Ramanujan-type congruences for 2-color partition triples

Shane Chern and Chun Wang

Abstract. Let \( p_{3,3}(n) \) denote the number of 2-color partition triples of \( n \) where one of the colors appears only in parts that are multiples of 3. In this paper, we shall establish some interesting Ramanujan-type congruences for \( p_{3,3}(n) \).

Keywords. Ramanujan-type congruences, 2-color partition triples, dissection identities.

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1. Introduction

A partition of a natural number \( n \) is a weakly decreasing sequence of positive integers whose sum equals \( n \). Let \( p(n) \) be the number of partitions of \( n \). We know that its generating function is

\[
\sum_{n \geq 0} p(n)q^n = \frac{1}{(q;q)_{\infty}},
\]

where for \(|q| < 1\), the shifted factorial is defined by

\[
(a; q)_{\infty} := \prod_{k \geq 0} (1 - aq^k).
\]

In 1919, Ramanujan [14] discovered the following celebrated congruences

\[
p(5n + 4) \equiv 0 \pmod{5},
\]
\[
p(7n + 5) \equiv 0 \pmod{7},
\]
\[
p(11n + 6) \equiv 0 \pmod{11}.
\]

As an analogue of the ordinary partition function, Chan [4] defined the cubic partition function \( a(n) \) by

\[
\sum_{n \geq 0} a(n)q^n := \frac{1}{(q; q)_{\infty}(q^2; q^2)_{\infty}},
\]

which enumerates the number of 2-color partitions of \( n \) where one of the colors appears only in multiples of 2. He also established the partition congruence

\[
a(3n + 2) \equiv 0 \pmod{3}.
\]

Subsequently, many authors studied the arithmetic properties of 2-color partitions with one of the colors appearing only in multiples of \( k \); see [1, 3, 7, 8, 9] for details.

Meanwhile, Chan and Cooper [5] studied the divisibility properties of the function \( c(n) \) defined by

\[
\sum_{n \geq 0} c(n)q^n := \frac{1}{(q; q)_{\infty}^2(q^3; q^3)_{\infty}}.
\]
and obtained the following partition congruence
\[ c(2n + 1) \equiv 0 \pmod{2}. \]
Here the partition function \( c(n) \) can be regarded as the number of 2-color partition pairs of \( n \) where one of the colors appears only in parts that are multiples of 3. Moreover, by considering the generalized partition function \( p_{[c^4d^m]}(n) \) defined by the generating function
\[
\sum_{n \geq 0} p_{[c^4d^m]}(n) q^n := \frac{1}{(q^c; q^c)_\infty (q^d; q^d)_\infty}
\]
and appealing to Ramanujan’s modular equations, Baruah and Ojah [2] presented new proofs of several formulas obtained by Chan and Toh [6] and established more Ramanujan-type congruences, including \( c(4n + 3) \equiv 0 \pmod{4} \).

Inspired by their work, we shall study the following 2-color partition triple function
\[
\sum_{n \geq 0} p_{3,3}(n) q^n := \frac{1}{(q;q)_\infty^3 (q^3; q^3)_\infty^3} \tag{1.1}
\]

\textbf{Theorem 1.1.} For \( n \geq 0 \), we have
\begin{align*}
p_{3,3}(12n + 6, 9) &\equiv 0 \pmod{2}, \quad (1.2) \\
p_{3,3}(6n + 4) &\equiv 0 \pmod{4}, \quad (1.3) \\
p_{3,3}(3n + 1) &\equiv 0 \pmod{3}, \quad (1.4) \\
p_{3,3}(3n + 2) &\equiv 0 \pmod{9}, \quad (1.5) \\
p_{3,3}(9n + 5, 8) &\equiv 0 \pmod{27}, \quad (1.6) \\
p_{3,3}(5n + 3) &\equiv 0 \pmod{5}. \quad (1.7)
\end{align*}

\textbf{Theorem 1.2.} For \( n \geq 0 \), \( \alpha \geq 1 \), and odd prime \( p \) with
\[
\left( \frac{-3}{p} \right) = -1,
\]
we have
\[
p_{3,3} \left( 9p^{2\alpha} n + p^{2\alpha - 1}(3p + 18j + 1) \right) \equiv 0 \pmod{27}, \tag{1.8}
\]
where \( j = 1, 2, \ldots, p - 1 \).

\section{2. Preliminaries}

Throughout this paper, we write \( f_k := (q^k; q^k)_\infty \) for positive integers \( k \) for notational convenience.

The following 2-dissections are necessary.

\textbf{Lemma 2.1.} It holds that
\begin{align*}
f_1 f_3 &= \frac{f_2 f_6 f_{12}^4}{f_2^4 f_6^2 f_{24}^4} - q \frac{f_4^4 f_6 f_{24}^2}{f_2^4 f_6 f_{24}^4}, \quad (2.1) \\
f_1 &= \frac{f_2 f_{16} f_{24}^2}{f_2^2 f_8 f_{48}} - q \frac{f_2 f_2 f_{12} f_{48}}{f_4 f_6 f_{16} f_{24}}, \quad (2.2) \\
f_2 &= \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q \frac{f_6 f_{24} f_{48}}{f_2^2 f_16 f_{24}}. \quad (2.3)
\end{align*}
Proof. Here (2.1), (2.2) and (2.3) are respectively (30.12.1), (30.10.1) and (30.10.3) in [11]. □

We also need the following 3-dissection identities.

Lemma 2.2. It holds that
\[ f_3^3 = P(q^3) - 3qf_3^3, \]  
\[ \frac{1}{f_3^3} = \frac{f_3^3}{f_3^2} \left( P(q^3)^2 + 3qf_3^3P(q^3) + 9q^2f_9^6 \right), \]  
where
\[ P(q) = \frac{f_2 f_3}{f_1^2 f_6} + 3qf_2 f_6^2 f_3^2. \]  

Proof. For (2.4), see [11, Eq. (21.3.3)]. One may obtain (2.5) by replacing \( q \) with \( \omega q \) and \( \omega^2 q \) in (2.4) and multiplying the two results. Finally, (2.6) follows from (21.3.7), (21.1.1) and (22.11.6) in [11]. □

Corollary 2.3. It holds that
\[ \frac{1}{f_3^3} = \frac{f_3^3}{f_3^2} \left( f_3^6 + 9qf_3^3 + 27q^2f_9^6 \right). \]

Proof. It follows by substituting \( P(q^3) = f_3^3 + 3qf_9^3 \) in (2.5). □

At last, we recall the \( p \)-dissection formula of \( f(-q) := (q; q)_\infty \).

Lemma 2.4 ([10, Theorem 2.2]). For any prime \( p \geq 5 \),
\[
f(-q) = (-1)^{\frac{p-1}{6}} q^{\frac{p^2-1}{24}} f(-q^3) + \sum_{k=-\frac{p-1}{2}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right).
\]

We further claim that for \( -(p-1)/2 \leq k \leq (p-1)/2 \) and \( k \neq (\pm p-1)/6 \),
\[
\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod{p}.
\]
Here for any prime \( p \geq 5 \),
\[
\frac{\pm p-1}{6} := \begin{cases} \frac{p-1}{6}, & p \equiv 1 \pmod{6}, \\ \frac{-p-1}{6}, & p \equiv -1 \pmod{6}. \end{cases}
\]

3. Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. From (1.1), we have
\[
\sum_{n \geq 0} p_{3,3}(n)q^n = \frac{1}{f_3^2 f_6^2} = \frac{1}{f_3^2 f_6^2} \left( f_2 f_2 f_2 f_2 f_2 f_2 - q f_2 f_2 f_2 f_2 f_2 f_2 \right) \pmod{4}.
\]
We now extract
\[
\sum_{n \geq 0} p_{3,3}(2n)q^n = \frac{1}{f_3^2 f_6^2} \cdot f_2 f_2 f_2 f_2 f_2 f_2 = \frac{1}{f_3^2 f_6^2} f_2 f_2 f_2 f_2 f_2 f_2 = f_3 f_3 f_3.
\]
This implies (1.5). We may further deduce from (3.4) that

\[ \sum_{n \geq 0} p_{3,3}(n)q^n = \frac{1}{f_1 f_3} \left( \frac{f_6}{f_3} \right)^2 = \left( \frac{f_6 f_5}{f_3 f_6} \right)^2 \] (mod 4). \hspace{1cm} (3.1)

Since there are no terms in which the power of \( q \) is 2 modulo 3, we arrive at (1.3).

On the other hand, we have

\[ \sum_{n \geq 0} p_{3,3}(n)q^n = \frac{1}{f_1 f_3} \left( \frac{f_6}{f_3} \right)^2 = \left( \frac{f_6 f_5}{f_3 f_6} \right)^2 \] (mod 4). \hspace{1cm} (3.2)

We extract

\[ \sum_{n \geq 0} p_{3,3}(3n)q^n = 3 \frac{f_6^3}{f_1} P(q). \] \hspace{1cm} (3.3)

This implies (1.4).

We also extract from (3.2) that

\[ \sum_{n \geq 0} p_{3,3}(3n + 1)q^n = 3 \frac{f_6^3}{f_1}. \] \hspace{1cm} (3.4)

This implies (1.5). We may further deduce from (3.4) that

\[ \sum_{n \geq 0} p_{3,3}(3n + 2)q^n = 9 \frac{f_6^3}{f_1^2}, \] \hspace{1cm} (mod 27). \hspace{1cm} (3.5)

Since there are no terms on the right in which the power of \( q \) is 1 or 2 modulo 3, we obtain (1.6).

Furthermore, we deduce from (3.2) that

\[ \sum_{n \geq 0} p_{3,3}(3n)q^n = \frac{f_3^3}{f_1^2} \frac{f_6}{f_3} P(q)^2 = \frac{f_3^3}{f_1^2} \left( \frac{f_6^6 f_3^6}{f_1^2 f_6^2} \right)^2 \] (mod 2). \hspace{1cm} (3.6)

We extract

\[ \sum_{n \geq 0} p_{3,3}(6n)q^n = \frac{f_3^3}{f_1^2} \frac{f_6^4}{f_3^2} \left( \frac{f_2}{f_6} + q \frac{f_5}{f_3} \right) = \frac{f_1}{f_3} \cdot \frac{f_2^6 f_6^6}{f_3^2 f_2} \left( \frac{f_2}{f_6} + q \frac{f_5}{f_3} \right) \] (mod 2). \hspace{1cm} (3.7)

Hence

\[ \sum_{n \geq 0} p_{3,3}(12n + 6)q^n = \frac{f_3^3}{f_1^2} \frac{f_6^4}{f_3^2} \left( \frac{f_2}{f_6} + q \frac{f_5}{f_3} \right) = \frac{f_1}{f_3} \cdot \frac{f_2^6 f_6^6}{f_3^2 f_2} \left( \frac{f_2}{f_6} + q \frac{f_5}{f_3} \right) \] (mod 2). \hspace{1cm} (3.8)
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Hence $p_{3,3}(12n + 6) \equiv 0 \pmod{2}$.

We may also extract from (3.6)

$$
\sum_{n \geq 0} p_{3,3}(6n + 3)q^n = \frac{f_3^2 f_6 f_8 f_{12}^2}{f_3 f_6 f_8 f_{12}} \left( \frac{f_2}{f_6} + q^{f_6^0/f_2^0} \right) = \frac{f_3}{f_1} \cdot \frac{f_3^2 f_6 f_8 f_{12}}{f_3 f_6 f_8 f_{12}} \left( \frac{f_2}{f_6} + q^{f_6^0/f_2^0} \right) \\
= \left( \frac{f_4 f_8 f_{16} f_{24}}{f_2 f_8 f_{12} f_{48}} + q f_6 f_8 f_{48} \right) \cdot \frac{f_3^2 f_6}{f_3^2 f_6} \left( \frac{f_2}{f_6} + q^{f_6^0/f_2^0} \right) \pmod{2}.
$$

(3.9)

Hence

$$
\sum_{n \geq 0} p_{3,3}(12n + 9)q^n = \frac{f_3^4 f_6^2 f_8 f_{12}^2}{f_3 f_6 f_8 f_{12} f_{24}} + \frac{f_3^4 f_6 f_8 f_{12}^2}{f_3 f_6 f_8 f_{12}} \\
= \frac{f_3^4 f_6^2 f_8 f_{12}^2}{f_3 f_6 f_8 f_{12} f_{24}} + \frac{f_3^4 f_6 f_8 f_{12}^2}{f_3 f_6 f_8 f_{12}} \\
= \frac{f_3^6}{f_3} + \frac{f_6^5}{f_1} \equiv 0 \pmod{2}.
$$

(3.10)

Hence $p_{3,3}(12n + 9) \equiv 0 \pmod{2}$.

At last, we show (1.7). It follows from (1.1) and (2.7) that

$$
\sum_{n \geq 0} p_{3,3}(n)q^n = \frac{1}{f_1^3 f_3} = \frac{1}{f_3^3} \left( \frac{f_3^3}{f_3^3} \left( f_6^0 + 9 q f_3^3 f_6^3 + 27 q^2 f_9^0 \right) \right) \\
= \frac{f_3^6 f_9^3}{f_3^3} + 9 q f_1 f_9 + f_3 f_9 \frac{f_9^0}{f_3^3} + 27 q^2 f_9^0 \\
= \frac{f_5}{f_1 15} f_1 f_9 + 4 q f_1 f_9 + f_3 f_9 f_1 f_9 + 2q^2 f_9 \frac{1}{f_1 15} f_9 \\
= f_3 f_1 f_9 \left( E_0 + E_1 + E_2 \right) \left( J_0^* + J_1^* \right) + 4 q f_1 f_9 \left( J_0 + J_1 \right) \left( E_0 + E_1 + E_2^* \right) \\
+ 2q^2 f_9 \frac{1}{f_1 15} \left( P_0^* + P_1^* + P_2^* + P_3^* \right) \pmod{5}.
$$

(3.11)

Here, $S_k$ and $S_k^*$ indicate series in which the powers of $q$ are congruent to $k$ modulo 5, whether $S$ is $E$ (for Euler), $J$ (for Jacobi) or $P$ (for partitions). Since there are no terms in which the power of $q$ is congruent to 3 modulo 5, we arrive at (1.7). We remark that the same technique is used in [11, §36.4].

**Proof of Theorem 1.2.** We know from (3.5) that

$$
\sum_{n \geq 0} p_{3,3}(3n + 2)q^n \equiv 9 f_3^1 \equiv 9 f_3 f_9 \pmod{27}.
$$

We therefore extract

$$
\sum_{n \geq 0} p_{3,3}(9n + 2)q^n \equiv 9 f_3 f_9 \pmod{27}.
$$

(3.12)

Given a prime $p \geq 5$, and integers $k$ and $m$ with $-(p - 1)/2 \leq k, m \leq (p - 1)/2$, we consider the following quadratic congruence:

$$
\frac{3k^2 + k}{2} + 3 \cdot \frac{3m^2 + m}{2} \equiv \frac{p^2 - 1}{6} \pmod{p},
$$
that is,
\[ 2(6k + 1)^2 + 6(6m + 1)^2 \equiv 0 \pmod{p}. \] (3.13)
We conclude that, for any odd prime \( p \) with
\[ \left( \frac{-3}{p} \right) = -1, \]
the solution to (3.13) is \( k = m = (\pm p - 1)/6 \).

It follows from Lemma 2.4 that
\[
\sum_{n \geq 0} p_{3,3} \left( 9 \left( pm + \frac{p^2 - 1}{6} \right) + 2 \right) q^n \equiv 9f(-q^p)f(-q^{3p}) \pmod{27},
\]
and
\[
\sum_{n \geq 0} p_{3,3} \left( 9 \left( p^2 n + \frac{p^2 - 1}{6} \right) + 2 \right) q^n \equiv 9f(-q)f(-q^3) \pmod{27}.
\]
At last, we induct on \( \alpha \geq 1 \) to obtain
\[
\sum_{n \geq 0} p_{3,3} \left( 9p^{2\alpha - 1}n + \frac{3p^{2\alpha} + 1}{2} \right) q^n \equiv 9f(-q^p)f(-q^{3p}) \pmod{27}.
\]
This implies that
\[
p_{3,3} \left( 9p^{2\alpha - 1}(pn + j) + \frac{3p^{2\alpha} + 1}{2} \right) \equiv 0 \pmod{27},
\]
where \( j = 1, 2, \ldots, p - 1 \). We arrive at (1.8). \( \square \)

4. Final remarks

Using an algorithm (which involves modular forms) due to Radu and Sellers [12, 13], we are able to prove the following congruences modulo 7 and 11:

**Theorem 4.1.** For \( n \geq 0 \), we have
\[
p_{3,3}(21n + 7, 10, 16, 18) \equiv 0 \pmod{7}, \] (4.1)
\[
p_{3,3}(121n + 39, 61, 72, 94, 105, 116) \equiv 0 \pmod{11}. \] (4.2)

However, it is still unclear if there exist any elementary proofs of these congruences.

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(S. Chern) Department of Mathematics, The Pennsylvania State University, University Park, PA 16802, USA

E-mail address: shanechern@psu.edu

(C. Wang) Department of Mathematics, East China Normal University, 500 Dongchuan Road, Shanghai 200241, PR China

E-mail address: wangchunmath@outlook.com