Nontopological first-order vortices in a gauged $CP(2)$ model with a dielectric function.

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We consider nontopological first-order solitons arising from a gauged $CP(2)$ model in the presence of the Maxwell term multiplied by a nontrivial dielectric function. We implement the corresponding first-order scenario by proceeding the minimization of the total energy, this way introducing the corresponding energy lower-bound, such a construction being only possible due to a differential constraint including the dielectric function itself and the self-interacting potential defining the model. We saturate the aforementioned bound by focusing our attention on those solutions fulfilling a particular set of two coupled first-order differential equations. In the sequel, in order to solve these equations, we choose the dielectric function explicitly, also calculating the corresponding self-interacting potential. We impose appropriate boundary conditions supporting nontopological solitons, from which we verify that the energy of final structures is proportional to the magnetic flux they engender, both quantities being not quantized, as expected. We depict the new numerical solutions, whilst commenting on the main properties they present.

PACS numbers: 11.10.Kk, 11.10.Lm, 11.27.+d

I. INTRODUCTION

In the context of classical field theories, vortices are planar solutions coming from highly nonlinear gauged models [1]. In general, these solutions are calculated directly from the second-order Euler-Lagrange equations. However, under very special circumstances, they can also be obtained via a particular set of first-order differential equations which minimize the energy of the overall system [2]. In this case, the energy bound is expected to be proportional to the quantized magnetic flux the resulting first-order vortices engender. In particular, first-order vortices supporting quantized flux were firstly found within the canonical Maxwell-Higgs scenario, which gives rise to topological configurations only [3]. In the sequel, both topological and nontopological vortices were also verified to exist in the Chern-Simons-Higgs electrodynamics [4].

Moreover, solitons inherent to nonstandard models were recently investigated, for instance, the ones arising from generalizations of the Abelian-Higgs systems [5], Lorentz-breaking scenarios [6] and gauged theories presenting unusual dynamics whose solutions were applied to study some interesting cosmological issues [7].

Therefore, it is certainly important to consider whether a gauged $CP(N-1)$ theory supports well-behaved first-order vortices, especially given that the $CP(N-1)$ model effectively mimics interesting properties of the Yang-Mills theories defined in four-dimensions [8].

In a recent work [9], vortex solutions inherent to a gauged $CP(2)$ model in the presence of the Maxwell term were considered, the corresponding solutions being obtained directly from the second-order Euler-Lagrange equations of motion. In addition, it was suggested the existence of configurations satisfying a particular set of first-order differential equations. In the sequel, in the contribution [10], it has been developed the self-dual framework giving rise to the aforementioned solutions, whilst introducing the corresponding first-order differential equations and the energy lower-bound explicitly. Moreover, it has been verified that the energy lower-bound the first-order solitons saturate is proportional to their magnetic flux, being quantized according the winding number rotulating such configurations, as expected. Here, it is important to say that, due to the boundary conditions fulfilling the finite-energy requirement, such first-order solitons present topological properties.

In the context of noncanonical models, a rather natural issue is the study of the gauged $CP(2)$ theory endowed with the Maxwell term multiplied by a dielectric function depending on the scalar field only, the motivation regarding this nontrivial coupling coming from supersymmetric scenarios, in which such a nonstandard kinetic term is necessary to support a gauged model with a noncompact gauge group [11]. Also, field models with a dielectric function have additionally been used to study quarks and gluons via soliton bag theories [12].

It is known that, under special circumstances, a gauged theory provided with a nontrivial dielectric function can support both topological or nontopological solitons. In this sense, the aim of the present manuscript is to investigate the way such a noncanonical $CP(2)$ model generates nontopological solitons satisfying a particular set of first-order differential equations.

In order to introduce our results, this manuscript is organized as follows: in the Section II, we introduce the overall $CP(N-1)$ model and the definitions inherent to it, from which we verify that $A^0 = 0$ holds as a legitimate gauge choice (the theory then engendering configurations with no electric field). In the sequel, for the sake
of simplicity, we particularize our study to the $N = 3$ case, whilst focusing on those static solutions possessing radial symmetry. We then proceed the minimization of the total energy, from which we find the corresponding first-order equations and the lower-bound for the total energy, such a theoretical construction being only possible due to a differential constraint including the dielectric function and the self-interacting potential defining the effective scenario. In the Sec. III, we use the first-order differential equations we have found previously to calculate genuine nontopological gauged solitons. The point to be highlighted here is the absence of nontopological profiles for $G (|\phi|) = 1$ (the energy of the resulting structures vanishing identically). We can contour this problem by considering convenient nontrivial forms of the dielectric function. Also, despite the apparent existence of two different solutions, we verify that the effective theory provides a unique phenomenology, at least regarding the nontopological solitons at the classical level. We present the solutions themselves in the Section IV, pointing out the main properties they engender. In the Section V, we end our manuscript by presenting the final comments and perspectives regarding future contributions.

In what follows, we use $\eta^{\mu\nu} = (+ − −)$ as the metric signature and the natural units system, for convenience.

II. THE OVERALL MODEL

We begin by presenting the Lagrange density of the gauged $CP(N − 1)$ model we consider in this manuscript,

$$
\mathcal{L} = -\frac{G}{4} F_{\mu\nu} F^{\mu\nu} + (P_{ab} D_{\mu} \phi_b)^* P_{ac} D^{\mu} \phi_c - V (|\phi|),
$$

(1)

where Greek indexes running over time-space coordinates, the Latin ones representing the internal indexes of the complex $CP(N − 1)$ field. Here, $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ is the standard electromagnetic field strength tensor, $P_{ab} = \partial_{\alpha} - h^{-1} \partial_{\beta} \delta_{ab}^\alpha$ being a projection operator introduced in a convenient way. Moreover, $D_{\mu} \phi_a = \partial_{\mu} \phi_a - ig A_{\mu} Q_{ab} \phi_b$ stands for the covariant derivative, $Q_{ab}$ being a real and diagonal charge matrix. The function $G (|\phi|)$ multiplying the Maxwell term stands for a dielectric function to be chosen conveniently later below, the resulting model standing for an effective action describing a system in a medium defined by such a dielectric function. The $CP(N − 1)$ field $\phi$ itself is constrained to satisfy $\phi^* \phi = h$.

The static Gauss law inherent to the model (1) reads

$$
\partial_j (G \phi^j \partial^0 A^0) = J^0,
$$

(2)

$j$ running over spatial coordinates only) with $J^0$, the charge density, given by

$$
J^0 = ig [(P_{ab} D^0 \phi_b)^* P_{ac} Q_{cd} \phi_d - P_{ab} D^0 \phi_b (P_{ac} Q_{cd} \phi_d)^*],
$$

(3)

where $D^0 \phi_b = -ig Q_{bc} \phi_c A^0$. It is evident that the gauge $A^0 = 0$ satisfies (2) identically. Therefore, it is possible to infer that the resulting time-independent solutions do not present electric field.

In what follows, we particularize our investigation to the case of the gauged $CP(2)$ model, for a sake of simplicity. Then, we focus our attention on those time-independent radially symmetric configurations defined by the following ansatz:

$$
A_i = -\frac{1}{gr} \epsilon^{ij} n^j A(r),
$$

(4)

$$
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix} = h \begin{pmatrix}
e^{im_1 \theta} \sin (\alpha(r)) \cos (\beta(r)) \\
e^{im_2 \theta} \sin (\alpha(r)) \sin (\beta(r)) \\
e^{im_3 \theta} \cos (\alpha(r))
\end{pmatrix},
$$

(5)

where $m_1$, $m_2$ and $m_3 \in \mathbb{Z}$ are winding numbers, $\epsilon^{ij}$ standing for the bidimensional Levi-Civita symbol (with $\epsilon^{12} = +1$), $n^j = (\cos \theta, \sin \theta)$ being the position unit vector.

In such a context, regular configurations possessing no divergences are attained via those profile functions $\alpha(r)$ and $A(r)$ fulfilling

$$
\alpha(r \to 0) \to 0 \text{ and } A(r \to 0) \to 0.
$$

(6)

Moreover, given that we are interested in nontopological solitons, the profile functions $\alpha(r)$ and $A(r)$ must satisfy

$$
\alpha(r \to \infty) \to 0 \text{ and } A'(r \to \infty) \to 0,
$$

(7)

with $A(r \to \infty) = A_{\infty}$ finite and arbitrary.

At this point, it is important to clarify that, regarding the charge matrix $Q_{ab}$ and the winding numbers $m_1$, $m_2$ and $m_3$, there are two different combinations supporting first-order solutions: (i) $Q = \lambda_8 / 2$ and $m_1 = -m_2 = m$, and (ii) $Q = \lambda_8 / 2$ and $m_1 = m_2 = m$ (both ones with $m_3 = 0$, $\lambda_3$ and $\lambda_8$ standing for the diagonal Gell-Mann matrices: $\lambda_3 = \text{diag}(1, -1, 0)$ and $\sqrt{3} \lambda_8 = \text{diag}(1, 1, -2)$). However, it is known that these two combinations are phenomenologically equivalent since they simply mimic each other, therefore existing only one effective scenario, as demonstrated in the Ref. [9]. Hence, in this work, we consider only the first choice (i.e. $m_1 = -m_2 = m$, $m_3 = 0$ and $Q = \lambda_3 / 2$), for convenience.

The second-order Euler-Lagrange equation for the additional profile function $\beta(r)$ is

$$
\frac{d^2 \beta}{dr^2} + \left( \frac{1 + 2}{r} \frac{d\alpha}{dr} \cot \alpha \right) \frac{d\beta}{dr} = \frac{\sin^2 \alpha \sin (4\beta)}{r^2} \left( m - \frac{A}{2} \right)^2.
$$

(8)

We are interested in solutions with $\beta$ being a constant so such solutions are ($k \in \mathbb{Z}$)

$$
\beta(r) = \beta_1 = \frac{\pi}{4} + \frac{\pi}{2} k \text{ or } \beta(r) = \beta_2 = \frac{\pi}{2} k,
$$

(9)

which apparently split our investigation in two distinct branches. However, it is important to say that, when concerning topological first-order solitons, these
two branches engender the very same phenomenon, being then physically equivalent. Below, we demonstrate that such equivalence also holds regarding nontopological solitons.

We now look for first-order differential equations providing genuine solutions of the model (1) by proceeding the minimization of its total energy. In this case, given the radially symmetric ansatz (4) and (5), the effective energy reads

\[
\frac{E}{2\pi} = \int \left( \frac{1}{2} GB^2 + V \right) r dr + h \int \left[ \left( \frac{d\alpha}{dr} \right)^2 + \frac{W}{r^2} \left( \frac{A}{2} - m \right)^2 \sin^2 \alpha \right] r dr, \tag{10}\]

with \( W = W(\alpha, \beta) = 1 - \sin^2 \alpha \cos^2 (2\beta) \), the function \( \beta(r) \) being necessarily one of those presented in (9), \( B(r) = -A'/gr \) standing for the magnetic field (here, prime denotes derivative with respect to the radial coordinate \( r \)). We then write the expression (10) in the form

\[
\frac{E}{2\pi} = \int \left[ \frac{G}{2} \left( B + \sqrt{\frac{2V}{G}} \right)^2 \pm B \sqrt{2GV} \right] r dr + h \int \left[ \left( \frac{d\alpha}{dr} \right)^2 + \sqrt{\frac{W}{G}} \right] r dr, \tag{11}\]

\[
\mp \int \left[ \frac{d(A-2m)}{dr} \sqrt{2GV} + (A-2m)h \sqrt{W} \frac{d\cos \alpha}{dr} \right] dr.
\]

In what follows, we impose the constraint

\[
\frac{d}{dr} \left( \sqrt{2GV} \right) = gh \sqrt{W} \frac{d}{dr} \cos \alpha, \tag{12}\]

via which we rewrite (11) as

\[
E = E_{bps} + \pi \int G \left( B \mp \sqrt{\frac{2V}{G}} \right)^2 r dr + 2\pi h \int \left[ \left( \frac{d\alpha}{dr} \right)^2 + \sqrt{\frac{W}{G}} \right] r dr, \tag{13}\]

with the energy bound \( E_{bps} \) given by

\[
E_{bps} = 2\pi \int r \varepsilon_{bps} dr, \tag{14}\]

where

\[
\varepsilon_{bps} = \mp \frac{2}{g} \frac{d}{dr} \left[ \left( \frac{A}{2} - m \right) \sqrt{2GV} \right], \tag{15}\]

the upper (lower) sign holding for positive (negative) values of \( m \). In this case, the lower-bound (14) can be easily calculated, i.e.

\[
E_{bps} = \mp \frac{2\pi}{g} \left[ (A_\infty - 2m) \sqrt{2G_\infty V_\infty} + 2m \sqrt{2G_0 V_0} \right], \tag{16}\]

where \( G_0 \equiv G(r \to 0) \), \( V_0 \equiv V(r \to 0) \), \( G_\infty \equiv G(r \to \infty) \) and \( V_\infty \equiv V(r \to \infty) \), with \( G_0V_0 \) and \( G_\infty V_\infty \) nonnegative and finite.

Now, given the expression (13), one concludes that the field solutions saturating the energy lower-bound (16) are those ones satisfying

\[
B = \pm \sqrt{\frac{2V}{G}}, \tag{17}\]

and

\[
\frac{d\alpha}{dr} = \pm \frac{\sin \alpha}{r} \left( \frac{A}{2} - m \right) \sqrt{1 - \sin^2 \alpha \cos^2 (2\beta)}, \tag{18}\]

which stand for the first-order equations related to the effective radially symmetric scenario. Here, it is important to highlight that the self-duality supporting the first-order equations (17) and (18) holds only in the presence of the constraint (12), such a constraint allowing us to rewrite the energy (10) in the form (13) via the Bogomol’nyi prescription [2].

It is also interesting to calculate the magnetic flux \( \Phi_B \) the radially symmetric configurations support. It reads

\[
\Phi_B = 2\pi \int r B(r) dr = \frac{2\pi}{g} A_\infty, \tag{19}\]

with \( A_\infty \equiv A(r \to \infty) \) being not necessarily proportional to the winding number \( m \), both the magnetic flux and the energy lower-bound (16) being not quantized.

In the next Section, we show how the first-order expressions we have introduced above can be used to generate well-behaved nontopological structures possessing finite-energy, these configurations satisfying the radially symmetric Euler-Lagrange equations, this way standing for legitimate solutions of the corresponding model. In this manuscript, for simplicity, we consider those cases fulfilling \( G_0V_0 = G_\infty V_\infty \), the energy of the resulting configurations equaling

\[
E = E_{bps} = \mp \frac{2\pi}{g} A_\infty \sqrt{2G_\infty V_\infty}, \tag{20}\]

with \( G_\infty V_\infty \) being positive and finite.

### III. NONTOPOLOGICAL FIRST-ORDER SCENARIOS

We now go further in our investigation by using the first-order framework we have introduced previously to obtain finite-energy nontopological solitons. We proceed according the prescription: firstly, we pick a particular solution for the function \( \beta(r) \) coming from (9), from which we solve the differential constraint (12) in order to get a concrete relation between the dielectric function \( G(|\phi|) \) and the self-interacting potential \( V(|\phi|) \). We then choose \( G(|\phi|) \) conveniently, this way getting the potential \( V(|\phi|) \) defining that particular model, also writing down
the corresponding first-order equations (17) and (18). We particularize the expression for the radially symmetric energy density coming from (10), the functions $\alpha(r)$ and $A(r)$ obeying the asymptotic boundary conditions (7). Finally, we use such conditions together with those in (6) in order to solve the first-order differential equations numerically, from which we depict the resulting profiles for $\alpha(r)$, $A(r)$, the magnetic field and energy density they engender. We also calculate the corresponding total energy (20) and magnetic flux (19) explicitly.

It is important to discuss the absence of nontopological solitons within the usual model, i.e. for $G(|\phi|) = 1$. In this case, the energy lower-bound (20) reduces to $E = E_{bps} = \mp 2\pi g^{-1}A_\infty\sqrt{2V_\infty}$. The point to be raised is that, in order to fulfill the finite-energy requirement $\varepsilon(r \to \infty) \to 0$, the self-interacting potential must satisfy $V_\infty \equiv V(r \to \infty) \to 0$, from which one also gets $E_{bps} = 0$, the corresponding solutions being energetically irrelevant.

Therefore, in this work, in order to avoid the aforementioned scenario, we consider nontrivial expressions for the dielectric function $G(|\phi|)$.

In the sequel, we study the cases with $\beta(r) = \beta_1$ and $\beta(r) = \beta_2$ separately.

**A. The $\beta(r) = \beta_1$ case**

It was demonstrated recently [10] that such a case gives rise to well-behaved first-order topological solitons with radial symmetry. Now, we go a little bit further by investigating the nontopological structures $\beta(r) = \beta_1$ supports. In this sense, we choose

$$\beta(r) = \beta_1 = \frac{\pi}{4} + \frac{\pi}{2} k,$$  

(21)

via which the differential constraint (12) can be reduced to

$$\frac{d}{dr} \left( \sqrt{2GV} \right) = \frac{d}{dr} \left( gh \cos \alpha \right),$$  

(22)

its solution defining the potential $V(\alpha)$ in terms of the dielectric function $G(\alpha)$, i.e.

$$V(\alpha) = \frac{g^2 h^2}{2G(\alpha)} \cos^2 \alpha,$$  

(23)

where the integration constant was conveniently set to be null.

Here, given the expression (23), one notes that $G = 1$ (standard case, absence of the dielectric function) leads to a self-interacting potential possessing no symmetric vacuum, therefore giving rise to topological configurations only. In this sense, the dielectric function in (23) must be chosen in order to engender a potential exhibiting a symmetric vacuum, such a symmetric point supporting the existence of nontopological solitons. We then proceed by fixing

$$G(\alpha) = \frac{(\cos \alpha)^{2-2M}}{1 - \cos \alpha},$$  

(24)

where $M = 1, 2, 3$ and so on. In this case, we get the potential

$$V(\alpha) = \frac{g^2 h^2}{2} (\cos \alpha)^{2M} (1 - \cos \alpha),$$  

(25)

which is positive for all values of the parameter $M$ (the resulting energy being itself positive, therefore justifying the way $M$ enters the definition (24)), the general first-order equations (17) and (18) being reduced to

$$\frac{1}{r} \frac{dA}{dr} = \pm \lambda^2 (\cos \alpha)^{2M-1} (\cos \alpha - 1),$$  

(26)

$$\frac{d\alpha}{dr} = \pm \frac{1}{r} \left( \frac{A}{2} - m \right),$$  

(27)

respectively, the parameter $\lambda$ standing for

$$\lambda = \sqrt{g^2 h}.$$  

(28)

We summarize the scenario as follows: given the dielectric function (24) and the self-interacting potential (25), the gauged $CP(N - 1)$ model (1) (with $N = 3$) supports radially symmetric time-independent solitons of the form (4) and (5) (with $\beta(r)$ as in (21)) satisfying the first-order equations (26) and (27), whilst behaving according the boundary conditions in (6) and (7). Here, it is worthwhile to point out that, for fixed values of $M$ and $\lambda$, the equations (26) and (27) support well-behaved solutions only for those values of $m$ for which the condition $0 \leq \alpha < \pi/2$ is fulfilled.

The energy density for such nontopological solitons satisfying the first-order differential equations is

$$\varepsilon(r) = \frac{g^2 h^2}{2} (1 - \cos \alpha) (\cos \alpha)^{2M}$$

$$+ 2\lambda \sin^2 \alpha \left( \frac{A}{2} - m \right)^2,$$  

(29)

being explicitly positive, whilst attaining the finite-energy condition $\varepsilon(r \to \infty) \to 0$.

It is interesting to investigate the way the profile functions $\alpha(r)$ and $A(r)$ behave near the boundaries. In this sense, we linearize the first-order equations (26) and (27), from which we calculate the behavior of these functions near the origin, i.e.,

$$\alpha(r) \approx C_m (\lambda r)^m - \frac{C_m^3}{16(m + 1)^2} (\lambda r)^{3m+2} + ...,$$  

(30)

$$A(r) \approx \frac{C_m^2}{4(m + 1)} (\lambda r)^{2m+2}$$

$$- \frac{(12M - 5)C_m^4}{48(2m + 1)} (\lambda r)^{4m+2} + ...,$$  

(31)

the asymptotic behavior (i.e. for $r \to \infty$) reading

$$\alpha(r) \approx \frac{C}{(\lambda r)^{\delta_m}} + ...,$$  

(32)

$$A(r) \approx 2m + 2\delta_m - \frac{C^2}{4(\delta_m - 1)} (\lambda r)^{2\delta_m - 2} + ...,$$  

(33)
where we have set $A_\infty \equiv A(r \to \infty) = 2m + 2\delta_m$, $C_m$, $C$ and $\delta_m$ standing for real and positive constants to be determined numerically by requiring the proper behavior near the origin and asymptotically, respectively.

It is known that the solution $\beta(r) = \beta_2 = \pi k/2$ gives rise to first-order topological configurations possessing finite-energy. Next, we discuss the way such a solution engenders nontopological solitons as well.

**B. The $\beta(r) = \beta_2$ case**

We now investigate the first-order nontopological configurations that $\beta(r) = \beta_2$ supports. In this sense, we choose

$$\beta(r) = \beta_2 = \frac{\pi}{2} k,$$

the constraint (12) being rewritten in the form

$$\frac{d}{dr} \left( \sqrt{2GV} \right) = \frac{d}{dr} \left( \frac{gh}{2} \cos^2 \alpha \right),$$

whose solutions is

$$V(\alpha) = \frac{g^2h^2}{32G(\alpha)} \cos^2 (2\alpha),$$

where the integration constant was chosen to be $-gh/4$. Again, we have found a relation between the dielectric function and the potential defining the model.

We proceed in the very same way as before, i.e. in order to have a potential with a symmetric vacuum (therefore supporting nontopological profiles, see the discussion just before the Eq. (24)), we choose the dielectric function as

$$G(\alpha) = \frac{\cos (2\alpha)^{2-2M}}{1 - \cos (2\alpha)},$$

then getting (as we demonstrate below, the factor $2\alpha$ was introduced in (37) in order to make the two a priori different scenarios phenomenologically equivalent via a suitable redefinition)

$$V(\alpha) = \frac{g^2h^2}{32} \cos (2\alpha)^{2M} (1 - \cos (2\alpha)),$$

the potential itself and its energy density being positive for all $M$, the corresponding first-order equations (17) and (18) thereby reading

$$\frac{1}{r} \frac{dA}{dr} = \pm \frac{g^2h}{4} \cos (2\alpha)^{2M-1} (\cos (2\alpha) - 1),$$

$$\frac{d\alpha}{dr} = \pm \frac{\sin (2\alpha)}{2r} \left( \frac{A}{2} - m \right),$$

respectively. In order to obtain nontopological structures, the above first-order equations must be solved according the boundary conditions (6) and (7), the energy density of the resulting solutions being

$$\varepsilon(r) = \frac{g^2h^2}{16} \cos (2\alpha)^{2M} (1 - \cos (2\alpha))$$

$$+ \frac{h \sin^2 (2\alpha)}{2r^2} \left( \frac{A}{2} - m \right)^2.$$

It is known that, regarding the first-order topological configurations, those two a priori different scenarios the solutions for $\beta(r)$ in (9) engender simply mimic each other, therefore existing only one effective case. Here, it is important to highlight that such correspondence also holds regarding the first-order nontopological structures we have introduced above, i.e., both the dielectric function in (24) and the self-interacting potential in (25) can be rewritten as those ones in (37) and (38), respectively, whether we implement the redefinitions $\alpha \to 2\alpha$ and $h \to h/4$, the corresponding first-order equations (26) and (27) then reducing to (39) and (40), this way existing only one effective scenario, at least regarding the nontopological radially symmetric first-order solitons in the presence of a nontrivial dielectric function.

Therefore, from now on, we investigate only the case defined by $\beta(r) = \beta_1$, the resulting first-order equations being (26) and (27).

**IV. THE SOLUTIONS**

It is instructive to point out that the first-order equations (26) and (27) support approximate analytical solu-
tions describing the corresponding nontopological configurations. In what follows, we investigate these solutions by choosing \( m > 0 \) only (i.e. the lower signs in the first-order expressions), for simplicity. We also suppose that \( \alpha(r) \ll 1 \) for all values of \( \lambda r \). In this sense, the first-order equations (26) and (27) reduce, respectively, to

\[
\frac{1}{r} \frac{dA}{dr} = \pm \frac{\lambda^2}{2} \alpha^2, \tag{42}
\]

\[
\frac{d\alpha}{dr} = \pm \frac{\alpha}{r} \left( \frac{A}{2} - m \right), \tag{43}
\]

which can be combined to each other into the Liouville’s equation for the profile function \( \alpha(r) \), i.e.

\[
\frac{d^2(\ln \alpha^2)}{dr^2} + \frac{1}{r} \frac{d(\ln \alpha^2)}{dr} + \frac{\lambda^2}{2} \alpha^2 = 0, \tag{44}
\]

whose general solution is [13]

\[
\alpha(r) = \frac{4C}{\lambda r_0} \left( \frac{r}{r_0} \right)^{\frac{1}{C} - 1} \left[ 1 + \left( \frac{r}{r_0} \right)^{\frac{1}{2C}} \right], \tag{45}
\]

with \( C \) and \( r_0 \) standing for integration constants. Here, it is interesting to note that this solution satisfies the conditions \( \alpha(r \to 0) \to 0 \) and \( \alpha(r \to \infty) \to 0 \) for \( C > 1 \) only.

In addition, given the solution (45), the first-order equation (43) provides

\[
A(r) = 2(m + 1) - 2C + \frac{4C \left( \frac{r}{r_0} \right)^{2C}}{1 + \left( \frac{r}{r_0} \right)^{2C}}, \tag{46}
\]

which fulfills \( A(r \to 0) \to 0 \) only for \( C = m + 1 \). Therefore, the approximate solutions (45) and (46) can be rewritten, respectively, as

\[
\alpha_m(r) = \frac{4(m + 1)}{\lambda r_0} \frac{\left( \frac{r}{r_0} \right)^m}{1 + \left( \frac{r}{r_0} \right)^{2m+2}}, \tag{47}
\]

\[
A_m(r) = 4(m + 1) \frac{\left( \frac{r}{r_0} \right)^{2m+2}}{1 + \left( \frac{r}{r_0} \right)^{2m+2}}, \tag{48}
\]

from which one also gets approximate expressions for the magnetic field

\[
B_m(r) = -\frac{g \hbar}{2} \alpha_m^2, \tag{49}
\]

and the energy density

\[
\varepsilon_{bps,m}(r) = \frac{g^2 \hbar}{2} \alpha_m^2 + 2\hbar \alpha_m^2 \left( \frac{A_m}{2} - m \right)^2, \tag{50}
\]

with \( \alpha_m(r) \) and \( A_m(r) \) being given by (47) and (48), respectively, the approximate value for \( A_{\infty,m} \equiv A(r \to \infty) \) being calculated exactly, i.e.

\[
A_{\infty,m} = A_m(r \to \infty) = 4(m + 1). \tag{51}
\]
The function (47) has its maximum value given by
\[
\alpha_m(r_{\text{max}}) = \frac{2 (m + 2)}{\lambda r_0} \left( \frac{m}{m + 2} \right)^{\frac{m}{2(m+1)}},
\]
where
\[
r_{\text{max}} = r_0 \left( \frac{m}{m + 2} \right)^{\frac{1}{2(m+1)}},
\]
which approximates \( r_0 \) for large values of \( m \), our previous assumption \( \alpha(r) \ll 1 \) holding whether the additional condition
\[
\lambda r_0 \gg 2 (m + 2) \left( \frac{m}{m + 2} \right)^{\frac{m}{2(m+1)}}
\]
is fulfilled. Therefore, for fixed values of \( g, h \) and \( r_0 \), only some values of the integer winding number \( m \) are allowed.

We highlight that, given the dielectric function (24), the self-interacting potential (25) and the boundary value (51), the resulting energy bound (20) can be calculated explicitly, being then equal to (remember that \( m > 0 \))
\[
E_{\text{bps}} = 8\pi (m + 1) h,
\]
where we have used \( G_0 V_0 = G_\infty V_\infty = g^2 h^2/2 \) (this way verifying our previous conjecture), the corresponding magnetic flux (19) reading
\[
\Phi_B = \frac{8\pi}{g} (m + 1),
\]
from which we get that \( E_{\text{bps}} = -gh\Phi_B \), the energy of the first-order solitons being then proportional to their magnetic flux, as expected.

In what follows, we proceed the numerical study of the first-order equations (26) and (27) by means of the finite-difference scheme, whilst using the boundary conditions (6) and (7). We adopt \( m > 0 \) (lower signs) and \( M = g = h = \lambda r_0 = 1 \), the profiles being again rings centered at the origin.

We begin our analysis by depicting the numerical profiles corresponding to the approximate expressions in (47) and (48). We choose \( m = 1 \), whilst varying \( \lambda r_0 \), the analytical solutions approximating the numerical ones for large values of such a parameter. Here, the dashed lines stand for the numerical solutions, the dotted lines representing the approximate ones (see Figs. 1-4).

In the Figure 1, we show the solutions to the profile function \( \alpha(r) \) for \( \lambda r_0 = 15 \) (blue lines), \( \lambda r_0 = 20 \) (red lines) and \( \lambda r_0 = 30 \) (black lines). We see that the resulting profiles are rings centered at the origin, their amplitudes (radii) decreasing (increasing) as \( \lambda r_0 \) itself increases, the numerical results fulfilling our previous assumption (i.e. that \( \alpha(r) \ll 1 \) for all \( \lambda r \)), the approximate solutions fitting relatively well.

The solutions to the field \( A(r) \) are those shown in the Figure 2, from which we see that these profiles approach the approximate boundary condition \( A_m(r \to \infty) = \)

![FIG. 4: Solutions to the energy density \( \varepsilon_{\text{bps}}(r) \), in units of \((gh)^2\). Conventions as in the Fig. 1, the approximate profiles standing for (50). The nontopological structures are well-localized in space.](image1)

![FIG. 5: Solutions to \( \alpha(r) \) for \( m = 1 \) (solid red line), \( m = 2 \) (dashed black line) and \( m = 3 \) (dash-dotted green line). Now, \( M = g = h = \lambda r_0 = 1 \), the profiles being again rings centered at the origin.](image2)
having as those for $\alpha(r)$, their amplitudes (radii) decreasing (increasing) as $\lambda r_0$ increases. The new solutions also vanish in the asymptotic limit $r \to \infty$, this way fulfilling the finite-energy requirement $\varepsilon(r \to \infty) \to 0$.

The solutions to the energy density $\varepsilon_{bps}(r)$ appear in the Figure 4, in units of $(gh)^2$, showing that the corresponding nontopological structures are localized in space. Here, we highlight the manner $\varepsilon_{bps}(r = 0)$ depends on $\lambda r_0$, that value increasing from 0 (zero) as this free parameter decreases.

It is also interesting to point out the existence of another class of numerical solutions that can not be predicted by an approximate analytical treatment, the condition $\alpha(r) \ll 1$ for all $\lambda r$ being not satisfied anymore. These new profiles are calculated for finite, but no large, values of the free parameter $\lambda r_0$, this way differing from the solutions presented above (see Figs. 5-8).

In order to introduce the aforementioned profiles, we again solve (26) and (27) numerically, for $m > 0$ and $M = g = h = 1$, via the conditions (6) and (7). However, now we choose $\lambda r_0 = 1$, whilst varying the winding number: $m = 1$ (solid red line), $m = 2$ (dashed black line) and $m = 3$ (dash-dotted green line), plotting the corresponding solutions with respect to the dimensionless variable $\lambda r$.

In this sense, the solutions to $\alpha(r)$ are those depicted in the Fig. 5, from which we see that these profiles behave in a similar manner as before, being rings centered at the origin, both amplitudes and radii increasing as the vorticity increases.

The Figure 6 shows the results to the function $A(r)$, the additional dotted blue line standing for $m = 4$. Here, it is worthwhile to note the appearance of an interesting internal structure for intermediarial values of the dimensionless variable $\lambda r$. It is also interesting to highlight that the new solutions do not fulfill the previous condition (51), the new values being $A_1(r \to \infty) \approx 9.64900$, $A_2(r \to \infty) \approx 15.01548$, $A_3(r \to \infty) \approx 20.78517$ and $A_4(r \to \infty) \approx 26.98683$.

The numerical profiles to the magnetic field $B(r)$ are plotted in the Fig. 7, in units of $gh$, from which we see that these solutions are drastically different from the ones appearing in the Fig. 3, the new configurations being double rings centered at $r = 0$, the magnetic field vanishing in the asymptotic limit.

Finally, the Fig. 8 shows the solutions to $\varepsilon_{bps}(r)$, again in units of $(gh)^2$, these profiles also standing for double rings centered at the origin. However, in this case, the amplitude of the inner ring is always greater than that of the outer one, $\varepsilon_{bps}(r = 0)$ vanishing for $m \neq 1$.

V. FINAL COMMENTS AND PERSPECTIVES

In this manuscript, we have studied the $CP(2)$ model endowed by the Maxwell term in the presence of an $a$ priori arbitrary dielectric function $G(|\phi|)$, from which
We have attained nontopological first-order vortices possessing finite-energy and nonquantized magnetic flux.

We have presented the particular model and the definitions inherent to it, from which we have verified that $A^0 = 0$ satisfies the static Gauss law identically, the resulting time-independent solutions supporting no electric field. We have particularized our investigation to case $N = 3$, for simplicity. Then, we have focused our attention on those radially symmetric configurations described by the Ansatz defined in (4) and (5) whilst satisfying the boundary conditions (6) and (7). We have introduced convenient choices regarding the charges and winding numbers inherent to the aforementioned Ansatz. In the sequel, we have calculated the solutions (9) for the additional profile function $\beta(r)$.

We have rewritten the expression for the radially symmetric energy (10) as that in (13), whilst defining the general first-order equations (17) and (18) satisfied by the fields $\alpha(r)$ and $A(r)$, additionally getting the corresponding lower-bound for the total energy (see (13) itself and (20)). The point to be highlighted here is that such construction was only possible due to the differential constraint (12) including $G(|\phi|)$ and the self-interacting potential $V(|\phi|)$. We have also calculated a general result for the magnetic flux $\Phi_B$ in (19).

We have discussed the absence of nontopological solitons for $G(|\phi|) = 1$, the energy of these structures vanishing. In order to avoid such scenario, we have considered only nontrivial forms for the dielectric function.

We have studied the case defined by $\beta(r) = \beta_1$ (21) firstly, from which we have written the constraint (12) as in (22), whose solution is (23). Then, we have chosen the dielectric function as showed in (24), from which we have obtained the potential (25) and the first-order equations (26) and (27). We have also linearized these equations in order to define the way the nontopological solutions behave near the boundaries.

We have implemented the same prescription also for $\beta(r) = \beta_2$ (34), from which we have calculated the corresponding first-order expressions. We have noticed that the two scenarios defined by the different solutions for $\beta(r)$ can be verified to mimic each other via the redefinitions $\alpha \to 2\alpha$, $\lambda \to \lambda/4$ and $h \to h/4$, therefore existing only one effective case. In this sense, we have focused our attention on the case $\beta(r) = \beta_1$ only.

We have supposed that $\alpha(r) \ll 1$ for all $\lambda r$ (with $\lambda^2 = g^2h$), from which we have combined the first-order equations (26) and (27), therefore verifying that the function $\alpha(r)$ satisfy the Liouville equation whose analytical solution has provided the profiles (47) and (48), also calculating the boundary value $A(r \to \infty) \to 4(m + 1)$. In addition, we have verified explicitly that the energy bound is proportional to the magnetic flux inherent to the resulting solitons, both quantities being not necessarily quantized, as expected.

We have depicted the numerical results we have found to $\alpha(r)$, $A(r)$, the magnetic field $B(r)$ and the energy density $\varepsilon_{bps}(r)$, for different values of the vorticity $m$ and the parameter $\lambda r_0$, from which we have pointed out the existence of two different classes of solutions: the ones coming from large values of $\lambda r_0$, being reasonably well described by the analytic expressions in (47) and (48), and those solutions related to small values of $\lambda r_0$, do not possessing an approximate counterpart.

Here, it is important to highlight that the results we have introduced in this work hold only for those time-independent solitons described by the Ansatz in (4) and (5). In this sense, it is not possible to say that the general model (1) supports regular first-order solutions outside the radially symmetric scenario, such question lying beyond the scope of the present investigation.

Moreover, ideas regarding future works include the study of the $CP(2)$ model in the presence of the Chern-Simons action (instead of the Maxwell’s one) and the first-order configurations the resulting theory possibly supports. This issue is now under consideration, and we hope relevant results for an incoming contribution.

Acknowledgments

The authors thank CAPES, CNPq and FAPEMA (Brazilian agencies) for partial financial support.
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