Extensions of the Moser-Scherck-Kemperman-Wehn Theorem

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Abstract

Let \( \Gamma = (V, E) \) be a reflexive relation having a transitive group of automorphisms and let \( v \in V \). Let \( F \) be a subset of \( V \) with \( F \cap \Gamma^{-}(v) = \{v\} \).

(i) If \( F \) is finite, then \( |\Gamma(F) \setminus F| \geq |\Gamma(v)| - 1 \).

(ii) If \( F \) is cofinite, then \( |\Gamma(F) \setminus F| \geq |\Gamma^{-}(v)| - 1 \).

In particular, let \( G \) be group, \( B \) be a finite subset of \( G \) and let \( F \) be a finite or a cofinite subset of \( G \) such that \( F \cap B^{-1} = \{1\} \). Then \( |(FB) \setminus F| \geq |B| - 1 \). The last result (for \( F \) finite), is famous Moser-Scherck-Kemperman-Wehn Theorem. Its extension to cofinite subsets seems new. We give also few applications.

1 Introduction

A problem of Moser solved also by Scherck in [22] states that in abelian group \( G \) two finite subsets \( A, B \) with \( |A \cap B^{-1}| = \{1\} \) verifies the Cauchy-Davenport inequality: \( |AB| \geq |A| + |B| - 1 \). The validity of this result in the non-abelian case was proved by Kemperman [16], mentioning that the result was independently proved by When [16]. In our work, \( A \) could be infinite. So we shall use the formulation \( |AB \setminus A| \geq |B| - 1 \).

All the known proofs of the Moser-Scherck-Kemperman-Wehn Theorem, used additive transforms. In this work, we obtain a completely different

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proof. The Moser-Scherck-Kemperman-Wehn Theorem has important applications in Number Theory and the reader may find several applications of this beautiful result to the Theory of Non-unique factorization in the text book of Geroldinger-Halter-Koch [8]. Recall that this result is used among other tools by Olson [21], to prove that a subset $S$ of a finite group $G$ with $|S| \geq 3\sqrt{|G|}$ contains a non-empty subset with a product (under some ordering) $= 1$. The Moser-Scherck-Kemperman-Wehn Theorem is a basic tool in the proof by Gao that a sequence of elements of an abelian group $G$ with length $|G| + d(G)$ contains a $|G|$-sub-sequence summing to 0, where $d(G)$ is the maximal size of a sequence of elements of $G$ having no non-empty zero-sum subsequence [7]. A recent generalization of Gao Theorem is contained in [15]. Notice that the Moser-Scherck-Kemperman-Wehn Theorem implies easily the following result of Shepherdson [23]:

For every nonempty subset $S$ of a finite abelian group $G$, there are $s_1, \ldots, s_k \in S$ such that $k \leq \lceil \frac{|G|}{|S|} \rceil$ and $\sum_{1 \leq i \leq k} s_i = 0$.

Some of the above results are related to questions investigated in Combinatorics. Let $\Gamma = (V, E)$ be a loopless finite (directed) graph with $\min\{|\Gamma(x)| : x \in V\} \geq 1$. It is well known that $D$ contains a directed cycle. The smallest cardinality of such a cycle is called the girth of $D$ and will be denoted by $g(D)$. In 1978, Caccetta and Häggkvist [3] conjectured that

$$|V| \geq \min(d^+ x : x \in V)(g(D) - 1) + 1.$$  

This conjecture is still largely open. The reader may find references and results about this question in [1].

This conjecture were proved by the author for graphs with a transitive group of automorphisms in [11]. This result applied to Cayley graphs shows the validity of Shepherdson’s result for all finite groups. Unfortunately we were not aware at that moment of Shepherdson’s result. Our proof [11] is based on the properties of atoms of a finite graph [9] and the Dirac-Menger’s Theorem [2, 19, 24]. Another proof of the Caccetta and Häggkvist Conjecture for the last class of graphs, based on the Moser-Scherck-Kemperman-Wehn Theorem and the representation of point-transitive graphs as coset graphs, was given by Nathanson in [20]. A third proof, using atoms but avoiding the Dirac-Menger’s was given later by the author in [14].

In an e-mail cited by Mader [18], Seymour suggested the possibility of the existence of $\min\{|\Gamma(x)| : x \in V\} \geq 1$ directed cycles having pairwise a fixed vertex in the intersection. Such a property implies clearly the Caccetta
and Häggkvist Conjecture. Mader obtained a positive answer if \( \Gamma \) has a transitive group of automorphisms [18].

Generalizing all the above results, we prove the following:

Let \( \Gamma = (V, E) \) be a reflexive relation having a transitive group of automorphisms. Let \( v \in V \) and let \( F \) be a subset of \( V \) with \( F \cap \Gamma^{-1}(v) = \{v\} \). Then

(i) If \( F \) is finite, then \( |\Gamma(F) \setminus F| \geq |\Gamma(v)| - 1 \).

(ii) If \( F \) is cofinite, then \( |\Gamma(F) \setminus F| \geq |\Gamma^{-1}(v)| - 1 \).

In particular, for \( i \) an integer \( j \geq 1 \) with \( \Gamma^j(v) \cap \Gamma^{-1}(v) = \{v\} \), we have \( |\Gamma(v)| \geq |\Gamma^{-1}(v)| + |\Gamma(v)| - 1 \).

The finite case of the last result is proved by Mader in [18]. The same conclusion was obtain under the stronger hypothesis \( \Gamma^j(v) \cap \Gamma^{-1}(v) = \{v\} \) in [14]. This weak form is enough to show Caccetta and Häggkvist Conjecture.

Let \( G \) be group and let \( B \) be a finite subset of \( G \). Applying this result to the Cayley graph defined on \( G \) by \( B \), we obtain Moser-Scherck-Kemperman-Wehn Theorem and its extension to cofinite subsets:

Let \( F \) be a finite or a cofinite subset of \( G \) such that \( F \cap B^{-1} = \{1\} \). Then \( |(FB) \setminus F| \geq |B| - 1 \).

Our extension to the cofinite case allows a very short proof for the following result, proved by Losonczy [17] in the abelian case and by Eliahou-Lecouvey [4] in the general case:

If \( 1 \notin B \), then here is a permutation \( \sigma \) of \( B \) such that \( x\sigma(x) \notin B \) for every \( x \in B \).

We conclude the paper by a simple proof of Mader’s result. According to our approach, Seymour’s question may be formulated as follows:

Under which condition, a graph satisfies the Moser-Scherck-Kemperman-Wehn property?

Our proofs require properties, developed in Sections 3 and 4, of weak atoms and Moser sets. The proofs we obtain for these properties are easier than the proofs of the corresponding properties of the atoms [9].

The present work is essentially self-contained. Our proof of Mader’s Theorem requires the Dirac-Menger’s Theorem. Also our proof of a result
due Losonczy and Eliahou-Lecouvey requires König-Hall’s Theorem. Let us state below these two results:

**Theorem A** *(The Dirac-Menger’s Theorem [2]*)

Let \( \Gamma = (V, E) \) be a finite reflexive graph. Let \( k \) be a nonnegative integer. Let \( x, y \in V \) such that \( y \notin \Gamma(x) \). If \( |\partial(\Gamma) \geq k \) for every subset \( X \) with \( x \in X \) and \( y \notin \Gamma(X) \), then there are \( k \) disjoint directed paths from \( \Gamma(x) \) to \( \Gamma^{-}(y) \).

**Theorem B** *(König-Hall’s Theorem [2]*) Let \( \Phi \subset V \times W \) be a relation with \( |V| = |W| < \infty \). Then the following conditions are equivalent:

- There is a bijection \( \phi: V \rightarrow W \) with \( \phi(x) \in \Phi(x) \), for every \( x \in V \).
- \( |\Phi(Y)| \geq |Y| \), for every subset \( Y \) of \( V \).

The reader may find proofs of the Dirac-Menger’s Theorem in the text books [19, 24] and some applications of this result to Additive Number Theory. Notice that the König-Hall’s Theorem follows by applying the Dirac-Menger’s Theorem to the graph obtained from \( \Phi \) by adding two distinct vertices one dominating \( V \) and the other dominated par \( W \).

### 2 Some Terminology

Let \( \Gamma = (V, E) \) be a relation and let \( X \subset V \). The subrelation \( \Gamma[X] \) induced on \( X \) is by definition \( (X, (X \times X) \cap E) \). A function \( f : V \rightarrow V \) will be called a homomorphism if for all \( x \in V \), we have \( \Gamma(f(x)) = f(\Gamma(x)) \). A bijective homomorphism is called an automorphism. The relation \( \Gamma \) will be called locally-finite if for all \( x \in V \), \( |\Gamma(x)| \) and \( |\Gamma^{-}(x)| \) are finite.

The relation \( \Gamma \) will be called point-transitive if for all \( x, y \in V \), there is an automorphism \( f \) such that \( y = f(x) \). Clearly a point-transitive relation is regular. Let \( S \) be a subset of \( G \). The relation \( (G, E) \), where \( E = \{(x, y) : x^{-1}y \in S\} \) is called a Cayley relation. It will be denoted by \( \text{Cay}(G, S) \).

Let \( \Gamma = \text{Cay}(G, S) \) and let \( F \subset G \). Clearly \( \Gamma(F) = FS \), where \( FS = \{xy : x \in F \text{ and } y \in S\} \). Cayley graphs are clearly point-transitive.

We identify graphs (directed graphs) and their relations. The reader may replace everywhere the term ”relation” by ”graph”.

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We shall write
\[ \partial_{\Gamma}(X) = \Gamma(X) \setminus X. \]
We also write
\[ \nabla_{\Gamma}(X) = V \setminus \Gamma(X). \]

When the context is clear the reference to \( \Gamma \) will be omitted.

Let \( F \) be a subset of \( V \). Clearly \( V = F \cup \nabla(F) \cup \partial(F) \) is a partition. Notice that \( \partial^{-}(\nabla(F)) \cap F = \emptyset \), otherwise there exist \( z \in F \) such that \( \Gamma(z) \cap (\nabla(F)) \neq \emptyset \), a contradiction. Hence
\[ \partial^{-}(\nabla(F)) \subset \partial(F). \quad (1) \]

The last observation, used extensively in the isoperimetric method [13], contains a useful duality. The next lemma contains a useful sub-modular relation:

**Lemma 1** [13] Let \( \Gamma = (V, E) \) be a locally finite reflexive graph. Let \( X, Y \) be finite subsets. Then

\[ |\partial(X \cup Y)| + |\partial(X \cap Y)| \leq |\partial(X)| + |\partial(Y)| \quad (2) \]

**Proof.**

Observe that
\[
|\Gamma(X \cup Y)| = |\Gamma(X) \cup \Gamma(Y)| \\
= |\Gamma(X)| + |\Gamma(Y)| - |\Gamma(X) \cap \Gamma(Y)| \\
\leq |\Gamma(X)| + |\Gamma(Y)| - |\Gamma(X \cap Y)|
\]

The result follows now by subtracting the equation \( |X \cup Y| = |X| + |Y| - |X \cap Y| \). \( \blacksquare \)

### 3 Weak atoms

The reader interested only in the finite case may ignore this section. Let \( \Gamma = (V, E) \) be a locally finite reflexive relation. We define the weak connectivity of \( \Gamma \) as

\[ \kappa(\Gamma) = \min\{|\partial(X)| : 1 \leq |X| < \infty\}. \quad (3) \]
A subset $X$ achieving the minimum in (3) is called a weak fragment of $\Gamma$. A weak fragment with minimum cardinality will be called a weak atom.

**Proposition 2** Let $\Gamma = (V, E)$ be a locally finite relation. If $F_1$ and $F_2$ are weak fragments with a nonempty intersection, then $F_1 \cup F_2$ and $F_1 \cap F_2$ are weak fragments. In particular, two distinct weak atoms are disjoint. Assume furthermore that $\Gamma$ is point–transitive and let $A$ be an atom of $\Gamma$. Then the weak atoms induce a partition of $V$. Moreover the subrelation $\Gamma[A]$ induced on $A$ is a point-transitive relation, for any weak atom $A$.

**Proof.** By (2), $2\kappa(\Gamma) \leq |\partial(F_1 \cap F_2)| + |\partial(F_1 \cup F_2)| \leq |\partial(F_1)| + |\partial(F_2)| \leq 2\kappa(\Gamma)$. Let $A$ be a weak atom and take $v \in A$. For every $y \in V$, there is an automorphism $\phi$ such that $\phi(v) = y$. Hence $y$ belongs to the atom $\phi(A)$. In particular, the set of atoms is a partition of $V$. If $y \in A$, then $\phi(A) = A$, and therefore $\phi/A$ is an automorphism of $\Gamma[A]$ with $\phi/A(v) = y$. Thus $\Gamma[A]$ is a point-transitive relation. \(\Box\)

4 Moser sets

The formalism we develop here is motivated from one side by the Moser-Scherck-Kempermann-Wehn Theorem and by the notion of vertex-fragment (a possibly empty set) in finite graphs considered by Mader in [18] from the other side. Our Moser sets are related to Mader’s vertex-fragments. But our concept is never empty, works in the infinite case too and leads to easier proofs. In particular, we do not need a duality between positive and negative vertex-fragments, in the spirit of the one introduced in [9], used extensively in Mader arguments. Also the Dirac-Menger’s Theorem, present in beginning of Mader’s formalism, is not needed to prove our main result, generalizing of the Moser-Scherck-Kempermann-Wehn Theorem. We use it only to give a simple proof of Mader’s result.

Let $\Gamma = (V, E)$ be a locally finite reflexive relation and let $v \in V$. A set $F$ is said to be a $v$–Moser set if $\Gamma^-(v) \cap F = \{v\}$. Put

$$\kappa_\Gamma(v) = \min\{|\partial(X)| : X \text{ is a } v \text{–Moser set}\}.$$

A Moser set $X$ with $|\partial(X)| = \kappa_v(\Gamma)$ will be called a $v$–fragment.
Lemma 3 Let $F_1$ and $F_2$ be two v–fragments. Then $F_1 \cap F_2$ and $F_1 \cup F_2$ are v–fragments. In particular, there exists a v–fragment $K_v$ contained in every v–fragment.

Proof. Notice that the intersection and union of two v–Moser sets are v–Moser sets. By (2), $\kappa \Gamma(v) \leq |\partial(F_1 \cap F_2)| + |\partial(F_1 \cup F_2)| \leq |\partial(F_1)| + |\partial(F_2)| \leq 2\kappa \Gamma(v)$. □

We shall denote the minimal v–fragment containing $v$ by $K_v$.

Consider the subgraph $\Theta = (V, F)$, where $F = \{(x, y) \in E : y \in K_x\}$. Set $\Psi(x) = \{y : K_y \subset K_x \text{ and } y \in \Theta(x)\}$.

Lemma 4 $\Theta^-(x) \setminus \Psi(x) \subset \partial(K_x) \setminus \Gamma(x)$.

Proof.

Take $y \in \Theta^-(x) \cap (V \setminus \Gamma(K_x))$. Since $x \in K_y$, we have by (1), $\Gamma^-(y) \subset \Gamma^-(V \setminus \Gamma(K_x)) \subset V \setminus K_x$. It follows that $\Gamma^-(y) \cap (K_x \cup K_y) = \Gamma^-(y) \cap K_y = \{y\}$. Thus $K_x \cup K_y$ is a y–Moser set. Noticing that $K_x \cap K_y$ is an x–Moser set, we have by (2),

$$\kappa \Gamma(y) + \kappa \Gamma(x) \leq |\partial(K_x \cap K_y)| + |\partial(K_x \cup K_y)| \leq |\partial(K_x)| + |\partial(K_y)| \leq \kappa \Gamma(x) + \kappa \Gamma(y).$$

It follows that $K_x \cap K_y$ is a x–Moser set and hence $K_x = K_x \cap K_y$. Thus $y \in \Psi(x)$. Therefore

$$\Theta^-(x) \cap (V \setminus \Gamma(K_x)) \subset \Psi(x).$$

Take $y \in \Theta^-(x) \setminus \Psi(x)$. Then $x \in \Theta(y) \subset \Gamma(y)$. By the last relation, we must have $y \in \Gamma(K_x)$. Since $K_x \cap \Gamma^-(x) = \{x\}$, we have $y \notin \Gamma(x)$. In particular $\Theta^-(x) \setminus \Psi(x) \subset \Gamma(x)$. The lemma follows now. □

5 The result and its applications

Theorem 5 Let $\Gamma = (V, E)$ be a point-transitive reflexive locally finite relation and let $v \in V$. Let $F$ be a subset of $V$ with $F \cap \Gamma^-(v) = \{v\}$.

(i) If $F$ is finite, then $|\Gamma(F) \setminus F| \geq |\Gamma(v)| - 1.$
(ii) If $F$ is cofinite, then $|\Gamma(F) \setminus F| \geq |\Gamma^-(v)| - 1$.

Proof. Recall that for every $x \in V$, $|\Gamma(v)| = |\Gamma(x)|$. This is an immediate consequence of the transitive action of automorphism group. Moreover if $V$ is finite, $|V||\Gamma(v)| = |E| = |V||\Gamma^-(v)|$. Thus $|\Gamma(v)| = |\Gamma^-(v)|$. In particular, (i) and (ii) are equivalent in the finite case. For every $x \in V$, there is an automorphism $\phi$ such that $\phi(v) = x$. Clearly $\phi(K_v) = \phi(K_x)$ and hence $|K_v| = |K_x|$. If $x \in \Gamma(v)$, then $v \in K_v \setminus K_x$, and thus we have $K_v \neq K_x$ and hence $K_v \not\subset K_x$. It follows that $\Psi(x) = \{x\}$.

Case 1 $V$ is finite. Since $\sum_{x \in V} |\Psi^-(x)| = \sum_{x \in V} |\Psi(x)|$, there is a $u \in V$ with $|\Theta(u)| \leq |\Theta^-(u)|$.

By Lemma 4 $\Theta^-(u) \setminus \{u\} = \Theta^-(u) \setminus \Psi(u) \subset \partial(K_u) \setminus \Gamma(u)$. Now we have

$$|\partial(K_u)| \geq |\Theta^-(u) \setminus \{u\}| + |\Gamma(u) \setminus K_u| \geq |\Theta(u) \setminus \{u\}| + |\Gamma(u) \setminus K_u| \geq |(\Gamma(u) \setminus K_u) \setminus \{u\}| + |\Gamma(u) \setminus K_u| = |\Gamma(u)| - 1.$$ 

Since $\{u\}$ is a Moser set, we have $|K_u| = 1$. It follows that $|K_v| = |K_u| = 1$. Since $F$ is a Moser set, we have

$$|\Gamma(F) \setminus F| \geq \kappa_v(\Gamma) = |\Gamma(K_v)| - 1 = |\Gamma(v)| - 1.$$ 

The result follows in this case.

Case 2 $V$ is infinite.

Subcase 2.1 $F$ is finite.

Let $A$ be an atom of $\Gamma$ containing $v$. Assume first that $\kappa(\Gamma) = 0$. Every point $x \in V \setminus A$ is contained in an atom $A_x$, disjoint from $A$ by Proposition 2 with $\partial(A_x) = 0$. It follows that $\Gamma(F \setminus v) \cap A = \emptyset$. Since $A$ is finite the result holds for $A$ by Case 1. It follows that $|\Gamma(F)| = |\Gamma(F \setminus A)| + |\Gamma(F \cap A)| \geq |F \setminus A| + |F \cap A| + |\Gamma(v)| - 1$. So we may assume that $\kappa(\Gamma) > 0$. Observe that $F \cap A$ is a $v$–Moser set of $A$. By case 1,

$$|\Gamma(F \cap A)| \geq |A \cap F| + |A \cap \Gamma(v)| - 1.$$ 

By the definition of $\kappa$, we have $|\partial(F \cup A)| \geq \kappa = \partial(A)$. Notice that $\partial(F \cup A) \setminus \partial(F) \subset \partial(A) \setminus \Gamma(v)$, and hence

$$|\partial(F)| \geq |\partial(F) \cap A| + |\partial(F \cap A) \cap \partial(F)| \geq |\partial(F) \cap A| + |\partial(F \cup A)| - |\partial(A) \setminus \Gamma(v)| \geq |A \cap \Gamma(v)| - 1 + |\partial(A)| - |\partial(A) \setminus \Gamma(v)| = |A \cap \Gamma(v)| - 1 + |\partial(A) \cap \Gamma(v)| = |\Gamma(v)| - 1.$$
Subcase 2.2 \( F \) is cofinite. Put \( R = (V \setminus \Gamma(F)) \cup \{v\} \). It follows that \( R \cap \Gamma(v) = \{v\} \cap \Gamma(v) = \{v\} \). In particular, \( R \) is a \( v \)-Moser set for the relation \( \Gamma^- \). Also we have using (1), \( \partial^- (R) \subset \partial^- (V \setminus \Gamma(F)) \cup (\partial^- (v) \setminus R) \subset \partial (F) \), since \( F \cap \Gamma^-(v) = \{v\} \).

By Subcase 2.1, \(|\partial(F)| \geq |\partial^-(V \setminus \Gamma(F))| \geq |\partial^-(R)| \geq |\Gamma^-(v)| - 1.\]

**Corollary 6** Let \( \Gamma = (V,E) \) be a point-symmetric reflexive locally finite relation and let \( v \in V \). Let \( j \geq 1 \) be an integer such that \( \Gamma^{j-1}(v) \cap \Gamma^-(v) = \{v\} \). Then

\[ |\Gamma^j(v)| \geq |\Gamma^{j-1}(v)| + |\Gamma(v)| - 1. \]

The proof is an immediate consequence of Theorem 5.

The finite case of the corollary is proved by Mader in [18]. The same conclusion was obtained under the stronger hypothesis \( \Gamma^j(v) \cap \Gamma^-(v) = \{v\} \) in [14]. This weak form is enough to show Caccetta and Häggkvist Conjecture.

**Corollary 7** (The Moser-Scherck-Kemperman-Wehn Theorem and its extension to cofinite sets)

Let \( G \) be group and let \( S \) be a finite subset of \( G \). Let \( F \) be a finite or a cofinite subset of \( G \) such that \( F \cap S^{-1} = \{1\} \). Then \(|(FS) \setminus F| \geq |S| - 1.\]

**Proof.** The corollary follows by applying Theorem 5 to the Cayley relation defined on \( G \) by \( S \).

Let \( G \) be a group and let \( X \subset G \). We shall write \( \overline{X} = G \setminus X \) and \( \tilde{X} = X \cup \{1\} \).

**Corollary 8** (Losonczy in the abelian case, Eliahou-Lecouvey in the general case)

Let \( G \) be group and let \( A \) be a finite subset of \( G \setminus \{1\} \). There is a permutation \( \sigma \) of \( S \) such that \( x \sigma (x) \notin A \) for every \( x \in A \).

**Proof.** Define a relation \( \Phi \) on \( A \), where \( \Phi(x) = (x^{-1}A) \cap A \). For every \( B \subset A^{-1} \), we have clearly \( \overline{A} \cap B = \{1\} \). Since \( \overline{A} \) is a cofinite subset, we have by Corollary 4 \(|B| \leq |(\overline{A}B) \setminus \overline{A}| = |(\overline{A}B) \setminus A| = |\Phi(B^{-1})| \). The existence \( \sigma \) follows now by Theorem 14.
Corollary 9 (Mader [18]) Let $\Gamma = (V, E)$ be a point-transitive loopless finite relation and let $v \in V$. There exist $|\Gamma(v)|$ directed cycles in a with a pairwise intersection $= \{v\}$.

Proof. Let $\Phi$ be the graph obtained by adding vertex $v' \notin V$, with $\Phi(v') = \Gamma^-(v)$. Let $\Theta$ be the reflexive closure $\Phi$ (obtained by adding loops everywhere. Let $F$ be a subset of $V$ with $v \in F$ and $v' \notin \Theta(F)$. Then clearly $F$ is a $v$–Moser set. By Theorem 5, $|\partial(F)| \geq |\Theta(v)|$. By the Dirac-Menger’s Theorem $\Phi$ contains $|\Gamma(v)|$ disjoint paths from $\Gamma(v)$ to $\Gamma^-(v') = \Gamma^-(v)$. Adding $v$ to each of these paths, we see the existence of $|\Gamma(v)|$ cycles verifying the Corollary.

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