A fresh look at the determination of $|V_{cb}|$ from $B \to D^* l \nu$

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We use recent Belle results on $\bar{B}^0 \to D^{\pm} l^- \bar{\nu}$ decays to extract the CKM element $|V_{cb}|$ with two different but well-founded parameterizations of the form factors. We show that the CLN and BGL parameterizations lead to quite different results for $|V_{cb}|$ and provide a simple explanation of this unexpected behaviour. A long lasting discrepancy between the inclusive and exclusive determinations of $|V_{cb}|$ may have to be thoroughly reconsidered.

I. INTRODUCTION

Semileptonic $B$ decays offer the most direct way to determine the element $|V_{cb}|$ of the Cabibbo-Kobayashi Maskawa (CKM) quark mixing matrix. This particular element plays a central role in the analyses of the CKM matrix unitarity and in the SM prediction of Flavour Changing Neutral Current transitions. For a long time the two available methods to extract $|V_{cb}|$ from experimental data, based on exclusive (single hadronic channel) and inclusive (sum of all hadronic channels) reconstruction of the semileptonic $B$ decays, have been in conflict. The two methods are based on very different theoretical foundations and while a new physics interpretation seems currently disfavoured on general grounds [1], it is not excluded [2] and is particularly interesting in view of the anomalies in $B \to D^{(*)} l \nu$ [3].

At present, the two most precise determinations are

\begin{equation}
|V_{cb}| = (38.71 \pm 0.75) \times 10^{-3},
\end{equation}

based on the HFAG global combination of $B \to D^* l \nu$ results [3] together with the FNAL-MILC Collaboration calculation [4] of the relevant form factor at zero-recoil, i.e., when the $D^*$ is produced at rest in the $B$ rest frame, and

\begin{equation}
|V_{cb}| = (42.00 \pm 0.65) \times 10^{-3},
\end{equation}

obtained in the Heavy Quark Expansion from a fit to the moments of various kinematic distributions in inclusive semileptonic decays [5]. The difference between (1) and (2) is 3.3$\sigma$, which becomes 3.1$\sigma$ once the QED corrections are treated in the same way in both cases. There are alternative calculations of the $B \to D^*$ zero-recoil form factor on the lattice [6] or based on Heavy Quark Sum Rules [7] [8] but they have larger uncertainties.

In a recent paper [9] we have reviewed and slightly updated the 20-years-old formalism to parameterize the form factors in $B \to D l \nu$ in a way that satisfies important unitarity constraints. Using up-to-date lattice calculations of the form factors and the available experimental results, we have shown that the parameterization dependence is small and obtained $|V_{cb}| = 40.49(97) \times 10^{-3}$, compatible with both (1) and (2) and only slightly less precise.

The purpose of this Letter is to perform a similar analysis for the $B \to D^* l \nu$ decay. We take advantage of the new Belle preliminary results [10] which, for the first time, include deconvoluted kinematic and angular distributions with complete statistical and systematic errors and correlations, without relying on a particular parameterization of the form factors. We first review the formalism and the data and then describe our fits and discuss the results.

II. FORM FACTOR PARAMETERIZATIONS

In the limit of massless leptons the fully differential decay rate is given by

\begin{equation}
\frac{d\Gamma(\bar{B} \to D^* l \bar{\nu})}{d\omega \ d\cos \theta_v \ d\cos \theta_l \ d\chi} = \frac{\eta_{EW}^2 m_B m_{D^*}}{4(4\pi)^4} \sqrt{\omega^2 - 1} \times
\end{equation}

\begin{equation}
(1 - 2\omega r + r^2) G_F^2 |V_{cb}|^2 \times \left\{ \begin{array}{l}
(1 - c_l)^2 s_c^2 H_2^2 + (1 + c_l)^2 s_c^2 H_2^2 \\
+4s_l^2 c_c^2 H_0^2 - 2s_l^2 s_c^2 \cos 2\chi H_+ H_- \\
-4s_l(1 - c_l) s_c c_v \cos \chi H_+ H_0 \\
+4s_l(1 + c_l) s_c c_v \cos \chi H_- H_0 \end{array} \right\},
\end{equation}

where $r = m_{D^*}/m_B$, $c_v \equiv \cos \theta_v$, $c_l \equiv \cos \theta_l$, and correspondingly for $\sin \theta_v$ and $\sin \theta_l$, $\theta_{v,l}$, and $\chi$ are the three angles that characterise the semileptonic decay. We also use the kinematic parameter

\begin{equation}
w = \frac{m_B^2 + m_{D^*}^2 - q^2}{2m_B m_{D^*}},
\end{equation}

where $q^2$ is the invariant mass of the lepton pair.

The helicity amplitudes $H_{\pm,0}$ in Eq. (3) are given in terms of three form factors, see e.g. Eqns. (3-5) of Ref. [10].
In the Caprini-Lellouch-Neubert (CLN) parameterization \[11\] one employs the form factor \(h_{A_1}(w)\) and the ratios \(R_{1,2}(w)\). Traditionally, the experimental collaborations use

\[
h_{A_1}(w) = h_{A_1}(1) \left[ 1 - 8y^2z + (53\rho^2 - 15)z^2 \right] - (231\rho^2 - 91)z^3,\]
\[
R_1(w) = R_1(1) - 0.12(w - 1) + 0.05(w - 1)^2,\]
\[
R_2(w) = R_2(1) + 0.11(w - 1) - 0.06(w - 1)^2,
\]
where \(z = (\sqrt{w + 1} - \sqrt{2})/(\sqrt{w + 1} + \sqrt{2})\) and there are four independent parameters in total. We will discuss the ingredients of this parameterization later on. After integration over the angular variables, the \(w\) distribution is proportional to \[11\]

\[
F^2(w) = h_{A_1}(w) \left( 1 + 4w/(w + 1) \right) \left[ 1 - 2wr + r^2 \right]^{-1} \times \left[ 2 - 2wr + r^2 \right] \left[ 1 + R_2^2(w) \right] \left[ w/(w + 1) \right] + \left( 1 + (1 - R_2(w)) \right) \left[ w/(w + 1) \right].
\]

An alternative parameterization is due to Boyd, Grinstein and Lebed (BGL) \[15\]. In their notation the helicity amplitudes \(H_i\) are given by

\[
H_0(w) = F_1(w)/\sqrt{q^2},
\]
\[
H_\pm(w) = f(w) \mp m_B m_D \cdot \sqrt{w-2} g(w).
\]

The relations between the relevant form factors in the CLN and BGL notation are

\[
f = \sqrt{m_B m_D}(1 + w) h_{A_1}, \quad g = h_V/\sqrt{m_B m_D},
\]
\[
F_1 = (1 + w)(m_B - m_D) \sqrt{m_B m_D} A_5,
\]
and

\[
R_1(w) = (w + 1) m_B m_D \cdot \frac{g(w)}{f(w)}.
\]
\[
R_2(w) = \frac{w - r}{w - 1} - \frac{F_1(w)}{m_B(w - 1)f(w)}.
\]

| Type | Mass (GeV) | References |
|------|------------|------------|
| \(1^-\) | 6.329 | \[12\] |
| \(1^-\) | 6.920 | \[12\] |
| \(1^-\) | 7.020 | \[13\] |
| \(1^-\) | 7.280 | \[14\] |
| \(1^+\) | 6.739 | \[12\] |
| \(1^+\) | 6.750 | \[13\] \[15\] |
| \(1^+\) | 7.145 | \[13\] \[15\] |
| \(1^+\) | 7.150 | \[13\] \[15\] |

TABLE I. Relevant \(B_c^{(*)}\) masses. The \(1^-\) resonances are as in Ref. \[9\].

The three BGL form factors can be written as series in \(z\),

\[
f(z) = \frac{1}{P_{1+}(z) \phi_f(z)} \sum_{n=0}^{\infty} a_n^f z^n,
\]
\[
F_1(z) = \frac{1}{P_{1+}(z) \phi_{F_1}(z)} \sum_{n=0}^{\infty} a_n^{F_1} z^n ,
\]
\[
g(z) = \frac{1}{P_{1+}(z) \phi_g(z)} \sum_{n=0}^{\infty} a_n^g z^n.
\]

In these equations the Blaschke factors \(P_{1+}\) are given by

\[
P_{1+}(z) = \prod_{p=1}^{n} \frac{z - z_P}{1 - z z_P},
\]

where \(z_P\) is defined as \((t_\pm = (m_B \pm m_D)^2)\)

\[
z_P = \sqrt{t_+ - m_P^2} - \sqrt{t_+ - m_P^2} / \sqrt{t_+ - m_P^2} + \sqrt{t_+ - m_P^2},
\]

and the product is extended to all the \(B_c\) resonances below the \(B-D^*\) threshold (7.29 GeV) with the appropriate quantum numbers \((1^+\) for \(f\) and \(F_1\), and \(1^-\) for \(g\)). We use the \(B_c\) resonances reported in Table \[4\] but do not include the fourth \(1^-\) resonance, which is too uncertain and close to threshold. The \(B_c\) resonances also enter the \(1^-\) unitarity bounds (see below) as single particle contributions. The outer functions \(\phi_i\) for \(i = g, f, F_1\), can be read from Eq. (4.23) in Ref. \[16\]:

\[
\phi_g(z) = \sqrt{\frac{n_f}{3 \pi \chi_{1-}^T(0)}} [(1 + r)(1 - z) + 2\sqrt{r} (1 + z)]^4,
\]
\[
\phi_f(z) = \frac{4r}{m_B^2} \sqrt{\frac{n_f}{3 \pi \chi_{1-}^T(0)}} [(1 + r) (1 - z) + 2\sqrt{r} (1 + z)]^4,
\]
\[
\phi_{F_1}(z) = \frac{4r}{m_B^2} \sqrt{\frac{n_f}{6 \pi \chi_{1-}^T(0)}} [(1 + r) (1 - z) + 2\sqrt{r} (1 + z)]^4,
\]

where \(\chi_{1-}^T(0)\) and \(\tilde{\chi}_{1-}^T(0)\) are constants given in Table II, and \(n_f = 2.6\) represents the number of spectator quarks (three), decreased by a large and conservative \(SU(3)\)
The decay experimental data determine the decay form factors in a stringent way: in the heavy quark limit they are all either proportional to the Isgur-Wise function or vanish. These relations can be used to make the unitarity bounds stronger, and to decrease the number of relevant parameters. The CLN parameterization is built out of these relations, improved with perturbative and $O(1/m)$ leading Heavy Quark Effective Theory (HQET) power corrections from QCD sum rules, and of the ensuing strong unitarity bounds. With respect to the original paper [11], the experimental analyses have an additional element of flexibility, as they fit the zero recoil value of $R_{1,2}$ directly from data, rather than fixing them at their HQET values $R_{1}(1) = 1.27$, $R_{2}(1) = 0.80$. It is quite obvious that the HQET relations employed in Ref. [11] have a non-negligible uncertainty. We will not discuss here how this was estimated and included in [11], but it should be recalled that the accuracy of the parameterization for $h_{A_{1}}(w)$ in Eq. (4) was estimated there to be better than 2%.

The CLN and BGL parameterizations are both constructed to satisfy the unitarity bounds. They differ mostly in the CLN reliance on next-to-leading order HQET relations between the form factors. In the following we are going to verify how important this underlying assumption is for the extraction of $V_{cb}$, remaining mainly agnostic on the validity of the HQET relations, a matter which ultimately will be decided by lattice QCD calculations.

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### Table III. Fit results using the BGL (a) and CLN (b) parameterizations. In the BGL fits $a_0^{F_1}$ is fixed by the value of $a_0^f$, see Eq. [9].

|                  | Data + lattice | Data + lattice + LCSR |
|------------------|----------------|-----------------------|
| BGL Fit: $\chi^2$/dof | 27.9/32        | 31.4/35               |
| $|V_{cb}|$            | 0.0417 ($^{+20}_{-21}$) | 0.0404 ($^{+16}_{-17}$) |
| $a_0^f$           | 0.01223 (18)   | 0.01224 (18)          |
| $a_1^f$           | $-0.054$ ($^{+45}_{-43}$) | $-0.052$ ($^{+45}_{-43}$) |
| $a_2^f$           | 0.2 ($^{+7}_{-12}$)   | 1.0 ($^{+0}_{-2}$)    |
| $a_1^{F_1}$       | $-0.0100$ ($^{+61}_{-56}$) | $-0.0070$ ($^{+54}_{-52}$) |
| $a_2^{F_1}$       | 0.12 (10)       | 0.089 ($^{+0}_{-96}$)  |
| $a_0^g$           | 0.012 ($^{+11}_{-10}$) | 0.0289 ($^{+57}_{-37}$) |
| $a_1^g$           | 0.7 ($^{+3}_{-4}$)   | 0.08 ($^{+8}_{-22}$)   |
| $a_2^g$           | 0.8 ($^{+2}_{-17}$)   | $-1.0$ ($^{+20}_{-0}$) |

|                  | Data + lattice | Data + lattice + LCSR |
|------------------|----------------|-----------------------|
| CLN Fit: $\chi^2$/dof | 34.3/36        | 34.8/39               |
| $|V_{cb}|$            | 0.0382 (15)     | 0.0382 (14)           |
| $\rho_D^*$        | 1.17 ($^{+15}_{-16}$) | 1.16 (14)            |
| $R_1(1)$          | 1.391 ($^{+92}_{-88}$) | 1.372 (36)          |
| $R_2(1)$          | 0.913 ($^{+73}_{-60}$) | 0.916 ($^{+65}_{-76}$) |
| $h_4(1)$          | 0.906 (13)      | 0.906 (13)            |

1 These points are also emphasized in [13], which appeared as we were about to publish this paper on the ArXiv.

2 As noted in [9], recent lattice calculations differ from the HQET ratios of form factors at the level of 10%.
III. FITS AND RESULTS

In our $\chi^2$ fits we use the unfolded differential decay rates measured in Ref. [10]. The Belle Collaboration provides the $w$, $\cos \theta_\ell$, $\cos \vartheta_1$, and $\chi$ distributions, measured in 10 bins each, for a total of 40 observables, and the relative covariance matrix. In addition, like Ref. [10], in the following we always use the value of the form factor $h_{A_1}$ calculated at zero-recoil on the lattice [3],

$$h_{A_1}(1) = 0.906 \pm 0.013. \quad (11)$$

This is the only form factor relevant at zero-recoil, and to the best of our knowledge this Fermilab/MILC calculation is the only published unquenched calculation. Among older quenched calculations, Ref. [19] extends up to the best of our knowledge this Fermilab/MILC calculation is the only published unquenched calculation. As far as the determination of $|V_{cb}|$ is concerned, the purpose of a fit to $B \to D^* \ell \nu$ observables is therefore simply to extrapolate the measurements to the zero-recoil point, where (11) provides the normalization. As the differential width vanishes like $\sqrt{w-1}$ as $w \to 1$, see Eq. (3) and Fig. 1(a) the extrapolation is not trivial. Like in the case of $B \to D \ell \nu$, the situation is set to improve significantly as soon as lattice calculations of the form factors at non-zero recoil will become available, but for the moment it is important to keep in mind that the extrapolation should be controlled by the low recoil behaviour of the form factors. In this context the angular observables provide very little information, as they are integrated over the full $w$ range and receive negligible contribution from the suppressed low-recoil region. Of course, the angular observables are very important to constrain new physics, see for instance Ref. [20], but their contribution in the determination of $|V_{cb}|$ is marginal.

The results of our BGL and CLN fits to the full data set and to (11) are given in the first columns of Tables III(a) and III(b). The results of the CLN fit are in perfect agreement with the one in Appendix B of Ref. [10]. Incidentally, Belle’s paper also reports the results of a fit performed without unfolding the distributions which gives $|V_{cb}| = 0.0374(13)$. The BGL fit in Table III(a) left column, has a 9% higher central value and a 40% larger uncertainty than the CLN fit. The fits are both good, and such a large shift in $|V_{cb}|$ comes quite unexpected. We believe it is related to the fact that the CLN parameterization has limited flexibility and that the angular observables dilute the sensitivity to the low recoil region, which is crucial for a correct extrapolation (they also decrease the overall normalisation of the rate by 0.8%). This is clearly seen in Fig.1b, where the bands corresponding to the BGL and CLN fits are compared with the data, and one can notice that the CLN band underestimates all the three low recoil points. Table IV shows $|V_{cb}|$ obtained from fits to the $w$ distribution and (11) only: the CLN fit is 7% higher and the two parameterizations give consistent results. Another fit which supports the simple explanation above is one where we give more flexibility to the CLN parameterization, by floating the slopes of the $R_{1,2}$ ratios. The result, shown in Table IV, is again very close to the BGL one.

Concerning the quality of the fits we show, one should take into account that all BGL fits are constrained fits where (11) are employed after truncation at order $N$. The effective number of degrees of freedom is therefore larger than the naive counting shown in the Tables (the number of degrees of freedom is not well-defined in a constrained fit, as the parameters are not allowed to take any possible value). This is well illustrated by the second fit in Table IV whose $\chi^2$/dof = 5.1/2 may look suspect. However, the unitarity constraints play an important role here: without them the best fit would have $\chi^2$/dof = 1.2/2.

It can be reasonably argued that the HQET input used in devising the CLN parameterization is important theoretical information that one should not neglect. A simple way to do that is to include HQET constraints on $R_{1,2}$ at specific values of $w$ with a conservative uncertainty. As an example, we have used the HQET values of $R_{1,2}$ at $w = 1.4$ with a 20% uncertainty in the BGL fit and observed a downward shift in the $V_{cb}$ central value, see Table IV. Lowering the uncertainty we observe very little effect: for a 10% uncertainty, $|V_{cb}| = 0.0407(19)$, and it turns out that the value of $|V_{cb}|$ depends most sensitively on that of $R_1$ at large $w$.

Alternatively, one can avoid HQET inputs altogether and employ instead information on the form factors at

| Additional fits                                      | $\chi^2$/dof | $|V_{cb}|$ |
|-----------------------------------------------------|--------------|-----------|
| CLN without angular bins                            | 7.1/6        | 0.0409(16) |
| BGL $(N = 2)$ without angular bins                  | 5.1/2        | 0.0428(17) |
| CLN only angular bins                               | 23.0/26      | 0.074(31)  |
| BGL $(N = 2)$ only angular bins                     | 22.3/32      | 0.058(32)  |
| CLN with $R_{1,2}$ slopes free                      | 28.1/34      | 0.0415(19) |
| BGL $(N = 2)$ fit with $R_{1,2}(w = 1.4) = HQET \pm 20\% \ (CLN \ Eq. \ (36))$ | 31.7/34      | 0.0407(17) |

TABLE IV. Additional fits. The lattice input (11) is always included and LCSR constraints are never included.
maximal recoil from Light Cone Sum Rules \[21\]:

\[
h_{A_1}(w_{\text{max}}) = 0.65(18),
R_1(w_{\text{max}}) = 1.32(4), \quad R_2(w_{\text{max}}) = 0.91(17).
\]

The results of the BGL and CLN fits with the complete Belle’s dataset and Eqs. (11,12) are given in Tables III(a) and III(b) right column. The CLN fit is unaffected by the LCSR constraints, while the BGL uncertainty is still somewhat larger.

For a particular parameterization (both give acceptable fits), but in the absence of new information from lattice on the slope and zero-recoil value of the form factors the central value we take the central value of the BGL fit see Eq. (6), at \( w = 1 \) were available from the lattice. For the central value we take the central value of the BGL fit with LCSR constraints. The results demonstrate the importance of a precise lattice determination of the slope to control the zero-recoil extrapolation. Indeed, the parameterization dependence becomes minimal and the LCSR constraints become much less important. The quality of the CLN fits deteriorates, while the BGL uncertainty is still somewhat larger.

IV. FINAL REMARKS

We have performed fits to the recent \( B \to D^*\ell \nu \) data by Belle [10] with the CLN and BGL parameterizations. The BGL results for \(|V_{cb}|\) are consistently higher than those obtained with the CLN parameterization. One cannot avoid noticing that the central values of all our BGL fits are perfectly compatible with [2]. However, one should be very careful in interpreting our results: we simply observed that the Belle data we have employed lead to different \(|V_{cb}|\) when they are analysed with two parameterizations which differ mainly in their reliance on HQET relations. The data do not show any preference for a particular parameterization (both give acceptable fits), but in the absence of new information from lattice the central value we take the central value of the BGL fit at \( w = 1 \) with the HQET predictions of the same quantities [11] (straight lines). They are perfectly compatible if one assumes a \( \sim 10\% \) uncertainty for the latter.

Finally, we show in Table V what would happen if a 5\% determination of the slope of the form factor \( F(w) \), see Eq. (4), at \( w = 1 \) were available from the lattice. For the central value we take the central value of the BGL fit with LCSR constraints. The results demonstrate the importance of a precise lattice determination of the slope to control the zero-recoil extrapolation. Indeed, the parameterization dependence becomes minimal and the LCSR constraints become much less important. The quality of the CLN fits deteriorates, while the BGL uncertainty is still somewhat larger.

| Future lattice fits          | \( \chi^2/\text{dof} \) | \(|V_{cb}| \)     |
|-----------------------------|-------------------------|-------------------|
| CLN                         | 56.4/37                 | 0.0407 (12)       |
| CLN+LCSR                    | 59.3/40                 | 0.0406 (12)       |
| BGL                         | 28.2/33                 | 0.0409 (15)       |
| BGL+LCSR                    | 31.4/36                 | 0.0404 (13)       |

TABLE V. Fits including an hypothetical future lattice calculation giving \( \frac{\partial F}{\partial w} |_{w=1} = -1.44 \pm 0.07 \).
FIG. 1. Comparison of fit results with different parametrizations.

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