ON MILNOR FIBRATIONS OF MIXED FUNCTIONS,
\(a_f\)-CONDITION AND BOUNDARY STABILITY

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Abstract

Convenient mixed functions with strongly non-degenerate Newton boundaries have a Milnor fibration ([9]), as the isolatedness of the singularity follows from the convenience. In this paper, we consider the Milnor fibration for non-convenient mixed functions. We also study geometric properties such as Thom’s \(a_f\)-condition, the transversality of the nearby fibers and stable boundary property of the Milnor fibration and their relations.

1. Preliminary

Let \(f(z, z)\) be a mixed function and write it as sum of real and imaginary part: \(f = g + ih\). Writing \(z = (z_1, \ldots, z_n)\) and \(z_j = x_j + iy_j\) \((j = 1, \ldots, n)\) with \(x_j, y_j \in \mathbb{R}\), the mixed hypersurface \(\{f = 0\}\) can be understood as the real analytic variety in \(\mathbb{R}^{2n}\) defined by \(\{g = h = 0\}\). The real and imaginary part \(g, h\) are also (real-valued) mixed functions and we also consider them as real analytic functions of variables \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\). By abuse of notations we use both notations \(g(z, z)\) and \(g(x, y)\) etc. We recall some notations. The real gradient vector for a real-valued mixed function \(k(x, y)\) is defined as

\[
\text{grad} \; k = (\text{grad}_x \; k, \; \text{grad}_y \; k) \in \mathbb{R}^{2n}
\]

\[
\text{grad}_x \; k = (k_{x_1}, \ldots, k_{x_n}), \quad \text{grad}_y \; k = (k_{y_1}, \ldots, k_{y_n}).
\]

Here \(k_{x_i}, k_{y_j}\) are respective partial derivatives. \(\mathbb{C}^n\) and \(\mathbb{R}^{2n}\) are identified by \(z \leftrightarrow z_{\mathbb{R}} = (x, y)\). Under this identification, the Euclidean inner product in \(\mathbb{R}^{2n}\) (denoted as \((*,*)_\mathbb{R}\)) and the hermitian inner product in \(\mathbb{C}^n\) (denoted as \((*,*)\)) are related as \((z_{\mathbb{R}}, z'_{\mathbb{R}})_\mathbb{R} = \Re(z, z')\). For a mixed function \(k\) (not necessarily real-valued), we define also holomorphic and anti-holomorphic gradients as
\[
\begin{align*}
\text{grad}_\partial k & = \left( \frac{\partial k}{\partial z_1}, \ldots, \frac{\partial k}{\partial z_n} \right), \\
\text{grad}_{\bar{\partial}} k & = \left( \frac{\partial k}{\partial \bar{z}_1}, \ldots, \frac{\partial k}{\partial \bar{z}_n} \right).
\end{align*}
\]

For simplicity of notations, we use the following notations:
\[
\begin{align*}
dk & := \text{grad} k, \quad d_x k := \text{grad}_x k, \quad d_y k := \text{grad}_y k, \\
q_k & := \text{grad} q_k, \quad q_{\bar{k}} := \text{grad}_{\bar{k}} k.
\end{align*}
\]

Note that if \( k \) is real-valued, \( q_k = q_{\bar{k}} \).

1.0.1. Tangent spaces. Let \( k(z, z) \) is a real valued mixed function. Then the tangent space of a regular point \( a \) of \( V_h := k^{-1}(\eta) \), \( \eta \in \mathbb{R} \) is described as follows. For a complex vector \( a \in \mathbb{C}^n \), we denote the corresponding real vector as \( a \in \mathbb{R}^{2n} \).

\[
T_a V_h = \{ v_R \in \mathbb{R}^{2n} \mid (v_R, dk(a_R)) = 0 \} = \{ v \in \mathbb{C}^n \mid \Re(v, \overline{k}(a)) = 0 \}.
\]

Consider the mixed hypersurface \( V_h = f^{-1}(\eta), \eta \neq 0 \). We introduce two vectors in \( \mathbb{C}^n \) which are more convenient to describe the Milnor fibration of the first type:
\[
\begin{align*}
v_1 & := \partial \log f(z, z) + \partial \log f(z, \bar{z}), \\
v_2 & := i(\overline{\partial \log f(z, z)} - \overline{\partial \log f(z, \bar{z})}).
\end{align*}
\]

These vectors describe the respective tangent spaces at a regular point \( a \) of the real codimension 1 varieties
\[
\begin{align*}
V_1 & := \{ z \mid |f(z, \bar{z})| = |f(a, \bar{a})| \}, \\
V_2 & := \{ z \mid \arg f(z, \bar{z}) = \arg \eta \}.
\end{align*}
\]

Namely, we have shown (Lemma 30, Observation 32, [9])
\[
\begin{align*}
T_a V_1 & := \{ v \mid \Re(v, v_1(a)) = 0 \} \\
T_a V_2 & := \{ v \mid \Re(v, v_2(a)) = 0 \}.
\end{align*}
\]

Note that \( V_h = V_1 \cap V_2 \). Observe that the two subspaces of dimension two
\[
\langle \partial g(a, \bar{a}), \partial h(a, \bar{a}) \rangle_R, \quad \langle v_1(a), v_2(a) \rangle_R
\]
are equal. In fact we have:
\[ \mathbf{v}_1 = \frac{\bar{\partial} f}{f} + \frac{\partial f}{f} = \frac{1}{|f|^2} (f(\bar{c}g - i\bar{c}h) + \bar{f}(\bar{c}g + i\bar{c}h)) = \frac{1}{|f|^2} (2g\bar{c}g + 2h\bar{c}h) \]

\[ \mathbf{v}_2 = i \frac{\bar{\partial} f}{f} + \frac{\partial f}{f} = \frac{i}{|f|^2} (f(\bar{c}g - i\bar{c}h) + \bar{f}(\bar{c}g + i\bar{c}h)) = \frac{1}{|f|^2} (-2h\bar{c}g - 2g\bar{c}h) \]

**Proposition 1** ([8]). Put \( f = g + hi \) as before. The next conditions are equivalent.

1. \( a \in C^n \) is a critical point of the mapping \( f : C^n \to C \).
2. \( dg(a_R), dh(a_R) \) are linearly dependent over \( R \).
3. \( \bar{c}g(a, \bar{a}), \bar{c}h(a, \bar{a}) \) are linearly dependent over \( R \).
4. There exists a complex number \( \alpha \) with \( |\alpha| = 1 \) such that \( \alpha \bar{f}(a, \bar{a}) = \bar{f}(a, \bar{a}) \).

Under the above equivalent conditions, we say that \( a \) is a mixed singular point of the mixed hypersurface \( f^{-1}(f(a)) \).

**Lemma 2** (cf [3]). Put \( \mathbf{V} = f^{-1}(\eta) \) and take \( \mathbf{p} \in S \cap \mathbf{V} \). Assume that \( \mathbf{p} \) is a non-singular point of \( \mathbf{V} \) and let \( k(\mathbf{z}, \overline{\mathbf{z}}) \) be a real valued mixed function. The following conditions are equivalent.

1. The restriction \( k|\mathbf{V} \eta \) has a critical point at \( \mathbf{p} \in \mathbf{V} \eta \).
2. There exists a complex number \( \alpha \in C^+ \) such that \( \bar{\partial}k(\mathbf{p}) = \bar{\alpha}f(\mathbf{p}, \overline{\mathbf{p}}) + \bar{\alpha}f(\mathbf{p}, \overline{\mathbf{p}}) \).
3. There exist real numbers \( c, d \) such that \( \bar{\partial}k(\mathbf{p}) = c\bar{g}(\mathbf{p}, \overline{\mathbf{p}}) + d\bar{h}(\mathbf{p}, \overline{\mathbf{p}}) \).
4. There exist real numbers \( c', d' \) such that \( \bar{\partial}k(\mathbf{p}) = c'v_1(\mathbf{p}, \overline{\mathbf{p}}) + d'v_2(\mathbf{p}, \overline{\mathbf{p}}) \).

**Proof.** As \( \mathbf{p} \in \mathbf{V} \) is assumed a non-singular point, (1) and (3) are equivalent. We show the implication (3) \( \Rightarrow \) (2). Assume \( \bar{\partial}k(\mathbf{p}) = c\bar{g}(\mathbf{p}, \overline{\mathbf{p}}) + d\bar{h}(\mathbf{p}, \overline{\mathbf{p}}), \quad \exists c, d \in R \).

We use the equality:

\( \bar{\partial}g(\mathbf{p}, \overline{\mathbf{p}}) = \frac{\bar{\partial}f(\mathbf{p}, \overline{\mathbf{p}}) + \partial f(\mathbf{p}, \overline{\mathbf{p}})}{2} \),

\( \bar{\partial}h(\mathbf{p}, \overline{\mathbf{p}}) = -\frac{i(\bar{\partial}f(\mathbf{p}, \overline{\mathbf{p}}) - \partial f(\mathbf{p}, \overline{\mathbf{p}}))}{2} \)

to obtain the equality:

\( \bar{\partial}k(\mathbf{p}) = \frac{c - di}{2} \bar{\partial}f(\mathbf{p}, \overline{\mathbf{p}}) + \frac{c + di}{2} \partial f(\mathbf{p}, \overline{\mathbf{p}}) \).
The implication $2 \Rightarrow 3$ can be shown similarly, using the equality
\begin{equation}
\partial f = \partial g + i \partial h, \quad \bar{\partial} f = \bar{\partial} g + i \bar{\partial} h.
\end{equation}

1.0.2. Newton boundary and strong non-degeneracy condition. Let $f(z, \bar{z}) = \sum v_{\mu} e_{\mu} z^\mu$ be a mixed polynomial. The Newton polygon $\Gamma_+(f)$ is defined by the convex hull of $\bigcup (v + \mu + R^+) \cap \mathbb{C}$, where the sum is taken for $v, \mu$ with $c_{\nu} \neq 0$. Newton boundary $\Gamma (f)$ is the union of compact faces of $\Gamma_+(f)$ as usual. $f$ is called convenient if for any $i = 1, \ldots, n$, $\Gamma(f)$ intersects with $z_i$-axis.

For any non-negative weight vector $P$, it defines a linear function $\ell_P$ on $\Gamma_+(f)$ by $\ell_P(\zeta) = p_1 \zeta_1 + \cdots + p_n \zeta_n$ where $P = (p_1, \ldots, p_n)$, $\zeta = (\zeta_1, \ldots, \zeta_n) \in \Gamma_+(f)$ and the minimal value is denoted as $d(P)$ and the face where this minimal value is taken is denoted by $\Delta(P)$. In other word, $\Delta(P) := \{ \zeta \in \Gamma_+(f) \mid \sum_{i=1}^n p_i \zeta_i = d(P) \}$. The face function associated by $P$ is defined as $f_P := f_{\Delta(P)}$. For any coordinate subspace $C_I$, we denote the restriction $f|_{C_I}$ as $f_I$ as usual. Note that if $P$ is strictly positive (i.e., $p_i > 0$, for any $i = 1, \ldots, n$), $\Delta(P)$ is a face of $\Gamma(f)$.

To treat the case of non-convenient functions, we define the modified Newton boundary $\Gamma_{nc}(f)$ by adding essential non-compact faces $\Xi$. Here $\Xi$ is called an essential non-compact face if there exists a semi-positive weight vector $P = (p_1, \ldots, p_n)$ such that

1. $\Delta(P) = \Xi$ with $\Xi$ being a non-compact face and $f^{I(P)} \equiv 0$ where $I(P) = \{ i \mid p_i = 0 \}$ and
2. for any $i \in I(P)$ and any point $v \in \Xi$, the half line starting from $v$, $v + R, E_i$ is contained in $\Xi$. Here $E_i$ is the unit vector in the direction of $i$-th coordinate axis.

The weight vector $P$ may not unique but $I(P)$ does not depend on $P$. Thus we denote it as $I(\Xi)$ and it is called the non-compact direction of $\Xi$. See

![Figure 1. Non-compact face](image-url)
Figure 1 which shows the modified Newton boundary of \( f = z_1^3 + z_2^3 + z_2z_3^2 \) in Example 3.

\( f \) is called strongly non-degenerate if (1) for any compact face \( \Delta \in \Gamma_{nc}(f) \), the face function \( f_{\Delta} := \sum_{v+\mu \in \Delta} c_v z^v \bar{z}^\mu \) has no critical point as a function \( f_{\Delta} : \mathbb{C}^{nm} \to \mathbb{C} \) and (2) for a non-compact face \( \Delta \in \Gamma_{nc}(f) \), \( f_{\Delta} : \mathbb{C}^{nm} \to \mathbb{C} \) has no critical point where \( \Delta_0 = \Delta \cap \Gamma(f) \).

Example 3. Consider a holomorphic function \( f = z_1^3 + z_2^3 + z_2z_3^2 \) of three variables. Note that \( \Gamma_{nc}(f) \) has three vertices \( A = (3,0,0) \), \( B = (0,3,0) \), \( C = (0,1,2) \) and the face \( \Delta := \{ AC + RE_3 \} = \Gamma_{nc}(f) \) where \( AC \) is the edge with endpoints \( A, C \). The non-compact faces with edge \( AB \) and \( BC \) are not essential. They are not vanishing coordinates i.e., \( f \) does not vanish on \( \{z_1 = z_3 = 0\} \) or \( \{z_2 = z_3 = 0\} \). See Figure 1.

2. Milnor fibration

Asume that \( f(z, \bar{z}) = \sum_{v,\mu} c_v z^v \bar{z}^\mu \) is a strongly non-degenerate mixed polynomial and let \( V = f^{-1}(0) \). In this section, we study the Milnor fibration of \( f \). If \( f(z, \bar{z}) \) has a convenient Newton boundary, the singularity is isolated and there exists a spherical Milnor fibration (= a Milnor fibration of the first type):

\[
\frac{f}{|f|} : S_r \to S^1, \quad K = V \cap S_r
\]

and also a tubular Milnor fibration (= a Milnor fibration of the second type):

\[
f : \partial E(r, \delta)^* \to S^1_3, \quad \partial E(r, \delta)^* = \{ z \in B_r | |f(z, \bar{z})| = \delta \}
\]

for sufficiently small \( r, \delta \) such that \( 0 < \delta < r \). They are \( C^\infty \)-equivalent (Theorems 19, 33, 37, [9]).

For non-convenient mixed function, the singularity need not be isolated. We have proved the same assertion under an extra condition “super strongly non-degenerate” (Theorem 52, [9]). In this paper, we prove the existence of Milnor fibrations for any strongly non-degenerate functions with a weaker assumption than the assumption “super”. We will study also some geometric properties behind the argument.

2.1. Smoothness of the nearby fibers. First we recall the following:

Lemma 4 (Lemma 28, [9]). Assume that \( f(z, \bar{z}) \) is a strongly non-degenerate mixed function. Then there exists a positive number \( r_0 \) and \( \delta \) such that the fiber \( V_\eta := f^{-1}(\eta) \) has no mixed singularity in the ball \( B_{r_0}^{2n} \) for any non-zero \( \eta \) with \( |\eta| \leq \delta \).

Proof. Though the proof is the same as that in [9], we repeat it for the beginner’s convenience. We show a contradiction, assuming that the assertion does not hold. Then using the Curve Selection Lemma ([6, 4]), we can find an analytic path \( z(t) \), \( 0 \leq t \leq 1 \) such that \( z(0) = O \) and \( f(z(t), \bar{z}(t)) \neq 0 \) and \( z(t) \).
is a critical point of the function \( f : \mathbb{C}^n \rightarrow \mathbb{C} \) for any \( t \neq 0 \). Using Proposition 1, we can find a real analytic family \( \lambda(t) \) in \( S^1 \subset \mathbb{C} \) such that

\[
(7) \quad \overline{\partial f}(z(t), \bar{z}(t)) = \lambda(t) \overline{\partial f}(z(t), \bar{z}(t)).
\]

Put \( I = \{ j \mid z_j(t) \neq 0 \} \). We may assume for simplicity that \( I = \{ 1, \ldots, m \} \) and we consider the restriction \( f^I = f \mid \mathbb{C}^I \). As \( f(z(t), \bar{z}(t)) = f^I(z(t), \bar{z}(t)) \neq 0 \), we see that \( f^I \neq 0 \). Consider the Taylor expansions of \( z(t) \) and \( \lambda(t) \):

\[
z_i(t) = b_i t^{a_i} + \text{(higher terms)}, \quad b_i \neq 0, \ a_i > 0, \ i = 1, \ldots, m
\]

\[
\lambda(t) = \lambda_0 + \text{(higher terms)}, \quad \lambda_0 \in S^1 \subset \mathbb{C}.
\]

Consider the weight vector \( A = \langle a_1, \ldots, a_m \rangle \) and a point in the torus \( b = (b_1, \ldots, b_m) \in \mathbb{C}^m \) and we consider the face function \( f^I_A \) of \( f^I(z, \bar{z}) \). Then we have for \( j \in I \)

\[
\frac{\partial f^I}{\partial z_j}(z(t), \bar{z}(t)) = \frac{\partial f^I}{\partial z_j}(b, \bar{b}) t^{d-a_j} + \text{(higher terms)},
\]

\[
\frac{\partial f^I}{\partial \bar{z}_j}(z(t), \bar{z}(t)) = \frac{\partial f^I}{\partial \bar{z}_j}(b, \bar{b}) t^{d-a_j} + \text{(higher terms)}
\]

where \( d = d(A; f^I) \). The equality (7) says that

\[
\overline{\partial f^I}(z(t), \bar{z}(t)) = \lambda(t) \overline{\partial f^I}(z(t), \bar{z}(t)), \quad j = 1, \ldots, m.
\]

which implies the next equality:

\[
\text{ord}_z \overline{\partial f^I}(z(t), \bar{z}(t)) = \text{ord}_z \frac{\partial f^I}{\partial z_j}(z(t), \bar{z}(t)), \quad j = 1, \ldots, m.
\]

Thus we get the equality:

\[
\overline{\partial f^I_A}(b, \bar{b}) = \lambda_0 \overline{\partial f^I_A}(b, \bar{b}), \quad b \in \mathbb{C}^m.
\]

This implies that \( b \) is a critical point of \( f^I_A : \mathbb{C}^m \rightarrow \mathbb{C} \), which is a contradiction to the strong non-degeneracy of \( f^I_A(z, \bar{z}) \).

### 2.2. Vanishing coordinate subspaces and essentially non-compact face functions.

We assume that \( f \) is a mixed polynomial (not only mixed analytic function). We denote by \( \mathcal{A}_m(f) \) the set of subset \( I \subset \{ 1, 2, \ldots, n \} \) such that \( f^I \neq 0 \) (we denoted this set as \( \mathcal{N} \mathcal{V} \) in [9]). We denote by \( \mathcal{H}(f) \) the subset of subset \( I \subset \{ 1, 2, \ldots, n \} \) such that \( f^I \equiv 0 \), and for \( I \in \mathcal{H}(f) \) and we consider also the set of non-compact faces \( \Delta \in \Gamma_m(f) \) such that there exists (possibly not unique) a non-negative weight \( P \) such that \( \Delta(P) = \Delta \) and \( I(P) = I \). Here \( I(P) = \{ i \mid p_i = 0 \} \). \( C^I \) is called a vanishing coordinate subspace. Note that \( C^I \subset V. \)
Definition 5. Let \( \pi_I : \mathbb{C}^n \to \mathbb{C}^I \) be the projection and put \( z_I = \pi_I(z) \). Take an essential non-compact face \( \Delta \in \Gamma_{nc}(f) \). Take a weight function \( P \) such that \( f_P = f_\Delta \) and \( I(P) = I(\Delta) \). We consider the function \( \rho_\Delta(z) := \|z_{I(\Delta)}\|^2 = \sum_{j \in I(\Delta)} |z_j|^2 \). An essential non-compact face function \( f_\Delta \) is \textit{locally tame} if there exists a positive number \( r_\Delta > 0 \) such that for any fixed \( z_{I(\Delta)} = a_{I(\Delta)} \in \mathbb{C}^{I(\Delta)} \) with \( \rho_\Delta(z) \leq r_\Delta^2 \), \( f_\Delta \) has no critical points in \( a_{I(\Delta)} \times \mathbb{C}^{I(\Delta)^I} \) as a mixed polynomial function of \( n - |I(\Delta)| \)-variables \( \{z_k \mid k \notin I(\Delta)\} \). We say that \( f \) is \textit{locally tame on the vanishing coordinate subspace} \( \mathbb{C}^I \) if any face function \( f_\Delta \) with \( I(\Delta) = I \) is locally tame.

Put \( r_I = \min\{r_\Delta \mid I(\Delta) = I\} \) for \( I \in \mathcal{I}(f) \) and \( r_{nc} = \min\{r_I \mid I \in \mathcal{I}(f)\} \). If \( f \) is convenient, \( r_{nc} = +\infty \).

Remark 6. We say that \( f \) is \textit{super strongly non-degenerate} if we can take \( r_\Delta = \infty \) in the above definition (\cite{9}).

2.3. Smoothness on the non-vanishing coordinate subspaces. Take \( I \subset \{1, \ldots, n\} \) and \( \mathbb{C}^I \) is called a non-vanishing coordinate subspace if \( f^I \neq 0 \). Put \( V^\# = \bigcup_{f \in \mathcal{P}(f)} V \cap \mathbb{C}^I \). Then there exists a \( r_0 > 0 \) so that \( V^\# \) and \( V^\# = V \cap \mathbb{C}^I \) are non-singular in the ball \( B_{r_0} \) and for any \( 0 < r \leq r_0 \), the sphere \( S_r \) and \( V^\# \) intersect transversely. The existence of such \( r_0 \) is shown in Theorem 16, \cite{9}.

2.4. Hamm-Lê type theorem. The following is a mixed function version of Lemma (2.1.4) (Hamm-Lê, \cite{5}). This enables us to prove the existence of Milnor fibration with locally tame behavior assumption.

Lemma 7. Assume that \( f(z, z) \) is a strongly non-degenerate mixed polynomial which behaves locally tamely along vanishing coordinate subspaces. Put \( p_0 = \min\{r_{nc}, r_0\} \) where \( r_{nc} \) and \( r_0 \) are described above. For any fixed positive number \( r_1 \leq p_0 \), there exists positive numbers \( \delta(r_1) \) (depending on \( r_1 \)) such that for any non-zero \( \eta, |\eta| \leq \delta(r_1) \),

1. the nearby fiber \( V_\eta := f^{-1}(\eta) \) has no mixed singularity in the ball \( B_{p_0}^{2n} \) and,
2. for any \( r, r_1 \leq r \leq p_0 \), the sphere \( S_r \) and the nearby fiber \( V_\eta = f^{-1}(\eta) \) intersect transversely.

Proof. We have already proved the assertion (1) (Lemma 4). So we will prove the assertion (2). Assume that the assertion is false. By the Curve Selection Lemma, we can find a real analytic curve \( z(t) \) and a complex valued function \( \alpha(t) \), \( 0 \leq t \leq 1 \)

\[
(8) \quad z_j(t) = \alpha(t) \frac{\partial f}{\partial z_j}(z(t)) + \beta(t) \frac{\partial f}{\partial \overline{z}_j}(z(t)), \quad \forall j
\]
where $z(t)$, $z(t)$ are expanded as
\[
z_j(t) = b_j t^{p_j} + \text{(higher terms)}, \quad b_j \neq 0 \text{ if } z_j(t) \neq 0,
\]
\[
z(t) = z_0 t^m + \text{(higher terms)}, \quad z_0 \neq 0.
\]
and $f(z(t)) \neq 0$ for $t \neq 0$. Obviously $z(t) \neq 0$.

Put $K = \{ i \mid z_j(t) \neq 0 \}$ and we consider the equality in $C^K$. Put $b = (b_j)$ and $P = (p_j)$, $I = \{ j \in K \mid p_j = 0 \}$, $I_1 = K - I$ and $\Delta = \Delta(P)$. In the following, we assume $K = \{ 1, \ldots, n \}$ as the argument is the same.

**Case 1.** Assume that $I \in \mathcal{A}(f)$. Then $f^I \neq 0$ and $b \in V^#$. We assumed that $V^#$ and $S\|b\|$ intersect transversely for any $b$, $\|b\| \leq p_0$ and thus $S\|z(t)\|$ is also transverse to $V_{f(z(t))}$ at $z(t)$ for a small $t \ll 1$, which is a contradiction.

**Case 2.** Assume that $I \in \mathcal{A}(f)$ and so $f^I \equiv 0$. In this case, $\Delta \in \Gamma_{nc}(f)$. The above equality (8) says:
\[
(9) \quad b_j t^{p_j} + \text{(higher terms)} = \left( z_0 \frac{\partial f_{\Delta(P)}}{\partial z_j} (b) t^{m + d(P) - p_j} + \text{(higher terms)} \right)
\]
\[
+ \left( z_0 \frac{\partial f_{\Delta(P)}}{\partial z_j} (b) t^{m + d(P) - p_j} + \text{(higher terms)} \right), \quad j \in K.
\]

We compare the order in $t$ (= the lowest degree) of the both side. The left side has order 0 and the order of the right side is at least $d(P) + m$ for $j \notin I$ and at least $d(P) + m$ for $j \in I$. Note that $b \in C^n$. If $d(P) + m > 0$, we get a contradiction $b_j = 0$ for $j \in I$. If $d(P) + m < 0$, we get
\[
0 = z_0 \frac{\partial f_{\Delta(P)}}{\partial z_j} (b) + z_0 \frac{\partial f_{\Delta(P)}}{\partial z_j} (b), \quad \forall j
\]
which says $b$ is a mixed critical point of $f_{\Delta}$, a contradiction to the strong non-degeneracy. Thus $d(P) + m = 0$ and
\[
(10) \quad b_j = z_0 \frac{\partial f_{\Delta(P)}}{\partial z_j} (b) + z_0 \frac{\partial f_{\Delta(P)}}{\partial z_j} (b), \quad j \in I
\]
\[
(11) \quad 0 = z_0 \frac{\partial f_{\Delta(P)}}{\partial z_j} (b) + z_0 \frac{\partial f_{\Delta(P)}}{\partial z_j} (b), \quad j \in K - I.
\]
The equality (11) says that the point $(b_j)_{j \in K - I}$ is a critical point of the face function $f_{\Delta}$ as a function of varianles $\{ z_j, j \in K - I \}$, fixing $z_i = b_i$, $i \in I$ with $p_{\Delta(b)} \leq p_0$. This is a contradiction on the assumption. \hfill \square

**Remark 8.** The assertion (2) also follows from $a_f$-condition (see Proposition 11 below.)
2.5. Tubular Milnor fibration. Put

\[ \begin{align*} D(\delta_0)^* &= \{ \eta \in \mathbb{C} \mid 0 < |\eta| \leq \delta_0 \}, \quad S_{10}^1 = \partial D(\delta_0)^* = \{ \eta \in \mathbb{C} \mid |\eta| = \delta_0 \} \\
E(r, \delta_0)^* &= f^{-1}(D(\delta_0)^*) \cap B_r^{2n}, \quad \partial E(r, \delta_0)^* = f^{-1}(S_{10}^1) \cap B_r^{2n}. \end{align*} \]

By Lemma 4 and the theorem of Ehresman ([16]), we obtain the following description of the tubular Milnor fibration (i.e., the Milnor fibration of the second type) ([5]).

**Theorem 9 (Tubular Milnor fibration).** Assume that \( f(z, \mathbf{z}) \) is a strongly non-degenerate mixed function which is locally tame along the vanishing coordinate subspaces. Take positive numbers \( r \leq \rho_0 \) and \( \delta_0 \leq \delta(r) \) as in Lemma 7. Then \( f : E(r, \delta_0)^* \to D(\delta_0)^* \) and \( f : \partial E(r, \delta_0)^* \to S_{10}^1 \) are locally trivial fibrations and the topological isomorphism class does not depend on the choice of \( \delta_0 \) and \( r \).

2.6. Spherical Milnor fibration. Consider the spherical Milnor fibration (i.e., Milnor fibration of the first kind):

\[ f / |f| : S_r - K \to S^1, \quad K = V \cap S_r. \]

In the proof of the existence of the spherical fibration and the equivalence to the tubular Milnor fibration (Theorem 52, [9]), we have assumed “super strongly non-degeneracy”. However this assumption is used only to prove the Hamm-Lê type assertion (Lemma 51, [9]). We have proved this Lemma with locally tameness assumption (Lemma 7). Thus we get

**Theorem 10.** Assume that \( f \) is a strongly non-degenerate mixed function which is locally tame along vanishing coordinate subspaces. For a sufficiently small \( r \), the spherical and tubular Milnor fibrations exist and they are equivalent each other.

3. Boundary stability, \( a_f \)-condition and transversality of the nearby fibers

In this section, we consider further geometric properties about mixed polynomials.

3.1. \( a_f \)-condition. Assume that \( f \) is a mixed polynomial and we assume that a Whitney regular stratification \( \mathcal{S} \) of \( \mathbb{C}^n \) is given so that \( V = f^{-1}(0) \) is a union of strata \( M \subset V \). We says that \( f \) satisfies Thom’s \( a_f \)-condition with respect to \( \mathcal{S} \) (locally at 0) if there exist positive numbers \( r \) and \( \delta \ll r \) which satisfies the following condition. \( V_\eta = f^{-1}(\eta) \) with \( \eta \neq 0, |\eta| \leq \delta \) is smooth in \( B_r \) and take any sequence \( z^{(n)} \) which converges to some \( w \neq 0, w \in M \), where \( M \) is a stratum in \( V \cap \mathcal{S} \) and suppose that the tangent space \( T_{z^{(n)}} f^{-1}(f(z^{(n)})) \) converges to some \( \tau \) in the suitable Grassmanian space. Then \( T_w M \) is a subspace of \( \tau \).
The following says that the nearby fiber’s transversality follows from $a_f$-condition.

**Proposition 11.** Assume that $f$ satisfies a $a_f$-condition at 0 and the nearby fibers are smooth. Then there exists a $r_0 > 0$ such that for any $0 < r_1 < r_0$, there exists a positive $\delta$ so that any nearby fiber $V_\eta$ intersects transversely with the sphere $S_r$ for $r_1 < r < r_0$ and $0 < |\eta| < \delta$.

**Proof.** Take $r_0$ so that for any $r > r_0$, the sphere $S_r$ intersects transversely with all strata $M \in V$. Note that $M$ and $S_r$ intersect transversely if and only if for any $a \in M \cap S_r$, $T_a M$ and $T_a S_r$ intersect transversely. That is $T_a M \neq T_a S_r$. Take a sequence of points $z^{(n)}$ converging to $a$. Assume $T_{z^{(n)}} f^{-1}(\eta_n)$ converges to $\tau$. Then $\tau \in T_a S_{r'}$. On the other hand, $a_f$-condition says that $T_a M \neq T_a S_{r'}$. This is a contradiction. □

### 3.2. Boundary stability condition.

Assume that $r_0 > 0$ is chosen so that $\varphi = f/|f| : S_r \setminus K \to S^1$ is a fibration for any $r < r_0$. We wish to consider the boundary condition $F_{\theta} \supseteq K$ is satisfied or not. This property is always true for holomorphic functions but not always true for mixed functions. For the argument’s simplicity, we consider as follows. Consider the Milnor fibration in an open ball:

$$\varphi_{\leq r} = f/|f| : B_r \setminus V \to S^1, \quad \varphi_{\leq r}(z) = f(z)/|f(z)|$$

and put $F_{\theta, \leq r} = \varphi_{\leq r}^{-1}(e^{i\theta})$. To distinguish this fibration with usual Milnor fibration on a sphere, we call this fibration an open ball Milnor fibration.

**Definition 12.** We say the open Milnor fibration satisfies the stable boundary condition if $F_{\theta, \leq r} \supseteq V \cap B_r$ for any $\theta$. Note that the Milnor fibration in a ball is homotopically equivalent to the one on a fixed sphere $f/|f| : S_r \setminus K \to S^1$.

Recall that a continuous mapping $\varphi : X \to Y$ is an open mapping along a subset $A \subset X$ if for any point $a \in A$ and any open neighborhood $U$ of $a$ in $X$, $\varphi(U)$ is a neighborhood of $\varphi(a)$ in $Y$. The following is an immediate consequence of the definition.

**Proposition 13.** The next two conditions are equivalent.

1. The boundary stability condition for the Milnor fibration of $f$ is satisfied.
2. $f : \mathbb{C}^n \to \mathbb{C}$ is an open mapping along $V \cap B_r$ for a sufficiently small $r > 0$. In particular, if $f$ is a holomorphic function, it satisfies the boundary stability condition.
Lemma 14. Assume that \( f(z, Z) \) is a strongly non-degenerate and locally tame along vanishing coordinate subspaces. Then the Milnor fibration satisfies the stable boundary property.

Proof. Take a point \( a = (a_1, \ldots, a_n) \in V \cap \text{Int}(B_r) \) and put \( I = \{ i | a_i \neq 0 \} \).

(i) Assume that \( I \in \mathcal{H}_n(f) \) so that \( a \) is a non-singular point of \( V^*I \). Then it is obvious that \( a \in F_0, \) as \( \{ V_\eta \}, |\eta| \leq \delta < r \) is a transversal family with the spheres \( S_{r''} \) for \( ||a||/2 \leq r'' \leq r \) and \( V_\eta \subset F_{0, \leq r} \) for \( \eta, \arg \eta = \theta \).

(ii) Assume that \( f^I \equiv 0 \). Take an essential non-compact face \( \Delta = \Delta(P) \) with \( I(\Delta) = I \) and consider the face function \( f_P(z, Z) \). Put \( f_{P,a_i} \) be the restriction of \( f_P \) on \( z_i = a_i, \ i \in I \). Thus we consider the polynomial mapping \( f_{P,a_i} : C^{n-I} \rightarrow C. \) As \( f_{P,a_i} \) is a strongly non-degenerate function for sufficiently small \( a_i, \) there exists \( b = (b_j) \not\in I \) such that \( f_{P,a_i}(b) = pe^{i\theta} \) for some \( p. \) Take an arc \( b(s), -\varepsilon \leq s \leq \varepsilon \) so that \( f_{P,a_i}(b(s)) = pe^{i(\theta+s)} \) and \( b(0) = b. \) This is possible as \( f_{P,a_i} : C^{n-I} \rightarrow C \) is a submersion. Consider the path:

\[
(t, s) \mapsto b(t, s) = (b_j(t, s))_j = 1_n, \quad b_j(t, s) = \begin{cases} b_j(t) & j \not\in I \\ a_j & j \in I. \end{cases}
\]

Then we have

\[
f(b(t, s)) = f_{P,a_i}(b(s))t^{d(P)} + \text{(higher terms)}
\]

\[
= pe^{i(\theta+s)}t^{d(P)} + \text{(higher terms)}.\n\]

Take a sequence \( t_v \rightarrow 0. \) As the arg \( f(b(t_v, s)) \rightarrow \theta + s, \) we can take a sequence \( s_v, -\varepsilon \leq s_v \leq \varepsilon \) such that \( \arg f(b(t_v, s_v)) = \theta \) for sufficiently small \( |t_v|. \) For example, assume that arg \( f(b(t, 0)) < \theta. \) Note that arg \( f(b(t, e)) > \theta \) as long as \( t \ll 1. \) Thus we use the mean value theorem to chose such a \( s_v. \) The point \( b(t_v, s_v) \in F_{0, \leq r} \) for sufficiently small \( |t_v| \) and it converges to \( a. \) This implies that the closure of \( F_{0, \leq r} \) contains \( V. \) \( \square \)

3.3. Strongly non-degenerate polynomials which is not locally tame. (1) Example 1. Consider the example of M. Tibar: \( f(z) = z_1|z_2|^2 \) \([13, 1, 2]\). This is a mixed weighted homogeneous polynomial. Thus it is strongly non-degenerate. A polar weight can be \( P = \{1, 0\}. \) \( S^1 \)-action is defined as \( \rho \circ (z_1, z_2) = (z_1\rho, z_2) \) for \( \rho \in S^1. \) Then for any \( r > 0, \) there exists a spherical Milnor fibration:

\[
\varphi = f/[f] : S_r \setminus K \rightarrow S^1.
\]

First we show that the boundary stability is not satisfied. Take a fiber \( F_0. \) \( K \) has two components, \( K_1 = \{ z_1 = 0 \} \) and \( K_2 = \{ z_2 = 0 \}. \) The closure of \( F_0 \) is given as \( F_0 = F_\emptyset \cup K_1 \cup \{ (re^{i\theta}, 0) \}. \) Thus the intersection \( F_\emptyset \cap K_2 \) is a single point \( (re^{i\theta}, 0) \) and this point \( (re^{i\theta}, 0) \) turns along \( K_2 \) once when \( \theta \) goes from 0 to \( 2\pi. \) Note that \( K_2 \) is a \( S^1 \)-orbit of the action. We call \( K_2 \) a rotating axis. The function \( f \) is not locally tame along the vanishing axis \( z_2 = 0 \) by Lemma 14. In fact, take a point \( (a, 0) \in K_2 \) and put \( a = pe^{i\alpha}. \) Take an open set \( U = \{ z_1 | z_1 - a | < \varepsilon \} \times \{ z_2 | z_2 < \varepsilon \} \) and put \( \alpha \) to be the small positive angle so
that $\tan x = \epsilon / \rho$. Then the image of $U$ by $f$ is contained in the closure of the angular region $\{ \eta \in \mathbb{C} | \theta - x \leq \arg \eta \leq \theta + x \}$ where 0 is on the boundary. Thus it is not an open mapping. More precisely we assert

**Assertion 15.** $F_\theta$ is homeomorphic to $\text{Cone}(K_1)$.

For example, taking $r = 1$, consider the mapping $\psi : F_\theta \to \text{Cone}(K_1)$, defined by $\psi(z_1, z_2) = (1 - |z_1|, \arg(z_2))$. Here we understand

$$\text{Cone}(K_1) = [0, 1] \times K_2 / \{0\} \times K_2, \quad K_2 \simeq S^1.$$

M. Tibar observed that $f$ does not have any stratification which satisfies the $a_f$-condition along $z_1$ axis ([11]). Put $f = g + i\mathbf{h}$ with $g = x_1(x_2^2 + y_2^2)$ and $h = y_1(x_2^2 + y_2^2)$. Then the Jacobian matrix is given as

$$J(g, h) = \begin{pmatrix} x_2^2 + y_2^2 & 0 & 2x_1x_2 & 2x_1y_2 \\ 0 & x_2^2 + y_2^2 & 2y_1x_2 & 2y_1y_2 \end{pmatrix}.$$

Note that the last $2 \times 2$ minor has rank one and this makes the problem at the limit. Take a point $p = (a_1 + ib_1, 0)$. Consider the rotated mixed polynomial $\tilde{f} := (b_1 + a_1i)f$ and write it as $\tilde{f} = \tilde{g} + i\tilde{h}$. Note that $f^{-1}(f(p)) = \tilde{f}^{-1}(\tilde{f}(p))$ and $\tilde{g} = b_1g - a_1h$. Then the normalized gradient of $\tilde{g}$ is given by

$$\text{grad} \tilde{g} = (b_1, -a_1, 0, 0).$$

Put $p = (a_1 + b_1i, z_2)$. Thus when $z_2 \to 0$,

$$T_p f^{-1}(f(p)) \subset T_p \tilde{g}^{-1}(\tilde{g}(p)) \not\subset \mathbb{C} \times \{0\}.$$

This implies, if there is a stratification which satisfies $a_f$-condition, the stratum of $\mathbb{C} \times \{0\}$ which contains $p$ cannot be two dimensional at $p \in \{z_2 = 0\}$. As this is the case at any point of $\{z_2 = 0\}$, there does not exist any stratification which satisfies $a_f$-condition. On the other hand, we assert that

**Proposition 16.** $f$ satisfies the transversality condition for the nearby fibers.

**Proof.** We may assume that the sphere has radius 1, by the polar homogeneity. Assume that there is a sequence $p_v = (u_v, v_v) \in S^1_1$ such that $f^{-1}(f(p_v))$ is not transverse to $S^1_1$ and $f(p_v) \to 0$. Then either $u_v \to 0$ or $v_v \to 0$ (equivalently either $|v_v| \to 1$ or $|u_v| \to 1$). We may assume that $p_v = z_v e^{i\theta} + \overline{z}_v e^{i\theta}$ by Lemma 2 which is equivalent to

$$\begin{cases} u_v = z_v |v_v|^2 \\ v_v = z_v \overline{u}_v v_v + \overline{z}_v u_v v_v. \end{cases}$$

From the first equality, we can put $u_v = r_v e^{i\theta}$, $z_v = p_v e^{i\theta}$. The second equality says that $1 = 2p_v r_v$ as $v_v \neq 0$. Thus $p_v \to 1/2$ if $r_v \to 1$ which implies $|v_v| \to 2$ and $|f(p_v)| \not\to 0$. Assume that $r_v \to 0$. Then $|v_v| = r_v / p_v = 2r_v^2 \to 0$. This is also impossible, as $|p_v| = 1$. \qed
This example shows that the transversality of nearby fibers does not imply either tameness or $a_f$-condition. On the other hand, tameness with strong non-degeneracy implies transversality of the nearby fibers, as we will see below.

(2) Example of A. Parusinski: $f = z_1(z_2 + z_3^2)z_2$ ([11], see also [1, 2]). Note that $f$ is strongly non-degenerate.

**Proposition 17 (A. Parusinski).** Consider $I = \{1\}$ and note that $f | C^I \equiv 0$. Then $f$ does not satisfy $a_f$-condition along $z_1$-axis $\{z_2 = z_3 = 0\}$.

**Proof.** The proof goes in the same line as that in Example 1. Consider the weight $P = ((0, 1, 3)$. Then $f_P = z_1|z_2|^2$ and $d(P) = 2$. Assume that there exists a stratification $\mathcal{S}$ satisfying $a_f$-condition. We show a contradiction. Take a point $p = (re^{i\theta}, 0, 0)$ and assume that $p \in M$ where $M$ is a real two dimensional stratum of $C^I$. Consider the modified function $\tilde{f} = (\sin \theta + i \cos \theta)f$. Then the real part $\tilde{g}$ of $\tilde{f}$ is given as

$$\tilde{g} = \sin \theta g - \cos \theta h = (x_1 \sin \theta - y_1 \cos \theta)|z_2|^2 + \Re(e^{i(\pi/2-\theta)}z_1z_2^2)$$

and the gradient vector of $\tilde{g}$ at $z(t) := (p, ta_2, t^3a_3)$ for $a_2, a_3 \in C^*$ fixed is given as

$$\text{grad} \tilde{g}(p, ta_2, t^3a_3) = (\sin \theta, -\cos \theta, 0, 0, 0)|a_3|^2t^2 + O(t^3).$$

Thus the normalized gradient vector converges to

$$v := (\sin \theta, -\cos \theta, 0, 0, 0).$$

This implies that

$$T_{z(t)}f^{-1}(f(z(t))) = T_{z(t)}\tilde{g}^{-1}(\tilde{g}(z(t))) \xrightarrow{t \to 0} v \perp \neq C^I.$$ 

This is a contradiction. 

**Remark 18.** We do not know (and do not care) if $f^{-1}(\eta)$, $\eta \neq 0$ is a transverse family for sufficiently small $\eta$.

(3) Example 3. Consider

$$f(z, z) = z_1k(z), \quad k(z) := \sum_{i=1}^{m} |z_i|^{2a_i} - \sum_{j=m+1}^{n} |z_j|^{2a_j}$$

for $2 \leq m < n$. Then $f$ is not strongly non-degenerate but polar weighted homogeneous and it has a Milnor fibration. However it is not locally tame along the vanishing coordinate subspaces and $f$ does not satisfy the $a_f$-condition.
In fact the link has two components $K_1 = \{z_1 = 0\}$ and $K_2 = \{k(z) = 0\}$. The component $K_2$ has **real codimension 1** and at any point of $K_2 \setminus K_1$, $f$ is not open mapping and thus

$$F_\theta = F_\theta \cup K_1 \cup \{z \in S_1 | \arg z_1 = \pm \theta\}$$

where sign is the same as that of $k(z)$. Thus $K_2$ is a rotation axis. The monodromy is the rotation around $z_1$ axis:

$$h_\theta : F_\theta \to F_\theta, \quad (z_1, z') \mapsto (z_1 e^{i\theta}, z').$$

The fiber $F_\theta$ has two components, $F^+_\theta = \{\arg z_1 = \theta, k(z) > 0\}$ and $F^-_\theta = \{\arg z_1 = -\theta, k(z) < 0\}$.

**Remark 19.** The function $k(z)$ is a real valued polynomial and the fibers $k^{-1}(\eta)$ are smooth for $\eta \neq 0$ and $k^{-1}(0)$ has an isolated singularity as a real hypersurface. However as a mixed function $k : \mathbb{C}^n \to \mathbb{C}$, it has no regular points.

### 3.4. Thom’s $a_f$-condition.

By analyzing above examples, we notice that the limit of two independent hyperplanes $T_{p_0} g^{-1}(g(p))$ and $T_{p_0} h^{-1}(h(p))$ may not independent when $p$ goes to some point of vanishing coordinate $C^I$, and this phenomena induces a failure of $a_f$-condition. This problem does not occur under the tameness condition.

**Theorem 20.** Assume that $f(z)$ is a strongly non-degenerate polynomial and assume that $f$ is locally tame along vanishing coordinate subspaces. We consider the canonical stratification $S_{can}$ which is defined by

$$S_{can} : \{V \cap C^I, C^I \setminus V \cap C^I | I \in \mathcal{S}_m(f)\} \cup \{C^I | I \in \mathcal{J}(f)\}.$$ 

Then $f$ satisfies $a_f$-condition with respect to $S_{can}$ in the ball $B_{\rho_0}^{2n}$ where $\rho_0$ is as in Lemma 7.

**Proof.** Take a point $q^I = (q_j)_{j \in I} \in V \cap C^I$. Using Curve Selection Lemma, it is enough to check the $a_f$-condition along an arbitrary analytic path. So take any analytic path $z(t)$ such that $z(0) = q^I$ and $z(t) \in C^{I,J}$ for $t \neq 0$ with $I \subset J$ with $I \neq J$. As the argument is precisely the same, we assume hereafter that $J = \{1, \ldots, n\}$. We will show that $a_f$-condition is satisfied for this curve. By non-degeneracy, we may assume that $I \in \mathcal{J}(f)$ so that $C^I$ is a vanishing coordinate. (Otherwise, $q^I$ is a smooth point of $V$ and the $a_f$-condition is obviously satisfied.) Consider the Taylor expansion:

$$z_j(t) = a_j t^{p_j} + \text{(higher terms)}, \quad \begin{cases} p_j = 0, & a_j = q_j, \quad j \in I \\
 p_j > 0, & j \notin I. \end{cases}$$
Put $P = (p_1, \ldots, p_n)$, $a = (a_1, \ldots, a_n)$, $d = d(P)$ and $\Delta = \Delta(P)$. For notation’s simplicity, we assume that $I = \{m + 1, \ldots, n\}$. Note that

$$\frac{\partial g}{\partial z_j}(z(t)) = \frac{\partial g_{\Delta}}{\partial z_j}(a) t^{d-p_j} + \text{(higher terms)}$$

$$\frac{\partial h}{\partial z_j}(z(t)) = \frac{\partial h_{\Delta}}{\partial z_j}(a) t^{d-p_j} + \text{(higher terms)}.$$

For simplicity, we assume that $p_1 \geq p_2 \geq \cdots \geq p_m$. For a vector $v = (v_1, \ldots, v_n)$ and $1 \leq \alpha \leq \beta \leq m$, we consider the truncation

$$v_{\alpha} := (v_{\alpha}, \ldots, v_{\beta}).$$

We choose $1 \leq \alpha \leq \beta \leq m$ as follows.

(A-1) For any $j < \alpha$, $\frac{\partial}{\partial z_j} g_{\Delta}(a) = 0$, $\frac{\partial}{\partial z_j} h_{\Delta}(a) = 0$ and

$$\left(\frac{\partial}{\partial z_j} g_{\Delta}(a), \frac{\partial}{\partial z_j} h_{\Delta}(a)\right) \neq (0, 0).$$

(A-2) Two complex vectors

$$(\tilde{g}_{\Delta}(a))_{\alpha} = \left(\frac{\partial}{\partial z_j} g_{\Delta}(a), \ldots, \frac{\partial}{\partial z_{\beta}} g_{\Delta}(a)\right)$$

$$(\tilde{h}_{\Delta}(a))_{\alpha} = \left(\frac{\partial}{\partial z_j} h_{\Delta}(a), \ldots, \frac{\partial}{\partial z_{\beta}} h_{\Delta}(a)\right)$$

are linearly independent over $\mathbb{R}$ and $(\tilde{g}_{\Delta}(a))_{\alpha}^\beta$, $(\tilde{h}_{\Delta}(a))_{\alpha}^\beta$ are linearly dependent over $\mathbb{R}$ for any $\beta < \beta'$. For simplicity, we use the notations:

$$v_{g}(t) := \tilde{g}(z(t)) = (v_{g,1}, \ldots, v_{g,m}), \quad v_{h}(t) := \tilde{h}(z(t)) = (v_{h,1}, \ldots, v_{g,m}).$$

We consider the order of $v_{g}(t) = \tilde{g}(z(t))$ and $v_{h}(t) = \tilde{h}(z(t))$. (Here the order is the lowest degree of in $t$.)

Suppose $\text{ord } v_{g} = r$ and the smallest index $1 \leq i \leq m$ with $\text{ord } v_{g,i} = r$ is called the leading index. Assume that $s$ is the leading index of $v_{g}(t)$. We call the coefficient of $t^s$ in the expansion of $v_{g,s}(t)$ the leading coefficient. Put $s'$ be the leading index of $v_{h}$.

For simplicity, we assume that $s \leq s'$ and if $s = s'$ we assume also $\text{ord } v_{g}(t) \leq \text{ord } v_{h}(t)$. This is possible by changing $g$ and $h$ considering if $(z, \bar{z})$, if necessary.

First we observe that

$$\text{ord } v_{g,i}(t), \text{ord } v_{h,i}(t) \geq d - p_i, \quad s \leq \alpha.$$

(If $s > \alpha$, this means $\frac{\partial}{\partial z_j} g_{\Delta}(a) = 0$ and $\frac{\partial}{\partial z_j} h_{\Delta}(a) \neq 0$ which is a contradiction to the assumption $s \leq s'$.)
Strategy. Put \( r = \text{ord} \, v_g(t) \), \( r' = \text{ord} \, v_h(t) \). We have three possible cases.

(1) \( s' > s \) or
(2-a) \( s = s' \) and the coefficients of \( t' \) of \( v_{g,s} \) and the coefficient of \( t'^r \) of \( v_{h,s} \) are linearly dependent over \( \mathbb{R} \) or
(2-b) \( s = s' \) and the coefficients of \( t' \) of \( v_{g,s} \) and the coefficient of \( t'^r \) of \( v_{h,s} \) are linearly independent over \( \mathbb{R} \).

For (1) or (1-a), we have nothing to do. In fact, write
\[
v_g(t) = v_g^\infty t^r + \text{(higher terms)}, \quad v_g^\infty \in \mathbb{C}^n
\]
\[
v_h(t) = v_h^\infty t'^r + \text{(higher terms)}, \quad v_h^\infty \in \mathbb{C}^n.
\]
Then the normalized limit of \( v_g(t) \), \( v_h(t) \) are given by \( v_g^\infty /\|v_g^\infty\| \), \( v_h^\infty /\|v_h^\infty\| \). In this case, the limit of \( v_g \) and \( v_h \) for \( t \to 0 \) are complex vectors \( v_g^\infty \), \( v_h^\infty \) (up to scalar multiplications) which are in \( \mathbb{C}^n \times \{0\} \). They are linearly independent over \( \mathbb{R} \). Thus the limit of \( T_{x(t)}f^{-1}(f(x(t))) \) is the real orthogonal complement \( \langle v_g^\infty, v_h^\infty \rangle = v_g^\perp \cap v_h^\perp \) which contains \( \mathbb{C}^j \).

Assume \( s = s' \) and the coefficients of \( t' \) in \( v_{g,s} \) and the coefficient of \( t'^r \) in \( v_{h,s} \) are linearly dependent over \( \mathbb{R} \). Then we consider the following operation.

Operation. Put \( r' = \text{ord} \, v_h \). We have assumed \( r' \geq r \). Take a unique real number \( \lambda \) and replace \( v_h \) by \( v_h = v_h - \lambda t'^r v_g \) with \( r = \text{ord} \, v_{g,s} \), \( r' = \text{ord} \, v_{h,s} \) to kill the coefficient of \( t'^r \) of \( v_{h,s} \). (We have assumed \( r \leq r' \).)

Note that after this operation, the vector changes into
\[
v_{h,j}'(t) = \left( \frac{\partial}{\partial z_j} h_A(a) - \lambda \frac{\partial}{\partial z_j} g_A(a) \right) t^{d-p_j} + \text{(higher terms)}
\]
where \( \varepsilon = 1 \) or \( 0 \) according to \( r' = r \) or \( r' > r \) respectively. We observe that if \( r' > r \), the leading term of \( v_{h,j}'(t) \) does not change. If \( r' = r \), the (leading)

\[
\text{coefficient } \frac{\partial}{\partial z_j} h_A(a) \text{ of } t^{d-p_j} \text{ in } v_{h,j} \text{ is changed into}
\]
\[
\frac{\partial}{\partial z_j} h_A(a) - \lambda \frac{\partial}{\partial z_j} g_A(a),
\]
the above two properties (A-1), (A-2) are unchanged.

We continue the operation as long as the leading index of \( v_{h}' \) is still \( s \). Suppose that this operation stops at \( k \)-th step. Then put \( s^{(k)} \) the leading index of \( v_h^{(k)} \) and \( r^{(k)} \) be the order of \( v_h^{(k)} \). By the above two properties, \( s < s^{(k)} \leq \beta \) and \( r^{(k)} \leq d - p_B \). This implies that the limit of the normalized gradient vectors \( v_g^{(k)} \) and \( v_h^{(k)} \), say \( v_g^\infty \), \( v_h^\infty \) are independent vectors in \( \mathbb{C}^m \times \{0\} = \mathbb{C}^j \) over \( \mathbb{R} \). On the other hand, by the definition of the above operations,
\[
T_{x(t)}f^{-1}(f(x(t))) = v_g(t)^\perp \cap v_h(t)^\perp
\]
\[
= v_g(t)^\perp \cap v_h'(t)^\perp = \cdots = v_g(t)^\perp \cap (v_h^{(k)}(t))^\perp.
\]
Thus the limit of $T_{z(t)}f^{-1}(f(z(t)))$ is nothing but $(v^\infty_g)^+ \cap (v^\infty_h)^+$. Note that $(v^\infty_g)^+ \cap (v^\infty_h)^+ \Rightarrow C^f$. This show that the $a_f$-property is satisfied along this curve. 

The following will be practically useful.

**Lemma 21.** Let $f_\Delta$ be a face function associated with an essential non-compact face $\Delta \in \Gamma_{nc}(f)$ with $I = I(\Delta)$. Assume that $I = \{m + 1, \ldots, n\}$.

1. For $f(z)$ a holomorphic function, the following is necessary and sufficient for $f_\Delta$ to be locally tame.

   $\left(\frac{\partial}{\partial z_1} f_\Delta(z), \ldots, \frac{\partial}{\partial z_m} f_\Delta(z)\right)$

is a non-zero vector for any $z$ with $\|z_t\| \leq \rho_0$.

2. For a mixed polynomial, $f_\Delta$ is locally tame if there exists a $j \in I^c$ such that two complex numbers $\frac{\partial g_\Delta}{\partial z_j}(z), \frac{\partial h_\Delta}{\partial z_j}(z)$ are linearly independent over $\mathbb{R}$. In other word,

   $\Im \left(\frac{\partial g_\Delta}{\partial z_j}(z) \frac{\partial h_\Delta}{\partial z_j}(z)\right) \neq 0.$

for any $z$ with $\|z_t\| \leq \rho_0$.

**Proof.** Recall that

$$\bar{\partial} g = \frac{1}{2} (\partial f + \partial \overline{f}), \quad \overline{\partial} h = i \left(\partial f - \partial \overline{f}\right).$$

If $f$ is holomorphic, $\bar{\partial} g = \frac{1}{2} \partial f$ and $\overline{\partial} h = -i \bar{\partial} g$ and they are perpendicular by the Euclidean inner product. Thus they are independent over $\mathbb{R}$. For the second assertion, note that the assumption is equivalent to the 2 minor

$$\det\begin{pmatrix} \frac{\partial g_\Delta}{\partial x_j}(a) & \frac{\partial g_\Delta}{\partial y_j}(a) \\ \frac{\partial h_\Delta}{\partial x_j}(a) & \frac{\partial h_\Delta}{\partial y_j}(a) \end{pmatrix} = -\Im \left(\frac{\partial g_\Delta}{\partial z_j}(a) \frac{\partial h_\Delta}{\partial z_j}(a)\right) \neq 0. \quad \square$$

### 3.4.1. Examples

**Example 1.** (Modification of Tibar's example) Consider the mixed monomial $f = z_1 z_2^a z_2$. Then we have

$$\bar{\partial} f = (0, z_1 z_2^a), \quad \overline{\partial} f = (z_2^a z_2, a z_1 z_2^a - z_2)$$

$$\bar{\partial} g = \frac{1}{2} (z_2^a z_2, z_1 z_2^a + a z_1 z_2^a - z_2)$$

$$\overline{\partial} h = \frac{i}{2} (-z_2^a z_2, z_1 z_2^a - a z_1 z_2^a - z_2)$$
Consider the vanishing coordinate $I = \{1\}$. Two complex numbers
\[ z_1 z_2^a + a z_1^n z_2^{a-1} z_2, \quad i(z_1 z_2^a - a z_1^n z_2^{a-1} z_2) \]
are linearly dependent over $\mathbb{R}$ if and only if $a = 1$ as
\[
(z_1 z_2^a + a z_1^n z_2^{a-1} z_2)(-i)(z_1 z_2^a - a z_1^n z_2^{a-1} z_2) \\
= -i(1 - a^2)|z_1|^2|z_2|^{2a} - ia(-z_1^2 z_2^{2a-1} z_2 + z_1^2 z_2^{2a-2} z_2) \\
= -i(1 - a^2)|z_1|^2|z_2|^{2a} - 2a^3 (z_1^2 z_2^{2a-1} z_2).
\]
Thus the imaginary part of the above complex number is zero if and only if $a = 1$. Note that $f$ is an open mapping along $z_2 = 0$ if and only if $a > 1$.

Example 2. Consider the mixed polynomial
\[ f(z, \bar{z}) = z_1^{a_1} z_2 + z_2^{a_2} z_3 + \cdots + z_n^{a_n} z_1, \quad a_1, \ldots, a_n \geq 2. \]
Then \( \{i\} \in \mathcal{I}(f) \) for all \( i = 1, \ldots, n \). Consider for example, \( I = \{n\} \). Then possible face functions are
\[ f_\Delta = z_2^{a_1} z_3 + \cdots + z_n^{a_n} z_1 \quad \text{and} \quad f_\Xi, \]
where $\Xi$ is a face of $\Delta$. Now we can see that
\[
\overline{\mathcal{g}_\Delta} = (z_n^{a_n}, 0, z_2^{a_2}, \ldots, z_{n-1}^{a_{n-1}}) \\
\overline{\mathcal{h}_\Delta} = (0, a_2 z_2^{a_2-1} z_3, \ldots, a_n z_n^{a_n-1} z_1) \\
\overline{\mathcal{g}_\Delta} = \frac{1}{2} (z_n^{a_n}, a_2 z_2^{a_2-1} z_3, \ldots, a_n z_n^{a_n-1} z_1) \\
\overline{\mathcal{h}_\Delta} = \frac{i}{2} (z_n^{a_n}, a_2 z_2^{a_2-1} z_3, \ldots, a_n z_n^{a_n-1} z_1).
\]
Thus
\[
(\overline{\mathcal{g}_\Delta})_1 \cdot (\overline{\mathcal{h}_\Delta})_1 = -\frac{i}{4} |z_n|^{2a_n}
\]
and its imaginary part is non-zero, which satisfies the condition of Lemma 21.

Now we consider a subset $\Xi \subset \Delta$. We consider the first monomial $z_j^{a_j} z_{j+1}$ so that
\[ z_n^{a_n} z_1, \ldots, z_{j-1}^{a_{j-1}} z_j \notin f_\Xi, \quad z_j^{a_j} z_{j+1} \in f_\Xi. \]
Then we have
\[
\Im(\overline{\mathcal{g}_\Xi})_j \cdot (\overline{\mathcal{h}_\Xi})_j = -\frac{1}{4} a_k^2 |z_k|^{2a_k-2} |z_{k+1}|^2 \neq 0.
\]
Thus by symmetry, we conclude that $f$ is locally tame along each vanishing coordinate axis $z_k$, $k = 1, \ldots, n$. 


4. Some application

4.1. Mixed cyclic coverings. Consider positive integer vectors

\[ a := (a_1, \ldots, a_n), \quad b = (b_1, \ldots, b_n) \]

such that \( a_j > b_j \geq 0, \ j = 1, \ldots, n \). We consider the mapping

\[ \varphi_{a,b} : \mathbb{C}^n \rightarrow \mathbb{C}^n, \quad (z_1, \ldots, z_n) \mapsto \left(z_1^{a_1} z_1^{b_1}, \ldots, z_n^{a_n} z_n^{b_n}\right). \]

This is a \( \prod_{j=1}^n (a_j - b_j) \)-fold multi-cyclic covering branched along the coordinate hyperplanes \{\( z_j = 0 \}\}, \ j = 1, \ldots, n \). Consider a holomorphic function \( f(z) \) which has a non-degenerate Newton boundary and the pull-back \( f(w, z) := f(\varphi_{a,b}(w, z)) \) of \( f \) by \( \varphi_{a,b} \). This gives a strongly non-degenerate mixed function \([10]\).

**Proposition 22.** Assume that \( f(z) \) is a non-degenerate holomorphic function which is locally tame along their vanishing coordinate subspaces. Then \( f(w, z) := f(\varphi_{a,b}(w, z)) \) is a non-degenerate mixed function. Its vanishing coordinate subspaces are the same as that of \( f(z) \) and it is locally tame along the vanishing coordinate subspaces.

**Proof.** Take a face function \( f_P(z) \) with weight \( P = (p_1, \ldots, p_n) \). Consider the weight \( \tilde{P} \) which is the primitive weight vector obtained by multiplying the least common multiple of the denominators of

\[ \frac{p_1}{a_1 + b_1}, \ldots, \frac{p_n}{a_n + b_n}. \]

Then \( \varphi_{a,b}^* f(w, z) \) is radially weighted homogeneous with respect to the weight \( \tilde{P} \). We observe also have \( f_P(w, z) = \varphi_{a,b}^* f_P(w, z) \). Thus we see that the Newton boundary \( \Gamma(f) \) corresponds bijectively to that of \( \Gamma(f) \) by this mapping. Suppose that \( I \in \mathcal{I}(f) = \mathcal{I}(\tilde{f}) \). We assume \( I = \{m+1, \ldots, n\} \) for simplicity. Take a non-compact face \( \Delta \) with \( I(\Delta) = I \) and let \( \Delta \) be the corresponding non-compact face of \( \tilde{f} \). We consider \( \tilde{f}_\Delta \) as the following composition, fixing \( (u_{m+1}, \ldots, u_n) \in \mathbb{C}^{l-1} \):

\[ \tilde{f}_\Delta : \mathbb{C}^{\infty m} \xrightarrow{\varphi_{a,b}^*} \mathbb{C}^{\infty m} \xrightarrow{f_\Delta} \mathbb{C} \]

where

\[ \varphi_{a,b}^* (w_1, \ldots, w_m) = (w_1^{a_1} w_1^{b_1}, \ldots, w_m^{a_m} w_m^{b_m}, u_{m+1}^{a_{m+1}} u_{m+1}^{b_{m+1}}, \ldots, u_n^{a_n} u_n^{b_n}). \]

As \( \varphi_{a,b}^* \) is an unbranched covering mapping, \( \tilde{f}_\Delta \) does not have any critical points.

\[ \square \]

4.2. Mixed functions with strongly mixed weighted homogeneous faces. We say a mixed polynomial \( h(z, \bar{z}) \) is mixed weighted homogeneous if it is radially weighted homogeneous and also polar weighted homogeneous. \( h(z, \bar{z}) \) is strongly mixed weighted homogeneous if the polar weight and the radial weight can be the
A mixed function \( f(z, \bar{z}) \) is called of strongly mixed weighted homogeneous face type if every face function \( f_A \) is strongly mixed weighted homogeneous polynomial \([10]\). Let \( \Gamma^*(f) \) be the Newton boundary and let \( \Sigma^* \) be an admissible regular subdivision of \( \Gamma^*(f) \) and let \( \hat{\pi} : X \rightarrow \mathbb{C}^n \) be the associated toric modification. Let \( S \) be the vertices of \( \Sigma^* \) which corresponds to the exceptional divisors as in \( \S 2, [10] \). Let \( \mathcal{S}_I \) be the set of \( |I| - 1 \) dimensional faces of \( \Gamma(f^I) \). It is shown that \( \hat{\pi} : X \rightarrow \mathbb{C}^n \) topologically resolves the mixed function \( f : \mathbb{C}^n \rightarrow \mathbb{C} \) \([10]\). Combining the existence of Milnor fibration and the argument in \([10]\), we can generalize Theorem 11 \([10]\) as follows. For \( I \in \mathcal{A}_{\text{hs}} \), we denote by \( \mathcal{S}_I \) the set of weight vectors which correspond to \( |I| - 1 \) dimensional faces of \( \Gamma(f^I) \).

The notations and definitions are the same as in Theorem 11 \([10]\).

**Theorem 23.** Let \( f(z, \bar{z}) \) a non-degenerate mixed polynomial of strongly mixed positive weighted homogeneous face type which is locally tame along vanishing coordinate subspaces. Let \( V = f^{-1}(0) \) be a germ of hypersurface at the origin and let \( \tilde{V} \) be the strict transform of \( V \) to \( X \). Then

1. \( \tilde{V} \) is topologically smooth and real analytic smooth variety outside of the union of the exceptional divisors \( \bigcup_{P \in \mathcal{S}_I} E(P) \).
2. The zeta function of the Milnor fibration of \( f(z, \bar{z}) \) is given by the formula

\[
\zeta(t) = \prod_I \zeta_I(t), \quad \zeta_I(t) = \prod_{P \in \mathcal{S}_I} (1 - t^{\text{pdeg}(P, f^I)})^{-\chi(P) / \text{pdeg}(P, f^I)}
\]

where \( \chi(P) \) is the Euler characteristic of the torus Milnor fiber of \( f^I \)

\[
F_P = \{ z_I \in \mathbb{C}^n \mid f^I(z_I) = 1 \}, \quad P \in \mathcal{S}_I.
\]

**4.2.1. Example:** 1. Curves with mixed Brieskorn faces. Consider a mixed polynomial

\[
f(z, \bar{z}) = z_1^2z_2^2(z_1^6z_2^3 + z_2^4z_1^3)(z_1^4z_2^2 + z_2^6z_1^2)
\]

\( f \) is strongly non-degenerate and has two faces which are strongly polar weighted homogeneous: Put \( P = \text{t}(2, 3), Q := \text{t}(3, 2) \). There are two faces corresponding to \( P \) and \( Q \).

\[
f_P(z, \bar{z}) = z_1^6z_2^2(z_1^6z_2^3 + z_2^4z_1^3), \quad f_Q(z, \bar{z}) = z_1^2z_2^6z_2^3z_1^4z_2^2 + z_2^6z_1^2
\]

and \( f_P, f_Q \) are strongly polar weighted with \( \text{pdeg} f_P = \text{pdeg} f_Q = 20 \). Thus the contribution of \( f_P \) to the zeta-function is \( (1 - t^{20})^{-\chi(P)/20} \) where \( \chi(P) \) is the Euler characteristic of

\[
F_P^* := \{ z \in \mathbb{C}^2 \mid f_P(z, \bar{z}) = 1 \}.
\]

\( F_P^* \) is diffeomorphic to

\[
F_P' := \{ z \in \mathbb{C}^2 \mid z_1^4z_2^2(z_1^3 + z_2^2) = 1 \}
\]

by Theorem 10 \([8]\). Thus \( \chi(P) = \chi(F_P^*) = -20 \). Thus using the symmetry of \( f_P \) and \( f_Q \), we get \( \zeta(t) = (1 - t^{20})^2 \).
In general, for a non-degenerate non-convenient mixed polynomial of two variables \( f(z, \bar{z}) \), consider the right end monomial \( z_1^{a-1}z_2^b \). Right end means that \( \Gamma(f) \) is in the space \( \{(v, \mu) \mid v \leq m + n, \mu \geq a + b\} \). If \( a + b \geq 1 \), \( z_1 \) axis is a vanishing coordinate. It is locally tame along \( z_1 \)-axis if and only if \( a - b \neq 0 \).

Example 2. Consider \( D_n \) singularity:
\[
D_n : f(z_1, z_2, z_3) = z_1^2 + z_2^2z_3 + z_3^{n-1}.
\]
Then the Milnor number \( \mu(f) \) of \( f \) is \( n \) and the zeta function is given as \( \zeta(t) = (t^{n-1} + 1)/(t^2 - 1) \). \( f \) has a vanishing axis \( z_2 \) but \( V \) is non-singular except at the origin. Consider
\[
\tilde{f}(w, \bar{w}) = \varphi_{2,1}f(w, \bar{w})
\]
\[
= w_1^4\bar{w}_1^2 + w_2^4\bar{w}_2^2\bar{w}_3 + w_3^{2(n-1)}w_3^{n-1}
\]
\( \tilde{f} \) has a vanishing coordinate axis \( w_2 \) but the data for the zeta function is exactly same as \( f \). As \( \varphi_{2,1} \) is a homeomorphism, \( \mu(\tilde{f}) = n \) and it has the same zeta functions as \( f \). See also Corollary 15, [10].

4.3. Join type polynomials. We consider the join type polynomial
\[
f(z, \bar{z}, w, \bar{w}) = f_1(z, \bar{z}) + f_2(w, \bar{w}), \quad (z, w) \in \mathbb{C}^n \times \mathbb{C}^m.
\]

**Proposition 24.** Assume that \( f_1 \) and \( f_2 \) are strongly non-degenerate mixed polynomial. Then \( f \) is also strongly non-degenerate. We assume that \( f_1, f_2 \) do not have any linear term so that they have a critical point at the respective origin. Then we have

(1) If \( f_1 \) and \( f_2 \) are locally tame along vanishing coordinate subspaces, \( f \) is also locally tame along vanishing coordinate subspaces. In particular, \( f \) satisfies \( a_f \)-condition.

(2) If \( f_1 \) or \( f_2 \) does not satisfy \( a_f \)-condition, \( f \) does not satisfy \( a_f \)-condition.

**Proof.** Assume that \( I_1 = I(f_1) \) and \( I_2 = I(f_2) \). Then \( f \mid C^{I_1} \times C^{I_2} \equiv 0 \). Take \( \Delta_1 \in \Gamma_{nc}(f_1) \) with \( I(\Delta_1) = I_1 \) and \( \Delta_2 \in \Gamma_{nc}(f_2) \) with \( I(\Delta_2) = I_2 \). Then \( \Delta := \Delta_1 \ast \Delta_2 \in \Gamma_{nc}(f) \) and \( f_\Delta(z, \bar{z}, w, \bar{w}) = f_{\Delta_1}(z, \bar{z}) + f_{\Delta_2}(w, \bar{w}) \) satisfies certainly the local tameness condition. (Here \( \Delta_1 \ast \Delta_2 \) is the convex polyhedron spanned by \( \Delta_1 \) and \( \Delta_2 \).) Conversely suppose that \( \Delta \in \Gamma_{nc}(f) \) with \( I = I(\Delta) \). Then \( f \mid C^{I} \equiv 0 \). Put \( I_1 = I \cap \{1, \ldots, n\} \) and \( I_2 = I \setminus I_1 \). Then \( I_1 \in \mathcal{J}(f_1) \) and \( I_2 \in \mathcal{J}(f_2) \). Take \( P \) so that \( I(P) = I \) and put \( \Delta_1 = \Delta \cap C^n \) and \( \Delta_2 = \Delta \cap C^m \). Then \( \Delta = \Delta_1 \ast \Delta_2 \). Let \( P_1, P_2 \) be the projection to \( C^n \) or \( C^m \) respectively. Then \( \Delta(P_1) = \Delta_1 \) and \( \Delta(P_2) = \Delta_2 \) and \( f_\Delta = f_{P_1} + f_{P_2} \) but it is certainly locally tame. This proves (1).

To prove the assertion (2), assume for example \( f_1 \) does not satisfy \( a_f \)-condition. Take a stratification \( \mathcal{S} \) such that its restriction to \( C^n \) and \( C^m \) are stratification \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \). Then \( \mathcal{S}_1 \) and \( \mathcal{S}_2 \) for \( f_1 \) and \( f_2 \) respectively. By the assumption, there
exists \( p \in V_1 = V(f_1) \) and a stratum \( M \) of \( \mathcal{S} \) with \( p \in M \) and an analytic curve \( z(t) \) in \( \mathbb{C}^n \setminus \{0\} \) such that \( z(0) = p \) and \( a_f \)-condition is not satisfied along this curve. Write \( f_1 = g_1 + ih_1 \), \( f_2 = g_2 + ih_2 \) and \( f = g + ih \). We may assume that

\[
\partial \bar{g}_1(z(t)) = v_{g_1} \bar{t}^d + (\text{higher terms})
\]

and it converges to \( v_{g_1}^\perp \). We assume \( \text{ord} \, \partial h_1(z(t)) \geq \text{ord} \, \partial g_1(z(t)) \). By the same technique as in the proof of Theorem 20, we take a new vector \( \partial h_1' := \partial h_1(t) - k(t) \bar{c}g_1(z(t)) \) so that

\[
\partial h_1'(t) = v_{h_1} \bar{t}^d + (\text{higher terms})
\]

so that the leading coefficient vectors \( v_{g_1}, v_{h_1} \) are linearly independent over \( \mathbb{R} \). Thus the limit of the tangent space \( T_{z(t)} f_1^{-1}(f(z(t))) \) is given by \( v_{g_1}^\perp \cap v_{h_1}^\perp \). By the assumption, we have that \( v_{g_1}^\perp \cap v_{h_1}^\perp \not= T_{z(0)} M \). Note that \( d \) is the order of \( \partial h_1'(z(t)) \). Consider the analytic path \( (z(t), w(t)) \) where \( w(t) = (t^d, \ldots, t^d) \).

Let us consider

\[
\partial h_2'(w(t)) := \partial h_2(w(t)) - k(t) \bar{c}g_2(w(t)).
\]

Then it is easy to see that \( \text{ord} \, \partial h_2'(w(t)), \text{ord} \, \partial h_2'(w(t)) \geq 2d \). Put

\[
\partial h'(z(t), w(t)) = \partial h(z(t), w(t)) - k(t) \bar{c}g(z(t), w(t)).
\]

Thus this implies that \( \text{ord} \, \partial g(z(t), w(t)) = \text{ord} \, \partial g_1(z(t)) \) and \( \text{ord} \, \partial h'(z(t), w(t)) = \text{ord} \, \partial h'_1(z(t)) \geq 2d \) and the normalized limits are given as

\[
\partial g(z(t), w(t)) \rightarrow (v_{g_1}, 0), \quad \partial h'(z(t), w(t)) \rightarrow (v_{h_1}, 0)
\]

which implies the limit of \( T_{z(t), w(t)} f^{-1}(f(z(t), w(t))) \) is \( (v_{g_1}^\perp \cap v_{h_1}^\perp) \times \mathbb{C}^m \). By the assumption, \( (v_{g_1}^\perp \cap v_{h_1}^\perp) \times \mathbb{C}^m \not= T_{z(0)} M \).

\[ \square \]

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