Methods of the Theory of Radiation Transfer for Bathymetry Problems

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Abstract. Based on the mathematical model of the propagation of an acoustic signal in a fluctuating medium, the inverse problem is formulated, which includes determination a function that describes the deviation of the bottom level from the average specified horizontal plane. In the double scattering approximation and the narrow directivity pattern of the receiving antenna, the solution of the direct problem is obtained. As a solution of the inverse problem, a nonlinear differential equation is obtained for the function describing the deviation of the seafloor relief. A numerical analysis of the solution is carried out.

1. Introduction

The study of the ocean is still a priority for the world community. A large number of research complexes are being developed to solve bathymetry problems (see e.g. [1-4]). Autonomous underwater robotic systems management vehicles [7]. Nowadays, the problem of mapping the ocean floor using side scan sonars (SSS) which were equipped of an autonomous unmanned underwater vehicle, is very relevant and promising [6, 8]. The study of the ocean properties by using SSS generates a very interesting problem of determining the relief of the seafloor. The sonar operation is based on the periodic emission of pulsed sound parcels and the detection of reflected echo signal from remote seabed areas. When the sonar antenna is moved, an acoustic image is formed on both sides of the underwater vehicle.

2. Formulation of the problem

The process of propagation of acoustic radiation is described by the transfer equation [1-5, 9]:

\[
\frac{1}{c} \frac{\partial j(r, k, t)}{\partial t} + k \cdot V_j l(r, k, t) + \mu l(r, k, t) = \frac{\sigma}{2\pi} \int_{\Omega} l(r, k', t) d\Omega + J(r, k, t),
\]

(1)

here \( r \in \mathbb{R}^2, t \in [0, T] \) and wave vector \( k \) belongs to the unique sphere \( \Omega = \{k \in \mathbb{R}^2; |k| = 1\} \). The function \( l(r, k, t) \) is interpreted as radiation intensity of wave in moment \( t \) in point \( r \), propagated in the direction \( k \) with constant velocity \( c \). The coefficients \( \mu \) and \( \sigma \) denote the attenuation and the scattering, correspondingly. \( J(r, k, t) \) describes the density of inner sources.

The process of echo signal propagation occurs in the domain \( G := \{r \in \mathbb{R}^2; \gamma_2 > -l + u(r_1)\} \), which is the upper half-space bounded from below by the curve,
\[ \partial G = \gamma = \{ y \in \mathbb{R}^2 : y_2 = -l + u(y_1) \} \], where the function \( u(y_1) \) describes the change of the ocean bottom relief.

We assume that the function \( J(r, k, t) \) describes a point isotropic sound source [7-10]:
\[
J(r, k, t) = J_0 \delta(r) \delta(t),
\]
where \( \delta \) – Dirac delta function and \( J_0 \) – source power.

Initial and boundary conditions for (1) [6,7]:
\[
I(y, k, t) = 2\sigma_d \int_{\Omega_+ (y)} \mid n(y) \cdot k' \mid I(y, k', t) dk',
\]
where \( \Omega_+ (y) = \{ k \in \Omega, sgn(k \cdot n(y)) = \pm 1 \} \), \( \sigma_d \) is the constant seabottom reflection coefficient, \( n(y) \) denotes external normal to \( \partial G \).

The solution of the initial-boundary problem (1), (3), (4) is deduced to the integral equation:
\[
I(r, k, t) = \int_0^{d(r, -k)} \exp(-\mu t') J_0 \left( r - t' k, k, t - \frac{t'}{c} \right) dt +
+ 2\sigma_d \exp(-\mu d(r, -k)) \int_{\Omega_+ (y-d(r, -k))} \mid n \cdot k' \mid I_{t'} \left( r - d(r, -k) k, k', t - \frac{d(r, -k)}{c} \right) dk' +
+ \int_0^{\frac{d(r, -k)}{c}} \exp(-\mu t') \frac{\sigma}{4\pi} \int_I \left( r - t' k, k, t - \frac{t'}{c} \right) dk' dt'
\]
where \( d(r, -k) \) – distance from the point \( r \in G \) in the \( -k \) direction to the boundary of the region \( G \).

Using approximation of non-scattering media (\( \sigma = 0 \)) last term is vanished.

For solving (5) authors construct a simple iteration method. Denote the initial approximation as
\[ I_0 = \int_0^{d(r, -k)} \exp(-\mu t') J_0 \left( r - t' k, k, t - \frac{t'}{c} \right) dt'. \]

Thus, signal, received by sonar in the moment \( t \), is modified to
\[
I(t) = \frac{\sigma_d \varepsilon c J_0 \exp(-\mu |y|) (y_1 u' + l - u)^2}{|y|^2 \sqrt{1 + u'^2} |(l - u) u' - y_1|}
\]
where \( |y| = \frac{ct}{s} \).

3. The inverse problem

Further, we consider the inverse problem, which consists in determining the function \( u \) from (6). We obtain a nonlinear equation with respect to \( u' \). To increase the stability of the solution, we use the expansion of the function \( \sqrt{1 + u'^2} \) in the Taylor series:
\[
\sqrt{1 + u'^2} = \sqrt{1 + v_0^2} + O((u' - v_0)^2),
\]
then
In the numerical algorithm, we represent the function \( v_{ei} = u'_i - 1 \). For an approximate solution of the differential equation (9), we need to set 2 initial conditions:

\[
\begin{align*}
\left. u \right|_{t=0} &= u_0, \\
\left. u' \right|_{t=0} &= \nu_0.
\end{align*}
\]

4. Computational experiment

In the case of a narrow reception antenna pattern, the problem of remote sensing of the side-scan sonar, moving with a constant velocity \( V \) along the axis \( r_3 \), is reduced to solving the problem (1) - (4) and is solved independently at each probing interval.

To carry out the computational experiment, the parameters from the echolocation sounding were taken from the values in Table I. [9,11] To solve the differential (7), the Euler method used.

Table 1. Probing parameters.

| \( \mu, \text{m}^{-1} \) | \( \sigma_d \) | \( c, \text{m/c} \) | \( l_0, \text{m} \) | \( y_1, \text{m} \) |
|------------------|----------------|-----------------|----------------|-----------------|
| 0.018            | 1              | 1500            | 1              | 20              |
|                  |                | [0, 300]        |                |                 |

Original bottom relief described by the following function:

\[
u(y_1, y_3) = \exp\left(-\frac{(y_1 - 100)^2 + (y_3 - 60)^2}{10^2}\right) - \exp\left(-\frac{(y_1 - 170)^2 + (y_3 - 60)^2}{10^2}\right)\]

Fig. 1 shows the solution of (7). As can be seen from the plot, the solution was obtained with an error of 1.8% (Fig. 2).
Figure 1. Restoration of the seabed relief for function (8).

In the following experiment, the bottom surface is given by a more variable function and has the form:

\[
u(y_1, y_3) = 1.5 \exp \left( - \frac{(y_1 - 120)^2 + (y_3 - 75)^2}{17^2} \right) - \right.
- 0.7 \exp \left( - \frac{(y_1 - 165)^2 + (y_3 - 70)^2}{25^2} \right) + \left. 0.5 \exp \left( - \frac{(y_1 - 155)^2 + (y_3 - 30)^2}{12^2} \right) - \right.
\left. 0.3 \exp \left( - \frac{(y_1 - 110)^2 + (y_3 - 40)^2}{15^2} \right) \right)
\]

(9)

Figure 3. Restoration of the seabed relief for function (9).

Figure 2. Observational error.

Figure 4. Observational error.
Fig. 3 shows the seabed profile based on the processing, the received signal, according to the (7). Maximum error $\delta u = 1.6\%$ (Fig. 4).

In the next experiment the input signal is computed in the case of finite directivity pattern of the receiving antenna with aperture $8^\circ$.

The seabottom relief is described as (10):

$$u(y_1, y_3) = \exp \left( -\frac{(y_1 - 100)^2 + (y_3 - 25)^2}{10^2} \right) - \exp \left( -\frac{(y_1 - 170)^2 + (y_3 - 25)^2}{10^2} \right)$$

(Fig. 5).

The Fig. 5, 6 are shown numerical error increase with increasing slant range, and objects is defocused along axes $y_2$. The fig. 7 are shown the maximum error is equal 40%.

(Fig. 6. View from above of Fig. 5.

(Fig. 7. Observational error.)
5. Conclusions
The series of computational experiments has shown that a small change in the bottom surface has little effect on the bottom profile recovery result in the case of a narrow beam pattern of the receiving antenna. A numerical algorithm is proposed for determining the characteristics of the bottom based on the signal obtained from the side-scan sonar, using formula (9). Also, the results of computational experiments prove its effectiveness. Thus, the authors carried out a study in the field of seabed mapping using the kinetic model of radiation transfer. During the work, the solution of the inverse problem in the form of a first-order differential equation is obtained, consisting in determining the function describing the deviation of the seabed relief from a given depth.

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