Orthogonally additive sums of powers of linear functionals

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Abstract. Let $E$ be a vector lattice, $\lambda_1, \lambda_2, \ldots, \lambda_k$ scalars and $\varphi_1, \ldots, \varphi_k$ pairwise independent regular linear functionals on $E$. We show that if $k < m$ then $\sum_{j=1}^{k} \lambda_j \varphi^m_j$ is orthogonally additive if and only if $\varphi_j$ or $-\varphi_j$ is a lattice homomorphism for each $j$, $1 \leq j \leq k$. Moreover, for each $m \geq 2$, we provide an example to show that this result does not extend to the case where $k = m$.

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1. Introduction. Let $E$ be a real vector space. An $m$-homogeneous polynomial $P: E \to \mathbb{R}$ is defined by $P(x) = A(x, \ldots, x)$, where $A$ is a symmetric $m$-linear mapping from $E^m$ to $\mathbb{R}$. We write $P = \hat{A}$. When $E$ is a Banach space the supremum norm on the space of $m$-homogeneous polynomials is given by $\|P\| = \sup_{\|x\| \leq 1} |P(x)|$. For further information on polynomials, we refer the reader to [7].

When $E$ is a vector lattice, we say that an $m$-homogeneous polynomial $P: E \to \mathbb{R}$ with associated symmetric $m$-linear mapping $A$ is positive if $A(x_1, \ldots, x_m) \geq 0$ for all $x_1, \ldots, x_m \geq 0$ in $E$. An $m$-homogeneous polynomial $P: E \to \mathbb{R}$ is said to be regular if it can be written as the difference of two positive polynomials.

An $m$-homogeneous polynomial $P: E \to \mathbb{R}$ is said to be orthogonally additive if $P(x + y) = P(x) + P(y)$ whenever $|x| \wedge |y| = 0$. Orthogonally additive polynomials were introduced by Sundaresan [12] and have been an area of active research in recent years. See [1, 3, 6, 9, 11]. If $A$ is the symmetric $m$-linear mapping associated with a regular $m$-homogeneous polynomial $P$, then it is well known (see [5, Lemma 5.1]) that $P$ is orthogonally additive if and only if
A(x_1, \ldots, x_m) = 0 \text{ whenever some pair, } x_i, x_j, \text{ of the arguments are disjoint } (|x_i| \land |x_j| = 0 \text{ for } i \neq j). \text{ An } m\text{-linear mapping with this property is said to be orthosymmetric.}

Let \( E \) and \( F \) be vector lattices. A linear operator \( T: E \to F \) is said to be a lattice homomorphism if it preserves the lattice operations, namely, \( T(x \lor y) = (Tx) \lor (Ty) \) and \( T(x \land y) = (Tx) \land (Ty) \) for all \( x \) and \( y \) in \( E \).

We have the following characterisation of lattice homomorphisms which we will use in this paper.

**Proposition 1.1** ([10, Proposition 1.3.11]). *Let \( E \) and \( F \) be vector lattices and \( T: E \to F \) be a linear operator. The following are equivalent:

(a) \( T \) is a lattice homomorphism.

(b) \( |Tx| = T|x| \) for all \( x \) in \( E \).

(c) \( Tx^+ \land Tx^- = 0 \) for all \( x \) in \( E \).

For more information on vector lattices and operators on vector lattices, we refer the reader to [2, 10].

2. Sums of powers of linear functionals. It is shown in [4, Proposition 2] that if \( E \) is a Banach lattice, \( m > 1 \) a positive integer, and \( \varphi \) a (continuous) linear functional, then \( \varphi^m \) is orthogonally additive if and only if \( \varphi \) or \( -\varphi \) is a lattice homomorphism.

With a slight adaptation, we can extend this result to regular linear functionals on vector lattices. We include the proof of the extension of this result to vector lattices for the sake of completeness.

**Proposition 2.1.** *Let \( E \) be a vector lattice, let \( \varphi \) be a regular linear functional on \( E \), and let \( m \geq 2 \). The \( m \)-homogeneous polynomial defined by \( P(x) = \varphi(x)^m \) is orthogonally additive if and only if either \( \varphi \) or \( -\varphi \) is a lattice homomorphism.*

**Proof.** If \( \pm \varphi \) is a lattice homomorphism, then it is obvious that \( P \) is orthogonally additive.

Conversely, suppose that \( P = \varphi^m \) is orthogonally additive. For every \( x \in E \), the vectors \( x^+ \) and \( tx^- \) are disjoint for all \( t \in \mathbb{R} \). Therefore

\[
\varphi(x^+)^m + t^m \varphi(x^-)^m = P(x^+ + tx^-) = \sum_{j=0}^{m} \binom{m}{j} \varphi(x^+)^{m-j} \varphi(x^-)^j t^j
\]

for every \( t \in \mathbb{R} \). Hence either \( \varphi(x^+) = 0 \) or \( \varphi(x^-) = 0 \).

If we can show that \( \varphi \) (or \( -\varphi \)) is positive, then it follows that \( \varphi \) (or \( -\varphi \)) is a lattice homomorphism.

Let \( a \) be a positive element of \( E \). The principal ideal \( E_a \) generated by \( a \), with the order unit norm, is isometrically Banach lattice isomorphic to a norm-dense sublattice of \( C(K) \) for some compact Hausdorff topological space \( K \) [8, Prop. 2.1]. As \( \varphi \) is regular, we have \(|\varphi(x)| \leq |\varphi(|x|)| \leq C|\varphi|(a)\) whenever \( x \in E_a \) with \(|x| \leq Ca\). It follows that \( \varphi \) is bounded for the order unit norm on \( E_a \). Therefore \( \varphi \) has a unique extension to a bounded linear functional \( \tilde{\varphi} \) on \( C(K) \).
We claim that \( \tilde{\varphi} \) has the same property as \( \varphi \); namely, that either \( \tilde{\varphi}(x^+) \) or \( \tilde{\varphi}(x^-) \) is zero for every \( x \in C(K) \). To see this, let \( (x_n) \) be a sequence in \( E_a \) that converges in norm to \( x \). Then the sequences \( (x_n^+) \), \( (x_n^-) \) converge to \( x^+ \), \( x^- \) respectively. Therefore \( (\varphi(x_n^+)) \), \( (\varphi(x_n^-)) \) converge to \( \tilde{\varphi}(x^+) \), \( \tilde{\varphi}(x^-) \) respectively. Now, either \( \varphi(x_n^+) \) or \( \varphi(x_n^-) \) must be zero for infinitely many values of \( n \) and it follows that either \( \tilde{\varphi}(x^+) \) or \( \tilde{\varphi}(x^-) \) is zero.

The functional \( \tilde{\varphi} \) is represented by a regular Borel signed measure \( \mu \) on \( K \) and the fact that \( \tilde{\varphi}(x^+) \) or \( \tilde{\varphi}(x^-) = 0 \) for all \( x \in C(K) \) implies that the support of \( \mu \) consists of a single point. It follows that either \( \varphi \) or \( -\varphi \) is positive on \( E_a \). Now \( E \) is the union of the principal ideals \( E_a \), which are upwards directed by inclusion. Thus, if \( \varphi \) (or \( -\varphi \)) is positive on one \( E_a \), then \( \varphi \) (or \( -\varphi \)) is positive on all of \( E \). \( \square \)

Let us now generalise this result to sums of powers of linear functionals. We start by extending the concept of higher order Fréchet derivatives of homogeneous polynomials on a Banach space to a vector space.

Given an \( m \)-homogeneous polynomial \( P \) on a vector space \( E \) with associated symmetric \( m \)-linear mapping \( A \) and \( x \) in \( E \), let us define the \( k \)th derivative of \( P \) at \( x \), \( \hat{d}^k P(x) \), to be \( \frac{m!}{(m-k)!} A(x, \ldots, x, \ldots) \). We observe that, when \( E \) is a Banach space and \( P \) is continuous, this definition of \( \hat{d}^k P(x) \) coincides with the \( k \)th Fréchet derivative of \( P \) at \( x \).

**Proposition 2.2.** Let \( E \) be a vector lattice, \( k, m \) positive integers with \( k < m \), and \( P \) a regular orthogonally additive \( m \)-homogeneous polynomial on \( E \). Then for every \( x \) in \( E \), \( \hat{d}^k P(x) \) is a regular orthogonally additive \( k \)-homogeneous polynomial on \( E \).

**Proof.** As \( P \) is orthogonally additive and regular, the symmetric \( m \)-linear mapping \( A \) associated with \( P \) is orthosymmetric. Since the symmetric \( k \)-linear mapping associated with \( \hat{d}^k P(x) \) is defined to be \( \frac{m!}{(m-k)!} A(x, \ldots, x, \ldots) \), we see that \( \hat{d}^k P(x) \) is a regular orthogonally additive \( k \)-homogeneous polynomial. \( \square \)

In the statement of the following theorem, we require an \( m \)-homogeneous polynomial written in the form \( \sum_{j=1}^{k} \lambda_j \varphi_j^m \) with \( \varphi_i \neq \lambda \varphi_j \). We note that given any \( m \)-homogeneous polynomial of the form \( \sum_{j=1}^{k} \lambda_j \varphi_j^m \), we can amalgamate terms and rewrite it as \( \sum_{i=1}^{l} \mu_i \psi_i^m \) with \( l \leq k \) where \( \psi_1, \ldots, \psi_l \) are pairwise independent.

**Theorem 2.3.** Let \( E \) be a vector lattice and let \( k, m \) be positive integers with \( k < m \). Let \( \lambda_1, \lambda_2, \ldots, \lambda_k \) be scalars and let \( \varphi_1, \varphi_2, \ldots, \varphi_k \) be pairwise independent regular linear functionals on \( E \) (\( \varphi_i \neq \lambda \varphi_j \) for \( i \neq j \) and \( \lambda \neq 0 \)). Then \( P = \sum_{j=1}^{k} \lambda_j \varphi_j^m \) is orthogonally additive if and only if \( \varphi_j \) or \( -\varphi_j \) is a lattice homomorphism for each \( j \), \( 1 \leq j \leq k \).
Proof. If for each \( j \), \( \varphi_j \) or \( -\varphi_j \) is a lattice homomorphism, then \( P \) is orthogonally additive.

To prove the converse, we use induction on \( m \). By Proposition 2.1, the result is true when \( m = 2 \) (and hence \( k = 1 \)).

Let us now suppose that the result is true for \( m \) and every \( k \) with \( k < m \). We will now establish the result for every \( 1 < l < m + 1 \).

Let us now suppose that \( P = \sum_{j=1}^{l} \lambda_j \varphi_j^{m+1} \) is an orthogonally additive polynomial. Given \( 1 \leq r \leq l \), choose \( s \), \( 1 \leq s \leq l \), so that \( s \neq r \). Since \( \varphi_r \) and \( \varphi_s \) are linearly independent, we can choose \( x \) in \( E \) so that \( \varphi_r(x) \neq 0 \) and \( \varphi_s(x) = 0 \). Using Proposition 2.2, we have that \( \hat{d}^m P(x) \) is orthogonally additive.

Then

\[
\hat{d}^m P(x) = (m + 1)! \sum_{j=1}^{l} \lambda_j \varphi_j(x) \varphi_j^{m}
\]

is a sum of at most \( l - 1 \) regular linear functionals each raised to the power of \( m \). Hence, by our induction hypothesis, we have that for each \( j \) with \( \varphi_j(x) \neq 0 \) either \( \varphi_j \) or \( -\varphi_j \) is a lattice homomorphism. In particular, we get that either \( \varphi_r \) or \( -\varphi_r \) is a lattice homomorphism. As \( r \) was arbitrary, we conclude that \( \varphi_j \) or \( -\varphi_j \) is a lattice homomorphism for \( 1 \leq j \leq l \).

We claim that the above theorem is sharp. This is particularly easy to see when \( k = m = 2 \) as the example below shows.

Example. Let us take \( E \) to be \( \mathbb{R}^2 \) with the standard order. Let \( \varphi_1(x) = x_1 + x_2 \) and \( \varphi_2(x) = x_1 - x_2 \). It is immediate that neither \( \varphi_1 \) nor \( \varphi_2 \) are lattice homomorphisms. However,

\[
\varphi_1^2(x) + \varphi_2^2(x) = 2(x_1^2 + x_2^2)
\]

is orthogonally additive.

More generally, we have:

**Proposition 2.4.** For every positive integer \( m \), there exist linear functionals \( \varphi_1, \ldots, \varphi_m \) on \( \mathbb{R}^2 \), none of which nor their negatives are lattice homomorphisms, such that the \( m \)-homogeneous polynomial \( \sum_{j=1}^{m} \lambda_j \varphi_j^m \) is orthogonally additive.

Proof. We will see that given any \( B_1, B_2 \) in \( \mathbb{R} \), it is possible to choose \( A_1, \ldots, A_n \) so that

\[
\sum_{r=1}^{n} A_r \left( (rx_1 + x_2)^{2n} + (rx_1 - x_2)^{2n} \right) = B_1 x_1^{2n} + B_2 x_2^{2n}.
\]

Let us first suppose that \( m = 2n \) is an even integer. Then we have

\[
(x_1 + x_2)^{2n} + (x_1 - x_2)^{2n} = 2 \sum_{j=0}^{n} \binom{2n}{2(n-j)} x_1^{2(n-j)} x_2^{2j},
\]
\[(2x_1 + x_2)^{2n} + (2x_1 - x_2)^{2n} = 2 \sum_{j=0}^{n} \left( \begin{array}{c} 2n \\ 2(n-j) \end{array} \right) 2^{2(n-j)} x_1^{2(n-j)} x_2^{2j},
\]

\[\vdots\]

\[(nx_1 + x_2)^{2n} + (nx_1 - x_2)^{2n} = 2 \sum_{j=0}^{n} \left( \begin{array}{c} 2n \\ 2(n-j) \end{array} \right) n^{2(n-j)} x_1^{2(n-j)} x_2^{2j}.
\]

Adding, we get that

\[
\sum_{r=1}^{n} A_r \left( (rx_1 + x_2)^{2n} + (rx_1 - x_2)^{2n} \right) = \sum_{r=1}^{n} A_r \left( 2 \sum_{j=0}^{n} \left( \begin{array}{c} 2n \\ 2(n-j) \end{array} \right) r^{2(n-j)} x_1^{2(n-j)} x_2^{2j} \right) = 2 \sum_{j=0}^{n} \left( \begin{array}{c} 2n \\ 2(n-j) \end{array} \right) \left( \sum_{r=1}^{n} A_r r^{2(n-j)} \right) x_1^{2(n-j)} x_2^{2j}.
\]

We now show that it is possible to choose \(A_1, \ldots, A_n\) so that

\[
\sum_{r=1}^{n} A_r r^{2n} \neq 0,
\]

and

\[
\sum_{r=1}^{n} A_r r^{2(n-j)} = 0
\]

for \(j = 1, \ldots, n - 1\).

Without loss of generality, let us suppose we wish to have \(2 \sum_{r=1}^{n} A_r r^{2n} = 1\). We rewrite these equations in matrix form as

\[
\left( \begin{array}{ccccc}
1 & 2^2 & 3^2 & \ldots & n^2 \\
1 & 2^4 & 3^4 & \ldots & n^4 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2^{2(n-1)} & 3^{2(n-1)} & \ldots & n^{2(n-1)} \\
1 & 2^{2n} & 3^{2n} & \ldots & n^{2n}
\end{array} \right) \left( \begin{array}{c} A_1 \\
A_2 \\
\vdots \\
A_{n-1} \\
A_n \end{array} \right) = \left( \begin{array}{c} 0 \\
0 \\
\vdots \\
0 \\
1 \end{array} \right).
\]

The transpose of the matrix

\[
\left( \begin{array}{ccccc}
1 & 2^2 & 3^2 & \ldots & n^2 \\
1 & 2^4 & 3^4 & \ldots & n^4 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2^{2(n-1)} & 3^{2(n-1)} & \ldots & n^{2(n-1)} \\
1 & 2^{2n} & 3^{2n} & \ldots & n^{2n}
\end{array} \right)
\]
As this matrix has the same determinant as the Vandermonde matrix
\[
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
1 & 1 & 1 & \ldots & 1 \\
1 & 2^2 & 2^4 & \ldots & 2^{2n} \\
1 & 3^2 & 3^4 & \ldots & 3^{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & (n-1)^2 & (n-1)^4 & \ldots & (n-1)^{2n} \\
n^2 & n^4 & \ldots & n^{2n}
\end{pmatrix},
\]
it is invertible. Therefore we can find \(A_1, \ldots, A_n\) so that
\[
\sum_{r=1}^{n} A_r \left( (rx_1 + x_2)^{2n} + (rx_1 - x_2)^{2n} \right) = x_1^{2n} + B_2 x_2^{2n} \tag{2.1}
\]
and hence \(\sum_{r=1}^{n} A_r \left( (rx_1 + x_2)^{2n} + (rx_1 - x_2)^{2n} \right)\) is orthogonally additive.

Taking the \((2n-1)^{st}\) derivative of both sides of (2.1) at the point \((1,1)\), we get that
\[
\sum_{r=1}^{n} A_r \left( (r+1)(rx_1 + x_2)^{2n-1} + (r-1)(rx_1 - x_2)^{2n-1} \right) = x_1^{2n-1} + B_2 x_2^{2n-1},
\]
or equivalently
\[
2A_1(x_1 + x_2)^{2n-1} + \sum_{r=2}^{n} A_r \left( (r+1)(rx_1 + x_2)^{2n-1} + (r-1)(rx_1 - x_2)^{2n-1} \right) = x_1^{2n-1} + B_2 x_2^{2n-1},
\]
giving us an orthogonally additive sum of \(2n-1\) linear functionals, each raised to the power of \(2n-1\), none of which, nor their negatives, by Proposition 1.1, is a lattice homomorphism.

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