FULL CAUSAL BULK VISCOUS COSMOLOGIES WITH TIME-VARYING CONSTANTS

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Abstract

We study the evolution of a flat Friedmann-Robertson-Walker Universe, filled with a bulk viscous cosmological fluid, in the presence of time varying “constants”. The dimensional analysis of the model suggests a proportionality between the bulk viscous pressure of the dissipative fluid and the energy density. On using this assumption and with the choice of the standard equations of state for the bulk viscosity coefficient, temperature and relaxation time, the general solution of the field equations can be obtained, with all physical parameters having a power-law time dependence. The symmetry analysis of this model, performed by using Lie group techniques, confirms the unicity of the solution for this functional form of the bulk viscous pressure. In order to find another possible solution we relax the hypotheses assuming a concrete functional dependence for the “constants”.

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I. INTRODUCTION

Since the pioneering work of Dirac\textsuperscript{1}, who proposed, motivated by the occurrence of large numbers in Universe, a theory with a time variable gravitational coupling constant $G$, cosmological models with variable $G$ and nonvanishing and variable cosmological term have been intensively investigated in the physical literature\textsuperscript{2–14}.

Dissipative thermodynamic processes in cosmology originating from a bulk viscosity are believed to play an important role in the dynamics and evolution of the Universe. Misner\textsuperscript{15} suggested that large-scale isotropy of the universe observed at the present epoch is due to the action of neutrino viscosity which was not negligible when the universe was less than a second old. There are a number of processes responsible for producing bulk viscosity in the early universe, such as the interaction between radiation and matter\textsuperscript{16}, gravitational string production\textsuperscript{17–18}, viscosity due to quark and gluon plasma, dark matter or particle creation\textsuperscript{24,33–34}.

In order to study these phenomena, the theories of dissipation in Eckart-Landau formulation\textsuperscript{19–20}, who made the first attempt at creating a relativistic theory of viscosity. However, these theories are now known to be pathological in several ways. Regardless of the choice of equation of state, all equilibrium states in these theories are unstable. In addition, as shown by Israel\textsuperscript{21}, signals may be propagated through the fluid at velocities exceeding the speed of light in contradiction with the principle of causality. These problems arise due to the first order nature of the theory since it considers only first-order deviations from the equilibrium, leading to parabolic differential equations, and hence to infinite speeds of propagation for heat flow and viscosity. While such paradoxes appear particularly glaring in relativistic theory, infinite propagation speeds already constitutes a difficulty at the classical level, since one does not expect thermal disturbances to be carried faster than some (suitably defined) mean molecular speed. Conventional theory is thus applicable only to phenomena which are “quasi-stationary” i.e. slowly varying on space and time scales characterized by mean free path and mean collision time\textsuperscript{21}. This is inadequate for many phenomena in high-energy astrophysics and relativistic cosmology involving steep gradients or rapid variations. These deficiencies can be traced to the fact that the conventional theories (both classical and relativistic) make overly restrictive hypothesis concerning the relation between the fluxes and densities of entropy, energy and particle number.
A relativistic second-order theory was found by Israel\textsuperscript{21} and developed by Israel and Stewart\textsuperscript{22} into what is called ‘transient’ or ‘extended’ irreversible thermodynamics. In this model, deviations from equilibrium (bulk stress, heat flow and shear stress) are treated as independent dynamical variables leading to a total of 14 dynamical fluid variables to be determined. The solutions of the full causal theory are well behaved for all times. Hence the advantages of the causal theories are the following\textsuperscript{23}: 1) for stable fluid configurations, the dissipative signals propagate causally 2) unlike Eckart-type’s theories, there is no generic short-wavelength secular instability in causal theories and 3) even for rotating fluids, the perturbations have a well-posed initial value problem. Therefore, the best currently available theory for analyzing dissipative processes in the Universe is the full Israel-Stewart causal thermodynamics.

Due to the complicated nonlinear character of the evolution equations, very few exact cosmological solutions of the gravitational field equations are known in the framework of the full causal theory. For a homogeneous Universe filled with a full causal viscous fluid source obeying the relation $\xi \sim \rho^{\frac{1}{2}}$, exact general solutions of the field equations have been obtained earlier\textsuperscript{24−33}. In this case the evolution of the bulk viscous cosmological model can be reduced to a Painleve-Ince type differential equation. It has also been proposed that causal bulk viscous thermodynamics can model on a phenomenological level matter creation in the early Universe\textsuperscript{24−33}.

In this paper, we consider the evolution of a causal bulk viscous fluid filled flat FRW type Universe, by assuming the standard equations of state for the bulk viscosity coefficient, temperature and relaxation time and in the presence of time varying constants ($G, c$ and $\Lambda$). In order to obtain some very general properties of this cosmological model with variable constants, we shall adopt a method based on the studies of the symmetries of the field equations. As a first step, we shall study the field equations from dimensional point of view. The dimensional method provides general relations between physical quantities and allows us to make some definite assumptions on the behavior of thermodynamical quantities as well as on the equation of state for the bulk viscous parameter. In particular, we find that, under the assumption of the conservation of the total energy of the Universe, the bulk viscous pressure of the cosmological fluid must be proportional to the energy density of the matter component and for $\gamma = 1/2$ (where $\gamma$ stands for the bulk viscous parameter), we find that $G/c^2$ must remain constant in spite of considering both “constants” as functions of
time \( t \). On using this assumption, the gravitational field equations can be integrated exactly, leading to a general solution in which all thermodynamical quantities have a power-law time dependence.

As we have been able to find a solution through dimensional analysis, it is possible that there are other symmetries of the model, since dimensional analysis is a reminiscent of scaling symmetries, which obviously are not the most general form of symmetries. Hence, we shall study the model through the method of Lie group symmetries, showing that under the assumed hypotheses of the proportionality of bulk viscous pressure to the energy density, there are no other solutions of the field equations.

The paper is organized as follows: In section (II), we outline the equations of the model, define all the quantities as well as their equations of state and fix the notation. In section (III), we study our equations from the dimensional point of view (subsection IIIA). This study allows us to make two simplifying hypotheses which we use to integrate the equations. In subsection (IIIB), we obtain the equation of state for the bulk viscosity through the dynamical approach. Section (IV) presents a naive method to study our model taking into account the previous hypotheses. This technique allows us to obtain a complete set of solutions as well as to arrive at some interesting conclusions. In section (V), we check the solutions obtained in the previous sections are correct and show that direct integration does not give more information about the solutions. As Dimensional Analysis is a manifestation of symmetries, in section (VI) we show that under the imposed hypothesis there are no more solutions for the field equations than ones already obtained. In section (VII), we try to relax the hypotheses in order to seek other solutions but (un)fortunately we only recover our previous solutions. We end the paper by summarising our findings in section (VIII).

II. THE MODEL

Following Maartens\textsuperscript{35}, we consider a Friedmann-Robertson-Walker (FRW) Universe with a line element

\[ ds^2 = c^2 dt^2 - f^2(t) \left( dx^2 + dy^2 + dz^2 \right), \]

(1)

filled with a bulk viscous cosmological fluid with the following energy-momentum tensor:

\[ T^k_i = (\rho + p + \Pi) u_i u^k - (p + \Pi) \delta^k_i, \]

(2)
where $\rho$ is the energy density, $p$ the thermodynamic pressure, $\Pi$ is the bulk viscous pressure and $u_i$ is the four velocity satisfying the condition $u_iu^i = 1$. The number 4-flux and the entropy 4-flux take the form:

$$N^i = nu^i, \quad (3)$$

$$S^i = sN^i - \left(\frac{\tau \Pi^2}{2\xi T}\right) u^i, \quad (4)$$

where $n$ is the number density, $s$ the specific entropy, $T \geq 0$ the temperature, $\xi$ the bulk viscosity coefficient and $\tau \geq 0$ the relaxation coefficient for transient bulk viscous effect (i.e. the relaxation time). The fundamental thermodynamic tensors (2-4) are subject to the dynamical laws of energy-momentum conservation, number conservation and the Gibb's equation:

$$T^k_{i;k} = 0, \quad (5)$$

$$N^i_{;i} = 0, \quad (6)$$

$$Tds = d\left(\frac{\rho}{n}\right) + p d\left(\frac{1}{n}\right). \quad (7)$$

The equations (2-4) and (5-7) imply:

$$TS^i_{;i} = -\Pi \left(3H + \frac{\dot{T}}{\xi} + \frac{1}{2} T\Pi \left(\frac{\tau}{\xi T}u^i\right)_{;i}\right), \quad (8)$$

by equations (5-7) and (8), the simplest way (linear in $\Pi$) to satisfy the $H$-theorem (i.e. for the entropy production to be non-negative, $S^i_{;i} = \frac{\Pi^2}{\xi T} \geq 0$) leads to the causal evolution equation for bulk viscosity given by:

$$\tau \dot{\Pi} + \Pi = -3\xi \dot{H} - \frac{\epsilon}{2} \tau \Pi \left(3H + \frac{\dot{T}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right), \quad (9)$$

In eq.(9), $\epsilon = 0$ gives the truncated theory (the truncated theory implies a drastic condition on the temperature), while $\epsilon = 1$ gives the full theory. The non-causal theory has $\tau = 0$.

The growth of the total commoving entropy $\Sigma$ over a proper time interval $(t_0, t)$ is given by:

$$\Sigma(t) - \Sigma(t_0) = -\frac{3}{k_B} \int_{t_0}^{t} \frac{\Pi H f^3}{T} dt, \quad (10)$$
where $k_B$ is the Boltzmann’s constant. The Einstein gravitational field equations with variable $G$, $c$ and $\Lambda$ are:

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8 \pi G(t)}{c^4(t)} T_{ik} + \Lambda(t) g_{ik}. \quad (11)$$

Applying the covariance divergence to the second member of equation (11) we get:

$$\text{div} \left( \frac{G}{c^4} T^j_i + \delta_i^j \Lambda \right) = 0, \quad (12)$$

$$T^j_i = \left( \frac{4 c_{,j}}{c} - \frac{G_{,j}}{G} \right) T^j_i - \frac{c^4 \delta_i^j \Lambda_{,j}}{8 \pi G}, \quad (13)$$

that simplifies to:

$$\dot{\rho} + 3 (\rho + p) H + 3 H \Pi = -\frac{\dot{\Lambda} c^4}{8 \pi G} - \rho \frac{\dot{G}}{G} - 4 \rho \frac{\dot{c}}{c}, \quad (14)$$

where $H$ stands for the Hubble parameter ($H = \dot{f}/f$). The last equation may be written in the following form:

$$\dot{\rho} + 3 (\rho + p) H + 3 H \Pi + \frac{\dot{\Lambda} c^4}{8 \pi G} + \rho \frac{\dot{G}}{G} - 4 \rho \frac{\dot{c}}{c} = 0. \quad (15)$$

Therefore, our model (with FRW symmetries) is described by the following equations:

$$2 \dot{H} + 3 H^2 = -\frac{8 \pi G}{c^2} (p + \Pi) + \Lambda c^2, \quad (16)$$

$$3 H^2 = \frac{8 \pi G}{c^2} \rho + \Lambda c^2, \quad (17)$$

$$\dot{\rho} + 3 (\rho + p + \Pi) H = -\frac{\dot{\Lambda} c^4}{8 \pi G} - \rho \frac{\dot{G}}{G} + 4 \rho \frac{\dot{c}}{c}, \quad (18)$$

$$\tau \dot{\Pi} + \Pi = -3 \xi H - \frac{\epsilon}{2} \tau \Pi \left( 3 H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T} \right). \quad (19)$$

In order to close the system of equations (16-19) we have to give the equation of state for $p$ and specify $T$, $\xi$ and $\tau$. As usual, we assume the following phenomenological (ad hoc) laws$^{35}$:

$$p = \omega \rho, \quad (20)$$

$$\xi = k_\gamma \rho^\gamma, \quad (21)$$

$$T = D_\delta \rho^\delta, \quad (22)$$

$$\tau = \xi \rho^{-1} = k_\gamma \rho^{\gamma-1}, \quad (23)$$
where $0 \leq \omega \leq 1$, and $k, \gamma \geq 0$, $D, \delta \geq 0$ are dimensional constants, $\gamma \geq 0$ and $\delta \geq 0$ are numerical constants. Eqs. (20) are standard in cosmological models whereas the equation for $\tau$ is a simple procedure to ensure that the speed of viscous pulses does not exceed the speed of light. These are without sufficient thermodynamical motivation, but, in absence of better alternatives, we use these equations and expect that they will at least provide an indication of the range of possibilities. For the temperature law, we take, $T = D, \delta \rho$, which is the simplest law guaranteeing positive heat capacity.

In the context of irreversible thermodynamics $p, \rho, T$ and the particle number density $n$ are equilibrium magnitudes which are related by equations of state of the form $p = p(T, n)$ and $p = p(T, n)$. From the requirement that the entropy is a state function, we obtain the equation

$$\left(\frac{\partial p}{\partial n}\right)_T = \frac{p + \rho}{n} - \frac{T}{n} \left(\frac{\partial p}{\partial T}\right)_n.$$  \hspace{1cm} (24)

For the equations of state (20-23) this relation imposes the constraint $\delta = \frac{\omega}{\omega + 1}$ so that $0 \leq \delta \leq 1/2$ for $0 \leq \omega \leq 1$, a range of values which is usually considered in the physical literature$^{38}$.

The Israel-Stewart-Hiscock theory is derived under the assumption that the thermodynamical state of the fluid is close to equilibrium, that is, the non-equilibrium bulk viscous pressure should be small when compared to the local equilibrium pressure $|\Pi| \ll p = \omega \rho$ $^{39-40}$. If this condition is violated then one is effectively assuming that the linear theory holds also in the nonlinear regime far from equilibrium. For a fluid description of the matter, this condition should be satisfied.

Therefore, with all these assumptions and taking into account the conservation principle, i.e., $\text{div}(T_i^j) = 0$, the resulting field equations are as follows:

$$2\dot{H} + 3H^2 = -\frac{8\pi G}{c^2} (p + \Pi) + \Lambda c^2,$$ \hspace{1cm} (25)

$$3H^2 = \frac{8\pi G}{c^2} \rho + \Lambda c^2,$$ \hspace{1cm} (26)

$$\dot{\rho} + 3 (\omega + 1) \rho H = -3H\Pi,$$ \hspace{1cm} (27)

$$\frac{\dot{\Lambda} c^4}{8\pi G} + \rho \frac{\dot{G}}{G} - 4\rho \frac{\dot{c}}{c} = 0,$$ \hspace{1cm} (28)

$$\dot{\Pi} + \frac{\Pi}{k, \gamma \rho^\gamma - 1} = -3\rho H - \frac{1}{2} \Pi \left(3H - W \frac{\dot{\rho}}{\rho}\right),$$ \hspace{1cm} (29)
where $W = \left( \frac{2\omega + 1}{\omega + 1} \right)$.

III. HYPOTHESES AND EQUATIONS OF STATE

In this section, we study the field equations in a dimensional way as well as with a dynamical systems approach in order to obtain relationships between various physical quantities, simplify the field equations and to obtain an adequate equation of state for the bulk viscous parameter.

We use dimensional considerations to re-write the field equations in a dimensionless way and show that some interesting relations can be obtained as a result. We find that the bulk viscous pressure has the same behaviour as the energy density and if $\gamma \neq 1/2$, the relationship $G/c^2$ should vary if we demand that our equations remain gauge and scale invariant. We find that with the special case $\gamma = 1/2$ the relationship $G/c^2 = \text{const.}$ but in such a way that both “constants” vary. Dimensional analysis suggests that $\gamma = 1/2$ is a very special value for the bulk viscous parameter. Using the dynamical systems approach, we shall show that for this value of $\gamma$, our model approximates the dynamics of a perfect fluid and for large time, the dynamics of the Universe follows that of a flat FRW model.

A. Dimensional considerations

The $\pi$ – monomia is the main object in dimensional analysis. It may be defined as the product of quantities which are invariant under change of fundamental units. $\pi$ – monomia are dimensionless quantities, their dimensions are equal to unity. The dimensional analysis has structure of Lie group$^{41-44}$. The $\pi$ – monomia are invariants under the action of the similarity group. We must mention here that the similarity group is only a special class of the group of all symmetries that can be obtained using the Lie method. For this reason, when one uses dimensional analysis only one of the several possible solutions to the problem is obtained.

The equations (25-29) and the equation of state (20-23) can be expressed in a dimension-
less way by the following $\pi - monomia$: 

$$\pi_1 = \frac{Gpt^2}{c^2} \quad \pi_2 = \frac{G\Pi t^2}{c^2} \quad \pi_3 = \frac{G\rho t^2}{c^2}$$  \hspace{1cm} (30)

$$\pi_4 = \frac{\Pi}{p} \quad \pi_5 = \frac{\xi}{\Pi t} \quad \pi_6 = \frac{\tau}{t} = \tau H$$  \hspace{1cm} (31)

$$\pi_7 = \frac{\xi}{k_\gamma \rho^\gamma} \quad \pi_8 = \frac{\xi}{\tau \rho} \quad \pi_9 = \frac{T}{D_8 \rho^b}$$  \hspace{1cm} (32)

$$\pi_{10} = \frac{\rho}{p} \quad \pi_{11} = \Lambda c^2 t^2 \quad \pi_{12} = \frac{\Lambda c^4}{G\rho}$$  \hspace{1cm} (33)

The following relations can be obtained from the $\pi - monomia$.

1. From $\pi_1, \pi_2, \pi_3, \pi_4$ and $\pi_{10}$, we see that

$$\rho \propto p \propto \Pi,$$  \hspace{1cm} (34)

and thus, $[\Pi] = [p] = [\rho]$. Therefore, these physical quantities have the same dimensional equation and exhibit the same behaviour in order of magnitude.

2. On using $\pi_7$ and $\pi_8$, we obtain the following relation

$$\tilde{\pi}_8 = \frac{k_\gamma \rho^{\gamma-1}}{t} \implies \rho = (k_\gamma^{-1} t)^b$$  \hspace{1cm} (35)

and from $\pi_3$ and $\tilde{\pi}_8$

$$\frac{G\rho t^2}{c^2} = \frac{k_\gamma \rho^{\gamma-1}}{t} \implies \frac{c^2}{Gt^2} = \left( \frac{t}{k_\gamma} \right)^{1/(\gamma-1)} \implies \frac{k_\gamma^b c^2}{G} = t^{b+2}$$  \hspace{1cm} (36)

where $b = 1/(\gamma - 1)$. When $\gamma = 1/2$, we obtain the relationship $k_\gamma^2 = c^2/G$, which indicates that these relations remain constant, i.e., $G$ and $c$ can vary but in such a way that $c^2/G$ remains constant. If $\gamma \neq 1/2$, the “constants” $G$ and $c$ must vary if we want that our equations to remain gauge invariant or scale invariant. In a reasonable physical approach, we need to impose the condition $G/c^2 = constant$, and in the process, uniquely fixing the value of the coefficient $\gamma$. The dimensional analysis not only fixes the value of $\gamma$, but also predicts the value of the proportionality coefficient as being $k_\gamma^2 = c^2/G$ (in natural units $k_\gamma^2 = 8\pi$).
Therefore, the solutions that Dimensional Analysis suggests us are the following:

\[ \rho \propto (k_{\gamma}^{-1}t)^{b}, \]  
(37)

\[ p = \omega \rho \propto (k_{\gamma}^{-1}t)^{b}, \]  
(38)

\[ \Pi \propto (k_{\gamma}^{-1}t)^{b} = \kappa \rho, \]  
(39)

\[ \Lambda \propto e^{-2t^{-2}}, \]  
(40)

\[ \frac{G}{c^{2}} \propto k_{\gamma}^{b}t^{-b-2}, \]  
(41)

where \( \kappa \) is a numerical constant. From the equation of state, we obtain

\[ \xi = k_{\gamma} \rho^{\gamma} \propto k_{\gamma} (k_{\gamma}^{-1}t)^{b\gamma}, \]  
(42)

\[ T = D_{\delta} \rho^{\delta} \propto (k_{\gamma}^{-1}t)^{\delta b} \text{ with } \delta = \frac{\omega}{\omega + 1}, \]  
(43)

\[ \tau = \xi \rho^{-1} = k_{\gamma} (k_{\gamma}^{-1}t)^{b(\gamma - 1)}, \text{ i. e. } \tau = t \]  
(44)

So far, we have seen that with this simple method we can obtain the behaviour of all the physical quantities. We would like to emphasize that we only have compared \( \pi \) – monomia and used the equation of state to obtain these results. In section IV, we will present a naive way to work with another dimensional method.

**B. Structural stability of bulk viscous cosmological models**

In this subsection, we consider the analysis of bulk viscous cosmological models using dynamical systems theory. Dynamical systems have already been used in the study of causal viscous fluids\(^ {28-30} \), non-causal viscous fluids\(^ {45} \), \( G_{2} \) cosmologies\(^ {46-47} \) and magnetic fields in scalar field cosmology\(^ {48} \) (for a review of dynamical systems in cosmology, see\(^ {49-50} \)). In this method, the governing equations of the model are considered as a finite system of autonomous ordinary differential equations. By using this method, we can determine the arbitrary exponent in the state equation of the bulk viscosity coefficient.

As a basic physical requirement, we assume that the causal bulk viscous model approximates the dynamics of a perfect fluid and for large time, the dynamics of the Universe follows that of a flat FRW model. Our assumption is supported by the fact that the viscous pressure will decay faster than the thermodynamic pressure\(^ {39} \) and hence the behaviour of our model is equivalent to that of a model with a perfect fluid. On a large scale, the Universe is well described by the classical flat FRW models. It is natural to assume here that
perturbations introduced by a dissipative parameter like bulk viscosity does not modify the cosmological evolution even at early times. Viscous dissipative terms can significantly modify the thermodynamical behaviour of the early cosmological fluid, but, we do not expect a major modification in the scale factor. With these assumptions, we can use mathematical techniques of dynamical systems theory to select a correct equation of state for the bulk viscosity coefficient, i.e. the values of the parameter \( \gamma \), in such a way that the viscous model describes a homeomorphic dynamics of the flat FRW model. A natural environment to study the first order autonomous ordinary differential equations

\[
\frac{dx}{dt} = P(x, y, z), \quad \frac{dy}{dt} = Q(x, y, z), \quad \frac{dz}{dt} = R(x, y, z),
\]

is a three-dimensional differential manifold \( M \) to every point \( p = (x, y, z) \) of which the tangent space \( T_pM \) is associated. The vector field \( X \in \mathfrak{X}(M) / X(p) = (P(p), Q(p), R(p)) \in T_pM \) of class \( C^1 \). The set \( \mathfrak{X}(M) \) of all such vector fields on \( M \) is called the space of differential equations on \( M \). The solution curves, \( \phi_t(p) \) (also called phase trajectories) of a vector field \( X(p) \) on \( M \) define a one-parameter group of transformations \( \phi_t : M \to M \) where \( t \in (a, b) \).

If \( M \) is compact, then \( t \in \mathbb{R} \) and \( \phi_t \) is called the dynamical system.

Two vector fields \( X, Y \in \mathfrak{X}(M) \) are said to be topologically equivalent if there is a homeomorphism \( h : M \to M \) preserving the orientation of the solution curves of \( X \) into those of \( Y \). Let \( D \subset M \) be a compact subset of \( M \). We say that \( X \in \mathfrak{X}(D) \) is structurally stable if there is a neighborhood \( N(X) \) of \( X \) in \( \mathfrak{X}(D) \) such that for every \( Y \subset N(X) \), \( Y \) is topologically equivalent to \( X \), i.e. there is an orientation preserving homeomorphism transforming the phase trajectories of \( X \) into those of \( Y \).

In the following, we seek spatially homogeneous and isotropic bulk viscous solutions of the gravitational field equations which, in a structurally stable way, approximate the dynamics of ordinary (perfect fluid) FRW models. More precisely, we will look for such spatially and isotropic solutions which, after being perturbed by the dissipative parameter, yield the dynamics topologically equivalent to those of FRW models. If we take into account the state equations \((20-23)\), the gravitational field and the bulk viscous pressure evolution equations
can be rewritten in the following form:

\[
\dot{H} = -H^2 - \frac{(3(\omega + 1) - 2)}{6} \rho - \frac{1}{2} \Pi + \frac{1}{3} \Lambda, \tag{46}
\]

\[
\dot{\rho} = -3H ((\omega + 1) \rho + \Pi), \tag{47}
\]

\[
\dot{\Pi} = - \left( \frac{\Pi}{k_\gamma \rho^{\gamma - 1}} + \frac{1}{2} \Pi \left[ 3H - W \frac{\dot{\rho}}{\rho} \right] \right), \tag{48}
\]

with

\[
H^2 = \frac{1}{3} \rho - \frac{3R}{6} + \frac{1}{3} \Lambda. \tag{49}
\]

Solving

\[
\dot{H} = 0, \quad \dot{\rho} = 0, \quad \dot{\Pi} = 0, \tag{50}
\]

for \(H, \rho\) and \(\Pi\), we obtain from (47)

\[
\Pi = -(\omega + 1)\rho,
\]

and taking into account this result, on using (46) we arrive at:

\[
H^2 = \frac{1}{3} (\rho + \Lambda). \tag{51}
\]

With the use of Eq. (51), the energy density \(\rho\) can be obtained from the equation (note that \(\dot{\rho} = 0\))

\[
\dot{\Pi} = - \left( \frac{\Pi}{k_\gamma \rho^{\gamma - 1}} + \frac{1}{2} \Pi \left[ 3H - W \frac{\dot{\rho}}{\rho} \right] \right) = 0, \tag{52}
\]

and is given by

\[
\left( \frac{(\omega + 1)^2}{k_\gamma^2} \rho^{1-2\gamma} - 3 \left( 1 - \frac{(\omega + 1)}{2} \right)^2 \right) \rho = 3 \Lambda \left( 1 - \frac{(\omega + 1)}{2} \right)^2, \tag{53}
\]

or alternately

\[
\rho \left( k_\gamma^{-2} (\omega + 1)^2 \rho^{1-2\gamma} - \beta \right) = \beta \Lambda, \tag{54}
\]

where \(\beta = 3 \left( 1 - \frac{(\omega + 1)}{2} \right)^2\). As we can see

\[
\rho = \left( \frac{k_\gamma^2 \beta}{(\omega + 1)^2} \right)^{\frac{1}{1-2\gamma}} = 0 \iff \gamma = \frac{1}{2}, \tag{55}
\]

since \(\frac{\beta}{(\omega + 1)^2} < 1, \forall \omega\). For this value, we have \(\rho = \Lambda = 0\).

Therefore, as has already been pointed out earlier\(^{24-26}\), causal bulk viscous cosmological models with \(\gamma = 1/2\) describe a two-phase evolution of the Universe. The Universe is
born from a zero energy density vacuum state and in the first phase the energy density is increasing to a maximum value. This period corresponds to a phase of matter creation. After reaching a maximum value, due to the expansion of the Universe, the energy density becomes a monotonically decreasing function of time and the second phase describes a standard evolution of the bulk viscous causal cosmological fluid. Hence, in this framework, bulk viscous processes can model matter creation in the early Universe. The ultra-stiff case of the Zel’dovich fluid, $\omega = 1$ also gives $\rho = 0$, but from this value we cannot find the value for $\gamma$.

The structurally stable approximation of the system perturbed by the viscous parameter must have the same critical points as the unperturbed system. The unperturbed system has no critical points except the point $(H, \rho) = (0, 0)$. Therefore it is observed that the only structurally stable approximations to the flat FRW solutions are those with $\xi = k_{\gamma} \rho^{1/2}$, i.e. $\gamma = 1/2$.

As already have been pointed out by M. Szydlowski and M. Heller in the study of non-causal viscous fluids it is possible to use a Lie group technique in order to determine an adequate equation of state, in particular theses authors find that there is a only one symmetry for the field equations if $\gamma = 1/2$.

IV. A NAIVE METHOD

In this section, we use dimensional analysis to obtain a complete set of solutions for the field equations with the assumptions $\text{div}(T) = 0$ and $\Pi = \kappa \rho$ and $\kappa \in \mathbb{R}^-$. This leads to

$$\rho = A_{\omega, \kappa} f^{-3(\omega + 1 + \kappa)} \quad \text{or} \quad \rho = A_{\omega, \kappa} f^{-\alpha},$$

valid for $\forall \gamma$. The set of governing parameters are $\mathfrak{M} = \{A_{\omega, \kappa}, k_{\gamma}, t\}$, where $[A_{\omega, \kappa}] = L^{3(\omega + 1 + \kappa) - 1} M T^{-2}$, $[k_{\gamma}] = L^{-1/2} M^{1/2}$ and $[t] = T$. Using these, we obtain the following
relations:

\[ G \propto A_{\omega, \alpha}^{\frac{3}{2}} k^{\frac{3 + \omega}{3(\alpha + 1)}} t^{-\frac{3 + \omega}{3(\alpha + 1)}}, \quad (57) \]
\[ c \propto A_{\omega, \alpha}^{\frac{1}{2}} k^{-\frac{1}{\alpha + 1}} t^{-1 - \frac{1}{3(\alpha + 1)}}, \quad (58) \]
\[ \rho \propto k_{\gamma}^{-b-1} t^{b-1} \propto \Pi, \quad (59) \]
\[ \xi = k_{\gamma} \rho^{\alpha}, \quad \tau = \xi \rho^{-1} \]
\[ f \propto A_{\omega, \alpha}^{\frac{3}{2}} k^{\frac{3}{(\alpha + 1)}} t^{-\frac{3}{3(\alpha + 1)}}, \quad (60) \]
\[ k_B \theta \propto A_{\omega, \alpha}^{\frac{3}{2}} k^{\frac{3}{(\alpha + 1)}} t^{-3 - \frac{3}{3(\alpha + 1)}} \]
\[ a^{-1/4} s \propto A_{\omega, \alpha}^{\frac{3}{2}} k^{\frac{3}{(\alpha + 1)}} t^{-2 - \frac{3}{3(\alpha + 1)}}, \quad (61) \]
\[ \Lambda \propto A_{\omega, \alpha}^{\frac{3}{2}} k^{\frac{3}{(\alpha + 1)}} t^{-2}, \quad (62) \]
\[ q = -b\alpha - b - 1 \]

where, \( \alpha = 3(\omega + 1 + \kappa) - 1 \), \( b = \gamma - 1 \), \( s \) is the entropy and \( q \) is the deceleration parameter. Note that the current method is very different from the method employed in the previous section.

It is observed that

\[ \frac{G}{c^2} = \text{const.} \iff b = -\frac{1}{2} \iff \gamma = \frac{1}{2}, \quad (66) \]

and for \( \gamma = \frac{1}{2} \), we obtain the following results:

\[ G \propto t^{-2(\frac{\alpha - 1}{\alpha + 1})}, \quad c \propto t^{-\frac{\alpha - 1}{\alpha + 1}}, \quad \rho \propto t^{-2} \propto \Pi, \quad (67) \]
\[ f \propto t^{-2(\frac{\alpha - 1}{\alpha + 1})}, \quad k_B \theta \propto t^{-2(\frac{\alpha - 1}{\alpha + 1})}, \quad a^{-1/4} s \propto t^{-\frac{3}{2(\alpha + 1)}}, \quad \Lambda \propto t^{-4(\frac{\alpha - 1}{\alpha + 1})}, \quad (68) \]
\[ q = -b\alpha - b - 1 \]

Thus, we see that (for \( \gamma = \frac{1}{2} \)) \( G, c \) and \( \Lambda \) are decreasing functions on time despite the fact that the relation \( G/c^2 \) remains constant. In a previous work\(^{52}\), where we only consider \( G \) and \( \Lambda \) as time functions, we found that for \( \gamma = 1/2 \), \( G \) behaves as a true constant i.e. \( G = \text{const.} \) and that \( \Lambda \) vanishes. In this new scenario, we find that \( G \) and \( c \) varies with time but in such a way that the relation \( G/c^2 = \text{const.} \) and \( \Lambda \) is a decreasing function on time. This consequence has some interesting implications in light of recent supernovae observations which suggest a small but non-zero value for the cosmological constant\(^{53}\).

We would like to point out that these results are very similar to those in\(^{54}\) where a perfect fluid cosmological model with time-varying constants in the presence of adiabatic matter
creation was considered. This striking similarity of our results is not surprising as it has been suggested by Zeldovich\textsuperscript{55} and later by Murphy\textsuperscript{56} and Hu\textsuperscript{57} that the introduction of viscosity in the cosmological fluid is a phenomenological description of the effect of creation of particles by the non-stationary gravitational field of the expanding cosmos. A non-vanishing particle production rate is equivalent to a bulk viscous pressure in the cosmological fluid or from a quantum point of view, with a viscosity of the vacuum. This is due to the simple circumstance that any source term in the energy balance of a relativistic fluid may be formally rewritten in terms of an effective bulk viscosity. Zimdahl and others\textsuperscript{40} have considered in detail the possibility that the bulk viscous pressure of the full Israel-Stewart-Hiscock theory may also be interpreted as an effective description for particle production processes. The particle creation process leads to considerable changes in the thermodynamical behavior of the Universe. If the chemical potential of the newly created particles is zero, $\mu = 0$, then the non-vanishing bulk pressure $\Pi$ associated with an increase in the number of fluid particles satisfies formally the same equation as in the case of presence of a real dissipative bulk viscosity\textsuperscript{24–33}.

This coincidence can be easily explained by observing the set of the governing parameters. In the case of a viscous fluid, this set is characterized by $\mathfrak{M} = \mathfrak{M} \{A_{\omega,\kappa}, k_\gamma, t\}$, where $[A_{\omega,\kappa}] = L^{3(\omega+1+\kappa)-1}MT^{-2}$, $[k_\gamma] = L^{-1/2}M^{1/2}$ and $[t] = T$, while in the case of adiabatic matter creation, this set is characterized by $\mathfrak{M} = \mathfrak{M} \{A_{\omega,\beta}, B, t\}$ with $[A_{\omega,\beta}] = L^{3(\omega+1)(1-\beta)-1}MT^{-2}$, $[G/c^2 = B] = LM^{-1}$ and $[t] = T$, see\textsuperscript{41} for the details. We can observe the extreme similarity between $A_{\omega,\kappa}$ and $A_{\omega,\beta}$ and the coincidence between $B$ and $k_\gamma^{-2}$, which has been shown in the previous section as $k_\gamma^2 = c^2/G$.

Another similarity of the present results with those obtained in\textsuperscript{54} is that for adiabatic matter creation, we cannot impose such a mechanism with $\omega = 0$ on the equation of state for matter predominance. In the same way, we observe from eq. (62) that if we fix $\omega = 0$ into the equation of state the fluid temperature increases indefinitely with time. Thus, we arrive at a similar conclusion that, in the matter predominance era, we cannot consider the effects of the bulk viscosity in our fluid. This fact is in agreement with a previous result obtained in\textsuperscript{58} where we studied a full causal bulk viscous model through the renormalization group method arriving to the conclusion that in the long-time asymptotics the cosmological model tends to a flat ideal (non-viscous) Friedmann type geometry.

We also notice that the $\kappa-$parameter, i.e. the causal bulk viscous effect, weakly perturbs
the FRW perfect fluid solution in such a way that the comoving entropy varies with \( t \), while in the perfect fluid case, i.e. \( \varkappa = 0 \), the comoving entropy is constant. The same considerations could be taken into account for the scale factor \( f(t) = f_0 t^{\frac{1}{(1-\gamma)(\omega+1+\varkappa)}} \), which is perturbed weakly for the bulk viscous parameter as we can see in the special case of \( \gamma = 1/2 \). The solutions obtained for the scale factor, energy density, entropy, etc. are similar “but different” to those obtained for a perfect fluid model where the \( \varkappa \)-parameter vanishes. We would like to point out that our model is thermodynamically consistent for the usual matter equations of state and valid for all \( \gamma \)-parameter, i.e. \( \forall \gamma \in (0,1) \). In the same way, we can see that the viscous parameter helps us to get rid of the so-called entropy problem since in this model entropy varies with \( t \).

The bulk viscous pressure and the energy density of the cosmological fluid are proportional to each other and hence, the general evolution of \( \Pi \) is qualitatively similar to the evolution of the thermodynamic pressure \( p \), both obeying a similar equation of state. In fact, by defining an effective coefficient \( \omega_{\text{eff}} = \omega - \varkappa \), the equation of state of the cosmological fluid can be formally written as \( p_{\text{eff}} = p + \Pi = (\omega_{\text{eff}} - 1) \rho \). However, since the bulk viscous pressure of the cosmological fluid must also satisfy the evolution equation (29), the resulting time evolution depends not only on \( \omega_{\text{eff}} \), but also, via the coefficients \( \delta \) and \( \gamma \), on the equations of state of the bulk viscosity coefficient, \( \xi = \xi (\rho) \) and of the temperature, \( T = T (\rho) \). Therefore, even that formally one can introduce an effective pressure (including both \( p \) and \( \Pi \)), obeying a \( \omega \)-law equation of state, due to the extra-constraints imposed by the requirements of the causal bulk evolution, the general dynamics of the present model cannot be reduced to a perfect fluid model evolution.

V. DIRECT INTEGRATION OF THE FIELD EQUATIONS

As we have seen in the previous section, dimensional analysis suggests us that \( \Pi \propto \rho \). We formalize this relationship in the following way, \( \Pi = \varkappa \rho \), and under physical considerations we take \( \varkappa \in \mathbb{R}^- \) and \( |\varkappa| \ll 1 \). In this section, we integrate the field equations with these assumptions and taking into account the conservation principle. This leads us to the following relationship

\[
\dot{\rho} + 3 (\omega + 1 + \varkappa) \rho H = 0. \tag{70}
\]

This trivially lead us to the well known relationship between the energy density \( \rho \) and
the scale factor $f$

$$\rho = A_\omega f^{-3(\omega+1+\kappa)} \quad \text{or} \quad \rho = A_\omega f^{-\alpha},$$

(71)

where $\alpha = 3(\omega + 1 + \kappa)$. Now, taking into account the eq. (19) and simplifying it, we obtain,

$$\kappa \dot{\rho} + \frac{3 \kappa \rho}{k_{\gamma} \rho^{\gamma-1}} = \frac{3}{\alpha} \dot{\rho} - \frac{1}{2} \kappa \dot{\rho} \left( -\frac{3}{\alpha} W \right),$$

(72)

thereby obtaining $\rho = \rho(t)$. On simplifying further, we obtain,

$$\frac{\dot{\rho}}{\rho^{2-\gamma}} = K \frac{k}{k_{\gamma}}, \quad \implies \quad \rho = dk_{\gamma}^{-b}t^{b},$$

(73)

where

$$K = \left( \frac{3}{\alpha} + \frac{3\kappa}{2\alpha} + \frac{W_\kappa}{2} - \kappa \right), \quad d = (\gamma - 1) K^b \quad \text{and} \quad b = \frac{1}{\gamma - 1}. \quad (74)$$

From equation (71) we obtain:

$$f = \left( \frac{A_\omega}{d} k_{\gamma} k_{\gamma}^{b-b} \right)^{1/\alpha}, \quad \text{i.e.} \quad f \propto t^{3(\omega+1+\kappa)(\gamma-1)}. \quad (75)$$

An important observational quantity is the deceleration parameter $q = \frac{d}{dt} \left( \frac{1}{H} \right) - 1$. The sign of the deceleration parameter indicates whether the model inflates or not. The positive sign of $q$ corresponds to “standard” decelerating models whereas the negative sign indicates inflation. In our model, the deceleration parameter behaves as:

$$q = -1 - \frac{\alpha}{b}. \quad (76)$$

Thus, for the parameters $\gamma, \omega$ and $\kappa$, the deceleration parameter indicates an inflationary behaviour when $\alpha/b > 0$ and a non-inflationary behaviour for $\alpha/b < -1$. We proceed with the calculation of the other physical quantities as under:

$$\xi = k_{\gamma} \rho^{\gamma} \propto k_{\gamma} (dk_{\gamma}^{-b}t^{-b})^{\gamma} = d^\gamma k_{\gamma}^{-b}t^{\gamma-b}, \quad (77)$$

$$T = D_\delta \rho^{\delta} = D_\delta (dk_{\gamma}^{-b}t^b)^{\delta} \quad \text{with} \quad \delta = \frac{\omega}{\omega + 1}, \quad (78)$$

$$\tau = \xi \rho^{-1} = k_{\gamma} (dk_{\gamma}^{-b}t^b)^{\gamma-1}, \quad \text{i.e.} \quad \tau = d^{\gamma-1}t \quad (79)$$

We see from $\tau = d^{\gamma-1}t$ that this result is in agreement with the theoretical result obtained in$^{35}$. For viscous expansion to be non-thermalizing, we should have $\tau < t$, or otherwise the basic interaction rate for viscous effects should be sufficiently rapid to restore the equilibrium as the fluid expands. The comoving entropy is

$$\Sigma(t) - \Sigma (t_0) = -\frac{3 \kappa \beta}{d^{\frac{3\kappa}{\alpha} + \delta} k_{\gamma} D_\delta} \left[ \frac{A_\omega^{1/\alpha}}{d^{\frac{3\kappa}{\alpha} + \delta} k_{\gamma} D_\delta} \frac{\nu}{b + \frac{3}{\alpha} + \delta} \nu_{b+\frac{3}{\alpha}+\delta} \right]^t \quad (80)$$

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We notice that the parameter \( \kappa \) weakly perturbs the perfect fluid FRW Universe. When \( \kappa = 0 \), the comoving entropy assumes a constant value and we recover the perfect fluid case.

Finally, we will use the equations

\[
3H^2 = \frac{8\pi G}{c^2} \rho + \Lambda c^2, \quad (81)
\]

\[
\frac{\dot{\Lambda}c^4}{8\pi G\rho} + \frac{\dot{G}}{G} - 4\frac{\dot{c}}{c} = 0, \quad (82)
\]

to obtain the behaviour of the "constants" \( G, c \) and \( \Lambda \). From (81), we obtain \( \Lambda \) and using it in eq. (82) we obtain

\[
-\frac{3\dot{c}c}{4\pi G\rho}H^2 + \frac{c^23H\dot{H}}{4\pi G\rho} - \frac{\dot{\rho}}{\rho} = 0 \quad (83)
\]

Considering the previous results

\[
H = \beta t^{-1} \quad \text{and} \quad \rho = dk_{\gamma}^{-b}t^{b}, \quad (84)
\]

where \( \beta = -b/\alpha \).

These results are used in eq. (83) to obtain the next ODE

\[
\frac{3\beta^2 c^2 t^{-2-b}}{4\pi d} \left[ \frac{\dot{c}}{c} + \frac{1}{t} \right] + b\frac{1}{t} = 0. \quad (85)
\]

We can see that the above equation verifies the relationship (a particular solution)

\[
\frac{k_{\gamma}b^2 c^2}{G} = gt^{b+2} \quad \text{therefore} \quad G = gk_{\gamma}c^2t^{-b-2}, \quad (86)
\]

\( g \) being a numerical constant. It is observed that if \( b = -2 \) i.e. \( \gamma = 1/2 \), we obtain \( G/c^2 = B \).

We use this relation in eq.(85) obtaining

\[
\frac{\dot{c}}{c} = -\left( 1 + \frac{b}{K_c} \right) t^{-1}, \quad (87)
\]

where \( K_c = \frac{3\beta^2 c^2}{4\pi d} \), and therefore,

\[
\frac{c}{K_c} = t^\kappa, \quad (88)
\]

where \( K_c \) is an integration constant and \( \kappa = \left( -1 - \frac{b}{K_c} \right) \).

If we impose a new assumption (which is supported by observational considerations of \( \Lambda \) decaying with time and assuming a small but non-zero value) as

\[
\Lambda = \frac{l}{c^2(t)^2} \quad \Rightarrow \quad \dot{\Lambda} = -\frac{2l}{c^2 t^2} \left( \frac{\dot{c}}{c} + \frac{1}{t} \right), \quad (89)
\]
we can see from eq. (81) that
\[
\frac{G}{c^2} = \left[ \frac{3\beta^2 - l}{8\pi d} \right] k_{\gamma} t^{-b-2} = B k_{\gamma} t^{-b-2}.
\] (90)

Taking all these relations into consideration, we use in eq. (82) to obtain
\[
-\frac{l}{4\pi B} \left( \frac{\dot{c}}{c} + \frac{1}{t} \right) - 2 \frac{\dot{c}}{c} - \frac{b + 2}{t} = 0,
\] (91)
whose trivial solution is
\[
c = \mathcal{R}_c t^{-\mu},
\] (92)
where \( \mathcal{R}_c \) is an integration constant and \( \mu = \left( \frac{l + 4\pi K(b+2)}{l + 8\pi K} \right) \).

We have solved our model under the assumptions: \( \text{div}(T) = 0, \Pi = \kappa \rho \), with \( \kappa \in \mathbb{R}^- \) and valid for \( \forall \gamma \). It is observed that if \( \gamma = 1/2 \) we obtain \( G/c^2 = \text{const.} \) as obtained earlier. Thus with the imposed hypotheses, we have obtained similar results.

VI. LIE METHOD

In this section, we study the field equations using the symmetry method. As we have discussed earlier, dimensional analysis is just a manifestation of scaling symmetry. However, this type of symmetry is not the most general form of symmetries\(^{59-61}\). Therefore, by studying the form of \( G(t) \) and \( c(t) \) for which the equations admit symmetries, we expect to uncover new integrable models. With these assumptions, we shall see that the solutions obtained in previous sections are recovered.

We start with the assumption \( \Pi = \kappa \rho \), with \( \kappa \in \mathbb{R}^- \) The bulk viscosity evolution equation can then be rewritten in the alternative form
\[
\delta \frac{\dot{\rho}}{\rho} + k_\gamma \rho^{1-\gamma} = 3\beta H,
\] (93)
where \( \beta = \left( \frac{1}{\kappa} - \frac{1}{2} \right) \) and \( \delta = \left( 1 - \frac{W}{2} \right) \).

Taking the derivative with respect to the time of this equation and with the use of the next equation
\[
\dot{H} = -4\pi \alpha \frac{G(t)}{c^2(t)} \rho,
\] (94)
obtained from the field equations (where \( \alpha = (1 + \omega + \kappa) \)), we obtain the following second order differential equation describing the time variation of the density of the cosmological fluid:

\[
\ddot{\rho} = \frac{\dot{\rho}^2}{\rho} - A \rho^s \dot{\rho} + B \frac{G(t)}{c^2(t)} \rho^2,
\]

(95)

where \( A = \frac{k^2}{(1-\gamma)} \), \( s = (1 - \gamma) \) and \( B = \frac{12\pi\alpha\beta}{\delta} \).

Equation (95) is of the general form.

\[
\ddot{\rho} = \psi(t, \rho, \dot{\rho}),
\]

(96)

where \( \psi(t, \rho, \dot{\rho}) = \frac{\dot{\rho}^2}{\rho} - A \rho^s \dot{\rho} + B \frac{G(t)}{c^2(t)} \rho^2 \).

We are going now to apply all the standard procedure of Lie group analysis to this equation (see\(^5\) for details and notation)

A vector field

\[
X = \xi(t, \rho) \partial_t + \eta(t, \rho) \partial_\rho,
\]

(97)

is a symmetry of (95) if

\[
-\xi \psi_t - \eta \psi_\rho + \eta_t + (2\eta_\rho - \xi_t) \dot{\rho} + (\eta_\rho \rho - 2\xi_t \rho) \dot{\rho}^2 - \xi_\rho \rho \dot{\rho}^3 +
\]

\[
+ (\eta_\rho - 2\xi_t - 3\dot{\rho} \xi_\rho) \psi - [\eta_t + (\eta_\rho - \xi_t) \dot{\rho} - \rho^2 \xi_\rho] \psi \dot{\rho} = 0.
\]

(98)

By expanding and separating (98) with respect to powers of \( \dot{\rho} \) we obtain the overdetermined system:

\[
\xi_{\rho \rho} + \rho^{-1} \xi_\rho = 0,
\]

(99)

\[
\eta \rho^{-2} - \eta_\rho \rho^{-1} + \eta_{\rho \rho} - 2\xi_t \rho + 2A \xi_\rho \rho^{1-s} = 0,
\]

(100)

\[
2\eta_\rho - \xi_{tt} + A \xi_t \rho^s - 3B \xi_\rho \frac{G}{c^2} \rho^2 - 2\eta_t \rho^{-1} + Asn \rho^{s-1} = 0,
\]

(101)

\[
-B \xi \left( \frac{\dot{G}}{c^2} - 2G \frac{\dot{c}}{c^2} \right) \rho^2 - 2B \eta \frac{G}{c^2} \rho + \eta_t + (\eta_\rho - 2\xi_t) \frac{G}{c^2} \rho^2 + A \eta_\rho \rho^{1-s} = 0.
\]

(102)

Solving (99-102), we find that

\[
\xi(\rho, t) = -ast + b, \quad \eta(\rho, t) = a\rho,
\]

(103)

with the constraint

\[
\frac{\dot{G}}{G} = \frac{2 \dot{c}}{c} + \frac{(1 - 2s)a}{b - ast},
\]

(104)

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Thus, we have found all the possible forms of $G$ and $c$ for which eq. (95) admits symmetries. There are two cases with respect to the values of the constant $a$; $a = 0$ which correspond to $G = \text{const}$, $c = \text{const}$, and $a \neq 0$ which correspond to

$$
\frac{G}{c^2} = K [ast + b]^{-2+\frac{1}{2}},
$$

(105)

with $K$ a constant of integration. If $s = \frac{1}{2}$, which corresponds to $\gamma = \frac{1}{2}$, then $G$ and $c$ remain constants. For this form of $G$ and $c$ eq. (95) admits a single symmetry

$$
X = (ast - b) \partial_t - (a \rho) \partial_\rho.
$$

(106)

The knowledge of one symmetry $X$ might suggest the form of a particular solution as an invariant of the operator $X$, i.e. the solution of

$$
\frac{dt}{\xi(t, \rho)} = \frac{d\rho}{\eta(t, \rho)}.
$$

(107)

This particular solution is known as an invariant solution (generalization of similarity solution). In this case

$$
\rho = \rho_0 t^{-\frac{1}{s}},
$$

(108)

where for simplicity we have taken $b = 0$, and $\rho_0$ is a constant of integration (note that $s = (1 - \gamma)$ being $\gamma$ the bulk viscous parameter).

We can apply a pedestrian method to try to obtain the same results. In this way, taking into account dimensional considerations, from the eq. (95) we obtain the following relationships between density, time and gravitational constant:

$$
k_{\gamma}^{-1} s \rho^s t \lesssim 1, \quad B \frac{G}{c^2} \rho t^2 \lesssim 1.
$$

(109)

This last relationship is also known as the relation for inertia obtained by Sciama. From these relationships, we obtain

$$
\rho \approx B^{-1} K^{-1} c^2 [ast + b]^{\frac{2s-1}{s}} t^{-2} \approx B^{-1} K^{-1} c^2 [ast]^{-\frac{1}{s}}.
$$

(110)

We can see that this result verifies the relation

$$
k_{\gamma}^{-1} s \rho^s t \lesssim 1.
$$

(111)
Once we have obtained $\rho$, we can obtain $f$ (the scale factor) from
\[ \rho = A_{\omega, \kappa} f^{-3(\omega+1+\kappa)} \implies f = (A_{\omega, \kappa} \rho_0^{-1} t)^{\frac{1}{3(\omega+1+\kappa)}} \],
\(112\)

In this way, we find $H$ and from eq. (26) the behaviour of $\Lambda$ is obtained as:
\[ \Lambda = \frac{3\kappa^2 - 8\pi K(\omega s)^{-2+\frac{1}{8}} \rho_0}{c^2 t^2} = \frac{l}{c^2 t^2}, \]
\(113\)

where, $\kappa = \frac{(A_{\omega, \kappa} \rho_0^{-1})^3}{3(\omega+1+\kappa)s}$. On replacing all these results into eq. (28), we obtain the exact behaviour for $c$, i.e.
\[ -\left(\frac{l}{4\pi K \rho_0} + \frac{1-2s}{s}\right) \frac{1}{t} = \left(\frac{l}{4\pi K \rho_0} + 2\right) \frac{\dot{c}}{c}, \]
\(114\)

therefore
\[ c = c_0 t^{-\varepsilon}, \]
\(115\)

where, $\varepsilon = \frac{(\lambda+\frac{1}{3})}{(\lambda+2)}$ and $\lambda = \frac{l}{4\pi K \rho_0}$.

Therefore, the assumption $\Pi = \kappa \rho$ together with the equation (29) is very restrictive. Hence, we conclude that under these assumptions, the field equations do not admit any other solution.

**VII. RELAXING THE HYPOTHESES**

In the previous section, we have seen that our assumption $\Pi = \kappa \rho$ together with the equation (29) does not give rise to any new solutions. This motivates us to investigate our assumptions further and relax the hypotheses we have used in obtaining solutions to the field equations. With the following assumptions obtained (suggested) from the sub-section III A, we would like to obtain an equation that helps us to determine the behaviour of the “constants”. The new hypotheses are:

1. Relationship between $G$ and $c$:
\[ \frac{G}{c^2} = K k^b H^{2+b}. \]
\(116\)

2. $\Lambda$ follows the law:
\[ \Lambda = \frac{lH^2}{c^2} \]
\(117\)
where \( K, l \in \mathbb{R} \) are numerical constants and \( b = 1/(\gamma - 1) \).

In this way, we allow the scale factor \( f \) to be an arbitrary function of time and it does not follow a power law i.e. \( f \propto t^\alpha \) as in previous sections. From the eq. (81) and taking into account the assumptions, we obtain

\[
\rho = dH^{-b},
\]

(118)

where, \( d = \left[ \frac{3-l}{8\pi K k^b} \right] \). Therefore from (27), we obtain

\[
\Pi = \frac{bd}{3} H^{-2-b} \dot{H} - d (\omega + 1) dH^{-b}
\]

(119)

If we introduce these results into the equation (29), then

\[
\ddot{H} + K_1 H \dot{H} + K_2 H^{-1} \dot{H}^2 + K_3 H^3 = 0
\]

(120)

where \( W = \left( \frac{2\omega + 1}{\omega + 1} \right) \) and

\[
K_1 = \left( 3 + k_{\gamma}^{-1} d_{\gamma}^{-1} \right), \quad K_2 = (-b - 2 + Wb), \quad K_3 = 3 \left( \frac{9 + 3\omega}{2b} - \frac{k_{\gamma}^{-1} d_{\gamma}^{-1} (\omega + 1)}{b} \right),
\]

(121)

In order to solve eq. (120), we use the Lie method which results in the following overdetermined system:

\[
-\xi_{HH} + \xi_H K_2 H^{-1} = 0
\]

(122)

\[
-\eta K_2 H^{-2} + \eta_{HH} - 2\xi_H + \eta_H K_2 H^{-1} + 2\xi_H K_1 H = 0
\]

(123)

\[
\eta K_1 + 2\eta_{HH} - \xi_{tt} + \xi_t K_1 H + 3\xi_H K_3 H^3 + 2\eta K_2 H^{-1} = 0
\]

(124)

\[
3\eta K_3 H^2 + \eta_H - \eta_H K_3 H^3 + 2\xi_t K_3 H^3 + \eta K_1 H = 0
\]

(125)

Solving (122-125), we find that

\[
\xi(t, H) = -at + b, \quad \eta(t, H) = aH
\]

(126)

\( a \) and \( b \) being numerical constants. For this form of \( \xi \) and \( \eta \), eq. (120) admits a single symmetry

\[
X = (at - b) \partial_t - (aH) \partial_H.
\]

(127)

Following the standard procedure, we find

\[
\frac{dt}{\xi(t, H)} = \frac{dH}{\eta(t, H)} \implies H = \frac{1}{at - b},
\]

(128)
which imply that

\[ f = (at - b)^{\frac{1}{2}} \]  

Thus, we obtain a power law solution as in previous section. To complete the set of equations, we have to take into account the next equation

\[-\frac{\dot{\Lambda}}{8\pi G\rho} - \frac{\dot{G}}{G} + 4\frac{\dot{c}}{c} = 0 \]  

and replace it into all our results thereby leading to the next relationship,

\[ c = KH^\alpha \]  

where \( \alpha = \frac{l + A(2+b)}{l + 2A} \) and \( A = 4\pi Kd \). In this way, we can see that

\[ G = KH^{2(\alpha + 1) + b} \quad \text{and} \quad \Lambda = lH^{2(1-\alpha)}. \]  

On using a straightforward calculation, we see that we have obtained the same solution even with different assumptions. The hypotheses considered earlier were relaxed in the above analysis with a view of obtaining a new set of solutions. This suggests that solutions obtained earlier are of a fairly general nature and do not really depend on any specific assumptions.

**VIII. CONCLUSIONS.**

In the present paper, we have studied a causal bulk viscous cosmological model with time varying constants. The dimensional analysis of the model allows us to make the following assumptions: the bulk viscous pressure is proportional to the energy density of the cosmological fluid and that for \( \gamma = 1/2 \), the relation \( G/c^2 \) remains constant in spite of considering “constants” \( G \) and \( c \) as functions of \( t \). In addition, we have shown that for \( \gamma = 1/2 \), the dynamics of our model is similar to that a perfect fluid FRW model. With the help of these assumptions, the general solution of the gravitational equations can be obtained in an exact form, leading to a power-law behaviour of the physical parameters on the cosmological time. These solutions show us that the “constants” \( G, c \) and \( \Lambda \) are decreasing functions of time and that for \( \gamma = 1/2 \), we obtain a model similar to a perfect fluid cosmological model with time-varying constants in the presence of adiabatic matter creation. This striking similarity of our results is not surprising as it has been pointed out by many
authors that the introduction of viscosity in the cosmological fluid is a phenomenological description of the effect of creation of particles by the non-stationary gravitational field of the expanding cosmos. We have shown that we cannot consider the bulk viscosity in the matter predominance era since the fluid temperature increases indefinitely with time. We arrive at the same conclusion in the case of adiabatic matter creation. This fact is in agreement with a previous result where we studied a full causal bulk viscous model through the renormalization group method arriving to the conclusion that in the long-time asymptotics the cosmological model tends to a flat ideal (non-viscous) Friedmann type geometry.

We also notice that the $\kappa$-parameter, i.e. the causal bulk viscous effect, weakly perturbs the FRW perfect fluid solution in such a way that the comoving entropy varies with $t$, while in the perfect fluid case, i.e. $\kappa = 0$, the comoving entropy is constant. The same considerations could be taken into account for the scale factor $f(t) = f_0 t^{\frac{\gamma - 1}{(\omega + 1 + \gamma)}}$, which is perturbed weakly for the bulk viscous parameter as we can see in the special case of $\gamma = 1/2$. The solutions obtained for the scale factor, energy density, entropy, etc. are similar “but different” to those obtained for a perfect fluid model where the $\kappa$-parameter vanishes. We would like to point out that our model is thermodynamically consistent for the usual matter equations of state and valid for all $\gamma$-parameter, i.e. $\forall \gamma \in (0, 1)$.

In the model considered, the evolution of the Universe starts from a singular state, with the energy density, bulk viscosity coefficient and cosmological constant tending to infinity. At the initial moment, $t = 0$, the relaxation time is $\tau = 0$. From the singular initial state, the Universe starts to expand, with the scale factor $f$ as a $\gamma$-dependent function. The “constants” $G, c$ and $\Lambda$ are all decreasing functions of time. In the radiation predominance era we can take into account the effects of the bulk viscosity in all the quantities but when the universe enters a matter predominance era, these effects vanish and in this case our model behaves as a perfect fluid model with variable constants. The present model is not defined for $s = 1$, showing that in the important limit of small densities a different approach is necessary.

The unicity of the solution is also proved by the investigation of the Lie group symmetries of the basic equation describing the time variation of the mass density of the Universe.

Bulk viscosity is expected to play an important role in the early evolution of the Universe,
when also the dynamics of the gravitational and cosmological constants and the speed of light could be different. Hence the present model, despite its simplicity, can lead to a better understanding of the dynamics of our Universe in its first moments of existence.

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