Stochastic Solutions for Dense Subgraph Discovery in Multilayer Networks

Yasushi Kawase
The University of Tokyo
Tokyo, Japan
kawase@mist.i.u-tokyo.ac.jp

Atsushi Miyauchi
The University of Tokyo
Tokyo, Japan
miyauchi@mist.i.u-tokyo.ac.jp

Hanna Sumita
Tokyo Institute of Technology
Tokyo, Japan
sumita@c.titech.ac.jp

ABSTRACT
Network analysis has played a key role in knowledge discovery and data mining. In many real-world applications in recent years, we are interested in mining multilayer networks, where we have a number of edge sets called layers, which encode different types of connections and/or time-dependent connections over the same set of vertices. Among many network analysis techniques, dense subgraph discovery, aiming to find a dense component in a network, is an essential primitive with a variety of applications in diverse domains. In this paper, we introduce a novel optimization model for dense subgraph discovery in multilayer networks. Our model aims to find a stochastic solution, i.e., a probability distribution over the family of vertex subsets, rather than a single vertex subset, whereas it can also be used for obtaining a single vertex subset. For our model, we design an LP-based polynomial-time exact algorithm. Moreover, to handle large-scale networks, we also devise a simple, scalable preprocessing algorithm, which often reduces the size of the input networks significantly and results in a substantial speed-up. Computational experiments demonstrate the validity of our model and the effectiveness of our algorithms.

CCS CONCEPTS
• Theory of computation → Graph algorithms analysis; Mathematical optimization.

KEYWORDS
network analysis, multilayer networks, dense subgraph discovery, stochastic solutions

ACM Reference Format:
Yasushi Kawase, Atsushi Miyauchi, and Hanna Sumita. 2023. Stochastic Solutions for Dense Subgraph Discovery in Multilayer Networks. In Proceedings of the Sixteenth ACM International Conference on Web Search and Data Mining (WSDM ’23), February 27–March 3, 2023, Singapore, Singapore. ACM, New York, NY, USA, 9 pages. https://doi.org/10.1145/3539597.3570444

1 INTRODUCTION
Network analysis has played a key role in knowledge discovery and data mining. In many real-world applications in recent years, we are interested in mining multilayer networks rather than ordinary (i.e., single-layer) networks, where we have a number of edge sets called layers, which encode different types of connections and/or time-dependent connections over the same set of vertices [7, 15, 30]. For example, in the Twitter network, there are various layers representing different types of connections, e.g., follower–followee relations, retweets, and mentions, among users. Moreover, each of those connections is time-dependent and therefore leads to multiple layers by itself. As another example, consider brain networks arising in neuroscience, where vertices correspond to small regions of a brain [19]. In this network, we can obtain at least two layers representing the structural connectivity and the functional connectivity (e.g., co-activation) among the small pieces of a brain.

Among many network analysis techniques, dense subgraph discovery, aiming to find a dense component in a network, is an essential primitive with a variety of applications in diverse domains [22, 32]. Examples include detecting communities and spam link farms in Web graphs [17, 21], experts extraction in crowdsourcing systems [28], real-time story identification in microblogging streams [1], and extracting molecular complexes in protein–protein interaction networks [3].

Recently, dense subgraph discovery has been extended from single-layer networks to multilayer networks. Jethava and Beerenwinkel [27] introduced the densest common subgraph problem, where given a multilayer network, we are asked to find a vertex subset that maximizes the minimum degree density (defined later) over the layers. They mentioned some concrete applications of this problem in biological networks. Moreover, we can see that the densest common subgraph problem may appear in the context of robust optimization. Indeed, multilayer networks can be seen as a model...
of single-layer networks with uncertain edges with a number of scenarios, where each layer corresponds to one scenario. If we can find a subgraph that is reasonably dense for all layers, the solution is more robust than that obtained by single-layer network analysis, which may be arbitrarily bad for some layers.

1.1 Our contribution

In this paper, we introduce a novel optimization model for dense subgraph discovery in multilayer networks. Given an edge-weighted multilayer network, our model aims to find a vertex subset that is dense for the layer selected by an adversary. Specifically, we employ a stochastic solution, i.e., a probability distribution over the family of vertex subsets, while existing work focuses only on a deterministic solution, i.e., a single vertex subset. More precisely, we stochastically choose a vertex subset according to a stochastic solution and then, knowing only the stochastic solution but not a realization, the adversary selects the worst layer for us. Note that the worst layer with respect to each realization of the stochastic solution may be different from that with respect to the stochastic solution. It is worth mentioning that our model can be seen as a zero-sum two-player Stackelberg game [2], where the leader is our algorithm and the follower is the adversary.

We measure the density of a vertex subset using the quality function called the degree density (or simply density) similarly to Jethava and Beerenwinkel [27]. The degree density of a vertex subset (in a single-layer network) is defined as half the average degree of the subgraph induced by the subset. This quality function is often used in the literature of dense subgraph discovery; in fact, this is the objective function of the well-known densest subgraph problem (see Section 2).

We evaluate the performance of a stochastic solution (or an algorithm) using the following three metrics for the layer selected by the adversary: (i) the density, i.e., the expected degree density, (ii) the robust ratio, i.e., the ratio of the expected density to the optimal density, and (iii) the regret, i.e., the difference between the optimal density and the expected density. Therefore, we address three optimization problems according to the metrics.

Our main algorithmic contribution is to design a polynomial-time exact algorithm for our optimization model, which is applicable to all of the above three metrics. Specifically, the algorithm first solves an LP, which is a generalization of the LP used for the densest common subgraph problem [27], and then computes an optimal probability distribution based on the LP’s optimal solution.

We observe that the output of our algorithm has a useful structure; the family of the vertex subsets with positive probabilities has a hierarchical structure. This leads to several practical benefits, e.g., the largest size subset contains all the other subsets and the optimal solution obtained by our algorithm has support size at most the number of vertices (although a solution of our optimization model may have support size exponential in the number of vertices). Moreover, we can demonstrate that the support size of the solution obtained is upper bounded by the number of layers. It is desirable to have a small support size for the understandability of stochastic solutions and for purposes of simple verification and validation.

For practical use of our proposed algorithm, we wish to speed up our algorithm so that it is applicable to larger-scale multilayer networks. The bottleneck is the computation cost of the LP, which has a lot of variables and constraints. To this end, we devise a simple, scalable preprocessing algorithm, which often reduces the size of the input networks significantly and results in a substantial speed-up. Specifically, the algorithm first computes an approximate solution by solving a much smaller LP than the aforementioned LP and then removes vertices from the original network using the information of the approximate solution obtained. Note that our preprocessing algorithm is a generalization of that proposed by Balalau et al. [4] for the densest subgraph problem. However, to verify that the preprocessing algorithm does not harm any optimal solution, we need a more sophisticated analysis, which is totally different from theirs.

Finally, we conduct thorough computational experiments using synthetic graphs and real-world networks to verify the validity of our model and to evaluate the performance of our algorithms.

2 RELATED WORK

The densest subgraph problem is one of the most popular optimization models for dense subgraph discovery. Let \( G = (V, E) \) be an undirected graph and \( w : E \rightarrow \mathbb{R}_+ \) a positive edge weight. For a vertex subset \( S \subseteq V \), the subgraph induced by \( S \) is denoted by \( G[S] \). The densest subgraph problem [27] is to find a vertex subset \( S \subseteq V \) that maximizes the density \( \delta(S) \) of \( S \), defined as \( \delta(S) = \frac{\sum_{(u,v) \in E[S]} w(u,v)}{|S|} \) (where we define the density of the empty set (i.e., 0/0) to be 0). In the densest subgraph problem, given a graph \( G = (V, E) \) with an edge weight \( w \), we are asked to find \( S \subseteq V \) that maximizes the density \( \delta(S) \).

It is well known that the densest subgraph problem can be solved exactly in polynomial time using a maximum-flow-based algorithm [23] or an LP-based algorithm [9]. Moreover, it was shown that a simple greedy algorithm called the greedy peeling admits 2-approximation in \( O(nm + n \log n) \) time [9, 31]. Recently, Boob et al. [8] designed an iterative greedy peeling algorithm, and demonstrated empirically that the output tends to be nearly optimal. Later, Chekuri, Quanrud, and Torres [11] proved the convergence to optimality of this algorithm (in a more general context).

Balalau et al. [4] introduced a simple preprocessing algorithm to improve the scalability of (exact) algorithms for the densest subgraph problem. Their preprocessing algorithm first computes
an approximate solution $S \subseteq V$ (using the greedy peeling), and then iteratively removes a vertex with the (weighted) degree less than the objective value of the approximate solution obtained. The validity of this preprocessing algorithm is guaranteed by the fact that any vertex with the (weighted) degree less than the optimal value is not contained in any optimal solution [4].

For the densest common subgraph problem, Jethava and Beerenwinkel [27] devised an LP-based polynomial-time heuristic and a valid value of this preprocessing algorithm is guaranteed by the fact that any vertex with the (weighted) degree less than the optimal value is not contained in any optimal solution [4].

Recently, Galimberti, Bonchi, and Gullo [20] introduced a generalization of the densest common subgraph problem, which they refer to as the multilayer densest subgraph problem. This problem exploits a trade-off between the minimum density value over layers and the number of layers considered. They proposed an approximation algorithm using a core decomposition technique for multilayer networks. Very recently, Hashemi, Behrouz, and Lakshmanan [24] designed a sophisticated core decomposition algorithm, which they call the FirmCore decomposition algorithm. Their algorithm finds the set of $(k,\lambda)$-FirmCores for all possible $k$ and $\lambda$ in polynomial time, where $(k,\lambda)$-FirmCore is a maximal subgraph in which every vertex has degree no less than $k$ in the subgraph for at least $\lambda$ layers. They demonstrated that the decomposition unfolds a better solution to the multilayer densest subgraph problem than the algorithm by Galimberti, Bonchi, and Gullo [20] for many instances.

Semertzidis et al. [38] introduced another generalization of the densest common subgraph problem, called the Best Friends Forever (BFF) problem, in the context of evolving graphs with a number of snapshots. The BFF problem is a series of optimization problems that maximize an aggregate density over snapshots, where the aggregate density is set to be the average/minimum value of the average/minimum degree of vertices over layers. They investigated the computational complexity of the problems and designed some approximation or heuristic algorithms.

Recalling that multilayer networks can also be seen as a model of networks with uncertainty, we can find some other optimization models related. Zou [44] studied the densest subgraph problem in uncertain graphs. An uncertain graph is a pair of $G = (V, E)$ and $p: E \rightarrow [0, 1]$, where $e \in E$ is present with probability $p(e)$ whereas $e \in E$ is absent with probability $1 - p(e)$. In the problem, given an uncertain graph $G = (V, E)$ with $p$, we seek $S \subseteq V$ that maximizes the expected density. Zou [44] showed that this problem can be reduced to the original densest subgraph problem, and designed a polynomial-time exact algorithm based on the reduction. Recently, Tsourakakis et al. [41] introduced a more general optimization problem called the risk-averse dense subgraph discovery.

As another example, Miyauuchi and Takeda [35] introduced an optimization problem called the robust densest subgraph problem. In this problem, given an undirected graph $G = (V, E)$ and an edge-weight space $I = \times_{e \in E} [l_e, r_e] \subseteq \times_{e \in E} [0, \infty]$, we are asked to find $S \subseteq V$ that maximizes $\min_{w \in I} \frac{\sum_{e \in S} w_e}{|S|}$, where $S_w$ is an optimal solution to the densest subgraph problem for $G$ with $w$. The intuition of this problem is the same as that of ours; this problem also seeks $S \subseteq V$ that is reasonably dense for any $w \in I$. However, they considered only deterministic solutions and gave a strong hardness result.

As well as dense subgraph discovery, many important primitives for single-layer network analysis have recently been extended to multilayer networks. Examples include community detection [6, 13, 25, 39], link prediction [13, 26], analyzing spreading processes [14, 37], and identifying central vertices [5, 16].

### 3 MODEL

In this section, we formally define our optimization model. Let $G = (V, (E_i))_{i \in [k]}$ be a multilayer network consisting of $k$ layers, and let $w_i: E_i \rightarrow \mathbb{R}_{++}$ be a positive edge weight for layer $i$. We denote by $E$ the union of all edge sets, i.e., $E := \bigcup_{i \in [k]} E_i$. Let $S_i^*$ be a densest subgraph for layer $i$, i.e., $S_i^* = \arg \max_{S \subseteq V} w_i(S)/|S|$. Our task is to find a vertex subset that is dense for the layer selected adversarially. As mentioned in the introduction, we consider a stochastic solution to compete with the adversary. Let $\Delta(2^V)$ be the set of probability distributions over $2^V$. For each $p \in \Delta(2^V)$, we denote by $p_S$ the probability of choosing $S \subseteq V$.

We aim to compute $p \in \Delta(2^V)$ that maximizes some metric (when the adversary selects the worst layer to $p$). We employ the following three metrics:

#### Density

The first metric is the degree density itself. Specifically, when we select a vertex subset according to a probability distribution $p \in \Delta(2^V)$ and the adversary selects a layer $i \in [k]$, our metric is defined as follows:

$$\mathbb{E}_{S \sim p} \left[ \frac{w_i(S)}{|S|} \right] = \sum_{S \subseteq V} p_S \frac{w_i(S)}{|S|}. \quad (1)$$

As the adversary selects the worst layer, we aim to find $p \in \Delta(2^V)$ that maximizes the minimum of the density $(1)$ among $i \in [k]$. Our optimization model with this metric can be seen as a stochastic version of the densest common subgraph problem introduced by Jethava and Beerenwinkel [27].

#### Robust ratio

The robust ratio is a metric based on the ratio of the expected degree to the optimal degree. For a nonempty $S \subseteq V$ and $i \in [k]$, let us consider a normalized density defined as $\frac{w_i(S)}{|S|}$. When we select a vertex subset according to a probability distribution $p \in \Delta(2^V)$ and the adversary selects a layer $i \in [k]$, the metric is defined as follows:

$$\mathbb{E}_{S \sim p} \left[ \frac{w_i(S)/|S|}{w_i(S^*_i)/|S^*_i|} \right] = \sum_{S \subseteq V} p_S \frac{w_i(S)/|S|}{w_i(S^*_i)/|S^*_i|} \quad (2)$$

In other words, the robust ratio is equivalent to the density for the multilayer network with weights $w'_1, \ldots, w'_k$ given by $w'_i(e) := \frac{w(e)}{w(S^*_i)/|S^*_i|}$ for each $i \in [k]$ and $e \in E_i$. As the adversary selects the worst layer, we aim to find $p \in \Delta(2^V)$ that maximizes the minimum of $(2)$ among $i \in [k]$. Note that the optimal robust ratio is contained

### 888
in the interval \([1/k, 1]\) because \(p \in \Delta(2^V)\) such that \(p_S = 1/k\) for each \(i \in [k]\) has the objective value of \(1/k\).

**Regret.** The regret is a metric based on the difference between the optimal density and the expected density. For \(S \subseteq V\) and \(i \in [k]\), the regret is defined as \(w_i(S^*_i)/|S^*_i| - w_i(S)/|S|\). When we select a vertex subset according to a probability distribution \(p \in \Delta(2^V)\) and the adversary selects a layer \(i \in [k]\), the metric is defined as follows:

\[
\mathbb{E}_{S \sim p} \left[ \frac{w_i(S^*_i)}{|S^*_i|} - \frac{w_i(S)}{|S|} \right] = \frac{w_i(S^*_i)}{|S^*_i|} - \sum_{S \subseteq V} p_S \frac{w_i(S)}{|S|}. \tag{3}
\]

As the adversary selects the worst layer, we aim to find \(p \in \Delta(2^V)\) that minimizes the maximum of (3) among \(i \in [k]\).

Here we explain how to select an appropriate metric. The density and regret metrics are useful when we are concerned with multilayer networks with homogeneous layers such as time-dependent follower-followee relations in the Twitter network. Although the density metric can be the first choice, the regret metric is more suitable for robust analysis. For example, consider the case where there are a number of layers consistent with each other together with some noisy (e.g., random) layers. The density metric would suffer from the effect of the noisy layers, but the regret metric would avoid it and find dense subgraphs in the other meaningful layers. On the other hand, the robust ratio metric is useful when we analyze multilayer networks with heterogeneous layers such as brain networks with structural and functional connectivity layers. From its definition, the robust ratio metric would find subgraphs that are reasonably dense for all layers. The density and regret metrics focus only on the layers with small optimal densities and the layers with large optimal densities, respectively.

**Unified concept: \((\alpha, \beta)\)-density.** Here we introduce a general metric, enabling us to deal with the above three metrics in a unified manner. An important fact is that the robust ratio and regret metrics can be obtained by affine transformations of the density. Specifically, when we select a subgraph according to a probability distribution \(p \in \Delta(2^V)\) and the adversary selects a layer \(i \in [k]\), we define the \((\alpha, \beta)\)-density using two vectors \(\alpha \in \mathbb{R}_k^k\) and \(\beta \in \mathbb{R}_k^k\) as follows:

\[
\mathbb{E}_{S \sim p} \left[ \alpha_i \frac{w_i(S)}{|S|} + \beta_i \right] = \alpha_i \sum_{S \subseteq V} p_S \frac{w_i(S)}{|S|} + \beta_i. \tag{4}
\]

Note that the above three metrics, the density, robust ratio, and regret, are equivalent to the \((1, 0)\)-density, \((|S^*_i|/w_i(S^*_i))_{i \in [k]}, 0)\)-density, and \((1, (-w_i(S^*_i)/|S^*_i|))_{i \in [k]})\)-density, respectively. Note that \(w_i(S^*_i)/|S^*_i|\) is polynomially computable for each \(i \in [k]\) [9, 23]. We refer to \((\alpha, \beta)\)-DENSITY as the problem of finding \(p \in \Delta(2^V)\) that maximizes the minimum of the \((\alpha, \beta)\)-density among \(i \in [k]\). Therefore, in the following, we aim to design an algorithm for \((\alpha, \beta)\)-DENSITY.

### 4 ALGORITHM

In this section, we provide an LP-based polynomial-time exact algorithm for \((\alpha, \beta)\)-DENSITY. Let \(((\hat{x}_v)_{v \in E}, (\hat{y}_v)_{v \in V}, t)\) be continuous

\[
\text{Algorithm 1: LP-based algorithm}
\]

1. Solve LP (4) to obtain an optimal solution \(((\hat{x}_v)_{v \in E}, (\hat{y}_v)_{v \in V}, t)\).
2. Let \(r_0, r_1, \ldots, r_t\) be reals such that \(\{r_0, r_1, \ldots, r_t\} = \{y_v \mid v \in V\} \cup \{0\}\) and \(r_0(= 0) < r_1 < r_2 < \cdots < r_t;\)
3. Let \(S_j = \{v \in V \mid y_v \geq r_j\} (j = 1, \ldots, t);\)
4. return \(\hat{p} \in \Delta(2^V)\) with \(\hat{p}_{S_j} = (r_j - r_{j-1}) \cdot |S_j| (j = 1, \ldots, t)\)
   (and \(\hat{p}_S = 0\) for the other \(S\)’s).

variables. We consider the following LP:

\[
\begin{align*}
\text{max} & \quad t \\
\text{s.t.} & \quad t \leq \alpha_i \sum_{e \in E} w_i(e) x_e + \beta_i \quad (\forall i \in [k]), \quad x_e \leq y_{uv}, \quad x_e \leq y_v \\
& \quad \sum_{v \in V} y_v = 1, \quad \sum_{e \in E} x_e \leq 1 \\
& \quad x_e, y_v \geq 0 \quad (\forall e \in E, \forall v \in V).
\end{align*}
\]

When \((\alpha, \beta) = (1, 0)\), this formulation coincides with the LP introduced by Jethava and Beerwinkel [27] for the densest common subgraph problem. However, they did not point out the connection between a solution of the LP and a distribution in \(\Delta(2^V)\).

Our algorithm first computes an optimal solution to LP (4), denoted by \(((\hat{x}_v)_{v \in E}, (\hat{y}_v)_{v \in V}, t)\). Then we set \(r_0, r_1, \ldots, r_t\) to be the reals such that \(\{y_v \mid v \in V\} = \{\hat{y}_v \mid v \in V\} \cup \{0\}\) and \(r_0(= 0) < r_1 < r_2 < \cdots < r_t\). In addition, let \(S_j = \{v \in V \mid y_v \geq r_j\} (j = 1, \ldots, t)\). The output of our algorithm is the probability distribution \(\hat{p} \in \Delta(2^V)\) defined as

\[
\hat{p}_{S_j} = (r_j - r_{j-1}) \cdot |S_j| (j = 1, \ldots, t) \quad \text{and} \quad \hat{p}_S = 0 \quad \text{for the other \(S\)’s}. \tag{5}
\]

We remark that \(\sum_{S \subseteq 2^V} \hat{p}_S = 1\) holds because

\[
1 = \sum_{v \in V} \sum_{j \in [t]} r_j \cdot \sum_{j \in [t]} \hat{p}_{S_j} = \sum_{j \in [t]} \hat{p}_{S_j},
\]

where \(S_{t+1} = \emptyset\) for ease of notation.

Our algorithm is formally described in Algorithm 1. Clearly, the algorithm runs in polynomial time.

### 5 ANALYSIS

In this section, we first demonstrate that the output \(\hat{p}\) of Algorithm 1 is optimal to \((\alpha, \beta)\)-DENSITY. Then we analyze the structure of the output \(\hat{p}\) with a special attention to its support size.

#### 5.1 Optimality of the output of Algorithm 1

We first prove that the \((\alpha, \beta)\)-density of \(\hat{p}\) is equal to the optimal value of LP (4):

\[
\text{Lemma 5.1. } \text{It holds that}
\min_{i \in [k]} \mathbb{E}_{S \sim p} \left[ \alpha_i \frac{w_i(S)}{|S|} + \beta_i \right] = \min_{i \in [k]} \left| \alpha_i \sum_{e \in E_i} w_i(e) \hat{x}_e + \beta_i \right|.
\]
Proof. To prove the lemma, we show that for any \( i \in [k] \),
\[
\mathbb{E}_{S \sim \hat{p}} \left[ \frac{\alpha_i}{S} \cdot w_i(S) \right] + \beta_i = \alpha_i \sum_{e \in E_i} w_i(e) \hat{x}_e + \beta_i.
\]
By the definition of \( S_j \) (\( j = 1, \ldots, t \)), for each \( e = \{u, v\} \in E_j \), we observe that \( e \in E_j[S_j] \iff u, v \in S_j \iff \hat{y}_u \geq r_j \) and \( \hat{y}_v \geq r_j \iff \hat{x}_e \geq r_j \), where the last equivalence follows from \( \hat{x}_e = \min(\hat{y}_u, \hat{y}_v) \).

Recall that \( \hat{p} \) is defined as (5). Then we have
\[
\mathbb{E}_{S \sim \hat{p}} \left[ \frac{\alpha_i}{S} \cdot w_i(S) \right] + \beta_i = \alpha_i \sum_{j \in [t]} w_i(S_j) \cdot \hat{x}_e + \beta_i
\]
\[
= \alpha_i \sum_{j \in [t]} w_i(S_j) \cdot (r_j - \hat{y}_j) \cdot \beta_i
\]
\[
= \alpha_i \sum_{j \in [t]} w_i(S_j) \cdot (r_j - \hat{y}_j) \cdot \beta_i
\]
\[
= \alpha_i \sum_{j \in [t]} w_i(S_j) \cdot (r_j - \hat{y}_j) \cdot \beta_i
\]
\[
= \alpha_i \sum_{j \in [t]} w_i(S_j) \cdot \beta_i,
\]
where the second last equality follows from the above equivalence and the last equality follows from the fact that for each \( e \in E \), \( \hat{x}_e = r_j = \sum_{j=1}^t (r_j - \hat{y}_j) \) for some \( j' \).

Next, we prove that the optimal value of LP (4) gives an upper bound on the optimal value of \((\alpha, \beta)\)-DENSITY:

**Lemma 5.2.** It holds that
\[
\min_{i \in [k]} \left\{ \alpha_i \sum_{e \in E_i} w_i(e) \hat{x}_e + \beta_i \right\} \geq \max_{p \in \Delta(2^V)} \min_{i \in [k]} \mathbb{E}_{S \sim \hat{p}} \left[ \frac{\alpha_i}{S} \cdot w_i(S) \right] + \beta_i.
\]

Proof. Let us take an arbitrary \( \hat{p} \in \Delta(2^V) \). We consider a solution \((\hat{x}_e)_{e \in E}, (\hat{y}_e)_{e \in V}, \hat{t}\) of LP (4) such that
\[
\hat{x}_e = \frac{\hat{p}_S}{|S|}, \quad \hat{y}_e = \frac{\hat{p}_S}{|S|}, \quad \hat{t} = \min_{i \in [k]} \left\{ \alpha_i \sum_{e \in E_i} w_i(e) \hat{x}_e + \beta_i \right\}.
\]

The solution \((\hat{x}_e)_{e \in E}, (\hat{y}_e)_{e \in V}, \hat{t}\) is feasible for LP (4), in fact, the only constraint is the third constraint but we see that
\[
\sum e \in V \hat{y}_e = \sum e \in S \frac{\hat{p}_S}{|S|} = \sum e \in S \frac{|S|}{|S|} = 1.
\]
Hence, the optimal value of LP (4) is at least \( \hat{t} \). Moreover, we have
\[
\hat{t} = \min_{i \in [k]} \left\{ \alpha_i \sum_{e \in E_i} w_i(e) \hat{x}_e + \beta_i \right\}
\]
\[
= \min_{i \in [k]} \left\{ \alpha_i \sum_{S \subseteq V} \hat{p}_S \cdot \frac{|S|}{|S|} \cdot w_i(S) + \beta_i \right\} = \min_{i \in [k]} \mathbb{E}_{S \sim \hat{p}} \left[ \frac{\alpha_i}{S} \cdot w_i(S) \right] + \beta_i.
\]
Recalling that \( \hat{p} \) is taken arbitrarily from \( \Delta(2^V) \), we see that the optimal value of LP (4) is at least that of \((\alpha, \beta)\)-DENSITY.

Combining Lemmas 5.1 and 5.2, we have the desired result:

**Theorem 5.3.** Algorithm 1 outputs an optimal solution to \((\alpha, \beta)\)-DENSITY.

### 5.2 Hierarchical structure and support size of the output of Algorithm 1

Here we observe some useful properties of the output of Algorithm 1. By the design of Algorithm 1, we see that the support of the output \( \hat{p} \) of the algorithm (i.e., an optimal solution to \((\alpha, \beta)\)-DENSITY) has a hierarchical structure. We denote the support of \( p \in \Delta(2^V) \) by \( \text{supp}(p) = \{S \subseteq V \mid |S| > 0\} \).

**Proposition 5.4.** Algorithm 1 outputs a hierarchical solution \( \hat{p} \), i.e., \( S \subseteq T \) or \( S \supseteq T \) for any \( S, T \in \text{supp}(\hat{p}) \).

From this proposition, the output \( \hat{p} \) has support size at most \( |V| - 1 \) (since the empty set and the singletons are useless). In addition, if we pick an optimal basic solution to LP (4) in Algorithm 1, the support size of \( \hat{p} \) becomes at most \( k \). For the definition of a basic solution, see e.g., Vanderbei’s book [42].

**Theorem 5.5.** If Algorithm 1 takes a basic optimal solution to LP (4), its output \( \hat{p} \) has support size at most \( k \).

Proof. Let \((\hat{x}_e)_{e \in E}, (\hat{y}_e)_{e \in V}, \hat{t}\) be a basic solution. Without loss of generality, we may assume that
\[
\hat{x}_e = \min(\hat{y}_u, \hat{y}_v) \quad (\forall e = \{u, v\} \in E).
\]
Recall that \( t \) denotes the number of different positive values in \( \{\hat{y}_u \mid \hat{y}_u > 0, u \in V\} \). Let \( V_0 = \{u \in V \mid \hat{y}_u = 0\} \). We divide \( V \setminus V_0 \) into \( t \) subsets of vertices sharing the same value of \( \hat{y}_u \), denoted by \( V_1, \ldots, V_t \).

Let us focus on the constraints in LP (4) that are satisfied with equality. For each \( j = 0, 1, \ldots, t \), let \( F_j \subseteq E[V_j] \) be a spanning forest in \( E[V_j] \). Let \( \beta \) be the number of connected components in \( E[V_1] \cup \cdots \cup E[V_t] \), and let \( \beta \) be that of \( E[V_0] \). We arbitrarily take \( \beta \) vertices, denoted by \( u_1, \ldots, u_\beta \), one from each connected component in \( E[V_0] \). We focus on the following constraints (satisfied with equality):
\[
t = \alpha_1 \sum_{e \in E_j} w_i(e) \hat{x}_e + \beta_i \quad (\forall i \in K'),
\]
\[
\hat{x}_e = y_u - x_v \quad (\forall e = \{u, v\} \in U_{j=0}^\beta E[V_j]),
\]
\[
\hat{x}_e = y_v \quad (\forall e = \{u, v\} \in E \setminus U_{j=0}^\beta E[V_j], \hat{y}_u < \hat{y}_v),
\]
\[
\sum_{e \in V} y_e = 1,
\]
\[
y_0 = 0 \quad (\forall u \in V_0),
\]
where \( K' = \{i \in [k] \mid t \alpha_1 \sum_{e \in E_i} w_i(e) x_e + \beta_i \} \).

We prove that the coefficient matrix of those constraints is full-rank. For ease of discussion, we remove constraints that are represented by a linear combination of others. Specifically, it is enough to focus on the following constraints:
\[
t = \alpha_1 \sum_{e \in E_j} w_i(e) \hat{x}_e + \beta_i \quad (\forall i \in K'), \quad \hat{x}_e = y_u - x_v \quad (\forall e = \{u, v\} \in U_{j=0}^\beta F_j), \quad \hat{x}_e = y_v \quad (\forall e = \{u, v\} \in E \setminus U_{j=0}^\beta F_j, u < v), \quad \hat{x}_e = y_v \quad (\forall e = \{u, v\} \in E \setminus U_{j=0}^\beta F_j, \hat{y}_u < \hat{y}_v),
\]
\[
\sum_{e \in V} y_e = 1,
\]
\[
y_0 = 0 \quad (\forall i \in \{\ell\}).
\]

There are two types of missing constraints. First, let \( e = \{u, v\} \in E[V_j] \setminus F_j \) (\( j \in \{\ell\} \)) such that \( x_e = y_e \) appears in (9). There exists a cycle \( C \in F_j \cup \{e\} \). By a telescoping sum of the constraints (8) along
C, i.e., $y_0 = x_0$, $x_0' = y_0'$, ..., $x_{11} = y_{11}$, we obtain $y_{12} = y_0$. Then, constraint $x_0 = y_0$ is obtained by summing $x_0 = y_0$ for $y_0 = y_0$. Next, let $v \in V_0$ belong to a connected component containing $u_j$ ($j \in [1])$. By summing (8) along a path from $v$ to $u_j$, i.e., $x_0 = x_0'$, $x_0' = y_0'$, ..., $x_0'' = y_0''$, we obtain $y_0 = y_{12}$. Then, constraint $y_0 = 0$ is obtained by summing $y_0 = y_{12}$ and $y_{12} = 0$ from (12).

The rank of the coefficient matrix is equal to the number of constraints (7)–(12). For (7), we have at most $k$ constraints; the number of constraints (8) is $2(\vert V \vert - \rho - \zeta)$; that for (9) and (10) is $\vert E \vert - \vert V \vert + \rho + \zeta$; that for (11) and (12) is $1 + \zeta$. Therefore, we have at most $\vert V \vert + \vert E \vert + \rho + 1 + k$ constraints.

We have $\vert V \vert + \vert E \vert + 1$ variables in LP (4). If $\rho > k$, then we have at most $\vert V \vert + \vert E \vert$ constraints, and hence the coefficient matrix of (7)–(12) cannot have rank $\vert V \vert + \vert E \vert + 1$. As the solution is basic, $\rho \leq k$ must hold. Recall that each $V_j$ ($j = 1, \ldots, l$) has at least one connected component. Therefore, we have at most $k$ different positive values of $y_0$, which implies that the output $\hat{\beta}$ has support size at most $k$.

\section{6 PREPROCESSING}

In this section, we present a simple, scalable preprocessing algorithm, which often reduces the size of the input networks significantly and results in a substantial speed-up. Specifically, the algorithm first computes an approximate solution by solving an LP, which is much smaller than LP (4) in practice, and then removes vertices from the original network using the information about the approximate solution obtained. We assume that $S_i^j$ for all $i \in [k]$ are known in advance because they can be computed efficiently using Charikar’s LP-based algorithm [9] together with the preprocessing algorithm introduced by Balalau et al. [4]. To describe our algorithm, we introduce some notations. For $S \subseteq V$, $v \in S$, and $i \in [k]$, let $d_i(S, v)$ denote the weighted degree of $v$ in the subgraph induced by $S$ in layer $i$, i.e., $d_i(S, v) := \sum_{e \in E_i[S]} w(e)$. When $S = V$, we simply write $d_i(v)$.

We first describe a fast algorithm for finding an approximate solution for $(\alpha, \beta)$-DENSITY. Specifically, we compute a probability distribution $q \in \Lambda(2^V)$ that maximizes the $(\alpha, \beta)$-density under the constraint that $q_S = 0$ for all $S \in 2^V \setminus \{S_1^1, \ldots, S_k^k\}$. The distribution can be found by solving the following LP:

$$\max \quad t$$
$$\text{s.t.} \quad t \leq \alpha \sum_{j \in [k]} \frac{w_l(S_j^i)}{|S_j^i|} q_j + \beta_i \quad (\forall i \in [k]),$$
$$\sum_{j \in [k]} q_j = 1,$n Noble q_j \geq 0 \quad (\forall j \in [k]).$$

Note that this LP has $k + 1$ variables and $2k + 1$ constraints. As $k$ is usually much smaller than $\vert V \vert$ and $\vert E \vert$, this LP is much smaller than LP (4) in practice.

Next we describe an algorithm for removing vertices using the information of the above approximate solution $q$. Let $t^*$ be the $(\alpha, \beta)$-density of $q$, i.e., the optimal value of LP (13). Note that this is a lower bound on the optimal value of $(\alpha, \beta)$-DENSITY. Our algorithm iteratively removes any vertex $v^*$ that satisfies $\max_{i \in [k]} (\alpha_i \cdot d_i(V', v^*) + \beta_i) < t^*$, where $V'$ is a remaining vertex set (initially $V' = V$), as long as there exists such a vertex. For reference, we describe the procedure in Algorithm 2. This algorithm can be implemented to run in $O(k|E| + |V| \log |V|)$ time.

From now on, we demonstrate that the algorithm does not remove any vertex that is contained in a subset of $(\alpha, \beta)$-DENSITY. The following is a key lemma in our analysis.

**Lemma 6.1.** Let $t^*$ be a lower bound on the optimal value of $(\alpha, \beta)$-DENSITY. If $\max_{i \in [k]} (\alpha_i \cdot d_i(v^*) + \beta_i) < t^*$, then $y_{v^*} = 0$ for any optimal solution to LP (4).

**Proof.** We prove the lemma by contradiction. We denote by $(\tilde{x}_{\cdot \cdot 0})_{e \in E}, (\tilde{y}_{\cdot \cdot 0})_{v \in V}, t)$ an optimal solution to LP (4) and let $v^* \in V$ be a vertex that satisfies $\alpha_i \cdot d_i(v^*) + \beta_i < t^*$ for all $i \in [k]$. Suppose for contradiction that $\tilde{y}_{v^*} > 0$.

We construct a solution $(x_{\cdot \cdot 0})_{e \in E}, (y_{\cdot \cdot 0})_{v \in V}, t)$ of LP (4) as follows:

$$x_e = \begin{cases} \frac{1}{1 - \tilde{y}_{v^*}} \cdot \tilde{x}_e & (e \ni v^*), \\ 0 & (e \ni v^*) \end{cases},$$

$$y_v = \begin{cases} \frac{1}{1 - y_{v^*}} \cdot \tilde{y}_v & (v \ni v^*), \\ 0 & (v \ni v^*). \end{cases}$$

$$t = \min_{i \in [k]} \left[ \alpha_i \sum_{e \in E_i} w_l(e) x_e + \beta_i \right].$$

It is easy to see that $(x_{\cdot \cdot 0})_{e \in E}, (y_{\cdot \cdot 0})_{v \in V}, t)$ is a feasible solution of LP (4). Moreover, we have:

$$t = \min_{i \in [k]} \left[ \alpha_i \sum_{e \in E_i} w_l(e) x_e + \beta_i \right]$$

$$= \min_{i \in [k]} \left[ \alpha_i \sum_{e \in E_i, v \in E} w_l(e) \tilde{x}_e - \sum_{e \in E_i, v \in E} w_l(e) \tilde{x}_e + \beta_i \right]$$

$$= \min_{i \in [k]} \left[ \alpha_i \sum_{e \in E_i} w_l(e) x_e - \sum_{e \in E_i} w_l(e) \tilde{x}_e + \beta_i \right]$$

$$= \min_{i \in [k]} \left[ \alpha_i \sum_{e \in E_i} w_l(e) x_e + \beta_i \right]$$

$$< \min_{i \in [k]} \left[ \alpha_i \sum_{e \in E_i} w_l(e) x_e + \beta_i \right] - t^* \cdot \tilde{y}_{v^*}$$

$$= \frac{t - t^*}{1 - \tilde{y}_{v^*}} \leq \frac{t - t^*}{1 - y_{v^*}} = t^*.$$
and \( y_{\sigma'} > 0 \), and the third inequality follows from Lemma 5.2. This
contradicts the optimality of \((\hat{x}_e)_{e \in E}, (\hat{y}_e)_{e \in E}, \hat{f}\).

**Theorem 6.2.** Let \( \ell' \) be a lower bound on the optimal value of
\((\alpha, \beta)\)-DENSITY. Then, any vertex \( \sigma' \) that satisfies \( \max_{e \in [k]} \left[ \alpha_i \cdot d_i(\sigma') + \beta_i \right] < \ell' \) is not contained in any subset in the support of any optimal solution to \((\alpha, \beta)\)-DENSITY.

**Proof.** Let \( \sigma' \) be any vertex with \( \max_{e \in [k]} \left[ \alpha_i \cdot d_i(\sigma') + \beta_i \right] < \ell' \). Let \( \hat{y} \) be any optimal solution to \((\alpha, \beta)\)-DENSITY. Construct \((\hat{x}_e)_{e \in E}, (\hat{y}_e)_{e \in E}, \hat{f}\) from \( \hat{y} \) as in (6). From Lemma 5.1 and the proof of Lemma 5.2, we see that \((\hat{x}_e)_{e \in E}, (\hat{y}_e)_{e \in E}, \hat{f}\) is an optimal solution to LP (4). Thus, by Lemma 6.1, we have \( y_{\sigma'} = 0 \). By the construction of \( \hat{y}_e \) in (6), \( \hat{y}_e = 0 \) for all \( S \subseteq V \) containing \( \sigma' \).

This theorem indicates that Algorithm 2 does not remove any vertex that is contained in a subset in the support of any optimal solution to \((\alpha, \beta)\)-DENSITY.

### 7 Experimental Evaluation

In this section, we conduct computational experiments using synthetic graphs and real-world networks to verify the validity of our proposed model and to evaluate the performance of our proposed algorithms. All experiments were conducted on a machine equipped with Intel Xeon W 10-core processor and 64GB RAM. Algorithms were implemented in Python using Gurobi Optimizer 9.0.2.

#### 7.1 Validity of our model

Here we aim to verify the validity of our model using synthetic graphs. To this end, we use a randomly generated multilayer network with a planted clique, and examine whether an optimal solution detects vertex subsets close to the clique.

We first explain our random procedure for generating multilayer networks. We produce an (unweighted) random power-law graph as a layer using the Chung–Lu model [12], where we first specify an expected degree \( d_v \) for each \( v \in V \) according to the power-law distribution with exponent \( \beta \), and then connect each pair of vertices \( \{u, v\} \) with probability \( \frac{d_u d_v}{\sum_{v' \in V} d_{v'}} \). Note that the graph becomes sparser as the exponent \( \beta \) increases. In such a multilayer network, we randomly select a vertex subset \( V_c \) with some size, and plant a clique on \( V_c \) (in some specified layers).

To evaluate the performance of optimal solution \( p \in \Lambda(2^V) \) to \((\alpha, \beta)\)-DENSITY in the above multilayer network, we introduce the following measure, which we refer to as the (expected) F measure:

\[
\text{F measure} = \frac{2 \cdot \text{precision} \cdot \text{recall}}{\text{precision} + \text{recall}},
\]

where \( \text{precision} := \sum_{S \subseteq V} \frac{|S \cap V_c|}{|S|} \) and \( \text{recall} := \sum_{S \subseteq V} \frac{|S \cap V_c|}{|V_c|} \). The F measure approaches 1 if \( p \) tends to be close to \( V_c \).

We first investigate the case where a clique is planted in all layers. We generate \( k = 1, 2, 3, 4, 5 \) power-law graphs (i.e., layers) with \( \beta = 3.0 \) on \( V \) with \( |V| = 1,000 \). Then we randomly select a subset \( V_c \subseteq V \) consisting of 10 vertices, and plant a clique on \( V_c \) in all layers. The performance of optimal solutions to \((\alpha, \beta)\)-DENSITY is shown in Figure 2(a). As can be seen, for any metric, the F measure is reasonably large for \( k \geq 2 \), meaning that our algorithm tends to detect \( V_c \) using the information of multiple layers.

Next we investigate the case where a clique is planted in only one layer. We generate \( k = 1, 2, 3, 4, 5 \) power-law graphs with \( \beta = 3.0 \) on \( V \) with \( |V| = 1,000 \). Then we randomly select \( V_c \subseteq V \) consisting of 20 vertices, and plant a clique on \( V_c \) in only one randomly selected layer. The performance of optimal solutions are described in Figure 2(b). As can be seen, the F measure becomes smaller as the number of layers increases. Among the three metrics, the regret performs particularly well because it concentrates on the layer containing the clique from its definition; therefore, the regret metric seems most suitable for robust analysis with noisy layers. The robust ratio performs second best; it also cares about the layer containing the clique.

Finally, we conclude this subsection by evaluating vertex subsets that are obtained from optimal solutions \( p \in \Lambda(2^V) \). To verify the validity of our model, we select a vertex subset attaining the highest probability in \( p \in \Lambda(2^V) \). As our algorithm does not know about \( V_c \), it cannot select a vertex subset using the F measure. For reference, we also run two baseline algorithms, DCS-LP and DCS-Greedy, designed by Jethava and Beerenwinkel [27]. Note that DCS-LP can be seen as the algorithm that selects a vertex subset with the largest minimum density value over layers from the support of \( p \in \Lambda(2^V) \), where \( p \in \Lambda(2^V) \) is an optimal solution to our algorithm with the density metric. The results are depicted in Figure 3, where we employed the same experimental settings as above and used the usual (deterministic) F measure for evaluation. As can be seen, the trend of our algorithm is similar to that observed in the above experiments; all metrics almost detect \( V_c \) for the all-layers setting, but only the regret metric is successful for the only-one-layer setting. As for the
7.2 Performance of our algorithm

Here we examine the performance of our proposed algorithm using publicly-available real-world multilayer networks. Table 1 shows the results together with the summary of the characteristics of the datasets. Even for large networks, our algorithm obtains an optimal solution in reasonable time. The preprocessing algorithm often reduces the size of the networks significantly using a reasonably large lower bound computed by LP (13) and results in a substantial speed-up. As an extreme example, for Cannes2013 with the density metric, while DCS-Greedy performs quite poorly. Another direction is to apply our algorithms for dense subgraph discovery in multilayer networks.

For example, the algorithm with the robust ratio admits a particularly large robust ratio value of 0.6652 for NYClimateMarch2014, meaning that the vertex subset obtained achieves an approximation ratio of 0.6652 for all layers. Moreover, the algorithm with the regret metric admits a particularly small regret value of 155.5000 for Cannes2013. For those instances, the algorithm with the density metric (i.e., DCS-LP) performs poorly in terms of those metrics, respectively. From the above, it seems quite important to select an appropriate metric depending on the practical purpose at hand.

8 CONCLUSION

In this paper, we have introduced a novel optimization model and algorithms for dense subgraph discovery in multilayer networks. There are several possible directions for future research. One direction is to improve the scalability of our algorithm, particularly for the regret metric, for which our preprocessing algorithm does not necessarily perform well. Another direction is to apply our model and algorithms to some real-world applications of multilayer-network analysis. Investigating multilayer-network counterparts of some existing generalizations of the densest subgraph problem (see e.g., [29, 34, 40, 43]) is also interesting future work.

ACKNOWLEDGMENTS

This work was partially supported by JST PRESTO Grant Numbers JPJMPR2212 and JSPS KAKENHI Grant Numbers JP17K12646, JP19K20218, JP20K19739, JP21K17708, and JP21H03397.
REFERENCES

[1] A. Angel, N. Sarkar, N. Koudas, and D. Srivastava. 2012. Dense subgraph maintenance under streaming edge weight updates for real-time story identification. In Proceedings of VLDB. 574–585.

[2] T. Basar and G. J. Olsder. 1999. Dynamic Noncooperative Game Theory. Classics in Applied Mathematics, Vol. 23. SIAM.

[3] G. D. Bader and C. W. V. Hogue. 2003. An automated method for finding molecular complexes in large protein interaction networks. BMC Bioinformatics 4, 1 (2003), 1–27.

[4] O. D. Balalau, F. Bonchi, and F. Gullo. 2017. Core decomposition and densest subgraph in multilayer networks. In Proceedings of WSDM. 379–388.

[5] P. Basaras, G. Iosifidis, D. Katsaros, and L. Tassiulas. 2019. Identifying influential spreaders in complex multilayer networks: A centrality perspective. IEEE Transactions on Network Science and Engineering 6, 1 (2019), 31–45.

[6] M. Bazzi, M. A. Porter, S. Williams, M. McDonald, D. J. Fenn, and S. D. Howison. 2016. Community detection in temporal multilayer networks, with an application to correlation networks. Multiscale Modeling & Simulation 14, 1 (2016), 1–41.

[7] S. Boccaletti, G. Bianconi, R. Criado, C. J. del Genio, J. Gómez-Gardeñes, M. Romance, I. Sendiña-Nadal, Z. Wang, and M. Zanin. 2014. The structure and dynamics of multilayer networks. Physics Reports 544, 1 (2014), 1–122.

[8] D. Boob, Y. Gao, R. Peng, S. Sawlani, C. E. Tsourakakis, D. Wang, and J. Wang. 2020. Flows: Extracting densest subgraphs without flow computations. In Proceedings of The Web Conference 2020. 573–583.

[9] M. Charikar. 2000. Greedy approximation algorithms for finding dense components in a graph. In Proceedings of APPROX. 84–95.

[10] M. Charikar, Y. Naamad, and J. Wu. 2018. On finding dense common subgraphs. arXiv preprint arXiv:1802.08631 (2018).

[11] C. Chekuri, K. Quanrud, and M. R. Torres. 2022. Densest subgraph. Supermodularity, iterative peeling, and flow. In Proceedings of SODA. 1531–1555.

[12] F. Chung and L. Lu. 2002. The average distances in random graphs with given expected degrees. Internet Mathematics 1 (2002), 15879–15882.

[13] C. De Bacco, E. A. Power, D. B. Larremore, and C. Moore. 2017. Community detection, link prediction, and layer independence in multilayer networks. Physical Review E 95 (2017), 042317.

[14] M. De Domenico, C. Granell, M. A. Porter, and A. Arenas. 2016. The physics of spreading processes in multilayer networks. Nature Physics 12, 10 (2016), 901–906.

[15] M. De Domenico, A. Solé-Ribalta, E. Cozzo, M. Kivelä, Y. Moreno, M. A. Porter, S. Gómez, and A. Arenas. 2013. Mathematical formulation of multilayer networks. Physical Review X 3 (2013), 041022.

[16] M. De Domenico, A. Solé-Ribalta, E. Omodei, S. Gómez, and A. Arenas. 2015. Ranking in interconnected multilayer networks reveals versatile nodes. Nature Communications 6 (2015), 6868.

[17] Y. Dourisboure, F. Geraci, and M. Pellegrini. 2007. Extraction and classification of dense communities in the web. In Proceedings of WWW. 461–470.

[18] J. A. Firth and B. C. Sheldon. 2015. Experimental manipulation of avian social structure reveals segregation is carried over across contexts. Proceedings of the Royal Society B. Biological Sciences 282, 1802 (2015), 20142350.

[19] K. J. Friston. 2011. Functional and effective connectivity: A review. Brain Connectivity 1, 1 (2011), 13–36.

[20] E. Galimberti, F. Bonchi, and F. Gullo. 2017. Core decomposition and densest subgraph in multilayer networks. In Proceedings of CIKM. 1807–1816.

[21] D. Gibson, R. Kumar, and A. Tomkins. 2005. Discovering large dense subgraphs in massive graphs. In Proceedings of VLDB. 721–732.

[22] A. Gionis and C. E. Tsourakakis. 2015. Dense subgraph discovery: KDD 2015 Tutorial. In Proceedings of KDD. 2313–2314.

[23] A. V. Goldberg. 1984. Finding a maximum density subgraph. Technical Report. University of California Berkeley.

[24] F. Hashemi, A. Behroz, and L. S. Lakshmanan. 2022. FirmCore decomposition of multilayer networks. In Proceedings of The Web Conference. 1589–1600.

[25] R. Interdonato, A. Tagarelli, D. Ienco, A. Sallaberry, and P. Pontelet. 2017. Local community detection in multilayer networks. Data Mining and Knowledge Discovery 31, 5 (2017), 1444–1479.

[26] M. Jahlí, Y. Orouchkhanli, M. Asgari, N. Alipourfard, and M. Perc. 2017. Link prediction in multiplex online social networks. Royal Society Open Science 4, 2 (2017), 160863.

[27] V. Jethava and N. Beerenwinkel. 2015. Finding dense subgraphs in relational graphs. In Proceedings of ECML PKDD. 641–654.

[28] Y. Kawase, Y. Kuroki, and A. Miyayuchi. 2019. Graph mining meets crowdsourcing: Extracting experts for answer aggregation. In Proceedings of IJCAI. 1272–1279.

[29] Y. Kawase and A. Miyayuchi. 2018. The densest subgraph problem with a convex/concave size function. Algorithmica 80, 12 (2018), 3461–3480.

[30] M. Kivelä, A. Arenas, M. Barthelemy, J. P. Gleeson, Y. Moreno, and M. A. Porter. 2014. Multilayer networks. Journal of Complex Networks 2, 3 (2014), 203–271.

[31] G. Kortsarz and D. Peleg. 1994. Generating sparse 2-spanners. Journal of Algorithms 17, 2 (1994), 222–236.

[32] V. E. Lee, N. Ruan, R. Jin, and C. Aggarwal. 2010. A survey of algorithms for dense subgraph discovery. 303–336.

[33] J. Leskovec, J. Kleinberg, and C. Faloutsos. 2005. Graphs over time: Densification laws, shrinking diameters and possible explanations. In Proceedings of KDD. 177–187.

[34] A. Miyayuchi and N. Kakimura. 2018. Finding a dense subgraph with sparse cut. In Proceedings of CIKM. 547–556.

[35] A. Miyayuchi and A. Takeda. 2018. Robust densest subgraph discovery. In Proceedings of RDM. 1188–1193.

[36] E. Omodei, M. De Domenico, and A. Arenas. 2015. Characterizing interactions in online social networks during exceptional events. Frontiers in Physics 3 (2015), 59.

[37] M. Salehi, R. Sharma, M. Marzolla, M. Magnani, P. Siami, and D. Montesi. 2015. Spreading processes in multilayer networks. IEEE Transactions on Network Science and Engineering 2, 2 (2015), 65–83.

[38] K. Semertzidou, E. Pitoura, E. Terzi, and P. Tsaparas. 2019. Finding lasting dense subgraphs. Data Mining and Knowledge Discovery 33, 5 (2019), 1417–1445.

[39] A. Tagarelli, A. Amelio, and F. Gullo. 2017. Ensemble-based community detection in multilayer networks. Data Mining and Knowledge Discovery 31, 5 (2017), 1506–1543.

[40] C. E. Tsourakakis. 2015. The k-clique densest subgraph problem. In Proceedings of WWW. 1122–1132.

[41] C. E. Tsourakakis, T. Chen, N. Kakimura, and J. Pachocki. 2019. Novel dense subgraph discovery primitives: Risk aversion and exclusion queries. In Proceedings of ECML PKDD. 378–394.

[42] R. J. Vanderbei. 2020. Linear Programming: Foundations and Extensions. International Series in Operations Research & Management Science, Vol. 196. Springer.

[43] N. Veldt, A. R. Benson, and J. Kleinberg. 2021. The generalized mean densest subgraph problem. In Proceedings of KDD. 1604–1614.

[44] Z. Zou. 2013. Polynomial-time algorithm for finding densest subgraphs in uncertain graphs. In Proceedings of MLG. No page numbers.