Superalgebras of Dirac operators on manifolds with special Killing-Yano tensors

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Abstract

We present the properties of new Dirac-type operators generated by real or complex-valued special Killing-Yano tensors that are covariantly constant and represent roots of the metric tensor. In the real case these are just the so called complex or hyper-complex structures of the Kählerian manifolds. Such a Killing-Yano tensor produces simultaneously a Dirac-type operator and the generator of a one-parameter Lie group connecting this operator with the standard Dirac one. In this way the Dirac operators are related among themselves through continuous transformations associated with specific discrete ones. We show that the group of these continuous transformations can be only $U(1)$ or $SU(2)$. It is pointed out that the Dirac and Dirac-type operators can form $\mathcal{N}=4$ superalgebras whose automorphisms combine isometries with the $SU(2)$ transformation generated by the Killing-Yano tensors. As an example we study the automorphisms of the superalgebras of Dirac operators on Minkowski spacetime.

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1 Introduction

The quantum physics in curved backgrounds uses operators acting on spaces of vector, tensor or spinor fields whose properties depend on the geometry of the manifolds where these objects are defined. A crucial problem is to find the symmetries having geometrical sources and the related operators. The problem is not trivial since, beside the evident geometrical symmetry given by isometries, there are different types of hidden symmetries frequently associated with supersymmetries that deserve to be carefully studied.

The isometries are related to the existence of the Killing vectors that give rise to the orbital operators of the scalar quantum theory commuting with that of the free field equation. In the theories with spin these operators get specific spin terms whose form is strongly dependent on the local non-holonomic frames we choose by fixing the gauge \[1, 2\]. Recently the theory of isometries was extended allowing one to pick up well-defined conserved quantities in theories with matter fields of any spin \[3, 4\].

Another type of geometrical objects related to several specific supersymmetries are the Killing-Yano (K-Y) tensors \[5\] and the Stäckel-Killing (S-K) tensors of any rank. The K-Y tensors play an important role in theories with spin and especially in the Dirac theory on curved spacetimes where they produce first order differential operators, called Dirac-type operators, which anticommute with the standard Dirac one, \(D\) \[1, 6\]. Another virtue of the K-Y tensors is that they enter as square roots in the structure of several second rank S-K tensors that generate conserved quantities in classical mechanics or conserved operators which commute with \(D\) \[7\]. The construction of Ref. \[1\] depends upon the remarkable fact that the S-K tensors must have square root in terms of K-Y tensors in order to eliminate the quantum anomaly and produce operators commuting with \(D\) \[7\]. These attributes of the K-Y tensors lead to an efficient mechanism of supersymmetry especially when the S-K tensor is just the metric tensor and the corresponding roots are covariantly constant K-Y tensors. Then each tensor of this type, \(f^i\), called from now unit root, gives rise to a Dirac-type operator, \(D^i\), representing a supercharge of a non-trivial superalgebra \(\{D^i, D^j\} \propto D^2\delta_{ij}\) \[8\]. The real-valued unit roots are nothing other than the complex or the hyper-complex structures of the
Kählerian manifolds. However, it was shown that operators $D^i$ can be produced by unit roots with complex-valued components \cite{9, 10}. This represents an extension of the Kählerian geometries that seems to be productive for the Dirac theory since it permits to construct superalgebras of Dirac-type operators even on the Minkowski spacetime which is not Kählerian, having only complex-valued unit roots \cite{10, 11}.

It is known that in four-dimensional manifolds the standard Dirac operator and the Dirac-type ones can be related among themselves through continuous or discrete transformations \cite{12, 10}. It is interesting that there are only two possibilities, namely either transformations of the $U(1)$ group or $SU(2)$ transformations \cite{12, 10, 11}. In the case of the real-valued unit roots, the first type of symmetry is proper to Kähler manifolds while the second largest one is characteristic for hyper-Kähler geometries \cite{12}. However, similar results can be obtained also in the non-Kählerian case of complex-valued unit roots. Recently we have shown that the symmetries of this type can not be larger than $SU(2)$ \cite{11}. Moreover, we pointed out that their transformations can be mixed with the isometries in order to obtain the automorphisms of the superalgebras of Dirac and Dirac-type operators of the hyper-Kähler manifolds \cite{13}. In the present paper we continue this study focusing on the automorphisms of the superalgebras of Dirac operators on non-Kählerian manifolds, including the Minkowski spacetime.

The paper is organized as a report including the previous results one needs for presenting the new topic. We start in the second section with the construction of a simple version of the Dirac theory in manifolds of any dimensions. In the next section we present the theory of external symmetry of the Dirac field \cite{3}. The unit roots are defined in the fourth section giving their main properties in the Kählerian case as well as in the non-Kählerian one. The main conclusion is that there are either single unit roots or triplets of unit roots which have special algebraic properties similar to those of the quaternion algebra. The Dirac-type operators produced by unit roots are introduced in the next section showing that these and the standard one, $D_i$, can be organized as superalgebras, with $\mathcal{N} = 2$ in the case of single unit roots and $\mathcal{N} = 4$ when one has triplets. The sixth section is devoted to the continuous symmetries generated by unit roots that relate the Dirac operators among themselves. We show that these are given by the group $U(1)$ when $\mathcal{N} = 2$ or by the group $SU(2)$ in the case of $\mathcal{N} = 4$. In the next one we study the effects of isometries pointing out that the isolated unit roots are invariant while the triplets transform according to a specific induced
representation of the isometry group. Using these elements we discuss the
automorphisms of our superalgebras in the section eight. Finally we study
the superalgebras of the Dirac and Dirac-type operators of the Minkowski
spacetime.

2 The Dirac theory in any dimensions

The gauge-covariant theory of the Dirac field can be formulated in any (non-
holonomic) orthogonal local frame of a (pseudo-)Riemannian manifold \( M^n \). Thus we must consider simultaneously a local frame and an usual natural
frame represented by the local chart where we introduce the coordinates,
\( x^\mu, \mu, \nu, ... = 1, 2, ...n \). The local frames are defined by the gauge fields (or
"vilbeins") \( e(x) \) and \( \hat{e}(x) \), whose components are labeled by the local indices
\( \hat{\alpha}, ..., \hat{\mu}, ... = 1, 2, ...n \). We remind the reader that the local indices are raised
or lowered by the (pseudo-)Euclidean metric \( \eta \) of the flat model of \( M^n \) while
for the natural indices we have to use the metric tensor \( g(x) \). The fields \( e \) and
\( \hat{e} \) accomplish the conditions
\[
e^\mu_{\hat{\alpha}} e^\nu_{\hat{\beta}} = \delta^\nu_{\mu}, \quad e^\mu_{\hat{\alpha}} \hat{e}^\beta_{\hat{\mu}} = \delta^\beta_{\hat{\alpha}},
\]
orthogonality relations,
\[
g_{\mu\nu} e^\mu_{\hat{\alpha}} e^\nu_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}, \quad \text{and give the metric tensor of } M_n \text{ as } g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} e^\hat{\alpha}_{\mu} e^\hat{\beta}_{\nu}.
\]

In what follows we shall focus only on the manifolds with an even number
of dimensions. In addition, we assume that the metric \( \eta \) has an arbitrary
signature, \((n_+, n_-) \) with \( n_+ + n_- = n = 2k \).

2.1 Clifford algebra and the gauge group

For the Dirac theory in manifolds \( M_n \) with \( n = 2k \) a \( 2k + 1 \)-dimensional
Clifford algebra \([14]\) is enough. This is defined on the \( 2^k \)-dimensional space
\( \Psi \) of the complex spinors \( \psi = \tilde{\varphi}_1 \otimes \tilde{\varphi}_2 ... \otimes \tilde{\varphi}_k \) built using complex two-
dimensional Pauli spinors \( \tilde{\varphi} \). According to our previous general results \([11]\),
we know that one can define a set of point-independent gamma matrices \( \gamma^{\hat{\mu}} \),
labeled by local indices, such that
\[
\{ \gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}} \} = 2\eta^{\hat{\alpha}\hat{\beta}} 1.
\]
where \( 1 \) is the identity matrix. In this form, the first \( n_+ \) matrices \( \gamma^{\hat{\mu}} \) remain
hermitian while the \( n_- \) last ones become anti-hermitian. The unitaryness
can be restored replacing the usual Hermitian adjoint with the generalized
Dirac adjoint \([11]\).
Definition 1 We say that $\bar{\psi} = \psi^+ \gamma$ is the generalized Dirac adjoint of the field $\psi$ if the hermitian matrix $\gamma = \gamma^+$ satisfies the condition $(\gamma)^2 = 1$ and all the matrices $\gamma^\mu$ are either self-adjoint or anti self-adjoint with respect to this operation, i.e. $\bar{\psi}^\mu = \gamma(\gamma^\mu)^+ \gamma = \pm \gamma^\mu$.

It is clear that the matrix $\gamma$ play here the role of metric operator giving the generalized Dirac adjoint of any square matrix $X$ as $\bar{X} = \gamma X^+ \gamma$.

The gauge group $G(\eta) = O(n_+, n_-)$ of $M_n$ defining the principal fiber bundle is a pseudo-orthogonal group that admits an universal covering group $\tilde{G}(\eta)$ which is simply connected and has the same Lie algebra we denote by $\mathfrak{g}(\eta)$. The group $G(\eta)$ is the model of the spinor fiber bundle that completes the spin structure we need. In order to avoid complications due to the presence of these two groups we consider here that the basic piece is the group $G(\tilde{\eta})$, denoting by $[\omega]$ their elements in the standard covariant parametrization given by the skew-symmetric real parameters $\omega_{\mu\nu} = -\omega_{\nu\mu}$. Then the identity element of $G(\eta)$ is $1 = [0]$ and the inverse of $[\omega]$ with respect to the group multiplication reads $[\omega]^{-1} = [-\omega]$.

Definition 2 We say that the gauge group is the vector representation of $G(\eta)$ and denote $G(\tilde{\eta}) = \text{vect}[G(\eta)]$. The representation $\text{spin}[G(\eta)]$ carried by the space $\Psi$ and generated by the spin operators

$$S^{\hat{\alpha}\hat{\beta}} = \frac{i}{4} [\gamma^{\hat{\alpha}}, \gamma^{\hat{\beta}}]$$

(2)

is called the spinor representation of $G(\tilde{\eta})$. The spin operators are the basis generators of the spinor representation $\text{spin}[\mathfrak{g}(\eta)]$ of the Lie algebra $\mathfrak{g}(\eta)$.

In general, the spinor representation is reducible. Its generators are self-adjoint, $S^{\hat{\alpha}\hat{\beta}} = S^{\hat{\beta}\hat{\alpha}}$, and satisfy

$$[S^{\hat{\alpha}\hat{\beta}}, \gamma^{\hat{\gamma}}] = i(\eta^{\hat{\alpha}\hat{\beta}} \gamma^{\hat{\gamma}} - \eta^{\hat{\beta}\hat{\alpha}} \gamma^{\hat{\gamma}}),$$

$$[S_{\hat{\alpha}\hat{\beta}}, S_{\hat{\gamma}\hat{\delta}}] = i(\eta_{\hat{\alpha}\hat{\gamma}} S_{\hat{\beta}\hat{\delta}} - \eta_{\hat{\beta}\hat{\alpha}} S_{\hat{\gamma}\hat{\delta}} + \eta_{\hat{\beta}\hat{\delta}} S_{\hat{\alpha}\hat{\gamma}} - \eta_{\hat{\alpha}\hat{\delta}} S_{\hat{\beta}\hat{\gamma}})$$

(3)

(4)

as it results from Eqs. (1) and (2). It is obvious that Eq. (4) gives just the canonical commutation rules of a Lie algebra isomorphic with that of the groups $G(\eta)$ or $G(\tilde{\eta})$ [11, 13]. The spinor and vector representations are related between themselves through the following
Theorem 1  For any real or complex valued skew-symmetric tensor $\omega_{\hat{\alpha}\hat{\beta}} = -\omega_{\hat{\beta}\hat{\alpha}}$ the matrix

$$T(\omega) = e^{-iS(\omega)}, \quad S(\omega) = \frac{1}{2} \omega_{\hat{\alpha}\hat{\beta}} S^{\hat{\alpha}\hat{\beta}},$$

transforms the gamma-matrices according to the rule

$$[T(\omega)]^{-1} \gamma^{\hat{\alpha}} T(\omega) = \Lambda^{\hat{\alpha}}_{\hat{\beta}} (\omega) \gamma^{\hat{\beta}}, \quad (6)$$

where

$$\Lambda^{\hat{\alpha}}_{\hat{\beta}} (\omega) = \delta^{\hat{\alpha}}_{\hat{\beta}} + \omega_{\hat{\alpha}\hat{\beta}} + \frac{1}{2} \omega_{\hat{\mu}\hat{\alpha}} \omega_{\hat{\beta}\hat{\mu}} + \ldots + \frac{1}{n!} \omega_{\hat{\mu}\hat{\nu}_{\hat{\beta}} \ldots \hat{\sigma}_{\hat{\alpha}}} \ldots.$$  \quad (7)

Proof: All these results can be obtained using Eqs. (1) and (2). \hspace{1cm} \blacksquare

The real components $\omega_{\hat{\alpha}\hat{\beta}}$ are the parameters of the covariant basis of the Lie algebra $g(\eta)$ giving all the transformation matrices $T(\omega) \in spin[G(\eta)]$ and $\Lambda(\omega) \in vect[G(\eta)]$. Hereby we see that the spinor representation $spin[G(\eta)]$ is unitary since for $\omega \in \mathbb{R}$ the generators $S(\omega) \in spin[g(\eta)]$ are self-adjoint, $S(\omega) = S(\omega)$, and the matrices $T(\omega)$ are unitary with respect to the Dirac adjoint satisfying $\overline{T}(\omega) = [T(\omega)]^{-1}$.

The covariant parameters $\omega$ can also take complex values. Then this parametrization spans the complexified group of $G(\eta)$, denoted by $G_c(\eta)$, and the corresponding vector and (non-unitary) spinor representations. Obviously, in this case the Lie algebra is the complexified algebra $g_c(\eta)$. We note that from the mathematical point of view $G(\eta) = vect[G(\eta)]$ is the group of automorphisms of the tangent fiber bundle $\mathcal{T}(M_n)$ of $M_n$ while the transformations of $G_c(\eta) = vect[G_c(\eta)]$ are automorphisms of the complexified tangent fiber bundle $\mathcal{T}(M_n) \otimes \mathbb{C}$.

2.2 The Dirac field

With these preparations, the gauge-covariant theory of the Dirac field $\psi$ can be formulated starting with the standard Dirac gauge invariant action [11]. This gives the Dirac equation $D\psi = m_0 \psi$ where $m_0$ is the fermion mass and the Dirac operator has the form

$$D = i \gamma^\mu \nabla_\mu, \quad \gamma^\mu(x) = \epsilon^\mu_{\hat{\alpha}} (x) \gamma^{\hat{\alpha}}.$$  \quad (8)
The covariant derivatives \( \nabla_{\mu} = \tilde{e}_{\mu}^\alpha \nabla_{\hat{a}} = \tilde{\nabla}_{\mu} + \Gamma_{\mu}^{\text{spin}} \) are formed by the usual ones, \( \tilde{\nabla}_{\mu} \) (acting in natural indices), and the spin connection

\[
\Gamma_{\mu}^{\text{spin}} = \frac{i}{2} e_\mu^\beta (\epsilon_\alpha^\gamma \Gamma_{\beta \mu}^{\alpha} - \epsilon_\beta^\sigma \Gamma_{\mu \alpha}^{\beta}) S_{\hat{\sigma}}^\gamma ,
\]

giving \( \nabla_{\mu} \psi = (\partial_{\mu} + \Gamma_{\mu}^{\text{spin}}) \psi \).

Now it is obvious that our definition of the generalized Dirac adjoint is correct since \( \overline{\gamma} = \gamma^\mu \) and \( \Gamma_{\mu}^{\text{spin}} = -\Gamma_{\mu}^{\text{spin}} \) such that the Dirac operator results to be self-adjoint, \( \overline{D} = D \). Moreover, the quantity \( \overline{\psi} \psi \) has to be derived as a scalar, i.e. \( \nabla_{\mu}(\overline{\psi} \psi) = \overline{\nabla}_{\mu} \psi \psi + \psi \nabla_{\mu} \psi = \partial_{\mu}(\overline{\psi} \psi) \), while the quantities \( \overline{\psi} \gamma^\alpha \gamma^\beta \ldots \psi \) behave as tensors of different ranks.

Thus we reproduce the main features of the familiar tetrad gauge covariant theories with spin in (1+3)-dimensions from which we can take over all the results arising from similar formulas. In this way we find that the point-dependent matrices \( \gamma^\mu(x) \) and \( S^{\mu\nu}(x) = e_\alpha^\mu(x)e_\beta^\nu(x)S_{\hat{\alpha}}^\hat{\beta} \) have similar properties as (1), (2), (3) and (4), but written in natural indices and with \( g(x) \) instead of the flat metric \( \eta \). Using this algebra and the standard notations for the Riemann-Christoffel curvature tensor, \( R_{\alpha\beta\mu\nu} \), Ricci tensor, \( R_{\alpha\beta} = R_{\alpha\mu\beta\nu}g^{\mu\nu} \), and scalar curvature, \( R = R_{\mu\nu}g^{\mu\nu} \), we recover the useful formulas

\[
\nabla_{\mu}(\gamma^\nu \psi) = \gamma^\nu \nabla_{\mu} \psi ,
\]

and the identity

\[
R_{\alpha\beta\mu\nu} \gamma^{\beta} \gamma^{\mu} = -2R_{\alpha\nu} \gamma^{\nu} \text{ that allow one to calculate}
\]

\[
D^2 = -\nabla^2 + \frac{1}{4} R 1 , \quad \nabla^2 = g^{\mu\nu} \nabla_{\mu} \nabla_{\nu} .
\]

It remains to complete the operator algebra with new observables from which we have to select complete sets of commuting observables for defining quantum modes. These can be obtained as conserved operators associated to the isometries of \( M_n \).

### 3 The external symmetry of the Dirac theory

The relativistic covariance in the sense of general relativity is too general to play the same role as the Lorentz or Poincaré covariance in special relativity.
In other respects, the gauge covariance of the theories with spin represents another kind of general symmetry that is not able to produce itself conserved observable. Therefore, if we look for sources of symmetries able to generate conserved quantities, we have to concentrate mainly on isometries that point out the spacetime symmetry giving us the specific Killing vectors. The physical fields minimally coupled with the gravitational one take over this symmetry, transforming according to different representations of the isometry group. In the case of the scalar vector or tensor fields these representations are completely defined by the well-known rules of the general coordinate transformations since the isometries are in fact particular coordinate transformations. However, the behavior under isometries of the fields with half integer spin is more complicated since their transformation rules explicitly depend on the gauge fixing. The specific theory of this type of transformations is the recent theory of external symmetry we present in this section.

### 3.1 The gauge and relativistic covariance

The use of the covariant derivatives assures the covariance of the whole theory under the gauge transformations,

\[
\begin{align*}
\hat{e}_\mu^\alpha(x) &\rightarrow \hat{e}_\mu'^\alpha = \Lambda_\mu_\beta [A(x)] \hat{e}_\mu^\beta(x), \\
e_\alpha(x) &\rightarrow e_\alpha'^\mu = \Lambda_\alpha_\beta [A(x)] e_\beta^\mu(x), \\
\psi(x) &\rightarrow \psi'(x) = T[A(x)] \psi(x),
\end{align*}
\]

produced by the mappings \( A : M_n \rightarrow G(\eta) \) the values of which are local transformations \( A(x) = [\omega(x)] \in G(\eta) \) determined by the set of real functions \( \omega_{\mu\tilde{\nu}} = -\omega_{\tilde{\nu}\mu} \) defined on \( M_n \). In other words \( A \) denotes sections of the spinor fiber bundle that can be organized as a group, \( \text{Sec}(M_n) \), with respect to the multiplication \( \times \) defined as \( (A' \times A)(x) = A'(x)A(x) \). We use the notations \( Id \) for the mapping identity, \( Id(x) = 1 \in G(\eta) \), and \( A^{-1} \) for the inverse of \( A \) which satisfies \( (A^{-1})(x) = [A(x)]^{-1} \).

The general gauge-covariant theory of the Dirac spinors must be also covariant under general coordinate transformation of \( M_n \) which, in the passive mode, \(^1\) can be seen as changes of the local charts corresponding to the same

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\(^1\)We prefer the term of coordinate transformation instead of diffeomorphism since we adopt this viewpoint.
domain of $M_n$. If $x$ and $x'$ are the coordinates of a given point in two different charts then there is a mapping $\phi$ between these charts giving the coordinate transformation $x \to x' = \phi(x)$. These transformations form the group $\text{Sec}(M_n)$ with respect to the composition of mappings, $\circ$, defined as usual, i.e. $(\phi' \circ \phi)(x) = \phi'[\phi(x)]$. We denote the identity map of this group by $id$ and the inverse mapping of $\phi$ by $\phi^{-1}$.

The coordinate transformations change all the components carrying natural indices including those of the gauge fields changing thus the positions of the local frames with respect to the natural ones. If we assume that the physical experiment makes reference to the axes of the local frame then it could appear situations when several correction of the positions of these frames should be needed before (or after) a general coordinate transformation. Obviously, these have to be made with the help of suitable gauge transformations associated to the coordinate ones.

**Definition 3** The combined transformation $(A, \phi)$ is the gauge transformation given by the section $A \in \text{Sec}(M_n)$ followed by the coordinate transformation $\phi \in \text{Aut}(M_n)$.

In this new notation the pure gauge transformations appear as $(A, id) \in \text{Sec}(M_n)$ while the coordinate transformations will be denoted from now by $(Id, \phi) \in \text{Aut}(M_n)$. The effect of a combined transformation $(A, \phi)$ upon our basic elements is, $x \to x' = \phi(x)$,

$$
\psi(x) \to \psi'(x') = T[A(x)]\psi(x),
$$

and $e(x) \to \hat{e}'(x')$ where $e'$ are the transformed gauge fields of the components

$$
e'_{\hat{\alpha}}^\mu[\phi(x)] = \Lambda_{\hat{\alpha}}^{\hat{\beta}}[A(x)] e_{\hat{\beta}}^\nu(x) \frac{\partial \phi^\mu(x)}{\partial x^\nu},$$

while the components of $\hat{e}'$ have to be calculated according to Eqs. (13). Thus we have written down the most general transformation laws that leave the theory invariant.

The association among the transformations of the gauge group and coordinate transformation leads to a new group with a specific multiplication. In order to find how looks this new operation it is convenient to use the composition among the mappings $A$ and $\phi$ (taken only in this order) giving the new mappings $A \circ \phi$ defined as $(A \circ \phi)(x) = A[\phi(x)]$. The calculation rules $Id \circ \phi = Id$, $A \circ id = A$ and $(A' \times A) \circ \phi = (A' \circ \phi) \times (A \circ \phi)$ are obvious. In this context one can demonstrate
Theorem 2 The set of combined transformations of $M_n$, $G(M_n)$, form a group with respect to the multiplication $\ast$ defined as

$$(A', \phi') \ast (A, \phi) = ((A' \circ \phi) \times A, \phi' \circ \phi).$$  \hfill (14)

Proof: First of all we observe that $(A, \phi) = (Id, \phi) \ast (A, id)$. Furthermore, one can verify the result calculating the effect of this product upon the field $\psi$. \hfill (12)

Now the identity is $(Id, id)$ while the inverse of a pair $(A, \phi)$ reads

$$(A, \phi)^{-1} = (A^{-1} \circ \phi^{-1}, \phi^{-1}).$$  \hfill (15)

In addition, one can demonstrate that the group of combined transformations is the semi-direct product $G(M_n) = \text{Sec}(M_n) \rtimes \text{Aut}(M_n)$ between the group of sections which plays the role of invariant subgroup and that of coordinate transformations \[3\]. The same construction but starting with the group $G_c(\eta)$ instead of $G(\eta)$ yields the complexified group of combined transformations, $G_c(M_n)$.

The use of combined transformations is justified only in theories where there are physical reasons requiring to use local frames since in natural frames the effect of the combined transformations on the vector and tensor fields reduces to that of coordinate transformations. However, the physical systems involving spinors can be described exclusively in local frames where our theory is essential.

Definition 4 The spinor representation of $G(M_n)$ has values in the space of the linear operators $U : \Psi \to \Psi$ such that for each $(A, \phi)$ there exists an operator $U(A, \phi) \in \text{spin}[G(M_n)]$ having the action

$$U(A, \phi)\psi = [T(A)\psi] \circ \phi^{-1} = [T(A \circ \phi^{-1})](\psi \circ \phi^{-1}).$$

This rule gives the transformations \[12\] in each point of $M_n$ if we put $\psi' = U(A, \phi)\psi$ and then calculate the value of $\psi'$ in the point $x' = \phi(x)$. The Dirac operator $D$ covariantly transforms as

$$(A, \phi) : D(x) \to D'(x') = T[A(x)]D(x)T[A(x)],$$

where $D' = U(A, \phi)D[U(A, \phi)]^{-1}$. In general, the combined transformations change the form of the Dirac operator which depends on the gauge one uses ($D' \neq D$). We note that for the gauge transformations with $\phi = id$ (when $x' = x$) the action of $U(A, id)$ reduces to the linear transformation given by the matrix $T(A) \in \text{spin}[G(\eta)]$.\hfill 10
3.2 Isometries and the external symmetry

In general, the symmetry of the manifold $M_n$ is given by its isometry group, $I(M_n) \subset \text{Aut}(M_n)$, whose transformations, $x \rightarrow x'(x)$, are coordinate transformation which leave the metric tensor invariant in any chart \[16, 17, 18\],

\[
g_{\alpha\beta}(x') \frac{\partial x'^\alpha}{\partial x^\mu} \frac{\partial x'^\beta}{\partial x^\nu} = g_{\mu\nu}(x). \tag{16}
\]

The isometry group is formed by sets of coordinate transformations, $x \rightarrow x' = \phi_\xi(x)$, depending on $N$ independent real parameters, $\xi^a$ ($a, b, c... = 1, 2, ..., N$), such that $\xi = 0$ corresponds to the identity map, $\phi_{\xi=0} = \text{id}$. These transformations form a Lie group equipped with the composition rule

\[
\phi_{\xi'} \circ \phi_\xi = \phi_p(\xi', \xi), \tag{17}
\]

where the functions $p$ define the group multiplication. These satisfy $p^a(0, \xi) = p^a(\xi, 0) = \xi^a$ and $p^a(\xi^{-1}, \xi) = p^a(\xi, \xi^{-1}) = 0$ where $\xi^{-1}$ are the parameters of the inverse mapping of $\phi_\xi$, $\phi_{\xi^{-1}} = \phi^{-1}_\xi$. Moreover, the functions $p$ give the structure constants $c_{abc}$ of this group \[19\] which define the commutation relations of the basis generators of the Lie algebra $i(M_n)$ of $I(M_n)$. For small values of the group parameters the infinitesimal transformations, $x^\mu \rightarrow x'^\mu = x^\mu + \xi^a k_a^\mu(x) + \cdots$, are given by the Killing vectors $k_a$ whose components,

\[
k_a^\mu = \left. \frac{\partial \phi_\xi^\mu}{\partial \xi^a} \right|_{\xi=0}, \tag{18}
\]

satisfy the Killing equations $k_a(\mu; \nu) \equiv k_a^{\mu; \nu} + k_a^{\nu; \mu} = 0$ and the identities

\[
k_a^\mu k_b^\nu - k_b^\mu k_a^\nu + c_{abc} k_c^\nu = 0. \tag{19}
\]

The simplest representation of $I(M_n)$ is the natural one carried by the space of the scalar fields $\vartheta$ which transform as $\vartheta \rightarrow \vartheta' = \vartheta \circ \phi^{-1}_\xi$. This rule defines the operator-valued representation of the group $I(M_n)$ generated by the operators,

\[
L_a = -ik_a^\mu \partial_\mu, \quad a = 1, 2, ..., N,
\]

which are completely determined by the Killing vectors. From Eq. \[19\] we see that they obey the commutation rules

\[
[L_a, L_b] = i c_{abc} L_c, \tag{19}
\]
given by the structure constants of the Lie algebra $i(M_n)$.

In the theories involving fields with spin, an isometry can change the relative positions of the local frames with respect to the natural ones. This fact may be an impediment when one intends to study the symmetries of these theories in local frames. For this reason it is natural to suppose that the good symmetry transformations we need are isometries preceded by appropriate gauge transformations which should assure that not only the form of the metric tensor would be conserved but the form of the gauge field components too. However, these transformations are nothing other than particular combined transformations whose coordinate transformations are isometries.

**Definition 5** The external symmetry transformations, $(A_\xi, \phi_\xi)$, are particular combined transformations involving isometries, $(\text{Id}, \phi_\xi) \in I(M_n)$, and corresponding gauge transformations, $(A_\xi, \text{id}) \in \text{Sec}(M_n)$, necessary to preserve the gauge.

This requirement is accomplished only if we assume that, for given gauge fields $e$ and $\hat{e}$, $A_\xi$ is defined by

$$\Lambda^{\tilde{\alpha}}{}_{\tilde{\beta}}[A_\xi(x)] = \hat{e}^{\tilde{\alpha}}{}_{\mu}[\phi_\xi(x)] \frac{\partial \phi_\xi^\mu(x)}{\partial x^\nu} e^{\nu}_{\tilde{\beta}}(x),$$

with the supplementary condition $A_{\xi=0}(x) = 1 \in G(\eta)$. Since $\phi_\xi$ is an isometry Eq. (16) guarantees that $\Lambda[A_\xi(x)] \in \text{vect}[G(\eta)]$ and, implicitly, $A_\xi(x) \in G(\eta)$. Then the transformation laws of our fields are

$$\begin{align*}
  x & \to x' = \phi_\xi(x) \\
  e(x) & \to e'(x') = e[\phi_\xi(x)] \\
  \hat{e}(x) & \to \hat{e}'(x') = \hat{e}[\phi_\xi(x)] \\
  \psi(x) & \to \psi'(x') = T[A_\xi(x)]\psi(x).
\end{align*}$$

The mean virtue of these transformations is that they leave invariant the form of the Dirac operator, $D' = D$.

**Theorem 3** The set of the external symmetry transformations $(A_\xi, \phi_\xi)$ form the Lie group $S(M_n) \subset G(M_n)$ with respect to the operation *. This group, will be called the group of external symmetry of $M_n$. 

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Proof: Starting with Eq. (20) we find that 
\[(A_{\xi'} \circ \phi_{\xi}) \times A_{\xi} = A_{p(\xi', \xi)},\]
and, according to Eqs. (14) and (17), we obtain
\[(A_{\xi'}, \phi_{\xi'}) \times (A_{\xi}, \phi_{\xi}) = (A_{p(\xi', \xi)}, \phi_{p(\xi', \xi)}),\] (22)
and 
\[(A_{\xi=0}, \phi_{\xi=0}) = (Id, id).\]
From Eq. (22) we understand that 
\(S(M_n)\) is locally isomorphic with 
\(I(M_n)\) and, therefore, the Lie algebra 
\(s(M_n)\) of the group 
\(S(M_n)\) is isomorphic with 
\(i(M_n)\) having the same structure constants. There are arguments that the group 
\(S(M_n)\) is isomorphic with the universal covering group of 
\(I(M_n)\) since it has anyway the topology induced by 
\(G(\eta)\) which is simply connected. In general, the number of group parameters of 
\(I(M_n)\) or 
\(S(M_n)\) (which is equal to the number of the independent Killing vectors of 
\(M_n\)) can be 
\[0 \leq N \leq \frac{1}{2} n(n + 1).\]

The last of Eqs. (21) giving the transformation law of the field \(\psi\) defines
the operator-valued representation 
\((A_{\xi}, \phi_{\xi}) \rightarrow U_{\xi} \equiv U(A_{\xi}, \phi_{\xi})\) of the group 
\(S(M_n)\),
\[(U_{\xi}\psi)[\phi_{\xi}(x)] = T[A_{\xi}(x)]\psi(x),\] (23)
which is the spinor representation 
\(spin[S(M_n)] \subset spin[G(M_n)]\) of the group 
\(S(M_n)\). This representation has unitary transformation matrices in the sense of the Dirac adjoint 
\((T = T^{-1})\) and its transformations leaves the operator 
\(D\) invariant,
\[U_{\xi}D U_{\xi}^{-1} = D.\]
Since 
\(A_{\xi}(x) \in G(\eta)\) we say that 
\(spin[S(M_n)]\) is induced by the representation 
\(spin[G(\eta)]\) (20, 21).

Theorem 4 The basis generators of the spinor representation 
\(spin[s(M_n)]\) of the Lie algebra 
\(s(M_n)\) are [1, 3]
\[X_a = -i k^\mu_a \nabla_\mu + \frac{1}{2} k_{a \mu, \nu} e^\mu_{\hat{a}} e^\nu_{\hat{b}} S^{\hat{a} \hat{b}}.\] (24)
Proof: The proof is given in [3] where we recover the important result of [1] derived for the Dirac field in 
\(M_4\).

Theorem 5 The operators (24) are self-adjoint with respect to the Dirac adjoint and satisfy the commutation rules
\[[X_a, X_b] = i c_{abc} X_c, \quad [D, X_a] = 0, \quad a, b... = 1, 2, ..., N,\] (25)
where 
\(c_{abc}\) are the structure constants of the isomorphic Lie algebras 
\(s(M_n) \sim i(M_n)\).
Proof: The proof is based on the identities given in [3].

The natural consequence is

Corollary 1 The operators $U_\xi \in \text{spin}[S(M_n)]$ transform the basis generators $X_a$ according to the adjoint representation of $S(M_n)$,

$$U_\xi X_a U_\xi^{-1} = \text{Adj}(\xi)_{ab} X_b,$$

defined as

$$\text{Adj}(\xi) = e^{i\xi_{\text{adj}}(X_a)}, \quad \text{adj}(X_a)_{bc} = -i\epsilon_{abc},$$

where $\text{adj}(X_a)$ are the basis generators of the adjoint representation of $s(M_n)$.

Proof: This is a general result of the group representation theory [20]. We note that here the phase factors are chosen such that the commutators

$$[\text{adj}(X_a), \text{adj}(X_b)] = i\epsilon_{abc} \text{adj}(X_c)$$

keep the form [20]. Whenever one can define a convenient relativistic scalar product in order to get only self-adjoint generators, the representation $\text{spin}[S(M_n)]$ is unitary. In this case one can define quantum modes correctly, using the set of commuting operators formed by the Casimir operators of $\text{spin}[s(M_n)]$, the generators of its Cartan subalgebra and the Dirac operator, $D$.

4 Special geometries

Apart from the standard Dirac operator, other operators of the same type may be defined with the help of suitable geometric objects as, for example, the K-Y tensors with some supplemental properties, known as complex structures or unit roots [11]. However, such objects arise only in geometries with particular features.

4.1 Kählerian geometries

A special type of geometries with many possible applications in physics is represented by the Kählerian manifolds. In general, these are manifolds $M_n$ of even dimension, $n = 2k$, equipped with special geometric objects called complex structures. The complex structures are particular automorphisms $h$.
\( \mathcal{T}(M_n) \to \mathcal{T}(M_n) \), of the tangent fiber bundle \( \mathcal{T}(M_n) \), which are covariantly constant and satisfy the algebraic property of a complex unit. This means that the matrix of \( h \) in a given basis, denoted by \( \langle h \rangle \), must satisfy the condition

\[
\langle h \rangle^2 = -1_n
\]  

(26)

(where \( 1_n \) is the \( n \times n \) identity matrix). The matrix \( \langle h \rangle \) in local frames is a pseudo-orthogonal point-dependent transformation of the gauge group \( G(\eta) = SO(n_+, n_-) \) of \( M_n \). With the help of the complex structure \( h \) one gives the following definition \cite{22, 23}:

**Definition 6** A Riemannian metric \( g \) on \( M_n \) is said Kählerian if \( h \) is pointwise orthogonal, i.e., \( g(\langle h \rangle X, \langle h \rangle Y) = g(X, Y) \) for all \( X, Y \in \mathcal{T}_x(M_n) \) at all points \( x \).

In natural frames the matrix \( \langle h \rangle \) has the elements \( h^\mu_\nu \), defining a skew-symmetric second rank tensor with real-valued covariant components \( h^\mu_\nu = -h^\nu_\mu \) which obey the condition \( g_{\mu\nu} h^\mu_\alpha h^\nu_\beta = g_{\alpha\beta} \) resulted from Eq. (26). The tensor \( h \) gives rise to the symplectic form \( \tilde{\omega} = \frac{1}{2} h^\nu_\mu dx^\nu \wedge dx^\mu \) (which is closed and non-degenerate). However, alternative definitions can be formulated starting with both, \( g \) and \( \tilde{\omega} \), which have to satisfy the Kähler relation \( \tilde{\omega}(X,Y) = g(X, \langle h \rangle Y) \) \cite{23}.

A hypercomplex structure on \( M_n \) is an ordered triplet \( h = \{ h^1, h^2, h^3 \} \) of complex structures on \( M_n \) satisfying

\[
\langle h^i \rangle \langle h^j \rangle = -\delta_{ij} 1_n + \varepsilon_{ijk} \langle h^k \rangle, \quad i, j, k... = 1, 2, 3.
\]  

(27)

In Lie algebraic terms, the matrices \( \frac{1}{2} \langle h^j \rangle \) realize the \( su(2) \) algebra.

**Definition 7** A hyper-Kähler manifold is a manifold whose Riemannian metric is Kählerian with respect to each different complex structures \( h^1, h^2 \) and \( h^3 \).

### 4.2 Isolated unit roots

The complex structures are real-valued K-Y tensors since the classical geometric objects are in general real fields. However, the theories with spin could involve even complex-valued K-Y tensors generating new Dirac-type operators. Based on this argument, we considered complex-valued complex structures called unit roots \cite{11}.
Definition 8  The non singular real or complex-valued K-Y tensor \( f \) of rank 2 defined on \( M_n \) which satisfies
\[
f^{\mu}_{\alpha} f_{\mu\beta} = g_{\alpha\beta}
\]
is called an unit root of the metric tensor \( g \) of \( M_n \), or simply an unit root of \( M_n \).

One can show that the unit roots are covariantly constant K-Y tensors which satisfy the algebraic condition \( \langle f \rangle^2 = -1 \), equivalent to Eq. (28).

It is known that the unit roots are allowed only by manifolds \( M_n \) with an even number of dimensions, \( n = 2k \). Our previous results show that the matrices of the unit roots are unimodular (i.e., \( \det \langle f \rangle = 1 \)) and completely reducible in \( 2 \times 2 \) diagonal blocks [11]. Hereby it results

Theorem 6  The unit roots are usual complex structures with real matrices only when \( n_+ \) and \( n_- \) are even numbers. In the case of odd values of \( n_+ \) and \( n_- \) the unit roots have complex-valued components.

Proof:  The proof is given in Ref. [11].

Thus we understand that the unit roots are defined in a similar way as the complex structures with the difference that the unit roots are automorphisms of the complexified tangent bundle, \( f : T(M_n) \otimes \mathbb{C} \to T(M_n) \otimes \mathbb{C} \). Therefore, when \( f \) has complex-valued components, the matrix \( \langle f \rangle \) is a transformation of the complexified group \( G_c(\eta) \).

Each unit root \( f \) can be seen as the basis of the one-dimensional linear real space \( L_f = \{ \rho | \rho = \alpha f, \alpha \in \mathbb{R} \} \) whose elements apart from 0 are called roots since \( \langle \rho \rangle^2 = -\alpha^2 1_n \). The quantity \( |\alpha| \) play here the role of the norm of the root \( \rho = \alpha f \). When the unit root \( f \) is complex-valued, there exists the adjoint root \( f^* \) which differs from \( f \) and satisfies \( \langle f^* \rangle, \langle f \rangle \rangle = 0 \). The unit root \( f^* \) is the basis of the linear space of roots \( L_{f^*} \), defined in the same manner as \( L_f \). As observed in Ref.[11] the spaces \( L_f \) and \( L_{f^*} \) can not be embedded in a larger linear structure.

4.3 Triplets of unit roots

The next step is to look for manifolds allowing families of unit roots \( f = \{ f^i | i = 1, 2, ..., N_f \} \) having supplementary properties which should guarantee that: (I) the linear space \( L_f = \{ \rho | \rho = \rho_i f^i, \rho_i \in \mathbb{R} \} \) is isomorphic with a real Lie algebra, and (II) each element of \( L_f - 0 \) is a root (of arbitrary norm). In these circumstances we have demonstrated the following theorem
Theorem 7 The unique type of family of unit roots with \( N_f > 1 \) having the properties (I) and (II) are the triplets \( f = \{ f^1, f^2, f^3 \} \) which satisfy

\[
\langle f^i \rangle \langle f^j \rangle = -\delta_{ij} 1_n + \varepsilon_{ijk} \langle f^k \rangle, \quad i, j, k... = 1, 2, 3.
\] (29)

Proof: We delegate the proof to Ref. [11].

Hereby it results that the matrices \( \langle f^i \rangle \) and \( 1_n \) generate a matrix representation of the quaternion algebra \( \mathbb{H} \). If the unit roots \( f^i \) have only real-valued components we recover the hypercomplex structures that obey Eq. (27) and define the hyper-Kähler geometry. An example of hyper-Kähler manifold is the Euclidean Taub-NUT space which is equipped with only one family of real unit roots [12] [11].

The manifolds with pseudo-Euclidean metric \( \eta \) of odd signature have only pairs of adjoint triplets, \( f \) and \( f^* \), the last one being formed by the adjoints of the unit roots of \( f \). The spaces \( L_f \) and \( L_{f^*} \) are isomorphic between themselves (as real vector spaces) and all the properties of \( f^* \) can be deduced from those of \( f \) using complex conjugation. Moreover, we must specify that the set \( L_f \cup L_{f^*} \) is no more a linear space since the linear operations among the elements of \( L_f \) and \( L_{f^*} \) are not allowed [11]. An example is the Minkowski spacetime which has a pair of conjugated triplets of complex-valued unit roots [10].

The mentioned examples of manifolds possessing triplets with the properties are of dimension four. The results we know indicate that similar properties may occur for other manifolds of dimension \( n = 4k \), \( k = 1, 2, 3, ... \) where we expect to find many such triplets [24]. The main geometric feature of all these manifolds is given by

Theorem 8 If a manifold \( M_n \) allows a triplet of unit roots then this must be Ricci flat (having \( R_{\mu \nu} = 0 \)).

Proof: As in the case of the hyper-Kähler manifolds, we start with the identity \( 0 = f_{\mu \nu \alpha \beta} - f_{\mu \nu \beta \alpha} = f_{\mu \sigma} R^\sigma_{\nu \alpha \beta} + f_{\sigma \nu} R^\sigma_{\mu \alpha \beta} \), giving

\[
R_{\mu \nu \alpha \beta} f^{\mu \nu}_{\sigma \tau} f^{\sigma \tau} = R_{\sigma \tau \alpha \beta}, \quad (30)
\]

and calculate \( R_{\mu \nu \alpha \beta} f^{1 \alpha \beta} = R_{\mu \nu \alpha \beta} f^{3 \sigma \alpha} \langle f^3 \rangle \langle f^1 \rangle = R_{\mu \nu \alpha \beta} f^{3 \sigma \alpha} f^{2 \alpha \beta} = -R_{\mu \nu \alpha \beta} f^{1 \alpha \beta} = 0 \). Then, permutating the first three indices of \( R \) we find the identity

\[
2R_{\mu \nu \alpha \beta} f^{1 \alpha \beta} = R_{\mu \nu \alpha \beta} f^{1 \alpha \beta} = 0. \quad (31)
\]
Finally, writing $R_{\mu\nu} = R_{\mu\sigma\nu\beta}f_1^{\alpha\cdot\beta}f_1^{\beta\cdot\sigma} = -R_{\mu\sigma\nu\beta}f_1^{\alpha\cdot\beta}f_1^{\beta\cdot\sigma} = 0$, we draw the conclusion that the manifold is Ricci flat. The same procedure holds for $f^2$ or $f^3$ leading to identities similar to (31). Note that the manifolds possessing only single unit roots (as the Kähler ones) are not forced to be Ricci flat.

The passing from the complex structures to unit roots has to be productive from the Dirac theory where the complex-valued K-Y tensors could be involved in the theory of the Dirac-type operators.

5 Dirac-type operators

First of all, the unit roots are complex-valued covariantly constant K-Y tensors. It is known that any second rank K-Y tensor $f$ (having the components $f_{\mu\nu} = -f_{\nu\mu}$ which satisfy the Killing equation, $f_{\mu\nu,\sigma} + f_{\mu\sigma,\nu} = 0$) gives rise to a specific first-order operator of the Dirac type acting in the space $\Psi$.

**Definition 9** The operators

$$D_f = if_{\mu,\nu}\gamma^\mu \nabla_\nu - \frac{1}{6}f_{\mu\nu,\rho}\gamma^\mu \gamma^\nu\gamma^\rho,$$  
(32)

constructed with the help of second rank K-Y tensors, $f$, are called Dirac-type operators.

These are non-standard Dirac operators which have the remarkable property to anticommute with the standard Dirac operator, $\{D_f, D\} = 0$. These operators can be involved in new types of (super)symmetries. Moreover, new interesting superalgebras of Dirac-type operators can be obtained when the second-order K-Y tensors we use are unit roots.

5.1 Operators produced by isolated unit roots

The Dirac-type operator generated by an unit roots $f$ has the simpler form

$$D_f = if_{\mu,\nu}\gamma^\mu \nabla_\nu,$$  
(33)

since the last term of Eq. (32) vanishes in the virtue of the fact that $f$ is now covariantly constant. The operators of this kind have an important property formulated in Ref. [9].
Theorem 9 The Dirac-type operator \(D_f\) produced by the K-Y tensor \(f\) satisfies the condition
\[(D_f)^2 = D^2,\] (34)
if and only if \(f\) is an unit root.

Proof: Arguments are given in [9] showing that Eq. (34) holds only for unit roots. Moreover, we note that for any unit root \(f\) the square of the Dirac-type operator \((33)\) has to be calculated exploiting the identity \(0 = f_{\mu;\alpha;\beta} - f_{\mu;\beta;\alpha} = f_{\mu;\alpha} R^\sigma_{\nu;\alpha;\beta} + f_{\sigma;\nu} R^\sigma_{\nu;\mu;\alpha\beta}\), which gives
\[R_{\mu;\alpha;\beta} f^\mu_{\sigma} f^\nu_{\tau} = R_{\sigma;\tau;\alpha;\beta}\] (35)
and leads to Eq. \((34)\). \(\blacksquare\)

Thus we conclude that the condition \((34)\) holds in any geometry of dimension \(n = 2k\) allowing unit roots (or complex structures).

Another interesting operator related to \(f\) can be defined as follows.

Definition 10 Given the unit root \(f\), the matrix
\[\Sigma_f = \frac{1}{2} f_{\mu;\nu} S^{\mu;\nu}\] (36)
is the spin-like operator associated to \(f\).

This is a matrix that acts on the space of spinors \(\Psi\) and, therefore, can be interpreted as a generator of the spinor representation \(spin[G(\eta)]\). From Eqs. \((9)\) and \((28)\) one can verify that this generator is covariantly constant in the sense that \(\nabla_\nu (\Sigma_f \psi) = \Sigma_f \nabla_\nu \psi\). Hereby we find that the Dirac-type operator \((33)\) can be written as
\[D_f = i [D, \Sigma_f],\] (37)
where \(D\) is the standard Dirac operator defined by Eq. \((8)\). In addition, we obtain the useful relations \([\Sigma_f, D^2] = [\Sigma_f, (D_f)^2] = 0\).

When there is a complex-valued unit root \(f\) then \(f^* \neq f\) and the corresponding spin-like operators are related to each other as \(\Sigma_{f^*} = \Sigma_f\) and, therefore, \(\Sigma_{f^*}\) commutes with \(D^2\). The Dirac-type operators \(D_f\) and \(D_{f^*} = i[D, \Sigma_{f^*}] = D_f^*\) satisfy \((D_f)^2 = (D_{f^*})^2 = D^2\) and
\[\{D_f, D\} = 0, \quad \{D_{f^*}, D\} = 0.\]

These properties suggest us to define \(\mathcal{N} = 2\) superalgebras of Dirac operators.
Definition 11 Given an isolated unit root \( f \), we say that the set \( d_f = \{ D(\lambda)|D(\lambda) = \lambda_0 D + \lambda_1 D_f; \lambda_0, \lambda_1 \in \mathbb{R} \} \) represent the \( \mathcal{N} = 2 \) real \( D \)-superalgebra generated by the unit root \( f \). The complex \( D \)-superalgebra with the same generators, \( D \) and \( D_f \), but having complex-valued coefficients will be denoted by \( (d_f)_c \).

When \( f^* \neq f \) the \( D \)-superalgebra \( d_{f^*} \) differs from \( d_f \) and, in general, these can not be embedded in a larger superalgebra.

5.2 Operators produced by triplets of unit roots

In the case of manifolds allowing triplets of unit roots, \( f = \{ f^1, f^2, f^3 \} \), satisfying Eq. (29), one can construct spin-like and Dirac-type operators for any unit root \( f^i \). Then it is convenient to introduce the simpler notations \( \Sigma_i \equiv \Sigma_{f^i} \) and \( D^i \equiv D_{f^i} = i[D, \Sigma^i], i = 1, 2, 3 \). These operators have the same properties as those produced by isolated unit roots. In addition, from Eqs. (29) we deduce that

\[
\left[ \Sigma^i, \Sigma^j \right] = \frac{i}{2} \left[ \langle f^i \rangle, \langle f^j \rangle \right] \mu \nu S^{\mu \nu} = 2i\varepsilon_{ijk} \Sigma^k. \tag{38}
\]

Taking into account that \( \Sigma^i \) are covariantly constant we find other important relations,

\[
\left[ D^i, \Sigma^j \right] = i\delta_{ij} D + i\varepsilon_{ijk} D^k.
\]

When we work with triplets of complex-valued unit roots, we must consider the operators generated by the adjoint triplet \( f^* \) whose properties have to be obtained simply using the Dirac conjugation as in the previous case of isolated unit roots. Thus it is not difficult to show that \( \Sigma_{(f^i)^*} = \Sigma^i \) and \( D_{(f^i)^*} = D^i \).

The triplets generates larger superalgebras whose algebraic structure is provided by

**Theorem 10** If a triplet \( f \) accomplishes Eqs. (29) then the corresponding Dirac-type operators satisfy

\[
\left\{ D^i, D^j \right\} = 2\delta_{ij} D^2, \quad \left\{ D^i, D \right\} = 0. \tag{39}
\]

**Proof:** If \( i = j \) we take over the result of Theorem 9. For \( i \neq j \) we take into account that \( M_n \) is Ricci flat finding that \( D^i \) and \( D^j \) anticommute. The
second relation was demonstrated earlier for any unit root. Thus it is obvious that the operators $D$ and $D^i$ ($i = 1, 2, 3$) form a basis of a four-dimensional superalgebra of Dirac operators.

**Definition 12** The set $\mathcal{d}_f = \{D(\lambda)|D(\lambda) = \lambda_0 D + \lambda_i D^i; \lambda_0, \lambda_i \in \mathbb{R}\}$ represent the $\mathcal{N} = 4$ real $D$-superalgebra generated by the triplet $f$. The complex $D$-superalgebra $(\mathcal{d}_f)_c$ has the same basis but complex-valued coefficients, $\lambda_0, \lambda_i \in \mathbb{C}$.

If the adjoint triplet $f^*$ differs from $f$ then the (real or complex) $D$-superalgebras generated by these triplets are different to each other and can not be seen as subspaces of a larger linear structure. The relation between these $D$-superalgebras is quite complicated and one must clear it up for each particular case separately.

### 6 Symmetries generated by unit roots

The $D$-superalgebras have the interesting property to present a new type of symmetries due only to the existence of the unit roots [13]. These give the spin-like operators which generate transformations relating the Dirac-type operators and the standard Dirac one among themselves. These symmetries mixed with the transformations produced by isometries give the complete set of automorphisms of the above defined $D$-superalgebras.

#### 6.1 Symmetries generated by isolated unit roots

In the simpler case of manifolds allowing only isolated unit roots $f$ we have to study the continuous symmetry generated by the Lie algebra $L_f$ which is able to relate $D$ and $D_f$ to each other. Let us observe that the roots $\rho = \alpha f \in L_f$ define the sections $A_\rho : M_n \rightarrow \mathbb{G}_c(\eta)$ of the complexified spinor fiber bundle such that $A_\rho(x) = [\rho(x)] \in \mathbb{G}_c(\eta)$.

**Definition 13** We say that $G_f = \{(A_\rho, id) | \rho = \alpha f, \alpha \in \mathbb{R}\} \subset [\mathbb{G}_c(M_n)]$ is the one-parameter Lie group associated to the unit root $f$.

The spinor representation of this group, $spin(G_f)$, is formed by all operators $U_\rho \in spin\mathbb{G}(M_n)$ with $\rho = \alpha f$ whose transformation matrices have the form

$$T(\alpha f) = e^{-ia\Sigma_f} \in spin[\mathbb{G}_c(\eta)]; \quad (40)$$
depending on the group parameter $\alpha \in \mathbb{R}$. Since the matrices (40) are just those defined by Eq. (5) where we replace $\omega$ by the roots $\rho = \alpha f \in L_f$, their action on the point-dependent Dirac matrices results from Eq. (6) to be,

$$[T(\alpha f)]^{-1} \gamma^\mu T(\alpha f) = \Lambda(\alpha f) \gamma^\nu,$$

(41)

where $\Lambda = e^\alpha \Lambda \cdot \cdot e^\beta$ are matrix elements with natural indices of the matrix

$$\Lambda(\alpha f) = e^\alpha \langle f \rangle = 1_n \cos \alpha + \langle f \rangle \sin \alpha,$$

(42)

calculated according to Eqs. (7) and (28). We note that this is a matrix representation of the usual Euler formula of the complex numbers. Now it is obvious that in local frames $\langle f \rangle = \Lambda(\pi f) \in \text{vect}[G(\eta)]$, as mentioned above.

**Theorem 11** The operators $U_\rho \in \text{spin}(G_f)$, with $\rho = \alpha f$, have the following action in the linear space spanned by the operators $D$ and $D_f$:

$$U_\rho D(U_\rho)^{-1} = T(\alpha f)D[T(\alpha f)]^{-1} = D \cos \alpha + D_f \sin \alpha,$$

$$U_\rho D_f(U_\rho)^{-1} = T(\alpha f)D_f[T(\alpha f)]^{-1} = -D \sin \alpha + D_f \cos \alpha.$$

(43)

(44)

**Proof:** From Eq. (42) we obtain the matrix elements $\Lambda(\alpha f) = \cos \alpha \delta^\mu_\nu + \sin \alpha f^\mu_\nu$ which lead to the above result since $\Sigma_f$ as well as $T(\alpha f)$ are covariantly constant. ■

From this theorem it results that $\alpha \in [0, 2\pi]$ and, consequently, the group $G_f \sim U(1)$ is compact. Therefore, it must be a subgroup of the maximal compact subgroup of $G(\eta)$. Note that the transformations (43) and (44) leave the operator $D^2 = (D_f)^2$ invariant because this commutes with the spin-like operator $\Sigma_f$ which generates these transformations.

In general, when $f$ has complex components (and $f^* \neq f$) then $L_f^* \sim so(2)$ is a different linear space representing the Lie algebra of $\text{vect}(G_f^*)$. These two Lie algebras are complex conjugated to each other and, therefore, remain isomorphic (as real or complex algebras). The relation among the transformation matrices of $\text{spin}(G_f)$ and $\text{spin}(G_f^*)$ is $T(\alpha f) = T(-\alpha f^*) = [T(\alpha f^*)]^{-1}$ which means that when $f^* \neq f$ the representation $\text{spin}(G_f)$ is no longer unitary in the sense of the generalized Dirac adjoint. However, if $f^* = f$ is a complex structure then $M_n$ is an usual Kähler manifold and the representation $\text{spin}(G_f)$ is unitary.
The conclusion is that a unit root \( f \) gives rise simultaneously to a Dirac-type operator \( D_f \) which satisfies Eq. (34) and the one-parameter Lie group \( G_f \) which relates \( D \) and \( D_f \) to each other. This general result can be extended for the triplets too.

### 6.2 Symmetries generated by triplets of unit roots

In manifolds with triplets \( f = \{f^1, f^2, f^3\} \) we shall find more complicated symmetries whose properties are provided by Eqs. (29) combined with the previous results (33)-(45).

**Definition 14** We say that \( G_f = \{(A_\rho, \text{id}) \mid \rho = \rho_i f_i \in L_f \} \sim SU(2) \subset G_c(\eta) \) is the Lie group associated to the triplet \( f \).

The spinor and the vector representations of this group are determined by the representations of its Lie algebra, \( g_f \), resulted from Theorem 7.

**Theorem 12** The basis-generators of \( \text{vect}(g_f) = L_f \) are \( J_i = \frac{i}{2} \langle f^i \rangle \) while the basis-generators of the algebra \( \text{spin}(g_f) \sim su(2) \) read \( \hat{s}_i = \frac{1}{2} \Sigma_i \), \((i = 1, 2, 3)\).

**Proof:** From Eqs. (29) and (38) we deduce that these generators satisfy the standard \( su(2) \) commutation rules. We note that they are Hermitian only in the Kählerian case when \( f^* = f \).

The group \( \text{vect}(G_f) = \{\Lambda(A_\rho) \mid \rho \in L_f, \|\rho\| \leq 2\pi\} \) of the vector representation is the compact group formed by the matrices

\[
\Lambda(\rho) = e^{\rho_i \langle f^i \rangle} = 1_n \cos \|\rho\| + \nu_i \langle f^i \rangle \sin \|\rho\|,
\]

where \( \|\rho\| = \sqrt{\rho_i \rho^i} \) and \( \nu_i = \rho_i / \|\rho\| \) for any \( \rho \in L_f \sim su(2) \sim so(3) \). The action of the operators of the spinor representation \( U_\rho \in \text{spin}(G_f) \) is defined by the transformation matrices

\[
T(\rho) = e^{-i\rho_i \Sigma^i} = e^{-2i\nu_i \hat{s}_i}, \quad \rho = \rho_i f_i \in L_f.
\]

These have to be calculated directly from Eq. (40) replacing \( \alpha = \pm \|\rho\| \) and \( f = \pm \rho / \|\rho\| \). Then the transformations (41) can be expressed in terms of parameters \( \rho_i \) using the matrices \( \Lambda(\rho) \).

Hereby we observe that \( \rho_i \) are nothing other than the analogous of the well-known Caley-Klein parameters but ranging in a larger spherical domain.
Proof: We consider the result of Theorem (11) for each one-parameter subgroup of $G_f$. These arguments lead to the conclusion that $\text{vect}(G_f) \sim SU(2) [12]$. On the other hand, since the rotations $\text{vect}(G_f)$ change the basis of $L_f$ leaving Eqs. (29) invariant, we understand that these form the group $\text{Aut}(L_f)$, of the automorphisms of the Lie algebra $L_f$ considered as a real algebra.

The most important result here is that the operator $D(\bar{\nu}) = \nu_i D^i$ defined by the unit vector $\bar{\nu}$ (with $\bar{\nu}^2 = 1$) can be related to $D$ through the transformations (13) and (44) of the one-parameter subgroup $G_{f_{\bar{\nu}}}$ associated to the unit root $f_{\bar{\nu}} = \nu_i f^i$. This is a subgroup of the group $G_f$ which embeds all the subgroups defined by the unit roots $f_{\bar{\nu}}$.

**Theorem 13** The operators $U_\rho \in \text{spin}(G_f)$, with $\rho = \rho_i f^i = \|\rho\| \nu_i f^i$, transform the Dirac operators $D$, $D^i$ ($i = 1, 2, 3$) as

\[
U_\rho D[U_\rho]^{-1} = T(\rho) D[T(\rho)]^{-1} = D \cos \|\rho\| + \nu_i D^i \sin \|\rho\|,
\]

\[
U_\rho D^i[U_\rho]^{-1} = T(\rho) D^i[T(\rho)]^{-1} = D^i \cos \|\rho\| - (\nu_i D + \varepsilon_{ijk} \nu_j D^k) \sin \|\rho\|.
\]

Proof: We consider the result of Theorem (11) for each one-parameter subgroup of $G_f$ generated by the unit root $f_{\bar{\nu}} = \nu_i f^i$. 

In the non-Kählerian manifolds equipped with pairs of adjoint triplets $f \neq f^*$, the corresponding Dirac-type operators $D^i$ and $\overline{D^i}$ are different to each other. In other respects, the generators $\Sigma^i$ are not Hermitian and this forces us to operate with non-unitary representations of the group $G_f \sim SU(2)$. In these conditions it may be necessary to extend the symmetry groups considering the complexified group $(G_f)_c \sim SL(2, \mathbb{C})$ of $G_f$ having the same transformations (45) and (49) but with complex-valued parameters $\rho_i \in \mathbb{C} [11]$. In this case one doubles the number of generators of the spinor or vector representations. More precisely, the basis generators of the spinor representation of the complexified algebra $\text{spin}(g_f)_c$ are $\hat{s}_i = \frac{i}{2} \Sigma^i$ and $\hat{r}_i = (\pm) \frac{i}{2} \Sigma^i$ while the corresponding vector representation, $\text{vect}(g_f)_c$, is generated by $J_i = \frac{i}{2} \langle f^i \rangle$ and $K_i = \pm \frac{i}{2} \langle f^i \rangle$. Obviously, according to Eqs. (29) and (38), these generators satisfy the standard $sl(2, \mathbb{C})$ commutation rules,

\[
[\hat{s}_i, \hat{s}_j] = i \varepsilon_{ijk} \hat{s}_k, \quad [\hat{s}_i, \hat{r}_j] = i \varepsilon_{ijk} \hat{r}_k, \quad [\hat{r}_i, \hat{r}_j] = -i \varepsilon_{ijk} \hat{s}_k,
\]
and similarly for $J_i$ and $K_i$. The spinor and vector representations of the group $(G_{f^*})_c$ are generated by $\tilde{s}_i$ and $\tilde{r}_i$ and respectively $J_i^*$ and $K_i^*$.

7 The effect of isometries

Beside the types of continuous symmetries we have studied, the isometries could have some non-trivial effects transforming the unit roots as well as the Dirac-type operators.

7.1 The invariance of isolated unit roots

Let us start with manifolds $M_n$ with isolated unit roots $f$ generating $\mathcal{N} = 2$ real superalgebras, $d_f$. The basis $(D, D_f)$ can be changed through the transformations (43) and (44) that preserve the anticommutation relations. Whenever the manifold has a non-trivial isometry group $I(M_n)$ then an arbitrary isometry $x \rightarrow x' = \phi_\xi(x)$ transforms $f$ as a second rank tensor,

$$f_{\mu\nu}(x) \rightarrow f'_{\mu\nu}(x') = f_{\alpha\beta}(x).$$

Since there is only one $f$ we are forced to put $f' = f$ which means that this remains invariant under isometries.

**Theorem 14** Aut($d_f$) = spin($G_f$) $\sim SO(2)$ is the group of automorphisms of the D-superalgebra $d_f$.

**Proof:** This group must be Abelian since $\mathcal{N} = 2$. Thus there are no transformations different from (43) and (44). This is in accordance with the invariance of $f$ under isometries. ■

A consequence is given by

**Corollary 2** If a manifold $M_n$ with the external symmetry group $S(M_n)$ has a single unit root, $f$, then every generator $X \in \text{spin}[s(M_n)]$ commutes with $D_f$.

**Proof:** We calculate first the derivatives with respect to $\xi^a$ of Eq. (51) for $f' = f$ and $\xi = 0$. Then, taking into account that $f$ is covariantly constant.
we can write \( f_{\alpha\lambda}k_{\beta}^\lambda = f_{\beta\lambda}k_{\alpha}^\lambda \) for each Killing vector field \( k \) defined by Eq. (18). This identity yields

\[
[X, \Sigma_f] = 0, \quad [X, D_f] = 0, \quad \forall X \in \text{spin}[s(M_n)],
\]

which means that the operators \( \Sigma_f \) and \( D_f \) are invariant under isometries.

### 7.2 The transformation of triplets under isometries

The case of the hyper-Kähler manifolds is more complicated since a triplet \( f \) gives rise to Dirac-type operators \( D^i = \overline{D}^i \) which anticommute with \( D \) and present the non-Abelian continuous symmetry discussed in the previous section. In this case we have to study the group of automorphisms of this D-superalgebra, \( \text{Aut}(d_f) \), and its Lie algebra, \( \text{aut}(d_f) \). The transformation matrices \( T(\rho) \) commute with \( D^2 \), leaving Eqs. (39) invariant under transformations (48) and (49) which appear thus as automorphisms of \( d_f \). Consequently, the group of these transformations, \( \text{spin}(G_f) \sim SU(2) \), is a subgroup of \( \text{Aut}(d_f) \). However, we need more automorphisms in order to complete the group \( \text{Aut}(d_f) \) with more ordinary or invariant subgroups, isomorphic with \( SU(2) \) or \( O(3) \). These supplemental automorphisms must transform the operators \( D_i \) among themselves preserving their anticommutators as well as the form of \( D \). Therefore, these may be produced by the isometries of \( M_n \) since these leave the operator \( D \) invariant.

In what concerns the transformation of the triplets \( f \) under isometries we have two possibilities, either to consider that all the unit roots \( f^i \in f \) are invariant under isometries or to assume that the isometries transform the components of the triplet among themselves, \( f^{i'} = \hat{R}_{ij} f^j \), such that Eqs. (29) remain invariant. The first hypothesis is not suitable since we need more transformations in order to fill in the group \( \text{Aut}(d_f) \) when we do not have other sources of symmetry. Therefore, we must adopt the second viewpoint assuming that the components of \( f \) are transformed as

\[
f_{\mu
u}^j(x') \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x'^\nu}{\partial x^\beta} = \hat{R}_{kj}(\xi, x) f_{\alpha\beta}^k(x),
\]

by \( 3 \times 3 \) real or complex-valued matrices \( \hat{R} \) which must be orthogonal in order to leave Eqs. (29) invariant. According to these equations, the matrix
elements can be put in the equivalent forms
\[ \hat{R}_{ij}(\xi, x) = \frac{1}{n} f^{i\alpha\beta}(x) \frac{\partial \phi^\mu_k(x)}{\partial x_\alpha} f^{j}_{\mu
u}[\phi(x)] \]
\[ = \frac{1}{n} f^{i\hat{\alpha}\hat{\beta}}(x) \Lambda^{\hat{\beta}}_{\hat{\alpha}}[A_\xi(x)] \Lambda^{\nu}_{\mu}[A_\xi(x)] f^{j}_{\mu\nu}[\phi(x)]. \] (53)

The last formula is suitable for calculations in local frames where we must consider the external symmetry using the gauge transformations \( \Lambda[A_\xi(x)] \in \text{vect}[G(\eta)] \) defined by Eq. (20) and associated to isometries for preserving the gauge.

The matrices \( \hat{R} \) might be point-dependent and depend on the parameters \( \xi^a \) of \( I(M_n) \). This means that the canonical covariant parameters \( \hat{\omega}_{ij} = -\hat{\omega}_{ji} \) giving the expansion \( \hat{R}_{ij}(\hat{\omega}) = \delta_{ij} + \hat{\omega}_{ij} + \cdots \) are also depending on these variables. Then, for small values of the parameters \( \xi^a \) the covariant parameters can be developed as \( \hat{\omega}_{ij} = \xi^a \hat{c}_{aij} + \cdots \) emphasizing thus the quantities \( \hat{c}_{aij} \) we shall see that do not depend on coordinates.

**Theorem 15** Let \( M_n \) be a manifold having the triplet of unit roots \( f = \{f^1, f^2, f^3\} \) and a non-trivial isometry group \( I(M_n) \) with parameters \( \xi^a \) and the corresponding Killing vectors
\[ k_a(x) = \frac{\partial \phi_\xi(x)}{\partial \xi^a} \bigg|_{\xi=0}. \]

Then the basis-generators (24) of the spinor representation \( \text{spin}[s(M_n)] \) and \( \Sigma^i \) satisfy
\[ [X_a, \Sigma^i] = i\hat{c}_{aij} \Sigma^j, \quad a = 1, 2, \ldots, N, \] (54)
where \( \hat{c}_{aij} \) are point-independent structure constants.

**Proof:** Deriving Eq. (52) with respect to \( \xi^a \) in \( \xi = 0 \) we deduce
\[ f^{i}_{\mu\lambda} k^{\lambda}_{a\;\nu} - f^{i\nu}_{\lambda\mu} k^{\lambda}_{a;\mu} = \hat{c}_{aij} f^{j}_{\mu\nu}, \] (55)
which leads to the explicit form
\[ \hat{c}_{aij} = -\frac{2}{n} \epsilon_{ijkl} \hat{k}^l_{a}, \quad \hat{k}^l_{a} = f^{l\mu\nu} k_{a\mu\nu}. \] (56)

Bearing in mind that \( f^{i\mu\nu}_{\rho\sigma} = 0 \) and using \( f^{i\mu\nu} k_{a\mu\nu;\rho} = R^{ij}_{\mu\nu\mu\nu} k_{a\mu\nu} f^{i\mu\nu} \) and Eq. (31), we find \( \nabla_\sigma \hat{k}^l_{a} = f^{i\mu\nu} k_{a\mu\nu;\sigma} = 0 \) which means that \( \partial_\sigma \hat{c}_{aij} = 0 \). Finally, from Eq. (55) we derive the commutation rules (54). ■

An important consequence is given by
Corollary 3  The basis generators $X_a \in \text{spin}[s(M_n)]$ and the Dirac-type operators of the $\mathcal{N} = 4$ superalgebra $d_\mathcal{F}$ obey

\[ [X_a, D^i] = i\hat{c}_{aij} D^j. \]

**Proof:** This formula results from commuting Eq. (54) with $D$.

We note that the structure constants $\hat{c}_{aij}$ take complex values in the non-Kählerian case when $f^* \neq f$. In other respects, the previous theorem provides the form of the matrices $\mathfrak{R}$.

Corollary 4  Eq. (55) defines the point-independent $3 \times 3$ matrices of the form,

\[ \hat{R}(\xi) = e^{\xi a X_a}, \quad (X_a)_{ij} = -i\hat{c}_{aij}, \]

whose generators satisfy the commutation rules of $i(M_n)$,

\[ [X_a, X_b] = i\hat{c}_{aij} X_c. \]  

**Proof:** These matrices are point-independent since $\hat{c}_{aij}$ are structure constants. If we commute Eq. (54) with $X_b$ using Eqs. (25) and (50) we obtain Eq. (58). The above results lead to the following conclusion:

Corollary 5  The group $O_\mathcal{F} = \{ \hat{R}(\xi) \mid (\text{Id}, \phi_{\xi}) \in I(M_n) \}$ is a representation of the group $I(M_n)$ induced by the group $O_c(3)$.

**Proof:** The matrices (57) are orthogonal since they are generated by the skew-symmetric generators defined by Eqs. (57) and (50). These matrices may have complex-valued components which means that the representation $O_\mathcal{F}$ is induced by $O_c(3)$. In the Kählerian case, when $f^* = f$ and $\hat{c}_{aij} \in \mathbb{R}$, this representation is induced by the orthogonal group $O(3)$.

Now we can point out how act the isometries $x \to x' = \phi_{\xi}(x)$ on the operators $D^i$.

Theorem 16  The Dirac-type operators $D^i$ produced by any triplet $f$ transform under isometries $(A_\xi, \phi_{\xi}) \in S(M_n)$ according to the transformation rule

\[ (U_\xi D^i U_\xi^{-1})[\phi_{\xi}(x)] = [T(A_\xi) D^i T(A_\xi)^{-1}](x) = \hat{R}_{ij}(\xi) D^j(x), \]

where the action of the operator $U_\xi \in \text{spin}[S(M_n)]$ is defined by Eq. (56).
Proof: From Eqs. (36) and (52) we derive
\[(U_\xi \Sigma^i U_\xi^{-1})[\phi_\xi(x)] = [T(A_\xi)\Sigma^i T(A_\xi)^{-1}](x) = \hat{R}_{ij}(\xi) \Sigma^j(x), \tag{60}\]
which leads to Eq. (59) after a commutation with \(D\) that is invariant under \(U_\xi\). □

8 The automorphisms of \(d_f\) and \((d_f)_c\)

Now we have a complete image of the symmetries that preserve the anticommutation rules of the real superalgebra \(d_f\) in a hyper-Kähler manifold. These are given by the transformations defined by Eqs. (48), (49) and (59). In what concerns the structure of the groups of automorphisms of the \(N=4\) D-superalgebras, the problem must be treated separately in the Kählerian and non-Kählerian cases.

Let us consider first the hyper-Kähler manifolds.

Theorem 17 Whenever \(f^* = f\) the group \(Aut(d_f) = \text{spin}[G_f \star S(M_n)]\) is a representation of the semi-direct product \(G_f \star S(M_n)\) where \(G_f\) is the invariant subgroup.

Proof: We observe that in this case \(O_f\) is an unitary representation of \(I(M_n)\) induced by \(O(3)\). This means that the transformations (59) lead only to real linear combinations of Dirac operators such that it is enough to study the automorphisms of the real D-superalgebra \(d_f\). The basis generators of the Lie algebra \(\text{aut}(d_f)\) of the group \(Aut(d_f)\) are the operators \(\hat{s}_i\) and \(X_a\) that obey the commutation relations (25), (50) and (54). These operators form a Lie algebra since \(\hat{c}_{aij}\) are point-independent. In this algebra \(g_f \sim su(2)\) is an ideal and, therefore, the corresponding \(SU(2)\) subgroup is invariant. However, this result can be obtained directly taking \((A_\xi, \phi_\xi) \in S(M_n)\) and \((A_\rho, id) \in G_f\) and evaluating \((A_\xi, \phi_\xi)*(A_\rho, id)*(A_\xi, \phi_\xi)^{-1} = ([A_\xi \times (A_\rho \times A_\xi^{-1})]\circ \phi_\xi^{-1}, id) = (A_\rho', id)\) where, according to (20), (45) and (52), we have \(\rho' = \rho_i \delta_{ij}(\xi)f^j\). Consequently, \((A_\rho', id) \in G_f\) which means that \(G_f \sim SU(2)\) is an invariant subgroup. □

An interesting restriction can be formulated as

Corollary 6 The minimal condition that \(M_n\) allows a hypercomplex structure is to have an isometry group that includes at least one \(O(3)\) subgroup.
Proof: The subgroup $O(3)$ of $\text{Aut}(d_f)$ needs at least three generators $X_a$ satisfying the $su(2)$ algebra. Thus we conclude that $S(M_n)$ must include one $SU(2)$ group for each different hypercomplex structure of $M_n$. ■

This restriction is known in four dimensions where there exists only three hyper-Kähler manifolds with only one hypercomplex structure and one subgroup $O(3) \subset I(M_4)$ [25]. These are given by the Atiyah-Hitchin [26], Taub-NUT and Eguchi-Hanson [27] metrics, the first one being only that does not admit more $U(1)$ isometries [25, 28].

The case of the non-Kählerian manifolds is more delicate since here the matrices $\hat{R}$ are complex-valued orthogonal matrices of the complexified group $O_c(3)$. Consequently, these lead to linear combinations of the operators $D^i$ with complex-valued coefficients. In these circumstances we must consider the automorphisms of the complex D-superalgebra $(d_f)_c$. However, this requires to use the complexified group $(G_f)_c \sim SL(2, \mathbb{C})$ instead of $G_f$. In this way we arrive at

**Theorem 18** If $f^* \neq f$ then the group $\text{Aut}(d_f)_c = \text{spin}[(G_f)_c \times S(M_n)]$ is the spin representation of the semi-direct product $(G_f)_c \times S(M_n)$ where $(G_f)_c$ is the invariant subgroup.

Proof: The proof is the same as for Theorem (17). ■

Moreover, following similar arguments as in the case of the Kählerian manifolds, one can deduce the minimal condition that $M_n$ allows a pair of adjoint triplets. This requires the group $S(M_n)$ to include at least one $SL(2, \mathbb{C})$ subgroup since we need six generators for building the representation $O_f \sim O(3)_c$. The Minkowski spacetime which has a pair of adjoint triplets and $O(3, 1)$ isometries is a typical example.

9 The Minkowski spacetime

Let us consider the Minkowski spacetime, $M$, with the metric $g = \eta = (1, -1, -1, -1)$ and Cartesian coordinates, $x^\mu (\mu, \nu, ... = 0, 1, 2, 3)$ representing the time $x^0 = t$ and the space coordinates $x^i, (i, j, ... = 1, 2, 3)$. We choose the usual gauge of inertial frames given by $\epsilon^\mu_\nu = \hat{\epsilon}^\mu_\nu = \delta^\mu_\nu$. In this gauge we take the chiral representation of the Dirac matrices (with off-diagonal $\gamma = \gamma^0$ [29]) where the standard Dirac operator reads

$$D = i\gamma^\mu \partial_\mu = \begin{pmatrix} 0 & i(\partial_t + \vec{\sigma} \cdot \vec{\partial}) \\ i(\partial_t - \vec{\sigma} \cdot \vec{\partial}) & 0 \end{pmatrix} = \begin{pmatrix} 0 & D^{(+)} \\ D^{(-)} & 0 \end{pmatrix},$$
and the generators $S_{\mu\nu}$ of the spinor representation of the group $G(\eta) = SL(2, \mathbb{C})$ have the form

$$S_{ij} = \varepsilon_{ijk}S_k = \frac{1}{2}\varepsilon_{ijk}\text{diag}(\sigma_k, \sigma_k), \quad S_{i0} = \frac{i}{2}\text{diag}(\sigma_i, -\sigma_i).$$

All these operators are self-adjoint with respect to the usual Dirac adjoint $(X = \gamma^0 X^+ \gamma^0)$.

The isometries of $M$ are the transformations $x' = \Lambda(\omega)x - a$ of the Poincaré group, $I(M) = P^+ = T(4) \rtimes SL^+(2, \mathbb{C})$. The group of external symmetry $S(M) = \tilde{P}^+ \sim T(4) \rtimes SL(2, \mathbb{C})$ is the universal covering group of $I(M)$. If we denote by $\xi^{(\mu\nu)} = \omega^{\mu\nu}$ the $SL(2, \mathbb{C})$ parameters and by $\xi^{(\mu)} = a^\mu$ those of the translation group $T(4)$, then we obtain the standard basis generators of the algebra $\text{spin}[s(M)]$,

$$X_{(\mu)} = i\partial_\mu,$$
$$X_{(\mu\nu)} = i(\eta_{\mu\alpha}x^\alpha \partial_\nu - \eta_{\nu\alpha}x^\alpha \partial_\mu) + S_{\mu\nu},$$

which show us that in this gauge the Dirac field $\psi$ transforms manifestly covariant. In applications it is convenient to denote $P_\mu = X_{(\mu)}$, $J_i = \frac{1}{2}\varepsilon_{ijk}X_{(jk)}$ and $K_i = X_{(0i)}$.

## 9.1 Dirac-type operators

The Minkowski spacetime possesses a pair of adjoint triplets $f \neq f^* \ [10]$. The unit roots of the first triplet, $f = \{f^1, f^2, f^3\}$, have the non-vanishing complex-valued components $\ [10]$

$$f^1_{23} = 1, \quad f^1_{01} = i, \quad f^2_{31} = 1, \quad f^2_{02} = i, \quad f^3_{12} = 1, \quad f^3_{03} = i.$$  

These triplets satisfy the standard algebra $\ [29]$ and, in addition, the unit roots of different triplets commute with each other,

$$\left[ \langle f^i \rangle, \langle f^j \rangle^* \right] = 0, \quad i, j = 1, 2, 3. \quad (61)$$

This property that holds in the Minkowski geometry seems to be particular since this could not be proved so far in the general case.
The first triplet, \( f \), gives rise to the spin-like operators

\[
\Sigma^i = \frac{1}{2} f^i_{\mu\nu} S^{\mu\nu} = \begin{pmatrix} \sigma_i & 0 \\ 0 & 0 \end{pmatrix},
\]

while the second one, \( f^* \), yields

\[
\Sigma^i = \frac{1}{2} \left( f^*_i S^{\mu\nu} \right)^* = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_i \end{pmatrix}.
\]

Because of the supplemental condition given by Eq. (61), the operators \( \Sigma^i \) and \( \Sigma^i \) have a special form. They act separately on orthogonal subspaces that are nothing other than the left (\( \Psi_L \)) and respectively right-handed (\( \Psi_R \)) chiral projections of the space of the Dirac spinors \( \Psi = \Psi_L \oplus \Psi_R \). Consequently, these subspaces will be the carrier spaces of the irreducible representations \( \text{spin}(G_f) \) and \( \text{spin}(G_f^*) \). In addition, it is interesting to observe that \( \Sigma^{(i)} + \Sigma^{(i)} = 2S^i \).

The Dirac-type operators of the first triplet are

\[
D^i = i [D, \Sigma^i] = \begin{pmatrix} 0 & -i\sigma_i D^{(+)i} \\ iD^{(-)i} & 0 \end{pmatrix},
\]

while the second triplet gives us \( D^i = i [D, \Sigma^i] \). The operators of each triplet and \( D \) obey the anticommutation rules (39) and anticommute with \( \gamma^0 \). The operators \( (D, D^i) \) form a basis for the D-superalgebra \( (d_f)_c \) while the conjugated algebra \( (d_f^*)_c \) has the basis \( (D, \overline{D^i}) \). It is remarkable that there are particular relations,

\[
\{D^i, \overline{D^j}\} = -2\delta_{ij}D^2 + 4P_iP_j + 4i\varepsilon_{ijk}P_0P_k
\]

involving operators from both different bases.

### 9.2 Symmetries

The next objective is to construct the groups of automorphisms of the complex D-superalgebras. The first group is \( \text{Aut}(d_f)_c = \text{spin}[(G_f)_c S(M_n)] \) where the isometries act according to Eq. (59) with \( \mathfrak{R} \in O_f \sim O_c(3) \). Our general results indicate that the generators of \( \text{spin}(G_f)_c \sim SL(2, \mathbb{C}) \)
are $\hat{s}_i = \frac{1}{2} \Sigma^{(i)}$ and $\hat{r}_i = \frac{i}{2} \Sigma^{(i)}$. These generators satisfy usual $sl(2, \mathbb{C})$ commutation rules \[ [\mathcal{P}_\mu, \hat{s}_i] = 0, \quad [\mathcal{J}_i, \hat{s}_j] = i \varepsilon_{ijk} \hat{s}_k, \quad [\mathcal{K}_i, \hat{s}_j] = i \varepsilon_{ijk} \hat{r}_k, \]
\[ [\mathcal{P}_\mu, \hat{r}_i] = 0, \quad [\mathcal{J}_i, \hat{r}_j] = i \varepsilon_{ijk} \hat{r}_k, \quad [\mathcal{K}_i, \hat{r}_j] = i \varepsilon_{ijk} \hat{s}_k. \quad (63) \]

The transformations matrices $T(\rho)$ (with $\rho = \rho_i f^{(i)}$ and $\rho_i = \rho_i' + i \rho_i''$ where $\rho_i', \rho_i'' \in \mathbb{R}$) that give the transformation laws \[ and \]
\[ are \]
\[ T(\rho) = e^{-i \rho_i \Sigma^{(i)}} = e^{-2i(\rho_i' \hat{s}_i + \rho_i'' \hat{r}_i)}. \]
as it results from Eq. \[ \] (63). Hereby we observe that the matrices $T(\rho)$ act only on the left-handed subspace $\Psi_L$ of the Dirac spinors.

The next step is to construct the complex-valued orthogonal matrices of the group $O_f \sim O_c(3)$ defined by Eq. \(\) (53). These have the general form
\[ \hat{R}_{ij}(\omega) = \frac{1}{4} f^{(i)}_{\alpha \beta} \Lambda^\mu_{\alpha}(\omega) \Lambda^\nu_{\beta}(\omega) f^{(j)}_{\mu \nu}, \]
and constitute a representation of the group $I(M)$ induced by the group $O_c(3)$. Of course, the translations have no effects in this representation such that we are left only with the transformations $\Lambda(\omega) \in O(3, 1)$. These give rise to non-trivial matrices $\hat{\mathcal{H}}(\omega)$ as, for example,
\[ \hat{\mathcal{H}}(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & \sin \varphi \\ 0 & -\sin \varphi & \cos \varphi \end{pmatrix}, \quad \hat{\mathcal{H}}(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh \alpha & i \sinh \alpha \\ 0 & -i \sinh \alpha & \cosh \alpha \end{pmatrix}, \]
calculated for non-vanishing parameters $\omega_{23} = \varphi$ (a rotation around $x^3$) and respectively $\omega_{01} = \alpha$ (a boost along $x^1$). Thus we have all the ingredients we need to write down the action of the transformations of the group $\text{Aut}(d_{13})_c$.

We note that Eqs. \(\) (63) show that $(G_f)_c$ is an invariant subgroup. The second group of automorphisms, $\text{Aut}(d_{13})_c$, can be constructed taking into account that the transformation matrices of $spin(G_{f_1})$ are $\mathcal{T}(\rho)$ and the group $O_{f_1}$ is formed by the conjugated matrices $\hat{\mathcal{H}}_c$. We observe that $\mathcal{T}$ are generated by the matrices $\hat{a}_i$ and $\hat{b}_i$ that act only on the right-handed subspace $\Psi_R$.

Hence it is clear that each chiral sector has its own set of unit roots defining Dirac-type operators and groups of automorphisms of the D-superalgebras for each chiral sector separately. The origin of this perfect balance
between the chiral sectors is the form of the operator \( D^+ D^- = D^- D^+ \) that commutes with \( \sigma_i \).

Finally we note that the principal problem that remains open is the physical significance of the new Dirac-type operators of the Minkowski spacetime we have studied here.

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