ON FUZZY BOUNDARY

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Abstract

In this paper, we introduce a new type fuzzy boundary and study some related set theoretic identities. Further, this new type of fuzzy boundary is compared with different existing fuzzy boundaries.

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1. Introduction

From general topology, we know crisp boundary of a set is the set of elements which are both in the closure of the set and closure of its complement. But in many real life situations, crisp boundary of a situation may not be well defined. For instance, boundary of ocean shared by two or more countries and boundary of the states in India to be recognized from satellite pictures etc. Such boundaries may be termed as fuzzy boundaries. Further, the importance of fuzzy boundary is found in generalization of modern control theory as discussed in [5] and [6].

Fuzzy boundary in the context of fuzzy topological space was defined by Warren [7], which was later modified by Cuchillo-Ibanez and Tarres in [3]. Subsequently, Pu and Lie [4] have defined fuzzy boundary as a generalization of the crisp boundary in general topology. Below are some definitions and results as discussed in [3], [4] and [7].

Definition 1.1. [7] Let A be a fuzzy set in an FTS (X, τ). The fuzzy boundary of A is defined as the infimum of all closed fuzzy sets D in X with the property that D(\(x\)) ≥ cl(A(\(x\))) for all \(x \in X\) for which (cl(A) \(\cap\) cl(A'))(\(x\)) > 0. We shall denote such boundary by bd\(_I\)A.

Warren verified the following properties of the fuzzy boundary:

(1) The boundary is closed.
(2) The closure is the supremum of the interior and boundary.
(3) The boundary reduces to the usual topological boundary when all fuzzy sets are crisp.
(4) The boundary operator is an equivalent way of defining a fuzzy topology.

However, Warren’s definition lacks the following properties:

(5) The boundary of a fuzzy set is identical to the boundary of the complement of the set.
(6) If a fuzzy set is closed (or open), then the interior of the boundary is empty.
(7) If a fuzzy set both open and closed then the boundary is empty.
Definition 1.2. Fuzzy boundary of a fuzzy set A is $\text{bd}_A = \text{cl}(A) \setminus \text{cl}(A^c)$.

Definition 1.3. Let A be a fuzzy set in an FTS $(X, \tau)$. The fuzzy boundary of A is defined as the infimum of all closed fuzzy sets $D$ in X with the property that $D(x) \geq \text{cl}(A(x))$ for all $x \in X$ for which $(\text{cl}(A) - \text{int}(A))(x) > 0$. We shall denote it by $\text{bd}_A$.

Cuchillo-Ibanez and Tarres established that their definition of fuzzy boundary satisfies the properties (1)-(4) and (7).

Later, Athar and Ahmad ([1], [2]) introduced and studied fuzzy semiboundary by generalizing fuzzy boundary through fuzzy semi-closed sets.

2. Fuzzy boundary

In this section, we introduce a new type of fuzzy boundary using the concepts of fuzzy closure and fuzzy interior. Our aim behind introducing the new definition is to move close to exactness.

Definition 2.1. Let A be a fuzzy set in an FTS $(X, \tau)$. The boundary of A, denoted by $\text{bd}A$ is a fuzzy set defined as $\text{bd}A = \text{int} \text{cl}A \setminus \text{cl} \text{int}A$.

From the definition, followings can easily be concluded:

- $\text{bd} 0_X = 0_X$ and $\text{bd} 1_X = 1_X$.
- The boundary of a set is same as the boundary of its complement.
- If a fuzzy set is clopen then the boundary is equal to itself.
- Boundary is not necessarily a closed fuzzy set.
- If the closures and interiors of two fuzzy sets are equal then their boundaries are also equal.

Example 2.2. Let $X = \{a, b\}$ and $\tau = \{0_X, \{a_{0.8}, b_{0.4}\}, \{a_{0.3}, b_{0.2}\}, \{a_{0.3}, b_{0.4}\}, \{a_{0.2}, b_{0.2}\}, 1_X\}$ be a fuzzy topology on X. Then boundary of the fuzzy set $\{a_{0.3}, b_{0.4}\}$ is $\{a_{0.3}, b_{0.4}\}$, which is not fuzzy closed.

Theorem 2.3. For any two fuzzy sets A and B in an FTS $(X, \tau)$, the following results hold.

(i) $\text{bd}(A \cup B) \geq \text{bd}A \cup \text{bd}B$;
(ii) $\text{bd}(A \cap B) \geq \text{bd}A \cap \text{bd}B$;
(iii) $\text{int}A \cup \text{bd}A = \text{bd}A$;
(iv) $\text{bd}A \leq \text{cl}A$;
(v) If A is clopen then $\text{bd}(\text{bd}A) = A$;
(vi) $\text{bd}(\text{cl}A) \geq \text{bd}A$;
(vii) $\text{bd}(\text{int}A) \leq \text{bd}A$;
(viii) If $A \leq B$ then $\text{bd}A < \text{bd}B$;
(ix) $A \cup \text{bd}A \leq \text{cl}A$;
(x) If $\text{int}A = \text{int}clA$ then $\text{bd}(\text{cl}A) = \text{bd}A$;
(xi) If $\text{cl}A = \text{cl}\text{int}A$ then $\text{bd}(\text{int}A) = \text{bd}A$;
(xii) A is closed $\iff$ $\text{bd}A \leq A$. 

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Proof. Let us consider fuzzy sets $A$ and $B$ in an FTS $(X, \tau)$. Then

(i) 
\[
\text{bd}(A \lor B) = \text{intcl}(A \lor B) \cap \text{clint}(A \lor B) \\
\geq [\text{int}(\text{cl}A \lor \text{cl}B)] \cap [\text{cl}(\text{int}A \lor \text{int}B)] \\
\geq [\text{intcl}A \lor \text{intcl}B] \cap [\text{clint}A \lor \text{clint}B] \\
= [\text{intcl}A \lor \text{clint}A] \lor [\text{intcl}B \lor \text{clint}B] \\
\lor [\text{intcl}B \lor \text{clint}A] \\
\geq \text{bd}A \lor \text{bd}B.
\]

(ii) 
\[
\text{bd}(A \land B) = [\text{intcl}(A \land B)] \cap [\text{clint}(A \land B)] \\
\leq [\text{int}(\text{cl}A \land \text{cl}B)] \cap [\text{cl}(\text{int}A \land \text{int}B)] \\
\leq [\text{intcl}A \land \text{intcl}B] \cap [\text{clint}A \land \text{clint}B] \\
= [\text{intcl}A \land \text{clint}A] \lor [\text{intcl}B \land \text{clint}B] \\
= \text{bd}A \lor \text{bd}B.
\]

(iii) 
\[
\text{int}A \lor \text{bd}A = \text{int}A \lor [\text{intcl}A \lor \text{clint}A] \\
= \text{int}A \lor \text{clint}A \quad \text{(since int}A \leq \text{clint}A) \\
= \text{bd}A.
\]

(iv) We know, intcl$A \leq$ cl$A$ and int$A \leq A \Rightarrow$ clint$A \leq$ cl$A$

i.e., intcl$A \land$ clint$A \leq$ cl$A$.

(v) As $A$ is clopen, $\text{bd}A = A \Rightarrow \text{bd}(\text{bd}A) = \text{bd}A = A$.

(vi) 
\[
\text{bd}(\text{cl}A) = \text{intcl}(\text{cl}A) \lor \text{clint}(\text{cl}A) \\
= \text{int}A \lor \text{clint}A \\
\geq \text{int}A \lor \text{clint}A \\
= \text{bd}A.
\]

(vii) 
\[
\text{bd}(\text{int}A) = \text{intcl}(\text{int}A) \lor \text{clint}(\text{int}A) \\
= \text{intcl}A \lor \text{clint}A \\
\leq \text{int}A \lor \text{clint}A \\
= \text{bd}A.
\]
(viii) \( A \leq B \Rightarrow clA \leq clB \text{ and } intA \leq intB. \)
So, \( intclA \leq intclB \text{ and } clintA \leq clintB. \)
\( \Rightarrow bdA \leq bdB. \)

(ix) Follows from \( bdA \leq clA \) and \( A \leq bdA. \)

(x) Straightforward.

(xi) Straightforward.

(xii) \( A \) is closed \( \Rightarrow clA = A \Rightarrow clintA = intclA \leq A \text{ and } clintA = clA = A \Rightarrow bdA \leq A. \)
On the other hand, it suffices to show \( clA \leq A. \) If possible let \( clA \nsubseteq A \)
\( \Rightarrow intclA \nsubseteq A \Rightarrow intclA \nsubseteq A \)
And \( clintA \leq clA \nsubseteq A \) implies \( bdA \nsubseteq A, \) a contradiction.

3. Comparative study

In this section, we compare the fuzzy boundary defined here with the different fuzzy boundaries which already exist.

**Theorem 3.1.** For any fuzzy set \( A, bdA \leq bd_{II}A. \)

**Proof.** For any fuzzy set \( A, intclA \leq clA \text{ and } clintA = cl(clA^c)^c = intclA^c \leq clA^c \)
So, \( intclA \wedge clintA \leq clA \wedge clA^c \text{ i.e., } bdA \leq bd_{II}A. \)

**Theorem 3.2.** For any fuzzy set \( A, bdA \leq bd_{I}A. \)

**Proof.** If \( clA \wedge clA^c = 0 \) then \( bd_{I}A = 0_{X}. \)
If \( clA \wedge clA^c \geq 0 \) then \( bd_{I}A = clA. \)
So, \( clA \wedge clA^c \leq bd_{I}A \) and hence \( bdA \leq bd_{I}A. \)

**Theorem 3.3.** For any fuzzy set \( A, bdA \leq bd_{III}A. \)

**Proof.** We have, if \( clA \rightarrow intA > 0_{X} \) then \( bd_{III}A = clA. \)
If \( clA \rightarrow intA = 0_{X} \) then \( A \) is clopen, so \( bdA = A. \) But we always have \( bd_{III}A \leq clA. \) Therefore, \( bdA \leq bd_{III}A. \)

4. Conclusion

In this paper, we introduced a new type of fuzzy boundary and studied a few properties of this fuzzy boundary. It is also concluded that, this fuzzy boundary is the smallest among different existing fuzzy boundaries. However, the components of exactness will be more in case of this new fuzzy boundary in fuzzy topological space.

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