SEMICLASSICAL ANALYSIS OF ELASTIC SURFACE WAVES

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Abstract. In this paper, we present a semiclassical description of surface waves or modes in an elastic medium near a boundary, in spatial dimension three. The medium is assumed to be essentially stratified near the boundary at some scale comparable to the wave length. Such a medium can also be thought of as a surficial layer (which can be thick) overlying a half space. The analysis is based on the work of Colin de Verdière [11] on acoustic surface waves. The description is geometric in the boundary and locally spectral “beneath” it. Effective Hamiltonians of surface waves correspond with eigenvalues of ordinary differential operators, which, to leading order, define their phase velocities. Using these Hamiltonians, we obtain pseudodifferential surface wave equations. We then construct a parametrix. Finally, we discuss Weyl’s formulas for counting surface modes, and the decoupling into two classes of surface waves, that is, Rayleigh and Love waves, under appropriate symmetry conditions.

1. Introduction. We carry out a semiclassical analysis of surface waves in a medium which is stratified near its boundary – with topography – at some scale comparable to the wave length. We discuss how the (dispersive) propagation of such waves is governed by effective Hamiltonians on the boundary and show that the system is displayed by a space-adiabatic behavior.

The Hamiltonians are non-homogeneous principal symbols of some pseudodifferential operators. Each Hamiltonian is identified with an eigenvalue in the point spectrum of a locally Schrödinger-like operator in dimension one on the one hand, and generates a flow identified with surface-wave bicharacteristics in the two-dimensional boundary on the other hand. The eigenvalues exist under certain assumptions reflecting that wave speeds near the boundary are smaller than in the deep interior. This assumption is naturally satisfied by the structure of Earth’s crust and mantle (see, for example, Shearer [36]). The dispersive nature of surface waves is manifested by the non-homogeneity of the Hamiltonians.

The spectra of the mentioned Schrödinger-like operators consist of point and essential spectra. The surface waves are identified with the point spectra while the essential spectra correspond with propagating body waves. We note, here, that the point and essential spectra for the Schrödinger-like operators may overlap.

Our analysis applies to the study of surface waves in Earth’s “near” surface in the scaling regime mentioned above. The existence of such waves, that is, propagating wave solutions which decay exponentially away from the boundary of a homogeneous (elastic) half-space was first noted by Rayleigh [34]. Rayleigh and (“transverse”) Love waves can be identified with Earth’s free oscillation triples \( nS_l \) and \( nT_l \) with \( n \ll l/4 \) assuming spherical symmetry. Love [23] was the first to argue that surface-wave dispersion is responsible for the oscillatory character of the main shock of an earthquake tremor, following the “primary” and “secondary” arrivals.

Our analysis is motivated by the (asymptotic) JWKB theory of surface waves developed in seismology by Woodhouse [42], Babich, Chichachev and Yanoskaya [3] and others. Tromp and Dahlen [40] cast this theory in the framework of a “slow” variational principle. The theory is also used in ocean acoustics [6] and is referred to as adiabatic mode theory. An early study of the propagation of waves in smoothly varying waveguides can be found in Bretherton [7]. Nomofilov [30] obtained the form of WKB solutions for Rayleigh waves in inhomogeneous, anisotropic elastic media using assumptions appearing in Proposition 4.2 in the main text. Many aspects of the propagation of surface waves in laterally inhomogeneous elastic media are discussed in the book of Malischewsky [24]. Here, we develop a comprehensive semiclassical analysis of elastic surface waves, generated by
interior (point) sources, with the corresponding estimates. This semiclassical framework was first formulated by Colin de Verdière [11] to describe surface waves in acoustics.

The scattering of surface waves by structures, away from the mentioned scaling regime, has been extensively studied in the seismology literature. This scattering can be described using a basis of local surface wave modes that depend only on the “local” structure of the medium, for example, with invariant embedding; see, for example, Odom [31]. Odom used a layer of variable thickness over a homogeneous half space to account for the interaction between surface waves and body waves by the topography of internal interfaces.

The outline of this paper is as follows. In Section 2, we carry out the semiclassical construction of general surface wave parametrices. In the process, we introduce locally Schrödinger-like operators in the boundary normal coordinate and their eigenvalues signifying effective Hamiltonians in the boundary (tangential) coordinates describing surface-wave propagation. In Section 3 we characterize the spectra of the relevant Schrödinger-like operators. That is, we study their discrete and essential spectra. In Section 4 we consider a special class of surface modes associated with exponentially decaying eigenfunctions. The existence of such modes is determined by a generalized Barnett-Lothe condition. In Section 5 we review conditions on the symmetry, already considered by Anderson [1], restricting the anisotropy to transverse isotropy with the axis of symmetry aligned with the normal to the boundary, allowing the decoupling of surface waves into Rayleigh and Love waves. In Section 6 we establish Weyl’s laws first in the isotropic (separately for Rayleigh and Love waves) and then in the anisotropic case. Finally, in Section 7 we relate the surface waves to normal modes viewing the analysis locally on conic regions, or more generally on Riemannian manifolds with a half cylinder structure. We give explicit formulas for the special case of a radial manifold.

2. Semiclassical construction of surface-wave parametrices. We consider the linear elastic wave equation in $\mathbb{R}^3$,

$$\frac{\partial^2 u}{\partial t^2} = \text{div} \frac{\sigma(u)}{\rho},$$

where $u$ is the displacement vector, and $\sigma(u)$ is the stress tensor given by Hooke’s law

$$\sigma(u) = C \varepsilon(u),$$

and $\varepsilon(u)$ denotes the strain tensor; $C$ is the fourth-order stiffness tensor with components $c_{ijkl}$, and $\rho$ is the density of mass. The componentwise expression of (2.2) is given by

$$\sigma_{ij}(u) = \sum_{k,l=1}^{3} c_{ijkl} \varepsilon_{kl}(u), \quad \varepsilon_{kl}(u) = \frac{1}{2} (\partial_k u_l + \partial_l u_k).$$

Equation (2.1) differs from the usual system given by

$$\rho \frac{\partial^2 u}{\partial t^2} = \text{div} \sigma(u)$$

in case $\rho$ is not a constant. However, the difference is in the lower terms. These can be accounted for considering (2.1).

We study the elastic wave equation (2.1) in the half space $X = \mathbb{R}^2 \times (-\infty, 0]$, with coordinates,

$$(x, z), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad z \in \mathbb{R}^- = (-\infty, 0].$$

We consider solutions, $u = (u_1, u_2, u_3)$, satisfying the Neumann boundary condition at $\partial X = \{z = 0\}$,

$$\frac{\partial^2 u_i}{\partial t^2} + M_{ii} u_i = 0,$$

$$u(t = 0, x, z) = 0, \quad \partial_t u(t = 0, x, z) = h(x, z),$$

$$\frac{c_{ijkl}}{\rho} \partial_k u_l(t, x, z = 0) = 0,$$

(2.3)
where

\[
M_{il} = -\frac{\partial}{\partial z} \frac{c_{33l}(x, z)}{\rho(x, z)} \frac{\partial}{\partial z} - \sum_{j,k=1}^{2} \frac{c_{ijkl}(x, z)}{\rho(x, z)} \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} - \sum_{j=1}^{2} \frac{\partial}{\partial x_j} \frac{c_{ij3l}(x, z)}{\rho(x, z)} \frac{\partial}{\partial z} - \sum_{k=1}^{2} \frac{\partial}{\partial x_k} \frac{c_{ik3l}(x, z)}{\rho(x, z)} \frac{\partial}{\partial z} - \sum_{j,k=1}^{2} \left( \frac{\partial}{\partial x_j} \frac{c_{ijkl}(x, z)}{\rho(x, z)} \right) \frac{\partial}{\partial x_k} - \sum_{j,k=1}^{2} \left( \frac{\partial}{\partial x_j} \frac{c_{ijkl}(x, z)}{\rho(x, z)} \right) \frac{\partial}{\partial x_k}.
\]

In the above, we assume that \(\rho(x, z) \in C^\infty(X)\) with \(\rho(x, z) \geq \rho_0 > 0\) for some \(\rho_0 > 0\) and \(c_{ijkl}(x, z) \in C^\infty(X)\) satisfies the following symmetries and strong convexity condition:

(symmetry) \(c_{ijkl} = c_{jikl} = c_{klij}\) for any \(i, j, k, l\);

(strong convexity) there exists \(\delta > 0\) such that for any nonzero \(3 \times 3\) real-valued symmetric matrix \((\varepsilon_{ij})\),

\[
\sum_{i,j,k,l=1}^{3} \frac{c_{ijkl}}{\rho} \varepsilon_{ij} \varepsilon_{kl} \geq \delta \sum_{i,j=1}^{3} \varepsilon_{ij}^2.
\]

We note that these are physically very natural assumptions. We invoke, additionally,

Assumption 2.1. The stiffness tensor and density obey the following scaling,

\[
\frac{c_{ijkl}}{\rho}(x, z) = C_{ijkl} \left( x, \frac{z}{\epsilon} \right), \quad \epsilon \in (0, \epsilon_0];
\]

\[
C_{ijkl}(x, Z) = C_{ijkl}(x, Z_I) = C_{ijkl}^I = \text{constant}, \quad \text{for } Z \leq Z_I < 0
\]

and, for any \(\hat{\xi} \in S^2\),

\[
\inf_{Z \leq 0} v_L(x, \hat{\xi}, Z) < v_L(x, \hat{\xi}, Z_I),
\]

where \(v_L\) is the so-called limiting velocity which will be defined in Section 3.

**2.1. Schrödinger-like operators.** Under Assumption 2.1 we make the following change of variables,

\[
u(t, x, z) = v \left( t, x, \frac{z}{\epsilon} \right);
\]

upon introducing \(Z = \frac{z}{\epsilon}\), the elastic wave equation in (2.3) takes the form

\[(\epsilon^2 \partial_t^2 + \hat{H})v = 0,
\]

where

\[
\hat{H}_{il} = -\frac{\partial}{\partial Z} C_{33l}(x, Z) \frac{\partial}{\partial Z} - \epsilon^2 \sum_{j,k=1}^{2} C_{ijkl}(x, Z) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} - \epsilon \sum_{j=1}^{2} \frac{\partial}{\partial x_j} \frac{c_{ij3l}(x, Z)}{\rho(x, z)} \frac{\partial}{\partial Z} - \epsilon \sum_{k=1}^{2} \frac{\partial}{\partial x_k} \frac{c_{ik3l}(x, Z)}{\rho(x, z)} \frac{\partial}{\partial Z}
\]

\[
- \epsilon^2 \sum_{j,k=1}^{2} C_{ijkl}(x, Z) \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_k} - \epsilon \sum_{j=1}^{2} \left( \frac{\partial}{\partial x_j} \frac{c_{ijkl}(x, Z)}{\rho(x, z)} \right) \frac{\partial}{\partial Z} - \epsilon \sum_{k=1}^{2} \left( \frac{\partial}{\partial x_k} \frac{c_{ij3l}(x, Z)}{\rho(x, z)} \right) \frac{\partial}{\partial Z}
\]

\[
- \epsilon^2 \sum_{j,k=1}^{2} \left( \frac{\partial}{\partial x_j} \frac{c_{ijkl}(x, Z)}{\rho(x, z)} \right) \frac{\partial}{\partial x_k}.
\]
With Definition \ref{def:1} in Appendix A, we consider $\hat{H}$ as a semiclassical pseudodifferential operator on $\mathbb{R}^2$, where $x$ belongs to; then (2.4) can be rewritten as

$$
e^2 \partial_t^2 v + \text{Op}_\epsilon(H(x, \xi)) v \sim 0,$$

where $\text{Op}_\epsilon(H(x, \xi))$ is a semiclassical pseudodifferential operator with symbol $H(x, \xi)$ defined by

$$H(x, \xi) = H_0(x, \xi) + H_1(x, \xi)$$

with

$$H_{0, \epsilon}(x, \xi) = -\frac{\partial}{\partial Z} C_{i33}(x, Z) \frac{\partial}{\partial Z} - i \sum_{j=1}^2 C_{ij3}(x, Z) \frac{\partial}{\partial Z} \xi_j - i \sum_{k=1}^2 \left( \frac{\partial}{\partial Z} C_{ik3k}(x, Z) \right) \xi_k + \sum_{j,k=1}^2 C_{ijkl}(x, Z) \xi_j \xi_k$$

and

$$H_{1, \epsilon}(x, \xi) = -\epsilon \sum_{j=1}^2 \left( \frac{\partial}{\partial Z} C_{ij3}(x, Z) \right) \frac{\partial}{\partial Z} + i \sum_{j,k=1}^2 \left( \frac{\partial}{\partial x_j} C_{ijkl}(x, Z) \right) \xi_k.$$

We view $H_0(x, \xi)$ and $H_1(x, \xi)$ as ordinary differential operators in $Z$, with domain

$$D = \left\{ v \in H^2(\mathbb{R}^-) \mid \sum_{i=1}^3 \left( C_{i33}(x, 0) \frac{\partial v_i}{\partial Z}(0) + i \sum_{k=1}^2 C_{ik3k}(x, 0) \xi_k v_i(0) \right) = 0 \right\}.$$

2.2. Effective Hamiltonians. We use eigenvalues and eigenfunctions of $H_0(x, \xi)$ to construct approximate solutions to (2.10). For fixed $(x, \xi)$, an eigenvalue $\Lambda(x, \xi)$ and the corresponding eigenfunction $V(x, \xi)$ of $H_0(x, \xi)$ are such that

$$H_0(x, \xi)V(x, \xi) = V(x, \xi)\Lambda(x, \xi),$$

where $V(x, \xi) \in D$. Since $H_0(x, \xi)$ is a positive symmetric operator in $L^2(\mathbb{R}^-)$ for $\xi \neq 0$, $\Lambda(x, \xi)$ is real-valued, and also positive. We note, here, that the contribution coming from $\xi$ confined to a compact set is negligible. This is why we can assume that $\xi \neq 0$.

We let $\mathcal{L}(H_1, H_2)$ denote the set of all bounded operators from a normed space $H_1$ to a normed space $H_2$.

Theorem 2.1. Let $\Lambda_{\alpha}(x, \xi)$ be an eigenvalue of $H_0(x, \xi)$, and $U \subset T^*\mathbb{R}^2 \setminus 0$ be open. Assume that $\Lambda_{\alpha}(x, \xi)$ has constant multiplicity $m_{\alpha}$ for all $(x, \xi) \in U$. There exist $\Phi_{\alpha, m}(x, \xi) \in \mathcal{L}(D, L^2(\mathbb{R}^-))$ and $a_{\alpha, m}(x, \xi) \in \mathcal{L}(L^2(\mathbb{R}^-), D)$ which admits asymptotic expansions

$$\Phi_{\alpha, \epsilon}(x, \xi) \sim \sum_{m=0}^{\infty} \Phi_{\alpha, m}(x, \xi) \epsilon^m,$$

$$a_{\alpha, \epsilon}(x, \xi) \sim \sum_{m=0}^{\infty} a_{\alpha, m}(x, \xi) \epsilon^m,$$

and satisfy

$$H \circ \Phi_{\alpha, \epsilon}(x, \xi) = \Phi_{\alpha, \epsilon} \circ a_{\alpha, \epsilon}(x, \xi) + \mathcal{O}(\epsilon^n)$$

for all $(x, \xi) \in U$. \hfill $\square$
where \( \circ \) denotes the composition of symbols (see Appendix A). Furthermore, \( a_{\alpha,0}(x, \xi) = \Lambda_{\alpha}(x, \xi) I \) and \( \Phi_{\alpha,0}(x, \xi) \) is the projection onto the eigenspace associated with \( \Lambda_{\alpha}(x, \xi) \).

**Proof.** First, we note that \( \Lambda_{\alpha}(x, \xi) \in C^\infty(U) \) (cf. Appendix B of [12]). By the composition of symbols (cf. Appendix A), we have

\[
H \circ \Phi_{\alpha,\epsilon} = H_0 \Phi_{\alpha,0} + \epsilon H_0 \Phi_{\alpha,1} + \epsilon H_1 \Phi_{\alpha,0} + \frac{\epsilon}{2i} \sum_{j=1}^{2} \frac{\partial H_0}{\partial \xi_j} \frac{\partial \Phi_{\alpha,0}}{\partial x_j} + O(\epsilon^2)
\]

and

\[
\Phi_{\alpha,\epsilon} \circ a_{\alpha,\epsilon} = \Phi_{\alpha,0} \Lambda_{\alpha} + \epsilon \Phi_{\alpha,0} a_{\alpha,1} + \epsilon \Phi_{\alpha,1} \Lambda_{\alpha} + \frac{\epsilon}{2i} \sum_{j=1}^{2} \frac{\partial \Phi_{\alpha,0}}{\partial \xi_j} \frac{\partial \Lambda_{\alpha}}{\partial x_j} + O(\epsilon^2).
\]

We construct the terms \( \Phi_{\alpha,m} \) by collecting terms of equal orders in the two expansions above. Terms of order \( O(\epsilon^0) \) give

\[
H_0 \Phi_{\alpha,0} = \Phi_{\alpha,0} \Lambda_{\alpha},
\]

which is consistent with \((2.9)\). Terms of order \( O(\epsilon^1) \) give

\[
(H_0(x, \xi) - \Lambda_{\alpha}(x, \xi)) \Phi_{\alpha,1}(x, \xi) = \Phi_{\alpha,0} a_{\alpha,1} + R(\Phi_{\alpha,0}),
\]

where \( R(\Phi_{\alpha,0}) \in \mathcal{L}(\mathcal{D}, L^2(\mathbb{R}^-)) \) denotes the remaining terms. We choose \( a_{\alpha,1} \), so that, for \( u \in \mathcal{D} \), \( (\Phi_{\alpha,0} a_{\alpha,1} + R(\Phi_{\alpha,0})) u \) is orthogonal to the eigenspace of \( \Lambda_{\alpha}(x, \xi) \). Then we let \( v = \Phi_{\alpha,1}(x, \xi) u \) be the unique solution of

\[
(H_0(x, \xi) - \Lambda_{\alpha}(x, \xi)) v = (\Phi_{\alpha,0} a_{\alpha,1} + R(\Phi_{\alpha,0})) u,
\]

which is orthogonal to the eigenspace of \( \Lambda_{\alpha}(x, \xi) \). Thus we have defined the operator \( \Phi_{\alpha,1}(x, \xi) \). Higher order terms \( \Phi_{\alpha,m}(x, \xi), m \geq 2 \), can be constructed successively by solving the equations,

\[
(H_0(x, \xi) - \Lambda_{\alpha}(x, \xi)) \Phi_{\alpha,m}(x, \xi) = R(\Phi_{\alpha,0}, \cdots, \Phi_{\alpha,m-1}),
\]

with \( R(\Phi_{\alpha,0}, \cdots, \Phi_{\alpha,m-1}) \in \mathcal{L}(\mathcal{D}, L^2(\mathbb{R}^-)) \), since \( \Phi_{\alpha,k}, a_{\alpha,k}, 0 \leq k \leq m - 1 \), in their respective topologies, depend continuously on \((x, \xi)\) by induction. Hence, we can choose \( a_{\alpha,m} = -R(\Phi_{\alpha,0}, \cdots, \Phi_{\alpha,m-1}) \) and solve for \( \Phi_{\alpha,m} \). This completes the construction of \( a_{\alpha,m} \) and \( \Phi_{\alpha,m} \).

For any bounded set \( U' \subset T^*\mathbb{R}^2 \setminus 0 \), we let \( U \subset T^*\mathbb{R}^2 \) be such that \( \overline{U'} \subset U \) and \( \chi \in C_0^\infty(U) \) with \( \chi \equiv 1 \) on \( \overline{U'} \). Then

\[
J_{\alpha,\epsilon} = \text{Op}_\epsilon \left( \frac{1}{\sqrt{\epsilon}} \chi(x, \xi) \Phi_{\alpha,\epsilon} \right)
\]

defines a linear map on \( \mathcal{M}(V) \) (the factor \( \frac{1}{\sqrt{\epsilon}} \) is to make the operator \( J_{\alpha,\epsilon} \) (microlocally) unitary), where

\[
V = \{(x, \xi, Z, \zeta) \in T^*X : (x, \xi) \in U'\}
\]

and \( \mathcal{M}(V) \) denotes the space of microfunctions on \( V \); see, Definition \( A.4 \) of Appendix A. Moreover, we find that, using Theorem \( 2.1 \)

\[
\hat{H} J_{\alpha,\epsilon} = J_{\alpha,\epsilon} \text{Op}_\epsilon(a_{\alpha,\epsilon}),
\]

where the left- and right-hand sides are considered as microfunctions.

**Remark 2.1.** We note that the principal symbol of \( a_{\alpha,\epsilon} \) is real-valued. Hence, the propagation of wavefront set of surface waves is purely on the surface. However, since the lower order terms of \( a_{\alpha,\epsilon} \) are operators in \( Z \), there exists coupling of surface waves with interior motion. There is no such coupling if the eigenvalue \( \Lambda_{\alpha} \) has multiplicity one. For this case \( a_{\alpha,\epsilon} \) is a classical pseudodifferential symbol.
2.3. Surface-wave equations and parametrices. With the results of the previous subsection, we construct approximate solutions of the system (2.3) with initial values

\[ h(x, \epsilon Z) = \sum_{\alpha=1}^{\aleph} J_{\alpha,\epsilon} W_{\alpha}(x, Z). \]

Then we construct solutions of the form

\[ u(t, x, z) = v \left( t, x, \frac{z}{\epsilon} \right) = \sum_{\alpha=1}^{\aleph} v_{\alpha,\epsilon} \left( t, x, \frac{z}{\epsilon} \right), \]

(2.14)

with

\[ v \left( 0, x, \frac{z}{\epsilon} \right) = 0, \quad \partial_t v \left( 0, x, \frac{z}{\epsilon} \right) = \sum_{\alpha=1}^{\aleph} J_{\alpha,\epsilon} W_{\alpha}(x, \frac{z}{\epsilon}), \]

\[ \frac{c_{33kl}}{\rho} \partial_k v_l \left( t, x, \frac{z}{\epsilon} \right) \bigg|_{z=0} = 0. \]

Here, \( \aleph \) is chosen such that for each \( (x, \xi) \in U \), there are at least \( \aleph \) eigenvalues for \( H_0(x, \xi) \). We assume that all eigenvalues \( \Lambda_1 < \cdots < \Lambda_\alpha < \cdots < \Lambda_\aleph \) are of constant multiplicities, \( m_1, \cdots, m_\alpha, \cdots, m_\aleph \). We let \( W_{\alpha,\epsilon} \) solve the initial value problems

\[ \epsilon^2 \frac{\partial^2}{\partial t^2} + \text{Op}_\epsilon(a_{\alpha,\epsilon})(., D_x) W_{\alpha,\epsilon}(t, x, Z) = 0, \]

(2.15)

\[ W_{\alpha,\epsilon}(0, x, Z) = 0, \quad \partial_t W_{\alpha,\epsilon}(0, x, Z) = J_{\alpha,\epsilon} W_{\alpha}(x, Z), \]

\[ \alpha = 1, \ldots, \aleph. \]

Equation (2.16) means that the initial values are in the span of the ranges of operators \( \text{Op}_\epsilon(\Phi_{\alpha,\epsilon}), \alpha = 1, \ldots, \aleph \).

We address the existence of eigenvalues of \( H_0(x, \xi) \) in later sections, under the Assumption 2.1. To construct the parametrix, we use the first-order system for \( W_{\epsilon} \) that is equivalent to (2.15).

\[ \frac{\epsilon}{\partial_t} \left( \begin{array}{c} W_{\alpha,\epsilon} \\ \epsilon \frac{\partial W_{\alpha,\epsilon}}{\partial t} \end{array} \right) = \left( \begin{array}{cc} 0 & 1 \\ -\text{Op}_\epsilon(a_{\alpha,\epsilon}) & 0 \end{array} \right) \left( \begin{array}{c} W_{\alpha,\epsilon} \\ \epsilon \frac{\partial W_{\alpha,\epsilon}}{\partial t} \end{array} \right). \]

(2.17)

We write \( \text{Op}_\epsilon(b_{\alpha,\epsilon}) = [\text{Op}_\epsilon(a_{\alpha,\epsilon})]^{1/2} \). The principal symbol of \( \text{Op}_\epsilon(b_{\alpha,\epsilon}) \) is given by \( \Lambda_\alpha^{1/2}(x, \xi) \). Then

\[ W_{\alpha,\epsilon,\pm} = \frac{1}{2} W_{\alpha,\epsilon} \pm \frac{i}{2} \epsilon \text{Op}_\epsilon(b_{\alpha,\epsilon})^{-1} \frac{\partial W_{\alpha,\epsilon}}{\partial t}, \]

(2.18)

satisfy the two first-order (“half wave”) equations

\[ P_{\alpha,\epsilon,\pm}(x, D_x, D_t) W_{\alpha,\epsilon,\pm} = 0, \]

(2.19)

where

\[ P_{\alpha,\epsilon,\pm}(x, D_x, D_t) = \epsilon \partial_t \pm i \text{Op}_\epsilon(b_{\alpha,\epsilon}), \]

(2.20)

supplemented with the initial conditions

\[ h_{\alpha,\epsilon,\pm} = \pm \frac{1}{2} i \text{Op}_\epsilon(b_{\alpha,\epsilon})^{-1} J_{\alpha,\epsilon} W_{\alpha,\epsilon}|_{z=0}. \]
The principal symbol of \( \frac{1}{\epsilon} P_{\alpha,\epsilon,\pm} = \epsilon D_t \pm \text{Op}_\epsilon(b_{\alpha,\epsilon}) \) is given by \( \omega \pm \Lambda_1^{1/2}(x, \xi) \) defining a Hamiltonian. The solution to the original problem is then given by

\[
W_{\alpha,\epsilon} = W_{\alpha,\epsilon,+} + W_{\alpha,\epsilon,-}.
\]

We construct WKB solutions for operator \( P_{\epsilon,+}(x, D_x, D_t) \), introducing a representation of the solution operator of the form,

\[
S_{\alpha,\epsilon}(t) h_{\alpha,\epsilon,+}(x) = \frac{1}{(2\pi \epsilon)^2} \int \int e^{i(\psi_{\alpha,\epsilon}(x, y, t) - \langle y, \eta \rangle)} A_{\alpha,+}(x, \eta, t) h_{\alpha,\epsilon,+}(y) dy d\eta.
\]

For more details of this construction, we refer to [4, 45].

**Construction of the phase function.** Surface-wave bicharacteristics are solutions, \((x, \xi)\), to the Hamiltonian system,

\[
\begin{align*}
\partial_x k(y, \eta, t) &= \frac{\partial \Lambda_1^{1/2}(x, \xi)}{\partial \xi_k}, \\
\partial_{\xi_k} k(y, \eta, t) &= -\frac{\partial \Lambda_1^{1/2}(x, \xi)}{\partial x_k}, \\
(x, \xi)|_{t=0} &= (y, \eta).
\end{align*}
\]

We define

\[
\kappa_t(y, \eta) = (x, \xi).
\]

For \((y, \eta)\) in \(U\), the map

\[
(x, \xi, y, \eta) = (\kappa_t(y, \eta), (y, \eta)) \mapsto (x, \eta)
\]

is surjective for small \(t\). We write \(\xi = \Xi(x, \eta, s)\) and introduce the phase function

\[
\psi_{\alpha,+}(x, \eta, t) = \langle x, \eta \rangle - \int_0^t \Lambda_1^{1/2}(x, \Xi(x, \eta, s)) ds,
\]

where \(\langle x, \eta \rangle\) denotes the inner product of \(x\) and \(\eta\). This phase function solves

\[
\partial_t \psi_{\alpha,+} + \Lambda_1^{1/2}(x, \partial_x \psi_{\alpha,+}) = 0, \quad \psi_{\alpha,+}(x, \eta, 0) = \langle x, \eta \rangle
\]

[45] Chapter 10. The group velocity, \(v\), is defined by

\[
v_k := \frac{\partial x_k}{\partial t} = \frac{\partial \Lambda_1^{1/2}(x, \xi)}{\partial \xi_k}.
\]

We note that \(\Lambda_1^{1/2}(x, \xi)\) is non-homogeneous in \(\xi\), thus the group velocity and the bicharacteristics depend on \(|\xi|\).

**Construction of the amplitude.** The amplitude \(A\) in (2.23) must satisfy

\[
(\epsilon D_t + b_{\alpha,\epsilon}(x, \epsilon D))(e^{i\psi_{\alpha,+}} A_{\alpha,+}) = O(\epsilon^\infty),
\]

or, equivalently,

\[
(\partial_t \psi_{\alpha,+} + \epsilon D_t + e^{i\psi_{\alpha,+}} b_{\alpha,\epsilon}(x, \epsilon D)e^{i\psi_{\alpha,+}})A_{\alpha,+} = O(\epsilon^\infty).
\]

We note that \(\eta\) appears in the construction of the phase function \(\psi_{\alpha,+}\). We construct solutions, \(A_{\alpha,+}\), via the expansion,

\[
A_{\alpha,+}(x, \eta, t) \sim \sum_{k=0}^{\infty} e^k A_{\alpha,+k}(x, \eta, t),
\]
and writing the symbol of \( b_{\alpha,\epsilon}(x, \epsilon D) \) as an expansion,

\[
b_{\alpha,\epsilon}(x, \xi) = \sum_{j=0}^{\infty} \epsilon^j b_{\alpha,j}(x, \xi).
\]

By the asymptotics of \( e^{-i\epsilon\alpha} b_{\alpha,\epsilon}(x, \epsilon D) e^{i\epsilon\alpha} \), the terms of order \( \mathcal{O}(\epsilon^0) \) yield the equation,

\[
(2.28) \quad \partial_t \psi_{\alpha,+} + b_{\alpha,0}(x, \partial_x \psi_{\alpha,+}) = \partial_t \psi_{\alpha,+} + \Lambda_\alpha^{1/2}(x, \partial_x \psi_{\alpha,+}) = 0,
\]

for the phase function \( \psi_{\alpha,+} \), which was introduced above. The terms of order \( \mathcal{O}(\epsilon^1) \) yield the transport equation for \( A_{\alpha,+0} \),

\[
(D_t + L)A_{\alpha,+0} = 0,
\]

\[
A_{\alpha,+0}(x, \eta, 0) = 1,
\]

where

\[
(2.29) \quad L = \sum_{j=1}^{2} \partial_{\xi_j} \Lambda_\alpha^{1/2}(x, \partial_x \psi_{\alpha,+}) D_{x_j} + b_{\alpha,1}(x, \partial_x \psi_{\alpha,+}) - \frac{i}{2} \sum_{j,k=1}^{2} \partial_{\xi_k \xi_j} \Lambda_\alpha^{1/2}(x, \partial_x \psi_{\alpha,+}) D_{x_k x_l} \psi_{\alpha,+},
\]

in which \( \partial_{\xi_k \xi_j} = \partial_{\xi_k} \partial_{\xi_j} \), \( D_{x_k x_l} = D_{x_k} D_{x_l} \). Denote

\[
c_{t,j}(x, \epsilon D, \eta) = e^{-i\epsilon \psi_{\alpha,+}(x, \eta, t)} b_{\alpha,j}(x, \epsilon D) e^{i\epsilon \psi_{\alpha,+}(x, \eta, t)} = \sum_{l=0}^{\infty} \epsilon^l c_{t,j}(x, \epsilon D, \eta).
\]

We can construct \( A_{\alpha,+k}, k = 1, 2, \ldots \), recursively by solving transport equations of the form \( (D_t + L)A_{\alpha,+k} = F(A_{\alpha,+0}, \cdots, A_{\alpha,+k-1}), k \geq 1 \),

\[
A_{\alpha,+k}(x, \eta, 0) = 0.
\]

Here,

\[
F(A_{\alpha,+0}, \cdots, A_{\alpha,+k-1}) = -i \partial_{\psi_{\alpha,+}} A_{\alpha,+k-1}(x, \eta, t) - i \sum_{j+l=k-1} c_{t,j}(x, \epsilon D, \eta) A_{\alpha,+j}(x, \eta, t).
\]

For solving the transport equations, we use the bicharacteristics determined by \( (2.24) \). We note that

\[
D_t + \sum_{j=1}^{2} \partial_{\xi_j} \Lambda_\alpha^{1/2}(x, \partial_x \psi_{\alpha,+}) D_{x_j} = D_t + \sum_{j=1}^{2} \frac{\partial x_j(y, \eta, t)}{\partial t} D_{x_j}
\]

so that

\[
\left( \frac{d}{dt} + i b_{\alpha,1}(x, \partial_x \psi_{\alpha,+}) + \frac{1}{2} \sum_{j,k=1}^{2} \partial_{\xi_k \xi_j} \Lambda_\alpha^{1/2}(x, \partial_x \psi_{\alpha,+}) D_{x_k x_l} \psi_{\alpha,+} \right) A_{\alpha,+k}(x(y, \eta, t), \eta, t)
\]

\[
= F(A_{\alpha,+0}, \cdots, A_{\alpha,+k-1})
\]

\[
A_{\alpha,+0}(x(y, \eta, 0), \eta, 0) = 1, \quad A_{\alpha,+k}(x(y, \eta, 0), \eta, 0) = 0, \quad \text{for } k = 1, 2, 3, \ldots
\]

The equations above can be solved using the standard theory of ordinary differential equations. For more details about semiclassical Fourier Integral Operators, we refer to Chapter 10 of Zworski’s book \( [45] \).
Trace. We can write the solution to (2.19), up to leading order, as
\[
\frac{1}{(2\pi\epsilon)^2} \int G_{\alpha,\pm,0}(x, t, \epsilon) h_{\epsilon,\pm}(y) \exp(-\frac{1}{\epsilon} \langle x - y, \eta \rangle) dy d\eta
\]
with
\[
G_{\alpha,\pm,0}(x, t, \eta, \epsilon) = \exp \left[ \pm \frac{i}{\epsilon} \int_0^t \Lambda_\alpha^{1/2}(x, \Xi(x, \eta, s)) ds \right] A_{\alpha,\pm,0}(x, \eta, t).
\]
Then, using (2.21), (2.22) and (2.23), we can write down the approximate Green's function (microlocalized in \(x\)), still up to leading order, for surface waves
\[
G_0(Z, x, t, Z', \eta; \epsilon) = \sum_{\alpha=1}^{2N} J^{(x)}_{\alpha,\epsilon}(Z, x, \eta)
\]
\[
\left( \frac{1}{2} G_{\alpha,+,0}(x, t, \eta, \epsilon) - \frac{1}{2} G_{\alpha,-,0}(x, t, \eta, \epsilon) \right) \Lambda_\alpha^{-1/2}(x, \eta) J^{(x)}_{\alpha,\epsilon}(Z', x, \eta)
\]
\[
= \sum_{\alpha=1}^{2N} J^{(x)}_{\alpha,\epsilon}(Z, x, \eta) \left( \frac{1}{2} \exp \left[ -\frac{i}{\epsilon} \int_0^t \Lambda_\alpha^{1/2}(x, \Xi(x, \eta, s)) ds \right] A_{\alpha,+,0}(x, \eta, t) \right.
\]
\[\left. - \frac{1}{2} \exp \left[ \frac{i}{\epsilon} \int_0^t \Lambda_\alpha^{1/2}(x, \Xi(x, \eta, s)) ds \right] A_{\alpha,-,0}(x, \eta, t) \right) \Lambda_\alpha^{-1/2}(x, \eta) J^{(x)}_{\alpha,\epsilon}(Z', x, \eta).
\]
Then
\[
\epsilon \partial_t G_0(Z, x, t, Z', \eta; \epsilon)
\]
\[
= \sum_{\alpha=1}^{2N} J^{(x)}_{\alpha,\epsilon}(Z, x, \eta) \left( \frac{1}{2} \exp \left[ -\frac{i}{\epsilon} \int_0^t \Lambda_\alpha^{1/2}(x, \Xi(x, \eta, s)) ds \right] A_{\alpha,+,0}(x, \eta, t) \right.
\]
\[\left. + \frac{1}{2} \exp \left[ \frac{i}{\epsilon} \int_0^t \Lambda_\alpha^{1/2}(x, \Xi(x, \eta, s)) ds \right] A_{\alpha,-,0}(x, \eta, t) \right) \Lambda_\alpha^{-1/2}(x, \eta) J^{(x)}_{\alpha,\epsilon}(Z', x, \eta).
\]
Now, we take the semiclassical Fourier transform in \(t\), multiply by the test function \(\chi(\omega) \in S(\mathbb{R})\) and apply the stationary phase formula to the leading order (see, for example, [18, Proposition 5.2). We get
\[
\int \epsilon \partial_t \tilde{G}_0(Z, x, \omega, Z', \eta; \epsilon) \chi(\omega) d\omega = \frac{1}{2\pi \epsilon} \int \epsilon \partial_t G_0(Z, x, t, Z', \eta; \epsilon) e^{-i \omega \epsilon} dt \chi(\omega) d\omega \sim \sum_{\alpha=1}^{2N} J^{(x)}_{\alpha,\epsilon}(Z, x, \eta)
\]
\[
\left( \frac{1}{2} A_{\alpha,+,0}(x, \eta, 0) \chi(-\Lambda_\alpha^{1/2}(x, \eta)) + \frac{1}{2} A_{\alpha,-,0}(x, \eta, 0) \chi(\Lambda_\alpha^{1/2}(x, \eta)) \right) J^{(x)}_{\alpha,\epsilon}(Z', x, \eta).
\]
As \(A_{\alpha,\pm,0}(x, \eta, 0) = 1\), we get
\[
\epsilon \partial_t \tilde{G}_0(Z, x, \omega, Z', \eta; \epsilon)
\]
\[
\sim \sum_{\alpha=1}^{2N} \frac{1}{2} J^{(x)}_{\alpha,\epsilon}(Z, x, \eta) \left[ \delta(\omega + \Lambda_\alpha^{1/2}(x, \eta)) + \delta(\omega - \Lambda_\alpha^{1/2}(x, \eta)) \right] J^{(x)}_{\alpha,\epsilon}(Z', x, \eta)
\]
\[
= \sum_{\alpha=1}^{2N} J^{(x)}_{\alpha,\epsilon}(Z, x, \eta) \delta(\omega^2 - \Lambda_\alpha(x, \eta)) \Lambda_\alpha^{1/2}(x, \eta) J^{(x)}_{\alpha,\epsilon}(Z', x, \eta)
\]
On the diagonal \( Z = Z' \), we get
\[
\epsilon \partial_t \mathcal{G}_0(Z, x, \omega, Z, \eta; \epsilon) \sim \sum_{\alpha=1}^{3N} \left( J_{\alpha,e}^{(x)}(Z, x, \eta) \right)^2 \delta (\omega^2 - \Lambda_{\alpha}(x, \eta)) \Lambda_{\alpha}^{1/2}(x, \eta).
\]

We remember here that \( J_{\alpha,e}^{(x)}(Z, x, \eta) = \frac{1}{i\epsilon} \left( \Phi_{\alpha,0}(Z, x, \eta) + \mathcal{O}(\epsilon^{-1}) \right) \) for \((x, \eta) \in U'\) (cf. (2.22)). Assuming the eigenvalues are all simple, we can simply take \( \Phi_{\alpha,0} \) as normalized eigenfunctions. Then,
\[
\int_{\mathbb{R}^-} \epsilon \partial_t \mathcal{G}_0(Z, x, \omega, Z, \eta; \epsilon) \, dx = \sum_{\alpha=1}^{3N} \delta (\omega^2 - \Lambda_{\alpha}(x, \eta)) \Lambda_{\alpha}^{1/2}(x, \eta) + \mathcal{O}(\epsilon^{-1}).
\]

The left-hand side can be viewed as a local trace in the boundary normal coordinate.

**Remark 2.2.** In the oscillatory integral (2.28), the phase variables are the components of \( \eta \). We can construct an alternative representation by using the frequency, \( \omega \), as a phase variable. This would result in the following form of solution operator
\[
(2.31) \quad S_{\alpha,e}(t)h_{\alpha,e,+}(x) := \frac{1}{(2\pi \epsilon)^2} \int e^{i\left( J_{\alpha,e}^{(x)}(x, \omega)\eta - (\omega, \eta) \right)} A_{\alpha,+}(x, x_0, \omega, \epsilon) h_{\alpha,e,+}(x_0) \, dx_0 \, d\omega.
\]

In a neighborhood of a fixed point, \( x_0 \), we construct \( \phi_{\alpha} = \phi_{\alpha}(x, x_0, \omega) \) satisfying the eikonal equation
\[
(2.32) \quad \Lambda_{\alpha}^{1/2}(x, \partial_x \phi_{\alpha}(x, x_0, \omega)) = \omega.
\]

Then the phase function is given by
\[
\tilde{\psi}_{\alpha}(x, x_0, t, \omega) = \phi_{\alpha}(x, x_0, \omega) - \omega t
\]
and satisfies (2.28),
\[
\partial_t \tilde{\psi}_{\alpha} + \Lambda_{\alpha}^{1/2}(x, \partial_x \tilde{\psi}_{\alpha}) = 0.
\]

We consider the bicharacteristics \((y(x_0, \xi_0, t), \eta(x_0, \xi_0, t))\) and let \((x, \xi) = (y, \eta)|_{t=t_x}\), while \(\partial_x \phi_{\alpha}(x, x_0, \omega) = \xi\). The generating function \(\phi_{\alpha}\) satisfies
\[
\frac{\partial}{\partial t} \phi_{\alpha}(y(t), x_0, \omega) = (\eta(x_0, \xi_0, t), \dot{y}(x_0, \xi_0, t)),
\]
and can then be written in the form
\[
\phi_{\alpha}(x, x_0, \omega) = \int_0^{t_x} (\eta(x_0, \xi_0, t), \dot{y}(x_0, \xi_0, t)) \, dt = \int_0^{t_x} \sum_{j=1}^{2 \eta} \dot{y}_j \, dy_j.
\]

We mention an alternative representation. For any \(\hat{\xi}_0 \in S^2\), we let \(\xi_0 = K_0 \hat{\xi}_0\), \(K_0(x_0, \omega, \hat{\xi}_0)\) be the solution of
\[
\Lambda_{\alpha}^{1/2}(x_0, K_0 \hat{\xi}_0) = \omega.
\]
This \(K_0\) is unique due to the monotonicity of eigenvalue in \(|\xi|\) with respect to the quadratic form associated with \(H_0\). We denote \(\eta(x_0, \xi_0, t) = K(x_0, \xi_0, t) \hat{\eta}\) with \(\hat{\eta} = \frac{\hat{\xi}_0}{|\hat{\xi}_0|} \in S^2\), and define the phase velocity, \(\mathfrak{V} = \mathfrak{V}(y, \omega, \hat{\eta})\), as
\[
(2.33) \quad \mathfrak{V} = \frac{\omega}{K}.
\]
Thus \(K(x_0, \xi_0, 0) = K_0\). Using (2.24) and (2.33) we find that
\[
(2.34) \quad \phi_{\alpha}(x, x_0, \omega) = \omega \int_0^{t_x} \sum_{j=1}^{2 \eta} \frac{\dot{y}_j}{\mathfrak{V}} \, dt,
\]
defined in a neighborhood of \(x_0\). Since \(\mathfrak{V}\) is frequency dependent, the geodesic distance is frequency dependent.
3. Characterization of the spectrum of $H_0$. In this section, we characterize the spectrum of $H_0$. In the case of constant coefficients, that is, a homogeneous medium, the spectrum was described in [22]. In the case of isotropic media, the spectrum was studied by Colin de Verdière [9]. We view

$$h_0(x, \hat{\xi}) = \frac{1}{|\xi|^2} H_0(x, \xi)$$

as a semiclassical pseudodifferential operator with $\frac{1}{|\xi|}$ as the semiclassical parameter, $h$ say. We write $\hat{\xi} = \frac{\xi}{|\xi|}$ and use $\zeta$ to denote the Fourier variable for $\frac{1}{|\xi|} \frac{\partial}{\partial Z}$. Then the principal symbol of $h_0(x, \hat{\xi})$ is given by

$$(3.1) \quad h_0(x, \hat{\xi})(Z, \zeta) = T\zeta^2 + (R + R^T)\zeta + Q,$$

where

$$(3.2) \quad T_{il}(x, Z) := C_{i33l}(x, Z),$$

$$R_{il}(x, \hat{\xi}, Z) := 2 \sum_{j=1}^2 C_{ij3l}(x, Z) \hat{\xi}_j \quad \text{and} \quad Q_{il}(x, \hat{\xi}, Z) := 2 \sum_{j,k=1}^2 C_{ijkl}(x, Z) \hat{\xi}_j \hat{\xi}_k.$$

We follow [38] to define so-called limiting velocities. Let $m$ and $n$ be orthogonal unit vectors in $\mathbb{R}^3$ which are obtained by rotating the orthogonal unit vectors $\hat{\xi}$ and $e_3$ around their vector product $\hat{\xi} \times e_3$ by an angle $\varrho$ ($-\pi \leq \varrho \leq \pi$):

$$m = m(\varrho) = \hat{\xi} \cos \varrho + e_3 \sin \varrho, \quad n = n(\varrho) = -\hat{\xi} \sin \varrho + e_3 \cos \varrho.$$

For any $v > 0$, we write

$$C_{ijkl}^v = C_{ijkl} - v^2 \hat{\xi}_j \hat{\xi}_k \delta_{il}.$$

We let

$$T(\varrho; v, x, \hat{\xi}, Z) = \sum_{j,k} C_{ijkl}^v(x, Z) m_j n_k,$$

$$R(\varrho; v, x, \hat{\xi}, Z) = \sum_{j,k} C_{ijkl}^v(x, Z) m_j n_k \quad \text{and} \quad Q(\varrho; v, x, \hat{\xi}, Z) = \sum_{j,k} C_{ijkl}^v(x, Z) m_j m_k$$

and note that

$$Q(\varrho + \frac{\pi}{2}; \cdot) = T(\varrho; \cdot), \quad R(\varrho + \frac{\pi}{2}; \cdot) = -R(\varrho; \cdot)^T, \quad T(\varrho + \frac{\pi}{2}; \cdot) = Q(\varrho; \cdot).$$

**Definition 3.1.** The limiting velocity $v_L = v_L(x, \hat{\xi}, Z)$ is the lowest velocity for which the matrices $Q(\varrho; v, x, \hat{\xi}, Z)$ and $T(\varrho; v, x, \hat{\xi}, Z)$ become singular for some angle $\varrho$:

$$v_L(x, \hat{\xi}, Z) = \inf \{ v > 0 : \exists \varrho : \det Q(\varrho; v, x, \hat{\xi}, Z) = 0 \} = \inf \{ v > 0 : \exists \varrho : \det T(\varrho; v, x, \hat{\xi}, Z) = 0 \}.$$

**Remark 3.1.** In the isotropic case, that is,

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$
where $\lambda, \mu$ are the two Lamé moduli, we have

$$C_{ijkl} = \hat{\lambda} \delta_{ij} \delta_{kl} + \hat{\mu} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}),$$

with $\hat{\mu} = \frac{\mu}{p}$ and $\hat{\lambda} = \frac{\lambda}{p}$. The limiting velocity is given by

$$v_L(x, \hat{\xi}, Z) = \sqrt{\hat{\mu}(x, Z)}.$$

The spectrum of $H_0(x, \xi)$ is composed of a discrete spectrum contained in $(0, v_L^2(x, \hat{\xi}, Z_I)|\xi|^2)$ and an essential spectrum, $[v_L^2(x, \hat{\xi}, Z_I)|\xi|^2, \infty)$ (Figure 1):

![Figure 1. Characterization of the spectrum](image)

**Theorem 3.2.** Under Assumption 2.1, the spectrum of $H_0(x, \xi)$ is discrete below $v_L^2(x, \hat{\xi}, Z_I)|\xi|^2$. The discrete spectrum is nonempty if $|\xi|$ is sufficiently large.

**Proof.** We need some notation and a few facts from operator theory for the proof of this theorem. A brief review of those prerequisites are provided in Appendix B. For more details, we refer to [35]. We adopt the classical Dirichlet-Neumann bracketing proof. Here we use $'$ to represent the partial derivative with respect to $Z$. Consider $h_0(x, \hat{\xi})$, and the corresponding quadratic form,

$$(3.3) \quad h_0(\varphi, \psi) = \int_{-\infty}^{0} \sum_{i,l=1}^{3} \left( h^2 C_{ijkl} \hat{\varphi}_i \hat{\varphi}_l - i h \sum_{j=1}^{2} C_{ijkl} \hat{\varphi}_j \hat{\varphi}_l + \sum_{k=1}^{2} \xi_k C_{ijkl} \varphi_i \varphi_l \right) dZ$$

on $H^1(\mathbb{R}^-)$, which is positive. We decompose $[Z_I, 0]$ into $m$ intervals $[Z_I, 0] = \cup_{p=1}^{m} I_q$, the interiors, $(I_q)^\circ$, of which are disjoint. We let $C^{q,+}_{ijkl}$ be a minimal element in the set,

$$\Pi^{q,+} = \{C_{ijkl}: C_{ijkl} \succeq C_{ijkl}(x, Z) \text{ for all } Z \in I_q\}.$$  

(The order is defined as $C^1_{ijkl} \succeq C^2_{ijkl}$ if $\sum_{i,j,k,l=1}^{3} (C^1_{ijkl} - C^2_{ijkl}) \epsilon_{ijkl} \geq 0$ for any symmetric matrix $\epsilon$. This is then a partially ordered set, and a minimal element exists by Zorn’s lemma.) Similarly, we define $C^{q,-}_{ijkl}$ to be a maximal element of

$$\Pi^{q,-} = \{C_{ijkl}: C_{ijkl} \preceq C_{ijkl}(x, Z) \text{ for all } Z \in I_q\}.$$  

Furthermore, we introduce $C^{m+1,\pm}_{ijkl} = C_{ijkl}(x, Z_I)$ on $I_{m+1} = (-\infty, Z_I]$. 

We consider

$$h_0^{-q}(\varphi, \psi) = \int I_q \sum_{i,j,l=1}^{3} \left( h^2 C_{ijkl}^{-q} \varphi_i \psi_l - i h \sum_{j=1}^{2} C_{ijkl}^{-q} \xi_j \varphi_i \psi_l - i h \sum_{k=1}^{2} \xi_k C^{+q}_{ijkl} \varphi_i \psi_l \right)$$

for \( q = 1, 2, \cdots, m + 1 \), and let \( h_0^{-q}(x, \xi) \) be the unique self-adjoint operator on \( L^2(I_q) \) associated to the quadratic form \( h_0^{-q}(\cdot, \cdot) \) (cf. Appendix B). This is equivalent to defining

\[
(h_0^{-q}(x, \xi))_{il} = -h^2 \frac{\partial}{\partial Z} C_{ijkl}^{-q} - i h \sum_{j=1}^{2} C_{ijkl}^{-q} \xi_j \frac{\partial}{\partial Z} \xi_k - \sum_{k=1}^{2} C_{ijkl}^{+q} \xi_k
\]

on \( \{ u \in H^2(I_q) : u \text{ with Neumann boundary condition} \} \). Then

\[ h_0(x, \xi) = \bigoplus_{q=1}^{m+1} h_0^{-q}(x, \xi). \]

It follows that for each \( q = 1, \cdots, m \), \( h_0^{-q}(x, \xi) \) has only a discrete spectrum, since it is a self-adjoint second-order elliptic differential operator on a bounded interval.

The spectrum of \( h_0^{-m+1}(x, \xi) \) is also discrete below \( v_1^2(x, \xi, Z_I) \) [35, Theorems 3.12, 3.13]. We note here that this property is related to the existence of surface waves for a homogeneous half space. Thus the spectrum of \( h_0(x, \xi) \) below \( v_1^2(x, \xi, Z_I) \) must be discrete (see the Lemma on pp. 270, Vol. IV of [35]).

We define \( h_0^{+q}(x, \xi) \) as the unique self-adjoint operator on \( L^2(I_q) \), the quadratic form of which is the closure of the form

\[
h_0^{+q}(\varphi, \psi) = \int I_q \sum_{i,j,l=1}^{3} \left( h^2 C_{ijkl}^{+q} \varphi_i \psi_l - i h \sum_{j=1}^{2} C_{ijkl}^{+q} \xi_j \varphi_i \psi_l - i h \sum_{k=1}^{2} \xi_k C_{ijkl}^{+q} \varphi_i \psi_l \right)
\]

with domain \( C_0^\infty(I_q) \). This is equivalent to defining

\[
(h_0^{+q}(x, \xi))_{il} = -h^2 \frac{\partial}{\partial Z} C_{ijkl}^{+q} - i h \sum_{j=1}^{2} C_{ijkl}^{+q} \xi_j \frac{\partial}{\partial Z} \xi_k - \sum_{k=1}^{2} C_{ijkl}^{+q} \xi_k
\]

on \( H_0^1(I_q) \cap H^2(I_q) \). Then

\[ h_0(x, \xi) \leq \bigoplus_{q=1}^{m+1} h_0^{+q}(x, \xi). \]
$h_0^{+,q}(x,\hat{\xi})$, for each $j = 1, \cdots, m$, has a discrete and real spectrum. The operator $h_0^{+,m+1}(x,\hat{\xi})$ also has a discrete spectrum below $v_L(x,\hat{\xi},Z_1)$, since $h_0^{+,m+1}(x,\hat{\xi}) \geq h_0^{+,m+1}(x,\hat{\xi})$.

We suppose now that the decomposition is fine enough, and that $C^{+q}_{ijkl}$ for some $q$ has a limiting velocity $v_L^{+,q}(x,\hat{\xi}) < v_L(x,\hat{\xi},Z_1)$. (There exists $Z^* \in (Z_1, 0]$ such that $v_L(x,\hat{\xi},Z^*) < v_L(x,\hat{\xi},Z_1)$.) Suppose $Z^* \in I_q$ for some $q$. For any $\varepsilon > 0$, if $|I_q|$ is small enough, $C_{ijkl}(x,Z^*) + \varepsilon(\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj}) \geq C_{ijkl}(x,Z), \forall Z \in I_q$. This show nonemptiness of $\Pi^{t+q}$. We can take $\varepsilon$ small enough, so that $C^{+q}_{ijkl} = C_{ijkl}(x,Z^*) + \varepsilon(\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{kl} + \delta_{il}\delta_{kj})$ has limiting velocity $v_L^{+,q}(x,\hat{\xi}) < v_L(x,\hat{\xi},Z_1)$.

We let $\lambda_{q,1}$ be the first eigenvalue of $h_0^{+,q}$, and claim that $\lambda_{q,1} < (v_L^{+,q}(x,\hat{\xi}))^2 + \mathcal{O}(h)$. We prove this claim next.

We let $I_q = [a, b]$. We note that

$$h_0^{+,q}u = ((v_L^{+,q}(x,\hat{\xi}))^2 + h)u$$

has a bounded solution, $u = ae^{\frac{h}{2}z}$ with $a \in \mathbb{C}^3$, $p \in \mathbb{R}$. To construct a solution of this form, we carry out a substitution and obtain

$$(3.6) \quad [T^{+q}p^2 + (R^{+q} + (R^{+q})^T)p + Q^{+q} - v_L^{+,q}(x,\hat{\xi})^2 - h]a = 0.$$  

Here, $T^{+q}, R^{+q}, Q^{+q}$ are defined as in $[32]$ for $C^{+q}_{ijkl}$. The above system has a non-trivial solution if and only if

$$(3.7) \quad \det[T^{+q}p^2 + (R^{+q} + (R^{+q})^T)p + Q^{+q} - v_L^{+,q}(x,\hat{\xi})^2 - h] = 0.$$  

Since $v_L^{+,q}(x,\hat{\xi}))^2 + h > v_L^{+,q}(x,\hat{\xi}))^2$, there exists a real-valued solution $p$ to $[37]$ (38), Lemma 3.2).

We let $v = u$ on $[a+h, b-h]$ and $v \in C^\infty_0(I_q)$; $v$ can be constructed such that $\|v\|L^2([a,a+h] \cup [b-h, b]) \leq C h^{1/2}$, $\|v\|L^2([a,a+h] \cup [b-h, b]) \leq C h^{1/2}$, $\|v\|L^2([a,a+h] \cup [b-h, b]) \leq C h^{-3/2}$. Then $C_1 < \|v\| < C_2$ for some $C_1, C_2 > 0$ independent of $h$. After renormalization, we can assume that $\|v\|_2 = 1$. Then

$$(3.8) \quad (h_0^{+,q}v, v) \leq \int_{a+h}^{a} h_0^{+,q}v \cdot \bar{v}dZ + \left( \int_{a}^{a+h} + \int_{b-h}^{b} \right) h_0^{+,q}v \cdot \bar{v}dZ$$

$$\leq (v_L^2(x,\hat{\xi},Z^*) + h)\|v\|L^2([a+h, b-h])^2 + Ch^2\|v\|L^2([a,a+h] \cup [b-h, b])\|v\|L^2([a,a+h] \cup [b-h, b])$$

$$+ Ch\|v\|L^2([a,a+h] \cup [b-h, b])\|v\|L^2([a,a+h] \cup [b-h, b]) + C\|v\|L^2([a,a+h] \cup [b-h, b])\|v\|L^2([a,a+h] \cup [b-h, b])$$

$$\leq (v_L^2(x,\hat{\xi},Z^*) + h)(1 - Ch) + Ch.$$  

When $h$ is small, that is, $|\xi|$ is large, $h_0^{+,q}$ has eigenvalues below $v_L^2(x,\hat{\xi},Z_1)$. Then the discrete spectrum of $h_0(x,\hat{\xi})$ is nonempty.

**Theorem 3.3.** The essential spectrum of $H_0(x,\hat{\xi})$ is given by $[v_L^2(x,\hat{\xi},Z_1)|\xi|^2, +\infty)$. 

**Proof.** We let $H_L(x,\xi)$ be the operator given by

$$(3.9) \quad (H_L(x,\xi))_{ij} = -h^2 \frac{\partial}{\partial Z} C_{ijkl}(x,Z_1) \frac{\partial}{\partial Z} \xi_j - ih \sum_{j=1}^{2} C_{ijkl}(x,Z_1) \xi_j \frac{\partial}{\partial Z} \xi_k - ih \sum_{k=1}^{2} C_{ijkl}(x,Z_1) \frac{\partial}{\partial Z} \xi_k$$

$$- ih \sum_{k=1}^{2} \left( \frac{\partial}{\partial Z} C_{ijkl}(x,Z_1) \right) \xi_k + \sum_{j,k=1}^{2} C_{ijkl}(x,Z_1) \xi_j \xi_k.$$
while any bounded subset of $L$ essential spectrum \cite[Corollary 3 in XIII.4]{35}.

{L maps of the eigenfunctions with eigenvalues above exponentially decays “immediately” from the surface. This is somewhat different from the behavior of $H$, proof, we use the simplified notation, $T$ existence of such an eigenvalue. (3.11) $\det[Tp^2 + (R + R^T)p + Q - v^2|a = 0$. The above system has a non-trivial solution if and only if

(3.11) $\det[Tp^2 + (R + R^T)p + Q - v^2] = 0$.

When $v^2 > v_L^2(x, \xi, Z_I)$, there exists a real-valued solution $p$ to (3.11) \cite[Lemma 3.2]{38}.

We take $u_k = a e^{i\frac{\xi \cdot k}{\eta}} \phi_k(Z)$, with $\phi_k \in C_0^\infty(\mathbb{R}^-)$, supp $\phi_k(Z) \subset [1, k]$, $\phi_k(Z) = 1$ on $[2, k - 1]$ and $|\partial^\alpha \phi_k| \leq C, |\alpha| \leq 2$. Then $\|u_k\|_2 \geq Ck$ for some constant $C > 0$. We note that $u_k \in \mathcal{D}$. With $v_k = \frac{u_k}{\|u_k\|}$ we find that

$$\|H_L(x, \xi)v_k - v^2|\xi|^2v_k\| \leq \frac{C}{k} \rightarrow 0.$$

Thus $v^2|\xi|^2$ is in the spectrum of $H_0(x, \xi)$.

This shows that any $\Lambda > v_L^2(x, \xi, Z_I)|\xi|^2$ is in the essential spectrum of $H_L(x, \xi)$. Then, by Theorem 3.22 $[v_L^2(x, \xi, Z_I)|\xi|^2, \infty)$ is the essential spectrum of $H_0(x, \xi)$. \[\]

We emphasize that, here, the essential spectrum is not necessarily a continuous spectrum. It may contain eigenvalues.

4. Surface-wave modes associated with exponentially decaying eigenfunctions.

When the limiting velocity is minimal at the surface, that is,

(4.1) $v_L(x, \xi, 0) = \inf_{Z \leq 0} v_L(x, \xi, Z),$

there may still exist an eigenvalue of $H_0(x, \xi)$ below $v_L^2(x, \xi, 0)|\xi|^2$. The corresponding eigenfunction exponentially decays “immediately” from the surface. This is somewhat different from the behavior of the eigenfunctions with eigenvalues above $v_L^2(x, \xi, 0)|\xi|^2$ (See Figure 2). Here, we discuss the existence of such an eigenvalue.

For the principal symbol (cf. 3.11) of $h_0(x, \xi)$, for $0 < v^2 < v_L^2(x, \xi, 0)$, we have

$$\det[T\zeta^2 + (R + R^T)\zeta + Q - v^2] = 0,$$
and viewed as a polynomial in $\zeta$ has 6 non-real roots that appear in conjugate pairs. Suppose that $\zeta_1, \zeta_2, \zeta_3$ are three roots with negative imaginary parts, and $\gamma \subset \mathbb{C}^-$ is a continuous curve enclosing $\zeta_1, \zeta_2, \zeta_3$. We let

$$S_1(x, Z, v, \hat{\xi}) = T^{-1/2}S_1(x, Z, v, \hat{\xi})T^{1/2},$$

where

$$\tilde{S}_1(x, Z, v, \hat{\xi}) := \left( \oint_{\gamma} \zeta \tilde{M}(x, Z, v, \hat{\xi}, \zeta)^{-1}d\zeta \right) \left( \oint_{\gamma} \tilde{M}(x, Z, v, \hat{\xi}, \zeta)^{-1}d\zeta \right)^{-1},$$

and

$$\tilde{M}(x, Z, v, \hat{\xi}, \zeta) = T^{-1/2}[T\zeta^2 + (R + R^T)\zeta + Q - v^2I]T^{-1/2}.$$  

Then we have [20, 38]

$$T\zeta^2 + (R + R^T)\zeta + Q - v^2I = (\zeta - \tilde{S}_1^*(x, Z, v, \hat{\xi}))T(\zeta - S_1(x, Z, v, \hat{\xi})).$$

with $\text{Spec}(S_1) \subset \mathbb{C}^-$, for $0 < v^2 < v_L^2(x, \hat{\xi}, 0)$.

We define an operator,

$$l_0(x, \xi, v)(Z, hD_Z) = (hD_Z - S_1)T(hD_Z - S_1).$$

A solution to $l_0(x, \xi, v)(Z, hD_Z)\varphi = 0$ in $H^2(\mathbb{R}^-)$ needs to satisfy

$$\varphi = 0.$$

This follows from letting $f = T(hD_Z - S_1)\varphi$, and noting that $(hD_Z - S_1)f = 0$. If $(hD_Z - S_1)\varphi = T^{-1}f(Z_0) \neq 0$ for some $Z_0 \in \mathbb{R}^-$, then

$$f(Z) = f(Z_0)e^{\frac{S_1^*((z - Z_0))}{\kappa}}$$

is exponentially growing as $Z \to -\infty$, since $\text{Spec}(S_1^*) \subset \mathbb{C}^+$. This contradicts the fact that $\varphi \in H^2(\mathbb{R}^-)$.

We now introduce the surface impedance tensor,

$$Z(x, v, \hat{\xi}) = -i (T(0, 0, \hat{\xi})S_1(x, 0, v, \hat{\xi}) + R^T(x, 0, \hat{\xi})).$$

The basic properties of this tensor are summarized in

**Proposition 4.1** [20]. For $0 \leq v < v_L(x, \hat{\xi}, 0)$ the following holds true.
The above Lyapunov-Sylvester equation has the unique solution

\[ Z(x, v, \hat{\xi}) \] is Hermitian.

(ii) \( Z(x, 0, \hat{\xi}) \) is positive definite.

(iii) The real part of \( Z(x, v, \hat{\xi}) \) is positive definite.

(iv) At most one eigenvalue of \( Z(x, v, \xi) \) is non-positive.

(v) The derivative, \( \frac{dZ(x,v,\hat{\xi})}{dv} \), is negative definite.

(vi) The limit \( \lim_{v \uparrow v_L(x,\xi,0)} Z(x,v,\hat{\xi}) \exists \).

To make this paper more self-contained, we include a proof of the above proposition. The properties of \( Z \) can be studied also by Stroh’s formalism and are important for the existence of subsonic Rayleigh wave in a homogeneous half-space; see \( [38] \) for a comprehensive overview.

**Proof.** We use the Riccati equation

\[ (4.7) \quad (Z(x,v,\hat{\xi}) - iR(x,\hat{\xi},0))T(x,0,\hat{\xi})^{-1}(Z(x,v,\hat{\xi}) + iR(x,\hat{\xi},0)^T) = Q(x,0,\hat{\xi}) - v^2I \]

as in \([27,20]\). We note that

\[ S_1(x,0,v,\hat{\xi}) = iT(x,0,\hat{\xi})^{-1}(Z(x,v,\hat{\xi}) + iR^T). \]

Taking the adjoint of (4.7), we find that

\[ (4.8) \quad (Z(x,v,\hat{\xi})^* - iR(x,\hat{\xi},0))T(x,0,\hat{\xi})^{-1}(Z(x,v,\hat{\xi})^* + iR(x,\hat{\xi},0)^T) = Q(x,0,\xi) - v^2I. \]

Subtracting (4.8) from (4.7), we obtain

\[ S_1(x,0,v,\hat{\xi})^*(Z(x,v,\hat{\xi}) - Z(x,v,\hat{\xi})^*) - (Z(x,v,\hat{\xi}) - Z(x,v,\hat{\xi})^*)S_1(x,0,v,\hat{\xi}) = 0. \]

The above Sylvester equation is nonsingular, because \( S_1(x,0,v,\xi) \) and \( S_1(x,0,v,\xi)^* \) have disjoint spectra. Therefore

\[ Z(x,v,\hat{\xi}) = Z(x,v,\hat{\xi})^*. \]

This proves (i).

Differentiating (4.7) in \( v \), we get

\[ iS_1(x,0,v,\hat{\xi})^* \frac{d}{dv}Z(x,v,\hat{\xi}) - i \frac{d}{dv}Z(x,v,\hat{\xi})S_1(x,0,v,\hat{\xi}) = -2vI. \]

The above Lyapunov-Sylvester equation has the unique solution

\[ \frac{d}{dv}Z(x,v,\hat{\xi}) = -2v \int_0^\infty \exp(-iS_1(x,0,v,\hat{\xi})^*) \exp(iS_1(x,0,v,\hat{\xi})) dt. \]

It is clear that \( \frac{d}{dv}Z(x,v,\hat{\xi}) \) is negative definite since \( S_1(x,0,v,\xi) \subset \mathbb{C}_- \). This proves (v).

In order to prove (ii), we assume that \( w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \) is a complex vector such that

\[ w^T Z(x,0,\hat{\xi}) w \leq 0. \]

Let

\[ u(x,\hat{\xi},Z) = e^{iZS_1(x,0,0,\hat{\xi})} w, \quad \text{on} \ Z \in (-\infty,0]. \]

Then

\[ -\left( hT(x,0,\hat{\xi}) \frac{\partial}{\partial Z} + iR(x,0,\hat{\xi})^T \right) u|_{Z=0} = Z(x,0,\hat{\xi}) w. \]
We have the energy identity

\[
\int_{-\infty}^{0} \sum_{i,l=1}^{3} \left( h^2 C_{i3j}(x,0)\hat{u}^*_i\hat{u}^*_j - i h \sum_{j=1}^{2} C_{i3j}(x,0)\hat{\xi}_j\hat{u}^*_j - i h \sum_{k=1}^{2} \xi_k C_{i3kl}(x,0)\hat{u}^*_l \hat{u}^*_k \right)
+ \sum_{j,k=1}^{2} C_{ijkl}(x,0)\hat{\xi}_j\hat{\xi}_k\hat{u}^*_l\hat{u}^*_k \right)dZ = w^T Z(x,0,\hat{\xi})w \leq 0.
\]

By positivity of the differential operator, \( u = 0 \). Thus \( w = 0 \), and we have proved (ii).

With (1.2), (3.3) and (4.0), we obtain

\[
iZ(x,v,\hat{\xi}) \left( \oint_{\gamma_R} M(x,0,v,\hat{\xi},\zeta)^{-1}d\zeta \right) = \oint_{\gamma_R} (\zeta T(x,0,\hat{\xi}) + R^T(x,0,\hat{\xi})) M(x,0,v,\hat{\xi},\zeta)^{-1}d\zeta,
\]

where \( \gamma_R \) is the closed contour consisting of \([-R,R] \) on the real line and the arc \( Re^{i\theta}, \theta \in [0,\pi], \) for \( R \) large enough. We note that

\[
T(x,0,\hat{\xi})M(x,0,v,\hat{\xi},\zeta)^{-1} = \zeta^{-2}I + O(|\zeta|^{-3}).
\]

We then have, by letting \( R \to \infty \),

\[
iZ(x,v,\hat{\xi}) \left( \int_{-\infty}^{+\infty} M(x,0,v,\hat{\xi},\zeta)^{-1}d\zeta \right) = i\pi I + \int_{-\infty}^{+\infty} (\zeta T(x,0,\hat{\xi}) + R^T(x,0,\hat{\xi})) M(x,0,v,\hat{\xi},\zeta)^{-1}d\zeta.
\]

The integral on the right-hand side is a real-valued matrix, and

\[
\int_{-\infty}^{+\infty} M(x,0,v,\hat{\xi},\zeta)^{-1}d\zeta
\]

is a positive definite matrix. Thus \( \text{Re} Z(x,v,\hat{\xi}) \) is positive definite. This proves (iii).

We prove (iv) by contradiction: Assume that \( Z(x,v,\hat{\xi}) \) has only one positive eigenvalue for some \( v \). The corresponding eigenspace would be one-dimensional. We can then choose a real-valued vector \( w \neq 0 \) orthogonal to the eigenspace such that \( w^T Z(x,v,\hat{\xi})w \leq 0 \). This contradicts the fact that \( \text{Re} Z(x,v,\hat{\xi}) \) is positive definite, and proves (iv).

Finally, we prove (vi). Let \( \| Z(x,v,\hat{\xi}) \| \) be the operator norm of \( Z(x,v,\hat{\xi}) \). Then \( \| Z(x,v,\hat{\xi}) \| \) is equal to the maximum of the absolute values of eigenvalues of \( Z(x,v,\hat{\xi}) \). By (v), we have

\[
Z(x,v,\hat{\xi}) \preceq Z(x,0,\hat{\xi}).
\]

Here \( A \preceq B \), with Hermitian matrices \( A \) and \( B \), means

\[
w^T(A - B)w \leq 0
\]

for any complex vector \( w \). Thus the eigenvalues of \( Z(x,v,\hat{\xi}) \) are not greater than \( \| Z(x,0,\hat{\xi}) \| \) for \( 0 \leq v \leq v_L(x,\hat{\xi},0) \). The sum of eigenvalues of \( Z(x,v,\hat{\xi}) \) is positive, since it is equal to trace(\( Z(x,v,\hat{\xi}) \)), which is in turn equal to trace(\( \text{Re} Z(x,v,\hat{\xi}) \)). Therefore, the possibly negative eigenvalue of \( Z(x,v,\hat{\xi}) \) is less than the sum of the two positive eigenvalues. Then

\[
\| Z(x,v,\hat{\xi}) \| \leq 2\| Z(x,0,\hat{\xi}) \|
\]

and we can take the limit in (vi). \( \square \)

We now introduce the generalized Barnett-Lothe condition [28]: \( (x,\xi) \) satisfies the generalized Barnett-Lothe condition if

\[
\lim_{v \uparrow v_L(x,\xi,0)} \det Z(x,v,\hat{\xi}) < 0,
\]
or
\[
\lim_{v \to v_L(x, \xi, 0)} \left[ (\text{trace} Z(x, v, \hat{\xi}))^2 - \text{trace} Z^2(x, v, \hat{\xi}) \right] < 0.
\]

We note that in the isotropic case, this condition is satisfied for all \((x, \hat{\xi})\).

We let \(\varphi\) be a solution of (1.3). Then the Neumann boundary value of \(\varphi\) is given by
\[
i (-hT(x, 0)D_Z - R(x, 0, \hat{\xi})^T)\varphi_0(0) = -i (T(x, 0)S_1(x, 0, v, \hat{\xi}) + R(x, 0, \hat{\xi})^T)\varphi_0(0)
= Z(x, v, \hat{\xi})\varphi_0(0).
\]

The generalized Barnett-Lothe condition guarantees that \(Z(x, v_L(x, \hat{\xi}, 0), \hat{\xi})\) has one negative eigenvalue. Therefore, with properties of \(Z(x, v, \hat{\xi})\) established in Proposition 4.1, we have

**Proposition 4.2.** There is a unique \(v_0\) with \(0 \leq v_0^2 < v_L^2(x, \hat{\xi}, 0)\) such that \(\det Z(x, v_0, \hat{\xi}) = 0\), if the generalized Barnett-Lothe condition holds.

At the \(v_0\) given in the proposition above, (4.3) has a solution \(\varphi_0\) that satisfies Neumann boundary condition, that is, \(\varphi_0 \in \mathcal{D}\). We further assume \(\|\varphi_0\|_2 = 1\). We emphasize, here, that the existence of such a \(v_0\) depends on the stiffness tensor at boundary only. This locality behavior was first observed by Petrovsky [34].

We find that
\[
(h_0(x, \hat{\xi})(Z, hD_Z) - v_0^2 - l_0(x, \hat{\xi}, v_0)(Z, hD_Z))\varphi_0 = h(D_Z(TS_1 + R^T))\varphi_0
\]
by straightforward calculations, noting that
\[
h_0(x, \hat{\xi})(Z, hD_Z)\varphi_0 = h^2D_Z(TD_Z\varphi_0) + h(R(D_Z\varphi_0 + D_Z(R^T\varphi_0)) + Q\varphi_0,
\]
\[
S_1^*TS_1 = Q - v_0^2
\]
and
\[
-S_1^*T - TS_1 = R + R^T.
\]
Then
\[
h_0(x, \hat{\xi})(\varphi_0, \varphi_0) = (h_0(x, \hat{\xi})(Z, hD_Z)\varphi_0, \varphi_0) \leq v_0^2 + Ch,
\]
since \(\varphi_0\) satisfies the Neumann boundary condition. Thus the first eigenvalue \(v_1^2(x, \xi)\) of \(h_0(x, \xi)(Z, hD_Z)\) satisfies
\[
v_1^2(x, \xi) = \min_{\|\varphi\|=1} h_0(x, \hat{\xi})(\varphi, \varphi) \leq v_0^2 + Ch.
\]

Therefore,

**Theorem 4.3.** Assume that (4.1) holds. If the generalized Barnett-Lothe condition holds, and \(|\xi|\) is sufficiently large, then there exists an eigenvalue \(\Lambda_1 < v_1^2(x, \xi, 0)|\xi|^2\) of \(H_0(x, \xi)\).

This particular eigenvalue may exist even when \(C_{ijkl}(x, \cdot)\) is constant (independent of \(x\)). It was first found by Rayleigh himself for the isotropic case. The uniqueness of such subsonic eigenvalue can be guaranteed if we further assume that \(C_{ijkl}(x, Z)\) is non-increasing as \(Z\) with respect to the order defined in the proof of Theorem 4.2.
5. Decoupling into Rayleigh and Love waves.

5.1. Isotropic case. For the case of isotropy,

\[ H_0 u = \begin{pmatrix} -\frac{\partial}{\partial Z}(\mu(x, Z)\frac{\partial}{\partial Z}) & 0 & 0 \\ 0 & -\frac{\partial}{\partial Z}(\mu(x, Z)\frac{\partial}{\partial Z}) & 0 \\ 0 & 0 & -\frac{\partial}{\partial Z}((\lambda + 2\mu)(x, Z)\frac{\partial}{\partial Z}) \end{pmatrix} \]

\[ -1 \begin{pmatrix} 0 & 0 & (\hat{\lambda}(x, Z)\xi_1 \frac{\partial}{\partial Z}) \\ 0 & 0 & (\hat{\mu}(x, Z)\xi_1 \frac{\partial}{\partial Z}) \\ \hat{\lambda}(x, Z)\xi_1 \frac{\partial}{\partial Z} & \hat{\mu}(x, Z)\xi_2 \frac{\partial}{\partial Z} & 0 \end{pmatrix} \]

\[ -i \begin{pmatrix} 0 & 0 & \left(\frac{\partial}{\partial Z}\hat{\mu}(x, Z)\xi_1\right) \\ 0 & 0 & \left(\frac{\partial}{\partial Z}\hat{\lambda}(x, Z)\xi_2\right) \\ (\hat{\lambda} + 2\hat{\mu})\xi_2^2 + \hat{\mu}\xi_2^2 & (\hat{\lambda} + 2\hat{\mu})\xi_1^2 + \hat{\mu}\xi_1^2 & 0 \end{pmatrix} \]

\[ \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \]

We introduce an orthogonal matrix

\[ P(\xi) = \begin{pmatrix} |\xi|^{-1}\xi_1 & |\xi|^{-1}\xi_2 & 0 \\ |\xi|^{-1}\xi_1 & -|\xi|^{-1}\xi_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

Using the substitution \( u = P(\xi)\varphi \), with \( \varphi = (\varphi_1, \varphi_2, \varphi_3) \), we obtain the equations for the eigenfunctions,

\[ -\frac{\partial}{\partial Z}\hat{\mu}\frac{\partial \varphi_2}{\partial Z} + \hat{\mu}|\xi|^2\varphi_2 = \Lambda\varphi_2, \]

\[ \frac{\partial \varphi_2}{\partial Z}(0) = 0, \]

for Love waves, and

\[ -\frac{\partial}{\partial Z}\hat{\mu}\frac{\partial \varphi_1}{\partial Z} - i|\xi|\left(\frac{\partial}{\partial Z}(\hat{\mu}\varphi_1) + \hat{\lambda}\frac{\partial}{\partial Z}\varphi_3\right) + (\hat{\lambda} + 2\hat{\mu})|\xi|^2\varphi_1 = \Lambda\varphi_1, \]

\[ -\frac{\partial}{\partial Z}(\hat{\lambda} + 2\hat{\mu})\frac{\partial \varphi_3}{\partial Z} - i|\xi|\left(\frac{\partial}{\partial Z}(\hat{\lambda}\varphi_1) + \hat{\mu}\frac{\partial}{\partial Z}\varphi_3\right) + \hat{\mu}|\xi|^2\varphi_3 = \Lambda\varphi_3, \]

\[ i|\xi|\varphi_3(0) + \frac{\partial \varphi_3}{\partial Z}(0) = 0, \]

\[ i\hat{\lambda}|\xi|\varphi_1(0) + (\hat{\lambda} + 2\hat{\mu})\frac{\partial \varphi_3}{\partial Z}(0) = 0, \]

for Rayleigh waves. So we have two types of surface-wave modes, Love and Rayleigh waves, decoupled up to principal parts in an isotropic medium. The lower-order terms can be constructed following the proof of Theorem 2.1 leading to coupling.
5.2. Transversely isotropic case. We consider a transversely isotropic medium having the surface normal direction as symmetry axis. Then the nonzero components of $C$ are

$$C_{1111}, C_{2222}, C_{3333}, C_{1122}, C_{1133}, C_{2233}, C_{2323}, C_{1313}, C_{1212}$$

and

$$C_{1111} = C_{2222}, \quad C_{1133} = C_{2233},$$

$$C_{2323} = C_{1313}, \quad C_{1212} = \frac{1}{2}(C_{1111} - C_{1122}).$$

Using the substitution $u = P(\xi)\varphi$, again, we obtain the equations for the eigenfunctions from $P^{-1}(\xi)H_0(x, \xi)P(\xi)\varphi = \Lambda\varphi$:

$$-\frac{\partial}{\partial Z}C_{1313}\frac{\partial \varphi_2}{\partial Z} + C_{1212}\frac{\partial \varphi_2}{\partial Z} = \frac{\partial \varphi_2}{\partial Z}(0) = 0,$$

for Love waves, and

$$-\frac{\partial}{\partial Z}C_{1313}\frac{\partial \varphi_1}{\partial Z} \pm i|\xi|\left(\frac{\partial}{\partial Z}(C_{1313}\varphi_3) + C_{1333}\frac{\partial \varphi_3}{\partial Z}\right) + C_{1111}|\xi|^2\varphi_1 = \Lambda\varphi_1,$$

for Rayleigh waves.

5.3. Directional decoupling. We assume that $\xi = (|\xi|, 0, 0)$, that is, the phase direction of propagation is $x_1$, and assume that the medium is monoclinic with symmetry plane $(x_1, Z)$. Then $C_{ijkl}$ is determined by 13 nonzero components:

$$C_{1111}, C_{1122}, C_{1133}, C_{2222}, C_{2233}, C_{3333}, C_{1223}, C_{1323}, C_{1212}, C_{1113}, C_{1333}, C_{1313}, C_{2213}.$$

The symmetry plane, $(x_1, Z)$, is called the sagittal plane. Then

$$H_0(x, \xi)u = \begin{pmatrix}
-\frac{\partial}{\partial Z}C_{1313}\frac{\partial \varphi_2}{\partial Z} & 0 & -\frac{\partial}{\partial Z}C_{1333}\frac{\partial \varphi_2}{\partial Z} \\
-\frac{\partial}{\partial Z}C_{2323}\frac{\partial \varphi_2}{\partial Z} & -\frac{\partial}{\partial Z}C_{2233}\frac{\partial \varphi_2}{\partial Z} & 0 \\
-\frac{\partial}{\partial Z}C_{1313}\frac{\partial \varphi_2}{\partial Z} & 0 & -\frac{\partial}{\partial Z}C_{1333}\frac{\partial \varphi_2}{\partial Z}
\end{pmatrix}$$

$$-i|\xi| \begin{pmatrix}
C_{1113}\frac{\partial \varphi_1}{\partial Z} & 0 & C_{1133}\frac{\partial \varphi_1}{\partial Z} \\
0 & C_{1223}\frac{\partial \varphi_1}{\partial Z} & 0 \\
C_{1313}\frac{\partial \varphi_1}{\partial Z} & 0 & C_{1333}\frac{\partial \varphi_1}{\partial Z}
\end{pmatrix} - i|\xi| \begin{pmatrix}
\frac{\partial}{\partial Z}(C_{1113}) & 0 & \frac{\partial}{\partial Z}(C_{1313}) \\
0 & \frac{\partial}{\partial Z}(C_{1113}) & 0 \\
\frac{\partial}{\partial Z}(C_{1313}) & 0 & \frac{\partial}{\partial Z}(C_{1333})
\end{pmatrix}$$

$$-i|\xi| \begin{pmatrix}
C_{1113}\frac{\partial \varphi_3}{\partial Z} & 0 & C_{1133}\frac{\partial \varphi_3}{\partial Z} \\
0 & C_{1223}\frac{\partial \varphi_3}{\partial Z} & 0 \\
C_{1313}\frac{\partial \varphi_3}{\partial Z} & 0 & C_{1333}\frac{\partial \varphi_3}{\partial Z}
\end{pmatrix} + |\xi|^2 \begin{pmatrix}
C_{1111} & 0 & C_{1113} \\
0 & C_{1212} & 0 \\
C_{1113} & 0 & C_{1313}
\end{pmatrix} \begin{pmatrix}
u_1 \\
u_2 \\
u_3\end{pmatrix}.$$
We obtain the equations for the eigenfunctions:

\[(5.14)\]

\[-\frac{\partial}{\partial Z} C_{2323} \frac{\partial}{\partial Z} u_2 - i C_{1223} |\xi| \frac{\partial}{\partial Z} u_2 - i |\xi| \frac{\partial}{\partial Z} (C_{1223} u_2) + |\xi|^2 C_{1212} u_2 = \Lambda u_2,\]

\[(5.15)\]

\[C_{2323} \frac{\partial u_2}{\partial Z}(0) - i |\xi| C_{1223} u_2(0) = 0\]

for Love waves, and

\[(5.16)\]

\[-\frac{\partial}{\partial Z} C_{1313} \frac{\partial}{\partial Z} u_1 - \frac{\partial}{\partial Z} C_{1333} \frac{\partial}{\partial Z} u_3 - i |\xi| C_{1113} \frac{\partial}{\partial Z} u_1 - i |\xi| C_{1333} \frac{\partial}{\partial Z} u_3

i |\xi| \frac{\partial}{\partial Z} (C_{1113} u_1) - i |\xi| \frac{\partial}{\partial Z} (C_{1333} u_3) + |\xi|^2 C_{1111} u_1 + |\xi|^2 C_{1333} u_3 = \Lambda u_1\]

\[(5.17)\]

\[-\frac{\partial}{\partial Z} C_{1333} \frac{\partial}{\partial Z} u_1 - \frac{\partial}{\partial Z} C_{3333} \frac{\partial}{\partial Z} u_3 - i |\xi| C_{1313} \frac{\partial}{\partial Z} u_1 - i |\xi| C_{3333} \frac{\partial}{\partial Z} u_3

i |\xi| \frac{\partial}{\partial Z} (C_{1313} u_1) - i |\xi| \frac{\partial}{\partial Z} (C_{3333} u_3) + |\xi|^2 C_{1111} u_1 + |\xi|^2 C_{1333} u_3 = \Lambda u_3\]

\[(5.18)\]

\[C_{1333} \frac{\partial u_1}{\partial Z}(0) + i |\xi| C_{1113} u_1(0) + C_{1333} \frac{\partial u_3}{\partial Z}(0) + i |\xi| C_{1333} u_3(0) = 0,\]

\[(5.19)\]

\[C_{1333} \frac{\partial u_1}{\partial Z}(0) + i |\xi| C_{1313} u_1(0) + C_{3333} \frac{\partial u_3}{\partial Z}(0) + i |\xi| C_{3333} u_3(0) = 0,\]

for Sloping-Rayleigh waves. We observe that the surface-wave modes decouple into Love waves and Sloping-Rayleigh waves when the direction of propagation is in the sagittal plane.

For “global” decoupling, decoupling takes place in every direction of \(\xi\). Thus transverse isotropy is the minimum symmetry to obtain global decoupling; for a further discussion, see \[11\]. For more generally anisotropic media, decoupling is no longer possible. Then there only exist “generalized” surface-wave modes propagating with elliptical particle motion in three dimensions \[14, 15\]. Again, only in particular directions of sagittal symmetry, however, the Sloping-Rayleigh and Love waves separate \[10\] up to the leading order term. This loss of polarization can be used to recognize anisotropy of the medium.

6. Weyl’s laws for surface waves. Weyl’s law was first established for an eigenvalue problems for the Laplace operator,

\[
\begin{align*}
-\Delta u &= \lambda u & \text{in } \Omega, \\
\quad u &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

for \(\Omega\) a two- or three-dimensional bounded domain. One defines the counting function, \(N(b) := \#\{\lambda_n \leq b\}\), where \(\lambda_n\) are the eigenvalues arranged in a non-decreasing order with counting multiplicity. Weyl \[11, 12, 13\] proved the following behaviors,

\[(6.1)\]

\[N(b) = \frac{\mid \Omega \mid}{4\pi^2} b + o(b), \quad b \to \infty,\]

for \(\Omega \subset \mathbb{R}^2\), and

\[(6.2)\]

\[N(b) = \frac{\mid \Omega \mid}{6\pi^2} b^{3/2} + o(b^{3/2}), \quad b \to \infty,\]

for \(\Omega \subset \mathbb{R}^3\). Formulae of these type are referred to as Weyl’s laws. For a rectangular domain, one can easily find explicitly the eigenvalues and verify the formulae. These imply that the leading order asymptotics of the number of eigenvalues is determined by the area/volume of the domain only. Extensive work has been done generalizing this result and deriving expressions for the remainder term. We refer to \[12\] for a comprehensive overview. Here, we present Weyl’s laws for surface waves.
6.1. Isotropic case.

**Love waves.** First, we consider the equations (5.2) and (5.3) for Love waves in isotropic media. We use the notation \( C_S^2(x,Z) := \hat{\mu}(x,Z) \). We have the following Hamiltonian,

\[
H_0(x,\xi)v := -\frac{\partial}{\partial Z} \left( C_S^2 \frac{\partial}{\partial Z} v \right) + C_S^2 |\xi|^2 v,
\]

where \( H_0(x,\xi) \) is an ordinary differential operator in \( Z \), with domain

\[
D = \{ v \in H^2(\mathbb{R}^-) : \frac{dv}{dZ}(0) = 0 \}.
\]

We make use of results in [11]. Although in [11], the boundary condition is of Dirichlet type, the spectral property is essentially the same. A straightforward extension of the argument in [19] suffices:

\[
\sigma(H_0(x,\xi)) = \sigma_p(H_0(x,\xi)) \cup \sigma_c(H_0(x,\xi)),
\]

where the point spectrum \( \sigma_p(H_0(x,\xi)) \), consists of a finite set of eigenvalues in

\[
\left( \inf_{Z \leq 0} C_S(x,Z)^2|\xi|^2, C_S(x,Z_1)^2|\xi|^2 \right)
\]

and the continuous spectrum is given by

\[
\sigma_c(H_0(x,\xi)) = [C_S(x,Z_1)^2|\xi|^2, \infty).
\]

Since the problem at hand is scalar and one-dimensional, the eigenvalues are simple. We note that, here, the essential spectrum is purely the continuous spectrum.

We write \( N(x,\xi,E) = \# \{ \Lambda_\alpha(x,\xi) \leq E |\xi|^2 \} \), where \( \Lambda_\alpha(x,\xi) \) is an eigenvalue of \( H_0(x,\xi) \). Weyl’s law gives a quantitative asymptotic approximation of \( N(x,\xi,E) \) in terms of \( |\xi| \) (Figure 3).

**Theorem 6.1** (Weyl’s law for Love waves). For any \( E < C_S^2(x,Z_1) \), we have

\[
N(x,\xi,E) = \frac{|\xi|}{2\pi} \left( \left| \{(Z,\zeta) : C_S^2(x,Z)(1 + \zeta^2) \leq E \} \right| + o(1) \right),
\]

where \( | \cdot | \) denotes the measure of the set.

The classical Dirichlet-Neumann bracketing technique could be used to prove this theorem [34]. However, we can also adapt the proof of Theorem 6.2 below.

We observe that \( \Lambda_\alpha(x,\xi) \) is a smooth function of \( (x,\xi) \in T^\ast \mathbb{R}^2 \) for all \( 1 \leq \alpha \leq N(x,\xi,E) \). Moreover, the corresponding eigenfunctions, \( \varphi_\alpha(x,\xi,Z) \), decay exponentially in \( Z \) for large \( Z \). More precisely, let \( Z_b = \sup \{ Z : C_S^2(x,Z) = b \} \), then for all \( \Lambda_\alpha < b \), the corresponding \( \varphi_\alpha \) are exponentially decaying from \( Z = Z_b \) to \( -\infty \). Hence, the energy of \( \varphi_\alpha \) is concentrated near a neighborhood of \( [Z_b,0] \). As \( b \to Z_0(x) \), then the energy of \( \varphi_\alpha \) with \( \Lambda_\alpha < b \) becomes more and more concentrated near the boundary. Thus those low-lying eigenvalues correspond to surface waves.

**Remark 6.1.** For Love waves in a monoclinic medium described by equations (5.14) – (5.15), Weyl’s law reads: For any \( E < C_{2121} - \frac{4C_{2233}^2}{C_{2323}} \), we have

\[
N(x,\xi,E) = \frac{|\xi|}{2\pi} \left( \left| \{(Z,\zeta) : C_{2323}^2 + 2C_{1223}\zeta + C_{1212} \leq E \} \right| + o(1) \right).
\]
Rayleigh waves. Here, we establish Weyl’s law for Rayleigh waves, cf. [5,6], [17]. Now,

\[
H_0(x, \xi) \left( \begin{array}{c} \varphi_1 \\ \varphi_3 \end{array} \right) = \left( \begin{array}{cc} -\frac{\partial}{\partial Z} (\hat{\mu} \frac{\partial \varphi_3}{\partial Z}) - i|\xi| \left( \frac{\partial}{\partial Z} (\hat{\mu} \varphi_3) + \hat{\lambda} \frac{\partial \varphi_3}{\partial Z} \right) + (\hat{\lambda} + 2\hat{\mu})|\xi|^2 \varphi_1 \right) \\
-\frac{\partial}{\partial Z} \left( (\hat{\lambda} + 2\hat{\mu}) \frac{\partial \varphi_3}{\partial Z} \right) - i|\xi| \left( \frac{\partial}{\partial Z} (\hat{\mu} \varphi_1) + \hat{\lambda} \frac{\partial \varphi_3}{\partial Z} \right) + \hat{\mu}|\xi|^2 \varphi_1 \end{array} \right).
\]

We use \( \zeta \) to denote the Fourier variable for \( \frac{1}{|\xi|} \frac{\partial}{\partial Z} \). For fixed \((x, \xi)\), we view \( H_0(x, \xi) \) as a semiclassical pseudodifferential operator with \( \frac{1}{|\xi|} \) as semiclassical parameter, \( h \), as before. We denote \( h_0(x, \xi) = h^2 H_0(x, \xi) \). Then the principal symbol for \( h_0(x, \xi) \) is

\[
h_0(x, \xi)(Z, \zeta) = \left[ \begin{array}{cc} \hat{\mu} \zeta^2 & 0 \\
0 & (\hat{\lambda} + 2\hat{\mu}) \zeta^2 \end{array} \right] + \left[ \begin{array}{cc} 0 & (\hat{\lambda} + \hat{\mu}) \zeta \\
(\hat{\lambda} + \hat{\mu}) \zeta & 0 \end{array} \right] + \left[ \begin{array}{cc} \hat{\lambda} + 2\hat{\mu} & 0 \\
0 & \hat{\mu} \end{array} \right].
\]

There exists \( \Pi(\zeta) \), namely,

\[
\Pi(\zeta) = \left[ \begin{array}{cc} 1 & -\zeta \\
\zeta & 1 \end{array} \right],
\]

such that

\[
h_0(x, \xi)(Z, \zeta) \Pi = \Pi \left[ \begin{array}{cc} (\hat{\lambda} + 2\hat{\mu})(1 + \zeta^2) & 0 \\
0 & \hat{\mu}(1 + \zeta^2) \end{array} \right].
\]

We write \( C^2_B(x, Z) = (\hat{\lambda} + 2\hat{\mu})(x, Z) \). We obtain

**Theorem 6.2 (Weyl’s law for Rayleigh waves).** For any \( E < \hat{\mu}(x, Z_I) \), define

\[
N(x, \xi, E) = \# \{ \Lambda_{\alpha} : \Lambda_{\alpha} \leq E|\xi|^2 \},
\]

where \( \Lambda_{\alpha} \) are the eigenvalues of \( H_0(x, \xi) \). Then

\[
(6.7) \quad N(x, \xi, E) = \frac{|\xi|}{2\pi} \left( \left| \{ (Z, \zeta) : C^2_B(x, Z)(1 + \zeta^2) \leq E \} \right| + \left| \{ (Z, \zeta) : C^2_B(x, Z)(1 + \zeta^2) \leq E \} \right| + o(1) \right).
\]
Proof. Let $\chi_1(Z)$, $\chi_2(Z) \in C^\infty(\mathbb{R})$ be real-valued, non-negative functions, such that $\chi_1(Z) = 0$ for $Z \geq -\frac{1}{2}$, $\chi_1(Z) = 1$ for $Z \leq -\delta$, and $\chi_1^2(Z) + \chi_2^2(Z) = 1$ for some $\delta > 0$. For any non-negative $f \in C_c^\infty(\mathbb{R})$, $supp f \subset (-\infty, \mu(x, Z))$, $f(h^2H_0)$ is a trace class operator; we have

$$\text{trace } f(h^2H_0) = \text{trace } (\chi_1^2 f(h^2H_0)) + \text{trace } (\chi_2^2 f(h^2H_0)).$$

We analyze the two terms successively. We note that

$$\text{trace } (\chi_1^2 f(h^2H_0)) = \text{trace } (\chi_1 f(h^2H_0)\chi_1),$$

and that $\chi_1 f(h^2H_0)\chi_1$ is a pseudodifferential operator on $\mathbb{R}$ with principal symbol,

$$\chi_1(Z)f(h_0(Z, \zeta))\chi_1(Z').$$

Revisiting the diagonalization of $h_0$ above, let $\Pi$ be the pseudodifferential operator with symbol

$$\frac{1}{\sqrt{1 + \zeta^2}}\Pi(\zeta);$$

it follows immediately that $\Pi$ is unitary on $L^2(\mathbb{R})$. Then

$$\chi_1 f(h^2H_0)\chi_1 = \chi_1 \Pi f(D_0)\Pi\chi_1 + O(h) = \Pi\chi_1 f(D_0)\chi_1^* + O(h),$$

with $D_0$ being the pseudodifferential operator with symbol

$$\left[ \begin{array}{cc} (\hat{\lambda} + 2\hat{\mu})(1 + \zeta^2) & 0 \\ 0 & \hat{\mu}(1 + \zeta^2) \end{array} \right].$$

Let $D_P$ and $D_S$ be the pseudodifferential operators with symbols $(\hat{\lambda} + 2\hat{\mu})(1 + \zeta^2)$ and $\hat{\mu}(1 + \zeta^2)$, respectively. Based on the discussion at the end of Appendix [B]

$$\text{(6.8) } \text{trace } (\chi_1 f(h^2H_0)\chi_1) = \text{trace}(\chi_1 f(D_0)\chi_1) + O(h)
= \text{trace}(\chi_1 f(D_S)\chi_1) + \text{trace}(\chi_1 f(D_P)\chi_1) + O(h)
= \frac{1}{2\pi\hbar} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{0} \chi_1(Z)^2 \left[ f(\hat{\mu}(Z)(1 + \zeta^2)) + f((\hat{\lambda} + 2\hat{\mu})(Z)(1 + \zeta^2)) \right] dZd\zeta + O(h) \right).$$

We then observe that

$$0 \leq \text{trace } (\chi_2^2 f(h^2H_0)) \leq \text{trace } f(h^2H_0, \delta),$$

where $H_{0,\delta}$ is the operator $H_0$ restricted to $[-\delta, 0]$. To show this, we choose the eigenvalues, $\mu_j$, and associated unit eigenvectors, $e_j$, of $h^2H_{0,\delta}$ subject to the Neumann boundary condition. Then $\{e_j\}_{j=1}^\infty$ forms an orthonormal basis of $L^2([-\delta, 0])$. We also consider an arbitrary orthonormal basis $\{f_j\}_{j=1}^\infty$ for $L^2([0, \infty])$. Clearly, $\{e_j, f_j\}_{i,j=1}^\infty$ form an orthonormal basis for $L^2(\mathbb{R})$. Therefore,

$$\text{(6.9) } \text{trace } (\chi_2^2 f(h^2H_0)) = \sum_{j=1}^\infty \langle \chi_2^2 f(h^2H_0)e_j, e_j \rangle + \sum_{j=1}^\infty \langle \chi_2^2 f(h^2H_0)f_j, f_j \rangle
= \sum_{j=1}^\infty \langle \chi_2^2 f(h^2H_0)e_j, e_j \rangle = \sum_{j=1}^\infty \langle \chi_2^2 f(\mu_j)e_j, e_j \rangle \leq \sum_{j=1}^\infty \langle f(\mu_j)e_j, e_j \rangle = \text{trace } f(h^2H_{0,\delta}).$$
For every small \( \epsilon > 0 \), we can take \( f(u) \leq 1_{[-\epsilon, E+\epsilon]}(u) \), where \( 1_{[-\epsilon, E+\epsilon]}(u) \) is the indicator function of \([-\epsilon, E+\epsilon]\), that is,

\[
1_{[-\epsilon, E+\epsilon]}(u) = \begin{cases} 
1, & \text{if } u \in [-\epsilon, E+\epsilon] \\
0, & \text{elsewhere.}
\end{cases}
\]

Then

\[
\text{trace } f(h^2 H_{0,\delta}) \leq \# \{E(h) \mid E(h) \leq E + \epsilon, \ E(h) \text{ is an eigenvalue of } h^2 H_{0,\delta}\} \leq C\delta \frac{1}{h}.
\]

(6.10)

To show this, we rescale \( z = \frac{z}{h} \). Then for \( \epsilon < 1 \),

\[
\# \{E(h) : E(h) \leq E + \epsilon, E(h) \text{ is an eigenvalue of } h^2 H_{0,\delta}\} \leq \# \{\lambda \mid \lambda \leq E + 1, \lambda \text{ is an eigenvalue of } A\},
\]

where

\[
A = \begin{pmatrix}
-\frac{\partial}{\partial z}(\hat{\mu} \frac{\partial}{\partial z}) + (\hat{\lambda} + 2\hat{\mu}) & -i\left(\frac{\partial}{\partial z}(\hat{\mu}) + \hat{\lambda} \frac{\partial}{\partial z}\right) \\
-i\left(\frac{\partial}{\partial z}(\hat{\lambda}) + \hat{\mu} \frac{\partial}{\partial z}\right) & -\frac{\partial}{\partial z}(\hat{\lambda} + 2\hat{\mu}) + \hat{\mu}
\end{pmatrix}
\]

(6.12)

on \([-\frac{\delta}{h}, 0]\) with the Neumann boundary condition applied. Without loss of generality, we can assume that \( \frac{\delta}{h} \) is an integer. Then we divide the interval \([-\frac{\delta}{h}, 0]\) into \( \frac{\delta}{h} \) intervals of the same length, 1, and let \( A_j \) be operator \( A \) restricted to \([-j, -j+1]\), with the Neumann boundary condition applied. Then, as in the proof of Theorem 3.2,

\[A \geq \bigoplus_{i=1}^{\frac{\delta}{h}} A_j.
\]

The number of eigenvalues of each \( A_j \) below \( E+1 \) can be bounded by a constant since the corresponding quadratic forms have a uniform lower bound. Hence, we obtain (6.11).

Finally, we combine (6.8) with (6.10), let \( \delta \to 0 \), and note that \( \chi_1 \to 1_{[0, \infty)} \), so that

\[
\text{trace } (f(h^2 H_0)) = \frac{1}{2\pi h} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[f(\hat{\mu}(Z)(1 + \zeta^2)) + f((\hat{\lambda} + 2\hat{\mu})(Z)(1 + \zeta^2))\right] dZ d\zeta + O(h) \right).
\]

Using a technique similar to the one used in the proof of [45] Theorem 14.11, we construct \( f^*_1, f^*_2 \in C_c^{\infty}(\mathbb{R}) \) by regularizing \( 1_{[0, E]} \), with \( f^*_1, f^*_2 \leq 1_{[-\epsilon, E+\epsilon]} \), such that

\[f^*_1(u) \leq 1_{[0, E]}(u) \leq f^*_2(u).
\]

More precisely, we may take

\[J(u) = \begin{cases} 
k \exp[-1/(1 - |u|^2)] & \text{if } |u| < 1 \\
0 & \text{if } |u| \geq 1,
\end{cases}
\]

where \( k \) is chosen so that \( \int_{\mathbb{R}} J(u) du = 1 \). Let \( J_\epsilon(u) = \frac{1}{k} J(\frac{u}{\epsilon}) \), and

\[f^*_1 = J_\epsilon * 1_{[-\epsilon, E-\epsilon]},
\]

\[f^*_2 = J_\epsilon * 1_{[-\epsilon, E+\epsilon]}.
\]

While observing that \( f^*_1, f^*_2 \to 1_{[0, E]} \), and using the estimates above, taking \( \epsilon \to 0 \) completes the proof. \( \square \)
6.2. Anisotropic case. We extend Weyl’s law from isotropic to anisotropic media. The proof for Rayleigh waves can be naturally adapted. We write the eigenvalue $s$ of the symmetric matrix-valued symbol defined in (3.1) as $C_1(x, \xi, Z, \zeta)$, $C_2(x, \xi, Z, \zeta)$ and $C_3(x, \xi, Z, \zeta)$.

Theorem 6.3. Assume the three eigenvalues $C_i(x, \xi, Z, \zeta)$, $i = 1, 2, 3$ are smooth in $(Z, \zeta)$, then for any $E < v_L(x, \xi, Z_I)|\xi|^2$, let

$$N(x, \xi, E) = \# \{ \Lambda_\alpha : \Lambda_\alpha(x, \xi) \leq E|\xi|^2 \}.$$

Then

$$N(x, \xi, E) = \frac{|\xi|}{2\pi} [ |\{(Z, \zeta) : C_1 \leq E\}| + |\{(Z, \zeta) : C_2 \leq E\}| + |\{(Z, \zeta) : C_3 \leq E\}| + o(1)].$$

7. Surface waves as normal modes. In this section, we identify surface wave modes with normal modes. We consider the Earth as a unit ball $B_1$. There is a global diffeomorphism, $\phi$, with

$$\phi : B_1 \setminus \{0\} \to S^2 \times \mathbb{R}^-,$$

$$\phi(B_r) = S^2 \times \left\{ \frac{1 - \frac{1}{r} }{r} \right\}, \ r \neq 0,$$

where $S^2$ is the unit sphere centered at the origin. For an open and bounded subset $U \subset S^2$, the cone region, $\{(\Theta, r) : \Theta \in U, \ 0 < r < 1\}$, is diffeomorphic to $U \times \mathbb{R}^-$; hence, we can find global coordinates for $U$ and we may consider our system on the domain $S^2 \times \mathbb{R}^-$. More generally, we consider the system on any Riemannian manifold of the form $M = \partial M \times \mathbb{R}^-$ with metric

$$g = \begin{pmatrix} g' & 0 & 0 \\ 0 & 1 \end{pmatrix}.$$
We represent \( \hat{H} \) on \( M \) in local coordinates, similarly as in (2.4). Now the displacement \( u_l \) is a covariant vector,

\[
(7.1) \quad \hat{H}_l u_l = - \frac{\partial}{\partial Z} C_i^{33l}(x, Z) \frac{\partial}{\partial Z} u_l - \epsilon \sum_{j=1}^{2} C_i^{33l}(x, Z) \frac{\partial}{\partial Z} u_j - \epsilon \sum_{k=1}^{2} \left( \frac{\partial}{\partial Z} C_i^{33l}(x, Z) \right) \nabla_k u_l - \epsilon \sum_{k=1}^{2} \left( \nabla_j C_i^{jkl}(x, Z) \right) \frac{\partial}{\partial Z} u_l - \epsilon \sum_{j,k=1}^{2} C_j^{jkl}(x, Z) \nabla_j \nabla_k u_l - \epsilon \sum_{j=1}^{2} \left( \nabla_j C_i^{jkl}(x, Z) \right) \nabla_k u_l.
\]

Here, \( \nabla_k \) is the covariant derivative associated with \( x_k \). Normal modes can be viewed as solutions to the eigenvalue problem for

\[
(7.2) \quad \hat{H}_l u_l = \omega^2 u_l,
\]

with Neumann boundary condition. In fact, (7.2) is asymptotically equivalent to

\[
(7.3) \quad a_{\alpha, \epsilon}(\cdot, \epsilon \nabla) \Psi(x) = \omega^2 \Psi(x),
\]

where \( a_{\alpha, \epsilon} \) is the pseudodifferential operator on \( \partial M \) defined in Theorem 2.1. By the estimates, \( \Lambda_\alpha(x, \xi) \geq C|\xi|^2 \), it follows that \( a_{\alpha, \epsilon}(x, \xi) \) is elliptic implying that

\[
\|a_{\alpha, \epsilon}(\cdot, \epsilon \nabla) \Psi\|_{L^2(\partial M)} \geq C\|\Psi\|_{H^2(\partial M)}. \]

Thus \( a_{\alpha, \epsilon}(\cdot, \epsilon \nabla)^{-1} \) is compact, and hence the spectrum of \( a_{\alpha, \epsilon}(\cdot, \epsilon \nabla) \) is discrete.

Now, let \( \omega \) be an eigenfrequency, then we construct asymptotic solutions of (7.3) of the form

\[
(7.4) \quad \Psi = \sum_{k=0}^{\infty} \epsilon^k B_k(x) e^{i\frac{\omega x}{\epsilon}}.
\]

Inserting this expression into (7.3), from the expansion, we find that

\[
\Lambda_\alpha(x, \partial \psi) = \omega^2
\]

and

\[
(7.5) \quad LB_0 = 0
\]

\[
(7.6) \quad LB_k = F(\psi, B_0, \cdots, B_{k-1}).
\]

Here, \( L \) and \( F \) are defined similar to those defined in Section 2.4. Thus

\[
u = \text{Op}_\epsilon(\Phi_{\alpha, \epsilon})(\Psi)
\]

will be an asymptotic solution of (7.2).

**Remark 7.1 (Radial manifolds).** Under the assumption of transverse isotropy and lateral homogeneity, that is, fixing an \( x \), and using the normal coordinates at \( x \), we have (2.3)-(2.4).
with $C_{ijkl}(x, Z) = C_{ijkl}(Z)$. Then we can construct asymptotic modes. Now $|\xi|^2$ corresponds to $-\Delta'_g = -\epsilon^2 \Delta'_g$, where $\Delta'_g$ is the Laplacian on $\partial M$. Then we consider the eigenvalue problem:

(7.7) \[ -\frac{\partial}{\partial Z} C_{1313} \frac{\partial}{\partial Z} u_2(x, Z) - C_{1212} \Delta'_g u_2(x, Z) = \omega^2 u_2(x, Z) \]
and

(7.8) \[ -\frac{\partial}{\partial Z} C_{1313} \frac{\partial}{\partial Z} u_1(x, Z) - i \left( \frac{\partial}{\partial Z} C_{1313} + C_{1133} \frac{\partial}{\partial Z} \right) \sqrt{-\Delta'_g} u_3(x, Z) - C_{1111} \Delta'_g u_1(x, Z) = \omega^2 u_1(x, Z) \]

(7.9) \[ -\frac{\partial}{\partial Z} C_{3333} \frac{\partial}{\partial Z} u_3(x, Z) - i \left( \frac{\partial}{\partial Z} C_{1313} + C_{1133} \frac{\partial}{\partial Z} \right) \sqrt{-\Delta'_g} u_1(x, Z) - C_{1313} \Delta'_g u_3(x, Z) = \omega^2 u_3(x, Z) \]

Then, if we have the eigenvalues and eigenfunctions of $-\Delta'_g$,

$$-\Delta'_g \Theta^l_n(x) = k^2 \Theta^l_n(x),$$

and solutions $\varphi(Z, k)$ for the system (5.8)-(5.13) with $|\xi| = k$, we find

$$u(k, x, Z) = \varphi(Z, k) \Theta^l_n(x)$$
as the solutions to (7.7)-(7.9). In spherically symmetric models of the earth, $\partial M = S^2$ and $\Theta^l_n(x)$ are the spherical harmonics.

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**Appendix A. Semiclassical pseudodifferential operators.**

Here, we give a summary of the basic definition and properties of semiclassical pseudodifferential operators which are used in the main text. Let $A(\cdot, \cdot) : T^*\mathbb{R}^n \to C^{m \times m}$ be a symbol that is smooth in $(x, \xi)$. We say that $A \in S(k)$, with $k \in \mathbb{Z}$, if

$$\forall \alpha, \beta \in \mathbb{N}^n, \quad |D_x^{\alpha} D_{\xi}^{\beta} A(x, \xi)| \leq C_{\alpha, \beta} \langle \xi \rangle^k,$$

with $\langle \xi \rangle = \sqrt{1 + |\xi|^2}$.

Let $A_j \in S(k)$ for $j = 0, 1, \cdots$, and $\epsilon \in (0, \epsilon_0)$ for some small $\epsilon_0 > 0$. One says that $A \in S(k)$ is asymptotic to $\sum_{j=0}^{\infty} \epsilon^j A_j$ and writes

$$A \sim \sum_{j=0}^{\infty} \epsilon^j A_j \quad \text{in} \quad S_k$$
if for any $N = 1, 2, \cdots$

$$|D^{\alpha} (A - \sum_{j=0}^{N-1} \epsilon^j A_j)| \leq C_{\alpha, N} \epsilon^N \langle \xi \rangle^k, \quad \text{for any} \alpha = 0, 1, 2, \cdots.$$
We refer to $A_0$ as the principal symbol of $A$. A semiclassical pseudodifferential operator associated with $A$ is defined as follows.

**Definition A.1.** Suppose that $A(x, \xi)$ is a symbol. We define the semiclassical pseudodifferential operator,

$$\text{Op}_\epsilon(A)w(x) = A(., \epsilon D)w(x) = \frac{1}{(2\pi \epsilon)^n} \int \int A(x, \xi) e^{i\langle \xi, x-y \rangle \epsilon} w(y) dyd\xi,$$

for any $w : \mathbb{R}^n \to \mathbb{C}^m$, which is compactly supported.

We have the following mapping property: For any $u \in H^s$,

$$\|A(x, \epsilon D)u\|_{H^{s+k}} \leq C_s \epsilon^k \|u\|_{H^s},$$

for some constant $C_s > 0$. Here, $s$ is an arbitrary real number, and $H^s$ denotes the $L^2$ Sobolev space in $\mathbb{R}^n$ with exponent $s$.

If $A(x, \xi)$ and $B(x, \xi)$ are two symbols, then

$$\text{Op}_\epsilon(A) \text{Op}_\epsilon(B) = \text{Op}_\epsilon(C)$$

with

$$C(x, \xi) \sim \sum_{\alpha \geq 0} \frac{|\alpha|^{|\alpha|}}{\alpha!} D_{\xi}^\alpha A(x, \xi) D_x^\alpha B(x, \xi).$$

We use the notation $\circ$ for the composition of two symbols: $C = A \circ B$.

**Definition A.2.** We call a family of functions (distributions) $u = \{u_\epsilon\}_{0 < \epsilon \leq \epsilon_0}$ admissible if there exist constants $k$ and $N \geq 0$ such that

$$\|\chi u_\epsilon\|_{H^k} = O(\epsilon^{-N}),$$

for any $\chi \in C_c^\infty(\mathbb{R}^n)$. We denote by $\mathcal{A}(\mathbb{R}^n)$ the space of such families.

**Definition A.3.** The wavefront set of $u_\epsilon$, denoted by $WF(u_\epsilon)$, of an admissible family $u = \{u_\epsilon\}_{0 < \epsilon \leq \epsilon_0}$ is the closed subset of $T^*\mathbb{R}^n$ which is defined as

$$(x_0, \xi_0) \notin WF(u_\epsilon)$$

if and only if $\exists \chi \in C_c^\infty(\mathbb{R}^n)$, $\chi(x_0) \neq 0$, such that

$$\mathcal{F}_\epsilon(\chi u_\epsilon)(\xi) = O(\epsilon^\infty)$$

for $\xi$ close to $\xi_0$. Here $\mathcal{F}_\epsilon$ is the semiclassical Fourier transform:

$$\mathcal{F}_\epsilon u_\epsilon(\xi) = \frac{1}{(2\pi \epsilon)^{n/2}} \int_{\mathbb{R}^n} e^{-i\langle \xi, x \rangle / \epsilon} u_\epsilon(x) dx.$$

**Definition A.4.** Let $U$ be an open set in $T^*\mathbb{R}^n$. The space of microfunctions, $\mathcal{M}(U)$, in $U$ is the quotient

$$\mathcal{M}(U) = \mathcal{A}(\mathbb{R}^n)/\{u_\epsilon : WF(u_\epsilon) \cap U = \emptyset\}.$$
Let \( p(x, \xi) \) be a symbol such that
\[
\nabla p \neq 0 \quad \text{on} \quad \{ p = 0 \}.
\]

Let \( (x, \xi) \) be the solution to the Hamilton system,
\[
\frac{\partial x_k(y, \eta, t)}{\partial t} = \frac{\partial p(x, \xi)}{\partial \xi_k}, \quad \frac{\partial \xi_k(y, \eta, t)}{\partial t} = -\frac{\partial p(x, \xi)}{\partial x_k}, \quad (x, \xi)\big|_{t=0} = (y, \eta).
\]

We denote by \( \exp(tH_p) \) the map such that \( \exp(tH_p)(y, \eta) = (x, \xi) \). The map \( \exp(tH_p) \) is called the Hamiltonian flow of \( p \). For each \( u_\epsilon \in A(\mathbb{R}^n) \) solving
\[
p(x, \epsilon D)u_\epsilon = f_\epsilon,
\]
where \( \{f_\epsilon\}_{0<\epsilon\leq\epsilon_0} \subset L^2(\mathbb{R}^n) \), we have \( WF(u_\epsilon) \setminus WF(f_\epsilon) \) is invariant under the Hamiltonian flow of \( p \).

**Appendix B. Operator theory.**

**Definition B.1.** Let \( \mathcal{H} \) be a Hilbert space, \( D \) a dense subspace of \( \mathcal{H} \), and \( A \) be an unbounded linear operator defined on \( D \).
1. The adjoint \( A^* \) of \( A \) is the operator whose domain is \( D^* \), where
\[
D^* = \{ v \in H : |\langle Au, v \rangle| \leq C(v)\|u\| \text{ for all } u \in D \},
\]

and
\[
\langle A^* v, u \rangle = \langle v, Au \rangle.
\]

2. \( A \) is called self-adjoint if \( D^* = D \) and \( A^* = A \).
3. \( A \) is called symmetric if \( D \subset D^* \) and \( Au = A^* u \) for all \( u \in D \).

**Definition B.2.**
1. Let \( A \) be an unbounded self-adjoint operator densely defined on \( \mathcal{H} \), with domain \( D \). The spectrum of \( A \) is
\[
\sigma(A) = \mathbb{R} \setminus \{ \lambda \in \mathbb{R} : (A - \lambda)^{-1} : \mathcal{H} \to \mathcal{H} \text{ is bounded} \}.
\]

2. The set of all \( \lambda \) for which \( A - \lambda \) is injective and has dense range, but is not surjective, is called the continuous spectrum of \( A \), denoted by \( \sigma_c(A) \).
3. If there exists a \( u \in D \) satisfying that \( Au = \lambda u \), \( \lambda \) is called an eigenvalue of \( A \). The set of all eigenvalues is called the point spectrum of \( A \), denoted by \( \sigma_p(A) \).
4. The discrete spectrum \( \sigma_{\text{disc}}(A) \) of \( A \) is the set of eigenvalues of \( A \) that have finite dimensional eigenspaces.
5. The essential spectrum \( \sigma_{\text{ess}}(A) \) of \( A \) is \( \sigma(A) \setminus \sigma_{\text{disc}}(A) \).

**Theorem B.3.** A number \( \Lambda \) is in the essential spectrum of \( A \) if and only if there exists a sequence \( \{u_k\} \) in \( \mathcal{H} \) (called singular sequence or Weyl sequence) such that
1. \( \lim_{k \to \infty} \langle u_k, v \rangle = 0 \), for all \( v \in H \);
2. \( \|u_k\| = 1 \);
3. \( u_k \in D \);
4. \( \lim_{k \to \infty} \| (A - \lambda) u_k \| = 0. \)

**Quadratic form.** A quadratic form is a map \( q : Q(q) \times Q(q) \to \mathbb{C} \), where \( Q(q) \) is a dense linear subset of Hilbert space \( \mathcal{H} \), such that \( q(\cdot, \psi) \) is linear and \( q(\varphi, \cdot) \) is skew-linear. If \( q(\varphi, \psi) = q(\psi, \varphi) \) we say that \( q \) is symmetric.

The map \( q \) is called semi-bounded if \( q(\varphi, \varphi) \geq -M\| \varphi \|^2 \) for some \( M > 0 \). A semi-bounded quadratic form \( q \) is called closed if \( Q(q) \) is complete under the norm

\[
\| \psi \|_1 = \sqrt{q(\psi, \psi) + (M + 1)\| \psi \|^2}.
\]

If \( q \) is a closed semi-bounded quadratic form, then \( q \) is the quadratic form of a unique self-adjoint operator \( A \), such that \( q(\varphi, \psi) = (A\varphi, \psi) \), for any \( \varphi, \psi \in D(A) \). Conversely, for any self-adjoint operator \( A \) on \( \mathcal{H} \), there exists a corresponding quadratic form \( q \). Then we denote \( Q(A) = Q(q) \). If \( A \) is merely symmetric, let \( q(\varphi, \psi) = (A\varphi, \psi) \) for any \( \varphi, \psi \in D(A) \). We can complete \( D(A) \) under the inner product

\[
(\varphi, \psi)_1 = q(\varphi, \psi) + (\varphi, \psi)
\]
to obtain a Hilbert space \( \mathcal{H}_{+1} \subset \mathcal{H} \). Then \( q \) extends to a closed quadratic form \( \mathcal{Q} \) on \( \mathcal{H}_{+1} \) (This is known as Friedrichs extension). We call \( \mathcal{Q} \) the closure of \( q \).

Let \( A_1, A_2 \) be two self-adjoint operators on Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \), with domains \( D_1(A_1) \) and \( D_2(A_2) \). Then denote \( A_1 \oplus A_2 \) to be the operator \( A \) on \( \mathcal{H}_1 \oplus \mathcal{H}_2 \), with domain \( D(A) = \{ (\varphi, \psi) | \varphi \in D(A_1), \phi \in D(A_2) \} \) with \( A(\varphi, \psi) = (A_1\varphi, A_2\phi) \). Now assume that \( A_1 \) and \( A_2 \) are nonnegative, and \( \mathcal{H}_2 \subset \mathcal{H}_1 \). We write \( A_1 \leq A_2 \) if and only if

1. \( Q(A_1) \supset Q(A_2) \).
2. For any \( \psi \in Q(A_2) \), \( (A_1\psi, \psi) \leq (A_2\psi, \psi) \).

**Lemma B.4.** If \( A_1 \) and \( A_2 \) are two self-adjoint operators, \( 0 \leq A_1 \leq A_2 \), then

\[
\lambda_n(A_1) \leq \lambda_n(A_2),
\]

where \( \lambda_n(A) \) denotes the \( n \)-th eigenvalue of \( A \), counted with multiplicity.

**Trace class.** Let \( T : \mathcal{H} \to \mathcal{H} \) be a compact operator, then \( T^*T : \mathcal{H} \to \mathcal{H} \) is a selfadjoint semidefinite compact operator, and hence it has a discrete spectrum,

\[
s_0(T)^2 \geq s_1(T)^2 \geq \cdots \geq s_k(T)^2 \to 0.
\]

We say that the nonnegative \( s_j(T), j = 1, 2, \ldots \) are singular values of \( T \). \( T \) is said to be of trace class, if

\[
\sum_{j=1}^{\infty} s_j(T) < \infty.
\]

**Theorem B.5.** Suppose that \( T \) is of trace class on \( \mathcal{H} \) and \( \{ f_n \}_{n=1}^{\infty} \) is any orthonormal basis of \( \mathcal{H} \), then

\[
\sum_{j=0}^{\infty} \langle T f_j, f_j \rangle
\]
converges absolutely to a limit that is independent of the choice of orthonormal basis. The limit is called the trace of $T$, denoted by $\text{trace}(T)$.

**Theorem B.6.** Suppose that $T$ is of trace class on $\mathcal{H}$ and $B$ is a bounded operator on $\mathcal{H}$, then $TB$ and $BT$ are of trace class on $\mathcal{H}$, and

$$\text{trace}(TB) = \text{trace}(BT).$$

**Theorem B.7.** If $K$ is an operator of trace class on $L^2(\Omega)$, and

$$Kf(x) = \int_{\Omega} K(x,y)f(y)dy$$

then

$$\text{trace}(K) = \int_{\Omega} K(x,x)dx.$$

Let $A(x,\epsilon D)$ be a semiclassical pseudodifferential operator with scalar symbol $A(x,\xi)$, then

$$A(x,\epsilon D)f = \int_{\mathbb{R}^n} K_A(x,y)f(y)dy,$$

with

$$K_A(x,y) = \frac{1}{(2\pi\epsilon)^n} \int_{\mathbb{R}^n} A(x,\xi) e^{i(x-y)\cdot\xi} d\xi.$$

If $A(x,\epsilon D)$ is of trace class, an immediate implication of the theorem above is that

$$\text{trace}(A(x,\epsilon D)) = \int_{\mathbb{R}^n} K_A(x,x)dx = \frac{1}{(2\pi\epsilon)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A(x,\xi)dx d\xi.$$

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