Noether Symmetries and Conservation Laws
For Non-Critical Kohn-Laplace Equations on
Three-Dimensional Heisenberg Group

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Abstract
We show which Lie point symmetries of non-critical semilinear Kohn-Laplace equations on the Heisenberg group $H^1$ are Noether symmetries and we establish their respective conservation laws.
1 Introduction and Main Results

In this paper we show which Lie point symmetries of the semilinear Kohn - Laplace equations on the three-dimensional Heisenberg group $H^1$,

$$\Delta_{H^1} u + f(u) = 0, \quad (1)$$

are Noether’s symmetries, and we establish their respective conservation laws.

The Kohn - Laplace operator on $H^1$ is defined by

$$\Delta_{H^1} := X^2 + Y^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 4(x^2 + y^2) \frac{\partial^2}{\partial t^2} + 4y \frac{\partial^2}{\partial x \partial t} - 4x \frac{\partial^2}{\partial y \partial t},$$

where

$$X = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} + 2x \frac{\partial}{\partial t}.$$  \hspace{1cm} (3)

Eq. (1) possesses variational structure and can be derived from the Lagrangian

$$\mathcal{L} = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2 + y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - F(u), \quad (2)$$

with $F'(u) = f(u)$.

The group structure, the left invariant vector fields on $H^1$ and their Lie algebra are given, respectively, by $\phi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$, where

$$\phi((x, y, t), (x_0, y_0, t_0)) := (x + x_0, y + y_0, t + t_0 + 2(xy_0 - yx_0)),$$

$$X = \frac{d}{ds} \phi((x, y, t), (s, 0, 0))|_{s=0} = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial t},$$

$$Y = \frac{d}{ds} \phi((x, y, t), (0, s, 0))|_{s=0} = \frac{\partial}{\partial y} + 2x \frac{\partial}{\partial t},$$

$$Z = \frac{d}{ds} \phi((x, y, t), (0, 0, s))|_{s=0} = \frac{\partial}{\partial t},$$

and

$$[X, T] = [Y, T] = 0, \quad [X, Y] = -4T.$$
In [2] a complete group classification for equation (1) is presented. It can be summarized as follows.

Let $G_f := \{T, R, \tilde{X}, \tilde{Y}\}$, where

$$T = \frac{\partial}{\partial t}, \quad R = y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}, \quad \tilde{X} = \frac{\partial}{\partial x} - 2y \frac{\partial}{\partial t}, \quad \text{and} \quad \tilde{Y} = \frac{\partial}{\partial y} + 2x \frac{\partial}{\partial t}. \quad (4)$$

For any function $f(u)$, the group $G_f$ is a (sub)group of symmetries. Its Lie algebra is summarized in Table 1.

|     | T | R | X | Y |
|-----|---|---|---|---|
| T   | 0 | 0 | 0 | 0 |
| R   | 0 | 0 | Y | -X|
| X   | 0 | -Y| 0 | 4T|
| Y   | 0 | X | -4T| 0 |

Table 1: Lie brackets of equation (1) with $f(u)$ arbitrary.

For special choices of function $f(u)$ in (1), the symmetry group can be enlarged. Below we exhibit these functions and their respective additional symmetries and Lie algebras.

- If $f(u) = 0$, the additional symmetries are

$$V_1 = (xt - x^2y - y^3) \frac{\partial}{\partial x} + (yt + x^3 + xy^2) \frac{\partial}{\partial y}$$

$$+ (t^2 - (x^2 + y^2)^2) \frac{\partial}{\partial t} - tu \frac{\partial}{\partial u}, \quad (5)$$

$$V_2 = (t - 4xy) \frac{\partial}{\partial x} + (3x^2 - y^2) \frac{\partial}{\partial y}$$

$$- (2yt + 2x^3 + 2xy^2) \frac{\partial}{\partial t} + 2yu \frac{\partial}{\partial u}, \quad (6)$$

$$V_3 = (x^2 - 3y^2) \frac{\partial}{\partial x} + (t + 4xy) \frac{\partial}{\partial y}$$

$$+ (2xt - 2x^2y - 2y^3) \frac{\partial}{\partial t} - 2xu \frac{\partial}{\partial u}, \quad (7)$$
\[ Z = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t}, \]

\[ U = u \frac{\partial}{\partial u}, \quad W_\beta = \beta(x, y, t) \frac{\partial}{\partial u}, \quad \text{where} \quad \Delta_{H^1}\beta = 0. \]  

(8)

|    | T | R | X | Y | U | W_\beta | V_1 | V_2 | V_3 | Z |
|----|---|---|---|---|---|--------|-----|-----|-----|---|
| T  | 0 | 0 | 0 | 0 | 0 | W_{T\beta} | V   | X   | Y   | 2T|
| R  | 0 | 0 | Y | -X| 0 | W_{R\beta} | 0   | V_3 | -V_2| 0 |
| X  | 0 | -Y| 0 | 4T| 0 | W_{\tilde{X}\beta} | V_2 | -6R | 2V | X |
| Y  | 0 | X | -4T| 0 | 0 | W_{\tilde{Y}\beta} | V_3 | -2V | -6R| Y |
| U  | 0 | 0 | 0 | 0 | 0 | 0        | 0   | 0   | 0   | 0 |
| W_\beta | -W_{T\beta} | -W_{R\beta} | -W_{\tilde{X}\beta} | -W_{\tilde{Y}\beta} | 0 | 0 | W_{V_1\beta} | W_{V_2\beta} | W_{V_3\beta} | W_{Z\beta} |
| V_1 | -V | 0 | -V_2 | -V_3 | 0 | -W_{V_1\beta} | 0   | 0   | 0   | -2V_1 |
| V_2 | -X | -V_3 | 6R | 2V | 0 | -W_{V_2\beta} | 0   | 0   | 4V_1 | -V_2 |
| V_3 | -Y | V_2 | -2V | 6R | 0 | -W_{V_3\beta} | 0   | -4V_1| 0   | -V_3 |
| Z  | -2T| 0 | -X | -Y | 0 | -W_{Z\beta} | 2V_1| V_2 | V_3 | 0 |

Table 2: Lie brackets of equation (1) with \( f(u) = 0 \). Here, \( V := Z - U \).

- If \( f(u) = u \), there are two additional symmetries, respectively, \( U \) and \( W_\beta \) as in Eq. (8), where \( \Delta_{H^1}\beta + \beta = 0 \).

|    | T | R | X | Y | U | W_\beta |
|----|---|---|---|---|---|--------|
| T  | 0 | 0 | 0 | 0 | 0 | W_{T\beta} |
| R  | 0 | 0 | Y | -X| 0 | W_{R\beta} |
| X  | 0 | -Y| 0 | 4T| 0 | W_{\tilde{X}\beta} |
| Y  | 0 | X | 4T| 0 | 0 | W_{\tilde{Y}\beta} |
| U  | 0 | 0 | 0 | 0 | 0 | 0        |
| W_\beta | -W_{T\beta} | -W_{R\beta} | -W_{\tilde{X}\beta} | -W_{\tilde{Y}\beta} | 0 | 0 |

Table 3: Lie brackets of equation (1) with \( f(u) = u \).
If \( f(u) = u^p \), \( p \neq 0, p \neq 1, p \neq 3 \), we have the generator of dilations
\[
D_p = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t} + \frac{2}{1-p} u \frac{\partial}{\partial u}.
\]
\hspace{1cm} (9)

| T | R | X | Y | \( D_p \) |
|---|---|---|---|----------|
| T | 0 | 0 | 0 | 0 | 2T |
| R | 0 | 0 | Y | -X | 0 |
| X | 0 | -Y | 0 | 4T | X |
| Y | 0 | X | -4T | 0 | Y |
| \( D_p \) | -2T | 0 | -X | -Y | 0 |

Table 4: Lie brackets of equation (1) with \( f(u) = u^p \), \( p \neq 0, p \neq 1, p \neq 3 \).

If \( f(u) = e^u \) the additional symmetry is
\[
E = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t} - 2 \frac{\partial}{\partial u}.
\]
\hspace{1cm} (10)

| T | R | X | Y | E |
|---|---|---|---|---|
| T | 0 | 0 | 0 | 0 | 2T |
| R | 0 | 0 | Y | -X | 0 |
| X | 0 | -Y | 0 | 4T | X |
| Y | 0 | X | -4T | 0 | Y |
| E | -2T | 0 | -X | -Y | 0 |

Table 5: Lie brackets of equation (1) with \( f(u) = e^u \).

In the critical case, \( f(u) = u^3 \), there are four additional generators, namely \( V_1, V_2, V_3 \) and \( D_3 \), given in (5), (6), (7) and (9) respectively. Their Lie algebra is presented in [4].
In [3] is showed that in the critical case, \( f(u) = u^3 \), all Lie point symmetries are Noether symmetries and then, by the Noether Theorem (see [1], pag. 275), in [4] is established the respective conservation laws for the symmetries \( T, R, \tilde{X}, \tilde{Y}, V_1, V_2, V_3 \) and \( D_3 \).

In this work, we show which Lie point symmetries of the other functions \( f(u) \) are Noether symmetries and then, we establish their respective conservation laws, concluding the work started in [3] and [4].

Let \( \mathbb{R} \ni u \mapsto F(u) \in \mathbb{R} \) be a differentiable function and

\[
    f(u) := F'(u).
\]

Our main results can be formulated as follows:

**Theorem 1** The group \( G_f \) is a Noether symmetry group for any function \( f(u) \) in (1).

**Theorem 2** The Noether symmetry group of (1), with \( f(u) = e^u \), is the group \( G_f \).

**Theorem 3** \( G_f \cup \{W_\beta\} \) is the Noether symmetry group of equation (1), with \( f(u) = u \) and \( \beta \) satisfies \( \Delta_{H^1}\beta + \beta = 0 \).

**Theorem 4** The Noether symmetry group of equation (1) with \( f(u) = 0 \) is generated by the group \( G_f \) and by symmetries \( W_\beta, V_1, V_2, V_3 \), where \( \beta \) satisfies \( \Delta_{H^1}\beta = 0 \).

As a consequence of theorems 1 - 4, we have the following conservation laws.

**Theorem 5** The conservation laws for the Noether symmetries of equation (1) for any \( f(u) \) are:

1. For the symmetry \( T \), the conservation law is \( \text{Div}(\tau) = 0 \), where \( \tau = (\tau_1, \tau_2, \tau_3) \) and

\[
    \tau_1 = -2yu^2 - u_xu_t, \\
    \tau_2 = 2xu_t^2 - u_yu_t, \\
    \tau_3 = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 - 2(x^2 + y^2)u_t^2 - F(u).
\]
2. For the symmetry $R$, the conservation law is $\text{Div}(\sigma) = 0$, where $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ and

\[
\begin{align*}
\sigma_1 &= -\frac{1}{2}yu_x^2 + \frac{1}{2}yu_y^2 + 2y(x^2 + y^2)u_t + xu_xu_y - yF(u), \\
\sigma_2 &= -\frac{1}{2}xu_x - \frac{1}{2}xu_y - 2x(x^2 + y^2)u_t - yu_xu_y + xu_xu_t + xF(u), \\
\sigma_3 &= -2y^2u_x^2 - 2x^2u_y^2 + 4xyuy - 4y(x^2 + y^2)u_xu_t + 4x(x^2 + y^2)u_yu_t.
\end{align*}
\]

3. For the symmetry $\tilde{X}$, the conservation law is $\text{Div}(\chi) = 0$, where $\chi = (\chi_1, \chi_2, \chi_3)$ and

\[
\begin{align*}
\chi_1 &= -\frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2 + 3y^2)u_t^2 + 2yu_xu_t - 2yu_yu_t - F(u), \\
\chi_2 &= -4xyu_x^2 - xu_xu_y + 2yu_xu_t + 2yu_yu_t, \\
\chi_3 &= -3yu_x^2 - xu_y^2 + 4y(x^2 + y^2)u_t^2 + 2yu_xu_y - 4(x^2 + y^2)u_xu_t + 2yF(u).
\end{align*}
\]

4. For the symmetry $\tilde{Y}$, the conservation law is $\text{Div}(v) = 0$, where $v = (v_1, v_2, v_3)$ and

\[
\begin{align*}
v_1 &= -4xyu_x^2 - xu_xu_y - 2yu_xu_t - 2yu_yu_t, \\
v_2 &= 2u_x^2 - \frac{1}{2}u_y^2 + 2(3x^2 + y^2)u_t^2 + 2yu_xu_t - 2yu_yu_t - F(u), \\
v_3 &= xu_x^2 + 3yu_y^2 - 4x(x^2 + y^2)u_t^2 - 2yu_xu_y - 4(x^2 + y^2)u_yu_t - 2xF(u).
\end{align*}
\]

**Theorem 6** If $f(u) = 0$ in (1), the conservation laws for the Noether symmetries are as follows.

1. For the symmetries $T$, $R$, $\tilde{X}$ and $\tilde{Y}$, the conservation laws are the same as in the Theorem 5 with $f(u) = 0$, in (1).

2. For the symmetry $V_1$, the conservation law is $\text{Div}(A) = 0$, where $A = (A_1, A_2, A_3)$ and

\[
\begin{align*}
A_1 &= -\frac{1}{2}(tx - x^2y - y^3)u_x^2 + \frac{1}{2}(tx - x^2y - y^3)u_y^2 + 2t(x^3 + xy^2 - ty)u_t^2 \\
&\quad - (x^3 + xy^2 + ty)u_xu_y - [t^2 - (x^2 + y^2)^2]u_xu_t - 2t(x^2 + y^2)u_yu_t \\
&\quad - tuu_x - 2tyuu_t + yu_x^2,
\end{align*}
\]
\[
A_2 = \frac{1}{2} (x^3 + ty + xy^2) u_x^2 - \frac{1}{2} (x^3 + ty + xy^2) u_y^2 + 2t (x^2 y + y^3 + tx) u_t^2
- (tx - x^2 y - y^3) u_x u_y + 2t (x^2 + y^2) u_x u_t - [t^2 - (x^2 + y^2)^2] u_y u_t
- t uu_y + 2tx uu_t - xu^2,
\]
\[
A_3 = \frac{1}{2} (t^2 - x^4 - 4txy + 2x^2 y^2 + 3y^4) u_x^2 + \frac{1}{2} (t^2 + 3x^4 + 4txy + 2x^2 y^2 - y^4) u_y^2
- 2(x^2 + y^2) [t^2 - (x^2 + y^2)^2] u_x^2 + 2[t(x^2 - y^2) - 2xy(x^2 + y^2)] u_x u_y
- 4(x^2 + y^2) (tx - x^2 y - y^3) u_x u_t - 4(x^2 + y^2) (x^3 + ty + xy^2) u_y u_t
- 2tuu_x + 2tx uu_y - 4t (x^2 + y^2) uu_t + 2(x^2 + y^2) u^2.
\]

3. For the symmetry \(V_2\), the conservation law is \(\text{Div}(B) = 0\), where \(B = (B_1, B_2, B_3)\) and
\[
B_1 = -\frac{1}{2} (t - 4xy) u_x^2 + \frac{1}{2} (t - 4xy) u_y^2 + [2t(x^2 + 3y^2) - 4xy(x^2 + y^2)] u_t^2
- (3x^2 - y^2) u_x u_y + 2(x^3 + ty + xy^2) u_x u_t - 2(tx - x^2 y - y^3) u_y u_t
+ 2yu u_x + 4y^2 u u_t,
\]
\[
B_2 = \frac{1}{2} (3x^2 - y^2) u_x^2 - \frac{1}{2} (3x^2 - y^2) u_y^2 + 2(x^4 - 2txy - y^4) u_t^2 - (t - 4xy) u_x u_y
+ 2(tx - x^2 y - y^3) u_x u_t + 2(x^3 + ty + xy^2) u_y u_t + 2yu u_y - 4xy uu_t - u^2,
\]
\[
B_3 = (7xy^2 - x^3 - 3ty) u_x^2 + (5x^3 - 3xy^2 - ty) u_y^2 + 4(x^2 + y^2) (x^3 + ty + xy^2) u_t^2
+ 2(tx - 7x^2 y + y^3) u_x u_y - 4(t - 4xy)(x^2 + y^2) u_x u_t - 4(3x^4 + 2x^2 y^2 - y^4) u_y u_t
+ 2x u^2 + 4y^2 uu_x - 4xy uu_y + 8y(x^2 + y^2) uu_t.
\]
4. For the symmetry \( V_3 \), the conservation law is \( \text{Div}(C) = 0 \), where 
\[ C = (C_1, C_2, C_3) \]
and
\[
C_1 = -\frac{1}{2}(x^2 - 3y^2)u_x^2 + \frac{1}{2}(x^2 - 3y^2)u_y^2 + (2x^4 - 4txy - 2y^4)u_t^2
\]
\[
-(t + 4xy)u_xu_y + (2tx - 2x^2y + 2y^3)u_xu_t - (2x^3 + 2ty + 2xy^2)u_yu_t
\]
\[-4xyuu_t - 2xuu_x + u^2;
\]
\[
C_2 = \frac{1}{2}(t + 4xy)u_x^2 - \frac{1}{2}(t + 4xy)u_y^2 + (6tx^2 + 4x^3y + 2ty^2 + 4xy^3)u_t^2
\]
\[
-(x^2 - 3y^2)u_xu_y + 2(x^3 + ty + xy^2)u_xu_t - 2(tx - x^2y - y^3)u_yu_t
\]
\[2xu_yu + 4x^2u_tu,
\]
\[
C_3 = (tx - 3x^2y + 5y^3)u_x^2 + (3tx + 7x^2y - y^3)u_y^2
\]
\[(-4tx^3 + 4x^4y - 4txy^2 + 8x^2y^3 + y^5)u_t^2 + 2(x^3 - ty - 7y^2)u_xu_y
\]
\[-2(2x^4 - 4x^2y^2 - 6y^4)u_xu_t - 4(x^2 + y^2)(t + 4xy)u_yu_t
\]
\[-8x^3uu_t - 8xy^2uu_t - 4x^2u_yu - 8xyuu_x + 2y^2.
\]

5. For the symmetry \( W_\beta \), the conservation law is \( \text{Div}(W) = 0 \), where 
\[ W = (W_1, W_2, W_3) \]
and
\[
W_1 = \beta(u_x + 2yu_t) - u(\beta_x + 2y\beta_t),
\]
\[
W_2 = \beta(u_y - 2xu_t) - u(\beta_y - 2x\beta_t),
\]
\[
W_3 = \beta[-2xu_y + 2yu_x + 4(x^2 + y^2)u_t
\]
\[+2u[x\beta_y - y\beta_x - 2(x^2 + y^2)]\beta_t].
\]
Theorem 7. If $f(u) = u$ in (1), the conservation laws for the Noether symmetries are as follows.

1. For the symmetries $T, R, \tilde{X}$ and $\tilde{Y}$, the conservation laws are the same as in the Theorem 5, with $f(u) = u$, in (1).

2. For the symmetry $W_\beta$, the conservation law is $\text{Div}(W) = 0$, where $W$ is given in (2).

The remaining of the paper is organized as follows. In section 2 we briefly present some of the main aspects of Lie point symmetries, Noether symmetries and conservation laws. In section 3 we prove theorems 1, 2 and 3. Theorem 4 is proved in section 4. Their respective conservation laws are discussed in section 5.

2 Lie point symmetries, Noether symmetries and conservation laws

Let $x \in M \subseteq \mathbb{R}^n$, $u : M \to \mathbb{R}$ a smooth function and $k \in \mathbb{N}$. $\partial^k u$ denotes the jet bundle correspondig to all $k$th partial derivatives of $u$ with respect to $x$. A Lie point symmetry of a partial differential equation (PDE) of order $k$, $F(x, u, \partial u, \cdots, \partial^k u) = 0$, is a vector field

$$S = \xi^i(x, u) \frac{\partial}{\partial x^i} + \eta(x, u) \frac{\partial}{\partial u}$$

on $M \times \mathbb{R}$ such that $S^{(k)} F = 0$ when $F = 0$ and

$$S^{(k)} = S + \eta^{(1)}_i(x, u, \partial u) \frac{\partial}{\partial u_i} + \cdots + \eta^{(k)}_{i_1 \cdots i_k}(x, u, \partial u, \cdots, \partial^k u) \frac{\partial}{\partial u_{i_1 \cdots i_k}}$$

is the extended symmetry on the jet space $(x, u, \partial u, \cdots, \partial^k u)$.

The functions $\eta^{(j)}(x, u, \partial u, \cdots, \partial^j u), 1 \leq j \leq k$, are given by

$$\eta^{(1)}_i = D_i \eta - (D_i \xi^j)u_j,$$

$$\eta^{(j)}_{i_1 \cdots i_j} = D_{i_j} \eta^{(j-1)}_{i_1 \cdots i_{j-1}} - (D_{i_j} \xi^l)u_{i_1 \cdots i_{j-1} l}, 2 \leq j \leq k.$$
We are using the Einstein sum convention.

If the PDE \( F = 0 \) can be obtained by a Lagrangian \( \mathcal{L} = \mathcal{L}(x, u, \partial u, \cdots, \partial^2 u) \) and if there exists some symmetry \( S \) of \( F \) and a vector \( \varphi = (\varphi_1, \cdots, \varphi_n) \) such that
\[
S^{(i)} \mathcal{L} + \mathcal{L} D_i \xi^i = D_i \varphi^i,
\]
where
\[
D_i = \frac{\partial}{\partial x^i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_j} + \cdots + u_{i_1 \cdots i_m} \frac{\partial}{\partial u_{i_1 \cdots i_m}} + \cdots
\]
is the total derivative operator of \( u \),
\[
u_i := \frac{\partial u}{\partial x^i}, \quad u_{ij} := \frac{\partial^2 u}{\partial x^i \partial x^j}, \quad \cdots, \quad u_{i_1 \cdots i_m} := \frac{\partial u}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_m}},
\]
the symmetry \( S \) is said to be a Noether symmetry. Then, the Noether’s Theorem asserts that the following conservation law holds
\[
D_i (\xi^i \mathcal{L} + W^i [u, \eta - \xi^j u_j] - \varphi^i) = 0.
\]
(14)
Above we have used the same notations and conventions as in [1]. (For the definition of \( W^i \) see [1], pp. 254-255.)

3 Proofs of theorems 1, 2 and 3

Lemma 8 Let \( u = u(x, y, t) \) be a smooth function. If a vector field \( V = (A, B, C) \) is a vector function of \( x, y, t, u, u_x, u_y, u_t \), its divergence necessarily depends on the second order derivatives of \( u \) with respect to \( x, y \) and \( t \).

Proof. Taking the divergence of vector field \( V \), we obtain
\[
\text{Div}(V) = A_x + B_y + C_t + u_x A_u + u_{xx} A_{ux} + u_{xy} A_{uy} + u_{xt} A_{ut}
\]
\[
+ u_y B_u + u_{xy} B_{ux} + u_{yy} B_{uy} + u_{yt} B_{ut}
\]
\[
+ u_t C_u + u_{xt} C_{ux} + u_{yt} C_{uy} + u_{tt} C_{ut}.
\]
Corollary 9 If the divergence of a vector field does not depend on the second order derivatives, then it does not depend on $u_x$, $u_y$ and $u_t$.

Lemma 10 The symmetry
\[
E = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t} - 2 \frac{\partial}{\partial u}
\]
is not a Noether symmetry.

Proof. In this case, $(\xi, \phi, \tau, \eta) = (x, y, 2t, -2)$. Then, $D_x \xi + D_y \phi + D_t \tau = 4$ and
\[
(\eta_x^{(1)}, \eta_y^{(1)}, \eta_t^{(1)}) = (-u_x, -u_y, -2u_t),
\]
which yields the following first order extension:
\[
E^{(1)} = E - u_x \frac{\partial}{\partial u_x} - u_y \frac{\partial}{\partial u_y} - 2u_t \frac{\partial}{\partial u_t}.
\]
Therefore,
\[
E^{(1)} \mathcal{L} + (D_x \xi + D_y \phi + D_t \tau) \mathcal{L} = u_x^2 + u_y^2 + 4(x^2 + y^2)u_t^2
\]
\[
+ 4yu_xu_t - 4xu_yu_t - 2e^u,
\]
where
\[
\mathcal{L} := \frac{1}{2} u_x^2 + \frac{1}{2} u_y^2 + 2(x^2 + y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - e^u.
\]
From Lemma 8 and equation (15), we conclude that there are not a potential $\phi$ which satisfies
\[
E^{(1)} \mathcal{L} + (D_x \xi + D_y \phi + D_t \tau) \mathcal{L} = \text{Div}(\phi).
\]
Thus, $E$ cannot be a Noether symmetry.

Lemma 11 The symmetry $U$ is not a Noether symmetry.

Proof. First one, note that $\eta = u$, $\xi = \phi = \tau = 0$. Then,
\[
U^{(1)} = u \frac{\partial}{\partial u} + u_x \frac{\partial}{\partial u_x} + u_y \frac{\partial}{\partial u_y} + u_t \frac{\partial}{\partial u_t}
\]
(16)
Aplying the operator obtained in (16) to the Lagrangian
\[ L_k := \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2 + y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t - \frac{k}{2}u^2, \] (17)
where \( k = 0 \) if \( f(u) = 0 \) or \( k = 1 \) if \( f(u) = u \), we find
\[ U^{(1)}L_k = u_x^2 + u_y^2 + 4(x^2 + y^2)u_t^2 + 4yu_xu_t - 4xu_yu_t - ku^2 = 2L_k. \]

From Lemma 8 and Corollary 9 we conclude that there is not a vector field such that equation (13) is true with \( S = U \). ■

**Lemma 12** The symmetry \( W_\beta \) is a Noether symmetry.

**Proof.** The first order extension \( W^{(1)} \) of \( W \) is
\[ W^{(1)} = \beta \frac{\partial}{\partial u} + \beta_x \frac{\partial}{\partial u_x} + \beta_y \frac{\partial}{\partial u_y} + \beta_t \frac{\partial}{\partial u_t}. \] (18)

From (18) and (17), we have
\[ W^{(1)}L_k = -\beta ku + (u_x + 2yu_t)\beta_x + (u_y - 2xu_t)\beta_y + (4(x^2 + y^2)u_t + 2yu_x - 2xu_y)\beta_t \]
\[ = \text{Div}((\beta_x + 2y\beta_t)u, (\beta_y - 2x\beta_t)u, (2y\beta_x - 2x\beta_y + 4(x^2 + y^2)\beta_t)u). \]

■

**Lemma 13** The symmetry
\[ Z = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + 2t \frac{\partial}{\partial t} \]
is not a Noether symmetry.

**Proof.** Since \( D_x\xi + D_y\phi + D_t\tau = 4 \),
\[ \mathcal{L} = \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + 2(x^2 + y^2)u_t^2 + 2yu_xu_t - 2xu_yu_t \] (19)
and
\[ Z^{(1)} = Z + u_x \frac{\partial}{\partial x} + u_y \frac{\partial}{\partial y} + 2u_t \frac{\partial}{\partial t} \] (20)
is a consequence of Eqs. (19) and (20), that
\[ Z^{(1)} \mathcal{L} + \mathcal{L}(D_x \xi + D_y \phi + D_t \tau) = 2 \mathcal{L}. \] (21)

By Lemma 8, there does not exist a vector field such that the right hand of (21) be its divergence. \[ \square \]

**Proof of Theorem 1** We will use four steps to prove this theorem. First, we obtain the first order extension of symmetries \( T, R, \tilde{X}, \tilde{Y} \). Next, we proof the theorem for each one of them.

1. Extensions:
   
   (a) Symmetry \( T \) The coefficients of \( T \) are \( \xi = \phi = \eta = 0 \) and \( \phi = 1 \). Then
   \[ T^{(1)} = T. \]

   (b) Symmetry \( R \) The coefficients of symmetry \( R \) are \( (\xi, \phi, \tau, \eta) = (y, -x, 0, 0) \). Then, we conclude that
   \[ R^{(1)} = R + u_y \frac{\partial}{\partial u_x} - u_x \frac{\partial}{\partial u_y}. \]

   (c) Symmetry \( \tilde{X} \) In this case, \( (\xi, \phi, \tau, \eta) = (1, 0, -2y, 0) \). Then
   \[ \eta_x^{(1)} = 0, \quad \eta_y^{(1)} = 2u_t, \quad \eta_t^{(1)} = 0 \]
   and
   \[ \tilde{X}^{(1)} = \tilde{X} + 2u_t \frac{\partial}{\partial u_y}. \]

   (d) Symmetry \( \tilde{Y} \) This case is analogous to case c and we present only its extension
   \[ \tilde{Y}^{(1)} = \tilde{Y} - 2u_t \frac{\partial}{\partial u_x}. \]

   **Corollary 14** The divergence of any symmetry \( S \in G_f \) is zero.

2. (a) Proof of theorem for the symmetry \( T \). Since \( \text{Div}(T) = 0 = T^{(1)} \mathcal{L} \) it is immediate that
   \[ T^{(1)} \mathcal{L} + \mathcal{L} \text{Div}(T) = 0. \]
(b) Proof of theorem for the symmetry $R$. We have

$$R^{(1)}\mathcal{L} = 0.$$  

Then, from Corollary 14

$$R^{(1)}\mathcal{L} + \mathcal{L}\text{div}(R) = 0.$$  

(c) Proof of theorem for the symmetries $\tilde{X}$ and $\tilde{Y}$. It is immediate that

$$\tilde{X}^{(1)}\mathcal{L} = 0.$$  

Again, by Corollary 14 we obtain

$$\tilde{X}^{(1)}\mathcal{L} + \mathcal{L}\text{div}(\tilde{X}) = 0.$$  

In the same way, we conclude that

$$\tilde{Y}^{(1)}\mathcal{L} + \mathcal{L}\text{div}(\tilde{Y}) = 0.$$  

Proof of Theorem 2: It is a consequence of Lemma 10 and Theorem 1. 

Proof of Theorem 3: From Lemma 11, $U$ is not a Noether symmetry. Then, by Theorem 1 and Lemma 12, $G_f \cup \{W_\beta\}$ is a Noether symmetry group. 

Proof of Theorem 4: By lemmas 11 and 13, the symmetries $Z$ and $U$ are not Noether symmetries. The proof that the symmetries $V_1$, $V_2$ and $V_3$ are Noether symmetries is obtained in same way that Bozhkov and Freire showed that $V_1$, $V_2$ and $V_3$ are Noether symmetries of (1) when $f(u) = u^3$, and can be found in [3]. Then, by Theorem 1 and Lemma 12 we conclude the proof.

4 Conservation Laws

The proof is by a straightforward calculation, which we shall not present here. However, a computer assisted proof can be obtained by means of the software Mathematica. It calculates the components of the conservation laws, which appear in the equation (14). The Mathematica notebook used for this purpose can be obtained form the author upon request.
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