Constrained control methods for lower extremity rehabilitation exoskeleton robot considering unknown perturbations

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Abstract In this paper, trajectory tracking control is investigated for lower extremity rehabilitation exoskeleton robot. Unknown perturbations are considered in the system which are inevitable in reality. The trajectory tracking control is constructively treated as constrained control issue. To obtain the explicit equation of motion and analytical solution of lower extremity rehabilitation exoskeleton robot, Udwadia–Kalaba theory is introduced. Lagrange multipliers and pseudo-variables are not needed in Udwadia–Kalaba theory, which is more superior than Lagrange method. On the basis of Udwadia–Kalaba theory, two constrained control methods including trajectory stabilization control and adaptive robust control are proposed. Trajectory stabilization control applies Lyapunov stability theory to modify the desired trajectory constraint equations. A leakage type of adaptive law is designed to compensate unknown perturbations in adaptive robust control. Finally, comparing with nominal control and approximate constraint-following control, simulation results demonstrate the superiority of trajectory stabilization control and adaptive robust control in trajectory tracking control.

Keywords Lower extremity rehabilitation exoskeleton robot · Trajectory tracking control · Udwadia–Kalaba theory · Unknown perturbations · Constraint control

1 Introduction

According to the statistics from the United Nations in 2019, the proportion of aged 65 years is 9% of the world’s population, which will be double in 2050 [1]. Lower extremity rehabilitation exoskeleton robot (LERER) is a boon for people with lower limb disabilities. It helps them with rehabilitation training and even assists them to walk normally [2]. Clinical medical researches, meanwhile, have shown that human nervous system can reestablish synaptic connections through rehabilitation training [3]. In recent years, researchers have attached remarkable attention to LERER and conducted numerous studies in related fields [4–6]. Chen et al. [4] propose the impedance control method to not only follow the desired trajectories but also to minimize the influence of external contact force in the swing phase of LERER. Adaptive seamless assist-as-needed control approach [5] is deployed in lower extremity rehabilitation robot to provide the...
assistance torque according to people’s needs. Hasan and Dhingra [6] formulate the model reference computed torque controller to estimate the joint torque requirements for tracking a trajectory.

The movement patterns of LERER are mainly divided into active and passive modes [7]. The trajectory tracking control is a necessary basic function, which is indispensable in LERER control system. The research of trajectory tracking control is necessary and significant. Establishing the kinematic or dynamical equations is the first stage in implementing trajectory tracking control. Generally, traditional Lagrangian or Newton–Euler is notable to describe constrained control system. However, Lagrangian multipliers or pseudo-variables are required. This makes the control process cumbersome and the calculation complex, which increases the challenge of control [8].

The control methods in this paper are developed on the ground of Udwadia–Kalaba (U–K) theory. It was proposed by Professors Udwadia and Kalaba of the University of Southern California in 1992 [9–12]. Form the view of constrained control, U–K theory can express the dynamic equations of the constrained system explicitly in the form of second order. The modeling process does not require any linearization and approximation, which greatly simplifies the computational process and control complexity.

In recent years, many researchers have achieved significant scientific results through U–K theory [13–17]. Chen et al. [18–20] realize the constraint tracking servo control of mechanical system based on U–K theory and complete the optimal design of fuzzy mechanical system. Sun et al. [21–23] adopt U–K theory to realize adaptive robust control of different mechanical systems considering the uncertainties; especially in [21], the vehicle dynamics are practically stable and lane following maneuvering is achieved by adaptive robust control, which can deal with nonlinear mechanical systems with both holonomic constraints and nonholonomic constraints. Zhen et al. [24] design a systematic control method for constrained mechanical systems and apply in coordinated robot. Yu et al. [25] extend U–K method to automated guided vehicle in order to solve the trajectory tracking control issue. For underactuated system with nonholonomic constraints, Chen et al. [26] present a novel control approach utilizing U–K approach, which is proved by Lyapunov approach.

Furthermore, Udwadia and Wanichanon apply U–K theory to the nonlinear multibody systems with perturbations, the “nominal system” that contains a control force is developed and the controller based on a concept of a sliding surface is formulated to handle the uncertainties [27–29]. In the frame of U–K theory, a formation-keeping control scheme with attitude constraints is proposed in the presence of uncertainties in the masses and moments of inertia of the satellites [30, 31]. The optimal tracking control of dynamical systems is considered by Udwadia [32], and the closed-form results are also obtained for the real-time optimal control. The practicality of U–K theory has been fully demonstrated. And U–K theory is not only applicable to the ideal situation, but also applicable to the situation with perturbations [33].

The main work of this paper is summarized as follows:

1. The unconstrained dynamics model of the LERER is constructed by the conventional Lagrangian equation. The explicit equations of motion are obtained by applying U–K theory to find the analytical solution. Moreover, the solution can be used as the nominal control.

2. Trajectory stabilization control (TSC) and adaptive robust control (ARC) are designed in this paper. TSC employs Lyapunov stability theory to modify the desired trajectory constraints. ARC which is designed with leakage-type adaptive laws corrects the control forces. The uniform boundedness and uniform ultimate boundedness of the LERER system can be guaranteed by ARC.

3. Simulation results are conducted to prove that TSC and ARC have excellent performance in trajectory tracking control. Their tracking effects are significantly better than those of the nominal control and approximate constraint-following control (ACFC) in [50].

The paper is structured as follows. Section 2 introduces the foundation knowledge of U–K theory. The dynamics of the unconstrained LERER system is modeled utilizing the Lagrangian method in Sect. 3. Section 4 introduces nominal control without unknown perturbations, TSC and ARC considering unknown disturbance. Simulation results are conducted in Sect. 5 to verify the effectiveness of trajectory tracking. Section 6 recaps all the works in the full paper.
2 Udvardy–Kalaba theory

Consider an unconstrained mechanical system described by $n$ generalized coordinates, whose equations of motion can be constructed by Lagrangian or Newton–Euler methods [34]:

$$M(x(t), \dot{x}(t), t)\ddot{x} + C(x(t), \dot{x}(t), t)\dot{x} + G(x(t), \dot{x}(t), t) = \tau(t)$$  
(1)

where $t \in \mathbb{R}$ expresses time, $x(t), \dot{x}(t)$ and $\ddot{x}(t) \in \mathbb{R}^n$ stand for generalized coordinates, velocity and acceleration, respectively. $\tau(t) \in \Sigma$ denotes parameters of perturbations. $\Sigma$ represents the boundary which is the range of $\delta$. $M \in \mathbb{R}^{n \times n}$ denotes the inertial mass matrix. $C \in \mathbb{R}^{n \times n}$ denotes the term of centrifugal force or coriolis force. $G \in \mathbb{R}^n$ denotes the term of gravity. $\tau \in \mathbb{R}^m$ is constraint force of the control input. When the system is free of constraint, $\tau = 0$.

Assumption 1  For an arbitrary $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, $\delta \in \Sigma$, $M(x, \delta, t) > 0$.

The coordinates $x(t)$ can be chosen according to the specific situation, which do not necessarily be generalized coordinates. Apply $n$ servo constraints to the unconstrained mechanical system described in Eq. (1) [35, 36]:

$$\sum_{i=1}^{m} B_i(x(t)) = d_i(x(t), t), \quad l = 1, \ldots, m$$  
(2)

where $B_i(\cdot)$ and $d_i(\cdot)$ are both $C^1$ and $1 \leq m \leq n$. Equation (2) means that the sum of the elements in the ith row of $B$ is equal to that of $d$, so the number of lines on the left and right sides in Eq. (2) is equal. It can be rewritten in the matrix form

$$B(x, t) = d(x, t)$$  
(3)

where $B = [B_i]_{m \times n} = \sum_{i=1}^{m} B_i(x, t)$ and $d = [d_1, d_2, \ldots, d_m]^T$.

Equation (2) may contain holonomic constraints or nonholonomic constraints. The forms of holonomic and nonholonomic constraints are shown as Eq. (4) and (5) [37]

$$\psi_i(x, t) = 0, \quad i = 1, 2, \ldots, k$$  
(4)

$$\psi_i(x, \dot{x}, t) = 0, \quad i = k + 1, k + 2, \ldots, m.$$  
(5)

By taking one derivative of Eq. (5) (nonholonomic constraint) or two derivatives of Eq. (4) (holonomic constraint), we can obtain the second-order constraint form of Eq. (6):

$$A(x, \dot{x}, t)\ddot{x} = b(x, \dot{x}, t)$$  
(6)

where $b = [b_1, b_2, \ldots, b_m]^T$. The acceleration is a linear function in the second-order constraint which is beneficial for further dynamic analysis. Almost all control problems can be written in the second-order constrained form, including stability, trajectory tracking and optimal control [38].

Assumption 2  The constraint in Eq. (6) is continuous and $A(x, \dot{x}, t)$ can be rank-deficient or full rank [39, 40].

Remark 1  For a given constraint, $A$ and $b$ are determined. There is at least one solution for $\ddot{x}$ in Eq. (6); then, Eq. (6) can be seen as continuous [41].

Theorem 1  (U–K theory) Assumptions 1–2 are satisfied, an unconstrained mechanical system is subjected to constraints as shown in Eq. (6), and then, the constraint force $Q^c$ is shown as [10, 42]

$$Q^c = M^{1/2}(AM^{-1/2})^+((b + AM^{-1}(C + G)).$$  
(7)

where “$+$” is the Moore–Penrose inverse.

The mechanical system considered in U–K theory is free of perturbations or the perturbations are known, in this case $\tau = Q^c$. However, the perturbations are available and undetermined in practical applications. Nominal control, also called Udvardy control, can be designed based on U–K theory.

3 Dynamic model of LERER

LERER considered in this paper has two rotational actuated DOFs of hip and knee joints and one passive DOF of ankle joint. The two-dimensional simplified model of LERER is shown in Fig. 1. The simplified model is composed of thigh and shank. $O$ is the origin of the plane coordinate system. $P$ and $Q$ represent hip and knee joints, respectively. The thigh and shank’s center of masses are $P'$ and $Q'$, respectively. $\theta_1$ and $\theta_2$ denote rotation angles of hip and knee joints, where counterclockwise direction is positive. $l_1$ and $l_2$ are lengths of thigh and shank, respectively.

The unconstrained ($\tau = 0$) dynamics equation of LERER can be obtained by Lagrangian method [43]:
\[ M(x, t)\ddot{x} + C(x, \dot{x}, t) + G(x, t) = \tau \]  
(8)

where

\[ x = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}, \dot{x} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}, \ddot{x} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \]  
(9)

\[ M(x, t) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \]  
(10)

\[ C(x, \dot{x}, t) = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \]  
(11)

\[ G(x, t) = \begin{bmatrix} G_{11} \\ G_{21} \end{bmatrix} \]  
(12)

with

\[ M_{11} = \frac{1}{4} m_1 l_1^2 + I_1 + m_2 l_2^2 \]  
(13)

\[ M_{12} = M_{21} = -\frac{1}{2} m_2 l_1 l_2 \cos(\theta_1 + \theta_2) \]  
(14)

\[ M_{22} = \frac{1}{4} m_2 l_2^2 + I_2 \]  
(15)

\[ C_{11} = \frac{1}{2} m_2 l_1 l_2 \sin(\theta_1 + \theta_2) \dot{\theta}_2 \]  
(16)

\[ C_{12} = C_{21} = -\frac{1}{2} m_2 l_1 l_2 \sin(\theta_1 + \theta_2) \left( \dot{\theta}_1 + \dot{\theta}_2 \right) \]  
(17)

\[ C_{22} = \frac{1}{2} m_2 l_1 l_2 \sin(\theta_1 + \theta_2) \dot{\theta}_1 \]  
(18)

\[ G_{11} = -\frac{1}{2} m_1 g l_1 \sin \theta_1 - m_2 g l_1 \sin \theta_1 \]  
(19)

\[ G_{21} = -\frac{1}{2} m_2 g l_2 \sin \theta_2 \]  
(20)

where \( m_1 \) and \( m_2 \) are mass of thigh and shank; \( I_1 \) and \( I_2 \) represent rotational inertia of thigh and shank; and \( g \) is the gravitational acceleration.

We can find the analytical solution according to U–K theory once the system is constrained. Equation (8) can provide required matrices \( M, C \) and \( G \) for Eq. (7). The desired trajectory is differentiated once or twice to obtain \( A \) and \( b \) in Eq. (6). Employing U–K theory, the trajectory tracking control problem can be regarded as constrained control.

### 4 Trajectory tracking control design

#### 4.1 Desired trajectory constraints

The designed LERER has two activated DOFs. Assume that each activated DOF follows certain trajectory constraints, which are obtained from clinical gait analysis data [44]:

\[ \theta_1 = 0.8 + 0.2 \sin(2\pi t), \]  
(21)

\[ \theta_2 = 1.2 - 0.2 \cos(2\pi t). \]  
(22)

Taking twice derivatives of Eqs. (21)–(22) with respect to time:

\[ \ddot{\theta}_1 = -0.8\pi^2 \sin(2\pi t), \]  
(23)

\[ \ddot{\theta}_2 = 0.8\pi^2 \cos(2\pi t). \]  
(24)

Rewriting Eqs. (23)–(24) into the form of Eq. (6):

\[ A(x, \dot{x}, t)\ddot{x} = b(x, \dot{x}, t) \]  
(25)

where

\[ A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, b = \begin{bmatrix} -0.8\pi^2 \sin(2\pi t) \\ 0.8\pi^2 \cos(2\pi t) \end{bmatrix}. \]  
(26)
4.2 Trajectory stabilization control

The desired trajectory constraint can be modified according to Lyapunov stability theory [45, 46]:

\[ \dot{\psi}(x, \dot{x}, t) = g(\psi, t; \xi) \]  \hspace{1cm} (27)

where \( g(\psi, t; \xi) \) is a vector of m-dimensional and contains parameter \( \xi \). Equation (27) should meet the below two qualifications:

1. One of the equilibrium points for the constraint equation is \( \psi = 0 \).
2. The equilibrium points are global asymptotic stability.

In practical implementation, when the constraint is in the form of Eqs. (4), (27) can be rewritten as:

\[ \dot{\psi}_i + \phi_1 \dot{\psi}_i + \phi_2 \psi_i = 0, \quad i = 1, 2, \ldots, k \]  \hspace{1cm} (28)

where \( \phi_1, \phi_2 > 0, \psi = \dot{\psi} = 0 \) is equilibrium point, which is global asymptotic stability. The above two qualifications are satisfied. Taking the modified Eq. (28) as the novel trajectory constraint. Deriving Eq. (28) can obtain the constraint equation in the form of Eqs. (6):

\[ A(x, \dot{x}, t)\ddot{x} = \bar{b}(x, \dot{x}, t) \]  \hspace{1cm} (29)

Therefore, the novel control force obtained by U–K theory:

\[ \bar{\tau} = \bar{Q}c = M^{1/2}(AM^{-1/2})^\top(\bar{b} + AM^{-1}(C + G)) \]  \hspace{1cm} (30)

4.3 Adaptive robust control

The equation of motion considering the unknown perturbation is given in Eq. (1). Decompose \( M, C \) and \( G \) into a “nominal part” and a “perturbed part”:

\[ M(x, \delta, t) = M(x, t) + \Delta M(x, \delta, t), \]  \hspace{1cm} (31)

\[ C(x, \dot{x}, \delta, t) = C(x, \dot{x}, t) + \Delta C(x, \dot{x}, \delta, t), \]  \hspace{1cm} (32)

\[ G(x, \delta, t) = G(x, t) + \Delta G(x, \delta, t) \]  \hspace{1cm} (33)

Assume the terms on the right-hand side of Eqs. (31)–(33) are all continuous and uninterrupted. When there are no unknown perturbations, then

\[ M(x, \delta, t) = M(x, t), C(x, \dot{x}, \delta, t) = C(x, \dot{x}, t), \]

\[ G(x, \delta, t) = G(x, t). \]  \hspace{1cm} (34)

Let \( H(x, \delta, t) := M(x, t) \times M^{-1} - I, R(x, t) := \bar{M}^{-1}(x, t) \) and \( \Delta R(x, \delta, t) := \Delta M^{-1}(x, \delta, t) - \bar{M}^{-1}(x, t) \), we can obtain that \( \Delta R(x, \delta, t) = R(x, t)H(x, \delta, t). \)

Assumption 3 Depending on Assumption 2, for a given matrix \( P \in R^{m \times m} \times R, P > 0 \), let:

\[ U(x, \delta, t) := PA(x, \dot{x}, t)R(x, t)h(x, \delta, t) \times \]

\[ + A^T(x, \dot{x}, t)(A(x, \dot{x}, t)A^T(x, \dot{x}, t))^{-1}P^{-1}, \]

\[ \frac{1}{2} \min_{\delta \in \sum} \lambda_m(U(x, \delta, t) + U^T(x, \delta, t)) \geq \rho_E \]  \hspace{1cm} (35)

where \( \lambda_m \) is the eigenvalues of matrix \( U(x, \delta, t) + U^T(x, \delta, t) \).

**Assumption 4**  (1) There exists an unknowable constant vector \( \alpha \in (0, \infty)^k \) and a known function \( \Pi(\cdot) : (0, \infty)^k \times R^n \times R^n \times R, \delta \in \sum \):\n
\[ (1 + \rho_E)^{-1} \max_{\delta \in \sum} (PA(x, t)\Delta R(x, \delta, t) \times \]

\[ (-C(x, \dot{x}, \delta, t)\Delta C(x, \dot{x}, \delta, t) + \tau_1(x, \dot{x}, t) \]

\[ + \tau_2(x, \dot{x}, t) - PA(x, \dot{x}, t)R(x, t) \times (\Delta C(x, \dot{x}, \delta, t) \dot{x} \]

\[ + \Delta G(x, \delta, t))) \leq \Pi(x, \dot{x}, t) \]. \hspace{1cm} (36)

(2) \( \Pi(x, \dot{x}, t) \) is a linear function with respect to \( \alpha \)

for all \((x, \dot{x}, t). \) And there exists a basis function \( \hat{\Pi}(\cdot) : R^n \times R^n \times R \to R_+ \):

\[ \Pi(x, \dot{x}, t) = \alpha^T \hat{\Pi}(x, \dot{x}, t). \]  \hspace{1cm} (37)

**Remark 2** The unknown constant vector \( \alpha \) depends on the bound of the perturbations. Furthermore, (35) implies that the unknown upper bound vector function of perturbations can be linearly parameterized with the basis function \( \hat{\Pi}(\cdot) \).

Based on U–K theory and Assumptions 2–4, ARC is proposed to modify the control force \( \tau \) as \( \bar{\tau} \).
\[ \tau(t) = \bar{\tau}(t) = \tau_1(x(t), \dot{x}(t), t) + \tau_2(x(t), \dot{x}(t), t) + \tau_3(x(t), \dot{x}(t), t), \]

\[ \tau_1(x, \dot{x}, t) := \bar{M}_{1/2}^{-1}(x, t)(A(x, \dot{x}, t)M_{-1/2}^{-1}(x, t))^+ \]

\[ \times \left( b(x, \dot{x}, t) + A(x, \dot{x}, t)M^{-1}(x, t) \right) \]

\[ (C(x, \dot{x}, t) + G(x, t)), \]

\[ \tau_2(x, \dot{x}, t) := -kM(x, t)A^T(x, \dot{x}, t) \]

\[ \times \left( A(x, \dot{x}, t)A^T(x, \dot{x}, t) \right)^{-1} \times P^{-1} \beta(x, \dot{x}, t) \]

\[ \tau_3(x, \dot{x}, t) := -\bar{M}(x, t)A^T(x, \dot{x}, t) \times \left( A(x, \dot{x}, t)A^T(x, \dot{x}, t) \right)^{-1} \]

\[ \times P^{-1} \gamma(x, \dot{x}, t) \]

where \( \gamma(x, \dot{x}, t) \) is a constant.

\[ \gamma(x, \dot{x}, t) = \begin{cases} \frac{1}{\mu(x, \dot{x}, t)} & \text{if } \mu(x, \dot{x}, t) > \dot{\epsilon} \\ \frac{1}{\dot{\epsilon}} & \text{if } \mu(x, \dot{x}, t) \leq \dot{\epsilon} \end{cases} \]

where \( \dot{\epsilon} > 0 \) is a constant.

\[ \mu(x, \dot{x}, t) = \beta(x, \dot{x}, t) \Pi(x, \dot{x}, t), \]

\[ \beta(x, \dot{x}, t) = A(x, \dot{x}, t)\dot{x} - c(x, t) + \eta(B(x, t) - d(x, t)), \]

where \( B(x, t) \) and \( d(x, t) \) are the same as in Eq. (3). \( B(x, t) - d(x, t) \) stands for the constraint error and \( \eta \) represents its weight.

\( \tau_1 \) is nominal control handling the absence of unknown perturbations in mechanical system; Initial condition offset problem is mainly solved by \( \tau_2 \); The role of \( \tau_3 \), the most essential item for ARC, is to compensate for unknown perturbations. Designing adaptive law \( \dot{\hat{z}} \) to control adaptive parameter \( \hat{z} \):

\[ \dot{\hat{z}} = k_1 \Pi(x, \dot{x}, t) \beta(x, \dot{x}, t) - k_2 \hat{z}. \]

**Remark 3** In this section, the generalized coordinates and velocities are still \( x(t) \) and \( \dot{x}(t) \) and their values are updated values considering the perturbations. In other words, we update the values of generalized coordinates and velocities to consider the uncertainty in the derivation of the control input.

**Theorem 2** Consider a constrained mechanical system of Eq. (1) met Assumptions 2–4, let \( \vartheta := \begin{bmatrix} \beta^T & (\dot{\hat{z}} - \hat{z})^T \end{bmatrix} \in \mathbb{R}^{m+k} \), ARC satisfies the uniform boundedness and uniform ultimate boundedness [47]:

1. Uniform boundedness: For all \( w > 0 \), there exists \( d(w) < \infty \). If \( \vartheta(t) \leq w \), then for all \( t \geq t_0 \), there exists \( \vartheta(t) \leq d(w) \).

2. Uniform ultimate boundedness: For all \( w > 0 \) and \( \vartheta(t_0) \leq w \), there exists \( \delta > 0 \); such that for all \( t \geq t_0 \), there exists \( \vartheta(t) \leq \delta \), where \( t \geq t_0 + T(d, w) \) and \( T(d, w) < \infty \).

**Proof** Choose a legitimate Lyapunov function candidate as follows [48, 49]:

\[ V(\beta, \dot{\hat{z}} - \hat{z}) = \beta^T \bar{P} \beta + k_1^{-1}(1 + \rho_E)(\dot{\hat{z}} - \hat{z})^T(\dot{\hat{z}} - \hat{z}). \]

(47)

In the procedure of proof, except for some key parameters, we basically leave the parameters of the function out. For a specific mechanical system with uncertainty and desired constraints, the \( \dot{V} \) is shown as:

\[ \dot{V} = 2\beta^T \bar{P} \dot{\beta} + 2k_1^{-1}(1 + \rho_E)(\dot{\hat{z}} - \hat{z})^T(\dot{\hat{z}} - \hat{z}). \]

(48)

By (8), (38) and (45), the first term of the right-hand side of (48) can be analyzed as:

\[ 2\beta^T \bar{P} \dot{\beta} = 2\beta^T \bar{P}(A\dot{x} - b + \eta(A\dot{x} - c)) \]

\[ = 2\beta^T \bar{P}(AM^{-1}(-C\dot{x} - \bar{G} + \tau) - b + \eta(A\dot{x} - c)) \]

\[ = 2\beta^T \bar{P}(AM^{-1}(-C\dot{x} - \bar{G} + \tau_1 + \tau_2 + \tau_3) \]

\[ + \eta(A\dot{x} - c) - b) \]

(49)

By (1) and (31)–(33),

\[ AM^{-1}(-C\dot{x} - \bar{G} + \tau_1 + \tau_2 + \tau_3) \]

\[ = A(R + \Delta R)(-C\dot{x} - \Delta C\dot{x} - \bar{G} - \Delta G + \tau_1 + \tau_2 + \tau_3) \]

\[ = A(R(-C\dot{x} - \bar{G} + \tau_1 + \tau_2) + R(-\Delta C\dot{x} - \Delta G) + \]

\[ AR(-C\dot{x} - \bar{G} + \tau_1) - b = 0 \]

(50)

Based on (39),

\[ AR(-C\dot{x} - \bar{G} + \tau_1) - b = 0. \]

(51)
According to Assumption 4–1,
\[
2\beta^TP(AR(-\Delta C\dot{x} - AG) + A\Delta R(-C\dot{x} - G + \tau_1 + \tau_2)) \\
\leq 2\beta PAR(-\Delta C\dot{x} - AG) + PARH(-C\dot{x} - G + \tau_1 \\
+ \tau_2) \leq 2(1 + \rho_E)\beta \Phi(x, x, \dot{x}, t).
\]
(52)

By (40), we obtain
\[
2\beta^TPAR\tau_2 = 2\beta^T PAR\left(-k_1MA^T(AA^T)^{-1}P^{-1}\beta\right) \\
= -2k_1\beta^T \beta = -2k\beta^2
\]
(53)

Based on (41), (44) and \(\Delta R(x, \delta, t) = R(x, t)H(x, \delta, t)\),
\[
2\beta^TPAR(\Delta R)\tau_3 = 2\beta^T PAR\left(-\left(\begin{array}{l}
\mu^T (AA^T)^{-1}P^{-1} \\
\gamma \mu \Pi(\dot{x}, x, \dot{x}, t)
\end{array}\right)\right)
\gamma \mu \Pi(\dot{x}, x, \dot{x}, t)
\]
\[
= -2\gamma \mu + 2\beta^T PAR\left(-\left(\begin{array}{l}
\mu^T (AA^T)^{-1}P^{-1} \\
\gamma \mu \Pi(\dot{x}, x, \dot{x}, t)
\end{array}\right)\right) \gamma \mu \Pi(\dot{x}, x, \dot{x}, t)
\]
(54)

According to Assumption 4 and Rayleigh’s principle [50, 51],
\[
2\beta^T PAR\left(-\left(\begin{array}{l}
\mu^T (AA^T)^{-1}P^{-1} \\
\gamma \mu \Pi(\dot{x}, x, \dot{x}, t)
\end{array}\right)\right) \gamma \mu \Pi(\dot{x}, x, \dot{x}, t)
\]
\[
= -2\gamma \mu^T \left(PARH\mu^T (AA^T)^{-1}P^{-1}\right) \mu
\]
\[
= -2\gamma \mu^T \left(PARH\mu^T (AA^T)^{-1}P^{-1} + P^{-1}(AA^T)^{-T} \right.
\]
\[
\left. AMH^T RA^TP \right) \mu = -2\gamma \mu^T (U + U^T) \mu
\]
\[
\leq -2\gamma \frac{1}{2} \lambda_m(U + U^T) \\|
\mu^2\| \leq -2\gamma \rho_E \| \mu^2\|
\]
(55)

Combining (54) and (55),
\[
2\beta^TPAR(\Delta R)\tau_3 \leq -2\gamma (1 + \rho_E) \| \mu^2\|
\]
(56)

If \(\mu > \hat{\mu}\), based on (53),
\[
-2\gamma (1 + \rho_E) \| \mu^2\| = -2(1 + \rho_E) \frac{1}{\hat{\mu}} \| \mu^2\|
\]
(57)

If \(\mu \leq \hat{\mu}\), based on (43),
\[
-2\gamma (1 + \rho_E) \| \mu^2\| = -2(1 + \rho_E) \frac{1}{\hat{\mu}} \| \mu^2\|
\]
(58)

In conclusion, for \(\mu > \hat{\mu}\), we have
\[
2\beta^TP\hat{\mu} =-2\gamma \mu \| \mu\|
\]
(59)

As for \(\mu \leq \hat{\mu}\),
\[
2\beta^TP\hat{\mu} \leq -2\gamma \mu \| \mu\|
\]
(60)

Based on Assumption 4 (2),
\[
2\beta^TP(\Delta R)\tau_3 = 2\beta^TPAR(\Delta R)\tau_3
\]
\[
= 2\beta^TPAR(\Delta R)\tau_2 - 2\beta^TPARh(\dot{x}, x, \dot{x}, t)
\]
\[
= 2\beta^TP^2(\dot{x}, x, \dot{x}, t)
\]
(61)

For all \(\mu\), we can obtain
\[
2\beta^TP\dot{\hat{\mu}} = -2\kappa \beta^2 + 2(1 + \rho_E) \| \mu\|
\]
(62)

Then, based on (46), we can analyze the second term of the right-hand side of (48) as
\begin{equation}
2k_1^{-1}(1 + \rho_E)(\dot{x} - x)^2 - 2k_1^{-1}(1 + \rho_E)(\dot{z} - a)^T (k_1 \| \beta \| - k_2 \ddot{z}) \\
= 2(1 + \rho_E)((\dot{x} - x)^T \| \beta \| - k_1^{-1}(\dot{x} - x)^T k_2 \ddot{z}) \\
= 2(1 + \rho_E)((\dot{x} - x)^T \| \beta \| - k_1^{-1}k_2(\dot{x} - x) \times (\dot{x} - a + x)) \\
= 2(1 + \rho_E)((\dot{x} - x)^T \| \beta \| - k_1^{-1}k_2(\dot{x} - x) \times (\ddot{x} - x) - k_1^{-1}k_2(\ddot{x} - x)^T x) \\
\leq 2(1 + \rho_E) \left( (\dot{x} - x)^T \| \beta \| - k_1^{-1}k_2(\dot{x} - x)^2 + k_1^{-1}k_3\| \beta \| - k_2 \| \gamma \| \right).
\end{equation}

Using (62) and (63) in (48),
\begin{equation}
\dot{V} \leq -2\kappa \| \beta \|^2 + (1 + \rho_E) \frac{d}{2} - 2k_1^{-1}k_2(1 + \rho_E) \times ||\ddot{x} - x^2|| + 2k_1^{-1}k_2(1 + \rho_E) ||\dot{z} - x|| ||\ddot{z}|| \\
\leq -k_1 \| \ddot{z} \|^2 + k_2 \| \ddot{z} \| + k_3
\end{equation}

where \( \| \ddot{z} \| := \| \beta \|^2 + \| \dot{x} - x^2 \|, \| \ddot{z} \| \geq \| \dot{z} - x \|, \).

The function \( d(w) \) in Theorem 2 (1) is shown as
\begin{equation}
d(w) = \begin{cases} \sqrt{\Psi_1} W & \text{if } w \leq W \\ \sqrt{\Psi_2} W & \text{if } w > W \end{cases}
\end{equation}

\begin{equation}
W = \frac{1}{2k_1} \left( k_2 + \sqrt{k_2^2 + 4k_1k_3} \right).
\end{equation}

Here, \( \Psi_1 = \min \{ \lambda_{min}(P), k_1^{-1}(1 + \rho_E) \} \) and \( \Psi_2 = \max \{ \lambda_{max}(P), k_1^{-1}(1 + \rho_E) \} \).

In addition, uniform ultimate boundedness in Theorem 2 (2) is also obedient to
\begin{equation}
d = \sqrt{\Psi_1}
\end{equation}

\begin{equation}
T(\mathcal{A}, w) = \begin{cases} 0 & \text{if } w \leq d \sqrt{\Psi_1} \\ \frac{\Psi_2w^2 - (\Psi_1^2/\Psi_2)d^2}{k_1d(\Psi_1/\Psi_2) - k_2d(\Psi_1/\Psi_2)^{1/2} - k_3} & \text{otherwise} \end{cases}
\end{equation}

5 Simulation results

In this section, simulation results are conducted to verify the superiority of TSC and ARC. The software platform is MATLAB R2019a. Table 1 describes the values of parameters in LERER.

5.1 Trajectory stabilization control results

The masses, friction and initial conditions of offset are taken as unknown perturbation terms. We select \( r(0) = 0 \) and the initial conditions of offset are described as \( \theta_1 = 1.2, \theta_2 = 1.2, \dot{\theta}_1 = 0, \dot{\theta}_2 = 0; \) the masses are introduced as \( m_1 = m_1 + \Delta m_1, \Delta m_1 = 0.1 \cos(0.1r), m_2 = m_2 + \Delta m_2, \Delta m_2 = 0.05 \cos(0.01r) \).

Furthermore, the friction is designed as \( F_j = A_1 \sin(o_1t) + A_2 \sin(3o_1t) + A_3 \sin(5o_1t) \) with \( A_1 = 3, A_2 = 2, A_3 = 1, o_1 = 2 \) [52]. The desired trajectory constraints are modified by applying the TRC, as shown in Eq. (69). Notice that \( \phi_i \) indicates nominal control.

\begin{equation}
\ddot{\psi}_j + \phi_j \dot{\psi}_j + \phi_j \psi_j = 0,
\end{equation}

where

\begin{equation}
\psi_1 = \begin{bmatrix} \psi_{11} \\ \psi_{12} \end{bmatrix} = \begin{bmatrix} \theta_1 - 0.8 - 0.2 \sin(2\pi t) \\ \theta_2 - 1.2 + 0.2 \cos(2\pi t) \end{bmatrix},
\end{equation}

\begin{equation}
\dot{\psi}_1 = \begin{bmatrix} \dot{\psi}_{11} \\ \dot{\psi}_{12} \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 - 0.4 \sin(2\pi t) \\ \dot{\theta}_2 - 0.4 \cos(2\pi t) \end{bmatrix},
\end{equation}

\begin{equation}
\ddot{\psi}_1 = \begin{bmatrix} \ddot{\psi}_{11} \\ \ddot{\psi}_{12} \end{bmatrix} = \begin{bmatrix} \ddot{\theta}_1 + 0.8 \pi^2 \sin(2\pi t) \\ \ddot{\theta}_2 + 0.8 \pi^2 \cos(2\pi t) \end{bmatrix}.
\end{equation}

Rewriting Eq. (69) in the form of Eq. (25), we can obtain that

\begin{equation}
A(x, x, t)\dot{x} = \bar{b}(x, x, t),
\end{equation}

where

\begin{equation}
A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\end{equation}

Table 1 Values of parameters in LERER

| Parameter | Value   | Parameter | Value   |
|-----------|---------|-----------|---------|
| \( m_1 \) | 36 kg   | \( I_1 \) | 2.5 kg \cdot m^2 |
| \( m_2 \) | 25 kg   | \( I_2 \) | 1.5 kg \cdot m^2 |
| \( l_1 \) | 0.5 m   | g         | 9.8 m/s^2    |
| \( l_2 \) | 0.45 m  |           |           |
\[
\vec{b} = \begin{bmatrix}
-0.8\pi^2 \sin(2\pi t) - \phi_1 \dot{\psi}_1 - \phi_2 \psi_2 \\
0.8\pi^2 \cos(2\pi t) - \phi_1 \dot{\psi}_2 - \phi_2 \psi_2
\end{bmatrix}.
\] (75)

Figures 2, 3, 4, 5, 6, 7 display the simulation results by choosing \(\phi_1 = \phi_2 = 0, 0.5, 1, 1.5\). The inspiration for choosing values of \(\phi_1\) and \(\phi_2\) comes from references [27] and [37]. Notice that all “error(s)” and “control torque” mentioned later are absolute values. Figures 2, 3 show the curves of \(\theta_1\) and \(\theta_2\) with respect to \(t\). It can be observed that once \(\phi_1 = \phi_2 = 0\), the actual joint angle trajectories deviate severely from the desired trajectory. It implies that nominal control is not applicable with unknown perturbations. For \(\phi_1 = \phi_2 = 0.5, 1, 1.5\), the actual joint angle trajectories can quickly track the desired trajectory and keep the error within a small range (less than 0.1 rad). Through the local magnification, the actual joint angle trajectory is closest to the desired trajectory when \(\phi_1 = \phi_2 = 1.5\). Moreover, through simulation results, we have found value of \(\phi_i(i = 1, 2)\) above 1.5 could not improve the tracking performance.

From Figs. 4 and 5, we can obtain the change curves of error1 and error2, which represent hip and knee angle trajectory error, respectively. Time history of errors is a further interpretation of hip and knee joints angle change curves. For \(\phi_1 = \phi_2 = 0\), the errors curves are approximately linear and the errors are increasing with respect to \(t\). When \(\phi_i(i = 1, 2)\) are 0.5, 1.0 or 1.5, errors can be stabilized around zero within 5 s. Among these curves, the trajectory errors are the smallest when \(\phi_1 = \phi_2 = 1.5\), and the errors

![Fig. 2 Time history of hip joint angle \(\theta_1\) where \(\phi\) means \(\phi_i(i = 1, 2)\)](image)

![Fig. 3 Time history of knee joint angle \(\theta_2\) where \(\phi\) means \(\phi_i(i = 1, 2)\)](image)

![Fig. 4 Time history of error1 where \(\phi\) means \(\phi_i(i = 1, 2)\)](image)

![Fig. 5 Time history of error2 where \(\phi\) means \(\phi_i(i = 1, 2)\)](image)
curves are more jittering than other curves when \( \phi = \phi_i (i = 1, 2) \).

Time history of control forces is illustrated in Figs. 6 and 7, where \( \tau_1 \) and \( \tau_2 \), respectively, mean control torque of hip and knee joints. Among \( \phi_i (i = 1, 2) = 0, 0.5, 1.0, 1.5 \), the larger the values of \( \phi_i (i = 1, 2) \), the smaller the control torques, which means lower control costs. From another perspective, Figs. 6, 7 describe that when \( \phi_i (i = 1, 2) = 0.5 \), the trajectory tracking effect is the best. Control torques of hip and knee joints provide a basis for the motor selection of the actual LERER control system.

5.2 Adaptive robust control and comparison simulation results

The unknown perturbations terms setups are as same as the simulation of trajectory stability control. The parameters of ARC are chosen as \( \kappa = 1, k_1 = 1, k_2 = 1, \ v = 0.001, \ \ddot{a}(0) = 0.2, \ P = I_2 \times 2 \). Then, Assumptions 1-2 can be easily satisfied. In order to meet Assumption 3, \( \Pi(x, q, \dot{q}, t) \) can be chosen as follows:

\[
\Pi(x, q, \dot{q}, t) = x_1\|q^2\| + x_2\|\dot{q}\| + x_3
\]

(75)

\[
= (x_1 \quad x_2 \quad x_3) [\|q^2\|]_1
\]

(76)

where \( x_i > 0 (i = 1, 2, 3) \). \( \Pi(x, q, \dot{q}, t) \) can also be chosen as follows to satisfy Assumption 3

\[
\pi_1q^2 + \pi_2\dot{q} + \pi_3 \leq \pi(\dot{q} + 1)^2
\]

(77)

where \( \pi = \max \left\{ x_1, x_2, x_3 \right\} \). Based on (76), we can also design the adaptive law as follows

\[
\dot{\pi} = k_1(\dot{q} + 1)^2 \beta - k_2\pi
\]

(78)

where \( k_1 \) and \( k_2 \) are constants.

Figures 8, 14 describe the simulation results of ARC and comparative simulation results. Here, trajectory stability control parameter \( \phi_i (i = 1, 2) \) is taken as 1.5. Figure 8 shows the illustration of \( \dot{\pi} \) with respect to time. \( \dot{\pi} \) increases quickly from the initial value \( \dot{\pi}(0) = 0.2 \) to the max value \( \dot{\pi}_{\text{max}} = 0.75 \), which is caused by the perturbations. Then, \( \dot{\pi} \) decreases rapidly and gradually stable for the compensation of the controller.

Figures 9 and 10 show the time history of hip and knee angles in different control methods. Curves of ACFC in [50] consistently deviate from desired trajectories. The TSC and ARC can both overcome the disturbances of unknown perturbations to achieve trajectory tracking control. Furthermore, the effectiveness of ARC is better than that of TSC.

Errors of hip and knee joints in different control methods are shown in Figs. 11, 12. Figures 11 and 12 obviously show that the errors of ACFC in [50] are divergent, its values are the largest and its convergence time is the longest. Similarly, the errors of ARC
are the smallest and the convergence time of ARC is also the shortest. At the same time, after a short fluctuation, TSC can also converge the errors around zero.

Figures 13, 14 demonstrate the time history of $\tau_1$ and $\tau_2$. The trends of ACFC is gradually increasing and divergent, which are not available in the LERER. In addition, the peak value of ARC is smaller than that of TSC, which means lower control costs. In summary, ARC has the best trajectory tracking effect, followed by TSC. It is worth mentioning that both ARC and TSC can meet the basic trajectory tracking requirements.
For the trajectory tracking control of LERER with unknown perturbations, this paper proposes two constrained control methods (TSC and ARC) based on U–K theory. U–K theory, which can derive the analytical form of the control force without linearization or approximation, provides explicit equations of motion for nonlinear constrained systems. It is less computationally intensive than traditional Lagrangian or Newton–Euler methods. In addition, it does not require the introduction of Lagrangian multipliers or pseudo-variables. The leakage type of adaptive law is designed, and the uniform boundedness and uniform ultimate boundedness are verified by Lyapunov method. Simulation results have shown that the trajectory tracking control effects of ARC and TSC are more excellent than that of comparative control methods, and the best is ARC.

The trajectory tracking control of LERER is of great significance for the implementation of its control system. It is more realistic considering the unknown perturbations. The parameter selection for both TSC and ARC is manual selection, which depends largely on manual experience. In the future, we will revolve around the adaptive selection of parameters. Furthermore, the verification of real machine experiment is also a very significant work in the future.

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**Code availability** Custom code is available upon request at Liang Yuan email address.

**Declarations**

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