Friedman vs Abel equations: A connection unraveled

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Abstract

We present an interesting connection between Einstein-Friedmann equations for the models of universe filled with scalar field and the special form of Abel equation of the first kind. This connection works in both ways: first, we show how, knowing the general solution of the Abel equation (corresponding to the given scalar field potential) one can obtain the general solution of the Friedman Equation (and use the former for studying such problems as existence of inflation with exit for particular models). On the other hand, one can invert the procedure and construct the Bäcklund auto-transformations for the Abel equation.

Keywords: Integrable evolution equations, Abel equation, Friedman equation

1. Introduction

Starting out from the classical work of Abel [1], the Abel equation has seen a lot of extensive studies and applications [2], [3] (see also [4], [5]). The reason for later lies in the fact that Abel equations rather frequently appear in the process of reduction of order for many second (and higher) order families [6], [7], and hence are often found in the modeling of real problems in varied areas.

In this work we present but another new field of application of the Abel equation: as we shall see, the general solution of the cosmological Einstein-Friedmann equations for the universe filled with scalar field for a given potential can be expressed via the general solution of the Abel equation of the first kind.

Let us start by writing the Friedmann equations describing homogeneous isotropic universe (in Friedmann-Lemaître-Robertson-Walker metric):

\[
\begin{align*}
\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} &= 0 \\
H^2 &= \frac{1}{2}\dot{\phi}^2 + V - \frac{k}{a^2}
\end{align*}
\]

(1)

where a is a scale factor, \(\phi\) - scalar field, \(V\) is the scalar field’s self-action potential, \(H\) - Hubble constant \(H = \dot{a}/a\), \(k = 0, \pm 1\) and we have incorporated the rescaling \(\frac{8\pi G}{3} = c = 1\).

The general study of the dynamics of the universe for a given self-action potential is an exceptionally difficult mathematical problem. Nevertheless, this paper is dedicated to the task of finding out a regular procedure of construction of the system (1)’s general solution for a given potential.

By definition, the general solution has to have two arbitrary integration constants: \(\phi = \phi(t; t_0, C_1, C_2)\).

\(^1\)Constant \(t_0\) is connected to the translation invariance of (1)
Restricting ourselves to the case of flat space-time $k = 0$, it would be enough to define the general solution for $\phi$, since for such model it is possible to exclude the Hubble constant from the first equation of (1) and rewrite it as
\[
\ddot{\phi} \pm 3\sqrt{\frac{1}{2}\dot{\phi}^2 + V \dot{\phi}} + \frac{dV}{d\phi} = 0
\]
Scale factor $a(t; t_0, C)$ then would be defined by simple integration of the second equation of (1), the process that will give birth to the second (multiplicative) integration constant, allowing for the scale factor rescaling.

**Remark 1.** For any twice differentiable scale factor function $a = a(t)$ and for any $k$ it is possible to find out the corresponding scalar field that will provide the necessary dynamics. This can be done via the following formulas:
\[
\phi(t) = \phi_0 \pm \sqrt{2 \int dt \frac{\dot{a}^2 - \ddot{a} + k}{a}}
\]
\[
V(t) = \frac{a \ddot{a} + 2 \dot{a}^2 + 2k}{3a^2}.
\]
Equations (2) describe the function $V = V(\phi)$ in the parametric form defined for any given function $a(t)$. Of course, this procedure has nothing to do with the searches for the general solution of (1). First of all, the described procedure might at best help to introduce the first of the constants ($\phi_0$, obtained via the integration of the first equation of (2)), but not the second one. It’s second (and worst from the physical point of view) fault lies in the fact that the potential has to be derived from the known scale factor, and not the other way around - in this sense, it can be called the "inverse" problem. As easy as the "inverse" problem goes, the "direct" problem is way more difficult - it would be enough to say that in most cases it is being solved only by the procedures of the numeric calculations. However, as we will show below, it is possible to reduce it to one of the most well-developed problem: to finding the solution of the particular Abel equation of the first kind.

2. The functional of full energy

In the rest of the article we’ll restrict ourselves to the case $k = 0$ as the one best fit to describe the observed universe. Following the ideas of the [8], let us introduce the functional of full energy (hamiltonian) $W$:
\[
W = \frac{1}{2} \dot{\phi}^2 + V(\phi).
\]
It is easy to see that, using (3) and assuming $k = 0$ system (1) can be rewritten as:
\[
\frac{dW}{d\phi} = -3H\dot{\phi}
\]
\[
H = \pm \sqrt{W}
\]

\footnote{It is by all means possible to choose the scale factor depending on the arbitrary amount of undefined constants \{C_1, ..., C_N\} so that they will appear in the solution $\phi = \phi(t, C_1, ..., C_N)$. But then, as follows from (2), the potential $V$ will depend on those constants as well.}
Moreover, knowledge of the function $W = W(\phi)$ provides an elementary way of finding all other quantities ($\phi = \phi(t)$, $a = a(t)$, $V = V(\phi)$). If $W \neq 0$ (cf. Remark 2), $\phi(t)$ can be derived from the ordinary differential equation

$$
\frac{d\phi}{dt} = \pm \frac{1}{3} \frac{W'(\phi)}{\sqrt{W(\phi)}}
$$

and potential $V$:

$$
V(\phi) = W(\phi) - \frac{1}{18} \frac{(W'(\phi))^2}{W(\phi)}.
$$

As for the scale factor $a(t) = \exp(\int H(t) dt)$, it can be obtained by substituting $\phi(t)$ from (5) into (4) and consequent integration.

**Example 1.** Choose $W(\phi) = \lambda \phi^4/4$, $\lambda > 0$. According to (6), this hamiltonian corresponds to the potential with the spontaneously broken symmetry

$$
V(\phi) = \frac{\lambda \phi^4}{4} - \frac{\mu^2}{2} \phi^2,
$$

whereas the dynamical variables would be:

$$
a(t) = a_0 \exp\left(\frac{3\phi_0^2}{8} \left[1 - e^{\pm \frac{\sqrt{\lambda}(t-t_0)}{4}}\right]\right)
$$

$$
\phi(t) = \phi_0 \exp\left(\pm \frac{2\sqrt{\lambda}}{3} (t - t_0)\right)
$$

$$
H(t) = \pm \frac{\sqrt{\lambda} \phi_0^2}{2} \exp\left(\pm \frac{4\sqrt{\lambda}(t-t_0)}{3}\right).
$$

Here the variables with zero subscript correspond to current ($t = t_0$) values of scalar field and scale factor.

Integration of (5) gives rise to the constant $t_0$, which appears because of the translational invariance of (11) and can be assimilated by the translation $t \rightarrow t + \text{const}$. In order for the solution $\phi = \phi(t; t_0, C)$ to be general it has to contain the second independent constant $C$. This constant can be obtained through the following calculations: consider (6) as a differential equation w.r.t. variable $W(\phi)$ with the given $V(\phi)$. General solution of this equation contains the thought after integration constant $C$. Thus, we can now formulate the following

**Proposition 1** If for a given $V(\phi)$ the general solution of equation (6) is $W = W(\phi, C)$, the general solution of the Friedmann equation (11) $\phi(t; t_0, C)$ exactly corresponds to the general solution of (5).

Hence, the problem is reduced to the task of finding the general solution of (6) for a given $V(\phi)$.

**Remark 2.** The case $W = 0$ shall be considered separately. It is easy to verify that for any given $V \leq 0$ one gets $H = 0$ ($a(t) = a_0 = \text{const}$ - stationary universe) and the general solution of the (11) contains the single integration constant:

$$
\left\{\begin{array}{l}
\int \frac{d\phi}{\sqrt{-2V(\phi)}} = \pm (t - t_0); \quad V \neq 0 \\
\phi = \phi_0; \quad V = 0
\end{array}\right.
$$

\[3\text{Note, that the value of constant } \mu \text{ in (7) isn’t arbitrary: } \mu^2 = 4\lambda/9.\]
So this model is meaningful for the non-positive potential $V(\phi)$ only [9]. It is also interesting to note that the parameter of equation of state $w = p/\rho = \infty$ where $\rho$ is the density and $p$ is the pressure of scalar field.

In what follows we’ll restrict ourselves to the case $W \neq 0$.

3. Main Theorem

The main result of this paper lies in the following theorem:

**Theorem 1** Let $x = 3\sqrt{2}\phi$, $\chi = \ln|V|$, $\kappa = \pm 1$. For a given $V(\phi)$ the corresponding hamiltonian $W = W(x, C)$ is defined as (cf. Remark 2):

$$W(x, C) = V(x) \left( \frac{(y + \sqrt{y^2 - 1})^2 + 1}{1 - (y + \sqrt{y^2 - 1})^2} \right)^2$$

(10)

where $y = y(x, C) \neq \pm 1$ is a general solution of Abel equation of 1st kind:

$$y' = -\frac{1}{2} (y^2 - 1) (\kappa - \chi' y).$$

(11)

Moreover, the special case $V = 0$ occurs if and only if $y = \pm 1$ and the hamiltonian $W$ has the form:

$$W = C e^{\kappa x}$$

(12)

The proof of the theorem can be performed by the direct calculations.

**Remark 3.** (10) defines a family of solutions of (1), parameterized by the constant $C$. Substituting $W(x, C)$ in (5) after the integration one will obtain $\phi = \phi(t; C, t_0)$, where $t_0$ is the second integration constant, connected to the invariance of the scalar field relative to translations $t \rightarrow t + \text{const}$. In other words, the suggested algorithm indeed allows to find the general solution of (1), which is a main result of the paper.

**Remark 4.** For the special case $y = \pm 1$ (i.e. $V = 0$) the general solution of the (1) has the form

$$\phi(t) = \phi_0 \pm \frac{\sqrt{2}}{3} \log (t - t_0),$$

the scale factor is

$$a(t) = a_0 (t - t_0)^{1/3}$$

and the case corresponds to the ”stiff” equation of state with $w = 1$.

**Remark 5.** If $\chi' \neq \text{const}$ (cf. Ch. 4, example D) the eq. (11) has two fixed points $y = \pm 1$. It means that for any initial $y_0 = y(x_0, C) \neq \pm 1$ the solution $y(x, C) \neq \pm 1$ for all values of $x$ excluding probably $x = \pm \infty$.

The general form of Abel equation of first kind is:

$$y' = \sum_{\eta=0}^{3} f_\eta y^\eta,$$

(13)
where, in our case:

\[
\begin{align*}
  f_0 &= \frac{\kappa}{2}, \\
  f_1 &= -\frac{1}{2} \chi', \\
  f_2 &= -\frac{\kappa}{2} = -f_0, \\
  f_3 &= \frac{1}{2} \chi' = -f_1.
\end{align*}
\]  

(14)

As known (cf., for example [5], [6]), if \( f_1 \) is continuous, \( f_2 \) and \( f_3 \) are continuously differentiable and \( f_3 \neq 0 \) then one can represent the equation (11) in normal form:

\[
\eta' = \eta^3 + J(x),
\]

(15)

where

\[
\begin{align*}
  y &= \omega(x)\eta(\xi) + \frac{\kappa V(x)}{3V'(x)}, \\
  \omega(x) &= \frac{1}{\sqrt{V(x)}} \exp \left( -\frac{1}{6} \int^x \frac{V(z)}{V'(z)} dz \right), \\
  \xi &= \frac{1}{2} \int \frac{V''(x)\omega^2(x)}{V(x)} dx, \\
  J &= \frac{2\kappa(9V'' - V)V}{(3V'\omega)^3}.
\end{align*}
\]  

(16)

Example 2. For the popular cosmological model with quadratic potential \(^4\)

\[
V(\phi) = \frac{m^2 \phi^2}{2},
\]

(17)

one can use the substitution (16) for the \( \phi > 0 \) (\( x > 0 \)). In this case

\[
\omega(x) = \frac{6}{mx} e^{-x^2/24}, \quad J(x) = \frac{\kappa m^3}{23328} x^4 \left( x^2 - 18 \right) e^{x^2/8},
\]

(18)

\[
\xi = \frac{3}{2m^2} \left[ \text{Ei} \left( 1, \frac{x^2}{12} \right) - \frac{12}{x^2} e^{-x^2/12} \right].
\]

Remark 6. In the case of polynomial potentials

\[
V = \frac{\lambda \phi^n}{n} = \frac{\lambda x^n}{18^{n/2} n},
\]

with positive coupling \( \lambda > 0 \), the Abel equation of 1st kind can be transformed into the particular case of the Abel equation of 2nd kind. In this case one gets \( \chi' = n/x \). Assuming \( |y_0| > 1 \) and \( \text{sign}(y_0) = -\kappa \), lets consider \( y \) as the independent variables and \( x = x(y) \) as a solution under the question. Introducing new function \( P = P(y) = \kappa x(y) - ny \) and substituting it into the (11) one get

\[
PP' = F_1(y)P + F_0(y),
\]

(19)

with

\[
F_1(y) = -n - \frac{2}{y^2 - 1}, \quad F_0(y) = -\frac{2ny}{y^2 - 1}.
\]

Lets define

\[
P = u(y) + F(y), \quad F(y) = \int_0^y F_1(z)dz = -ny + \log \left| \frac{y + 1}{y - 1} \right|.
\]

\(^4\)In a quantum field theory this model describes noninteracting massive scalar particles. We shall carefully consider this model in the Sec.5
Then the equation (19) will be reduced to
\[
(u + F)u' = F_0. \tag{20}
\]
Finally, if we'll assume that \( |y_0| \geq 1 \) then \( F_0 \neq 0 \) and one can introduce a new independent variable
\[
\xi = \int_0^y F_0(z)dz = -n \log (y^2 - 1), \quad u(y) = \eta(\xi).
\]
with whom the equation (20) will has the normal form
\[
(\eta + F) \eta' = 1.
\]
Another form of the Abel equation of 1st kind (11) can be obtained if we know at least one of it’s exact solutions. For (11) there are two such solutions \( y = \pm 1 \equiv K \). Calculating the function
\[
E(x) = \exp \int (3f_3K^2 + 2f_2K + f_1) dx = V(x)e^{-\kappa K x},
\]
and using the substitution
\[
y = K + \frac{E(x)}{z(x)},
\]
one get
\[
z' + \frac{\Phi_1}{z} + \Phi_2 = 0, \tag{21}
\]
where
\[
\Phi_1 = \frac{1}{4} (V^2)' e^{-2\kappa K x}, \quad \Phi_2 = \frac{1}{2} e^{-\kappa K x} (3KV' - \kappa V).
\]
For the case
\[
V = V_0 e^{\kappa K x/3}
\]
\( \Phi_2 = 0 \) and the equation (21) can is exactly solvable. This example will be thoroughly examined in the next section.

**Remark 7.** There have been many works recently that treat either modified gravity or \( f(R) \)-gravity [10], [11], [12], [13], [14]. All these models suggest an alternatives for the origin of dark energy. It may be naturally expected that gravitational action contains some extra terms which became relevant recently due to the significant decrease of the universe curvature.

The \( f(R) \) models can in general be describes via the action
\[
S = \frac{1}{k^2} \int d^4x \sqrt{-g} [R + f(R)],
\]
where \( f(R) \) is the proposed additional term. After the introduction of the Friedmann-Lemaitre-Robertson-Walker metric
\[
dS^2 = -dt^2 + a^2(t) \sum_{i=1}^{3} (dx^i)^2, \tag{22}
\]
one gets new equations, naturally differing from the generic Friedmann equations (1). Clearly, the general solutions problem in this new framework gets even more difficult then before.
However, one can still reduce any given $f(R)$ model to a special usual Einstein-Friedmann universe filled with scalar field with non minimal coupling. In fact, letting $a(t)$ be the particular solution of some $f(R)$-model in metric (22) and substituting it into the (2) one will obtain the function $V = V(\phi)$ in the parametric form defined for the given scalar factor. Therefore, we end up with the following

**Proposition 2** For any solution $a(t)$ of any $f(R)$-model in metric (22) there exist an Einstein-Friedmann model with scalar field having exactly the same scale factor.

In other words, one can reduce $f(R)$-models to usual Friedmann equation (1) in order to find their general solutions.

4. Some Examples

Let’s consider a couple of examples of potentials that are frequently used in cosmological studies.

a) 

$$V(\phi) = \frac{\lambda \phi^n}{n}.$$ 

Here we have the following set of functions $f_\eta$:

$$f_3 = -f_1 = \frac{n}{2x}, \quad f_2 = -f_0 = -\frac{\kappa}{2}.$$ 

b) 

$$V(\phi) = \frac{\lambda \phi^4}{4} + \frac{m^2}{2} \phi^2.$$ 

This potential describes the popular field model of scalar particles with the coupling constant $\lambda > 0$ and the mass $m$.

$$f_3 = -f_1 = \frac{2(\lambda x^2 + 18m^2)}{x(\lambda x^2 + 36m^2)}.$$ 

c) 

$$V = \Lambda = \text{const.}$$

$$f_3 = -f_1 = 0.$$ 

if $\phi = \text{const}$ this case corresponds to the model with the cosmological constant.

d) 

$$V = V_0 e^{6\sqrt{2} \alpha \phi}, \quad \alpha = \text{const.}$$

$$f_3 = -f_1 = \alpha.$$ 

Let us now consider the cases c) and d) in details. 

If $V = \Lambda = \text{const}$, equation (11) has the solution:

$$y(x, x_0) = e^{\frac{\kappa (x-x_0)}{2}} + 1$$

and the functional $W$ has the form:

$$W(x, x_0) = \Lambda \cosh^2 \left( \frac{\kappa}{2}(x - x_0) \right).$$
After substitution into (5) we obtain the following differential equation:

\[ \dot{\phi} = \pm \kappa \sqrt{2 \Lambda} \sinh \left( \frac{3\kappa}{\sqrt{2}} (\phi - \phi_0) \right). \]  

(25)

The general solution of (25) \( \phi(x; x_0, t_0) \) will be parameterized by two arbitrary constants \( x_0, t_0 \) and will have a form:

\[ \phi = \phi_0 + \sqrt{\frac{2}{3\kappa}} \arccosh \left( \cot \left[ 3\sqrt{\Lambda} |t - t_0| \right] \right) \]  

(26)

The scale factor

\[ a(t) = a_0 \left[ \sinh \left( 3\sqrt{\Lambda} |t - t_0| \right) \right]^{\pm 1/3}. \]  

(27)

Note, that if \( t \to \infty \) then \( \phi \to \phi_0 \) (see (26)) so the density \( \rho \to \Lambda \) and the pressure \( p \to -\Lambda \). In the case of positive power in (27) we obtain the well-known de-Sitter (dS) solution as \( t \to \infty \):

\[ a(t) \to a_0 \frac{2}{1/3} e^{\sqrt{\Lambda} t}, \]

while the negative power case contribute the solution with the so called Big Rip singularity at \( t = t_0 \) which means that one deals with the phantom cosmology here [15], [16], [17], [18].

For the case d) equation (11) takes the form

\[ y' = \alpha (y - 1)(y + 1)(y - s), \]  

(28)

where \( s = \kappa/(2\alpha) \). This equation has three fixed points: \( y = \pm 1 \) and \( y = s \). For simplicity one choose \( s > 1 \). Therefore if the initial value \( 1 < y_0 < s \) then this will be the case for \( y(x; C) \) for any \( x \). The solution of the (28) has the form

\[ \left( \frac{y + 1}{y - 1} \right)^s \frac{(y - s)^2}{y^2 - 1} = Ce^{2\alpha(s^2 - 1)x}, \]  

(29)

where \( C > 0 \) is an integration constant. One can see that

\[
\begin{align*}
y &\to s \quad \text{at} \quad x \to +\infty \\
y &\to 1 \quad \text{at} \quad x \to -\infty
\end{align*}
\]

for \( \alpha < 0 \) and

\[
\begin{align*}
y &\to s \quad \text{at} \quad x \to -\infty \\
y &\to 1 \quad \text{at} \quad x \to +\infty
\end{align*}
\]

for \( \alpha > 0 \).

5. The \( \frac{m^2\phi^2}{2} \) model: inflation and slow-rolling approximation

The Abel representation (11) of Friedmann equation (1) can be extremely useful even in the cases when one cannot find its exact solution. To demonstrate this let us consider the popular cosmological model with the quadratic potential (17). It is known that Friedmann equation with this potential is non-integrable. Same goes for the Abel equation. However, by studying this model it is still possible to draw some interesting conclusions, namely: that the equation (11) may result in inflation with natural exit from it.
The standard approach to this task (cf., for example, [19]) would be to use the slow-rolling approximation, i.e. assume that
\[ K = \frac{\dot{\phi}^2}{2} \ll |V|. \] (30)
If this is true, the equation of state is\[ p \sim -\rho \] and one gets inflation. For polynomial potential (akin to the one in question) condition (30) may be valid if \( \phi \gg 1 \). It is then assumed that during the inflation process the kinetic term \( K \) increases until the slow-rolling approximation (30) is no longer applicable, which heralds the (spontaneous) natural exit from the inflation. This reasoning, being rather simple as an idea proved to be quite a challenge when it came to the rigorous proofs in particular cases. In the last 14 years this difficult problem has drawn a lot of interest and attention. The investigations conducted in these field (see [20], [21], [22], [24] and list of references) have led to the following conclusions:

1. The idea of slow rolling is true. Inflation does indeed occur under an extremely broad range of self-acting potential, and there is hence no need to fix a certain form of the potential to obtain an inflationary universe.
2. The exit from inflation, on the contrary, turned out to be a problem. For many model potentials the universe never stops inflating. In general, the exit has to be achieved only by the means of fine-tuning or, in other words, by the parameter fitting.

In this Section, by using the Abel equation we show that the popular non-integrable model with quadratic potential (17) is free from the later problem, which lies in total accordance with the conclusions achieved so far.

Let us start by introducing the quantity \( \theta^2(y) \) (cf. (10)):
\[ \theta^2(y) = \left( \frac{y + \sqrt{y^2 - 1}}{1 - (y + \sqrt{y^2 - 1})^2} + 1 \right)^2. \] (31)
For \( y \geq 1 \) this is the monotonously decreasing function of \( y \), such that:
\[ \lim_{y \to 1} \theta^2(y) = +\infty, \quad \lim_{y \to \infty} \theta^2(y) = 1. \]
The plot of \( \theta^2(y) \) is represented on the Fig.1. The slow-rolling approximation (30) can be rewritten in the form
\[ \frac{K}{|V|} = \theta^2(y) - 1 \ll 1. \] (32)
For the \( y \to \infty \) we have \( \theta^2(y) \sim 1 + 1/y^2 \) therefore the approximation (32) will be valid if \( y \gg 1 \).

The inflation take place whenever \( \ddot{a}(t)/a(t) > 0 \) (which is identical to condition \( \rho + 3p < 0 \)). This will be the case if \( \theta^2(y) < 3/2 \) or \( y > \sqrt{3} \). The pressure will be negative if \( \theta^2(y) < 2 \) or \( y > \sqrt{2} \). All these results are presented in Tabl. 1.

Now consider the Abel equation (11) with the quadratic potential (17), i.e. with \( \chi = 2/x \). For simplicity let us choose \( \kappa = +1 \). The equation (11) takes the form:
\[ y' = -\frac{1}{2} \left( y^2 - 1 \right) \left( 1 - \frac{2y}{x} \right). \] (33)

Remark 8. Since \( y = \pm 1 \) are fixed points of the equation (33) the range of an arbitrary solution of (33) \( y(x, C) \) will be splitted into the three distinct regions: \( I_1 = \{ y \mid y > +1 \}; I_2 = \{ y \mid -1 < y < +1 \}; I_3 = \{ y \mid y < -1 \} \). In this chapter for the sake of simplicity we'll refer to the solutions of (33) as just \( y(x) \), omitting the constant \( C \).
Figure 1: The plot of $\theta^2(y)$.

|        | Slow-rolling ($\frac{\dot{\theta}^2}{\theta^2} \ll |V|$) | Inflation ($\rho + 3p < 0$) | Negative pressure |
|--------|---------------------------------------------------------|-----------------------------|-------------------|
| I: $1 \ll y_* < y < \infty$ | yes | yes | yes |
| II: $\sqrt{3} < y < y_*$ | no | yes | yes |
| III: $\sqrt{2} < y < \sqrt{3}$ | no | no | yes |
| IV: $y < \sqrt{2}$ | no | no | no |

Table 1:

$y < +1$}; and $I_3 = \{y|y < -1\}$. Curve $y(x)$ can cross the lines $y = \pm 1$ in two possible cases: (i) at $x = 0$, which is a singular point of (33) and (ii) if $y(x)$ is singular in at least one point $x = x_s$. In the framework of the discussed applicability of the Abel equations to the cosmological models with real scalar field, the most interesting regions would be $I_1$ and $I_3$. In this work we’ll concentrate on studying $I_1$.

Before we actually start tackling the equation (33), it would be worthwhile to spend some time on analyzing the easier but still somewhat similar Riccati equation:

$$y' = -\frac{1}{2} \left( y - 1 \right) \left( 1 - \frac{2y}{x} \right).$$

Similarly to (33), equation (34) also has a fixed point at $y = 1$ as well as the singular point at $x = 0$. However, contrary to (33), this equation can be solved explicitly; it’s general solution has a form:

$$y(x, C) = \frac{x + 2 + Ce^{x/2}}{2 + Ce^{x/2}},$$

where $C$ is an integration constant. (35) is obviously regular whenever $C > 0$, and $C < 0$ implies existence of singularity at a certain point $x_s$ (see Fig. 2). Existence and location of singularity $x_s$ is uniquely determined by the initial conditions $\{x_0, y_0\}$ as follows:

$$x_s = x_0 - 2 \log \left[ \frac{2y_0 - x_0 - 2}{2(y_0 - 1)} \right].$$
Figure 2: The plot of solution (35) at $c = 1$ (left) and $c = -1$ (right). Horizontal line corresponds to the fixed point $y = 1$. Note, that $y(x)$ intersects it at $x = 0$.

If $x_s$, defined via (36) is real, solution (35) will have a singularity there. In particular, choice of initial conditions lying in $I_1$ and such that $y_0 \gg x_0$ will result in singularity (Fig. 2, right) at $x_s \sim x_0$. We would also like to point out that this choice in general leads to the violation of Lipshitz condition that in our case has the form

$$|2(y_2 + y_1) - x_0 - 2| \leq 2x_0 L,$$

where $L$ is a positive constant.

Note that in order to obtain the aforementioned estimate one can actually avoid the usage of solution (35), but instead start from (34), assuming that $y \to +\infty$ at $x \to x_s > 0$ and integrate the reduced version of (34) $y' = y^2/x_s$ to get

$$y(x \sim x_s) = \frac{x_s}{x_s - x}.$$  \hspace{1cm} (37)

(37) can be obtained from (35) with $C = -2e^{-x_s/2}$ by the Taylor expansion around $x = x_s$. Finally, the usage of the initial conditions $\{x_0, y_0\}$ together with (37) results in a crude estimate for $x_s$ (cf. (36)), that gets more and more accurate for $y_0 \gg 1$:

$$x_s = \frac{y_0 x_0}{y_0 - 1} \to x_0,$$  \hspace{1cm} (38)

in a good accordance with the results obtained by studying the exact solution (35).

As we have mentioned already, equation (33) is not the one allowing for explicit integration, hence the study for existence and location of possible singularities has to be performed using the approximate approach that we have tested on the Riccatti equation (34). The analog of (37) for (33) has the form

$$y(x \sim x_s) = \sqrt{\frac{x_s}{2(x_s - x)}},$$  \hspace{1cm} (39)

i.e. unlike the discussed singularities of solutions of (35) there might exist at most one left-sided discontinuity point of second order, while for $x > x_s$ the solution is either finite or complex (cf., for example, solution of a simple ODE $y' = y^2$). By analogy with (38) we’ll obtain for (33)

$$x_s = \frac{2x_0 y_0^2}{2y_0^2 - 1} \to x_0,$$
when $y_0 \gg 1$. Note that this condition together with $y_0 \gg 1$ are also violating the Lipschitz condition, therefore it is expected for the singular point to appear near the critical point $x_s \sim x_0$. Contrary, the choice $x_0 \gg 1$, a $y_0 = 1 + o(1)$ is deemed to render the nonsingular solution (at least on the right half axes), by analogy with Fig. 2, left. For what follows we’ll assume that the initial conditions are chosen in such a fashion that ensures continuity of solution of (33) at $x > 0$.

**Proposition 3** Solution of (33), continuous at $x \neq 0$ is infinitely differentiable there.

The proof of the proposition. Continuity of $y(x)$ is postulated on the chosen class of initial conditions (cf. [23]). Differentiability of $y(x)$ at $x \neq 0$ follows directly from (33). After $n - 1$ differentiation of (33) w.r.t. $x$ one gets the following:

$$\frac{d^n y}{dx^n} = \sum_{\mu=1}^{k_n} \alpha_{\mu}(x)y^{\mu},$$

where $k_n = 2^n + 1$, and coefficients $\alpha_{\mu}(x)$ are continuous and differentiable at $x \neq 0$, which concludes the proof.

**Theorem 2** Solution of (33) that belongs to $I_1$ has one and only one maximum on the right half plane $x > 0$.

The proof of the theorem. First, let us show that existence of extremum of solution $y(x)$ on $x > 0$ implies the uniqueness of maximum.

Indeed, assume that $x = x_m > 0$ is an extremum of $y(x)$, i.e. $y'(x_m) = 0$. According to (33) this means that either $y = \pm 1$ or $y_m = y(x_m) = x_m/2$. Since the second derivative at $x_m$ is equal to

$$y''(x_m) = -\frac{y_m(y_m^2 - 1)}{x_m^2},$$

and $x_m > 0$ implies $y_m > 0$ it follows that inside of the $I_1$ region $y''(x_m) < 0$, hence $x_m$ can only be the maximum. Uniqueness of maximum then follows straight from the analyticity of $y(x)$ (cf. Proposition 3).

Now consider the line $y_1(x) = x/2$. According to all the above-said, if $y(x)$ and $y_1(x)$ intersects at any point $x_m > 0$, at this $y(x)$ suffers a maximum and, moreover, this maximum is unique. Let us now prove that existence of such intersection point. Indeed, let us assume by contradiction, that $y(x)$ and $y_1(x)$ does not intersect at $x > 0$. This means either one of two possibilities:

A. $y(x) < y_1(x)$ at $x > 0$ or
B. $y(x) > y_1(x)$ at $x > 0$.

Case A shall be excluded due to the fact that whenever $0 < x < 2$ we have $y(x) > 1$ (region $I_1$), but $y_1(x) < 1$. Therefore, we are left with the case B, corresponding to the unbounded increase of function $y(x)$.

It is easy to see that function $y(x)$ has no oblique asymptote at $x > 0$. Indeed, existence of such asymptote implies the existence of limit of left hand side of (33), corresponding to the angular

\[^6\text{Note that when } x \to 0 \text{ the solutions of (33) has a behaves as } d^n y/dx^n \to 0 \text{ – a very interesting property, differing it from solution (35) of Riccatti equation (34).} \]
coefficient of the asymptote, which contradicts the $x^2$ divergence of the right hand side of (33). Thus, $y(x)$ has to increase fast enough, so that for large values of $x$ the contributions of unities in both parentheses in (33) will be negligibly small, which, in turn, leads to a simple differential equation $y' = y^3/x$. General solution of this equation contains an unavoidable singularity at a certain point $x_s(C) > 0$, whose location depends on the arbitrary integration constant, and for $x > x_s(C)$ the equation has no real solutions. □

This theorem allows for examinations of the asymptotic behavior of $y(x)$ at $x \to +\infty$ and at $x \to +0$ provided that $y(x)$ belongs to the region $I_1$.

**Proposition 4** For $y(x)$ from $I_1$ for $x > 0$ the following limits exists:

\[
\lim_{x \to +\infty} y(x) = 1, \tag{40}
\]

\[
\lim_{x \to +0} y(x) = 1, \tag{41}
\]

The proof of the proposition. The (40) is a consequence of the following three facts: (a) the function $y(x)$ has a maximum at $x > 0$; (b) the function $y(x)$ belongs to $I_1$ and (c) $y = +1$ is a fixed point for equation (33).

The limit (41) is somewhat more interesting. First, note, that despite the fact that $y = 1$ is a fixed point, since $x = 0$ is a singular point of (33) (by analogy with the Riccatti Equation, cf. Fig. 2, left), (41) doesn’t lead to a contradiction. In order to establish (41) it should be kept in mind that, according to the Theorem 2, $y(x)$ is a bounded function, hence for sufficiently small values of $x$ it is possible to neglect the unity at the right parenthesis of (33) in order to get the ODE:

\[
y' = \frac{(y^2 - 1)y}{2x},
\]

whose general solution is

\[
y(x) = \frac{1}{\sqrt{1 - cx^2}}.
\]

The (41) follows immediately.

All these results can also be justified by the numerical integration; the results of one can be seen on Fig. 3.

The initial value of field $x \gg 1$ so $y \sim 1$ and we have the ”stiff” equation of state. During the dynamical evolution the field $x$ decreases. The function $y(x)$ has one point of maximum $y_{\text{max}} = x_m/2$. If $y_{\text{max}} > \sqrt{3}$ then this evolution results in inflation (the region II) (if $y_{\text{max}} \gg 1$, the evolution results instead in slow-rolling regime I). Then the field decreases and starting out from the $y = \sqrt{3}$ we have the natural exit from the inflation (the region III) - inflation indeed ends spontaneously.

**Remark 9.** The same types of reasonings can be conveyed for another cosmologically and quantum field theory popular model with the potential $V = \lambda \phi^4/4$, where $\lambda > 0$. Evidently, all the qualitative properties of its dynamics (inflation’s spontaneous start and exit) will remain the same. Moreover, as follows from the analysis, the beginning of the inflation, its exit together with the presence of the slow-rolling regime doesn’t depend on the model’s physical parameters, be it either scalar particle’s mass $m$ (for quadratic potential) or the coupling $\lambda$ for the $\phi^4$ model. This demonstrate the universality of inflation, at least for the class of the power potentials.
Figure 3: The plot of numerical solution $y(x)$ of the Eq. (11) with initial data $y(10) = 1.2$.

Remark 10. We have used above the analyticity of solutions of (33) at $x > 0$. This condition will be of crucial importance in the studies on the (becoming very popular recently) effect of the crossing of the phantom divide line. In a series of works (cf., for example, [24], [25]) it has been shown that during the cosmic evolution it is possible to for kinetic term to become negative valued. In order for this to happen there should exist a region where kinetic term becomes equal to zero, i.e. $\dot{\phi} = 0$. This implies the existence of a point $x_*$, such that $W(x_*) = V(x_*)$. Comparing this with (10) it is possible to show that $x_*$ has to be a second order break point for the function $y(x)$. Hence, for the discussed power potentials the effect of the crossing of the phantom divide line is out of the question.

6. The Bäcklund auto-transformations for the Abel equation

Throughout this paper we have used the Abel equation for construction of the general solution of the Einstein-Friedmann equations for the models of universe filled with scalar field with the given potentials. In this Section we’ll show that one can use the established connection to study some interesting mathematical properties of the Abel equation itself.

Theorem 3 The Abel equation (11) admits a set of Bäcklund auto-transformations:

$$\chi(x) \to \chi^{(1)}(x), \quad y(x) \to y^{(1)}(x), \quad (42)$$

satisfying the following properties:

Property 1. If $y(x, C)$ is a general solution of the (11) with given $\chi(x)$, then (42) applied to $y$ and $\chi$ generates a general solution

$$\chi(x) \to \chi^{(1)}(x), \quad y(x, C) \to y^{(1)}(x, C),$$

of a transformed equation

$$\left( y^{(1)} \right)' = -\frac{1}{2} \left( y^{(1)} \right)^2 (1 - \left( \chi^{(1)} \right)' y^{(1)}), \quad (43)$$
Property 2. This transformation can be applied arbitrarily many times; after the \( n \)-th step the \( n \)-times transformed function \( y^{(n)}(x, C) \) will contain an arbitrary constant \( C \), while \( \chi^{(n)} = \chi^{(n)}(x) \).

Property 3. The transformations can be inverted by setting the integration constant equal to zero. For example, for \( n = 2 \)

\[ \chi(x) \rightarrow \chi^{(2)} = \chi^{(2)}(x, C^{(1)} = 0) = \chi(x) , \]

where \( C^{(1)} \) is a certain intermediate parameter.

The proof of the Theorem 3. It was shown that using exact solution of the Abel equation (11) \( y(x, C) \) it is possible to find the exact solutions \( x(t, t_0, C) \) (i.e. \( \phi(t, t_0, C) \)) and \( a(t, t_0, C) \) of the Einstein-Friedmann equations in the flat space-time. Define function \( \psi(t, t_0, C) = a^3(t, t_0, C) \). One can show that \( \psi \) is the solution of the Schrödinger equation [8], [21] (with the notation \( \ddot{\psi} = \frac{d^2 \psi}{dt^2} \)):

\[ \ddot{\psi} = 9U_C(t)\psi, \tag{44} \]

where \( U_C(t) = V(x(t, C)) \).

Remark 11. \( U_C(t) \), unlike to \( V(x) \) does depend on constant \( C \) (for simplicity we’ll omit constant \( t_0 \) for everything that follows). In fact, using equation (44) it is possible to present \( t \) as a function of \( x \) and \( C \):

\[ t = t(x, C) \pm \frac{1}{6} \int dx \frac{\sqrt{W(x, C)}}{W'(x, C)}. \tag{45} \]

Inverting the (45) (this can be done explicitly, as we will show below) we’ll get \( x = x(t, C) \), i.e. \( V(x) = U_C(t) \). This observation gives rise to what we’ll call the reparametrization invariance of function \( U \):

\[ U_C(t) = U_C(t(x, C)) = V(x). \tag{46} \]

This property will prove to be of utmost importance for the proof of the Property 1.

Equation (44) admits Darboux transformation [26], [27]:

\[ U_C(t) \rightarrow U_C^{(1)}(t) = U_C(t) - \frac{2}{9} \frac{d^2}{dt^2} \log \psi(t, C), \tag{47} \]

\[ \psi(t, C) \rightarrow \psi^{(1)}(t, C, C^{(1)}) = \frac{1}{\psi(t, C)} \left( 1 + C^{(1)} \int dt \psi^2(t, C) \right), \tag{48} \]

where \( C^{(1)} \) is a new arbitrary constant of integration [1]. Using the equations (11) it is possible to define a new field variable

\[ x^{(1)} = x^{(1)}_0 \pm 2 \int dt \sqrt{\frac{d^2}{dt^2} \log \frac{1}{\psi^{(1)}}}, \tag{49} \]

as a function of time \( x^{(1)} = x^{(1)}(t, C, C^{(1)}) \). Inverting this relation, we obtain \( t = t(x^{(1)}, C^{(1)}) \). Of course, the inversion of function in general is a highly nontrivial task, but the one easily accomplished in our particular case.

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7Another one of new integration constants in (48) is introduced multiplicatively and cancels for all the formulas below
Proposition 5 The relationship \( x = x(t) \) can be explicitly inverted (as \( t = t(x) \)) for all explicitly known solutions of equation (11) (or (4)).

In order to justify this statement one has to refer to (45), which is a result of inversion of \( x = x(t) \), defined by (49) with the substitution \( \psi(1) \rightarrow \psi \), \( x(1) \rightarrow x_0 \).

Using the relation \( t = t(x(1), C, C(1)) \) it is possible to calculate the quantities
\[
V^{(1)}(1) = U^{(1)}(t(x(1), C)).
\]

I.e., the reparametrization invariance assures that a new potential \( V^{(1)} \) depends on the new independent variable \( x(1) \), but not on \( C \). This is by all means true also for \( \chi^{(1)} = \chi^{(1)}(x(1)) \).

On the next one is to calculate the quantity
\[
W^{(1)}(x(1), C) = \left( H^{(1)}(t(x(1), C)) \right)^2,
\]
where
\[
H^{(1)}(t) \equiv \frac{1}{3} \frac{d}{dt} \log \psi^{(1)},
\]
and \( \psi^{(1)} \) shall be defined from (48) with regards of the substitution \( C^{(1)} = C \). Finally, using the formula (10)
\[
W^{(1)}(x(1), C) = V^{(1)}(x(1)) \theta^2(y^{(1)}(x(1), C)),
\]
where \( \theta^2(y) \) is defined by (31) one derives the thought-after "dressed" solution of Abel equation
\[
y^{(1)}(x(1), C) = \sqrt{\frac{W^{(1)}(x(1), C)}{W^{(1)}(x(1), C) - V^{(1)}(x(1))}}.
\]

Formulas (50) and (51) defines the Bäcklund auto-transformations for the Abel equation (11) and proves the Property 1. The proof of Property 2 is based on the usage of n-fold consequent Darboux transformations and the Crum determinant formulas [28]. The Property 3 can be checked directly: it is enough to let \( C^{(1)} = 0 \) (but before one lets this parameter be equal to \( C \), because at this step \( C^{(1)} \) and \( C \) are still being treated as the independent constants) in (48) and perform the second Darboux transformation:
\[
U^{(1)}_C \rightarrow U^{(1)}_C - \frac{2}{9} \frac{d^2}{dt^2} \log \psi^{(1)}(t, C, C^{(1)} = 0) = U_C(t),
\]
Q.E.D. \( \Box \)

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References

[1] N.H. Abel. Oeuvres Complètes II. S.Lie and L.Sylow, Eds., Christiana 1881.

[2] P. Appell. Sur les invariants de quelques équations différentielles. Journal de Math. (4) 5 (1889) 361-423.
[3] R. Liouville. Sur une équation différentielle du premier ordre. *Acta Math.* **27** (1903) 55-78.

[4] G.M. Murphy. Ordinary Differential Equations and Their Solution. *Princeton, NJ: Van Nostrand* 1960.

[5] D. Zwillinger. Handbook of Differential Equations, 3rd ed. *Boston, MA: Academic Press* p. 120, 1997.

[6] E. Kamke. Differentialgleichungen: Lösungsmethoden und Lösungen. *Chelsea Publishing Co, New York* 1959.

[7] P.L. Sachdev. A Compendium of Nonlinear Ordinary Differential Equations. *John Wiley & Sons* 1997.

[8] S.V. Chervon, V.M. Zhuravlev, V.K. Shchigolev. New exact solutions in standard inflationary models. *Phys.Lett.* **B398** (1997) 269-273.

[9] G. Felder, A. Frolov, L. Kofman, A. Linde. Cosmology With Negative Potentials. *Phys.Rev.* **D66** (2002) 023507.

[10] S. Nojiri, S.D. Odintsov. Modified f(R) gravity consistent with realistic cosmology: from matter dominated epoch to dark energy universe. *Phys.Rev.* **D74** (2006) 086005.

[11] S. Capozziello, S. Nojiri, S.D. Odintsov, A. Troisi. Cosmological viability of f(R)-gravity as an ideal fluid and its compatibility with a matter dominated phase. *Phys.Lett.* **B639** (2006) 135-143.

[12] S. Nojiri, S.D. Odintsov. Introduction to Modified Gravity and Gravitational Alternative for Dark Energy. *Int.J.Geom.Meth.Mod.Phys.* **4** (2007) 115-146.

[13] A.A. Starobinsky. Disappearing cosmological constant in f(R) gravity. *JETP Lett.* **86** (2007) 157-163; *Pisma Zh. Eksp. Teor. Fiz.* **86** (2007) 183-189.

[14] B. Li, J.D. Barrow. The Cosmology of f(R) Gravity in the Metric Variational Approach, *Phys. Rev.* **D75** (2007) 084010.

[15] R.R. Caldwell, M. Kamionkowski, N.N. Weinberg. Phantom Energy and Cosmic Doomsday. *Phys.Rev.Lett.* **91** (2003) 071301.

[16] F. Briscese, E. Elizalde, S. Nojiri, S. D. Odintsov. Phantom scalar dark energy as modified gravity: understanding the origin of the Big Rip singularity. *Phys.Lett.* **B646** (2007) 105-111.

[17] A.V. Yurov, P.M. Moruno, P.F. Gonzalez-Diaz. New “Bigs” in cosmology. *Nucl.Phys.* **B759** (2006) 320-341.

[18] P.F. Gonzalez-Diaz. You need not be afraid of phantom energy. *Phys.Rev.* **D68** (2003) 021303.

[19] A. Linde. Particle Physics and Inflationary Cosmology. *Contemp.Concepts Phys.* **5** (2005) 1-362.

[20] J.D. Barrow. Exact Inflationary Universes With Potential Minima. *Phys.Rev.* **D49** (1994) 3055.
[21] R. Maartens, D.R. Taylor, Roussos. Exact Inflationary Cosmologies With Exit. Phys.Rev. D52 (1995) 3358.

[22] A. V. Yurov. Exact Cosmologies With Exit: From An Inflaton Complex Field To Anti-inflaton One. Class. Quantum Grav. 18 (2001) 3753-3766.

[23] A.V. Yurov, S.D. Vereshchagin. The Darboux Transformation And Exactly Solvable Cosmological Models. Theoretical and Mathematical Physics 139 (2004) 787-800.

[24] A.A. Andrianov, F. Cannata, A.Y. Kamenshchik. Smooth dynamical (de)-phantomization of a scalar field in simple cosmological models. Phys. Rev. D72 (2005) 043531.

[25] G. Darboux. C. R. Acad. Sci. (Paris) 94 (1882) 1456-1459 (French); Darboux G. Théorie générale des surfaces. New York: Chelsea 1972.

[26] V.B. Matveev, M.A. Salle. Darboux Transformation and Solitons. Springer Verlag, Berlin–Heidelberg 1991.

[27] M. M. Crum. Associated Sturm-Liouville equations. Q. J. Math. 6 (1955) 121-127.

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