Solving polynomial equations and applications

Simon Telen

Workshop on Solving polynomial equations and applications
October 5, 2022
## Workshop program

https://event.cwi.nl/semester-programs/2022/PolOpt/indexW2.html
This morning

1. Systems of polynomial equations
2. Applications
3. Number of solutions
4. Computational methods
5. A case study
Notes

Solving Polynomial Equations and Applications

Simon Telen

September 3, 2022

Abstract

These notes accompany an introductory lecture given by the author at the workshop on solving polynomial equations & applications at CWI Amsterdam in the context of the 2022 fall semester programme on polynomial optimization & applications. We introduce systems of polynomial equations and the main approaches for solving them. We also discuss applications and solution counts. The theory is illustrated by many examples.

https://simontelen.webnode.page/l/workshop-on-solving-polynomial-equations-and-applications/
1. Systems of polynomial equations
The polynomial ring

... with coefficients in a field $K$ is denoted by

$$ R = K[x_1, \ldots, x_n] $$

Its elements are of the form

$$ f(x) = \sum_{\alpha \in \mathbb{N}^n} c_{(\alpha_1, \ldots, \alpha_n)} x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sum_{\alpha \in \mathbb{N}^n} c_\alpha x^\alpha $$

**Example:** the Motzkin polynomial $f(x, y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1$

is an important nonnegative polynomial on $\mathbb{R}^2$

**Example:** $f(x, y, z) = 81(x^3 + y^3 + z^3) - 189(x^2y + x^2z + y^2x + y^2z + xz^2 + yz^2) + 54xyz + 126(xy + xz + yz) - 9(x^2 + y^2 + z^2) - 9(x + y + z) + 1$
Affine algebraic varieties

Find $x \in \overline{K}^n$ such that

$$f_1(x) = f_2(x) = \cdots = f_s(x) = 0, \quad f_i \in K[x_1, \ldots, x_n]$$

This is a polynomial system whose set of solutions is the algebraic variety

$$V(f_1, \ldots, f_s) = \{x \in \overline{K}^n \mid f_1(x) = \cdots = f_s(x) = 0\}$$

Example

$$f = -7x - 9y - 10x^2 + 17xy + 10y^2 + 16x^2y - 17xy^2,$$

$$g = 2x - 5y + 5x^2 + 5xy + 5y^2 - 6x^2y - 6xy^2.$$ 

$$f(x, y) = 0$$

$$g(x, y) = 0$$

$$f(x, y) = g(x, y) = 0$$
Univariate and linear systems

**Example:** the variety $V(f)$ of $f = c_0 + c_1 x + \cdots + c_d x^d \in K[x]$ consists of the roots of $f$

$$C_f = \begin{pmatrix} 1 & -c_0/c_d \\ \vdots & \ddots \\ 1 & -c_{d-1}/c_d \end{pmatrix}$$

$C_f \cdot \mathbf{v} = \lambda \mathbf{v}$

**Example:** the variety $V(f_1, \ldots, f_s)$ of $f_i = \sum_{i=1}^{n} a_{ij} x_j - b_i \in K[x_1, \ldots, x_n]$ is an affine linear space

$A \mathbf{x} = \mathbf{b}$

These are problems from **linear algebra**
The Clebsch surface

\[ f(x, y, z) = 81(x^3 + y^3 + z^3) - 189(x^2y + x^2z + y^2x + y^2z + xz^2 + yz^2) \\
+ 54xyz + 126(xy + xz + yz) - 9(x^2 + y^2 + z^2) - 9(x + y + z) + 1 \]

More algebraic surfaces by H. Hauser:

https://homepage.univie.ac.at/herwig.hauser/bildergalerie/gallery.html
Chebyshev curves

\[ T_0(x) = 1, \quad T_1(x) = x, \quad T_{k+1}(x) = 2x \cdot T_k(x) - T_{k-1}(x) \]

\[ f_i = \sum_{|\alpha|<d} c_{\alpha,i} T_{\alpha_1}(x_1) \cdot T_{\alpha_2}(x_2) \cdot \cdots \cdot T_{\alpha_n}(x_n) \]

\[ f(x, y) = 0 \]
\[ g(x, y) = 0 \]
\[ f(x, y) = g(x, y) = 0 \]
Solutions over the ground field

Example: \( f(x, y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1 = 0 \)

\[ V_\mathbb{R}(f) = \{(1,1), (1, -1), (-1,1), (-1, -1)\} \]

\[ V_\mathbb{C}(f) = \text{a complex curve with } 4 \text{ real points} \]

\[ f(x, y) = 0 \implies x^2 - 1 = y^2 - 1 = 0 \]

Example:

\[ f = -7x - 9y - 10x^2 + 17xy + 10y^2 + 16x^2y - 17xy^2, \]
\[ g = 2x - 5y + 5x^2 + 5xy + 5y^2 - 6x^2y - 6xy^2. \]
On the field $K$

- Homotopy continuation, particle physics
- Optimization, chemical reaction networks, robotics
- Grassmannians, discriminants, resultants, number theory

- Cryptography
- Tropical geometry
2. Applications
Polynomial optimization

\[ K = \mathbb{R} \]

\[
\begin{align*}
\min_{x \in \mathbb{R}^k} & \quad g(x_1, \ldots, x_k), \\
\text{subject to} & \quad h_1(x_1, \ldots, x_k) = \cdots = h_\ell(x_1, \ldots, x_k) = 0, \\
& \quad h_{\ell+1} \geq 0, \ldots, h_{\ell+\ell'} \geq 0 \quad \text{SDP}
\end{align*}
\]

Lagrangian \( L = g - \lambda_1 h_1 - \cdots - \lambda_\ell h_\ell \)

\[
\frac{\partial L}{\partial x_1} = \cdots = \frac{\partial L}{\partial x_k} = h_1 = \cdots = h_\ell = 0.
\]

\( s = k + \ell \) equations in \( n = k + \ell \) variables \( x_1, \ldots, x_k, \lambda_1, \ldots, \lambda_\ell \)
Find the point on a variety that minimizes the Euclidean distance to a given point

\[ g(x_1, \ldots, x_k) = \| x - u \|^2 \]
Chemical reaction networks

Law of **mass action**: time derivatives of the concentrations of species in a chemical reaction network are polynomials

\[
A + B
\]

\[
\begin{align*}
D & \quad \xrightarrow{\kappa_{23}} \quad C \\
& \quad \xleftarrow{\kappa_{21}} \quad C \\
& \quad \xleftarrow{\kappa_{12}} \quad D
\end{align*}
\]

\[
f_A = \frac{dx_A}{dt} = -\kappa_{12} x_A x_B + \kappa_{21} x_C + \kappa_{31} x_D,
\]

\[
f_B = \frac{dx_B}{dt} = -\kappa_{12} x_A x_B + \kappa_{21} x_C + \kappa_{31} x_D,
\]

\[
f_C = \frac{dx_C}{dt} = \kappa_{12} x_A x_B - \kappa_{21} x_C - \kappa_{23} x_C,
\]

\[
f_D = \frac{dx_D}{dt} = \kappa_{23} x_C - \kappa_{31} x_D.
\]

Stoichiometric compatibility class: \( x_a + x_b + x_c + x_d = \text{const} \)

For which parameter values do we have multistationarity?
Robotics

\[ x^2 + y^2 - L_1^2 = (a - x)^2 + (b - y)^2 - L_2^2 = 0 \]

Wampler, C., & Sommese, A. (2011). Numerical algebraic geometry and algebraic kinematics. *Acta Numerica*, 20, 469-567. doi:10.1017/S0962492911000067
Alt’s problem (1923):

Determine all four-bar linkages whose coupler curve passes through 9 fixed points

Morgan - Sommese - Wampler (1992):

This can be formulated as a polynomial system of 8 equations in 8 unknowns with at least 8652 solutions

Jonathan D. Hauenstein & Martin Helmer (2022) Probabilistic Saturations and Alt's Problem, Experimental Mathematics, 31:3, 975-987, DOI: 10.1080/10586458.2020.1740835
3. Number of solutions
The univariate case

\[ F(d) = \{ a_0 + a_1 x + \cdots + a_d x^d \mid (a_0, \ldots, a_d) \in \mathbb{C}^{d+1} \} \cong \mathbb{C}^{d+1} \]

A polynomial \( f \in F(d) \) has \( d \) roots if it lies outside the variety \( V(\Delta_d) \)

\[ \Delta_1 = a_1, \]
\[ \Delta_2 = a_2 \cdot (a_1^2 - 4 a_0 a_2), \]
\[ \Delta_3 = a_3 \cdot (a_1^2 a_2^2 - 4 a_0 a_2^3 - 4 a_1^3 a_3 + 18 a_0 a_1 a_2 a_3 - 27 a_0^2 a_3^2), \]
\[ \Delta_4 = a_4 \cdot (a_1^2 a_2^3 a_3 - 4 a_0 a_2^3 a_3 - 4 a_1^3 a_3^2 + 18 a_0 a_1 a_2 a_3^2 + \cdots + 256 a_0^3 a_4^3) \]
Bézout's theorem

... is a generalisation for $n$ polynomials in $n$ variables of fixed degrees

$$R = K[x_1, \ldots, x_n], \text{ with } K \text{ algebraically closed}$$

$$R_d = \{ f \in R : \deg(f) \leq d \}$$

$$\mathcal{F}(d_1, \ldots, d_n) = R_{d_1} \times \cdots \times R_{d_n} \cong K^D$$

**Theorem 3.1** (Bézout). For any $F = (f_1, \ldots, f_n) \in \mathcal{F}(d_1, \ldots, d_n)$, the number of isolated solutions of $f_1 = \cdots = f_n = 0$, i.e., the number of isolated points in $V_K(F)$, is at most $d_1 \cdots d_n$. Moreover, there exists a proper subvariety $\nabla_{d_1,\ldots,d_n} \subsetneq K^D$ such that, when $F \in \mathcal{F}(d_1, \ldots, d_n) \setminus \nabla_{d_1,\ldots,d_n}$, the variety $V_K(F)$ consists of precisely $d_1 \cdots d_n$ isolated points.

Many systems from applications lie in $\nabla_{d_1,\ldots,d_n}$.
Bézout's theorem

\[ f = -7x - 9y - 10x^2 + 17xy + 10y^2 + 16x^2y - 17xy^2, \]
\[ g = 2x - 5y + 5x^2 + 5xy + 5y^2 - 6x^2y - 6xy^2. \]

\((f, g) \in \mathcal{F}(3,3)\)

\[ 7 < 3 \cdot 3 = 9 \]

The monomials \(x^3, y^3\) do not appear in \(f, g\)

\[ \text{supp} \left( \sum_{\alpha} c_\alpha x^\alpha \right) = \{ \alpha : c_\alpha \neq 0 \} \subset \mathbb{N}^n. \]
Kushnirenko’s theorem

\[ \mathcal{A} \subset \mathbb{N}^n \ \text{finite subset} \]

\[ \mathcal{F}(\mathcal{A}) = \{ (f_1, \ldots, f_n) \in \mathbb{R}^n : \text{supp}(f_i) \subset \mathcal{A}, \ i = 1, \ldots, n \} \cong K^{n \cdot |\mathcal{A}|} \]

\[ \text{Conv}(\mathcal{A}) = \left\{ \sum_{\alpha \in \mathcal{A}} \lambda_\alpha \cdot \alpha : \lambda_\alpha \geq 0, \sum_{\alpha \in \mathcal{A}} \lambda_\alpha = 1 \right\} \subset \mathbb{R}^n \]

Example:

\[ f = -7x - 9y - 10x^2 + 17xy + 10y^2 + 16x^2y - 17xy^2, \]
\[ g = 2x - 5y + 5x^2 + 5xy + 5y^2 - 6x^2y - 6xy^2. \]

Theorem 3.3 (Kushnirenko). For any \( F = (f_1, \ldots, f_n) \in \mathcal{F}(\mathcal{A}) \), the number of isolated solutions of \( f_1 = \cdots = f_n = 0 \) in \( (K \setminus \{0\})^n \), i.e., the number of isolated points in \( V_K(F) \cap (K \setminus \{0\})^n \), is at most \( \text{vol}(\mathcal{A}) \). Moreover, there exists a proper subvariety \( \nabla_\mathcal{A} \subseteq K^{n \cdot |\mathcal{A}|} \) such that, when \( F \in \mathcal{F}(\mathcal{A}) \setminus \nabla_\mathcal{A} \), \( V_K(F) \cap (K \setminus \{0\})^n \) consists of precisely \( \text{vol}(\mathcal{A}) \) isolated points.
Kushnirenko’s theorem

Kushnirenko recovers Bézout when
\[ d_1 = \cdots = d_n = d \]
and
\[ \mathcal{A} = \{ \alpha \in \mathbb{N}^n : \deg(x^\alpha) \leq d \} \]

Kushnirenko’s number is the degree of the embedded projective toric variety \( X_{\mathcal{A}} \)

Bernstein’s theorem (BKK) allows different supports:

12 solutions
4. Computational methods
Two approaches

We work over an algebraically closed field $K$

We assume $f_1(x) = f_2(x) = \cdots = f_s(x) = 0$ has $\delta < \infty$ solutions

| Normal form methods                                                                 | Homotopy continuation methods                                      |
|------------------------------------------------------------------------------------|-------------------------------------------------------------------|
| Reduce the problem to the univariate case using algebraic manipulations             | Solve a simpler system first, then deform continuously into the target system |
| Linear algebra                                                                      | Ordinary differential equations                                   |
| Any field $s \geq n$                                                                | $K = \mathbb{C}$                                                   |
| $\delta \lesssim 10,000$                                                            | $\delta \lesssim 1,000,000$                                       |
Normal form methods

Our polynomials generate an ideal

\[ I = \{g_1f_1 + \cdots + g_sf_s : g_i \in R\} \subset R = K[x_1, \ldots, x_n] \]

We denote the solutions by \( V_K(I) = V_K(f_1, \ldots, f_s) = \{z_1, \ldots, z_\delta\} \)

\( R/I = \) quotient ring from equivalence relation \( f \sim g \iff f - g \in I \)

**Theorem:** if the ideal \( I \) is radical, the evaluation map

\[
\text{ev}([f]) = (f(z_1), \ldots, f(z_\delta))
\]

is an isomorphism of \( K \)-vector spaces \( R/I \cong K^\delta \)
**Multiplication maps**

\[ M_g : R/I \rightarrow R/I, \quad [f] \mapsto [f \cdot g] \]

Once we fix a basis of \( R/I \), these are \( \delta \times \delta \) matrices

\[ \Lambda_g = \begin{pmatrix} g(z_1) \\ g(z_2) \\ \vdots \\ g(z_\delta) \end{pmatrix} \]

**Theorem:** If \( I \) is radical, the eigenvalues of \( M_g^T \) are \( g(z_1), \ldots, g(z_\delta) \), with corresponding eigenvectors

\[ [f] \mapsto f(z_i) \]
A familiar example

\[ f = c_0 + c_1 x + \cdots + c_d x^d \in K[x], \quad I = \langle f \rangle \]

The quotient ring has basis \([1], [x], \ldots, [x^{d-1}]\)

We construct ‘multiplication with \(g = x\)’

\[
x \cdot 1 = x \\
x \cdot x = x^2 \\
x \cdot x^2 = x^3 \\
\vdots \\
x \cdot x^{d-2} = x^{d-1} \\
x \cdot x^{d-1} = -\frac{1}{c_d} (c_0 + c_1 x + \cdots + c_{d-1} x^{d-1}) = \text{the normal form of} \ x^d
\]

\[
M_x = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & -c_0/c_d \\
1 & 0 & 0 & \cdots & 0 & -c_1/c_d \\
0 & 1 & 0 & \cdots & 0 & -c_2/c_d \\
0 & 0 & 1 & \cdots & 0 & -c_3/c_d \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & -c_{d-1}/c_d
\end{pmatrix}
\]
Normal forms

(A) Compute multiplication matrices $M_g$ and

(B) extract the coordinates of $z_i$ from their eigenvectors or eigenvalues

Let $[b_1], \ldots, [b_δ]$ be a basis of $R/I$ and consider the map $\mathcal{N} : R \to B$

$$\mathcal{N}(f) = c_1(f) \cdot b_1 + \cdots + c_δ(f) \cdot b_δ$$

with $c_i(f) \in K$ such that $f - \mathcal{N}(f) \in I$

This map is the identity on $B$, and its kernel is $I$ —> NORMAL FORM

Normal forms compute multiplication matrices: $(M_g)_{ij} = c_i(g \cdot b_j)$
Macaulay matrices

\[ f = x^2 + y^2 - 2, \quad g = 3x^2 - y^2 - 2, \quad I = \langle f, g \rangle \subset \mathbb{Q}[x, y] \]

\[ b_1 = 1, \quad b_2 = x, \quad b_3 = y, \quad b_4 = xy \]

\[ M = \begin{pmatrix}
    \begin{array}{cccc|ccccc}
    f & x^3 & x^2 y & x y^2 & y^3 & x^2 & y^2 & 1 & x & y & xy \\
    x f & 1 & 1 & 1 & 1 & -2 & -2 & -2 & -2 \\
    y f & 1 & 1 & 1 & 1 & -2 & -2 & -2 & -2 \\
    g & 3 & -1 & 3 & -1 & -2 & -2 & -2 & -2 \\
    x g & 3 & -1 & 3 & -1 & -2 & -2 & -2 & -2 \\
    y g & 3 & -1 & 3 & -1 & -2 & -2 & -2 & -2 \\
    \end{array}
    \end{pmatrix} \]
Macaulay matrices

\[ f = x^2 + y^2 - 2, \quad g = 3x^2 - y^2 - 2, \quad I = \langle f, g \rangle \subset \mathbb{Q}[x, y] \]

\[ b_1 = 1, \quad b_2 = x, \quad b_3 = y, \quad b_4 = xy \]

\[
\tilde{M} = \begin{bmatrix}
    x^3 - x & 1 & 1 & 1 & -1 \\
    x^2y - y & 1 & 1 & 1 & -1 \\
    xy^2 - x & 1 & 1 & 1 & -1 \\
    y^3 - y & 1 & 1 & 1 & -1 \\
    x^2 - 1 & 1 & 1 & 1 & -1 \\
    y^2 - 1 & 1 & 1 & 1 & -1 \\
\end{bmatrix}
\]

\[ \mathcal{N}(x^2y) = y \]
Macaulay matrices

\[
\tilde{M} = \begin{bmatrix}
x^3 & x^2 y & x y^2 & y^3 & x^2 & y^2 & 1 & x & y & x y \\
x^3 - x & 1 & & & & & -1 & & & -1 \\
x^2 y - y & 1 & & & & & & -1 & & -1 \\
x y^2 - x & 1 & & & & & & & -1 & -1 \\
y^3 - y & & & & & & & & & -1 \\
x^2 - 1 & & & & & & & & & -1 \\
y^2 - 1 & & & & & & & & & -1 \\
\end{bmatrix}
\]

\[
M_x = \begin{bmatrix}
[x] & [x^2] & [x y] & [x^2 y] \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad M_y = \begin{bmatrix}
[y] & [x y] & [y^2] & [x y^2] \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Euclidean division

\[ n = 1 \]

\[ f = c_0 + c_1 x + \cdots + c_d x^d \in K[x], \quad I = \langle f \rangle \]

\[ g = q \cdot f + r \quad \text{deg}(r) < \text{deg}(f) \]

\[ \implies r = \mathcal{N}(g) \]

\[ n > 1 \]

\[ I = \langle f_1, \ldots, f_s \rangle \subset K[x_1, \ldots, x_n] \]

fix a monomial order \( \preceq \)

\[ g = q_1 f_1 + \cdots + q_s f_s + r \quad \text{LT}(f_i) \nmid \text{LT}(r) \]

\[ \implies r = \mathcal{N}(g) \]

... unless \( f_1, \ldots, f_s \) is a Gröbner basis of \( I \)
Gröbner bases and elimination

\[ f_1, \ldots, f_s \] is a \textbf{Gröbner basis} of \( I \)

if \( \langle \text{LT}(f_1), \ldots, \text{LT}(f_s) \rangle = \langle \text{LT}(f) : f \in I \rangle \)

Taking remainder upon division by a Gröbner basis is a normal form

Computing a Gröbner basis can be seen as Gaussian elimination on a Macaulay matrix

With the right monomial order, Gröbner bases can be used to eliminate variables

**Example:** intersect the Clebsch surface with the sphere \( g = x^2 + y^2 + z^2 - 1 = 0 \), then project onto the xy-plane
Homotopy continuation

Start system:

\[
\begin{align*}
  x^2 - 1 &= 0 \\
  y^2 - 1 &= 0
\end{align*}
\]

Target system:

\[
\begin{align*}
  x^2 + y^2 - 9 &= 0 \\
  0.25x^2 + 2y^2 - 6 &= 0
\end{align*}
\]

Idea: solve the target system by deforming the start system in a continuous manner and by ‘tracking’ the solutions.
Homotopy continuation
Homotopy continuation

\[ G(x_1, \ldots, x_n) = (g_1, \ldots, g_n) \longrightarrow F(x_1, \ldots, x_n) = (f_1, \ldots, f_n) \]

\[ H(x_1, \ldots, x_n; t) = (1 - t) \cdot G(x_1, \ldots, x_n) + t \cdot F(x_1, \ldots, x_n) \]

Example: \( G(x) = x^{12} - 1, \quad F(x) = (x - 1) \cdot (x - 2) \cdot \cdots \cdot (x - 12) \)
Homotopy continuation

A solution path satisfies $H(x(t); t) = 0$. This leads to Davidenko’s ODE:

$$\frac{dH(x(t), t)}{dt} = J_x \cdot \dot{x}(t) + \frac{\partial H}{\partial t}(x(t), t) = 0,$$

with $J_x = \left( \frac{\partial h_j}{\partial x_i} \right)_{j,i}$.

We compute a discretization of these paths using predict-correct routines.
Implementations

| Normal form methods | Homotopy continuation methods |
|---------------------|-------------------------------|
| Macaulay2           | Homotopy Continuation.jl      |
| SINGULAR            |                               |
| MAGMA               | NAG4M2 -- NumericalAlgebraicGeometry |
| The msolve library  |                               |
| Mathematica         | Hom4PS-3                      |
| OSCAR               |                               |

Maple
5. A case study: 27 lines on the Clebsch surface
The equations

Every smooth cubic surface over an algebraically closed field contains exactly 27 lines (Cayley, Salmon, 1849)

\[ f(x, y, z) = 81(x^3 + y^3 + z^3) - 189(x^2y + x^2z + y^2x + y^2z + xz^2 + yz^2) \]
\[ + 54xyz + 126(xy + xz + yz) - 9(x^2 + y^2 + z^2) - 9(x + y + z) + 1 \]

\[ L = \{(a_1 + b_1 \cdot t, a_2 + b_2 \cdot t, a_3 + b_3 \cdot t) : t \in \mathbb{C}\} \]

\[ f|_L = f_1(a, b) \cdot t^3 + f_2(a, b) \cdot t^2 + f_3(a, b) \cdot t + f_4(a, b) \]

\[ f_1(a, b) = f_2(a, b) = f_3(a, b) = f_4(a, b) = 0 \]

\[ s = 4, n = 6 \]
The equations

Every smooth cubic surface over an algebraically closed field contains exactly 27 lines (Cayley, Salmon, 1849)

\[ f_1(a, b) = f_2(a, b) = f_3(a, b) = f_4(a, b) = 0 \]

The space of lines in three space has dimension 4, not 6

\[ a_3 = -(7 + a_1 + 3a_2)/5, \quad b_3 = -(11 + 3b_1 + 5b_2)/7 \]

We solve a system of 4 equations in 4 unknowns with 27 real solutions
Gröbner bases in Maple

```maple
> f := 81*(x^3 + y^3 + z^3) - 189*(x^2*y + x^2*z + x*y^2 + x*z^2 + y^2*z + y*z^2) + 54*x*y*z + z^2) - 9*(x + y + z) + 1:

> f := expand(subs({x = t*b[1] + a[1], y = t*b[2] + a[2], z = t*b[3] + a[3]}, f)):

> ff := coeffs(f, t):

> with(Groebner): GB := Basis({ff}, grlex(a[1], a[2], b[1], b[2])):

> ns, rv := NormalSet(GB, grlex(a[1], a[2], b[1], b[2])):

> Ma1 := MultiplicationMatrix(a[1], ns, rv, GB, grlex(a[1], a[2], b[1], b[2]));
```

```
0 0
0 0
0 0
0 0
0 0
0 0
0 0
0 0

11737/4992 2057/416
1306460381129659465245061281859651164265847/838725858593765076524006727331634580633120
17778387427951441822273723829083308/199696632998515494410477792221817
0
```

44
Gröbner bases in The msolve library

In [32]:
1. using Oscar
2. R, (a₁, a₂, b₁, b₂) = PolynomialRing(QQ, ["a₁", "a₂", "b₁", "b₂"])
3. I = ideal(R, [-189*b₂*b₁^2 - 189*b₂^2*b₁ + 27*(11 + 3*b₁ + 5*b₂)*b₁^2 + 27*(11 + 3*b₁ + 211266/245 + (-143748/245)*a₁ + (-202554/245)*a₂ + (-367488/245)*b₁ + (-444
4. -135432/175 + (-218592/175)*a₁ + (-295416/175)*a₂ + (-134208/175)*b₁ + (-158
5. -28288/125 + (-81792/125)*a₁ + (-106416/125)*a₂ + (-69696/125)*b₁ + (-393
6. A, B = msolve(I);

In [33]:
1. B

Out[33]:
27-element Vector{Vector{fmpq}}:

[169020080304350868662270940501/507062400912917605986812821504, 2704321280486889389859
6335048017/507062400912917605986812821504, 1/507062400912917605986812821504, 27888312
05021046832927470518267/507062400912917605986812821504, 16391265005705735037
4175801344, 0, 3486039150627630854115933814779/633825300114114700748351602688]

[2863311531/8589934592, 28633115307/8589934592, 0, 11/2]

[448959587580831246363415718571/158456325028528675187087900672, 211275100038038233582783
867565/633825300114114700748351602688, 1743019575313815427057966907393/63382530011411470
0748351602688, 3/633825300114114700748351602688]

[343322037561812129572023784791/158456325028528675187087900672, 1/792281625142643375935
(2) 27 27-지요대장리터리 오토소리 나오는 유망한 음악해시 수학 확정결과 45 27-지요대장리터리 오토소리 나오는 유망한 음악해시 수학 확정결과

In [34]:
1. [convert.(Float64, convert.(Rational{BigInt}, b)) for b in B]

Out[34]:
27-element Vector{Vector{Float64}}:

[0.3333333333333333, 5.333333333333333, 1.9721522630525295e-31, 5.5]

[1.5777218104420236e-30, 4.333333333333333, 0.0, 5.5]

[-0.33333333333721384, 3.3333333333721384, 0.0, 5.5]

[2.833333333333333, 0.333333333333333, 2.75, 4.733165431326071e-30]

[2.166666666666666, 1.26217748353619e-29, 2.75, 1.8932641672530423e-29]

[1.5, -0.333333333333333, 2.75, -9.466330862652142e-30]

[5.333333333333333, -4.666666666666667, 5.5, -5.5]

[4.0, -3.666666666666667, 5.5, -5.5]
Homotopy continuation in

```
1 INPUT
2 variable_group a1, a2, b1, b2;
3 function f1, f2, f3, f4;
4
5 a3 = -(7+a1+3*a2)/5;
6 b3 = -(11+3*b1+5*b2)/7;
7 f1 = -189*b1*b3^2 - 189*b1^2*b3 - 189*b2*b3^2 - 189*b2^2*b3;
8 f2 = 126*b1*b3 + 243*b1^2*a1 - 189*b1^2*a2;
9 f3 = -9*b1 - 9*b2 - 9*b3 - 18*b1*a1 + 243;
10 f4 = 1 - 9*a1 - 9*a2 - 9*a3 + 126*a1*a2 -
11
12 END;
```

Tracking path 0 of 81
Tracking path 20 of 81
Tracking path 40 of 81
Tracking path 60 of 81
Tracking path 80 of 81

Finite Solution Summary

NOTE: nonsingular vs singular is based on condition number and identical endpoints

|                  | Number of real solns | Number of non-real solns | Total |
|------------------|-----------------------|--------------------------|-------|
| Non-singular     | 27                    | 0                        | 27    |
| Singular         | 0                     | 0                        | 0     |
| Total            | 27                    | 0                        | 27    |

Finite Multiplicity Summary

| Multiplicity | Number of real solns | Number of non-real solns |
|--------------|-----------------------|--------------------------|
| 1            | 27                    | 0                        |
Homotopy continuation in

In [38]:

```python
using HomotopyContinuation
@var x y z t a[1:3] b[1:3]
f = 81*(x^3 + y^3 + z^3) - 189*(x^2*y + x^2*z + x*y^2 + x*z^2 + y^2*z + y*z^2) + 54*x*y*
fab = subs(f, [x;y;z] => a+t*b)
E, C = exponents_coefficients(fab,[t])
F = subs(C,[a[3];b[3]]) => [-7*a[1]+3*a[2])/5; -(11+3*b[1]+5*b[2])/7])
R = solve(F)
```

Out[38]: Result with 27 solutions
---------------------------
- 45 paths tracked
- 27 non-singular solutions (27 real)
- random_seed: 0xe9946777
- start_system: :polyhedral

- Find the **Eckardt points** of the Clebsch surface

- How many lines are contained in a smooth quintic threefold?
Thank you!