(\xi, \eta)\psi_\ell(\eta/2^{j\rho}) = 1, \quad (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n,
\]

in the first equality of (3.5), and the facts

\[
\Psi_j(\xi, \eta)\psi_\ell(\eta/2^{j\rho}) = 0, \quad \ell > j(1-\rho)+2,
\]
and
\[ \psi_{\ell}(\eta/2^{j\rho}) = \psi_{\ell}(\eta/2^{j\rho}) \tilde{\psi}_{\ell}(\eta/2^{j\rho}), \quad \ell \geq 0, \]
in the second equality of (3.5). We write the partial inverse Fourier transform of 
\( \sigma_{j,\ell}(x, \xi, \eta) \) with respect to \( (\xi, \eta) \) as
\[ K_{j,\ell}(x, y, z) = \frac{1}{(2\pi)^{2n}} \int_{(\mathbb{R}^n)^2} e^{i(y \cdot \xi + z \cdot \eta)} \sigma_{j,\ell}(x, \xi, \eta) \, d\xi \, d\eta, \quad x, y, z \in \mathbb{R}^n, \]
and then
\[ T_{\sigma_{j,\ell}}(a, g_{j,\ell})(x) = \int_{(\mathbb{R}^n)^2} K_{j,\ell}(x, x - y, x - z)a(y)g_{j,\ell}(z) \, dydz. \]

Notice that \( \sigma_{j,\ell} \) satisfies the following:
\[
\begin{align*}
\text{supp } \sigma_{j,\ell}(x, \xi, \eta) & \subset \{|\xi| \leq 2^{j+1}, |\eta| \leq 2^{j\rho+\ell+1}\}, \\
1 + |\xi| + |\eta| & \approx 2^j \text{ on supp } \sigma_{j,\ell}(x, \xi, \eta), \\
|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma_{j,\ell}(x, \xi, \eta)| & \lesssim 2^{jm} 2^{j\rho(|\alpha| - |\beta| - |\gamma|)}.
\end{align*}
\]

**Proof of (3.2)** Let \( x \notin Q^* \). Using the moment condition on \( a \) and Taylor’s formula, we have
\[
T_{\sigma_{j,\ell}}(a, g_{j,\ell})(x) = \int_{(\mathbb{R}^n)^2} K_{j,\ell}(x, x - y, x - z)a(y)g_{j,\ell}(z) \, dydz
\]
\[
= \int_{(\mathbb{R}^n)^2} \left( K_{j,\ell}(x, x - y, x - z) - \sum_{|\alpha| < L} \frac{(c_Q - y)^\alpha}{\alpha!} K_{j,\ell}^{(\alpha,0)}(x, x - c_Q, x - z) \right)
\times a(y)g_{j,\ell}(z) \, dydz
\]
\[
= L \sum_{|\alpha| = L} \int_{\substack{y \in Q \, z \in \mathbb{R}^n \, \text{ s.t. } 0 < t < 1}} (1 - t)^{L-1} \frac{(c_Q - y)^\alpha}{\alpha!} K_{j,\ell}^{(\alpha,0)}(x, x - [c_Q, y]_t, x - z)
\times a(y)g_{j,\ell}(z) \, dydz \, dt,
\]
where \( K_{j,\ell}^{(\alpha,0)}(x, y, z) = \partial_y^\alpha K_{j,\ell}(x, y, z) \) and \([c_Q, y]_t = c_Q + t(y - c_Q)\). It follows from the size condition on \( a \) that
\[
|T_{\sigma_{j,\ell}}(a, g_{j,\ell})(x)| \lesssim \ell(Q)^{L-n/p}
\]
\[
\sum_{|\alpha| = L} \int_{\substack{y \in Q \, z \in \mathbb{R}^n \, \text{ s.t. } 0 < t < 1}} |K_{j,\ell}^{(\alpha,0)}(x, x - [c_Q, y]_t, x - z)g_{j,\ell}(z)| \, dydz \, dt.
\]
Let $M$ and $M'$ be integers satisfying $M > n/p - n/2$ and $M' > n/2$. Since $|x - [c_Q, y]_t| \approx |x - [c_Q, y]|_t$ for $x \notin Q^*$, $y \in Q$ and $0 < t < 1$, Schwarz’s inequality with respect to the $z$-variable gives

$$(1 + 2^{ip}|x - c_Q|)^M |T_{\sigma, j, \ell}(a, g_{j, \ell})(x)| \lesssim \ell(Q)^{L - n/p} \sum_{|\alpha| = L} \int_{y \in Q |z|_t < 1} \left(1 + 2^{ip}|x - [c_Q, y]_t|\right)^M K_{j, \ell}^{(a, 0)}(x, x - [c_Q, y], x - z) dy dz dt \leq \ell(Q)^{L - n/p} \sum_{|\alpha| = L} \int_{y \in Q |z|_t < 1} \left(1 + 2^{ip}|x - z|\right)^{M'} K_{j, \ell}^{(a, 0)}(x, x - [c_Q, y], x - z) \left(1 + 2^{ip}|x - z|\right)^{-M} g_{j, \ell}(z) dy dt.$$

Thus, by writing

$$h_{j, \ell}^{(Q, L)}(x) = 2^{-j\rho/2} \ell(Q)^{L - n/p} \sum_{|\alpha| = L} \sum_{|\beta| \leq M} \sum_{|\gamma| \leq M'} \int_{y \in Q |z|_t < 1} \left(2^{j\rho}(x - [c_Q, y])\right)^{\beta} \left(2^{j\rho}z\right)^{\gamma} K_{j, \ell}^{(a, 0)}(x, x - [c_Q, y], z) \parallel_{L^2_x} dy dt$$

and

$$\bar{g}_{j, \ell}(x) = 2^{j\rho/2} \parallel (1 + 2^{ip}|x - z|)^{-M} g_{j, \ell}(\cdot) \parallel_{L^2_x},$$

we have

$$|T_{\sigma, j, \ell}(a, g_{j, \ell})(x)| \lesssim (1 + 2^{ip}|x - c_Q|)^{-M} h_{j, \ell}^{(Q, L)}(x) \bar{g}_{j, \ell}(x).$$

We shall estimate the $L^2$-norm of $h_{j, \ell}^{(Q, L)}$. By Minkowski’s inequality for integrals,

$$\parallel h_{j, \ell}^{(Q, L)} \parallel_{L^2} \leq 2^{-j\rho/2} \ell(Q)^{L - n/p} \sum_{|\alpha| = L} \sum_{|\beta| \leq M} \sum_{|\gamma| \leq M'} \int_{y \in Q |z|_t < 1} \left(2^{j\rho}(x - [c_Q, y])\right)^{\beta} \left(2^{j\rho}z\right)^{\gamma} K_{j, \ell}^{(a, 0)}(x, x - [c_Q, y], z) \parallel_{L^2_{x,z}} dy dt.$$

The function in the above $\parallel \cdot \parallel_{L^2_{x,z}}$ can be written as

$$\left(2^{j\rho}(x - [c_Q, y])\right)^{\beta} \left(2^{j\rho}z\right)^{\gamma} K_{j, \ell}^{(a, 0)}(x, x - [c_Q, y], z)$$
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and

which implies

\[ R \]

Hence, the Calderón–Vainilacourt theorem on \( \mathbb{R}^2 \) (3.4) with \( d = n \) and we used (3.6). From (3.7) and (3.8), we see that

\[
\left\| \partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma (2j^p \partial_\xi)^\beta (2j^p \partial_\eta)^\gamma \left[ \xi^\alpha \sigma_{j,\ell}(x, \xi, \eta) \right] \right\|_{L^2_x}\n\]

\[
\lesssim 2j(|\alpha|+m)(1+|\xi|+|\eta|)^\rho(|\alpha'|-|\beta'|-|\gamma'|). \]

Hence, the Calderón–Vainilacourt theorem on \( \mathbb{R}^2n \) (3.3) and Plancherel’s theorem give

\[
\left\| (2j^p (x-[c_Q,y]_r))^{\beta}(2j^p z)^\gamma K_{j,\ell}^{(\alpha,0)}(x,x-[c_Q,y]_r,z) \right\|_{L^2_{x,z}} \n\]

\[
\lesssim 2j(|\alpha|+m) \left\| e^{-i[c_Q,y]_r \cdot \xi} \psi_0(\xi/2^{j+1}) \psi_0(\eta/2j^{p+\ell+1}) \right\|_{L^2_{\xi,\eta}} \n\]

\[
\approx 2j(|\alpha|+m+n/2)2(j^{p+\ell}n/2), \]

which implies

\[
\| h_{j,\ell}^{(Q,L)} \|_{L^2} \lesssim 2^{-j^p\rho/2} \| (Q)^{-n/p} \sum_{|\alpha|=L} 2j(|\alpha|+m+n/2)2(j^{p+\ell}n/2) |Q| \]

\[
\approx (2j \ell(Q))^{-n/p+n} 2j^{\rho(1/p-1/2)}2^{(\ell-(1-\rho)n/2)}. \]  (3.12)

If we do not use the moment condition on \( a \) in (3.9), a similar argument yields

\[
|T_{\sigma,j,\ell}(a,g_{j,\ell})(x)| \lesssim (1+2j^p |x-c_Q|)^{-M} h_{j,\ell}^{(Q,0)}(x) g_{j,\ell}(x) \]  (3.13)

with

\[
h_{j,\ell}^{(Q,0)}(x) = 2^{-j^p\rho/2} \| (Q)^{-n/p} \sum_{|\beta|\leq M} \sum_{|\gamma|\leq M'} \int_{y \in Q} \]

\[
(2j^p (x-y))^{\beta}(2j^p z)^\gamma K_{j,\ell}(x,x-y,z) \|_{L_x^2} \]

\[
\times \left\| \left(2j^p (x-y))^{\beta}(2j^p z)^\gamma K_{j,\ell}(x,x-y,z) \right) \right\|_{L_x^2} dy \]

and

\[
\| h_{j,\ell}^{(Q,0)} \|_{L^2} \lesssim (2j \ell(Q))^{-n/p+n} 2j^{\rho(1/p-1/2)}2^{(\ell-(1-\rho)n/2)}. \]  (3.14)

Combining (3.11) and (3.13), we have

\[
|T_{\sigma,j,\ell}(a,g_{j,\ell})(x)| \lesssim u_{j,\ell}(x) g_{j,\ell}(x) \]
with
\[
u_{j,\ell}(x) = (1 + 2^{j\rho}|x - c_Q|)^{-M} \min \left\{ h_{j,\ell}^{(Q,L)}(x), h_{j,\ell}^{(Q,0)}(x) \right\}.
\]

We take an \(\epsilon\) satisfying \(0 < \epsilon < n/2\) and set
\[
u(x) = \left( \sum_{\ell \leq j(1-\rho)+2} 2^{-(\ell-j(1-\rho))2\epsilon} u_{j,\ell}(x)^2 \right)^{1/2}
\]
and
\[
\tilde{v}(x) = \left( \sum_{\ell \leq j(1-\rho)+2} 2^{(\ell-j(1-\rho))2\epsilon} \tilde{g}_{j,\ell}(x)^2 \right)^{1/2}.
\]
(3.15)

(The number \(\epsilon\) can be chosen arbitrarily in the range \(0 < \epsilon < n/2\). For example \(\epsilon = n/4\) suffices.) Then Schwarz's inequality gives
\[
|T_\sigma(a, g)(x)| \leq \sum_{\ell \leq j(1-\rho)+2} |T_{\sigma_{j,\ell}}(a, g_{j,\ell})(x)| \lesssim \sum_{\ell \leq j(1-\rho)+2} u_{j,\ell}(x) \tilde{g}_{j,\ell}(x) \leq u(x)v(x)
\]
for \(x \notin Q^*\). Certainly the function \(u\) depends only on \(a\) and the function \(v\) depends only on \(g\). In the rest of the argument, we shall prove that \(\|u\|_{L^p} \lesssim 1\) and \(\|v\|_{L^2} \lesssim \|g\|_{L^2}\), which will complete the proof of (3.2).

First we shall prove \(\|u\|_{L^p} \lesssim 1\). By Hölder's inequality with \(1/p = 1/q + 1/2\) and by (3.12) and (3.14),
\[
\|u_{j,\ell}\|_{L^p} \leq \|(1 + 2^{j\rho}|x - c_Q|)^{-M}\|_{L^q} \|\min \left\{ h_{j,\ell}^{(Q,L)}, h_{j,\ell}^{(Q,0)} \right\}\|_{L^2}
\lesssim 2^{-j\rho n/q} \min \left\{ \|h_{j,\ell}^{(Q,L)}\|_{L^2}, \|h_{j,\ell}^{(Q,0)}\|_{L^2} \right\}
\lesssim 2^{(\ell-j(1-\rho))n/2} \min \left\{ \left(2^j \ell(Q)^{L-n/p+n}\right)^{L-n/p+n}, \left(2^j \ell(Q)^{-n/p+n}\right)^{-n/p+n} \right\}.
\]

Thus
\[
\|u\|_{L^p}^p = \left( \sum_{\ell \leq j(1-\rho)+2} 2^{-(\ell-j(1-\rho))2\epsilon} u_{j,\ell}^2 \right)^{1/2} \|u\|_{L^p}^p
\leq \sum_{\ell \leq j(1-\rho)+2} \left(2^{-(\ell-j(1-\rho))\epsilon} \|u_{j,\ell}\|_{L^p}\right)^p
\lesssim \sum_{j=0}^\infty \left( \min \left\{ \left(2^j \ell(Q)^{L-n/p+n}\right)^{L-n/p+n}, \left(2^j \ell(Q)^{-n/p+n}\right)^{-n/p+n} \right\} \right)^p
\]
Proof of (3.3) Take an $M' > n/2$. By Schwarz’s inequality,

$$|T_{\sigma_j,\ell}(a, g_j, \ell)(x)| \leq |Q|^{-1/p} \int_{(\mathbb{R}^n)^2} |K_j,\ell(x, x - y, x - z)g_j,\ell(z)| \, dydz$$

where the last $\lesssim$ holds because $L - n/p + n > 0$ and $-n/p + n < 0$.

Next, to prove $\|v\|_{L^2}^2 \lesssim \|g\|_{L^2}^2$, observe that $\|g_j,\ell\|_{L^2} \approx \|g_j,\ell\|_{L^2}$. Hence

$$\|v\|_{L^2}^2 = \sum_{\ell \leq j(1-\rho)+2} 2^{(\ell-j(1-\rho))2\epsilon} \|g_j,\ell\|_{L^2}^2 \approx \sum_{j=0}^{\infty} \sum_{\ell=0}^{\infty} 2^{(\ell-j(1-\rho))2\epsilon} \|g_j,\ell\|_{L^2}^2$$

We divide the sum over $\ell$ into two parts $\ell = 0$ and $\ell \geq 1$. For the terms with $\ell = 0$, Young’s inequality gives $\|g_j,0\|_{L^2} \leq \|\mathcal{F}^{-1}\tilde{\varphi}_0\|_{L^1}\|g\|_{L^2} \approx \|g\|_{L^2}$ and thus

$$\sum_{j=0}^{\infty} 2^{-j(1-\rho)2\epsilon} \|g_j,0\|_{L^2}^2 \lesssim \sum_{j=0}^{\infty} 2^{-j(1-\rho)2\epsilon} \|g\|_{L^2}^2 \approx \|g\|_{L^2}^2$$

since $\rho < 1$. For the terms with $\ell \geq 1$, we have $g_j,\ell = \tilde{\varphi}(D/2^{[j\rho]+\ell})g$ and hence, by a change of variables, we have

$$\sum_{j=0}^{\infty} \sum_{\ell=1}^{[j(1-\rho)]+2} 2^{(\ell-j(1-\rho))2\epsilon} \|g_j,\ell\|_{L^2}^2$$

$$= \sum_{j=0}^{\infty} \sum_{k=[j\rho]+1}^{[j\rho]+[j(1-\rho)]+2} 2^{(k-j\rho-j(1-\rho))2\epsilon} \|\tilde{\varphi}(D/2^k)g\|_{L^2}^2$$

$$\lesssim \sum_{j=0}^{\infty} \sum_{k=1}^{j+2} 2^{(k-j)2\epsilon} \|\tilde{\varphi}(D/2^k)g\|_{L^2}^2 = \sum_{k=1}^{\infty} \sum_{j=\max(0, k-2)}^{\infty} 2^{(k-j)2\epsilon} \|\tilde{\varphi}(D/2^k)g\|_{L^2}^2$$

$$\approx \sum_{k=1}^{\infty} \|\tilde{\varphi}(D/2^k)g\|_{L^2}^2 \lesssim \|g\|_{L^2}^2,$$

where the last $\lesssim$ follows from the fact supp $\tilde{\varphi}$ is included in an annulus. Therefore, we obtain $\|v\|_{L^2} \lesssim \|g\|_{L^2}$.

\[\square\]
\[ \leq |Q|^{-1/p} \left\| (1 + 2^{j\rho}|x - y|)^{M'} (1 + 2^{j\rho}|x - z|)^{M'} K_{j,\ell}(x, x - y, x - z) \right\|_{L^2_{y,z}} \]
\[ \times \left\| (1 + 2^{j\rho}|x - y|)^{-M'} (1 + 2^{j\rho}|x - z|)^{-M'} g_{j,\ell}(z) \right\|_{L^2_{y,z}} \]

For the first \( L^2_{y,z} \) norm above, we use Plancherel’s theorem, (3.8) and (3.6) to obtain
\[ \left\| (1 + 2^{j\rho}|y|)^{M'} (1 + 2^{j\rho}|z|)^{M'} K_{j,\ell}(x, y, z) \right\|_{L^2_{y,z}} \approx \sum_{|\beta| \leq M'} \sum_{|\gamma| \leq M'} \left\| (2^{j\rho}\partial_\xi)^\beta (2^{j\rho}\partial_\eta)^\gamma \sigma_{j,\ell}(x, \xi, \eta) \right\|_{L^2_{\xi,\eta}} \lesssim 2^{j(m+n)/2} 2^{(j\rho+\ell)n/2} \]
for all \( x \in \mathbb{R}^n \). As for the second \( L^2_{y,z} \) norm, we have
\[ \left\| (1 + 2^{j\rho}|x - y|)^{-M'} (1 + 2^{j\rho}|x - z|)^{-M'} g_{j,\ell}(z) \right\|_{L^2_{y,z}} \approx 2^{-j\rho n/2} \left\| (1 + 2^{j\rho}|x - z|)^{-M'} g_{j,\ell}(z) \right\|_{L^2_{z}} = 2^{-j\rho n} \tilde{g}_{j,\ell}(x), \]
where \( \tilde{g}_{j,\ell} \) is defined by (3.10). Thus
\[ |T_{\sigma_j,\ell}(a, g_{j,\ell})(x)| \lesssim |Q|^{-1/p} 2^{j(m+n)/2} 2^{(j\rho+\ell)n/2} 2^{-j\rho n} \tilde{g}_{j,\ell}(x) = |Q|^{-1/p} 2^{-j(1-\rho)n(1/p-1)} 2^{(\ell-j(1-\rho))(n/2)} \tilde{g}_{j,\ell}(x). \] (3.16)

Since
\[ \sum_{\ell \leq j(1-\rho)+2} \left( 2^{-j(1-\rho)n(1/p-1)} 2^{(\ell-j(1-\rho))(n/2-\epsilon)} \right)^2 \]
\[ = \sum_{j=0}^{\infty} 2^{-2j(1-\rho)n(1/p-1)} \left( \sum_{\ell=0}^{[j(1-\rho)]+2} 2^{(\ell-j(1-\rho))(n-2\epsilon)} \right) \]
\[ \approx \sum_{j=0}^{\infty} 2^{-2j(1-\rho)n(1/p-1)} \approx 1, \]
the estimate (3.16) together with Schwarz’s inequality gives
\[ |T_\sigma(a, g)(x)| \leq \sum_{\ell \leq j(1-\rho)+2} |T_{\sigma_j,\ell}(a, g_{j,\ell})(x)| \lesssim |Q|^{-1/p} v(x), \]
where \( v \) is the function defined by (3.15). In particular
\[ |T_\sigma(a, g)(x)| 1_{Q^*}(x) \lesssim |Q|^{-1/p} 1_{Q^*}(x) v(x). \]
In Proof of (3.2), we have proved that \( \|v\|_{L^2} \lesssim \|g\|_{L^2} \). Therefore, we can take \( |Q|^{-1/p} \mathbf{1}_{Q^*} \) and \( v \) as \( u' \) and \( v' \) in (3.3). \( \square \)

4 Proof of Theorem 1.3

In this section, we shall prove Theorem 1.3, namely the \( H^p \times L^\infty \to L^p \) boundedness of \( T_{\sigma} \), where \( 0 \leq \rho < 1, 0 < p < 1, m = -(1 - \rho)n/p \), and \( \sigma \in BS^m_{\rho, \rho} \).

By the usual argument using the atomic decomposition for \( H^p \), the desired boundedness follows if we prove the estimate

\[
\|T_{\sigma}(a, g)\|_{L^p} \lesssim \|g\|_{L^\infty}
\]

for all \( H^p \)-atoms \( a \). Moreover, by virtue of the translation invariance,

\[
T_{\sigma}(a, g)(x + x_0) = T_{\sigma_{x_0}}(a(\cdot + x_0), g(\cdot + x_0))(x),
\]

where \( \sigma_{x_0}(x, \xi, \eta) = \sigma(x + x_0, \xi, \eta) \), it is sufficient to treat \( H^p \)-atoms supported in cubes centered at the origin.

Let \( g \in L^\infty \) and let \( a \) be an \( H^p \)-atom satisfying (2.1) with a cube \( Q \) centered at the origin and with \( L > n/p - n \). We divide the \( p \)th power of the \( L^p \)-norm in the left hand side of (4.1) into

\[
\|T_{\sigma}(a, g)\|_{L^p(Q^*)}^p + \|T_{\sigma}(a, g)\|_{L^p((Q^*)^c)}^p.
\]

For the former term, it follows from Theorem B with \( (p, q, r) = (2, \infty, 2) \) that

\[
\|T_{\sigma}(a, g)\|_{L^p(Q^*)} \leq |Q^*|^{1/p - 1/2} \|T_{\sigma}(a, g)\|_{L^2} \lesssim |Q|^{1/p - 1/2} \|a\|_{L^2} \|g\|_{L^\infty} \leq \|g\|_{L^\infty},
\]

where we used the fact

\[
BS_{\rho, \rho}^{-(1-\rho)n/p} \subset BS_{\rho, \rho}^{-(1-\rho)n/2}.
\]

In the rest of this section, we shall estimate the latter term in (4.2). The method will be similar to the one used in Sect. 3.

Let \( \Psi_j, j \geq 0 \), be as in (3.4) with \( d = 2n \). This time we do not need a delicate decomposition such as (3.5) for the proof of Theorem 1.3 and decompose \( \sigma \) as

\[
\sigma(x, \xi, \eta) = \sum_{j \geq 0} \sigma_j(x, \xi, \eta)
\]

with

\[
\sigma_j(x, \xi, \eta) = \sigma(x, \xi, \eta)\Psi_j(\xi, \eta).
\]
We also write the partial inverse Fourier transform of $\sigma_j(x, \xi, \eta)$ with respect to $(\xi, \eta)$ as

$$K_j(x, y, z) = \frac{1}{(2\pi)^{2n}} \int_{(\mathbb{R}^n)^2} e^{i(y \cdot \xi + z \cdot \eta)} \sigma_j(x, \xi, \eta) \, d\xi d\eta, \quad x, y, z \in \mathbb{R}^n,$$

and then

$$T_{\sigma_j}(a, g)(x) = \int_{(\mathbb{R}^n)^2} K_j(x, x - y, x - z)a(y)g(z) \, dydz.$$

Notice that $\sigma_j$ satisfies the following:

\begin{align*}
\text{supp } \sigma_j(x, \cdot, \cdot) &\subset \{|\xi| \leq 2^{j+1}, \ |\eta| \leq 2^{j+1}\}, \\
1 + |\xi| + |\eta| &\approx 2^j \quad \text{on } \text{supp } \sigma_j(x, \cdot, \cdot), \\
|\partial_x^\alpha \partial_\xi^\beta \partial_\eta^\gamma \sigma_j(x, \xi, \eta)| &\lesssim 2^{jm} 2^j |x - ty| |x - ty||y||\eta|^{\gamma}.
\end{align*}

Let $x \not\in Q^*$. By the same argument using the moment condition on $a$ as in (3.9) with $c_Q = 0$, we have

$$T_{\sigma_j}(a, g)(x) = L \sum_{|\alpha| = L} \int_{y \in Q} (1 - t)^{L-1} \frac{(-y)^\alpha}{\alpha!} K_j^{(\alpha, 0)}(x, x - ty, x - z)$$

$$\times a(y)g(z) \, dydzdt,$$

where $K_j^{(\alpha, 0)}(x, y, z) = \partial_y^\alpha K_j(x, y, z)$. We take integers $M$ and $M'$ satisfying $M > n/p - n/2$ and $M' > n/2$. Since $|x| \approx |x - ty|$ for $x \not\in Q^*$, $y \in Q$ and $0 < t < 1$, Schwarz’s inequality with respect to the $z$-variable gives

\begin{align*}
(1 + 2^j |x|)^M |T_{\sigma_j}(a, g)(x)| &\lesssim \|g\|_{L^\infty} \ell(Q)^{L-n/p} \sum_{|\alpha| = L} \int_{y \in Q} \|K_j^{(\alpha, 0)}(x, x - ty, x)\|_{L^2} \, dydzdt \\
&\lesssim \|g\|_{L^\infty} 2^{-jpn/2} \ell(Q)^{L-n/p} \sum_{|\alpha| = L} \int_{y \in Q} (1 + 2^j |x - ty|)^M |K_j^{(\alpha, 0)}(x, x - ty, z)| \, dydzdt.
\end{align*}

Thus, by writing

$$h_j^{(Q, L)}(x) = 2^{-jpn/2} \ell(Q)^{L-n/p} \sum_{|\alpha| = L} \sum_{|\beta| \leq M} \sum_{|\gamma| \leq M'} \int_{y \in Q} (1 + 2^j |x - ty|)^M |K_j^{(\alpha, 0)}(x, x - ty, z)| \, dydzdt,$$
we have
\[
|T_{\sigma_j}(a, g)(x)| \lesssim (1 + 2^{j\rho} |x|)^{-M} h_j^{(Q, L)}(x) \|g\|_{L^\infty}. \tag{4.7}
\]

We shall make a slight modification on the argument (3.12) to estimate the \(L^2\)-norm of \(h_j^{(Q, L)}\). By Minkowski’s inequality for integrals,
\[
\|h_j^{(Q, L)}\|_{L^2} \leq 2^{-j\rho n/2} \ell(Q)^{L-n/p} \sum_{|\alpha| = L} \sum_{|\beta| \leq M} \sum_{|\gamma| \leq M'} \int_{0 < t < 1} \|e^{[x \cdot \xi + z \cdot \eta]} (2^{j\rho} \partial_\xi)^\beta (2^{j\rho} \partial_\eta)^\gamma \left[ \xi^\alpha \sigma_j(x, \xi, \eta) \right] \|_{L^2_{\xi, \eta}} d\xi d\eta, \]
where \(\psi_0\) is as in (3.4) with \(d = n\) and we used (4.3). From (4.4) and (4.5), we see that
\[
\left| \partial_\xi^\alpha \partial_\eta^\beta \partial_\eta^\gamma (2^{j\rho} \partial_\xi)^\beta (2^{j\rho} \partial_\eta)^\gamma \left[ \xi^\alpha \sigma_j(x, \xi, \eta) \right] \right| \\
\lesssim 2^j(|\alpha| + m)(1 + |\xi| + |\eta|)^\rho(|\alpha'| - |\beta'| - |\gamma'|).
\]
Hence, the Calderón–Vaillancourt theorem on \(\mathbb{R}^{2n}\) ([3]) and Plancherel’s theorem give
\[
\left\| (2^{j\rho} (x - ty))^{\beta} (2^{j\rho} z)^{\gamma} K_j^{(\alpha, 0)}(x, x - ty, z) \right\|_{L^2_{x, z}} \\
\lesssim 2^j(|\alpha| + m) \left\| e^{-ity \cdot \xi} \psi_0(\xi/2^{j+1}) \psi_0(\eta/2^{j+1}) \right\|_{L^2_{\xi, \eta}} \\
\approx 2^j(|\alpha| + m + n),
\]
which implies
\[
\|h_j^{(Q, L)}\|_{L^2} \lesssim 2^{-j\rho n/2} \ell(Q)^{L-n/p} \sum_{|\alpha| = L} 2^j(|\alpha| + m + n) |Q| \\
\approx \left( 2^j \ell(Q) \right)^{L-n/p+n} 2^{j\rho n(1/p-1/2)}. \tag{4.8}
\]
If we do not use the moment condition on \(a\) in (4.6), a similar argument yields

\[
|T_{\sigma_j}(a, g)(x)| \lesssim (1 + 2^{j\rho}|x|)^{-M} h_j^{(Q,0)}(x)\|g\|_{L^\infty} \tag{4.9}
\]

with

\[
h_j^{(Q,0)}(x) = 2^{-j\rho n/2} \ell(Q)^{-n/p} \sum_{|\beta| \leq M} \sum_{|\gamma| \leq M'} \int_{y \in Q} (2^{j\rho}(x-y))^\beta (2^{j\rho}z)^\gamma K_j(x, x - y, z) dy
\]

and

\[
\|h_j^{(Q,0)}\|_{L^2} \lesssim \left(2^j \ell(Q)^{n/p+n}\right)^{-n/p+n} 2^{j\rho(1/p-1/2)} \tag{4.10}
\]

Combining (4.7) and (4.9), we have

\[
|T_{\sigma_j}(a, g)(x)| \lesssim (1 + 2^{j\rho}|x|)^{-M} \min \left\{ h_j^{(Q,L)}(x), h_j^{(Q,0)}(x) \right\} \|g\|_{L^\infty}.
\]

Using (4.8), (4.10) and Hölder’s inequality with \(1/p = 1/q + 1/2\), we have

\[
\left\| (1 + 2^{j\rho} \cdot )^{-M} \min \left\{ h_j^{(Q,L)}, h_j^{(Q,0)} \right\} \right\|_{L^p} \lesssim \left(2^j \ell(Q)^{n/p+n}\right)^{-n/p+n} \min \left\{ \left(2^j \ell(Q)^{L-n/p+n}\right), \left(2^j \ell(Q)^{-n/p+n}\right) \right\}.
\]

Therefore,

\[
\|T_{\sigma}(a, g)\|_{L^p((Q^*)^c)}^p \lesssim \sum_{j=0}^\infty \|T_{\sigma_j}(a, g)\|_{L^p((Q^*)^c)}^p
\]

\[
\lesssim \left( \sum_{2^j \ell(Q) \leq 1} \left(2^j \ell(Q)^{L-n/p+n}\right)^p \right) \|g\|_{L^\infty}^p \lesssim \|g\|_{L^\infty}^p,
\]

which is the desired estimate for the latter term in (4.2).

References

1. Bényi, Á., Bernicot, F., Maldonado, D., Naibo, V., Torres, R.: On the Hörmander classes of bilinear pseudodifferential operators II. Indiana Univ. Math. J. 62, 1733–1764 (2013)
2. Bényi, Á., Maldonado, D., Naibo, V., Torres, R.: On the Hörmander classes of bilinear pseudodifferential operators. Integral Equ. Oper. Theory 67, 341–364 (2010)
3. Calderón, A.P., Vaillancourt, R.: A class of bounded pseudo-differential operators. Proc. Nat. Acad. Sci. U.S.A. 69, 1185–1187 (1972)
4. Grafakos, L., Kalton, N.: Multilinear Calderón–Zygmund operators on Hardy spaces. Collect. Math. 52, 169–179 (2001)
5. Michalowski, N., Rule, D., Staubach, W.: Multilinear pseudodifferential operators beyond Calderón–Zygmund operators. J. Math. Anal. Appl. 414, 149–165 (2014)
6. Miyachi, A., Tomita, N.: Minimal smoothness conditions for bilinear Fourier multipliers. Rev. Mat. Iberoam. 29, 495–530 (2013)
7. Miyachi, A., Tomita, N.: Calderón-Vaillancourt type theorem for bilinear operators. Indiana Univ. Math. J. 62, 1165–1201 (2013)
8. Miyachi, A., Tomita, N.: Bilinear pseudo-differential operators with exotic symbols, To appear in Ann. Inst. Fourier (Grenoble), arXiv:1801.06744
9. Naibo, V.: On the $L^\infty \times L^\infty \to BMO$ mapping property for certain bilinear pseudodifferential operators. Proc. Am. Math. Soc. 143, 5323–5336 (2015)
10. Stein, E.M.: Harmonic Analysis, Real Variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton, NJ (1993)