On Matrix Realizations of the Contact Superconformal Algebra $\hat{K}'(4)$ and the Exceptional $N = 6$ Superconformal Algebra

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Abstract. The superalgebra $K'(4)$ and the exceptional $N = 6$ superconformal algebra have “small” irreducible representations in the superspaces $V^\mu = t^\mu C[t, t^{-1}] \otimes \Lambda(N)$, where $N = 2$ and 3, respectively. For $\mu \in \mathbb{C}\setminus\mathbb{Z}$ they are associated to the embeddings of these superalgebras into the Lie superalgebras of pseudodifferential symbols on the supercircle $S^{1|N}$. In this work we describe $K'(4)$ and the exceptional $N = 6$ superconformal algebra in terms of matrices over a Weyl algebra. Correspondingly, we obtain realizations of their representations in $V^\mu$ for $\mu = 0$.

Keywords. Superconformal algebra, pseudodifferential symbols, Poisson superalgebra, Weyl algebra.

AMS (MOS) subject classification: 17B68, 17B65, 81R10

1 Introduction

This work is a continuation of [20, 21].

Recall that a superconformal algebra is a simple complex Lie superalgebra spanned by the coefficients of a finite family of pairwise local fields

$$a(z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1},$$

one of which is the Virasoro field $L(z)$ [3, 9–11]. Superconformal algebras play an important role in the string theory and conformal field theory. They can also be described in terms of derivations of the associative superalgebra $\mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$, where $\Lambda(N)$ is the Grassmann algebra in $N$ variables. The Lie superalgebra $K(N)$ of contact vector fields with Laurent polynomials as coefficients is spanned by $2^N$ fields [3, 6, 7, 10]. It is also known as the $SO(N)$ superconformal algebra [1]. $K(N)$ is simple if $N \neq 4$, if $N = 4$, then the derived Lie superalgebra $K'(4)$ is simple. The nontrivial central extensions of $K(1)$, $K(2)$, and $K'(4)$ are well-known: they are isomorphic to the Neveu-Schwarz superalgebra, the “$N = 2$” superconformal algebra, and the “big $N = 4$” superconformal algebra [1]. $K(6)$ contains the exceptional $N = 6$ superconformal algebra, also denoted by $CK_6$, as a subsuperalgebra. Note that $CK_6$ is “one half” of $K(6)$: it is spanned by 32 fields [3, 4, 6, 12, 15, 22–24].

In [16, 17] Martinez and Zelmanov obtained $CK_6$ as a particular case of their construction of superalgebras $CK(R, d)$, where $R$ is an associative commutative superalgebra with an even derivation $d$.

Our approach is based on the realization of $K(2N)$ in terms of pseudodifferential symbols on the circle extended by $N$ odd variables. It is well-known that a Lie algebra of contact vector fields can be realized as a subalgebra of Poisson algebra $\mathfrak{g}$ [2]. Analogously, $K(2N)$ can be embedded into the Poisson superalgebra $P(2N)$ of pseudodifferential symbols on the supercircle $S^{1|N}$ [20, 21]. There exists a family $P_\mu(2N)$ of Lie superalgebras of pseudodifferential symbols on $S^{1|N}$, which contracts to $P(2N)$. There is no embedding of $K(2N)$ into $P_\mu(2N)$ if $N \geq 3$. It is remarkable that a nontrivial central extension $K'(4)$ of $K'(4)$ and $CK_6$ can be embedded into $P_\mu(2N)$, where $N = 2$ and 3, respectively [20, 21].

Associated to these embeddings, there are “small” irreducible representations of $K'(4)$ and $CK_6$ in the superspaces $V^\mu = t^\mu C[t, t^{-1}] \otimes \Lambda(N)$, where $(\partial/\partial t)^{-1}$ acts as an antiderivative. This requires that $\mu \in \mathbb{C}\setminus\mathbb{Z}$. Nevertheless, the representations of $K'(4)$ and $CK_6$ in $V^\mu$ can be defined if $\mu = 0$. In this work we describe these superalgebras in terms of matrices over the Weyl algebra $W = \sum_{i \geq 0} Ad^i$, where $A = \mathbb{C}[t, t^{-1}]$ and $d = t\partial/\partial t$ (Theorems 1 and 2). This gives realizations of the representations in $V^\mu$ for $\mu = 0$.

2 Contact and Poisson superalgebras

A superconformal algebra is a complex Lie superalgebra $\mathfrak{g}$ such that

1. $\mathfrak{g}$ is simple,
2. $\mathfrak{g}$ contains the centerless Virasoro algebra $\mathfrak{vir} = \oplus_{n \in \mathbb{Z}} C \mathfrak{L}_n$ with the commutation relations
$$[L_n, L_m] = (m - n)L_{n+m}$$

as a subalgebra,
3. $adL_0$ is diagonalizable with finite-dimensional eigenspaces:
$$\mathfrak{g} = \oplus_{i} \mathfrak{g}_i, \quad \mathfrak{g}_i = \{x \in \mathfrak{g} \mid [L_0, x] = ix\},$$

so that $\dim \mathfrak{g}_i < C$, where $C$ is a constant independent of $i$ [7].
Let $\Lambda(2N)$ be the Grassmann algebra in $2N$ variables $\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N$, and let

\[
\Lambda(1,2N) = \mathbb{C}[t,t^{-1}] \otimes \Lambda(2N)
\]

be the associative superalgebra with natural multiplication and with the following parity of generators:

\[
p(t) = 0, \quad p(\xi_i) = p(\eta_i) = 1 \quad \text{for} \quad i = 1, \ldots, N.
\]

Let $W(2N)$ be the Lie superalgebra of all derivations of $\Lambda(1,2N)$. By definition,

\[
K(2N) = \{ D \in W(2N) | D\Omega = f\Omega \text{ for some } f \in \Lambda(1,2N) \}
\]

where $\Omega = dt + \sum_{i=1}^{N} (\xi_i d\tau_i + \eta_i d\tau_i)$ is a differential contact 1-form [3–7, 10, 22–24]. There is a one-to-one correspondence between the differential operators $D \in K(2N)$ and the functions $f \in \Lambda(1,2N)$. Let $\partial_1, \partial_2$, and $\partial_3$ stand for $\partial/\partial \tau, \partial/\partial \xi$, and $\partial/\partial \eta$, respectively. The correspondence $f \leftrightarrow D_f$ is given by

\[
D_f = \Delta(f)\partial_t + (\partial_t f)E - H_f,
\]

where

\[
E = \sum_{i=1}^{N} (\xi_i \partial_{\xi_i} + \eta_i \partial_{\eta_i}), \quad \Delta = 2 - E,
\]

\[
H_f = (-1)^{p(f)+1} \sum_{i=1}^{N} (\partial_{\xi_i} f \partial_{\eta_i} + \partial_{\eta_i} f \partial_{\xi_i}).
\]

The Poisson algebra $P$ of pseudodifferential symbols on the circle is formed by the formal series

\[
A(t, \tau) = \sum_{n=-\infty}^{\infty} a_n(t) \tau^n,
\]

where $a_n(t) \in \mathbb{C}[t,t^{-1}]$, and the even variable $\tau$ corresponds to $\partial_\tau$. The Poisson bracket is defined as follows:

\[
\{ A(t, \tau), B(t, \tau) \} = \partial_\tau A(t, \tau) \partial_\tau B(t, \tau) - \partial_\tau A(t, \tau) \partial_\tau B(t, \tau).
\]

An associative algebra $P_h$, where $h \in (0,1]$ is a deformation of $P$. The multiplication in $P_h$ is given as follows:

\[
A(t, \tau) \circ_h B(t, \tau) = \sum_{n=0}^{\infty} \frac{h^n}{n!} \partial_\tau^n A(t, \tau) \partial_\tau^n B(t, \tau).
\]

The Lie algebra structure on the vector space $P_h$ is given by

\[
[A, B]_h = A \circ_h B - B \circ_h A,
\]

so that

\[
\lim_{h \to 0} \frac{1}{h} [A, B]_h = \{ A, B \},
\]

see [13, 14, 18, 19]. The Poisson superalgebra of pseudodifferential symbols on $S^{1|N}$ is $P(2N) = P \otimes \Lambda(2N)$. The Poisson bracket is defined as follows:

\[
\{ A, B \} = \partial_\tau A \partial_\tau B - \partial_\tau A \partial_\tau B + (-1)^{p(A)+1} \sum_{i=1}^{N} (\partial_{\xi_i} A \partial_{\eta_i} B + \partial_{\eta_i} A \partial_{\xi_i} B).
\]

Let $\Lambda_h(2N)$ be an associative superalgebra with generators $\xi_1, \ldots, \xi_N, \eta_1, \ldots, \eta_N$ and relations

\[
\xi_i \xi_j = -\xi_j \xi_i, \quad \eta_i \eta_j = -\eta_j \eta_i, \quad \eta_i \xi_j = h \delta_{i,j} - \xi_j \eta_i.
\]

Let $P_h(2N) = P_h \otimes \Lambda_h(2N)$ be a superalgebra with the product given by

\[
(A_1 \otimes X)(B_1 \otimes Y) = (A_1 \circ_h B_1) \otimes (XY),
\]

where $A_1, B_1 \in P_h$ and $X, Y \in \Lambda_h(2N)$. The Lie bracket of $A = A_1 \otimes X$ and $B = B_1 \otimes Y$ is

\[
[A, B]_h = AB - (-1)^{p(A)p(B)} BA,
\]

and (3) is satisfied. $P_h(2N)$ is the Lie superalgebra of pseudodifferential symbols on $S^{1|N}$. There exist embeddings of $K'(4)$ and $CK_h$ into $P_h(2N)$, where $N = 2$ and $N = 3$, respectively [20, 21].

3 Case $\hat{K}'(4)$

The derived superalgebra

\[
K'(4) = [K(4), K(4)]
\]

is a simple ideal in $K(4)$ of codimension one, defined from the exact sequence

\[
0 \to K'(4) \to K(4) \to \mathbb{C}D_{-1} - \xi_1 \xi_2 \eta_1 \eta_2 \to 0.
\]

The superalgebra $K'(4)$ is spanned inside $P(4)$ by the 12 fields:

\[
L_n = t^{n+1} \tau, \quad X_n^j = t^{n+1} \tau \xi_j, \quad Y_n^j = t^{n+1} \tau \xi_j \xi_1, \quad (4)
\]

\[
L^j_n = t^n \eta_i, \quad X^j_n = t^n \xi_j \eta_i, \quad Y^j_n = t^n \xi_j \xi_1 \eta_i, \quad (5)
\]

where $i, j = 1, 2$, and 4 fields

\[
F^0_n = t^{n-1} \tau^{-1} \eta_1 \eta_2, \quad F^i_n = t^{n-1} \tau^{-1} \xi_1 \eta_i \eta_2, \quad i = 1, 2, \quad F^3_n = t^{n-1} \tau^{-1} \xi_1 \xi_2 \eta_2, \quad n \neq 0.
\]

Note that $L_n$ is a Virasoro field [20, 21]. Let $\hat{K}'(4)$ be one of three independent central extensions of $K'(4)$, such that the corresponding 2-cocycle is

\[
c(L_n, F^3_k) = \delta_{n+k,0}, \quad n \neq 0,
\]

\[
c(X^i_n, F^j_k) = (-1)^{1} \delta_{n+k,0}, \quad 1 \leq i \neq j \leq 2,
\]

\[
c(Y^i_n, F^3_k) = \delta_{n+k,0}.
\]

The superalgebra $\hat{K}'(4) \subset P_h(4)$ is spanned by the 12 fields (4)–(5) and 4 fields:

\[
F^0_{n,h} = \tau^{-1} \circ_h t^{n-1} \eta_1 \eta_2, \quad (6)
\]

\[
F^i_{n,h} = \tau^{-1} \circ_h t^{n-1} \eta_1 \eta_2 \xi_i, \quad i = 1, 2, \quad (7)
\]

\[
F^3_{n,h} = \tau^{-1} \circ_h t^{n-1} \eta_1 \eta_2 \xi_1 \xi_2 + \frac{h}{n} t^n, \quad n \neq 0, \quad (8)
\]
and the central element $h \in P_h(4)$, so that
\[
\lim_{h \to 0} \hat{K}'(4) = K'(4) \subset P(4).
\]

Let $V^\mu = t^\mu C[t, t^{-1}] \otimes \Lambda(\xi_1, \xi_2)$, where $\mu \in \mathbb{C} \setminus \mathbb{Z}$. We fix $h = 1$, and define a representation of $\hat{K}'(4)$ in $V^\mu$ accordingly to the formulas (4)–(8). Namely, $\xi_i$ is the operator of multiplication in $\Lambda(\xi_1, \xi_2)$, $\eta_i$ is identified with $\delta_i^j \tau^{-1}$ if $\tau^{-1}$ is identified with an antiderivative, and the central element 1 acts by the identity operator. Consider the following basis in $V^\mu$:
\[
\begin{align*}
v_{m0}^0(\mu) &= \frac{1}{m + \mu} t^{m+\mu}, & v_{m1}^1(\mu) &= t^{m+\mu} \xi_1, \\
v_{m2}^2(\mu) &= t^{m+\mu} \xi_2, & v_{m3}^3(\mu) &= t^{m+\mu} \xi_1 \xi_2, & m \in \mathbb{Z}.
\end{align*}
\]

Explicitly, the action of $\hat{K}'(4)$ on $V^\mu$ is given as follows:
\[
\begin{align*}
L_n(v_{m0}^0(\mu)) &= (n + m + \mu)v_{m+n}^0(\mu), \\
L_n(v_{m1}^1(\mu)) &= (m + \mu)v_{m+n}^1(\mu), & i \neq 1, 2, 3, \\
X_n^0(v_{m0}^0(\mu)) &= v_{m+n}^0(\mu), & i = 1, 2, \\
X_n^1(v_{m1}^1(\mu)) &= (m + \mu)v_{m+n}^1(\mu), & i = 1, 2, \\
X_n^2(v_{m2}^2(\mu)) &= -(m + \mu)v_{m+n}^2(\mu), \\
Y_n(v_{m0}^0(\mu)) &= v_{m+n}^0(\mu), \\
L_n^+(v_{m1}^1(\mu)) &= (n + m + \mu)v_{m+n}^1(\mu), & i = 1, 2, \\
L_n^0(v_{m2}^2(\mu)) &= v_{m+n}^2(\mu), \\
X_n^0(v_{m3}^3(\mu)) &= v_{m+n}^3(\mu), & i = 1, 2, 3.
\end{align*}
\]

Naturally, $V^\mu = \bigoplus_m V^\mu_m$, where $V^\mu_m = t^{m+\mu} \otimes \Lambda(\xi_1, \xi_2)$. A $\mathbb{Z}$-grading on $\hat{K}'(4)$ is defined by the element $L_0 = t r\tau$ of the Virasoro algebra according to (2). We have that
\[
\mathfrak{g}_1(V^\mu_m) \subset V^\mu_{m+i}, \quad (9)
\]
and $\mathfrak{g}_0 \cong \mathfrak{sl}(2|2)$, where the central element is $L_0$. Note that $\mathfrak{sl}(2|2)$ has the following one-parameter family spin$_{\lambda}$ of (2|2)-dimensional irreducible representations:
\[
\text{spin}_\lambda : \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \otimes \mathbb{C} L_0 \rightarrow \left( \begin{array}{cc} A & B + \lambda\tilde{C} \\ C & D \end{array} \right) \otimes \mathbb{C}\Lambda_{1|2},
\]
where $A, B, C, D \in \mathfrak{gl}(2|2)$, $\text{tr}A = \text{tr}D$, $\lambda \in \mathbb{C}$. Let $E_{ij}$ be an elementary $2 \times 2$-matrix. $\tilde{C}$ is determined by the following conditions:
\[
\text{if } C = E_{ii}, \text{ then } \tilde{C} = E_{ij}, \text{ where } i \neq j, \\
\text{if } C = E_{ij}, \text{ if } \neq j, \text{ then } \tilde{C} = -E_{ij}.
\]
According to (9), there is a representation of $\mathfrak{g}_0$ in $V^\mu_m$ for each $m \in \mathbb{Z}$, and $V^\mu \cong \text{spin}_{\lambda_{m+\mu}}$ as $\mathfrak{g}_0$-modules.

Note that if $\mu = 0$, we cannot formally define a representation of $\hat{K}'(4)$ in $V^\mu$. Nevertheless, all the formulas for the action of $\hat{K}'(4)$ on vectors $v^\mu_m(\mu)$, where $i = 0, 1, 2, 3$ and $m \in \mathbb{Z}$, remain true to $\mu = 0$. Thus a representation of $\hat{K}'(4)$ in the superspace
\[
V = \text{Span}(v^i_m(0) \mid i = 0, 1, 2, 3 \text{ and } m \in \mathbb{Z}) \quad (11)
\]
is well-defined. To obtain a realization of this representation, at first we will describe $\hat{K}'(4)$ in terms of matrices over a Weyl algebra. By definition, a Weyl algebra is
\[
W = \bigoplus_{i \geq 0} \mathbb{R} A_i^i, \quad (12)
\]
where $A$ is an associative commutative algebra and $d : A \rightarrow A$ is a derivation of $A$ with the relations
\[
da = d(a) + ad, \quad a \in A.
\]
Set
\[
A = \mathbb{C}[t, t^{-1}], \quad \mathcal{D} = L_0 = t r\tau. \quad (13)
\]
Replacing $\lambda$ by $d$ in the formulas for spin$_{\lambda}$, we obtain the following theorem.

**Theorem 1.** Let $\hat{K}'(4) = \bigoplus \mathfrak{g}_i$, where the $\mathbb{Z}$-grading is given by $L_0 = t r\tau$. Then
\[
1) \mathfrak{g}_0 \cong \mathfrak{sl}(2|2) \text{ is realized as a Lie superalgebra of } 4 \times 4 \text{ matrices over } \mathbb{C}[d] \text{ of the type}
\left( \begin{array}{cc} A & B + d\tilde{C} \\ C & D \end{array} \right) \otimes \mathcal{D} \cdot 1_{2|2},
\]
where $A, B, C, D$ are $2 \times 2$ matrices over $\mathbb{C}$, $trA = trD$ and $\tilde{C}$ is determined by the conditions (10). The central element in $\mathfrak{sl}(2|2)$ is $L_0 = d \cdot 1_{2|2}$, and the central element in $\hat{K}'(4)$ is $1_{2|2}$.

2) $\hat{K}'(4)$ is a subsuperalgebra of $4 \times 4$ matrices over $W$ generated by $\mathfrak{sl}(2|2)$ and by all matrices
\[
\left( \begin{array}{cc} E_{ij}(a) & 0 \\ 0 & 0 \end{array} \right), \quad \left( \begin{array}{cc} 0 & E_{ij}(a) \\ 0 & 0 \end{array} \right),
\]
where $a \in A$ and $1 \leq i \neq j \leq 2$.

3) The standard representation of $\hat{K}'(4)$, realized as matrices over $W$, in $(2|2)$-dimensional vector superspace over $A$ is isomorphic to the above-mentioned representation in the superspace $V$ in the case when $\mu = 0$, see (11).

### 4 Case $\mathbf{CK}_6$

The exceptional superconformal algebra $\mathbf{CK}_6$ is spanned by the following 32 fields inside $K(6) \subset P(6)$:
\[
\begin{align*}
L_n &= t^{n+1} r, & G_n^i &= t^{n+1} r \xi_i, & i = 1, 2, 3, \\
\tilde{G}_n^i &= t^n \eta_i - nt^{n-1} r^{-1} \xi_i \eta_j \eta_j, & i &= 1, 2, 3, \\
T_{ij} &= t^n \xi_i \eta_j - nt^{n-1} r^{-1} \xi_j \xi_i \eta_j \eta_j, & i &= 1, 2, 3, \\
T_i &= -t^n (\xi_i \eta_j + \xi_j \eta_i) + nt^{n-1} r^{-1} \xi_j \xi_i \eta_j \eta_j, & i &= 1, 2, 3, \\
S_n^i &= -t^n \xi_i \eta_j + \xi_j \eta_i + nt^{n-1} r^{-1} \xi_j \xi_i \eta_j \eta_j, & i &= 1, 2, 3, \\
\tilde{S}_n^i &= t^{n-1} r^{-1} (\xi_j \eta_i - \xi_i \eta_j) \eta_i, & i &= 1, 2, 3,
\end{align*}
\]
where \( n \in \mathbb{Z} \), and \((i,j,k)\) is the cycle (1,2,3) in the formulas for \( \tilde{v}_i \). We have that

\[
\tilde{v}_m(\mu) = \frac{m+\mu}{m+\mu} \Pi(\xi_i), \quad 1 \leq i \leq 3,
\]

where \( m \in \mathbb{Z} \) and \((i,j,k)\) is the cycle (1,2,3) in the formulas for \( \tilde{v}_m(\mu) \). Explicitly, the action of \( C K_6 \) on \( V^\mu \) is given as follows:

\[
L_n(\tilde{v}_m(\mu)) = (m+n+\mu)\tilde{v}_m+n+m(\mu),
\]

\[
T_i^{\pm j}(\tilde{v}_m(\mu)) = \begin{cases} \pm v_{i\pm j}(\mu), & i \neq j, \\ \pm \tilde{v}_{i\pm j}(\mu), & i = j. \end{cases}
\]

We have that

\[
V^\mu = \oplus_m V_m^\mu, \quad V_m^\mu = t^{m+\mu} \boxdot \Pi(\Lambda(\xi_1, \xi_2, \xi_3)).
\]

A Z-grading in \( CK_6 \) is defined by the element \( L_0 = tr \) of the Virasoro algebra according to (2), so that (9) holds. Note that \( g_0 = \hat{\mathcal{P}}(4) \), where the central element is \( L_0 \) and \( \mathcal{P}(4) \) is a simple Lie superalgebra defined as follows. Let \( \mathcal{P}(4) \) be the Lie superalgebra, which preserves the odd nondegenerate supersymmetric bilinear form antidiag(14,14) on the (4|4)-dimensional complex superspace. Thus

\[
\mathcal{P}(4) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \right\} | A \in \mathfrak{gl}(4, \mathbb{C}), B^t = B, C^t = -C. \]

\( \mathcal{P}(4) \) is a subsuperalgebra of \( \hat{\mathcal{P}}(4) \) such that \( A \in \mathfrak{sl}(4, \mathbb{C}) \), see [8]. \( \hat{\mathcal{P}}(4) \) is a nontrivial central extension of \( \mathcal{P}(4) \). It is known that \( \mathcal{P}(4) \) has a family \( \text{spin}_\lambda \) of (4|4)-dimensional irreducible representations:

\[
\text{spin}_\lambda : \left( \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \right) \boxplus CL_0 \rightarrow \left( \begin{pmatrix} A & B - \lambda \hat{C} \\ C & -A^t \end{pmatrix} \right) \boxplus CL_1 4|4,
\]

where \( \lambda \in \mathbb{C}, \lambda \in 4|4 \) is the identity matrix, and \( \hat{C} \) is determined by the following condition:

\[
\text{if } C_{ij} = E_{ij} - E_{ji}, \text{ then } \hat{C}_{ij} = C_{ij}, \text{ so that the permutation } (1,2,3,4) \rightarrow (i,j,k,l) \text{ is even}
\]

cf. [6] and [22–24]. According to (9), there is a representation of \( g_0 \) in \( V_m^\mu \) for each \( m \in \mathbb{Z} \), and \( V_m^\mu \cong \text{spin}_{\lambda = m+\mu} \) as \( g_0 \)-modules.

Similarly to the case of \( \hat{K}'(4) \), all the formulas for the action of \( C K_6 \) on vectors \( v_m^\mu(\mu) \), \( \tilde{v}_m(\mu) \), where \( 1 \leq i \leq 4 \) and \( m \in \mathbb{Z} \), remain true to \( \mu = 0 \). Thus a representation of \( C K_6 \) in the superspace

\[
V = \text{Span}(v_m^\mu(0), \tilde{v}_m(0)) \quad | 1 \leq i \leq 4 \text{ and } m \in \mathbb{Z}
\]
is well-defined. To obtain a realization of this representation, we will use the Weyl algebra \( W \) defined in (12) and (13). Replacing \( \lambda \) by \( d \) in the formulas for \( \text{spin}_\lambda \), we obtain the following theorem, cf. [17].

**Theorem 2.** Let \( C K_6 = \oplus g_0 \), where the Z-grading is given by \( L_0 = tr \). Then

1) \( g_0 = \mathcal{P}(4) \) is realized as a Lie superalgebra of \( 8 \times 8 \) matrices over \( \mathbb{C}[d] \) of the type

\[
\left( \begin{array}{cc} A & B - d \hat{C} \\ C & -A^t \end{array} \right) \boxplusCd \cdot 1_{4|4},
\]

where \( A, B, \) and \( C \) are \( 4 \times 4 \) matrices over \( \mathbb{C} \), \( trA = 0 \), \( B^t = B, C^t = -C \), and \( \hat{C} \) is determined by the condition (15). The central element in \( \mathcal{P}(4) \) is \( L_0 = d \cdot 1_{4|4} \).

2) \( C K_6 \) is a subsuperalgebra of \( 8 \times 8 \) matrices over \( W \) generated by \( \mathcal{P}(4) \) and by all matrices

\[
\left( \begin{array}{cc} E_{ij}(a) & 0 \\ 0 & -E_{ij}(a) \end{array} \right), \text{ where } a \in A \text{ and } 1 \leq i \neq j \leq 4.
\]

3) The standard representation of \( C K_6 \), realized as matrices over \( W \), in (4|4)-dimensional vector superspace over \( A \) is isomorphic to the above-mentioned representation in the superspace \( V \) in the case when \( \mu = 0 \), see (16).
5 Acknowledgements

This material is based upon work supported by the National Science Foundation under agreement No. DMS—0111298. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author and do not necessarily reflect the views of the National Science Foundation.

The author is grateful to the Institute for Advanced Study for the hospitality and support during term II of the academic year 2006–2007. She wishes to thank the organizers of the 5th International Conference on Differential Equations and Dynamical Systems. She is also grateful to V. Serganova for very useful discussions.

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