Photon position observable

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In biorthogonal quantum mechanics, the eigenvectors of a quasi-Hermitian operator and those of its adjoint are biorthogonal and complete and the probability for a transition from a quantum state to any one of these eigenvectors is positive definite. We apply this formalism to the long standing observable problem can be extended to photons.

The KG field is sometimes used as a simple model for a particle’s state vectoronto the eigenvectors of the Newton Wigner (NW) position operator. NW found a position operator for KG particles, but they concluded that the only photon position operator is the Pryce operator whose vector components do not commute, making the simultaneous determination of photon position in all three directions of space impossible. They had assumed spherically symmetrical position eigenstates for photons, while photon position eigenvectors have an axis of symmetry like twisted light.

Following the NW method with omission of the spherical symmetry axiom, a photon position operator with commuting components and eigenvectors that are cylindrically symmetrical in k-space can be constructed. Since spin and orbital angular momentum are not separately observable, its eigenvectors have only definite total angular momentum along some fixed but arbitrary axis.

Here all calculations will be performed in the physical Hilbert space of solutions to the wave equation using a scalar product of the form derived in and biorthogonal QM summarized here as follows: The eigenvectors of a quasi-Hermitian operator $\hat{O}$ and its adjoint $\hat{O}^\dagger$ are not orthogonal, as is the case for conventional Hermitian operators, but biorthogonal. This means that, given the eigenvector equations

\begin{equation}
\hat{O}|\omega_i\rangle = \omega_i|\omega_i\rangle, \\
\hat{O}^\dagger|\tilde{\omega}_j\rangle = \omega_j|\tilde{\omega}_j\rangle
\end{equation}

we have $\langle \tilde{\omega}_j|\omega_i\rangle = \delta_{ij}\langle \tilde{\omega}_i|\omega_i\rangle$ and the completeness relation $\hat{1} = \sum_i |\omega_i\rangle \langle \omega_i| / \langle \omega_i|\omega_i\rangle$. An arbitrary state $|\psi\rangle$ has an associated state $\tilde{\psi}$. If an arbitrary state vector...
is expanded as \( |\psi\rangle = \sum_j c_j |\omega_j\rangle \) in the Hilbert space \( \mathcal{H} \). Then in biorthogonal QM its associated state is \( |\tilde{\psi}\rangle = \sum_i c_i |\tilde{\omega}_i\rangle \in \mathcal{H}^* \) where \( c_i = \langle \tilde{\omega}_i |\psi\rangle = \langle \omega_i |\tilde{\psi}\rangle \). Using these expansions it is straightforward to verify that \( \langle \tilde{\psi}_1 |\tilde{\psi}_2\rangle = \langle \psi_1 |\psi_2\rangle \). The probability for a transition from a quantum state \( |\psi\rangle \) to an eigenvector \( |\omega_i\rangle \) of \( \hat{O}^\dagger \) is

\[
p_i = \frac{\langle |\tilde{\omega}_i |\psi\rangle |^2}{\langle \psi |\psi\rangle \langle \omega_i |\omega_i\rangle}.
\]

A generic operator can be written in the form

\[
\hat{F} = \sum_{i,j} f_{ij} |\omega_i\rangle \langle \omega_j|
\]

where \( f_{ij} \) can be viewed as a matrix \([24]\). In Section III we will apply this formalism to the biorthogonal position eigenvectors \( |\phi(x)\rangle = \tilde{\phi}(x) |0\rangle \) and \( |\bar{\phi}(x)\rangle = |\pi(x)\rangle = \tilde{\pi}(x) |0\rangle \) where \( x^\mu = (ct, \mathbf{x}) \), \( \tilde{\phi}(x) \) is a field operator, \( \tilde{\pi}(x) \) is its conjugate momentum operator, and \(|0\rangle \) is the vacuum state.

The rest of this paper is organized as follows: In Section II KG wave mechanics, with the field rescaled here to facilitate application to particles with zero mass, is reviewed. In Section III biorthogonality of the one-particle states created by the field operator and its conjugate momentum are examined and the covariant position operator and positive definite probability density are derived. A second quantized formalism is used to facilitate future application to multiparticle problems such as entanglement. In Section II the KG position observable discussed in Sections II and III is extended to photons. In Section V the wave function of the photon emitted by an atom is discussed and in Section VI we conclude.

The configuration space scalar field and four-potential will be called \( \phi(x) \) and \( A^\mu(x) \). State vectors such as \( |\psi(t)\rangle \) and position eigenvectors such as \( |\phi(x)\rangle \) and \( |\mathbf{E}_\lambda(x)\rangle \) introduced in Sections III and IV are given as expansions in Fourier space. The function \( \psi(x) \) used in \([27]\) to represent the scalar field will be reserved for the wave function that equals the projection of a particle’s state vector onto a basis of position eigenvectors. The KG and photon wave functions, \( \psi(x) \) and \( \psi_\perp(x) \) respectively, are proportional to probability amplitudes, with units that differ from those of \( \phi(x) \), \( A^\mu(x) \) and their spacetime derivatives.

II. KLEIN-GORDON WAVE MECHANICS

We will start with a review of the KG position observable problem. The KG equation

\[
\partial_\mu \partial^\mu \phi(x) + \frac{m^2 c^2}{\hbar^2} \phi(x) = 0
\]

describes charged and neutral particles with zero spin (pions). Here covariant notation and the mostly minus convention are used in which \( x^\mu = x = (ct, \mathbf{x}) \), \( \partial_\mu = (\partial_t, \nabla) \), \( m \) is the mass of the KG particle, \( c \) is the speed of light, \( 2\pi \hbar \) is Planck’s constant and \( f_1 \partial^\mu f_2 \equiv f_1 (\partial_\mu f_2) - (\partial_\mu f_1) f_2 \). The function \( \phi(x) \) is any scalar field that satisfies the KG equation \([3]\). The four-density

\[
J^\mu_{KG}(x) = i g \phi(x)^* \tilde{\partial}^\mu \phi(x),
\]

satisfies a continuity equation. Plane wave normal mode solutions to \([4]\) proportional to \( \exp(-i\omega t) \) are referred to as positive frequency solutions, while those proportional to \( \exp(i\omega t) \) are negative frequency. Completeness requires that both positive and negative frequency modes be included. Their contributions to \( J^\mu_{KG}(x) \) are of opposite sign, so \( J^\mu_{KG}(x) \) is interpreted as charge density and the quantity \( g \) in \([4]\) is set equal to \( qc/\hbar \) for particles of charge \( q \).

If only particles, as opposed to both particles and antiparticles, are to be considered, then the KG field can be restricted to positive frequencies and the scalar product

\[
(\phi_1, \phi_2)_{KG} = \frac{i}{\hbar} \int_0^t d\tau \phi_1(x)^* \tilde{\partial}_\tau \phi_2(x)
\]

is positive definite. Here \( t \) denotes a local hyperplane of simultaneity at instant \( t \). The integrand of \([7]\) looks like a particle density but this is misleading since, as noted in the Introduction, \( J^\mu_{KG}(x) \) is not positive definite for positive frequency fields.

The problem of a probability interpretation for KG particles has a long history. Lack of a probability interpretation led Dirac to derive his celebrated equation for spin half particles, but this does not solve the problem for KG fields. In a seminal paper intended to clarify the confusion about relativistic wave mechanics, Feshbach and Villars reviewed the two component formalism that separates the wave function into its particle and antiparticle parts for charged or for neutral particles \([25]\). Since then various strategies have been employed to derive a positive definite probability density. The four-current density can be redefined so that its zeroth component is positive definite \([28]\), but this construction has no apparent physical basis and it fails if \( m = 0 \) \([29]\). It has been proposed that for charged pions only positive definite eigenstates of the Hamiltonian are physical \([30]\). A new \( J^\mu(x) \) was derived that does not require separation of the field into positive and negative frequency parts \([31]\) so it can be applied to the real fields that describe neutral pions. If \( \phi(x) \) is restricted to positive frequencies it reduces to \([4]\) so this \( J^\mu(x) \) that describes an arbitrary linear combination of positive and negative frequency fields, including real fields, will be used here.

Working in the two component formalism with a pseudo-Hermitian Hamiltonian Mostafazadeh \([3]\) defined the positive-definite Hermitian operator

\[
\hat{D} = -\nabla^2 + m^2 c^2 / \hbar^2
\]
in terms of which the KG equation is \((\hat{D} + \partial_{ct}^2) \psi = 0\). He derived the conjugate field
\[
\phi_c(x) = i\hat{D}^{-1/2} \partial_{ct} \phi(x)
\] (9)
such that if \(\phi = \phi^+ + \phi^-\) then \(\phi_c = \phi^+ - \phi^-\). This implies that \(\phi_c\) is a scalar. It is then straightforward to verify that \(\phi_c\) satisfies the KG equation and that
\[
J^\mu(x) = \frac{i}{\hbar} \phi(x) s^{\mu} \partial_{\mu} \phi_c(x)
\] (10)
satisfies the continuity equation \(\partial_{\mu} J^\mu = 0\). Up to a constant that just scales \(J^\mu\), (10) is the expression derived in [31]. Like (9), \(J^\mu(x)\) is manifestly covariant. It was proved in [3] that the scalar product
\[
(\phi_1, \phi_2) = \frac{1}{\hbar} \int d^3x \phi_1(x)^* \frac{\partial}{\partial t} \phi_2 c(x) \tag{11}
\]
is positive definite, time independent and can be written in covariant form as
\[
(\phi_1, \phi_2) = \frac{2c}{\hbar} \sum_{\epsilon = \pm} \int d\sigma n_\mu \phi_1(x)^* \frac{\partial}{\partial \mu} \phi_2 c(x) \tag{12}
\]
where \(n\) is an arbitrary spacelike hyperplane with normal \(n^\mu\), in other words, it is a Cauchy surface. The infinitesimal volume elements \(d\sigma \equiv dx\) are invariant. When restricted to positive frequencies \([10]\) reduces to (9) and (11) reduces to (11). Using (3), (9) and \(\phi_c = \phi^+ - \phi^-\), the scalar product (11) can be written as
\[
(\phi_1, \phi_2) = \frac{2c}{\hbar} \sum_{\epsilon = \pm} \int d\sigma n_\mu \phi_1(x)^* \frac{\partial}{\partial \mu} \phi_2 c(x) = \left( \sum_{\epsilon = \pm} \int d\sigma n_\mu \phi_1(x)^* \frac{\partial}{\partial \mu} \phi_2 c(x) \right) \tag{13}
\]
where
\[
\langle \chi_1 | \chi_2 \rangle = \int d^3x \chi_1^*(x) \chi_2(x) \tag{14}
\]
The non-relativistic Hilbert space is the vector space of square integrable continuous functions with the scalar product (14). In the relativistic Hilbert space the scalar product used here is (13). These scalar products can be evaluated in configuration space or in \(k\)-space. The covariant Fourier transform is
\[
\phi^s(x) = \int \frac{dk}{(2\pi)^3 2\omega_k} \pi^- (k) e^{-i(kt-kx)} \tag{15}
\]
Since \(\phi^s(x)\) is a scalar and \(dk/(2\pi)^3 2\omega_k\) is invariant, \(\pi^- (k) \in \mathcal{H}^*\) is a scalar, analogous to the transformation properties of photons [28]. For \(\omega_k = \sqrt{k^2 c^2 + m^2 c^4}/\hbar^2\) the function \(\phi^s(x)\) satisfies the KG equation and (15) can be written as
\[
(\phi_1, \phi_2) = \frac{1}{\hbar} \sum_{\epsilon = \pm} \int \frac{dk}{(2\pi)^3 2\omega_k} \pi_1^* \pi_2^s (k) \tag{16}
\]
In the biorthogonal formalism the bases \(\{\phi_{x_i}(k)\} = \{\pi_{x_i}^* (k)/\omega_k\} \in \mathcal{H} \) and \(\{\pi_{x_j} (k)\} \in \mathcal{H}^*\) are biorthogonal and complete and the Hermitian adjoint of an operator is its complex conjugate transpose [24]. Since the scalar product (13) is positive definite, standard QM can be recovered if a nontrivial metric operator \(\langle . | . \rangle\) is introduced [29]. In this metric formulation the basis is \(\{\pi_{x_i}^* (k)\}\) and operators are Hermitian. With the flat metric \(\langle . , . \rangle\) operators representing observables can be non-Hermitian with biorthogonal eigenvectors. The norm and orthogonality of the elements of the Hilbert space and the concept of Hermiticity are determined by the definition of scalar product. Newton and Wigner defined the KG fields \(\pi_{x_i} (k) \propto e^{i\mathbf{k} \cdot \mathbf{x}} \omega_k^{1/2}\) that satisfy \((\phi_1, \phi_2) \propto \delta(\mathbf{x}_1 - \mathbf{x}_2)\) and are eigenvectors of the position operator \(i\nabla_k - \frac{ic}{2\hbar} k\) [17]. Hermiticity of the NW position operator with eigenvectors of this form is discussed by Pike and Sarkar [32]. The NW position operator has played a central role in the discussion relativistic particle position since its publication in 1949 [17], but this operator is not covariant and its eigenvectors are not localized. We will show in the next section that the covariant commutation relations are consistent with (13) and that the formalism of biorthogonal QM leads to a covariant position operator. A second quantized version of the biorthogonal formulation in [24] will be used to facilitate applications in quantum optics and understanding of the relationship of the classical wave equation to quantum field theory (QFT).

### III. KG POSITION EIGENVECTORS

In QFT particles are created at a point in spacetime by a field operator or its canonical conjugate. The interaction picture (IP) scalar field operators \(\hat{\phi}(x)\) and \(\hat{\pi}(x) = \partial_t \hat{\phi}(x)\) will be written as
\[
\hat{\phi}(x) = \sqrt{\hbar} \int \frac{dk}{(2\pi)^3 2\omega_k} e^{i(k \cdot x - \omega_k t)} \hat{a}^\dagger (k) + \text{H.c.}, \tag{17a}
\]
\[
\hat{\pi}(x) = i\sqrt{\hbar} \int \frac{dk}{(2\pi)^3 2\omega_k} e^{i(k \cdot x - \omega_k t)} \hat{a}^\dagger (k) + \text{H.c.} \tag{17b}
\]
where H.c. is the Hermitian conjugate and the covariant normalization condition is [33]
\[
[\hat{a}(k), \hat{a}^\dagger (q)] = (2\pi)^3 2\omega_k \delta (k - q). \tag{18}
\]
On the \(t\) hyperplane the field operators satisfy the commutation relations
\[
[\hat{\phi}(x), \hat{\pi}(y)] = i\hbar \delta(x - y). \tag{19}
\]
If the vacuum state \(|0\rangle\) is defined by the condition \(\forall k \hat{a}(k) |0\rangle = 0\) then the field operators create one-particle states in this vacuum. In the IP the basis vectors are time dependent [21]. To accommodate positive
and negative frequency wavefunctions, \( \epsilon = \pm \) states will be defined as

\[
|\phi^\epsilon(x)\rangle = \sqrt{\hbar} \int \frac{dk}{(2\pi)^3 2\omega_k} e^{i(\omega_k t - k \cdot x)} |1_k\rangle,
\]
(20a)

\[
|\pi^\epsilon(x)\rangle = \sqrt{\hbar} \int \frac{dk}{(2\pi)^3 2} e^{i(\omega_k t - k \cdot x)} |1_k\rangle,
\]
(20b)

where \( |\phi^\epsilon(x)\rangle \equiv \tilde{\phi}^\epsilon(x) |0\rangle \) and \( |\pi^\epsilon(x)\rangle \equiv e^{\hat{D}^{1/2}} |\phi^\epsilon(x)\rangle \)

so that the phase factor \( i \) is absorbed into the bases and

\[
i\partial_t |\pi^\epsilon(x)\rangle = -\epsilon c \hat{D}^{1/2} |\pi^\epsilon(x)\rangle.
\]
(21)

With these definitions \( \langle \pi^\epsilon(x) | \psi^+ \rangle \) is positive frequency while

\[
\langle \pi^- (x) | \psi^- \rangle = \langle \pi^+ (x) | \psi^+ \rangle^* = \langle \psi^+ | \pi^+ (x) \rangle
\]
(22)

is negative frequency where \( \epsilon = + \) refers to a particle arriving from the past and absorbed on \( n \), while \( \epsilon = - \) refers to a particle emitted on \( n \) and propagating into the future. These basis vectors are biorthogonal in the sense that

\[
\langle \pi^\epsilon (x) | \phi^\epsilon' (y) \rangle = \hbar \delta_n (x - y) \delta \epsilon \epsilon'.
\]
(23)

The notation \( \delta_n (x - y) \) is defined to select \( x \) and \( y \) such that \( x^\mu = y^\mu \) on the hyperplane with normal \( n_\mu \). Since \( |\phi^\epsilon(x)\rangle \) and \( |\pi^\epsilon(x)\rangle \) are biorthogonal, they satisfy the completeness relation

\[
\hat{1} = \frac{2}{\hbar} \sum_{\epsilon = \pm} \int dx |\phi^\epsilon(x)\rangle \langle \pi^\epsilon (x) |
\]
(24)

where the factor \( 2/\hbar \) is due to normalization (see (23)).

There is a direct correspondence between the scalar product \( \langle \rangle \) and the vacuum expectation value of the QFT commutator. Substitution of \( |\phi^\epsilon(x)\rangle \) and \( |\pi^\epsilon(x)\rangle \) in the vacuum expectation value of \( \langle \rangle \) gives

\[
\langle \phi (x) | \pi (y) \rangle = \langle 0 | \phi^\epsilon (x) \pi^- (y) - \pi^+ (y) \phi^- (x) |0\rangle
\]
\[
= \langle \phi^\epsilon (x) | \pi^+ (y) \rangle + \langle \pi^+ (y) | \phi^\epsilon (x) \rangle
\]
\[
= \langle \phi^\epsilon (x) | \pi^\epsilon (y) \rangle + \langle \phi^- (x) | \pi^- (y) \rangle.
\]
(25)

On the \( t \) hyperplane events \( x^\mu = (ct, x) \) and \( y^\mu = (ct, y) \) appear simultaneous but an inertial observer with velocity \( c \beta \) will see these events as time ordered. Since the Fourier space integrand of \( \langle \phi^\epsilon (x) | \pi^\epsilon (y) \rangle \) is proportional to \( e^{-i\omega_k (t_x - t_y)} \) while that of \( \langle \pi^- (y) | \phi^- (x) \rangle \) will be seen as proportional to \( e^{i\omega_k (t_x - t_y)} \), if \( t_x > t_y \) the first term is positive frequency (\( \epsilon = + \)) while the second is negative frequency (\( \epsilon = - \)). This assignment is not unique, since an observer with velocity \(-c\beta\) will see the opposite time order. Thus the covariant scalar product \( \langle 25 \rangle \) should be a sum over \( \epsilon = \pm \), consistent with \( \langle 14 \rangle \) but inconsistent with \( \langle 7 \rangle \). Based on \( \langle 25 \rangle \) or \( \langle 12 \rangle \) a particle is either emitted or absorbed at \( x \) on \( n \) and there are no \( \epsilon = +/\epsilon = - \) cross terms in the scalar product.

It can be verified by substitution that the basis states \( \langle 20 \rangle \) are eigenvectors of a position operator of the form \( \langle 41 \rangle \),

\[
\hat{x} = \frac{2}{\hbar} \sum_{\epsilon = \pm} \int dx x |\phi^\epsilon (x)\rangle \langle \pi^\epsilon (x) |
\]
(26)

and its adjoint, that is

\[
\hat{x} |\phi^\epsilon (x)\rangle = x |\phi^\epsilon (x)\rangle,
\]
(27a)

\[
\hat{x} \dagger |\pi^\epsilon (x)\rangle = x |\pi^\epsilon (x)\rangle,
\]
(27b)

consistent with their biorthogonality. Any one-particle state can therefore be projected onto the position bases as

\[
|\psi(t)\rangle = \hat{1} |\psi(t)\rangle
\]
\[
= \frac{2}{\hbar} \sum_{\epsilon = \pm} \int dx |\phi^\epsilon (x)\rangle \langle \pi^\epsilon (x) | \psi(t)\rangle,
\]
(28a)

\[
|\psi(t)\rangle = \frac{2}{\hbar} \sum_{\epsilon = \pm} \int dx |\pi^\epsilon (x)\rangle \langle \phi^\epsilon (x) | \psi(t)\rangle.
\]
(28b)

The wave function

\[
\psi^\epsilon (x) = \langle \pi^\epsilon (x) | \psi(t)\rangle
\]
(29)

completely describes the one-photon state \( |\psi(t)\rangle \) in the \( \{ |\phi^\epsilon (x)\rangle \} \) basis of position eigenvectors. It may have positive frequency and negative frequency components. According to the rules of biorthogonal QM outlined in Section \( \langle 4 \rangle \) we have the equality

\[
\langle \phi^\epsilon (x) | \psi(t)\rangle = \langle \pi^\epsilon (x) | \psi(t)\rangle.
\]
(30)

Using \( \langle 28 \rangle \), \( \langle 28 \rangle \) and \( \langle 30 \rangle \) the squared norm of \( |\psi(t)\rangle \),

\[
\langle \psi | \psi \rangle = \frac{2}{\hbar} \sum_{\epsilon = \pm} \int dx \langle \pi^\epsilon (x) | \psi^\epsilon (t)\rangle^2,
\]
(31)

and the probability density,

\[
p^\epsilon (x) = \frac{2}{\hbar} \langle \psi | \psi \rangle \langle \pi^\epsilon (x) | \psi(t)\rangle^2,
\]
(32)

are positive definite. It is the invariant \( p (x) \) given by \( \langle 32 \rangle \), not the zeroth component of the four-current, \( j^0 (x) \), that describes particle density.

Since this application of biorthogonal QM is based on an invariant scalar, product the QM that it describes is covariant. In particular, the wave function of a plane wave, \( |1_k\rangle \), is \( |1_k| 1_q \rangle = (2\pi)^3 2\omega k \delta (k - q) \) in Fourier space and \( \langle \phi (x) | 1_q \rangle = \sqrt{\hbar} e^{-i(\omega_k t - k \cdot x)} \) in configuration space and a localized state, \( |\phi (y)\rangle \), is \( |1_k| 1_q \rangle \langle \phi (y) | 1_q \rangle = \sqrt{\hbar} e^{-i(\omega_k t - k \cdot y)} \) in Fourier space and \( \langle \pi (x) | \phi (y)\rangle = \)
\[ \hat{x}_n (x-y) \) in configuration space. Eqs. (24) and (26) can be generalized to
\[
\hat{1} = \frac{2}{\hbar} \int d\sigma |\phi(x)\rangle \langle -i\epsilon \sigma \partial^\mu \phi(x)|,
\]
and
\[
\hat{x}_i = \frac{2}{\hbar} \int d\sigma x_i |\phi(x)\rangle \langle -i\epsilon \sigma \partial^\mu \phi(x)|
\]
respectively where \( x_i \) is in the hyperplane.

The relationship of the relativistic position operator to the nonrelativistic position operator i\( \nabla_k \) and the NW position operator can be seen by transforming to Fourier space. The position operator (28) is in the IP while the position operator \( \hat{\mathcal{O}} \) is quasi-Hermitian [26]. In (16) the metric is Lorentz invariant. The photon scalar product that relates the classical Maxwell wave equation \((\vec{D} + \partial_2^2)A_\lambda = 0\) where \( m = 0 \) is Lorentz invariant. The photon scalar product that relates the classical Maxwell wave equation \((\vec{D} + \partial_2^2)A_\lambda = 0\) where \( m = 0 \) is Lorentz invariant. The photon scalar product that relates the classical Maxwell wave equation \((\vec{D} + \partial_2^2)A_\lambda = 0\) where \( m = 0 \) is Lorentz invariant. The photon scalar product that relates the classical Maxwell wave equation \((\vec{D} + \partial_2^2)A_\lambda = 0\) where \( m = 0 \) is Lorentz invariant. The photon scalar product that relates the classical Maxwell wave equation \((\vec{D} + \partial_2^2)A_\lambda = 0\) where \( m = 0 \) is Lorentz invariant. The photon scalar product that relates the classical Maxwell wave equation \((\vec{D} + \partial_2^2)A_\lambda = 0\) where \( m = 0 \) is Lorentz invariant. The photon scalar product that relates the classical Maxwell wave equation \((\vec{D} + \partial_2^2)A_\lambda = 0\) where \( m = 0 \) is Lorentz invariant. The photon scalar product that relates the classical Maxwell wave equation \((\vec{D} + \partial_2^2)A_\lambda = 0\) where \( m = 0 \) is Lorentz invariant. The photon scalar product that relates the classical Maxwell wave equation \((\vec{D} + \partial_2^2)A_\lambda = 0\) where \( m = 0 \) is Lorentz invariant. The photon scalar product that relates the classical Maxwell wave equation \((\vec{D} + \partial_2^2)A_\lambda = 0\) where \( m = 0 \) is Lorentz invariant. The photon scalar product that relates the classical Maxwell wave equation \((\vec{D} + \partial_2^2)A_\lambda = 0\) where \( m = 0 \) is Lorentz invariant.
in the factor \( \exp(\mathbf{i} \mathbf{k} \cdot \mathbf{x}) \) in the wave function but the factors \( \omega_k^{\pm 1/2} \) must be eliminated before the nonrelativistic position operator \( \mathbf{i} \nabla_k \) can be used to extract this information. For the transverse fields that describe photons an additional unitary transformation \( \hat{E} \) that rotates the field vectors to axes fixed in space is needed. A position eigenvector has a vortex structure like twisted light [19, 22] with nonzero spatial extension in which the photon position eigenvalue \( \mathbf{x} \) is the center of internal angular momentum \([23]\). Here the NW-like position operator derived in [20] will not be used, but the definite helicity basis vectors \( e_\lambda(\mathbf{k}) \) that are defined for all \( \mathbf{k} \) are still needed to describe position eigenvectors.

Following the derivation in Section III the IP photon position operator is

\[
\hat{x} = \frac{2}{\hbar} \sum_{c, \lambda = \pm} \int \mathbf{d}x \mathbf{x} \langle \mathbf{A}_\lambda^c(\mathbf{x}) \rangle \cdot \langle \mathbf{E}_\lambda^c(\mathbf{x}) \rangle ,
\]

the position eigenvector equations are

\[
\hat{x} \mathbf{A}_\lambda^c(\mathbf{x}) = \mathbf{x} \mathbf{A}_\lambda^c(\mathbf{x}) ,
\]

and position basis states are

\[
|\mathbf{A}_\lambda(\mathbf{x}) \rangle \equiv \hat{A}_\lambda(\mathbf{x}) |0 \rangle ,
\]

\[
|\mathbf{E}_\lambda(\mathbf{x}) \rangle \equiv c \hat{D}^{1/2} \hat{A}_\lambda(\mathbf{x}) |0 \rangle .
\]

Here the potential operator reads

\[
\hat{A}_\lambda(\mathbf{x}) = \sqrt{\frac{\hbar}{\epsilon_0}} \int \frac{\mathbf{d}k}{(2\pi)^3 2\omega_k} e_\lambda(\mathbf{k}) e^{-i(k \cdot x - \omega_k t)} \delta_\lambda^c(\mathbf{k}) .
\]

The Fourier space canonical commutation relations and orthogonality relations are

\[
\left[ \hat{\alpha}_\lambda(\mathbf{k}) , \hat{\alpha}_\lambda^\dagger(\mathbf{q}) \right] = (2\pi)^3 2\omega_k \delta(\mathbf{k} - \mathbf{q}) \delta_{\lambda\sigma} ,
\]

\[
\langle 1_{\lambda, \mathbf{k}} | 1_{\sigma, \mathbf{q}} \rangle = (2\pi)^3 2\omega_k \delta(\mathbf{k} - \mathbf{q}) \delta_{\lambda\sigma} ,
\]

where \( |1_{\lambda, \mathbf{k}} \rangle \equiv \alpha_\lambda^c(\mathbf{k}) |0 \rangle \).

With \( \epsilon = \pm \) states defined as in Section III the photon position eigenvectors are biorthogonal since their commutation relations imply that

\[
\sum_{i=1}^3 \langle E_{\lambda i}^c(\mathbf{x}) | A_{\epsilon i}^c(y) \rangle = \frac{\hbar}{2\epsilon_0} \delta_\epsilon(x-y) \delta_{\lambda\sigma} \delta_{\epsilon\epsilon'} ,
\]

where the subscripts \( i \) denote Cartesian components of the three-vectors and \( \epsilon = + \) for absorption at \( x \) while \( \epsilon = - \) for emission at \( x \). For these position eigenvectors the scalar product (49) is \( \langle A_{\lambda}^c(\mathbf{x}) , A_{\epsilon}^c(\mathbf{y}) \rangle = \delta_\epsilon(x-y) \delta_{\lambda\sigma} \).

For free photons described by transverse fields the completeness relation is

\[
\hat{1}_\perp = \frac{2\epsilon_0}{\hbar} \sum_{\epsilon, \lambda = \pm} \int \mathbf{d}x |A_\lambda^c(\mathbf{x})\rangle \cdot \langle \mathbf{E}_\lambda^c(\mathbf{x})| ,
\]

where we have defined

\[
|\mathbf{A}_\lambda^c(\mathbf{x})\rangle \cdot \langle \mathbf{E}_\lambda^c(\mathbf{x})| \equiv \sum_{i=1}^3 |A_{\lambda i}^c(\mathbf{x})\rangle \langle E_{\lambda i}^c(\mathbf{x})| .
\]

The identity operator \( \hat{1}_\perp \) on the space of transverse photons is closely connected to the so-called ‘transverse Dirac delta’ of QED [21]. For any transverse one-photon state we can hence write

\[
|\psi_\perp(\mathbf{t})\rangle = \frac{2\epsilon_0}{\hbar} \sum_{\epsilon, \lambda = \pm} \int \mathbf{d}x |\mathbf{A}_\lambda(\mathbf{x})\rangle \cdot \langle \mathbf{E}_\lambda(\mathbf{x})| |\psi^\epsilon(\mathbf{t})\rangle
\]

and the wave function

\[
\psi_\perp(\mathbf{x}) = \langle \mathbf{E}_\lambda(\mathbf{x}) | \psi^\epsilon(\mathbf{t}) \rangle = \langle A_\lambda(\mathbf{x}) | \psi^\epsilon(\mathbf{t}) \rangle
\]

completely describes the one-photon state \( |\psi_\perp(\mathbf{t})\rangle \) in either basis of position eigenvectors. The dual state vector is

\[
\psi_\perp^\epsilon(\mathbf{x}) = \langle A_\lambda(\mathbf{x}) | \psi^\epsilon(\mathbf{t}) \rangle ,
\]

the squared norm is

\[
\langle \psi_\perp | \psi_\perp^\epsilon \rangle = \frac{2\epsilon_0}{\hbar} \sum_{\epsilon, \lambda = \pm} \int \mathbf{d}x |\psi_\perp(\mathbf{x})|^2
\]

and the probability density per photon for a transition from \( |\psi_\perp(\mathbf{t})\rangle \) to \( \epsilon \)-frequency position eigenvector with helicity \( \lambda \)

\[
p^\epsilon_\lambda(\mathbf{x}) = \frac{2\epsilon_0}{\hbar} \langle \psi_\perp | \psi_\perp^\epsilon \rangle |\psi_\perp(\mathbf{x})|^2
\]

is positive definite.

A one-photon state can be Fourier expanded as

\[
|\psi_\perp(\mathbf{t})\rangle = \sum_{\lambda, \mathbf{k}} \int \frac{\mathbf{d}k}{(2\pi)^3 2\omega_k} c_\lambda(\mathbf{k}, \mathbf{t}) |1_{\lambda, \mathbf{k}}\rangle .
\]

Eqs. (49), (50) and (51) give \( \langle \mathbf{E}_\lambda(\mathbf{x}) | 1_{\lambda, \mathbf{k}} \rangle / \omega_k = \langle A_\lambda(\mathbf{x}) | 1_{\lambda, \mathbf{k}} \rangle \) so that \( |1_{\lambda, \mathbf{k}} / \omega_k \rangle \in \mathcal{H} \) and \( |1_{\lambda, \mathbf{k}} \rangle \in \mathcal{H}^\epsilon \). Substitution in (52) then gives the dual state vector

\[
|\psi_\perp(\mathbf{t})\rangle = \sum_{\lambda, \mathbf{k}} \int \frac{\mathbf{d}k}{(2\pi)^3 2\omega_k} c_\lambda(\mathbf{k}, \mathbf{t}) |1_{\lambda, \mathbf{k}}\rangle .
\]

The probability amplitude for a transition to a \( \epsilon \)-frequency plane wave state with wave vector \( \mathbf{k} \) and helicity \( \lambda \) is proportional to \( (1_{\lambda, \mathbf{k}} | \psi^\epsilon(\mathbf{t}) \rangle = \langle 1_{\lambda, \mathbf{k}} / \omega_k | \psi^\epsilon(\mathbf{t}) \rangle = c_\lambda(\mathbf{k}, \mathbf{t}) \). According to the rules of
bioriental QM outlined in the Introduction the probability density per photon for this transition is

$$p_x^t(k) = \frac{|\langle 1_{x,k} | \psi(t) \rangle|^2}{\langle \psi(t) | \psi(t) \rangle (2\pi)^3 2} = \frac{c^2_x(k,t)}{\sum_{\lambda=\pm} \int dk |c^2_x(k,t)|^2}. \quad (57)$$

Time dependence of $c^2_x(k,t)$ indicates the presence of a source. When a photon is emitted by an atom, the expectation value of the photon number is smaller than one and approaches unity as $t \to \infty$. If $|\psi(t)\rangle$ is normalized so that $n(t) = \langle \psi(t) | \psi(t) \rangle$ is the number of photons, the probability density for $k^\mu = (\omega_k,k)$ is $|c^2_x(k,t)|^2 / [(2\pi)^3 2]$ while the probability density to find a photon at $x$ on the hyperplane $\sigma$ is $|\psi^2_x(x)|^2 2\epsilon_0/\hbar$. In the SP the photon position operator $\hat{x}$ can be written as

$$\hat{x}^{\text{SP}}(k) = \hat{E} i \nabla_k \hat{E}^{-1}. \quad (58)$$

The scalar product $\langle \mathbf{E}_x'(x) | \psi^+(t) \rangle = \langle \mathbf{E}_x(x) | \psi(t) \rangle$ that leads to an invariant probability to count a photon is proportional to probability amplitude, not the electric field. Glauber defined an ideal photon detector as a system of negligible size with a frequency-independent photon absorption probability $|\mathbf{A} \cdot \mathbf{p}|^2$. For the positive frequency one-photon state $|\psi(t)\rangle$ he found that the probability to count a photon is proportional to $|\langle \mathbf{E}_x(x) | \psi(t) \rangle|^2$. Glauber considered photodetection to be a square law process and interpreted it to be responsive to the density of electromagnetic energy, but number density gives an invariant probability to count a photon while energy density does not. Indeed, the biorthogonal completeness relation implies that a basis of ideal Glauber detectors can be defined provided the state vector $|\psi\rangle \in \mathcal{H}$ of the photon at hand has been created by the $\hat{A} \cdot \mathbf{p}$ minimal coupling Hamiltonian. In that case, $|\langle \mathbf{E}_x(x) | \psi(t) \rangle|^2$ is proportional to photon probability density.

Here a positive definite particle density is obtained in the physical Hilbert space according to the rules of bioriental QM summarized in the Introduction. An alternative approach is to transform the physical fields to the Foldy representation using the nonlocal operator $\hat{D}^{-1/4}$ and its inverse $\hat{F}$. For photons this nonlocal transformation leads to the Landau-Peierls (LP) wave function $\mathbf{E} \mathbf{E}$. The disadvantage to this approach is that the relationship between the LP wave function and a current source is nonlocal. Here the fields $A^x(x)$ due to the local interaction Hamilton $j_x(x) A^x(x)$ are calculated first and the probability amplitude for a transition to a position eigenvector is then obtained using the invariant scalar product. These fields have well defined Lorentz and gauge transformation properties. The positive definiteness of the probability follows then directly from the mathematical rules of bioriental QM.

An advantage of the second quantized formalism used here is that multiphoton wave functions can be introduced as in [36, 37, 38]. For example, to the two photon state $|\psi_2\rangle$ can be associated the wave function

$$\psi_{\lambda_1,\lambda_2}(x_1,x_2,t) = \langle \mathbf{E}_{\lambda_1}(x_1) \mathbf{E}_{\lambda_2}(x_2) | \psi_2(t) \rangle \quad (59)$$

with $|\mathbf{E}_{\lambda_1}(x_1) \mathbf{E}_{\lambda_2}(x_2)\rangle \equiv \hat{\mathbf{E}}_{\lambda_1}(x_1) \hat{\mathbf{E}}_{\lambda_2}(x_2) |0\rangle$. This wave function localizes the photons at spatial points $x_1$ and $x_2$ at time $t$ and can describe entangled two-photon states.

V. WAVE FUNCTION OF A PHOTON Emitted BY AN ATOM

The wave function for a photon emitted by a two-level atom initially in its excited state was derived in [41, 42]. To first order in the IP minimal coupling interaction Hamiltonian $\hat{H}_I = (\epsilon/m_e) \hat{A} (\hat{\mathbf{x}}_e,t) \cdot \hat{\mathbf{p}}_e(t)$. For a two-level atom initially in its excited state $|e\rangle$ with no photons present, the positive frequency IP wave function describing decay to its ground state $|g\rangle$ while emitting one photon is

$$|\psi(t)\rangle = c_e(t) |e,0\rangle + \sum_{\lambda=\pm} \int \frac{dk}{(2\pi)^3 2\epsilon_k} c_{e,\lambda}(k,t) |g,1_{\lambda,k}\rangle \quad (60)$$

where

$$c_{e,\lambda}(k,t) = \frac{\epsilon}{m_e} \left[ e^{i(\omega_k - \omega_0)t} \hat{A}_{\lambda} (\hat{\mathbf{x}}_e,t) \cdot \hat{\mathbf{p}}_e(t) |e,0\rangle \int \frac{dk}{(2\pi)^3 2} c_{g,\lambda}(k,t) |1_{\lambda,k}\rangle \right]. \quad (61)$$

Here $\hbar\omega_0$ is the level separation between the ground and excited states and $\mathbf{x}_e$ and $\mathbf{p}_e$ are the electron position and momentum operators. For $|\psi_\perp\rangle$ given by [59], the transverse single-photon state and its dual are thus given by

$$|\psi_\perp(t)\rangle = \sum_{\lambda=\pm} \int \frac{dk}{(2\pi)^3 2} c_{g,\lambda}(k,t) |1_{\lambda,k}\rangle, \quad (62a)$$

$$|\bar{\psi}_\perp(t)\rangle = \sum_{\lambda=\pm} \int \frac{dk}{(2\pi)^3 2} c_{g,\lambda}(k,t) |1_{\lambda,k}\rangle. \quad (62b)$$

In [41, 42] the minimal coupling Hamiltonian created the photon state vector $|\psi_\perp\rangle$ in the $|\mathbf{A}_\lambda(x)\rangle$ basis so the appropriate wave function is $\langle \mathbf{E}_\lambda(x) | \psi_\perp \rangle$. Indeed we have from [31] and [60]

$$\psi_\lambda(x) = i \sqrt{\frac{\hbar}{\epsilon_0}} \int \frac{dk}{(2\pi)^3 2} e^{-i k_x x \epsilon_0} e_{\lambda}(k) e^{-i k_x x \epsilon_0} c_{g,\lambda}(k,t). \quad (63)$$

The positive frequency wave function of the emitted photon is calculated, but causal solutions that include negative frequencies are also considered. Taking into account
a factor $-i$ in the electric field operator used in $\mathbf{41, 42}$, $c_{\mathbf{x}, \lambda} = -ic_{\lambda}^{\dagger}$ in (42). Substitution of the wave function $\psi_{\lambda}(x) = \langle \mathbf{E}_{\lambda}(x) | \psi_{\perp}(t) \rangle$ in (53) gives the probability density in space to count a photon at time $t$. Since in $\mathbf{41, 42}$ the wave function is normalized as $\langle \psi_{\perp} | \psi_{\perp} \rangle = 1$, the factor $\langle \psi_{\perp} | \psi_{\perp} \rangle$ in (54) and (57) approaches $1/\omega_0$ as $t \to \infty$.

If the standard (dipolar) $\mathbf{E} \cdot \mathbf{x}$ Hamiltonian were to be used instead, the photon would be created in the $\mathbf{41}$ photon position operator. Here, the Hermitian conjugate of each other. Our formalism of two different position operators, which are the vector potential (and its canonically conjugate Wigner-Bargmann quantum field operator (for photons, the factor $\lambda(x)$ in (54) and (57) approaches $1/\omega_0$ as $t \to \infty$.

VI. CONCLUSION

The formalism of biorthogonal systems can be, as we saw, called in action in relativistic quantum mechanics. It is particularly well-matched to the relativistic scalar product. In the biorthogonal formalism, both the Wigner-Bargmann quantum field operator (for photons, the vector potential) and its canonically conjugate momentum (for photons, the electric field) are put on an equal footing, and they generate respectively the direct and the dual basis of position eigenvalues of two different position operators, which are the Hermitian conjugate of each other. Our formalism further clarifies the meaning of the free parameter $\alpha$ in the photon position operator. Here, the freedom in the choice of $\alpha$ allows us to use both $\mathbf{x}$ and $\mathbf{x}^\dagger$.

The probability density (54) suggests a resolution of the apparent dichotomy between photon number counting and the sensitivity of a detector to energy density. The wave function $\langle \mathbf{E}_{\lambda}(x) | \psi(t) \rangle$ together with the state vector (60) describes creation of a photon in the time interval $0 \leq t \leq t_f$ followed by its detection at time $t_f$. Since it is created in the $| \mathbf{A}_{\lambda}(x) \rangle$ basis and observed in the dual $| \mathbf{E}_{\lambda}(x) \rangle$ basis, that wave function is proportional to a probability amplitude. The probability density for a transition from $| \psi_{\perp} \rangle$ to the position eigenvector at $\mathbf{x}$, given by $2\epsilon_0 | \langle \mathbf{E}_{\lambda}(x) | \psi(t) \rangle |^2 / \hbar$, is of the Glauber form (32). However, in contrast to theories of photodetection based on energy density, we have proposed, through the position amplitude $\langle \mathbf{E}_{\lambda}(x) | \psi(t) \rangle$ in the dual basis, a true position measurement that describes an array of ideal photon counting detectors.

For a state vector that is an arbitrary linear combination of positive and negative frequency terms the probability density for a transition to the position eigenvector at $\mathbf{x}$ is positive definite. This particle plus antiparticle probability density describes a particle at spatial location $\mathbf{x}$ independent of whether it was absorbed or emitted. Thus (54) can be interpreted as probability density even if the wave function (51) is real as in classical electromagnetism. This application of biorthogonal QM is based on an invariant positive definite scalar product so transition probabilities are invariant and positive definite, the position operator is covariant, and there is no NW $\omega_k^{3/2}$ nonlocality in the wave function.

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