ON THE BOHR INEQUALITY WITH A FIXED ZERO COEFFICIENT

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ABSTRACT. In this paper, we introduce the study of the Bohr phenomenon for a quasi-subordination family of functions, and establish the classical Bohr’s inequality for the class of quasisubordinate functions. As a consequence, we improve and obtain the exact version of the classical Bohr’s inequality for bounded analytic functions and also for $K$-quasiconformal harmonic mappings by replacing the constant term by the absolute value of the analytic part of the given function. We also obtain the Bohr radius for the subordination family of odd analytic functions.

1. Introduction and Preliminaries

In this article, our primary concern is to study Bohr’s phenomenon for the class of quasi-subordination functions and obtain the exact version of the classical Bohr’s inequality for the case of analytic functions and also for the case of harmonic functions defined on the open unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$. The classical result of H. Bohr [9], which in the final form was proved independently by M. Riesz, I. Schur and N. Wiener, is as follows:

**Theorem A.** Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be analytic in $D$ and $|f(z)| \leq 1$ for all $z \in D$. Then

$$\left|a_0\right| + \sum_{k=1}^{\infty} |a_k| r^k \leq 1 \quad \text{for all } r \leq \frac{1}{3}$$

and the constant $1/3$, called the Bohr radius, cannot be improved.

In 1956, Ricci [26] initiated the investigation of the Bohr radius with fixed zero-coefficient $a_0$, and in 1962, Bombieri [10] solved the problem for $|a_0| \geq \frac{1}{2}$. In this paper in the later part of our investigation (see Theorem 2.7), we strengthen these results and furthermore, in Theorem 2.9, we extend it for sense-preserving $K$-quasiconformal harmonic mappings of the unit disk.

In the recent years, the problem about the Bohr radius attracted the attention of many researchers in various directions in functions of one and several complex variables: to planar harmonic mappings, to polynomials, to domains in several complex variables, to solutions of elliptic partial differential equations and to more...
abstract settings. For more information about Bohr's inequality stated above and further related investigations, we refer the reader to the recent survey articles on the Bohr radius from \cite{11, 17, 16} Chapter 8], and the references therein. See also \cite{10, 11}. In particular, Boas and Khavinson \cite{8}, Aizenberg \cite{2, 3}, and Aizenberg and Tarkhanov \cite{4} have extended the Bohr inequality for holomorphic functions on certain specific domains (such as complete Reinhardt domain) in $\mathbb{C}^n$.

Recently, Kayumov et al. \cite{21} investigated Bohr’s radius for locally univalent planar harmonic mappings. Several improved versions of the classical Bohr’s inequality were given by Kayumov and Ponnusamy in \cite{19} (see also \cite{20}) whereas Evdoridis et al. \cite{15} have presented several improved versions of Bohr’s inequality for harmonic mappings. In \cite{18}, Kayumov and Ponnusamy also discussed Bohr’s radius for the class of analytic functions $g$, when $g$ is subordinate to a member of the class of odd univalent functions. For certain recent results, we refer to \cite{3, 8, 19}. In particular, Kayumov and Ponnusamy \cite{20} established the following theorem which settled the open problem proposed by Ali et al. \cite{5}.

**Theorem B.** If a function $f(z) = \sum_{k=1}^{\infty} a_{2k-1} z^{2k-1}$ is odd analytic in $\mathbb{D}$ and $|f(z)| \leq 1$ in $\mathbb{D}$, then

$$\sum_{k=1}^{\infty} |a_{2k-1}| r^{2k-1} \leq 1 \quad \text{for all } r \leq r_0,$$

where $r_0 \simeq 0.789991...$ is the maximal positive root of the equation

$$8r^4 + r^2 - 6r + 1 = 0$$

and the constant $r_0$ cannot be improved.

In 1970, Robertson \cite{24} introduced and developed the concept of quasi-subordination which combines the principles of subordination and majorization.

If $f$ and $g$ are analytic in $\mathbb{D}$, $\omega$ is a Schwarz function (i.e. $\omega$ is analytic in $\mathbb{D}$, $\omega(0) = 0$ and $|\omega(z)| \leq 1$ for $|z| < 1$) and all three satisfy $f(z) = g(\omega(z))$ for $z \in \mathbb{D}$, then we write $f(z) \prec g(z)$ in $\mathbb{D}$ and say that $f$ is subordinate to $g$. The importance of the principle of subordination stems from the fact that when $f$ is subordinate to $g$, $f(\mathbb{D}) \subset g(\mathbb{D})$ and this has been extensively used in the literature. We say that $f(z)$ is majorized by $g(z)$ in $\mathbb{D}$ if $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{D}$.

**Definition 1.1.** For any two analytic functions $f$ and $g$ in $\mathbb{D}$, we say that the function $f$ is quasi-subordinate to $g$ (relative to $\Phi$), denoted by $f(z) \prec_{\Phi} g(z)$ in $\mathbb{D}$, if there exist two functions $\Phi$ and $\omega$, analytic in $\mathbb{D}$, satisfying $\omega(0) = 0$, $|\Phi(z)| \leq 1$ and $|\omega(z)| \leq 1$ for $|z| < 1$ such that

$$f(z) = \Phi(z)g(\omega(z)).$$

There are two special cases which are of particular interest. The choice $\Phi(z) = 1$ corresponds to subordination, whereas $\omega(z) = z$ gives majorization, i.e. \eqref{1.2} reduces to the form $f(z) = \Phi(z)g(z)$. In other words, if either $f \prec g$ or $|f(z)| \leq |g(z)|$ in $\mathbb{D}$, then $f(z) \prec_{\Phi} g(z)$ in $\mathbb{D}$. Thus, the notion of quasi-subordination generalizes both the concept of subordination and the principle of majorization. Several theorems exist in the literature that relate with these two concepts and are widely used in function theory, and some of the known results continue to hold in the setting of quasi-subordination. See \cite{22, 24}. Note also that \eqref{1.2} is equivalent to saying that the quotient $f(z)/\Phi(z)$ is analytic and is subordinate to $g(z)$ in $\mathbb{D}$.
Remark 1.2. On the linear space $H(D)$ consisting of complex-valued analytic functions $g$ defined on $D$, let $\omega$ denote self-map of $D$. The composition operator $C_\omega$ with symbol $\omega$ is defined as

$$C_\omega g = g \circ \omega \quad \text{for} \quad g \in H(D).$$

Similarly, a weighted composition operator $W_{\omega, \Phi}$ is an operator that maps $g \in H(D)$ into $W_{\omega, \Phi}(f) = \Phi(z)g(\omega(z))$, where $\Phi$ and $\omega$ are analytic defined on $D$ such that $\omega(D) \subset D$. Note also that for a given complex-valued function $\Phi$ defined on $D$, the multiplication operator with symbol $\omega$ is defined by

$$M_\Phi g = \Phi g \quad \text{for} \quad g \in H(D).$$

These operators appear in a natural way, for example, in the study of a number of questions about the boundedness and compactness of operators on various function spaces in a more general setting. See [13]. Thus, it is worth pointing out that the concept of ‘subordination’ is nothing but a composition (operator) with a function mapping $D$ into itself, and the concept of ‘quasi-subordination’ is nothing but a weighted composition (operator). Note also that the multiplication operator is related to the majorization.

The paper is organized as follows. Section 2 is devoted to state our main results whose proofs will be presented in Section 3. First we show that (Theorem 2.1) the radius $1/3$ of Bohr inequality remains the same even when the functions $f$ and $g$ are related with a quasi-subordination relation (1.2) which clearly reveals the fact that the classical Bohr inequality continues to hold in a more general setting. Secondly, as a consequence of Theorem 2.1 we present in Corollary 2.6 the exact version of Theorem A. Thirdly, we show in Theorem 2.4 that the Bohr radius for the subordinating family of odd functions is $1/\sqrt{3}$. In Theorem 2.5, we present a sharp version of Bohr’s inequality for sense-preserving $K$-quasiconformal harmonic mappings. Finally, in Theorems 2.7 and 2.9 we essentially investigate the Bohr phenomenon by replacing the constant term by the function itself in the case of analytic functions, and by the analytic part in the case of harmonic functions, respectively.

2. Main Results and their consequences

First we state an improved version of Bohr’s inequality for a quasi-subordinating family of functions.

**Theorem 2.1.** Let $f(z)$ and $g(z)$ be two analytic functions in $D$ with the Taylor series expansions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ for $z \in D$. If $f(z) \prec_q g(z)$, then

$$\sum_{k=0}^{\infty} |a_k|^r \leq \sum_{k=0}^{\infty} |b_k|^r \quad \text{for all} \quad r \leq \frac{1}{3}.$$

We are now ready to state two simple corollaries which are of independent interest, and the first of which was obtained recently by Bhowmik and Das [7].

**Corollary 2.2.** Let $f(z)$ and $g(z)$ be two analytic functions in $D$ such that $f(z) = \sum_{k=0}^{\infty} a_k z^k$, and $g(z) = \sum_{k=0}^{\infty} b_k z^k$. If $f(z) \prec g(z)$ in $D$, then

$$\sum_{k=0}^{\infty} |a_k|^r \leq \sum_{k=0}^{\infty} |b_k|^r \quad \text{for all} \quad r \leq \frac{1}{3}.$$
and the constant 1/3 cannot be improved.

**Corollary 2.3.** Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be two analytic functions in $\mathbb{D}$. If $f(z)$ is majorized by $g(z)$, i.e. $|f(z)| \leq |g(z)|$ in $\mathbb{D}$, then

$$
\sum_{k=0}^{\infty} |a_k|^r \leq \sum_{k=0}^{\infty} |b_k|^r \quad \text{for all } r \leq \frac{1}{3},
$$

and the constant 1/3 cannot be improved.

These corollaries play a crucial role in establishing generalized versions of Theorem A.

According to Theorem B, the Bohr radius for the class of odd functions is 0.789991... and thus, it is natural to ask for the Bohr radius for the subordinating family of odd analytic functions. Our next result answers this question.

**Theorem 2.4.** Let $f(z)$ and $g(z)$ be odd analytic functions in $\mathbb{D}$ with Taylor expansions $f(z) = \sum_{k=1}^{\infty} a_{2k-1} z^{2k-1}$ and $g(z) = \sum_{k=1}^{\infty} b_{2k-1} z^{2k-1}$, respectively. If $f(z) < g(z)$, then

$$
(2.1) \quad \sum_{k=1}^{\infty} |a_{2k-1}| r^{2k-1} \leq \sum_{k=1}^{\infty} |b_{2k-1}| r^{2k-1} \quad \text{for } |z| = r \leq \frac{1}{\sqrt{3}},
$$

A harmonic mapping in $\mathbb{D}$ is a complex-valued function $f$ in $\mathbb{D}$, which satisfies the Laplace equation $\Delta f = 4f_x f_y = 0$. It follows that $f$ admits the canonical representation $f = h + \overline{g}$, where $h$ and $g$ are analytic in $\mathbb{D}$ with $f(0) = h(0)$. The Jacobian $J_f$ of $f$ is given by $J_f = |h'|^2 - |g'|^2$. We say that $f$ is sense-preserving in $\mathbb{D}$ if $J_f(z) > 0$ in $\mathbb{D}$. Consequently, $f$ is locally univalent and sense-preserving in $\mathbb{D}$ if and only if $J_f(z) > 0$ in $\mathbb{D}$; or equivalently if $h' \neq 0$ in $\mathbb{D}$ and the dilatation $\omega_f = \omega = g'/h'$ has the property that $|\omega(z)| < 1$ in $\mathbb{D}$. For a detailed treatment of the geometric point of view of planar harmonic mappings of the unit disk, we refer to [14] and also [13, 23].

In order to state our result about the Bohr radius for quasiconformal harmonic mappings, we recall that a sense-preserving homeomorphism $f$ from the unit disk $\mathbb{D}$ onto $\Omega'$, contained in the Sobolev class $W^{1,2}_{\text{loc}}(\mathbb{D})$, is said to be a $K$-quasiconformal mapping if, for $z \in \mathbb{D}$,

$$
\left| \frac{f_z + \overline{f_{\overline{z}}}}{f_z - \overline{f_{\overline{z}}}} \right| = \frac{1 + |\omega_f(z)|}{1 - |\omega_f(z)|} \leq K, \quad \text{i.e., } |\omega_f(z)| = \left| \frac{g'(z)}{h'(z)} \right| \leq K - 1 \quad \text{for } \frac{1}{K+1}.
$$

where $K \geq 1$ so that $k \in [0,1)$. We now state a new version of Bohr’s inequality for harmonic mappings.

**Theorem 2.5.** Suppose that $f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$ is a sense-preserving $K$-quasiconformal harmonic mapping of the disk $\mathbb{D}$ (or more generally $|\omega_f(z)| \leq k$ for some $k \in [0,1]$), where $|h(z)| \leq 1$ in $\mathbb{D}$. Then the following sharp inequalities hold:

$$
(2.2) \quad \frac{1 - r|a_0| + (k + 1)(1 - |a_0|^2)}{1 - r|a_0|} + \sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leq 1 \quad \text{for } r \leq \frac{1}{3}.
$$

The functions

$$
\frac{z + a_0}{1 + \overline{a_0} z} + \lambda \frac{z + a_0}{1 + \overline{a_0} z}
$$
with $\lambda \to 1$ demonstrate that the inequality (2.2) is sharp for all $a_0 \in \mathbb{D}$ and all $r \leq \frac{1}{3}$.

Before we continue the discussion, let us remark that the classical Bohr inequality is not sharp for any individual function. Namely, it is easy to show that for any given function the Bohr radius is always greater than $1/3$. As a result of Theorem 2.5, here is the sharp result which shows that $1/3$ cannot be improved even in the case of individual functions.

**Corollary 2.6.** Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in $\mathbb{D}$ and $|f(z)| \leq 1$ for all $z \in \mathbb{D}$. Then the following sharp inequality holds:

$$
(2.3) \quad \frac{1 - (1 + |a_0| - |a_0|^2)r}{1 - |a_0|r} + \sum_{n=1}^{\infty} |a_n|r^n \leq 1 \quad \text{for all } r \leq 1/3,
$$

The function $g(z) = (z + a_0)/(1 + a_0z)$ shows that equality holds for all $a_0 \in \mathbb{D}$ and $r \leq 1/3$.

**Proof.** The result follows if we let $K = 1$ (i.e. $k = 0$) in Theorem 2.5 so that $g(z) \equiv 0$ in $\mathbb{D}$. \hfill \square

**Theorem 2.7.** Suppose that $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is an analytic function in $\mathbb{D}$ and $|f(z)| < 1$ for all $z \in \mathbb{D}$ and $0 \leq |a_0| = a < 1$. Then

$$
(2.4) \quad |f(z)| + \sum_{k=1}^{\infty} |a_k|r^k \leq 1
$$

for all $a \geq 2\sqrt{3} - 3 \approx 0.4641016$ and $|z| = r \leq r_a$, where

$$
r_a = \frac{\sqrt{(1 + a)^2 + a^2} - (1 + a)}{a^2} = \frac{1}{\sqrt{(1 + a)^2 + a^2} + 1 + a}
$$

and the radius $r_a$ is sharp.

**Remark 2.8.** From the proof of Theorem 2.7, it can be easily seen that for $r \leq \sqrt{3} - 2$ the inequality (2.4) continues to hold for all $a < 1$.

We now generalize Theorem 2.7 in order to present a generalized version of Bohr’s inequality with analytic part of the corresponding harmonic function.

**Theorem 2.9.** Suppose that $f(z) = h(z) + \overline{g(z)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$ is a sense-preserving $K$-quasiconformal harmonic mapping of the disk $\mathbb{D}$, where $|h(z)| < 1$ in $\mathbb{D}$ and $0 \leq a = |a_0| < 1$. Then the following sharp inequalities hold:

$$
(2.5) \quad |h(z)| + \sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=1}^{\infty} |b_n|r^n \leq 1
$$

for all $a \geq \alpha_k$ and $|z| = r \leq r_{a,k}$, where

$$
\alpha_k = \frac{\sqrt{k^2 + 12k + 12} - (2k + 3)}{k + 1} \quad \text{and} \quad r_{a,k} = \frac{B_{a,k} - (k + 2)(1 + a)}{2a^2(k + 1) + 2ak}
$$

with $B_{a,k} = \sqrt{a^2(k^2 + 8k + 8) + 2a(k^2 + 6k + 4) + (k + 2)^2}$. The radius $r_{a,k}$ is sharp.
3. Proofs of the Main Results

In the proofs of Theorems 2.1 and 2.4, we will use some approaches used in [25] (see also [7, Proof of Lemma 1]).

3.1. Proof of Theorem 2.1. Suppose that \( f \prec_q g \). Then there exist two analytic functions \( \Phi \) and \( \omega \) satisfying \( \omega(0) = 0 \), \( |\omega(z)| \leq 1 \) and \( |\Phi(z)| \leq 1 \) for all \( z \in \mathbb{D} \) such that

\[
(3.1) \quad f(z) = \Phi(z) g(\omega(z)).
\]

Now for the analytic function \( \omega(z) = \sum_{n=1}^{\infty} \alpha_n z^n \), the Taylor expansion of the \( k \)-th power of \( \omega \), where \( k \in \mathbb{N} \), can be written as

\[
(3.2) \quad \omega^k(z) = \sum_{n=k}^{\infty} \alpha_n^{(k)} z^n.
\]

We observe that, since \( \omega(0) = 0 \) and \( |\omega(z)| \leq 1 \), it follows from Theorem A that

\[
(3.3) \quad \sum_{n=k}^{\infty} |\alpha_n^{(k)}| r^{n-k} \leq 1 \quad \text{for all} \quad r \leq \frac{1}{3}.
\]

For the analytic function \( \Phi(z) \), we may write \( \Phi(z) = \sum_{m=0}^{\infty} \phi_m z^m \) and thus, by Theorem A, we have

\[
(3.4) \quad \sum_{m=0}^{\infty} |\phi_m| r^m \leq 1 \quad \text{for all} \quad r \leq \frac{1}{3}.
\]

Also, from the equality (3.1), taking into consideration from (3.2) that

\[
\omega^0(z) = 1 = \sum_{n=0}^{\infty} \alpha_n^{(0)} z^n, \quad \text{where} \quad \alpha_n^{(0)} = 1, \quad \alpha_n^{(0)} = 0 \quad \text{for} \quad n \geq 1,
\]

we can rewrite the quasi-subordinate relation (1.2) with the help of (3.2) in series form as

\[
\sum_{k=0}^{\infty} a_k z^k = \sum_{m=0}^{\infty} \phi_m z^m \left( \sum_{k=0}^{\infty} b_k \sum_{n=k}^{\infty} \alpha_n^{(k)} z^n \right)
\]

\[
= \sum_{m=0}^{\infty} \phi_m z^m \left( \sum_{k=0}^{\infty} \left( \sum_{n=0}^{k} b_n \alpha_k^{(n)} \right) z^k \right)
\]

\[
= \sum_{m=0}^{\infty} \phi_m z^m \sum_{k=0}^{\infty} B_k z^k,
\]

where \( B_k = \sum_{n=0}^{k} b_n \alpha_k^{(n)} \). Thus, the last relation takes the form

\[
\sum_{k=0}^{\infty} a_k z^k = \sum_{m=0}^{\infty} \left( \sum_{m+j=k} \phi_m B_j \right) z^k,
\]

which by equating the coefficients of \( z^k \) on both sides gives

\[
(3.5) \quad a_k = \sum_{m+j=k} \phi_m B_j \quad \text{for each} \quad k \geq 0.
\]
Applying the triangle inequality to the last relation shows that
\[
\sum_{k=0}^{\infty} |a_k|^r \leq \sum_{k=0}^{\infty} \left( \sum_{m+j=k} |\phi_m||B_j| \right)^r = \sum_{k=0}^{\infty} \sum_{m+j=k} |\phi_m|^m |B_j|^j r^j
\]
\[
= \left( \sum_{m=0}^{\infty} |\phi_m|^m \right) \sum_{k=0}^{\infty} |B_k|^r
\]
\[
\leq \sum_{k=0}^{\infty} |B_k|^r \text{ for all } r \leq \frac{1}{3}, \text{ (by (3.4))}.
\]

Also, because \(|B_k| \leq \sum_{n=0}^{k} |b_n| |\alpha_k^{(n)}|\), we obtain that
\[
\sum_{k=0}^{\infty} |B_k|^r \leq \sum_{k=0}^{\infty} \sum_{n=0}^{k} |b_n| |\alpha_k^{(n)}|^r \leq \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} |\alpha_n^{(k)}|^r r^n
\]
\[
= \sum_{k=0}^{\infty} b_k \left( \sum_{n=k}^{\infty} |\alpha_n^{(k)}|^r r^{n-k} \right) r^k
\]
\[
\leq \sum_{k=0}^{\infty} b_k r^k \text{ for all } r \leq \frac{1}{3}, \text{ (by (3.3))}
\]
and hence, we obtain that
\[
\sum_{k=0}^{\infty} |a_k|^r \leq \sum_{k=0}^{\infty} |B_k|^r \leq \sum_{k=0}^{\infty} |b_k|^r \text{ for all } r \leq \frac{1}{3}.
\]

The proof of Theorem 2.1 is complete. \(\square\)

3.2. Improved version of the classical Bohr inequality for odd and \(p\)-symmetric functions. Our next result is indeed a simple consequence of Corollary 2.2 and we state it in this form because of its independent interest.

**Lemma 3.1.** Let \(f(z)\) and \(g(z)\) be analytic and \(p\)-symmetric in \(D\) with the Taylor expansions \(f(z) = \sum_{k=0}^{\infty} a_{pk} z^{pk}\) and \(g(z) = \sum_{k=0}^{\infty} b_{pk} z^{pk}\), respectively. If \(f(z) \prec g(z)\), then
\[
\sum_{k=0}^{\infty} |a_{pk}| r^{pk} \leq \sum_{k=0}^{\infty} |b_{pk}| r^{pk} \text{ for all } r \leq \frac{1}{\sqrt[3]{3}}.
\]

The constant \(1/\sqrt[3]{3}\) cannot be improved.

**Proof.** It suffices to set \(\zeta = z^p\), and consider the functions
\[
f_1(\zeta) = \sum_{k=0}^{\infty} a_{pk} \zeta^k \text{ and } g_1(\zeta) = \sum_{k=0}^{\infty} b_{pk} \zeta^k.
\]

Then \(f_1(\zeta) \prec g_1(\zeta)\) for \(|\zeta| < 1\) and, by Corollary 2.2, we obtain that
\[
\sum_{k=0}^{\infty} |a_{pk}| |\zeta|^k \leq \sum_{k=0}^{\infty} |b_{pk}| |\zeta|^k \text{ for all } |\zeta| = |z|^p \leq \frac{1}{3}.
\]

The desired conclusion follows. \(\square\)
3.3. Proof of Theorem 3.3. Let \( f \prec g \), where \( f \) and \( g \) are as in the statement. Then there exists a function \( \omega \), analytic in \( D \), satisfying \( \omega(0) = 0 \) and \( |\omega(z)| \leq 1 \) for all \( |z| < 1 \) such that \( f(z) = g(\omega(z)) \) which in terms of series can be written as

\[
(3.6) \quad \sum_{k=1}^{\infty} a_{2k-1} z^{2k-1} = \sum_{k=1}^{\infty} b_{2k-1} \omega(z)^{2k-1},
\]

where, as usual, we write \( \omega(z) = \sum_{n=1}^{\infty} a_n z^n \) and the Taylor expansion of the \((2k-1)\)-th power of \( \omega \), where \( k \in \mathbb{N} \), has the form

\[
(3.7) \quad \omega^{2k-1}(z) = \sum_{n=2k-1}^{\infty} a_n^{(2k-1)} z^n.
\]

Now we plug the equality (3.7) into the right hand side of the relation (3.6), and obtain

\[
\sum_{k=1}^{\infty} a_{2k-1} z^{2k-1} = \sum_{k=1}^{\infty} b_{2k-1} \left( \sum_{n=2k-1}^{\infty} a_n^{(2k-1)} z^n \right) = \left( b_1 a_1^{(1)} z + b_1 a_2^{(1)} z^2 + b_1 a_3^{(1)} z^3 + \cdots \right) + \left( b_3 a_3^{(3)} z^3 + b_3 a_4^{(3)} z^4 + b_3 a_5^{(3)} z^5 + \cdots \right) + \left( b_5 a_5^{(5)} z^5 + b_5 a_6^{(5)} z^6 + b_5 a_7^{(5)} z^7 + \cdots \right) + \cdots.
\]

Clearly, the coefficients of \( z^{2n} \) have to be zero and thus, \( a_n^{(2k-1)} = 0 \) for \( m = 1, 2, \ldots \). Thus, we can write the last equation as

\[
\sum_{k=1}^{\infty} a_{2k-1} z^{2k-1} = \sum_{k=1}^{\infty} \left( \sum_{n=2k-1}^{\infty} b_{2n-1} a_n^{(2k-1)} \right) z^{2k-1}
\]

and equating the coefficients of \( z^{2k-1} \) on both sides, we have

\[
a_{2k-1} = \sum_{n=1}^{k} b_{2n-1} a_n^{(2k-1)} \quad \text{for any } k \geq 1.
\]

Applying the triangle inequality to the last relation shows that

\[
(3.8) \quad \sum_{k=1}^{m} |a_{2k-1}| r^{2k-1} \leq \sum_{k=1}^{m} \left( \sum_{n=1}^{k} |b_{2n-1}| |a_n^{(2k-1)}| \right) r^{2k-1} = \sum_{n=1}^{m} |b_{2n-1}| \left( \sum_{k=n}^{m} |a_n^{(2k-1)}| r^{2k-1} \right).
\]

Now for the series \( \sum_{k=n}^{m} |a_n^{(2k-1)}| r^{2k-1} \), since \( \omega^n(z)/z^n \) is bounded in \( D \), Lemma 3.1 yields that

\[
(3.9) \quad \sum_{k=n}^{m} |a_n^{(2k-1)}| r^{2(k-n)} \leq \sum_{k=n}^{m} |a_n^{(2k-1)}| r^{2(k-n)} \leq 1 \quad \text{for } r \leq \frac{1}{\sqrt{3}}.
\]

Consequently,

\[
\sum_{k=n}^{m} |a_n^{(2k-1)}| r^{2k-1} \leq r^n \quad \text{for } r \leq \frac{1}{\sqrt{3}}.
\]
Lemma 3.2. 

Theorem 2.5, we need the following lemma.

3.4. Improved version of Bohr’s inequality for harmonic mappings. For the proof of the new version of Bohr’s inequality for harmonic mappings, namely, Theorem 2.5 we need the following lemma.

Lemma 3.2. Suppose that \( f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n \) is harmonic such that \( |g'(z)| \leq k|h'(z)| \) in \( \mathbb{D} \) and for some \( k \in [0, 1] \), where \( |h(z)| \leq 1 \) in \( \mathbb{D} \). Then

\[
\sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leq (1 + k) r \frac{1 - |a_0|^2}{1 - r |a_0|} \quad \text{for all} \quad r \leq \frac{1}{3}.
\]

Proof. It suffices to assume that \( |h(z)| < 1 \) in \( \mathbb{D} \), and thus, hypotheses imply that the function \( h \) is subordinate to \( \varphi \), where

\[
\varphi(z) = \frac{z + a_0}{1 + a_0 z} = a_0 + \sum_{k=1}^{\infty} \varphi_k z^k, \quad \varphi_k = (-1)^{k-1} (1 - |a_0|^2) a_0^{k-1},
\]

and it is easy to see that

\[
\sum_{k=1}^{\infty} |\varphi_k| r^k = r \frac{1 - |a_0|^2}{1 - r |a_0|}.
\]

Because \( h(z) \prec \varphi(z) \), by using Corollary 2.2 and the last fact, we deduce that

\[
(3.9) \quad \sum_{n=1}^{\infty} |a_n| r^n \leq r \frac{1 - |a_0|^2}{1 - r |a_0|} = 1 - \left[ \frac{1 - r |a_0| - r (1 - |a_0|^2)}{1 - r |a_0|} \right] \quad \text{for all} \quad r \leq \frac{1}{3}.
\]

Next, by Corollary 2.3 it follows from the condition \( |g'(z)| \leq k|h'(z)| \) that

\[
\sum_{n=1}^{\infty} n |b_n| r^{n-1} \leq k \sum_{n=1}^{\infty} n |a_n| r^{n-1}.
\]

Integrating this inequality we obtain

\[
\sum_{n=1}^{\infty} |b_n| r^n \leq k \sum_{n=1}^{\infty} |a_n| r^n
\]

and as a consequence of it, we have

\[
\sum_{n=1}^{\infty} |a_n| r^n + \sum_{n=1}^{\infty} |b_n| r^n \leq (1 + k) \sum_{n=1}^{\infty} |a_n| r^n \leq (1 + k) r \frac{1 - |a_0|^2}{1 - r |a_0|},
\]

for all \( 0 \leq r \leq 1/3 \), where the last inequality is a consequence of (3.9). □

3.5. Proof of Theorem 2.5. The proof easily follows from Lemma 3.2. □
3.6. **Proof of Theorem 2.7.** By assumption \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) is analytic in \( \mathbb{D} \) and \( |f(z)| < 1 \) in \( \mathbb{D} \). Since \( f(0) = a_0 \), by assumption, the Schwarz-Pick lemma (often referred as Lindelöf’s inequality) applied to the function \( f \) shows that

\[
|f(z)| \leq \frac{r + a}{1 + ar} \quad \text{for } |z| = r,
\]

where \( a = |a_0| \). By using Corollary 2.6 we can write

\[
\sum_{k=1}^{\infty} |a_k r^k| \leq \frac{(1 - a^2)r}{1 - ar} \quad \text{for all } r \leq \frac{1}{3}.
\]

Combining the last two inequalities, we have

\[
|f(z)| + \sum_{k=1}^{\infty} |a_k r^k| \leq \frac{r + a}{1 + ar} + \frac{(1 - a^2)r}{1 - ar} = \frac{2(1 - a^2)r}{1 - a^2r^2} + a \quad \text{for all } r \leq \frac{1}{3},
\]

which is less than or equal to 1 if

\[
r^2 a^2 + 2ra + 2r - 1 \leq 0.
\]

Solving this inequality, we obtain that

\[
r \leq r_a = \frac{\sqrt{(1 + a)^2 + a^2} - (1 + a)}{a^2} = \frac{1}{\sqrt{(1 + a)^2 + a^2} + 1 + a}.
\]

We are restricted by the inequality \( r(a) \leq 1/3 \), which gives the condition \( a \geq 2\sqrt{3} - 3 \). This means that for \( a \geq 2\sqrt{3} - 3 \) and \( r \leq r_a \), the desired inequality, namely (2.4), holds. The first part of the theorem is proved.

To show the sharpness of the radius \( r_a \), we let \( a = |a_0| \in [0, 1) \) and consider the function

\[
f(z) = \frac{a_0 - z}{1 - a_0 z} = a_0 - (1 - |a_0|^2) \sum_{k=1}^{\infty} (a_0 z)^{k-1} z^k, \quad z \in \mathbb{D}.
\]

For this function, we observe that for \( z = -r \) and \( a_0 \geq 0 \)

\[
|f(z)| + \sum_{k=1}^{\infty} |a_k r^k| = \frac{r + a}{1 + ar} + \frac{r - a^2}{1 - ar}
\]

which shows the sharpness of \( r_a \). This completes the proof of the theorem. \( \square \)

3.7. **Proof of Theorem 2.9.** We follow the method of proof of Theorem 2.7. Accordingly, the hypotheses imply that

\[
|h(z)| \leq \frac{r + a}{1 + ar}, \quad |z| = r,
\]

where \( a = |a_0|, h(0) = a_0. \) The last inequality and Lemma 3.2 yield that

\[
|h(z)| + \sum_{n=1}^{\infty} |a_n r^n| + \sum_{n=1}^{\infty} |b_n r^n| \leq \frac{r + a}{1 + ar} + (1 + k) r \frac{1 - a^2}{1 - ra} \quad \text{for all } r \leq \frac{1}{3}.
\]

By making the right hand side less than or equal to 1, we get

\[
(1 + k) r \frac{1 - a^2}{1 - ra} \leq 1 - \frac{r + a}{1 + ar} = \frac{(1 - a)(1 - r)}{1 + ar},
\]

which upon simplification gives

\[
a(a + k + ka)r^2 + (k + 2)(a + 1)r - 1 \leq 0;
\]
or equivalently,
\begin{equation}
(3.13) \quad r^2(k+1)a^2 + r(kr + k + 2)a + r(k+2) - 1 \leq 0.
\end{equation}

Solving the inequality (3.13), we get that
\[
r \leq r_{a,k} = \frac{B_{a,k} - (a+1)(k+2)}{2a^2(k+1) + 2ak}
\]
where
\[
B_{a,k} = \sqrt{a^2(k^2 + 8k + 8) + 2a(k^2 + 6k + 4) + (k + 2)^2}.
\]

We have to consider those values of \(a\) for which the inequality \(r \leq 1/3\) holds. A little algebra shows that the inequality \(r \leq 1/3\) holds for \(a \geq \alpha_k\) and hence in this case for \(r \leq r_{a,k}\) the desired inequality (2.5) holds. Here \(\alpha_k\) is as in the statement of Theorem 2.9.

To show the sharpness of the radius \(r_{a,k}\), we consider the function
\[
f(z) = h(z) + \lambda h'(z), \quad h(z) = \frac{z + a_0}{1 + a_0z},
\]
with \(\lambda \to 1\). For this function, we get that (for \(z = r\) and \(a_0 \geq 0\))
\[
|h(z)| + \sum_{n=1}^{\infty} |a_n|r^n + \sum_{n=1}^{\infty} |b_n|r^n = \frac{r + a}{1 + ar} + (\lambda + 1)r \frac{1 - a^2}{1 - ra}
\]
and the last expression shows the sharpness of \(r_{a,k}\). This completes the proof of the theorem. \(\square\)

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