Block Crossings in Storyline Visualizations

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Abstract. Storyline visualizations help visualize encounters of the characters in a story over time. Each character is represented by an $x$-monotone curve that goes from left to right. A meeting is represented by having the characters that participate in the meeting run close together for some time. In order to keep the visual complexity low, rather than just minimizing pairwise crossings of curves, we propose to count block crossings, that is, pairs of intersecting bundles of lines.

Our main results are as follows. We show that minimizing the number of block crossings is NP-hard, and we develop, for meetings of bounded size, a constant-factor approximation. We also present two fixed-parameter algorithms and, for meetings of size 2, a greedy heuristic that we evaluate experimentally.

1 Introduction

A storyline visualization is a convenient abstraction for visualizing the complex narrative of interactions among people, objects, or concepts. The motivation comes from the setting of a movie, novel, or play where the narrative develops as a sequence of interconnected scenes, each involving a subset of characters. See Fig. 1 for an example.

The storyline abstraction of characters and events occurring over time can be used as a metaphor for visualizing other situations, from physical events involving groups of people meeting in corporate organizations, political leaders managing global affairs, and groups of scholars collaborating on research to abstract co-occurrences of “topics” such as a global event being covered on the front pages of multiple leading news outlets, or different organizations turning their attention to a common cause.

A storyline visualization maps a set of characters of a story to a set of curves in the plane and a sequence of meetings between the characters to regions in the plane where the corresponding curves come close to each other. The current form of storyline visualizations seems to have been invented by Munroe \cite{munroe} (compare

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Fig. 1), who used it to visualize, in a compact way, which subsets of characters meet over the course of a movie. Each character is shown as an x-monotone curve. Meetings occur at certain times from left to right. A meeting corresponds to a point in time where the characters that meet are next to each other with only small gaps between them. Munroe highlights meetings by underlaying them with a gray shaded region, while we use a vertical line for that purpose. Hence, a storyline visualization can be seen as a drawing of a hypergraph whose vertices are represented by the curves and whose edges come in at specific points in time.

A natural objective for the quality of a storyline visualization is to minimize unnecessary “crossings” among the character lines. The number of crossings alone, however, is a poor measure: two blocks of “locally parallel” lines crossing each other are far less distracting than an equal number of crossings randomly scattered throughout the drawing. Therefore, instead of pairwise crossings, we focus on minimizing the number of block crossings, where each block crossing involves two arbitrarily large sets of parallel lines forming a crossbar, with no other line in the crossing area; see Fig. 1 for an example.

Previous Work. Kim et al. [6] used storylines to visualize genealogical data; meetings correspond to marriages and special techniques are used to indicate child–parent relationships. Tanahashi and Ma [12] computed storyline visualizations automatically and showed how to adjust the geometry of individual lines to improve the aesthetics of their visualizations. Muelder et al. [10] visualized clustered, dynamic graphs as storylines, summarizing the behavior of the local network surrounding user-selected foci.

Only recently a more theoretical and principled study was initiated by Kostitsyna et al. [8], who considered the problem of minimizing pairwise (not block) crossings in storylines. They proved that the problem is NP-hard in general, and showed that it is fixed-parameter tractable with respect to the (total) number of characters. For the special case of 2-character meetings without repetitions, they developed a lower bound on the number of crossings, as well as an upper bound of $O(k \log k)$ when the meeting graph—whose edges describe the pairwise meetings of characters—is a tree.

Our work builds on the problem formulation of Kostitsyna et al. [8] but we considerably extend their results by designing (approximation) algorithms for
general meetings—for a different optimization goal: we minimize the number of block crossing rather than the number of pairwise line crossings. Block crossings were introduced by Fink et al. [5] for visualizing metro maps.

**Problem Definition.** A storyline \( S \) is a pair \((C, M)\) where \( C = \{1, \ldots, k\} \) is a set of characters and \( M = \{m_1, m_2, \ldots, m_n\} \) with \( m_i \subseteq C \) and \(|m_i| \geq 2\) for \( i = 1, 2, \ldots, n \) is a sequence of meetings of at least two characters. We call any set \( g \subseteq C \) of characters that has at least one meeting, a group. We define the group hypergraph \( H = (C, \Gamma) \) whose vertices are the characters and whose hyperedges are the groups that are involved in at least one meeting. The group hypergraph does not include the temporal aspect of the storyline—it models only the graph-theoretical structure of groups participating in the storyline meetings; it can be built by lexicographically sorting the meetings in \( M \) in \( O(\mathit{nk} \log n) \) time.

Note that we do not encode the exact times of the meetings: In a given visualization, at any time \( t \), there is a unique vertical order \( \pi \) of the characters. Without changing \( \pi \) by crossings, we can increase or decrease vertical gaps between lines. If a group \( g \) forms a contiguous interval in \( \pi^t \), then we can bring \( g \)'s lines within a short distance \( \delta_{\text{group}} \) without any crossing, and also make sure that all other lines are at a larger distance of at least \( \delta_{\text{sep}} \). Since any group must be supported at a time just before its meeting starts, computing an output drawing consists mainly of changing the permutation of characters over time so that during a meeting its group is supported by the current permutation. We therefore focus on changing the permutation by crossings over time, and only have to be concerned about the order of meetings; the final drawing can be obtained by a simple post-processing from this discrete set of permutations.

If \( \{\pi_1, \pi_2, \ldots, \pi_k\} = \{1, 2, \ldots, k\} \), then \( \langle \pi_1, \pi_2, \ldots, \pi_k \rangle \) is a permutation of length \( k \) of \( C \). For \( a \leq b < c \), a block crossing \((a, b, c)\) on the permutation \( \pi = \langle 1, \ldots, k \rangle \) is the exchange of two consecutive blocks \( \langle a, \ldots, b \rangle \) and \( \langle b+1, \ldots, c \rangle \); see Fig. 2. A meeting \( m \) fits a permutation \( \pi \) (or a permutation \( \pi \) supports a meeting \( m \)) if the characters participating in \( m \) form an interval in \( \pi \). In other words, there is a permutation of \( m \) that is part of \( \pi \). If we apply a sequence \( B \) of block crossings to a permutation \( \pi \) in the given order, we denote the resulting permutation by \( B(\pi) \).

**Problem 1 (Storyline Block Crossing Minimization (SBCM)).** Given a storyline instance \((C, M)\) find a solution consisting of a start permutation \( \pi^0 \) of \( C \) and a sequence \( B \) of (possibly empty) sequences of block crossings \( B_1, B_2, \ldots, B_n \) such that the total number of block crossings is minimized and \( \pi^1 = B_1(\pi^0) \) supports \( m_1 \), \( \pi^2 = B_2(\pi^1) \) supports \( m_2 \), etc.

We also consider \(d\)-SBCM, a special case of SBCM where meetings involve groups of size at most \( d \), for an arbitrary constant \( d \). E.g., 2-SBCM allows only 2-character meetings, a setting that was also studied by Kostitsyna et al. [8].
Our Results. We observe that a storyline has a crossing-free visualization if and only if its group hypergraph is an interval hypergraph. A hypergraph can be tested for the interval property in $O(n^2)$ time, where $n$ is the number of hyperedges. We show that 2-SBCM is NP-hard (see Sect. 3) and that SBCM is fixed-parameter tractable with respect to $k$ (Sect. 4). The latter can be modified to handle pairwise crossings, where its runtime improves on Kostitsyna et al. [8]. We present a greedy algorithm for 2-SBCM that runs in $O(k^3n)$ time for $k$ characters. We do some preliminary experiments where we compare greedy solutions to optimal solutions; see Sect. 5. One of our main results is a constant-factor approximation algorithm for $d$-SBCM for the case that $d$ is bounded and that meetings cannot be repeated; see Sect. 6. Our algorithm is based on a solution for the following NP-complete hypergraph problem, which may be of independent interest. Given a hypergraph $H$, we want to delete the minimum number of hyperedges so that the remainder is a interval hypergraph. We develop a $(d + 1)$-approximation algorithm, where $d$ is the maximum size of a hyperedge in $H$; see Sect. 7. Finally, we list some open problems in Appendix H.

2 Preliminaries

First, we consider the special case where every meeting consists of two characters. For these restricted instances, every meeting can be realized from any permutation by a single block crossing. This raises the question whether there is also an optimal solution that fulfills this condition. The answer is negative—if we may prescribe the start permutation; see Appendix A for details.

Observation 2. Given an instance of 2-SBCM, there is a solution with at most one block crossing before each of the meetings. In particular, there is a solution with at most $n$ block crossings in total.

Detecting Crossing-Free Storylines. If a storyline admits a crossing-free visualization, then the vertical permutation of the character lines remains the same over time, and all meetings involve groups that form contiguous subsets in that permutation. (The visualization can be obtained by placing characters along a vertical line in the correct permutation and for each meeting bringing its lines together for the duration of the meeting and then separating them apart again.) In other words, a single permutation supports each group of $H = (C, \Gamma)$. This holds if and only if $H$ is an interval hypergraph. This is the case if there exists a permutation $\pi = (v_1, \ldots, v_k)$ of $C$ such that each hyperedge $e \in \Gamma$ corresponds to a contiguous block of characters in this permutation. As an anonymous reviewer pointed out, this is equivalent to the hypergraph having path support [11]. An interval hypergraph can be visualized by placing all of its vertices on a line, and drawing each of its hyperedges as an interval that includes all vertices of $e$ and no vertex of $V \setminus e$. Checking whether a $k$-vertex hypergraph is an interval hypergraph takes $O(k^2)$ time [13]. Recall that we can build $H$ in $O(nk \log n)$ time.
Theorem 3. Given the group hypergraph $H$ of an instance of SBCM with $k$ characters, we can check in $O(k^2)$ time whether a crossing-free solution exists.

For 2-SBCM we only need to check (in $O(k)$ time) whether $H$ is a collection of vertex-disjoint paths; this is dominated by the time ($O(n)$) for building $H$.

3 NP-Completeness of SBCM

In this section we prove that SBCM is NP-complete. This is known for BCM. But SBCM is not simply a generalization of BCM because in SBCM we can choose an arbitrary start permutation. Therefore, the idea of our hardness proof is to force a certain start permutation by adding some characters and meetings. We reduce from Sorting by Transpositions (SBT), which has also been used to show the hardness of BCM [5]. In SBT, the problem is to decide whether there is a sequence of transpositions (which are equivalent to block crossings) of length at most $k$ that transforms a given permutation $\pi$ to the identity. SBT was recently shown NP-hard by Bulteau et al. [2].

We show hardness for 2-SBCM, which also implies that SBCM is NP-hard. It is easy to see that SBCM is in NP: Obviously, the maximum number of block crossings needed for any number of characters and meetings is bounded by a polynomial in $k$ and $n$. Therefore also the size of the solutions is bounded by a polynomial. To test the feasibility of a solution efficiently, we simply test whether the permutations between the block crossings support the meetings in the right order from left to right. We will use the following obvious fact.

Observation 4 If permutation $\pi$ needs $c$ block crossings to be sorted, any permutation containing $\pi$ as subsequence needs at least $c$ block crossings to be sorted.

Theorem 5. 2-SBCM is NP-complete.

Proof. It remains to show the NP-hardness. We reduce from SBT. Given an instance of SBT, that is, a permutation $\pi$ of $\{1, \ldots, k\}$, we show how to use a hypothetical, efficient algorithm for 2-SBCM to determine the minimum number of transpositions (i.e., block crossings) that transforms $\pi$ to the identity $\iota = \langle 1, 2, \ldots, k \rangle$. Note that $\pi$ can be sorted by at most $k$ block crossings. So $k$ is an upper bound for an optimal solution of instance $\pi$ of SBT.

We extend the set of characters $\{1, 2, \ldots, k\}$ to $C = \{1, \ldots, k, c_1, c_2, \ldots, c_{2k}\}$. Correspondingly, we extend $\pi = \langle \pi_1, \pi_2, \ldots, \pi_k \rangle$ to $\pi' = \langle c_1, \ldots, c_{2k}, \pi_1, \ldots, \pi_k \rangle$ and $\iota$ to $\iota' = \langle c_1, c_2, \ldots, c_{2k}, 1, 2, \ldots, k \rangle$. Let $M_\pi'$ and $M_\iota'$ be the sequences of meetings of all neighboring pairs in $\pi'$ and $\iota'$, respectively. Let $M_1$ and $M_2$ be the concatenations of $k + 1$ copies of $M_\pi'$ and $M_\iota'$, respectively. By repeating we get $M_1 = M_{k+1}'$ and $M_2 = M_{k+1}'$. This yields the instance $\mathcal{S} = (C, M)$ of 2-SBCM, where $M$ is the concatenation of $M_1$ and $M_2$; see Fig. 3.

We show that the number of block crossings needed for the 2-SBCM instance $\mathcal{S}$ equals the number of block crossings to sort instance $\pi$ of SBT.

First, let $B$ be a shortest sequence of block crossings to sort $\pi$. Then, $(\pi', B)$ is a feasible solution for $\mathcal{S}$. The start permutation $\pi'$ supports all meetings in
$M_1$ without any block crossing. Using $B$, the lines are sorted to $ι'$, and this permutation supports all meetings in $M_2$ without any further block crossings; see Fig. 3. Hence, the number of block crossings in any solution of $π$ is an upper bound for the minimum number of block crossings needed for $S$.

For the other direction, let $(π^*, B^*)$ be an optimal solution for $S$. Any solution of 2-SBCM gives rise to a symmetric solution that is obtained by reversing the order of the characters. Without loss of generality, we assume that $π'$ (rather than the reverse permutation $π'^R$) occurs somewhere in $M_1$.

Next, we show that the start permutation $π'$ occurs somewhere in $M_1$ and that $ι'$ occurs somewhere in $M_2$. If there is a sequence $M_{π'}$ of meetings between which there is no block crossing, the permutation at this position can only be the start permutation $π'$ or its reverse. For a contradiction, assume that $π'$ does not occur during $M_1$ in the layout induced by $(π^*, B^*)$. Then there is no such sequence without any block crossing in it. As this sequence is repeated $k + 1$ times, the solution would need at least $k + 1$ block crossings. This contradicts our upper bound, which is $k$. Analogously, we can show that the permutation $ι'$ or its reverse occurs in $M_2$.

We now want to show that the unreversed version of $ι'$ occurs in $M_2$. For a contradiction, assume the opposite. We forget about the lines $1, \ldots, k$ and only consider the sequence $π'' = ⟨c_1, \ldots, c_{2k}⟩$ in $π'$ which is reversed to $ι'' = ⟨c_{2k}, \ldots, c_1⟩$ in $ι'^R$. Eriksson et al. [4] showed that we need $\lceil (l + 1) / 2 \rceil$ block crossings to reverse a permutation of $l$ elements. This implies that we need $k + 1$ block crossings to transform $π''$ to $ι''$. As $π'$ and $ι'^R$ contain these sequences as subsequences, Observation 4 implies that the transformation from $π'$ to $ι'^R$ also needs at least $k + 1$ block crossings. As the optimal solution uses at most $k$ block crossings, we know that we cannot reach $ι'^R$ and thus the sequence of permutations contains $π'$ and $ι'$.

The sequence of block crossings that transforms $π'$ to $ι'$ yields a sequence $B$ of block crossings of the same length that transforms $π$ to $ι$. This shows that the length of a solution for $S$ is an upper bound for the length of an optimal solution of the corresponding SBT instance $π$. Thus, the two are equal.

\[ \square \]

**Hardness Without Repetitions.** With arbitrarily large meetings, SBCM is hard even without repeating meetings. We can emulate a repeated sequence of 2-character meetings by gradually increasing group sizes; see Appendix B.

### 4 Exact Algorithms

We present two exact algorithms. Conceptually, both build up a sequence of block crossings while keeping track of how many meetings have already been
accomplished. The first uses polynomial space; the second improves the runtime at the cost of exponential space.

We start with a data structure that keeps track of permutations, block crossings and meetings. It is initialized with a given permutation and has two operations. The Check operation returns whether a given meeting fits the current permutation. The BlockMove operation performs a given block crossing on the permutation and then returns whether the most-recently CHECKed meeting now fits. See Appendix C for a detailed description.

**Lemma 6.** A sequence of arbitrarily interleaved BlockMove and Check operations can be performed in \(O(\beta + \mu)\) time, where \(\beta\) is the number of block crossings and \(\mu\) is sum of cardinalities of the meetings given to Check. Space usage is \(O(k)\).

A block crossing can be represented by indices \((a, b, c)\) with \(1 \leq a \leq b < c \leq k\); hence, there are \(\frac{k^3 - k}{6}\) distinct block crossings on a permutation of length \(k\).

Now we provide an output-sensitive algorithm for SBCM whose runtime depends on the number of block crossings required by the optimum.

**Theorem 7.** An instance of SBCM can be solved in \(O(k! \cdot \frac{(k^3 - k)^\beta}{6} \cdot (\beta + \mu))\) time and \(O(\beta k)\) working space if a solution with \(\beta\) block crossings exists, where \(\mu = \sum_{i \in M} |m_i|\).

**Proof.** Consider a branching algorithm that starts from a permutation of the characters and keeps trying all possible block crossings. This has branching factor \(\frac{k^3 - k}{6}\) and we can enumerate the children of a node in constant time each by enumerating triples \((a, b, c)\). While applying block crossings, the algorithm keeps track of how many meetings fit this sequence of permutations using the data structure from Lemma 6. We use depth-first iterative-deepening search [7] from all possible start permutations until we find a sequence of permutations that fulfills all meetings. Correctness follows from the iterative deepening: we want an (unweighted) shortest sequence of block crossings. The runtime and space bounds follow from the standard analysis of iterative-deepening search, observing that a node uses \(O(k)\) space and it takes \(O(\beta + \mu)\) time in total to evaluate a path from root to leaf.

We have that \(\mu\) is \(O(kn)\) since there are \(n\) meetings and each consists of at most \(k\) characters. At the cost of exponential space, we can improve the runtime and get rid of the dependence on \(\beta\), showing the problem to be fixed parameter linear for \(k\). We note that the following algorithm can easily be adapted to handle pairwise crossings rather than block crossings; in this case the runtime improves upon the original result of Kostisyna et al. [8] by a factor of \(k!\).

**Theorem 8.** An instance of SBCM can be solved in \(O(k! \cdot k^3 \cdot n)\) time and \(O(k! \cdot k \cdot n)\) space.
Proof. Let \( f(\pi, \ell) \) be the optimal number of block crossings in a solution to the given instance when restricted to the first \( \ell \) meetings and to have \( \pi \) as its final permutation. Note that by definition the solution for the actual instance is given by \( \min_{\pi'} f(\pi', n) \), where the minimum ranges over all possible permutations. As a base case, \( f(\pi, 0) = 0 \) for all \( \pi \), since the empty set of meetings is supported by any permutation. Let \( \pi \) and \( \pi' \) be permutations that are one block crossing apart and let \( 0 \leq \ell \leq \ell' \). If the meetings \{\( m_{\ell+1}, \ldots, m_{\ell'} \)\} fit \( \pi' \), then \( f(\pi', \ell') \leq f(\pi, \ell) + 1 \): if we can support the first \( \ell \) meetings and end on \( \pi \), then with one additional block crossing we can support the first \( \ell' \) meetings and end with \( \pi' \).

We now model this as a graph. Let \( G \) be an unweighted directed graph on nodes \((\pi, \ell)\) and call a node start node if \( \ell = 0 \). There is an arc from \((\pi, \ell)\) to \((\pi', \ell')\) if and only if \( \pi \) and \( \pi' \) are one block crossing apart, \( \ell \leq \ell' \), and the meetings \{\( m_{\ell+1}, \ldots, m_{\ell'} \)\} fit \( \pi' \). Note that we allow \( \ell = \ell' \) since we may need to allow block crossings that do not immediately achieve an additional meeting (cf. Proposition 18), so \( G \) is not acyclic. In the constructed graph, \( f(\pi, \ell) \) equals the graph distance from the node \((\pi, \ell)\) to the closest start node. Call a path to a start node that realizes this distance optimal.

In \( G \), consider any path \([(\pi_1, \ell_1), (\pi_2, \ell_2), (\pi_3, \ell_3)]\) with \( \ell_3 > \ell_2 \). If meeting \( \ell_3 + 1 \) fits \( \pi_2 \), then \([(\pi_1, \ell_1), (\pi_2, \ell_2 + 1), (\pi_3, \ell_3)]\) is also a path. Repeating this transformation shows that for all \( \pi \), the node \((\pi, n)\) has an optimal path in which every arc maximally increases \( \ell \). Let \( G' \) be the graph where we drop all arcs from \( G \) that do not maximally increase \( \ell \). Note that \( G' \) still contains a path that corresponds to the global optimum.

The graph \( G' \) has \( O(k! \cdot n) \) nodes and each node has outdegree \( O(k^3) \). Then a breadth-first search from all start nodes to any node \((\pi^*, n)\) achieves the claimed time and space bounds, assuming we can enumerate the outgoing arcs of a node in constant time each.

For a given node \((\pi, \ell)\) we can enumerate all possible block crossings in constant time each, as before. In \( G' \), we also need to know the maximum \( \ell' \) such that all meetings \( \ell + 1 \) up to \( \ell' \) fit \( \pi' \). Note that \( \ell' \) only depends on \( \ell \) and \( \pi' \). We precompute a table \( M(\pi, \ell) \) that gives this value. Computing \( M(\pi, \ell) \) for given \( \pi \) and all \( \ell \) takes a total of \( O(kn) \) time: first compute for every \( m_i \) whether it fits \( \pi \), then compute the implied ‘forward pointers’ using a linear scan. So using \( O(k! \cdot k \cdot n) \) preprocessing time and \( O(k! \cdot n) \) space, we have an efficient implementation of the breadth-first search. The theorem follows. \( \square \)

5 SBCM with Meetings of Two Characters

A Greedy Algorithm. To quickly draw good storyline visualizations for 2-SBCM, we develop an \( O(kn) \)-time greedy algorithm. Given an instance \( S = (C, M) \), we reserve a list \( B = [ \] \) that the algorithm will use to store the block crossings. The algorithm starts with an arbitrary permutation \( \pi^0 \) of \( C \). In every step the algorithm removes all meetings from the beginning of \( M \) that fit the current permutation \( \pi^i \) of the algorithm. Subsequently, the algorithm picks a block crossing \( b \) such that the resulting permutation \( \pi^{i+1} = b(\pi^i) \) supports the maxi-
minimum number of meetings from the beginning of $M$. Then $b$ is appended to the list $B$. This process repeats until $M$ is empty. The algorithm returns $(\pi^0, B)$.

Note that there are at most $O(k^3)$ possible block crossings. Thus to find the appropriate block crossings, the algorithm could simply check all of them. Many of those, however, will result in permutations that do not even support the next meeting, which would be a bad choice. Hence, our algorithm considers only relevant block crossings, i.e., block crossings yielding a permutation that supports the next meeting. Let $\{c, c'\}$ be the next meeting in $M$. If $x$ and $y$ are the positions of $c$ and $c'$ in the current permutation, i.e., $\pi_x = c$ and $\pi_y = c'$ (without loss of generality, assume $x < y$), the relevant block crossings are:

$$\{(z, x, y-1): 1 \leq z \leq x\} \cup \{(x, z, y): x \leq z < y\} \cup \{(x+1, y-1, z): y \leq z \leq k\}.$$  

So the number of relevant block crossings in each step is $k + 1$. Let $n_i$ be the maximum number of meetings at the beginning of $M$ we can achieve by one of these block crossings. We use the data structure in Lemma 6 and check for each relevant block crossing how many meetings can be done with this permutation. Hence, we can identify a block crossing achieving the maximum number in $O(kn_i)$ time since we have to check $k + 1$ paths containing up to $n_i$ meetings each. Clearly, the numbers of meetings $n_i$ in each iteration of the algorithm sum up to $n$ and therefore the algorithm runs in $O(kn)$ total time.

The way we described the greedy algorithm, it starts with an arbitrary permutation. Instead, we could start with a permutation that supports the maximum number of meetings before the first block crossing needs to be done. In other words, we want to find a maximal prefix $M'$ of $M$ such that $(C, M')$ can be represented without any block crossings. We can find $M'$ in $O(kn)$ time: we start with an empty graph and add the meetings successively. In each step we check whether the graph is still a collection of paths, which can be done in $O(k)$ time. It is easy to construct a permutation that supports all meetings in $M'$. While this is a sensible heuristic, we do not prove that this reduces the total number of block crossings. Indeed, we experimentally observe that while the heuristic is generally good, this is not always the case; see Fig. 4 for an example that uses the heuristic start permutation.

Note that the greedy algorithm yields optimal solutions for special cases of 2-SBCM. The proof for the following theorem can be found in Appendix D.

**Theorem 9.** For $k = 3$, the greedy algorithm produces optimal solutions.
Experimental Evaluation. In this section, we report on some preliminary experimental results. We only consider 2-SBCM. We generated random instances as follows. Given $n$ and $k$, we generate $n$ pairs of characters as meetings, uniformly at random using rejection sampling to ensure that consecutive meetings are different. (Repeated meetings are not sensible.)

First, we consider the exact algorithm of Theorem 7. As expected, its runtime depends heavily on $k$ (Fig. 5, left). Perhaps unexpectedly, we observe exponential runtime in $n$. This is actually a property of our random instances, in which $\beta$ tends to increase linearly with $n$. Note that this does not invalidate the algorithm since we may be interested in instances for which $\beta$ is indeed small.

Since the exact algorithm is feasible only for rather small instances, we now shift our focus to the greedy algorithm. Recall that it starts with an arbitrary permutation and proceeds greedily. The histogram in Fig. 5 (right) shows the number of block crossings used by the greedy algorithm depending on the start permutation, for a single random instance: this bell curve is typical. We see that there are “rare” start permutations that do strictly better than almost all others. Indeed, for the reported instance, a random start permutation does block crossings worse in expectation than the best possible start permutation.

We call the best possible result of the greedy algorithm over all start permutations $\text{BestGreedy}$, which we calculate by brute force. Let $\text{RandomGreedy}$ start with a permutation chosen uniformly at random, and let $\text{HeuristicGreedy}$ start with the heuristic start permutation that we have described above. The histogram in Fig. 6 (left) shows how many more block crossings $\text{HeuristicGreedy}$ uses than $\text{BestGreedy}$ on random instances. This distribution is heaviest near zero, but there are instances where performance is poor. Note that we do not know how to compute $\text{BestGreedy}$ efficiently. Compared to $\text{RandomGreedy}$, we see that $\text{HeuristicGreedy}$ fares well (Fig. 6, right).

Lastly, we compare the greedy algorithm to the optimum, which we can only do for small $k$ and $n$. On 1000 random instances with $k = 5$ and $n = 12$,
Fig. 6. Left: histogram of HeuristicGreedy minus BestGreedy, 200 instances with with $k = 7$ and $n = 100$. Right: histogram of RandomGreedy minus HeuristicGreedy, 1000 instances with $k = 30$ and $n = 200$.

HeuristicGreedy was optimal 56% of the time. It was sometimes off by one (38%), two (5%), or three (1%), but never worse. This is a promising behavior, but clearly cannot be extrapolated verbatim to larger instances.

Based on these experiments, we recommend HeuristicGreedy as an efficient, reasonable heuristic.

6 Approximation Algorithm

We now develop a constant-factor approximation algorithm for $d$-SBCM where $d$ is a constant. We initially assume that each group meeting occurs exactly once, but later show how to extend our results to the setting where the same group can meet a bounded number of times.

Overview. Our approximation algorithm has the following three main steps.

1. Reduce the input group hypergraph $H = (C, \Gamma)$ to an interval hypergraph $H_f = (C, \Gamma \setminus \Gamma_p)$ by deleting a subset $\Gamma_p \subseteq \Gamma$ of the edges of $H$.
2. Choose a permutation $\pi^0$ of the characters that supports all groups of this interval hypergraph $H_f$. Thus, $\pi^0$ is the order of characters at the beginning of the timeline.
3. Incrementally create support for each deleted meeting of $\Gamma_p$ in order of increasing time, as follows. Suppose that $g \in \Gamma_p$ is the group meeting to support. Keep one of the character lines involved in this meeting fixed and bring, for the duration of the meeting, the remaining (at most $d-1$) lines close to it. Then retract those lines to their original position in $\pi^0$; see Fig. 7.

Step 2 is straightforward: Section 2 shows how to find a permutation supporting all the groups for an interval hypergraph. In Step 3, we introduce at most $2(d-1)$ block crossings for each meeting $g \in \Gamma_p$ not initially supported. The main technical parts of the algorithm are Step 1 and an analysis to charge at most a constant number of block crossings in Step 3 to a block crossing.
in the optimal visualization. Step 1 requires solving a hypergraph problem; this is technically the most challenging part, and consumes the entire Section 7.

**Bounds and Analysis.** We call $\Gamma_p$ paid edges, and the remainder $\Gamma_f = \Gamma \setminus \Gamma_p$ free edges. Intuitively, free edges can be realized without block crossings because $\mathcal{H}_f$ is an interval hypergraph, while the edges of $\Gamma_p$ must be charged to block crossings of the optimal drawing. We initialize the drawing by placing the characters in the vertical order $\pi^0$, which supports all the groups in $\Gamma_f$. Now we consider the paid edges in left-to-right order. Suppose that the next meeting involves a group $g' \in \Gamma_p$. We have $|g'| \leq d$. We arbitrarily fix one of its characters, leaving its line intact, and bring the remaining $(d-1)$ lines in its vicinity to realize the meeting. This creates at most $(d-1)$ block crossings, one per line. When the meeting is over, we again use up to $(d-1)$ block crossings to revert the lines back to their original position prescribed by $\pi^0$; see Fig. 7.

We do this for each paid hyperedge, giving rise to at most $2(d-1)|\Gamma_p|$ block crossings. We now prove that this bound is within a constant factor of optimal. We first establish a lower bound on the optimal number of block crossings assuming that $\pi^0$ is the optimal start permutation.

**Lemma 10.** Let $\pi$ be a permutation of the characters, let $\Gamma_f$ be the groups supported by $\pi$, and let $\Gamma_p = \Gamma \setminus \Gamma_f$. Any storyline visualization that uses $\pi$ as the start permutation has at least $4|\Gamma_p|/(3d^2)$ block crossings.

**Proof.** Let $g \in \Gamma_p$. Since $g$ is not supported by $\pi$, the optimal drawing does not contain the characters of $g$ as a contiguous block initially. However, in order to support this meeting, these characters must eventually become contiguous before the meeting starts. The order changes only through (block) crossings; we bound the number of groups that can become supported after each block crossing.

After a block crossing, at most three pairs of lines that were not neighbors before can become neighbors in the permutation: after the blocks $C_1, C_2 \subseteq C$ cross, there is one position in the permutation where a line of $C_1$ is next to a line of $C_2$, and two positions with a line of $C_1$ ($C_2$, respectively) and a line of $C \setminus (C_1 \cup C_2)$. Any group that was not supported, but is supported after the block crossing, must contain one of these pairs. We can describe each such group in the new permutation by specifying the new pair and the numbers $d_1$ and $d_2$ of characters of the group above and below the new pair in the permutation. Since the group size is at most $d$, we have $d_1 + d_2 \leq d$. The product $d_1(d-d_1)$ achieves its maximum value for $d_1 = d_2 = d/2$, and so there are at most $d^2/4$ possible groups for each new pair. Thus, the total number of newly supported groups after a block crossing is at most $3d^2/4$, which shows that the optimal number of block crossings is at least $4|\Gamma_p|/(3d^2)$, completing the proof. \(\square\)

We now bound the loss of optimality caused by not knowing the initial permutation used by the optimal solution. The key idea here is to use a constant-factor approximation for the problem of deleting the minimum number of hyperedges from $\mathcal{H}$ so that it becomes an interval hypergraph (**Interval Hypergraph Edge Deletion**). We prove the following theorem in Section 7.
**Theorem 11.** We can find a \((d+1)\)-approximation for **Interval Hypergraph Edge Deletion** on group hypergraphs with \(n\) meetings of rank \(d\) in \(O(n^2)\) time.

Let \(\Gamma_{\text{OPT}}\) be the set of paid edges in the optimal solution, and \(\Gamma_p\) the set of paid edges in our algorithm. By Theorem 11, we have \(|\Gamma_p| \leq (d+1)|\Gamma_{\text{OPT}}|\). Let \(\text{ALG}\) and \(\text{OPT}\) be the numbers of block crossings for our algorithm and the optimal solution, respectively. By Lemma 10, we have \(\text{OPT} \geq 4|\Gamma_{\text{OPT}}|/(3d^2)\), which gives \(|\Gamma_{\text{OPT}}| \leq 3d^2/4 \cdot \text{OPT}\). On the other hand, we have \(\text{ALG} \leq 2(d-1)|\Gamma_p| \leq 2(d-1)(d+1)|\Gamma_{\text{OPT}}|\). Combining the two inequalities, we get \(\text{ALG} \leq 3(d^2 - 1)d^2/2 \cdot \text{OPT}\), which establishes our main result.

**Theorem 12.** \(d\)-**SBCM** admits a \(3(d^2 - 1)d^2/2\)-approximation algorithm.

**Remark.** We assumed that each group meets only once, but we can extend the result if each group can meet \(c\) times, for constant \(c\). Our algorithm then yields a \((c \cdot 3(d^2 - 1)d^2/2)\)-factor approximation; each repetition of a meeting may trigger a constant number of block crossings not present in the optimal solution.

**Runtime Analysis.** We have to consider the permutation (of length \(k\)) of characters before and after each of the \(n\) meetings, as well as after each of the \(O(n)\) block crossings. This results in \(O(kn)\) time for the last part of the algorithm, but this is dominated by the time \((O(n^2))\) needed for finding \(\Gamma_p\) and for determining the start permutation.

We can improve the running time to \(O(kn)\) by a slight modification: using the approximation algorithm for **Interval Hypergraph Edge Deletion** is only necessary for sparse instances. If \(H\) has sufficiently many edges, any start permutation will yield a good approximation. Since no meeting involves more than \(d\) characters, no start permutation can support more than \(dk\) meetings. If \(n \geq 2dk\), then even the optimal solution must therefore remove at least half of the edges. Hence, taking an arbitrary start permutation yields an approximation factor of at most \(2 < d + 1\).

We now change the algorithm to use an arbitrary start permutation if \(n \geq 2dk\) and only use the approximation for **Interval Hypergraph Edge Deletion** otherwise, i.e., especially only if there are \(O(k)\) edges. Hence, for sparse instances we have \(O(n^2) = O(k^2)\), and for dense instances, the \(O(n^2)\) runtime is not necessary. We get the following improved result. (The runtime is worst-case optimal since the output complexity is of the same order.)

**Theorem 13.** \(d\)-**SBCM** admits an \(O(kn)\)-time \((3(d^2 - 1)d^2/2)\)-approximation algorithm.

Using some special properties of the 2-character case, we can improve the approximation factor for 2-**SBCM** from 18 to 12; see Appendix E.
**Fig. 8.** Forbidden subhypergraphs for interval hypergraphs (edges represent pairwise hyperedges, circles/ellipses show hyperedges of higher cardinality).

### 7 Interval Hypergraph Edge Deletion

We now describe the main missing piece from our approximation algorithm: how to approximate the minimum number of edges whose deletion reduces a hypergraph to an interval hypergraph, i.e., how to solve the following problem.

**Problem 14 (Interval Hypergraph Edge Deletion).** Given a hypergraph \( \mathcal{H} = (V, E) \) find a smallest set \( E_p \subseteq E \) such that \( \mathcal{H}_f = (V, E \setminus E_p) \) is an interval hypergraph.

Note that a graph contains a Hamiltonian path if and only if one can remove all but \( n - 1 \) edges so that only vertex-disjoint paths (here, a single path) remain; hence, our problem is hard even for graphs.

**Theorem 15.** **Interval Hypergraph Edge Deletion** is NP-hard.

We now present a \((d + 1)\)-approximation algorithm for rank-\(d\) hypergraphs, in which each hyperedge has at most \(d\) vertices. In this section we give all main ideas. Detailed proofs can be found in Appendix E; they are mostly not too hard to obtain, but require the distinction of many cases.

For our algorithm, we use the following characterization: A hypergraph is an interval hypergraph if and only if it contains none of the hypergraphs shown in Fig. 8 as a subhypergraph [13,9]. Due to the bounded rank, the families of \( F_k \) and \( M_k \) are finite with \( F_{d-2} \) and \( M_{d-1} \) as largest members. Cycles are the only arbitrarily large forbidden subhypergraphs in our setting. Let \( \mathcal{F} = \{O_1, O_2, F_1, \ldots, F_{d-2}, M_1, \ldots, M_{d-1}, C_3, \ldots, C_{d+1}\} \). A hypergraph is \( \mathcal{F} \)-free if it does not contain any hypergraph of \( \mathcal{F} \) as a subhypergraph. Note that a cycle in a hypergraph consists of hyperedges \( e_1, \ldots, e_k \) so that there are vertices \( v_1, \ldots, v_k \) with \( v_i \in e_i \cap e_{i-1} \) for \( 2 \leq i \leq k \) (and \( v_1 \in e_1 \cap e_k \)) and no edge \( e_i \) contains a vertex of \( v_1, \ldots, v_k \) except for \( v_i \) and \( v_{i+1} \).

Our algorithm consists of two steps. First, we search for subhypergraphs contained in \( \mathcal{F} \), and remove all edges involved in these hypergraphs. In the second step, we break remaining (longer) cycles by removing some more hyperedges after carefully analyzing the structure of connected components. Subhypergraphs in \( \mathcal{F} \) consist of at most \( d + 1 \) hyperedges. A given optimal solution must remove...
at least one of the hyperedges; removing all of them instead yields a factor of at most $d + 1$. The second step will not negatively affect this approximation factor.

Intuitively, allowing long cycles, but forbidding subhypergraphs of $\mathcal{F}$, results in a generalization of interval hypergraphs where the vertices may be placed on a cycle instead of a vertical line. This is not exactly true, but we will see that the connected components after the first step have a structure similar to this, which will help us find a set of edges whose removal destroys all remaining long cycles.

Lemma 22 (Appendix F) shows that any vertex is contained in at most three hyperedges of a cycle, where the case of three hyperedges with a common vertex occurs only if a hyperedge is contained in the union of its two neighbors in the cycle. Assume that $e_1, e_2,$ and $e_3$ are consecutive edges of a cycle $C$. If all three edges are present in an interval representation, we know that we will first encounter vertices that are only contained in $e_1$, then vertices that are in $(e_1 \cap e_2) \setminus e_3$, then vertices in $e_1 \cap e_2 \cap e_3$, followed by vertices of $(e_2 \cap e_3) \setminus e_1$, and vertices of $e_3 \setminus (e_1 \cup e_2)$. Some of the sets (except for pairwise intersections) may be empty. We do not know the order of vertices within one set, but we know the relative order of any pair of vertices of different sets. By generalizing this to the whole cycle, we get a cyclic order—describing the local order in a possible interval representation—of sets defined by containment in 1, 2, or 3 hyperedges. We call these sets cycle-sets and their cyclic order the cycle-order of $C$.

We can analyze how an edge $e \not\in C$ relates to the order of cycle-sets; $e$ can contain a cycle-set completely, can be disjoint from it, or can contain only part of its vertices. We call a consecutive sequence of cycle-sets contained in edge $e$—potentially starting and ending with cycle-sets partially contained in $e$—an interval of $e$ on $C$. The following lemma shows that every edge forms only a single interval on a given cycle.

**Lemma 16.** If a hyperedge $e \in E$ intersects two cycle-sets of a cycle $C$, then $e$ fully contains all cycle-sets lying in between in one of the two directions along $C$.

We now know that by opening the cycle at a single position within a cycle-set not contained in $e$, $C + e$ forms an interval hypergraph. Edge $e$ adds further information: If only part of the vertices of a cycle-set are contained in $e$ and also vertices of the next cycle-set in one direction, we know that the vertices of $e$ in the first cycle-set should be next to the second cycle-set. We use this to refine the cycle-sets to a cyclic order of cells, the cell order (a cell is a set of vertices that should be contiguous in the cyclic order). Initially, the cells are the cycle-sets. In each step we refine the cell-order by inserting an edge containing vertices of more than one cell, possibly splitting two cells into two subcells each. The following lemma shows that during this process of refinements, as an invariant each remaining edge forms a single interval on the cell order.

**Lemma 17.** If a hyperedge $e \in E$ intersects two cells, then $e$ fully contains all cells lying in between in one of the two directions along the cyclic order.

After refining cells as long as possible, each edge of the connected component that we did not insert lies completely within a single cell. Several edges can lie
within the same cell, forming a hypergraph that imposes restrictions on the order of vertices within the cell. However, the cell contains fewer than \(d\) vertices. Hence, this small hypergraph cannot contain any cycles, since we removed all short cycles, and must be an interval hypergraph.

With this cell-structure, it is not too hard to show that the following strategy to make the connected component an interval hypergraph is optimal (see Lemmas 24, 25, and 26 in Appendix F): For each pair of adjacent cells we determine the number of edges containing both cells, select the pair minimizing that number, and remove all edges containing both. The cell order then yields an order of the connected component’s vertices that supports all remaining edges. Since this last step of the algorithm is done optimally, we do not further change the approximation ratio, which, overall, is \(d + 1\), because we never remove more than \(d + 1\) edges for at least one edge that the optimal solution removes.

**Runtime.** Our algorithm can be implemented to run in \(O(m^2)\) time for \(m\) hyperedges. We give the main ideas here and present details in Appendix G. When searching for forbidden subhypergraphs, we first remove all cycles of length \(k \leq d\) using a modified breadth-first search in \(O(m^2)\) time. The remaining types of forbidden subhypergraphs each contain an edge that contains all but one (\(O_2\) and \(F_k\)), two (\(M_k\)), or three (\(O_1\)) vertices of the subhypergraph. We always start searching from such an edge and use that all short cycles have already been removed. In the second phase, we determine the connected components and initialize the cell order for each of them, in \(O(n + m)\) time. Stepwise refinement requires \(O(m^2)\) time. Counting hyperedges between adjacent cells, determining optimal splitting points, and finding the final order can all be done in linear time.

**Theorem 11** We can find a \((d+1)\)-approximation for INTERVAL HYPERGRAPH EDGE DELETION on hypergraphs with \(m\) hyperedges of rank \(d\) in \(O(m^2)\) time.

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Appendix

A Preliminaries: Proofs

Observation 2. Given an instance of 2-SBCM, there is a solution with at most one block crossing before each of the meetings. In particular, there is a solution with at most $n$ block crossings in total.

Proof. Let $\pi'$ be an arbitrary permutation and $m = \{c, c'\} \in M$ the next meeting. Let $i$ and $j$ be the positions of the characters in the permutation, that is, $\pi'_i = c$ and $\pi'_j = c'$. Without loss of generality, assume $i < j$. If $\pi'$ does not support $m$, we can realize it using the block crossings $(i, i, j − 1)$, that is, moving the line of $c$ directly above that of $c'$.

Proposition 18. There is an instance $S$ of 2-SBCM and a start permutation $\pi^0$ such that there is no optimal solution $(\pi^0, B)$ of $S$ that starts with $\pi^0$ and uses at most one block crossing before the first and between each pair of consecutive meetings.

Proof (by contradiction). Consider the instance $S = (C, M)$ with

$$C = \{1, 2, 3, 4, 5, 6, 7, 8\} \text{ and } M = \{(6, 3), (7, 2), (1, 5), (5, 6), (6, 3), (3, 4), (4, 8), (8, 7)\}.$$  

Let $\pi^0 = \langle 1, 2, 3, 4, 5, 6, 7, 8 \rangle$ be the start permutation. There is a solution that performs only two block crossings, namely $(\pi^0, B)$ with $B = [(2, 4, 7), (4, 5, 8)]$, see Fig. 9. Let $\pi^1$ be the permutation after the first block crossing of $B$ on $\pi^0$, and $\pi^2$ the permutation after both block crossings. The permutation $\pi^2$ supports all meetings in $M$. The first meeting $\{6, 3\}$ in $M$ does not fit $\pi^0$ or $\pi^1$, that is, both block crossings occur before the first meeting.

Now assume there is another solution $(\pi^0, B')$ with $|B'| \leq 2$ that has at most one block crossing before each meeting. Starting from $\pi^0$ there are exactly nine feasible block crossings that allow the first meeting. They yield the following permutations:

![Fig. 9. Optimal solution for $S$ from the proof of Proposition 18.](image-url)
None of these permutations supports the second meeting \{7, 2\}. So we need the second block crossing before this meeting. This second block crossing needs to prepare all of the remaining meetings, because otherwise \(|B'| > 2\). These meetings can only be supported by the permutation \(\sigma = (1, 5, 6, 3, 4, 8, 7, 2)\) or its reverse permutation \(\sigma^R\). It remains to show that none of the permutations yielded by the feasible first block crossing can be transformed to \(\sigma\) or \(\sigma^R\) by one additional block crossing. All permutations containing \(\langle 3, 6 \rangle\) as a subsequence are infeasible because there is only one block crossing that swaps two neighboring characters and it does not produce \(\sigma\). For permutations starting with \(\langle 1, 2 \rangle\) there is only one possible block crossing to bring 2 to the end of the permutation while 1 stays at the first position, which also does not yield \(\sigma\). Similarly, we can show that there is also no block crossing after any of the feasible block crossing for the first step that leads to \(\sigma^R\).

\[\Box\]

## B  NP-Hardness without Repetitions

With arbitrarily large meetings, we can slightly modify our hardness proof, and show that minimizing the number of block crossings is also hard without repeating the same meeting many times. The idea to change our reduced instance, is to replace the repeated sequence of 2-character meetings so that in each repetition the group size is increased by one for all meetings; see Fig. 10.

Due to the overlapping structure of the groups in a single sequence, they can only be all supported at the same time if also the 2-character meetings that they replaced are supported. The only thing that we have to be careful about is that when the groups get larger than \(k/2\) there is a growing set of characters in the middle that are contained in exactly the same groups, and their relative order does not matter. We will avoid that this happens.

Fig. 10. Simulating repeated 2-character meetings using groups of increasing size
Since we have \( k + 1 \) sequences of repeated meetings at the beginning as well as at the end of the timeline, and we keep increasing the group sizes, we have groups of \( 2k + 3 \) characters in the end. We replace \( c_1, \ldots, c_{2k} \) by a new sequence \( c_1, \ldots, u_{5k} \) of characters without changing anything else on the structure. Then, we can increase the group size up to \( 2k + 3 \) while in the end still less than half of all characters are involved in each group. Since the growing meetings completely simulate the desired 2-character meetings, the rest of the reduction and its proof stay the same, and we get the following result.

**Theorem 19.** \( SBCM \) is \( NP \)-hard even if meetings are not repeated.

### C Exact Algorithms: Proofs

**Lemma 6** A sequence of arbitrarily interleaved \textsc{BlockMove} and \textsc{Check} operations can be performed in \( O(\beta + \mu) \) time, where \( \beta \) is the number of block crossings and \( \mu \) is sum of cardinalities of the meetings given to \textsc{Check}. Space usage is \( O(k) \).

*Proof.* Represent the permutation as a doubly-linked list. Then it takes constant time to check whether a 2-meeting fits: check the previous/next pointers. Since a block crossing changes at most 6 adjacencies, a \textsc{BlockMove} can update the linked list in constant time.

Now we look at a meeting of cardinality \( m \). Interpret the linked list as a path and consider the subgraph induced by the nodes in the meeting. If the meeting fits the permutation, this subgraph is connected and, being a path, has \( m - 1 \) edges; if the meeting does not fit, this subgraph has more components and therefore fewer edges. The \textsc{Check} operation on a meeting of size \( m \) can be performed in \( O(m) \) time by counting at every node in the meeting whether zero, one or two of its neighbors are also in the meeting. For the amortized runtime over a sequence of operations, remember this count: \textsc{BlockMove} can update it in constant time, since again at most 6 adjacencies change.

In terms of space, there is only the doubly linked list and the count. \( \square \)

### D SBCM with Meetings of Two Characters: Proofs

For the following lemma, we assume that no two subsequent meetings in the input are the same. We call an instance *normal* if this is the case. An instance can be normalized by simply dropping the repeated meetings. This does not affect the optimum number of block crossings or the behavior of the greedy algorithm, but note that it does lower \( n \).

**Lemma 20.** A normal instance of 2-SBCM with \( k = 3 \) can be solved using at most \( \lceil n/2 \rceil - 1 \) block crossings.
Proof. Note that there are only three possible meetings, namely \{1, 2\}, \{1, 3\}, and \{2, 3\}. Any permutation supports precisely two of these and not the third, and is equivalent in this sense to its reverse. For example, the permutation \langle 1, 2, 3 \rangle and its reverse support the meetings \{1, 2\} and \{2, 3\}, but not \{1, 3\}. Let \(\pi\) and \(\pi'\) be distinct permutations. Case distinction shows that it is always possible in a single block crossing to get from \(\pi\) to either \(\pi'\) or its reverse.

For the analysis, we partition the sequence of meetings into epochs as follows. We start from the first meeting and keep going until the third distinct meeting occurs: these meetings form the first epoch. That is, an epoch alternates between two different meetings. Repeating this process partitions the entire sequence of meetings into epochs, possibly with a single remaining meeting as final epoch. A solution can choose the start permutation \(\pi_0\) that supports the first epoch. After that it can always get to a permutation that supports the entire next epoch in one block crossing. In the worst case all epochs have length 2, and we need \(\lceil n/2 \rceil - 1\) block crossings.

\[\Box\]

**Theorem 9** For \(k = 3\), the greedy algorithm produces optimal solutions.

Proof. We look at the epochs from Lemma 20 again. The greedy algorithm produces one block crossing fewer than the number of epochs.

Consider any epoch except the last one and include the meeting after it. By construction, this is the third distinct meeting and therefore these meetings together cannot fit a single permutation. Then in any solution to the problem, a block crossing must occur after at least one of the meetings in the epoch. This holds for all epochs except the last one and since they are disjoint, the number of epochs reduced by one is a lower bound for the optimum number of block crossings. The result of the greedy algorithm realizes this bound.

\[\Box\]

**E Improved Approximation for 2-SBCM**

By using specific structures for 2-character meetings we can improve approximation factor and runtime (the general algorithm yields an 18-approximation).

Note that for 2-character meetings the group hypergraph is a graph, and an interval hypergraph here is a collection of vertex-disjoint paths. Our algorithm for INTERVAL HYPERGRAPH EDGE DELETION for \(d = 2\) yields a 3-approximation. We develop a better approximation using the following observation. Consider a character \(c\) in the collection of paths supported in the beginning of some solution. If \(c\) has two neighbors \(c_1\) and \(c_2\) in its path, but \(c\)'s first meeting is with a character \(c_3 \notin \{c_1, c_2\}\), then at the beginning of that meeting \(c\) can only be neighbor to one of the two, say, to \(c_1\), even in an optimal solution; the meeting with \(c_2\) then must later be reconstructed by block crossings. Hence, the effective set of meetings supported in the beginning is in fact a collection of paths with the additional restriction that each character is adjacent to at most one character except for the one he meets first. Without changing the rest of the analysis, we can approximate this new problem for finding the start permutation.
We first consider, for each vertex $c$, all edges incident to $c$ except for the one describing $c$’s first meeting. If there are $\ell \geq 2$ such edges, we know that even the optimal solution can support at most one of them and, hence, has to remove $\ell - 1$ of them. We remove all $\ell$ of them, which yields an approximation factor of $\ell/(\ell - 1) \leq 2$. Eventually, all vertices have degree 2 or less and the connected components are paths and cycles. For each cycle, we remove one arbitrary edge, so that we end up with a collection of paths. This second step does not change the approximation factor since the optimal solution has to remove at least one edge per cycle as well. This algorithm easily runs in linear time, which speeds up the runtime of the complete algorithm to $O(kn)$.

**Theorem 21.** We can find a $12$-approximate solution for 2-SBCM without repetitions in $O(kn)$ time.

## F Interval Hypergraph Edge Deletion

**Lemma 22.** Let $\mathcal{H} = (V, E)$ be an $\mathcal{F}$-free hypergraph. Let $C$ be a cycle appearing as a subhypergraph in $\mathcal{H}$. Then two edges of $C$ have a common vertex if and only if they are consecutive in $C$ or they share a common neighbor in $C$.

**Proof.** No edge of the $C$ can fully contain another edge of $C$. Let $e_1, e_2, e_3 \in C$ be three edges of $C$, and assume that $e_1 \cap e_2 \cap e_3 \supseteq \{v\} \neq \emptyset$. If there are vertices $v_1 \in e_1 \setminus (e_2 \cup e_3)$, $v_2 \in e_2 \setminus (e_1 \cup e_3)$, and $v_3 \in e_3 \setminus (e_1 \cup e_2)$, the three hyperedges form a subhypergraph of type $M_1$ (with $v_1, v_2, v_3$, and $v$ serving as vertices); see Fig. 11a.

On the other hand, if one of the three, say, $v_2$ does not exist, we have $e_2 \subseteq e_1 \cup e_3$ and one easily checks that this can only be the case if the three edges are consecutive on the cycle, since every vertex of $e_2$ must also be a vertex of $e_1$ or $e_3$; see Fig. 11b.

Now, assume that there are two edges $e, e' \in C$ with $e \cap e' \supseteq \{v\} \neq \emptyset$ that are neither consecutive nor have a common neighboring hyperedge in $C$. As we have seen, $v$ can be contained in none of the neighbors of $e$ and $e'$ in $C$. Let $e_1$ and $e_2$ be the neighbors of $e$ in $C$. If either of the two intersects with $e'$, we find $C_3$ as a subhypergraph, a contradiction. Hence, there are elements $v_1 \in e_1, v_2 \in e_2,
and \( v' \in e' \) so that each of the vertices is contained in no other of the four involved hyperedges. With these vertices, we have found \( O_1 \) as a subhypergraph; see Fig. 11c.

With this lemma, we know about the structure of the vertices contained in hyperedges of a cycle: A vertex can be contained in at most three hyperedges of the cycle, where the case of three hyperedges with a common vertex occurs only if a hyperedge is contained in the union of its two neighbors in the cycle.

Assume that \( e_1, e_2, \) and \( e_3 \) are three consecutive edges of a cycle \( C \). If all three edges are present in an interval representation for part of the edges of \( H \), we know that in the order we will first encounter vertices that are only contained in \( e_1 \), then vertices that are in \((e_1 \cap e_2) \setminus e_3\), followed by vertices of \((e_2 \cap e_3) \setminus e_1\), and vertices of \((e_3 \setminus (e_1 \cup e_2))\). Some of these sets (except for the pairwise intersections) may be empty. We do not know anything about the relative order of vertices within one of these sets, but we know the relative order of any pair of vertices of different sets; see Fig. 12. By generalizing this to the whole cycle, we get a cyclic order—describing the local order in a possible interval representation—of sets defined by containment in 1, 2, or 3 hyperedges. We call these sets cycle-sets, and their cyclic order the cycle-order of \( C \).

**Lemma 23.** Let \( H = (V, E) \) be an \( F \)-free hypergraph and let \( C \) be a cycle appearing as a subhypergraph in \( H \). There is no hyperedge \( e \in E \) that contains both vertices of edges of \( C \) and at least one vertex \( v \notin \bigcup_{e' \in C} e' \).

**Proof.** Assume to the contrary that such a hyperedge \( e \) exists. If \( e \) contains at least one vertex that lies in the intersection of two edges of \( C \), then we find a subhypergraph \( M_k \) (with a \( k \leq d - 1 \)) as follows. Assume \( v' \in e \cap e_1 \cap e_2 \) with edges \( e_1, e_2 \) consecutive on \( C \). From \( v' \) on we follow \( C \) in both directions as long as we find vertices in the intersection of consecutive cycle edges that also belong to \( v \). This process must stop eventually, since \( e \) can contain at most \( d - 1 \) vertices of cycle edges, while \( C \) has length at least \( d + 2 \). Together with two more vertices of the next intersections of cycle edges (that are not in \( e \)), we have found a path that, with \( e \) and \( v \), forms a subhypergraph of the type \( M_k \); see Fig. 13.

Now, we know that \( e \) cannot contain a vertex that lies in two cycle edges, but there can still be an edge \( e' \) of \( C \) with a vertex \( v' \in e \cap e' \). However, by using \( e, e' \), and the two neighbors of \( e' \) in \( C \) we immediately find \( O_1 \) as a subhypergraph (just as in Fig. 11c).
Fig. 13. $M_k$ as a subhyperedge if $e'$ contains vertices involved in the cycle.

As a consequence of the previous lemma, the hyperedges of two different cycles either cover the exactly same set of vertices, or their sets of vertices are disjoint. This also means that each connected component is either acyclic, or forms a ground for a set of cycles. We now try to analyze the structure of cycles on such a connected component in order to break all remaining cycles optimally.

If two cycles share their vertex sets, we can analyze how an edge of the one cycle relates to the structure—the cycle sets and their order—of the other cycle. Recall that we know about the relative order of the cycle-sets, but not of the internal order of vertices within the same cycle-set. Another edge can contain a cycle-set completely, can be disjoint from it, or can contain only part of its vertices. We call a consecutive sequence of cycle-sets contained in edge $e$—potentially starting and ending with cycle-sets partially contained in $e$—an interval of $e$ on $C$. The following lemma shows that every edge forms only a single interval on a given cycle.

**Lemma 16** Let $H = (V,E)$ be an $F$-free hypergraph. Let $C$ be a cycle appearing as a subhypergraph in $H$ and let $e \in E$ be a hyperedge on the same vertex set $\bigcup_{e' \in C} e'$. If $e$ intersects two cycle-sets, then $e$ must fully contain the vertices of all cycle-sets lying in between in one of the two directions along the cycle.

**Proof.** Assume that the claim is not true, that is, $e$ consists of a collection of at least two intervals of (partially) contained cycle-sets, where any two such intervals are separated by a vertex not in $e$ lying in a cycle-set. We distinguish cases similar to the proof of the previous lemma. First, assume that one such interval contains a vertex of the cycle. We follow the cycle in both directions from that vertex, as long as we find a vertex of $e$ in the intersection of the current edge with the next one along the cycle. Since $e$ has at most $d$ vertices but $C$ has length at least $d + 2$, this process will eventually stop, thus forming a path of length at least two, whose first and last vertices are not in $e$, but all internal vertices are. Now, assume that there is another vertex $v \in e$ that is contained in none of the edges of the path. Then, we have found $M_k$ as a subhypergraph.

On the other hand, if there is no such vertex $v$, we still know that there must be more than one interval formed by $e$. Hence, there especially must be a vertex $v'$ in a cycle-set separating two consecutive internal vertices $v_1, v_2$ of the path that is not contained in $e$. Let $e'$ be the edge of $C$ connecting $v_1$ and $v_2$; see Fig. 14a. If none of the neighbors of $v_1$ and $v_2$ along the cycle lies in $e$, we have found a $M_2$-subhypergraph as in Fig. 14a. If the neighbor of only one of
them, say, \( v_1 \) is in \( e \) but the neighbor of \( v_2 \) isn’t, then by disregarding \( v_1 \) we find an \( M_1 \)-subhypergraph centered on \( v_2 \); see Fig. 14a. On the other hand, if both neighbors lie in \( e \), then we have \( O_2 \) as a subhypergraph; see Fig. 14c.

If \( e \) contains no element in the intersection of any two consecutive cycle edges, then we take vertices \( v \) and \( v' \) from two different intervals; \( v \in e_1 \) and \( v' \notin e_1 \) for a cycle edge \( e_1 \). Let \( e_0 \) and \( e_2 \) be the neighbors of \( e_1 \) in \( C \). We have, \( v' \notin e_0 \cup e_2 \) since otherwise there would be a triangle. Then, \( e_0, e_1, e_2, \) and \( e \) (via \( v' \)) form \( O_1 \) as a subhypergraph.

Since \( e \) forms only a single interval of cycle-sets, we know that by opening the cycle at a single position within a cycle-set not contained in \( e \), \( C + e \) forms an interval hypergraph. \( e \) adds further information on the relative order within some cycle-sets. If only part of the vertices of a cycle-set are contained in \( e \) and also vertices of the next cycle-set in one direction, we know that the vertices of \( e \) in the first cycle-set should be next to the second cycle-set.

We use this to refine the cycle-sets to a cell structure with a cyclic order of cells, the cell order. A cell is just a set of vertices that must be contiguous in the cyclic order prescribed by hyperedges. Initially, the cells are the cycle-sets. Then, in each step we refine the cell-order by inserting an edge containing vertices of more than one cell, possibly splitting two cells into two subcells each. If after refining the cell order, it is still true that each remaining edge forms a single interval, then this results in a final refined cell order, where each remaining edge of the connected component must be fully contained in one of the cells. The following lemma shows that the interval property is indeed preserved during the process of refinements.

**Lemma 17.** Let \( \mathcal{H} = (V, E) \) be an \( F \)-free hypergraph. Let \( C \) be a cycle appearing as a subhypergraph in \( \mathcal{H} \). If we initialize the cell order with the cycle-sets of \( C \) and keep refining the structure by considering edges that contain vertices of at least two different cells, then the following interval property holds for any hyperedge \( e \in \mathcal{H} \) on the vertex set \( \bigcup_{e' \in C} e' \):

If \( e \) intersects two cells, then \( e \) must fully contain the vertices of all cells lying in between in one of the two directions along the cyclic order.

**Proof.** We show the property by induction over the insertions. Due to Lemma [16] it holds in the beginning. Now, assume that the interval property holds for the
cell order after inserting a set of edges. We show that after refining the cells by considering another edge $e'$, the property still holds.

Assume that for the refined cells the interval property does not hold for an edge $e$. Since the property did hold for the cells of the previous step, the only problem can occur in a cell $c$ of the previous step that is only partially contained by both $e$ and $e'$. Without loss of generality, we can assume that $e'$ also contains elements of the cell right of $c$; let $c_1$ and $c_2$ in this order be the (nonempty) cells resulting from splitting $c$, i.e., $c_1 = c \setminus e'$ and $c_2 = c \cap e'$. There are two basic cases in which the interval property could be violated for $e$.

First, if $e$ contains also elements of the cell right of $c$, then we have a violation only if there are vertices $v_1 \in c_1 \cap e$ and $v_2 \in c_2 \setminus e$. We distinguish cases based on the right boundary of cell $c$, which—from left to right—can either be closing or opening one (or more) hyperedge $\bar{e}$.

First assume that it is closing $\bar{e}$; $\bar{e}$ fully contains $c$ and at least also the cell left of $c$. If there is a common vertex of $e$ and $e'$ in the cell right of $c$, then we find $C_3$ as a subhypergraph with $e$, $e'$, and $\bar{e}$; see Fig. 15a. Otherwise, there are vertices in the next cell that are unique for $e$ and $e'$, respectively. Since we never inserted an edge completely contained in cells, this must also have held for $\bar{e}$. Therefore, there must be an edge (apart from $e$ and $e'$) containing some (but not all) cells from $\bar{e}$ and cells either left or right of $\bar{e}$. If such an edge $\bar{e}$ contains cells to the right, then it must especially contain cell $c$ and the cell right of it. Together with a vertex in $\bar{e}$ not contained in $\bar{e}$, we have found $O_2$ as a subhypergraph; see Fig. 15b. On the other hand, if $\bar{e}$ contains cells of $\bar{e}$ and cells left of it, then we find $O_1$ as a subhypergraph by adding a vertex in $\bar{e} \cap \bar{\bar{e}}$ (not in $c$) and a vertex in $\bar{e} \setminus \bar{\bar{e}}$; see Fig. 15c.

Now, assume that $\bar{e}$ is opening on the right boundary of $c$. If the cell right of $c$ contains no common element of $e$ and $e'$, the situation is symmetric to the one we had before by exchanging the role of $c$ with the cell right of $c$; see Fig. 16a. Otherwise, there is an element of $e \cap e'$ in the next cell. If $\bar{e}$ contains an element not in $e \cup e'$, then we have found $M_1$ as a subhypergraph; see Fig. 16b. We know that there must be at least one previously inserted hyperedge $\bar{e}$ overlapping with $\bar{e}$. Assume that $\bar{e}$ is overlapping from the left. If there is a vertex of $e \cap e'$ in $\bar{e} \setminus \bar{\bar{e}}$, we have found a $C_3$-subhypergraph; see Fig. 16c. Otherwise, there must be a vertex in $\bar{e} \setminus \bar{\bar{e}}$ that is contained in only one of $e$ and $e'$, say, in $e$, and we find $F_1$ as a subhypergraph; see Fig. 17a. Now, assume that $\bar{e}$ is overlapping with $\bar{e}$ coming from the right. If $\bar{e} \cap \bar{\bar{e}}$ contains a vertex of only one of the sets, say, $e$, we consider $\bar{e} \setminus \bar{\bar{e}}$. If there is a vertex not in $e$ (and not in $e'$), we have found $M_2$; see
(a) Symmetric situation.

(b) $M_1$.

(c) $C_3$.

**Fig. 16.** $e$ and $e'$ overlapping in cell $c$ from the same direction.

(a) $F_1$.

(b) $M_2$.

(c) $M_k$.

**Fig. 17.** $e$ and $e'$ overlapping in cell $c$ from the same direction.

Fig. 17b. (If $\tilde{e} \cap \overline{e}$ contains a vertex of $e \cap e'$, we find $M_1$ instead). Otherwise, since $\tilde{e} \cup \overline{e}$ must overlap with at least one more edge, we can continue to explore more edges. As long as there is a hyperedge overlapping with the hyperedges starting from $\tilde{e}$ and $\overline{e}$ to the right, we choose the one ending rightmost, thus forming a path of hyperedges that is extending to the right. If this process eventually finds a vertex that is neither in $e$ nor in $e'$, we find $M_1$ as a subhypergraph; see Fig. 17c. If we do not reach a vertex not in $e$ nor $e'$ with the path because there are no more edges overlapping from the right, we know that there must be an edge overlapping the whole path from the left (otherwise, the edges of the path would not have been inserted before). Let $\overline{e}'$ be this edge. Now, if there is a vertex of $e \cap e'$ not contained in $\overline{e}'$, we have found $C_3$ as a subhypergraph; see Fig. 18a. Otherwise, the part of the path outside of $\overline{e}'$ contains a vertex that is only in one of the hyperedges, say, in $e$. Then, the forbidden subhypergraph that we find is $F_k$ with $\overline{e}'$ and $e$ as the big hyperedges; see Fig. 18b.

Now, we can consider the second case in which we get a contradiction to the interval property after inserting $\overline{e}'$: Again, let $\overline{e}'$ split a cell $c$ into $c_1$ and $c_2$ as before. Then, $e$ contains vertices from the cell left of $c$, at least one vertex $v_2$ of $c_2$, but there is also a vertex $v_1 \in c_1 \setminus e$, i.e., $e$ does not completely contain $c_1$. We know that there must be at least one edge containing cell $c$. First, assume

(a) $C_3$.

(b) $F_k$.

**Fig. 18.** $e$ and $e'$ overlapping in cell $c$ from the same direction.
that such an edge $\tilde{e}$ exists and there are vertices $v \in e \setminus \tilde{e}$ and $v' \in e' \setminus \tilde{e}$. Then, we find $M_1$ as a subhypergraph; see Fig. 19a.

If no such $\tilde{e}$ exists, we know that any edge containing $c$ must fully contain at least one of $e$ and $e'$ as a subset. On the other hand, we know that there must be at least one edge overlapping with $e$ and one edge overlapping with $e'$ (and by now, these two edges must be different). Assume that there are edges $\tilde{e}$ overlapping with $e$ and fully containing $e'$ and $\bar{e}$ overlapping with $e'$ and fully containing $e$. Then we find $F_1$ as a subhypergraph; see Fig. 19b.

Now, assume that there is only a hyperedge $\tilde{e}$ overlapping with $e$ and fully containing $e'$; among these edges let $\tilde{e}$ be the one ending leftmost and (among the ones ending leftmost) the shortest one. We know that there must be at least one edge overlapping with $e'$, but no such edge can go to the right (and contain $c$), otherwise we would be in one of the previous cases. Let $\bar{e}$ be the hyperedge overlapping with $e'$ and ending leftmost. If $\bar{e}$ contains a vertex not contained in $\tilde{e}$, then we have found $M_2$ as a subhypergraph; see Fig. 20a. Otherwise, we continue searching for the leftmost starting hyperedge overlapping with $\tilde{e}$, forming a path of hyperedges reaching to the left. If eventually we reach at a vertex not contained in $\tilde{e}$, then we have found an $M_k$-subhypergraph; see Fig. 20b. On the other hand, if the path ends before reaching out of $\tilde{e}$, by considering the union of the path hyperedges starting from $\tilde{e}$, we know that there must be a hyperedge overlapping from the right. Due to the choice of $\tilde{e}$, this hyperedge may or may not overlap with $e$, but it must contain an element of $e$ that is not contained in $\tilde{e}$. Therefore, we find $F_k$ as a subhypergraph; see Fig. 21a.

In the remaining case, each edge containing cell $c$ must fully contain both $e$ and $e'$. Let $\tilde{e}$ be the edge containing $c$ that is shortest and starts leftmost.
Both for \( e \) and \( e' \) we know that there is at least one edge previously inserted that overlaps with them. Similarly to the argument before, we can start with the leftmost overlapping for \( e' \) and the rightmost for \( e \) and build paths of overlapping edges into these directions until we reach a vertex outside of \( \tilde{e} \), or we find no further hyperedge to extend the path. If both paths leave \( \tilde{e} \), we find an \( M_k \)-subhypergraph; see Fig. 21b. Now, assume only the one for \( e \) reaches out of \( \tilde{e} \), but the one for \( e' \) doesn’t (the other case is symmetric). Since the hyperedges of the path for \( e' \) have been inserted, there must still be a hyperedge overlapping with them. The only remaining possibility is then that this hyperedge \( \bar{e} \) extends to the right and contains \( c \) and fully contains both \( e \) and \( e' \). Due to the choice of \( \tilde{e} \) being the shortest hyperedge containing \( c \), \( \bar{e} \) must also contain at least one cell right of \( \tilde{e} \). Hence, we find an \( F_k \)-subhypergraph; see Fig. 22a. The remaining case is that neither path reaches out of \( \tilde{e} \). Then, apart from \( \bar{e} \), with the symmetric argument we find a hyperedge \( \bar{e}' \) that overlaps with the path for \( e \), fully contains \( e \) and \( e' \), and reaches out of \( \tilde{e} \) to the left. By using \( \bar{e}' \) in place of \( \tilde{e} \), we again find an \( F_k \)-subhypergraph; see Fig. 22b. This completes the proof.

The lemma shows that we can keep refining the cell-structure by inserting edges that contain vertices of at least two different cells. We end up with a cyclic order of cells so that each edge of the connected component that we did not insert lies completely within a single cell. Several edges can lie within the same cell, sharing vertices, and forming a small hypergraph that imposes further restrictions on the relative order of vertices within the cell. However, the cell contains fewer than \( d \) vertices. Hence, this small hypergraph cannot contain any long cycles and, since we removed all other forbidden subhypergraphs, must be an interval hypergraph.
Lemma 24. If for any two adjacent cells there is a hyperedge containing the vertices of both cells, we can find a cycle as a subhypergraph.

Proof. We start at an arbitrary cell \( c \). There must be a hyperedge \( e_1 \) containing both \( c \) and the next cell in clockwise order. We iteratively form a path by considering the rightmost cell explored so far and finding a hyperedge that contains that cell as well as the cell right of it. Since the number of cells is finite, we eventually reach the first cell. By dropping edges fully contained in other edges found, if necessary, we have a complete cycle. \( \square \)

Lemma 25. In any interval hypergraph that is obtained from the connected component there is at least one pair of neighboring cells so that all edges containing both cells have been removed.

Proof. The lemma is a direct corollary from Lemma 24 since if there is no such pair of cells, the condition of that lemma holds. \( \square \)

Lemma 26. Given a cyclic cell-order, let \( c \) and \( c' \) be a neighboring pair of cells in clockwise order. Removing all edges that contain both \( c \) and \( c' \) results in an interval hypergraph.

Proof. We number the cells \( c' = c_1, c_2, \ldots, c_k = c \) in clockwise order. Next, we place the vertices on a straight line so that vertices of each cell form an interval on the line and the cells appear as \( c_1, \ldots, c_k \) from top to bottom. Since the edges falling completely within a cell form an interval hypergraph, we put the vertices within a cell into an order that supports this interval hypergraph; recall that this is an interval hypergraph of constant size. Hence, each edge falling within a cell is supported.

Now consider an edge \( e \) that spans over several cells. If the interval that \( e \) spans is over cells \( c_i, \ldots, c_j \) with \( 1 \leq i < j \leq n \), it is supported by our order of vertices. On the other hand, if the cyclic interval of \( e \) is of the type \( c_i, \ldots, c_k, c_1, \ldots, c_j \) with \( 1 \leq j < j \leq k \), then \( e \) also contains the cells \( c = c_k \) and \( c' = c_1 \) and, therefore, has been removed. \( \square \)

G Interval Hypergraph Edge Deletion – Implementation in \( O(m^2) \) Time

The first phase of our algorithm consists mainly of searching for given subhypergraphs. In general, searching for a subhypergraph of parameterized size \( k \) is hard to achieve in time \( n^{o(k)} \) since this includes the hard search for \( k \)-cliques \([3]\). However, the structure of our problem allows us to do the search in \( O(m^2) \) time as follows. First, we check for cycles by considering any edge \( e \), choosing any pair \( v_1, v_2 \) of its up to \( d \) vertices (we have to try every pair), removing all edges containing both vertices, and then trying to find a shortest path from \( v_1 \) to \( v_2 \) using breadth-first search. If there is such a path of length \( k \leq d \), we have found \( C_{k+1} \), and we remove all its edges. Since any edge has to be considered only
once—it is then either removed or cannot be part of a short cycle—this part takes \(O(m^2)\) time.

For destroying the remaining types of forbidden subhypergraphs, we make use of the fact, that each of them contains an edge that contains all but 1 \((O_2\) and \(F_k\)), 2 \((M_k)\), or 3 \((O_1)\) vertices of the subhypergraph. We try each edge \(e\) to play that role. Since \(e\) has at most \(d\) vertices (constant), we can try each combination of its vertices for the vertices of the forbidden subhypergraph in the edge as shown in Fig. 8. Since there are only up to three more vertices not in \(e\) required, we could try all combinations for these and end up with an \(O(m^2n^3)\)-time algorithm. However, we can get rid of the factor \(n^3\) as follows. Suppose there are vertices \(v_1, v_2 \in e\) and hyperedges \(e_1, e_2\) so that \(v_1 \in e_1, v_2 \in e_2\) but \(v_1 \notin e_2\) and \(v_2 \notin e_1\). If there is a vertex \(v \in (e_1 \cap e_2) \setminus e\) in the intersection of \(e_1\) and \(e_2\) outside of \(e\), then \(v, v_1, v_2\) with the hyperedges \(e, e_1,\) and \(e_2\) form a \(C_3\)-subhypergraph; however, we have already removed short cycles, a contradiction.

Now, consider the search for \(O_1\). If for each of the three involved vertices in the larger hyperedge \(e\) we find a hyperedge containing vertices not in \(e\), then we must have found \(O_1\), otherwise the above argument yields \(C_3\). For the other forbidden subhypergraphs we must additionally check whether there is at least one hyperedge realizing exactly each of the necessary pairwise adjacencies within \(e\). For \(M_k, k \leq d - 1\), this suffices to check for an occurrence. For \(O_2\) we must also check whether there is a hyperedge containing the two nonadjacent vertices of \(e\) and an element not in \(e\). For \(F_k, k \leq d - 2\), we need a vertex in the intersection of a hyperedge \(e_1\) connecting the rightmost path-vertex to something outside of \(e\) with the second hyperedge \(e_2\) containing \(k + 2\) vertices. This can be checked in \(O(m)\) time by searching all feasible hyperedges and marking vertices outside of \(e\) if they lie in one such vertex. Note that no hyperedge realizing one of the pairwise adjacencies of \(F_k\) can contain such a vertex of \(e_1 \cap e_2 \setminus e\) since our above argument yields \(C_3\) in that case.

Summing up, we can test in \(O(m)\) time whether a given edge is the “large edge”—the edge of highest cardinality—of any of the forbidden subhypergraphs in \(O(m)\) time. Since after considering an edge it is either removed, or we know that it is not contained as large edge in any forbidden subhypergraph, we can make \(\mathcal{H} F\)-free in \(O(m^2)\) time.

Then, we determine the connected components in linear time, find a cycle for each of them and initialize the cell order, in \(O(n + m)\) time in total. For all components, the stepwise refinement can be done in \(O(m^2)\) time in total. Counting the numbers of hyperedges between adjacent cells, determining the optimum splitting point, as well as finding the final order, can all be done in linear time (since the size of edges is constant).

**Theorem 11** We can find a \((d+1)\)-approximation for INTERVAL HYPERGRAPH EDGE DELETION on hypergraphs with \(m\) hyperedges of rank \(d\) in \(O(m^2)\) time.
H Open Problems

While our paper yields insight into the complexity of several aspects of SBCM, several interesting problems remain open.

– Does the greedy algorithm yield an approximation for 2-SBCM? Can it be reasonably generalized to more than two characters per meeting? Can we find an optimal starting permutation in polynomial time?

– It is open if there always is an optimal solution for 2-SBCM that uses at most one block crossing between two meetings when the start permutation is not fixed. Our experiments strongly suggest some relations between $n$, $k$ and the optimum in random instances, but we have not properly investigated this.

– Can we get better results for any variant of the problem if we consider the start permutation part of the input and fixed?

– Can similar approximation results be obtained for simple crossings rather than block crossings? Since our analysis and algorithms heavily depend on the extended powers of block crossings, it seems hard to adjust our approach.