Quasilocal Thermodynamics of Kerr de Sitter Spacetimes and the dS/CFT Correspondence

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We consider the quasilocal thermodynamics of rotating black holes in asymptotic de Sitter spacetimes. Using the minimal number of intrinsic boundary counterterms, we carry out an analysis of the quasilocal thermodynamics of Kerr-de Sitter black holes for virtually all possible values of the mass, rotation parameter and cosmological constant that leave the quasilocal boundary inside the cosmological event horizon. Specifically, we compute the quasilocal energy, the conserved charges, the temperature and the heat capacity for the (3 + 1)-dimensional Kerr-dS black holes. We perform a quasilocal stability analysis and find phase behavior that is commensurate with previous analysis carried out through the use of Arnowitt-Deser-Misner (ADM) parameters. Finally, we investigate the non-rotating case analytically.

I. INTRODUCTION

The recent astrophysical data which indicate a positive cosmological constant [1] and the recent dS conformal field theory (CFT) correspondence [2, 3] are two of the main motivations for studying the thermodynamics of asymptotic de Sitter (dS) black holes [4]. Since these black holes have a cosmological event horizon and there is no well-known thermodynamic for outside the cosmological horizon, it is worthwhile to study the quasilocal thermodynamic of asymptotic de Sitter black holes inside the cosmological event horizon.

Among the quantities associated with gravitational thermodynamics are the physical entropy $S$ and the temperature $\beta^{-1}$, where these quantities are, respectively, proportional to the area and surface gravity of the event horizon(s) [5, 6]. One of the surprising and impressive features of this area law of entropy is its universality. It applies to all kinds of black holes and black strings [7, 8, 9, 10]. It also applies to the cosmological event horizon of the asymptotic de Sitter black holes [11]. Other black hole properties such as energy, angular momentum and conserved charge, can also be given a thermodynamic interpretation.
But as is known, these quantities typically diverge for asymptotic flat, AdS, and dS spacetimes.

A common approach to evaluating them has been to carry out all computations relative to some other spacetime that is regarded as the ground state for the class of spacetimes of interest. This could be done by taking the original action for gravity coupled to matter fields and subtracting from it a reference action, which is a functional of the induced metric $\gamma$ on the boundary $\partial M$. Conserved and/or thermodynamic quantities are then computed relative to this boundary, which can then be taken to infinity if desired.

This approach has been widely successful in providing a description of gravitational thermodynamics in regions of both finite and infinite spatial extent [13, 14]. Unfortunately, it suffers from several drawbacks. The choice of reference spacetime is not always unique [15], nor is it always possible to embed a boundary with a given induced metric into the reference background. Indeed, for Kerr spacetimes this latter problem creates a serious barrier against calculating the subtraction energy, and calculations have only been performed in the slow-rotating regime [16].

An extension of this approach which addresses these difficulties was developed based on the conjectured AdS/CFT correspondence for asymptotic AdS spacetimes [17, 18] and recently for asymptotic de Sitter spacetimes [19, 20, 21]. It is believed that appending a counterterm, $I_{ct}$, to the action which depends only on the intrinsic geometry of the boundary(ies) can remove the divergences. This requirement, along with general covariance, implies that these terms are functionals of curvature invariants of the induced metric and have no dependence on the extrinsic curvature of the boundary(ies). An algorithmic procedure exists for constructing $I_{ct}$ for asymptotic AdS [22] and dS spacetimes [20], and so its determination is unique. Addition of $I_{ct}$ will not affect the bulk equations of motion, thereby eliminating the need to embed the given geometry in a reference spacetime. Hence thermodynamic and conserved quantities can now be calculated intrinsic for any given asymptotically AdS or dS spacetime. The efficiency of this approach has been demonstrated in a broad range of examples for both the asymptotic AdS and dS spacetimes [20, 21, 23].

Recently, we have considered the effects of including $I_{ct}$ for quasilocal gravitational thermodynamics of Kerr and Kerr-AdS black holes situations in which the region enclosed by $\partial M$ was spatially finite and we performed a quasilocal stability analysis and found phase behavior that was commensurate with a previous analysis carried out at infinity [24]. In this
paper, we investigate the thermodynamic properties of the class of Kerr-dS black holes in the context of a spatially finite boundary with radius less than the radius of the cosmological horizon of the black hole, with the boundary action supplemented by $I_{ct}$. With their lower degree of symmetry relative to spherically symmetric black holes, these spacetimes allow for a more detailed study of the consequences of including $I_{ct}$ for spatially finite boundaries. There are several reasons for considering this. First, there is no good interpretation of the thermodynamics of the hole outside the cosmological horizon and thereby at spatial infinity. Furthermore, the inclusion of $I_{ct}$ eliminates the embedding problem from consideration, whether or not the spatially infinite limit is taken, and so it is of interest to see what its impact is on quasilocal thermodynamics.

The outline of our paper is as follows. We review the basic formalism in Sec. II. In Sec. III we consider the Kerr-dS spacetime and compute the entropy associated with the cosmological event horizon by generalizing the Gibbs-Duhem relation to this situation. In the next sections, we compute the quasilocal energy, mass, angular momentum, temperature, and heat capacity for Kerr-de Sitter black holes. These quantities must be computed numerically for general values of the parameters, and we present our results graphically. We perform a simple stability analysis and compare to the previous analysis using the Arnowitt-Deser-Misner (ADM) parameters [26]. The nonrotating case is considered in Sec. IV. We finish the paper with some concluding remarks.

II. GENERAL FORMALISM

The gravitational action of asymptotic de Sitter spacetimes in four dimensions is

$$I_G = -\frac{1}{16\pi} \int_M d^4x \sqrt{-g} \left( R - 2\Lambda \right) + \frac{1}{8\pi} \int_{\partial M} d^3x \sqrt{-\gamma} \Theta(\gamma),$$

(1)

where the first term is the Einstein-Hilbert volume (or bulk) term with positive cosmological constant $\Lambda = 3/l^2$, and the second term is the Gibbons-Hawking boundary term which is chosen such that the variational principle is well-defined. The Euclidean manifold $M$ has metric $g_{\mu\nu}$, covariant derivative $\nabla_{\mu}$, and time coordinate $\tau$ which foliates $M$ into nonsingular hypersurfaces $\Sigma_\tau$ with unit normal $u_\mu$ over a real line interval $\gamma$. $\Theta$ is the trace of the extrinsic curvature $\Theta^{\mu\nu}$ of any boundary(ies) $\partial M$ of the manifold $M$, with induced metric(s) $\gamma_{ij}$. In general, $I_G$ of Eq. (1) is divergent when evaluated on solutions, as is the Hamiltonian
and other associated conserved quantities. Rather than eliminating these divergences by incorporating a reference term in the spacetime, a new term $I_{ct}$ is added to the action which is a functional only of boundary curvature invariants [21]. Although there may exist a very large number of possible invariants, one could add only a finite number of them in a given dimension. Quantities such as energy, mass, etc. can then be understood as intrinsically defined for a given spacetime, as opposed to being defined relative to some abstract (and nonunique) background, although this latter option is still available. In four dimensions the counterterm is [21]

$$I_{ct} = \frac{2}{l^2} \int_{\partial M} d^3x \sqrt{-\gamma} \left( 1 - \frac{l^2}{4} \mathcal{R}(\gamma) \right),$$

(2)

where $\mathcal{R}$ is the Ricci scalar of the boundary metric $\gamma_{ab}$. We shall study the implications of including the terms in (2) for Kerr-dS black holes in spatially finite regions. Although other counterterms (of higher mass dimension) may be added to $I_{ct}$, they will make no contribution to the evaluation of the action or Hamiltonian due to the rate at which they decrease toward infinity, and we shall not consider them in our analysis here.

Thorough discussion of the quasilocal formalism has been given elsewhere [13, 14, 19? ] and so we only briefly recapitulate it here. The action is the linear combination of the gravity action $I_G$ [1] and the counterterm $I_{ct}$ given by (2). Under the variation of the metric, one obtains

$$\delta I = \text{[terms that vanish when the equations of motion hold]} \mu^\nu \delta g_{\mu\nu} + \int_{\partial M} d^3x (P^{ij} + Q^{ij}) \delta \gamma_{ij},$$

(3)

where $P^{ij}$ is related to the variation of the Hawking-Gibbons boundary term,

$$P^{ij} = \frac{\sqrt{-\gamma}}{8\pi} (\Theta^{ij} - \Theta \gamma^{ij}),$$

(4)

and $Q^{ij}$ is due to the variation of the counterterm [2] given as

$$Q^{ij} = \frac{\sqrt{-\gamma}}{8\pi} \left\{ -\frac{n-1}{l} \gamma^{ij} + \frac{l}{n-2} (R^{ij} - \frac{1}{2} R \gamma^{ij}) \right\}.$$

(5)

Decomposing the metric $\gamma_{ij}$ on the timelike boundary (the radius of the boundary is less than the cosmological horizon and therefore the boundary is timelike) $\partial \Sigma \times \Upsilon = \mathcal{B} \times \Upsilon = \mathcal{T}$ which connects the initial and final hypersurfaces

$$\gamma_{ij} dx^i dx^j = -N^2 dt^2 + \sigma_{ab} (dx^a + V^a) \left( dx^b + V^b \right),$$

(6)
yields after some manipulation

\[ \delta I|_\mathcal{T} = \int \mathcal{T} d^3x \sqrt{\sigma} (\varepsilon \delta N + j_a \delta V^a + \frac{1}{2} N s^{ab} \delta \sigma_{ab}), \quad (7) \]

where the coefficients of the varied fields are

\[ \varepsilon = \frac{2}{N \sqrt{\sigma}} (P^{ij} + Q^{ij}) u_i u_j, \quad (8) \]

\[ j_a = -\frac{2}{N \sqrt{\sigma}} \sigma_{ai} (P^{ij} + Q^{ij}) u_j, \quad (9) \]

\[ s^{ab} = \frac{2}{N \sqrt{\sigma}} \sigma^a_i \sigma^b_j (P^{ij} + Q^{ij}), \quad (10) \]

and we see that the counterterm variation \( Q^{ij} \) supplants terms which come from a reference action in the original formulation of the quasilocal technique. Since \( -\sqrt{\sigma} \varepsilon \) is the time rate of change of the action, \( \varepsilon \) is identified with the energy density on the surface \( \mathcal{B} \), and the total quasilocal energy for the system is therefore

\[ E = \int \mathcal{B} d^2x \sqrt{\sigma} \varepsilon, \quad (11) \]

and this quantity can be meaningfully associated with the thermodynamic energy of the system \[27\]. Using similar reasoning, the quantities \( j_a \) and \( s^{ab} \) are, respectively, referred to as the momentum surface density and the spatial stress.

When there is a Killing vector field \( \xi \) on the boundary \( \mathcal{T} \), an associated conserved charge is defined by

\[ Q(\xi) = \int \mathcal{B} d^2x \sqrt{\sigma} (\varepsilon u^i + j^i) \xi_i, \quad (12) \]

provided there is no matter stress energy in the neighborhood of \( \mathcal{T} \) (this assumption can be dropped, allowing one to compute the time rate of change of \( Q \) if desired [28]). When this holds, the value of \( Q \) is independent of the particular hypersurface \( \mathcal{B} \), a property not shared by the energy \( E \). For boundaries with timelike \( [\xi^a = (\partial/\partial t)^a/\Xi] \) and rotational Killing vector fields \( (\varsigma = \partial/\partial \phi) \), we obtain

\[ M = \int \mathcal{B} d^2x \sqrt{\sigma} (\varepsilon u^i + j^i) \xi_i, \quad (13) \]

\[ J = \int \mathcal{B} d^2x \sqrt{\sigma} j^i \varsigma_i, \quad (14) \]

provided the surface \( \mathcal{B} \) contains the orbits of \( \varsigma \). These quantities are, respectively, the conserved mass and angular momentum of the system enclosed by the boundary. Note that
they will both be dependent on the location of the boundary $B$ in the spacetime, although each is independent of the particular choice of foliation $B$ within the surface $T$.

In the context of the dS/CFT correspondence, the limit in which the boundary $B$ becomes infinite is taken, and the counterterm prescription $[21]$ ensures that the conserved charges $[12]$ are finite. No embedding of the surface $T$ into a reference spacetime is required. This is of particular advantage for the class of Kerr spacetimes in which it is not possible to embed an arbitrary two-dimensional boundary surface into a flat (or constant-curvature) spacetime $[16, 24]$.

### III. THE KERR-dS$_4$ SPACETIME

The class of metrics we shall consider are the Kerr-dS family of solutions in four dimension, whose general form is

$$ds^2 = -\frac{\Delta^2_r}{\rho^2} \left( dt - \frac{a}{\Xi} \sin^2 \theta d\phi \right)^2 + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Delta^2_\theta \sin^2 \theta}{\rho^2} \left( adt - \frac{(r^2 + a^2)}{\Xi} d\phi \right)^2,$$  \hspace{1cm} (15)

where

$$\Delta^2_r = (r^2 + a^2)(1 - r^2/l^2) - 2mr,$$

$$\Delta^2_\theta = 1 + \frac{a^2}{l^2} \cos^2 \theta,$$

$$\Xi = 1 + \frac{a^2}{l^2},$$

$$\rho^2 = r^2 + a^2 \cos^2 \theta.$$  \hspace{1cm} (16)

For $m = 0$, the metric (15) and (16) is that of pure dS$_4$ spacetime (or flat spacetime if $l \to \infty$), and for $a = 0$ the metric is that of Schwarzschild-dS$_4$ spacetime, which has zero angular momentum. Hence we expect the parameters $m$ and $a$ to be associated with the mass and angular momentum of the spacetime, respectively. The metric of Eqs. (15) and (16) has three horizons located at $r_-$, $r_+$, and $r_c$, provided the parameters $m$, $l$, $a$ are chosen suitably. For given values of $l$ and $a$, $m$ should lie in the range between $m_{1,\text{crit}} \leq m \leq m_{2,\text{crit}}$, where $m_{1,\text{crit}}$ and $m_{2,\text{crit}}$ are the positive real solutions of the following equation:

$$a^{10} + 4l^2 a^8 + 4l^6 a^4 + 6l^4 a^6 - 33a^2 l^6 m^2 + 33a^4 l^4 m^2 + 27m^4 l^6 - m^2 l^8 + m^2 l^2 a^6 = 0.$$  \hspace{1cm} (17)
It is worthwhile to mention that when \( m = m_{1, \text{crit}} \), \( r_- \) and \( r_+ \) are equal and we have an extreme black hole. For the case in which \( m = m_{2, \text{crit}} \), the radius of all three horizons are equal and we have a critical hole. Also, one may note that in the limit \( l \to \infty \), \( m_{1, \text{crit}} = a \), \( m_{2, \text{crit}} \to \infty \), \( r_\pm = m \pm \sqrt{m^2 - a^2} \), and \( r_c \to \infty \). For given values of \( a \) and \( m \), the parameter \( l \) should be greater than \( l_{\text{crit}} \), where \( l_{\text{crit}} \) is the positive real solution of Eq. (17). For given values of \( l \) and \( m \), the parameter \( a \) should be less than a critical value \( a_{\text{crit}} \), where it is the largest real value solution of Eq. (17).

The total action of the system, \( I = I_G + I_{ct} \), can be calculated through the use of Eqs. (1) and (2):

\[
I = -\beta_c \frac{\Xi}{2l^2} [ml^2 + r_c(r_c^2 + a^2)],
\]

where \( r_c \) is the radius and \( \beta_c \) is the inverse of the Hawking temperature of the cosmological event horizon given by

\[
\beta_c = -\frac{4\pi l^2 r_c(r_c^2 + a^2)}{3r_c^4 + a^2 r_c^2 - l^2 r_c^2 + a^2 l^2}.
\]

The conserved mass and angular momentum of the hole calculated on the boundary \( B \) at infinity calculated from Eqs. (12) are given by (21):

\[
M_\infty = -\frac{m}{\Xi},
\]

\[
J = \frac{ma}{\Xi^2}.
\]

Now generalizing the Gibbs-Duhem relation for asymptotic de Sitter black holes (8),

\[
S = I - \beta_c(M + \Omega J),
\]

the entropy associated with the cosmological event horizon can be calculated as

\[
S = \frac{\pi(r_c^2 + a^2)}{\Xi}.
\]

As we see, the entropy (23) is in agreement with the area law of entropy even for the cosmological horizon (11).
IV. THE QUASILOCAL THERMODYNAMICS OF THE BLACK HOLE

Now we consider the quasilocal thermodynamics of Kerr-dS black holes. The energy of the system can be written as

\[ E = E_1 + E_2; \quad E_i = \int_B d^2 x \sqrt{\sigma} \epsilon_i, \]  

(24)

where \( B \) is a two dimensional surface defined by setting the radial coordinate to a constant value \( r < r_c \), and \( \epsilon_1 \) and \( \epsilon_2 \) are given by

\[
\sqrt{\sigma} \epsilon_1 = \frac{2}{N} P^{ij} u_i u_j = \frac{r \Delta r}{8 \pi \rho \Xi \Delta \theta [(r^2 + a^2)^2 \Delta \theta - a^2 \Delta \pi^2 \sin^2 \theta]^{1/2}} \sin \theta, \]

\[
\epsilon_2 = \frac{2}{N} Q^{ij} u_i u_j = \frac{\sin \theta}{8 \pi l \rho \xi \Delta \theta [(r^2 + a^2)^2 \Delta \theta - a^2 \Delta \pi^2 \sin^2 \theta]^{1/2}} \times \]

\[
\left\{ \left[ \Xi (r^2 + a^2) - 2mr \right] a^6 \cos^6 \theta + \left[ 2 \Xi r^2 (r^2 + a^2) - mr (3l^2 - a^2) \right] a^6 \cos^6 \theta \right. \]

\[
+ \left[ \Xi r^2 (a^2 + a^2) (r^2 - l^2) + 2mr (4l^2 a^4 - 2r^2 a^4 + 7r^2 l^2 a^2 + 5l^2 r^4 \right. \]

\[
+ 2a^2 r^4 - 2mr^2 l^2 - 2a^6 \right] a^2 \cos^2 \theta + \Xi r^6 (r^2 + a^2) (2r^2 - l^2) - \]

\[
2ma^2 r^3 (l^2 a^2 + l^2 r^2 - 2r^4) \right\}. \]  

(25)

(26)

We first consider special cases in which the integral in Eqs. (23) and (24) can be evaluated. For \( r = r_+ \), where \( r_+ \) is the radius of the outer event horizon, we find that the total energy can be explicitly computed, yielding

\[
E(r_+) = \frac{l}{2m} \left( 1 - \frac{a^2}{l^2} - 2 \frac{r_+}{l^2} \right). \]  

(27)

For small values of angular momentum, \( E \) can be integrated easily. Then the total energy for small \( a \) is

\[
E = \frac{l}{2} \left( 1 - 2 \frac{r_+^2}{l^2} \right) - r \sqrt{1 - \frac{r^2}{l^2} - 2 \frac{m}{r} - \left\{ \frac{2}{3l} \left( 1 - \frac{r^2}{l^2} + \frac{m}{r} \right) \right. \]

\[
+ \frac{1}{3r} \left( 1 - \frac{r^2}{l^2} - 2 \frac{m}{r} \right)^{-1/2} \left( 1 - \frac{3r^2}{l^2} + 2 \frac{r^4}{l^4} + 5 \frac{mr}{l^4} + 2 \frac{m^2}{r^4} \right) a^2 + O(a^4). \]  

(28)

For general values of the parameters, however, we cannot analytically integrate (25) and (26) exactly, and so we resort to numerical integration. To do this we set the radius of
the boundary equal to \( r = 0.56 \), and we apply the restrictions discussed in Sec. I on the parameters \( a \), \( m \) and \( l \). Note that for \( r = 0.56 \), each of these parameters has two critical values, one due to the extreme Kerr-dS black hole and the other due to the case in which this radius is equal to the radius of the three event horizons (in this case the radius of the three event horizons are equal). In the first four figures we plot the quasilocal energy as a function of \( a \), \( m \), and \( l \). The \( a \) dependence of the energy is given in Figs. 1 and 2. We see that for small values of \( l \), the energy is a slowly increasing function of \( a \), while for larger values of \( l \), the energy is a decreasing function of \( a \) as in the case of the Kerr metric considered in [24]. Figures 3 and 4 show the \( m \) and \( l \) dependence of the energy. For large values of \( l \), the energy is asymptotic to a linear function of \( l \). This is due to the fact that the counterterm used for the Kerr-dS spacetime is proportional to \( l \) for large values of this parameter.

Next we consider the quasilocal conserved charges of the Kerr-dS metric, which are the mass and \( \phi \) component of the angular momentum of the system defined in Eqs. (13) and (14). The \( \phi \) component of the angular momentum due to the counterterm is zero and one can evaluate the total angular momentum as

\[
J = \frac{am}{\Xi^2},
\]

which is therefore valid for any quasilocal surface of fixed radius \( r \), and yields the angular momentum of the Kerr metric as \( l \) goes to infinity.
Using the normalized Killing vector \( \zeta^a = (\partial/\partial t)^a / \Xi \) instead of \( (\partial/\partial t)^a \) in Eq. (13), the total mass of the system can be integrated:

\[
M = \frac{1}{\Xi^2} \left( \frac{\Delta^2}{2a} \tan^{-1} \left( \frac{a}{r} \right) + \frac{r}{2} \left( 1 - \frac{r^2}{l^2} - \frac{m^2}{r^2} \right) - \frac{r^2 \Delta r}{8la} \ln \left( \frac{\sqrt{r^2 + a^2} + a}{\sqrt{r^2 + a^2} - a} \right) - \Delta \left[ \frac{a^2 + 3r^2 - 2l^2}{4l(r^2 + a^2)^{1/2}} + \frac{mla^2}{3r(r^2 + a^2)^{3/2}} \right] \right).
\]  
(30)
FIG. 4: $E$ versus $l$ for $m = 0.2$, $a = 0.1$ (solid), and 0.2 (dashed).

For $r = r_+$, one obtains

$$M(r_+) = \frac{r_+}{2\Xi^2} \left(1 - \frac{r_+^2}{l^2} - 2 \frac{m}{r_+}\right) = \frac{a^2}{2\Xi^2 r_+} \left(1 - \frac{r_+^2}{l^2}\right).$$  \(31\)

Note that in the limit of slow rotation, $M(r_+)$ vanishes, a strictly different feature from that of the energy as given in Eq. (27), which in this limit equals $(l/2)[1 - 2(r_+^2/l^2)]$.

We turn next to a thermodynamics consideration of the Kerr-dS black hole. The interior of the quasilocal surface can be regarded as a thermodynamic system, which is corotating with an exterior heat bath whose angular velocity is $\Omega_+$, where $\Omega_+ = a(r_+^2 + a^2)^{-1}$ is the angular velocity of the outer event horizon. The quasilocal thermodynamic quantities are generally given by surface integrals over the boundary data rather than products of constant surface data. In particular, it is not possible to choose the quasilocal surface to simultaneously be both isothermal and a surface of constant $\Omega$, nor is this possible for the quasilocal surface we have chosen (one at fixed $r$). Furthermore, it is by now well-recognized that the entropy of a black hole depends only on the geometry of its horizon in the classical approximation, and is conversely independent of the asymptotic behavior of the gravitational field or of the presence of external matter fields. For stationary black holes, the entropy is one-quarter of the horizon area, yielding $S = \pi(r_+^2 + a^2)/\Xi$ in this case. We shall therefore regard the energy $E$ in Eq. (24) as the thermodynamic internal energy for the Kerr black hole within the boundary with radius $r$, view it as a function of $r$, the entropy $S$, and
the angular momentum $J$, and define the temperature and the angular velocity from their ensemble derivatives, integrated over the quasilocal boundary data.

For the temperature we obtain

$$T = \left( \frac{\partial E}{\partial S} \right)_{r,J} = \left( \frac{\partial E}{\partial M} \right)_{r,J} \left( \frac{\partial S}{\partial M} \right)_{r,J}^{-1} = \beta_+ \left( \frac{\partial E}{\partial M} \right)_{r,J},$$

(32)

where $\beta_+$ is the inverse of Hawking temperature associated with the outer horizon,

$$\beta_+ = -\frac{3r^4_+ + r^2_+ (a^2 - l^2) + a^2 l^2}{4\pi l^2 r_+ (r^2_+ + a^2)}. \quad (33)$$

It is worthwhile to mention that unlike the nonrotating case, the factor $\beta_+$ does not give the Tolman redshift factor because a surface of fixed radius $r$ is not a surface of constant redshift (it is not an isotherm). The derivative $(\partial E/\partial M)_{r,J}$ is obtained by taking the derivatives of the terms in Eqs. (25) and (26) above and then performing the integration with respect to $\theta$.

Now we investigate the $a$, $m$, and $l$ dependence of the temperature of a Kerr-dS black hole. For the general case, the temperature must be computed numerically. The $a$ dependence of the temperature is shown in Fig. 5. We find that for small values of $m$, the temperature is a weakly decreasing function of $a$, and it decreases rapidly as $a$ approaches its maximum allowed value $a_{\text{crit}}$ (extreme black hole). For larger values of $m$ ($l = 1$ and $m = 0.2$), the temperature increases slowly as $a$ increases, but again it goes to zero rapidly as $a$ approaches its critical value. The $m$ dependence of $T$ is shown in Fig. 6. We find that for small values of $a$ ($a = 0.1$) the function $T(m)$ has a maximum at $m \simeq 0.116$ and a minimum at $m \simeq 0.188$ and therefore $(\partial T/\partial M)_{r,J}$ is negative between these two values. The $l$ and $r$ dependence of the temperature can be seen from Figs. 7 and 8. We find that $T$ is an initially strongly decreasing function of $l$, but as $l$ increases the temperature increases linearly. Note that for $m = 0.2$ and $a = 0.2$ (a closely extreme hole), $T$ goes to zero as $l$ goes to infinity, which is commensurable to our previous results in [24].

The heat capacity at constant surface area, $4\pi r^2$, is defined by

$$C_r = \left( \frac{\partial E}{\partial T} \right)_{r,J}, \quad (34)$$

where the energy is expressed as a function of $r$ and $T$. Expressing the energy and temperature as functions of $r$, $M$, and $J$, the heat capacity can be written as

$$C_r = \left( \frac{\partial E}{\partial M} \right)_{r,J} \left( \frac{\partial T}{\partial M} \right)_{r,J}^{-1}. \quad (35)$$
For small values of \( m \), the heat capacity is not positive for all the allowed values of \( a \). For example, as one can see from Fig. 5, for \( l = 1 \) and \( m = 0.1 \) the heat capacity is positive only for \( a \gtrsim 0.07 \), while for larger values of \( m \) (\( m = 0.2 \)) there are two stable phases, a slowly rotating black hole (\( a \lesssim 0.11 \)) and a nearly extreme hole (\( 0.17 \lesssim m \lesssim 0.21 \)). These two stable phases are separated by an intermediate angular momentum unstable phase (\( 0.11 \lesssim m \lesssim 0.17 \)). Indeed, for this value of \( m \) there exist two \( a \)'s for which the
heat capacity passes through an infinite discontinuity, characteristic of a second-order phase transition. We see from Fig. 10 that for small values of $a$ there is a phase of stable small black holes and a phase of stable large black holes, separated by a regime of intermediate unstable black holes. However, for sufficiently large $a$, the heat capacity is positive for all allowed values of $m$, and there is a single phase consisting of a nearly extreme black hole. The stability analysis for these black holes is in qualitative agreement with that of Davies.
[26], who performed a thorough analysis of the thermodynamic properties of asymptotic de
Sitter Kerr-Newman black holes. However, he used the ADM mass parameters, whereas we
are considering the quasilocal energy $E$ in Eq. (24) as a function of $r$, $S$, and $J$.

The angular velocity can be written as

$$\Omega = \left(\frac{\partial E}{\partial J}\right)_{r, S}.$$  \hspace{1cm} (36)

The $a$ dependence of the angular velocity can be seen from Fig. 11.
V. THE QUASILOCAL THERMODYNAMICS OF THE NONROTATING CASE

Now we investigate the nonrotating case in which all the thermodynamic quantities can be integrated easily. For \( a = 0 \), one obtains the quasi-local energy of a Schwarzschild-dS black hole as

\[
E = \frac{l}{2} \left( 1 - 2 \frac{r^2}{l^2} \right) - r \sqrt{1 - \frac{r^2}{l^2} - 2 \frac{m}{r}}. \tag{37}
\]

Using Eq. (32), the temperature of the black hole is

\[
T = \frac{1 - 3 \frac{r^2}{l^2}}{4 \pi r_+ \sqrt{1 - \frac{r^2}{l^2} - 2 \frac{m}{l^2}}}. \tag{38}
\]

It is worthwhile to mention that the temperature given by Eq. (38) can be obtained by a Wick rotation \( l \rightarrow il \) from the expression given in Ref. [14] for a Schwarzschild AdS black hole. The heat capacity can be obtained from (35) as

\[
C_r = -\frac{4 \pi r_+^2 \left( 1 - \frac{r^2}{l^2} - 2 \frac{m}{r} \right) \left( 1 - 3 \frac{r^2}{l^2} \right)}{r_+ \left( 3 - 2 \frac{r^2}{l^2} + 3 \frac{r^4}{l^4} \right) + 2r \left( 1 + 3 \frac{r^2}{l^2} \right) \left( 1 - \frac{r^2}{l^2} \right)}. \tag{39}
\]

One may note that the heat capacity \( C \) passes through an infinite discontinuity, characteristic of a second-order phase transition, when the denominator of the heat capacity vanishes, i.e.,

\[
3r_+^5 - 2l^2 r_+^3 - 6r(l^2 - r^2) r_+^2 + 3l^4 r_+ - 2rl^2 (l^2 - r^2) = 0. \tag{40}
\]
For a given $l$ and a fixed value of $r$ in the range $r_+ < r < r_c$, Eq. (40) has a single real solution for $r_+$, which belongs to a value for the parameter $m$ denoted by $m_0$. The heat capacity (39) for black holes with $m > m_0$ is positive and therefore they are stable. Note that as $l \rightarrow \infty$, $m_0 \rightarrow r/3$.

VI. CLOSING REMARKS

In this paper, we have computed the entropy associated with the cosmological event horizon of the Kerr-dS black hole by generalizing the Gibbs-Duhem relation for asymptotic de Sitter spacetimes. The result is in agreement with the well-known area law of the entropy. Since there is no well-defined thermodynamics for outside the cosmological horizon, an investigation of the quasilocal thermodynamics is necessary. Our investigation of the boundary-induced counterterm prescription (2) at finite distances has allowed us to obtain a number of interesting results, which we recapitulate here.

We have been able to compute the energy, mass, angular momentum, temperature, and heat capacity quasilocally for arbitrary values of the parameters of the Kerr-dS$_4$ solution, apart from the mild reality restrictions considered in Sec. III. These quasilocal quantities are intrinsically calculable numerically at fixed $r$ without any reference to a background spacetime, a marked improvement over the methods given in Ref. [13]. We also found that quasilocal angular momentum is independent of the radius of the boundary, a fact which is not true for the total mass of the spacetime. This remarkable result holds because the angular momentum due to the counterterm integrates to zero, and not because it vanishes identically.

We found that the entropy $S$ and angular momentum $J$ were not given by surface integrals over quasilocal boundary data, but rather were constants dependent on the parameters $a$, $m$ and $l$ of the black hole. We therefore chose to regard the interior of the quasilocal surface as a thermodynamic system whose energy $E$ is a function of $S$ and $J$. We found that the temperature goes to zero as the black hole approaches its extreme case, where the radii of the inner and outer horizons are equal, and goes to infinity as the radius of the boundary approaches the radius of the outer or cosmological horizons.

A stability analysis yielded results that are in qualitative agreement with previous investigations carried out by using the ADM mass parameter [26]. This stability analysis is,
of course, local, and phase transitions to other spacetimes of lower free energy might exist. Finally, we have considered the case of nonrotating black holes and found that at a fixed value of $r$ there exists a value for the mass parameter ($m_0$) for which the heat capacity passes an infinite discontinuity. For $m > m_0$, the black hole is stable while there exists a nonstable phase for $m < m_0$.

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