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Key-Words: Chiral perturbation theory, Pion, Electromagnetic corrections.

Contribution to the Eighth International Symposium on Meson-Nucleon Physics and the Structure of the Nucleon, Zuoz, Engadine, Switzerland, August 15-21 1999
Electromagnetic Corrections to Low-Energy $\pi\pi$ Scattering

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INTRODUCTION

The Chiral Perturbation Theory (ChPT) analyses, in both the generalized $[1,2]$ and the standard $[3,4]$ frameworks, of two-loop effects in low-energy $\pi\pi$ scattering lead to strong interaction corrections which are rather small as compared to the leading order and one-loop contributions, of the order of 5% in the case of the S wave scattering lengths $a_0^0$ and $a_2^0$, for instance. This leads one to expect that higher order corrections to these quantities are well under control and can be safely neglected. However, these calculations were undertaken without taking into account isospin breaking effects, coming either from the mass difference between the $u$ and $d$ quarks, or from the electromagnetic interaction. The smallness of the two-loop corrections naturally raises the question of how they compare to these isospin breaking effects. The quark mass difference induces corrections of the order $O((m_d - m_u)^2)$, which are expected to be negligible, as already known to be the case for the pion mass difference $M_{\pi^\pm} - M_{\rho^0}$, the latter being in fact dominated by electromagnetic effects due to the virtual photon cloud $[5]$.

In the present contribution, we shall review the status of radiative corrections to the amplitude $\pi^+\pi^- \rightarrow \pi^0\pi^0$ $[6]$. The reason why we focus on the latter comes from the fact that it directly appears in the expression of the lifetime of the $\pi^+\pi^-$ dimeson atom $[7,8,9,10]$ (see in particular the last of these references), that will be measured by the DIRAC experiment at CERN $[11,12]$.

VIRTUAL PHOTONS IN ChPT: THE GENERAL FRAMEWORK

The general framework for a systematic study of radiative corrections in ChPT has been described in $[3,4]$. It consists in writing down a low-momentum representation for the generating functional of QCD Green’s functions of quark bilinears in the presence of the electromagnetic field,

$$e^{i\mathcal{Z}[\nu, a_\mu, s, p, Q_L, Q_R]} = \int \mathcal{D}[\mu]_{QCD} \mathcal{D}[A_\mu] e^{i \int d^4x \mathcal{L}},$$

(1)

with

$$\mathcal{L} = \mathcal{L}_{QCD}^0 + \mathcal{L}_\gamma^0 + \bar{q}\gamma^\mu[\nu_\mu + \gamma_5 a_\mu]q - \bar{q}[s - i\gamma_5 p]q + A_\mu[\bar{q}L\gamma^\mu Q_L q_L + \bar{q}R\gamma^\mu Q_R q_R].$$

(2)

Here $\mathcal{L}_{QCD}^0$ is the QCD lagrangian with $N_f$ flavours of massless quarks, while $\mathcal{L}_\gamma^0$ is the Maxwell lagrangian of the photon field. The coupling of the latter to the left-handed and right-handed quark fields, $q_{L,R} = \frac{1+i\gamma_5}{2} q$, occurs via the spurion sources $Q_{L,R}(x)$. Under local $SU(N_F)_L \times SU(N_F)_R$ chiral transformations ($g_L(x), g_R(x)$), they transform as (the transformation properties of the vector ($\nu_\mu$), axial ($a_\mu$), scalar ($s$) and pseudoscalar ($p$) sources can be found in Ref.$[13]$)

$$q_I(x) \rightarrow g_I(x)q_I(x), \quad Q_I(x) \rightarrow g_I(x)Q_I(x)g_I(x)^+, \quad I = L, R,$$

(3)

so that the generating functional $\mathcal{Z}$ remains invariant (up to the usual Wess-Zumino term). Thus, although the electromagnetic interaction represents an explicit breaking of chiral

*Work supported in part by TMR, EC-contract No. ERBFMRX-CT980169 (EURODAPHNE).
symmetry, this breaking occurs in a well defined way, which is precisely the information encoded in the transformation properties of Eq. [3], much in the same way as the transformation properties of the scalar source \( s(x) \) conveys the information on how the quark masses break chiral symmetry. At the end of the calculation, the sources \( v_\mu(x) \), \( a_\mu(x) \) and \( p(x) \) are set to zero, \( s(x) \) becomes the diagonal quark mass matrix, while the electromagnetic spurions are turned into the diagonal charge matrix of the quarks. Additional symmetries of \( Z \) consist of the discrete transformations like parity and charge conjugation. Finally, \( \mathcal{L} \) is invariant under an additional charge conjugation type symmetry, which however affects only the photon field and the electromagnetic spurion sources,

\[
Q_{L,R} \rightarrow -Q_{L,R}(x), \quad A_\mu(x) \rightarrow -A_\mu(x).
\]

The low-energy representation of \( Z \) is constructed systematically in an expansion in powers of momenta, of quark masses and of the electromagnetic coupling, by computing tree and loop graphs with an effective lagrangian \( \mathcal{L}_{\text{eff}} \) involving the \( N_f \times N_f \) matrix \( U(x) \) of pseudoscalar fields, and constrained by the chiral symmetry properties as well as the above discrete symmetries.

At lowest order, in the counting scheme where the electric charge \( e \) and the spurions \( Q_{L,R}(x) \) count as \( \mathcal{O}(p) \), the effective lagrangian is thus simply given by (for the notation, we follow [13,6])

\[
\mathcal{L}^{(2)}_{\text{eff}} = \frac{F^2}{4} \langle d^{\mu} u_{\mu} U + \chi^+ U + U^+ \chi \rangle - \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + C \langle Q_R U Q L U^+ \rangle.
\]

The effect of the electromagnetic interaction is contained in the covariant derivative \( d_\mu \), defined as \( d_\mu U = \partial_\mu U - i (v_\mu + Q_R A_\mu + a_\mu) U + i U (v_\mu + Q_L A_\mu - a_\mu) \), and in the low-energy constant \( C \), which at this order is responsible for the mass difference of the charged and neutral pions,

\[
\Delta_\pi \equiv M^2_{\pi^\pm} - M^2_{\pi^0} = 2Ce^2/F^2.
\]

In fact, for the case of two light flavours \( (N_f = 2) \), to which we restrict ourselves from now on, this is the only direct effect induced by this counterterm. Of course, this mass splitting will in turn modify the kinematics of the low-energy \( \pi \pi \) amplitudes and the corresponding scattering lengths. The details of this lowest order analysis can be found in Ref. [6]. Here, we shall rather consider the structure of the effective theory at next-to-leading order. Besides the counterterms described by the well known low-energy constants \( l_i \), there are now, if we restrict ourselves to constant spurion sources, 11 additional counterterms at order \( \mathcal{O}(e^2 p^2) \), and three more at order \( \mathcal{O}(e^4) \). The latter contribute only to the scattering amplitudes involving charged pions alone. The complete list of these counterterms \( k_i \), \( i = 1, ..., 14 \) and of their \( \beta \)-function coefficients can be found in Refs. [3,4].

### RADIATIVE CORRECTIONS TO THE ONE LOOP \( \pi^+ \pi^- \rightarrow \pi^0 \pi^0 \) AMPLITUDE

The computation of the amplitude \( A^{+-,00}(s,t,u) \) for the process \( \pi^+ \pi^- \rightarrow \pi^0 \pi^0 \), including corrections of order \( \mathcal{O}(e^2 p^2) \) and of order \( \mathcal{O}(e^4) \), is then a straightforward exercise in quantum field theory. The explicit expressions can be found in Ref. [3] and will not be reproduced here. Let us rather discuss some features of the one-photon exchange graph of Fig. 1, which induces an electromagnetic correction to the strong vertex. This graph contains both the long range Coulomb interaction between the charged pions, and an infrared singularity. The latter is treated in the usual way, the physical, infrared finite, observable being the cross section for \( \pi^+ \pi^- \rightarrow \pi^0 \pi^0 \) with the emission of soft photons (one soft photon is enough at the order at which we are working here). The Coulomb force leads to a singular behaviour of the amplitude \( A^{+-,00}(s,t,u) \) at threshold \( (q \) denotes the momentum of the charged pions in the center of mass frame),

\[
\text{Re}A^{+-,00}(s,t,u) = -\frac{4M^2_{\pi^\pm} - M^2_{\pi^0}}{F^2} \cdot \frac{e^2}{16} \cdot \frac{M_{\pi^\pm}}{q} + \text{Re}A^{+-,00}_{\text{thr}} + \mathcal{O}(q),
\]  

(7)
with

\[ ReA_{\text{thr}}^{+,-00} = 32\pi \left[ -\frac{1}{3}(a_0^0)_{\text{str}} + \frac{1}{3}(a_2^0)_{\text{str}} \right] - \frac{\Delta_+}{F_\pi^2} + \frac{e^2M_{\pi^0}^2}{32\pi^2F_\pi^2} (30 - 3\kappa_{1+0}^+ + \kappa_{2+0}^+) \]

\[ - \frac{\Delta_+}{48\pi^2F_\pi^4} \left[ M_{\pi^0}^2(1 + 4\bar{l}_1 + 3\bar{l}_3 - 12\bar{l}_4) - 6F_\pi^2 e^2(10 - \kappa_{1+0}^+) \right] \]

\[ + \frac{\Delta_+^2}{480\pi^2F_\pi^4} \left[ 212 - 40\bar{l}_1 - 15\bar{l}_3 + 180\bar{l}_4 \right], \tag{8} \]

and \((a_0^0)_{\text{str}}\) and \((a_2^0)_{\text{str}}\) denote the S wave scattering lengths in the presence of the strong interactions only, but expressed, for convention reasons, in terms of the charged pion mass \[16\].

\[ (a_0^0)_{\text{str}} = \frac{7M_{\pi^0}^2}{32\pi^2F_\pi^2} \left\{ 1 + \frac{5}{84\pi^2} \frac{M_{\pi^0}^2}{F_\pi^2} \left[ \bar{l}_1 + 2\bar{l}_2 - \frac{3}{8}\bar{l}_3 + \frac{21}{10}\bar{l}_4 + \frac{21}{8} \right] \right\} \]

\[ (a_2^0)_{\text{str}} = -\frac{M_{\pi^0}^2}{16\pi^2F_\pi^2} \left\{ 1 - \frac{1}{12\pi^2} \frac{M_{\pi^0}^2}{F_\pi^2} \left[ \bar{l}_1 + 2\bar{l}_2 - \frac{3}{8}\bar{l}_3 - \frac{3}{2}\bar{l}_4 + \frac{3}{8} \right] \right\}, \tag{9} \]

corresponding to the numerical values \(F_\pi = 92.4\,\text{MeV}\) \((a_0^0)_{\text{str}} = 0.20 \pm 0.01\) and \((a_2^0)_{\text{str}} = -0.043 \pm 0.004\), respectively \[13\]. The quantity \(ReA_{\text{thr}}^{+,-00}\), which is by itself free of the infrared divergence mentioned above, appears directly in the lifetime of the pionium atom \[14\], the long range Coulomb interaction being, in that case, absorbed by the bound state dynamics. The contributions of the low-energy constants \(k_i\) are contained in the two quantities \(\kappa_{1+0}^+\) and \(\kappa_{2+0}^+\). Naive dimensional estimates lead to \((e^2F_\pi^2/M_{\pi^0}^2)\kappa_{1+0}^+ = 1.8 \pm 0.9\) and \((e^2F_\pi^2/M_{\pi^0}^2)\kappa_{2+0}^+ = 0.5 \pm 2.2\). With these estimates, one obtains \[8\]

\[ \frac{1}{32\pi} ReA_{\text{thr}}^{+,-00} - \left[ -\frac{1}{3}(a_0^0)_{\text{str}} + \frac{1}{3}(a_2^0)_{\text{str}} \right] = (-1.2 \pm 0.7) \times 10^{-3}, \tag{10} \]

whereas the two-loop correction to the same combination of scattering lengths appearing between brackets amounts to \(\sim -4 \times 10^{-3}\). For a more careful evaluation of the contribution of the counterterms \(k_i\) to \(ReA_{\text{thr}}^{+,-00}\), see \[18\].

Fig. 1. The one photon exchange electromagnetic correction to the strong vertex.

**CONCLUSION**

We have evaluated the radiative corrections of order \(O(e^2p^2)\) and of order \(O(e^4)\) to the amplitude \(A^{+,-00}(s, t, u)\), which is relevant for the description of the pionium lifetime. Similar results for the scattering process involving only neutral pions can be found in Refs. \[14,17\]. The formalism presented here for the case \(N_f = 2\) has also been applied to the study of radiative corrections to the pion form factors \[13\]. Radiative corrections for the scattering amplitudes involving only charged pions, which would be relevant for the \(2p - 2s\) level-shift of pionium, for instance, have however not been worked out so far.

Information on the low-energy scattering of pions can only be obtained in an indirect way, either from the pionium lifetime, or from \(K_{\ell4}\) decays. This last process, however,
has electromagnetic corrections of its own, which are only partly covered by the present analysis. A systematic framework devoted to the study of radiative correction for the semi-leptonic processes has been presented in Refs. 20,21.

REFERENCES

1. M. Knecht, B. Moussallam, J. Stern and N.H. Fuchs, “The low-energy ππ amplitude to one and two loops”, Nucl. Phys. B457, 513 (1995).
2. M. Knecht, B. Moussallam, J. Stern and N.H. Fuchs, “Determination of two-loop ππ scattering amplitude parameters”, Nucl. Phys. B471, 445 (1996).
3. J. Bijnens, G. Colangelo, G. Ecker, J. Gasser and M. Sainio, “Elastic ππ scattering to two loops”, Phys. Lett. B374, 210 (1996).
4. J. Bijnens, G. Colangelo, G. Ecker, J. Gasser and M. Sainio, “Pion-pion scattering at low energy”, Nucl. Phys. 508, 263 (1997); ibid B517, 639 (1998) (erratum).
5. J. Gasser and H. Leutwyler, “Quark masses”, Phys. Rep. 87, 77 (1982).
6. M. Knecht and R. Urech, “Virtual photons in low-energy ππ scattering”, Nucl. Phys. B519, 329 (1998).
7. H. Jallouli and H. Sazdjian, “Relativistic effects in the pionium lifetime”, Phys. Rev. D58, 014011 (1998); ibid. D58, 099901 (1998) (erratum).
8. M.A. Ivanov, V.E. Lyubovitsky, E.Z. Lipartia and A. Rusetsky, “π+π− atom in chiral perturbation theory”, Phys. Rev. D58, 0904024 (1998).
9. D. Eiras and J. Soto, “Effective field theory approach to pionium”, hep-ph/9905543, and J. Soto, contribution to these proceedings.
10. A. Gall, J. Gasser, V.E. Lyubovitsky and A. Rusetsky, “On the lifetime of the π+π− atom”, Phys. Lett. B462, 335 (1999).
V. E. Lyubovitsky, “Hadronic atoms in QCD”, contribution to these proceedings.
A. Rusetsky, “Spectrum and decay of hadronic atoms”, contribution to these proceedings.
11. B. Adeva et al., Proposal to the SPSLC, CERN/SPSLC 95-1 (1995).
12. A. Lanaro, “Lifetime measurement of dimeson atoms with the DIRAC experiment”, contribution to these proceedings.
13. R. Urech, “Virtual photons in chiral perturbation theory”, Nucl. Phys. B433, 234 (1995).
14. H. Neufeld and H. Rupertsberger, “Isospin breaking in chiral perturbation theory and the decays η → πνν and τ → ηπν”, Z. Phys. C68, 91 (1995).
H. Neufeld and H. Rupertsberger, “The electromagnetic interaction in chiral perturbation theory”, Z. Phys. C71, 131 (1995).
15. J. Gasser and H. Leutwyler, “Chiral perturbation theory: expansion in the mass of the strange quark”, Nucl. Phys. B250, 465 (1985).
16. J. Gasser and H. Leutwyler, “Chiral perturbation theory to one loop”, Ann. Phys. 158, 142 (1984).
17. U.-G. Meißner, G. Müller and S. Steininger “Virtual photons in SU(2) chiral perturbation theory and electromagnetic corrections to ππ scattering”, Phys. lett. B406, 154 (1997); ibid B407, 454 (1997) (erratum).
18. J. Gasser, V.E. Lyubovitsky and A. Rusetsky, “Numerical analysis of the π+π− atom lifetime in ChPT”, preprint BUTP-99/20 and hep-ph/9910435.
19. B. Kubis and U.-G. Meißner, “Virtual photons in the pion form factors and the energy-momentum tensor”, hep-ph/990820.
20. M. Knecht, H. Neufeld, H. Rupertsberger and P. Talavera, “Chiral perturbation theory with virtual photons and leptons”, hep-ph/9909284 and Eur. Phys. J. C, to appear.
21. H. Neufeld, “Chiral perturbation theory with photons and leptons”, contribution to these proceedings.