An analogue of the Hom functor and a generalized nuclear democracy theorem

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Abstract. We give an analogue of the Hom functor and prove a generalized form of the nuclear democracy theorem of Tsuchiya and Kanie by using a notion of tensor product for two modules for a vertex operator algebra.

1 Introduction

The notion of vertex operator algebra ([B], [FHL], [FLM]) is the algebraic counterpart of the notion of what is now usually called "chiral algebra" in conformal field theory, and vertex operator algebra theory generalizes the theories of affine Lie algebras, the Virasoro algebra and representations (cf. [B], [DL], [FLM], [FZ]). It has been well known (cf. [FZ], [L1]) that the irreducible highest weight modules (usually called the vacuum representations) $L(\ell, 0)$ for an affine Lie algebra $\hat{\mathfrak{g}}$ of level $\ell$ and $L(c, 0)$ for the Virasoro algebra with central charge $c$ have natural vertex operator algebra structures. If $\ell$ is a positive integer, it was proved ([DL], [FZ], [L1]) that the category of $L(\ell, 0)$-modules is a semi-simple category whose irreducible objects are irreducible highest weight integrable $\hat{\mathfrak{g}}$-modules of level $\ell$ (cf. [K]). If $c = 1 - \frac{6(p-q)^2}{pq}$, where $p, q \in \{2, 3, \cdots\}$ are relatively prime, it was proved ([DMZ], [W]) that the category of $L(c, 0)$-modules is also a semi-simple category whose irreducible objects are exactly those irreducible Virasoro algebra modules $L(c, h)$ listed in [BPZ]. These give the rationality (defined in Section 2) of $L(\ell, 0)$ and $L(c, 0)$.

To state our results, let us start with definitions of intertwining operator. In the minimal models, an intertwining operator from $L(c, h_2)$ to $L(c, h_3)$ was defined in [BPZ]
to be a primary field operator $\Phi(x)$ of weight $h$, i.e., $\Phi(x) \in \text{Hom}_{\mathbb{C}}(L(c,h_2), L(c,h_3))\{x\}$ satisfying the following relation:

$$[L(m), \Phi(x)] = x^m \left( (m+1)h + x \frac{d}{dx} \right) \Phi(x)$$  \hspace{1cm} (1.1)

for $m \in \mathbb{Z}$. For WZW models with $g = sl_2$, an intertwining operator of type \( \left( \begin{array}{c} j_3 \\ j_1 \\ j_2 \end{array} \right) \) was defined (cf. [TK]) as a linear map $\Phi(u, x) \in \text{Hom}(L(\ell, j_2), L(\ell, j_3))\{x\}$ such that (1.1) holds with $h = \frac{j(j+2)}{4(\ell+2)}$ and

$$[a(m), \Phi(u, x)] = x^m \Phi(au, x) \text{ for } m \in \mathbb{Z}, a \in g, u \in L(j),$$  \hspace{1cm} (1.2)

where $L(j)$ is the irreducible $sl_2$-module with highest weight $j$. By employing singular vectors, Tsuchiya and Kanie proved in [TK] that such an intertwining operator $\Phi(\cdot, x)$ on $L(j)$ can be uniquely and naturally extended to an intertwining operator on $L(\ell, j)$. This is the so-called nuclear democracy theorem of Tsuchiya and Kanie.

On the other hand, in the context of vertex operator algebra, an intertwining operator of type $\left( \begin{array}{c} W_3 \\ W_1 \\ W_2 \end{array} \right)$, where $W_i$ ($i = 1, 2, 3$) are modules for a vertex operator algebra $V$, is defined in [FHL] to be a linear map $I(\cdot, x)$ from $W_1$ to $(\text{Hom}(W_2, W_3))\{x\}$ satisfying the $L(-1)$-bracket formula (1.1) with $m = -1$ and the Jacobi identity (2.1) (together with the truncation condition (I1) in Section 2).

An intertwining operator in the sense of [FHL] restricted to $W_1(0)$ gives an intertwining operator on $W_1(0)$ in the sense of [TK] and [BPZ] for the WZW and minimal models. For WZW models, Tsuchiya and Kanie’s nuclear democracy theorem implies that the two definitions define the same fusion rules. The question is: do we have a generalized form of the nuclear democracy theorem for an arbitrary vertex operator algebra? If $V$ is not rational, the answer is negative. (See the appendix for a counterexample.) As the main result of this paper we prove a generalized form of the nuclear democracy theorem for a rational vertex operator algebra so that for all rational models, the fusion rules defined
in the context of vertex operator algebra coincide with those defined in the context of
craneal field theory.

For WZW models, one has an affine Lie algebra \( \hat{\mathfrak{g}} \) available so that one can make
use of the notion of Verma module and singular vectors. To any vertex operator alge-
bra \( V \), we associate a \( \mathbb{Z} \)-graded Lie algebra \( \mathfrak{g}(V) = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}(V)_n \) with generators \( t^n \otimes a \)
(linearly in \( a \)) for \( a \in V, n \in \mathbb{Z} \) and with Borcherds’ commutator formula (2.4) and the
\( L(-1) \)-bracket formula as its defining relations (see also [B],[FFR]). Since \( L(0) \) is a central
element in \( \mathfrak{g}(V)_0 \), using the triangular decomposition with respect to the \( \mathbb{Z} \)-grading we
have the notions of generalized Verma \( g(V) \)-module [Le] (or Weyl module) and lowest
weight module. Then any \( V \)-module \( M \) is a natural \( g(V) \)-module such that any weight
space \( M(h) \) is a natural \( g(V)_0 \)-module where \( t^{n-1} \otimes a \) is represented by \( a_{n-1} \) for \( a \in V \).
But a lowest weight, or even an irreducible lowest weight \( g(V) \)-module is not necessarily
a weak \( V \)-module.

To formulate a nuclear democracy theorem for arbitrary rational vertex operator al-
gebra, we notice that (1.2) is a special case of the general commutator formula (2.4). Since
(1.2) does not hold if \( a \) is not a weight-one element, we have to use a certain cross product
[FLM]. Here is our generalized form of the nuclear democracy theorem or briefly GNDT:
Let \( V \) be a rational vertex operator algebra and \( W_i \ (i = 1, 2, 3) \) be three irreducible \( V \-
modules with lowest weights \( h_i \), respectively. Let \( W_i(0) \) be the lowest weight subspace
of \( W_i \) (with weight \( h_i \)). Let \( \Phi(\cdot, x) \) be a linear map from \( W_1(0) \) to \( \text{Hom}_\mathbb{C}(W_2,W_3)\{x\} \)
satisfying the \( L(-1) \)-bracket formula and

\[
(x_1 - x_2)^{n-1} Y(a, x_1) \Phi(u, x_2) - (-x_2 + x_1)^{n-1} \Phi(u, x_2) Y(a, x_1)
= \ x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) \Phi(a_{n-1} u, x_2)
\]

for any \( a \in V(n), u \in W_1(0) \). Then there exists a unique intertwining operator \( I(\cdot, x) \) from
\( W_1 \otimes W_2 \) to \( W_3 \) in the sense of [FHL], which extends \( \Phi(\cdot, x) \).

To prove this GNDT, we notice that if it is true, then \( I(\cdot, x) \) will be an injective map
on $W_1$ so that $W_1(0)$ can be identified as the space $\Phi(W_1(0), x)$ consisting of $\Phi(u, x)$ for $u \in U$ because $W_1$ is an irreducible $V$-module. For any $u \in W_1, a \in V$, $I(u, x)$ satisfies the $L(-1)$-bracket formula, but (1.3) is not true for an arbitrary $u \in W_1$. However, the local property holds, i.e., for any $a \in V, u \in W_1$, there is a positive integer $k$ such that

$$(x_1 - x_2)^k Y(a, x_1) \Phi(u, x_2) = (x_1 - x_2)^k \Phi(u, x_2) Y(a, x_1)$$

(cf. [DL, formula (9.37)]). A field operator $\Phi(x)$ from $W_2$ to $W_3$ satisfying the $L(-1)$-bracket formula and the local property is called a generalized intertwining operators.

Collecting all generalized intertwining operators $\Phi(x)$ from $W_2$ to $W_3$ we get a vector space $G(W_2, W_3)$. Then we prove (Theorem 4.6) that $G(W_2, W_3)$ becomes a $V$-module under a natural action that comes from the Jacobi identity. Then GNDT follows. We also prove that $G(W_2, W_3)$ satisfies the universal property: For any $V$-module $W$ and any intertwining operator $I(\cdot, x)$ from $W \otimes W_2$ to $W_3$, there exists a unique $V$-homomorphism $\psi$ from $W$ to $G(W_2, W_3)$ such that $I(u, x) = \psi(u)(x)$ for $u \in W$. It follows from the universal property that there is a natural linear isomorphism from $\text{Hom}_V(W, G(W_2, W_3))$ onto $I \left( \begin{array}{c} W_3 \\ WW_2 \end{array} \right)$, the space of intertwining operators of the indicated type.

For WZW models, there is another notion of intertwining operator involving homomorphisms from the tensor product module of a loop module with a highest weight module to another highest weight module for an affine Lie algebra $\hat{g}$. By using the generalized form of the nuclear democracy theorem we prove (Proposition 4.15) that this notion is essentially equivalent to the notion in [FHL].

The notion of $G(W_2, W_3)$ is clearly analogous to the notion of “Hom”-functor. In Lie algebra theory, if $U_i$ ($i = 1, 2, 3$) are modules for a Lie algebra $g$, the space $\text{Hom}_C(U_1, U_2)$ is a natural $g$-module and we have the following natural inclusion relations:

$$(U_1)^* \otimes U_2 \longrightarrow \text{Hom}_C(U_1, U_2) \longrightarrow (U_1 \otimes (U_2)^*)^*.$$

If both $U_1$ and $U_2$ are finite-dimensional, the arrows are isomorphisms so that the space
of linear homomorphisms gives a construction of tensor product modules.

In vertex operator algebra theory, a tensor product theory has been recently developed [HL0-4]. (In the affine Lie algebra level, a theory of tensor product was developed in [KL0-2] for modules of certain levels for an affine Lie algebra and part of this theory was extended to positive integral levels in [F].) In [HL0-4], in addition to the notion of intertwining operator, a notion called intertwining map was also used. An intertwining map was proved to be essentially equivalent to an intertwining operator and could be viewed as an operator-valued functional instead of a formal series of operators. As one of our results in this paper we give a definition and a construction of tensor product in terms of formal variable language.

Motivated by the classical tensor product theory, we formulate a definition of tensor product of an ordered pair of two $V$-modules in terms of intertwining operators and a certain universal property. As an analogue of the construction of the classical tensor product we give a construction of tensor product for a rational vertex operator algebra $V$. Roughly speaking, our tensor product module $T(W_1, W_2)$ is constructed as the quotient space of the tensor product vector space $\mathbb{C}[t, t^{-1}] \otimes W_1 \otimes W_2$ (symbolically the linear span of all coefficients of $Y(u_1, x)u_2$ for $u_i \in W_i$) modulo all the axioms for an intertwining operator of a certain type. It is very natural that the tensor product vector space $\mathbb{C}[t, t^{-1}] \otimes W_1 \otimes W_2$ modulo all the axioms for an intertwining operator of a certain type is a weak $V$-module. By using universal properties, it can be proved that the tensor product module from this construction is isomorphic to those (depending on $z \in \mathbb{C}^\times$) constructed in [HL0-4] in the category of $V$-modules.

Analogous to the classical result, if $V$ satisfies certain “finiteness” and “semisimplicity” conditions, we prove that there exists a unique maximal submodule $\Delta(W_1, W_2)$ inside the weak module $G(W_1, W_2)$ (Proposition 4.9) such that $\Delta(W_1, W_2)'$ is a tensor product module for the ordered pair $(W_1, W_2)'$ (Theorem 4.10).
This paper is organized as follows: Section 2 is preliminary. In Section 3 we formulate a definition of tensor product and give a construction of a tensor product. In Section 4, we prove a generalized form of the nuclear democracy theorem by using an analogue of “Hom”- functor.

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2 Preliminaries

In this section we first review some necessary definitions from [B], [FHL] and [FLM]. Then we present some elementary results about certain Lie algebras and modules related to a vertex (operator) algebra. We use standard notations and definitions of [FHL], [FLM] and [FZ].

Definition 2.1 A vertex operator algebra is a quadruple \((V, Y, 1, \omega)\) where \(V = \oplus_{n \in \mathbb{Z}} V(n)\) is a \(\mathbb{Z}\)-graded vector space, \(Y(\cdot, x)\) is a linear map from \(V\) to \((\text{End} V)[[x, x^{-1}]]\), \(1\) and \(\omega\) are fixed elements of \(V\) such that the following conditions hold:

\[(V0) \dim V(n) < \infty \text{ for any } n \in \mathbb{Z} \text{ and } V(n) = 0 \text{ for } n \text{ sufficiently small};\]

\[(V1) \quad Y(1, x) = 1;\]

\[(V2) \quad Y(a, x)1 \in (\text{End} V)[[x]] \text{ and } \lim_{x \to 0} Y(a, x)1 = a \text{ for any } a \in V;\]

\[(V3) \quad \text{For any } a, b \in V, Y(a, x)b \in V((x)) \text{ and for any } a, b, c \in V, \text{ the following Jacobi identity holds:}\]

\[x^{-1}_0 \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(a, x_1)Y(b, x_2)c - x^{-1}_0 \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y(b, x_2)Y(a, x_1)c\]

\[= x^{-1}_2 \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(a, x_0)b, x_2)c.\]

(2.1)

For \(a \in V\), \(Y(a, x) = \sum_{n \in \mathbb{Z}} a_n x^{-n-1}\) is called the vertex operator associated to \(a\);

\[(V4) \quad \text{Set } Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-1}. \text{ Then we have}\]

\[[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12}\delta_{m+n,0}\text{rank}V\]

(2.2)

for \(m, n \in \mathbb{Z}\), where \(\text{rank}V\) is a fixed complex number, called the rank of \(V\);

\[Y(L(-1)a, x) = \frac{d}{dx}Y(a, x) \quad \text{for any } a \in V;\]

(2.3)

and \(L(0)u = nu := (\text{wt}u)u\) for \(u \in V(n), n \in \mathbb{Z}\).

This completes the definition of vertex operator algebra.
Remark 2.2 If a triple \((V,Y,1)\) satisfies the axioms (V1)-(V3) (without assuming the \(\mathbb{Z}\)-grading and the existence of Virasoro algebra), \((V,Y,1)\) is called a vertex algebra. It can be proved (cf. [L1]) that this definition is equivalent to Borcherds’ definition in [B].

As a consequence of the Jacobi identity we have the following commutator formula [B]:

\[
[Y(a,x_1), Y(b,x_2)] = \text{Res}_{x_0} x_0^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(a,x_0)b, x_2).
\tag{2.4}
\]

Definition 2.3 A module for a vertex operator algebra \(V\) is a pair \((M,Y_M)\) where \(M = \bigoplus_{h \in \mathbb{C}} M(h)\) is a \(\mathbb{C}\)-graded vector space and \(Y_M(\cdot, x)\) is a linear map from \(V\) to \((\text{End}M)[[x, x^{-1}]]\) satisfying the following conditions:

\((M0)\) For any \(h \in \mathbb{C}\), \(L(0)u = hu\) for \(u \in M(h)\), \(\dim M(h) < \infty\) and \(M(n+h) = 0\) for \(n \in \mathbb{Z}\) sufficiently small;

\((M1)\) \(Y_M(1, x) = 1\);

\((M2)\) \(Y_M(L(-1)a, x) = \frac{d}{dx} Y_M(a, x)\) for any \(a \in V\);

\((M3)\) \(Y_M(a,x)u \in M((x))\) for any \(a \in V, u \in M\) and for any \(a,b \in V, u \in M\), the following Jacobi identity holds:

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_M(a,x_1)Y_M(b,x_2)u - x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y_M(b,x_2)Y_M(a,x_1)u = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y_M(Y(a,x_0)b,x_2)u.
\tag{2.5}
\]

By a weak \(V\)-module we mean a pair \((M,Y_M)\) satisfying the axioms (M1)-(M3). A weak \(V\)-module \(M\) is said to be \(\mathbb{N}\)-gradable if there exists an \(\mathbb{N}\)-gradation \(M = \bigoplus_{n \in \mathbb{N}} M(n)\) such that

\[
a_n M(k) \subseteq M(m + n - 1 + k) \quad \text{for} \ m, n, k \in \mathbb{Z}, \ a \in V(m),
\tag{2.6}
\]

where \(\mathbb{N}\) is the set of nonnegative integers and \(M(n) = 0\) for \(n < 0\) by definition. The notions of submodule, irreducible module, quotient module and module homomorphism
can be defined in the obvious way. A vertex operator algebra \( V \) is said to be rational if any \( \mathbb{N} \)-gradable weak \( V \)-module is a direct sum of irreducible \( \mathbb{N} \)-gradable weak \( V \)-modules. If \( V \) is rational, it was proved [DLM1] that there are only finitely many irreducible modules up to equivalence and that any irreducible \( \mathbb{N} \)-gradable weak \( V \)-module is a module so that \( L(0) \) acts semisimply on any \( \mathbb{N} \)-gradable weak \( V \)-module. Then this definition of rationality is equivalent to Zhu’s definition [Z] of rationality. There are also other variant definitions of rationality. For example, the definition of rationality in [HL0-4] is different from the current definition.

Let \( M = \bigoplus_{h \in \mathbb{C}} M(h) \) be a \( V \)-module. Set \( M' = \bigoplus_{h \in \mathbb{C}} M^*(h) \) and define

\[
\langle Y(a, x) u', v \rangle = \langle u', Y(e^{xL(1)}(-x^2)L(0)a, x^{-1})v \rangle
\]

for \( u' \in M', v \in M \). Then it was proved in [FHL] that \( M' \) is a \( V \)-module, called the contragredient module, and that \( (M')' = M \). If \( f \) is a \( V \)-homomorphism from a \( V \)-module \( W \) to \( M \), then we have a \( V \)-homomorphism \( f' \) from \( M' \) to \( W' \) such that

\[
\langle f'(u'), v \rangle = \langle u', f(v) \rangle \quad \text{for} \quad u' \in M', v \in W.
\]

Furthermore, we have \((f')' = f\) [HL0-4].

**Definition 2.4** Let \( W_1, W_2 \) and \( W_3 \) be three weak \( V \)-modules. An intertwining operator of type \( \begin{pmatrix} W_3 \\ W_1 \end{pmatrix} \) is a linear map

\[
I(\cdot, x) : \quad W_1 \to (\text{Hom}(W_2, W_3)) \{x\},
\]

\[
u \mapsto I(u, x) = \sum_{\alpha \in \mathbb{C}} u_\alpha x^{-\alpha - 1}
\]

satisfying the following conditions:

(1) For any fixed \( u \in W_1, v \in W_2, \alpha \in \mathbb{C}, u_{\alpha + n}v = 0 \) for \( n \in \mathbb{Z} \) sufficiently large;

(2) \( I(L(-1)u, x)v = \frac{d}{dx}I(u, x)v \) for \( u \in W_1, v \in W_2 \);

(3) For \( a \in V, u \in W_1, v \in W_2 \), the modified Jacobi identity (2.7) where \( Y(b, x_2) \) and \( Y(Y(a, x_0)b, x_2) \) are replaced by \( I(u, x_2) \) and \( I(Y(a, x_0)u, x_2) \), respectively, holds.
We denote by \( I \left( \begin{array}{c} W_3 \\ W_1W_2 \end{array} \right) \) the vector space of all intertwining operators of the indicated type and we call the dimension of this vector space the fusion rule of the corresponding type.

The following proposition was proved in [FHL] and [FZ]:

**Proposition 2.5** Let \( W_i = \bigoplus_{n=0}^{\infty} W_i(n) \) \((i = 1, 2, 3)\) be weak \( V \)-modules such that \( L(0)_{W_i(n)} = (h_i + n)\text{id} \) \((i = 1, 2, 3)\) and let \( I(\cdot, x) \) be an intertwining operator of type \( \begin{array}{c} W_3 \\ W_1W_2 \end{array} \). Then

\[
I^0(u, x) := x^{h_1+h_2-h_3}I(u, x) \in (\text{Hom}(W_2, W_3))[x, x^{-1}]. \tag{2.10}
\]

Set \( I^0(u, x) = \sum_{n \in \mathbb{Z}} I_u(n)x^{-n-1} \). Then for any \( k \in \mathbb{N}, u \in W_1(k), m, n \in \mathbb{N}, \)

\[
I_u(n)W_2(m) \subseteq W_3(m + k - n - 1). \tag{2.11}
\]

In particular,

\[
I_u(k + m + i)W_2(m) = 0 \quad \text{for all } i \geq 0. \tag{2.12}
\]

Let \( W_i \) \((i = 1, 2, 3)\) be \( V \)-modules and let \( I(\cdot, x) \) be an intertwining operator of type \( \begin{array}{c} W_3 \\ W_1W_2 \end{array} \). The transpose operator \( I'(\cdot, x) \) is defined by:

\[
I'(\cdot, x) : W_2 \otimes W_1 \rightarrow W_3\{x\}
\]

\[
I'(u_2, x)u_1 = e^{xL(-1)}I(u_1, e^{\pi i}x)u_2 \tag{2.13}
\]

for \( u_1 \in W_1, u_2 \in W_2 \). The adjoint operator \( I'(\cdot, x) \) is defined by:

\[
I'(\cdot, x) : W_1 \otimes W_3' \rightarrow W_2'\{x\}
\]

\[
\langle I'(u_1, x)u'_3, u_2 \rangle = \langle u'_3, I(e^{xL(1)}(e^{\pi i}x^{-2})L(0)u_1, x^{-1})u_2 \rangle \tag{2.14}
\]

for \( u_1 \in W_1, u_2 \in W_2, u'_3 \in W_3' \). The following proposition was proved in [HL0-4] (see also [FHL], [L2]).
Proposition 2.6 The transpose operator $I_t(\cdot, x)$ and the adjoint operator $I'(\cdot, x)$ are intertwining operators of corresponding types.

Notice that the transpose operator $I_t(\cdot, x)$ can be defined more generally for weak $V$-modules $W_i$ for $i = 1, 2, 3$ and it follows from the same proof that it is an intertwining operator.

The following Borcherds’ examples of vertex algebras [B] show that the notion of vertex algebra is really a generalization of the notion of commutative associative algebra.

Example 2.7 Let $A$ be a commutative associative algebra with identity together with a derivation $d$. Define

$$Y(a, x)b = \left(e^{xd}a\right)b \quad \text{for any } a, b \in A. \quad (2.15)$$

Then $(A, Y, 1)$ is a vertex algebra. Furthermore, let $M$ be a module for $A$ viewed as an associative algebra. Define $Y_M(a, x)u = \left(e^{xd}a\right)u$ for $a \in V, u \in M$. Then $(M, Y_M)$ is a module for the vertex algebra $(A, Y, 1)$. In particular, let $A = \mathbb{C}((t))$ and $d = \frac{d}{dt}$. Then $(\mathbb{C}((t)), Y, 1)$ is a vertex algebra. By definition, we have

$$Y(f(t), x) = e^{x\frac{d}{dt}}f(t) = f(t + x) \quad \text{for } f(t) \in \mathbb{C}((t)). \quad (2.16)$$

It is clear that the Laurent polynomial ring $\mathbb{C}[t, t^{-1}]$ is a vertex subalgebra.

For convenience in the following we shall associate Lie algebras $g_0(V)$ and $g(V)$ to a vertex algebra $V$. The following lemma could be found in [B]:

Lemma 2.8 Let $(V, Y, 1)$ be a vertex algebra and let $d$ be the endomorphism of $V$ defined by $d(a) = a_{-2}1$ for $a \in V$. Then the quotient space $g_0(V) := V/dV$ is a Lie algebra with the bilinear product: $[\bar{a}, \bar{b}] = a_0b$ for $a, b \in V$. Furthermore, any $V$-module $M$ is a $g_0(V)$-module with the action given by: $au = a_0u$ for $a \in V, u \in M$. 

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Let $V$ be any vertex algebra. Then from [FHL] (see also [B]), $\hat{V} := \mathbb{C}[t, t^{-1}] \otimes V$ has a vertex algebra structure with $Y(f(t) \otimes u, x) = Y(f(t), x) \otimes Y(u, x)$ for any $f(t) \in \mathbb{C}[t, t^{-1}], u \in V$, and $1 = 1 \otimes 1_V$. (The affinization of a vertex operator algebra has also been used in [HL0-4].) Set $\hat{d} := \frac{d}{dt} \otimes 1 + 1 \otimes d_V$. Then $\hat{d}(u) = u - 2$ for $u \in \hat{V}$. Then from Lemma 2.8 $g_0(\hat{V}) = \hat{V}/\hat{d}\hat{V}$ is a Lie algebra. For any $m, n \in \mathbb{Z}, a \in V$, by definition we have

$$(t^m \otimes a)_n = \text{Res}_x x^n Y(t^m \otimes a, x)$$

$$= \text{Res}_x x^n (t + x)^n \otimes Y(a, x)$$

$$= \sum_{i=0}^{\infty} \binom{m}{i} t^{m+n-i} \otimes a_i. \quad (2.17)$$

Thus

$$[(t^m \otimes a), (t^n \otimes b)] = (t^m \otimes a)_0(t^n \otimes b) = \sum_{i=0}^{\infty} \binom{m}{i} t^{m+n-i} \otimes a_i b \quad (2.18)$$

for any $a, b \in V, m, n \in \mathbb{Z}$, where “bar” denotes the natural quotient map from $\hat{V}$ to $g_0(\hat{V})$. Therefore, we have (see also [B])

**Proposition 2.9** Let $V$ be any vertex algebra. Then the quotient space $g(V) := g_0(\hat{V})$ is a Lie algebra with the bilinear operation:

$$[t^m \otimes a, t^n \otimes b] = \sum_{i=0}^{\infty} \binom{m}{i} t^{m+n-i} \otimes a_i b. \quad (2.19)$$

(This Lie algebra $g(V)$ has been also studied in [FFR].) We also use $a(m)$ for $t^m \otimes a$ through the paper. It is clear that $1(-1)$ is a central element of $g(V)$. If $1(-1)$ acts as a scalar $k$ on a $g(V)$-module $M$, we call $M$ a $g(V)$-module of level $k$. (This corresponds to level for affine Lie algebras.) A $g(V)$-module $M$ is said to be restricted if for any $a \in V, u \in M, a(n)u = 0$ for $n$ sufficiently large. Then any weak $V$-module $M$ is a restricted $g(V)$-module of level one, where $a(n)$ is represented by $a_n$. (However, a restricted $g(V)$-module is not necessarily a weak $V$-module.) Then we obtain a functor
\[ \mathcal{F} \] from the category of weak \( V \)-modules to the category of restricted \( g(V) \)-modules. For any restricted \( g(V) \)-module \( M \), we define \( J(M) \) to be the intersection of all \( \ker f \), where \( f \) runs through all \( g(V) \)-homomorphisms from \( M \) to weak \( V \)-modules. Then \( M \) is a weak \( V \)-module if and only if \( J(M) = 0 \). Furthermore, \( M/J(M) \) is a weak \( V \)-module and \( M/J(M) \) is a universal from \( M \) to the functor \( \mathcal{F} [J] \).

To summarize, for any vertex algebra \( V \) we have two Lie algebras \( g_0(V) \) and \( g(V) \) which are related by the following inclusion relations:

\[ g_0(V) \subseteq g(V) \cong g(V) \subseteq g(V) \subseteq \cdots . \quad (2.20) \]

Let \( V \) be a vertex operator algebra. For any \( a \in V(m), m, n \in \mathbb{Z} \), we define

\[ \deg a(n) = \deg (t^n \otimes a) = \text{wta} - n - 1 = m - n - 1. \quad (2.21) \]

Then \( g(V) \) becomes a \( \mathbb{Z} \)-graded Lie algebra. Denote by \( g(V)_0 \) the degree-zero subalgebra. Then we obtain a triangular decomposition \( g(V) = g(V)_+ \oplus g(V)_0 \oplus g(V)_- \).

**Lemma 2.10** Let \( V \) be a vertex algebra, let \( M \) be a \( V \)-module and let \( z \) be any nonzero complex number. For any \( a \in V, u \in M, m, n \in \mathbb{Z} \), define

\[ a(m)(t^n \otimes u) = \sum_{i=0}^{\infty} \binom{m}{i} z^{n-i} (t^{m+n-i} \otimes a_i u). \quad (2.22) \]

Then this defines a \( g(V) \)-module (of level zero) structure on \( \hat{M} := \mathbb{C}[t, t^{-1}] \otimes M \).

**Proof.** Let \( \psi \) be the automorphism of the associative algebra \( \mathbb{C}[t, t^{-1}] \) such that \( \psi(f(t)) = f(zt) \) for \( f(t) \in \mathbb{C}[t, t^{-1}] \). Set \( \mathbb{C}[t, t^{-1}]^\psi = \mathbb{C}[t, t^{-1}] \). Then \( \mathbb{C}[t, t^{-1}]^\psi \) is a \( \mathbb{C}[t, t^{-1}] \)-module with the following action:

\[ f(t)u = \psi(f(t))u = f(zt)u \quad \text{for } f(t) \in \mathbb{C}[t, t^{-1}], u \in \mathbb{C}[t, t^{-1}]^\psi. \]

By Example 2.7 \( \mathbb{C}[t, t^{-1}]^\psi \) is a module for the vertex algebra \( \mathbb{C}[t, t^{-1}] \) such that

\[ Y(f(t), x)u = \psi(e^{xt} \hat{f}(t))u = f(zt + x)u \quad (2.23) \]
for \( f(t) \in \mathbb{C}[t, t^{-1}], u \in \mathbb{C}[t, t^{-1}]^\psi \). Then \( \mathbb{C}[t, t^{-1}]^\psi \otimes M \) is a \( \hat{V} \)-module, so that it is a \( g(V) \) (= \( g_0(\hat{V}) \))-module (of level zero). Then the lemma follows (2.17) immediately. \( \square \)

Let \( V \) be a vertex algebra and let \( M \) be a \( V \)-module. For any nonzero complex number \( z \), let \( \mathbb{C}_z \) be the evaluation module for the associative algebra \( \mathbb{C}[t, t^{-1}] \) with \( t \) acting as a scalar \( z \). Then from Example 2.7 \( \mathbb{C}_z \) is a module for vertex algebra \( \mathbb{C}[t, t^{-1}] \), so that \( \mathbb{C}_z \otimes M \) is a \( \hat{V} \)-module. Therefore \( \mathbb{C}_z \otimes M \) is a \( g(V) = g_0(\hat{V}) \)-module (by Lemma 2.1).

From (2.17) we have

\[
a(m) \cdot (1 \otimes u) = \sum_{i=0}^{\infty} \binom{m}{i} z^{m-i} (1 \otimes a_i u) \quad \text{for } a \in V, u \in M. \tag{2.24}
\]

Denote this \( g(V) \)-module by \( M_z \). Then we obtain

**Proposition 2.11** Let \( V \) be a vertex algebra, let \( M \) be a \( V \)-module and let \( z \) be any nonzero complex number. Define \( \rho : g(V) \to \text{End}_\mathbb{C} M \) as follows:

\[
\rho(a(m)) = \sum_{i=0}^{\infty} \binom{m}{i} z^{m-i} a_i u \quad \text{for } a \in V, u \in M. \tag{2.25}
\]

Then \( \rho \) is a representation of \( g(V) \) (of level zero) on \( M \).

Noticing that \( \sum_{i=0}^{\infty} \binom{m}{i} z^{m-i} a_i \) is an infinite sum (although it is a finite sum after applied to each vector \( u \) of \( M \)), we may consider a certain completion of \( g(V) \). By considering the tensor product vertex algebra \( \mathbb{C}((t)) \otimes V \) we obtain a Lie algebra \( g_0(\mathbb{C}((t)) \otimes V) \) (from Lemma 2.1). It is clear that this Lie algebra is the completion of \( g(V) \) with respect to a certain topology for \( g(V) \). We denote this Lie algebra by \( \tilde{g}(V) \).

For any \( f(t) = \sum_{m \geq k} c_m t^m \in \mathbb{C}((t)) \), since the following sum:

\[
\sum_{m \geq k} c_m \left( \sum_{i=0}^{\infty} \binom{m}{i} z^{m-i} t^i \right)^i = \sum_{i=0}^{\infty} \left( \sum_{m \geq k} \binom{m}{i} c_m z^{m-i} \right) t^i \tag{2.26}
\]

may not be a well-defined element of \( \mathbb{C}((t)) \), we cannot extend an evaluation \( g(V) \)-module \( M_z \) to a \( \tilde{g}(V) \)-module.
Define a linear map $\Delta_z$ as follows:

$$\Delta_z : \mathbb{C}[t,t^{-1}] \otimes V \rightarrow (\mathbb{C}((t)) \otimes V) \otimes (\mathbb{C}((t)) \otimes V);$$

$$f(t) \otimes a \mapsto 1 \otimes (f(t) \otimes a) + (f(z + t) \otimes a) \otimes 1.$$  \hspace{1cm} (2.27)

**Proposition 2.12** $\Delta_z$ induces an associative algebra homomorphism from $U(g(V))$ to $U(\tilde{g}(V)) \otimes U(\tilde{g}(V)).$

**Proof.** Define a linear map $\Delta_1^z$ from $\mathbb{C}[t,t^{-1}] \otimes V$ to $\mathbb{C}((t)) \otimes V$ as follows:

$$\Delta_1^z(f(t) \otimes a) = f(z + t) \otimes a \quad \text{for } f(t) \in \mathbb{C}[t,t^{-1}], a \in V.$$ \hspace{1cm} (2.28)

Then $\Delta_z = \Delta_1^z \otimes 1 + 1 \otimes \text{id}$. Therefore it suffices to prove that $\Delta_1^z$ induces a Lie algebra homomorphism from $g(V)$ to $\tilde{g}(V)$. Let $\psi_z$ be the algebra homomorphism from $\mathbb{C}[t,t^{-1}]$ to $\mathbb{C}((t))$ defined by: $\psi_z(f(t)) = f(z + t)$ for $f(t) \in \mathbb{C}((t))$. From Examples 2.4 $\psi_z$ is a vertex algebra homomorphism from $\mathbb{C}[t,t^{-1}]$ to $\mathbb{C}((t))$, so that $\psi_z \otimes \text{id}$ is a vertex algebra homomorphism from $\mathbb{C}[t,t^{-1}] \otimes V$ to $\mathbb{C}((t)) \otimes V$. By definition $\Delta_1^z = \psi_z \otimes \text{id}$. Therefore $\Delta_1^z$ induces a Lie algebra homomorphism from $g(V)$ to $\tilde{g}(V)$. \hspace{1cm} $\square$

**Remark 2.13** The Hopf-like algebra $(U(g(V)), U(\tilde{g}(V)), \Delta_z)$ is implicitly used in many references such as [HL0-4], [KL0-2] and [MS].

## 3 A definition of tensor product and a construction

In this section we shall first formulate a definition of a tensor product in terms of a certain universal property as an analogue of the notion of the classical tensor product. Then we give a construction of a tensor product for an ordered pair of modules for a rational vertex operator algebra.

Throughout this section, $V$ will be a fixed vertex operator algebra. Let $\mathcal{C}$ be the category of weak $V$-modules where a morphism $f$ from $W$ to $M$ is a linear map such that
\[ f(Y(a,x)u) = Y(a,x)f(u) \] for any \( a \in V, u \in W \). Let \( \mathcal{C}_0 \) be the subcategory of \( \mathcal{C} \) where objects of \( \mathcal{C}_0 \) are weak \( V \)-modules satisfying all the axioms of a module except that in (M0), infinite-dimensional homogeneous subspaces are allowed.

**Definition 3.1** Let \( \mathcal{D} \) be either the category \( \mathcal{C} \) or \( \mathcal{C}_0 \) and let \( W_1 \) and \( W_2 \) be objects of \( \mathcal{D} \). A tensor product for the ordered pair \((W_1,W_2)\) is a pair \((M,F(\cdot,x))\) consisting of an object \( M \) of \( \mathcal{D} \) and an intertwining operator \( F(\cdot,x) \) of type \( \left( \begin{array}{c} M \\ W_1W_2 \end{array} \right) \) satisfying the following universal property: For any object \( W \) of \( \mathcal{D} \) and any intertwining operator \( I(\cdot,x) \) of type \( \left( \begin{array}{c} W \\ W_1W_2 \end{array} \right) \), there exists a unique \( V \)-homomorphism \( \psi \) from \( M \) to \( W \) such that \( I(\cdot,x) = \psi \circ F(\cdot,x) \). (Here \( \psi \) extends canonically to a linear map from \( M\{x\} \) to \( W\{x\} \).)

**Remark 3.2** Just as in the classical algebra theory, it follows from the universal property that if there exists a tensor product \((M,F(\cdot,x))\) in the category \( \mathcal{C} \) or \( \mathcal{C}_0 \) for the ordered pair \((W_1,W_2)\), then it is unique up to \( V \)-module isomorphism, i.e., if \((W,G(\cdot,x))\) is another tensor product, then there is a \( V \)-module isomorphism \( \psi \) from \( M \) to \( W \) such that \( G(\cdot,x) = \psi \circ F(\cdot,x) \). Conversely, let \((M,F(\cdot,x))\) be a tensor product for the ordered pair \((W_1,W_2)\) and let \( \sigma \) be an automorphism of the \( V \)-module \( M \). Then \((M,\sigma \circ F(\cdot,x))\) is a tensor product for \((W_1,W_2)\).

**Lemma 3.3** Let \((W,F(\cdot,x))\) is a tensor product in the category \( \mathcal{C} \) or \( \mathcal{C}_0 \) for the ordered pair \((W_1,W_2)\). Then \( F(\cdot,x) \) is surjective in the sense that all the coefficients of \( F(u_1,x)u_2 \) for \( u_i \in W_i \) linearly span \( W \).

**Proof.** Let \( \overline{W} \) be the linear span of all the coefficients of \( F(u_1,x)u_2 \) for \( u_i \in W_i \). Then it follows from the commutator formula (2.4) that \( \overline{W} \) is a submodule of \( W \) and \( F(\cdot,x) \) is an intertwining operator of type \( \left( \begin{array}{c} \overline{W} \\ W_1W_2 \end{array} \right) \). It follows from the universal property of \((W,F(\cdot,x))\) that there is a unique \( V \)-module homomorphism \( \psi \) from \( W \) to \( \overline{W} \) such that

\[
F(u_1,x)u_2 = \psi(F(u_1,x)u_2) \quad \text{for} \; u_1 \in W_1, u_2 \in W_2.
\] (3.1)
Since $\bar{W}$ is a submodule of $W$, $\psi$ may be viewed as a $V$-homomorphism from $W$ to $W$. It follows from the universal property of $(W,F(\cdot,x))$ that $\psi = 1$. Thus $W = \bar{W}$. Then the proof is complete. $\square$

**Corollary 3.4** If $(M,F(\cdot,x))$ is a tensor product in the category $\mathcal{C}$ or $\mathcal{C}_0$ for the ordered pair $(W_1,W_2)$, then for any weak $V$-module $W_3$ in the same category, $\text{Hom}_V(M,W_3)$ is naturally isomorphic to the space of intertwining operators of type $(W_3,W_1W_2)$.

**Proof.** Let $\phi$ be any $V$-homomorphism from $M$ to $W_3$. Then $\phi F(\cdot,x)$ is an intertwining operator of type $(W_3,W_1W_2)$. Thus we obtain a linear map $\pi$ from $\text{Hom}_V(M,W_3)$ to $I(W_3,W_1W_2)$ defined by $\pi(\phi) = \phi F(\cdot,x)$. Since $F(\cdot,x)$ is surjective (Lemma 3.3), $\pi$ is injective. On the other hand, the universal property of $(W,F(\cdot,x))$ implies that $\pi$ is surjective. $\square$

**Remark 3.5** If $(M,F(\cdot,x))$ is a tensor product in the category $\mathcal{C}$ or $\mathcal{C}_0$ for the ordered pair $(W_1,W_2)$, then one can show that $(M,F^t(\cdot,x))$ is a tensor product in the same category for the ordered pair $(W_2,W_1)$. This gives a sort of commutativity of tensor product. It is important to notice that it should not be confused with the symmetric property of a tensor category. As a matter of fact, the tensor category of $V$-modules is not a symmetric tensor category [HL0-4]. If $(M,Y_M(\cdot,x))$ is a $V$-module, one can show that $(M,Y_M(\cdot,x))$ is a tensor product for $(V,M)$. This shows that the adjoint module $V$ satisfies a certain unital property.

Next toward a construction of a tensor product we shall construct an $\mathbb{N}$-gradable weak $V$-module $T(W_1,W_2)$ for an ordered pair $(W_1,W_2)$ of $\mathbb{N}$-gradable weak $V$-modules. First form the vector space

$$F_0(W_1,W_2) = \mathbb{C}[t,t^{-1}] \otimes W_1 \otimes W_2$$ (3.2)
and set
\[ Y_t(u, x) = \sum_{n \in \mathbb{Z}} (t^n \otimes u) x^{-n-1} \quad \text{for } u \in W_1. \] (3.3)

Then \( \mathbb{C}[t, t^{-1}] \otimes W_1 \) is linearly spanned by the coefficients of all \( Y_t(u, x) \) for \( u \in W_1 \).

Fix a gradation \( W_i = \bigoplus_{n \in \mathbb{N}} W_i(n) \) for each \( W_i \) (\( i = 1, 2 \)). Later we will show that if \( V \) is rational, there is a canonical gradation for any \( \mathbb{N} \)-gradable weak \( V \)-module.

We define a \( \mathbb{Z} \)-grading for \( F_0(W_1, W_2) \) as follows: For \( k \in \mathbb{Z}; m, n \in \mathbb{N}, u \in W_1(m), v \in W_2(n) \), we define
\[ \deg (t^k \otimes u \otimes v) = m + n - k - 1. \] (3.4)

Define an action of \( \hat{V} = \mathbb{C}[t, t^{-1}] \otimes V \) on \( F_0(W_1, W_2) \) as follows: For \( a \in V, u \in W_1, v \in W_2 \), we define
\[ Y_t(a, x_1)(Y_t(u, x_2) \otimes v) \]
\[ = Y_t(u, x_2) \otimes Y(a, x_1)v + \text{Res}_{x_0}x_2^{-1}\delta \left( \frac{x_1 - x_0}{x_2} \right) Y_t(Y(a, x_0)u, x_2) \otimes v. \] (3.5)

**Proposition 3.6** Under the above defined action of \( \hat{V} \), \( F_0(W_1, W_2) \) becomes a \( \mathbb{Z} \)-graded \( g(V) \)-module of level one, i.e.,
\[ Y_t(1, x) = 1, \ Y_t(L(-1)a, x) = \frac{d}{dx}Y_t(a, x) \quad \text{for } a \in V; \] (3.6)
\[ \deg a(n) = \text{wt } a - n - 1 \quad \text{for each homogeneous } a \in V, n \in \mathbb{Z}; \] (3.7)
\[ [Y_t(a, x_1), Y_t(b, x_2)] = \text{Res}_{x_0}x_2^{-1}\delta \left( \frac{x_1 - x_0}{x_2} \right) Y_t(Y(a, x_0)b, x_2) \quad \text{for } a, b \in V. \] (3.8)

**Proof.** Writing (3.5) into components we get
\[ (t^m \otimes a)(t^n \otimes u \otimes v) \]
\[ = t^n \otimes u \otimes a_m u \]
\[ + \text{Res}_{x_0}\text{Res}_{x_1}\text{Res}_{x_2}x_1^m x_2^n x_1^{-1} x_2^{-1}\delta \left( \frac{x_2 + x_0}{x_1} \right) Y_t(Y(a, x_0)u, x_2) \otimes v \]
\[
= t^n \otimes u \otimes a_m u + \text{Res}_{x^2} \sum_{i=0}^{\infty} \binom{m}{i} x^{m+n-i}Y_t(a_i u, x_2) \otimes v \\
= t^n \otimes u \otimes a_m u + \sum_{i=0}^{\infty} \binom{m}{i} (t^{m+n-i} \otimes a_i u \otimes v).
\]

(3.9)

It follows from Lemma \ref{Lemma2.10} that (3.5) defines a \(g(V)\)-module structure on \(\mathbb{C}[t, t^{-1}] \otimes W_1 \otimes W_2\), which is a tensor product module of level-zero \(g(V)\)-module \(\mathbb{C}[t, t^{-1}] \otimes W_1\) with the level-one \(g(V)\)-module \(W_2\).

Define \(J_0\) to be the \(g(V)\)-submodule of \(F_0(W_1, W_2)\) generated by the following subspaces:

\[t^{m+n+i} \otimes W_1(m) \otimes W_2(n) \text{ for } m, n, i \in \mathbb{N}.
\]

(3.10)

Set

\[F_1(W_1, W_2) = F_0(W_1, W_2)/J_0.
\]

(3.11)

**Remark 3.7** The space \(F_1(W_1, W_2)\) is an \(\mathbb{N}\)-gradable \(g(V)\)-module of level one, so that the axioms (M1), (M2) and the commutator formula (2.4) automatically hold. Furthermore, for any \(a \in V, w \in F_1(W_1, W_2)\), \(Y_t(a, x)w\) involves only finitely many negative powers of \(x\).

**Remark 3.8** Notice that the action (3.3) of \(g(V)\) on \(F_0(W_1, W_2)\) only reflects the commutator formula (2.4), which is weaker than the Jacobi identity, unlike the situation in the classical Lie algebra theory. In the next step, we consider the whole Jacobi identity relation for an intertwining operator. This step in our approach might be related to the “compatibility condition” in Huang and Lepowsky’s approach [HL0-4].

Let \(\pi_1\) be the quotient map from \(F_0(W_1, W_2)\) onto \(F_1(W_1, W_2)\) and let \(J_1\) be the subspace of \(F_1(W_1, W_2)\), linearly spanned by all coefficients of monomials \(x_0^m x_1^n x_2^k\) in the
following expressions:

\[ x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_t(a, x_1) \pi_1(Y_t(u, x_2) \otimes v) \]

\[-x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \pi_1(Y_t(u, x_2) \otimes Y(a, x_1)v) \]

\[-x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \pi_1(Y_t(Y(a, x_0)u, x_2)) \otimes v \]  

(3.12)

for \( a \in V, u \in W_1, v \in W_2. \)

**Proposition 3.9** The subspace \( J_1 \) is a graded \( g(V) \)-submodule of \( F_1(W_1, W_2). \)

**Proof.** For \( a, b \in V, u \in W_1, v \in W_2, \) we have

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_t(b, x_3)Y_t(a, x_1)\pi_1(Y_t(u, x_2) \otimes v) \]

\[= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_t(a, x_1)Y_t(b, x_3)\pi_1(Y_t(u, x_2) \otimes v) \]

\[+ \text{Res}_x x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) x_1^{-1} \delta \left( \frac{x_3 - x_4}{x_1} \right) Y_t(Y(b, x_4)a, x_1)\pi_1(Y_t(u, x_2) \otimes v) \]

\[= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_t(a, x_1)\pi_1(Y_t(u, x_2) \otimes Y(b, x_3)v) \]

\[+ \text{Res}_x x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) x_2^{-1} \delta \left( \frac{x_3 - x_4}{x_2} \right) Y_t(a, x_1)\pi_1(Y_t(Y(b, x_4)u, x_2) \otimes v) \]

\[+ \text{Res}_x x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) x_1^{-1} \delta \left( \frac{x_3 - x_4}{x_1} \right) Y_t(Y(b, x_4)a, x_1)\pi_1(Y_t(u, x_2) \otimes v) \]

(3.13)

\[Y_t(b, x_3)\pi_1(Y_t(u, x_2) \otimes Y(a, x_1)v) \]

\[= \pi_1(Y_t(u, x_2) \otimes Y(b, x_3)Y(a, x_1)v) \]

\[+ \text{Res}_x x_2^{-1} \delta \left( \frac{x_3 - x_4}{x_2} \right) \pi_1(Y_t(Y(b, x_4)u, x_2) \otimes Y(a, x_1)v) \]

\[= \pi_1(Y_t(u, x_2) \otimes Y(a, x_1)Y(b, x_3)v) \]

\[+ \text{Res}_x x_2^{-1} \delta \left( \frac{x_3 - x_4}{x_1} \right) \pi_1(Y_t(u, x_2) \otimes Y(Y(b, x_4)a, x_1)v) \]

\[+ \text{Res}_x x_2^{-1} \delta \left( \frac{x_3 - x_4}{x_2} \right) \pi_1(Y_t(Y(b, x_4)u, x_2) \otimes Y(a, x_1)v) \]  

(3.14)
Then it is clear that $J_1$ is stable under the action of $Y_1(b, x)$ for any $b \in V$. □

**Theorem 3.10** The quotient space $F_2(W_1, W_2) := F_1(W_1, W_2)/J_1$ is an $\mathbb{N}$-gradable weak $V$-module.

**Proof.** We only need to prove the Jacobi identity. Let $\pi_2$ be the natural quotient
map from $F_0(W_1, W_2)$ onto $F_2(W_1, W_2)$. For $a, b \in V, u \in W_1, v \in W_2,$ we have

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_t(a, x_1) Y_t(b, x_2) \pi_2(Y_t(u, x_3) \otimes v)$$

$$= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_t(a, x_1) \pi_2 \left( Y_t(u, x_3) \otimes Y(b, x_2)v \right)$$

$$+ \text{Res}_x x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) Y_t(a, x_1) \pi_2 \left( Y_t(Y(b, x_4)u, x_3) \otimes v \right)$$

$$= x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \pi_2(Y_t(u, x_3) \otimes Y(a, x_1) Y(b, x_2)v)$$

$$+ \text{Res}_x x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) \pi_2(Y_t(Y(a, x_4)u, x_3) \otimes Y(b, x_2)v)$$

$$+ \text{Res}_x x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) Y_t(a, x_1) \pi_2(Y_t(Y(b, x_4)u, x_3) \otimes v)$$

(3.16)

$$+ \text{Res}_x x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) Y_t(a, x_1) \pi_2(Y_t(Y(b, x_4)u, x_3) \otimes v);$$

(3.17)

$$x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) Y_t(b, x_2) Y_t(a, x_1) \pi_2(Y_t(u, x_3) \otimes v)$$

$$= x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) \pi_2(Y_t(u, x_3) \otimes Y(b, x_2) Y_t(a, x_1)v)$$

$$+ \text{Res}_x x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) \pi_2(Y_t(Y(b, x_4)u, x_3) \otimes Y(a, x_1)v)$$

$$+ \text{Res}_x x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) x_3^{-1} \delta \left( \frac{x_1 - x_4}{x_3} \right) Y_t(b, x_2) \pi_2(Y_t(Y(a, x_4)u, x_3) \otimes v);$$

(3.19)

$$+ \text{Res}_x x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) x_3^{-1} \delta \left( \frac{x_1 - x_4}{x_3} \right) Y_t(b, x_2) \pi_2(Y_t(Y(a, x_4)u, x_3) \otimes v);$$

(3.20)

and

$$x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y_t(Y(a, x_0)b, x_2) \pi_2(Y_t(u, x_3) \otimes v)$$

$$= x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \pi_2(Y_t(u, x_3) \otimes Y(Y(a, x_0)b, x_2)v)$$

$$+ \text{Res}_x x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) \pi_2(Y_t(Y(Y(a, x_0)b, x_4)u, x_3) \otimes v).$$

(3.22)

(3.23)

It follows from the Jacobi identity of $W_2$ that $\text{(3.16) - (3.17)} = \text{(3.22)}$. Since

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right)$$
Similarly, we obtain

\[
\begin{align*}
&= x_0^{-1} \delta \left( \frac{x_1 - x_3 - x_4}{x_0} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) \\
&= (x_1 - x_3)^{-1} \delta \left( \frac{x_0 + x_4}{x_1 - x_3} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) \\
&= (x_0 + x_4)^{-1} \delta \left( \frac{x_1 - x_3}{x_0 + x_4} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right); \\
&= x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) \\
&= x_0^{-1} \delta \left( \frac{-x_3 - x_4 + x_1}{x_0} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) \\
&= (x_0 + x_4)^{-1} \delta \left( \frac{-x_3 + x_1}{x_0 + x_4} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right),
\end{align*}
\]

by the $J_1$-defining relation (3.12), we have

\[
(3.18) - (3.21)
\]

\[
= \text{Res}_{x_4} x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) x_3^{-1} \delta \left( \frac{x_1 - x_0 - x_4}{x_3} \right) \\
\quad \cdot \pi_2(Y_t(Y(a, x_0 + x_4)Y(b, x_4)u, x_3) \otimes v) \\
= \text{Res}_{x_4} \text{Res}_{x_5} x_5^{-1} \delta \left( \frac{x_0 + x_4}{x_5} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) x_3^{-1} \delta \left( \frac{x_1 - x_5}{x_3} \right) \\
\quad \cdot \pi_2(Y_t(Y(a, x_5)Y(b, x_4)u, x_3) \otimes v) \\
= \text{Res}_{x_4} \text{Res}_{x_5} x_0^{-1} \delta \left( \frac{x_5 - x_4}{x_0} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) x_3^{-1} \delta \left( \frac{x_1 - x_5}{x_3} \right) \\
\quad \cdot \pi_2(Y_t(Y(a, x_5)Y(b, x_4)u, x_3) \otimes v).
\]

Similarly, we obtain

\[
(3.17) - (3.21)
\]

\[
= -\text{Res}_{x_4} x_3^{-1} \delta \left( \frac{x_1 - x_4}{x_3} \right) x_3^{-1} \delta \left( \frac{x_1 + x_0 - x_4}{x_3} \right) \\
\quad \cdot \pi_2(Y_t(Y(a, -x_0 + x_4)Y(b, x_4)u, x_3) \otimes v) \\
= -\text{Res}_{x_4} \text{Res}_{x_5} x_5^{-1} \delta \left( \frac{-x_0 + x_4}{x_5} \right) x_3^{-1} \delta \left( \frac{x_1 - x_4}{x_3} \right) x_3^{-1} \delta \left( \frac{x_2 - x_5}{x_3} \right) \\
\quad \cdot \pi_2(Y_t(Y(b, x_4)Y(a, x_5)u, x_3) \otimes v) \\
= -\text{Res}_{x_4} \text{Res}_{x_5} x_0^{-1} \delta \left( \frac{-x_5 + x_4}{x_0} \right) x_3^{-1} \delta \left( \frac{x_1 - x_4}{x_3} \right) x_3^{-1} \delta \left( \frac{x_2 - x_5}{x_3} \right).
\]
\[ \cdot \pi_2(Y_t(Y(b, x_4)Y(a, x_5)u, x_3) \otimes v) \]
\[ = -\text{Res}_{x_4} \text{Res}_{x_5} x_0^{-1} \delta \left( \frac{-x_4 + x_5}{x_0} \right) x_3^{-1} \delta \left( \frac{x_1 - x_5}{x_3} \right) x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) \]
\[ \cdot \pi_2(Y_t(Y(b, x_5)Y(a, x_4)u, x_3) \otimes v). \]

Therefore, we have

\[ (3.17) + (3.18) - (3.20) - (3.21) \]
\[ = \text{Res}_{x_4} \text{Res}_{x_5} x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \pi_2(Y_t(Y(a, x_0)b, x_4)u, x_3) \otimes v) \]
\[ = (3.23). \] (3.28)

Here we have used the following fact:

\[ \text{Res}_{x_4} x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \]
\[ = \text{Res}_{x_3} x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) x_3^{-1} \delta \left( \frac{x_1 - x_4 - x_0}{x_3} \right) x_5^{-1} \delta \left( \frac{x_4 + x_0}{x_5} \right) \]
\[ = x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) x_1^{-1} \delta \left( \frac{x_2 - x_4 + x_4 + x_0}{x_1} \right) \]
\[ = x_3^{-1} \delta \left( \frac{x_2 - x_4}{x_3} \right) x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right). \] (3.29)

Then the Jacobi identity is proved. \( \square \)

Since \( F_2(W_1, W_2) \) is a weak \( V \)-module, we will freely use \( Y(a, x) \) for \( Y_t(a, x) \). Recall from Section 2 that any weak \( V \)-module \( M \) is a restricted \( g(V) \)-module and that for any restricted \( g(V) \)-module \( M \), \( \tilde{M} := M/J(M) \) is a weak \( V \)-module, where \( J(M) \) is the intersection of all \( \ker f \) with \( f \) running through all \( g(V) \)-homomorphisms from \( M \) to weak \( V \)-modules. Then Theorem 3.10 says that \( J_1 = J(F_1(W_1, W_2)) \).

**Remark 3.11** If \( W_i \) for \( i = 1, 2, 3 \) are just restricted \( g(V) \)-modules, we can also define intertwining operator by using the same axioms as in Definition 2.2. Then following the
proof given in [FHL] one can easily see that the transpose operator $I^t(\cdot, x)$ is well defined and it is an intertwining operator.

**Proposition 3.12** Let $W_1$ and $W_3$ be weak $V$-modules, let $M$ be a restricted $g(V)$-module and let $I(\cdot, x)$ be an intertwining operator of type $\begin{pmatrix} W_3 \\ W_1M \end{pmatrix}$. Then $I(\cdot, x)J(M) = 0$, so that we obtain an intertwining operator of type $\begin{pmatrix} W_3 \\ W_1\bar{M} \end{pmatrix}$.

**Proof.** In the proof of Theorem 3.10, replace $W_2$, $F_2(W_1, W_2)$ and $Y(\cdot, x)$ by $M$, $W_3$ and $I(\cdot, x)$, respectively. Then the $J_1$-defining relation (3.12) or Jacobi identity for $I(\cdot, x)$ and the Jacobi identity for $W_3$:

$$\begin{align*}
(3.17) + (3.18) - (3.20) - (3.21) &= (3.23)
\end{align*}$$

are given. Following the proof of Theorem 3.10, we obtain $(3.16) - (3.19) = (3.22)$. That is, $I(\cdot, x)J(M) = 0$. Then the proof is complete. □

Symmetrically, we have

**Proposition 3.13** Let $W_2$ and $W_3$ be weak $V$-modules, let $M$ be a restricted $g(V)$-module and let $I(\cdot, x)$ be an intertwining operator of type $\begin{pmatrix} W_3 \\ MW_2 \end{pmatrix}$. Then $I(J(M), x) = 0$, so that we obtain an intertwining operator of type $\begin{pmatrix} W_3 \\ \bar{M}W_2 \end{pmatrix}$.

**Proof.** The proof of this proposition does not directly follow from the proof of Theorem 3.10, but it follows from Proposition 3.12 and the notion of transpose intertwining operator. Since $I^t(\cdot, x)$ is an intertwining operator of type $\begin{pmatrix} W_3 \\ W_2M \end{pmatrix}$, by Proposition 3.12 we get $I^t(\cdot, x)J(M) = 0$. Thus $I(J(M), x) = 0$. □

To construct a tensor product out of the weak $V$-module $F_2(W_1, W_2)$, we shall study the universal property. For simplicity, from now on we assume that $W_1$ and $W_2$ are weak $V$-modules in the category $C_0$ such that $W_i = \oplus_{n \in \mathbb{N}} (W_i)_{(n+h_i)}$ for $i = 1, 2$. 25
Let $W$ be a weak $V$-module in the category $C_0$ such that $W = \oplus_{n=0}^{\infty} W(n+h)$ for some $h$. Let $I(\cdot, x)$ be an intertwining operator of type $\left( \begin{array}{c} W \\ W_1 W_2 \end{array} \right)$. Let $I^o(\cdot, x) = x^{h_1+h_2-h} I(\cdot, x)$ be the normalization. Then we define

$$
\psi_I : F_0(W_1, W_2) \to W, \ t^n \otimes u \otimes v \mapsto I_u(n)v
$$

for $u \in W_1, v \in W_2, n \in \mathbb{Z}$. In terms of generating elements, $\psi_I$ can be written as:

$$
\psi_I(Y_t(u, x) \otimes v) = I^o(u, x)v \quad \text{for } u \in W_1, v \in W_2.
$$

Lemma 3.14 The linear map $\psi_I$ is a $g(V)$-homomorphism. In other words,

$$
\psi_I(Y_t(a, x)w) = Y(a, x)\psi_I(w) \quad \text{for } a \in V, w \in F_0(W_1, W_2).
$$

Proof. For $a \in V, u \in W_1, v \in W_2$, we have

$$
\begin{align*}
\psi_I(Y_t(a, x_1)(Y_t(u, x_2) \otimes v)) &= \psi_I(Y_t(u, x_2) \otimes Y(a, x_1)v) + \text{Res}_{x_0} x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) \psi_I(Y_t(Y_t(a, x_0)u, x_2) \otimes v) \\
&= I^o(u, x_2)Y(a, x_1)v + \text{Res}_{x_0} x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) I^o(Y(a, x_0)u, x_2)v \\
&= Y(a, x_1)I^o(u, x_2)v \\
&= Y(a, x_1)\psi_I(Y_t(u, x_2) \otimes v). \quad \square
\end{align*}
$$

Corollary 3.15 The linear map $\psi_I$ induces a $V$-homomorphism $\tilde{\psi}_I$ from $F_2(W_1, W_2)$ to $W$ such that $\tilde{\psi}_I$ preserves the $\mathbb{N}$-gradation and $\tilde{\psi}_I = \pi_2 \psi_I$, where $\pi_2$ is the quotient map from $F_0(W_1, W_2)$ to $F_2(W_1, W_2)$.

Proof. From Proposition 2.5 and the Jacobi identity for a $V$-module and for an intertwining operator we get: $J_0 + J_1 \subseteq \ker \psi_I$. Then we have an induced linear map $\tilde{\psi}_I$ from $F_2(W_1, W_2)$ to $W$. From (3.32) $\tilde{\psi}_I$ is a $V$-homomorphism. \quad \square
Let $W = \oplus_{n \in \mathbb{N}} W_{n+h}$ be given as before. Let $\text{Hom}^0_V(F_2(W_1, W_2), W)$ be the space of all $V$-homomorphisms from $F_2(W_1, W_2)$ to $W$ which preserve the given $\mathbb{N}$-gradation. Then we define the following linear map:

$$\bar{\psi}: I \left( \begin{array}{c} W \\ W_1W_2 \end{array} \right) \rightarrow \text{Hom}^0_V(F_2(W_1, W_2), W)$$

$$I(\cdot, x) \mapsto \bar{\psi}_I,$$  \hspace{1cm} (3.33)

**Proposition 3.16** The map $\bar{\psi}: I \left( \begin{array}{c} W \\ W_1W_2 \end{array} \right) \rightarrow \text{Hom}^0_V(F_2(W_1, W_2), W); I \mapsto \bar{\psi}_I$ is a linear isomorphism.

**Proof.** It is clear that $\bar{\psi}$ is injective. Let $f$ be a $V$-homomorphism from $F_2(W_1, W_2)$ to $W$ that preserves the $\mathbb{Z}$-grading. Define a linear map $I(\cdot, x)$ from $W_1$ to $\text{Hom}(W_2, W)\{x\}$ as follows:

$$I(u_1, x)u_2 = x^{h-h_1-h_2}f\pi_2(Y_t(u_1, x) \otimes u_2)$$ \hspace{1cm} (3.34)

for any $u_i \in W_i$. It follows from the defining relations $J_0$ and $J_1$ that $I(\cdot, x)$ satisfies the axioms (I1) and (I3). If we prove (I2), then $I(\cdot, x)$ is an intertwining operator satisfying $\bar{\psi}_I = f$. For $k \in \mathbb{Z}, m, n \in \mathbb{N}; u \in W_1(m), v \in W_2$, we have

$$\text{deg } (t^k \otimes u \otimes v) = m + n - k - 1.$$  

Therefore

$$L(0)f\pi(t^k \otimes u \otimes v) = (h + m + n - k - 1)f\pi(t^k \otimes u \otimes v).$$ \hspace{1cm} (3.35)

By formula (3.3), we obtain

$$L(0)(t^k \otimes u \otimes v)$$

$$= t^k \otimes u \otimes L(0)v + t^{k+1} \otimes u \otimes v + t^k \otimes L(0)u \otimes v$$

$$= t^{k+1} \otimes L(-1)u \otimes v + (h_1 + h_2 + m + n)t^k \otimes u \otimes v.$$

\hspace{1cm} (3.36)
Therefore
\[
f_\pi (t^{k+1} \otimes L(-1)u \otimes v + (h_1 + h_2 - h + k + 1)t^k \otimes u \otimes v) = 0.
\] (3.37)

This is exactly the axiom (I2) in terms of components. Then the proof is complete. \qed

For any nonzero \(\mathbb{N}\)-gradable weak \(V\)-module with a fixed a gradation \(M = \oplus_{n \in \mathbb{N}} M(n)\) such that \(M(0) \neq 0\), we define the radical of \(M\) to be the maximal graded submodule \(R(M)\) such that \(R(M) \cap M(0) = 0\).

**Definition 3.17** Define \(T(W_1, W_2)\) to be the quotient module of \(F_2(W_1, W_2)\) modulo the radical of \(F_2(W_1, W_2)\) with respect to the given gradation.

As a corollary of Proposition 3.16 we get

**Corollary 3.18** The linear isomorphism \(\bar{\psi}: I \left(\begin{array}{c} W \\ W_1W_2 \end{array}\right) \rightarrow \text{Hom}^0_V(F_2(W_1, W_2), W); I \mapsto \bar{\psi}_I\) gives rise to a linear isomorphism from \(I \left(\begin{array}{c} W \\ W_1W_2 \end{array}\right)\) to \(\text{Hom}^0_V(T(W_1, W_2), W)\), the space of all \(V\)-homomorphisms which preserve the given gradation.

From now on we shall assume that \(V\) is rational. Then up to equivalence, \(V\) has only finitely many irreducible modules. Let \(\lambda_1, \ldots, \lambda_k\) be all the distinct lowest weights of irreducible \(V\)-modules. For any \(\mathbb{N}\)-gradable weak \(V\)-module \(W\), let \(W^{(i)}\) be the sum of all irreducible submodules of \(W\) with lowest weight \(\lambda_i\). Then we obtain a canonical decomposition \(W = \oplus_{i=1}^k W^{(i)}\). Since \(W^{(i)}\) is a direct sum of irreducible \(V\)-modules with lowest weight \(\lambda_i\), any submodule of \(W^{(i)}\) is a direct sum of irreducible modules with lowest weight \(\lambda_i\). Thus \(W^{(i)} = \oplus_{n \in \mathbb{N}} W^{(i)}_{(n+\lambda_i)} := \oplus_{n \in \mathbb{N}} W^{(i)}(n)\). Then \(W = \oplus_{n \in \mathbb{N}} \oplus_{i=1}^k W^{(i)}_{(n+\lambda_i)}\).

Then \(W\) has a canonical \(\mathbb{N}\)-gradation with \(W(n) = \oplus_{i=1}^k W^{(i)}_{(n+\lambda_i)}\) for \(n \in \mathbb{N}\). It is clear that the radical of \(W\) is zero with respect to this gradation. In particular,

\[
T(W_1, W_2) = T(W_1, W_2)^{(1)} \oplus \cdots \oplus T(W_1, W_2)^{(k)}.
\]
Since $F_2(W_1, W_2)$ is completely reducible, $T(W_1, W_2)$ is isomorphic to the submodule generated by the degree-zero subspace $F_2(W_1, W_2)(0)$. Let $P_i$ be the projection map of $T(W_1, W_2)$ onto $T(W_1, W_2)^{(i)}$ and let $\pi$ be the natural quotient map from $F_0(W_1, W_2)$ onto $T(W_1, W_2)$. Then we define

$$F(\cdot, x) : W_1 \to (\text{Hom}_C(W_2, T(W_1, W_2)))\{x\};$$

$$u_1 \mapsto F(u_1, x) \quad \text{for } u_1 \in W_1$$

where $F(u_1, x)u_2 = \sum_{i=1}^{k} x^{\lambda_i-h_1-h_2} P_i \pi(Y_i(u_1, x) \otimes u_2)$ for $u_1 \in W_1, u_2 \in W_2$.

**Proposition 3.19** Suppose that $V$ is rational. Then the defined map $F(\cdot, x)$ is an intertwining operator of type $\left( \frac{T(W_1, W_2)}{W_1 W_2} \right)$.

**Proof.** Let $F_i(u_1, x)u_2 = x^{\lambda_i-h_1-h_2} P_i \pi(Y_i(u_1, x) \otimes u_2)$. Then it follows from Proposition 3.16 that each $F_i(\cdot, x)$ is an intertwining operator of type $\left( \frac{T(W_1, W_2)^{(i)}}{W_1 W_2} \right)$. Then it follows immediately. $\Box$

**Theorem 3.20** If $V$ is rational and $W_i \; (i=1,2,3)$ are irreducible weak $V$-modules in the category $C_0$, then the pair $(T(W_1, W_2), F(\cdot, x))$ is a tensor product in the category $C_0$ for the ordered pair $(W_1, W_2)$.

**Proof.** Let $W$ be a $V$-module and let $I(\cdot, x)$ be any intertwining operator of type $\left( \frac{W}{W_1 W_2} \right)$. Let $D_i$ be the projection of $W$ onto $W^{(i)}$ for $i = 1, \cdots, k$. Then $D_i I(\cdot, x)$ is an intertwining operator of type $\left( \frac{W^{(i)}}{W_1 W_2} \right)$. By Corollary 3.16, we obtain a $V$-homomorphism $g_i$ from $T(W_1, W_2)$ to $W^{(i)}$ satisfying the condition:

$$g_i \pi(Y_i(u_1, x) \otimes u_2) = D_i I^o(u_1, x)u_2 \quad \text{for } u_1 \in W_1, u_2 \in W_2.$$ (3.39)

Since $g_i P_j = 0$ for $j \neq i$, we obtain $g_i \circ F(u_1, x)u_2 = D_i I(u_1, x)u_2$ for $u_1 \in W_1, u_2 \in W_2$. Set $g = g_1 \oplus \cdots \oplus g_k$. Then

$$g \circ F(u_1, x)u_2 = I(u_1, x)u_2 \quad \text{for } u_1 \in W_1, u_2 \in W_2.$$
From the construction of $T(W_1, W_2)$, $F(\cdot, x)$ is surjective in the sense of Lemma 3.3, i.e., all the coefficients of $F(u_1, x)u_2$ for $u_i \in W_i$ linearly span $T(W_1, W_2)$. Thus such a $g$ is unique. Then the pair $(T(W_1, W_2), F(\cdot, x))$ is a tensor product for the ordered pair $(W_1, W_2)$. □

4 An analogue of the “Hom”-functor and a generalized nuclear democracy theorem

In this section we shall introduce the notion of what we call “generalized intertwining operator” from a $V$-module $W_1$ to another $V$-module $W_2$. The notion of generalized intertwining operator can be considered as a generalization of the physicists’ notion of “primary field” (cf. [BPZ], [MS] and [TK]) to the notion of general (non-primary) field. On the other hand, it exactly reflects the main features of $I(u, x)$ for $u \in M$, where $M$ is a $V$-module and $I(\cdot, x)$ is an intertwining operator of type $\left( \frac{W_2}{MW_1} \right)$. We prove that $G(W_1, W_2)$, the space of all generalized intertwining operators, is a weak $V$-module (Theorem 4.3), which satisfies a certain universal property in terms of the space of intertwining operators of a certain type (Theorem 4.7). If the vertex operator algebra $V$ satisfies certain finiteness and semisimplicity conditions, we prove that there exists a unique maximal submodule $\Delta(W_1, W_2)$ of $G(W_1, W_2)$ so that the contragredient module of $\Delta(W_1, W_2)$ is a tensor product module for the ordered pair $(W_1, W'_2)$ (Theorem 4.10). Using Theorem 4.7 we derive a generalized form of the nuclear democracy theorem of Tsuchiya and Kanie [TK] (Theorem 4.12). All these results show that the notion of $G(W_1, W_2)$ is an analogue of the classical “Hom”-functor.

Throughout this section, $V$ will be a fixed vertex operator algebra.

Definition 4.1 Let $W_1$ and $W_2$ be $V$-modules. A generalized intertwining operator from $W_1$ to $W_2$ is an element $\Phi(x) = \sum_{\alpha \in \mathbb{C}} \Phi_\alpha x^{-\alpha - 1} \in (\text{Hom}(W_1, W_2)) \{x\}$ satisfying the following conditions (G1)-(G3):
For any $\alpha \in \mathbb{C}, u_1 \in W_1, \Phi_{\alpha + n}u_1 = 0$ for $n \in \mathbb{Z}$ sufficiently large;

$[L(-1), \Phi(x)] = \Phi'(x) \left( = \frac{d}{dx} \Phi(x) \right)$;

For any $a \in V$, there exists a positive integer $k$ such that

$$(x_1 - x_2)^k Y_{W_2}(a, x_1) \Phi(x_2) = (x_1 - x_2)^k \Phi(x_2) Y_{W_1}(a, x_1).$$  \hspace{1cm} (4.1)$$

Denote by $G(W_1, W_2)$ the space of all generalized intertwining operators from $W_1$ to $W_2$. A generalized intertwining operator $\Phi(x)$ is said to be homogeneous of weight $h$ if it satisfies the following condition:

$$[L(0), \Phi(x)] = \left( h + x \frac{d}{dx} \right) \Phi(x).$$ \hspace{1cm} (4.2)$$

A generalized intertwining operator $\Phi(x)$ of weight $h$ is said to be primary if the following condition holds:

$$[L(m), \Phi(x)] = x^m \left( (m + 1)h + x \frac{d}{dx} \right) \Phi(x) \quad \text{for } m \in \mathbb{Z}. \hspace{1cm} (4.3)$$

Denote by $G(W_1, W_2)_{(h)}$ the space of all weight-$h$ homogeneous generalized intertwining operators from $W_1$ to $W_2$ and set

$$G^0(W_1, W_2) = \bigoplus_{h \in \mathbb{C}} G(W_1, W_2)_{(h)}. \hspace{1cm} (4.4)$$

Let $W(W_1, W_2)$ be the space consisting of each element $\Phi(x) \in \text{Hom}_\mathbb{C}(W_1, W_2)$ which satisfies the condition (G1) and let $E(W_1, W_2)$ be the space consisting of each element $\Phi(x) \in \text{Hom}_\mathbb{C}(W_1, W_2)$ which satisfies the conditions (G1) and (G3). For any $a \in V$, we define the left and the right actions of $\hat{V}$ on $W(W_1, W_2)$ as follows:

$$Y_t(a, x_0) \ast \Phi(x_2) : = \text{Res}_{x_0} x_1^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_{W_2}(a, x_1) \Phi(x_2)$$

$$= Y_{W_2}(a, x_0 + x_2) \Phi(x_2). \hspace{1cm} (4.5)$$

$$\Phi(x_2) \ast Y_t(a, x_0) : = \text{Res}_{x_0} x_1^{-1} \delta \left( -\frac{x_2 + x_1}{x_0} \right) \Phi(x_2) Y_{W_1}(a, x_1)$$

$$= \Phi(x_2) (Y_{W_1}(a, x_0 + x_2) - Y_{W_1}(a, x_2 + x_0)). \hspace{1cm} (4.7)$$
Proposition 4.2  a) $W(W_1, W_2)$ is a left $g(V)$-module of level one under the defined left action.

b) $W(W_1, W_2)$ is a right $g(V)$-module of level zero under the defined right action.

Proof. a) First we check that $W(W_1, W_2)$ is closed under the left action. For any $a \in V, m \in \mathbb{Z}, \Phi(x) \in W(W_1, W_2), u \in W_1$, by definition we have:

$$((t^m \otimes a) \ast \Phi(x))u = \text{Res}_{x_0} x_0^m Y_{W_2}(a, x_0 + x)\Phi(x)u = \sum_{i=0}^{\infty} \left(-m + i - 1\right) x^i a_{m-i} \Phi(x) u. \quad (4.9)$$

Then it is clear that $(t^m \otimes a) \ast \Phi(x)$ satisfies (G1). Next, we check the defining relations for $g(V)$. By definition we have

$$Y_t(1, x_0) \ast \Phi(x_2) = Y_{W_2}(1, x_0 + x_2)\Phi(x_2) = \Phi(x_2) \quad (4.10)$$

and

$$Y_t(L(-1)a, x_0) \ast \Phi(x_2) = Y_{W_2}(L(-1)a, x_0 + x_2)\Phi(x_2) = \frac{\partial}{\partial x_0} Y_{W_2}(a, x_0 + x_2)\Phi(x_2) = \frac{\partial}{\partial x_0} Y_t(a, x_0) \ast \Phi(x_2). \quad (4.11)$$

Furthermore, for any $a, b \in V$, we have

$$Y_t(a, x_1) \ast Y_t(b, x_2) \ast \Phi(x_3) = Y_t(a, x_1) \ast (Y_{W_2}(b, x_2 + x_3)\Phi(x_3)) = Y_{W_2}(a, x_1 + x_3)Y_{W_2}(b, x_2 + x_3)\Phi(x_3). \quad (4.12)$$

Similarly, we have

$$Y_t(b, x_2) \ast Y_t(a, x_1) \ast \Phi(x_3) = Y_{W_2}(b, x_2 + x_3)Y_{W_2}(a, x_1 + x_3)\Phi(x_3). \quad (4.13)$$
Therefore

\[ Y_t(a, x_1) \ast Y_t(b, x_2) \ast \Phi(x_3) - Y_t(b, x_2) \ast Y_t(a, x_1) \ast \Phi(x_3) \]

\[ = \text{Res}_{x_0}(x_2 + x_3)^{-1} \delta \left( \frac{x_1 + x_3 - x_0}{x_2 + x_3} \right) Y_{W_2}(Y(a, x_0)b, x_2 + x_3) \Phi(x_3) \]

\[ = \text{Res}_{x_0} x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) Y_{W_2}(Y(a, x_0)b, x_2 + x_3) \Phi(x_3) \]

\[ = \text{Res}_{x_0} x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y_t(Y(a, x_0)b, x_2) \ast \Phi(x_3). \quad (4.14) \]

Then a) is proved.

The proof of b) is similar to the proof of a), but for completeness, we also write the details. For any \( a \in V, \Phi(x) \in W(W_1, W_2) \), by definition we have

\[ \Phi(x_2) \ast Y_t(L(-1)a, x_0) \]

\[ = \Phi(x_2)(Y_{W_1}(L(-1)a, x_0 + x_2) - Y_{W_1}(L(-1)a, x_2 + x_0)) \]

\[ = \frac{\partial}{\partial x_0} (\Phi(x_2)(Y_{W_1}(a, x_0 + x_2) - Y_{W_1}(a, x_2 + x_0)) \]

\[ = \frac{\partial}{\partial x_0} \Phi(x_2) \ast Y_t(a, x_0). \quad (4.15) \]

For any \( a, b \in V \), we have

\[ \Phi(x_3) \ast Y_t(a, x_1) \ast Y_t(b, x_2) \]

\[ = \text{Res}_{x_4} x_1^{-1} \delta \left( \frac{-x_3 + x_4}{x_1} \right) (\Phi(x_3)Y_{W_1}(a, x_4)) \ast Y_t(b, x_2) \]

\[ = \text{Res}_{x_4} \text{Res}_{x_5} x_1^{-1} \delta \left( \frac{-x_3 + x_4}{x_1} \right) x_2^{-1} \delta \left( \frac{-x_3 + x_5}{x_2} \right) \Phi(x_3)Y_{W_1}(a, x_4)Y_{W_1}(b, x_5). \quad (4.16) \]

Similarly, we have

\[ \Phi(x_3) \ast Y_t(b, x_2) \ast Y_t(a, x_1) \]

\[ = \text{Res}_{x_4} \text{Res}_{x_5} x_1^{-1} \delta \left( \frac{-x_3 + x_4}{x_1} \right) x_2^{-1} \delta \left( \frac{-x_3 + x_5}{x_2} \right) \Phi(x_3)Y_{W_1}(b, x_5)Y_{W_1}(a, x_4). \quad (4.17) \]
Thus

$$
\Phi(x_3) * Y_t(a, x_1) * Y_t(b, x_2) - \Phi(x_3) * Y_t(b, x_2) * Y_t(a, x_1)
= \text{Res}_{x_4} \text{Res}_{x_5} \text{Res}_{x_0} x_1^{-1} \delta \left( \frac{-x_3 + x_4}{x_1} \right) x_2^{-1} \delta \left( \frac{-x_3 + x_5}{x_2} \right) x_5^{-1} \delta \left( \frac{x_4 - x_0}{x_5} \right)
\cdot \Phi(x_3) Y_{W_1} (Y(a, x_0)b, x_5)
= \text{Res}_{x_5} \text{Res}_{x_0} x_1^{-1} \delta \left( \frac{-x_3 + x_5 + x_0}{x_1} \right) x_2^{-1} \delta \left( \frac{-x_3 + x_5}{x_2} \right) \Phi(x_3) Y_{W_1} (Y(a, x_0)b, x_5)
= \text{Res}_{x_5} \text{Res}_{x_0} x_1^{-1} \delta \left( \frac{x_2 + x_0}{x_1} \right) x_2^{-1} \delta \left( \frac{-x_3 + x_5}{x_2} \right) \Phi(x_3) Y_{W_1} (Y(a, x_0)b, x_5)
= \Phi(x_3) * \text{Res}_{x_0} x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y_t(Y(a, x_0)b, x_2).
$$

Then the proof is complete.  \(\Box\)

For any \(a \in V, \Phi(x) \in W(W_1, W_2)\), we define

$$
Y_t(a, x_0) \circ \Phi(x_2) :=
= Y_t(a, x_0) * \Phi(x_2) - \Phi(x_2) * Y_t(a, x_0)
= \text{Res}_{x_1} \left( x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_{W_2}(a, x_1) \Phi(x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \Phi(x_2) Y_{W_1}(a, x_1) \right),
$$

or equivalently

$$
a(m) \circ \Phi(x_2) = \text{Res}_{x_1} ((x_1 - x_2)^m Y_{W_2}(a, x_1) \Phi(x_2) - (-x_2 + x_1)^m \Phi(x_2) Y_{W_1}(a, x_1))
$$

for any \(m \in \mathbb{Z}\). From the classical Lie algebra theory, we have

**Corollary 4.3** Under the defined action "\(\circ\)" , \(W(W_1, W_2)\) becomes a \(g(V)\)-module (of level one).

**Lemma 4.4** Let \(\Phi(x) \in W(W_1, W_2)\) satisfying \((4.2)\) for some complex number \(h\) and let \(a\) be any homogeneous element of \(V\). Then

$$
[L(0), Y_t(a, x_0) \circ \Phi(x_2)] = \left( wt a + h + x_0 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_2} \right) Y_t(a, x_0) \circ \Phi(x_2).
$$

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Proof. By definition we have

\[ [L(0), Y_t(a, x_0) \circ \Phi(x_2)] \]

\[ = \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) [L(0), Y(a, x_1) \Phi(x_2)] \]

\[ - \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{-x_2 + x_1}{x_0} \right) [L(0), \Phi(x_2) Y(a, x_1)] \]

\[ = \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \left( wt + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) Y(a, x_1) \Phi(x_2) \]

\[ - \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) \left( wt + x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2} \right) \Phi(x_2) Y(a, x_1) \]

\[ = (wt + h) Y_t(a, x_0) \circ \Phi(x_2) \]

\[ - \text{Res}_{x_1} \left( \frac{\partial}{\partial x_1} x_1 x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \right) Y(a, x_1) \Phi(x_2) \]

\[ + \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) x_2 \frac{\partial}{\partial x_2} Y(a, x_1) \Phi(x_2) \]

\[ - \text{Res}_{x_1} x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) x_2 \frac{\partial}{\partial x_2} \Phi(x_2) Y(a, x_1) \]

\[ + \text{Res}_{x_1} \left( \frac{\partial}{\partial x_1} x_1 x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) \right) \Phi(x_2) Y(a, x_1). \]

(4.22)

Since

\[
\frac{\partial}{\partial x_0} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) = \frac{\partial}{\partial x_2} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) = - \frac{\partial}{\partial x_1} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right),
\]

(4.23)

we have

\[
\frac{\partial}{\partial x_1} \left( x_1 x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \right) = \frac{\partial}{\partial x_1} \left( (x_0 + x_2) x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) \right)
\]

\[ = x_0 \frac{\partial}{\partial x_1} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) + x_2 \frac{\partial}{\partial x_1} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right)
\]

\[ = -x_0 \frac{\partial}{\partial x_0} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) - x_2 \frac{\partial}{\partial x_2} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right). \]

(4.24)

Similarly, we have

\[
\frac{\partial}{\partial x_1} \left( x_1 x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) \right) = - \left( x_0 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_2} \right) x_1 x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right).
\]

(4.25)
Therefore, we obtain

\[
\begin{align*}
&[L(0), Y_t(a, x_0) \circ \Phi(x_2)] \\
=& \left( wta + h + x_0 \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_2} \right) Y_t(a, x_0) \circ \Phi(x_2). \quad \square
\end{align*}
\] 

(4.26)

**Proposition 4.5** The subspaces \(E(W_1, W_2)\) and \(G(W_1, W_2)\) are restricted \(g(V)\)-submodules of \(W(W_1, W_2)\) and \(G^0(W_1, W_2)\) is a \(C\)-graded \(g(V)\)-module.

**Proof.** For any \(a \in V, m \in \mathbb{Z}, \Phi(x) \in E(W_1, W_2)\), it follows from the proof of Proposition 3.2.7 in [L1] (for an analogous result) that \(a(m) \circ \Phi(x) \in E(W_1, W_2)\). Thus \(E(W_1, W_2)\) is a submodule of \(W(W_1, W_2)\). For \(\Phi(x) \in G(W_1, W_2)\), since

\[
\begin{align*}
&[L(-1), Y_t(a, x_0) \ast \Phi(x_2)] \\
=& \ [L(-1), Y_{W_2}(a, x_0 + x_2) \Phi(x_2)] \\
=& \ [L(-1), Y_{W_2}(a, x_0 + x_2)] \Phi(x_2) + Y_{W_2}(a, x_0 + x_2)[L(-1), \Phi(x_2)] \\
=& \left. \frac{\partial}{\partial x_2} \right| (Y_{W_2}(a, x_0 + x_2) \Phi(x_2)) \\
=& \left. \frac{\partial}{\partial x_2} \right| Y_t(a, x_0) \ast \Phi(x_2),
\end{align*}
\]

(4.27)

\(a(m) \ast \Phi(x_2)\) satisfies (G2). Thus \(a(m) \ast \Phi(x_2) \in G(W_1, W_2)\). Similarly, since

\[
\begin{align*}
&\left. \frac{\partial}{\partial x_2} \right| (\Phi(x_2) \ast Y_t(a, x_0)) \\
=& \left. \frac{\partial}{\partial x_2} \right| (\Phi(x_2)(Y_{W_1}(a, x_0 + x_2) - Y_{W_1}(a, x_2 + x_0))) \\
=& \Phi'(x_2)(Y_{W_1}(a, x_0 + x_2) - Y_{W_1}(a, x_2 + x_0)) \\
&+ \Phi(x_2)(Y_{W_1}(L(-1)a, x_0 + x_2) - Y_{W_1}(L(-1)a, x_2 + x_0)) \\
=& \ [L(-1), \Phi(x_2) \ast Y_t(a, x_0)],
\end{align*}
\]

(4.28)

we obtain \(\Phi(x_2) \ast a(m) \in G(W_1, W_2)\). Therefore \(a(m) \circ \Phi(x) \in G(W_1, W_2)\). Thus \(G(W_1, W_2)\) is a submodule. That \(G^0(W_1, W_2)\) is a \(C\)-graded \(g(V)\)-module follows from
Lemma 4.4. It follows from (4.20) and (G3) that \( E(W_1, W_2) \) is a restricted \( g(V) \)-module and so are \( G(W_1, W_2) \) and \( G^0(W_1, W_2) \). Then the proof is complete. \( \square \)

Define a linear map \( F(\cdot, x) \) from \( E(W_1, W_2) \) to \( \text{Hom}(W_1, W_2) \{ x \} \) as follows:

\[
F(\Phi, x)u_1 = \Phi(x)u_1 \quad \text{for } \Phi \in E(W_1, W_2), u_1 \in W_1. \tag{4.29}
\]

For \( a \in V, \Phi \in E(W_1, W_2) \), we have

\[
F(Y(a, x_0)\Phi, x_2) = (Y(a, x_0)\Phi)(x)|_{x=x_2} = \text{Res}_{x_1} \left( x_0^{-1}\delta \left( \frac{x_1 - x}{x_0} \right) Y(a, x_1)\Phi - x_0^{-1}\delta \left( \frac{-x + x_1}{x_0} \right) \Phi(x)Y(a, x_1) \right)|_{x=x_2} = \text{Res}_{x_1} \left( x_0^{-1}\delta \left( \frac{x_1 - x_2}{x_0} \right) Y(a, x_1)F(\Phi, x_2) - x_0^{-1}\delta \left( \frac{x_2 - x_1}{x_0} \right) F(\Phi, x_2)Y(a, x_1) \right). \tag{4.30}
\]

It is well-known that this iterate formula implies the associativity [FHL]. Furthermore, (G3) gives the commutativity for \( F(\cdot, x) \). Therefore, \( F(\cdot, x) \) satisfies the Jacobi identity ([DL], [FHL], [L1]). Thus \( F(\cdot, x) \) is a weak intertwining operator. It is clear that \( F(\cdot, x) \) is injective in the sense that \( F(\Phi, x) = 0 \) implies \( \Phi = 0 \) for \( \Phi \in E(W_1, W_2) \). Furthermore, if \( \Phi(x) \in G(W_1, W_2) \), by definition we have

\[
F(L(-1)\Phi, x)u_1 = (L(-1)\Phi)(x)u_1 = (L(-1) \circ \Phi(x))u_1 = \frac{d}{dx}\Phi(x)u_1 = \frac{d}{dx}F(\Phi, x)u_1. \tag{4.31}
\]

Therefore, \( F(\cdot, x) \) is an intertwining operator of type \( \begin{pmatrix} W_2 \\ G(W_1, W_2)W_1 \end{pmatrix} \) after restricted to \( G(W_1, W_2) \).

**Theorem 4.6** The \( g(V) \)-module \( E(W_1, W_2) \) and \( G(W_1, W_2) \) are weak \( V \)-modules.

**Proof**[1]. By Proposition 3.12, we get

\[
F(\Phi, x) = 0 \quad \text{for any } \Phi \in J(E(W_1, W_2)).
\]

\[1\]This was proved directly in [L1].
Since $F(\cdot, x)$ injective, $J(E(W_1, W_2)) = 0$. That is, $E(W_1, W_2)$ is a weak $V$-module. □

Let $M$ be another $V$-module and let $f \in \text{Hom}_V(M, G(W_1, W_2))$. Then $F(f \cdot, x)$ is an intertwining operator of type $\left(\begin{array}{c} W_2 \\ MW_1 \end{array}\right)$. Since $F(\cdot, x)$ is injective, we obtain an injective linear map

$$\theta : \text{Hom}_V(M, G(W_1, W_2)) \to I\left(\begin{array}{c} W_2 \\ MW_1 \end{array}\right)$$

$$f \mapsto F(f \cdot, x). \quad (4.32)$$

On the other hand, for any intertwining operator $I(\cdot, x)$ of type $\left(\begin{array}{c} W_2 \\ MW_1 \end{array}\right)$, it is clear that $I(u, x) \in G(W_1, W_2)$ for any $u \in M$. Then we obtain a linear map $f_I$ from $M$ to $G(W_1, W_2)$ defined by $f_I(u) = I(u, x)$. For any $a \in V, u \in M$, from the definition of $Y(a, x_0) \circ I(u, x)$ we get

$$f_I(Y(a, x_0)u)$$

$$= I(Y(a, x_0)u, x)$$

$$= \text{Res}_{x_1} \left(x_0^{-1} \delta \left(\frac{x_1 - x}{x_0}\right) Y(a, x_1)I(u, x) - x_0^{-1} \delta \left(\frac{x - x_1}{-x_0}\right) I(u, x)Y(a, x_1)\right)$$

$$= Y(a, x_0) \circ I(u, x)$$

$$= Y(a, x_0)f_I(u). \quad (4.33)$$

Thus $f_I$ is a $V$-homomorphism such that $F(f_I \cdot, x) = I(\cdot, x)$. Since $F(\cdot, x)$ is injective, such an $f_I$ is unique. Therefore we obtain

**Theorem 4.7** Let $W_1$ and $W_2$ be $V$-modules. Then (a) For any $V$-module $M$ and any intertwining operator $I(\cdot, x)$ of type $\left(\begin{array}{c} W_2 \\ MW_1 \end{array}\right)$, there exists a unique $V$-homomorphism $f$ from $M$ to $G(W_1, W_2)$ such that $I(u, x) = F(f(u), x)$ for $u \in M$.

(b) The linear space $\text{Hom}_V(M, G(W_1, W_2))$ is naturally isomorphic to $I\left(\begin{array}{c} W_2 \\ MW_1 \end{array}\right)$ for any $V$-module $M$.

The universal property in Theorem 4.7 looks very much like the universal property for a tensor product in Definition 3.1 and also in [HL1]. Next, we study the relation between
G(W_1, W_2) and the contragredient module of tensor product of W_1 and W'_2.

**Remark 4.8** Let M be any V-module. Then it was proved in [L1] that G(V, M) ≃ M. If M = V, then V = G(V, V). That is, any generalized intertwining operator is a vertex operator. In this special case, this has been proved in [G].

For any two V-modules W_1 and W_2, let Δ(W_1, W_2) be the sum of all V-modules inside the weak V-module G(W_1, W_2).

**Proposition 4.9** Let V be a vertex operator algebra satisfying the following conditions:
(1) There are finitely many inequivalent irreducible V-modules. (2) Any V-module is completely reducible. (3) Any fusion rule for three modules is finite. Then for any V-modules W_1 and W_2, Δ(W_1, W_2) is the unique maximal V-module inside the weak module G(W_1, W_2).

**Proof.** It follows from the condition (2) that Δ(W_1, W_2) is a direct sum of irreducible V-modules. It follows from Theorem 1.7 and the condition (3) that the multiplicity of each irreducible V-module in Δ(W_1, W_2) is finite. Therefore Δ(W_1, W_2) is a direct sum of finitely many irreducible V-modules. That is, Δ(W_1, W_2) is a V-module. By the definition of Δ(W_1, W_2), it is clear that Δ(W_1, W_2) is the unique maximal V-module inside the weak V-module G(W_1, W_2). □

Let V be a vertex operator algebra satisfying the conditions (1)-(3) of Proposition 1.9 and let W_1 and W_2 be any two V-modules. Let F(·, x) be the restriction of F(·, x) on Δ(W_1, W_2) so that F(·, x) is an intertwining operator of type \( \left( \begin{array}{c} W_2 \\ \Delta(W_1, W_2)W_1 \end{array} \right) \) such that

\[
F(\Phi, x) = \Phi(x) \quad \text{for any } \Phi \in \Delta(W_1, W_2).
\]

Then by Proposition 2.6, the transpose operator \( F^t(·, x) \) of F(·, x) is an intertwining operator of type \( \left( \begin{array}{c} W_2 \\ W_1\Delta(W_1, W_2) \end{array} \right) \). Furthermore, it follows from Proposition 2.6 that

\[\text{[A similar result has also been obtained in [HL0-4].]}\]
$(\bar{F}^t)'(\cdot, x)$ is an intertwining operator of type $\left(\frac{(\Delta(W_1, W_2))'}{W_1 W_2'}\right)$.

**Theorem 4.10** If $V$ satisfies the conditions (1)-(3) of Proposition 4.9, then the pair $((\Delta(W_1, W_2)'), (\bar{F}^t)'(\cdot, x))$ is a tensor product for the ordered pair $(W_1, W_2')$ in the category of $V$-modules.

**Proof.** Let $W$ be any $V$-module and let $I(\cdot, x)$ be any intertwining operator of type $\left(\frac{W'}{W_1 W_2'}\right)$. It follows from Proposition 2.6 that $(I')'(\cdot, x)$ is an intertwining operator of type $\left(\frac{W'}{W W_1'}\right)$. From Theorem 4.6, there exists a (unique) $V$-homomorphism $\psi$ from $W'$ to $G(W_1, W_2)$ such that $(I')'(w', x) = \bar{F}(\psi(w'), x)$ for any $w' \in M'$. It follows from the definition of $\Delta(W_1, W_2)$ that $\psi$ is a $V$-homomorphism from $W'$ to $\Delta(W_1, W_2)$. Therefore, we obtain a $V$-homomorphism $\psi'$ from $(\Delta(W_1, W_2))'$ to $W$. For any $w' \in W'$, $u_1 \in W_1$, $u_2' \in W_2'$, by using FLM’s conjugation formulas [FHL] we obtain

$$
\langle w', \psi'(\bar{F}^t)'(u_1, x)u_2' \rangle \\
= \langle \bar{F}^t(e^{x L(1)}(e^{\pi i x}x^{-1})L(0))u_1, x^{-1})\psi w', u_2' \rangle \\
= \langle I'(e^{x L(1)}(e^{\pi i x}x^{-1})L(0))u_1, x^{-1})w', u_2' \rangle \\
= \langle w', I(e^{-x L(1)}(e^{\pi i x}x^{-1})L(0))e^{x L(1)}(e^{\pi i x}x^{-1})L(0))u_1, x)u_2' \rangle \\
= \langle w', I(e^{2\pi i L(0)}u_1, x)u_2' \rangle.
$$

(4.35)

For any $V$-module $M$, we define a linear endomorphism $t_M$ of $M$ by: $t_M(u) = e^{2\pi i L(0)}u$ for $u \in M$. Then one can easily prove that $t_M$ is a $V$-automorphism of $M$ so that $t_M$ is a scalar if $M$ is irreducible. Let $t_{W_1} = \alpha$. Then $\alpha^{-1}\psi'(\bar{F}^t)'(\cdot, x) = I(\cdot, x)$. The uniqueness of $\alpha^{-1}\psi'$ follows from the uniqueness of $\psi$. Then the proof is complete. \[\square\]

**Remark 4.11** It was proved in [DLM2] that the category $C$ of all weak $V$-modules is a semisimple category for vertex operator algebras $L(\ell, 0)$, associated to an integrable highest weight module of level $\ell$ for an affine Lie algebra, $L(c_{\rho, q}, 0)$, associated to the
irreducible highest weight module for the Virasoro algebra with central charge $c_{p,q} = 1 - \frac{6(p-q)^2}{pq}$, the moonshine module vertex operator algebra $V^2$ and $V_L$, associated to any even positive-definite lattice $L$. Thus, for these vertex operator algebras, we have $G(W_1, W_2) = \Delta(W_1, W_2)$.

Let $U$ be an irreducible $g(V)_{0}$-module on which $L(0)$ acts as a scalar $h$. Define $g(V)_{-}U = 0$. Then $U$ becomes a $(g(V)_{-} + g(V)_{0})$-module. Form the induced $g(V)$-module $\text{Ind}(U) = U(g(V)) \otimes_{U(g(V)_{-} + g(V)_{0})} U$. Set $V(U) = \text{Ind}(U)/J(\text{Ind}(U))$. Then $V(U)$ is a lowest weight weak $V$-module. If $V$ is rational, it follows from the complete reducibility of $V(U)$ that $V(U)$ is irreducible. The following is our generalized nuclear democracy theorem of Tsuchiya and Kanie [TK].

**Theorem 4.12** Let $W_1$ and $W_2$ be $V$-modules. Let $U$ be a $g(V)_{0}$-module on which $L(0)$ acts as a scalar $h$ and let $I_0(\cdot, x)$ be a linear injective map from $U$ to $(\text{Hom}_{C}(W_1, W_2)) \{x\}$ such that for any $u \in U, I_0(u, x)$ satisfies the truncation condition $(G1)$, the $L(-1)$-bracket formula $(G2)$ and the following condition:

\[
(x_1 - x_2)^{\text{wta}-1}Y_{W_2}(a, x_1)I_0(u, x_2) - (-x_2 + x_1)^{\text{wta}-1}I_0(u, x_2)Y_{W_1}(a, x_1) = x_1^{-1}\delta \left( \frac{x_2}{x_1} \right) I_0(a, x_2) (4.36)
\]

for any $a \in V, u \in U$. Then there exists a lowest weight weak $V$-module $W$ with $U$ as its lowest weight subspace generating $W$ and there is a unique intertwining operator $I(\cdot, x)$ of type \( \left( \begin{array}{c} W_2 \\ W \end{array} \right) \) extending $I_0(\cdot, x)$. In particular, if $V$ is rational and $U$ is irreducible, $W$ is irreducible.

**Proof.** Since $(x_1 - x_2)\delta \left( \frac{x_2}{x_1} \right) = 0$, we have

\[
(x_1 - x_2)^{\text{wta}+i}Y_{W_2}(a, x_1)I_0(u, x_2) = (-x_2 + x_1)^{\text{wta}+i}I_0(u, x_2)Y_{W_1}(a, x_1) (4.37)
\]
for $a \in V, u \in U, i \in \mathbb{N}$. Then by definition $I_0(u, x) \in G(W_1, W_2)$ for any $u \in U$ and

$$a_m \circ I_0(u, x) = 0 \quad \text{for} \quad m \geq \text{wt} a, \quad (4.38)$$

$$a_{\text{wt} a - 1} \circ I_0(u, x) = I_0(a_{\text{wt} a - 1} u, x). \quad (4.39)$$

Set $\bar{U} := \{I(u, x) | u \in U\} \subseteq G(W_1, W_2)$. Then $\bar{U}$ is a $g(V)_0$-submodule of $G(W_1, W_2)$ and $\bar{U}$ as a $g(V)_0$-module is isomorphic to $U$. Let $W = U(g(V))\bar{U}$ be the $V$ or $g(V)$-submodule of $G(W_1, W_2)$. Then $W = U(g_+)\bar{U}$ is a lower-truncated $\mathbb{Z}$-graded weak $V$-module generated by $U$. Then we have a natural intertwining operator of type $\begin{pmatrix} W_2 \\ W W_1 \end{pmatrix}$. The uniqueness is clear. Then the proof is complete. $\square.$

Let $g$ be a finite-dimensional simple Lie algebra, let $h$ be a Cartan subalgebra, let $\Delta$ be the root system of $g$ and let $\langle \cdot, \cdot \rangle$ be the normalized Killing form on $g$ [K]. For any linear functional $\lambda \in h^*$, we denote by $L(\lambda)$ the irreducible highest weight $g$-module with highest weight $\lambda$.

Let $\hat{g} = \mathbb{C}[t, t^{-1}] \otimes g \oplus \mathbb{C}c$ be the corresponding affine Lie algebra and let $\hat{g} = \hat{g} \oplus \mathbb{C}d$ be the extended affine algebra. For any $\ell \in \mathbb{C}, \lambda \in h^*$, let $L(\ell, \lambda)$ be the irreducible highest weight $\hat{g}$-module of level $\ell$. For any $g$-module $U$, let $\hat{U}$ be the loop $\hat{g}$-module $\mathbb{C}[t, t^{-1}] \otimes U$ of level $0$. It is well known (cf. [FZ], [L1]) that each $L(\ell, 0)$ has a vertex operator algebra structure except when $\ell$ is the negative dual Coxeter number. Then we have the following nuclear democracy theorem of Tsuchiya and Kanie. (To be precise, this was proved only for $g = sl_2$ in [TK].)

**Proposition 4.13** Let $\ell$ be a positive integer and let $W_2, W_3$ be $L(\ell, 0)$-modules. Let $\lambda$ be a linear functional on $h$, let $L(\lambda)$ be the irreducible highest weight $g$-module with highest weight $\lambda$ and let $\Phi(\cdot, x)$ be a nonzero linear map from $L(\lambda)$ to $\text{Hom}(W_2, W_3)\{x\}$ such that

$$[a(m), \Phi(u, x)] = x^m \Phi(a(0) u, x); \quad (4.40)$$

$$[L(-1), \Phi(u, x)] = \frac{d}{dx} \Phi(u, x) \quad (4.41)$$

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for any \( a \in g \subseteq L(\ell,0), u \in L(\lambda), m \in \mathbb{Z} \). Then \( L(\ell, \lambda) \) is an irreducible \( L(\ell,0) \)-module and there is a unique intertwining operator \( I(\cdot, x) \) on \( L(\ell, \lambda) \) in the sense of [FHL] extending \( \Phi(\cdot, x) \).

**Proof.** Writing (4.40) in terms of generating functions, we obtain

\[
[Y(a, x_1), \Phi(u, x_2)] = x_2^{-1} \delta \left( \frac{x_1}{x_2} \right) \Phi(a_0 u, x_2). \tag{4.42}
\]

Since \( (x_1 - x_2) \delta \left( \frac{x_1}{x_2} \right) = 0 \), we get

\[
(x_1 - x_2)[Y(a, x_1), \Phi(u, x_2)] = 0 \tag{4.43}
\]

for any \( a \in g, u \in L(\lambda) \). Since \( g \) generates \( L(\ell,0) \) as a vertex operator algebra, similar to the proof of Proposition 4.5 it follows from the proof of Proposition 3.2.7 in [L1] that \( I(u, x) \) satisfies (G3) for any \( a \in L(\ell,0) \). Furthermore, (4.40) implies (G2). Therefore \( \Phi(u, x) \in G(W_2, W_3) \) for \( u \in L(\lambda) \). From (4.20) and (4.43) we obtain

\[
a(0) \circ \Phi(u, x) = [a(0), \Phi(u, x)] = \Phi(a(0) u, x); \tag{4.44}
\]

\[
a(m) \circ \Phi(u, x) = 0 \quad \text{for any } a \in g, m > 0, u \in L(\lambda). \tag{4.45}
\]

Then \( \Phi \) is a \( g \)-homomorphism. Consequently, \( L(\lambda) \) is embedded into \( G(W_2, W_3) \) by \( \Phi \). Let \( W \) be the \( V \)-submodule generated by \( L(\lambda) \). Then \( W \) is a certain quotient module of \( M(\ell, \lambda) \). From the rationality of \( L(\ell,0) \), we get \( W = L(\ell, \lambda) \). By Theorem 4.7, we obtain an intertwining vertex operator \( I(\cdot, x) \) of type \( \begin{pmatrix} W_3 \\ L(\ell, \lambda) W_2 \end{pmatrix} \). The uniqueness is clear. Then the proof is complete. \( \square \)

**Remark 4.14** Under the conditions of Proposition 4.13, we obtain an intertwining operator \( I(\cdot, x) \) of type \( \begin{pmatrix} W_3 \\ L(\ell, \lambda) W_2 \end{pmatrix} \). It follows from commutator formula (2.4) that

\[
[L(m), I(u, x)] = x^m \left( (m + 1)h + x \frac{d}{dx} \right) I(u, x)
\]
for $u \in L(\lambda)$, where $h$ is the lowest weight of $L(\ell, \lambda)$. Thus

$$[L(m), \Phi(u, x)] = x^m \left((m+1)h + x \frac{d}{dx}\right) \Phi(u, x) \quad \text{for } u \in L(\lambda), m \in \mathbb{Z}. \quad (4.46)$$

In many references the notion of loop $\hat{g}$-module was used to define intertwining operators. Next we shall discuss this issue.

Suppose $L(\ell, \lambda_i) \; (i = 1, 2, 3)$ are $L(\ell, 0)$-modules. Let $I(\cdot, x)$ be an intertwining operator of type $\left( \frac{L(\ell, \lambda_3)}{L(\ell, \lambda_1)L(\ell, \lambda_2)} \right)$. As before, we set

$$I^a(u_1, x) = x^{h_1+h_2-h_3}I(u_1, x) = \sum_{n \in \mathbb{Z}} I_{u_1}(n)x^{-n-1} \quad \text{for any } u_1 \in L(\ell, \lambda_1). \quad (4.47)$$

Then (the second identity follows from Proposition 2.5)

$$[a(m), I_u(n)] = I_{au}(m+n); \quad (4.48)$$

$$[L(0), I_u(n)] = (h_3 - h_2 - n - 1)I_u(n) \quad (4.49)$$

for $a \in \mathfrak{g}, u \in L(\lambda_1), m, n \in \mathbb{Z}$. Then $I^a(\cdot, x)$ naturally gives rise to a linear map $R_I$ from $\mathbb{C}[t, t^{-1}] \otimes L(\lambda_1) \otimes L(\ell, \lambda_2)$ to $L(\ell, \lambda_3)$ such that

$$R_I(t^n \otimes u_1 \otimes u_2) = I_{u_1}(n)u_2 \quad \text{for } u_1 \in L(\lambda_1), u_2 \in L(\ell, \lambda_2), n \in \mathbb{Z}. \quad (4.50)$$

(4.48) is equivalent to say that the map $R_I$ is a $\hat{g}$-homomorphism from $\hat{L}(\lambda_1) \otimes L(\ell, \lambda_2)$ to $L(\ell, \lambda_3)$. From (4.49) we get

$$L(0)(t^n \otimes u_1 \otimes u_2) = t^n \otimes u_1 \otimes L(0)u_2 + (h_3 - h_2 - n - 1)(t^n \otimes u_1 \otimes u_2) \quad (4.50)$$

for $u_1 \in L(\lambda_1), u_2 \in L(\ell, \lambda_2), n \in \mathbb{Z}$. Then

$$(h_3 - L(0))(t^n \otimes u_1 \otimes u_2) = t^n \otimes u_1 \otimes (h_2 - L(0))u_2 + (n+1)(t^n \otimes u_1 \otimes u_2). \quad (4.51)$$

View $\hat{L}(\lambda_1)$ as a $\hat{g}$-module with $d = (1 + t\frac{d}{dt}) \otimes 1$ and view $L(\ell, \lambda)$ as a $\hat{g}$-module with $d = h - L(0)$ where $h$ is the lowest weight. Then it follows from (4.51) that $R_I$ is a
\( g \)-homomorphism. Then we obtain a linear map:

\[
R : I \left( \begin{array}{c} L(\ell, \lambda_3) \\ L(\ell, \lambda_1) L(\ell, \lambda_2) \end{array} \right) \to \text{Hom}_{\hat{g}}(\hat{L}(\lambda_1) \otimes L(\ell, \lambda_2), L(\ell, \lambda_3));
\]

\[ I(\cdot, x) \mapsto R_I. \]

(4.52)

In some references, an intertwining operator of type \( (L(\ell, \lambda_3) L(\ell, \lambda_1) L(\ell, \lambda_2)) \) is defined to be a \( \tilde{g} \)-module homomorphism from \( \hat{L}(\lambda_1) \otimes L(\ell, \lambda_2) \) to \( L(\ell, \lambda_3) \). The following proposition asserts that this definition is equivalent to FHL’s definition.

**Proposition 4.15** The intertwining operator space \( I \left( \begin{array}{c} L(\ell, \lambda_3) \\ L(\ell, \lambda_1) L(\ell, \lambda_2) \end{array} \right) \) is naturally isomorphic to the space of \( \tilde{g} \)-homomorphisms from \( \hat{L}(\lambda_1) \otimes L(\ell, \lambda_2) \) to \( L(\ell, \lambda_3) \).

**Proof.** From the above discussion we see that for any intertwining operator \( I(\cdot, x) \) of type \( \left( \begin{array}{c} L(\ell, \lambda_3) \\ L(\ell, \lambda_1) L(\ell, \lambda_2) \end{array} \right) \), we obtain a \( \tilde{g} \)-homomorphism \( R_I \). Conversely, let \( f \) be a \( \tilde{g} \)-homomorphism from \( \hat{L}(\lambda_1) \otimes L(\ell, \lambda_2) \) to \( L(\ell, \lambda_3) \). Then we define a linear map \( \Phi(\cdot, x) \) from \( L(\lambda_1) \) to \( \text{Hom}(L(\ell, \lambda_2), L(\ell, \lambda_2)) \{ x \} \) such that

\[
\Phi(u_1, x) u_2 = x^{h_3 - h_1 - h_2} \sum_{n \in \mathbb{Z}} f(t^n \otimes u_1 \otimes u_2) x^{-n-1}
\]

(4.53)

for \( u_1 \in L(\lambda_1), u_2 \in L(\ell, \lambda_2) \). Then \( \Phi(\cdot, x) \) satisfies (4.40) and

\[
[L(0), \Phi(u_1, x)] = \left( h_1 + x \frac{d}{dx} \right) \Phi(u_1, x) \quad \text{for} \quad u_1 \in L(\lambda_1).
\]

(4.54)

Then \( \Phi(u_1, x) \in E(L(\ell, \lambda_2), L(\ell, \lambda_3)) \) for any \( u_1 \in L(\lambda_1) \). Similar to the proof of Proposition 1.13, \( L(\ell, \lambda_1) \) is a submodule of \( E(L(\ell, \lambda_2), L(\ell, \lambda_3)) \) generated by \( L(\lambda_1) \) and there is a weak intertwining operator \( I(\cdot, x) \) from \( L(\ell, \lambda) \) to \( \text{Hom}(L(\ell, \lambda_2), L(\ell, \lambda_3)) \{ x \} \). It is well known (cf. [HL0-4], [FLM]) that under the commutator formula (2.4), the \( L(-1) \)-bracket formula (I2) is equivalent to the \( L(0) \)-bracket formula. Thus \( \Phi(u_1, x) \in G(L(\ell, \lambda_2), L(\ell, \lambda_3)) \) for \( u_1 \in L(\lambda_1) \). Since \( L(\lambda_1) \) generates \( L(\ell, \lambda_1) \) by \( U(\hat{g}) \), it follows from Proposition 1.5 that \( L(\ell, \lambda_1) \subseteq G(L(\ell, \lambda_2), L(\ell, \lambda_3)) \). Thus \( I(\cdot, x) \) is an intertwining operator. Then the proof is complete.  \( \square \)
Let $L(c, h)$ be the irreducible module of the Virasoro algebra $\text{Vir}$ with central charge $c$ and lowest weight $h$. It is well known (cf. [FZ], [H1], [L1]) that $L(0,0)$ is a vertex operator algebra. Suppose that $L(c, h_1)$ and $L(c, h_2)$ are two modules for the vertex operator algebra $L(c, 0)$. Let $\Phi(x) \in (\text{Hom}_C(L(c, h_1), L(c, h_2))) \{x\}$ such that

$$[L(m), \Phi(x)] = x^m (m + 1) h + x \frac{d}{dx} \Phi(x)$$

for some complex number $h$. That is,

$$[Y(\omega, x_1), \Phi(x_2)] = x_1^{-1} \delta \left( \frac{x_2}{x_1} \right) \frac{d}{dx_2} \Phi(x_2) + h x_1^{-2} \delta' \left( \frac{x_2}{x_1} \right) \Phi(x_2).$$

Similarly to the proof of Proposition 4.9 we get $\Phi(x) \in G(L(c, h_1), L(c, h_2))$ and $\Phi(x)$ generates a $L(c, 0)$-module $M$ which is a lowest weight Virasoro algebra module with lowest weight $h$ in $G(L(c, h_1), L(c, h_2))$. If $c = 1 - \frac{6(p-q)^2}{pq}$, where $p, q \in \{2, 3, \cdots\}$ are relatively prime, $L(c, 0)$ is rational ([DMZ], [W]). Therefore $M = L(c, h)$. Then we obtain an intertwining vertex operator of type $\left( \frac{L(c, h_2)}{L(c, h)L(c, h_2)} \right)$. Thus we have

**Proposition 4.16** If $c = 1 - \frac{6(p-q)^2}{pq}$, where $p, q \in \{2, 3, \cdots\}$ are relatively prime, let $L(c, h_1), L(c, h_2)$ be $L(c, 0)$-modules and let $\Phi(x)$ satisfy (4.55). Then there exists a unique intertwining vertex operator of type $\left( \frac{L(c, h_2)}{L(c, h)L(c, h_2)} \right)$ extending $\Phi(x)$.

**5 Appendix**

The main purpose of this appendix is to give an example to show that the generalized form of the nuclear democracy theorem may not be true if $V$ is not rational. We use the same notions as in Section 4. Let $\ell$ be a positive integer and let $\mathbb{C}_\ell$ be the $(\mathbb{C}[t] \otimes g + Cc)$-module such that $c$ acts as $\ell$ and $\mathbb{C}[t] \otimes g$ acts as zero. Set

$$M(\ell, C) = U(g)U(\mathbb{C}[t] \otimes g + Cc)\mathbb{C}_\ell.$$
rational, we may choose an $\lambda$ such that $L(\ell, \lambda)$ is not a $L(\ell, 0)$-module. Let $\Phi(x)$ be the identity map from $L(\ell, \lambda)$ to $L(\ell, \lambda)$. Then $\Phi$ satisfies all the conditions in Theorem 4.12. If we could extend $\Phi$ to an intertwining operator on $L(\ell, 0)$, then we would have an intertwining operator of type $\left( \frac{L(\ell, \lambda)}{L(\ell, 0)L(\ell, \lambda)} \right)$ so that $L(\ell, \lambda)$ would be a $L(\ell, 0)$-module. This would contradict the assumption that $L(\ell, \lambda)$ is not a $L(\ell, 0)$-module.

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