A reanalysis of Finite Temperature SU(N) Gauge Theory

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Abstract

We revise the SU(N\textsubscript{c}), N\textsubscript{c} = 3, 4, 6, lattice data on pure gauge theories at finite temperature by means of a quasi-particle approach. In particular we focus on the relation between the quasi-particle effective mass and the order of the deconfinement transition, the scaling of the interaction measure with N\textsuperscript{2}-1, the role of gluon condensate, the screening mass.

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I. INTRODUCTION

A careful investigation of the quark-gluon plasma phase needs an understanding of the details of the deconfinement transition which occurs above a critical temperature T\textsubscript{c} and in this respect the data provided by lattice simulations represent the best tool for testing various dynamical models close to T\textsubscript{c}. Besides the full QCD simulations, interesting indications on the gluonic sector can be drawn from lattice studies on the high temperature transition of pure non-abelian gauge theories SU(N\textsubscript{c}), where all difficulties related to the presence of fermions and to the details of the chiral symmetry breaking are absent. Recent lattice simulations on SU(N\textsubscript{c}) gauge theories at finite temperature T and for large number of colors N\textsubscript{c} [1, 2] are now available and they show a peculiar scaling of the interaction measure, ∆ = (\epsilon - 3p)/T\textsuperscript{4} (\epsilon = energy density and p = pressure), with the number of gluons N\textsuperscript{2}-1. Moreover, in the range 1.1 T\textsubscript{c} < T < 4 T\textsubscript{c}, the interaction measure has a O(1/T\textsuperscript{2}) behavior which implies a O(T\textsuperscript{2}) contribution in ε - 3p.

The previous features have interesting consequences on the number of effective degrees of freedom and on the role of the gluon condensate above the critical temperature. The observed approximate scaling of ∆ with N\textsuperscript{2}-1 for N\textsubscript{c} ≥ 3, suggests: 1) a quasi-particle behavior of the effective degrees of freedom, with the typical degeneracy, N\textsuperscript{2}-1, of the gluons and with an effective, temperature dependent, mass that turns out to be divergent or, at least, very large at T\textsubscript{c}; 2) the presence of the same degeneracy factor in the gluon condensate contribution, if any, to ∆; 3) the same proportionality to the number of gluons N\textsuperscript{2}-1 of the dynamical mechanism which produces the O(T\textsuperscript{2}) contribution.

However, a more accurate analysis shows that the scaling of interaction measure and of the other thermodynamic observables with N\textsuperscript{2}-1 is not exact [2]. More precisely one can consider two different ranges of temperature. Near the transition temperature , i.e. for T\textsubscript{c} < T < (1.05 - 1.1)T\textsubscript{c}, the scaling with N\textsuperscript{2}-1 is clearly violated as shown in Fig.1, whereas above the ∆ peak temperature the scaling is almost exact, see Fig.2.

In this brief report we reconsider the SU(3, 4, 6) lattice data in [2, 8], by means of a quasi-particle approach in order to discuss the origin of the deviations from the exact scaling in relation to the critical behavior. The general formalism is discussed Sec. II; the comparison with lattice data is presented in Sec. III; the role of the gluon condensate is addressed in Sec. IV and in Sec. V we draw our conclusions.
II. GENERAL FORMALISM

The partition function for the very simple case of free quasi-particles in a volume $V$, at temperature $T$ and with temperature dependent mass $m(T)$ is

$$\ln Z(T, V) = 2V(N_c^2 - 1) \times \int \frac{d^3k}{(2\pi)^3} \ln \left[ f_T(k) \exp \left( \sqrt{k^2 + m^2(T)/T} \right) \right]$$

where $f_T(k)$ is the distribution

$$f_T(k) = \left[ \exp \left( \sqrt{k^2 + m^2(T)/T} \right) - 1 \right]^{-1}$$

and all thermodynamical quantities are obtained by deriving Eq. (1). For our purpose, the energy density $\epsilon$, the pressure $p$, the entropy density $s$ are respectively, $\epsilon = (T^2/V) \partial \ln Z/\partial T$, $p = T \partial \ln Z/\partial V$ and $s = (\epsilon + p)/T$. The interaction measure $\Delta$ is directly obtained from $\epsilon$ and $p$ as $\Delta = (\epsilon - 3p)/T^4$.

Obviously the temperature dependence of the mass must be taken into account when differentiating with respect to $T$. Note also that the additional effect of a temperature independent bag pressure (gluon condensate) $B$ corresponds to the changes $p \rightarrow p - B$, and $\epsilon \rightarrow \epsilon + B$, with no change in the entropy density $s$.

The factor 2 in front of the color multiplicity factor $N_c^2 - 1$ in Eq. (1), corresponds to the number of polarization degrees of freedom. In general, the representations of the Poincaré group of massive and massless particles in this case would suggest that our massive physical constituents carry three, rather than two, spin degrees of freedom. However, this is valid for free particles and, in fact, the comparison of all predicted thermodynamic quantities with the observed high temperature lattice results clearly shows much better agreement when only two polarization states are considered.

In particular all lattice QCD results, as shown also in Fig.5, hint at an asymptotic limit of the $\epsilon/T^4$ consistent with $2(N_c^2 - 1)$ degrees of freedom. In other words, a simple shift to massive gluons with three spin degrees of freedom cannot satisfactorily explain the effects of the interaction apparently still present in the gluon gas above the critical temperature.

It is well known that gauge symmetry forbids a mass term in the lagrangian for the elementary gluons and, in order to preserve the symmetry, one can expect to observe the generation of mass through a dynamical mechanism, such as the Schwinger mechanism [9] in which the mass comes from the appearance of a pole in the self-energy. In fact this effect has been explicitly pointed out and it has been argued that the longitudinal polarization component could be canceled by the scalar massless pole [10–12].

On the other hand, in the modified Hard Thermal Loop perturbation theory approach, where each order already includes some aspects of gluon dressing and which leads to a rather rapid convergence of the expansion, the contribution of longitudinal gluons vanishes in the limit $g \rightarrow 0$, and, in particular, one also obtains
the right number of degrees of freedom for the Stefan-Boltzmann form [13].

Moreover, from a comparison of the lattice glueball spectrum with the predictions of constituent models it has recently been argued [14] that massive gluons should in fact be transversely polarized, since two massless gluons cannot combine to form a longitudinally polarized massive gluon [15]. According to these indications we limit ourselves to consider just to polarization degrees of freedom for the effective quasiparticle in Eq. (1).

Let us now turn to the most important ingredient in our approach, that is the effective temperature dependent mass \( m(T) \) which contains the non-perturbative dynamics. Previous analyses [4–6] show that \( m(T) \) strongly increases near \( T_c \) and a qualitative explanation of it has been outlined in [16]. To illustrate this aspect one describes the mass of the quasi-gluon in the strongly coupled region as the energy contained in a region of volume \( V_{\text{cor}} \) whose characteristic size is given by the correlation range \( \xi \), so that in three spatial dimensions one gets (\( \eta \) is the anomalous dimension and \( t \) is the reduced temperature \( t = T/T_c \)):

\[
m(t) \simeq \epsilon(t)V_{\text{cor}} = \epsilon(t) \int dr \frac{r^2 \exp[-r/\xi(t)]}{r^{1-\eta}} \quad (3)
\]

In the case of a second order phase transition, the correlation length shows the power law divergence \( \xi(t) = (t-1)^{-\nu} \) at \( t = 1 \), which indicates that the associated fluctuations have an infinite range at criticality, and the corresponding component of the energy density vanishes as \( \epsilon(t) \simeq (t-1)^{1-\alpha} \) where \( \alpha \) is the specific heat critical exponent. In this case Eq. (3) predicts a power law divergence of the mass \( m(1) \). Then, by focussing on the 3D Ising model universality class, which includes the critical behavior of the \( SU(2) \) gauge theory deconfinement transition, one finds the value of the critical exponent for our mass: \( m(t) \simeq (t-1)^{0.41} \). In addition the \( SU(2) \) critical behavior suggests that very close to \( T_c \), \( \Delta \) should have the form [17]

\[
\Delta = A\tau(1-\tau)^{0.89} + B\tau \quad (4)
\]

with \( \tau = t^{-1} \) and \( A, B \) constants and the exponent is given by 0.89 = 1 - \( \alpha \).

In such transitions \( m(t) \) close to \( t = 1 \) has a power law divergent behavior which has to be considered as an approximation for \( T \) near \( T_c \). By recalling that for \( t >> 1 \), the mass is expected to grow linearly with the temperature, which is the only dimensionful scale available, a suitable ansatz for \( m(t) \) is

\[
m(t) = \frac{a}{(t-1)^c} + bt, \quad (5)
\]

where \( a, b, c \) are constant parameters.

For the gauge groups \( SU(N_c) \), with \( N_c = 3, 4, 6 \) here considered, a first order phase transition and consequently a finite correlation length, is expected at \( T_c \) and power law behavior at criticality is modified by the finite scale \( \xi \). However, in the case of weak first order transitions one should expect a behavior of the thermodynamical quantities at \( T_c \) not totally different from that observed in second order transitions, and therefore a finite but large correlation length and a corresponding large \( m(1) \).

In particular, as discussed in [18], the thermodynamical quantities approach \( T_c \) (from larger values of the temperature \( t > 1 \) ) as in a second order phase transition with the critical temperature shifted to a lower value: \( t \rightarrow \delta \) with \( 0 \sim (1-\delta) << 1 \). According to this suggestion the ansatz (5) for weakly first order phase transition must be changed into

\[
m(t) = \frac{a}{(t-\delta)^c} + bt \quad (6)
\]

which yields a large but finite value of the effective mass and non-vanishing interaction measure at the transition point \( t = 1 \).

The quasi-particle mass \( m(t) \) should not to be confused with the screening mass \( m_D(T) \). The relation between \( m(t) \) and \( m_D(T) \), has been clarified in [4] where it is shown that

\[
m_D = \frac{g^2 N_c}{\pi^2 T} \int_0^\infty dk \frac{f_T^2(k) \exp\sqrt{k^2 + m(T)^2}}{k^2} \quad (7)
\]

and the leading order QCD coupling \( g^2 \) is evaluated at the average, \( M^2 \), over the squared quasi-particle momenta, i.e.

\[
M^2(T) = \frac{4}{3} \int dk \frac{k^4 f_T(k)}{k^2 f_T(k)} \quad (8)
\]
To illustrate this point in the next Section we display \( m(t) \) and \( m_D(T) \) which show totally different behaviors when approaching \( T_c \).

### III. ANALYSIS OF THE LATTICE DATA

Now we check the simple model outlined in Sect. II against the lattice data for \( SU(3, 4, 6) \) \[2, 8\]. These theories undergo a weakly first order transition and we shall resort to the large but finite mass in Eq. (6) for computing the various thermodynamical quantities.

In addition, as discussed above, the singular behavior of the 3D Ising model corresponds to the value or the critical exponent in Eq. (5) \( c = 0.41 \), which is close to the mean field exponent \( c = 0.5 \). Therefore, it seems reasonable to verify whether the mean field behavior produces a good fit to the data.

In Table I we collect the values of the mass \( m(1) \), of the shift \( \delta \) and of the \( \chi^2/dof \) per degree of freedom (dof), obtained by fitting the data with \( a, b, \delta \) free parameters and \( c = 0.5 \) fixed. As an example the \( SU(3) \) values of the other parameters \( a \) and \( b \), turn out to be \( a/T_c = 1.42 \) and \( b/T_c = 0.53 \). The critical mass decreases when \( N_c \) is increased and the values of the \( \chi^2 \) indicate a reasonably good agreement with the data.

However no clear dependence on \( N_c \) can be traced in the values of \( m(t) \) and this is expected because the dependence of the interaction measure \( \Delta \), shown in Figs. 1, 2, on \( m(t) \) is highly non-linear. On the other hand the different behavior of \( \Delta \) in Fig. 1, i.e. near the critical temperature, is essentially due to the dependence of \( m(T_c) \) on \( N_c \) given in Table I.

It is then remarkable that the simple

| \( N_c \) | \( c \) | \( m(T_c)/T_c \) | \( \delta \) | \( \chi^2/dof \) |
|-------|---|------------|-----|----------|
| 3     | 0.5 (0.46) | 6.33 (6.64) | 0.940 (0.952) | 2.6 (2.0) |
| 4     | 0.5 (0.35) | 6.02 (7.68) | 0.944 (0.982) | 7.7 (0.8) |
| 6     | 0.5 (0.33) | 4.33 (5.91) | 0.888 (0.969) | 6.5 (5.8) |

TABLE I: Mass, shift \( \delta \), and \( \chi^2/dof \) from the fit to \( SU(3, 4, 6) \) lattice data with \( a, b, \delta \) as free parameters and \( c = 0.5 \) fixed. In brackets, the same quantities from the fit with \( a, b, c, \delta \) as free parameters.

FIG. 3: The interaction measure as obtained from a fit to the \( SU(3, 4, 6) \) lattice data with \( c = 0.5 \) fixed.

FIG. 4: The same fit as in Fig.3 but optimizing also on the parameter \( c \).

FIG. 5: Fit to the pressure of \( SU(3, 4, 6) \) optimizing also on the parameter \( c \).
parametrization of \( m(t) \) in Eq.(5,6) produces the correct behavior of the interaction measure \( \Delta \) which is plotted in Fig. 3.

An improvement on these results is obviously obtained by releasing the constraint on \( c \) and leaving it as another free parameter of the fit. Results are again reported in Table 1 (in brackets) and the interaction measure \( \Delta \) and the pressure \( p \) are plotted in Figs. 4, 5. In this case the \( \chi^2/dof \) shows an excellent agreement with the data.

We note that even if we have not forced any specific dependence of the mass on the coupling \( g \) and on \( N_c \) the results of the fit shown in Fig. 6 manifests an independence of \( m(T) \) on the the \( SU(N_c) \) gauge group for temperature above the peak in the interaction measure. This could be expected if one considers the parametric dependence of the mass in a perturbative approach, \( m_0^2 \approx g^2 N_c T^2 \), along with the t’Hooft scaling of the coupling, \( g^2 \approx 1/N_c \).

Moreover, the comparison between the gluon effective mass \( m(t) \) and the Debye screening mass \( m_D(T) \) according to Eqs. (7,8), is displayed in Fig. 6. Finally in Fig. 7 \( c_s^2 = \partial p/\partial \epsilon \) is reported.

As mentioned in the Introduction, lattice results very close to \( T_c \) seem to scale approximately with \( N_c (N_c^2 - 1) \) rather than with \( (N_c^2 - 1) \). Accordingly also the behavior of \( m(t) \) very close to \( T_c \) breaks its independence on \( N_c \), even if not clearly visible from Fig. 6.

This signals that the t’Hooft condition of QCD at large \( N_c \), i.e. \( g^2 \approx 1/N_c \), is violated at the transition region. The effective mass at \( T_c \) depends on \( N_c \) in a way which is not consistent with the standard \( O(1/N_c^2) \) corrections in QCD at large \( N_c \). This interesting aspect will be reconsidered in the next Section.

IV. THE ROLE OF THE GLUON CONDENSATE

A fundamental ingredient of the non perturbative QCD regime is the gluon condensate which has been evaluated by lattice simulations at zero and finite temperature, in quenched and unquenched QCD [19]. It turns out that:

1) for \( T < T_c \) the gluon condensate is almost \( T \) independent;

2) for \( T > T_c \) the chromo-electric part of the gluon condensate quickly decreases to zero whereas the magnetic one is constant. This corresponds to the deconfinement transition.

In QCD the gluon condensate is related to the trace anomaly of the energy-momentum tensor by the general expression:

\[
\theta^\mu = 4B = \frac{\beta(g)}{g} < G^a_{\mu\nu} G^a_{\nu\mu} >
\]  

where \( B \) is the bag pressure, \( \beta(g) \) is the QCD \( \beta \)-function, \( a = 1, 2, ..., (N_c^2 - 1) \).

Above \( T_c \) its contribution to interaction measure is about a half of the zero temperature
value (because the electric-part melts) and, as discussed in ref. [16], a temperature independent gluon condensate/bag pressure is not able to fit lattice data and in particular the behavior $\Delta \propto T^2 \simeq \text{const.}$ observed in the range $1.1T_c < T < 4T_c$.

To explain this behavior one has to include in $B$ a term proportional to $T^2$, as already suggested in [3]

$$B(T) = B_0 + B_1 T^2$$

Then, it is probably possible to fit the lattice data because $B_1 T^2$ gives the correct $O(1/T^2)$ behavior above the peak and at the same time $B_0$ can be adjusted to optimize the curve below the peak.

But, as $\Delta$ scales with $\simeq N_c(N_c^2 - 1)$ near $T_c$ and with $(N_c^2 - 1)$ for larger temperature, the scaling behavior of $B_0$ and $B_1$ with $N_c$ should be different. In particular the scaling near the critical point can be understood as a violation of the 't Hooft limit condition $g^2 \simeq 1/N_c$ because

$$B_0 \simeq \frac{\beta(g)}{g} < G_{\mu\nu} G_{\mu\nu} > \simeq g^2 \gamma N_c(N_c^2 - 1)$$

where the constant $\gamma$ contains numerical factors and an average, $N_c$ independent, condensate per gluon. Therefore, $\Delta \simeq N_c(N_c^2 - 1)$ near $T_c$ implies $g^2 \neq O(1/N_c)$ as seen before for the mass term. However it must be remarked that there is no indication in lattice computation [15] of the $T^2$ behavior in Eq. (10) up to $T \simeq 1.5 T_c$.

The points discussed above indicate that it is very unlikely that the condensate alone could explain the behavior of the interaction measure $\Delta$. On the contrary, it is reasonable to treat the condensate as a small additional piece which has to be added to the quasi-particle contribution to $\Delta$, discussed in the previous Sections. In fact lattice data allow the insertion of a physically acceptable constant $B_0$, with a (minor) role only in the region near $T_c$, without qualitatively changing the fit to the data shown in Section III.

V. COMMENTS AND CONCLUSIONS

The previous results show that a quasi-particle approach, where the effective mass is related with the features of the deconfinement transition, gives a very good description of the interaction measure and of the thermodynamical quantities for weak first order phase transition.

In our opinion this is not so surprising. Indeed, a quasi-particle approach means that the relevant dynamics is contained in the two-point function and fluctuations have a minor role. A general framework to describe this behavior in quantum field theory is the effective potential for composite operators (CJT) [20] in the so-called Gaussian approximation. It has been extensively applied at finite temperature for scalar, fermion and gauge theories and naturally leads to a first order phase transition to restore the symmetries, although this conclusion has to be confirmed by other more reliable methods (e.g., expansion, lattice simulations, etc).

We find that the scaling of the interaction measure with $(N_c^2 - 1)$ is observed for $T > 1.1T_c$ and clearly violated near $T_c$. Accordingly, above the peak temperature of the interaction measure, the mass behavior is independent on $N_c$ in agreement with a perturbative parametric dependence on $g^2 N_c$ and the $1/\sqrt{N_c}$ 't Hooft scaling of the coupling which, on the other hand, is broken very close to $T_c$. Moreover by combining this aspect with the almost constant behavior of $\Delta \propto T^2$ in the range $1.1T_c < T < 4T_c$, it turns out difficult, in our opinion, to describe the interaction measure by a temperature dependent bag pressure and/or gluon condensate above $T_c$. An interesting point is to find the connection with the confined phase below $T_c$ which can be described by a glueball gas plus bag pressure [21].

It would be also useful to analyze the $SU(2)$ case because there is a second order phase transition , with a corresponding divergent effective mass at the critical point, and a small number of colors. However the $SU(2)$ data are quite old and, unfortunately, there does not seem to exist any lattice study allowing an extrapolation to the continuum, thus eliminating finite lattice size effects [17].

The meaning of the effective gluon mass as discussed in Sect. II, which is similar to a "colored glue-lump", is probably related with the dynamical mechanism of the transition, bubble nucleation for example, and a deeper understand-
ing in this direction would be extremely appealing.

Following the same strategy here applied, the next step will be the analysis of full QCD to study the role of fermions in modifying the effective gluon properties related to the deconfined phase cross-over.

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[1] M. Panero, Phys. Rev. Lett. 103 (2009) 23001.
[2] S. Datta and S. Gupta, Phys. Rev. D 82 (2010) 114505.
[3] R. Pisarski, Progr. Theoret. Phys. Suppl. 168 (2007) 276.
[4] A. Peshier, B. Kampfer, O. P. Pavlenko and G. Soff, Phys. Rev. 54 (1996) 2399.
[5] P. Castorina and M. Mannarelli, Phys. Rev. C75 (2007) 054901, arXiv: hep-ph/0701206.
[6] M. Bluhm, B. Kampfer, K. Redlich, Nucl. Phys. A 830 (2009) 737C, arXiv:0907.3841 [hep-ph]; M. Bluhm, B. Kampfer, K. Redlich, ”Ratio of bulk to shear viscosity in a quasigluon plasma: from weak to strong coupling”, arXiv:1101.3072 [hep-ph].
[7] S. Plumari, W. M. Alberico, V. Greco, C. Ratti, arXiv:1103.5611 [hep-ph].
[8] G. Boyd et al., Nucl. Phys. B 469 (1996) 419.
[9] J. Schwinger, Phys Rev. 128, (1962) 2425.
[10] R. Jackiw and K. Johnson, Phys Rev. D8, (1973) 2386.
[11] J.M. Cornwall, Phys Rev. D26, (1982) 1453.
[12] D. Binosi and J. Papavassiliou, Phys. Rept. 479 (2009) 1.
[13] For reviews, see J.-P. Blaizot, Nucl. Phys. A 702 (2002) 99c; A. Rebhan, Nucl. Phys. A 702 (2002) 111c; A. Peshier, Nucl. Phys. A 702 (2002) 128c.
[14] V. Mathieu, ”How Many Degrees of Freedom Has the Gluon?”, PoS QCD-TNT09:024, (2009), arXiv:0910.4855 [hep-ph].
[15] C.-N. Yang, Phys Rev. 77 (1950) 242.
[16] P. Castorina, D. E. Miller and H. Satz, ”Trace Anomaly and Quasi-Particles in Finite Temperature SU (N) Gauge Theory”, arXiv:1101.1253 [hep-ph].
[17] J. Engels, J. Fingberg, K. Redlich, H. Satz, M. Weber, Z. Phys. C42 (1989) 341.
[18] K. Binder, Rep. Prog. Phys. 50 (1987) 783.
[19] M.D’Elia, A.Di Giacomo and E.Meggiolaro, Phys. Lett.B 408(1997) 315; Phys. Rev. D 67(2003)114504.
[20] J.M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. D10 (1974) 2428.
[21] F. Buisseret, Eur. Phys. J. C 68 (2010) 473, arXiv:0912.0678 [hep-ph].