RIGIDITY OF COMMUTING AFFINE ACTIONS ON REFLEXIVE
BANACH SPACES

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ABSTRACT. We give a simple argument to show that if $a$ is an affine isometric action
of a product $G \times H$ of topological groups on a reflexive Banach space $X$ with linear
part $\pi$, then either $\pi(H)$ fixes a unit vector or $a|_{\pi}$ almost fixes a point on $X$.

It follows that any affine isometric action of an abelian group on a reflexive Banach
space $X$, whose linear part fixes no unit vectors, almost fixes points on $X$.

1. LINEAR REPRESENTATIONS AND COCYCLES

When $X$ is a vector space, the group of bijective affine transformations of $X$, $\text{Aff}(X)$, can be decomposed as a semidirect product
\[
\text{Aff}(X) = GL(X) \ltimes X,
\]
with respect to the natural action of $GL(X)$ on $X$. The product in $GL(X) \ltimes X$ is then simply $(T, x) \cdot (S, y) = (TS, Ty + x)$, while the corresponding action of $(T, x) \in GL(X) \ltimes X$ on $X$ is given by $(T, x) \cdot y = Ty + x$.

Thus, an action $\alpha$ of a group $G$ by affine transformations of the vector space $X$ can be viewed as a homomorphism of $G$ into $\text{Aff}(X)$, which thus can be split into a linear representation $\pi : G \to GL(X)$, called the linear part of $\alpha$, and an associated cocycle $b : G \to X$ such that the following cocycle identity holds,
\[
b(gf) = \pi(g)b(f) + b(g),
\]
for all $g, f \in G$.

If, moreover, $X$ is a reflexive Banach space and $\pi : G \to GL(X)$ is a fixed isometric linear representation of a topological group $G$ on $X$ that is strongly continuous, i.e., such that for every $x \in X$ the map $g \in G \to gx \in X$ is continuous, we can consider the corresponding vector space $Z^1(G, \pi)$ of continuous cocycles $b : G \to X$ associated to $\pi$. The subspace $B^1(G, \pi) \subseteq Z^1(G, \pi)$ consisting of those cocycles $b$ for which the corresponding affine action $\alpha$ fixes a point on $X$, i.e., for which there is some $x \in X$ such that $b(g) = x - \pi(g)x$ for all $g \in G$, is called the set of coboundaries. Note that if $b$ is a coboundary, then $b(G)$ is a bounded subset of $X$. Conversely, if $b(G)$ is a bounded set, then any orbit $\Theta$ of the corresponding affine action is bounded and so, by reflexivity of $X$, its closed convex hull $C = \overline{\text{conv}(\Theta)}$ is a weakly compact convex set on which $G$ acts by affine isometries. It follows by the Ryll-Nardzewski fixed point theorem [4] that $G$ fixes a point on $C$, meaning that $b$ must be a coboundary.

Every compact set $K \subseteq G$ determines a seminorm $\|\cdot\|_K$ on $Z^1(G, \pi)$ by $\|b\|_K = \sup_{g \in K} \|b(g)\|$ and the family of seminorms thus obtained endows $Z^1(G, \pi)$ with a locally convex topology. With this topology, one sees that a cocycle $b$ belongs to the closure $\overline{B^1(G, \pi)}$ if and only if the corresponding affine action $\alpha = (\pi, b)$ almost has

The author’s research was partially supported by NSF grants DMS 0901405 and DMS 1201295.
fixed points, that is, if for any compact set \( K \subseteq G \) and \( \epsilon > 0 \) there is some \( x = x_{K,\epsilon} \in X \) verifying
\[
\sup_{g \in K} \| (\pi(g)x + b(g)) - x \| = \sup_{g \in K} \| b(g) - (x - \pi(g)x) \| < \epsilon.
\]

If, for any \( K \), we can choose \( x = x_{K,1} \) above to have arbitrarily large norm, we see that the supremum
\[
\sup_{g \in K} \| \pi(g)\frac{x}{\|x\|} - \frac{x}{\|x\|} \| < \frac{\sup_{g \in K} \| b(g) \| + 1}{\|x\|}
\]
can be made arbitrarily small, which means that the linear action \( \pi \) almost has invariant unit vectors. If, on the other hand, for some \( K \) the choice of \( x_{K,1} \) is bounded (but non-empty), then the same bound holds for any compact \( K' \supseteq K \), whereby we find that \( b(G) \subseteq X \) is a bounded set, i.e., that \( b \in B^1(G,\pi) \). Thus, this shows that if \( \pi \) does not almost have invariant unit vectors, the set \( B^1(G,\pi) \) will be closed in \( Z^1(G,\pi) \). In fact, if \( b \in Z^1(G,\pi) \setminus B^1(G,\pi) \) and \( \pi \) does not almost have invariant unit vectors, then for any constant \( \epsilon \) there is a compact set \( K \subseteq G \) such that no vector is \((\alpha(K),\epsilon)\)-invariant, where \( \alpha = (\pi,b) \).

Conversely, a result of A. Guichardet [3], valid for locally compact \( \sigma \)-compact \( G \), states that if \( \pi \) does not have invariant unit vectors and \( B^1(G,\pi) \) is closed in \( Z^1(G,\pi) \), then \( \pi \) does not almost have invariant unit vectors.

We define the first cohomology group of \( G \) with coefficients in \( \pi \) to be the quotient space \( H^1(G,\pi) = Z^1(G,\pi)/B^1(G,\pi) \), while the reduced cohomology group is \( \overline{H^1}(G,\pi) = Z^1(G,\pi)/\overline{B^1}(G,\pi) \).

2. AFFINE ACTIONS OF PRODUCT GROUPS ON REFLEXIVE SPACES

In the following, let \( X \) be a reflexive Banach space, \( G \) and \( H \) be topological groups and \( \pi \) be a strongly continuous linear isometric representation of \( G \times H \) on \( X \). We also fix a cocycle \( b \in Z^1(G \times H,\pi) \) and let \( \alpha \) be the corresponding affine isometric action of \( G \times H \) on \( X \).

**Proposition 1.** One of the following must hold,

1. there is a \( \pi(H) \)-invariant unit vector;
2. for any closed convex \( \alpha(H) \)-invariant sets \( C \subseteq X \), \( \alpha|_C \) almost has fixed points on \( C \).

**Proof.** Assume that there are no \( \pi(H) \)-invariant unit vectors in \( X \). Then, if \( \pi^n : H \to GL(X^n) \) denotes the diagonal representation on \( X^n = (X \oplus \ldots \oplus X)_2 \), \( \pi^n(H) \) has no invariant unit vectors on \( X^n \). By reflexivity, for any \( x \in X^n \), \( C = \text{conv}(\pi^n(H)x) \) is a \( \pi^n(H) \)-invariant weakly compact convex subset of \( X^n \) and thus, by the Ryll-Nardzewski fixed point theorem, \( \pi^n(H) \) fixes a point on \( C \), whereby \( 0 \in \text{conv}(\pi^n(H)x) \).

Therefore, for any \( \epsilon > 0 \) and \((y_1,\ldots,y_n) \in X^n \) there are \( h_i \in H \) and \( \lambda_i > 0 \), \( \sum \lambda_i = 1 \), such that for all \( k = 1,\ldots,n \),
\[
\| \sum \lambda_i \pi(h_i)y_k \| < \epsilon.
\]

In particular, if \( C \subseteq X \) is a closed convex \( \alpha(H) \)-invariant set, \( \epsilon > 0 \) and \( K \subseteq G \) compact, fix \( y \in C \) and find \( g_1,\ldots,g_n \in K \) such that \( (\alpha(g_1)y,\ldots,\alpha(g_n)y) \) is \( \frac{\epsilon}{2} \)-dense in \( \alpha(K)y \). Choose now \( h_i \) and \( \lambda_i \) as above such that
\[
\| \sum \lambda_i \pi(h_i)(y - \alpha(g_k)y) \| < \frac{\epsilon}{2}
\]
for all $k = 1, \ldots, n$. Thus, if $g \in K$, pick $k$ such that $\|a(g)y - a(g_k)y\| < \frac{\epsilon}{2}$. Then, since $\|\sum_i \lambda_i \pi(h_i)\| < 1$,
\[
\|\sum_i \lambda_i a(h_i)y - a(g)\| = \left\| \sum_i \lambda_i a(h_i)y - \left( \sum_i \lambda_i a(g)\|a(h_i)y\| \right) \right\| \\
= \left\| \sum_i \lambda_i a(h_i)y - \left( \sum_i \lambda_i a(h_i)a(g)\right) \right\| \\
< \sum_i \lambda_i \pi(h_i)(y - a(g)y) \\
< \sum_i \lambda_i \pi(h_i)(y - a(g_k)y) + \frac{\epsilon}{2} \\
< \epsilon.
\]

In other words, the point $\sum_i \lambda_i a(h_i)y \in C$ is $(a(K), \epsilon)$-invariant. \hfill $\square$

**Corollary 2.** Let $\pi$ be a strongly continuous isometric linear representation of an abelian topological group $G$ on a reflexive Banach space $X$ and suppose that $\pi(G)$ has no fixed unit vectors. Then $H^1(G, \pi) = 0$, i.e., any affine isometric action with linear part $\pi$ almost has fixed points on $X$.

**Proof.** It suffices to consider the linear representation of $G \times G$ given by $\pi$ separately on the first and second factor and then apply Proposition 1. \hfill $\square$

Let us also note that Corollary 2 fails for more general Banach spaces, e.g., for $\ell_1$. To see this, let $\pi$ denote the left regular representation of $Z$ on $\ell_1(Z)$ and let $b \in Z^1(Z, \pi)$ be given by $b(n) = e_0 + e_1 + \ldots + e_{n-1}$. Then $\pi$ has no invariant unit vectors. Also, if $x = \sum_{n=-k}^k a_n e_n$ is any finitely supported vector, we have
\[
\|x - a(1)x\| = |a_{-k} + |a_{-k+1} - a_{-k}| + \ldots + |a_{-1} - a_{-2}| + |a_0 - a_{-1} + 1|
+ |a_1 - a_0| + \ldots + |a_k - a_{k-1}| + |a_k| \\
> 1.
\]

So $\|x - a(1)x\| > 1$ for all $x \in \ell_1(Z)$ and $b \in B^1(G, \pi)$.

**Corollary 3.** If $a(G \times H)$ has no fixed point on $X$ and $\pi(G)$ and $\pi(H)$ no invariant unit vectors, then
\begin{enumerate}
\item $a|G$ and $a|H$ almost have fixed points, and
\item $\pi|G$ and $\pi|H$ almost have invariant unit vectors.
\end{enumerate}

**Proof.** Item (1) follows directly from Proposition 1 which means that $b|G \in B^1(G, \pi|G)$ and $b|H \in B^1(H, \pi|H)$. However, neither $a(G)$ nor $a(H)$ have fixed points, i.e., $b|G \notin B^1(G, \pi|G)$ and $b|H \notin B^1(H, \pi|H)$. For if, e.g., $a(H)$ fixed a point $x \in X$, then $C = \{x\}$ would be a closed convex $a(H)$-invariant set on which $a|G$ would have almost fixed points, i.e., $x$ would be fixed by $a(G)$ and so $x$ would be a fixed point for $a(G \times H)$, contradicting our assumptions. Thus, neither $B^1(G, \pi|G)$ nor $B^1(H, \pi|H)$ is closed, whereby (2) follows. \hfill $\square$

**Corollary 4.** Suppose $G = G_1 \times \ldots \times G_n$ is a product of topological groups and $\pi : G \rightarrow GL(X)$ is a linear isometric representation on a separable reflexive space $X$. Then $X$
admits a decomposition into $\pi(G)$-invariant linear subspaces $X = V \oplus Y_1 \oplus \ldots \oplus Y_n \oplus W$, such that

1. $V$ is the space of $\pi(G)$-invariant vectors,
2. any $b \in Z^1(G, \pi^W)$ factors through a cocycle defined on $G_i$, 
3. $Z^1(G, \pi^W) \subseteq B^1(G_1, \pi^W) \oplus \ldots \oplus B^1(G_n, \pi^W)$,

where $\pi^W$ denotes the restriction of $\pi$ to the invariant subspace $W$ and similarly for $Y_i$.

Proof. By Theorem 4.10 of [2], for any group of linear isometries of a separable reflexive space $Y$ there is an invariant decomposition of $Y$ into the subspace of fixed points and a canonical complement. Thus, by recursion on the size of $s \subseteq \{1, \ldots, n\}$, we obtain a $\pi(G)$-invariant decomposition

$$X = \sum_{s \subseteq \{1, \ldots, n\}} X_s,$$

where every non-zero $x \in X_s$ is fixed by $\pi\{\prod_{i \in s} G_i\}$ and by none of $\pi(G_i)$ for $i \in s$. So if $b \in Z^1(G, \pi^X)$ and $g \in \prod_{i \in s} G_i$, then for any $h \in \prod_{i \in s} G_i$,

$$b(h) + b(g) = \pi(g)b(h) + b(g) = b(gh) = b(hg) = \pi(h)b(g) + b(h),$$

i.e., $\pi(h)b(g) = b(g)$, which implies that $b(g) = 0$. It follows that if $s \neq \emptyset$, then any $b \in Z^1(G, \pi^X)$ factors through a cocycle defined on $\prod_{i \in s} G_i$.

Also, if $|s| > 2$, then by Corollary 3 we see that any $b \in Z^1(G, \pi^X)$ can be written as $b = b_1 \oplus \ldots \oplus b_n$, where $b_i \in B^1(G_i, \pi^X_i)$. Thus, if we set $V = X_\emptyset$, $Y_i = X_{\{i\}}$ and $W = \sum_{|s| > 2} X_s$, the result follows.

Proposition 1 was shown by Y. Shalom [5] in the special case of locally compact $\sigma$-compact $G$ and $H$ and $X = H$ a Hilbert space, but by different methods essentially relying on the local compactness of $G$ and $H$ and the euclidean structure of $X$. This also provided the central lemma for the rigidity results of [5] via the following theorem, whose proof we include for completeness.

**Theorem 5** (Shalom [5] for locally compact $G$ and $H$). Let $\pi: G \times H \rightarrow GL(\mathcal{H})$ be a strongly continuous isometric linear representation of a product of topological groups on a Hilbert space $\mathcal{H}$ and assume that neither $\pi(G)$ nor $\pi(H)$ have invariant unit vectors. Then $Z^1(G \times H, \pi) = B^1(G \times H, \pi)$ and so $H^1(G \times H, \pi) = 0$.

Proof. Let $b \in Z^1(G \times H, \pi)$ be given with corresponding affine isometric action $a$ and fix compact subsets $K \subseteq G$, $L \subseteq H$ and an $\varepsilon > 0$. Then, by Proposition 1, the closed convex $a(G)$-invariant set $C \subseteq \mathcal{H}$ of $(a(L), c/2)$-invariant points is non-empty. Similarly, there is an $(a(K), c/2)$-invariant point in $\mathcal{H}$.

Now, by the euclidean structure of $\mathcal{H}$, for any $y \in \mathcal{H}$, there is a unique point $P(y) \in C$ closest to $y$ and, as $a(G)$ acts by isometries on $\mathcal{H}$ leaving $C$ invariant, the map $P$ is $a(G)$-equivariant, i.e., $P(a(g)y) = a(g)P(y)$. Moreover, using the euclidean structure again, $P$ is 1-Lipschitz, whereby

$$\|P(y) - a(g)P(y)\| = \|P(y) - P(a(g)y)\| \leq \|y - a(g)y\|,$$

for all $y \in \mathcal{H}$ and $g \in G$. In particular, if $y \in \mathcal{H}$ is $(a(K), c/2)$-invariant, then $P(y)$ is both $(a(K), c/2)$- and $(a(L), c/2)$-invariant, i.e., $P(y)$ is $(a(K \times L), c)$-invariant. Since $K$, $L$ and $c$ are arbitrary, we have that $b \in B^1(G \times H, \pi)$. □
U. Bader, T. Gelander, A. Furman and N. Monod [1] studied the structure of affine actions of product groups on uniformly convex spaces (a subclass of the reflexive spaces) and in this setting obtained a slightly weaker result than Shalom. Namely, if $\pi: G \times H \to GL(X)$ is a strongly continuous isometric linear representation of a product of topological groups on a uniformly convex space $X$ such that neither $\pi(G)$ nor $\pi(H)$ have invariant unit vectors, then either

(a) $\pi$ almost has invariant unit vectors, or

(b) $Z^1(G \times H, \pi) = B^1(G \times H, \pi)$.

Proposition [1] is somewhat independent of their statement and shows that one can add that $\alpha|_G$ and $\alpha|_H$ almost have fixed points to (a) above.

References

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