Parametric localized patterns and breathers in dispersive quadratic cavities

P. Parra-Rivas,1,2 C. Mas-Arabí,1 and F. Leo1
1Service OPERA-photonics, Université libre de Bruxelles, 50 Avenue F. D. Roosevelt, CP 194/5, B-1050 Bruxelles, Belgium
2Laboratory of Dynamics in Biological Systems, KU Leuven Department of Cellular and Molecular Medicine, University of Leuven, B-3000 Leuven, Belgium

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We study the formation of localized patterns arising in doubly resonant dispersive optical parametric oscillators. They form through the locking of fronts connecting a continuous-wave and a Turing pattern state. This type of localized state can be seen as a slug of the pattern embedded in a homogeneous surrounding. They are organized in terms of a homoclinic snaking bifurcation structure, which is preserved under the modification of the control parameters of the system. We show that, in the presence of phase mismatch, localized patterns can undergo oscillatory instabilities which make them breathe in a complex manner.

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I. INTRODUCTION

The formation of temporal dissipative structures in passive nonlinear resonators is currently attracting renewed attention. These robust nonlinear attractors correspond, in the spectral domain, to optical frequency combs, i.e., coherent light sources composed of a set of equidistant spectral lines that find applications in metrology and spectroscopy [1]. While mostly generated in Kerr cavities at first [2,3], their emergence in quadratically nonlinear cavities has been intensely studied in the past few years because they offer advantages that may help overcome some of the limitations of Kerr combs [4]. Indeed, quadratic combs allow us to address spectral regions far from that of the continuous-wave pump and may facilitate self-referencing [5].

Dissipative structures can be either localized or extended [6] and their dynamics and shape determine the breadth and coherence of the underlying comb [7]. Localized dissipative structures, hereafter LSs, are particularly attractive as they correspond to denser combs. The LSs arising in Kerr cavities are very well known [7–11]. Bright and dark temporal cavity solitons, for example, have been shown to correspond to ultra-coherent frequency combs [3,12]. Conversely, the bifurcation structure of quadratic resonators is still poorly understood.

LSs have been first studied in quadratic diffractive cavities, where they form within the transverse plane to the propagation direction of light [13–21]. In this context solitary waves appear in frequency doubling cavities [13], and domain walls (DWs) between two plane-wave states exist in optical parametric oscillators (OPOs), leading to the formation of more complex states and dynamics [14–22].

LSs have been also recently investigated in dispersive quadratic cavities, where they arise along the propagation direction. In this situation such states have been studied in cavity-enhanced second-harmonic generation [23,24], in degenerate OPOs (DOPOs) with pure quadratic nonlinearity [25,26], and in the context of competing nonlinearities [27].

In all these studies LSs either consist in a solitary wave, i.e., a cavity soliton [24,27], or they form due to the locking of DWs connecting two different but coexisting continuous-wave (cw) states [23,25,26]. However, there is yet another mechanism that can lead to the formation of LSs: the locking of fronts or DWs connecting a subcritical Turing pattern and a stable cw state. In this context a LS consists in a portion of a pattern embedded in the cw state. Turing patterns emerge from a modulational instability (MI), and although its dynamical implications have been studied by different authors in this context [28–30] its potential for the generation of LSs has not been yet analyzed. In this paper, we focus on the study of parametric LSs emerging in dispersive doubly resonant DOPOs in the framework of this last mechanism. Our investigation reveals the presence of multistability between LSs of different widths, the result of a particular bifurcation structure known as homoclinic snaking [31]. These states and their bifurcation structure persist through the modification of different parameters of the system, and, moreover, they can undergo instabilities which make them oscillate in a complex fashion.

The paper is organized as follows. In Sec. II we briefly introduce the model and the methods that we apply in our paper. Section III is devoted to the linear stability analysis of the cw state and the emergence of Turing patterns. Later, in Sec. IV, we investigate the origin and formation of the LSs, analyze their bifurcation structure and stability, and classify the different dynamical regimes in a phase diagram. In Sec. V the oscillatory dynamics of the LSs is studied, and their dynamical origin determined. Finally, in Sec. VI we present a short discussion and the conclusions.

II. THE MODEL AND METHODS

We consider a doubly resonant DOPO cavity as the one shown in Fig. 1. High finesse cavities of this type can be described by the mean-field model:

\[ \partial_t A = -(1 + i \Delta_1)A - i \eta_1 \partial_x^2 A + i B \bar{A}, \]
\[ \partial_t B = -(\alpha + i \Delta_2)B - (\partial_x \eta_2 \partial_x^2)B + i A^2 + S, \]

where \( A \) and \( B \) are the slowly varying envelopes of the normalized signal electric field centered at frequency \( \omega_0 \), and pump
field centered at the frequency $2\omega_0$, where $\hat{A}$ is the complex conjugate of $A$ [25]. In this context, $t$ corresponds to the normalized slow time describing the evolution of fields after every round trip, and $x$ is the normalized fast time. The parameters $\Delta_{1,2}$ are the normalized cavity phase detunings, with $\Delta_2 = 2\Delta_1 + \varrho$ and $\varrho$ the normalized wave-vector mismatch between the fields $A$ and $B$ over one round trip; $\alpha$ is the ratio of the round-trip losses $\alpha_{1,2}$ associated with the propagation of the signal and pump fields; $\eta_{1,2}$ are the normalized group-velocity dispersion (GVD) parameters of $A$ and $B$; $d$ is the normalized rate of temporal walk-off between both fields; and $S$ is the driving field amplitude or pump at frequency $2\omega_0$. Here $\eta_1 = +1 (-1)$ denotes normal (anomalous) GVD, and $\eta_2$ can take any positive or negative value.

The model (1) is formally equivalent to those describing diffractive spatial cavities [14,20,32], where $\eta_{1,2} > 0$ are the diffraction parameters and $x$ represents a transverse spatial dimension.

Stationary states of this system satisfy the set of ordinary differential equations:

$$-i\eta_1 \hat{\psi}_A - (1 + i\Delta_1)A + iB\hat{A} = 0, \quad (2a)$$
$$-(d\hat{\psi}_A + i\eta_2 \hat{\psi}_B - (\alpha + i\Delta_2)B + iA^2 + S = 0. \quad (2b)$$

Note that the steady states can also be studied in terms of a single integro-differential equation for $A$, as done in [25,26].

To reveal the dynamics and the bifurcation structure of the different LSs circulating in this cavity we combine direct numerical simulations through a pseudospectral split-step scheme [33] for the integration of Eqs. (1), and numerical parameter continuation algorithms to track the steady-state solutions of Eqs. (2) [34,35].

To calculate the stability of the different LSs we solve numerically the eigenvalue problem:

$$\mathcal{L}\psi = \sigma \psi, \quad (3)$$

where $\sigma$ is the eigenvalue associated with the eigenmode $\psi$, and $\mathcal{L}$ represents the linear operator obtained from the linearization of Eq. (1) around a given stationary state. A stationary state is stable whenever all the eigenvalues satisfy $\text{Re}[\sigma] < 0$ and unstable otherwise. Due to the periodic nature of the resonator we consider periodic boundary conditions in $A$ and $B$, and a domain size $L = 70$.

In practice, the fields $A$ and $B$ exhibit a strong temporal walk-off [28], which dominates against other dispersion effects. However, when $A$ and $B$ have GVD of different sign, the walk-off can vanish through dispersion management [23], allowing the emergence of a large variety of LSs, mostly absent otherwise [24,26]. For this reason, we focus on a vanishing walk-off scenario ($\varrho = 0$), and we fix normal GVD for $A$ ($\eta_1 = 1$), and anomalous GVD for $B$. In particular here we choose $\eta_2 = -0.25$, although similar results are found for different values of $|\eta_2|$. A detailed study of the effect of temporal walk-off on the dynamics of the LSs is beyond the scope of this paper. In what follows, we also consider that both the pump and signal fields exhibit the same losses and fix $\alpha = 1$.

Initially, in Secs. III and IV we consider perfect phase matching ($\varrho = 0$); the implications of phase mismatch ($\varrho \neq 0$) on the LSs are presented in Sec. V.

III. STABILITY OF THE CONTINUOUS-WAVE STATES AND PERIODIC PATTERNS

The cw state solutions of the system are obtained by setting the derivatives in Eqs. (2) to zero, i.e., $\left(\hat{\psi}_A, \hat{\psi}_B, \hat{\psi}_B^2\right) = (0, 0, 0)$. Then, the cw states can be completely determined through the signal field $A$, and correspond to the trivial state $A_0 = 0$ and the nontrivial one $A = |A|^2 e^{i\phi^*}$, with intensity

$$|A|^2 = \alpha(\Delta_1 + \Delta_2 - 1) \pm \sqrt{\Delta_1^2 + \Delta_2 + 1}, \quad (4)$$

and phase

$$\phi^* = \frac{1}{2} \arccos \left[ \frac{|A|^2 + \alpha(1 + \Delta_2^2)}{\sqrt{1 + \Delta_2^2}} \right]. \quad (5)$$

The $B$ component of the cw state can be obtained through the equation

$$B = \frac{i\hat{A}^2 + S}{\alpha + i\Delta_2}. \quad (6)$$

The nontrivial state emerges from a pitchfork bifurcation in $A_0$ occurring at $S_p = \alpha \sqrt{1 + \Delta_1^2}(1 + \Delta_2^2)$. When $\Delta_2\Delta_1 - 1 < 0$, $A^+$ arises supercritically. In contrast, if $\Delta_2\Delta_1 - 1 > 0$, $A^-$ emerges subcritically, and eventually undergoes a fold at $S_f = \sigma(\Delta_1 + \Delta_2)$, where it becomes $A^-$. In the following, we focus on a configuration where the cw state arises subcritically, as shown in Fig. 2(a) for $\Delta_1 = -5$ and $\varrho = 0$. This bifurcation diagram shows the energy of $A$:

$$||A||^2 = \frac{1}{L} \int_{-L/2}^{L/2} |A(x)|^2 dx,$$

as a function of the pump intensity $S$. Note that for the cw states $||A||^2 = ||A||^2 \equiv I_0$.

At this stage we can calculate the linear stability of the cw state against plane-wave perturbations of the form $e^{i\phi}\psi_k + \text{c.c.}$, where $\phi$ is the growth rate of the perturbation and $\psi_k$ is the mode associated with the wave number $k$. This analysis amounts to solving the eigenvalue problem (3), with $\mathcal{L}$ being evaluated at the cw state.
For the set of parameters considered in this paper, $A_0$ is stable for $S < S_t$ and unstable otherwise.

The stability of $A^\pm$ is characterized by the marginal instability curve (MIC) obtained from the condition $\text{Re}[\sigma] = 0$ [26]. The MIC defines the band of unstable modes $\psi_l$ and is plotted in Fig. 2(b) for the same parameter values as Fig. 2(a). Thus, $A^\pm$ is unstable against a given mode $\psi_l$ if the intensity $I_c$ of the cw state lies within the MIC [see gray region in Fig. 2(b)], and stable otherwise. In correspondence, Fig. 2(a) shows stable (unstable) cw states using solid (dashed) lines. While $A^+$ is always unstable, $A^-$ is only unstable between the fold SN, and the MI occurring at $S_c$. At this point the perturbation of the cw state slowly evolves to a periodic Turing pattern characterized by a critical wave number $k_c$. We refer to this pattern as the primary pattern $P$. Note that this instability corresponds to the maximum of the MIC, occurring at $(k, I_c) = (k_c, I_c)$.

The primary pattern can be then tracked in the parameter $S$ applying numerical continuation algorithms, and as a result one obtains the solution branches plotted in red in Fig. 2(a). For our particular choice of parameters, the pattern arises subcritically from the MI at $S_c$ (i.e., unstably) and stabilizes in the saddle node $\text{SN}_{P^*}$. After that the periodic state remains stable until $\text{SN}_{P^*}$, where it changes stability once more and eventually connects back to $A^-$ in a spatial resonance [36]. Within the range of parameters studied in this paper, the primary pattern always arises subcritically.

The subcriticality of the pattern defines a hysteresis loop with the cw $A^+$, resulting in the coexistence of the pattern and the cw state in a given interval of the pump intensity $S$. This bistability region is marked with a gray box in Fig. 2(a). Figure 2(c) shows schematically such coexistence, where the green line represents the cw $A^+$, and the subcritical pattern $P$ is plotted in red.

We need to point out that together with the primary pattern arising from the MI there are many others that emerge all along $A^+$ and that connect back to $A^-$. The orange line in Fig. 2(a) shows one of such type of secondary patterns. A similar scheme has been described in detail in the context of Kerr cavities [36].

**IV. LOCALIZED PATTERNS: BIFURCATION STRUCTURE AND STABILITY**

Within the bistability region depicted in Fig. 2(a), fronts or DWs like the ones shown in Fig. 2(d) may arise, interact, and eventually lock, forming LSs of different widths, as the one plotted in Fig. 2(e). This type of LS consists of a slug of the pattern state embedded in the cw $A^+$, which is why they are called localized patterns (LPs). To properly understand the formation of these LPs, one has to approach the problem from a geometrical perspective as discussed in detail in [8,31,37].

The bifurcation structure associated with this type of states is shown in Figs. 3(a) and 3(b) where we plot the energy $||A||^2$ as a function of $S$ for $\Delta l = -5$. Panel (b) shows a close-up view of panel (a) that highlights the different branches of the structure. Representative LP profiles along these branches are plotted in panels (i)–(v), where the first (second) column shows the real and imaginary part of $A$ $(B)$, and their Fourier transforms $F \left[ i.e., F(T) = 10\log_{10}(\left|F(T)\right|^2) \right]$ are plotted in panels (i’)–(v’). To calculate these LPs and track them as a function of $S$ we have applied numerical continuation algorithms starting from a suitable initial guess.

This bifurcation diagram, known as homoclinic snaking, consists in a sequence of LP solution branches (see lines in blue) which oscillate back and forth within the snaking or pinning region defined by the parameter interval $S_1 < S < S_2$ (see gray region in Figs. 3(a) and 3(b)). This oscillation reflects the successive addition of a pair of pattern rolls, one on each side of the state, as one follows the diagram downwards (i.e., decreasing energy). Furthermore, these branches undergo a sequence of saddle-node bifurcations where the LPs gain or lose stability. We have labeled these bifurcations $\text{SN}_i^{\pm}$ where $i$ indicates the number of peaks in the LP and $l$ ($r$) stands for the left (right) side of the bifurcation diagram. In what follows we refer to a state with $i$ peaks as LP$_l$.

Figure 3(b) shows that the limits of the snaking region (i.e., $S_<$ and $S_>$) are well determined by the positions of any saddle node $\text{SN}_i^{\pm}$ with $i > 1$, such as $\text{SN}_3^{\pm}$. The stability of these LSs is indicated using solid (dashed) lines for the
FIG. 3. Homoclinic snaking and LPs for $\Delta_1 = -5$. Panel (a) shows the cw state, two pattern branches arising from it (red and orange lines), and the solution branches corresponding to the LPs (in blue). Solid (dashed) lines represent stable (unstable) states. The gray area within the interval $[S_-, S_+]$ shows the pinning region. Panel (b) is a close-up view of panel (a) around the homoclinic snaking structure (blue set of branches). The red pattern branch $P$ represents the primary pattern emerging from the MI located at $(S_c, I_c)$; the orange line shows another periodic pattern arising from the cw from a point below $I_c$. The labels (i)–(v) stand for the LPs shown on the right. The first column shows the real and imaginary part of the signal field $A$, while the second column shows the pump field $B$. Panels (i')–(v') correspond to their frequency spectrum $FT(\cdot) = 10\log_{10}(\|F(\cdot)\|^2)$ in orange and blue, respectively. Here we consider $L = 70$ and phase matching ($\varrho = 0$).

stable (unstable) states [see Figs. 3(a) and 3(b)]. The homoclinic snaking structure reflects the presence of multistability between LPs of different widths within the pinning region $S \in [S_-, S_+]$.

The LPs shown here are composed of an odd number of pattern peaks, and they emerge subcritically from the MI together with the primary pattern [see Fig. 3(b)]. Together with these families of solutions there is another set of LPs, characterized by an even number of pattern peaks, which also undergo homoclinic snaking [11,31,38]. To avoid further confusion we do not show these states.

In finite domains, the roll (i.e., peak) adding process must terminate at some point. Furthermore, in periodic domains like ours the snaking branches terminate on one of the many branches of periodic patterns that are present [see orange branches in Figs. 3(a) and 3(b)].

We have to point out that this scenario is not intrinsic to the field of nonlinear optics, but generic, appearing in many different contexts, from hydrodynamics and material sciences to plant ecology [11,31,38–45].

At this point one may wonder how the snaking region, i.e., the region of existence of LPs, and shape of the states vary for different values of the cavity phase detuning $\Delta_1$. To answer this question we perform a two-parameter continuation of $SN_{1\rightarrow 3}$ (which define the region of existence of LP$_1$, and $SN_{3\rightarrow 3}$ (which define the limit points $S_-$ and $S_+$) in $(\Delta_1,S)$-parameter space. As a result, we obtain the phase diagram shown in Fig. 4 where $SN_{1\rightarrow 3}$ are plotted in red, and $SN_{3\rightarrow 3}$ in
orange. The snaking region is shown in light red. The region of existence of LP 3 overlaps with the previous one, and is shown in light orange. For completeness we also plot the lines $S_P$ and $S_t$ corresponding to the pitchfork and fold bifurcations of the cw state, and the MI (see purple line).

Decreasing $|\Delta_1|$ the snaking region shrinks: $SN_{1}^{l}$ and $SN_{1}^{r}$ approach each other and eventually collide in a cusp bifurcation $C_1$ occurring at $(S^{C_1}, \Delta_1^{C_1}) \approx (21.81, -4.3446)$, where LP 1 disappears. Decreasing $|\Delta_1|$ even further, $SN_{3}^{l}$ and $SN_{3}^{r}$ annihilate one another in a second cusp $C_2$ at $(S^{C_2}, \Delta_1^{C_2}) \approx (14.0762, -3.4696)$, where LP 3 disappears. In a similar way, a cascade of such collisions marks the successive destruction of the rest of the LPs.

At this stage of the paper we can identify five different dynamical regimes (see Fig. 4) that we describe as follows:

(I) Only $A_0$ exists and is stable for pump intensities below $S_t$.

(II) The $A^+$ cw state is modulationally unstable within the MI at $S_t$ and $S_r$, and coexists with stable $A_0$. In this region Turing patterns exist.

(III) Between the MI and the pitchfork $S_P$, $A^+$ is stable and coexists with the stable $A_0$. In this region one may expect the formation of LSs through the locking of DWs between $A_0$ and $A^+$, as described in [26].

(IV) This area corresponds to the snaking region (i.e., $S \in [S_r, S_{t_{\perp}}] \approx [SN_{1}^{l}, SN_{1}^{r}]$), where the system exhibits multistability of LPs.

(V) In this region, $A^+$ and $A_0$ coexist although $A_0$ is unstable.

Within the limits of region IV, LPs undergo standard homoclinic snaking. However, although LPs persist for larger values of $|\Delta_1|$, their bifurcation structure becomes much more complex, involving the formation of isolas, and the reconnection with solution branches of different unstable states.

Furthermore, we have also verified that the LPs presented here and the LSs formed through the locking of domain walls studied in [26] can coexist and interact when modifying some of the control parameters. As a result one may expect the emergence of new types of LSs and bifurcation scenarios, which has been reported in other fields [46,47]. Nevertheless, the study of these complex scenarios is beyond the scope of the present paper, and will be presented in detail in another work.

V. OSCILLATORY DYNAMICS IN THE PRESENCE OF PHASE MISMATCH

So far we have considered perfect phase matching between the signal and pump fields (i.e., $\varrho = 0$). However, in practice, this condition is difficult to achieve, and non-negligible phase mismatch $\varrho$ between $A$ and $B$ may arise, which can influence the dynamics and stability of LSs [24,48].

In this section we investigate how the LPs, their bifurcation structure, and their stability are modified when varying $\varrho$. The modification of the pinning region as a function of the phase mismatch is shown in the $(\varrho, S)$-phase diagram of Fig. 5 for $\Delta_1 = -6$, where $SN_{1}^{l,r}$ and $SN_{11}^{l,r}$ are plotted using orange and red solid lines (in particular here we identify $S_{\perp}$ with $SN_{11}^{l,r}$). Increasing $|\varrho|$, the pinning region shifts to larger values of $S$, although it maintains its extension. Here we just show the range $[-3.5, -1.9]$. Note that these bifurcation lines continue all the way until $\varrho \geq 0$.

Figure 6(a) shows the homoclinic snaking corresponding to $\varrho = -2.5$ (see vertical dashed line in Fig. 5). The stability analysis of these structures reveals that the previously stable
LPs undergo a sequence of Hopf bifurcations where they become dynamically unstable to oscillatory LPs or LP breathers like the one shown in Fig. 7. These Hopf bifurcations are labeled $H_i$, where $i$ represents the number of peaks of the LP. Proceeding downwards in energy, the first Hopf that appears is $H_7$ followed by $H_9$ to $H_{17}$. We can notice that, increasing the number of peaks, the $H_i$ bifurcations asymptotically reach a single constant value of $S \approx 66.2$. We suspect that this phenomenon may be related with a Hopf bifurcation undergone by the pattern, although the characterization of this scenario needs further investigation.

In Fig. 6(b) we show a detailed view of a portion of the bifurcation diagram of Fig. 6(a) around $H_{11}$, where the minimum of $|A|^2$ is plotted as a function of the pump intensity $S$. The red dots show the modification that the central peak’s extrema undergo when decreasing the pump intensity $S$ from $H_{11}$.

An example of a LP$_{11}$ breather is shown in Fig. 7 for $S \approx 64.9$ [see orange vertical dashed line in Fig. 6(b)]. The colormaps in panels (a) and (b) show the temporal evolution of the breather in $|A|^2$ and Re($B$), respectively, and on top of them we plot the profiles of the fields for $t = 10$; the panels below (a) and (b) show a close-up view of the temporal evolution which allows us to appreciate the oscillatory behavior more in detail. An interesting feature of these dynamical states is that the ensemble of the peaks does not oscillate synchronously as a whole, but rather contiguous peaks oscillate out of phase. This behavior is shown in Fig. 7(c) where the temporal evolution of peaks $p_0$ and $p_1$ [see top profile in Fig. 7(a)] is plotted over several periods [see blue and orange lines in Fig. 7(c)]. The oscillations occur close to the inner part of the structure, and far from the localization boundaries which remain stationary, as one can easily appreciate in Figs. 7(a) and 7(b). In Fig. 7(d) we show the limit cycle described by the projection of the breather dynamics onto the two-dimensional phase space defined by Re($A$)($p_0$) and Im($A$)($p_0$).

The breather arises supercritically from $H_{11}$ [see Fig. 6(b)], and therefore with a very small amplitude of oscillation. Decreasing the input pump $S$ this amplitude increases, and the breather persists until reaching its maximal amplitude value at $S_{11}$. Soon after passing that fold the stable LP$_{11}$ breather disappears, and the system jumps to another oscillatory attractor (not shown here).

The blue line in the $(\varphi, S)$-phase diagram of Fig. 5 corresponds to $H_{11}$. The LP$_{11}$ breather exists within the light blue area between $H_{11}$ and $S_{11}$. For the range of parameters studied here, $H_{11}$ is always supercritical. The stability analysis reveals that the Hopf bifurcation emerges from a Gavrilov-Guckenheimer codimension-2 bifurcation [49]. At this point $S_{11}$ and $H_{11}$ collide, and therefore this bifurcation is characterized by the three eigenvalues $\sigma_1 = 0$ and $\sigma_2, \sigma_3 = \pm i\Omega$, with $\Omega > 0$ [49]. For this reason this point is also known as a fold-Hopf bifurcation. This scenario is similar to the one appearing in the context of Kerr cavities with anomalous GVD [7], and can be related with the presence of temporal chaos [50]. However, for the range of parameters studied in this paper, we have not observed any route leading to chaos. This oscillatory behavior is not related either with the presence of a Hopf instability in the cw state or with the occurrence of a Turing-Hopf crossing [51,52], but is a direct consequence of spatial coupling i.e., chromatic dispersion [53].

VI. DISCUSSION AND CONCLUSIONS

In this paper we have presented a detailed study of the dynamics and bifurcation structure of dissipative LSs arising in dispersive doubly resonant DOPO.

We have considered a mean-field model for the description of the signal $A$ and pump $B$ fields, which has been derived in [25]. We have neglected the velocity mismatch between the signal and pump fields (i.e., $\delta = 0$). In practice, this situation can be achieved through dispersion engineering as shown in [23].

The LSs studied here form due to the locking of fronts or DWs connecting a cw and a Turing pattern in a parameter region where both states are stable. Thus, such a state can be seen as a portion of the pattern embedded in a cw surrounding. We have to point out that these states, commonly known as localized patterns, are different to those formed through the locking of DWs connecting two cw states [26].
FIG. 7. LP₁₁ breather for $S = 64.9$, $\Delta_1 = -6$, and $\varrho = -2.5$. Panel (a) shows the temporal evolution of the intensity of the signal field, i.e., $|A|^2$, for an interval of time $t = 10$. The bottom panel shows an enlarged view of the oscillatory behavior, and the top panel shows the intensity profile of the LP at $t = 10$. Panel (b) shows the temporal evolution of the real part of the pump field $B$ for the same interval of time. The top and bottom panels show the profile at time $t = 10$, and the close-up view of the oscillatory behavior. Panel (c) shows the temporal evolution of the intensity of the peaks labeled $p_0$ and $p_1$ in panel (a). In panel (d) we show the limit cycle defined by the projection of the breather evolution on the variables $\text{Re}(A)(p_0)$ and $\text{Im}(A)(p_0)$.

LPs undergo a particular bifurcation structure known as homoclinic snaking: for a fixed detuning $\Delta_1$, the LP solution branches oscillate back and forth in the driving field intensity $S$ within a well-defined snaking or pinning region $[S_-, S_+]$. As a direct consequence of this structure, the system exhibits multistability between LPs of different widths. The homoclinic snaking phenomenon is not intrinsic to this system, but generic, arising in many different fields ranging from optics to plant ecology [11,31,38–45]. For the set of $(\Delta_1, S)$ parameters studied here, homoclinic snaking is well preserved. Decreasing $|\Delta_1|$, LPs suffer a series of cusp bifurcations, and eventually disappear. Increasing $|\Delta_1|$, although LPs persist, their bifurcation structure may eventually become more complex.

In most of the physical situations, perfect phase matching is difficult to achieve, and one has to take into account how phase mismatch between the signal and pump fields ($\varrho \neq 0$) may perturb the existence and stability of the aforementioned states. We have verified that, for a wide range of $\varrho$, LPs are preserved and undergo the same type of bifurcation structure, although their stability may be strongly modified. Indeed for $\varrho < 0$ we have found that LPs of different extensions undergo Hopf instabilities, leading to the appearance of LP breathers, where contiguous peaks oscillate out of phase. This Hopf instability emerges from a Gavrilov-Guckenheimer bifurcation as was reported in the case of Kerr cavities [7]. As far as we know, this type of dynamical LPs has not been yet reported in the context of dispersive cavities.

These LSs are excellent candidates for coherent frequency comb generation in optical parametric oscillators. The coexistence of multiple LPs allows tuning the shape of the comb without changing the parameters of the system. Furthermore, the multistability shown by the LPs studied here, where the region of existence of every LP has the same extension, is different from the one of the LSs underpinned by the coexistence of stable cw states [10,26]. In this last scenario, LSs undergo collapse snaking, making the existence region of most of the wider states very narrow and thus very difficult to observe experimentally.

For clarity, two interesting points have been left out of this paper. First, the LPs presented here and the LS analyzed in [26] may coexist and interact, forming much more complex states the dynamics of which are yet unexplored. One plausible situation consists in a smooth transition between the homoclinic and collapsed snaking as was described in [46].

The second point focuses on the impact that walk-off may have on the current LPs. The walk-off breaks the $x \to -x$ symmetry, inducing a drift at constant speed, and leading to the destruction of the homoclinic snaking through either the formation of a stack of isolas [54,55] or more complex configurations [56].

The clarification of these two scenarios needs further investigation, and a detailed analysis will appear elsewhere.

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