Numerical Simulation of Coupled Fractional Differential-Integral Equations Utilizing the Second Kind Chebyshev Wavelets

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In order to solve coupled fractional differential-integral equations more effectively and to deal with the problem that the huge algebraic equations lead to considerable computational complexity and large data storage requirements in the calculation process, this paper approximates the function of the unknown solution based on the Chebyshev wavelet of the second kind and then combines the collocation method to solve the numerical solution of nonlinear fractional Fredholm integral-differential equations. By using the proposed method, the original problem can be reduced to a system of linear algebraic equations, which can be easily solved by some mathematical techniques. In addition, the convergence analysis of the system based on the second kind of Chebyshev wavelet is studied. Several numerical test problems are presented, and the absolute error values under different fractional orders are given, which proves the superiority and effectiveness of the proposed method. It provides support for improving the precision and reliability of the system.

1. Introduction

Fractional differential-integral belongs to the field of applied mathematics and can handle integrals and derivatives of any order. It has the advantage of being a practical tool for reasonably explaining the memory and genetic quality of various materials and processes. With the development of natural science and social economy, it is found that integer order differential equation cannot be used to solve complex problems, but these problems can be solved by fractional calculus equation [1]. Therefore, differential-integral equation plays a very important role in various problems of natural science and related fields of engineering technology [2]. There are different types of integral equations, such as Fredholm integral equations, Volterra integral equations, and Volterra-Fredholm integral equations, depending on the structure of the integral [3–5]. Nonlinear integral equations have been widely studied in many different disciplines, such as vehicular traffic, biology, optimal control theory, economics. [6, 7]. In this paper, numerical calculations are carried out for a class of system of coupled fractional Fredholm differential-integral equations. The general form of the research question is as follows:

\[
\begin{align*}
D^\alpha_t Y_1 (t) &= \lambda_{11} \int_0^t k_{11} (x, t) [Y_1 (x)]^p \, dx + \lambda_{12} \int_0^t k_{12} (x, t) [Y_2 (x)]^q \, dx + g_1 (t) \\
D^\beta_t Y_2 (t) &= \lambda_{21} \int_0^t k_{21} (x, t) [Y_1 (x)]^p \, dx + \lambda_{22} \int_0^t k_{22} (x, t) [Y_2 (x)]^q \, dx + g_2 (t)
\end{align*}
\]
where \( p_1, p_2, q_1, q_2 > 1 \), and the initial conditions are given as follows:

\[
Y_1^{(i)}(0) = \delta_i, i = 0, 1, \ldots, r - 1, \quad r - 1 < \alpha < r, r \in \mathbb{N}^*,
\]

\[
Y_2^{(j)}(0) = \gamma_j, j = 0, 1, \ldots, m - 1, \quad m - 1 < \beta < m, m \in \mathbb{N}^*.
\]

(2)

where \( g_1, g_2 \in L^2([0,1]), k_{11}, k_{12}, k_{21}, k_{22} \in L^2([0,1] \times [0,1]) \) are given functions, \( Y_1(t), Y_2(t) \) are the solutions to be determined, \( D^\alpha_0 Y_1, D^\beta_0 Y_2 \) refer to fractional derivatives in the Caputo sense, and \( p, q \) are positive integers.

In recent years, different types of numerical methods have been proposed for solving fractional differential and integral equations. These methods mainly include the variation of parameters method [8], ADM [9, 10], VIM and HPM [11], CAS wavelet [12], and Taylor expansion method [13]. The wavelet methods stand out among these methods and have been adopted in various scientific or engineering applications. The wavelet methods make the accurate representation of different functions and operators and the establishment of the connection with fast numerical algorithms possible [14]. Wavelet methods, including CAS wavelet [15], Legendre wavelet [16], Haar wavelet [17], Chebyshev wavelet [18], and others, are widely used in solving linear or nonlinear differential equations, integral equations, and integro-differential equations. In [12], the numerical method of CAS wavelet is applied to solve the fractional integral and differential equations, and the numerical results are compared with the analytical results. In [19], the second kind of Chebyshev wavelet method is utilized to obtain the numerical solutions of fractional nonlinear Fredholm integral-differential equations. In recent years, wavelet methods develop rapidly [20–22]. At present, some scholars have done in-depth research on the time delay and the existence and controllability of equations for fractional differential-integral system [23–27]. In view of the problem, the huge algebraic equations will lead to considerable computational complexity and large data storage requirements in the calculation process. Because of the structural redundancy of the second kind of Chebyshev wavelet, the computational complexity of the algebraic system can be reduced. In order to more effectively solve the coupled fractional differential-integral equations [28, 29], the second kind of Chebyshev wavelet method is introduced in this paper, and the original problem can be reduced to a system of linear algebraic equations, which can be easily solved by some mathematical techniques. In the study, the main system of fractional Fredholm integral-differential equations is discussed in detail by the second kind Chebyshev wavelet, and the convergence analysis of the system is investigated.

The main structure of the article is as follows: in Section 2, some necessary definitions of the fractional integral-differential are introduced. In Section 3, the second kind Chebyshev wavelet and its operational matrix of the fractional integration are derived. In Section 4, the main computing steps are described in detail. In Section 5, the convergence analysis of the system is investigated. In Section 6, the effectiveness of the proposed method is verified by several test problems. Finally, a conclusion is drawn in Section 7.

2. Fractional Calculus

In the section, some necessary definitions and mathematical preliminaries about the fractional integral-differential theory are given [30, 31].

**Definition 1.** Riemann-Liouville fractional integral operator \( \mathcal{I}^\alpha \) of order \( \alpha \) defined in the region \([a,b]\) is given by

\[
(\mathcal{I}^\alpha Y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha - 1} Y(\tau) d\tau, \quad \alpha > 0,
\]

(3)

**Definition 2.** Caputo definition of a fractional differential operator is given by

\[
(D^\alpha Y)(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t (t - \tau)^{n-\alpha - 1} Y^{(n)}(\tau) d\tau,
\]

(4)

\[n - 1 < \alpha \leq n, n \in \mathbb{N}^*.
\]

Caputo fractional differential operator has some basic properties:

\[(D^\alpha \mathcal{I}^\alpha Y)(t) = Y(t),\]

(5)

and

\[(\mathcal{I}^\alpha D^\alpha Y)(t) = Y(t) - \sum_{k=0}^{n-1} Y^{(k)}(0^+) \frac{(t-a)^k}{k!}, t > 0.\]

(6)

3. The Second Kind Chebyshev Wavelet

3.1. Definition. The second kind Chebyshev wavelet \( \psi_{mn}(t) = \psi(k,n,m,t) \) has four arguments, \( n = 1,2,\ldots,2^k-1, k \in \mathbb{N}^* \). \( t \) is the normalized time. They are defined on the interval \([0,1]\) as follows [32, 33]:

\[
\psi_{mn}(t) = \begin{cases} 2^{k/2} \bar{U}_m(2^k t - 2n + 1), & 0 \leq t < \frac{n}{2^k-1} \\ 0, & \text{a.w.} \end{cases}
\]

(7)

with

\[
\bar{U}_m(t) = \sqrt{n} U_m(t), m = 0, 1, 2, \ldots, M - 1.
\]

(8)

Here, \( U_m(t) \) is the second kind Chebyshev polynomials with the weight function \( w_n(t) = \sqrt{1-t^2} \) and satisfies the following recursive formula

\[U_0(t) = 1, U_1(t) = 2t, U_{m+1}(t) = 2tU_m(t) - U_{m-1}(t), m = 1, 2, \ldots.\]

A function \( Y(t) \) defined over \( L^2_{w_n}([0,1]) \) may be expanded as
proximated as

\[ Y(t) = \sum_{m=0}^{\infty} \sum_{n=0}^{m} Y_{nm}(t) = Y^T \Psi(t) = \bar{Y}(t), \]

where

\[
\begin{align*}
Y &= \begin{bmatrix} Y_{10} & \cdots & Y_{1(m-1)} & \cdots & Y_{12} & \cdots & Y_{12(m-1)} \end{bmatrix}^T \\
\Psi(t) &= \begin{bmatrix} \psi_{10}(t) & \cdots & \psi_{11}(t) & \cdots & \psi_{12}(t) & \cdots & \psi_{12(m-1)} \end{bmatrix}^T
\end{align*}
\]

Taking the collocation points as follows:

\[ t_i = \frac{2i - 1}{2^M}, i = 1, 2, \ldots, 2^{k-1}M, \bar{m} = 2^{k-1}M. \]

We define the second kind Chebyshev wavelet matrix \( \Phi_{mom} \) as

\[ \Phi_{mom} = \begin{bmatrix} \psi_{10} & \cdots & \psi_{1(m-1)} & \cdots & \psi_{12} & \cdots & \psi_{12(m-1)} \end{bmatrix}. \]

Any function \( k(x,t) \in L^2_w([0,1] \times [0,1]) \) can be approximated as

\[ k_{ij}(x,t) = \psi_i(x) \bar{K}_{ij} \psi_j(t), \quad i,j = 1,2, \]

where \( \bar{K}_{ij} \) is \( \bar{m} \times \bar{m} \) matrix with

\[ \bar{K}_{ij} = \begin{pmatrix} \psi_i(x), \{ k(x,t), \psi_j(t) \} \end{pmatrix}. \]

3.2. Fractional Integration Operational Matrix. In the interval \([0,1]\), a \( \bar{m} \)-set of block-pulse functions (BPFs) is defined

\[ B_i(t) = \begin{cases} 1, \text{if } \bar{m} \leq t < \frac{i+1}{\bar{m}}, \\ 0, \text{o.w.} \end{cases} \]

The functions \( B_i(t) \) are disjoint and orthogonal

\[
\begin{bmatrix} B_i(t)B_j(t) \end{bmatrix} = \begin{cases} 1, \text{if } i \neq j \\ 0, \text{o.w.} \end{cases} \int_0^1 B_i(s)B_j(s)ds = \begin{cases} 0, \text{if } i \neq j \\ 1/\bar{m}, \text{if } i = j \end{cases}
\]

The second kind Chebyshev wavelet vector can be expressed by an \( \bar{m} \)-term block-pulse functions as

\[ \Psi(t) = \Phi_{mom} B_{\bar{m}}(t). \]

In the previous study, the fractional integral operational matrix of block-pulse functions can be expressed as

\[ y_{nm} = \int_0^{\infty} \omega_n(t)Y(t)\psi_{nm}(t)dt, \]

and the weight function \( \omega_n(t) = \omega(2^k t - 2n + 1) \). Moreover, \( Y \) and \( \Psi(t) \) are \( \bar{m} = (2^{k-1}M) \) column vectors given by

\[ \left( \mathcal{F}^\alpha B_{\bar{m}} \right)(t) = \mathcal{F}^\alpha B_{\bar{m}}(t), \]

where

\[ B_{\bar{m}}(t) = [B_0(t), B_1(t), \ldots, B_{\bar{m}-1}(t)]^T, \]

and

\[ \mathcal{F}^\alpha = \begin{bmatrix} 1 & \xi_1 & \xi_2 & \cdots & \xi_{\bar{m}-1} \\ 0 & 1 & \xi_1 & \cdots & \xi_{\bar{m}-2} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \]

The fractional integration of the second kind Chebyshev wavelet can be given as

\[ \left( \mathcal{F}^\alpha \Psi \right)(t) = \mathcal{F}^\alpha \Phi_{mom} \Psi(t), \]

where \( \mathcal{F}^\alpha \Phi_{mom} \) is called fractional integral operational matrix of the second kind Chebyshev wavelet, and it can be given by

\[ \mathcal{F}^\alpha \Phi_{mom} = \Phi_{mom} \mathcal{F}^\alpha \Phi_{mom}^{-1}. \]

For more details, see [32].

4. Numerical Implementation

In the section, the second kind Chebyshev wavelet method is presented to numerically solve the system of nonlinear fractional Fredholm integral-differential equation (1). Firstly, the functions \( k_{11}(x,t), k_{12}(x,t), k_{21}(x,t), \) and \( k_{22}(x,t) \) can be approximated by the second kind Chebyshev as
\[
\begin{aligned}
\begin{aligned}
\mathbf{K}_{11} &= \left[ k_{n_i}, l_{m_x}, v_x \right], & \mathbf{K}_{12} &= \left[ k_{n_i}, l_{m_y}, v_y \right] \quad (25), \\
\mathbf{K}_{21} &= \left[ k_{n_i}, l_{m_x}, v_x \right], & \mathbf{K}_{22} &= \left[ k_{n_i}, l_{m_y}, v_y \right].
\end{aligned}
\end{aligned}
\]

where
\[
\begin{aligned}
k_{n_i, l_{m_x}, v_x} &= \langle \psi_{n_i, l_{m_x}}, \phi_{n_i, l_{m_x}} \rangle, \\
k_{n_i, l_{m_y}, v_y} &= \langle \psi_{n_i, l_{m_y}}, \phi_{n_i, l_{m_y}} \rangle.
\end{aligned}
\]

Moreover,
\[
\begin{aligned}
g_1(t) &= G_T\Psi(t), \\
g_2(t) &= G_T\Psi(t).
\end{aligned}
\]

Suppose that
\[
\begin{aligned}
\mathbf{D}_Y^1 Y_1(t) &= \mathbf{Y}_2^1 \Psi(t) \\
\mathbf{D}_Y^2 Y_2(t) &= \mathbf{Y}_2^2 \Psi(t).
\end{aligned}
\]

For simplification, assuming \( \delta_i = y_i = 0 \), then by equations (2), (7), (24), and (28), we can obtain
\[
\begin{aligned}
\begin{aligned}
Y_1(t) &= \mathbf{Y}_2^1 \mathbf{P}_m^{\mathbf{B}_m} - \mathbf{P}_m^{\mathbf{B}_m}(t), \\
Y_2(t) &= \mathbf{Y}_2^2 \mathbf{P}_m^{\mathbf{B}_m} - \mathbf{P}_m^{\mathbf{B}_m}(t).
\end{aligned}
\end{aligned}
\]

Let
\[
\begin{aligned}
r &= \left[ r_0, r_1, \ldots, r_{m-1} \right] = \mathbf{Y}_2^1 \mathbf{P}_m^{\mathbf{B}_m} - \mathbf{P}_m^{\mathbf{B}_m}, \\
s &= \left[ s_0, s_1, \ldots, s_{m-1} \right] = \mathbf{Y}_2^2 \mathbf{P}_m^{\mathbf{B}_m} - \mathbf{P}_m^{\mathbf{B}_m},
\end{aligned}
\]

then \( Y_1(t) = r \mathbf{B}_m - t \). According to the property of BPFs, we have
\[
\begin{aligned}
(Y_1(t))^2 &= \left[ r \mathbf{B}_m - t \right]^2 \\
&= \left[ r_0 B_0(t) + r_1 B_1(t) + \cdots + r_{m-1} B_{m-1}(t) \right]^2 \\
&= \left[ r_0 B_0(t) + r_1 B_1(t) + \cdots + r_{m-1} B_{m-1}(t) \right]^2 \\
&= r_2 \mathbf{B}_m - t,
\end{aligned}
\]

and
\[
\begin{aligned}
(Y_2(t))^2 &= \left[ s_0 B_0(t) + s_1 B_1(t) + \cdots + s_{m-1} B_{m-1}(t) \right]^2 \\
&= \left[ s_0 B_0(t) + s_1 B_1(t) + \cdots + s_{m-1} B_{m-1}(t) \right]^2 \\
&= \left[ s_2 B_0(t) + s_1 B_1(t) + \cdots + s_{m-1} B_{m-1}(t) \right]^2 \\
&= \left[ s_2 B_0(t) + s_1 B_1(t) + \cdots + s_{m-1} B_{m-1}(t) \right]^2.
\end{aligned}
\]
where

\[
\begin{align*}
\int_{0}^{1} B_m(x) B_m^T(x) \bar{s}_{m} d\bar{s} &= \int_{0}^{1} \left[ \begin{array}{c}
B_0(x) \\
B_1(x) \\
\vdots \\
B_{m-1}(x)
\end{array} \right] \left[ \begin{array}{c}
\bar{s}_{0} \\
\bar{s}_{1} \\
\vdots \\
\bar{s}_{m-1}
\end{array} \right] d\bar{s} \\
&= \int_{0}^{1} \left[ s_{0} B_0(x), s_{1} B_1(x), \ldots, s_{m-1} B_{m-1}(x) \right]^T dx \\
&= \frac{1}{m} \left[ s_{0}, s_{1}, \ldots, s_{m-1} \right]^T,
\end{align*}
\]

\[
\begin{align*}
\int_{0}^{1} B_{m}^T(x) B_{m} (x) \bar{s}_{m} d\bar{s} &= \int_{0}^{1} \left[ \begin{array}{c}
B_0(x) \\
B_1(x) \\
\vdots \\
B_{m-1}(x)
\end{array} \right] \left[ \begin{array}{c}
\bar{s}_{0} \\
\bar{s}_{1} \\
\vdots \\
\bar{s}_{m-1}
\end{array} \right] d\bar{s} \\
&= \int_{0}^{1} \left[ s_{0} B_0(x), s_{1} B_1(x), \ldots, s_{m-1} B_{m-1}(x) \right]^T dx \\
&= \frac{1}{m} \left[ s_{0}, s_{1}, \ldots, s_{m-1} \right]^T,
\end{align*}
\]

\[
\begin{align*}
\int_{0}^{1} B_{m}^T(x) B_{m} (x) \bar{r}_{m} d\bar{r} &= \int_{0}^{1} \left[ \begin{array}{c}
B_0(x) \\
B_1(x) \\
\vdots \\
B_{m-1}(x)
\end{array} \right] \left[ \begin{array}{c}
r_{0} \\
r_{1} \\
\vdots \\
r_{m-1}
\end{array} \right] d\bar{r} \\
&= \int_{0}^{1} \left[ r_{0}^B B_0(x), r_{1}^B B_1(x), \ldots, r_{m-1}^B B_{m-1}(x) \right]^T dx \\
&= \frac{1}{m} \left[ r_{0}^B, r_{1}^B, \ldots, r_{m-1}^B \right]^T,
\end{align*}
\]

and
Obviously, (39) is a system of linear algebraic equations. Dispersing the unknown variable \( t \) using Matlab software, we can obtain \( [r_0, r_1, \ldots, r_{m-1}] \) and \( [s_0, s_1, \ldots, s_{m-1}] \), and then, by (29), the solutions of (1) can be obtained.

5. Convergence Analysis

In this section, we focus on the convergence analysis of the system. In fact, by (9), \( \bar{Y}(t) \) converge (in the mean) to \( Y(t) \) as \( k \) approaches to \( \infty \). If the function \( Y(t) \) is several times continuously differentiable, we can bound the error, as established by the following theorem.

**Theorem 1.** Suppose that the function \( Y : [0, 1] \rightarrow R \) is \( m \) times continuously differentiable and \( Y \in \mathbb{C}^m[0, 1] \). Then, \( \bar{Y}(t) \) approximate \( Y(t) \) with mean error bounded as follows:
\[
\|Y(t) - \bar{Y}(t)\|_2 \leq \frac{2}{2^{m(k-1)}4^m m!} \sup_{t \in [0,1]} |Y^m(t)|. \tag{40}
\]

**Proof.** See [37]. Assume that \((C[J], \| \cdot \|)\) is the Banach space of all continuous functions on the interval \(J = [0, 1]\) with norm
\[
\|Y_1(t)\|_\infty = \max_{t \in J} |Y_1(t)|, \quad \|Y_2(t)\|_\infty = \max_{t \in J} |Y_2(t)|. \tag{41}
\]

Let
\[
\begin{align*}
F_{p_1}(Y_1(x)) &= [Y_1(x)]^{p_1} \\
G_{p_2}(Y_2(x)) &= [Y_2(x)]^{p_2} \\
F_{q_1}(Y_1(x)) &= [Y_1(x)]^{q_1} \\
G_{q_2}(Y_2(x)) &= [Y_2(x)]^{q_2}
\end{align*}
\tag{42}
\]
\[
\begin{align*}
\left\{ \begin{array}{l}
\xi_{11}(x,t) \leq M_{11} \\
\xi_{12}(x,t) \leq M_{12} \\
\xi_{21}(x,t) \leq M_{21} \\
\xi_{22}(x,t) \leq M_{22}
\end{array} \right. \quad \forall 0 < x, t \leq 1.
\end{align*}
\] (43)

Suppose that the nonlinear terms \( F_{p1}(Y_1(x)), F_{q1}(Y_1(x)), G_{p2}(Y_2(x)), \) and \( G_{q2}(Y_2(x)) \) satisfy the Lipschitz conditions:

\[
\begin{align*}
\left| F_{p1}(Y_1(x)) - F_{p1}(\bar{Y}_1(x)) \right| & \leq L_{11} \| Y_1(x) - \bar{Y}_1(x) \| \\
\left| F_{q1}(Y_1(x)) - F_{q1}(\bar{Y}_1(x)) \right| & \leq L_{12} \| Y_1(x) - \bar{Y}_1(x) \| \\
\left| G_{p2}(Y_2(x)) - G_{p2}(\bar{Y}_2(x)) \right| & \leq L_{12} \| Y_2(x) - \bar{Y}_2(x) \| \\
\left| G_{q2}(Y_2(x)) - G_{q2}(\bar{Y}_2(x)) \right| & \leq L_{22} \| Y_2(x) - \bar{Y}_2(x) \|
\end{align*}
\] (44)

where \( Y_1(t), Y_2(t) \) are the \( m \) times continuous differentiable analytical functions of the system, and \( \bar{Y}_1(t), \bar{Y}_2(t) \) are the approximate functions of the system, respectively.

Then, we have

\[
\| Y_1(t) - \bar{Y}_1(t) \|_\infty \leq \max_{\omega \in [0,1]} \left| Y_1(t) - \bar{Y}_1(t) \right|
\]

\[
\leq \max_{\omega \in [0,1]} \left( \left| \theta_{11} \right| \int_0^1 \left| \xi_{11}(x,t) \right| \left| F_{p1}(Y_1(x)) - F_{p1}(\bar{Y}_1(x)) \right| dx \right)
\]

\[
\leq \left| \theta_{11} \right| \frac{M_{11}L_{11}}{\Gamma(1 + \alpha)} \max_{\omega \in [0,1]} Y_1(t) - \bar{Y}_1(t) \| + \left| \theta_{12} \right| \frac{M_{12}L_{12}}{\Gamma(1 + \alpha)} \max_{\omega \in [0,1]} Y_2(t) - \bar{Y}_2(t) \| 
\]

\[
\leq \left| \theta_{11} \right| \frac{M_{11}L_{11}}{\Gamma(1 + \alpha)} \| Y_1(t) - \bar{Y}_1(t) \|_2 + \left| \theta_{12} \right| \frac{M_{12}L_{12}}{\Gamma(1 + \alpha)} \| Y_2(t) - \bar{Y}_2(t) \|_2 
\]

\[
\leq \left| \theta_{11} \right| \frac{M_{11}L_{11}}{\Gamma(1 + \alpha)} \frac{2^{m(k-1)/4} m!}{\Gamma(m)} Y_1^m(t) \| + \left| \theta_{12} \right| \frac{M_{12}L_{12}}{\Gamma(1 + \alpha)} \frac{2^{m(k-1)/4} m!}{\Gamma(m)} \sup_{\omega \in [0,1]} Y_2^m(t),
\]

and

\[
\| Y_2(t) - \bar{Y}_2(t) \|_\infty = \max_{\omega \in [0,1]} \left| Y_2(t) - \bar{Y}_2(t) \right|
\]

\[
\leq \max_{\omega \in [0,1]} \left( \left| \theta_{21} \right| \int_0^1 \left| \xi_{21}(x,t) \right| \left| F_{p2}(Y_2(x)) - F_{p2}(\bar{Y}_1(x)) \right| dx \right)
\]

\[
\leq \left| \theta_{21} \right| \frac{M_{21}L_{21}}{\Gamma(1 + \beta)} \| Y_1(t) - \bar{Y}_1(t) \|_2 + \left| \theta_{22} \right| \frac{M_{22}L_{22}}{\Gamma(1 + \beta)} \| Y_2(t) - \bar{Y}_2(t) \|_2 
\]

\[
\leq \left| \theta_{21} \right| \frac{M_{21}L_{21}}{\Gamma(1 + \beta)} \| Y_1(t) - \bar{Y}_1(t) \|_2 + \left| \theta_{22} \right| \frac{M_{22}L_{22}}{\Gamma(1 + \beta)} \| Y_2(t) - \bar{Y}_2(t) \|_2 
\]

\[
\leq \left| \theta_{21} \right| \frac{M_{21}L_{21}}{\Gamma(1 + \beta)} \frac{2^{m(k-1)/4} m!}{\Gamma(m)} \sup_{\omega \in [0,1]} Y_1^m(t) \| + \left| \theta_{22} \right| \frac{M_{22}L_{22}}{\Gamma(1 + \beta)} \frac{2^{m(k-1)/4} m!}{\Gamma(m)} \sup_{\omega \in [0,1]} Y_2^m(t).
\]
In order to verify the effectiveness of the proposed method via the following examples, we defined the 2-norm error as

\[
\|Y_i(t) - \overline{Y}_i(t)\|_2 = \left( \int_0^t (Y_i(t) - \overline{Y}_i(t))^2 \, dt \right)^{1/2}
\]

and

\[
\|\overline{Y}_1(t) - \overline{Y}_2(t)\|_2 = \left( \frac{1}{N} \sum_{i=0}^{N-1} (Y_1(t_i) - \overline{Y}_1(t_i))^2 \right)^{1/2}
\]

for \(0 \leq t \leq T\) and \(i = 1, \ldots, N, j = 1, 2, \ldots, N\), with \(m = 12, 32, 80\).

where \(Y_i(t)\) is the analytical solution of this system, and \(\overline{Y}_i(t)\) is the approximate solution of this system.

**Test Problem 1.** Considering the following nonlinear system of fractional-order Fredholm integral-differential equations

\[
D_t^{2.5}Y_1(t) = \int_0^t (x + t) [Y_1(x)]^2 \, dx + \int_0^t xt [Y_2(x)]^2 \, dx
+ 12t^{0.5} \sqrt{\pi} \frac{15t}{56} \frac{1}{8}
\]

\[
D_t^{2.5}Y_2(t) = \int_0^t (x - t) [Y_1(x)]^2 \, dx + \int_0^t xt^2 [Y_2(x)]^2 \, dx
+ 12t^{0.5} \sqrt{\pi} \frac{t}{7} - \frac{1}{8} \frac{t^2}{8}
\]

with the zero initial conditions: \(Y_1(0) = Y_2(0) = Y_1' (0) = Y_2' (0) = Y_1'' (0) = Y_2'' (0) = 0\). The analytical solutions of the system are \(Y_1(t) = t^3\) and \(Y_2(t) = -t^3\).

The numerical results of the system with some value of \(k, M\) are shown in Tables 1 and 2. From Tables 1 and 2, it can be concluded that the numerical solutions approach to the analytical solutions as \(k, M\) increases. Tables 3 and 4 show the absolute error tables of \(Y_1\) and \(Y_2\), respectively. It can be...
seen from the error that the error decreases with the increase of the discrete term.

\[
\begin{align*}
D_t^{5/3}Y_1(t) &= 5 \int_0^1 xt[Y_1(x)]^2 \, dx + 6 \int_0^1 x^3 t[Y_2(x)]^{1.5} \, dx + \frac{\Gamma(5/2)}{\Gamma(7/6)} t^{1/6} - 2t \\
D_t^{5/3}Y_2(t) &= 3 \int_0^1 t^2 x[Y_1(x)]^2 \, dx + 6 \int_0^1 x^2 t^2 [Y_2(x)]^{1.5} \, dx + \frac{2}{\Gamma(4/3)} t^{1/3} - t^2
\end{align*}
\]

with the initial conditions:

\[Y_1(0) = Y_1'(0) = 0, \quad Y_2(0) = Y_2'(0) = 0. \quad (50)\]

The analytical solutions of the system are \(Y_1(t) = t^{3/2}\) and \(Y_2(t) = t^2\). When \(m = 4, k = 3, 4, 5\), the obtained numerical results are shown in Tables 5 and 6. From Tables 5 and 6, it can be concluded that the numerical solutions approach to the analytical solutions as \(k, M\) increases.

**Test Problem 2.** Considering the following nonlinear system of fractional-order Fredholm differential-integral equations

\[
\begin{align*}
Y_1'(t) &= \int_0^1 x t[Y_1(x)]^{1.5} \, dx + \pi \cos(\omega t) - 0.241213254623034t \\
Y_2'(t) &= \int_0^1 x^3 t[Y_1(x)]^3 \, dx + \cosh(t) - \frac{2t^2}{3\pi}
\end{align*}
\]

subject to the initial conditions \(Y_1(0) = Y_2(0) = 0\). The analytical solutions of the system are \(Y_1(t) = \sin(\pi t)\) and \(Y_2(t) = \sinh(t)\).

When \(m = 12 (k = M = 3), m = 32 (k = M = 4), \) and \(m = 80 (k = M = 5)\), the 2-norm errors are listed in Table 7. Table 7 shows that the proposed method has a good convergence precision for solving these kinds of problems.

Through the analysis of the above two experimental examples, it can be concluded that the numerical results approach to the analytical results as \(m\) grows. The results show that the absolute error between the analytical and numerical results can achieve a high convergence precision.

Figures 1–3 show the absolute error of \(Y_1\) and \(Y_2\) under different fractional orders. It can be seen from the figure that the error is increasing with the increase of the order, but it is not an absolute increase. Due to some calculation factors, the error will also fluctuate to a certain extent.

### 7. Conclusions

In this paper, in view of the problem that the huge algebraic equations will lead to considerable computational complexity and large data storage requirements in the calculation process, the second kind Chebyshev wavelet method is applied to obtain the numerical solutions of nonlinear system of fractional Fredholm integral-differential equations. Using this method, the system of differential-integral equations has been reduced to a system of algebraic equations. The results show that the second Chebyshev wavelet method is used; the accuracy is improved with the increase of \(K\) and \(M\) values. In addition, the convergence analysis of the system based on the second kind of Chebyshev wavelet is studied. The analysis is carried out by several numerical experiments, and the absolute error values under different fractional orders are given, which proves the superiority and effectiveness of the proposed method. It provides support for improving the precision and reliability of the system.

### Data Availability

All data generated or analysed during this study are included in this published article.

### Conflicts of Interest

The authors declare that there are no conflicts of interest.

### Authors’ Contributions

W.S. provided conceptualization; W.S. and Y.J.Z. developed the methodology; W.S., J.Q.X., T.W., and Y.J.Z. performed validation; W.S. developed software; W.S. and Y.J.Z. performed formal analysis; W.S. performed investigation; W.S. helped to write and to prepare the original draft; W.S., J.Q.X., T.W., and Y.J.Z. written the article and reviewed and edited the article; T.W. performed supervision. All authors have read and agreed to the published version of the manuscript.
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