EIGENVALUE ASYMPTOTICS, INVERSE PROBLEMS AND A TRACE FORMULA FOR THE LINEAR DAMPED WAVE EQUATION

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Abstract. We determine the general form of the asymptotics for Dirichlet eigenvalues of the one-dimensional linear damped wave operator. As a consequence, we obtain that given a spectrum corresponding to a constant damping term this determines the damping term in a unique fashion. We also derive a trace formula for this problem.

1. Introduction

Consider the one-dimensional linear damped wave equation on the interval \((0, 1)\), that is,

\[
\begin{aligned}
&w_{tt} + 2a(x)w_t = w_{xx} + b(x)w, \\
&w(0, t) = w(1, t) = 0, \\
&w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad x \in (0, 1)
\end{aligned}
\]

(1.1)

The eigenvalue problem associated with (1.1) is given by

\[
\begin{aligned}
&u_{xx} - (\lambda^2 + 2\lambda a - b)u = 0, \\
&u(0) = u(1) = 0,
\end{aligned}
\]

(1.2) (1.3)

and has received quite a lot of attention in the literature since the papers of Chen et al. [CFNS] and Cox and Zuazua [CZ]. In the first of these papers the authors derived formally an expression for the asymptotic behaviour of the eigenvalues of (1.2), (1.3) in the case of a zero potential \(b\), which was later proved rigorously in the second of the above

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papers. Following this, there were several papers on the subject which, among other things, extended the results to non-vanishing $b$ [BR], and showed that it is possible to design damping terms which make the spectral abscissa as large as desired [CC]. In [F2] the second author of the present paper addressed the inverse problem in arbitrary dimension giving necessary conditions for a sequence to be the spectrum of an operator of this type in the weakly damped case. As far as we are aware, these are the only results for the inverse problem associated with (1.2), (1.3). Other results for the $n$-dimensional problem include, for instance, the fact that in that case the decay rate is no longer determined solely by the spectrum [L], a study of some particular cases where the role of geometric optics is considered [AL], the asymptotic behaviour of the spectrum [S] and the study of sign-changing damping terms [F1].

The purpose of the present paper is twofold. On the one hand, we show that problem (1.2), (1.3) may be addressed in the same way as the classical Sturm–Liouville problem in the sense that, although this is not a self-adjoint problem, the methods used for the former problem may be applied here with similar results. This idea was already present in both [CFNS] and [CZ]. Here we take further advantage of this fact to obtain the full asymptotic expansion for the eigenvalues of (1.2), (1.3) (Theorem 1). Based on these similarities, we were also led to a (regularized) trace formula in the spirit of that for the Sturm–Liouville problem (Theorem 4).

On the other hand, the idea behind obtaining further terms in the asymptotics was to use this information to address the associated inverse spectral problem of finding all damping terms that give a certain spectrum. Our main result along these lines is to show that in the case of constant damping there is no other smooth damping term yielding the same spectrum (Corollary 2). Namely, we obtain the criterion for the damping term to be constant. Note that this is in contrast with the inverse (Dirichlet) Sturm–Liouville problem, where for each admissible spectrum there will exist a continuum of potentials giving the same spectrum [PT]. In particular this result shows that we should expect the inverse problem to be much more rigid in the case of the wave equation than it is for the Sturm–Liouville problem. This should be understood in the sense that, at least in the case of constant damping, it will not be possible to perturb the damping term without disturbing the spectrum, as is the case for the potential in the Sturm–Liouville problem.

The plan of the paper is as follows. In the next section we set the notation and state the main results of the paper. The proof of the
asymptotics of the eigenvalues is done in Sections 3 and 4, where in the first of these we derive the form of the fundamental solutions of equation (1.2), while in the second we apply a shooting method to these solutions to obtain the formula for the eigenvalues as zeros of an entire function – the idea is the same as that used in [CZ]. Finally, in Section 5 we prove the trace formula.

2. Notation and results

It is easy to check that if \( \lambda \) is an eigenvalue of the problem (1.2), (1.3), then \( \lambda \) is also an eigenvalue of the same problem. In view of this property, we denote the eigenvalues of this problem by \( \lambda_n, n \neq 0 \), and order them as follows

\[
\lambda_{n-2} \leq \lambda_{n-1} \leq \lambda_1 \leq \lambda_2 \leq \ldots
\]

while assuming that \( \lambda_{-n} = \overline{\lambda}_n \). We also suppose that possible zero eigenvalues are \( \lambda_{\pm 1} = \lambda_{\pm 2} = \ldots = \lambda_{\pm p} = 0 \). If \( p = 0 \), the problem (1.2), (1.3) has no zero eigenvalues. For any function \( f = f(x) \) we denote \( \langle f \rangle := \int_0^1 f(x) \, dx \).

**Theorem 1.** Suppose \( a \in C^{m+1}[0,1], b \in C^m[0,1], m \geq 1 \). The eigenvalues of (1.2), (1.3) have the following asymptotic behaviour as \( n \to \pm \infty \):

\[
\lambda_n = \pi n i + \sum_{j=0}^{m-1} c_j n^{-j} + O(n^{-m}),
\]

were the \( c_j \)'s are numbers which can be determined explicitly. In particular,

\[
c_0 = -\langle a \rangle, \quad c_1 = \frac{\langle a^2 + b \rangle}{2\pi i}, \quad c_2 = \frac{1}{2\pi^2} \left[ \langle a(a^2 + b) \rangle - \langle a \rangle \langle a^2 + b \rangle + \frac{a'(1) - a'(0)}{2} \right].
\]

A straightforward consequence of the fact that the spectrum determines the average as well as the \( L^2 \) norm of the damping term (assuming \( b \) fixed) is that the spectrum corresponding to the constant damping determines this damping uniquely.

**Corollary 2.** Assume that \( a \in C^3[0,1], \lambda_n \) are the eigenvalues of the problem (1.2), (1.3), the function \( b \in C^2[0,1] \) is fixed, and the formula (2.7) gives the asymptotics for these eigenvalues. Then the function \( a(x) \) is constant, if and only if

\[
c_0^2 = 2\pi ic_1 - \langle b \rangle,
\]
in which case \( a(x) \equiv -c_0 \).

In the same way, the asymptotic expansion allows us to derive other spectral invariants in terms of the damping term \( a \). However, these do not have such a simple interpretation as in the case of the above constant damping result.

**Corollary 3.** Suppose \( b \equiv 0 \), \( a_i(x) = a_0(x) + \tilde{a}_i(x) \), \( i = 1, 2 \), where \( a_0(1 - x) = a_0(x) \), \( \tilde{a}_i(1 - x) = -\tilde{a}_i(x) \), \( \tilde{a}_i, a_0 \in C^4[0, 1] \), and for \( a = a_i \) the problems (1.2), (1.3) have the same spectra. Then

\[
\langle \tilde{a}_1^2 \rangle = \langle \tilde{a}_2^2 \rangle, \quad \langle a_0 \tilde{a}_1^2 \rangle = \langle a_0 \tilde{a}_2^2 \rangle
\]

is valid.

From Theorem 1 we have that the quantity \( \text{Re}(\lambda_n - c_0) \) behaves as \( O(n^{-2}) \) as \( n \to \infty \). This means that the series

\[
\sum_{n=0}^{\infty} (\lambda_n - c_0) = 2 \sum_{n=1}^{\infty} \text{Re}(\lambda_n - c_0)
\]

converges. In the following theorem we express the sum of this series in terms of the function \( a \). This is in fact the formula for the regularized trace.

**Theorem 4.** Let \( a \in C^3[0, 1] \), \( b \in C^2[0, 1] \). Then the identity

\[
\sum_{n=-\infty}^{\infty} (\lambda_n - c_0) = \frac{a(0) + a(1)}{2} - \langle a \rangle
\]

holds.

### 3. Asymptotics for the Fundamental System

In this section we obtain the asymptotic expansion for the fundamental system of the solutions of the equation (1.2) as \( \lambda \to \infty \), \( \lambda \in \mathbb{C} \). This is done by means of the standard technique described in, for instance, [E, Ch. IV, Sec. 4.2, 4.3], [Fe, Ch. II, Sec. 3].

We begin with the formal construction assuming the asymptotics to be of the form

\[
(3.1) \quad u_{\pm}(x, \lambda) = e^{\pm \lambda x \pm \frac{\lambda}{2} \phi_{\pm}(t, \lambda)} dt,
\]

where

\[
(3.2) \quad \phi_{\pm}(x, \lambda) = \sum_{i=0}^{m} \phi_{\pm}^{(i)}(x) \lambda^{-i} + O(\lambda^{-m-1}), \quad m \geq 1.
\]
In what follows we assume that $a \in C^{m+1}[0, 1], b \in C^m[0, 1]$. We substitute the series (3.1), (3.2) into (1.2) and equate the coefficients of the same powers of $\lambda$. It leads us to a recurrent system of equations determining $\phi_i^{(\pm)}$ which read as follows:

(3.3) \[ \phi_0^{(\pm)} = a, \]
(3.4) \[ \phi_1^{(\pm)} = -\frac{1}{2}(\pm a' + a^2 + b), \]
(3.5) \[ \phi_i^{(\pm)} = -\frac{1}{2} \left( \pm \phi_{i-1}^{(\pm)} + \sum_{j=0}^{i-1} \phi_j^{(\pm)} \phi_{i-j-1}^{(\pm)} \right), \quad i \geq 2. \]

The main aim of this section is to prove that there exist solutions to (1.2) having the asymptotics (3.1), (3.2). In other words, we are going to justify these asymptotics rigorously. We will do this for $u_+$, the case of $u_-$ following along similar lines.

Let us write

\[ U_m(x, \lambda) = e^{\lambda x + \sum_{i=0}^{m} \phi_i^{(\pm)}(0) \lambda^{-i}}, \]

In view of the assumed smoothness for $a$ and $b$ we conclude that $U_m \in C^2[0, 1]$. It is also easy to check that

\[ U''_m - \lambda^2 U_m - 2\lambda a U_m + b U_m = \lambda^{-m} e^{\lambda x} f_m(x, \lambda), \quad x \in [0, 1], \]

where the function $f_m$ satisfies the estimate

\[ |f_m(x, \lambda)| \leq C_m \]

uniformly for large $\lambda$ and $x \in [0, 1]$. We consider first the case $\text{Re} \lambda \geq 0$. Differentiating the function $u_+$ formally we see that

\[ u'_+(0, \lambda) = \lambda + \phi_+(0, \lambda) = \lambda + \sum_{i=0}^{m} \phi_i^{(\pm)}(0) \lambda^{-i} + O(\lambda^{-m-1}). \]

Let

\[ A_0(\lambda) = \lambda + \sum_{i=0}^{m} \phi_i^{(\pm)}(0) \lambda^{-i}, \]

and $u_+ (x, \lambda)$ be the solution to the Cauchy problem for the equation (1.2) subject to the initial conditions

\[ u_+(0, \lambda) = 1, \quad u'_+(0, \lambda) = A_0(\lambda). \]
We introduce one more function \( w_m(x, \lambda) = u_+(x, \lambda)/U_m(x, \lambda) \). This function solves the Cauchy problem
\[
(U_m^2 w'_m)' + \lambda^{-m} U_m e^{\lambda x} f_m w_m = 0, \quad x \in [0, 1],
\]
\[
w_m(0, \lambda) = 1, \quad w'_m(0, \lambda) = 0.
\]
The last problem is equivalent to the integral equation
\[
w_m(x, \lambda) + \lambda^{-m}(K_m(\lambda)w_m)(x, \lambda) = 1,
\]
\[
(K_m(\lambda)w_m)(x, \lambda) := \int_0^x U_m^{-2}(t_1) \int_0^{t_1} U_m(t_2) e^{\lambda t_2} f_m(t_2, \lambda)w_m(t_2, \lambda) \, dt_2 \, dt_1.
\]

Since \( \text{Re} \lambda \geq 0 \) for \( 0 \geq t_2 \geq t_1 \geq 1 \), the estimate
\[
|U_m^{-2}(t_1, \lambda)U_m(t_2, \lambda)e^{\lambda t_1}| \leq C_m
\]
holds true, where the constant \( C_m \) is independent of \( \lambda, t_1, t_2 \). Hence, the integral operator \( K_m : C[0, 1] \to C[0, 1] \) is bounded uniformly in \( \lambda \) large enough, \( \text{Re} \lambda \geq 0 \). Employing this fact, we conclude that
\[
w_m(x) = 1 + \mathcal{O}(\lambda^{-m}), \quad \lambda \to \infty, \quad \text{Re} \lambda \geq 0,
\]
in the \( C^2[0, 1] \)-norm. Hence, the formula (3.7), where
\[
\phi_+(x, \lambda) = \sum_{i=0}^{m-1} \phi_+^{(i)}(x) \lambda^{-i} + \mathcal{O}(\lambda^{-m}),
\]
gives the asymptotic expansion for the solution of the Cauchy problem (1.2), (3.6) as \( \lambda \to \infty \), \( \text{Re} \lambda \geq 0 \).

Suppose now that \( \text{Re} \lambda \leq 0 \). Let \( A_1(\lambda), A_2(\lambda) \) be functions having the asymptotic expansions
\[
A_1(\lambda) = \lambda + \sum_{i=0}^{m} \lambda^{-i} \int_0^1 \phi_+^{(i)}(x) \, dx, \quad A_2(\lambda) = \lambda + \sum_{i=0}^{m} \phi_+^{(i)}(1) \lambda^{-i}.
\]
We define the function \( \widetilde{u}_+(x, \lambda) \) as the solution to the Cauchy problem for equation (1.2) subject to the initial conditions
\[
\widetilde{u}_+(1, \lambda) = e^{A_1(\lambda)}, \quad \widetilde{u}'_+(1, \lambda) = A_2(\lambda)e^{A_1(\lambda)}.
\]
In a way analogous to the arguments given above, it is possible to check that the function \( \widetilde{u}_+ \) has the asymptotic expansion (3.1) in the \( C^2[0, 1] \)-norm as \( \lambda \to +\infty \), \( \text{Re} \lambda \leq 0 \). Hence, \( \widetilde{u}_+(0, \lambda) = 1 + \mathcal{O}(\lambda^{-m}) \) for each \( m \geq 1 \). In view of this identity we conclude that the function \( u_+(x, \lambda) := \widetilde{u}_+(x, \lambda)/\widetilde{u}_+(0, \lambda) \) is a solution to (1.2), satisfies the condition \( u_+(0, \lambda) = 1 \), and has the asymptotic expansion (3.1), where the asymptotics for \( \phi_+ \) is given in (3.7).
For convenience we summarize the obtained results in

**Lemma 3.1.** Let \( a \in C^{m+1}[0,1], b \in C^{m}[0,1] \). There exist two linear independent solutions to the equation (1.2) satisfying the initial condition \( u_\pm(0, \lambda) = 1 \) and having the asymptotic expansions (3.1) in the \( C^2[0,1] \)-norm as \( \lambda \to \infty, \lambda \in \mathbb{C} \), where

\[
\phi_\pm(x, \lambda) = \sum_{i=0}^{m-1} \phi_i^{(\pm)}(x)\lambda^{-i} + O(\lambda^{-m}).
\]

### 4. Asymptotics of the eigenvalues

This section is devoted to the proof of Theorem 1 and Corollaries 2 and 3. We assume that \( a \in C^{m+1}[0,1], b \in C^{m}[0,1], m \geq 1 \).

Let \( u = u(x, \lambda) \) be the solution to (1.2) subject to the initial conditions \( u(0, \lambda) = 0, u'(0, \lambda) = 1 \). Denote \( \gamma_0 := u(1, \lambda) \). The function \( \gamma_0 \) is entire, and its zeros coincide with the eigenvalues of the problem (1.2), (1.3). It follows from Lemma 3.1 that, for \( \lambda \) large enough the function \( u(x, \lambda) \) can be expressed in terms of \( u_\pm \) by

\[
u(x, \lambda) = \frac{u_+(x, \lambda) - u_-(x, \lambda)}{u'_+(0, \lambda) - u'_-(0, \lambda)}.
\]

The denominator is non-zero, since due to (3.1)

\[
u'_+(0, \lambda) - \nu'_-(0, \lambda) = 2\lambda + 2\langle a \rangle + O(\lambda^{-1}), \quad \lambda \to \infty.
\]

Thus, for \( \lambda \) large enough

\[
(4.1) \quad \gamma_0(\lambda) = \frac{u_+(1, \lambda) - u_-(1, \lambda)}{u'_+(0, \lambda) - u'_-(0, \lambda)}.
\]

**Lemma 4.1.** For \( n \) large enough, the set

\[
Q := \{ \lambda : |\Re \lambda| < \pi n + \pi/2, |\Im \lambda| < \pi n + \pi/2 \}
\]

contains exactly \( 2n \) eigenvalues of the problem (3.1), (3.3).

**Proof.** Let \( \gamma_1(\lambda) := \gamma_0(\lambda)e^{\lambda a(0)} \). The zeros of \( \gamma_1 \) are those of \( \gamma_0(\lambda) \). For \( \lambda \) large enough we represent the function \( \gamma_1(\lambda) \) as

\[
\gamma_1(\lambda) = \gamma_2(\lambda) + \gamma_3(\lambda), \quad \gamma_2 := \frac{e^{2(\lambda + a(0))} - 1}{2(\lambda + a(0))},
\]

\[
\gamma_3(\lambda) = -\gamma_2(\lambda) \frac{2\lambda(\lambda + a(0)) - e^{\lambda(\phi_+(.,\lambda))} - e^{-\lambda(\phi_-(.,\lambda))}}{2(\lambda + a(0)) + \phi_+(0, \lambda) + \phi_-(0, \lambda)}.
\]

\[
+ \frac{e^{2(\lambda + a(0))} - 1}{2(\lambda + a(0)) + \phi_+(0, \lambda) + \phi_-(0, \lambda)}.
\]

\[
\]
\[ \tilde{\phi}_\pm(x, \lambda) := \lambda^{-1}(\phi_\pm(x, \lambda) - a(x)). \]

It is clear that for \( \lambda \) large enough the function \( \gamma_3(\lambda) \) satisfies an uniform in \( \lambda \) estimate
\[
|\gamma_3(\lambda)| \leq C|\lambda|^{-2}(|\gamma_2(\lambda)| + 1).
\]

One can also check easily that
\[
|\gamma_2(\lambda)| \geq C|\lambda|, \quad \lambda \in \partial K, \quad \text{if } n \text{ is large enough.}
\]

These two last estimates imply that
\[
|\gamma_3(\lambda)| \leq |\gamma_2(\lambda)| \quad \text{as } \lambda \in \partial K, \quad \text{if } n \text{ is large enough.}
\]

By Rouché theorem we conclude that for such \( n \) the function \( \gamma_1 \) has the same amount of zeros inside \( Q \) as the function \( \gamma_2 \) does. Since the zeros of the latter are given by \( \pi n i - \langle a \rangle, \ n \neq 0 \), this completes the proof. \( \Box \)

**Proof of Theorem 1.** Assume first that \( a \in C^2[0,1], \ b \in C^1[0,1] \). As was mentioned above, the eigenvalues of problem (1.2), (1.3) are the zeros of the function \( \gamma_0(\lambda) = 0 \). It follows from Lemma 4.1 that these eigenvalues tend to infinity as \( n \to \infty \). By Lemma 3.1, for \( \lambda \) large enough the equation \( \gamma_0(\lambda) = 0 \) becomes
\[
e^{2\lambda + \langle \phi_+(\cdot, \lambda) + \phi_-(\cdot, \lambda) \rangle} = 0
\]
which may be rewritten as
\[
2\lambda + 2\langle a \rangle + O(\lambda^{-1}) = 2\pi ni, \quad n \in \mathbb{Z}.
\]
If we now replace \( \phi_\pm \) by the leading terms of their asymptotic expansions we obtain
\[
2\lambda + 2\langle a \rangle + O(\lambda^{-1}) = 2\pi ni,
\]
\[
\lambda = \pi ni - \langle a \rangle + o(1), \quad n \to \infty.
\]
Hence, the eigenvalues behave as \( \lambda \sim \pi ni - \langle a \rangle \) for large \( n \). Moreover, it follows from Lemma 4.1 that it is exactly the eigenvalue \( \lambda_n \) which behaves as
\[
\lambda_n = \pi ni - \langle a \rangle + o(1), \quad n \to \infty.
\]
It follows from this identity and (4.3) that
\[
\lambda_n = \pi ni - \langle a \rangle + O(n^{-1}), \quad n \to \infty,
\]
and we complete the proof in the case \( m = 1 \). If \( m = 2 \), we substitute the above identity and (3.1) into (4.2) and get
\[
\lambda_n + \langle a \rangle + \frac{1}{\lambda_n^2}(\phi_1^{(+) \phi_1^{(-)})} + O(\lambda_n^{-2}) = \pi ni,
\]
\[
\lambda_n = \pi ni - \langle a \rangle - \frac{\langle \phi_1^{(+) \phi_1^{(-)} \rangle}}{\pi ni} + O(n^{-2}).
\]
The last formula and the identities (3.4) yield formulas (2.2) for \( c_0 \) and \( c_1 \). Repeating the described procedure one can easily check that the asymptotics (2.1), (2.2) hold true.

**Proof of Corollary 2.** The coefficients \( c_0, c_1 \) in the asymptotics (2.1) are determined by the formulas (2.2) and, by the Cauchy-Schwarz inequality, we thus obtain

\[
c_0^2 = \langle a \rangle^2 \leq \langle a^2 \rangle = 2\pi ic_1 - \langle b \rangle,
\]

with equality if and only if \( a(x) \) is a constant function. This fact completes the proof.

**Proof of Corollary 3.** It follows from (2.2), (2.3) that

\[
\langle a_1^2 \rangle = \langle a_2^2 \rangle, \quad \langle a_1^3 \rangle = \langle a_3^2 \rangle.
\]

Now we check that

\[
\langle a_i^2 \rangle = \langle a_0^2 \rangle + \langle \tilde{a}_i^2 \rangle, \quad \langle a_i^3 \rangle = \langle a_0^3 \rangle + 3\langle a_0 \tilde{a}_i^2 \rangle, \quad i = 1, 2,
\]

and arrive at the statement of the theorem.

### 5. Regularized Trace Formulas

In this section we prove Theorem 4. We follow the idea employed in the proof of the similar trace formula for the Sturm-Liouville operators in [LS, Ch. I, Sec. 14].

We begin by defining the function

\[
\Phi(\lambda) := \lambda^{2p} \prod_{n=p+1}^{\infty} \left( 1 - \frac{\lambda}{\lambda_n} \right) \left( 1 - \frac{\lambda}{\bar{\lambda}_n} \right).
\]

The above product converges, since

\[
\left( 1 - \frac{\lambda}{\lambda_n} \right) \left( 1 - \frac{\lambda}{\bar{\lambda}_n} \right) = 1 + \frac{\lambda^2 - 2\lambda \Re \lambda_n}{|\lambda_n|^2},
\]

and by Theorem 2 we have

\[
|\lambda_n|^2 = \pi^2 n^2 - 2\pi ic_1 + c_0^2 + \mathcal{O}(n^{-2}), \quad \Re \lambda_n = c_0 + \mathcal{O}(n^{-2})
\]

as \( n \to +\infty \). Proceeding in the same way as in the formulas (14.8), (14.9) in [LS, Ch. I, Sec. 14], we obtain

\[
\Phi(\lambda) = \frac{C_0 \Psi(\lambda) \sinh \lambda}{\lambda},
\]

\[
\Psi(\lambda) := \prod_{n=1}^{\infty} \left( 1 - \frac{\pi^2 n^2 - |\lambda_n|^2 + 2\lambda \Re \lambda_n}{\pi^2 n^2 + \lambda^2} \right),
\]
\[ C_0 := (\pi n)^{2p} \prod_{n=p+1}^{\infty} \frac{\pi n^2}{|\lambda_n|^2}. \]

In what follows we assume that \( \lambda \) is real, positive and large. In the same way as in [LS, Ch. I, Sec. 14] it is possible to derive the formula
\[
(5.2) \quad \ln \Psi(\lambda) = - \sum_{k=1}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \left( \frac{\pi^2 n^2 - |\lambda_n|^2 + 2 \lambda \text{Re} \lambda_n}{\pi^2 n^2 + \lambda^2} \right)^k.
\]

Our aim is to study the asymptotic behaviour of \( \ln \Psi(\lambda) \) as \( \lambda \to +\infty \). Employing the same arguments as in the proof of Lemma 14.1 and in the equation (14.11) in [LS, Ch. I, Sec. 14], we arrive at the estimate
\[
(5.3) \quad \sum_{k=3}^{\infty} \frac{1}{k} \sum_{n=1}^{\infty} \left( \frac{\pi^2 n^2 - |\lambda_n|^2 + 2 \lambda \text{Re} \lambda_n}{\pi^2 n^2 + \lambda^2} \right)^k = O(\lambda^{-3}), \quad \lambda \to +\infty,
\]
where \( c \) is a constant independent of \( k \) and \( n \). Let us analyze the asymptotic behaviour of the first two terms in the series (5.2). As \( k = 1 \), we have
\[
(5.4) \quad \sum_{n=1}^{\infty} \frac{\pi^2 n^2 - |\lambda_n|^2 + 2 \lambda \text{Re} \lambda_n}{\pi^2 n^2 + \lambda^2} = \sum_{n=1}^{\infty} \frac{\pi^2 n^2 - |\lambda_n|^2 - 2 \pi i c_1 + c_0^2}{\pi^2 n^2 + \lambda^2} + (2 \pi i c_1 - c_0^2 + 2 \lambda c_0) \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2 + \lambda^2} + 2 \lambda^{-1} S - 2 \lambda^{-1} \sum_{n=1}^{\infty} \frac{\pi^2 n^2 (\text{Re} \lambda_n - c_0)}{\pi^2 n^2 + \lambda^2},
\]
where \( S := \sum_{n=1}^{\infty} (\text{Re} \lambda_n - c_0) \).

Taking into account (5.1), we have
\[
\left| \sum_{n=1}^{\infty} \frac{\pi^2 n^2 - |\lambda_n|^2 + 2 \pi i c_1 + c_0^2}{\pi^2 n^2 + \lambda^2} \right| \leq C \sum_{n=1}^{\infty} \frac{1}{n^2 (\pi^2 n^2 + \lambda^2)} = \frac{\pi^2 3 + \lambda^2 - 3 \coth \lambda}{6} \leq C \lambda^{-2},
\]
\[
\left| \sum_{n=1}^{\infty} \frac{(\text{Re} \lambda_n - c_0)\pi^2 n^2}{\pi^2 n^2 + \lambda^2} \right| \leq C \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2 + \lambda^2} \leq C\lambda^{-1},
\]
where the constant \( C \) is independent of \( \lambda \). Here we have also used the formula
\[
\sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2 + \lambda^2} = \frac{\lambda \coth \lambda - 1}{2\lambda^2} = \frac{\lambda^{-1} - \lambda^{-2}}{2} + O(\lambda^{-1} e^{-2\lambda}),
\]
as \( \lambda \to +\infty \). We employ this formula to calculate the remaining terms in (5.4) and arrive at the identity
\[
\sum_{n=1}^{\infty} \frac{\pi^2 n^2 - |\lambda_n|^2 + 2\lambda \text{Re} \lambda_n}{\pi^2 n^2 + \lambda^2} = c_0 + \left( 2S - c_0 - \frac{\zeta_0^2}{2} + i\pi c_1 \right) \lambda^{-1} + O(\lambda^{-2}),
\]
as \( \lambda \to +\infty \). For \( k = 2 \) we proceed in the similar way,
\[
\sum_{n=1}^{\infty} \left( \frac{\pi^2 n^2 - |\lambda_n|^2 + 2\lambda \text{Re} \lambda_n}{\pi^2 n^2 + \lambda^2} \right)^2 = \sum_{n=1}^{\infty} \frac{(\pi^2 n^2 - |\lambda_n|^2)^2}{(\pi^2 n^2 + \lambda^2)^2} - 2\lambda \sum_{n=1}^{\infty} \frac{(\pi^2 n^2 - |\lambda_n|^2) \text{Re} \lambda_n}{(\pi^2 n^2 + \lambda^2)^2} + 4\lambda^2 c_0^2 \sum_{n=1}^{\infty} \frac{1}{(\pi^2 n^2 + \lambda^2)^2} + 4\lambda^2 \sum_{n=1}^{\infty} \frac{(\text{Re} \lambda_n)^2 - c_0^2}{(\pi^2 n^2 + \lambda^2)^2}.
\]
By differentiating (5.5) we obtain
\[
\sum_{n=1}^{\infty} \frac{1}{(\pi^2 n^2 + \lambda^2)^2} = \frac{\lambda \coth \lambda - 2 - \lambda^2(1 - \coth^2 \lambda)}{4\lambda^4}.
\]
This identity and (5.1) yield that as \( \lambda \to +\infty \)
\[
\sum_{n=1}^{\infty} \frac{(\pi^2 n^2 - |\lambda_n|^2)^2}{(\pi^2 n^2 + \lambda^2)^2} \leq C \sum_{n=1}^{\infty} \frac{1}{(\pi^2 n^2 + \lambda^2)^2} \leq C\lambda^{-3},
\]
\[
\sum_{n=1}^{\infty} \frac{(\pi^2 n^2 - |\lambda_n|^2) \text{Re} \lambda_n}{(\pi^2 n^2 + \lambda^2)^2} \leq C \sum_{n=1}^{\infty} \frac{1}{(\pi^2 n^2 + \lambda^2)^2} \leq C\lambda^{-3},
\]
\[
\sum_{n=1}^{\infty} \frac{(\text{Re} \lambda_n)^2 - c_0^2}{(\pi^2 n^2 + \lambda^2)^2} \leq C \sum_{n=1}^{\infty} \frac{1}{(\pi^2 n^2 + \lambda^2)^2} \leq C\lambda^{-3}.
\]
Hence,
\[
\sum_{n=1}^{\infty} \left( \frac{\pi^2 n^2 - |\lambda_n|^2 + 2\lambda \text{Re} \lambda_n}{\pi^2 n^2 + \lambda^2} \right)^2 = c_0^2 \lambda^{-1} + O(\lambda^{-2})
\]
as \( \lambda \to +\infty \). It follows from (5.2), (5.3), (5.6), (5.7) that
\[
\ln \Psi(\lambda) = -c_0 - (2S - c_0 + i\pi c_1)\lambda^{-1} + O(\lambda^{-2}),
\]
(5.8) \( \Phi(\lambda) = \frac{C_0 e^{-c_0} \sinh \lambda}{\lambda} \left[ 1 - (2S - c_0 + i\pi c_1)\lambda^{-1} + O(\lambda^{-2}) \right] \)
\[
\text{as } \lambda \to +\infty. \text{ It follows from (4.1) and Lemma 3.1 that for } \lambda \text{ large enough the estimate}
\]
\[
|\gamma_0(\lambda)| \leq C|\lambda|^{-1}e^{|\lambda|}
\]
holds true. Hence, the order of the entire function \( \gamma_0(\lambda) \) is one. In view of Theorem 4 we also conclude that the series \( \sum_{n=p+1}^{\infty} |\lambda_n|^{-2} \) converges and therefore the genus of the canonical product associated with \( \gamma_0 \) is one. We apply Hadamard’s theorem (see, for instance, [Le, Ch. I, Sec. 10, Th. 13]) and obtain that
\[
\gamma_0(\lambda) = e^{P(\lambda)} \Phi(\lambda), \quad P(\lambda) = \alpha_1 \lambda + \alpha_0 + 2 \sum_{n=p+1}^{\infty} |\lambda_n|^{-2} \text{Re } \lambda_n,
\]
where \( \alpha_1, \alpha_0 \) are some numbers. Hence, due to (5.8), it follows that \( \gamma_0 \) behaves as
\[
\gamma_0(\lambda) = \frac{C_0 e^{P(\lambda)} \sinh \lambda}{\lambda} \left[ 1 - (2S - c_0 + i\pi c_1)\lambda^{-1} + O(\lambda^{-2}) \right],
\]
as \( \lambda \to +\infty \). On the other hand, Lemma 3.1 and (4.1) imply that
\[
\gamma_0(\lambda) = \frac{e^{\lambda + (a)}}{2\lambda} \left[ 1 + (\langle \phi_1^{(+)} \rangle - a(0))\lambda^{-1} + O(\lambda^{-2}) \right] + O(\lambda^{-1}e^{-\lambda}),
\]
as \( \lambda \to +\infty \). Comparing the last two identities yields \( \alpha_1 = 0 \),
\[
C_0 e^{\lambda_0 - c_0 + 2} \sum_{n=1}^{\infty} |\lambda_n|^{-2} \text{Re } \lambda_n = e^{(a)}
\]
and
\[
-(2S - c_0 + i\pi c_1) = \langle \phi_1^{(+)} \rangle - a(0).
\]
It now follows from (2.2), (3.4) that
\[
\sum_{n=n_0}^{\infty} (\lambda_n - c_0) = 2S = c_0 + a(0) - \langle \phi_1^{(+)} \rangle - i\pi c_1 = \frac{a(0) + a(1)}{2} - \langle a \rangle,
\]
completing the proof of Theorem 4.

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