Deformed $W$–Algebras as Symmetries of Generalized Integrable Hierarchies

Carlos R. Fernández-Pousa$^1$, Manuel V. Gallas$^2$, J. Luis Miramontes$^3$, and Joaquín Sánchez Guillén$^4$

Departamento de Física de Partículas, Facultad de Física, Universidad de Santiago, E-15706 Santiago de Compostela, Spain

ABSTRACT

A unified description of the relationship between the Hamiltonian structure of a large class of integrable hierarchies of equations and $W$-algebras is discussed. The main result is an explicit formula showing that the former can be understood as a deformation of the latter.
Integrability and non-linear extensions of the conformal algebra (\(W\)-algebras) play an important role both in quantum field theory and statistical mechanics. However, since they involve non-linear structures, a systematic study is almost indispensable for distinguishing their characteristics from the technicalities involved in their mathematical description. The outstanding value of the Drinfel’d and Sokolov’s work \([1]\) is precisely to provide a systematic algebraic method for the construction of integrable hierarchies of equations that unifies a large variety of previous and dispersed results. More recently, and following \([2]\), a very general approach that includes the original Drinfel’d-Sokolov construction has been proposed in \([3,4]\), and one of the main features of the resulting generalized KdV-type hierarchies is their early recognized relation with classical realizations of (deformed) \(W\)-algebras through their Hamiltonian structure. In this letter, we discuss the general pattern of that relationship, worked out in detail in \([5,6]\), explaining in a simple and precise way the main results and their implications. The crucial property to be explored is the possibility of embedding the phase space of the \(W\)-algebras into the phase space of the hierarchies. Then, by comparing the reduction processes involved in both constructions, it is possible to investigate their equivalence, and, this way, to open the possibility of obtaining new extensions of the conformal algebra.

In \([3,4]\), generalized Drinfel’d–Sokolov (DS) hierarchies of equations are constructed in terms of the matrix Lax operator \(L\) associated to the data \(\{\tilde{g}, \mathcal{H}[w], s_\omega, s, \Lambda\}\). Here, \(\tilde{g}\) is the loop algebra of a finite dimensional (complex) simple Lie algebra \(g\), even though the construction could be easily generalized to the case of reductive Lie algebras too. \(s\) and \(s_\omega\) are two vectors of rank(\(g\)) + 1 non-negative integers defining gradations of \(\tilde{g}\) such that \(s \preceq s_\omega\) with respect to the partial ordering introduced in \([3,4]\); \(s_\omega\) also gives a gradation of a Heisenberg subalgebra \(\mathcal{H}[w]\) of \(\tilde{g}\). Finally, \(\Lambda\) is a constant element in \(\mathcal{H}[w]\) with positive \(s_\omega\)-grade \(i\) that satisfies \([\Lambda, \tilde{g}_0^0] \neq 0\),$ which will be called the non-degeneracy condition. It is worth mentioning that \(\Lambda\) can be equivalently characterized as a constant semisimple graded element in \(\tilde{g}\) constrained by the latter condition. The corresponding Lax operator \(L = \partial_x + \Lambda(z) + q_{\geq 0}^i(x, z)\) is defined in terms of periodic currents (or potentials) of the form \(\Lambda(z) + q_{\geq 0}^i(x, z)\), where \(x \in S^1\), and the dependence on \(z\), the affine parameter of the loop algebra, is explicitly indicated. These Lax operators admit gauge transformations preserving the form of the potential \(q\),

\[
q \rightarrow \tilde{q} = \Phi \partial_x \Phi^{-1} + \Phi(\Lambda + q_{\geq 0}^i)\Phi^{-1} , \quad \Phi \in G^*;
\]

they are generated by the gauge group \(G^* = \exp(P)\) with \(P = \tilde{g}_0^0\), which is a nilpotent subalgebra of \(\tilde{g}_0\). Gauge transformations preserve the infinite set of commuting flows on \(L\),

\[\text{From now on, superscripts and subscripts will indicate } s_\omega\text{- and } s\text{-grades, respectively.} \]
which enables their restriction to gauge equivalence classes. Moreover, the non-degeneracy condition ensures the possibility of performing a DS gauge-fixing thus leading to gauge invariant currents which are polynomials on the original currents.

The hierarchy also has a Hamiltonian description where the flows are defined by means of a Poisson bracket and a set of infinite Hamiltonians associated to the elements of \( \mathcal{H}[w]_{\geq 0} \). Even more, if \( s \) is the homogeneous gradation, the hierarchy admits two coordinated Poisson structures, but, in general, there is only one that is usually called the “second”. Its definition can be achieved through a Poisson reduction procedure where the relevant algebraic object is a classical \( R \) matrix \([7]\), i.e., an endomorphism of \( \g\) defined in terms of the gradation \( s \) as \( R = \frac{1}{2}[\Pi_{\geq 0} - \Pi_{< 0}] \), with \( \Pi_{\geq 0} \) and \( \Pi_{< 0} \) being the projectors onto \( \g_{\geq 0} \) and \( \g_{< 0} \), respectively. \( R \) induces a different Lie algebraic structure on \( \g \), whose Lie bracket will be denoted by \([\cdot, \cdot]_R\). The corresponding Kirillov-Kostant bracket on the space of maps of \( S^1 \) onto \( \g \) is

\[
\{\phi, \psi\}[u] = ([d\phi, d\psi]_R \mid u) + \omega_R(d\phi \mid d\psi) \\
= ([\partial + u, d\phi_{\geq 0}] \mid d\psi_{\geq 0}) - ([u, d\phi_{< 0}] \mid d\psi_{< 0}),
\]

where \((A \mid B) = \sum_{k \in \mathbb{Z}} \int_{S^1} dx \langle A_k(x), B_{-k}(x) \rangle\) is the generalization of the invariant Killing form \( \langle \cdot, \cdot \rangle \) of \( g \) to the affine Lie algebra, \( u(x, z) = \sum_{k \in \mathbb{Z}} z^k u_k(x) \) is a generic \( \g \)-current, and \( \omega_R \) is the associated \( R \)-cocycle \([7]\). This bracket, restricted to the gauge invariant functionals of constrained currents of the form \( u(x, z) = \Lambda(z) + q_{\geq 0}^\Xi(x, z) \), is precisely the “second” Poisson bracket \([4,6]\).

The infinite set of flow equations of the hierarchy is invariant under a (global) scale transformation where the components of the potential transform according to their \( s_{\omega} \)-grades. This result can be generalized to arbitrary conformal transformations which are Poisson symmetries of the second Poisson bracket \([4]\); thus suggesting a relationship between the second Poisson bracket algebra and extended conformal algebras. Nevertheless, to establish rigourously such relation one has to show that there exists a gauge invariant energy-momentum tensor \( T_\epsilon[q] \) generating the conformal transformation, \( \delta_q \epsilon(x) = \{T_\epsilon, q(x)\}\). Actually, the generator for the components of \( q_0 \) has already been obtained in \([8,5]\), but the existence of a generator for those components lying on \( \g_{>0} \) is unclear. In particular, it is known that some of them are centres of the second Poisson bracket and, hence, no energy-momentum tensor can generate their conformal transformations.

Our purpose is to relate the second Poisson bracket algebra with the \( \mathcal{W} \)-algebra associated to the finite reductive subalgebra \( \g_0 \) and to its nilpotent element \( \Lambda_0 \), which specifies an \( sl(2, \mathbb{C}) \) subalgebra of \( \g_0 \) \([9,10]\). This \( \mathcal{W} \)-algebra is defined on the set of invariant \( \g_0 \)-currents with respect to the group of transformations generated by some first class constraints; we
will show that it is a Poisson substructure of the second Poisson bracket algebra. It is important to realize that the restriction of the bracket (2) to the currents on $\tilde{g}_0$ is just the Kirillov-Poisson bracket associated to $\tilde{g}_0$, which does not involve the R-matrix at all. Then, the main difficulty in relating the reduction procedures leading to the $\mathcal{W}$ and the Poisson bracket algebras is that the gauge transformations considered in the latter generally mix the components $q_0$ and $q_{>0}$ while those in the former are transformations of $q_0$ only. However, we will be able to give a very precise account of that relation by assuming an additional restriction on the algebraic data defining the hierarchy. To be specific, we will consider in more detail those cases where $\Lambda + q^{\leq i}_{\geq 0} \in \tilde{g}_0 \oplus \tilde{g}_1$; then the bracket of two gauge invariant functionals of $\Lambda + q^{\leq i}_{\geq 0}$ reduces to

$$\{\phi, \psi\}[\Lambda + q] = ([\partial + \Lambda_0 + q^{<i}_{\geq 0}, d\phi_0] | d\psi_0),$$

which is just the Kirillov-Poisson bracket on the reduced currents $\Lambda_0 + q_0 \in \tilde{g}_0$.

In the Hamiltonian reduction approach, $\mathcal{W}$-algebras are defined by means of the Dirac bracket associated to some second class constraints on the set of currents associated to a finite (reductive) Lie algebra [9]. This Dirac bracket can be equivalently described as the result of the reduction of the Kirillov-Kostant bracket of the finite Lie algebra by first class constraints, i.e., of its Hamiltonian reduction [10]. In contrast, the second Hamiltonian structure of the generalized Drinfel’d-Sokolov hierarchies of [3,4] has a totally different origin. It corresponds to a bracket on the set of gauge invariant functionals of the Lax operator, which can be understood as the reduction of the Kirillov-Kostant bracket associated to the R-dependent algebraic structure defined now on a loop algebra [4]. To exhibit the differences between both brackets, we start by describing a non-standard reduction scheme that is general enough to accomodate both reduction procedures; it has been introduced by Fehér who names it “hybrid reduction” [11].

Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold and $\phi_1, \ldots, \phi_r$ a set of first class constraints (FCC), $\{\phi_i, \phi_j\} |_{\phi_k=0} = 0$, such that their zero set $N = \{ x \in M \mid \phi_k(x) = 0 \}$ is an embedded submanifold of $M$. Under general conditions, the corresponding Hamiltonian vector fields $X_k(\cdot) = \{\phi_k, \cdot\}$ can be integrated to form a group $G$ of transformations on $M$ preserving $N$; such transformations are Hamiltonian by construction, i.e., $\delta_k f = \{\phi_k, f\}$. Let us assume that they are also Poisson transformations regular enough to ensure the existence of a space of orbits in $N$, $N/G$, and of a convenient gauge slice $N/G \simeq \hat{M} \subset N$. Then, for any function $f$ on $\hat{M}$ there exists an unique gauge invariant extension $\hat{f}$ onto $N$, which defines the isomorphism of algebras $\rho : C^\infty(\hat{M}) \to C^\infty_G(N), \rho(f) = \hat{f}$. Correspondingly, this isomorphism induces the following Poisson structure on $\hat{M}$:

$$\{f, g\}^* = \rho^{-1}(\{\hat{f}^\phi, \hat{g}^\phi\} | N),$$

(4)
where $\hat{f}^\phi$ and $\hat{g}^\phi$ are arbitrary extensions of $\hat{f}$ and $\hat{g}$ onto $M$, and the resulting bracket is independent of the choice of the extension as a consequence of the first class character of the constraints. If the FCC do not generate transformations on $N$, notice that $\hat{M} \equiv N$ is a Poisson submanifold. Therefore, since any subset of constraints not generating transformations can be trivially imposed on $M$, we will assume that all the FCC do generate a $r$-parameter group of transformations on $N$.

Let us now suppose that the submanifold $\hat{M}$ can be completely fixed by an extended set of $2r$ constraints, $\{\psi_i\} = \{\phi_k\} \cup \{\sigma_k\}$, $k = 1, \ldots, r$, where the $\sigma_k$’s will be called gauge fixing constraints. Then, the constraint matrix $\Delta_{ij}(x) = \{\psi_i, \psi_j\}(x)$ is non-degenerate on $\hat{M}$, and, hence, it is invertible. In this case, a convenient local form of the Poisson structure is provided by the Dirac bracket

\[
\{f, g\}^* \equiv \{f, g\}^D = \{f^*, g^*\} \big|_{\hat{M}} - \{f^*, \psi_i\} \Delta^{ij} \{\psi_j, g^*\} \big|_{\hat{M}}
\]

(5)

where $\Delta^{ij}(x)$ is the inverse of the constraint matrix, and $f^*$ and $g^*$ are arbitrary extensions of $f$ and $g$ onto $M$. Since the gauge group is Hamiltonian by construction, this reduction procedure is usually known as Hamiltonian reduction. Before proceeding, let us point out that any centre $c$ of the Poisson structure is invariant with respect to any Hamiltonian transformation, i.e., $\delta_k c = \{\phi_k, c\} = 0$. Recall also that the Dirac bracket can be generally defined in terms of any set of constraints whose constraint matrix is nondegenerate (second class constraints); the interpretation as a Hamiltonian reduction is \textit{a posteriori}.

However, to describe the second Poisson bracket algebra we have to generalize the usual Hamiltonian reduction procedure. Suppose that we enlarge the group of transformations from $G$ to $G^*$, $G \subset G^*$, such that $G^*$ induces another Poisson action on $M$ preserving $N$. Assuming again regularity conditions for the existence of $N/G^* \simeq \hat{M}^*$, the same bracket (4) now defines a Poisson structure on $\hat{M}^*$. When $G$ is trivial, notice that this is just the Poisson reduction of $N = M$ by means of the transformations generated by $G^*$. Nevertheless, the most interesting case for our purposes occurs when the extended group of transformations $G^*$ is not Hamiltonian, which implies that the resulting bracket will not be of Dirac type in general.

Let us specialize the previous discussion to the reduction of the currents involved in the construction of the second Poisson bracket algebra:

\[
\begin{align*}
 u(x, z) & \in \tilde{g} \longrightarrow \Lambda(z) + q^i_{\geq 0}(x, z).
\end{align*}
\]

The set of currents $\Lambda + q^i_{\geq 0}$ (the analogue of the manifold $N$) can be viewed as the result of imposing the linear constraints $\phi_i[u(x)] = \left\langle \theta_i, u(x) - \Lambda \right\rangle = 0$ on the set of $\tilde{g}$-currents.
(corresponding to $M$) for any

$$\theta_i \in \tilde{g}_{>0} \oplus \tilde{g}_{\leq -i} \oplus \tilde{g}_{<0} \equiv \Gamma_{>0} \oplus \Gamma_0 \oplus \Gamma_{<0}. \quad (6)$$

It can be easily checked that these constraints are first class, and that only those associated to the nilpotent subalgebra $\Gamma_0$ generate transformations on $N$ [6]. However, in general, the corresponding Hamiltonian group of transformations is only a subgroup of the group of gauge transformations generated by $P = \tilde{g}_{<0}, \Gamma_0 = \tilde{g}_{\leq -i} \subseteq P$.

Since the finite subalgebra $\tilde{g}_0$ is graded by $s_\omega$, let $n$ be the highest $s_\omega$-grade of the elements of $\tilde{g}_0$; for instance, when $s = (1,0,\ldots,0)$ is the homogeneous gradation\(^6\) $n = N_{s_\omega} - s_0^0$ is the $s_\omega$-grade of the highest root step operator of $g$ [5]. Then, depending on the values of $n$ and $i$, the $s_\omega$-grade of $\Lambda$, the following cases can be distinguished:

(i) $i = 1$, which means that $\Gamma_0 = P$. Then, the group of transformations is fully generated by FCC and the reduction is just an example of Hamiltonian reduction. Moreover, and as a special feature of this particular case, the restriction of the Poisson bracket (2) to $N$ is well defined and, hence, the second Poisson bracket algebra can be equivalently understood as a Poisson reduction by means of the group $G = \exp(P)$.

(ii) $1 < i \leq n$ indicates that $\{0\} \neq \Gamma_0 \subset P$, but $\Gamma_0 \neq P$. Consequently, the set of gauge transformations is larger than those generated by the FCC and, therefore, they will not be Hamiltonian in general. This case is a particular example of hybrid reduction.

(iii) Finally, $i > n$ implies that $\Gamma_0 = \{0\}$. Therefore, since the constraints do not generate any transformation, the constrained manifold $N$ is again a Poisson submanifold and the second Poisson bracket algebra follows from a Poisson reduction. Moreover, in this case, the projection of $\Lambda$ onto $\tilde{g}_0$ vanishes, $\Lambda_0 = 0$.

Within the standard Hamiltonian reduction approach, $\mathcal{W}$-algebras are constructed as follows [9,10]. Let $\tilde{g}_0$ be a finite reductive Lie algebra and consider the set of currents on $\tilde{g}_0$ equipped with the usual Kirillov–Kostant bracket. Then, there is a $\mathcal{W}$-algebra for each embedded $sl(2, \mathbb{C})$ subalgebra of $\tilde{g}_0$\(^7\), which is given by the Dirac bracket associated to the second class constraints leading to constrained currents whose components are lowest weights in the decomposition of $\tilde{g}_0$ under the adjoint action of the $sl(2, \mathbb{C})$ subalgebra. In a completely equivalent way, this Dirac bracket can be derived by means of a Hamiltonian

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\(^6\) If $r = \text{rank}(g)$, recall that the gradations of the loop algebra $\tilde{g}$ are specified by a set of $r + 1$ integers, $s = (s^0, s^1, \ldots, s^r)$, and that $N_s = \sum_{j=0}^r k_j s^j$, where $k_0, \ldots, k_r$ are the (Kac) labels of the Dynkin diagram of $g$.

\(^7\) By an embedding of $sl(2, \mathbb{C})$ into a reductive Lie algebra $\tilde{g}_0$ we mean a direct sum of embeddings into each simple ideal.
reduction, *i.e.*, the second class constraints can be split into first class and gauge fixing constraints, an explicit decomposition technically known as a “halving”[10].

Our method to compare the second Poisson bracket algebra of the integrable hierarchies of [3,4] with \( W \)-algebras is the following. When \( \Lambda_0 \neq 0 \), it is possible to choose the gauge slice for the gauge transformations generated by \( G^* = \exp(P) \) such that its components on \( \tilde{g}_0 \) are lowest weights in the decomposition of \( \tilde{g}_0 \) under the adjoint action of the \( sl(2, \mathbb{C}) \) subalgebra specified by \( \Lambda_0 \). This way, the set of generators of the \( W \)-algebra can be embedded into the set of gauge invariant currents, and the corresponding Poisson structures can be compared.

From now on, we will assume that \( \Lambda^+_{\geq i} \subset \tilde{g}_0 \oplus \tilde{g}_1 \), which, in particular, implies that \( \Lambda \) can be uniquely decomposed as \( \Lambda = \Lambda_0 + \Lambda_1 \). Let us first consider the case when \( \Lambda_0 \neq 0 \), which requires \( i \leq n \). Then, \( J_+ = \Lambda_0 \) is a nilpotent element characterizing a \( sl(2, \mathbb{C}) \) subalgebra of \( \tilde{g}_0 \), \((J_+, J_0, J_-)\), which induces the direct sum decomposition \( P = \bar{P} \oplus P^* \) with \( P^* = \ker(\text{ad } J_+) \cap P \) and \( \bar{P} \cap \ker(\text{ad } J_+) = \{0\} \).

Since gauge transformations act independently on the components of the currents in \( \tilde{g}_0 \) and \( \tilde{g}_1 \), the transformations generated by the elements of \( \bar{P} \) can be used to gauge fix the components in \( \tilde{g}_0 \), while the remaining \( P^* \) fix those in \( \tilde{g}_1 \). This way, the gauge slice \( q^{\text{can}} \) can be chosen such that \( q^{\text{can}} \cap \tilde{g}_0 = \ker(\text{ad } J_-) \), *i.e.*, such that the components of \( q^{\text{can}} \) in \( \tilde{g}_0 \) are lowest weights [6]. This gauge fixing amounts to impose the linear constraints associated to certain subspace \( \theta \subset \tilde{g}_1 \), on \( \tilde{g}_0 \), and to \( \Gamma_0 \oplus \Gamma_1 \oplus \Gamma_2 \subset \tilde{g}_0 \), with

\[
\begin{align*}
\Gamma_0 &= \text{Im}(\text{ad } J_-) \cap \tilde{g}_0^{\leq-i}, \\
\Gamma_1 &= \text{Im}(\text{ad } J_-) \cap \tilde{g}_0^{>i} \cap \tilde{g}_0^{<0}, \quad \text{and} \quad \Gamma_2 = \text{Im}(\text{ad } J_-) \cap \tilde{g}_0^{>0},
\end{align*}
\]

(7)
on \( \tilde{g}_0 \). It is important to notice that the constraints associated to \( \theta \) can be considered independently of the others, since they lead just to a Poisson subalgebra [6]. Therefore, the reduction of the phase space of currents on \( \tilde{g} \) by the gauge transformations generated by \( G^* = \exp(P) \) can be viewed as a two steps process

\[
u \rightarrow \Lambda + q_0 + q_1^{\text{can}} \rightarrow \Lambda + q_0^{\text{can}} + q_1^{\text{can}},
\]

where the first reduction is trivial, *i.e.*, it leads just to a Poisson subalgebra. According to our previous discussion, we recognize the constraints associated to \( \Gamma_0 \) as the only FCC generating transformations on the reduced phase space; moreover, by identifying \([J_+, \Gamma_0]\) with \( \Gamma_2^\vee \), the dual space of \( \Gamma_2 \), the constraints associated to \( \Gamma_2 \) are precisely the gauge-fixing constraints of those associated to \( \Gamma_0 \). In contrast, the origin of the constraints generated by \( \Gamma_1 \) is that \( P - \Gamma_0 \neq \{0\} \), and, therefore, they have to be considered only if \( i > 1 \).
The set of linear constraints induced by \( \Gamma_0 \oplus \Gamma_1 \oplus \Gamma_2 \) on \( \tilde{g}_0 \) is precisely the same set involved in the Hamiltonian reduction construction of the \( \mathcal{W} \)-algebra associated to the \( sl(2, \mathbb{C}) \) subalgebra \( \{ J_+ = \Lambda_0, J_0, J_- \} \) of \( \tilde{g}_0 \). Consequently, not only the corresponding constraint matrix \( \Delta_{ij} \) is non-degenerate, but also its inverse \( \Delta^{ij} \) exists everywhere on the phase space and it depends polynomially on the reduced currents. Even more, since \( \Delta_{ij} \) always admits a “halving” [10], the group of transformations \( \tilde{G} \) generated by the FCC that specify the halving has to be a subgroup of our \( G^* \), although their actions on the currents will be different (in general, \( \tilde{G} \) does not transform \( q_1 \) while \( G^* \) does).

Therefore, our first result is that the phase space of the hierarchies is an extension of the phase space of the \( \mathcal{W} \)-algebra associated to \( \tilde{g}_0 \) and \( J_+ = \Lambda_0 \). Then, one can define two \textit{a priori} different brackets on the gauge invariant currents: the second Poisson bracket corresponding to the \( G^* \)-invariant extensions, and the Dirac bracket giving the \( \mathcal{W} \)-algebra. With respect to the latter, the components of \( q^\text{can}_1 \) are just centres, as those of \( q_1 \) before the reduction, but, in general, this will not be the case with respect to the former.

In more geometrical terms, the Dirac bracket gives the Poisson structure resulting from the Hamiltonian reduction of the currents \( \Lambda_0 + q^{\text{can}}_0 \) plus some trivial components (centres) \( q_1 \) by means of \( \tilde{G} \). Then, for generic functionals \( \phi, \psi \) of \( \Lambda + q^\text{can} \), the Dirac bracket is given by

\[
\{ \phi, \psi \}^D = \{ \phi^*, \psi^* \} |_{\Lambda + q^\text{can}} - \sum_{i,j} \int_{S^1} dx \, dy \{ \phi^*, \gamma_i(x) \} \Delta^{i,j}(x,y) \{ \gamma_j(y), \psi^* \} |_{\Lambda + q^\text{can}},
\]

where \( \phi^*, \psi^* \) are arbitrary extensions onto \( \tilde{g} \), and \( \{ \gamma_i \} \) is some basis for the vector subspace \( \Gamma_0 \oplus \Gamma_1 \oplus \Gamma_2 \). The constraint matrix \( \Delta_{ij} \) is a local differential operator whose block form is

\[
\Delta_{i,j}(x,y) [q^\text{can}] = \left\langle [\gamma_i, \gamma_j], J_+ + q^\text{can}_0(x) \right\rangle \delta(x-y) + \left\langle \gamma_i, \gamma_j \right\rangle \partial_x \delta(x-y)
\]

\[
\begin{pmatrix}
0 & 0 & * \\
0 & B(x) \delta(x-y) & * \\
* & * & *
\end{pmatrix},
\]

where the *’s stand for some matrix differential operators whose form is irrelevant in the following, and, for \( \gamma_i, \gamma_j \in \Gamma_1 \), \( B_{i,j}(x) = \left\langle [\gamma_i, \gamma_j], J_+ + q^\text{can}_0(x) \right\rangle \) is a \( q^\text{can}_0 \)-dependent matrix.
Consequently, the block form of the inverse matrix is just

\[
\Delta_{i,j}(x,y)[q^{\text{can}}] = \begin{pmatrix}
\Gamma_0 & \Gamma_1 & \Gamma_2 \\
\Gamma_0 & * & * & * \\
\Gamma_1 & * & \begin{pmatrix} B^{-1}(x)\delta(x-y) \end{pmatrix} & 0 \\
\Gamma_2 & * & 0 & 0
\end{pmatrix}.
\]

(10)

To relate the two Poisson structures, let us choose \(\phi^* = \hat{\phi}\) and \(\psi^* = \hat{\psi}\) being the \(G^*\)-gauge invariant extensions of \(\phi\) and \(\psi\); gauge invariance implies that

\[
0 = \delta_S \hat{\psi} = \{\hat{\psi}, \phi_S\}[q] + \left((d\hat{\psi})_{-1}, [S(x), \Lambda_1 + q_1(x)]\right),
\]

(11)

with \(\phi_S[q] = (S(x), q(x))\) for any \(S(x) \in P\), and this identity already exhibits that some gauge transformations are not Hamiltonian. Now, using (11) in (8), and taking into account the explicit form (10) of the inverse constraint matrix, one obtains

\[
\{\phi, \psi\}[q^{\text{can}}] = \{\hat{\phi}, \hat{\psi}\}^D[q] + \mathcal{C}(\hat{\phi}, \hat{\psi})[q], \quad \text{where}
\]

\[
\mathcal{C}(\hat{\phi}, \hat{\psi})[q] = \sum_{\gamma_i, \gamma_j \in \Gamma_1} \int_{S^1} dx \left\langle (d\hat{\phi})_{-1} \big|_{\Lambda^q_{\gamma_i}, \Lambda_1 + q_1^{\text{can}}(x)} \right\rangle \left\langle (d\hat{\psi})_{-1} \big|_{\Lambda^q_{\gamma_j}, \Lambda_1 + q_1^{\text{can}}(x)} \right\rangle,
\]

(12)

and we have explicitly indicated that \(B_{i,j}(x)\) depends only on \(q_0^{\text{can}}\). This last equation is our main result; it shows that the second Poisson bracket algebra of the integrable hierarchies of [3,4] is the \(\mathcal{W}\)-algebra corresponding to \(\{\cdot, \cdot\}^D\) deformed by \(\mathcal{C}(\cdot, \cdot)\), which is antisymmetric and depends polynomially on the components of \(q^{\text{can}}\).

The resulting form of the second Poisson bracket algebra is clarified by splitting the set of generators in \(W_a(x)\)'s and \(B_a(x)\)'s, associated to the components of \(q^{\text{can}}\) with \(s\)-grade zero and one, respectively. Then, since \(B_a(x)\) is always a centre of the Dirac bracket \(\{\cdot, \cdot\}^D\) but not of the second Poisson bracket, one gets

\[
\{W_a(y), W_b(z)\} = \{W_a(y), W_b(z)\}^D + \mathcal{C}(W_a(y), W_b(z))
\]

\[
\{W_a(y), B_b(z)\} = \mathcal{C}(W_a(y), B_b(z))
\]

\[
\{B_a(y), B_b(z)\} = \mathcal{C}(B_a(y), B_b(z)).
\]

(13)

We have already anticipated that the phase space of the integrable hierarchies of [3,4] includes non-dynamical components (centres) that should not be considered as actual degrees of freedom; their elimination amounts to a trivial further reduction of the Hamiltonian structure. Since they are functionals of the components of \(q^{\text{can}}\), setting them to zero
provides additional (polynomial) relations that allow one to express certain components in terms of the others. To be precise [4,5], when \( i > 1 \) there is a centre of \( \{\cdot,\cdot\}^* \) for each \( b \)
in the set

\[
\mathcal{Z}^* = \left[ \text{Ker}(\text{ad} \Lambda) \cap \tilde{g}_{1-i}(s_\omega) \right] \cup \left[ \text{Cent} \left( \text{Ker}(\text{ad} \Lambda) \right) \cap \bigoplus_{j=1-i}^{i-1} \tilde{g}_j(s_\omega) \right].
\]  

(14)

Then [6], if \((b)_{-1} \neq 0\) it is possible to express some components of \(q_{1}^{\text{can}}\) in terms of those of \(q_{0}^{\text{can}}\). In contrast, when \((b)_{-1} = 0\), which is only possible if \(\text{Ker}(\text{ad} \Lambda) \neq \mathcal{H}[w]\) and, hence, \(\Lambda\) is not regular, the elimination of this centre implies that some of the \textit{a priori} generators of the \(\mathcal{W}\)-algebra have to be expressed in terms of the others; this latter possibility is quite suggestive from the point of view of looking for new extensions of the conformal algebra.

In the following, we produce some examples to illustrate the power of eq.(12):

\textbf{(i)} \(\Lambda_{0} \neq 0\) and \(P^* = \{0\}\). This is equivalent to \(\text{Ker}(\text{ad} \Lambda_{0}) \cap P = \{0\}\), which is a stronger version of our non-degeneracy condition that naturally arises in the context of the Hamiltonian reduction approach to \(\mathcal{W}\)-algebras [10]. In this case, the condition \(\Lambda + q \in \tilde{g}_0 \oplus \tilde{g}_1\), assumed to derive eq.(12), is actually unnecessary, and one can obtain an analogous formula when, instead, \(\Lambda + q \in \tilde{g}_0 \oplus \cdots \oplus \tilde{g}_p\) but, still, \(\Lambda_{0} \neq 0\). Then, the generalization of (12) is given by

\[
\mathcal{C}(\hat{\phi}, \hat{\psi})[q] = \sum_{\gamma_i, \gamma_j \in \Gamma_1} \int_{S^1} dx \left\langle \left. (d\hat{\phi})_{<0} \right|_{\Lambda + q_{\text{can}}^{\text{can}}} , [\gamma_i, \Lambda_{>0} + q_{>0}^{\text{can}}(x)] \right\rangle
\]

\[
B^{i,j}[q_{0}^{\text{can}}(x)] \left\langle \left. (d\hat{\psi})_{<0} \right|_{\Lambda + q_{\text{can}}^{\text{can}}} , [\gamma_j, \Lambda_{>0} + q_{>0}^{\text{can}}(x)] \right\rangle.
\]  

(15)

According to [5], this stronger version of the non-degeneracy condition ensures that the gauge fixing can be performed in terms of the components of \(q_{0}\) only. Then, the \(W_a(x)\)'s only depend on \(q_{0}\) and, hence, \((dW_a(x))_{<0} = 0\). Correspondingly, the generators \(B_a(x)\), associated to the components of \(q_{>0}^{\text{can}}\), have the general form

\[
B_a(x) = \left\langle \beta_a , q_1(x) + [S_{\text{can}}[q_0(x)] , q_1(x)] + \cdots \right\rangle,
\]  

(16)

where \(S_{\text{can}}[q_0(x)]\) is the \(q_0\)-dependent gauge transformation taking an arbitrary potential to its canonical form on the gauge slice, and \(\beta_a\) is an arbitrary constant element of the subspace of \(\tilde{g}_{<0}\) that is dual to \(q_{>0}^{\text{can}}\). Then, using that \(S_{\text{can}}[q_0(x)]\) vanishes for \(q_0(x) \in q_{0}^{\text{can}}(x)\), it follows that

\[
(dB_a(x))_{<0} \big|_{\Lambda + q_{\text{can}}^{\text{can}}} = \beta_a \delta(x - y).
\]

Taking all this into account, the second Poisson bracket algebra gets decoupled as [6]

\[
\{W_a(y), W_b(z)\} = \{W_a(y), W_b(z)\}^D
\]

\[
\{W_a(y), B_b(z)\} = 0
\]

\[
\{B_a(y), B_b(z)\} = \mathcal{C}(B_a(y), B_b(z))
\].
and, hence, the $W_a(x)$’s, i.e., the functionals of $q_{0\text{can}}$, form a $\mathcal{W}$-algebra [5].

At this point, it is possible to address one of the relevant questions regarding the connection between $\mathcal{W}$-algebras and integrable soliton equations: is any $\mathcal{W}$-algebra the Poisson bracket algebra of some Hamiltonian integrable hierarchy of equations? So far, the answer seems to be negative. In fact, in the particular case of the integrable hierarchies associated to $g = A_n$ and $s = (1, 0, \ldots, 0)$ (the homogeneous gradation) such that $\tilde{g}_0 \equiv g = A_n$, this problem has been considered in [5] where the following result is obtained:

Within the $\mathcal{W}$-algebras constructed in terms of $A_n$, only those associated to the embeddings of $A_1$ into $A_n$ labelled by partitions of the form

$$n + 1 = k(a) + q(1) \quad \text{or} \quad n + 1 = k(a + 1) + k(a) + q(1), \quad \text{for} \quad a, k, q \in \mathbb{Z} \geq 0,$$

correspond to the second Poisson bracket algebra of some of the integrable hierarchies of [3,4]. Then, they involve the Heisenberg subalgebras $\mathcal{H}[w] \subset \tilde{A}_n$ associated to the conjugacy classes of the Weyl group of $A_n$ specified by $[w] = [k(a), q(1)]$ and $[w] = [k(2a + 1), (q - k)(1)]$, and $\Lambda$ has $s_\omega$-grade 1 and 2, respectively (see theorem 3 of [5] for details).

Since the class of hierarchies considered in [3,4] is large enough to accommodate practically all the generalizations of the Drinfeld-Sokolov construction so far considered, we find this result particularly relevant.

Another general feature observed in [5] is that some $\mathcal{W}$-algebras are associated to more than one integrable hierarchy. For example, the $\mathcal{W}$-algebras specified by the partitions $n + 1 = k(2) + q(1)$ are shared as second Poisson bracket algebras by the hierarchies associated to the conjugacy classes $[w] = [k(2), q(1)]$ and $[w] = [k(3), (q - k)(1)]$.

Particular examples are provided by the fractional $[2N + 1]^{(2)}$ generalized KdV $A_{2N}$ hierarchies\(^8\), where the second Poisson bracket algebra is just the $\mathcal{W}$-algebra associated to the $sl(2, \mathbb{C})$ subalgebra labelled by the partition $2N + 1 = (N + 1) + (N)$, which is nothing else than the fractional $W_N^{(2)}$ algebra of [13]; this constitutes a generalization of the results of [12,4,14].

(ii) $\Lambda_0 \neq 0$ and $\dim(P^*) = 1$. Then, since $P^*$ is one dimensional, the gauge fixing involves a unique component of $q_1(x)$ and, hence, $(dW_a(x))_{-1}$ is a function of $x$ taking values on certain one-dimensional subspace of $\tilde{g}_{-1}$. Consequently, since $\mathcal{C}(W_a(y), W_b(z))$

---

\(^8\) Following the terminology of [12], the fractional $[N]^{(i)}$ generalized KdV $A_{N-1}$ hierarchy is associated to the principal Heisenberg subalgebra $\mathcal{H}[w] = \mathcal{H}[N]$ of $\tilde{A}_{N-1}$, the homogeneous gradation $s = (1, 0, \ldots, 0)$, and the principal gradation $s_\omega = (1, 1, \ldots, 1)$. Then, $1 < i < N$ is the principal grade of $\Lambda$. 

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- 10 -
is antisymmetric, this term vanishes identically and the \( W_a \) generators again form a \( \mathcal{W} \)-subalgebra of the second Poisson bracket algebra [6]. Nevertheless, in general, \( \mathcal{C} (B_a(y), W_b(z)) \neq 0 \) and, hence, the \( W_a \)'s and the \( B_a \)'s are not decoupled.

As an example of this second case, the fractional \([N]^{(3)}\) generalized KdV \( A_{N-1} \) hierarchies have been discussed in [6]. Then, the Poisson bracket algebra is just the \( \mathcal{W} \)-algebra associated to the \( \mathfrak{sl}(2, \mathbb{C}) \) subalgebra labelled by the partition \( N = (N+2) + (N-1) + (N-1) \) for \( N \in 1 + 3\mathbb{Z} \), and \( N = (N+1) + (N+1) + (N-2) \) if \( N \in 2 + 3\mathbb{Z} \). In particular, \([4]^{(3)}\) corresponds just to the “\( W_4^{(3)} \)” algebra of [15].

(iii) \( \Lambda_0 \neq 0 \) and \( \dim(P^*) > 1 \). This is the most general and interesting case but, so far, a systematic analysis is still to be done. In particular, it is not known if some of the corresponding Poisson bracket algebras give rise to new extensions of the conformal algebra, and, in fact, the mere existence of a generator for the conformal symmetry has not been investigated yet. On the contrary, the only example discussed in [6] leads again to a \( \mathcal{W} \)-algebra. It is the fractional \([N]^{(N-1)}\) generalized KdV \( A_{N-1} \) hierarchy, and the restriction of its second Poisson bracket algebra to the \( W_a \)'s is the \( \mathcal{W} \)-algebra associated to the partition \( N = 2 + 1 + \cdots + 1 \) [6] (see also [11]).

Finally, let us briefly discuss the form of the second Poisson bracket algebra when \( \Lambda_0 = 0 \); always assuming that \( \Lambda + q \in \tilde{g}_0 \oplus \tilde{g}_1 \). Then, since \([\Lambda, P] = [\Lambda_1, \tilde{g}^{<0}_0] \subset \tilde{g}_1 \), the gauge fixing involves only the components of \( q_1 \) and, therefore, the gauge invariant currents associated to the components of \( q_0^{\text{can}} \) are of the form

\[
W_a(x) \equiv W_{\alpha_a}(x) = \langle \alpha_a, q_0^{\text{can}}(x) \rangle = \langle \alpha_a, e^{S[q_1(x)]} (\partial + q_0(x)) e^{-S[q_1(x)]} \rangle \\
= \langle \alpha_a, q_0(x) - \partial S[q_1(x)] + [S[q_1(x)], q_0(x)] + \cdots \rangle ,
\]

for any \( \alpha_a \in \tilde{g}_0 \), which means that \((dW_{\alpha_a}(x))_0 \big|_{\Lambda+q^{\text{can}}} = \alpha_\delta(x-y) \). Consequently, and according to (3), the restriction of the Poisson bracket to these currents is

\[
\{W_{\alpha_a}(x), W_{\alpha_b}(y)\} = W_{[\alpha_a, \alpha_a]}(x) \delta(x-y) - \langle \alpha_a, \alpha_b \rangle \partial_x \delta(x-y) ,
\]

which is just the Kirillov-Kostant bracket associated to \( \tilde{g}_0 \), i.e., an affine Kac–Moody algebra. On the contrary, this result is not valid when \( \Lambda + q \not\subset \tilde{g}_0 \oplus \tilde{g}_1 \) even if the condition \( \Lambda_0 = 0 \) is satisfied, which was not noticed in [5].

To sum up, we have compared two different Poisson structures that can be defined on the phase space of the generalized Drinfel’d-Sokolov hierarchies of [3,4]: the second Poisson bracket structure corresponding to their Hamiltonian formalism, and the Dirac bracket defining some \( \mathcal{W} \)-algebra by means of Hamiltonian reduction. Both are Poisson structures on the same phase space, the difference being the \( R \)-matrix origin of the Poisson
structure of the hierarchy that changes the Lie-algebraic character of $\tilde{g}$ and, hence, the Hamiltonian nature of the relevant group of transformations. Eq. (12) follows precisely from the breakdown of the Hamiltonian property for some of the gauge transformations involved in the construction of the hierarchies, and gives the precise relation between both Poisson structures. In particular, it shows that the second Poisson bracket algebra is a deformation of some $\mathcal{W}$-algebra. Although we have succeed in the explanation of practically all the (polynomial) $\mathcal{W}$-structures appearing in the literature in the context of generalized Drinfel’d-Sokolov hierarchies, a more detailed analysis of the second Poisson bracket algebra is still to be done. In particular, it is still unclear if the resulting class of Poisson structures always provide extensions, eventually new, of the conformal algebra; an important question that eq.(12) should help to answer.

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