Klein-Gordon oscillator in a cosmic spacetime with a cosmic space-like dislocation: Conditional solvability and position-dependent mass settings

Omar Mustafa

Department of Physics, Eastern Mediterranean University, G. Magusa, north Cyprus, Mersin 10 - Turkey.

Abstract: We reformulate, recycle, and discuss Klein-Gordon (KG) oscillators in a cosmic spacetime with a space-like dislocation. We emphasize that the notion of "KG-oscillator" is a reflection of its exact resemblance with the Schrödinger oscillator, and, therefore, should not copy the parametric characterizations of the Schrödinger one. We also consider a confined (in a Cornell-type potential) KG-oscillator and report/discuss the torsion (space-like dislocation) effect on its "conditionally" exact energy levels. We observe shifts/dislocations of the energy levels along the torsion’s parameter, which manifestly yield energy levels crossings (i.e., occasional degeneracies). To find out parallel systems that admit invariance and isospectrality with the confined KG-oscillator, we discuss the KG-oscillator in a pseudo-cosmic spacetime with a cosmic space-like dislocation. Moreover, we argue that analogous to textbook treatment that the momentum operator \( \hat{p}_j = -i \partial_j \) (for constant mass particles) copies itself into the relativistic wave equations, so should be the case with the position-dependent mass (PDM) momentum operator, \( \hat{p}_j (r) = -i \left( \partial_j - \frac{\partial_{m(r)}}{m(r)} \right) ; j = 1, 2, 3 \). We use such a new recipe to describe PDM KG-oscillator (PDM KG-particles in general) and consider two PDM illustrative examples, a power-law type PDM, and an exponentially growing PDM. For the exponentially growing PDM, we show that such a PDM introduces a Cornell-type confinement as its own byproduct. We observe clustering trends of the energy levels.

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I. INTRODUCTION

In the early universe and as a consequence of vacuum symmetry breaking phase transitions, the grand unified theories have predicted possible topological defects [1-4] that have been investigated in many areas of physics. For example, in condensed matter physics [5], in gravitation [6,8] (where the linear defects are due dislocation (torsion) and curvature (disclinations)), in domain wall [2,3], in cosmic string [8,10], in global monopole [11], etc. However, in their work on Volterra distortions and cosmic defects, Puntigam and Soleng [6] have generalized the Volterra distortion to (3+1)-dimensions using differential geometric and gauge theoretical methods and introduced the concept of Volterra defected/distorted spacetime. They have constructed solutions of the Einstein-Cartan field equation that match with the Volterra defected spacetime, where the resulting matter distributions are interpreted as cosmic strings and cosmic dislocations. Hereby, distortions are line-like defects characterized by a delta-function-valued curvature (classified as

*Electronic address: omar.mustafa@emu.edu.tr*
disclination) and torsion (classified as dislocation) distributions that result in rotational and translational holonomy. Dislocation may be in the form of a spiral-type or a screw-type . In the current study, we investigate the dislocation (torsion) effects on some effective position-dependent mass (PDM) Klein-Gordon (KG) oscillators in a cosmic spacetime with a cosmic space-like dislocation.

The KG-oscillator has been a subject of research studies in recent years ever since the introduction of the Dirac oscillator by Moshinsky and Szczepaniak . It has been studied in the Gödel and Gödel-type spacetime background (e.g., ), in cosmic string spacetime and Kaluza-Klein theory backgrounds (e.g., ), in Som-Raychaudhuri backgrounds (e.g., ), in the (2+1)-dimensional Gödel spacetime backgrounds (e.g., ).

The concept of position-dependent effective mass (PDM in short) settings of Mathews-Lakshmanan oscillator , on the other hand, has sparked research interest on PDM in both classical and quantum mechanics . Such a PDM concept is, in fact, a metaphoric manifestation of coordinate deformation/transformation . The coordinate transformation, in effect, changes the form of the canonical momentum in classical and the momentum operator in quantum mechanics (e.g., , and related references therein). In classical mechanics, for example, negative the gradient of the potential force field is no longer the time derivative of the canonical momentum operator in quantum mechanics (e.g., , and related references therein).

Nevertheless, attempts were made to include PDM settings in the Dirac and KG relativistic equations (e.g., ) through the assumption that \( m \rightarrow m + m(r) + S(r) \), where \( m \) now denotes the rest mass energy, \( S(r) \) is the Lorentz scalar potential (i.e., transforms like the rest mass energy), and \( m(r) \) is effectively an amendment on the Lorentz scalar potential \( S(r) \) (but not a dimensionless scalar multiplier as that in ). Hereby, it should be noted that in relativistic quantum mechanics the Lorentz scalar potential \( \hat{S}(r) \) manifestly introduces nothings but a trivial amendment to the Lorentz scalar potential \( \hat{S}(r) \), and not PDM settings as in the non-relativistic PDM Schrödinger equation. Therefore, in the current study, we do not follow this practice. We shall rather argue that...
analogous to textbook recipe, where the constant mass momentum operator $p_j = -i\partial_j$ is used in the relativistic wave equations, the PDM-momentum operator (1) should be used to describe effective PDM-relativistic quantum particles (e.g., [35, 39, 40, 44]). In the most simplistic language, for effectively PDM particles (non-relativistic/relativistic) the PDM-momentum operator (1) should replace the constant mass textbook momentum operator $p_j = -i\partial_j$.

The organization of the current methodical proposal is in order. We start, in section 2, with a confined (in a Cornell-type potential, commonly used in heavy quarkonium spectroscopy [57, 58]) KG-oscillator in a cosmic spacetime with space-like dislocation and report/discuss the dislocation (torsion) effect on the conditionally exact energy levels. Therein, we argue that the notion of "KG-oscillator" is a metaphoric manifestation of the exact mathematical resemblance to the Schrödinger harmonic oscillator. It should never, therefore, copy the parametric characterizations of the Schrödinger harmonic oscillator, but rather it would mathematically inherit its eigenvalues and eigenfunctions. In the same section, we found that the space-like dislocation results in energy levels shifts along the dislocation parameter axis, and consequently energy levels crossings are unavoidable in the process. Energy levels crossings, nevertheless, is a phenomenon responsible for electron transfer in protein, it underlies stability analysis in mechanical engineering, and appears in algebraic geometry (e.g., [59] and references cited therein). Moreover, clusterings of energy levels are found feasible for $|\delta| >> 1$, where $\delta$ denotes dislocation (torsion) parameter. To find out systems that admit invariance and isospectrality with the confined KG-oscillator, we discuss (in section 3) the KG-oscillator in a pseudo-cosmic spacetime with space-like dislocation. Such invariant and isospectral systems are found to copy the same effects as above. Moreover, we suggest (in section 4) an alternative PDM setting for the KG-particles (relativistic particles in general). Wherein, we use the PDM-momentum operator (1) constructed by Mustafa and Algadhi [39] and discuss the effects of torsion and PDM settings on the KG-oscillator in a cosmic spacetime with space-like dislocation. Two PDM-KG models are used as illustrative examples. A power-law type PDM is found to have similar trend of behavior as that for the confined KG-oscillator of section 2. However, for an exponentially growing PDM, we found that the KG-oscillator is confined in its own PDM-byproduced Cornell-type confinement. Yet, an obvious energy levels clustering is observed as the PDM parameter, $\xi \geq 0$, grows up from zero, but no energy levels crossings are found feasible for a fixed value of the torsion parameter $\delta$. However, the effect of the torsion parameter $\delta$ on the energy levels, for a fixed PDM parameter $\xi$, is found to maintain the same trend of behavior as that in section 2. Our concluding remarks are given in section 5.

II. CONFINED KG-OSCILLATOR IN A COSMIC SPACETIME WITH SPACE-LIKE DISLOCATION: REFORMULATED AND RECYCLED

In this section, we consider a cosmic spacetime with space-like dislocation (i.e., a Volterra-type spacetime with space-like dislocation [6, 60, 61] (in $\hbar = c = 1$ units) described by the line element

$$ds^2 = -dt^2 + dr^2 + r^2d\varphi^2 + (dz + \delta d\varphi)^2,$$

(3)
where $\delta$ denotes space-like dislocation parameter (i.e., torsion parameter). The covariant and contravariant metric tensors in this case, respectively, read

$$
g_{\mu \nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & (r^2 + \delta^2) & \delta \\
0 & 0 & \delta & 1
\end{pmatrix} \quad \iff \quad g^{\mu \nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{\delta} & -\frac{\delta}{r^2} \\
0 & 0 & -\frac{\delta}{r^2} & (1 + \frac{\delta^2}{r^2})
\end{pmatrix}; \quad \det (g) = -r^2. \quad (4)
$$

On the other hand, the Klein-Gordon equation (KG), with a Lorentz scalar potential $S(r)$ (i.e., $m \longrightarrow m + S(r)$) \[54, 55\], is given by

$$
\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \sqrt{-g} g^{\mu \nu} \partial_{\nu} \Psi \right) = (m + S(r))^2 \Psi. \tag{5}
$$

Moreover, we may now use Mirza et al.’s recipe \[62\] or it generalized form by \[20, 58\] for the KG-oscillator and consider

$$
p_{\mu} \longrightarrow p_{\mu} + i \eta \chi_{\mu}, \tag{6}
$$

where $\chi_{\mu} = (0, r, 0, 0)$. This would, in effect, transform KG-equation \[5\] into

$$
\frac{1}{\sqrt{-g}} \partial_{\mu} \left( \partial_{\mu} r^2 \right) = \frac{1}{r} \partial_{\mu} \partial_{\nu} - \frac{1}{r^2} \partial_{\nu} r^2 - \eta^2 r^2 - 2 \eta - (m + S(r))^2 \Psi, \tag{7}
$$

and yields

$$
\left\{ -\partial^2_t + \left( \partial^2_{r} + \frac{1}{r} \partial_{r} \right) + \frac{1}{r^2} \partial^2_{\varphi} + \left( 1 + \frac{\delta^2}{r^2} \right) \partial^2_z - \frac{2 \delta}{r^2} \partial_{\varphi} \partial_{z} - \eta^2 r^2 - 2 \eta - (m + S(r))^2 \right\} \Psi = 0. \tag{8}
$$

A substitution in the form of

$$
\Psi (t, r, \varphi, z) = \exp \left( i \left[ \ell \varphi + k_z z - Et \right] \right) \psi (r) = \exp \left( i \left[ \ell \varphi + k_z z - Et \right] \right) \frac{R (r)}{\sqrt{r}}, \tag{9}
$$

would result in

$$
R'' (r) + \left[ \lambda - \left( \frac{\ell^2 - 1}{r^2} \right) - \eta^2 r^2 - 2 m S (r) - S (r)^2 \right] R (r) = 0, \tag{10}
$$

where

$$
\lambda = E^2 - k_z^2 - 2 \eta - m^2; \quad \ell^2 = (\ell - k_z \delta)^2. \tag{11}
$$

Obviously, equation \[10\] resembles, with $S(r) = 0$, the 2-dimensional radial harmonic oscillator (and should not be confused with the Schrödinger harmonic oscillator, e.g., \[63\]) with an effective irrational oscillation frequency $\eta$, say, (i.e., positive and negative values for $\eta$ are allowed unlike the frequency of the Schrödinger harmonic oscillator) and consequently mathematically inherits its textbook eigenvalues

$$
\lambda = 2 \eta \left( 2 n_r + |\ell| + 1 \right) \iff E^2 = 2 \eta \left( 2 n_r + |\ell - k_z \delta| + 2 \right) + k_z^2 + m^2 \tag{12}
$$

and radial eigenfunctions

$$
\psi (r) \sim r^{|\ell - k_z \delta|} \exp \left( -\frac{\eta r^2}{2} \right) L_{n_r}^{(|\ell - k_z \delta|)} (\eta r^2) \iff \psi (r) \sim r^{|\ell - k_z \delta|} \exp \left( -\frac{\eta r^2}{2} \right) L_{n_r}^{(|\ell - k_z \delta|)} (\eta r^2), \tag{13}
$$
FIG. 1: We plot the energy levels of (19) versus the torsion parameter $\delta$, for $m = k_z = \eta = 1$, $a = b = 2$ and for (a) $\ell = 0$, $n_r = 0, 1, 2, 3$, (b) $n_r = 1$, $\ell = 0, \pm 1, \pm 2$, and (c) $n_r = 3$, $\ell = 0, \pm 3, \pm 5$.

where $L_{n_r}^{\ell + k_z \delta}(\eta r^2)$ are the associated Laguerre polynomial.

Let us now consider the above KG-oscillator confined in a Cornell type potential

$$S(r) = ar + \frac{b}{r},$$

(14)

to imply that equation (10) reads

$$R''(r) + \left[\tilde{\lambda} - \frac{(\tilde{\gamma}^2 - 1/4)}{r^2} - \tilde{\omega}^2 r^2 - 2ma + \frac{2mb}{r}\right] R(r) = 0,$$

(15)

where

$$\tilde{\lambda} = E^2 - k_z^2 - 2\eta - m^2 - 2ab; \quad \tilde{\gamma}^2 = (\ell - k_z \delta)^2 + b^2; \quad \tilde{\omega}^2 = \eta^2 + a^2.$$

(16)

Equation (15) admits a finite/bounded solution in the form of biconfluent Heun polynomials in the form of

$$\psi(r) = \frac{R(r)}{\sqrt{r}} \sim r^{\vert \tilde{\gamma} \vert} \exp \left(-\frac{\tilde{\omega}^2 r^2 + 2amr}{2\tilde{\omega}}\right) H_B \left(2\vert \tilde{\gamma} \vert, \frac{2ma}{\tilde{\omega}^{3/2}}, \frac{a^2 m^2 + \tilde{\lambda} \tilde{\omega}^2}{\tilde{\omega}^3}, \sqrt{\frac{4mb}{\tilde{\omega}}}, \sqrt{\omega r}\right),$$

(17)

where $H_B(\alpha,\beta,\gamma,\delta,r)$ is the biconfluent Heun polynomial of degree $2n_r \geq 0$. This would immediately suggest that

$$\gamma = 2(2n_r + 1) + \alpha \quad \text{(this choice is manifested by the fact that when } a = b = 0 \text{ the energies in (12) should naturally be recovered, see e.g., [30] for more details on this issue)},$$

hence

$$\frac{a^2 m^2 + \tilde{\lambda} \tilde{\omega}^2}{\tilde{\omega}^3} = 2(2n_r + \vert \tilde{\gamma} \vert + 1) \quad \iff \quad \tilde{\lambda} = 2\tilde{\omega}(2n_r + \vert \tilde{\gamma} \vert + 1) - \frac{m^2 a^2}{\tilde{\omega}^2}.$$  

(18)

In this case, we get the relation for the energy eigenvalues as

$$E^2 = 2 \left(\sqrt{\eta^2 + a^2}\right) \left(2n_r + \left(\sqrt{(\ell - k_z \delta)^2 + b^2} + 1\right) - \frac{m^2 a^2}{\eta^2 + a^2} + 2\eta + k_z^2 + m^2 + 2ab.\right.$$  

(19)

At this point, one should be aware that this result, along with that in (17), belong to the set of the so called conditionally exactly solvable problems (the comprehensive analysis of which is reported by Fernandez [63]). Moreover, it is obvious that for the case when $\tilde{\omega} = 0 = a$, the biconfluent Heun polynomial (17) fails to provide any information
FIG. 2: We plot the energy levels of (19) versus the torsion parameter $\delta$, without the Cornell confinement, for $m = k_z = \eta = 1, a = b = 0$ and for (a) $\ell = 0, n_r = 0, 1, 2, 3$, (b) $\ell = 10, n_r = 0, 1, 2, 3$, and (c) $\ell = -10, n_r = 0, 1, 2, 3$.

on the spectrum and/or the radial wave functions. Yet, instead of collapsing into the spectrum of the KG-Coulombic problem, the reported spectrum (19) collapses into the free relativistic particle energies $E^2 = m^2 + k_z^2$. This is a clear drawback of the biconfluent Heun polynomial (17). Hence, for the PDM-KG Coulombic particle one has to restart again from the differential equation (15) to get its exact solution. Nevertheless, we continue with the available conditionally exact solution (17) and (19).

In Figures 1 and 2, we show the effect of the torsion related parameter $\delta$ on the energy levels of a confined KG-oscillator in a cosmic spacetime with cosmic space-like dislocation. We clearly observe that the first term under the square root determines the shifts/dislocations in the energy levels at $\delta = \ell / k_z$, on the $\delta$-axis. That is, for negative $\ell$ values the shifts/dislocations will be in the negative $\delta$ region, whereas for positive $\ell$ values the shifts will be in the positive $\delta$ region. This would, in effect, manifestly yield energy levels crossings (i.e., occasional degeneracies, as shown in figures 1(a), 1(b), and 1(c), with the Cornell confinement). Moreover, in Figures 2(a), 2(b), and 2(c), we observe eminent energy levels clusterings when $|\delta| >> 1$, for each value of the magnetic quantum number $\ell = 0, \pm 1, \pm 2, \cdots$.

The effect of the torsion parameter on the energy levels of the confined KG-oscillator in a cosmic spacetime with cosmic space-like dislocation is clear, therefore.

III. KG-OSCILLATORS IN A PSEUDO-COSMIC SPACETIME WITH SPACE-LIKE DISLOCATION: ISOSPECTRALITY AND INVARIANCE

Let metric (3) that describes a cosmic spacetime with space-like dislocation be transformed/deformed in such a way that

$$\begin{align*}
    ds^2 \longrightarrow d\tilde{s}^2 &= -dt^2 + d\tilde{r}^2 + \tilde{r}^2 d\tilde{\varphi}^2 + (d\tilde{z} + \delta d\tilde{\varphi})^2,
\end{align*}$$

where

$$\begin{align*}
    d\tilde{r} &= \sqrt{m(r)} dr, \quad \tilde{r} = \sqrt{Q(r)} r, \quad d\tilde{\varphi} = d\varphi, \quad d\tilde{z} = dz \quad d\tilde{t} = dt,
\end{align*}$$

and hence

$$\begin{align*}
    \frac{d\tilde{r}}{dr} \Rightarrow \sqrt{m(r)} = \sqrt{Q(r)} \left[ 1 + \frac{Q'(r)}{2Q(r)} \right]^{-1},
\end{align*}$$
to govern the correlation between the scalar multipliers \( m(r) \) and \( Q(r) \). Then the covariant and contravariant metric tensors (with \( f(r) = Q(r) r^2 \) for economy of notations) in this case, respectively, read

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & m(r) & 0 & 0 \\
0 & 0 & (f(r) + \delta^2) & \delta \\
0 & 0 & \delta & 1
\end{pmatrix}
\quad \leftrightarrow \quad
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & {1 \over m(r)} & 0 & 0 \\
0 & 0 & {1 \over f(r)} & -\delta \\
0 & 0 & -\delta f(r) & (1 + f^2(r))
\end{pmatrix} ; \quad \det(g) = -m(r) f(r) .
\quad (23)
\]

We now include the PDM KG-oscillator using the momentum operator \( [61] \) of Mirza et al.’s recipe \( [62] \) and suggest that \( \chi_\mu = \left( 0, \sqrt{m(r)f(r)}, 0 \right) \) to accommodate the new PDM-settings. This would, in effect, transform the KG-oscillator equation \( [17] \) into

\[
\begin{aligned}
& \left[ \frac{\partial_r}{\sqrt{m(r)f(r)}} \left( \sqrt{f(r) \over m(r)} \partial_r \right) \right] U(r) + \left[ \lambda - \frac{\vec{r}^2}{f(r)} - \eta^2 f(r) - 2mS(r) - S(r)^2 \right] U(r) .
\end{aligned}
\quad (24)
\]

Which upon the substitution

\[
\Psi(t, r, \varphi, z) = \exp \left(i \left[ \ell \varphi + k_z z - Et \right] \right) U(r) ,
\quad (25)
\]

yields

\[
\begin{aligned}
& \frac{\partial_r}{\sqrt{m(r)f(r)}} \left( \sqrt{f(r) \over m(r)} \partial_r \right) U(r) + \left[ \lambda - \frac{\vec{r}^2}{f(r)} - \eta^2 f(r) - 2mS(r) - S(r)^2 \right] U(r) .
\end{aligned}
\quad (26)
\]

Obviously, the first term can be rewritten, with \( f(r) = Q(r) r^2 = \vec{r}^2 \) and \( \partial_r = {1 \over \sqrt{m(r)}} \partial_r \), as

\[
\begin{aligned}
& \frac{1}{f(r)} \frac{1}{\sqrt{m(r)}} \partial_r \left( \sqrt{f(r) \over m(r)} \partial_r \right) U(r) = \frac{1}{f(r)} \frac{\partial \partial_r}{\partial \partial_r} \left( \frac{\partial \partial_r}{\partial \partial_r} \right) U(\vec{r}) = \left( \frac{\partial^2}{\partial \vec{r}^2} + \frac{1}{\vec{r}} \frac{\partial}{\partial \vec{r}} \right) U(\vec{r}) .
\end{aligned}
\quad (27)
\]

To remove the first derivative we may define \( U(\vec{r}) = R(\vec{r}) / \sqrt{\vec{r}} \) to eventually imply

\[
\begin{aligned}
& \frac{d^2}{d\vec{r}^2} R(\vec{r}) + \left[ \lambda - V_{eff}(\vec{r}(r)) \right] R(\vec{r}) = 0 ; \quad V_{eff}(\vec{r}) = \left( \frac{\vec{r}^2 - 1/4}{\vec{r}^2} \right) + \eta^2 \vec{r}^2 + 2mS(\vec{r}) + S(\vec{r})^2 .
\end{aligned}
\quad (28)
\]

This would in turn result, with \( R(\vec{r}) = R(\vec{r}(r)) = m(r)^{-1/4} \phi(r) \),

\[
\begin{aligned}
& \left( \frac{1}{\sqrt{m(r)}} \frac{d}{dr} \frac{1}{\sqrt{m(r)}} \frac{d}{dr} \right) m(r)^{-1/4} \phi(r) + \left[ \lambda - V_{eff}(\vec{r}(r)) \right] m(r)^{-1/4} \phi(r) = 0 .
\end{aligned}
\quad (29)
\]

Now multiplying from the left by \( m(r)^{1/4} \) we get

\[
\begin{aligned}
& \left( m(r)^{-1/4} \frac{d}{dr} m(r)^{-1/2} \frac{d}{dr} m(r)^{-1/4} \right) \phi(r) + \left[ \lambda - V_{eff}(r) \right] \phi(r) = 0 ,
\end{aligned}
\quad (30)
\]

where

\[
V_{eff}(r) = \left( \frac{\vec{r}^2 - 1/4}{Q(r) r^2} \right) + \eta^2 Q(r) r^2 + 2mS(\vec{r}) + S(\vec{r})^2 .
\quad (31)
\]

One should be reminded that equation \( [30] \) is known in the literature as the von Roos PDM-Schrödinger equation \( [32] \), with the parametric ordering of Mustafa-Mazharimousavi \( [33] \). Furthermore, it is obvious that for \( S(\vec{r}) = 0 \)
equation (28) is isospectral and invariant with that in (10) and admits the same energies in (12). Yet, for a confining potential in the form of a deformed Cornell type
\[ S(\tilde{r}(r)) = a\sqrt{Q(r)}r + \frac{b}{\sqrt{Q(r)}r}. \] (32)
the KG-oscillator in the transformed/deformed cosmic spacetime with a cosmic space-like dislocation (20) would consequently inherit the eigenvalues (19) and eigenfunctions (17) of (15). The two systems (15) and (28) are invariant and are isospectral, therefore. Moreover, under such PDM transformation settings, our transformed/deformed cosmic spacetime metric \( d\tilde{s}^2 \) in (20) may be classified as a pseudo-cosmic spacetime with space-like dislocation.

IV. PDM KG-OSCILLATOR IN A COSMIC SPACETIME WITH A COSMIC SPACE-LIKE DISLOCATION: AN ALTERNATIVE KG-OSCILLATOR

Above, we have mentioned that Mustafa and Algadhi [39] have shown that an effective PDM-momentum operator is given by (1). In this section, we shall use such PDM-momentum operator to describe PDM KG-oscillators in a cosmic spacetime with a space-like dislocation (3). Therefore, the corresponding inverse metric tensor \( g_{\mu\nu} \) is readily given in (4). Moreover, we shall use the assumption that \( m(r) = m(r) \) (i.e., the PDM function is only radially dependent). Under such settings, the momentum operator in (6), with \( \eta = 0 \) would take the PDM form so that
\[ \tilde{p}_\mu \rightarrow -i\partial_\mu + iF_\mu = \left( 0, \frac{m'(r)}{4m(r)}, 0, 0 \right) \Longrightarrow F_r = \frac{m'(r)}{4m(r)}. \] (33)
is used to construct the PDM KG-oscillator through
\[ \frac{1}{\sqrt{-g}} (\partial_\mu + F_\mu) \left[ \sqrt{-g}g^{\mu\nu} (\partial_\nu - F_\nu) \Psi \right] = (m + S(r))^2 \Psi. \] (34)
Then the PDM KG-oscillator equation (34), with the contravariant metric tensors in (4), would yield
\[ \left[ \frac{1}{r} \partial_r r \partial_r - \partial_\ell^2 + \frac{1}{r^2} \partial_\varphi^2 + \left( 1 + \frac{\delta^2}{r^2} \right) \partial_z^2 - \frac{2\delta}{r^2} \partial_z \partial_\varphi - \frac{F_r}{r} - \frac{F_\varphi}{r} - (m + S(r))^2 \right] \Psi = 0. \] (35)
We may now use \( \Psi(t, r, \varphi, z) \) of (9) to obtain
\[ R''(r) + \left[ \lambda - \frac{(\ell^2 - 1/4)}{r^2} - 2mS(r) - S(r)^2 + M(r) \right] R(r) = 0 \] (36)
where \( \lambda = E^2 - k_z^2 - m^2, \ell^2 = (\ell - k_z \delta)^2 \) as in (11), and
\[ M(r) = \frac{3}{16} \left( \frac{m'(r)}{m(r)} \right)^2 - \frac{1}{4} \left( \frac{m''(r)}{m(r)} \right) - \frac{m'(r)}{4rm(r)}. \] (37)
Obviously, for constant mass settings \( m(r) = 1 \), this equation collapses into that of (11) as should be. Yet, one should notice that when the KG-oscillator irrational frequency \( \eta = 0 \), equation (36) would describe KG-particles in a cosmic spacetime with a cosmic space-like dislocation, in general.

In connection with torsion effect on such PDM KG-oscillators, we provide two illustrative examples: a power-law PDM function \( m(r) = Ar^\sigma \), and an exponentially growing PDM function \( m(r) = B \exp(\xi r) \).
A. Example 1: A power law PDM

A power-law PDM in the form of \( m(r) = Ar^\alpha \) would, through (37), imply
\[
M(r) = -\frac{\sigma^2}{16r^2} - \frac{\sigma}{2}\eta.
\] (38)

Which, in turn, yields
\[
R''(r) + \left[ \mathcal{E} - \frac{(\zeta^2 - 1/4)}{r^2} - \eta^2 r^2 - 2mS(r) - S'(r)^2 \right] R(r) = 0,
\] (39)

where
\[
\mathcal{E} = E^2 - k_z^2 - 2\eta - m^2 - \frac{\sigma}{2}\eta; \quad \zeta^2 = (\ell - k_z \delta)^2 + \frac{\sigma^2}{16}.
\] (40)

With \( S'(r) = 0 \), this equation resembles that of the two-dimensional radial Schrödinger oscillator discussed in section 2 (namely, equations (10), (12), and (13)) and admits eigenvalues
\[
\mathcal{E} = 2\eta(2n_r + |\zeta| + 1) \iff E^2 = 2\eta \left( 2n_r + \left\lvert \ell - k_z \delta \right\rvert^2 + \frac{\sigma^2}{16} + 2 \right) + k_z^2 + m^2 + \frac{\sigma}{2}\eta,
\] (41)

and radial eigenfunctions
\[
R(r) \sim r^{|\zeta|+1/2} \exp \left(-\frac{\eta^2}{2} \right) L_{n_r}^{\zeta} (\eta^2) \iff \psi(r) \sim r^{|\zeta|} \exp \left(-\frac{\eta^2}{2} \right) L_{n_r}^{\zeta} (\eta^2).
\] (42)

Next, we now consider the PDM KG-oscillators confined in the Cornell-type potential of (14). This would, in effect, imply that equation (39) be rewritten as
\[
R''(r) + \left[ \tilde{\mathcal{E}} - \frac{(\tilde{\zeta}^2 - 1/4)}{r^2} - \tilde{\omega}^2 r^2 - 2mar - \frac{2mb}{r} \right] R(r) = 0,
\] (43)

where
\[
\tilde{\mathcal{E}} = \mathcal{E} - 2ab; \quad \tilde{\zeta}^2 = \zeta^2 + b^2; \quad \tilde{\omega}^2 = \eta^2 + a^2.
\] (44)

Under such settings, equation (43) admits a finite/bounded radial solution in the form of biconfluent Heun polynomials as
\[
\psi(r) \sim r^{|\zeta|} \exp \left(-\frac{\tilde{\omega}^2 r^2 + 2mar}{2\tilde{\omega}} \right) H_B \left( 2\left| \zeta \right|, \frac{2ma}{\tilde{\omega}^3}, \frac{\tilde{\omega}^2}{\tilde{\omega}^3}, \frac{4mb}{\sqrt{\tilde{\omega}^3}} \right),
\] (45)

where again \( H_B(\alpha, \beta, \gamma, \delta, r) \) is the biconfluent Heun polynomial of degree \( 2n_r \geq 0 \). This would immediately suggest that \( \gamma = 2(2n_r + 1) + \alpha \) in (45) (this choice is again manifested by the fact that when \( a = b = 0 \) the energies in (12) should naturally be recovered), hence
\[
\frac{a^2m^2 + \tilde{\mathcal{E}} \tilde{\omega}^2}{\tilde{\omega}^3} = 2 \left( 2n_r + \left| \zeta \right| + 1 \right) \iff \tilde{\mathcal{E}} = 2\tilde{\omega} \left( 2n_r + \left| \zeta \right| + 1 \right) - \frac{m^2a^2}{\tilde{\omega}^2}.
\] (46)

In this case, we get the relation for the energy eigenvalues as
\[
E^2 = 2 \left( \sqrt{\eta^2 + a^2} \right) \left( 2n_r + \sqrt{(\ell - k_z \delta)^2 + \frac{\sigma^2}{16} + b^2} + 1 \right) - \frac{m^2a^2}{\eta^2 + a^2} + 2\eta + k_z^2 + m^2 + 2ab + \frac{\sigma}{2}\eta.
\] (47)
We plot the energy levels (52) of the exponentially growing PDM for $m = k_z = \eta = 1$. We show in (a) the effect of the PDM parameter $\xi$ for $n_r = \ell = 0$, (b) the effect of the torsion parameter $\delta$, for $n_r = 2, \xi = 4, \ell = 0, \pm 1, \pm 2$, and (c) the effect of the torsion parameter $\delta$, for $\ell = 2, \xi = 4, n_r = 0, 1, 2, 3, 4$.

Obviously, such energy levels inherit the behavior of those of (19) discussed in section 2. Namely, Figures 1 and 2. That is, one may rewrite this energy equation as

$$E^2 = 2\tilde{\eta} \left(2n_r + \sqrt{(|\ell - k_z\delta|^2 + \tilde{b}^2)} + 1\right) - \frac{m^2a^2}{\eta^2} + \left(2 + \frac{\sigma^2}{2}\right) \eta + k_z^2 + m^2 + 2ab. \quad (48)$$

where $\tilde{\eta} = \sqrt{\eta^2 + a^2}$ and $\tilde{b}^2 = b^2 + \sigma^2/16$, to observe that similar trends of behavior are manifested here.

**B. Example 2: A Cornell-type confinement as a byproduct of an exponentially growing PDM**

An exponentially growing PDM function $m(r) = B \exp(\xi r)$ would yield

$$M(r) = -\frac{\xi^2}{16} - \frac{1}{4} \frac{\xi}{r} - \frac{1}{2} \xi \eta r. \quad (49)$$

Consequently, the PDM KG-oscillator’s equation (36), with $S(r) = 0$, reads

$$R''(r) + \left[\Sigma - \left(\frac{\tilde{E}^2 - 1/4}{r^2}\right) - \eta^2 r^2 - \frac{1}{4} \frac{\xi}{r} - \frac{1}{2} \xi \eta r - 2mS(r) - S(r)^2\right] R(r) = 0, \quad (50)$$

where $\Sigma = E^2 - k_z^2 - 2\eta - m^2 - \xi^2/16$, and $\tilde{E}^2 = (\ell - k_z\delta)^2$. It is clear that a Cornell-type confinement (i.e., $\xi/4r + \xi \eta r/2$) is introduced as a byproduct of the PDM settings at hand. We, therefore, continue with $S(r) = 0$. This equation (50) admits a finite/bounded radial solution in the form of biconfluent Heun polynomials (as reported in the preceding example above) so that

$$\psi(r) \sim r^{\ell/2} \exp\left(-\frac{1}{2} m^2 - \frac{1}{4} \xi r\right) H_B\left(2\left|\ell\right|, \frac{\xi}{2\sqrt{\eta}}, \frac{\xi^2 + 16\Sigma}{16\eta}, \frac{\xi}{2\sqrt{\eta}}\sqrt{\eta} r\right). \quad (51)$$

Hence, the corresponding energy levels are given by

$$\frac{\xi^2 + 16\Sigma}{16\eta} = 2\left(2n_r + \left|\ell\right| + 1\right) \implies E^2 = 2\eta \left(2n_r + |\ell - k_z\delta| + 2\right) + + k_z^2 + m^2 + \frac{1}{4} \eta \xi^2. \quad (52)$$

The energy levels are shown in Figure 3. In Figure 3(a), we show the energy levels against the PDM parameter $\xi \geq 0$ and observe eminent clustering of the energy levels as $\xi$ grows up, but no energy levels crossing are found feasible.
the other hand, the torsion’s parameter $\delta$ effect on the energy levels, for some fixed values of $\xi$, maintains the same trend of behavior as that associated with (19) and discussed in section 2.

V. CONCLUDING REMARKS

In this work, we have studied the KG-oscillator in a cosmic spacetime with a cosmic space-like dislocation (torsion). We have started with a confined (in a Cornell-type Lorentz scalar potential) KG-oscillator and reported/discussed the torsion effect on the conditionally exact energy levels. This effect may very well be summarized as torsion yields shifts/dislocations of the energy levels along the torsion’s parameter $\delta$-axis by $\delta = \ell/k_z$; $\ell = 0, \pm 1, \pm 2, \ldots$ (documented in Figures 1(b), 1(c), 2(b), 2(c), 3(b), and 3(c)). That is, for negative $\ell$ values the shifts/dislocations will be in the negative $\delta$ region, whereas for positive $\ell$ values the shift will be in the positive $\delta$ region. This in turn manifestly resulted in energy levels crossings (i.e., occasional degeneracies, as shown in figures 1(b), 1(c), and 3(b)). Moreover, in Figures 2(a), 2(b), and 2(c), we have observed eminent energy levels clusterings when $|\delta| >> 1$, for each value of the magnetic quantum number $\ell = 0, \pm 1, \pm 2, \ldots$. In order to find parallel systems that admit invariance and isospectrality with the confined KG-oscillator, we have discussed (in section 3) KG-oscillator in pseudo-cosmic spacetime with a cosmic space-like dislocation. Such parallel systems are found to inherit the same effects discussed above.

Yet, we have suggested (in section 4) a new recipe for the PDM KG-oscillator in a cosmic spacetime with a cosmic space-like dislocation. We have used the PDM-momentum operator constructed by Mustafa and Al gadhi [39] and discussed the effects of PDM setting on the confined KG-oscillator, through two illustrative examples. For a power-law type PDM, the energy levels are shown to have similar trend of behavior as those of (19) discussed in section 2. Whereas, for the exponentially growing PDM, we found that such a PDM setting introduces a Cornell-like confinement. Such a KG-oscillator endowed with an exponentially growing PDM created its own byproducted confinement. Hereby, we have observed the effect of the exponential PDM parameter $\xi \geq 0$ on the energy levels. Obvious clustering of the energy levels are observed, as the PDM parameter $\xi$ grows up, but no energy levels crossing are found feasible for a fixed torsion parameter $\delta$ value (documented in figure 3(a)). Moreover, the effect of the torsion parameter $\delta$ on the energy levels, for a fixed PDM parameter $\xi$, is found to maintain the same trend of behavior as that associated with (19) and discussed in section 2.

Finally, the current methodical proposal may very well be extended to cover a more general case of PDM KG-particles and PDM Dirac-particles in different spacetime and topological defects. To the best of our knowledge, such a PDM KG-oscillator methodical proposal has never been reported elsewhere.

Data Availability Statement Authors can confirm that all relevant data are included in the article and/or its supplementary information files

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