Converting Zitterbewegung oscillation to directed motion

Q. Zhang\textsuperscript{1}, J. B. Gong\textsuperscript{2,3} and C. H. Oh\textsuperscript{1,4}

\textsuperscript{1} Centre of Quantum Technologies and Department of Physics, National University of Singapore
Singapore 117542, Republic of Singapore
\textsuperscript{2} Department of Physics and Centre for Computational Science and Engineering, National University of Singapore
Singapore 117542, Republic of Singapore
\textsuperscript{3} NUS Graduate School for Integrative Sciences and Engineering - Singapore 117597, Republic of Singapore
\textsuperscript{4} Institute of Advanced Studies, Nanyang Technological University - Singapore 639798, Republic of Singapore

received 28 May 2011; accepted in final form 10 August 2011
published online 15 September 2011

PACS 03.75.-b – Matter waves
PACS 32.80.Qk – Coherent control of atomic interactions with photons
PACS 71.70.Ej – Spin-orbit coupling, Zeeman and Stark splitting, Jahn-Teller effect

Abstract – Zitterbewegung (ZB) oscillation, namely, the jittering center-of-mass motion predicted by free-space Dirac (or Dirac-like) equations, has been studied in several different contexts. It is shown here that ZB can be converted to directed center-of-mass motion by a modulation of the Dirac-like equation, if the modulation is on resonance with the ZB frequency. Tailored modulation may also stop, re-launch or even reverse the directed motion of a wavepacket with negligible distortion. The predictions may be examined by current ZB experiments using trapped-ion systems.

Copyright © EPLA, 2011

Introduction. – Zitterbewegung (ZB) oscillation originally refers to the trembling motion of a free relativistic particle (e.g., electron) as described by the Dirac equation \cite{1}. ZB in this context has an extremely large frequency and a very small amplitude, so it is hardly observable. Nevertheless, the physics of ZB becomes highly relevant to a number of interesting situations where the quantum dynamics is governed by artificial Dirac or Dirac-like equations. Examples include band electrons in graphene \cite{2}, cavity electrodynamics \cite{3}, ultracold atoms in designed laser fields \cite{4–8}, macroscopic sonic crystals and photonic superlattices \cite{9}, as well as trapped-ion systems \cite{10,11}. For a review of recent fruitful studies on ZB physics using electrons in semiconductors, see ref. \cite{12}.

Many quantum dynamical phenomena induced by a control field have been explored using the Schrödinger equation. Much less is known in the case of driven Dirac or Dirac-like equations. This work is concerned with ZB in a simple version of a driven Dirac-like equation. Previously, we showed that ZB in the cold-atom context as a quantum coherence phenomenon can be coupled with some mechanical oscillations, resulting in the control of the amplitude, the frequency, and the damping of the ZB associated with a wavepacket \cite{8}. Going one step further, it should be of interest to study what happens to ZB if one term in a Dirac-like equation is modulated with a frequency on-resonance with the ZB frequency. Indeed, in both classical mechanics and quantum mechanics, it is always expected that an oscillation phenomenon can behave remarkably differently when it is subject to an on-resonance or almost one-resonance driving field. A latest example is the Bloch oscillation subject to an almost on-resonance driving field, where the Bloch oscillation may be converted to “super-Bloch oscillation” or even to directed motion \cite{13–15}.

In this work we start from a Dirac-like equation that is already experimentally realized in the context of trapped-ion systems \cite{11}. We then introduce a modulation to the Dirac-like equation, whose frequency is on resonance with the ZB frequency. Due to the on-resonance driving, it is shown, both analytically and numerically, that ZB associated with a wavepacket can be converted to directed wavepacket motion. By tailoring the modulation, we may also stop, re-launch or even reverse the directed motion of a wavepacket with negligible distortion. These results indicate that ZB is not a mysterious oscillation, and it provides an extra useful control knob for manipulating quantum motion.

Theoretical analysis. – Let us consider a simple version of a two-component-spinor Dirac-like equation in
one dimension (taking $\hbar = 1$ throughout),
\[
\frac{d}{dt} |\psi\rangle = H |\psi\rangle = \frac{1}{2} \left( C \sigma_y + D p_x \sigma_z \right) |\psi\rangle,
\]  
(1)
where $p_x$ is the momentum in the $x$-direction; $C$ and $D$ are two $\epsilon$-number coefficients, $\sigma_x$, $\sigma_y$ are Pauli matrices in the $x$ and $y$ directions. In a trapped-ion realization of the above equation, parameters $C$ and $D$ are directly connected with the laser intensity of various laser fields. In particular, $C$ is proportional to the strength of a carrier resonance laser field and $D$ is proportional to the common strength of four sideband laser fields. We assume the product state of a one-dimensional wavepacket in $x$-direction (taking $\hbar = 1$)

\[
\text{We now examine the motion of an initial state } (t = 0) \text{ as a product state of a one-dimensional wavepacket in } x \text{ and an internal two-component spinor state,}
\]

\[
|\psi\rangle = G(x) \begin{pmatrix} a \\ b \end{pmatrix} e^{ip_x x},
\]

(2)
where $G(x)$ is a broad Gaussian in real space centered at $x_0 = 0$; $p_0$ (assumed to be zero below) is the central momentum of the wavepacket along $x$. To better understand ZB, the dynamics can be analyzed in the momentum space. Hence we carry out the Fourier transformation of the wave function to get a narrow wavepacket in momentum representation (the required narrowness will be explained later),

\[
|\psi_p(0)\rangle = \langle p_x |\psi\rangle = g(p_x) \begin{pmatrix} a \\ b \end{pmatrix} e^{-ip_0 p_x},
\]

(3)
where $g(p_x)$ is also a Gaussian as the Fourier transformation of $G(x)$.

Explicit initial spinor state will be specified in eq. (6). In general, an arbitrary internal two-component spinor state will evolve because it effectively experiences two “magnetic fields”: one along $y$ of strength $C_0$ and the other along $x$ with strength $Dp_x$ for the component $p_x$ (see eq. (1)). The total effective magnetic-field strength is $\sqrt{C_0^2 + p_x^2 D^2}$ and the direction of the total magnetic field is characterized by an angle $\arctan(Dp_x/C_0)$. To obtain analytical results, we keep effects up to the first order of $Dp_x/C_0$, thus making approximations $\arctan(p_x D/C_0) \approx p_x D/C_0$ and $\sqrt{C_0^2 + p_x^2 D^2} \approx C_0$. Physically, our approximation is to assume that for different $p_x$ components, their effective Zeeman splitting is assumed to be almost the same, but with the internal state precessing around slightly different directions characterized by $\arctan(Dp_x/C_0)$. With our approximation we then obtain the following two eigenstates of Hamiltonian $H$ (see eq. (1), for $C = C_0$, i.e.:

\[
|\psi_+(p_x)\rangle \approx \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i e^{i\frac{p_x D}{C_0}} \end{pmatrix}, \quad E_+ \approx C_0
\]

(4)
\[
|\psi_-(p_x)\rangle \approx \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i e^{-i\frac{p_x D}{C_0}} \end{pmatrix}, \quad E_- \approx -C_0,
\]
where $E_{\pm}$ are the associated energy eigenvalues. Exploiting the explicit expressions of the above eigenstates, the analytical solution to the Dirac-like equation (1) can be directly obtained. That is,

\[
|\psi_p(t)\rangle = \langle p_x |\psi_p(0)\rangle e^{-iE_+t} |\psi_+\rangle + \langle p_x |\psi_p(0)\rangle e^{-iE_-t} |\psi_-\rangle
\]

\[
= a \cos \left( \frac{1}{2} C_0 t \right) g(p_x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\alpha_p p_x} + a \sin \left( \frac{1}{2} C_0 t \right) g(p_x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i(x_0 + \frac{C_0}{D} p_x)}
\]

\[
+ b \cos \left( \frac{1}{2} C_0 t \right) g(p_x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\alpha_p p_x} - b \sin \left( \frac{1}{2} C_0 t \right) g(p_x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i(x_0 - \frac{C_0}{D} p_x)}. \]  
(5)
Here the extra $D$-dependent phase factors $e^{i\left[\mp \left(D/C_0\right) \alpha_p \right]}$ in the second and fourth terms arise from the $p_x$-dependence of the spinor precession axis (or of the eigenstates in eq. (4)). The meaning of our early assumption of a narrow wavepacket in the momentum space is also clear: the main part of the wavepacket should ensure that $|p_x D| \ll C_0$. Considering effects due to $(p_x D/C_0)^2$ or high orders may capture some insignificant deformation of the wavepacket and the damping of ZB, but as demonstrated by our full numerical results below, these high-order effects will not change the essence of the physics under discussion and can be neglected here.

Interestingly, the analytical solution in eq. (5) can be interpreted as a superposition of four sub-wavepackets: two centered at $x_0 = 0$, one at $x = x_0 + D/C_0 = D/C_0$ and another one at $x = x_0 - D/C_0 = -D/C_0$. In terms of the mean position of the sub-wavepackets, the first two terms and the last two in eq. (5) are symmetric with respect to $x_0 = 0$, we therefore examine only the first two terms without loss of generality. With this in mind and for convenience we choose below the initial product state as $a = 1$, $b = 0$ such that

\[
|\psi_p(0)\rangle = \langle p_x |\psi\rangle = g(p_x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\alpha_p p_x}.
\]

(6)
As a result only the first two terms in eq. (5) arise, with $a = 1$ and the corresponding two sub-wavepackets centered at $x_0 = 0$ and $x = x_0 + D/C_0 = D/C_0$. The occupation probabilities of the two sub-wavepackets are given by
Converting Zitterbewegung oscillation to directed motion

Fig. 1: Schematic plot of wavepacket motion satisfying a Dirac-like equation. In case (a), a Dirac-like equation is not modulated. Cyclic population transfer between two sub-wavepackets gives rise to ZB oscillation. In case (b), a Dirac-like equation is periodically modulated, with the modulation frequency on-resonance with the ZB frequency. Due to the on-resonance modulation, sub-wavepackets successively re-emerge in different positions, yielding directed motion.

\[
\begin{align*}
\cos^2(C_0 t/2) & \quad \sin^2(C_0 t/2) \\
\text{trembling motion} & \\
\text{initial wavepacket} & \quad \text{current wavepacket} \\
\end{align*}
\]

The occupation probabilities oscillate, thus giving rise to a time-independent positions, yielding directed motion. The frequency is on-resonance with the ZB frequency. Dueto the on-resonance modulation, sub-wavepackets successively re-emerge in different positions, yielding directed motion.

\[
\begin{align*}
\langle x \rangle &= \cos^2(C_0 t/2)x_0 + \sin^2(C_0 t/2)(x_0 + D/C_0) \\
&= \frac{D}{2C_0}[1 - \cos(C_0 t)],
\end{align*}
\]

which is nothing but a ZB phenomenon, with the ZB frequency \(C_0\), the ZB period \(T_{ZB} = 2\pi/C_0\), and the “equilibrium point” of the oscillation at \(x = D/(2C_0)\). In particular, when \(t = \pm \pi/C_0\), the sub-wavepacket located at \(x = 0\) disappears and the other one located at \(x = D/C_0\) reaches its maximal population. Figure 1(a) schematically depicts this ZB mechanism.

Given this interpretation of ZB in terms of the population cycling between two sub-wavepackets, we next investigate what if the sign of the \(\sigma_y\) term is changed at \(t = \pi/C_0 = T_{ZB}/2\), with the modified Hamiltonian given by

\[
H = \frac{1}{2}(-C_0 \sigma_y + Dp_x \sigma_x).
\]

At that instant, the state of the system is given by

\[
|\psi_p(\pi/C_0)\rangle = \langle p_x|\psi(t = \pi/C_0)\rangle = g(p_x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i(\frac{2\pi}{C_0})p_x}.
\]

Adopting the same approximations as introduced above, we analytically obtain

\[
|\psi_p(t)\rangle = \cos(C_0(t - \pi/C_0)/2)g(p_x) \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i(\frac{2\pi}{C_0})p_x} + \sin(C_0(t - \pi/C_0)/2)g(p_x) \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i(\frac{2\pi}{C_0})p_x}.
\]

Remarkably, eq. (10) represents a superposition of a sub-wavepacket located at \(x = D/C_0\) and a new one at \(x = 2D/C_0\). The new equilibrium position of ZB is hence shifted to \(x = 3D/(2C_0)\). Loosely speaking, this indicates that if a sign change of the \(C_\sigma_y\) term occurs at the right moment, then the \(D\)-dependent extra phase factor (see eqs. (5) and (10)) in the time-evolving state makes an opposite contribution and consequently a sub-wavepacket at a new location emerges. From this new solution, it is also observed that if at \(t = 2\pi/C_0 = T_{ZB}\), the occupation probability of the sub-wavepacket centered at \(x = D/C_0\) will totally vanish and that of the other centered at \(x = 2D/C_0\) will become unity. The average position of the system moves from \(x = D/C_0\) at \(t = T_{ZB}/2\) to \(x = 2D/C_0\) at \(t = T_{ZB}\).

Repeating this strategy, i.e., abruptly changing the sign of \(C\) after every interval of \(T_{ZB}/2\), a directed transport emerges from a modulated ZB. That is, if we let

\[
C = \begin{cases} 
C_0, \; \eta \in [0, \frac{1}{2}), \\
-C_0, \; \eta \in [\frac{1}{2}, 1), 
\end{cases}
\]

where \(\eta \equiv \text{mod}(\frac{t}{T_{ZB}}, 1)\), then ZB physics no longer induces oscillations. Instead, ZB is forced to become directed motion, as schematically illustrated in fig. 1(b). Certainly, modulating the \(C\sigma_y\) term in a slightly different manner may also result in directed transport in the \(-x\)-direction. We will not repeat the similar theoretical analysis. In addition, as shown below, tailored modulation of \(C\) can lead to more complicated control over the wavepacket dynamics.

**Numerical simulation.** – In this section, we verify our analytical results by numerical simulations without any approximation. Two representative modulation schemes are considered (the initial state is specified by eq. (6)). In scheme i), we introduce the modulation of \(C\) as described by eq. (11) first and then stop the modulation for \(T_{stop} = NT_{ZB}\) (\(N\) integer), followed by the same modulation again (see fig. 2(a)). In scheme ii), everything is the same except that \(2T_{stop}/T_{ZB}\) is an odd integer (see fig. 2(b)).

For scheme i), fig. 2(c) shows that the wavepacket first moves in the \(x\)-direction, then undergoes ZB once the modulation is stopped, and finally the directed motion continues after the modulation is switched on again. The wavepacket profile in real space at different times is plotted in fig. 2(c). For scheme ii), fig. 2(d) shows that the wavepacket first moves in the same way as in the first scheme. However, once the modulation is re-launched, the wavepacket transport is reversed. As also shown in figs. 2(e) and (f), in both schemes the distortion of the wavepacket profile for the shown time scale is invisible to our naked eyes. The numerical results here confirm our theoretical analysis, demonstrating that ZB can be indeed converted to directed wavepacket motion in a well-controlled fashion. Indeed, the strategy here is analogous
Fig. 2: (Color online) Numerical results for a modulated Dirac-like equation. Panels (a) and (b) illustrate two schemes of modulation of the sign of system parameter $C$. When modulation is on, the sign of $C$ is reversed after each time interval of $T_{2\pi B}/2$ (see eq. (11)). The modulation off-period $T_{stop}$ is an integer multiple of $T_{2\pi B}$ in (a) and a half-integer multiple of $T_{2\pi B}$ in (b). Panels (c) and (d) depict the time dependence of the expectation value of $\langle x \rangle$, for modulation schemes in (a) and (b), respectively. It is seen that in (c), the wavepacket continues its directed motion after the modulation is switched on again; but in (d), the wavepacket eventually reverses its directed motion. Panels (e) and (f) show the wavepacket profile, i.e., probability density distribution in $x$, at three different times, for modulation schemes in (a) and (b), respectively. It is seen that wavepacket distortion is hardly visible. Note that in (f), the wavepacket at $t = 125.6$ is on top of that at $t = 0$. All plotted quantities and system parameters are in dimensionless units, with $C_0 = D = 1$.

Fig. 3: (Color online) Same as in fig. 2, except for a sinusoidal modulation of $C$, i.e., $C = C_0 \sin(\omega(t - t_0))$, with $\omega = 1$, $C_0 = \pi/2$ and $D = 1$. Note that in the numerical calculations for the ending or restarting regime of this modulation, we change the value of $C$ linearly with time and also choose appropriate $t_0$ values such that throughout the C parameter can be a rather smooth function of time. The pattern shown in panels (c)–(f) here are much similar to those seen in fig. 2(c)–(f), but the distance traveled by the wavepackets is different.

to how Bloch oscillations may be controlled by periodically altering the tilt of a lattice [14].

In real experiments, it should be more straightforward to introduce a smooth modulation to system parameters. Consider then a sinusoidal modulation of $C$, $C = C_0 \sin(\omega t)$. This case is somewhat complicated because the magnitude of $C$, and hence the ZB frequency, is also time dependent. To resolve this technical issue, we define a time-averaged effective ZB angular frequency over one half-period of modulation, i.e.,

$$\omega_{\text{eff}}^{\text{ZB}} \equiv \frac{\omega}{\pi} \int_{t=0}^{\pi/\omega} C_0 \sin(\omega t) dt = \frac{2C_0}{\pi}. \quad (12)$$

In the light of our previous interpretation of the ZB oscillation, it is expected that after the time interval $\pi/\omega_{\text{eff}}^{\text{ZB}}$, one of the two sub-wavepackets has vanished and the other has an occupation probability of unity. In accord with our theoretical insights above, the sign of $C$ must be changing at this moment (from $0^-$ to $0^+$ or vice versa) in order to convert ZB oscillation to directed motion. This suggests the following modified resonance condition: $\omega = \omega_{\text{eff}}^{\text{ZB}} = \frac{2C_0}{\pi}$. It should be also noted that for a sinusoidal modulation, the magnitude of $C$ can be very small during certain time windows and hence our early treatment in terms of the first-order expansion of $D/C$ can be somewhat problematic. Nevertheless, as seen in fig. 3, the numerical results confirm that $\omega = \omega_{\text{eff}}^{\text{ZB}}$ is the required resonance condition under a sinusoidal modulation so that ZB oscillation is again converted to directed motion. This also implies that the controlled dynamics is mainly contributed by the time segments during which $C$ is appreciably nonzero. As seen in fig. 3, the pattern of the results presented in fig. 3(c)–(f) is essentially the same as those seen in fig. 2(c)–(f), with again little wavepacket distortion. Note also that here the wavepacket transport distance per modulation cycle is less than that achieved by a mere discontinuous sign modulation of $C$.

Discussion. – The Dirac-like equation in eq. (1) can be realized in cold-atom systems or in trapped-ion systems. According to the concrete experimental realization in ref. [11], the $C_{xy}$ term can be realized through two on-resonance laser fields, with the system parameter $C$ connected with the phase and the strength of the laser fields. Modulation of $C$ can hence be realized by modulating the laser phase, via a laser phase modulator or even an oscillating mirror [8]. The ZB frequency in ref. [11] can be several tens of kHz, and the required phase modulation frequency should match this ZB frequency. Again using the experimental parameters in ref. [11], it can be estimated

Q. Zhang et al.
that for a ZB frequency of 40 kHz, a wavepacket can be transported by about $10^{-8}$ m per modulation cycle.

Though our analysis is based on a Dirac-like equation in one dimension, the basic mechanism of converting ZB to directed motion can work in two-dimensional systems as well. However, there the amplitude of ZB oscillations decays rapidly and for this reason, applying just one simple on-resonance driving field can only yield slower and slower directed motion. This can be overcome by use of additional strong constraint potential in the transverse direction [6] or other control fields [8] that can effectively convert a two-dimensional Dirac-like equation to the one-dimensional problem studied here.

In summary, using a simple Dirac-like equation that is already experimentally realized, we show that ZB can be converted to directed motion via a periodic driving of the Dirac-like equation, if the driving is on resonance with the ZB oscillation. Our results can motivate further studies of quantum control in driven Dirac-like equations.

***

This work is supported by National Research Foundation and Ministry of Education, Singapore (Grant No. WBS: R-710-000-008-271).

REFERENCES

[1] Schrödinger E. et al., *Sitzungber. Preuss. Akad. Wiss. Phys.-Math. Kl.*, 24 (1930) 418.
[2] Novoselov K. S. et al., *Science*, 306 (2004) 666.
[3] Larson J. and Levin S., *Phys. Rev. Lett.*, 103 (2009) 013602.
[4] Juzeliunas G. et al., *Phys. Rev. A*, 77 (2008) 011802(R).
[5] Vaishnav J. Y. and Clark C. W., *Phys. Rev. Lett.*, 100 (2008) 153002.
[6] Merkl M. et al., *EPL*, 83 (2008) 54002.
[7] Zhang Q., Gong J. B. and Oh C. H., *Ann. Phys. (N.Y.)*, 325 (2010) 1219.
[8] Zhang Q., Gong J. B. and Oh C. H., *Phys. Rev. A*, 81 (2010) 023608.
[9] Zhang X. and Liu Z., *Phys. Rev. Lett.*, 101 (2008) 264303; Dreesow F. et al., *Phys. Rev. Lett.*, 105 (2010) 143902.
[10] Lamata L., Leon J., Schatz T. and Solano E., *Phys. Rev. Lett.*, 98 (2007) 253005.
[11] Gerritsma R. et al., *Nature*, 463 (2010) 68.
[12] Zawadzki W. and Rusin T. M., *J. Phys.: Condens. Matter*, 23 (2011) 143201.
[13] Korsch H. J. and Mossmann S., *Phys. Lett. A*, 317 (2003) 54; Wilkinson S. R., Bharucha C. F., Madison K. W., Niu Q. and Raizen M. G., *Phys. Rev. Lett.*, 76 (1996) 4512; Ivanov V. V. et al., *Phys. Rev. Lett.*, 100 (2008) 043602; Sias C. et al., *Phys. Rev. Lett.*, 100 (2008) 040404; Alberti A., Ivanov V. V., Tino G. M. and Ferrari G., *Nat. Phys.*, 5 (2009) 547; Haller E. et al., *Phys. Rev. Lett.*, 104 (2010) 200403; Kudo K. and Monteiro T. S., *Phys. Rev. A*, 83 (2011) 053627.
[14] Hartmann T., Keck F., Korsch H. J. and Mossmann S., *New J. Phys.*, 6 (2004) 2; Breid B. M., Wittmann D. and Korsch H. J., *New J. Phys.*, 9 (2007) 62.
[15] Creffield C. E. and Sols F., *Phys. Rev. A*, 84 (2011) 023630; Thommen Q., Claude Garreau J. and Zehnle V., arXiv:1104.3254 (2011).