On the size scaling of the nearest level spacing at criticality

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It is conjectured that the size scaling of the nearest level spacing in the critical spectral region, \( S(N) \propto N^{-\lambda} \), remains qualitatively the same within phases of extended and critical states. The exponent \( \lambda \) is therefore identical to that for the bare level spacing (at zero disorder). Our calculation of the scaling for the one-dimensional model with diagonal disorder and long-range power-like interaction confirms the conjecture.

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Localization-delocalization transition (LDT) in disordered systems, predicted by Anderson for 3D in 1958 [1], still remains a thoroughly investigated problem (see Refs. [2, 3] for an overview). A remarkable progress has been achieved in understanding of critical systems, in particular, the multifractal nature of critical wave functions has been discovered [4, 5, 6, 7, 8]. Statistics of energy levels within disordered models have also been intensively investigated during the recent years. It was demonstrated that statistical properties of level spacing are closely related to general localization properties [9, 10, 11].

The spacing between two energy levels is determined by the overlap of the corresponding wave functions. Thus, weak overlap of localized states results in Poisson statistics of energy levels in the thermodynamic limit, which implies that the level spacing can be arbitrary small and does not depend on the system size. Contrary to that, strong overlap of the extended eigenstates results in strong level repulsion. Therefore, Wigner-Dyson statistics describe well energy levels in good metal [9, 12]. The level spacing for extended states depends on the system size. In this contribution we consider the level spacing in a critical spectral regions, i.e. in a region where the localization-delocalization transition takes place or at the mobility edge.

Within the phase of the extended states, the exponent \( \lambda \) of the size scaling of the level spacing, \( S(N) \propto N^{-\lambda} \), is expected to be independent of disorder, otherwise the statistics of the energy levels spacing could change with disorder, which would be in direct contradiction to the robustness of the Wigner-Dyson statistics within the phase of the extended states [2, 12, 13]. At criticality, e.g. close to the transition point or the mobility edge, the level statistics, although manifesting strong level repulsion, is different from the Wigner-Dyson one [10]. Moreover, reasonings based on the consideration of wave function overlap should be used with caution, because critical eigenstates are very sparse and it is not obvious how much they overlap. Fyodorov and Mirlin addressed this problem within the framework of the power law random-banded model (see Refs. [14, 15] and references therein for the detailed description of the model). They showed analytically that two close critical eigenstates are strongly correlated, in the sense that the overlap of the two states is of the order of the self-overlap of a state [16]. It was conjectured in Ref. [16] that the strong correlation of critical states is a general property. This correlation results in strong level repulsion up to the critical point [11, 17, 18, 19, 20, 21]. Because the level spacing is determined by the overlap of wave functions, we conjecture that the strong overlap of critical eigenfunctions preserves the size scaling of the level spacing within the phase of the critical states also. Thus, the scaling exponent \( \lambda \) is expected to be the same throughout both phases of extended and critical states. The latter phase extends up to the magnitude of disorder at which the localization length becomes equal to the linear system size. The exponent is therefore identical to that for the bare level spacing (at zero disorder) in the considered energy region.

To prove the conjecture on the independence of the scaling exponent of disorder we consider the level spacing within the framework of the tight-binding Hamiltonian on a one-dimensional (1D) regular lattice of \( N \) sites with power-like inter-site interaction:

\[
\mathcal{H} = \sum_{n=1}^{N} \varepsilon_n |n\rangle\langle n| + \sum_{m,n=1}^{N} J_{mn} |m\rangle\langle n| ,
\]

where \(|n\rangle\) is the ket vector of a state with on-site energy \( \varepsilon_n \). The energies are stochastic variables, uncorrelated for different sites and distributed uniformly around zero within the interval of width \( W \). The hopping integrals are \( J_{mn} = J/|m-n|^\mu \), \( J_{nn} = 0 \) with \( 1 < \mu \leq 3/2 \). We set \( J > 0 \), then the LDT occurs at the upper band edge provided \( 1 < \mu < 3/2 \) [22, 23]. The latter fact makes the model very advantageous from the viewpoint of numerical calculations because efficient algorithms, such as the Lanczos, can be used to find a few extreme eigenstates of interest. In all calculations we consider three uppermost states under open boundary conditions for system sizes in the range \( N = 4096 \div 65536 \). All results are averaged over more than \( 5 \times 10^3 \times (65536/N) \) disorder realizations. The lattice constant is set to unity.

Recently, the model [11] was studied numerically for
two values of interaction exponent $\mu$: $\mu = 3/2$ and $\mu = 4/3$ [24]. In the former marginal case no signs of transition was found, while at $\mu = 4/3$ the model was found to reveal the LDT at $W_c = 10.9 \pm 0.2$. In this contribution we focus on the analysis of the level spacing. In particular, we study the size scaling of the level spacing at the top of the band in the vicinity of the critical value of disorder $W_c$ for $\mu = 4/3$.

We are particularly interested in the scaling of the level spacing within the phase of the critical states. Because we deal with finite systems, the phase of critical states is of finite width, so we determine the width first. The critical phase extends from the critical point $W = W_c$ up to the disorder magnitude $W = W_L$ at which the localization length becomes equal to the linear system size: $\xi(W_c) = L$ (within the considered 1D model $L \equiv N$). To determine the disorder magnitude $W_L$ we analyze the size scaling of the typical value of the inverse participation ratio (IPR). The IPR for the state $\alpha$ is defined in a standard way:

$$I_2^{(\alpha)} = \sum_{n=1}^{N} |\psi_{\alpha n}|^4,$$

where $\psi_{\alpha n}$ is the $\alpha$-th normalized eigenstate of the Hamiltonian $H$. The typical value of the IPR is defined as $I_2 = \exp(\langle \ln I_2^{(\alpha)} \rangle)$, where the angle brackets stand for averaging over disorder realizations and $\alpha$. We use the typical value of the IPR rather than the mean value of the IPR because, unlike the latter, the former is a self-averaged quantity [23].

Within the phase of extended states the IPR size scales as $I_2 \propto L^{-1}$, while for the localized states the scaling reads: $I_2 \propto L^{D_2}$. Wegner found that at criticality $I_2$ size scales anomalously [3]: $I_2 \propto L^{-D_2}$ where the fractal dimension $D_2$ is smaller than the spacial dimensionality, which results from the multifractal structure of the eigenfunctions [4, 5, 6, 7, 8]. Above the transition point the scaling of the IPR is governed by the localization length $\xi$ rather than by the system size: $I_2 \propto \xi^{D_2}$ [27]. The localization length depends on the magnitude of disorder in the vicinity of the critical point $\xi(W) = \xi_0|W/W_c - 1|^{-\nu}$ ($\nu > 0$), therefore, two scaling regimes are expected above the critical point. Within the phase of the critical states, i.e. for $W > W_c$ and system sizes $L < \xi(W)$, the IPR scaling is still determined by the size of the system: $I_2 \propto L^{-D_2}$. Contrary to that, within the phase of the localized states, i.e. for $L > \xi(W)$, the IPR scaling is determined by the localization length $\xi(W)$ and does not depend on the system size.

Fig. 1 shows the size scaling of the quantity $\tilde{I}_2 = I_2 L^{D_2}$. The value of the correlation dimension $D_2$ was obtained by studying the size scaling of the typical value of the IPR at criticality: $D_2 = 0.66$ for $\mu = 4/3$ (a detailed analysis of the correlation dimension is to be published elsewhere). Within the phase of the extended states the quantity $\tilde{I}_2$ decreases with the system size: $\tilde{I}_2 \propto L^{D_2 - 1}$ ($D_2 < 1$). At the transition point, $\tilde{I}_2$ is expected to be constant. Above the transition, two scaling regimes are expected: $\tilde{I}_2$ remains constant for $L < \xi(W)$, while for systems that are larger than the localization length, $\tilde{I}_2$ is expected to increase with the size: $\tilde{I}_2 \propto \xi^{-D_2} L$. The above transparent scenario suggests that for a given set of system sizes the phase of the critical states extends from the critical point $W = W_c$ up to the disorder magnitude $W = \tilde{W} > W_c$ at which the size scaling of $\tilde{I}_2$ starts to deviate from a constant (as a matter of fact, $\tilde{W} = W_L$ for $L = L_{max}$). Fig. 1 demonstrates that, for the considered system sizes ($L \equiv N = 4096 + 65536$), the critical phase extends up to the $\tilde{W} = 11.5 J$.

Having obtained the disorder at which the uppermost eigenstates are critical we turn to the analyses of the size scaling of the level spacing for these states. Under periodic boundary conditions the level spacing at the upper edge of the bare band (for non-disordered system) depends on the system size $N$ ($N \gg 1$) as follows:

$$S_0(N) = \frac{C(\mu)}{N^{\mu - 1}} + O\left(\frac{1}{N^\mu}\right),$$

where $C(\mu) = 2 \Gamma(2 - \mu) \cos(\pi (\mu - 1)/2)$. Under open boundary conditions, which we used in all calculations, no analytical expression for the spacing at the upper band edge can be obtained. Nevertheless, its size scaling is close to Eq. (3) and the leading (non-zero) power of $N$ in the expansion is also $1 - \mu$. In particular, for $\mu = 4/3$ the spacing reads:

$$S(N) = \frac{C_{4/3}}{N^{1/3}} + O\left(\frac{1}{N^{4/3}}\right),$$
the constant $C_{4/3}$ was obtained numerically: $C_{4/3} \approx 3.58$. We calculated the size scaling of the mean level spacing at the upper band edge for $\mu = 4/3$ and different values of disorder magnitude $W$. For each value of $W$, we fit the formula $CN^{-\lambda}$ to the calculated size dependence of the mean spacing, varying the system size within the range $N = 4096 \div 65536$. The result of the fit is presented in Fig. 2: squares show the disorder dependence of the exponent $\lambda$ normalized to its value for non-disordered system, 1/3, while triangles show the disorder scaling of the factor $C$ normalized to $C_{4/3}$ (see Eq. 4). Vertical lines demonstrate the critical disorder $W = 10.7 \div 11.1 J$. Error bars represent confidence intervals of the parameters $C$ and $\lambda$. The statistical error is of the order of the symbol size.

![Diagram](image-url)

FIG. 2: Disorder scaling of the parameters $\lambda$ and $C$ for $\mu = 4/3$ in the vicinity of the critical point.

The figure demonstrates that within the phase of the extended and critical states, that is for $W < 11.5 J$, the exponent that determines the size dependence of the mean level spacing in the critical energy region (i.e. at the upper band edge) is the same as that for the bare band: $\lambda = 1/3$. The disorder reduces the level spacing, as expected, but does not affect the qualitative dependence of the spacing on the system size. Within the phase of the localized states, that is for $W > 11.5 J$, the localization length becomes smaller than the system size, and the level spacing is not determined by the size of the system. Therefore, the formula $CN^{-\lambda}$ becomes irrelevant within the phase of the localized states; the increase of the confidence intervals for the parameters $C$ and $\lambda$ in the range $W > 11.5 J$ manifests the irrelevance.

To conclude, we have demonstrated that within both phases of the extended and critical states the size scaling of the level spacing remains qualitatively unchanged: the exponent of the size scaling does not depend on disorder. The exponent is therefore the same as that in the considered energy region of the bare band (for a non-disordered system). The above result is probably general as it is a consequence of strong overlap of critical eigenstates that was reported on recently [16]. The independence of the scaling exponent of the disorder has several important consequences. It suggests, in particular, that the bare scaling of the energy levels can be used for qualitative arguments that are applied to disordered systems (see, for example, Ref. 23 and references therein). Properties of critical states are usually studied by considering all states within an energy window in the vicinity of the critical point. The mean level spacing $\Delta \propto N^{-1}$ is often used to estimate the number of critical states, which can lead to an over-estimation of this number, because the spacing in the critical region is much greater than $\Delta$. In particular, for the standard three-dimensional Anderson model the critical spectral region is the center of the band, where the spacing scales as $S \propto N^{-1/3}$; the relevant spacing is therefore much greater than the average one. The over-estimation of the number of the critical states can lead to consideration of not only critical states but localized states also, which would result in the underestimation of the correlation dimension, affect the energy level statistics, etc.

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