Possible precise measurements of the $X(3872)$ mass with the $e^+e^- \rightarrow \pi^0\gamma X(3872)$ and $p\bar{p} \rightarrow \gamma X(3872)$ reactions

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Abstract

It was recently proposed that the $X(3872)$ binding energy, the difference between the $D^0\bar{D}^{*0}$ threshold and the $X(3872)$ mass, can be precisely determined by measuring the $\gamma X(3872)$ line shape from a short-distance $D^*\bar{D}^*\bar{D}D^*$ source produced at high-energy experiments. Here, we investigate the feasibility of such a proposal by estimating the cross sections for the $e^+e^- \rightarrow \pi^0\gamma X(3872)$ and $p\bar{p} \rightarrow \gamma X(3872)$ processes considering the $D^*\bar{D}^*\bar{D}D^*$ triangle loops. These loops can produce a triangle singularity slightly above the $D^*\bar{D}^*$ threshold. It is found that the peak structures originating from the $D^*\bar{D}^*$ threshold cusp and the triangle singularity are not altered much by the energy dependence introduced by the $e^+e^- \rightarrow \pi^0\gamma X(3872)$ and $p\bar{p} \rightarrow \bar{D}\bar{D}D^*$ production parts or by considering a finite width for the $X(3872)$. We find that $\sigma(e^+e^- \rightarrow \pi^0\gamma X(3872)) \times B(X(3872) \rightarrow \pi^+\pi^- J/\psi)$ is about 0.03 fb to 0.2 fb with the $\gamma X(3872)$ invariant mass integrated from 4.01 to 4.02 GeV and the c.m. energy of the $e^+e^-$ pair fixed at 4.23 GeV. The cross section $\sigma(p\bar{p} \rightarrow \gamma X(3872)) \times B(X(3872) \rightarrow \pi^+\pi^- J/\psi)$ is estimated to be of $O(10 \text{ pb})$. Our results suggest that a precise measurement of the $X(3872)$ binding energy can be done at PANDA.

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I. INTRODUCTION

Among many charmonium-like states listed in the Review of Particle Physics (RPP) [1], special attention has been paid to the $X(3872)$.\(^1\) The mass of $X(3872)$ is consistent with the $D^0\bar{D}^{*0}$ threshold energy, $m_X = (3871.69 \pm 0.17)$ MeV, and only an upper bound is provided for its small width, $\Gamma_X < 1.2$ MeV [1]. The latest experimental development comes from the LHCb Collaboration that reported precise determinations of the mass and width [2, 3]. In particular, a detailed analysis of the $X(3872)$ line shape using the Flatté parametrization [4], which is more proper than the Breit–Wigner (BW) form for states near an $S$-wave strongly-coupled threshold, is performed in Ref. [2]. The closeness of its mass and the $D^0\bar{D}^{*0}$ threshold invokes the hadronic molecular description of $X(3872)$: the $X(3872)$ is treated as a shallow $S$-wave bound state of $D^0\bar{D}^{*0}$, e.g., in Refs. [5–13]. Such a description can successfully explain the large branching ratio of the isospin forbidden $X(3872) \to \pi^+\pi^-J/\psi$ relative to the isospin allowed $\pi^+\pi^-\pi^0J/\psi$ mode [14], and the strong coupling of the molecular state to its constituents in the molecular description, i.e., $X(3872)$ to $D\bar{D}^*$, would naturally explain the large branching fractions of the $X(3872)$ to $\pi^0D^0\bar{D}^{*0}/D^0\bar{D}^{*0}$ [1, 15, 16]. The strong coupling of the $X(3872)$ to the $D^0\bar{D}^{*0}$ in an $S$-wave implies that there must be a strong cusp exactly at the threshold [17], complicating the line shape analysis. The line shapes of the $\pi^+\pi^-J/\psi$ and/or $D^0\bar{D}^{*0}$ distributions were analyzed with the Flatté parametrization [2, 18–21] or the effective range expansion [22, 23] in which the threshold effect is incorporated by requiring unitarity; however, no conclusive results have not been achieved so far. See, e.g., Refs. [24–27] and references therein for further information on works related to $X(3872)$, in particular from the hadronic molecular point of view.

Recently, a possible way to precisely determine the $X(3872)$ binding energy, which is defined as the difference between the $D^0\bar{D}^{*0}$ threshold and the $X(3872)$ mass\(^2\)

$$\delta = m_{D^0} + m_{\bar{D}^{*0}} - m_X,$$

was proposed in Ref. [28]. This can be done by measuring the $\gamma X(3872)$ distribution instead of the $X(3872)$ line shape in its decay products like $\pi^+\pi^-J/\psi$ or $D^0\bar{D}^{*0}$. Consider a triangle diagram for the transition of an $S$-wave $D^{*0}\bar{D}^{*0}$ pair, produced at short distances in some high-energy experiment, into $\gamma X(3872)$. The $D^{*0}$ ($\bar{D}^{*0}$) subsequently decays

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\(^{1}\) In this paper, the $\chi_{c1}(3872)$ in the RPP [1] is denoted by $X(3872)$ or merely $X$, and $Z_c(4020)$ or $Z_c$ stands for the $X(4020)$ in the RPP.

\(^{2}\) A negative $\delta$ corresponds to a mass above the threshold and thus a resonant state in this paper.
into $\gamma D^0$ ($\gamma D^0$), and the $X(3872)$ is produced by merging the $D^0\bar{D}^{*0} + D^0\bar{D}^{*0}$ pair at the last step. The process thus proceeds via a $D^{*0}\bar{D}^{*0}D^0$ triangle loop. This loop can have a triangle singularity (TS) due to the simultaneous on-shellness of all three intermediate mesons, which leads to a peak in the $\gamma X(3872)$ distribution just above the $D^{*0}\bar{D}^{*0}$ threshold. With the Landau equation \[29\] or with a simple equation for the TS position derived with a refined formulation \[30\], one sees that the TS position is sensitive to the $X(3872)$ mass: the TS is located at $4015.14$ MeV with $\delta = -180$ keV and $4015.64$ MeV with $\delta = -50$ keV. For the $X(3872)$ mass within $\{4015.17, 4016.40\}$ MeV which can be obtained by using Eqs. (55) and (60) in Ref. \[17\]. While the TS, at which the amplitude diverges logarithmically, is turned into a finite peak due to the width of the internal particles, the peak originating from the TS of the $D^*\bar{D}^*D$ loop should be still clear thanks to the tiny width of the $D^{*0}$, which is only $55.3 \pm 1.4$ keV \[28, 31\]. Then, one expects that the $X(3872)$ binding energy can be determined well with the precise measurement of the TS peak in the $\gamma X(3872)$ distribution.

The role of the TS stemming from the $D^*\bar{D}^*D$ loop on the $X(3872)$ production has been studied in some papers. The $e^+e^- \rightarrow \gamma X(3872)$ transition is studied in Refs. \[32, 33\]. In Ref. \[34\], the energy dependence of the $Z_c(4050)^0 \rightarrow \gamma X(3872)$ branching fraction is studied. One can see the difference of the energy dependence by changing the $X(3872)$ binding energy. In addition to the radiative reactions, decays emitting a pion with the $D^{*0}\bar{D}^{*0}D^0$ loop has also been considered \[34–36\]. While the TS appears in a smaller range of the $\pi X(3872)$ energy compared with the $\gamma X(3872)$ case, the asymmetry of the $\pi X(3872)$ line shape may be used to extract the $X(3872)$ binding energy. The decay process $B \rightarrow (J/\psi\pi^+\pi^-)K\pi$ with the $J/\psi\pi^+\pi^-$ produced by the $D^0\bar{D}^{*0}$ rescattering considering the $D^{**}D^{*0}D^0/D^{*-}D^{*0}D^0$ loop is studied in Ref. \[37\]. For more works related to the TS, we refer to Ref. \[17\].

In this paper, we investigate two promising reactions in which the proposal of precisely measuring the $X(3872)$ binding energy by virtue of the TS mechanism may be realized: the $e^+e^- \rightarrow \pi^0\gamma X(3872)$ and $p\bar{p} \rightarrow \gamma X(3872)$ reactions. In these reactions, the $D^*\bar{D}^*$ pair can be produced in an $S$ wave. In the case of the $e^+e^-$ collisions, the isovector resonance $Z_c(4020)$ seen in the $D^*\bar{D}^*$ distribution of the $e^+e^- \rightarrow \pi^0(D^*\bar{D}^*)^0$ process \[38\] is expected to be a good source of the $S$-wave $D^*\bar{D}^*$ pair, and high-statistics data can be expected for the $p\bar{p}$ reaction by the PANDA experiment at the Facility for Antiproton and Ion Research (FAIR) in the near future.
FIG. 1. Triangle diagram contributing to the $e^+e^- \to \pi^0\gamma X(3872)$ process considered here.

This paper is organized as follows. In Sec. II, the formalism for calculating the $e^+e^- \to \pi^0\gamma X(3872)$ and $p\bar{p} \to \gamma X(3872)$ amplitudes is provided with the effect of the $X(3872)$ width is taken into account. The results of our calculation, the $\gamma X(3872)$ invariant mass distributions in these reactions and the estimated cross sections, are given in Sec. III. A brief summary is given in Sec. IV. Detailed expressions of the amplitudes used in Sec. II are relegated to Appendix A.

II. FORMALISM

A. $e^+e^- \to \pi^0\gamma X(3872)$

First, we consider the $e^+e^- \to \pi^0\gamma X(3872)$ amplitude with the $D^*\bar{D}^* D/\bar{D}^* D\bar{D}$ loops. The diagram is given in Fig. 1. Only the neutral $D^*\bar{D}^* D/\bar{D}^* D\bar{D}$ loops are accounted for the process because we focus on the TS peak of the $\gamma X(3872)$ invariant mass distribution near the $D^*\bar{D}^*$ threshold and the $X(3872)$ appears near the $D^0\bar{D}^0$ threshold as a narrow peak. As found in Ref. [38], the $(D^*\bar{D}^*)^0$ distribution of $e^+e^- \to \pi^0(D^*\bar{D}^*)^0$ at the c.m. energies $\sqrt{s} = 4.23$ and 4.26 GeV can be described well by including a resonance with $J^P = 1^+$, and the $(D^*\bar{D}^*)^0$ pair is predominantly produced by the resonance around the $D^*\bar{D}^*$ threshold. Here, we also assume that the $Z_c(4020)$ is the $J^P = 1^+$ exotic state which can decay into an $S$-wave $D^*\bar{D}^*$ pair. The $\pi Z_c(4020)$ pair is produced by the $\psi(4230)$ resonance, which is seen in some hidden- and open-charm productions [1] and would be needed to describe the dependence of the cross section on the $e^+e^-$ c.m. energy because the $e^+e^- \to \pi D^*\bar{D}^*$ cross section at $\sqrt{s} = 4.26$ GeV is smaller than that of $\sqrt{s} = 4.23$ GeV [38]. We use the central values of the mass and width of the $\psi(4230)$ given in the RPP [1], $m_\psi = (4220 \pm 15)$ MeV and $\Gamma_\psi = (60 \pm 40)$ MeV. Note that, while the width of the $\psi(4230)$ is not fixed well, the $\gamma X(3872)$ invariant mass distribution at a given $\sqrt{s}$, which will be
considered in this work, is not affected by the details of the $\psi(4230)$ properties.

The $e^+e^- \to \gamma^*, \gamma^* \to \psi(4230)$, $\psi(4230) \to \pi^0 Z_c(4020)^0$, and $Z_c(4020)^0 \to D^*\bar{D}^*$ amplitudes are written as follows:

$$-i t_{e^+e^-} = i e g_{\mu\nu} \bar{v}^{\nu} u(\epsilon_{\psi}^*)^\gamma,$$

$$-i t_{\gamma,\psi} = i e g_{\mu\nu} (\epsilon_{\gamma})^{\mu}(\epsilon_{\psi}^*)^{\nu},$$

$$-i t_{\psi,\pi^0 Z_c} = i g_{\mu\nu} (\epsilon_{\psi}^{\nu})^{\mu}(\epsilon_{Z_c}^*)^{\nu},$$

$$-i t_{Z_c, D^*\bar{D}^*} = i g_{\mu\nu\rho\sigma}(p_{Z_c})_{\mu}(\epsilon_{Z_c})_{\nu}(\epsilon_{D^*})_{\rho}(\epsilon_{\bar{D}^*})_{\sigma},$$

where $e(e > 0)$ denotes the electric charge unit, $\bar{v}$ and $u$ are the spinors for the positron and electron, respectively, and the $\epsilon$’s are the polarization vectors of the involved spin-1 particles. With the isospin symmetry and the phase convention $|D^{(s^+)}\rangle = - |I = 1/2, I_z = 1/2\rangle$, a minus sign is needed for the $Z_c(4020)^0 \to D^{*+}D^{*-}$ coupling constant relative to the $Z_c(4020)^0 \to D^{*0}\bar{D}^{*0}$ coupling. Constant amplitudes are used for the $S$-wave vertices of the $\psi(4230) \to \pi^0 Z_c(4020)^0$ and $Z_c(4020)^0 \to D^*\bar{D}^*$ because the lowest angular momentum gives the dominant contribution in the near-threshold region. Then, the $e^+e^- \to \pi^0 D^*\bar{D}^*$ amplitude is given by

$$-i M_{e^+e^-\pi^0 D^*\bar{D}^*} = i e g_{\mu\nu} g_{12} D_{\gamma}^{-1}(s) D_{\psi}^{-1}(s) D_{Z_c}^{-1}(m_{D^*}\bar{D}^*)$$

$$\times \bar{v}^{\nu} u^{\mu} \bar{P}_{\psi} \bar{P}_{Z_c} \bar{P}_{D^*} \epsilon_{D^*}(\epsilon_{\psi}^*)^{\gamma}(\epsilon_{Z_c}^*)^{\nu}(\epsilon_{D^*}^*)^{\delta},$$

with $D_R(s) = s - m_R^2 + i m_R \Gamma_R$ and $[P_R]^{\mu\nu} = -g^{\mu\nu} + \frac{\delta^{\mu\nu} \delta}{m_R^2}$. The energy dependence of the width is taken into account as done in Ref. [38] (see also the review on the resonances of Ref. [1]):

$$\Gamma_{Z_c}(m_{D^*}\bar{D}^*) = \frac{\Gamma_{Z_0}}{2} \left( \frac{p_{D^{*0}}(m_{D^*}\bar{D}^*)}{p_{D^{*0}}(m_{Z_0})} + \frac{p_{D^{*+}}(m_{D^*}\bar{D}^*)}{p_{D^{*+}}(m_{Z_0})} \right),$$

$$p_{D^{*+}}(m_{D^*}\bar{D}^*) = \frac{1}{2m_{D^*}\bar{D}^*} \lambda^{1/2}(m_{D^*}\bar{D}^*, m_{D^*}, m_{D^*}),$$

where $\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$. The central values of $m_{Z_0}$ and $\Gamma_{Z_0}$ in Ref. [38], $m_{Z_0} = (4031.7 \pm 2.1)$ MeV and $\Gamma_{Z_0} = (25.9 \pm 8.8)$ MeV, are used. With the amplitude in Eq. (2), the differential cross section of $e^+e^- \to \pi^0 Z_c(4020)^0 \to \pi^0(D^*\bar{D}^*)^0$, $d\sigma_{e^+e^-\pi^0(D^*\bar{D}^*)^0}/dm_{D^*}\bar{D}^*$, is given by

$$\frac{d\sigma_{e^+e^-\pi^0(D^*\bar{D}^*)^0}}{dm_{D^*}\bar{D}^*} = \sum_{D^*\bar{D}^*} \frac{P_{\pi^0} P_{D^*}}{(4\pi)^2 p_e s} \int d\Omega_{\pi^0} \int d\Omega_{D^*} |M_{e^+e^-\pi^0(D^*\bar{D}^*)^0}|^2,$$
with \( p_{s^0} = \lambda^{1/2}(s, m_{s^0}^2, m_{D^*}^2)/(2\sqrt{s}) \), \( p_{D^*} = \lambda^{1/2}(m_{D^*}^2, m_{D^*}^2, m_{s^0}^2)/(2m_{s^0}^2) \), and \( p_c = \lambda^{1/2}(\sqrt{s}, m_c^2, m_c^2)/(2\sqrt{s}) \). The sum of \( D^* \bar{D}^* \) takes care of both the \( D^{*0} \bar{D}^{*0} \) and \( D^{*+} \bar{D}^{*-} \) that are included in the \( (D^* \bar{D}^*)^0 \) final state observed by BESIII [38]. The solid angles \( \Omega_{s^0} \) and \( \Omega_{D^*} \) are those in the \( e^+e^- \) c.m. frame and \( D^* \bar{D}^* \) c.m. frame, respectively. The overlined quantities are those after the spin sum and average. With the \( e^+e^- \rightarrow \pi^0 Z_c(4020)^0 \rightarrow \pi^0(D^* \bar{D}^*)^0 \) cross section in Ref. [38], (61.6 ± 8.2) pb at \( \sqrt{s} = 4.23 \text{ GeV} \), the product of the coupling constant \( g_0g_1g_2 \) is fixed to be \( g_0g_1g_2 = 0.68 \text{ GeV}^3 \).

Now, we move to the \( D^{*0} \bar{D}^{*0} D^0 \) triangle loop amplitude. The \( P \)-wave \( D^{*0} \rightarrow \gamma D^0 \) transition amplitude is given by [39]

\[
-iM_{D^{*0} \rightarrow \gamma D^0} = e g_3 \epsilon^{\mu\nu\rho\sigma}(p_{D^{*0}})_{\mu}(p_\gamma)_{\nu}(\epsilon_{D^{*0}})_{\rho}(\epsilon_\gamma)_{\sigma},
\]

and the parameter \( g_3 \) is fixed to be \( g_3 = 1.77 \text{ GeV}^{-1} \) with the \( D^{*0} \rightarrow \gamma D^0 \) branching ratio 35.3\% [1] and the \( D^{*0} \) full width \( \Gamma_{D^{*0}} = 55.3 \text{ keV} \) [28], which can be obtained by using isospin symmetry to relate to the \( D^{*+} \) full width and the \( D^{*+} \rightarrow \pi^+ D^0 \) and \( D^{*0} \rightarrow \pi^0 D^0 \) branching ratios [28, 40]. The \( \bar{D}^{*0} \rightarrow \gamma \bar{D}^0 \) amplitude needs one minus sign that comes from the \( C \) parity of the photon and the convention of the \( C \) transformation, \( CD^{*0} = + \bar{D}^{*0} \).

The \( S \)-wave transition amplitude of the \( D^0 \bar{D}^{*0} \rightarrow X(3872) \) transition is written as

\[
-it_{D^0 \bar{D}^{*0} \rightarrow X} = ig_4 \epsilon_{D^{*0}}^\mu(\epsilon_X^\sigma)^\nu,
\]

and the coupling constant of \( \bar{D}^0 D^{*0} \rightarrow X(3872) \) is the same. We estimate the coupling constant \( g_4 \) with two different ways for the \( X(3872) \) mass above or below the \( D^0 \bar{D}^{*0} \) threshold. When the \( X(3872) \) mass is below the \( D^0 \bar{D}^{*0} \) threshold, the coupling constant can be evaluated assuming the \( X(3872) \) be an \( S \)-wave \( D^0 \bar{D}^{*0} \) molecule [41–43],

\[
g_X^2 = \frac{16\pi m_X^2}{\mu_{D^0 \bar{D}^{*0}}^2} \sqrt{2\mu_{D^0 \bar{D}^{*0}}} \delta,
\]

with \( \mu_{D^0 \bar{D}^{*0}} \) and \( \delta \) being the \( D^0 \bar{D}^{*0} \) reduced mass and the \( X(3872) \) binding energy given by Eq. (1), respectively. In Eq. (6), \( g_X \) is the coupling constant of \( X(3872) \) to the \( D \bar{D}^* \) pair of \( J^{PC} = 1^{++} \), and \( g_4 \) and \( g_X \) are related with \( g_4 = g_X/2 \) [36]. When the \( X(3872) \) mass is above the \( D^0 \bar{D}^{*0} \) threshold, \( g_4 \) can be obtained by using the \( X(3872) \rightarrow D^0 \bar{D}^{*0} \) branching

3 The coupling constant of \( D^{*+} \rightarrow \gamma D^+, g_3^3 \), evaluated with the measured full width and branching ratio is \( g_3^3 = 0.47 \text{ GeV}^{-1} \), which is less than 1/3 of the \( D^{*0} \rightarrow \gamma D^0 \) coupling. This makes the charged \( D^* \bar{D}^* D \)-loop contribution even less important.
FIG. 2. The coupling constant $g_4$ as a function of the $X(3872)$ binding energy, $\delta$. The black and red lines in $\delta > 0$ and $\delta < 0$ correspond to the cases with the $X(3872)$ mass below and above the $D^0\bar{D}^{*0}$ threshold, and $g_4$ is evaluated with Eqs. (6) and (7), respectively.

The ratio \cite{44}; using Eq. (5), we have

$$g_4^2 = \frac{1}{2} \Gamma_X \text{Br}[X(3872) \to D^{*0}D^0 + \text{c.c.}] \frac{8\pi m_X^2/p_{D^{*0}}}{\frac{2}{3} \left(1 + \frac{E_{D^{*0}}}{2m_{D^{*0}}}ight)}$$  \hspace{1cm} (7)

with $p_{D^{*0}} = \lambda^{1/2}(m_X^2, m_{D^0}^2, m_{D^{*0}}^2)/(2m_X)$ and $E_{D^{*0}} = (m_X^2 + m_{D^{*0}}^2 - m_{D^0}^2)/(2m_X)$. In this work, the mass of $X(3872)$ is treated as a parameter, and it will be changed to see the difference of the $\gamma X(3872)$ invariant mass distribution. The width of $X(3872)$, $\Gamma_X$, is currently not known and the upper bound is provided \cite{1}. Here, $\Gamma_X$ is assumed to be 100 keV, which is the value expected from the calculation of the $X(3872) \to \pi^0 D^0\bar{D}^0$ partial width in the hadronic molecular picture \cite{8, 45, 46} and the $X(3872) \to \pi^0 D^0\bar{D}^0$ branching ratio \cite{1, 15, 16}. The coupling constant $g_4$ as a function of $\delta$ is shown in Fig. 2. Note that the values of $g_4$ from both the $\delta > 0$ and $\delta < 0$ sides are similar if we neglect the part with $\delta$ in the vicinity of 0. In that special region, the absolute value of the imaginary of the pole position cannot be approximated by half the width computed using Eq. (7). Furthermore, the coupling of the $X(3872)$ to the charged and neutral $DD^*$ can be computed from the residue of the coupled-channel $D^0\bar{D}^{*0} - D^+D^{*-} T$-matrix. It is found that the couplings of the $X(3872)$ to $D^0\bar{D}^{*0}$ and to $D^+D^{*-}$ are approximately the same \cite{14, 45}, and are consistent with the values shown in Fig. 2 (see also the discussion in Ref. [36]). For an estimate of the cross sections, in the following we will show the results with $\delta = \pm 50, \pm 180$ keV and use the coupling shown in Fig. 2.

Then, with the amplitudes Eqs. (2), (4), and (5), the $e^+e^- \to \pi^0\gamma X(3872)$ production
amplitude considering the $D^{*0}\bar{D}^{*0}D^0$ and $\bar{D}^{*0}D^{*0}\bar{D}^0$ triangle loops in Fig. 1 is given by

$$-i\mathcal{M}_{e^+e^-,\pi^0\gamma X} = 2\int \frac{d^4l}{(2\pi)^4}(-i\mathcal{M}_{e^+e^-,\pi^0\gamma X}^\delta) e g_3 g_4 D_\Delta^{-1}[P_{D^{*0}}]\gamma [P_{D^{*0}}]_\delta \tau \times \epsilon^\mu \epsilon^\nu \epsilon^\sigma \epsilon^\tau,$$

with

$$D_\Delta = [l^2 - m_{D^{*0}}^2 + i\epsilon][(k_1 + l)^2 - m_{D^{*0}}^2 + i\epsilon][(k_2 + l)^2 - m_{D^{*0}}^2 + i\epsilon].$$

The factor of 2 in the above equation comes from the same contribution from the charge-conjugated loops. The library LoopTools is used for the evaluation of the one-loop integral [47]. The width of the particles are taken into account by replacing the mass of $D^{*0}$ and $\bar{D}^{*0}$, with $m_{D^{*0}} - i\Gamma_{D^{*0}}/2$ in the propagator. See the Appendix A for the details of $\mathcal{M}_{e^+e^-,\pi^0\gamma X}$.

With the $e^+e^- \rightarrow \pi^0\gamma X(3872)$ amplitude in Eq. (8), the $\gamma X(3872)$ invariant mass distribution is given by

$$\frac{d\sigma_{e^+e^-,\pi^0\gamma X}}{dm_{\gamma X}} = \frac{p_{\pi^0}p_{\gamma}}{(4\pi)^5 s_p} \int d\Omega_{\pi^0} \int d\Omega_{\gamma} |\mathcal{M}_{e^+e^-,\pi^0\gamma X}|^2,$$

where $p_{\pi^0}$ and $p_{\gamma}$ are given by the expressions below Eq. (3) changing $m_{D^0}^2$ to $m_{X}^2$, and $p_{\gamma} = \lambda^{1/2}(m_{\pi^0}^2, 0, m_{X}^2)/(2m_{\gamma X})$. $\Omega_{\pi^0}$ and $\Omega_{\gamma}$ are the solid angles of the $\pi^0$ in the $e^+e^-$ c.m. frame and of the photon in the $\gamma X(3872)$ c.m. frame, respectively.

**B. $p\bar{p} \rightarrow \gamma X(3872)$**

The $p\bar{p} \rightarrow \gamma X(3872)$ amplitude is considered in this part. The diagram of the $p\bar{p} \rightarrow \gamma X(3872)$ transition with the $D^{*0}\bar{D}^{*0}D^0$/$\bar{D}^{*0}D^{*0}\bar{D}^0$ loops is shown in Fig. 3.

The $\bar{D}^{*0}D^{*0}$ pair can be produced from $p\bar{p}$ by exchanging a $\Lambda_c$ as depicted in Fig. 4. Possible $\Sigma_c^{(*)}$ contributions are ignored as argued in Ref. [48] based on the flavor SU(4)
model. With the effective Lagrangian for the $p\Lambda_cD^*0$ coupling [49],

$$\mathcal{L}_{N\Lambda_cD^*0} = g_v\bar{\Lambda}_c\gamma^\mu(D^*0)_{\mu\nu} + h.c.,$$

the $p\bar{p} \to \bar{D}^*0D^*0$ transition amplitude with the $\Lambda_c$ exchange is written as

$$-i\mathcal{M}_{pp,\bar{D}^*0D^*0} = \bar{v}(ig_v\gamma^\mu)\frac{iF^2_{p,\bar{D}^*0\Lambda_c}}{p - k - m_{\Lambda_c} + i\epsilon}(ig_v\gamma^\mu)u(\epsilon^*_{D^*0})_{\mu'}(\epsilon^*_{D^*0})_{\mu},$$

where $u$ and $\bar{v}$ are the spinors of the proton and antiproton, and a form factor $F_{p,\bar{D}^*0\Lambda_c}$ is introduced. For the parameter $g_v$, we take the value in Refs. [49, 50] obtained by using the SU(4) model, $g_v = -5.20$. For the form factor $F_{p,\bar{D}^*0\Lambda_c}$, we use

$$F^2_{p,\bar{D}^*0\Lambda_c} = \frac{\Lambda^4}{(m^2 - m^2_{\Lambda_c})^2 + \Lambda^4};$$

where $\Lambda$ is used, e.g., in Refs. [43, 51], and the cut off is typically set to be around $\Lambda = 2$ GeV. Here, since the aim is to get an order-of-magnitude estimate of the cross section for the $p\bar{p} \to \gamma X(3872)$, it suffices to take a value used in the literature, and we take $\Lambda = 2.0$ GeV. The dependence of our results on this parameter will be checked.

We are interested in the manifestation of the TS in the $\gamma X(3872)$ invariant mass distribution. As shown in Ref. [52], the TS emerges when the process can occur classically, i.e., the internal particles of the loop are simultaneously placed on shell and all the momenta are collinear. At this time, the exchanged $\Lambda_c$ in the $\bar{D}^*0D^*0$ production is far away from on shell. Then, Eq. (10) can be approximated by taking the leading term of the expansion in powers of $1/m_{\Lambda_c}$. The $p\bar{p} \to \bar{D}^*0D^*0$ production amplitude is reduced to

$$-i\mathcal{M}_{pp,\bar{D}^*0D^*0} = \frac{ig^2_vF^2_{p,\bar{D}^*0\Lambda_c}}{m_{\Lambda_c}}\bar{v}\gamma^\mu\gamma^\mu u(\epsilon^*_{D^*0})_{\mu'}(\epsilon^*_{D^*0})_{\mu} = -i\mathcal{M}_{pp,\bar{D}^*0D^*0}(\epsilon^*_{D^*0})_{\mu'}(\epsilon^*_{D^*0})_{\mu}.$$
Because the internal particles are close to on shell in the vicinity of the TS energies, the 4-momentum transfer \((p - k)^2\) in \(F_{p, D^* A_c}^2\) can be approximated by

\[
(p - k)^2 = m_p^2 + m_{D^* A_c}^2 - 2m_{D^* A_c} E_p,
\]

where the spatial momentum of the \(\bar{D}^*\) is ignored because the TS energy is close to the \(D^* \bar{D}^*\) threshold.

The part of the triangle loop in Fig. 3 is the same as the \(e^+ e^- \rightarrow \pi^0 \gamma X(3872)\) reaction given in Sec. II A. The \(p\bar{p} \rightarrow \gamma X(3872)\) amplitude with the \(D^* \bar{D}^* D^0\) loop is written as

\[
-i \mathcal{M}_{p\bar{p}, \gamma X}^{(D^* \bar{D}^* D^0)} = \int \frac{d^4 l}{(2\pi)^4} (-i M_{\mu\mu}^{(l)} \epsilon g_3 g_4 D_{\Delta}^{-1} [P_{D^*}] \mu \gamma [P_{\bar{D}^*}] \mu \tau) \epsilon^\alpha \epsilon^\beta \epsilon^\gamma \epsilon^\delta (\epsilon_X^* \tau), \tag{12}
\]

and the \(\bar{D}^* D^* \bar{D}^0\) loop gives

\[
-i \mathcal{M}_{\bar{p}p, \gamma X}^{(D^* \bar{D}^* D^0)} = \int \frac{d^4 l}{(2\pi)^4} (-i M_{\mu\mu}^{(l)} \epsilon g_3 g_4 D_{\Delta}^{-1} [P_{\bar{D}^*}] \mu \gamma [P_{D^*}] \mu \tau) \epsilon^\alpha \epsilon^\beta \epsilon^\gamma \epsilon^\delta (\epsilon_X^* \tau). \tag{13}
\]

The details of \(\mathcal{M}_{p\bar{p}, \gamma X}^{(D^* \bar{D}^* D^0/\bar{D}^* D^* D^0)}\) can be found in Appendix A. Finally, the amplitude of the \(p\bar{p} \rightarrow \gamma X(3872)\) with the \(D^* \bar{D}^* D^0/\bar{D}^* D^* D^0\) loops, \(\mathcal{M}_{p\bar{p}, \gamma X}\), is

\[
\mathcal{M}_{p\bar{p}, \gamma X} = \mathcal{M}_{\text{ISI}} \left( \mathcal{M}_{p\bar{p}, \gamma X}^{(D^* \bar{D}^* D^0)} + \mathcal{M}_{p\bar{p}, \gamma X}^{(\bar{D}^* D^* D^0)} \right), \tag{14}
\]

where \(\mathcal{M}_{\text{ISI}}\) is a factor to take into account the \(p\bar{p}\) initial-state interaction (ISI). In Ref. [49], this factor \(|\mathcal{M}_{\text{ISI}}|^2\) is about 0.25 at \(\sqrt{s} = 5\) GeV and moderately increases along with \(\sqrt{s}\). Here we treat \(\mathcal{M}_{\text{ISI}}\) as a constant and take \(|\mathcal{M}_{\text{ISI}}|^2 = 0.2\) for an estimation of the ISI effect.

With the \(p\bar{p} \rightarrow \gamma X(3872)\) amplitude given in Eq. (14) and the phase-space factor, the cross section of the \(p\bar{p} \rightarrow \gamma X(3872)\), \(\sigma_{p\bar{p}, \gamma X}\), as a function of \(\sqrt{s}\), which is now the \(p\bar{p}\) c.m. energy, is given by

\[
\sigma_{p\bar{p}, \gamma X} = \int d\Omega \frac{1}{64\pi^2 \sqrt{s} p} \frac{k}{|\mathcal{M}_{p\bar{p}, \gamma X}|^2},
\]

with \(k = \lambda^{1/2}(s, 0, m_X^2)/(2\sqrt{s})\) and \(p = \lambda^{1/2}(s, m_p^2, m^2_\pi)/(2\sqrt{s})\).
C. Width effect of the $X(3872)$

To take into account the width of the $X(3872)$, the cross sections need to be convolved with the spectral function of the $X(3872)$. The spectral function may be parametrized using either the BW or the Flatté form. Although the latter is more proper for analyzing the $X(3872)$ line shape in order to extract the pole position, they do not make much difference when convolved with cross sections. We consider both forms in the following.

First, we take the BW amplitude with a constant width for the $X(3872)$ spectral function,

$$\rho_X(\tilde{m}_X) = -\frac{1}{\pi} \text{Im} \left( \frac{1}{D_X} \right), \quad D_X = \tilde{m}_X - m_X + i\Gamma_X/2,$$

As mentioned above, the $X(3872)$ width $\Gamma_X = 100$ keV is used in this calculation and the $X(3872)$ mass will be changed within $\pm 180$ keV with respect to the $D^0\bar{D}^{*0}$ threshold energy, which covers the range of the uncertainty of the $X(3872)$ mass given in Ref. \[1\].

For comparison, the results smeared with a spectral function of the Flatté type will also be provided. In this case, the spectral function is given by \[2, 18\]

$$\rho_X(\tilde{m}_X) = -\frac{1}{\pi} \text{Im} \left( \frac{1}{D_X} \right), \quad D_X = \tilde{m}_X - m_X + i\Gamma_X(\tilde{m}_X)/2,$$

$$\Gamma_X(\tilde{m}_X) = g(k_1 + k_2) + \Gamma_{X,\rho}(\tilde{m}_X) + \Gamma_{X,\omega}(\tilde{m}_X) + \Gamma_{X0},$$

$$k_1 = \sqrt{2\mu_{D^0\bar{D}^{*0}}(\tilde{m}_X - m_{D^0} - m_{\rho})},$$

$$k_2 = \sqrt{2\mu_{D^+D^-}(\tilde{m}_X - m_{D^+} - m_{D^-})},$$

$$\Gamma_{X,\rho}(\tilde{m}_X) = \int_{2m_{\pi}}^{m_{X}-m_{J/\psi}} \frac{dm'}{2\pi} \frac{\Gamma_{\rho}}{(m' - m_{\rho})^2 + \Gamma_{\rho}^2/4},$$

$$\Gamma_{X,\omega}(\tilde{m}_X) = \int_{3m_{\pi}}^{m_{X}-m_{J/\psi}} \frac{dm'}{2\pi} \frac{\Gamma_{\omega}}{(m' - m_{\omega})^2 + \Gamma_{\omega}^2/4},$$

$$q(\tilde{m}_X, m') = \frac{1}{2\tilde{m}_X} \lambda^{1/2}(\tilde{m}_X^2, m'^2, m_{J/\psi}^2),$$

with $\Gamma_{\rho}$ and $\Gamma_{\omega}$ being the widths of the $\rho$ and $\omega$ mesons, respectively. The nonrelativistic momenta $k_{1,2}$ are analytically continued below the threshold. In the case with the Flatté amplitude, due to the scaling property \[54\] which hinders a determination all free parameters, here we only make a qualitative discussion on the line shape of the $\gamma X(3872)$ distribution expecting the magnitude to be of the same order as that in the BW case. We make use of

\footnote{See Ref. \[53\] for a detailed discussion on the smearing effect of the experimental energy resolution, and see also Ref. \[28\] for arguments for the sensitivity of the TS peak on the $X(3872)$ binding energy, where the binning of the $\gamma X(3872)$ energy is considered.}
the Flatté parameters, \(m_{X_0}, \Gamma_{X_0}, g, f_\rho, \) and \(f_\omega\) from Ref. [2] which fixes \(m_{X_0}\) and fits the other parameters to the data.

As pointed out in Ref. [28], for determining the \(X(3872)\) binding energy from the \(\gamma X(3872)\) line shape, the \(X(3872)\) needs to be reconstructed from decay modes other than the \(\pi^0 D^0 \bar{D}^0\) one; otherwise, one has to consider the tree-level contribution of \(D^{*0} \bar{D}^{*0} \rightarrow \pi^0 D^0 \bar{D}^0\), which has a subtle interference with the triangle diagrams and cannot be treated as a smooth background near the TS energies [55–58]. In Ref. [33], the \(e^+ e^- \rightarrow \gamma D^* \bar{D}^0\) process is studied, and it is found that the \(D^0 \bar{D}^{*0}\) distribution with a fixed \(\sqrt{s}\) is completely dominated by the tree-level contribution, which increases rapidly at the TS energy. In this work, we consider the \(\pi^+ \pi^- J/\psi\) mode for reconstructing the \(X(3872)\). Then, we make the convolution as follows:

\[
\bar{F}(m_{\gamma X}) = \int_{m_X - 2\Gamma_X}^{m_X + 2\Gamma_X} d\tilde{m}_X \frac{1}{2\pi \mathcal{N}} \frac{\Gamma_{X,\rho}(\tilde{m}_X)}{|D_X(\tilde{m}_X)|^2} F(m_{\gamma X}, \tilde{m}_X),
\]

\[
\mathcal{N} = \int_{m_X - 2\Gamma_X}^{m_X + 2\Gamma_X} d\tilde{m}_X \rho_X(\tilde{m}_X),
\]

with \(F\) being \(d\sigma_{e^+ e^- \rightarrow \pi^0 \gamma X}/dm_{\gamma X}\) or \(\sigma_{pp,\gamma X}\). The coupling constant \(g_4\) is kept fixed to the value evaluated at the central value \(m_X\) in the spectral function, and the \(X(3872)\) mass appearing in the loop amplitude is changed to \(\tilde{m}_X\) in the convolution. \(\Gamma_{X,\rho}(\tilde{m}_X)\) is the \(\pi^+ \pi^- J/\psi\) part of the \(X(3872)\) decay width. In the BW case, the \(\Gamma_{X,\rho}\) is given by the \(X(3872) \rightarrow \pi^+ \pi^- J/\psi\) partial width with the branching ratio \((4.1 \pm 1.3)\%\) extracted by the BaBar Collaboration [44], and Eq. (17) is used for \(\Gamma_{X,\rho}\) in the Flatté case [2, 18]. The integration range for the convolution with the Flatté amplitude is chosen to be \(\pm 400\) keV from the \(D^0 \bar{D}^{*0}\) threshold which is twice of the half-maximum width of the peak. See Fig. 5 for a plot of \((\Gamma_{X,\rho}/|D_X|^2)/(2\pi \mathcal{N})\).

Finally, the parameters used in this calculation are summarized in Table I.

III. RESULTS

A. \(e^+ e^- \rightarrow \pi^0 \gamma X(3872)\)

First, we show the \(\gamma X(3872)\) invariant mass distribution in the \(e^+ e^- \rightarrow \pi^0 \gamma X(3872)\) reaction, where \(X(3872)\) decays further into \(\pi^+ \pi^- J/\psi\), denoted by \(d\sigma_{e^+ e^- \rightarrow \pi^0 \gamma X}/dm_{\gamma X}\) (here and in the following, we use \(\sigma\) to denote cross sections convolved with the \(X(3872)\) spectral
FIG. 5. \((\Gamma_{X,\rho}/|D_X|^2)/(2\pi N)\) as functions of \(\tilde{m}_X\) with Eqs. (15) and (16). The vertical line is the \(D^0\bar{D}^{*0}\) threshold.

TABLE I. Parameters used in this work.

| Parameter | Value |
|-----------|-------|
| \(m_{D^0}\) (GeV) | 1.86483 |
| \(m_{D^{*0}}\) (GeV) | 2.00685 |
| \(\Gamma_{D^{*0}}\) (keV) | 55.3 |
| \(m_{D^{*+}}\) (GeV) | 2.01026 |
| \(m_{\pi^0}\) (GeV) | 0.13498 |
| \(m_\psi\) (GeV) | 4.22 |
| \(\Gamma_\psi\) (GeV) | 0.06 |
| \(m_p\) (GeV) | 0.93827 |
| \(g_0\) | 0.68 |
| \(g_1\) | 1.77 |
| \(g_2\) (GeV\(^3\)) | 4.0317 |
| \(m_{Z,0}\) (GeV) | 0.0259 |
| \(\Gamma_{Z,0}\) (GeV) | -5.20 |
| \(\Lambda\) (GeV) | 2.0 |

function of the \(J/\psi\pi^+\pi^-\) mode. The \(\gamma X(3872)\) distribution with a few different masses (\(\delta\) values) of the \(X(3872)\) is shown in the left panel of Fig. 6. The BW distribution Eq. (15) is used in the convolution of the \(X(3872)\) spectral function for the plot of Fig. 6, and the distributions with \(\delta = \pm 180\) and \(\pm 50\) keV are plotted (the \(X(3872)\) mass is characterized by \(\delta\) as Eq. (1)). The plot of the \(\gamma X(3872)\) distribution of the \(e^+e^-\to \pi^0\gamma X(3872)\) cross section normalized to the value at \(m_{\gamma X} = m_{D^{*0}} + m_{\bar{D}^{*0}}\) with \(\delta = 180\) keV is also given in the right panel of Fig. 6 to make the comparison of the line shapes easier.

The distribution \(d\sigma_{e^+e^-\to \pi^+\pi^-\gamma X}/dm_{\gamma X}\), which involves the \(X(3872)\) decay into the \(\pi^+\pi^- J/\psi\) mode, is the order 0.01 pb/GeV within \(\delta = \pm 180\) keV. As one can see in the left panel of Fig. 6, the magnitude is bigger with larger \(\delta\). This is because the \(D^0\bar{D}^{*0}\to X(3872)\) coupling is bigger with larger \(\delta\) except for the vicinity of \(\delta = 0, |\delta| < 50\) keV (see Fig. 2 for \(g_4\) as a function of \(\delta\)). In the right panel of Fig. 6, one can see that the peak of the TS looks more clear with a negative \(\delta\) compared with that with a positive \(\delta\). The peak positions for the \(\delta = -50\) keV and \(-180\) keV cases are 4.0155 GeV and 4.015 GeV, respectively, which
are dictated by the TS whose location can be easily obtained using the master formula in Ref. [30]. On the other hand, the peak around \( m_{\gamma X} = 4.016 \text{ GeV with } \delta > 0 \) is a remnant of the TS because the TS is in the complex plane even when the \( D^*\bar{D}^* \) width is neglected in this case.

Other than the TS peak, one can see a cusp of the \( D^*\bar{D}^* \) threshold slightly below \( m_{\gamma X} = 4.014 \text{ GeV as a consequence of the } S\text{-wave production of } D^*\bar{D}^* \). The two relevant singularities, the cusp at the \( D^*\bar{D}^* \) threshold and the peak caused by the TS, fix the line shape, and the peak is sensitive to the binding energy as can be seen from the figure. As studied in Ref. [53], even after considering the energy resolution, the shapes can still be distinguished for different binding energies. The distribution shows slightly increasing behavior along with increasing \( m_{\gamma X} \). This is because of the \( Z_c(4020) \) resonance included in the \( D^*\bar{D}^* \) production mechanism. Yet, its inclusion does not change the TS peak structures in the \( \gamma X(3872) \) distribution.

Notice that for the \( e^+e^- \rightarrow \gamma X(3872) \) cross section [32], three is no \( D^*\bar{D}^* \) threshold cusp as the \( D^*\bar{D}^* \) pair is produced in \( P \) wave in that case, and only the TS peak can be seen in the \( \gamma X(3872) \) distribution.

To see the impact of the smearing of the cross section with the \( X(3872) \) mass distribution, we show in Fig. 7 the \( \gamma X(3872) \) invariant mass distribution of \( \delta = 50 \text{ keV with and} \)
FIG. 7. The $\gamma X(3872)$ distribution for the $e^+e^- \rightarrow \pi^0 \gamma X(3872)$ with and without smearing the $X(3872)$ mass ($\sqrt{s} = 4.23$ GeV). The $X(3872)$ binding energy $\delta$ is fixed to $\delta = 50$ keV in the plot.

FIG. 8. The $\gamma X(3872)$ distribution for $e^+e^- \rightarrow \pi^0 \gamma X(3872)$ with the Flatté distribution, Eq. (16), with the error band given by the parameter errors of the Flatté distribution [2] in an arbitrary unit. The $e^+e^-$ c.m. energy is fixed at $\sqrt{s} = 4.23$ GeV. The vertical dash-dotted line is the $D_s^0 \bar{D}_s^{*0}$ threshold.

without smearing. The cross section without smearing is given by Eq. (9) multiplied by the $X(3872) \rightarrow \pi^+\pi^- J/\psi$ branching fraction to compare with the smeared distribution. As one can see in Fig. 7, the TS peak position is slightly shifted to a lower energy by the smearing. This tendency is the same for different $\delta$ values.

The $\gamma X(3872)$ distribution smeared with the Flatté distribution Eq. (16) is given in Fig. 8. The black solid line is the plot with the best-fit parameters of the Flatté analysis in Ref. [2], and the gray band is given by the parameter uncertainties (the statistical and systematic errors are summed in quadrature). The shape looks similar to the line of $\delta = 50$ keV in Fig. 6, but the peak position is slightly lower, about 4.015 GeV, and the peak is more obscure. This is a consequence of the smearing: the original position of the TS peak without
smearing would be about 4.016 GeV as in the case of $\delta > 0$ of Fig. 6 because the $X(3872)$ is seen in the $\pi^+\pi^-J/\psi$ distribution as a peak at the $D^0\bar{D}^{*0}$ threshold (see Fig. 5). In the smearing with the Flatté distribution, the cross section is averaged in the range of $\tilde{m}_X \in m_{D^0} + m_{D^{*0}} \pm 400$ keV to cover the $X(3872)$ peak region in Fig. 6. Hence, the smearing effect is larger than the smearing with the BW distribution with $\Gamma_X = 100$ keV. The parameter uncertainties give relatively large uncertainties in the magnitude as seen from the gray band in Fig. 8, but the peak position is not changed.

Let us make a comment on the uncertainties of the order of magnitude. The uncertainty of the $e^+e^- \rightarrow \pi^0(D^*\bar{D}^*)^0$, which is used to fix the parameter $g_0g_1g_2$, is about 10%, and the uncertainty of the $D^* \rightarrow \gamma D$ part would be a few percent referring to the relative errors of the $D^{**}$ full width and the $D^*$ branching ratios [1]. Then, the uncertainties of the cross section is expected be about 10% which mainly comes from the $e^+e^- \rightarrow \pi^0\gamma X(3872)$ input.

Integrating the differential cross section in Fig. 6 over the $m_{\gamma X}$ region between 4.01 and 4.02 GeV, we get about 0.2 fb for $\delta = 180$ keV and 0.03 fb for $\delta = -180$ keV. Such a small cross section implies that measuring the $\gamma X(3872)$ line shape of the $e^+e^- \rightarrow \pi^0\gamma X(3872)$ process would be very challenging.

### B. $p\bar{p} \rightarrow \gamma X(3872)$

The plot of the $p\bar{p} \rightarrow \gamma X(3872)$ cross section, $\sigma_{p\bar{p},\gamma X}$ (note that the $X(3872) \rightarrow \pi^+\pi^-J/\psi$ branching fraction has been taken into account as before), as a function of the $p\bar{p}$ c.m. energy, $\sqrt{s}$, is given in the left panel of Fig. 9, and the right panel of Fig. 9 is the plot with all line shapes normalized to that of $\delta = 180$ keV at the $D^{*0}\bar{D}^{*0}$ threshold as the right panel of Fig. 6. The $\gamma X(3872)$ invariant mass distribution of the $p\bar{p} \rightarrow \gamma X(3872)$ process is qualitatively the same as the $e^+e^- \rightarrow \pi^0\gamma X(3872)$ case, since the singularities are the same. The cross section increases with larger $\delta$, and the peak structure looks more significant with $\delta < 0$. The lines of $\delta = -50$ keV and $-180$ keV show a clear peak structure due to the TS in the physical region: the peak of $\delta = -50$ keV is at 4.0155 GeV and that of $\delta = -180$ keV is at 4.015 GeV as in Fig. 6. Comparing the distributions of $\delta = 50$ keV and $\delta = 180$ keV, the enhancement at $m_{\gamma X} = 4.016$ GeV in the $\gamma X(3872)$ distribution with $\delta = 50$ keV is more

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5 Due to the asymmetric shape of the Flatté distribution, using the parameters from the LHCb analysis [2] leads to a bit broader line shape for the $X(3872)$ than that using the BW form with a width of 100 keV; see Fig. 5.
FIG. 9. Left: The $\gamma X(3872)$ distribution for the $p\bar{p} \rightarrow \gamma X(3872)$ with different $X(3872)$ binding energies as a function of the $p\bar{p}$ c.m. energy $\sqrt{s}$. The $X(3872) \rightarrow \pi^+ \pi^- J/\psi$ branching fraction has been taken into account. Right: The $p\bar{p} \rightarrow \gamma X(3872)$ cross section normalized with the value at $m_{\gamma X} = m_{D^{*0}} + m_{\bar{D}^{*0}}$ of $\delta = 180$ keV. In both panels, the vertical line is the $D^{*0}\bar{D}^{*0}$ threshold.

FIG. 10. The $\gamma X(3872)$ distribution for the $p\bar{p} \rightarrow \gamma X(3872)$ smeared with the Flatté distribution. The unit of the cross section is arbitrary. The gray error band is given by the parameter errors of the Flatté amplitude [2]. The $D^{*0}\bar{D}^{*0}$ threshold is shown with the gray dash-dotted line.

clear since the TS is closer to the physical region.

The plot of the $p\bar{p} \rightarrow \gamma X(3872)$ cross section convolved with the Flatté distribution in Eq. (16) is given in Fig. 10. The distribution is similar to the analogous one for the $e^+e^- \rightarrow \pi^0\gamma X(3872)$ in Fig. 8, peaking at $m_{\gamma X} = 4.015$ GeV.

About the cut off $\Lambda$ in the form factor Eq. (11) for the $p\bar{p} \rightarrow \bar{D}^*D^*$ transition, $\Lambda = 2$ GeV is used in the plot of Figs. 9 and 10. Varying the cut off $\Lambda$ within $\Lambda = 2.0 \pm 0.2$ GeV, the cross section changes by a factor of 2 compared to the value with $\Lambda = 2$ GeV with the
same line shape, indicating a large uncertainty in the estimate of the $p\bar{p} \rightarrow \gamma X(3872)$ cross section. Nevertheless, the order of magnitude should be reliable, and we expect $\sigma_{p\bar{p},\gamma X}$ to be of $\mathcal{O}(10 \text{ pb})$ for $\sqrt{s} \sim 4015$ MeV. Considering an integrated luminosity of 2 fb$^{-1}$ in about five months for PANDA [59], $\mathcal{O}(2 \times 10^4)$ events are expected to be collected for the $X(3872)$ in the $J/\psi\pi^+\pi^-$ mode. Considering further the reconstruction of the $J/\psi$ from the $e^+e^-$ and $\mu^+\mu^-$ pairs, each of which has about a branching fraction of about 6% [1], we expect that $\mathcal{O}(2 \times 10^3)$ events can be collected at PANDA. According to the Monte Carlo simulation in Ref. [17], a high-precision measurement of the $X(3872)$ binding energy is foreseen even after further smearing due to the energy resolution is taken into account [53]. In particular, such a smearing effect at PANDA will be very small since the energy resolution can reach the level of 100 keV [59, 60].

IV. SUMMARY

In this paper, we have estimated the cross sections for the production of $\gamma X(3872)$ from a short-distance $D^{*0}\bar{D}^{*0}$ source. A measurement of the $\gamma X(3872)$ line shape was proposed to achieve an unprecedented precision in determining the $X(3872)$ binding energy [28]. We focused on two processes in this paper: $e^+e^- \rightarrow \pi^0\gamma X(3872)$ and $p\bar{p} \rightarrow \gamma X(3872)$. The $\gamma X(3872)$ invariant mass distributions for these two processes were computed, which clearly show a special peak sandwiched between the $D^{*0}\bar{D}^{*0}$ threshold and the triangle singularity of the $D^{*0}\bar{D}^{*0}D^{0}/\bar{D}^{*0}D^{*0}\bar{D}^{0}$ loops. The obtained line shapes with different $X(3872)$ binding energies can be distinguished from each other in both the $e^+e^-$ and $p\bar{p}$ processes: the peak is more narrow when the $X(3872)$ mass is above the $D^{0}\bar{D}^{*0}$ threshold. Convolving the distributions with the spectral function of the $X(3872)$ does not change the conclusion, and the effect of smearing is marginal considering a width of 100 keV for the $X(3872)$.

In the $e^+e^- \rightarrow \pi^0\gamma X(3872)$ reaction, the $Z_c(4020)$ resonance is introduced, and it is found that this resonance does not essentially change the peak structure caused by the TS. For the c.m. energy of the $e^+e^-$ pair fixed at 4.23 GeV, with inputs from the BESIII measurements of the $e^+e^- \rightarrow \pi^0(D^{*}\bar{D}^*)^0$ [38], we find that the cross section $\sigma(e^+e^- \rightarrow \pi^0\gamma X(3872)) \times B(X(3872) \rightarrow \pi^+\pi^-J/\psi)$ is about 0.03 fb to 0.2 fb with the $\gamma X(3872)$

\[6\] The beam energy resolutions for the high luminosity and high resolution modes of the High Energy Storage Ring are 167.8 keV and 33.6 keV, respectively [59, 60].
invariant mass integrated from 4.01 to 4.02 GeV. For the $p\bar{p} \rightarrow \gamma X(3872)$, the cross section is much larger. Considering a $\Lambda_c$ exchange to produce $D^{*0}\bar{D}^{*0}$ from the $p\bar{p}$ collisions, it is estimated to be $\sigma(p\bar{p} \rightarrow \gamma X(3872)) \times B(X(3872) \rightarrow \pi^+\pi^-J/\psi) = O(10 \text{ pb})$. This result indicates that while it is hard to measure $e^+e^- \rightarrow \pi^0\gamma X(3872)$, plenty of events can be collected for $p\bar{p} \rightarrow \gamma X(3872)$ at the PANDA experiment. A precise determination of the $X(3872)$ binding energy is foreseen, which can definitely shed new light into understanding this most mysterious charmonium-like particle.

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**Appendix A: $e^+e^- \rightarrow \pi^0\gamma X(3872)$ and $p\bar{p} \rightarrow \gamma X(3872)$ amplitudes**

With the $e^+e^- \rightarrow \pi^0\gamma X(3872)$ amplitude in Eq. (8) and the momentum assignment in Fig. 1, we have

$$
\mathcal{M}_{e^+e^- \rightarrow \pi^0\gamma X(3872)} = -2e^3g_0g_1g_3g_4D^{-1}_\gamma(s)D^{-1}_\psi(s)D^{-1}_Z(m^2_{\gamma X})\bar{v}\gamma\beta'\gamma' u [P_\psi]^{\beta''\beta'} [P_{Z_c}]_{\beta'\beta} \epsilon^{\alpha\beta\gamma\delta} (p_{Z_c})_\alpha \\
\times \int \frac{d^4l}{(2\pi)^4} \frac{D^{-1}_\Delta}{m_{D^{*0}}} \left[ -g_{\gamma\rho} + \frac{(p_{D^{*0}})_\gamma (p_{D^{*0}})_\rho}{m_{D^{*0}}} \right] \left[ -g_{\delta\tau} + \frac{(p_{\bar{D}^{*0}})_\delta (p_{\bar{D}^{*0}})_\tau}{m_{\bar{D}^{*0}}} \right] \\
\times \epsilon^{\mu\nu\rho\sigma} (p_{D^{*0}})_\mu (p_{\gamma})_\nu (k_1^*)_\sigma (k_2^*)_\tau \\
= -2e^3g_0g_1g_3g_4D^{-1}_\gamma(s)D^{-1}_\psi(s)D^{-1}_Z(m^2_{\gamma X})\bar{v}\gamma\beta'\gamma' u [P_\psi]^{\beta''\beta'} [P_{Z_c}]_{\beta'\beta} \epsilon^{\alpha\beta\gamma\delta} (k_1)_{\alpha} \\
\times \int \frac{d^4l}{(2\pi)^4} \frac{D^{-1}_\Delta}{m_{D^{*0}}} \left( -g_{\gamma\rho} - \frac{l_{\beta'\gamma}}{m_{D^{*0}}} \right) \epsilon^{\mu\nu\rho\sigma} (k_1 + l)_{\mu} (k_1 - k_2)_{\nu} (k_2^*)_{\sigma} (k_2^*)_{\tau}.
$$
The $p\bar{p} \rightarrow \gamma X(3872)$ amplitude Eq. (12) with the particle momenta assigned as in Fig. 3 is reduced to

$$\mathcal{M}_{p\bar{p},\gamma X}^{(D^{*0},D^{0})} = \int \frac{d^4 l}{(2\pi)^4} \frac{eg_3g_4F^2}{m_{\Lambda_c}} \bar{\epsilon}(ig_\gamma \gamma^\mu)(ig_\gamma \gamma^\mu)u$$

$$\times D_\Lambda^{-1}(-g_{\mu\gamma}) \left[ -g_{\mu\tau} + \frac{(p_{D^{*0}})_{\mu}(p_{D^{*0}})_{\tau}}{m_{D^{*0}}^2} \right] \epsilon^\alpha\beta\gamma\delta(p_{D^{*0}})_\alpha(p_{\gamma})_\beta(\epsilon^*_\gamma)_\delta(\epsilon^*_X)^\tau$$

$$= - \int \frac{d^4 l}{(2\pi)^4} \frac{g_2^2eg_3g_4F^2}{m_{\Lambda_c}} \bar{\epsilon}g\gamma^\mu g\gamma^\mu u$$

$$\times D_\Lambda^{-1}(-g_{\mu\gamma})(-g_{\mu\tau} + \frac{l_{\mu\tau}}{m_{D^{*0}}^2}) \epsilon^\alpha\beta\gamma\delta(k_1 + l)_\alpha(k_1 - k_2)_\beta(\epsilon^*_\gamma)_\delta(\epsilon^*_X)^\tau,$$

and the amplitude of the $D^{*0}D^{0}\bar{D}^0$ loop, Eq. (13), gives

$$\mathcal{M}_{p\bar{p},\gamma X}^{(D^{*0},D^{0},\bar{D}^{0})} = - \int \frac{d^4 l}{(2\pi)^4} \frac{g_2^2eg_3g_4F^2}{m_{\Lambda_c}} \bar{\epsilon}(ig_\gamma \gamma^\mu)(ig_\gamma \gamma^\mu)u$$

$$\times D_\Lambda^{-1}(-g_{\mu\gamma} + \frac{(p_{D^{*0}})_{\mu}(p_{D^{*0}})_{\gamma}}{m_{D^{*0}}^2}) \left[ -g_{\mu\tau} + \frac{(p_{D^{*0}})_{\mu}(p_{D^{*0}})_{\tau}}{m_{D^{*0}}^2} \right]$$

$$= \int \frac{d^4 l}{(2\pi)^4} \frac{g_2^2eg_3g_4F^2}{m_{\Lambda_c}} \bar{\epsilon}g\gamma^\mu g\gamma^\mu u$$

$$\times D_\Lambda^{-1}(-g_{\mu\gamma})(-g_{\mu\tau} + \frac{l_{\mu\tau}}{m_{D^{*0}}^2}) \epsilon^\alpha\beta\gamma\delta(k_1 + l)_\alpha(k_1 - k_2)_\beta(\epsilon^*_\gamma)_\delta(\epsilon^*_X)^\tau.$$

Adding these two terms, we get

$$\mathcal{M}_{p\bar{p},\gamma X}^{(D^{*0},D^{0})} + \mathcal{M}_{p\bar{p},\gamma X}^{(D^{*0},D^{0},\bar{D}^{0})} = \int \frac{d^4 l}{(2\pi)^4} \frac{g_2^2eg_3g_4F^2}{m_{\Lambda_c}} \bar{\epsilon}g\gamma^\mu g\gamma^\mu u$$

$$\times D_\Lambda^{-1}(-g_{\mu\gamma} + \frac{l_{\mu\tau}}{m_{D^{*0}}^2}) \epsilon^\alpha\beta\gamma\delta(k_1 + l)_\alpha(k_1 - k_2)_\beta(\epsilon^*_\gamma)_\delta(\epsilon^*_X)^\tau.$$
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