On two-qubit states ordering with quantum discords

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Abstract – The counterintuitive effect of non-unique ordering of two-qubit states with quantum entanglement measures was discovered over ten years ago. More precisely, it was shown by Monte Carlo simulations that there exist states for which the entanglement of formation and the negativity do not impose the same ordering of states, i.e. \( E_F(\rho_{AB}) \leq (\geq) E_F(\rho'_{AB}) \) is not equivalent to \( N(\rho_{AB}) \leq (\geq) N(\rho'_{AB}) \). Recently, it was discovered that quantum discord and the geometric quantum discord do not necessarily imply the same ordering of two-qubit X-states, which means that the lack of the unique ordering of states with quantum entanglement measures goes beyond entanglement. Inspired by this observation, we study the problem of the states ordering with quantum discords, considering two-qubit Bell-diagonal states for analytical simplicity. In particular, we identify some classes of states for which the states ordering with quantum discords is preserved as long as the states belong to the same class and give a few illustrative examples.

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Introduction. – In quantum information theory, the problem of characterization of correlations present in a quantum state has been intensively studied during the last two decades (for review, see \[1,2\]). The most significant progress has been made in this subject in the framework of paradigm based on the entanglement-separability dichotomy introduced by Werner \[3\]. Within this paradigm, the quantum correlations are identified with entanglement which is quantified by many entanglement measures. However, it has become clear gradually that quantum correlations cannot be only limited to entanglement because separable quantum states can also have non-classical correlations \[4–10\], and therefore it has become clear that the entanglement-separability paradigm is too narrow and needs reconsideration.

The first step in this direction was the introduction of the concept of quantum discord, the difference of two natural extensions of the classical mutual information, which can be used as a measure of non-classical correlations beyond quantum entanglement \[11,12\].

After the recent discovery \[13–15\] that non-classical correlations other than entanglement can be responsible for the quantum computational efficiency of deterministic quantum computation with one pure qubit \[4\], quantum discord became a subject of intensive study in different contexts, such as complete positivity of reduced quantum dynamics \[16,17\], broadcasting of quantum states \[18\], random quantum states \[19\], dynamics of quantum discord \[20–25\], operational interpretation of quantum discord \[26,27\], connection between quantum discord and entanglement irreversibility \[28\], relation between quantum discord and distillable entanglement \[29\], relation between quantum discord and distributed entanglement of formation \[30,31\], and monogamy of quantum discord \[32,33\].

For pure states, quantum correlations characterized by quantum discord can be identified with quantum entanglement as measured by the entanglement of formation. However, for mixed states it was shown that quantum entanglement, as measured by the entanglement of formation, may be smaller or larger than quantum discord \[34–36\].

Because evaluation of quantum discord involves the complicated optimization procedure, another measure of non-classical correlations beyond quantum entanglement was needed. Recently, Dakić et al. \[37\] introduced the geometric quantum discord, which involves a simpler optimization procedure than quantum discord. The geometric quantum discord was studied in different contexts, such as the quantum computational efficiency of deterministic quantum computation with one pure qubit \[37\], dynamics of the geometric quantum discord \[38–42\], relation between the geometric quantum discord and other measures of non-classical correlations \[36,43–45\].
The geometric quantum discord could be used instead of quantum discord, if one showed that they give consistent results, taking into account that both quantum discord are observable measures of quantum correlations [46,47].

Recently, Yeo et al. [40] discovered, quite unexpectedly, that quantum discord is not consistent, namely they showed that quantum discord and the quantum geometric discord do not necessarily imply the same ordering of two-qubit X-states which means that the following condition,

\[ D_A(\rho_{AB}) \leq (\geq) D_A(\rho'_{AB}) \iff D_A^G(\rho_{AB}) \leq (\geq) D_A^G(\rho'_{AB}), \tag{1} \]

is not satisfied for arbitrary states \( \rho_{AB} \) and \( \rho'_{AB} \). Therefore, the lack of the unique ordering of states with quantum entanglement measures [48–58] goes beyond entanglement.

In this paper, we investigate the problem of the states ordering with quantum discord considering a large family of two-qubit states, namely two-qubit Bell-diagonal states. In particular, we identify wide classes of states for which quantum discord gives consistent results.

**Quantum discord and geometric quantum discord.** – The quantum mutual information of a state \( \rho_{AB} \),

\[ I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{2} \]

is a measure of the total correlations present in a state \( \rho_{AB} \), where \( \rho_{A(B)} \) is the reduced state of the system \( A(B) \), and \( S(\rho) = -\text{Tr}(\rho \log \rho) \) is the von Neumann entropy. The quantum conditional entropy, \( S(\rho_{AB}) = S(\rho_{AB}) - S(\rho_A) \), allows one to rewrite the quantum mutual information in the following form:

\[ I(\rho_{AB}) = S(\rho_B) - S(\rho_{AB}). \tag{3} \]

The fact that the quantum conditional entropy quantifies the ignorance about the system \( B \) that remains even if we make measurements on the system \( A \) allows one to find an alternative expression for the quantum conditional entropy, and thereby the quantum mutual information.

If the von Neumann projective measurement, described by a complete set of one-dimensional orthogonal projectors, \( \{\Pi_i^A\} \), corresponding to outcomes \( i \), is performed, then the state of the system \( B \) after the measurement is given by \( \rho_{B|i} = \text{Tr}_A[\Pi_i^A \otimes I]\rho_{AB}\Pi_i^A \otimes I]/p_i^A \), where \( p_i^A = \text{Tr}[(\Pi_i^A \otimes I)\rho_{AB}] \). The von Neumann entropies \( S(\rho_{B|i}) \) weighted by probabilities \( p_i^A \) lead to the quantum conditional entropy of the system \( B \) given the complete measurement \( \{\Pi_i^A\} \) on the system \( A \),

\[ S_{\{\Pi_i^A\}}(\rho_{B|A}) = \sum_i p_i^A S(\rho_{B|i}), \tag{4} \]

and thereby the quantum mutual information, induced by the von Neumann measurement performed on the system \( A \), is defined by

\[ J_{\{\Pi_i^A\}}(\rho_{AB}) = S(\rho_B) - S_{\{\Pi_i^A\}}(\rho_{B|A}). \tag{5} \]

The measurement-independent quantum mutual information \( J_A(\rho_{AB}) \), interpreted as a measure of classical correlations [12,59], is defined by

\[ J_A(\rho_{AB}) = \sup_{\{\Pi_i^A\}} J_{\{\Pi_i^A\}}(\rho_{AB}). \tag{6} \]

In the general case, \( I(\rho_{AB}) \) and \( J_A(\rho_{AB}) \) may differ and the difference, interpreted as a measure of quantum correlations, is called quantum discord [12]

\[ D_A(\rho_{AB}) = I(\rho_{AB}) - J_A(\rho_{AB}). \tag{7} \]

Despite the simplicity of this definition, the analytical expressions for quantum discord are known only for two-qubit Bell-diagonal states [34], for seven-parameter two-qubit X-states [35], for two-mode Gaussian states [60,61], and for a class of two-qubit states with parallel non-zero Bloch vectors [62].

Since evaluation of quantum discord involves the complicated optimization procedure, another measure of non-classical correlations beyond quantum entanglement was needed. Recently, Dakić et al. introduced the geometric quantum discord [37]

\[ D_A^G(\rho_{AB}) = \inf_{\chi_{AB}} \|\rho_{AB} - \chi_{AB}\|^2, \tag{8} \]

where the infimum is over all zero-discord states, \( D_A(\rho_{AB}) = 0 \), and \( \|\cdot\| \) is the Hilbert–Schmidt norm. Since the geometric quantum discord involves a simpler optimization procedure than quantum discord, the analytical expression for geometric quantum discord was obtained for arbitrary two-qubit states [37] as well as for arbitrary bipartite states [63].

Recently, Luo and Fu [64] showed that the geometric quantum discord is a measurement-based measure of non-classical correlations closely related to quantum discord.

**Two-qubit Bell-diagonal states ordering.** – Recently, Yeo et al. [40] discovered, studying the non-Markovian effects on quantum-communication protocols, an explicit example of two-qubit X-states for which the states ordering with respect to quantum discord significantly differs from that given by the geometric quantum discord. More recently, Girolami and Adesso [44] and independently Batle et al. [43] provided a numerical comparison between quantum discord and the geometric quantum discord for general two-qubit states, from which one can infer that there exist other states violating the states ordering with quantum discord.

Inspired by this observation, we investigate the problem of the states ordering with quantum discord.

For analytical simplicity, let us consider two-qubit Bell-diagonal states [34]

\[ \rho_{AB} = \frac{1}{4} \left( I \otimes I + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i \right), \tag{9} \]
Fig. 1: Two-qubit Bell-diagonal states belong to the tetrahedron, each of the twelve triangles is a set of states for which the states ordering is preserved.

where matrices $\sigma_i$ are the Pauli spin matrices and real numbers $c_i$ fulfill the following conditions:

$$0 \leq \frac{1}{4}(1 - c_1 - c_2 - c_3) \leq 1,$$  \hspace{1cm} (10a)
$$0 \leq \frac{1}{4}(1 + c_1 + c_2 + c_3) \leq 1,$$  \hspace{1cm} (10b)
$$0 \leq \frac{1}{4}(1 + c_1 - c_2 + c_3) \leq 1,$$  \hspace{1cm} (10c)
$$0 \leq \frac{1}{4}(1 + c_1 + c_2 - c_3) \leq 1.$$  \hspace{1cm} (10d)

The above inequalities describe a tetrahedron with vertices $(1, 1, -1), (-1, -1, -1), (1, -1, 1)$ and $(-1, 1, 1)$ (see fig. 1) corresponding to the Bell states

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$  \hspace{1cm} (11a)
$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle),$$  \hspace{1cm} (11b)
$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$  \hspace{1cm} (11c)
$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$  \hspace{1cm} (11d)

respectively.

For two-qubit Bell-diagonal states, the quantum discord is given by [34]

$$D_A(\rho_{AB}) = \frac{1}{4}[(1 - c_1 - c_2 - c_3) \log_2(1 - c_1 - c_2 - c_3) + (1 - c_1 + c_2 + c_3) \log_2(1 - c_1 + c_2 + c_3) + (1 + c_1 - c_2 + c_3) \log_2(1 + c_1 - c_2 + c_3) + (1 + c_1 + c_2 - c_3) \log_2(1 + c_1 + c_2 - c_3)] - \frac{1}{2}[(1 - c) \log_2(1 - c) + (1 + c) \log_2(1 + c)],$$  \hspace{1cm} (12)

with $c = \max(|c_1|, |c_2|, |c_3|)$, whereas the geometric quantum discord is given by [37]

$$D^2_A(\rho_{AB}) = \frac{1}{4}(c_1^2 + c_2^2 + c_3^2 - c^2).$$  \hspace{1cm} (13)

It can be verified that for two-qubit Bell-diagonal states, if any two such states $\rho_{AB}$ and $\rho'_{AB}$ belong to one of twelve triangles (see fig. 1) with vertices

$$(0, -1, 0), \quad (0, -0.5, 0.5), \quad (0, -0.5, -0.5),$$  \hspace{1cm} (14a)
$$(0, 0, 1), \quad (0, 0.5, 0.5), \quad (0, 0.5, -0.5),$$  \hspace{1cm} (14b)
$$(0, 0, -1), \quad (0, -0.5, -0.5), \quad (0, 0.5, -0.5),$$  \hspace{1cm} (14c)
$$(0, 0, 1), \quad (0, 0.5, 0.5), \quad (0, -0.5, 0.5),$$  \hspace{1cm} (14d)
$$(0, -1, 0), \quad (-0.5, 0, 0.5), \quad (-0.5, 0, -0.5),$$  \hspace{1cm} (14e)
$$(1, 0, 0), \quad (0.5, 0, 0.5), \quad (0.5, 0, -0.5),$$  \hspace{1cm} (14f)
$$(0, 0, -1), \quad (0.5, 0, -0.5), \quad (-0.5, 0, -0.5),$$  \hspace{1cm} (14g)
$$(0, 0, 1), \quad (0.5, 0, 0.5), \quad (-0.5, 0, 0.5),$$  \hspace{1cm} (14h)
$$(1, 0, 0), \quad (0.5, -0.5, 0), \quad (0.5, 0.5, 0),$$  \hspace{1cm} (14i)
$$(0, -1, 0), \quad (0.5, -0.5, 0), \quad (-0.5, -0.5, 0),$$  \hspace{1cm} (14j)
$$(1, 0, 0), \quad (0.5, 0.5, 0), \quad (-0.5, 0.5, 0),$$  \hspace{1cm} (14k)
then the states ordering (1) is preserved, otherwise one can find states for which it is violated.

In other words, if $\rho_{AB}$ and $\rho'_{AB}$ belong to one of twelve two-parameter families of states corresponding to triangles (14)

$$c_1 = 0, \quad -1 \leq c_2 \leq -0.5, \quad |c_3| \leq 1 + c_2,$$  \hspace{1cm} (15a)
$$c_1 = 0, \quad 0.5 \leq c_2 \leq 1, \quad |c_3| \leq 1 - c_2,$$  \hspace{1cm} (15b)
$$c_1 = 0, \quad -1 \leq c_3 \leq -0.5, \quad |c_2| \leq 1 + c_3,$$  \hspace{1cm} (15c)
$$c_1 = 0, \quad 0.5 \leq c_3 \leq 1, \quad |c_2| \leq 1 - c_3,$$  \hspace{1cm} (15d)
$$c_2 = 0, \quad -1 \leq c_1 \leq -0.5, \quad |c_3| \leq 1 + c_1,$$  \hspace{1cm} (15e)
$$c_2 = 0, \quad 0.5 \leq c_1 \leq 1, \quad |c_3| \leq 1 - c_1,$$  \hspace{1cm} (15f)
$$c_2 = 0, \quad -1 \leq c_3 \leq -0.5, \quad |c_1| \leq 1 + c_3,$$  \hspace{1cm} (15g)
$$c_2 = 0, \quad 0.5 \leq c_3 \leq 1, \quad |c_1| \leq 1 - c_3,$$  \hspace{1cm} (15h)
Let us notethat forthestates (15a) the ordering of states is preserved because for given\( c_2 \) quantum discords (16) are convex functions of\( c_3 \) with local minimum at the same point and \( D_A(\rho_{AB}) \geq D^G_A(\rho_{AB}) \) (see fig. 2). In a similar way, it can be shown that the ordering of states is preserved in the case of other families of states (15).

**Example 1.** Let us consider states \( \rho_{AB} \) and \( \rho'_{AB} \) belonging to the triangle with vertices (14a), i.e. states belonging to the two-parameter family (15a). It can be shown via eqs. (12) and (13) that for these states we have

\[
D_A(\rho_{AB}) = \frac{1}{4}(1-c_2-c_3) \log_2(1-c_2-c_3) \\
+ (1+c_2+c_3) \log_2(1+c_2+c_3) \\
+ (1-c_2+c_3) \log_2(1-c_2+c_3) \\
+ (1+c_2-c_3) \log_2(1+c_2-c_3) \\
- \frac{1}{2}[(1-c_2) \log_2(1-c_2) \\
+ (1+c_2) \log_2(1+c_2)],
\]

\[
D^G_A(\rho_{AB}) = \frac{1}{4}c_3.
\]

Let us notethat for the states (15a) the ordering of states is preserved because for given \( c_2 \) quantum discords (16) are convex functions of \( c_3 \) with local minimum at the same point and \( D_A(\rho_{AB}) \geq D^G_A(\rho_{AB}) \) (see fig. 2). In a similar way, it can be shown that the ordering of states is preserved in the case of other families of states (15).

**Example 2.** Let us consider two states, namely \( \rho_{AB} \) with

\[
c_1 = 0.1, \quad c_2 = 0, \quad c_3 = -0.75
\]

and \( \rho'_{AB} \) with

\[
c_1 = 0.1, \quad c_2 = 0, \quad c_3 = 0.9
\]

It can be verified via eqs. (15g) and (15h) that \( \rho_{AB} \) belongs to the triangle with vertices (14g), while \( \rho'_{AB} \) belongs to the triangle with vertices (14h). It can be shown directly via eqs. (12) and (13) that for these states the ordering of states is violated because

\[
D_A(\rho_{AB}) \simeq 0.0169,
\]

\[
D_A^G(\rho_{AB}) = 0.0025,
\]

and

\[
D_A(\rho'_{AB}) \simeq 0.0519,
\]

\[
D_A^G(\rho'_{AB}) = 0.0025.
\]

**Example 3.** Let us consider a one-parameter family of states with

\[
c_1 = -0.5, \quad c_2 = 0.5, \quad 0 < c_3 \leq 1.
\]

It can be verified that these states do not belong to any of the triangles with vertices (14), i.e. families of states (15), and moreover it can be shown via eqs. (12) and (13) that for these states the ordering of states is violated (see fig. 3).

**Example 4.** Let us consider a one-parameter family of states with

\[
c_1 \neq 0, \quad c_2 = -c_1, \quad c_3 = 1.
\]

It can be verified that these states do not belong to any of triangles with vertices (14), i.e. families of states (15), and moreover it can be shown via eqs. (12) and (13) that for these states the ordering of states is preserved (see fig. 4).
Summary. – We have investigated the problem of the states ordering with quantum discord and the geometric quantum discord. For analytical simplicity, we have considered two-qubit Bell-diagonal states. We have identified twelve two-parameter families of states for which the states ordering with quantum discords is preserved as long as the states belong to the same family, and we have found that otherwise, one can find both the states for which it is preserved. We have also shown that counterintuitively the ordering of states can be violated in the case of states belonging to different families, which explains why a general solution of the problem of two-qubit states ordering with quantum discords is a challenging issue worth further investigations. Moreover, a few explicit examples have been given to illustrate the results.

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