Convenient parameterizations of matrices in terms of vectors

M. Bruschi* and F. Calogero+
Dipartimento di Fisica, Università di Roma "La Sapienza", 00185 Roma, Italy
Istituto Nazionale di Fisica Nucleare, Sezione di Roma

Abstract

Convenient parameterizations of matrices in terms of vectors transform (certain classes of) matrix equations into covariant (hence rotation-invariant) vector equations. Certain recently introduced such parameterizations are tersely reviewed, and new ones introduced.

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*Corresponding author. Fax: +39-06-4454749
*mario.bruschi@roma1.infn.it
+frenesca.calogero@roma1.infn.it, fresca.calogero@uniroma1.it
1 Introduction

A technique to identify classical (i.e., nonquantal nonrelativistic) many-body problems amenable to exact treatments in $S$-dimensional space (with $S > 1$) is to firstly identify suitable matrix evolution equations amenable to exact treatments, and then to parameterize matrices via vectors so that these matrix evolution equations become rotation-invariant equations of motion of Newtonian type ("acceleration equal force") [1] [2] [3]. It is therefore important to identify parameterizations of matrices in terms of vectors which are suitable to implement this approach. In this paper we tersely review some representations of this kind that have been recently used in this context, [4] and we introduce new, more general and convenient, ones. The exploitation of these latter representations to identify integrable systems of linear plus cubic oscillators in $S$-dimensional space (with $S = 2, S = 3$ as well as arbitrary $S$) – analogous yet different from those treated in [2] [3] – is reported in a separate paper [5].

Notation: heafer matrices are identified by underlined characters, and vectors by superimposed arrows; their dimensions should in each case be clear from the context.

2 Parameterizations

Hereafter we denote with the symbol $\equiv$ the one-to-one correspondence that the parameterization under consideration institutes among matrices and $S$-vectors (and, in some cases, scalars). For instance a well-known parameterization for $(2 \otimes 2)$-matrices reads

$$\begin{align*}
\mathbf{M} &= \rho \mathbf{I} + i \mathbf{r} \cdot \mathbf{\sigma} \quad (1a)
\end{align*}$$

where $\rho$ is a scalar, $\mathbf{r}$ is a 3-vector, $\mathbf{I}$ is the unit matrix and the 3 matrices $\mathbf{\sigma}_x, \mathbf{\sigma}_y, \mathbf{\sigma}_z$ are the standard Pauli matrices,

$$\begin{align*}
\mathbf{I} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{\sigma}_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \mathbf{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (1b)
\end{align*}$$

So in this case, in correspondence to (1a), we write

$$\begin{align*}
\mathbf{M} \equiv (\rho, \mathbf{r}) \quad (1c)
\end{align*}$$

and, via standard calculations, we also have, in self-evident notation,

$$\begin{align*}
\mathbf{M}^{-1} \equiv \frac{(\rho, -\mathbf{r})}{\rho^2 + \mathbf{r}^2} \quad (1d)
\end{align*}$$

$$\begin{align*}
\mathbf{M}^{(1)} \mathbf{M}^{(2)} \equiv \left( \rho^{(1)} \rho^{(2)} - \mathbf{r}^{(1)} \cdot \mathbf{r}^{(2)}, \rho^{(1)} \mathbf{r}^{(2)} + \rho^{(2)} \mathbf{r}^{(1)} - \mathbf{r}^{(1)} \wedge \mathbf{r}^{(2)} \right) \quad (1e)
\end{align*}$$
matrices to introduce (new) parameterizations in terms of $S$.

In this parameterization (see (2)) the matrices $\tilde{V}$ of the (sparse) $(K \times N)$ matrices, more specifically the matrices $\tilde{V}^{i,j,k}$, are identically vanishing square matrices, more specifically the matrices $\tilde{W}^{(j,k)}$ are $(L \times L)$-matrices and the matrices $\tilde{W}^{j,k}$ are $(S \times S)$-matrices. The consisteney of this block structure of the (sparse) $(K \times K)$-matrix $U$, with $K = N (L + S)$, is plain.

\[ M^{(1)} M^{-1} M^{(2)} = (\rho^{(1)} \rho^{(2)} + \rho^{(1)} (\tilde{T} \cdot \tilde{T}^{(2)}) + \rho^{(2)} (\tilde{T} \cdot \tilde{T}^{(1)})
- \rho (\tilde{T}^{(1)} \cdot \tilde{T}^{(2)}) + \tilde{T} \cdot (\tilde{T}^{(1)} \wedge \tilde{T}^{(2)})
+ \tilde{T}^{(1)} (\rho^{(2)} + \tilde{T} \cdot \tilde{T}^{(2)}) + \tilde{T}^{(2)} (\rho^{(1)} + \tilde{T} \cdot \tilde{T}^{(1)})
- \tilde{T} (\rho^{(1)} \rho^{(2)} + \tilde{T}^{(1)} \cdot \tilde{T}^{(2)})
+ \rho^{(1)} (\tilde{T} \cdot \tilde{T}^{(1)} + \tilde{T}^{(2)}) - \rho^{(2)} \tilde{T}^{(1)} \wedge \tilde{T}^{(2)} - \rho \tilde{T}^{(1)} \wedge \tilde{T}^{(2)} \right)^{-1}. \] (1f)

And of course introducing $N \otimes N$ block matrices whose elements are $(2 \otimes 2)$-matrices of type (1a), a parameterization is automatically introduced of $(4N) \otimes (4N)$ matrices in terms of $N^2$ 3-vectors and of $N^2$ scalars.

Our approach is analogous but more general: we use (appropriate) block matrices to introduce (new) parameterizations in terms of $S$-vectors. The basic block structure of the matrices we parameterize reads as follows:

\[
U = \begin{pmatrix}
V^{(1)} & V^{(11)} & V^{(12)} & V^{(1N)} \\
W^{(1)} & W^{(11)} & W^{(12)} & W^{(1N)} \\
W^{(1)} & W^{(11)} & W^{(12)} & W^{(1N)} \\
\vdots & \vdots & \vdots & \vdots \\
V^{(N)} & V^{(N1)} & V^{(N2)} & V^{(NN)} \\
W^{(N)} & W^{(N1)} & W^{(N2)} & W^{(NN)} \\
W^{(N)} & W^{(N1)} & W^{(N2)} & W^{(NN)} \\
\vdots & \vdots & \vdots & \vdots
\end{pmatrix}
\] (2)

The dimensions and structure of the (generally rectangular) matrices $V^{(j,k)}$, $W^{(j,k)}$, $V^{(j,k)}$, $W^{(j,k)}$ ($j, k = 1, 2, ..., N$) are detailed below, characterizing the different parameterizations we introduce.

\subsection{2.1 Parameterization 1 (P1)}

In this parameterization (see (2)) the matrices $V^{(j,k)}$ are (generally rectangular) $(L \otimes S)$-matrices, namely matrices with $L$ lines and $S$ columns, and, conversely, the matrices $V^{(j,k)}$ are $(S \otimes L)$-matrices, namely matrices with $S$ lines and $L$ columns, while the matrices $W^{(j,k)}$, $\tilde{W}^{(j,k)}$ are identically vanishing square matrices, more specifically the matrices $W^{(j,k)}$ are $(L \otimes L)$-matrices and the matrices $W^{(j,k)}$ are $(S \otimes S)$-matrices. The consisteney of this block structure of the (sparse) $(K \otimes K)$-matrix $U$, with $K = N (L + S)$, is plain.
We obtain a parameterization of the matrix $U$ in terms of the $2N^2L$ $S$-vectors $\vec{\tau}^{(jk)}(\ell), \vec{\tau}^{(j)}(\ell)$,

$$U \doteq \left( \vec{\tau}^{(jk)}(\ell), \vec{\tau}^{(j)}(\ell) \right) ,$$

(3a)

where of course (here and below, in this section) the indices $j, k$ range from 1 to $N$ and the index $\ell$ from 1 to $L$ ($j, k = 1, 2, \ldots, N$; $\ell = 1, 2, \ldots, L$), via the following identifications:

$$\left( \vec{U}^{(jk)} \right)_{\ell s} = r^{(jk)}(\ell) ,$$

(3b)

$$\left( \vec{U}^{(j)} \right)_{s \ell} = s^{(j)}(\ell) .$$

(3c)

In the last two formulas the quantities $r_s^{(jk)}$ respectively $s_s^{(jk)}$, with the index $s$ ranging of course from 1 to $S$, are the components of the $N^2L$ $S$-vectors $\vec{\tau}^{(jk)}(\ell)$ respectively of the $N^2L$ $S$-vectors $\vec{\tau}^{(jk)}(\ell)$. Now it is straightforward to verify the following (remarkable) formula:

$$U^{[1]}U^{[2]}U^{[3]} \doteq \left( \vec{R}^{(jk)}(\ell), \vec{R}^{(j)}(\ell) \right) ,$$

(3d)

with the $N^2L$ $S$-vectors $\vec{R}^{(jk)}(\ell)$, respectively the $N^2L$ $S$-vectors $\vec{R}^{(j)}(\ell)$, defined by the following covariant expressions:

$$\vec{R}^{(jk)}(\ell) = \sum_{\mu, \nu=1}^{N} \sum_{\lambda=1}^{L} \left( \vec{\tau}^{(1)}(\mu\nu)(\lambda) , \vec{\tau}^{(2)}(\mu\nu)(\lambda) \right) \vec{\tau}^{(3)}(\nu\ell)(\lambda) ,$$

(3e)

$$\vec{R}^{(j)}(\ell) = \sum_{\mu, \nu=1}^{N} \sum_{\lambda=1}^{L} \vec{\tau}^{(1)}(\mu\nu)(\lambda) \left( \vec{\tau}^{(2)}(\mu\nu)(\lambda) , \vec{\tau}^{(3)}(\nu\ell)(\lambda) \right) .$$

(3f)

Here and throughout a dot sandwiched among two $S$-vectors denotes the standard scalar product in $S$-dimensional space.

There hold moreover the formulas

$$AU \doteq \left( \vec{R}^{(A)(jk)}(\ell), \vec{R}^{(A)(j)}(\ell) \right) ,$$

(3g)

$$UA \doteq \left( \vec{R}^{(jk)}(\ell)(A) , \vec{R}^{(j)}(\ell)(A) \right)$$

(3h)

with

$$\vec{R}^{(A)(jk)}(\ell) = \sum_{\mu=1}^{N} a^{(j\mu)} \vec{\tau}^{(\mu k)}(\ell) , \quad \vec{R}^{(A)(j)}(\ell) = \sum_{\mu=1}^{N} a^{(j\mu)} \vec{\tau}^{(\mu)}(\ell) ,$$

(3i)

$$\vec{R}^{(jk)}(\ell)(A) = \sum_{\mu=1}^{N} \vec{\tau}^{(j\mu)}(\ell) a^{(\mu k)} , \quad \vec{R}^{(j)}(\ell)(A) = \sum_{\mu=1}^{N} \vec{\tau}^{(j\mu)}(\ell) a^{(\mu)} ,$$

(3j)
provided the \((K \otimes K)\)-matrix \(A\) (with \(K = N (L + S)\), as above) has again the structure \(2\) but now with the \(N^2 (L \otimes S)\)-matrices \(V_{(j,k)}\), as well as the \(N^2 (S \otimes L)\)-matrices \(W_{(j,k)}\) are given by
\[
W_{(j,k)}^{(jk)} = a^{(jk)} I, \quad (I \text{ being of course here the } (L \otimes L) \text{ identity matrix}),
\]
and the \(N^2 (S \otimes S)\)-matrices \(\tilde{W}_{(j,k)}\) are given by
\[
\tilde{W}_{(j,k)}^{(jk)} = \tilde{a}^{(jk)} I, \quad (I \text{ being of course here the } (S \otimes S) \text{ identity matrix}).
\]
As suggested by this notation, the \(N^2\) quantities \(a^{(jk)}\), as well as the \(N^2\) quantities \(\tilde{a}^{(jk)}\), are supposed to play the role of scalars.

This parameterization was already introduced in \([2]\); note however that we use here a somewhat different – and, we believe, more convenient – notation.

2.2 Parameterization 2 (P2)

In this parameterization (see \([2]\)) the \(N^2\) matrices \(V_{(j,k)}\) are \((1 \otimes L)\)-matrices, namely row matrices, and, conversely, the \(N^2\) matrices \(\tilde{V}_{(j,k)}\) are \((L \otimes 1)\)-matrices, namely column matrices, while the matrices \(W_{(j,k)}\), \(\tilde{W}_{(j,k)}\) are identically vanishing square matrices, more specifically, the \(N^2\) (vanishing!) quantities \(W_{(j,k)}^{(jk)} \equiv W_{(j,k)}\) are \((1 \otimes 1)\)-matrices (i. e., scalars) while the \(N^2\) (identically vanishing!) matrices \(\tilde{W}_{(j,k)}^{(jk)} \equiv \tilde{W}_{(j,k)}\) are \((L \otimes L)\)-matrices. The consistency of this block structure of the (sparse) \((K \otimes K)\)-matrix \(U\), with \(K = L (N + 1)\), is plain (\(N, L\) being again arbitrary positive integers). Note that in this parameterization \(S = N\). Note that here, and below, the indices \(j, k, s\) range from 1 to \(N\), while the index \(\ell\) ranges from 1 to \(L\).

We obtain a parameterization of the matrix \(U\) in terms of the \(2NL\) \(N\)-vectors \(\overrightarrow{r}^{(nt)}\), \(\overrightarrow{\tilde{r}}^{(nt)}\)
\[
U = \left( \overrightarrow{r}^{(nt)}, \overrightarrow{\tilde{r}}^{(nt)} \right),
\]
via the following identifications:
\[
\left( \overrightarrow{r}^{(jn)} \right)_{1\ell} = r^{(nt)}_{j} \quad j, n = 1, 2, \ldots, N; \quad \ell = 1, 2, \ldots, L, \quad (4b)
\]
\[
\left( \overrightarrow{\tilde{r}}^{(nj)} \right)_{\ell 1} = \tilde{r}^{(nt)}_{j} \quad j, n = 1, 2, \ldots, N; \quad \ell = 1, 2, \ldots, L. \quad (4c)
\]
In the last two formulas the quantities \(r^{(nt)}_{j}\) respectively \(\tilde{r}^{(nt)}_{j}\) are of course the components of the \(N\)-vectors \(\overrightarrow{r}^{(nt)}\) respectively \(\overrightarrow{\tilde{r}}^{(nt)}\).
Now it is straightforward to verify the following relation:

\[ L^{[1]} U^{[2]} L^{[3]} = \begin{pmatrix} \vec{R}^{(n\ell)} \end{pmatrix}, \quad (4d) \]

with

\[ \vec{R}^{(n\ell)} = \sum_{\nu=1}^{N} \sum_{\lambda=1}^{L} \vec{r}^{[1]}(\nu\lambda) \left( \vec{r}^{[2]}(\nu\lambda) \cdot \vec{r}^{[3]}(n\ell) \right), \quad n = 1, 2, \ldots, N; \ \ell = 1, 2, \ldots, L, \quad (4e) \]

\[ \vec{R}^{(n\ell)} = \sum_{\nu=1}^{N} \sum_{\lambda=1}^{L} \left( \vec{r}^{[1]}(n\ell) \cdot \vec{r}^{[2]}(\nu\lambda) \right) \vec{r}^{[3]}(\nu\lambda), \quad n = 1, 2, \ldots, N; \ \ell = 1, 2, \ldots, L. \quad (4f) \]

The covariant structure of these expressions of the \(2NLN\)-vectors \( \vec{R}^{(n\ell)} \), \( \vec{R}^{(n\ell)} \) is again remarkable.

And there hold moreover the relations

\[ AU = \begin{pmatrix} \vec{R}^{(A)(n\ell)} \end{pmatrix}, \quad (4g) \]

\[ UA = \begin{pmatrix} \vec{R}^{(n\ell)(A)} \end{pmatrix}, \quad (4h) \]

with the \(2NL\) \(N\)-vectors \( \vec{R}^{(A)(n\ell)} \), \( \vec{R}^{(A)(n\ell)} \) defined as follows:

\[ \vec{R}^{(A)(n\ell)} = \vec{r}^{(n\ell)} \quad (A) \quad (4i) \]

\[ \vec{R}^{(A)(n\ell)} = \sum_{\nu=1}^{N} \sum_{\lambda=1}^{L} \vec{a}^{(n,\nu)(\ell,\lambda)} \vec{r}^{(\nu\lambda)}, \quad n = 1, 2, \ldots, N; \ \ell = 1, 2, \ldots, L, \quad (4j) \]

provided the \((K \otimes K)\)-matrix \( A \) (with \(K = L(N+1)\)) has again the structure \( (A) \) but now with the \((1 \otimes L)\)-matrices \( V^{(jk)} \) as well as the \((L \otimes 1)\)-matrices \( \tilde{V}^{(j,k)} \) identically vanishing, while the \((1 \otimes 1)\)-matrices \( W^{(jk)} \) are given by

\[ W^{(jk)} = \alpha \delta_{jk} \quad j, k = 1, 2, \ldots, N, \quad (4k) \]

\((\delta_{jk}\) being of course the Kronecker symbol), and for convenience we denote as

\[ \tilde{a}^{(jk)(\ell,\lambda)} = \left( \tilde{W}^{(jk)} \right)_{\lambda \ell}, \quad j, k = 1, 2, \ldots, N; \ \lambda, \ell = 1, 2, \ldots, L \quad (4l) \]

the elements of the \((L \otimes L)\)-matrices \( \tilde{W}^{(jk)} \)
Remark 1 Note that with a 'transposed' assignment of the matrices \( \overrightarrow{V}, \overrightarrow{\tilde{V}} \) in the above parameterizations (namely, the choice of \((S \otimes L)\)-matrices \( \overrightarrow{V} \) and \((L \otimes S)\)-matrices \( \overrightarrow{\tilde{V}} \) in the parameterization \( P1 \), of column matrices \( \overrightarrow{V} \) and row matrices \( \overrightarrow{\tilde{V}} \) in the parameterization \( P2 \)), by changing accordingly the structure of the matrices \( \overrightarrow{W}, \overrightarrow{\tilde{W}}, \overrightarrow{A} \) one obtains essentially the same formulas: i.e. the parameterizations are basically the same, up to a trivial reindexing of the vectors.

For the readers convenience, we exhibit below the explicit formulas of the parameterization \( P2 \) in the simple case \( L = 1 \) for 2-vectors, 3-vectors and arbitrary \( N \)-vectors, adding also in each case the important parameterization for the inverse matrix. To do so, and in order to have a compact notation for some formulas below, it is convenient to introduce the external (antisymmetric) product \( \overrightarrow{\tau}^{(n)} \) of the \( N-1 \) \( N \)-vectors obtained excluding the vector \( \overrightarrow{\tau}^{(n)} \) in a set of \( N \) \( N \)-vectors \( \overrightarrow{\tau}^{(k)} = (\overrightarrow{r}_1^{(k)}, \overrightarrow{r}_2^{(k)}, \ldots, \overrightarrow{r}_N^{(k)}) \), \( k = 1, 2, \ldots, N \):

\[
\overrightarrow{\tau}^{(n)} = \overrightarrow{\tau}^{(1)} \wedge \overrightarrow{\tau}^{(2)} \wedge \ldots \wedge \overrightarrow{\tau}^{(n-1)} \wedge \overrightarrow{\tau}^{(n+1)} \wedge \ldots \wedge \overrightarrow{\tau}^{(N-1)} \wedge \overrightarrow{\tau}^{(N)} \quad (5a)
\]

\[
\begin{vmatrix}
\overrightarrow{r}_1^{(1)} & \overrightarrow{r}_2^{(1)} & \ldots & \overrightarrow{r}_{N-1}^{(1)} & \overrightarrow{r}_N^{(1)} \\
\overrightarrow{r}_1^{(2)} & \overrightarrow{r}_2^{(2)} & \ldots & \overrightarrow{r}_{N-1}^{(2)} & \overrightarrow{r}_N^{(2)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\overrightarrow{r}_1^{(n-1)} & \overrightarrow{r}_2^{(n-1)} & \ldots & \overrightarrow{r}_{N-1}^{(n-1)} & \overrightarrow{r}_N^{(n-1)} \\
\overrightarrow{r}_1^{(n+1)} & \overrightarrow{r}_2^{(n+1)} & \ldots & \overrightarrow{r}_{N-1}^{(n+1)} & \overrightarrow{r}_N^{(n+1)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\overrightarrow{r}_1^{(N)} & \overrightarrow{r}_2^{(N)} & \ldots & \overrightarrow{r}_{N-1}^{(N)} & \overrightarrow{r}_N^{(N)}
\end{vmatrix}
\]

where the set of \( N \) \( N \)-vectors \( \{ \overrightarrow{\tau}^{(n)} \} \), \( n = 1, 2, \ldots, N \), provides the standard orthonormal basis in the \( N \)-vectors space,

\[
\left( \overrightarrow{\tau}^{(n)} \right)_j = \delta_{nj} \quad (5c)
\]

Of course for \( N = 3 \) the usual vector product for two 3-vectors is recovered but note that the above definition is valid also for \( N = 2 \). Also note that, with this definition, the scalar product \( \overrightarrow{\tau}^{(n)} \cdot \overrightarrow{\tau}^{(n)} \) is independent of the index \( n \), and it coincides with the standard determinant associated with the set of \( N \) \( N \)-vectors \( \{ \overrightarrow{\tau}^{(k)} \} \),

\[
\Delta = \overrightarrow{\tau}^{(n)} \cdot \overrightarrow{\tau}^{(n)} = \begin{vmatrix}
\overrightarrow{r}_1^{(1)} & \ldots & \overrightarrow{r}_N^{(1)} \\
\vdots & \ddots & \vdots \\
\overrightarrow{r}_1^{(N)} & \ldots & \overrightarrow{r}_N^{(N)}
\end{vmatrix}
\]

which has of course a well-known geometrical significance.

The relevant formulas in this parameterization read, in self-evident notation, as follows.
2.2.1 \((4 \otimes 4)\)-matrices in terms of four 2-vectors

\[
\mathbf{U} \doteq \left( \begin{array}{cc} \vec{r}^{(1)} & \vec{r}^{(2)} \\ \vec{r}^{(1)} & \vec{r}^{(2)} \end{array} \right) ,
\]
\(\vec{r}^{(n)} \equiv \left( x^{(n)}, y^{(n)} \right), \quad \vec{r}^{(n)} \equiv \left( \tilde{x}^{(n)}, \tilde{y}^{(n)} \right), \quad n = 1, 2 ,
\]
\[
\vec{r}^{(1)} \equiv \left( y^{(2)}, -x^{(2)} \right), \quad \vec{r}^{(2)} \equiv \left( -y^{(1)}, x^{(1)} \right)
\]
(with analogous formulas for the tilded vectors),

\[
\Delta = x^{(1)} y^{(2)} - x^{(2)} y^{(1)}, \quad \Delta = \tilde{x}^{(1)} \tilde{y}^{(2)} - \tilde{x}^{(2)} \tilde{y}^{(1)},
\]

\[
\mathbf{U} \doteq \begin{pmatrix} 0 & x^{(1)} & 0 & x^{(2)} \\ \tilde{x}^{(1)} & 0 & \tilde{y}^{(1)} & 0 \\ 0 & y^{(1)} & 0 & y^{(2)} \\ \tilde{x}^{(2)} & 0 & \tilde{y}^{(2)} & 0 \end{pmatrix} ,
\]

\[
\mathbf{U}^{-1} \doteq \begin{pmatrix} \vec{r}^{(1)} \Delta, & \vec{r}^{(2)} \Delta, & \vec{r}^{(1)} \Delta, & \vec{r}^{(2)} \Delta \end{pmatrix},
\]

\[
\mathbf{U}^{[1]} \mathbf{U}^{[2]} \mathbf{U}^{[3]} \doteq \left( \vec{R}^{(1)}, \vec{R}^{(2)}; \vec{R}^{(1)}, \vec{R}^{(2)} \right) ,
\]

\[
\vec{R}^{(1)} = \vec{r}^{[1]}(1) \left( \vec{r}^{[2]}(1), \vec{r}^{[3]}(1) \right) + \vec{r}^{[1]}(2) \left( \vec{r}^{[2]}(2), \vec{r}^{[3]}(1) \right),
\]

\[
\vec{R}^{(2)} = \vec{r}^{[1]}(1) \left( \vec{r}^{[2]}(2), \vec{r}^{[3]}(2) \right) + \vec{r}^{[1]}(2) \left( \vec{r}^{[2]}(2), \vec{r}^{[3]}(2) \right),
\]

\[
\vec{R}^{(1)} = \vec{r}^{[1]}(1) \left( \vec{r}^{[2]}(1), \vec{r}^{[3]}(1) \right) + \vec{r}^{[1]}(2) \left( \vec{r}^{[2]}(1), \vec{r}^{[3]}(2) \right),
\]

\[
\vec{R}^{(2)} = \vec{r}^{[1]}(2) \left( \vec{r}^{[2]}(1), \vec{r}^{[3]}(2) \right) + \vec{r}^{[1]}(2) \left( \vec{r}^{[2]}(2), \vec{r}^{[3]}(2) \right),
\]

\[
\mathbf{UA} \doteq \left( \tilde{\alpha}^{(11)} \vec{r}^{(1)} + \tilde{\alpha}^{(21)} \vec{r}^{(2)}, \tilde{\alpha}^{(12)} \vec{r}^{(1)} + \tilde{\alpha}^{(22)} \vec{r}^{(2)}, \alpha \vec{r}^{(1)}, \alpha \vec{r}^{(2)} \right),
\]

\[
\mathbf{AU} \doteq \left( \alpha \vec{r}^{(1)}, \alpha \vec{r}^{(2)}, \tilde{\alpha}^{(11)} \vec{r}^{(1)} + \tilde{\alpha}^{(21)} \vec{r}^{(2)}, \tilde{\alpha}^{(12)} \vec{r}^{(1)} + \tilde{\alpha}^{(22)} \vec{r}^{(2)} \right),
\]

with

\[
\mathbf{A} = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \tilde{\alpha}^{(11)} & 0 & \tilde{\alpha}^{(12)} \\ 0 & 0 & \alpha & 0 \\ 0 & \tilde{\alpha}^{(21)} & 0 & \tilde{\alpha}^{(22)} \end{pmatrix}.
\]
2.2.2 \((6 \otimes 6)\)-matrices in terms of six 3-vectors

\(\mathcal{U} = \begin{pmatrix} \varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)}; \varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)} \end{pmatrix},\) \hspace{1cm} (7a)

\(\varphi^{(n)}(x^{(n)}, y^{(n)}, z^{(n)}), \varphi^{(n)}(x^{(n)}, y^{(n)}, z^{(n)}), \quad n = 1, 2, 3,\) \hspace{1cm} (7b)

\(\varphi^{(n)} = \varphi^{(n+1)} \wedge \varphi^{(n+2)} \wedge \varphi^{(n-1)} \wedge \varphi^{(n+2)}, \quad n = 1, 2, 3 \mod(3),\) \hspace{1cm} (7c)

\(\Delta = \varphi^{(1)}, \varphi^{(2)} \wedge \varphi^{(3)}, \quad \tilde{\Delta} = \varphi^{(1)}, \varphi^{(2)} \wedge \varphi^{(3)},\) \hspace{1cm} (7d)

\[
\mathcal{U} = \begin{pmatrix}
0 & x^{(1)} & 0 & x^{(2)} & 0 & x^{(3)} \\
\tilde{x}^{(1)} & 0 & \tilde{y}^{(1)} & 0 & \tilde{z}^{(1)} & 0 \\
\tilde{x}^{(2)} & 0 & \tilde{y}^{(2)} & 0 & \tilde{z}^{(2)} & 0 \\
0 & \tilde{x}^{(2)} & 0 & \tilde{z}^{(2)} & 0 & \tilde{z}^{(3)} \\
\tilde{x}^{(3)} & 0 & \tilde{y}^{(3)} & 0 & \tilde{z}^{(3)} & 0
\end{pmatrix},
\] \hspace{1cm} (7e)

\[
\mathcal{U}^{-1} = \begin{pmatrix}
\tilde{R}^{(1)} & \tilde{R}^{(2)} & \tilde{R}^{(3)}; \tilde{R}^{(1)} & \tilde{R}^{(2)} & \tilde{R}^{(3)} \end{pmatrix},
\] \hspace{1cm} (7f)

\(\tilde{R}^{(n)} = \tilde{\Delta}^{-1} \varphi^{(n+1)} \wedge \varphi^{(n+2)}, \quad n = 1, 2, 3 \mod(3),\) \hspace{1cm} (7g)

\(\tilde{R}^{(n)} = \Delta^{-1} \varphi^{(n+1)} \wedge \varphi^{(n+2)}, \quad n = 1, 2, 3 \mod(3),\) \hspace{1cm} (7h)

\[
\mathcal{U}^{[1]} \mathcal{U}^{[2]} \mathcal{U}^{[3]} = \begin{pmatrix}
\tilde{R}^{(1)} & \tilde{R}^{(2)} & \tilde{R}^{(3)}; \tilde{R}^{(1)} & \tilde{R}^{(2)} & \tilde{R}^{(3)} \end{pmatrix},
\] \hspace{1cm} (7i)

\(\tilde{R}^{(n)} = \sum_{k=1}^{3} \varphi^{[1(k)]} (\varphi^{[2(k)]} \varphi^{[3(n)]}), \quad n = 1, 2, 3,\) \hspace{1cm} (7j)

\(\tilde{R}^{(n)} = \sum_{k=1}^{3} \left( \varphi^{[1(n)]} \varphi^{[2(k)]} \right) \varphi^{[3(k)]}, \quad n = 1, 2, 3,\) \hspace{1cm} (7k)

\[
\mathcal{A} = \begin{pmatrix}
\tilde{R}^{(A)(1)}, \tilde{R}^{(A)(2)}, \tilde{R}^{(A)(3)}; \tilde{R}^{(A)(1)}, \tilde{R}^{(A)(2)}, \tilde{R}^{(A)(3)} \end{pmatrix},
\] \hspace{1cm} (7l)

\[
\mathcal{U} \mathcal{A} = \begin{pmatrix}
\tilde{R}^{(1)(A)}, \tilde{R}^{(2)(A)}, \tilde{R}^{(3)(A)}; \tilde{R}^{(1)(A)}, \tilde{R}^{(2)(A)}, \tilde{R}^{(3)(A)} \end{pmatrix},
\] \hspace{1cm} (7m)

\(\tilde{R}^{(A)(n)} = \alpha \varphi^{(n)} = \sum_{\nu=1}^{3} \hat{\alpha}^{(n\nu)} \varphi^{(\nu)}, \quad n = 1, 2, 3,\) \hspace{1cm} (7n)
\( \vec{R}^{(n)}(A) = \sum_{\nu=1}^{3} \vec{\nu}^{(\nu)} \vec{a}^{(\nu n)} \), \( \vec{R}^{(n)} = \alpha \vec{r}^{(n)} \), \( n = 1, 2, 3 \), \hspace{1cm} (70)

with

\[
\vec{A} = \begin{pmatrix}
\alpha & 0 & 0 & 0 & 0 & 0 \\
0 & \tilde{\alpha}^{(11)} & \tilde{\alpha}^{(12)} & 0 & \tilde{\alpha}^{(13)} \\
0 & 0 & \alpha & 0 & 0 \\
0 & \tilde{\alpha}^{(21)} & \tilde{\alpha}^{(22)} & 0 & \tilde{\alpha}^{(23)} \\
0 & 0 & 0 & \alpha & 0 \\
0 & \tilde{\alpha}^{(31)} & \tilde{\alpha}^{(32)} & 0 & \tilde{\alpha}^{(33)}
\end{pmatrix}.
\hspace{1cm} (7p)

2.2.3 \((2N \otimes 2N)\)-matrices in terms of \(2N\) \(N\)-vectors

\[
\vec{U} \equiv \left( \vec{\nu}^{(1)}, ..., \vec{\nu}^{(N)}; \vec{r}^{(1)}, ..., \vec{r}^{(N)} \right),
\hspace{1cm} (8a)
\]

\[
\vec{\nu}^{(n)} \equiv \left( r_1^{(n)}, r_2^{(n)}, ..., r_N^{(n)} \right), \hspace{0.5cm} \vec{r}^{(n)} \equiv \left( \hat{r}_1^{(n)}, \hat{r}_2^{(n)}, ..., \hat{r}_N^{(n)} \right), \hspace{0.5cm} n = 1, ..., N
\hspace{1cm} (8b)
\]

\[
\vec{U} = \begin{pmatrix}
0 & r_1^{(1)} & 0 & r_1^{(2)} & 0 & ... & 0 & r_1^{(N)} \\
r_1^{(1)} & 0 & \hat{r}_1^{(1)} & 0 & r_3^{(1)} & ... & \hat{r}_3^{(1)} & 0 \\
0 & r_2^{(1)} & 0 & r_2^{(2)} & 0 & ... & 0 & r_2^{(N)} \\
\hat{r}_1^{(2)} & 0 & \hat{r}_2^{(2)} & 0 & \hat{r}_3^{(2)} & ... & \hat{r}_N^{(2)} & 0 \\
0 & r_3^{(1)} & 0 & r_3^{(2)} & 0 & ... & 0 & r_3^{(N)} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & r_N^{(1)} & 0 & r_N^{(2)} & 0 & ... & 0 & r_N^{(N)} \\
\hat{r}_1^{(N)} & 0 & \hat{r}_2^{(N)} & 0 & \hat{r}_3^{(N)} & ... & \hat{r}_N^{(N)} & 0
\end{pmatrix},
\hspace{1cm} (8c)
\]

\[
\vec{U}^{-1} \equiv \left( \frac{\vec{\nu}^{(1)}}{\Delta}, ..., \frac{\vec{\nu}^{(N)}}{\Delta}; \frac{\vec{r}^{(1)}}{\Delta}, ..., \frac{\vec{r}^{(N)}}{\Delta} \right)
\hspace{1cm} (8d)
\]

(see \#5), and of course analogous formulas hold for the tilded vectors,

\[
\vec{U}^{[1]} \vec{U}^{[2]} \vec{U}^{[3]} \equiv \left( \vec{R}^{(1)}, ..., \vec{R}^{(N)}; \vec{R}^{(1)}, ..., \vec{R}^{(N)} \right),
\hspace{1cm} (8e)
\]

\[
\vec{R}^{(n)} = \sum_{k=1}^{N} \vec{\nu}^{[1](k)} \left( \vec{\nu}^{[2](k)} ; \vec{\nu}^{[3](n)} \right), \hspace{0.5cm} n = 1, 2, ..., N
\hspace{1cm} (8f)
\]

\[
\vec{R}^{(n)} = \sum_{k=1}^{N} \left( \vec{r}^{[1](n)} ; \vec{r}^{[2](k)} \right) \vec{r}^{[3](k)}, \hspace{0.5cm} n = 1, 2, ..., N
\hspace{1cm} (8g)
\]
\[ A U = \left( \overrightarrow{R}(A)(1), ..., \overrightarrow{R}(A)(N), \overleftarrow{R}(A)(1), ..., \overleftarrow{R}(A)(N) \right), \quad (8h) \]

\[ U A = \left( \overrightarrow{R}(1)(A), ..., \overrightarrow{R}(N)(A), \overleftarrow{R}(1)(A), ..., \overleftarrow{R}(N)(A) \right), \quad (8i) \]

\[ \overrightarrow{R}(A)(n) = \alpha \overrightarrow{p}(n), \quad \overrightarrow{R}(A)(n) = \sum_{\nu=1}^{N} \overrightarrow{\tilde{a}}(\nu) \overrightarrow{p}(\nu), \quad n = 1, 2, ..., N, \quad (8j) \]

\[ \overrightarrow{R}(n)(A) = \sum_{\nu=1}^{N} \overrightarrow{p}(\nu) \overrightarrow{\tilde{a}}(\nu n), \quad \overrightarrow{R}(n)(A) = \alpha \overrightarrow{p}(n), \quad n = 1, 2, ..., N, \quad (8k) \]

with

\[
A = \begin{pmatrix}
\alpha & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & \tilde{\alpha}^{(11)} & 0 & \tilde{\alpha}^{(12)} & \ldots & \tilde{\alpha}^{(1 N-1)} & 0 & \tilde{\alpha}^{(1N)} \\
0 & 0 & \alpha & 0 & \ldots & 0 & 0 & 0 \\
0 & \tilde{\alpha}^{(21)} & 0 & \tilde{\alpha}^{(22)} & \ldots & \tilde{\alpha}^{(2 N-1)} & 0 & \tilde{\alpha}^{(2N)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & \tilde{\alpha}^{(N-1 1)} & 0 & \tilde{\alpha}^{(N-1 2)} & \ldots & \tilde{\alpha}^{(N N-1)} & 0 & \tilde{\alpha}^{(NN)} \\
0 & 0 & 0 & 0 & \ldots & 0 & \alpha & 0 \\
0 & \tilde{\alpha}^{(N1)} & 0 & \tilde{\alpha}^{(N2)} & \ldots & \tilde{\alpha}^{(NN)} & 0 & \alpha \end{pmatrix} \quad (8l)\]

3 Concluding remarks

The matrix parameterizations in terms of vectors reported in this paper have the property to be preserved – in terms of vectors yielded by covariant expressions – if the matrices are multiplied by appropriate matrices with scalar matrix elements (indicated as \(A\), see above), and as well for the product of three matrices, hence, by iteration, for the product of any odd number of these matrices (possibly interspersed by matrices of type \(A\)). They are therefore appropriate to transform matrix equations that only involve such products, into rotation-invariant vector equations (for examples see \[2\] \[3\] \[5\]); of course in such a context it may also be possible, and interesting, to also consider reductions, characterized by the presence of a smaller number of vectors than is naturally yielded by these parameterizations – because some vectors can be set to zero and/or be linearly related to each other (provided this is compatible with the time evolution under consideration).
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