Trees with exponential height dependent weight

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Trees with exponential height dependent weight

Bergfinnur Durhuus · Meltem Ünel

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Abstract
We consider planar rooted random trees whose distribution is even for fixed height \( h \) and size \( N \) and whose height dependence is of exponential form \( e^{-\mu h} \). Defining the total weight for such trees of fixed size to be \( Z_N^{(\mu)} \), we determine its asymptotic behaviour for large \( N \), for arbitrary real values of \( \mu \). Based on this we identify the local limit of the corresponding probability measures and find a transition at \( \mu = 0 \) from a single spine phase to a multi-spine phase. Correspondingly, there is a transition in the volume growth rate of balls around the root as a function of radius from linear growth for \( \mu < 0 \) to the familiar quadratic growth at \( \mu = 0 \) and to cubic growth for \( \mu > 0 \).

Keywords Random trees · Height coupled trees · Local limits of BGW trees

Mathematics Subject Classification 60B10 · 05C05 · 60J80

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1 Introduction

Random trees have been at the stage of research in theoretical probability for decades, their relationship with important classes of branching processes being a strong motivating factor. In particular, the class of Bienaymé–Galton–Watson (BGW) processes and associated probability measures on trees have been intensively studied [1, 8, 17]. In recent years, important additional motivation for studying random trees, as well as more general random graphs, originates from theoretical physics, e.g. by serving as models of statistical mechanical systems in random environments and by providing a framework for investigating non-perturbative aspects of quantum gravity [6]. Of particular relevance for two-dimensional systems are topics relating to random surfaces and random planar maps, on which significant progress has been obtained, both on the combinatorial side [11, 35], sometimes involving nontrivial correspondences between the maps in question and classes of labelled planar trees, and on the analytic side yielding constructions of interesting local limits [7, 13, 15, 26, 33] and scaling limits [4, 14, 27–29, 31, 34].

A particularly simple and concrete correspondence between planar rooted trees and planar maps is provided by the so-called causal triangulations of the disc [19, 30]. The infinite volume limit of this ensemble can via the mentioned correspondence be identified with the local limit as \( N \to \infty \) of the uniform distribution of rooted planar trees of size \( N \) [19], which is called the Uniform Infinite Planar Tree (UIPT) and is a special case of a more general construction of local limits of BGW measures conditioned on size [5, 23, 24]. From a physical point of view, it is natural in this context to consider the weight of a given tree \( T \) to be given by

\[
w(T) = e^{-\Lambda|T|},
\]

where \( \Lambda \) is a real constant and \( |T| \) denotes the size of \( T \), which also equals half the area (number of triangles) of the corresponding triangulation. We shall use the notation \( g = e^{-\Lambda} \) in the main text below and will see that the total volume \( X(g) \) of the measure on the space of all finite planar rooted trees defined by (1) is finite if and only if \( g \leq \frac{1}{4} \). Written as a power series in \( g \), it equals the generating function for the number \( A_N \) of all rooted planar trees of size \( N \), while for the particular "critical" value \( g = \frac{1}{4} \) the measure equals a critical BGW measure up to a factor 2, denoted by \( \rho \) in the following [see Eq. (49) below]. When restricting the measure defined by (1) to the set \( T_N \) of trees of fixed size \( N \) and normalising, one evidently obtains the uniform distribution \( \nu_N \) on \( T_N \), independently of \( g \), whose explicit form is given by (20) below. As already mentioned, the local limit of \( \nu_N \) as \( N \to \infty \) equals the UIPT, whose main features were first uncovered by Kesten [25]. In particular, the fact that
the local limiting measure is supported on trees with a single spine, i.e. a unique linear
infinite subgraph emerging from the root, and that the branches attached to the spine
are independently distributed according to $\rho$, are of importance for the subsequent
discussion.

The form (1) lends itself to a natural generalisation,

$$w(T) = e^{-\Lambda |T| - \mu h(T)},$$

where $\mu$ is a real constant and $h(T)$ denotes the height of $T$. Indeed, this form of
weight turns out to be of relevance, when considering certain types of loop models
on random causal triangulations as realised in [20]. It is worth noting that, for $\mu \neq 0$,
weight functions of the form (2) do not define simply generated trees [32], i.e. $w(T)$
cannot be written as a product over vertices in $T$ of a local weight function depending
only on the vertex degree. Hence, analytic tools depending on this feature, typically in
the form of recursion relations, are not readily available in case $\mu \neq 0$. The goal of this
paper is to present a generalisation of the local limit result for $\mu = 0$ to arbitrary $\mu \in \mathbb{R}$
and to give a basic characterisation of the corresponding random trees, including a
determination of volume growth exponents. In particular, we show that the single
spine feature persists for $\mu < 0$ but the branches become subcritical BGW trees,
whereas for $\mu > 0$ the spine seizes to be one-ended and becomes a random tree of its
own with statistically dependent (infinite) branches and whose $n$’th generation size is
Poisson distributed with mean $n \mu$. Moreover, the full local limit is in this case obtained
by grafting independent critical BGW trees onto the random spine. These measures
also occur as local limits of BGW trees conditioned on the asymptotic behavior of
generation size, see [3].

The paper is organized as follows. In Sect. 2.1, we give a combinatorial definition
of rooted planar trees, fix some notation, and introduce a convenient metric defining
a topology and an associated Borel $\sigma$-algebra on the space of rooted planar trees, on
which the probability measures in question will be defined, providing an appropriate
setting for discussing local (or weak) limits. In Sect. 2.2, the analytic structure of the
familiar generating functions $X_m$ for the number of trees of height at most $m$ is deter-
mined and used to derive closed expressions for their Taylor coefficients $A_{m,N}$, that
will be of importance for the analysis in Sect. 4. Moreover, the probability measures
$\nu_{N}^{(\mu)}$, $N \in \mathbb{N}$, obtained by restricting the measure defined by (2) to trees of fixed
size $N$ and normalising, are introduced. As in the case $\mu = 0$, they are independent
of $\Lambda$. Choosing $\Lambda = 0$, the relevant normalisation factors are denoted by $Z_{N}^{(\mu)}$ and
are referred to as finite size partition functions. Their asymptotic behaviour for large
$N$ is of crucial importance for the determination of the local limit. Section 3 covers
the case $\mu < 0$. A rather simple analysis of the analytic properties of the generat-
ing function $Z^{(\mu)}(g)$ for the partition functions $Z_{N}^{(\mu)}$ allows a determination of their
asymptotic behavior in Sect. 3.1. Subsequently, lower bounds on ball volumes are
established in Sect. 3.2, that are strong enough to prove existence of the limit and
to identify the limiting measure $\nu^{(\mu)}$ with the local limit of the measures obtained
by conditioning a subcritical BGW measure on $\{h(T) \geq n\}$, $n \in \mathbb{N}$, as detailed in

1 In this paper $\mathbb{N} = \{1, 2, \ldots \}$.  

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Theorem 3.5. In Sect. 4, the case $\mu > 0$ is considered. A saddle point approach is applied to determine the asymptotic behavior of $Z_N^{(\mu)}$ for large $N$ in Sect. 4.1, while in Sect. 4.2 this approach is extended to obtain estimates on ball volumes that ultimately allow us to identify the local limit $\nu(\mu)$ with the infinite Poisson tree introduced in [3]. The properties of $\nu(\mu)$ are investigated in Sect. 4.3 where we first establish a decomposition result in Theorem 4.4, which also allows us to identify the measure governing the spine, including the mentioned Poisson distribution of its generation size, see Theorem 4.5. Finally, the statistical behaviour of the volume of the ball of radius $r$ around the root of a tree is investigated as a function of $r$. It is shown that its expectation value is quadratic in $r$ for the spine while it is cubic for the full measure $\nu(\mu)$, see Corollaries 4.7(ii) and 4.8(ii). The corresponding almost sure statements are formulated in Corollary 4.7(iv) and Theorem 4.9, while some technical arguments are deferred to an appendix.

2 Preliminaries

2.1 Metric spaces of trees

Combinatorially, a planar tree $T$ is given by a sequence $D = (D_0, D_1, D_2, \ldots)$ of (disjoint) ordered, finite sets whose elements are the vertices of $T$,

$$V(T) = \bigcup_{r=0}^{\infty} D_r,$$

and a sequence $\phi = (\phi_1, \phi_2, \phi_3, \ldots)$ of order preserving maps $\phi_r : D_r \to D_{r-1}$, called parent maps, such that the edges of $T$ are of the form $\{i, \phi_r(i)\}$, i.e.

$$E(T) = \{\{i, \phi_r(i)\} \mid r \in \mathbb{N}, i \in D_r\}.$$

Moreover, we assume $D_0 = \{i_0\}$ and $D_1 = \{i_1\}$ are one-point sets and call $i_0$ the root and $\{i_0, i_1\}$ the root edge of $T$. Vertices in $\phi_{r-1}^{-1}(j)$ are referred to as offspring of $j \in D_{r-1}$ and they inherit an ordering from $D_r$. The notion of ancestor and descendant are defined in terms of the parent maps in the obvious way. Two trees $T = (D, \phi)$ and $T' = (D', \phi')$ are considered equal if there exist order preserving bijective maps $\psi_r : D_r \to D'_r$ such that $\psi_{r-1} \circ \phi_r = \phi'_r \circ \psi_r$, for each $r \in \mathbb{N}$. It is easy to see that, by choosing a fixed orientation of $\mathbb{R}^2$ and a right-handed coordinate system, one can embed any such tree as a graph in $\mathbb{R}^2$ such that the vertices in $D_r$ are mapped into the horizontal line through $(0, r)$ and ordered according to their first coordinate where edges are represented by straight line segments, and two embedded trees are identical if and only if one can be mapped onto the other by an orientation preserving homeomorphism of $\mathbb{R}^2$. Hence, we shall frequently refer to the ordering of $D_r$ as being from left to right. Moreover, given a tree $T$, we shall denote the corresponding $D_r$ by $D_r(T)$. 

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The neighbours of a vertex \( i \in D_r, r \geq 1 \), consist of its offspring together with its parent. Hence, the degree \( \sigma_i \) of \( i \) in \( T \) is given by

\[
\sigma_i = |\phi_{r+1}^{-1}(i)| + 1,
\]

where we write \( |A| \) for the cardinality of a set \( A \). Ordering the offspring \( i_1, i_2, \ldots, i_{\sigma_i-1} \) from left to right, the oriented edges \((i, i_n), n = 1, \ldots, \sigma_i - 1\) will be called outgoing from \( i \). Together with \((i, \phi_r(i))\), these edges divide a small disc around \( i \) into \( \sigma_i \) angular sectors that will be denoted \( S_{i,1}, \ldots, S_{i,\sigma_i} \), ordered consistently with the ordering of the offspring, and we will call them the sectors around the vertex \( i \). Combinatorially, the sector \( S_{i,n} \) may be identified with the pair of oriented edges \((i, i_{n-1}), (i, i_n)\), for \( n = 1, \ldots, \sigma_i \), with the convention \( i_0 = i_{\sigma_i} = \phi_r(i) \).

By \( T_N \) we shall denote the set of planar trees \( T \) of size \( N \), i.e. \(|T| := |E(T)| = N\). Here \( N \in \mathbb{N} \) or \( N = \infty \). Thus, the set of finite trees is

\[
T_{\text{fin}} := \bigcup_{N=1}^{\infty} T_N,
\]

and we define

\[
T = T_{\text{fin}} \cup T_{\infty}.
\]

Given \( T \) as above and letting \( d_T \) denote the standard graph distance on \( T \), it is clear that \( D_r \) is the set of vertices at distance \( r \) from the root, which we shall also call the height of those vertices in \( T \). The height \( h(T) \) of a tree \( T \in T_{\text{fin}} \) is defined as the maximal height of any vertex in \( T \), or

\[
h(T) = \max \{ r \mid D_r(T) \neq \emptyset \}.
\]

If \( T \in T_{\infty} \) we set \( h(T) = \infty \).

With the aim of discussing local limits of sequences of measures on \( T \) we shall equip it with a natural metric as follows. Given \( r \in \mathbb{N} \), let \( B_r(T) \) denote the ball of radius \( r \) around the root \( i_0 \) defined as the subtree of \( T \) spanned by the vertices at distance at most \( r \) from \( i_0 \), i.e.

\[
V(B_r(T)) = \bigcup_{s=0}^{r} D_s(T).
\]

For \( T, T' \in T \) we then set

\[
dist(T, T') = \inf \left\{ \frac{1}{r} \mid r \in \mathbb{N}, \ B_r(T) = B_r(T') \right\}.
\]
It is then easy to see that dist is a metric on $T$, in fact an ultrametric. We shall denote by $B_a(T_0)$ the ball of radius $a > 0$ around $T_0 \in T$,

$$B_a(T_0) := \{ T \in T \mid \text{dist}(T, T_0) \leq a \}.$$

The measures on $T$ discussed in the following are all Borel measures, i.e. they are defined on the Borel $\sigma$-algebra $\mathcal{F}$ generated by the open sets. By definition, a sequence $\nu_N$, $N \in \mathbb{N}$, of probability measures converges weakly to a probability measure $\nu$ on $T$, if

$$\int_T F \, d\nu_N \to \int_T F \, d\nu \quad \text{as } N \to \infty$$

for all real valued bounded continuous functions $F$ on $T$. This requirement is equivalent to the statement that

$$\nu_N(B) \to \nu(B) \quad \text{as } N \to \infty \quad (3)$$

for any ball $B$ in $T$ as further detailed in the following remark, listing some basic properties of the metric space $T$. They are easily verifiable and will be used repeatedly in the subsequent discussion, see e.g. [13, 18] for more details.

**Remark 2.1**

(i) If $T \neq T'$, then $\text{dist}(T, T') = \frac{1}{r}$, where $r \geq 1$ is the radius of the largest ball around their roots shared by $T$ and $T'$, and in this case we have

$$B_{\frac{1}{r}}(T) = B_{\frac{1}{r}}(T') = B_{\frac{1}{r}}(T_0), \quad \text{where } T_0 = B_r(T) \text{ and hence } h(T_0) = r.$$

Noting that $B_{\frac{1}{s}}(T_0) = \{ T_0 \}$ for $s > r$, it follows that any finite tree is an isolated element of $T$.

If integers $s, r$ fulfill $1 \leq s < r$ and $T_1$ is a finite tree of height $s$, there is a unique decomposition of $B_{\frac{1}{s}}(T_1)$ into disjoint balls of radius $\frac{1}{r}$,

$$B_{\frac{1}{r}}(T_1) = \bigcup_{T_0 \in \mathcal{T}_{\text{fin}}, \ h(T_0) \leq r, \ B_r(T_0) = T_1} B_{\frac{1}{r}}(T_0).$$

(ii) Any ball in $T$ is both open and closed and any two balls are either disjoint or one is contained in the other. Since $\mathcal{T}_{\text{fin}}$ is a countable dense subset of $T$, it follows immediately by use of Theorem 2.2 in [10] that a sequence of probability measures $\nu_N$, $N \in \mathbb{N}$, converges weakly to a probability measure $\nu$ on $T$, if (3) holds for all balls $B$.

(iii) It is easy to show that $T$ is complete. Moreover, from separability and the first statement in (ii) it follows that every open subset of $T$ can be written as a countable union of pairwise disjoint balls.

(iv) For later reference we define $\mathcal{F}_r$ to be the collection of subsets of $T$ that can be written as a countable union of balls of radius $\frac{1}{r}$. It follows from i) above
that \( F_r, \ r \in \mathbb{N}, \) is a filtration of \( \sigma \)-algebras consisting of sets that are both open and closed. Moreover, since \( T \) is separable, the set algebra \( F_\infty := \bigcup_{r \in \mathbb{N}} F_r \) generates the Borel \( \sigma \)-algebra \( F \) of \( T \).

Given \( T \in \mathcal{T}_\infty \), we say that a vertex \( i \) of \( T \) is of infinite type if it has infinitely many descendants, and otherwise it is of finite type. Clearly, if \( i \) is of infinite type then so are all its ancestors and, since \( T \) is locally finite, \( i \) has at least one off-spring of infinite type. It follows that the vertices of infinite type span a subtree of \( T \) with the same root and root edge and with no leaves, i.e. no vertices of degree 1. We call this subtree the spine of \( T \). The mapping \( \chi : \mathcal{T}_\infty \to \mathcal{T}_\infty \) will be called the spine map, and we shall need the following fact about it.

**Lemma 2.1**  The spine map \( \chi : \mathcal{T}_\infty \to \mathcal{T}_\infty \) is Borel measurable.

**Proof** Let \( \chi_r : \mathcal{T}_\infty \to \mathcal{T}_\infty \) be defined such that \( \chi_r(T) \) is obtained by deleting the leaves in \( D_r(T) \) from \( T \) together with the edges containing them. Then \( \chi_r(T) \) is a subtree of \( T \) with the same spine, since only vertices of finite type are deleted and only finitely many of them. Note also that \( \chi_r \) is continuous, since \( \chi_r(B_{\frac{1}{s}}(T)) = B_{\frac{1}{s}}(\chi_r(T)) \) for \( s > r \). It is thus sufficient to show that

\[
\chi(T) = \lim_{n \to \infty} \chi_2 \circ \chi_3 \circ \cdots \circ \chi_n(T).
\]

For this purpose, let \( i \) be any vertex of finite type in \( T \) at height \( r \) and consider the finite subtree \( T_0 \) of \( T \) spanned by the descendants of \( i \) together with \( i \) and \( j = \phi_r(i) \), and where \( j \) is the root of \( T_0 \). Observe that, if \( k \) denotes the height of \( T_0 \), then \( \chi_2 \circ \chi_3 \circ \cdots \circ \chi_n(T) \) does not contain \( i \) when \( n \geq r + k - 1 \). Given \( s \in \mathbb{N} \), there are only finitely many vertices in \( T \) of height \( \leq s \), so it follows from this observation that if \( n \) is large enough then \( \chi_2 \circ \chi_3 \circ \cdots \circ \chi_n(T) \) does not contain any vertices of finite type at heights \( \leq s \), and hence coincides with \( \chi(T) \) up to height \( s \). This proves the claim. \( \square \)

Trees belonging to \( \chi(\mathcal{T}_\infty) \) will be called spine trees in the following and we will use the notation \( \mathcal{T}^s \) for this set.

**Lemma 2.2** The subset \( \mathcal{T}^s \) of \( \mathcal{T}_\infty \) is closed.

**Proof** Note, that a tree \( T \in \mathcal{T} \) belongs to the complement of \( \mathcal{T}^s \) if and only if it has at least one leaf, and if \( T \) has a leaf at height \( r \) then the trees in \( B_{\frac{1}{s}}(T) \) also have a leaf at height \( r \) for \( s \geq r \). Hence the complement of \( \mathcal{T}^s \) is open, which proves the lemma. \( \square \)

We shall make use of the process of **grafting** a tree \( T_1 \in \mathcal{T} \) onto another tree \( T_0 \in \mathcal{T} \) on several occasions in the following. Although intuitively rather clear, it will be useful to specify this notion and the associated notation in some detail. Let \( i \) be some vertex in \( T_0 \) at height \( r_1 \geq 1 \) and let \( 1 \leq n \leq \sigma_i \). The grafted tree \( T := gr(T_0; (i, n); T_1) \) is then defined by setting

\[
D_r(T) = \begin{cases} D_r(T_0) & \text{if } r \leq r_1 \\ D_r(T_0) \cup D_{r-r_1+1}(T_1) & \text{if } r > r_1 \end{cases}
\]

(4)
with ordering induced by those of \( T_0 \) and \( T_1 \) such that, if \( r > r_1 \) and \( j \in D_r(T_0) \), then \( j \) is to the left of \( D_{r_1 + 1}(T_1) \) if the ancestor of \( j \) in \( D_{r_1}(T_0) \) is to the left of \( i \) or if it equals \( i \) and \( j \) is among the first \( n - 1 \) offspring of \( i \) or a descendant thereof. Otherwise, \( j \) is to the right of \( D_{r_1 + 1}(T_0) \). Moreover, the parent maps of \( T \) are defined in the obvious way in terms of those of \( T_0 \) and \( T_1 \), with the specification that the parent of vertices in \( D_2(T_1) \subseteq D_{r_1 + 1}(T) \) is defined to be \( i \). In this way, \( T_1 \) can be considered as a subtree of \( T \) whose root edge is identified with \( \{ i, \phi_{r_1}(i) \} \). Pictorially, one can think of \( T \) as being obtained by identifying the outgoing root edge of \( T_1 \) with \( \phi_{r_1}(i, i) \) and drawing the remaining part of \( T_1 \) in the \( n \)’th sector \( S_{i,n} \) of the plane around \( i \), see Fig. 1. Likewise, \( T_0 \) is a subtree of \( T \) with the same root and root edge and with vertices of identical degrees in both, except for \( i \) in case \( T_1 \) has more than one edge. We say that \( T \) is obtained by grafting \( T_1 \) onto \( T_0 \) at \( (i, n) \) or in sector \( S_{i,n} \). In case \( i \) is a leaf, there is only one sector \( S_{i,n} \) and we say that \( T_1 \) is grafted onto \( T_0 \) at \( i \).

It is easily seen that, for fixed \( T_0 \) and pairs \((i_1, n_1), \ldots, (i_K, n_K)\) labelling different vertex sectors, successive grafting of trees \( T_1, \ldots, T_K \) at \((i_1, n_1), \ldots, (i_K, n_K)\), respectively, is well defined and independent of the order of grafting. We denote the so obtained tree by \( gr(T_0; (i_1, n_1), \ldots, (i_K, n_K); T_1, \ldots, T_K) \).

The following result on the process of grafting will be needed.

**Lemma 2.3** For fixed \( T_0 \in \cal{T} \) and different pairs \((i_1, n_1), \ldots, (i_K, n_K)\) as above, the mapping

\[
G : (T_1, \ldots, T_K) \to gr(T_0; (i_1, n_1), \ldots, (i_K, n_K); T_1, \ldots, T_K)
\]

has the following properties.

(i) \( G \) maps \( \cal{T}^K \) homeomorphically onto a closed subset of \( \cal{T} \). If \( T_0 \in \cal{F}_\infty \), the image is also open.

(ii) If \( T_0 \) is finite and \( i_1, \ldots, i_K \) denote the vertices (leaves) at maximal height \( r := h(T_0) \), then \( G \) maps \( \cal{T}^K \) homeomorphically onto \( B_{\bar{r}}(T_0) \).

**Proof** It is clear from the definition of the grafting operation that

\[
G(B_{\frac{1}{r}}(T_1) \times \cdots \times B_{\frac{1}{r}}(T_K)) \subseteq B_{\frac{1}{r}}(G(T_1, \ldots, T_K)).
\]

This shows, in particular, that \( G \) is a contraction and so is continuous. Letting \( r_k \) be the height of \( i_k \) in \( T_0 \) for \( k = 1, \ldots, K \) and letting \( \bar{r} \) denote the maximal of these heights, it also follows that \( G(T_1, \ldots, T_K) \) and \( G(T_1', \ldots, T_K') \) have a common ball around the root of radius \( r \geq \bar{r} \) if and only if \( T_k \) and \( T_k' \) have a common ball of radius \( r - r_k + 1, k = 1, \ldots, K \). This implies that

\[
\text{dist}(G(T_1, \ldots, T_K), G(T_1', \ldots, T_K')) \geq \frac{1}{\bar{r}} \max \{ \text{dist}(T_k, T_k') \mid k = 1, \ldots, K \},
\]

from which we conclude that \( G \) is injective with a continuous inverse defined on the image of \( G \), which is closed by the completeness of \( \cal{T} \). This establishes the first part of (i).
Assume now that $T_0$ is finite. The claim in (ii) then follows by observing that for any tree $T \in B^r_s(T_0)$ we have $T = G(T_1, \ldots, T_K)$, where $T_k$ is the subtree of $T$ spanned by $i_k$ and its descendants together with its root $\phi_r(i_k)$, for $k = 1, \ldots, K$.

In order to verify the last statement in (i), let $T = G(T_1, \ldots, T_K)$. If $T$ is finite it is isolated in $T$ and hence an open subset. On the other hand, if $T$ is infinite, some $T_k$ must be infinite. Considering $T'_0 := B_s(T)$ for some integer $s > h(T_0)$ and letting $j_1, \ldots, j_M$ denote the vertices at maximal height $s$ in $T'_0$, we have that $T$ is contained in the ball of radius $\frac{1}{s}$ around $T'_0$ and by (ii) this ball is contained in the image of $G$, since any of the vertices $j_1, \ldots, j_M$ is contained in some $T_k$. This completes the proof of the lemma.

\[\square\]

### 2.2 Generating functions

In this subsection we introduce the quantities needed in order to define the measures on $T$ whose local limits will be investigated in subsequent sections. The use of generating function techniques to deal with the combinatorial properties of those quantities is a main theme of the discussion below.

The generating function for the number $A_{m,N}$ of trees in $T_{\text{fin}}$ of height at most $m$ and size $N$ is defined by

$$X_m(g) = \sum_{h(T) \leq m} g^{|T|} = \sum_{N=1}^{\infty} A_{m,N} g^N.$$ 

Similarly, we define

$$X(g) = \sum_{T \in T_{\text{fin}}} g^{|T|} = \sum_{N=1}^{\infty} A_N g^N,$$ 

where $A_N$ is the number of trees in $T_N$. In order to study the convergence properties of the sums involved, it is convenient to make use of the well known recursion relation, see e.g. [17, 18],
\[ X_{m+1}(g) = \frac{g}{1 - X_m(g)}, \quad m \geq 1, \quad X_1(g) = g, \]  
(6)
as well as the equation satisfied by \( X \),
\[ X(g) = \frac{g}{1 - X(g)}. \]  
(7)
The unique solution to (7) with \( X(0) = 0 \) is
\[ X(g) = \frac{1 - \sqrt{1 - 4g}}{2}, \]  
(8)
from which the values of its Taylor coefficients \( A_N \) can be deduced, yielding
\[ A_N = C_{N-1} := \frac{(2N - 2)!}{N!(N - 1)!} \cdot \frac{1}{\sqrt{\pi}} N^{-\frac{3}{2}} 4^{N-1} \left( 1 + O(N^{-1}) \right), \]  
(9)
where \( C_N \) is known as the \( N \)'th Catalan number. In particular, \( X \) is analytic in the disc
\[ \mathbb{D} = \{ g \in \mathbb{C} | |g| < \frac{1}{4} \} . \]
Moreover, \( X \) equals the limit of \( X_m \) as \( m \to \infty \) on \( \mathbb{D} \), as can be seen from the explicit formula for \( X_m \) stated below. This convergence will also be discussed in more detail in Sect. 3.1.

Equation (6) can be rewritten in linear form and solved explicitly with the result (see e.g. [17])
\[ X_m(g) = 2g \frac{(1 + \sqrt{1 - 4g})^m - (1 - \sqrt{1 - 4g})^m}{(1 + \sqrt{1 - 4g})^{m+1} - (1 - \sqrt{1 - 4g})^{m+1}}, \quad m \geq 1. \]  
(10)
Recalling that the Chebychev polynomials \( U_m \) of the second kind are defined by
\[ U_m(\cos \theta) = \frac{\sin(m + 1)\theta}{\sin \theta} \]  
(11)
and setting
\[ e^{i\theta} = \frac{1 + \sqrt{1 - 4g}}{2\sqrt{g}}, \]
on one finds that
\[ X_m(g) = \sqrt{g} \frac{U_{m-1}(\frac{1}{\sqrt{g}})}{U_m(\frac{1}{2\sqrt{g}})}. \]  
(12)
The roots $x_{m,k}$ of $U_m$ are given by

$$x_{m,k} = \cos \theta_{m,k}, \quad \text{where} \quad \theta_{m,k} = \frac{\pi k}{m+1} \quad \text{and} \quad k = 1, \ldots, m.$$ 

Recalling that $U_m$ is of degree $m$ and has the same parity as $m$, it follows that if $m = 2l$ is even we have that $g^l U_m\left(\frac{1}{\sqrt{g}}\right)$ is a polynomial in $g$ of degree $l$ with non-vanishing constant term, while if $m = 2l + 1$ is odd then the same holds for $\sqrt{g} g^l U_m\left(\frac{1}{\sqrt{g}}\right)$. Hence, we conclude from (10) that $X_m$ is a rational function of $g$ of the form

$$X_m(g) = \frac{g P_m(g)}{Q_m(g)},$$

where both $P_m$ and $Q_m$ are polynomials of degree $\left\lfloor \frac{m}{2} \right\rfloor$ if $m$ is odd, whereas they are of degree $\frac{m}{2} - 1$ and $\frac{m}{2}$, respectively, if $m$ is even. Moreover, the roots of $Q_m$ are in both cases of the form $g_{m,k} = \frac{1}{4x_{m,k}}$, where $x_{m,k}$ is a nonvanishing root of $U_m$ as given above. Since $x_{m,k} = -x_{m,m+1-k}$ we obtain precisely $\left\lfloor \frac{m}{2} \right\rfloor$ different roots $g_{m,k}$ given by

$$g_{m,k} = \frac{1}{4} \left( 1 + \tan^2 \frac{\pi k}{m+1} \right), \quad k = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor.$$ 

These are all simple poles of $X_m$ whose corresponding residues $r_{m,k}$ we calculate below. In particular, it follows that $X_m$ is an analytic function in the disc

$$D_m := \{ g \in \mathbb{C} \mid |g| \leq g_m \},$$

where

$$g_m := g_{m,1} = \frac{1}{4} \left( 1 + \tan^2 \frac{\pi}{m+1} \right)$$

is the radius of convergence for the power series (5) defining $X_m$.

Using (12), we obtain by differentiating the denominator on the right-hand side that

$$r_{m,k} = -\frac{4g_{m,k}^2 U_{m-1}(\frac{1}{\sqrt{g_{m,k}}})}{U'_m(\frac{1}{\sqrt{g_{m,k}}})}.$$ 

From the defining relation (11) one obtains

$$U_{m-1}(\cos \theta_{m,k}) = (-1)^{k+1}$$
and
\[ U'_m(\cos \theta_{m,k}) = (-1)^{k+1} \frac{m+1}{\sin^2 \frac{\pi k}{m+1}}. \]

Inserting these into (14) then gives
\[ r_{m,k} = - \frac{1}{4(m+1)} \tan^2 \frac{\pi k}{m+1} \left( 1 + \tan^2 \frac{\pi k}{m+1} \right), \quad k = 1, \ldots, \left\lfloor \frac{m}{2} \right\rfloor. \quad (15) \]

Taking into account (13) and the relative degrees of \( P_m \) and \( Q_m \), we have
\[ X_m(g) = \sum_{k=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{r_{m,k}}{g - g_{m,k}} + c_m + c'_m g, \quad (16) \]

where the constants \( c_m \) and \( c'_m \) are fixed by the requirements \( X_m(0) = 0 \) and \( X'_m(0) = 1 \) for \( m \geq 1 \). If \( m \) is even, the numerator and denominator in (13) have the same degree, hence \( c'_m = 0 \), while a simple calculation yields \( c'_m = \frac{2}{m+1} \) if \( m \) is odd.

By expanding the pole terms on the right-hand side of (16) as geometric series and using (15), we obtain the power series expansion of \( X_m \), and hence its Taylor coefficients \( A_{m,N} \) in the form
\[ A_{m,N} = 4^N \sum_{k=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \frac{1}{m+1} \tan^2 \frac{\pi k}{m+1} \left( 1 + \tan^2 \frac{\pi k}{m+1} \right)^{-N}, \quad N \geq 2, \quad (17) \]

which will constitute the basis of much of the discussion in Sect. 4. This formula can also be found in [16].

**Remark 2.2** Note that, \( A_{m,N} \) is constant for fixed \( N \) and \( m \geq N \), since the height of a tree obviously cannot exceed its size. On the other hand, the sum on the right-hand side of (17) approximates the integral
\[ I_N = \int_0^{\frac{1}{2}} \tan^2 \pi x \left( 1 + \tan^2 \pi x \right)^{-N} dx \]
as \( m \to \infty \). It is straightforward to calculate \( I_N \) by recursion and showing that it equals the Catalan number \( C_{N-1} \) up to the factor \( 4^N \) in accordance with (9).

The goal of the subsequent discussion is to study in some detail a one-parameter family of probability measures \( \nu^{(\mu)} \) obtained as local limits of finite size measures \( \nu_N^{(\mu)} \), which are uniform in size for fixed height, defined by
\[ \nu_N^{(\mu)}(T) = \frac{e^{-\mu h(T)}}{Z_N^{(\mu)}}, \quad \text{for } T \in T_N, \quad (18) \]
with normalisation factor (partition function) $Z^{(\mu)}_N$ given by

$$Z^{(\mu)}_N = \sum_{m=1}^{\infty} e^{-\mu m} (A_{m,N} - A_{m-1,N}),$$  \hspace{1cm} (19)$$

where $A_{0,N} = 0$ by convention. We shall consider these as measures on the space $\mathcal{T}$ and obtain the local limits as measures supported on $\mathcal{T}_\infty$. For $\mu = 0$, Eq. (19) yields the uniform probability measure on $\mathcal{T}_N$ given by

$$\nu^{(0)}_N(T) = \frac{1}{C_{N-1}}, \text{ for } T \in \mathcal{T}_N,$$  \hspace{1cm} (20)$$
as a consequence of (9). As previously mentioned, its local limit as $N \to \infty$, called the UIPT, is well studied, and a brief account of its main features is contained in the discussion in Sect. 3.3 below, see in particular Remark 3.3. Our primary focus is on the cases $\mu < 0$ and $\mu > 0$ which, as it turns out, need to be treated by quite different techniques.

3 The case $\mu < 0$

3.1 Partition function

In order to determine the asymptotic behaviour of $Z^{(\mu)}_N$ for $N$ large, as a prerequisite for establishing the existence of the local limit of $\nu^{(\mu)}_N$, $N \in \mathbb{N}$, some more detailed information on the rate of convergence of $X_m$ towards $X$ will be needed. This is provided by the next lemma.

**Lemma 3.1** Defining

$$c(g) = \frac{g}{(1 - X(g))^2}$$  \hspace{1cm} (21)$$

we have that

$$|c(g)| \leq c(|g|) < 1 \text{ for } g \in \mathbb{D},$$  \hspace{1cm} (22)$$

and the following statements hold.

(i) $|X_m(g) - X(g)| \leq |g| c(|g|)^m, \text{ } m \in \mathbb{N}, g \in \mathbb{D}.$  \hspace{1cm} (23)$$

(ii) The product $\prod_{l=1}^{\infty} \frac{1 - X(g)}{1 - X_l(g)}$ converges and equals an analytic function $f$ on $\mathbb{D}$ with no zeroes and fulfilling

$$\left| \prod_{l=1}^{m} \frac{1 - X(g)}{1 - X_l(g)} - f(g) \right| \leq \text{cst} \cdot c(|g|)^m, \text{ } m \in \mathbb{N}, |g| \leq a,$$  \hspace{1cm} (24)$$
for each fixed $a < \frac{1}{4}$, where the constant on the right hand side depends on $a$.

**Proof** From the definition (5) of $X$ it is clear that $|X(g)| \leq X(|g|)$, whenever the sum in (5) is absolutely convergent, i.e. when $|g| \leq \frac{1}{4}$. Moreover, $X$ is strictly increasing on $[0, \frac{1}{4}]$ with $X(\frac{1}{4}) = \frac{1}{2}$. Hence $c$ is likewise strictly increasing with $c(0) = 0$ and $c(\frac{1}{4}) = 1$. From this (22) follows.

In order to verify (i), we note that

$$|X(g) - X_m(g)| = \left| \sum_{T \in T} g^{\left| T \right|} - \sum_{T \in T, h(T) \leq m} g^{\left| T \right|} \right| \leq X(|g|) - X_m(|g|), \quad (25)$$

as well as the identity

$$X(g) - X_m(g) = \frac{g (X(g) - X_{m-1}(g))}{(1 - X(g))(1 - X_{m-1}(g))},$$

which is a consequence of (6) and (7) and by iteration gives

$$X(g) - X_m(g) = g \cdot c(g)^m \prod_{l=1}^{m-1} \frac{1 - X_l(g)}{1 - X(g)}.$$

Here we note that $0 \leq X_l(g) \leq X(g) < \frac{1}{2}$ for $0 \leq g < \frac{1}{4}$, and hence the last product is bounded by 1. Thus (23) follows by use of (25).

As a consequence of (22) and (23) we have that $\sum_{m=1}^{\infty} |X_m(g) - X(g)|$ is uniformly convergent on compact subsets of $\mathbb{D}$. Hence, the first statements of (ii) follow from Theorem 15.6 of [36], since $X_m$ and $X$ are analytic on $\mathbb{D}$. Furthermore, applying standard estimates, see e.g. the proof of Theorem 15.4 of [36], we have

$$\left| \prod_{l=1}^{m} \frac{1 - X_l(g)}{1 - X_l(g)} - f(g) \right| = |f(g)| \left| \prod_{l=m+1}^{\infty} \frac{1 - X_l(g)}{1 - X(g)} - 1 \right| \leq |f(g)| \left( \sum_{l=m+1}^{\infty} \left| \frac{X(g) - X_l(g)}{1 - X_l(g)} \right| - 1 \right),$$

from which (24) follows when taking into account (23) and the continuity of $f$. \hfill $\square$

We next consider the generating function for $Z_{N}^{(\mu)}$ given by

$$Z^{(\mu)}(g) := \sum_{N=1}^{\infty} Z_{N}^{(\mu)} g^{N} = \sum_{m=1}^{\infty} e^{-\mu m} (X_m(g) - X_{m-1}(g)).$$
Assuming $\mu < 0$, we let $g_c(\mu)$ denote the unique value of $g \in \mathbb{D}$ fulfilling $e^{-\mu} c(g) = 1$. Using (21) and (8) one finds that

$$g_c(\mu) = \frac{e^{\mu}}{(1 + e^{\mu})^2}. \quad (26)$$

The following theorem shows that the singularity of $Z^{(\mu)}$ closest to 0 is shifted from $g = \frac{1}{4}$ for $Z^{(0)} = X$, to $g_c(\mu) < \frac{1}{4}$ for $\mu < 0$, and that the singularity becomes a simple pole instead of a square root branch point.

**Theorem 3.2** For fixed $\mu < 0$, there exists $b > g_c(\mu)$ such that $Z^{(\mu)}(g)$ is analytic in

$$\{g \in \mathbb{C} \mid |g| < b, g \neq g_c(\mu)\}, \quad (27)$$

and has a simple pole at $g_c(\mu)$.

**Proof** Using (6), we have

$$e^{-\mu m} (X_{m+1}(g) - X_m(g)) = \frac{e^{-\mu} g^2}{1 - X(g)} \left( e^{-\mu} c(g) \right)^{m-1} \frac{1 - X(g)}{1 - X_m(g)} \prod_{l=1}^{m-1} \left( \frac{1 - X(g)}{1 - X_l(g)} \right)^2, \quad (28)$$

which by Lemma 3.1 can be written as

$$e^{-\mu m} (X_{m+1}(g) - X_m(g)) = \frac{e^{-\mu} (gf(g))^2}{1 - X(g)} \left( e^{-\mu} c(g) \right)^{m-1} + h_m(g), \quad (29)$$

where $h_m$ is analytic in $\mathbb{D}$ and fulfills

$$|h_m(g)| \leq \text{cst} \cdot \left( e^{-\mu} c(|g|)^2 \right)^m \quad \text{for } |g| \leq b \text{ and } m \in \mathbb{N},$$

for any fixed $b < \frac{1}{4}$. Since $c(g_c(\mu)) < 1$, we can choose $b > g_c(\mu)$ such that $e^{-\mu} c(|g|)^2 < 1$ for $|g| \leq b$ which ensures that $\sum_{m=1}^{\infty} h_m(g)$ converges to an analytic function $h(g)$ for $|g| < b$. By summing over $m$ in (29), we conclude that

$$Z^{(\mu)}(g) = \frac{e^{-\mu} (gf(g))^2}{(1 - X(g)) (1 - e^{-\mu} c(g))} + h(g)$$

is analytic for $|g| < b$ except at $g = g_c(k)$, which is a simple zero of the denominator $1 - e^{-\mu} c(g)$. This completes the proof. \hfill $\Box$

**Corollary 3.3** There exists $d > 0$ such that

$$Z_N^{(\mu)} = r \cdot g_c(\mu)^{-(N+1)} \left( 1 + O(e^{-dN}) \right) \quad (30)$$

for $N$ large, where $r$ is the residue of $-Z^{(\mu)}(g)$ at $g = g_c(\mu)$. \hfill $\Box$
Proof By Theorem 3.2 we may write

\[
Z(\mu)(g) = \frac{r}{g_c(\mu) - g} + \tilde{h}(g),
\]

where \( \tilde{h} \) is analytic in a disc centered at 0 of radius \( b > g_c(\mu) \). Expanding the pole term as a geometric series in \( \frac{g}{g_c(\mu)} \) then yields the dominant term in (30), while the subdominant part arises from the Taylor coefficients of \( \tilde{h} \) which are bounded by \( \text{cst} \cdot (g_c(\mu)e^{d})^{-N} \), where \( d > 0 \) fulfills \( g_c(\mu)e^{d} < b \). \( \square \)

3.2 Lower bounds on ball volumes

We now use the asymptotic results for \( Z_N(\mu) \) to establish lower bounds on ball volumes with respect to \( \nu_N(\mu) \).

Lemma 3.4 Let \( \mu < 0 \) and let \( T_0 \in T_{\text{fin}} \) have height \( r \) with \( K \) vertices in \( D_r(T_0) \). For each \( M \in \mathbb{N} \) there exists \( d > 0 \) such that

\[
\nu_N(\mu)(B_1(T_0)) \geq K \cdot e^{-\mu(r-1)} g_c(\mu)^{|T_0|-K} \left( \sum_{S=1}^{M} C_{S-1} g_c(\mu)^S \right)^{K-1} \left( 1 + O(e^{-dN}) \right),
\]

(31)

where \( g_c(\mu) \) is given by (26).

Proof Given \( T_0 \) as stated, let \( i_1, \ldots, i_K \) denote the vertices at maximal height \( r = h(T_0) \). By Lemma 2.3(ii), any tree \( T \) in \( B_1(T_0) \) can be obtained by grafting \( K \) trees \( T_1, \ldots, T_K \in \mathcal{T} \) onto \( T_0 \) at \( i_1, \ldots, i_K \), respectively, which we shall refer to as the branches of \( T \), see Fig. 2. If \( |T| = N \), we then have

\[
N = |T_0| + |T_1| + \cdots + |T_K| - K,
\]

and hence a branch \( T_j \) of maximal size among \( T_1, \ldots, T_K \) must fulfill

\[
|T_j| \geq \frac{N - |T_0| + K}{K}.
\]

Given \( M \in \mathbb{N} \) and \( j \in \{1, \ldots, K\} \) and imposing the constraint \( |T_i| \leq M \) for \( i \neq j \), it follows that \( T_j \) is the unique branch of maximal size, provided \( N > K(M - 1) + |T_0| \). Imposing the additional condition that \( h(T_j) > M \), we obtain that \( T_j \) also has maximal height among \( T_1, \ldots, T_K \) and that

\[
h(T) = h(T_j) + r - 1.
\]
Trees with exponential height dependent weight

**Fig. 2** Structure of a tree belonging to the the ball $B_{\frac{1}{\sigma}}(T_0)$ around $T_0$ with root $i_0$ and leaves $(i_1, i_2, i_3, i_4)$ at height $h(T_0) = 3$, obtained by grafting 4 trees $T_1, T_2, T_3, T_4 \in T$ onto $T_0$ at $i_1, i_2, i_3, i_4$, respectively.

Hence, setting $|T_i| = N_i$, it holds that

$$v_N^{(\mu)}(B_{\frac{1}{\sigma}}(T_0)) \geq e^{-\mu(r-1)} \sum_{j=1}^{K} \sum_{i \neq j, N_i \leq M} (Z_N^{(\mu)})^{-1} Z_{M,N,j} \prod_{i \neq j} C_{N_i} \geq 1 \quad (32)$$

for $N$ large enough, where

$$Z_{M,N} := \sum_{m=M+1}^{\infty} e^{-\mu m} (A_{m,N} - A_{m-1,N}).$$

Clearly, the generating function for the $Z_{M,N}$ for fixed $M$ equals $Z^{(\mu)}(g)$ minus the function $\sum_{m=1}^{M-1} e^{-\mu m} (X_m(g) - X_{m-1}(g))$, which is analytic in $D$. As a consequence, $Z_{N,M}$ has the same asymptotic form (30) for $N$ large as $Z_N^{(\mu)}$. For any given $j$ and fixed values of $N_i, i \neq j$, it follows that the corresponding term in (32) fulfills

$$(Z_N^{(\mu)})^{-1} Z_{M,N,j} = g_c(\mu)^{i \neq j} \left(1 + O(e^{-dN})\right).$$

Inserting this into (32), we obtain (31).

We note that (31) immediately implies that

$$\liminf_{N \to \infty} v_N^{(\mu)}(B_{\frac{1}{\sigma}}(T_0)) \geq \Lambda(T_0), \quad (33)$$

where $\Lambda(T_0)$ equals the large-$N$ limit of the right-hand side of (31) followed by the $M \to \infty$ limit, i.e.

$$\Lambda(T_0) := K \cdot e^{-\mu(r-1)} g_c(\mu)^{|T_0| - K} K (g_c(\mu))^{K-1}$$

$$= K \cdot e^{-\mu r} \left(\frac{e^{\mu}}{1 + e^{\mu}}\right)^{|T_0|} (1 + e^{\mu})^{K+1}, \quad (34)$$
where \( X(g_c(\mu)) = \frac{e^\mu}{1+e^\mu} \) has also been used.

### 3.3 Local limit and its properties

In this section we establish the existence of the limit measures \( \nu^{(\mu)} \), \( \mu < 0 \), and identify them with certain measures that are well known from the theory of branching processes; additionally, we discuss a few results on volume (or population) growth for such measures.

We first give a brief account of some aspects of Bienaymé–Galton–Watson (BGW) branching process (with one type of individual) and their local limits, while referring, e.g., [1, 8] for further details. Such a BGW process is defined in terms of an offspring probability distribution \( p(n), n = 0, 1, 2, \ldots \), fulfilling

\[
\sum_{n=0}^{\infty} p(n) = 1. \tag{35}
\]

The corresponding BGW tree is the probability measure \( \lambda \) on \( T \) defined by setting

\[
\lambda(B_1(T)) = \prod_{v \in \bigcup_{s=1}^{r-1} D_s(T)} p(\sigma(v) - 1), \tag{36}
\]

for any \( T \in T \). Indeed, due to (35) it is straightforward to show by induction that this formula defines, for each \( r \in \mathbb{N} \), a measure \( \lambda_r \) on \( F_r \), see Remark 2.1(iii), and that they are compatible, i.e. the restriction of \( \lambda_s \) to \( F_r \) equals \( \lambda_r \), if \( r < s \). In particular, they define a unique finitely additive measure \( \lambda_\infty \) on \( F_\infty \). By a Kolmogoroff type of argument one can show that \( \lambda_\infty \) has a unique extension \( \lambda \) to a Borel measure on \( T \).

It is well known that \( \lambda \) is supported on \( T_{\text{fin}} \) if and only if \( p \) is subcritical or critical, i.e. if the average offspring \( m \) satisfies

\[
m := \sum_{n=0}^{\infty} np(n) \leq 1.
\]

Moreover, in this case it is a fundamental result of Kesten [25] that if \( \lambda_{\geq N} \) denotes \( \lambda \) conditioned on \( \{ T \mid h(T) \geq N \} \), then the local limit of \( \lambda_{\geq N} \) as \( N \to \infty \) exists and equals a BGW tree \( \hat{\lambda} \) with two types of individuals called special and normal, respectively, and whose offspring probabilities are restricted such that normal individuals can have \( n = 0, 1, 2, \ldots \) normal offspring with probability \( p(n) \), but no special offspring, while special individuals can have exactly one special off-spring and in addition a number of \( n \geq 0 \) normal offspring with probability \( p^*(n) \), given by

\[
p^*(n) = \frac{(n+1)p(n+1)}{m}.
\]
More precisely, \( \hat{\lambda} \) is supported on \( T_\infty \) and its value on any ball \( B_1^\infty(T_0) \), where \( T_0 \in T_\text{fin} \) has height \( r \), is determined as follows. First, name each vertex of \( T_0 \) different from the root \( i_0 \) as either special or normal, such that the vertex \( i_1 \) next to the root in special and such that the restrictions on the offspring probabilities mentioned are respected, i.e. a special vertex has exactly one special offspring and a normal vertex has no special offspring. Clearly, the special vertices will form a path of length \( r - 1 \) from \( i_1 \) to a vertex \( i \) at height \( r \). We call this path \( \omega(i) \) since it is uniquely determined by \( i \).

Finally, we sum the resulting product over paths \( \omega \), thus defining

\[
\hat{\lambda}(B_1^\infty(T_0)) = \sum_{i \in D_r(T_0)} \prod_{v \in \cup_{s=1}^{r-1} D_s(T_0) \setminus V(\omega(i))} p(\sigma(v) - 1) \prod_{v \in V(\omega(i)) \setminus \{i\}} p^s(\sigma(v) - 2),
\]

which characterises the measure \( \hat{\lambda} \) on \( T_\infty \) uniquely, called the Kesten tree corresponding to the offspring distribution \( p \). It is known \([25]\) that \( \hat{\lambda} \) is supported on the subset of \( T_\infty \) consisting of trees with a single spine, i.e. the spine map \( \chi \) is in this case a.s. constant and equals the infinite linear path emerging from the root (spanned by the special vertices). Due to the multiplicative structure of (38), the branches grafted onto the vertices next to the spine, to the left and right, are independent with identical distribution equal to the BGW tree with offspring probability \( p \), while the degrees \( \sigma(v) \) of the spine vertices \( v \) are likewise independently and identically distributed according to \( p^s(\sigma(v) - 2), \sigma(v) \geq 2 \). In the following we shall denote single spine trees distributed according to \( \hat{\lambda} \) by \( \hat{T} \).

We can now formulate the main limit result of this section.

**Theorem 3.5** For each \( \mu < 0 \) the sequence \( v^{(\mu)}_N, N \in \mathbb{N} \), is weakly convergent to a probability measure \( v^{(\mu)} \) on \( T_\infty \), and it equals the Kesten tree corresponding to subcritical geometric offspring distribution given by

\[
p(n) = \left( \frac{1}{1 + e^{-\mu}} \right)^n \frac{1}{1 + e^{\mu}}, \quad n = 0, 1, 2, \ldots.
\]

**Proof** Since \( \frac{1}{1 + e^{\mu}} < \frac{1}{2} \), it is clear that \( p \) given by (39) defines a probability distribution with mean

\[
m = e^\mu < 1,
\]

\( \Box \)
and hence defines a subcritical BGW process. Moreover, by (26) the corresponding distribution $p^*$ has the form

$$p^*(n) = \frac{1}{(1 + e^\mu)^2} \left( \frac{1}{1 + e^{-\mu}} \right)^n.$$ 

Using these in (38), it is seen for a given $T_0$ of height $r$ and with $|D_r(T_0)| = K$ that all summands have the same value given by

$$\left( \frac{1}{1 + e^\mu} \right)^{r-1} \left( \frac{1}{1 + e^{-\mu}} \right)^{|T_0| - r} \left( \frac{1}{1 + e^\mu} \right)^{|T_0| - K - r + 1} = e^{-\mu r} \left( \frac{e^\mu}{(1 + e^\mu)^2} \right)^{|T_0|} (1 + e^\mu)^K - 1.$$

Upon multiplication by $K$, the last expression is seen to coincide with $\Lambda(T_0)$. Hence by (33) we have

$$\liminf_{N \to \infty} \nu^*(B_{\hat{T}}(T_0)) \geq \hat{\lambda}(B_{\hat{T}}(T_0)).$$

By Remark 2.1(iii) this inequality extends from balls to arbitrary open sets and hence we conclude by the Portmanteau theorem [10, Theorem 2.1] that $\nu^*(\mu) \to \hat{\lambda}$ as $N \to \infty$.

Remark 3.1 The proof of Theorem (3.5) could alternatively have been based on the collection of sets of the form $T(x, T_0) := \{ T \in \mathcal{T} \mid T = gr(T_0; (x, 1); T'), T' \in \mathcal{T} \}$, where $x$ is a leaf of $T_0 \in \mathcal{T}_{\text{fin}}$ instead of the set of balls used above, and making use of Lemma 2.1 in [2].

Letting $E_\mu$ denote the expectation w.r.t. $\nu^*(\mu)$, the following result on the average growth of balls around the root of $\hat{T}$ as a function of radius is an easy consequence of familiar results about BGW processes.

Corollary 3.6 For $\mu < 0$, the following asymptotic relations hold,

$$E_\mu(|D_r|) = \frac{1 + m}{1 - m} + O(m^{-1}r^{-1}), \quad (41)$$

$$E_\mu(|B_r|) = \frac{1 + m}{1 - m} \cdot r + O(1), \quad (42)$$

where $m$ is given by (40).

Proof Let $T_s^L$ and $T_s^R$ be the branches of $\hat{T}$ grafted to the left and right, respectively, at the spine vertex at distance $s$ from the root. It follows from Theorem 3.5 and the preceding remarks that these are i.i.d. according to the subcritical BGW tree with offspring distribution given by (39). In particular, we have (see e.g. Ch. 1 of [8])

$$E_\mu(|D_r(T_s^L)|) = E_\mu(|D_r(T_s^R)|) = m^{-1}r^{-1}, \quad r \geq 1.$$
Since
\[ |D_r(\hat{T})| = 1 + \sum_{s=1}^{r-1} \left( |D_{r-s+1}(T_s^L)| + |D_{r-s+1}(T_s^R)| \right), \tag{43} \]
it follows that
\[ E_\mu(|D_r(\hat{T})|) = 1 + 2m \frac{1 - m^{r-1}}{1 - m}, \]
from which the first relation follows. Using
\[ |B_r(\hat{T})| = \sum_{s=1}^{r} |D_s(\hat{T})|, \tag{44} \]
Eq. (42) follows from (41). \hfill \Box

The following result on a.s. asymptotic growth of individual trees is presumably not optimal but sufficient for our purposes. A proof is included in the appendix.

**Proposition 3.7** Let \( \mu < 0 \). There exist constants \( C_1, C_2 > 0 \) and for \( \nu^{(\mu)} \)-a.e. \( \hat{T} \) a number \( r_0(\hat{T}) \in \mathbb{N} \), such that
\[ 1 \leq |D_r(\hat{T})| \leq C_1 \cdot \ln r \quad \text{and} \quad r \leq |B_r(\hat{T})| \leq C_2 \cdot r \ln r \tag{45} \]
for all \( r \geq r_0(\hat{T}) \).

**Remark 3.2** The volume growth exponent \( d_h \) of a tree \( T \in T_\infty \), defined by
\[ d_h := \lim_{r \to \infty} \frac{\ln |B_r(T)|}{\ln r}, \tag{46} \]
is commonly referred to as the Hausdorff dimension of \( T \), provided the limit exists. Proposition 3.7 shows that \( d_h = 1 \) \( \nu^{(\mu)} \)-a.s. for \( \mu < 0 \).

**Remark 3.3** The local limit result of Kesten described above still holds if \( \lambda_N \) is replaced by \( \lambda \) conditioned on \( \{h(T) = N\} \), and in case \( \lambda \) is critical, i.e. if \( m = 1 \), the same conclusion holds if conditioning on \( \{|T| = N\} \) is used instead, see [1] for a more detailed account. For the critical BGW process with offspring distribution
\[ p_n = 2^{-(n+1)}, \quad n = 0, 1, 2, \ldots, \tag{47} \]
we shall denote by \( \rho \) the corresponding BGW measure on \( T \) as defined by (36), which in this case reads
\[ \rho(B_{\hat{T}}(T_0)) = 4^{-|T_0|} 2^{K+1}, \tag{48} \]
for any given tree $T_0 \in \mathcal{T}_{\text{fin}}$ of height $r$ with $K$ vertices (leaves) at height $r$. As mentioned earlier, $\rho$ is supported on $\mathcal{T}_{\text{fin}}$ and is alternatively given by

$$\rho(T) = 2 \cdot 4^{-|T|}, \ T \in \mathcal{T}_{\text{fin}}. \quad (49)$$

By conditioning on size, it follows that

$$\rho(\cdot \mid |T| = N) = \nu_N^{(0)},$$

and hence the local limit $\nu^{(0)}$ in this case is supported on single spine trees $\hat{T}$ as before, but with branches grafted onto the spine vertices that are i.i.d. according to $\rho$ [18]. Since $\rho$ is associated with a critical BGW process, the volume growth exponent in this case turns out to be $d_h = 2$. More precisely, it is proven in [19], see also [9] for related results, that there exist constants $C_1', C_2' > 0$ and $r_0(\hat{T}) \in \mathbb{N}$ for $\nu^{(0)}$-a.e. $\hat{T}$, such that

$$C_1' \cdot r^2 (\ln r)^{-2} \leq |B_r(\hat{T})| \leq C_2' \cdot r^2 \ln r, \quad \text{for } r \geq r_0(\hat{T}).$$

4 The case $\mu > 0$

4.1 The partition function

In this subsection we determine the asymptotic behaviour of the partition functions $Z_N^{(\mu)}$ for large $N$, in case $\mu > 0$, which has previously been obtained in [21] (see Sect. 7) in a slightly different context. For later purposes we provide a proof of this result. Note, however, that the order of our remainder term deviates from the one stated in [21].

**Proposition 4.1** For each $\mu > 0$ it holds for any $\delta \in [0, \frac{1}{6}]$ that

$$Z_N^{(\mu)} = (e^\mu - 1) \sqrt{\frac{\pi}{2}} e^{-AN^3 N^{-\frac{5}{6}} 4N (1 + O(N^{-\delta}))}, \quad (50)$$

for $N$ large, where

$$A = 3 \left( \frac{\pi \mu}{2} \right)^{\frac{2}{3}} \quad \text{and} \quad B = 3 \left( \frac{\mu^2}{4\pi} \right)^{\frac{2}{3}}.$$

**Proof** Rewriting

$$Z_N^{(\mu)} = (1 - e^{-\mu}) W_N + C_N e^{-\mu(N+1)}, \quad (51)$$
where

$$W_N = \sum_{m=1}^{N} e^{-\mu m} A_{m,N},$$

and noting that the last term in (51) is exponentially suppressed compared to the right-hand side of (50), as a consequence of (9), we need to show that the asymptotic behaviour of $W_N$ is given by (50) with the factor $1 - e^{\mu}$ replaced by $e^{\mu}$. In order to achieve this we use a saddlepoint argument the details of which are as follows.

We first consider the contribution $\tilde{W}_N$ to $W_N$ obtained by retaining only the first term in the sum (17) defining $A_{m,N}$, and write it as

$$\tilde{W}_N := 4^N \sum_{m=2}^{N} \frac{e^{\mu m}}{m+1} \tan^2 \frac{\pi}{m+1} e^{-f_N(m+1)},$$

where

$$f_N(t) = \mu t + N \ln \left(1 + \tan^2 \frac{\pi}{t}\right), \quad t > 2.$$  

Since

$$f_N'(t) = \mu - 2\pi N \frac{1}{t^2} \tan \frac{\pi}{t}$$

is a strictly increasing function of $t$ that maps $]-\infty, 2[$ onto $]-\infty, \mu[$ it follows that $f_N$ has a unique minimum $t_0$ determined as a smooth function of $\frac{\mu}{N}$ by

$$\frac{\mu}{N} = 2\pi \frac{1}{t_0^2} \tan \frac{\pi}{t_0}.$$  

Clearly, $t_0 \to \infty$ as $N \to \infty$ for fixed $\mu$ and by Taylor expanding $\tan \frac{\pi}{t_0}$ on the right-hand side, we get

$$\frac{\mu}{2\pi^2 N} = \frac{1}{t_0^3} \left(1 + \frac{\pi^2}{3t_0^2} + O\left(\frac{1}{t_0^4}\right)\right)$$

which gives

$$t_0 = \left(\frac{2\pi^2 N}{\mu}\right)^{\frac{1}{3}} + O\left(N^{-\frac{1}{3}}\right).$$  

Here, the leading term on the right-hand side is the unique minimum of the function

$$g_N(t) = \mu t + N \frac{\pi^2}{t^2}, \quad t > 0,$$
which is obtained by Taylor expanding the ln-term in (54) to first order in $\frac{1}{t}$, and we thus have

$$f_N(t) = g_N(t) + NO \left(\frac{1}{t^4}\right)$$  \hspace{1cm} (56)

for $t$ large. Let $x_0$ denote the minimum of $g_1$, i.e.

$$x_0 = \left(\frac{2\pi^2}{\mu}\right)^{\frac{1}{3}},$$  \hspace{1cm} (57)

and observe that

$$g_N(t) = N^{\frac{1}{3}} g_1(N^{-\frac{1}{3}} t).$$  \hspace{1cm} (58)

Given $\delta \in [0, \frac{1}{6}]$, we now first estimate the contribution to the sum in (53) from terms with $m + 1 \geq t_0 + N^{\frac{1}{3}} - \delta$. For such $m$ we have

$$f_N(m + 1) \geq f_N(t_0 + N^{\frac{1}{3}} - \delta) = N^{\frac{1}{3}} g_1 \left(x_0 + N^{-\delta} + O(N^{-\frac{2}{3}})\right) + NO(N^{-\frac{4}{3}}),$$

by use of (55), (56) and (58). Inserting the Taylor expansion

$$g_1(x) = A + B(x - x_0)^2 + O(|x - x_0|^3),$$  \hspace{1cm} (59)

where

$$A = g_1(x_0) = 3 \left(\frac{\pi \mu}{2}\right)^{\frac{2}{3}} \quad \text{and} \quad B = \frac{1}{2} g''_1(x_0) = 3 \left(\frac{\mu^2}{4\pi}\right)^{\frac{2}{3}},$$

we thus obtain

$$f_{N+1}(m + 1) \geq AN^{\frac{1}{3}} + BN^{\frac{1}{3} - 2\delta} + O(N^{-\frac{1}{3}}) + O(N^{\frac{1}{3} - 3\delta}).$$

Assuming $\delta > \frac{1}{9}$, it follows that

$$\sum_{t_0 + N^{\frac{1}{3}} - \delta \leq m + 1 \leq N+1} \frac{1}{m+1} \tan^2 \frac{\pi}{m+1} e^{-f_N(m+1)} \leq \text{cst} \cdot Ne^{-AN^{\frac{1}{3}} - BN^{\frac{1}{3} - 2\delta}}.$$  \hspace{1cm} (60)

By the same arguments, this bound holds for the sum over $m + 1 \leq t_0 - N^{\frac{1}{3}} - \delta$ as well.

In the remaining range $t_0 - N^{\frac{1}{3}} - \delta < m + 1 < t_0 + N^{\frac{1}{3}} - \delta$ we set

$$x_m = N^{-\frac{1}{3}} (m + 1),$$  \hspace{1cm} (61)
such that \( x_m = x_0 + O(N^{-\delta}) \) by (55). Using (56) and (59), we then get
\[
f_N(m + 1) = N^\frac{1}{2} g_1(x_m) + NO(N^{-\frac{3}{4}})
= N^\frac{1}{2} \left( A + B(x_m - x_0)^2 + O(N^{-3\delta}) \right) + O(N^{-\frac{1}{2}}).
\]
Since \( \delta > \frac{1}{6} \) this implies
\[
e^{-f_N(m+1)} = e^{-AN^\frac{1}{3}} e^{-B(N^\frac{1}{6}(x_m-x_0))^2} \left( 1 + O(N^{\frac{1}{3}-3\delta}) \right).
\]
Now, note that for \( m \) in the range under consideration it holds that
\[
\frac{1}{m+1} \tan^2 \frac{\pi}{m+1} = \frac{1}{N^3 x_0^3} \left( 1 + O(N^{-\delta}) \right), \tag{62}
\]
such that we obtain
\[
\sum_{|m+1-t_0|<N^{\frac{1}{3}-\delta}} \frac{1}{m+1} \tan^2 \frac{\pi}{m+1} e^{-f_N(m+1)}
= \frac{\pi^2}{x_0^3} \frac{1}{N} e^{-AN^\frac{1}{3}} \sum_{|m+1-t_0|<N^{\frac{1}{3}-\delta}} e^{-B(N^\frac{1}{6}(x_m-x_0))^2} \left( 1 + O(N^{\frac{1}{3}-3\delta}) \right), \tag{63}
\]
}\[
\sum_{|m+1-t_0|<N^{\frac{1}{3}-\delta}} e^{-B(N^\frac{1}{6}(x_m-x_0))^2} - \sqrt{\frac{\pi}{B}} \leq O(N^{-\delta}),
\]
and so (63) implies that
\[
\sum_{|m+1-t_0|<N^{\frac{1}{3}-\delta}} \frac{1}{m+1} \tan^2 \frac{\pi}{m+1} e^{-f_N(m+1)}
= \sqrt{\frac{\pi}{B} x_0^3} e^{-AN^\frac{1}{3}} N^{-\frac{5}{6}} \left( 1 + O(N^{\frac{1}{3}-3\delta}) \right). \tag{64}
\]
Taking into account (60) and (64) we have shown that
\[
\tilde{W}_N = e^{\mu} \sqrt{\frac{\pi}{B} x_0^3} e^{-AN^\frac{1}{3}} N^{-\frac{5}{6}} 4^N \left( 1 + O(N^{\frac{1}{3}-3\delta}) \right).
\]
Noting that
\[ \frac{\pi^2}{x_0^3} = \frac{\mu}{2}, \] (66)
and that the function \( \delta \to 3\delta - \frac{1}{3} \) maps \([\frac{1}{9}, \frac{1}{6}]\) onto \([0, \frac{1}{6}]\), this is precisely of the form claimed for \( W_N \).
It remains to show that \( W_N - \tilde{W}_N = o(\tilde{W}_N) \) as \( N \to \infty \). For this purpose, note that
\[
\sum_{2 \leq k \leq \lfloor \frac{m}{2} \rfloor} \frac{1}{m+1} \tan^2 \frac{\pi k}{m+1} \left( 1 + \tan^2 \frac{\pi k}{m+1} \right)^{-N} \leq \left( 1 + \tan^2 \frac{2\pi}{m+1} \right)^{-(N-1)},
\]
such that, by setting
\[
\tilde{W}_N = 4N \sum_{m=4}^N \left( 1 + \tan^2 \frac{2\pi}{m+1} \right)^{-(N-1)},
\]
we have
\[
\tilde{W}_N \leq W_N \leq \tilde{W}_N + U_N.
\] (67)
Evidently, we can apply the same procedure as above to estimate \( U_N \). By inspection, it is easily verified that this yields the bound
\[
U_N \leq \text{cst} \cdot e^{-h(y_0)N^{\frac{1}{3}}} N^{\frac{1}{3}} 4^N,
\] (68)
where
\[
h(y) = \mu y + \frac{(2\pi)^2}{y^2}, \quad y > 0,
\]
and \( y_0 \) is the unique minimum of \( h \) determined by
\[
\mu = 2\frac{(2\pi)^2}{y_0^3}.
\]
One finds that
\[
h(y_0) = 3(\pi \mu)^{\frac{2}{3}} > A,
\]
and hence it follows from (65) and (68) that
\[
U_N \leq \tilde{W}_N \cdot e^{-cN^{\frac{1}{3}}}.
\]
Trees with exponential height dependent weight

for \( N \) large, where \( c \) is a positive constant. Using (67), we conclude that \( \tilde{W}_N \) and \( W_N \) have the same asymptotic behaviour given by (65), and this completes the proof of the theorem. \( \square \)

4.2 Lower bounds on ball volumes and the local limit

We next establish lower bounds on the \( \nu_N^{(\mu)} \)-volume of balls that will allow us to prove weak convergence.

Lemma 4.2 Assume \( \mu > 0 \) and that \( T_0 \in T_{\text{fin}} \) has height \( r \), and set \( K = |D_r(T_0)| \). Given \( 0 < \epsilon < \frac{1}{K} \) and \( M \in \mathbb{N} \), it holds for any \( \delta \in ]0, \frac{1}{6}[ \) that

\[
\nu_N^{(\mu)}(B_{\frac{1}{\tau}}(T_0)) \geq e^{-\mu(r-1)} \frac{1}{4^{i(T_0) - K}} \sum_{R=1}^{K} \binom{K}{R} \frac{1}{(R-1)!} \left( \frac{\mu}{2} \right)^{R-1} \left( \sum_{S=1}^{M} C_{S-1}4^{-S} \right)^{K-R} (1 - \epsilon K)^{K} (1 + O(N^{-\delta}))
\]

for \( N \) large.

Proof Below we use \( O \) to indicate a generic \( O \)-function satisfying \( |O(x)| \leq \text{cst} \cdot |x| \) for \( x \) small enough, where the constant may depend on \( K \), \( M \) and \( \epsilon \).

By Lemma 2.3, the elements \( T \) of \( B_{\frac{1}{\tau}}(T_0) \) are obtained by grafting \( K \) trees \( T_1, \ldots, T_K \) onto \( T_0 \) at the \( K \) vertices of maximal height, see Fig. 2. We call \( T_1, \ldots, T_K \) the branches of \( T \). For \( T \in T_N \) we then have

\[
N = \sum_{i=1}^{K} |T_i| + |T_0| - K,
\]

and we shall call the branch \( T_i \) small if \( |T_i| \leq M \), while we call it large if \( |T_i| > \epsilon N \).

Note that, if \( N > \frac{M}{\epsilon} \), no \( T_i \) can be both small and large, and if \( N > K \cdot M + |T_0| - K \) there must be at least one large branch \( T_i \). Assuming \( N \) fulfills these inequalities in the following, let \( \Omega_{T_0,R} \), \( 1 \leq R \leq K \), denote the subset of \( B_{\frac{1}{\tau}}(T_0) \) consisting of trees \( T \) of size \( N \) whose large branches are precisely \( T_1, \ldots, T_R \). Since \( \nu_N^{(\mu)} \) restricted to \( B_{\frac{1}{\tau}}(T_0) \) is invariant under permutation of the branches we have

\[
\nu_N^{(\mu)}(B_{\frac{1}{\tau}}(T_0)) = \sum_{R=1}^{K} \binom{K}{R} \nu_N^{(\mu)}(\Omega_{T_0,R}) .
\]

Hence, we proceed to estimate \( \nu_N^{(\mu)}(\Omega_{T_0,R}) \).

Imposing the restriction \( h(T) \geq M + r - 1 \) on \( T \in \Omega_{T_0,R} \) ensures that the highest branch must be among the first \( R \) branches. Denoting the smallest \( i \) such that \( T_i \) has
maximal height by \( j \) and \( |T_i| \) by \( N_i \), we have

\[
v^{(\mu)}_N(\Omega_{T_0,R}) \geq \left( Z^{(\mu)}_N \right)^{-1} \sum_{m=M}^{\infty} e^{-\mu(m+r-1)} \sum_{N_{R+1},\ldots,N_K \leq M} C_{N_{R+1}} \cdots C_{N_K}
\]

\[
\sum_{j=1}^{R} \sum_{N_{1},\ldots,N_{R} = N-N_{R+1},\ldots,-N_{K} + |T_0|}^{N-N_{R+1},\ldots,-N_{K} + K} A_{m-1,N_1} \cdots A_{m-1,N_{j-1}} (A_{m,N_{j}} - A_{m-1,N_j}) A_{m,N_j+1} \cdots A_{m,N_R}
\]

\[
= e^{-\mu(r-1)} Z^{(\mu)}_N \sum_{m=M}^{\infty} \sum_{N_{R+1},\ldots,N_K \leq M} C_{N_{R+1}} \cdots C_{N_K}
\]

\[
\sum_{N_{1},\ldots,N_{R} = N-N_{R+1},\ldots,-N_{K} + |T_0|}^{N-N_{R+1},\ldots,-N_{K} + K} e^{-\mu^n} (A_{m,N_1} \cdots A_{m,N_K} - A_{m-1,N_1} \cdots A_{m-1,N_K}).
\]

(71)

Now, consider the last sum for fixed values of \( N_{R+1}, \ldots, N_K \) and set

\[ N' = N_{R+1} + \cdots + N_K + |T_0| - K. \]

With notation as in Sect. 4.1, a lower bound on this sum is obtained by restricting the sum to values of \( m \) fulfilling \( |x_m - x_0| \leq N^0 \), where \( 0 < \delta < \frac{1}{6} \). Hence, we proceed to further estimate this lower bound

\[
V_N := \sum_{|x_m - x_0| \leq N^{-\delta}} \sum_{N_{1},\ldots,N_{R} = N-N'}^{N_{1},\ldots,N_{R} > \epsilon N} e^{-\mu m} \prod_{s=1}^{N_{R+1}} A_{m,N_s} \left( 1 - \prod_{t=1}^{R} \frac{A_{m-1,N_t}}{A_{m,N_t}} \right).
\]

(72)

Recalling the definition (17) of \( A_{m,N} \), we write

\[
A_{m,N_s} = 4N_s \frac{1}{m+1} \tan^2 \frac{\pi}{m+1} \left( 1 + \tan^2 \frac{\pi}{m+1} \right)^{-N_s}
\]

\[
\left( 1 + \sum_{k=2}^{\lfloor \frac{m}{2} \rfloor} \frac{\tan^2 \frac{\pi}{m+1}}{\tan^2 \frac{\pi}{m+1} + \frac{\pi}{m+1}} \frac{\tan^2 \frac{\pi}{m+1}}{1 + \tan^2 \frac{\pi}{m+1}} N_s \right). \quad (73)
\]

Using that

\[
\frac{1 + \tan^2 \frac{\pi}{m+1}}{1 + \tan^2 \frac{\pi}{m+1}} \leq \frac{1 + \tan^2 \frac{\pi}{m+1}}{1 + \tan^2 \frac{2\pi}{m+1}} = e^{-\frac{3\pi^2}{(m+1)^2} \left( 1 + O\left( \frac{1}{m^2} \right) \right)}
\]

for \( 2 \leq k \leq \lfloor \frac{m}{2} \rfloor \), the sum in (73) is bounded from above by

\[
(m + 1)^3 e^{-\frac{3\pi^2(N_s-1)}{(m+1)^2} \left( 1 + O\left( \frac{1}{m^2} \right) \right)}.
\]
Since $m = x_0 N^{\frac{3}{4}} + O(N^{\frac{1}{3} - \delta})$ for the range of $m$ considered and $N_s > \epsilon N$, we get that

$$A_{m,N_s} = 4^{N_s} \frac{1}{m+1} \tan^2 \frac{\pi}{m+1} \left( 1 + \tan^2 \frac{\pi}{m+1} \right)^{-N_s} \left( 1 + O(e^{- \epsilon c N^{\frac{1}{3}}}) \right), \quad (74)$$

where $c > 0$ is a numerical constant. In particular, the first product in (72) takes the form

$$\prod_{s=1}^{R} A_{m,N_s} = 4^{N-N'} \left( \frac{1}{m+1} \tan^2 \frac{\pi}{m+1} \right)^R \left( 1 + \tan^2 \frac{\pi}{m+1} \right)^{-(N-N')} \left( 1 + O(e^{- \epsilon c N^{\frac{1}{3}}}) \right)^R. \quad (75)$$

To deal with the second product, we use

$$\frac{1 + \tan^2 \frac{\pi}{m+1}}{1 + \tan^2 \frac{\pi}{m}} = e^{\frac{\pi^2}{2m^2} - \frac{\pi^2}{2(m+1)^2} + O(\frac{1}{m^4})} = e^{- \frac{2\pi^2}{m(m+1)^2} + O(\frac{1}{m^4})} = e^{- \frac{\pi^2}{m} + O(N^{-1-\delta})} = e^{- \frac{\mu}{N} + O(N^{-1-\delta})},$$

which together with

$$\frac{m+1}{m} \tan^2 \frac{\pi}{m+1} = 1 + O(N^{-\frac{1}{3}})$$

and (74) yields

$$\prod_{t=1}^{R} \frac{A_{m-1,N_t}}{A_{m,N_t}} = e^{- \mu N_1 + \cdots + N_R} \left( 1 + O(N^{-\delta}) \right) = e^{- \mu} \left( 1 + O(N^{-\delta}) \right). \quad (76)$$

Inserting (75) and (76) into (72), the sum over $N_1, \ldots, N_R$ can be performed and yields a combinatorial factor

$$\binom{N - N' - R \lfloor \epsilon N \rfloor + R - 1}{R - 1} \geq \frac{N^{R-1}}{(R-1)!} \left( 1 - \frac{N' + R \lfloor \epsilon N \rfloor}{N} \right)^{R-1} \geq \frac{N^{R-1}}{(R-1)!} (1 - \epsilon R)^{R-1} \left( 1 + O(N^{-1}) \right).$$

Thus, we obtain

$$V_N \geq 4^{N-N'} \frac{1}{(R-1)!} e^{- \mu} \sum_{\lfloor x_m - x_0 \rfloor \leq N^{-\delta}} N^{R-1} e^{- \mu m} \left( \frac{1}{m+1} \tan^2 \frac{\pi}{m+1} \right)^R.$$
\[
\left(1 + \tan^2 \frac{\pi}{m+1}\right)^{-(N-N')} (1 - \epsilon K)^K \left(1 + O(N^{-\delta})\right) .
\]

Using now (62), the sum can be estimated by repeating the arguments leading to (64), and we arrive at
\[
V_N \geq 4^{N-N'} \frac{e^\mu - 1}{(R-1)!} \sqrt{\frac{\pi}{B}} \left(\frac{\pi^2}{x_0^2}\right)^R \cdot \frac{e^{-AN^{1/2}N^{-5/6}}}{(1 - \epsilon K)^K} K \cdot (1 - \epsilon K)^K \left(1 + O(N^{1/3-3\delta})\right) ,
\]
provided \(\delta > \frac{1}{9}\). Using this estimate in (71) as well as Theorem 4.1 and (66), the claimed inequality (69) follows from (70).

In the following, let \(\Xi(T_0; M, \epsilon)\) denote the large-\(N\) limit of the right-hand side of (69),
\[
\Xi(T_0; M, \epsilon) = e^{-\mu(r-1)4^{-|T_0|}2^{K+1}} \sum_{R=1}^{K} \left(\frac{K}{R}\right) \frac{1}{(R-1)!} \left(\sum_{S=1}^{M} C_{S-1}4^{-S}\right)^{K-R} (1 - \epsilon K)^K ,
\]
and set
\[
\Xi(T_0) := \lim_{\epsilon \to 0} \Xi(T_0; M, \epsilon) = e^{-\mu(r-1)4^{-|T_0|}2^{K+1}} \sum_{R=1}^{K} \left(\frac{K}{R}\right) \frac{\mu^{R-1}}{(R-1)!} , \tag{77}
\]
where we have also used that
\[
\sum_{S=1}^{\infty} C_{S-1}4^{-S} = X \left(\frac{1}{4}\right) = \frac{1}{2}.
\]

By comparison with Lemma 5.1 of [3] it is seen that
\[
\Xi(T_0) = \nu_P^{(\mu)}(B_1^r(T_0)) , \tag{78}
\]
where \(\nu_P^{(\mu)}\) is a Borel probability measure, called the infinite Poisson tree with parameter \(\mu\) in [3]. We shall provide more details on this measure below, but for now we only need to know of its existence to establish the following main result of this section.

**Theorem 4.3** For each \(\mu > 0\), the sequence \((\nu_N^{(\mu)}(\cdot))\) is weakly convergent to a Borel probability measure \(\nu^{(\mu)}\) on \(T\), characterized by
\[
\nu^{(\mu)}(B_1^r(T_0)) = e^{-\mu(r-1)4^{-|T_0|}2^{K+1}} \sum_{R=1}^{K} \left(\frac{K}{R}\right) \frac{\mu^{R-1}}{(R-1)!} , \tag{79}
\]
for any tree \(T_0 \in T_{}\) of height \(r \geq 1\), where \(K = |D_r(T_0)|\), and which equals the infinite Poisson tree with parameter \(\mu\) of Abraham et al. [3].
Proof For arbitrary $T_0 \in T_{\text{fin}}$ it follows from Lemma 4.2 and (78) that
\[
\liminf_{N \to \infty} \nu_N^{(\mu)}(B_{\frac{1}{r}}(T_0)) \geq \nu_{\text{Poi}}^{(\mu)}(B_{\frac{1}{r}}(T_0)).
\]
The claim now follows by applying the Portmanteau theorem as in the proof of Theorem 3.5. \hfill \Box

4.3 Properties of the local limit

Before formulating the basic decomposition result for the measure $\nu^{(\mu)}$, some further notational conventions are needed.

Given $n \in \mathbb{N}$, let the standard $(n-1)$-simplex $\Delta_n$ be defined by
\[
\Delta_n := \{ (x_1, \ldots, x_n) \mid x_1 + \cdots + x_n = 1, x_1, \ldots, x_n > 0 \}.
\]
Scaling by $\mu > 0$ we obtain the simplex
\[
\mu \cdot \Delta_n := \{ (\mu_1, \ldots, \mu_n) \mid \mu_1 + \cdots + \mu_n = \mu, \mu_1, \ldots, \mu_n > 0 \}.
\]
By $d\omega(\mu_1, \ldots, \mu_n)$ we shall denote the normalised Lebesgue measure on $\mu \cdot \Delta_n$.

For $(\mu_1, \ldots, \mu_n) \in \mu \cdot \Delta_n$, consider the product measure $\prod_{k=1}^n \nu^{(\mu_k)}$ on $T^n$ and note that, if $B_1, \ldots, B_n$ are balls in $T$, then $\prod_{k=1}^n \nu^{(\mu_k)}(B_1 \times \cdots \times B_n)$ is a continuous function of $(\mu_1, \ldots, \mu_n)$ by (79). For $r \in \mathbb{N}$, let $\mathcal{F}_r^n$, $r \in \mathbb{N}$, denote the collection of subsets of $T^n$ that can be written as countable unions of products of balls in $T$ with radius $\frac{1}{r}$ and set $\mathcal{F}_\infty^n = \bigcup_{r \in \mathbb{N}} \mathcal{F}_r^n$. As in the case $n = 1$ discussed in Remark 2.1(iii), $\mathcal{F}_\infty^n$ is a set algebra consisting of sets with empty boundary, and it generates the Borel $\sigma$-algebra $\mathcal{F}_\infty^n$ of $T^n$. It follows that $\prod_{k=1}^n \nu^{(\mu_k)}(A)$ is a measurable function of $(\mu_1, \ldots, \mu_n)$ for any $A \in \mathcal{F}_\infty^n$, and we obtain a well defined finitely additive set function $\lambda_0$ on $\mathcal{F}_\infty^n$ by setting
\[
\lambda_0(A) := \int_{\mu \cdot \Delta_n} \prod_{k=1}^n \nu^{(\mu_k)}(A) d\omega(\mu_1, \ldots, \mu_n), \quad A \in \mathcal{F}_\infty^n.
\]
Moreover, it follows from the monotone convergence theorem that $\lambda_0$ is countably additive on $\mathcal{F}_\infty^n$, and hence extends to a unique Borel probability measure on $T^n$ (see e.g. Theorem A in §13 in [22]). We shall denote this extension by
\[
\int_{\mu \cdot \Delta_n} d\omega \prod_{k=1}^n \nu^{(\mu_k)}.
\]
The right-hand side of Eq. (80) then equals the measure of $A$ for more general sets, such as open and closed sets, but we shall only need this expression for products of balls.
In the following theorem we make use of the measures just introduced as well as of the BGW measure \( \rho \) introduced in Remark 3.3. Note that, under the identification of \( B_{1}^{r}(T_{0}) \) with \( T_{0}^{D_{r}(T_{0})} \) implied by Lemma 2.3(ii) where \( T_{0} \in T_{\text{fin}} \) has height \( r \), the identity
\[
\rho(\cdot \mid B_{1}^{r}(T_{0})) = \prod_{i \in D_{r}(T_{0})} \rho_{i}, \quad \rho_{i} = \rho, \quad (81)
\]
holds as a consequence of (48) and (49), expressing that in the corresponding BGW process the events of birth by different individuals in any generation are i.i.d. according to \( \rho \).

**Theorem 4.4** Assume \( \mu > 0 \) and that \( T_{0} \in T_{\text{fin}} \) has height \( r \), and let \( K = |D_{r}(T_{0})| \).

Under the identification \( B_{1}^{r}(T_{0}) = T_{0}^{D_{r}(T_{0})} \) provided by Lemma 2.3(ii), the measure \( \nu(\mu) \) restricted to \( B_{1}^{r}(T_{0}) \) is given by
\[
\nu(\mu) \big|_{B_{1}^{r}(T_{0})} = e^{-\mu(r-h-2)4^{-|T_{0}|2^{K'+1}}} \sum_{D \subseteq D_{r}(T_{0})} \frac{\mu^{|D| - 1}}{(|D| - 1)!} \left( \int_{\mu \cdot \Delta_{|D|}} d\omega \prod_{i \in D} \nu(\mu_{i}) \right) \times \prod_{j \in D_{r}(T_{0}) \setminus D} \rho_{j}, \quad (82)
\]

where \( \rho_{j} = \rho \).

**Proof** It is sufficient to show that both sides of (82) coincide when applied to products of balls \( A = B_{1}^{r_{1}}(T_{1}) \times \cdots \times B_{1}^{r_{K}}(T_{K}) \) for arbitrary finite trees \( T_{1}, \ldots, T_{K} \), where \( h_{i} = h(T_{i}) \). Letting \( h = \max\{h_{1}, \ldots, h_{K}\} \) and noting that \( A \) can be decomposed into a disjoint union of balls of radius \( r + h - 1 \) around trees \( T \in B_{1}^{r}(T_{0}) \) of height \( r + h - 1 \) containing \( G(T_{1}, \ldots, T_{K}) \) as a subtree, it follows that we may assume \( h_{1} = \cdots = h_{K} = h \). Setting \( T = G(T_{1}, \ldots, T_{K}) \) and \( K' = |D_{r+h-1}(T)| \) we then have, by use of (79),
\[
\nu(\mu) \big|_{B_{1}^{r}(T_{0})} (A) = e^{-\mu(r-h-2)4^{-|T|2^{K'+1}}} \sum_{R=1}^{K'} \binom{K'}{R} \frac{\mu^{R-1}}{(R-1)!},
\]

which we rewrite as
\[
\nu(\mu) \big|_{B_{1}^{r}(T_{0})} (A) = e^{-\mu(r-h-2)4^{-|T|2^{K'+1}}} \sum_{D' \subseteq D_{r+h-1}(T) \setminus \emptyset} \frac{\mu^{|D'| - 1}}{(|D'| - 1)!}. \quad (83)
\]

For a given \( D' \), let \( D \subseteq D_{r}(T_{0}) \) denote the set of ancestors at height \( r \) in \( T \) of elements in \( D' \) and, for each \( i \in D \), let \( D_{i} \subseteq D_{h}(T_{i}) \) consist of the descendants of \( i \) in \( D' \). Then
\[ D_i \neq \emptyset \text{ and } D' = \bigcup_{i \in D} D_i . \] Using the relation

\[
\int_{\Delta_n} d\omega(x) \prod_{i=1}^n x_i^{m_i} = \frac{(n-1)!}{(m_1 + \cdots + m_n + n-1)!} \prod_{i=1}^n m_i !.
\]

for non-negative integers \( m_1, \ldots, m_n \), which is easy to establish by induction, a simple scaling argument, with \( \mu_i = \mu \cdot x_i \), yields

\[
\frac{\mu^{\mid D' \mid - 1}}{( \mid D' \mid - 1)!} = \frac{\mu^{\mid D \mid - 1}}{( \mid D \mid - 1)!} \int_{\mu^{-1}\Delta_n} d\omega \prod_{i=1}^{|D|} \mu_i^{\mid D_i \mid - 1}. 
\]

Substituting the last expression into (83) and using \( \mid T \mid + K = \mid T_0 \mid + \mid T_1 \mid + \cdots + \mid T_K \mid \) as well as \( K' = \mid D_h(T_1) \mid + \cdots + \mid D_h(T_K) \mid \), we obtain

\[
\nu(\mu) |_{B_{\frac{1}{r}}(T_0)}(A) = e^{-\mu(r-1)}4^{-\mid T_0 \mid}2^{k+1} \sum_{D \subseteq D_r(T), D \neq \emptyset} \frac{\mu^{\mid D \mid - 1}}{(\mid D \mid - 1)!} \int_{\mu^{-1}\Delta_n} d\omega \prod_{i \in D} \left( \sum_{D_i \subseteq D_h(T_i), D_i \neq \emptyset} e^{-\mu_i(h-1)}4^{-\mid T_i \mid}2^{\mid D_h(T_i) \mid} \frac{\mu_i^{\mid D_i \mid - 1}}{(\mid D_i \mid - 1)!} \right) \prod_{i \notin D} 4^{-\mid T_i \mid}2^{\mid D_h(T_i) \mid + 1}, \quad (84)
\]

which by comparison with (79) and (48) is seen to coincide with the measure of \( A \) with respect to the measure on the right-hand side of (82). This completes the proof of Theorem 4.4.

\[ \square \]

As a first consequence of Theorem 4.4 we note that, since \( \nu(\mu) \) is supported on \( T_\infty \) while \( \rho \) is supported on \( T_{\text{fin}} \), formula (82) provides an explicit decomposition of \( \nu(\mu) \) conditioned on the event \( \{ T \mid B_r(T) = T_0 \} \) into measures supported on trees that have a fixed spine up to height \( r \) labelled by the set \( D \) of vertices of infinite type at height \( r \). In order to elaborate further on this decomposition, we first determine the measure \( \tilde{\nu}(\mu) \) induced by \( \nu(\mu) \) on the set \( T_s \) of spine trees, i.e. \( \tilde{\nu}(\mu) \) is the pushforward of \( \nu(\mu) \) by the spine map \( \chi \),

\[
\tilde{\nu}(\mu)(A) = \nu(\mu)(\chi^{-1}(A)), \quad A \subseteq T_s \text{ a Borel set}. \quad (85)
\]

We shall use the notation

\[
B^s_a(T_s) := B_a(T_s) \cap T_s
\]

for the ball in \( T_s \) of radius \( a > 0 \) around a spine tree \( T_s \). Given \( T_s \) and \( r \in \mathbb{N} \), we have by Remark 2.1(i) that \( B^s_{\frac{1}{r}}(T^s) = B^-_{\frac{1}{r}}(T_0^s) \cap T^s \), where \( T_0^s = B_r(T^s) \) is a finite tree of height \( r \), all of whose leaves are at height \( r \). Moreover, a tree \( T \) in \( \chi^{-1}(B^s_{\frac{1}{r}}(T^s)) \) is
obtained by grafting arbitrary infinite trees onto the leaves of $T^s_0$ while grafting finite trees $T(i, n)$ in each sector $(i, n)$ of any vertex at height $< r$ different from the root, see Fig. 3. In this way, we have by Lemma 2.3(i) a homeomorphism

$$G : \chi^{-1}(B^s_\ell(T^s_0)) \to T^R_\infty \times T^\sigma_{\text{fin}},$$

(86)

where $R = |D_r(T^s)|$ is the number of leaves in $T^s_0$ and

$$\sigma = 2|T^s_0| - R - 1$$

(87)

is the total number of sectors associated with $T^s$, excluding those adjacent to the leaves.

**Theorem 4.5** Assume $\mu > 0$ and let $T^s_0$ be any finite tree of height $r$ with $R$ leaves, all of which are at height $r$. Then the following two statements hold.

(i)

$$\tilde{\nu}^{(\mu)}(B^s_\ell(T^s_0)) = e^{-(r-1)\mu} \frac{\mu^{R-1}}{(R-1)!}, \quad r \geq 1.$$  

(88)

(ii) Under the identification (86) we have

$$\nu^{(\mu)}(\cdot \mid \chi^{-1}(B^s_\ell(T^s_0))) = \left( \int_{\mu \Delta R} d\omega \prod_{j=1}^R \nu^{(\mu_j)} \right) \times \prod_{(i, n)} \rho(i, n),$$

(89)

where $\rho(i, n) = \rho$. Equivalently, conditioning on the event $\{T \mid B_r(\chi(T)) = T^s_0\}$ renders the finite branches $T(i, n)$ i.i.d. according to $\rho$, whereas the joint distribution of the infinite branches is independent thereof and determined by $\int_{\mu \Delta R} d\omega \prod_{j=1}^R \nu^{(\mu_j)}$.

**Proof** In order to compute the left-hand side of (88), we use the preceding notation and set $T'(i, n) = B_{r-h_i+1}(T(i, n))$, where $h_i$ is the height of $i$ in $T^s_0$. Letting $T_0$ be the tree of height $r$ obtained by grafting $T'(i, n)$ onto $T^s_0$ at $(i, n)$ instead of $T(i, n)$ in each sector, Theorem 4.4 shows, for fixed $T'(i, n)$, that the branches of $T'(i, n)$ grafted on the vertices of $T'(i, n)$ at (maximal) height $r - h_i + 1$ are i.i.d. according to $\rho$. If $l(i, n)$ denotes the number of vertices in $T'(i, n)$ at height $r - h_i + 1$, the number of vertices in $T_0$ at height $r$ equals $K = R + \sum (i, n) l(i, n)$. Together with (87) this implies, with notation as in Theorem 4.4, that

$$4^{-|T_0|} 2^{K+1} = 4^{-|T_0'|} 2^{R+1} \prod_{(i, n)} 4^{-|T'_{i, n}|+1} 2^{l(i, n)} = \prod_{(i, n)} 4^{-|T'_{i, n}|} 2^{l(i, n)+1}.$$
Trees with exponential height dependent weight

Taking into account \((48)\) and \((81)\) and the identification \((86)\), it then follows from Theorem 4.4 that

\[
\nu(\mu) \big|_{\chi^{-1}(B_{1/\tau}(T_0^s))} = e^{-(r-1)\mu} \frac{\mu^{R-1}}{(R-1)!} \left( \int_{\mu \Delta R} d\omega \prod_{j=1}^{R} \nu(\mu_j) \right) \times \prod_{(i,n)} \rho(i,n),
\]

which clearly implies both statements of the theorem.

It is worth observing that the right-hand side of \((88)\) only depends on the height \(r\) and the number of vertices at height \(r\) in \(T_0^s\), but not on the structure of \(T_0^s\) below height \(r\). As a consequence it follows, in particular, that the random variables \(|D_r(T^s)|\), \(r \in \mathbb{N}\), constitute a Markov process. The following corollary shows that it is a random walk on the positive integers with Poisson distributed increments.

**Corollary 4.6** The random variables \(\zeta_r\), \(r \in \mathbb{N}\), on \(T^s\) defined by

\[
\zeta_r(T^s) = |D_{r+1}(T^s)| - |D_r(T^s)|
\]

are i.i.d. according to

\[
\tilde{\nu}(\mu)(\zeta_r = n) = e^{-\mu} \cdot \frac{\mu^n}{n!}, \quad n = 0, 1, 2, \ldots.
\]

In particular, the number \(|D_r(T^s)|\) of vertices at height \(r \geq 2\) with respect to \(\tilde{\nu}(\mu)\) is Poisson distributed with mean \((r-1)\mu\), i.e.

\[
\tilde{\nu}(\mu)(|D_r(T^s) = R|) = e^{-(r-1)\mu} \frac{((r-1)\mu)^{R-1}}{(R-1)!}, \quad R \geq 1.
\]
Proof By (88) we have
\[ \tilde{\nu}(\mu)(|D_r(T^s)| = R) = e^{-(r-1)\mu} \frac{\mu^{R-1}}{(R-1)!} \cdot M_{R,r}, \]
where \( M_{R,r} \) is the number of finite trees \( \tilde{T}_0 \) with height \( r \) and with \( R \) leaves, all of which are at height \( r \). This number can be determined recursively by noting that, given \( R \geq 1 \) vertices at height \( r \) and \( R' \geq 1 \) vertices at height \( r - 1 \), the number of ways of connecting them with non-intersecting edges such that every vertex at height \( r - 1 \) has at least one offspring is given by
\[ \frac{R-1}{R'-1} \cdot \tilde{\nu}(\mu)(|D_{r-1}(T^s)| = R'), \]
interpreted as 0 when \( R < R' \). Hence, (88) implies that
\[ \tilde{\nu}(\mu)(|D_r(T^s)| = R, |D_{r-1}(T^s)| = R') = e^{-(r-1)\mu} \frac{\mu^{R-1}}{(R-1)!(R-R')!} \cdot M_{R',r-1} \]
and consequently
\[ \tilde{\nu}(\mu)(|D_r(T^s)| = R) = e^{-\mu} \frac{\mu^{R-R'}}{(R-R')!} \tilde{\nu}(\mu)(|D_{r-1}(T^s)| = R'), \]
Since \( |D_r(T^s)|, r \in \mathbb{N} \), is a Markov process as already mentioned, the first statement of the corollary follows. The second one then follows by taking an \((r-1)\)-fold convolution of the Poisson distribution (90) and using that \( |D_1(T^s)| = 1 \). This completes the proof.

\[ \square \]

Remark 4.1 As mentioned, the infinite Poisson tree \( \nu(\mu) \) first appears in [3], constructed as a limit of the critical BGW tree with offspring distribution given by (47) conditioned on the generation size \( |D_n| = a_n \) for a sequence \( a_n, n \in \mathbb{N} \), such that \( \lim_{n \to \infty} a_n n^{-2} = \mu \), see Proposition 5.3 in [3]. The Poisson nature of \( |D_{r+1}| - |D_r|, r \in \mathbb{N} \), is also exploited there. We remark that, with slightly more work, Lemma 4.2 above also yields tightness of the sequence \( \nu_N(\mu), N \in \mathbb{N} \), and hence can be used to establish existence of the limit \( \nu(\mu) \) without relying on results of [3].

Denoting expectation w.r.t. \( \tilde{\nu}(\mu) \) by \( \tilde{\mathbb{E}}_\mu \), the following result on the average volume growth of \( T^s \) now easily follows.

Corollary 4.7 The following relations hold.

(i) \( \tilde{\mathbb{E}}_\mu(|D_{r}|) = \mu(r-1), \)
(ii) \( \tilde{\mathbb{E}}_\mu(|B_{r}|) = \frac{1}{2} \mu r (r-1) + 1, \)
(iii) \( \lim_{r \to \infty} \frac{|D_r(T^s)|}{r} = \mu \text{ for } \tilde{\nu}(\mu)-a.e. \ T^s, \)
(iv) \( \lim_{r \to \infty} \frac{|B_r(T^s)|}{r^2} = \frac{1}{2} \mu \text{ for } \tilde{\nu}(\mu)-a.e. \ T^s. \)

Proof The relation (i) follows immediately from (90), and (iii) follows by applying the strong law of large numbers (see e.g. [12]). The second relation follows from (i) and (44), and similarly (iv) follows from (iii). \[ \square \]
Remark 4.2 Corollary 4.7(iii) shows that, for $\mu > 0$, it holds that $d_h = 2$ for $\tilde{\nu}(\mu)$-a.e. $T^s$.

Next we note the following result on the average volume growth of $T$ with respect to $\nu(\mu)$.

Corollary 4.8 The following statements hold.

(i) $\mathbb{E}_\mu(|D_r|) = \mu r^2 \left(1 + O\left(\frac{1}{r}\right)\right)$,
(ii) $\mathbb{E}_\mu(|B_r|) = \frac{1}{3} \mu r^3 \left(1 + O\left(\frac{1}{r}\right)\right)$.

Proof It is clear that (ii) follows from $i)$ and (44). To establish (i), we use that $\mathbb{E}_\rho(|D_r|) = 1$ for all $r$, since $\rho$ is associated with a critical BGW process. Using this in (89) together with $D_r(T) = D_r(\chi(T)) + \sum_{(i,n)} |D_{r-h_i+1}(T_{(i,n)})|$, with notation as above, we get

$$\mathbb{E}_\mu(|D_r| | \chi^{-1}(B^*_r(T^s))) = \sigma(B_r(T^s)) + |D_r(T^s)|,$$

where $\sigma$ is given by (87). Integrating over $T^s$ then gives

$$\mathbb{E}_\mu(|D_r|) = \mathbb{E}_\mu(2|B_r| - 1),$$

and so relation (i) follows from Corollary (4.7)(ii).

We conclude by stating the following almost sure result on the volume growth of $T$ w.r.t. $\nu(\mu)$, whose proof is deferred to the appendix.

Theorem 4.9 For each $\mu > 0$, there exist constants $C_1''$, $C_2'' > 0$ and for $\nu(\mu)$-almost every $T \in T$ a number $r_0(T) \in \mathbb{N}$, such that

$$C_1'' \cdot r^3 \leq |B_r(T)| \leq C_2'' \cdot r^3 \ln r, \quad r \geq r_0(T).$$

(91)

In particular, it holds that $d_h = 3$ for $\nu(\mu)$-a.e. tree $T$.

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Appendix

Proof of Proposition 3.7

Note first, that the lower bounds in (45) are obvious and that the second upper bound follows from the first one and (44). Hence, it suffices to establish the first upper bound.

Let \( f \) be the generating function for the offspring probabilities given by (39),

\[
    f(x) = \sum_{n=0}^{\infty} p(n)x^n = \frac{1 - X(g_c(k))}{1 - xX(g_c(k))} = \frac{1}{1 - m(x-1)},
\]

where the last equality follows from (40), and let \( f_r \) denote the probability generating function for the size of the \( r \)th generation of the corresponding BGW process,

\[
    f_r(x) = \mathbb{E}(x^{D_r}), \quad r \geq 1.
\]

It is a standard result (see e.g. [8]) that

\[
    f_{r+1} = f \circ f_r \quad \text{for} \quad r \geq 2 \quad \text{and} \quad f_2 = f. \tag{93}
\]

Similarly, let \( g_r \) be the probability generating function for the size of the \( r \)’th generation w.r.t. the local limit measure \( \nu(\mu) \). Using the same notation as in the proof of Corollary 3.6 for the corresponding single spine tree \( \hat{T} \) and its branches grafted to the left and right along the spine, it follows from (43) that

\[
    g_r(x) = \mathbb{E}_\mu(x^{D_r}) = x \prod_{k=2}^{r} f_k(x)^2. \tag{94}
\]

We claim there exist constants \( b > 1 \) and \( 0 < c < 1 \) such that

\[
    f_r(x) \leq 1 + c^{r-1}(x - 1), \quad 1 \leq x \leq b, \quad r \geq 2. \tag{95}
\]

Indeed, by (92) we have

\[
    f(x) = 1 + m(x - 1) + \frac{m^2(x - 1)^2}{1 - m(x - 1)},
\]

and since \( 0 < m < 1 \) we can choose \( b > 1 \) such that

\[
    0 < \frac{m^2(b - 1)}{1 - m(b - 1)} < 1 - m,
\]

and hence

\[
    f(x) \leq 1 + c(x - 1) \quad \text{for} \quad 1 \leq x \leq b,
\]
where

\[ c := m + \frac{m^2(b-1)}{1-m(b-1)} < 1. \]

A simple inductive argument using (93) then implies (95).

Combining (94) and (95), we obtain

\[ gr(x) \leq x \prod_{k=2}^{r} e^{2ck^{k-1}(x-1)} \leq b \cdot e^{2 \frac{x-1}{1-c}} \]

for \( 1 \leq x \leq b \). By the Chebychev inequality

\[ \mathbb{E}_\mu(e^{\theta|D_r|}) \geq e^{\theta \lambda \nu(\mu)(\{|D_r| \geq \lambda\})}, \]

it therefore follows by choosing \( \theta = \ln b \) that

\[ \nu(\mu)(\{|D_r| \geq \lambda\}) \leq gr(b) \cdot b^{-\lambda} \leq e^{2 \frac{b-1}{1-c}} \cdot b^{1-\lambda}, \]

for any \( \lambda > 0 \). Thus, choosing \( \lambda = C_1 \cdot \ln r \), we get that

\[ \sum_{r=1}^{\infty} \nu(\mu)(\{|D_r| \geq C_1 \cdot \ln r\}) < \infty, \]

provided \( C_1 > (\ln b)^{-1} \). Invoking the Borel-Cantelli lemma, the first upper bound in (45) then holds for \( r \) sufficiently large for a.e. \( \hat{T} \), which concludes the proof.

**Proof of Theorem 4.9**

**Upper bound.** We employ a method similar to the one of the previous proof. Note first that the generating function \( f \) for the offspring probabilities (47) (corresponding to \( m = 1 \) in the previous proof) fulfills

\[ f(x) = \frac{1}{2-x} = x + (x-1)^2 + \frac{(x-1)^2}{2-x} \leq x + 3(x-1)^3, \quad \text{for } 1 \leq x \leq \frac{3}{2}. \]

The probability generating function for \(|D_r|\) w.r.t. the corresponding BGW process is then given by (93). It is easily shown by induction that

\[ f_r(x) \leq x + 12r(x-1)^2, \quad 0 \leq x-1 \leq \frac{1}{12(r-1)}. \] (96)

Consider now \( T \in \chi^{-1}(B_1(T^s)) \) for given \( r \geq 1 \) and \( T^s \in T^s \). As previously, let \( T_{(i,n)} \) denote the finite branch of \( T \) grafted in sector \((i, n)\) of \( T^s_0 = B_r(T^s) \) and let
$D^f_r(T)$ denote the set of vertices of finite type in $T$ at height $r$, i.e.

$$|D^f_r(T)| = \sum_{(i,n)} |D_{r-h_i+1}(T_{(i,n)})| .$$

Using that the branches $T_{(i,n)}$ are i.i.d. according to $\rho$ by Theorem 4.5, it follows that

$$\mathbb{E}_{\mu}(x^{|D^f_r|}) = \tilde{\mathbb{E}}_{\mu} \left( \prod_{k=1}^{r-1} f_{r-h+1}(x)^{|D_h(T^s)|+|D_{h+1}(T^s)|} \right),$$

since the number of sectors associated to $T^s$ at height $h$ equals $|D_h(T^s)|+|D_{h+1}(T^s)|$.

Writing $|D_h| = 1 + \xi_1 + \xi_2 + \cdots + \xi_{h-1}$ on $T^s$, then gives

$$\mathbb{E}_{\mu}(x^{|D^f_r|}) = \left( \prod_{k=2}^{r} f_k(x)^2 \right) \mathbb{E}_{\mu} \left( \prod_{l=1}^{r-2} \left( f(x) \prod_{m=l+1}^{r-1} f_{r-m+2}(x) f_{r-m+1}(x) \right)^{\xi_l} f(x)^{\xi_{r-1}} \right)$$

Here, the right-hand side can be computed by use of Corollary 4.6 and

$$\tilde{\mathbb{E}}_{\mu}(r^\xi) = e^{\mu(t-1)}, \quad t \in \mathbb{R},$$

yielding the result

$$\mathbb{E}_{\mu}(x^{|D^f_r|}) = e^{\mu(f(x)-1)} \left( \prod_{k=2}^{r} f_k(x)^2 \right) \prod_{l=1}^{r-2} e^{\mu(f_{r-l+1}(x) \prod_{m=l+2}^{r-1} f_m(x)^2 - 1)} . \quad (97)$$

Setting $x = e^\theta$ and using Chebychev’s inequality, we have

$$v^{(\mu)}(|D^f_r| \geq \lambda r^2) \leq e^{-\theta\lambda r^2} \mathbb{E}_{\mu}(e^{\theta|D^f_r|}) , \quad (98)$$

for $\theta, \lambda > 0$. From this, a useful bound is obtained by choosing $\theta = r^{-2}$ and bounding the factors on the right-hand side of (97) as follows. Noting that (96) implies

$$f_k(x) \leq 1 + 2(x - 1) \leq e^{2(x-1)} , \quad 0 \leq x - 1 \leq \frac{1}{12r} \quad \text{and} \quad 2 \leq k \leq r , \quad (99)$$

we get

$$f_k(e^{r^{-2}}) \leq e^{4r^{-2}} \quad \text{for} \quad 2 \leq k \leq r \quad \text{and} \quad r \geq 24.$$
Hence,

\[
f_{r-l+1}(e^{-s}) \prod_{m=2}^{r-l} f_m(e^{-s})^2 - 1 \leq e^{8r^{-1}} - 1 \leq 16r^{-1}, \quad 1 \leq l \leq r - 2,
\]

and consequently we obtain for the last product in (97) the bound

\[
\prod_{l=1}^{r-2} e^\mu \left( f_{r-l+1}(e^{-s}) \prod_{m=2}^{r-l} f_m(e^{-s})^2 - 1 \right) \leq e^{16\mu},
\]

for \( r \geq 24 \). Since these estimates likewise show that the first two factors in (97) are bounded by constants for \( x = e^{-s} \), it follows from (97) and (98) that

\[
\nu(\mu) \left( |D_r f| \geq \lambda r^2 \right) \leq C \cdot e^{-\lambda},
\]

where \( C > 0 \) is a constant independent of \( r \).

Choosing \( \lambda = a \log r \), where \( a > 1 \), this implies that

\[
\sum_{r=1}^{\infty} \nu(\mu) \left( |D_r f| \geq ar^2 \log r \right) < \infty.
\]

Hence, by the Borel–Cantelli lemma, there exists \( r_0(T) \in \mathbb{N} \) for \( \nu(\mu) \)-a.e. \( T \) such that

\[
|D_r f(T)| \leq ar^2 \log r \quad \text{for all} \quad r \geq r_0(T).
\]

Since \( |D_r(T)| = |D_r f(T)| + |D_r(\chi(T))| \), it follows from Corollary 4.7(iii) and (44) that the upper bound in (91) holds for any \( C''_2 > \frac{1}{3} \).

**Lower bound.** For \( N \in \mathbb{N} \), let

\[
\mathcal{A}_N^x = \{ T^x \in T^x \mid \frac{1}{2} \mu(r + 1) \leq |D_r(T^x)| \leq 3\mu r \quad \text{for all} \quad r \geq N \}.
\]

Then \( \mathcal{A}_1^x \subseteq \mathcal{A}_2^x \subseteq \mathcal{A}_3^x \subseteq \ldots \) and

\[
\tilde{\nu}(\mu)(\mathcal{A}_N^x) \to 1 \quad \text{as} \quad N \to \infty
\]

as a consequence of Corollary 4.7(iii).

Let \( N \) be fixed, as well as \( r \geq 3N \). Given \( T^x \in \mathcal{A}_N^x \), set \( T_0^x = B_r(T^x) \) and note that

\[
\frac{1}{2} \mu \left( \frac{r}{3} \right) \leq |D_r(T_0^x)| \leq \mu r.
\]
Since $T^s_0$ has no leaves at heights less than $r$ there exist disjoint paths $\omega_i$, $i \in D_{\lfloor \frac{r}{3} \rfloor}(T^s_0)$, in $T^s_0$ of length $\lfloor \frac{r}{3} \rfloor$ such that $\omega_i$ connects $i$ to a vertex $j_i \in D_{\lfloor \frac{r}{3} \rfloor}(T^s_0)$. Let us denote the vertices in $\omega_i$ by $v^i_0, v^i_1, \ldots, v^i_{\lfloor \frac{r}{3} \rfloor}$, such that $v^i_0 = i$ and $v^i_{\lfloor \frac{r}{3} \rfloor} = j_i$. Now, consider $T \in \chi^{-1}(B^s_{\frac{r}{3}}(T^s_0))$ and denote, for each $i \in D_{\lfloor \frac{r}{3} \rfloor}(T^s_0)$, by $T^i_k$, $0 \leq k \leq \lfloor \frac{r}{3} \rfloor$, the branches of $T$ grafted in sectors $(v^i_k, 1)$ along $\omega_i$, respectively. With respect to $\nu(\mu)$, conditioned on the set $\chi^{-1}(B^s_{\frac{r}{3}}(T^s_0))$, the branches $T^i_k$ are i.i.d. according to $\rho$, by Theorem (4.5)(ii). Hence, setting

$$q(\lambda) = \nu(\mu) \left( \sum_{k=0}^{\lfloor \frac{r}{3} \rfloor} \left| B_{\lfloor \frac{r}{3} \rfloor}(T^i_k) \right| \leq \lambda \left( \left\lfloor \frac{r}{3} \right\rfloor \right)^2 \right) \chi^{-1}(B^s_{\frac{r}{3}}(T^s_0)) \right) \right)$$

for $\lambda > 0$, we have that $q(\lambda)$ is independent of $i \in D_{\lfloor \frac{r}{3} \rfloor}(T^s_0)$ and that

$$q(\lambda) \leq \left( \rho \left( \left\{ \left| B_{\lfloor \frac{r}{3} \rfloor} \right| \leq \lambda \left( \left\lfloor \frac{r}{3} \right\rfloor \right)^2 \right\} \right) \right)^{\lfloor \frac{r}{3} \rfloor}.$$ (103)

We next establish an upper bound on the last expression by making use of the fact that there exists a constant $c' > 0$ such that

$$\rho(\{ |B_R| \geq R^2 \}) \geq \frac{c'}{R}, \quad R \geq 1.$$ (104)

This is a consequence of the following estimates:

$$\rho(\{ |B_R| \geq R^2 \}) = 2 \sum_{|T| \geq R^2} 4^{-|T|} + 2 \sum_{h(T) < R} 4^{-|T|} \left( h(T) \right)^2$$

$$\geq 2 \sum_{|T| \geq R^2} 4^{-|T|}$$

$$= 2 \sum_{N \geq R^2} A_{R,N} 4^{-N}$$

$$\geq 2 \sum_{N \geq R^2} \frac{1}{R+1} \tan^2 \frac{\pi}{R+1} \left( 1 + \tan^2 \frac{\pi}{R+1} \right)^{-N}$$

$$\geq \frac{2}{R+1} \sum_{N \geq R^2} \frac{\pi^2}{(R+1)^2} \left( 1 + \tan^2 \frac{\pi}{R+1} \right)^{-N},$$ (105)

where we have used (48) and (17), and we note that the last sum is clearly finite for all $R \geq 1$ and converges to $\int_{\pi^2}^{\infty} e^{-x} \, dx = e^{-\pi^2}$ as $R \to \infty$, thus proving (104).
For $0 < \lambda < 1$ it then follows that

$$\rho(||B_R| \leq \lambda R^2|) \leq \rho(||B_{[\sqrt{\lambda}R]}^+| \leq \lambda R^2|) \leq 1 - \frac{c'}{\sqrt{\lambda}R + 1}$$

for $R \geq 1$, and hence (103) implies

$$q(\lambda) \leq e^{-\frac{c'}{2\lambda R}},$$

(106)

if $\lfloor \frac{r}{\lambda} \rfloor \geq \lambda^{-\frac{1}{2}}$.

Returning to $T \in \chi^{-1}(B^s_{\frac{r}{\lambda}}(T^s_{0}))$, we have

$$|B_r(T)| \leq \frac{\lambda}{4} \left\lfloor \frac{r}{\lambda} \right\rfloor^3$$

$$\Rightarrow \sum_{i \in D_{\lfloor \frac{r}{\lambda} \rfloor}(T^s_{0})} \sum_{k=0}^{\lfloor \frac{r}{\lambda} \rfloor} |B_{\lfloor \frac{r}{\lambda} \rfloor}(T^s_k)| \leq \frac{\lambda}{4} \left\lfloor \frac{r}{\lambda} \right\rfloor^3$$

$$\Rightarrow \left\{ i \in D_{\lfloor \frac{r}{\lambda} \rfloor}(T^s_{0}) \left| \sum_{k=0}^{\lfloor \frac{r}{\lambda} \rfloor} |B_{\lfloor \frac{r}{\lambda} \rfloor}(T^s_k)| \leq \frac{\lambda}{\mu} \left\lfloor \frac{r}{\lambda} \right\rfloor^2 \right\} \geq \frac{\mu}{4} \left\lfloor \frac{r}{\lambda} \right\rfloor,$$

(107)

where the last implication follows from the first inequality of (101). Hence, it follows from the independence of the events $|B_{\lfloor \frac{r}{\lambda} \rfloor}(T^s_k)| \leq \lambda \left(\left\lfloor \frac{r}{\lambda} \right\rfloor^2, i \in D_{\lfloor \frac{r}{\lambda} \rfloor}(T^s_{0})\right)|$, and the definition (102) of $q$, that

$$v^{(\mu)}\left(\left\{ |B_r| \leq \frac{\lambda}{4} \left\lfloor \frac{r}{\lambda} \right\rfloor^3 \right\} \left| \chi^{-1}(B^s_{\frac{r}{\lambda}}(T^s_{0})) \right) \leq \sum_{m \geq \frac{1}{\lambda} \mu |\frac{r}{\lambda}|} \left(\left| D_{\lfloor \frac{r}{\lambda} \rfloor}(T^s_{0}) \right| \right) q(\lambda, \mu)^m \left(1 - q(\lambda, \mu)^{-1}\right)^{\left| D_{\lfloor \frac{r}{\lambda} \rfloor}(T^s_{0}) \right| - m}$$

$$\leq \sum_{m \geq \frac{1}{\lambda} \mu |\frac{r}{\lambda}|} \left(\frac{\mu r}{m} \right) q(\lambda, \mu)^m \leq \text{cst} \cdot q(\lambda, \mu)^{-1} \frac{1}{12} \cdot 2^{\mu r},$$

(108)

where the upper bound in (101) has also been used. Recalling (106), we may choose $\lambda = \lambda(\mu)$ small enough, independently of $r$ and $T^s_0$, such that $q_0 := 2q \left(\frac{\lambda(\mu)}{\mu}\right)^{\frac{1}{12}} < 1$, yielding

$$v^{(\mu)}\left(\left\{ |B_r| \leq \frac{\lambda(\mu)}{4} \left\lfloor \frac{r}{\lambda} \right\rfloor^3 \right\} \left| \chi^{-1}(B^s_{\frac{r}{\lambda}}(T^s_{0})) \right) \leq \text{cst} \cdot q_0^{\mu r},$$

(109)
provided $|r_f^3| \geq \max\{N, \left(\frac{\mu}{\lambda(\mu)}\right)^{\frac{1}{2}}\}$. Since this holds uniformly in $T^s \in \mathcal{A}_N^s$, with $T_0^s = B_r(T^s)$, for $r$ large enough, we conclude for such $r$ that

$$v(\mu) \left( \left| B_r \right| \leq \frac{\lambda(\mu)}{4} \left\lfloor \frac{r}{3} \right\rfloor^3 \cap \chi^{-1}(\mathcal{A}_N^s) \right) \leq \text{cst} \cdot q_0^{\mu r}.$$  

In particular, it follows that

$$\sum_{r=1}^{\infty} v(\mu) \left( \left| B_r \right| \leq \frac{\lambda(\mu)}{4} \left\lfloor \frac{r}{3} \right\rfloor^3 \cap \chi^{-1}(\mathcal{A}_N^s) \right) < \infty.$$  

Hence, the Borel-Cantelli lemma implies that $\left\{ \left| B_r \right| \leq \frac{\lambda(\mu)}{4} \left\lfloor \frac{r}{3} \right\rfloor^3 \text{ i.o.} \} \cap \chi^{-1}(\mathcal{A}_N^s)$ is a nullset. Since this holds for arbitrary $N$, it follows from (100) that $\left| B_r \right| \leq \frac{\lambda(\mu)}{4} \left\lfloor \frac{r}{3} \right\rfloor^3 \text{ i.o.}$ is a nullset, which hence completes the proof of the lower bound in (91) with $0 < C''_1 < \frac{\lambda(\mu)}{108}$.

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