Quantum Thermodynamic Uncertainty Relation for Continuous Measurement

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We derive a thermodynamic uncertainty relation in Markovian open quantum systems, which bounds the fluctuation of continuous measurements, through quantum estimation theory. The derived quantum thermodynamic uncertainty relation holds for arbitrary continuous measurements satisfying a scaling condition. This is in contrast to the classical counterparts, which require specification on how we measure systems. We derive two relations; the first relation bounds the fluctuation by the dynamical activity and the second one by the entropy production. We apply our bounds to a two-level atom driven by laser field and a three-level quantum thermal machine with jump and diffusion measurements. Our result shows that there exists the universal bound on the fluctuations regardless of continuous measurements.

Introduction.—Uncertainty relations distinguish the possible from the impossible, which have played fundamental roles in physics. Recently, thermodynamic uncertainty relations (TURs) have been found in stochastic thermodynamics, which argue that the fluctuation of time-integrated observables is lower bounded by the thermodynamic costs, such as the entropy production and the dynamical activity [1–23] (see [24] for review). TURs predict the fundamental limit of biomolecular processes and thermal machines, and they have been applied to infer the entropy production [25–27].

In contrast to classical systems, studies of TURs in a quantum regime are in early stage. One of the distinguishing properties of quantum systems is measurement. In stochastic thermodynamics, it is naturally assumed that we can measure stochastic trajectories of the system. In quantum systems, output is obtained through measurements but the measurements themselves alter the system state. Moreover, in addition to the freedom on how we compute the current in stochastic thermodynamics, we have an extra degree of freedom on how we measure the system in quantum systems. Although TURs have been recently studied in quantum systems [28–32], these works do not consider the measurement effects explicitly or specify a type of measurements in advance.

In this Letter, we derive a quantum thermodynamic uncertainty relation (QTUR) for Markovian open quantum dynamics by using quantum estimation theory [33–35]. In Ref. [18], we have derived a TUR for Langevin dynamics via the Cramér–Rao inequality. Extending this line to quantum dynamics, we derive a QTUR for continuous measurements with the quantum Cramér–Rao inequality. The quantum Cramér–Rao inequality holds for arbitrary measurements, while the classical one is satisfied for specific measurements, indicating that the quantum version is more general. By virtue of this generality, obtained QTUR holds for arbitrary continuous measurements satisfying a scaling condition (cf. Eq. (5)). Our QTUR has two variants; the first relation is bounded by the dynamical activity, and the second one by the entropy production. We demonstrate the QTUR with a two-level atom and a quantum thermal machine under jump and diffusion measurements.

Methods.—The TURs in classical stochastic thermodynamics consider the fluctuation of currents, which are time-integration of stochastic trajectories. Analogously, we wish to bound the fluctuation of time-integration of continuous measurements in quantum dynamics. In continuous measurements, we consider a principal system $S$ and an environment $E$. Consider a Kraus operator $V_m$ on the principal system, which satisfies $\sum_m V_m^\dagger V_m = I_S$, where $I_S$ denotes the identity operator of the principal system. We can describe time-evolution induced by the Kraus operator $V_m$ on the principal system by a unitary operator $U$ acting on the composite system $S + E$. Let $|e_k\rangle$ be an orthonormal basis for $E$. We can define the unitary operator $U$ such that [36]

$$|\psi'\rangle = U |\psi_S\rangle \otimes |e_0\rangle = \sum_m V_m |\psi_S\rangle \otimes |e_m\rangle,$$  \hspace{1cm} (1)

where $|e_0\rangle$ is some standard state of the environment and $|\psi_S\rangle$ is the initial state of the principal system. When applying the projective measurement $|e_m\rangle$ on the environment, the principal system becomes $|\psi'_S\rangle \propto V_m |\psi_S\rangle$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{FIG_1.png}
\caption{Quantum trajectories and measurements of (a) jump measurement (photon counting) and (b) diffusion measurement (homodyne detection) in a two-level atom. Upper panels are quantum trajectories of $\rho_{ee} \equiv \langle e_e | \rho | e_e \rangle$ and lower panels are outputs.}
\end{figure}
Therefore, the operator $V_m$ is associated with the output $m$ and constitutes a measurement operator. We sequentially repeat this procedure to describe the continuous measurement during a time interval $[0, T]$. We discretize time by dividing the time interval $[0, T]$ into $N$ equipartitioned intervals, where the time resolution is $\Delta t = T/N$. At each time interval, we consider Eq. (1). Then the state of the composite system at time $t = T$ is

$$|\psi(T)\rangle = \sum_{m} V_{mN-1} \cdots V_{m0} |\psi_S\rangle \otimes |e_{mN-1}, \ldots, e_{m0}\rangle,$$  

(2)

where $\mathbf{m} = [m_0, \ldots, m_{N-1}]$. In Eq. (2), we assumed that $V_m$ is time-independent, which leads to Markovian dynamics. We hereafter consider the limit of $N \to \infty$, where $\mathbf{m}$ becomes a record of the continuous measurement. For instance, in the case of jump measurement, $m$ corresponds to either “detection” or “no detection” of jump within $\Delta t$. Depending on $\mathbf{m}$, the state of the principal system $|\psi_S(T)\rangle \propto V_{mN-1} \cdots V_{m0} |\psi_S\rangle$ is determined, which is referred to as a quantum trajectory. For example, in Fig. 1, we show quantum trajectories and their corresponding measurement records for the jump [Fig. 1(a)] and the diffusion [Fig. 1(b)] measurements.

The time evolution of the density operator $\rho$ is $\dot{\rho} = \mathcal{L}(\rho) \equiv -i [H, \rho] + \sum_c \mathcal{D}(\rho, L_c)$,  

(3)

where $\mathcal{L}$ is the Lindblad operator, $[\cdot, \cdot]$ is the commutator, $H$ is a Hamiltonian, $\mathcal{D}(\rho, L) = L \rho L^\dagger - \{L L^\dagger, \rho\}/2$ is the dissipator with $\{\cdot, \cdot\}$ being the anti-commutator, and $L_c$ is a jump operator. Although the Kraus operator $V_m$ depends on measurements, the Lindblad equation does not depend on performed continuous measurements. In Eq. (3), the first and the second terms are referred to as coherent dynamics and dissipation, respectively. We assume that the Hamiltonian $H$ and the jump operators $L_c$ are parameterized by a parameter $\theta \in \mathbb{R}$, which we denote $H_\theta$ and $L_{c, \theta}$, respectively. We define $\mathcal{L}_\theta$, which is the Lindblad operator consisting of $H_\theta$ and $L_{c, \theta}$. We consider the estimation of the parameter $\theta$ from the continuous measurement. Let $\Theta$ be an observable and $\mathbb{E}_{\theta}[\Theta]$ be the expectation of $\Theta$ with a parameter $\theta$. For arbitrary positive-operator valued measure (POVM), according to the quantum Cramér–Rao inequality, Ref. [37] proved the following inequality: $\text{Var}_\theta[\Theta] / \langle \text{Tr} \mathbb{E}_\theta[\Theta] \rangle^2 \geq 1 / \mathcal{I}_Q(\theta)$, where $\text{Var}_\theta[\Theta]$ is the variance of $\Theta$ and $\mathcal{I}_Q(\theta)$ is a quantum Fisher information (see [33, 34] for its review). This expression is a generalization of the conventional quantum Cramér–Rao inequality [38]. Let $\mathcal{I}_c(\theta; \mathcal{M}_m)$ be the classical Fisher information obtained through POVM elements $\mathcal{M}_m$. Then $\mathcal{I}_Q(\theta) = \max_m \mathcal{I}_c(\theta; \mathcal{M}_m)$, indicating that the quantum Cramér–Rao inequality is satisfied by any quantum measurements [33, 34].

Recently, Ref. [35] obtained the quantum Fisher information for continuous measurements. Reference [35] took advantage of the fact that performing a continuous measurement is equivalent to a projective measurement $|e_{mN-1}, \ldots, e_{m0}\rangle$ on $|\psi(T)\rangle$ in Eq. (2). For $T \to \infty$, Ref. [35] showed that $\mathcal{I}_Q(\theta) = 4T \partial_{\theta_1} \partial_{\theta_2} \Re[\tilde{\lambda}_{\theta_1, \theta_2}]_{\theta_1 = \theta_2 = \theta}$, where $\Re(z)$ returns a real part of $z \in \mathbb{C}$ and $\tilde{\lambda}_{\theta_1, \theta_2}$ is a dominant eigenvalue of a modified Lindblad operator $\mathcal{L}_{\mu_0, \rho} \equiv -i H_{\mu_0, \rho} + \rho L_{\mu_0}^\dagger L_{\mu_0} - \frac{1}{2} \sum_c [1_{c, \theta} L_{c, \theta} \rho + \rho L_{c, \theta}^\dagger L_{c, \theta}]$. For $\theta_1 \to \theta$ and $\theta_2 \to \theta$, $\mathcal{L}_{\theta_1, \theta_2} \to \mathcal{L}_{\theta}$ and $\tilde{\lambda}_{\theta_1, \theta_2} \to 0$.

QTUR of dynamical activity.—We now derive a QTUR by using the quantum Cramér–Rao inequality. We hereafter assume that the density matrix of the system is in a single steady state $\rho^\infty$ and only consider the limit of $T \to \infty$. In Ref. [18], a TUR was derived via the classical Cramér–Rao inequality by considering a virtual perturbation [16], which only affects the time-scale of the dynamics while keeping the steady-state distribution unchanged. Analogously, we consider the following modified Hamiltonian and jump operator in Eq. (3):

$$H_\theta = (1 + \theta) H, \quad L_{c, \theta} = \sqrt{1 + \theta} L_c.$$  

(4)

Since the Lindblad operator corresponding to Eq. (4) is given by $\mathcal{L}_\theta = (1 + \theta) \mathcal{L}_{\theta=0}$, the dynamics of $\mathcal{L}_\theta$ is identical to that of the unmodified dynamics (i.e., the dynamics of $\theta = 0$) except for the time scale. Let us consider a time-integrated observable $\Theta(\mathbf{m})$ satisfying

$$\mathbb{E}_{\theta}[\Theta(\mathbf{m})] = h(\theta) \mathbb{E}_{\theta=0}[\Theta(\mathbf{m})],$$  

(5)

where $h(\theta)$ is a scaling function independent of $\mathbf{m}$ [$h(0) = 1$ should be satisfied]. Typically, it is given by $h(\theta) = 1 + \theta$. $\Theta(\mathbf{m})$ can be an arbitrary function of $\mathbf{m}$ as long as Eq. (5) is satisfied. For instance, suppose an estimator that counts the number of photon emission during $[0, T]$. Because the system is assumed to be in steady state, the average number of photon emission for $\mathcal{L}_\theta$ is $1 + \theta$ times larger than that of $\mathcal{L}_{\theta=0}$ and hence this observable satisfies Eq. (5) with $h(\theta) = 1 + \theta$. Combining the quantum Cramér–Rao inequality and Eq. (5), we find $\text{Var}[\Theta] / \mathbb{E}[\Theta]^2 \geq h'(0)^2 / 4 \mathcal{I}_Q(0)$. $\mathcal{I}_Q(\theta)$ can be calculated by the differentiation of a dominant eigenvalue of $\tilde{\mathcal{L}}_{\theta_1, \theta_2}$. Using the eigenvalue differentiation [35, 39], we obtain

$$\frac{\text{Var}[\Theta]}{\mathbb{E}[\Theta]^2} \geq \frac{h'(0)^2}{T (Y + \Psi)}.$$  

(6)

Here

$$Y \equiv \text{Tr} \left[ \sum_c L_c \rho^\infty L_c^\dagger \right],$$  

(7)

$$\Psi = -4 \Re \left[ \sum_{i,j \in \{1,2\} : i \neq j} \text{Tr} \left[ K_i \circ L^\dagger_{\mathcal{P}_j} \circ K_j (\rho^\infty) \right] \right],$$  

(8)
where $K_1 \equiv -iH\rho + \frac{1}{2} \sum_c (L_c \rho L_c^\dagger - L_c^\dagger L_c \rho)$ and $K_2 \equiv i\rho H + \frac{1}{2} \sum_c (L_c \rho L_c^\dagger - \rho L_c^\dagger L_c)$, and $L_+^\dagger$ is a subspace of $L^+$ which is complementary to the steady-state subspace, where $L^+$ is the Moore-Penrose pseudo inverse of $L$ (see [40] for an explicit expression). Equation (6) is the first result of this Letter, which holds for arbitrary continuous measurements in Markovian open quantum systems.

For simplicity, let us consider the following case:

$$L_{ji} = \sqrt{\eta_{ji}}|\epsilon_j\rangle \langle \epsilon_i|, \quad \rho_{ii}^0 = 0 \quad (i \neq j),$$

where $|\epsilon_i\rangle$ is the eigenbasis of the Hamiltonian $H$, $\eta_{ji}$ is a transition rate from $|\epsilon_i\rangle$ to $|\epsilon_j\rangle$ (we redefined the subscript of the jump operator from $L_c$ to $L_{ji}$), and $\rho_{ij}^0 \equiv \langle \epsilon_i|\rho^0|\epsilon_j\rangle$. Off-diagonal elements of the steady-state density matrix in the energy eigenbasis are zero. These assumptions are often satisfied for quantum thermal machines [41]. Then we obtain $\Psi = \sum_{i \neq j} \rho_{ij}^0 |\eta_{ji}|$, which corresponds to the dynamical activity in classical Markov process, implying that $\Psi$ is a quantum analogue of the dynamical activity [40]. Moreover, we can obtain a simpler lower bound by scaling the jump operator alone. In this case, $\Psi$ in Eq. (6) becomes 0, which re-derived the classical TUR. This shows that $\Psi$ quantifies the degree of the coherent dynamics in the Lindblad equation, which is also shown in a two-level atom. Therefore, Eq. (6) is a quantum generalization of a TUR [10, 17], which is bounded by the dynamical activity. Recently, Ref. [30] proved a similar bound for quantum jump processes. The bound of Ref. [30] was derived given quantum trajectories. Therefore, their bound is obtained for a specified continuous measurement. Reference [32] derived a TUR in a quantum nonequilibrium steady state using the classical Cramér–Rao inequality. Since their TUR bounds the fluctuation of instantaneous currents (current measurement operators), the measurement effects are not explicitly incorporated.

As an example of the QTUR, we consider a two-level atom driven by a classical laser field. Let $|\epsilon_g\rangle$ and $|\epsilon_e\rangle$ denote the ground and the excited states, respectively. A Hamiltonian is given by $H = \Delta |\epsilon_g\rangle \langle \epsilon_e| + \frac{1}{2}(|\epsilon_g\rangle \langle \epsilon_g| + |\epsilon_e\rangle \langle \epsilon_e|)$, where $\Delta$ is a detuning between the laser field frequency and the atomic transition frequency, and $\Omega$ is the Rabi oscillation frequency. A jump operator is $L = \sqrt{\kappa} |\epsilon_g\rangle \langle \epsilon_e|$, where $\kappa$ is the decay rate, and it induces the jump from $|\epsilon_e\rangle$ to $|\epsilon_g\rangle$. We obtain the dynamical activity $\Psi = \kappa \rho_{ee}^\infty = \kappa \Omega^2/(4\Delta^2 + \kappa^2 + 2\Omega^2)$ and the coherent dynamics contribution

$$\Psi = \frac{8\Omega^4}{\kappa} \frac{4\Delta^4 + \Delta^2(\kappa^2 + 8\Omega^2) + (\kappa^2 + 2\Omega^2)^2}{(4\Delta^2 + \kappa^2 + 2\Omega^2)^3}. \quad (10)$$

We first consider a jump measurement (photon detection). The quantum trajectory is given by a stochastic Schrödinger equation [42] (corresponding $V_m$ is shown in [40]):

$$d\rho = \left(-i[H, \rho] - \frac{1}{2} \{L^\dagger L, \rho\} + \rho \text{Tr}[L L^\dagger]\right) dt$$

$$+ \left(\frac{L L^\dagger}{\text{Tr}[L L^\dagger]} - \rho\right) d\mathcal{N}, \quad (11)$$

where $d\mathcal{N}$ is a noise increment and $d\mathcal{N} = 1$ when photon is detected between $t$ and $t + dt$ and $d\mathcal{N} = 0$ otherwise. $m = [m_0, ..., m_{N-1}]$ in Eq. (2) corresponds to $[\Delta N_0, ..., \Delta N_{N-1}]$. Its average reads $\mathbb{E}[d\mathcal{N}] = \text{Tr}[L L^\dagger] dt$. We consider an observable $\Theta_N \equiv \delta_{00}$. In Fig. (2(b), the triangles denote Var$[\Theta_N]/\mathbb{E}[\Theta_N]$ as a function $T\eta$. In Fig. 2(a), when $\kappa$ becomes larger (i.e., more frequent jump), the dynamical activity $\Psi$ is dominant in the quantum Fisher information $I_Q(0)$. For $\kappa \to 0$, $\Psi \to 0$ and the coherent dynamics contribution $\Psi$ becomes the major portion of $I_Q(0)$. We numerically check the QTUR for the jump measurement by generating $\kappa$, $\Omega$, and $\Delta$ randomly (ranges of the parameters are shown in the caption of Fig. 2(b)) and calculate Var$[\Theta_N]/\mathbb{E}[\Theta_N]^2$. In Fig. 2(b), the circles denote Var$[\Theta_N]/\mathbb{E}[\Theta_N]^2$ as a function of $I_Q(0)$ and the lower bound of the Eq. (6) is shown by the dashed line. We confirm that all of the realizations satisfy the QTUR, which verify Eq. (6). In a classical case [10, 17], the bound is lowered bounded by the dynamical activity alone (i.e., $T\Psi$). Thus we also check whether Var$[\Theta_N]/\mathbb{E}[\Theta_N]^2$ can be bounded only by $T\Psi$. In Fig. 2(b), the triangles denote Var$[\Theta_N]/\mathbb{E}[\Theta_N]^2$ as a function $T\Psi$, where the dashed line now describes...
FIG. 2. Quantum Fisher information and results of computer simulation for the jump measurement. (a) The quantum Fisher information \( L_Q(0) = T(\Upsilon + \Psi) \) (solid line), \( TT \) (dashed line), and \( T\Psi \) (dotted line) as a function \( \kappa \), where \( \Omega = 1 \) and \( \Delta = 1 \). (b) \( \text{Var}[\Theta_C]/E[\Theta_C] \) as a function of \( T(\Upsilon + \Psi) \) (circles) and \( T\Psi \) (triangles) for the jump measurement, where \( \Delta \in [0.1, 10.0] \), \( \Omega \in [0.1, 10.0] \), \( \kappa \in [0.1, 10.0] \), and \( T = 1000 \). The dashed line corresponds to \( 1/[T(\Upsilon + \Psi)] \) for the circles and \( 1/[TT\Psi] \) for the triangles.

FIG. 3. Illustration of the model and results of computer simulation for the thermal machine. (a) Thermal machine consists of three levels \(|\epsilon_A\rangle\), \(|\epsilon_B\rangle\), and \(|\epsilon_C\rangle\). The transition between each of the states is coupled with heat reservoir with the inverse temperature \( \beta_r \) \((r = 1, 2, 3)\). (b) \( \text{Var}[\Theta_C]/E[\Theta_C] \) as a function of \( T\Sigma \). The dashed line corresponds to the lower bound \( 1/[T\Sigma] \). Clearly, some realizations are below \( 1/[T\Sigma] \), indicating that the lower bound of the QTUR is lower than the classical bound \([10, 17]\). Similar enhancement of precision has been reported for quantum jump processes \([30]\), and for classical systems with periodic driving \([13]\) or magnetic fields \([11]\). We also performed a computer simulation for the diffusion measurement and verified the bound (see \([40]\)).

**QTUR of entropy production.**—Employing a scaling different from Eq. (4), we can bound \( \text{Var}[\Theta]/E[\Theta]^2 \) by the entropy production. Again, we assume that the system satisfies the conditions of Eq. (9). Moreover, we assume that whenever \( \eta_{ij} > 0 \), \( \eta_{ij} > 0 \) should be satisfied. Inspired by Ref. \([19]\), we consider the following modified process instead of Eq. (4):

\[
L_{ji,\theta} = \sqrt{\eta_{ji}} \left[ 1 + \theta \left( 1 - \frac{\eta_{ji} \rho_{ji}}{\eta_{ji} \rho_{ji}^{\beta_r}} \right) \right] |\epsilon_j\rangle \langle \epsilon_i| \ (i \neq j),
\]

With Eq. (13), the steady state density remains unchanged. Repeating a similar calculation to the dynamical activity case (see \([40]\) for details), an observable \( \Theta \) satisfying Eq. (5) obeys

\[
\frac{\text{Var}[\Theta]}{E[\Theta]^2} \geq \frac{2h'(0)^2}{T\Sigma},
\]

where \( \Sigma \equiv \sum_{i\neq j} \rho_{ii}^\alpha \eta_{ij} \ln \left[ \rho_{ii}^\alpha \eta_{ij} / \left( \rho_{ii}^\beta \eta_{ij} \right) \right] \). Equation (14) is the second result of this Letter. Expression of \( \Sigma \) is identical to the entropy production in stochastic thermodynamics \([46]\). Therefore, our approach re-derives the classical TUR \([1, 3]\) but its applicability is broader than the classical counterpart as detailed below.

As an example, we consider a quantum thermal machine. It is a basics for a quantum clock and thus considering the precision is of importance \([28, 41]\). Specifically, we employ a three-level thermal machine powered by three heat reservoirs at different inverse temperatures \( \beta_r \) \((r = 1, 2, 3)\) \([41, 47]\). Each transition is coupled with each of the heat reservoirs (Fig. 3(a)). The Hamiltonian is \( H = \omega_1 |\epsilon_B\rangle \langle \epsilon_B| + \omega_2 |\epsilon_A\rangle \langle \epsilon_A| \), where \( \omega_1, \omega_2, \omega_3 = \omega_1 + \omega_2 \) are energy gaps between \(|\epsilon_A\rangle\leftrightarrow|\epsilon_B\rangle\), \(|\epsilon_B\rangle\leftrightarrow|\epsilon_A\rangle\), and \(|\epsilon_B\rangle\leftrightarrow|\epsilon_C\rangle\), respectively. Let \( \dot{Q}_r \) be the heat current from \( r \)th reservoir with temperature \( \beta_r \). We assume that the dynamics of the density operator \( \rho \) obeys the Lindblad equation:

\[
\dot{\rho} = -i[H, \rho] + \sum_{i\neq j} D(\rho, L_{ji}),
\]

where \( L_{ji} \) is defined in Eq. (9) with \( \eta_{AB} = \gamma_{ij}^{1h} + 1 \), \( \eta_{BA} = \gamma_{ij}^{1h} \), \( \eta_{AB} = \gamma_{ij}^{2h} + 1 \), \( \eta_{BA} = \gamma_{ij}^{2h} \), \( \eta_{AB} = \gamma_{ij}^{3h} \) \((n_{ij}^{\beta} = (e_{ij}^{\beta})^{r - 1} \) and \( \gamma \) is the decay rate). The entropy production is \( \Sigma_{TM} = -\sum_{r=1}^{3} \beta_r \dot{Q}_r \), where \( \dot{Q}_r \) is the heat flux entering from the \( r \)th reservoir \([48, 49]\), and it satisfies \( \Sigma_{TM} = \Sigma \ [40] \).

We first consider a standard jump measurement. The quantum trajectory is given by a stochastic Schrödinger equation:

\[
d\rho = \left(-i[H, \rho] - \sum_{i\neq j} \left( L_{ji} L_{ji} \rho L_{ji} \right) / 2 + \rho \text{Tr} \left[ L_{ji} \rho L_{ji} \right] \right) dt + \sum_{i\neq j} \left( L_{ji} \rho L_{ji} \right) / \text{Tr}[L_{ji} \rho L_{ji}] dN_{ij}.
\]

We consider the following observable:

\[
\Theta_C \equiv \sum_{i\neq j} R_{ji} \int_0^T dN_{ji},
\]

where \( R_{ji} = -R_{ij} \) and \( R_{ij} \in \mathbb{R} \). \( \Theta_C \) satisfies the scaling condition of Eq. (5) and thus it satisfies the QTUR of Eq. (14). Because the dynamics of Eq. (15) are jumps between the energy eigenstates which
equivalent to classical dynamics, $\Theta_C$ trivially satisfies Eq. (14).

We next consider a generalized jump measurement [42]. The Lindblad equation is invariant under the following transformation: $L'_j = (L_{ji} + \zeta_j I)S$ and $H' = H - \frac{1}{2} \sum_{ij} [\zeta_j L_{ji} - \zeta_i L_{ij}]$, where $\zeta_j \in \mathbb{C}$ is a parameter. $\zeta_{ji} = 0$ for all $i$ and $j$ recovers the standard jump measurement. Thus we can consider a transformed stochastic Schrödinger equation, where $H$ and $L_{ji}$ are replaced with $H'$ and $L'_{ji}$, respectively, in Eq. (15), and we define $dN'_{ji}$ as a noise increment in the transformed equation. Quantum trajectories are not simple jump processes between the energy eigenstates anymore [40]. We consider an observable $\Theta_C' = \sum_{i \neq j} R_{ji} I_{ji}^T dN'_{ji}$, where $R_{ji} = -R_{ij}$, for the transformed equation. When $|\zeta_{ji}| = |\zeta_{ij}|$ for all $i$ and $j$, $\Theta_C'$ satisfies the QTUR of Eq. (14) [40].

We verify the QTUR of Eq. (14) for the generalized jump measurement (i.e., $\zeta_{ji} \neq 0$ and $|\zeta_{ji}| = |\zeta_{ij}|$) with a computer simulation. We numerically check the QTUR by generating $\beta_{ij}$, $\omega_{ij}$, $R_{ij}$, and $\zeta_{ij}$ randomly (parameters are shown in the caption of Fig. 3(b)) and calculate Var$[\Theta_C'/\Theta_C']^2$. In Fig. 3(b), the circles denote Var$[\Theta_C'/\Theta_C']^2$ as a function of the entropy production $\Sigma$ and the lower bound of the Eq. (14) is shown by the dashed line. We confirm that all of the realizations satisfy the QTUR, which verify Eq. (14). Although the bound of Eq. (14) itself is identical to the classical TUR [1, 3], our QTUR provides the lower bound for arbitrary measurements. No matter how we measure the thermal machine, an observable satisfying the scaling relation [Eq. (5)] should obey the QTUR of Eq. (14), which cannot be deduced by the classical TURs.

Conclusion.—In this Letter, we have derived the QTUR from the quantum Cramér-Rao inequality. In contrast to classical counterparts, the QTUR hold for arbitrary continuous measurements satisfying the scaling condition. We expect that the present study can be a basis for obtaining uncertainty relations in a quantum regime.

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This supplementary material describes the calculations introduced in the main text. Equation and figure numbers are prefixed with S (e.g., Eq. (S1) or Fig. S1). Numbers without this prefix (e.g., Eq. (1) or Fig. 1) refer to items in the main text.

S1. BASICS

First, we introduce the vectorization of quantum operators following [1]. Let $\rho$ be an arbitrary density matrix

$$\rho = \sum_{i,j} \rho_{ij} |i \rangle \langle j|,$$

where $|i \rangle$ is some orthonormal basis. We introduce a vectorized form of $\rho$ by

$$\text{vec}(\rho) \equiv \sum_{i,j} \rho_{ij} |j \rangle \otimes |i \rangle.$$  \hspace{1cm} (S1)

Let $A$, $B$, and $C$ be matrices. With Eq. (S1), the following relation holds

$$\text{vec}(ABC) = (C^\top \otimes A)\text{vec}(B),$$ \hspace{1cm} (S2)

where $\top$ is the matrix transpose. With the vectorization, the Hilber–Schmidt inner product becomes

$$\langle A, B \rangle \equiv \text{Tr}[A^\dagger B] = \text{vec}(A)^\dagger \text{vec}(B) = \langle \text{vec}(A), \text{vec}(B) \rangle,$$

which is the inner product between two vectors $\text{vec}(A)$ and $\text{vec}(B)$. Consider the following Lindblad equation:

$$\dot{\rho} = \mathcal{L}(\rho) \equiv -i[H, \rho] + \sum_c \left[ L_c \rho L_c^\dagger - \frac{1}{2} \{ L_c^\dagger L_c \rho + \rho L_c^\dagger L_c \} \right],$$ \hspace{1cm} (S3)

where $\mathcal{L}$ is the Lindblad operator, $H$ is the Hamiltonian, and $L_c$ is the jump operator. By using Eq. (S1), Eq. (S3) is converted into

$$\frac{d}{dt} \text{vec}(\rho) = \hat{\mathcal{L}} \text{vec}(\rho),$$

where $\hat{\mathcal{L}}$ is a matrix representation of $\mathcal{L}$ obtained through Eq. (S2):

$$\hat{\mathcal{L}} \equiv -i(\mathbb{I} \otimes H - H^\top \otimes \mathbb{I}) + \sum_c \left[ L_c^* \otimes L_c - \frac{1}{2} \mathbb{1} \otimes L_c^\dagger L_c - \frac{1}{2} (L_c^\dagger L_c)^\top \otimes \mathbb{1} \right].$$ \hspace{1cm} (S4)

Here, a superscript $*$ denotes complex conjugate and $\mathbb{I}$ is the identity operator. Now the Lindblad equation becomes a linear differential equation and we can compute the steady state density matrix from $\hat{\mathcal{L}}$. The right and left eigenvectors corresponding to a zero eigenvalue are

$$\hat{\mathcal{L}} \text{vec}(\rho^\text{ss}) = 0,$$ \hspace{1cm} (S5)

$$\text{vec}(\mathbb{I})^\dagger \hat{\mathcal{L}} = 0,$$ \hspace{1cm} (S6)

where $\rho^\text{ss}$ is the steady state density matrix.
Next, we show expression of the eigenvalue derivative based on Refs. [2, 3]. Let us consider an eigenvalue problem. Let \( A(\theta) \) be a matrix parametrized by a vector parameter \( \theta \). We assume that eigenvalues are not degenerate. Right and left eigenvectors (\( u(\theta) \) and \( v(\theta) \), respectively) of an eigenvalue \( \lambda(\theta) \) satisfy
\[
A(\theta)u(\theta) = \lambda(\theta)u(\theta), \quad A^\dagger(\theta)v(\theta) = \lambda^*(\theta)v(\theta).
\] (S7)  (S8)

We wish to find the derivative of \( \lambda(\theta) \) around some chosen value \( \theta_0 \). Specifically, we focus on an eigenvalue which vanishes at \( \theta_0 \), i.e., \( \lambda(\theta_0) = 0 \). We impose the following normalization constraints:
\[
\langle u(\theta_0), u(\theta_0) \rangle = 1, \quad \langle v(\theta_0), v(\theta_0) \rangle = 1.
\] (S9)  (S10)

We introduce the following notation:
\[
A_i(\theta_0) \equiv \frac{\partial}{\partial \theta_i} A(\theta) \bigg|_{\theta = \theta_0}, \quad A_{ij}(\theta_0) \equiv \frac{\partial^2}{\partial \theta_i \partial \theta_j} A(\theta) \bigg|_{\theta = \theta_0}.
\]

From Ref. [2, 3], the derivative of \( \lambda(\theta) \) is
\[
\frac{\partial^2}{\partial \theta_i \partial \theta_j} \lambda(\theta) \bigg|_{\theta = \theta_0} = \mathcal{X} + Z_1 + Z_2, \quad \text{(S11)}
\]

where
\[
\mathcal{X} \equiv \langle v(\theta_0), A_{ij}(\theta_0)u(\theta_0) \rangle, \quad Z_1 \equiv -\langle v(\theta_0), A_i(\theta_0)\mathbb{P}A(\theta_0)^+\mathbb{P}A_j(\theta_0)u(\theta_0) \rangle, \quad Z_2 \equiv -\langle v(\theta_0), A_j(\theta_0)\mathbb{P}A(\theta_0)^+\mathbb{P}A_i(\theta_0)u(\theta_0) \rangle. \quad \text{(S12)} \quad \text{(S13)} \quad \text{(S14)}
\]

Here \( A^+ \) is the Moore–Penrose pseudo inverse of \( A \) and \( \mathbb{P} \) is a projector defined by
\[
\mathbb{P} \equiv I - u(\theta_0)v(\theta_0)\dagger. \quad \text{(S15)}
\]

**S2. DERIVATIONS**

**A. Bound by dynamical activity**

We derive the QTUR bounded by the dynamical activity. We consider the following scaling in the main text [Eq. (4)]:
\[
H_\theta = (1 + \theta)H, \quad L_{c, \theta} = \sqrt{1 + \theta}L_c.
\] (S16)

The corresponding modified Lindblad operator is
\[
\tilde{L}_{\theta_1, \theta_2}(\rho) = -i \left( [1 + \theta_1] H \rho - (1 + \theta_2) \rho H \right) + \sqrt{(1 + \theta_1)(1 + \theta_2)} \sum_c L_c \rho L_c^\dagger
\]
\[
- \frac{1 + \theta_1}{2} \sum_c L_c^\dagger L_c \rho - \frac{1 + \theta_2}{2} \sum_c \rho L_c^\dagger L_c. \quad \text{(S17)}
\]

Let \( \tilde{L}_{\theta_1, \theta_2} \) be a matrix representation of the modified Lindblad operator, which is obtained by Eq. (S2). We wish to obtain the derivative of an eigenvalue of \( \tilde{L}_{\theta_1, \theta_2} \) through Eq. (S11). For \( A(\theta) = \tilde{L}_{\theta_1, \theta_2} \), we find the following relation from Eqs. (S5), (S6), (S9), and (S10):
\[
u(\theta_0) = k_v \text{vec}(\rho^m), \quad v(\theta_0) = k_v \text{vec}(I),
\]
where $k_u \equiv 1/\sqrt{\langle \vec{\rho}_{ss}, \vec{\rho}_{ss} \rangle}$ and $k_v \equiv \sqrt{\langle \vec{\rho}_{ss}, \vec{\rho}_{ss} \rangle}$ are normalization constants. From Eq. (S11), the eigenvalue differentiation is given by

$$
\frac{\partial^2}{\partial \theta_1 \partial \theta_2} \tilde{\lambda}_{\theta_1, \theta_2} \bigg|_{\theta_1=\theta_2=\theta} = \mathcal{X} + Z_1 + Z_2,
$$

where $\tilde{\lambda}_{\theta_1, \theta_2}$ is a dominant eigenvalue of a modified Lindblad operator $\tilde{\mathcal{L}}_{\theta_1, \theta_2}$. $\tilde{\lambda}_{\theta_1, \theta_2}$ smoothly converges to 0 for $\theta_1 \to 0$ and $\theta_2 \to 0$. We obtain

$$
\mathcal{X} = \text{Tr} \left[ \frac{\partial^2}{\partial \theta_1 \partial \theta_2} \tilde{\mathcal{L}}_{\theta_1, \theta_2} (\rho_{ss}) \right]_{\theta_1=\theta_2=\theta} = \frac{1}{4} \sum_c \text{Tr} \left[ L_c \rho_{ss} L_c^\dagger \right].
$$

Since $\text{Tr}[L_c \rho_{ss} L_c^\dagger] = \text{Tr}[(L_c \rho_{ss} L_c^\dagger)] = \text{Tr}[L_c \rho_{ss} L_c^\dagger]$, we find that $\mathcal{X}$ is real. $Z_1$ and $Z_2$ are given by

$$
Z_1 = - \langle \text{vec}(1), \hat{K}_1 (1 - \text{vec}(\rho_{ss}) \text{vec}(1)) \hat{L}^+ (1 - \text{vec}(\rho_{ss}) \text{vec}(1)) \hat{K}_2 \text{vec}(\rho_{ss}) \rangle,
$$

$$
Z_2 = - \langle \text{vec}(1), \hat{K}_2 (1 - \text{vec}(\rho_{ss}) \text{vec}(1)) \hat{L}^+ (1 - \text{vec}(\rho_{ss}) \text{vec}(1)) \hat{K}_1 \text{vec}(\rho_{ss}) \rangle,
$$

where $\hat{K}_1$ and $\hat{K}_2$ are matrix representations of

$$
\hat{K}_1 \equiv -iH\rho + \frac{1}{2} \sum_c (L_c \rho L_c^\dagger - L_c^\dagger L_c \rho),
$$

$$
\hat{K}_2 \equiv i\rho H + \frac{1}{2} \sum_c (L_c \rho L_c^\dagger - \rho L_c^\dagger L_c).
$$

As denoted in the main text, for $T \to \infty$, Ref. [3] found that the quantum Fisher information for continuous measurements is given as follows:

$$
\mathcal{I}_Q(\theta = 0) = 4T \frac{\partial^2}{\partial \theta_1 \partial \theta_2} \text{Re} \left[ \tilde{\lambda}_{\theta_1, \theta_2} \right]_{\theta_1=\theta_2=0},
$$

where $\text{Re}(z)$ returns a real part of $z \in \mathbb{C}$. Substituting Eq. (S18) into (S20), we obtain

$$
\mathcal{I}_Q(\theta = 0) = T [4\mathcal{X} + 4\text{Re} (Z_1 + Z_2)] = T [\mathcal{T} + \Psi],
$$

which yields the first main result of the main text [Eq. (6)]:

$$
\frac{\text{Var}[\Theta]}{\mathbb{E}[\Theta]^2} \geq \frac{h'(0)^2}{T (\mathcal{T} + \Psi)}.
$$

Next, we limit our discussion to the following case [Eq. (9)]:

$$
L_{ji} = \sqrt{n_{ji}} \langle \epsilon_i | \epsilon_j \rangle, \quad \rho_{ij}^{ss} = 0 \quad (i \neq j),
$$

where $|\epsilon_i\rangle$ is the eigenbasis of the Hamiltonian $H$, $n_{ji}$ is a transition rate from $|\epsilon_i\rangle$ to $|\epsilon_j\rangle$ (we redefined the subscript of the jump operator from $L_c$ to $L_{ji}$), and $\rho_{ij}^{ss} \equiv \langle \epsilon_i | \rho^{ss} | \epsilon_j \rangle$. The dynamics of this system is jumps between energy eigenstates. Therefore, this system corresponds to classical Markov chains and thus we can derive the classical TUR through the derivation above. Now we only have to consider a scaling for the jump operator. $\mathcal{X}$ in Eq. (S18) remains unchanged with this specific case, i.e., $\mathcal{X}$ is given by Eq. (S19). To calculate $Z_1$ (and $Z_2$), we focus on $A_j(\theta_0)u(\theta_0)$ in Eq. (S13), which is calculated into

$$
A_j(\theta_0)u(\theta_0) \propto \frac{\partial \tilde{\mathcal{L}}_{\theta_1, \theta_2}}{\partial \theta_1} (\rho_{ss})
$$

$$
= \frac{1}{2} \sum_{i \neq j} \eta_{ji} (\rho_{ii}^{ss} | \epsilon_j \rangle \langle \epsilon_j | - | \epsilon_i \rangle \langle \epsilon_i | \rho_{ss}^{ss}).
$$

Since we have assumed that the density matrix does not have off-diagonal elements in the energy eigenbasis, we substitute $\rho_{ss} = \sum_i \rho_{ii}^{ss} | \epsilon_i \rangle \langle \epsilon_i |$ to obtain

$$
A_j(\theta_0)u(\theta_0) = \sum_{i \neq j} \eta_{ji} (\rho_{ii}^{ss} | \epsilon_j \rangle \langle \epsilon_j | - \rho_{ii}^{ss} | \epsilon_i \rangle \langle \epsilon_i |).
$$

(S24)
Multiplying $\langle \epsilon_k \rangle$ and $|\epsilon_k\rangle$ to Eq. (S24) from left and right, respectively, we obtain
\[
\langle \epsilon_k | \sum_{i \neq j} \eta_{ji} (\rho_{ii}^{ss} |\epsilon_j\rangle \langle \epsilon_j| - \rho_{ii}^{ss} |\epsilon_i\rangle \langle \epsilon_i|) |\epsilon_k\rangle = \sum_i \eta_{ki} \rho_{ii}^{ss} - \sum_j \eta_{jk} \rho_{kk}^{ss} = 0.
\] (S25)

The last line of Eq. (S25) holds because $\rho^{ss}$ is the steady-state solution of the Lindblad equation. From Eq. (S25), we obtain
\[
\frac{\text{Var}[\Theta]}{\mathbb{E}[\Theta]^2} \geq \frac{h'(0)^2}{TT}.
\] (S26)

Equation (S26) re-derives the classical TUR bounded by the dynamical activity.

**B. Bound by entropy production**

We next derive the QTUR bounded by the entropy production. We again limit our discussion to Eq. (S22). Inspired by Ref. [4], we consider the following modified jump operator in the main text [Eq. (13)]:
\[
L_{ji,\theta} = \sqrt{\eta_{ji}} \left[ 1 + \theta \left( 1 - \sqrt{\frac{\eta_{jj} \rho_{jj}^{ss}}{\eta_{jj} \rho_{ii}^{ss}} \right) |\epsilon_j\rangle \langle \epsilon_i| \right. (i \neq j),
\] (S27)

The derivative of $L_{ji,\theta}$ is
\[
\frac{\partial}{\partial \theta} L_{ji,\theta} \bigg|_{\theta=0} = \frac{1}{2} \sqrt{\eta_{ji}} \left( 1 - \sqrt{\frac{\eta_{jj} \rho_{jj}^{ss}}{\eta_{jj} \rho_{ii}^{ss}}} \right) |\epsilon_j\rangle \langle \epsilon_i|.
\] (S28)

Similar to the dynamical activity case considered above, the derivative of $\tilde{X}_{\theta_1, \theta_2}$ is given by Eq. (S18), where $\mathcal{X}$ is
\[
\mathcal{X} \equiv \text{Tr} \left[ \frac{\partial^2}{\partial \theta_1 \partial \theta_2} \hat{L}_{\theta_1, \theta_2} (\rho^{ss}) \right]_{\theta_1 = \theta_2 = 0} = \text{Tr} \left[ \frac{\partial L_{ji,\theta_1}}{\partial \theta_1} \rho L_{ji,\theta_2} \right]_{\theta_1 = \theta_2 = 0} = \frac{1}{4} \sum_{i,j} \left( \sqrt{\eta_{jj} \rho_{jj}^{ss}} - \sqrt{\eta_{ij} \rho_{jj}^{ss}} \right)^2.
\]

By using the following inequality
\[
(a - b)^2 \leq \frac{1}{2} (a^2 - b^2) \ln \frac{a}{b} \quad (a > 0, b > 0),
\] (S29)

$\mathcal{X}$ is upper bounded by
\[
\mathcal{X} \leq \frac{\Sigma}{8},
\] (S30)

where
\[
\Sigma = \sum_{i \neq j} \eta_{ji} \rho_{ii}^{ss} \ln \frac{\eta_{ji} \rho_{ii}^{ss}}{\eta_{jj} \rho_{jj}^{ss}}.
\] (S31)

$\Sigma$ corresponds to the entropy production in classical stochastic thermodynamics.

Next, we calculate $Z_1$ (and $Z_2$). We focus on $A_1(\theta_0) u(\theta_0)$ in Eq. (S13), which is calculated into
\[
A_1(\theta_0) u(\theta_0) \propto \frac{\partial \tilde{L}_{\theta_1, \theta_2}}{\partial \theta_1} (\rho^{ss})
\]
\[
= \frac{1}{2} \sum_{i \neq j} \eta_{ji} \left( 1 - \sqrt{\frac{\eta_{jj} \rho_{jj}^{ss}}{\eta_{jj} \rho_{ii}^{ss}}} \right) |\epsilon_j\rangle \langle \epsilon_i| \rho^{ss} |\epsilon_i\rangle \langle \epsilon_i| - \frac{1}{2} \sum_{i \neq j} \eta_{ji} \left( 1 - \sqrt{\frac{\eta_{jj} \rho_{jj}^{ss}}{\eta_{jj} \rho_{ii}^{ss}}} \right) |\epsilon_i\rangle \langle \epsilon_i| \rho^{ss}
\]
\[
= \frac{1}{2} \sum_{i \neq j} \eta_{ji} \left( 1 - \sqrt{\frac{\eta_{jj} \rho_{jj}^{ss}}{\eta_{jj} \rho_{ii}^{ss}}} \right) \rho_{ii}^{ss} |\epsilon_j\rangle \langle \epsilon_j| - \rho_{ii}^{ss} |\epsilon_i\rangle \langle \epsilon_i| \rho^{ss}.
\]
We substitute $\rho_{ss} = \sum_i \rho_{ii}^{ss} |\epsilon_i\rangle \langle \epsilon_i|$ to obtain

$$A_1(\theta_0)u(\theta_0) \propto \sum_{i \neq j} \eta_{ji} \left( 1 - \sqrt{\frac{\eta_{ij}^{ss}}{\eta_{ji}^{ss}}} \right) (\rho_{ii}^{ss} |\epsilon_j\rangle \langle \epsilon_j| - \rho_{jj}^{ss} |\epsilon_i\rangle \langle \epsilon_i|)$$

$$= \sum_{i \neq j} \eta_{ji} (\rho_{ii}^{ss} |\epsilon_j\rangle \langle \epsilon_j| - \rho_{jj}^{ss} |\epsilon_i\rangle \langle \epsilon_i|)$$

$$= 0,$$

where the second line is identical to Eq. (S24). From Eqs. (S29) and (S31), $Z_1 = Z_2 = 0$ and hence we obtain

$$\frac{\text{Var} [\Theta]}{\mathbb{E} [\Theta]^2} \geq \frac{2h'(0)^2}{T\Sigma}.$$  

Equation (S32) is the second result of the main text [Eq. (14)].

S3. CONTINUOUS MEASUREMENTS

A. Jump measurement

We introduce a jump measurement. The Kraus operator $V_m$ for the jump measurement is

$$V_0 = I_S - i \left( H - \frac{i}{2} \sum_c L_c L_c^\dagger \right) dt,$$

$$V_c = L_c \sqrt{dt},$$

Equations (S33) and (S34) satisfy the completeness relation:

$$V_0^\dagger V_0 + \sum_c V_c^\dagger V_c = I_S.$$

B. Diffusion measurement

We introduce a diffusion measurement following Ref. [5]. For simplicity, we consider a one-dimensional case, because the multidimensional generalization is straightforward. For the diffusion measurement, the Kraus operator $V_m$ is given by

$$V_{\Delta Y} = \sqrt{\mathcal{P}(\Delta Y)} \left[ I_S - iH \Delta t - \frac{1}{2} L_c^\dagger L_c \Delta t + L \Delta Y \right],$$

where $\Delta Y$ is the output of the measurement and $\mathcal{P}(\cdot)$ is a Gaussian distribution with zero mean and the variance $\Delta t$. Equation (S35) satisfies the completeness relation:

$$\int d\Delta Y V_{\Delta Y}^\dagger V_{\Delta Y} = I_S.$$

Therefore, $V_{\Delta Y}$ constitutes a valid Kraus operator. The probability of observing the output $\Delta Y$ is

$$p(\Delta Y) = \text{Tr} \left[ V_{\Delta Y} \rho V_{\Delta Y}^\dagger \right]$$

$$= \mathcal{P}(\Delta Y) \text{Tr} \left[ \rho + (L \rho + \rho L^\dagger) \Delta Y \right].$$

From Eq. (S36), the mean and the variance of $\Delta Y$ are

$$\mathbb{E} [\Delta Y] = \int d\Delta Y p(\Delta Y) \Delta Y = \text{Tr} \left[ L \rho + \rho L^\dagger \right] \Delta t,$$

$$\mathbb{E} [\Delta Y^2] = \int d\Delta Y p(\Delta Y) \Delta Y^2 = \Delta t.$$

(S38)
This implies that the dynamics of $\Delta Y$ is given by the following Ito stochastic differential equation:

$$\Delta Y = \text{Tr} \left[ L \rho + \rho L^\dagger \right] \Delta t + \Delta W,$$

where $W$ is the standard Wiener process. The time-evolution of $\rho(t)$ given the output $\Delta Y$ is

$$\rho(t + \Delta t) = V_{\Delta Y} \rho(t) V_{\Delta Y}^\dagger$$

$$= \mathcal{P}(\Delta Y) \left[ \rho - i[H, \rho] t + \frac{1}{2} \left[ L^\dagger L, \rho \right] t + L \rho \Delta Y + \frac{1}{2} \rho L^\dagger L \Delta t \right].$$  \hfill (S40)

In Eq. (S40), $dY^2 = dt$ since $dW^2 = dt$ for any non-anticipating functions [6]. Introducing an unnormalized density operator $\tilde{\rho}$, for $\Delta t \to 0$, $\tilde{\rho}$ is governed by

$$d\tilde{\rho} = \left( -i[H, \tilde{\rho}] + L\tilde{\rho}L^\dagger - \frac{1}{2} L^\dagger L \tilde{\rho} - \frac{1}{2} \tilde{\rho}L^\dagger L \right) dt + (L\tilde{\rho} + \tilde{\rho}L^\dagger) dY.$$

The normalized density is given by $\rho = \tilde{\rho}/\text{Tr}[\tilde{\rho}]$, which yields

$$d\rho = \left( -i[H, \rho] - \frac{1}{2} L^\dagger L \rho - \frac{1}{2} \rho L^\dagger L + L \rho L^\dagger \right) dt + [L \rho + \rho L^\dagger - \rho \text{Tr}(L \rho + \rho L^\dagger)] dW,$$

$$dY = \text{Tr}[L \rho + \rho L^\dagger] dt + dW.$$  \hfill (S41)

Equations (S41) and (S42) are known as the quantum state diffusion.

**S4. EXAMPLES**

**A. Two-level atom**

The Lindblad equation of the two-level atom is given by

$$\frac{d\rho}{dt} = -i [H, \rho] + L \rho L^\dagger - \frac{1}{2} \{L^\dagger L, \rho \},$$

where the Hamiltonian $H$ and the jump operator $L$ are defined by

$$H = \Delta |\epsilon_e \rangle \langle \epsilon_e| + \frac{\Omega}{2} (|\epsilon_e \rangle \langle \epsilon_g| + |\epsilon_g \rangle \langle \epsilon_e|),$$

$$L = \sqrt{\kappa} |\epsilon_g \rangle \langle \epsilon_e|.$$
Here, $|\epsilon_e\rangle$ and $|\epsilon_g\rangle$ are excited and ground states, respectively, $\Delta$ is a detuning between the laser field frequency and the atomic transition frequency, $\Omega$ is the Rabi oscillation frequency, and $\kappa$ is the decay rate. The steady-state density matrix is

$$\rho^{ss} = \begin{bmatrix} \rho^{ss}_{ee} & \rho^{ss}_{eg} \\ \rho^{ss}_{ge} & \rho^{ss}_{gg} \end{bmatrix} = \frac{1}{4\Delta^2 + \kappa^2 + 2\Omega^2} \begin{bmatrix} 4\Delta^2 + \kappa^2 + \Omega^2 & -2\Delta\Omega + i\kappa \Omega \\ -2\Delta\Omega - i\kappa \Omega & \Omega^2 \end{bmatrix},$$

where $\rho^{ss}_{ij} \equiv \langle \epsilon_i | \rho^{ss} | \epsilon_j \rangle$. In the main text, we performed a computer simulation for the jump measurement. We first show a trajectory of $\mathcal{Y}(t)$ as a function of $t$. As denoted in the main text, it is given by

$$\mathcal{Y}(t) = \int_0^t d\mathcal{Y} = \int_0^t dt' [L\rho + \rho L^\dagger] + \int_0^t dW,$$

and its average is

$$\mathbb{E} [\mathcal{Y}(t)] = t[L\rho^{ss} + \rho^{ss} L^\dagger] = -\frac{4\sqrt{\kappa}\Delta\Omega}{4\Delta^2 + \kappa^2 + 2\Omega^2} t.$$  

We plot an example of $\mathcal{Y}(t)$ as a function of $t$ in Fig. S1(a), where the solid line is a random realization of $\mathcal{Y}(t)$ and the dashed line is its expectation $\mathbb{E} [\mathcal{Y}(t)]$ shown by Eq. (S44).

We also check the QTUR for the diffusion measurement by generating $\kappa$, $\Omega$, and $\Delta$ randomly (ranges of the parameters are shown in the caption of Fig. S1(b)) and calculate $\text{Var}[\Theta_{\mathcal{Y}}]/\mathbb{E} [\Theta_{\mathcal{Y}}]^2$. As mentioned in the main text, the lower bound of the diffusion measurement is $1/4$ times smaller than that of the jump measurement case, since $\hbar'(0) = 1/2$. Figure S1(b) plots $\text{Var}[\Theta_{\mathcal{Y}}]/\mathbb{E} [\Theta_{\mathcal{Y}}]^2$ as a function of $I_Q(0) = T(\Upsilon + \Psi)$ with circles. In Fig. S1(b), we plot the lower bound $1/(4I_Q(0))$ with the dashed line. We also plot $1/(I_Q(0))$, which is the lower bound of the jump measurement case, with the dotted line. As can be seen, all the realizations are located above $1/(4I_Q(0))$, which verify the QTUR of Eq. (S21) [Eq. (6) in the main text] while some realizations are below $1/(I_Q(0))$, supporting that the bound of the diffusion measurement is $1/4$ times smaller than the jump measurement case.

### B. Three-level thermal machine

We also consider a three-level thermal machine in the main text. We follow a description in Ref. [7]. The Lindblad equation of the system is given by

$$\frac{d\rho}{dt} = -i[H, \rho] + \sum_{i \neq j} \left[ L_{ji} \rho L_{ji}^\dagger - \frac{1}{2} \left\{ L_{ji}^\dagger, L_{ji}, \rho \right\} \right],$$

where the Hamiltonian $H$ and the jump operator $L_{ji}$ are defined by

$$H = \omega_3 |\epsilon_B\rangle \langle \epsilon_B| + \omega_1 |\epsilon_A\rangle \langle \epsilon_A|,$$

$$L_{ji} = \sqrt{\gamma_{ji}} |\epsilon_j\rangle \langle \epsilon_i| \quad (i \neq j).$$

Here, $\omega_3$ and $\omega_1$ are the atomic transition frequencies, $\Omega_B$ is the Rabi oscillation frequency, and $\gamma$ is the decay rate. The equation of the system is given by

$$\frac{d\rho}{dt} = -i[H, \rho] + \sum_{i \neq j} \left[ L_{ji} \rho L_{ji}^\dagger - \frac{1}{2} \left\{ L_{ji}^\dagger, L_{ji}, \rho \right\} \right].$$

We also check the QTUR for the diffusion measurement by generating $\kappa$, $\Omega$, and $\Delta$ randomly (ranges of the parameters are shown in the caption of Fig. S1(b)) and calculate $\text{Var}[\Theta_{\mathcal{Y}}]/\mathbb{E} [\Theta_{\mathcal{Y}}]^2$. As mentioned in the main text, the lower bound of the diffusion measurement is $1/4$ times smaller than that of the jump measurement case, since $\hbar'(0) = 1/2$. Figure S1(b) plots $\text{Var}[\Theta_{\mathcal{Y}}]/\mathbb{E} [\Theta_{\mathcal{Y}}]^2$ as a function of $I_Q(0) = T(\Upsilon + \Psi)$ with circles. In Fig. S1(b), we plot the lower bound $1/(4I_Q(0))$ with the dashed line. We also plot $1/(I_Q(0))$, which is the lower bound of the jump measurement case, with the dotted line. As can be seen, all the realizations are located above $1/(4I_Q(0))$, which verify the QTUR of Eq. (S21) [Eq. (6) in the main text] while some realizations are below $1/(I_Q(0))$, supporting that the bound of the diffusion measurement is $1/4$ times smaller than the jump measurement case.
Here, $|\epsilon_g\rangle$, $|\epsilon_A\rangle$, and $|\epsilon_B\rangle$ are energy levels, and $\eta_{ji}$ is the transition rate from $|\epsilon_i\rangle$ to $|\epsilon_j\rangle$. $\omega_1$, $\omega_2$, and $\omega_3 = \omega_1 + \omega_2$ are energy gaps between $|\epsilon_A\rangle \leftrightarrow |\epsilon_g\rangle$, $|\epsilon_B\rangle \leftrightarrow |\epsilon_A\rangle$, and $|\epsilon_B\rangle \leftrightarrow |\epsilon_g\rangle$, respectively. The transition rate fulfills the detailed balance condition:

$$\frac{\eta_{gA}}{\eta_{Ag}} = \frac{\gamma (n_{1}^{th} + 1)}{\gamma n_{1}^{th}} = e^{\beta_1 \omega_1},$$

$$\frac{\eta_{AB}}{\eta_{BA}} = \frac{\gamma (n_{2}^{th} + 1)}{\gamma n_{2}^{th}} = e^{\beta_2 \omega_2},$$

$$\frac{\eta_{BB}}{\eta_{BG}} = \frac{\gamma (n_{3}^{th} + 1)}{\gamma n_{3}^{th}} = e^{\beta_3 \omega_3},$$

where $n_{r}^{th} = (e^{\beta_r \omega_r} - 1)^{-1}$, $\beta_r$ is the inverse temperature of $r$th heat reservoir, and $\gamma$ is the decay rate. The entropy production of the thermal machine is

$$\Sigma_{TM} = -3 \sum_{r=1}^{3} \beta_r \dot{Q}_r,$$

where $\dot{Q}_r$ is the heat flux entering from the $r$th reservoir. $\dot{Q}_r$ is represented by

$$\dot{Q}_1 = \text{Tr}[(D(\rho, L_{Ag}) + D(\rho, L_{AA})) H] = \omega_1 \left( \eta_{Ag} \rho_{gg}^{ss} - \eta_{gA} \rho_{AA}^{ss} \right),$$

$$\dot{Q}_2 = \text{Tr}[(D(\rho, L_{BA}) + D(\rho, L_{BB})) H] = \omega_2 \left( \eta_{BA} \rho_{AA}^{ss} - \eta_{AB} \rho_{BB}^{ss} \right),$$

$$\dot{Q}_3 = \text{Tr}[(D(\rho, L_{Bg}) + D(\rho, L_{gB})) H] = \omega_3 \left( \eta_{BG} \rho_{gg}^{ss} - \eta_{gB} \rho_{BB}^{ss} \right).$$

Combining Eqs. (S48)–(S54), we obtain

$$\Sigma_{TM} = -\beta_1 \omega_1 \left( \eta_{gA} \rho_{gg}^{ss} - \eta_{Ag} \rho_{AA}^{ss} \right) - \beta_2 \omega_2 \left( \eta_{BA} \rho_{AA}^{ss} - \eta_{AB} \rho_{BB}^{ss} \right) - \beta_3 \omega_3 \left( \eta_{BG} \rho_{gg}^{ss} - \eta_{gB} \rho_{BB}^{ss} \right)$$

which is identical to the heat dissipation rate in stochastic thermodynamics. For steady-state, we obtain $\Sigma_{TM} = \Sigma$.

We first consider a standard jump measurement [Eq. (15) in the main text]. The dynamics of the density matrix is given by the stochastic Schrödinger equation:

$$d\rho = \left( -i[H, \rho] - \frac{1}{2} \sum_{i \neq j} \left\{ L_{ji}^\dagger L_{ji}, \rho \right\} + \rho \text{Tr} \left[ L_{ji}^\dagger \rho L_{ji} \right] \right) dt + \sum_{i \neq j} \left( \frac{L_{ji} \rho L_{ji}^\dagger}{\text{Tr}[L_{ji} \rho L_{ji}^\dagger]} - \rho \right) dN_{ji},$$

where $dN_{ji}$ is a noise increment. $dN_{ji} = 1$ when the jump from $|\epsilon_i\rangle$ to $|\epsilon_j\rangle$ occurs in each time interval $dt$ and $dN_{ji} = 0$ otherwise. Its expectation is given by $\mathbb{E}[dN_{ji}] = \text{Tr}[L_{ji} \rho \sigma_{i}^{ss} L_{ji}^\dagger] dt = \eta_{ji} \sigma_{i}^{ss} dt$. The corresponding Kraus operator $V_m$ is given by

$$V_{ji} = L_{ji} \sqrt{\Delta t}, \quad (i \neq j),$$

$$V_{0} = I_{2} - i \left[ H - \frac{i}{2} \sum_{i,j} L_{ji}^\dagger L_{ji} \right] \Delta t.$$

In the main text, we consider the following observable:

$$\Theta_C \equiv \sum_{i \neq j} R_{ji} \int_{0}^{T} dN_{ji},$$

where $R_{ji} \in \mathbb{R}$ and $R_{ji} = -R_{ij}$ for all $i$ and $j$. Using Eq. (S27), we find that the following relation holds:

$$\mathbb{E}_\theta[dN_{ji}] - \mathbb{E}_\theta[dN_{ij}] = \text{Tr} \left[ L_{ji,\theta} \rho^{ss} L_{ji,\theta}^\dagger \right] - \text{Tr} \left[ L_{ij,\theta} \rho^{ss} L_{ij,\theta}^\dagger \right] dt$$

$$= \eta_{ji} \left[ 1 + \theta \left( 1 - \sqrt{\frac{\eta_{ji} \rho_{ij}^{ss}}{\eta_{ij} \rho_{jj}^{ss}}} \right) \right] \rho_{ii}^{ss} dt - \eta_{ij} \left[ 1 + \theta \left( 1 - \sqrt{\frac{\eta_{ij} \rho_{jj}^{ss}}{\eta_{ji} \rho_{ jj}^{ss}}} \right) \right] \rho_{jj}^{ss} dt$$

$$= (1 + \theta) \left( \eta_{ji} \rho_{ii}^{ss} - \eta_{ij} \rho_{jj}^{ss} \right) dt$$

$$= (1 + \theta) \left[ \mathbb{E}_{\theta=0}[dN_{ji}] - \mathbb{E}_{\theta=0}[dN_{ij}] \right] dt.$$

(S59)
Therefore, $\Theta_C$ satisfies the scaling condition of Eq. (5) and thus the QTUR of Eq. (14) holds for $\Theta_C$.

In the main text, we also consider a generalized jump measurement. First, note that the Lindblad equation is invariant under the following transformation:

$$L'_{ji} = (L_{ji} + \zeta_{ji} I_S), \quad (S60)$$

$$H' = H - \frac{\partial}{2} \sum_{i \neq j} \left[ \zeta_{ji}^* L_{ji} - \zeta_{ji} L_{ji}^* \right], \quad (S61)$$

where $\zeta_{ji} \in \mathbb{C}$ is a parameter. $\zeta_{ji} = 0$ for all $i$ and $j$ recovers the standard jump measurement. The Lindblad equation remains unchanged when we replace $H$ and $L_{ji}$ with $H'$ and $L'_{ji}$, respectively. With this transformation, the stochastic Schrödinger equation is given by

$$d\rho = \left( -i[H', \rho] - \frac{1}{2} \sum_{i \neq j} \left[ L_{ji}^\dagger L_{ji}' + L_{ji}' L_{ji}^\dagger \right] \rho + \rho \text{Tr} \left[ L_{ji}' \rho L_{ji}^\dagger \right] \right) dt + \sum_{i \neq j} \left( \frac{L_{ji}' \rho L_{ji}^\dagger}{\text{Tr}[L_{ji}' \rho L_{ji}^\dagger]} - \rho \right) dN'_{ji}, \quad (S62)$$

where $dN'_{ji}$ is a noise increment, whose meaning is identical to $dN_{ji}$. The expectation of $dN'_{ji}$ is $\mathbb{E}[dN'_{ji}] = \text{Tr}[L_{ji}' \rho \rho^S L_{ji}^\dagger] dt$. The corresponding Kraus operators are given by

$$V_{ji}' = L_{ji}' \sqrt{\Delta t}$$

$$= (L_{ji} + \zeta_{ji} I_S) \sqrt{\Delta t}, \quad (S63)$$

$$V_0' = I_S - i \left[ H' - \frac{i}{2} \sum_{i \neq j} L_{ji}^\dagger L_{ji}' \right] \Delta t$$

$$= I_S - i \left[ H - \frac{i}{2} \sum_{i \neq j} \left[ \zeta_{ji}^* L_{ji} - \zeta_{ji} L_{ji}^* \right] - \frac{i}{2} \sum_{i \neq j} (L_{ji}^\dagger + \zeta_{ji}^* I_S)(L_{ji}' + \zeta_{ji} I_S) \right] \Delta t. \quad (S64)$$

We can easily show that Eqs. (S63) and (S64) satisfies $V_0' V_{ji}' + \sum_{i \neq j} V_{ji}' V_{ji}' = I$.

As mentioned above, the Lindblad equation of Eq. (S45) remains unchanged under the transformation of Eqs. (S60) and (S61). However, a quantum trajectory of the transformed stochastic Schrodinger equation becomes different from the standard jump measurement case. Quantum trajectories of the standard jump measurement [Eq. (S55)] are transitions between the energy eigenstates $|\epsilon_A\rangle$, $|\epsilon_B\rangle$, and $|\epsilon_g\rangle$ (Fig. S2(a)). On the other hand, for the generalized jump measurement [Eq. (S62)], quantum trajectories are not jumps between the energy eigenstates (Fig. S2(b)).

From Eq. (S60), the mean of the noise increment $dN'_{ji}$ is

$$\mathbb{E}[dN'_{ji}] = \text{Tr} \left[ L_{ji}' \rho \rho^s L_{ji}^\dagger \right] dt$$

$$= \left( \text{Tr} \left[ L_{ji} \rho \rho^s L_{ji}^\dagger \right] + \text{Tr} \left[ L_{ji} \rho \zeta_{ji}^* \right] + \text{Tr} \left[ \zeta_{ji} \rho \rho^s L_{ji}^\dagger \right] + \text{Tr} \left[ \zeta_{ji} \rho \zeta_{ji}^* \right] \right) dt$$

$$= \left[ \text{Tr} \left[ L_{ji} \rho \rho^s L_{ji}^\dagger \right] dt + |\zeta_{ji}|^2 dt \right]$$

$$= \mathbb{E}[dN_{ji}] + |\zeta_{ji}|^2 dt. \quad (S65)$$

In the second line, we used the fact that $\rho^s$ is diagonal in the energy eigenbasis. Again, we consider an observable $\Theta'_C$, where $dN_{ji}$ is replaced with $dN'_{ji}$:

$$\Theta'_C \equiv \sum_{i \neq j} R_{ji} \int_0^T dN'_{ji},$$

where $R_{ji} = -R_{ij}$. Using Eq. (S59) and (S65), we calculate Eq. (S59) for $dN'_{ji}$ as follows:

$$\mathbb{E}_\theta \left[ dN'_{ji} \right] - \mathbb{E}_\theta \left[ dN_{ij} \right] = \mathbb{E}_\theta \left[ dN_{ji} \right] - \mathbb{E}_\theta \left[ dN_{ij} \right] - |\zeta_{ji}|^2 dt - |\zeta_{ji}|^2 dt$$

$$= (1 + \theta) \left( \mathbb{E}_{\theta=0} \left[ dN_{ji} \right] - \mathbb{E}_{\theta=0} \left[ dN_{ij} \right] \right) + |\zeta_{ji}|^2 dt - |\zeta_{ji}|^2 dt$$

$$= (1 + \theta) \left( \mathbb{E}_{\theta=0} \left[ dN'_{ji} \right] - \mathbb{E}_{\theta=0} \left[ dN_{ij} \right] - |\zeta_{ji}|^2 dt + |\zeta_{ij}|^2 dt \right) + |\zeta_{ij}|^2 dt - |\zeta_{ij}|^2 dt.$$
Therefore when $|ζ_{ji}| = |ζ_{ij}|$ for all $i$ and $j$, the observable $Θ'_C$ satisfies the scaling condition and thus it obeys the QTUR of Eq. (S32) [Eq. (14)].

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