On birational involutions of $\mathbb{P}^3$

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Abstract. We study elements $\tau$ of order two in the birational automorphism groups of rationally connected three-dimensional algebraic varieties such that there exists a non-uniruled divisorial component of the $\tau$-fixed point locus. Using the equivariant minimal model program, we give a rough classification of such elements.

Keywords: rational map, Cremona group, Fano variety.

Dedicated to Igor Rostislavovich Shafarevich with admiration

§ 1. Introduction

Let $k$ be an algebraically closed field of characteristic zero. The Cremona group $\text{Cr}_n(k)$ is the group of birational transformations of $\mathbb{P}^n_k$ or, equivalently, the group of $k$-automorphisms of the field $k(x_1, \ldots, x_n)$. For $n = 2$ the Cremona group and its subgroups were studied extensively in the classical literature as well as in many recent works. In particular, a classification of finite subgroups in $\text{Cr}_2(k)$ was started in the works of Bertini and almost completed recently by Dolgachev and Iskovskikh [1]. The situation is much more complicated for the Cremona group in three variables. There are only a few partial results in this direction (see, for example, [2], [3]).

The modern approach to the classification of finite subgroups in Cremona groups is based on the simple observation that any finite subgroup $G \subset \text{Cr}_n(k)$ is conjugate to a biregular action on some rational projective variety $X$. For every biregular action of a finite cyclic group $G = \langle \tau \rangle$ on a smooth projective threefold $X$ one can define a subvariety $F(\tau) \subset X$, a non-uniruled component of codimension 1 in the locus of fixed points. The birational type of $F(\tau)$ (if this set is non-empty) does not depend on the choice of the model $X$ and, therefore, is an invariant of the subgroup $G \subset \text{Cr}_n(k)$ up to conjugacy.

For $n = 2$, the invariant $F(\tau)$ is sufficient to distinguish the conjugacy classes of involutions $\tau$ in $\text{Cr}_2(k)$.

Theorem 1.1 [4]. Every element $\tau \in \text{Cr}_2(k)$ of order 2 is conjugate to one and only one of the following involutions.

AMS 2010 Mathematics Subject Classification. 14E07, 20B25.
(i) A linear involution acting on \( \mathbb{P}^2 \). Here \( F(\tau) = \emptyset \).

(ii) A de Jonquières involution. Here \( F(\tau) \) is a hyperelliptic (or elliptic) curve of genus \( g \geq 1 \).

(iii) A Geiser involution. Here \( F(\tau) \) is a non-hyperelliptic curve of genus 3.

(iv) A Bertini involution. Here \( F(\tau) \) is a non-hyperelliptic curve of genus 4 whose canonical model lies on a singular quadric.

Moreover, two elements \( \tau, \tau' \in \text{Cr}_2(\mathbb{k}) \) of order 2 are conjugate if and only if \( F(\tau) \simeq F(\tau') \). In particular, \( \tau \in \text{Cr}_2(\mathbb{k}) \) is linearizable (that is, conjugate to a linear involution acting on the projective plane) if and only if \( F(\tau) = \emptyset \).

In higher dimensions, the last assertion of Theorem 1.1 does not hold: \( F(\tau) \) does not distinguish the conjugacy classes (see Example 4.6). In particular, the field of invariants \( \mathbb{k}(x_1, x_2, x_3)^\tau \) need not be rational. Thus the isomorphism class of this field is another invariant distinguishing the conjugacy classes of involutions.

In this paper we give a rough classification of elements \( \tau \) of order 2 with \( F(\tau) \neq \emptyset \) in the groups of birational automorphisms not only of rational varieties but also of arbitrary rationally connected varieties (not necessarily rational). Our main result is the following theorem.

**Theorem 1.2.** Let \( Y \) be a three-dimensional rationally connected variety, and let \( \tau \in \text{Bir}(Y) \) be an element of order 2 with \( F(\tau) \neq \emptyset \). Then \( \tau \) is conjugate to one of the following actions on a threefold \( X \) (birationally equivalent to \( Y \)).

(C) \( X \) is given by the equation \( \varphi_0(u, v)x_0^2 + \varphi_1(u, v)x_1^2 + \varphi_2(u, v)x_3^2 = 0 \) in \( \mathbb{P}^2_{x_0, x_1, x_2} \times \mathbb{A}^2_{u, v} \) and \( \tau \) acts as \( (x_0, x_1, x_2, u, v) \mapsto (-x_0, x_1, x_2, u, v) \).

(D) \( X \) is a \( \tau \)-equivariant del Pezzo fibration over a rational curve with trivial action of \( \tau \). The action on a generic fibre \( X_\eta \) satisfies the condition \( \text{Pic}(X_\eta)^\tau \simeq \mathbb{Z} \) and is conjugate to one of the following involutions.

(a) The Bertini involution \( \tau_\eta: X_\eta \to X_\eta \) with \( K_{X_\eta}^2 \) = 1.

(b) The Geiser involution \( \tau_\eta: X_\eta \to X_\eta \) with \( K_{X_\eta}^2 \) = 2.

(c) An action \( \tau_\eta: X_\eta \to X_\eta \) with \( \text{Fix}(\tau_\eta, X_\eta) = C_\eta \cup \Lambda_\eta \), where \( C_\eta \) is an elliptic curve, \( C_\eta \subset |-K_{X_\eta}| \), \( \Lambda_\eta \) is a 0-cycle of degree 4 \(- K_{X_\eta}^2 \) and \( 1 \leq K_{X_\eta}^2 \leq 4 \).

(F\text{a}) \( X \) is a Fano threefold with terminal \( G \mathbb{Q} \)-factorial singularities, \( \text{Pic}(X)^\tau \simeq \mathbb{Z} \) and one of the following conditions holds.

(a) \( \dim |-K_X| \leq 0 \).

(b) \( \dim |-K_X| = 1 \) and \( |-K_X| \) determines a birational structure of a K3-fibration over \( \mathbb{P}^1 \) (see Proposition 6.4).

(c) \( \dim |-K_X| = 2 \) and \( |-K_X| \) determines a birational structure of an elliptic curve fibration over \( \mathbb{P}^2 \) (see Proposition 6.4).

(F\text{c}) \( X \) is a Fano threefold with canonical Gorenstein singularities:

(a) \( X = X_6 \subset \mathbb{P}(1, 1, 1, 1, 3) \) is given by the equation \( y^2 = \varphi_6(x_1, \ldots, x_4) \), the singularities of \( X \) are cDV, and \( \tau \) acts by one of the following rules:

(i) \( y \mapsto -y \),

(ii) \( x_1 \mapsto -x_1 \) (and \( x_1 \) appears in \( \varphi_6 \) only in even degrees);
(b) $X$ is a double covering of a smooth quadric $W_2 \subset \mathbb{P}^4$ branched over a surface $B \subset W_2$ of degree 8, the singularities of $X$ are terminal and $\tau$ is equal to

(i) either the Galois involution $X \to W_2$,
(ii) or the involution in Example 7.5;

(c) $X \subset \mathbb{P}^4$ is a quartic with terminal singularities and $\tau$ is the involution in Example 7.6;

(d) $X \subset \mathbb{P}^5$ is a smooth intersection of a quadric and a cubic cone and $\tau$ is the involution in Example 7.7;

(e) $X \subset \mathbb{P}^6$ is a smooth intersection of three quadrics and $\tau$ is the involution in Example 7.8.

Remark 1.3. The surface $F(\tau)$ can be described in all the cases of Theorem 1.2 except $(F^q)$:

| Type          | $F(\tau)$                                                                 |
|---------------|---------------------------------------------------------------------------|
| $(C)$         | double covering of $\mathbb{P}^2$                                        |
| $(D), (a)$    | fibration over $\mathbb{P}^1$ into non-hyperelliptic curves of genus 4 whose canonical model is contained in a singular quadric |
| $(D), (b)$    | fibration over $\mathbb{P}^1$ into non-hyperelliptic curves of genus 3   |
| $(D), (c)$    | elliptic curve fibration over $\mathbb{P}^1$                             |
| $(F^c), (a), (i)$ | $\{ \varphi_6 = 0 \} \subset \mathbb{P}^3$                           |
| $(F^c), (b), (i)$ | $B = W_2 \cap U_4 \subset \mathbb{P}^4$, where $U_4$ is a quartic         |
| $(F^c)$ (other cases) | K3-surface                                                              |

Remark 1.4. Examples in the corresponding sections show that all the cases $(C); (D), (a)-(c); (F^q), (a)-(c); (F^c), (a)-(e)$ really occur.

We note that our classification is indeed ‘rough’. First, we do not provide detailed descriptions, especially in the case $(F^q)$. The reason is the lack of a classification of singular Fano threefolds. The involutions of type $(F^q), (a)$ seem to be the most difficult to study. Next, our cases can overlap and at the moment we cannot control this. Finally, in many cases we cannot select rational varieties from our list to deduce a classification of involutions in the Cremona group. For example, generic elements $X$ of the series $(F^c), (a)-(d)$ and all elements of the series $(F^c), (e)$ are non-rational [5], [6]. However there are many examples of specific (singular) Fano threefolds of the series $(F^c), (a)-(c)$ which are rational.

The paper is organized as follows. §2 is preliminary. In §3 we prove some easy technical facts concerning the set $F(\tau)$. In §§4–6 and §8 we prove Theorem 1.2 in the cases $(C), (D), (F^q)$ and $(F^c)$ respectively. In §7 we give some examples of involutions of type $(F^c)$.

§2. Preliminaries

2.1. Notation. We introduce the following notation: $\mathbb{P}(a_1, \ldots, a_n)$ is the weighted projective space, $X_d \subset \mathbb{P}(a_1, \ldots, a_n)$ stands for a hypersurface of weighted degree $d$, and $X_{d_1,d_2,\ldots,d_r} \subset \mathbb{P}(a_1, \ldots, a_n)$ stands for a weighted complete intersection of type $(d_1,d_2,\ldots,d_r)$. 
We work over an algebraically closed field $k$ of characteristic zero. By saying that a variety has, for example, terminal singularities, we mean that the singularities are no worse than that.

### 2.2. $G$-varieties

Let $G$ be a finite group. In our paper, a $G$-variety is an algebraic variety $X$ over $k$ endowed with a biregular action of $G$. The morphisms (resp. rational maps) between $G$-varieties are usually supposed to be $G$-equivariant.

A normal $G$-variety $X$ is said to be $G\mathbb{Q}$-factorial if every $G$-invariant Weil divisor on $X$ is $\mathbb{Q}$-Cartier.

A $G$-Fano–Mori fibration is a projective $G$-equivariant morphism $f: X \to Z$ such that $f_* \mathcal{O}_X = \mathcal{O}_Z$, $\dim Z < \dim X$, $X$ has only terminal $G\mathbb{Q}$-factorial singularities, the anticanonical divisor $-K_X$ is ample over $Z$ and the relative $G$-invariant Picard number $\rho(X/Z)^G$ is equal to 1. When $\dim X = 3$, we have the following possibilities:

- (C) $Z$ is a rational surface and the generic fibre $X_\eta$ is a conic,
- (D) $Z \simeq \mathbb{P}^1$ and the generic fibre $X_\eta$ is a smooth del Pezzo surface,
- (F) $Z$ is a point and $X$ is a so-called $G\mathbb{Q}$-Fano threefold.

In these situations we say that $X/Z$ is of type (C), (D), (F) respectively.

**Remark 2.1.** In case (C) there is a non-empty Zariski open subset $U \subset Z$ such that the restriction $f_U: X_U \to U$ is a conic bundle. In case (D) one can similarly find $U \subset Z$ such that $f_U: X_U \to U$ is a smooth del Pezzo surface.

**Remark 2.2.** We shall also consider a class of Fano $G$-varieties different from (F):

- $(F^c)$ Fano threefolds with canonical Gorenstein (not necessarily $G\mathbb{Q}$-factorial) singularities.

This is sometimes more convenient for classification purposes. We write $(F^q)$ for the subclass of (F) consisting of $G\mathbb{Q}$-Fano threefolds $X$ such that $K_X$ is not Cartier.

**Proposition 2.3.** Let $G$ be a finite group and let $X$ be a rationally connected $G$-variety. Then one can find a $G$-Fano–Mori fibration $f: X \to Z$ and a $G$-equivariant birational map $X \dashrightarrow \overline{X}$.

**Proof.** Using the standard argument (see, for example, [3]), we can replace $X$ by a non-singular projective model such that the action of $G$ on $X$ is biregular. Then we run the $G$-equivariant minimal model program ($G$-MMP): $X \dashrightarrow \overline{X}$ (in higher dimensions we can run $G$-MMP with scaling, see [7], Corollary 1.3.3). Note that [7] deals with varieties without group actions, but adding an action of a finite group does not make a big difference (see [8], Example 2.18). Running this program, we always stay in the category of projective varieties with only terminal $G\mathbb{Q}$-factorial singularities. Since $X$ is rationally connected, the canonical divisor cannot be numerically effective at the final step ([9], Theorem 1). Hence we get a $G$-Fano–Mori fibration $f: X \to Z$. □

**Definition 2.4.** With the notation of Proposition 2.3 and under the assumption that $\dim X = 3$, we say that the original $G$-variety belongs to the class (C) (resp. (D), (F)) if $X/Z$ does. Clearly, the choice of the birational map $X \dashrightarrow \overline{X}$ is non-unique, so the classes (C), (D), (F) can overlap.
2.3. G-minimal model program for pairs [10]. Let $X$ be a normal $G$-variety and let $\mathcal{M}$ be an invariant linear system of Weil divisors without fixed components. Then we say that $(X, \mathcal{M})$ is a $G$-pair. The discrepancy $a(E, X, \mathcal{M})$ of a prime divisor $E$ with respect to $(X, \mathcal{M})$ can be defined in the standard way. We say that the pair $(X, \mathcal{M})$ is terminal (resp. canonical) if $a(E, X, \mathcal{M}) > 0$ (resp. $a(E, X, \mathcal{M}) \geq 0$) for all exceptional divisors $E$ over $X$. One can run the G-minimal model program in the category of $G$-pairs $(X, \mathcal{M})$ such that $X$ is $G\mathbb{Q}$-factorial and $(X, \mathcal{M})$ is terminal. In particular, for every $G$-pair $(X, \mathcal{M})$ there is a terminal model $f: (X', M') \to (X, \mathcal{M})$, where $f$ is a $G$-equivariant morphism and $(X', M')$ is a terminal $G$-pair such that $K_{X'} + M'$ is $f$-numerically effective, $M'$ is the birational transform of $\mathcal{M}$ and $X'$ is $G\mathbb{Q}$-factorial. We can also write

$$K_{X'} + M' = f^*(K_X + \mathcal{M}) - \sum_i a_i E_i,$$

where the $E_i$ are $f$-exceptional divisors, $a_i \geq 0$ for all $i$, and $a_i = 0$ for all $i$ if and only if $(X, \mathcal{M})$ is canonical. In the last case we say that $f$ is log crepant.

2.4. Varieties of minimal degree. For convenience of reference we recall the following well-known fact.

**Theorem 2.5.** Let $W \subset \mathbb{P}^N$ be an $n$-dimensional projective variety not lying in a hyperplane. Then $\deg W \geq \text{codim} W + 1$ and equality holds if and only if $W$ is one of the following varieties.

1) $W = \mathbb{P}^N$.

2) $W = W_2 \subset \mathbb{P}^N$ is a quadric.

3) $W$ is the image of $\mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$, where $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2}(2) \oplus \bigoplus_{i=1}^{n-2} \mathcal{O}_{\mathbb{P}^2}$, under the (birational) morphism given by $|\mathcal{O}(1)|$.

4) $W$ is the image of $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})$, where $\mathcal{E} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(a_i)$, $a_i \geq 0$, under the (birational) morphism given by $|\mathcal{O}(1)|$.

2.5. Fano threefolds. Let $X$ be a Fano threefold with only canonical Gorenstein singularities. By the Riemann–Roch formula and the Kawamata–Vieweg vanishing theorem, we have $\dim |-K_X| = g + 1$, where $g = g(X)$ is a positive integer such that $-K_X^3 = 2g - 2$. This integer is called the genus of $X$. Let $\Phi: X \dashrightarrow \mathbb{P}^{g+1}$ be the anticanonical map. All such Fano varieties are divided in the following groups (see [11]–[14]).

(i) $\text{Bs} |-K_X| \neq \emptyset$ and $\Phi(X)$ is a surface as in Theorem 2.5.

(ii) (hyperelliptic case) $\text{Bs} |-K_X| = \emptyset$ and $\Phi$ is a double covering of a threefold $W \subset \mathbb{P}^{g+1}$ as in Theorem 2.5.

(iii) (trigonal case) $\Phi$ is an embedding and the image $\Phi(X)$ is not an intersection of quadrics. In this case the quadrics passing through $\Phi(X)$ cut out a fourfold as in Theorem 2.5.

(iv) (main series) $\Phi$ is an embedding and the image $\Phi(X)$ is an intersection of quadrics.

In the above notation, $X$ is called a del Pezzo threefold if its anticanonical class is divisible by $2$ in $\text{Pic}(X)$, that is, $-K_X \sim 2A$ for some $A \in \text{Pic}(X)$. The degree of $X$ is defined as $d = d(X) := A^3$. 
Theorem 2.6 ([11], Ch. 2, §1, [15], Ch. 1, §6). Let $X$ be a del Pezzo threefold and let $A := -\frac{1}{2}K_X$. Then $\dim |S| = d(X) + 1$ and $d(X) \leq 8$. Moreover, the following assertions hold.

1) If $d = 1$, then $X \cong X_6 \subset \mathbb{P}(1^3, 2, 3)$. The linear system $|2A|$ determines a double covering $X \to \mathbb{P}(1^3, 2)$ whose branch locus $B \subset \mathbb{P}(1^3, 2)$ is a surface of weighted degree 6.

2) If $d = 2$, then $X \cong X_4 \subset \mathbb{P}(1^4, 2)$. The linear system $|A|$ determines a double covering $X \to \mathbb{P}^3$ whose branch locus $B \subset \mathbb{P}^3$ is a surface of degree 4.

3) If $d = 3$, then $X \cong X_3 \subset \mathbb{P}^4$.

4) If $d = 4$, then $X \cong X_{2,2} \subset \mathbb{P}^5$.

§ 3. The locus of fixed points

In what follows $G$ stands for a finite cyclic group generated by an element $\tau$. Let $X$ be a $G$-variety of dimension $n$. We denote the set of fixed points by $\text{Fix}(\tau, X)$ and write $\text{Fix}(\tau)$ when no confusion is likely.

Remark 3.1. Assume that $X$ has only $G\mathbb{Q}$-factorial terminal singularities and let $F = \bigcup F_i$ be the union of all $(n-1)$-dimensional components $F_i \subset \text{Fix}(\tau)$. Then $\text{Sing}(F) \subset \text{Sing}(X)$, $F$ is the disjoint union of its irreducible components, and each component $F_i$ is normal. Indeed, for every point $P \in F$, the induced action of $\tau$ on the Zariski tangent space $T_{P,F}$ is trivial. On the other hand, the action on $T_{P,X}$ is non-trivial. Hence $T_{P,F} \not\subset T_{P,X}$ and, therefore, $\text{Sing}(F) \subset \text{Sing}(X)$. In particular, distinct components $F_i, F_j$ cannot meet each other outside $\text{Sing}(X)$. Note that $\text{codim} \text{Sing}(X) \geq 3$ since the singularities of $X$ are terminal. On the other hand, $F_i$ and $F_j$ are $\mathbb{Q}$-Cartier divisors because $X$ is $G\mathbb{Q}$-factorial. Therefore $F_i \cap F_j = \emptyset$. The same argument shows that each $F_i$ is smooth in codimension 1. Since $F_i$ is Cohen–Macaulay (see, for example, [8], Corollary 5.25), it is normal.

Remark 3.2. Assume that $X$ has the structure of a $G$-Fano–Mori fibre space $f: X \to Z$ and there is a non-uniruled irreducible component $S \subset \text{Fix}(\tau, X)$ of dimension $n - 1$. Then $S$ dominates $Z$ (see, for example, [16], Corollary 1.3). Therefore the action of $\tau$ on $Z$ is trivial.

Proposition 3.3. Assume that $X$ is rationally connected.

1) Then there is at most one irreducible component $S \subset \text{Fix}(\tau, X)$ of dimension $n - 1$ which is not uniruled.

2) Let $\psi: X \dasharrow X'$ be a birational map, where $X'$ is a projective variety with Kawamata log terminal singularities (with respect to some boundary). Assume that there is a divisorial component $S \subset \text{Fix}(\tau, X)$ which is not uniruled. Then $\psi_* S$ is a divisor on $X'$.

Proof. To prove part 1), assume the contrary: there are two such components $S_1$ and $S_2$. As in Proposition 2.3, we may assume that $X$ is projective and smooth. Run the equivariant MMP $X \dasharrow \overline{X}$. The non-uniruled divisors $S_i$ cannot be contracted (see, for example, [16], Corollary 1.3). Therefore their images $\overline{S}_1$ and $\overline{S}_2$ are also divisors contained in $\text{Fix}(\tau, \overline{X})$. Since $X$ is rationally connected, it cannot have a minimal model, that is, the canonical divisor $K_{\overline{X}}$ cannot be numerically...
effective ([9], Theorem 1). Thus $\overline{X}$ has the structure of a $G$-Fano–Mori fibre space $f: \overline{X} \rightarrow Z$. By Remark 3.1, the divisors $\overline{S}_1, \overline{S}_2$ are disjoint. By Remark 3.2, they are $f$-ample and the action of $G$ on $Z$ is trivial. Restricting $\overline{S}_1 + \overline{S}_2$ to a generic fibre $F = f^{-1}(z)$, we get an ample disconnected divisor. This is possible only for $\dim F = 1$, whence $F \simeq \mathbb{P}^1$. Since the action is non-trivial on $F$, we see that $\text{Fix}(\tau, \overline{X}) \cap F$ consists of two points: $\overline{S}_1 \cap F$ and $\overline{S}_2 \cap F$. On the other hand, $Z$ is rationally connected and, therefore, $\overline{S}_i$ cannot generically be a section of $f$, a contradiction.

Part 2) follows from [16], Corollary 1.6. □

We denote the non-uniruled component of codimension 1 of $\text{Fix}(\tau, X)$ by $F(\tau, X)$ or simply by $F(\tau)$ (if no confusion is likely). By Proposition 3.3, (ii), the birational type of $F(\tau, X)$ does not depend on the choice of the projective model $X$ (with only log terminal singularities) acted on biregularly by $G$. In particular, we can define the Kodaira dimension $\kappa(\tau, X)$ to be equal to $\kappa(F(\tau, X))$ if $F(\tau, X) \neq \emptyset$ and to $-\infty$ otherwise.

§ 4. Conic bundles

In what follows $G = \{1, \tau\}$ is a group of order 2.

**Lemma 4.1.** Let $f: X \rightarrow Z$ be a $G$-Fano–Mori fibre space over a rational surface $Z$. Assume that the action of $G$ on $Z$ is trivial. Then $f$ is $G$-birationally equivalent to the action

$$\tau: (x_0, x_1, x_2; u, v) \mapsto (-x_0, x_1, x_2; u, v)$$

(4.1)

on the hypersurface in $\mathbb{P}^2_{x_0, x_1, x_2} \times \mathbb{A}^2_{u, v}$ given by

$$\varphi_0(u, v)x_0^2 + \varphi_1(u, v)x_1^2 + \varphi_2(u, v)x_2^2 = 0.$$  (4.2)

Moreover, the quotient $X/G$ is rational.

**Proof.** First, by shrinking $Z$, we may assume that $Z \subset \mathbb{A}^2_{u, v}$ is a Zariski open subset, $X$ is smooth and $f$ is a conic bundle over $Z$. Then the linear system $|-K_X|$ determines a $G$-equivariant embedding $X \hookrightarrow \mathbb{P}^2_{x_0, x_1, x_2} \times \mathbb{A}^2_{u, v}$ such that the fibres of $f$ are conics in $\mathbb{P}^2_{x_0, x_1, x_2}$ (see [5], Proposition 1.2). Clearly, the action on $\mathbb{P}^2 \times \mathbb{A}^2$ can be written in the form (4.1). Then, by changing the coordinate system, we may assume that $X$ is given by (4.2). Finally, $X/G$ is rational because $k(X/G) \simeq k(u, v, x_1/x_2)$. □

**Corollary 4.2.** Let $Y$ be a rationally connected $G$-threefold with $G = \{1, \tau\}$ and $\kappa(\tau, Y) \geq 0$. Assume that $(Y, G)$ belongs to the class (C) and let $f: X \rightarrow Z$ be a conic bundle model, that is, a $G$-Fano–Mori fibration with $\dim Z = 2$ such that there is a $G$-equivariant birational map $Y \dasharrow X$. Then the action on $Z$ is trivial and the restriction $f_{F(\tau)}: F(\tau) \rightarrow Z$ is generically a double covering. Hence the action of $G$ is conjugate to the action given by (4.1), (4.2).

**Proof.** Put $S := F(\tau)$. By Remark 3.2, the action on $Z$ is trivial and $f(S) = Z$. This proves the first assertion. Furthermore, for a generic fibre $F$ we have $F \simeq \mathbb{P}^1$.
and the action of $G$ on $F$ is not trivial. Then $F \cap S$ consists of two points and, therefore, $f|_S: S \to Z$ is generically a double covering. □

**Remark 4.3.** We recall that the *irrationality degree* of an algebraic variety $V$ is the minimal degree of a dominant rational map from $V$ to $\mathbb{P}^n$ [17]. By Corollary 4.2, the irrationality degree of $F(\tau)$ is equal to 2 if $\tau$ is an involution with $F(\tau) \neq \emptyset$ acting on a conic bundle. The following construction shows that *any* algebraic surface of irrationality degree 2 is birationally equivalent to $F(\tau)$ for some involution in $\text{Cr}_3(k)$.

**Construction 4.4.** Let $\pi: S \to Z$ be a generically finite rational map of degree 2, where $Z$ is a rational surface. By shrinking $Z$, we may assume that $\pi$ is a morphism, $Z$ is a Zariski open subset of $\mathbb{A}^2_{u,v}$, and $S$ is given in $\mathbb{A}^2_{u,v} \times \mathbb{P}^1_x$ by the equation $x^2 = \varphi(u,v)$.

We define an action of $\tau$ on $X := \mathbb{A}^2_{u,v} \times \mathbb{P}^1_x$ by the formula

$$\tau: (u, v, x) \mapsto (u, v, \varphi(u,v)x^{-1}).$$

Clearly, $\text{Fix}(\tau) = S$.

We now give a series of examples of involutions with $F(\tau) = \emptyset$ in the groups of birational self-maps of conic bundles. In particular, this yields examples of non-linearizable involutions in $\text{Cr}_3(k)$ with $F(\tau) = \emptyset$ (compare Theorem 1.1).

**Construction 4.5.** Let $h: Y \to Z$ be a conic bundle over a surface. Assume that there is a rational surface $S \subset Y$ such that the restriction $h|_S: S \to Z$ is a generically finite morphism of degree 2 (hence $S$ is a birational 2-section of $h$). By shrinking $Z$, we may assume that the morphism $h|_S$ is finite. Consider the base change $f: X = Y \times_Z S \to S$. Then $f$ is a locally trivial fibration into rational curves. In particular, $X$ is rational. Then the Galois involution $\tau$ of the double covering $X \to Y$ determines an element of $\text{Cr}_3(k)$. Note that if $Y = X/\tau$ is not rational, then $\tau$ is not linearizable. Indeed, otherwise $X/\tau$ is birationally equivalent to the rational variety $\mathbb{P}^3/\tau'$, where $\tau'$ is a linear involution of $\mathbb{P}^3$ conjugate to $\tau$.

**Example 4.6.** Let $V \subset \mathbb{P}^4$ be a smooth cubic hypersurface, $L \subset V$ a line, $Y \to V$ the blow-up of $L$, and $S$ the exceptional divisor. The projection away from $L$ induces a conic bundle structure $Y \to \mathbb{P}^2$, and $S$ is a rational 2-section. The cubic $V$ is not rational [18]. By Lemma 4.1, our construction gives an example of a non-linearizable involution in $\text{Cr}_3(k)$ with $F(\tau) = \emptyset$.

§ 5. del Pezzo fibrations

In this section we study the case (D).

**Lemma 5.1.** Let $\Gamma$ be any finite group. Then there is a natural bijection between the following sets:

1) the set of all $\Gamma$-del Pezzo fibrations $f: X \to \mathbb{P}^1$ with trivial action on $\mathbb{P}^1$ modulo $\Gamma$-equivariant birational transformations over $\mathbb{P}^1$,

2) the set of all del Pezzo surfaces $V$ over $k(t)$ admitting an effective $\Gamma$-action with $\text{Pic}(V)^\Gamma \simeq \mathbb{Z}$ modulo $\Gamma$-isomorphisms.
Proof. Write \( k(\mathbb{P}^1) = k(t) \) and let \( \eta = \text{Spec} k(t) \) be the generic point. Let \( f : X \to \mathbb{P}^1 \) be a \( \Gamma \)-del Pezzo fibration with trivial action on \( \mathbb{P}^1 \). Then \( \Gamma \) acts on the generic fibre \( V := X_\eta \), which is a del Pezzo surface over \( k(t) \) with \( \text{Pic}(V)^\Gamma \simeq \mathbb{Z} \). Conversely, if \( \Gamma \) acts on a del Pezzo surface \( V \) defined over \( k(t) \) and \( \text{Pic}(V)^\tau \simeq \mathbb{Z} \), then we can take a subring \( k \subset R \subset k(t) \) such that the quotient field of \( R \) coincides with \( k(t) \), \( Z := \text{Spec} R \) is a rational affine normal curve, and there is a variety \( X_Z \) over \( R \) such that \( X_Z \times_{\text{Spec} R} \text{Spec} k(t) = V \). Then, as in Proposition 2.3, we get a \( \Gamma \)-del Pezzo fibration \( X/\mathbb{P}^1 \). □

We recall that \( G = \{1, \tau\} \) in our case.

Proposition 5.2. Let \( f : X \to Z \) be a \( G \)-equivariant del Pezzo fibration over a rational curve \( Z \) with \( \text{Pic}(X/Z)^G \simeq \mathbb{Z} \) and \( F(\tau) \neq \emptyset \). We denote the generic fibre by \( X_\eta \) and put \( d := K^2_{X_\eta} \). Then one of the following assertions holds.

(i) \( d = 1 \) and \( \tau \) is a Bertini involution on \( X_\eta \).
(ii) \( d = 2 \) and \( \tau \) is a Geiser involution on \( X_\eta \).
(iii) \( 1 \leq d \leq 4 \) and \( \tau \) has an elliptic curve of fixed points on \( X_\eta \).

Moreover, \( \text{Pic}(X_\eta/G) \simeq \mathbb{Z} \).

Proof. We recall that the action on \( Z \) is trivial by Remark 3.2. Put \( S := F(\tau, X) \). Let \( X_\eta \) be the generic fibre and let \( S_\eta \subset X_\eta \) be the generic fibre of \( S/Z \). Thus \( S_\eta \) is a curve of \( \tau \)-fixed points. Since \( \kappa(S) \geq 0 \), we have \( p_a(S_\eta) > 0 \). If \( p_a(S_\eta) > 1 \), then by [4] (compare with Theorem 1.1) we have cases (i) and (ii). Assume that \( p_a(S_\eta) = 1 \). Then \( S_\eta \) is an elliptic curve and \( S_\eta \in |-K_{X_\eta}| \). In this case the statement follows from Lemma 5.3 below.

The last assertion is clear since \( \rho(X_\eta/G) = \rho(X_\eta)^G = 1 \). □

Lemma 5.3. Let \( V \) be a del Pezzo surface over a field \( \mathbb{L} \) of characteristic 0, \( \tau \in \text{Aut}_\mathbb{L}(V) \) an element of order 2, and \( W := V/\tau \). Assume that \( F(\tau, V) \) is a (smooth) elliptic curve. Then \( W \) is a del Pezzo surface with only Du Val singularities of type \( A_1 \), \( K^2_W = 2K^2_V \), and

\[
\# \text{Sing}(W) + K^2_V = 4.
\]

In particular, \( K^2_V \leq 4 \).

Proof. We may assume that \( \mathbb{L} = \mathbb{C} \). Let \( \pi : V \to W \) be the quotient map. Put \( s := \# \text{Sing}(W) \) and \( d := K^2_V \). Near each fixed point \( P \in V \), the action of \( \tau \) can be written in suitable (analytic) coordinates as \( \tau : (x_1, x_2) \mapsto (-x_1, x_2) \) or \( \tau : (x_1, x_2) \mapsto (-x_1, -x_2) \). In the first case, the point \( \pi(P) \) is smooth. In the second, \( \pi(P) \) is of type \( A_1 \). Let \( C \) be the sum of the one-dimensional components of \( \text{Fix}(\tau) \) and let \( C_1 := F(\tau, V) \subset C \).

We claim that \( C_1 = C \sim -K_V \). Indeed, assume that \( C_1 \not\sim -K_V \). Since \( (K_V + C_1) \cdot C_1 = 2p_a(C_1) - 2 = 0 \) and \( K_V \cdot C_1 < 0 \), we have \( C_1^2 > 0 \). By the Hodge index theorem,

\[
(K_V + C_1) \cdot K_V = (K_V + C_1)^2 < 0.
\]

Hence \( d = K^2_V < -K_V \cdot C_1 \). On the other hand, the action of \( \tau \) on the linear system \( |-r_dK_V| \) is non-trivial, where \( r_1 = 2 \) and \( r_d = 1 \) if \( d > 1 \). Indeed, otherwise
the morphism given by $|−rdK_V|$ factors through $W$, and then $\tau$ must be either a Bertini involution or a Geiser involution. Hence $C$ is a component of a divisor $D \subseteq |−rdK_V|$. Therefore $d < −K_V \cdot C_1 \leq −K_V \cdot D = rd$. This is possible only when $d = 1$ and $D = C_1$. Hence $p_a(C_1) = 2$, a contradiction. Thus, $C_1 \sim −K_V$. Note that $C$ is smooth (because $C \subset \text{Fix}(\tau)$). In particular, $C$ is the disjoint union of its irreducible components. Since $C_1 \sim −K_V$ is ample, we have $C = C_1$.

Now, by the Hurwitz formula, $K_V = \pi^*K_W + C$, whence $2K_V = \pi^*K_W$. Therefore the divisor $−K_W$ is ample, that is, $W$ is a del Pezzo surface with only Du Val singularities of type $A_1$ and $K_W^2 = 2d$. Applying the Noether formula to the minimal resolution of singularities of $W$, we get

$$\rho(W) + s = 10 − K_W^2 = 10 − 2d.$$ 

On the other hand, by the Lefschetz fixed-point formula,

$$s = 2 + \text{Tr}_{H^2(V,R)}(\tau) = 2 + \rho(W) − (\rho(V) − \rho(W)) = 2 + 2\rho(W) − 10 + d.$$ 

Thus $s + d = 4$. □

We now give two examples of involutions in $\text{Cr}_3(\mathbb{k})$ of the types described in Proposition 5.2, (i), (ii).

**Example 5.4** (compare [19], Examples 7.11, 8.10, 4.6, 5.5). Let $X = X_4 \subset \mathbb{P}(1^4, 2)x_1, \ldots, x_4,y$ be a hypersurface of degree 4 with only terminal singularities, and let $\pi: X \rightarrow \mathbb{P}^3x_1, \ldots, x_4$ be the projection. Then $\pi$ is a double covering branched over a quartic $B \subset \mathbb{P}^3$ and $X$ is a del Pezzo threefold of degree 2. Let $\tau$ be the Galois involution. We call it the Geiser involution of $X$. One can write the equation of $X$ in the form

$$y^2 = \varphi_4(x_1, \ldots, x_4),$$

where $\varphi_4 = 0$ is the equation of $B$ and $\tau$ acts by the formula $x_i \mapsto x_i$, $y \mapsto −y$. Let $l_1$ and $l_2$ be general linear forms in $x_1, \ldots, x_4$. Then the map

$$X \rightarrow \mathbb{P}^1, \quad (x_1, \ldots, x_4, y) \rightarrow (l_1, l_2),$$

determines a birational $\tau$-equivariant structure of del Pezzo fibration on $X$ as in Proposition 5.2, (ii). Clearly, $\text{Fix}(\tau) \simeq B$ and this surface is either K3 with only Du Val singularities or birationally ruled. For a generic choice of $B$, the variety $X$ is not rational. However, $X$ can be rational for some special $B \subset \mathbb{P}^3$. For example, let $\mathbb{P}^3 \rightarrow \mathbb{P}^3$ be the blow-up of 6 points in general position and let $\mathbb{P}^3 \rightarrow X$ be the contraction of $K$-trivial curves. Then $X$ is a del Pezzo threefold as above. This variety $X$ is rational and has exactly 16 singular points. Hence $B$ is a Kummer surface. See [19] for more examples.

**Example 5.5** (compare [19], Examples 7.13, 5.5). As above, starting with a hypersurface $X = X_6 \subset \mathbb{P}(1^3, 2, 3)$, we get the Bertini involution on $X$. A general projection determines a birational $\tau$-equivariant structure of a del Pezzo fibration on $X$ as in Proposition 5.2, (i). In this case $\text{Fix}(\tau)$ is a hypersurface of degree 6 in $\mathbb{P}(1^3, 2)$, a surface of general type.
Involutions of type 5.2 (iii) can be given by the following series of examples.

**Example 5.6.** Let $Q \subset \mathbb{P}^3_{k(t)}$ be a smooth quadric defined over $k(t)$. Assume that $\text{Pic}(Q) \simeq \mathbb{Z}$. For example, we can take $Q = \{x_0x_1 + x_2^2 + tx_3^2 = 0\}$. Let $V \to Q$ be the double covering branched over a smooth curve $C \in |-K_Q|$ (which is also defined over $k(t)$). Then $V$ is a del Pezzo surface of degree 4. It is easy to see that $V$ is an intersection of two quadrics $Q_1, Q_2$ in $\mathbb{P}^4_{k(t)}$ such that $Q_1$ is a cone over $Q$ and $Q_2 \cap Q = C$. Let $\tau$ be the Galois involution of $V/Q$. Since $\text{Pic}(Q) \simeq \mathbb{Z}$, we have $\text{Pic}(V)^{\tau} \simeq \mathbb{Z}$. We get an action of $\tau$ on a del Pezzo fibration as in Proposition 5.2, (iii).

**Example 5.7.** Let $V \subset \mathbb{P}^3_{k(t)}$ be a smooth cubic surface given by the equation $x_0^2x_1 + x_1\varphi_2(x_1, x_2, x_3) + \varphi_3(x_1, x_2, x_3) = 0$, where $\varphi_d$ is a homogeneous polynomial of degree $d$ and $\varphi_3(x_1, x_2, x_3)$ is irreducible over $k(t)$. Thus $(1 : 0 : 0 : 0)$ is an Eckardt point, and the lines passing through it are conjugate under $\text{Gal}(k(t)/k(t))$. The involution $\tau$ acts by the formula $x_0 \mapsto -x_0$, and the quotient $V/\tau$ is a del Pezzo surface of degree 6 with a point of type $A_1$. We get an action of $\tau$ on a del Pezzo fibration as in Proposition 5.2, (iii).

**Example 5.8.** Let $V \subset \mathbb{P}_{k(t)}(1,1,1,2)$ be a smooth hypersurface of weighted degree 4. We can write its equation as $y^2 = \varphi_4(x_1, x_2, x_3)$. Assume that $\varphi_4$ contains only terms of even degree in $x_1$. Then $V$ is invariant under the involution $\tau: x_1 \mapsto -x_1$. The quotient $V/\tau$ is a hypersurface of weighted degree 4 in $\mathbb{P}_{k(t)}(1,1,2,2)$. This is a del Pezzo surface of degree 4 with two points of type $A_1$. As above, we get an action of $\tau$ on a del Pezzo fibration as in Proposition 5.2, (iii).

**Example 5.9.** Let $V \subset \mathbb{P}_{k(t)}(1,1,2,3)$ be a smooth hypersurface of weighted degree 6. We can write its equation as $z^2 = \varphi_6(x_1, x_2, y)$. Assume that $\varphi_6$ contains only terms of even degree in $x_1$. The involution $\tau$ acts on $V$ by the formula $\tau: x_1 \mapsto -x_1$. As above, we get an action of $\tau$ on a del Pezzo fibration as in Proposition 5.2, (iii).

§ 6. Non-Gorenstein Fano threefolds

In this section we study the case (F4). Since we assert nothing in the case when $\dim |-K_X| \leq 0$, we may assume that $\dim |-K_X| \geq 1$. The essential result is that for a Fano threefold with $\dim |-K_X| \geq 3$ admitting an involution with $\text{F}(\tau) \neq \emptyset$ there is an equivariant birational transformation to a threefold with the structure of a fibration of type (C) or (D), or to a Fano threefold with canonical Gorenstein singularities. We need the following simple but very useful lemma.

**Lemma 6.1.** Let $X$ be a normal projective $G$-threefold and let $\mathcal{M}$ be a $G$-invariant linear system without fixed components such that $-K_X + \mathcal{M} \sim_\mathbb{Q} D$, where $D$ is an effective $\mathbb{Q}$-divisor. If either $D \neq 0$ or the pair $(X, \mathcal{M})$ is not canonical, then $(X, G)$ is of type (C) or (D).

**Proof.** Take a $G$-invariant pencil $\mathcal{L} \subset \mathcal{M}$ (without fixed components) and replace $(X, \mathcal{L})$ by its terminal $G\mathbb{Q}$-factorial model (see § 2.3). Then we can write $-K_X \sim_\mathbb{Q} \mathcal{L} + D$, where $D > 0$. Run the $G$-equivariant $(K_X + \mathcal{L})$-MMP. At each step
we contract a \((K + \mathcal{L})\)-negative extremal ray, and \(-(K + \mathcal{L}) \sim_{\mathbb{Q}} D\). Hence \(D\) is not contracted and \(K + \mathcal{L}\) cannot be numerically effective. If \((X, G)\) is not of type (C) or (D), then at the end we get a pair \((X', \mathcal{L}')\) such that \(\text{Pic}(X')^G \simeq \mathbb{Z}\) and \(-K_{X'} \sim_{\mathbb{Q}} \mathcal{L}' + D'\), where \(D' > 0\). We can write \(-K_X \sim_{\mathbb{Q}} a' \mathcal{L}'\) for some \(a' > 1\). By our construction, the pair \((X', \mathcal{L}')\) is terminal. Hence a general element \(L' \in \mathcal{L}'\) is smooth and lies in the smooth locus of \(X'\) ([10], Lemma 1.22). By the adjunction formula, \(-K_{L'} = (a' - 1)L'|_{L'}\). Hence \(L'\) is a del Pezzo surface. In particular, \(L'\) is rational and, therefore, \(X\) has a \(G\)-invariant pencil of rational surfaces. Resolving the base locus and running the relative \(G\)-MMP, we get a \(G\)-Fano–Mori fibration of type (D) or (C). □

**Corollary 6.2.** Let \(X\) be a \(G\mathbb{Q}\)-Fano threefold. Assume that \(-K_X \sim D + M\), where \(D\) and \(M\) are effective divisors with \(D \neq 0\) and \(\dim |M| > 0\). Then \((X, G)\) is of type (C) or (D).

**Corollary 6.3** (compare [10], §4). Let \(X\) be a \(G\mathbb{Q}\)-Fano threefold. Assume that \(\dim |-K_X| \geq 1\) and \((X, G)\) is neither of type (C) nor of type (D). Then the pair \((X, |-K_X|)\) is canonical. In particular, \(|-K_X|\) has no fixed components.

**Proposition 6.4** (compare [10], Theorem 4.5). Let \(X\) be a non-Gorenstein \(G\mathbb{Q}\)-Fano threefold. Assume that \(\dim |-K_X| \geq 1\) and \((X, G)\) is neither of type (C) nor of type (D). Then there are \(G\)-equivariant birational morphisms

\[ X \leftarrow f \tilde{X} \xrightarrow{h} Y, \]

where \(f\) is a log crepant terminal model of the pair \((X, |-K_X|)\) and the singularities of \(\tilde{X}\) are terminal Gorenstein. Furthermore, there are the following possibilities:

(i) \(Y \simeq \mathbb{P}^1\), \(\dim |-K_X| = 1\) and \(h\) is a K3-fibration,

(ii) \(Y \simeq \mathbb{P}^2\), \(\dim |-K_X| = 2\) and \(h\) is an elliptic fibration,

(iii) \(Y\) is a Fano threefold with only Gorenstein canonical singularities and \(h\) is a birational \(K\)-crepant morphism.

**Proof** (compare [10], Theorem 4.5). By Corollary 6.3, the pair \((X, |-K_X|)\) is canonical. Let \(f\) be a log crepant terminal model for \((X, |-K_X|)\). Thus \((\tilde{X}, |-K_{\tilde{X}}|)\) is terminal and \(\tilde{X}\) is \(G\mathbb{Q}\)-factorial. Then \(-K_{\tilde{X}}\) is base-point free (see the proof of Theorem 4.5 in [10]) and, in particular, \(K_{\tilde{X}}\) is a Cartier divisor. Let \(\tilde{f}: \tilde{X} \to \tilde{X} \subset \mathbb{P}^n\) be the morphism determined by \(|-K_{\tilde{X}}|\), and let \(\tilde{X} \to Y \to \tilde{X}\) be the Stein factorization. We have a \(G\)-equivariant diagram

\[ \begin{array}{ccc}
\tilde{X} & \xrightarrow{f} & X \\
\downarrow h & & \downarrow f \\
Y & \xrightarrow{j} & \tilde{X} \subset \mathbb{P}^n
\end{array} \]

Let \(\tilde{H} \in |-K_{\tilde{X}}|\) be a generic element. Then \(\tilde{H}\) is a smooth K3-surface. Since the restriction map \(H^0(\tilde{X}, -K_{\tilde{X}}) \to H^0(\tilde{H}, -K_{\tilde{X}})\) is surjective, we see from [20] that
the image $\tilde{f}(\tilde{H}) \subset \mathbb{P}^n$ is either

1) a rational normal curve,

2) a rational normal surface of degree $n - 1$ (see Theorem 2.5), or

3) a K3-surface with Du Val singularities.

Consider the case $\dim X = 1$. Then $X = \tilde{f}(\tilde{H})$ is a rational normal curve and

$$-K_X = \tilde{f}^* \mathcal{O}(1) = n\tilde{f}^*\tilde{P},$$

where $\tilde{P} \in X$ is a point. By Corollary 6.2, $n = 1$ and we get (i).

Consider the case $\dim X = 2$. Then we are in one of situations 1), 2). Both possibilities yield that $X \subset \mathbb{P}^n$ is a rational normal surface of degree $n - 1$. Since $-K_X = \tilde{f}^* \mathcal{O}(1)$, it follows from Corollary 6.2 that the linear system $|\mathcal{O}_X(1)|$ cannot be decomposed into a sum of movable linear systems. The only possibility is (ii) (see Theorem 2.5).

Finally, in the case $\dim X = 3$ we get (iii) as in [10], Corollary 4.6. □

Example 6.5. Consider Fletcher’s famous list of 95 terminal weighted Fano hypersurfaces [21]. The condition $\dim \langle -K_X \rangle \leq 2$ holds for 92 of them. Many of these varieties admit a biregular involution $\tau$ with $\kappa(\tau, X) \geq 0$.

For example, the hypersurface $X = X_{66} \subset \mathbb{P}(1, 5, 6, 22, 33)$ can be given in a suitable coordinate system $(x_1, x_5, x_6, x_{22}, x_{33})$ by the equation $x_{33}^2 = \varphi(x_1, x_5, x_6, x_{22})$, where $\varphi$ is a quasihomogeneous polynomial of weighted degree 66. The projection $X \to \mathbb{P}(1, 5, 6, 22)$ is a double covering, whose Galois involution $\tau$ satisfies $\text{Fix}(\tau) \simeq \{\varphi = 0\} \subset \mathbb{P}(1, 5, 6, 22)$. We easily see that for a general choice of $\varphi$ the subvariety $S := \text{Fix}(\tau)$ is a normal surface with only cyclic quotient singularities of types $\frac{1}{2}(1, 1)$ and $\frac{1}{5}(1, 2)$. By the adjunction formula, $K_S = \mathcal{O}_S(32)$. Hence $K_S^2 = \frac{512}{5}$. Let $\mu: \tilde{S} \to S$ be a minimal resolution of singularities. Then $K_{\tilde{S}} = \mu^*K_S - \Delta$, where $\Delta$ is the codiscrepancy divisor. In our case, $\Delta$ is supported over the point of type $\frac{1}{5}(1, 2)$, whence $\Delta^2 = -\frac{2}{5}$ and $K_{\tilde{S}}^2 = \frac{512}{5} - \frac{2}{5} = 102$. Therefore $\tilde{S}$ is a surface of general type and $\kappa(\tau, X) = 2$.

Note that a general Fano hypersurface $X$ in Fletcher’s list is not rational [22]. Hence, for general $X$ our construction produces no involution in the Cremona group. It is therefore interesting to investigate special varieties in this list.

§ 7. Gorenstein Fano threefolds. Examples

In this section we collect several examples of involutions acting on Gorenstein canonical Fano threefolds. We are interested only in those involutions which are not conjugate to actions on conic bundles or del Pezzo fibrations. First, every hyperelliptic Fano threefold has the Galois involution.

Example 7.1. Let $\pi: X \to \mathbb{P}^3$ be a double covering branched over a divisor $B \subset \mathbb{P}^3$ of degree 6 such that the pair $(\mathbb{P}^3, \frac{1}{2}B)$ is Kawamata log terminal. Then $X$ is a hyperelliptic Fano threefold with $g(X) = 2$. Let $\tau$ be the Galois involution. Then $\text{Sing}(X) = \pi^{-1}(\text{Sing}(B))$ and $\text{Fix}(\tau) = \pi^{-1}(B)$. If $B$ is smooth or has only Du Val singularities, then $\kappa(\tau, X) = 2$. Note that if $X$ has only isolated cDV-singularities and is $\mathbb{Q}$-factorial, then it is not rational [23]. In particular, being interested in elements of order 2 in $\text{Cr}_3(\mathbb{k})$, we should consider singular sextics $B \subset \mathbb{P}^3$. 
For example, let $B \subset \mathbb{P}^3$ be the Barth sextic. This surface is given by the equation
\[ 4(\varepsilon^2 x_1^2 - x_2^2)(\varepsilon^2 x_2^2 - x_3^2)(\varepsilon^2 x_3^2 - x_1^2) - x_4^2(1 + 2\varepsilon)(x_1^2 + x_2^2 + x_3^2 - x_4^2)^2 = 0, \]
where $\varepsilon = \frac{1}{2}(1 + 2\sqrt{5})$. Then $X$ has exactly 65 nodes and no other singularities. Moreover, $X$ is rational (see [24], Example 3.7). This yields an example of an involution $\tau \in \text{Cr}_3(\mathbb{k})$ with $\kappa(\tau, X) = 2$.

**Example 7.2.** Let $W = W_2 \subset \mathbb{P}^4$ be a smooth quadric let $\pi: X \to W$ be a double covering branched over a divisor $B \subset |\mathcal{O}_W(4)|$ such that the pair $(W, \frac{1}{2}B)$ is Kawamata log terminal. Then $X$ is a hyperelliptic Fano threefold with $g(X) = 3$. Let $\tau \in \text{Aut}(X)$ be the Galois involution. As above, $\kappa(\tau, X) = 2$ if the branch divisor has ‘mild’ singularities. However, all smooth varieties $X$ of this type are non-rational [6].

A series of examples comes from double coverings of del Pezzo threefolds.

**Construction 7.3.** Let $Y$ be a del Pezzo threefold with only canonical Gorenstein singularities. By definition, $-K_Y \sim 2A$ for some $A \in \text{Pic}(Y)$. Let $B \subset |-K_Y|$ be such that the pair $(Y, \frac{1}{2}B)$ is Kawamata log terminal. Consider the double covering $\pi: X \to Y$ branched over $B$. By the Hurwitz formula,
\[ -K_X = -\pi^* \left( K_Y + \frac{1}{2}B \right) = \pi^* A. \]

Hence $X$ is a Fano threefold with canonical Gorenstein singularities. We denote the Galois involution by $\tau$ and put $S := \tau^{-1}(B)$. Then $S \cong B$, $\text{Sing}(X) = \text{Sing}(S) \cup \tau^{-1} \text{Sing}(Y)$ and $\text{Fix}(\tau) = S$. Since $K_B = 0$, we have $\kappa(\tau, X) \leq 0$ and $\kappa(\tau, X) = 0$ if and only if $B$ is irreducible and normal and has only Du Val singularities. Let $d = A^3$ be the degree of $Y$. Then $-K_X^3 = 2A^3$. Hence $g(X) = d + 1$.

Here are some particular cases of Construction 7.3.

**Example 7.4** ($g = 2$). Let $Y$ be a del Pezzo threefold of degree 1. Then $Y$ is a hypersurface of degree 6 in $\mathbb{P}(1^3, 2, 3)$ and the projection $\delta: Y \to \mathbb{P}(1^3, 2) =: \mathbb{P}$ is a double covering. Consider another double covering $\gamma: \mathbb{P}^3 \to \mathbb{P}$, the quotient by a reflection. Put $X := Y \times_{\mathbb{P}} \mathbb{P}^3$. Then $\tau$ is the Galois involution of the projection $X \to Y$, which is a double covering. In other words, $X$ is a hypersurface of degree 6 in $\mathbb{P}(1^4, 3)$ given by the (quasihomogeneous) equation $z^2 = \varphi(x_0^2, x_1, x_2, x_3)$, and $\tau$ is the reflection $x_0 \mapsto -x_0$.

**Example 7.5** ($g = 3$). Let $Y$ be a del Pezzo threefold of degree 2. Then $Y$ is a hypersurface of degree 4 in $\mathbb{P}(1^4, 2)$. The projection $\delta: Y \to \mathbb{P}(1^4) = \mathbb{P}^3$ is a double covering. As above, consider another double covering $\gamma: W_2 \to \mathbb{P}$, the projection of a quadric $W_2 \subset \mathbb{P}^4$ away from a point $P \notin W_2$. Put $X := Y \times_{\mathbb{P}^3} W_2$. Then $\tau$ is the Galois involution of the projection $X \to Y$, which is a double covering.

**Example 7.6** ($g = 3$). Let $X \subset \mathbb{P}^4$ be a quartic given by the (homogeneous) equation $ax^4_0 + y_0^2 \varphi_2(x_1, x_2, x_3, x_4) + \varphi_4(x_1, x_2, x_3, x_4) = 0$. Then $\tau$ is the reflection $x_0 \mapsto -x_0$. The quotient is given by the equation $ay^2 + y_0 \varphi_2(x_1, x_2, x_3, x_4) + \varphi_4(x_1, x_2, x_3, x_4) = 0$ in $\mathbb{P}(1^4, 2)$. If $a \neq 0$, then this quotient is a del Pezzo threefold of degree 2. If $a = 0$, then $\tau$ is linearizable.
Example 7.7 \((g = 4)\). Let \(X = X_{2,3} \subset \mathbb{P}^5\) be the intersection of a quadric \(Q\) and a cubic cone \(V\) with vertex at \(P \in V\) such that \(P \notin Q\). The projection \(\pi: X \to Y \subset \mathbb{P}^4\) away from \(P\) is a double covering of the cubic \(Y \subset \mathbb{P}^4\). Then \(\tau\) is the Galois involution of this projection.

Example 7.8 \((g = 5)\). Let \(X = X_{2,2,2} \subset \mathbb{P}^6\) be the intersection of three quadrics \(Q_1, Q_2, Q_3\) such that \(Q_1\) and \(Q_2\) are cones with vertices at the same point \(P \in Q_1 \cap Q_2\), where \(P \notin Q_3\). The projection \(\pi: X \to Y \subset \mathbb{P}^5\) away from \(P\) is a double covering of the intersection \(Y \subset \mathbb{P}^5\) of two quadrics. As above, \(\tau\) is the Galois involution of this projection.

§ 8. Gorenstein Fano threefolds. Proof of the main theorem

Assumption 8.1. Throughout this section, \(X\) is a Fano threefold with only canonical Gorenstein singularities. Let \(g := g(X) \geq 2\) be the genus of \(X\). Thus \(−K_X^3 = 2g - 2\) and \(\dim |-K_X| = g + 1\). Let \(\Phi = \Phi|_{-K_X}|: X \dasharrow \mathbb{P}^{g+1}\) be the anticanonical map. Assume that \((X, G)\) is neither of type (C) nor of type (D) (otherwise we are in the situation of § 4 or § 5).

Our aim is to replace \((X, G)\) by another birational model satisfying Assumption 8.1 with minimal possible genus.

Lemma 8.2. The singularities of \(X\) are of type cDV.

Proof. We recall that a three-dimensional canonical Gorenstein singularity is of type cDV if and only if the centre of every crepant exceptional divisor is one-dimensional. Let \(P \in X\) be a non-cDV singularity. There are two possibilities for its orbit \(\Lambda:\{P\}\) and \(\{P, P'\}\), where \(P' \neq P\). Let \(\mathcal{L} \subset |-K_X|\) be the subsystem of all elements passing through \(\Lambda\). Since \(\dim |-K_X| = g + 1 \geq 3\), we have \(\dim \mathcal{L} \geq 1\). Write \(\mathcal{L} = D + \mathcal{M}\), where \(D\) (resp. \(\mathcal{M}\)) is the fixed (resp. movable) part of \(\mathcal{L}\). By our assumption, \((X, G)\) is not of type (C) or (D), and by Lemma 6.1 we have \(D = 0\) and \(\mathcal{M} = \mathcal{L}\). Let \(E\) be a crepant exceptional divisor centred at \(P\). Since \(P \in Bs \mathcal{L}\), the discrepancy of \(E\) satisfies \(a(E, X, \mathcal{L}) < a(E, X, 0) = 0\). Hence the pair \((X, \mathcal{L})\) is not canonical. This contradicts Lemma 6.1. □

Proposition 8.3. Assume that \(|-K_X|\) is not very ample. Then one of the following assertions holds.

(i) \(X\) is a double covering of \(\mathbb{P}^3\) branched over the sextic \(B \subset \mathbb{P}^3\).

(ii) \(X\) is a double covering of a smooth quadric \(W_2 \subset \mathbb{P}^4\) branched over a surface cut out on \(X\) by a quartic hypersurface.

Proof. We first assume that \(Bs |-K_X| \neq \emptyset\). Then the image \(W := \Phi(X) \subset \mathbb{P}^{g+1}\) is a surface of minimal degree (see § 2.5). We apply Theorem 2.5. Since \(\dim |-K_X| = g + 1 \geq 3\), we see that \(W\) is either a Hirzebruch surface \(\mathbb{F}_n\) (a rational ruled surface) or a cone \(\mathbb{P}(1,1,n)\) over a rational normal curve. In both cases we get a contradiction by Corollary 6.2 because \(|-K_X| = \Phi^*O_W(1)|\) is decomposable as a sum of two movable linear systems.

Thus \(X\) is hyperelliptic and the linear system \(|-K_X|\) determines a double covering \(\Phi: X \to W\), where \(W \subset \mathbb{P}^{g+1}\) is a (normal) variety of degree \(g - 1\) (see § 2.5).
We apply Theorem 2.5. In case 3) the surface $W$ is a Veronese cone, $W \simeq \mathbb{P}(1^3, 2)$. Thus $-K_X = 2\Phi^*O_{\mathbb{P}(1^3, 2)}(1)$, where $|O_{\mathbb{P}(1^3, 2)}(1)|$ is a movable linear system. This contradicts Corollary 6.2. In case 4) of Theorem 2.5 we consider the diagram

$$
\begin{array}{ccc}
X & \leftarrow & \tilde{X} \\
\downarrow & & \downarrow \\
W & \leftarrow & \mathbb{P}(\mathcal{O}) \rightarrow \mathbb{P}^1
\end{array}
$$

where $\tilde{X}$ is the normalization of the product $X \times_W \mathbb{P}(\mathcal{O})$. Then $\tilde{X}$ is a weak Fano threefold with canonical Gorenstein singularities ([14], Lemma 3.6) and the morphism $\tilde{X} \rightarrow X$ is crepant and $G$-equivariant. Hence the composite $\tilde{X} \rightarrow \mathbb{P}^1$ is a $G$-equivariant del Pezzo fibration (of degree 2). This contradicts our assumptions. In case 1) of Theorem 2.5 we get (i) (see [14]). Finally, in case 2) we have a double covering as in (ii). It remains to prove that the quadric $W_2 \subset \mathbb{P}^4$ is smooth. Indeed, otherwise the linear system $|O_{W_2}(1)|$ of its hyperplane sections admits a decomposition into the sum of two $G$-invariant movable linear systems of Weil divisors and, therefore, $|-K_X| = \Phi^*|O_{W_2}(1)|$ also admits such a decomposition. This contradicts Corollary 6.2. □

**Proposition 8.4.** Assume that the Fano variety $X$ is trigonal. Then $X$ is one of the following:

(i) $Y \simeq Y_4 \subset \mathbb{P}^4$;
(ii) $Y \simeq Y_{2,3} \subset \mathbb{P}^5$.

**Proof.** We have a $G$-equivariant anticanonical embedding $X = X_{2g-2} \subset \mathbb{P}^{g+1}$. Assume that $X$ is not an intersection of quadrics. Then the quadrics through $X$ in $\mathbb{P}^{g+1}$ cut out a fourfold $W \subset \mathbb{P}^{g+1}$ of degree $g-2$ (see §2.5). As above, we apply Theorem 2.5. If $W = \mathbb{P}^4$ or $W = W_2 \subset \mathbb{P}^5$, then we get (i) or (ii) respectively. In the remaining cases, let $\sigma : \tilde{W} \rightarrow W$ be the blow-up of the singular locus (we put $\sigma = \text{id}$ if $W$ is smooth) and let $\tilde{X} \subset \tilde{W}$ be the proper transform of $X$. Then $\tilde{W}$ is smooth, $\tilde{X}$ is a weak Fano threefold with canonical Gorenstein singularities ([14], Lemmas 4.4, 4.7) and the morphism $\tilde{X} \rightarrow X$ is crepant (and $G$-equivariant). The variety $\tilde{W}$ is either a $\mathbb{P}^2$-bundle over $\mathbb{P}^2$ or a $\mathbb{P}^3$-bundle over $\mathbb{P}^1$. By the adjunction formula, the projection to the base accordingly endows $\tilde{X}$ with a $G$-equivariant structure of either a conic bundle or a del Pezzo fibration. This contradicts our assumptions. □

**Lemma 8.5.** Assume that $|-K_X|$ is very ample. Then there are no lines through a generic point $P \in X$.

**Proof.** Indeed, otherwise the family of lines on $X$ has dimension at least 2. Let $L \subset X$ be a line through $P$, $\delta : X' \rightarrow X$ a terminal crepant model, and $L' \subset X'$ the proper transform of $L$. If $L' \cap \text{Sing}(X') \neq \emptyset$, then we easily see from the inequality $\dim \text{Sing}(X') \leq 0$ that $X$ is a cone over $S$. The base of this cone is a hyperplane section, a K3-surface with only Du Val singularities. Then the vertex is not a canonical singularity, a contradiction. Therefore $L'$ is contained in the smooth locus of $X'$. Since $L'$ belongs to a covering family of rational
curves, we have \( N_{L'/X'} \simeq \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \), where \( a, b \geq 0 \). On the other hand, \( \deg N_{L'/X'} = -K_{X'} \cdot L' - 2 = -1 \), a contradiction. □

**Lemma 8.6.** Assume that \( g(X) \geq 3 \) and there is a \( G \)-fixed singular point \( P \in X \). Then there is a \( G \)-equivariant birational map \( X \dashrightarrow Y \), where \( Y \) is a Fano threefold with canonical Gorenstein singularities and \( g(Y) < g(X) \).

**Proof.** By Proposition 8.3, either \( -K_X \) is very ample or case (ii) of that proposition holds, that is, \( X \) is a hyperelliptic Fano variety and \( \Phi(X) \) is a smooth quadric \( W_2 \subset \mathbb{P}^4 \). In the second case, \( P \) lies on the ramification divisor because \( \Phi \) cannot be étale over \( \Phi(P) \).

Let \( \mathcal{M} \subset |-K_X| \) be the subsystem of all elements passing through \( P \). Thus \( \text{Bs} |\mathcal{M}| = \{P\} \). We claim that the map \( h: X \dashrightarrow X \subset \mathbb{P}^9 \) determined by \( \mathcal{M} \) is generically finite. Indeed, if \( -K_X \) is very ample, then \( h \) is nothing but the projection away from \( P \). Since \( X \) is not a cone, \( h \) must be generically finite. Therefore we may assume that \( \mathcal{M} = \Phi^* \mathcal{L} \), where \( \mathcal{L} \) is the linear system of hyperplane sections of \( W_2 \) passing through \( P' := \Phi(P) \). We have the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{h} & X' \\
\downarrow \Phi & & \downarrow \\
W_2 & \xrightarrow{\Phi} & \mathbb{P}^3 \\
\end{array}
\]

where \( W_2 \dashrightarrow \mathbb{P}^3 \) is the projection away from \( P' \). Hence \( h \) is generically finite.

If the pair \((X, \mathcal{M})\) is not canonical, then Lemma 6.1 yields that the pair \((X, G)\) is of type (C) or (D). Thus \((X, \mathcal{M})\) is canonical (but not terminal at \( P \)). Consider the log crepant terminal model \( f: (\tilde{X}, \tilde{\mathcal{M}}) \rightarrow (X, \mathcal{M}) \) (see §2.3). Then the pair \((\tilde{X}, \tilde{\mathcal{M}})\) is terminal and

\[
K_{\tilde{X}} + \tilde{\mathcal{M}} = f^*(K_X + \mathcal{M}) \sim 0,
\]

whence \( \tilde{\mathcal{M}} \subset |-K_{\tilde{X}}| \). We write

\[
K_{\tilde{X}} = f^*K_X + \Theta, \quad \tilde{\mathcal{M}} = f^*\mathcal{M} - \Theta,
\]

where \( \Theta \) is an effective exceptional divisor with \( \text{Supp}(\Theta) = f^{-1}(P) \). For a generic element \( D \in |-K_{\tilde{X}}| \) we have \( f_*D \in |-K_X| = |-f^*K_X - \Theta| \). Since \( \Theta \) cannot be numerically trivial over \( X \), we have \( \text{Supp}(\Theta) \cap \text{Supp}(D) \neq \emptyset \), \( P \in f(D) \) and \( f_*D \in \mathcal{M} \). Therefore \( \tilde{\mathcal{M}} = |-K_{\tilde{X}}| \).

Since \((\tilde{X}, \tilde{\mathcal{M}})\) is terminal, the base locus of \( \tilde{\mathcal{M}} \) consists of at most finitely many points ([10], Lemma 1.22). In particular, \( -K_{\tilde{X}} \) is a numerically effective Cartier divisor. By what was said above, the divisor \( -K_{\tilde{X}} \) is also big, that is, \( \tilde{X} \) is a weak Fano threefold. Consider the anticanonical model

\[
Y := \text{Proj} \bigoplus_{n \geq 0} H^0(\tilde{X}, -nK_{\tilde{X}}).
\]
Then $Y$ is a Fano threefold with only canonical Gorenstein singularities. Let $\tilde{X} \to Y \to X$ be the Stein factorization of $h \circ f$. Thus we have a diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{f} & X \\
\downarrow h & & \downarrow X \\
Y & \to & X
\end{array}
\] (8.1)

By Proposition 8.3, the linear system $|-K_Y|$ is base-point free and hence so is $|-K_{\tilde{X}}|$. Our construction is $G$-equivariant. Finally,

\[
dim |-K_Y| = \dim |-K_{\tilde{X}}| = \dim \mathcal{M} = \dim X - 1.
\]

Therefore $g(Y) < g(X)$. □

**Lemma 8.7.** Assume that $g(X) \geq 4$ and $X$ is singular. Then there is a $\tau$-equivariant birational map $X \to Y$, where $Y$ is a Fano threefold with only canonical Gorenstein singularities and $g(Y) < g(X)$.

**Proof.** By Proposition 8.3, the linear system $|-K_X|$ determines an embedding $X = X_{2g-2} \subset \mathbb{P}^{g+1}$. By Lemma 8.6 we may assume that $\tau$ has no fixed singular points. Hence there are points $P_1, P_2$ that are switched by $\tau$. As in the proof of Lemma 8.6, we consider the subsystem $\mathcal{M} \subset |-K_X|$ of all elements passing through $P_1$ and $P_2$. It determines a map $h: X \to \tilde{X} \subset \mathbb{P}^{g-1}$ which is the projection away from the line $L \subset \mathbb{P}^{g+1}$ passing through $P_1$ and $P_2$. We claim that $h$ is generically finite. Indeed, assume that its generic fibre contains a curve $C$. Then $C \subset \Pi \cap X$, where $\Pi$ is a plane passing through $L$. Since $(X, G)$ is not of type (C), the curve $C$ is not rational. In this case $X$ cannot be an intersection of quadrics. By Proposition 8.4, the only possibility is that $X$ is the intersection of a quadric $Q$ and a cubic $Y$ in $\mathbb{P}^5$. Since $C \subset \Pi \cap X$ is a non-rational curve, we have $\Pi \subset Q$. If $L \not\subset \text{Sing}(Q)$, then there are only finitely many planes through $L$ in $Q$. Thus we may assume that $L \subset \text{Sing}(Q)$. But then $L \cap Y$ either coincides with $L$ or is an invariant 0-cycle of degree 3. In both cases $\tau$ has fixed points on $L$, which must be singular points of $X$. This contradicts our assumption.

Therefore $h$ is generically finite. As in the proof of Lemma 8.6, we get the diagram (8.1) and we can show that $\dim |-K_Y| < \dim |-K_X|$. □

**Lemma 8.8.** Assume that $S := F(\tau) \neq \emptyset$ and $g(X)$ is minimal among all birational models of $(X, G)$ satisfying Proposition 8.1. Then one of the following assertions holds.

(i) $X$ is a hyperelliptic Fano variety and $\tau$ is the hyperelliptic involution.

(ii) $S \in |-K_X|$ and $S$ has only Du Val singularities.

**Proof.** By Proposition 8.3 we may assume that $|-K_X|$ determines a morphism $\Phi: X \to \mathbb{P}^{g+1}$. If the natural action of $\tau$ on $H^0(X, -K_X)$ is trivial, then $X$ must be hyperelliptic and we get the case (i). Thus we may assume that $\tau$ acts faithfully on $H^0(X, -K_X)$. Then $\Phi(S)$ is contained in a hyperplane $\mathbb{P}^g \subset \mathbb{P}^{g+1}$. Hence we can write

\[
\Phi^{-1}(\mathbb{P}^g) = S + R \sim -K_X,
\]
where \( R \) is an effective Weil divisor. Let \( \nu: S' \to S \) be the normalization. By the subadjunction formula (see, for example, [25], Ch.16) we have

\[
K_{S'} + \text{Diff}_{S'}(R) = \nu^*(K_X + S + R) = 0,
\]

where Diff is the so-called different, an effective \( \mathbb{Q} \)-divisor. By Corollary 3 in [9] we have \( \text{Diff}_{S'}(R) = 0 \). Therefore \( \dim S \cap R \leq 0 \) and \( S \) is smooth in codimension 1. Since \( S + R \) is Cohen–Macaulay, it follows that \( R = 0 \) and \( S \) is normal. By considering a minimal resolution of singularities of \( S \), we obtain that the singularities of \( S \) are Du Val. \( \square \)

**Lemma 8.9.** Assume that \( g(X) \geq 6 \) and \( F(\tau) \neq \emptyset \). Then there is a \( G \)-equivariant birational map \( X \dashrightarrow Y \), where \( Y \) is a Fano threefold with canonical Gorenstein singularities and \( g(Y) < g(X) \).

**Proof.** By Lemma 8.5, the union of all the lines on \( V \) is a closed proper subset of \( V \). Since the surface \( F(\tau) \) is not uniruled, it is not a component of \( V \). Take a point \( P \in F(\tau) \setminus V \). Let \( f: \widetilde{X} \to X \) be the blow-up of \( P \), and let \( E := f^{-1}(P) \) be the exceptional divisor. We claim that the divisor \( -K_{\widetilde{X}} = -f^*K_X - 2E \) is numerically effective. Indeed, if \( -K_{\widetilde{X}} \cdot C < 0 \) for some curve \( C \), then \( C \) is contained in \( Bs \langle -K_{\widetilde{X}} \rangle \).

Note that \( \mathcal{M} := f_*|{-K_{\widetilde{X}}}| \subset |{-K_X}| \) is the subsystem of all elements that are singular at \( P \). Therefore \( C := f(\widetilde{C}) \) satisfies \( C \subset X \cap T_{P,X} \). By Proposition 8.4, the variety \( X \) is an intersection of quadrics. Hence \( C \) is a line through \( P \). This contradicts our choice of \( P \). Thus the divisor \( -K_{\widetilde{X}} \) is numerically effective. Since \( -K_{\widetilde{X}}^3 = -K_X^3 - 8 > 0 \), this divisor is big. By the base-point-free theorem (see, for example, [8], Theorem 3.3), the linear system \( |-nK_{\widetilde{X}}| \) determines a morphism \( h: \widetilde{X} \to Y \subset \mathbb{P}^n \) and we may assume that \( Y \) is normal and has only canonical Gorenstein singularities, \( K_{\widetilde{X}} = h^*K_Y \) and \( -K_Y \) is ample. Thus we get a new Fano threefold \( Y \) with \( -K_Y^3 = -K_{\widetilde{X}}^3 - 8 > 0 \). \( \square \)

The assertions in this section yield the following description of our Fano threefold \( X \).

**Corollary 8.10.** Replacing \( (X, \tau) \) by a birational model, we may assume that \( g(X) \leq 5 \) and one of the following assertions holds.

(i) \( X \) is a double covering of \( \mathbb{P}^3 \) branched over a sextic \( B \subset \mathbb{P}^3 \). The singularities of \( X \) are cDV.

(ii) \( X \) is a double covering of a smooth quadric \( W_2 \subset \mathbb{P}^4 \) branched over a surface \( B \subset W_2 \) of degree 8. The singularities of \( X \) are terminal.

(iii) \( X \subset \mathbb{P}^4 \) is a quartic with terminal singularities.

(iv) \( X \subset \mathbb{P}^5 \) is the smooth intersection of a quadric and a cubic cone.

(v) \( X \subset \mathbb{P}^6 \) is the smooth intersection of three quadrics.

**Proof.** By Lemma 8.2, the singularities of \( X \) are cDV. We recall that \( g(X) \geq 2 \) (because \( -K_X^3 = 2g(X) - 2 > 0 \)). If \( g(X) = 2 \), then \( \dim |-K_X| = 3 \) (see Proposition 8.1) and we get case (i) by Proposition 8.3. Thus we can assume that \( g(X) \geq 3 \). Using the classification and applying Lemma 8.9, we get a model \((X, G)\) with \( g(X) \leq 5 \).
In the cases $g(X) = 5, 4$ we may assume by Lemma 8.7 that $X$ is smooth. In the case $g(X) = 3$ we may assume by Lemma 8.6 that $\text{Fix}(\tau, X) \cap \text{Sing}(X) = \emptyset$ and, in particular, $X$ has only isolated cDV-singularities. □

In case $(F^c), (a)$ of Theorem 1.2 we have an embedding $X \hookrightarrow \mathbb{P}(1, 1, 1, 3)$, which is equivariant (because this embedding is given by sections of $H^0(X, -nK_X)$). In suitable coordinates we get the desired subcases (i) and (ii). It remains to show that the action of $\tau$ in cases $(F^c), (b), (ii), (c), (d)$ is given by Construction 7.3. This is a consequence of the following lemma.

**Lemma 8.11.** Let $X$ be as in Corollary 8.10, (ii)–(v). Assume that $\text{F}(\tau, X) \neq \emptyset$ and the action of $G$ on $H^0(X, -K_X)$ is non-trivial. Assume also that $g(X)$ is minimal among all birational models of $(X, G)$ satisfying Proposition 8.1. Then the quotient $X/G$ is a (Gorenstein) del Pezzo threefold of degree $g(X) - 1$.

**Proof.** Let $\pi: X \to Y = X/G$ be the quotient map. We put $S := \text{F}(\tau, X)$ and $R := \pi(S)$. By Lemma 8.8, $S \sim -K_Y$. The divisor $-K_X + S = -\pi^*K_Y$ is ample. Hence $Y$ is a (log terminal) Fano threefold. By the adjunction formula, $0 = K_R = (K_Y + R)|_R$. Hence $K_Y + R \sim_Q 0$ and $-K_Y^3 = 4(-K_X)^3 = 8(g - 1)$. Moreover, the branch divisor $R \subset Y$ is divisible by 2 in $\text{Cl}(Y)$. Thus it suffices to show that $Y$ has only terminal Gorenstein singularities.

By Lemma 8.6, $\text{Fix}(\tau, X) \cap \text{Sing}(X) = \emptyset$. Therefore $S$ is smooth, whence $Y$ is smooth along $R$. Assume that $\text{Fix}(\tau, X) \neq S$. Then $\text{Fix}(\tau, X) \setminus S$ consists of finitely many smooth points $P_1, \ldots, P_k \in X$ and, by what was said above, every point $\pi(P_i) \in Y$ is of type $\frac{1}{2}(1, 1, 1)$. Then the linear span of $\Phi(S)$ is a hyperplane $\mathbb{P}^g$ and $\Phi(P_i) = \cdots = \Phi(P_k) := Q \notin \Phi(S)$. Here $\text{Fix}(\tau, \mathbb{P}^{g+1}) = \mathbb{P}^g \cup \{Q\}$. In particular, $k \leq 2$. An easy computation with the orbifold Riemann–Roch formula shows that the group $\text{Cl}(Y)$ is torsion-free (see, for example, [26], Proposition 2.9). Since $K_Y + R \sim_Q 0$, we see that $K_Y$ is a Cartier divisor, a contradiction. Thus $\text{Fix}(\tau, X) = S$ and the singularities of $Y$ are terminal Gorenstein. □

The author is grateful to the referee for numerous helpful comments.

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Received 22/JUN/12  
30/NOV/12  
Translated by THE AUTHOR