A DETERMINANT OF GENERALIZED FIBONACCI NUMBERS

CHRISTIAN KRATTENTHALER† AND ANTONIO M. OLLER-MARCÉN

Abstract. We evaluate a determinant of generalized Fibonacci numbers, thus providing a common generalization of several determinant evaluation results that have previously appeared in the literature, all of them extending Cassini’s identity for Fibonacci numbers.

1. Introduction

The well-known Fibonacci sequence is given by $f_n = f_{n-1} + f_{n-2}$ with $f_0 = f_1 = 1$. Numerous properties of this sequence are known. We refer the reader to the monograph [9] for a wealth of information on this sequence. One of these properties is the so called Cassini identity, given by

$$f_nf_{n+2} - f_{n+1}^2 = (-1)^n,$$

which can be written in matrix form as

$$\det \begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix} = (-1)^n. \quad (1.1)$$

Miles [6] introduced $k$-generalized Fibonacci numbers $f_n^{(k)}$ by

$$f_n^{(k)} = \sum_{i=0}^k f_{n-i}^{(k)},$$

with $f_n^{(k)} = 0$ for every $0 \leq n \leq k-2$, $f_k^{(k)} = 1$, and he gave the following generalization of (1.1):

$$\det \begin{pmatrix} f_n^{(k)} & f_{n+1}^{(k)} & \cdots & f_{n+k-1}^{(k)} \\ f_{n+1}^{(k)} & f_{n+2}^{(k)} & \cdots & f_{n+k}^{(k)} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n+k-1}^{(k)} & f_{n+k}^{(k)} & \cdots & f_{n+2k-2}^{(k)} \end{pmatrix} = (-1)^{(2n+k)(k-1)/2}. \quad (1.2)$$

More recently, Stakhov [8] has generalized Cassini’s identity for sequences of the form $f_n = f_{n-1} + f_{n-p-1}$.

Hoggat and Lind [4] consider the so called “dying rabbit problem”, previously introduced in [1] and studied in [2] or [3], which modifies the original Fibonacci setting by letting rabbits die. In previous work by one of the authors [7], the sequence arising in

2010 Mathematics Subject Classification. Primary 11B39; Secondary 05A10 05A19 11C20.

Key words and phrases. generalized Fibonacci numbers, Cassini’s identity, determinant evaluation.

†Research partially supported by the Austrian Science Foundation FWF, grants Z130-N13 and S9607-N13, the latter in the framework of the National Research Network “Analytic Combinatorics and Probabilistic Number Theory”.

1
this setting was studied in detail. For instance, the recurrence relation for this sequence depends on two parameters \( k, \ell \geq 2 \) and is given by

\[
C_n^{(k,\ell)} = C_{n-\ell}^{(k,\ell)} + C_{n-\ell-1}^{(k,\ell)} + \cdots + C_{n-k-\ell+1}^{(k,\ell)},
\]

where \( C_0^{(k,\ell)}, \ldots, C_{k+\ell-2}^{(k,\ell)} \) are initial values which will be specified below. It was also proved that, if \( r_1, \ldots, r_{k+\ell-1} \) are the distinct roots of \( g_{k,\ell}(x) = x^{k+\ell-1} - \frac{x^k-1}{x-1} \), then the general term of the sequence is given by \( C_n^{(k,\ell)} = \sum_{i=1} a_i r_i \), with

\[
a_i = \frac{(-1)^{k+\ell+i-1}}{\prod_{j>i} (r_j - r_i) \prod_{j<i} (r_i - r_j)} \times \left( \sum_{t=0}^{k-2} C_t^{(k,\ell)} \frac{r_{t+1}^i - 1}{r_i^t + (r_i - 1)} + \sum_{t=k-1}^{k+\ell-3} C_t^{(k,\ell)} \frac{r_{t+1}^i - 1}{r_i^t + (r_i - 1)} + C_{k+\ell-2}^{(k,\ell)} \right). \tag{1.3}
\]

Given the previous sequence, for every \( j \geq 0 \) we can define a matrix \( A_{j,k,\ell} \) by

\[
A_{j,k,\ell} = \begin{pmatrix}
C_j^{(k,\ell)} & C_{j+\ell}^{(k,\ell)} & \cdots & C_{j+k+2\ell-3}^{(k,\ell)} \\
C_{j+1}^{(k,\ell)} & C_{j+\ell+1}^{(k,\ell)} & \cdots & C_{j+k+2\ell-2}^{(k,\ell)} \\
\vdots & \vdots & \ddots & \vdots \\
C_{j+k+\ell-2}^{(k,\ell)} & C_{j+k+2\ell-2}^{(k,\ell)} & \cdots & C_{j+2k+3\ell-5}^{(k,\ell)}
\end{pmatrix}
\]

The main goal of this paper will be to find an explicit expression for \( \det(A_{j,k,\ell}) \), thus extending (1.1) and (1.2).

2. Extending Cassini’s identity

Before we proceed, we have to fix our initial conditions. In the original setting [7], when we start with a pair of rabbits that become mature \( \ell \) months after their birth and die \( k \) months after their maturness, the \( k+\ell-1 \) initial conditions are given by \( C_0^{(k,\ell)} = \cdots = C_{\ell-1}^{(k,\ell)} = 1 \) and \( C_n^{(k,\ell)} = C_{n-1}^{(k,\ell)} + C_{n-2}^{(k,\ell)} \) for every \( \ell \leq n \leq k+\ell-2 \). Instead, in what follows we will consider the following initial conditions:

\[
\widetilde{C}_0^{(k,\ell)} = 1,
\]

\[
\widetilde{C}_1^{(k,\ell)} = \cdots = \widetilde{C}_{k-1}^{(k,\ell)} = 0,
\]

\[
\widetilde{C}_k^{(k,\ell)} = \cdots = \widetilde{C}_{k+\ell-2}^{(k,\ell)} = 1.
\]

Note that this change in the initial conditions results only in a shift of indices. Namely, if \( C_n^{(k,\ell)} \) denotes the original sequence and \( \widetilde{C}_n^{(k,\ell)} \) denotes the sequence given by the same recurrence relation and these new initial conditions, then for every \( n \geq 0 \) we have

\[
C_n^{(k,\ell)} = \widetilde{C}_{n+k-1}^{(k,\ell)}.
\]

Thus, if \( \widetilde{A}_{j,k,\ell} \) is the corresponding matrix (defined in the obvious way), we have \( A_{j,k,\ell} = \widetilde{A}_{j+k+1,k,\ell} \). Hence, we can focus on finding a formula for \( \det(\widetilde{A}_{j,k,\ell}) \).
First of all, observe that \( \det(\tilde{A}_{j,k,\ell}) = (-1)^{k+\ell-2}\det(\tilde{A}_{j-1,k,\ell}) \) because \( \tilde{A}_{j,k,\ell} \) is obtained from \( \tilde{A}_{j-1,k,\ell} \) by replacing the first row by the sum of the first \( k \) rows of the matrix, and then permuting the rows so that the first row becomes the last one. If we apply this idea repeatedly, we obtain that \( \det(\tilde{A}_{j,k,\ell}) = (-1)^{(k+\ell-2)} \det(\tilde{A}_{0,k,\ell}) \). Hence, it is sufficient to compute this latter determinant.

We shall focus now on computing this determinant, which explicitly is

\[
\det(\tilde{A}_{0,k,\ell}) = \det \left( \begin{array}{cccc}
\hat{C}^{(k,\ell)}_0 & \hat{C}^{(k,\ell)}_1 & \cdots & \hat{C}^{(k,\ell)}_{k+2\ell-3} \\
\hat{C}^{(k,\ell)}_1 & \hat{C}^{(k,\ell)}_2 & \cdots & \hat{C}^{(k,\ell)}_{k+2\ell-2} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{C}^{(k,\ell)}_{k+2\ell-2} & \hat{C}^{(k,\ell)}_{k+2\ell-1} & \cdots & \hat{C}^{(k,\ell)}_{2k+3\ell-5}
\end{array} \right).
\]

To do so, recall that we have \( \hat{C}^{(k,\ell)}_n = \sum_{s=1}^{k+\ell-1} a_s r_s^n \), where the \( a_s \)'s are given by (1.3).

We substitute this in the above determinant and use multilinearity in the columns to expand it into the sum

\[
\sum_{1 \leq s_1, \ldots, s_{k+\ell-1} \leq k+\ell-1} \left( \prod_{j=1}^{k+\ell-1} a_{s_j} \right) \det_{1 \leq i \leq k+\ell-1} \left( r_{s_1}^{i-1} r_{s_2}^{i+\ell-1} r_{s_3}^{i+\ell} \cdots r_{s_{k+\ell-1}}^{i+k+2\ell-4} \right).
\]

Now, if in this sum two of the \( s_j \)'s should equal each other, then the corresponding two columns in the determinant would be dependent so that the determinant would vanish. We can therefore restrict the sum to permutations of \( \{1, 2, \ldots, k + \ell - 1\} \). With \( S_{k+\ell-1} \) denoting the set of these permutations, this leads to

\[
\det(\tilde{A}_{0,k,\ell}) = \sum_{\sigma \in S_{k+\ell-1}} \left( \prod_{j=1}^{k+\ell-1} a_{\sigma(j)} \right) \det_{1 \leq i \leq k+\ell-1} \left( r_{\sigma(1)}^{i-1} r_{\sigma(2)}^{i+\ell-1} r_{\sigma(3)}^{i+\ell} \cdots r_{\sigma(k+\ell-1)}^{i+k+2\ell-4} \right)
\]

\[
= \left( \prod_{j=1}^{k+\ell-1} a_j \right) \sum_{\sigma \in S_{k+\ell-1}} \left( \prod_{j=2}^{k+\ell-1} r_{\sigma(j)}^{j-1+j-2} \right) \det_{1 \leq i, j \leq k+\ell-1} \left( r_{\sigma(j)}^{i-1} \right)
\]

\[
= \left( \prod_{j=1}^{k+\ell-1} a_j \right) \left( \prod_{1 \leq i < j \leq k+\ell-1} (r_j - r_i) \right) \sum_{\sigma \in S_{k+\ell-1}} \left( \prod_{j=2}^{k+\ell-1} r_{\sigma(j)}^{j+\ell-2} \right)
\]

\[
= \left( \prod_{j=1}^{k+\ell-1} a_j \right) \left( \prod_{1 \leq i < j \leq k+\ell-1} (r_j - r_i) \right) \left( \prod_{i=1}^{k+\ell-1} \det_{1 \leq i \leq k+\ell-1} \left( r_i^{k+2\ell-3} \right) \right)
\]

\[
\times \left( \prod_{i=1}^{k+\ell-1} \det_{1 \leq i \leq k+\ell-1} \left( r_i^{-k+2\ell+3} r_i^{-k+\ell+3} \cdots r_i \right) \right)
\]
\[
\left( \prod_{j=1}^{k+\ell-1} a_j \right) \left( \prod_{1 \leq i < j \leq k+\ell-1} (r_j - r_i)(r_i^{-1} - r_j^{-1}) \right) \left( \prod_{i=1}^{k+\ell-1} r_i \right)^{k+2\ell-3} \times h_{\ell-1}(r_1^{-1}, \ldots, r_{k+\ell-1}^{-1}).
\]

(2.1)

In the last line we have used the following notations and facts: first of all, \(h_m(x_1, \ldots, x_N)\) denotes the \(m\)-th complete homogeneous symmetric function in \(N\) variables \(x_1, \ldots, x_N\), explicitly given by

\[
h_m(x_1, \ldots, x_N) = \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq N} x_{i_1} \cdots x_{i_m}.
\]

Furthermore, the Schur function indexed by a partition \(\lambda = (\lambda_1, \ldots, \lambda_N)\) in the variables \(x_1, \ldots, x_N\) is defined by

\[
s_\lambda(x_1, \ldots, x_N) = \frac{\det \left( x_i^{\lambda_j + N - j} \right)_{1 \leq i, j \leq N}}{\det \left( x_i^{N - j} \right)_{1 \leq i, j \leq N}} \prod_{1 \leq i < j \leq N} (x_i - x_j).
\]

It is well-known (cf. [5, p. 41, Eq. (3.4)]) that for \(\lambda = (m, 0, \ldots, 0)\) the Schur function \(s_\lambda(x_1, \ldots, x_N)\) reduces to \(h_m(x_1, \ldots, x_N)\). These facts together explain the last line in the above computation.

To proceed further, let us first observe that, by reading off the constant coefficient of \(g_{k,\ell}(x)\), we obtain

\[
\prod_{i=1}^{k+\ell-1} r_i = (-1)^{k+\ell}.
\]

Furthermore, we have

\[
g_{k,\ell}(x) = x^{k+\ell-1} - \frac{x^k - 1}{x - 1} = \prod_{i=1}^{k+\ell-1} (x - r_i) = (-1)^{k+\ell-1} \prod_{i=1}^{k+\ell-1} r_i(1 - r_i^{-1}x)
\]

\[
= - \prod_{i=1}^{k+\ell-1} (1 - r_i^{-1}x).
\]

Hence, we obtain

\[
\sum_{m=0}^{\infty} h_m(r_1^{-1}, \ldots, r_{k+\ell-1}^{-1}) x^m = \frac{1}{x^{k+\ell-1}} \prod_{i=1}^{k+\ell-1} (1 - r_i^{-1}x)
\]

\[
= \frac{1}{x^{k+\ell-1} - x^{k+\ell}}
\]

\[
= \frac{1 - x}{1 - x^{k+\ell-1} + x^{k+\ell}}
\]

\[
= 1 - x + x^k - x^{k+1} + \cdots + O(x^{k+\ell-1}).
\]

In order to evaluate \(h_{\ell-1}(r_1^{-1}, \ldots, r_{k+\ell-1}^{-1})\), we just have to extract the coefficient of \(x^{\ell-1}\) in the expansion on the right-hand side. This is easy: if \(\ell - 1\) equals a multiple of \(k\).
then we obtain 1, if \( \ell - 2 \) equals a multiple of \( k \) then we obtain \(-1\), and in all other cases we obtain 0.

We continue evaluating the other factors in (2.1). We have

\[
\prod_{1 \leq i < j \leq k + \ell - 1} (r_j - r_i)(r_i^{-1} - r_j^{-1}) = \prod_{1 \leq i < j \leq k + \ell - 1} \frac{(r_j - r_i)^2}{r_ir_j} = \frac{1}{\prod_{i=1}^{k+\ell-1} r_i} \left( \prod_{1 \leq i < j \leq k + \ell - 1} (r_j - r_i)^2 \right) = \frac{1}{(-1)^{k+\ell}} \left( \prod_{i=1}^{k+\ell-1} r_i \right)^2.
\]

Furthermore, we must compute \( \prod_{j=1}^{k+\ell-1} a_j \). To begin with, recall the formula (1.3) and the fact that \( \tilde{C}_0^{(k,\ell)} = \tilde{C}_k^{(k,\ell)} = \cdots = \tilde{C}_{k+\ell-2}^{(k,\ell)} = 1 \) and \( \tilde{C}_1^{(k,\ell)} = \cdots = \tilde{C}_{k-1}^{(k,\ell)} = 0 \). With this in mind, we get

\[
\prod_{j=1}^{k+\ell-1} a_j = \prod_{1 \leq i < j \leq k + \ell - 1} (-1)^{k+\ell+j-1} \prod_{j=1}^{k+\ell-1} \left( \frac{r_j - 1}{r_j(r_j - 1)} + \sum_{i=1}^{\ell-2} \frac{r_j^k - 1}{r_j^{k+i}(r_j - 1)} + 1 \right) = (-1)^{\frac{(3k+3\ell-2)(k+\ell-1)}{2}} \prod_{1 \leq i < j \leq k + \ell - 1} \left( \frac{1}{r_j} + \sum_{i=1}^{\ell-2} \frac{r_j^k - 1}{r_j^{k+i}(r_j - 1)} + 1 \right).
\]

Moreover, observe that

\[
\frac{1}{r_j} + \sum_{i=1}^{\ell-2} \frac{r_j^k - 1}{r_j^{k+i}(r_j - 1)} + 1 = \frac{1}{r_j} + \frac{r_j^k - 1}{r_j^k - 1 - 1} + 1 = \frac{1}{r_j} + \frac{r_j^{k+\ell-1} - 1}{r_j - 1} + 1 = \frac{r_j^{\ell} - 1}{r_j(r_j - 1)}.
\]

Here, to obtain the second line, we have used the fact that \( 1 \neq r_j \) is a root of \( x^{k+\ell-1} - \frac{x^k}{x-1} \).
Finally observe that for all integers $k, \ell \geq 2$, we have

$$\det(\tilde{A}_{0,k,\ell}) = \begin{cases} (-1)^{\frac{(k+\ell)(k+\ell-1)}{2} + 1}, & \text{if } \ell - 1 = \alpha k \text{ and } \gcd(\ell, k - 1) = 1, \\ (-1)^{\frac{(k+\ell)(k+\ell-1)}{2}}, & \text{if } \ell - 2 = \beta k \text{ and } \gcd(\ell, k - 1) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Corollary. Let $k_0, \ell_0 \geq 2$ be any integers. Then the following hold:

i) The sequence $\{\alpha_\ell\}_{\ell \geq 2}$ given by $\alpha_\ell = |\det(\tilde{A}_{0,k_0,\ell})|$ is periodic, and its period is a divisor of $k_0 \cdot \text{rad}(k_0 - 1)$.

ii) The sequence $\{\beta_\ell\}_{\ell \geq k}$ given by $\beta_\ell = |\det(\tilde{A}_{0,k_0,\ell})|$ is eventually zero.

Proof. i) Clearly $\gcd(\ell, k_0 - 1) > 1$ implies that $\gcd(\ell + k_0 \cdot \text{rad}(k_0 - 1), k_0 - 1) > 1$. In the same way, if $\ell - 1$ and $\ell - 2$ are not multiples of $k_0$, then neither are $\ell + k_0 \cdot \text{rad}(k_0 - 1) - 1$ or $\ell + k_0 \cdot \text{rad}(k_0 - 1) - 2$. Consequently, if $\alpha_\ell = 0$, also $\alpha_{\ell + k_0 \cdot \text{rad}(k_0 - 1)} = 0$ as claimed.

ii) If $k \geq \ell_0$ obviously neither $\ell - 1$ nor $\ell - 2$ can be multiples of $k$ and therefore $\beta_\ell = 0$ for every $k \geq \ell_0$. 

References

[1] Alfred B.U., Exploring Fibonacci numbers, Fibonacci Quart. 1 (1963), no. 1, 57–63.
[2] Alfred B.U., Dying rabbit problem revived, Fibonacci Quart. 1 (1963), no. 4, 482–487.
[3] Cohn, J.H.E., Letter to the editor, Fibonacci Quart. 2 (1964), 108.
[4] **Hoggat V.E., Lind D.A.**, *The dying rabbit problem*, Fibonacci Quart. 7 (1969), no. 5, 482–487.

[5] **Macdonald, I.G.**, *Symmetric functions and Hall polynomials*, 2nd ed., Oxford University Press, New York/London, 1995.

[6] **Miles, E.**, *Generalized Fibonacci numbers and related matrices*, Amer. Math. Monthly 67 (1960), 745–752.

[7] **Oller-Marcén, A.M.**, *The dying rabbit problem revisited*, Integers 9 (2009), 129–138.

[8] **Stakhov, A.P.**, *Fibonacci matrices, a generalization of the “Cassini formula”, and a new coding theory*, Chaos Solitons Fractals 30 (2006), no. 1, 1108–1109.

[9] **Vajda, S.**, *Fibonacci and Lucas numbers, and the golden section*, John Wiley & Sons, New York, 1989.

**Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Vienna, Austria.** WWW: [http://www.mat.univie.ac.at/~kratt](http://www.mat.univie.ac.at/~kratt)

**Centro Universitario de la Defensa, Ctra. de Huesca s/n, 50090 Zaragoza (España)**

**E-mail address:** oller@unizar.es