Triangular inequality for 3d Euclidean simplicial complex in loop quantum gravity

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Abstract. Triangular inequality is an important relation in geometry such that this relation, intuitively, is a statement that the direct line connecting two points is the shortest one. Loop quantum gravity is presented after a reformulation of gravity using Ashtekar variables. The quantization follows the Dirac procedures, which results in the existence of state of quanta of 3d space as an element of Hilbert space. Spin network states has become the basis state for quanta of space in loop quantum gravity. In loop quantum gravity space is discrete and the geometrical quantity is quantized at the Planck scale. In 3d space, we can define triangular discretization of the hypersurface. In this article we discuss the length spectrum and check whether the triangular inequality is satisfied by the quantum length. The answer to the question is positive, such that even at the Planck scale the triangular inequality is still valid.

1. Introduction

Loop quantum gravity (LQG) has become one of the candidates for quantum theory of gravity. In this theory the geometrical quantity of space is quantized [1, 2, 3]. The main concern of the kinematical part of this theory is the spatial part of the spacetime. The quantization procedure of LQG can be summarised in three parts: first, one introduce the new set of variables for GR phase space [4] so that it resembles the form of the gauge theory, then one discretize the spatial hypersurface into chunks of space such that the phase space variables are discretized and finally one constructs the kinematical Hilbert space where the geometrical operators act upon. The geometrical operator we are going to discuss is restricted to the length operator.

One expects quantum geometries to posses quantum behaviour. This raises the question whether the quantum geometry satisfies the classical geometry relation. In these article we study the triangular inequality relation of a 3-dimensional discretized space in which the metric has Euclidean signature. We use the dual picture of LQG where triangles are described by nodes and segments by edges [5]. With this dual picture, a quanta of space contains one node and three edges. This quanta of space is described by spin network states in the kinematical Hilbert space with each edges labeled by spin $j > 0$.

In this article we give the derivation of the discrete spectrum of length for 3d space with Euclidean signature. The discreteness of area in 3d space with Euclidean signature was shown...
in [6], and for 3d space with Lorentzian signature in [7]. The analysis for length operator in 4d Lorentzian was given in [8, 9, 10]. Since the discussion here is restricted to triangular inequality, we only need to consider a quanta of space. Spin network states in 3d is diagonalized by length operator [7] so that the representation of spin network states is edges length. In this article we show that even at the level of quanta of space, the triangular inequality is still valid.

2. Elements of LQG in 3d Euclidean

Here we describe the kinematical LQG for 3d Euclidean manifold. As already described in the previous section the LQG quantization program can be summarised in three steps:

(i) New phase space variables

The usual metric formulation of GR is transformed into the formulation using connection $\omega$ which is a $su(2)$ element. The action for 3d Euclidean gravity can be written as

$$S[e, \omega] = \frac{1}{16\pi G} \int e_{ijk} e^i \wedge F^{jk}[\omega], \quad (1)$$

with $e_i$ is the triad and $F^{jk}$ is the curvature. The formulation of gravity using $e$ and $\omega$ as independent variables is called as first order formulation.

(ii) Discretization of 2d manifold

The concern of the kinematical LQG is restricted to the hypersurface foliation of the whole spacetime namely the $\Sigma$ part of $M = \Sigma \times \mathbb{R}$. For 3d Euclidean manifold the hypersurface $\Sigma$ is 2d manifold. In this step we discretized the hypersurface into triangles. The dual picture of LQG [11, 12] is then used, such that triangle is described by a node and a segment is described by an edge. The set of nodes and edges is defined as a graph $\gamma$, which is dual to the manifold. By the discretization, the triad and the connection are also discretized along the edge. Discretization of triad along an edge gives $L_e \in so(3)$ and discretization of connection along an edge gives holonomy $h_e$, such that our phase space now is $(h_e, L_e) \in so(3) \times SO(3)$.

(iii) Spin network states

The state of a graph is described by the spin network states. Each quanta of space contains one node and three edges, so that the corresponding spin-network basis can be written as a tensor product of three angular momentum states such that their edges are labeled by spin numbers.

The canonical formulation of GR gives three constraints: diffeomorphism constraint, Hamilton constraint and Gauss constraint. The first two give the dynamics of the theory and the third gives the kinematics of the theory. Kinematical LQG then only consider the Gauss constraint which can be written as

$$\hat{L}_{1i} + \hat{L}_{2i} + \hat{L}_{3i} = 0, \quad (2)$$

with $\hat{L} \in su(2)$. Using this constraint, the value for $j_3$ is restricted to

$$|j_1 - j_2| \leq j_3 \leq j_1 + j_2. \quad (3)$$

Spin network basis can be written in the direct product basis, namely $\otimes$ basis as follows:

$$|\psi_{3l}\rangle_{\otimes} = |j_1, m_1, n_1\rangle |j_2, m_2, n_2\rangle |j_3, m_3, n_3\rangle \quad (4)$$

or in the direct sum basis, namely $\oplus$ basis

$$|\psi_{3l}\rangle_{\oplus} = \bigotimes_{i=1}^{3} |j_{1i}, j_{2i}, j_{3i}; j_{12} = j_3, 0, 0; n_i\rangle. \quad (5)$$
Moreover, we can transform $\bigoplus$ basis to $\bigotimes$ basis using Clebsch-Gordon coefficient:

$$|\psi_3 l\rangle \oplus = \sum_{m_1, m_2, m_3} i^{m_1 m_2 m_3} |j_1, m_1, n_1\rangle |j_2, m_2, n_2\rangle |j_3, m_3, n_3\rangle.$$  \hspace{1cm} (6)

The $i^{m_1 m_2 m_3}$ are the symmetric form of the Clebsch-Gordon coefficient and known as Wigner 3$j$-symbols or intertwiner.

3. Length operator

Having defined the basis states for kinematical Hilbert space, now we can define the length operator that will work on this basis. Classical expression for length can be written as

$$\int_c ds = \int_c \sqrt{e^a(t) dx^a(t) dt}.$$  \hspace{1cm} (7)

To define length operator, we promote the triad $e^i_a(x)$ to be an operator

$$\hat{e}^i_a = -ihG\epsilon_{ab}\delta \frac{\delta}{\delta \omega^i_b(x)}.$$  \hspace{1cm} (8)

Since the triad $e^i_a(x)$ is a 1-form, then the regularized version of the operator (8) is given by

$$\hat{e}^i = \int^t_l \hat{e}^i_a dx^a = \int^t_l \frac{dx^a(t)}{dt} \hat{e}^i_a dt.$$  \hspace{1cm} (9)

The action of this operator to the spin-$j$ irreducible matrix representation of the holonomy $D^{(j)}[h_e]$ along an edge $e$ then gives

$$\hat{e}^i D^{(j)}[h_e[\omega]] = i\hbar G \sqrt{j(j+1)} D^{(j)}[h_e(s, 0)],$$  \hspace{1cm} (10)

where $D^{(j)}[h_e] = (h_e|j, m, n)$. If we contract the relation above we get

$$\hat{e}^i \hat{e}^j D^{(j)}[h_e[\omega]] = h^2 G^2 \epsilon_{ab} \int \frac{dx^a(l)}{dl} \frac{\delta D^{(j)}[h_e(1, s)]}{\delta \omega^i_b(l)} \tau_i D^{(j)}[h_e(s, 0)] dl$$

$$= D^{(j)}[h_e(1, s)] \tau^2 D^{(j)}[h_e(s, 0)] h^2 G^2$$

$$= h^2 G^2 j(j+1) D^{(j)}[h_e[\omega]].$$  \hspace{1cm} (11)

Since the length operator is

$$\hat{L} = \sqrt{\hat{e}^i \hat{e}^i},$$  \hspace{1cm} (12)

then the action of it on $D^{(j)}[h_e]$ is given by

$$\hat{L} D^{(j)}[h_e] = hG \sqrt{j(j+1)} D^{(j)}[h_e],$$  \hspace{1cm} (13)

or in the ket state representation $|j, m, n\rangle$

$$\hat{L} |j, m, n\rangle = hG \sqrt{j(j+1)} |j, m, n\rangle.$$  \hspace{1cm} (14)

From (14) we obtain two facts: the length of an edge is quantized and $|j, m, n\rangle$ are eigenstate of length operator. This shows us that the spin-$j$ irreducible representation of the holonomy or the spin-network state is the state of the length of an edge.
4. Triangular inequality

Now consider the triangulation of the foliation $\Sigma$ with spin-labelling on each edge $j_1$, $j_2$ and $j_3$. The state of the three angular momentum coupling can be written as

$$|\psi_{3i}\rangle = \bigotimes_{i=1}^{3} |j_1, j_2, j_3; j_{12} = j_3, 0, 0; n_i\rangle,$$

(15)

where we use gauge invariance condition $j_{12} = j_3$. By using (14) for each edge we get

$$\hat{L}_1^2 |\psi_{3i}\rangle = \hbar^2 G^2 j_1 (j_1 + 1) |\psi_{3i}\rangle = l_1^2 |\psi_{3i}\rangle$$
$$\hat{L}_2^2 |\psi_{3i}\rangle = \hbar^2 G^2 j_2 (j_2 + 1) |\psi_{3i}\rangle = l_2^2 |\psi_{3i}\rangle$$
$$\hat{L}_3^2 |\psi_{3i}\rangle = \hbar^2 G^2 j_3 (j_3 + 1) |\psi_{3i}\rangle = l_3^2 |\psi_{3i}\rangle,$$

(16)

and since $j_{12} = j_3$ then the interval allowed for $j_3$ is

$$|j_1 - j_2| \leq j_{12} = j_3 \leq j_1 + j_2.$$  

(17)

By assuming $j_1 > j_2$ we get

$$(j_1 - j_2)(j_1 - j_2 + 1) \leq l_3^2 \leq (j_1 + j_2)(j_1 + j_2 + 1).$$  

(18)

Now we check the relation of the three square of length above whether they satisfy the classical triangular inequality. Firstly, we check the relation for the maximum value of $j_3$ in the interval (17), if it is satisfied by the maximum value then it will be satisfied by the entire interval. For $j_{3\text{max}} = j_1 + j_2$ then

$$|l_3|^2 = \hbar^2 G^2 j_{3\text{max}} (j_{3\text{max}} + 1)$$
$$= \hbar^2 G^2 (j_1 + j_2)(j_1 + j_2 + 1)$$
$$= \hbar^2 G^2 [j_1(j_1 + 1) + j_2(j_2 + 1) + 2j_1j_2].$$

(19)

From (19) and (16) we conclude that

$$|l_1|^2 + |l_2|^2 + 2|l_1||l_2| > |l_3|^2,$$

(20)

or equivalently

$$|l_1| + |l_2| > |l_3|.$$  

The result shows that the triangular inequality relation is satisfied by the maximum value allowed for $j_3$, so that it is also satisfied by the entire interval. Classical triangular inequality has a degenerate condition in which the area is zero or $|l_1| + |l_2| = |l_3|$. This situation occurs when the angle between $|l_1|$ and $|l_2|$ is $180^\circ$. Now we check for quantum triangle with $j_3$ written as $j_3 = j_1 + j_2 - n$ and $0 \leq n \leq 2j_2$ is an integer. For $j_3 = j_1 + j_2 - n$ then

$$|l_3|^2 = \hbar^2 G^2 j_3 (j_3 + 1)$$
$$= |l_1|^2 + |l_2|^2 + 2j_1j_2 - 2n(j_1 + j_2) + n(n - 1).$$

(21)
If we set $|l_1|^2 + |l_2|^2 + 2|l_1||l_2| = |l_3|^2$, we obtain

\[-2j_1j_2 + 2n(j_1 + j_2) + n - n^2 + 2\sqrt{j_1j_2(j_1 + 1)(j_2 + 1)} = 0\]

\[n^2 - (2j_1 + 2j_2 + 1)n + 2[j_1j_2 - \sqrt{j_1j_2(j_1 + 1)(j_2 + 1)}] = 0. \quad (22)\]

By solving the quadratic equation above we obtain:

\[n(j_1, j_2) = \frac{1}{2}[(2j_1 + 2j_2 + 1) \pm \sqrt{1 + 4j_1(j_1 + 1) + 4j_2(j_2 + 1) + 8\alpha(j_1, j_2)}]\]

\[\alpha(j_1, j_2) = \sqrt{j_1j_2(j_1 + 1)(j_2 + 1)}. \quad (23)\]

By substituting $j_i = \frac{m_i}{2}$ with $m_i = 1, 2, ..., m$, we get

\[n(m_1, m_2) = \frac{1}{2}[(m_1 + m_2 + 1) \pm \sqrt{1 + m_1(m_1 + 2) + m_2(m_2 + 2) + 2\alpha(m_1, m_2)}]\]

\[\alpha(m_1, m_2) = \sqrt{m_1m_2(m_1 + 1)(m_2 + 2)}. \quad (25)\]

The condition for equality is achieved if $n$ is integer. Our numerical calculation reveal that no integer exist up to $m_1, m_2 = 100$. This gives us a hint that the triangular inequality for quantum triangle is:

\[|l_1| + |l_2| > |l_3|. \quad (27)\]

This is due to the fact that in quantum triangle, once we set the spin value of $j_1$ and $j_2$, the spin value of $j_3$ is restricted. This situation also restrict the possible value of length $l_3$, as we can see in (18).

5. Conclusion

We have constructed the length operator in 3d Euclidean in a similar way the length operator constructed in 3d Lorentzian as described in [7]. The difference between the two is that in [7] one has timelike and spacelike curve, while here we only have spacelike curve since 3d Euclidean metric is positive semi-definite. The calculation shows that the length operator is the Casimir of the spin network states in 3d Euclidean in a similar way the area operator is the Casimir of the spin network states in 4d Lorentzian.

We consider a quanta of space in our calculation to check the triangular inequality. Our result above shows that even at the level of quantum geometry, triangular inequality is still valid for the Euclidean manifold. The difference is that, the inequality in quantum triangle does not contain the degenerate condition, namely, the condition where $|l_1| + |l_2| = |l_3|$. This is due
to the fact that in classical triangle, once we fix the length of the two segments \( l_1 \) and \( l_2 \), the allowed value of the length for the third segment is

\[
0 \leq |l_3|^2 \leq |l_1| + |l_2| + 2|l_1||l_2|.
\]  

(28)

While for quantum triangle the allowed value is

\[
|l_1|^2 + |l_2|^2 - 2j_2(j_1 + 1) \leq |l_3|^2 \leq |l_1|^2 + |l_2|^2 + 2j_1j_2,
\]  

(29)

which is more stricter compared to the classical triangle case. The upper bound for \( |l_3|^2 \) in (29) is smaller than in (28). Because the upper bound for classical triangle is the degenerate one, then we conclude the degenerate quantum triangle does not exists. Another difference is that the interval in (28) is continuous, while in (29) is discrete, with gap

\[
|l_{3(n-1)}|^2 - |l_{3(n)}|^2 = 2(j_1 + j_2 + 1 - n).
\]  

(30)

\( n \) is defined as \( n = j_1 + j_2 - j_3 \) and has the interval \( 0 \leq n \leq 2j_2 \) for \( j_1 > j_2 \). We still working on the validity of this inequality for the 4d Lorentzian case.

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