COABELIAN IDEALS IN $\mathbb{N}$-GRADED LIE ALGEBRAS AND APPLICATIONS TO RIGHT ANGLED ARTIN LIE ALGEBRAS

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ABSTRACT. We consider homological finiteness properties $FP_n$ of certain $\mathbb{N}$-graded Lie algebras. After proving some general results, see Theorem A, Corollary B and Corollary C, we concentrate on a family that can be considered as the Lie algebra version of the generalized Bestvina-Brady groups associated to a graph $\Gamma$. We prove that the homological finiteness properties of these Lie algebras can be determined in terms of the graph in the same way as in the group case.

1. INTRODUCTION

In this paper we study homological properties of certain $\mathbb{N}$-graded Lie algebras. Recall that a Lie algebra $L$ is $\mathbb{N}$-graded if $L = \oplus_{i \geq 1} L_i$ for $L_i$ such that $[L_i, L_j] \subseteq L_{i+j}$ for $i, j \geq 1$. If not otherwise stated the Lie algebras we consider are over a field $K$ of arbitrary characteristic.

In Section 2 we study homological finiteness properties of $\mathbb{N}$-graded Lie algebras $L$. Recall that a Lie algebra is of homological type $FP_m$ if the trivial $U(L)$-module $K$ has a projective resolution, where all projectives in dimension smaller or equal to $m$ are finitely generated. If such a projective resolution exists then there is a free resolution with the same property. We note that a Lie algebra $L$ is of type $FP_1$ precisely when $L$ is finitely generated and if $L$ is finitely presented (in terms of generators and relations) then $L$ is of type $FP_2$. It is an open problem whether there exist a Lie algebra that is of type $FP_2$ but is not finitely presented. But by [22, (2.13)] for $\mathbb{N}$-graded Lie algebras the properties $FP_2$ and finite presentability coincide, see for more details Lemma 2.1. We show in Proposition 2.2 that $\mathbb{N}$-graded Lie algebras also behave like pro-$p$ groups in the sense that $L$ is of homological type $FP_m$ if and only if $H_i(N, K)$ is finite dimensional over $K$ for every $i \leq m$.

Using homological methods we show the following surprising result.

**Theorem A.** Let $L = \oplus_{i \geq 1} L_i$ be an $\mathbb{N}$-graded Lie algebra that is $FP_n$ such that $[L, L] = \oplus_{i \geq 2} L_i$ and $M$ be a proper ideal of $L$ such that $[L, L] \subseteq M$. Then $M$ is of type $FP_n$ if and only if for every Lie subalgebra $N$ of $L$ of codimension one such that $M \subseteq N$ we have that $N$ is $FP_n$.

Since an $\mathbb{N}$-graded Lie algebra $L$ is of type $FP_1$ (resp. $FP_2$) if and only if it is finitely generated (resp. finitely presented) Theorem A implies immediately the following corollary.

**Corollary B.** Let $L = \oplus_{i \geq 1} L_i$ be an $\mathbb{N}$-graded Lie algebra that is finitely generated (resp. finitely presented) such that $[L, L] = \oplus_{i \geq 2} L_i$ and $M$ be a proper ideal of $L$ such that $[L, L] \subseteq M$. Then $M$ is finitely generated (resp. finitely presented) if and
only if for every Lie subalgebra $N$ of $L$ of codimension one such that $M \subseteq N$ we have that $N$ is finitely generated (resp. finitely presented).

Theorem A together with a result of Wasserman [21], whose proof uses the HNN-construction for Lie algebras, imply another surprising result.

**Corollary C.** Let $L = \oplus_{i=1}^n L_i$ be an $\mathbb{N}$-graded Lie algebra that is finitely presented (in terms of generators and relations). Assume that $[L,L] = \oplus_{i=2}^n L_i$ and that $L$ does not contain an ordinary non-abelian free Lie subalgebra. Then $[L,L]$ is a finitely generated Lie algebra.

There are examples of finitely presented metabelian Lie algebras $L$ with infinitely generated $[L,L]$ if $\text{char}(K) \neq 2$ (for example see Claim 1 from subsection 5.3 in [11] but such examples are not $\mathbb{N}$-graded). Thus the hypothesis of Corollary C that $L$ is $\mathbb{N}$-graded is indispensable.

Bestvina and Brady constructed in [3] groups of type $\text{FP}_2$ that are not finitely presented. For a finite graph $\Gamma = (V(\Gamma), E(\Gamma))$ (graphs are assumed to be without loops or multiple edges all throughout the paper) the right angled Artin group $\Gamma$ is generated by the vertices $V(\Gamma)$ of $\Gamma$ with relations $[v_1, v_2] = 1$, whenever $v_1$ and $v_2$ are linked by an edge in $E(\Gamma)$. The Bestvina-Brady group $H_\Gamma$ is the kernel of the map $\Gamma \rightarrow \mathbb{Z}$ that sends every vertex to 1. Let $\Delta_\Gamma$ be the flag complex of $\Gamma$, i.e., the complex obtained from $\Gamma$ after gluing a simplex to every clique (complete subgraph) of $V(\Gamma)$. By the main result in [3] $H_\Gamma$ is finitely presented if and only if $\Delta_\Gamma$ is 1-connected. A combinatorial proof of the backward direction was given by Dicks and Leary in [7]. Furthermore by [3] $H_\Gamma$ is of type $\text{FP}_m$ if and only if $\Delta_\Gamma$ is $(m-1)$-acyclic. These results were later generalized to kernels of arbitrary homomorphisms (often called real characters) $G \rightarrow \mathbb{R}$ by Meier, Meinert and van Wyk ([16], see below).

A graph $\Gamma$ as before also determines a Lie ring (over $\mathbb{Z}$) defined as

$$\mathfrak{g}_\Gamma = Fr(V(\Gamma))/\langle\langle v, w \rangle\rangle = 0 \text{ if } v, w \text{ are end points of an edge from } E\rangle,$$

where $Fr(V(\Gamma))$ is the free Lie ring (over $\mathbb{Z}$) with a free basis the set $V(\Gamma)$ and $\langle\langle R\rangle\rangle$ denotes the ideal generated by $R$. The lower central series of a group $G$ yields a Lie ring $\mathfrak{gr}(G)$ (over $\mathbb{Z}$)

$$\mathfrak{gr}(G) = \oplus_{k=1}^\infty \mathfrak{k}_k(G)/\mathfrak{k}_{k+1}(G)$$

which is $\mathbb{N}$-graded and generated by elements in degree one. In the case of a right angled Artin group $G_\Gamma$, there is a natural isomorphism of Lie rings $\mathfrak{gr}(G_\Gamma) \simeq \mathfrak{g}_\Gamma$, see [8, 9, 17, Thm. 3.4]. Moreover, Papadima and Suciu showed in [19] Cor. 9.6 that if $H_1(\Delta_\Gamma, \mathbb{Q}) = 0$, the inclusion map $H_\Gamma \rightarrow G_\Gamma$ induces a group isomorphism $H_1(\Gamma) \rightarrow G_\Gamma$ and an isomorphism of the derived subalgebras of the $\mathbb{N}$-graded Lie algebras $\mathfrak{gr}(G_\Gamma) \otimes \mathbb{Q}$ and $\mathfrak{gr}(H_\Gamma) \otimes \mathbb{Q}$. The approach of Papadima and Suciu uses the theory of 1-formality [18, 17, 19]. We will not use techniques involving 1-formality in this paper but observe that 1-formality requires $\text{char}(K) = 0$. The results in this paper are proved for a field $K$ of an arbitrary characteristic.

Consider the $K$-Lie algebra $L_\Gamma = \mathfrak{gr}(G_\Gamma) \otimes K = \mathfrak{g}_\Gamma \otimes K$ for a fixed field $K$. Let $\pi : L_\Gamma \rightarrow L_\Gamma/[L_\Gamma, L_\Gamma]$ be the canonical projection and consider a non-zero linear map $\chi : L_\Gamma/[L_\Gamma, L_\Gamma] \rightarrow K$. Put $I_\chi = \pi^{-1}(\text{Ker} \chi)$. The living graph $\Gamma_\chi$ is the full subgraph of $\Gamma$ spanned by the vertices with non-zero $\chi$-value. Let $\Delta_\chi$ be the flag complex of $\Gamma_\chi$. For a (possibly empty) clique $w \subseteq \Gamma$, $\text{lk}_{\Delta\chi}(w)$ is defined as either
$\Delta^*_\Gamma$ if $w = \emptyset$ or the link in $\Delta^*_\Gamma$ of the simplex associated to $w$ otherwise. We also set

$$\text{lk}_{\Delta^*_\Gamma}(w) := \text{lk}_{\Delta^*_\Gamma}(w) \cap \Delta^*_\Gamma.$$  

Note that any codimension 1 ideal of $L^*_\Gamma$ is of the form $I^*_\chi$ for some $\chi$. In the following result we use homological methods to classify when those ideals are of type $FP_n$. A similar computation for Bestvina-Brady groups can be found in [12] and for cocyclic normal subgroups of certain Artin groups in [6].

**Theorem D.** The Lie algebra $N = I^*_\chi$ is of type $FP_n$ if and only if $\text{lk}_{\Delta^*_\Gamma}(w)$ is $(n-1-|w|)$-acyclic over the field $K$ for every (possibly empty) clique $w \subseteq \Gamma \setminus \Gamma^*_\chi$. For $w = \emptyset$ this translates to the flag complex $\Delta^*_\Gamma$ is $(n-1)$-acyclic over $K$.

This result together with Theorem A above yield a a complete classification of the coabelian ideals $M$ of $L = L^*_\Gamma$ which are of type $FP_n$. A group theoretic version for right angled Artin groups was proved by Meier, Meinert and van Wyk in [16] as a corollary of their description of the Bieri-Neumann-Strebel-Renz $\Sigma$-invariants for right angled Artin groups. In Theorem 3.5 in Subsection 3.1 we give a group theoretic version of Theorem A for the right angled Artin group $G^*_\Gamma$ that involves the Bieri-Neumann-Strebel-Renz $\Sigma$-theory.

We state two corollaries of Theorem D. The first of them, Corollary E below, characterizes when the ideal $I^*_\chi$ is finitely generated as a Lie algebra. Although it can be deduced from Theorem A, in Section 4 we show how to prove it using only elementary combinatorial methods. For the statement, recall that we say that a subgraph $\Gamma_1$ of $\Gamma$ is dominant in $\Gamma$ if for every $v \in V(\Gamma) \setminus V(\Gamma_1)$ there is $w \in V(\Gamma_1)$ that is linked with $v$ by an edge in $\Gamma$. This definition is important for the description of the Bieri-Neumann-Strebel-Renz $\Sigma^1$-invariant of a right angled Artin group $G$ given by Meier and van Wyk in [15], which was later generalised in [16].

**Corollary E.** The ideal $I^*_\chi$ of $L^*_\Gamma$ is finitely generated as a Lie algebra if and only if $\Gamma^*_\chi$ is connected and dominant in $\Gamma$. In this case $I^*_\chi$ is generated as a Lie algebra by $\text{Ker}(\chi)$.

Our second corollary to Theorem D can be viewed, in the specific case $m = 2$, as a Lie algebra version of the results of Dicks and Leary in [7].

**Corollary F.** The $\mathbb{N}$-graded Lie algebra $\text{gr}(H^*_\Gamma) \otimes_\mathbb{Z} K$ is of type $FP_m$ if and only if the flag complex $\Delta^*_\Gamma$ is $(m-1)$-acyclic over $K$, i.e. $H_i(\Delta^*_\Gamma, K) = 0$ for $i \leq m - 1$.

In particular, Corollary F classifies when $\text{gr}(H^*_\Gamma) \otimes_\mathbb{Z} K$ is a finitely presented Lie algebra since for $\mathbb{N}$-graded Lie algebra $FP_2$ and finite presentability are the same. Note that by [13] we know that the group $H^*_\Gamma$ is finitely presented if and only if the flag complex $\Delta^*_\Gamma$ is 1-connected and by Corollary F, $\text{gr}(H^*_\Gamma) \otimes_\mathbb{Z} K$ is a finitely presented Lie algebra if and only if the flag complex $\Delta^*_\Gamma$ is 1-acyclic. In the case $K = \mathbb{Q}$ this was earlier proved using 1-formality by Papadima and Suciu in [19].

Finally in Section 5 we use geometric methods to prove the following result, which gives a sufficient condition for a coabelian ideal $M$ of a right angled Artin Lie algebra $L^*_\Gamma$ to be of type $FP_n$.

**Theorem G.** Let $L = L^*_\Gamma$ and $[L,L] \leq M \triangleleft L$ be an ideal of codimension $k$. Let $X$ be the set of all subsets of $V(\Gamma)$ which generate complete subgraphs and $Y \subseteq X$ the set of those $A \in X$ such that $M \cap L_A$ has corank $k$ in $L_A$. Assume that $\text{lk}_{V(\Gamma)}(Z)$ is $(n-i-1)$-acyclic for any $Z \in X \setminus Y$ with $|Z| = i$. Then $M$ is of type $FP_n$. 

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2. ON THE HOMOLOGICAL FINITENESS PROPERTY $FP_m$ OF $\mathbb{N}$-GRADED LIE ALGEBRAS

We denote by $\mathbb{N}$ the set of positive natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. An $\mathbb{N}$-graded Lie algebra $L$ is the direct sum $\bigoplus_{i \in \mathbb{N}L_i}$, where each $L_i$ is $K$-vector subspace of $L$ and $[L_i, L_j] \subseteq L_{i+j}$. Furthermore if each $L_i$ is finite dimensional we call $L$ an $\mathbb{N}$-graded Lie algebra of finite type. Note that every finitely generated $\mathbb{N}$-graded Lie algebra $L$ is of finite type. Furthermore an $\mathbb{N}$-graded Lie algebra $L$ is finitely generated if and only $L/[L, L] \cong H_1(L, K)$ is finite dimensional.

Note that when $L$ is an $\mathbb{N}$-graded Lie algebra there is a natural $\mathbb{N}_0$-grading on the universal enveloping algebra $R = U(L)$ as $\bigoplus_{i \in \mathbb{N}_0R_i}$ where $R_0 = K.1$ and $R_i = \sum_{i_1 + \ldots + i_j = i}L_{i_1}\ldots L_{i_j}$. An $\mathbb{N}_0$-graded $R$-module $V = \bigoplus_{i \in \mathbb{N}_0}V_i$ has the property that $V_iR_j \subseteq V_{i+j}$. From now we write graded $R$-module for $\mathbb{N}_0$-graded $R$-module and a homomorphism $\phi : V = \bigoplus_{i \geq 0}V_i \rightarrow W = \bigoplus_{i \geq 0}W_i$ between graded $R$-modules is called graded if it is a homomorphism of $R$-modules and preserves the grading i.e. $\phi(V_i) \subseteq W_i$. Here and all along the paper, module means right module.

One important property of a graded $R$-module $V$ is that we have the following version of the Nakayama Lemma: For an $R$-module $V$, $V = 0$ if and only if $V \otimes_R K = 0$ (see [22], note that although in [22] all graded Lie algebras and modules are assumed to be of finite type this property remains true in general, i.e., if the graded objects are not of finite type). Furthermore, $V$ is finitely generated as $R$-module if and only if $V \otimes_R K$ is finite dimensional (over $K$), see [22] Section 2).

Lemma 2.1. Let $L$ be an $\mathbb{N}$-graded Lie algebra. Then $L$ is of type $FP_2$ if and only if $L$ is finitely presented (in terms of generators and relations).

Proof. The non-trivial direction is to show that when $L$ is $FP_2$ then $L$ is finitely presented. Note that for every Lie algebra $L$ of type $FP_2$ the homology groups $H_1(L, K) \cong L/[L, L]$ and $H_2(L, K)$ are finite dimensional. In particular, since $L/[L, L]$ is finite dimensional, $L$ is finitely generated and hence $L$ is an $\mathbb{N}$-graded Lie algebra of finite type. By [22] (2.13) if an $\mathbb{N}$-graded Lie algebra $L$ of finite type has finite dimensional $H_2(L, K) = \text{Tor}_2^U(L, K)$ then $U(L)$ is a finitely presented graded associative algebra, hence $L$ is a finitely presented Lie algebra.

Let $L$ be an $\mathbb{N}$-graded Lie algebra. We say that $L$ is graded $FP_m$ if there is a graded projective resolution (i.e. of graded $U(L)$-modules with graded homomorphisms) of the trivial $U(L)$-module $K$, where all projective modules in dimension smaller or equal to $m$ are finitely generated.

Let $L = \bigoplus_{i \geq 1}L_i$ be an $\mathbb{N}$-graded Lie algebra. An ideal $N$ of $L$ is a graded ideal if $N = \bigoplus_{i \geq 1}N \cap L_i$.

The following is a Lie graded algebra version of the pro-$p$-groups result [10] Thm. A).

Proposition 2.2. Let $L$ be an $\mathbb{N}$-graded Lie algebra over a field $K$ and $N$ be a graded ideal such that $U(L/N)$ is left and right Noetherian. Then $L$ is graded $FP_m$. 


if and only if $H_i(N, K)$ is finitely generated as $U(L/N)$-module for every $i \leq m$.

This implies that both graded $FP_m$ and ordinary $FP_m$ are the same property.

**Proof.** Suppose first that $L$ is graded $FP_m$. Then $L$ is ordinary $FP_m$. Any projective resolution $\mathcal{P}$ of the trivial $U(L)$-module $K$ with finitely generated modules in dimension up to $m$ can be used to calculate the homology groups $H_i(N, K) = H_i(\mathcal{P} \otimes U(N) K)$. Since $\mathcal{P} \otimes U(N) K$ is a complex of projective $U(L/N)$-modules and up to dimension $m$ all modules are finitely generated, using Noetherianess we deduce that $H_i(N, K)$ is finitely generated as $U(L/N)$-module for every $i \leq m$.

For the converse we built by induction on $m \geq 0$ a free graded resolution $\mathcal{Q}$ of the trivial $U(L)$-module $K$ with finitely generated modules in dimension up to $m$. The case $m = 0$ is obvious. Suppose we have built the resolution up to dimension $m - 1$, then we have an exact graded complex of right $U(L)$-modules (i.e. all modules and homomorphisms are graded)

$$\mathcal{Q}' : 0 \to B_m \to P_{m-1} \to \ldots \to P_1 \to P_0 \to K \to 0,$$

where $P_0, \ldots, P_{m-1}$ are free $U(L)$-modules. Consider the complex $\mathcal{Q}' \otimes U(N) K$. We claim that

$$(1) \quad \text{Ker}(B_m \otimes U(N) K \to P_{m-1} \otimes U(N) K) \simeq H_m(N, K).$$

Indeed let

$$\mathcal{Q} : \ldots \to Q_{m+1} \to Q_m \to P_{m-1} \to \ldots \to P_1 \to P_0 \to K \to 0$$

be a free graded resolution of $U(L)$-modules (i.e. all homomorphisms are graded and the modules are free). Since tensoring is right exact, the exactness of $Q_{m+1} \to Q_m \to B_m \to 0$ implies that

$$Q_{m+1} \otimes U(N) K \xrightarrow{\alpha} Q_m \otimes U(N) K \xrightarrow{\beta} B_m \otimes U(N) K \to 0$$

is exact,

hence

$$\text{Im}(\alpha) = \text{Ker}(\beta).$$

Note that

$$H_m(N, K) = \text{Ker}(Q_m \otimes U(N) K \to P_{m-1} \otimes U(N) K) / \text{Im}(\alpha) = \text{Ker}(Q_m \otimes U(N) K \to P_{m-1} \otimes U(N) K) / \text{Ker}(\beta) \simeq$$

$$\beta(\text{Ker}(Q_m \otimes U(N) K \to P_{m-1} \otimes U(N) K)) = \text{Ker}(B_m \otimes U(N) K \to P_{m-1} \otimes U(N) K),$$

hence (1) holds.

Note that the $U(L/N)$-module $P_{m-1} \otimes U(N) K$ is finitely generated and since $U(L/N)$ is Noetherian $\text{Im}(B_m \otimes U(N) K \to P_{m-1} \otimes U(N) K)$ is finitely generated too. This combined with (1) implies that $B_m \otimes U(N) K$ is a finitely generated $U(L/N)$-module, hence

$$(2) \quad B_m \otimes U(L) K \text{ is finite dimensional over } K.$$

Recall that all modules and homomorphisms in $\mathcal{Q}$ are graded, in particular $B_m$ is a graded $U(L)$-module. Thus (2) implies that $B_m$ is a finitely generated graded $U(L)$-module. Then we can choose $P_m$ as any finitely generated graded free $U(L)$-module together with a graded epimorphism $P_m \to B_m$.

Finally if $L$ is ordinary $FP_m$ then the first paragraph of the proof implies that $H_i(N, K)$ is finitely generated as $U(L/N)$-module for every $i \leq m$. Then $L$ is graded $FP_m$.

Applying the proposition for $N = L$ we obtain
Corollary 2.3. Let $L$ be an $\mathbb{N}$-graded Lie algebra over a field $K$. Then $L$ is of homological type $FP_m$ if and only if $H_i(L,K)$ is finite dimensional for every $i \leq m$.

2.1. Cohomological finiteness of coabelian ideals.

Let $M$ be a coabelian ideal of an $\mathbb{N}$-graded Lie algebra $L$, i.e., an ideal such that $[L,L] \leq M$. In this subsection we will prove that the cohomological finiteness properties of $M$ can be determined by the cohomological finiteness properties of all those codimensional one ideals $N$ of $L$ such that $M \leq N$.

Proposition 2.4. Let $R = K[x_1, \ldots, x_k]$ and $S = K[x_2, \ldots, x_k]$ be commutative polynomial rings. We view $S$ as a left $R$-module via a surjective homomorphism of $K$-algebras $\theta : R \to S$ that maps $Kx_1 + \ldots + Kx_k$ surjectively to $Kx_2 + \ldots + Kx_k$. Let $\mathcal{P}$ be a complex

$$\mathcal{P} : \ldots \to P_i \xrightarrow{\partial_i} P_{i-1} \to \ldots \to P_0 \to 0$$

of free $R$-modules. Then $H_1(\mathcal{P}) \otimes_R S$ embeds in $H_1(\mathcal{P} \otimes_R S)$.

Proof. Note that since $P_i$ is a free $R$-module we have an exact sequence

$$0 = \text{Tor}_1^R(P_i, S) \to \text{Tor}_1^R(\text{Im}(\partial_i), S) \to \text{Ker}(\partial_i) \otimes_R S \xrightarrow{\alpha_i} P_i \otimes_R S \xrightarrow{\beta_i} \text{Im}(\partial_i) \otimes_R S \to 0$$

which is part of the long exact sequence in Tor applied to the short exact sequence of $R$-modules

$$0 \to \text{Ker}(\partial_i) \to P_i \to \text{Im}(\partial_i) \to 0.$$

Using $\theta$ we can construct a free resolution of $S$ as $R$-module

$$0 \to R \xrightarrow{\mu} S \to 0,$$

to see it note that $\text{Ker}(\theta) = yR \neq 0$ for some $y \in Kx_1 + \ldots + Kx_k$ and set $\mu(r) = yr$. We use this resolution to calculate $\text{Tor}_1^R(\text{Im}(\partial_i), S)$ i.e.

$$\text{Tor}_1^R(\text{Im}(\partial_i), S) = \text{Ker}(\text{Im}(\partial_i) \xrightarrow{\nu} \text{Im}(\partial_i)),$$

where $\nu_i$ is given by multiplication with $y$. Since $\text{Im}(\partial_i)$ is an $R$-submodule of the free $R$-module $P_{i-1}$ we deduce that $\text{Ker}(\nu_i) = 0$, hence $\text{Tor}_1^R(\text{Im}(\partial_i), S) = 0$ and $\alpha_i$ is injective.

The short exact sequence of $R$-modules $0 \to \text{Im}(\partial_{i+1}) \to \text{Ker}(\partial_i) \to H_i(\mathcal{P}) \to 0$ induces an exact sequence

$$\text{Im}(\partial_{i+1}) \otimes_R S \xrightarrow{\gamma_i} \text{Ker}(\partial_i) \otimes_R S \to H_i(\mathcal{P} \otimes_R S) \to 0.$$

Then

$$H_i(\mathcal{P} \otimes_R S) \simeq (\text{Ker}(\partial_i) \otimes_R S)/\text{Im}(\gamma_i).$$

Let $\{d_i = \partial_i \otimes id_S\}$ be the differentials of $\mathcal{P} \otimes_R S$. Then

$$d_{i+1} = \alpha_i \gamma_i \beta_{i+1}$$

and using that $\alpha_{i-1}$ is injective and $\beta_{i+1}$ is surjective we get

$$H_i(\mathcal{P} \otimes_R S) = \text{Ker}(d_i)/\text{Im}(d_{i+1}) = \text{Ker}(\alpha_{i-1} \gamma_{i-1} \beta_i)/\text{Im}(\alpha_i \gamma_i \beta_{i+1}) =$$

$$\text{Ker}(\gamma_{i-1} \beta_i)/\text{Im}(\alpha_i \gamma_i) \cong \text{Ker}(\beta_i)/\text{Im}(\alpha_i \gamma_i) = \text{Im}(\alpha_i)/\text{Im}(\alpha_i \gamma_i) \cong$$

$$(\text{Ker}(\partial_i) \otimes_R S)/\text{Im}(\gamma_i) \cong H_i(\mathcal{P}) \otimes_R S.$$

□

Let $R = K[x_1, \ldots, x_k]$ be a commutative polynomial ring. We can see $R$ as a graded ring, where $x_1, \ldots, x_k$ have weight 1.
Proposition 2.5. Let $i \geq 0$ be a fixed natural number. Let $R = K[x_1, \ldots, x_k]$ be a commutative polynomial ring and $\mathcal{P}$ be a graded complex

$$\mathcal{P} : \ldots \rightarrow P_i \xrightarrow{\partial_i} P_{i-1} \rightarrow \ldots \rightarrow P_0 \rightarrow 0$$

of free $R$-modules, where each $P_j$ is finitely generated for $j \leq i$. Suppose that for every $K$-algebra epimorphism $\theta : R \rightarrow S_0 = K[v]$, where $\theta(x_i) \in K[v]$ for $1 \leq s \leq k$, we have that $H_i(\mathcal{P} \otimes_K S_0)$ is finite dimensional over $K$, where we view $S_0$ as a left $R$-module via $\theta$. Then the homology group $H_i(\mathcal{P})$ is finite dimensional over $K$.

Proof. From the very beginning we can assume that $K$ is an algebraically closed field, otherwise we consider the complex $\mathcal{P} \otimes_K \overline{K}$ over $\overline{K}[x_1, \ldots, x_k]$, where $\overline{K}$ is the algebraic closure of $K$. Then $H_i(\mathcal{P} \otimes_K \overline{K}) \simeq H_i(\mathcal{P}) \otimes_K \overline{K}$ is finite dimensional over $\overline{K}$ if and only if $H_i(\mathcal{P})$ is finite dimensional over $K$.

In order to prove the proposition we proceed by induction on $k$. The case $k = 1$ is obvious as $S_0 = R$ and we can choose $\theta$ to be the identity map. Assume from now on that $k \geq 2$ and that the proposition holds for all polynomial rings on at most $k - 1$ variables.

Let $R_0 = R$ and

$$R_i = K[x_2, \ldots, x_k], \quad R_i \xrightarrow{\theta_i} R_{i+1} = K[x_{i+2}, \ldots, x_k]$$

be epimorphisms of $K$-algebras, where for each epimorphism $\theta_j$, we have $\theta_j(Kx_{j+1} + \ldots + Kx_k) = Kx_{j+2} + \ldots + Kx_k$. Then by Proposition 2.4 $H_i(\mathcal{P} \otimes_R R_j) \otimes_R R_{j+1}$ embeds in $H_i(\mathcal{P} \otimes_R R_{j+1})$. By the assumptions of Proposition 2.5 for $S_0 = R_{k-1}$ and $\theta = \theta_{k-2} \ldots \theta_0$ we deduce that $H_i(\mathcal{P} \otimes_R R_{k-1})$ is finite dimensional over $K$.

On the other hand, if $j \geq 1$ we consider $\mathcal{P} \otimes_R R_j$ as a complex of free $R_j$-modules and as $R_j$ is a polynomial ring on less than $k$ variables and for such rings by induction Proposition 2.5 holds we conclude that $H_i(\mathcal{P} \otimes_R R_j)$ is finite dimensional over $K$. We claim that this implies the result. To see it, note that by Proposition 2.4 $H_i(\mathcal{P}) \otimes_R R_1$ embeds in the finite dimensional $K$-vector space $H_i(\mathcal{P} \otimes_R R_1)$. Then the short exact sequence

$$0 \rightarrow H_i(\mathcal{P}) \otimes_R R_1 \rightarrow H_i(\mathcal{P} \otimes_R R_1) \rightarrow Q \rightarrow 0$$

where $Q = H_i(\mathcal{P} \otimes_R R_1)/H_i(\mathcal{P}) \otimes_R R_1$ is finite dimensional, induces a long exact sequence

$$\ldots \rightarrow \text{Tor}^1_{R_k}(Q, R_{k-1}) \rightarrow (H_i(\mathcal{P}) \otimes_R R_1) \otimes_{R_1} R_{k-1} \rightarrow$$

$$H_i(\mathcal{P} \otimes_R R_1) \otimes_{R_1} R_{k-1} \rightarrow Q \otimes_{R_1} R_{k-1} \rightarrow 0.$$

Note that since $R_{k-1}$ is a cyclic $R_1$-module, by Noetherianess we can find a free resolution of $R_{k-1}$ as $R_1$-module, where each module is finitely generated. Using this resolution to calculate $\text{Tor}^1_{R_k}(Q, R_{k-1})$ together with the fact that $Q$ is finite dimensional we deduce that $\text{Tor}^1_{R_k}(Q, R_{k-1})$ is finite dimensional. Since $R_{k-1}$ is a cyclic $R_1$-module and $H_i(\mathcal{P} \otimes_R R_1)$ is finite dimensional over $K$ we deduce that $H_i(\mathcal{P} \otimes_R R_1) \otimes_{R_1} R_{k-1}$ is finite dimensional over $K$. This combined with the above long exact sequence implies that $(H_i(\mathcal{P}) \otimes_R R_1) \otimes_{R_1} R_{k-1} \simeq H_i(\mathcal{P}) \otimes_R R_{k-1}$ is finite dimensional over $K$ too.

Suppose that $H_i(\mathcal{P})$ is not finite dimensional. By Noetherianess $H_i(\mathcal{P})$ is a finitely generated $R$-module and as the complex $\mathcal{P}$ is graded, $H_i(\mathcal{P})$ is a graded $R$-module. Hence $H_i(\mathcal{P})$ has an infinite dimensional graded quotient $M = R/I$, i.e. $I$ is a graded ideal in $R$. Moreover, again by Noetherianess, $I$ is finitely generated.
so it is generated by a finite set of homogeneous polynomials \( f_1, \ldots, f_t \). For each \( 1 \leq s \leq k \) consider the multiplicatively closed set \( \Delta_s = \{ x_i^z \mid z \geq 0 \} \). Then

\[
M\Delta_s^{-1} = K[y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_k, y_s^{\pm1}] / \Delta_s^{-1},
\]

where \( y_j = x_j/x_s \) for \( j \neq i \) and \( y_s = x_s \). Then since each \( f_j \) is homogeneous we can write \( f_j = y_i^d g_j(y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_k) \) for some positive integer \( d \) and some polynomial \( g_j \) for every \( 1 \leq j \leq t \). Write \( J_i \) for the ideal of \( K[y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_k] \) generated by \( g_j \) for \( 1 \leq j \leq t \). Then

\[
M\Delta_s^{-1} \cong (K[y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_k]/J_i) \otimes_K y_s^{\pm1}.
\]

If the localization \( M\Delta_s^{-1} \) is not zero, we embed \( J_i \) in a maximal ideal of \( K[y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_k] \) and since \( K \) is algebraically closed any maximal ideal is generated by \( \{ y_j - \lambda_j \}_{j \neq s} \) for some \( \lambda_j \in K \). Hence there is an epimorphism of \( K \)-algebras

\[
\rho_s : M\Delta_s^{-1} \to K[y_s^{\pm1}]
\]

that sends \( y_j \) to \( \lambda_j \) for \( j \neq s \) and \( y_s \) to \( y_s \). Note that this map sends \( x_j \) to \( \lambda_j x_s \) for \( j \neq s \) and is the identity on \( x_s = y_s \). The canonical map \( M \to M\Delta_s^{-1} \) composed with \( \rho_s \) gives a graded epimorphism

\[
\overline{\rho}_s : M \to K[y_s] = K[x_s].
\]

Consider the epimorphism \( \gamma = \overline{\rho}_s \circ \pi : R \to K[x_s] \), where \( \pi : R \to M = R/I \) is the canonical projection. Thus we can view \( K[x_s] \) as \( R \)-module via \( \gamma \) and there is an epimorphism of \( K \)-vector spaces

\[
\overline{\rho}_s \otimes id : M \otimes_R K[x_s] \to K[x_s] \otimes_R K[x_s].
\]

Furthermore consider the epimorphism

\[
\pi \otimes id : R \otimes_R K[x_s] \to M \otimes_R K[x_s]
\]

and note that after identifying \( R \otimes_R K[x_s] \) with \( K[x_s] \) and identifying \( K[x_s] \otimes_R K[x_s] \) with \( K[x_s] \) (via the multiplication in \( K[x_s] \)), we get that \( \pi \otimes id \) is the inverse of \( \overline{\rho}_s \otimes id \). In particular \( M \otimes_R K[x_s] \cong K[x_s] \) and \( M \otimes_R K[x_s] \) is infinite dimensional as \( K \)-vector space.

Recall that \( H_l(\mathcal{P}) \otimes_R K[x_s] \cong H_l(\mathcal{P}) \otimes_R K[x_s] \) is finite dimensional. By permuting the variables \( x_1, \ldots, x_k \), we have that \( H_l(\mathcal{P}) \otimes_R K[x_s] \) is finite dimensional, where \( K[x_s] \) is a left \( R \)-module via an arbitrary epimorphism \( R \to K[x_s] \) that sends \( Kx_1 + \ldots + Kx_k \) onto \( Kx_s \). Note that \( M \otimes_R K[x_s] \cong K[x_s] \) is a quotient of the finite dimensional \( H_l(\mathcal{P}) \otimes_R K[x_s] \), a contradiction.

Hence for every \( 1 \leq s \leq k \) we have that \( M\Delta_s^{-1} = 0 \). Thus there exists a positive integer \( z_s \) such that \( x_i^{z_s} \in I \), so \( x_i \in \sqrt{I} \). Then

\[
(x_1, \ldots, x_k) \subseteq \sqrt{I},
\]

hence the radical \( \sqrt{I} \) has finite codimension in \( R = K[x_1, \ldots, x_k] \) (in fact codimension 1). Note that by Noetherianess there is a positive integer \( z \) such that \( \sqrt{I}^z \subseteq I \) and each \( \sqrt{I}^z / \sqrt{I}^{z+1} \) is a finitely generated \( R/\sqrt{I} \)-module, hence is finite dimensional. Thus each \( R/\sqrt{I}^z \) is finite dimensional and so \( M = R/I \) is finite dimensional, a contradiction.

We can now prove the main result of this section, which is Theorem A from the introduction.
Theorem 2.6. (Theorem A) Let $L$ be an $\mathbb{N}^\text{-graded}$ Lie algebra of type $\text{FP}_n$ such that $L = \oplus_{i \geq 1} L_i$ and $[L, L] = \oplus_{i \geq 2} L_i$ and $M$ be a proper ideal of $L$ such that $[L, L] \leq M$. Then $M$ is of type $\text{FP}_n$ if and only if for every Lie subalgebra $N$ of $L$ such that $M \subseteq N$ and $\dim_K ([L, N]) = 1$ we have $N$ is $\text{FP}_n$.

Proof. The easy part of the proof is to show that when $M$ is $\text{FP}_n$ then each $N$ is $\text{FP}_n$. Indeed since $N/M$ is a finite dimensional abelian Lie algebra, it is $\text{FP}_n$, in particular is $\text{FP}_n$. This follows from the fact that $U(N/M)$ is a Noetherian ring. Then $N$ is an extension of $M$, an $\text{FP}_n$ Lie algebra, by $N/M$. As $N/M$ is also of type $\text{FP}_n$, $N$ is $\text{FP}_n$ as claimed.

Suppose now that each $N$ is of type $\text{FP}_n$. Note that since $M$ is an ideal of $L$ such that $\oplus_{i \geq 2} L_i = [L, L] \leq M$ then $M$ is also $\mathbb{N}^\text{-graded}$ via $M = (L_1 \cap M) \oplus (\oplus_{i \geq 2} L_i)$. By Corollary 2.7, $M$ is of type $\text{FP}_n$ if and only if $H_i(M, K)$ is finite dimensional for $i \leq n$.

Let $R$ be a free graded resolution of the trivial $U(L)$-module $K$ with finitely generated modules in dimension $\leq n$. Set

$$\mathcal{P} = R^\text{del} \otimes_{U(M)} K,$$

where the upper index del means that we have substituted the module in dimension -1 with the zero module. This $\mathcal{P}$ is a complex of free $R$-modules, where $R = U(L/M)$ is a commutative polynomial ring. Then for $i \geq 0$

$$H_i(M, K) \simeq H_i(\mathcal{P})$$

and

$$H_i(N, K) \simeq H_i(\mathcal{P}^\text{del} \otimes_{U(N)} K) \simeq H_i(\mathcal{P} \otimes_{U(M)} K) \simeq H_i(\mathcal{P} \otimes_{U(L)} U(L/N)).$$

Since $N$ is $\text{FP}_n$ we have that $H_i(\mathcal{P} \otimes_{U(L)} U(L/N))$ is finite dimensional for every $i \leq n$ and every $N$ of codimension 1 in $L$. Then by Proposition 2.5 this implies that $H_i(\mathcal{P})$ is finite dimensional for $i \leq n$. \qed

Corollary 2.7. (Corollary C) Let $L$ be a finitely presented $\mathbb{N}^\text{-graded}$ Lie algebra $L = \oplus_{i \geq 1} L_i$ that does not contain an ordinary non-abelian free Lie subalgebra and $[L, L] = \oplus_{i \geq 2} L_i$. Then $[L, L]$ is a finitely generated Lie algebra.

Proof. By [21, Cor. 9.2] if $L$ is a finitely presented soluble Lie algebra then every ideal $N$ of codimension 1 is finitely generated as a Lie subalgebra. The proof of [21, Cor. 9.2] needs solubility only to exclude the possibility that $L$ contains an ordinary non-abelian free Lie subalgebra. Thus we can use the same argument here and we can apply Theorem 2.6 for $M = [L, L]$ and $n = 1$ together with the fact that a Lie algebra is finitely generated if and only if it is of type $\text{FP}_1$. \qed

3. Lie subalgebras of type $\text{FP}_n$ in $L_\Gamma$

Let $\Gamma$ be a finite graph with no loops or double edges. Denote by $x_1, \ldots, x_m$ the set of vertices $V(\Gamma)$ of $\Gamma$ and let $G = G_\Gamma$ be the right angled Artin group associated with $\Gamma$ and

$$L = L_\Gamma = \text{gr}(G_\Gamma) \otimes_{\mathbb{Z}} K.$$ 

Let

$$V = Kx_1 + \ldots + Kx_m$$

be a $K$-vector space with a basis $x_1, \ldots, x_m$. We identify $V$ with $L/[L, L]$ and let

$$\chi : V \rightarrow K$$
be a non-zero $K$-linear map. Put

$$I\chi = \pi^{-1}(\text{Ker}(\chi)),$$

where $\pi : L \to L/[L, L]$ is the canonical epimorphism. For example if $\chi(x_i) = 1$ for all $i$ and $\Gamma$ is connected then $I\chi$ is precisely $\text{gr}(H_{\Gamma}) \otimes K$, where $H_{\Gamma}$ is the kernel the epimorphism $G \to \mathbb{Z}$ that sends each $x_i$ to 1. This follows from the fact that for a connected finite graph $\Gamma$ by [13] Thm. 5.6 the inclusion map $H_{\Gamma} \to G_{\Gamma}$ induces an isomorphism of Lie algebras $[\text{gr}(H_{\Gamma}), \text{gr}(H_{\Gamma})] \simeq [\text{gr}(G_{\Gamma}), \text{gr}(G_{\Gamma})]$. And in the general case $I\chi$ is a codimension one ideal of $L$. Moreover, since $L = \oplus_{i \geq 1} L_i$ and $[L, L] = \oplus_{i \geq 2} L_i$ we deduce that $I\chi = (I\chi \cap L_1) \oplus (\oplus_{i \geq 1} L_i)$ is an $\mathbb{N}$-graded Lie algebra.

We fix an order in the set the vertices of $\Gamma$. Recall that the flag complex $\Delta_{\Gamma}$ of $\Gamma$ is the complex obtained from $\Gamma$ after gluing a simplex to every non-empty clique of $\Gamma$, i.e. an $n$-cell of $\Delta_{\Gamma}$ is $(v_1, \ldots, v_n)$, where the vertices $v_1, \ldots, v_n$ are all pairwise linked in $\Gamma$ and $v_1 < v_2 < \ldots < v_n$.

The following complex is the well known minimal resolution of the trivial $U(L)$-module $K$ (see [11] Subsection 2.4).

$$(3) \quad \mathcal{P}_{\Gamma} : \ldots \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \xrightarrow{\partial_{n-1}} \ldots \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} K \to 0,$$

where

$$P_n = \bigoplus_{\sigma \subseteq \Gamma} c_{\sigma} U(L)$$

and the sum is over all the cliques $\sigma$ of $\Gamma$ with $|\sigma| = n$. Here each $c_{\sigma} U(L)$ is a copy of the free right $U(L)$-module, $c_{\emptyset} = 1_K$, $P_0 = c_{\emptyset} U(L) = U(L)$ and $\partial_0$ is the augmentation map. For higher degrees the differential is

$$\partial_n(c_{\sigma}) = \sum_r (-1)^{r-1} c_{\sigma \setminus \{v_r\}} v_r,$$

where $\sigma = (v_1, \ldots, v_n)$.

The link of a vertex $v$ in a graph is the subgraph spanned by all those vertices different from $v$ which are linked to it. This definition extends to subsets of vertices by taking the intersection of all the links of the vertices in the subset. In the case of the empty set, the link is the graph itself. This definition can be extended to simplicial complexes: the link of a simplex $s$ in a simplicial complex $C$ is the subcomplex of $C$ consisting of the simplices $t$ that are disjoint from $s$ and such that both $s$ and $t$ are faces of some higher-dimensional simplex in $C$, equivalently, such that $s \cup t$ is also a simplex in $C$.

Consider the living graph $\Gamma_{\chi}$ i.e. the subgraph of $\Gamma$ spanned by the vertices with non-zero $\chi$-value. Let $\Delta_{\Gamma_{\chi}}$ be the flag complex of $\Gamma_{\chi}$ and $\Delta_{\Gamma}$ be the flag complex of $\Gamma$. The flag complex of the link of a non-empty clique $w$ in $\Delta_{\Gamma}$ is the simplicial link of the simplex spanned by the clique. We denote this complex by $\text{lk}_{\Delta_{\Gamma}}(w)$ (also in the case when $w = \emptyset$). Fix a possibly empty clique $w \subseteq \Gamma \setminus \Gamma_{\chi}$ and set

$$\text{lk}_{\Delta_{\Gamma_{\chi}}}(w) = \text{lk}_{\Delta_{\Gamma}}(w) \cap \Delta_{\Gamma_{\chi}}.$$

At this point, we can prove Theorem C. Recall that a space $W$ is $m$-acyclic over a field $K$ if $H_i(W, K) = 0$ for all $0 \leq i \leq m$. 


Theorem 3.1. (Theorem D) The Lie algebra $N = I_X$ is of type $FP_n$ if and only if $\text{lk}_{\Delta X}(w)$ is $(n - 1 - |w|)$-acyclic over $K$ for every clique $w \subseteq \Gamma \setminus X$. For $w = \emptyset$ this translates to the flag complex $\Delta X$ is $(n - 1)$-acyclic over $K$.

Proof. For $i \geq 1$ the homology $H_i(N, K)$ is the homology of the complex $\mathcal{P}_\Gamma \otimes_{U(N)} K$, where $\mathcal{P}_\Gamma$ is the complex in (3). To describe this complex, note first that

$$P_n \otimes_{U(N)} K = \bigoplus_{\sigma \subseteq \Gamma \text{ clique, } |\sigma| = n} c_\sigma U(L) \otimes_{U(N)} K \simeq C_n \otimes_K U(L/N),$$

where

$$C_n = \bigoplus_{\sigma \subseteq \Gamma \text{ clique, } |\sigma| = n} K c_\sigma.$$

As for the differential $d_n := \partial_n \otimes 1_d$, identifying $U(L) \otimes_{U(N)} K \simeq U(L/N)$ with the polynomial ring $K[v]$, where for each $v_i \in V(\Gamma)$, $v_i \otimes 1 \in U(L) \otimes_{U(N)} K$ is sent to $\chi(v_i)v$, we have

$$d_n(c_\sigma \otimes 1) = \sum_r (-1)^{r-1} c_{\sigma \setminus \{v_r\}} v_r \otimes 1 = \sum_r (-1)^{r-1} c_{\sigma \setminus \{v_r\}} \otimes \chi(v_r)v,$$

where $\sigma = (v_1, \ldots, v_n)$.

To get an easier description of $d_n$, we renormalize the $K[v]$-modules $C_n K[v]$ by setting for each clique $\sigma \subseteq \Gamma$

$$c_\sigma = \tilde{c}_\sigma = \prod_{v \in \sigma, \chi(v) \neq 0} \chi(v).$$

From now on we write $C_n K[v]$ instead of $C_n \otimes_{\mathbb{Z}} K[v]$, and think of $C_n K[v]$ as a free $K[v]$-module with free basis $\{c_w\}$, i.e. the free basis of $C_n$. Then

$$d_n(\tilde{c}_\sigma) = \sum_{\chi(v_r) \neq 0} (-1)^{r-1} \tilde{c}_{\sigma \setminus \{v_r\}} v.$$  
Define $D_n = \text{Im}(d_{n+1})$ and $A_n = \text{Ker}(d_n)$. Then

$$H_n(N, K) = A_n/D_n.$$

Consider the chain complex

$$\mathcal{C} : \ldots \rightarrow C_i \rightarrow C_{i-1} \rightarrow \ldots \rightarrow C_0 = K \rightarrow 0,$$

with differential $d_n : C_n \rightarrow C_{n-1}$

given by

$$d_n(\tilde{c}_w) = \sum_{\chi(v_r) \neq 0} (-1)^{r-1} \tilde{c}_{\sigma \setminus \{v_r\}},$$

thus $d_n(\tilde{c}_w)v = d_n(\tilde{c}_w)$. Denote by $\mathcal{C}_*$ the complex obtained from $\mathcal{C}$ by shifting the index by -1 i.e. in $\mathcal{C}_*$ the module $C_n$ is in dimension $n - 1$ and we write $\tilde{d}_n$ for its differential, thus $\tilde{d}_{n-1} = \tilde{d}_n$. Note that

$$D_n = \text{Im}(\tilde{d}_{n+1})vK[v]$$ and $A_n = \text{Ker}(\tilde{d}_n)K[v]$.

Define $B_n = \text{Im}(\tilde{d}_{n+1})K[v]$. Then we have a short exact sequence of $K$-vector spaces

$$(4) \quad 0 \rightarrow B_n/D_n \rightarrow A_n/D_n \rightarrow A_n/B_n \rightarrow 0.$$
Note that $B_n/D_n \simeq \text{Im}(d_{n+1}) = \text{Im}(d_n)$ is a $K$-vector subspace of the finite dimensional $K$-vector space $C_n$, hence $\text{Im}(d_n)$ is finite dimensional. Moreover

$$A_n/B_n \simeq (\text{Ker}(d_{n-1})/\text{Im}(d_n)) \otimes_K K[v] \simeq H_{n-1}(\mathcal{C}_\bullet) \otimes_K K[v].$$

Hence the short exact sequence (4) gives a short exact sequence

$$0 \to \text{Im}(d_n) \to H_n(N, K) \to H_{n-1}(\mathcal{C}_\bullet) \otimes_K K[v] \to 0$$

so $H_n(N, K)$ is finite dimensional if and only if $H_{n-1}(\mathcal{C}_\bullet) = 0$.

As $N$ is $\mathbb{N}$-graded, Corollary 2.3 implies that $N$ is of type $FP_n$ if and only if $H_i(N, K)$ is finite dimensional for $i \leq n$. By the previous paragraph this is equivalent to the $(n-1)$-acyclicity of the complex $\mathcal{C}_\bullet$.

Fix a clique $w = (v_{k+1}, \ldots, v_n) \subseteq \Gamma \setminus \Gamma_\chi$, i.e., all its vertices have zero $\chi$-value and consider the subcomplex $\mathcal{C}_w$ of $\mathcal{C}_\bullet$ spanned by those $\tilde{c}_\sigma$, where

$$\sigma = (v_1, \ldots, v_k, v_{k+1}, \ldots, v_n)$$

is a clique in $\Gamma$ with $v_1, \ldots, v_k$ satisfying $\chi(v_i) \neq 0$ and $w = (v_{k+1}, \ldots, v_n)$ the fixed clique. Then $\mathcal{C}_w$ is a direct sum of the subcomplexes $\mathcal{C}_w$ over all possible cliques $w \subseteq \Gamma \setminus \Gamma_\chi$.

Note that $\mathcal{C}_w$ is the chain complex of the flag complex $\Delta_\chi$, of the living graph $\Gamma_\chi$ and that $\tilde{c}_\sigma \in \mathcal{C}_w$ if and only if $\sigma_0 = (v_1, \ldots, v_k)$ is a simplex in $lk_{\Delta_\chi}(w)$. Hence

$$H_{n-1}(\mathcal{C}_w) \simeq H_{n-1-|w|}(lk_{\Delta_\chi}(w), K)$$

and

$$H_{n-1}(\mathcal{C}_\bullet) \simeq \oplus_w H_{n-1-|w|}(lk_{\Delta_\chi}(w), K),$$

where the sum is over cliques $w \subseteq \Gamma \setminus \Gamma_\chi$.

Therefore $\mathcal{C}_\bullet$ is $(n-1)$-acyclic if and only if $H_i(lk_{\Delta_\chi}(w)) \otimes_Z K = 0$ for $i \leq n - 1 - |w|$ for any clique $w \subseteq \Gamma \setminus \Gamma_\chi$ i.e. if and only if $lk_{\Delta_\chi}(w)$ is $(n-1-|w|)$-acyclic over $K$ for any clique $w \subseteq \Gamma \setminus \Gamma_\chi$.

\[\square\]

Remark 3.2. The condition on the links of the statement of Theorem 3.1 (= Theorem D) is equivalent with the acyclicity condition used in the statement of [16 Main Thm.] that classifies when $\chi : G_{\Gamma} \to \mathbb{Z}$ belongs to the Bieri-Neumann-Renz-Strebel invariant $L^*(G_{\Gamma}, \mathbb{Z})$. Indeed, there is an obvious modification of the proof of Theorem 3.1 for groups, where $N$ is a subgroup of $G_{\Gamma}$ with $G_{\Gamma}/N \simeq \mathbb{Z}$. For a field $K$ this implies that $H_i(N, K)$ is finite dimensional for $i \leq n$ precisely when the acyclicity condition used in the statement of Theorem 3.1 holds. By [19 Thm. 7.3] $H_i(N, K)$ is finite dimensional for $i \leq n$ if and only if $N$ is of type $FP_n$ (note this is true in this specific case for a subgroup $N$ satisfying $G_{\Gamma}/N \simeq \mathbb{Z}$ and is not a general statement). But by [16 Cor. A] $N$ is of type $FP_n$ if and only if the acyclicity condition used in the statement of [16 Main Thm.] holds.

Corollary 3.3. (Corollary F) Let $N$ be the Lie algebra $\text{gr}(H_{\Gamma}) \otimes_Z K$ and $\Delta_{\Gamma}$ be the flag complex of $\Gamma$. Then $N$ is of type $FP_n$ if and only if $\Delta_{\Gamma}$ is $(n-1)$-acyclic over $K$ i.e. $H_i(\Delta_{\Gamma}, K) = 0$ for $i \leq n - 1$.

\[\text{Proof.}\] Note that $N = I_{\chi}$, where $\chi(v) = 1$ for every $v \in V(\Gamma)$. Then $\Gamma = \Gamma_\chi$ and by Theorem 3.1 $N$ is $FP_n$ if and only if the flag complex $\Delta_{\Gamma} = \text{lk}_{\Delta_{\Gamma}}(\emptyset)$ is $(n-1)$-acyclic over the field $K$. \[\square\]
We recall next the statement of Theorem 2.6 for right angled Artin Lie algebras $L_F$. This result together with Theorem 3.1 gives a complete classification of the coabelian ideals of $L_F$ which are of type $FP_n$.

**Corollary 3.4.** Let $L = L_F = \text{gr}(G_F) \otimes_{\mathbb{Z}} K$ and $M$ be a proper ideal of $L$ such that $[L, L] \leq M$. Then $M$ is of type $FP_n$ if and only if for every Lie subalgebra $I_{\chi}$ of $L$ such that $\chi(M) = 0$ we have that $\text{lk}_{\chi}^1(w)$ is $(n - 1 - |w|)$-acyclic over $K$ for any clique $w \subseteq \Gamma \setminus \Gamma_{\chi}$.

### 3.1. Kernels of higher codimension: the group case.

Next we prove the group theoretical version of Corollary 3.4 which involves the Bieri-Neumann-Strebel-Renz $\Sigma$-invariants of a group $G$. By definition the $\Sigma$-invariants are subsets of the character sphere $S(G) = \text{Hom}(G, \mathbb{R})/\sim$, where two characters (non-zero homomorphisms) $\chi_1, \chi_2 : G \to \mathbb{R}$ are equivalent if and only if there is a positive real number $r$ such that $\chi_1 = r \chi_2$. The equivalence class of $\chi$ is denoted by $[\chi]$. If $G/[G, G]$ has torsion-free rank $n$ it is easy to see that $S(G)$ can be identified with the unit sphere $S^{n-1}$ in $\mathbb{R}^n$. We omit details about the $\Sigma$-invariants but note that there are known for few classes of groups including the class of right angled Artin groups, see [16, Main Thm.]. Another fact that we will need in the proof of the following theorem is that the equivalence of items i) and iii) is precisely the statement of [4, Thm. B]. In this subsection we use the $\Sigma$-invariants as a technical tool we need in order to prove that items i), ii) and iv) from Theorem 3.3 are equivalent.

Recall that a group $G$ is of type $FP_n$ if the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ has a projective resolution with finitely generated modules in dimensions smaller or equal to $n$. If such projective resolution exists then there is a free resolution with the same property. In the case when $G_F/M \simeq \mathbb{Z}$ the equivalence between i) and ii) in Theorem 3.3 was proved in [19, Thm. 7.3].

**Theorem 3.5.** For a normal subgroup $M$ of the right angled Artin group $G_F$ that contains the commutator the following conditions are equivalent:

- i) $M$ is of homological type $FP_n$.
- ii) $H_i(M, K)$ is finite dimensional (over $K$) for $i \leq n$ and every field $K$.
- iii) $[\chi] \in \Sigma^n(G_F, \mathbb{Z})$ for any character $\chi : G_F \to \mathbb{R}$ that vanishes on $M$.
- iv) for every subgroup $N$ of $G$ such that $M \subseteq N$ and $G_F/N \simeq \mathbb{Z}$ we have that $N$ is $FP_n$.

**Proof.** Note that i) is equivalent to iii) by [4, Thm. B].

The fact that i) implies ii) is obvious. Indeed, if $\mathcal{P}$ is a free resolution of the trivial $\mathbb{Z}M$-module $\mathbb{Z}$ with finitely generated modules in dimensions smaller or equal to $n$ then $H_i(M, K) \simeq H_i(\mathcal{P} \otimes_{\mathbb{Z}M} K)$ is finite dimensional (over $K$) since the modules of $\mathcal{P} \otimes_{\mathbb{Z}M} K$ in dimensions smaller or equal to $n$ are all finite dimensional (over $K$).

We check that ii) implies iv). For any discrete character $\chi : G_F \to \mathbb{Z}$ such that $\chi(M) = 0$, consider the Lyndon-Hochshild-Serre spectral sequence

$$E^2_{p,q} = H_p(\text{Ker}(\chi)/M, H_q(M, K))$$

that converges to $H_{p+q}(\text{Ker}(\chi), K)$. Since $\text{Ker}(\chi)/M$ is a finitely generated abelian group and $H_q(M, K)$ is finite dimensional for $q \leq n$, we deduce that $H_i(\text{Ker}(\chi), K)$ is finite dimensional for all $i \leq n$. By Theorem 7.3 from [19], $N = \text{Ker}(\chi)$ is $FP_n$, i.e. iv) holds.
Finally, we claim that iv) implies iii). If $\chi : G_\Gamma \to \mathbb{R}$ is a character (i.e. non-zero homomorphism) such that $\chi(M) = 0$, let $\chi_0 : G_\Gamma \to \mathbb{Z}$ be a discrete character such that $\chi_0(M) = 0$ and $\Gamma_\chi = \Gamma_{\chi_0}$ i.e. for $v \in V(\Gamma)$ we have $\chi(v) = 0$ if and only if $\chi_0(v) = 0$. By [4] Thm. B the fact that $N = \text{Ker}(\chi_0)$ is $FP_n$ implies $[\chi_0] \in \Sigma^n(G_\Gamma, \mathbb{Z})$. Then by the description of $\Sigma^n(G_\Gamma, \mathbb{Z})$ in [16] Main Thm.] the property $\Gamma_\chi = \Gamma_{\chi_0}$ implies $[\chi] \in \Sigma^n(G_\Gamma, \mathbb{Z})$ if and only if $[\chi_0] \in \Sigma^n(G_\Gamma, \mathbb{Z})$. Thus iii) holds.

4. ELEMENTARY APPROACH FOR FINITE GENERATION OF $I_\chi$

In this Section we give a proof of the criterion when $I_\chi$ is finitely generated as a Lie algebra, i.e. Corollary E, using more elementary combinatorial methods and avoiding homological arguments, i.e. not using Theorem D.

As before $L_\Gamma = \text{gr}(G_\Gamma) \otimes_\mathbb{Z} K$ for a graph $\Gamma$.

Lemma 4.1. Let $\Gamma$ be a graph with $m \geq 2$ vertices and no edges and

$$\chi : V \to K$$

be a non-zero $K$-linear map for $V = L_\Gamma/[L_\Gamma, L_\Gamma]$. Then $I_\chi$ is not finitely generated as Lie algebra.

**Proof.** Note that in this case $G_\Gamma$ is the free group of rank $m$ and $\text{gr}(G_\Gamma) \otimes_\mathbb{Z} K$ is the free Lie algebra $F_n$ (over $K$) on $m$ elements. And no proper ideal in $F_n$ is finitely generated as a Lie algebra [11] Theorem 3].

Recall that $\Gamma_\chi$ is the subgraph of $\Gamma$ spanned by all vertices with non-zero $\chi$-value. We call $\Gamma_\chi$ the living subgraph with respect to $\chi$. We say that a subgraph $\Gamma_1$ of $\Gamma$ is dominant in $\Gamma$ if for every $v \in V(\Gamma) \setminus V(\Gamma_1)$ there is $w \in V(\Gamma_1)$ that is linked with $v$ by an edge in $\Gamma$.

**Lemma 4.2.** Let $\Gamma$ be a finite graph.

a) Suppose that $\Gamma_\chi$ is not connected. Then $I_\chi$ is not finitely generated as a Lie algebra.

b) Suppose that $\Gamma_\chi$ is connected but is not dominant in $\Gamma$. Then $I_\chi$ is not finitely generated as a Lie algebra.

**Proof.** a) Let $\Gamma_0$ be the set of connected components in $\Gamma_\chi$. We view $\Gamma_0$ as a graph without edges and denote by $[v]$ the connected component in $\Gamma_0$ of a vertex $v \in \Gamma_\chi$. For every $[v] \in \Gamma_\chi$ choose one vertex $v_0 \in [v]$. Define an epimorphism of Lie algebras

$$\pi_0 : L_\Gamma \to L_{\Gamma_0}$$

given by $\pi_0(w) = \frac{\chi(w)}{\chi(v_0)}[v]$ if $w \in [v] \in \Gamma_0$ and $\pi_0(w) = 0$ if $\chi(w) = 0$. Then $\text{Ker}(\pi_0) \subseteq I_\chi$.

Note that $\Gamma_0$ is a finite set of at least two disjoint points, hence we can apply Lemma 4.1 to deduce that $I_{\chi_0}$ is not finitely generated, where

$$\chi_0 : \bigoplus_{v \in \Gamma_0} K[w] \to K$$

is a $K$-linear map induced by $\chi$, i.e. $\chi_0([v]) = \chi((v_0))$. Note that $I_\chi/\text{Ker}(\pi_0) \simeq I_{\chi_0}$, hence $I_\chi$ is not finitely generated.

b) Let $v_1$ be a vertex of $\Gamma \setminus \Gamma_\chi$ that is not linked with any vertex from $\Gamma_\chi$. Consider the epimorphism

$$\pi_1 : L_\Gamma \to L_{\Gamma_1},$$
where $\Gamma_1$ is the subgraph spanned by $\Gamma_\chi$ and $v_1$ and $\pi_1$ sends every vertex of $\Gamma \setminus (\Gamma_\chi \cup \{v_1\})$ to 0. Then $\ker(\pi_1) \subseteq I_\chi$ and $I_\chi / \ker(\pi_1) \simeq I_{\chi_1}$, where

$$\chi_1 : L_{\Gamma_1}[L_{\Gamma_1}, L_{\Gamma_1}] \to K$$

is the restriction of $\chi$.

Recall that $\Gamma_\chi$ is a connected graph. Let $\Gamma_2$ be the graph with two vertices $u_1$ and $u_2$ and no edges. Fix a vertex $v_0 \in \Gamma_\chi$ and define the epimorphism of Lie algebras

$$\pi_2 : L_{\Gamma_1} \to L_{\Gamma_2}$$

$$v_1 \mapsto u_1, \quad v \mapsto \frac{\chi(v)}{\chi(v_0)} u_2 \text{ for any } v \in \Gamma_\chi.$$

Note that $\ker(\pi_2) \subseteq I_{\chi_1}$. Finally we define a $K$-linear map

$$\chi_2 : L_{\Gamma_2}[L_{\Gamma_2}, L_{\Gamma_2}] \to K$$

as induced by $\chi$ i.e. $\chi_2(u_1) = 0$, $\chi_2(u_2) = \chi(v_0)$. Then $I_{\chi_1} / \ker(\pi_2) \simeq I_{\chi_2}$ and by Lemma 4.1 $I_{\chi_2}$ is not finitely generated. Hence $I_{\chi_1}$ and $I_{\chi}$ are not finitely generated.

\begin{proposition}
If $\Gamma_{\chi}$ is connected and dominant in $\Gamma$ then $I_{\chi}$ is generated as a Lie algebra by a basis of the $K$-linear space $\ker(\chi)$, in particular $I_{\chi}$ is a finitely generated Lie algebra.
\end{proposition}

\begin{proof}
Let $T$ be a maximal tree in $\Gamma_{\chi}$. Fix a vertex $x_1$ in $T$ and decompose $T$ as a union of geodesics $\gamma$ that start at $x_1$. Let $v_{i,1} = x_1, v_{i,2}, \ldots, v_{i,t}$ be the consecutive vertices of the geodesics $\gamma$. Then define

$$\gamma_{i,j} = \chi(v_{i,j-1} - v_{i,j}) \in \ker(\chi) \subseteq I_\chi$$

and note that the coefficients $\chi(v_{i,j-1})$ and $\chi(v_{i,j})$ are non-zero since $v_{i,j-1}, v_{i,j}$ are vertices in $\Gamma_{\chi}$. Then by (5) we can write $v_{i,j}$ as a linear combination $\frac{1}{\chi(v_{i,j-1})} v_{i,j-1} + \frac{\chi(v_{i,j})}{\chi(v_{i,j-1})} v_{i,j-1}$ and then repeat this for $v_{i,j-1}, v_{i,j-2}$ and so on until $v_{i,2}$. This gives

$$v_{i,j} = \lambda_{i,j} x_1 + \sum_{2 \leq t \leq j} \lambda_{i,j,t} y_{i,t},$$

where all coefficients $\lambda_{i,j}, \lambda_{i,j,t} \in K \setminus \{0\}$. Note that by (5) $[v_{i,j-1}, v_{i,j}] = 0$ implies that $[v_{i,j-1}, y_{i,j}] = 0$ and using (6) for $v_{i,j-1}$ we get

$$[\lambda_{i,j-1} x_1 + \sum_{2 \leq t \leq j-1} \lambda_{i,j-1,t} y_{i,t}, y_{i,j}] = 0.$$ 

Thus

$$[x_1, y_{i,j}] \in \langle \{y_{i,j}\} \rangle.$$ 

On the other hand for $x_s \in V(\Gamma) \setminus V(\Gamma_\chi)$ we have that $\chi(x_s) = 0$ and there is a vertex $v_{i,j} \in V(\Gamma_{\chi})$ such that there is an edge between $x_s$ and $v_{i,j}$. Then $[v_{i,j}, x_s] = 0$, hence by (6)

$$[\lambda_{i,j} x_1 + \sum_{2 \leq t \leq j} \lambda_{i,j,t} y_{i,t}, x_s] = 0.$$ 

Then

$$[x_1, x_s] \in \langle \{y_{i,t}, x_s\} \rangle.$$ 

Note that $\{y_{i,t}\}_t \cup \{x_s\}_{x_s \in V(\Gamma) \setminus V(\Gamma_\chi)}$ is a basis of $\ker(\chi)$ as a $K$-vector space and by (7) and (8) the Lie algebra $\langle \ker(\chi) \rangle$ is an ideal in $L_\Gamma$ i.e it is the ideal $I_{\chi}$.
\end{proof}
Finally note that Lemma 4.2 and Proposition 4.3 imply Corollary E.

5. Kernels of higher degrees : a sufficient topological condition

In this Section we explain a more geometric approach that gives a sufficient condition for the Lie algebra $M$ from Theorem 2.6 to be of type $FP_n$ in the case $L = L^\Gamma$. Recall that as the universal enveloping algebra $U(L)$ is a Hopf algebra, the $K$-tensor product of two right $U(L)$-modules is a right $U(L)$-module via the comultiplication $\Delta: U(L) \to U(L) \otimes U(L)$ that sends $a \in L$ to $1 \otimes a + a \otimes 1$. Moreover, if $T \leq L$ is a Lie subalgebra, $W$ a $U(T)$-module and $V$ a $U(L)$-module we have the following Mackey-type formula i.e. an isomorphism of right $U(L)$-modules

\[
(W \otimes_K V) \otimes_{U(T)} U(L) \simeq (W \otimes_{U(T)} U(L)) \otimes_K V
\]

that sends $w \otimes v \otimes \lambda$ to $\sum_i (w \otimes \lambda_{i,1}) \otimes v \lambda_{i,2}$, where $\Delta(\lambda) = \sum_i \lambda_{i,1} \otimes \lambda_{i,2}$, (see [III], pages 8 and 9).

We use the isomorphism (9) to show

**Lemma 5.1.** Let $M \triangleleft L$ be an ideal of the Lie algebra $L$ and $T \leq L$ a subalgebra. Assume that the inclusion of $T$ in $L$ induces an isomorphism of Lie algebras $T/(T \cap M) \simeq L/M$. Then there is an isomorphism of $U(L)$-modules

\[
(K \otimes_{U(T)} U(L)) \otimes_K (K \otimes_{U(M)} U(L)) \simeq K \otimes_{U(T \cap M)} U(L).
\]

**Proof.** Note there is an isomorphism of right $U(T)$-modules

\[
K \otimes_{U(M \cap T)} U(T) \simeq U(T / T \cap M) \simeq U(L / M) \simeq K \otimes_{U(M)} U(L).
\]

Then using (9) for $W = K, V = K \otimes_{U(M)} U(L)$ we obtain

\[
(K \otimes_{U(M)} U(L)) \otimes_K (K \otimes_{U(T)} U(L)) \simeq (K \otimes_{U(M)} U(L)) \otimes_{U(T)} U(L) \simeq (K \otimes_{U(M \cap T)} U(T)) \otimes_{U(T)} U(L) \simeq K \otimes_{U(T \cap M)} U(L).
\]

□

**Lemma 5.2.** Let $\Lambda$ be any associative ring and $W$ a $\Lambda$-module fitting in an exact chain complex

\[
\text{Ker}(\delta) \to D_n \overset{\delta}{\to} D_{n-1} \to \ldots \to D_0 \to W \to 0
\]

such that each $D_i$ is a $\Lambda$-module of type $FP_n$. Then $W$ is of type $FP_n$.

**Proof.** By induction on $n$ we may assume that $\text{Ker}(D_0 \to W)$ is of type $FP_{n-1}$. Then there is a short exact sequence of $\Lambda$-modules $0 \to \text{Ker}(D_0 \to W) \to D_0 \to W \to 0$ with $D_0$ of type $FP_n$ and $\text{Ker}(D_0 \to W)$ of type $FP_{n-1}$ and [5] Prop. 1.4 implies that $W$ is of type $FP_n$. □

**Lemma 5.3.** ([III, Lemma 2.8]) Let $S \leq L$ be a subalgebra of an arbitrary Lie algebra $L$. Then $S$ is of type $FP_n$ if and only if the induced $U(L)$-module $K \otimes_{U(S)} U(L)$ is of type $FP_n$.

Recall that in this paper $\Gamma$ always denotes a finite graph and $L^\Gamma$ is $\text{gr}(Gr^\Gamma) \otimes_{\mathbb{Z}} K$.

**Theorem 5.4.** Let $[L, L] \leq M \triangleleft L$ be a codimension $k$ ideal of the right angled Artin Lie algebra $L = L^\Gamma$. Assume that there is an exact chain complex of $U(L)$-modules

\[
\text{Ker}(\delta) \to C_n \overset{\delta}{\to} C_{n-1} \to \ldots \to C_0 \to K \to 0
\]
such that $K$ is the trivial $U(L)$-module and each $C_i$ is a finite sum of $U(L)$-modules of the form

$$K \otimes_{U(T)} U(L)$$

for $T \leq L$ Lie subalgebras such that $T \cap M$ has codimension $k$ in $T$ and $T \cap M$ is of type $FP_n$. Then $M$ is also of type $FP_n$.

Proof. Tensoring the exact chain complex in the statement with $K \otimes_{U(M)} U(L)$ we get an exact chain complex

$$\text{Ker}(\delta) \rightarrow D_n \rightarrow D_{n-1} \rightarrow \ldots \rightarrow D_0 \rightarrow K \otimes_{U(M)} U(L) \rightarrow 0,$$

where each $D_i$ is a finite sum of modules of the form $(K \otimes_{U(T)} U(L)) \otimes_K (K \otimes_{U(M)} U(L))$. As $T/T \cap M$ is a vector space of dimension $k$, we see that $T/T \cap M \cong L/M$ and thus using Lemma 5.1 we deduce that each $D_i$ is a finite sum of modules of the form $K \otimes_{U(T;M)} U(L)$. By the hypothesis that $T \cap M$ is $FP_n$ together with Lemma 5.2 these modules are all of type $FP_n$ as $U(L)$-modules. Then Lemma 5.2 implies that $K \otimes_{U(M)} U(L)$ is of type $FP_n$ as $U(L)$-module and it suffices to apply Lemma 5.2 again to deduce that $K$ is of type $FP_n$ as $U(M)$-module i.e. $M$ is $FP_n$. \qed

To construct a suitable chain complex to which we can apply the previous Theorem we will use “parabolic” Lie subalgebras. Let $\Omega \subseteq V(\Gamma)$ be a (possibly empty) subset. The parabolic algebra of $L_\Gamma$ associated to $\Omega$ is the subalgebra of $L_\Gamma$ generated by $\Omega$. It is isomorphic to the right angled Artin Lie algebra $L_{\Gamma_0}$ associated to the full subgraph $\Gamma_0 \subseteq \Gamma$ generated by the vertices in $\Omega$. Indeed, by [20, Thm. 6.3] there is an isomorphism $\text{gr}(G_\Gamma) \cong L'(U_\Gamma)$ of $\mathbb{N}$-graded Lie algebras over $\mathbb{Z}$, where $U_\Gamma$ is an associative ring defined as the free $\mathbb{Z}$-module with basis the free partially commutative monoid $M_\Gamma$ generated by $V(\Gamma)$ and $L'(U_\Gamma)$ is the Lie ring (over $\mathbb{Z}$) obtained from $U_\Gamma$ by defining the Lie operation as $[u_1, u_2] = u_1u_2 - u_2u_1$. Note that by [13] the embedding of $\Gamma_0$ in $\Gamma$ induces an embedding of $G_{\Gamma_0}$ in $G_\Gamma$ and this induces an embedding of $M_{\Gamma_0}$ in $M_\Gamma$. From now on, we will use the same notation, $L_\Omega$ or $L_{\Gamma_0}$, for subsets of vertices $\Omega$ or for subgraphs $\Gamma_0$.

Now consider the poset of all subsets of $V(\Gamma)$ ordered by inclusion and assume that $H$ is a subposet. Let $|H|$ be the simplicial realization of $H$, that is, the simplicial complex with $k$-simplices $\sigma : A_0 \subset A_1 \subset \ldots \subset A_k$, where $A_0, \ldots, A_k \in H$. We write $|\sigma| = k$, call $A_0, \ldots, A_k$ vertices of $\sigma$ and $A_k$ the biggest vertex of $\sigma$. Then we associate to the simplex $\sigma$ the subalgebra $L_\sigma := L_{|\sigma|}$, i.e., the subalgebra generated by the smallest of these subsets. Let $m$ be the dimension of $|H|$. We define next a chain complex of $U(L)$-modules depending on $H$ as follows. We set

$$C^l(H) : 0 \rightarrow C^l_m(H) \rightarrow \ldots \rightarrow C^l_k(H) \rightarrow \ldots \rightarrow C^l_0(H) \rightarrow C^l_{-1}(H) = K \rightarrow 0,$$

where

$$C^l_k(H) = \oplus_{\sigma \in |H|, |\sigma| = k} c_\sigma K \otimes_{U(L_\sigma)} U(L)$$

and the differential is given by

$$\delta(c_\sigma) = \sum_{i=0}^k (-1)^i c_{\sigma_i}$$

where $\sigma_i = \sigma \setminus \{A_i\}$ for $k > 0$ and by $\delta(c_\sigma) = 1$ for $k = 0$. We call $C^l(H)$ the coset poset complex of $H$ in $L$. 
Lemma 5.5. Let $\Gamma$ be a complete graph, $L = L_\Gamma$ and $X$ the poset of all the subsets of $Z := V(\Gamma)$. Then $C^L(X)$ is exact and there is a short exact sequence of chain complexes

$$0 \to C^L(X \setminus Z) \to C^L(X) \to E(Z) \to 0,$$

where $E(Z)$ is a complex with zero homology everywhere except of the top dimension $|Z|$. 

Proof. 1) We show first that $C^L(X)$ is exact. We will define a homotopy

$$s_k : C^L_k(X) \to C^L_{k+1}(X),$$

which will be a homomorphism of right $U(L)$-modules. For $\sigma : A_0 \subset A_1 \subset \ldots \subset A_k$ we write $\sigma = (A_0, \ldots, A_k)$ and we define

$$s_k(c_{(A_0, \ldots, A_k)}) = (-1)^{k+1} c_{(A_0, \ldots, A_k, Z)} \text{ if } A_k \neq Z \text{ and } s_k(c_{(A_0, \ldots, A_k, Z)}) = 0.$$ 

In dimension -1 we define $s_{-1}(1) = c_Z$. Then for $k \geq 0$ and $A_k \neq Z$ we have

$$(\partial_{k+1}s_k + s_{k-1}\partial_k)(c_{(A_0, \ldots, A_k)}) =$$

$$(-1)^{k+1}\partial_{k+1}(c_{(A_0, \ldots, A_k, Z)}) + \sum_{0 \leq i \leq k} (-1)^i s_{k-1}(c_{(A_0, \ldots, \hat{A}_i, \ldots, A_k, Z)}) =$$

$$\sum_{0 \leq i \leq k} (-1)^{k+i+1} c_{(A_0, \ldots, \hat{A}_i, \ldots, A_k, Z)} + \sum_{0 \leq i \leq k} (-1)^i s_{k-1}(c_{(A_0, \ldots, \hat{A}_i, \ldots, A_k, Z)}).$$

And for $k \geq 0$ and $A_k = Z$ we have

$$(\partial_{k+1}s_k + s_{k-1}\partial_k)(c_{(A_0, \ldots, A_k, Z)}) =$$

$$\sum_{0 \leq i \leq k-1} (-1)^i s_{k-1}(c_{(A_0, \ldots, \hat{A}_i, \ldots, A_k, Z)}).$$

Finally $\partial_0 s_{-1}(1) = \partial_0(c_Z) = 1$, hence

$$\partial_{k+1}s_k + s_{k-1}\partial_k = id.$$ 

This completes the proof of the fact that $C^L(X)$ is exact.

2) We will show now that $E(Z)$ has zero homology except at the top dimension $|Z|$. Suppose $Z = V(\Gamma) = \{x_1, \ldots, x_m\}$ and let $X_{i_1, \ldots, i_d}$ be the poset of all subsets of $Z \setminus \{x_{i_1}, \ldots, x_{i_d}\}$. Denote by $L_{i_1, \ldots, i_d}$ the Lie subalgebra of $L$ generated by $Z \setminus \{x_{i_1}, \ldots, x_{i_d}\}$. Denote by $R^\text{del}$ the complex obtained from a complex $R$ by substituting the module in dimension -1 with 0. Since $C^{L_{i_1, \ldots, i_d}}(X_{i_1, \ldots, i_d})$ is an exact complex, the complex

$$R^\text{del}_{\{i_1, \ldots, i_d\}} = C^{L_{i_1, \ldots, i_d}}(X_{i_1, \ldots, i_d}) \otimes_{U(L_{i_1, \ldots, i_d})} U(L)$$

is a subcomplex of $C^L(X \setminus Z)^\text{del}$ with

$$R^\text{del}_{\{i_1, \ldots, i_d\}} \cap R^\text{del}_{\{j_1, \ldots, j_h\}} = R^\text{del}_{\{i_1, \ldots, i_d\} \cup \{j_1, \ldots, j_h\}}.$$ 

This induces an exact sequence of complexes

$$0 \to R \to R_{\{x_1, \ldots, x_m\}} \to \cdots \to \oplus_{1 \leq |J| \leq Z} R_J \to \oplus_{|J| = 1 \leq Z} R_{|J| = 1} \to$$

$$\cdots \to \oplus_{1 \leq |J| \leq m} R_{\{x_1\}} \to C^L(X \setminus Z) \to 0,$$

where for $I = \{x_{i_1}, \ldots, x_{i_k}\}$ with $j_1 < \ldots < j_l$, $c_{\sigma} \in R_I$ is send to $\sum_{1 \leq l \leq k} (-1)^l c_{\sigma} \in \oplus_{1 \leq l \leq k} R_{I \setminus \{x_{i_l}\}}$ and $R$ is a complex concentrated in dimension -1.
By dimension shifting argument since $5$ is an exact sequence of complexes and all complexes in $5$ except the first and the last are exact, we deduce that

$$H_j(\mathcal{P}) \simeq H_{j+m}(C^L(X \setminus Z)).$$

Finally using that $C^L(X)$ is an exact complex and dimension shifting argument for the short exact sequence

$$0 \to C^L(X \setminus Z) \to C^L(X) \to E(Z) \to 0,$$

we obtain that

$$H_{i-1}(C^L(X \setminus Z)) \simeq H_i(E(Z)).$$

Thus

$$H_i(E(Z)) \simeq H_{i-1-m}(\mathcal{P})$$

and since the complex $\mathcal{P}$ is concentrated in dimension $-1$, $H_i(E(Z))$ is concentrated in dimension $m$, as required.

\begin{proposition}
Let $X$ be the poset of all the cliques of $\Gamma$. Then the chain complex $C(X) := C^L(X)$ is exact.
\end{proposition}

\begin{proof}
The case when the graph $\Gamma$ is complete is Lemma $5.5$. In the case when $\Gamma$ is not complete, choose $v_1, v_2 \in \Gamma$ not linked by an edge. Let $\Gamma_1$ be the subgraph of $\Gamma$ spanned by $V(\Gamma) \setminus \{v_2\}$ and $\Gamma_2$ be the subgraph of $\Gamma$ spanned by $V(\Gamma) \setminus \{v_1\}$. Then $L_{\Gamma} = L_{\Gamma_1} \star L_{\Gamma_2}$ is the amalgamated product of Lie algebras, where $\Lambda = \Gamma_1 \cap \Gamma_2$. Consider the short exact sequence associated to the amalgamated product.

\begin{equation}
0 \to K \otimes_{U(\Lambda)} U(L) \to K \otimes_{U(\Gamma_1)} U(L) \oplus K \otimes_{U(\Gamma_2)} U(L) \to K \to 0.
\end{equation}

Note that arguing by induction we may assume that if $X_1$ and $X_2$ are the posets of all the subsets of $\Gamma_1$, resp. $\Gamma_2$, that generate complete subgraphs, then the complexes $C(X_1) := C^{L_{\Gamma_1}}(X_1)$ and $C(X_2) := C^{L_{\Gamma_2}}(X_2)$ are exact. Moreover, $X_1 \cap X_2$ is the poset of all the subsets of $\Lambda$ that generate a complete subgraph, so we may also assume that $C(X_1 \cap X_2) := C^{L_{\Lambda}}(X_1 \cap X_2)$ is exact. From these complexes we can get the exact complexes $C(X_1) \otimes_{U(\Gamma_1)} U(L)$, $C(X_2) \otimes_{U(\Gamma_2)} U(L)$ and $C(X_1 \cap X_2) \otimes_{U(\Lambda)} U(L)$. At this point we claim that using $(11)$ one can show that there is a short exact sequence of chain complexes

$$0 \to C(X_1 \cap X_2) \otimes_{U(\Lambda)} U(L) \to (C(X_1) \otimes_{U(\Gamma_1)} U(L)) \oplus (C(X_2) \otimes_{U(\Gamma_2)} U(L)) \to C(X) \to 0$$

which implies that $C(X)$ is exact. To see this note that at each dimension $k$ we have a short exact sequence

$$0 \to C_k(X_1 \cap X_2) \otimes_{U(\Lambda)} U(L) \xrightarrow{\partial_k} (C_k(X_1) \otimes_{U(\Gamma_1)} U(L)) \oplus (C_k(X_2) \otimes_{U(\Gamma_2)} U(L)) \xrightarrow{\partial_k} C_k(X) \to 0.$$

Here, $\partial_k$ maps the copy of $(K \otimes_{U(\Lambda)} U(L_\Lambda)) \otimes_{U(\Lambda)} U(L) = K \otimes_{U(\Lambda)} U(L)$ corresponding to each $\sigma : A_0 \subset \ldots \subset A_k$ in $X_1 \cap X_2$ to the corresponding sum of copies of $(K \otimes_{U(L_\Gamma)} U(L_\Gamma)) \otimes_{U(L_\Gamma)} U(L) = K \otimes_{U(L_\Gamma)} U(L)$, $i = 1, 2$ via $a \mapsto (a, -a)$ and $\partial_k$ maps each of the copies in either $C_k(X_1) \otimes_{U(\Gamma_1)} U(L)$ or $C_k(X_2) \otimes_{U(\Gamma_2)} U(L)$ identically to the corresponding copy in $C_k(X)$. The facts that $|X| = |X_1| \cup |X_2|$ and that $|X_1 \cap X_2| = |X_1| \cap |X_2|$ imply that this is a short exact sequence. At degree $-1$ we have the short exact sequence $(11)$. Then one easily checks that these maps commute with the differentials of the chain complexes, so the claim follows.
\end{proof}
Proposition 5.7. Let $X$ be the poset of all subsets of $V(\Gamma)$ that generate complete subgraphs and $Y \subseteq X$ a subposet with the property that for any $A \in Y$, $B \in X$ with $A \subseteq B$, we have $B \in Y$. Then there is a finite tower of chain complexes

$$C^k(Y) := C(Y) = D^0 \subseteq D^1 \subseteq \ldots \subseteq D^r = C^k(X) := C(X)$$

such that for each $i$ there is a short exact sequence

$$0 \to D^{i-1} \to D^i \to T^i \to 0,$$

where

$$T^i = \bigoplus_{Z \in X \setminus Y, |Z| = i} T(Z),$$

$$T(Z) \simeq (E(Z) \otimes_{U(L_Z)} U(L)) \otimes_K \check{C}_K(\text{lk}_{|Y|}(Z)),$$

$\check{C}_K$ is the augmented (ordinary) chain complex over the field $K$ of a simplicial complex and $E(Z)$ fits in a short exact sequence

$$0 \to C^k_0(X \setminus Z) \to C^k(Z) \to E(Z) \to 0,$$

where $X$ is the poset of all subsets of $Z$ and by definition $E(\emptyset) = K$. Moreover, if $\text{lk}_{|Y|}(Z)$ is $(n+1)$-acyclic over $K$ for any $Z \in X \setminus Y$ with $|Z| = i$, then $T(Z)$ is $(n-1)$-acyclic.

Proof. Let $D^i$ be the complex associated to the simplices $\sigma : A_0 \subseteq \ldots \subseteq A_k$ such that for $0 \leq j \leq k$, $|A_j| \geq i$ implies $A_j \in Y$. Then

$$D^0 \subseteq D^1 \subseteq \ldots \subseteq D^i \subseteq D^{i+1} \subseteq \ldots$$

Each simplex $\sigma : A_0 \subseteq \ldots \subseteq A_k$ defines $\sigma_{X \setminus Y} : A_0 \subseteq \ldots \subseteq A_{i-1}$ and $\sigma_Y : A_i \subseteq \ldots \subseteq A_k$, where each $A_0, \ldots, A_{i-1}$ is in $X \setminus Y$ and each $A_i, \ldots, A_k$ is in $Y$. Then $D^i$ is the complex of those simplices such that the biggest vertex $A_{i-1}$ of the simplex $\sigma_{X \setminus Y}$ is a set of cardinality at most $i - 1$. This implies that if $\sigma \notin D^i$ but $\sigma \in D^{i+1}$ then the biggest vertex of $\sigma_{X \setminus Y}$ has cardinality exactly $i$. And from this we see that we may decompose the quotient chain complex $D^{i+1}/D^i$ as a direct sum of

$$D^{i+1}/D^i = \bigoplus_{Z \in X \setminus Y, |Z| = i} T(Z),$$

where $T(Z)$ is a sum of the summands corresponding to cells $\sigma$ having precisely $Z$ as the biggest vertex in $\sigma_{X \setminus Y}$. Note that the boundary of such a cell $\sigma$ will be either a cell in the same set or a cell that lies in $D^i$ and thus vanish in the quotient $D^{i+1}/D^i$.

Now, the chain complex $T(Z)$ is the following

$$T(Z) = (E(Z) \otimes_{U(L_Z)} U(L)) \otimes_K \check{C}(\text{lk}_{|Y|}(Z)).$$

Recall that $\text{lk}_{|Y|}(Z)$ is a combinatorial subcomplex of $|Y|$ that contains a cell $\sigma_Y : A_i \subseteq \ldots \subseteq A_k$ from $|Y|$ if $\sigma_0 : Z \subseteq A_i \subseteq \ldots \subseteq A_k$ is a cell in $|X|$.

Note that Lemma 5.5 implies that $\check{H}_j(E(Z)) = 0$ for $j \neq i = |Z|$. At this point, using Küneth Theorem and the fact that $U(L)$ is flat as $U(L_Z)$-module we have

$$\check{H}_n(T(Z)) = \bigoplus_{0 \leq j \leq n} \check{H}_j(E(Z) \otimes_{U(L_Z)} U(L)) \otimes_K \check{H}_{n-j}(\text{lk}_{|Y|}(Z), K) =$$

$$\bigoplus_{0 \leq j \leq n} \check{H}_j(E(Z)) \otimes_{U(L_Z)} U(L) \otimes_K \check{H}_{n-j}(\text{lk}_{|Y|}(Z), K).$$

Thus

$$\check{H}_n(T(Z)) = 0$$

for $n \leq i - 1$.
and
\[ H_n(T(Z)) = H_i(E(Z)) \otimes_{\mathcal{U}(L_2)} U(L) \otimes_K H_{n-i}(\text{lk}_{|Y|}(Z), K) \]
for \( n \geq i \),
so if \( \text{lk}_{|Y|}(Z) \) is \((n - i - 1)\)-acyclic over \( K \), then \( T(Z) \) is \((n - 1)\)-acyclic.
\[ \square \]

**Corollary 5.8. (Theorem G)** Let \( L = L_F \) and \( [L, L] \leq M \triangleleft L \) be an ideal of codimension \( k \). Let \( X \) be the set of all subsets of \( V(\Gamma) \) which generate complete subgraphs and \( Y \subseteq X \) be the set of those \( A \in X \) such that \( M \cap L_A \) has corank \( k \) in \( L_A \). Assume that \( \text{lk}_{|Y|}(Z) \) is \((n - i - 1)\)-acyclic for any \( Z \in X \setminus Y \) with \( |Z| = i \). Then \( M \) is of type \( \text{FP}_n \).

**Proof.** By Proposition 5.7 the complex \( C^L(Y) = C(Y) \) is \((n - 1)\)-acyclic. Then by Theorem 3.1 \( M \) is of type \( \text{FP}_n \).
\[ \square \]

Now we want to relate the condition of Corollary 5.8 with the condition of Theorem 3.1 in the case when \( M \) has codimension 1 in \( L = L_F = \text{gr}(G_T) \otimes Z K \). To do that, let \( \pi : L \rightarrow L/[L, L] \) be the canonical homomorphism and \( \chi : L/[L, L] \rightarrow K \) be a non-zero \( K \)-linear map such that \( \pi^{-1}(\text{Ker}(\chi)) = M \) and as in Theorem 3.1 denote by \( \Gamma' \) the living subgraph of \( \Gamma \). Let \( X \) and \( Y \) be as in Corollary 5.8.

Let \( Z \in X \setminus Y \) and for consistency with the notation of Theorem 3.1 we set \( n - k \) for the cardinality of \( Z \). Put
\[ P_1 = \{ A \in X \mid A \cap V(\Gamma'_X) \neq \emptyset, Z \subseteq A \} \]
and
\[ P_2 = \{ \emptyset \neq B \in X \mid B \subseteq V(\Gamma'_X), Z \cup B \in X \}. \]
Let \( f : P_1 \rightarrow P_2 \) be given by \( f(A) = A \cap V(\Gamma'_X) \) and \( g : P_2 \rightarrow P_1 \) be given by \( g(B) = B \cup Z \). Then \( fg(B) = B \) and \( g(f(A)) = (A \cap V(\Gamma'_X)) \cup Z \subseteq A \). Using 2.4 Lemma 6.4.5 this implies that the simplicial realizations \( |P_1| \) and \( |P_2| \) are homotopy equivalent.

Note that \( |P_1| = \text{lk}_{|Y|}(Z) \) and \( |P_2| = \text{lk}_{\Delta_T'}(Z) = \text{lk}_{\Delta_T}(Z) \cap \Delta_T \). This implies that the hypothesis of Corollary 5.8 are equivalent to the hypothesis of Theorem 3.1.

**References**

[1] Baumslag, B.; Free Lie algebras and free groups. J. London Math. Soc. (2) 4 (1971/72), 523 - 532.
[2] Benson, D. J.; Representations and cohomology. Vol. 2. Cambridge university press, 1991.
[3] Bestvina, M.; Brady, N.; Morse theory and finiteness properties of groups, Inventiones mathematicae (1997), 129, Iss. 3, pp 445 - 470.
[4] Bieri R., Renz B., Valuations on free resolutions and higher geometric invariants of groups, Comment. Math. Helv. 63 (1988), no. 3, 464 - 497.
[5] Bieri, R.; Homological dimension of discrete groups, Queen Mary College Mathematical Notes, Queen Mary College Department of Pure Mathematics, London, second edition, 1981.
[6] Blasco-García, R., Cogolludo-Agustín, J. I., Martínez-Pérez, C.; On the Sigma invariants of even Artin groups of FC-type. Preprint.
[7] Dicks, W.; Leary, I. J.; Presentations for subgroups of Artin groups. Proc. Amer. Math. Soc. 127, (1999), no. 2, 343 - 348.
[8] Duchamp, G.; Krob, D.; The free partially commutative Lie algebra: bases and ranks, Adv. Math. 95, (1992), 92 - 126.
[9] Duchamp, G.; Krob, D.; The lower central series of the free partially commutative group, Semigroup Forum 45, (1992), 385 - 394.
[10] King, J. D.; Homological finiteness conditions for pro-p groups. Comm. Algebra 27 (1999), no. 10, 4969 - 4991.
[11] Kochloukova, D.; Martínez-Pérez, C.; Bass-Serre Theory for Lie Algebras: A Homological approach.

[12] Leary, I., Saadetoglu, M.; The cohomology of Bestvina-Brady groups. Groups, Geometry and Dynamics, 5 (1) (2011), 121-138.

[13] van der Lek, H.; The homotopy type of complex hyperplane complements, Ph.D. thesis, University of Nijmegen, Nijmegen, 1983

[14] Lichtman, A. I., Shirvani, M.; HNN-extensions for Lie algebras, Proc. AMS, Vol. 125, No. 12, 3501 - 3508, 1997

[15] Meier, J.; VanWyk, L.; The Bieri-Neuman-Strebel invariants for graph groups. Proc. London Math. Soc. (3), 71 (1995), 263 - 280

[16] Meier, J.; Meinert, H.; VanWyk, L.; Higher generation subgroup sets and the \( \Sigma \)-invariants of graph groups. Comment. Math. Helv. 73 (1998), no. 1, 22 - 44

[17] Papadima, S.; Suciu, A.; Algebraic invariants for right-angled Artin groups, Math. Annalen, 334, (2006), 533 - 555

[18] Papadima, S.; Suciu, A.; Algebraic invariants for Bestvina-Brady groups, J. Lond. Math. Soc. (2), 76 (2007), no. 2, 273 - 292

[19] Papadima, S.; Suciu, A.; Toric complexes and Artin kernels, Adv. Math. 220 (2009), no. 2, 441 - 477

[20] Wade, R. D., The lower central series of a right-angled Artin group. Enseign. Math. 61 (2015), no. 3-4, 343 - 371

[21] Wasserman, A.; A derivation HNN construction for Lie algebras. Israel J. of Math. 106 (1998), 79 - 92

[22] Weigel, T.; Graded Lie algebras of type FP, Israel J. of Math. 205, (1) (2015), 185 - 209

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