Wavelet Numerical Solutions for a Class of Elliptic Equations with Homogeneous Boundary Conditions

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Abstract: This paper focuses on a method to construct wavelet Riesz bases with homogeneous boundary condition and use them to a kind of second-order elliptic equation. First, we construct the splines on the interval [0, 1] and consider their approximation properties. Then we define the wavelet bases and illustrate the condition numbers of stiffness matrices are small and bounded. Finally, several numerical examples show that our approach performs efficiently.

Keywords: B-spline wavelets; elliptic equation; homogeneous boundary condition; numerical solutions; convergence

1. Introduction

Let Ω be a Lebesgue measurable open subset of \( \mathbb{R} \), \( L^p(\Omega) \) denotes the functions satisfying \( \| f \|_{L^p(\Omega)} < \infty \). The Fourier transform \( \hat{f}(\xi) \) for \( f(x) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \) is defined by

\[
\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx, \quad \xi \in \mathbb{R}.
\]

Let \( \mu \geq 0 \), \( H^\mu(\mathbb{R}) \) denotes the Sobolev space such that \( \| f \|_{H^\mu(\mathbb{R})} := \left( \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \xi^{2\mu}d\xi \right)^{1/2} < \infty \). The inner product given by

\[
\langle f, g \rangle_{H^\mu(\mathbb{R})} := \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}(1 + \xi^{2\mu})d\xi, \quad f, g \in H^\mu(\mathbb{R}),
\]

and the norm is given by \( \| f \|_{H^\mu(\mathbb{R})} := (\langle f, f \rangle_{H^\mu(\mathbb{R})})^{1/2} \). We use \( C_0^\infty(\Omega) \) to denote the closure of \( C_0^\infty(\Omega) \) in \( H^\mu(\mathbb{R}) \).

In this paper let \( \Omega = (0, 1) \). We are interested in the following elliptic variable coefficient equation with the homogeneous boundary conditions:

\[
\begin{aligned}
\left\{ \begin{array}{l}
-u''(x) + q(x)u(x) = f(x), \quad x \in \Omega, \\
u(0) = u(1) = 0,
\end{array} \right.
\end{aligned}
\]

where \( f(x) \) is a given function in \( L_2(\Omega) \), \( 0 \leq q(x) \leq c, c \) is a constant.

For \( u, v \in H_0^1(\Omega) \), define a bilinear form \( a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R} \),

\[
a(u, v) := \langle u', v' \rangle_{L_2(\Omega)} + \langle qu, v \rangle_{L_2(\Omega)}. \tag{2}
\]

Hence, there exists a unique \( u \in H_0^1(\Omega) \) such that

\[
a(u, v) = \langle f, v \rangle_{L_2(\Omega)}, \quad \forall v \in H_0^1(\Omega), \tag{3}
\]

thanks to Lax-Milgram theorem.
To solve the variational problem (3), we use finite-dimensional subspaces \( V = \text{span}\{v_1, v_2, \cdots, v_m\} \) to approximate \( H^1_0(\Omega) \). Suppose \( x_k \in \mathbb{C}, k = 1, 2, \cdots, m, \) such that

\[ u(x) := \sum_{k=1}^{m} x_k v_k(x) \]

satisfies the following equation

\[ a(u, v_j) = \langle f, v_j \rangle_{L^2(\Omega)}, \quad \forall v_j \in H^1_0(\Omega), \quad j = 1, 2, \cdots, m, \]

or equivalently,

\[ \sum_{k=1}^{m} a_{jk} x_k = b_j, \quad j = 1, 2, \cdots, m, \]

where \( a_{jk} := a(v_j, v_k), b_j := \langle f, v_j \rangle, \quad j, k \in \{1, \cdots, m\}. \)

In 1992, Chui and Wang [1] studied semi-orthogonal wavelets generated from cardinal spline. Dahmen, Kunoth and Urban [2] gave biorthogonal spline wavelets in 1999. In 2006, Jia and Liu [3] constructed wavelet bases on the interval \([0, 1]\) and applied them to the Sturm-Liouville Equation with the Dirichlet boundary condition. In 2011, Jia and Zhao [4] applied the wavelets bases on the unit square to the biharmonic equation and extended the method to general elliptic equation of fourth-order. However, to our knowledge, there is no numerical schemes based on wavelet bases to be derived.

This paper is organized as follows. In Section 2, we construct splines on the interval \([0, 1]\) with homogeneous boundary condition and then investigate their approximation properties. The sufficient condition for norm equivalence is provided in Section 3. In Section 4, we describe the wavelet method and show that the condition number of the wavelet stiffness matrix is not only relatively small but also uniformly bounded. Finally, some numerical examples are given in Section 5 so as to demonstrate that our wavelet bases are very useful and efficient.

2. Splines and Approximation Property

In this section, we construct splines which satisfy the homogeneous boundary conditions on the interval \([0, 1]\) and then investigate their some properties.

Let \( j, d \in \mathbb{N} \) and \( \mathcal{I} \) a countable index set. Suppose that \( \mathbf{t} := (t_k^j) \) is the sequence such that \( t_k^j < t_{k+1}^j \) for all \( k \in \mathcal{I} \). The B-spline of order \( d \) is given by

\[ B_{k,d}^j(x) := (t_{k+d}^j - t_k^j)[t_{k+1}^j, \cdots, t_{k+d}^j; (t - x)_+^{d-1}]_t, \quad x \in \mathbb{R}, \]

where \( [t_{k+1}^j, \cdots, t_{k+d}^j; f(t)]_t \) denotes the \( d \)-th order divided difference at the points \( t_{k+1}^j, \cdots, t_{k+d}^j \) and \( x_+ := \max\{x, 0\}, \quad x^m_+ := (x_+)^m \).

From now on, suppose that \( d = 3, j \geq 2, \mathbf{t} := (t_k^j)_{k=1}^{2j+3} \) is given by

\[ t_k^j := \begin{cases} 
    k - 1, & k = 1, \\
    k - 2, & 2 \leq k < 2j + 2, \\
    k - 3, & k = 2j + 3.
\end{cases} \tag{5} \]

Many useful properties of B-spline can be established. For example, there exist complex numbers \( c_k, (k \in \mathbb{Z}) \) such that \( p(x) = \sum_{k \in \mathbb{Z}} c_k B_{k,3}^j(x), j \geq 3, k = 2j + 3. \) i.e., a polynomial \( p \) whose degree is at most 2 can be represented as a B-spline series.

Moreover, the above properties, we also have the following Lemma.

**Lemma 1** ([5]). Let \( j \geq 2, I_j = \{1, 2, \cdots, 2j\}, \mathbf{t} := (t_k^j)_{k=1}^{2j+3} \) be given by (5), then one has

(i) \( \text{supp} B_{k,3}^j(x) = [t_{k+3}^j, t_{k+3}^j], k \in I_j; \)

(ii) \( B_{k,3}^j(x) = B_{2j-k+1,3}^j(2j - x), k \in I_j, x \in \mathbb{R}; \)
(iii) $B_{k,3}^j(x) = B_{2,3}^j(x - k + 2), k \in I_j \setminus \{1, 2^j\}, x \in \mathbb{R};$

(iv) $B_{k,3}^j(x) = B_{k,3}^{j+1}(x), k \in I_j \setminus \{2^j\}, x \in \mathbb{R}.$

From the properties (ii), (iii), (iv) of Lemma 1, we can see that B-splines defined by (5) are divided into two kinds: $B_{k,3}^j(x), k = 1, 2^j$, are symmetric boundary functions. The others are interior functions and can be obtained by shifting function $B_{2,3}^j(x).$ So we only need to discuss $B_{1,3}^2(x)$ and $B_{2,3}^2(x)$ especially, where

$$B_{1,3}^2(x) := 2[0, 0, 1, 2; (t - x)^2] = \begin{cases} -\frac{3}{2}x^2 + 2x, & 0 < x \leq 1, \\ \frac{1}{2}x^2 - 2x + 2, & 1 < x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

$$B_{2,3}^2(x) := 3[0, 1, 2, 3; (t - x)^2] = \begin{cases} \frac{1}{2}x^2, & 0 < x \leq 1, \\ -x^2 + 3x - \frac{3}{2}, & 1 < x \leq 2, \\ \frac{1}{2}x^2 - 3x + \frac{9}{2}, & 2 < x \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

**Lemma 2.** If $j \geq 2$, $v^j := (t_k^j)_{k=1}^{2^j+3}$ is given by (5), then

(i) $B_{2,3}^j(x) = \frac{1}{4}B_{2,3}^2(2x) + \frac{3}{4}B_{2,3}^2(2x - 1) + \frac{3}{4}B_{2,3}^2(2x - 2) + \frac{1}{4}B_{2,3}^2(2x - 3), x \in \mathbb{R};$

(ii) $B_{1,3}^j(x) = \frac{1}{2}B_{2,3}^2(2x) + \frac{1}{2}B_{2,3}^2(2x - 1), x \in \mathbb{R};$

(iii) $B_{2,3}^j(x) \in H^\mu(\mathbb{R}), B_{1,3}^j(x) \in H^\sigma(\mathbb{R}),$ where $0 < \mu < \frac{5}{2}, 0 < \sigma < \frac{3}{2}.$

**Proof.** It is easy to check (i), (ii) are established.

(iii) Let $\phi(x) := B_{2,3}^j(x),$ one obtains

$$\hat{\phi}(\xi) = \left(\frac{\sin \frac{\xi}{2}}{\xi}\right)^3$$

by the Fourier transform of $\phi(x).$

Since

$$\int_{\mathbb{R}} |\hat{\phi}(\xi)|^2 |\xi|^2 |\xi|^\mu d\xi = \int_{|\xi| < 1} |\hat{\phi}(\xi)|^2 |\xi|^2 |\xi|^\mu d\xi + \int_{|\xi| > 1} |\hat{\phi}(\xi)|^2 |\xi|^2 |\xi|^\mu d\xi \leq \int_{|\xi| < 1} |\hat{\phi}(\xi)|^2 d\xi + \int_{|\xi| > 1} \left(\frac{\sin \frac{\xi}{2}}{\xi}\right)^6 |\xi|^\mu d\xi \leq \int_{|\xi| < 1} |\hat{\phi}(\xi)|^2 d\xi + \int_{|\xi| > 1} \frac{1}{|\xi|^3} |\xi|^\mu d\xi.$$

Hence, for $0 < \mu < \frac{5}{2},$ one obtains

$$\int_{\mathbb{R}} |\hat{\phi}(\xi)|^2 |\xi|^2 |\xi|^\mu d\xi < \infty.$$ i.e., $\phi(x) \in H^\mu(\mathbb{R}), 0 < \mu < \frac{5}{2}$. An analogous argument shows that $B_{1,3}^j(x) \in H^\sigma(\mathbb{R}), 0 < \sigma < \frac{3}{2}.$

**Lemma 3.** If $\phi_{j,k}(x) := 2^jB_{k,3}^j(2^j x), k \in I_j$, then one has

(i) $\text{supp } \phi_{j,k} = [2^{-j}l_{k+3}^j, 2^{-j}l_{k+3}^j] \subset \Omega;$

(ii) There exists a constant $C > 0$ which is independent of $j$ such that $\|\phi_{j,k}\|_{L^2(\Omega)} \leq C;$

(iii) $\text{for } k \in I_j \setminus \{1, 2^j\}, \phi_{j,k} \in H^\mu(\mathbb{R})$ with $0 < \mu < \frac{5}{2};$

(iv) $\text{for } k \in \{1, 2^j\}, \phi_{j,k} \in H^\sigma(\mathbb{R})$ with $0 < \sigma < \frac{3}{2}.$
(v) Let $\Phi_j := \{\phi_{j,k}(x) := 2^j B_{k,3}^j(2^j x) \mid k \in I_j, x \in \mathbb{R}\}$, then

$$\Phi_j = \frac{1}{\sqrt{2}} M_j^T \Phi_{j+1},$$

where

$$M_j^T = \begin{pmatrix}
\frac{1}{2} & \frac{3}{4} & \frac{1}{4} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & \cdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & \frac{3}{4} & \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{4} & \frac{3}{4} & \frac{3}{4} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{4} & \frac{3}{4} & \frac{1}{2} \\
& & & & & & & & & & & \\
\end{pmatrix}_{(2^j+1) \times (2^j+1)}$$

(vi) $\Phi_j$ is the Riesz sequence in the space $L_2(\Omega)$, and the Riesz bounds are independent of $j$.

**Proof.** The conclusions (i)–(v) can be obtained easily.

(vi) It is necessary to show that there exist $C_1 > 0, C_2 > 0$ which are independent of $j$ such that

$$C_1 \|c_j\|_{L^2(I_j)} \leq \| \sum_{k \in I_j} c_{j,k} \phi_{j,k} \|_{L^2(\Omega)} \leq C_2 \|c_j\|_{L^2(I_j)},$$

In fact,

$$\| \sum_{k \in I_j} c_{j,k} \phi_{j,k} \|_{L^2(\Omega)}^2 = c_j^T G_j c_j,$$

where $G_j =: (\langle \phi_{j,k}, \phi_{j,m} \rangle_{L^2(\Omega)})_{k,m \in I_j}$ is Gramian matrix. i.e.,

$$G_j = \begin{pmatrix}
\frac{10}{3} & \frac{5}{24} & \frac{120}{7} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{5}{24} & \frac{7}{120} & \frac{13}{60} & \frac{1}{120} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{120} & \frac{13}{60} & \frac{7}{10} & \frac{13}{60} & \frac{1}{120} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{120} & \frac{13}{60} & \frac{13}{60} & \frac{1}{120} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & \frac{13}{60} & \frac{7}{10} & \frac{13}{60} & \frac{1}{120} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & \frac{1}{120} & \frac{13}{60} & \frac{7}{10} & \frac{13}{60} & \frac{1}{120} & 0 \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{120} & \frac{5}{24} & \frac{10}{3} & 0 & 0 \\
\end{pmatrix}_{2^j \times 2^j}$$

Thanks to the properties of the Rayleigh quotient [6], one has

$$\lambda_{\min}(G_j) \|c_j\|_{L^2(I_j)}^2 \leq \| \sum_{k \in I_j} c_{j,k} \phi_{j,k} \|_{L^2(\Omega)}^2 \leq \lambda_{\max}(G_j) \|c_j\|_{L^2(I_j)}^2,$$

where $\lambda_{\min}(G_j)$ and $\lambda_{\max}(G_j)$ denote the minimal and maximal eigenvalue (in absolute value) of the matrix $G_j$ respectively.
To estimate the eigenvalues of $G_j$, one uses Gerschgorin’s theorem [7] to obtain that

$$\sigma(G_j) \subseteq B_{\frac{10}{3}} \left( \frac{10}{3} \right) \cup B_{\frac{7}{10}} \left( \frac{7}{10} \right),$$

where $B_r(x) := \{ y \in \mathbb{R} \mid |x - y| \leq r \}$, $\sigma(G_j)$ denotes the spectrum of the matrix $G_j$. Therefore

$$\lambda_{\min}(G_j) \geq \min \left\{ \frac{10}{3} - \frac{13}{60} \cdot \frac{7}{10} - \frac{9}{20} \right\} = \min \left\{ \frac{187}{60} \cdot \frac{1}{4} \right\} = \frac{1}{4},$$

$$\lambda_{\max}(G_j) \leq \max \left\{ \frac{10}{3} + \frac{13}{60} \cdot \frac{7}{10} + \frac{9}{20} \right\} = \max \left\{ \frac{71}{20} \cdot \frac{23}{20} \right\} = \frac{71}{20}.$$

Hence, one obtains

$$\frac{1}{2} \| \xi_j \|_{L^2(\Omega)} \leq \| \sum_{k \in I_j} c_{ijk} \Phi_j \|_{L^2(\Omega)} \leq \sqrt{\frac{71}{20}} \| \xi_j \|_{L^2(\Omega)} ,$$

i.e., $\Phi_j$ is a Riesz sequence in $L^2(\Omega)$, and the Riesz bounds are independent of the level $j$. \(\square\)

**Theorem 1.** Let $V_j := \text{span} \Phi_j, j \geq 2$, then one obtains

(i) $V_j$ is a closed subspace of Sobolev space $H^\mu_0(\Omega), 0 < \mu < \frac{3}{2}$;

(ii) $\dim V_j = 2^j$;

(iii) $V_j \subset V_{j+1}$;

(iv) $P_2(\Omega) \subset V_j$, where $P_2(\Omega)$ denotes the space of all polynomials $p$ of degree at most 2 on $\Omega$.

For $j \geq 2$, since $\{ B^j_{k,m}(x), k \in I_j \}$ are local linear independence, one can find a continuous function $\bar{B}^j_{m,3}(x)$, where $\text{supp} \bar{B}^j_{m,3}(x) \subset \text{supp} B^j_{k,3}$ such that

$$\langle B^j_{k,3}, \bar{B}^j_{m,3} \rangle_{L^2(\Omega)} = \delta_{k,m} = \begin{cases} 1, & k = m, \\ 0, & k \neq m. \end{cases}$$

Obviously, there exists a positive constant $M$ which is independent of $j$, such that $\| \bar{B}^j_{m,3} \|_{L^2(\Omega)} \leq M$. Use the notation

$$\tilde{\phi}_{j,k}(x) := 2^j \bar{B}^j_{k,3}(2^j x), \quad k \in I_j, \quad x \in \mathbb{R},$$

then $\text{supp} \tilde{\phi}_{j,k} \subset \text{supp} \Phi_{j,k} \subset \Omega$. $\langle \phi_{j,k}, \tilde{\phi}_{m,l} \rangle_{L^2(\Omega)} = \delta_{k,m}$.

**Theorem 2.** For any $f \in H^2(\Omega) \cap H^\mu_0(\Omega)$, one obtains

$$\| P_j f - f \|_{L^2(\Omega)} \lesssim 2^{-j} \| f \|_{H^2(\Omega)}, \quad j \geq 2,$$

where $0 < \mu < \frac{3}{2}$. Here and throughout, the notation $A \lesssim B$ indicates that $A \leq cB$ with a positive constant $c$ which is independent of $A$ and $B$. If $A \lesssim B$ and $B \lesssim A$, we call $A$ and $B$ are equivalent, denoted by $A \sim B$. 
Theorem 3. For any $j \geq 2$, one has
(i) $\psi(k) = \psi_b(k) = 0$, where $k \in \mathbb{Z}$;
(ii) $\text{supp } \psi = [0, 3]$, $\text{supp } \psi_b = [0, 2]$;
(iii) $\text{supp } \psi_j(k, x) = \begin{cases} [0, 2^{-j+1}], & k = 1, \\ [2^{-j}(k-2), 2^{-j}(k+1)], & k = 2, 3, \ldots, 2^j - 1, \\ [2^{-j+1}, 1], & k = 2^j. \end{cases}$
(iv) $\Psi_j = \frac{1}{\sqrt{2}} M^T_{j,1} \Phi_{j+1}$, where

$$
M^T_{j,0} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0
\\
0 & 1 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0
\\
0 & 0 & 0 & 1 & -1 & -1 & \cdots & 0 & 0 & 0 & 0 & 0
\\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -1 & -1 & 1 & 0
\\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 & 1
\\
\end{pmatrix}_{2^j \times 2^j+1}
$$

For all $j \geq 2$, let $Q_j$ be the linear projection from $V_{j+1}$ onto $V_j$ given as follows: for $f_{j+1} \in V_{j+1}$, $f_j := Q_j f_{j+1}$ is the unique element in $V_j$ determined by the interpolation condition [4]

$$
f_j \left( \frac{k}{2^j} \right) = f_{j+1} \left( \frac{k}{2^j} \right), \quad k = 1, 2, \ldots, 2^j - 1,
$$

then $\psi_{j,k} \in V_{j+1}$, and $Q_j \psi_{j,k} = 0$. Define

$$
W_j := \text{span } \Psi_j,
$$

then $\dim W_j = 2^j$. Moreover, $\Psi_j$ is a Riesz basis of $W_j$ and its Riesz bounds are independent of $j$. Since $\dim V_j + \dim W_j = \dim V_{j+1}$, one obtains

$$
V_{j+1} = V_j \oplus W_j.
$$

Therefore

$$
V_{j+1} = V_2 \oplus \bigoplus_{i=2}^j W_i.
$$

Suppose that

$$
M_j := (M_{j,0}, M_{j,1}),
$$

then from Lemma 3 (v) and Theorem 3 (iv), one obtains

$$
\begin{pmatrix}
\Phi_{j-1} \\
\Psi_{j-1}
\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}
M^T_{j-1,0} \\
M^T_{j-1,1}
\end{pmatrix} \Phi_j = \frac{1}{\sqrt{2}} M^T_{j-1,j} \Phi_j (\forall j \geq 3).
$$

Recursively, one has

$$
\begin{pmatrix}
\Phi_2 \\
\Psi_2 \\
\Psi_3 \\
\vdots \\
\Psi_{j-1}
\end{pmatrix} = \begin{pmatrix}
I_3 \\
I_4 \\
\ddots \\
I_{j-1}
\end{pmatrix} \left( \begin{pmatrix}
\frac{1}{\sqrt{2}} M^T_{2,0} \\
\frac{1}{\sqrt{2}} M^T_{2,1} \\
\ddots \\
\frac{1}{\sqrt{2}} M^T_{j-2,j-2} \\
\frac{1}{\sqrt{2}} M^T_{j-1,0}
\end{pmatrix} \right) \frac{1}{\sqrt{2}} M^T_{j-1,j} \Phi_j.
That is
\[ \Psi^j = T_j \Phi_j, \]  
where
\[ T_j := \begin{pmatrix} \frac{1}{\sqrt{2}} M^j_1 & I_3 & \cdots & \frac{1}{\sqrt{2}} M^j_{j-1} \\ I_3 & I_4 & \cdots & I_{j-1} \end{pmatrix}, \]  
\[ (8) \]

\( I_j \) is a \( 2^j \times 2^j \) unity matrix. Since the determinant of \( M_j \) is equal to 0.5, one obtains \( \Psi^{j+1} := \Phi_2 \cup \Psi_2 \cup \Psi_3 \cup \cdots \cup \Psi_j \) is a basis of the space \( V_{j+1} \).

3. Characterization Theorem

In this section, we use wavelet bases to characterize Sobolev space \( H^\mu_0(\Omega) \), where \( 0 < \mu < \frac{3}{2} \).

Let \( \nabla_y f(\cdot) := f(\cdot) - f(\cdot - y) \), \( \nabla_y^m := \nabla_y \nabla_y^{m-1}, (m > 1) \). For \( x, y \in \mathbb{R} \), we use \([x, y]\) to denote the line segment \( \{(1-t)x + ty : 0 \leq t \leq 1\} \). \( \Omega_y := \{ x \in \Omega : [x - y, x] \subset \Omega \} \). The modulus of continuity of \( f \) is given by
\[ \omega(f, h)_p := \sup_{|y| < h} \| \nabla_y f \|_{L_p(\Omega_y)}, \quad h > 0. \]

The \( m \)th modulus of smoothness of \( f \) is given by
\[ \omega_m(f, h)_p := \sup_{|y| < h} \| \nabla_y^m f \|_{L_p(\Omega_{my})}, \quad h > 0. \]

Let \( \mu > 0, 1 \leq p, q \leq \infty \), the Besov space \( B^\mu_{p,q}(\Omega) \) is the collection of the functions \( f \in L_p(\Omega) \) satisfying
\[ |f|_{B^\mu_{p,q}(\Omega)} := \begin{cases} \left( \sum_{k \in \mathbb{Z}} \| 2^{k\mu} \omega_m(f, 2^{-k})_p \|_q^q \right)^{\frac{1}{q}}, & 1 < q < \infty, \\ \sup_{k \in \mathbb{Z}} \{ 2^{k\mu} \omega_m(f, 2^{-k})_p \}, & q = \infty, \end{cases} \]
where \( m \) is the least integer greater than \( \mu \). The norm for \( B^\mu_{p,q}(\Omega) \) is defined by
\[ \|f\|_{B^\mu_{p,q}(\Omega)} := \|f\|_{L_p(\Omega)} + |f|_{B^\mu_{p,q}(\Omega)}. \]

If \( p = q = 2 \), the space \( B^\mu_{2,2}(\mathbb{R}) \) is the same as the Sobolev space \( H^\mu(\mathbb{R}) \), and the semi-norm \( |\cdot|_{B^\mu_{2,2}(\mathbb{R})} \) and \( |\cdot|_{H^\mu(\mathbb{R})} \) are equivalent [9].

In the following theorem, we give a characterization of the space \( H^\mu_0(\Omega) \) via the B-spline wavelets constructed before.

**Theorem 4.** For any \( f \in H^\mu_0(\Omega), 0 < \mu < \frac{3}{2} \), there exists two constants \( C_3 > 0, C_4 > 0 \) such that
\[ C_3 |f|_{H^\mu_0(\Omega)} \leq ((2^{2\mu}\|P_2 f\|_{L^2(\Omega)})^2 + \sum_{j=3}^{\infty} (2^{2\mu}\|(P_j - P_{j-1})f\|_{L^2(\Omega)})^2)^{\frac{1}{2}} \leq C_4 |f|_{H^\mu_0(\Omega)}. \]  
\[ (9) \]

**Proof.** According to paper [10], for any \( f \in L_2(\Omega) \), one has
\[ \|P_j f - f\|_{L^2(\Omega)} \lesssim \omega_3(f, 2^{-j}). \]
Let $g_2 := P_2 f, g_j := P_j f - P_{j-1} f, j \geq 3$, then one obtains $f = \sum_{j=3}^{\infty} g_j$ is convergent in $L_2(\Omega)$.

(i) First, we show that the left part in (9) is established. Since $g_j \in V_j$, one obtains

$$g_j = \sum_{k \in I_j} b_{j,k} \phi_{j,k}, \quad j \geq 2.$$ 

Please note that $\{2^{-j\mu} \phi_{j,k} \mid j \geq 2, k \in I_j\}$ is a Bessel sequence in space $H^\mu_0(\Omega)$ [11]. Therefore,

$$|f|_{H^\mu_0(\Omega)} = \sum_{j=2}^{\infty} \sum_{k \in I_j} 2^{j\mu} |b_{j,k} 2^{-j\mu} \phi_{j,k}|_{H^\mu_0(\Omega)} \leq C_1 \sum_{j=2}^{\infty} \sum_{k \in I_j} |2^{j\mu} b_{j,k}|^2 \frac{1}{2^j}.$$ 

Since $\Phi_j$ is a Riesz bases of $V_j$ and the Riesz bounds are independent of $j$, one obtains

$$\sum_{k \in I_j} |b_{j,k}|^2 \leq C_2^2 \|g_j\|_{L^2(\Omega)}.$$ 

Therefore,

$$|f|_{H^\mu_0(\Omega)} \leq C_1 C_2 \left( \sum_{j=2}^{\infty} (2^{j\mu} \|g_j\|_{L^2(\Omega)})^2 \right)^{\frac{1}{2}}.$$ 

(ii) Next we show that the right part of inequality (9) is established. For any $f \in H^\mu_0(\Omega)$, according to Poincare inequality, one has

$$\|g_2\|_{L^2(\Omega)} = \|P_2 f\|_{L^2(\Omega)} \leq \|P_2 \| f\|_{L^2(\Omega)} \leq \tilde{C} |f|_{H^\mu_0(\Omega)}.$$ 

For $j \geq 3$, one obtains

$$\|g_j\|_{L^2(\Omega)} = \|P_j f - P_{j-1} f\|_{L^2(\Omega)} \leq \|P_j f - f\|_{L^2(\Omega)} + \|f - P_{j-1} f\|_{L^2(\Omega)} \leq \omega_3(f, 2^{-j}) + \omega_3(f, 2^{-j-1}) \leq \omega_3(f, 2^{-j}).$$ 

From the definition of Besov space, one obtains

$$\left( \sum_{j=3}^{\infty} (2^{j\mu} \|g_j\|_{L^2(\Omega)})^2 \right)^{\frac{1}{2}} \leq \left( \sum_{j=3}^{\infty} (2^{j\mu} \omega_3(f, 2^{-j}))^2 \right)^{\frac{1}{2}} \leq |f|_{H^\mu_0(\Omega)}.$$

$\square$

4. Wavelet Preconditioning

In this section, we show that our wavelet bases constructed in the previous section are very useful and efficient.

Let $u_j \in V_j$ such that

$$a(u_j, v_j) = (f, v_j)_{L^2(\Omega)}, \quad \forall v_j \in V_j. \quad (10)$$ 

Recall that $\Phi_j$ is the base of $V_j$, then the above Equation (10) is equivalent to the systems

$$A_j \bar{u}_j = f_j,$$

where $A_j := (a(\phi_{j,k}, \phi_{j,m}))_{k,m \in I_j}$, $f_j := (f, \phi_{j,m})_{L^2(\Omega)}_{m \in I_j}$, $\bar{u}_j := (u_{j,k})_{k \in I_j}$.
However, the following example shows that the condition number of the stiffness matrix \(A_j\) is almost \(O(2^j)\) \((j \to \infty)\). Hence, the system (11) is very difficult to solve without preconditioning.

**Example 1.** In Equation (1), let the coefficient function \(q(x) \equiv c, x \in \Omega\). For \(c \in \{0, 0.1, 1, 10, 100\}\), the condition numbers of the matrix \(A_j\) are shown in Table 1. It is clearly seen that the condition numbers increase exponentially with respect to the level \(j\).

| \(j/c\) | \(0\)   | \(0.1\) | \(1\)   | \(10\)  | \(100\)|
|------|------|------|------|------|------|
| 2    | 2.7836 | 2.7653 | 2.6168 | 1.9363 | 2.8893 |
| 3    | 10.0465 | 9.9519 | 9.1768 | 5.2604 | 1.8349 |
| 4    | 39.2159 | 38.8271 | 35.6489 | 19.6774 | 3.9295 |
| 5    | 155.7240 | 154.1682 | 141.4521 | 77.5918 | 14.3248 |
| 6    | 619.2913 | 613.1178 | 562.6423 | 308.6641 | 56.2159 |
| 7    | 2469.1144 | 2444.1941 | 2239.5977 | 1228.3905 | 223.5504 |
| 8    | 9857.4455 | 9756.4896 | 8935.0989 | 4896.3646 | 890.2448 |

Table 1. Condition numbers of the stiffness matrix \(A_j\).

In Table 2, the factors between two successive levels are indicated. The numbers show that the growth is by a factor of 4. Moreover, the increase rate is independent on the particular choice of \(c\).

| \(j/c\) | \(0\)   | \(0.1\) | \(1\)   | \(10\)  | \(100\)|
|------|------|------|------|------|------|
| 2    | —    | —    | —    | —    | —    |
| 3    | 3.6092 | 3.5988 | 3.5068 | 2.7167 | 0.6351 |
| 4    | 3.9034 | 3.9015 | 3.8847 | 3.7404 | 2.1416 |
| 5    | 3.9709 | 3.9706 | 3.9679 | 3.9432 | 3.6454 |
| 6    | 3.9769 | 3.9769 | 3.9776 | 3.9780 | 3.9244 |
| 7    | 3.9870 | 3.9865 | 3.9805 | 3.9797 | 3.9766 |
| 8    | 3.9923 | 3.9917 | 3.9896 | 3.9860 | 3.9823 |

Table 2. Factors of condition numbers between level \(j\) and \(j - 1\).

To overcome the above difficulty, we use the wavelets preconditioning method. In fact, as we know, the collection of functions \(\Psi^j := \Phi_2 \cup \Psi_2 \cup \Psi_3 \cup \cdots \cup \Psi_{j-1}\) is also a basis of \(V_j\), so we suppose that

\[
    u_j = \sum_{k \in I_2} a_{2,k} \phi_{2,k} + \sum_{i=2}^{j-1} \sum_{k \in I_i} b_{i,k} \psi_{i,k} := \vec{b}_j^T \Psi^j,
\]

where the vector \(\vec{b}_j^T = \{a_{2,k}, b_{i,m} | k \in I_2, i = 2, \ldots, j - 1, m \in I_i\}\) is the solution of the following equation

\[
    B_j \vec{b}_j = \vec{F}_j,
\]

where the matrix \(B_j := (a(\sigma, \eta))_{\sigma, \eta \in \Psi^j}\), \(\vec{F}_j^T := (\langle f, \eta \rangle)_{\eta \in \Psi^j}\).

In fact, we use the PCG (Preconditioned Conjugate Gradient) algorithm (please see, e.g., [12] pp. 94–95) to solve the system (13), that is

\[
    D_j^{-1} B_j D_j^{-1} \vec{b}_j = D_j^{-1} \vec{F}_j^T.
\]
where the preconditioner is a diagonal matrix

\[
D_j := \text{diag}(2^2, \ldots, 2^2, 2^3, \ldots, 2^3, 2^4, \ldots, 2^4, \ldots, 2^{j-1}, \ldots, 2^{j-1}).
\]  

(14)

Therefore, the system (13) is equivalent to the following equation

\[
B_j \bar{b}_j = \bar{F}_j,
\]

(15)

where \(B_j := D_j^{-1}B_jD_j^{-1}\), \(\bar{b}_j := D_j\bar{b}_j\), \(\bar{F}_j := D_j^{-1}\bar{F}_j\).

To show the condition number of the matrix \(B_j\) is uniformly bounded, we give the following example.

Example 2. In Equation (1), let the coefficient function \(q(x) \equiv c, x \in \Omega\). For \(c \in \{0, 0.1, 1, 10, 100\}\), the condition numbers of the matrix \(B_j\) are shown in Table 3. Comparing with the values in Table 1, it is clearly seen that the condition numbers of the matrix \(B_j\) are uniformly bounded with respect to the level \(j\). Therefore, using \(\Psi_j\) as a basis for \(V_j\) yields an asymptotically preconditioned system (15).

### Table 3. Condition numbers of the stiffness matrix \(B_j\).

| \(j/c\) | 0      | 0.1    | 1      | 10     | 100    |
|-------|--------|--------|--------|--------|--------|
| 3     | 537.2908 | 537.0562 | 535.2687 | 533.2748 | 651.7660 |
| 4     | 540.3562 | 539.5721 | 538.9766 | 536.2379 | 656.3201 |
| 5     | 545.8523 | 541.2897 | 540.3523 | 539.9642 | 663.8994 |
| 6     | 548.1359 | 546.0578 | 543.4826 | 542.8217 | 670.1432 |
| 7     | 550.9631 | 550.3573 | 548.2370 | 546.3572 | 673.3134 |
| 8     | 555.8472 | 553.9882 | 551.3587 | 550.3356 | 681.3874 |

However, Figure 1 shows that the matrix \(B_j\) is not sparse. The so-called finger structure is visible. Now we are facing the following situation:

- the stiffness matrix \(A_j\) with respect to the single-scale basis \(\Phi_j\) is sparse but ill conditioned;
- the wavelet stiffness matrix \(B_j\) is asymptotically optimal preconditioned, but not sparse.

Till now, both methods cannot be used immediately. However, \(B_j = a(\Psi_j, \Psi_j) = a(T_j\Phi_j, T_j\Phi_j) = T_ja(\Phi_j, \Phi_j)T_j^T = T_jA_jT_j^T\), \(\bar{F}_j = \langle f, \Psi_j \rangle_{L^2(\Omega)} = \langle f, T_j\Phi_j \rangle_{L^2(\Omega)} = T_j\langle f, \Phi_j \rangle_{L^2(\Omega)} = T_j\bar{f}_j\), where the matrix \(T_j\) and \(D_j\) given by (8) (14) respectively are both sparse. Therefore, we know that the matrix Hence \(B_j := D_j^{-1}T_jA_jT_j^TD_j^{-1}\), \(\bar{F}_j := D_j^{-1}T_j\bar{f}_j\), can be expressed as the product of some sparse matrices. that is Equation (15) can be written as

\[
D_j^{-1}T_jA_jT_j^TD_j^{-1} = D_j^{-1}T_j\bar{f}_j.
\]

(16)

In fact, the above Equation (16) combines both positive effects.
5. Numerical Examples

In this Section, we focus on some numerical tests.
Supposed that \( u \) is the exact solution of Equation (1), \( u_j \) is the Galerkin approximation solution given by (12), then according to paper [6] and Theorem 2, one has

\[
\|u - u_j\|_{L^2(\Omega)} \lesssim \inf_{v_j \in V_j(\Omega)} \|u - v_j\|_{L^2(\Omega)} \lesssim 2^{-2j} |u|_{H^2(\Omega)}. \tag{17}
\]

The following examples show that combining with the preconditional Galerkin method, our wavelet bases constructed before are very useful and efficient.

**Example 3.** In Equation (1), let \( q(x) = 0, f(x) = \sin(2\pi x), x \in \Omega \). Then the exact solution \( u_1 \) is given by

\[
u_1(x) = \frac{1}{(2\pi)^2} \sin(2\pi x), \quad x \in \Omega.
\]

**Example 4.** In Equation (1), let \( q(x) = \sin(2\pi x), f(x) = -\pi^2 \cos(2\pi x) + \frac{1}{4} \sin(2\pi x) - \frac{1}{4} \sin(2\pi x) \cos(2\pi x), x \in \Omega \). Then the exact solution \( u_2 \) is given by

\[
u_2(x) = \frac{1 - \cos(2\pi x)}{4}, \quad x \in \Omega.
\]

The exact solutions for the above two examples and their numerical solutions with \( j = 6 \) are showed respectively in Figure 2. The error estimates in \( L_2 \) norm and convergence factor between successive level are given in Table 4 which demonstrates that the growth factor is almost by 4 which is the same as in (17).

**Table 4.** Error estimates in \( L_2 \) norm and convergence factors.

| \( j \) | \( L_2 \) - Errors | Convergence Factors | \( L_2 \) - Errors | Convergence Factors |
|-------|-----------------|-----------------|-----------------|-----------------|
| 4     | \( 4.4245 \times 10^{-4} \) | —               | \( 1.7798 \times 10^{-2} \) | —               |
| 5     | \( 1.1480 \times 10^{-4} \) | 3.854           | \( 4.5754 \times 10^{-3} \) | 3.752           |
| 6     | \( 2.8794 \times 10^{-5} \) | 3.987           | \( 1.1752 \times 10^{-3} \) | 3.893           |
| 7     | \( 7.2003 \times 10^{-6} \) | 3.999           | \( 2.9709 \times 10^{-4} \) | 3.956           |
| 8     | \( 1.8644 \times 10^{-6} \) | 3.862           | \( 7.5905 \times 10^{-3} \) | 3.914           |
Figure 2. The figure of the Galerkin approximation solution ($j = 6$) and the exact solution $u_1$ (left), $u_2$ (right).

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