A CHARACTERIZATION OF MODULATION SPACES BY SYMPLECTIC ROTATIONS

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ABSTRACT. This note contains a new characterization of modulation spaces $M^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, by symplectic rotations. Precisely, instead to measure the time-frequency content of a function by using translations and modulations of a fixed window as building blocks, we use translations and metaplectic operators corresponding to symplectic rotations. Technically, this amounts to replace, in the computation of the $M^p(\mathbb{R}^n)$-norm, the integral in the time-frequency plane with an integral on $\mathbb{R}^n \times U(2n, \mathbb{R})$ with respect to a suitable measure, $U(2n, \mathbb{R})$ being the group of symplectic rotations. More conceptually, we are considering a sort of polar coordinates in the time-frequency plane. In this new framework, the Gaussian invariance under symplectic rotations yields to choose Gaussians as suitable window functions. We also provide a similar characterization with the group $U(2n, \mathbb{R})$ being reduced to the $n$-dimensional torus $T^n$.

1. Introduction

The objective of this study is to find a new characterization of modulation spaces using symplectic rotations. Precisely, we are interested in those metaplectic operators $\hat{S} \in Mp(n, \mathbb{R})$, such that the corresponding projection $S := \pi(\hat{S})$ onto the symplectic group $Sp(n, \mathbb{R})$ is a symplectic rotation. Let us recall that the symplectic group $Sp(n, \mathbb{R})$ is the subgroup of $2n \times 2n$ invertible matrices $GL(2n, \mathbb{R})$, defined by

\[ Sp(n, \mathbb{R}) = \{ S \in GL(2n, \mathbb{R}) : SJS^T = J \}, \]

where $J$ is the orthogonal matrix

\[ J = \begin{pmatrix} 0_n & I_n \\ -I_n & 0_n \end{pmatrix}, \]

($I_n$, $0_n$ are the $n \times n$ identity matrix and null matrix, respectively). Here we consider the subgroup

\[ U(2n, \mathbb{R}) := Sp(n, \mathbb{R}) \cap O(2n, \mathbb{R}) \cong U(n) \]
of symplectic rotations (cf., e.g. [15, Section 2.3]), namely

\[ U(2n, \mathbb{R}) = \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} : AA^T + BB^T = I_n, AB^T = B^T A \right\} \subset Sp(n, \mathbb{R}), \]

endowed with the normalized Haar measure \( dS \) (the group \( U(2n, \mathbb{R}) \), being compact, is unimodular).

In the 80’s H. Feichtinger [16] introduced modulation spaces to measure the time-frequency concentration of a function/distribution on the time-frequency space (or phase space) \( \mathbb{R}^{2n} \). They are nowadays become popular among mathematicians and engineers because they have found numerous applications in signal processing [6, 17, 18], pseudodifferential and Fourier integral operators [7, 8, 9, 26, 27], partial differential equations [1, 2, 3, 4, 10, 13, 11, 11, 28, 29, 30] and quantum mechanics [12, 15].

To recall their definition, we need a few time-frequency tools. First, the translation \( T_x \) and modulation \( M_\xi \) operators are defined by

\[ T_x f(t) = f(t - x), \quad M_\xi f(t) = e^{2\pi i t \cdot \xi} f(t), \quad t, x, \xi \in \mathbb{R}^n, \]

for any function \( f \) on \( \mathbb{R}^n \).

The time-frequency representation which occurs in the definition of modulation spaces is the short-time Fourier Transform (STFT) of a distribution \( f \in S'(\mathbb{R}^n) \) with respect to a function \( g \in S(\mathbb{R}^n) \setminus \{0\} \) (so-called window), given by

\[ V_g f(x, \xi) = \langle f, M_\xi T_x g \rangle = \int_{\mathbb{R}^n} f(t) \overline{g(t - x)} e^{-2\pi i t \cdot \xi} dt, \quad x, \xi \in \mathbb{R}^n. \]

The short-time Fourier transform is well-defined whenever the bracket \( \langle \cdot, \cdot \rangle \) makes sense for dual pairs of function or distribution spaces, in particular for \( f \in S'(\mathbb{R}^n), g \in S(\mathbb{R}^n) \), or for \( f, g \in L^2(\mathbb{R}^n) \).

**Definition 1.1 (Modulation spaces).** Given \( g \in S(\mathbb{R}^n) \), and \( 1 \leq p \leq \infty \), the modulation space \( M^p(\mathbb{R}^n) \) consists of all tempered distributions \( f \in S'(\mathbb{R}^n) \) such that \( V_g f \in L^p(\mathbb{R}^{2n}) \). The norm on \( M^p(\mathbb{R}^n) \) is

\[ \|f\|_{M^p} = \|V_g f\|_{L^p} = \left( \int_{\mathbb{R}^{2n}} |V_g f(x, \xi)|^p dx d\xi \right)^{1/p} = \left( \int_{\mathbb{R}^{2n}} |\langle f, M_\xi T_x g \rangle|^p dx d\xi \right)^{1/p} \]

(with obvious modifications for \( p = \infty \)).

The spaces \( M^p(\mathbb{R}^n) \) are Banach spaces, and every nonzero \( g \in M^1(\mathbb{R}^n) \) yields an equivalent norm in [4], so that their definition is independent of the choice of \( g \in M^1(\mathbb{R}^n) \) (see [16, 20]).

We now provide an equivalent norm to (4) by using translations \( T_x \) (or modulations \( M_\xi \)) and the operators \( \hat{S} \), with \( S \in U(2n, \mathbb{R}) \) as follows.
Theorem 1.2. Consider the Gaussian function \( \varphi(t) = 2^{d/4}e^{-\pi|t|^2} \).

(i) For \( 1 \leq p < \infty \) and \( f \in M^p(\mathbb{R}^n) \), we have

\[
\|f\|_{M^p(\mathbb{R}^n)} \asymp \left( \int_{\mathbb{R}^n \times U(2n,\mathbb{R})} |x|^n |\langle f, \hat{T}_x \varphi \rangle|^p dx dS \right)^{1/p},
\]

where \( dx \) is the Lebesgue measure on \( \mathbb{R}^n \) and \( dS \) the Haar measure on \( U(2n,\mathbb{R}) \).

Similarly,

\[
\|f\|_{M^p(\mathbb{R}^n)} \asymp \left( \int_{\mathbb{R}^n \times U(2n,\mathbb{R})} |\xi|^n |\langle f, \hat{S}_\xi \varphi \rangle|^p d\xi dS \right)^{1/p},
\]

with \( d\xi \) being the Lebesgue measure on \( \mathbb{R}^n \) and \( dS \) the Haar measure on \( U(2n,\mathbb{R}) \).

(ii) For \( p = \infty \), \( f \in M^\infty(\mathbb{R}^n) \), it occurs

\[
\|f\|_{M^\infty(\mathbb{R}^n)} \asymp \sup_{S \in U(2n,\mathbb{R})} \sup_{x \in \mathbb{R}^n} |\langle f, \hat{T}_x \varphi \rangle|,
\]

or, similarly,

\[
\|f\|_{M^\infty(\mathbb{R}^n)} \asymp \sup_{S \in U(2n,\mathbb{R})} \sup_{\xi \in \mathbb{R}^n} |\langle f, \hat{S}_\xi \varphi \rangle|.
\]

The interpretation of the integral (5) above is as follows. The metaplectic operator \( \hat{S} \) produces a time-frequency rotation of the shifted Gaussian \( T_x \varphi \). In this way, the operator

\[
f \mapsto \langle f, \hat{T}_x \varphi \rangle
\]

detects the time-frequency content of \( f \) in an oblique strip, see Figure 1. All the contributions are then added together with a weight \( |x|^n \) which takes into account the underlapping of the strips as \( |x| \to \infty \) and the overlapping as \( |x| \to 0 \).

Formulas (6), (7) and (8) have similar meanings.

\[\text{Figure 1. The time-frequency content of } f \text{ in the oblique strip is detected by the operator } f \mapsto \langle f, \hat{T}_x \varphi \rangle.\]
Observe that in dimension $n = 1$, $U(2, \mathbb{R}) \simeq U(1)$ and the above formula is essentially a transition to polar coordinates with $|x|$ being the Jacobian.

Comparing (4) and (5) we observe that in (5) the modulation operator $M_\xi$ is replaced by the metaplectic operator $\hat{S}$ and the integral on the phase space $\mathbb{R}^{2n}$ has become an integral on the cartesian product $\mathbb{R}^n \times U(2n, \mathbb{R})$. The integration parameters $(x, \xi)$ of (4) live in $\mathbb{R}^{2n}$, with dim $\mathbb{R}^{2n} = 2n$, whereas the parameters $(x, S)$ of (5) live in $\mathbb{R}^n \times U(2n, \mathbb{R})$. Recall that dim $U(2n, \mathbb{R}) = n^2$ [15]; this suggests that a formula similar to (5) should hold when $U(2n, \mathbb{R})$ is reduced to a suitable subgroup $K \subset U(2n, \mathbb{R})$ of dimension $n$. This is indeed the case, as shown in the subsequent Theorem 1.3.

Consider the $n$-dimensional torus

\begin{align}
T^n = \left\{ S = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix} : \theta_1, \ldots, \theta_n \in \mathbb{R} \right\} \subset U(n)
\end{align}

with the Haar measure $dS = d\theta_1 \ldots d\theta_n$. The torus is isomorphic to a subgroup $K \subset U(2n, \mathbb{R})$, via the isomorphism $\iota$ in formula (16) below (see the subsequent section).

We exhibit the following characterization for $M^p$-spaces.

**Theorem 1.3.** Let $\varphi$ be the Gaussian of Theorem 1.2.

(i) For $1 \leq p < \infty$, $f \in M^p(\mathbb{R}^n)$, we have

\begin{align}
\|f\|_{M^p(\mathbb{R}^n)} \asymp \left( \int_{\mathbb{R}^n \times T^n} \left| x_1 \ldots x_n \right|^p |\langle f, \hat{S}T_x \varphi \rangle|^p dx dS \right)^{\frac{1}{p}},
\end{align}

or, similarly,

\begin{align}
\|f\|_{M^p(\mathbb{R}^n)} \asymp \left( \int_{\mathbb{R}^n \times T^n} \left| \xi_1 \ldots \xi_n \right|^p |\langle f, \hat{S}M_\xi \varphi \rangle|^p d\xi dS \right)^{\frac{1}{p}}.
\end{align}

(ii) For $p = \infty$,

\begin{align}
\|f\|_{M^\infty(\mathbb{R}^n)} \asymp \sup_{S \in T^n} \sup_{x \in \mathbb{R}^n} |\langle f, \hat{S}T_x \varphi \rangle|,
\end{align}

or

\begin{align}
\|f\|_{M^\infty(\mathbb{R}^n)} \asymp \sup_{S \in T^n} \sup_{\xi \in \mathbb{R}^n} |\langle f, \hat{S}M_\xi \varphi \rangle|.
\end{align}

The above results for the groups $U(2n, \mathbb{R})$ and $T^n$ can be interpreted, in a sense, as two extreme cases, and it would be interesting to find, more generally, for which compact subgroups $K \subset U(2n, \mathbb{R})$ similar characterizations hold. We conjecture that they should be precisely the subgroups $K \subset U(2n, \mathbb{R})$ such that every orbit for their action on $\mathbb{R}^{2n}$ intersects $\{0\} \times \mathbb{R}^n$ (up to subsets of measure zero), with a corresponding weighted measure on $\mathbb{R}^n \times K$ to be determined.
Another open problem which is worth investigating is the study of discrete versions of the above characterizations.

The paper is organized as follows: in Section 2 we collected some preliminary results, whereas Section 3 is devoted to the proof of Theorems 1.2 and 1.3. In Section 4 we rephrase more explicitly Theorem 1.3 in terms of the partial fractional Fourier transform.

2. Notation and Preliminaries

**Notation.** We write $x \cdot y$ for the scalar product on $\mathbb{R}^n$ and $|t|^2 = t \cdot t$, for $t, x, y \in \mathbb{R}^n$. For expressions $A, B \geq 0$, we use the notation $A \lesssim B$ to represent the inequality $A \leq cB$ for a suitable constant $c > 0$, and $A \asymp B$ for the equivalence $c^{-1}B \leq A \leq cB$.

The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^n)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^n)$. We use the brackets $\langle f, g \rangle$ to denote the extension to $\mathcal{S}'(\mathbb{R}^n) \times \mathcal{S}(\mathbb{R}^n)$ of the inner product $\langle f, g \rangle = \int f(t)g(t)dt$ on $L^2(\mathbb{R}^n)$.

**Metaplectic Operators.** The metaplectic representation $\mu$ of $Mp(n, \mathbb{R})$, the two-sheeted cover of the symplectic group $Sp(n, \mathbb{R})$, defined in (1) arises as intertwining operator between the standard Schrödinger representation $\rho$ of the Heisenberg group $H^d$ and the representation that is obtained from it by composing $\rho$ with the action of $Sp(n, \mathbb{R})$ by automorphisms on $H^d$ (see, e.g., [15, 19, 21]). Let us recall the main points of a direct construction.

The symplectic group $Sp(n, \mathbb{R})$ is generated by the so-called free symplectic matrices

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n, \mathbb{R}), \quad \det B \neq 0.$$ 

To each such a matrix the associated generating function is defined by

$$W(x, x') = \frac{1}{2}DB^{-1}x \cdot x - B^{-1}x \cdot x' + \frac{1}{2}B^{-1}Ax' \cdot x'.$$

Conversely, to every polynomial of the type

$$W(x, x') = \frac{1}{2}Px \cdot x - LX \cdot x' + \frac{1}{2}Qx' \cdot x'$$

with

$$P = P^T, Q = Q^T$$

and

$$\det L \neq 0$$

it can be associated a free symplectic matrix, namely

$$S_W = \begin{pmatrix} L^{-1}Q & L^{-1} \\ PL^{-1}Q - L^T & PL^{-1} \end{pmatrix}.$$
Given $S_W$ as above and $m \in \mathbb{Z}$ such that
\[ m\pi \equiv \text{arg det } L \mod 2\pi, \]
the related operator $\hat{S}_{W,m}$ is defined by setting, for $\psi \in \mathcal{S}(\mathbb{R}^n)$,
\[ (14) \quad \hat{S}_{W,m}\psi(x) = \frac{1}{i^{n/2}} \Delta(W) \int_{\mathbb{R}^n} e^{2\pi i W(x,x')} \psi(x') dx' \]
(with $i^{n/2} = e^{i\pi n/4}$) where
\[ \Delta(W) = i^m \sqrt{|\det L|}. \]

The operator $\hat{S}_{W,m}$ is named quadratic Fourier transform associated to the free symplectic matrix $S_W$. The class modulo 4 of the integer $m$ is called Maslov index of $\hat{S}_{W,m}$. Observe that if $m$ is one choice of Maslov index, then $m + 2$ is another equally good choice: hence to each function $W$ we associate two operators, namely $\hat{S}_{W,m}$ and $\hat{S}_{W,m+2} = -\hat{S}_{W,m}$.

The quadratic Fourier transform corresponding to the choices $S_W = J$ and $m = 0$ is denoted by $\hat{J}$. The generating function of $J$ is simply $W(x,x') = -x \cdot x'$. It follows that
\[ (15) \quad \hat{J}\psi(x) = \frac{1}{i^{n/2}} \int_{\mathbb{R}^n} e^{-2\pi i x \cdot x'} \psi(x') dx' = \frac{1}{i^{n/2}} F\psi(x) \]
for $\psi \in \mathcal{S}(\mathbb{R}^n)$, where $F$ is the usual unitary Fourier transform.

The quadratic Fourier transforms $\hat{S}_{W,m}$ form a subset of the group $U(L^2(\mathbb{R}^n))$ of unitary operators acting on $L^2(\mathbb{R}^n)$, which is closed under the operation of inversion and they generate a subgroup of $U(L^2(\mathbb{R}^n))$ which is, by definition, the metaplectic group $Mp(n, \mathbb{R})$. The elements of $Mp(n, \mathbb{R})$ are called metaplectic operators.

Hence, every $\hat{S} \in Mp(n, \mathbb{R})$ is, by definition, a product
\[ \hat{S}_{W_1,m_1} \cdots \hat{S}_{W_k,m_k} \]
of metaplectic operators associated to free symplectic matrices.

Indeed, it can be proved that every $\hat{S} \in Mp(n, \mathbb{R})$ can be written as a product of exactly two quadratic Fourier transforms: $\hat{S} = \hat{S}_{W,m}\hat{S}_{W',m'}$. Now, it can be shown that the mapping
\[ \hat{S}_{W,m} \mapsto S_W \]
extends to a group homomorphism
\[ \pi : Mp(n, \mathbb{R}) \to Sp(n, \mathbb{R}), \]
which is in fact a double covering.

We also observe that each metaplectic operator is, by construction, a unitary operator in $L^2(\mathbb{R}^n)$, but also an automorphism of $\mathcal{S}(\mathbb{R}^n)$ and of $\mathcal{S}'(\mathbb{R}^n)$. 
We are interested in its restriction \( \hat{S} = \pi(S) \), with \( S \in U(2n, \mathbb{R}) \), the symplectic rotations in \((2)\).

Observe that \( U(n) := U(n, \mathbb{C}) \), the complex unitary group (the group of \( n \times n \) invertible complex matrices \( V \) satisfying \( VV^* = V^*V = I_n \)) is isomorphic to \( U(2n, \mathbb{R}) \). The isomorphism \( \iota \) is the mapping \( \iota : U(n) \to U(2n, \mathbb{R}) \) given by

\[
\iota(A + iB) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix},
\]

for details see [15, Chapter 2.3].

We present here some results related to the group \( U(2n, \mathbb{R}) \), which will be used in the sequel to attain the characterization of Theorem 1.2. First, we recall a well-known result, see for instance [20, Lemma 9.4.3]:

**Lemma 2.1.** For \( f, g \in L^2(\mathbb{R}^n) \) and \( S \in Sp(n, \mathbb{R}) \), the STFT \( V_g f \) satisfies

\[
|\hat{V}_g(S f)(x, \xi)| = |V_g f(S^{-1}(x, \xi))|, \quad (x, \xi) \in \mathbb{R}^{2n}.
\]

This second issue is contained in [5], we sketch the proof for the sake of consistency.

**Lemma 2.2.** For \( \varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \) and \( S \in U(2n, \mathbb{R}) \), the STFT \( V_\varphi(\hat{S} \psi) \) is a Schwartz function, with seminorms uniformly bounded when \( S \in U(2n, \mathbb{R}) \).

**Proof.** Since \( \varphi \in \mathcal{S}(\mathbb{R}^n) \), the STFT \( V_\varphi \) is a continuous mapping from \( \mathcal{S}(\mathbb{R}^n) \) into \( \mathcal{S}(\mathbb{R}^{2n}) \) (see [16]). Hence, it is enough to show that

\[
\{ \hat{S} \varphi : S \in U(2n, \mathbb{R}) \}
\]

is a bounded subset of the Schwartz class \( \mathcal{S}(\mathbb{R}^n) \), i.e., every Schwartz seminorm is bounded on it. Since the group \( U(2n, \mathbb{R}) \) is compact, it is sufficient to show that every seminorm is locally bounded, that is, we can limit ourselves to consider \( S \) in a sufficiently small neighbourhood for any fixed \( S_0 \in U(2n, \mathbb{R}) \). Equivalently, we can consider \( S \) of the form \( S = S_1 J^{-1} S_0 \) where \( S_1 \) belongs to a enough small neighbourhood of \( J \) in \( U(2n, \mathbb{R}) \). Using the representation of metaplectic operators recalled at the beginning of this section, we can write

\[
\hat{S} \varphi(x) = \pm \hat{S}_1 [\hat{J}^{-1} \hat{S}_0 \varphi](x)
\]

\[
= c \sqrt{\det L} \int_{\mathbb{R}^n} e^{2\pi i (\frac{1}{4} P x \cdot x - L x \cdot y + \frac{1}{4} Q y \cdot y)} \hat{\hat{S}_0 \varphi}(y) dy
\]

where \( |c| = 1 \) and, we might say, \( \|P\| < \epsilon, \|Q\| < \epsilon, \|L - I\| < \epsilon \). If \( \epsilon < 1 \), it is straightforward to check that \( \hat{S} \varphi \) belongs to a bounded subset of \( \mathcal{S}(\mathbb{R}^n) \), as desired. \( \square \)
Lemma 2.3. Let \( B = (b_{i,j})_{i,j=1,...,n} \) be the \( n \times n \) submatrix in (2). The subset \( \Sigma \subset U(2n,\mathbb{R}) \) obtained by setting \( b_{i,1} = 0, i = 1,\ldots,n \) (i.e., the first column of \( B \) is set to zero), is a submanifold of codimension \( n \).

Proof. We have to verify that the coordinates \( b_{1,1},\ldots,b_{n,1} \) are independent on the subset \( \Sigma \), namely the projection \( (b_{1,1},\ldots,b_{n,1}) : U(2n,\mathbb{R}) \rightarrow \mathbb{R}^n \) has rank \( n \) on \( \Sigma \).

Let us first show that for every \( S_0 \in \Sigma \) there exists a \( U(2n,\mathbb{R}) \)-valued smooth function \( S(b_1,\ldots,b_n) \), defined in a neighbourhood of 0 \( \in \mathbb{R}^n \), such that \( S(0) = S_0 \) and the first column “of its submatrix \( B \)” is precisely \( (b_1,\ldots,b_n)^T \).

Let \( S_0 = A + iB = (V_1,\ldots,V_n) \in \Sigma \), with \( V_j \) being a \( n \times 1 \) complex vector, \( j = 1,\ldots,n \), so that by assumption \( (b_{i,1})_{i=1,...,n} = \text{Im} V_1 = 0 \). We consider any smooth function \( V_1(b_1,\ldots,b_n) \), defined in a neighbourhood of 0 \( \in \mathbb{R}^n \), valued in the unit sphere of \( \mathbb{C}^n \), such that

\[
\text{Im} V_1(b_1,\ldots,b_n) = (b_1,\ldots,b_n)^T, \quad V_1(0) = V_1.
\]

Then, we apply the Gram-Schmidt orthonormalization procedure in \( \mathbb{C}^n \) to the set of vectors \( (V_1(b_1,\ldots,b_n), V_2,\ldots,V_n) \). This provides the desired \( U(n) \)-valued function \( S(b_1,\ldots,b_n) \). In particular \( S(0) = S_0 \).

Now, the composition of the mapping \( (b_1,\ldots,b_n) \mapsto S(b_1,\ldots,b_n) \) followed by the projection \( (b_{1,1},\ldots,b_{n,1}) : U(2n,\mathbb{R}) \rightarrow \mathbb{R}^n \) is therefore the identity mapping in a neighbourhood of 0 and has rank \( n \). Hence the same is true for the projection \( (b_{1,1},\ldots,b_{n,1}) : U(2n,\mathbb{R}) \rightarrow \mathbb{R}^n \) at \( S_0 \).

\[ \square \]

Lemma 2.4. For every \( \epsilon > 0 \), define

\[ \chi_{\epsilon}(x,\xi) = \frac{1}{\epsilon^n} 1_Q \left( \frac{\xi}{\epsilon} \right), \]

where

\[ Q = \left[ -\frac{1}{2},\frac{1}{2} \right]^n \subset \mathbb{R}^n \quad \text{and} \quad 1_Q = \begin{cases} 1, & \xi \in Q \\ 0, & \xi \not\in Q \end{cases} \]

and

\[ \tilde{\chi}_{\epsilon}(z) = \frac{\chi_{\epsilon}(z)}{\int_{U(2n,\mathbb{R})} \chi_{\epsilon}(Sz) \, dS}, \quad z \in \mathbb{R}^{2n}. \]

Then we have

\[ \int_{U(2n,\mathbb{R})} \tilde{\chi}_{\epsilon}(Sz) \, dS = 1, \quad \forall z \in \mathbb{R}^{2n}. \]
and

\begin{equation}
\lim_{\epsilon \to 0^+} \int_{\mathbb{R}^{2n}} \tilde{\chi}_\epsilon(x, \xi) \Phi(x, \xi) dx d\xi = C \int_{\mathbb{R}^n} |x|^n \Phi(x, 0) dx,
\end{equation}

for some $C > 0$ and for every continuous function $\Phi$ on $\mathbb{R}^{2n}$ with a rapid decay at infinity.

**Proof.** We will show in a moment that, for $z = (x, \xi) \in \mathbb{R}^{2n}$,

\begin{equation}
\int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS \gtrsim \min\{\epsilon^{-n}, |z|^{-n}\}
\end{equation}

(with the convention, at $z = 0$, that $\min\{\epsilon^{-n}, +\infty\} = \epsilon^{-n}$). In particular, $\int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS \neq 0$, for every $z \in \mathbb{R}^{2n}$. Formula (20) then follows, because

\begin{equation}
\int_{U(2n, \mathbb{R})} \tilde{\chi}_\epsilon(Sz) dS = \int_{U(2n, \mathbb{R})} \frac{\chi_\epsilon(Sz)}{\int_{U(2n, \mathbb{R})} \chi_\epsilon(Uz) dU} dS = 1
\end{equation}

for every $z \in \mathbb{R}^{2n}$, since the Haar measure is right invariant.

Let us now prove (22). For $z = 0$ we have

\begin{equation}
\int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS = \frac{1}{\epsilon^n} \int_{U(2n, \mathbb{R})} dS = \frac{C_0}{\epsilon^n},
\end{equation}

with $C_0 = \text{meas}(U(2n, \mathbb{R})) > 0$. Consider now $z \neq 0$. Observe that the function

\begin{equation}
\Psi_\epsilon(z) := \int_{U(2n, \mathbb{R})} \chi_\epsilon(Sz) dS
\end{equation}

is constant on the orbits of $U(2n, \mathbb{R})$ in $\mathbb{R}^{2n}$, so that we can suppose

\begin{equation}
z = (x, 0), \quad x = (x_1, 0, \ldots, 0), \quad x_1 = |x| = |z| > 0.
\end{equation}

Now, by the definition of $\chi_\epsilon$ and $\Psi_\epsilon$,

\begin{equation}
\Psi_\epsilon(z) = \epsilon^{-n} \text{meas} \left\{ S = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in U(2n, \mathbb{R}) : |b_{i,1}| < \frac{\epsilon}{2|z|}, \quad i = 1, \ldots, n \right\},
\end{equation}

where $(b_{i,1})_{i=1,\ldots,n}$ is the first column of the matrix $B = (b_{i,j})_{i,j=1,\ldots,n}$.

Define, for $\mu > 0$,

\begin{equation}
f(\mu) = \text{meas} \left\{ S = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in U(2n, \mathbb{R}) : |b_{i,1}| < \mu, \quad i = 1, \ldots, n \right\}.
\end{equation}

Observe that $f(\mu)$ is non-decreasing and constant for $\mu \geq 1$. Moreover, from Lemma 2.3 we know that by setting $b_{i,1} = 0$, $i = 1, \ldots, n$, in $U(2n, \mathbb{R})$, we get a submanifold $\Sigma$ of codimension $n$, and the function $f(\mu)$ is the measure
of a tubular neighbourhood of Σ in \( U(2n, \mathbb{R}) \). Hence we have the asymptotic behaviour
\[
(24) \quad \mu^{-n} f(\mu) \to C_0 > 0, \quad \text{as} \quad \mu \to 0^+
\]
and in particular
\[
(25) \quad f(\mu) \gtrsim \min\{1, \mu^n\}.
\]
We then infer
\[
(26) \quad \Psi_\epsilon(z) = \epsilon^{-n} f\left(\frac{\epsilon}{2|z|}\right) \to \frac{C_1}{|z|^n}, \quad \text{as} \quad \epsilon \to 0^+
\]
locally uniformly in \( \mathbb{R}^{2n} \setminus \{0\} \), with \( C_1 = 2^{-n}C_0 \), and
\[
(27) \quad \Psi_\epsilon(z) \gtrsim \epsilon^{-n} \min\left\{1, \left(\frac{\epsilon}{|z|}\right)^n\right\} = \min\{\epsilon^{-n}, |z|^{-n}\},
\]
which is (22).

Let us finally prove (21). We are interested in the limit \( \epsilon \to 0^+ \), so we can assume \( \epsilon \leq 1 \). Consider a continuous function \( \Phi \) on \( \mathbb{R}^{2n} \) with rapid decay at infinity. By definition of \( \tilde{\chi}_\epsilon(z) \) in (19) we have
\[
\tilde{\chi}_\epsilon(x, \xi) = \frac{\epsilon^{-n}}{\Psi_\epsilon(x, \xi)} 1_{[-\epsilon/2, \epsilon/2]^n} (\xi)
\]
so that, by (27),
\[
|\tilde{\chi}_\epsilon(x, \xi)\Phi(x, \xi)| \lesssim \epsilon^{-n} (1 + |x|^n) 1_{[-\epsilon/2, \epsilon/2]^n} (\xi) |\Phi(x, \xi)| \in L^1(\mathbb{R}^{2n})
\]
for \( 0 < \epsilon \leq 1 \). Fubini’s Theorem then allows one to look at the first integral in (21) as an iterated integral
\[
I_\epsilon := \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \tilde{\chi}_\epsilon(x, \xi)\Phi(x, \xi) d\xi \right) dx
\]
and we apply the dominated convergence theorem to the integral with respect to the \( x \) variable as follows. Setting
\[
\Upsilon_\epsilon(x) := \int_{\mathbb{R}^n} \tilde{\chi}_\epsilon(x, \xi)\Phi(x, \xi) d\xi = \epsilon^{-n} \int_{[-\epsilon/2, \epsilon/2]^n} \frac{1}{\Psi_\epsilon(x, \xi)}\Phi(x, \xi) d\xi,
\]
by (26) we have, for every fixed \( x \neq 0 \),
\[
\Upsilon_\epsilon(x) \to C|x|^n \Phi(x, 0);
\]
for some constant \( C > 0 \). On the other hand \( \Upsilon_\epsilon(x) \) is dominated, using (27), by
\[
(1 + |x|^n) \sup_{\xi \in \mathbb{R}^n} |\Phi(x, \xi)| \in L^1(\mathbb{R}^n).
\]
Hence
\[
\lim_{\epsilon \to 0^+} I_\epsilon = \int_{\mathbb{R}^n} \lim_{\epsilon \to 0^+} \Upsilon_\epsilon(x) dx = C \int_{\mathbb{R}^n} |x|^n \Phi(x, 0) dx.
\]
This concludes the proof.

Remark 2.5. Observe that there are no conditions on the derivatives of the function $\Phi$ in (21).

3. Proofs of the main results

In what follows we prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. (i) First Step. Let us start with showing that formula (5) is true for any function $\psi$ in the Schwartz class $\mathcal{S}(\mathbb{R}^n) \subset M^p(\mathbb{R}^n)$, $1 \leq p < \infty$. Using the Gaussian $\varphi(t) = \frac{2^d}{d!} e^{-\pi |t|^2}$ as window function, we compute the $M^p$-norm of $\psi$ as in (4) and then use Lemma 2.4 so that

$$\|\psi\|_{M^p} = \int_{\mathbb{R}^{2n}} |\hat{\varphi}(z)|^p dSdz = \int_{\mathbb{R}^{2n}} \int_{U(2n,\mathbb{R})} \tilde{\chi}_\epsilon(Sz)|V_\varphi \psi(z)|^p dSdz$$

where in the last equality we used Lemma 2.1. Observe that, since $S$ is unitary and $\varphi$ is a Gaussian, $\hat{S}\varphi = c\varphi$, for some phase factor $c \in \mathbb{C}$, with $|c| = 1$ (see [15, Proposition 252]) and this phase factor is killed by the modulus obtaining $|V_{S\varphi} \hat{S}\psi(z)| = |V_\varphi \hat{S}\psi(z)|$. Continuing the above computation we infer

$$\|\psi\|_{M^p}^p = \int_{\mathbb{R}^{2n}} \tilde{\chi}_\epsilon(z) \int_{U(2n,\mathbb{R})} |V_\varphi \hat{S}\psi(z)|^p dSdz.$$

Set

$$\Phi(z) = \int_{U(2n,\mathbb{R})} |V_\varphi \hat{S}\psi(z)|^p dS.$$

The dominated convergence theorem guarantees that $\Phi$ is continuous on $\mathbb{R}^{2n}$, moreover $\Phi$ has rapid decay at infinity. This follows from Lemma 2.2.

Letting $\epsilon \to 0^+$ and using (21) we obtain

$$\|\psi\|_{M^p} = C \int_{\mathbb{R}^n} |x|^n \int_{U(2n,\mathbb{R})} |V_\varphi \hat{S}\psi(x,0)|^p dSdx$$

$$= C \int_{\mathbb{R}^n} |x|^n \int_{U(2n,\mathbb{R})} |\langle \hat{S}\psi, T_x \varphi \rangle|^p dSdx$$

$$= C \int_{\mathbb{R}^n} |x|^n \int_{U(2n,\mathbb{R})} |\langle \psi, \hat{S} T_x \varphi \rangle|^p dSdx.$$
The last equality is due to \( \langle \hat{S}\psi, T_x \varphi \rangle = \langle \psi, \hat{S}^{-1}T_x \varphi \rangle \) and the invariance of the Haar measure with respect to the change of variable \( S \to S^{-1} \).

**Second Step.** Consider \( f \in M^p(\mathbb{R}^n), \ 1 \leq p < \infty \). Using the density of the Schwartz class \( \mathcal{S}(\mathbb{R}^n) \) in \( M^p(\mathbb{R}^n) \) (cf. e.g., \[20, Chapter 12\]), there exists a sequence \( \{ \psi_k \}_k \subset \mathcal{S}(\mathbb{R}^n) \) such that \( \psi_k \to f \) in \( M^p(\mathbb{R}^n) \). This implies that \( \psi_k \to f \) in \( \mathcal{S}'(\mathbb{R}^n) \) and

\[
\langle \psi_k, \hat{S}T_x \varphi \rangle \to \langle \psi, \hat{S}T_x \varphi \rangle
\]

pointwise for every \( x \in \mathbb{R}^n, \ S \in U(2n, \mathbb{R}) \). Let us define, for every \( f \in M^p(\mathbb{R}^n) \),

\[
|||f||| = \left( \int_{\mathbb{R}^n \times U(2n, \mathbb{R})} |x|^n \langle f, \hat{S}T_x \varphi \rangle^p \, dx \, dS \right)^{\frac{1}{p}}.
\]

By Fatou's Lemma, for any \( f \in M^p(\mathbb{R}^n) \):

\[
|||f||| \leq \liminf_{k \to \infty} |||\psi_k||| \lesssim \liminf_{k \to \infty} \|\psi_k\|_{M^p} = \|f\|_{M^p}.
\]

It is easy to check that \( |||f||| \) is a seminorm on \( M^p(\mathbb{R}^n) \). Applying (29) to the difference \( f - \psi_k \) we obtain \( |||f - \psi_k||| \to 0 \) and hence \( |||\psi_k||| \to |||f||| \). By assumption we also have \( \|\psi_k\|_{M^p} \to \|f\|_{M^p} \), and the desired norm equivalence in (5) then extends from \( \mathcal{S}(\mathbb{R}^n) \) to \( M^p(\mathbb{R}^n) \).

**Third Step.** We will show that (6) easily follows from (5). Indeed, the Fourier transform \( \hat{J} = \mathcal{F} \) is a metaplectic operator and we recall that the Fourier transform is a topological isomorphism \( \mathcal{F} : M^p(\mathbb{R}^n) \to M^p(\mathbb{R}^n), \ 1 \leq p \leq \infty, \ [16] \). Furthermore, by the definition of the symplectic group (10), for any \( S \in U(2n, \mathbb{R}) \),

\[
J^{-1}S = (S^T)^{-1}J^{-1} = JS^{-1}
\]

for \( S^{-1} = S^T \). For any \( f \in M^p(\mathbb{R}^n) \), \( \|f\|_{M^p} \asymp \|\hat{f}\|_{M^p} \), and using (15),

\[
|\langle \hat{f}, ST_x \varphi \rangle| = |\langle f, \hat{J}^{-1}\hat{S}T_x \varphi \rangle| = |\langle f, \hat{S}\mathcal{F}^{-1}T_x \varphi \rangle| = |\langle f, \hat{S}M_x\mathcal{F}^{-1} \varphi \rangle| = |\langle f, \hat{S}M_x \varphi \rangle|
\]

since the Gaussian is an eigenvector of \( \mathcal{F}^{-1} \) with eigenvalue equal to 1. This immediately yields (6).

(ii) Case \( p = \infty \). Observe that any \( z \in \mathbb{R}^{2n} \) can be written as

\[
z = S^{-1} \begin{pmatrix} x \\ 0 \end{pmatrix},
\]
for some \( x \in \mathbb{R}^n \), \( S \in U(2n, \mathbb{R}) \), so that, for any \( f \in M^\infty(\mathbb{R}^n) \),

\[
\|f\|_{M^\infty(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n} |V_\varphi f(z)| \simeq \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} \left| V_\varphi \left( S^{-1} \begin{pmatrix} x \\ 0 \end{pmatrix} \right) \right|
\]

\[
= \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} |V_\varphi (Sf)(x,0)| = \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} |\langle Sf, T_x \varphi \rangle|
\]

\[
= \sup_{S \in U(2n, \mathbb{R})} \sup_{x \in \mathbb{R}^n} |\langle f, ST_x \varphi \rangle|,
\]

as desired.

We now prove the similar result, with the group \( U(2n, \mathbb{R}) \) replaced by the subgroup \( T^n \) (up to isomorphisms).

**Proof of Theorem 1.3.** (i) The proof uses a similar pattern to Theorem 1.2, replacing the group \( U(2n, \mathbb{R}) \) by \( T^n \). The preparation of Lemma 2.3 is no longer necessary. Lemma 2.4 must be revisited in this context as follows. Using the same notation, to estimate the function \( \Psi_\epsilon(z) \) we again observe that it is constant on the orbits of the torus \( T^n \), so that we can suppose

\[
z = (x,0), \quad x = (x_1, \ldots, x_n), \quad x_j = |x_j| = |z_j| > 0, \quad j = 1, \ldots, n.
\]

Then

\[
\Psi_\epsilon(z) = \epsilon^{-n} \text{meas} \left\{ S = \begin{pmatrix} e^{i\theta_1} & \cdots & \cdot & e^{i\theta_n} \\
 & \ddots & \cdots & \\
 & & \ddots & \\
 & & & \epsilon^{n} \\
\end{pmatrix} : \left| \sin \theta_j \right| < \frac{\epsilon}{2|z_j|}, \ j = 1, \ldots, n \right\}
\]

\[
= \epsilon^{-n} f \left( \frac{\epsilon}{2|z_1|} \right) \cdots f \left( \frac{\epsilon}{2|z_n|} \right),
\]

where now we set, for \( \mu > 0 \),

\[
f(\mu) = \text{meas} \left\{ \theta \in [0, 2\pi] : \left| \sin \theta \right| < \mu \right\}.
\]

We have

\[
\mu^{-1} f(\mu) \to C > 0 \quad \text{as} \quad \epsilon \to 0^+
\]

and

\[
f(\mu) \gtrsim \min \{ 1, \mu \}
\]

which gives

\[
\Psi_\epsilon(z) \to \frac{C}{|z_1 \cdots z_n|}, \quad \text{as} \quad \epsilon \to 0^+
\]

locally uniformly for \( z_1, \ldots, z_n \in \mathbb{R}^2 \setminus \{0\} \), and

\[
\Psi_\epsilon(z) \gtrsim \min \{ \epsilon^{-1}, |z_1|^{-1} \cdots |z_n|^{-1} \}.
\]

Using these estimates in place of (26) and (27), one can proceed as in the proofs of Lemma 2.4 and Theorem 1.2 and obtain the desired conclusion for \( f \in S(\mathbb{R}^n) \).
Next, using the same argument as in the Step 2 of the previous proof, one infers (34).

The characterization (35) has the same proof as the corresponding formula (6).

(ii) The $M^\infty$ case uses the same argument as in the proofs of (7) and (8), with the group $U(2n, \mathbb{R})$ replaced by $T^n$. \hfill \Box

4. Integral representations for the torus in terms of the factional Fourier transform

Observe that the symplectic matrix in $U(2n, \mathbb{R})$ corresponding to the complex matrix $S \in T^n$ in (9) via the isomorphism $\iota$ in (16) is given by

$$\iota(S) = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

with

$$A = \begin{pmatrix} \cos \theta_1 \\ & \ddots \\ & & \cos \theta_n \end{pmatrix}, \quad B = \begin{pmatrix} \sin \theta_1 \\ & \ddots \\ & & \sin \theta_n \end{pmatrix}.$$

Consider the case $\theta_i \neq k\pi$, $k \in \mathbb{Z}$, $i = 1, \ldots, n$. The matrix $\iota(S)$ is a free symplectic matrix and the related metaplectic operator possesses the integral representation (14). Since

$$AB^{-1} = B^{-1}A = \begin{pmatrix} \cos \theta_1 \\ & \ddots \\ & & \cos \theta_n \end{pmatrix},$$

the polynomial $W(x, x')$ becomes

$$W(x_1, \ldots, x_n, x'_1, \ldots, x'_n) = \sum_{i=1}^n \frac{1}{2 \sin \theta_i} (\cos \theta_i x_i^2 - 2x_i x'_i + \cos \theta_i x'_i^2)$$

and

$$\Delta(W) = \frac{c}{\sqrt{|\sin \theta_1 \cdots \sin \theta_n|}}$$

for some phase factor $c \in \mathbb{C}$, with $|c| = 1$. Hence we obtain, for $\psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\widehat{\iota(S)}\psi(x) = \frac{c}{\sqrt{|\sin \theta_1 \cdots \sin \theta_n|}} \int_{\mathbb{R}^n} e^{2\pi i W(x, x')} \psi(x') dx',$$

with $W(x, x')$ in (30). From (31) we deduce that $\widehat{\iota(S)}$ can be written as the composition of the operators

$$\iota(S) = \pm \iota(S_1) \cdots \iota(S_n),$$
where, for some phase factor $c$,

$$\hat{\iota}(S_i)\psi(x) = \frac{c}{\sqrt{|\sin \theta_i|}} \int_{\mathbb{R}} e^{\frac{mi}{2}(\cos \theta_i x_i^2 - 2x_i x'_i + \cos \theta_i x'_i^2)} \psi(x'_1, \ldots, x'_i, \ldots, x'_n) dx'_i.$$

Indeed if $\theta_i = \pi/2$, then $\hat{\iota}(S_i) = \pm \hat{J}$ is the Fourier transform with respect to the variable $x_i$. Otherwise, $\hat{\iota}(S_i) = \pm \mathcal{F}_{\theta_i}$, the $\theta_i$-angle partial fractional Fourier transform (again referred to the variable $x_i$).

Alternatively, the same conclusion (32) can be drawn by writing

$$(33) \quad S = \begin{pmatrix} e^{i\theta_1} & \cdots & \cdots & e^{i\theta_n} \\ \vdots & & & \vdots \\ e^{i\theta_1} & \cdots & \cdots & e^{i\theta_n} \end{pmatrix} = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & & & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta_1} & \cdots & \cdots & e^{i\theta_n} \\ \vdots & & & \vdots \\ e^{i\theta_1} & \cdots & \cdots & e^{i\theta_n} \end{pmatrix},$$

that is

$$S = S_1 \cdots S_i \cdots S_n,$$

with

$$S_i = \begin{pmatrix} 1 & \cdots & \cdots & 1 \\ \vdots & & & \vdots \\ e^{i\theta_i} & \cdots & \cdots & e^{i\theta_i} \\ \vdots & & & \vdots \\ 1 & \cdots & \cdots & 1 \end{pmatrix}, \quad i = 1, \ldots, n$$

so that

$$\hat{\iota}(S) = \hat{\iota}(S_1) \cdots \hat{\iota}(S_i) = \pm \hat{\iota}(S_1) \cdots \hat{\iota}(S_n).$$

If $\theta_i = 2k\pi$ for some $k \in \mathbb{Z}$, $\hat{\iota}(S_i) = \pm I$ with $I$ the identity operator. If $\theta_i = (2k + 1)\pi$ for some $k \in \mathbb{Z}$, $\hat{\iota}(S_i)\psi(x) = \pm \psi(x_1, \ldots, -x_i, \ldots, x_n)$.

Hence using the $\theta_i$-angle partial fractional Fourier transform $\mathcal{F}_{\theta_i} = \pm \hat{\iota}(S_i)$ we can rephrase Theorem 1.3 as follows.

**Theorem 4.1.** Let $\varphi$ be the Gaussian of Theorem 1.2.

(i) For $1 \leq p < \infty$, $f \in M^p(\mathbb{R}^n)$, we have

$$(34) \quad \|f\|_{M^p(\mathbb{R}^n)} \asymp \left( \int_{\mathbb{R}^n \times \mathbb{T}^n} |x_1 \ldots x_n| \langle f, \mathcal{F}_{\theta_1} \ldots \mathcal{F}_{\theta_n} T_{x \varphi} \rangle^p dx d\theta_1 \ldots d\theta_n \right)^{\frac{1}{p}},$$

or, similarly,

$$(35) \quad \|f\|_{M^p(\mathbb{R}^n)} \asymp \left( \int_{\mathbb{R}^n \times \mathbb{T}^n} |\xi_1 \ldots \xi_n| \langle f, \mathcal{F}_{\theta_1} \ldots \mathcal{F}_{\theta_n} M_{\xi \varphi} \rangle^p d\xi d\theta_1 \ldots d\theta_n \right)^{\frac{1}{p}}.$$
(ii) For $p = \infty$,
\[
\|f\|_{M^\infty(\mathbb{R}^n)} \simeq \sup_{S \in T^n} \sup_{x \in \mathbb{R}^n} |\langle f, \mathcal{F}_{\theta_1} \ldots \mathcal{F}_{\theta_n} T_x \varphi \rangle|.
\]

or
\[
\|f\|_{M^\infty(\mathbb{R}^n)} \simeq \sup_{S \in T^n} \sup_{\xi \in \mathbb{R}^n} |\langle f, \mathcal{F}_{\theta_1} \ldots \mathcal{F}_{\theta_n} M_\xi \varphi \rangle|.
\]

This concludes our study.

For sake of completeness, let us recall that integral representations involving metaplectic operators that do not arise from free symplectic matrices were studied in \cite{14, 22}.

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