METRIC LINES IN THE JET SPACE.

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Abstract. Given a sub-Riemannian manifold, a relevant question is: what are the metric lines (isometric embedding of the real line)? The space of \( k \)-jets of a real function of one real variable \( x \), denoted by \( J^k(\mathbb{R}, \mathbb{R}) \), admits the structure of a Carnot group, as every Carnot group \( J^k(\mathbb{R}, \mathbb{R}) \) is a sub-Riemannian Manifold. This work is devoted to provide a partial result about the classification of the metric lines in \( J^k(\mathbb{R}, \mathbb{R}) \).

The method to prove the main Theorems is to use an intermediate 3-dimensional sub-Riemannian space \( \mathbb{R}^3_F \) lying between the group \( J^k(\mathbb{R}, \mathbb{R}) \) and the Euclidean space \( \mathbb{R}^2 \cong J^k(\mathbb{R}, \mathbb{R})/\text{[}J^k(\mathbb{R}, \mathbb{R}), J^k(\mathbb{R}, \mathbb{R})\text{]} \).

1. Introduction

The space of \( k \)-jets of a real function of one real variable \( x \), denoted by \( J^k(\mathbb{R}, \mathbb{R}) \), admits the structure of a Carnot group, as every Carnot group \( J^k(\mathbb{R}, \mathbb{R}) \) is a sub-Riemannian Manifold. This work is the first of a sequence of papers, where we attempt to make a full classification of the metric lines in \( J^k(\mathbb{R}, \mathbb{R}) \). Let us introduce the definition of a metric line in the context of sub-Riemannian geometry.

Definition 1. Let \( M \) be a sub-Riemannian manifold, we denote by \( \text{dist}_M(\cdot, \cdot) \) the sub-Riemannian distance on \( M \). Let \( | \cdot | : \mathbb{R} \to [0, \infty) \) be the absolute value. We say that a geodesic \( \gamma : \mathbb{R} \to M \) is a metric line if

\[
|a - b| = \text{dist}_M(\gamma(a), \gamma(b)) \text{ for all compact set } [a, b] \subset \mathbb{R}.
\]

See Definition [1, sub-sub-Chapter 4.7.2] or [18, sub-Chapter 1.4] for the formal definition of a sub-Riemannian geodesic. An alternative term for “metric line” is “globally minimizing geodesic”.

In [12, Background Theorem], we showed a bijection between the set of pairs \((F, I)\) and the set of geodesics in \( J^k(\mathbb{R}, \mathbb{R}) \), where \( F \) is a polynomial of degree \( k \) or less, and \( I \) is a closed interval, called the hill interval, see Definition 3 below. The polynomial \( F \) defines a reduced Hamiltonian system \( H_F \), see equation (2.2) below, provided by the symplectic reduction of sub-Riemannian geodesic flow on \( J^k(\mathbb{R}, \mathbb{R}) \). In addition, we classified the geodesic in \( J^k(\mathbb{R}, \mathbb{R}) \) according to their reduced dynamics, that is, the geodesics in \( J^k(\mathbb{R}, \mathbb{R}) \) are line, \( x \)-periodic, homoclinic, heteroclinic of the direct-type.

Key words and phrases. Carnot group, Jet space, Global minimizing geodesic, sub-Riemannian geometry.
or heteroclinic of the turn-back, see sub-Section 2.1.2 or Figure 1.1. The Conjecture concerning metric lines in $J^k(\mathbb{R}, \mathbb{R})$ is the following.

**Conjecture 2.** The metric lines in $J^k(\mathbb{R}, \mathbb{R})$ are precisely geodesics of the type: line, homoclinic and the heteroclinic of the direct-type.

It is well known that the line geodesics are metric lines, see Corollary 9. In [12, Theorem 1], we proved geodesics of type $x$-periodic and heteroclinic turn-back are not metric lines. Theorem A is the first main result of this work and proves Conjecture 2 for the case of heteroclinic of the direct-type geodesics.

**Theorem A.** Heteroclinic of the direct-type geodesics are metric lines in $J^k(\mathbb{R}, \mathbb{R})$.

Conjecture 2 remains open for homoclinic geodesics. Theorem B is the second principal result of this work and provides a family of homoclinic geodesics that are metric lines.

**Theorem B.** The homoclinic-geodesic defined by the polynomial $F(x) = \pm(1 - bx^{2n})$ and hill interval $[0, \sqrt{\frac{2}{b}}]$ is a metric line in $J^k(\mathbb{R}, \mathbb{R})$ for all $k \geq 2n$ and $b > 0$.

**Previous Results.** In [6, 5, 7, 8], A. Andertov and Y. Sachkov proved Conjecture 2 for the case $k = 1$ and $k = 2$ using optimal synthesis. In [12, Theorem 2], we showed that a family of heteroclinic of the direct-type geodesics are metric lines.

The case $k = 1$ corresponds to $J^1(\mathbb{R}, \mathbb{R})$ being the Heisenberg group where the geodesics are $x$-periodic or geodesic lines. The case $k = 2$ corresponds to $J^2(\mathbb{R}, \mathbb{R})$ being Engel’s group, denoted by Eng. Besides geodesic lines, up to a Carnot translation and dilation Eng has a unique metric line such that its projection to the plane $\mathbb{R}^2 \simeq \text{Eng} / [\text{Eng}, \text{Eng}]$ is the Euler-soliton. The family of metric lines defined by Theorem B is the generalization of A. Andertov and Y. Sachkov’s result from [6, 5, 7, 8]. More specific, when $n = 1$ then the geodesic defined by the polynomial $F(x) = \pm(1 - bx^2)$ is the one whose projection to the plane $\mathbb{R}^2$ is the Euler-soliton. See [10, Section 4] for more details about the Euler-Elastica and geodesics in Eng. In addition, see [1, sub-sub-chapter 7.8.3] or [15, Chapter 14] for the relation of Euler-Elastica and some sub-Riemannian geodesics as the rolling problem and the Euclidean group.

**Our Method To Prove A Geodesic Is A Metric Line.** The optimal synthesis and the weak KAM theory are the two classical methods to prove that a geodesic is a metric line. See [15, sub-Chapter 9.4], [1, sub-Chapter 13.4] or [2, Chapter 13] for an introduction to the optimal synthesis. See [14] for more details of weak KAM theory on the Riemannian context, and see [18, sub-sub-chapter 1.9.2] or [12, Section 5] on the sub-Riemannian context. The optimal synthesis requires the explicit integration of the geodesic...
The images show the projection to $\mathbb{R}^2 \cong J^k(\mathbb{R}, \mathbb{R})/[J^k(\mathbb{R}, \mathbb{R}), J^k(\mathbb{R}, \mathbb{R})]$, with coordinates $(x, \theta_0)$, of geodesics in $J^k(\mathbb{R}, \mathbb{R})$. The first panel presents a generic $x$-periodic geodesic, the second panel displays the projection of a homoclinic geodesic, which is the Euler-soliton solution to the Euler-Elastica problem and correspond to case $n = 1$ from Theorem B. The third panel presents the projection of a turn-back geodesic, the forth panel displays the projection of a heteroclinic of the direct-type geodesic.

equations, and the weak KAM theory needs a global Calibration function. In both cases, the integrability of the flows is a necessary condition. Although the sub-Riemannian geodesic flow in $J^k(\mathbb{R}, \mathbb{R})$ is integrable, see [10, Theorem 1.1], these methods cannot prove the Conjecture 2. On one side, the explicit integration of the equation of motion is impossible in the general case. On the second side, the local Calibration functions, found in [12, Section 5] or [11, sub-Section 3.2], do not have a global extension.

Besides Theorem A and B, the main contribution of this work is the formalization of the method used in [12]. We will consider a sub-Riemannian manifold $\mathbb{R}^3_F$, called the magnetic space, and a sub-Riemannian submersion $\pi_F : J^k(\mathbb{R}, \mathbb{R}) \to \pi_F$. Thanks to the fact that lift of metric line is a metric line (Proposition 8), it is enough to prove that if $\gamma(t)$ is a sub-Riemannian geodesic corresponding to a polynomial $F$ and satisfying the conditions of Theorems A or B, then the projection $\pi_F(\gamma(t))$ is a metric line in $\mathbb{R}^3_F$. In other words, we reduce the problem of studying metric lines in $J^k(\mathbb{R}, \mathbb{R})$ to studying metric lines in the magnetic space $\mathbb{R}^3_F$. Theorems 29 and 40 show that the curve $c(t) := \pi_F(\gamma(t))$ is a metric line, where $\gamma(t)$ is sub-Riemannian geodesic given by Theorems A and B, respectively. To prove Theorems 29 and 40, we consider a sequence of minimizing sub-Riemannian geodesics $c_n(t)$ joining every time farther away points on the geodesic $c(t)$, see Figure 3.1. We show that the sequence has a convergent subsequence.
c_{n_j}(t)$ converging to a minimizing geodesic $c_{\infty}(t)$ corresponding to the polynomial $F$, since every two sub-Riemannian geodesics corresponding to the polynomial $F$ are related by sub-Riemannian isometry we conclude that $c(t)$ is a metric line.

**Outline.** Section 2 introduces the preliminary result necessary to prove Theorem A and B. Sub-Section 2.1 briefly describes $J^k(\mathbb{R}, \mathbb{R})$ as a sub-Riemannian manifold and summarizes some previous results from [12]. Between them, the most important are: the Background Theorem establishing the correspondence between sub-Riemannian geodesics and the pair $(F, I)$, the classification of sub-Riemannian geodesic, the formal definition of a sub-Riemannian submersion, and Proposition 8. Sub-Section 2.2 presents the magnetic space $\mathbb{R}_F^3$ and some previous results from [12]: The correspondence between sub-Riemannian geodesic in the magnetic space $\mathbb{R}_F^3$ and the pair $(G, I)$ where $G$ is a polynomial in a two-dimensional space $Pen_F$, the period map $\Theta(G, I)$, and an upper bound for the cut time. In addition, sub-Section 2.2 provides the relation between the sub-Riemannian geodesic in $\mathbb{R}_F^3$ and $J^k(\mathbb{R}, \mathbb{R})$, the cost function definition, and some sequence of sub-Riemannian geodesics’ properties.

The main goal of Section 3 is to prove Theorem 29. Sub-Section 3.1 presents a particular magnetic space for each heteroclinic geodesic of the direct-type and some essential properties of this space. In particular, Corollary 30 says that the Period map $\Theta(G, I)$ is one-to-one when restricted to space of heteroclinic geodesics of the heteroclinic direct-type. Sub-sub-Section 3.2.1 sets up the proof, sub-sub-Section 3.2.2 presents the proof of Theorem 29 and sub-sub-Section 3.2.3 provides the formal proof of Theorem A.

The main goal of Section 4 is to prove Theorem 40 and has a similar structure to Section 3. In addition, Section 4 introduces Theorem 41, which says that the sub-Riemannian geodesic corresponding to the polynomial $F(x) = 1 - 2x^{2n+1}$ is not a metric line in the magnetic space $\mathbb{R}_F^3$.

**Acknowledgment.** I want to acknowledge the mathematics department at the University of California Santa Cruz for allowing me to be in the Ph.D. program and helping me succeed in my degree. This work resulted from years of working, talks, and conversations in the mathematics department. In particular, I express my gratitude to my advisor, Richard Montgomery, for setting up the problem and his ideas to approach it. I thank Andrei Ardentov, Yuri Sachkov, and Felipe Monroy-Perez, for e-mail conversations throughout this work and whose work inspired me. This paper was developed with the support of the scholarship (CVU 619610) from ”Consejo de Ciencia y Tecnologia” (CONACYT)

2. **Preliminary**

Here we will introduce the necessary results to prove Theorems A and B.
2.1. $J^k(\mathbb{R}, \mathbb{R})$ as a sub-Riemannian manifold. We describe here briefly the sub-Riemannian structure on $J^k(\mathbb{R}, \mathbb{R})$. For more details about $J^k(\mathbb{R}, \mathbb{R})$ as a Carnot group and sub-Riemannian manifolds, see [1, sub-Chapter 10.2], [12, sub-Section 2.1], [10, Section 2] or [20, Section 3]. We see $J^k(\mathbb{R}, \mathbb{R})$ as $\mathbb{R}^{k+2}$, using $(x, \theta_0, \ldots, \theta_k)$ as global coordinates, then $J^k(\mathbb{R}, \mathbb{R})$ is endowed with a natural rank 2 distribution $\mathcal{D} \subset T J^k(\mathbb{R}, \mathbb{R})$ characterized by the $k$-Pfaffian equations

$$0 = d\theta_i - \frac{1}{i!} x^i d\theta_0, \quad i = 1, \ldots, k.$$ 

$\mathcal{D}$ is globally framed by two vector fields

$$X = \frac{\partial}{\partial x} \quad \text{and} \quad Y = \sum_{i=0}^{k} \frac{x^i}{i!} \frac{\partial}{\partial \theta_i}. \quad (2.1)$$

We declare these two vector fields to be orthonormal to define the sub-Riemannian structure on $J^k(\mathbb{R}, \mathbb{R})$. The sub-Riemannian metric is given by restricting $ds^2 = dx^2 + d\theta_0^2$ to $\mathcal{D}$. $J^k(\mathbb{R}, \mathbb{R})$ is endowed with a canonical projection $\pi : J^k(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^2 \simeq J^k(\mathbb{R}, \mathbb{R})/[J^k(\mathbb{R}, \mathbb{R}), J^k(\mathbb{R}, \mathbb{R})]$, which in coordinates is given by $\pi(x, \theta_0, \ldots, \theta_k) = (x, \theta_0)$.

2.1.1. Reduced System. As we proved in [12, Appendix A], a geodesic in $J^k(\mathbb{R}, \mathbb{R})$ is determined by the pair $(F, I)$. Let $F$ be a polynomial $F$ of degree $k$ or less, then the reduced Hamiltonian system $H_F$ is given by

$$H_F = \frac{1}{2}(p_x^2 + F^2(x)). \quad (2.2)$$

The condition $\frac{1}{2} = H_F$ implies that the reduced dynamics occur in the hill interval $I$ and the sub-Riemannian geodesic is parametrized by arc length. Let us formalize the hill interval definition.

**Definition 3.** We say that a closed interval $I$ is a hill interval associated to $F(x)$, if $|F(x)| < 1$ for every $x$ in the interior of $I$ and $|F(x)| = 1$ for every $x$ in the boundary of $I$. If $I$ is of the form $[x_0, x_1]$, then we call $x_0$ and $x_1$ the endpoints of the hill interval. We say that $\text{hill}(F)$ is the hill region of $F$ if $\text{hill}(F)$ is union of all the hill intervals of $F$.

By definition, if $F(x)$ is not a constant polynomial then $I$ is compact. In contrast, the constant polynomial $F(x)$ has hill interval $I = \mathbb{R}$ if $|F(x)| \leq 1$ and $I$ is equal to a single point if $F(x) = \pm 1$.

Here, we prescribe the method to build a sub-Riemannian geodesic: first, find a solution to the reduced system (2.2) with energy $\frac{1}{2} = H_F$. Second, having found the solution $(p_x(t), x(t))$, we define a curve $\gamma(t)$ in $J^k(\mathbb{R}, \mathbb{R})$ by the following equation

$$\dot{\gamma}(t) = \dot{x}(t) X(\gamma(t)) + F(x(t)) Y(\gamma(t)) \quad \text{where} \quad \dot{x}(t) = p(t).$$

The **Background Theorem** establishes the correspondent between the pair $(F, I)$ and the sub-Riemannian geodesics on $J^k(\mathbb{R}, \mathbb{R})$. 

**METRIC LINES IN THE JET SPACE.**
Background Theorem. The above prescription yields a geodesic in $J^k$ parameterized by arc length. Conversely, any arc length parameterized geodesic in $J^k$ can be achieved by this prescription applied to some polynomial $F(x)$ of degree $k$ or less.

The Background Theorem was proved first in [3, 4, 17], later we gave an alternative proof in [12, Appendix A].

2.1.2. Classification Of Geodesic In Jet Space. Let $\gamma(t)$ be a geodesic in $J^k(\mathbb{R}, \mathbb{R})$ corresponding to the pair $(F, I)$, in the case when $F(x)$ is no constant polynomial let assume $I = [x_0, x_1]$, then $\gamma(t)$ is only one of the following options:

- We say that a geodesic $\gamma(t)$ is a line if the projected curve $\pi(\gamma(t))$ is a line in $\mathbb{R}^2$. Line geodesics correspond to constant polynomial or trivial solutions of the reduced dynamics.
- We say $\gamma(t)$ is $x$-periodic if its reduced dynamics is periodic. The reduced dynamics is periodic if and only if $x_0$ and $x_1$ are regular points of $F(x)$.
- We say $\gamma(t)$ is homoclinic if its reduced dynamics is a homoclinic orbit. The reduced dynamics has a homoclinic orbit if and only if one of the points $x_0$ and $x_1$ is regular and the other is a critical point of $F(x)$.
- We say $\gamma(t)$ is heteroclinic if its reduced dynamics is a heteroclinic orbit. The reduced dynamics has a heteroclinic orbit if and only if both points $x_0$ and $x_1$ are critical of $F(x)$.
- We say a heteroclinic geodesic $\gamma(t)$ is turn-back if $F(x_0)F(x_1) = -1$.
- We say a heteroclinic geodesic $\gamma(t)$ is direct-type if $F(x_0)F(x_1) = 1$.

See figure 1.1 for a better understanding of these names.

2.1.3. Unitary Geodesics. To prove Theorem A and B, we will introduce the concept of a unitary geodesic.

Definition 4. We say a geodesic $\gamma(t)$ in $J^k(\mathbb{R}, \mathbb{R})$ corresponding to the pair $(F, I)$ is unitary if $I = [0, 1]$. We say a heteroclinic of the direct-type geodesic (or homoclinic) $\gamma(t)$ is unitary, if in addition $F(x(t)) \to 1$ when $t \to \pm\infty$.

Let us denote by $Iso(J^k(\mathbb{R}, \mathbb{R}))$, the isometry group of $J^k(\mathbb{R}, \mathbb{R})$. The reflection $R_{\theta_0}(x, \theta_0, \theta_1, \ldots, \theta_k) = (x, -\theta_0, \theta_1, \ldots, \theta_k)$ is in $Iso(J^k(\mathbb{R}, \mathbb{R}))$. Then, to classify metric lines it is enough to study unitary geodesics, since if $\gamma(t)$ is a heteroclinic of the direct-type or homoclinic geodesic such that $F(x(t)) \to -1$ when $t \to \pm\infty$, then $R_{\theta_0}(\gamma(t))$ is such that $F(x(t)) \to 1$ when $t \to \pm\infty$.

Corollary 5. Let $\gamma(t)$ be a unitary heteroclinic of the direct-type geodesic for $F(x)$, then there exists $q(x)$ such that $F(x) = 1 - x^{k_1}(1 - x)^{k_2}q(x)$, where $1 < k_1$, $1 < k_2$, and $q(x)$ is polynomial of degree $k - k_1 - k_2$ such that $0 < x^{k_1}(1 - x)^{k_2}q(x) < 2$ if $x$ is in $(0, 1)$. 

Proof. By construction, $F(x)$ is such that

$$F(0) = F(1) = 1, \quad F'(0) = F'(1) = 0 \quad \text{and} \quad |F(x)| < 1 \text{ if } x \text{ is in } (0, 1),$$

then using the Euclidean algorithm we find the desired result. □

The following Proposition tells us that every geodesic in $J^k(\mathbb{R}, \mathbb{R})$ is related to unitary geodesic by a Carnot dilatation and translation.

**Proposition 6.** Let $\gamma(t)$ be a geodesic in $J^k(\mathbb{R}, \mathbb{R})$ associated to the pair $(F,I)$ and let $h(\tilde{x}) = x_0 + u\tilde{x}$ be the affine map taking $[0,1]$ to $I = [x_0, x_1]$ with $u := x_1 - x_0$. If $\hat{F}(h(\tilde{x})) = F(x)$ and $\hat{\gamma}(t)$ is the geodesic in $J^k(\mathbb{R}, \mathbb{R})$ corresponding to the pair $(\hat{F}, [0,1])$. Then $\gamma(t)$ is related to $\hat{\gamma}(t)$ by Carnot dilatation and translation, that is

$$\gamma(t) = \delta_u \hat{\gamma}(\frac{t}{u}) * (x_0, 0 \ldots, 0),$$

where $\delta_u$ is the Carnot dilatation.

For more details about the Carnot dilatation see [18, sub-Chapter 8.2].

Proposition 6 and the reflection $R_{\theta_0}$ imply that it is enough to prove Theorem A and B for the unitary case.

2.1.4. **Sub-Riemannian Submersion.** Let us formalize the definition of sub-Riemannian submersion and present Proposition 8.

**Definition 7.** Let $(M, D_M, g_M)$ and $(N, D_N, g_N)$ be two sub-Riemannian manifolds and let $\phi: M \to N$ a submersion ($\dim(M) \geq \dim(N)$). We say that $\phi$ is a sub-Riemannian submersion if $\phi_* D_M = D_N$ and $\phi^* g_N = g_M$.

We remark that the projection $\pi: J^k(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^2$, defined in sub-Section 2.1, is a sub-Riemannian submersion. As a consequence, a curve $\gamma(t)$ in $J^k(\mathbb{R}, \mathbb{R})$ and its projection $\pi(\gamma(t))$ have the same arc length.

A classic result on metric lines is the following

**Proposition 8.** Let $\phi: M \to N$ be a sub-Riemannian submersion and let $c(t)$ be a metric line in $N$, then the horizontal lift of $c(t)$ is a metric line in $M$.

The proof of Proposition 8 is given in [12, p. 154]. The following corollary is an immediate result to the Proposition 8.

**Corollary 9.** Geodesic lines are metric lines in $J^k(\mathbb{R}, \mathbb{R})$.

2.2. **The 3-Dimensional Magnetic Space.** In [12], we introduced the 3-dimensional sub-Riemannian manifold, denoted by $\mathbb{R}^3_F$, and called “magnetic sub-Riemannian structure” or “magnetic space”, whose geometry depends on the choice of a polynomial $F(x)$. To endow $\mathbb{R}^3_F$ with the sub-Riemannian structure we use global coordinates $(x, y, z)$. Then, we define the two rank non-integrable distribution $D_F$ and the sub-Riemannian metric on the distribution $D_F$ by the Pfaffian equation $dz - F(x)dy = 0$ and
Let $ds^2_{\mathbb{R}_F^3} = (dx^2 + dy^2)|_{D_F}$, respectively. We provided a sub-Riemannian submersion $\pi_F$ factoring the sub-Riemannian submersion $\pi : J^k(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^2$, that is, $\pi = pr \circ \pi_F$, where the target of $\pi_F$ is $\mathbb{R}_F^3$ and the target of $pr$ is $\mathbb{R}^2$. If $F(x) = \sum_{i=0}^{3} a_i x^i$, then the projections $\pi_F$ and $pr$ are given in coordinates by

$$\pi_F(x, \theta) = (x, \theta_0, \sum_{\ell=0}^{m-1} a_\ell \theta_\ell) = (x, y, z), \quad \text{and} \quad pr(x, y, z) := (x, y).$$

It follows that $\pi_F$ maps the frame $\{X, Y\}$ defined in (2.1) into the frame $\{\tilde{X}, \tilde{Y}\}$, that is,

$$\tilde{X} := \frac{\partial}{\partial x} = (\pi_F)_* X \quad \text{and} \quad \tilde{Y} := \frac{\partial}{\partial y} + F(x) \frac{\partial}{\partial z} = (\pi_F)_* Y.$$

We conclude $D_F$ is globally framed by the orthonormal vector fields $\{\tilde{X}, \tilde{Y}\}$.

An explanation of the names “magnetic sub-Riemannian structure” or “magnetic space” is given in [12, sub-Section 4.1].

2.2.1. Geodesics In The Magnetic Space. The Hamiltonian function governing the sub-Riemannian geodesic flow in $\mathbb{R}_F^3$ is

$$H_{\pi_F^*}(p_x, p_y, p_z, x, y, z) = \frac{1}{2} \sum_{i=1}^{n} p_{x_i}^2 + \frac{1}{2} (p_y + F(x) p_z)^2. \tag{2.4}$$

We say a curve $c(t) = (x(t), y(t), z(t))$ is a $\mathbb{R}_F^3$-geodesic parametrized by arc length in $\mathbb{R}_F^3$, if it is the projection of the sub-Riemannian geodesic flow with the condition $H_{\pi_F^*} = \frac{1}{2}$. Since $H_{\pi_F^*}$ does not depend on the coordinates $y$ and $z$, they are cycle coordinates, so the momentum $p_y$ and $p_z$ are constant of motion, see [16, p. 162] or [9, p. 67] for the definition of cycle coordinates and their properties. Since $y$ and $z$ are cycle coordinates, then the translation $\varphi_{(y_0, z_0)}(x, y, z) = (x, y + y_0, z + z_0)$ is an isometry.

**Definition 10.** We denote by $dist_{\mathbb{R}_F^3}(\cdot, \cdot)$ and $Iso(\mathbb{R}_F^3)$, the sub-Riemannian distance and the isometry group in $\mathbb{R}_F^3$.

For more details about the definition of sub-Riemannian distance the sub-Riemannian group of isometries see [18, Chapter 1.4] or [1, sub-Chapter 3.2]. Then the translation $\varphi_{(y_0, z_0)}$ is in $Iso(\mathbb{R}_F^3)$.

Setting $p_y = a$ and $p_z = b$ inspired the following definition.

**Definition 11.** We say that the two-dimensional linear space $Pen_F$ is the pencil of $F(x)$, if $Pen_F := \{G(x) = a + b F(x) : (a, b) \in \mathbb{R}^2\}$.

We define the lift of a curve in $\mathbb{R}_F^3$ to a curve in $J^k(\mathbb{R}, \mathbb{R})$.

**Definition 12.** Let $c(t)$ be a curve in $\mathbb{R}_F^3$. We say that a curve $\gamma(t)$ in $J^k(\mathbb{R}, \mathbb{R})$ is the lift of $c(t) = (x(t), y(t), z(t))$ if $\gamma(t)$ solves

$$\dot{\gamma}(t) = \dot{x}(t) X(\gamma(t)) + G(x(t)) Y(\gamma(t)).$$
Now we describe the \( R^3_P \)-geodesics, their lifts, and their relation with the sub-Riemannian geodesics in \( J^k(\mathbb{R}, \mathbb{R}) \).

**Proposition 13.** Let \( c(t) \) be a \( R^3_P \)-geodesic for \( G(x) \) in \( P \epsilon_F \), then the component \( x(t) \) satisfies the 1-degree of freedom Hamiltonian equation
\[
H_{(a,b)}(p_x, x) := \frac{1}{2}p_x^2 + \frac{1}{2}(a + bF(x))^2 = \frac{1}{2}p_x^2 + \frac{1}{2}G^2(x).
\]

Having found a solution component \( x(t) \) given by equation (2.5), the coordinates \( y(t) \) and \( z(t) \) satisfy
\[
\dot{y} = G(x(t)) \quad \text{and} \quad \dot{z} = G(x(t))F(x(t)).
\]

Moreover, every \( R^3_P \)-geodesic is the \( \pi_F \)-projection of a geodesic in \( J^k(\mathbb{R}, \mathbb{R}) \) corresponding to \( G(x) \) in \( P \epsilon_F \). Conversely, the lifts of a \( R^3_P \)-geodesic are precisely those geodesics corresponding to polynomials in \( P \epsilon_F \).

The proof was presented in [12, sub-Section 4.1].

The sub-Riemannian geometry has two type of geodesics: normal and abnormal. The sub-Riemannian geodesic flow defines the normal geodesics, while, the endpoint map determines the abnormal geodesics, the following Lemma characterizes the abnormal geodesics in \( R^3_P \).

**Lemma 14.** A curve \( c(t) \) in \( R^3_P \) is an abnormal geodesic if and only if \( c(t) \) is tangent to the vector field \( \tilde{Y} \) and \( x(t) = x^* \) is a constant point in \( \mathbb{R} \) such that \( F'(x^*) = 0 \).

For more details about the endpoint map and abnormal geodesics, see [18, Chapter 3], [1, sub-sub-Chapter 4.3.2] or [13].

**Corollary 15.** Let \( \gamma(t) \) be a sub-Riemannian geodesic in \( J^k(\mathbb{R}, \mathbb{R}) \) corresponding to the polynomial \( F(x) \) and let \( c(t) \) be the curve given by \( \pi_F(\gamma(t)) \), then \( c(t) \) is a \( R^3_P \)-geodesic corresponding to the pencil \((a,b) = (0,1)\).

**Proof.** By construction, the pencil \((a,b) = (0,1)\) correspond to the polynomial \( F(x) \).

We classify the sub-Riemannian geodesics in \( R^3_P \) according to their reduce dynamics defined by the reduced Hamiltonian \( H_{(a,b)} \), equation (2.4), in the same way as we did in sub-sub-Section 2.1.2.

**2.2.2. Cost Map In Magnetic Space.** In [12, sub-Section 7.2], we defined the Cost map and used it to prove the main result. Here, we introduce Cost, an auxiliary function to show Theorems A and B.

**Definition 16.** Let \( c(t) \) be a \( R^3_P \)-geodesic defined on the interval \([t_0, t_1]\). We define the function \( \Delta : (c, [t_0, t_1]) \rightarrow [0, \infty] \times \mathbb{R}^2 \) given by
\[
\Delta(c, [t_0, t_1]) := (\Delta t(c, [t_0, t_1]), \Delta y(c, [t_0, t_1]), \Delta z(c, [t_0, t_1]))
\]
(2.6)

And the function \( \text{Cost} : (c, [t_0, t_1]) \rightarrow [0, \infty] \times \mathbb{R} \) given by
\[
\text{Cost}(c, [t_0, t_1]) := (\text{Cost}_t(c, [t_0, t_1]), \text{Cost}_y(c, [t_0, t_1])
\]
(2.7)
where
\[
\begin{align*}
\text{Cost}_t(c, [t_0, t_1]) &= \Delta t(c, [t_0, t_1]) - \Delta y(c, [t_0, t_1]) \\
\text{Cost}_y(c, [t_0, t_1]) &= \Delta y(c, [t_0, t_1]) - \Delta z(c, [t_0, t_1]).
\end{align*}
\]

Let us prove that \(\text{Cost}(c, [t_0, t_1])\) is well-defined:

**Proof.** By construction,
\[
|\Delta y(c, [t_0, t_1])| \leq \Delta t(c, [t_0, t_1]), \quad \text{so } 0 \leq \text{Cost}_t(c, [t_0, t_1]).
\]

We interpret \(\text{Cost}_t(c, [t_0, t_1])\) as the cost that takes to the geodesic \(c(t)\) travel through the \(y\)-component in the positive direction. To give more meaning to this interpretation, we present the following Lemma.

**Lemma 17.** Let \(c(t)\) and \(\tilde{c}(t)\) be two \(\mathbb{R}^3_E\)-geodesics. Let us assume that they travel from a point \(A\) to a point \(B\) in a time interval \([t_0, t_1]\) and \([\tilde{t}_0, \tilde{t}_1]\), respectively. If \(\text{Cost}_t(c_1, [t_0, t_1]) < \text{Cost}_t(c_2, [\tilde{t}_0, \tilde{t}_1])\), then the arc length of \(c(t)\) is shorter than the arc length of \(\tilde{c}(t)\).

**Proof.** We need to show that \(\Delta t(c_1, [t_0, t_1]) < \Delta t(c_2, [\tilde{t}_0, \tilde{t}_1])\). Since \(A = c(t_0) = \tilde{c}(\tilde{t}_0)\) and \(B = c(t_1) = \tilde{c}(\tilde{t}_1)\), it follows that
\[
\Delta y(c_1, [t_0, t_1]) = \Delta y(c_2, [\tilde{t}_0, \tilde{t}_1])
\]
which implies
\[
\Delta t(c_1, [t_0, t_1]) - \text{Cost}_t(c_1, [t_0, t_1]) = \Delta t(c_2, [\tilde{t}_0, \tilde{t}_1]) - \text{Cost}_t(c_2, [\tilde{t}_0, \tilde{t}_1]),
\]
so \(0 < \text{Cost}_t(c_2, [\tilde{t}_0, \tilde{t}_1]) - \text{Cost}_t(c_1, [t_0, t_1]) = \Delta t(c_2, [\tilde{t}_0, \tilde{t}_1]) - \Delta t(c_1, [t_0, t_1]).\)

**Remark.** The following Proposition is a classical result from classical mechanics.

**Proposition 18.** Let \(c(t)\) be a \(x\)-periodic \(\mathbb{R}^3_E\)-geodesic for the pencil \((a, b)\) with a hill interval \(I\) the period is given by
\[
L(G, I) := 2 \int_I \frac{dx}{\sqrt{1 - G^2(x)}}.
\]

Moreover, the changes \(\Delta y(c, [t, t + L]) = \Delta y(G, I)\) and \(\Delta z(c, [t, t + L]) = \Delta y(G, I)\) are given by
\[
\begin{align*}
\Delta y(G, I) &= 2 \int_I \frac{G(x)dx}{\sqrt{1 - G^2(x)}} \quad \text{and} \quad \Delta z(G, I) = 2 \int_I \frac{G(x)F(x)dx}{\sqrt{1 - G^2(x)}}.
\end{align*}
\]

In [12, sub-Section 4.3], we proved Proposition 18 using classical mechanics, see [16, Section 11]. In [11, Section 2], we showed an equivalent statement, in the context of \(J^k(\mathbb{R}, \mathbb{R})\), using a generating function of the second type, see [9, Section 50]. \(L(G, I)\) and \(\Delta y(G, I)\) are smooth functions with respect to the parameters \((a, b)\) if and only if the corresponding geodesic \(c(t)\) for \((G, I)\) is \(x\)-periodic. We define an auxiliary map that will help us to prove Theorems A and B.
Definition 19. The period map \( \Theta : (G, I) \to [0, \infty] \times \mathbb{R} \) is given by
\[
\Theta(G, I) := (\Theta_1(G, I), \Theta_2(G, I))
\]
\[
:= 2\int_I \frac{1 - G(x)}{1 + G(x)} dx, \int_I G(x) \frac{1 - F(x)}{\sqrt{1 - G^2(x)}} dx.
\]

\( \Theta_1(G, I) \) is a smooth function with respect the parameters \( (a, b) \) not only when the corresponding geodesic \( c(t) \) for \( (G, I) \) is \( x \)-periodic, \( \Theta_1(G, I) \) is also smooth when \( c(t) \) is a heteroclinic of the direct-type or homoclinic geodesic such that \( G(x(t)) \to 1 \) when \( t \to \pm \infty \).

Corollary 20. Let \( G(x) \) be in Pen_F. Then:
(1) \( \Theta_1(G, I) = 0 \) if and only if \( G(x) = 1 \).
(2) If \( I = [x_0, x_1] \) is compact, then \( \Theta_1(G, I) \) is finite if and only if \( x_0 \) and \( x_1 \) are not critical point of \( G(x) \) with value \(-1 \).

We introduce an important concept called the travel interval.

Definition 21. Let \( c(t) \) be a \( \mathbb{R}^3_F \)-geodesic traveling during the time interval \([t_0, t_1]\). We say that \( \mathcal{I}[t_0, t_1] := x([t_0, t_1]) \) is the travel interval of \( c(t) \), counting multiplicity.

For instance, if \( c(t) \) is a \( \mathbb{R}^3_F \)-geodesic with hill interval \( I \) such that its coordinate \( x(t) \) is \( L \)-periodic, then \( \mathcal{I}[t, t + L] = 2I \).

Corollary 22. Let \( c(t) \) be a \( \mathbb{R}^3_F \)-geodesic for \( G(x) \) in Pen_F with travel interval \( \mathcal{I} \). Then \( \Delta(c, [t_0, t_1]) \) from Definition 16 can be rewritten in terms of polynomial \( G(x) \) and the travel interval \( \mathcal{I} \) as follows;
\[
\Delta(c, [t_0, t_1]) = \Delta(G, \mathcal{I})
\]
\[
:= (\int_I \frac{dx}{\sqrt{1 - G^2(x)}}, \int_I \frac{G(x)dx}{\sqrt{1 - G^2(x)}}, \int_I \frac{G(x)F(x)dx}{\sqrt{1 - G^2(x)}}).
\]
In the same way, the map \( \text{Cost}(c, [t_0, t_1]) \) from Definition 16 can be rewritten as follows:
\[
\text{Cost}(c, [t_0, t_1]) = \text{Cost}(G, \mathcal{I}) := (\int_I \frac{1 - G(x)}{\sqrt{1 - G^2(x)}} dx, \int_I \frac{(1 - F(x))G(x)}{\sqrt{1 - G^2(x)}} dx)
\]

The proof is the same as that for Proposition 18. We remark that \( \text{Cost}(G, \mathcal{I}) \) and \( \Theta(G, I) \) are equal if and only if \( I = \mathcal{I} \).

Corollary 23. Let \( c(t) \) be a \( \mathbb{R}^3_F \)-geodesic, then
\[
\lim_{n \to \infty} \text{Cost}_n(c, [-n, n]) \text{ is finite if and only if } \lim_{t \to \pm \infty} G(x(t)) = 1.
\]

2.2.4. Upper Bound of the Cut Time. Here we introduce the cut time definition in the context of \( J_k^k(\mathbb{R}, \mathbb{R}) \).

Definition 24. Let \( \gamma : \mathbb{R} \to J_k^k(\mathbb{R}, \mathbb{R}) \) be a sub-Riemannian geodesic parameterized by arc length. The cut time of \( \gamma \) is
\[
t_{\text{cut}}(\gamma) := \sup\{t > 0 : |\gamma|_{[0, t]} \text{ is length-minimizing}\}.
\]
2.2.5. Sequence Of Geodesics On The Magnetic Space. Let us present two classical results on metric spaces.

Lemma 26. Let $c_n(t)$ be a sequence of minimizing geodesics on the compact interval $\mathcal{T}$ converging uniformly to a geodesic $c(t)$, then $c(t)$ is minimizing in the interval $\mathcal{T}$.

Proof. Let $[t_0, t_1] \subset \mathcal{T}$, since $c_n(t)$ is sequence of minimizing geodesic then $\text{dist}_{\mathbb{R}^3_3}(c_n(t_0), c_n(t_1)) = |t_1 - t_0|$ for all $n$. By the uniformly convergence, if $n \to \infty$ then $\text{dist}_{\mathbb{R}^3_3}(c(t_0), c(t_1)) = |t_1 - t_0|$. \hfill $\square$

Proposition 27. Let $K$ be a compact subset of $\mathbb{R}^3_3$ and let $\mathcal{T}$ be a compact time interval. Let us define the following set of $\mathbb{R}^3_3$-geodesics

$$\text{Min}(K, \mathcal{T}) := \{\text{geodesics } c(t) : c(\mathcal{T}) \subset K \text{ and } c(t) \text{ is minimizing in } \mathcal{T}\}.$$ 

Then $\text{Min}(K, \mathcal{T})$ is a sequentially compact set with respect to the uniform topology.

Proof. Let $c_n(t)$ be an arbitrary sequence in $\text{Min}(K, \mathcal{T})$, we must prove $c_n(t)$ has a uniformly convergent subsequence converging to $c(t)$ in $\text{Min}(K, \mathcal{T})$. The space of geodesics $\text{Min}(K, \mathcal{T})$ is uniformly bounded and smooth in compact interval $\mathcal{T}$, then $\text{Min}(K, \mathcal{T})$ is a equi-continuous family of geodesics. By Arzela-Ascoli theorem, every sequence $c_n(t)$ in $\text{Min}(K, \mathcal{T})$ has a convergent subsequence $c_{n_k}(t)$ converging uniformly to a smooth curve $c(t)$. By Lemma 26 $c(t)$ is minimizing in $\mathcal{T}$. \hfill $\square$

A useful tool for the proof of Theorem A and B is the following.

Corollary 28. Let $c_1(t)$ be a $\mathbb{R}^3_3$-geodesic in $\text{Min}(K, \mathcal{T})$ and let $c_2(t)$ be a $\mathbb{R}^3_3$-geodesic. If $\varphi(x, y, z)$ is an isometry such that $c_2(\mathcal{T}') \subset \varphi(c_1(\mathcal{T}))$, then $c_2(t)$ is in $\text{Min}(\varphi(K), \mathcal{T}')$.

3. Heteroclinic Of The Direct-Type Geodesic

This section is devoted to proving Theorem A. Let $\gamma_d(t)$ be an arbitrary heteroclinic of the direct-type geodesic in $J^k(\mathbb{R}, \mathbb{R})$ for a polynomial $F_d(x)$. We will consider the space $\mathbb{R}^3_{F_d}$ and the $\mathbb{R}^3_{F_d}$-geodesic $c_d(t) := \pi_{F_d}(\gamma_d(t))$. Then we will prove the following Theorem.

Theorem 29. Let $\gamma_d(t)$ be an arbitrary heteroclinic of the direct-type geodesic in $J^k(\mathbb{R}, \mathbb{R})$ for a polynomial $F_d(x)$. If $c_d(t) := \pi_{F_d}(\gamma_d(t))$, then $c_d(t)$ is a metric line $\mathbb{R}^3_{F_d}$. 
Without loss of generality, let us assume $\gamma_d(t)$ is a unitary geodesic and let $F_d(x)$ has the form given by Corollary 5. The goal is to show that for arbitrary $T$ the geodesic is minimizing in the interval $[-T,T]$, the strategy to verify the goal is the following: For all $n > T$, we will take a sequence of geodesics $c_n(t)$ minimizing in the interval $[-n,n]$ and joining the points $c_d(-n)$ and $c_d(n)$, see 3.1. Then, we will find a convergent subsequence $c_{n_j}(t)$ converging to a $\mathbb{R}^3_{F_d}$-geodesic $c_\infty(t)$ in $\text{Min}(K,T)$ and isometry $\varphi$ in $\text{Iso}(\mathbb{R}^3_{F_d})$ such that $c([-T,T]) \subseteq \varphi(c_\infty(T))$, where $K$ is a compact subset of $\mathbb{R}^3_{F_d}$ and $T$ is a compact interval. By corollary 28, $c_d(t)$ is minimizing in $[-T,T]$. Since $T$ is arbitrary, $c_d(t)$ is a metric line.

Let $c_d(t) = (x(t), y(t), z(t))$. Without loss of generality, we can assume that $0 \leq x(t)$ and $c_d(0) = (x(0), 0, 0)$ for some $x(0)$ in $(0,1)$ since the proof for the case $0 \geq x(t)$ is similar and we can use the $t$, $y$, and $z$ translations.

3.1. The Magnetic Space For Heteroclinic Geodesic.

Corollary 30. Let $q_{\text{max}}$ be equal to $\max_{x \in [0,1]} \{x^{k_1}(1-x)^{k_2}q(x)\}$, where $q(x)$, $k_1$ and $k_2$ are given by Corollary 5. The set of all the heteroclinic of the direct-type $\mathbb{R}^3_{F_d}$-geodesic with hill interval $[0,1]$ is given by

$$\text{Pen}_d := \text{Pen}_d^+ \cup \text{Pen}_d^-,$$

where

$$\text{Pen}_d^+ := \{(a,b) = (s, 1-s): s \in (\frac{2}{q_{\text{max}}}, 1)\},$$

$$\text{Pen}_d^- := \{(a,b) = (-s, s-1): s \in (\frac{2}{q_{\text{max}}}, 1)\}.$$

Moreover, the map $\Theta_2(G, [0,1]) : \text{Pen}_d^+ \rightarrow \mathbb{R}$ is one to one, and the cost map $\text{Cost}(c_d, [t_0, t_1])$ is bounded by $\Theta_d := \Theta_1(F_d, [0,1])$ for all $[t_0, t_1]$.

Proof. Since $F_d(x) \neq -1$ if $x$ is in $[0,1]$, the constant $\Theta_d$ is finite. Let us prove that $\text{Cost}(c_d, [t_0, t_1])$ is bounded by $\Theta_d$ for all $[t_0, t_1]$. Using Corollary (22) and the condition $|F_d(x)| \leq 1$ for $x$ in $[0,1]$, we find that:

$$|\text{Cost}_y(c_d, [t_0, t_1])| < \text{Cost}_t(c_d, [t_0, t_1])$$

$$< 2 \int_{[0,1]} \sqrt{\frac{1 - F_d(x)}{1 + F_d(x)}} dx = : \Theta_1(F_d, [0,1]).$$

To prove that $\Theta_2(G, [0,1]) : \text{Pen}_d^+ \rightarrow \mathbb{R}$ is one to one, we consider the one-parameter family of polynomials $G_s(x) = s + (1-s)F_d(x)$. Thus, $\Theta_2(G_s, [0,1]) : (0, q_{\text{max}}) \rightarrow \mathbb{R}$ is one variable function, let us calculate its derivative:

$$\frac{d}{ds} \Theta_2(G_s, [0,1]) = \frac{d}{ds} \int_{[0,1]} \frac{(1 - F_d(x))G_s(x)}{\sqrt{1 - G_s^2(x)}} dx = \int_{[0,1]} \frac{1 - F_d(x)}{(1 - G_s^2(x))^{\frac{3}{2}}} dx.$$

Since $0 < 1 - F_d(x)$, then $0 < \frac{d}{ds} \Theta_2(G_s, [0,1])$. \qed
We remark that $Pen^+_d$ defines the heteroclinic geodesics of the direct-type such that $\lim_{t \to \pm \infty} y(t) = \infty$, while $Pen^-_d$ defines the heteroclinic geodesics of the direct-type such that $\lim_{t \to \pm \infty} y(t) = -\infty$.

**Lemma 31.** Let $\Omega(F_d) = \text{hill}(F_d) \times \mathbb{R}^2$ be the region, then $c_d(t)$ is minimizing between the curves that lay in the region $\Omega(F_d)$.

The proof is consequence of the calibration function defined on the region $\Omega(F_d)$ and provided in [12, Section 5].

**Corollary 32.** There exist $T^*_d > 0$ such that $y_d(t) > 0$ if $T^*_d < t$, and $y_d(t) < 0$ if $-T^*_d > t$.

**Proof.** By construction, $\lim_{t \to \infty} y_d(t) = \infty$ and $\lim_{t \to -\infty} \Delta y_d(0) = -\infty$. \qed

**Definition 33.** If $T := [t_0, t_1]$, we define the following set

$$\text{Com}([0, 1]) := \{(c(t), T) : c(t) \text{ is a } \mathbb{R}^3_{F_d}\text{-geodesic,}$$

$$x(t_0) \in [0, 1] \text{ and } x(t_1) \in [0, 1]\}.$$  

**Lemma 34.** Let us consider a sequence $(c_n(t), [-n, n])$ in $\text{Com}([0, 1])$. If $Cost(c_n, [-n, n])$ is uniformly bounded, then there exists a compact subset $K_x$ of $\mathbb{R}$ such that $I_n[-n, n] \subset K_x$ for all $n$.

The proof is Appendix A.1.

### 3.2. Proof of Theorem 29.

#### 3.2.1. Set Up The Proof Of Theorem 29.

Let $T$ be arbitrarily large and consider the sequence of points $c_d(-n)$ and $c_d(n)$ where $T < n$ and $n$ is in $\mathbb{N}$. Let $c_n(t) = (x_n(t), y_n(t), z_n(t))$ be a sequence of minimizing $\mathbb{R}^3_{F_d}$-geodesics, in the interval $[0, T_n]$ such that:

$$c_n(0) = c_d(-n), \quad c_n(T_n) = c_d(n) \quad \text{and} \quad T_n \leq n.$$  

We call the equations and inequality from (3.2) the endpoint conditions and the shorter condition, respectively. If $c_n(t)$ is geodesic for the polynomial $G_n(x)$ and a hill interval $I_n$, then Proposition 25 implies $T_n \leq L(G_n, I_n)$. Since the endpoint condition holds for all $n$, then the sequence $c_n(t)$ holds asymptotic conditions:

$$\lim_{n \to \infty} c_n(0) = (0, -\infty, -\infty), \quad \lim_{n \to \infty} c_n(T_n) = (1, \infty, \infty),$$

and the asymptotic period condition:

$$\lim_{n \to \infty} Cost_y(c_n, [0, T_n]) = \frac{1}{2} \Theta_2(F_d, [0, 1]).$$

**Corollary 35.** The sequence of $\mathbb{R}^3_{F_d}$-geodesics $c_n(t)$ is not a sequence of geodesic lines and does not converge to a geodesic line. In particular, $c_n(t)$ does not converge to an abnormal geodesic.
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Figure 3.1. The images show the projection to $\mathbb{R}^2$, with coordinates $(x, y)$, of a heteroclinic of the direct-type geodesic $c_d(t)$ and the sequence of geodesics $c_n(t)$.

Proof. Lemma 31 implies that if $c_n(t)$ is shorter than $c_d(t)$, then $c_n(t)$ must leave the region $\Omega(F_d)$ and come back. Thus $c_n(t)$ is a geodesic for non-constant polynomial $G_n(x)$, and $c_n(t)$ is not a geodesic line.

Let $I_n$ travel interval of $c_n(t)$, then $c_n(t)$ cannot converge to a geodesic line, since $\lim_{n \to \infty} I_n = [0, 1]$ and the only line in the plane $(x, y)$ that travel from $y = -\infty$ into $y = \infty$ in a fine travel interval is the vertical line, but the travel interval of the vertical line is a single point. In particular, Lemma 14 implies $c_n(t)$ cannot converge to an abnormal geodesic.

The construction of the $\mathbb{R}^3_{F_d}$-geodesic $c_n(t)$ is such that the initial condition $c_n(0)$ is not bounded. The following Proposition provides a bounded initial condition.

**Proposition 36.** Let $n$ be a natural number larger than $T^*_d$, where $T^*_d$ is given by Corollary 32, and let $K_0 := K_x \times [-1, 1] \times [-\Theta_d, \Theta_d]$ be a compact set, where $K_x$ is the compact set from Lemma 34 and $\Theta_d$ the constant provided by Corollary 30. Then there exist a time $t^*_n \in (0, T_n)$ such that $c_n(t^*_n)$ is in $K_0$ for all $n > T^*_d$.

Proof. Let $n$ be a natural number larger than $T^*_d$. By construction, $y_n(0) < 0$ and $y_n(T_n) > 0$, the intermediate value theorem implies that exist a $t^*_n$ in $(0, T_n)$ such that $y_n(t^*_n) = 0$. Since $\text{Cost}(c_n, [0, T_n])$ is bounded, by Lemma 34, there exists a compact set $K_x$ such that $x_n(t)$ is in $K_x$ for all $t$ in $[0, T_n]$.

Let us prove that $|z_n(t^*_n)| \leq \Theta_d$: the endpoint conditions imply

$$\Delta y(c_d, [-n, n]) = \Delta y(c_n, [0, T_n]) \quad \text{and} \quad \Delta z(c_d, [-n, n]) = \Delta z(c_n, [0, T_n]).$$
So $\text{Cost}_g(c_d, [-n, n]) = \text{Cost}_g(c_n, [0, T_n])$ and by definition of $\text{Cost}_g$, it follows that:

$$
\begin{align*}
\Delta z(c_n, [0, t^*_n]) &= \Delta y(c_n, [0, t^*_n]) - \text{Cost}_g(c_n, [0, t^*_n]), \\
\Delta z(d, [-n, 0]) &= \Delta y(d, [-n, 0]) - \text{Cost}_g(d, [-n, 0]).
\end{align*}
$$

By construction, $\Delta y(c_n, [0, t^*_n]) = \Delta y(d, [-n, 0])$, $z_d(0) = 0$ and $z_n(0) = z_d(-n)$, then

$$
|z_n(t^*_n)| = |\text{Cost}_g(c_n, [0, t^*_n]) - \text{Cost}_g(d, [-n, 0])| \leq \Theta_d,
$$

since Corollary 30 says $\text{Cost}_g(c_n, [0, T_n])$ is bounded by $\Theta_d$. We just proved $c_n(t^*_n)$ is in $K_0$.

Let us reparametrize the sequence of minimizing $\mathbb{R}^3_{F_d}$-geodesics $c_n(t)$. Let $\tilde{c}_n(t)$ be a minimizing $\mathbb{R}^3_{F_d}$-geodesic in the interval $T_n := [-t^*_n, T_n - t^*_n]$ given by $\tilde{c}_n(t) := c_n(t + t^*_n)$. Then, $\tilde{c}_n(0)$ is bounded and $\tilde{c}_n(t)$ is a minimizing $\mathbb{R}^3_{F_d}$-geodesics in the interval $T_n$.

**Corollary 37.** There exists a subsequence $T_{n_j}$ such that $T_{n_j} \subset T_{n_{j+1}}$.

**Proof.** On one side $\tilde{c}_n(0)$ is bounded, on the other side $c(-t^*_n)$ and $c(T_n - t^*_n)$ are unbounded. Then $[-t^*_n, T_n - t^*_n] \to [-\infty, \infty]$ when $n \to \infty$, and we can take a subsequence of intervals $T_{n_j}$ such that $T_{n_j} \subset T_{n_{j+1}}$. \hfill \Box

For simplicity, we will use the notation $T_n$ for the subsequence $T_{n_j}$.

**Lemma 38.** Let $N$ be a natural number larger than $T^*_d$. Then there exist compact set $K_N \subset \mathbb{R}^3_F$ such that $c_n(t)$ is in $\text{Min}(K_N, T_N)$ if $n > N$.

**Proof.** Since $\tilde{c}_n(t)$ is minimizing on the interval $T_n$, it follows that $\tilde{c}_n(t)$ is minimizing on the interval $T_N \subset T_n$ if $n > N$. Moreover, there exists a compact set $K_N$ such that $\tilde{c}_n(T_N) \subset K_N$, since $c_n(0)$ is in $K_0$ and $c_n(t)$ is a family of smooth functions defined on the compact set $T_N$. \hfill \Box

Therefore, $\tilde{c}_n(t)$ has a convergent subsequence $\tilde{c}_{n_j}(t)$ converging to a $\mathbb{R}^3_{F_d}$-geodesic $c_\infty(t)$ in $\text{Min}(K_N, T_N)$. Corollary 35 implies that $c_\infty(t)$ is a normal $\mathbb{R}^3_{F_d}$-geodesic, then we can associate $c_\infty(t)$ to a polynomial $G(x)$ in $\text{Pen}_{F_d}$. The following Lemma tells the $G(x) = F_d(x)$.

**Lemma 39.** $G(x) = F_d(x)$ is the unique polynomial in the pencil of $F_d(x)$ satisfying the asymptotic conditions given by (3.3) and (3.4).

**Proof.** By Proposition 26, $\tilde{c}_n(t)$ has a convergent subsequence $\tilde{c}_{n_j}(t)$ converging to a minimizing geodesic $\tilde{c}(t)$ on the interval $T_N$. Being a $\mathbb{R}^3_{F_d}$-geodesic, $c(t)$ is associated to a polynomial $G(x) = a + bF_d(x)$. $G(0) = a + b$ must be equal 1, to satisfy the asymptotic conditions given by (3.3). Then $(a, b)$ is in $\text{Pen}^+_d$, the set defined in Corollary 30. Since the map $\Theta_1(a, b) : \text{Pen}^+_d \to \mathbb{R}$ is one to one, the unique polynomial in $\text{Pen}^+_d$ satisfying the condition (3.3) and (3.4) is $G(x) = F_d(x)$. \hfill \Box
3.2.2. Proof of Theorem 29.

Proof. Let \( \tilde{c}_n(t) \) be the sequence of geodesics defined by the endpoint conditions (3.2). By Lemma 38, for all \( N > T_d^* \) there exist a compact set \( K_N \) such that \( \tilde{c}(t) \) is in \( \text{Min}(K_N, T_N) \) if \( n > N \). By Proposition 27, there exists a subsequence \( \tilde{c}_{n_j}(t) \) converging to a \( \mathbb{R}^3_{F_d} \)-geodesic \( c_\infty(t) \) in \( \text{Min}(K_N, T_N) \). Corollary 35 implies that \( c_\infty(t) \) is a normal geodesic for a polynomial \( G(x) \) in \( \mathop{Pen}_{F_d}^2 \). Lemma 38 tells that \( G(x) = F_d(x) \).

Since \( c_\infty(t) \) and \( c_d(t) \) are \( \mathbb{R}^3_{F_d} \)-geodesics for \( F_d(x) \) with the same hill interval, there exists a translation \( \varphi_{(y_0,z_0)} \) in \( \text{Iso}(\mathbb{R}^3_{F_d}) \) sending \( c_\infty(t) \) to \( c_d(t) \).

Using \( N \) is arbitrary and \( c_d([-T,T]) \) is bounded, we can find compact sets \( K := K_N \) and \( T := T_N \) such that \( c_d([-T,T]) \subset \varphi_{(y_0,z_0)}(c_\infty(T)) \) and \( c_\infty \) is in \( \text{Min}(K,T) \). Corollary 28 implies that \( c_d(t) \) is minimizing in \([-T,T] \) and \( T \) is arbitrarily. Therefore, \( c_d(t) \) is a metric line in \( \mathbb{R}^3_{F_d} \).

\[ \square \]

3.2.3. Proof of Theorem A.

Proof. Let \( \gamma_d(t) \) be an arbitrary heteroclinic of the direct-type geodesic. By Theorem 29, \( c_d(t) := \pi_{F_d}(\gamma_d(t)) \) is a metric line in \( \mathbb{R}^3_{F_d} \). Since \( \pi_{F_d} \) is a sub-Riemannian submersion and \( \gamma_d(t) \) is the lift of \( c_d(t) \), then Proposition 8 implies \( \gamma_d(t) \) is a metric line in \( J^k(\mathbb{R}, \mathbb{R}) \).

\[ \square \]

4. Homoclinic Geodesics In Jet Space

This chapter is devoted to proving Theorem B. Let \( \gamma_h(t) \) be the homoclinic geodesic in \( J^k(\mathbb{R}, \mathbb{R}) \) for \( F_h(x) := \pm(1 - bx^{2n}) \). We will consider the space \( \mathbb{R}^3_{F_h} \) and the geodesic \( c_h(t) := \pi_{F_h}(\gamma_h(t)) \), then we will prove the following theorem.

**Theorem 40.** Let \( \gamma_h(t) \) be an arbitrary homoclinic geodesic in \( J^k(\mathbb{R}, \mathbb{R}) \) for the polynomial \( F_h(x) := \pm(1 - bx^{2n}) \). If \( c_h(t) := \pi_{F_h}(\gamma_h(t)) \), then \( c_h(t) \) is a metric line \( \mathbb{R}^3_{F_h} \).

Without loss of generality, we will consider the polynomial \( F_h(x) := 1 - 2x^{2n} \) with hill interval \([0,1] \). The strategy to prove Theorem 29 is the same as the one used for Theorem 40.

Before prove Theorem 40, we present the following.

**Theorem 41.** Let \( \mathbb{R}^3_{F_h} \) be the magnetic space for the polynomial \( F_h(x) := 1 - 2x^{2n+1} \) and let \( c(t) \) be the homoclinic \( \mathbb{R}^3_{F_h} \)-geodesic corresponding to \( F_h(x) \). Then \( c(t) \) is not a metric line \( \mathbb{R}^3_{F_h} \).

Theorem 41 say that we cannot use the the magnetic space \( \mathbb{R}^3_{F_h} \) to the Conjecture 2 for the general homoclinic case. Since the method used to prove Theorem 40 does not work for the odd case \( F(x) := 1 - 2x^{2n+1} \). The proof of Theorem 41 is in Appendix B.
4.1. The Magnetic Space For the Homoclinic Geodesics. Without loss of generality, \( c_h(0) = (1,0,0) \), by use of the \( t, y \) and \( z \) translations. By the time reversibility of the reduced Hamiltonian \( H_F \) given by \((2.2)\), it follows that \( x(-n) = x(n) \) and \( \Delta x(c_h,[-n,n]) := x(n) - x(-n) = 0 \) for all \( n \).

Lemma 42. Let \( c_h(t) \) be the homoclinic \( \mathbb{R}^3_{F_h} \)-geodesic for \( F_h(x) := 1 - 2x^{2n} \), then

\[
\Theta_2(F_h,[0,1]) < 0.
\]

Proof. By construction, \( -xF'_h(x) = (2n-1)(1-F_h(x)) \). Using integration by parts it follows that

\[
\Theta_2(F_h,[0,1]) = \frac{-2}{2n-1} \int_{[0,1]} \frac{x F'_h(x) F(x) dx}{\sqrt{1 - F_h^2(x)}} = \frac{2}{2n-1} x \sqrt{1 - F_h^2(x)} \big|_{0}^{1} - \frac{2}{2n-1} \int_{[0,1]} \sqrt{1 - F_h^2(x)} dx.
\]

The desired result follows by \( x \sqrt{1 - F_h^2(x)} \big|_{0}^{1} = 0 \). \(\square\)

Corollary 43. The set of all the homoclinic \( \mathbb{R}^3_{F_h} \)-geodesics is given by

\[
\text{Pen}_h := \text{Pen}^+_h \cup \text{Pen}^-_h,
\]

where

\[
\text{Pen}^+_h := \{(a,b) = (s,1-s) : s \in (1,\infty)\}
\]

\[
\text{Pen}^-_h := \{(a,b) = (-s,s-1) : s \in (1,\infty)\}.
\]

Moreover, the map \( \Theta_2(G,[0,1]) : \text{Pen}^+_h \to \mathbb{R} \) is one to one and the cost map \( \text{Cost}(c_h,[t_0,t_1]) \) is bounded by \( \Theta_1(F_h,[0,1]) := \Theta_h \) for all \([t_0,t_1]\).

Proof. The proof’s first part is the same as the one from 30. To prove that \( \Theta_1(a,b) : \text{Pen}^-_h \to \mathbb{R} \) is one to one, we consider the one-parameter family of homoclinic polynomial \( G_s(x) := s - (1-s)F_h(x) \) with hill interval \([0, \sqrt{2n(1-s)}]\).

Thus, \( \Theta_1(G_s,[0, \sqrt{2n(1-s)}]) : (0,\infty) \to \mathbb{R} \) is a one variable function and it is enough to show it is a monotone increasing function. Let us set up the change of variable \( x = \sqrt{\frac{2n}{1-s}} \), then \( F(\tilde{x}) = 1 - 2\tilde{x}^{2n} = F_h(\tilde{x}) \) and

\[
\Theta_2(G_s,[0, \sqrt{2n(1-s)}]) = \int_{[0, \sqrt{2n(1-s)}]} \frac{2x^{2n}G_s(x)}{\sqrt{1-G_s^2(x)}} dx = (\frac{2n}{1-s})^{n+1} \Theta_2(F_h,[0,1]).
\]

Since \( \frac{1}{s} \) is monotone decreasing and \( \Theta_2(F_h,[0,1]) \) is negative. Then, we conclude \( \Theta_2(G_s,[0, \sqrt{2n(1-s)}]) \) is a monotone increasing function with respect to \( s \). \(\square\)
We remark that $\text{Pen}_h^+$ defines the homoclinic geodesics such that
\[ \lim_{t \to \infty} y(t) = \infty \quad \text{and} \quad \lim_{t \to -\infty} y(t) = -\infty, \]
while $\text{Pen}_h^-$ defines the homoclinic geodesics such that
\[ \lim_{t \to \infty} y(t) = -\infty \quad \text{and} \quad \lim_{t \to -\infty} y(t) = \infty. \]

**Corollary 44.** There exist $T_h^* > 0$ such that $y_h(t) > 0$ if $T_h^* < t$ and $y_h(t) < 0$ if $-T_h^* > t$. Moreover, $\text{Cost}_y(c_h, [-t, t]) < 0$ if $T_h^* < t$.

**Proof.** Since $\text{Cost}_y(c_h, [-t, t]) \to \Theta_2(F_h, [0, 1])$ as $t \to \infty$ and $\Theta_2(F_h, [0, 1]) < 0$, we can find the desired $T_h^*$. The rest of the proof is equal to Corollary 32. \(\square\)

### 4.2. Set Up The Proof Of Theorem 40

Let $T$ be arbitrarily large and consider the sequence of points $c_h(-n)$ and $c_h(n)$ where $T < n$ and $n$ is in $\mathbb{N}$. Let $c_n(t) = (x_n(t), y_n(t), z_n(t))$ be a sequence of minimizing $\mathbb{R}_F^3$-geodesics in the interval $[0, T_n]$ such that:

\[ c_n(0) = c_h(-n), \quad c_n(T_n) = c_h(n) \quad \text{and} \quad T_n \leq n. \]

We call the equations and inequality from (4.1) the endpoint conditions and the shorter condition, respectively. If $c_n(t)$ is geodesic for the polynomial $G_n(x)$ and a hill interval $I_n$, then Proposition 25 implies $T_n \leq L(G_n, I_n)$. Since the endpoint condition holds for all $n$, the sequence $c_n(t)$ has the asymptotic conditions:

\[ \lim_{n \to \infty} c_n(0) = (0, -\infty, -\infty), \quad \lim_{n \to \infty} c_n(T_n) = (0, \infty, \infty), \]

and the asymptotic period condition

\[ \lim_{n \to \infty} \text{Cost}_y(c_n, [0, T_n]) = \Theta_2(F_h, [0, 1]). \]

The following Corollary tells us $c_n(t)$ is not a sequence of line geodesics. We remark that applying the calibration function found in [12, Section 5] is only possible for every sub-interval of the time intervals $(-\infty, 0)$ or $(0, \infty)$, in other words the calibration method does not work on an interval containing the time $t = 0$, which correspond to the point when the $x$ coordinate bounce on the point $x = 1$, for more details see [12, Section 5].

**Corollary 45.** Let $n$ be larger than $T_h^*$, where $T_h^*$ is given by Corollary 44, then the sequence of geodesics $c_n(t)$ neither is a sequence of geodesic lines, nor converge to a geodesics line. In particular, $c_n(t)$ does not converge to an abnormal geodesic.

**Proof.** Let us assume that $c_n(t)$ is a sequence of geodesic lines. Since $\Delta x(c_n, [-t, t]) = 0$ for all $n$ and $\Delta y(c_h, [-t, t]) > 0$ for all $n > T_h^*$, the unique geodesic line satisfying these conditions is the vertical line, which is
generated by the polynomial \( G_n(x) = 1 \). Since \( 1 - F_h(x) > 0 \) for all \( x \), then \((1 - F_h(x))G_n(x) > 0\) for all \( x \) and it follows that:

\[
\text{Cost}_y(c_n, [0, T_n]) = \int_0^T (1 - F_h(x(t)))G_n(x(t))dt > 0.
\]

This contradicts the endpoint conditions given by (4.1) since if \( T_n^* < t \) then \( \text{Cost}_y(c_n, [-t, t]) < 0 \). The same proof follows if \( c_n(t) \) converges to a geodesics line \( c(t) \) generated by \( G(x) = 1 \), since there exists \( N \) big enough that \( G_n(x) > \frac{1}{2} \) for \( n > N \). \( \square \)

Notice that this proof cannot be done in the case \( F_h(x) = 1 - 2x^{2n+1} \). In Appendix B under the hypothesis \( F_h(x) = 1 - 2x^{2n+1} \), we will find a sequence of curves \( c_n(t) \) shorter than \( c_h(t) \) that converges to the abnormal geodesic.

The following Proposition provides the bounded initial condition.

**Proposition 46.** Let \( n \) be a natural number larger than \( T_n^* \), where \( T_n^* \) is given by Corollary 44, and let \( K_0 = K_x \times [-1, 1] \times [-C_h, C_h] \) be a compact set, where \( K_x \) is the compact set from Lemma 34 and \( C_h \) is constant provided by Corollary 43. Then there exist a time \( t_n^* \in (0, T_n) \) such that \( c_n(t_n^*) \) is in \( K_0 \) for all \( n > T_n^* \).

Same proof as Proposition 36.

Let us reparametrize the sequence of minimizing \( \mathbb{R}^3_{F_h} \)-geodesics \( c_n(t) \). Let \( \tilde{c}_n(t) \) be a minimizing \( \mathbb{R}^3_{F_h} \)-geodesic in the interval \( T_n := [-t_n^*, T_n - t_n^*] \) given by \( \tilde{c}_n(t) := c_n(t + t_n^*) \). Then, \( \tilde{c}_n(0) \) is bounded and \( \tilde{c}_n(t) \) is a minimizing \( \mathbb{R}^3_{F_h} \)-geodesics in the interval \( T_n \).

**Corollary 47.** There exists a subsequence \( T_{n_j} \) such that \( T_{n_j} \subset T_{n_{j+1}} \).

The proof of Corollary 47 is equal to the of Corollary 37. For simplicity, we will use the notation \( T_n \) for the subsequence \( T_{n_j} \).

**Lemma 48.** There exist compact set \( K_N \subset \mathbb{R}^3_{F} \) such that if \( n > N \) then \( c_n(t) \) is in Min\( (K_N, T_N) \).

The proof of Lemma 48 is equal to the of Lemma 38. Therefore, \( \tilde{c}_j(t) \) has a convergent subsequence \( \tilde{c}_{j_i}(t) \) converging to a \( \mathbb{R}^3_{F_h} \)-geodesic \( c_\infty(t) \).

Corollary 45 implies that \( c_\infty(t) \) is a normal \( \mathbb{R}^3_{F_h} \)-geodesic for a polynomial \( G(x) \) in \( \text{Pen}_{F_h} \). The following Lemma tells \( G(x) = F_h(x) \).

**Lemma 49.** \( G(x) = F_h(x) \) is the unique polynomial in the pencil of \( F_h(x) \) satisfying the asymptotic conditions given by (4.2) and (4.3).

**Proof.** By Proposition 27 \( \tilde{c}_n(t) \) has a convergent subsequence \( \tilde{c}_{n_i}(t) \) converging to a minimizing geodesic \( c(t) \) on the interval \( T_N \). Being a geodesic in \( \mathbb{R}^3_{F_h} \), \( c(t) \) is associated to a polynomial \( G(x) = a + bF_h(x) \). \( G(0) = a + b \) must be equal 1, to satisfy the asymptotic conditions given by (4.2). Then \((a, b)\) is in \( \text{Pen}_{F_h}^+ \), the set defined in Corollary 43. Since the map \( \Theta_1(G, I) : \text{Pen}_{F_h}^+ \to \mathbb{R} \)
is one to one, the unique polynomial in \( Pen_h^* \) satisfying the condition (4.2) is \( G(x) = F_h(x) \).

The proof of Theorems 40 and B are the same as the proof of Theorems 29 and A, respectively.

5. Conclusion

We formalized the method used in [12] to prove that a particular geodesic is a metric line. Theorem A proves the Conjecture 2 for the heteroclinic of the direct-type case, and the problem remains open for the homoclinic case. Theorem 41 says we cannot use the space \( \mathcal{R}_F^3 \) to prove the Conjecture for the homoclinic case. However, Theorem 41 does not imply that the Conjecture is false. The homoclinic case can be solved by showing the corresponding period map in \( J^h(\mathbb{R}, \mathbb{R}) \) restricted to the homoclinic geodesics is one-to-one.

Appendix A. Proof Of Lemma 34

Definition 50. Let \( \mathcal{P}(k) \) be the vector space of polynomial on \( \mathbb{R} \) of degree bounded by \( k \), and let \( \|F\|_\infty := \sup_{x \in [0, 1]} |F(x)| \) be the uniform norm. We denote by \( B(k) \) the closed ball of radius 1.

Proposition 51. \( B(k) \) is a compact set.

Proof. Since \( B(k) \) is a bounded subset of the finite-dimensional space \( \mathcal{P}(k) \), it is enough to prove that \( B(k) \) is closed, indeed, by Arzela-Ascoli theorem we just need to prove that \( B(k) \) is an equi-continuous set: let \( F(x) \) be a polynomial in \( C(k) \), then the Markov brothers’ inequality implies \( |F'(x)| \leq k^2 \), so \( |F(x_1) - F(x_2)| < k^2 |x_1 - x_2| \).

Definition 52. We say a polynomial \( F \) is unitary if \( F \) has a hill interval \([0, 1]\), and let \( \mathcal{P}_N(k) \) be the set of unitary polynomials.

Corollary 53. If \( G_n(x) \) is a sequence of non-constant polynomials in \( Pen_F \) with hill interval \( I_n = [x_n, x'_n] \) such that \( G_n(x_n) = G_n(x'_n) = 1 \), \( \lim_{n \to \infty} x_n = -\infty \) and \( \lim_{n \to \infty} x'_n = \infty \), then \( F(x) \) must be even degree.

Proof. Let \( G_n(x) \) be equal to \( a_n + b_n F(x) \). There exists \( K_x \) a compact set containing all the roots of \( F(x) \), and let \( n \) be large enough that \( K_x \subset I_n \). Let us assume \( F(x'_n) > 0 \) and \( F(x_n) < 0 \), then \( 0 = G(x'_n) - G(x_n) = b_n (F(x'_n) - F(x_n)) \) and \( b_n = 0 \) since \( F(x'_n) - F(x_n) > 0 \), which is a contradiction to the assumption that \( G_n(x) \) is a sequence of non-constant polynomials.

A.1. Proof Of Lemma 34.

Proof. Let \( c_n(t) = (x_n(t), y_n(t), z_n(t)) \) be a sequence of \( \mathbb{R}_F^3 \)-geodesics traveling during a time interval \([t_0, t_1]_n\) and with travel interval \( \mathcal{I}_n([t_0, t_1]_n) \) such that \( x_n((t_0)_n) \) and \( x_n((t_1)_n) \) are in \([0, 1]\) for all \( n \). We will prove that if \( \mathcal{I}_n \) is unbounded, then \( \Theta(c, [t_0, t_1]) \) is unbounded.
The sequence of \( c_n(t) \) of \( \mathbb{R}^3_F \)-geodesics, induces a sequence of \( G_n(x) \) polynomials. We use the sequence \( G_n(x) \) to define a sequence of unitary polynomials \( \hat{G}_n(\tilde{x}) := G_n(h_n(\tilde{x})) \) where \( h_n(\tilde{x}) = (x_0)_n + u_n \tilde{x} \) with \( u_n := (x_0)_n - (x_1)_n \). Since \( \hat{G}_n(\tilde{x}) \) is in \( C(k) \). There exists a subsequence \( \hat{G}_{n_j}(\tilde{x}) \) converging to \( \hat{G}(\tilde{x}) \). Let us proceed by the following cases: case \( \hat{G}(\tilde{x}) \neq 1 \) or case \( \hat{G}(\tilde{x}) = 1 \).

Case \( \hat{G}(\tilde{x}) \neq 1 \): by Fatou’s lemma \( 0 < \text{Cost}(\hat{G}) \leq \liminf_{n_j \to \infty} \text{Cost}(\hat{G}_n) \). Then \( u_{n_j} \to \infty \) implies \( \text{Cost}(c, \mathcal{I}_{n_j}) \) is unbounded.

Case \( \hat{G}(\tilde{x}) = 1 \): let \( K'_x \) be a compact set such that all the roots of \( 1 - F(x) \) are in \( K'_x \). There exists \( n^* > 0 \) such that \( \hat{G}(\tilde{x}) > \frac{1}{2} \) for all \( \tilde{x} \) in \([0, 1]\) if \( n_s > n^* \). We split the integral for \( \Delta z(c, \mathcal{I}_n) \) given by Corollary 22 in the following way

\[
\int_{\mathcal{I}_{n_j}} \frac{(1 - F(x))G_{n_j}(x)}{\sqrt{1 - G_{n_j}^2(x)}} \, dx = \int_{K'_x \cap \mathcal{I}} \frac{(1 - F(x))G_{n_j}(x)}{\sqrt{1 - G_{n_j}^2(x)}} \, dx
\]

(A.1)

Since the first integral of the right side is finite, it is enough to focus on the second integral.

We proceed by cases: Case 1, \((x_0)_{n_j}\) and \((x_1)_{n_j}\) are both unbounded. Case 2, \((x_0)_{n_j}\) is bounded and \((x_1)\) is unbounded. Case 3, \((x_0)_{n_j}\) is unbounded and \((x_1)_{n_j}\) is bounded.

Case 1: by Corollary 53 implies \( F(x) \) is even, then the condition \( \hat{G}(\tilde{x}) > \frac{1}{2} \) implies \( |G_{n_j}(x)| > \frac{1}{2} \) in the travel interval \( \mathcal{I}_{n_j} \) and \((1 - F(x))G_{n_j}(x)\) does not change sign in the set \( \mathcal{I}_{n_j} \setminus K'_x \), therefore

\[
|\int_{\mathcal{I}_{n_j} \setminus K'_x} \frac{(1 - F(x))G_{n_j}(x)}{\sqrt{1 - G_{n_j}^2(x)}} \, dx| > \frac{1}{2} \int_{\mathcal{I}_{n_j} \setminus K'_x} |F(x)| \, dx \to \infty \text{ when } n_j \to \infty.
\]

A similar proof follows for Cases 2 and 3.

\[\Box\]

**APPENDIX B. PROOF OF THEOREM 41**

For simplicity, we will prove Theorem 41 for the case \( F(x) = 1 - 2x^3 \). Let \( c(t) \) be a \( \mathbb{R}^3_F \)-geodesic for \( F(x) = 1 - 2x^3 \) with initial point \( c(0) = (1, 0, 0) \) and hill interval \([0, 1]\). Let us consider the time interval \([-n, n]\), since the reduced system is reservable it follows \( x_n := x(n) = x(-n) \) and the travel interval \( \mathcal{I}_n = [0, 1] \). By Corollary 22, the relation between the travel interval and \( n \) is given by

\[
n = \int_{[x_n, 1]} \frac{dx}{\sqrt{1 - F^2(x)}}.
\]
Both images show the projection of the geodesic \(c(t)\) for \(F(x) = 1 - 2x^3\) and the curve \(\tilde{c}(t)\) to the \((x,y)\) and \((x,z)\) planes, respectively.

In addition, the change in \(\Delta y(c, n)\) and \(\Delta z(c, n)\) are given by

\[
\Delta y(c, n) := 2 \int_{[x_n, 1]} \frac{F(x)dx}{\sqrt{1 - F^2(x)}} \quad \text{and} \quad \Delta z(c, n) = 2 \int_{[x_n, 1]} \frac{F^2(x)dx}{\sqrt{1 - F^2(x)}}.
\]

Therefore

\[
c(-n) = (x(-n), \frac{-\Delta y(F, n)}{2}, \frac{-\Delta z(F, n)}{2})
\]

and

\[
c(n) = (x(n), \frac{\Delta y(F, n)}{2}, \frac{\Delta z(F, n)}{2}).
\]

**Corollary 54.** If \(F(x) = 1 - 2x^3\) and \(n\) is large enough, then

\[
\Delta y(F, n) < \Delta z(F, n) \quad \text{and} \quad \lim_{n \to \infty} \frac{\Delta z(F, n)}{\Delta y(F, n)} = 1.
\]

**Proof.** If \(F(x) = 1 - 2x^3\), then the same integration by parts, used to prove Corollary 42, implies the inequality \(\Delta y(F, n) - \Delta z(F, n) > 0\). L'Hôpital rules shows \(\lim_{n \to \infty} \frac{\Delta z(F, n)}{\Delta y(F, n)} = 1\).

**B.1. Proof Of Theorem 41.**

**Proof.** For every large enough \(n\) we can find \(0 < \epsilon_n\) and \(0 < \delta_n\) such that

\[
\Delta z(F, n) = (1 + \epsilon_n) \Delta y(F, n) \quad \text{and} \quad F(-\delta_n) = 1 + \epsilon_n.
\]

If \(T_1 = x_n + \delta_n, \ T_2 = T_1 + \Delta y(F, n)\) and \(T_3 = T_1 + T_2\), then for every \(n\) we define the following curve \(\tilde{c}_n(t)\) in \(\mathbb{R}^3_F\) in the interval \([0, T_3]\) as follows

\[
\tilde{c}_n(t) = \begin{cases} 
  c(-n) + (-t, 0, 0) & \text{where } t \in [0, T_1] \\
  c(-n) + (-T_1, t - T_1, 0) & \text{where } t \in [T_1, T_2] \\
  c(-n) + (-T_1 + t - T_2, \Delta y(F, n), \Delta z(F, n)) & \text{where } t \in [T_2, T_3].
\end{cases}
\]

See figure B.1. By construction, \(c(-n) = \tilde{c}_n(0)\) and \(c(n) = \tilde{c}_n(T_3)\), the relation between the \(n\) and \(\Delta y(F, n)\) is given by

\[
2n = \Delta y(F, n) + \text{Cost}_I(F, [-n, n]),
\]
while the relation between $T_3$ and $\Delta y(F, n)$ is given by

$$T_3 = \Delta y(F, n) + 2(\delta_n + x_n).$$

If $n \to \infty$, then $\text{Cost}_t(F, [-n, n]) \to \Theta_1(F, [0, 1]) > 0$ and $2(\delta_n + x_n) \to 0$. Thus there exists an $n$ such that $\text{Cost}_t(F, [-n, n]) > 2(\delta_n + x_n)$, in other words $T_3 < 2n$ and we conclude that $\tilde{c}_n(t)$ is shorter than $c(t)$.

□

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