DISCRETE FRAMES ON FINITE DIMENSIONAL QUATERNION HILBERT SPACES

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Abstract. A general theory of frames on finite dimensional quaternion Hilbert spaces is demonstrated along the lines of their complex counterpart.

1. Introduction

Frames were first introduced by Duffin and Schaeffer in a study of non-harmonic Fourier series [9]. However, among many others, the pioneering works of Daubechies et al. brought the proper attention to frames [7, 8]. Wavelets and coherent states of quantum optics are specific classes of continuous frames [3]. The study of frames has exploded in recent years, partly because of their applications in digital signal processing [4, 11] and other areas of physical and engineering problems. In particular, they are an integral part of time-frequency analysis. In this note we are primarily interested in frames on finite dimensional quaternion Hilbert spaces. There has been a constant surge in finding finite tight frames, largely as a result of several important applications such as internet coding, wireless communication, quantum detection theory, and many more [5, 6, 10, 4, 13]. It is crucial to find a specific class of frame to fit to a specific physical problem, because there is no universal class of frame that fit to all problems. As technology advances, physicists and engineers will face new problems and thereby our search for tools to solve them will continue.

A Separable Hilbert space possesses an orthonormal basis and each vector in the Hilbert space can be uniquely written in terms of this orthonormal basis. Despite orthonormal bases are hard to find, this uniqueness restricted flexibility in applications and pleaded for an alternative. As a result frames entered to replace orthonormal bases. Frames are classes of vectors in Hilbert spaces. In a finite dimensional Hilbert space a typical frame possesses more vectors than the dimension of the space, and thereby each vector in the space can have infinitely many representations with respect to the frame. This redundancy of frames is the key to their success in applications. The role of redundancy varies according to the requirements of the application at hand. In fact, redundancy gives greater design flexibility which allows frames to be constructed to fit a particular problem in a manner not possible by a set of linearly independent vectors [3, 4, 7, 2].

Hilbert spaces can be defined over the fields $\mathbb{R}$, the set of all real numbers, $\mathbb{C}$, the set of all complex numbers, and $\mathbb{H}$, the set of all quaternions only $[1]$. The fields $\mathbb{R}$ and $\mathbb{C}$ are
associative and commutative and the theory of functional analysis is a well formed theory over real and complex Hilbert spaces. But the quaternions form a non-commutative associative algebra and this feature highly restricted mathematicians to work out a well-formed theory of functional analysis on quaternionic Hilbert spaces. Further, due to the noncommutativity there are two types of Hilbert spaces on quaternions, called right quaternion Hilbert space and left quaternion Hilbert space. In assisting the study of frames the functional analytic properties of the underlying Hilbert space are essential. In the sequel we shall prove the necessary functional analytic properties as needed.

To the best of our knowledge a general theory of frames on quaternionic Hilbert spaces is not formulated yet. In this part of the thesis we shall construct frames on finite dimensional left quaternionic Hilbert spaces following the lines of [4]. Since non-commutativity of quaternions does not play a bigger role in the construction of frames on finite dimensional quaternion Hilbert spaces, most of the results follow their complex counterparts. While the complex numbers are two dimensional the quaternions are four dimensional; the increase in the dimension expected to give greater flexibility in applications. We are also expected to demonstrate this issue in applications as the thesis progress further.

2. Quaternion Algebra

In this section we shall define quaternions and some of their properties as needed here. For details one may consult [1, 14, 12].

2.1. Quaternions. Let $H$ denote the field of quaternions. Its elements are of the form $q = x_0 + x_1 i + x_2 j + x_3 k$, where $x_0, x_1, x_2$ and $x_3$ are real numbers, and $i, j, k$ are imaginary units such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$ and $ki = -ik = j$. The quaternionic conjugate of $q$ is defined to be $q^\dagger = x_0 - x_1 i - x_2 j - x_3 k$.

Quaternions can also be represented by using $2 \times 2$ complex matrices. It can be written as the linear combination of the matrices

$$
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}, \quad
\begin{pmatrix}
0 & i \\
i & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
i & 0 \\
0 & -i
\end{pmatrix}
$$

where $\sigma_1, \sigma_2$ and $\sigma_3$ are the usual Pauli matrices. In this notations the quaternions can be written as

$$(2.1) \quad q = x_0 \sigma_0 + i x_1 \sigma_1 + j x_2 \sigma_2 + k x_3 \sigma_3$$

with $x_0 \in \mathbb{R}, x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $\sigma = (\sigma_1, -\sigma_2, \sigma_3)$. The quaternionic imaginary units are identified as, $i = \sqrt{-1} \sigma_1, j = \sqrt{-1} \sigma_2, k = \sqrt{-1} \sigma_3$. Thereby

$$(2.2) \quad q = x_0 \sigma_0 + i x_1 \sigma_1 - i x_2 \sigma_2 + i x_3 \sigma_3 = \begin{pmatrix}
x_0 + i x_3 & -x_2 + i x_1 \\
x_2 + i x_1 & x_0 - i x_3
\end{pmatrix}$$

and $\overline{q} = q^\dagger$ (matrix adjoint). Introducing the polar coordinates:

$$
x_0 = r \cos \theta \\
x_1 = r \sin \theta \sin \phi \cos \psi \\
x_2 = r \sin \theta \sin \phi \sin \psi \\
x_3 = r \sin \theta \sin \phi \cos \psi
$$
where \( r \in [0, \infty), \theta, \phi \in [0, \pi], \) and \( \psi \in [0, 2\pi), \) we may write

\[
q = A(r)e^{i\theta\sigma(\tilde{n})},
\]

where

\[
A(r) = r\sigma_0 \quad \text{and} \quad \sigma(\tilde{n}) = \begin{pmatrix}
\cos \phi & \sin \phi e^{i\psi} \\
\sin \phi e^{-i\psi} & -\cos \phi
\end{pmatrix}.
\]

The matrices \( A(r) \) and \( \sigma(\tilde{n}) \) satisfy the conditions,

\[
A(r) = A(r)^\dagger, \quad \sigma(\tilde{n})^2 = \sigma_0, \quad \sigma(\tilde{n})^\dagger = \sigma(\tilde{n}), \quad [A(r), \sigma(\tilde{n})] = 0
\]

2.2. Properties of Quaternions: The quaternion product allows the following properties. For \( q, r, s \in H, \) we have

(a) \( q(rs) = (qr)s \) (associative)
(b) \( q(r + s) = qr + qs. \)
(c) For each \( q \neq 0, \) there exists \( r \) such that \( qr = 1 \)
(d) If \( qr = qs \) then \( r = s \) whenever \( q \neq 0 \)

The quaternion product is not commutative.

3. Frames in Quaternion Hilbert Space

Definition 3.1. Let \( V^L_H \) is a vector space under left multiplication by quaternionic scalars, where \( H \) stands for the quaternion algebra. For \( f, g, h \in V^L_H \) and \( q \in H, \) the inner product

\[
\langle \cdot | \cdot \rangle : V^L_H \times V^L_H \rightarrow H
\]

satisfies the following properties:

(a) \( \overline{\langle f | g \rangle} = \langle g | f \rangle \)
(b) \( \|f\|^2 = \langle f | f \rangle > 0 \) unless \( f = 0, \) a real norm
(c) \( \langle f | g + h \rangle = \langle f | g \rangle + \langle f | h \rangle \)
(d) \( \langle qf | g \rangle = q \langle f | g \rangle \)
(e) \( \langle f | qg \rangle = \langle f | g \rangle q. \)

Assume that the space \( V^L_H \) is together with \( \langle \cdot | \cdot \rangle \) is a separable Hilbert space. Properties of left quaternion Hilbert spaces as needed here can be listed as follows: For \( f, g \in V^L_H \) and \( p, q \in H, \) we have

(a) \( pf + qg \in V^L_H \)
(b) \( p(f + g) = pf + qg \)
(c) \( (pq)f = p(qf) \)
(d) \( (p + q)f = pf + qf. \)

Proposition 3.2. \[ \text{Schwartz inequality} \quad \langle f | g \rangle \langle g | f \rangle \leq \|f\|^2 \|g\|^2, \] for all \( f, g \in V^L_H \)
Proof. Let \( f, g \in V^L_H \) and \( p, q \in H \), then \( \langle pf - qg|pf - qg \rangle > 0 \) unless \( pf - qg = 0 \). Now

\[
\langle pf - qg|pf - qg \rangle = \\
= \langle pf - qg|pf \rangle - \langle pf - qg|qg \rangle \\
= \langle pf|pf - qg \rangle - \langle qg|pf - qg \rangle \\
= \langle pf|pf \rangle - \langle pf|qg \rangle - \langle qg|pf \rangle + \langle qg|qg \rangle \\
= p \langle f|f \rangle p - p \langle f|g \rangle q - q \langle g|f \rangle p + q \langle g|g \rangle q \\
= p \langle f|f \rangle p - q \langle g|f \rangle p - p \langle f|g \rangle q + q \langle g|g \rangle q \\as \overline{pq} = \overline{qp} \text{ on quaternion.}
\]

Hence \( \langle pf - qg|pf - qg \rangle = [\langle g|g \rangle \langle f|f \rangle - \langle f|g \rangle \langle g|f \rangle] \langle g|g \rangle > 0 \), which implies, as \( \langle g|g \rangle > 0 \), \( \langle g|g \rangle \langle f|f \rangle - \langle f|g \rangle \langle g|f \rangle > 0 \) and \( \|g\|^2 \|f\|^2 > \langle f|g \rangle \langle f|g \rangle = \|\langle f|g \rangle\|^2 \). Thereby

\[(3.1) \quad |\langle f|g \rangle|^2 < \|f\|^2 \|g\|^2, \quad \text{for all } f, g \in V^L_H
\]

Equality:

\[
pf - qg = 0 \\
\iff \langle pf - qg|pf - qg \rangle = 0 \\
\iff \langle g|g \rangle \langle f|f \rangle - \langle f|g \rangle \langle g|f \rangle = 0 \quad \text{[by above part]} \\
\iff \|g\|^2 \|f\|^2 = \langle f|g \rangle \langle g|f \rangle = \langle f|f \rangle \langle g|g \rangle = \|\langle f|g \rangle\|^2. \quad \text{Thereby}
\]

\[(3.2) \quad |\langle f|g \rangle|^2 = \|f\|^2 \|g\|^2, \quad \text{for all } f, g \in V^L_H.
\]

From (3.1) and (3.2),

\[(3.3) \quad |\langle f|g \rangle|^2 \leq \|f\|^2 \|g\|^2, \quad \text{for all } f, g \in V^L_H.
\]

\[\square\]

For an enhanced explanation of quaternions and quaternion Hilbert spaces one may consult [11, 12] and the many references listed there.

### 3.1. Some basic facts in left quaternion Hilbert space.

**Definition 3.3.** *(Basis)* Let \( V^L_H \) be a finite dimensional left quaternion Hilbert space, equipped with an inner product \( \langle \cdot, \cdot \rangle \) which we choose to be linear in the second entry. A sequence \( \{e_k\}_{k=1}^m \) in \( V^L_H \) is a basis for \( V^L_H \) if the following two conditions are satisfied:

1. \( V^L_H = \text{left span} \{e_k\}_{k=1}^m \);
2. \( \{e_k\}_{k=1}^m \) is a linearly independent set.
As a consequence of this definition, for every $f \in V_H^L$ there exist unique scalar coefficients \( \{c_k\}_{k=1}^m \subseteq H \) such that 
\[
f = \sum_{k=1}^m c_k e_k.
\]
If \( \{e_k\}_{k=1}^m \) is an orthonormal basis, that is 
\[
\langle e_k | e_j \rangle = \delta_{kj}, \quad \text{then} \quad \langle f | e_j \rangle = \left( \sum_{k=1}^m c_k e_k \right) e_j = \sum_{k=1}^m c_k \langle e_k | e_j \rangle = c_j.
\]
Thereby 
\[
f = \sum_{k=1}^m \langle f | e_k \rangle e_k.
\]
We now introduce the frames on finite dimensional left quaternion Hilbert spaces. We shall show that the complex treatment adapt to the quaternions as well. In this chapter left span means left span over the quaternion scalar field, $H$. We shall also prove the functional analytic properties for quaternions as needed here, and these proofs are the adaptation of the proofs of the complex cases given in [15]. The theory of frames offered here, more or less, follows the lines of [4].

**Definition 3.4.** (Frames) A countable family of elements \( \{f_k\}_{k \in I} \) in \( V_H^L \) is a frame for \( V_H^L \) if there exist constants \( A, B > 0 \) such that 
\[
A \|f\|^2 \leq \sum_{k \in I} |\langle f | f_k \rangle|^2 \leq B \|f\|^2,
\]
for all \( f \in V_H^L \).

The numbers \( A \) and \( B \) are called frame bounds. They are not unique. The optimal lower frame bound is the supremum over all lower frame bounds, and the optimal upper frame bound is the infimum over all upper frame bounds. Note that the optimal frame bounds are actually the frame bounds. A frame is said to be normalized if \( \|f_k\| = 1 \), for all \( k \in I \). In this chapter we shall only consider finite frames \( \{f_k\}_{k=1}^m, \ m \in \mathbb{N} \). With this restriction, Schwartz inequality shows that 
\[
\sum_{k=1}^m |\langle f | f_k \rangle|^2 \leq \sum_{k=1}^m \|f_k\|^2 \|f\|^2,
\]
for all \( f \in V_H^L \).

From (3.6) it is clear that the upper frame condition is always satisfied with 
\[
A = \sum_{k=1}^m \|f_k\|^2.
\]
In order for the lower condition in (3.5) to be satisfied, it is necessary that \( \text{leftspan}\{f_k\}_{k=1}^m = V_H^L \). Let us see this in the following.

**Lemma 3.5.** Let \( \{f_k\}_{k=1}^m \) be a sequence in \( V_H^L \) and \( W := \text{leftspan}\{f_k\}_{k=1}^m \). Then a mapping \( \varphi : W \rightarrow \mathbb{R} \) is continuous if and only if for any sequence \( \{x_n\} \) in \( W \) which converges to \( x_0 \) as \( n \rightarrow \infty \) then \( \varphi(x_n) \) converges to \( \varphi(x_0) \) as \( n \rightarrow \infty \).

**Proof.** Suppose that \( \varphi \) is continuous on \( W \) then \( \varphi \) is continuous at \( x_0 \in W \). Thereby, for given \( \epsilon > 0 \) there exists \( \delta > 0 \) such that 
\[
\|\varphi(x) - \varphi(x_0)\| < \epsilon, \quad \text{for all} \quad x \in W \quad \text{with} \quad \|x - x_0\| < \delta.
\]
Since \( x_n \) converges to \( x_0 \) as \( n \rightarrow \infty \), there exists \( N \in \mathbb{N} \) such that 
\[
\|x_n - x_0\| < \delta, \quad \text{for all} \quad n > N.
\]
From (3.7) and (3.8), there exists \( N \in \mathbb{N} \) such that 
\[
\|\varphi(x_n) - \varphi(x_0)\| < \epsilon, \quad \text{for all} \quad n > N.
\]
Thereby for given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that

\[(3.10) \quad ||\varphi(x_n) - \varphi(x_0)|| < \epsilon, \quad \text{for all } n > N\]

From (3.10), $\varphi(x_n)$ converges to $\varphi(x_0)$ as $n \to \infty$.

Conversely suppose that, for a sequence $\{x_n\}$ in $W$ which converges to $x_0$ as $n \to \infty$ then $\varphi(x_n)$ converges to $\varphi(x_0)$ as $n \to \infty$. Assume that $\varphi$ is not continuous, then for given $\delta > 0$, there exists $\epsilon > 0$ such that

\[(3.11) \quad ||\varphi(x_n) - \varphi(x_0)|| \geq \epsilon, \quad \text{for all } x_n \in W \text{ with } ||x_n - x_0|| < \delta.\]

Take $\delta = \frac{1}{n}$, in (3.11) then $||x_n - x_0|| < \frac{1}{n}$. Hence $x_n \to x_0$ as $n \to \infty$, but $||\varphi(x_n) - \varphi(x_0)|| \geq \epsilon$. This is a contradiction with our supposition. Hence $\varphi$ is continuous. \[\square\]

**Lemma 3.6.** Let $\varphi : W \to \mathbb{R}$ be a continuous mapping and $M$ is a compact subset of $W$, then $\varphi$ assumes a maximum and a minimum at some points of $M$.

**Proof.** First we want to prove that $\varphi(M)$ is compact. For, let $y_n \in \varphi(M)$, then we have $y_n = \varphi(x_n)$, for some $x_n \in M$. Since $M$ is compact, $x_n$ contains a sub sequence $x_{n_k}$ which converges in $M$. Since $\varphi$ is continuous, $\{\varphi(x_{n_k})\}$ is a subsequence of $\{y_n\}$ which converges in $\varphi(M)$. Thereby every sequence in $\varphi(M)$ has a convergent subsequence, as $y_n$ is arbitrary. Hence $\varphi(M)$ is compact. Therefore $\varphi(M)$ is closed and bounded. So that $\inf_{x \in M} \varphi(x) \in \varphi(M)$, $\sup_{x \in M} \varphi(x) \in \varphi(M)$, and the inverse images of these points consist of points of $M$ at which $\varphi(x)$ is minimum or maximum respectively. \[\square\]

**Proposition 3.7.** Let $\{f_k\}_{k=1}^m$ be a sequence in $V_H^L$. Then $\{f_k\}_{k=1}^m$ is a frame for left span $\{f_k\}_{k=1}^m$.

**Proof.** From the Schwartz inequality the upper frame condition is satisfied with $B = \sum_{k=1}^m \|f_k\|^2$. Thereby

\[(3.12) \quad \sum_{k=1}^m |\langle f|f_k \rangle|^2 \leq B \|f\|^2.\]

Let $W := \text{left span} \{f_k\}_{k=1}^m$ and consider the mapping

$$\varphi : W \to \mathbb{R}, \quad \varphi(f) := \sum_{k=1}^m |\langle f|f_k \rangle|^2.$$
Now we want to prove that \( \varphi \) is continuous. Let \( \{g_n\} \) be a sequence in \( W \) such that \( g_n \to g \) as \( n \to \infty \). Now,

\[
\|\varphi(g_n) - \varphi(g)\| = \left\| \sum_{k=1}^{m} |\langle g_n|f_k \rangle|^2 - \sum_{k=1}^{m} |\langle g|f_k \rangle|^2 \right\|
\]

\[
\leq \sum_{k=1}^{m} \left| |\langle g_n|f_k \rangle|^2 - |\langle g|f_k \rangle|^2 \right|
\]

\[
= \sum_{k=1}^{m} \|\langle g_n|f_k \rangle - |\langle g|f_k \rangle \| \|g||f_k\|
\]

\[
= \sum_{k=1}^{m} \||\langle g_n|f_k \rangle \langle f_k|g_n \rangle - \langle g|f_k \rangle \langle f_k|g \rangle ||
\]

\[
= \sum_{k=1}^{m} \||\langle g_n|f_k \rangle - \langle g|f_k \rangle || \langle f_k|g_n \rangle + \langle g|f_k \rangle \langle f_k|g_n \rangle - \langle f_k|g \rangle ||
\]

\[
= \sum_{k=1}^{m} \||\langle f_k|g_n \rangle - \langle f_k|g \rangle || \langle f_k|g_n \rangle + \langle g|f_k \rangle \langle f_k|g_n \rangle - \langle f_k|g \rangle ||
\]

\[
= \sum_{k=1}^{m} \||\langle f_k|g_n - g \rangle| \langle f_k|g_n \rangle + \langle g|f_k \rangle \langle f_k|g_n - g \rangle ||
\]

\[
\to 0 \text{ as } n \to \infty \quad [:: g_n \to g \text{ as } n \to \infty]
\]

Thereby \( \varphi(g_n) \) converges to \( \varphi(g) \) as \( n \to \infty \). From the lemma \[\text{[4.5]}\], \( \varphi \) is continuous. Since the closed unit ball in \( W \) is compact, from the lemma \[\text{[3.6]}\], we can find \( g \in W \) with \( \|g\| = 1 \) such that

\[
A := \sum_{k=1}^{m} |\langle g|f_k \rangle|^2 = \inf \left\{ \sum_{k=1}^{m} |\langle f|f_k \rangle|^2 : f \in W, \|f\| = 1 \right\}.
\]

It is clear that \( A > 0 \) as not all \( f_k \) are zero. Now given \( f \in W \), \( f \neq 0 \), we have

\[
\left\| \frac{f}{\|f\|} \right\| = 1 \text{, so } \sum_{k=1}^{m} \left| \left\langle \frac{f}{\|f\|}, f_k \right\rangle \right|^2 \geq A. \text{ Hence}
\]

\[
\sum_{k=1}^{m} |\langle f|f_k \rangle|^2 = \sum_{k=1}^{m} \left| \left\langle \frac{f}{\|f\|}, f_k \right\rangle \right|^2 \|f\|^2 \geq A \|f\|^2.
\]

Thereby

\[
(3.13) \quad \sum_{k=1}^{m} |\langle f|f_k \rangle|^2 \geq A \|f\|^2.
\]
From (3.12) and (3.13),
\[ A \|f\|^2 \leq \sum_{k=1}^{m} |\langle f|f_k \rangle|^2 \leq B \|f\|^2, \]
for all \( f \in W \).

Hence \( \{f_k\}_{k=1}^{m} \) is a frame for leftspan \( \{f_k\}_{k=1}^{m} \).

**Corollary 3.8.** A family of elements \( \{f_k\}_{k=1}^{m} \) in \( V_H^L \) is a frame for \( V_H^L \) if and only if leftspan \( \{f_k\}_{k=1}^{m} = V_H^L \).

**Proof.** Suppose that \( \{f_k\}_{k=1}^{m} \) is a frame for \( V_H^L \). Then there exist \( A, B > 0 \) such that
\[ (3.14) \]
\[ A \|f\|^2 \leq \sum_{k=1}^{m} |\langle f|f_k \rangle|^2 \leq B \|f\|^2, \]
for all \( f \in V_H^L \). If there exists \( f \in V_H^L \) such that \( f \notin \text{leftspan}\{f_k\}_{k=1}^{\infty} \). Then \( f \neq \sum_{k=1}^{m} c_k f_k \)
for all sequences \( \{c_k\}_{k=1}^{m} \subset H \). That is, \( \|f\|^2 \neq \sum_{k=1}^{m} |c_k|^2 \|f_k\|^2 \) for any sequence \( \{c_k\}_{k=1}^{m} \subset H \). Set \( c_k = \left( \frac{f_k}{B \|f_k\|} \right) \in H \) for all \( k = 1, 2, ..., m \). Thereby
\[ \|f\|^2 \neq \sum_{k=1}^{m} \left( \frac{f_k}{\sqrt{B}} \right)^2 \|f_k\|^2 \]
\[ = \frac{1}{B} \sum_{k=1}^{m} |\langle f|f_k \rangle|^2 \]
\[ \leq \frac{1}{B} B \|f\|^2 = \|f\|^2 \]
by (3.14),
which is a contradiction. Thereby \( V_H^L \subseteq \text{leftspan}\{f_k\}_{k=1}^{m} \). Clearly \( \text{leftspan}\{f_k\}_{k=1}^{m} \subseteq V_H^L \). Thereby the conclusion follows.

Conversely suppose that \( V_H^L = \text{leftspan}\{f_k\}_{k=1}^{m} \). From proposition (3.7) \( \{f_k\}_{k=1}^{m} \) is a frame for \( \text{leftspan}\{f_k\}_{k=1}^{m} \), thereby \( \{f_k\}_{k=1}^{m} \) is a frame for \( V_H^L \).

From the above corollary it is clear that a frame is an over complete family of vectors in a finite dimensional Hilbert space.

### 3.2. Frame operator in left quaternion Hilbert space.

#### 3.2.1. Operators on left quaternion Hilbert spaces. Let \( \mathcal{O} : V_H^L \rightarrow V_H^L \) be a quaternion linear operator. In this case, the operators always act from the left as \( \mathcal{O}|f \rangle \) and the scalar multiple of the operator is taken from the left as \( q \mathcal{O} \). Further the operators obey the following rules:

(i) \( \mathcal{O}|qf \rangle = q(\mathcal{O}|f \rangle) \).

(ii) \( \langle f|\mathcal{O}g \rangle = \langle \mathcal{O}^\dagger f|g \rangle \); \( \mathcal{O}^\dagger \) is the adjoint of \( \mathcal{O} \).

(iii) \( (q\mathcal{O})|f \rangle = \mathcal{O}|qf \rangle \).

For a detail explanation we refer the reader to [1].
3.2.2. Frame operators. Consider now a left quaternion Hilbert space, $V_H^L$ with a frame \( \{ f_k \}_{k=1}^m \) and define a linear mapping

\[
T : H^m \rightarrow V_H^L, \quad T \{ c_k \}_{k=1}^m = \sum_{k=1}^m c_k f_k, \quad c_k \in H.
\]

$T$ is usually called the pre-frame operator, or the synthesis operator. The adjoint operator

\[
T^* : V_H^L \rightarrow H^m, \quad \text{by } T^* f = \{ \langle f | f_k \rangle \}_{k=1}^m
\]

is called the analysis operator. By composing $T$ with its adjoint we obtain the frame operator

\[
S : V_H^L \rightarrow V_H^L, \quad S f = T T^* f = \sum_{k=1}^m \langle f | f_k \rangle f_k.
\]

Note that in terms of the frame operator, for $f \in V_H^L$

\[
\langle S f | f \rangle = \left\langle \sum_{k=1}^m \langle f | f_k \rangle f_k, f \right\rangle
\]

\[
= \sum_{k=1}^m \langle f | f_k \rangle \langle f_k | f \rangle
\]

\[
= \sum_{k=1}^m |\langle f | f_k \rangle|^2.
\]

Thereby

\[
\langle S f | f \rangle = \sum_{k=1}^m |\langle f | f_k \rangle|^2, \quad f \in V_H^L.
\]

A frame $\{ f_k \}_{k=1}^m$ is tight if we can choose $A = B$ in the definition (3.4), hence (3.5) gives

\[
\sum_{k=1}^m |\langle f | f_k \rangle|^2 = A \| f \|^2, \quad \text{for all } f \in V_H^L.
\]

Thereby $\langle S f | f \rangle = A \| f \|^2$, for all $f \in V_H^L$.

**Proposition 3.9.** Assume that $\{ f_k \}_{k=1}^m$ is a tight frame for $V_H^L$ with frame bound $A$. Then $S = AI$ (here $I$ is the identity operator on $V_H^L$), and

\[
f = \frac{1}{A} \sum_{k=1}^m \langle f | f_k \rangle f_k, \quad \text{for all } f \in V_H^L.
\]

**Proof.** The frame operator $S$ is given by

\[
S : V_H^L \rightarrow V_H^L; \quad S f = \sum_{k=1}^m \langle f | f_k \rangle f_k.
\]
Let \( f \in V^L_H \), then
\[
\langle Sf | f \rangle = \left\langle \sum_{k=1}^{m} \langle f | f_k \rangle f_k | f \right\rangle
= \sum_{k=1}^{m} |\langle f | f_k \rangle|^2.
\]

Since the frame \( \{ f_k \}_{k=1}^{m} \) is tight for \( V^L_H \),
\[
\langle Sf | f \rangle = A \| f \|^2, \text{ for all } f \in V^L_H.
\]

The identity operator on \( V^L_H \) is given by
\[ I : V^L_H \rightarrow V^L_H, \text{ } I f = f. \]

Now,
\[
\langle Sf | f \rangle = A \| f \|^2
= A \langle f | f \rangle
= A \langle If | f \rangle
= \langle AIf | f \rangle.
\]

Thereby \( \langle Sf | f \rangle = \langle AIf | f \rangle \), for all \( f \in V^L_H \). Hence \( S = A I \). Since \( \{ f_k \}_{k=1}^{m} \) is a frame for \( V^L_H \), from corollary (3.8),
\[
(3.19) \quad V^L_H = \text{leftspan} \{ f_k \}_{k=1}^{m}.
\]

Therefore for given \( f \in V^L_H \), there exists \( c_k \in H \) such that
\[
(3.20) \quad f = \sum_{k=1}^{m} c_k f_k.
\]

Now define \( c_k = \langle f | g_k \rangle \) and \( g_k = \frac{1}{A} f_k \), here \( g_k \in V^L_H \). Then (3.20) becomes
\[
f = \sum_{k=1}^{m} \langle f | g_k \rangle f_k
= \sum_{k=1}^{m} \left( f \frac{1}{A} f_k \right) f_k
= \sum_{k=1}^{m} \langle f | f_k \rangle \left( \frac{1}{A} \right) f_k
= \frac{1}{A} \sum_{k=1}^{m} \langle f | f_k \rangle f_k \text{ as } A \text{ is real.}
\]

Hence
\[
f = \frac{1}{A} \sum_{k=1}^{m} \langle f | f_k \rangle f_k, \text{ for all } f \in V^L_H.
\]

\( \square \)
**Definition 3.10.** Let $V^L_H$ be any left quaternion Hilbert space. A mapping $S : V^L_H \rightarrow V^L_H$ is said to be **linear** if,

$$S(\alpha f + \beta g) = \alpha S(f) + \beta S(g),$$

for all $f, g \in V^L_H$ and $\alpha, \beta \in H$.

**Definition 3.11.** A linear operator $S : V^L_H \rightarrow V^L_H$ is said to be **bounded** if,

$$\|Sf\| \leq K \|f\|,$$

for some constant $K \geq 0$ and all $f \in V^L_H$.

**Definition 3.12.** (Adjoint operator) Let $S : V^L_H \rightarrow V^L_H$ be a bounded linear operator on a left quaternion Hilbert space $V^L_H$. We define its adjoint to be the operator $S^* : V^L_H \rightarrow V^L_H$ that has the property

$$(3.21) \quad \langle f | Sg \rangle = \langle S^* f | g \rangle,$$

for all $f, g \in V^L_H$.

**Lemma 3.13.** The adjoint operator $S^*$ is linear and bounded.

**Proof.** For arbitrary $f_1, f_2 \in V^L_H$ and scalars $\alpha, \beta \in H$, we have the following computation for every $g \in V^L_H$,

$$
\langle S^*(\alpha f_1 + \beta f_2) | g \rangle = \langle \alpha f_1 + \beta f_2 | Sg \rangle \\
= \langle Sg | (\alpha f_1 + \beta f_2) \rangle \\
= \langle Sg | \alpha f_1 \rangle + \langle Sg | \beta f_2 \rangle \\
= \alpha \langle f_1 | Sg \rangle + \beta \langle f_2 | Sg \rangle \\
= \alpha \langle S^* f_1 | g \rangle + \beta \langle S^* f_2 | g \rangle \\
= \alpha \langle S^* f_1 | g \rangle + \beta \langle S^* f_2 | g \rangle \\
= \langle g \alpha S^* f_1 \rangle + \langle g \beta S^* f_2 \rangle \\
= \langle g | \alpha S^* f_1 + \beta S^* f_2 \rangle \\
= \langle \alpha S^* f_1 + \beta S^* f_2 | g \rangle.
$$

Since this holds for all $g \in V^L_H$, we conclude that

$$S^*(\alpha f_1 + \beta f_2) = \alpha S^* f_1 + \beta S^* f_2,$$

completing the proof that $S^*$ is linear. For $f \in V^L_H$, we have the computation

$$
\|S^* f\|^2 = \langle S^* f | S^* f \rangle \\
= \langle f | S(S^* f) \rangle \\
\leq \|f\| \|S(S^* f)\| \quad \text{by Schwarz inequality} \\
\leq \|f\| \|S\| \|S^* f\| \quad \text{as $S$ is bounded.}
$$

Hence either $S^* f = 0$ or $\|S^* f\| \leq \|S\| \|f\|$. In either case

$$\|S^* f\| \leq \|S\| \|f\|.$$

This proves that $S^*$ is bounded, and furthermore, that $\|S^*\| \leq \|S\|$. \hfill $\square$

**Definition 3.14.** (Self-adjoint operator) Let $V^L_H$ be a left quaternion Hilbert space. A bounded linear operator $S$ on $V^L_H$ is called **self-adjoint**, if $S = S^*$. 
Lemma 3.15. Let \( S : V_H^L \rightarrow V_H^L \) and \( T : V_H^L \rightarrow V_H^L \) be bounded linear operators on \( V_H^L \). Then for any \( f, g \in V_H^L \), we have

(a) \( \langle Sf \mid g \rangle = \langle f \mid S^*g \rangle \).

(b) \( (S + T)^* = S^* + T^* \).

(c) \( (ST)^* = T^*S^* \).

(d) \( (S^*)^* = S \).

(e) \( I^* = I \), where \( I \) is an identity operator on \( V_H^L \).

(f) If \( S \) is invertible then \( (S^{-1})^* = (S^*)^{-1} \).

Proof. (a) For arbitrary \( f, g \in V_H^L \),

\[
\langle f \mid S^*g \rangle = \overline{\langle S^*g \mid f \rangle} = \overline{\langle g \mid Sf \rangle} \text{ by definition of adjoint operator} = \langle Sf \mid g \rangle.
\]

(b) For arbitrary \( f, g \in V_H^L \),

\[
\langle (S + T)^*f \mid g \rangle = \langle f \mid (S + T)g \rangle = \langle f \mid Sg + Tg \rangle = \langle f \mid Sg \rangle + \langle f \mid Tg \rangle = \langle S^*f \mid g \rangle + \langle T^*f \mid g \rangle = \overline{\langle g \mid S^*f \rangle} + \overline{\langle g \mid T^*f \rangle} = \langle g \mid S^*f \rangle + \langle g \mid T^*f \rangle = \langle g \mid S^*f + T^*f \rangle = \langle S^*f + T^*f \mid g \rangle.
\]

Hence \( (S + T)^*f = S^*f + T^*f \), for all \( f \in V_H^L \). Thereby \( (S + T)^* = S^* + T^* \).

(c) For arbitrary \( f, g \in V_H^L \),

\[
\langle (ST)^*f \mid g \rangle = \langle f \mid STg \rangle = \overline{\langle STf \mid g \rangle} \text{ by definition of adjoint operator of } S = \overline{\langle T^*S^*f \mid g \rangle} \text{ by definition of adjoint operator of } T.
\]

Hence \( (ST)^*f = T^*S^*f \), for all \( f \in V_H^L \). Thereby \( (ST)^* = T^*S^* \).

(d) For arbitrary \( f, g \in V_H^L \),

\[
\langle (S^*)^*f \mid g \rangle = \langle f \mid S^*g \rangle \text{ by definition of adjoint operator} = \langle Sf \mid g \rangle \text{ by lemma 2.13(a)}.
\]

Hence \( (S^*)^*f = Sf \), for all \( f \in V_H^L \). Thereby \( (S^*)^* = S \).

(e) For arbitrary \( f, g \in V_H^L \),

\[
\langle I^*f \mid g \rangle = \langle f \mid Ig \rangle \text{ by definition of adjoint operator} = \langle f \mid g \rangle \text{ as } Ig = g = \langle If \mid g \rangle \text{ as } If = f.
\]

Hence \( I^*f = If \), for all \( f \in V_H^L \). Thereby \( I^* = I \).

(f) We have \( SS^{-1} = S^{-1}S = I \). Hence,

\[
(SS^{-1})^* = (S^{-1}S)^* = I^* \Rightarrow (S^{-1})^*S = S^*(S^{-1})^* = I.
\]

Thereby \( (S^*)^{-1} = (S^{-1})^* \). \( \square \)
Lemma 3.16. Let $U^L_H$ and $V^L_H$ be finite dimensional left quaternion Hilbert spaces and $S: U^L_H \to V^L_H$ be a linear mapping then $\ker S$ is a subspace of $U^L_H$.

Proof. We have $\ker S = \{u \in U^L_H : Su = 0\}$. Let $u_1, u_2 \in \ker S$ and $\alpha, \beta \in H$, then
\[ S(\alpha u_1 + \beta u_2) = \alpha S(u_1) + \beta S(u_2) = 0. \]
Therefore $\alpha u_1 + \beta u_2 \in \ker S$. Hence $\ker S$ is a subspace of $U^L_H$. \hfill \Box

Lemma 3.17. Let $U^L_H, V^L_H$ be finite dimensional left quaternion Hilbert spaces and $S: U^L_H \to V^L_H$ be a linear mapping then
\[ \dim R_S + \dim N_S = \dim U^L_H, \]
where $R_S := \text{image of } S$, $N_S := \ker S$.

Proof. Since $N_S$ is a subspace of $U^L_H$,
\[ \dim N_S \leq \dim U^L_H. \]
Let $\dim N_S = m$ and $\dim U^L_H = n$. Let $B = \{u_1, u_2, \ldots, u_m\}$ be a basis for $N_S$. Since $u_i \in N_S$,
\[ Su_i = 0, \text{ for each } i = 1, 2, \ldots, m. \]
But $B$ is linearly independent in $N_S$ and therefore in $U^L_H$, we can extend this linearly independent set of $U^L_H$ to a basis for $U^L_H$. Let $B_1 = \{u_1, u_2, \ldots, u_m, u_{m+1}, \ldots, u_n\}$ be a basis for $U^L_H$. Now consider the set $A = \{Su_{m+1}, Su_{m+2}, \ldots, Su_n\}$ and claim that $A$ is a basis for $R_S$. For, Since $U^L_H = \text{leftspan } B_1$, $R_S = \text{leftspan } \{Su_1, Su_2, \ldots, Su_m, \ldots, Su_n\}$. But $Su_i = 0$, for $i = 1, 2, \ldots, m$. Hence
\[ (3.22) \quad R_S = \text{leftspan } \{Su_{m+1}, Su_{m+2}, \ldots, Su_n\}. \]
Let $\alpha_i \in H$, consider
\[ (3.23) \quad \alpha_{m+1}Su_{m+1} + \alpha_{m+2}Su_{m+2} + \cdots + \alpha_n Su_n = 0, \]
then $S(\alpha_{m+1}u_{m+1} + \alpha_{m+2}u_{m+2} + \cdots + \alpha_n u_n) = 0$. Hence
\[ \alpha_{m+1}u_{m+1} + \alpha_{m+2}u_{m+2} + \cdots + \alpha_n u_n \in N_S = \text{leftspan } B. \]
Thus
\[ \alpha_{m+1}u_{m+1} + \alpha_{m+2}u_{m+2} + \cdots + \alpha_n u_n = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_m u_m, \]
for some $\beta_i \in H$. Therefore
\[ \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_m u_m + (-\alpha_{m+1})u_{m+1} + (-\alpha_{m+2})u_{m+2} + \cdots + (-\alpha_n)u_n = 0. \]
Hence
\[ (3.24) \quad \beta_1 = \beta_2 = \cdots = \beta_m = \alpha_{m+1} = \alpha_{m+2} = \cdots = \alpha_n = 0, \]
as $B_1 = \{u_1, u_2, \ldots, u_m, u_{m+1}, \ldots, u_n\}$ is linearly independent. From (3.23) and (3.24), $A = \{Su_{m+1}, Su_{m+2}, \ldots, Su_n\}$ is linearly independent. Hence $A = \{Su_{m+1}, Su_{m+2}, \ldots, Su_n\}$ is a basis for $R_S$. Therefore $\dim R_S = n - m = \dim U^L_H - \dim N_S$. Hence
\[ \dim R_S + \dim N_S = \dim U^L_H. \]

Lemma 3.18. Let $S: U^L_H \to V^L_H$ be a linear mapping. $S$ is one to one if and only if $N_S = \{0\}$. \hfill \Box
Lemma 3.20. Let $V$ and $W$ be finite dimensional left quaternion Hilbert spaces with the same dimension. Let $S : U^L_H \rightarrow V^L_H$ be a linear mapping. If $S$ is one to one then $S$ is onto.

Proof. Suppose that $S$ is one to one. Let $u \in N_S$ then $S(u) = 0$. So $S(u) = 0 = S(0)$ as $S$ is linear. Since $S$ is one to one, $u = 0$. Hence $N_S = \{0\}$. Conversely suppose that $N_S = \{0\}$. Let $u_1, u_2 \in U^L_H$ such that $S(u_1) = S(u_2)$. Since $S$ is linear, $u_1 - u_2 \in N_S = \{0\}$. Thereby $u_1 = u_2$ Hence $S$ is one to one. \qed

Lemma 3.19. Let $U^L_H, V^L_H$ are finite dimensional left quaternion Hilbert spaces with same dimension. Let $S : U^L_H \rightarrow V^L_H$ be a linear mapping. If $S$ is one to one then $S$ is onto.

Proof. Suppose that $S$ is one to one. From the lemma (3.18) $N_S = \{0\}$. Hence $\dim N_S = 0$, as dimension of null space is zero. From the lemma (3.17) $\dim R_S = \dim U^L_H (= \dim V^L_H)$. So $R_S = V^L_H$. Hence $S$ is onto. \qed

Lemma 3.20. (Pythagoras’ law) Suppose that $f$ and $g$ is an arbitrary pair of orthogonal vectors in the left quaternion Hilbert space $V^L_H$. Then we have Pythagoras’ formula

$$(3.25) \quad \|f + g\|^2 = \|f\|^2 + \|g\|^2.$$  

Proof. We have

$$\|f + g\|^2 = \langle f + g, f + g \rangle = \langle f, f \rangle + \langle f, g \rangle + \langle g, f \rangle + \langle g, g \rangle = \langle f, f \rangle + \langle g, g \rangle.$$  

Since $\langle f, f \rangle = \|f\|^2$ and $\langle g, g \rangle = \|g\|^2$, the above equation implies

$$\|f + g\|^2 = \|f\|^2 + \|g\|^2,$$

as $\langle f, g \rangle = \langle g, f \rangle = 0$, if $f$ and $g$ are orthogonal. \qed

Lemma 3.21. Let $T : H^m \rightarrow V^L_H$ be a linear mapping and $T^* : V^L_H \rightarrow H^m$ be its adjoint operator. Then $N_T = R_{T^*}$, where $N_T := \ker T$ and $R_{T^*} := \text{range of } T^*.$

Proof. Since $T : H^m \rightarrow V^L_H$ and $T^* : V^L_H \rightarrow H^m$, we can define

$$(3.26) \quad N_T = \{x \in H^m : Tx = 0\}.$$  

$$(3.27) \quad R_{T^*} = \{T^*y : y \in V^L_H\}.$$  

Let $x \in N_T$, then $x \in H^m$ such that $Tx = 0$. This implies

$$(3.28) \quad \langle y, Tx \rangle = 0, \text{ for all } y \in V^L_H.$$  

Hence by the definition of adjoint operator,

$$(3.29) \quad \langle T^*y, x \rangle = 0, \text{ for all } y \in V^L_H.$$  

Since $R_{T^*} = \{T^*y : y \in V^L_H\}$, (3.29) shows that $x \in R_{T^*}^\perp$, hence

$$(3.30) \quad N_T \subseteq R_{T^*}^\perp.$$  

Conversely $x \in R_{T^*}^\perp$, then $\langle x, T^*y \rangle = 0$, for all $y \in V^L_H$. Thereby $\langle T^*y, x \rangle = 0$, for all $y \in V^L_H$. By the definition (3.12), $\langle y, Tx \rangle = 0$, for all $y \in V^L_H$. Thereby $\langle Tx, y \rangle = 0$, for all $y \in V^L_H$. It follows that $Tx = 0$. Hence $x \in N_T$. Therefore,

$$(3.31) \quad \text{R}_{T^*}^\perp \subseteq N_T.$$
From (3.30) and (3.31),

\[ N_T = R_{T^*}^\perp. \]

\[ \Box \]

**Theorem 3.22.** Let \( \{ f_k \}_{k=1}^m \) be a frame for \( V_H^L \) with frame operator \( S \). Then

1. \( S \) is invertible and self-adjoint.
2. Every \( f \in V_H^L \), can be represented as

\[ f = \sum_{k=1}^m \langle f | S^{-1} f_k \rangle f_k = \sum_{k=1}^m \langle f | f_k \rangle S^{-1} f_k. \]

3. If \( f \in V_H^L \), and has the representation \( f = \sum_{k=1}^m c_k f_k \) for some scalar coefficients \( \{ c_k \}_{k=1}^m \), then

\[ \sum_{k=1}^m |c_k|^2 = \sum_{k=1}^m |\langle f | S^{-1} f_k \rangle|^2 + \sum_{k=1}^m |c_k - \langle f | S^{-1} f_k \rangle|^2. \]

**Proof.** (1) \( S : V_H^L \rightarrow V_H^L \), by \( S f = TT^* f = \sum_{k=1}^m \langle f | f_k \rangle f_k \), for all \( f \in V_H^L \). Now

\[ S^* = (TT^*)^* = (T^*)^* T^* = TT^* = S. \]

It follows that \( S \) is self-adjoint. We have \( \ker S = \{ f : S f = 0 \} \). Let \( f \in \ker S \), then \( S f = 0 \). Therefore

\[ 0 = \langle S f | f \rangle = \left( \sum_{k=1}^m \langle f | f_k \rangle f_k \right) f \]

\[ = \sum_{k=1}^m |\langle f | f_k \rangle|^2. \]

Thereby \( \sum_{k=1}^m |\langle f | f_k \rangle|^2 = 0 \). Since \( \{ f_k \}_{k=1}^m \) be a frame for \( V_H^L \), by definition (3.3)

\[ A \| f \|^2 \leq \sum_{k=1}^m |\langle f | f_k \rangle|^2 \leq B \| f \|^2, \]

for all \( f \in V_H^L \).

Hence \( A \| f \|^2 \leq \sum_{k=1}^m |\langle f | f_k \rangle|^2 \leq B \| f \|^2 \), for all \( f \in V_H^L \) and \( A, B > 0 \). So \( \| f \|^2 = 0 \). Thereby \( f = 0 \), for all \( f \in V_H^L \), it follows that \( N_S = 0 \). Hence \( S \) is one to one. Since \( V_H^L \) is of the finite dimension, from the lemma (3.19) \( S \) is onto. Therefore \( S \) is invertible.

(2) If \( S \) is self-adjoint then \( S^{-1} \) is self-adjoint. For, Consider

\[ (S^{-1})^* = (S^*)^{-1} = (S)^{-1} \text{ as } S \text{ is self adjoint}. \]

Thereby \( S^{-1} \) is self-adjoint. If \( S : V_H^L \rightarrow V_H^L \), is linear and bijection then \( S^{-1} \) is linear. For, Since \( S \) is onto, \( S^{-1} : V_H^L \rightarrow V_H^L \). Let \( f, g \in V_H^L \) then there exists \( k, h \in V_H^L \) such
that $S^{-1}(f) = k$ and $S^{-1}(g) = h$. Thereby $f = S(k)$ and $g = S(h)$. Let $\alpha, \beta \in H$, then

\[ S^{-1}(\alpha f + \beta g) = S^{-1}(\alpha S(k) + \beta S(h)) = S^{-1}(\alpha k + \beta h) = \alpha S^{-1}(f) + \beta S^{-1}(g) \]

Thereby for all $f, g \in V^L_H$ and $\alpha, \beta \in H$,

\[ S^{-1}(\alpha f + \beta g) = \alpha S^{-1}(f) + \beta S^{-1}(g) \]

Hence $S^{-1}$ is linear. Let $f \in V^L_H$, then

\[ f = SS^{-1}f = TT^*S^{-1}f = \sum_{k=1}^{m} \langle S^{-1}f | f_k \rangle f_k = \sum_{k=1}^{m} \langle f | (S^{-1})^* f_k \rangle f_k, \text{ by definition (3.12)} \]

\[ = \sum_{k=1}^{m} \langle f | S^{-1}f_k \rangle f_k, \text{ as } S^{-1} \text{ is self adjoint.} \]

Thereby for every $f \in V^L_H$,

(3.33)  \[ f = \sum_{k=1}^{m} \langle f | S^{-1}f_k \rangle f_k. \]

Similarly we have

\[ f = S^{-1}Sf = S^{-1}TT^*f = S^{-1} \left( \sum_{k=1}^{m} \langle f | f_k \rangle f_k \right) = \sum_{k=1}^{m} S^{-1}(\langle f | f_k \rangle f_k), \text{ as } S^{-1} \text{ is linear.} \]

\[ = \sum_{k=1}^{m} \langle f | f_k \rangle S^{-1}f_k. \]

Thereby for every $f \in V^L_H$,

(3.34)  \[ f = \sum_{k=1}^{m} \langle f | f_k \rangle S^{-1}f_k. \]

From (3.33) and (3.34), for every $f \in V^L_H$,

\[ f = \sum_{k=1}^{m} \langle f | S^{-1}f_k \rangle f_k = \sum_{k=1}^{m} \langle f | f_k \rangle S^{-1}f_k. \]
(3) Let \( f \in V_H^L \), from corollary 3.8

\[
f = \sum_{k=1}^{m} c_k f_k, \quad \text{for some } c_k \in H.
\]

From the part(1),

\[
f = \sum_{k=1}^{m} c_k f_k = \sum_{k=1}^{m} \langle f | S^{-1} f_k \rangle f_k.
\]

Hence

\[
\sum_{k=1}^{m} \left( c_k - \langle f | S^{-1} f_k \rangle \right) f_k = 0.
\]

Thereby \( \sum_{k=1}^{m} d_k f_k = 0 \), for some \( d_k = \left( c_k - \langle f | S^{-1} f_k \rangle \right) \in H \). From (3.15), \( T : H^m \to V_H^L \), is defined by \( T \{ d_k \}_{k=1}^{m} = \sum_{k=1}^{m} d_k f_k, \ d_k \in H \). We have \( N_T = \{ \{ d_k \}_{k=1}^{m} | T \{ d_k \}_{k=1}^{m} = 0 \} \),

\[
\{ \{ d_k \}_{k=1}^{m} = \{ c_k \}_{k=1}^{m} - \{ \langle f | S^{-1} f_k \rangle \}_{k=1}^{m} \in N_T. \]

From lemma (3.21), \( N_T = R_T^{\perp}, \) then

\[
\{ c_k \}_{k=1}^{m} - \{ \langle f | S^{-1} f_k \rangle \}_{k=1}^{m} \in R_T^{\perp}.
\]

From (3.15) and (3.16) we have \( T^* : V_H^L \to H^m, \ T^* f = \{ \langle f | f_k \rangle \}_{k=1}^{m}, \) and \( S : V_H^L \to V_H^L, \ Sf = TT^* f = \sum_{k=1}^{m} \langle f | f_k \rangle f_k. \) Hence \( T^*(S^{-1} f) = \{ \langle S^{-1} f | f_k \rangle \}_{k=1}^{m}. \) Therefore

\[
\{ \langle S^{-1} f | f_k \rangle \}_{k=1}^{m} \in R_{T^*}.
\]

Since \( S^{-1} \) is self adjoint,

\[
\{ \langle S^{-1} f | f_k \rangle \}_{k=1}^{m} = \{ \langle f | S^{-1} f_k \rangle \}_{k=1}^{m}.
\]

Hence

\[
\{ \langle f | S^{-1} f_k \rangle \}_{k=1}^{m} = \{ \langle S^{-1} f | f_k \rangle \}_{k=1}^{m} \in R_{T^*}.
\]

Now we can write,

\[
\{ c_k \}_{k=1}^{m} = \{ c_k \}_{k=1}^{m} - \{ \langle f | S^{-1} f_k \rangle \}_{k=1}^{m} + \{ \langle f | S^{-1} f_k \rangle \}_{k=1}^{m}.
\]

From (3.38), (3.39), (3.40) and lemma (3.20),

\[
\sum_{k=1}^{m} |c_k|^2 = \sum_{k=1}^{m} |\langle f | S^{-1} f_k \rangle|^2 + \sum_{k=1}^{m} |c_k - \langle f | S^{-1} f_k \rangle|^2.
\]

Theorem (3.22) is one of the most important results about frames, and

\[
f = \sum_{k=1}^{m} \langle f | S^{-1} f_k \rangle f_k = \sum_{k=1}^{m} \langle f | f_k \rangle S^{-1} f_k.
\]
is called the frame decomposition. If \( \{ f_k \}_{k=1}^m \) is a frame but not a basis, there exists non-zero sequences \( \{ g_k \}_{k=1}^m \) such that \( \sum_{k=1}^m g_k f_k = 0 \). Thereby \( f \in V_H^L \) can be written as

\[
f = \sum_{k=1}^m \langle f | S^{-1} f_k \rangle f_k + \sum_{k=1}^m g_k f_k
\]

\[
= \sum_{k=1}^m (\langle f | S^{-1} f_k \rangle + g_k) f_k
\]

showing that \( f \) has many representations as superpositions of the frame elements.

**Corollary 3.23.** Assume that \( \{ f_k \}_{k=1}^m \) is a basis for \( V_H^L \). Then there exists a unique family \( \{ g_k \}_{k=1}^m \) in \( V_H^L \) such that

\[
(3.42) \quad f = \sum_{k=1}^m \langle f | g_k \rangle f_k, \text{ for all } f \in V_H^L.
\]

In terms of the frame operator, \( \{ g_k \}_{k=1}^m = \{ S^{-1} f_k \}_{k=1}^m \). Furthermore \( \langle f_j | g_k \rangle = \delta_{j,k} \).

**Proof.** Let \( f \in V_H^L \), from the Theorem (3.22),

\[
(3.43) \quad f = \sum_{k=1}^m \langle f | S^{-1} f_k \rangle f_k.
\]

Now take \( \{ g_k \}_{k=1}^m = \{ S^{-1} f_k \}_{k=1}^m \), in (3.43) then,

\[
(3.44) \quad f = \sum_{k=1}^m \langle f | g_k \rangle f_k.
\]

Hence there exists a family \( \{ g_k \}_{k=1}^m \) in \( V_H^L \) such that

\[
(3.45) \quad f = \sum_{k=1}^m \langle f | g_k \rangle f_k, \text{ for all } f \in V_H^L.
\]

**Uniqueness:** Assume that there is another family \( \{ h_k \}_{k=1}^m \) in \( V_H^L \) such that

\[
(3.46) \quad f = \sum_{k=1}^m \langle f | h_k \rangle f_k, \text{ for all } f \in V_H^L.
\]

Then \( \sum_{k=1}^m \langle f | g_k \rangle f_k = \sum_{k=1}^m \langle f | h_k \rangle f_k \).

\[
\implies \sum_{k=1}^m (\langle f | g_k \rangle - \langle f | h_k \rangle) f_k = 0.
\]

\[
\implies \sum_{k=1}^m \langle f | g_k - h_k \rangle f_k = 0.
\]

\[
\implies \langle f | g_k - h_k \rangle = 0, \text{ for all } k = 1, 2, \cdots, m, \text{ as } \{ f_k \}_{k=1}^m \text{ is a basis for } V_H^L.
\]

\[
\implies \langle f | g_k - h_k \rangle = 0, \text{ for all } k = 1, 2, \cdots, m, \text{ for all } f \in V_H^L.
\]

\[
\implies g_k - h_k = 0, \text{ for all } k = 1, 2, \cdots, m.
\]
\[ g_k = h_k, \text{ for all } k = 1, 2, \cdots, m. \]

Hence there exists a unique family \( \{ g_k \}_{k=1}^{m} \) in \( V_H^L \) such that

\[ f = \sum_{k=1}^{m} \langle f | g_k \rangle f_k, \text{ for all } f \in V_H^L. \]  

(3.46)

Since \( f = \sum_{k=1}^{m} \langle f | g_k \rangle f_k, \text{ for all } f \in V_H^L, \text{ for fixed } f_j \in V_H^L, \)

\[ f_j = \sum_{k=1}^{m} \langle f_j | g_k \rangle f_k. \]  

(3.47)

Since \( \{ f_k \}_{k=1}^{m} \) is a basis for \( V_H^L, \{ f_k \}_{k=1}^{m} \) is linearly independent. Therefore in (3.47), \( \langle f_j | g_k \rangle = \delta_{j,k} \).

(3.48)

We can give a perceptive clarification of why frames are important in signal transmission. Let us assume that we want to transmit the signal \( f \) belonging to a left quaternion Hilbert space from a transmitter \( A \) to a receiver \( R \). If both \( A \) and \( R \) have knowledge of frame \( \{ f_k \}_{k=1}^{m} \) for \( V_H^L \), this can be done if \( A \) transmits the frame coefficients \( \{ \langle f | S^{-1} f_k \rangle \}_{k=1}^{m} \); based on knowledge of these numbers, the receiver \( R \) can reconstruct the signal \( f \) using the frame decomposition. Now assume that \( R \) receives a noisy signal, meaning a perturbation \( \{ \langle f | S^{-1} f_k \rangle + c_k \}_{k=1}^{m} \) of the correct frame coefficients. Based on the received coefficients, \( R \) will assert that the transmitted signal was

\[ \sum_{k=1}^{m} \langle f | S^{-1} f_k \rangle + c_k \]  

\[ = \sum_{k=1}^{m} \langle f | S^{-1} f_k \rangle f_k + \sum_{k=1}^{m} c_k f_k \]

\[ = f + \sum_{k=1}^{m} c_k f_k \]

this differs from the correct signal \( f \) by the noise \( \sum_{k=1}^{m} c_k f_k \). Minimizing this noise for various signals with different types of noises has been a hot topic in signal processing. We shall touch this issue in later chapters. For now, if \( \{ f_k \}_{k=1}^{m} \) is over complete, parts of the noise contribution might add up to zero and cancel. This will never happen if \( \{ f_k \}_{k=1}^{m} \) is an orthonormal basis. In that case

\[ \| \sum_{k=1}^{m} c_k f_k \| = \sum_{k=1}^{m} |c_k|^2, \]

so each noise contribution will make the reconstruction worse.

**Definition 3.24.** For \( 0 < p < \infty \),

\[ \ell^p = \left\{ x = \{ x_n \} \subset H \mid \sum_n |x_n|^p < \infty \right\}. \]

If \( p \geq 1 \), \( \| x \|_p = \left( \sum_n |x_n|^p \right)^{1/p} \) defines a norm in \( \ell^p \). In fact \( \ell^p \) is a complete metric space with respect to this norm.
We have already seen that, for \( f \in V^L_H \), the frame coefficients \( \{ \langle f | S^{-1} f_k \rangle \}_{k=1}^m \) have minimal \( \ell^2 \) norm among all sequences \( \{ c_k \}_{k=1}^m \) for which \( f = \sum_{k=1}^m c_k f_k \). In the next theorem, let us see that the existence of coefficients minimizing the \( \ell^1 \) norm.

**Theorem 3.25.** Let \( \{ f_k \}_{k=1}^m \) be a frame for a finite-dimensional left quaternion Hilbert space \( V^L_H \). Given \( f \in V^L_H \), there exist coefficients \( \{ d_k \}_{k=1}^m \in H^m \) such that \( f = \sum_{k=1}^m d_k f_k \), and

\[
\sum_{k=1}^m |d_k| = \inf \left\{ \sum_{k=1}^m |c_k| : f = \sum_{k=1}^m c_k f_k \right\}.
\]

**Proof.** Fix \( f \in V^L_H \). It is clear that we can choose a set of coefficients \( \{ c_k \}_{k=1}^m \), \( c_k \in H \) such that \( f = \sum_{k=1}^m c_k f_k \). Let \( r := \sum_{k=1}^m |c_k| \). Since we want to minimize the \( \ell^1 \) norm of the coefficients, we can now restrict our search for a minimizer to sequences \( \{ d_k \}_{k=1}^m \) belonging to the compact set

\[
M := \{ \{ d_k \}_{k=1}^m \in H^m : |d_k| \leq r, \ k = 1, 2, ..., m \}.
\]

Now,

\[
\left\{ \{ d_k \}_{k=1}^m \in M : f = \sum_{k=1}^m d_k f_k \right\}
\]

is compact.

Define a function

\[
\varphi : H^m \rightarrow \mathbb{R}, \ \varphi \{ d_k \}_{k=1}^m := \sum_{k=1}^m |d_k|.
\]

We can prove \( \varphi \) is continuous by similar proof of proposition (3.7). From (3.50) and lemma (3.6),

\[
\sum_{k=1}^m |d_k| = \inf \left\{ \sum_{k=1}^m |c_k| : f = \sum_{k=1}^m c_k f_k \right\}.
\]

Hence for given \( f \in V^L_H \), there exist coefficients \( \{ d_k \}_{k=1}^m \in H^m \) such that \( f = \sum_{k=1}^m d_k f_k \), and

\[
\sum_{k=1}^m |d_k| = \inf \left\{ \sum_{k=1}^m |c_k| : f = \sum_{k=1}^m c_k f_k \right\}.
\]

\( \square \)

In Proposition (3.7), every finite set of vectors \( \{ f_k \}_{k=1}^m \) is a frame for its span. If left span \( \{ f_k \}_{k=1}^m \neq V^L_H \), the frame decomposition associated with \( \{ f_k \}_{k=1}^m \) gives convenient expression for the orthogonal projection onto the subspace left span \( \{ f_k \}_{k=1}^m \).

**Theorem 3.26.** Let \( \{ f_k \}_{k=1}^m \) be a frame for a subspace \( W \) of the left quaternion Hilbert space \( V^L_H \). Then the orthogonal projection of \( V^L_H \) onto \( W \) is given by

\[
P f = \sum_{k=1}^m \langle f | S^{-1} f_k \rangle f_k.
\]
Proof. Define a function $P$ from $V^L_H$ onto $W$ by

$$P : V^L_H \rightarrow W \text{ by } Pf = \sum_{k=1}^{m} \langle f | S^{-1} f_k \rangle f_k, \text{ for all } f \in V^L_H.$$  

First let us prove that $P$ is onto. For, let $f_1 \in W$ and $S : W \rightarrow W$ be a frame operator in $W$. Since $\{f_k\}_{k=1}^{m}$ be a frame for the subspace $W$, we have

$$f_1 = \sum_{k=1}^{m} \langle f_1 | S^{-1} f_k \rangle f_k$$

But $W \subseteq V^L_H$, thereby $f_1 \in V^L_H$. Since $f_1$ is arbitrary, for given $f \in W$, there exists $g \in V^L_H$ such that $Pg = f$. Thereby $P$ is onto. Now we want to prove that $P$ is an orthogonal projection. Hence our claims are

(i) $Pf = f$, for $f \in W$.
(ii) $Pf = 0$, for $f \in W^\perp$.

For,

(i) The mapping $P : V^L_H \longrightarrow W$ is given by

$$Pf = \sum_{k=1}^{m} \langle f | S^{-1} f_k \rangle f_k.$$  

Since $\{f_k\}_{k=1}^{m}$ be a frame for a subspace $W$ of the left quaternion Hilbert space $V^L_H$, from (3.15) the frame operator $S$ is given by

$$S : W \rightarrow W, \text{ by } Sf = \sum_{k=1}^{m} \langle f | f_k \rangle f_k, \text{ for all } f \in W.$$  

From the Theorem (3.22), every $f \in W$, can be represented as

$$f = \sum_{k=1}^{m} \langle f | S^{-1} f_k \rangle f_k.$$  

From (3.57) and (3.59),

$$Pf = f, \text{ for all } f \in W.$$  

(ii) Let $f \in W^\perp$. The mapping $P : V^L_H \longrightarrow W$ is given by

$$Pf = \sum_{k=1}^{m} \langle f | S^{-1} f_k \rangle f_k.$$  

From the Theorem (3.22), the frame operator $S : W \rightarrow W$ is bijective. Hence the range of $S^{-1}$ is $W$. That is,

$$S^{-1} : W \rightarrow W.$$
So $S^{-1}f_k = g$ for some $g \in W$. Hence (3.61) gives

$$Pf = \sum_{k=1}^{m} \langle f | S^{-1}f_k \rangle f_k$$

$$= \sum_{k=1}^{m} \langle f | g \rangle f_k, \text{ for some } g \in W$$

$$= 0 \text{ as } g \in W \text{ and } f \in W^\perp.$$ 

Therefore

$$Pf = 0 \text{ for all } f \in W^\perp.$$ 

From (3.60) and (3.62), $P$ is an orthogonal projection. \qed

**Definition 3.27.** The numbers

$$\langle f | S^{-1}f_k \rangle, \ k = 1, \cdots, m$$

are called **frame coefficients**. The frame \( \{S^{-1}f_k\}_{k=1}^{m} \) is called the **canonical dual** of \( \{f_k\}_{k=1}^{m} \).

**Example 3.28.** Let \( \{e_k\}_{k=1}^{2} \) be an orthonormal basis for a two dimensional left quaternion Hilbert space \( V_H^L \). Let

$$f_1 = e_1, \ f_2 = e_1 - e_2, \ f_3 = e_1 + e_2.$$ 

Then \( \{f_k\}_{k=1}^{3} \) is a frame for \( V_H^L \). Now the frame operator $S$ is given by

$$Sf = \sum_{k=1}^{3} \langle f | f_k \rangle f_k.$$ 

We obtain that

$$Se_1 = \langle e_1 | f_1 \rangle f_1 + \langle e_1 | f_2 \rangle f_2 + \langle e_1 | f_3 \rangle f_3$$

$$= \langle e_1 | e_1 \rangle e_1 + \langle e_1 | e_1 - e_2 \rangle (e_1 - e_2) + \langle e_1 | (e_1 + e_2) \rangle (e_1 + e_2)$$

$$= \|e_1\|^2 e_1 + (\|e_1\|^2 - \langle e_1 | e_2 \rangle)(e_1 - e_2) + (\|e_1\|^2 + \langle e_1 | e_2 \rangle)(e_1 + e_2)$$

$$= e_1 + e_1 - e_2 + e_1 + e_2 \text{ as } \{e_k\}_{k=1}^{2} \text{ be an orthonormal basis for } V_H^L$$

$$= 3e_1.$$ 

Thereby

$$Se_1 = 3e_1.$$ 

Similarly we can get,

$$Se_2 = 2e_2.$$ 

Thus

$$S^{-1}e_1 = \frac{1}{3}e_1, \ S^{-1}e_2 = \frac{1}{2}e_2.$$ 

Therefore the canonical dual frame is

$$\{S^{-1}f_k\}_{k=1}^{3} = \left\{ \frac{1}{3}e_1, \frac{1}{3}e_1 - \frac{1}{2}e_2, \frac{1}{3}e_1 + \frac{1}{2}e_2 \right\}.$$
From Theorem (3.22), the representation of \( f \in V^L_H \) in terms of the frame is given by

\[
f = \sum_{k=1}^{3} \langle f | S^{-1} f_k \rangle f_k = \frac{1}{3} \langle f | e_1 \rangle e_1 + \left( \langle f | \frac{1}{3} e_1 - \frac{1}{2} e_2 \rangle (e_1 - e_2) + \langle f | \frac{1}{3} e_1 + \frac{1}{2} e_2 \rangle (e_1 + e_2) \right).
\]

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