SOME REMARKS ON QUANTIZED LIE SUPERALGEBRAS OF
CLASSICAL TYPE

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ABSTRACT. In this paper we use the Etingof-Kazhdan quantization of Lie bi-superalgebras to investigate some interesting questions related to Drinfeld-Jimbo type superalgebra associated to a Lie superalgebra of classical type. It has been shown that the D-J type superalgebra associated to a Lie superalgebra of type A-G, with the distinguished Cartan matrix, is isomorphic to the E-K quantization of the Lie superalgebra. The first main result in the present paper is to extend this to arbitrary Cartan matrices. This paper also contains two other main results: 1) a theorem stating that all highest weight modules of a Lie superalgebra of type A-G can be deformed to modules over the corresponding D-J type superalgebra and 2) a super version of the Drinfeld-Kohno Theorem.

1. INTRODUCTION

Let us start by recalling that Kac [13] showed that Lie superalgebras of type A-G are characterized by their associated Dynkin diagrams or equivalently Cartan matrices. A Cartan matrix associated to a Lie superalgebra is a pair consisting of a matrix $A$ and a set $\tau$ determining the parity of the generators. Let $(A, \tau)$ be such a Cartan matrix and $g$ be the Lie superalgebra arising from $(A, \tau)$.

The Drinfeld-Jimbo algebra associated to a semi-simple Lie algebra is defined by generators and relations. The higher order relations of this algebra are called the quantum Serre relations. Many authors (see for example: [8, 15, 20]) have studied generalizations of this algebra to the setting of Lie superalgebras. This generalization introduces defining relations (e.g. (3.6)) that are not of the form of the quantum Serre relations. These new relations depend directly on the Cartan matrix $(A, \tau)$. Let $U_{h}^{DJ}(g, A, \tau)$ be the Drinfeld-Jimbo type superalgebra associated to the triple $(g, A, \tau)$.

In [4] Drinfeld asked: “Does there exist a universal quantization for Lie bialgebras?” Etingof and Kazhdan [6] gave a positive answer to this question. In [9] the author extended this quantization from the setting of Lie bialgebras to the setting of Lie bi-superalgebras. The triple $(g, A, \tau)$ has a natural Lie bi-superalgebra structure. Let $U_{h}(g, A, \tau)$ be the Etingof-Kazhdan quantization of this Lie bi-superalgebra. Let $\mathfrak{h}$ be the Cartan sub-superalgebra of $g$.

Theorem 1.1. (proof in [4.3]) There exists an isomorphism of quantized universal enveloping (QUE) superalgebras:

$$\alpha : U_{h}^{DJ}(g, A, \tau) \rightarrow U_{h}(g, A, \tau)$$

such that $\alpha|_{\mathfrak{h}} = id.$
In [9] the author proves the above theorem in the case when g is a Lie superalgebra of type A-G with the distinguished Cartan matrix. Most of the arguments given in [9] can be adapted to the present situations. However, in [9] the author checks by hand that the quantum Serre relations (which depend on the Cartan matrix) are in the kernel of a certain bilinear form. Here we appeal to the work of Yamane [20] to show that these relations are in the desired kernel.

In the remainder of this section we state the other main results of this paper. In the case when g is a semi-simple Lie algebra, the following two theorems are analogous to results of Drinfeld [2, 5]. Drinfeld’s proof uses deformation theoretic arguments based on the fact that $H^i(g, U(g)) \neq 0$, $i = 1, 2$, for semisimple Lie algebras. In general, this vanishing result is not true for Lie superalgebras (for example $H^1(sl(2|1), U(sl(2|1))) \neq 0$). Our proof is based on a different approach than Drinfeld’s, utilizing Theorem 1.1 and the general theory of the Etingof-Kazhdan quantization of Lie (super)bialgebras (see Equation 1.1).

1.1. Deformations of weight modules. Lusztig proved that each irreducible dominant integral weight module of a Kac-Moody algebra can be deformed to a module over the corresponding Drinfeld-Jimbo algebra. In [4] Drinfeld asked if arbitrary weight modules over a Kac-Moody algebra can be deformed. Etingof and Kazhdan gave a positive answer to this question in [7]. The following theorem gives a positive answer to this question for Lie superalgebras of type A-G.

For the definition of highest weight modules over g and $U_{DJ}^g(A, \tau)$ see subsection 6.1.

**Theorem 1.2.** (proof in §6.1) For $\Lambda \in \mathfrak{h}^*$, let $V(\Lambda)$ be the irreducible highest weight module over g of highest weight $\Lambda$. Then there exists a highest weight $U_{DJ}^g(A, \tau)$-module $\tilde{V}(\Lambda)$ of weight $\Lambda$ which is a deformation of $V(\Lambda)$. Moreover, the characters of $V(\Lambda)$ and $\tilde{V}(\Lambda)$ are equal. In other words, if

$$V(\Lambda) = \bigoplus_{\Lambda \in \mathfrak{h}^*} V_{\Lambda},$$

then

$$\tilde{V}(\Lambda) = \bigoplus_{\mu \in \mathfrak{h}^*[\mathfrak{h}]} \tilde{V}_{\mu},$$

where $\tilde{V}_{\mu} := \{v : av = \mu(a)v \text{ for all } a \in \mathfrak{h}\}$ and $\tilde{V}_{\mu} \equiv V_{\mu}^{\mathfrak{h}^*[\mathfrak{h}]}$.

Theorem 1.2 allows one to derive properties about $U_{DJ}^g(A, \tau)$-modules by working with the underlying g-module. In particular, character formulas, tensor product decompositions, and other properties about g-modules lead to analogous properties for the corresponding $U_{DJ}^g(A, \tau)$-module. This procedure is very useful in knot theory. For example, it is used to construct generalized multivariable Alexander link invariants arising from Lie superalgebras (see [10] [11]).

Let us now say a few words about the proof of Theorem 1.2 (a detailed proof is given in Section 6.1). In the case when g is equal to $osp(1|2n)$, $osp(2|2n)$ and $sl(m|n)$ it has been shown that $H^2(g, U(g)) = 0$ (see [18] [19]). Therefore, for such a Lie superalgebra g it follows that any deformation of $U(g)$ is trivial as an associative superalgebra. This allow g-modules to be deformed to $U_{DJ}^g(A, \tau)$-modules. However, this does not imply that a weight g-module of weight $\Lambda$ will be deformed to a weight $U_{DJ}^g(A, \tau)$-module whose weight is equal (relative to $\mathfrak{h}$) to $\Lambda$. In the case of finite dimensional Lie algebras, Drinfeld gives further argument (using the vanishing of the first cohomology) to show that weight modules can be
Some Remarks on Quantized Lie Superalgebras of Classical Type

As mentioned above such arguments will not work in the present situation. Instead our proof is based on the fact that the E-K quantization \( U_h(g, A, \tau) \) is by construction the twist of a quasi-Hopf superalgebra whose underlying superalgebra is \( U(g)[[h]] \). Combining this fact with Theorem 1.1 it follows that we have an isomorphism of superalgebras

\[
f : U^{DJ}_h(g, A, \tau) \to U(g)[[h]]
\]

such that \( f|_h = id \). We will show Theorem 1.2 follows from the observation that highest weight \( g \)-modules can be deformed to highest weight \( U(g)[[h]] \)-modules and the fact that \( f \) preserves weights.

1.2. The Drinfeld-Kohno Theorem. Here we state the Drinfeld-Kohno theorem for Lie superalgebras. In the coming sections we elaborate on the definitions of the objects involved in this statement.

Let \( V(\Lambda) \) be an irreducible highest weight module of \( g \) and let \( \tilde{V}(\Lambda) \) be the \( U^{DJ}_h(g, A, \tau) \)-module given in Theorem 1.2. Let \( B_n = \langle \sigma_i \rangle \) be the braid group. Define \( \rho_n \) to be the representation of \( B_n \) on \( \tilde{V}(\Lambda)^{\otimes n} \) given by

\[
\sigma_i \mapsto \tau_{i,i+1} R_{ii+1}
\]

where \( \tau_{i,i+1} \) is the super permutation of the \( i \)-th and the \((i+1)\)-th component and \( R \) is the universal \( R \)-matrix of \( U^{DJ}_h(g, A, \tau) \). Finally, let \( \rho^KZ_n \) be the monodromy representation of \( B_n \) arising from the KZ system of differential equations.

The proof of the following theorem can be found in subsection 6.2.

**Theorem 1.3.** (The Drinfeld-Kohno theorem for Lie superalgebras) The representations \( \rho_n \) and \( \rho^KZ_n \) are equivalent.

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2. Preliminaries

In this section we recall facts and definitions related to Lie super(bi)algebras, for more details see [13].

Let \( k \) be a field of characteristic zero. A **superspace** is a \( \mathbb{Z}_2 \)-graded vector space \( V = V_0 \oplus V_1 \) over \( k \). We denote the parity of a homogeneous element \( x \in V \) by \( \bar{x} \in \mathbb{Z}_2 \). We say \( x \) is even (odd) if \( x \in V_0 \) (resp. \( x \in V_1 \)). In this paper the tensor product will have the natural induced \( \mathbb{Z}_2 \)-grading. Throughout, all modules will be \( \mathbb{Z}_2 \)-graded modules, i.e. module structures which preserve the \( \mathbb{Z}_2 \)-grading (see [13]).

A **Lie bi-superalgebra** is a Lie superalgebra \( g \) with a linear map \( \delta : g \to \wedge^2 g \) that preserves the \( \mathbb{Z}_2 \)-grading and satisfies both the super-coJacobi identity and cocycle condition (see [1]). A triple \((g, g_+, g_-)\) of finite dimensional Lie superalgebras is a finite dimensional **super Manin triple** if \( g \) has a non-degenerate super-symmetric invariant bilinear form \( \langle , \rangle \), such that \( g \cong g_+ \oplus g_- \) as superspaces, and \( g_+ \) and \( g_- \) are isotropic Lie sub-superalgebras of \( g \). There is a one-to-one correspondence between finite dimensional super Manin triples and finite dimensional Lie bi-superalgebras (see [1] Proposition 1).
Now we give the notion of the double of a finite dimensional Lie bi-superalgebra. Let $(g_+, [\cdot,\cdot], \delta)$ be a finite dimensional Lie bi-superalgebra and $(g, g_+, g_-)$ its corresponding super Manin triple. Then $g := g_+ \oplus g_-$ has a natural structure of a quasitriangular Lie bi-superalgebra, see [9]. We call $g$ the double of $g_+$ and denote it by $D(g_+)$. We will now define the Casimir element of $g$. Let $p_1, \ldots, p_n$ be a homogeneous basis of $g_+$. Using the isomorphism $g_- \rightarrow g^*_+$ pick a homogeneous basis $m_1, \ldots, m_n$ of $g_-$ that is dual to $p_1, \ldots, p_n$, i.e. $\langle m_i, p_j \rangle = \delta_{ij}$. Notice $m_1, \ldots, m_n, p_1, \ldots, p_n$ is a basis of $g$ that is dual to the basis $p_1, \ldots, p_n, (-1)^{m_1} m_1, \ldots, (-1)^{m_n} m_n$, with respect to $(\cdot, \cdot)$. Define the Casimir element to be

$$\Omega = \sum p_i \otimes m_i + \sum (-1)^{m_i} m_i \otimes p_i. \quad (2.1)$$

The element $\Omega$ is even, invariant and super-symmetric. Moreover, it is independent of the choice of basis.

3. The Drinfeld-Jimbo type quantization of Lie superalgebras of type A-G

In this section we recall some basic facts related to complex Lie superalgebras of type A-G and their quantum analogue. For the purposes of this paper Lie superalgebras of type A-G will be complex and include the Lie superalgebra $D(2,1,\alpha)$.

Any two Borel subalgebras of a semisimple Lie algebra are conjugate. Moreover, semisimple Lie algebras are determined by their root systems or equivalently their Dynkin diagrams. Not all Borel sub-superalgebras of classical Lie superalgebras are conjugate. As shown by Kac [13] a Lie superalgebra can have more than one Dynkin diagram depending on the choice of Borel. However, using Dynkin type diagrams Kac gave a characterization of Lie superalgebras of type A-G. The constructions of this paper depend on the choice of Borel sub-superalgebra.

Let $h$ be an indeterminate.

3.1. Lie superalgebras of type A-G. Let $g$ be a Lie superalgebra of type A-G. Let $\Phi = \{\alpha_1, \ldots, \alpha_s\}$ be a simple root system of $g$ and let $(A, \tau)$ be the corresponding Cartan matrix, where $A$ is a $s \times s$ matrix and $\tau$ is a subset of $\{1, \ldots, s\}$ determining the parity of the generators. The matrix $A = (a_{ij})$ is symmetrizable, i.e. there exists nonzero rational numbers $d_1, \ldots, d_s$ such that $d_i a_{ij} = d_j a_{ji}$. By rescaling, if necessary, we may and will assume that $d_1 = 1$. For notational convenience we set $I = \{1, \ldots, s\}$.

From Propositions 2.5.3 and 2.5.5 of [13] there exists a unique (up to constant factor) non-degenerate supersymmetric invariant bilinear form $(\cdot, \cdot)$ on $g$. Moreover, the restriction of this form to the Cartan sub-superalgebra $h$ is non-degenerate. Let $h_i, i \in I$, be defined by $(a, h_i) = d_i^{-1} a_i(a)$ for all $a \in h$.

Yamane [20] showed that $g$ is given by generators and relations which depend on $(A, \tau)$. We will now recall this presentation.

The Lie superalgebra $g$ is generated by $h_i, e_i$, and $f_i$ for $i \in I$ (whose parities are all even except for $e_i$ and $f_i$ if $i \in \tau$ which are odd) such that the relations satisfy:

$$[h_i, h_j] = 0, \quad [h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j \quad [e_i, f_j] = \delta_{ij} h_i \quad (3.1)$$

and the super Serre relations

$$[e_i, e_i] = [f_i, f_i] = 0 \text{ for } i \in \tau$$
From the discussion above we see that plus nonstandard super Serre-type relations which depend on \((A, \tau)\). The results of this subsection are straightforward generalizations of the non-super results have previously been considered by other authors (see for example \([12, 17]\)).

3.2. Lie bi-superalgebra structure. In this subsection we will recall that Lie superalgebras of type \(A\)-\(G\) has a natural Lie bi-superalgebra structure. Similar results have previously been considered by other authors (see for example \([12, 17]\)). The results of this subsection are straightforward generalizations of the non-super case.

Let \((\mathfrak{g}, A, \tau)\) be a Lie superalgebra of type \(A\)-\(G\). Let \(\mathfrak{h} = \langle h_i \rangle_{i \in I}\) be the Cartan subalgebra of \(\mathfrak{g}\). Let \(\mathfrak{n}_+\) (resp., \(\mathfrak{n}_-\)) be the nilpotent Lie sub-superalgebra of \(\mathfrak{g}\) generated by \(e_i\)’s (resp., \(f_i\)’s). Let \(\mathfrak{b}_\pm := \mathfrak{n}_\pm \oplus \mathfrak{h}\) be the Borel Lie sub-superalgebra of \(\mathfrak{g}\).

Let \(\eta_\pm : \mathfrak{b}_\pm \to \mathfrak{g} \oplus \mathfrak{h}\) be defined by

\[
\eta_\pm(x) = x \oplus (\pm \bar{x}),
\]

where \(\bar{x}\) is the image of \(x\) in \(\mathfrak{h}\). Using this embedding we can regard \(\mathfrak{b}_+\) and \(\mathfrak{b}_-\) as Lie sub-superalgebras of \(\mathfrak{g} \oplus \mathfrak{h}\).

As above let \((\_\,)\) be the unique non-degenerate supersymmetric invariant bilinear form on \(\mathfrak{g}\). Let \((\_\,)_{\mathfrak{g} \oplus \mathfrak{h}} := (\_\,) - (\_\,)_{\mathfrak{h}}\), where \((\_\,)_{\mathfrak{h}}\) is the restriction of \((\_\,)\) to \(\mathfrak{h}\).

**Proposition 3.1.** \((\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{b}_+, \mathfrak{b}_-)\) is a super Manin triple with \((\_\,)_{\mathfrak{g} \oplus \mathfrak{h}}\).

*Proof.* Under the embedding \(\eta_\pm\) the Lie subsuperalgebra \(\mathfrak{b}_\pm\) is isotropic with respect to \((\_\,)_{\mathfrak{g} \oplus \mathfrak{h}}\). Since \((\_\,)\) and \((\_\,)_{\mathfrak{h}}\) both are invariant super-symmetric nondegenerate bilinear forms then so is \((\_\,)_{\mathfrak{g} \oplus \mathfrak{h}}\). Therefore the Proposition follows. \(\square\)

The Proposition implies that \(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{b}_+, \mathfrak{b}_-\) are Lie bi-superalgebras. Moreover, we have that \(\mathfrak{b}_+^* \cong \mathfrak{b}_-^{op}\) as Lie bi-superalgebras, where \(^{op}\) is the opposite cobracket.

We will now compute the formulas for the cobrackets of these Lie bi-superalgebras. The cobracket on \(\mathfrak{b}_+ \subset \mathfrak{g} \oplus \mathfrak{h}\) is induced by \(\mathfrak{b}_-\) under the pairing \((\_\,)_{\mathfrak{g} \oplus \mathfrak{h}} : \mathfrak{b}_+ \otimes \mathfrak{b}_- \to \mathbb{C}\). In other words, this pairing induces a Lie superalgebra structure on \(\mathfrak{b}_+^*\) and thus a Lie supercoalgebra structure on \(\mathfrak{b}_+^{**} \cong \mathfrak{b}_+.\) Using these facts we compute the cobracket on \(\mathfrak{b}_+\).

Let \(f_i\) and \(K_j\) be a basis of \(\mathfrak{b}_-\) so that

\[
(h_i \oplus h_i, K_j)_{\mathfrak{g} \oplus \mathfrak{h}} = \delta_{ij}
\]

where \(h_i \oplus h_i \in \mathfrak{h} \subset \mathfrak{b}_+\). Let \(\pi : \mathfrak{g} \oplus \mathfrak{h} \to \mathfrak{g}\) be the natural projection. Set \(k_j := \pi(K_j)\). Then

\[
(h_i \oplus h_i, K_j)_{\mathfrak{g} \oplus \mathfrak{h}} = 2(h_i, k_j) = 2d_i^{-1}\alpha_i(k_j),
\]

implying \(\alpha_i(k_j) = \delta_{ij}d_i/2\). By definition

\[
[h^*_j, e^*_i] := [k_j, f_i] = -\alpha_i(k_j)f_i = -\delta_{ij}d_i/2 e^*_i.
\]

From the discussion above we see that

\[
\delta(e_i) = d_i/2 (e_i \otimes h_i - h_i \otimes e_i) = d_i/2 e_i \wedge h_i.
\]
Similarly,

\[ \delta(f_i) = -\frac{d_i}{2} f_i \wedge h_i \quad \delta(a) = 0 \text{ for } a \in \mathfrak{b} \subset \mathfrak{b}_\pm \] (3.3)

These formulas define a Lie bi-superalgebra structure on the Lie superalgebras \( \mathfrak{g}, \mathfrak{b}_+ \) and \( \mathfrak{b}_- \). Moreover, we have that \( \mathfrak{g}, \mathfrak{r} \) is a quasitriangular Lie bi-superalgebra where \( \mathfrak{r} \) is the image of the canonical element \( \mathfrak{r} \) in the double \( D(\mathfrak{b}_+) \cong \mathfrak{g} \otimes \mathfrak{h} \) under the natural projection.

3.3. The Drinfeld-Jimbo type superalgebra \( U_{h}^{DJ}(\mathfrak{g}, A, \tau) \). Khoroshkin-Tolstoy [16] and Yamane [20] used the quantum double notion to define a quasitriangular QUE superalgebra \( U_{h}^{DJ}(\mathfrak{g}, A, \tau) \). In this subsection we recall their results which are needed in this paper.

Set \( q = e^{h/2} \) and \( q_i = q^{d_i} \).

**Theorem 3.2** ([16] [20]). Let \( (\mathfrak{g}, A, \tau) \) be a Lie superalgebra of type A-G. There exists an explicit quasitriangular QUE superalgebra \( U_{h}^{DJ}(\mathfrak{g}, A, \tau), R \). The superalgebra \( U_{h}^{DJ}(\mathfrak{g}, A, \tau) \) is defined as the \( \mathbb{C}[[h]] \)-superalgebra generated by \( \mathfrak{h} \) and the elements \( E_i \) and \( F_i \), \( i \in I \) (all generators are even except \( E_i \) and \( F_i \) for \( i \in \tau \) which are odd)

\[
[a, a'] = 0, \quad [a, E_i] = \alpha_i(a) E_i, \quad [a, F_i] = -\alpha_i(a) F_i, \quad \text{for } a, a' \in \mathfrak{h} \quad (3.4)
\]

\[
[E_i, F_j] = \delta_{i,j} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q_i - q_i^{-1}}, \quad (3.5)
\]

plus the super quantum Serre-type relations (see [20]). The coproduct and counit are given by

\[
\Delta(E_i) = E_i \otimes q^{d_i h_i} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + q^{-d_i h_i} \otimes F_i,
\]

\[
\Delta(a) = a \otimes 1 + 1 \otimes a, \quad \epsilon(a) = \epsilon(E_i) = \epsilon(F_i) = 0
\]

for all \( a \in \mathfrak{h} \). Moreover, the \( R \)-matrix is given by explicit formulas.

We call \( U_{h}^{DJ}(\mathfrak{g}, A, \tau) \) the Drinfeld-Jimbo type quantization of \( (\mathfrak{g}, A, \tau) \).

**Remark 3.3.** The super quantum Serre-type relations depend directly on \((A, \tau)\). For example,

\[
E_m E_{m-1} E_{m+1} + E_m E_{m+1} E_m E_{m-1} + E_{m-1} E_m E_{m+1} E_m + E_{m+1} E_m E_{m-1} E_m - (q + q^{-1}) E_m E_{m-1} E_{m+1} E_m = 0 \quad (3.6)
\]

if \( m - 1, m + 1 \in I, \ a_{nm} = 0 \) and the Cartan matrix \( A \) is not of type C or D.

**Remark 3.4.** In defining \( U_{h}^{DJ}(\mathfrak{g}, A, \tau) \), Yamane constructed a bilinear form on a free algebra whose kernel is the so called super quantum Serre-type relations. We will now give the properties of this form, which we will use later.

Let \( B_+ \) be the Hopf superalgebra generated over \( \mathbb{C}[[h]] \) by \( \mathfrak{h} \) and \( E_i, i \in I \), where \( E_i \) is odd if \( i \in \tau \) and all other generators are even, with relations satisfying

\[
[a, a'] = 0, \quad [a, E_i] = \alpha_i(a) E_i,
\]

and coproduct defined by

\[
\Delta(E_i) = E_i \otimes q^{d_i h_i} + 1 \otimes E_i \quad \Delta(a) = a \otimes 1 + 1 \otimes a
\]
for \( a, a' \in \mathfrak{h} \) and \( i \in I \). In [20] Yamane defined a \( \mathbb{C}(\mathfrak{h}) \)-valued bilinear form \( C \) on \( \mathcal{B}_+ \) with the following properties:

\[
C(xy, z) = C(x \otimes y, \Delta(z)), \quad C(x, yz) = C(\Delta(x), y \otimes z)
\]

(where \( C(x \otimes y, z \otimes w) := (-1)^{\delta_x \delta_z} C(x, z) C(y, w) \)) and

\[
C(E_i, E_j) = \begin{cases} (q_i - q_i^{-1})(q^{d_a} - q^{-d_a})^{-1} & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}
\]

where \( d_a = \frac{1}{2} \) if \( a_i = 0 \) and \( d_a = d_a a_i / 2 \) otherwise. Yamane showed that \( \text{Ker}(C) \) is generated by the super quantum Serre-type relations (see sections 4.2 and 10.4 of [20]). Moreover, the morphism \( \mathcal{B}_+ \to \mathcal{U}_h^{DJ}((\mathfrak{g}, \mathfrak{A}, \tau)) \) which is the identity on \( \mathfrak{h} \) and maps \( E_i \to E_i \) induces an isomorphism \( \mathcal{B}_+ / \text{Ker}(C) \to \mathcal{U}_h^{DJ}(\mathfrak{b}_+) \), where \( \mathcal{U}_h^{DJ}(\mathfrak{b}_+) \) is the sub-superalgebra of \( \mathcal{U}_h^{DJ}((\mathfrak{g}, \mathfrak{A}, \tau)) \) generated by \( \mathfrak{h} \) and \( E_i \), \( i \in I \).

### 4. The Etingof-Kazhdan quantization

In this section, we will show that for Lie superalgebras of type A-G, the Etingof-Kazhdan quantization is isomorphic to the Drinfeld-Jimbo quantization. As in [7] we will show that the E-K quantization is given by the desired generators and relations. In particular, we show that the E-K quantization of the Borel sub-superalgebra is isomorphic to a superalgebra given by generators and relations modulo the kernel of an appropriate bilinear form. It then follows from Yamane’s work that the E-K quantization is in fact the D-J quantization. Throughout this section we will use the notation of section 3.

Let \( \mathfrak{g}_0 \) be the Lie superalgebra generated by \( e_i, f_i \) and \( h_i \) for \( i \in I \) satisfying relation (3.1) where all generators are even except \( e_i \) and \( f_i \) for \( i \in \tau \) which are odd. Let \( \mathfrak{b}_+ \) and \( \mathfrak{b}_- \) be the Borel sub-superalgebras of \( \mathfrak{g}_0 \) generated by \( e_i, h_i \) and \( f_i, h_i \), respectively. The formulas (4.2) and (4.3) define Lie bi-superalgebra structures on \( \mathfrak{g}_0 \) and \( \mathfrak{b}_+ \).

Let \( U_h(\mathfrak{g}) \) be the Etingof-Kazhdan quantization of a Lie bi-superalgebra \( \mathfrak{g} \) defined in [9]. This quantization has two important properties that we will use here. The first is that the quantization is functorial. The second is that it commutes with taking the double, i.e.

\[
D(U_h(\mathfrak{g})) \cong U_h(D(\mathfrak{g}))
\]

where \( D(U_h(\mathfrak{g})) \) is the quantum double and \( D(\mathfrak{g}) \) is the double of \( \mathfrak{g} \) (see [2]). When \( \mathfrak{g} \) is a Lie superalgebra of type A-G its natural bi-superalgebra structure depends on the choice of Cartan matrix (see [4.2]). For this reason we denote the E-K quantization of such an Lie superalgebra by \( U_h(\mathfrak{g}, \mathfrak{A}, \tau) \).

#### 4.1. Generators and relations for \( U_h(\mathfrak{b}_+) \)

**Theorem 4.1.** The quantized universal enveloping (QUE) superalgebra \( U_h(\mathfrak{b}_+) \) is isomorphic to the QUE superalgebra \( \tilde{\mathcal{U}}_+ \) generated over \( \mathbb{C}[\hbar] \) by \( \mathfrak{h} \) and the elements \( E_i \) for \( i \in I \) (all generators are even except for \( E_i \), \( i \in \tau \) which are odd) satisfying the relations

\[
[a, a'] = 0, \quad \quad [a, E_i] = \alpha_i(a) E_i,
\]

with coproduct

\[
\Delta(a) = a \otimes 1 + 1 \otimes a, \quad \quad \Delta(E_i) = E_i \otimes q^{d_i h_i} + 1 \otimes E_i,
\]
for all \( a, a' \in \mathfrak{h} \) and \( i, j \in I \).

The theorem follows from the following two lemmas.

**Lemma 4.2.** The QUE superalgebra \( U_h(\tilde{b}_+) \) is isomorphic to the QUE superalgebra generated over \( \mathbb{C}[[h]] \) by \( \mathfrak{h} \) and the elements \( E_i, i \in I \) (all generators are even except for \( E_i, i \in \tau \) which are odd) satisfying the relations
\[
[a, a'] = 0 \tag{4.1}
\]
\[
[a, E_i] = \alpha_i(a) E_i, \tag{4.2}
\]
with coproduct
\[
\Delta(a) = 1 \otimes a + a \otimes 1, \tag{4.3}
\]
\[
\Delta(E_i) = E_i \otimes q^i + 1 \otimes E_i, \tag{4.4}
\]
for all \( a, a' \in \mathfrak{h} \) and \( i, j \in I \) and suitable elements \( \gamma_i \in \mathfrak{h}[[h]] \).

**Proof.** Since the E-K quantization is functorial, the embedding of Lie bi-superalgebras \( \mathfrak{h} \to \tilde{\mathfrak{b}}_+ \) induces an embedding of QUE superalgebras \( U_h(\mathfrak{h}) \to U_h(\tilde{\mathfrak{b}}_+) \). Note that this embedding of QUE superalgebras restricted to \( \mathfrak{h} \) is the identity. We will use this observation later.

By construction \( U_h(\mathfrak{h}) \) is equal to \( U(\mathfrak{h})[[h]] \). It follows that \( U_h(\tilde{\mathfrak{b}}_+) \) has a sub-superalgebra generated by \( \mathfrak{h} \) which satisfies relation \( (4.1) \) and whose coproduct is given by \( (4.3) \). Since \( \tilde{\mathfrak{b}}_+ = \mathfrak{b}_+ \oplus \tilde{n}_+ \) where \( \tilde{n}_+ \) is the free Lie bi-superalgebra generated by \( e_i, i \in I \) and \( U_h(\tilde{\mathfrak{b}}_+) \cong U(\mathfrak{b})[[h]] \) as a superalgebra, to complete the proof, it suffices to show that there exists \( E_i \) in \( U_h(\tilde{\mathfrak{b}}_+) \) which satisfies relations \((4.2)\) and \((4.4)\).

The Lie bi-superalgebra \( \tilde{\mathfrak{b}}_+ \) has a natural \( \mathbb{Z}_+^\tau \)-grading given by \( \deg_i(h_j) = 0 \) and \( \deg_i(e_j) = \delta_{ij} \). The functorality of the quantization implies \( U_h(\tilde{\mathfrak{b}}_+) \) has a \( \mathbb{Z}_+^\tau \)-grading. It follows that \( U_h(\tilde{\mathfrak{b}}_+) = \bigoplus_{m \in \mathbb{Z}_+^\tau} U_h(\tilde{\mathfrak{b}}_+)[[m]] \) where \( U_h(\tilde{\mathfrak{b}}_+)[[m]] \) is a free \( U_h(\mathfrak{b}) \)-module of finite rank. In particular, let \( 1_i \in \mathbb{Z}_+^\tau \) be given by \( 1_i(j) = \delta_{ij} \) then \( U_h(\tilde{\mathfrak{b}}_+)|1_i \) has rank 1. Let \( E_i' \) be an element in \( U_h(\tilde{\mathfrak{b}}_+)|1_i \) such that \( E_i' = e_i \mod h \).

The proof is completed by showing that there exist an element \( x \) in \( 1 + hU(\mathfrak{b})[[h]] \subset U_h(\mathfrak{b}) \) such that \( E_i = E_i'x \) satisfies \( (4.4) \). After replacing the ordinary tensor product with the super-tensor product, the construction of \( x \) follows as in the proof of Proposition 3.1 of [7]. There are no new signs introduced. For the most part, this is true because the arguments of the proof are based on the purely even Cartan subalgebra \( \mathfrak{h} \) and the functorality of the quantization. \qed

**Lemma 4.3.** \( \gamma_i = d_i h_i \)

**Proof.** By definition we have the natural projection \( \tilde{\mathfrak{b}}_+ \to \mathfrak{b}_+ \). Then the functorality of the quantization implies that there is an epimorphism of Hopf superalgebras \( U_h(\tilde{\mathfrak{b}}_+) \to U_h(\mathfrak{b}_+) \). Therefore, \( U_h(\mathfrak{b}_+) \) is generated by \( \mathfrak{h} \) and \( E_i \) satisfying the relations \( (4.1), (4.4) \) (and possibly other relations). So it suffices to show that \( \gamma_i = d_i h_i \) in \( U_h(\mathfrak{b}_+) \).

Next we show that \( U_h(\mathfrak{b}_+) \cong U_{-\mathfrak{g}}(\mathfrak{b}_+)^{op} \) where \( ^{op} \) denotes the quantum dual with the opposite coproduct (see [3]). From the definition of \( \mathfrak{g} \) the Lie bi-superalgebra \( \mathfrak{b}_+ \) is self dual, i.e. \( \mathfrak{b}_+ \cong \mathfrak{b}_+^* \). Again from functoriality we have that \( U_h(\mathfrak{b}_+) \cong \)
$U_h(b^+_+)$. From Proposition 3.1 we have $b^+_+ \cong b^{op}$. Then equation (45) of [9] imply that $U_h(b^+_+)^{op} \cong U_h(b^{op})$. Substituting $b^{op}$ for $b_+$ we have $U_h(b^{op})^{op} \cong U_h(b^+_+)$. Finally from relation (7) of [9] it follows that $U_h(b^+_+) \cong U_h(b^+_+)^{op}$ which implies that $U_h(b^+_+)^{op} \cong U_h(b^+_+)$. Thus, we have shown that $U_h(b^+_+) \cong U_h(b^+_+)^{op}$.

This isomorphism gives rise to the bilinear form $B : U_h(b^+_+) \otimes U_h(b^+_+) \rightarrow \mathbb{C}(h)$ which satisfies the following conditions

$$B(xy, z) = B(x \otimes y, \Delta(z)), \quad B(x, yz) = B(\Delta(x), y \otimes z) \quad (4.5)$$

Let $a \in h$ and $i \in I$. Set $B_i = B(E_i, E_i)$, which is nonzero. Using (4.5) we have

$$B(E_i, q^a E_i) = B(E_i \otimes q^{\gamma_i} + 1 \otimes E_i, q^a \otimes E_i) = B(E_i, q^a)B(q^{\gamma_i}, E_i) + B(1, q^a)B(E_i, E_i) = B_i$$

since $B(E_i, q^a) = 0$. Similarly, we have $B(E_i, q^a E_i q^{-a}) = B(E_i, q^a E_i)B(q^{\gamma_i}, q^{-a})$ implying

$$B_i q^{(a, \gamma_i)} = B(E_i, q^a E_i q^{-a}). \quad (4.6)$$

To complete the proof we need the following relation:

$$q^a E_i q^{-a} = q^{\alpha_i(a)} E_i \quad (4.7)$$

This relation is equivalent to $q^{b_i} E_i q^{-h_i} = q^{\alpha_i(h_i)} E_i$ which follows from expanding $q = e^A$ and using the relation $[a, E_i] = \alpha_i(a) E_i$. From (4.6) and (4.7) we have

$$B_i q^{(a, \gamma_i)} = B(E_i, q^a E_i q^{-a}) = B(E_i, q^{\alpha_i(a)} E_i) = B_i q^{\alpha_i(a)}.$$

Thus, $(a_i, \gamma_i) = \alpha_i(a_i)$, but $\alpha_i(a) = d_i(a_i, h_i)$, and so $\gamma_i = d_i h_i$, which completes the proof. \hfill \square

4.2. The quantized universal enveloping superalgebra $U_h(b^+_+)$. In this subsection we show that there exist a bilinear form on $U_h(b^+_+)$ such that $U_h(b^+_+)$ modulo the kernel of the form is isomorphic to $U_h(b^+_+)$. 

**Theorem 4.4.** There exists a unique bilinear form on $U_h(b^+_+)$ which takes values in $\mathbb{C}(h)$ with the following properties

$$B(xy, z) = B(x \otimes y, \Delta(z)), \quad B(x, yz) = B(\Delta(x), y \otimes z)$$

$$B(q^a, q^b) = q^{-(a, b)}, \quad a, b \in h,$$

$$B(E_i, E_j) = \begin{cases} (q_i - q_i^{-1})(q^{d_{a_i}} - q^{-d_{a_i}})^{-1} & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$$

where $d_{a_i} = 1$ if $a_i = 0$ and $d_{a_i} = d_i a_i / 2$ otherwise. Moreover $U_h(b^+_+) \cong U^+_+ := \tilde{U}_+ / \text{Ker}(B)$ as QUE superalgebras.

**Proof.** The existence and uniqueness follows from the fact that the superalgebra generated by the $E_i$ is free.

We will show that there is a nondegenerate bilinear form on $U_h(b^+_+)$ with the same properties as $B$. From the proof of Lemma 4.3 we have that $U_h(b^+_+) \cong U_h(b^+_+)^{op}$. But the even homomorphism $U_h(b^+_+)^{op} \rightarrow U_h(b^+_+)$ given by conjugation by $q^{-\sum x_i^2/2}$, where $x_i$ is an orthonormal basis for $h$, is an isomorphism. Therefore
we have a even isomorphism $U_h(b_+) \cong U_h(b_+)^\ast$. This isomorphism gives rise to the desired form on $U_h(b_+)$. 

So the form $B$ is the pull back of the form on $U_h(b_+)$. Implying that the kernel of the form on $U_h(b_+)$ is contained in the image of the kernel of $B$ under natural projection.

But the kernel of the form on $U_h(b_+)$ is zero since the form is nondegenerate. Thus we have $U_h(b_+) \cong \tilde{U}_+/\ker(B)$. □

**Corollary 4.5.** Let $U^D_J(b_+)$ be the sub-superalgebra of $U^D_J(g, A, \tau)$ generated by $\mathfrak{g}$ and the elements $E_i$, $i \in I$. Then the map $$g : U^D_J(b_+) \to U_h(b_+) \text{ given by } g|_\mathfrak{h} = id \text{ and } E_i \mapsto E_i$$ is an isomorphism of QUE superalgebras.

**Proof.** The corollary follows directly from Theorem 4.4 and Remark 3.4. □

### 4.3. Proof of Theorem 1.1

**Proof of Theorem 1.1** It suffices to show that the QUE superalgebra $U_h(g, A, \tau)$ is isomorphic to the quotient of the double $D(\tilde{U}_+)$ by the ideal generated by the identification of $\mathfrak{h} \subset U_+$ and $\mathfrak{h}^* \subset U_+^\ast$.

Recall from §3.2 that the Lie bi-superalgebra structure of $\mathfrak{g}$ comes from identifying $\mathfrak{h}$ and $\mathfrak{h}^*$ in $\mathfrak{g} \oplus \mathfrak{h} = b_+ \oplus b_+^\ast$. Also since the quantization commutes with the double we have $$U_h(D(b_+)) \cong D(U_h(b_+)) = U_h(b_+) \otimes U_h(b_+)^\ast.$$ 

Therefore, we have $U_h(g, A, \tau)$ is isomorphic to $D(U_h(b_+)) = U_h(b_+) \otimes U_h(b_+)^\ast$ modulo the ideal generated by the identification of $\mathfrak{h} \subset U_h(b_+)$ and $\mathfrak{h}^* \subset U_h(b_+)^\ast$. But from Theorem 4.4 we have that $D(U_h(b_+)) \cong D(\tilde{U}_+)$ and then Corollary 4.5 implies result. □

### 4.4. Twisting of Drinfeld-Jimbo superalgebras.

In this subsection we give a corollary of Theorem 1.1.

Khoroshkin and Tolstoy [15] showed that any two isomorphic Lie superalgebras with different Cartan matrices have isomorphic deformations (as associative superalgebras) and their coproducts are connected by a twisting of a factor of the universal $R$-matrix. It is not clear if the definition of the quantized superalgebra associated to a Lie superalgebra of type A-G is correct in [15] (as some relations appear to be missing). However, after adding the missing relations it becomes apparent that the results of [15] hold.

In any case, we give an alternative proof that any two isomorphic Lie superalgebras with different Cartan matrices have isomorphic Drinfeld-Jimbo type superalgebras (as associative superalgebras) and their coproducts are connected by a twist. The primary difference in our approach, as opposed to [15], is to construct the twist using the E-K quantization rather than as a factor of the universal $R$-matrix.

**Corollary 4.6.** Let $g$ and $g'$ be two isomorphic Lie superalgebras of type A-G with associated Cartan matrices $(A, \tau)$ and $(A', \tau')$. Let $J$ and $J'$ be the element of $U(g)[[h]]^{\otimes 2}$ defined in (32) of [12] using $(g, A, \tau)$ and $(g', A', \tau')$, respectively. Then the QUE superalgebra $U^D_J(g, A, \tau)$ is isomorphic to $U^D_{J'}(g', A', \tau')$ twisted by the element $(J')^{-1}J$. In particular, $U^D_J(g, A, \tau)$ and $U^D_{J'}(g', A', \tau')$ are isomorphic as associative superalgebras.
**Proof.** Let $\Omega$ and $\Omega'$ be the Casimir elements of $(\mathfrak{g}, A, \tau)$ and $(\mathfrak{g}', A', \tau')$, respectively. Since the Casimir element is independent of the choice of basis we have $A_g,\Omega$ and $A_{g'},\Omega'$ are isomorphic quasitriangular quasi-Hopf superalgebras. Recall that by construction $U_h(\mathfrak{g}', A', \tau') = (A_{g',\Omega'})_{\mathfrak{g}'}$, and so 

$$(U_h(\mathfrak{g}', A', \tau'))_{\mathfrak{g}'} = A_{g',\Omega'}.$$  

Similarly, $U_h(g, A, \tau) = (A_{g,\Omega})_{g}$ implying

$$U_h(g, A, \tau) \cong ((U_h(g, A', \tau'))_{\mathfrak{g}'}^{-1})_{g}$$

as quasitriangular quasi-Hopf superalgebras. But $U_h(g, A, \tau)$ is a quasitriangular Hopf superalgebra and so the result follows from Equation (4.8) and Theorem 1.1. 

As mentioned above the relations of the D-J type superalgebra depend on the choice of the Cartan matrix. For this reason it is not apparent from the definition that $U_h^{D_{\mathcal{J}}} (g, A, \tau)$ and $U_h^{D_{\mathcal{J}}}(g', A', \tau')$ are isomorphic as associative superalgebras.

5. A THEOREM OF DRINFELD’S

Let $(g, A, \tau)$ be a Lie superalgebra of type A-G. Recall from section 2 that for each super Manin triple there exists a Casimir element. Let $\Omega$ be this element associated to the triple $(\mathfrak{g}, \mathfrak{b}_+, \mathfrak{b}_-)$. For each Lie algebra and symmetric invariant 2-tensor Drinfeld [3] defined a quasitriangular quasi-Hopf quantized universal enveloping algebra:

$$\{U(\mathfrak{g})[[h]], \Delta_0, \epsilon_0, R_{KZ} = e^{th/2}, \Phi_{KZ}\}.$$ 

The morphisms $\Delta_0$ and $\epsilon_0$ are the standard coproduct and counit of $U(\mathfrak{g})[[h]]$. The element $\Phi_{KZ}$ is the KZ-associator. Setting $t = \Omega$ let $A_{g,t}$ be the analogous topologically free quasitriangular quasi-Hopf superalgebra (for more details see [9, 21]). Also recall the definition $U_h^{D_{\mathcal{J}}}(g, A, \tau)$ given in [8].

Here we show that the categories of topologically free modules over $A_{g,t}$ and $U_h^{D_{\mathcal{J}}}(g, A, \tau)$ are braided tensor equivalent. We do this in two steps: (1) we show that $U_h(g, A, \tau)$ and $A_{g,t}$ have equivalent module categories, (2) we use the fact the $U_h^{D_{\mathcal{J}}}(g, A, \tau)$ and $U_h(g, A, \tau)$ are isomorphic to prove the desired result. For more on braided tensor categories see [14].

5.1. **The E-K quantization** $U_h(g, A, \tau)$ and $A_{g,t}$. In this subsection we show that $U_h(g, A, \tau)$-Mod and $A_{g,t}$-Mod are equivalent braided tensor categories. To this end we recall the following definitions.

Let $(A, \Delta, \epsilon, \Phi, R)$ be a quasitriangular quasi-bi-superalgebra (see §4.1 of [9]). An invertible element $J \in A \otimes A$ is a gauge transformation on $A$ if

$$(\epsilon \otimes id)(J) = (id \otimes \epsilon)(J) = 1.$$ 

Using a gauge transformation $J$ on $A$, one can construct a new quasitriangular quasi-bi-superalgebra $A_J$ with coproduct $\Delta_J$, R-matrix $R_J$ and associator $\Phi_J$ defined by

$$\Delta_J = J^{-1} \Delta J, \ R_J = (J^{op})^{-1} R J,$$

$$\Phi_J = J_{23}^{-1}(id \otimes \Delta)(J^{-1}) \Phi(\Delta \otimes id)(J) J_{12}.$$ 

As is the case of quasitriangular (quasi-)bialgebra, the category of modules over a quasitriangular (quasi-)bi-superalgebra is a braided tensor category. Let $X$ be a
topological (quasi-)bi-superalgebra and let $X-Mod$ be the category of topologically free $X$-modules.

**Theorem 5.1.** Let $A$ and $A'$ be a quasitriangular quasi-bi-superalgebra. Suppose that $J$ is a gauge transformation on $A'$ and $\alpha : A \to A'$ is an isomorphism of quasitriangular quasi-bi-superalgebras. Then $\alpha$ induces an equivalence between the braided tensor categories $A'-Mod$ and $A-Mod$.

**Proof.** Let $\alpha^*: A'-Mod \to A-Mod$ be the functor defined as follows. On objects, the functor $\alpha^*$ is defined by sending the module $W$ to the same underlying vector space with the action given via the isomorphism $\alpha$. For any morphism $f: W \to X$ in $A'-Mod$ let $\alpha^*(f)$ be the image of $f$ under the isomorphism $\text{Hom}_{A'}(W,X) \cong \text{Hom}_A(W,X)$.

A standard categorical argument shows that this functor is an equivalence of braided tensor categories (see §XV.3 of [14]). □

Let $J$ be the element of $U(\mathfrak{g})[[h]]\otimes \mathbb{C}$ defined in (32) of [9]. The definition of the element $J$ uses the associator $\Phi$. By construction the E-K quantization is the twist of $A_{\mathfrak{g},t}$ by the element $J$, i.e. $U_h(\mathfrak{g},A,\tau) = (A_{\mathfrak{g},t})_J$. For exact formulas of the coproduct and R-matrix of $U_h(\mathfrak{g},A,\tau)$ see Proposition 16 and the end of section 5.2 of [9].

### 5.2. A braided tensor equivalence.

The following theorem was first due to Drinfeld [5] in the case of semi-simple Lie algebras.

**Theorem 5.2.** The braided tensor categories $A_{\mathfrak{g},t}-Mod$ and $U^D_J(h,\mathfrak{g},A,\tau)-Mod$ are equivalent.

**Proof.** As mentioned at the end of the last subsection $U_h(\mathfrak{g},A,\tau) = (A_{\mathfrak{g},t})_J$. Combining this fact with Theorem 1.1 we have that there exists an isomorphism of quasitriangular quasi-bi-superalgebra

$$\alpha : U^D_J(h,\mathfrak{g},A,\tau) \to (A_{\mathfrak{g},t})_J.$$

Now as a consequence of Theorem 5.1 we have that $A_{\mathfrak{g},t}-Mod$ and $U^D_J(h,\mathfrak{g})-Mod$ are braided tensor equivalent. □

**Remark 5.3.** Drinfeld’s proof of Theorem 5.2 in the case of semi-simple Lie algebras uses deformation theoretic arguments to show the existence of $\alpha$. Our proof constructs the isomorphism $\alpha$ explicitly.

### 6. Proofs

We will now give the proofs of Theorems 1.2 and 1.3.

6.1. **Proof of Theorem 1.2.** In this subsection we give the definitions of highest weight modules and a proof of Theorem 1.2.

Let $(\mathfrak{g},A,\tau)$ be a Lie superalgebra of type A-$G$ and let $\mathfrak{h}$ and $\mathfrak{n}_+$ be its Cartan and nilpotent sub-superalgebras, respectively. Let $\Lambda$ be an element of $\mathfrak{h}^*$. Let $c_\Lambda$ be the one dimensional $\mathfrak{b}_+$-module generated by $v_\Lambda$ with the following action:

$$n_+ v_\Lambda = 0, \quad a v_\Lambda = \Lambda(a) v_\Lambda \text{ for } a \in \mathfrak{h}$$
and where we set $\bar{v}_\Lambda = 0$. Set $\hat{V}(\Lambda) = \text{Ind}^{\mathcal{E}}_{\mathcal{R}^+}c_{\Lambda}$. Then $\hat{V}(\Lambda)$ contains a unique maximal submodule $I(\Lambda)$. We call $V(\Lambda) := \hat{V}(\Lambda)/I(\Lambda)$ the irreducible weight module with highest weight $\Lambda$.

Next we define a similar notion for $U_h^{DJ}(\mathfrak{g}, A, \tau)$-module. A topologically free $U_h^{DJ}(\mathfrak{g}, A, \tau)$-module $V$ is call a highest weight module with highest weight $\Lambda \in \mathfrak{h}^*$ if there exists a non-zero even generating vector $v_\Lambda$ such that

$$U_h^{DJ}(n_+)v_\Lambda = 0, \quad av_\Lambda = \Lambda(a)v_\Lambda \text{ for } a \in \mathfrak{h}$$

where $U_h^{DJ}(n_+)$ is the sub-superalgebra of $U_h^{DJ}(\mathfrak{g}, A, \tau)$ generated by $E_i$, $i \in I$.

**Proof of Theorem 1.2.** Consider the $U(\mathfrak{g})[[\hbar]]$-module $V(\Lambda)[[\hbar]]$. From Theorem 5.2 we know that $\alpha$ induces a $U_h^{DJ}(\mathfrak{g}, A, \tau)$-module structure on $V(\Lambda)[[\hbar]]$, denote this $U_h^{DJ}(\mathfrak{g}, A, \tau)$-module by $\hat{V}(\Lambda)$. In addition, from Corollary 3.5 we have $\alpha|_\hbar = \text{Id}$ and $\alpha$ restricted to $U(\mathfrak{h})[[\hbar]]$ is an isomorphism between $U(\mathfrak{h})[[\hbar]]$ and $U_h^{DJ}(\mathfrak{h})$. Thus, $\hat{V}(\Lambda)$ is a highest weight $U_h^{DJ}(\mathfrak{g}, A, \tau)$-module such that $V(\Lambda)/\hbar V(\Lambda) \cong V(\Lambda)$.

**6.2. Proof of Theorem 1.3.** Here we give the proof of the Drinfeld-Kohno Theorem for Lie superalgebras. Before proving the theorem we will define the $KZ$ monodromy representation of the braid group.

Let $(\mathfrak{g}, A, \tau)$ be a Lie superalgebra of type A-G. Let $V$ be a irreducible highest weight module of $\mathfrak{g}$. Consider the Knizhnik-Zamolodchikov system of differential equations with respect to a function $\omega(z_1, ..., z_n)$ of complex variables $z_1, ..., z_n$, with values in $V^{\otimes n}[[\hbar]]$:

$$\frac{\partial \omega}{\partial z_i} = \frac{\hbar}{2\pi i} \sum_{i \neq j} \Omega_{ij} \omega(z_1, ..., z_n) = \frac{\hbar}{2\pi i} \sum_{i \neq j} \Omega_{ij} \frac{\partial \omega}{\partial z_j}.$$  \hspace{1cm} (6.1)

We have that this system of equations defines a flat connection on the trivial bundle $Y_n \times V^{\otimes n}[[\hbar]]$ where $Y_n = \{(z_1, ..., z_n) | i \neq j \text{ implies } z_i \neq z_j \} \subset \mathbb{C}^n$. This connection determines a monodromy representation from $\pi_1(Y_n)$ to $\text{Aut}_{\mathbb{C}[[\hbar]]}(V^{\otimes n}[[\hbar]])$. Moreover, since the system of equations (6.1) is invariant under the action of the symmetric group we obtain a monodromy representation

$$\rho^KZ_n : \pi_1(X_n, p) \rightarrow \text{Aut}_{\mathbb{C}[[\hbar]]}(V^{\otimes n}[[\hbar]])$$

where $X_n = Y_n/S_n$ and $p = (1, 2, ..., n) \in \mathbb{C}^n$. Finally, we identify $\pi_1(X_n, p)$ with the braid group $B_n$ to get a monodromy representation of $B_n$.

**Proof of Theorem 1.3.** Let $\rho^R_{KZ}$ be the representation of $B_n$ on $V(\Lambda)^{\otimes n}[[\hbar]]$ induced by the $R$-matrix $R_{KZ} = e^{\hbar^2/2}$. From Theorem 5.2 and Theorem 1.2 we have that $\rho^R_{KZ}$ and $\rho^KZ$ correspond to each other under the braided tensor functor $\alpha^*$. The theorem follows since $\rho^KZ_n$ coincides with $\rho^R_{KZ}$.

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