ARITHMETIC SPARSITY IN MIXED HODGE SETTINGS

KENNETH CHUNG TAK CHIU

Abstract. Let $V \to X$ be a surjective quasi-projective morphism between irreducible quasi-projective varieties over a number field, where $X$ is smooth. We prove that there exists a non-empty Zariski open subset of $X$ such that the $S$-integral points in it are covered by subpolynomially many geometrically irreducible subvarieties, each lying in a fiber of the mixed period mapping. We also prove analogous theorems for semi-simplicial varieties and abstract variations of mixed Hodge structures. They are based on recent works by Brunebarbe-Maculan and Ellenberg-Lawrence-Venkatesh. As an application, we prove that there are subpolynomially many $S$-integral Laurent polynomials with fixed reflexive Newton polyhedron $\Delta$ and fixed non-zero principal $\Delta$-determinant. Our results answer a question in Ellenberg-Lawrence-Venkatesh.

1. Introduction

Faltings theorem \cite{Faltings} states that curves of genus $\geq 2$ over $\mathbb{Q}$ have only finitely many rational points. In higher dimensions, Bombieri-Lang conjecture states that rational points of a variety of general type defined over a number field $K$ are not Zariski-dense, in other words, finitely many irreducible algebraic $K$-subvarieties properly contained in the variety are enough to cover these rational points. One could also look at weaker statements where rational points are replaced by $S$-integral points, where $S$ is a finite set of primes. Same questions can also be asked for $S$-integral points in moduli spaces of varieties with fixed set of properties. There were affirmative answers to these questions in the case for abelian varieties of fixed dimension \cite{Faltings}, curves of fixed genus \cite{Faltings}, hypersurfaces of fixed large degree \cite{Faltings}, and smooth hypersurfaces representing an ample class in the Neron-Severi group of an abelian variety \cite{Faltings}.

We aim at proving statements where non-density is weakened to sparsity, i.e. subpolynomial growth rate, in terms of the heights of the $S$-integral points in a dense open subset. Given a smooth irreducible quasi-projective $K$-variety equipped with a surjective quasi-projective morphism onto it, we will prove that subpolynomially many certain $K$-subvarieties are sufficient to cover the $S$-integral points of bounded height in certain dense open subset of this variety. The $K$-subvarieties to be counted are lying in fibers of a mixed period mapping which depends on the given morphism. Mixed Hodge theory and in particular simplicial
methods have to be used since the morphism is not assumed to be smooth and projective.

We will apply this result to prove sparsity of $S$-integral Laurent polynomials with fixed reflexive Newton polyhedron $\Delta$ and fixed non-zero principal $\Delta$-determinant. The notion of principal $\Delta$-determinant was defined by Gelfand-Kapranov-Zelevinsky \[12\]. Properties of principal $\Delta$-determinant and Batyrev’s infinitesimal Torelli theorem \[1\] will be used to prove our result. It was shown by Batyrev that reflexivity of the polyhedron is a necessary and sufficient condition for the characterization of $\Delta$-regular affine hypersurfaces in tori that have Calabi-Yau projective closure in the toric variety with only canonical singularities \[1\] Theorem 12.2].

Our paper is motivated by and based on recent works of Ellenberg-Lawrence-Venkatesh \[8\] and Brunebarbe-Maculan \[5\]. The question of what happens in the mixed Hodge setting was asked by Ellenberg-Lawrence-Venkatesh \[8\] p. 4]. Ellenberg-Lawrence-Venkatesh \[8\] proved that given a variety over $K$ admitting a geometric variation of pure Hodge structures on it, its $S$-integral points with height at most $B$ are covered by $O_\varepsilon(B^\varepsilon)$ geometrically irreducible $K$-subvarieties, each lying in a single fiber of the period mapping arising from the variation. They also deduced from their result that there are $O_\varepsilon(B^\varepsilon)$ regular $S$-integral homogeneous polynomials with fixed number of variables, degree $d \geq 3$, and discriminant, up to the action by an arithmetic group. The condition on the degree in their sparsity results are relaxed in comparison with the non-density result in Lawrence-Venkatesh \[19\]. In our case for Laurent polynomials, we do not have to count up to the action by a group. Brunebarbe-Maculan \[5\] proved that given a quasi-projective variety over $K$ with large geometric étale fundamental group, its number of $S$-integral points of bounded height, with respect to line bundles on the projective closure, grow subpolynomially.

1.1. Arithmetic sparsity in mixed Hodge settings. Let $K$ be a finite extension of $\mathbb{Q}$ with an embedding into $\mathbb{C}$. Let $X$ be a smooth irreducible quasi-projective algebraic variety over $K$ with an embedding into the projective space $\mathbb{P}^m_K$ for some positive integer $m$. Let $S$ be a finite set of primes of $K$ such that $X$ has a good integral model over the ring $\mathcal{O}_{K,S}$ of $S$-integers. Our goal is to prove the following main theorem, which is a mixed Hodge analogue of \[8\] Theorem 1.2]:

**Theorem 1.1.** Let $\pi : V \to X$ be a surjective quasi-projective morphism over $K$ from an irreducible algebraic variety $V$. There exists a non-empty Zariski open subset (constructed via resolutions and generic smoothness) $X^*$ of $X$ such that for any $i$, the higher direct image with compact support $(R^i(\pi|_{V^*})\cdot \mathbb{Q}_{V^*})|_{X^*}$
underlies an admissible graded-polarized variation of $\mathbb{Q}$-mixed Hodge structures, and such that for any $\varepsilon > 0$, the $S$-integral points of $X^*$ with height at most $B$ are covered by $O_\varepsilon(B^*)$ geometrically irreducible $K$-subvarieties, each lying in a single fiber of the mixed period mapping $\Phi$ arising from the variation.

The analogous theorem over $\mathbb{C}$ without the condition about integral points is proved by Brosnan-El Zein [4, Cor. 8.1.22] and Fujino-Fujisawa [11, Theorem 4.13]. Due to Theorem 1.1 to count points in $X$, we only have to count points in each fiber of a period mapping. This will become useful when the period mapping is not a constant map, i.e. the variation of mixed Hodge structure is not trivial, say for example in situations where the infinitesimal Torelli theorems hold.

Theorem 1.1 will follow from an analogous theorem for morphisms of semi-simplicial varieties over $K$ that we now state. We refer the reader to Section 3.1 for the notion of semi-simplicial varieties. Suppose we have the following commutative diagram of semi-simplicial varieties over $K$ with finitely many non-empty faces:

$$
\begin{array}{ccc}
D_\bullet & \xrightarrow{\varphi} & Y_\bullet \\
\downarrow{g} & & \downarrow{f} \\
X & & 
\end{array}
$$

such that $f$ and $g$ are smooth and projective. Base changing to $\mathbb{C}$ gives a similar commutative diagram of semi-simplicial varieties. Let $C(\varphi^{-1}_{an})$ be the cone of the canonical morphism $\varphi^{-1}_{an}: R(f_{C,an})_*\mathbb{Z}_{Y_\bullet, C} \to R(g_{C,an})_*\mathbb{Z}_{D_\bullet, C}$ of complexes (see Section 3.1 for definitions). For any $i$, the local system $H^i(C(\varphi^{-1}_{an}))$ underlies an admissible graded-polarized variations of $\mathbb{Q}$-mixed Hodge structures [11, Lemma 4.12].

**Theorem 1.2.** With the above set-up, for any $\varepsilon > 0$, the $S$-integral points of $X$ with height at most $B$ are covered by $O_\varepsilon(B^*)$ geometrically irreducible $K$-subvarieties, each lying in a single fiber of the mixed period mapping $\Phi$ arising from the variation for which $H^i(C(\varphi^{-1}_{an}))$ underlies.

Theorem 1.2 will be proved using the following analogous theorem for abstract variations of mixed Hodge structures:

**Theorem 1.3.** Let $\mathcal{L}_{an}$ be a local system of $\mathbb{Z}$-modules on the analytification $X^n_{\mathbb{C}}$. Suppose $\mathcal{L}_{an}$ underlies an admissible graded-polarized variation of $\mathbb{Q}$-mixed Hodge structures (VMHS). Let $p$ be a prime for which $\mathcal{L}_{an}$ is $p$-torsion-free. Suppose for each positive integer $n$, there exists on $X_K$ an étale local system $\mathcal{L}_{n, \text{ét}}$ of $\mathbb{Z}/p^n\mathbb{Z}$-modules such that $(\mathcal{L}_{n, \text{ét}, \mathbb{C}})_{an} \simeq \mathcal{L}_{an} \otimes (\mathbb{Z}/p^n\mathbb{Z})$, where $\mathcal{L}_{n, \text{ét}, \mathbb{C}}$ is the pullback of $\mathcal{L}_{n, \text{ét}}$ to $X_{\mathbb{C}}$. Then for any $\varepsilon > 0$, the $S$-integral points of $X$ with height at most
are covered by $O_{\epsilon}(B^\epsilon)$ geometrically irreducible $K$-subvarieties, each lying in a single fiber of the mixed period mapping $\Phi$ arising from the variation.

The techniques of the proof of Theorem 1.3 will be based on the recent papers by Brunebarbe-Maculan [5] and Ellenberg-Lawrence-Venkatesh [8]: the geometric portion of the argument is to construct covers of $X$ under which the preimages of certain subvarieties have large degrees, by using a result in [5] about small degree normal cycles; the arithmetic portion of the argument is to apply Broberg’s theorem [3] (which builds on fundamental ideas of Bombieri-Pila [2] and Heath-Brown [15]) about the number of divisors of bounded degrees required to contain the rational points of bounded height of an arbitrary irreducible closed subvariety of fixed dimension and degree.

1.2. Sparsity of $S$-integral Laurent polynomials with fixed data. We discuss an application of Theorem 1.1. Let $n$ be a positive integer. Let $L$ be the Laurent polynomial ring $\mathbb{C}[X_1^\pm, \ldots, X_n^\pm]$. Let $T := \text{Spec} L \cong (\mathbb{C}^\times)^n$. An $n$-dimensional convex polyhedron $\Delta$ in $\mathbb{R}^n$ whose interior contains the origin is said to be integral if all vertices of $\Delta$ belong to the lattice $\mathbb{Z}^n$. Such polyhedron is said to be reflexive if its dual polyhedron

$$\Delta^* := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i y_i \geq -1 \text{ for all } (y_1, \ldots, y_n) \in \Delta\}$$

is integral. Let $M$ be the free abelian group of rank $n$. The Newton polyhedron $\Delta(L)$ of a Laurent polynomial $L = \sum a_m X^m \in L$ is the convex hull of integral points $m \in M$ such that $a_m \neq 0$. Let $L(\Delta)$ be the space of all Laurent polynomials with the Newton polyhedron $\Delta$.

Every Laurent polynomial $L$ defines the affine hypersurface $Z_L := \{X \in T : L(X) = 0\}$. For any $L = \sum a_m X^m \in L(\Delta)$ and any $l$-dimensional face $\Delta'$ of $\Delta$, let

$$L^{\Delta'}(X) := \sum_{m \in \Delta'} a'_m X^{m'}.$$  

A Laurent polynomial $L \in L(\Delta)$ and its affine hypersurface $Z_L$ are said to be $\Delta$-regular if for every $l$-dimensional edge $\Delta' \subset \Delta$ ($l > 0$), the polynomial equations

$$L^{\Delta'}(X) = X_1 \frac{\partial}{\partial X_1} L^{\Delta'}(X) = \cdots = X_n \frac{\partial}{\partial X_n} L^{\Delta'}(X) = 0.$$  

have no common solutions in $T$.

Gelfand, Kapranov, and Zelevinski introduced the principal $\Delta$-determinant $\text{Disc}_\Delta(L)$ of a Laurent polynomial $L$ with Newton polyhedron $\Delta$. It is a certain complex number attached to $L$. Since its definition is complicated to state and we are only using its properties, we refer the interested reader to Chapter 10 of
their book [12]. A Laurent polynomial $L$ is $\Delta$-regular if and only if its principal $\Delta$-determinant $\text{Disc}_\Delta(L) \neq 0$ [1 Prop. 4.16].

In Section 2 we will use Theorem 1.1 to prove the following theorem:

**Theorem 1.4.** Let $S$ be a finite set of rational primes. Let $\Delta := \Delta \cap \mathbb{Z}^n$. Let $N \in \mathbb{Q}^\times$. For any $\varepsilon > 0$, there are $O_{\Delta,N,S,\varepsilon}(B^\varepsilon)$ Laurent polynomials $L = \sum_{m \in A} \{a_m\} X^m$ with Newton polyhedron $\Delta$ and principal $\Delta$-determinant $N$, and with $S$-integral coefficients such that

$$\max_{m \in A, v \in S \cup \{\infty\}} |a_m|_v \leq B.$$

1.3. **Acknowledgements.** I would like to thank Jacob Tsimerman for helpful discussions.

## 2. Reduction of Theorem 1.4 to Theorem 1.1

Let $S$ be a finite set of rational primes. Let $\Delta$ be an $n$-dimensional reflexive polyhedron. Let $A := \Delta \cap \mathbb{Z}^n$. Let $\mathbb{C}^A$ be the affine complex space parametrizing coefficients of Laurent polynomials $L = \sum_{m \in A} a_m X^m$. The multiplicative group $\mathbb{C}^\times$ acts on $\mathbb{C}^A$ by componentwise multiplication. The complex tori $T := (\mathbb{C}^\times)^n$ acts on $\mathbb{C}^A$ by

$$t \cdot \left( \sum_{m \in A} a_m X^m \right) = \sum_{m \in A} t^m a_m X^m$$

for any $t \in T$ and $\sum a_m X^m \in \mathbb{C}^A$. The $\mathbb{C}^\times$-action and the $T$-action preserve $\Delta$-regularity [1 Prop. 11.2 or Prop. 4.6].

### 2.1. Lemmas on $S$-integral Laurent polynomials.

**Lemma 2.1.** Let $N \in \mathbb{Q}^\times$. Suppose $L = \sum_{m \in A} a_m X^m$ is a Laurent polynomial with principal $\Delta$-determinant $N$ and $S$-integral coefficients such that

$$\max_{m \in A, v \in S \cup \{\infty\}} |a_m|_v \leq B.$$

Let $m'$ be a vertex of $\Delta$. Then $a_{m'}$ can only attain $O_{\Delta,N,S,m'}((\log B)^{|S|})$ possible values.

**Proof.** By [12 Cor. 2.5, p. 318], since $m'$ is a vertex of $\Delta$, we have $\text{Disc}_\Delta(L) = c \cdot a_{m'}^k \cdot h$, where $c \in \mathbb{Q}$, $k$ is some non-negative integer, and $h$ is the value at $(a_m)_{m \in A}$ of some polynomial $\eta$ in the ring $\mathbb{Z}[A]$. Here $c$ depends only on $\Delta$, while $k$ and $\eta$ depend only on $\Delta$ and $m'$. Write $c = r/q$, $a_{m'} = r'/q'$, $h = r''/q''$, $a_m = r_m/q_m$, and $N = r_N/q_N$, which are fractions in the lowest terms with $q, q', q'', q_m, q_N > 0$. We have

$$\frac{r}{q} \cdot \left( \frac{r'}{q'} \right)^k \cdot \frac{r''}{q''} = \text{Disc}_\Delta(f) = N = \frac{r_N}{q_N},$$

where $f$ is a polynomial in $\mathbb{Z}[A]$. This shows that $a_{m'}$ can only attain $O((\log B)^{|S|})$ different values.
so $r'$ divides $r_Nq q''$. There are only $O_{\Delta,N}(1)$ choices for the divisors of $r_N$ and $q$. We also know that $q''$ divides a monomial $\prod f_m^{e_m}$, where the powers $f_m$ depends only on $\eta$, which in turn depends only on $\Delta$ and $m'$. Write $S = \{v_1, \ldots, v_\ell\}$ and $q_m = v_1^{e_{m,1}} \cdots v_\ell^{e_{m,\ell}}$. The monomial is then equal to

$$\prod_{i=1}^\ell v_i^{\sum_{m \in A} f_m e_{m,i}}.$$ 

For all $i = 1, \ldots, \ell$ and $m \in A$, we have $v_i^{e_{m,i}} \leq B$, so

$$\sum_{m \in A} f_m e_{m,i} \leq \sum_{m \in A} f_m \log B = O_{\Delta,m',S}(\log B),$$

thus there are only $O_{\Delta,m',S}(\log B)|S)$ choices for the divisors of $q''$. Similarly, $q' = q_m$ can only attain $O_{S}(\log B)|S)$ possible values. Multiplying all the bounds together, the proof is completed. \qed

**Lemma 2.2.** Let $N \in \mathbb{Q}^\times$. Each $(\mathbb{C}^\times \times T)$-orbit has

$$O_{\Delta,N,S}(\log B)^{(n+1)|S|)}$$

Laurent polynomials with Newton polyhedron $\Delta$, principal $\Delta$-determinant $N$, and $S$-integral coefficients $a_m$ such that

$$\max_{m \in A, v \in S \cup \{\infty\}} |a_m|_v \leq B.$$

**Proof.** Let $L = \sum_{m \in A} a_m X^m$ be such Laurent polynomial in the orbit. Let $(\alpha, t) \in \mathbb{C}^\times \times T$. Suppose $(\alpha, t) \cdot L$ is again such Laurent polynomial. Write $\alpha = r_\alpha e^{i \theta_\alpha}$ and $t = (r_1 e^{i \theta_1}, \ldots, r_n e^{i \theta_n})$, where $r_\alpha, r_1, \ldots, r_n > 0$ and $0 \leq \theta_\alpha, \theta_1, \ldots, \theta_n < 2\pi$. Write $m = (m_1, \ldots, m_n)$. We have

$$(\alpha, t) \cdot L = \sum_{m \in A} \alpha a_m t^{m} X^m = \sum_{m \in A} r_\alpha a_m r_1^{m_1} \cdots r_n^{m_n} e^{i(\theta_\alpha + m_1 \theta_1 + \cdots + m_n \theta_n)} X^m.$$ 

Since it has $S$-integral coefficients, $\theta_\alpha + m_1 \theta_1 + \cdots + m_n \theta_n = \pi$ or 0, modulo $2\pi$, for each $m \in A$; so this sum can only attain finitely many possible values since it is bounded by $2\pi(1 + |m_1| + \cdots + |m_n|)$. Denote this finite set of possible values by $\Omega_m$, for each $m \in A$. Since $\Delta$ is $n$-dimensional, we can pick vertices $m^{(1)}, \ldots, m^{(n)} \in A$ such that the $n \times n$ matrix $(m_{ij})$, where $m_{ij} = m^{(i)}_j$, has rank $n$. Since $\Delta$ has at least $n+1$ vertices, we can pick a vertex $m^{(n+1)}$ pairwise distinct from $m^{(1)}, \ldots, m^{(n)}$. Since $m^{(n+1)}$ cannot lie in the simplex $[m^{(1)}, \ldots, m^{(n)}]$, the system

$$\begin{cases} 
  s_1 + \cdots + s_n = 1 \\
  s_1 m^{(1)} + \cdots + s_n m^{(n)} = m^{(n+1)}
\end{cases}$$
has no solution. Therefore, the \((n+1) \times (n+1)\) matrix

\[
R := \begin{pmatrix}
1 & (m_{ij}) \\
0 & m^{(n+1)}
\end{pmatrix}
\]

has rank \(n+1\). For each \(i = 1, \ldots, n+1\), since \(m^{(i)}\) is a vertex, the coefficient of the term \(X^{m^{(i)}}\) in \((\alpha, t) \cdot L\) can only attain \(O_{\Delta, N, S, m^{(i)}}((\log B)^{[S]})\) possible values by Lemma 2.1. Let \(d_i\) be one of such possible value for each \(i\). Since \((\alpha, t) \cdot L\) has Newton polyhedron \(\Delta\), we know \(d_i \neq 0\). Since \(R\) has full rank, the system

\[
\log \alpha + m^{(i)}_1 \log r_1 + \cdots + m^{(i)}_n \log r_n = -\log |a_{m^{(i)}}| + \log |d_i|, \quad i = 1, \ldots, n+1
\]

has a unique solution for \((r_\alpha, r_1, \ldots, r_n)\). Let \((\theta^{(1)}, \ldots, \theta^{(n+1)}) \in \Omega_{m^{(1)}} \times \cdots \times \Omega_{m^{(n+1)}}\). Since \(R\) has full rank, the system

\[
\theta_\alpha + m^{(i)}_1 \theta_1 + \cdots + m^{(i)}_n \theta_n = \theta^{(i)}, \quad i = 1, \ldots, n+1
\]

has a unique solution for \((\theta_\alpha, \theta_1, \ldots, \theta_n)\).

\(\square\)

2.2. Jacobian ideals and Jacobian rings. We first recall the notions of Jacobian ideals and Jacobian rings of Laurent polynomials in Batyrev’s paper [1]. Let \(S_\Delta\) be the subalgebra of \(L[X] = \mathbb{C}[X_0, X_1^\pm, \ldots, X_n^\pm]\) generated as a \(\mathbb{C}\)-vector space by elements of \(\mathbb{C}\) and all monomials \(X_0^a X_1^{m_1} \cdots X_n^{m_n}\) such that the the rational point \((m_1/k, \ldots, m_n/k)\) belongs to \(\Delta\). The standard grading of \(L[X]\) induces the grading of \(S_\Delta\). Let \(S_\Delta^i\) be the \(i\)-th homogeneous component. Let \(S_\Delta^+\) be the maximal homogeneous ideal in \(S_\Delta\). For any \(L \in L\), define \(L(X_0, X) := X_0 L(X) - 1\). For any \(i = 0, \ldots, n\),

\[
L_i(X_0, X) := X_i \frac{\partial}{\partial X_i} L(X_0, X).
\]

The ideal \(J_{L, \Delta}\) of \(S_\Delta\) generated by \(L_0, L_1, \ldots, L_n\) is called the Jacobian ideal of \(L\). The quotient ring \(R_L := S_\Delta/J_{L, \Delta}\) is called the Jacobian ring of \(L\). The grading of \(S_\Delta\) induces a grading of \(R_L\). Let \(R_L^i\) be the \(i\)-th homogeneous component. Let \(R_L^+\) be the maximal homogeneous ideal in \(R_L\).

2.3. Reduction of Theorem 1.3 to Theorem 1.1. Let \(\mathbb{C}_{\Delta, \text{reg}}^A\) be Zariski open subset of \(\mathbb{C}^A\) parametrizing \(\Delta\)-regular Laurent polynomials with Newton polyhedron \(\Delta\). For generic \(L_\Delta = \sum_{m \in A} a_m X^m\) with Newton polyhedron \(\Delta\), the principal \(\Delta\)-determinant \(\text{Disc}_\Delta(L_\Delta)\) is a polynomial over \(\mathbb{Q}\) in the indeterminates \(a_m\) [12, Cor. 2.5, p. 318]. Let \(Y\) be the Zariski open subset of \(\mathbb{Q}^A\) defined by \(\text{Disc}_\Delta(L_\Delta) \neq 0\) and \(a_m' \neq 0\) for any vertex \(m'\) of \(\Delta\). We have \(Y_\mathbb{C} = \mathbb{C}_{\Delta, \text{reg}}^A\). Let \(f : V \to Y\) be the universal family of the affine hypersurfaces defined by these Laurent polynomials.

By results of Batyrev [1, Theorem 8.2, Cor. 3.14], the dimensions of the weight filtrations and the Hodge filtrations for the \((n-1)\)-th cohomology stay
the same as $L$ varies in $Y_C$. By computation of the Gauss-Manin connection of the universal family $f_C : V_C \to Y_C$ by Batyrev \cite{Batyrev1994} Prop. 11.5, Theorem 11.6, Theorem 7.13, the differential of the period mapping at any $L \in Y_C$ is induced by the composition $S^1_\Delta \to R^1_L \to \text{End} R^+_L$, where the first map comes from quotienting $J^1_L := J_{L,\Delta} \cap S^1_\Delta$, and the second map comes from the $R^1_L$-module multiplication $R^1_L \otimes R^+_L \to R^+_L$. As $\Delta$ is reflexive, $R^1_L \to \text{End} R^+_L$ is injective by \cite{Batyrev1994} Theorem 12.2 (vi). By \cite{Batyrev1994} Prop. 11.2, the Jacobian ideal $J^1_L$ is isomorphic to the tangent space of the orbit $(\mathbb{C}^\times \times T) \cdot L$ at $L$. Therefore, a tangent vector at $L$ is in the kernel of the differential of $\Phi$ if and only if it is tangent to $(\mathbb{C}^\times \times T) \cdot L$.

One one hand, $\Phi$ has zero differential at every point in $(\mathbb{C}^\times \times T) \cdot L$, so $\Phi$ is constant on $(\mathbb{C}^\times \times T) \cdot L$. On the other hand, dimension a fiber of $\Phi$ is at most the dimension of the kernel of the differential of $\Phi$ at a generic point in the fiber. This dimension is in turn smaller than the dimension of a $(\mathbb{C}^\times \times T)$-orbit contained in the fiber by what we have proved. Hence, any connected component of a fiber of $\Phi$ is a $(\mathbb{C}^\times \times T)$-orbit.

By Theorem \cite{Batyrev1994} there exists a non-empty Zariski open subset $Y^*$ of $Y$ such that for any $\epsilon > 0$, the $S$-integral points of $Y^*$ with height at most $B$ are covered by $O_{\Delta,N,\epsilon}(B^\epsilon)$ geometrically irreducible $\mathbb{Q}$-subvarieties, whose collection is denoted by $\{Y_a\}$, each lying in a single fiber of the period mapping restricted to $Y_C^*$.

Since the affine hypersurfaces in the universal family $f$ are smooth, and since we are looking at the middle cohomology, the sheaf $(R^{n-1}(f|_{V_C})_* \mathcal{Q}_{V_C})|_{Y_C^*}$ is dual to $(R^{n-1}(f|_{V_C})_* \mathcal{Q}_{V_C})|_{Y_C^*}$ by \cite{Deligne1994} Lemma-Def. 6.25, Cor. 6.26]. Hence, we can replace the period mapping attached to higher direct image with compact support in the previous paragraph by the period mapping attached to the usual higher direct image.

Since each $Y_a$ is geometrically irreducible and is contained in a fiber of the period mapping, it is contained in a $(\mathbb{C}^\times \times T)$-orbit. Therefore, the $S$-integral points of $Y^*$ with height at most $B$ are covered by $O_{\Delta,N,\epsilon}(B^\epsilon)$ $(\mathbb{C}^\times \times T)$-orbits.

By applying Theorem \cite{Batyrev1994} again, with irreducible components of $Y \setminus Y^*$ instead of $Y$; and repeat with irreducible subvarieties of smaller and smaller dimensions, we know that the $\Delta$-regular Laurent polynomials with Newton polyhedron $\Delta$ and $S$-integral coefficients of heights at most $B$, are in $O_{\Delta,N,\epsilon}(B^\epsilon)$ $(\mathbb{C}^\times \times T)$-orbits. By Lemma \cite{Deligne1994} each orbit has $O_{\Delta,N,S,\epsilon}(B^\epsilon)$ such Laurent polynomials with principal $\Delta$-determinant $N$. Theorem \cite{Batyrev1994} follows.

### 3. Reduction of Theorem 1.2 to Theorem 1.3

We will first review the notion of sheaves on semi-simplicial schemes and their images under derived functors. Then we will prove comparison theorems for...
cones in the relative set-up in Theorem 1.2. They will then be used together with Theorem 1.3 to deduce Theorem 1.2.

3.1. Sheaves on semi-simplicial schemes. Let $\Delta^+$ be the category with objects the ordered sets $[n] := \{0, \ldots, n\}$, $n \in \mathbb{Z}_{\geq 0}$, and with morphisms the strictly increasing maps. A semi-simplicial object in a category $\mathcal{C}$ is a contravariant functor $C_\bullet : \Delta^+ \to \mathcal{C}$. Write $C_n := C_\bullet[n]$. We say a semi-simplicial object $C_\bullet$ has finitely many faces if there is a non-negative integer $n_0$ such that $C_n = \emptyset$ for all $n \geq n_0$. A morphism of between two semi-simplicial objects $C_\bullet \to C'_\bullet$ is a natural transformation of the functors. In particular, every $[n]$ gives a morphism $C_n \to C'_n$.

Let $B$ be a scheme. We will mainly take $B$ to be a number field $K$ or its ring $\mathcal{O}_{K,S}$ of $S$-integers. A semi-simplicial $B$-scheme is a semi-simplicial object in the category of $B$-scheme. A $B$-scheme can be regarded as a constant semi-simplicial $B$-scheme. A morphism $Y_\bullet \to X_\bullet$ between semi-simplicial $B$-schemes is said to be smooth (or projective) if $Y_n \to X_n$ is smooth (or projective) for all $n$. A (étale) sheaf on a semi-simplicial $B$-scheme is a semi-simplicial object in the category of pairs $(X, \mathcal{F})$, where $X$ is a $B$-scheme, and $\mathcal{F}$ is a (étale) sheaf on $X$, and whose morphisms are pairs $(f, f^\#) : (Y, \mathcal{G}) \to (X, \mathcal{F})$, with $f : Y \to X$ a morphism of $B$-schemes, $f^\# : \mathcal{G} \to f_\ast \mathcal{F}$ a sheaf homomorphism.

Let $f : Y \to X$ be a morphism of $B$-schemes. Let $\mathcal{F}$ be an abelian (resp. étale) sheaf on $Y$. We denote by $Rf_\ast \mathcal{F}$ (resp. $R_\text{ét} f_\ast \mathcal{F}$) the complex $f_\ast C^\bullet_{\text{Gdm}} \mathcal{F}$, where $C^\bullet_{\text{Gdm}}$ is the Godement resolution, see [21 B.2.1].

Let $f : Y_\bullet \to X_\bullet$ be a morphism of semi-simplicial $B$-schemes. Let $\mathcal{F}^\bullet$ be an (resp. étale) abelian sheaf on $Y_\bullet$. Write $f_k : Y_k \to X$. We have a double complex whose columns are the complexes $Rf_k \mathcal{F}^k$, $k \geq 0$, and horizontal maps are given by Čech-type morphisms, see [7, 5.2.6.1]. We denote by $Rf_\ast \mathcal{F}^\bullet$ (resp. $R_\text{ét} f_\ast \mathcal{F}^\bullet$) the single complex attached to this double complex. Equivalently, $Rf_\ast \mathcal{F}^\bullet$ is the direct image of $\mathcal{F}^\bullet$, regarded as an object in the derived category.

Given a commutative diagram as in Section 1.1 the morphism $\varphi^{-1} Z_{Y_k, \mathcal{C}} \to Z_{D_k, \mathcal{C}}$ to the terminal object induces the morphism $f_k \ast C^\bullet_{\text{Gdm}} Z_{Y_k, \mathcal{C}} \to g_k \ast C^\bullet_{\text{Gdm}} Z_{D_k, \mathcal{C}}$ for any $k$; patching all these morphisms for all $k$, we have a morphism of double complexes that induces the canonical morphism $\varphi_{an}^{-1} : R(f_{\mathcal{C}, an}) \ast Z_{Y_\bullet, \mathcal{C}} \to R(g_{\mathcal{C}, an}) \ast Z_{D_\bullet, \mathcal{C}}$ of complexes.

For any complex $\mathcal{A}^\bullet$, denote $\mathcal{A}^\bullet[1]$ to be the complex with $\mathcal{A}[1]^i = \mathcal{A}(i+1)$. The cone $C^\bullet = C(\phi)$ of a morphism $\phi : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ of complexes is the complex $\mathcal{A}^\bullet[1] \oplus \mathcal{B}^\bullet$ with the differential $d_C(a, b) = (d_A(a), (-1)^{\deg a} \phi(a) + d_B(b))$.

A morphism of semi-simplicial varieties $\varepsilon : X_\bullet \to Y$ is said to be of cohomological descent [7 §5.3] if the natural map $\varepsilon : Z_Y \to R\varepsilon_\ast Z_{X_\bullet}$ is a quasi-isomorphism.
3.2. **Comparison theorems for cones.** By [23 Section 0DE7], the commutative diagram in Section 1.1 induces a commutative diagram

$$
\begin{array}{ccc}
D_{\text{ét}} & \xrightarrow{\varphi_{\text{ét}}} & Y_{\text{ét}} \\
\downarrow g_{\text{ét}} & & \downarrow f_{\text{ét}} \\
X_{\text{ét}} & & \\
\end{array}
$$

of semi-simplicial étale sites and their corresponding ringed topoi. Let \( p \) be a prime number. Let \( n \) be a non-negative integer. Let \( C_{\text{ét}}(\varphi_n^{-1}) \) be the cone of the canonical morphism \( \varphi_n^{-1} : R_{\text{ét}}f_* (\mathbb{Z} / p^n \mathbb{Z}) \to R_{\text{ét}}g_* (\mathbb{Z} / p^n \mathbb{Z}) \) of complexes.

**Lemma 3.1.** The sheaves \( \mathcal{H}_{n,\text{ét}}^i := H^i(C_{\text{ét}}(\varphi_n^{-1})) \) are locally constant for all \( i \) and \( n \).

*Proof.* Let \( E_{1}^{k,i} = R_{\text{ét}}^i(f_k)_* (\mathbb{Z} / p^n \mathbb{Z}) \). By smooth proper base change theorem, \( E_{1}^{k,i} \) are locally constant for all \( k, i \). We have the spectral sequence of double complex ([23 Lemma 0130 (3) and Lemma 0132])

$$E_{1}^{k,i} \Rightarrow R_{\text{ét}}^{k+i} f_* (\mathbb{Z} / p^n \mathbb{Z}).$$

Since subsheaves of locally constant sheaves are locally constant, sheaves are still locally constant after turning pages in the spectral sequence. Hence, \( R_{\text{ét}}^{i} f_* (\mathbb{Z} / p^n \mathbb{Z}) \), and similarly \( R_{\text{ét}}^{i} g_* (\mathbb{Z} / p^n \mathbb{Z}) \), are locally constant for all \( i \). By an equivalent description of locally constant sheaves [20 V.1.10, p. 162], for any geometric points \( x_0, x_1 \) of \( X \), with \( x_0 \) in the closure of \( x_1 \), and a map \( \mathcal{O}_{X,x_0} \to \mathcal{O}_{X,x_1} \), the cospecialization map

$$R_{\text{ét}}^{i} f_* (\mathbb{Z} / p^n \mathbb{Z})_{x_0} \to R_{\text{ét}}^{i} f_* (\mathbb{Z} / p^n \mathbb{Z})_{x_1}$$

is an isomorphism. Similarly, we have an isomorphism with \( f \) replaced by \( g \). Then by the long exact sequence of the cone ([13 Lemma 3.4]) and the Five Lemma of homological algebra, we have an isomorphism

$$H^i(C_{\text{ét}}(\varphi_n^{-1}))_{x_0} \simeq H^i(C_{\text{ét}}(\varphi_n^{-1}))_{x_1}$$

By [20 V.1.10, p. 162], the lemma follows. \( \square \)

Let \( i \) be a non-negative integer. Let

- \( \mathcal{F}_{\text{an}}^i := R^i(f_{\text{C}, \text{an}})_* \mathbb{Z}, \)
- \( \mathcal{F}_{n,\text{an}}^i := R^i(f_{\text{C}, \text{an}})_* (\mathbb{Z} / p^n \mathbb{Z}), \)
- \( \mathcal{F}_{n,\text{C},\text{ét}}^i := R_{\text{ét}}^i (f_{\text{C}})_* (\mathbb{Z} / p^n \mathbb{Z}), \)
- \( \mathcal{G}_{\text{an}}^i := R^i(g_{\text{C}, \text{an}})_* \mathbb{Z}, \)
- \( \mathcal{G}_{n,\text{an}}^i := R^i(g_{\text{C}, \text{an}})_* (\mathbb{Z} / p^n \mathbb{Z}), \)
- \( \mathcal{G}_{n,\text{C},\text{ét}}^i := R_{\text{ét}}^i (g_{\text{C}})_* (\mathbb{Z} / p^n \mathbb{Z}). \)
Let $C_{\eta}(\varphi_{C,n}^{-1})$ be the cone of the canonical morphism

$$\varphi_{C,n}^{-1} : R_{\eta}(f_{C})_*(\mathbb{Z}/p^n\mathbb{Z}) \to R_{\eta}(g_{C})_*(\mathbb{Z}/p^n\mathbb{Z}).$$

By derived proper base change, the long exact sequence of the cone, the exactness of inverse image functor (for abelian sheaves), and the Five Lemma, we know $H^i(C_{\eta}(\varphi_{C,n}^{-1}))$ is isomorphic to the inverse image of $\mathcal{H}_{n,\text{ét}}$ under $X_C \to X$. Let $\mathcal{H}_{n,\text{ét}} := H^i(C_{\eta}(\varphi_{C,n}^{-1}))$. Let $C(\varphi_{n,an}^{-1})$ be the cone of the canonical morphism

$$\varphi_{n,an}^{-1} : R(f_{\mathbb{C},an})_* (\mathbb{Z}/p^n\mathbb{Z}) \to R(g_{\mathbb{C},an})_* (\mathbb{Z}/p^n\mathbb{Z}).$$

Let $\mathcal{H}_{n,an} := H^i(C(\varphi_{n,an}^{-1}))$.

**Lemma 3.2.** We have isomorphisms $(\mathcal{H}_{n,\text{ét},an})^i \simeq \mathcal{H}_{n,an}^i$ for all $i, n$.

**Proof.** By a derived version of étale-analytic comparison theorem ([11, Theorem 11.6]), we have isomorphisms $(\mathcal{F}_{n,\text{ét},an})^i \simeq \mathcal{F}_{n,an}^i$ and $(\mathcal{G}_{n,\text{ét},an})^i \simeq \mathcal{G}_{n,an}^i$. By the long exact sequence of the cone ([13, Lemma 3.4]) and the Five Lemma of homological algebra, the lemma follows.

Since $\mathcal{H}_{n,\text{an}} := H^i(C(\varphi_{n,an}^{-1}))$, $\mathcal{F}_{an}^i$, and $\mathcal{G}_{an}^i$ are locally constant, we can choose a prime $p$ such that $\mathcal{H}_{an}^i$, $\mathcal{F}_{an}^i$, and $\mathcal{G}_{an}^i$ are $p$-torsion-free for all $i$.

**Lemma 3.3.** We have $H^i_{an} \otimes (\mathbb{Z}/p^n\mathbb{Z}) \simeq \mathcal{H}_{n,an}^i$.

**Proof.** Since $\mathcal{F}_{an}^i$ and $\mathcal{G}_{an}^i$ are $p$-torsion-free, we have $\mathcal{F}_{an}^i = \mathcal{F}_{an}^i \otimes (\mathbb{Z}/p^n\mathbb{Z})$ and $\mathcal{G}_{an}^i = \mathcal{G}_{an}^i \otimes (\mathbb{Z}/p^n\mathbb{Z})$ by the universal coefficient theorem. We have the long exact sequence of the cone ([13, Lemma 3.4])

$$\mathcal{F}_{an}^i \to \mathcal{G}_{an}^i \to \mathcal{H}_{an}^i \to \mathcal{F}_{an}^{i+1} \to \mathcal{G}_{an}^{i+1}$$

This long exact sequence breaks into short exact sequences involving the images, and they remain exact after tensoring with $\mathbb{Z}/p^n\mathbb{Z}$ because of $p$-torsion-freeness. We have $\text{im}(\mathcal{F}_{an}^i \otimes (\mathbb{Z}/p^n\mathbb{Z})) \simeq (\text{im}\mathcal{F}_{an}^i) \otimes (\mathbb{Z}/p^n\mathbb{Z})$. Similarly for $\text{im}\mathcal{G}_{an}^i$ and $\text{im}\mathcal{H}_{an}^i$. We have the long exact sequence of the cone ([13, Lemma 3.4])

$$\mathcal{F}_{an}^i \to \mathcal{G}_{an}^i \to \mathcal{H}_{an}^i \to \mathcal{F}_{an}^{i+1} \to \mathcal{G}_{an}^{i+1}$$

for all $i$. By the Five Lemma, $H^i_{an} \otimes (\mathbb{Z}/p^n\mathbb{Z}) \simeq \mathcal{H}_{n,an}^i$. □

### 3.3. Reduction of Theorem 1.2 to Theorem 1.3

To apply Theorem 1.3, we take $\mathcal{L}_{an} = \mathcal{H}_{an}^i$, which is $p$-torsion-free. For each $n$, we also take $\mathcal{L}_{n,\text{ét}} = \mathcal{H}_{n,\text{ét}}^i$, which is an étale local system of $(\mathbb{Z}/p^n\mathbb{Z})$-modules by Section 3.2. By Lemma 3.2 and Lemma 3.3

$$(\mathcal{L}_{n,\text{ét},\mathbb{C}})_{\text{an}} = (\mathcal{H}_{n,\text{ét},\mathbb{C}})_{\text{an}} \simeq \mathcal{H}_{n,\text{an}}^i = H^i_{an} \otimes (\mathbb{Z}/p^n\mathbb{Z}) = \mathcal{L}_{an} \otimes (\mathbb{Z}/p^n\mathbb{Z})$$

for all $n$. Theorem 1.2 then follows from Theorem 1.3.
4. Proofs of Theorem 1.1 and Theorem 1.3

4.1. Proof of Theorem 1.3. The techniques are based on [5] and [8]. Let $\overline{X}$ be the Zariski closure of $X$ in $\mathbb{P}^m_K$. Let $Z = \overline{X}\setminus X$. Let $L = \mathcal{O}_{\overline{X}}(1)$ be the hyperplane line bundle on $\overline{X}$. Since the statement of Theorem 1.3 is only about smooth variety $X$, we can assume that $\overline{X}$ is smooth by resolution of singularities [16]. By enlarging $S$ if necessary, we can choose a smooth $\mathcal{O}_{K,S}$-model $X$ of $\overline{X}$.

A normal cycle on a normal variety $W$ over a field of characteristic 0 is a finite morphism $T \to W$ which is birational onto its image, where $T$ is a geometrically irreducible normal variety. For any complete variety $Q$ over $K$ with an ample line bundle $J$, we let $\deg(Q, J) := \lim_{k \to \infty} \frac{\dim_K \Gamma(Q, J^\otimes k)}{k^{\dim Q}}$.

For any $x \in X(\mathbb{C})$, the local systems $\mathcal{L}_{an}$ and $\mathcal{L}_{n,an} := \mathcal{L}_{an} \otimes (\mathbb{Z}/p^n\mathbb{Z})$ induce the monodromy representation $\pi^\text{top}_1(X_{\mathbb{C}}, x) \to \text{Aut} \mathcal{L}_{an,x}$ and the mod $p^n$ monodromy representation $\pi^\text{top}_1(X_{\mathbb{C}}, x) \to \text{Aut} \mathcal{L}_{n,an,x}$ respectively. For any $x \in X(\overline{K})$, the étale local system $\mathcal{L}_{n,\text{ét}}$ induces an étale monodromy representation $\pi^\text{ét}_1(X_\mathbb{C}, x) \to \text{Aut} \mathcal{L}_{n,\text{ét},\mathbb{C},x}$.

Lemma 4.1. Let $V$ be a complex irreducible subvariety of $\overline{X}_{\mathbb{C}}$ such that $V$ is not contained in $Z_{\mathbb{C}}$ and that $V^\circ := V \cap X_{\mathbb{C}}$ is not contained in a fiber of $\Phi$. Let $V^{\circ,\text{an}}$ be the smooth locus of $V^\circ$. Then $\pi^\text{top}_1(V^{\circ,\text{an}}) \to \text{Aut} \mathcal{L}_{an,x}$ has infinite image.

Proof. Suppose $\pi^\text{top}_1(V^{\circ,\text{an}}) \to \text{Aut} \mathcal{L}_{an,x}$ has finite image. Restrict the VMHS on $V^{\circ,\text{an}}$. Passing to a finite cover $V^{\circ,\text{an}}$, we get a VMHS on $V^{\circ,\text{an}}$ with trivial monodromy. By rigidity [6, Theorem 7.12], this VMHS is trivial, i.e. $\overline{\Phi}(V^{\circ,\text{an}})$ is a point, which contradicts that $V^\circ$ is not contained in a fiber of $\Phi$. □

Lemma 4.2. For any $D \geq 1$, there exist a finite group $G$, a finite morphism $\tau : \overline{X} \to \overline{X}$ of $K$-varieties, and an embedding $G \hookrightarrow \text{Aut}(\overline{X}/\overline{X})$ such that

- $\tau|_X$ is finite étale Galois with deck group $G$ and it extends to a finite étale cover of the smooth integral model $X$.
- Let $U$ be an irreducible closed complex subvariety of $\overline{X}_{\mathbb{C}}$. Let $U^\circ := U \cap X_{\mathbb{C}}$. Suppose $U$ is not contained in $Z_{\mathbb{C}}$ and $U^{\circ,\text{an}}$ is not contained in a single fiber of $\Phi$. Let $Q$ be any irreducible component of $\tau^{-1}U$, endowed
with the reduced structure, such that the induced finite map $\tau|_Q: Q \to U$ is dominant. Then degree $\deg(Q, \tau^*L_Q) \geq D$.

Proof. By \cite{5} Lemma 2.9, up to conjugation, there are only finitely many subgroups of $\pi^\et_1(X_\C)$ obtained as the image of $\pi^\et_1(f^{-1}(X_\C)) \to \pi^\et_1(X_\C)$ with $f: T \to \overline{X}_\C$ a normal cycle such that

\begin{itemize}
  \item $\deg(T, f^*L) \leq D$,
  \item $f(T)$ is not contained in $\Z_\C$,
  \item $f(T)^0 := f(T) \cap \overline{X}_\C$ is not contained in a single fiber of $\Phi$.
\end{itemize}

Let $E_1, \ldots, E_r$ be such subgroups. For any $i = 1, \ldots, r$, let $f_i: T_i \to \overline{X}_\C$ be normal cycles that induce these $E_i$ and have the listed properties above. Let $F_i$ be the étale fundamental group of the smooth locus $f_i(T_i)^{\circ,s}$ of $f_i(T_i)^0$. Let $M_{i,n}$ be the image of $F_i$ under the étale monodromy representation $\pi^\et_i(X_\C, x) \to \text{Aut } \mathcal{L}_{n,\et,\C,x}$. Let $F_i^{\text{top}}$ be the topological fundamental group of $f_i(T_i)^{\circ,s}$. For any $i, n$, let $M_i^{\text{top}}$ and $M_{i,n}^{\text{top}}$ be the image of $F_i^{\text{top}}$ under the monodromy representation $\pi^\et_i(X_{\et}^\an, x) \to \text{Aut } \mathcal{L}_{\an,x}$ and the mod $p^n$ representation respectively. By Lemma \ref{4.3}, $M_i^{\text{top}}$ are infinite for all $i$. Hence, the cardinalities of $M_{1,n}, \ldots, M_{r,n}^{\text{top}}$ can be made arbitrarily large uniformly (here we are using the finiteness) when $n \to \infty$.

It follows from the commutative diagram in the beginning of this section that the same is true for the cardinalities of $M_{1,n}, \ldots, M_{r,n}$.

By constructibility, $X$ is open in $\overline{X}$, so $f_i(T_i)^{\circ,s}$ is open in $f_i(T_i)$. Since $T_i$ is irreducible, $f_i(T_i)^{\circ,s}$ is irreducible. Since $f_i^{-1}(f_i(T_i)^{\circ,s})$ is open in $T_i$, it is irreducible and normal. Then by \cite{17} Lemma 11, the image under $\pi^\et_i(f_i^{-1}(f_i(T_i)^{\circ,s})) \to F_i$ has finite index in $F_i$.

Therefore, by fixing a large $n$, the cardinalities of the images of $E_1, \ldots, E_r$ under the mod $p^n$ étale monodromy representations can be made arbitrarily large uniformly, say $\geq (\dim X)! \cdot D$.

The étale local system $\mathcal{L}_{n,\et}$ induces a finite étale Galois cover of $X$ with deck group denoted by $G$. By enlarging $S$ if necessary, this cover extends to a finite étale cover of the smooth integral model $X$. Let $\tau : \overline{X} \to \overline{X}$ be normalization of $\overline{X}$ in this cover. The $G$-action on the cover extends uniquely to a $G$-action on $\tau$.

Let $\deg(\tau|_Q)$ be the degree of the finite map $\tau|_Q: Q \to U$. Suppose $\deg(U, L|_U) < D$. Let $\nu : U' \to U$ be the normalization. By the projection formula,

$$\deg(U', \nu^*L|_{U'}) = \deg(U, L|_U) < D.$$  

The image of $\pi^\et_1(\nu^{-1}(U^\circ)) \to \pi^\et_1(U^\circ)$ is conjugated to some $E_i$. By \cite{5} Lemma 2.10, $\deg(\tau|_Q)$ is equal to the cardinality of the image of the homomorphism $\pi^\et_1(\nu^{-1}(U^\circ)) \to G$. This cardinality is equal to the cardinality of the image of
\[ \pi_1^\text{et}(E_i) \to G, \text{ while this cardinality is } \geq (\dim X)! \cdot D. \] By asymptotic Riemann-Roch,
\[ \deg(U, L|_U) := \lim_{k \to \infty} \frac{\dim_K \Gamma(U, L^{\otimes k}_{|U})}{k^{\dim U}} = \frac{(L|_U)^{\dim U}}{(\dim U)!}. \]
The intersection number \( (L|_U)^{\dim U} \) is a positive integer. By the projection formula,
\[ \deg(Q, \tau^*L|_Q) = \deg(\tau|_Q) \deg(U, L|_U). \]
Therefore,
\[ \deg(Q, \tau^*L|_Q) \geq (\dim X)! \cdot D \cdot \frac{(L|_U)^{\dim U}}{(\dim U)!} \geq D. \]

In the case where \( \deg(U, L|_U) \geq D \), we also have \( \deg(Q, \tau^*L|_Q) \geq D. \) \( \square \)

The remaining proof of Theorem 1.3 is the same as the proof of \([8, \text{Lemma 4.2}]\) and Section 4.2 in \( \text{op. cit.} \). For completeness, we will give a sketch of it.

Let \( \ell, d \geq 1 \). Let \( V \) be a geometrically irreducible closed subvariety of \( \overline{X} \) over \( K \) of dimension \( \ell \) and degree \( d \) such that \( V_C \) is not contained in \( Z_C \) and \((V_C \cap X_C)^{an} \) is not contained in a fiber of \( \Phi \).

Choose \( D \) such that \((\ell + 1)/D^{1/\ell} < \varepsilon\). Taking this \( D \) in Lemma \([1,2]\) we obtain a finite group \( G \) and a finite morphism \( \tau : \overline{X} \to \overline{X} \) satisfying the two properties therein. Let \( X' := \tau^{-1}(X) \).

Firstly, there are finitely many covers \( X'_j \to X \) such that every \( x \in X(\mathcal{O}_{K,S}) \) lifts to a rational point in one of such covers: A family of covers that satisfies this lifting property can be obtained by twisting the cover \( X' \to X \) and using \([22, \text{Theorem 8.4.1}]\). Finiteness of such twists is due to Hermite-Minkowski theorem, see lines 3-14 of p. 16 of \([8]\) for details. Let \( \tau_j : \overline{X}_j \to \overline{X} \) be the normalization of \( \overline{X} \) in the cover \( X'_j \to X \).

For a large enough integer \( e \), the pullback \((\tau_j^*L)^{\otimes e}\) is very ample for all \( j \). Use these line bundles to get projective embeddings \( \overline{X}_j \hookrightarrow \mathbb{P}^{M_j} \). There exists \( c_{d,\varepsilon} > 0 \) such that for any integral point of \( X \cap V \) with height \( \leq B \), it is of the form \( \tau_j(P) \) for some \( P \in X'_j(K) \cap \tau_j^{-1}(V)(K) \) of height \( \leq c_{d,\varepsilon}B^e \), see lines 15-27 of p. 16 of \([8]\) for details.

Let \( V^s \) be the smooth locus of \( V \). Let \( V^{s,\circ} := V^s \cap X \). Since \( \tau_j^{-1}(V^{s,\circ}) \) is a finite étale cover of the geometrically irreducible smooth \( K \)-variety \( V^{s,\circ} \), its geometric components are pairwise distinct by \([14, \text{Exp. 1., Cor. 10.8}]\) and permuted by \( \text{Gal}(\overline{K}/K) \). The geometric components of \( \tau_j^{-1}(V^{s,\circ}) \) having a \( K \)-rational point are thus defined over \( K \), and the number of such components is bounded by the cardinality of \( G \). Let \( Q^\circ \) be one of such components. The \( K \)-Zariski closure \( Q \) of \( Q^\circ \) is geometrically irreducible. The map \( \tau_j : X'_j \to X \) induces a map \( \tau_j : Q \to V \). Since \( \tau_j \) is étale over \( V^{s,\circ} \), the image \( \tau_j(Q) \) contains an open subset, so \( \tau_j : Q \to V \) is dominant. By the second property of Lemma \([3,2]\) and the fact
that $\tau_j$ is the twist of $\tau$, $\deg(Q, \tau^*L|_Q) \geq D$. Then as in lines 4-16 of p. 17 of [8] (with the only difference that $\varepsilon$ is rescaled to $\varepsilon/2e$ instead of $\varepsilon/2$ at the very end because we were taking a slightly different approach to bound the degree of $Q$; also note that $\varepsilon$ is independent of $B$ and $V$, and can be chosen depending only on $\varepsilon$ and $\ell$), using Broberg’s theorem [3] (which builds on fundamental ideas of Bombieri-Pila [2] and Heath-Brown [15]), we can obtain the following lemma:

**Lemma 4.3.** Let $V$ be a geometrically irreducible closed subvariety of $\overline{X}$ over $K$ of dimension $\ell$ and degree $d$ such that $V_C$ is not contained in $Z_C$ and $(V_C \cap X_C)^m$ is not contained in a fiber of $\Phi$. Then all integral points of $X \cap V$ of height $\leq B$ can be covered by $O_{d,\varepsilon,\ell}(B)$ irreducible subvariety over $K$ of dimension $\leq \ell - 1$ and degree $O_{d,\varepsilon,\ell}(1)$.

For the irreducible subvarieties obtained in Lemma 4.3 that are not geometrically irreducible, their rational points can be covered by $O_{d,\varepsilon,\ell}(1)$ subvarieties of smaller dimensions and of degree $O_{d,\varepsilon,\ell}(1)$ using [8, Lemma 2.4(c)]. For the geometrically irreducible ones that cover the integral points but does not contained in $Z_C$ and not contained in a fiber of $\Phi$, we can apply Lemma 4.3 again on them. By starting from $X$ instead and repeating this procedure, we can deduce Theorem 1.3, as in [8, Section 4.2].

4.2. **Proof of Theorem 1.1** By compactification theorem, $\pi$ is the composition of an open embedding $\iota : V \to \overline{V}$ and a projective surjective morphism $f : \overline{V} \to X$. Let $D := \overline{V} \setminus V$ be equipped with the reduced induced scheme structure. Let $\varphi : D \to \overline{V}$ be the closed embedding. Let $g = f \circ \varphi$. As in p. 122 and the paragraph before Cor. 5.30 of [21], using resolution of singularities [16], there is a commutative diagram

$$
\begin{array}{ccc}
D_\bullet & \overset{\varphi_\bullet}{\longrightarrow} & \overline{V}_\bullet \\
\downarrow & & \downarrow \\
D & \overset{\varphi}{\longrightarrow} & \overline{V},
\end{array}
$$

where $D_\bullet$ and $\overline{V}_\bullet$ are smooth and have only finitely many non-empty faces, the mappings $\overline{V}_\bullet \to \overline{V}$ and $D_\bullet \to D$ are proper, and after base change to $\mathbb{C}$ the mappings $\varepsilon_{\overline{V}_\bullet} : (\overline{V}_\bullet)_\mathbb{C} \to \overline{V}_\mathbb{C}$ and $\varepsilon_{D_\bullet} : (D_\bullet)_\mathbb{C} \to D_\mathbb{C}$ are of cohomological descent by p. 123-124 of [21], i.e.

$$R(f_{\mathbb{C}} \varepsilon_{\overline{V}_\bullet})_* Q_{(\overline{V}_\bullet)_\mathbb{C}} \simeq R(f_{\mathbb{C}})_* Q_{\overline{V}_\mathbb{C}}$$

and

$$R(g_{\mathbb{C}} \varepsilon_{D_\bullet})_* Q_{(D_\bullet)_\mathbb{C}} \simeq R(g_{\mathbb{C}})_* Q_{D_\mathbb{C}}.$$

Let $C((\varphi_\bullet)^{-1}_\mathbb{C})$ be the cone of the canonical morphism

$$(\varphi_\bullet)^{-1}_\mathbb{C} : R(f_{\mathbb{C}} \varepsilon_{\overline{V}_\bullet})_* Q_{\overline{V}_\bullet,\mathbb{C}} \to R(g_{\mathbb{C}} \varepsilon_{D_\bullet})_* Q_{D_\bullet,\mathbb{C}}$$
of complexes, see Section 3.1. By [23, Lemma 02UT], we have an exact sequence

\[ 0 \to (\iota_\mathcal{C})_! \mathbb{Q}\mathcal{V}_c \to \mathbb{Q}\mathcal{V}_c \to (\varphi_{\mathcal{C}})_* \mathbb{Q}\mathcal{D}_c \to 0, \]

which induces an exact sequence

\[ 0 \to R(f_{\mathcal{C}})_*(\iota_\mathcal{C})_! \mathbb{Q}\mathcal{V}_c \to R(f_{\mathcal{C}})_* \mathbb{Q}\mathcal{V}_c \to R(f_{\mathcal{C}})_*(\varphi_{\mathcal{C}})_* \mathbb{Q}\mathcal{D}_c \to 0 \]

of complexes. By [13, Lemma 3.4], we have a quasi-isomorphism \( \ker((\varphi_{\mathcal{C}})_*)^{-1}_C \simeq C((\varphi_{\mathcal{C}})_C)^{-1}[-1] \). Combining what we have proved, we get a quasi-isomorphism \( R(\pi|_{\mathcal{V}_c})_! \mathbb{Q}\mathcal{V}_c \to C((\varphi_{\mathcal{C}})_C)^{-1}[-1] \) of complexes. By generic smoothness, choose a non-empty Zariski open subset \( X^* \) of \( X \) such that the compositions \( D_{\bullet} \to D \to X \) and \( V_{\bullet} \to V \to X \) are smooth over the preimages of \( X^* \). For all \( i \),

\[ (R^i(\pi|_{\mathcal{V}_c})_! \mathbb{Q}\mathcal{V}_c)|_{X^*_c} \simeq H^{i-1}_c(C((\varphi_{\mathcal{C}})_C)^{-1})|_{X^*_c}, \]

which underlies an admissible graded polarized variations of \( \mathbb{Q} \)-mixed Hodge structures by [11, Lemma 4.12]. By proper base change theorem, the restriction to \( X^*_c \) of the cone of complexes of sheaves under consideration is the cone of the restriction of the complexes to \( X^*_c \). By Theorem [12] for any \( \varepsilon > 0 \), the \( S \)-integral points of \( X^* \) with height at most \( B \) are covered by \( O_{\varepsilon}(B^\varepsilon) \) geometrically irreducible \( K \)-subvarieties, each lying in a single fiber of the mixed period mapping \( \Phi \) arising from the variation. This finishes the proof of Theorem [11].

**References**

[1] V. V. Batyrev, *Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori*, Duke Math. J. 69(2) (1993), 349–409.

[2] E. Bombieri and J. Pila, *The number of integral points on arcs and ovals*, Duke Math. J. 59(2) (1989), 337–357.

[3] N. Broberg, *A note on a paper by R. Heath-Brown: “The density of rational points on curves and surfaces”*, J. reine angew. Math. 571 (2004), 159–178.

[4] P. Brosnan and F. El Zein, *Variation Of Mixed Hodge Structures*, in Hodge theory (edited by E. Cattani, F. El Zein, P. A. Griffiths, D. T. Lê), Princeton University Press, 2014.

[5] Y. Brunebarbe and M. Maculan, *Counting integral points of bounded height on varieties with large fundamental group*, arXiv:2205.05436v2.

[6] J.-L. Brylinski and S. Zucker, *An overview of recent advances in Hodge theory*, in Complex Manifolds, Springer, 1998.

[7] P. Deligne, *Théorie de Hodge : III*, Publ. Math. I.H.E.S. 44 (1974), 5-77.

[8] J. S. Ellenberg, B. Lawrence, and A. Venkatesh, *Sparsity of Integral Points on Moduli Spaces of Varieties*, arXiv:2109.01043v1.

[9] G. Faltings, *Endlichkeitssätze für abelsche Varietäten über Zahlkörpern*, Invent. Math. 73(3) (1983), 349–366.

[10] E. Freitag and R. Kiehl, *Étale cohomology and the Weil conjecture*, Springer-Verlag, 1988.

[11] O. Fujino and T. Fujisawa, *Variations of mixed Hodge structure and semi-positivity theorems*, Publ. RIMS Kyoto Univ. 50 (2014), 589–661.
[12] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, *Discriminants, Resultants, and Multidimensional Determinants*, Birkhäuser, 1994.

[13] M. Goresky, *Lecture notes on sheaves and perverse sheaves*, arXiv:2105.12045v3.

[14] A. Grothendieck and M. Raynaud, *Revêtements Étales et Groupe Fondamental (SGAI)*, Séminaire de géométrie algébrique du Bois Marie, 1960/61, in Lecture notes in mathematics 224, Springer-Verlag, 1971.

[15] D. R. Heath-Brown, *The Density of Rational Points on Curves and Surfaces*, Ann. of Math. **155**(2) (2002), 553–598.

[16] H. Hironaka, *Resolution of Singularities of an Algebraic Variety Over a Field of Characteristic Zero: I*, Ann. of Math. **79**(1) (1964), 109–203.

[17] J. Kollár, *Rationally Connected Varieties and Fundamental Groups*, in Higher Dimensional Varieties and Rational Points, p. 69–92, Royal Society Mathematical Studies 12, Springer, 2003.

[18] B. Lawrence and W. Sawin, *The Shafarevich conjecture for hypersurfaces in abelian varieties*, arXiv:2004.09046v2.

[19] B. Lawrence and A. Venkatesh, *Diophantine problems and p-adic period mappings*, Invent. Math. **221**(3) (2020), 893–999.

[20] J. S. Milne, *Étale cohomology*, Princeton University Press, 1980.

[21] C. A. M. Peters and J. H. M. Steenbrink, *Mixed Hodge Structures*, Springer, 2008.

[22] B. Poonen, *Rational Points on Varieties*, Graduate Studies in Mathematics 186, American Mathematical Society, 2017.

[23] The Stacks Project Authors, *Stacks Project* (2018), https://stacks.math.columbia.edu

Department of Mathematics, University of Toronto, Toronto, Canada.

Email address: kennethct.chiu@mail.utoronto.ca