String corrections to four point functions
in the AdS/CFT correspondence

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Abstract
In a string calculation to order $\alpha'^3$, we compute an eight-derivative four-dilaton term in the type IIB effective action. Following the AdS prescription, we compute the order $(g_{YM}^2 N_c)^{-3/2}$ correction to the four-point correlation function involving the operator $trF^2$ in four dimensional $N = 4$ super Yang-Mills using the string corrected type IIB action extending the work of Freedman et al. (hep-th/9808006). In the limit where two of the Yang-Mills operators approach each other, we find that our correction to the four-point correlation functions develops a logarithmic singularity. We discuss the possible cancelation of this logarithmic singularities by conjecturing new terms in the type IIB effective action.

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1. Introduction

Despite the fact that our universe is presumed not to have a negative cosmological constant, there has been a great deal of interest lately in anti-de Sitter gravity. This is because of the remarkable connection between $d$ dimensional superconformal field theories decoupled from gravity and $d+1$ dimensional gravitational theories in the AdS background [1][2][3]. Because the degrees of freedom of the bulk gravitational theory are described by a field theory in one less dimension living on its boundary, the AdS/CFT correspondence is an example of the holographic principle put forward by [4][5]. (see also [6]). In [2][3] it was shown how to compute two-point correlation functions of operators in the conformal field theory by doing a corresponding supergravity calculation in the AdS background. The prescription given for computing n-point correlation functions is to use the bulk supergravity action as the generating function for correlation functions in the boundary conformal field theory

$$< O(x_1)...O(x_n) > = \frac{\delta}{\delta \phi_0(x_1)}...\frac{\delta}{\delta \phi_0(x_n)} S_{eff}^{IIB},$$

(1)

where $O(x)$ is an operator in the conformal field theory and $\phi_0$ is the bulk field value at the boundary. Subsequently, three-point correlation functions were computed in [7][8][9][10] from the correspondence, while leading order in $\alpha'$ contributions to four-point correlation functions were examined in [11][12][13]. Unlike two-point and three-point correlation functions which are determined by the conformal symmetries, four-point functions are determined only up to a function of cross ratios

$$\rho = \frac{x_{12}x_{34}}{x_{14}x_{23}},$$

$$\eta = \frac{x_{12}x_{34}}{x_{13}x_{24}}.$$

(2)

In [14] it was shown that non-perturbative corrections to four-point-correlation functions of four dimensional $N = 4$ super Yang-Mills can be calculated in the AdS/CFT correspondence by including stringy $R^4$ corrections to the effective type IIB supergravity action

$$S_{eff}^{IIB} = \frac{1}{\alpha'^4} \int d^{10}x \sqrt{g} (e^{-2\phi} R + K \alpha'^3 e^{-\phi} f_4(\tau, \bar{\tau}) R^4)$$

(3)

since $AdS_5 \times S^5$ is still a solution of (3). The $f_4$ prefactor to the $R^4$ term was argued in [15] to be a generalized Eisenstein series

$$f_4(\tau, \bar{\tau}) = \sum_{(m,n) \neq (0,0)} \frac{x_2^{3/2}}{|m + n\tau|^3}$$

(4)
which for large $e^{-\phi}$ has the asymptotic expansion

$$
\tau_2^{1/2} f_4 = 2\zeta(3)(\tau_2)^2 + \frac{2\pi^2}{3} + 4\pi^{3/2} \sum_M (M\tau_2)^{1/2} \sum_{M|m} \frac{1}{m^2} \\
\times \left(e^{2\pi iM\tau} + e^{-2\pi iM\bar{\tau}}\right) \left(1 + \sum_{k=1}^\infty (4\pi M\tau_2)^{-k} \frac{\Gamma(k-1/2)}{\Gamma(-k-1/2)k!}\right),
$$

(5)

where $\tau = \chi + ie^{-\phi}$ and $\chi$ is the RR scalar. The first term in (5) comes from a string tree calculation, the second from a 1-loop calculation, and the infinite series from charge $M$ D-instantons. The prefactor (5) has recently been proven by [16] to be the exact coefficient to the $R^4$ term. Since in the AdS/CFT correspondence the energy-momentum tensor $T_{\mu\nu}$ of the $d$ dimensional boundary conformal field theory maps to the graviton in the $d+1$ dimensional bulk theory, the four-energy-momentum-tensor correlation function in four dimensional $N = 4$ $SU(N_c)$ SYM is related to the $R^4$ term of IIB superstring theory via (1) after substituting (3). The prefactor (5) expressed in terms of Yang-Mills parameters, $\lambda = g_{YM}^2 N_c = \frac{L^4}{\alpha'^2}$ and $g_{YM}^2 = 4\pi g_s$ takes the following form

$$
f_4(\lambda, N_c) = \frac{2\zeta(3)}{4\pi \lambda^{3/2}} + \frac{\pi^2 \lambda^{1/2}}{2N_c^2(4\pi)^{1/2}} + \frac{(4\pi)^{3/2}}{N_c^{3/2}} \sum_M Z_M M^{1/2}(e^{-M(\frac{8\pi^2}{\lambda_{YM}^2}+i\theta)} + e^{-M(\frac{8\pi^2}{\lambda_{YM}^2}-i\theta)}).
$$

(6)

Note that the last term in (6) comes from Yang-Mills instantons and this indicates that D-instantons in $AdS_5$ can be identified with instantons in the SYM [17], [18].

Interestingly there are many other terms in the IIB effective action that are related to the $R^4$ term by supersymmetry. We will show that one such term involving only the scalar dilaton field is

$$
\frac{1}{\alpha'^4} \int d^{10}x \sqrt{-g} \alpha'^3 e^{-\phi/2} f_4(\tau, \bar{\tau}) D_\mu D_\nu \phi(x) D^\mu D^\nu \phi(x) D_\alpha D_\beta \phi(x) D^\alpha D^\beta \phi(x).
$$

(7)

where $\phi(x)$ is the dilaton. In this note we calculate the $(g_{YM}^2 N_c)^{-3/2}$ correction to the four-point correlation function of the trace of the $N = 4$ Yang-Mills field strength squared, $tr(F^2)$, from the term (7) in the IIB effective action expanded about the $AdS_5 \times S^5$ background. In section 2, we derive (7) by doing a string theory calculation in Minkowski space. In section 3, we compactify IIB string theory on $AdS_5 \times S^5$ and in the Maldacena limit [11] we calculate the result for the four point correlation function of $tr(F^2)$ using the AdS/CFT correspondence. In section 4, we show that in the limit where two operators in the CFT approach each other the contribution to the four-point functions calculated in section 3
exhibit logarithmic singularities. If these logarithmic singularities were really present this would have rather dramatic consequences for the conformal field theory; namely, it would imply that $N = 4$ SYM is a logarithmic CFT \[19\]. In section 5, we discuss how including terms in the effective action containing the Riemann curvature and five form field strenth and four scalars could cancel the logarithmic singularities.

2. Four-point functions in Minkowski space

The tree four-point scattering amplitudes for the massless states of type IIB superstring theory can be expressed in a compact form

$$A = cA_{\text{tree}}(s, t, u)K(\zeta_1, k_1, \cdots, \zeta_4, k_4)K(\bar{\zeta}_1, k_1, \cdots, \bar{\zeta}_4)$$

where the tree level amplitude is given by

$$A_{\text{tree}} = \frac{1}{g_5^2} \frac{\Gamma(-s/4)\Gamma(-t/4)\Gamma(-u/4)}{\Gamma(1 + s/4)\Gamma(1 + t/4)\Gamma(1 + u/4)}.$$  \hspace{1cm} (9)

The kinematic factor $K\bar{K}$ is the same for the tree and one loop amplitude and is the product of two kinematic factor for the open string four point function. This reflects the fact that the closed string can be viewed as the product of a left and rightmoving open string. In the kinematic factor $K$ for the open string the wavefunction $\zeta_i$ for the $i$-th particle can either be a vector $\xi_\mu$ or a Majorana-Weyl spinor $u_a$. For four vectors the kinematic factor takes the form

$$K(\xi_1, \xi_2, \xi_3, \xi_4) = t_8^{\mu\nu\lambda\kappa\rho\sigma\tau\omega} \xi_1^\mu k_1^\nu \xi_2^\lambda k_2^\kappa \xi_3^\rho k_3^\sigma \xi_4^\tau k_4^\omega$$

$$= -\frac{1}{4} (st\xi_1 \cdot \xi_3 \xi_2 \cdot \xi_4 + su\xi_2 \cdot \xi_3 \xi_1 \cdot \xi_4 + \cdots)$$

Where the detailed expression for $t_8$ can be found in \[20\]. The massless states of the IIB superstring are given by the tensor product $\zeta_L \otimes \zeta_R$. The bosonic states are given by $\xi_\mu \otimes \xi_\nu = h_{(\mu\nu)} + b_{[\mu\nu]}$ where $b$ denotes the NS-NS antisymmetric tensor and $h$ contains the transverse and traceless graviton together with the transverse dilaton. In the following we will be particularly interested in the four point function for the dilaton $\phi$.

The scattering amplitude (8) contains an infinite number of poles in the s,t and u channel, coming from massive intermediate string states. In a low energy effective field approximation the effect of this stringy behavior can be seen by an $\alpha'$ expansion of (8) which effectively integrates out all the massive intermediate string states at tree-level and
leads to an infinite series of higher derivative contact terms. This expansion was performed in [21]

\[
\frac{\Gamma(-1-s/8)\Gamma(-1-t/8)\Gamma(-1-u/8)}{\Gamma(2+s/8)\Gamma(2+t/8)\Gamma(2+u/8)} = \frac{1}{stu} + 2\alpha'^{3}\zeta(3) + O(\alpha'^{5})
\] (11)

Since the kinematic tensor in (8) contain eight momenta, it is easy to see on dimensional grounds that the first term in (11) corresponds to a two derivative contact term and does not give a new term in the effective action. This is produced by the lowest order terms in the action like the Einstein term. The next term in (11) corresponds to a new eight-derivative string theory correction in the IIB supergravity effective action. We will focus on the eight derivative four dilaton amplitude which will be related to the four point function of \(\text{tr}(F^2)\) in the AdS/CFT correspondence.

The dilaton polarization tensor is given by \(\xi_{\mu} \otimes \xi_{\nu} = \phi/\sqrt{8}(\eta_{\mu\nu} - n_{\mu}k_{\nu} - n_{\nu}k_{\mu})\). The \(n_{\mu}\) satisfy \(n_{\mu}^2 = 0, n \cdot k = 1\) and are necessary to make the dilaton polarization vector transverse, i.e. \(\zeta_{\mu\nu}k^\nu = 0\). Plugging this into \(K\bar{K}\) with \(K\) and \(\bar{K}\) given by (10) leads to

\[
K\bar{K} = s^8a_1a_3a_4a_5a_6a_7a_8s^8b_1b_2b_3b_4b_5b_6b_7b_8k^1_{a_1}k^1_{b_1}k^1_{a_2}k^1_{b_2} \cdots k^4_{a_7}k^4_{b_7}k^4_{a_8}k^4_{b_8}
\]

(12)

the normalization \(\phi\) of the dilaton vertex can be fixed by comparing the three point functions involving two dilatons and one graviton with the three point function coming from the kinetic term for the dilaton in the IIB effective action. The eight derivative term which reproduces this S-matrix element is given by

\[
S^{(1)}_{\phi^4} = c\frac{\zeta(3)\alpha'^{3}}{g_s^2} \int d^{10}x\sqrt{g}D_\mu D_\nu \phi D^\mu D^\nu \phi D_\rho D_\lambda \phi D^\rho D^\lambda \phi
\] (13)

Apart from the tree level contribution (3), the four point amplitudes at order \(\alpha'^{3}\) receives a one-loop and non-perturbative D-instanton contributions. The exact form was conjectured in [13] to be

\[
A_4 = K\bar{K}\tau_2^{1/2}f_4(\tau, \bar{\tau})
\] (14)

where \(f_4\) is a nonholomorphic Eisenstein series. In the weak coupling expansion (3) of (14) the first two term are the tree level and one loop contributions, whereas the rest is an infinite series of D-instanton contributions and perturbative fluctuations around the D-instantons.
The form of the terms in the IIB effective action which obey a non renormalization theorem \[16\] can be encoded in an integral over half the IIB superspace \[22\] of an appropriate power of a a constrained linearized superfield $\Phi(x, \theta^a)$ where $\theta^a, a = 1, \cdots, 16$. The superfield satisfies

$$\bar{D}\Phi = 0, \quad D^4\Phi = \bar{D}^4\Phi$$

and has an expansion up to order $\theta^8$ given by

$$\Phi = \tau + \bar{\theta}\lambda + \cdots \theta^8 D^4\bar{\tau} \quad (16)$$

where the complex scalar $\tau$ contains the dilaton and RR-scalar $\tau = \chi + i e^{-\phi}$. The four point function can the be expressed as an integral over half the superspace

$$\text{Re} \left( \int d^{16} \theta \Phi^4 \right) = \frac{1}{2} \left( \tau^2 D^4\bar{\tau}D^4\bar{\tau} + \bar{\tau}^2 D^4\tau D^4\tau \right)$$

where we integrated by parts.

3. Four point function in AdS/CFT

The AdS/CFT correspondence allows one to calculate correlation functions of operators in the $N = 4$ SYM using the effective action of IIB superstring theory. We will focus on terms in the effective action which involve eight derivatives on four dilatons. The dilaton $\phi$ of IIB on $AdS_5 \times S_5$ corresponds to the marginal operator $\text{tr}(F^2)$.

We will concentrate on an $\alpha'3$ four dilaton term \[13\] compactified on $AdS_5 \times S_5$ which is given by

$$S^{(1)}_{\phi^4} = \int \frac{d^5z}{z_0^4} \sqrt{g} D_\mu D_\nu \phi D_\rho D_\lambda \phi$$

where we drop the $S_5$ dependence (which only gives a volume factor). The prescription to calculate CFT correlators using the IIB effective action is given by

$$W_{CFT}(\phi_{CFT}) = S_{IIB}(\phi_{bulk})$$

where $W_{CFT}$ in the generating function for the conformal field theory. Here $S_{IIB}$ is the IIB effective action compactified on $AdS_5 \times S_5$ and the bulk and CFT fields are related (for scalars of dimension $\Delta$) by

$$\phi_{bulk}(z) = c \int d^4x K_\Delta(z, x) \phi_{CFT}(x)$$
A correlator of operators in the CFT associated with $\phi_{CFT}$ can then be calculated by
\[
I(x_1, x_2, x_3, x_4) = \langle O(x_1)O(x_2)O(x_3)O(x_4) \rangle_{CFT} = \frac{\delta \delta \phi(x_1) \delta \phi(x_2) \delta \phi(x_3) \delta \phi(x_4)}{\delta \phi(x_1) \delta \phi(x_2) \delta \phi(x_3) \delta \phi(x_4)} W(\phi) \tag{21}
\]
In order to calculate the correction to the correlator $I = \langle \text{tr}(F^2)\text{tr}(F^2)\text{tr}(F^2)\text{tr}(F^2) \rangle$ coming from the higher derivative term (18), and use the prescription (21) which gives
\[
I^s = \int \frac{dz_0}{z_0^5} D_\mu D_\nu K_\Delta(z, x_1)D^\mu D^\nu K_\Delta(z, x_2)D_\rho D_\lambda K_\Delta(z, x_3)D^\rho D^\lambda K_\Delta(z, x_4) \tag{22}
\]
Here $I^s$ denotes the 's-channel' part of the amplitude. There are also $I^t$ and $I^u$ which are obtained from (22) by exchanging $x_2 \leftrightarrow x_4$ and $x_2 \leftrightarrow x_3$ respectively, and the complete correlator is given by $I = I^s + I^t + I^u$.

The metric on the five dimensional euclidean AdS space is given by
\[
g_{\mu\nu} = \frac{1}{z_0^2} \delta_{\mu\nu}, \quad \mu, \nu = 0, \ldots, 4 \tag{23}
\]
Hence the nonvanishing components of the metric connection are given by
\[
\Gamma^0_{ij} = \frac{1}{z_0} \delta_{ij}, \quad \Gamma^i_{0j} = -\frac{1}{z_0} \delta_{ij}, \quad \Gamma^0_{00} = -\frac{1}{z_0} \tag{24}
\]
where $i, j = 1...4$. The bulk to boundary Greens function for a field of conformal dimension $\Delta$ is given by
\[
K_\Delta = \lambda_\Delta \left( \frac{z_0}{z_0^2 + (z-x)^2} \right)^\Delta \tag{25}
\]
For notational simplicity we leave out the normalization factor
\[
\lambda_\Delta = \Gamma(\Delta)/(\pi^{d/2}\Gamma(\Delta - d/2)) \tag{26}
\]
from (25) in the following which can be easily reinstated at the end. The second order covariant derivatives with respect to the connection (24) of the propagtor $D_\mu D_\nu K_\Delta(z, x)$ are given by
\[
D_0 D_0 K_\Delta = \Delta^2 \left( \frac{z_0^{\Delta-2}}{(z_0^2 + (z-x)^2)^\Delta} - 4 \Delta(\Delta + 1) \frac{z_0^\Delta}{(z_0^2 + (z-x)^2)^{\Delta+1}} \right)
\]
\[
+ 4 \Delta(\Delta + 1) \frac{z_0^{\Delta+2}}{(z_0^2 + (z-x)^2)^{\Delta+2}},
\]
\[
D_0 D_i K_\Delta = -2 \Delta(\Delta + 1) \frac{z_0^{\Delta-1}}{(z_0^2 + (z-x)^2)^{\Delta+1}} (z-x)^i
\]
\[
- 2 \Delta(\Delta + 1) \frac{z_0^{\Delta+1}}{(z_0^2 + (z-x)^2)^{\Delta+2}} (z-x)^i,
\]
\[
D_i D_j K_\Delta = -\Delta \delta_{ij} \frac{z_0^{\Delta-2}}{(z_0^2 + (z-x)^2)^\Delta} + 4 \Delta(\Delta + 1) \frac{z_0^\Delta}{(z_0^2 + (z-x)^2)^{\Delta+2}} (z-x)^i(z-x)^j. \tag{27}
\]
To calculate \( g^{\mu\nu}g^{\rho\lambda}D_\mu D_\rho K_\Delta(z,x_i)D_\nu D_\lambda K_\Delta(z,x_j) \) we use the formulas (27) and the metric (23). The resulting expression can be simplified using \((z-x_i) \cdot (z-x_j) = 1/2\{(z-x_i)^2 + (z-x_j)^2 - (x_i-x_j)^2\}\). The result is given by the identity

\[
g^{\mu\nu}g^{\rho\lambda}D_\mu D_\rho K_\Delta(z,x_i)D_\nu D_\lambda K_\Delta(z,x_j) = \Delta^2(\Delta^2 + d)K_\Delta(z,x_i)K_\Delta(z,x_j) - 4\Delta^2(\Delta + 1)^2x_{ij}^2K_{\Delta+1}(z,x_i)K_{\Delta+1}(z,x_j) + 4\Delta^2(\Delta + 1)^2x_{ij}^4K_{\Delta+2}(z,x_i)K_{\Delta+2}(z,x_j)
\]

(28)

Note that once (28) is expressed in terms of \(K_\Delta\)'s, all terms with extra factors of \(z_0\) cancel. This fact will be important for the conformal invariance of the resulting CFT correlators.

Using (28) \(I^s\) can be brought into the following form

\[
I^s = \Delta^4(\Delta^2 + d)^2I_{\Delta\Delta\Delta\Delta} - 4\Delta^4(\Delta + 1)^2(\Delta^2 + d)\left\{x_{12}^2I_{\Delta+1\Delta+1\Delta+1\Delta+1} + x_{34}^2I_{\Delta\Delta\Delta+1\Delta+1}\right\}
\]

\[
- x_{12}^4I_{\Delta+2\Delta+2\Delta+2\Delta+2} + 16\Delta^4(\Delta + 1)^4\left\{x_{12}^4x_{34}^4I_{\Delta+2\Delta+2\Delta+2\Delta+2} - x_{12}^2x_{34}^4I_{\Delta+1\Delta+1\Delta+1\Delta+1}\right\}
\]

(29)

where \(I_{\Delta_1\Delta_2\Delta_3\Delta_4}\) is defined by

\[
I_{\Delta_1\Delta_2\Delta_3\Delta_4} = \int \frac{d^5z}{z_0^3}K_{\Delta_1}(z,x_1)K_{\Delta_2}(z,x_2)K_{\Delta_3}(z,x_3)K_{\Delta_4}(z,x_4)
\]

(30)

\(I^t\) can be obtained from (29) by permuting \(x_2 \leftrightarrow x_4\) and the arguments \(\Delta_2 \leftrightarrow \Delta_4\) in \(I_{\Delta_1\Delta_2\Delta_3\Delta_4}\) and similarly \(I^u\) can be obtained by \(x_2 \leftrightarrow x_3\) and the arguments \(\Delta_2 \leftrightarrow \Delta_3\) in \(I_{\Delta_1\Delta_2\Delta_3\Delta_4}\).

4. Logarithmic singularities in the four point function

We want to investigate the corrections to correlators of the four dimensional boundary conformal field theory coming from the eight derivative terms \(I^s, I^t\) and \(I^u\) of the bulk supergravity theory. In particular we want to analyze the behavior of these four point functions when two operators come close, i.e, \(x_{ij} \rightarrow 0\) with all the other separations remaining finite. The most useful representation of \(I_{\Delta_1\Delta_2\Delta_3\Delta_4}\) for this purpose is given by (51) in the appendix

\[
I_{\Delta_1\Delta_2\Delta_3\Delta_4} = C\int d\beta_2 d\beta_4 \frac{\beta_2^{\Delta_2-1}\beta_4^{\Delta_4-1}}{(x_{13}^2 + \beta_2 x_{23}^2 + \beta_4 x_{34}^2)^{\Delta_3}(x_{12}^2 + \beta_2 x_{14}^2 + \beta_4 x_{24}^2)^{\Delta_2-\Delta_3}}
\]

(31)
Here we denote \( \Delta = \sum_{i=1}^{4} \Delta_i \). It is easy to see that if \( x_{12} \to 0 \) and all other \( x_{ij} \) remain finite the only region of the integration which can give a singularity is \( \beta_4 \to 0 \). Extracting \( \beta_2 \) from the second term in the denominator and redefining \( \beta'_4 = \beta_4 (x_{14}^2 + \beta_2 x_{24}^2) / \beta_2 \) gives

\[
I_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} = C \int d\beta_2 d\beta_4 \frac{d\beta_4^{\Delta_4-1}}{(x_{12}^2 + \beta_4)^{\Delta/2-\Delta_3}} \frac{\beta_2^{\Delta_2+\Delta_3-\Delta_4-\Delta/2-1}}{(x_{14}^2 + \beta_2 x_{24}^2)^{\Delta_4}} \times \frac{1}{(x_{13}^2 + \beta_2 x_{23}^2 + \beta_4 \beta_2 x_{34}^2/(x_{14}^2 + \beta_2 x_{24}^2))^{\Delta_3}} \]

(32)

It is easy to see that (32) displays a divergence \( 1/(x_{12}^2)^k \) as \( x_{12}^2 \to 0 \) if \( \Delta_3 + \Delta_4 = \Delta/2 + k \) and it displays a logarithmic divergence \( \ln (x_{12}^2 / x_{13} x_{14}) \) if \( \Delta_3 + \Delta_4 = \Delta/2 \). Note that if there is a power singularity of certain order in (32) then expanding the rest of the integral as a power series in \( \beta_4 \) also produces lower order singularities.

As we shall see only the logarithmic singularities will be important in the calculation of the four point function. Setting \( \Delta_3 + \Delta_4 = \Delta/2 \), the singular part of the integral then becomes

\[
I_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \sim C \frac{1}{(x_{13}^2)^{\Delta_3}} \frac{1}{(x_{14}^2)^{\Delta_4}} \ln \left( \frac{x_{12}^2}{x_{13} x_{14}} \right) \int d\beta_2 \frac{\beta_2^{\Delta_2-1}}{(1 + \beta_2)^{\Delta/2}} + o(1) \]

(33)

In this limit the integral over \( \beta_2 \) simply gives a Euler beta function and together with the value of \( C \) given in (52) the coefficient of the log singularity is given by

\[
I_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \sim \frac{\pi^2}{2} \frac{1}{(\Delta/2 - 1)(\Delta/2 - 2)} \frac{1}{(x_{13}^2)^{\Delta_3}} \frac{1}{(x_{14}^2)^{\Delta_4}} \ln \left( \frac{x_{12}^2}{x_{13} x_{14}} \right) + o(1) \]

(34)

The behavior of \( I^s \) in (29) as \( x_{12} \to 0 \) is easy to analyze, all \( I_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} \) which produce power singularities, i.e. have \( \Delta_4 + \Delta_4 > \Delta/2 \) are multiplied by factors of \( x_{12}^2 \) such that the result is nonsingular. In addition the only logarithmic singularity comes from the first term in (29) since all the other logarithmic divergent terms are also multiplied by factors of \( x_{12}^2 \).

On the other hand one finds that all the terms in \( I^t \) and \( I^u \) display logarithmic singularities as \( x_{12}^2 \to 0 \), and the complete logarithmic singularity is given by summing over all terms taking the prefactors appearing in (29) and (34) into account. The final result is

\[
I \sim c_1 \frac{1}{(x_{13}^2)^{\Delta_3}} \frac{1}{(x_{14}^2)^{\Delta_4}} \ln (x_{12}^2 / x_{13} x_{14}) \]

(35)

\[ We drop nonsingular terms which are important for conformal invariance.\]
where the numerical constant $c_1$ is given by
\[ c_1 = \frac{\pi^2 \lambda^4}{2} \left\{ \frac{3\Delta^4(\Delta^2 + d)^2}{(2\Delta - 1)(2\Delta - 2)} - \frac{32\Delta^4(\Delta + 1)^2(\Delta^2 + d)}{(2\Delta + 1)(2\Delta)(2\Delta - 1)} + \frac{16 \cdot 12\Delta^4(\Delta + 1)^4}{(2\Delta + 3)(2\Delta + 2)(2\Delta + 1)2\Delta} \right\} \]
where we reinstated the normalization $\lambda_\Delta$ defined in (26) of the propagators (25). It is easy to see that $c_1$ does not vanish, in particular for the case of the dilaton and $AdS_5$ we set $d = 4$ and $\Delta = 4$ and get
\[ c_1 = \frac{1}{\pi^6} \frac{152985600}{77} \] (37)

5. Possible cancellations of logarithmic terms

In the last section we found we found a logarithmic singularity in the four point function of $N = 4$ SYM in the large $N_c$ limit at order $1/(g_{YM}^2 N_c)^{3/2}$. Similar logarithmic singularities at leading order in four-point functions were found in [12] coming from the tree level two derivative part of the action. In [12], it was speculated that by adding all diagrams the logarithmic singularities might cancel. We will now address this possibility of cancelation of logarithmic singularities in our case. We found a nonzero logarithmic contribution from summing the $s, t$ and $u$ channel of diagrams coming from (13). Hence additional four point functions which might cancel the logarithmic singularities in (35) must come from other terms in the effective action. This is possible because in contrast to the flat Minkowski background the $AdS_5$ background has a nonvanishing curvature tensor as well as a nonvanishing five-form field strength.

\[ R_{\mu\nu\lambda\rho} = -\frac{1}{L^2} \left( g_{\mu\lambda} g_{\nu\rho} - g_{\mu\rho} g_{\nu\lambda} \right) \] (38)
\[ F_{\mu\nu\lambda\rho\sigma} = \frac{1}{L} \epsilon_{\mu\nu\lambda\rho\sigma} \]

Hence there is the possibility that terms of the same order in $\alpha'$ as the eight derivative term (13) which involve curvature tensors or five form field strengths together with four scalars contribute to the four point function in the conformal field theory once the constant background values (38) are substituted. Examples of such terms are four dilatons with six derivatives together with one Riemann tensor or two $F_5$ as well as terms with two curvature tensors, four $F_5$ or one Riemann tensor and two $F_5$ and four dilatons with four derivatives. Such terms should be in the supersymmetric completion of the $R^4$ term like

\[ \text{We are grateful to A. Rajaraman for raising this possibility.} \]
the eight derivative term of four scalars. After using (38) the generic form of the additional four-scalar terms in the effective action is given by

\[ S_{\phi^4}^{(2)} \sim \int \frac{d^5 z}{z_5^3} D_\mu \phi D^{\mu} \phi D_\nu \phi D^{\nu} \phi \]
\[ S_{\phi^4}^{(3)} \sim \int \frac{d^5 z}{z_5^3} D_\mu \phi D^{\mu} \phi D_\nu \phi D^{\nu} \phi \]

(39)

Note that there is an additional the eight derivative term where the indices are contracted cyclicly. In flat space this contraction of the derivatives is related to (13) by integration by parts. In the AdS background the issue of the noncommutative nature of the covariant derivatives becomes important and an integration by parts relates the two eight derivative terms up to terms involving the curvature and two less derivatives on the scalars, since

\[ [D_\mu, D_\nu] D_\rho \phi = R^\lambda_{\mu\nu\rho} D_\lambda \phi. \]

(40)

It is easy to repeat the analysis of section 4 for the terms (39) of the effective action and it follows that the logarithmic singularity of the four point correlation function of the CFT as \( x_{12}^2 \to 0 \) induced by these terms in exactly of the same form as (35). The result for the numerical coefficients \( c_2 \) and \( c_3 \) multiplying the singularities is

\[ c_2 = \frac{\pi^2 \lambda^4_\Delta}{2} \left\{ \frac{(8 - 2\Delta)(\Delta^6 + \Delta^4 d)}{(2\Delta - 1)(2\Delta - 2)2\Delta} + \frac{32\Delta^4(\Delta + 1)^2(\Delta - 2)}{(2\Delta + 2)(2\Delta + 1)2\Delta(2\Delta - 1)} \right\} \]
\[ c_3 = \frac{\pi^2 \lambda^4_\Delta}{2} \left\{ \frac{3\Delta^4}{(2\Delta - 1)(2\Delta - 2)} - \frac{8\Delta^4}{2\Delta(2\Delta - 1)} + \frac{8\Delta^4}{(2\Delta + 1)2\Delta} \right\} \]

(41)

For the four dilaton amplitude in \( AdS_5 \) inserting \( \Delta = 4, d = 4 \) into (41) gives

\[ c_2 = \frac{1}{\pi^6} \frac{368640}{7}, \quad c_3 = \frac{1}{\pi^6} \frac{46080}{7} \]

(42)

The supersymmetric completion of the \( R^4 \) term in the ten dimensional IIB effective action should in principle uniquely determine the relative coefficients of these terms. Unfortunately the details of this are not known and either the supersymmetrization of the \( R^4 \) term or a calculation of five point and six point S-matrix elements in string perturbation theory involving one or two gravitons and four dilatons around flat space to determine the relative normalizations of the three terms seems to be difficult.\footnote{23} We are not able to confirm or defute the cancellation of the logarithmic singularities coming from the \( \alpha'^3 \) terms in the IIB effective action at this moment.

\footnote{23} for a discussion of the supersymmetrizations of \( R^4 \) in the \( d = 10, N=1 \) context.
6. Conclusions

In this note, we computed the corrections to the four-point correlation function of the operator $trF^2$ from the eight-derivative-four-dilaton term in the IIB effective action using the AdS/CFT correspondence. We found that as two of the Yang-Mills operators approach each other, the four point function yields a logarithmic singularity, like the logarithmic singularities found at a lower order in \cite{12}. This may indicate that the theory is a logarithmic conformal field theory. Logarithmic conformal field theories in two-dimensional are non-unitary. If this were the case in four-dimensions, it would certainly be a shocking result. It would imply that $AdS$ gravity is described holographically by a non-unitary conformal field theory! Clearly, the most satisfying solution would be if the logarithmic singularities in our four-point calculation \cite{35} were cancelled by other terms in the effective action. However, the coefficients required for cancellation seem peculiar and unfortunately are difficult to calculate. However, the fact that the logarithmic singularities do appear in our calculations and in \cite{12} seems to suggest that they are a generic feature of the AdS/CFT correspondence. At present there is the possibility that the CFT on the boundary of AdS either maybe logarithmic if a four dimensional analog of \cite{19} exists or that a new scale is introduced and the theory ceases to be conformal.

Assuming that the logarithmic singularities do cancel, the $I_{s,t,u}$ terms will modify the CFT four-point function with finite terms which is in agreement with the intuition that the massive string modes which when integrated out do not modify the structure of the CFT significantly.

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Appendix
The integrals which appear in the calculation of the four point amplitudes can be partially done using the method of Feynman parameters. The basic formula is

\[
\frac{1}{X_1 \cdots X_n} = \int d\alpha_1 \cdots d\alpha_n \delta(1 - \sum \alpha_i) \frac{1}{(\alpha_1 X_1 + \cdots + \alpha_n X_n)^n}
\]  

(43)

Specializing to \( n = 4 \) and taking derivatives with respect to \( X_i \) the following formula easily follows

\[
\frac{1}{X_1^{\Delta_1} \cdots X_4^{\Delta_4}} = \frac{(\Delta - 1)!}{6(\Delta_1 - 1)!\cdots(\Delta_4 - 1)!} \int d\alpha_1 \cdots d\alpha_4 \delta(1 - \sum \alpha_i) \frac{\alpha_1^{\Delta_1-1} \cdots \alpha_4^{\Delta_4-1}}{(\alpha_1 X_1 + \cdots + \alpha_4 X_4)^\Delta}
\]

(44)

Where \( \Delta = \sum_{i=1}^{4} \Delta_i \). In the supergravity amplitudes the factors \( X_i \) are given by \( X_i = z_0^2 + (z - x_i)^2 \). It is easy to see that

\[
\alpha_1 X_1 + \cdots + \alpha_4 X_4 = z_0^2 + z^2 + \sum_{i<j} \alpha_i \alpha_j x_{ij}^2
\]

(45)

where \( x_{ij}^2 = (x_i - x_j)^2 \). The introduction of Feynman parameters makes it straightforward to do the integral over the \( AdS_5 \) in the four point function. The general form of such integrals is

\[
I(N, M) = \int \frac{dz_0 d^4 z}{z_0^5} \frac{z_0^{2N+1}}{(z_0^2 + z^2 + \sum_{i<j} \alpha_i \alpha_j x_{ij}^2)^M}
\]

\[
= \frac{\pi^2}{2} \frac{N!(M - N - 4)!}{(M - 1)!} \frac{1}{(\sum_{i<j} \alpha_i \alpha_j x_{ij}^2)^{M - N - 3}}
\]

(46)

The integrals which we want to evaluate in the AdS bulk are in general of the following form

\[
I_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} = \int \frac{dz_0 d^4 z}{z_0^5} K_{\Delta_1}(z, x_1) K_{\Delta_2}(z, x_2) K_{\Delta_3}(z, x_3) K_{\Delta_4}(z, x_4)
\]

(47)

where \( K_{\Delta} \) is defined in (25). Using (44) gives

\[
I_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} = \frac{(\Delta - 1)!}{6(\Delta_1 - 1)!\cdots(\Delta_4 - 1)!} \int d\alpha_1 \cdots d\alpha_4 \delta(1 - \sum \alpha_i) \times \int \frac{dz_0 d^4 z}{z_0^5} z^\Delta \frac{\alpha_1^{\Delta_1-1} \cdots \alpha_4^{\Delta_4-1}}{(z_0^2 + z^2 + \sum_{i<j} \alpha_i \alpha_j x_{ij}^2)^\Delta}
\]

(48)

Hence the integral over the AdS bulk variable \( z \) can be done using (46) giving

\[
I_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} = \frac{\pi^2}{2} \frac{(\Delta/2 - 3)!(\Delta/2 - 1)!}{(\Delta_1 - 1)!(\Delta_2 - 1)!(\Delta_3 - 1)!(\Delta_4 - 1)!} \times \int d\alpha_1 \cdots d\alpha_4 \delta(1 - \sum \alpha_i) \frac{\alpha_1^{\Delta_1-1} \cdots \alpha_4^{\Delta_4-1}}{(\sum_{i<j} \alpha_i \alpha_j x_{ij}^2)^{\Delta/2}}
\]

(49)
In [13] such integrals were evaluated by introducing new variables \( \beta_1 = \alpha_1, \beta_2 = \alpha_1 \alpha_2, \beta_3 = \alpha_1 \alpha_3, \beta_4 = \alpha_1 \alpha_4 \). The integral over \( \beta_1 \) can then be done by using the \( \delta \)-function constraint.

\[
\int d\alpha_1 \cdots d\alpha_4 \delta (1 - \sum \alpha_i) \frac{\alpha_1^{\Delta_1 - 1} \cdots \alpha_4^{\Delta_4 - 1}}{(\sum_{i<j} \alpha_i \alpha_j x_{ij}^2)^{\Delta/2}}
\]

\[
= \int d\beta_2 d\beta_3 d\beta_4 \frac{\beta_2^{\Delta_2 - 1} \beta_3^{\Delta_3 - 1} \beta_4^{\Delta_4 - 1}}{(\beta_2 x_{12}^2 + \beta_3 x_{13}^2 + \beta_4 x_{14}^2 + \beta_2 \beta_3 x_{23}^2 + \beta_2 \beta_4 x_{24}^2 + \beta_3 \beta_4 x_{34}^2)^{\Delta/2}}
\]

\[
= \frac{\Delta_3 - 1)!((\Delta/2 - \Delta_3 - 1)!}{(\Delta/2 - 1)!} \int d\beta_2 d\beta_4 \frac{\beta_2^{\Delta_2 - 1} \beta_4^{\Delta_4 - 1}}{(x_{13}^2 + \beta_2 x_{23}^2 + \beta_4 x_{34}^2)^{\Delta_3}}
\times \frac{1}{(\beta_2 x_{12}^2 + \beta_4 x_{14}^2 + \beta_2 \beta_4 x_{24}^2)^{\Delta/2 - \Delta_3}}
\]

(50)

Where the elementary integral over \( \beta_3 \) was done to get the third line. A useful integral representation for \( I \) is therefore given by

\[
I_{\Delta_1 \Delta_2 \Delta_3 \Delta_4} = C \int d\beta_2 d\beta_4 \frac{\beta_2^{\Delta_2 - 1} \beta_4^{\Delta_4 - 1}}{(x_{13}^2 + \beta_2 x_{23}^2 + \beta_4 x_{34}^2)^{\Delta_3} (\beta_2 x_{12}^2 + \beta_4 x_{14}^2 + \beta_2 \beta_4 x_{24}^2)^{\Delta/2 - \Delta_3}}
\]

(51)

where the numerical constant \( C \) depends on \( \Delta_i, i = 1, \ldots, 4 \).

\[
C = \frac{\pi^2 (\Delta/2 - 3)!((\Delta/2 - \Delta_3 - 1)!}{2(\Delta_1 - 1)!((\Delta_2 - 1)!((\Delta_4 - 1)!}
\]

(52)

The representation (51) will be most useful for the analysis of the logarithmic singularities involving the four-point functions. The conformal covariance properties of these integrals can be made manifest by expressing one integral in terms of a hypergeometric function which depends only on the conformally invariant cross ratios (2)(see [13][12]).
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