On holomorphic functions on a strip in the complex plane

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Abstract

Let $f$ be a holomorphic function on the strip \( \{ z \in \mathbb{C} : -\alpha < \Im z < \alpha \}, \alpha > 0 \), belonging to the class $H(\alpha, -\alpha; \varepsilon)$ defined below. It is shown that there exist holomorphic functions $w_1$ on \( \{ z \in \mathbb{C} : 0 < \Im z < 2\alpha \} \) and $w_2$ on \( \{ z \in \mathbb{C} : -2\alpha < \Im z < 2\alpha \} \) such that $w_1$ and $w_2$ have boundary values of modulus one on the real axis and satisfy the relation $w_1(z) = f(z - \alpha i) w_2(z - 2\alpha i)$ and $w_2(z + 2\alpha i) = \overline{f(z+\alpha i)} w_1(z)$ for $0 < \Im z < 2$, where $\overline{f(z)} := f(\overline{z})$. This leads to a "polar decomposition" $f(z) = u_f(z+\alpha i) g_f(z)$ of the function $f(z)$, where $u_f(z+\alpha i)$ and $g_f(z)$ are holomorphic functions for $-\alpha < \Im z < \alpha$ such that $|u_f(x)| = 1$ and $g_f(x) \geq 0$ a.e. on the real axis. As a byproduct, an operator representation of a $q$-deformed Heisenberg algebra is developed.

1 Introduction and Main Results

Let $\varepsilon$ be a positive number and $\alpha$ and $\beta$ be real numbers such that $\alpha > \beta$. Let $\mathcal{H}(\alpha, \beta; \varepsilon)$ denote the set of all holomorphic functions $h(z)$ on the strip $I(\alpha, \beta) := \{ z \in \mathbb{C} : \alpha > \Im z > \beta \}$ such that

$$\sup_{\alpha > y > \beta} \int_{-\infty}^{\infty} |h(x+y\iota)|^2 e^{-2\gamma x^2} \, dx < \infty$$

for all numbers $\gamma > \varepsilon$. As stated in Lemma 2 below, each function $h \in \mathcal{H}(\alpha, \beta; \varepsilon)$ admits boundary values $h(x+\beta \iota)$ and $h(x+\alpha \iota), x \in \mathbb{R}$, which satisfy

$$\lim_{y \downarrow \beta} \int |h(x+y\iota) - h(x+\beta \iota)|^2 e^{-2\gamma x^2} \, dx = 0,$$

$$\lim_{y \uparrow \alpha} \int |h(x+y\iota) - h(x+\alpha \iota)|^2 e^{-2\gamma x^2} \, dx = 0$$

for all $\gamma > \varepsilon$. Throughout this paper, $\iota$ denotes the complex unit. By some slight abuse of notation, we denote functions in $\mathcal{H}(\alpha, \beta; \varepsilon)$ and their boundary values by the same symbol.

Our main results are contained in the following

**Theorem 1.** Let $\varepsilon$ and $\alpha$ be positive real numbers and let $f \neq 0$ a function of the class $\mathcal{H}(\alpha, -\alpha; \varepsilon)$ such that

$$\inf \{|f(x-\alpha \iota)|; x \in \mathbb{R} \} > 0.$$
Then there exist functions \( w_1 \in \mathcal{H}(2\alpha,0;\varepsilon) \) and \( w_2 \in \mathcal{H}(2\alpha,-2\alpha;\varepsilon) \) such that \( |w_1(x)| = |w_2(x)| = 1 \) a.e. on \( \mathbb{R} \) and

\[
\begin{align*}
    w_1(x) &= f(x-\alpha)w_2(x-2\alpha), \\
    w_2(x) &= \tilde{f}(x-\alpha)w_1(x-2\alpha)
\end{align*}
\]  

(4)  

a.e. on \( \mathbb{R} \). If \( \tilde{w}_1 \) and \( \tilde{w}_2 \) are two other functions with these properties, then there is a constant \( c \) of modulus one such that \( \tilde{w}_1(x) = cw_1(x) \) and \( \tilde{w}_2(x) = cw_2(x) \) a.e. on \( \mathbb{R} \).

We briefly discuss some consequences of the preceding result. Since both sides of the equality (4) are boundary values of holomorphic functions on the strip \( J(2\alpha,0) \), we conclude that

\[
    w_1(z+2\alpha) = f(z+\alpha)w_2(z), \quad z \in J(0,-2\alpha).
\]

(6)

Since \( \tilde{f}(z-\alpha)w_1(z-2\alpha) \) is holomorphic on \( J(2\alpha,0) \), it follows from (5) that the function \( w_2(x) \) on \( \mathbb{R} \) is boundary value of a holomorphic function \( w_2(z) \) on \( J(2\alpha,0) \) and

\[
    w_2(z+2\alpha) = \tilde{f}(z+\alpha)w_1(z), \quad z \in J(2\alpha,0).
\]

(7)

If the function \( f \in \mathcal{H}(\alpha,-\alpha;\varepsilon) \) from Theorem 1 is holomorphic on the whole upper half-plane, then it follows from (5) and (6) that \( w_1(z) \) and \( w_2(z) \) are holomorphic on the upper half-plane and that the relations (5) and (6) hold for all \( z \in \Phi, \text{Im } z > -2\alpha \).

Moreover, both relations then imply that

\[
    w_1(z+4\alpha) = f(z+3\alpha)\tilde{f}(z+\alpha)w_1(z),
\]

\[
    w_2(z+4\alpha) = \tilde{f}(z+3\alpha)f(z+\alpha)w_2(z)
\]

for \( \text{Im } z > -2\alpha \).

Theorem 1 is a result on holomorphic functions, but its proof is based on operator-theoretic tools. This technique might be also of interest in itself. Let \( P \) be the operator \(-i\frac{d}{dx}\) on \( L^2(\mathbb{R}) \). For \( f \in \mathcal{H}(\alpha,-\alpha;\varepsilon) \), let \( L_f \) and \( R_f \) be the closures of the linear operators \( \tilde{L}_f \) and \( \tilde{R}_f \) on the Hilbert space \( L^2(\mathbb{R}) \) defined by the formulas

\[
    \tilde{L}_f = f(x-\alpha)e^{2\alpha P} \quad \text{and} \quad \tilde{R}_f = e^{2\alpha P} \tilde{f}(x+\alpha).
\]

The operators \( L_f \) and \( R_f \) are crucial in the proof of Theorem 1. The polar decomposition of the operator \( L_f \) is described by the following theorem. Some more properties of these operators can be found in Sections 2 and 3 and in our previous papers [5] and [6].

**Theorem 2.** Retain the assumptions and notations from Theorem 1 and define holomorphic functions \( u_f \) and \( g_f \) on the strips \( J(2\alpha,0) \) and \( J(\alpha,-\alpha) \), respectively, by

\[
    u_f(z) = w_1(z)\overline{w_2(z)}, \quad g_f(z) = w_2(z+\alpha)\overline{w_2(z-\alpha)}.
\]

(8)

Then the polar decomposition of the closed linear operator \( L_f \) is given by \( L_f = u_f L_g f \) and we have \( L_f^* = R_f \).
From the relation \( L_f = u_f L_{g_f} \) we immediately obtain that \( f(x-\alpha i) = u_f(x)g_f(x-\alpha i) \) a.e. on \( \mathbb{R} \). This in turn implies that
\[
f(z) = u_f(z+\alpha i)g_f(z), \quad z \in \mathcal{J}(\alpha,-\alpha),
\]
where \( u_f(z+\alpha i) \) and \( g_f(z) \) are holomorphic functions on the strip \( \mathcal{J}(\alpha,-\alpha) \) such that
\[
|u_f(x)| = 1 \quad \text{a.e. on } \mathbb{R}
\]
\[
g_f(x) = |w_2(x+i\alpha)|^2 \geq 0 \quad \text{on } \mathbb{R}.
\]

Because of the two latter properties and the fact that it originates from the polar decomposition of the operator \( L_f \), we refer to the decomposition (9) as the polar decomposition of the holomorphic function \( f \in H(\alpha,-\alpha; \varepsilon) \) with respect to the strip \( \mathcal{J}(\alpha,-\alpha) \).

The proofs of Theorems 1 and 2 will be given in Section 3. Some necessary operator-theoretic tools will be collected in Section 2. The example \( f(z) = z \) is treated in Section 4. As an application we construct in Section 5 an interesting operator representation of a \( q \)-deformed Heisenberg algebra.

## 2 Technical Preliminaries

Throughout, \( P \) denotes the self-adjoint operator \(-i \frac{d}{dx}\) on the Hilbert space \( L^2(\mathbb{R}) \) and \( \alpha \) is a positive real number. The following lemma describes the action and the domain of the operator \( e^{\alpha P} \). Its proof given in [6] is essentially based on the classical Paley-Wiener theorem [2]. The domain of an operator \( T \) is denoted by \( \mathcal{D}(T) \).

**Lemma 1.** Suppose that \( \alpha > 0 \). Let \( g(z) \) be a holomorphic function on the strip \( \mathcal{I}(0,-\alpha) \) such that
\[
\sup_{0 < y < \alpha} \int_{-\infty}^{\infty} |g(x-iy)|^2 \, dx < \infty.
\]
Then there exist functions \( g(x) \in L^2(\mathbb{R}) \) and \( g_{-\alpha}(x) \in L^2(\mathbb{R}) \) such that \( \lim_{y \uparrow 0} g_y = g \) and \( \lim_{y \downarrow -\alpha} g_y = g_{-\alpha} \) in \( L^2(\mathbb{R}) \), where \( g_y(x) := g(x+iy) \) for \( x \in \mathbb{R} \) and \( y \in (0,-\alpha) \). Setting \( g(x-i\alpha) := g_{-\alpha}(x), x \in \mathbb{R} \), we have
\[
\lim_{n \to \infty} g \left( x - n^{-2}i \right) = g(x) \quad \text{and} \quad \lim_{n \to \infty} g \left( x - (\alpha + n^{-2})i \right) = g(x-i\alpha) \quad \text{a.e. on } \mathbb{R}.
\]

The function \( g \) belongs to the domain \( \mathcal{D}(e^{\alpha P}) \) and \( (e^{\alpha P}g)(x) = g(x-i\alpha) \). Conversely, each function \( g \in \mathcal{D}(e^{\alpha P}) \) arises in this way.

**Proof.** [6], Lemma 1.

For \( \delta \geq 0 \), we define a dense linear subspace \( \mathcal{D}_{\delta} \) of the Hilbert space \( L^2(\mathbb{R}) \) by
\[
\mathcal{D}_{\delta} = \text{Lin}\{e^{-\gamma x^2 + \beta x} : \gamma > \delta, \beta \in \mathbb{C}\}.
\]
Lemma 2. (i) Suppose that \( h \in \mathcal{H}(\alpha, \beta; \epsilon) \). Then there exist measurable functions \( h(x+\beta) \) and \( h(x+\alpha i) \) on \( \mathbb{R} \) both contained in the domain \( \mathcal{D}(e^{-\gamma x^2}) \) for each \( \gamma > \epsilon \) such that (4) and (5) are satisfied. If \( \varphi \in L^2(\mathbb{R}) \) such that \( h \varphi \in \mathcal{D}(e^{\alpha P}) \) (in particular, if \( \varphi \in \mathcal{D}_\epsilon \)), we have

\[
(e^{\alpha P} h \varphi)(x) = h(x-\alpha i) \varphi(x-\alpha i) = h(x-\alpha i) (e^{\alpha P} \varphi)(x). \tag{11}
\]

(ii) For any \( \alpha \in \mathbb{R} \) and \( \beta \geq 0 \), \( \mathcal{D}_\beta \) is a core for \( e^{\alpha P} \).

Proof. [6], Lemma 2 and Lemma 3(ii).

Lemma 3. Let \( \alpha > 0 \). Suppose that \( f \neq 0 \) is a function of the class \( \mathcal{H}(\alpha, -\alpha; \epsilon) \) satisfying condition (3). Then we have:

(i) \( L_f = L_f \equiv f(x-\alpha i)e^{2\alpha P} \).
(ii) \( \ker L_f = \{0\} \).
(iii) \( L_f^* = R_f \).

Proof. (i): We have to show that the operator \( \tilde{L}_f \equiv f(x-\alpha i)e^{2\alpha P} \) is closed. For let \( \{\eta_n\} \) be a sequence of vectors \( \eta_n \in \mathcal{D}(\tilde{L}_f) \) such that \( \eta_n \to \eta \) and \( f(x-\alpha i)e^{2\alpha P} \eta_n \to \zeta \) in the Hilbert space \( L^2(\mathbb{R}) \). Since \( \{f(x-\alpha i)e^{2\alpha P} \eta_n\} \) is a Cauchy sequence, condition (3) implies that \( \{e^{2\alpha P} \eta_n\} \) is also a Cauchy sequence in \( L^2(\mathbb{R}) \), so that \( e^{2\alpha P} \eta_n \to \xi \) for some \( \xi \in L^2(\mathbb{R}) \).

Because the operator \( e^{2\alpha P} \) is closed, we conclude that \( \xi \in \mathcal{D}(e^{2\alpha P}) \) and \( e^{2\alpha P} \eta_n \to f(x-\alpha i)e^{2\alpha P} \eta_n \to f(x-\alpha i) \xi = \zeta \).

(ii) follows immediately from (i).

(iii): Using the relation \( \overline{f(x-\alpha i)} = f(x+\alpha i) \) one easily verifies that \( R_f \subseteq L_f^* \). We now prove the opposite inclusion. Since the function \( \overline{f(x+\alpha i)}^{-1} \) is bounded because of assumption (3), we have \( e^{2\alpha P} \subseteq R_f f(x+\alpha i)^{-1} \). Using this fact we conclude that

\[
f(x-\alpha i)^{-1} R_f^* \subseteq (R_f \overline{f(x+\alpha i)}^{-1})^* \subseteq (e^{2\alpha P})^* = e^{2\alpha P}
\]

which in turn implies that

\[
R_f^* \subseteq f(x-\alpha i)e^{2\alpha P} = L_f.
\]

Applying the adjoint to the latter we get \( L_f^* \subseteq (R_f)^{**} = R_f \), because the operator \( R_f \) is closed. Both inclusion together yield the desired equality \( L_f^* = R_f \).

\[
3 \text{ Proof of the Theorems}
\]

Consider the self-adjoint operator \( \mathcal{A} \) and the one parameter unitary group \( U(t), t \in \mathbb{R} \), on the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) given by the operator matrices

\[
A_f = \begin{pmatrix} 0 & L_f \\ L_f^* & 0 \end{pmatrix}, \quad U(t) = \begin{pmatrix} e^{itQ} & 0 \\ 0 & e^{itQ} \end{pmatrix}.
\tag{12}
\]
Since \( e^{-itQ} e^{2\alpha t} e^{itQ} = e^{2\alpha t} \), we have \( e^{-itQ} L_f e^{itQ} = e^{2\alpha t} L_f \) and hence \( e^{-itQ} A_f e^{itQ} = e^{2\alpha t} L_f^* \) for \( t \in \mathbb{R} \). These relations immediately imply that
\[
U(-t) A_f U(t) = e^{2\alpha t} A_f, \quad t \in \mathbb{R},
\] (13)

Let us write \( A_f = A_f^+ \oplus (-A_f^-) \), where \( A_f^+ \) and \( A_f^- \) are the positive and the negative parts of the self-adjoint operator \( A_f \). The corresponding reducing subspaces of the Hilbert space \( \mathcal{H} \) are denoted by \( \mathcal{H}_+ \) and \( \mathcal{H}_- \). We first verify that \( A_f^+ \neq 0 \) and \( A_f^- \neq 0 \). Let \( L_f = u_f|L_f| \) be the polar decomposition of the closed operator \( L_f \). Since ker \( L_f = \{0\} \) and ker \( L_f^* = \{0\} \) as noted above, \( u_f \) is unitary. We have \( L_f^* = |L_f|u_f^* \) and hence
\[
\langle A_f(\eta_1 \oplus \eta_2), \eta_1 \oplus \eta_2 \rangle = \langle L_f \eta_2, \eta_1 \rangle + \langle L_f^* \eta_1, \eta_2 \rangle
\]
\[
= \langle |L_f| \eta_2, u_f^* \eta_1 \rangle + \langle |L_f| u_f^* \eta_1, \eta_2 \rangle
\] (14)

for \( \eta_1 \in \mathcal{D}(L_f^*) \) and \( \eta_2 \in \mathcal{D}(L_f) \). Choosing \( \eta_2 = u_f^* \eta_1, \eta_1 \neq 0 \), the expression (14) becomes positive. For \( \eta_2 = -u_f^* \eta_1, \eta_1 \neq 0 \), it is negative. This implies that \( A_f^+ \neq 0 \) and \( A_f^- \neq 0 \).

From relation (13) it follows that \( U(-t) A_f^+ U(t) = e^{2\alpha t} A_f^+ \) and \( U(t) \mathcal{H}_\pm \subseteq \mathcal{H}_\pm \) for real \( t \). Recall that \( A_f^+ \) and \( A_f^- \) are positive self-adjoint operators with trivial kernels. Therefore, we conclude that \( U(-t)(A_f^+) s U(t) = (e^{2\alpha t} A_f^+) s \) for \( s, t \in \mathbb{R} \). That is, the unitary groups \( U_\pm(t) := U(t)[\mathcal{H}_\pm] \) and \( V_\pm(s) := (A_f^+) s \) on the Hilbert space \( \mathcal{H}_\pm \) satisfy the Weyl relation \( V_\pm(s) U_\pm(t) = e^{2\alpha t} U_\pm(t) V_\pm(s), s, t \in \mathbb{R} \). By construction, the unitary group \( U(t) = U_+(t) \oplus U_-(t) \) has uniform spectral multiplicity two. Consequently, since \( A_f^+ \neq 0 \) and hence \( \mathcal{H}_\pm \neq \{0\} \), both unitary groups \( U_+(t) \) and \( U_-(t) \) have spectral multiplicity one. Therefore, it follows from the Stone–von Neumann uniqueness theorem (see, for instance, [3]) that each pair \( \{U_\pm(t), A_f^\pm\} \) on \( \mathcal{H}_\pm, \pm = \pm1 \), is unitarily equivalent to the pair \( \{e^{itQ}, e^{2\alpha t}\} \) acting on the Hilbert space \( L^2(\mathbb{R}) \). Hence the pair \( \{U(t) = U_+(t) \oplus U_-(t), A_f = A_f^+ \oplus (-A_f^-)\} \) is unitarily equivalent to the pair \( \{e^{itQ} \oplus e^{itQ}, e^{2\alpha t} \oplus (-e^{2\alpha t})\} \) on \( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \). For the subsequent considerations it is convenient to transform the latter pair by means of the unitary symmetry
\[
\frac{1}{\sqrt{2}} \left( \begin{array}{cc} I & I \\ I & -I \end{array} \right).
\]

Putting the preceding together, it follows that there exists a unitary 2×2-operator matrix
\[
W = \begin{pmatrix} w_1 & w_3 \\ w_4 & w_2 \end{pmatrix}
\]
of the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \) such that
\[
W^* U(t) W = \begin{pmatrix} e^{itQ} & 0 \\ 0 & e^{itQ} \end{pmatrix}, \quad t \in \mathbb{R},
\] (15)
\[
W^* A_f W = \begin{pmatrix} e^{2\alpha t} & 0 \\ 0 & e^{-2\alpha t} \end{pmatrix} := B.
\] (16)

Relation (15) means that \( w_j e^{itQ} = e^{itQ} w_j, t \in \mathbb{R} \), for \( j = 1, 2, 3, 4 \). This implies that the entries \( w_j \) of the operator matrix \( W \) are multiplication operators by essentially bounded
measurable functions \( w_j(x), x \in \mathbb{R} \). Equation (16) is equivalent to the two relations
\[ W B \subseteq A_f W \text{ and } W^* A_f \subseteq BW^* \]. Applying the relation \( WB \subseteq A_f W \) to vectors \((0, \eta)\), and \((\eta, 0)\), where \( \eta \in \mathcal{D}(e^{2\alpha P}) \), in the domain of \( B \) we obtain that
\[ w_1 e^{2\alpha P} \subseteq L_f w_2, \quad w_4 e^{2\alpha P} \subseteq L_f^* w_3, \quad w_1 e^{2\alpha P} \subseteq L_f w_1, \quad w_3 e^{2\alpha P} \subseteq L_f^* w_3, \quad w_2 e^{2\alpha P} \subseteq L_f^* w_1. \]
respectively. Similarly, the relation \( W^* A_f \subseteq BW^* \) applied to vectors \((0, \eta)\), \( \eta \in \mathcal{D}(L_f) \), and \((\eta, 0)\), \( \eta \in \mathcal{D}(L_f^*) \), yields
\[ w_1 e^{4\alpha P} \subseteq L_f L_f^* w_1, \quad w_2 e^{4\alpha P} \subseteq L_f L_f^* w_2, \quad w_3 e^{4\alpha P} \subseteq L_f L_f^* w_3, \quad w_4 e^{4\alpha P} \subseteq L_f L_f^* w_4. \]
respectively. In fact, (17) and (18) are equivalent to the inclusion \( WB \subseteq A_f W \), while (19) and (20) are equivalent to \( W^* A_f \subseteq BW^* \).

From (17) and (18) it follows at once that
\[ w_1 e^{4\alpha P} \subseteq L_f L_f^* w_1, \quad w_2 e^{4\alpha P} \subseteq L_f L_f^* w_2, \quad w_3 e^{4\alpha P} \subseteq L_f L_f^* w_3, \quad w_4 e^{4\alpha P} \subseteq L_f L_f^* w_4. \]
Let us fix one of the relations \( w B_1 \subseteq B_2 w \) of (21) or (22), where \( w = w_j, B_1 = e^{4\alpha P} \) and \( B_2 \) is one of the self-adjoint operator \( L_f L_f^* \) or \( L_f^* L_f \), respectively. Since \( B_1 \) and \( B_2 \) are self-adjoint and \( w \) is bounded, we obtain \( B_1 w^* = (w B_1)^* \supseteq (B_2 w)^* \supseteq w^* B_2 \) and hence \( w^* w B_1 \subseteq w^* B_2 w \subseteq B_1 w^* w \). That is, we have \( |w_j(x)|^2 e^{4\alpha P} \subseteq e^{4\alpha P} |w_j(x)|^2 \). From the latter we conclude that the bounded operator \( |w_j(x)|^2 \) commutes with all functions of the unbounded self-adjoint operator \( e^{4\alpha P} \), so \( |w_j(x)|^2 \) commutes in particular with the unitary group \( e^{i s P}, s \in \mathbb{R} \), on \( L^2(\mathbb{R}) \). Therefore, the function \( |w_j(x)| \) is almost everywhere constant on \( \mathbb{R} \), say \( |w_j(x)| = c_j \) a.e. on \( \mathbb{R} \).

Since the \( 2 \times 2 \)-matrix \( W \) is a unitary operator on \( \mathcal{H} \), we conclude that \( c_1 = c_2, c_3 = c_4 \), and \( c_1^2 + c_3^2 = c_2^2 + c_4^2 = 1 \). Without loss of generality let us suppose that \( c_1 \neq 0 \). (If \( c_1 = 0 \), then \( c_3 \neq 0 \) and we replace the function \( w_1, w_2 \) by \( w_3, w_4 \) in what follows.) Then, upon replacing \( w_1 \) by \( w_1 c_1^{-1} \) and \( w_2 \) by \( w_2 c_1^{-1} \) we can assume that the functions \( w_1(x) \) and \( w_2(x) \) satisfying (17) and (18) are of modulus one on the real line.

From the first relations of (17) and (19) and the second relations of (18) and (20) we now easily obtain
\[ L_f = w_1 e^{2\alpha P} \overline{w_2} \quad \text{and} \quad L_f^* = w_2 e^{2\alpha P} \overline{w_1}. \]
Thus, we have \( L_f^* L_f = w_2 e^{4\alpha P} \overline{w_2} \) and \( L_f L_f^* = w_1 e^{4\alpha P} \overline{w_1} \). For the operators \( |L_f| = (L_f L_f^*)^{1/2} \) and \( |L_f^*| = (L_f^* L_f)^{1/2} \) we therefore obtain
\[ |L_f| = w_2 e^{2\alpha P} \overline{w_2} \quad \text{and} \quad |L_f^*| = w_1 e^{2\alpha P} \overline{w_1}. \]

**Remarks.** 1. Recall that \( |w_1(x)| = |w_2(x)| = 1 \) a.e. on \( \mathbb{R} \). Using this fact and repeating the above reasoning it follows that the two relations (23) are equivalent to
\[ \begin{pmatrix} \overline{w}_1 & 0 \\ 0 & \overline{w}_2 \end{pmatrix} \begin{pmatrix} 0 & L_f \\ L_f^* & 0 \end{pmatrix} \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix} = \begin{pmatrix} 0 & e^{2\alpha P} \\ e^{2\alpha P} & 0 \end{pmatrix}. \]
That is, the unitary matrix $W$ satisfying (13) and (14) can be chosen to be diagonal with functions $w_1$ and $w_2$ as diagonal entries.

2. Note that in equations (23) and (24) we have strict equality of the corresponding unbounded operators on both sides.

Let us begin with the proof of Theorem 1. By Lemma 3(i), the operator $\tilde{L}_f = f(x-\alpha i)e^{2\alpha P}$ is closed because of assumption (3), so it coincides with $L_f$. From the first equality of (23) we obtain

$$f(x-\alpha i)e^{2\alpha P} = w_1 e^{2\alpha P} \frac{w_2}{|w_2|^2} \quad \text{and} \quad f(x-\alpha i)e^{2\alpha P} w_2 = w_1 e^{2\alpha P}. \quad (26)$$

Set $\eta_\gamma(x) := e^{-\gamma x^2}$ for $\gamma > \varepsilon$. Since $f \in \mathcal{H}(\alpha, -\alpha; \varepsilon)$ by assumption, $\eta_\gamma \in \mathcal{D}(f(x-\alpha i)e^{2\alpha P})$. Therefore, by the first relation of (23), $w_2 \eta_\gamma \in \mathcal{D}(e^{2\alpha P})$. Since $\eta_\gamma$ is also in the domain of $w_1 e^{2\alpha P}$, the second equality of (26) yields $w_2 \eta_\gamma \in \mathcal{D}(e^{2\alpha P})$. By the characterization of the domain $\mathcal{D}(e^{2\alpha P})$ given in Lemma 1, the facts that $w_2 \eta_\gamma$ and $w_2 \eta_\gamma$ for arbitrary $\gamma > \varepsilon$ are in $\mathcal{D}(e^{2\alpha P})$ imply that $w_2 \in \mathcal{H}(0, -2\alpha; \varepsilon)$ and $w_2 \in \mathcal{H}(0, -2\alpha; \varepsilon)$. Obviously, the fact that $w_2 \in \mathcal{H}(0, -2\alpha; \varepsilon)$ leads to $w_2 \in \mathcal{H}(2\alpha, 0; \varepsilon)$. Since $w_2$ is holomorphic on the union $\mathcal{J}(2\alpha, 0) \cup \mathcal{J}(0, 2\alpha)$ and has boundary values of modulus one on the real axis, it follows from Morera’s theorem that $w_2$ is holomorphic on the strip $\mathcal{J}(2\alpha, -2\alpha)$. Having this it is clear that $w_2 \in \mathcal{H}(2\alpha, -2\alpha; \varepsilon)$. Applying the second equality of (26) to the vector $\eta_\gamma$ and using (11), we get

$$f(x-\alpha i) w_2 (x-2\alpha i) \eta_\gamma(x-2\alpha i) = w_1(x) \eta_\gamma(x-2\alpha i).$$

Since $\eta_\gamma(x-2\alpha i) \neq 0$ on $\mathbb{R}$, this gives equation (4).

Next we prove formula (1) and the fact that $w_1 \in \mathcal{H}(2\alpha, 0; \varepsilon)$. Combining Lemma 3(iii) and the second equality of (23) we get

$$e^{2\alpha P} \overline{f}(x+\alpha i) \subseteq R_f = L_f^* = w_2 e^{2\alpha P} \overline{w_1}. \quad (27)$$

Note that $\overline{f} \in \mathcal{H}(\alpha, -\alpha; \varepsilon)$ because of the assumption $f \in \mathcal{H}(\alpha, -\alpha; \varepsilon)$. Therefore, $\eta_\gamma$ is in the domain of the operator $e^{2\alpha P} \overline{f}(x+\alpha i)$ and hence $w_1 \eta_\gamma \in \mathcal{D}(e^{2\alpha P})$ by (27) for any $\gamma > \varepsilon$. From the latter fact we conclude that $\overline{w_1} \in \mathcal{H}(0, -2\alpha; \varepsilon)$ and so $w_1 \in \mathcal{H}(2\alpha, 0; \varepsilon)$. Applying (27) to the vector $\eta_\gamma$, we obtain

$$\overline{f}(x-\alpha i) \eta_\gamma(x-2\alpha i) = w_2(x) \overline{w_1}(x-2\alpha i) \eta_\gamma(x-2\alpha i).$$

Since $|w_1(x)| = 1$ a.e. on $\mathbb{R}$ and hence $w_1(z) = \overline{w_1(z)}^{-1}$ for $z \in \mathcal{J}(0, -2\alpha)$, the preceding equation implies (3).

Finally, we prove the uniqueness assertion of Theorem 1. Suppose that $\tilde{w}_1$ and $\tilde{w}_2$ are two other functions having the properties of $w_1$ and $w_2$, respectively, stated Theorem 1. The crucial step of this part of the proof is to show that then

$$\tilde{w}_1 e^{2\alpha P} \subseteq L_f \tilde{w}_2 \quad \text{and} \quad \tilde{w}_2 e^{2\alpha P} \subseteq L_f^* \tilde{w}_1. \quad (28)$$

Let $\eta \in \mathcal{D}_\varepsilon$. Since $\tilde{w}_2 \in \mathcal{H}(0, -2\alpha; \varepsilon)$ by assumption, we have $\tilde{w}_2 \eta \in \mathcal{D}(e^{2\alpha P})$ by Lemma 1 and $(e^{2\alpha P} \tilde{w}_2 \eta)(x) = \tilde{w}_2(x-\alpha \varepsilon) \eta(x-2\alpha i)$ by (11). From the relation $\tilde{w}_1(x) =
Because \( \tilde{w}_2(z+2\alpha) \), \( f(z+\alpha) \) and \( \tilde{w}_1(z) \) are holomorphic on the strip \( J(0,-2\alpha) \) by assumption, it follows from (3) that

\[
\tilde{w}_2(x+2\alpha) = f(x+\alpha)\tilde{w}_1(x) \quad \text{a.e. on } \mathbb{R}.
\]  

(29)

Since \( \tilde{w}_2 \in \mathcal{H}(2\alpha,0;\varepsilon) \), \( \tilde{w}_2\eta_i \in \mathcal{D}(e^{\alpha P}) \) for \( \eta_i \in \mathcal{D}_\varepsilon \). Therefore, using formulas (14) and (24) and the fact \( L_f^* = R_f \) we obtain

\[
\tilde{w}_2(x)e^{\alpha P} \eta = e^{\alpha P}\tilde{w}_2(x+2\alpha)\eta = e^{\alpha P}f(x+\alpha)\tilde{w}_1(x)\eta = R_f\tilde{w}_1\eta = L_f^*\tilde{w}_1\eta.
\]

Arguing as in the preceding paragraph, the latter implies that \( \tilde{w}_2 e^{\alpha P} \subseteq L_f^*\tilde{w}_1 \) which proves the second inclusion of (28).

Let \( \tilde{W} \) denote the 2x2 diagonal matrix which has \( \tilde{w}_1 \) and \( \tilde{w}_2 \) in the main diagonals. Note that (28) is nothing but relations (17) and (18) with \( w_3 = w_4 = 0 \) and \( w_j \) replaced by \( \tilde{w}_j, j = 1, 2 \). As noted after formula (20), relations (28) are equivalent to the inclusion \( \tilde{W}B \subseteq A_f\tilde{W} \) and so to \( \tilde{W}B\tilde{W}^* \subseteq A_f \), where the matrix \( B \) has been defined by (16). Since \( \tilde{W}B\tilde{W}^* \) and \( A_f \) are both self-adjoint operators, the latter inclusion implies that \( \tilde{W}B\tilde{W}^* = A_f \). On the other hand, by Remark 2 the diagonal matrix \( W \) with diagonal entries \( w_1 \) and \( w_2 \) also satisfies the relation \( WBW^* = A_f \). Consequently, we have \( VBV^* = B \), where \( V = \tilde{W}W^{-1} \). Set \( v_j := \tilde{w}_jw_j \) for \( j = 1, 2 \). Then we obtain \( VB^2V^* = B^2 \) which means that \( v_j e^{i\alpha P}v_j = e^{i\alpha P}, j = 1, 2 \). From this we conclude that \( v_j e^{i\alpha P}v_j = e^{i\alpha P}, t \in \mathbb{R} \). Thus the function \( v_j(x) \) commutes with the translation group on the real line. Therefore, \( v_j(x) \) is constant a.e. on \( \mathbb{R} \), say \( v_j(x) = \gamma_j \). Then, \( \tilde{w}_j = \gamma_j w_j \) for \( j = 1, 2 \). Inserting this into relation (3), applied to \( \tilde{w}_j \) and to \( w_j \), and remembering that \( f \neq 0 \) we obtain \( \gamma_1 = \gamma_2 =: c \). This completes the proof of Theorem 1.

Next let us turn to the proof of Theorem 2. Since \( w_1 \in \mathcal{H}(2\alpha,0;\varepsilon) \) and \( w_2 \in \mathcal{H}(2\alpha,-2\alpha;\varepsilon) \), the functions \( u_f(z) \) and \( g_f(z) \) are indeed holomorphic on the strips \( J(2\alpha,0) \) and \( J(\alpha,-\alpha) \), respectively. The relation \( L_f^* = R_f \) has been proved by Lemma 3(iii).

We verify that \( |L_f| = L_{g_f} \). As already noted above, we have \( L_f = f(x-\alpha)i e^{2\alpha P} \) by Lemma 3(i). The first relation of (23) implies that \( \tilde{w}_1 L_f = e^{2\alpha P} \tilde{w}_2 \). Using the first equality of (24) and the preceding facts we conclude that

\[
|L_f| = w_2 e^{2\alpha P} \tilde{w}_2 = w_2 \tilde{w}_1 f(x-\alpha) e^{2\alpha P}.
\]  

(30)

On the other hand, since \( \tilde{w}_1 f(x-\alpha)w_2(x-2\alpha) = 1 \) a.e. on \( \mathbb{R} \) by (1) and \( \tilde{w}_2(x-2\alpha) = w_2(x-2\alpha)^{-1} \), we obtain

\[
w_2(x)\tilde{w}_1(x) f(x-\alpha) = w_2(x) \tilde{w}_2(x-2\alpha) = g_f(x-\alpha).
\]

Inserting this into (31) it follows that \( |L_f| = L_{g_f} \). By (23) and the definition of the function \( u_f \) we have \( L_f = \tilde{w}_1 w_2 L_f = u_f L_f \). Therefore, the relation \( L_f = u_f L_{g_f} \) must be the polar decomposition of the closed operator \( L_f \). This finishes the proof of Theorem 2.
4 The Example \( f(z) = z \)

In this brief section we treat the simplest case \( f(z) = z \) and express the corresponding functions \( w_1 \) and \( w_2 \) in terms of the Weierstraß Delta function (see, for instance, [4])

\[
\Delta(z) = z e^{cz} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}.
\]

Here \( c = \lim_{n \to \infty} (1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log(n+1)) \) denotes the Euler-Mascheroni constant. It is well-known that \( \Delta(z) \) is an entire function which satisfies the functional equation

\[
\Delta(z) = z \Delta(z+1), \quad z \in \mathbb{C}. \quad (31)
\]

Let us abbreviate

\[
\beta := - \frac{i}{4\alpha}, \quad \gamma = - \frac{1}{2\alpha} \log 4\alpha \quad (32)
\]

and define two meromorphic functions \( w_1 \) and \( w_2 \) by

\[
w_1 = e^{i\gamma z} \frac{\Delta(\beta z + \frac{1}{4})}{\Delta(-\beta z + \frac{1}{4})}, \quad w_2 = i e^{i\gamma z} \frac{\Delta(\beta z + \frac{3}{4})}{\Delta(-\beta z + \frac{3}{4})}. \quad (33)
\]

From the definition of the Delta function it is clear that \( |w_1(x)| = |w_2(x)| = 1 \) for real \( x \). Further, one easily verifies that \( w_1 \in \mathcal{H}(2\alpha, 0; \varepsilon) \) and \( w_2 \in \mathcal{H}(2\alpha, -2\alpha; \varepsilon) \) for large \( \varepsilon \). Using formulas (32) and (33) we conclude that

\[
\frac{w_1(z)}{w_2(z-2\alpha i)} = \frac{1}{i e^{2\alpha \gamma}} \frac{\Delta(\beta z + \frac{1}{4})}{\Delta(-\beta z + \frac{1}{4})} \frac{\Delta(-\beta z - 2\alpha i + \frac{1}{4})}{\Delta(\beta z - 2\alpha i + \frac{3}{4})} = -4\alpha i \frac{\Delta(\beta z + \frac{1}{4})}{\Delta(\beta z + \frac{3}{4})} = -4\alpha i \left(\beta z + \frac{1}{4}\right) = z - \alpha i,
\]

\[
\frac{w_2(z)}{w_1(z-2\alpha i)} = \frac{i}{e^{2\alpha \gamma}} \frac{\Delta(\beta z + \frac{3}{4})}{\Delta(-\beta z + \frac{3}{4})} \frac{\Delta(-\beta z - 2\alpha i + \frac{1}{4})}{\Delta(\beta z - 2\alpha i + \frac{3}{4})} = 4\alpha i \frac{\Delta(-\beta z - \frac{1}{4})}{\Delta(-\beta z + \frac{1}{4})} = 4\alpha i \left(-\beta z - \frac{1}{4}\right) = z - \alpha i.
\]

Therefore, by the uniqueness assertion of Theorem 1, \( w_1(z) \) and \( w_2(z) \) are the functions \( w_1 \) and \( w_2 \) for the holomorphic function \( f(z) = z \). In particular we see that \( w_1(z) \) has zeros at \((4n+1)\alpha i, n \in \mathbb{N}_0\), and poles at \(-(4n+1)\alpha i, n \in \mathbb{N}_0\), while \( w_2(z) \) has zeros at \((4n+3)\alpha i, n \in \mathbb{N}_0\), and poles at \(-(4n+3)\alpha i, n \in \mathbb{N}_0\). All these zeros and poles are simple and there are no other zeros and poles of \( w_1 \) and \( w_2 \). Inserting \( w_1 \) and \( w_2 \) into (8) we obtain explicit expressions for the components \( u_f \) and \( g_f \) of the polar decomposition of \( f(z) = z \). We omit the details.
5 An Operator Representation of a $q$-Deformed Heisenberg Algebra

As a by product of the preceding considerations we obtain an interesting operator representation of the $q$-deformed Heisenberg algebra introduced in [1]. First let us recall the definition of this algebra.

For a positive real number $q \neq 1$, let $\mathcal{A}(q)$ be the complex unital algebra with generators $p, x, u, u^{-1}$ and defining relations

\[ uu^{-1} = q^{-1}u, \quad uu^{-1} = u^{-1}u = 1, \quad px - qp = i_q^{1/2}(q - q^{-1})u^{-1}. \tag{35} \]

If we replace (35) by the relations

\[ px = i_q^{1/2}u^{-1} - i_q^{-1/2}u, \quad xp = i_q^{-1/2}u^{-1} - i_q^{1/2}u, \tag{36} \]

then we obtain an equivalent set of defining relations (34) and (36). The algebra $\mathcal{A}(q)$ becomes a $*$-algebra with respect to the involution determined on the generators by

\[ p = p^*, \quad x = x^*, \quad u^* = u^{-1}. \]

Let us write $q = e^{2\gamma}$ with $\gamma$ real and fix two real numbers $\alpha$ and $\beta$ such that $\alpha \beta = \gamma$. In order to be in accordance with the preceding sections, one may assume that $\alpha > 0$ as well.

Representations of the $*$-algebra $\mathcal{A}(q)$ by Hilbert space operators have been constructed and studied in [1] and [7]. We now give an operator representation of $\mathcal{A}(q)$ in terms of the "usual" position and momentum operators $P = -i\frac{d}{dx}$ and $Q = x$ on $L^2(\mathbb{R})$ such that $p$ and $u$ are both represented by self-adjoint operators. On the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, we define three operators $\rho(u), \rho(p), \rho(x)$ by the operator matrices

\[ \rho(u) = \begin{pmatrix} e^{i\beta x} & 0 \\ 0 & e^{i\beta x} \end{pmatrix}, \quad \rho(p) = \begin{pmatrix} 0 & e^{-2\alpha P} \\ e^{-2\alpha P} & 0 \end{pmatrix}, \]

\[ \rho(x) = \begin{pmatrix} 0 & 2 \sin \beta(x+\alpha i)e^{2\alpha P} \\ e^{2\alpha P}2 \sin \beta(x + \alpha i) & 0 \end{pmatrix}. \]

Obviously, $\rho(u)$ is a unitary and $\rho(p)$ is an unbounded self-adjoint operator. Since $\inf \{ |\sin \beta(x-\alpha i)|; \ x \in \mathbb{R} \} > 0$, the operator $2\sin \beta(x-\alpha i)e^{2\alpha P}$ is closed by Lemma 3(i) and so $2\sin \beta(x-\alpha i)e^{2\alpha P} = L_{2\sin \beta x}$. Further, since the multiplication operator by the function $2\sin \beta(x-\alpha i)$ is bounded, we have $(L_{2\sin \beta x})^* = e^{2\alpha P}2 \sin \beta(x+\alpha i)$. (Note that because of the boundedness of the function $2\sin \beta(x-\alpha i)$ we do not need the full strength of Lemma 3(iii) here.) Therefore, $\rho(x)$ coincides with the operator $A_{2\sin \beta x}$ defined by (12). In particular we conclude in this manner that $\rho(x)$ is a self-adjoint operator.

Next we check that the operators $\rho(u), \rho(p)$ and $\rho(x)$ satisfy the relations (34) and (36). Let $\eta \in \mathcal{D}(e^{-2\alpha P})$. Then, obviously $e^{2\alpha P}\eta \in \mathcal{D}(e^{-2\alpha P})$. From the characterization of the domain $\mathcal{D}(e^{-2\alpha P})$ given by Lemma 1 (more precisely, from the corresponding assertion
for $\alpha < 0$ it follows that $2 \sin \beta (x-\alpha i)e^{2\alpha P} \eta$ is also in $\mathcal{D}(e^{-2\alpha P})$. By (11), we have
\[ e^{-2\alpha P} 2 \sin \beta (x-\alpha i)e^{2\alpha P} = 2 \sin \beta (x+\alpha i) \eta. \]
Thus we shown that
\[ e^{-2\alpha P} 2 \sin \beta (x-\alpha i)e^{2\alpha P} \subseteq 2 \sin \beta (x+\alpha i). \]
Using this equation and the fact that $q^{\pm 1/2} = q^{\pm \alpha \beta}$ we easily get
\[ \rho(p) \rho(x) \subseteq iq^{1/2} \rho(u)^{-1} - iq^{-1/2} \rho(u). \]
In a similar manner we conclude that
\[ \rho(x) \rho(p) \subseteq iq^{-1/2} \rho(u)^{-1} - iq^{1/2} \rho(u), \]
\[ \rho(u) \rho(p) \rho(u)^{-1} = q \rho(p), \quad \rho(u) \rho(x) \rho(u)^{-1} = q^{-1} \rho(x) \]
That is, the operators $\rho(u), \rho(p)$ and $\rho(x)$ fulfill the defining relations (34) and (36) of the algebra $\mathcal{A}(q)$. There is also an invariant dense core for these operators; for instance, one may take the domain $\mathcal{D}_0 \oplus \mathcal{D}_0$, where $\mathcal{D}_0$ is defined by (10). Thus, we get indeed a $*$-representation [8] of the $*$-algebra $\mathcal{A}(q)$ on the invariant dense domain $\mathcal{D}_0 \oplus \mathcal{D}_0$.

Now let $w_1$ and $w_2$ be the functions from Theorem 1 in the case $f(z) = 2 \sin \beta z$. Let $W$ and $V$ denote unitary diagonal operator on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ defined by $W(\eta_1, \eta_2) = (w_1 \eta_1, w_2 \eta_2)$ and $V(\eta_1, \eta_2) = (V-\eta_1, V-\eta_2)$ for $\eta_1, \eta_2 \in L^2(\mathbb{R})$, where $(V-\eta_j)(x) := \eta_j(-x), j = 1, 2$. From formula (25) (or equivalently (10)) in the proof of Theorem 1 we then obtain
\[ (WV)^* \rho(x) WV = V^*W^*A_{2\sin \beta z}WV = V^*W^* \begin{pmatrix} 0 & L_{2\sin \beta z} \\ (L_{2\sin \beta z})^* & 0 \end{pmatrix} WV \]
\[ V^* \begin{pmatrix} 0 & e^{2\alpha P} \\ e^{-2\alpha P} & 0 \end{pmatrix} V = \begin{pmatrix} 0 & e^{-2\alpha P} \\ e^{2\alpha P} & 0 \end{pmatrix} = \rho(p). \]
That is, the self-adjoint operators $\rho(x)$ and $\rho(p)$ are unitarily equivalent via the unitary operator $WV$. Thus, if we think of $\rho(x)$ and $\rho(p)$ as $q$-analogues of the position and momentum operators, then the unitary $WV$ takes the role of the Fourier transform in these considerations. Being the main ingredients of the unitary operator $WV$ the two holomorphic functions $w_1$ and $w_2$ from Theorem 1 are crucial in this context.

References

[1] Hebecker, A., Schreckenberg, S., Schwenk, J., Weich, W. and Wess, J., Representations of a $q$-deformed Heisenberg algebra. Z. Phys. C 64(1994), 335–359.
[2] Katznelson, V., An Introduction to Harmonic Analysis. Dover Publ., New York, 1976.
[3] Putnam, C.R., Commutation Properties of Hilbert Space Operators. Springer–Verlag, Berlin, 1967.
[4] Remmert, R., Funktionentheorie II. Springer-Verlag, Berlin, 1991.
[5] Schmüdgen, K., Integrable operator representations of $\mathbb{R}_q^2$, $X_{\gamma}^q$, and $SL_q(2, \mathbb{R})$. Commun. Math. Phys. 159 (1994), 217–237.

[6] Schmüdgen, K., On operator-theoretic approach to a cocycle problem in the complex plane. Bull. London Math. Soc. 27 (1995), 341–346.

[7] Schmüdgen, K., Operator representations of a $q$-deformed Heisenberg algebra. J. Math. Phys., to appear.

[8] Schmüdgen, K., *Unbounded Operator Algebras and Representation Theory*. Birkhäuser-Verlag, Basel, 1990.

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