Off-axis vortex in a rotating dipolar Bose–Einstein condensate

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Abstract
We consider a singly quantized off-axis straight vortex in a rotating dipolar ultracold gas in the Thomas–Fermi (TF) regime. We derive analytic results for small displacements and perform numerical calculations for large displacements within the TF regime. We prove that the dipolar interaction energy increases (decreases) as the vortex moves from the trap centre to the edge in an oblate (a prolate) trap. We show that the angular velocity representing the onset of metastability is lowered (raised) in an oblate (a prolate) trap. We find that the effect of the dipole–dipole interaction is the lowered (raised) precession velocity of an off-centre straight vortex line in an oblate (a prolate) trap.

1. Introduction

The first experimental detection of a vortex in a dilute alkali-atomic gas Bose–Einstein condensate (BEC) was made by Matthews et al in 1999 using $^{87}$Rb atoms [1], and theoretical predictions on the main features of the vortex states have been shown to agree with experiments [2] (and references therein). $^{87}$Rb has a small dipole moment, while chromium atoms possess a larger permanent magnetic dipole moment, which leads to significant dipolar interactions in addition to the usual short-range interactions. The successful Bose–Einstein condensation of $^{52}$Cr atoms has stimulated a growing interest in the study of the BEC with nonlocal dipole–dipole interactions [3–5]. This nonlocal character has remarkable consequences for the physics of rotating dipolar gases [6–20]. It has been shown that the critical angular frequency for vortex creation may be significantly affected by the dipolar interaction [6]. In addition, dipolar gases under fast rotation develop vortex lattices, which due to the dipolar interaction may be severely distorted [12], and even may change their configuration from the usual triangular Abrikosov lattice into other arrangements [13, 14]. It was shown that the dipolar interaction may significantly modify the vortex line stability. Under appropriate conditions, the dispersion law for transverse modes shows a rotonlike minimum, which for sufficiently large dipolar interaction may reach zero energy, destabilizing the Kelvin waves [15]. In the Thomas–Fermi (TF) limit, the dipole–dipole interaction changes the stability and instability conditions and the possibility of vortex lattice formation for a rotating dipolar BEC in an elliptical trap [18]. The long-range and anisotropic interactions introduce rich physical effects, as well as opportunities to control BECs. In a prolate dipolar gas with the dipoles aligned along the $z$-axis, the dipolar interaction is attractive, whereas it is repulsive for an oblate dipolar gas. As a result, the sign of the dipolar mean-field energy can be controlled via the trap aspect ratio. In this paper, we consider an off-axis vortex line in an oblate dipolar BEC with the dipoles aligned in the $z$-direction by an external field. In an oblate condensate, the vortex line can be approximated as straight. This is not the case for a prolate condensate. In that case, vortex lines are twisted. In the case of short-range contact interaction, an off-axis vortex in a BEC was studied in detail [21–26]. We analyse the effects of the dipolar interaction on the physics of an off-axis vortex. Specifically, we assume that a straight singly quantized vortex line in an oblate trap is displaced from the trap centre of the $^{52}$Cr condensate with the transverse coordinates $x_0$ and $y_0$. This paper is structured as follows. Section 2 reviews the TF solution for a dipolar condensate. Section 3 investigates the singly quantized straight vortex line in the presence of the dipole–dipole interaction. The last section discusses the results.

2. TF solution

In this section, we review the TF solution for a dipolar gas. Interparticle interaction potential in dipolar gases includes both a short-range van der Waals term and a long-range dipole–dipole term. Because of the long-range character
of the dipole–dipole interaction, scattering properties at low energies are significantly changed. In the case of a short-range interaction, only the s-wave scattering is important at low energies. However, in the case of a long-range interaction, all partial waves contribute to scattering. Within the mean-field description of the condensate, the interaction potential is well described by the following model potential [27–31]:

\[ V = g \delta(r) + \frac{\mu^2}{r^3}(1 - 3 \cos^2 \theta), \]

where \( g = \frac{4\pi\hbar^2}{m}, \) \( \alpha_i \) is the scattering length, \( d \) is the electric dipole moment (the results are equally valid for magnetic dipoles), \( r \) is the vector connecting two dipolar particles and \( \theta \) is the angle between \( r \) and the dipole orientation. In this study, we suppose that the dipoles are polarized along the \( z \)-axis.

For dipolar condensates, it is useful to introduce a dimensionless parameter that measures the relative strength of the dipolar and s-wave interactions:

\[ \epsilon_{dd} \equiv \frac{C_{dd}}{3g}, \]

where the coupling \( C_{dd} = \mu_0 \mu^2 \). Chromium atoms possess an anomalously large magnetic dipole moment \( \mu_{Cr} = 6 \mu_B \mu_B \) (the Bohr magneton), while \(^{87}\text{Rb} \) has a dipole moment equal to \( \mu_{Rb} = 1 \mu_B \) [31]. It has been shown in [32] that in the TF limit a dipolar BEC is also stable as long as \( 0 < \epsilon_{dd} < 1 \).

Consider a dipolar BEC of \( N \) particles with mass \( m \) and electric dipole \( d \) oriented in the \( z \)-direction by a sufficiently large external field. At sufficiently low temperatures, the description of the ground state of the condensate is provided by the solution of the Gross–Pitaevskii (GP) equation

\[ \frac{i\hbar}{\partial t} \Psi(r,t) = \left( -\frac{\hbar^2}{2m} \nabla^2 + V_T + g|\Psi(r,t)|^2 + d^2 \Phi_{dd}(r) \right) \Psi(r,t), \]

where \( V_T \) is the trap potential

\[ V_T = \frac{m}{2} \omega_s^2 (\rho^2 + \gamma^2 z^2), \]

where \( \rho^2 = x^2 + y^2 \) and \( \gamma = \frac{m}{\omega_z} \) is the trap aspect ratio, and \( \Phi_{dd}(r) = \int d^3r' \frac{1 - 3 \cos^2 \theta}{|r - r'|} |\Psi(r',t)|^2 \) is the mean-field potential due to dipole–dipole interactions.

Equation (3) is an integro-differential equation since it has both integrals and derivatives of an unknown wavefunction. This equation can be solved analytically if we assume that the zero-point kinetic energy associated with the density variation becomes negligible in comparison to both the trap energy and the interaction energy between atoms. In this case, the kinetic term can be omitted in the equation. This approximation is known as TF approximation. Eberlein et al showed that a parabolic density profile remains an exact solution for a harmonically trapped vortex-free dipolar condensate in the TF limit [32]:

\[ n_{bg}(r) = n_0 \left( 1 - \frac{\rho^2}{R^2} - \frac{z^2}{L^2} \right) \]

In the absence of the dipolar interaction, the condensate aspect ratio, \( \kappa \equiv \frac{\theta}{\gamma} \), and the trap aspect ratio, \( \gamma \), match. However, the presence of the dipolar interaction changes the condensate aspect ratio. It satisfies the following equation:

\[ 3\kappa^2 \epsilon_{dd} \left[ \frac{(\gamma^2 + 1)}{2} \frac{f(\kappa)}{1 - \kappa^2} - (\epsilon_{dd} - 1)(\kappa^2 - \gamma^2) \right] = 0, \]

where \( f(\kappa) \) for oblate case \( (\kappa > 1) \) is given by

\[ f(\kappa) = \frac{2 + \kappa^2 \left( 4 - 6 \frac{|\gamma^2 - 1|}{\sqrt{\gamma^2 - 1}} \right)}{2(1 - \kappa^2)}. \]

In the TF regime, the mean-field potential integral due to dipole–dipole interactions, \( \Phi_{dd}(r) \), can be evaluated in the spherical coordinates [32]. The result can be expressed in cylindrical coordinates:

\[ \Phi_{dd}^{bg}(\rho, z) = \frac{n_0 C_{dd}}{3} \left( \frac{\rho^2 - \frac{2z^2}{L^2}}{\rho^2} - f(\kappa) \left( 1 - \frac{3}{2} \frac{\rho^2 - 2z^2}{\rho^2 - L^2} \right) \right). \]

Let us now write the energy expression for a dipolar gas. The total energy for a dipolar gas can be written as

\[ E_{tot} = E_{kin} + E_{trap} + E_{sw} + E_{dd}, \]

where \( E_{kin} \) is the kinetic energy

\[ E_{kin} = -\frac{\hbar^2}{2m} \int d^3r \nabla^2 |\Psi(r)|^2, \]

\( E_{trap} \) is the trap energy

\[ E_{trap} = \frac{1}{2} \int d^3r |\Psi(r)|^4, \]

\( E_{sw} \) is the energy due to the short-range interaction

\[ E_{sw} = \frac{g}{2} \int d^3r |\Psi(r)|^4 \]

and \( E_{dd} \) is the energy due to the long-range dipole–dipole interaction

\[ E_{dd} = \frac{1}{2} \int d^3r d^3r' |\Psi(r)|^2 U_{dd}(r - r') |\Psi(r')|^2 \]

\[ = \frac{1}{2} \int d^3r \sigma(r) \Phi_{dd}(r). \]

In the TF approximation, the kinetic energy term (10) is neglected. The total energy associated with the vortex-free Thomas–Fermi solution is given by [32]

\[ E_{tot} = N \frac{\hbar^2}{14m} \omega_s^2 R^2 \left( 2 + \frac{\gamma^2}{\kappa^2} \right) + \frac{15}{28\pi} N^2 g \left( 1 - \epsilon_{dd} f(\kappa) \right). \]

### 3. Dipolar condensate with a vortex

Consider a single straight vortex line at a position \( \rho_0 \) along the \( z \)-axis. In this case, the wavefunction, normalized to the total number of atoms \( \int |\Psi|^2 d^3r = N \), is given by

\[ \Psi(\rho, \phi, z) = \sqrt{n(r)} e^{iS(\rho_0, \rho)}, \]

where \( n(r) \equiv |\Psi(r)|^2 \) is the density. The expression of the phase \( S(\rho, \rho_0) \) characterizing the circulating flow around the vortex line is given by [2]

\[ S(\rho, \rho_0) = \arctan \left( \frac{\gamma - \gamma_0}{\kappa - \kappa_0} \right). \]
where \( \rho_0 = \frac{\rho_0}{L_z} \). The corresponding irrotational flow velocity is given by

\[
\vec{v} = \frac{\hbar}{m} \nabla S(\rho, \rho_0).
\]

There is a singularity on the vortex line, \( \rho = \rho_0 \), where the velocity diverges. However, the particle current density, \( J = n \vec{v} \), vanishes as \( \rho \to \rho_0 \). When a quantized vortex is present at the position \( \rho_0 \), the density drops to zero at the centre of the vortex core whose size is determined by the parameter \( \beta \). For a centred vortex in a BEC without the dipole–dipole interaction, the parameter \( \beta \) is given by

\[
\beta_{sw}(\rho) = \frac{\beta_{sw}(\rho_0)}{\sqrt{1 - \rho_0^2/R^2}},
\]

where \( d_\perp = \sqrt{\hbar/m|\rho_0|} \) is the mean oscillator strength. The TF length scale reads \( \rho \ll d_\perp \ll R \). The vortex core size increases with \( \rho_0 \). The parameter \( \beta_{sw}(\rho_0) \) characterizing the small vortex core at the position \( \rho_0 \) in a BEC without the dipole–dipole interaction is [22]

\[
\beta_{sw}(\rho_0) = \frac{\beta_{sw}}{\sqrt{1 - \rho_0^2/R^2}}.
\]

Having written the expression of the phase for a straight off-centre vortex, let us now find the density \( n(\vec{r}) \). The repulsive interactions and the repulsive dipolar interaction (for oblate case) significantly expand the condensate, so that the kinetic energy associated with the density variation becomes negligible compared to the trap energy and interaction energies. In the TF regime, the density profile of a condensate with a straight off-axis vortex line at \( \rho_0 \) is given by [21]

\[
n(\vec{r}) = n_0 \left(1 - \frac{\rho^2}{R^2} - \frac{\rho^2}{L^2} \right) \left( \frac{|\rho - \rho_0|^2}{|\rho - \rho_0|^2 + \beta^2} \right),
\]

where \( n(\vec{r}) = 0 \) when the right-hand side is negative and \( \rho, R \) and \( L \) are the variational parameters that describe the size of the vortex core, and the radial and the axial sizes, respectively. Note that the density function (20) behaves like \( |\rho - \rho_0|^2/\beta^2 \) when \( \rho \ll \beta \) and like \( (1 - \rho_0^2/\rho^2 - \rho^2/R^2) \) when \( \rho \gg \beta \). These parameters will be calculated by minimizing the energy functional. The central density \( n_0 \) can be found using the normalization condition

\[
n_0 = \frac{N}{32\pi R^2} \left(60R^2 + 25\beta^2 - 4R^2 \left(3 \ln\left(\frac{2}{\beta}\right) - 4\right) + 9\rho_0^2 \left(3 - 2\ln\left(\frac{2}{\beta}\right)\right)\right).
\]

We do not include an image vortex because the form of the TF condensate density ensures that the particle current density vanishes at the surface.

Let the total angular momentum for a singly quantized vortex line along the trap axis at the position \( \rho_0 \) be \( L_z \) \( (L_z = m \int r \psi(\vec{r}) \psi^*(\vec{r}) d\vec{r}) \). Then the corresponding energy of the system in the rotating frame is \( E' = E - \Omega L_z \), where \( E \) is the energy in the non-rotating frame. If we denote the energy of the BEC in its ground state without a vortex by \( E_0 \) and the extra energy needed to generate a vortex by \( \Delta E \), then \( E = E_0 + \Delta E \). We can now write the energy of the vortex state in the rotating frame as

\[
E' = E_0 + \Delta E - \Omega L_z.
\]

A vortex is generated if \( E' \) is smaller than \( E_0 \). In other words, a vortex is formed above a certain critical value of the rotation frequency. The critical rotational velocity is given by

\[
\Omega_c = \frac{\Delta E}{L_z}.
\]

It should be noted that a vortex lattice starts to appear when the rotation frequency is further increased.

We proceed by minimizing the total energy with respect to the three variational parameters \( R, \kappa \) and \( \beta \). Let us now calculate the kinetic, trap, s-wave and dipole–dipole interaction energies separately. Since \( \beta \) is small, we neglect the terms of order \( \beta^3 \) and higher. The energy integral can be evaluated analytically up to the second order of \( \rho_0 \). Below we obtain analytical expression for small \( \rho_0 \). In the following section, we perform numerical calculations for large \( \rho_0 \) in the TF limit.

Let us firstly obtain the kinetic energy

\[
E_{\text{kin}} = \frac{\hbar^2 \pi n_0 R}{9\kappa m} \left(-22 + 12(1 + 3\beta^2) \ln\left(\frac{2}{\beta}\right) - 27\beta^2 + 18 \left(2 - (1 + 2\beta^2) \ln\left(\frac{2}{\beta}\right)\right) \beta_0^2\right),
\]

where we have defined

\[
\beta = \frac{\beta_{sw}}{R}, \quad \beta_0 = \frac{\rho_0}{R}.
\]

Using expression (11), the trapping energy is straightforwardly evaluated to be

\[
E_{\text{trap}} = \frac{4\pi m n_0 R^2 \kappa^2}{15\kappa^3} \left(\frac{2}{\beta} \right)^2 - \frac{\beta^2}{15} \left(15\kappa^2 - y^2 \left(23 - 15\ln\left(\frac{2}{\beta}\right)\right)\right) + \frac{5\beta^2}{12} \left(28\kappa^2 - 11y^2 + 6\ln\left(\frac{2}{\beta}\right) \left(y^2 - 2\kappa^2\right)\right) \beta_0^2.
\]

In the similar way, formula (12) yields the s-wave interaction energy

\[
E_{sw} = \frac{8\pi g R^2 n_0^2}{15\kappa} \left(\frac{2}{7} + \frac{107}{15} \beta^2 - 4\beta^2 \ln\left(\frac{2}{\beta}\right) - \frac{5\beta^2}{6} \left(25 - 12\ln\left(\frac{2}{\beta}\right)\right) \beta_0^2\right).
\]

Let us now calculate the dipole–dipole interaction energy. Since \( \beta \) is small, the dipolar energy function can be approximated as [6]

\[
E_{dd} \approx \frac{1}{2} \int d^3 r n_{bg}(\vec{r}) \Phi_{dd}(\vec{r}) + \int d^3 r n_v(\vec{r}) \Phi_{dd}^*(\vec{r}).
\]

where \( n_{bg} \) was defined in (5) and \( n_v \) is defined as

\[
n_v(\vec{r}) = -n_0 \frac{\beta^2}{|\rho - \rho_0|^2 + \beta^2} \left(1 - \frac{\rho^2}{R^2} - \frac{\rho^2}{L^2}\right).
\]
Note that $n(r) = n_{bg}(r) + n_{s}(r)$. Hence, the dipolar interaction energy becomes $E_{dd} = E_{odd} + ΔE_{dd}$, where

$$E_{odd} = \frac{4πg_{dd}R^{3}n_{0}^{2}}{225κ} \left( -\frac{60}{7} f(κ) + \frac{1}{κ^{2}} \left( 62 - 245κ^{2} + 30(5κ^{2} - 2) \ln \left( \frac{2}{κ} \right) \right) \right) - 122,$$

$$ΔE_{dd} = \frac{2πg_{dd}R^{3}\beta^{2}n_{0}^{2}}{9κ} \left( 50 - 24 \ln \left( \frac{2}{κ} \right) \right) + \frac{f(κ)}{κ^{2} - 1} \left( 6 + 69κ^{2} - 36κ^{2} \ln \left( \frac{2}{κ} \right) \right) \rho_{0}^{2}. \quad (30)$$

We have obtained the energy expressions up to the second order of fractional vortex displacement, $ρ_{0}$. Note that the central density $n_{0}$ in these expressions also includes $ρ_{0}^{2}$ (21). Up to the order of $β^{2}$, they agree with the results [6] in the limit $ρ_{0} → 0$.

As can be seen, the kinetic energy decreases with $ρ_{0}$. The kinetic energy goes to zero as $ρ_{0} → R$ since TF density vanishes at the surface. Note that the description of a vortex close to the boundary is outside the scope of the present approach since the TF approach does not work close to the surface. The dipole–dipole interaction increases with $ρ_{0}$ for an oblate trap while decreases with $ρ_{0}$ for a prolate trap. The kinetic energy depends on $ρ_{0}^{2}$, while the dipolar, trap and the s-wave interaction energies depend on $β^{2} ρ_{0}^{2}$.

Before embarking on a specific example, let us study the energy expressions qualitatively for an oblate trap. Firstly, let us investigate roughly how the total energy is distributed among kinetic, dipolar, trap and s-wave interaction energies. The ratio between the kinetic energy and the trap energy is of order $β^{2}$; $E_{kin} ≈ β^{2}E_{trap}$. The trap and the s-wave interaction energies are comparable to each other; $E_{sw} ≈ E_{trap}$. The ratio between dipolar and the s-wave interaction energies is of order $ε_{dd}$: $E_{dd} ≈ ε_{dd}E_{sw}$.

Secondly, let us investigate how the excess energy $ΔE$ needed to generate a vortex is distributed. Consider first a central vortex, $ρ_{0} = 0$. The excess energy for the trap, dipolar and s-wave interaction energies varies as $β^{2}$. However, the excess kinetic energy is of the order of the kinetic energy, $ΔE_{kin} ≈ E_{kin} ≈ β^{2}E_{trap}$. Hence, $ΔE_{kin} ≈ ΔE_{trap}$. We emphasize that the excess energy for dipole–dipole and s-wave interaction energies is negative. Hence, the effect of increasing the dipole moment and scattering length is the decreased critical angular velocity $Ω_{c}$. The relations between the excess energies for dipolar, trap and s-wave terms are given by $ΔE_{dd} ≈ ε_{dd}ΔE_{sw}$ and $ΔE_{trap} ≈ −ΔE_{sw}$.

Finally, let us mention how the energy changes with the position of an off-axis vortex, $ρ_{0}$. The kinetic energy and the trap energy decrease with $ρ_{0}$, while the dipole–dipole and the s-wave interaction energies increase. Furthermore, the total energy decreases with $ρ_{0}$.

Let us briefly study qualitatively the energy expressions for a prolate trap with a straight vortex line. If we neglect the vortex bending effect, we can use the above energy expressions. In this case, the dipolar interaction energy is negative. However, the excess energy for the dipolar interaction is positive. Hence, the effect of increasing the dipole moment for a prolate trap is the increased critical angular velocity $Ω_{c}$. Finally, the dipole–dipole interaction energy decreases with increasing $ρ_{0}$. As a result, the effect of dipolar interaction is to repel an off-axis vortex away from the trap centre for a prolate trap and to attract it to the trap centre for an oblate trap.

In what follows, we give an explicit example for a straight off-axis vortex for an oblate trap.

4. Results

We study a dipolar BEC with a single vortex in an oblate trap including 150 000 $^{52}$Cr atoms. We take the numerical values used in [6] to compare the off-centre vortex to the central vortex. The trap frequencies are $ω_{z} = 2π × 200$ rad s$^{-1}$ and $ω_{r} = 2π × 1000$ rad s$^{-1}$ for $γ = 5$. The harmonic oscillator length of the trap along the radial direction is $d_{z} = 0.986$ μm. The magnitude of the magnetic dipole interaction for $^{52}$Cr is $C_{dd} = μ_{0}(6μ_{B})^{2}$. For small values of $ρ_{0}$, we use the analytical results obtained in the previous section. For large values $ρ_{0}$, numerical computation within the TF limit is performed.

Let us firstly analyse the three variational parameters $β$, $κ$ and $R$. The density of the condensate drops to zero at the centre of the vortex core whose size is equal to $β$. It is very small compared to the radial size of the condensate. The smallness of $β$ ensures that the vortex affects the density only in the immediate vicinity of the core. Figure 1 depicts the fractional vortex core size, $β = β/R$, versus the scattering length. The solid curve corresponds to a central vortex, while the dashed curve to an off-centre vortex with $ρ_{0} = 0.4R$. The parameter $β$ is bigger in the presence of an off-axis vortex. As can be seen from the figure, the fractional vortex core size decreases with increasing scattering length. This can be understood as follows. The radial size increases as scattering length is enlarged. The vortex core size is inversely proportional to the radial size. So we conclude that $β$ decreases with $α_{s}$.

Similarly, figures 2 and 3 show the aspect ratio $κ$ and the radial size of the condensate $R$ versus the scattering length, respectively, for $ρ_{0} = 0$ and $ρ_{0} = 0.4R$. Contrary to the case of the vortex core size $β$, the parameters $κ$ and $R$ do not change appreciably with $ρ_{0}$ when $α_{s} > 50α_{0}$, where $α_{0}$ is the Bohr radius. Hence, the curves lie on top of each other in figures 2 and 3.

Having discussed the three variational parameters, let us now study the total energy of a dipolar condensate. Figures 4 and 5 show the total energy as a function of the vortex position for fixed $α_{s} = 100α_{0}$ and scattering length for fixed $ρ_{0} = 0.2$ in a non-rotating oblate trap ($Ω = 0$), respectively. The solid curve corresponds to a condensate with s-wave plus dipolar interactions, while the dashed line corresponds to a condensate with a pure s-wave interaction. The total energy is bigger when $ε_{dd} ≠ 0$. This is because the dipole–dipole interaction energy is positive in an oblate trap. The total energy of the system attains a maximum when $ρ_{0} = 0$ for both cases. As the off-axis vortex moves to the edge of the condensate, the total energy
Figure 1. The fractional vortex core size for central ($\tilde{\rho} = 0$) and off-axis ($\tilde{\rho} = 0.4$) vortices versus the scattering length for a dipolar BEC with a vortex in an oblate trap with $\gamma = 5$. The scattering length is measured in units of Bohr radius, $a_0$.

Figure 2. For $\tilde{\rho} = 0$ and $\tilde{\rho} = 0.4$, the aspect ratio of a condensate with a vortex in an oblate trap with $\gamma = 5$ as a function of the scattering length. The scattering length is measured in units of Bohr radius, $a_0$.

Figure 3. The radial size of a dipolar BEC with a vortex in an oblate trap with $\gamma = 5$ for $\tilde{\rho} = 0$ and $\tilde{\rho} = 0.4$. $R$ is measured in units of $d_\perp$ and the scattering length is measured in units of Bohr radius, $a_0$.

decreases. More specifically, the kinetic and trap energies decrease with $\tilde{\rho}_0$, while the dipolar and s-wave interaction energies increase with $\tilde{\rho}_0$. In fact, for higher than a specific value of $\epsilon_{dd}$, dipolar interaction becomes more dominant, so the total energy increases with $\tilde{\rho}_0$. We calculate that this happens when $\epsilon_{dd} > 1$. As mentioned in [6], however, the condensate enters an instability region when $\epsilon_{dd} > 1$. As can be seen from figure 5, the energy differences between the two cases decrease when the scattering length is increased. This is because $\epsilon_{dd}$ is decreased with increasing scattering length (2).
For the investigation of the vortex generation, not the total energy but the excess energy $\Delta E$ associated with the presence of an off-axis straight vortex is more important. Figure 6 compares the excess energy of the condensates with $\epsilon_{dd} = 0.15$ (solid curves) and $\epsilon_{dd} = 0$ (dashed curves) as a function of a fractional vortex displacement. Different curves represent different fixed values of the external angular velocity $\Omega$. The top solid and dashed curves correspond to $\Omega = 0$, where $\Omega$ increases as one moves towards the lowest curve with $\Omega = 0.08\omega_\perp$ and $\Omega = 0.140\omega_\perp$. Note that the critical rotation frequency is $\Omega_c = 0.124\omega_\perp$ ($\Omega_c = 0.119\omega_\perp$) when $\epsilon_{dd} = 0$ ($\epsilon_{dd} = 0.15$). As can be seen from the figure, the dipolar interaction lowers $\Delta E$ compared to the pure contact interaction. It is of great importance to note that although dipolar interaction is positive for an oblate trap, the excess dipolar energy is negative. As $\bar{\rho}_0$ is increased, the curves for $\epsilon_{dd} = 0.15$ and $\epsilon_{dd} = 0$ start to coincide. The top two curves show that the excess energy $\Delta E$ decreases monotonically with increasing $\bar{\rho}_0$, with negative curvature at $\bar{\rho}_0 = 0$. So a central vortex is unstable to infinitesimal displacements. The presence of dissipation will move an off-axis vortex toward the edge of the condensate. If the trap is rotated with the angular velocity $\Omega$, then the energy of a vortex decreases. Inspection of figure 6 reveals that with increasing rotation speed, the function $\Delta E$ flattens. At a special value of rotation frequency, $\Omega_m$, the curvature of the function $\Delta E$ becomes zero at $\bar{\rho}_0 = 0$. Hence, above an angular velocity $\Omega_m$, the vortex attains a local minimum. The central position is not globally stable but locally stable. One of the results of this paper is that the presence of dipolar interaction lowers $\Omega_m$ for an oblate trap. Let us look at the lowest curves in figure 6. In this case, the appearance of a vortex becomes energetically favourable since $\Delta E < 0$. The central vortex is both locally and globally stable relative to the vortex-free state. A vortex initially placed off-centre will follow a path of constant energy under the action of the Magnus force, which is proportional to the gradient of the energy in the radial direction. The precession velocity of a displaced vortex of a nonrotating trap increases with the vortex displacement. Hence, a vortex near the surface precesses more rapidly than one near the centre. Another result of this paper is that the precession velocity of a displaced vortex is lowered in the presence of the dipolar interaction.
Figure 6. The increased energy $\Delta E$ in units of $N\hbar\omega_\perp$ in the rotating frame associated with the presence of an off-axis straight vortex as a function of a fractional vortex displacement in an oblate trap. The solid (dashed) curves correspond to $\epsilon_{dd} = 0.15$ ($\epsilon_{dd} = 0$). Different curves represent different fixed values of the external angular velocity $\Omega$. The top solid and dashed curves corresponds to $\Omega = 0$, where $\Omega$ increases as one moves towards the lowest curve with $\Omega = 0.08\omega_\perp$. The solid (dashed) curves correspond to $\epsilon_{dd} = 0$ ($\epsilon_{dd} = 0.15$). Different curves represent different fixed values of the external angular velocity $\Omega$. The top solid and dashed curves corresponds to $\Omega = 0$, where $\Omega$ increases as one moves towards the lowest curve with $\Omega = 0.08\omega_\perp$.

Figure 7. The critical angular velocity of a condensate with a vortex for $\gamma = 5$ and $\gamma = 10$ as a function of the scattering length. $\Omega_c$ is measured in units of $\omega_\perp$.

interaction in an oblate trap. In contrast, it is raised in a prolate trap (ignoring vortex bending effect). Note that the precession velocity around the centre for a nonrotating trap, $\omega$, can be calculated using $\omega = \frac{\partial E}{\partial L} = \frac{\partial E}{\partial \rho_0} \frac{\partial L}{\partial \rho_0}$, where $E$ is the energy and $L$ is the angular momentum [2, 22]. For a condensate in rotational equilibrium at an angular velocity $\Omega$, the original precession frequency is altered to $\omega \rightarrow \omega - \Omega$ [2].

Finally, in figure 7, we have examined the critical angular velocity of the condensate for $\gamma = 5$ and $\gamma = 10$. The critical angular velocity above which a vortex state is energetically favourable depends on $\gamma$. As can be seen, $\Omega_c$ increases with decreasing $\gamma$. For stirring frequencies below $\Omega_c$, no vortex can be nucleated. The presence of the dipole–dipole interaction decreases $\Omega_c$ for an oblate trap.

We have found that $\Omega_c$, $\Omega_m$ and precession velocity decrease (increase) in an oblate (a prolate) trap. This can be understood simply as follows. The dipolar mean-field potential has a parabolic profile $\Phi^{dd}_{\rho_0}(\rho, z) = n_0 C_{dd}/3L^2(1.21L^2 - 0.04\rho^2 - 1.83z^2)$ when $\epsilon_{dd} = 0.15$ and $\gamma = 5$ (8). This potential has the same inverted parabola shape as in the case of contact interactions. So we conclude that there is a similarity between the dipolar and non-dipolar BEC in the TF regime. The difference is in the expressions for the radial and axial size. It is well known that the contact interaction with a positive scattering length decreases the critical angular frequency $\Omega_c$ ($\Omega_c = \omega_\perp$ for a noninteracting trapped gas). In the similar way, $\Omega_m/\omega_\perp$ and precession velocity decrease with increasing scattering length. So we conclude that the inclusion of the dipolar interaction in an oblate trap reduces $\Omega_c$, $\Omega_m$ and precession velocity in the TF regime. Furthermore, if we ignore the vortex bending effect, the mean-field dipolar potential for a prolate trap has the same form as the mean-field contact potential with a negative scattering length. In contrast to the case for repulsive interactions, $\Omega_c$, $\Omega_m$ and precession velocity increase in the presence of attractive contact interactions. Analogously, we conclude that dipolar interactions increase them in a prolate trap.

In this paper, off-axis vortex lines in an oblate $^{52}$Cr BEC polarized along the $z$-direction have been studied. The effects of the dipolar interaction on the physics of an off-axis vortex have been analysed. It was shown that the condensate aspect

\[ C_{\rho_0} = \frac{1}{3} \frac{n_0}{L^2} \left( 1.21L^2 - 0.04\rho^2 - 1.83z^2 \right) \]
ratio, $\kappa$, and the radial size, $R$, remain almost the same in the presence of an off-axis vortex when $a_t > 50a_0$. The dipolar interaction raises (lowers) the total energy in an oblate (a prolate) trap. In contrast, the excess dipolar energy needed to generate a vortex decreases (increases) in an oblate (a prolate) trap. It was found that the angular velocity $\Omega_m$ representing the onset of metastability and the critical angular velocity of the condensate $\Omega_c$ are lowered (raised) in an oblate (a prolate) trap. Finally, it was proven that the effect of the dipole–dipole interaction is the lowered (raised) precession velocity of an off-axis straight vortex line around the centre in an oblate (a prolate) trap.

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