NODAL DEFICIENCY OF RANDOM SPHERICAL HARMONICS IN PRESENCE OF BOUNDARY

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Dedicated to the memory of Jean Bourgain

ABSTRACT. We consider a random Gaussian model of Laplace eigenfunctions on the hemisphere satisfying the Dirichlet boundary conditions along the equator. For this model we find a precise asymptotic law for the corresponding zero density functions, in both short range (around the boundary) and long range (far away from the boundary) regimes. As a corollary, we were able to find a logarithmic negative bias for the total nodal length of this ensemble relatively to the rotation invariant model of random spherical harmonics.

Jean Bourgain’s research, and his enthusiastic approach to the nodal geometry of Laplace eigenfunctions, has made a crucial impact in the field and the current trends within. His works on the spectral correlations \cite{20, Theorem 2.2} and joint with Bombieri \cite{6} have opened a door for an active ongoing research on the nodal length of functions defined on surfaces of arithmetic flavour, like the torus or the square. Further, Bourgain’s work \cite{7} on toral Laplace eigenfunctions, also appealing to spectral correlations, allowed for inferring deterministic results from their random Gaussian counterparts.

1. INTRODUCTION

1.1. Nodal length of Laplace eigenfunctions. The nodal line of a smooth function \( f : \mathcal{M} \to \mathbb{R} \), defined on a smooth compact surface \( \mathcal{M} \), with or without a boundary, is its zero set \( f^{-1}(0) \). If \( f \) is non-singular, i.e. \( f \) has no critical zeros, then its nodal line is a smooth curve with no self-intersections. An important descriptor of \( f \) is its nodal length, i.e. the length of \( f^{-1}(0) \), receiving much attention in the last couple of decades, in particular, concerning the nodal length of the eigenfunctions of the Laplacian \( \Delta \) on \( \mathcal{M} \), in the high energy limit.

Let \((\phi_j, \lambda_j)_{j \geq 1}\) be the Laplace eigenfunctions on \( \mathcal{M} \), with energies \( \lambda_j \) in increasing order counted with multiplicity, i.e.

\[
\Delta \phi_j + \lambda_j \phi_j = 0,
\]

endowed with the Dirichlet boundary conditions \( \phi|_{\partial \mathcal{M}} \equiv 0 \) in presence of nontrivial boundary. In this context Yau’s conjecture asserts that the nodal length \( \mathcal{L}(\phi_j) \) of \( \phi_j \) is commensurable with \( \sqrt{\lambda_j} \), in the sense that

\[
c_M \cdot \sqrt{\lambda_j} \leq \mathcal{L}(\phi_j) \leq C_M \cdot \sqrt{\lambda_j},
\]

with some constants \( C_M > c_M > 0 \). Yau’s conjecture was resolved for \( \mathcal{M} \) analytic \cite{9, 10, 14}, and, more recently, a lower bound \cite{23} and a polynomial upper bound \cite{22, 24} were asserted in full generality (i.e., for \( \mathcal{M} \) smooth).

1.2. (Boundary-adapted) random wave model. In his highly influential work \cite{4} Berry proposed to compare the high-energy Laplace eigenfunctions on generic chaotic surfaces and their nodal lines to random monochromatic waves and their nodal lines respectively. The random monochromatic waves...
(also called Berry’s “Random Wave Model” or RWM) is a centred isotropic Gaussian random field 
\(u : \mathbb{R}^2 \to \mathbb{R}\) prescribed uniquely by the covariance function
\[
\mathbb{E}[u(x) \cdot u(y)] = J_0(\|x - y\|),
\]
with \(x, y \in \mathbb{R}^2\) and \(J_0(\cdot)\) the Bessel \(J\) function.

Let
\[
K^u_n(x) = \phi_{u(x)}(0) \cdot \mathbb{E}[\|\nabla u(x)\| | u(x) = 0]
\]
be the zero density, also called the “first intensity” function of \(u\), with \(\phi_{u(x)}\) the probability density function of the random variable \(u(x)\). In this isotropic case, it is easy to directly evaluate
\[
K^u_1(x) = \frac{1}{2\sqrt{2}},
\]
and then appeal to the Kac-Rice formula, valid under the easily verified non-degeneracy conditions on the random field \(u\), to evaluate the expected nodal length \(\mathcal{L}(u; R)\) of \(u(\cdot)\) restricted to a radius-\(R\) disc \(\mathcal{B}(R) \subseteq \mathbb{R}^2\) to be precisely
\[
\mathbb{E}[\mathcal{L}(u; R)] = \int_{\mathcal{B}(R)} K^u_n(x)dx = \frac{1}{2\sqrt{2}} \cdot \text{Area}(\mathcal{B}(R)).
\]

Berry [5] found that, as \(R \to \infty\), the variance \(\text{Var}(\mathcal{L}(u; R))\) satisfies the asymptotic law
\[
\text{Var}(\mathcal{L}(u; R)) = \frac{1}{256} \cdot R^2 \log R + O(R^2),
\]
much smaller than the a priori heuristic prediction \(\text{Var}(\mathcal{L}(u; R)) \approx R^3\) made based on the natural scaling of the problem, due to what is now known as “Berry’s cancellation” [35] of the leading non-oscillatory term of the 2-point correlation function (also known as the “second zero intensity”).

Further, in the same work [5], Berry studied the effect induced on the nodal length of eigenfunctions satisfying the Dirichlet condition on a nontrivial boundary, both in its vicinity and far away from it. With the (infinite) horizontal axis \(\{(x_1, x_2) : x_2 = 0\} \subseteq \mathbb{R}^2\) serving as a model for the boundary, he introduced a Gaussian random field \(v(x_1, x_2) : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}\) of boundary-adapted (non-stationary) monochromatic random waves, forced to vanish at \(x_2 = 0\). Formally, \(v(x_1, x_2)\) is the limit, as \(J \to \infty\), of the superposition
\[
\frac{2}{\sqrt{J}} \sum_{j=1}^{J} \sin(x_2 \sin(\theta_j)) \cdot \cos(x_1 \cos(\theta_j) + \phi_j)
\]
of \(J\) plane waves of wavenumber \(1\) forced to vanish at \(x_2 = 0\). Alternatively, \(v\) is the centred Gaussian random field prescribed by the covariance function
\[
r_v(x, y) := \mathbb{E}[v(x) \cdot v(y)] = J_0(\|x - y\|) - J_0(\|x - \tilde{y}\|),
\]
x = \((x_1, x_2)\), y = \((y_1, y_2)\), and \(\tilde{y} = (y_1, -y_2)\) is the mirror symmetry of \(y\); the law of \(v\) is invariant w.r.t. horizontal shifts
\[
v(\cdot, \cdot) \mapsto v(a + \cdot, \cdot),
\]
a \(\in \mathbb{R}\), but not the vertical shifts.

By comparing (1.2) to (1.7), we observe that, far away from the boundary (i.e. \(x_2, y_2 \to \infty\)), \(r_v(x, y) \approx J_0(\|x - y\|)\), so that, in that range, the (covariance of) boundary-adapted waves converge to the (covariance of) isotropic ones (1.2), though the decay of the error term in this approximation is slow and of oscillatory nature. Intuitively, it means that, at infinity, the boundary has a small impact on the random waves, though it takes its toll on the nodal bias, as it was demonstrated by Berry, as follows.
Let \( K_1^\lambda(x) = K_1^\lambda(x_2) \) be the zero density of \( v \), defined analogously to (1.3), depending on the height \( x_2 \) only, independent of \( x_1 \) by the inherent invariance (1.8). Berry showed\(^1\) that, as \( x_2 \to 0 \),

\[
(1.9) \quad K_1^\lambda(x_2) \to \frac{1}{2\pi},
\]

and attributed this “nodal deficiency” \( \frac{1}{2\pi} < \frac{1}{2\sqrt{2}} \), relatively to (1.4), to the a.s. orthogonality of the nodal lines touching the boundary [13, Theorem 2.5].

Further, as \( x_2 \to \infty \),

\[
(1.10) \quad K_1^\lambda(x_2) = \frac{1}{2\sqrt{2}} \cdot \left( 1 + \frac{\cos(2x_2 - \pi/4)}{\sqrt{\pi x_2}} - \frac{1}{32\pi x_2} + E(x_2) \right),
\]

with some prescribed error term\(^2\) \( E(\cdot) \). In this situation a natural choice for expanding domains are the rectangles \( \mathcal{D}_R := [-1, 1] \times [0, R] \), \( R \to \infty \) (say). As an application of the Kac-Rice formula (1.5) in this case, it easily follows that

\[
(1.11) \quad \mathbb{E}[\mathcal{L}(v; \mathcal{D}_R)] = \frac{1}{2\sqrt{2}} \cdot \text{Area}(\mathcal{D}_R) - \frac{1}{32\sqrt{2}\pi} \log R + O(1)
\]

e.g., a logarithmic “nodal deficiency” relatively to (1.5), impacted by the boundary infinitely many wave lengths away from it. The logarithmic fluctuations (1.6) in the isotropic case \( u \), possibly also holding for \( v \), give rise to a hope to be able to detect the said, also logarithmic, negative boundary impact (1.11) via a single sample of the nodal length, or, at least, very few ones.

1.3. Random spherical harmonics. The (unit) sphere \( \mathcal{M} = S^2 \) is one of but few surfaces, where the solutions to the Helmholtz equation (1.1) admit an explicit solution. For a number \( \ell \in \mathbb{Z}_{\geq 0} \), the space of solutions of (1.1) with \( \lambda = \ell(\ell + 1) \) is the \((2\ell + 1)\)-dimensional space of degree-\( \ell \) spherical harmonics, and conversely, all solutions to (1.1) are spherical harmonics of some degree \( \ell \geq 0 \). Given \( \ell \geq 0 \), let \( \mathcal{E}_\ell := \{ \eta_{\ell,1}, \ldots, \eta_{\ell,2\ell+1} \} \) be any \( L^2 \)-orthonormal basis of the space of spherical harmonics of degree \( \ell \). The random field

\[
(1.12) \quad \tilde{T}_\ell(x) = \sqrt{\frac{4\pi}{2\ell + 1}} \sum_{k=1}^{2\ell + 1} a_k \cdot \eta_{\ell,k}(x),
\]

with \( a_k \) i.i.d. standard Gaussian random variables, is the degree-\( \ell \) random spherical harmonics. The law of \( \tilde{T}_\ell \) is invariant w.r.t. the chosen orthonormal basis \( \mathcal{E}_\ell \), uniquely defined via the covariance function

\[
(1.13) \quad \mathbb{E}[\tilde{T}_\ell(x) \cdot \tilde{T}_\ell(y)] = P_\ell(\cos d(x, y)),
\]

with \( P_\ell(\cdot) \) the Legendre polynomial of degree \( \ell \), and \( d(\cdot, \cdot) \) is the spherical distance between \( x, y \in S^2 \). The random fields \( \{ \tilde{T}_\ell \} \) are the Fourier components of the \( L^2 \)-expansion of every isotropic random field [25], of interest, for instance, in cosmology and the study of Cosmic Microwave Background radiation (CMB).

Let \( \mathcal{L}(\tilde{T}_\ell) \) be the total nodal length of \( \tilde{T}_\ell \), of high interest for various pure and applied disciplines, including the above. Berard [3] evaluated the expected nodal length to be precisely

\[
(1.14) \quad \mathbb{E}[\mathcal{L}(\tilde{T}_\ell)] = \sqrt{2\pi} \cdot \sqrt{\ell(\ell + 1)},
\]

and, as \( \ell \to \infty \) its variance is asymptotic [35] to

\[
(1.15) \quad \text{Var}(\mathcal{L}(\tilde{T}_\ell)) \sim \frac{1}{32} \log \ell,
\]

\(^1\) Though a significant proportion of the details of the computation were omitted, we validated Berry’s assertions for ourselves.

\(^2\) Here \( E(x_2) \) is of order \( \frac{1}{16\pi x_2^2} \), so not smaller by magnitude than \( \frac{1}{32\pi x_2^2} \), but of oscillatory nature, and will not contribute to the Kac-Rice integral along expanding domains, as neither the term \( \frac{\cos(2x_2 - \pi/4)}{\sqrt{\pi x_2}} \).
in accordance with Berry’s (1.6), save for the scaling, and the invariance of the nodal lines w.r.t. the symmetry \( x \mapsto -x \) of the sphere, resulting in a doubled leading constant in (1.15) relatively to (1.6) suitably scaled. A more recent proof [26] of the Central Limit Theorem for \( \mathcal{L}(\bar{T}_t) \), asserting the asymptotic Gaussianity of

\[
\mathcal{L}(\bar{T}_t) - \mathbb{E}[\mathcal{L}(\bar{T}_t)] \sqrt{\frac{1}{2\ell} \log \ell},
\]

is sufficiently robust to also yield the Central Limit Theorem, as \( R \to \infty \) for the nodal length \( \mathcal{L}(u; R) \) of Berry’s random waves, as it was recently demonstrated [32], also claimed by [28].

1.4. Principal results: nodal bias for the hemisphere, at the boundary, and far away. Our principal results concern the hemisphere \( \mathcal{H}^2 \subseteq S^2 \), endowed with the Dirichlet boundary conditions along the equator. We will widely use the spherical coordinates

\[
\mathcal{H}^2 = \left\{ (\theta, \phi) : \theta \in [0, \pi/2], \phi \in [0, 2\pi] \right\},
\]

with the equator identified with \( \{ \theta = \pi/2 \} \subseteq \mathcal{H}^2 \). Here all the Laplace eigenfunctions are necessarily spherical harmonics restricted to \( \mathcal{H}^2 \), subject to some extra properties. Recall that a concrete (complex-valued) orthonormal basis of degree \( \ell \) are the Laplace spherical harmonics \( \{Y_{\ell,m}\}_{m=-\ell}^{\ell} \), given in the spherical coordinates by

\[
Y_{\ell,m}(\theta, \phi) = e^{im\phi} \cdot P_{\ell}^m(\cos \theta),
\]

with \( P_{\ell}^m(\cdot) \) the associated Legendre polynomials of degree \( \ell \) on order \( m \). For \( \ell \geq 0 \), \( |m| \leq \ell \) the spherical harmonic \( Y_{\ell,m} \) obeys the Dirichlet boundary condition on the equator, if and only if \( m \not\equiv \ell \mod 2 \), spanning a subspace of dimension \( \ell \) inside the \((2\ell + 1)\)-dimensional space of spherical harmonics of degree \( \ell \) [17, Example 4]. (Its \((\ell + 1)\)-dimensional orthogonal complement is the subspace satisfying the Neumann boundary condition.) Conversely, every Laplace eigenfunction on \( \mathcal{H}^2 \) is necessarily a spherical harmonic of some degree \( \ell \geq 0 \) that is a linear combination of \( Y_{\ell,m} \) with \( m \not\equiv \ell \mod 2 \).

The principal results of this paper concern the following model of boundary-adapted random spherical harmonics

\[
T_{\ell}(x) = \sqrt{\frac{8\pi}{2\ell + 1}} \sum_{m=-\ell \atop m \not\equiv \ell \mod 2} a_{\ell,m} Y_{\ell,m}(x),
\]

where the \( a_{\ell,m} \) are the standard (complex-valued) Gaussian random variables subject to the constraint \( a_{\ell,-m} = \overline{a_{\ell,m}} \), so that \( T_{\ell}(\cdot) \) is real-valued. Our immediate concern is for the law of \( T_{\ell} \), which, as for any centred Gaussian random field, is uniquely determined by its covariance function, claimed by the following proposition.

**Proposition 1.1.** The covariance function of \( T_{\ell} \) as in (1.16) is given by

\[
r_{\ell}(x, y) := \mathbb{E}[T_{\ell}(x) \cdot T_{\ell}(y)] = P_{\ell}(\cos d(x, y)) - P_{\ell}(\cos d(x, \overline{y})),
\]

where \( \overline{y} \) is the mirror symmetry of \( y \) around the equator, i.e. \( y = (\theta, \phi) \mapsto \overline{y} = (\pi - \theta, \phi) \) in the spherical coordinates.

It is evident, either from the definition or the covariance, that the law of \( T_{\ell} \) is invariant w.r.t. rotations of \( \mathcal{H}^2 \) around the axis orthogonal to the equator, that is, in the spherical coordinates,

\[
T_{\ell}(\theta, \phi) \mapsto T_{\ell}(\theta, \phi + \phi_0), \quad \phi \in [0, 2\pi).
\]

The boundary impact of (1.17) relatively to (1.13) is in perfect harmony with the boundary impact of the covariance (1.7) of Berry’s boundary-adapted model relatively to the isotropic case (1.2), except that the mirror symmetry \( y \mapsto \overline{y} \) relatively to the \( x \) axis in the Euclidean situation is substituted by mirror symmetry \( y \mapsto \overline{y} \) relatively to the equator for the spherical geometry. These generalize to 2 dimensions the boundary impact on the ensemble of stationary random trigonometric
polynomials on the circle [3, 1] resulting in the ensemble of non-stationary random trigonometric polynomials vanishing at the endpoints [15, 1].

Let 
\[
K_{1, \ell}(x) = \frac{1}{\sqrt{2\pi} \cdot \sqrt{\text{Var}(T_{\ell}(x))}} \mathbb{E} \left[ \| \nabla T_{\ell}(x) \| : T_{\ell}(x) = 0 \right],
\]
be the zero density of \(T_{\ell}\), that, unlike the rotation invariant the spherical harmonics (1.12), genuinely depends on \(x \in \mathcal{H}\). More precisely, by the said invariance w.r.t. (1.18), the zero density \(K_{1, \ell}(x)\) depends on the polar angle \(\theta\) only. We rescale by introducing the variable
\[
\psi = \ell(\pi - 2\theta),
\]
and, with a slight abuse of notation, write
\[
K_{1, \ell}(\psi) = K_{1, \ell}(x).
\]

Our principal result deals with the asymptotics of \(K_{1, \ell}(\cdot)\), in two different regimes, in line with (1.9) and (1.10) respectively.

**Theorem 1.2.**

(1) For \(C > 0\) sufficiently large, as \(\ell \to \infty\), one has
\[
K_{1, \ell}(\psi) = \frac{\sqrt{\ell(\ell + 1)}}{2\sqrt{2}} \left[ 1 + \sqrt{\frac{2}{\pi}} \frac{1}{\psi} \cos \left\{ (\ell + 1/2)\psi/\ell - \pi/4 \right\} - \frac{1}{16\pi\psi} \right.
\]
\[
+ \frac{15}{16\pi\psi} \cos \left\{ (\ell + 1/2)2\psi/\ell - \pi/2 \right\} + O(\psi^{-3/2}\ell^{-2})
\]
uniformly for \(C < \psi < \pi\ell\), with the constant involved in the ‘\(O\)’-notation absolute.

(2) For \(\ell \geq 1\) one has the uniform asymptotics
\[
K_{1, \ell}(\psi) = \frac{\ell}{2\pi} \left[ 1 + O(\ell^{-1}) + O(\psi^2) \right],
\]
with the constant involved in the ‘\(O\)’-notation absolute.

Clearly, the statement (1.22) is asymptotic for \(\psi\) small only, otherwise yielding the mere bound \(K_{1, \ell}(\psi) = O(\ell)\), which is easy. As a corollary to Theorem 1.2 one may evaluate the asymptotic law of the total expected nodal length of \(T_{\ell}\), and detect the negative logarithmic bias relatively to (1.14), in full accordance with Berry’s (1.11).

**Corollary 1.3.** As \(\ell \to \infty\), the expected nodal length has the following asymptotics:
\[
\mathbb{E}[L(T_{\ell})] = 2\pi \frac{\sqrt{\ell(\ell + 1)}}{2\sqrt{2}} - \frac{1}{32\sqrt{2}} \log(\ell) + O(1).
\]

**Acknowledgements.** We are grateful to Ze’ev Rudnick for raising the question addressed within this manuscript. V.C. has received funding from the Istituto Nazionale di Alta Matematica (INdAM) through the GNAMPA Research Project 2020 “Geometria stocastica e campi aleatori”. D.M. is supported by the MIUR Departments of Excellence Program Math@Tov.

2. **Discussion**

2.1. **Toral eigenfunctions and spectral correlations.** Another surface admitting explicit solutions to the Helmholtz equation (1.1) is the standard torus \(T^2 = \mathbb{R}^2 / \mathbb{Z}^2\). Here the Laplace eigenfunctions with eigenvalue \(4\pi^2 n\) all correspond to an integer \(n\) expressible as a sum of two squares, and are given by a sum
\[
f_n(x) = \sum_{||\mu||^2 = n} a_\mu e(\langle \mu, x \rangle)
\]
over all lattice points \( \mu = (\mu_1, \mu_2) \in \mathbb{Z}^2 \) lying on the radius-\( \sqrt{n} \) centred circle, \( n \) is a sum of two squares with \( e(y) := e^{2\pi iy} \), \( \langle \mu, x \rangle = \mu_1 x_1 + \mu_2 x_2, x = (x_1, x_2) \in \mathbb{T}^2 \). Following \([29]\), one endows the eigenspace of \( \{f_n\} \) with a Gaussian probability measure with the coefficients \( a_\mu \) standard (complex-valued) i.i.d. Gaussian, save for \( a_{-\mu} = \overline{a_\mu} \), resulting in the ensemble of “arithmetic random waves”.

The expected nodal length of \( f_n \) was computed \([30]\) to be
\[
\mathbb{E}[\mathcal{L}(f_n)] = \sqrt{2\pi^2 \cdot \sqrt{n}},
\]
and the useful upper bound
\[
\text{Var}(\mathcal{L}(f_n)) \ll \frac{n}{\sqrt{r_2(n)}}
\]
was also asserted, with \( r_2(n) \) the number of lattice points lying on the radius-\( \sqrt{n} \) circle, or, equivalently, the dimension of the eigenspace \( \{f_n\} \) as in \((2.1)\). A precise asymptotic law for \( \text{Var}(\mathcal{L}(f_n)) \) was subsequently established \([20]\), shown to fluctuate, depending on the angular distribution of the lattice points. A non-central non-universal limit theorem was asserted \([27]\), also depending on the angular distribution of the lattice points.

An instrumental key input to both the said asymptotic variance and the limit law was Bourgain’s first nontrivial upper bound \([20\) Theorem 2.2\] of \( \alpha_{r_2(n) \to \infty} (r_2(n))^4 \) for the number of length-6 spectral correlations, i.e. 6-tuples of lattice points \( \{\mu : \|\mu\|^2 = n\} \) summing up to 0. Bourgain’s bound was subsequently improved and generalized to higher order correlations \([6]\), in various degrees of generality, conditionally or unconditionally. These results are still actively used within the subsequent and ongoing research, in particular, \([7]\) and its followers.

2.2. **Boundary impact.** It makes sense to compare the torus to the square with Dirichlet boundary, and test what kind of impact it would have relatively to \((2.2)\) on the expected nodal length, as the “boundary-adapted arithmetic random waves”, that were addressed in \([11]\). It was concluded, building on Bourgain-Bombieri’s \([6]\), and by appealing to a different notion of spectral correlation, namely, the spectral semi-correlations, that, even at the level of expectation, the total nodal bias is fluctuating from nodal deficiency (negative bias) to nodal surplus (positive bias), depending on the angular distribution of the lattice points and its interaction with the direction of the square boundary, at least, for generic energy levels. A similar experiment conducted by Gnutzmann-Lois for cuboids of arbitrary dimensions, averaging for eigenfunctions admitting separation of variables belonging to different eigenspaces, revealed consistency with Berry’s nodal deficiency ansatz stemming from \([1,11]\).

It would be useful to test whether different Gaussian random fields on the square would result in different limiting nodal bias around the boundary corresponding to \((1.22)\), that is likely to bring in a different notion of spectral correlation, not unlikely “quasi-semi-correlation” \([8\) \([18]\). Another question of interest is “de-randomize” any of these results, i.e. infer the corresponding results on deterministic eigenfunctions following Bourgain \([7]\). We leave all of these to be addressed elsewhere.

3. **Joint distribution of \((f_n(x), \nabla f_n(x))\)**

In the analysis of \(K_{1,\ell}(x)\) we naturally encounter the distribution of \(T_\ell(x)\), determined by
\[
\text{Var}(T_\ell(x)) = 1 - P_\ell(\cos d(x, \bar{x}));
\]
and the distribution of \(\nabla T_\ell(x)\) conditioned on \(T_\ell(x) = 0\), determined by its \(2 \times 2\) covariance matrix
\[
\Omega_\ell(x) = \mathbb{E}[\nabla T_\ell(x) \cdot \nabla^\top T_\ell(x)|T_\ell(x) = 0].
\]
Let \(x\) correspond to the spherical coordinates \((\theta, \phi)\). An explicit computation shows that the covariance matrix \(\Omega_\ell(x)\) depends only on \(\theta\), and below we will often abuse notation to write \(\Omega_\ell(\theta)\) instead, and also, when convenient, \(\Omega_\ell(\psi)\) with \(\psi\) as in \((1.20)\). A direct computation shows that:

**Lemma 3.1.** The \(2 \times 2\) covariance matrix of \(\nabla T_\ell(x)\) conditioned on \(T_\ell(x) = 0\) is the following real symmetric matrix
\[
\Omega_\ell(x) = \frac{\ell(\ell + 1)}{2} \left[ I_2 + S_\ell(x) \right],
\]
where
\[
S_\ell(x) = \begin{pmatrix} S_{11,\ell}(x) & 0 \\ 0 & S_{22,\ell}(x) \end{pmatrix},
\]
and for \( x = (\theta, \phi) \)
\[
S_{11,\ell}(x) = -\frac{2}{\ell(\ell+1)} \left[ \cos(2\theta) P''_\ell(\cos(\pi - 2\theta)) + \sin^2(2\theta) P'_\ell(\cos(\pi - 2\theta)) \right.
\]
\[
+ \left. \frac{1}{1 - P(\cos(\pi - 2\theta))} \sin^2(2\theta) [P'_\ell(\cos(\pi - 2\theta))]^2 \right],
\]
\[
S_{22,\ell}(x) = -\frac{2}{\ell(\ell+1)} P'_\ell(\cos(\pi - 2\theta)).
\]

In the next two sections we prove Lemma 3.1 that is, we evaluate the \(2 \times 2\) covariance matrix of \( \nabla T_\ell(x) \) conditioned upon \( T_\ell(x) = 0 \). First, in section 3.1 we evaluate the unconditional \(3 \times 3\) covariance matrix \( \Sigma_\ell(x) \) of \( (T_\ell(x), \nabla T_\ell(x)) \) and then, in section 3.2 we apply the standard procedure for conditioning multivariate Gaussian random variables.

3.1. The unconditional covariance matrix. The covariance matrix of
\[
(T_\ell(x), \nabla T_\ell(x)),
\]
which could be expressed as
\[
\Sigma_\ell(x) = \begin{pmatrix} A_\ell(x) & B_\ell(x) \\ B_\ell(x)^T & C_\ell(x) \end{pmatrix},
\]
where
\[
A_\ell(x) = \text{Var}(T_\ell(x)),
\]
\[
B_\ell(x) = \mathbb{E}[T_\ell(x) \cdot \nabla_y T(y)]_{x=y},
\]
\[
C_\ell(x) = \mathbb{E}[\nabla_x T_\ell(x) \otimes \nabla_y T_\ell(y)]_{x=y}.
\]
The \(1 \times 2\) matrix \( B_\ell(x) \) is
\[
B_\ell(x) = \begin{pmatrix} B_{\ell,1}(x) & B_{\ell,2}(x) \end{pmatrix},
\]
where \( B_\ell(x) \) depends only on \( \theta \), and by an abuse of notation we write
\[
B_{\ell,1}(x) = \left. \frac{\partial}{\partial \theta_y} r_\ell(x, y) \right|_{x=y} = -\sin(2\theta) \cdot P'_\ell(\cos(\pi - 2\theta)),
\]
\[
B_{\ell,2}(x) = \left. \frac{1}{\sin \theta_y} \frac{\partial}{\partial \phi_y} r_\ell(x, y) \right|_{x=y} = 0.
\]
The entries of the \(2 \times 2\) matrix \( C_\ell(x) \) are
\[
C_\ell(x) = \begin{pmatrix} C_{\ell,11}(x) & C_{\ell,12}(x) \\ C_{\ell,21}(x) & C_{\ell,22}(x) \end{pmatrix},
\]
where again recalling that \( x = (\theta, \phi) \) we write
\[
C_{\ell,11}(x) = \left. \frac{\partial}{\partial \theta_x} \frac{\partial}{\partial \theta_y} r_\ell(x, y) \right|_{x=y} = P''_\ell(1) - \cos(2\theta) \cdot P'_\ell(\cos(\pi - 2\theta)) - \sin^2(2\theta) \cdot P''_\ell(\cos(\pi - 2\theta)),
\]
\[
C_{\ell,12}(x) = C_{\ell,21}(x) = \left. \frac{1}{\sin \theta_y} \frac{\partial}{\partial \phi_y} \frac{\partial}{\partial \theta_x} r_\ell(x, y) \right|_{x=y} = 0,
\]
\[
C_{\ell,22}(x) = \left. \frac{1}{\sin \theta_y} \frac{1}{\sin \theta_x} \frac{\partial}{\partial \phi_y} \frac{\partial}{\partial \theta_x} r_\ell(x, y) \right|_{x=y} = P'_\ell(1) - P'_\ell(\cos(\pi - 2\theta)).
\]
3.2. Conditional covariance matrix. The conditional covariance matrix of the Gaussian vector $(\nabla T_\ell(x)|T_\ell(x) = 0)$ is given by the standard Gaussian transition formula:

$$\Omega_\ell(x) = C_\ell(x) - \frac{1}{\text{Var}(T_\ell(x))}B'_\ell(x)B_\ell(x).$$

Again taking $x = (\theta, \phi)$ and observing that

$$\frac{B'_\ell(x)B_\ell(x)}{\text{Var}(T_\ell(x))} = \frac{1}{1 - P_\ell(\cos(\pi - 2\theta))} \left( \begin{array}{cc} \sin^2(2\theta) \cdot [P_\ell'(\cos(\pi - 2\theta))]^2 & 0 \\ 0 & 0 \end{array} \right),$$

and

$$P_\ell'(1) = \frac{\ell(\ell + 1)}{2},$$

we have

$$\Omega_\ell(x) = \frac{\ell(\ell + 1)}{2} I_2 - \left( \begin{array}{cc} \cos(2\theta) \cdot P_\ell'(\cos(\pi - 2\theta)) + \sin^2(2\theta) \cdot P_\ell''(\cos(\pi - 2\theta)) & 0 \\ 0 & P_\ell'(\cos(\pi - 2\theta)) \end{array} \right)$$

$$- \frac{1}{1 - P_\ell(\cos(\pi - 2\theta))} \left( \begin{array}{cc} \sin^2(2\theta) \cdot [P_\ell'(\cos(\pi - 2\theta))]^2 & 0 \\ 0 & 0 \end{array} \right),$$

that is the statement of Lemma 3.1

4. Proof of Theorem 1.21: Perturbative analysis away from the boundary

4.1. Perturbative analysis. The asymptotic analysis (1.21) is in two steps. First, we evaluate the variance $\text{Var}(T_\ell(x))$ and each entry in $S_\ell(x)$ using the high degree asymptotics of the Legendre polynomials and its derivatives (Hilb’s asymptotics). In the second step, performed within Proposition 4.3, we exploit the analyticity of the Gaussian expectation (1.19) as a function of the entries of the non-singular covariance matrix, to Taylor expand $K_{1,\ell}(x)$ where both $\text{Var}(T_\ell(x)) - 1$ and the entries of $S_\ell(x)$ are assumed to be small.

Lemma 4.1 (Hilb’s asymptotics).

$$P_\ell(\cos \varphi) = \left( \frac{\varphi}{\sin \varphi} \right)^{1/2} J_0((\ell + 1/2)\varphi) + \delta_\ell(\varphi),$$

uniformly for $0 \leq \varphi \leq \pi - \varepsilon$, where $J_0$ is the Bessel function of the first kind. For the error term we have the bounds

$$\delta_\ell(\varphi) \ll \begin{cases} \varphi^2 O(1), & 0 < \varphi \leq C/\ell, \\ \varphi^{1/2} O(\ell^{-3/2}), & C/\ell \leq \varphi \leq \pi - \varepsilon, \end{cases}$$

where $C$ is a fixed positive constant and the constants involved in the $O$-notation depend on $C$ only.

Lemma 4.2. The following asymptotic representation for the Bessel functions of the first kind holds:

$$J_0(x) = \left( \frac{2}{\pi x} \right)^{1/2} \cos(x - \pi/4) \sum_{k=0}^{\infty} (-1)^k g(2k) (2x)^{-2k}$$

$$+ \left( \frac{2}{\pi x} \right)^{1/2} \cos(x + \pi/4) \sum_{k=0}^{\infty} (-1)^k g(2k + 1) (2x)^{-2k-1},$$

where $\varepsilon > 0$, $|\arg x| \leq \pi - \varepsilon$, $g(0) = 1$ and

$$g(k) = \frac{(-1)(-1)^2 \cdots (-2k-1)^2}{2^{2k} k!} = \left( -1 \right)^k \frac{(2k)!}{2^{2k} k!}. $$
For a proof of Lemma 4.1 and Lemma 4.2 we refer to [31, Theorem 8.21.6] and [21, section 5.11] respectively.

Recall the scaled variable \( \psi \) related to \( \theta \) via (1.20), so that an application of lemmas 4.1 and 4.2 yields that, for \( \ell \geq 1 \) and \( C < \psi < \ell \pi \),

\[
\begin{align*}
P_\ell(\cos(\psi/\ell)) &= \sqrt{\frac{2}{\pi}} \frac{\ell^{-1/2}}{\sin^{1/2}(\psi/\ell)} \left[ \cos((\ell + 1/2)\psi/\ell - \pi/4) - \frac{1}{2\ell\psi/\ell} \cos((\ell + 1/2)\psi/\ell + \pi/4) \right] \\
&+ O((\psi/\ell)^{1/2} \ell^{-3/2}).
\end{align*}
\]

Observing that

\[
\frac{\ell^{-1/2}}{\sin^{1/2}(\psi/\ell)} = \frac{1}{\sqrt{\psi/\ell}} + O((\psi/\ell)^{1/2} \ell^{-3/2}),
\]

we write

\[
(4.1) \quad P_\ell(\cos(\psi/\ell)) = \sqrt{\frac{2}{\pi}} \frac{1}{\sqrt{\psi}} \cos((\ell + 1/2)\psi/\ell - \pi/4) + O(\psi^{-3/2} + O(\psi^{3/2} \ell^{-2})).
\]

A repeated application of lemmas 4.1 and 4.2 also yields an asymptotic estimate for the first couple of derivatives of the Legendre Polynomials [12, Lemma 9.3]:

\[
\begin{align*}
P'_\ell(\cos(\psi/\ell)) &= \sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{1-1/2}(\psi/\ell)} \left[ \sin((\ell + 1/2)\psi/\ell - \pi/4) - \frac{1}{8\ell\psi/\ell} \sin((\ell + 1/2)\psi/\ell + \pi/4) \right] \\
&+ O(\ell^{-1/2} \psi^{-5/2} \ell^4),
\end{align*}
\]

and

\[
\begin{align*}
P''_\ell(\cos(\psi/\ell)) &= \sqrt{\frac{2}{\pi}} \frac{\ell^{2-1/2}}{\sin^{2+1/2}(\psi/\ell)} \left[ - \cos((\ell + 1/2)\psi/\ell - \pi/4) + \frac{1}{8\ell\psi/\ell} \cos((\ell + 1/2)\psi/\ell + \pi/4) \right] \\
&- \sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{3+1/2}(\psi/\ell)} \left[ \cos((\ell - 1 + 1/2)\psi/\ell + \pi/4) + \frac{1}{8\ell\psi/\ell} \cos((\ell - 1 + 1/2)\psi/\ell - \pi/4) \right] \\
&+ O(\psi^{-7/2} \ell^4).
\end{align*}
\]

Since we have that

\[
\frac{\ell^{1-1/2}}{\sin^{1+1/2}(\psi/\ell)} = \ell^{1-1/2} \left[ \frac{1}{(\psi/\ell)^{1/2}} + O((\psi/\ell)^{1/2}) \right] = \ell^{2} \psi^{-1/2} + O(\psi^{1/2}),
\]

we have

\[
(4.2) \quad P'_\ell(\cos(\psi/\ell)) = \sqrt{\frac{2}{\pi}} \frac{\ell^{1-1/2}}{\sin^{1+1/2}(\psi/\ell)} \sin((\ell + 1/2)\psi/\ell - \pi/4) + O(\psi^{-5/2} \ell^2),
\]

and observing that

\[
\begin{align*}
\frac{\ell^{2-1/2}}{\sin^{2+1/2}(\psi/\ell)} &= \ell^{2-1/2} \left[ \frac{1}{(\psi/\ell)^{5/2}} + O((\psi/\ell)^{-1/2}) \right] = \frac{\ell^4}{\psi^{5/2}} + O(\psi^{-1/2} \ell^2) \\
\frac{\ell^{2-1/2}}{\sin^{2+1/2}(\psi/\ell)} &= O(\psi^{-7/2} \ell^4)
\end{align*}
\]
On \((\psi/\ell)^{3/2}\) we obtain

\[ \frac{\psi}{\ell^{1/2} \sin^{3+1/2}(\psi/\ell)} = \ell^{1-1/2} \left[ \frac{1}{(\psi/\ell)^{1/2}} + O((\psi/\ell)^{-3/2}) \right] = \frac{\ell^4}{\psi^{7/2}} + O(\psi^{-3/2} \ell^2) \]

we obtain

\[ P''_\ell(\cos(\psi/\ell)) = -\sqrt{\frac{2}{\pi}} \frac{\ell^{2-1/2}}{\sin^{2+1/2}(\psi/\ell)} \cos((\ell + 1/2)\psi/\ell - \pi/4) + O(\psi^{-3/2}\ell^2) + O(\psi^{-7/2}\ell^4). \]

(4.3)

The estimates in (4.1), (4.2) and (4.3), imply that for \(\ell \geq 1\) and uniformly for \(C < \psi < \ell\pi\), with \(C > 0\), we have

\[
\begin{align*}
P'_\ell(\cos(\psi/\ell)) &= \sqrt{\frac{2}{\pi}} \frac{1}{\psi} \cos((\ell + 1/2)\psi/\ell - \pi/4) + O(\psi^{-3/2} + O(\psi^{3/2}\ell^{-2}), \\
\{P'_\ell(\cos(\psi/\ell))\}_\ell^2 &= \frac{2}{\psi^2} \cos^2((\ell + 1/2)\psi/\ell - \pi/4) + O(\psi^{-2} + O(\psi^{-2}\ell^2),
\end{align*}
\]

(4.4)

With the same abuse of notation as above, we write \(S_\ell(\psi) := S_\ell(x)\) as in Lemma 3.1 and in analogous manner for its individual entries \(S_{ij,\ell}(\psi) := S_{11,\ell}(x)\). We have

\[
\begin{align*}
S_{11,\ell}(\psi) &= 2\sqrt{\frac{2}{\pi}} \frac{1}{\psi} \cos((\ell + 1/2)\psi/\ell - \pi/4) - \frac{4}{\pi} \sin^2((\ell + 1/2)\psi/\ell - \pi/4) + O(\psi^{-3/2} + O(\psi^{3/2}\ell^{-2}), \\
S_{22,\ell}(\psi) &= -2\sqrt{\frac{2}{\pi}} \frac{1}{\psi^3} \sin((\ell + 1/2)\psi/\ell - \pi/4) + O(\psi^{1/2}\ell^{-2} + O(\psi^{-5/2}).
\end{align*}
\]

(4.5) (4.6)

The next proposition prescribes a precise asymptotic expression for the density function \(K_{1,\ell}(\cdot)\) via a Taylor expansion of the relevant Gaussian expectation as a function of the associated covariance matrix entries.

**Proposition 4.3.** For \(C > 0\) sufficiently large we have the following expansion on \(C < \psi < \ell\pi\):

\[ K_{1,\ell}(\psi) = \frac{\sqrt{\ell(\ell + 1)}}{2\sqrt{2}} L_\ell(\psi) + E_\ell(\psi), \]

with the leading term

\[ L_\ell(\psi) = \frac{\sqrt{\ell(\ell + 1)}}{4\sqrt{2}} \left[ s_\ell(\psi) + \frac{1}{2} \text{tr} S_\ell(\psi) + \frac{3}{4} s_\ell^2(\psi) + \frac{1}{4} s_\ell(\psi) \text{tr} S_\ell(\psi) - \frac{1}{16} \text{tr} S_\ell^2(\psi) - \frac{1}{32} (\text{tr} S_\ell(\psi))^2 \right], \]

where \(s_\ell(\psi) = P_\ell(\cos(\psi/\ell))\), and the error term \(E_\ell(\psi)\) is bounded by

\[ |E_\ell(\psi)| = O(\ell \cdot (|s_\ell(\psi)|^3 + |S_\ell(\psi)|^3)), \]

with constant involved in the \(O\)-notation absolute.

**Proof.** To prove Proposition 4.3, we perform a precise Taylor analysis for the density function \(K_{1,\ell}(\psi)\), assuming that both \(s_\ell(\psi)\) and the entries of \(S_\ell(\psi)\) are small. We introduce the scaled covariance matrix (see (3.1))

\[ \Delta_\ell(\psi) = \frac{2}{\ell(\ell + 1)} \Omega_\ell(\psi) = I_2 + S_\ell(\psi). \]

The density function \(K_{1,\ell}(\cdot)\) could be expressed as

\[ K_{1,\ell}(\psi) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1 - s_\ell(\psi)}} 2\pi \sqrt{\det \Delta_\ell(\psi)} \frac{\sqrt{\ell(\ell + 1)}}{\sqrt{2}} \int_{\mathbb{R}^2} \|z\| \exp \left\{ -\frac{1}{2} z^T \Delta_\ell^{-1}(\psi) z \right\} dz, \]

On \((C, \pi\ell)\), with \(C\) sufficiently large, we Taylor expand

\[ \frac{1}{\sqrt{1 - s_\ell(\psi)}} = 1 + \frac{1}{2} s_\ell(\psi) + \frac{3}{8} s_\ell^2(\psi) + O(s_\ell^3(\psi)), \]
since, using the high degree asymptotics of the Legendre polynomials (Hilb’s asymptotics), we see that \( |P_{\ell}(\cos(\psi/\ell))| \) is bounded away from 1. Next, we consider the Gaussian integral

\[
\mathcal{J}(S_{\ell}(\psi)) = \int\int_{\mathbb{R}^2} |z||z| \exp \left\{ -\frac{1}{2} z(I_2 + S_{\ell}(\psi))^{-1} z^t \right\} dz,
\]

observing that on \((C, \pi \ell)\), for \(C\) sufficiently large, we can Taylor expand

\[
(I_2 + S_{\ell}(\psi))^{-1} = I_2 - S_{\ell}(\psi) + S_{\ell}^2(\psi) + O(S_{\ell}^3(\psi)),
\]

and the exponential as follows

\[
\exp \left\{ -\frac{1}{2} z(I_2 + S_{\ell}(\psi))^{-1} z^t \right\} = \exp \left\{ -\frac{z z^t}{2} \right\} \left[ 1 + \frac{1}{2} z(S_{\ell}(\psi) - S_{\ell}^2(\psi) + O(S_{\ell}^3(\psi))) z^t + \frac{1}{8} \left(z(S_{\ell}(\psi)) z^t\right)^2 + O\left(z(S_{\ell}(\psi) - S_{\ell}^2(\psi) + O(S_{\ell}^3(\psi))) z^t\right)^3 \right],
\]

so that

\[
\mathcal{J}(S_{\ell}(\psi)) = \int\int_{\mathbb{R}^2} |z||z| \exp \left\{ -\frac{z z^t}{2} \right\} \left[ 1 + \frac{1}{2} z(S_{\ell}(\psi) - S_{\ell}^2(\psi) + O(S_{\ell}^3(\psi))) z^t + \frac{1}{8} \left(z(S_{\ell}(\psi)) z^t\right)^2 \right] dz + O(S_{\ell}^3(\psi)).
\]

We introduce the following notation:

\[
\mathcal{J}_0(S_{\ell}(\psi)) = \int\int_{\mathbb{R}^2} |z||z| \exp \left\{ -\frac{z z^t}{2} \right\} dz = 2\pi \int_0^\infty \rho \exp \left\{ -\frac{1}{2} \rho^2 \right\} \rho d\rho = 2\pi \sqrt{\frac{\pi}{2}} = \sqrt{2\pi^{3/2}},
\]

\[
\mathcal{J}_1(S_{\ell}(\psi)) = \frac{1}{2} \int\int_{\mathbb{R}^2} |z||z| \exp \left\{ -\frac{z z^t}{2} \right\} z S_{\ell}(\psi) z^t dz = \frac{3}{2^{3/2}} \pi^{3/2} tr S_{\ell}(\psi),
\]

and

\[
\mathcal{J}_2(S_{\ell}(\psi)) = -\frac{1}{2} \int\int_{\mathbb{R}^2} |z||z| \exp \left\{ -\frac{z z^t}{2} \right\} z S_{\ell}^2(\psi) z^t dz
\]
\[
= -\frac{1}{2} \int\int_{\mathbb{R}^2} |z||z| \exp \left\{ -\frac{z z^t}{2} \right\} \left(S_{11,\ell}(\psi) z_1^2 + S_{22,\ell}(\psi) z_2^2\right) dz
\]
\[
= -\frac{3}{2^{3/2}} \pi^{3/2} tr S_{\ell}(\psi).
\]

We also define

\[
\mathcal{J}_3(S_{\ell}(\psi)) = \frac{1}{8} \int\int_{\mathbb{R}^2} |z||z| \exp \left\{ -\frac{z z^t}{2} \right\} \left(z S_{\ell}(\psi) z^t\right)^2 dz
\]
\[
= \frac{1}{8} \int\int_{\mathbb{R}^2} |z||z| \exp \left\{ -\frac{z z^t}{2} \right\} \left(S_{11,\ell}(\psi) z_1^4 + S_{22,\ell}(\psi) z_2^4 + 2 S_{11,\ell}(\psi) S_{22,\ell}(\psi) z_1^2 z_2^2\right) dz,
\]

and note that

\[
\int\int_{\mathbb{R}^2} |z||z| \exp \left\{ -\frac{z z^t}{2} \right\} (z_1^2 + z_2^2)^2 dz = 2\pi \int_0^\infty \rho \exp \left\{ -\frac{\rho^2}{2} \right\} \rho^3 d\rho = 2 \frac{15}{\sqrt{2}} \pi^{3/2},
\]

\[
\int\int_{\mathbb{R}^2} |z||z| \exp \left\{ -\frac{z z^t}{2} \right\} z_1^4 dz = \frac{15}{\sqrt{2}} \frac{3}{4} \pi^{3/2},
\]

and that

\[
\int\int_{\mathbb{R}^2} |z||z| \exp \left\{ -\frac{z z^t}{2} \right\} z_1^2 z_2^2 dz
\]
\[
= \frac{1}{2} \int\int_{\mathbb{R}^2} |z||z| \exp \left\{ -\frac{z z^t}{2} \right\} (z_1^2 + z_2^2)^2 dz - \int\int_{\mathbb{R}^2} |z||z| \exp \left\{ -\frac{1}{2} z z^t \right\} z_1^4 dz = \frac{15}{\sqrt{2}} \frac{1}{4} \pi^{3/2}.
\]
Substituting (4.9) and (4.10) into (4.8), we obtain
\[
\mathcal{I}_3(S_\ell(\psi)) = \frac{15}{8\sqrt{2}}(3S_{11,\ell}(\psi) + 3S_{22,\ell}^2(\psi) + 2S_{11,\ell}(\psi)S_{22,\ell}(\psi))
\]
\[
= \frac{15\sqrt{2}}{64} \pi^{3/2} \{2\text{tr } S_\ell^2(\psi) + [\text{tr } S_\ell^2(\psi)]^2\}.
\]

Write
\[
\mathcal{I}(S_\ell(\psi)) = \mathcal{I}_0(S_\ell(\psi)) + \mathcal{I}_1(S_\ell(\psi)) + \mathcal{I}_2(S_\ell(\psi)) + \mathcal{I}_3(S_\ell(\psi)) + O(S_\ell^3(\psi))
\]
\[
= \sqrt{2}\pi^{3/2} + \frac{3}{2}\pi^{3/2} \text{ tr } S_\ell(\psi) - \frac{9}{16\sqrt{2}} \pi^{3/2} \text{ tr } S_\ell^2(\psi) + \frac{15\sqrt{2}}{64} \pi^{3/2} [\text{tr } S_\ell(\psi)]^2 + O(S_\ell^3(\psi)).
\]

We finally expand
\[
\frac{1}{\sqrt{\det \Delta_\ell(\psi)}} = \frac{1}{\sqrt{\det(I_2 + S_\ell(\psi))}};
\]
note that
\[
\det(I_2 + S_\ell(\psi)) = [1 + S_{11,\ell}(\psi)][1 + S_{22,\ell}(\psi)] = 1 + \text{tr } S_\ell(\psi) + \det S_\ell(\psi),
\]
and so,
\[
\frac{1}{\sqrt{\det \Delta_\ell(\psi)}} = 1 - \frac{1}{2} \{\text{tr } S_\ell(\psi) + \det S_\ell(\psi)\}^2 + O(S_\ell^3(\psi))
\]
\[
= 1 - \frac{1}{2} \text{ tr } S_\ell(\psi) + \frac{1}{4} \text{ tr } S_\ell^2(\psi) + \frac{1}{8} [\text{tr } S_\ell(\psi)]^2 + O(S_\ell^3(\psi)),
\]
where we have used the fact that $S_{11,\ell}(\psi)$ and $S_{22,\ell}(\psi)$ are the eigenvalues of $S_\ell^2(\psi)$, and we have written $\det S_\ell(\psi)$ as follows:
\[
\det S_\ell(\psi) = \frac{1}{2} \{S_{11,\ell}(\psi) + S_{22,\ell}(\psi)\}^2 - [S_{11,\ell}(\psi) + S_{22,\ell}(\psi)] = \frac{1}{2} \{[\text{tr } S_\ell(\psi)]^2 - \text{tr } S_\ell^2(\psi)\}.
\]

In conclusion, we have:
\[
K_{1,\ell}(\psi) = \frac{\ell(\ell + 1)}{2^2\pi^2} \left[1 + \frac{1}{2} s_\ell(\psi) + \frac{3}{8} s_\ell^2(\psi) + O(s_\ell^3(\psi))\right]
\]
\[
\times \left[\sqrt{2}\pi^{3/2} + \frac{3}{2}\pi^{3/2} \text{ tr } S_\ell(\psi) - \frac{9}{16\sqrt{2}} \pi^{3/2} \text{ tr } S_\ell^2(\psi) + \frac{15\sqrt{2}}{64} \pi^{3/2} [\text{tr } S_\ell(\psi)]^2 + O(S_\ell^3(\psi))\right]
\]
\[
\times \left[1 - \frac{1}{2} \text{ tr } S_\ell(\psi) + \frac{1}{4} \text{ tr } S_\ell^2(\psi) + \frac{1}{8} [\text{tr } S_\ell(\psi)]^2 + O(S_\ell^3(\psi))\right]
\]
\[
= \frac{\sqrt{\ell(\ell + 1)}}{2^2\sqrt{2}} \left[2 + s_\ell(\psi) + \frac{1}{2} \text{ tr } S_\ell(\psi) + \frac{3}{4} s_\ell^2(\psi) + \frac{1}{4} s_\ell(\psi) \text{ tr } S_\ell(\psi) - \frac{1}{16} \text{ tr } S_\ell^2(\psi) - \frac{1}{32} [\text{tr } S_\ell(\psi)]^2\right]
\]
\[
+ O(\ell \cdot s_\ell^3(\psi)) + O(\ell \cdot S_\ell^3(\psi)).
\]

\[\Box\]

4.2. Proof of Theorem 1.2(1).

**Proof.** Substituting the estimates (4.4), (4.5) and (4.6) into (4.7) we obtain
\[
K_{1,\ell}(\psi) = \frac{\sqrt{\ell(\ell + 1)}}{2^2\sqrt{2}} \left[2 + 2\sqrt{\frac{2\pi}{\sqrt{\psi}}} \cos((\ell + 1/2)\psi/\ell - \pi/4)\right]
\]
\[
= \frac{\sqrt{\ell(\ell + 1)}}{2^2\sqrt{2}} \left[1 + \frac{1}{4} s_\ell(\psi) + \frac{3}{2} s_\ell^2(\psi) + \frac{1}{2} s_\ell(\psi) \text{ tr } S_\ell(\psi) + \frac{1}{4} s_\ell^2(\psi) [\text{tr } S_\ell(\psi)]^2\right]
\]
\[
+ O(\ell \cdot s_\ell^3(\psi)) + O(\ell \cdot S_\ell^3(\psi)).
\]
and, since \( \cos^2(x) = \frac{1}{2}[1 + \cos(2x)] \) and \( \sin^2(x) = \frac{1}{2}[1 - \cos(2x)] \), we can write
\[
\frac{7}{4\pi} \cos^2((\ell + 1/2)\psi/\ell - \pi/4) - \frac{2}{\pi} \sin^2((\ell + 1/2)\psi/\ell - \pi/4) = \frac{7}{4\pi} \left[1 + \cos((\ell + 1/2)2\psi/\ell - \pi/2)\right] - \frac{2}{\pi} \left[1 - \cos((\ell + 1/2)2\psi/\ell - \pi/2)\right] = \frac{7}{4\pi} \frac{1}{2} + \left[\frac{7}{4\pi} \frac{1}{2} + \frac{2}{\pi} \frac{1}{2}\right] \cos((\ell + 1/2)2\psi/\ell - \pi/2) = -\frac{1}{8\pi} + \frac{15}{8\pi} \cos((\ell + 1/2)2\psi/\ell - \pi/2).
\]

The above implies
\[
K_{1,\ell}(\psi) = \frac{\sqrt{\ell(\ell + 1)}}{2\sqrt{2}} \left[2 + 2\sqrt{\frac{2}{\pi}} \cos((\ell + 1/2)\psi/\ell - \pi/4) - \frac{1}{8\pi} + \frac{15}{8\pi} \cos((\ell + 1/2)2\psi/\ell - \pi/2)\right] + O(\psi^{-3/2}\ell^{-2}).
\]

The statement (1.21) of Theorem 1.2(1).

5. Proof of Theorem 1.2(2): Perturbative analysis at the boundary

The aim of this section is to study the asymptotic behaviour of the density function \( K_{1,\ell}(\psi) \) for \( 0 < \psi < \epsilon_0 \) with \( \epsilon_0 > 0 \) sufficiently small. We have
\[
K_{1,\ell}(\psi) = \frac{1}{\sqrt{2\pi} \sqrt{1 - P_\ell(\cos(\psi/\ell))}} \frac{1}{2\pi \sqrt{\det \Delta_\ell(\psi)}} \sqrt{\ell(\ell + 1)} \int_{\mathbb{R}^2} ||z|| \exp \left\{ -\frac{1}{2} z'(\Delta_\ell^{-1}(\psi)z) \right\} dz,
\]
where \( \Delta_\ell(\psi) \) is the scaled conditional covariance matrix
\[
\Delta_\ell(\psi) = C_\ell(\psi) - \frac{B_\ell'(\psi)B_\ell(\psi)}{1 - P_\ell(\cos(\psi/\ell))}.
\]

We have that
\[
1 - P_\ell(\cos(\psi/\ell)) = \frac{\ell(\ell + 1)\psi^2}{\ell^2} - \frac{(\ell - 1)\ell(\ell + 1)(\ell + 2)\psi^4}{4\ell^4} + \frac{1}{36} \frac{(\ell - 3)(\ell - 1)\ell(\ell + 1)(\ell + 2)(\ell + 3)\psi^6}{\ell^6} + O(\psi^8),
\]
with constant involved in the 'O'-notation absolute. We also have
\[
B_\ell'(\psi) = \begin{pmatrix} -\sin(\psi/\ell) P_\ell'(\cos(\psi/\ell))(-\frac{1}{\ell}) & 0 \\ \frac{\ell(\ell + 1)}{2} \frac{\psi^2}{\ell^2} - \frac{(\ell - 1)(\ell + 1)(\ell + 2)\psi^4}{2\ell^4} & \frac{1}{12} \frac{(\ell - 2)(\ell - 1)\ell(\ell + 1)(\ell + 2)(\ell + 3)\psi^6}{\ell^6} + O(\psi^7) \end{pmatrix},
\]
and \( C_\ell(\psi) \) is the \( 2 \times 2 \) symmetric matrix with entries
\[
C_{\ell,11}(\psi) = \left[ P_\ell'(1) + \cos(\psi/\ell) P_\ell'(\cos(\psi/\ell)) - \sin^2(\psi/\ell) P_\ell''(\cos(\psi/\ell)) \right] \left( -\frac{1}{\ell} \right)^2.
\]
we define the zero density (first intensity) of $F$ and

$$\delta_{11,\ell}(\psi) = \frac{1}{2^8 \ell^3} \frac{(\ell - 2)(\ell - 1)(\ell + 1)(\ell + 2)(\ell + 3)}{\ell^6} \psi^4 + O(\psi^6)$$

(5.2)

and

$$\delta_{22,\ell}(\psi) = \frac{(\ell - 1)(\ell + 1)(\ell + 2)}{4 \ell^4} \psi^2 + O(\psi^4) = \frac{\psi^2}{16} + O(\ell^{-1} \psi^2) + O(\psi^4).$$

(5.3)

We introduce the change of variable $\xi = \Delta_{\ell}^{-1/2}(\psi) z$, and we write

$$K_{1,\ell}(\psi) = \frac{1}{\sqrt{2\pi} \sqrt{1 - P_{\ell}(\cos(\psi/\ell))}} \frac{1}{2\pi} \sqrt{\ell(\ell + 1)} \int_{\mathbb{R}^2} \sqrt{\delta_{11,\ell}(\psi) \xi_1^2 + \delta_{22,\ell}(\psi) \xi_2^2} \exp \left\{ - \frac{\xi_1^2 \xi_2^2}{2} \right\} d\xi.$$

Using the expansions in (5.1), (5.2) and (5.3), we write

$$K_{1,\ell}(\psi) = \frac{1}{\sqrt{2\pi} \sqrt{\psi^2/4 + O(\ell^{-1} \psi^2) + O(\psi^4)}} \sqrt{\ell(\ell + 1)} \left[ \frac{\psi^2}{4} + O(\ell^{-1} \psi) + O(\psi^3) \right] \frac{1}{2\pi} + O(1) + O(\ell^{-1} \psi^2),$$

which is (1.22).

6. PROOF OF COROLLARY 1.3: EXPECTED NODAL LENGTH

6.1. Kac-Rice formula for expected nodal length. The Kac-Rice formula is a meta-theorem allowing one to evaluate the moments of the zero set of a random field satisfying some smoothness and non-degeneracy conditions. For $F : \mathbb{R}^d \to \mathbb{R}$, a sufficiently smooth centred Gaussian random field, we define

$$K_{1,F}(x) := \frac{1}{\sqrt{2\pi} \sqrt{\text{Var}(F(x))}} \cdot \mathbb{E}[\|\nabla F(x)\| F(x) = 0]$$

the zero density (first intensity) of $F$. Then the Kac-Rice formula asserts that for some suitable class of random fields $F$ and $\overline{D} \subseteq \mathbb{R}^d$ a compact closed subdomain of $\mathbb{R}^d$, one has the equality

(6.1) \[ \mathbb{E}[\text{Vol}_{d-1}(F^{-1}(0) \cap \overline{D})] = \int_{\overline{D}} K_{1,F}(x) dx. \]

We would like to apply (6.1) to the boundary-adapted random spherical harmonics $T_\ell$ to evaluate the asymptotic law of the total expected nodal length of $T_\ell$. Unfortunately the aforementioned non-degeneracy conditions fail at the equator

$$\mathcal{E} = \{ (\theta, \phi) : \theta = \pi/2 \} \subseteq \mathbb{H}^2.$$

Nevertheless, in a manner inspired by [11 Proposition 2.1], we excise a small neighbourhood of this degenerate set, and apply the Monotone Convergence Theorem so to be able to prove that (6.1) holds
Proposition 6.1. The expected nodal length of $T_\ell$ satisfies

\begin{equation}
\mathbb{E}[\mathcal{L}(T_\ell)] = \int_{\mathbb{H}^2} K_{1,\ell}(x) dx + 2\pi,
\end{equation}

where $K_{1,\ell}(\cdot)$ is the zero density of $T_\ell$.

Proof. One way justify the Kac-Rice formula outside the equator is by using [2, Theorem 6.8], that assumes the non-degeneracy of the $3 \times 3$ covariance matrix at all these points, a condition we were able to verify via an explicit, though somewhat long, computation, omitted here. Alternatively, to validate the Kac-Rice formula it is sufficient [19, Lemma 3.7] that the Gaussian distribution of $T_\ell$ is non-degenerate for every $x \in \mathbb{H}^2 \setminus \mathcal{E}$, which is easily satisfied.

We construct a small neighbour of the equator $\mathcal{E}$, i.e. the set

\begin{equation}
\mathcal{E}_\varepsilon = \left\{(\theta, \phi) : \theta \in \left[\frac{\pi}{2}, \frac{\pi}{2} - \varepsilon\right]\right\},
\end{equation}

and we denote $\mathcal{H}_\varepsilon = \mathcal{H} \setminus \mathcal{E}_\varepsilon$.

Since Kac-Rice formula holds for $T_\ell$ restricted to $\mathcal{H}_\varepsilon$, the expected nodal length for $T_\ell$ restricted to $\mathcal{H}_\varepsilon$ is

\begin{equation}
\mathbb{E}[\mathcal{L}(T_\ell|\mathcal{H}_\varepsilon)] = \int_{\mathcal{H}_\varepsilon} K_{1,\ell}(x) dx.
\end{equation}

Since the restricted nodal length $\{\mathcal{L}(T_\ell|\mathcal{H}_\varepsilon)\}_{\varepsilon > 0}$ is an increasing sequence of nonnegative random variables with a.s. limit

\begin{equation}
\lim_{\varepsilon \to 0} \mathcal{L}(T_\ell|\mathcal{H}_\varepsilon) = \mathcal{L}(T_\ell) - 2\pi,
\end{equation}

the Monotone Convergence Theorem yields

\begin{equation}
\lim_{\varepsilon \to 0} \mathbb{E}[\mathcal{L}(T_\ell|\mathcal{H}_\varepsilon)] = \mathbb{E}[\mathcal{L}(T_\ell)] - 2\pi.
\end{equation}

Moreover, by the definition

\begin{equation}
\lim_{\varepsilon \to 0} \int_{\mathcal{H}_\varepsilon} K_{1,\ell}(x) dx = \int_{\mathcal{H}} K_{1,\ell}(x) dx.
\end{equation}

The equality of the limits in (6.3) and (6.4) show that Proposition 6.1 holds. \hfill \Box

6.2. Expected nodal length.

Proof of Corollary 1.3. To analyse asymptotic behaviour of the expected nodal length, we separate the contribution of the following three subregions of the hemisphere $\mathcal{H}$ in the Kac-Rice integral on the r.h.s of (6.2):

\begin{align*}
\mathcal{H}_C &= \{(\psi, \phi) : 0 < \psi < \varepsilon_0\}, \quad \mathcal{H}_I = \{(\psi, \phi) : \varepsilon_0 < \psi < C\}, \quad \mathcal{H}_F = \{(\psi, \phi) : C < \psi < \pi\ell\};
\end{align*}

note that we express the three subregions of $\mathcal{H}$ in terms of the scaled variable $\psi$. In what follows we argue that $\mathcal{H}_F$ gives the main contribution.

In the (scaled) spherical coordinates we may rewrite the Kac-Rice integral (6.2) as

\begin{equation}
\mathbb{E}[\mathcal{L}(T_\ell)] - 2\pi = \frac{\pi}{\ell} \int_0^{\pi} K_{1,\ell}(\psi) \sin \left(\frac{\pi}{2} - \frac{\psi}{2\ell}\right) d\psi,
\end{equation}

and the contribution of the third range $\mathcal{H}_F$ as

\begin{equation}
\mathbb{E}[\mathcal{L}(T_\ell|\mathcal{H}_F)] = \frac{\pi}{\ell} \int_C^{\pi} K_{1,\ell}(\psi) \sin \left(\frac{\pi}{2} - \frac{\psi}{2\ell}\right) d\psi.
\end{equation}
We are now going to invoke the asymptotics of $K_1,ℓ(ψ)$, prescribed by (1.21) for this range. The first term in (1.21) contributes

$$\frac{π}{ℓ} \sqrt{\frac{ℓ(ℓ+1)}{2\sqrt{2}}} \int_C^{π} \sin \left( \frac{π}{2} - \frac{ψ}{2ℓ} \right) dψ = \frac{π}{ℓ} \sqrt{\frac{ℓ(ℓ+1)}{2\sqrt{2}}} 2\ell \left[ 1 - \sin \left( \frac{C}{2ℓ} \right) \right] = 2π \sqrt{\frac{ℓ(ℓ+1)}{2\sqrt{2}}} \left[ 1 - \frac{C}{2ℓ} + O \left( \frac{C}{ℓ} \right) \right].$$

(6.6)

to the integral (6.5). The second term in (1.21) gives

$$\frac{π}{ℓ} \sqrt{\frac{ℓ(ℓ+1)}{2\sqrt{2}}} \int_C^{π} \frac{1}{ψ} \cos \left( \left( ℓ + \frac{1}{2} \right)ψ/ℓ - π/4 \right) \sin \left( \frac{π}{2} - \frac{ψ}{2ℓ} \right) dψ = O(ℓ^{-1/2}),$$

(6.7)
since, upon transforming the variables $w = ψ/ℓ$, this term is bounded by

$$\sqrt{ℓ} \int_{C/ℓ}^{π} \frac{1}{w} \cos \left( \left( ℓ + \frac{1}{2} \right)w - π/4 \right) dw = \sqrt{ℓ} \int_{C/ℓ}^{π} \frac{1}{w} \cos \left( \left( ℓ + \frac{1}{2} \right)w \right) + \sin \left( \left( ℓ + \frac{1}{2} \right)w \right) dw$$

$$= \frac{\sqrt{ℓ}}{2} \left\{ \frac{1}{w} \sin \left( \left( ℓ + \frac{1}{2} \right)w \right) \right\}_{C/ℓ}^{π} + \frac{1}{2} \int_{2π/ℓ}^{π} w^{-3/2} \sin \left( \left( ℓ + \frac{1}{2} \right)w \right) \ell + 1/2 dw$$

$$+ \frac{\sqrt{ℓ}}{2} \left\{ \frac{1}{w} \cos \left( \left( ℓ + \frac{1}{2} \right)w \right) \right\}_{C/ℓ}^{π} - \frac{1}{2} \int_{2π/ℓ}^{π} w^{-3/2} \cos \left( \left( ℓ + \frac{1}{2} \right)w \right) \ell + 1/2 dw$$

$$= O(1/\sqrt{ℓ}).$$

The logarithmic bias is an outcome of

$$\frac{π}{ℓ} \sqrt{\frac{ℓ(ℓ+1)}{2\sqrt{2}}} \int_C^{π} \left( - \frac{1}{16πψ} \right) \sin \left( \frac{π}{2} - \frac{ψ}{2ℓ} \right) dψ = -\frac{1}{16ℓ} \sqrt{\frac{ℓ(ℓ+1)}{2\sqrt{2}}} \left[ -\log \left( \frac{C}{2ℓ} \right) + O(1) \right]$$

$$= -\frac{1}{16ℓ} \sqrt{\frac{ℓ(ℓ+1)}{2\sqrt{2}}} \log(ℓ) + O(1).$$

(6.8)

Consolidating all of the above estimates (6.6), (6.7) and (6.8), and the contribution of the error term in (1.21), we finally obtain

$$\mathbb{E}[\mathcal{L}(T_ℓ|H_C)] = 2π \sqrt{\frac{ℓ(ℓ+1)}{2\sqrt{2}}} - \frac{1}{16ℓ} \sqrt{\frac{ℓ(ℓ+1)}{2\sqrt{2}}} \log(ℓ) + O(1).$$

The contribution to the Kac-Rice integral on the r.h.s of (6.2) of the set $H_C$ is bounded by the straightforward

$$\mathbb{E}[\mathcal{L}(T_ℓ|H_C)] = \frac{π}{ℓ} \int_0^{π} K_{1,ℓ}(ψ) \sin \left( \frac{π}{2} - \frac{ψ}{2ℓ} \right) dψ = O(1),$$

on recalling the uniform estimate (1.22). Finally, we may bound the contribution of the intermediate range $H_I$ as follows. We first write

$$\mathbb{E}[\mathcal{L}(T_ℓ|H_I)] = \frac{1}{\sqrt{2π}} \int_{H_I} \frac{1}{\sqrt{1 - P_ℓ(\cos(ψ/ℓ))}} \cdot \mathbb{E} \left[ \|\nabla T_ℓ(ψ/ℓ)\| \left| T_ℓ(ψ/ℓ) = 0 \right. \right] dψ$$

then we observe that on the intermediate range

$$H_I = \left\{ \left( ψ/ℓ, ϕ \right) : ϵ_0 < ψ < C \right\},$$

the variance at the denominator, i.e. $1 - P_ℓ(\cos(ψ/ℓ))$, is bounded away from 0, and moreover the diagonal entries of the unconditional covariance matrix $C_ℓ$ of the Gaussian vector $\nabla T_ℓ$ are $O(ℓ^2)$, and so are the diagonal entries of the conditional matrix $Ω_ℓ$, since they are bounded by the unconditional
ones, as it follows directly from (3.42), or, alternatively, from the vastly general Gaussian Correlation Inequality [44]. This easily gives the following upper bound:

\[ \mathbb{E}[\|\nabla T_\ell(\psi/\ell)\|] \leq \left( \mathbb{E}[\|\nabla T_\ell(\psi/\ell)\|^2] T_\ell(\psi/\ell) = 0 \right)^{1/2} \leq \left( \mathbb{E}[\|\nabla T_\ell(\psi/\ell)\|^2] \right)^{1/2} = O(\ell). \]

Since the area of \( \mathcal{H}_C \) is \( O(\ell^{-1}) \), it follows that the total contribution this range to the expected nodal length is \( O(1) \).

\[ \square \]

**APPENDIX A. PROOF OF PROPOSITION 1.1**

We have that

\[
\mathbb{E}[T_\ell(x) \cdot T_\ell(y)] = \frac{8\pi}{2\ell + 1} \sum_{m=1}^{\ell} Y_{\ell,m}(x) \overline{Y}_{\ell,m}(y)
\]

\[
= \frac{1}{2} \frac{8\pi}{2\ell + 1} \left[ \sum_{m=-\ell}^{\ell} Y_{\ell,m}(x) \overline{Y}_{\ell,m}(y) + \sum_{m=1}^{\ell} (-1)^{m+1} Y_{\ell,m}(x) \overline{Y}_{\ell,m}(y) \right]
\]

\[
= \frac{1}{2} \frac{8\pi}{2\ell + 1} \left[ \sum_{m=-\ell}^{\ell} Y_{\ell,m}(x) \overline{Y}_{\ell,m}(y) - \sum_{m=1}^{\ell} Y_{\ell,m}(x) \overline{Y}_{\ell,m}(y) \right],
\]

where we have used the fact that \( Y_{\ell,m}(\theta, \phi) = (-1)^m Y_{\ell,m}(\pi - \theta, \phi) \). We apply now the Addition Theorem for Spherical Harmonics:

\[
P_\ell(\cos d(x,y)) = \frac{4\pi}{2\ell + 1} \sum_{m=-\ell}^{\ell} Y_{\ell,m}(x) \overline{Y}_{\ell,m}(y),
\]

so that

\[
\mathbb{E}[T_\ell(x) \cdot T_\ell(y)] = P_\ell(\cos d(x,y)) - P_\ell(\cos d(\overline{x},y)).
\]

**Remark A.1.** In particular, we note that,

\[
\mathbb{E}[T_\ell^2(x)] = P_\ell(\langle x, x \rangle) - P_\ell(\langle \overline{x}, x \rangle) = 1 - P_\ell(\cos(\pi - 2\theta)),
\]

this implies

\[
\text{Var}(T_\ell(x)) = \begin{cases} 1 - P_\ell(\cos(\pi)) = 1 - (-1)^\ell & \text{if } \theta = 0, \\ 1 - P_\ell(1) = 0 & \text{if } \theta = \pi/2, \\ 1 & \text{as } \ell \to \infty & \text{if } \theta \neq 0, \pi/2. \end{cases}
\]

Moreover, as \( \ell \to \infty \), for \( \theta \neq 0, \pi/2 \),

\[
\frac{\mathbb{E}[T_\ell(x) \cdot T_\ell(y)]}{P_\ell(\cos d(x,y))} \to 1.
\]

**REFERENCES**

[1] Azaïs, J.M., Dalmao, F. and León, J.R. CLT for the zeros of classical random trigonometric polynomials. Annales de l’Institut Henri Poincaré, Probabilités et Statistiques, 52, no. 2, 804-820 (2016)

[2] Azaïs, J.-M., Wschebor, W. Level Sets and Extrema of Random Processes and Fields. John Wiley & Sons Inc., Hoboken, NJ, 2009

[3] Béard, P. Volume des ensembles nodaux des fonctions propres du laplacien. Séminaire de théorie spectrale et géométrie, 3, 1-9 (1985)

[4] Berry, M. V. Regular and irregular semiclassical wavefunctions. J. Phys. A, 10, no. 12, 2083-2091 (1977)

[5] Berry, M. V. Statistics of nodal lines and points in chaotic quantum billiards: perimeter corrections, fluctuations, curvature. J. Phys. A, 35, 3025-3038 (2002)

[6] Bombieri, E., Bourgain, J. A problem on sums of two squares. Int. Math. Res. Notices (IMRN), 11, 3343-3407 (2015)

[7] Bourgain, J. On toral eigenfunctions and the random wave model. Israel Journal of Mathematics, 201(2), 611-630 (2014)
[8] Benatar, J., Marinucci, D., Wigman, I. Planck-scale distribution of nodal length of arithmetic random waves. J. d’Anal. Math., to appear. Available online [https://arxiv.org/abs/1710.06153]

[9] Brüning, J. Über Knoten Eigenfunktionen des Laplace-Beltrami Operators. Math. Z. 158, 15-21 (1978)

[10] Brüning, J. and Gromes, D. Über die Länge der Knotenlinien schwingender Membranen. Math. Z. 124, 79-82 (1972)

[11] Cammarota, V., Klurman, O. and Wigman, I., Boundary effect on the nodal length for Arithmetic Random Waves, and spectral semi-correlations. Communications in Mathematical Physics, (2020)

[12] Cammarota, V., Marinucci, D. and Wigman, I., On the distribution of the critical values of random spherical harmonics. The Journal of Geometric Analysis, 26(4), 3252-3324 (2016)

[13] Cheng, S. Y. Eigenfunctions and nodal sets. Comm. Math. Helv., 51, 43-55 (1976)

[14] Donnelly, H., Fefferman, C. Nodal sets of eigenfunctions on Reimannian manifolds. Inventiones Mathematicae, 93(1), 161-183 (1988)

[15] Dunnage, J.E.A. The number of real zeros of a random trigonometric polynomial. Proceedings of the London Mathematical Society, 3(1), 53-84 (1966)

[16] Granville, A., Wigman, I. The distribution of the zeros of random trigonometric polynomials. American Journal of Mathematics, 133(2), 295-357 (2011)

[17] Hassell, A., Tao, T. Upper and lower bounds for normal derivatives of Dirichlet eigenfunctions. Math. Res. Letters, 9, 289-305 (2002)

[18] Klurman, O., Sartori, A. Research in progress.

[19] Kabluchko, Z., Wigman, I. Asymptotics for the expected number of nodal components for random lemniscates. Int. Math. Res. Not. (IMRN), to appear.

[20] Krishnapur, M., Kurlberg, P., Wigman, I. Nodal length fluctuations for arithmetic random waves. Ann. Math., 177 no. 2, 699-737 (2013)

[21] Logunov, A. Nodal sets of Laplace eigenfunctions: estimates of the Hausdorff measure in dimensions two and three. 50 Years with Hardy spaces, Oper. Theory Adv. Appl. 261, 333-344 (2018)

[22] Logunov, A. Nodal sets of Laplace eigenfunctions: proof of Nadirashvili’s conjecture and of the lower bound in Yau’s conjecture. Ann. of Math., 187 no. 1, 241-262 (2018)

[23] Logunov, A. Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure. Ann. of Math., 187 no. 1, 221-239 (2018)

[24] Marinucci, D., Peccati, G. Random fields on the sphere: representation, limit theorems and cosmological applications. Cambridge University Press, 2011

[25] Marinucci, D., Peccati, G., Rossi, M., Wigman, I. The asymptotic equivalence of the sample trispectrum and the nodal length for random spherical harmonics. Annales de l’Institut Henri Poincaré, Probabilités et Statistiques, 56, no. 1, 374-390 (2020)

[26] Marinucci, D., Peccati, G., Rossi, M., Wigman, I. Non-universality of nodal length distribution for arithmetic random waves. Geometric and Functional Analysis, 26, no. 3, 926-960 (2016)

[27] Nourdin, I., Peccati, G., Rossi, M. Nodal statistics of planar random waves. Communications in Mathematical Physics, 369(1), 99-151 (2019)

[28] Oravecz, F., Rudnick, Z., Wigman, I. The Leray measure of nodal sets for eigenfunctions on the torus. In Annales de l’institut Fourier, 58, no. 1, 299-335 (2008)

[29] Rudnick, Z., Wigman, I. On the volume of nodal sets for eigenfunctions of the Laplacian on the torus. Ann. Henri Poincaré, 9, no. 1, 109-130 (2008)

[30] Szégo, G. Orthogonal Polynomials. Fourth edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. Providence, R.I., 1975

[31] Vidotto, A. A Note on the Reduction Principle for the Nodal Length of Planar Random Waves. arXiv preprint arXiv:2007.04228 (2020)

[32] Qualls, C. On the number of zeros of a stationary Gaussian random trigonometric polynomial. Journal of the London Mathematical Society, 2(2), 216-220 (1970)

[33] Royen, T. A simple proof of the Gaussian correlation conjecture extended to multivariate gamma distributions. arXiv preprint arXiv:1408.1028 (2014)

[34] Wigman, I. Fluctuations of the nodal length of random spherical harmonics. Communications in Mathematical Physics, 298 (3), 787-831 (2010). Erratum published Comm. Math. Phys., 309 no. 1, 293-294 (2012)