GENERALIZED TRANSFORMS AND GENERALIZED
CONVOLUTION PRODUCTS ASSOCIATED WITH GAUSSIAN
PATHS ON FUNCTION SPACE

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ABSTRACT. In this paper we define a more general convolution product (associated with Gaussian processes) of functionals on the function space $C_{a,b}[0,T]$. The function space $C_{a,b}[0,T]$ is induced by a generalized Brownian motion process. Thus the Gaussian processes used in this paper are non-centered processes. We then develop the fundamental relationships between the generalized Fourier–Feynman transform associated with the Gaussian process and the convolution product.

1. Introduction. For $f \in L_2(\mathbb{R})$, let the Fourier transform of $f$ be given by

$$\mathcal{F}(f)(u) = \int_{\mathbb{R}} e^{iuv} f(v) dm_L^n(v)$$

and for $f, g \in L_2(\mathbb{R})$, let the convolution of $f$ and $g$ be given by

$$(f \ast g)(u) = \int_{\mathbb{R}} f(u-v) g(v) dm_L^n(v),$$

where $dm_L^n(v)$ denotes the normalized Lebesgue measure $(2\pi)^{-1/2} dv$ on $\mathbb{R}$. The Fourier transform $\mathcal{F}$ satisfies Parseval’s relation in the form

$$\int_{\mathbb{R}} f(v) g(v) dm_L^n(v) = \int_{\mathbb{R}} \mathcal{F}(f)(v) \mathcal{F}(g)(v) dm_L^n(v).$$

Furthermore $\mathcal{F}$ acts like a homomorphism with convolution $\ast$ and ordinary multiplication on $L_2(\mathbb{R})$. More precisely, one can see that for $f, g \in L_2(\mathbb{R})$,

$$\mathcal{F}(f \ast g) = \mathcal{F}(f) \mathcal{F}(g).$$

(1.1)

Given a positive real number $T > 0$, let $C_0[0,T]$ denote the one-parameter Wiener space, that is, the space of all real-valued continuous functions $x$ on the compact interval $[0,T]$ with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable

$\mathcal{M}$

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subsets of $C_0[0,T]$ and let $m$ denote the Wiener measure which is a Gaussian measure on $C_0[0,T]$ with mean zero and covariance function $r(s,t) = \min\{s,t\}$. Then, as it is well-known, $(C_0[0,T], M, m)$ is a complete measure space.

The concept of the analytic Fourier–Feynman transform (FFT) on the Wiener space $C_0[0,T]$, initiated by Brue [1], has been developed in the literature. For instance see [2, 16]. This transform and its properties are similar in many respects to the ordinary Fourier transform $F$. For an elementary introduction to the analytic FFT, see [21] and the references cited therein. We refer to [21] for the concepts, the precise definitions, and the notations of the scale-invariant measurability, the $L_p$ analytic FFT, and the convolution product (CP) on the classical Wiener space $C_0[0,T]$.

In [12], Huffman, Park and Skoug defined the CP for functionals on $C_0[0,T]$, and they then obtained various results for the analytic FFT and the CP, see also [13, 14, 20]. In previous researches involving [12, 13, 14, 20], the authors established a relationship between the analytic FFT and the corresponding CP of functionals $F$ and $G$ on $C_0[0,T]$ of the form

$$T_q^{(p)}((F \ast G)_{q,h})(y) = T_q^{(p)}(F) \left( \frac{y}{\sqrt{2}} \right) T_q^{(p)}(G) \left( \frac{y}{\sqrt{2}} \right)$$

(1.2)

for scale-almost every $y \in C_0[0,T]$, where $T_q^{(p)}(F)$ and $(F \ast G)_q$ denote the analytic FFT and the CP of functionals on $C_0[0,T]$, respectively.

The concepts of the generalized Wiener integral (namely, the Wiener integral associated with Gaussian paths) and the generalized analytic Feynman integral (namely, the analytic Feynman integral associated with Gaussian paths) on $C_0[0,T]$ were introduced by Chung, Park and Skoug [11] and further developed in [10, 15, 19]. In [10, 11, 15, 19], the generalized Wiener integral was defined by the Wiener integral

$$\int_{C_0[0,T]} F(\mathcal{Z}_h(x, \cdot))dm(x),$$

where $\mathcal{Z}_h : C_0[0,T] \times [0,T] \rightarrow \mathbb{R}$ is the Gaussian process given by

$$\mathcal{Z}_h(x, t) = \int_0^t h(s) \tilde{d}x(s),$$

(1.3)

where $h$ is a non-zero function in $L_2[0,T]$ and $\int_0^t h(s) \tilde{d}x(s)$ denotes the Paley–Wiener–Zygmund stochastic integral [18].

In [15], Huffman, Park and Skoug introduced a generalized FFT (GFFT) and a generalized CP (GCP) associated with the Gaussian process $\mathcal{Z}_h$ given by (1.3) above, and they developed a relationship between the GFFT and the GCP for functionals in the Banach algebra $S(L_2[0,T])$ introduced by Cameron and Storvick in [3]. Also, in [15], the authors examined several relationships between the GFFT and the GCP. The basic relationship investigated is as follows:

$$T_{q,h}^{(p)}((F \ast G)_{q,h})(y) = T_{q,h}^{(p)}(F) \left( \frac{y}{\sqrt{2}} \right) T_{q,h}^{(p)}(G) \left( \frac{y}{\sqrt{2}} \right)$$

(1.4)

for scale-almost every $y \in C_0[0,T]$, where $T_{q,h}^{(p)}(F)$ and $(F \ast G)_{q,h}$ denote the analytic GFFT and the GCP on $C_0[0,T]$, respectively. In [10], the authors presented a more general CP of functionals on $C_0[0,T]$ and established equation (1.4).
On the other hand, in [5, 6, 8], the authors introduced the FFTs on the very general function space \( C_{a,b}[0, T] \) (rather than the Wiener space \( C_0[0, T] \)), and studied their properties and related topics. The function space \( C_{a,b}[0, T] \) induced by a generalized Brownian motion process (GBMP), was introduced by Yeh [22, 23] and was used extensively in [4, 5, 6, 7, 8, 9].

A GBMP on a probability space \((\Omega, \Sigma, P)\) and a time interval \([0, T]\) is a Gaussian process \( Y = \{ Y_t \}_{t \in [0, T]} \) such that \( Y_0 = c \) almost surely for some constant \( c \in \mathbb{R} \), and for any set of time moments \( 0 = t_0 < t_1 < \cdots < t_n \leq T \) and any Borel set \( B \subset \mathbb{R}^n \), the measure \( P(I_{t_1,\ldots,t_n,B}) \) of the cylinder set \( I_{t_1,\ldots,t_n,B} \) of the form

\[
I_{t_1,\ldots,t_n,B} = \{ \omega \in \Omega : (Y(t_1,\omega),\ldots,Y(t_n,\omega)) \in B \}
\]

is equal to

\[
\left( (2\pi)^n \prod_{j=1}^n (b(t_j) - b(t_{j-1})) \right)^{-1/2} \times \int_B \exp \left\{ -\frac{1}{2} \sum_{j=1}^n \frac{(n_j - a(t_j)) - (n_{j-1} - a(t_{j-1}))^2}{b(t_j) - b(t_{j-1})} \right\} \, d\eta_1 \cdots d\eta_n,
\]

where \( \eta_0 = c, a(t) \) is a continuous real-valued function on \([0, T]\), and \( b(t) \) is an increasing continuous real-valued function on \([0, T]\). Thus, the GBMP \( Y \) is determined by the continuous functions \( a(\cdot) \) and \( b(\cdot) \). For more details, see [22, 23]. Note that when \( c = 0 \), \( a(t) \equiv 0 \) and \( b(t) = t \) on \([0, T]\), the GBMP is a standard Brownian motion (Wiener process). In this paper we set \( c = a(0) = b(0) = 0 \). Then the function space \( C_{a,b}[0, T] \) induced by the GBMP \( Y \) determined by the \( a(\cdot) \) and \( b(\cdot) \) can be considered as the space of continuous sample paths of \( Y \).

In view of equation (1.1), it is worthwhile to study a fundamental relation between the GFFT and the GCP, such as (1.2) and (1.4), on the function space \( C_{a,b}[0, T] \). However, by an effect of the drift \( a(t) \) of the GBMP \( Y \), the analytic GFFT on \( C_{a,b}[0, T] \) has unusual behaviors. In particular, the relations such as (1.2) and (1.4) do not hold between the GFFT and the GCP on the function space \( C_{a,b}[0, T] \). For some previous work on the GFFT and the GCP, see [7], and for a more detailed study of the effect of drift on the GBMP, see [4] and the references cited therein.

In this paper, we will establish fundamental relationships between the GFFT and the GCP in view of a concept of the rotation of Gaussian processes on the function space \( C_{a,b}[0, T] \). This paper is organized as follows. In Section 2, we recall a brief background on the function space \( C_{a,b}[0, T] \). In Sections 3 and 4, we expound the Gaussian processes which are defined as integral processes on \( C_{a,b}[0, T] \) and the concept of the GFFT associated with the Gaussian paths \( Z_a(x,\cdot) \). In Section 5, we investigate the existence of the GFFT associated with the Gaussian paths of exponential-type functionals on \( C_{a,b}[0, T] \). We also establish that the transforms used in this paper are onto transforms on the class \( \mathcal{E}(C_{a,b}[0, T]) \), the space of exponential-type functionals. In Section 6, we define a more general CP of functionals on \( C_{a,b}[0, T] \). It turns out, as noted in Remark 6.5 below, that the class \( \mathcal{E}(C_{a,b}[0, T]) \) forms a noncommutative complex algebra with the convolution. We then proceed to derive a fundamental relationship between the generalized transform and the generalized convolution on the function space \( C_{a,b}[0, T] \).

The Wiener process used in [1, 2, 3, 12, 13, 14, 16, 20] is stationary in time and is free of drift, while the Gaussian process used in [10, 11, 15, 19] is non-stationary.
in time and is free of drift. However the stochastic processes used in this paper, as well as in [4, 5, 6, 7, 8, 9], are non-stationary in time and are subject to a drift \( a(t) \), and can be used to explain the position of the Ornstein–Uhlenbeck process in an external force field [17]. But, by choosing \( a(t) \equiv 0 \) and \( b(t) \equiv t \) on \([0, T]\), the function space \( C_{a,b}[0,T] \) reduces to the Wiener space \( C_0[0,T] \), and so the expected results on \( C_0[0,T] \) are immediate corollaries of our results in this paper.

2. Preliminaries. In this section, we present the brief backgrounds which are needed in the following sections.

Let \( a(t) \) be an absolutely continuous real-valued function on \([0, T]\) with \( a(0) = 0 \) and \( a'(t) \in L^2[0, T] \), and let \( b(t) \) be a strictly increasing, continuously differentiable real-valued function with \( b(0) = 0 \) and \( b'(t) > 0 \) for each \( t \in [0, T] \). The GBMP \( Y \) determined by \( a(t) \) and \( b(t) \) is a Gaussian process with mean function \( a(t) \) and covariance function \( r(s, t) = \min\{b(s), b(t)\} \). For more details, see [4, 6, 8, 22, 23]. As illustrated in Section 1 above, applying [23, Theorem 14.2], one can construct a probability measure space \((C_{a,b}[0,T], B(C_{a,b}[0,T]), \mu)\) where \( C_{a,b}[0,T] \) is the space of continuous sample paths of (a separable version of) the GBMP \( Y \) (it is equivalent to the Banach space of continuous functions \( x \) on \([0, T]\) with \( x(0) = 0 \) under the sup norm) and \( B(C_{a,b}[0,T]) \) is the Borel \( \sigma \)-field of \( C_{a,b}[0,T] \) induced by the sup norm. We then complete this function space to obtain the complete probability measure space \((C_{a,b}[0,T], \mathcal{W}(C_{a,b}[0,T]), \mu_0)\) where \( \mathcal{W}(C_{a,b}[0,T]) \) is the set of all \( \mu \)-Carathéodory measurable subsets of \( C_{a,b}[0,T] \).

**Remark 2.1.** The coordinate process \( e : C_{a,b}[0,T] \to \mathbb{R} \) defined by \( e(x, t) = e_t(x) = x(t) \) is also the GBMP determined by \( a(t) \) and \( b(t) \), i.e., for each \( t \in [0, T] \), \( e_t(x) \sim N(a(t), b(t)) \), and the process \( \{e_t : 0 \leq t \leq T\} \) has non-stationary and independent increments.

**Remark 2.2.** Recall that the process \( \{e_t : 0 \leq t \leq T\} \) on \( C_{a,b}[0,T] \) is a continuous process. The function space \( C_{a,b}[0,T] \) reduces to the Wiener space \( C_0[0,T] \), considered in papers [1, 2, 3, 10, 11, 12, 13, 14, 15, 16, 19, 20] if and only if \( a(t) \equiv 0 \) and \( b(t) = t \) for all \( t \in [0, T] \).

A subset \( B \) of \( C_{a,b}[0,T] \) is said to be scale-invariant measurable provided \( \rho B \) is \( \mathcal{W}(C_{a,b}[0,T]) \)-measurable for all \( \rho > 0 \), and a scale-invariant measurable set \( N \) is said to be scale-invariant null provided \( \mu(\rho N) = 0 \) for all \( \rho > 0 \). A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s.a.e.). A functional \( F \) is said to be scale-invariant measurable provided \( F \) is defined on a scale-invariant measurable set and \( F(\rho \cdot) \) is \( \mathcal{W}(C_{a,b}[0,T]) \)-measurable for every \( \rho > 0 \). If two functionals \( F \) and \( G \) defined on \( C_{a,b}[0,T] \) are equal s.a.e., we write \( F \approx G \). Note that the relation \( \approx \) is an equivalence relation.

Let \( L^2_{a,b}[0,T] \) be the space of functions on \([0,T]\) which are Lebesgue measurable and square integrable with respect to the Lebesgue–Stieltjes measures on \([0,T]\) induced by \( a(\cdot) \) and \( b(\cdot) \); i.e.,

\[
L^2_{a,b}[0,T] = \left\{ v : \int_0^T v^2(s)db(s) < +\infty \quad \text{and} \quad \int_0^T v^2(s)d|a|(s) < +\infty \right\}
\]

where \( |a|(\cdot) \) denotes the total variation function of \( a(\cdot) \). Then \( L^2_{a,b}[0,T] \) is a separable Hilbert space with inner product defined by

\[
(u, v)_{a,b} = \int_0^T u(t)v(t)\,dm_{|a|,b}(t) \equiv \int_0^T u(t)v(t)d[b(t) + |a|(t)],
\]
where $m_{|a|,b}$ denotes the Lebesgue–Stieltjes measure induced by $|a|\cdot(\cdot)$ and $b(\cdot)$. In particular, note that $\|u\|_{a,b} = \sqrt{\langle u, u \rangle_{a,b}} = 0$ if and only if $u(t) = 0$ a.e. on $[0,T]$. Furthermore, $(L^2_{a,b}[0,T], \| \cdot \|_{a,b})$ is a separable Hilbert space.

Throughout the rest of this paper, we consider the linear space

$$C'_{a,b}[0,T] = \left\{ w \in C_{a,b}[0,T] : w(t) = \int_0^t z(s)dB(s) \text{ for some } z \in L^2_{a,b}[0,T] \right\}.$$  

For $w \in C'_{a,b}[0,T]$, with $w(t) = \int_0^t z(s)dB(s)$ for $t \in [0,T]$, let $D : C'_{a,b}[0,T] \to L^2_{a,b}[0,T]$ be defined by the formula

$$Dw(t) = z(t) = \frac{w'(t)}{b'(t)}. \quad (2.1)$$

Then $C'_{a,b} \equiv C_{a,b}[0,T]$ with inner product

$$(w_1, w_2)_{C_{a,b}} = \int_0^T Dw_1(t) Dw_2(t) dB(t)$$

is also a separable Hilbert space.

Note that the two separable Hilbert spaces $L^2_{a,b}[0,T]$ and $C'_{a,b}[0,T]$ are topologically homeomorphic under the linear operator given by equation (2.1). The inverse operator of $D$ is given by

$$(D^{-1} z)(t) = \int_0^t z(s)dB(s)$$

for $t \in [0,T]$.

In this paper, in addition to the conditions put on $a(t)$ above, we now add the condition

$$\int_0^T |a'(t)|^2 dB(t) |a'(t)| < +\infty. \quad (2.2)$$

Then, the function $a : [0,T] \to \mathbb{R}$ satisfies the condition (2.2) if and only if $a(\cdot)$ is an element of $C'_{a,b}[0,T]$. Under the condition (2.2), we observe that for each $w \in C'_{a,b}[0,T]$ with $Dw = z$,

$$(w, a)_{C_{a,b}} = \int_0^T Dw(t) Da(t) dB(t) = \int_0^T z(t) \frac{a'(t)}{b'(t)} db(t) = \int_0^T z(t) da(t).$$

3. **Gaussian processes.** In order to present our results involving the analytic GFFT and the GCP, we follow the exposition of [5].

For each $w \in C'_{a,b}[0,T]$ and $x \in C_{a,b}[0,T]$, we let $(w, x)^\sim$ denote the Paley–Wiener–Zygmund (PWZ) stochastic integral [5, 9]. It is known that for each $w \in C'_{a,b}[0,T]$, the PWZ stochastic integral $(w, x)^\sim$ exists for s.a.e. $x \in C_{a,b}[0,T]$ and it is a Gaussian random variable with mean $(w, a)_{C_{a,b}}$ and variance $\|w\|_{C'_{a,b}}^2$. Furthermore, if $Dw = z \in L^2_{a,b}[0,T]$ is of bounded variation on $[0,T]$, then the PWZ stochastic integral $(w, x)^\sim$ equals the Riemann–Stieltjes integral $\int_0^T z(t) dx(t)$. Also we note that for $w,x \in C'_{a,b}[0,T]$, $(w, x)^\sim = (w, x)_{C'_{a,b}}$.

For each $t \in [0,T]$, let $\chi_{[0,t]}$ denote the indicator function of the interval $[0,t]$, and for a function $k \in C'_{a,b}[0,T]$ with $Dk = h$ and with $\|k\|_{C'_{a,b}} = \int_0^T h^2(t) db(t) \leq 0$, let $Z_k(x,t)$ be the PWZ stochastic integral

$$Z_k(x,t) = (D^{-1}(h\chi_{[0,t]}), x)^\sim. \quad (3.1)$$
Let
\[
\gamma_k(t) = \int_0^t Dk(u)da(u) = \int_0^t h(u)da(u) \tag{3.2}
\]
and let
\[
\beta_k(t) = \int_0^t (Dk(u))^2db(u) = \int_0^t h^2(u)db(u). \tag{3.3}
\]
Then the stochastic process \(Z_k : C_{a,b}[0,T] \times [0,T] \rightarrow \mathbb{R}\) given by \((x,t) \mapsto Z_k(x,t)\) is Gaussian with mean function
\[
\int_{C_{a,b}[0,T]} Z_k(x,t)d\mu(x) = \int_0^t h(u)da(u) = \gamma_k(t)
\]
and covariance function
\[
\int_{C_{a,b}[0,T]} (Z_k(x,s) - \gamma_k(s))(Z_k(x,t) - \gamma_k(t))d\mu(x)
= \int_0^{\min\{s,t\}} h^2(u)db(u) = \beta_k(\min\{s,t\}).
\]
In addition, by [23, Theorem 21.1], \(Z_k(\cdot , t)\) is stochastically continuous in \(t\) on \([0,T]\).
Furthermore, for any \(\rho \in \mathbb{R}\), \(s,t \in [0,T]\), and \(k,k_1,k_2 \in C'_{a,b}[0,T]\),
\[
\rho Z_k(x,t) = Z_{\rho k}(x,t) = Z_k(\rho x,t) \tag{3.4}
\]
and
\[
\int_{C_{a,b}[0,T]} Z_{k_1}(x,s)Z_{k_2}(x,t)d\mu(x)
= \int_0^{\min\{s,t\}} Dk_1(u)Dk_2(u)db(u) + \int_0^s Dk_1(u)da(u) \int_0^t Dk_2(u)da(u). \tag{3.5}
\]
If \(Dk\) is of bounded variation on \([0,T]\), then, for all \(x \in C_{a,b}[0,T]\), \(Z_k(x,t)\) is continuous in \(t\). Of course if \(k(t) \equiv b(t)\), then \(Z_b(x,t) = x(t)\), the sample paths of the coordinate process \(Y = \{x_t : 0 \leq t \leq T\}\) referred in Section 2 above. Also, as mentioned in Section 2, if \(a(t) \equiv 0\) and \(b(t) = t\) on \([0,T]\), the function space \(C_{a,b}[0,T]\) reduces to the classical Wiener space \(C_0[0,T]\) and the Gaussian process (3.1) with \(k(t) \equiv t\) is an ordinary Wiener process.

Let \(C^*_{a,b}[0,T]\) be the set of functions \(k\) in \(C'_{a,b}[0,T]\) such that \(Dk\) is continuous except for a finite number of finite jump discontinuities and is of bounded variation on \([0,T]\). For any \(w \in C_{a,b}[0,T]\) and \(k \in C^*_{a,b}[0,T]\), let the operation \(\odot\) between \(C'_{a,b}[0,T]\) and \(C^*_{a,b}[0,T]\) be defined by
\[
w \odot k = D^{-1}(DwDk), \ \text{i.e.,} \ D(w \odot k) = DwDk, \tag{3.5}
\]
where \(DwDk\) denotes the pointwise multiplication of the functions \(Dw\) and \(Dk\). In this case, \((C^*_{a,b}[0,T], \odot)\) forms a commutative algebra with the identity \(b\). For more details, see [5].

Given any \(w \in C'_{a,b}[0,T]\) and \(k \in C^*_{a,b}[0,T]\), it follows that
\[
(w, Z_k(x, \cdot))^{\sim} = \int_0^T Dw(t)d\left(\int_0^t Dk(s)dx(s)\right)
= \int_0^T Dw(t)Dk(t)dx(t) = (w \odot k, x)^{\sim} \tag{3.6}
\]
for s.a.e \( x \in C_{a,b}[0,T] \). Thus, throughout the rest of this paper, we require \( k \) to be in \( C_{a,b}^*[0,T] \) for each process \( Z_k \). This will ensure that the Lebesgue–Stieltjes integrals

\[
\|w \circ k\|_{C_{a,b}}^2 = \int_0^T ( Dw(t) )^2 ( Dk(t) )^2 db(t)
\]

and

\[
(w \circ k, a)_{C_{a,b}} = \int_0^T Dw(t) Dk(t) da(t) = \int_0^T Dw(t) Dk(t) da(t)
\]

will exist for all \( w \in C_{a,b}^*[0,T] \) and \( k \in C_{a,b}^*[0,T] \).

Using equation (3.6) and the change of variable theorem, the function space integration formula

\[
\int_{C_{a,b}[0,T]} \exp\{ \rho(w, Z_k(x, \cdot)) \} d\mu(x) = \exp \left\{ \frac{\rho^2}{2} \|w \circ k\|_{C_{a,b}}^2 + \rho(w, a)_{C_{a,b}} \right\}
\]

(3.7)

holds for every \( w \in C_{a,b}^*[0,T] \), \( k \in C_{a,b}^*[0,T] \) and \( \rho \in \mathbb{C} \).

Let \( (C_{a,b}^2[0,T], W(C_{a,b}^2[0,T]), \mu^2) \) be the product function space, where \( C_{a,b}^2[0,T] = C_{a,b}[0,T] \times C_{a,b}[0,T] \), \( W(C_{a,b}^2[0,T]) \equiv W(C_{a,b}[0,T]) \otimes W(C_{a,b}[0,T]) \) denotes the \( \sigma \)-field generated by measurable rectangles \( A \times B \) with \( A, B \in W(C_{a,b}[0,T]) \), and \( \mu^2 \equiv \mu \times \mu \).

For \( k_1, k_2 \in C_{a,b}^*[0,T] \) with \( \|k_j\|_{C_{a,b}} > 0, j \in \{1,2\} \), let \( Z_{k_1} \) and \( Z_{k_2} \) be the Gaussian processes given by (3.1), respectively. Then the process

\[
\mathbf{x}_{k_1, k_2} : C_{a,b}^2[0,T] \times [0,T] \to \mathbb{R}
\]

given by

\[
\mathbf{x}_{k_1, k_2}(x_1, x_2, t) = Z_{k_1}(x_1, t) + Z_{k_2}(x_2, t)
\]

is also a continuous Gaussian process with mean function

\[
m_{k_1, k_2}(t) = \int_{C_{a,b}^2[0,T]} \mathbf{x}_{k_1, k_2}(x_1, x_2, t) d\mu^2(x_1, x_2) = \gamma_{k_1}(t) + \gamma_{k_2}(t)
\]

and variance function

\[
v_{k_1, k_2}(t) = \int_{C_{a,b}^2[0,T]} (\mathbf{x}_{k_1, k_2}(x_1, x_2, t) - m_{k_1, k_2}(t))^2 d\mu^2(x_1, x_2) = \beta_{k_1}(t) + \beta_{k_2}(t)
\]

where \( \gamma_k \) and \( \beta_k \) are given by (3.2) and (3.3), respectively. In fact, the covariance function of \( \mathbf{x}_{k_1, k_2}(x_1, x_2, \cdot) \) is given by

\[
\int_{C_{a,b}^2[0,T]} (\mathbf{x}_{k_1, k_2}(x_1, x_2, s) - m_{k_1, k_2}(s)) \times (\mathbf{x}_{k_1, k_2}(x_1, x_2, t) - m_{k_1, k_2}(t)) d\mu^2(x_1, x_2)
\]

\[
= v_{k_1, k_2}(\min\{s, t\}).
\]

Let \( k_1 \) and \( k_2 \) be functions in \( C_{a,b}^*[0,T] \) with \( Dk_j = h_j, j \in \{1,2\} \). Then there exists a function \( s \in C_{a,b}^*[0,T] \) such that

\[
(Ds(t))^2 = h_1^2(t) + h_2^2(t) = (Dk_1(t))^2 + (Dk_2(t))^2
\]

for \( m_{a,b} \text{ a.e. } t \in [0,T] \). Note that the function ‘s’ satisfying (3.8) is not unique. We will use the symbol \( s(k_1, k_2) \) for the functions ‘s’ that satisfy (3.8) above.
Next we will consider a stochastic process associated with \( Z_{\alpha(k_1,k_2)} \). Given non-zero functions \( k_1 \) and \( k_2 \) in \( C^*_a[0,T] \), define a process
\[ R_{k_1,k_2} : C_{a,b}[0,T] \times [0,T] \to \mathbb{R} \]
by
\[ R_{k_1,k_2}(x,t) = Z_{\alpha(k_1,k_2)}(x,t) + A(k_1,k_2)(t), \]
where
\[ A(k_1,k_2)(t) = \int_0^t \left( Dk_1(u) + Dk_2(u) - Ds(k_1,k_2)(u) \right) da(u). \]
(3.9)
Then \( R_{k_1,k_2} \) is a Gaussian process with mean function
\[ \int_{C_{a,b}[0,T]} R_{k_1,k_2}(x,t) d\mu(x) = m_{k_1,k_2}(t) \]
and covariance function
\[ \int_{C_{a,b}[0,T]} \left( R_{k_1,k_2}(x,s) - m_{k_1,k_2}(s) \right) \left( R_{k_1,k_2}(x,t) - m_{k_1,k_2}(t) \right) d\mu(x) \]
\[ = v_{k_1,k_2}(\min\{s,t\}). \]
Also, \( R_{k_1,k_2} \) is a continuous process.

Given non-zero functions \( k_j \in C^*_a[0,T], j \in \{1, \ldots, m\} \), let \( Z_{k_j} \) be the Gaussian processes given by (3.1), and let \( R_{k_1,\ldots,k_m} : C_{a,b}[0,T] \times [0,T] \to \mathbb{R} \) be the Gaussian process given by
\[ R_{k_1,\ldots,k_m}(x,t) = Z_{\alpha(k_1,\ldots,k_m)}(x,t) + A(k_1,\ldots,k_m)(t), \]
where
\[ A(k_1,\ldots,k_m)(t) = \int_0^t \left[ \sum_{j=1}^m Dk_j(u) - Ds(k_1,\ldots,k_m)(u) \right] da(u), \]
and where \( s(k_1,\ldots,k_m) \) is a function in \( C^*_a[0,T] \) which satisfies the condition
\[ |Ds(k_1,\ldots,k_m)|^2 = \sum_{j=1}^m |Dk_j|^2 \]
(3.11)
for \( m_{a,b} \)-a.e. on \([0,T] \). Then \( R_{k_1,\ldots,k_m} \) is a continuous Gaussian process with mean function
\[ m_{k_1,\ldots,k_m}(t) = \int_{C_{a,b}[0,T]} R_{k_1,\ldots,k_m}(x,t) d\mu(x) = \sum_{j=1}^m \gamma_{k_j}(t) \]
and covariance function
\[ v_{k_1,\ldots,k_m}(\min\{s,t\}) \]
\[ = \int_{C_{a,b}[0,T]} \left( R_{k_1,\ldots,k_m}(x,s) - m_{k_1,\ldots,k_m}(s) \right) \times \left( R_{k_1,\ldots,k_m}(x,t) - m_{k_1,\ldots,k_m}(t) \right) d\mu(x) \]
\[ = \sum_{j=1}^n \beta_{k_j}(\min\{s,t\}). \]
It is clear that for any permutation \( \pi \) of the set \( \{1, \ldots, m\} \), \( R_{k_1,\ldots,k_m} = R_{\pi(k_1),\ldots,\pi(k_m)}. \)
In view of the argument above and using an induction argument, it follows that for a finite sequence \(\{k_1, \ldots, k_m\}\) of non-zero functions in \(C^{\ast}_{a,b}[0,T]\) and any \(l \in \{1, \ldots, m\}\), the three processes
\[
\sum_{j=1}^{m} \mathcal{Z}_{k_j} \text{ on } C^{\ast}_{a,b}[0,T] \times [0,T],
\]
\[
\mathcal{R}_{k_1, \ldots, k_m} \text{ on } C_{a,b}[0,T] \times [0,T]
\]
and
\[
\sum_{j=1}^{l} \mathcal{Z}_{k_j} + \mathcal{R}_{k_{l+1}, \ldots, k_m} \text{ on } (C^{\ast}_{a,b}[0,T] \times C_{a,b}[0,T]) \times [0,T]
\]
are equivalent in law.

4. Generalized Fourier–Feynman transform associated with Gaussian paths. Let \(\mathcal{G}\) be a continuous Gaussian process on \(C_{a,b}[0,T] \times [0,T]\). We define the \(\mathcal{G}\)-function space integral (namely, the function space integral associated with the Gaussian paths \(\mathcal{G}(x, \cdot)\)) for functionals \(F\) on \(C_{a,b}[0,T]\) by the formula
\[
I_{\mathcal{G}}[F] \equiv I_{\mathcal{G},x}[F(\mathcal{G}(x, \cdot))] = \int_{C_{a,b}[0,T]} F(\mathcal{G}(x, \cdot))d\mu(x)
\]
whenever the integral exists.

Throughout the rest of this paper, let \(\mathbb{C} \), \(\mathbb{C}_{+}\) and \(\mathbb{C}_{+}\) denote the set of complex numbers, complex numbers with positive real part, and non-zero complex numbers with nonnegative real part, respectively. Furthermore, for each \(\lambda \in \mathbb{C}_{+}\), \(\lambda^{1/2}\) denotes the principal square root of \(\lambda\); i.e., \(\lambda^{1/2}\) is always chosen to have positive real part, so that \(\lambda^{-1/2} = (\lambda^{-1})^{1/2}\) is in \(\mathbb{C}_{+}\).

**Definition 4.1.** Let \(\mathcal{G}\) be a continuous Gaussian process on \(C_{a,b}[0,T] \times [0,T]\), and let \(F\) be a \(\mathbb{C}\)-valued scale-invariant measurable functional on \(C_{a,b}[0,T]\) such that
\[
J_{F}(\mathcal{G};\lambda) = I_{\mathcal{G},x}[F(\mathcal{G}(x, \cdot))]\]
exists and is finite for all \(\lambda > 0\). If there exists a function \(J_{F}(\mathcal{G};\lambda)\) analytic on \(\mathbb{C}_{+}\) such that \(J_{F}(\mathcal{G};\lambda) = J_{F}(\mathcal{G};\lambda)\) for all \(\lambda > 0\), then \(J_{F}(\mathcal{G};\lambda)\) is defined to be the analytic \(\mathcal{G}\)-function space integral (namely, the analytic function space integral associated with the paths \(\mathcal{G}(x, \cdot)\)) of \(F\) over \(C_{a,b}[0,T]\) with parameter \(\lambda\), and for \(\lambda \in \mathbb{C}_{+}\) we write
\[
I^{an}_{\mathcal{G}}[F] \equiv I^{an}_{\mathcal{G},x}[F(\mathcal{G}(x, \cdot))] \equiv \int_{C_{a,b}[0,T]} F(\mathcal{G}(x, \cdot))d\mu(x) = J_{F}(\mathcal{G};\lambda). \tag{4.1}
\]

Let \(q\) be a non-zero real number and let \(F\) be a measurable functional whose analytic \(\mathcal{G}\)-function space integral \(I^{an}_{\mathcal{G}}[F]\) exists for all \(\lambda \in \mathbb{C}_{+}\). If the following limit exists, we call it the generalized analytic \(\mathcal{G}\)-Feynman integral (namely, the generalized analytic Feynman integral associated with the paths \(\mathcal{G}(x, \cdot)\)) of \(F\) with parameter \(q\), and we write
\[
I^{an}_{\mathcal{G}}[F] \equiv I^{an}_{\mathcal{G},x}[F(\mathcal{G}(x, \cdot))] = \lim_{\lambda \rightarrow -iq} I^{an}_{\mathcal{G},x}[F(\mathcal{G}(x, \cdot))], \tag{4.2}
\]
where \(\lambda\) approaches \(-iq\) through values in \(\mathbb{C}_{+}\).

Next we state the definition of the GFET associated with Gaussian paths on function space.
Definition 4.2. Let $q$ be a non-zero real number. Given a continuous Gaussian process $\mathcal{G}$ on $C_{a,b}[0,T] \times [0,T]$, let $F$ be a scale-invariant measurable functional on $C_{a,b}[0,T]$ such that for all $\lambda \in \mathbb{C}_+$ and $y \in C_{a,b}[0,T]$, the following analytic $\mathcal{G}$-function space transform

$$T_{\lambda, \mathcal{G}}(F)(y) = I_{\lambda}^{an\ell}[F(y + \mathcal{G}(x, \cdot))],$$

exists. For $p \in (1, 2]$, we define the $L_p$ analytic $\mathcal{G}$-GFFT (namely, the GFFT associated with the Gaussian paths $\mathcal{G}(x, \cdot)$), $T_{q, \mathcal{G}}^{(p)}(F)$ of $F$, by the formula

$$T_{q, \mathcal{G}}^{(p)}(F)(y) = \lim_{\lambda \to -iq, \lambda \in \mathbb{C}_+} T_{\lambda, \mathcal{G}}(F)(y) = I_{k, \mathcal{G}}^{an\ell}[F(y + \mathcal{G}(x, \cdot))],$$

if it exists; i.e., for each $\rho > 0$,

$$\lim_{\lambda \to -iq, \lambda \in \mathbb{C}_+} \int_{C_{a,b}[0,T]} \left| T_{\lambda, \mathcal{G}}(F)(\rho y) - T_{q, \mathcal{G}}^{(p)}(F)(\rho y) \right|^p \rho^q d\mu(y) = 0,$$

where $1/p + 1/p' = 1$. We define the $L_1$ analytic $\mathcal{G}$-GFFT, $T_{q, \mathcal{G}}^{(1)}(F)$ of $F$, by the formula

$$T_{q, \mathcal{G}}^{(1)}(F)(y) = \lim_{\lambda \to -iq, \lambda \in \mathbb{C}_+} T_{\lambda, \mathcal{G}}(F)(y) = I_{q, \mathcal{G}}^{an\ell}[F(y + \mathcal{G}(x, \cdot))],$$

for s.a.e. $y \in C_{a,b}[0,T]$ whenever this limit exists.

Remark 4.3. Note that if $k \equiv b$ on $[0,T]$, then the generalized analytic $\mathcal{Z}_b$-Feynman integral, $I_{q, \mathcal{Z}_b}^{an\ell}[F]$, and the $L_p$ analytic $\mathcal{Z}_b$-GFFT, $T_{q, \mathcal{Z}_b}^{(p)}(F)$ agree with the previous definitions of the generalized analytic Feynman integral and the analytic GFFT, respectively, [6, 7, 8].

5. Transforms on the class of exponential-type functionals. Let $\mathcal{E}$ be the class of all functionals which have the form

$$\Psi_w(x) = \exp\{(w, x)^-\}$$

for some $w \in C_{a,b}^c[0,T]$ and for s.a.e. $x \in C_{a,b}[0,T]$. More precisely, since we shall identify functionals which coincide s.a.e. on $C_{a,b}[0,T]$, the class $\mathcal{E}$ can be regarded as the space of all s-equivalence classes of functionals of the form (5.1).

A functional given by (5.1) is called an exponential functional. Also, given $q \in \mathbb{R} \setminus \{0\}, \tau \in C_{a,b}^c[0,T]$, and $k \in C_{a,b}^c[0,T]$, let $\mathcal{E}_{q, \tau, k}$ be the class of all functionals having the form

$$\Psi_{\tau, \tau, k}^q(x) = K_{\tau, \tau, k}^a \Psi_w(x)$$

for s.a.e. $x \in C_{a,b}[0,T]$, where $\Psi_w$ is given by equation (5.1) and $K_{\tau, \tau, k}^a$ is a complex number given by

$$K_{\tau, \tau, k}^a = \exp \left\{ \frac{i}{2q} \| \tau \otimes k \|_{C_{a,b}^c}^2 + (-iq)^{-1/2} (\tau \otimes k, a)_{C_{a,b}^c} \right\}.$$
The functionals given by equation (5.2) and linear combinations (with complex coefficients) of the \( \Psi_{w}^{q,\tau,k} \)'s are called the (partially) exponential-type functionals on \( C_{a,b}[0,T] \). The functionals given by (5.1) are also partially exponential-type functionals because \( \Psi_{w}^{q,\tau,0}(x) = \Psi_{w}^{q,\tau,k}(x) = \Psi_{w}(x) \) for s-a.e. \( x \in C_{a,b}[0,T] \).

For notational convenience, we let
\[
\Psi_{(w)}^{q,\cdot,k} = \Psi_{w}^{q,\cdot,k} = K_{q,w,k}^{a}\Psi_{w}.
\]
(5.4)

Also, let \( \Psi_{w}^{0,\tau,k}(x) = \Psi_{w}(x) \) and let \( \mathcal{E}_{0,\tau,k} = \mathcal{E} \). Then for any \((q,\tau,k) \in \mathbb{R} \times C_{a,b}[0,T] \times C_{a,b}^{*}[0,T]\), the class \( \mathcal{E}_{q,\tau,k} \) is dense in \( L_{2}(C_{a,b}[0,T]) \). Next, let \( \mathcal{E}(C_{a,b}[0,T]) = \text{Span} \mathcal{E}. \) Then, using the fact that
\[
\mathcal{E} = \mathcal{E}_{0,\tau,k} \subseteq \bigcup_{q \in \mathbb{R}} \bigcup_{\tau \in C_{a,b}[0,T]} \bigcup_{k \in C_{a,b}[0,T]} \mathcal{E}_{q,\tau,k} \subset \mathcal{E}(C_{a,b}[0,T]),
\]
one can see that \( \mathcal{E}(C_{a,b}[0,T]) = \text{Span} \mathcal{E}_{q,\tau,k} \) for every \((q,\tau,k) \in \mathbb{R} \times C_{a,b}^{*}[0,T] \times C_{a,b}[0,T]\).

**Remark 5.1.** The linear space \( \mathcal{E}(C_{a,b}[0,T]) \) of partially exponential-type functionals is a commutative (complex) algebra under the pointwise multiplication and with identity \( \Psi_{0} \equiv 1 \) because
\[
\Psi_{q,w_{1}}^{q_{1},\tau_{1},k_{1}}(x) \Psi_{q_{2},w_{2}}^{q_{2},\tau_{2},k_{2}}(x) = K_{q_{1},w_{1}}^{a} K_{q_{2},w_{2}}^{a} \Psi_{w_{1}+w_{2}}^{q_{2},\tau_{2},k_{2}}(x)
\]
for \( \mu \)-a.e. \( x \in C_{a,b}[0,T] \).

Note that every exponential-type functional is scale-invariant measurable. Since we shall identify functionals which coincide s-a.e. on \( C_{a,b}[0,T] \), \( \mathcal{E}(C_{a,b}[0,T]) \) can be regarded as the space of all s-equivalence classes of exponential-type functionals.

Using (4.3) and (4.4), and applying (3.7), we obtain the following theorem.

**Theorem 5.2.** Let \( \Psi_{w} \in \mathcal{E} \) be given by equation (5.1). Then for all \( p \in [1,2] \), any non-zero real \( q \), and each non-zero function \( k \) in \( C_{a,b}^{*}[0,T] \), the \( L_{p} \) analytic \( \mathcal{Z}^{-}\mathcal{GFFT} \) of \( \Psi_{w} \), \( T_{q,\mathcal{Z}^{k}}^{(p)}(\Psi_{w}) \) exists and is given by the formula
\[
T_{q,\mathcal{Z}^{k}}^{(p)}(\Psi_{w}) \approx \Psi_{(w)}^{q,\cdot,k},
\]
(5.5)
where \( \Psi_{(w)}^{q,\cdot,k} \) is given by equation (5.4). Thus, \( T_{q,\mathcal{Z}^{k}}^{(p)}(\Psi_{w}) \) is an element of \( \mathcal{E}(C_{a,b}[0,T]) \).

Since \( \mathcal{E}(C_{a,b}[0,T]) = \text{Span} \mathcal{E} \), for each exponential-type functional \( F \) in \( \mathcal{E}(C_{a,b}[0,T]) \), it can be written as
\[
F \approx \sum_{j=1}^{n} c_{j} \Psi_{w_{j}}
\]
(5.6)
for a finite sequence \( \{w_{1}, \ldots, w_{n}\} \) of functions in \( C_{a,b}[0,T] \), and a sequence \( \{c_{1}, \ldots, c_{n}\} \in \mathbb{C} \setminus \{0\} \). We note that for every \((q, w, k) \in \mathbb{R} \times C_{a,b}[0,T] \times C_{a,b}^{*}[0,T]\), the complex number \( K_{q,w,k}^{a} \) given by (5.3) with \( \tau \) replaced with \( w \) is non-zero. Thus, using the linearity of the analytic \( \mathcal{Z}^{-}\mathcal{GFFT} \) \( T_{q,\mathcal{Z}^{k}}^{(p)} \), (5.5), (5.4), and (5.3), it follows that for every \((q, w, k) \in \mathbb{R} \times C_{a,b}[0,T] \times C_{a,b}^{*}[0,T]\),
\[
T_{q,\mathcal{Z}^{k}}^{(p)}((K_{q,w,k}^{a})^{-1} \Psi_{w}) \approx (K_{q,w,k}^{a})^{-1} T_{q,\mathcal{Z}^{k}}^{(p)}(\Psi_{w}) \approx (K_{q,w,k}^{a})^{-1} \Psi_{(w)}^{q,\cdot,k} \approx \Psi_{w},
\]
where
Thus, \( \Psi \) where \( \mathbb{E} \) non-zero real numbers \( \mathbb{F} \) linearity of the analytic GFFT \( \mathbb{K} \) where \( (\mathbb{C}, \mathbb{E}) \) given by equation (5.6),
\[
T_{q, Z_k}^{(p)} \left( \sum_{j=1}^{n} c_j^{-1}(K_{q, w, k}^a)^{-1}\Psi_{w_j} \right) \approx F.
\]

In view of these observation and Theorem 5.2, we obtain the following theorem.

**Theorem 5.3.** Let \( F \in \mathcal{E}(C_{a,b}[0,T]) \) be given by equation (5.6). Then for all \( p \in [1, 2] \), any non-zero real \( q \), and each non-zero function \( k \) in \( C_{a,b}^*[0,T] \), the \( L_p \) analytic \( Z_k \)-GFFT of \( F \), \( T_{q, Z_k}^{(p)}(F) \) exists and is given by the formula
\[
T_{q, Z_k}^{(p)}(F) \approx \sum_{j=1}^{n} c_j T_{q, Z_k}^{(p)}(\Psi_{w_j}) \approx \sum_{j=1}^{n} c_j \Psi_{(w_j)}^{q:a,b,k},
\]
where \( \Psi_{w_j}^{q:a,b,k} \) is given by equation (5.4) with \( w \) replaced with \( w_j \) for each \( j \in \{1, \ldots, n\} \).

Thus, \( T_{q, Z_k}^{(p)}(F) \) is an element of \( \mathcal{E}(C_{a,b}[0,T]) \). Moreover, the \( L_p \) analytic \( Z_k \)-GFFT, \( T_{q, Z_k}^{(p)} : \mathcal{E}(C_{a,b}[0,T]) \rightarrow \mathcal{E}(C_{a,b}[0,T]) \) is an onto transform.

Corollary 5.4 below follows easily from (5.5), (5.4), (5.7), and the fact that \( \mathcal{E}(C_{a,b}[0,T]) \) is the linear span of the exponential functionals.

**Corollary 5.4.** Let \( \{k_1, \ldots, k_m\} \) be a set of non-zero functions in \( C_{a,b}^*[0,T] \), and let \( F \in \mathcal{E}(C_{a,b}[0,T]) \) be given by equation (5.6). Then for all \( p \in [1, 2] \) and all non-zero real numbers \( q_1, \ldots, q_n \), the iterated analytic GFFT of \( F \),
\[
T_{q_m, Z_{k_m}}^{(p)}(T_{q_{m-1}, Z_{k_{m-1}}}^{(p)}(\cdots(T_{q_1, Z_{k_1}}^{(p)}(F))\cdots))(y)
\]
exists and is an element of \( \mathcal{E}(C_{a,b}[0,T]) \). In particular, given an exponential functional \( \Psi_w \) in \( \mathcal{E} \),
\[
T_{q_m, Z_{k_m}}^{(p)}(T_{q_{m-1}, Z_{k_{m-1}}}^{(p)}(\cdots(T_{q_1, Z_{k_1}}^{(p)}(\Psi_w))\cdots))(y) = \exp\left(\{(w, y)^{\sim}\} \prod_{l=1}^{m} K_{q_l, w, k_l}^a \right)
\]
for s.a.e. \( y \in C_{a,b}[0,T] \), where \( K_{q_l, w, k_l}^a \) is given by equation (5.3) with \( (q_l, \tau, k_l) \) replaced with \( (q_l, w, k_l) \) for each \( l \in \{1, \ldots, m\} \). Thus for each functional \( F \in \mathcal{E}(C_{a,b}[0,T]) \) given by equation (5.6), it follows that
\[
T_{q_m, Z_{k_m}}^{(p)}(T_{q_{m-1}, Z_{k_{m-1}}}^{(p)}(\cdots(T_{q_1, Z_{k_1}}^{(p)}(F))\cdots))(y) = \exp\left(\{\{w, y\}^{\sim}\} \sum_{j=1}^{n} c_j \prod_{l=1}^{m} K_{q_j, w, k_l}^a \right)
\]
for s.a.e. \( y \in C_{a,b}[0,T] \).

In our next theorem we show that the composition of GFFTs associated with different Gaussian processes can be reduced to a single GFFT.
Theorem 5.5. Given non-zero functions $k_1$ and $k_2$ in $C^*_a[0,T]$, let $s(k_1,k_2)$ be a function in $C^*_a[0,T]$ which satisfies the relation (3.8), and let $F$ be an exponential-type functional in $\mathcal{E}(C^*_a[0,T])$. Then, for all $p \in [1,2]$ and any non-zero real $q$, the $L_p$ analytic $R_{k_1,k_2}$-GFFT, $T^{(p)}_{q,R_{k_1,k_2}}(F)$ of $F$ exists and

$$T^{(p)}_{q,R_{k_1,k_2}}(F) \approx T^{(p)}_{q,Z_{k_2}}(T^{(p)}_{q,Z_{k_1}}(F)).$$

(5.9)

Proof. In view of Theorem 5.3, it will suffice to show that for each exponential functional $\Psi_w \in \mathcal{E}$,

$$T^{(p)}_{q,R_{k_1,k_2}}(\Psi_w) \approx T^{(p)}_{q,Z_{k_2}}(T^{(p)}_{q,Z_{k_1}}(\Psi_w)).$$

(5.10)

Using equations (5.8) and (5.3), it first follows that

$$T^{(p)}_{q,Z_{k_2}}(T^{(p)}_{q,Z_{k_1}}(\Psi_w))(y) = K^a_{q,w,k_1}K^a_{q,w,k_2}\exp\{(w,y)\sim\}$$

$$= K^a_{q,w,k_1}K^a_{q,w,k_2}\Psi_w(y)$$

(5.11)

for s-a.e. $y \in C^*_a[0,T]$.

On the other hand, using (3.9) together with (3.10), (5.1), the Fubini theorem, and (3.7), it follows that for all $\lambda > 0$ and s-a.e. $y \in C^*_a[0,T]$,

$$T_{\lambda,R_{k_1,k_2}}(\Psi_w)(y)$$

$$= I_{R_{k_1,k_2}}(\Psi_w(y + \lambda^{-1/2}R_{k_1,k_2}(x,.))$$

$$= \exp\{(w,y)^\sim + \lambda^{-1/2}(w,A(k_1,k_2))^\sim\}$$

$$\times I_{Z_{s(k_1,k_2)}}[\exp\{\lambda^{-1/2}(w,Z_{s(k_1,k_2)}(x,.))^\sim\}]$$

(5.12)

$$= \exp\{(w,y)^\sim + \lambda^{-1/2}(w,A(k_1,k_2))^\sim\}$$

$$+ \frac{1}{2\lambda}\|w \circ s(k_1,k_2)\|^2_{C^*_a} + \lambda^{-1/2}(w \circ s(k_1,k_2),a)_{C^*_a}.$$

But, using (3.8), (3.5), and (3.10), it follows that

$$\|w \circ s(k_1,k_2)\|^2_{C^*_a} = \|w \circ k_1\|^2_{C^*_a} + \|w \circ k_2\|^2_{C^*_a}$$

and

$$(w \circ s(k_1,k_2),a)_{C^*_a} + (w,A(k_1,k_2))^\sim = (w \circ k_1,a)_{C^*_a} + (w \circ k_2,a)_{C^*_a}.$$

(5.13)

(5.14)

Thus, using equation (5.12) together with (5.13) and (5.14), it follows that

$$T_{\lambda,R_{k_1,k_2}}(\Psi_w)(y)$$

$$= \exp\{(w,y)^\sim + \frac{1}{2\lambda}\sum_{j=1}^{2}\|w \circ k_j\|^2_{C^*_a} + \lambda^{-1/2}\sum_{j=1}^{2}(w \circ k_j,a)_{C^*_a}\}$$

(5.15)

for s-a.e. $y \in C^*_a[0,T]$. Now, by an analytic continuation, it follows that for s-a.e. $y \in C^*_a[0,T]$,

$$T^{(p)}_{q,R_{k_1,k_2}}(\Psi_w)(y) = K^a_{q,w,k_1}K^a_{q,w,k_2}\Psi_w(y).$$

Equation (5.10) follows from equations (5.11) and (5.15).

Proof complete.

Using an induction argument, we obtain the following corollary.
Corollary 5.6. Given a finite sequence \( \{k_1, \ldots, k_m\} \) of non-zero functions in \( C^*_{a,b}[0,T] \), let \( s(k_1, \ldots, k_m) \) be a function in \( C^*_{a,b}[0,T] \) which satisfies the relation (3.11) and let \( F \) be an exponential-type functionals in \( \mathcal{E}(C_{a,b}[0,T]) \). Then, for all \( p \in [1,2] \) and any non-zero real \( q \), the \( L_p \) analytic \( \mathcal{R}_{k_1, \ldots, k_m} \)-GFFT, \( T^{(p)}_{q,k_1, \ldots, k_m}(F) \) of \( F \) exists and

\[
T^{(p)}_{q,k_1, \ldots, k_m}(F)(y) = T^{(p)}_{q,z_m} \left( \cdots \left( T^{(p)}_{q,z_1}(F) \cdots \right) \right)(y)
\]

for s-a.e. \( y \in C_{a,b}[0,T] \).

Remark 5.7. In view of Corollary 5.6 and Theorem 5.3 with an induction argument, \( T^{(p)}_{q,k_1, \ldots, k_m}(F) \) is an element of \( \mathcal{E}(C_{a,b}[0,T]) \) for each functional \( F \) in \( \mathcal{E}(C_{a,b}[0,T]) \). Moreover, the \( L_p \) analytic \( \mathcal{R}_{k_1, \ldots, k_m} \)-GFFT,

\[
T^{(p)}_{q,k_1, \ldots, k_m} : \mathcal{E}(C_{a,b}[0,T]) \to \mathcal{E}(C_{a,b}[0,T])
\]

is an onto transform.

6. Generalized convolution product with respect to Gaussian process. In this section we define a GCP of functionals on \( C_{a,b}[0,T] \), and investigate the fundamental relationships between the GFFT and the GCP. We first give the definition of the GCP with respect to Gaussian process on the function space \( C_{a,b}[0,T] \).

Definition 6.1. Let \( G \) be a continuous Gaussian process on \( C_{a,b}[0,T] \times [0,T] \). Let \( F \) and \( G \) be scale-invariant measurable functionals on \( C_{a,b}[0,T] \). For \( \lambda \in \mathbb{C}_+ \), we define their GCP \( (F \ast G)_{\lambda,G} \) with respect to the Gaussian process \( G \) (if it exists) by

\[
(F \ast G)_{\lambda,G}(y) = \int_{C_{a,b}[0,T]} F(x) G(y-x) \mu(dx), \quad \lambda \in \mathbb{C}_+
\]

(6.1)

when \( \lambda = -iq \), \( q \in \mathbb{R} \), \( q \neq 0 \).

Remark 6.2. When \( a(t) \equiv 0 \), then GCP with respect to the Gaussian process \( Z_k \) given by (3.1) is commutative. That is to say, \( (F \ast G)_{q,z_k} = (G \ast F)_{q,z_k} \). However, from (6.1) and (3.4), one can see that

\[
(F \ast G)_{q,z_k} = (G \ast F)_{q,-z_k}.
\]

It generally does not hold that \( (F \ast G)_{q,z_k} = (F \ast G)_{q,-Z_k} \), because

\[
\int_{C_{a,b}[0,T]} F(-Z_k(x,\cdot)) \mu(dx) \neq \int_{C_{a,b}[0,T]} F(Z_k(x,\cdot)) \mu(dx)
\]

for almost every functional \( F \) on \( C_{a,b}[0,T] \).

Our first theorem gives an expression for the GCP \( (\Psi_{w_1} \ast \Psi_{w_2})_{q,z_k} \) of exponential functionals \( \Psi_{w_1} \) and \( \Psi_{w_2} \) in \( \mathcal{E} \).

Theorem 6.3. Let \( \Psi_{w_1} \) and \( \Psi_{w_2} \) be exponential functionals in \( \mathcal{E} \). Then the GCP of \( \Psi_{w_1} \) and \( \Psi_{w_2} \), \( (\Psi_{w_1} \ast \Psi_{w_2})_{q,z_k} \) exists for all non-zero real numbers \( q \) and each
non-zero function $k$ in $C_{a,b}^*[0,T]$, and is given by the formula
\[
(\Psi_{w_1} \ast \Psi_{w_2})_{q,z_k} (y) = \exp \left\{ \left( \frac{w_1 + w_2}{\sqrt{2}}, y \right) \right\} + \frac{i}{4q} \left\| (w_1 - w_2) \odot k \right\|^2_{C_{a,b}^*} \right) C_{a,b}^* \right) \right\}
\]
\[
= K^a_{q,(w_1 - w_2)/\sqrt{2},k} \Psi_{(w_1 + w_2)/\sqrt{2}}(y)
\]
for s.a.e. $y \in C_{a,b}[0,T]$. From the third expression of (6.2), we assert that the GCP with respect to $Z_k$, $(\Psi_{w_1} \ast \Psi_{w_2})_{q,z_k}$ is an element of the class $\mathcal{E}(C_{a,b}[0,T])$ for each non-zero function $k$ in $C_{a,b}^*[0,T]$.

**Proof.** Using (3.7) we first observe that for all $\lambda > 0$ and s.a.e. $y \in C_{a,b}[0,T],$
\[
(\Psi_{w_1} \ast \Psi_{w_2})_\lambda z_k (y)
\]
\[
= \int_{C_{a,b}[0,T]} \Psi_{w_1} \left( y + \lambda^{-1/2} Z_k(x, \cdot) \right) \Psi_{w_2} \left( y - \lambda^{-1/2} Z_k(x, \cdot) \right) d\mu(x)
\]
\[
= \exp \left\{ \left( \frac{w_1 + w_2}{\sqrt{2}}, y \right) \right\} \int_{C_{a,b}[0,T]} \exp \left\{ \left( \frac{1}{2\lambda} \left\| (w_1 - w_2) \odot k, x \right\|^2 \right) \right\} d\mu(x)
\]
\[
= \exp \left\{ \left( \frac{w_1 + w_2}{\sqrt{2}}, y \right) \right\} \cdot \exp \left\{ \left( \frac{1}{2\lambda} \left\| (w_1 - w_2) \odot k \right\|^2_{C_{a,b}^*} + (2\lambda)^{-1/2}(\left\| (w_1 - w_2) \odot k, a \right\|_{C_{a,b}^*} \right) \right\}
\]
\[
= \exp \left\{ \left( \frac{w_1 + w_2}{\sqrt{2}}, y \right) \right\} \cdot \exp \left\{ \left( \frac{1}{2\lambda} \left\| (w_1 - w_2) \odot k \right\|^2_{C_{a,b}^*} + \lambda^{-1/2}(\left\| (w_1 - w_2) \odot k, a \right\|_{C_{a,b}^*} \right) \right\}.
\]

But the last expression is an analytic function of $\lambda$ throughout $C_+$. Hence, in view of Definition 6.1, $(\Psi_{w_1} \ast \Psi_{w_2})_{q,z_k}$ exists and is given by the second expression of equation (6.2) for all $q \in \mathbb{R} \setminus \{0\}$. The third expression of equation (6.2) follows from the conventions (5.3) and (5.2). Next, by the fact that $\mathcal{E}(C_{a,b}[0,T])$ forms a complex algebra under the pointwise multiplication (see Remark 5.1 above), one can see that
\[
(\Psi_{w_1} \ast \Psi_{w_2})_{q,z_k} \approx K^a_{q,(w_1 - w_2)/\sqrt{2},k} \Psi_{(w_1 + w_2)/\sqrt{2}} \approx K^a_{q,(w_1 - w_2)/\sqrt{2},k} \Psi_{w_1/\sqrt{2}} \Psi_{w_2/\sqrt{2}}
\]
is an element of the complex algebra $\mathcal{E}(C_{a,b}[0,T])$.

Throughout the rest of this paper, for any functionals $F$ and $G$ in $\mathcal{E}(C_{a,b}[0,T])$, we will always express $F$ by (5.6), and $G$ by
\[
G \approx \sum_{l=1}^m c_l \Psi_{v_l}
\]
where $\{v_1, \ldots, v_m\}$ is a finite sequence of functions in $C_{a,b}^*[0,T]$, $\{e_1, \ldots, e_m\}$ is a sequence in $\mathbb{C} \setminus \{0\}$, and $\Psi_{v_l}$ is an exponential functional given by (5.1) with $w$ replaced with $v_l$ for each $l \in \{1, \ldots, m\}$. 

\[
(\Psi_{w_1} \ast \Psi_{w_2})_{q,z_k} \approx K^a_{q,(w_1 - w_2)/\sqrt{2},k} \Psi_{(w_1 + w_2)/\sqrt{2}} \approx K^a_{q,(w_1 - w_2)/\sqrt{2},k} \Psi_{w_1/\sqrt{2}} \Psi_{w_2/\sqrt{2}}
\]
Thus, we assert that \( \Psi \) define an operation

\[
(F * G)_{q,z_k} = \sum_{j=1}^{n} \sum_{l=1}^{m} c_j c_l (\Psi_{w_j} * \Psi_{v_l})_{q,z_k}.
\]

Thus, we assert that \((F * G)_{q,z_k}\) exists and is an element of the class \(E(C_{a,b}[0,T])\).

**Remark 6.5.** Given a non-zero real \( q \) and a non-zero function \( k \) in \( C_{a,b}^{*}[0,T] \), define an operation \(*_{q,k}\) on \( E(C_{a,b}[0,T]) \) as follows: for any functionals \( F \) and \( G \) in \( E(C_{a,b}[0,T]) \), let

\[
*_{q,k}(F,G) = (F * G)_{q,z_k}.
\]

Then, by Theorem 6.4, the operation \(*_{q,k}\) is well-defined and thus the linear space \( E(C_{a,b}[0,T]) \) is a noncommutative algebra with the operation \(*_{q,k}\).

Using the next two theorems, we establish interesting relationships between the GFFT and the GCP defined on the function space \( C_{a,b}[0,T] \). These results are improvements of (1.2) and (1.4) holding on the Wiener space \( C[0,T] \), to the function space \( C_{a,b}[0,T] \).

**Theorem 6.6.** Let \( \Psi_{w_1} \) and \( \Psi_{w_2} \) be exponential functionals in \( E \). Then for all \( p \in [1,2] \), any non-zero real \( q \), and each non-zero function \( k \) in \( C_{a,b}^{*}[0,T] \),

\[
T_{q,z_k}^{(p)}((\Psi_{w_1} * \Psi_{w_2})_{q,z_k})(y) = T_{q,\mathcal{R}_{-k}/\sqrt{2},k/\sqrt{2}}^{(p)}(\Psi_{w_1})(y) \quad (6.4)
\]

and

\[
T_{q,z_k}^{(p)}((\Psi_{w_1} * \Psi_{w_2})_{q,z_k})(y) = T_{q,\mathcal{R}_{-k}/\sqrt{2},k/\sqrt{2}}^{(p)}(\Psi_{w_2})(y) \quad (6.5)
\]

for s.a.e. \( y \in C_{a,b}[0,T] \), respectively.

**Proof.** Using the third expression of (6.2), (5.5) with \( \Psi_w \) and \( \Psi_{w_1}^{q,k} \) replaced with \( \Psi_{w_1+w_2}/\sqrt{2} \) and \( \Psi_{w_1}^{q,k} \) replaced with \( \Psi_{w}(w_1+w_2)/\sqrt{2} \), respectively, and (5.4) with \( w \) replaced with \((w_1+w_2)/\sqrt{2} \), it follows that for s.a.e. \( y \in C_{a,b}[0,T] \),

\[
T_{q,z_k}^{(p)}((\Psi_{w_1} * \Psi_{w_2})_{q,z_k})(y) = K_{q,(w_1-w_2)/\sqrt{2},k/\sqrt{2}}^{(p)}(\Psi_{w_1+w_2}/\sqrt{2})(y)
\]

\[
= K_{q,(w_1-w_2)/\sqrt{2},k}^{a} \Psi_{w_1}^{q,k}((w_1+w_2)/\sqrt{2})(y)
\]

\[
= K_{q,(w_1-w_2)/\sqrt{2},k}^{a} K_{q,(w_1+w_2)/\sqrt{2},k}^{a} \Psi_{w_1+w_2}/\sqrt{2}(y)
\]

\[
= \exp \left\{ \frac{i}{2q} \left( \frac{w_1-w_2}{\sqrt{2}} \right) \otimes k \right\}_{C_{a,b}^{*}}^{2} + (-iq)^{-1/2} \left( \frac{w_1-w_2}{\sqrt{2}} \right) \otimes k, a \right\}_{C_{a,b}^{*}}
\]

\[
\times \Psi_{w_1}/\sqrt{2}(y) \Psi_{w_2}/\sqrt{2}(y). \]
Corollary 6.8. Given a finite sequence \( \{k_1, \ldots, k_n\} \) of non-zero functions in \( C_{a,b}^*[0,T] \), let \( s(k_1, \ldots, k_n) \) be a function in \( C_{a,b}^*[0,T] \) which satisfies equation (3.11), and let 

Next, using the parallelogram equality 

\[
\|w_1 + w_2\|^2_{C_{a,b}^*} + \|w_1 - w_2\|^2_{C_{a,b}^*} = 2(\|w_1\|^2_{C_{a,b}^*} + \|w_2\|^2_{C_{a,b}^*}),
\]

it follows that for s.a.e. \( y \in C_{a,b}[0,T] \), 

\[
T_{q,z_k}(w_1 * w_2)(y) = \exp \left\{ \frac{i}{2q} \left( \|w_1 \circ k\|^2_{C_{a,b}^*} + \|w_2 \circ k\|^2_{C_{a,b}^*} - 2(-2iq)^{-1/2}(w_1 \circ k, a)_{C_{a,b}^*} \right) \right\} \times \Psi_{w_1}(y/\sqrt{2}) \Psi_{w_2}(y/\sqrt{2}).
\]

On the other hand, applying equations (5.9) with \( F \) replaced with \( \Psi_{w_j}, j \in \{1,2\} \), and (5.11) replaced with \( \Psi_{w_j}, j \in \{1,2\} \), one can see that for s.a.e. \( y \in C_{a,b}[0,T] \), 

\[
T_{q,z_k}^{(p)}(\Psi_{w_1}^{\alpha} \Psi_{w_2}) \left( \frac{y}{\sqrt{2}} \right) T_{q,z_k}^{(p)}(\Psi_{w_1}^{\alpha} \Psi_{w_2}) \left( \frac{y}{\sqrt{2}} \right) = K_{q,w_1,k/\sqrt{2}}^{a} K_{q,w_2,k/\sqrt{2}}^{a} \Psi_{w_1}(y/\sqrt{2}) K_{q,w_2,k/\sqrt{2}}^{a} \Psi_{w_2}(y/\sqrt{2}).
\]

Next, using (5.3), the last expression of (6.7) also is given by the right hand side of equation (6.6). Using similar methods as those used in the proof of equation (6.4), it also follows (6.5).

**Theorem 6.7.** Let \( F \) and \( G \) be exponential-type functionals in \( E(C_{a,b}[0,T]) \). Then for all \( p \in [1,2] \), any non-zero real \( q \), and each non-zero function \( k \) in \( C_{a,b}^*[0,T] \), 

\[
T_{q,z_k}^{(p)}(F \ast G) \left( \frac{y}{\sqrt{2}} \right) = T_{q,z_k}^{(p)}(F) \left( \frac{y}{\sqrt{2}} \right) T_{q,z_k}^{(p)}(G) \left( \frac{y}{\sqrt{2}} \right)
\]

and 

\[
T_{q,z_k}^{(p)}(s(F \ast G)) \left( \frac{y}{\sqrt{2}} \right) = T_{q,z_k}^{(p)}(s(F)) \left( \frac{y}{\sqrt{2}} \right) T_{q,z_k}^{(p)}(s(G)) \left( \frac{y}{\sqrt{2}} \right)
\]

for s.a.e. \( y \in C_{a,b}[0,T] \), respectively.

**Proof.** Let \( \Psi_{\alpha}, \Psi_{\beta} \) and \( \Psi_{\gamma} \) be exponential functionals. Then it follows that 

\[
\Psi_{\gamma} \ast (\Psi_{\alpha} \ast \Psi_{\beta}) \approx \Psi_{\gamma} \ast (\Psi_{\alpha} \ast \Psi_{\beta}).
\]

Thus, for any functionals \( F \) and \( G \) in \( E(C_{a,b}[0,T]) \), using their representations of linear combinations of exponential functionals given by (5.6) and (6.3), and (6.10), and applying the linearity of the GFFTs \( T_{q,k}^{(p)} \) and \( T_{q,-k}^{(p)} \), it follows that equations (6.8) and (6.9) hold with \( \Psi_{w_1} \) and \( \Psi_{w_2} \) replaced with \( F \) and \( G \), respectively.

Using equation (5.16) with \( F \) replaced with \( (F \ast G)_{q,R_{k_1},\ldots,k_n} \), and an induction argument, we obtain the following corollary.

**Corollary 6.8.** Given a finite sequence \( \{k_1, \ldots, k_n\} \) of non-zero functions in \( C_{a,b}^*[0,T] \), let \( s(k_1, \ldots, k_n) \) be a function in \( C_{a,b}^*[0,T] \) which satisfies equation (3.11), and let
$F$ and $G$ be exponential-type functionals in $\mathcal{E}(C_{a,b}[0,T])$. Then, for all $p \in [1,2]$ and all real $q$, it follows that

$$
T^{(p)}_{q,Z_{k_n}} \left( T^{(p)}_{q,Z_{k_{n-1}}} \left( \cdots \left( T^{(p)}_{q,Z_{k_1}} \left( (F \ast G)_{q,R_{k_1},\ldots,k_n} \right) \right) \cdots \right) \right)(y)
$$

and

$$
T^{(p)}_{p,Z_{-k_n}} \left( T^{(p)}_{p,Z_{-k_{n-1}}} \left( \cdots \left( T^{(p)}_{p,Z_{-k_1}} \left( (F \ast G)_{q,R_{k_1},\ldots,k_n} \right) \right) \cdots \right) \right)(y)
$$

for s.a.e. $y \in C_{a,b}[0,T]$, respectively.

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