Analytic calibration in Andreasen-Huge SABR model.

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Abstract

We derive analytic formulae which link \( \alpha \), \( \nu \) and \( \rho \) parameters in Andreasen-Huge style SABR model to the ATM price and option prices at four equally spaced strikes on the sides of ATM. We give two applications. First, we give a characterisation for the SABR parameters in terms of the swap rate forward probability density function. Second, we show how Andreasen-Huge SABR can be combined with other well know analytic SABR formulae to allow consistent use in the non-arbitrage region.

1 Introduction

20 years ago Patrik Hagan, Deep Kumar, Andrew Lesniewski and Diana Woodward published a paper [1] which revolutionised interest rate volatility modelling. They derived a closed form analytic approximation for the implied volatility of a forward swap rate assuming it satisfies a system of stochastic differential equations:

\[
\begin{align*}
    dF_t &= \alpha_t F_t^{\beta} dW_t; \\
    d\alpha_t &= \nu \alpha_t dZ_t; \\
    \rho &= <W_t, Z_t>.
\end{align*}
\]

(1)

The analytic approximation gave interest rate traders a simple tool to manage the swap rate probability density distributions which are skewed and smiled over the traditional Gaussian normal distribution. The model got a name - SABR model.

Various improvements were made to the analytical formula from [1] in works [2, 3] and even more recently in [7].

One of the issues which became transparent in the industrial use of SABR model was its approximation character. As it was an approximation for the forwards behavior around the ATM level, it couldn’t possibly give the terminal swap rate probability density at every point. Polynomial expansions used in the formula were unbounded at infinity and, thus, led to negative values of the implied density function.

A striking approach to address the problem of constructing terminal swap rate probability density valid along the whole real line appeared 10 years later in another seminal paper [8]. Jasper Andreasen

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and Brian Huge showed how within SABR model assumptions one can reduce the construction of the forward swap rate probability density function to a solution of a single tridiagonal system. The construction not only resolved the issue of the validity of the probability density function at every point, but also prompted more widespread use of the density functions for modelling light exotics like CMSs and CMS spreads, which were often done by a slower replication based methods. While some market participants went for the approach [3], other developments also made into production like CMSs and CMS spreads, which were often done by a slower replication based methods. While point, but also prompted more widespread use of the density functions for modelling light exotics

An important technical problem which accompanies the use of SABR model is the parameters’ calibration. The parameter \( \beta \) can be estimated from the log-log plot of the forward variance with respect to the forward [1] and \( \alpha \) could be found as a root of a cubic equation [9]. The skew parameters \( \nu \) and \( \rho \) were often manually calibrated. Multi dimensional solvers were introduced to address parameter mapping while migrating to the shifted SABR model which became a necessity as the rates entered negative territory. Extensive studies have been made on how to chose the initial guess to optimise solver convergence [10].

Andreasen-Huge construction of the non-arbitrage SABR did not contain explicit analytics to invert, and, thus, required use of solvers to imply not only the skew parameters, but the parameter alpha as well. As Andreasen-Huge construction gives considerable computational speed up in comparison with the other arbitrage free SABR methods, the use of solvers did not represent a limitation for building infrastructure around their version of SABR model. The goal of the present paper is to show that Andreasen-Huge construction of SABR model, in fact, allows one to have exact analytic expressions of all three model parameters: \( \alpha \), \( \nu \) and \( \rho \), in terms of option prices close to ATM. In the paper, we show that if call prices \( c(F - 2h), c(F - h), c(F), c(F + h), c(F + 2h) \) are given at five consecutive grid points of an equally spaced Andreasen-Huge shifted SABR grid with the step \( h \) and \( c(F) = ATM \) - the ATM option price, then:

\[
\alpha = \frac{h}{(F + shift)^2 \sqrt{\frac{1}{T} \left[ c(F - h) + c(F + h) - 2ATM \right]}} \tag{2}
\]

\[
\nu^2 = \frac{z_+ h^2}{T \kappa_h \alpha^2 (y(F - h) - y(F + h))} - \frac{1}{y(F - h) y(F + h)}, \tag{3}
\]

\[
\rho = \frac{1}{2 \nu} \left( \nu^2 y(F - h) - \frac{1}{y(F - h) \left( T \kappa_h \alpha^2 (F - h + shift)^2 \beta - 1 \right)} \right), \tag{4}
\]

where \( y(k) \) is the shifted SABR local volatility diffusion distance from the forward to the strike, \( \kappa_h \) is the one half of the value of the Andreasen-Huge local volatility one time step ODE adjustment at the strikes \( F \pm h \) and

\[
z_+ = \frac{c(F + h) - c(F + 2h) + ATM - 2c(F + h)}{c(F + 2h) + ATM - 2c(F + h)}, \quad z_- = \frac{c(F - h) - c(F - 2h) + ATM - 2c(F - h)}{c(F - 2h) + ATM - 2c(F - h)}. \tag{5}
\]

We discuss two applications of the analytic expressions. 1) We study the limiting behavior when \( h \to 0 \), and derive a characterisation for SABR parameters in terms of probability density function of the forward swap rate distribution. 2) We show how these formulae can be used for migrating from/between analytic expansions [1] and the Andreasen-Huge construction.

The structure of the paper is as follows. In Section 1 we remind the Andreasen-Huge construction for SABR model. In Section 2 we derive analytic formulae for (shifted) SABR parameters in the Andreasen-Huge construction. In Section 3 we study the limiting behavior of the analytic formulae. In Section 4 we show how the Andreasen-Huge construction can be calibrated to the Hagan et al. original construction. We summarise our findings in the conclusion.
In this section we remind the Andreasen-Huge one time step ODE reduction for SABR SDE [8]. It rests on their other remarkable paper [11] where they showed that the option pricing heat equation:

\[
\frac{c_t(t,k)}{2} + \frac{\vartheta(k)}{\sqrt{\kappa(k)}} \frac{c_{kk}(t,k)}{2} = (F - k)^+, \quad (6)
\]

with the local volatility coefficient \(\vartheta(k)\) dependent only on the strike \(k\) and the terminal condition given by the intrinsic value of a call option, can be replaced by an ODE:

\[
c(t,k) - \frac{1}{2}(T-t)\vartheta(k)^2 c_{kk}(t,k) = (F - k)^+. \quad (7)
\]

Based on this reduction, Andreasen and Huge derived [8] the relation between \(\theta(k)\) and \(\vartheta(k)\) in terms of the standard normal distribution:

\[
\theta(k)^2 = \frac{\vartheta(k)^2 c_t(t,k) - (F - k)^+}{(T-t)c_t(t,k)} = 2\vartheta(k)^2 \left(1 - \xi \frac{\Phi(-\xi)}{\phi(\xi)}\right), \quad (8)
\]

where \(\Phi(\xi)\) is the CDF of the standard normal random variable, \(\phi(\xi)\) is the PDF of the standard normal random variable and \(\xi = |F - k|/\sigma\), with \(\sigma\) being ATM normal volatility. In what follows we shall be calling the coefficient

\[
\kappa(k) = 2 \left(1 - \xi \frac{\Phi(-\xi)}{\phi(\xi)}\right) \quad (9)
\]

as Andreasen-Huge local volatility one time step adjustment.

The ODE (7) is solved by inverting a tridiagonal matrix corresponding to the finite difference equation [11]:

\[
\left(1 + \frac{T\theta(k)^2}{2h^2}\right) c(0,k) - \frac{T\theta(k)^2}{2h^2} [c(0,k+h) + c(0,k-h)] = (F - k)^+. \quad (10)
\]

In the identical way we can solve the option pricing problem in terms of puts:

\[
\left(1 + \frac{T\theta(k)^2}{2h^2}\right) p(0,k) - \frac{T\theta(k)^2}{2h^2} [p(0,k+h) + p(0,k-h)] = (k - F)^+. \quad (11)
\]

Note, that when reducing (10) or (11) to solving the linear system with a tridiagonal matrix we need to set an absorbing boundary condition [11]:

\[
c_{kk}(0,k_{-\infty}) = c_{kk}(0,k_{\infty}) = 0, \quad (12)
\]

\[
p_{kk}(0,k_{-\infty}) = p_{kk}(0,k_{\infty}) = 0. \quad (13)
\]

In Andreasen-Huge SABR construction [8] we capitalise on the analytic form of the local volatility \(\vartheta(k)\) for SABR SDE [11]. Using shifted version of the SABR SDE [11] we can write [8, 12]:

\[
\vartheta(k) = \alpha J(y)(k + shift)^{\beta}, \quad (14)
\]

\[
y(k) = \frac{1}{\alpha} \int_k^F (u + shift)^{-\beta} du, \quad (15)
\]

\[
J(y) = \sqrt{1 - 2\rho \nu y + \nu^2 y^2}. \quad (16)
\]
Thus, in order to solve for option prices in (shifted) SABR SDE (1) we can solve linear tridiagonal system (10) or (11), whose coefficients are populated using reduction (8) and analytic formulae (14–16).

In further sections, we will need explicitly the values of the local volatility one time step adjustments for \( k = F - h, F, F + h \) respectively, where \( F \) is the forward and \( h \) is the step of the uniform grid in (10) (or (11)). They can be calculated as below:

\[
\kappa(F) = 2, \quad \kappa(F \pm h) = 2 \left( 1 - \frac{h}{\sigma \phi(h/\sigma)} \left( \frac{1}{2} - \int_0^{h/\sigma} \phi(s) ds \right) \right)
\]

\[
= 2 \left( 1 - \frac{h}{\sigma \phi(h/\sigma)} \left( \frac{1}{2} - \frac{h}{\sigma} \phi(\zeta) \right) \right), \quad \zeta \in [0; h/\sigma].
\]

In what follows we shall be using a notation \( \kappa_h = \kappa(F \pm h)/2 \).

### 3 Analytic calibration

In this section we use Andreasen-Huge one time step ODE reduction of the (shifted) SABR SDE to derive exact formulae which relate SABR SDE coefficients \( \alpha, \nu \) and \( \rho \) to the prices of five options close to ATM.

First we write the the one time step approach of the previous section explicitly in terms of put option prices (11). The values of all OTM puts \( p_0, \ldots, p_{n-1} \), ATM put \( p_n \) and the first ITM put \( p_{n+1} \) satisfy the following tridiagonal system:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-z_1 & 1 + 2z_1 & -z_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -z_2 & 1 + 2z_2 & -z_2 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -z_{n-2} & 1 + 2z_{n-2} & -z_{n-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -z_{n-1} & 1 + 2z_{n-1} & -z_{n-1} \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -z_n & 1 + 2z_n & -z_n \\
\end{pmatrix}
\begin{pmatrix}
p_0 \\
p_1 \\
p_2 \\
\vdots \\
p_{n-1} \\
p_n \\
p_{n+1} \\
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
p_0 \\
p_1 \\
\vdots \\
p_{n-1} \\
p_n \\
p_{n+1} \\
\end{pmatrix},
\]

where the coefficients \( z_j, j = 0, \ldots, n \), are given by (11):

\[
z_j = \frac{T \theta(k)^2}{2h^2}.
\]

The value \( ATM = p_n \) is (typically) observed from the market and is

\[
ATM = p_n = \sqrt{\frac{T}{2\pi} \sigma_{ATM}},
\]

whith \( \sigma_{ATM} \) - the implied normal volatility of the ATM option (put or call). We can reduce the
tridiagonal system by one dimension to

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-\tilde{z}_1 & 1 + 2\tilde{z}_1 & -\tilde{z}_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -\tilde{z}_2 & 1 + 2\tilde{z}_2 & -\tilde{z}_2 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -\tilde{z}_{n-2} & 1 + 2\tilde{z}_{n-2} & -\tilde{z}_{n-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -\tilde{z}_{n-1} & 1 + 2\tilde{z}_{n-1} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\tilde{z}_n & -\tilde{z}_n \\
\end{pmatrix}
\begin{pmatrix}
p_0 \\
p_1 \\
p_2 \\
\vdots \\
p_{n-1} \\
p_{n+1}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
\tilde{z}_{n-1}\text{ATM} \\
-(1 + 2\tilde{z}_n)\text{ATM}
\end{pmatrix}
\]

The last equation of the system:

\[ z_n(p_{n-1} + p_{n+1}) = (1 + 2z_n)\text{ATM} \]  \hspace{1cm} (21)

can be solved separately after solving

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
-\tilde{z}_1 & 1 + 2\tilde{z}_1 & -\tilde{z}_1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & -\tilde{z}_2 & 1 + 2\tilde{z}_2 & -\tilde{z}_2 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & -\tilde{z}_{n-2} & 1 + 2\tilde{z}_{n-2} & -\tilde{z}_{n-2} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & -\tilde{z}_{n-1} & 1 + 2\tilde{z}_{n-1} & 0 \\
\end{pmatrix}
\begin{pmatrix}
p_0 \\
p_1 \\
p_2 \\
\vdots \\
p_{n-1} \\
p_{n+1}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
\tilde{z}_{n-1}\text{ATM}
\end{pmatrix}
\]

The last row of the later gives a further condition:

\[ -\tilde{z}_{n-1}p_{n-2} + (1 + 2\tilde{z}_{n-1})p_{n-1} = \tilde{z}_{n-1}\text{ATM}. \]  \hspace{1cm} (22)

The same approach can be repeated for all OTM calls \(c_{n+1}, \ldots, c_{n+m}\) together with ATM call \(c_n = \text{ATM}\) and the first ITM call \(c_{n-1}\). Working in the same vein with (10) we use the first equation of the (reduced) call based system to get:

\[ -\tilde{z}_{n+1}c_{n+2} + (1 + 2\tilde{z}_{n+1})c_{n+1} = \tilde{z}_{n+1}\text{ATM}. \]  \hspace{1cm} (23)
The ATM term $z_n$ of the put (11) or the call (10) systems, is particularly simple in the shifted SABR approach:

\[
\begin{align*}
  z_n &= \frac{T\theta(F)^2}{2h^2}, \\
  &= \frac{T\vartheta(F)^2\kappa(F)}{2h^2}, \\
  &= \frac{T}{h^2}\alpha^2(F + shift)^{2\beta}.
\end{align*}
\]  

(24)

We use it together with (21) to derive an approximation for $\alpha$

\[
\alpha = \frac{h}{(F + shift)^\beta} \sqrt{\frac{1}{T} \sqrt{\frac{ATM}{p_{n-1} + p_{n+1} - 2ATM}}},
\]

(25)

Two equations (22) and (23) can be used to imply parameters $\rho$ and $\nu$ from option prices close ATM. More explicitly, we have:

\[
\begin{align*}
  z_{n-1} &= \frac{p_{n-1}}{p_{n-2} + ATM - 2p_{n-1}}, \\
  z_{n+1} &= \frac{c_{n+1}}{c_{n+2} + ATM - 2c_{n+1}},
\end{align*}
\]

(26)

with

\[
\begin{align*}
  z_i &= \frac{T\vartheta(k_i)^2\kappa(k_i)}{2h^2}, \\
  \vartheta(k) &= \alpha J(y)(k + shift)^\beta, \\
  y(k) &= \frac{1}{\alpha} \int_k^F (u + shift)^{-\beta} du, \\
  J(y) &= \sqrt{1 - 2\rho\nu y + \nu^2 y^2}.
\end{align*}
\]

(27)\quad(28)\quad(29)

Moving all the terms in (27) independent of $\nu$ and $\rho$ to one side we obtain a system:

\[
\begin{align*}
  \nu^2 y(k_{n-1}) - 2\rho \nu &= \frac{1}{y(k_{n-1})} \left( \frac{2z_{n-1}h^2}{T\kappa(k_{n-1})\alpha^2(k_{n-1} + shift)^{2\beta}} - 1 \right), \\
  \nu^2 y(k_{n+1}) - 2\rho \nu &= \frac{1}{y(k_{n+1})} \left( \frac{2z_{n+1}h^2}{T\kappa(k_{n+1})\alpha^2(k_{n+1} + shift)^{2\beta}} - 1 \right).
\end{align*}
\]

(30)\quad(31)

This system can be solved analytically for $\nu$ and $\rho$. We obtain the following result:

**Theorem 1** In Andreasen-Huge one time step approximation for the shifted SABR SDE, the parameters $\alpha$, $\nu$ and $\rho$ have the following analytic expressions:

\[
\begin{align*}
  \alpha &= \frac{h}{(F + shift)^\beta} \sqrt{\frac{1}{T} \sqrt{\frac{ATM}{p_{n-1} + p_{n+1} - 2ATM}}},
  \\
  \nu^2 &= \frac{\frac{z_{n-1}h^2}{y(F-h)(F-h+shift)^{2\beta}} - \frac{z_{n+1}h^2}{y(F+h)(F+h+shift)^{2\beta}}}{T\kappa(h)\alpha^2(y(F-h) - y(F+h))} - \frac{1}{y(F-h)y(F+h)},
  \\
  \rho &= \frac{1}{2\nu} \left( \frac{\frac{y(F+h)z_{n-1}h^2}{y(F-h)(F-h+shift)^{2\beta}} - \frac{y(F-h)z_{n+1}h^2}{y(F+h)(F+h+shift)^{2\beta}}}{T\kappa(h)\alpha^2(y(F-h) - y(F+h))} - \frac{y(F-h) - y(F+h)}{y(F-h)y(F+h)} \right),
\end{align*}
\]

(32)\quad(33)\quad(34)
where

\[
\begin{align*}
\begin{aligned}
z_{n-1} &= \frac{p_{n-1}}{p_{n-2} + ATM - 2p_{n-1}}, \\
z_{n+1} &= \frac{c_{n+1}}{c_{n+2} + ATM - 2c_{n+1}}, \\
\kappa_h &= 1 - \frac{h}{\sigma \phi(h/\sigma)} \left( \frac{1}{2} - \int_0^{h/\sigma} \phi(s)ds \right).
\end{aligned}
\end{align*}
\]

(35)

4 Limiting behavior

In this section we study the limiting behavior of (32-34) as the grid step $h$ goes to zero. First, we derive an analytic expression for $\alpha$:

\[
\begin{align*}
\alpha &= \lim_{h \to 0} \frac{h}{(fwd + shift)^{1/2} \sqrt{\frac{1}{T} \sqrt{\frac{ATM}{p_{n-1} + p_{n+1} - 2ATM}}}}, \\
&= \frac{1}{(fwd + shift)^{1/2} \sqrt{\frac{1}{T} \sqrt{\frac{ATM}{pdf_{ATM}}}}}. \\
\end{align*}
\]

(36)

(37)

If we assume that the forward swap rate PDF is approximately Gaussian normal we can recover a popular approximation:

\[
\alpha \approx \frac{1}{(fwd + shift)^{1/2} \sqrt{\frac{1}{T} \sqrt{\frac{\sigma_{ATM}^2}{pdf_{ATM}}}}} = \frac{\sigma_{ATM}}{(fwd + shift)^{1/2}}.
\]

(38)

Next, we look at the term $\rho \nu$ in (30, 31). We shall be using the following approximations:

\[
\begin{align*}
\begin{aligned}
z_{n-1}h^2 &= \frac{p_{n-1}h^2}{p_{n-2} + ATM - 2p_{n-1}} \approx \frac{p(F-h)}{pdf(F-h)}, \\
z_{n+1}h^2 &= \frac{c_{n+1}h^2}{c_{n+2} + ATM - 2c_{n+1}} \approx \frac{p(F+h) - h}{pdf(F+h)}.
\end{aligned}
\end{align*}
\]

(39)

(40)

The transformation (28) is a change of coordinates with a Jacobian:

\[
\frac{\partial y(k)}{\partial k} = -\frac{1}{\alpha(k + shift)^{1/2}}, \quad y(F) = 0.
\]

(42)

We use this change of coordinates to factor out a derivative with respect to $y(k)$ first from (30) by using (37):

\[
\rho \nu = \frac{1}{2} \lim_{h \to 0} \frac{1}{y(k_{n-1})} \left( 1 - \frac{2z_{n-1}h^2}{Tk(k_{n-1})\alpha^2(k_{n-1} + shift)^2} \right),
\]

(43)

\[
= \frac{1}{T\alpha^2} \lim_{h \to 0} \frac{1}{y(k_{n-1})} \left( \frac{p(F)}{\kappa(F)pdf(F)(F + shift)^{2\beta}} - \frac{p(F-h)}{\kappa(F-h)pdf(F-h)(F-h + shift)^{2\beta}} \right),
\]

\[
= -\frac{1}{T\alpha^2} \frac{\partial}{\partial y} \frac{p(k(y))}{\kappa(k(y))pdf(k(y))(k(y) + shift)^{2\beta}} \bigg|_{k(y) = F},
\]

(44)
where by \( \partial_- \) we denoted the left derivative with respect to \( y \). Similarly, using (31) we get

\[
\rho \nu = -\frac{1}{T} \frac{\partial_+}{\partial y} \kappa(y) \text{pdf}(k(y))(k(y) + \text{shift})^{2\beta} \bigg|_{k(y)=F},
\]

with \( \partial_+ \) being the right derivative with respect to \( y \). We can glue the functions

\[
-\frac{1}{T} \frac{\partial_-}{\partial y} \kappa(y) \text{pdf}(k(y))(k(y) + \text{shift})^{2\beta} - \frac{1}{T} \frac{\partial_+}{\partial y} \kappa(y) \text{pdf}(k(y))(k(y) + \text{shift})^{2\beta}
\]

at \( k(y) = F \). Using (33) to see that the result is a differentiable function (as \( y(k_{n-1}) \) and \( y(k_{n+1}) \) are independent), we arrive at:

\[
\nu^2 = \frac{2}{T} \frac{\partial_-}{\partial y} \kappa(y) \text{pdf}(k(y))(k(y) + \text{shift})^{2\beta} \bigg|_{k=F}
\]

\[
\nu^2 = \frac{2}{T} \frac{\partial_+}{\partial y} c(k(y)) \bigg|_{k=F}
\]

\[
\rho = -\frac{1}{\alpha \sqrt{2T}} \frac{\partial_-}{\partial y} \kappa(y) \text{pdf}(k(y))(k + \text{shift})^{2\beta} \bigg|_{k=F}
\]

\[
\rho = -\frac{1}{\alpha \sqrt{2T}} \frac{\partial_+}{\partial y} c(k(y)) \bigg|_{k=F}
\]

We summarise our findings in the next theorem:

**Theorem 2** In the short maturity approximation of Andreasen-Huge SSABR model the following analytic expressions for model parameters \( \alpha, \nu \) and \( \rho \) hold:

\[
\alpha = \frac{1}{(\text{fwd} + \text{shift})^\beta} \sqrt{\frac{1}{T} \int \frac{ATM}{\text{pdf}_{ATM}}},
\]

\[
\nu = \frac{1}{\alpha \sqrt{2T}} \int \frac{\partial_-}{\partial y} \kappa(y) \text{pdf}(k(y))(k + \text{shift})^{2\beta} \bigg|_{k=F}
\]

\[
\rho = -\frac{1}{\alpha \sqrt{2T}} \int \frac{\partial_+}{\partial y} c(k(y)) \bigg|_{k=F}
\]

where

\[
y(k) = \frac{1}{\alpha} \int_k^F (u + \text{shift})^{-\beta} du.
\]

5 Parameter recalibration

In this section we show how analytic formulae from Section 3 can be used for recalibration of different versions of SABR to each other.

First, we consider an example of EUR 10y 10y Swaption and demonstrate how the analytic skew calibration improves matching between Hagan and Andreasen-Huge SSABR approximations. We use Hagan SSABR parameters:

\[
\text{shift} = 3\%, \quad \beta = 40\%, \quad \alpha = 2.17\%, \quad \rho = -23.78\%, \quad \nu = 26.12\%
\]
Using formulae from Section 3 we find calibrated skew parameters to use in Andreasen-Huge SSABR as

\[\text{shift} = 3\%, \quad \beta = 40\%, \quad \alpha = 2.06\%, \quad \rho = -26.84\%, \quad \nu = 27.54\%. \tag{55}\]

The comparison is plotted in Figure 1.

Next we show how to recalibrate Andreasen Huge SSABR following a change of the parameter beta. We use the same Hagan data as in the previous example, but change beta from 40% to 60%. The recalibrated skew parameters for Andreasen-Huge SSABR are:

\[\text{shift} = 3\%, \quad \beta = 60\%, \quad \alpha = 4.08\%, \quad \rho = -35.88\%, \quad \nu = 29.50\% \tag{56}\]

The comparison is plotted in Figure 2.
Similarly we consider beta change from 40% to 20%. The recalibrated skew parameters for Andreasen-Huge SSABR are:

\[
\begin{align*}
\text{shift} &= 3\%, \quad \beta = 20\%, \quad \alpha = 1.05\%, \quad \rho = -16.27\%, \quad \nu = 25.84\% \\
\end{align*} \tag{57}
\]

The comparison is plotted in Figure 3.

![Figure 3: EUR 10y 10y Swaption Skew, Hagan beta 40% vs Andreasen-Huge beta 20%](image)

We shall now consider a change of shift from 3% to 3.25%. The recalibrated skew parameters for Andreasen-Huge SSABR are:

\[
\begin{align*}
\text{shift} &= 3.25\%, \quad \beta = 40\%, \quad \alpha = 2.01\%, \quad \rho = -25.57\%, \quad \nu = 27.17\% \\
\end{align*} \tag{58}
\]

The comparison is plotted in Figure 4.

![Figure 4: EUR 10y 10y Swaption Skew, Hagan shift 3% vs Andreasen-Huge shift 3.25%](image)

If we also change Andreasen-Huge beta from 40% to 50% we obtain recalibrated skew parameters for Andreasen-Huge SSABR as:

\[
\begin{align*}
\text{shift} &= 3.25\%, \quad \beta = 50\%, \quad \alpha = 2.80\%, \quad \rho = -30.10\%, \quad \nu = 28.01\% \\
\end{align*} \tag{59}
\]
The comparison is plotted in Figure 5.

Figure 5: EUR 10y 10y Swaption Skew, Hagan shift 3%, beta 40% vs Andreasen-Huge shift 3.25%, beta 50%.

While analytic skew calibration allows us to make a closer match between Hagan and Andreasen-Huge SSABR implementations, we cannot achieve a perfect match between the two due to numerical limitations. Firstly, the calibration only operates in a small neighborhood of ATM and there is an unavoidable divergence at the wings. Secondly, the use of various numerical procedures to interpolate and extrapolate Andreasen-Huge SSABR in to non-grid points leads to numerical noise that prevents us from having an exact comparison.

Conclusion

We derived analytic formulae which express SABR parameters $\alpha$, $\nu$ and $\rho$ in Andreasen-Huge 1 time step framework in terms of five option prices close ATM. We used these formulae to give a characterisation for SABR parameters in terms of terminal swap rate distribution. We showed how the formulae can be used for calibrating Andreasen-Huge SSABR model to analytical approximation for the solution of the SSABR SDE, as well as how Andreasen-Huge SSABR model can be recalibrated after changes of shift or beta parameters. The later procedure can be used for constructing consistent risk in the multi parametric SSABR model where different regions of the strike space are governed by different SABR regimes.

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