Risk-sensitive Markov Control Processes with
Strictly Convex Risk Maps*

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Abstract

We fully develop the Lyapunov approach to optimal control problems of Markov control processes on general Borel spaces equipped with risk maps, especially, with strictly convex risk maps including the entropic map. To ensure the existence and uniqueness of a solution to the associated nonlinear Poisson equation with possibly unbounded costs, we propose a new set of conditions: 1) Lyapunov-type conditions on both risk maps and cost functions that control the growth speed of iterations, and 2) Doeblin’s conditions that generalize the known conditions for Markov chains. In the special case of the entropic map, we show that the above conditions can be replaced by the existence of a Lyapunov function, a local Doeblin’s condition for the underlying Markov chain, and a growth condition for cost functions.

Key words. Markov control processes, Poisson equation, risk-sensitive control, risk measures, stability of nonlinear operators, Doeblin’s condition, Lyapunov Stability

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1 Introduction

In recent years, many authors [29, 3, 33] have developed a general framework of risk-sensitive sequential decision-making problems on Borel spaces by applying coherent/convex risk measures [1, 14], which were originally employed in mathematical finance, to the classical discrete-time risk-neutral Markov control processes (MCPs, see, e.g., [20, 21], and [28] under the name Markov decision processes). Within the framework, two infinite-horizon risk-sensitive criteria, discounted total risk and average risk, are optimized under various settings. Among them, we applied in our previous work [33] weighted norm spaces to incorporate possibly unbounded costs and stated Lyapunov-type stability conditions that generalized known conditions for

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Markov chains to ensure the existence of solutions to the optimality equation for the average-risk criterion.

More specifically, we introduced the concept of upper modules [33, Subsection 3.1] for risk maps and assumed [33, Assumption 3.1] that 1) a Lyapunov function exists to the upper module, and 2) the Doeblin’s condition holds over a compact subspace. However, given a strictly convex (i.e., convex but not coherent) risk map, the upper module may be infinite for unbounded cost functions. Hence, a (real-valued) Lyapunov function need not exist and the theory developed in our previous work remains valid only if 1) costs are bounded and 2) the Doeblin’s condition holds over the whole space. This excludes the applicability of our framework to Markov models with non-compact state spaces, where Doeblin’s condition does not hold over the whole space (see, e.g., [26, Chapter 5]).

As an example, consider the entropic map

\[
R_x(c) := \frac{1}{\lambda} \log \left( \int_X e^{\lambda c(y)} P_x(dy) \right), \quad x \in X, \lambda > 0,
\]

which is a widely applied risk map in the literature of risk sensitive MCPs (see, e.g., [22, 7, 4, 11, 13, 18, 5]). Here, \( c : X \to \mathbb{R} \) is a cost function, \( P \) denotes a transition kernel of a time-homogeneous Markov chain on \((X, B(X))\) and the positive constant \( \lambda \) is used to control the risk sensitivity. Then, the upper module of \( R \), \( \bar{R} \), is defined as

\[
\bar{R}_x(c) := \sup_{\gamma \in \mathbb{R}, \gamma \neq 0} \frac{1}{\gamma} R_x(\gamma c) = P_x^{\text{esssup}} c.
\]

Its value becomes infinite if \( c \) is unbounded from above and the probability measure \( P_x(\cdot) \) has full support on \( X \) (e.g., the autoregressive model of order one with Gaussian noise [26, Section 2.1.1]). Given a cost function \( c \) and a risk map \( R \), the nonlinear Poisson equation

\[
T(h) := c + R(h) = \rho + h,
\]

where \( \rho \in \mathbb{R} \) and \( h : X \to \mathbb{R} \) are unknown, plays a central role in solving the optimal control problem associated to risk-sensitive MCPs with the average criterion. In our previous work [33], the Poisson equation is solved by an iterative technique, and the sequence of iterations \( \{T^n, n = 1, 2, \ldots\} \) is required to be uniformly bounded under a weighted norm determined by a Lyapunov function \( w \) satisfying \( \bar{R}_x(w) \leq \gamma w(x) + K, \forall x \in X \), with some constants \( \gamma \in (0, 1) \) and \( K > 0 \). However, as discussed above, a real-valued \( w \) need not exist for the entropic map.

Hence, the theory developed in our previous work can only allow for bounded costs with compact state spaces, when applying strictly convex risk maps like the entropic map. However, in the last four decades, the entropic map is the most widely applied risk map for risk-sensitive control in the framework of MCPs (see, e.g., [22, 7, 4, 11, 13, 18, 5]). It is, therefore, of great importance to cover this special type of risk maps in our framework of risk-sensitive MCPs on general Borel spaces with possibly unbounded costs.

The purpose of this paper is to solve the above mentioned problems within the same framework by introducing 1) constraints on the cost functions to control the
growth of iterations (Assumption 3.1(i)), and 2) additional minorization properties on small sets (Assumption 3.1(ii)) that generalize the original Doeblin’s condition. Under these conditions, we show the existence of a bounded forward invariant subset that covers the whole iterations. Restricted to the bounded subset, we assume the existence of the Lyapunov function (Assumption 3.7) for the weaker type of upper module, which is called upper envelope in this paper, to ensure the existence of a unique solution to the optimality equation for the average-risk criterion. As a special case, we show in Section 4 that, when applying the entropic map, the above conditions are satisfied, if 1) a Lyapunov function exists for the entropic map, 2) the local Doeblin’s condition holds for the underlying Markov chain, and 3) a growth condition for cost functions.

Most of the existing literature on risk-sensitive MCPs, especially that applies the entropic map, considers finite or countable state spaces (see, e.g., [2, 4, 8, 13, 18, 5]), or bounded cost functions (see, e.g., [3]). Comparing with the few literature [10, 11] (for detailed comparisons, see Remark 4.16) of the same general settings, i.e., Borel spaces and unbounded cost functions, we provide in this paper a more general framework which can be applied to all types of risk maps, and more importantly, with a conceptually simpler proof, whereas the methods developed in [10, 11] can be only applied to the entropic map. Moreover, the conditions we stated in Section 4 for the entropic map are easier to verify than the conditions stated in [10, 11].

The paper is organized as follows. In Section 2, we briefly review the definitions and basic properties of the weighted norm space, risk measures and risk maps. In Section 3, we develop a general theory of nonlinear Poisson equation for risk maps. As a special case, we show in Section 4 that under proper assumptions, the entropic map fits the theoretical framework developed in the previous section. In Section 5, the theory is applied to solve the optimal control problems within the framework of risk-sensitive MCPs, and an example with the entropic map applied to discretized ergodic diffusions is presented in Subsection 5.3.

2 Preliminaries

Let $X$ be a Borel space, which is a Borel subset of a complete separable metric space, and its Borel $\sigma$-algebra is denoted by $\mathcal{B}(X)$.

2.1 Weighted norm

Let $w: X \to [1, \infty)$ be a given real-valued $\mathcal{B}(X)$-measurable function. Consider the $w$-norm $\|u\|_w := \sup_{x \in X} \frac{|u(x)|}{w(x)}$. Let $\mathcal{B}_w$ be the space of real-valued $\mathcal{B}(X)$ measurable functions with bounded $w$-norm. It is obvious that $\mathcal{B} \subset \mathcal{B}_w$, where $\mathcal{B}$ denotes the space of bounded $\mathcal{B}(X)$-measurable functions. Let $\mu$ be a signed measure on $\mathcal{B}(X)$. Define $\|\mu\|_w := \sup_{\|u\|_w \leq 1} \int_X ud\mu = \int_X \mu \leq \|\mu\|_{TV}$, where $\|\cdot\|_{TV}$ denotes the total variation norm of probability measures.
The following \( w \)-**seminorm** is used throughout this paper:

\[
\|v\|_{s,w} := \sup_{x \neq y} \frac{|v(x) - v(y)|}{d_w(x, y)}, \text{ where } d_w(x, y) := \begin{cases} 
0 & x = y \\
\|w(x) + w(y)\| & x \neq y
\end{cases}
\]

This seminorm is originally used by Hairer and Mattingly (2011) \cite{17} to study the ergodicity of Markov chains. In particular, when restricting to the space \( B \), i.e., setting \( w \equiv 1 \), the seminorm is called *span-norm* in \cite{19} or *Hilbert seminorm* in \cite{16}.

**Lemma 2.1** (see Lemma 2.1, \cite{17}). \( \|v\|_{s,w} = \min_{c \in \mathbb{R}} \|v + c\|_w, \forall v \in B_w. \)

### 2.2 Risk maps without control

The partial ordering \( \leq \) between elements in \( B_w \) is defined as \( v \leq u \) if \( v(x) \leq u(x) \forall x \in X \). A real number \( u \in \mathbb{R} \) can be viewed as a constant-valued function which belongs also to \( B_w \). We now define risk measures on \( B_w \). A mapping \( \nu : B_w \to \mathbb{R} \) is said to be a **risk measure** (cf. \cite{1} \cite{14}) if (I) (Monotonicity) \( \nu(v) \leq \nu(u) \) whenever \( v \leq u \in B_w \); (II) (Translation invariance) \( \nu(v + u) = \nu(v) + u \), for any \( u \in \mathbb{R} \); (III) (Centralization) \( \nu(0) = 0 \).

Risk measures can be categorized as follows: a risk measure \( \nu \) is said to be **convex**, if for all \( \alpha \in [0,1] \), \( v, u \in B_w \), \( \nu(\alpha v + (1 - \alpha) u) \leq \alpha \nu(v) + (1 - \alpha) \nu(u) \); **concave**, if \( \nu(\cdot) := -\nu(-\cdot) \) is a convex risk measure; **homogeneous**, if for all \( \lambda \in \mathbb{R}_+ \) and \( v \in B_w \), \( \nu(\lambda v) = \lambda \nu(v) \); **coherent**, if \( \nu \) is convex and homogeneous; and **strictly convex**, if it is convex but not homogeneous.

**Proposition 2.2.** A convex risk measure \( \nu \) satisfies \( |\nu(v)| \leq \nu(|v|), \forall v \in B_w. \)

**Proof.** By monotonicity, \( \nu(v) \leq \nu(|v|) \). Next we show \( -\nu(v) \leq \nu(-v) \leq \nu(|v|) \).

Indeed, due to the convexity, we obtain that \( \frac{1}{2}(\nu(v) + \nu(-v)) \geq \nu(\frac{1}{2}(v - v)) = 0 \).

Let \( \{X_t, t = 0, 1, 2, \ldots\} \) be a time-homogeneous Markov chain on \((X, B(X))\) with \( P \) as its transition kernel. We generalize the idea of risk measures to **risk maps** equipped with a Markov chain. A mapping \( R(x, v) : X \times B_w \to \mathbb{R} \) is said to be a **risk map** (cf. \cite{33}) on the Markov chain \( \{X_t\} \) with transition kernel \( P \), if (i) for each \( x \in X \), \( R_x(\cdot) := R(x, \cdot) \) is a risk measure; and (ii) \( R(\cdot, v) \in B_w \) for each \( v \in B_w \). Analogously, a risk map \( R \) is said to be **convex** (resp. **concave, homogeneous, coherent, strictly convex**) if \( R_x \) is convex (resp. concave, homogeneous, coherent, strictly convex), for all \( x \in X \).

**Remark 2.3.** Comparing with the standard literature \cite{1} \cite{14} \cite{39}, we define risk measures/maps on the weighted space \( B_w \), rather than the space of bounded random variables, \( L^\infty \), since the weighted space is more suitable for investigating the stability properties of the underlying Markov chain (see, e.g., \cite{26} \cite{17} \cite{33}) and is also more general than \( L^\infty \). We will specify later in Section 3 and 4 the choice of \( w \), depending on the form of risk maps and the properties of the underlying Markov chain as well.
### 2.3 Examples

The standard conditional expectation

\[ \mathcal{R}_x(v) = \mathbb{E}_x^P[v] := \int vdP_x \]

with a transition kernel \( P \) is obviously an example of a risk map. Besides it, we present two examples in this paper. For more examples of risk measures/maps applied in mathematical finance see [32, Section 2] and for those applied in other fields see [33, Subsection 4.3].

**Entropic map:**

\[ \mathcal{R}_x(v) = \frac{1}{\lambda} \log \left( \int e^{\lambda v} dP_x \right), \lambda > 0, \]

where the parameter \( \lambda \) controls risk-sensitivity. It is easy to check that \( \mathcal{R} \) is strictly convex. This risk map is intensively studied in the field of optimal control (see, e.g., [22, 7, 11, 8, 13, 18, 5]).

**Mean-semideviation trade-off** [27, 31] consider the trade-off between the one-step conditional mean and semideviation,

\[
\mathcal{R}_x(v) := \mathbb{E}_x^P[v] + \lambda \left[ \mathbb{E}_x^P(v - \mathbb{E}_x^P[v])^r \right]^{1/r}
\]

where \( r \geq 1 \) and \( \lambda \in [0,1] \) denotes the risk-sensitivity parameter. Moreover, this map is coherent (for the proof of convexity, see, e.g., [31]).

### 3 Nonlinear Poisson Equations

Let \( c \in \mathcal{B}_w \) be a fixed cost function. In this section, we shall prove under some sufficient conditions the existence of a unique solution to the following (nonlinear) Poisson equation

\[
c + \mathcal{R}(h) = \rho + h,
\]

and the existence of an invariant risk measure, \( \nu \), satisfying

\[
\nu(c + \mathcal{R}(v)) = \nu(v) + \rho, \forall v \in \mathcal{B}_w : \|v\|_{s,w} \leq C.
\]

with properly chosen weight function \( w \) and constant \( C > 0 \). As in the theory of MCPs, both Poisson equation and invariant risk measures play important roles in studying the stability properties of risk maps and the optimization of the average risk (see Section 5).

#### 3.1 Bounded forward invariant subset

Define an operator \( \mathcal{T} : \mathcal{B}_w \to \mathcal{B}_w \),

\[
\mathcal{T}(v) := c + \mathcal{R}(v).
\]
and its $n$th iteration, $T^n(v) := T \left( T^{n-1}(v) \right)$, $n = 2, 3, \ldots$. In this subsection, we state a set of sufficient conditions that guarantee the existence of a bounded (under $w$-seminorm) forward invariant subset covering the whole sequence of $\{T^n(v)\}$. More specifically, we consider subspaces of the following form

$$\mathcal{B}_w^{(C)} := \{ v \in \mathcal{B}_w \| v \|_{w, w} \leq C \}.$$ 

Here, we choose the $w$-seminorm, since we shall prove the contraction property under the $w$-seminorm as in the standard literature of Markov chain and MCPs (see, e.g., [17]).

**Assumption 3.1.** There exist a $\mathcal{B}(X)$-measurable function $w_0 : X \rightarrow [0, \infty)$ and constants $\gamma_0 \in (0, 1)$, $K_0 > 0$ and $\tilde{K}_0 > K_0$ such that (i)

$$\gamma_0 w_0(x) + K_0, \forall x \in X,$$

and (ii) for all $x, y \in \mathcal{B} := \{ x \in X | w_0(x) \leq R_0 := \frac{2K_0}{1-\gamma_0} \}$, the following inequality

$$R_x(v) - R_y(v) \leq 2(\tilde{K}_0 - K_0) + R_x(w_0) - R_y(-w_0)$$

holds for all $v \in \mathcal{B}_{1+w_0}$ satisfying $\| v \| \leq w_0 + \tilde{K}_0$.

**Remark 3.2.** (a) If $\mathcal{R}$ is a convex risk map, a sufficient condition to guarantee the assumption (i) is

$$|c| + \mathcal{R}(w_0) \leq \gamma_0 w_0 + K_0,$$

since by Proposition 2.2, $-\mathcal{R}(-w_0) \leq \mathcal{R}(w_0)$.

(b) The assumption (i) can be replaced by two conditions for the cost function $c$ and risk map $\mathcal{R}$ separately. 1) $w_0$ is a Lyapunov function satisfying $\mathcal{R}(w_0) \vee (-\mathcal{R}(-w_0)) \leq \gamma_0 w_0 + \tilde{K}_0$, and 2) $|c| \leq \gamma_0 w_0 + C_0$ with some constants $\gamma_0 \in (0, 1 - \eta_0)$ and $C_0 > 0$. Hence, comparing with our previous work [33], the assumption on the cost function $c$ is more restrictive in the present work.

(c) The assumption (ii) is more general than the Doeblin’s condition assumed in [33 Assumption 3.1(ii)]. Indeed, given a convex risk map $\mathcal{R}$, the Doeblin’s condition implies that for all $\tilde{K}_0 > 0$, there exists a constant $\alpha \in (0, 1)$ such that $\mathcal{R}_x(v) - \mathcal{R}_y(v) \leq 2(1-\alpha) \tilde{K}_0 + \mathcal{R}_x(w_0) - \mathcal{R}_y(-w_0)$, which implies (6).

(d) Applying the entropic map, we will show in Section 4 some sufficient conditions ensuring (ii) based on properties of the underlying Markov chain.

**Theorem 3.3.** Suppose Assumption 3.1 holds. Then $\| c + \mathcal{R}(v) \|_{s,1+\tilde{K}_0^{-1}w_0} \leq \tilde{K}_0$, whenever $\| v \|_{s,1+\tilde{K}_0^{-1}w_0} \leq \tilde{K}_0$.

**Proof.** Let $\beta_0 := \tilde{K}_0^{-1}$. Note that adding a constant to $v$ will not change the required inequality. Due to Lemma 2.4 we assume that $|v| \leq \beta_0^{-1} + w_0$. By the definition of $w$-seminorm, the task is to prove $|c(x) + \mathcal{R}_x(v) - c(y) - \mathcal{R}_y(v)| \leq 2\beta_0^{-1} + w_0(x) + $
$w_0(y), \forall x \neq y \in X$. Note that since switching $x$ and $y$ will not change the right-hand side of the inequality, it is sufficient to show that

$$\begin{equation}
(8) \quad c(x) + R_x(v) - c(y) - R_y(v) \leq 2\beta_0^{-1} + w_0(x) + w_0(y), \forall x \neq y \in X.
\end{equation}$$

We consider the following two cases. Case I: $w_0(x) + w_0(y) \geq R_0$. By (5), we have for all $\beta_0 > 0$,

$$\begin{align*}
&c(x) + R_x(v) \leq c(x) + \beta_0^{-1} + R_x(w_0) \leq \beta_0^{-1} + \gamma_0 w_0(x) + K_0, \\
&\quad \quad \quad \quad \quad -c(y) - R_y(v) \leq -c(y) + \beta_0^{-1} - R_y(-w_0) \leq \beta_0^{-1} + \gamma_0 w_0(y) + K_0.
\end{align*}$$

By the choice of $R_0$, $2\beta_0^{-1} + \gamma_0 (w_0(x) + w_0(y)) + 2K_0 \leq 2\beta_0^{-1} + w_0(x) + w_0(y)$ holds. Hence, (8) holds for this case.

Case II: $w_0(x) + w_0(y) \leq R_0$. Then both $x$ and $y$ are in the subset $B$. By (5),

$$\begin{align*}
&c(x) + R_x(v) - c(y) - R_y(v) \\
&\leq (c(x) - c(y)) + 2(K_0 - K_0) + R_x(w_0) - R_y(-w_0) \\
&\leq 2(K_0 - K_0) + \gamma_0 w_0(x) + \gamma_0 w_0(y) + 2K_0 \\
&\leq 2\beta_0^{-1} + w_0(x) + w_0(y).
\end{align*}$$

Combining I and II, we obtain the required inequality.

As a special case, if $R$ is a (strictly) convex risk map, by Proposition 2.2 and repeating the above proof, we immediately obtain the following corollary.

**COROLLARY 3.4.** Suppose $R$ is a convex risk map satisfying that there exist a $B(X)$-measurable function $w_0 : X \to [0, \infty)$ and constants $\gamma_0 \in (0, 1)$, $K_0 \geq 0$ and $\tilde{K}_0 > K_0$ such that (i) $|c(x)| + R(w_0) \leq \gamma_0 w_0(x) + K_0, \forall x \in X$, and (ii) for all $x, y \in B := \{x \in X : w_0(x) \leq R_0 := \frac{2K_0}{1 - \gamma_0}\}$, the following inequality

$$\begin{align*}
R_x(v) - R_y(v) &\leq 2(\tilde{K}_0 - K_0) + R_x(w_0) + R_y(w_0)
\end{align*}$$

holds for all $v \in B_{1+w_0}$ satisfying $|v| \leq w_0 + \tilde{K}_0$. Then $\|c + R(v)\|_{s,1+\tilde{K}_0^{-1}w_0} \leq \tilde{K}_0$, whenever $\|v\|_{s,1+\tilde{K}_0^{-1}w_0} \leq \tilde{K}_0$.

Theorem 3.3 shows that (together with Lemma 2.1) starting from some $v$ satisfying $|v| \leq w_0 + \tilde{K}_0$, for each $n \in \mathbb{N}$, there exists a real number $\sigma \in \mathbb{R}$ such that $|T^n(v) + A| \leq w_0 + K_0$. Hence, we may set $w = 1 + \tilde{K}_0^{-1}w_0$ and the corresponding forward invariant subset is $B_w^{(\tilde{K}_0)}$.

### 3.2 Geometric contraction

Given a risk map satisfying Assumption 3.1, we can then restrict ourselves to the invariant subset $B_w^{(C)}$ (with $C = \tilde{K}_0$) rather than the whole set $B_w$. We introduce in the following the concept of *upper envelope*, which weakens the sub- and uppermodules employed in our previous work [33].
Assumption 3.6. Apparently, if \( \nu \) (resp. \( \mathcal{R} \)) is coherent, then \( \nu \) (resp. \( \mathcal{R} \)) is an upper envelope of itself for all bounds \( C > 0 \) and \( \beta > v \).

Remark 3.6. We now prove the contraction property based on the following assumption, which is similar to Assumption 3.1 in [33].

Assumption 3.7. There exist two real-valued \( \mathcal{B}(X) \)-measurable functions, \( w_0 : X \to [0, \infty) \) and \( w : X \to [1, \infty) \) satisfying that (i) \( \mathcal{B}_{1+w_0} = \mathcal{B}_w \); (ii) there exist constants \( \gamma \in (0, 1) \), \( K > 0 \) and an upper envelope \( \mathcal{R}^{(w,C)} \) such that

\[
\mathcal{R}^{(w,C)}(w_0) \leq \gamma w_0 + K;
\]

and (iii) for all \( v \geq u \in \mathcal{B}_{1+w_0} \), there exist a constant \( \alpha \in (0, 1) \) and a probability measure \( \mu \) on \( (X, \mathcal{B}(X)) \) such that

\[
\mathcal{R}^{(w,C)}(v) - \mathcal{R}^{(w,C)}(u) \geq \alpha \int (v(x) - u(x)) \mu(dx), \forall x \in \mathcal{B},
\]

where \( \mathcal{B} := \{ x \in X | w_0(x) \leq R \} \) for some \( R > \frac{2K}{\gamma} \).

Theorem 3.8. Suppose Assumption 3.7 holds. Then there exist constants \( \tilde{\alpha} \in (0, 1) \) and \( \tilde{\beta} > 0 \) such that \( \| \mathcal{R}(v) - \mathcal{R}(u) \|_{s,1+\beta w_0} \leq \tilde{\alpha} \| v - u \|_{s,1+\beta w_0}, \forall v,u \in \mathcal{B}_w^{(C)} \).

Proof. Define \( w' := 1 + \beta w_0 \) for some \( \beta \in \mathbb{R}_+ \), whose value will be specified later. Suppose \( \| v - u \|_{s,w'} = A \in \mathbb{R}_+ \). Due to Lemma 2.1 and the fact that adding any constant to \( v \) and \( u \) will not change the values of both sides of the required inequality, we may assume that \( \| v - u \|_{w'} = A \).

By the definition of upper envelope, we then have

\[
\mathcal{R}_x(v) - \mathcal{R}_x(u) \leq \mathcal{R}^{(w,C)}(v) - \mathcal{R}^{(w,C)}(u) \leq \mathcal{R}^{(w,C)}(v - u), \forall x \in X,
\]

where the last inequality is due to Proposition 2.2. Switching \( v \) and \( u \), we obtain

\[
|\mathcal{R}_x(v) - \mathcal{R}_x(u)| \leq \mathcal{R}^{(w,C)}(v - u) \leq \| v - u \|_{w'} \mathcal{R}^{(w,C)}(w'), \forall x \in X.
\]

Case I: \( w_0(x) + w_0(y) \geq R \) and set \( \gamma_0 := \gamma + \frac{2K}{\beta} < 1 \) and \( \gamma_1 := \frac{2 + \beta \gamma_0}{2 + \beta \gamma_0} \) for some \( \beta > 0 \).

It is easy to verify that \( \gamma_1 \in (0, 1) \). Then (10) yields

\[
|\mathcal{R}_x(v) - \mathcal{R}_x(u) - \mathcal{R}_y(v) + \mathcal{R}_y(u)| \leq A(2 + \beta \mathcal{R}_x^C(w_0) + \beta \mathcal{R}_y^C(w_0)) \leq A(2 + \gamma \gamma_0 w_0(x) + \gamma \gamma_0 w_0(y) + 2\beta K)
\]

\[
|\mathcal{R}_x(v) - \mathcal{R}_x(u)| \leq A\gamma_1(w'(x) + w'(y)).
\]
We select \(\beta\) (12). Note that since (1.Hence, \(\tilde{B}\) is a valid coherent risk measure on \(\mathcal{B}_{1+\beta w_0} = \mathcal{B}_1 = \mathcal{B}_w\) for all \(x \in \mathcal{B}\) and \(R\). Indeed, the monotonicity is satisfied due to Assumption 3.7(iii). Hence, \(\tilde{R}_x(v) - \tilde{R}_x(u) \leq \tilde{R}_x^{(w,C)}(v - u)\), which indicates that \(\tilde{R}_x^{(w,C)}\) is an upper envelope of \(\tilde{R}_x\) for all \(x \in \mathcal{B}\). Hence,

\[
\tilde{R}_x(v) := \frac{1}{1 - \alpha}R_x(v) - \frac{\alpha}{1 - \alpha}\mu(v), \quad \text{and}
\tilde{R}_x^{(w,C)}(v) := \frac{1}{1 - \alpha}R_x^{(w,C)}(v) - \frac{\alpha}{1 - \alpha}\mu(v).
\]

It is easy to verify that \(\tilde{R}_x^{(w,C)}\) is a valid coherent risk measure on \(\mathcal{B}_{1+\beta w_0} = \mathcal{B}_1 + w_0 = \mathcal{B}_w\) for all \(x \in \mathcal{B}\). We set

\[
\bar{1} \rightarrow \infty
\]

\[
\bar{2} \rightarrow \infty \quad \text{as in Theorem 3.8,}
\]

\[
\text{we obtain}
\]

\[
|\tilde{R}_x(v) - \tilde{R}_x(u) - \tilde{R}_y(v) + \tilde{R}_y(u)|
\]

\[
= (1 - \alpha)|\tilde{R}_x(v) - \tilde{R}_x(u) - \tilde{R}_y(v) + \tilde{R}_y(u)|
\]

\[
\leq (1 - \alpha)|\tilde{R}_x(v) - \tilde{R}_x(u)| + (1 - \alpha)|\tilde{R}_y(v) - \tilde{R}_y(u)|
\]

\[
\leq (1 - \alpha)\tilde{R}_x^{(w,C)}(|v - u|) + (1 - \alpha)\tilde{R}_y^{(w,C)}(|v - u|)
\]

\[
\leq 2A(1 - \alpha) + A(1 - \alpha)\beta \left(\tilde{R}_x^{(w,C)}(w_0) + \tilde{R}_y^{(w,C)}(w_0)\right).
\]

Note that since (1 - \(\alpha\))\(\tilde{R}_x^{(w,C)}(w_0) \leq \tilde{R}_x^{(w,C)}(w_0)\) holds for all \(x \in \mathcal{B}\), we obtain

\[
|\tilde{R}_x(v) - \tilde{R}_x(u) - \tilde{R}_y(v) + \tilde{R}_y(u)|
\]

\[
\leq 2A(1 - \alpha) + A\beta \left(\tilde{R}_x^{(w,C)}(w_0) + \tilde{R}_y^{(w,C)}(w_0)\right)
\]

\[
\leq 2A(1 - \alpha) + A\beta (\gamma w_0(x) + \gamma w_0(y) + 2\mathcal{K}).
\]

We select \(\beta := \frac{\alpha}{\mathcal{K}}\) for some \(\alpha_0 \in (0, \alpha)\). Setting \(\gamma_2 := (1 - \alpha + \alpha_0) \vee \gamma \in (0, 1)\) yields for all \(x \neq y\)

\[
|\tilde{R}_x(v) - \tilde{R}_x(u) - \tilde{R}_y(v) + \tilde{R}_y(u)|
\]

\[
\leq 2A(1 - \alpha + \alpha_0) + A\gamma_2 \tilde{w}(w_0(x) + w_0(y)) \leq A\gamma_2 (w'(x) + w'(y)).
\]

Hence, setting \(\bar{\alpha} := \gamma_1 \vee \gamma_2 < 1\), (11) and (13) imply for all \(x \neq y\)

\[
|\tilde{R}_x(v) - \tilde{R}_x(u) - \tilde{R}_y(v) + \tilde{R}_y(u)| \leq \|v - u\|_{s,w} \bar{\alpha}(w'(x) + w'(y)),
\]

the required inequality.

**3.3 Poisson equation**

We set \(w' = 1 + \beta w_0\) as in Theorem 3.8, \(w = 1 + \tilde{K}_0^{-1}w_0\) and \(C = \tilde{K}_0\) as in Theorem 3.8. Hence, apparently \(\mathcal{B}_w = \mathcal{B}_w\).

**Lemma 3.9.** Suppose Assumption 3.7 hold. Then for any \(v, u \in \mathcal{R}_w^{(C)}\),

\[
\lim_{n \to \infty} \frac{1}{n} \|T^n(v) - T^n(u)\|_w = 0.
\]
Proof. It is sufficient to show that $\|T^n(v) - T^n(u)\|_w$ is uniformly bounded, which is equivalent to requiring that $\|T^n(v) - T^n(u)\|_{w'}$ is uniformly bounded.

Indeed, by Assumption 3.7(ii), setting $K := \beta K + 1 - \gamma$, we have

$$|T(v) - T(u)| \leq \mathcal{R}(w,C)(|v - u|) \leq \|v - u\|_{w'}(\gamma w' + K')$$

where the first inequality is due to Proposition 2.2.

On the other hand, by Theorem 3.3 $\|T^n(v)\|_{s,w} \leq C$ holds for all $n \in \mathbb{N}_+$. Hence, by induction w.r.t. $n$, we have for $n = 2, 3, \ldots$

$$|T^n(v) - T^n(u)| \leq \mathcal{R}(w,C)(|T^{n-1}(v) - T^{n-1}(u)|)$$

$$\leq \|v - u\|_w \mathcal{R}(w,C) \left( \gamma^{n-1}w' + K' \sum_{k=0}^{n-2} \gamma^k \right)$$

$$\leq \|v - u\|_w \left( \gamma^n w' + K \sum_{k=0}^{n-1} \gamma^k \right),$$

which implies that $\|T^n(v) - T^n(u)\|_{w'} \leq \frac{K'}{1 - \gamma}$.

Let $\tilde{B}_w' = B_w' / \sim$ be the quotient space, which is induced by the equivalence relation $\sim$ on $B_w'$ defined by $v \sim u$ if and only if there exists some constant $A \in \mathbb{R}$ such that $v(x) - u(x) = A \forall x \in X$, endowed with the quotient norm induced by the weighted seminorm.

**Theorem 3.10.** Suppose Assumption 3.7 and 3.8 hold. Then there exist (i) a solution $(\rho, h) \in \mathbb{R} \times B_w$ to the Poisson equation (1), where $\rho$ is unique and (ii) a risk measure $\nu$ satisfying $\nu(c + \mathcal{R}(v)) = \nu(v) + \rho, \forall v \in \mathcal{R}(C)$.

**Proof.** Starting from any $v$ satisfying $\|v\|_{s,w} \leq C$, $\{v_n := T^n(v)\}$ is a Cauchy sequence in $\tilde{B}_w'$ under the $w'$-seminorm due to Theorem 3.3 and Theorem 3.8. Then by the fixed point argument w.r.t. the $w'$-seminorm applied in the proof of Theorem 3.14, there exists a fixed point $h \in \tilde{B}_w'(= \tilde{B}_w)$ such that $\|T(h) - h\|_{s,w'} = 0$. Hence, by Lemma 2.1 there exists a constant $\rho \in \mathbb{R}$ such that $T(h) = c + \mathcal{R}(h) = h + \rho$.

Uniqueness of $\rho$. Suppose there are two solutions $(\rho, h)$ and $(\rho', h')$ in $\mathbb{R} \times B_w$. Then, $T^n(h) = h + n\rho$ and $T^n(h') = h' + n\rho'$. By Lemma 3.9

$$\frac{1}{n} \|T^n(h) - T^n(h')\|_w = \frac{1}{n} \|h - h' + n(\rho - \rho')\|_w \to 0$$

as $n \to \infty$ implies that $\rho = \rho'$.

(ii) Let $\mu_0 \in \mathcal{M}_{w'}$ be a probability measure and $h$ be one solution in $B_w$. We show first that

$$\lim_{m \to \infty} \sup_{n \geq m} |\mu_0[T^n(v) - T^n(h)] - \mu_0[T^m(v) - T^m(h)]| = 0, \forall v \in B(C).$$
Indeed, define \( v_n := T^n(v) \) and \( h_n := T^n(h) \), \( n = 1, 2, \ldots \) and we have

\[
\sup_{\|v-h\|, \|v_1-h_1\| \leq A} |\mu_0[v_n - h_n] - \mu_0[v_m - h_m]| \\
\leq \sup_{\|v_1-h_1\| \leq A} |\mu_0[T^{n-1}(v_1) - T^{n-1}(h_1)] - \mu_0[T^{m-1}(v_1) - T^{m-1}(h_1)]| \\
\leq \sup_{\|v_m-h_m\| \leq A} |\mu_0[T^{n-m}(v_m) - T^{n-m}(h_m)] - \mu_0[v_m - h_m]|,
\]

by which (14) follows immediately.

Define \( D(\cdot) := T(\cdot) - \rho \) and \( \mu_n(\cdot) := \mu_0(D^n(\cdot)). \) (14) is equivalent to

\[
\lim_{m \to \infty} \sup_{n \geq m} |\mu_0[D^n(v) - D^m(v)]| = \lim_{m \to \infty} \sup_{n \geq m} |\mu_n(v) - \mu_m(v)| = 0, \forall v \in \mathcal{B}_w(C).
\]

Hence, \( \mu_n \) converges to a mapping \( \mu_\infty : \mathcal{B}_w \to \mathbb{R} \) satisfying \( \mu_\infty(D(v)) = \mu_\infty(v) \), \( \forall v \in \mathcal{B}_w(C) \). On the other hand, for each \( n \), \( \mu_n \) satisfies the axioms of risk measures except the axiom of centralization. Hence, \( \mu_\infty \) preserves two axioms of risk measures and by setting \( \nu(\cdot) := \mu_\infty(\cdot) - \mu_\infty(0) \) we obtain the required risk measure.

**Remark 3.11.** If \( \mathcal{R} \) is coherent, its upper envelope \( \bar{\mathcal{R}}_w^{(C)} \) becomes \( \mathcal{R} \) itself. In this case, Assumption 3.1 is no longer needed to determine a priori the size of the bounded forward invariant subset, \( C \). Moreover, 1) Assumption 3.7(iii) implies Assumption 3.1(ii) due to (12), and 2) Theorem 3.10(ii) holds for all \( v \in \mathcal{B}_w \). For instance, the mean-semideviation map defined in (1) is coherent. It is shown in [33, Section 6] that the mean-semideviation map with \( r = 2 \) in a 1-dimensional linear model satisfy Assumption 3.7(ii) and (iii), based on which the existence and uniqueness of a solution to the Poisson equation is guaranteed.

4 Entropic Map

Recall that, given a Markov transition kernel \( P \), the entropic map is defined as

\[
\mathcal{R}_x(v) := \frac{1}{\lambda} \log \left( \int e^{\lambda v} dP_x \right), \lambda > 0.
\]

Without loss of generality, in the remaining part of this paper, we set \( \lambda = 1 \).

4.1 Upper envelope

We now derive the upper envelope for entropic measures.

**Proposition 4.1.** Let \( \nu(v) := \log \left( \int e^v d\mu \right) \) with a probability measure \( \mu \) on \( (X, \mathcal{B}(X)) \).

Suppose that for all \( v \in \mathcal{B}_w \), \( \int e^v d\mu < \infty \) holds. Then (i) \( \nu(v) \leq \frac{\int e^v u d\mu}{\mu(e^v)} \), and (ii) \( \bar{\nu}^{(w,C)}(u) := \sup_{v \in \mathcal{B}_w(C)} \frac{e^{v u} d\mu}{\int e^{v} d\mu} \) is an upper envelope for \( \nu \) given \( C \).
Proof. Given any two \( u, v \in \mathcal{B}_w \), we obtain
\[
\nu(v) - \nu(u) = \log \frac{\mu(e^v)}{\mu(e^u)} = \log \frac{\mu(e^v e^{v-u})}{\mu(e^u)} \geq \log \frac{\mu(e^u (v - u))}{\mu(e^u)},
\]
where the last inequality is due to Jensen’s inequality. Hence,
\[
\log \left[ \mu(e^v) \right] \geq \frac{\mu(e^v)}{\mu(e^u)} \left( u - \log \left[ \mu(e^{v-u}) \right] \right), \forall u, v \in \mathcal{B}_w.
\]
Define \( \xi_u := \frac{e^v}{\mu(e^v)} \). Restricting \( u \) and \( v \) to be in the subset \( \mathcal{B}_w^{(C)} \), the above inequality yields
\[
\log \left[ \mu(e^v) \right] \geq \sup_{\xi = \frac{e^v}{\mu(e^v)}, u \in \mathcal{B}_w^{(C)}} \mu(\xi v) - \mu(\xi \log(\xi)).
\]
Since the equality holds by taking \( \xi^* := \frac{e^v}{\mu(e^v)} \), we obtain
\[
\log \left[ \mu(e^v) \right] = \sup_{\xi = \frac{e^v}{\mu(e^v)}, u \in \mathcal{B}_w^{(C)}} \mu(\xi v) - \mu(\xi \log(\xi)).
\]
The second term \( \mu(\xi \log(\xi)) \) on the right-hand side of the above equation is the relative entropy and is always nonnegative (for proof see, e.g., [25, Section 5.1]). Hence, we obtain (i). Finally, (ii) is followed by
\[
\log \left[ \mu(e^v) \right] - \log \left[ \mu(e^u) \right] \leq \sup_{\xi = \frac{e^v}{\mu(e^v)}, f \in \mathcal{B}_w^{(C)}} \mu(\xi (v - u)) = \sup_{f \in \mathcal{B}_w^{(C)}} \frac{\int e^{\xi f(v - u)} d\mu}{\int e^{\xi f} d\mu},
\]
and it is easy to verify that \( \bar{\nu}(w,C)(u) = \sup_{f \in \mathcal{B}_w^{(C)}} \frac{\int e^{\xi f} u d\mu}{\int e^{\xi f} d\mu} \) is a valid coherent risk measure. \( \square \)

Remark 4.2. The inequality in (17) is similar to the dual representation of convex risk measures on \( L^\infty \) [14, 15] or on more general spaces such as Orlicz hearts [6]. However, since we consider a different functional space, namely, the weighted norm space \( \mathcal{B}_w \), the existing result cannot be directly applied here. On the other hand, for other types of convex risk measures, their dual representation provide us with a generic approach to calculate their upper envelopes, as shown in the above proposition.

By Proposition 4.1 we obtain one upper envelope for the entropic map:
\[
\bar{\mathcal{R}}^{(C)}_x(u) = \sup_{f \in \mathcal{B}_w : \|f\| \leq C} \frac{\int e^{\xi f} u dP_x}{\int e^{\xi f} dP_x},
\]
provided that \( P_x(e^f) < \infty \) holds for all \( f \in \mathcal{B}_w \) and \( x \in X \).
4.2 Lyapunov functions

Now we investigate properties of Lyapunov functions w.r.t. the entropic map.

**Definition 4.3.** A function \( w \) is said to be a Lyapunov function w.r.t. a risk map \( R \), if (i) \( w : X \to [0, \infty) \) is \( \mathcal{B}(X) \)-measurable and unbounded from above, and (ii) there exist constants \( \gamma \in (0, 1) \) and \( K > 0 \) satisfying \( R_x(w) \leq \gamma w(x) + K, \forall x \in X \).

We also introduce the following notation of level-sets. For any unbounded non-negative \( \mathcal{B}(X) \)-measurable function \( w \) and any real number \( R \in \mathbb{R} \), we define

\[
\mathcal{B}_w(R) := \{ x \in X | w(x) \leq R \}
\]

and \( \mathcal{B}_w^c(R) \) its complementary set. We then make the following assumption.

**Assumption 4.4.** There exists a Lyapunov function \( w_1 \geq 1 \) w.r.t. the entropic map \( R \), with constants \( \gamma_1 \in (0, 1) \) and \( K_1 > 0 \).

If the above assumption holds and setting \( w_0 := w_1^p \) with any \( p \in (0, 1) \), then for all \( f \in \mathcal{H}_{w_0} \), there exists a constant \( K_f \) (depending on \( p \) and \( \|f\|_{w_0} \)) satisfying \( |f(x)| \leq \|f\|_{w_0} w_0(x) \leq w_1(x) + K_f, \forall x \in X \). We immediately have \( P_x(e^{f}) \leq P_x(e^{w_1^p}) \leq e^{K_f e^{\gamma_1 w_1^p} + K_1} < \infty, \forall x \in X \) and therefore, the upper envelope for the entropic map in \( \mathcal{H} \) is well defined. In the following theorem, we show that if \( w_1 \) is a Lyapunov function w.r.t. \( R \), then \( w_0 = w_1^p \) with any \( p \in (0, 1) \) is a Lyapunov function w.r.t. the upper envelope of \( R \).

**Theorem 4.5.** Suppose that Assumption 4.4 holds. Let \( w_0 := w_1^p \) with \( p \in (0, 1) \). Then, for any constant \( C > 0 \), there exist constants \( \gamma_2 \in (0, 1) \) (depending only on \( p \) and \( \lambda_1 \)) and \( K_2 > 0 \) (depending on \( p \), \( C \), \( \lambda_1 \) and \( K_1 \)) such that

\[
\sup_{f : f \in \mathcal{H}_{w_0}, \|f\|_{w_0} \leq w_0 + C} \frac{P_x(e^{f})}{P_x(e^{f})} \leq \gamma_2 w_0(x) + K_2.
\]

**Proof.** Due to Assumption 4.4, for any \( \lambda \in (\gamma_1, 1) \), we have

\[
R_x(w_1) \leq \lambda w_1(x), \forall x \in B^c_{w_1}(A), A := \frac{K_1}{\lambda - \gamma_1}.
\]

It implies that for all \( x \in B^c_{w_1}(A) \),

\[
(19) \quad \int_{B^c_{w_1}(\lambda w_1(x))} P_x(dy) \left(e^{w_1(y) - \lambda w_1(x)} - 1 \right) \leq \int_{B^c_{w_1}(\lambda w_1(x))} P_x(dy) \left(1 - e^{w_1(y) - \lambda w_1(x)} \right).
\]

Taking some \( \gamma_2 \in (\lambda^p, 1) \), by the definition of \( w_0 \), we then have

\[
(20) \quad B^c_{w_0}(\gamma_2 w_0(x)) \subseteq B^c_{w_1}(\lambda w_1(x)), \forall x \in X.
\]

Indeed, for any \( y \in B^c_{w_0}(\gamma_2 w_0(x)) \), it satisfies \( w_0(y) > \gamma_2 w_0(x) \), which is equivalent to \( w(y) > (\gamma_2)^{1/p} w_1(x) > \lambda w_1(x) \). Hence, \( y \in B^c_{w_1}(\lambda w_1(x)) \) as well. We will need the following two lemmas (Lemma 4.6 and 4.7) to complete the proof of Theorem 4.5.
Lemma 4.6. For any \( \eta \in (0, 1 - \lambda) \), \( p \in (0, 1) \) and \( \gamma_2 \in ((\lambda + \eta)^p, 1) \), there exists a constant \( R_1 > 0 \) such that for all \( y \in \mathcal{B}_{w_0}^c(\gamma_2w_0(x)) \), \( x \in \mathcal{B}_{w_1}^c(R) \) and \( R \geq R_1 \),
\[
e^{w_0(y)+\eta w_1(x)}(w_0(y) - \gamma_2w_0(x)) \leq e^{w_1(y)-\lambda w_1(x)} - 1.
\]

Proof. It is sufficient to show that there exists a constant \( R_1 > 0 \) satisfying
\begin{equation}
(21) \quad w_0(y) + \log w_0(y) + C + \log 2 + \eta w_1(x) \leq w_1(y) - \lambda w_1(x)
\end{equation}
for all \( y \in \mathcal{B}_{w_0}^c(\gamma_2w_0(x)), x \in \mathcal{B}_{w_1}^c(R) \) and \( R \geq R_1 \). Note that for any \( p \in (0, 1) \) and \( \epsilon \in (0, 1) \), there exists a constant \( D \) (depending on \( p \) and \( \epsilon \)) satisfying
\[
x^p + p \log x \leq cx + D, \forall x \geq 1,
\]
which implies that \( w_0(x) + \log w_0(x) \leq \epsilon w_1(x) + D, \forall x \in X \). Hence, for all \( y \in \mathcal{B}_{w_0}^c(\gamma_2w_0(x)) \), we have
\[
w_1(y) - w_0(y) - \log w_0(y) - (\lambda + \eta)w_1(x) \\
\geq (1 - \epsilon)w_1(y) - (\lambda + \eta)w_1(x) - D \geq \left((1 - \epsilon)\lambda_2^{1/p} - \lambda - \eta\right)w_1(x) - D.
\]
Choosing \( \gamma_2 \in ((\lambda + \eta)^p, 1) \), \( \epsilon < 1 - \frac{\lambda + \eta}{\lambda_2^{1/p}} \) and \( R_1 := \frac{D + C + \log 2}{(1-\epsilon)\lambda_2^{1/p} - \lambda - \eta} \) (21) holds for all \( y \in \mathcal{B}_{w_0}^c(\gamma_2w_0(x)), x \in \mathcal{B}_{w_1}^c(R) \) and \( R \geq R_1 \). \( \square \)

Lemma 4.7. For any \( \eta > 0 \), \( p \in (0, 1) \) and \( C \geq 0 \), there exists a constant \( R_2 \) such that for all \( y \in \mathcal{B}_{w_1}(\lambda w_1(x)), x \in \mathcal{B}_{w_1}^c(R) \) and \( R \geq R_2 \),
\begin{equation}
(22) \quad e^{-w_0(y)+\eta w_1(x)-C} (\gamma_2w_0(x) - w_0(y)) \geq 1 - e^{w_1(y)-\lambda w_1(x)}.
\end{equation}

Proof. It is sufficient to show that \( e^{-w_0(y)+\eta w_1(x)-C} (\gamma_2w_0(x) - w_0(y)) \geq 1 \) under the same condition. Note that there exists a constant \( D > 0 \) such that
\[
\frac{\gamma_2}{\eta} x^p \leq x + D, \forall x \geq 1,
\]
which yields \(-w_0(y) + \eta w_1(x) - C \geq -w_0(y) + \gamma_2w_0(x) - C - D \) and hence,
\[
e^{-w_0(y)+\eta w_1(x)-C} (\gamma_2w_0(x) - w_0(y)) \geq e^{\gamma_2w_0(x)-w_0(y)-C-D} (\gamma_2w_0(x) - w_0(y)).
\]
For all \( y \in \mathcal{B}_{w_1}(\lambda w_1(x)) \), we have \( \gamma_2w_0(x) - w_0(y) \geq (\gamma_2 - \lambda^p)w_0(x) \). Hence,
\[
e^{\gamma_2w_0(x)-w_0(y)-C-D} (\gamma_2w_0(x) - w_0(y)) \geq e^{(\gamma_2 - \lambda^p)w_0(x)-C-D} (\gamma_2 - \lambda^p)w_0(x).
\]
Due to the fact that \( g(x) = e^{x} \cdot x \) is an increasing function on \( \mathbb{R}_+ \), we can choose \( \tilde{R}_2 > 0 \) such that \( e^{\tilde{R}_2} \cdot \tilde{R}_2 = e^{C+D} \). Hence, we have for all \( y \in \mathcal{B}_{w_1}(\lambda w_1(x)), x \in \mathcal{B}_{w_0}(\hat{R}) \) and \( \hat{R} \geq \tilde{R}_2 \), \( e^{-w_0(y)+\eta w_1(x)-C} (\gamma_2w_0(x) - w_0(y)) \geq 1 \) holds. Finally, setting \( R_2 = \tilde{R}_2^{1/p} \), the assertion is obtained. \( \square \)
Proof of Theorem 4.5 continued: hence, by Lemma 4.6 and 4.7 for all $x \in B_{w_1}(R_1 \lor R_2 \lor A)$,

\[
\int_{B_{w_1}(\gamma_2 w_0(x))} P_x(dy) e^{w_0(y)+C+\eta w_1(x)} (w_0(y) - \gamma_2 w_0(x)) \leq \int_{B_{w_1}(\gamma_2 w_0(x))} P_x(dy) \left( e^{w_1(y)-\lambda w_1(x)} - 1 \right)
\]

(Lemma 4.6) \leq \int_{B_{w_1}(\lambda w_1(x))} P_x(dy) \left( e^{w_1(y)-\lambda w_1(x)} - 1 \right)

(Lemma 4.7) \leq \int_{B_{w_1}(\lambda w_1(x))} P_x(dy) \left( 1 - e^{w_1(y)-\lambda w_1(x)} \right)

which implies that for all $f \in \mathcal{B}_{w_0}$ satisfying $|f| \leq w_0 + C$,

(23) \int P_x(dy) e^{f(y)} (w_0(y) - \gamma_2 w_0(x)) \leq 0, \forall x \in B_{w_1}(R_1 \lor R_2 \lor A).

Finally, for all $x \in B_{w_1}(R_1 \lor R_2 \lor A)$ and $f \in \mathcal{B}_{w_0}$ satisfying $|f| \leq w_0 + C$,

\[
\frac{P_x(e^{f}w_0)}{P_x(e^{f})} \leq \frac{P_x(e^{w_0+C}w_0)}{P_x(e^{-w_0-C})} \leq e^{2C} P_x(e^{w_0}w_0) \cdot P_x(e^{w_0})
\]

Using again the fact that there exists some constant $D > 0$ satisfying

\[
x^p + p \log x \leq x + D, \forall x \geq 1,
\]

we obtain that $P_x(e^{w_0}w_0) \leq e^D P_x(e^{w_1})$ which is upper bounded on $B_{w_1}(R_1 \lor R_2 \lor A)$. Hence, there exists a $K_2 > 0$ such that for all $f \in \mathcal{B}_{w_0}$ satisfying $|f| \leq w_0 + C$,

\[
\frac{P_x(e^{f}w_0)}{P_x(e^{f})} \leq K_2, \forall x \in B_{w_1}(R_1 \lor R_2 \lor A),
\]

which together with (23) implies the required inequality. \qed

Remark 4.8. The statement of Theorem 4.5 can be easily generalized as follows: for any positive $C$ and $A$, there exist constants $\gamma_2 \in (0,1)$ and $K_2 \in \mathbb{R}_+$ such that

\[
\sup_{f: f \in \mathcal{B}_{w_0}, |f| \leq A w_0 + C} \frac{P_x(e^{f}w_0)}{P_x(e^{f})} \leq \gamma_2 w_0(x) + K_2.
\]

Corollary 4.9. Suppose that Assumption 4.4 holds. Then, for any $p \in (0,1)$, $w_0 := w_1^p$, there exist constants $\tilde{\gamma}_0 \in (0,1)$ (depending on $p$ and $\gamma_1$) and $K_0$ (depending on $p$, $\gamma_1$ and $K_1$) satisfying $R_x(w_0) \leq \tilde{\gamma}_0 w_0(x) + K_0, \forall x \in X.$
Proof. By Proposition 4.1(i), \( R_x(w_0) = \log P_x(e^{w_0}) \leq \frac{P_x(e^{w_0} w_0)}{P_x(e^{w_0})}, \forall x \in X \). Then, by Theorem 4.5, there exist constants \( \hat{\gamma}_0 \in (0, 1) \) (depending on \( p, \gamma_1 \) and \( K_1 \)) and \( \hat{\gamma}_0 > 0 \) (depending on \( p, \gamma_1 \) and \( K_1 \)) such that

\[
\frac{P_x(e^{w_0} w_0)}{P_x(e^{w_0})} \leq \sup_{f \in \mathcal{F}_{w_0}: |f| \leq w_0 \frac{P_x(e^{f w_0})}{P_x(e^{w_0})}} \leq \hat{\gamma}_0 w_0 + \hat{\lambda}_0,
\]

which yields the required inequality.

In summary, if Assumption 4.4 holds, then

(a) by Corollary 4.9, \( w_0 \) is a Lyapunov function w.r.t. the entropic map with constants \( \hat{\gamma}_0 \) and \( \hat{\lambda}_0 \);

(b) by Theorem 4.5, the same \( w_0 \) is also a Lyapunov function with constants \( \gamma_2 \) and \( K_2 \), which satisfies satisfying Assumption 3.7(i) (see Remark 3.2(a) and (b) ) if the cost function \( c \) satisfies \( |c| \leq \hat{\gamma}_0 w_0 + C_0 \) with some \( \hat{\gamma}_0 \in (0, 1 - \gamma_2) \) and \( C_0 > 0 \);

(c) combining (a) and (b), Assumption 3.7(i) also holds.

### 4.3 Minorization properties

We investigate now the properties of the entropic map restricted to bounded level-sets. We introduce first the local Doeblin’s condition (see [12] and references therein) as follows.

**Assumption 4.10.** Let \( w_0 : X \rightarrow [0, \infty) \) be a \( \mathcal{B}(X) \)-measurable function. For any level-set \( C := B_{w_0}(R), R > 0, \) there exist a measure \( \mu_C \) and constants \( \lambda_C^-, \lambda_C^+ > 0 \) such that \( \mu_C > 0 \) and

\[
\lambda_C^- \mu_C(A \cap C) \leq P_x(A \cap C) \leq \lambda_C^+ \mu_C(A \cap C), \forall x \in C, A \in \mathcal{B}(X).
\]

The following proposition indicates the connection to the standard Doeblin’s condition.

**Proposition 4.11.** The following two conditions are equivalent:

(i) there exist a measure \( \mu_C \) and a constant \( \lambda_C^- > 0 \) such that \( \mu_C(C) > 0 \) and

\[
P_x(A \cap C) \geq \lambda_C^- \mu_C(A \cap C), \forall x \in C, A \in \mathcal{B}(X).
\]

(ii) there exist a probability measure \( \mu \) and a constant \( \alpha > 0 \) such that

\[
P_x(A) \geq \alpha \mu(A), \forall x \in C, A \in \mathcal{B}(X).
\]

**Proof.** First, it is clear that (25) implies (24). Conversely, assume that (24) holds. Then, \( \mu(\cdot) := \frac{\mu_C(\cdot)}{\mu_C(C)} \), and \( \alpha := \lambda_C^- \mu_C(C) \) satisfy (25). \( \Box \)
Theorem 4.12. Suppose Assumption 4.10 and Assumption 4.4 hold. Let \( w_1 \) be the Lyapunov function and \( B = B_{w_0}(R_0) \) be a bounded levels-set with some \( R_0 > 0 \), where \( w_0 := w_1^p, p \in (0, 1) \). Then for any positive constant \( K_0 > 0 \), there exists a positive constant \( \tilde{K}_0 > K_0 \) such that for all \( v \in B_{w_0} \) satisfying \( |v| \leq w_0 + K_0 \), the following inequality holds \( R_x(v) - R_y(v) \leq 2(\tilde{K}_0 - K_0) + R_x(w_0) - R_y(-w_0), \forall x, y \in B \).

Proof. Let \( C := B_{w_0}(R) \supset B_{w_0}(R_0) \) with \( R > R_0 \). Then

\[
(26) \quad \frac{P_x(e^v)}{P_y(e^v)} = \frac{P_x(e^v 1_C) + P_x(e^v 1_C^c)}{P_y(e^v 1_C) + P_y(e^v 1_C^c)} \leq \frac{P_x(e^v 1_C) + P_x(e^v 1_C^c)}{P_y(e^v 1_C)}.
\]

We first consider the quotient \( \frac{P_x(e^v 1_C)}{P_y(e^v 1_C)} \). By Assumption 4.10, we immediately obtain

\[
\theta(x, C) := \frac{P_x(e^v 1_C)}{P_y(e^v 1_C)} \leq e^{2 \tilde{K}_0} \frac{P_x(e^v 0 1_C)}{P_y(e^v 0 1_C)} = e^{2 \tilde{K}_0} \frac{\theta(x, C) P_x(e^v 0)}{\theta(y, C) P_y(e^v 0)}
\]

where we define \( \theta(x, C) := \frac{P_x(e^v 0 1_C)}{P_y(e^v 0 1_C)} \) and \( \theta(y, C) := \frac{P_y(e^v 0 1_C)}{P_y(e^v 0 1_C)} \). By Theorem 4.5 there exist some constants \( \gamma_2 \in (0, 1) \) and \( K_2 > 0 \) such that

\[
\theta(x, C) \leq \|1_C\|_{w_0} \sup_{|v| \leq w_0} \frac{P_x(e^v 0 1_C)}{P_y(e^v 0 1_C)} \leq \|1_C\|_{w_0} \sup_{|v| \leq w_0} \frac{P_x(e^v 0)}{P_y(e^v 0)} \leq \|1_C\|_{w_0} (\gamma_2 w_0(x) + K_2).
\]

Hence, \( \theta(x, C) \leq \|1_C\|_{w_0} \sup_{x \in B} (\gamma_2 w_0(x) + K_2) \leq 2 \tilde{R}_0 + K_0 \). Similarly, we have

\[
\theta'(y, C) = 1 - \frac{P_y(e^{-w_0 1_C})}{P_y(e^{-w_0 1_C})} \geq 1 - \frac{\gamma_2 R_0 + K_2}{R}
\]

Hence, \( \sup_{x, y \in B} \frac{\theta(x, C)}{\theta'(y, C)} \to 0 \) as \( R \to \infty \), which implies that for any \( K_0 > 0 \), we can select sufficiently large \( R \) such that

\[
(27) \quad \log \frac{\theta(x, C)}{\theta'(y, C)} \leq -2 \tilde{K}_0 - \log 2, \forall x, y \in B.
\]

Thus for any \( \tilde{K}_0 > K_0 > 0 \), there exists a sufficiently large \( R \) (depending on \( K_0 \)) such that

\[
(28) \quad \frac{P_x(e^v 1_C)}{P_y(e^v 1_C)} \leq e^{2(\tilde{K}_0 - K_0) + R_x(w_0) - R_y(-w_0) - \log 2}.
\]

Now we consider the first quotient in (26). By Assumption 4.10 we immediately have \( \frac{P_x(e^v 1_C)}{P_y(e^v 1_C)} \leq \frac{\lambda^+}{\lambda^-} \). Hence, setting

\[
(29) \quad \tilde{K}_0 := K_0 + \frac{1}{2} \log 2 + \log(\frac{\lambda^+}{\lambda^-}),
\]

we obtain \( \frac{P_x(e^v 1_C)}{P_y(e^v 1_C)} \leq e^{2(\tilde{K}_0 - K_0) + R_x(w_0) - R_y(-w_0) - \log 2} \). Together with (28), it yields the required inequality: \( \frac{P_x(e^v)}{P_y(e^v)} \leq e^{2(\tilde{K}_0 - K_0) + R_x(w_0) - R_y(-w_0)} \), where \( \tilde{K}_0 \) is chosen according to (29), while \( R \) is determined by (27). \qed
We investigate now the minorization property of the upper envelope \( \bar{R}^{(w,C)} \) of the entropic map \( R \), which is required in Assumption 3.7(iii).

**Proposition 4.13.** Let \( w : X \to [1, \infty) \) be a \( \mathcal{B}(X) \)-measurable function and \( B := \mathcal{B}_{w}(R) \) with some \( R > 0 \). Suppose Assumption 4.10 holds. Assume further that \( \bar{R}^{(w,C)}(w_0) < \infty \) for all \( x \in B \). Then there exist a constant \( \alpha \in (0,1) \) and a probability measure \( \mu \) on \( (X, \mathcal{B}(X)) \) satisfying

\[
\bar{R}^{(w,C)}(v) - \bar{R}^{(w,C)}(u) \geq \alpha \mu(v - u), \forall x \in B, v \geq u \in \mathcal{B}_{1+w_0}.
\]

**Proof.** Note that since \( \bar{R}^{(w,C)}(w_0) < \infty \), we have for all \( v \in \mathcal{B}_{1+w_0} \) and \( x \in B \),

\[
|\bar{R}^{(w,C)}(v)| \leq \bar{R}^{(w,C)}(|v|) \leq \|v\|_{1+w_0} \bar{R}^{(w,C)}(1 + w_0) < \infty.
\]

By Proposition 4.11, Assumption 4.10 implies that there exist a probability measure \( \mu_B \) and \( \alpha_B \) such that \( P_x(v) \geq \alpha_B \mu_B(v) \) for all nonnegative measurable function \( v \). Hence, for all \( x \in B \) and \( h' \in \mathcal{B}_w^{(C)} \), we have

\[
P_x(e^{h'(v - u)}) \geq \alpha_B \mu_B(e^{h'(v - u)}) \geq \frac{\alpha_B \mu_B(e^{-Cw}(v - u))}{\max_{x \in B} P_x(e^{Cw})} \mu_B(e^{-Cw}(v - u))
\]

Hence, \( \alpha := \frac{\alpha_B \mu_B(e^{-Cw})}{\max_{x \in B} P_x(e^{Cw})} \) and the probability measure \( d\mu := \frac{e^{-Cw}d\mu_B}{\int e^{-Cw}d\mu_B} \) are the required constant and probability measure respectively.

The following theorem shows that applying the entropic map, together with an additional growth condition for cost functions (see (30) below), Assumption 4.4 and 4.10 are sufficient for Assumption 3.7 and 3.7.

**Theorem 4.14.** Let \( R \) be the entropic map with \( \lambda = 1 \). Assume that Assumption 4.4 and 4.10 hold with a Lyapunov function \( w_1 \), and that the cost function \( c \) satisfies

\[
c \in \mathcal{B}_{w_1^q} \quad \text{with some } q \in (0,1).
\]

Then Assumption 3.7 holds with \( w_0 = w_1^p \) for any \( p \in (q,1) \), and some \( K_0 > 0 \), and Assumption 3.7 holds with \( w_0 \) and \( w = 1 + K_0^{-1}w_0 \).

18
Proof. Fix one $p \in (q, 1)$ and let $w_0 = w_1^p$. Then by Corollary 4.9, there exists $\gamma_0 \in (0, 1)$ and $K_0 > 0$ satisfying $R_x(w_0) \leq \gamma_0 w_0(x) + K_0$. By assumption, there exists some $C > 0$ and $q \in (0, 1)$ such that $|c| \leq CW_1^q$. Choosing one $\gamma_0'(c) \in (0, 1 - \gamma_0)$, there exists a constant $K_0'(c) > 0$ satisfying $CW_1^q(x) \leq \gamma_0'(c) w_0(x) + K_0'(c)$. Hence, Assumption 3.1(i) holds with $\gamma_0 := \gamma_0 + \gamma_0'(c) \in (0, 1)$ and $K_0 := K_0 + K_0'(c)$. Due to Proposition 4.11, Assumption 3.1(ii) holds with some constant $\tilde{K}_0 > 0$. Next, by Theorem 3.3 and 4.5, Assumption 3.7(i) and (ii) hold with $w := 1 + K_0^{-1} w_0$ and $C := K_0^{-1}$. Assumption 3.7(iii) holds due to Proposition 4.13.

By Theorem 3.6 and 3.10, the above theorem implies the following corollary.

Corollary 4.15. Let $R$ be the entropic map with $\lambda = 1$. Suppose Assumption 4.4 and 4.10 hold and $w_1$ is the Lyapunov function. Assume further that the cost function $c$ satisfies (50). Then for any $p \in (q, 1)$, there exist (i) constants $\tilde{a} \in (0, 1)$ and $\beta > 0$ such that $\|R(v) - R(u)\|_{s,w} \leq \hat{a}\|v - u\|_{s,w}$ with $w := 1 + \beta w_1^p$, and (ii) a solution $(\rho, h) \in R \times R_w$ to the Poisson equation $c + R(h) = \rho + h$, where $\rho$ is unique.

Remark 4.16. Hence, for the entropic map, the required sufficient conditions in Assumption 3.1 and 3.7 can replaced by the existence of Lyapunov function in Assumption 4.4, the local Doeblin’s condition in Assumption 4.11 and the growth condition for the cost function in (50). We compare our results with two mostly related results in the literature.

Comparison with [17]. (a) The assumption (A4) in [17] Section 4] requires a positive continuous density, i.e., there exists a positive function $q$ satisfying $Q(dy|x,a) = q(x,a,y)\mu(dy)$ for some reference probability measure $\mu$, which implies the local Doeblin’s condition in Assumption 4.11. Hence, our assumption is more general than its counterpart in [17].

(b) The assumption (A3) set in [17] Section 3] for the cost function $c$ is implicit and difficult to be verified. On the contrary, the sufficient growth condition for $c$, (50), is explicit in form of the Lyapunov function $w_1$ w.r.t. the entropic map. Note that, in the example provided by [17], the assumption (A3) is also verified with the help of a Lyapunov function.

(c) As an advantage, in comparison with [17], the convergence rate of iterations towards the solution to the Poisson equation is explicitly specified by $\tilde{a}$ in Theorem 3.6 under the chosen seminorm.

Comparison with [24]. Among others, Kontoyiannis and Meyn (2005) developed in [24] (see also their earlier work on the same topic [23]) a spectral theory of multiplicative Markov processes, where the Poisson equation w.r.t. the entropic map (called multiplicative Poisson equation in [24]) plays the central role. Though our assumptions are less general than the assumptions stated in [24, 23], our proof that generalizes the Hairer-Mattingly approach [17] is conceptually simpler than the one provided in [24, 23], and can also be applied to other types of risk maps. Note again that, in our approach, the convergence rate of iterations towards the solution to the Poisson equation is explicitly specified by $\tilde{a}$ in Theorem 3.6 under the chosen seminorm.
5 Optimal Risk-sensitive Control

5.1 Markov control processes

In this subsection, we introduce the framework of Markov control processes, where we mostly follow the notations of Hernández-Lerma & Lasserre (1999) [21].

A Markov control process, \((X, A, \{A(x)|x \in X\}, Q, c)\), consists of the following components: state space \(X\) and action space \(A\), which are Borel spaces; the feasible action set \(A(x)\), which is a nonempty Borel space of \(A\); for a given state \(x \in X\); the transition model \(Q(B|x, a), B \in B(X), (x, a) \in K\); a stochastic kernel on \(X\) given \(K\), where \(K\) denotes the set of feasible state-action pairs \(K := \{(x, a)|x \in X, a \in A(x)\}\), which is a Borel subset of \(X \times A\); and the cost function \(c: K \rightarrow \mathbb{R}, B(K)\)-measurable. Random variables are denoted by capital letters, e.g., \(X_t\) and \(A_t\), whereas realizations of the random variables are denoted by lowercase letters, e.g., \(x_t\) and \(a_t\).

We consider in this paper Markov policies, \(\pi = [\pi_0, \pi_1, \pi_2, \ldots]\), where each single-step policy \(\pi_t(\cdot|x_t)\), which denotes the probability of choosing action \(a_t\) at \(x_t\), \((x_t, a_t) \in K\), is Markov (independent of the states and actions before \(t\)) and, therefore, a stochastic kernel on \(A\) given \(X\). We use the boldface to represent a sequence of policies while using the normal typeface for a single-step policy. Let \(\Delta\) denote the set of all stochastic kernels on \(A\) given \(X\), \(\mu\), such that \(\mu(A(x)|x) = 1\) and \(\Pi_M = \Delta\) denotes the set of all Markov policies. A policy \(f \in \Delta\) is deterministic if for each \(x \in X\), there exists some \(a \in A(x)\) such that \(f(\{a\}|x) = 1\). Let \(\Delta_D \subset \Delta\) denote the set of all deterministic single-step policies. A policy \(\pi\) is said to be stationary, if \(\pi = \pi^\infty\) for some \(\pi \in \Delta\). For each \(x \in X\) and single-step policy \(\pi \in \Delta\), define

\[
(31) \quad c^\pi(x) := \int_{A(x)} c(x, a)\pi(da|x), P^\pi(B|x) := \int_{A(x)} Q(B|x, a)\pi(da|x), B \in B(X).
\]

The following average cost is used as an objective:

\[
(32) \quad S := \limsup_{T \rightarrow \infty} \frac{1}{T}S_T, \text{ where } S_T := \sum_{t=0}^{T} c(X_t, A_t).
\]

The optimization problem is then to minimize the expected objective

\[
(33) \quad \inf_{\pi \in \Pi_M} \mathbb{E}^\pi [S | X_0 = x]
\]

by selecting a policy \(\pi\). We notice that the finite-stage objective function can be decomposed as follows,

\[
(34) \quad \mathbb{E}^{\pi}_{X_0}[S_T] = c^{\pi_0}(X_0) + \mathbb{E}^{\pi_0}_{X_0}\left[c^{\pi_1}(X_1) + \mathbb{E}^{\pi_1}_{X_1}\left[c^{\pi_2}(X_2) + \ldots + \mathbb{E}^{\pi_T}_{X_{T-1}}[c^{\pi_T}(X_T)] \ldots \right]\right],
\]

where \(\mathbb{E}^{\pi_{t+1}}_{X_{t+1}}[v(X_{t+1})] := \int v(X_{t+1})P^{\pi_{t+1}}(dX_{t+1}|X_t)\) denotes the conditional expectation of the function \(v\) of the successive state \(X_{t+1}\) given current state \(X_t\).
5.2 Average risk-sensitive MCPs

To incorporate risk as in our previous work [33], we directly replace the conditional expectation $E_{X_t}^v$, with a risk map $R_{X_t}^v$, which is similar to the risk mapping defined in [30] and is formally defined as follows.

A mapping $R(v|x,a): K \times \mathcal{B}_w \to \mathbb{R}$ (simply written as $R$) is said to be a risk map on an MCP $(X, A, \{A(x)|x \in X\}, Q)$, if (i) for each $(x,a) \in K$, $R(\cdot|x,a): \mathcal{B}_w \to \mathbb{R}$ is a risk measure; and (ii) for each $v \in \mathcal{B}_w$, $R(v|\cdot)$ is a real-valued $\mathcal{B}(K)$-measurable function. Furthermore, we define for any $\pi \in A$, $R^\pi(x|\cdot) := \int_{A(x)} \pi(da|x)R(v|x,a)$.

For convenience, we sometimes write $R_{x,a}(v) := R(v|x,a)$ and $R^\pi_x(v) := R^\pi(x|v)$.

Replacing the conditional expectation in (34) with a risk map $R$, we obtain

$$J_T(x, \pi) := c^{T_0}(x) + R^\pi_{T_0}(c^{T_1} + R^\pi_{T_1}(c^{T_2} + \ldots + R^\pi_{T_2}(c^{T_T} \ldots)))$$, $\pi \in \Pi_M, x \in X$,

and the risk-sensitive objective considered in this paper is the average risk (AR):

$$(35) \quad J(x, \pi) := \limsup_{T \to \infty} \frac{1}{T} J_T(x, \pi).$$

Remark 5.1. Applying the same constructive approach as above, other two widely used objectives in the literature of MCPs, the finite-stage total cost and the discounted cost, can be analogously extended to risk-sensitive objectives [33], the finite-stage total risk and the discounted risk, respectively. Among them, the finite-stage risk can be optimized by dynamic programming [29]. For the discounted case, we refer to our previous work [33] Subsection 5.1], where the same problem for strictly convex risk maps, i.e., a Lyapunov function w.r.t. the upper module of the risk map need not exist, can be easily solved by replacing the upper module by the upper envelope defined in this paper.

In the rest of this section, for convenience, the AR objective can be considered as functions on $X$ within the space $\mathcal{B}_w$ by using the notation $J(\pi)$, as well as $J^*$. Analogous to classical MCPs, we need further assumptions to guarantee the existence of the “selector” in the optimization problem.

Assumption 5.2. For each $x \in X$,

(i) the cost function $c(x,a)$ is lower semi-continuous on $A(x)$,

(ii) the action space $A(x)$ is compact, and

(iii) the function $u'(x,a) := R_{x,a}(u)$ is continuous in $a \in A(x)$ for any $u \in \mathcal{B}_w$.

Define the following operators

$$F^\pi(v) := c^\pi + R^\pi(v), \quad F(v) := \inf_{\pi \in \Pi_M} F^\pi(v), \quad v \in \mathcal{B}_w.$$ (36)

Proposition 5.3 (see Proposition 5.1 in [33]). Suppose $R$ is a risk map satisfying Assumption 5.2. Then, for all $v \in \mathcal{B}_w$ and $x \in X$, there exists a deterministic policy $f \in \Delta_D$, such that

$$c^f(x) + R^f(v|x) = F(v|x) = \inf_{\pi \in \Delta} \{c^\pi(x) + R^\pi(v|x)\}.$$
We now extend Assumption 3.1 and 3.7 to the MCP-framework.

**Assumption 5.4.** There exist a \( B(X) \)-measurable function \( w_0 : X \to [0, \infty) \), constants \( \gamma_0, \gamma \in (0, 1) \), \( K_0 > K > 0 \), and \( \alpha \in (0, 1) \) such that (i)
\[
(c(x, a) + R_{x,a}(w_0)) \vee (-c(x, a) - R_{x,a}(-w_0)) \leq \gamma_0 w_0(x) + K_0, \forall (x, a) \in K;
\]
(ii) for all \( x, x' \in B_0 := \{ x \in X | w_0(x) \leq R_0 := \frac{2K_0}{1-\gamma_0} \} \), \( a \in A(x), a' \in A(x') \), the inequality
\[
R_{x,a}(v) - R_{x',a}(v) \leq 2(\hat{K}_0 - K_0) + R_{x,a}(w_0) - R_{x',a}(w_0)
\]
holds for all real-valued \( B(X) \)-measurable function \( v \) satisfying \( |v| \leq w_0 + \hat{K}_0 \); (iii) letting \( w := 1 + \hat{K}_0^{-1}w_0 \), there exists an upper envelope \( \bar{R}^{w,\hat{K}_0} \) such that
\[
\bar{R}^{w,\hat{K}_0}_{x,a}(w_0) \leq \gamma w_0(x) + K, \forall (x, a) \in K;
\]
and (iv) for all \( x, x' \in B := \{ x \in X | w_0(x) \leq R, R > \frac{2K}{1-\gamma_0} \} \), \( a \in A(x), a' \in A(x') \),
\[
\bar{R}^{w,\hat{K}_0}_{x,a}(v) - \bar{R}^{w,\hat{K}_0}_{x,a}(u) \geq \alpha \int (v(x) - u(x)) \mu(dx), \forall v, u \in B^{\hat{K}_0}_w.
\]

**Lemma 5.5.** Suppose Assumption 5.2 and 5.4 hold. Then there exists \( \hat{\alpha} \in (0, 1) \) and \( \beta > 0 \) such that \( \| F(v) - F(u) \|_{s,1+\beta w_0} \leq \hat{\alpha} \| v - u \|_{s,1+\beta w_0} \) for all \( v, u \in B^{\hat{K}_0}_w \).

**Proof.** By Proposition 5.3, there exist deterministic policies \( f_v, f_u \in \Delta_D \) such that \( F(v) = F^{f_v}(v) \) and \( F(u) = F^{f_u}(u) \). Thus
\[
F(v) - F(u) \leq F^{f_v}(v) - F^{f_u}(u) = R^{f_v}(v) - R^{f_u}(u)
\]
and
\[
F(u) - F(v) \leq F^{f_u}(u) - F^{f_v}(v) = R^{f_u}(u) - R^{f_v}(v)
\]
yield \( F_x(v) - F_x(u) + F_y(u) - F_y(v) \leq R^{f_v}_x(v) - R^{f_u}_x(v) + R^{f_u}_y(u) - R^{f_v}_y(v) \) for all \( x, y \in X \). By Assumption 5.4 and repeating the proof in Theorem 5.8 we obtain the required inequality.

**Theorem 5.6.** Suppose Assumption 5.2 and 5.4 hold. Then there exists a unique \( \rho^* \in \mathbb{R} \) and \( h \in \mathcal{B}_w \) satisfying the average risk optimality equation (AROE)
\[
(37) \quad \rho^* + h(x) = F(h|x) = \inf_{a \in A(x)} \{ c(x, a) + R(h|x, a) \}.
\]

and furthermore, \( \rho^* = J^*(x) = J(x, f^\infty) \) for all \( x \in X \), where \( f \) denotes the optimal selector in the right hand side of the AROE (37).

**Proof.** The existence of a unique solution to the AROE is simply due to Lemma 5.3 and Theorem 5.10(i). For the proof of the remaining part, we refer to the proof of Theorem 5.10.

**Remark 5.7.** When applying the entropic map with \( \lambda = 1 \), analogous to the without-control case stated in Corollary 3.15, Assumption 5.4 can be replaced by the following conditions:
(i) there exist a function \( w_1 : X \in [1, \infty) \), constants \( \gamma_1 \in (0, 1) \) and \( K_1 > 0 \) such that \( \mathcal{R}_{x,a}(w_1) \leq \gamma_1 w_1(x) + K_1, \forall (x,a) \in \mathcal{K} \),

(ii) for all \( p \in (0,1) \) and all level-sets \( C := \mathcal{B}_{w_1^p}(R), R > 0 \), there exist a measure \( \mu_C \) and constants \( \lambda_C^+ > \lambda_C^- > 0 \) such that \( \mu_C(C) > 0 \) and \( \lambda_C^- \mu_C(A \cap C) \leq Q_{x,a}(A \cap C) \leq \lambda_C^+ \mu_C(A \cap C), \forall x \in C, a \in \mathcal{A}(x) \) and \( \forall \mathcal{A} \in \mathcal{B}(X) \), and

(iii) the cost function \( c \) satisfies that \( \tilde{c}(x) := \sup_{a \in \mathcal{A}(x)} |c(x,a)| \) belongs to \( \mathcal{B}_{w_1^p} \) for some \( q \in (0, 1) \).

In the following example, we present one MCP that satisfies the above conditions (i) and (ii) with the entropic map.

### 5.3 Entropic map with discretized ergodic diffusions

Let \( X = \mathbb{R}^d \). Consider the following discretized ergodic diffusion \( \{x_n \in \mathbb{R}^d\} \) (cf. the example in [11] Section 6):

\[
x_{n+1} = Ax_n + b(x_n, a_n) + D(x_n, a_n)w_n,
\]

where \( \{w_n \in \mathbb{R}^d\} \) is a sequence of i.i.d. standard white noise, \( D : \mathcal{K} \to \mathbb{R}^{d \times d} \) is a continuous bounded matrix-valued function which is uniformly elliptic, i.e., there exists a constant \( L > 0 \) such that

\[
(38) \quad L^{-1}\|\xi\|^2 \leq \xi^T D(x,a)D^T(x,a)\xi \leq L\|\xi\|^2, \forall (x,a) \in \mathcal{K}, \xi \in \mathbb{R}^d,
\]

and \( b : \mathcal{K} \to \mathbb{R}^d \) is a continuous bounded vector function, and \( A \) is a matrix satisfying that there exists a constant \( \gamma \in (0,1) \) such that \( \xi^T A^T A\xi \leq \gamma\|\xi\|^2, \forall \xi \in \mathbb{R}^d \). Then the transition kernel \( Q(dy|x,a) \) has the following density w.r.t. the Lebesgue measure,

\[
(39) \quad q(y|x,a) = (2\pi)^{-d/2}|\Sigma|^{1/2}e^{-\frac{1}{2}(y-Ax-b)^T \Sigma(y-Ax-b)},
\]

where \( \Sigma = (DD^T)^{-1} \). Take one \( \gamma \in (\hat{\gamma},1) \) and consider the following weight function

\[
(40) \quad \tilde{w}_1(x) = \frac{\epsilon}{2}\|x\|^2, \text{ with some positive } \epsilon \leq \frac{\gamma - \hat{\gamma}}{\gamma} L^{-1} < L^{-1}.
\]

Hence, \( \Sigma(x,a) - \epsilon I \) is positive definite for all \( (x,a) \in \mathcal{K} \). We show that \( \tilde{w}_1 \) is a Lyapunov function w.r.t. the entropic map satisfying the condition (i) in Remark [27] as follows. By setting \( \tilde{x} := Ax + b \), we obtain

\[
\int Q(dy|x,a)e^{\tilde{w}_1(y)} = (2\pi)^{-d/2}|\Sigma|^{1/2} \int e^{-\frac{1}{2}(y^T (\Sigma - \epsilon I)y - 2y^T \Sigma \tilde{x} + \tilde{x}^T \Sigma \tilde{x})} dy
\]

which yields

\[
\log\left(Q_{x,a}[e^{\tilde{w}_1}]\right) = \log\left(\frac{|\Sigma|^{1/2}}{|\Sigma - \epsilon I|^{1/2}}\right) + \frac{1}{2}(Ax + b)^T \Sigma ((\Sigma - \epsilon I)^{-1} - \Sigma^{-1}) \Sigma (Ax + b).
\]
By (35) and the choice of $\epsilon$ in (40), we have
\[
\frac{1}{2} x^\top A^\top \Sigma \left( (\Sigma - \epsilon I)^{-1} - \Sigma^{-1} \right) \Sigma A x \leq \frac{\gamma \epsilon}{2} \|x\|^2 = \gamma \hat{w}_1(x), \forall (x,a) \in K.
\]

Finally, due to the uniform boundedness of $b$ and $\log \left( \frac{\|\Sigma\|_1}{\|\Sigma - \epsilon I\|_1} \right)$, we can always select a $\gamma_1 \in (\gamma, 1)$ and $\hat{K}_1 > 0$ such that
\[
\log \int Q(dy|x,a) e^{\hat{w}_1(y)} \leq \gamma_1 \hat{w}_1(x) + \hat{K}_1, \forall (x,a) \in K,
\]
which confirms that $\hat{w}_1 \geq 0$ is a Lyapunov function w.r.t. the entropic map. Hence, the condition (i) in Remark 5.7 holds with $w_1 := \hat{w}_1 + 1$, $\gamma_1$ and $K_1 := \hat{K}_1 + 1 - \gamma_1$.

Next, since the transition kernel $Q$ has a positive continuous density function, $q$ (see (39)), w.r.t. the Lebesgue measure, the local Doeblin’s condition (ii) in Remark 5.7 is obviously satisfied.

References

[1] P. Artzner, F. Delbaen, J.M. Eber, and D. Heath, Coherent measures of risk, Mathematical Finance, 9 (1999), pp. 203–228.

[2] G. Avila-Godoy and E. Fernández-Gaucherand, Controlled Markov chains with exponential risk-sensitive criteria: modularity, structured policies and applications, in Proceedings of the 37th IEEE Conference on Decision and Control, 1998, pp. 778–783.

[3] N. Bäuerle and U. Rieder, More risk-sensitive Markov decision processes, Mathematics of Operations Research, 39 (2013).

[4] V.S. Borkar and S.P. Meyn, Risk-sensitive optimal control for Markov decision processes with monotone cost, Mathematics of Operations Research, (2002), pp. 192–209.

[5] R. Cavazos-Cadena, Optimality equations and inequalities in a class of risk-sensitive average cost Markov decision chains, Mathematical Methods of Operations Research, 71 (2010), pp. 47–84.

[6] P. Cheridito and T. Li, Risk measures on Orlicz hearts, Mathematical Finance, 19 (2009), pp. 189–214.

[7] K.J. Chung and M.J. Sobel, Discounted MDPs: distribution functions and exponential utility maximization, SIAM Journal on Control and Optimization, 25 (1987), p. 49.

[8] S.P. Coraluppi and S.I. Marcus, Mixed risk-neutral/minimax control of discrete-time, finite-state Markov decision processes, IEEE Transactions on Automatic Control, 45 (2000), pp. 528–532.

[9] F. Delbaen, Coherent risk measures on general probability spaces, Advances in Finance and Stochastics Essays in Honour of Dieter Sondermann, (2000), pp. 1–37.

[10] G.B. Di Masi and L. Stettner, Infinite horizon risk sensitive control of discrete time Markov processes with small risk, Systems & control letters, 40 (2000), pp. 15–20.
[1] G.B. Di Masi and L. Stettner, Infinite horizon risk sensitive control of discrete time Markov processes under minorization property, SIAM Journal on Control and Optimization, 46 (2008), p. 231.

[2] R. Douc, G. Fort, E. Moulines, and P. Priouret, Forgetting the initial distribution for hidden Markov models, Stochastic processes and their applications, 119 (2009), pp. 1235–1256.

[3] W.H. Fleming and D. Hernández-Hernández, Risk sensitive control of finite state machines on an infinite horizon. I, in Proceedings of the 36th IEEE Conference on Decision and Control, IEEE, 1997, pp. 3407–3412.

[4] H. Föllmer and A. Schied, Convex measures of risk and trading constraints, Finance and Stochastics, 6 (2002), pp. 429–447.

[5] ———, Stochastic Finance, Walter de Gruyter & Co., Berlin, 2004. Extended edition.

[6] S. Gaubert and J. Gunawardena, The Perron-Frobenius theorem for homogeneous, monotone functions, Transactions American Mathematical Society, 356 (2004), pp. 4931–4950.

[7] M. Hairer and J.C. Mattingly, Yet another look at Harris’ ergodic theorem for Markov chains, in Seminar on Stochastic Analysis, Random Fields and Applications VI, Springer, 2011, pp. 109–117.

[8] D. Hernández-Hernández and S.I. Marcus, Risk sensitive control of Markov processes in countable state space, Systems & Control Letters, 29 (1996), pp. 147–155.

[9] O. Hernández-Lerma, Adaptive Markov Control Processes, Springer, 1989.

[10] O. Hernández-Lerma and J.B. Lasserre, Discrete-time Markov Control Processes: Basic Optimality Criteria, Springer, 1996.

[11] ———, Further Topics on Discrete-Time Markov Control Processes, Springer Verlag, 1999.

[12] R.A. Howard and J.E. Matheson, Risk-sensitive Markov decision processes, Management Science, 18 (1972), pp. 356–369.

[13] I. Kontoyiannis and S.P. Meyn, Spectral theory and limit theorems for geometrically ergodic Markov processes, Annals of Applied Probability, (2003), pp. 304–362.

[14] ———, Large deviations asymptotics and the spectral theory of multiplicatively regular Markov processes, Electron. J. Probab., 10 (2005), pp. 61–123.

[15] M. Ledoux, The Concentration of Measure Phenomenon, American Mathematical Society, 2001.

[16] S.P. Meyn and R.L. Tweedie, Markov Chains and Stochastic Stability, Springer-Verlag London Ltd., London, 1993.

[17] W. Ogryczak and A. Ruszczyński, From stochastic dominance to mean-risk models: Semideviations as risk measures, European Journal of Operational Research, 116 (1999), pp. 33–50.

[18] M.L. Puterman, Markov Decision Processes: Discrete Stochastic Dynamic Programming, John Wiley & Sons, Inc., 1994.

[19] A. Ruszczyński, Risk-averse dynamic programming for Markov decision processes, Mathematical Programming, (2010), pp. 1–27.

[20] A. Ruszczyński and A. Shapiro, Conditional risk mappings, Mathematics of Operations Research, 31 (2006), pp. 544–561.
[31] ——, *Optimization of risk measures*, Probabilistic and Randomized Methods for Design under Uncertainty, (2006), pp. 119–157.

[32] A. Schied, H. Föllmer, and S. Weber, *Robust preferences and robust portfolio choice*, Handbook of Numerical Analysis, 15 (2009), pp. 29–87.

[33] Y. Shen, W. Stannat, and K. Obermayer, *Risk-sensitive Markov Control Processes*, SIAM Journal on Control and Optimization, (2013), pp. 3652–3672.