UNIFIED MICROSCOPIC THEORY OF A SYSTEM OF INTERACTING BOSONS

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Abstract

This paper reports the unified microscopic theory of a system of interacting bosons, such as liquid $^4$He. Each particle in the system represents a $(q, -q)$ pair moving with a centre of mass momentum $K$. Particles form bound pairs below $\lambda$-point and have a kind of collective binding between them. The binding is identified as an energy gap between the superfluid and normal states of the system. The $\lambda$-transition is a consequence of inter-particle quantum correlations. It follows an order-disorder of particles in their phase space as well as the onset of Bose Einstein condensation in the state of $q = \pi/d$ and $K = 0$. In addition to the well known modes of collective motions such as phonons, rotons, maxons, etc., the superfluid state also exhibits a new kind of quantum quasi-particle, $omon$, characterised by a phononlike wave of the oscillations of momentum coordinates of the particles. The theory explains the properties of $He - II$ at quantitative level and vindicates two fluid theory of Landau. The paper, finally, describes the way this theory could help in understanding the superfluidity of 1-D and 2-D systems. It also analyses the possibility of applying this

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approach to develop similar framework for a fermion system including an atomic nucleus.
1. INTRODUCTION

A system of interacting bosons, such as Liquid $^4$He, forms an important subject of study for its low temperature ($T$) behaviour. Liquid $^4$He transforms from its normal (N) phase ($He - I$) to superfluid (S) phase ($He - II$) at $T_{\lambda} = 2.17K$; the latter shows zero viscosity ($\eta = 0$) when it flows through narrow channels. The phenomenon has been investigated extensively because it provides a unique opportunity to study quantum behaviour of a system at macroscopic level. Different aspects of the subject have been reviewed in several articles and books, e.g. [1-8]. Widely varying ideas and mathematical tools have been used to develop possible microscopic theories of the system but the desired theory, that explains the properties of liquid $^4$He, could not emerge. As per the contemporary belief superfluidity arises due to the existence of zero momentum ($p = 0$) condensate, $n(o)$, representing a macroscopically large fraction of particles having Bose Einstein Condensation (BEC) in a single particle state of $p = 0$. However, this premise could also not help in achieving the goal. Analysis of certain experimental results reveals $n(o) \approx 0.12$ [6] but not with unquestionable certainty. This motivated us to develop the present theory by using an entirely new approach to the problem. Here we present the salient aspects of our theory [9,10].

2. BASIC ASPECTS OF THEORY

2.1 The System

The Hamiltonian of the system can be expressed as

$$H(N) = -\frac{\hbar^2}{2m} \sum_i \nabla_i^2 + \sum_{i<j} V_{ij}(r = |r_i - r_j|).$$

(1)

where notations have their usual meaning. To a good approximation $V_{ij}(r)$ is the sum of: (i) the hard core (HC) repulsion (i.e. $V_{ij}(r < \sigma) = \infty$ and $V_{ij}(r \geq \sigma) = 0$ with $\sigma =$ HC diameter), and (ii) a relatively long range weak attraction. Since the latter can be replaced by a constant negative external potential, we have only the HC to deal with. We note that two particles of momenta $k_1$ and $k_2$ (as seen in the laboratory frame) see each other as particles of equal and opposite momenta of the value, $k = |k_1 - k_2|$. In fact in a frame attached to their centre of mass (CM), their momenta are found to be $q$ and $-q$ with $q$
\[ \frac{k}{2}. \text{ Naturally, a state function of the system should be in conformity with this basic fact. To this effect we first examine the wave mechanics of two HC particles.} \]

2.2 Wave Mechanics of Two HC Particles

In the CM coordinate system a pair of HC particles is described by

\[ \left[ \frac{\hbar^2}{4m} \nabla_R^2 + \frac{\hbar^2}{2m} \nabla_r^2 + V_{HC}(r) \right] \Psi(R,r) = E \Psi(R,r). \tag{2} \]

The notations \( R \), and \( r \) have their usual meaning through \( R = \frac{r_i + r_j}{2} \), and \( r = \frac{r_i - r_j}{} \). \( \Psi(R,r) \) has a form \( \psi_k(r) \exp(i\mathbf{K}.R) \) with \( \mathbf{K} \) being the CM momentum. The form of \( \psi_k(r) \) describing the relative motion of two particles, depends on the way we deal with \( V_{HC}(r) \) in solving Eqn.(2). While other studies \[11\] use a boundary condition, i.e. \( \psi_k(r < \sigma) = 0 \) and \( \psi_k(r \geq \sigma) \neq 0 \) or its equivalent (e.g. Jastrow type correlation \[12\]), we use a new condition. Accordingly, the separation \( d \) between two HC particles should satisfy \( d \geq \lambda/2 \) (with \( \lambda = 2\pi/q \)). In this context, we note that a particle in wave mechanics manifests itself as a wave packet (WP) of size \( \lambda/2 \); as two HC particles do not overlap, their WPs should do likewise. If two particles, somehow, have \( \lambda/2 > d \) they experience mutual repulsion until increased \( d \) satisfies \( d = \lambda/2 \); in case the system does not allow desired increase in \( d \), particles would absorb necessary energy from the interacting surroundings so that \( \lambda/2 \) gets squeezed to satisfy this condition. Since two particles, in a physically possible state, always have \( r \geq \sigma \), i.e. \( V_{HC}(r) = 0 \), the solutions, \( \Psi(R,r) \), of Eqn.(2) take the form of \( U^\pm(R,r) \equiv U^\pm \). We have

\[ U^\pm = (\sqrt{2}).\cos[(\alpha + \mathbf{k}.r)/2].\exp[i\mathbf{K}.R].\exp[-i(\varepsilon(K) + \varepsilon(k))t/\hbar] \tag{3} \]

with \( E = \varepsilon(K) + \varepsilon(k) = \hbar^2[K^2 + k^2]/4m = \hbar^2[k_1^2 + k_2^2]/2m \), \( \alpha = 0 \) for \( U^+ \) and \( \alpha = \pi \) for \( U^- \). A critical analysis of \( U^\pm \) reveals that : (a). The probability that two particles have a phase separation \( \phi = \mathbf{k}.r \) is

\[ |U^\pm|^2 = [1 + \cos(\alpha + \mathbf{k}.r)] = [1 + \cos \phi'] = |\psi_k(r)^\pm|^2 \tag{4} \]

Since \( |U^\pm|^2 \) is independent of \( \mathbf{K} \), the relative configuration of the pair can be derived by using \( \mathbf{K} = 0 \), i.e. \( \mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{q} \), without loss of generality. Eqn.(3) gives \( \psi_k(r)^\pm \equiv \psi^\pm = \sqrt{2}.\cos[(\alpha + \mathbf{k}.r)/2].\exp[-i.\varepsilon(k)t/\hbar] \) that represents a \( (\mathbf{q}, -\mathbf{q}) \) pair. As \( \psi^\pm \) is a kind of stationary matter wave (SMW) that modulates the probability of finding two particles in \( \phi \) space, the \( (\mathbf{q}, -\mathbf{q}) \)
pair can be known as SMW pair. Note that $\psi^+$ and $\psi^-$ differ only in the locations of the origin $\phi' = 0$. For $\psi^+$, it is located at $\phi = 0$, -the central point of an antinodal region (AR), while for $\psi^-$ at $\phi = \pi$ (the nodal point) of a SMW. Obviously, $\psi^+$ and $\psi^-$ are equivalent. (b). Each AR of $\psi^\pm$ is of $\lambda/2$ size and $< r >$ of a particle evaluted over an AR is found to be its central point. This gives $< r_1 > - < r_2 > = \lambda/2$ if we assume that each AR is occupied, exclusively, by one particle. The fact, that this configuration ensures no overlap of two WPs, implies that particles satisfying $\lambda/2 \leq d$ render $< V_{HC}(r) >= 0$. (c). Since each particle of the pair is a representative of $U^\pm$ state it needs to be expressed by $U^\pm$. In this context we note that a particle in a closed enclosure has two fields: $u_{k'}(r') = (1/\sqrt{V}) \exp(ik'.r')$ and $u_{k''}(r'') = (1/\sqrt{V}). \exp(ik''.r'')$; the latter is the reflection of the former.

To account for the reflected fields, we arrange

$$H(2) = -\frac{1}{2} \sum_{j=1}^{2} \frac{\hbar^2}{2m} \left[\nabla_{r'_j}^2 + \nabla_{r''_j}^2 \right] = -\frac{1}{2} \sum_{j=1}^{2} \left[\frac{\hbar^2}{4m} \nabla_{R_j}^2 + \frac{\hbar^2}{m} \nabla_{r_j}^2 \right]. \quad (5)$$

A $U^\pm$ of j-th particle can be the superposition of any two of the four $u_{k}(r)$. The choices are :$(1'1'')$, $(1'2'')$, $(1''2')$, $(2'1'')$, $(2'2'')$, and $(1''2'')$ with numbers referring to particles in superposition. The possible state functions rendered by these $U^\pm$ are: $\Psi^{(1)}_2 = (1'1'')(2'2'')$, $\Psi^{(2)}_2 = (1'2'')(2'1'')$ and $\Psi^{(3)}_2 = (1'2')(1''2'')$. (d). A pair waveform such as $(1', 1'')$ is a case of self superposition (SS) of a particle where $V_{HC}(r)$ does not operate; naturally, $U^+$ can be used rightly to represent the situation. But the other cases such as $(1', 2')$, $(1', 2'')$, etc. represent mutual superposition (MS) of two particles. The situation is best represented by $U^-$ because the waveform is expected to vanish at $r = 0$. However, in a state of wave mechanical superposition (i.e. $\lambda \geq 2\sigma$), there is no way to determine whether two particles have SS or MS. But then we use either $U^+$ or $U^-$ for all particles to keep a single frame of observation. Use of $U^-$ for two HC bosons is consistent with the fact that such bosons do not have common $r$ coordinates and they behave like fermions in this respect. In momentum space, however, while any number of bosons can have equal $K$, two identical fermions have different values of $K$.

2.3 State Functions of $N$ HC Particles

Following the arguments of Section (2.2) for taking care of $V_{HC}(r)$, we can express $H(N)$ the way Eqn.(5) expresses $H(2)$, and describe each particle by
a pair waveform $U^\pm$. For $N$ particles, we have $2N$ plane waves and $N(2N-1)$ different $U^\pm$ rendering $S = 1.3.5...(2N-1)$ different $\Psi_N$ (state functions of equal $E$). Using $E(K) = \sum_i^N \varepsilon(K)_i$ and $E(k) = \sum_i^N \varepsilon(k)_i$, we have

$$\Psi_N = \phi_N(q) \phi_N(K).$$

$$\phi_N(q) = \left[ \left( \frac{2}{\sqrt{N}} \right)^N \prod_{j=1}^N \cos(\alpha + 2q_j \cdot r_j)/2 \right] \exp[-iE(k)t/\hbar].$$

$$\phi_N(K) = \left[ A \left( \frac{1}{\sqrt{N}} \right)^N \sum_{pK}^N (\pm 1)^p \prod_{j=1}^N \exp[i(K_j \cdot R_j)] \right] \exp[-iE(K)t/\hbar]$$

with $A = \sqrt{1/N}!$. Here $\sum_{pK}^N (\pm 1)^p$ refers to the sum of different permutations of $K$ over all particles. While the choice of $(+1)^p$ or $(-1)^p$ in Eqn.(6b) distinguishes between the systems of bosons and fermions, use of the restriction $q_j \geq \pi/d$ in Eqn.(6a) treats the fermion type behaviour due to HC nature of bosons and fermions alike. $S$ different $\Psi_N$ counted above take care of the permutation of $k$. We have

$$\Phi_N = \frac{1}{\sqrt{S}} \sum_i^S \Psi_N^{(i)}.$$

which represents the general form of a state function that should reveal the physics of the system. Note that $\Phi_N$ represents a state where each particle (as a WP of size $= \pi/q$) has a plane wave motion of momentum $K$ and in conformity with our expectation (cf. Section 2.1) truly has $(q, -q)$ configuration. As such a particle has two motions: (i) the $q$ motion of energy $\varepsilon(q) = \hbar^2 q^2 / 2m$, and (ii) the $K$ motion of energy $\varepsilon(K) = \hbar^2 K^2 / 2(4m)$.

### 2.4 Ground State Energy

The condition $\lambda/2 \leq d$ implies that a spherical volume of diameter $\lambda/2$ belongs exclusively to an HC particle of $\lambda/2 > \sigma$. We also note that each particle in the ground state has lowest possible energy, i.e. largest possible $\lambda/2$, the net ground state energy of the system should be

$$E_o = \sum_i^N \frac{\hbar^2}{8mV_i^{2/3}} \quad \text{and} \quad \sum_i^N v_i = V \quad \text{(constant)}$$

$$=$$
if particles are assumed to occupy different $v_i$. Simple algebra reveals that $E_o$ has its minimum value for $v_1 = v_2 = \ldots v_N = V/N$. Obviously,

$$E_o = N\hbar^2/8md^2 = N\varepsilon_o$$

(9)

In sharp contrast with $E_o$ obtained from conventional theories [11, 12] our $E_o$ does not depend on $\sigma$. The accuracy of this aspect of our result is well evident because two particles of $\lambda/2 > \sigma$ cannot resolve the HC structure within the larger size WPs of each other. The belief in such possibility would contradict the basic principle of image resolution.

2.5 Evolution of the System with Decreasing $T$

For constant particle density ($d - \lambda/2$) decreases with decreasing $T$. In the process at certain $T = T_c$, when $d - \lambda/2$ vanishes at large, $q$ motions get freezed into zero point motions of $q = q_o = \pi/d$. Evidently, the system moves from a state of $\lambda/2 \leq d$ to that of $\lambda/2 = d$. While the former state of $\phi(=2qd) \geq 2\pi$ represents randomness of $\phi$ positions and, therefore, a disorder in $\phi$ space, the latter of $\phi = 2\pi$ defines an ordered state. Thus the particles in the system move from their disordered to ordered state in $\phi$ space at $T_c$. With all $q_j = q_o$, different $\Psi^{(i)}_N$ of Eqn.(7) become identical and $\Phi_N$ attains the form of a single $\Psi_N$ (cf. Eqn.6). As such all the $S$ microstates merge into one at $T_c$ indicating that the entire system attains a kind of oneness [13]; the system at $T \leq T_c$ is, therefore, described by

$$\Phi_N(S) = \phi^o_N(q_o).\phi^c_N(K)$$

(10)

obtained by replacing all $q_j, r_j$ in Eqn.(6) by $q_o, r$ as for given $r$ and $q, r = \pi$ (i.e. $r \cos \theta = \lambda/2$), the lowest energy configuration demands $\theta = 0$.

3. QUANTUM CORRELATION POTENTIAL

The interparticle quantum correlation potential (QCP), originating from the wave nature of particles, is obtained by comparing the partition function (under the quantum limits of the system), $Z_q = \sum_n \exp(-E_n/k_BT).|\Phi_n|^2$ and its classical equivalent, $Z_c = \sum_n \exp(-E_n/k_BT).\exp(-U_n/k_BT)$. Here $\Phi_n$ is a $\Phi_N(S)$ (Eqn.(10)) of $n$-th state. The procedure is justified because our theory describes the system by symmetrised plane waves and our assumption that only one particle occupies a single AR of these SMWs screens out the HC potential. Simplifying $U_n$, one easily finds that pairwise QCP has two
components [9]. The $U_{ij}^s$ pertaining to $k$ motion controls the $\phi = kr$ position of a particle and we have
\[
U_{ij}^s = -k_B T_o \ln[2 \cos^2(\phi'/2)] \quad \text{with} \quad \phi' = (\alpha + \phi), \quad (11a)
\]
where $T$ has been replaced by $T_o$ because $T$ equivalent of $k$ motion energy at all $T \leq T_\lambda$ is $T_o$. $U_{ij}^s$ has its minimum value ($-k_B T_o \ln 2$) at $\phi' = 2n\pi$ and maximum value ($= \infty$) at $\phi' = (2n+1)\pi$ occurring periodically at $\Delta \phi = 2n\pi$ (with $n = 1, 2, 3, \ldots$). Since $U_{ij}^s$ increases as $\frac{1}{2} C (\delta \phi)^2$ (with $C = \frac{1}{2} k_B T_o$) for small change $\delta \phi$ in $\phi$ around a point of its minimum, it generates a force $= -C \delta \phi$ which tries to maintain $\delta \phi = 0$ and hence the order of particles in $\phi$ space. Since $U_{ij}^s$ is not a real interaction, in general, it cannot manipulate $d$.

The $\phi$-space order is, therefore, achieved by driving all $q$ towards $q_o$. In the $S$-phase, however, the interdependence of $r$ and $q$ through $\phi = 2qr = 2n\pi$ renders $V_{ij}$ a function of $\phi$ and $q$ values. More accurately (cf. Section 5.4), particles in $S$-state acquire self energy (a kind of additional potential energy) that depends on $q$ values. Evidently, $V_{ij}$ in $S$ state can be replaced by $U_{ij}^s$.

The second component pertaining to $K$ motion is expressed by
\[
U_{ij} = -k_B T \ln \left[1 + \exp \left(\frac{-2\pi |\mathbf{R}' - \mathbf{R}''|^2}{\lambda_T^2}\right)\right], \quad (11b)
\]
with $\lambda_T = \hbar / \sqrt{2\pi(4m)k_B T}$. $U_{ij}$ may be seen as the origin of a force that facilitates BEC in the state of $K = 0$ by driving particles in $K$ space towards $K = 0$ where it has its minimum value ($-k_B T_o \ln 2$).

4. THE TRANSITION

To a good approximation, the lower bound of $T_c$ is $T_o$ (the $T$ equivalent of $\varepsilon_o$ or that of $\lambda = 2d$). This gives $T_c = T_o = \hbar^2 / 8\pi mk_B d^2$, by using $\lambda_T = \hbar / \sqrt{(2\pi mk_B T)}$. To reveal the real $T_c = T_\lambda$, we note that with $T$ moving below $T_\lambda$ particles not only have $q = q_o$, but also start attaining $K = 0$. Evidently, the $\lambda$-transition follows two processes simultaneously: (i) an order-disorder of particles in $\phi$ space rendering $q = q_o$ and (ii) the BEC of particles (as SMW pairs) driving them towards $K = 0$. In view of these aspects it is evident that $\lambda$-transition is a second order transition. This also gives
\[
T_\lambda = T_o + \frac{1}{4} T_{BEC} = \frac{\hbar^2}{8\pi mk_B} \left[\frac{1}{d^2} + \left(\frac{N}{2.61V}\right)^{\frac{2}{3}}\right] \quad (12)
\]
where $T_{\text{BEC}}$ is usual BEC temperature [4]. The factor of $\frac{1}{4}$ appears because the plane wave $K$ motion of a particle has $\hbar^2 K^2/2(4m)$ energy and $T_{\text{BEC}}$ varies as $\frac{1}{m}$. The sharpness of the transition is well evident.

5. PROPERTIES

5.1 Configuration of Particles

Our key results, $\Delta\phi = 2n\pi$ and $q = q_o$, revealing uniformly equal $d$, define $\phi$, $q$, and $r$ space configurations. The system can be identified as a close packed arrangement of WPs. The packing leaves no freedom for particles to move across each other. They can move in the order of their locations maintaining $\Delta\phi = 2n\pi$. Naturally, the motion has coherence.

Since $U^\phi_{ij}$ restores $\phi = 2n\pi$ and the system is a close packed arrangement of WPs, we visualise waves of $\phi$ oscillations. To reveal their frequency dispersion, $\omega_\phi(Q)$, we consider a linear chain of atoms and only nearest neighbour interactions as the responsible forces. We have

$$\omega_\phi(Q) = \sqrt{(4C)/\beta.|\sin(Qd/2)|}$$  \hspace{1cm} (13)

where $Q$ is the wave vector and $\beta$ is the measure of inertia for $\phi$ motion. However, $\phi$ oscillations can appear as the oscillations of $r$ and $q$ because $\delta\phi = 2q.d_r + 2\delta q.r$. We have phonons when $q = q_o$, and omons (a new kind of quantum quasiparticle representing a phononlike wave of the oscillations of momentum) when $r = d$. We note that a system like liquid $^4\text{He}$ is expected to exhibit: (i) no transverse mode because the shear forces between particles are negligibly small, and (ii) only one branch of longitudinal mode because the system is isotropic. Evidently, $\omega_r(Q)$ of phonons can be represented, to a good approximation, by the dispersion of the elastic waves in a chain of identical atoms and it can be obtained from Eqn.(13) by replacing $\beta$ by $m$ and $C$ by $C \left[= 4\pi^2.C/d^2 = 2\pi^2.k_BT_o/d^2\right]$. However, for a better accuracy $d$ and $C$ should, respectively, be considered descending and ascending functions of $Q$ because increase in the energy of particles affected by an excitation reduces WP size and this renders a decrease in $d$ and an increase in $C$. As such $\omega_r(Q)$ should be expressed as

$$\omega_r(Q) = \sqrt{4C(Q)/m.|\sin(Qd(Q)/2)|}$$  \hspace{1cm} (14)

5.2 Thermal Excitations
which also reveals that the phase and group velocities of low Q modes are

\[ v_p = v_g = \sqrt{\pi \hbar/2md} \]  

(15)

The energy dispersion \( E(Q) \) should be phononlike till the excitation wavelength \( \Lambda > d \). However, since the momentum and energy of an excitation of \( \Lambda < \sigma \) is shared only by a single atom, we expect \( E(Q) = \hbar^2 Q^2/2m^* \) (with \( m^* \) = effective mass of an atom). Obviously, the transition of \( E(Q) \) from phonon-like curve to \( E(Q) = \hbar^2 Q^2/2m^* \) takes place over the range, \( 2\pi/d < Q < 2\pi/\sigma \). Evidently, \( E(Q) \) does not touch the Q-axis at any \( Q \neq 0 \). Its maxon point lies at \( \approx \pi/\sigma \) and the roton minimum around the middle point of \( 2\pi/d \) and \( 2\pi/\sigma \). As a whole \( E(Q) \) represents Landau type spectrum. The \( E(Q) \) of N-phase is, obviously, expected to differ from this spectrum. To obtain \( \omega_q(Q) \) of omons, we note that the equation of motion of \( r_s \) (the \( r \) of \( s \)-th atom), i.e., \( \partial_t^2 r_s = -\frac{1}{4}\omega_o^2[2r_s - r_{s-1} - r_{s+1}] \), transforms into a similar equation for \( p_s = hq_s \) by operating \( m.\partial_t \). This reveals \( \omega_r(Q) = \omega_q(Q) \). As concluded in Section 5.4, an omon is an anti-phonon quantum quasiparticle. The anomalous behaviour of \( \omega_r(Q) \) at low \( Q \) arises due to \( Q \) dependence of \( C \) and \( d \). Using this observation to explain the anomalous nature of \( E(Q) \) of \( He-II \) [14], we found that moderately \( Q \) dependent \( d \) and \( C \) become nearly \( Q \) independent when \( Q \) approaches \( \pi/\sigma \). It seems obvious because \( d \neq \sigma \). Considering \( \psi = \sum_i f(r_i)\phi \) as the wave function of an excited state and \( \phi \) that of ground state, Feynman [15] showed that the excited state energy is minimum for \( f(r_i) = \exp(iK.r_i) \) and \( E(Q) = \hbar^2 Q^2/2mS(Q) \) (with \( S(Q) = \) structure factor); this \( E(Q) \) for \( He-II \) is, however, found to be about two times of the experimental value. Feynman and Cohen [16] later obtained better results but with considerable discrepancy at higher \( Q \). Since our state function (cf. Eqn.(10) and (6)) satisfies all considerations of Feynman, we obtained \( E(Q) = h^2 Q^2/4mS(Q) \) by using the fact that a SMW pair (rather than a single particle) forms the basic unit of the system until \( Q > 2\pi/\sigma \). This explains the observed \( E(Q) \) for \( He-II \) to a good accuracy [14].

5.3 Two Fluid Behaviour

Since \( \phi_N^q(q_o) \) and \( \phi_N^K(K) \) deal separately with \( q \) and \( K \) motions of particles they represent two different components in the system. The component \( \phi_N^q(q_o) \) representing particles in their ground state, obviously, has zero entropy (S=0); this also has \( \eta = 0 \) because particles are constrained to move in
the order of their locations (cf. Section 5.1). In the rotating fluid, however, particles moving on the neighbouring cocentric circular paths have relative velocity and the system shows its natural viscous behaviour. The excitations are the effects that can propagate from one end of the system to the other against the closely packed WPs in the background. They, obviously, form a kind of gas (as envisaged by Landau [17]) that accounts for the total S and other thermal properties of the system. They also render a $\eta \neq 0$ because their effects can lead to frictional movement of particles. As such $\phi^\circ_N(q_o)$ has the basic properties of S-fluid, while $\phi^\circ_N(K)$ has those of N-fluid; however, these fluids are inseparable since each particle participates in $\phi^\circ_N(q_o)$ as well as $\phi^\circ_N(K)$. All these aspects vindicate two fluid theory of Landau [17].

5.4 Energy Gap and Self Energy

With $T$ moving below $T_\lambda$, the WPs of neighbouring atoms have an overlap because $\lambda/2$ tends to increase by pushing particles to have increased $d$ against interparticle attraction. This perturbs $\psi^\pm$ and its energy $\varepsilon_o$. Considering a pair of particles and following the standard method dealing with similar situation (e.g. the formation of a molecular orbital from two identical atomic orbitals), we find that the energy of perturbed state becomes $\varepsilon_o \pm |v|$; here $|v|$ is the expectation value of interatomic attraction. $|v|$ could better be replaced by $|v(T)|$ as the overlap may depend on $T$. The state of lower energy ($\varepsilon_o - |v(T)|$) represents a kind of bonding (or paired) state while that of $\varepsilon_o + |v(T)|$ an antibonding (or unpaired) state. The lower energy state can thus be regarded as a state of bound pairs. Since this happens to all particles, their ground state energy falls from $N\varepsilon_o(T_\lambda)$ at $T = T_\lambda$ to a new value $N\varepsilon_o(T) = N\varepsilon_o(T_\lambda) - N|v_N(T)| = N\varepsilon_o(T_\lambda) - E_g(T)$ at $T < T_\lambda$; note that $|v_N(T)|$ may differ from $|v(T)|$ obtained for an isolated pair. This gives

$$E_g(T) = N[\varepsilon_o(T_\lambda) - \varepsilon_o(T)] \approx N\hbar^2(d_T - d_\lambda)/4md_\lambda^3$$ (16)

$E_g(T)$ can be identified as: (i) an order parameter of the transition, (ii) a collective binding among all particles (clearly different from the binding of two electrons in a Cooper pair) rendering the system to behave like a single molecule [18], and (iii) an energy gap between S and N phases in a sense that S phase becomes N phase if $E_g(T)$ energy is supplied from outside. The system retains fluidity since $|v_N(T)| << \varepsilon_o$. To determine the factor that controls the overlap of neighbouring WPs and hence the value of
$E_g(T)$, we note that the increased size of WPs in the bonding state forces the system to expand. This requires energy (to be managed from within the system) to work against inter-atomic attraction. In this context we find that the system at $T_\lambda$ has thermally excited $N^*(T_\lambda)$ particles devoid of quantum correlation for their $\Lambda$ being $< \sigma$. The transition of such particles to their ground state releases an additional $\Delta \epsilon(T) = [N^*(T_\lambda) - N^*(T)]k_BT_o\ln 2$ energy when the system is cooled from $T_\lambda$ to $T$. This is because the lower energy level of the transition lies below the zero line of the kinetic energy by $-k_BT_o\ln 2$. Evidently, the desired controlling factor is $\Delta \epsilon(T)$ energy available for the expansion. The process of expansion pushes the particles to their higher potential energy rendering a net increase by $\Delta V_s(T)$ termed as self energy. Naturally, $\Delta V_s(T) = \Delta \epsilon(T)$. In this state of new equilibrium the zero-point repulsion ($= \hbar^2/4md^2$) equalises with interparticle attraction. This gives $\Delta V_s(T) = |E_g(T)|$. We note that: (i) $\Delta V_s(T)$ is a function of $q$ because it depends on the size of WPs, (ii) it provides a basis for the propagation of omons in the S-phase, and (iii) it can be recognised as the energy of omon field. Since $\Delta V_s(T)$ increases with decreasing $T$, it implies that omon field intensity increases when phonon field intensity decreases and vice versa. Evidently, $\Delta V_s(T)$ could either be identified as the energy of phonon field absorbed by the particles or the omon is considered as an anti-phonon quantum quasi-particle. Since $\Delta V_s(T)$ attains its maximum value at $T = 0$, it can serve as a source of energy for creating collective motions even at $T = 0$.

5.5 Consequences of Energy Gap

If two heads $X$ and $Y$ in the system have small $T$ and $P$ (pressure) differences, the equation of state is $E_g(X) = E_g(Y) + S\Delta T - V\Delta P$. Using $E_g(X) = E_g(Y)$ for equilibrium, we get

$$S\Delta T = V\Delta P$$

(17)

This reveals that the system should exhibit thermo-mechanical and mechano-caloric effects. This also concludes that measurement of $\eta$ by capillary flow method performed under the condition $\Delta T = 0$ and that of thermal conductivity ($\Theta$) performed under $\Delta P = 0$ should reveal $\eta = 0$ and $\Theta \approx \infty$. This explains why $He - II$ is a superfluid of infinitely high $\Theta$. Equating $E_g(T)$
with the energy of flowing system we obtain the upper bound of critical velocity for which the S-phase becomes N-phase. We have

\[ v_c(T) = \sqrt{\frac{2E_g(T)}{Nm}} \] (18)

The lower \( v_c \), at which the superfluid may show signs of viscous behaviour, arises due to creation of quantized vortices, but this does not destroy superfluidity. Similarly, we find the following relation for coherence length (not to be confused with healing length [4]),

\[ \xi = \hbar \sqrt{\frac{N}{2mE_g(T)}} \] (19)

Correlating the superfluid density, \( \rho_s \), as the order parameter of the transition, with \( E_g(T) \) we find a new relation

\[ \rho_s(T) = \frac{E_g(T)}{E_g(0)} \rho \] (20)

to determine \( \rho_s \) and normal density, \( \rho_n = \rho - \rho_s \).

5.6 Quantized Vortices

Using the symmetry property of a state of bosons, Feynman [15] showed that the circulation, \( \kappa \), of the velocity field should be quantized, i.e., \( \kappa = \frac{n \hbar}{m} \) with \( n = 1, 2, 3, ... \) However, Wilks [1] has rightly pointed out that this account does not explain the fact that \( He - I \) to which Feynman’s argument applies equally well, does not exhibit quantized vortices. We find that particles in S-phase maintain phase coherence, not existing in N-phase, and for this reason the quantized vortices can be observed in S-phase only.

5.7 Single Particle Density Matrix and ODLRO

Using Eqns.(10), and (6), we have [8] the single particle density matrix,

\[ \rho(R^* - R) = \left[ \frac{N(o)}{V} + \frac{N}{\lambda_T^2} \exp \left( -2\pi \frac{\|R^* - R\|^2}{\lambda_T^2} \right) \right] \cdot \left( \frac{2}{V} \cos^2 \left[ \frac{\pi (r'' - r')}{d} \right] \right) \] (21)

where \( \lambda_T = \hbar / \sqrt{2\pi (4m)k_B T} \) represents thermal wavelength attributed to \( K \) motions. We used \( q_o.(r'' - r')^* = 2n\pi + q_o.(r'' - r') \). The term in big (..) represents the variation of density over a single AR. The fact that under the
limit $|\mathbf{R}' - \mathbf{R}|$ tends to $\infty$, $\rho(\mathbf{R}' - \mathbf{R})$ has nonzero value for $T < T_\lambda$ and zero for $T \geq T_\lambda$ since $N(o)$ (= number of particles representing the SMW pairs of $K = 0$) is $\neq 0$ for $T < T_\lambda$ and $= 0$ for $T \geq T_\lambda$. It clearly proves the existence of off diagonal long range order (ODLRO) in the S-phase of the system.

5.8 Logarithmic Singularity of Specific Heat

The specific heat $C_p(T)$ of the system is expected to show usual cusp at $T_\lambda$ if BEC of SMW pairs is considered as the only mechanism of the transition. But our system at its $\lambda$-point also has an onset of ordering of particles in phase space rendering widely different changes in $\phi'$-position of different particles.

To determine the corresponding change in energy $\Delta \varepsilon$ we, however, assume for simplicity that of the $N^\ast(T_\lambda)$ uncorrelated particles in their excited states, $N_\lambda$ make significant contribution to $\Delta \varepsilon$ and they move from their $\phi' = 2n\pi \pm (\pi - \delta \phi_\lambda)$ at $T_\lambda^+$ (just above $T_\lambda$) to $\phi' = 2n\pi$ at $T_\lambda^-$ (just below $T_\lambda$). This gives

$$\Delta \varepsilon = -N_\lambda k_B T_o \left[ \ln 2 \cos^2 \left( \frac{2n\pi \pm (\pi - \delta \phi_\lambda)}{2} \right) - \ln 2 \right]$$

(22)

Following the theories [19] of critical phenomenon we may define

$$\delta \phi_\lambda = \delta \phi_\lambda(o).|\zeta|^\nu \left[ 1 + a_2.|\zeta|^2 + a_3.|\zeta|^3 \right]$$

(23)

with $\zeta = \frac{T - T_\lambda}{T_\lambda}$. To a good approximation we have

$$\Delta \varepsilon = -N \left( \frac{T - T_\lambda}{T_\lambda} \right).k_B T_o \ln \left( \frac{\delta \phi_\lambda(o).|\zeta|^\nu}{2} \right)^2$$

(24)

by using $\delta \phi_\lambda = \delta \phi_\lambda(o).|\zeta|^\nu$ and $N_\lambda = N.\frac{T - T_\lambda}{T_\lambda}$; the latter expression is so chosen to ensure that $\Delta \varepsilon$ does not diverge at $T_\lambda$ and it decreases with decreasing $T$ through $T_\lambda$. Eqn.(24) gives

$$C_p(T \approx T_\lambda) \approx -\frac{N}{T_\lambda}.k_B T_o.[2\nu. \ln |\zeta| + \ln (\delta \phi_\lambda(o)^2) - \ln 4 + 2\nu]$$

(25)

5.9 Properties of $He – II$

All important conclusions of our theory that the S-state of the system should
exhibit: (i) two fluid behaviour, (ii) negative volume expansion, (iii) \( \eta = 0 \) for its capillary flow, (iv) \( \eta \neq 0 \) in the state of its rotation, (v) infinitely high \( \Theta \), (iv) quantized vortices, (v) coherence of particle motion, etc., are well known properties of He-II. That our theory provides an accurate account of the thermodynamic and hydrodynamical properties of He-II is well evident since its theoretical \( E(Q) \), \( \rho_n(T) \) and \( \rho_s(T) \) [9,14] match closely with experiments. Our estimates [20] of: (i) \( T_\lambda \approx 2.03 \text{K} \), (ii) upper bound \( v_c \) varying smoothly from 0 (at \( T_\lambda \)) to \( \approx 9 \text{m/sec} \) (at \( T = 0 \)), (iii) anomalous variation of \( v_p \) and \( v_g \) and their low Q value \( \approx 227 \text{m/sec} \), (iv) \( \xi \) varying smoothly from \( \approx 10^{-6} \text{cm} \), at \( T = 0 \) to infinitely large value at \( T = T_\lambda \), etc. agree with experiments (see [4] for (i-ii), [6] for (iii), and [21] for (iv)). For the first time our theory explains the origin of the experimentally observed logarithmic singularity in \( C_p(T) \) at \( T_\lambda \). Using the parameters of liquid \(^4\text{He}\) in Eqn.(25), we have

\[
C_p(\text{J/mole.K}) \approx -5.24. \ln |\zeta| - 9.5 = A. \ln |\zeta| + B
\]

(26)

where we use \( \nu = 0.55 \) and \( \delta \phi_\lambda(o) = \pi \), while the experimental results reveal \( A = 5.355 \) and \( B = -7.77 \) for \( T > T_\lambda \) and \( A = 5.1 \) and \( B = 15.52 \) for \( T < T_\lambda \) [22]. The fact that our A value agrees closely with experiments speaks of the accuracy of our theoretical result. With respect to our choice of \( \nu = 0.55 \) and \( \delta \phi_\lambda(o) = \pi \), we note that: (i) \( \delta \phi_\lambda \) originates basically from change in momentum \( \Delta k \approx \xi^{-1} \) and \( \xi \) varies around \( T_\lambda \) as \( |T - T_\lambda|^{-\nu} \); note that critical exponent \( \nu \) lies in the range 0.55 to 0.7. While we have no definite reason for our choice of \( \delta \phi_\lambda(o) = \pi \), however, this is the largest possible value by which phase position of a particle can change. While our value (\( \approx -21 \text{K} \)) of potential energy per \(^4\text{He}\) atom does not differ from others [12], our zero point kinetic energy (\( \approx 3.93 \text{K} \)) is much lower than \( \approx 14 \text{K} \) estimated by others [12]. Evidently, the configuration revealed by our theory is energetically favourable one. Using the values of \( \xi \) and \( E_g(T) \) and following the standard method [23] we obtained the time of persistent currents for He-II as \( \approx 10^{34} \text{sec} \), which is much larger than the life of our universe, i.e. \( \approx 10^{18} \text{sec} \).

5.10 Low Dimensional System

A 1-D (or 2-D) system that could be realised in a laboratory is nothing but a 3-D system having 2 (or 1) dimension(s) reduced to the size of the
order of $d$ (or $\lambda/2$); the $k, K, r$ and $R$, obviously, retain their 3-D character. The only important aspect in which these systems differ from a real 3-D system is that $k$ can be controlled by the smallest dimension of these systems. Our theory, obviously, has no difficulty in explaining the superfluid behaviour of such systems because contrary to the conventional theories presuming particles to have $\lambda \approx \infty$, it presumes $\lambda/2 \leq d$ and finds $\lambda/2 = d$ as the basic condition of superfluidity. While particles in a low dimensional system can not have $\lambda \approx \infty$ (characteristic of $p = 0$ condensate), they can certainly satisfy $\lambda/2 = d$ if the density of particles is high enough to satisfy $d \leq t$ where $t$ represents the diameter of the narrow channel (or the thickness of the film); here $d$ and $t$ are defined by excluding the solidified atomic layers on the surface of the channel or the substrate. If $t < d$, We can only have $\lambda/2 \leq t$ since $\lambda/2$ is now controlled by $t$. Naturally, $\lambda/2$ is always $< d$ and the system would not exhibit superfluidity. For this case we also find that the WPs of neighbouring particles fail to connect each other to develop a macroscopic chain (or 2-D network) of SMWs. Evidently, $d \leq t$ identified as the connectivity condition has to be satisfied by the system for superfluidity. In other words the particle number density for a given $t$ of 1-D (or 2-D) system should be $> t^{-1}$ (or $t^{-2}$). Note that: (i) the surface (in contact with the sample) of channels (or substrates) used to realise 1-D (or 2-D) systems are very rough at the scale of $d$, (ii) the diameter of the channels has fluctuations of the order of $d$, and (iii) particles moving near the substrate surface have increased effective mass because they experience surface potential of attractive nature. Naturally, the $\lambda$- transition in different parts of the sample occurs at different $T_\lambda(d) < T_\lambda$ of the bulk system. Further since the system attains maximum density at $T > T_\lambda$, and superfluidity can be observed only when the process of transition is completed in the entire sample, it is evident that $T$ of: (i) the peaks of maximum density, ($T_m$) and specific heat ($T_{sh}$), and (ii) the onset of superfluidity, ($T_{sf}$) would satisfy $T_m > T_h > T_{sf}$ which agrees with experiments [24]. The locations of these temperatures would also not be so well defined as $T_\lambda$ of the bulk $^4$He. Note that this analysis does not apply to a real case of 1-D (or 2-D) systems in which 2 (or 1) dimension(s) are reduced to zero exactly.

5.11 Similarity with Photons in Laser Beam

We note that the system below $T_\lambda$ defines a 3-D network of SMWs ex-
tending from one end to the other without any discontinuity. In lasers too these are the standing waves of electromagnetic field that modulate the probability of finding a photon at a chosen phase point. The basic difference between the two lies in the number of bosons in a single AR. In case of lasers this could be any number since photons are non-interacting but for the systems like $^4He$ and $^{87}Rb$ one AR can only one atom.

6. CONCLUDING REMARK

Our theory is consistent with the volume exclusion condition [25], as well as (ii) the microscopic and macroscopic uncertainty [4]. It vindicates: (i) the two fluid theory of Landau [17], (ii) London’s idea of macroscopic wave function of the S-state [26], (iii) the observation of Bogoliubov [27] that superfluidity is an interplay of interparticle interactions, etc. The theory, however, does not reveal the existence of $p = 0$ condensate. When the interactions are switched off, a particle does not experience the existence of the other. The states of the system are, obviously, single particle states and its ground state is defined by $K = 0$ and $q_o = \pi/L(\approx 0)$; $L =$ size of the container. Using $q = 0$ alongwith $\alpha = 0$, we note that Eqn.(6) becomes the usual wave function of plane waves. It is evident that our theory has the basic foundation that should help in explaining the behaviour of 1-D and 2-D analogs of our system and that of $^{87}Rb$ [28] like samples. Our preliminary results [14] corroborate this point. In order to extend our formulation to a fermi system, we can use Eqn.(6) by noting that : (i) in view of the condition $\lambda/2 \leq d$ all HC fermions in their ground state can have $q = \pi/d$ and (ii) they need to follow fermi distribution over the possible states of $K$ representing a free particle of mass $2m$ placed in a 3-D box; note that a SMW pair is the basic unit of the system unless the excitation wavelength $\Lambda < \sigma$ in which case the excitations should be single particle excitations. We may require to replace $m$ by effective mass $m^*$ due to interparticle interaction. In case of an atomic nucleus the allowed values of $K$ would correspond to a free particle of mass $2m^*$ kept in a spherical shell; the shell may have some deformations in its shape depending on its volume or related parameters. As such our theory finds no difficulty in identifying an atomic nucleus as a tiny drop of fermi superfluid. In view of our inferences (c.f. Section 5.1) which are applicable HC bosons and fermions alike this drop should represent
a close packing of the WPs of different nucleons, obviously, forbidden to have random motion; they should also have an orderly arrangement in phase space. These new inferences should certainly help in developing a better understanding of the physics of an atomic nucleus. For example the origin of collective oscillations of its nucleons as well as the single particle excitations can be better conceived. Using these observations we are working out the basic aspects of these systems and we hope to report our findings soon. As such we hope that our theory would help in developing a unified model of superfluidity and superconductivity and a better understanding of the physics of atomic nucleus and neutron star.

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20. To a good approximation, the system being a close packed arrangement of WPs could be assigned a symmetry such as hcp, fcc, etc. Naturally, \( d \) can be as large as \( 1.122a \) for fcc arrangement and as small as \( a = (V/N)^{1/3} \) for sc. In our estimates we use \( d = 1.091a \) of bcc arrangement.

21. The \( \xi \) (Eqn. 19) should be related to the spatial separation between two particles in SMW configuration keeping definite phase correlation and it appears to be the origin of: (i) nearly constant thickness \( (\approx 3 \times 10^{-6} \text{ cm}) \) of the film observed in the beaker film flow experiment (see L.J. Campbell in Ref. [8], Chapter 4), (ii) the radius of vortex rings \( (\approx 5 \times 10^{-6} \text{ to } 10^{-4} \text{ cm}) \) observed for vortices created by moving ions (see Ref. [1], Chapter 12), etc.

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