Gröbner-Shirshov bases for Lie $\Omega$-algebras and free Rota-Baxter Lie algebras

Jianjun Qiu and Yuqun Chen
School of Mathematical Sciences, South China Normal University
Guangzhou 510631, P. R. China
jianjunqiu@126.com
yqchen@scnu.edu.cn

Abstract: In this paper, we generalize the Lyndon-Shirshov words to Lyndon-Shirshov $\Omega$-words on a set $X$ and prove that the set of all non-associative Lyndon-Shirshov $\Omega$-words forms a linear basis of the free Lie $\Omega$-algebra on the set $X$. From this, we establish Gröbner-Shirshov bases theory for Lie $\Omega$-algebras. As applications, we give Gröbner-Shirshov bases for free $\lambda$-Rota-Baxter Lie algebras, free modified $\lambda$-Rota-Baxter Lie algebras and free Nijenhuis Lie algebras and then linear bases of such three free algebras are obtained.

Key words: Lie $\Omega$-algebra; Nijenhuis Lie algebra; Rota-Baxter Lie algebra; Lyndon-Shirshov word; Gröbner-Shirshov basis.

AMS 2010 Subject Classification: 16S15, 13P10, 17A50, 16T25

1 Introduction

Let $(R, \cdot)$ be an algebra over a field $k$, $\lambda \in k$ and $P : R \to R$ a linear map satisfying

$$P(x) \cdot P(y) = P(P(x) \cdot y) + P(x \cdot P(y)) + \lambda P^n(x \cdot y), \quad x, y \in R,$$

where $n$ is a nonnegative integer. If $n = 1$ (resp. $n = 0$, $n = 2$ and $\lambda = -1$), then the operator $P$ is called a $\lambda$-Rota-Baxter (resp. modified $\lambda$-Rota-Baxter, Nijenhuis) operator on the algebra $(R, \cdot)$ and the triple $(R, \cdot, P)$ is called a $\lambda$-Rota-Baxter (resp. modified $\lambda$-Rota-Baxter, Nijenhuis) algebra.

Rota-Baxter operator on an associative algebra was introduced by G. Baxter to solve an analytic problem in probability [6] and was studied by G.-C. Rota [51] in combinatorics later. The Rota-Baxter operator of weight $\lambda = 0$ on Lie algebra is also called the operator form of the classical Yang-Baxter equation

\*Supported by the NNSF of China (11171118, 11571121).

\*Corresponding author.
due to Semenov-Tian-Shansky’s work [52]. Modified Rota-Baxter operator on an associative algebra was introduced by K. Ebrahimi-Fard [33] with motivation from modified classical Yang-Baxter equation on a Lie algebra [52]. The Nijenhuis operator on an associative algebra was introduced by J. Cariñena et al. [24] to study quantum bi-Hamiltonian systems. In [56], Nijenhuis operators are constructed by analogy with Poisson-Nijenhuis geometry, from relative Rota-Baxter operators. Nijenhuis operators on Lie algebras play an important role in the study of integrability of nonlinear evolution equations [28]. Recently, there are some results on Rota-Baxter operators on Lie algebras and related topic, for example, see [1, 4, 5, 47, 49].

There are many constructions of free $\lambda$-Rota-Baxter associative algebras, free modified $\lambda$-Rota-Baxter associative algebras and free Nijenhuis associative algebras by using different methods, for example, [3, 14, 25, 34–42, 46, 51]. However, as we know, there are no constructions of free $\lambda$-Rota-Baxter Lie algebras, free modified $\lambda$-Rota-Baxter Lie algebras and free Nijenhuis Lie algebras. We will apply Gröbner-Shirshov bases method to construct such three free Lie algebras.

Gröbner bases and Gröbner-Shirshov bases were invented independently by A.I. Shirshov for ideals of free (commutative, anti-commutative) non-associative algebras [54, 55], free Lie algebras [53, 55] and implicitly free associative algebras [53, 55] (see also [7, 9]), by H. Hironaka [44] for ideals of the power series algebras (both formal and convergent), and by B. Buchberger [21] for ideals of the polynomial algebras. Gröbner bases and Gröbner-Shirshov bases theories have been proved to be very useful in different branches of mathematics, including commutative algebra and combinatorial algebra. It is a powerful tool to solve the following classical problems: normal form; word problem; conjugacy problem; rewriting system; automaton; embedding theorem; PBW theorem; extension; homology; growth function; Dehn function; complexity; etc. See, for example, the books [2, 18, 22, 23, 27, 30], the papers [7–9, 13, 29, 37, 48], and the surveys [11, 15–17].

The concept of $\Omega$-algebra was introduced by A.G. Kurosh [45] under an influence of the concept of multioperator group of P.J. Higgins [43]. An $\Omega$-algebra over a field $k$ is a $k$-algebra $A$ with a set of multilinear operators $\Omega$, where $\Omega = \bigcup_{m=1}^{\infty} \Omega_m$ and each $\Omega_m$ is a set of $m$-ary multilinear operators on $A$. $\lambda$-Rota-Baxter Lie algebras, modified $\lambda$-Rota-Baxter Lie algebras and Nijenhuis Lie algebras are Lie $\Omega$-algebras with one operator. In [14], the associative $\Omega$-words on a set $X$ are introduced, it is shown that the set of all associative $\Omega$-words on $X$ forms a linear basis of the free associative $\Omega$-algebra on $X$, and then Gröbner-Shirshov bases theory for associative $\Omega$-algebras is established. For Gröbner-Shirshov bases theory of various $\Omega$-algebras and their applications, see [12, 13, 32, 37, 50].

The first linear basis of a free Lie algebra $\text{Lie}(X)$ on a set $X$ had been given by M. Hall 1950 [31], K.-T. Chen, R.H. Fox, R.C. Lyndon 1958 [26] and A.I. Shirshov 1958 [53] introduced non-associative Lyndon-Shirshov words on $X$ and proved that they form a linear basis of $\text{Lie}(X)$, independently. One of the main applications of Lyndon-Shirshov basis is the Shirshov’s theory of Gröbner-Shirshov bases theory for Lie algebras [53].
The paper is organized as follows. In section 2, we review the concept and some related properties of Lyndon-Shirshov words on a set $X$, generalize associative (resp. non-associative) Lyndon-Shirshov words to associative (resp. non-associative) Lyndon-Shirshov $\Omega$-words on a set $X$ and show that the set of all non-associative Lyndon-Shirshov $\Omega$-words forms a linear basis of the free Lie $\Omega$-algebra on the set $X$. In section 3, we review Gröbner-Shirshov bases theory for associative $\Omega$-algebras, give definition of Gröbner-Shirshov bases for Lie $\Omega$-algebras and establish Composition-Diamond lemma for Lie $\Omega$-algebras. In section 4, we give Gröbner-Shirshov bases for free $\lambda$-Rota-Baxter Lie algebras, free modified $\lambda$-Rota-Baxter Lie algebras, free Nijenhuis Lie algebras and then linear bases of such three free algebras are obtained by Composition-Diamond lemma for Lie $\Omega$-algebras.

2 Free Lie $\Omega$-algebras

2.1 Lyndon-Shirshov words

In this subsection, we review the concept and some properties of Lyndon-Shirshov words, which can be found in [10, 53, 55].

For any set $X$, we define the following notations:

$S(X)$: the set of all nonempty associative words on $X$.

$X^*$: the set of all associative words on $X$ including the empty word $1$.

$X^{**}$: the set of all non-associative words on $X$.

For any $u \in X^*$, denote $\deg(u)$ to be the degree (length) of $u$. Let $>$ be a well order on $X$. Define the lex-order $>_\text{lex}$ and the deg-lex order $>_{\text{deg-lex}}$ on $X^*$ with respect to $>$ by:

(i) $1 >_{\text{lex}} u$ for any nonempty word $u$, and if $u = x_i u'$ and $v = x_j v'$, where $x_i, x_j \in X$, then $u >_{\text{lex}} v$ if $x_i > x_j$, or $x_i = x_j$ and $u' >_{\text{lex}} v'$ by induction.

(ii) $u >_{\text{deg-lex}} v$ if $\deg(u) > \deg(v)$, or $\deg(u) = \deg(v)$ and $u >_{\text{lex}} v$.

A nonempty associative word $w$ is called an associative Lyndon-Shirshov word on $X$, if $w = uv >_{\text{lex}} vu$ for any decomposition of $w = uv$, where $1 \neq u, v \in X^*$.

A non-associative word $(u) \in X^{**}$ is said to be a non-associative Lyndon-Shirshov word on $X$ with respect to the lex-order $>_{\text{lex}}$, denoted by $\lbrack u \rbrack$, if

(a) $u$ is an associative Lyndon-Shirshov word on $X$;

(b) if $(u) = ((v)(w))$, then both $(v)$ and $(w)$ are non-associative Lyndon-Shirshov words on $X$;

(c) if $(v) = ((v_1)(v_2))$, then $v_2 \leq_{\text{lex}} w$.

Denote the set of all associative (resp. non-associative) Lyndon-Shirshov words on $X$ with respect to the lex-order $>_{\text{lex}}$ by $\text{ALSW}(X)$ (resp. $\text{NLSW}(X)$). It is well known that for any $u \in \text{ALSW}(X)$, there exists a unique Shirshov standard bracketing way $\lbrack u \rbrack$ such that $\lbrack u \rbrack \in \text{NLSW}(X)$. Then $\text{NLSW}(X) = \{\lbrack u \rbrack | u \in \text{ALSW}(X)\}$. 

3
Let $k(X)$ be the free associative algebra on $X$ over a field $k$ and $\text{Lie}(X)$ be the Lie subalgebra of $k(X)$ generated by $X$ under the Lie bracket $(uv) = uv - vu$. It is well known that $\text{Lie}(X)$ is a free Lie algebra on the set $X$ with a linear basis $\text{NLSW}(X)$.

For any $f \in k(X)$, let $\bar{f}$ be the leading word of $f$ with respect to the deg-lex order $>_{\text{deg-lex}}$ on $X^*$.

**Lemma 2.1** ([10, 53, 55]) For any non-associative word $(u) \in X^{**}$, $(u)$ has a representation $$(u) = \sum \alpha_i [u_i],$$ in $\text{Lie}(X)$, where each $\alpha_i \in k$ and $u_i \in \text{ALSW}(X)$.

**Lemma 2.2** ([10, 53, 55]) If $u \in \text{ALSW}(X)$, then $[u] = u$.

**Lemma 2.3** ([10, 53, 55]) For any $u \in X^*$, there exists a unique decomposition $$u = u_1u_2\cdots u_m,$$ where each $u_i \in \text{ALSW}(X)$ and $u_j \leq_{\text{lex}} u_{j+1}$, $1 \leq j \leq m - 1$.

**Lemma 2.4** ([10, 53, 55]) Let $u = avb$, where $u, v \in \text{ALSW}(X)$ and $a, b \in X^*$. Then $$[u] = [a[v]c],$$ where $b = cd$ for some $c, d \in X^*$. Let $$[u]_v = [u[vc] \cdots (((v)c_1)]c_2] \cdots [c_m]]$$ where $c = c_1c_2 \cdots c_m$ with each $c_i \in \text{ALSW}(X)$ and $c_i \leq_{\text{lex}} c_{i+1}$. Then, $$[u]_v = a[v]b + \sum \alpha_i a_i[v]b_i,$$ where each $\alpha_i \in k$, $a_i, b_i \in X^*$ and $a_i v b_i <_{\text{deg-lex}} avb = u$. It follows that $[u]_v = u$.

### 2.2 Lyndon-Shirshov $\Omega$-words

In this subsection, we define the Lyndon-Shirshov $\Omega$-words on a set $X$.

Let $$\Omega = \bigcup_{m=1}^{\infty} \Omega_m,$$ where $\Omega_m$ is a set of $m$-ary operators for any $m \geq 1$. For any set $Y$, denote $$\Omega(Y) := \bigcup_{m=1}^{\infty} \left\{ \omega^{(m)}(y_1, y_2, \cdots, y_m) | y_i \in Y, 1 \leq i \leq m, \omega^{(m)} \in \Omega_m \right\}.$$
Let $X$ be a set. Define $⟨Ω; X⟩_0 := S(X)$ and $(Ω; X)_0 := X^{**}$. Assume that we have defined $⟨Ω; X⟩_{n-1}$ and $(Ω; X)_{n-1}$. Define

$$
⟨Ω; X⟩_n := S(X ∪ Ω(⟨Ω; X⟩_{n-1})),
(Ω; X)_n := (X ∪ Ω(⟨Ω; X⟩_{n-1}))^{**}.
$$

Then it is easy to see that for any $n ≥ 0$,

$$
⟨Ω; X⟩_n ⊆ ⟨Ω; X⟩_{n+1}, (Ω; X)_n ⊆ (Ω; X)_{n+1}.
$$

Denote

$$
⟨Ω; X⟩ := \bigcup_{n=0}^{∞} ⟨Ω; X⟩_n, (Ω; X) := \bigcup_{n=0}^{∞} (Ω; X)_n.
$$

The elements of $⟨Ω; X⟩$ (resp. $(Ω; X)$) are called the associative (resp. non-associative) $Ω$-words on $X$.

If $u ∈ X ∪ Ω(⟨Ω; X⟩)$, then $u$ is called prime. Therefore, for any $u ∈ ⟨Ω; X⟩$, $u$ can be expressed uniquely in the canonical form

$$
u = u_1 u_2 · · · u_n, n ≥ 1,
$$

where each $u_i$ is prime. The number $n$ is called the breath of $u$, which is denoted by $bre(u)$. The degree of $u$, denoted by $deg(u)$, is defined to be the total number of all occurrences of all $x ∈ X$ and $θ ∈ Ω$ in $u$. For any associative $Ω$-word $u$, define the depth of $u$ to be

$$
dep(u) := \min\{n|u ∈ ⟨Ω; X⟩_n\}.
$$

Let $(u)$ be a non-associative $Ω$-word. Define the depth of $(u)$ by $dep((u)) = dep(u)$.

For example, if $u = ω^{(3)}(x_2 x_1 x_1, x_1, ω^{(1)}(x_2 x_2 x_1)) x_2 x_1$, where $x_1, x_2 ∈ X$ and $ω^{(3)}, ω^{(1)} ∈ Ω$, then $deg(u) = 11$, $bre(u) = 3$ and $dep(u) = 2$.

Let $u = u_1 u_2 · · · u_n, n ≥ 1$, where each $u_i$ is prime. Denote

$$
wt(u) := \langle deg(u), bre(u), u_1, u_2, · · · , u_n \rangle.
$$

Let $X$ and $Ω$ be well-ordered sets with the orders $>_X$ and $>_Ω$, respectively. Define the Deg-lex order $>_p$ on $(Ω; X)$ as follows. For any $u = u_1 u_2 · · · u_n, v = v_1 v_2 · · · v_m$, where $u_i, v_j$ are prime, define

$$
u>_p v \text{ if } wt(u) > wt(v) \text{ lexicographically,}
$$

where if $u_i = ω(u_{i_1}, u_{i_2}, · · · , u_{i_d}), v_i = \theta(v_{i_1}, v_{i_2}, · · · , v_{i_d})$ and $deg(u_i) = deg(v_i)$, then $u_i > v_i$ if

$$(ω, u_{i_1}, u_{i_2}, · · · , u_{i_d}) > (θ, v_{i_1}, v_{i_2}, · · · , v_{i_d}) \text{ lexicographically.}
$$

Let $≻$ be the restriction of $>_p$ on $X ∪ Ω(⟨Ω; X⟩)$. We define the Lyndon-Shirshov $Ω$-words on the set $X$ by induction on the depth of the $Ω$-words.
For $n = 0$, define

$$ALSW(Ω; X)_0 := ALSW(X),$$

$$NLSW(Ω; X)_0 := NLSW(X) = \{[u]|u \in ALSW(Ω; X)_0\}$$

with respect to the lex-order $\succ_{lex}$ on $X^*$.

Assume that we have defined $ALSW(Ω; X)_{n-1}$ and $NLSW(Ω; X)_{n-1} = \{[u]|u \in ALSW(Ω; X)_{n-1}\}$.

Define

$$ALSW(Ω; X)_n := ALSW(X \cup Ω(ALSW(Ω; X)_{n-1}))$$

with respect to the lex-order $\succ_{lex}$.

For any $u \in X \cup Ω(ALSW(Ω; X)_{n-1})$, define the bracketing way on $u$ by

$$[u] := \begin{cases} 
    u, & \text{if } u \in X, \\
    \omega^{(m)}([u_1], [u_2], \cdots, [u_m]), & \text{if } u = \omega^{(m)}(u_1, u_2, \cdots, u_m).
\end{cases}$$

Denote

$$[X \cup Ω(ALSW(Ω; X)_{n-1})] := \{[u]|u \in X \cup Ω(ALSW(Ω; X)_{n-1})\}.$$  

Then, the order $\succ$ on $X \cup Ω(ALSW(Ω; X)_{n-1})$ induces an order (still denoted by $\succ$) on $[X \cup Ω(ALSW(Ω; X)_{n-1})]$ by $[u] \succ [v]$ if $u \succ v$ for any $u, v \in X \cup Ω(ALSW(Ω; X)_{n-1})$. If $u = u_1u_2\cdots u_m \in ALSW(Ω; X)_n$, where each $u_i \in X \cup Ω(ALSW(Ω; X)_{n-1})$, then we define

$$[u] := [[u_1][u_2]\cdots[u_m]]$$

is the Shirshov standard bracketing way on $\{[u_1], [u_2], \cdots, [u_m]\}$, which means $[u]$ a non-associative Lyndon-Shirshov word on $\{[u_1], [u_2], \cdots, [u_m]\}$ with respect to the lex-order $\succ_{lex}$. Denote

$$NLSW(Ω; X)_n := \{[u]|u \in ALSW(Ω; X)_n\}.$$  

It is easy to see that

$$NLSW(Ω; X)_n = NLSW([X \cup Ω(ALSW(Ω; X)_{n-1})]).$$

Denote

$$ALSW(Ω; X) := \bigcup_{n=0}^\infty ALSW(Ω; X)_n,$$

$$NLSW(Ω; X) := \bigcup_{n=0}^\infty NLSW(Ω; X)_n.$$
Then
\[ \text{NLSW}(\Omega; X) = \{[u]|u \in \text{ALSW}(\Omega; X)\}. \]

The elements of \( \text{ALSW}(\Omega; X) \) (resp. \( \text{NLSW}(\Omega; X) \)) are called the associative (resp. non-associative) Lyndon-Shirshov \( \Omega \)-words. By the above definitions, for any associative Lyndon-Shirshov \( \Omega \)-word \( u \), \( u \) is associated a unique non-associative Lyndon-Shirshov \( \Omega \)-word \([u]\). For example, if \( u = \omega^{(3)}(x_2x_1x_1, x_1, \omega^{(1)}(x_2x_2x_1))x_2x_1 \), where \( x_1, x_2 \in X \) with \( x_2 > x_1 \), and \( \omega^{(3)}, \omega^{(1)} \in \Omega \), then \( u \in \text{ALSW}(\Omega; X) \) and
\[ [u] = (\omega^{(3)}((x_2x_1)x_1), x_1, \omega^{(1)}((x_2(x_2x_1)))(x_2x_1)) \in \text{NLSW}(\Omega; X). \]

### 2.3 Free Lie \( \Omega \)-algebras

In this subsection, we prove that the set \( \text{NLSW}(\Omega; X) \) of all non-associative Lyndon-Shirshov \( \Omega \)-words is a linear basis of the free Lie \( \Omega \)-algebra on the set \( X \).

An associative (resp. non-associative, Lie) \( \Omega \)-algebra over a field \( k \) is an associative (resp. non-associative, Lie) algebra \( A \) with multilinear operators \( \Omega \).

For any associative \( \Omega \)-algebra \( (A, \cdot, \Omega) \), it is easy to see that \( (A, [\cdot, \cdot], \Omega) \) is a Lie \( \Omega \)-algebra, where
\[ [a, a'] = a \cdot a' - a' \cdot a, \quad a, a' \in A. \]

Let \( X \) be a set. An associative (resp. non-associative, Lie) \( \Omega \)-algebra \( F(X) \) together with an injective map \( i : X \to F(X) \) is called a free associative (resp. non-associative, Lie) \( \Omega \)-algebra on \( X \), if for any associative (resp. non-associative, Lie) \( \Omega \)-algebra \( A \) and any map \( \sigma : X \to A \), there exists a unique associative (resp. non-associative, Lie) \( \Omega \)-algebra homomorphism \( \tilde{\sigma} : F(X) \to A \) such that \( \tilde{\sigma}i = \sigma \).

Let \( k\langle \Omega; X \rangle \) be the \( k \)-linear space spanned by \( \langle \Omega; X \rangle \). Then \( k\langle \Omega; X \rangle \) is a free associative \( \Omega \)-algebra on the set \( X \), see [14]. Denote \( \text{Lie}(\Omega; X) \) the Lie \( \Omega \)-subalgebra of \( k\langle \Omega; X \rangle \) generated by \( X \) under the Lie bracket \( (uv) = uv - vu \).

The elements of \( k\langle \Omega; X \rangle \) (resp. \( \text{Lie}(\Omega; X) \)) are called associative (resp. Lie) \( \Omega \)-polynomials on \( X \).

**Lemma 2.5** For any \((u) \in (\Omega; X)\), \((u)\) has a representation
\[ (u) = \sum \alpha_i[u_i], \]
in \( \text{Lie}(\Omega; X) \), where each \( \alpha_i \in k \) and \([u_i] \in \text{NLSW}(\Omega; X)\).

**Proof.** Induction on \( \text{dep}((u)) \). If \( \text{dep}((u)) = 0 \), by Lemma 2.1 we have \((u) = \sum \alpha_i[u_i] \), where each \( \alpha_i \in k \) and \([u_i] \in \text{NLSW}(X) \subseteq \text{NLSW}(\Omega; X) \).

Assume that the result is true for any \((u)\) with \( \text{dep}((u)) \leq n - 1 \).

Let \( \text{dep}((u)) = n \geq 1 \). There are two cases to consider.
Case 1. If \( (u) = \theta((u_1), (u_2), \cdots, (u_m)) \), then \( \text{dep}((u_i)) \leq n - 1, 1 \leq i \leq m \). Thus, by induction, we may assume that \( (u_i) = [u_i] \), where each \( [u_i] \in \text{NLSW}(\Omega; X) \). Therefore,
\[
(u) = \theta([u_1], [u_2], \cdots, [u_m] \} = \theta(u_1, u_2, \cdots, u_m) \in \text{NLSW}(\Omega; X).
\]

Case 2. If \( (u) = (a_1 a_2 \cdots a_m), m \geq 2 \), where each \( a_i \) is prime, then by Case 1, we may assume that \( (u) = ([a_1][a_2] \cdots [a_m]) \), where each \( a_i \in X \cup \Omega(\text{ALSW}(\Omega; X)_{n-1}) \). Thus, \( (u) \) is a non-associative word on \( \{[a_1], [a_2], \cdots, [a_m]\} \).

By Lemma 2.1 we have
\[
([a_1][a_2] \cdots [a_m]) = \sum \alpha_i [u_i],
\]
where each \( \alpha_i \in k \) and \( [u_i] \in \text{NLSW}(\{[a_1], [a_2], \cdots, [a_m]\}) \subseteq \text{NLSW}(\Omega; X) \). □

For any \( f \in k(\Omega; X) \), let \( \overline{f} \) be the leading \( \Omega \)-word of \( f \) with respect to the order \( >_{D_l} \) on \( \langle \Omega; X \rangle \).

**Lemma 2.6** If \( u \in \text{ALSW}(\Omega; X) \), then \( \overline{u} = u \) with respect to the order \( >_{D_l} \) on \( \langle \Omega; X \rangle \).

**Proof.** Induction on \( \text{dep}(u) \). If \( \text{dep}(u) = 0 \), then \( \overline{u} = u \) by Lemma 2.2. Assume that the result is true for any \( u \in \text{ALSW}(\Omega; X) \) with \( \text{dep}(u) \leq n - 1 \).

Let \( \text{dep}(u) = n \geq 1 \). There are two cases to consider.

Case 1. If \( \text{bre}(u) = 1 \) and \( u = \theta(u_1, u_2, \cdots, u_m) \), then \( [u] = \theta([u_1], [u_2], \cdots, [u_m]) \).

By induction, we have \( [u_i] = u_i, 1 \leq i \leq m \). Thus,
\[
\overline{u} = \theta([u_1], [u_2], \cdots, [u_m]) = \theta(u_1, u_2, \cdots, u_m) = u.
\]

Case 2. If \( \text{bre}(u) > 1 \) and \( u = a_1 a_2 \cdots a_m \), where each \( a_i \) is prime, then by Lemma 2.2 we have
\[
[u] = ([a_1][a_2] \cdots [a_m]) = [a_1][a_2] \cdots [a_m] + \sum \alpha_i [a_i],
\]
where each \( \{i_1, i_2, \ldots, i_m\} = \{1, 2, \ldots, m\} \) and \( a_i a_{i_2} \cdots a_{i_m} >_{D_l} a_1 a_2 \cdots a_m \).

By Case 1, we have \( [a_i] = a_i \) and \( [a_i] = a_i \). It follows that \( \overline{u} = a_1 a_2 \cdots a_m = u \). □

**Lemma 2.7** \( \text{NLSW}(\Omega; X) \) is a linear basis of \( \text{Lie}(\Omega; X) \).

**Proof.** Suppose \( \sum_{i=1}^{m} \alpha_i[u_i] = 0 \) in \( \text{Lie}(\Omega; X) \), where each \( \alpha_i \in k, u_i \in \text{ALSW}(\Omega; X) \) and \( u_i >_{D_l} u_{i+1} \). If \( \alpha_1 \neq 0 \), then by Lemma 2.6 we have \( \sum_{i=1}^{m} \alpha_i[u_i] = u_1 \), a contradiction. Therefore, \( \text{NLSW}(\Omega; X) \) is linear independent set. By Lemma 2.5 \( \text{NLSW}(\Omega; X) \) is a linear basis of \( \text{Lie}(\Omega; X) \). □
Let \( k(\Omega; X) \) be the \( k \)-linear space spanned by \( (\Omega; X) \). It is easy to see that \( k(\Omega; X) \) is a free non-associative \( \Omega \)-algebra on the set \( X \). Denote \( R_\Omega \) the set consisting of the following relations in \( k(\Omega; X) \):

\[
((u)(v)) = -(v)(u),
\]

\[
(((u)(v))(w)) = (((u)(w))(v)) + ((u)((v)(w))),
\]

where \((u), (v), (w) \in (\Omega; X)\). Then \( FL_\Omega(X) := k(\Omega; X)/Id(R_\Omega) \) is a free Lie \( \Omega \)-algebra on the set \( X \).

**Theorem 2.8** \( Lie(\Omega; X) \) is a free Lie \( \Omega \)-algebra on the set \( X \) with a linear basis \( NLSW(\Omega; X) \).

**Proof.** Let \( i : X \rightarrow FL_\Omega(X), x \mapsto x + Id(R_\Omega) \) and \( \varphi : X \rightarrow k(\Omega; X), x \mapsto x \). Since \( FL_\Omega(X) \) is a free Lie \( \Omega \)-algebra on \( X \), there is a unique Lie \( \Omega \)-algebra homomorphism \( \tilde{\varphi} : FL_\Omega(X) \rightarrow k(\Omega; X) \) such that \( \tilde{\varphi}i = \varphi \). It is easy to see that

\[
\tilde{\varphi}(FL_\Omega(X)) = Lie(\Omega; X).
\]

Similar to proof of Lemma 2.5, we have for any \( (u) \in (\Omega; X) \),

\[
(u) + Id(R_\Omega) = \sum \beta_j[u_j] + Id(R_\Omega),
\]

where each \( \beta_j \in k \) and \( u_j \in ALSW(\Omega; X) \). If

\[
\tilde{\varphi}\left(\sum_{i=1}^{m} \alpha_i[u_i] + Id(R_\Omega)\right) = \sum_{i=1}^{m} \alpha_i[u_i] = 0
\]

in \( k(\Omega; X) \), where each \( \alpha_i \in k \), \( u_i \in ALSW(\Omega; X) \), then by Lemma 2.7, we have each \( \alpha_i = 0 \). Thus, \( \tilde{\varphi} \) is injective. It follows that \( FL_\Omega(X) \cong Lie(\Omega; X) \), i.e., \( Lie(\Omega; X) \) is a free Lie \( \Omega \)-algebra on the set \( X \). \( \square \)

## 3 Gröbner-Shirshov bases for Lie \( \Omega \)-algebras

### 3.1 Composition-Diamond lemma for associative \( \Omega \)-algebras

In this subsection, we review Gröbner-Shirshov bases theory for associative \( \Omega \)-algebras, which can be found in [14].

Let \( k(\Omega; X) \) be the free associative \( \Omega \)-algebra on \( X \) and \( * \not\in X \). By a \(*\)-\( \Omega \)-word we mean any expression in \( (\Omega; X \cup \{*\}) \) with only one occurrence of \(*\). The set of all \(*\)-\( \Omega \)-word on \( X \) is denoted by \( (\Omega; X)^* \).

Let \( \pi \) be a \(*\)-\( \Omega \)-word and \( s \in k(\Omega; X) \). Then we call

\[
\pi|_s := \pi|_{s \mapsto s}
\]

an \( s \)-\( \Omega \)-word.
Now, we assume that \( \langle \Omega; X \rangle \) is equipped with a monomial order \( > \). This means that \( > \) is a well order on \( \langle \Omega; X \rangle \) such that for any \( v, w \in \langle \Omega; X \rangle \) and \( \pi \in \langle \Omega; X \rangle^* \), if \( w > v \), then \( \pi|_w > \pi|_v \).

For every \( \Omega \)-polynomial \( f \in k\langle \Omega; X \rangle \), let \( \bar{f} \) be the leading \( \Omega \)-word of \( f \) with respect to the order \( > \). If the coefficient of \( \bar{f} \) is 1, then we call that \( f \) is monic. We also call a set \( S \subseteq k\langle \Omega; X \rangle \) monic if each \( s \in S \) is monic.

Let \( f, g \in k\langle \Omega; X \rangle \) be monic. Then we define two kinds of compositions.

(I) If there exists an associative \( \Omega \)-word \( w = \bar{f}a = b\bar{g} \) for some \( a, b \in \langle \Omega; X \rangle \) such that \( \text{bre}(w) < \text{bre}(\bar{f}) + \text{bre}(\bar{g}) \), then we call \( (f, g)_w := f a - bg \) the intersection composition of \( f \) and \( g \) with respect to the ambiguity \( w \).

(II) If there exists an associative \( \Omega \)-word \( w = \bar{f} = \pi|_g \) for some \( \pi \in \langle \Omega; X \rangle^* \), then we call \( (f, g)_w := f - \pi|_g \) the inclusion composition of \( f \) and \( g \) with respect to the ambiguity \( w \).

Let \( S \subseteq k\langle \Omega; X \rangle \) be monic. The composition \( (f, g)_w \) is called trivial modulo \( (S, w) \) if
\[
(f, g)_w = \sum \alpha_i \pi_i|_{s_i},
\]
where each \( \alpha_i \in k \), \( \pi_i \in \langle \Omega; X \rangle^* \), \( s_i \in S \) and \( \pi_i|_{\overline{s_i}} < w \). If this is the case, we write
\[
(f, g)_w \equiv_\text{ass} 0 \mod(S, w).
\]

In general, for any two associative \( \Omega \)-polynomials \( p \) and \( q \), \( p \equiv_\text{ass} q \mod(S, w) \) means that \( p - q = \sum \alpha_i \pi_i|_{s_i} \), where each \( \alpha_i \in k \), \( \pi_i \in \langle \Omega; X \rangle^* \), \( s_i \in S \) and \( \pi_i|_{\overline{s_i}} < w \).

A monic set \( S \) is called a Gröbner-Shirshov basis in \( k\langle \Omega; X \rangle \) if any composition \( (f, g)_w \) of \( f, g \in S \) is trivial modulo \( (S, w) \).

**Lemma 3.1** (Composition-Diamond lemma for associative \( \Omega \)-algebras)

Let \( S \subseteq k\langle \Omega; X \rangle \) be monic, \( > \) a monomial order on \( \langle \Omega; X \rangle \) and \( \text{Id}_{\text{ass}}(S) \) the ideal of \( k\langle \Omega; X \rangle \) generated by \( S \). Then the following statements are equivalent:

(i) \( S \) is a Gröbner-Shirshov basis in \( k\langle \Omega; X \rangle \).

(ii) \( f \in \text{Id}_{\text{ass}}(S) \) \( \Rightarrow \bar{f} = \pi|_{\overline{s}} \) for some \( \pi \in \langle \Omega; X \rangle^* \) and \( s \in S \).

(iii) The set \( \text{Irr}(S) = \{ w \in \langle \Omega; X \rangle|w \neq \pi|_{\overline{s}}, \pi \in \langle \Omega; X \rangle^*, s \in S \} \) is a linear basis of the associative \( \Omega \)-algebra \( k\langle \Omega; X \rangle|S := k\langle \Omega; X \rangle/\text{Id}_{\text{ass}}(S) \).

### 3.2 Composition-Diamond lemma for Lie \( \Omega \)-algebras

In this subsection, we establish Composition-Diamond lemma for Lie \( \Omega \)-algebras, which is a generalization of the Shirshov’s Composition-Diamond lemma for Lie algebras.

Let \( >_{\text{Di}} \) be the order defined as before. It is easy to see that \( >_{\text{Di}} \) is a monomial order on \( \langle \Omega; X \rangle \). We always use this order in this subsection, in
particular, a Gröbner-Shirshov basis in \(k(\Omega; X)\) is with respect to the order \(>_{DL}\).

The following lemma follows from Lemma 2.3

**Lemma 3.2** Let \(u = u_1 u_2 \cdots u_m\), where each \(u_i \in X \cup \Omega(ALSW(\Omega; X))\). Then there exists a unique decomposition

\[ u = c_1 c_2 \cdots c_t, \]

where each \(c_i \in ALSW(\Omega; X)\) and \(c_j \preceq_{lex} c_{j+1}, 1 \leq j \leq t - 1\).

**Lemma 3.3** Let \(\pi \in \langle \Omega; X \rangle^*\) and \(v, \pi_v \in ALSW(\Omega; X)\). Then there is a \(\pi_i \in \langle \Omega; X \rangle^*\) and \(c \in \langle \Omega; X \rangle\) such that

\[ [\pi_v] = [\pi_i]_{[vc]}, \]

where \(c\) may be empty. Let

\[ [\pi_v]_v := [\pi_i]_{[vc]}[vc] \cdots [\pi]_{[vc]} \cdots [c_m], \]

where \(c = c_1 c_2 \cdots c_m\) with each \(c_i \in ALSW(\Omega; X)\) and \(c_i \preceq_{lex} c_{i+1}, 1 \leq i \leq m - 1\). Then,

\[ [\pi_v]_v = \pi_v |_{[vc]} + \sum \alpha_i \pi_i |_{[vc]}, \]

where each \(\alpha_i \in k\) and \(\pi_i |_{[vc]} <_{DL} \pi |_{[vc]}\). It follows that \([\pi_v]_v = \pi_v|_{[vc]}\) with respect to the order \(>_{DL}\).

**Proof.** Induction on the depth of \(\pi_v\). If \(\text{dep}(\pi_v) = 0\), then the result is true by Lemma 2.3. Assume that the result is true for any \(\pi_v\) with \(\text{dep}(\pi_v) \leq n - 1\).

Let \(\text{dep}(\pi_v) = n \geq 1\). There are two cases to consider.

Case 1. If \(\pi_v = avb\), where \(a, b \in \langle \Omega; X \rangle\). Let \(a = a_1 a_2 \cdots a_l, l \geq 0\) and \(b = b_1 b_2 \cdots b_t, t \geq 0\), where \(a_i\) and \(b_j\) are prime. By Lemma 2.3, we can obtain

\[ [\pi_v]_v = [a_1][a_2] \cdots [a_l][vc][b_{j+1}] \cdots [b_t] = [a|vc]d, \]

where \(c = b_1 b_2 \cdots b_j\) and \(d = b_{j+1} b_{j+2} \cdots b_t\). Let \(c = c_1 c_2 \cdots c_m\), where each \(c_i \in ALSW(\Omega; X)\) and \(c_i \preceq_{lex} c_{i+1} \cdots \preceq_{lex} c_m\). Then,

\[ [\pi_v]_v = [a_1][a_2] \cdots [a_l][c_1][c_2] \cdots [c_m][b_{j+1}] \cdots [b_t] \]

and by Lemma 2.4, we have

\[ [\pi_v]_v = [a_1][a_2] \cdots [a_l][v][b_1][b_2][b_t] + \sum \alpha_i d_i + [d_{i_p}][v][d_{i_p+1}] \cdots [d_{i_p+t}]. \]

Therefore, by Lemma 2.6, we have

\[ [\pi_v]_v = a[v]b + \sum \alpha_j a_j [v]b_j, \]

where each \(a_j b_j <_{DL} avb\).
Case 2. If \( \pi|_v = a\theta(u_1, u_2, \cdots, \pi'|_v, \cdots, u_q)b \), where \( a, b \in \langle \Omega; X \rangle \) and may be empty, then we have

\[
[\pi|_v] = [a\theta(u_1, u_2, \cdots, \pi'|_v, \cdots, u_q)b].
\]

By induction,

\[
[\pi'|_v] = [\pi''|_{\bar{v}v}], \quad [\pi'|_v] = [\pi'|_v] + \sum \alpha_i \pi'_i|_{\bar{v}v},
\]

where \( \pi'_i|_v <_{\text{di}} \pi'|_v \). Therefore,

\[
[\pi|_v] = [a\theta(u_1, u_2, \cdots, [\pi''|_{\bar{v}v}], \cdots, u_q)b]
\]

and

\[
[\pi|_v] = [a\theta(u_1, \cdots, \pi'|_v, \cdots, u_q)b]_v
= [a\theta([u_1], \cdots, [\pi'|_v], \cdots, [u_q])b]
= a\theta([u_1], \cdots, [\pi'|_v], \cdots, [u_q])b + \sum \alpha_i a_i\theta([u_1], \cdots, [\pi'|_v], \cdots, [u_q])b_i
= a\theta(u_1, \cdots, \pi'|_v, \cdots, u_q)b + \sum \beta_j a'_j \theta(u'_j_1, u'_j_2, \cdots, \pi''|_v, \cdots, u'_j_q)b'_j
= \pi|_v + \sum \beta_j a'_j \theta(u'_j_1, u'_j_2, \cdots, \pi''|_v, \cdots, u'_j_q)b'_j,
\]

where each \( a'_j \theta(u'_j_1, \cdots, \pi''|_v, \cdots, u'_j_q)b'_j <_{\text{di}} \pi|_v \). It follows that \( [\pi|_v] = [\pi|_v] \).

**Definition 3.4** Let \( \pi \in \langle \Omega; X \rangle^* \) and \( f \in \text{Lie}(\Omega; X) \subseteq k\langle \Omega; X \rangle \) be monic. If \( \pi|_f \in \text{ALSW}(\Omega; X) \), then

\[
[\pi|_f] := [\pi|_f]|_{f \rightarrow f}
\]

is called a special normal \( f \)-word.

**Corollary 3.5** Let \( f \in \text{Lie}(\Omega; X) \) and \( \pi|_f \in \text{ALSW}(\Omega; X) \). Then

\[
[\pi|_f] = \pi|_f + \sum \alpha_i \pi_i|_f,
\]

where each \( \alpha_i \in k \) and \( \pi_i|_f <_{\text{di}} \pi|_f \).

Let \( f, g \in \text{Lie}(\Omega; X) \) be monic. There are two kinds of compositions.

(a) If \( w = f \alpha = bg \), where \( a, b \in \langle \Omega; X \rangle \) with \( \text{bre}(w) < \text{bre}(\bar{f}) + \text{bre}(\bar{g}) \), then

\[
\langle f, g \rangle_w := [fa]_{\bar{f}} - [bg]_{\bar{g}}
\]

is called the intersection composition of \( f \) and \( g \) with respect to the ambiguity \( w \).
(b) If \( w = \bar{f} = \pi|\bar{g} \), then

\[
\langle f, g \rangle_w := f - [\pi|g]|_{\bar{g}}
\]

is called the inclusion composition of \( f \) and \( g \) with respect to the ambiguity \( w \).

If \( S \) is a monic subset of \( \text{Lie}(\Omega; X) \), then the composition \( \langle f, g \rangle_w \) is called trivial modulo \( (S, w) \) if

\[
\langle f, g \rangle_w = \sum \alpha_i [\pi_i|s_i]_{\bar{s}_i}
\]

where each \( \alpha_i \in k \), \( s_i \in S \), \([\pi_i|s_i]_{\bar{s}_i}\) is a special normal \( s_i \)-word and \( \pi_i|s_i <_{Dl} w \).

If this is the case, then we write

\[
\langle f, g \rangle_w \equiv 0 \mod (S, w).
\]

In general, for any two Lie \( \Omega \)-polynomials \( p \) and \( q \), \( p \equiv q \mod (S, w) \) means that

\[
p - q = \sum \alpha_i [\pi_i|s_i]_{\bar{s}_i}\]

where each \( \alpha_i \in k \), \( s_i \in S \), \([\pi_i|s_i]_{\bar{s}_i}\) is a special normal \( s_i \)-word and \( \pi_i|s_i <_{Dl} w \).

**Definition 3.6** A monic set \( S \) is called a Gröbner-Shirshov basis in \( \text{Lie}(\Omega; X) \) if any composition \( \langle f, g \rangle_w \) of \( f, g \in S \) is trivial modulo \((S, w)\).

**Lemma 3.7** Let \( f, g \in \text{Lie}(\Omega; X) \subset k(\Omega; X) \) be monic. Then

\[
\langle f, g \rangle_w - (f, g)_w \equiv_{ass} 0 \mod (\{f, g\}, w).
\]

**Proof.** If \( \langle f, g \rangle_w \) and \( (f, g)_w \) are compositions of intersection, where \( w = \bar{f}a = b\bar{g} \), then

\[
\langle f, g \rangle_w = [fa]_{\bar{f}} - [bg]_{\bar{g}} = fa + \sum \alpha_i a_i f a'_i - bg - \sum \beta_j b_j g b'_j,
\]

where \( a_i f a'_i, b_j g b'_j <_{Dl} w \). It follows that

\[
\langle f, g \rangle_w - (f, g)_w \equiv_{ass} 0 \mod (\{f, g\}, w).
\]

If \( \langle f, g \rangle_w \) and \( (f, g)_w \) are compositions of inclusion, where \( w = \bar{f} = \pi|\bar{g} \), then

\[
\langle f, g \rangle_w = f - [\pi|g]|_{\bar{g}} = f - \pi|g - \sum \alpha_i \pi_i|g,
\]

where \( \pi_i|\bar{g} <_{Dl} w \). It follows that

\[
\langle f, g \rangle_w - (f, g)_w \equiv_{ass} 0 \mod (\{f, g\}, w). \quad \square
\]

**Lemma 3.8** Let \( S \subset \text{Lie}(\Omega; X) \subset k(\Omega; X) \) be monic. With the order \( >_{Dl} \) on \( (\Omega; X) \), the following two statements are equivalent:

(i) \( S \) is a Gröbner-Shirshov basis in \( \text{Lie}(\Omega; X) \).

(ii) \( S \) is a Gröbner-Shirshov basis in \( k(\Omega; X) \).
Proof. (i) ⇒ (ii). Suppose that $S$ is a Gröbner-Shirshov basis in $\text{Lie}(\Omega; X)$. Then, for any composition $(f,g)_w$, we have $(f,g)_w = \sum \alpha_i |\pi_i|_{s_i} <_{dl} w$, where each $\alpha_i \in k$, $\pi_i |\pi_i|_{s_i} <_{dl} w$, $\pi_i \in \langle \Omega; X \rangle^*$, $s_i \in S$. By Corollary 3.5 we have $(f,g)_w = \sum \beta_i |\pi_i|_{s_i} <_{dl} w$. Therefore, by Lemma 3.7, we can obtain that $(f,g)_w \equiv_{ass} 0$ mod$(S,w)$. Thus, $S$ is a Gröbner-Shirshov basis in $k\langle \Omega; X \rangle$.

(ii) ⇒ (i). Assume that $S$ is a Gröbner-Shirshov basis in $k\langle \Omega; X \rangle$. Then, for any composition $(f,g)_w$ in $S$, we have $(f,g)_w \in \text{Lie}(\Omega; X)$ and $(f,g)_w \in \text{Id}_{ass}(S)$. By Composition-Diamond lemma for associative $\Omega$-algebras, $(f,g)_w = \pi_1 |\pi_1|_{w} \in \text{ALSW}(\Omega; X)$. Let

$$h_1 = (f,g)_w - \alpha_1 |\pi_1|_{s_1}$$

where $\alpha_1$ is the coefficient of $(f,g)_w$. Then, $h_1 <_{dl} (f,g)_w$, $h_1 \in \text{Id}_{ass}(S)$ and $h_1 \in \text{Lie}(\Omega; X)$.

Lemma 3.9 Let $S \subset \text{Lie}(\Omega; X)$ be monic and

$$\text{Irr}(S) := \{[w]|w \in \text{ALSW}(\Omega; X), w \neq \pi|s, s \in S, \pi \in \langle \Omega; X \rangle^*\}.$$  

Then, for any $h \in \text{Lie}(\Omega; X)$, $h$ can be expressed as

$$h = \sum \alpha_i |u_i| + \sum \beta_j |\pi_j|_{s_j}$$

where each $u_i \in \text{ALSW}(\Omega; X), u_i \leq_{dl} \bar{h}$ and $s_i \in S, \pi_j |\pi_j|_{s_j} \leq_{dl} \bar{h}$.

Proof. Since $f \in \text{Lie}(\Omega; X)$, $h = \sum \alpha_i |u_i|$, where each $u_i \in \text{ALSW}(\Omega; X)$ and $u_i >_{dl} u_{i+1}$. If $[u_1] \in \text{Irr}(S)$, then let $h_1 = h - \alpha_1 |u_1|$.

Otherwise, there exists $s_1 \in S$ such that $u_1 = \pi |\pi|_{s_1}$. Let $h_1 = h - |\pi|_{\bar{s}_1}$. In both of the above two cases, we have $h_1 \in \text{Lie}(\Omega; X)$ and $h_1 <_{dl} \bar{h}$. Then, by induction on $\bar{h}$, we can obtain the result.

The following theorem is Composition-Diamond lemma for Lie $\Omega$-algebras. It is an analogue of Shirshov’s Composition lemma for Lie algebras [53], which was specialized to associative algebras by L. A. Bokut [9], see also G.M. Bergman [7] and B. Buchberger [19,21].

Theorem 3.10 (Composition-Diamond lemma for Lie $\Omega$-algebras) Let $S \subset \text{Lie}(\Omega; X)$ be a non-empty monic set and $\text{Id}_{\text{Lie}}(S)$ the ideal of $\text{Lie}(\Omega; X)$ generated by $S$. Then the following statements are equivalent:

(i) $S$ is a Gröbner-Shirshov basis in $\text{Lie}(\Omega; X)$.

(ii) $f \in \text{Id}_{\text{Lie}}(S) \Rightarrow \bar{f} = \pi |\pi| \in \text{ALSW}(\Omega; X)$ for some $s \in S$ and $\pi \in \langle \Omega; X \rangle^*$.

(iii) The set

$$\text{Irr}(S) = \{[w]|w \in \text{ALSW}(\Omega; X), w \neq \pi|s, s \in S, \pi \in \langle \Omega; X \rangle^*\}$$

is a linear basis of the Lie $\Omega$-algebra $\text{Lie}(\Omega; X|S) := \text{Lie}(\Omega; X)/\text{Id}_{\text{Lie}}(S)$. 

14
Proof. $(i) \Rightarrow (ii)$ Since $f \in \text{Id}_{\text{Lie}}(S) \subseteq \text{Id}_{\text{ass}}(S)$, by Lemmas 3.8 and 3.1 we have $f = \pi_s \in \text{ALSW}(\Omega; X)$ for some $s \in S$ and $\pi \in (\Omega; X)^*$. 

$(ii) \Rightarrow (iii)$ Suppose that $\sum \alpha_i[u_i] = 0$ in $\text{Lie}(\Omega; X)$, where each $[u_i] \in \text{Irr}(S)$ and $u_i >_{\nu_1} u_i+1$. That is, $\sum \alpha_i[u_i] \in \text{Id}_{\text{Lie}}(S)$. Then each $\alpha_i$ must be 0. Otherwise, say $\alpha_1 \neq 0$, since $\sum \alpha_i[u_i] = u_1$ and by (ii), we have $u_1 \notin \text{Irr}(S)$, a contradiction. Therefore, $\text{Irr}(S)$ is linear independent. By Lemma 3.9 $\text{Irr}(S)$ is a linear basis of $\text{Lie}(\Omega; X|S)$.

$(iii) \Rightarrow (i)$ For any composition $(f, g)_w$ of $f, g \in S$, we have $(f, g)_w \in \text{Id}_{\text{Lie}}(S)$. Then, by (iii) and Lemma 3.9

$$
\langle f, g \rangle_w = \sum \beta_j[\pi_j|s_j],
$$

where each $\beta_j \in k, \pi_j \in (\Omega; X)^*, s_j \in S, \pi_j <_{\nu_1} w$. This proves that $S$ is a Gröbner-Shirshov basis in $\text{Lie}(\Omega; X)$.

\[\square\]

4 Applications

In this section, as applications of Theorem 3.10, we give Gröbner-Shirshov bases for free $\lambda$-Rota-Baxter Lie algebras, free modified $\lambda$-Rota-Baxter Lie algebras and free Nijenhuis Lie algebras and then linear bases of such three free algebras are obtained.

4.1 Free $\lambda$-Rota-Baxter Lie algebras

A Lie algebra $L$ equipped with a linear map $P : L \rightarrow L$ satisfying

$$
[P(x)P(y)] = P([P(x)y]) + P([xP(y)]) + \lambda P([xy]), \quad x, y \in L
$$

is called a Rota-Baxter Lie algebra of weight $\lambda$ or $\lambda$-Rota-Baxter Lie algebra. It is easy to see that any $\lambda$-Rota-Baxter Lie algebra $(L, P)$ is a Lie $\Omega$-algebra, where $\Omega = \{P\}$.

Let $\text{Lie}(\{P\}; X)$ be the free Lie $\{P\}$-algebra on a set $X$. Denote $S$ the set consisting of the following Lie $\{P\}$-polynomials in $\text{Lie}(\{P\}; X)$:

$$
(f_{u,v} = (P([u]P([v])) - P(\mu([u][v])) - \lambda P([u][v])),
$$

where $u, v \in \text{ALSW}(\{P\}; X)$ and $u >_{\nu_1} v$.

It is easy to see

$$
\text{RBL}(X) := \text{Lie}(\{P\}; X|S) = \text{Lie}(\{P\}; X)/\text{Id}_{\text{Lie}}(S)
$$

is the free $\lambda$-Rota-Baxter Lie algebra on the set $X$.

Theorem 4.1 With the order $\nu_1$ on $(\{P\}; X)$, the set $S$ is a Gröbner-Shirshov basis in $\text{Lie}(\{P\}; X)$. It follows that the set

$$
\text{Irr}(S) = \left\{ [w] \in \text{NLSW}(\{P\}; X) \mid w \neq \pi[P(u)]P(v), \quad \pi \in (\{P\}; X)^*, \quad u, v \in \text{ALSW}(\{P\}; X), \quad u >_{\nu_1} v \right\}
$$

15
is a linear basis of the free $\lambda$-Rota-Baxter Lie algebra $RBL(X) = \text{Lie}\{\{P\}; X|S\}$ on $X$.

**Proof.** All the possible compositions of Lie $\{P\}$-polynomials in $S$ are listed as below:

$$\langle f_{u,v}, f_{v,w}\rangle_{w_1}, w_1 = P(u)P(v)P(w), u >_{D_1} v >_{D_1} w,$$

$$\langle f_{\pi|P(u)v, w}, f_{u,v}\rangle_{w_2}, w_2 = P(\pi|P(u)v)P(w), u >_{D_1} v, \pi|P(u)v >_{D_1} w,$$

$$\langle f_{u, \pi|P(v)w}, f_{v,w}\rangle_{w_3}, w_3 = P(u)P(\pi|P(v)w), v >_{D_1} w, u >_{D_1} \pi|P(v)w).$$

We check that all the compositions are trivial.

$$\langle f_{u,v}, f_{v,w}\rangle_{w_1} = [f_{u,v}P(w)]_{f_{u,v}} - [P(u)f_{v,w}]_{f_{u,v}}$$

$$= (P([u]P([v]))P([w])) - (P((P([u])[v]))P([w])) - (P(([u]P([v])))P([w]))$$

$$- \lambda(P(([u][v]))P([w])) + (P([u])P((P([v])P([w])))) + (P([u])P(([v]P([w]))))$$

$$+ \lambda(P([u])P(([v][w]))).$$

By direct computation, we have $\text{mod}(S, w_1)$:

$$((P([u]P([v]))P([w]))$$

$$\equiv P((P([u])P([v]))P([w])) + P(([u]P([v]))) + \lambda P(([u][v]))P([v]))$$

$$\equiv P((P([u]P([v]))[v])P([w])) + P(([u]P([v]))P([v])) + \lambda P(([u]P([v]))[v])$$

$$+ P((P([u]P([v]))[v])P([w])) + P(([u]P([v]))P([v])) + \lambda P(([u]P([v]))P([w]))$$

$$+ \lambda P((P([u][v]))[v]) + \lambda P(([u][v])P([v])) + \lambda P(([u][v])P([w])) + \lambda P(([u][v])P([v])) + \lambda P(([u][v])P([w])),$$

$$P((P([u])P([v]))P([w]))$$

$$\equiv P((P([u])P([v]))P([w])) + P(([u]P([v]))P([w])) + \lambda P(([u]P([v]))P([w]))$$

$$\equiv P((P([u]P([v]))[v])P([w])) + P(([u]P([v]))P([v])) + \lambda P(([u]P([v]))[v])$$

$$+ P((P([u]P([v]))[v])P([w])) + P(([u]P([v]))P([v])) + \lambda P(([u]P([v]))P([w]))$$

$$+ \lambda P((P([u][v]))[v]) + \lambda P(([u][v])P([v])) + \lambda P(([u][v])P([w])) + \lambda P(([u][v])P([v])) + \lambda P(([u][v])P([w])),$$

$$(P([u][v])P([w]) \equiv P((P([u][v]))[w]) + P(([u][v])P([w])) + \lambda P(([u][v])P([w]))).$$

16
\[(P([u])P([v])[w]))\]
\[\equiv P((P([u])(P([v])[w])))) + P(([u]P([v])[w])) + \lambda P((([u]P([v])[w])))\]
\[\equiv P((P([u])P([v])[w])) + P((P([v])(P([u])[w]))) + P(([u]P((P([v])[w])))\]
\[\quad + \lambda P((([u]P([v])[w])))\]
\[\quad + P((P([u])(P([v])[w]))) + P((P([v])(P([u])[w]))) + \lambda P((([u]P((P([v])[w])))\]
\[\quad + P((P([v])(P([u])[w]))) + P(([u]P((P([v])[w]))) + \lambda P((([u]P((P([v])[w])))\]
\[\quad + P((P([u])P([v])[w])))\]
\[\equiv P((P([u])([v]P([w]))) + P(([u]P([v])[w]))) + \lambda P((([u]P([v])[w])))\]
\[(P([u])P([v])[w])) \equiv P((P([u])([v]P([w]))) + P(([u]P([v])[w]))) + \lambda P((([u]P([v])[w])))\]

Therefore, we have
\[\langle f_{u,v}, f_{v,w} \rangle_{w_1} = [f_{u,v}P(w)|_{\pi w_1} - [P(u)f_{v,w}]|_{\pi w_1} = 0 \ mod(S, w_1).\]

Denote
\[r(\pi|_{f_{u,v}}) = [\pi|_{f_{u,v}} - [\pi|_{f_{u,v}}].\]

Then,
\[\langle f_{u,v}P_{u,v}, w, f_{u,v} \rangle_{w_2}\]
\[= f_{u,v}P_{u,v} - [P(\pi|_{f_{u,v}})P(w)|_{\pi(P_{u,v})} \]
\[= f_{u,v}P_{u,v} - (P(\pi|_{f_{u,v}})P(w))\]
\[= P((P([u])P([v])[w])) + P(([\pi|_{f_{u,v}})P([w]))) + \lambda P((([u]P([v])[w])))\]
\[\quad - (P(\pi|_{f_{u,v}})P([w])))\]
\[\equiv P((r(\pi|_{f_{u,v}})[w])) + P((r(\pi|_{f_{u,v}})[w])) + \lambda P((r(\pi|_{f_{u,v}})[w]))\]
\[\quad - P((r(\pi|_{f_{u,v}})[w])) - P((r(\pi|_{f_{u,v}})[w])) - \lambda P((r(\pi|_{f_{u,v}})[w]))\]
\[\equiv 0 \ mod(S, w_2),\]

\[\langle f_{u,v}|_{P_{u,v}(P_{w}), f_{u,v} \rangle_{w_3}\]
\[= f_{u,v}|_{P_{u,v}(P_{w})} - [P(u)P(\pi|_{f_{u,v}})]P(P(w))\]
\[= f_{u,v}|_{P_{u,v}(P_{w})} - (P(u)P(\pi|_{f_{u,v}}))P(P(w))\]
\[= P((P([u])P([v])[w])) + P(([u]P(\pi|_{f_{u,v}})P(P(w)))) + \lambda P((([u]P(\pi|_{f_{u,v}})P(P(w))))\]
\[\quad - (P([u])P(r(\pi|_{f_{u,v}})))\]
\[\equiv P((P([u])r(\pi|_{f_{u,v}})) + P(([u]P(r(\pi|_{f_{u,v}}))) + \lambda P((([u]r(\pi|_{f_{u,v}})))\]
\[\quad - P((P([u])r(\pi|_{f_{u,v}})) - P(([u]P(r(\pi|_{f_{u,v}}))) - \lambda P((([u]r(\pi|_{f_{u,v}})))\]
\[\equiv 0 \ mod(S, w_3).\]

Therefore, \(S\) is a Gr"obner-Shirshov basis in \(Lie\{P\}; X\). By Composition-Diamond lemma for Lie \(\Omega\)-algebras, the set \(Irr(S)\) is a linear basis of the free \(\lambda\)-Rota-Baxter Lie algebra \(RBL(X)\).
4.2 Free modified $\lambda$-Rota-Baxter Lie algebras

A Lie algebra $L$ equipped with a linear map $P : L \to L$ satisfying

$$[P(x)P(y)] = P([P(x)y]) + P([xP(y)]) + \lambda [xy], \quad x, y \in L$$

is called a modified $\lambda$-Rota-Baxter Lie algebra. If $\lambda = -1$, the modified $\lambda$-Rota-Baxter Lie algebra is also called a Baxter Lie algebra in [20]. It is easy to see that any modified $\lambda$-Rota-Baxter Lie algebra $(L, P)$ is a Lie $\Omega$-algebra, where $\Omega = \{P\}$.

Let $Lie(\{P\}; X)$ be the free $\{P\}$-algebra on a set $X$. Denote $S_B$ the set consisting of the following $\{P\}$-polynomials in $Lie(\{P\}; X)$:

$$f_{u,v}^B = (P([u])P([v])) - P((P([u])[v])) - P((|P([u])|v)) - \lambda([u][v]),$$

where $u, v \in ALSW(\{P\}; X)$ and $u \geq_D v$.

It is easy to see

$$MRBL(X) := Lie(\{P\}; X|S_B) = Lie(\{P\}; X)/Id_{Lie}(S_B)$$

is the free modified $\lambda$-Rota-Baxter Lie algebra on $X$.

**Theorem 4.2** With the order $\geq_D$ on $\{P\}; X)$, the set $S_B$ is a Gröbner-Shirshov basis in $Lie(\{P\}; X)$. It follows that the set

$$Irr(S_B) = \left\{ [w] \in NLSW(\{P\}; X) \mid w \neq \pi_1 P(\pi_2 P(v)), \quad \pi \in \{\{P\}; X^* \}, \quad u, v \in ALSW(\{P\}; X), \quad u \geq_D v \right\}$$

is a linear basis of the free modified $\lambda$-Rota-Baxter Lie algebra $MRBL(X) = Lie(\{P\}; X|S_B)$ on $X$.

**Proof.** All the possible compositions of Lie $\{P\}$-polynomials in $S$ are listed as below:

$$\langle f_{u,v}^B, f_{v,w}^B \rangle, \quad w_1 = P(u)P(v)P(w), \quad u \geq_D v \geq_D w;$$

$$\langle f_{\pi,P(u)P(v)}, f_{u,v}^B \rangle, \quad w_2 = P(\pi P(u)P(v)), \quad u \geq_D v, \quad \pi \geq_D P(u) \geq_D w;$$

$$\langle f_{u,\pi,P(u)P(v)}, f_{v,w}^B \rangle, \quad w_3 = P(u)P(\pi P(v)P(w)), \quad v \geq_D w, \quad u \geq_D \pi \geq_D P(v)P(w).$$

We check that all the compositions are trivial.

$$\langle f_{u,v}^B, f_{v,w}^B \rangle = [P(u)f_{v,w}^B]_{f_{u,v}^B} = [f_{u,v}^B P(u)]_{f_{v,w}^B} = [P(u)f_{v,w}^B]_{f_{u,v}^B}$$

$$= ([P([u])P([v])] - P([P([u])[v]]) - P([u][P([v])]) - \lambda([u][v]))P([w]) - (P(u)((P([v])P([w]))) - (P([v][P([w])])) - P([u][P([v][w])]))$$

$$= (P([u])P([v]))P([w]) - (P([P([u])[v]])P([w])) - (P([u][P([v])])P([w])) - \lambda([u][v])P([w]) + (P([u])P([P([v][w])])) + (P([u])P([v][w]))$$

$$+ \lambda(P([u])(|v|w))).$$
By direct computation, we have $\text{mod}(S, w_1)$:

$$(P(\lambda P(u))P(v))$$

$$= ((P(P(u))P(v)) + P((\lambda P(u))P(v)))$$

$$= P(P(P(u))P(v)) + P(P((\lambda P(u))P(v))) + \lambda((P(u))P(v))$$

$$+ P(P((\lambda P(u))P(v))) + P((\lambda P(u))P(v))) + \lambda((P(u))P(v))$$

$$(P(\lambda P(u))P(v))$$

$$= P(P(P(u))P(v)) + P((\lambda P(u))P(v))) + \lambda((P(u))P(v))$$

$$= P(P(P(u))P(v)) + P((\lambda P(u))P(v))) + \lambda((P(u))P(v))$$

$$= P((\lambda P(u))P(v)) + P((\lambda P(u))P(v))) + \lambda((P(u))P(v))$$

$$= P((\lambda P(u))P(v)) + P((\lambda P(u))P(v))) + \lambda((P(u))P(v))$$

Therefore, we have

$$\langle f_u^B, f_v^B \rangle_{w_1} = [f_u^B P(u)]_{f_u^B} - [P(u) f_v^B]_{f_v^B} \equiv 0 \text{ mod}(S, w_1).$$

Denote

$$r(\pi | f_u^B) = [\pi | f_u^B]_{f_u^B} - [\pi | f_u^B]_{f_u^B}.$$
Then,
\[
\langle f^B_{\pi|P(u)} f^B_{v,w} \rangle_{N_{\pi}}
= f^B_{\pi|P(u)} - [P(\pi|f^B_{\pi,w})P(w)]_{P(u)P(v)\pi}
= f^B_{\pi|P(u)} - (P(\pi|f^B_{\pi,w})P(w))
= P(((\pi|P(u))P(l))) + \lambda(\pi|P(u))P(\pi(l)))
- (P(\pi|f^B_{\pi,w})P(l))
\equiv 0 \mod(S, w_2),
\]

\[
\langle f^B_{u,v} f^B_{v,w} \rangle_{N_{\pi}}
= f^B_{u,v} - [P(u)|P(v)P(w)]_{P(u)P(v)w_2}
= f^B_{u,v} - (P(u)|P(v)P(w))
= P(((u|P(v))P(w))) + \lambda(\pi|P(u))P(v)\pi)
- (P(\pi|f^B_{\pi,w})P(v))
\equiv 0 \mod(S, w_3).
\]

Therefore, \( S_B \) is a Gröbner-Shirshov basis in \( \text{Lie}(\{P\}; X) \). By Composition-Diamond lemma for Lie \( \Omega \)-algebras, the set \( \text{Irr}(S_B) \) is a linear basis of the free modified \( \lambda \)-Rota-Baxter Lie algebra \( \text{MRBL}(X) \). \( \square \)

### 4.3 Free Nijenhuis Lie algebras

A Lie algebra \( L \) equipped with a linear map \( P : L \to L \) satisfying

\[
[P(x)P(y)] = P([P(x)y] + P(xP(y))) - P^2([xy]), \quad x, y \in L
\]
is called a Nijenhuis Lie algebra.

Let \( \text{Lie}(\{P\}; X) \) be the free Lie \( \{P\} \)-algebra on a set \( X \). Denote \( S_N \) the set consisting of the following Lie \( \{P\} \)-polynomials in \( \text{Lie}(\{P\}; X) \):

\[
f^N_{u,v} = (P([u]P([v])) - P(\pi([u]P([v]))) - P([u]P([v])) + P^2([u][v]),
\]

where \( u, v \in \text{ALSW}([\{P\}; X) \) and \( u \triangleright v, v \).

It is easy to see that

\[
\text{NL}(X) := \text{Lie}(\{P\}; X|S_N) = \text{Lie}(\{P\}; X)/\text{Id}_{\text{Lie}}(S_N)
\]
is the free Nijenhuis Lie algebra on \( X \).
Theorem 4.3  With the order \(\succ_{D_1}\) on \(\langle\{P\}; X\rangle\), the set \(S_N\) is a Gröbner-Shirshov basis in \(\text{Lie}\{\{P\}; X\}\). It follows that the set

\[ \text{Irr}(S_N) \equiv \left\{ \left[ w \right] \in \text{NLSW}(\{P\}; X) \mid w \neq \pi|_{\bar{P}(u)} P(v), \quad \pi \in \{\{P\}; X\}^* \right\} \]

is a linear basis of the free Nijenhuis Lie algebra \(NL(X) = \text{Lie}\{\{P\}; X| S_N\}\) on \(X\).

Proof. All the possible compositions of Lie \(\{P\}\)-polynomials in \(S\) are listed as below:

\[
\begin{align*}
(f^N_{u,v}, f^N_{v,w}, w_1, w_1) &= P(u)P(v)P(w), u \succ_{D_1} v \succ_{D_1} w, \\
(f^N_{u,v}, f^N_{v,w}, w_2, w_2) &= P(\pi|_{\bar{P}(u)} P(v))P(w), u \succ_{D_1} v, \pi|_{\bar{P}(u)} P(v) \succ_{D_1} w, \\
(f^N_{u,v}, f^N_{v,w}, w_3, w_3) &= P(u)P(\pi|_{\bar{P}(v)} P(w)), v \succ_{D_1} w, u \succ_{D_1} \pi|_{\bar{P}(v)} P(w).
\end{align*}
\]

We check that all the compositions are trivial.

\[
\begin{align*}
&\langle f^N_{u,v}, f^N_{v,w}, w_1 \rangle = [f^N_{u,v}P(w)]_{T_{u,v}} - [P(u)f^N_{v,w}]_{T_{v,w}} \\
&= ((P([u]P([w]))P([v])) - (P((P([u])P([v]))P([w]))) - (P(([u]P([w]))P([v]))) \\
&+ (P^2(([u]P([v]))P([w]))) + (P([u]P((P([v])P([w])))) + (P([u]P(([v]P([w]))))) \\
&- (P([u]P^2([v]P[w]))). \\
\end{align*}
\]

By direct computation, we have \(mod(S, w_1)\):

\[
\begin{align*}
&(P([u]P([v])))P([v]) \\
\equiv& (P((P([u])P([w])))P([v])) + (P(([u]P([w])))P([v])) - (P^2(([u]P([w])))P([v])) \\
\equiv& P((P([u]P([w])))P([v])) + P(([u]P([w])))P([v])) - P^2(([u]P([w])))P([v])) \\
&+ P((P([u]P([w])))P([v])) + P(([u]P([w])))P([v])) - P^2(([u]P([w])))P([v])) \\
&- P((P^2(([u]P([w])))P([v]))) - P((P(([u]P([w])))P([v]))) + P^2(([u]P([w])))P([v])) \\
\equiv& P((P([u]P([w])))P([v])) + P(([u]P([w])))P([v])) - P^2(([u]P([w])))P([v])) \\
&+ P((P([u]P([w])))P([v])) + P(([u]P([w])))P([v])) - P^2(([u]P([w])))P([v])) \\
&- P((P^2(([u]P([w])))P([v]))) - P((P(([u]P([w])))P([v]))) + P^2(([u]P([w])))P([v])) \\
&+ P^3(([u]P([w])))P([v])) + P^2((P(([u]P([w])))P([v]))) \\
\end{align*}
\]

\[
\begin{align*}
&(P([u]P([v])))P([v]) \\
\equiv& P((P([u]P([w])))P([v])) + P(([u]P([w])))P([v])) - P^2(([u]P([w])))P([v])) \\
\equiv& P((P([u]P([w])))P([v])) + P(([u]P([w])))P([v])) + P([u]P([v]))P([w])) \\
&- P^2(([u]P([w])))P([v])) \\
\equiv& P((P([u]P([w])))P([v])) + P(([u]P([w])))P([v])) + P([u]P((P([v])P([w]))) \\
&+ P(([u]P((P([v])P([w])))P([v]))) - P((P[u]P^2([v]P[w]))) - P^2(([u]P([v])))P([w])).
\end{align*}
\]
Denote $P(P([u][v])P([w]))$

$\equiv P((P([u][v])P([w])) + P((P([u][v])P([w]))) - P^2((P([u][v])P([w])))$

$\equiv P((P([u][v])P([w])) + P((P([u][v])P([w]))) + P((P([u][v])P([w])))$

$\quad - P^2((P([u][v])P([w])))$

$\equiv P((P([u][v])P([w])) + P((P([u][v])P([w]))) + P((P([u][v])P([w])))$

$\quad - P^2((P([u][v])P([w])))$

$\quad - P^2((P([u][v])P([w])))$

Therefore, we have

$\langle f_{u,v}^N, f_{v,w}^N \rangle_{w_1} = [f_{u,v}^N P(w)]\frac{1}{\tau_{k,\omega}} - [P(u)f_{v,w}^N]\frac{1}{\tau_{k,\omega}} \equiv 0 \mod(S, w_1)$.

Denote

$r(\pi | f_{u,v}^N) = [\pi | f_{u,v}^N] - [\pi | f_{u,v}^N] \frac{1}{\tau_{k,\omega}}$.
Then,
\[
(f^N_{\pi|P(u),P(v),w}, f^N_{u,v})_{w_2}
\]
\[=
\]  
\[
f^N_{\pi|P(u),P(v),w} - [P(\pi|f^N_{u,v})P(w)]P(u)P(v)
\]
\[=
\]  
\[
f^N_{\pi|P(u),P(v),w} - (P(\pi|f^N_{u,v})P(w))
\]
\[=
\]  
\[
P((P[\pi|P(u)]P(v)][w])) + P((\pi|P(u)P(v)][w])) - P^2((\pi|P(u)P(v)][w]))
\]
\[=
\]  
\[
0 \mod(S, w_2),
\]
\[
(f^N_{u,\pi|P(v),P(\omega)}, f^N_{u,v})_{w_3}
\]
\[=
\]  
\[
f^N_{u,\pi|P(v),P(\omega)} - [P(u)P(\pi|f^N_{u,v})P(w)]
\]
\[=
\]  
\[
f^N_{u,\pi|P(v),P(\omega)} - (P(u)P(\pi|f^N_{u,v})P(w))
\]
\[=
\]  
\[
P((P[u]|P(v)]P(u)][w])) + P([u]P(\pi|P(v)[P(w)])]) - P^2(([u]|P(v)]P(u)][w]))
\]
\[=
\]  
\[
0 \mod(S, w_3).
\]

Therefore, $S_N$ is a Gröbner-Shirshov basis in $\text{Lie}(\{P\}; X)$. By Composition-Diamond lemma for Lie Omega-algebras, the set $\text{Irr}(S_N)$ is a linear basis of the free Nijenhuis Lie algebra $NL(X)$. □

**Acknowledgement:** The authors would like to express their deepest gratitude to Professor L.A. Bokut for his kind guidance, useful discussions and enthusiastic encouragements.

**References**

[1] K. Abdaoui, S. Mabrouk, A. Makhlouf, Rota-Baxter operators on pre-lie superalgebras and beyond, [arXiv:1512.08043](https://arxiv.org/abs/1512.08043)

[2] W.W. Adams, P. Loustaunau, An introduction to Gröbner bases, Graduate Studies in Mathematics, Vol. 3, American Mathematical Society (AMS), 1994.

[3] M. Aguiar, W. Moreira, Combinatorics of free Baxter algebra, *Electr. J. Comb.*, 13 (2006), R17.
[4] H. An, C. Bai, From Rota-Baxter algebras to pre-Lie algebras, *J. Phys. A: Math. Theor.*, **41** (2008), 015201.

[5] C. Bai, L. Guo, X. Ni, Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and PostLie algebras, *Commun. Math. Phys.*, **297** (2010), 553-596.

[6] G. Baxter, An analytic problem whose solution follows from a simple algebraic identity, *Pacific J. Math.*, **10**(1960), 731-742.

[7] G.M. Bergman, The diamond lemma for ring theory, *Adv. in Math.*, **29**(1978), 178-218.

[8] L.A. Bokut, Unsolvability of the word problem, and subalgebras of finitely presented Lie algebras, *Izv. Akad. Nauk. SSSR Ser. Mat.*, **36**(1972), 1173-1219.

[9] L.A. Bokut, Imbeddings into simple associative algebras, *Algebra i Logika*, **15**(1976), 117-142.

[10] L.A. Bokut, Y.Q. Chen, Gröbner-Shirshov bases: after A.I. Shirshov, *Southeast Asian Bull. Math.*, **31**(2007) 1057-1076.

[11] L.A. Bokut, Y.Q. Chen, Gröbner-Shirshov bases and their calculation, *Bull. Math. Sci.*, **4**(2014), 325-395.

[12] L.A. Bokut, Y.Q. Chen, J.P. Huang, Gröbner-Shirshov bases for L-algebras, *Inter. J. Alge. Comput.*, **23**(3)(2013), 547-571.

[13] L.A. Bokut, Y.Q. Chen, Q.H. Mo, Gröbner-Shirshov bases and embeddings of algebras, *Inter. J. Alge. Comput.*, **20**(2010), 875-900.

[14] L.A. Bokut, Y.Q. Chen, J.J. Qiu, Gröbner-Shirshov bases for associative algebras with multiple operations and free Rota-Baxter algebras, *J. Pure Appl. Alg.*, **214**(2010), 89-100.

[15] L.A. Bokut, Y.Q. Chen, K.P. Shum, Some new results on Gröbner-Shirshov bases, in: Proceedings of International Conference on Algebra 2010, Advances in Algebraic Structures, 2012, 53-102.

[16] L.A. Bokut, Y. Fong, W.-F. Ke, P.S. Kolesnikov, Gröbner and Gröbner-Shirshov bases in algebra and conformal algebras, *Fund. App. Math.*, **6**(3)(2000), 669-706.

[17] L.A. Bokut, P.S. Kolesnikov, Gröbner-Shirshov bases: from their incipiency to the present, *J. Math. Sci.*, **116**(1)(2003), 2894-2916.

[18] L.A. Bokut, G. Kukin, Algorithmic and Combinatorial algebra, Kluwer Academic Publ., Dordrecht, 1994.
[19] B. Buchberger, An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal, Ph.D. thesis, University of Innsbruck, Austria, 1965 (in German).

[20] M. Bordemann, Generalized Lax pairs, the modified classical Yang-Baxter equation, and affine geometry of Lie groups, Commun. Math. Phys., 135(1990), 201-216.

[21] B. Buchberger, An algorithmical criteria for the solvability of algebraic systems of equations, Aequationes Math., 4(1970), 374-383.

[22] B. Buchberger, G.E. Collins, R. Loos and R. Albrecht, Computer algebra, symbolic and algebraic computation, Computing Supplementum, Vol.4, New York: Springer-Verlag, 1982.

[23] B. Buchberger, F. Winkler, Gröbner bases and applications, London Mathematical Society Lecture Note Series, Vol.251, Cambridge: Cambridge University Press, 1998.

[24] J. Cariñena, J. Grabowski, G. Marmo, Quantum bi-Hamiltonian systems, Int. J. Mod. Phys. A, 15(2000), 4797-4810.

[25] P. Cartier, On the structure of free Baxter algebras, Adv. in Math., 9(1972), 253-265.

[26] K.-T. Chen, R.H. Fox, R.C. Lyndon, Free differential calculus. IV. The quotient groups of the lower central series, Ann. Math., 68(2)(1958) 81-95.

[27] D.A. Cox, J. Little, D. O’Shea, Ideals, varieties and algorithms: An introduction to computational algebraic geometry and commutative algebra, Undergraduate Texts in Mathematics, New York: Springer-Verlag, 1992.

[28] I. Dorfman, Dirac structures and integrability of nonlinear evolution equations, Wiley, Chichester, 1993.

[29] V. Dotsenko, A. Khoroshkin, Gröbner bases for operads, Duke Math. J., 153(2010), 363-396

[30] D. Eisenbud, Commutative algebra with a view toward algebraic geometry, Graduate Texts in Math., Vol.150, Berlin and New York: Springer-Verlag, 1995.

[31] M. Hall, A basis of free Lie rings and higher commutators in free groups, Proc. Amer. Math. Soc., 1(1950) 575-581.

[32] V. Drensky, R. Holtkamp, Planar trees, free nonassociative algebras, invariants, and elliptic integrals, Alg. Discrete Math., 2(2008), 1-41.

[33] K. Ebrahimi-Fard, Loday-type algebras and the Rota-Baxter relation, Lett. Math. Phys. 61(2002), 139-147.
[34] K. Ebrahimi-Fard, On the associative nijenhuis relation, *Electr. J. Comb.*, 11(2004), R38.

[35] K. Ebrahimi-Fard, L. Guo, Rota-Baxter algebras and dendriform dialgebras, *J. Pure Appl. Alg.*, 212(2008), 320-339.

[36] K. Ebrahimi-Fard, L. Guo, Free Rota-Baxter algebras and rooted trees, *J. Alg. Appl.*, 7(2008), 167-194.

[37] X. Gao, L. Guo, M. Rosenkranz, Free integro-differential algebras and Gröbner-Shirshov bases, *J. Alg.*, 442(2015), 354-396.

[38] L. Guo, An introduction to Rota-Baxter algebra, *Higher education press*, Beijing, 2012.

[39] L. Guo, Operated semigroups, Motzkin paths and rooted trees, *J. Algebr. Comb.*, 29(2009), 35-62.

[40] L. Guo, W. Keigher, Baxter algebras and Shuffle products, *Adv. in Math.*, 150(2000), 117-149.

[41] L. Guo, W. Keigher, On free Baxter algebras: completions and the internal construction, *Adv. in Math.*, 151(2000), 101-127.

[42] L. Guo, W. Keigher, On differential Rota-Baxter algebras, *J. Pure Appl. Alg.*, 212(2008), 522-540.

[43] P.J. Higgins, Groups with multiple operators, *Proc. London Math. Soc.*, 6(1956), 366-416.

[44] H. Hironaka, Resolution of singulatities of an algebtaic variety over a field if characteristic zero, I, II, *Ann. Math.*, 79(1964), 109-203, 205-326.

[45] A.G. Kurosh, Free sums of multiple operator algebras, *Siberian. Math. J.*, 1(1960), 62-70.

[46] P. Lei, L. Guo, Nijenhuis algebras, NS algebras, and N-dendriform algebras, *Front. Math. China*, 7(5)(2012), 827-846.

[47] X. Li, D. Hou, C. Bai, Rota-Baxter operators on pre-lie algebras, *J. non-linear math. phys.*, 14(2)(2007), 269-289.

[48] A.A. Mikhalev, The composition lemma for color Lie superalgebras and for Lie $p$-superalgebras, *Contemp. Math.*, 131(2)(1992), 91-104.

[49] J. Pei, C. Bai, L. Guo, Rota-Baxter operators on $sl(2, C)$ and solutions of the classical Yang-Baxter equation, *J. Math. Phys.*, 55(2)(2014), 021701.

[50] J.J. Qiu, Y.Q. Chen, Composition-Diamond lemma for $\lambda$-differential associative algebras with multiple operators, *J. Alg. Appl.*, 9 (2010), 223-239.
[51] G.-C. Rota, Baxter algebras and combinatorial identities I, *Bull. Amer. Math. Soc.*, 5(1969), 325-329.

[52] M.A. Semenov-Tian-Shansky, What is a classical $R$-matrix? *Funct. Anal. Appl.*, (1983) 259-272.

[53] A.I. Shirshov, Some algorithmic problem for Lie algebras, *Sibirsk. Mat. Z.*, 3(1962), 292-296 (in Russian); English translation in SIGSAM Bull., 33(2) (1999), 3-6.

[54] A.I. Shirshov, Some algorithmic problem for $\varepsilon$-algebras, *Sibirsk. Mat. Z.*, 3 (1962), 132-137.

[55] Selected works of A.I. Shirshov, Eds L.A. Bokut, V. Latyshev, I. Shestakov, E. Zelmanov, Trs M. Brenner, M. Kochetov, Birkhäuser, Basel, Boston, Berlin, 2009.

[56] K. Uchino, Twisting on associative algebras and Rota-Baxter type operators, *J. Noncommut. Geom.*, 4(2010), 349-379.