Strong-weak coupling duality in anisotropic current interactions

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Abstract

The recently proposed all orders βeta function for current interactions in two dimensions is further investigated. By using a strong-weak coupling duality of the βeta function, and some added topology of the space of couplings we are able to extend the flows to arbitrarily large or small scales. Using a non-trivial RG invariant we are able to identify sine-Gordon, sinh-Gordon and Kosterlitz-Thouless phases. We also find an additional phase with cyclic or roaming RG trajectories.

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I. INTRODUCTION

Current-current interactions in two dimensions arise in many important physical systems, for example in non-Fermi liquids, Kosterlitz-Thouless transitions, and disordered Dirac fermions. For the generally non-integrable anisotropic cases, the renormalization group (RG) to low orders in perturbation theory has been one of the main tools for understanding these theories. Recently, an all-orders $\beta$ function was proposed \cite{1}. Some difficulties were encountered in understanding completely the consequences of these $\beta$ functions, due mainly to the existence of poles \cite{2}. In this paper we resolve some of these difficulties by using the duality pointed out in \cite{2} combined with some added topology in the space of couplings. We focus only on the anisotropic $su(2)$ case.

The $\beta$eta functions were originally computed by using the current-algebra Ward identities to isolate the log divergences at each order and summing the perturbative series. These series converge in a neighborhood of the origin which we identify as the perturbative domain. The $\beta$eta functions turn out to be rational functions of the couplings and possess pole singularities at the boundary of this perturbative domain. We extend the $\beta$eta functions away from this domain by analytical continuation. Although simple, this ansatz is a hypothesis which may hide some dynamical effects, for example the possibility that degrees of freedom become relevant at the singularities. Assuming that the proposed $\beta$eta functions capture all perturbative contributions, the analytical continuation prescription provides a non-perturbative definition of them. One of the aims of this paper is to check the consistency of the proposed $\beta$eta functions upon analytical continuation and to decipher the consequences.

It is useful to recall a few properties of the renormalization group transformations we expect, and to discuss pathologies that may be encountered. RG transformations, defined for example in the Wilsonian scheme by successive integrations of the fast modes, induce a map on the space of effective actions. In this space, of a priori huge dimension, the RG transformations may be extended to arbitrarily large scales. The space of effective actions contains many irrelevant directions, corresponding to what are called stable manifolds, which are washed out in the continuum limit. The remaining directions, hopefully finite in number, label the continuous field theories if they exist. For renormalizable theories the RG transformations may be projected onto this parameter space. The renormalized coupling constants may then be viewed as the values of these parameters at the renormalization length scale. For renormalizable theories, this then defines RG flows which may be extended to arbitrarily large or small scales as they simply correspond to space dilatations in the continuous field theories.

In perturbation theory one computes $\beta$eta functions in some domains which may not cover all of phase space. The sizes of these domains, as well as the analytical properties of the $\beta$eta functions, depend on renormalization prescriptions, i.e. on choices of coordinates. As a consequence, some RG trajectories cannot be extended from perturbative domains to arbitrarily large scales. For example, one of the perturbative coordinates may blow up after a finite scale transformation. This signals that either some relevant degrees of freedom or some non-perturbative effects have been missed, or that the perturbative domains have to be extended. Thus, as a rule we shall look for extensions of the perturbative phase spaces such that all RG trajectories can be followed up to infinitely large scales. In this situation
we may call the RG phase space complete in analogy with General Relativity where one looks for geodesically complete manifolds. Similarly, if the RG flows defined in this way cannot be extended to small scales, but break at some ultraviolet cut-off, this may indicate the absence of a non-perturbatively defined renormalizable continuous theory.

II. DUALITY IN THE $\beta$eta FUNCTION

The models we consider are perturbations of the $su(2)_k$ WZW models, the latter being conformal field theories with $su(2)$ affine symmetry [3]. Let us normalize the $su(2)$ level $k$ current algebra as follows

$$J_3(z)J_3(0) \sim \frac{k}{2z^2}, \quad J_3(z)J^\pm(0) \sim \pm \frac{1}{z} J^\pm(0), \quad J^+(z)J^-(0) \sim \frac{k}{2z^2} + \frac{1}{z} J_3(0)$$

(2.1)

and similarly for $\bar{J}$, and consider the anisotropic left-right current-current perturbation:

$$S = S_{su(2)_k} + \int \frac{d^2 x}{2\pi} \left( g_1 (J^+ \bar{J}^- + J^- \bar{J}^+) + g_2 J_3 \bar{J}_3 \right)$$

(2.2)

The $\beta$eta functions proposed in [1] are

$$\beta_{g_1} = \frac{g_1 (g_2 - g_1^2 k/4)}{(1 - k^2 g_1^2 /16)(1 + k g_2/4)} \quad (2.3)$$

$$\beta_{g_2} = \frac{g_1^2 (1 - k g_2/4)^2}{(1 - k^2 g_1^2 /16)^2} \quad (2.4)$$

The $su(2)$ invariant isotropic lines correspond to $g_1 = g_2$, and also to $g_1 = -g_2$ due to the $su(2)$ automorphism $J^\pm \to -J^\pm$ which can be performed on the left-moving currents only sending $g_1 \to -g_1$.

By rescaling the currents $J \to \sqrt{k} J$ in the isotropic case one easily sees that

$$\beta_g = \frac{1}{k} F(C_{\text{adj}}/k, kg)$$

(2.5)

where $C_{\text{adj}}$ is the quadratic Casimir in the adjoint representation, and $F$ is a function of the combinations $C_{\text{adj}}/k, kg$. The log divergences that contribute to the $\beta$eta function found in [1] are linear in $C_{\text{adj}}/k$ to any order in perturbation theory, which explains the simple $k$ dependence in (2.3). The manner in which log divergences were isolated in [1] led to no corrections which are higher powers of $C_{\text{adj}}/k$, and thus corrections to higher powers of $1/k$, however the argument given does not constitute a proof that there are no such corrections. The work [1] also did not deal with potential infra-red divergences. In this work we do not address these issues but simply assume the above $\beta$eta function is correct and see if it can be made sense of.

In [2] various regimes of behavior of the RG flows were identified. The main difficulty encountered in interpreting these flows was due to the fact that in some regimes after a finite scale transformation one flows to the poles and one cannot extend the flows reliably beyond the poles numerically. Another difficulty encountered in [3] is that for some RG trajectories
one of the coupling constants \( g_{1,2} \) blows up after a finite scale transformation. In this paper we propose a resolution of this difficulty based on a strong-weak coupling duality of the \( \beta \eta \) function and some added topology of the space of couplings.

Consider the isotropic \( \beta \eta \) function for \( g_1 = g_2 = g \) (for \( k = 1 \)) with a double pole at \( g = -4 \):

\[
\beta_g = \frac{16g^2}{(g + 4)^2} \tag{2.6}
\]

It exhibits a duality [2]. Define a dual coupling \( g^* = 16/g \) and let \( \beta^*(g^*) = (\partial g^*/\partial g) \beta(g) \), then

\[
\beta^*(g^*) = -\beta(g \rightarrow g^*) \tag{2.7}
\]

Suppose the flow starts at \( g < -4 \) toward the pole. Then at some scale \( r_0, g = g^* = -4 \equiv g_0 \), where \( g_0 \) is the self-dual coupling. The duality of the \( \beta \eta \) function implies that a coupling \( g \) at a scale \( r \) flows to \( g^* \) at a scale \( r_0/r \). Thus the ultraviolet and infrared values of \( g \) are simply related by duality. This can be seen from the analytical solution:

\[
g - g^* + 8 \log(g/g_0) = 16 \log(r/r_0) \tag{2.8}
\]

The anisotropic \( \beta \eta \) function also exhibits the duality (2.7) with

\[
g_1^* = \frac{16}{k^2 g_1}, \quad g_2^* = \frac{16}{k^2 g_2} \tag{2.9}
\]

If \( g_{1,2}(r) \) is a solution of the RG equations, so is \( g_{1,2}^*(r_0/r) \) for any \( r_0 \). To simplify the notation, unless explicitly stated we rescale \( g_{1,2} \) to absorb the \( k \) dependence.

The precise shape of the RG trajectories can be determined by using the RG invariant

\[
Q = \frac{g_1^2 - g_2^2}{(g_2 - 4)^2(g_1^2 - 16)} \tag{2.10}
\]

It is simple to show

\[
\sum_g \beta_g \partial_g Q = 0 \tag{2.11}
\]

In consistency with the duality of the \( \beta \eta \) functions, one finds that \( Q \) is self-dual, i.e. \( Q(g) = Q(g^*) \). Eq.(2.10) may be inverted to express \( g_1^2 \) as a function of \( Q \) and \( g_2 \). This gives the equations for the RG trajectories. The \( \beta \eta \) function for \( g_2 \) may be written as a function of \( g_2 \) and \( Q \) only:

\[
\beta_{g_2} = 16 \frac{(g_2^2 - 16Q(g_2 - 4)^2)(1 - Q(g_2 - 4)^2)}{(g_2 + 4)^2}
\]

The remaining pole at \( g_2 = -4 \) reflects the instability of the theory at that value of \( g_2 \); see its bosonized form (3.2) in the following section. The zeroes of \( \beta_{g_2} \) are then found to be either on the \( g_2 \)-axis with \( g_1 = 0 \) or at infinity \( g_1 = \infty \).
We propose to use the duality (2.9) to extend the flows through the poles. Namely, if at a scale \( r \) the couplings are \( g_{1,2}(r) \), and the flow reaches the pole at a scale \( r_0 \), then at the scale \( r_0/r \) the couplings are given by \( g_{1,2}^*(r) \). This procedure for example allows us to make sense of all trajectories in the shaded domains in Figure 1. The beta functions (2.3) are ill-defined at the self-dual points and as a consequence there are many trajectories going through these points. To resolve this paradox one has to locally change the description of the phase space. Namely, points of the phase space will be described by coordinates \((g_1, g_2, Q)\) with the identification \((2.11)\). This is a redundant description away from the self-dual points but not in the neighborhood of these points where it amounts to choosing for example \((g_2, Q)\) as coordinates. This procedure, which is related to the mathematical construction known as local blowing up, changes slightly the topology of the phase space.

It turns out one additional ingredient is needed for a global interpretation of the phase diagram. Consider the region \( g_2 > 4, g_1 < 4 \). Using \( Q \), one sees that here one is attracted to \( g_2 \rightarrow \infty \) with \( 0 < g_1 < 4 \). Using duality arguments, it is easy to convince oneself that these trajectories reach \( g_2 = \infty \) in a finite RG time \( \log r \). Without further information one cannot extend the RG flows to arbitrarily large scales on the two dimensional plane with coordinates \( g_1, g_2 \). Since these trajectories are not flowing to a self-dual point, one cannot use duality to extend the flows. However at \(|g_2| = \infty \), the beta function satisfies

\[
\beta(g_1, g_2) = \beta(g_1, -g_2) \quad \text{when} \quad |g_2| \rightarrow \infty
\]  

(2.12)

Thus if a flow goes to \( g_2 = \infty \) after a finite scale transformation, one can consistently continue the flow at \( g_2 = -\infty \). This is also consistent with the fact that \( Q(g_2) = Q(-g_2) \) at \( g_2 = \infty \). Thus we propose to identify the points \( g_2 = \pm \infty \) which endows the coupling constant space with the topology of a cylinder. Alternatively, instead of identifying the two infinities \( g_2 = \pm \infty \), one can view the procedure as gluing multiple patches of the coupling constant space together at \( g_2 = \pm \infty \), with (2.12) as a consistent gluing condition. The \( +\infty \) of one patch is identified with \(-\infty \) of the next patch. In this alternative construction the phase space will also acquire a non-trivial topology.

With the above hypotheses all RG flows can be extended to arbitrarily large or small length scales. All flows either begin on the \( g_2 \) axis and terminate at \( g_1 = \infty \), or vice versa. Since both \( g_1 = 0 \) and \( g_1 = \infty \) are simple zeroes of the \( \beta \)eta functions, it takes an infinite RG time for the trajectories to reach these loci. Consider the flows that originate from \( g_1 = 0 \), \( 0 < g_2 < 4 \). They are attracted to the self-dual point \((g_1, g_2) = (4, 4)\), and then by duality end up at \( g_1 = \infty \), \( 4 < g_2 < \infty \). Similarly, the flows that originate from \( g_1 = 0, g_2 < -4 \) end up at \( g_1 = \infty, -4 < g_2 < 0 \). The flows that end up on the \( g_2 \) axis with \(-4 < g_2 < 0\) originate from \( g_1 = \infty \), \(-\infty < g_2 < -4 \) by duality around the self-dual point \((4, -4)\). Consider now the flows that originate from \( g_1 = 0, g_2 > 4 \). After a finite RG time \( \log r \), they are attracted to \( g_2 = \infty \) with \( 0 < g_1 < 4 \). These flows then continue from \( g_2 = -\infty \) of the next patch to the self-dual point \((g_1, g_2) = (4, -4)\) and end up at \( g_1 = \infty, 0 < g_2 < 4 \). Again, since the flow went through a self-dual point, the couplings in the UV and IR are related by duality. The resulting phase diagram is shown in figure 1. For \( g_1 < 0 \), the phase diagram is the reflection of the figure about the \( g_1 \) axis since \( \beta_{g_1} = -\beta(-g_1) \), \( \beta_{g_2} = \beta(-g_1) \).

Interestingly one phase exhibits roaming or cyclic RG trajectories. Namely, flows that begin in a patch at \( g_2 = -\infty, 4 < g_1 < \infty \), flow through the poles at \((g_1, g_2) = (4, -4)\), then through the poles at \((4, 4)\), then off to \( g_2 = \infty, 4 < g_1 < \infty \) where they start over again at
\( g_2 = -\infty \) in the next patch. It takes a finite RG time to go from \( g_2 = -\infty \) to \( g_2 = +\infty \) following these trajectories. With the points \( g_2 = \infty \) and \( g_2 = -\infty \) identified, these trajectories are cyclic, but in the alternative construction of the phase space by gluing multiple patches, these trajectories explore all patches. Note that these flows pass by the perturbative domain, \( |g_1| < 4, |g_2| < 4 \), in which the perturbative expansions of the \( \beta \)eta functions converge. If one assumes that field theories with initial couplings in the perturbative domain exist, one is naturally lead to include these RG trajectories into the phase space.

In the perturbative domain \( 0 < g_{1,2} < 4 \) one observes flows toward the \( su(2) \) invariant isotropic line, but this line is unstable beyond the pole and the above duality arguments indicate that the \( su(2) \) symmetry does not continue to be restored at strong coupling. The phases above or below the isotropic line have then very different large scale asymptotics. As we describe in the next section, if the flows continued along the isotropic line, this would be inconsistent with the believed existence of a sine-Gordon phase. These issues were recently addressed at one loop in [4].

### III. SINE-GORDON AND SINH-GORDON PHASES

In this section we interpret the RG flows in a bosonized description. When \( k = 1 \), the current algebra can be bosonized as

\[
J^\pm = \frac{1}{\sqrt{2}} \exp \left( \pm i \sqrt{2} \varphi \right), \quad J_3 = \frac{i}{\sqrt{2}} \partial_z \varphi
\]

where \( \varphi(z) \) is the \( z \)-dependent part of a free massless scalar field \( \phi = \varphi(z) + \bar{\varphi}(\bar{z}) \). Viewing the \( g_2 \) coupling as a perturbation of the kinetic term and rescaling the field \( \phi \) one obtains the sine-Gordon (sG) action

\[
S = \frac{1}{4\pi} \int d^2 x \left[ \frac{1}{2}(\partial \phi)^2 + g_1 \cos(b\phi) \right]
\]

where \( b(g_1, g_2) \) is to be determined.

When \( g_1 \approx 0, \beta_{g_1} \approx 4g_2g_1/(4 + g_2) \). Since this is linear in \( g_1 \), we can identify the dimension \( \Gamma \) of \( \cos b\phi \). Generally, if \( \beta(g_c) = 0 \), then

\[
\beta(g) = (2 - \Gamma)(g - g_c) + ....
\]

where \( \Gamma \) is the dimension of the perturbing operator. Thus \( 2 - \Gamma = 4g_2/(4 + g_2) \). Since \( \Gamma = b^2 \) we find

\[
\frac{b^2}{2} = \frac{1 - g_2/4}{1 + g_2/4}, \quad \text{when } g_1 \approx 0
\]

With this identification, the \( \beta \)eta function matches the known two-loop result [2]. Since \( g_2 \) flows under RG, the above formula cannot be valid for all \( g \) since the exact S-matrix for the

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\(^1b\) is related the conventional sine-Gordon coupling \( \beta \) by \( b = \beta/\sqrt{4\pi} \), where \( \beta = \sqrt{4\pi} \) corresponds to the free-fermion point and \( \sqrt{8\pi} \) is the Kosterlitz-Thouless point.
sG theory depends on an RG invariant $b$ with $0 < b^2 < 2$. It is natural then that in general $b$ is a function of the RG invariant $Q$. Matching (3.4) at $g_1 = 0$ one finds

$$\frac{b^2}{2} = \frac{1}{1 + 8\sqrt{Q}}$$

(3.5)

We propose the above formula is valid for all $g$. A non-trivial check is at $g_1 = \infty$ where $\beta_{g_1} \approx 16g_1/(4 + g_2)$. Again identifying $b^2$ through the dimension of $\cos b\phi$, one finds

$$\frac{b^2}{2} = \frac{g_2 - 4}{g_2 + 4} \quad \text{when} \quad g_1 \approx \infty$$

(3.6)

One sees that the expression (3.5) matches both (3.4) and (3.6) and thus has the right behavior at $g_1 = 0, \infty$. This is of course consistent with duality (2.9).

The flows that originate from $g_1 = 0, 0 < g_2 < 4$, have $0 < b^2 < 2$; we thus identify this as a sG phase. Thus we see that the very existence of a sine-Gordon phase relies on the existence of the RG invariant $Q$. The flows that terminate at $g_1 = 0, -4 < g_2 < 0$, have $2 < b^2 < \infty$. The operator $\cos b\phi$ is thus irrelevant and this is the massless Kosterlitz-Thouless phase.

The other flows that originate from the $g_2$ axis correspond to a negative $b^2$. Letting $b = ib_{shG}$, we interpret this as a sinh-Gordon phase (shG) with the action (3.2) where $\cos b\phi \to \cosh b_{shG}\phi$. This analytical continuation is compatible with the interpretation of the singularities at $g_2 = -4$ as consequences of instabilities of the kinetic term in eq.(3.2). The region $4 < g_2 < \infty$ corresponds to $0 < b^2_{shG} < 2$ whereas $g_2 < -4$ corresponds to $2 < b^2_{shG} < \infty$, so the whole range of $b^2_{shG}$ is covered. The flows all originate from the $g_2$ axis since $\cosh b_{shG}$ is relevant for $b_{shG}$. Repeating the above argument we find

$$\frac{b^2_{shG}}{2} = \frac{1}{8\sqrt{Q} - 1}$$

(3.7)

Consider now the roaming or cyclic RG flows of the last section. In this region $Q$ is negative, $-\infty < Q < 0$. Assuming it is valid in this regime, eq. (3.5) implies then that $b^2$ has both real and imaginary parts so that it is neither sine-Gordon nor sinh-Gordon. This is actually reminiscent of Zamolodchikov’s staircase model [5]. The staircase model corresponds to $b^2_{shG}/(2 + b^2_{shG}) = 1/2 \pm i\theta_0/2\pi$, which implies $b^2_{shG}/2 = e^{i\alpha}$ where $\cos \alpha = (1 - (\theta_0/\pi)^2)/(1 + (\theta_0/\pi)^2)$. From (3.7) one sees that for small $\alpha$, $i\alpha = 8\sqrt{Q}$, which is compatible with $Q$ being negative. Of course, more investigation is required for a better characterization of this phase.

The above flows do not have non-trivial fixed points in both the UV and IR that can be compared with known results. However, by letting $g_1$ be imaginary, there are such flows[6]. The resulting imaginary potential sine-Gordon theory is known to possess both an ultraviolet and infra-red fixed point, both of which are $c = 1$ gaussian conformal field theories at different radius of compactification [7] [8]. The anisotropic $su(2)$ beta function precisely confirms this. To see this, let $g_1 \to ig_1$. There is still a line of fixed points at $g_1 = 0$. At

\[\text{[6] This check was suggested to us by Al. Zamolodchikov [5].}\]
small coupling \( Q \approx (g_2^2 + g_1^2)/256 \), thus the RG contours are circles rather than hyperbolas. This implies that flows can originate in the ultra-violet along the positive \( g_2 \) axis and end up at a different fixed point on the negative \( g_2 \) axis. The values of \( g_2 \) in the UV and IR can be determined from the RG invariance of \( Q \). Since \( Q = (g_2/(g_2 - 4))^2/16 \) at \( g_1 = 0 \), one finds

\[
\frac{g_2^{IR}}{g_2^{IR} - 4} = -\frac{g_2^{UV}}{g_2^{UV} - 4}
\]

Let us express this result in terms of the anomalous dimension of the perturbation at the fixed points. As discussed above, using eq. (3.3) this gives \( \Gamma = 2(4 - g_2)/(4 + g_2) \) at \( g_1 = 0 \) fixed points. Using (3.8) one finds

\[
\Gamma_{IR} = \frac{\Gamma_{UV}}{\Gamma_{UV} - 1}
\]

This agrees precisely with the statements made in [8]. Note that for \( 1 < \Gamma_{UV} < 2 \) relevant, \( \Gamma_{IR} \) is irrelevant as it should be. Since (3.9) turns out to be exact, this strongly suggests there are no further corrections to the \( \beta \)eta function.

For higher \( k \) the above results are generalized to the fractional supersymmetric sine-Gordon model [6]. This serves as a mild check of the \( k \) dependence of the \( \beta \)eta function. In these models, the interaction \( \cos b\phi \) is replaced with \( \psi_1 \bar{\psi}_1 \cos b_{fssg}\phi \) where \( \psi_1 \) are \( Z_k \) parafermions of dimension \((k-1)/k\). The case \( k = 2 \) is the supersymmetric sine-Gordon model. Since \( \beta_{g_1} \approx g_2 g_2/(1 + k g_2/4) \) when \( g_1 \approx 0 \), repeating the above arguments we find

\[
b_{fssg}^2 = \frac{2(4 - k g_2)}{k(4 + k g_2)}, \quad \text{when} \quad g_1 \approx 0
\]

From the \( \beta \)eta functions we find that the fractional supersymmetric sine-Gordon regime corresponds to \( 0 < g_2 < 4/k \) which corresponds to \( 0 < b_{fssg}^2/2 < 1/k \), in agreement with [6]. This is not a strong check of the \( k \) dependence since we have input the dimension of the parafermions by hand.

**IV. CONCLUSION**

In summary, by using duality and endowing the coupling constant space with some additional topological properties, we managed to interpret the RG flows at strong and weak coupling based on the \( \beta \)eta functions proposed in [1]. This supports the analytic continuation of the \( \beta \)eta function to strong coupling.

The cyclic or roaming RG trajectory we found needs further investigation. We point out that cyclic RG flows are ruled out under the assumptions of the \( c \)-theorem which shows that the central charge \( c \) always decreases [10]. Though the action written in terms of currents is hermitian, the bosonized form indicates that in some domains the kinetic term can be negative. Thus, presumably the assumption of positivity breaks down in the cyclic domain. We also remark that cyclic RG trajectories have previously occurred in certain problems in nuclear physics [11].

We have extended the scheme described in this paper to the \( \beta \)eta functions for the network model computed in [2] and this will be described in a separate publication.
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FIG. 1. Phase Diagram for anisotropic $su(2)$. 