Tuning the stochastic background of gravitational waves using the WMAP data

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Abstract

The cosmological bound of the stochastic background of gravitational waves is analyzed with the auxilium of the WMAP data, differently from lots of works in literature, where the old COBE data were used. From our analysis, it will result that the WMAP bounds on the energy spectrum and on the characteristic amplitude of the stochastic background of gravitational waves are greater than the COBE ones, but they are also far below frequencies of the earth-based antennas band. At the end of this letter a lower bound for the integration time of a potential detection with advanced LIGO is released and compared with the previous one arising from the old COBE data. Even if the new lower bound is minor than the previous one, it results very long, thus for a possible detection we hope in the LISA interferometer and in a further growth in the sensitivity of advanced projects.

The design and construction of a number of sensitive detectors for gravitational waves (GWs) is underway today. There are some interferometers like the VIRGO detector, being built in Cascina near Pisa, Italy, by a joint Italian-French collaboration, the GEO 600 detector being built in Hanover, Germany by a joint Anglo-Germany collaboration, the two LIGO detectors being built in the United States (one in Hanford, Washington and the other in Livingston, Louisiana) by a joint Caltech-Mit collaboration, and the TAMA 300 detector, being built near Tokyo, Japan. There are many bar detectors currently in operation too, and several interferometers and bars are in a phase of planning and proposal stages (for the current status of GWs experiments see [1, 2]).
The results of these detectors will have a fundamental impact on astrophysics and gravitation physics. There will be several experimental data to be analyzed, and theorists will be forced to interact with lots of experiments and data analysts to extract the physics from the data stream.

In this letter the stochastic background of GWs (SBGWs) [3, 4, 5], which is, in principle, a possible target of these experiments, is analyzed with the auxilium of the WMAP data [6, 7]. We emphasize that, in general, in previous works in literature about the SBGWs, old COBE data were used (see [3, 4, 5, 8, 9] for example).

From our analysis, it will result that the WMAP bounds on the energy spectrum and on the characteristic amplitude of the SBGWs are greater than the COBE ones, but they are also far below frequencies of the earth-based antennas band. At the end of this letter a lower bound for the integration time of a potential detection with advanced LIGO is released and compared with the previous one arising from the old COBE data. Even if the new lower bound is minor than the previous one, it results very long, thus for a possible detection we hope in the LISA interferometer and in a further growth in the sensitivity of advanced projects.

The strongest constraint on the spectrum of the relic SBGWs in the frequency range of ground based antennas like bars and interferometers, which is the range $10^3 \text{Hz} \leq f \leq 10^4 \text{Hz}$, comes from the high isotropy observed in the Cosmic Background Radiation (CBR).

The fluctuation $\Delta T$ of the temperature of CBR from its mean value $T_0 = 2.728 \text{ K}$ varies from point to point in the sky [6, 7], and, since the sky can be considered the surface of a sphere, the fitting of $\Delta T$ is performed in terms of a Laplace series of spherical harmonics

$$\frac{\Delta T}{T_0}(\hat{\Omega}) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\hat{\Omega}),$$

and the fluctuations are assumed statistically independent ($< a_{lm} > = 0$, $< a_{lm} a^*_{l'm'} > = C_l \delta_{ll'} \delta_{mm'}$). In eq. (1) $\hat{\Omega}$ denotes a point on the 2-sphere while the $a_{lm}$ are the multipole moments of the CBR. For details about the definition of statistically independent fluctuations in the context of the temperature fluctuations of CBR see [10].

The WMAP data [6, 7] permit a more precise determination of the rms quadrupole moment of the fluctuations than the COBE data

$$Q_{rms} = T \left( \sum_{m=-2}^{2} \frac{|a_{2m}|^2}{4\pi} \right)^{\frac{1}{2}} = 8 \pm 2 \mu K,$$

while in the COBE data we had [3, 13]

$$Q_{rms} = 14.3^{+5.2}_{-3.3} \mu K.$$

A connection between the fluctuation of the temperature of the CBR and the SBGWs derives from the *Sachs-Wolfe effect* [5, 9]. Sachs and Wolfe showed that
variations in the density of cosmological fluid and GWs perturbations result in the fluctuation of the temperature of the CBR, even if the surface of last scattering had a perfectly uniform temperature \[9\]. In particular the fluctuation of the temperature (at the lowest order) in a particular point of the space is

\[
\frac{\Delta T}{T_0}(\hat{\Omega}) = \frac{1}{2} \int_{null\text{geodesic}} d\lambda \frac{\partial}{\partial \eta} h_{rr}.
\] (4)

The integral in eq. (4) is taken over a path of null geodesic which leaves the current spacetime point heading off in the direction defined by the unit vector $\hat{\Omega}$ and going back to the surface of last scattering of the CBR.

Here $\lambda$ is a particular choice of the affine parameter along the null geodesic.

By using conformal coordinates, we have for the metric perturbation

\[
\delta g_{ab} = R^2(\eta) h_{ab},
\] (5)

and $r$ in eq. (4) is a radial spatial coordinate which goes outwards from the current spacetime point. The effect of a long wavelength GW is to shift the temperature distribution of CBR from perfect isotropy. Because the fluctuations are very small ($< \Delta T/T_0 > \leq 5 \times 10^{-5}$ \[6, 7\]), the perturbations caused by the relic SBGWs cannot be too large.

The WMAP results give rather tight constraints on the spectrum of the SBGWs at very long wavelengths. In \[3\] we find a constraint on $\Omega_{gw}(f)$ which derives from the COBE observational limits, given by

\[
\Omega_{gw}(f) h_{100}^2 < 7 \times 10^{-11} \left(\frac{H_0}{f}\right)^2 \text{ for } H_0 < f < 30H_0.
\] (6)

Now the same constraint will be obtained from the WMAP data. Because of its specific polarization properties, relic SBGWs should generate particular polarization pattern of CBR anisotropies, but the detection of CBR polarizations is not fulfilled today \[12\]. Thus an indirect method will be used. We know that relic GWs have very long wavelengths of Hubble radius size, so the CBR power spectrum from them should manifest fast decrease at smaller scales (high multipole moments). But we also know that scalar modes produce a rich CBR power spectrum at large multipole moments (series of acoustic peaks, ref. \[6, 7\]). Then the properties of tensor modes of cosmological perturbations of spacetime metric can be extract from observational data using angular CBR power spectrum combined with large scale structure of the Universe. One can see (fig. 1) that in the range $2 \leq l \leq 30$ (the same used in \[3\], but with the old COBE data \[13\]) scalar and tensor contributions are comparable. From \[6, 7\] the WMAP data give for the tensor/scalar ratio $r$ the constraint $r < 0.9$, ($r < 0.5$ in the COBE data, ref. \[13\]; Novosyadly and Apunevych obtained $\Omega_{\text{scalar}}(H_0) < 2.7 \times 10^{-9}$ \[12\]. Thus, if one remembers that, at order of Hubble radius, the tensorial spectral index is $-4 \leq n_t \leq -2$, it results

\[
\Omega_{gw}(f) h_{100}^2 < 1.6 \times 10^{-9} \left(\frac{H_0}{f}\right)^2 \text{ for } H_0 < f < 30H_0,
\] (7)
which is greater than the COBE data result of eq. (6).

We emphasize that the limit of eq. (7) is not a limit for any GWs, but only for relic ones of cosmological origin, which were present at the era of the CBR decoupling. Also, the same limit only applies over very long wavelengths (i.e. very low frequencies) and it is far below frequencies of the Virgo - LIGO band.

The primordial production of the relic SBGWs has been analyzed in [3, 4, 5, 14, 15], where it has been shown that in the range $10^{-15} \, \text{Hz} \leq f \leq 10^{10} \, \text{Hz}$ the spectrum is flat and proportional to the ratio

$$\frac{\rho_{\text{ds}}}{\rho_{\text{Planck}}} \approx 10^{-12}. \quad (8)$$

WMAP observations put strongly severe restrictions on the spectrum, as we discussed above. In fig. 2 the spectrum $\Omega_{gw}$ is mapped: the amplitude (determined by the ratio $\frac{\rho_{ds}}{\rho_{\text{Planck}}}$) has been chosen to be as large as possible, consistent with the WMAP constraint (7). Nevertheless, because the spectrum falls off $\propto f^{-2}$ at low frequencies [3, 4, 5], this means that today, at Virgo and
LISA frequencies, indicated in fig. 2,

\[ \Omega_{gw}(f) h_{100}^2 < 9 \times 10^{-13}, \]  

(9)

while using the COBE data it was

\[ \Omega_{gw}(f) h_{100}^2 < 8 \times 10^{-14} \text{(refs. [3, 13])}. \]

It is interesting to calculate the correspondent strain at \( \approx 100 \text{Hz} \), where interferometers like Virgo and LIGO have a maximum in sensitivity. The well known equation for the characteristic amplitude [3, 8] can be used:

\[ h_\text{c}(f) \approx 1.26 \times 10^{-18} \left( \frac{1 \text{Hz}}{f} \right) \sqrt{h_{100}^2 \Omega_{gw}(f)}, \]  

(10)

obtaining

\[ h_\text{c}(100 \text{Hz}) < 1.7 \times 10^{-26}. \]  

(11)

Then, because for ground-based interferometers a sensitivity of the order of \( 10^{-22} \) is expected at \( \approx 100 \text{Hz} \), four order of magnitude have to be gained in the signal to noise ratio [16, 17]. Let us analyze smaller frequencies too. The sensitivity of the Virgo interferometer is of the order of \( 10^{-21} \) at \( \approx 10 \text{Hz} \) [16] and in that case it is

\[ h_\text{c}(100 \text{Hz}) < 1.7 \times 10^{-25}. \]  

(12)

For a better understanding of the difficulties on the detection of the SBGWs a lower bound for the integration time of a potential detection with advanced LIGO is released. For a cross-correlation between two interferometers the signal to noise ratio (SNR) increases as [3, 4, 5, 18]

\[ (SNR) = \sqrt{\frac{2T H_0^2}{5\pi^2}} \sqrt{\int_0^\infty \frac{\Omega_{gw}^2(f) \gamma^2(f)}{f_0 P_1(|f|) P_2(|f|)}}. \]  

(13)

where \( P_i(|f|) \) is the one-sided power spectral density of the \( i \) detector [18] and \( \gamma(f) \) the well known overlap-reduction function [18, 19]. Assuming two coincident coaligned detectors (\( \gamma(f) = 1 \)) with a noise of order \( 10^{-48}/\text{Hz} \) (i.e. a typical value for the advanced LIGO sensitivity [20]) one gets (SNR) \( \sim 1 \) for \( \sim 3 \times 10^5 \text{years} \) using our result \( \Omega_{gw}(f) h_{100}^2 \sim 9 \times 10^{-13} \) while it is (SNR) \( \sim 1 \) for \( \sim 3 \times 10^7 \text{years} \) using previous COBE result \( \Omega_{gw}(f) h_{100}^2 \sim 8 \times 10^{-14} \). Since the overlap reduction function degrades the SNR, these results can be considered a solid upper limit for the advanced LIGO configuration for the two different values of the spectrum.

The sensitivity of the LISA interferometer will be of the order of \( 10^{-22} \) at \( 10^{-3} \approx H \text{z} \) [21] and in that case it is
\[ h_c(100 \text{Hz}) < 1.7 \times 10^{-21}. \] (14)

Then a stochastic background of relic gravitational waves could be in principle detected by the LISA interferometer. We also hope in a further growth in the sensitivity of advanced projects.

We emphasize that the assumption that all the tensorial perturbation in the Universe are due to a SBGWs is quit strong, but our results \((9), (11), (12)\) and \((14)\) can be considered like upper bounds.

Reasuming in this letter the SBGWs has been analyzed with the auxilium of the WMAP data, while previous works in literature, used the old COBE data, seeing that the predicted signal for these relic GWs is very weak. From our analysis it resulted that the WMAP bound on the energy spectrum and on the characteristic amplitude of the SBGWs are greater than the COBE ones, but they are also far below frequencies of the earth-based antennas band. In fact the integration time of a potential detection with advanced interferometers is very long, thus, for a possible detection we have to hope in a further growth in the sensitivity of advanced ground based projects and in the LISA interferometer.
Figure 2: The spectrum of relic gravitons in inflationary models is flat over a wide range of frequencies. The horizontal axis is $\log_{10}$ of frequency, in Hz. The vertical axis is $\log_{10} \Omega_{gr}$. The inflationary spectrum rises quickly at low frequencies (wave which reentered in the Hubble sphere after the Universe became matter dominated) and falls off above the (appropriately redshifted) frequency scale $f_{max}$ associated with the fastest characteristic time of the phase transition at the end of inflation. The amplitude of the flat region depends only on the energy density during the inflationary stage; we have chosen the largest amplitude consistent with the WMAP constrain discussed earlier: $\Omega_{gr}(f)h_{100}^2 < 1.6 \times 10^{-9}$ at $10^{-18}$ Hz. This means that at Virgo and Lisa frequencies, $\Omega_{gr}(f)h_{100}^2 < 9 \times 10^{-13}$

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