Are There Incongruent Ground States in $2D$ Edwards-Anderson Spin Glasses?

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Abstract

We present a detailed proof of a previously announced result [1] supporting the absence of multiple (incongruent) ground state pairs for $2D$ Edwards-Anderson spin glasses (with zero external field and, e.g., Gaussian couplings): if two ground state pairs (chosen from metastates with, e.g., periodic boundary conditions) on $\mathbb{Z}^2$ are distinct, then the dual bonds where they differ form a single doubly-infinite, positive-density domain wall. It is an open problem to prove that such a situation cannot occur (or else to show — much less likely in our opinion — that it indeed does happen) in these models. Our proof involves an analysis of how (infinite-volume) ground states change as (finitely many) couplings vary, which leads us to a notion of zero-temperature excitation metastates, that may be of independent interest.

KEY WORDS: spin glass; ground state; incongruence; metastate; excitation.

1 Introduction

The decades-old challenge of understanding the physical nature of laboratory spin glasses and the mathematical nature of spin glass models at low temperature continues. It is a paradigm of the wider effort to analyze the many novel features that occur in disordered systems generally. One can only hope that this effort will achieve some fraction of the successes that

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have been reached in understanding homogeneous systems — in and out of equilibrium —
and that are epitomized by the work of Joel Lebowitz and his many collaborators. It is
indeed an honor to contribute to this celebration of Joel’s first 70 years; may he live to 120.

Our focus here is entirely on the Edwards-Anderson (EA) [2] model on $\mathbb{Z}^d$, simplest of
the short-ranged Ising spin glasses, with Hamiltonian

$$\mathcal{H}_\mathcal{J}(\sigma) = - \sum_{\langle x,y \rangle} J_{xy} \sigma_x \sigma_y$$  \hspace{1cm} (1)

Here $\mathcal{J}$ denotes a specific realization of the couplings $J_{xy} = J_{(x,y)}$, the spins $\sigma_x = \pm 1$ and
the sum is over nearest-neighbor pairs $\langle x, y \rangle$ only, with the sites $x, y$ on the square lattice $\mathbb{Z}^d$. The $J_{xy}$'s are independently chosen from a symmetric, continuous distribution with unbounded support, such as Gaussian with mean zero; we denote by $\nu$ the overall disorder
distribution for $\mathcal{J}$.

In this paper, we restrict attention entirely to ground states, and further, to the lowest
interesting dimension, $d = 2$. Of course, for $d = 1$, and assuming as we do that the $J_{xy}$'s
are continuously distributed, it is easy to see that the multiplicity of infinite-volume ground
states is exactly two — i.e., a single ground state pair (GSP) of spin configurations related
to each other by a global spin flip — since, in the absence of frustration, every bond can be
satisfied in a ground state.

We are interested in the question of whether there are infinitely many observable GSP’s.
By “observable” we mean that these states can be generated without using special $\mathcal{J}$-
dependent boundary conditions. This means that by using, say, periodic boundary condi-
tions on the $L \times L$ squares $S_L$ centered at the origin, for a sequence of $L$’s tending to
infinity, also chosen in a $\mathcal{J}$-independent way, the corresponding sequence of finite-volume GSP’s for the finite-volume Hamiltonians $\mathcal{H}_\mathcal{J}^{(L)}$ (when restricted to a fixed, but arbitrarily
large window about the origin) will generate an empirical distribution, i.e., a histogram, that
in the limit is dispersed over many GSP’s.

2 Main Result

2.1 Preliminaries: Metastates

To state a precise theorem about the GSP’s that arise in this way, we need to explain the
notion of a metastate [3, 4, 5, 6] in this zero-temperature context. We will do this in the
briefest possible way here, using empirical distributions, while delaying to later sections of
the paper a discussion of the fact that there are alternative definitions giving rise to the same mathematical object.

First, we note that for a given $J$, with all couplings nonzero, a GSP $\alpha$ may be identified with the collection of unsatisfied bonds, which we regard as edges in the dual lattice. Now suppose that $L_j \to \infty$ is a sequence of scale sizes, not depending on $J$, such that for $\nu$-almost every $J$, there is a probability measure (called a metastate) $\kappa_J$, defined on the configurations $\alpha$ of GSP’s on all of $\mathbb{Z}^2$, which is the limit of the empirical distributions of the finite volume GSP’s $\alpha_J^{(L_j)}$ along the sequence $L_j$ as follows: Let $D_1$ and $D_2$ be disjoint finite sets of dual edges, let $A(D_1, D_2)$ denote the event that every edge in $D_1$ is unsatisfied and every edge in $D_2$ is satisfied; let $F_J^{(M)}(D_1, D_2)$ denote the fraction of the indices $j \in \{1, \ldots, M\}$ such that all the edges of $D_1$ and $D_2$ are within the square $S_{L_j}$ and such that the GSP $\alpha_J^{(L_j)}$ obeys all the requirements of $A(D_1, D_2)$; then for every such $D_1$ and $D_2$,

$$\lim_{M \to \infty} F_J^{(M)}(D_1, D_2) = \kappa_J(A(D_1, D_2)) \quad (2)$$

Thus a metastate for $T = 0$ is an ensemble of infinite-volume GSP’s that describes the asymptotic fractions of squares, along a subsequence $L_j$, for which the various GSP’s are observed (when restricted to windows of fixed, but arbitrarily large, size) within the finite-volume systems. It can be shown by compactness arguments \cite{5, 6} that such subsequences $L_j$ exist; in fact every subsequence has such an $L_j$ as a further sub-subsequence. Although it is a reasonable conjecture that any two metastates are in fact the same for almost every $J$, no general result has been proved. However, this would be an immediate corollary of the following conjecture, at least for $d = 2$, which would also imply that the metastate is supported on a single GSP for almost every $J$. We note that recent numerical results are consistent with the existence of only a single GSP in two dimensions \cite{7, 8}.

**Conjecture 1.** Let $J$ be chosen from the disorder distribution $\nu$ and let $\alpha$ and $\beta$ be GSP’s chosen independently from $d = 2$ periodic boundary condition metastates, $\kappa_J$ and $\kappa'_J$ (coming from subsequences $L_j$ and $L'_k$). Then, with probability one, $\alpha = \beta$.

### 2.2 Theorem

The main result of this paper is the proof of the following theorem, which we regard as partial verification of the above Conjecture — see the Remark below. Equality of two GSP’s, $\alpha$ and $\beta$, is of course equivalent to the vanishing of the symmetric difference $\alpha \Delta \beta$, the collection of bonds that are satisfied in one of the two GSP’s and unsatisfied in the other. It is not hard to
show (see Proposition 1 below) that, at least for periodic boundary conditions, the symmetric
difference must consist either of a single domain wall (i.e., a doubly-infinite self-avoiding path
in the dual lattice) with strictly positive density or else multiple nonintersecting domain walls
which have altogether strictly positive density, but may have zero density individually. A
priori, we felt (and still feel) that on a heuristic level, the former scenario for GSP multiplicity
is the less plausible of the two. The next theorem rigorously eliminates the latter scenario.

**Theorem 1.** Let \( J \) be chosen from the disorder distribution \( \nu \) and let \( \alpha \) and \( \beta \) be
GSP’s chosen independently from \( d = 2 \) periodic boundary condition metastates, \( \kappa_J \) and
\( \kappa'_J \) (coming from subsequences \( L_j \) and \( L'_k \)). Then, with probability one, either \( \alpha = \beta \) or else
\( \alpha \Delta \beta \) is a single domain wall with strictly positive density.

**Proof.** This theorem will be an immediate consequence of three propositions, given in
Section 4 of the paper.

**Remark.** Although Theorem 1 does not eliminate the scenario of multiple GSP’s whose
symmetric differences are *single* positive density domain walls, we suspect that such domain
walls do not in fact occur. The proof of Theorem 1 is based on showing that the presence
of two or more \( \alpha \beta \) domain walls would create an instability for both \( \alpha \) and \( \beta \) with respect
to the flip of a large droplet whose boundary consists of two long segments from adjacent
domain walls, connected by two short “rungs” between the walls. The stability of \( \alpha \) and
\( \beta \) to such flips is controlled by the infimum \( E' \) of the necessarily positive rung energies —
see Equation (11). Proposition 3 of Sect. 4 proves instability by showing that \( E' = 0 \), while
Proposition 2 there shows that such unstable GSP’s cannot actually occur with nonzero
probability. If there were a single domain wall, it would be natural to expect that, like the
rungs in Proposition 3, the “pseudo-rungs” that connect sections of the domain wall that are
close in Euclidean distance, but greatly separated in distance along the domain wall, could
also have arbitrarily low positive energies. If these pseudo-rungs connected long pieces of the
domain wall containing some fixed bond (and we emphasize that these properties have not
been proved), then single domain walls would be ruled out by an analogue of Proposition 2.
The consequence would be that the periodic boundary condition metastate in the 2D EA
Ising spin glass would be unique and supported on a single GSP.

### 2.3 Extension to Other Boundary Conditions

The restriction to periodic boundary conditions in Theorem 1 can in fact be relaxed to
allow other boundary conditions that *do not depend on \( J \).* For boundary conditions such as
antiperiodic that are flip-related to periodic ones, nothing needs to be done, since they yield the same metastate — see Section IV of [4].

To explain how other boundary conditions can be handled, we begin by noting that the significance of periodic boundary conditions is that they yield translation-invariance of various infinite-volume objects, which in turn is a crucial ingredient in the propositions of the next section. With periodic boundary conditions, translation-invariance is already valid for finite volume. For example, from the random pair \((\mathcal{J}, \alpha^{(L)}_J)\), the finite dimensional distributions of finitely many coupling values and finitely many bond satisfaction variables are unchanged under translation by \(y\), as long as \(y\) does not translate any of the finitely many bonds in question beyond \(S_L\). On the other hand, in the spirit of the empirical distribution construction of the metastate described above, one could rather consider the random pair \((\mathcal{J}, \alpha^{(L)}_J)\), with \(L\) chosen, uniformly at random, from \(L_1, \ldots, L_M\). In that case, there is in a certain sense only approximate translation invariance for finite \(M\), since the bonds typically do get translated out of \(S_{L_j}\) for small \(j\). But full translation-invariance is restored in the limit \(M \to \infty\).

For non-periodic, but still \(J\)-independent, boundary conditions, one can somewhat similarly obtain infinite-volume translation-invariance, as follows. For each \(L\) and \(x\), let \(\alpha^{(L,x)}_J\) denote the GSP in the translated square \(S_L + x\) with some \(J\)-independent boundary condition, such as free or plus. Next, let \(\mathcal{X}(L)\) denote a uniformly random site in \(S_{L'(L)}\), where the deterministic \(L'(L) \to \infty\) with, say, \(L - L'(L) \to \infty\) (e.g., \(L'(L) = \sqrt{L}\)). Then the random pair \((\mathcal{J}, \alpha^{(L,\mathcal{X}(L))}_J)\) or, alternatively, \((\mathcal{J}, \alpha^{(L,\mathcal{X}(L))}_J)\), has approximate translation-invariance, which becomes exact as \(L \to \infty\), or, alternatively, \(M \to \infty\). Using such an “average over translates” construction, one can obtain metastates coming from, e.g., free or plus boundary conditions, for which the analogue of Theorem 1 will be valid. Such averaging over translates can also be used to obtain translation-invariance for the extended notions of metastates we describe next.

3 The Excitation Metastate

An important part of the proof of Theorem 1 is based on extending the notion of metastates so as to describe how a given GSP changes as the couplings in \(\mathcal{J}\) vary. Of course, if Conjecture 1 were true, then, at least for \(d = 2\), there would be, for almost every \(\mathcal{J}\), a GSP \(\alpha_\mathcal{J}\), uniquely determined as being the one on which the periodic boundary condition metastate is supported; thus one would know how \(\alpha_\mathcal{J}\) changes even when infinitely many of the couplings
in \( J \) vary. But in general, since there might be many GSP’s and perhaps even many metastates, it is not so obvious how to formulate the dependence of a given GSP in the support of a metastate even on finitely many couplings.

Neither the statement of Theorem 1 nor that of our three main propositions requires this extension of metastates, but it will be needed for the proofs of the latter two of the main propositions. This extension will be presented in detail in Section 5 of the paper, but we present a short exposition here, since it seems to be of independent interest. Roughly speaking, the extension requires that we keep track of not only the GSP itself, but also of all its excitations in which finitely many spins are forced to take specified values, modulo a global flip. We note that recent numerical studies of spin glasses have analyzed excitations induced in this way \([9]\) and in more novel ways \([10]\). There are two types of information about our excitations that one might wish to keep track of: (a) the minimum energy cost required to force the spins, and (b) the pair of spin configurations that does the minimizing — i.e., the excited state. It actually suffices to keep track only of (a), but it is perhaps conceptually simpler to keep track of (b) as well, and we will take that tack.

Suppose \( A \) is a finite subset of \( \mathbb{Z}^2 \) (in this discussion, we only take \( d = 2 \) for convenience), \( \eta \) is a spin configuration on \( A \) and \( L \) is sufficiently large so that \( A \subset S_L \). We denote by \( \alpha^A_{\eta;L} \) the pair of periodic boundary condition spin configurations on \( S_L \) with minimum energy subject to the constraint that they equal \( \pm \eta \) on \( A \). If \( A \) is empty or a singleton site, this is just the ordinary finite-volume ground state \( \alpha^{(L)}_{J} \). We also define the excitation energy \( \Delta E^{A,\eta;L}_{J} \) to be the energy of \( \alpha^{A,\eta;L}_{J} \) minus the ground state energy of \( \alpha^{(L)}_{J} \). Let \( B \) be a finite set of bonds \( b = \{x, y\} \) and let \( J^B \) denote a realization of the couplings \( J_b \) for all \( b \in B \). To see how \( \alpha^{(L)}_{J} \) and eventually \( \alpha^{A,\eta;L}_{J} \) vary with \( J^B \) when all other couplings are fixed, we begin by letting \( A = A(B) \) denote the set of sites that are endpoints of bonds in \( B \) and considering the excitation energies \( \Delta E^{A,\eta;L}_{J} \) and corresponding excited states \( \alpha^{A,\eta;L}_{J} \), for all possible spin configurations \( \eta \) on \( A \). We also define

\[
\mathcal{H}_{J^B}(\eta) = -\sum_{(x,y)\in B} J^B_{xy} \eta_x \eta_y, \quad \mathcal{H}_{J}(\eta; B) = -\sum_{(x,y)\in B} J_{xy} \eta_x \eta_y, \quad (3)
\]

and denote by \( \mathcal{J}_{[J^B]} \) the coupling configuration in which each coupling \( J_b \) of \( \mathcal{J} \) with \( b \in B \) is replaced by \( J^B_b \) and all other couplings are left unchanged. Then, for fixed \( \eta \), \( \alpha^{A,\eta;L}_{J} \) does not depend on \( J^B \) and

\[
\mathcal{H}^{(L)}_{J_{[J^B]}(\alpha^{A,\eta;L}_{J})} - \mathcal{H}_{J}^{(L)}(\alpha^{A,\eta;L}_{J}) = \mathcal{H}^{(L)}_{J_{[J^B]}(\alpha^{A,\eta;L}_{J})} - \mathcal{H}_{J}^{(L)}(\alpha^{A,\eta;L}_{J}) = \mathcal{H}_{J}^{(L)}(\alpha^{A,\eta;L}_{J}) - \mathcal{H}_{J}^{(L)}(\alpha^{A,\eta;L}_{J})(4)
\]

\[
= (\mathcal{H}_{J^B}(\eta) - \mathcal{H}_{J}(\eta; B)) - (\mathcal{H}_{J^B}(\eta) - \mathcal{H}_{J}(\eta; B)) + \Delta E^{A,\eta;L}_{J} - \Delta E^{A,\eta;L}_{J}. (4)
\]
Note that $\Delta E_{J}^{A,\eta}(L)$ depends on $J$ but not on $J^B$ while $H_{J^B}(\eta)$ depends on $J^B$ but not on $J$. Consider now the finitely many functions, as $\eta$ varies on $A$,

$$h_{\eta}^{(L)}(J^B) \equiv \Delta E_{J}^{A,\eta}(L) + H_{J^B}(\eta) - H_{J}(\eta; B).$$

These are affine functions of $J^B$, and if we define $\eta^*(L)(J^B)$ to be the $\eta$ that minimizes $h_{\eta}^{(L)}(J^B)$, it follows that

$$\alpha_{J}[J^B] = \alpha_{J}^{\eta^*(L)(J^B),(L)}.$$ (6)

When letting $L \to \infty$, we will do so for the ground state $\alpha_{J}$ and simultaneously for the excitation energies $\Delta E_{J}^{A,\eta}$ and excited states $\alpha_{J}^{\eta}$ for all choices of finite $A$ and spin configurations $\eta$ on $A$; a superscript $\sharp$ will denote that collection of choices. Of course, this needs to be done via a metastate construction that extends the “ground metastate” $\kappa_{J}$ described earlier, to what we will call the excitation metastate $\kappa_{J}^{\sharp}$. The excitation metastate is a probability measure on infinite-volume excitation energies and states for the given $J$, $(\Delta E^\sharp, \alpha^\sharp)$, which includes the ground metastate since the ground state $\alpha$ can be obtained by restricting $\alpha^\sharp$ to $A$ being the empty set (or a singleton, since we are dealing with periodic boundary conditions that do not break spin-flip symmetry). To see how the ground state $\alpha$ changes to $\alpha_{J}[J^B]$ when the couplings in a fixed finite $B$ vary, we can then use the infinite-volume extensions of our last two displayed equations (where $H_{J^B}(\eta)$ and $H_{J}(\eta; B)$ are as before):

$$h_{\eta}(J^B) \equiv \Delta E^{A(B),\eta} + H_{J^B}(\eta) - H_{J}(\eta; B),$$

and

$$\alpha_{J}[J^B] = \alpha^{A(B),\eta^*(J^B)},$$

where $\eta^*(J^B)$ is the $\eta$ on $A(B)$ that minimizes $h_{\eta}(J^B)$.

4 The Main Propositions

In this section, we present the three central propositions leading immediately to Theorem 1. The proof of the first of these, a direct application to spin glasses of general 2D percolation results of Burton and Keane [11], will be given in this section. The proof of the second and third propositions will be given in Sect. 6. We begin with a somewhat more detailed discussion of ground metastates than given in the last section. For simplicity, we continue to restrict the discussion to periodic boundary condition metastates, as in Sect. 2.
An (infinite-volume) ground state pair or GSP for a given coupling realization $\mathcal{J}$ is a pair of spin configurations $\pm \sigma$ on $\mathbb{Z}^d$, whose energy, governed by Eq. (1), cannot be lowered by flipping any finite subset of spins. That is, it must satisfy the constraint
\[
\sum_{(x,y) \in \mathcal{C}} J_{xy} \sigma_x \sigma_y \geq 0 \quad (9)
\]
along any closed loop $\mathcal{C}$ in the dual lattice. Infinite-volume ground states are always the limits of finite volume ground states, but, in general, the finite-volume boundary conditions may need to be carefully chosen, depending on $\mathcal{J}$ and/or the limiting ground state. In a disordered system, if there are many distinct GSP’s for typical fixed $\mathcal{J}$, then in general, as noted in [12], the limit $\lim_{L \to \infty} \alpha^{(L)}_{\mathcal{J}}$ doesn’t exist, if the $L$’s are chosen in a coupling-independent way. This phenomenon was called chaotic size dependence [12]. The ground metastate, a probability measure $\kappa_{\mathcal{J}}$ on the infinite-volume ground states $\alpha_{\mathcal{J}}$, was proposed in [5] as a means of analyzing the way in which $\alpha_{\mathcal{J}}^{(L)}$ samples from its various possible limits as $L \to \infty$. (The metastate was introduced and defined for both zero and positive temperatures, but we confine the discussion here to zero temperature.) The same metastate can be constructed by at least two distinct approaches. The first, introduced earlier by Aizenman and Wehr (AW) [13], directly employs the randomness of the $\mathcal{J}$’s, while the “empirical distribution” approach of [5] and subsequent papers was motivated by, but doesn’t require, the potential presence of chaotic size dependence for fixed $\mathcal{J}$.

The empirical distribution point of view (and its natural extension to excitation metastates) will be the primary one used throughout this paper. However, we briefly describe the AW construction, since it is the one that directly gives, for, e.g., periodic boundary conditions, the translation invariance that will be crucial in our first proposition; for more details see [13]. Here one considers, for each $L$, the random pair $(\mathcal{J}, \alpha_{\mathcal{J}}^{(L)})$ (where $\alpha_{\mathcal{J}}^{(L)}$ is the finite-volume periodic boundary condition GSP obtained using the restriction $\mathcal{J}^{(L)}$ of $\mathcal{J}$ to $S_L$), and takes the limit of the finite-dimensional distributions along a $\mathcal{J}$-independent subsequence of $L$’s, using compactness. This yields a probability distribution $\mathcal{K}$ on infinite-volume $(\mathcal{J}, \alpha)$’s which is translation invariant, under simultaneous lattice translations of $\mathcal{J}$ and $\alpha$, because of the periodic boundary conditions, and is such that the conditional distribution $\tilde{\kappa}_{\mathcal{J}}$ of $\alpha$ given $\mathcal{J}$ is supported entirely on GSP’s for that $\mathcal{J}$. The conditional distribution $\tilde{\kappa}_{\mathcal{J}}$ is the AW ground metastate.

It is easy to show that there is sequential compactness leading to convergence for $\mathcal{J}$-independent subsequences of $L$’s, as described above. We have conjectured [8] that all...
subsequence limits are the same; i.e., that existence of a limit does not require taking a
subsequence. Proving this conjecture remains an open problem.

The empirical distribution approach of [3, 5, 6], as described in Sect. 2, takes a fixed
and, roughly speaking, replaces the “$J$-randomness” used in the AW construction of $\tilde{\kappa}$ with
“$L$-randomness” — i.e., with chaotic size dependence. The empirical distributions along a
subsequence $(L_1, L_2, \ldots)$ are the measures

$$\kappa^M_J = (1/M) \sum_{k=1}^M \delta_{\alpha^{(L_k)}}$$

(10)

where $\delta_{\alpha}$ denotes the Dirac delta measure at the state $\alpha$ and where for convenience we regard
the finite-volume GSP $\alpha^{(L)}_J$ as defined in infinite volume by, e.g., taking all bonds outside
$S_L$ as satisfied. We say that $\kappa^M_J$ has a limit $\kappa_J$ if the probability of any event $A(D_1, D_2)$
(that every edge in $D_1$ is unsatisfied and every edge in $D_2$ is satisfied, where $D_1$ and $D_2$ are
disjoint finite sets of dual edges) converges to the $\kappa_J$-probability of that event.

It was shown in [6] that there exists a $J$-independent subsubsequence where the limits
$\tilde{\kappa}_J$ and $\kappa_J$ are the same. For more details and proofs, see [3, 5, 6]. Also see [4] for additional
properties of the metastate, particularly invariance with respect to gauge-related boundary
conditions.

Before we state Proposition 1, some additional definitions are needed. Consider a periodic
boundary condition metastate $\kappa_J$ (in some fixed dimension, not necessarily two) and two
GSP’s $\alpha$ and $\beta$ chosen from $\kappa_J$. Then their symmetric difference $\alpha \Delta \beta$, as introduced in
Sect. 2, is the set of edges in the dual lattice $\mathbb{Z}^d$ that are satisfied in $\alpha$ and not $\beta$ or
vice-versa. If $\mathcal{B}$ is the graph whose edge set is $\alpha \Delta \beta$ and whose vertices are all sites in $\mathbb{Z}^d$
touching $\alpha \Delta \beta$, then a domain wall, defined relative to the two GSP’s, is a cluster (i.e.,
a maximal connected component) of $\mathcal{B}$. (In two dimensions, according to Proposition 1,
domain walls are generically doubly-infinite self-avoiding paths in the dual lattice.) The
symmetric difference $\alpha \Delta \beta$ is the union of all $\alpha \beta$ domain walls and may consist of a single
domain wall or of multiple domain walls that are site-disjoint and hence also edge-disjoint.

Two distinct GSP’s $\alpha$ and $\beta$ are said to be incongruent if $\alpha \Delta \beta$ has a well-defined nonvanishing density within the set of all edges in $\mathbb{Z}^d$; if the density is zero, they are regionally
congruent. We do not consider here the case where the density is not well-defined; we will
see from Proposition 1 that in fact this cannot happen in two dimensions. In Proposition 1,
we will also see that, if there are multiple GSP’s, the “observable” ones are incongruent.
Our primary interest is therefore in the question of existence of these “physical” incongruent
states, which should be observable by using coupling-independent boundary conditions. As mentioned in Sect. 2, incongruent states may consist of a single positive-density wall, or else of multiple domain walls, which individually may or may not have positive density, but collectively have strictly positive density.

In all our propositions, $J$ is chosen from the disorder distribution $\nu$ and then $\alpha$ and $\beta$ are GSP’s chosen independently from periodic boundary condition metastates $\kappa_J$ and $\kappa'_J$ (which may be the same), as described above.

**Proposition 1.** Distinct $\alpha$ and $\beta$ in any dimension must, with probability one, be incongruent. In two dimensions, all domain walls comprising $\alpha \Delta \beta$ have the following properties with probability one: (i) they are infinite and contain no loops or dangling ends; (ii) they cannot branch and thus are doubly-infinite self-avoiding paths; (iii) they together partition $\mathbb{Z}^2$ into at most two topological half-spaces and/or a finite or infinite number of doubly-infinite topological strips (that also cannot branch — i.e., each strip has two boundary domain walls and exactly one neighboring strip or half-space on each side). (iv) Moreover, each domain wall has a well-defined density and there cannot simultaneously be positive-density and zero-density walls.

**Proof of Proposition 1.** Let us denote by $D_J$ the probability measure on configurations of $\alpha \Delta \beta$ corresponding to choosing $\alpha$ and $\beta$ independently from $\kappa_J$ and $\kappa'_J$, and denote by $D$ the measure then obtained by integrating out the couplings $J$ with respect to the disorder distribution $\nu$. We claim that $D$ is translation-invariant. To see this, begin with the translation-invariant measures on joint configurations of couplings and GSP’s $\mathcal{K} (= \nu \kappa_J)$ and $\mathcal{K}' (= \nu \kappa'_J)$ and note that the natural coupling $\nu \kappa_J \kappa'_J$, a measure on $(J, \alpha, \beta)$ configurations, retains translation-invariance. $D$ is then translation-invariant since it is just the distribution of $\alpha \Delta \beta$ with $(\alpha, \beta)$ distributed as the marginal of this coupled measure. The translation-invariance of $D$ in turn implies by the ergodic theorem with respect to $\mathbb{Z}^2$-translations that any “geometrically defined event”, such as a bond belonging to a domain wall, occurs either nowhere or else with strictly positive density. This proves the first claim.

To prove property (i), we note that a domain wall taken from $\alpha \Delta \beta$ separates regions in which the spins of $\alpha$ and $\beta$ agree from regions where they disagree. A domain wall therefore cannot end at a point in any finite region. To rule out loops, note that the sum $\sum_{(x,y)} J_{xy} \sigma_x \sigma_y$ along any such loop must have opposite signs in the two GSP’s, violating Eq. (9), unless the sum vanishes. But this occurs with zero probability because the couplings are chosen independently from a continuous distribution.
Claims (ii), (iii), and (iv) are proven in [11], using percolation-theoretic arguments first presented in [14]; we sketch the arguments. To prove (ii), suppose that a domain wall branches at some site \( z \) in the dual lattice. (We note, although it’s not needed for the proof, that the number of branches emanating from \( z \) must be even, again because domain walls separate regions of spin configuration agreement from regions of disagreement. Hence the minimal branching at \( z \) is four.) None of these branches may intersect somewhere else, by property (i). By the translation-invariance of \( D \), there must then be a positive density of branch points, so that the domain wall would have a treelike structure. That implies the existence of an \( \epsilon > 0 \) such that the boundary of \( S_L \) is intersected by a number of distinct branches that grows as \( \epsilon L^2 \) as \( L \to \infty \), which is impossible.

The proof of (iii) uses a similar argument to rule out branching of the strips — see Theorem 2 of [11] for details. Property (iv) is not needed for subsequent arguments, but is included for completeness; it is proven in Theorem 4 of [11] and follows readily from the properties just proven. If zero-density and positive-density clusters coexist, then for some \( p > 0 \), there is positive \( D \)-probability that the origin of the dual lattice is contained in a zero-density domain wall with an adjacent wall of density at least \( p \). Let \( S_p \) be the set of all walls with density greater than or equal to \( p \). Then there can be no more than \((1/p)\) walls in \( S_p \). The maximum number of walls of density zero that are adjacent to walls belonging to \( S_p \) (i.e., if every \( S_p \)-wall is surrounded by two zero-density walls whose other adjacent wall does not belong to \( S_p \)) is therefore \( 2/p \). But then the union of such zero-density walls has density zero and so the probability of the event that the origin is contained in a zero-density wall adjacent to a wall in \( S_p \) is zero, leading to a contradiction. This completes the proof of the proposition.

So the picture we now have of the symmetric difference \( \alpha \Delta \beta \) is a union of one or more doubly infinite domain walls. These domain walls do not branch or have any internal loops, and they divide the plane into strips or (if there are positive-density domain walls) half-planes. In all cases where there is more than a single domain wall, translation-invariance of \( D \) implies that distinct domain walls mostly remain within an \( O(1) \) distance of one another. E.g., there can be no “hourglass”, “martini glass”, etc., domain wall configurations; these can be ruled out by arguments similar to those used in the proof of part (ii) of Proposition 1.

The essential idea behind the proof of Theorem 1 is contained in the next two propositions. Before we state these propositions, we need to introduce the notion of a “rung” between adjacent domain walls. A rung \( R \), defined with respect to \( \alpha \Delta \beta \), is a path of edges in \( \mathbb{Z}^2 \) connecting two distinct domain walls, with only the first and last sites in \( R \) on any domain.
wall. So $\mathcal{R}$ can contain only edges that are not in $\alpha \Delta \beta$, and the corresponding couplings are therefore either both satisfied or both unsatisfied in $\alpha$ and $\beta$. The energy $E_\mathcal{R}$ of $\mathcal{R}$ is defined to be

$$E_\mathcal{R} = \sum_{(xy) \in \mathcal{R}} J_{xy} \sigma_x \sigma_y,$$

(11)

with $\sigma_x \sigma_y$ taken from $\alpha$ or equivalently $\beta$. It must be that $E_\mathcal{R} > 0$ with probability one for the following reasons, which we sketch here and make precise later in the proof of Proposition 2. Suppose that a rung could be found with negative energy (there is zero probability of a zero-energy rung); by translation-invariance there would need to be many such rungs between some fixed pair of adjacent domain walls. Consider the “rectangle” formed by two such negative-energy rungs and the connecting segments of the two adjacent domain walls. The sum of $J_{xy} \sigma_x \sigma_y$ along the couplings in the domain wall segments would be positive in one GSP (say, $\alpha$), and would therefore be negative in the other (say, $\beta$). Therefore, the loop formed by the boundary of this rectangle would violate Eq. (9) in GSP $\beta$.

It is then natural to ask the deeper question of whether rung energies along any strip are strictly bounded away from zero, or whether their infimum is exactly zero. Propositions 2 and 3 address this question.

**Proposition 2.** The rung energies $E_{\mathcal{R}'}$ between two fixed (adjacent) domain walls cannot be arbitrarily small; i.e., there is zero probability that $E' = \inf_{\mathcal{R}'} E_{\mathcal{R}'} = 0$.

**Proposition 3.** There is zero probability that $E' > 0$.

The contradiction between Propositions 2 and 3 leads directly to Theorem 1. These propositions will be proved in Section 6.

## 5 Transition Values and Flexibilities

In this section, we present two auxiliary propositions. They will be used in the next section to prove Propositions 2 and 3. These auxiliary propositions involve two notions, transition value and flexibility, that arise in the analysis of how a GSP changes when a single coupling, $J_b$, varies. Since this is a restricted case of the dependence of $\alpha_{\mathcal{J}^B}$ on a finite collection $\mathcal{J}^B$ of couplings, we begin the section by providing a more detailed exposition of the excitation metastate than that given in Sect. 3 above.

Along with an empirical distribution construction of the excitation metastate $\kappa_{\mathcal{J}}^\sharp$ as a probability measure, defined for $\nu$-almost every $\mathcal{J}$, on configurations $(\Delta E^\sharp, \alpha^\sharp)$ of excitation
energies and states for the given $J$, there is an alternative AW-type construction, as follows. For each $L$, consider $(J, \Delta E_J^{L,\eta}, \alpha_J^{L,\eta})$, where $\Delta E_J^{L,\eta}$ and $\alpha_J^{L,\eta}$ denote the excitation energies and states in $S_L$, with periodic boundary conditions, when the spin configuration on $A \subset S_L$ is constrained to be $\pm \eta$ (for all allowed $A$’s and $\eta$’s). As in the AW ground metastate construction, one has sequential compactness of the corresponding probability measures, $K_J^{\eta,L}$, leading to convergence of the finite dimensional distributions (involving finitely many couplings, finitely many finite $A$’s and finitely many $\eta$’s) to those of a limiting translation-invariant measure $K_J^\eta$ on infinite-volume configurations $(J, \Delta E_J^\eta, \alpha_J^\eta)$ along deterministic subsequences of $L$’s.

The marginal distribution of $J$ from this $K_J^\eta$ is of course just $\nu$ and the conditional distribution of $(\Delta E_J^\eta, \alpha_J^\eta)$ given $J$ is then an excitation metastate $\tilde{\kappa}_J^\eta$, which, like in the ground metastate case, can be shown for $\nu$-almost every $J$ to equal the $\kappa_J^\eta$ constructed via empirical distributions, as the limit along a subsubsequence of

$$(1/M) \sum_{k=1}^{M} \delta_{(\Delta E_J^{L_k,\eta_k}, \alpha_J^{L_k,\eta_k})},$$

where $\Sigma_A$ denotes the sum over bonds $\langle x, y \rangle$ with either $x$ or $y$ or both in $A$, together with the fact that the distribution of the $J_{xy}$’s does not change with $L$.

As explained in Sect. 3, for a given $J$, we can extract from $(\Delta E_J^\eta, \alpha_J^\eta)$ not only the GSP $\alpha$, but also $\alpha_{[J,b]}$, which describes how the GSP changes when the couplings in a fixed finite set $B$ of bonds vary. When $B$ consists of a single bond $b = \langle x, y \rangle$, we write $\alpha(K'_b; b)$ for the ground state that results when $J_b$ is replaced by $K'_b$ with all other couplings of $J$ left unchanged. It should be clear from Equations (4) and (8) that as $K'$ varies in $(-\infty, +\infty)$, the GSP $\alpha(K'_b; b)$ changes exactly once (this is particularly easy to see in finite volume and the property is preserved in the excitation metastate), from its original configuration $\alpha$ when $K' = J_b$ to a new configuration

$$\alpha^b = \alpha^{(x,y),\tilde{\eta}},$$

where $\tilde{\eta}$ is one of the two spin configurations on $\{x, y\}$ of opposite parity to the original GSP $\alpha$ (so that $\sigma_x \sigma_y$ is $+1$ in one of $\alpha$ and $\alpha^b$ and $-1$ in the other, or equivalently $J_b$ is satisfied
in one and unsatisfied in the other). We call the value of $K'$ where this change happens the transition value and denote it by $K_b$.

For a given $b$, the transition value $K_b$ and the unordered set of two GSP’s $\{\alpha, \alpha^b\}$ do not depend on the value of $J_b$, with all other couplings held fixed (again, this is clear for finite volume, and is preserved in the limit). This means that with respect to the probability measure $\mathcal{K}^b$ on infinite-volume configurations $(\mathcal{J}, \Delta E^b, \alpha^b)$, the random variables $K_b$ and $J_b$ are independent. The next proposition is an immediate consequence of this independence.

**Proposition 4.** With probability one, no coupling $J_b$ is exactly at its transition value $K_b$.

**Proof of Proposition 4.** From the independence of $J_b$ and $K_b$, and the continuity of the distribution of $J_b$, it follows that there is probability zero that $J_b - K_b = 0$.

As in the proof of the last proposition, we continue to work on the probability space of $(\mathcal{J}, \Delta E^b, \alpha^b)$ configurations with probability measure $\mathcal{K}^b$. When the value of $J_b$ is moved from its original value past the transition value $K_b$, the change from the original ground state of $\alpha$ to the new ground state, and originally excited state, of $\alpha^b$ may involve the flipping of a finite droplet (region of $\mathbb{Z}^2$) or one or more infinite droplets. Thus the symmetric difference $\alpha \Delta \alpha^b$, representing the dual bonds which change from satisfied to unsatisfied or vice-versa, may consist of a single finite loop or else of one or more infinite disconnected paths, but in all cases some part must pass through $b$ since its satisfaction status clearly changes. To help analyze what other bonds $\alpha \Delta \alpha^a$ may or may not pass through, we introduce the notion of flexibility. The flexibility of a bond $b = \langle x, y \rangle$ is defined as

$$F_b \equiv |K_b - J_b| = (1/2)|\Delta E^{(x,y)},\hat{\eta}|$$

and thus is proportional to the excitation energy needed to flip the relative sign of the spins at $x$ and $y$; it is a measure of the stability of the ground state $\alpha$ with respect to fluctuations of the single coupling $J_b$.

**Proposition 5.** For two bonds $a$ and $b$, there is zero probability that $F_b > F_a$ and simultaneously $\alpha \Delta \alpha^a$ passes through $b$.

**Proof of Proposition 5.** For finite $L$, and a bond $e$ in $S_L$, let us denote by $F_e^{(L)} \equiv |J_e - K_e^{(L)}|$ the finite-volume flexibility. Now $F_e^{(L)}$ is clearly the minimum, over all droplets in $S_L$, with periodic boundary conditions, whose boundary passes through $e$, of (half the) droplet flip energy cost in the GSP $\alpha^{(L)}$. Since this is the case for both $e = a$ and $e = b$, 14
it is an immediate consequence that the finite-volume droplet boundary \( \alpha^{(L)} \Delta \alpha^{a,(L)} \) cannot pass through \( b \) if \( F_b^{(L)} > F_a^{(L)} \). After \( L \to \infty \), the characterization of \( F_{\epsilon} \) as a minimum over finite droplets may be lost, but we claim that the conclusion of the proposition still holds. This is because, although the convergence of \( K^{\Delta,(L)} \) along a subsequence to \( K^{\Delta} \) is not sufficient to imply, e.g., that the probability of \( F_b^{(L)} > F_a^{(L)} \) converges along the subsequence to the limiting probability of \( F_b > F_a \), it is sufficient to imply that the probability of the event in the proposition is less than or equal to the the \( \lim \inf \) of the (zero) probability of the corresponding finite-volume events. This completes the proof of the proposition.

6 Proof of Propositions 2 and 3

Proof of Proposition 2. Suppose that there are two adjacent domain walls from the GSP’s \( \alpha \) and \( \beta \), \( W_1 \) and \( W_2 \), with \( W_1 \) passing through the origin of the dual lattice, and suppose further that the infimum \( E' \) of rung energies \( E_{R'} \) for rungs \( R' \) between \( W_1 \) and \( W_2 \) is zero. Our object is to prove that this event has zero probability. If the probability is nonzero, then for every \( \epsilon > 0 \) there is some \( \ell(\epsilon) < \infty \) so that, with nonzero probability, there is a rung \( R' \) between \( W_1 \) and \( W_2 \), with the property \( \mathcal{P}(\epsilon) \), that its length, defined as the number of bonds, is below \( \ell(\epsilon) \) and its energy \( E_{R'} \) is below \( \epsilon \). But then, by translation-invariance and the lemma given right after this proof, there must, with nonzero probability, be infinitely many such rungs with property \( \mathcal{P}(\epsilon) \) with starting points on \( W_1 \) in both directions from the origin along \( W_1 \). Thus we can find two such rungs \( R \) and \( R' \), one in each direction, and sufficiently far apart that they do not touch each other.

Consider the “rectangular” region of \( \mathbb{Z}^2 \) whose boundary is the union of these two rungs and the connecting segments, \( C_1 \) and \( C_2 \) of \( W_1 \) and \( W_2 \). The energy cost of flipping the spins in this region in \( \alpha \) (respectively, in \( \beta \)) is \( +E(C_1, C_2) + E_{R} + E_{R'} \) (respectively, \( -E(C_1, C_2) + E_{R} + E_{R'} \)). Both these quantities must be positive since both \( \alpha \) and \( \beta \) are GSP’s; hence \(|E(C_1, C_2)|\) is bounded by \( E_{R} + E_{R'} < 2\epsilon \) and the energy costs in both ground states are bounded by \( 4\epsilon \). This implies that every bond \( b \) that \( W_1 \) (or \( W_2 \)) passes through has flexibility less than \( 2\epsilon \). Since \( \epsilon \) is arbitrary, the flexibilities must be zero, but that would contradict Proposition 4. This, together with the following lemma, completes the proof.

Lemma 1. Suppose \( \mathcal{P} \) is a translation-invariant property of rungs, e.g., the property that the rung energy is below a certain value and/or the rung length is below a certain value. There is zero probability that there exist two adjacent domain walls, \( W_1 \) and \( W_2 \), such that
the set of starting points on $W_1$ of rungs between $W_1$ and $W_2$ that satisfy $\mathcal{P}$ is nonempty without being doubly infinite, i.e., along both directions of $W_1$.

**Proof of Lemma 1.** The proof is based entirely on the translation invariance of the measure $K^\#$. Suppose the claim of the lemma is false. Then for each site $x$ in the dual lattice, there is nonzero probability for the event $A_x$ that there is a domain wall $W$ passing through $x$ and an adjacent wall $W'$ such that $x$ is the last site in one of the two directions along $W$ such that there is a rung from that site to $W'$ satisfying $\mathcal{P}$. Since every domain wall has two directions and at most two adjacent domain walls, there can be at most four sites on any domain wall for which this event occurs. Every domain wall that intersects the square $S_L$, sitting inside the infinite lattice, must touch the boundary of the square and thus there are at most $cL$ such domain walls for some constant $c < \infty$, and consequently at most $4cL$ sites $x$ in $S_L$ for which $A_x$ occurs. But by the ergodic theorem for spatial translations, there is nonzero probability that the number of such sites exceeds $c'L^2$ for some constant $c' > 0$. This contradiction completes the proof.

**Proof of Proposition 3.**

For the proof, we need the notion of a “super-satisfied” bond $b = \langle x, y \rangle$. It is easy to see, for a given $\mathcal{J}$, that $b$ is satisfied in every ground state if $|J_{xy}| > \min\{M_x, M_y\}$, where $M_x$ is the sum of the three other coupling magnitudes $|J_{xz}|$ touching $x$, and $M_y$ is defined similarly. Such a bond or its dual, called super-satisfied, clearly cannot be part of a domain wall between any two GSP’s.

As in the proof of Proposition 1, but using the excitation metastates $\kappa^\#_\mathcal{J}$ and $\kappa'^\#_\mathcal{J}$ that extend the ground metastates from which $\alpha$ and $\beta$ are chosen, we work in the probability space with the coupled measure $\nu\kappa^\#_\mathcal{J}\kappa'^\#_\mathcal{J}$. On this space, we can consider the modified ground states $\alpha_{[J\mathcal{V}]}$ and $\beta_{[J\mathcal{V}]}$ as any finitely many couplings are varied as well as the transition values and flexibilities for both $\alpha$ and $\beta$ for all bonds $b$.

Now suppose that the rung energy infimum $E'$ between some pair $W_1, W_2$ of domain walls satisfies $E' > 0$ with positive probability; we show this leads to a contradiction. First we find, as in Fig. 1, a rung $\mathcal{R}$ and two dual bonds $b_1, b_2$ whose locations on $W_1$ are respectively in opposite directions from the starting site of $\mathcal{R}$, and such that $E_\mathcal{R} - E'$, which we denote by $\delta$, is strictly less than the flexibility values for both $\alpha$ and $\beta$ of both $b_1, b_2$. The existence with positive probability of such an $\mathcal{R}$, $b_1$ and $b_2$ follows from the non-vanishing of flexibilities given by Proposition 4 and translation-invariance (e.g., Lemma 1).

But we also want a situation, as in Fig. 1, where all the dual lattice non-domain-wall bonds that touch $W_1$ between $b_1$ and $b_2$, other than the first bond $a$ in $\mathcal{R}$, are super-satisfied,
Figure 1: A rung $\mathcal{R}$ with $E_{\mathcal{R}} = E' + \delta$. The dots are sites in $\mathbb{Z}^2$, and bonds are drawn in the dual lattice. Two domain walls are solid lines and $\mathcal{R}$ is the dashed line. The bonds $b_1$ and $b_2$ have flexibility $> \delta$. The ten dotted line bonds are super-satisfied.

and remain so regardless of changes of $J_a$ (by a bounded amount). We will call these bonds, numbering ten in Fig. 1, the “special” bonds. How do we know that such a situation will occur with nonzero probability? If necessary, we can first adjust the signs and then increase the magnitudes (in an appropriate order) of the couplings of the special bonds, so that they first become satisfied and then super-satisfied. This can be done in an “allowed” way because of our assumption that the distribution of individual couplings has unbounded support. Also, this can be done so that $\alpha_{[J_B]}$ and $\beta_{[J_B]}$ remain unchanged from $\alpha$ or $\beta$, and without changing $E_{\mathcal{R}}$, without decreasing any other $E_{\mathcal{R}'}$ (and thus without changing $E'$ or $E_{\mathcal{R}} - E' = \delta$) and without decreasing the flexibilities of $b_1$ or $b_2$. Starting from a nonzero probability event, such an allowed change of finitely many couplings in $J$ yields an event which still has nonzero probability.

Next, suppose we move $J_a$ toward its transition value $K_a$ by an amount slightly greater than $\delta$. The geometry — see, e.g., Fig. 1 — and Proposition 5 forbid the replacement of either $\alpha$ or $\beta$ by $\alpha_a$ or $\beta_a$, because it is impossible, under the conditions given, for $\alpha \Delta \alpha_a$ or $\beta \Delta \beta_a$ to connect to the end of bond $a$ touching $W_1$. But this change of $J_a$ reduces $E_{\mathcal{R}}$ below $E_{\mathcal{R}'}$ for any $\mathcal{R}'$ not containing $a$, yielding a nonzero probability event that contradicts translation-invariance (i.e., Lemma 1). This completes the proof.
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