The Conley conjecture for Hamiltonian systems on the cotangent bundle and its analogue for Lagrangian systems

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Abstract

In this paper, the Conley conjecture, which were recently proved by Franks and Handel [FrHa] (for surfaces of positive genus), Hingston [Hi] (for tori) and Ginzburg [Gi] (for closed symplectically aspherical manifolds), is proved for $C^1$-Hamiltonian systems on the cotangent bundle of a $C^3$-smooth compact manifold $M$ without boundary, of a time 1-periodic $C^2$-smooth Hamiltonian $H : \mathbb{R} \times T^*M \to \mathbb{R}$ which is strongly convex and has quadratic growth on the fibers. Namely, we show that such a Hamiltonian system has an infinite sequence of contractible integral periodic solutions such that any one of them cannot be obtained from others by iterations. If $H$ also satisfies $H(-t, q, -p) = H(t, q, p)$ for any $(t, q, p) \in \mathbb{R} \times T^*M$, it is shown that the time-one map of the Hamiltonian system (if exists) has infinitely many periodic points sitting in the zero section of $T^*M$. If $M$ is $C^5$-smooth and dim $M > 1$, $H$ is of $C^4$ class and independent of time $t$, then for any $\tau > 0$ the corresponding system has an infinite sequence of contractible periodic solutions of periods of integral multiple of $\tau$ such that any one of them cannot be obtained from others by iterations or rotations. These results are obtained by proving similar results for the Lagrangian system of the Fenchel transform of $H$, $L : \mathbb{R} \times TM \to \mathbb{R}$, which is proved to be strongly convex and to have quadratic growth in the velocities yet.

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1 Introduction and main results

Recently, a remarkable progress in Symplectic geometry and Hamiltonian dynamics is that the Conley conjecture [Co, SaZe] were proved by Franks and Handel [FrHa] (for surfaces of positive genus, also see [Le] for generalizations to Hamiltonian homeomorphisms), Hingston [Hi] (for tori) and Ginzburg [Gi] (for closed symplectically aspherical manifolds). See [FrHa, Le, Gi] and references therein for a detailed history and related studies.

In this paper we always assume that $M$ is an $n$-dimensional, connected $C^3$-smooth compact manifold without boundary without special statements. For a time 1-periodic $C^2$-smooth Hamiltonian $H : \mathbb{R} \times T^*M \rightarrow \mathbb{R}$, let $X_H$ be the Hamiltonian vector field of $H$ with respect to the standard symplectic structure on $T^*M$, $\omega_{\text{can}} := -dq \wedge dp$ in local coordinates $(q, p)$ of $T^*M$, that is, $\omega(X_H(t, q, p), \xi) = -dH(t, q, p)(\xi) \forall \xi \in T_{(q, p)}T^*M$. Unlike the case of compact symplectic manifolds we only consider subharmonic solutions of the Hamiltonian equations

$$\dot{x}(t) = X_H(t, x(t)) \quad (1.1)$$

for $C^2$-smooth Hamiltonians $H : \mathbb{R} \times T^*M \rightarrow \mathbb{R}$ satisfying the following conditions (H1)-(H3):

(H1) $H(t + 1, q, p) = H(t, q, p)$ for all $(t, q, p) \in \mathbb{R} \times T^*M$.

In any local coordinates $(q_1, \cdots , q_n)$, there exist constants $0 < C_1 < C_2$, depending on the local coordinates, such that

(H2) $C_1 |u|^2 \leq \sum_{ij} \frac{\partial^2 H}{\partial p_i \partial p_j}(t, q, p) u_i u_j \leq C_2 |u|^2 \quad \forall u = (u_1, \cdots , u_n) \in \mathbb{R}^n$,
A class of important examples of such Hamiltonians are Physical Hamiltonian (including 1-periodic potential and electromagnetic forces in time) of the form

\[ H(t, q, p) = \frac{1}{2} \| p - A(t, q) \|^2 + V(t, q) \]  

(1.2)

For \( C^r \)-smooth Hamiltonians \( H : \mathbb{R} \times T^*M \to \mathbb{R} \) satisfying the conditions (H1)-(H3), \( r \geq 2 \), by the inequality in the left side of the condition (H2), we can use the inverse Legendre transform to get a fiber-preserving \( C^{r-1} \)-diffeomorphism

\[ \mathcal{L}_H : \mathbb{R}/\mathbb{Z} \times T^*M \to \mathbb{R}/\mathbb{Z} \times TM, \quad (t, q, p) \mapsto (t, q, D_pH(t, q, p)), \]  

(1.3)

and a \( C^r \)-smooth function \( L : \mathbb{R} \times TM \to \mathbb{R} \):

\[ L(t, q, v) = \max_{p \in T_qM} \{ \langle p, v \rangle - H(t, q, p) \} = \langle p(t, q, v), v \rangle - H(t, q, p(t, q, v)), \]  

(1.4)

where \( p = p(t, q, v) \) is a unique point determined by the equality \( v = D_pH(t, q, p) \).

(See [F a, Prop.2.1.6]). By (1.4) we have

(L1) \( L(t+1, q, v) = L(t, q, v) \) for all \( (t, q, v) \in \mathbb{R} \times TM \).

It is easily checked that the corresponding \( L \) with the physical Hamiltonian in (1.2) is given by

\[ L(t, q, v) = \frac{1}{2} \| v \|^2 + \langle A(t, q), v \rangle - V(t, q). \]

In Appendix we shall prove

**Proposition A.** Under the condition (H1), (H2) is equivalent to the following (L2) plus the third inequality in (L3), and (H2) + (H3) \( \iff \) (L2) + (L3).

In any local coordinates \( (q_1, \cdots, q_n) \), there exist constants \( 0 < c < C \), depending on the local coordinates, such that

(L2) \[ \sum_{ij} \left| \frac{\partial^2 L}{\partial v_i \partial v_j} (t, q, v) u_i u_j \right| \geq c | u |^2 \quad \forall u = (u_1, \cdots, u_n) \in \mathbb{R}^n, \]

(L3) \[ \left| \frac{\partial^2 L}{\partial q_i \partial q_j} (t, q, v) \right| \leq C(1 + |v|^2), \quad \left| \frac{\partial^2 L}{\partial q_i \partial v_j} (t, q, v) \right| \leq C(1 + |v|), \quad \text{and} \]

\[ \left| \frac{\partial^2 L}{\partial v_i \partial v_j} (t, q, v) \right| \leq C. \]

(One can also write these two conditions in the free coordinates, see [AbSc, \$2].) So Proposition A shows that the conditions (L2)-(L3) have the same properties as (H2)-(H3). (Note: we do not claim that the condition (H2) (resp. (H3)) is equivalent to (L2) (resp.(L3)).) By (L2), the Legendre transform produces the inverse of \( \mathcal{L}_H \),

\[ \mathcal{L}_L : \mathbb{R}/\mathbb{Z} \times TM \to \mathbb{R}/\mathbb{Z} \times T^*M, \quad (t, q, v) \mapsto (t, q, D_vL(t, q, v)), \]  

(1.5)

and \( H \) and \( L \) are related by:

\[ H(t, q, p) = \langle p, v(t, q, p) \rangle - L(t, q, v(t, q, p)), \]
where $v = v(t,q,p)$ is a unique point determined by the equality $p = D_v L(t,q,v)$. In this case, it is well-known that a curve $\mathbb{R} \to T^* M$, $t \mapsto x(t) = (\gamma(t), \dot{\gamma}(t))$ is a solution of (1.1) if and only if $\gamma^s(t) = D_v L(t, \gamma(t), \dot{\gamma}(t)) \forall t \in \mathbb{R}$ and $\gamma$ is a solution of the Lagrangian system on $M$:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0$$

(1.6)

in any local coordinates $(q_1, \ldots, q_n)$.

Hence we only need to study the existence of infinitely many distinct integer periodic solutions of the system (1.6) under the assumptions (L1)-(L3). To describe our results we introduce the following notations and notions.

For any $T > 0$, each map in $C(\mathbb{R}/TZ, M)$ represent a homotopy class of free loops in $M$. As topological spaces $C(\mathbb{R}/TZ, M)$ and $C(\mathbb{R}/Z, M)$ are always homeomorphic. For a homotopy class $\alpha$ of free loops in $M$, denote by $C(\mathbb{R}/TZ, M; \alpha)$ the subset of maps in $C(\mathbb{R}/TZ, M)$ representing $\alpha$. For $k \in \mathbb{N}$, if we view $\gamma \in C(\mathbb{R}/TZ, M; \alpha)$ as a $T$-periodic map $\gamma : \mathbb{R} \to M$, it is also viewed as a $kT$-periodic map from $\mathbb{R}$ to $M$ and thus yields an element of $C(\mathbb{R}/kTZ, M)$, called the $k$-th iteration of $\gamma$ and denoted by $\gamma^k$. This $\gamma^k \in C(\mathbb{R}/kTZ, M)$ represents a free homotopy class in $M$, denoted by $\alpha^k$.

So $\gamma^k \in C(\mathbb{R}/kTZ, M; \alpha^k)$. Note also that topological spaces $C(\mathbb{R}/TZ, M; \alpha)$ and $C(\mathbb{R}/Z, M; \alpha)$ are always homeomorphic yet. For $m \in \mathbb{N}$ let $C^m(\mathbb{R}/TZ, M)$ denote the subset of all $C^m$-loops $\gamma : \mathbb{R}/TZ \to M$.

A periodic map $\gamma : \mathbb{R} \to M$ is called reversible (or even) if $\gamma(-t) = \gamma(t)$ for any $t \in \mathbb{R}$. Note that such a map is always contractible! For $\gamma \in C(\mathbb{R}/TZ, M)$ we define rotations of $\gamma$ via $s \in \mathbb{R}$ as maps $s \cdot \gamma : \mathbb{R} \to M$ defined by $s \cdot \gamma(t) = \gamma(t + s)$ for $t \in \mathbb{R}$. Then $s \cdot \gamma \in C(\mathbb{R}/TZ, M)$ and $(s \cdot \gamma)^m = s \cdot \gamma^m$ for any $s \in \mathbb{R}$ and $m \in \mathbb{N}$. We call the set

$$\{\gamma^m\}_{m \in \mathbb{N}} \quad \text{(resp. } \{s \cdot \gamma^m\}_{s \in \mathbb{R}} \text{)}$$

a $T$-periodic map tower (resp. $T$-periodic orbit tower) based on $\gamma$ (a $T$-periodic map from $\mathbb{R}$ to $M$). A $T_1$-periodic map tower $\{\gamma^m_1\}_{m \in \mathbb{N}}$ (resp. $T_1$-periodic orbit tower $\{s \cdot \gamma^m_1\}_{s \in \mathbb{R}}$) based on a $T_1$-periodic map $\gamma_1 : \mathbb{R} \to M$ is called distinct with $\{\gamma^m\}_{m \in \mathbb{N}}$ (resp. $\{s \cdot \gamma^m\}_{m \in \mathbb{N}}$) if there is no $T$-periodic map $\beta : \mathbb{R} \to M$ such that $\gamma = \beta^p$ and $\gamma_1 = \beta^q$ for some $p,q \in \mathbb{N}$ (resp. $\gamma = s \cdot \beta^p$ and $\gamma_1 = s' \cdot \beta^q$ for some $p,q \in \mathbb{N}$ and $s,s' \in \mathbb{R}$). When $\gamma$ is contractible as a map from $\mathbb{R}/TZ$ to $M$, we call the $T$-periodic map tower $\{\gamma^m\}_{m \in \mathbb{N}}$ (resp. $T$-periodic orbit tower $\{s \cdot \gamma^m\}_{s \in \mathbb{R}}$) contractible.

For $\tau \in \mathbb{N}$, if $\gamma : \mathbb{R} \to M$ is a $\tau$-periodic solution of (1.6), we call the set $\{\gamma^m\}_{m \in \mathbb{N}}$ a $\tau$-periodic solution tower of (1.6) based on $\gamma$. Two periodic solution towers of (1.6) are said to be distinct if they are distinct as periodic map towers. Furthermore, if $s \cdot \gamma$ is also a $\tau$-periodic solution of (1.6) for any $s \in \mathbb{R}$, (for example, in the case $L$ is independent of $t$), we call $\{s \cdot \gamma^m\}_{m \in \mathbb{N}}$ a $\tau$-periodic solution orbit tower of (1.6).

When two periodic solution orbit towers are distinct as periodic orbit towers we call them distinct periodic solution orbit towers of (1.6) based on $\gamma$. Clearly, the existence of infinitely many distinct integer periodic solution towers (resp. solution orbit towers) of (1.6) implies that there exist an infinite sequence of integer periodic
solutions of (1.6) such that each of them cannot be obtained from others by iterations (resp. iterations or rotations). The following is the first main result of this paper.

**Theorem 1.1** Let $M$ be a $C^3$-smooth compact $n$-dimensional manifold without boundary, and $C^2$-smooth map $L : \mathbb{R} \times TM \to \mathbb{R}$ satisfy the conditions (L1)-(L3). Then:

(i) Suppose that for a homotopy class $\alpha$ of free loops in $M$ and an abelian group $\mathbb{K}$ the singular homology groups $H_r(C(\mathbb{R}/\mathbb{Z}, M; \alpha^k); \mathbb{K})$ have nonzero ranks for some integer $r \geq n$ and all $k \in \mathbb{N}$. Then either for some $l \in \mathbb{N}$ there exist infinitely many distinct $l$-periodic solutions of (1.6) representing $\alpha^l$, or there exist infinitely many positive integers $l_1 < l_2 < \cdots$, such that for each $i \in \mathbb{N}$ the system (1.6) has a periodic solution with minimal period $l_i$ and representing $\alpha^{l_i}$.

(ii) Suppose that for some abelian group $\mathbb{K}$ and integer $r \geq n$ the singular homology groups $H_r(C(\mathbb{R}/\mathbb{Z}, M); \mathbb{K})$ have nonzero ranks. Then either for some $l \in \mathbb{N}$ there exist infinitely many distinct $l$-periodic solutions of (1.6), or there exist infinitely many positive integers $l_1 < l_2 < \cdots$, such that for each $i \in \mathbb{N}$ the system (1.6) has a periodic solution with minimal period $l_i$.

Let $0$ denote the free homotopy class of contractible loops in $M$, i.e., $C(\mathbb{R}/\mathbb{Z}, M; 0)$ consists of all contractible loops $\gamma : \mathbb{R}/\mathbb{Z} \to M$. The obvious inclusion $\iota : M \to C(\mathbb{R}/\mathbb{Z}, M; 0)$ and the evaluation

$\text{EV} : C(\mathbb{R}/\mathbb{Z}, M; 0) \to M, \gamma \mapsto \gamma(0)$

satisfy $\text{EV} \circ \iota = \text{id}_M$. It easily follows that

$\iota_* : H_k(M; \mathbb{Z}_2) \to H_k(C(\mathbb{R}/\mathbb{Z}, M; 0); \mathbb{Z}_2)$

is injective for any $k \in \mathbb{N}$. Since $H_n(M, \mathbb{Z}_2) = \mathbb{Z}_2$ for $n = \dim M$, we get

$\text{rank}H_n(C(\mathbb{R}/\mathbb{Z}, M; 0); \mathbb{Z}_2) \neq 0$. \hspace{1cm} (1.7)

**Corollary 1.2** Let $M$ be a $C^3$-smooth compact $n$-dimensional manifold without boundary, and $C^2$-smooth map $L : \mathbb{R} \times TM \to \mathbb{R}$ satisfy the conditions (L1)-(L3). Then the system (1.6) possesses infinitely many distinct contractible integer periodic solution towers.

**Remark 1.3** 1° When $M$ has finite fundamental group, Benci [Be] first proved that the system (1.6) has infinitely many distinct contractible 1-periodic solutions for $C^2$-smooth Lagrangian $L$ satisfying the conditions (L1)-(L3) and

$$\left| \frac{\partial L}{\partial q_i}(t, q, v) \right| \leq C(1 + |v|^2), \quad \left| \frac{\partial L}{\partial v_i}L(t, q, v) \right| \leq C(1 + |v|)$$

in some local coordinates $(q_1, \cdots, q_n)$ for some constant $C > 0$. Recently, under weaker assumptions than (L1)-(L3), i.e. Tonelli conditions and (L5) below, Abbondandolo and Figalli [AbF] Cor.3.2] showed that the system (1.6) has an infinite sequence of 1-periodic contractible solutions with diverging action and diverging Morse
The key in [Be, AbF] is the fact that the space of free loops in a compact simply connected manifold has infinitely many nonzero (co)homology groups with real coefficients [Su]. A new technique in [AbF] is to modify their Tonelli Lagrangian $L$ to one satisfying (L1)-(L3).

2° On $n$-dimensional torus $T^n$, for the Lagrangian of the form

$$L(t, q, v) = \frac{1}{2}g_q(v, v) + U(t, q)$$

for all $(t, q, v) \in \mathbb{R} \times TT^n = \mathbb{R} \times T^n \times \mathbb{R}^n$, where $g$ is a $C^3$-smooth Riemannian metric on $T^n$ and $U \in C^3(\mathbb{R}/\mathbb{Z} \times T^n, \mathbb{R})$, (such a $L$ satisfies the conditions (L1)-(L3)), Yiming Long [Lo2] proved that the system (1.6) possesses infinitely many distinct contractible integer periodic solution towers.

We refer the reader to [Lo2] and the references given there for the detailed history on the integer periodic solutions of the Lagrangian system.

If $L : \mathbb{R} \times TM \to \mathbb{R}$ also satisfies

(L4) $L(-t, q, -v) = L(t, q, v)$ for any $(t, q, v) \in \mathbb{R} \times TM$,

we can improve Corollary 1.2 as follows.

**Theorem 1.4** Let $M$ be a $C^3$-smooth compact $n$-dimensional manifold without boundary, and $C^2$-smooth map $L : \mathbb{R} \times TM \to \mathbb{R}$ satisfy the conditions (L1)-(L4). Then the system (1.6) possesses infinitely many distinct contractible integer periodic solution towers based on reversible periodic solutions.

This result was proved by the author and Mingyan Wang [LuW2] in the case that $M = T^n$ and that $L$ has the form (1.8) and satisfies (L4), i.e. $U(-t, q) = U(t, q)$ for any $(t, q) \in \mathbb{R} \times T^n$. In particular, we have a generalization of [LuW2, Th.1.6].

**Corollary 1.5** If $L \in C^2(TM, \mathbb{R})$ satisfies (L2)-(L4), then for any real number $\tau > 0$, the following three claims have at least one to be true:

- $L$ has infinitely many critical points sitting in $M = 0_{TM}$ and thus the system (1.6) possesses infinitely many different constant solutions in $M$;
- there exists some positive integer $k$ such that the system (1.6) possesses infinitely many different nonconstant $k\tau$-periodic solution orbit towers based on reversible periodic solutions of (1.6);
- there exist infinitely many positive integers $k_1 < k_2 < \cdots$, such that for each $k_m$ the system (1.6) possesses a reversible periodic solution with minimal period $k_m\tau$, $m = 1, 2, \cdots$.

When $M = T^n$ and $L$ has the form (1.8) with real analytic $g$ and nonconstant, autonomous and real analytic $U$, the author and Mingyan Wang [LuW1] observed that suitably improving the arguments in [CaTa] can give a simple proof of Corollary 1.5. It should also be noted that even if $M$ is simply connected the methods in [Be, AbF] cannot produce infinitely many reversible integer periodic solutions because the space
of reversible loops in \( M \) can contract to the zero section of \( TM \) and therefore has no infinitely many nonzero Betti numbers.

If \( L \in C^2(TM, \mathbb{R}) \) only satisfies (L2)-(L3), it is possible that two distinct solutions \( \gamma_1 \) and \( \gamma_2 \) obtained by Theorem 1.1 only differ a rotation, i.e., \( \gamma_1(t) = \gamma_2(s + t) \) for some \( s \in \mathbb{R} \) and any \( t \in \mathbb{R} \). However, we can combine the proof of Theorem 1.1 with the method in [LoLu] to improve the results in Theorem 1.1 as follows:

**Theorem 1.6** Let \( M \) be a \( C^5 \)-smooth compact \( n \)-dimensional manifold without boundary, and \( C^4 \)-smooth map \( L : TM \to \mathbb{R} \) satisfy the conditions (L2)-(L3). Then for any \( \tau > 0 \) the following results hold:

(i) Suppose that for a homotopy class \( \alpha \) of free loops in \( M \) and an abelian group \( \mathbb{K} \) the singular homology groups \( H_r(C(\mathbb{R}/\mathbb{Z}, M; \alpha^k); \mathbb{K}) \) have nonzero ranks for some integer \( r \geq n \) and all \( k \in \mathbb{N} \). If either \( r \geq n + 1 \) or \( r = n > 1 \), then either for some \( l \in \mathbb{N} \) there exist infinitely many distinct periodic solution orbit towers based on \( l \tau \)-periodic solutions of (1.7) representing \( \alpha_l \), or there exist infinitely many positive integers \( l_1 < l_2 < \cdots \), such that for each \( i \in \mathbb{N} \) the system (1.6) has a periodic solution orbit tower based on a periodic solution with minimal period \( l_i \tau \) and representing \( \alpha_{l_i} \).

(ii) Suppose that the singular homology groups \( H_r(C(\mathbb{R}/\mathbb{Z}, M); \mathbb{K}) \) have nonzero ranks for some integer \( r \geq n \) and some abelian group \( \mathbb{K} \). If either \( r \geq n + 1 \) or \( r = n > 1 \), then either for some \( l \in \mathbb{N} \) there exist infinitely many distinct periodic solution orbit towers based on \( l \tau \)-periodic solutions of (1.7), or there exist infinitely many positive integers \( l_1 < l_2 < \cdots \), such that for each \( i \in \mathbb{N} \) the system (1.6) has a periodic solution orbit tower based on a periodic solution with minimal period \( l_i \tau \).

By (1.7) we immediately get:

**Corollary 1.7** Let \( M \) be a \( C^5 \)-smooth compact manifold of dimension \( n > 1 \) and without boundary, and \( C^4 \)-smooth map \( L : TM \to \mathbb{R} \) satisfy the conditions (L2)-(L3). Then for any \( \tau > 0 \) the system (1.6) possesses infinitely many distinct periodic solution orbit towers based on contractible periodic solutions of integer multiple periods of \( \tau \).

Clearly, when (L4) is satisfied Corollary 1.5 seems to be stronger than Corollary 1.7. If \( n = 1 \) and (L4) is not satisfied, we do not know whether Corollary 1.7 is still true. Moreover, the reason that we require higher smoothness in Theorem 1.6 and Corollary 1.7 is to assure that the normal bundle of a nonconstant periodic orbit is \( C^2 \)-smooth.

When \( M = T^n \) and \( L \) has the form (1.8) with flat \( g \) and autonomous \( U \), Yiming Long and the author [LoLu] developed the equivariant version of the arguments in [Lo2] to prove Corollary 1.7. Even if \( g \) is not flat, the author and Mingyan Wang [LuW2, Th.1.6] also derived a stronger result than Corollary 1.7 in the case that \( M = T^n \). Campos and Tarallo [CaTa] obtained a similar result provided that the
metric $g$ is real analytic, and that the potential $U$ is autonomous, real analytic and nonconstant.

Even if $L = \frac{1}{2} g$ for a $C^4$-Riemannian metric $g$ on $M$, it seems that Theorem 1.6 or Corollary 1.7 cannot yield infinitely many geometrically distinct closed geodesics.

Assume that $L$ also satisfies

(L5) For any $(q, v) \in TM$ there exists an unique solution of (1.6), $\gamma: \mathbb{R} \to M$, such that $(\gamma(0), \dot{\gamma}(0)) = (q, v)$.

By [AbE, §2], this assumption can be satisfied if

$$- \partial_t L(t, q, v) \leq c (1 + D_v L(t, q, v)[v] - L(t, q, v)) \quad \forall (t, q, v) \in \mathbb{R} \times TM. \quad (1.9)$$

(Clearly, the left side may be replaced by $\text{const} - \partial_t L(t, q, v)$ since (L5) is also satisfied up to adding a constant to $L$. Moreover, that $L$ satisfies (L1)-(L3) is equivalent to that the Fenchel transform $H$ of $L$ given by (1.4) satisfies the assumptions (H1)-(H3) below.

In this case (1.9) is equivalent to (1.11) below. Hence (1.9) holds if $L$ is independent of $t$ as noted below (1.11).)

Under the assumption (L5), we have an one-parameter family of $C^1$-diffeomorphisms $\Phi^t_L \in \text{Diff}(TM)$ satisfying $\Phi^t_L(\gamma(0), \dot{\gamma}(0)) = (\gamma(t), \dot{\gamma}(t))$.

(See [Fa, Th.2.6.5]). Following [Lo2], the time-1-map $\Phi^1_L = \Phi^L$ is called the Poincaré map of the system (1.6) corresponding to the Lagrangian function $L$. Every integer periodic solution $\gamma$ of (1.6) gives a periodic point $(\gamma(0), \dot{\gamma}(0))$ of $\Phi_L$. If $\gamma$ is even, then the periodic point $(\gamma(0), \dot{\gamma}(0))$ sits in the zero section $0_{TM}$ of $TM$. So Corollary 1.2 and Theorem 1.3 yield the following

**Corollary 1.8** Let $M$ be a $C^3$-smooth compact $n$-dimensional manifold without boundary, and $C^2$-smooth map $L : \mathbb{R} \times TM \to \mathbb{R}$ satisfy the conditions (L1)-(L3) and (L5). Then the Poincaré map $\Phi_L$ has infinitely many distinct periodic points. Furthermore, if (L4) is also satisfied then the Poincaré map $\Phi_L$ has infinitely many distinct periodic points sitting in the zero section $0_{TM}$ of $TM$.

If $L$ is independent of $t$, for a periodic point $(\gamma(0), \dot{\gamma}(0))$ of $\Phi_L$ generated by a $\tau$-periodic solution $\gamma$, then all points of $\{(\gamma(s), \dot{\gamma}(s)) | s \in \mathbb{R}\}$ are periodic points of $\Phi_L$. We call such period points orbitally same. By remarks below (1.9), using Corollary 1.7 we can improve Corollary 1.8 as follows:

**Corollary 1.9** Let $M$ be a $C^5$-smooth compact manifold of dimension $n > 1$ and without boundary, and $C^4$-smooth map $L : TM \to \mathbb{R}$ satisfy the conditions (L2)-(L3). Then the Poincaré map $\Phi_L$ has infinitely many orbitally distinct periodic points.

It is easily checked that the assumption (L4) is equivalent to the following:

(H4) $H(-t, q, -p) = H(t, q, p)$ for any $(t, q, p) \in \mathbb{R} \times T^*M$.

In this case, $v = v(t, q, p)$ uniquely determined by the equality $p = D_v L(t, q, v)$ satisfies

$$v(-t, q, -p) = -v(t, q, p) \quad \forall (t, q, p) \in \mathbb{R} \times T^*M. \quad (1.10)$$
So if a solution $\gamma : \mathbb{R} \to M$ of (L6) satisfies $\gamma(-t) = \gamma(t) \forall t \in \mathbb{R}$, then $\gamma^*(-t) = -\gamma^*(t)$ for all $t \in \mathbb{R}$.

With the same way as the definition of solution towers and solution orbit towers to (L1) we can define solution towers to (L1), and solution orbit towers to (L1) in the case $H$ is independent of $t$. Then the Hamiltonian versions from Theorem 1.1 to Corollary 1.7 can be obtained immediately. For example, from Corollary 1.2, Theorem 1.4 and Corollary 1.7 we directly derive:

**Theorem 1.10** 1°) Let $M$ be a $C^3$-smooth compact $n$-dimensional manifold without boundary, and $C^2$-smooth map $H : \mathbb{R} \times T^*M \to \mathbb{R}$ satisfy the conditions (H1)-(H3). Then the system (L1) possesses infinitely many distinct contractible integer periodic solution towers. Furthermore, if (H4) is also satisfied then the system (L1) possesses infinitely many distinct contractible integer periodic solution towers based on periodic solutions with reversible projections to $M$.

2°) Let $M$ be a $C^5$-smooth compact manifold of dimension $n > 1$ and without boundary, and $C^4$-smooth map $H : \mathbb{R} \times T^*M \to \mathbb{R}$ satisfy the conditions (H2)-(H3). Then for any $\tau > 0$ the system (L1) has infinitely many distinct periodic solution orbit towers based on contractible periodic solutions of integer multiple periods of $\tau$.

**Remark 1.11** If $\pi_1(M)$ is finite, Cieliebak [Ci] showed that the system (L1) has infinitely many contractible 1-periodic solutions (with unbounded actions) provided that $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times T^*M, \mathbb{R})$ satisfies

(HC1) $dH(t, q, p) \left[ p \frac{\partial}{\partial p} \right] - H(t, q, p) \geq h_0 \|p\|^2 - h_1,$

(HC2) $\left| \frac{\partial^2 H}{\partial p_i \partial p_j} (t, q, p) \right| \leq d$ and $\left| \frac{\partial^2 H}{\partial q_i \partial p_j} (t, q, p) \right| \leq d,$

for all $(t, q, p) \in \mathbb{R} \times T^*M$, with respect to a suitable metric on the bundle $T^*M \to M$ and constants $h_0 > 0, h_1$ and $d$. Here $q_1, \ldots, q_n, p_1, \ldots, p_n$ are coordinates on $T^*M$ induced by geodesic normal coordinates $q_1, \ldots, q_n$ on $M$.

Recently, Abbondandolo and Figalli stated in [AbF] Remark 7.4 that the same result can be derived from [AbF Th.7.3] if the assumptions (HC1)-(HC2) are replaced by

(HAF1) $dH(t, q, p) \left[ p \frac{\partial}{\partial p} \right] - H(t, q, p) \geq a(|p|_q)$ for some function $a : [0, \infty) \to \mathbb{R}$ with $\lim_{s \to +\infty} a(s) = +\infty,$

(HAF2) $H(t, q, p) \geq h(|p|_q)$ for some function $h : [0, \infty) \to \mathbb{R}$ with $\lim_{s \to +\infty} \frac{h(s)}{s} = +\infty$ and all $(t, q, p) \in \mathbb{R} \times T^*M,$

and (H5) below. Note that no convexity assumption on $H$ was made in [Ci] and therefore that their results cannot be obtained from one on Lagrangian system via the Legendre transform.

It is easily seen that the assumption (L5) is equivalent to the following:

(H5) For any $(q, p) \in T^*M$ there exists an unique solution of $\dot{x}(t) = X_H(t, x(t)),$ $x : \mathbb{R} \to M,$ such that $x(0) = (q, p).$
The assumption can be satisfied under the following equivalent condition of (1.9):
\[ \partial_t H(t, q, p) \leq c (1 + H(t, q, p)) \quad \forall (t, q, p) \in \mathbb{R} \times T^*M, \quad (1.11) \]
see [AbF, pp.629]. Since (H2) implies that \( H \) is superlinear on the fibers of \( T^*M \), (1.11) holds clearly if \( H \) is independent of time \( t \). The condition (H5) guarantees that the global flow of \( X_H \) exists on \( T^*M \). Thus we have an one-parameter family of Hamiltonian diffeomorphisms \( \Psi^H_t \in \text{Ham}(T^*M, \omega_{\text{can}}) \) satisfying \( \Psi^H_t(\gamma(0), \dot{\gamma}(0)) = (\gamma(t), \dot{\gamma}(t)) \). As usual, the time-1-map \( \Psi^H = \Psi^H_1 \) is called the Poincaré map of the system (1.1) corresponding to the Hamiltonian function \( H \). For each \( t \in \mathbb{R} \) recall that the Legendre transform associated with \( L_t : TM \to T^*M \):
\[ (q, v) \mapsto (q, D_v L_t(q, v)) \]
It is easy to check that
\[ \Psi^H_t \circ \mathcal{L}_t = \mathcal{L}_t \circ \Phi^H_t \quad \text{for any} \ t \in \mathbb{R}. \quad (1.12) \]
From this one immediately gets the following equivalent Hamiltonian versions of Corollary 1.8 and Corollary 1.9.

**Theorem 1.12** 1°) Let \( M \) be a \( C^3 \)-smooth compact \( n \)-dimensional manifold without boundary, and \( C^2 \)-smooth map \( H : \mathbb{R} \times T^*M \to \mathbb{R} \) satisfy the conditions (H1)-(H3) and (H5). Then the Poincaré map \( \Psi^H \) has infinitely many distinct periodic points. Furthermore, if (H4) is also satisfied then the Poincaré map \( \Psi^H \) has infinitely many distinct periodic points sitting in the zero section \( 0_{T^*M} \) of \( T^*M \).

2°) Let \( M \) be a \( C^5 \)-smooth compact manifold of dimension \( n > 1 \) and without boundary, and \( C^4 \)-smooth map \( H : \mathbb{R} \times T^*M \to \mathbb{R} \) satisfy the conditions (H2)-(H3). Then the Poincaré map \( \Psi^H \) has infinitely many orbitally distinct periodic points. (That is, any two do not sit the same Hamiltonian orbit.)

Theorems 1.10, 1.12 may be viewed a solution for the Conley conjecture for Hamiltonian systems on cotangent bundles, and Corollary 1.8 and Corollary 1.9 may be viewed as confirm answers of Lagrangian systems analogue of the Conley conjecture for Hamiltonian systems.

The main proof ideas come from [Lo2]. We shall prove Theorems 1.11, 1.6 in the case \( r = n \), and Theorem 1.3 by generalizing the variational arguments in [Lo2], [LoLu] and [LuW2] respectively. Some new ideas are needed because we do not lift to the universal cover space of \( M \) as done in [Lo2] LoLu LuW2 for the tori case. We also avoid using finite energy homologies used in [Lo2] LoLu LuW2. Let us outline the variational setup and new ideas as follows. For \( \tau > 0 \), let
\[ S_\tau := \mathbb{R}/\tau\mathbb{Z} = \{ [s]_\tau \mid [s]_\tau = s + \tau \mathbb{Z}, s \in \mathbb{R} \}, \quad \text{and} \quad E_\tau = W^{1,2}(S_\tau, M) \]
denote the space of all loops \( \gamma : S_\tau \to M \) of Sobolev class \( W^{1,2} \). For a homotopy class \( \alpha \) of free loops in \( M \), let
\[ H_\tau(\alpha), \quad H_\tau = H_\tau(0), \quad EH_\tau \]
respectively denote the subset of loops of $E_{\tau}$ representing $\alpha$, that of all contractible loops in $E_{\tau}$, and that of all reversible loops in $E_{\tau}$. Then $EH_{\tau} \subset H_{\tau}$.

For integer $m \geq 2$, if $M$ is $C^{m}$-smooth, all these spaces $E_{\tau}$, $H_{\tau}(\alpha)$ and $EH_{\tau}$ have $C^{m-1}$-smooth Hilbert manifold structure [K], and the tangent space of $E_{\tau}$ at $\gamma$ is $T_{\gamma}E_{\tau} = W^{1,2}(\gamma^{*}TM)$. Moreover, any $(C^{m-1})$ Riemannian metric $\langle \cdot, \cdot \rangle$ on $M$ induces a complete Riemannian metric on $E_{\tau}$:

$$\langle \xi, \eta \rangle_{\tau} = \int_{0}^{T} \left( \langle \xi(t), \eta(t) \rangle_{\gamma(t)} + \langle \nabla_{\xi}t \xi(t), \nabla_{\eta}t \eta(t) \rangle_{\gamma(t)} \right) dt \quad (1.13)$$

Here $\nabla_{t}$ denotes the covariant derivative in direction $\dot{\gamma}$ with respect to the Levi-Civita connection $\nabla$ of $\langle \cdot, \cdot \rangle$. Let $||\xi||_{\tau} = \sqrt{\langle \xi, \xi \rangle_{\tau}}$, $\forall \xi \in T_{\gamma}E_{\tau}$. Then the distance on $E_{\tau}$ induced by $|| \cdot ||_{\tau}$ is complete and also compatible with the manifold topology on $E_{\tau}$. Consider the functional $L_{\tau} : E_{\tau} \rightarrow \mathbb{R}$

$$L_{\tau}(\gamma) = \int_{0}^{T} L(t, \gamma(t), \dot{\gamma}(t)) dt \quad \forall \gamma \in E_{\tau}. \quad (1.14)$$

For integer $m \geq 3$, if $M$ is $C^{m}$-smooth and $C^{m-1}$-smooth $L : \mathbb{R} \times TM \rightarrow \mathbb{R}$ satisfies the assumptions (L1)-(L3), then the functional $L_{\tau}$ is $C^{2}$-smooth, bounded below, satisfies the Palais-Smale condition, and all critical points of it have finite Morse indexes and nullities, (see [Ab, Prop.4.1, 4.2] and [Ba]). By [Fa, Th.3.7.2], all critical points of $L_{\tau}$ are all of class $C^{m-1}$ and therefore correspond to all $\tau$-periodic solutions of (1.6).

Let $L_{\tau}^{E}$ denote the restriction of $L_{\tau}$ on $EH_{\tau}$. When $L$ satisfies (L4), it is not hard to prove that a map $\gamma : \mathbb{R} \rightarrow M$ is a $\tau$-periodic even solution to (1.6) if and only if $\gamma$ is a critical point of $L_{\tau}^{E}$ on $EH_{\tau}$, cf. [LuW2 Lem.1.7].

When we attempt to prove Theorem 1.1 by the method of [Lo2], we first need to know how to relate the Morse index and nullity of a critical point $\gamma \in E_{\tau}$ of $L_{\tau}$ to those of the $k$-th iteration $\gamma^{k} \in E_{k\tau}$ as a critical point of $L_{k\tau}$ on $E_{k\tau}$. Since we do not assume that $M$ is orientable or $\gamma$ is contractible, the bundle $\gamma^{*}TM \rightarrow S_{\tau}$ might not be trivial. However, for the 2-th iteration $\gamma^{2}$, the pullback bundle $(\gamma^{2})^{*}TM \rightarrow S_{2\tau}$ is always trivial. Since our proof is indirect by assuming that the conclusion does not hold, the arguments can be reduced to the case that all $\tau$-periodic solutions have trivial pullback bundles (as above Lemma 3.2). For such periodic solutions we can choose suitably coordinate charts around them on $E_{k\tau}$ so that the question is reduced to the case $M = \mathbb{R}^{n}$ as in Lemma 3.2. Hence we can get expected iteration inequalities as in Theorem 3.1. The second new idea is that under the assumption each $L_{k\tau}$ has only isolated critical points we show in Lemma 5.2 how to use an elementary arguments as above Corollary 1.2 and the Morse theory to get a non-minimal saddle point with nonzero th-n critical module with $\mathbb{Z}_{2}$-coefficient; the original method in [Lo2] Lemma 4.1] is to use Lemma II.5.2 on the page 127 of [Ch] to arrive at this goal, which seems to be difficult for me generalizing it to manifolds. It is worth noting that we avoid using finite energy homologies used in [Lo2, LoLu, LuW2]. That is based on an observation, that is, the composition $(j_{k\tau}) \circ \psi_{k}$ in (5.14) has a good
decomposition \((J_k)_* \circ (\psi^k)_* \circ (I_1)_*\) as in (5.15) such that for each \(\omega \in C_n(L_\tau, \gamma; \mathbb{K})\), \((I_1)_*(\omega)\) is a singular homology class of a \(C^1\)-Hilbert manifold and hence has a \(C^1\)-singular cycle representative. It is the final claim that allows us to use the singular homology to complete the remained arguments in Long’s method of \([Lo2]\). A merit of this improvement is to reduce the smoothness of the Largangi \(L\). That is, we only need to assume that \(L\) is of class \(C^2\). However, a new problem occurs, i.e. \(\tilde{\Theta}_{k\tau}\) in (4.12) is only a homeomorphism. It is very fortunate that \(\tilde{\alpha}_{k\tau}\) is also of class \(C^2\) as noted at the end of proof of Theorem 5.1 (the generalized Morse lemma) on the page 44 of \([Ch]\). Using the image of Gromoll-Meyer of \(\tilde{\alpha}_{k\tau}(\eta) + \tilde{\beta}_{k\tau}(\xi)\) under \(\tilde{\Theta}_{k\tau}\), called topological Gromoll-Meyer, to replace a Gromoll-Meyer of \(\tilde{L}_{k\tau}\) at \(\tilde{\gamma}^k\), we construct topological Gromoll-Meyer pairs of \(L_\tau\) at \(\gamma \in H_\tau(\alpha)\) and of \(L_{k\tau}\) at \(\gamma^k \in H_{k\tau}(\alpha^k)\), to satisfy Theorem 4.6 which is enough to complete our proof of Theorem 1.1. For the proof of Theorem 1.6 we need to complete more complex arguments as in §4.3. But the ideas are similar.

The paper is organized as follows. Section 2 will review some basic facts concerning the Maslov-type indices and relations between them and Morse indexes. In Section 3 we give some iteration inequalities of the Morse indexes. Section 4 studies changes of the critical modules under iteration maps. In Sections 5, 6 and 7, we give the proofs of Theorems 1.1, 1.4 and 1.6 respectively. Motivated by the second claim in Theorem 1.10, a more general question than the Conley’s conjecture and a program in progress are proposed in Section 8. In Appendix of Section 9 we prove Proposition A and a key Lemma A.4, which is a generalization of \([Lo2\text{, Lemma 2.3]}\).

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2 Maslov-type indices and Morse index

2.1. A review on Maslov-type indices. Let \(\text{Sp}(2n, \mathbb{R}) = \{M \in \mathbb{R}^{2n \times 2n} \mid M^T J_0 M = J_0\}\), where \(J_0 = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}\). For \(\tau > 0\), denoted by

\[
P_\tau(2n) = \{\Psi \in C([0, \tau], \text{Sp}(2n, \mathbb{R})) \mid \Psi(0) = I_{2n}\},
\]

\[
P_*^\tau(2n) = \{\Psi \in P_\tau(2n) \mid \text{det}(\Psi(\tau) - I_{2n}) \neq 0\}.
\]

The paths in \(P_*^\tau(2n)\) are called nondegenerate. The Maslov-type index (or Conley-Zehnder index) theory for the paths in \(P_*^\tau(2n)\) was defined by \([CoZe]\, \[Lo1\] and \([Vi2]\).
Yiming Long [Lo4] extended this theory to all paths in $\mathcal{P}_r(2n)$. The Maslov-type index of a path $\Psi \in \mathcal{P}_r(2n)$ is a pair of integers $(i_r(\Psi), \nu_r(\Psi))$, where

$$
\nu_r(\Psi) = \dim \ker (\Psi(\tau) - I_{2n}) \quad \text{and} \quad i_r(\Psi) = \inf \{i_r(\beta) \mid \beta \in \mathcal{P}_r(2n) \text{ is sufficiently } C^0 \text{ close to } \Psi \text{ in } \mathcal{P}_r(2n) \},
$$

with $i_r(\beta)$ defined as in [CoZe]. Clearly, the map $i_r : \mathcal{P}_r(2n) \to \mathbb{Z}$ is lower semi-continuous. For any paths $\Psi_k \in \mathcal{P}_r(2n)$, $k = 0, 1$, $(i_r(\Psi_k), \nu_r(\Psi_k)) = (i_r(\Psi_1), \nu_r(\Psi_1))$ if and only if there exists a homotopy $\Psi_s, 0 \leq s \leq 1$ from $\Psi_0$ to $\Psi_1$ in $\mathcal{P}_r(2n)$ such that $\Psi_s(0) = I_{2n}$ and $\nu_r(\Psi_s(\tau)) = \nu_r(\Psi_0)$ for any $s \in [0, 1]$.

For $a < b$ and any path $\Psi \in C([a, b], \text{Sp}(2n, \mathbb{R}))$, choose $\beta \in \mathcal{P}_1(2n)$ with $\beta(1) = \Psi(a)$, and define $\phi \in \mathcal{P}_1(2n)$ by $\phi(t) = \beta(2t)$ for $0 \leq t \leq 1/2$, and

$$
\phi(t) = \Psi(a + (2t - 1)(b - a)) \quad \text{for} \quad 1/2 \leq t \leq 1.
$$

It was showed in [Lo4] that the difference $i_1(\phi) - i_1(\beta)$ only depends on $\Psi$, and was called the **Maslov-type index** of $\Psi$, denoted by

$$
i(\Psi, [a, b]) := i_1(\phi) - i_1(\beta).
$$

Clearly, $i(\Psi, [0, 1]) = i_1(\Psi)$ for any $\Psi \in \mathcal{P}_1(2n)$.

Let $(F, \langle \cdot, \cdot \rangle)$ be the symplectic space with $F = \mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ and

$$
\{u, v\} = \langle J u, v \rangle \quad \forall u, v \in F, \quad \text{where} \quad J = \begin{pmatrix} -J_0 & 0 \\ 0 & J_0 \end{pmatrix}.
$$

**All vectors are understand as column vectors in this paper without special statements.** Let $\text{Lag}(F)$ be the manifold of Lagrangian Grassmannian of $(F, \langle \cdot, \cdot \rangle)$, and $\mu^{\text{CLM}}$ be the Cappell-Lee-Miller index characterized by properties I-VI of [CLM, pp. 127-128]. There exists the following relation between $\mu^{\text{CLM}}$ and the index defined by (2.1),

$$
i(\Psi, [a, b]) = \mu^{\text{CLM}}(W, \text{Gr}(\Psi), [a, b]) - n, \quad (2.2)
$$

where $W = \{(x^T, x^T)^T \in \mathbb{R}^{4n} \mid x \in \mathbb{R}^{2n}\}$.

With $U_1 = \{0\} \times \mathbb{R}^n$ and $U_2 = \mathbb{R}^n \times \{0\}$, two new Maslov-type indices for any path $\Psi \in C([a, b], \text{Sp}(2n, \mathbb{R}))$ were defined in [LoZZ] as follows:

$$
\mu_k(\Psi, [a, b]) = \mu_{2n}^{\text{CLM}}(U_k, \Psi U_k, [a, b]), \quad k = 1, 2. \quad (2.3)
$$

Let $\Psi(b) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A, B, C, D \in \mathbb{R}^{n \times n}$. In terms of [LoZZ, (2.21)], define

$$
\nu_1(\Psi, [a, b]) = \dim \ker (B) \quad \text{and} \quad \nu_2(\Psi, [a, b]) = \dim \ker (C). \quad (2.4)
$$

In particular, for $\Psi \in \mathcal{P}_r(2n)$ and $k = 1, 2$ we denote by

$$
\mu_{k, r}(\Psi) = \mu_k(\Psi, [0, \frac{r}{2}]) \quad \text{and} \quad \nu_{k, r}(\Psi) = \nu_k(\Psi, [0, \frac{r}{2}]). \quad (2.5)
$$
**Assumption B**. (B1) Let $B \in C(\mathbb{R}, \mathbb{R}^{2n \times 2n})$ be a path of symmetric matrix which is $\tau$-periodic in time $t$, i.e., $B(t + \tau) = B(t)$ for any $t \in \mathbb{R}$.

(B2) Let $B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}$, where $B_{11}, B_{22}, t \mapsto \mathbb{R}^{n \times n}$ are even at $t = 0$ and $\tau/2$, and $B_{12}, B_{21}, t \mapsto \mathbb{R}^{n \times n}$ are odd at $t = 0$ and $\tau/2$.

Under the assumption (B1), let $\Psi$ be the fundamental solution of the problem

$$\dot{\Psi}(t) = J_0 B(t) \Psi(t), \quad \Psi(0) = I_{2n}. \quad (2.6)$$

By the classical Floquet theory, $\nu_r(\Psi)$ is the dimension of the solution space of the linear Hamiltonian system

$$\dot{u}(t) = J_0 B(t) u(t) \quad \text{and} \quad u(t + \tau) = u(t).$$

Similarly, under the assumptions (B1) and (B2), it was also shown in [LoZZ, Prop.1.3]) that $\nu_{1,\tau}(\Psi)$ and $\nu_{2,\tau}(\Psi)$ are the dimensions of the solution spaces of the following two problems respectively,

$$\left\{ \begin{array}{l}
\dot{u}(t) = J_0 B(t) u(t), \\
u(t + \tau) = u(t), \quad u(-t) = Nu(t),
\end{array} \right. \quad \text{where } N = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix}.$$ 

Let $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ denote the coordinates in $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. Denote by $\omega_0 = \sum_{k=1}^n dx_k \wedge dy_k$ the standard symplectic structure on $\mathbb{R}^{2n}$, i.e. $\omega_0(u, v) = \langle J_0 u, v \rangle \forall u, v \in \mathbb{R}^{2n}$. Here $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{2n}$. Define $H : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ by $H(t, u) = \frac{1}{2} \langle B(t) u, u \rangle$. Let $X_H$ be the corresponding Hamiltonian vector field defined by

$$\omega_0(X_H(t, u), v) = -d_u H(t, u)(v). \quad (2.7)$$

Then $X_H(t, u) = J_0 B(t) u$ for any $u \in \mathbb{R}^{2n}$.

For $\Psi \in \mathcal{P}_r(2n)$, extend the definition of $\Psi$ to $[0, +\infty)$ by

$$\Psi(t) = \Psi(t - j\tau) \Psi(\tau)^j, \quad \forall j\tau \leq t \leq (j + 1)\tau, \quad j \in \mathbb{N}, \quad (2.8)$$

and define the $m$-th iteration $\Psi^m$ of $\Psi$ by

$$\Psi^m = \Psi|[0,m\tau]. \quad (2.9)$$

It was proved in [Lo3, pp. 177-178] that the mean index per $\tau$ of $\Psi \in \mathcal{P}_r(2n)$,

$$\hat{i}_\tau(\Psi) := \lim_{m \to +\infty} \frac{i_{m\tau}(\Psi^m)}{m} \quad (2.10)$$

always exists.
Lemma 2.1  (i) For any $\Psi \in \mathcal{P}_\tau(2n)$ it holds that

$$\max \left\{ 0, m \hat{\nu}_\tau(\Psi) - n \right\} \leq i_{m\tau}(\Psi^m) \leq m \hat{\nu}_\tau(\Psi) + n - \nu_{m\tau}(\Psi^m), \quad \forall m \in \mathbb{N}.$$  

(ii) $|\mu_1(\Psi) - \mu_2(\Psi)| \leq n$ for any $\Psi \in \mathcal{P}_\tau(2n)$ with $\tau > 0$.

(iii) Under Assumption B, let $\Psi : [0, +\infty) \to \text{Sp}(2n, \mathbb{R})$ be the fundamental solution of the problem (2.7). (It must satisfy (2.5)). Then

$$\mu_{1,m\tau}\left(\Psi|_{[0, \frac{2}{\tau}]}\right) + \mu_{2,m\tau}\left(\Psi|_{[0, \frac{2}{\tau}]}ight) = i_{m\tau}\left(\Psi|_{[0, m\tau]}\right) + n \quad \forall m \in \mathbb{N},$$

(2.11)

(or equivalently $\mu_1(\Psi, [0, m\tau]) + \mu_2(\Psi, [0, m\tau]) = i_{m\tau}(\Psi|_{[0, m\tau]}) + n \forall m \in \mathbb{N}$). Moreover, for $k = 1, 2$ the mean indices of $\Psi$ per $\tau$ defined by

$$\hat{\mu}_{k,\tau}(\Psi) := \lim_{m \to +\infty} \frac{\mu_{k,m\tau}\left(\Psi|_{[0, m\tau]}\right)}{m}$$

(2.12)

always exist and equal to $\frac{1}{2} \hat{\nu}_\tau(\Psi)$.

(i) comes from [LiLo] or [Lo3, p. 213, (17)], (ii) is [LoZZ, Th.3.3], and (iii) is [LoZZ, Prop.C, Cor.6.2] (precisely is derived from the proof of [LoZZ, Prop.C, Cor.6.2]). It is easily checked that (i) implies $|i_{m\tau} - m\tau| \leq (m + 1)n$ for any $m \in \mathbb{N}$. A similar inequality to the latter was also derived in [DDP, (12)] recently.

2.2. Relations between Maslov-type indices and Morse indices.

Lemma 2.2. Let the Lagrangian $L : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$ be given by

$$L(t, y, v) = \frac{1}{2} P(t)v \cdot v + Q(t)y \cdot v + \frac{1}{2} R(t)y \cdot y,$$

where $P, Q, R : \mathbb{R} \to \mathbb{R}^{n \times n}$ are $C^1$-smooth and $\tau$-periodic, $R(t) = R(t)^T$, and each $P(t) = P(t)^T$ is also positive definite. The corresponding Lagrangian system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v}(t, y, \dot{y})\right) - \frac{\partial L}{\partial y}(t, y, \dot{y}) = (P\dot{y} + Qy)\dot{y} - Q^T \dot{y} - Ry = 0.$$  

(2.13)

Let $\dot{y}$ be a critical point of the functional

$$f_\tau(y) = \int_0^\tau L(t, y(t), \dot{y}(t)) \, dt$$

on $W^{1,2}(S_\tau, \mathbb{R}^n)$, and the second differential of $f_\tau$ at it be given by

$$d^2f_\tau(\bar{y})(y, z) = \int_0^\tau \left[ (P\dot{y} + Qy) \cdot \dot{z} + Q^T \dot{y} \cdot z + Ry \cdot z \right] dt.$$

The linearized system of (2.13) at $\bar{y}$ is the Sturm system:

$$-(P\dot{y} + Qy)^\prime + Q^T \dot{y} + Ry = 0.$$  

Let

$$S(t) = \begin{pmatrix} P(t)^{-1} & -P(t)^{-1}Q(t) \\ -Q(t)^TP(t)^{-1} & Q(t)^TP(t)^{-1}Q(t) - R(t) \end{pmatrix},$$

(2.14)
and $\Psi : [0, +\infty) \to \text{Sp}(2n, \mathbb{R})$ be the fundamental solution of the problem
\[
\dot{u}(t) = J_0 S(t) u
\] (2.15)
with $\Psi(0) = I_{2n}$. Suppose that each $P(t)$ is symmetric positive definite, and that each $R(t)$ is symmetric. Then $f_{\tau}$ at $\tilde{y} \in W^{1,2}(S_{\tau}, \mathbb{R}^n)$ has finite Morse index $m_{\tau}(f_{\tau}, \tilde{y})$ and nullity $m^0_{\tau}(f_{\tau}, \tilde{y})$, and
\[
m_{\tau}(f_{\tau}, \tilde{y}) = i_{\tau}(\Psi) \quad \text{and} \quad m^0_{\tau}(f_{\tau}, \tilde{y}) = \nu_{\tau}(\Psi). \quad (2.16)
\]

**Remark 2.3** Since $L_{\nu \tau}(t, y, v) = P(t)$ is invertible for every $t$, $L$ has the Legendre transform $H : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}$:
\[
H(t, x, y) = x \cdot v(t, x, y) - L(t, x, v(t, x, y)),
\]
where $v(t, x, y) \in \mathbb{R}^n$ is determined by $L_{\nu}(t, y, v(t, x, y)) = x$. Precisely, $v(t, x, y) = P(t)^{-1}[x - Q(t)y]$ and
\[
H(t, x, y) = \frac{1}{2} P(t)^{-1} x \cdot x - P(t)^{-1} x \cdot Q(t)y + \frac{1}{2} P(t)^{-1} Q(t)y \cdot Q(t)y - \frac{1}{2} R(t)y \cdot y.
\]
Then $X_H(t, x, y) = J_0 S(t) u$ with $u = (x^T, y^T)^T$, and $\dot{u} = (\tilde{x}^T, \tilde{y}^T)^T$ is a $\tau$-periodic solution of (2.15).

Let
\[
\begin{align*}
EW^{1,2}(S_{\tau}, \mathbb{R}^n) &= \{ y \in W^{1,2}(S_{\tau}, \mathbb{R}^n) \mid y(-t) = y(t) \forall t \in \mathbb{R} \}, \\
OW^{1,2}(S_{\tau}, \mathbb{R}^n) &= \{ y \in W^{1,2}(S_{\tau}, \mathbb{R}^n) \mid y(-t) = -y(t) \forall t \in \mathbb{R} \}.
\end{align*}
\]

**Lemma 2.4** ([LuW2 Th.3.4]) Under the assumptions of Lemma 2.2, suppose furthermore that
\[
\begin{align*}
P(t + \tau) &= P(t) = P(t)^T = P(-t) \forall t \in \mathbb{R}, \\
R(t + \tau) &= R(t) = R(t)^T = R(-t) \forall t \in \mathbb{R}, \\
Q(t + \tau) &= Q(t) = -Q(-t) \forall t \in \mathbb{R},
\end{align*}
\] (2.17)
and thus $L$ in Lemma 2.2 satisfies (LA). So the present $S(t)$ in (2.14) also satisfies the Assumption B. Let $\tilde{y}$ be a critical point of the restriction $f_{\tau}^L$ of the functional $f_{\tau}$ to $EW^{1,2}(S_{\tau}, \mathbb{R}^n)$. (It is also a critical point of the functional $f_{\tau}$ on $W^{1,2}(S_{\tau}, \mathbb{R}^n)$ because $f_{\tau}$ is even). As in Lemma 2.1, let $\Psi$ denote the fundamental solution of (2.15). Let
\[
\begin{align*}
EW^{1,2}(S_{\tau}, \mathbb{R}^n) &= EW^{1,2}(S_{\tau}, \mathbb{R}^n)^+ \oplus EW^{1,2}(S_{\tau}, \mathbb{R}^n)^0 \oplus EW^{1,2}(S_{\tau}, \mathbb{R}^n)^- , \\
OW^{1,2}(S_{\tau}, \mathbb{R}^n) &= OW^{1,2}(S_{\tau}, \mathbb{R}^n)^+ \oplus OW^{1,2}(S_{\tau}, \mathbb{R}^n)^0 \oplus OW^{1,2}(S_{\tau}, \mathbb{R}^n)^- .
\end{align*}
\]
be respectively $d^2 f_\tau(y)$-orthogonal decompositions according to $d^2 f_\tau(y)$ being positive, null, and negative definite. Then

$$\dim EW^{1,2}(S_\tau, \mathbb{R}^n)^- = m^-_\tau(f^E_\tau, y) = \mu_1(\Psi),$$ \hspace{1cm} (2.18)

$$\dim EW^{1,2}(S_\tau, \mathbb{R}^n)^0 = m^0_\tau(f^E_\tau, y) = \nu_1(\Psi),$$ \hspace{1cm} (2.19)

$$\dim OW^{1,2}(S_\tau, \mathbb{R}^n)^- = \mu_2(\Psi) - n,$$ \hspace{1cm} (2.20)

$$\nu_\tau(\Psi) = \nu_1(\Psi) + \nu_2(\Psi).$$ \hspace{1cm} (2.21)

For conveniences we denote by

$$m^-_{2,\tau}(f_\tau, y) := \dim OW^{1,2}(S_\tau, \mathbb{R}^n)^-,$$ \hspace{1cm} (2.22)

$$m^0_{2,\tau}(f_\tau, y) := \dim OW^{1,2}(S_\tau, \mathbb{R}^n)^0.$$ \hspace{1cm} (2.23)

Then under the assumptions of Lemma 2.1, Lemma 2.1(ii)(iii) and (2.21) become

$$|n + m^-_{2,\tau}(f_\tau, y) - m^-_\tau(f^E_\tau, y)| \leq n,$$ \hspace{1cm} (2.24)

$$m^-_{2,\tau}(f_\tau, y) + m^-_\tau(f^E_\tau, y) = m^-_\tau(f_\tau, y),$$ \hspace{1cm} (2.25)

$$m^0_{\tau}(f_\tau, y) = m^0_\tau(f^E_\tau, y) + m^0_{2,\tau}(f_\tau, y).$$ \hspace{1cm} (2.26)

### 3 Iteration inequalities of the Morse index

#### 3.1 The case of general periodic solutions

In this subsection we always assume: $M$ is $C^3$-smooth, $L$ is $C^2$-smooth and satisfies (L1)-(L3). Let $\gamma \in E_\tau$ be a critical point of the functional $L_\tau$ on $E_\tau$. It is a $\tau$-periodic map from $\mathbb{R}$ to $M$. For each $k \in \mathbb{N}$, $\gamma : \mathbb{R} \to M$ is also $k\tau$-periodic map and therefore determines an element in $E_{k\tau}$, denoted by $\gamma^k$ for the sake of clearness. It is not difficult to see that $\gamma^k$ is a critical point of $L_{k\tau}$ on $E_{k\tau}$. Let

$$m^-_{k\tau}(\gamma^k) \text{ and } m^0_{k\tau}(\gamma^k)$$

denote the Morse index and nullity of $L_{k\tau}$ on $E_{k\tau}$ respectively. Note that

$$0 \leq m^0_{k\tau}(\gamma^k) \leq 2n \quad \forall k \in \mathbb{N}.$$ (This can be derived from (2.16) and Lemma 3.2 below). A natural question is how to estimate $m^-(\gamma^k)$ in terms of $m^-_\tau(\gamma)$, $m^0_\tau(\gamma)$ and $m^0_{k\tau}(\gamma^k)$. The following theorem gives an answer.

**Theorem 3.1** For a critical point $\gamma$ of $L_\tau$ on $E_\tau$, assume that $\gamma^*TM \to S_\tau$ is trivial. Then the mean Morse index

$$\hat{m}^-_\tau(\gamma) := \lim_{k \to \infty} \frac{m^-_{k\tau}(\gamma^k)}{k}$$ \hspace{1cm} (3.1)

always exists, and it holds that

$$\max \{0, k\hat{m}^-_\tau(\gamma) - n\} \leq m^-_{k\tau}(\gamma^k) \leq k\hat{m}^-_\tau(\gamma) + n - m^0_{k\tau}(\gamma^k) \quad \forall k \in \mathbb{N}.$$ \hspace{1cm} (3.2)
Consequently, for any critical point $\gamma$ of $L_\tau$ on $E_\tau$, $\hat{m}_2(\gamma^2)$ exists and
\[
\max \{0, k\hat{m}_2(\gamma^2) - n\} \leq m_{2k\tau}(\gamma^2) \leq k\hat{m}_2(\gamma) + n - m_{2k\tau}(\gamma^2) \quad \forall k \in \mathbb{N}
\]  
(3.3) because $(\gamma^2)^*TM \to S_{2\tau}$ is always trivial.

Before proving this result it should be noted that the following special case is a direct consequence of Lemma 2.1(i) and Lemma 2.2.

**Lemma 3.2** Under the assumptions of Lemma 2.2, for each $k \in \mathbb{N}$, $\bar{y}$ is also a $k\tau$-periodic solution of (2.13), denoted by $\bar{y}^k$. Then $\bar{y}^k$ is a critical point of the functional
\[
f_{k\tau}(y) = \int_0^{k\tau} L(t, y(t), \dot{y}(t))dt
\]
on $W^{1,2}(S_{k\tau}, \mathbb{R}^m)$, and
\[
\hat{m}_-^k(f_{k\tau}, \bar{y}^k) := \lim_{k \to +\infty} \frac{m_{k\tau}^-(f_{k\tau}, \bar{y}^k)}{k} = \lim_{k \to +\infty} \frac{i_{k\tau}(\Psi^k)}{k} = \hat{i}_\tau(\Psi),
\]
(3.4)
max$\{0, k\hat{m}_-^k(f_{k\tau}, \bar{y}^k) - n\} \leq m_{k\tau}^-(f_{k\tau}, \bar{y}^k) \leq k\hat{m}_-^k(f_{k\tau}, \bar{y}^k) + n - m_{k\tau}^0(f_{k\tau}, \bar{y}^k)
\]
(3.5)
with $0 \leq m_{k\tau}^0(f_{k\tau}, \bar{y}^k) \leq 2n$ for any $k \in \mathbb{N}$.

This result was actually used in [Lo2, LoLu, LuW2]. In the following we shall show that Theorem 3.1 can be reduced to the special case.

**Proof of Theorem 3.1** Step 1. Reduce to the case $M = \mathbb{R}^n$. Let $\gamma \in E_\tau$ be a critical point $\gamma$ of $L_\tau$ on $E_\tau$ with trivial pullback $\gamma^*TM \to S_\tau$. Take a $C^2$-smooth loop $\gamma_0 : \tau \to M$ such that $\max_{\tau} d(\gamma(t), \gamma_0(t)) < \rho$, where $d$ and $\rho$ are the distance and injectivity radius of $M$ with respect to some chosen Riemannian metric on $M$ respectively. (Actually we can choose $\gamma_0 = \gamma$ because $\gamma_0$ is $C^2$-smooth under the assumptions of this subsection). Clearly, $\gamma$ and $\gamma_0$ are homotopic, and thus $\gamma_0^*TM \to S_\tau$ is trivial too. Since $\gamma_0$ is $C^2$-smooth, we can choose a $C^2$-smooth orthogonal trivialization
\[
S_\tau \times \mathbb{R}^n \to \gamma_0^*TM, (t, q) \mapsto \Phi(t)q.
\]
(3.6)
It naturally leads to a smooth orthogonal trivialization of $(\gamma_0^k)_{TM}$ for any $k \in \mathbb{N}$,
\[
S_{k\tau} \times \mathbb{R}^n \to (\gamma_0^k)_{TM}, (t, q) \mapsto \Phi(t)q.
\]
(3.7)
Let $B^m_\rho(0)$ denote an open ball in $\mathbb{R}^m$ centered at 0 with radius $\rho$. Then for each $k \in \mathbb{N}$, we have a coordinate chart on $E_{k\tau}$ containing $\gamma^k$,
\[
\phi_{k\tau} : W^{1,2}(S_{k\tau}, B^m_\rho(0)) \to E_{k\tau}, \quad \phi_{k\tau}(\tilde{\alpha})(t) = \exp_{\gamma_0(t)}(\Phi(t)\tilde{\alpha}(t)).
\]
(3.8)
Clearly, $\phi_{k\tau}(\tilde{\alpha})$ has a period $\tau$ if and only if $\tilde{\alpha}$ is actually $\tau$-periodic. Thus we have a unique $\tilde{\gamma} \in W^{1,2}(S^1, B^{n}_{\rho}(0))$ such that $\phi_{k\tau}(\tilde{\gamma}^k) = \gamma^k$ for any $k \in \mathbb{N}$. Denote by the iteration maps

\[
\psi^k : E_{\tau} \to E_{k\tau}, \quad \alpha \mapsto \alpha^k,
\]

\[
\psi^k : T_{\alpha} E_{\tau} \to T_{\alpha^k} E_{k\tau}, \quad \xi \mapsto \xi^k,
\]

\[
\tilde{\psi}^k : W^{1,2}(S^1, \mathbb{R}^n) \to W^{1,2}(S_{k\tau}, \mathbb{R}^n), \quad \tilde{\alpha} \mapsto \tilde{\alpha}^k.
\]

It is easy to see that

\[
\phi_{k\tau} \circ \tilde{\psi}^k = \psi^k \circ \phi_{\tau} \quad \forall k \in \mathbb{N}.
\]  

(3.9)

For any $k \in \mathbb{N}$, set

\[
\tilde{L}_{k\tau} : W^{1,2}(S_{k\tau}, B^{n}_{\rho}(0)) \to \mathbb{R}, \quad \tilde{L}_{k\tau} = L_{k\tau} \circ \phi_{k\tau}.
\]  

(3.10)

Then $\tilde{\gamma} = \phi_{\tau}^{-1}(\gamma)$ is a critical point of $\tilde{L}_{\tau}$, and therefore $\tilde{\gamma}^k = \phi_{k\tau}^{-1}(\gamma^k) = \tilde{\psi}^k(\tilde{\gamma})$ is a critical point of $\tilde{L}_{k\tau}$ for any $k \in \mathbb{N}$. Moreover, the Morse indexes and nullities of these critical points satisfy the relations:

\[
m_{k\tau}^{-}(\tilde{\gamma}^k) = m_{k\tau}^{-}(\gamma^k) \quad \text{and} \quad m_{k\tau}^{0}(\tilde{\gamma}^k) = m_{k\tau}^{0}(\gamma^k), \quad \forall k \in \mathbb{N}.
\]  

(3.11)

Viewing $\gamma_0$ a $\tau$-periodic map from $\mathbb{R} \to M$, consider the $C^2$-smooth map

\[
\Xi : \mathbb{R} \times B^{n}_{\rho}(0) \to M, \quad (t, \tilde{q}) \mapsto \exp_{\gamma_0(t)}(\Phi(t)\tilde{q}).
\]  

(3.12)

Then $\Xi(t + \tau, \tilde{q}) = \Xi(t, \tilde{q})$ for any $(t, \tilde{q}) \in \mathbb{R} \times M$. Clearly,

\[
\phi_{k\tau}(\tilde{\alpha})(t) = \Xi(t, \tilde{\alpha}(t)) \quad \text{and} \quad \frac{d}{dt}(\phi_{k\tau}(\tilde{\alpha}))(t) = \frac{d}{dt}\Xi(t, \tilde{q})|_{\tilde{q} = \tilde{\alpha}(t)} + d_{\tilde{q}}\Xi(t, \tilde{q})(\tilde{\alpha}(t))
\]  

(3.13)

for any $t \in \mathbb{R}$ and $\tilde{\alpha} \in W^{1,2}(S_{k\tau}, B^{n}_{\rho}(0))$. Define $\tilde{L} : \mathbb{R} \times B^{n}_{\rho}(0) \times \mathbb{R}^n \to \mathbb{R}$ by

\[
\tilde{L}(t, \tilde{q}, \tilde{v}) = L \left( t, \Xi(t, \tilde{q}), \frac{d}{dt}\Xi(t, \tilde{q}) + d_{\tilde{q}}\Xi(t, \tilde{q})(\tilde{v}) \right).
\]  

(3.15)

Then $\tilde{L}(t + \tau, \tilde{q}, \tilde{v}) = \tilde{L}(t, \tilde{q}, \tilde{v}) \forall (t, \tilde{q}, \tilde{v}) \in \mathbb{R} \times B^{n}_{\rho}(0) \times \mathbb{R}^n$, and $\tilde{L}$ also satisfies the conditions (L2')-(L3') (up to changing the constants). For $\tilde{\alpha} \in W^{1,2}(S_{k\tau}, B^{n}_{\rho}(0))$, by (3.10) we have

\[
\tilde{L}_{k\tau}(\tilde{\alpha}) = L_{k\tau}(\phi^k(\tilde{\alpha})) = \int_{0}^{k\tau} L \left( t, \phi^k(\tilde{\alpha})(t), \frac{d}{dt}(\phi^k(\tilde{\alpha}))(t) \right) dt = \int_{0}^{k\tau} \tilde{L}(t, \tilde{\alpha}(t), \dot{\tilde{\alpha}}(t)) dt.
\]  

(3.16)

Therefore we may assume $M = \mathbb{R}^n$. That is, by (3.11) we only need to prove

\[
\hat{m}_{\tau}(\tilde{\gamma}) := \lim_{k \to \infty} \frac{m_{k\tau}^{-}(\tilde{\gamma}^k)}{k} \quad \text{exists},
\]

\[
\max \left\{ 0, k\hat{m}_{\tau}(\tilde{\gamma}) - n \right\} \leq m_{k\tau}^{-}(\tilde{\gamma}^k) \leq k\hat{m}_{\tau}(\tilde{\gamma}) + n - m_{k\tau}^{0}(\tilde{\gamma}^k) \quad \forall k \in \mathbb{N}.
\]  

(3.17)
Step 2. Reduce to the case of Lemma 3.2. Note that

\[
d\tilde{L}_\tau(\tilde{\gamma})(\tilde{\xi}) = \int_0^\tau \left( D_q\tilde{L} (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) (\tilde{\xi}(t)) + D_{\tilde{\xi}}\tilde{L} (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) (\dot{\tilde{\xi}}(t)) \right) dt
\]

\[
= \int_0^\tau \left( D_q\tilde{L} (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) - \frac{d}{dt} D_{\tilde{\xi}}\tilde{L} (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) \right) \cdot \dot{\tilde{\xi}}(t) dt
\]

for any \( \tilde{\xi} \in W^{1,2}(S_\tau, \mathbb{R}^n) \). Since \( d\tilde{L}_\tau(\tilde{\gamma}) = 0 \), we have also

\[
d^2\tilde{L}_\tau(\tilde{\gamma})(\tilde{\xi}, \tilde{\eta}) = \int_0^\tau \left( D_{\tilde{\xi}\tilde{\eta}}\tilde{L} (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) (\tilde{\xi}(t), \tilde{\eta}(t)) + D_{\tilde{\xi}}\tilde{L} (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) (\tilde{\xi}(t), \tilde{\eta}(t)) + D_{\tilde{\xi}\tilde{\eta}}\tilde{L} (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) (\tilde{\xi}(t), \tilde{\eta}(t)) \right) dt
\]

for any \( \tilde{\xi}, \tilde{\eta} \in W^{1,2}(S_\tau, \mathbb{R}^n) \). Set

\[
\begin{align*}
\hat{P}(t) &= D_{\tilde{\xi}\tilde{\eta}}\tilde{L} (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)), \\
\hat{Q}(t) &= D_{\tilde{\xi}}\tilde{L} (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)), \\
\hat{R}(t) &= D_{\tilde{\xi}\tilde{\eta}}\tilde{L} (t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t))
\end{align*}
\tag{3.19}
\]

and

\[
\hat{L}(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) = \frac{1}{2} \hat{P}(t) \tilde{\gamma} + \hat{Q}(t) \dot{\tilde{\gamma}} + \frac{1}{2} \hat{R}(t) \tilde{\gamma}(t) \cdot \dot{\tilde{\gamma}}. \tag{3.20}
\]

Clearly, they satisfy the conditions of Lemma 2.2 and \( \tilde{y} = 0 \in W^{1,2}(S_\tau, \mathbb{R}^n) \) is a critical point of the functional

\[
\hat{f}_\tau(\tilde{y}) = \int_0^\tau \hat{L}(t, \tilde{\gamma}(t), \dot{\tilde{\gamma}}(t)) dt
\]

on \( W^{1,2}(S_\tau, \mathbb{R}^n) \). It is also easily checked that

\[
d^2\hat{f}_\tau(0)(\tilde{\xi}, \tilde{\eta}) = d^2\tilde{L}_\tau(\tilde{\gamma})(\tilde{\xi}, \tilde{\eta}) \quad \forall \tilde{\xi}, \tilde{\eta} \in W^{1,2}(S_\tau, \mathbb{R}^n).
\]

It follows that

\[
m_{k\tau}^- (\hat{f}_\tau, 0) = m_{k\tau}^- (\gamma^k) \quad \text{and} \quad m_{k\tau}^0 (\hat{f}_\tau, 0) = m_{k\tau}^0 (\gamma^k) \quad \forall k \in \mathbb{N}.
\]

These and Lemma 3.2 together give the desired (3.17) and (3.18) \( \square \).

3.2. The case of even periodic solutions. Let \( M \) and \( L \) be as in §3.1. But we also assume that \( L \) satisfies (L4). Note that the even periodic solutions are always contractible. Let \( \mathcal{L}_{k\tau}^E \) denote the restriction of \( \mathcal{L}_{k\tau} \) on \( EH_{k\tau} \). As noted in the introduction, if \( \gamma \in EH_{\tau} \) is a critical point of \( \mathcal{L}_{\tau}^E \) on \( EH_{\tau} \) then \( \gamma^k \) is a critical point of \( \mathcal{L}_{k\tau} \) on \( H_{k\tau} \) for each \( k \in \mathbb{N} \). Let

\[
m_{1,k\tau}^-(\gamma^k) \quad \text{and} \quad m_{1,k\tau}^+(\gamma^k)
\]

denote the Morse index and nullity of \( \mathcal{L}_{k\tau}^E \) on \( EH_{k\tau} \) respectively. Then \( 0 \leq m_{1,k\tau}^-(\gamma^k) \leq m_{k\tau}^0 (\gamma^k) \leq 2n \) for any \( k \). We shall prove
Theorem 3.3 Let $L$ satisfy the conditions $\text{(L1)-(L4)}$. Then for any critical point $\gamma$ of $L^E_k$ on $EH_\tau$, the mean Morse index

$$
\hat{m}_{1,\tau}(\gamma) := \lim_{k \to \infty} \frac{m_{-1,k\tau}(\gamma^k)}{k} \quad (3.21)
$$

exists, and it holds that

$$
m_{-1,k\tau}(\gamma^k) + m_{-1,k\tau}(\gamma^k) \leq n \quad \forall k \in \mathbb{N} \quad \text{if} \quad \hat{m}_{1,\tau}(\gamma) = 0. \quad (3.22)
$$

Firstly, by (2.10) and (2.16) the mean Morse index

$$
\hat{m}_{1,\tau}(f_\tau, \tilde{y}) := \lim_{k \to \infty} \frac{m_{-1,k\tau}(f_\tau, \tilde{y}^k)}{k} = \hat{\mu}_{1,\tau}(\Psi) = \frac{1}{2} \hat{m}_{1,\tau}(f_\tau, \tilde{y}), \quad (3.24)
$$

exists and equals to $\hat{\tau}(\Psi)$. Under the assumptions of Lemma 2.4 for each $k \in \mathbb{N}$, $\tilde{y}^k$ is a critical point of the restriction $f^E_\tau$ of the functional $f_\tau$ to $EW^{1,2}(S_\tau, \mathbb{R}^n)$, and it follows from (2.12), (2.18), (2.20) and (2.22) that

$$
\hat{m}_{1,\tau}(f^E_\tau, \tilde{y}) := \lim_{k \to \infty} \frac{m_{-1,k\tau}(f^E_\tau, \tilde{y}^k)}{k} = \hat{\mu}_{1,\tau}(\Psi) = \frac{1}{2} \hat{m}_{1,\tau}(f_\tau, \tilde{y}). \quad (3.25)
$$

Moreover, by (2.25) and (2.26), for any $k \in \mathbb{N}$ it holds that

$$
m_{-2,k\tau}(f_\tau, \tilde{y}^k) = m_{2,k\tau}(f_\tau, \tilde{y}^k) = m_{-1,k\tau}(f_\tau, \tilde{y}^k),
$$

$$m_{0,k\tau}(f_\tau, \tilde{y}^k) = m_{0,k\tau}(f^E_\tau, \tilde{y}^k) + m_{0,k\tau}(f_\tau, \tilde{y}^k).$$

From these we derive that (3.5) becomes

$$
\max\{0, 2k\hat{m}_{1,\tau}(f^E_\tau, \tilde{y}) - n\} \leq m_{-2,k\tau}(f_\tau, \tilde{y}^k) + m_{-1,k\tau}(f^E_\tau, \tilde{y}^k)
$$

$$\leq 2k\hat{m}_{1,\tau}(f^E_\tau, \tilde{y}) + n - m_{0,k\tau}(f^E_\tau, \tilde{y}^k) - m_{0,k\tau}(f_\tau, \tilde{y}^k) \quad (3.26)
$$

for any $k \in \mathbb{N}$. In particular, if $\hat{m}_{1,\tau}(f^E_\tau, \tilde{y}) = 0$, then

$$m_{-2,k\tau}(f_\tau, \tilde{y}^k) + m_{0,k\tau}(f_\tau, \tilde{y}^k) \leq n \quad \forall k \in \mathbb{N}. \quad (3.27)
$$

([LuW2 Th.3.7]).

Proof of Theorem 3.3 Since $\gamma$ is even we can still choose $\gamma_0$ and $\Phi$ in (3.6) to be even, i.e. $\gamma_0(-t) = \gamma_0(t)$ and $\Phi(-t) = \Phi(t)$ for any $t \in \mathbb{R}$. These imply

$$
\Xi(-t, \tilde{q}) = \Xi(t, \tilde{q}), \quad \frac{d}{dt}\Xi(-t, \tilde{q}) = -\frac{d}{ds}\Xi(s, \tilde{q})|_{s=-t} = \frac{d}{dt}\Xi(t, \tilde{q}). \quad (3.28)
$$

It follows that the coordinate chart $\phi_{k\tau}$ in (3.8) naturally restricts to a coordinate chart on $EH_{k\tau}$,

$$
\phi_{k\tau}^E : EW^{1,2}(S_\tau, B^{n}_\rho(0)) \to EH_{k\tau} \quad (3.29)
$$
which also satisfies

$$\phi_{k\tau}^E \circ \tilde{\psi}^k = \psi^k \circ \phi_{\tau}^E \quad \forall k \in \mathbb{N}. \quad (3.30)$$

By (L4), (3.15) and (3.28) we have

$$\tilde{L}(-t, \tilde{q}, -\tilde{v}) = L\left(-t, \Xi(-t, \tilde{q}), \frac{d}{dt} \Xi(-t, \tilde{q}) - d_q \Xi(-t, \tilde{q})(-\tilde{v})\right)$$

$$= L\left(-t, \Xi(t, \tilde{q}), -\frac{d}{dt} \Xi(t, \tilde{q}) - d_q \Xi(t, \tilde{q})(\tilde{v})\right)$$

$$= L\left(t, \Xi(t, \tilde{q}), \frac{d}{dt} \Xi(t, \tilde{q}) + d_q \Xi(t, \tilde{q})(\tilde{v})\right)$$

$$= L\left(t, \Xi(t, \tilde{q}), \frac{d}{dt} \Xi(t, \tilde{q}) + d_q \Xi(t, \tilde{q})(\tilde{v})\right)$$

$$= \tilde{L}(t, \tilde{q}, \tilde{v}). \quad (3.31)$$

That is, $\tilde{L}$ also satisfies (L4). It follows that for any $k \in \mathbb{N}$, the functional

$$\tilde{\mathcal{L}}^E_{k\tau} : EW^{1,2}(S_{k\tau}, B^a_0(0)) \to \mathbb{R}, \quad \tilde{\mathcal{L}}^E_{k\tau} = \mathcal{L}^E_{k\tau} \circ \phi_{k\tau}$$

is exactly the restriction of the functional $\tilde{\mathcal{L}}_{k\tau}$ in (3.10) to $EW^{1,2}(S_{k\tau}, B^a_0(0))$. Hence the question is reduced to the case $M = \mathbb{R}$ again. That is, we only need to prove

$$m_{1,\tau}^-(\tilde{\gamma}) := \lim_{k \to \infty} \frac{m_{1,\tau}^-(\tilde{\gamma}^k)}{k} \quad \text{exists,} \quad (3.33)$$

$$m_{1,\tau}^-(\tilde{\gamma}^k) + m_{0,\tau}^-(\tilde{\gamma}^k) \leq n \quad \forall k \in \mathbb{N} \quad \text{if} \quad \tilde{m}_{1,\tau}^-(\tilde{\gamma}) = 0. \quad (3.34)$$

By (3.31) we have

$$D_{\tilde{q}\tilde{v}} \tilde{L}(-t, \tilde{q}, -\tilde{v}) = D_{\tilde{q}\tilde{v}} \tilde{L}(t, \tilde{q}, \tilde{v}),$$

$$D_{\tilde{q}\tilde{v}} \tilde{L}(-t, \tilde{q}, -\tilde{v}) = -D_{\tilde{q}\tilde{v}} \tilde{L}(t, \tilde{q}, \tilde{v}),$$

$$D_{\tilde{q}\tilde{v}} \tilde{L}(-t, \tilde{q}, -\tilde{v}) = D_{\tilde{q}\tilde{v}} \tilde{L}(t, \tilde{q}, \tilde{v})$$

for any $(t, \tilde{q}, \tilde{v}) \in \mathbb{R} \times B^a_0(0) \times \mathbb{R}$. Since $\tilde{\gamma}(-t) = \tilde{\gamma}(t)$ and $\tilde{\dot{\gamma}}(-t) = -\tilde{\dot{\gamma}}(t)$, it follows from this that $\hat{P}$, $\hat{Q}$ and $\hat{R}$ in (3.19) satisfy (2.17). For $\hat{L}$ in (3.20) and the functionals

$$\hat{f}^E_{k\tau}(\tilde{y}) := \int_0^{k\tau} \hat{L}(t, \tilde{y}(t), \tilde{\dot{y}}(t)) \, dt$$

on $EW^{1,2}(S_{k\tau}, \mathbb{R}^n)$, $k = 1, 2, \cdots$, we have

$$m_{k\tau}^-(\hat{f}^E_{k\tau}, 0) = m_{1,\tau}^-(\tilde{\gamma}^k) \quad \text{and} \quad m_{0,\tau}^-(\hat{f}^E_{k\tau}, 0) = m_{1,\tau}^-(\tilde{\gamma}^k) \quad \forall k \in \mathbb{N}. \quad (3.35)$$

By (3.24) and (3.27) we get

$$\hat{m}_{\tau}^-(\hat{f}^E_{\tau}, 0) := \lim_{k \to \infty} \frac{m_{k\tau}^-(\hat{f}^E_{k\tau}, 0)}{k} \quad (3.36)$$

exists, and if $\hat{m}_{\tau}^-(\hat{f}^E_{\tau}, 0) = 0,$

$$m_{k\tau}^-(\hat{f}^E_{k\tau}, 0) + m_{0,\tau}^-(\hat{f}^E_{k\tau}, 0) \leq n \quad \forall k \in \mathbb{N}. \quad (3.37)$$

Now (3.33) - (3.37) give (3.33) and (3.34), and therefore the desired (3.21) and (3.22).
4 Critical modules under iteration maps

In this section we shall study relations of critical modules under iteration maps in three different cases. We first recall a few of notions. Let $\mathcal{M}$ be a $C^2$ Hilbert-Riemannian manifold and $f \in C^1(\mathcal{M}, \mathbb{R})$ satisfies the Palais-Smale condition. Denote by $\mathcal{K}(f)$ the set of critical points of $f$. Recall that a connected submanifold $N$ of $\mathcal{M}$ is a critical submanifold of $f$ if it is closed, consists entirely of critical points of $f$ and $f|_N = \text{constant}$. Let $N \subset \mathcal{M}$ be an isolated critical submanifold of $f$ with $f|_N = c$, and $U$ be a neighborhood of $N$ such that $U \cap \mathcal{K}(f) = N$. For $q \in \mathbb{N} \cup \{0\}$, recall that the $q^{th}$ critical group with coefficient group $\mathbb{K}$ of $f$ at $N$ is defined by

$$C_q(f,N; \mathbb{K}) := H_q(\{f \leq c\} \cap U, (\{f \leq c\} \setminus N) \cap U; \mathbb{K}).$$

Hereafter $H_*(X,Y; \mathbb{K})$ stands for the relative singular homology with the abelian coefficient group $\mathbb{K}$ without special statements. The group $C_q(f,N; \mathbb{K})$ does not depend on a special choice of such neighborhoods $U$ up to isomorphisms. There also exists another equivalent definition of critical groups, which is convenient in many situations.

Let $V : (\mathcal{M} \setminus \mathcal{K}(f)) \to T\mathcal{M}$ be a pseudo-gradient vector field for $f$ on $\mathcal{M}$. According to [Ch pp.48, 74] and [Wa Def.2.3] or [GM1], a pair of topological subspaces $(W,W^-)$ of $\mathcal{M}$ is called a Gromoll-Meyer pair with respect to $V$ for $N$, if

1. $W$ is a closed neighborhood of $N$ possessing the mean value property, i.e., $\forall t_1 < t_2$, $\eta(t_i) \in W$, $i = 1, 2$, implies $\eta(t) \in W$ for all $t \in [t_1, t_2]$, where $\eta(t)$ is the decreasing flow with respect to $V$. And there exists $\epsilon > 0$ such that $W \cap f_{c-\epsilon} = f^{-1}[c-\epsilon, c) \cap \mathcal{K}(f) = \emptyset$, $W \cap \mathcal{K}(f) = N$;
2. the set $W^- = \{p \in W \mid \eta(t,p) \notin W, \forall t > 0\}$;
3. $W^-$ is a piecewise transversal submanifold, and the flow $\eta$ is transversal to $W^-$. By [Ch pp.74] or [Wa §2], there exists an (arbitrarily small) Gromoll-Meyer pair for $N$, $(W,W^-)$, and for such a pair it holds that

$$H_*(W,W^-; \mathbb{K}) \cong C_*(f,N; \mathbb{K}).$$

Hence $H_*(W,W^-; \mathbb{K})$ may be used to give an equivalent definition of $C_*(f,N; \mathbb{K})$. We need the following fact which seems to be obvious, but is often neglected.

**Lemma 4.1** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be $C^2$ Hilbert-Riemannian manifolds, and $\Theta : \mathcal{M}_1 \to \mathcal{M}_2$ be a homeomorphism. Suppose that $f_i \in C^1(\mathcal{M}_i, \mathbb{R})$, $i = 1, 2$, satisfy the Palais-Smale condition and $f_2 = f_1 \circ \Theta$. Let $N_1 \subset \mathcal{M}_1$ and $N_2 = \Theta(N_1) \subset \mathcal{M}_2$ be isolated critical submanifolds of $f_1$ and $f_2$ respectively. Assume that $(W_1,W_1^-)$ is a Gromoll-Meyer pair of $N_1$ of $f_1$. Then

$$C_*(f_2,N_2; \mathbb{K}) \cong H_*(\Theta(W_1), \Theta(W_1^-); \mathbb{K})$$

though $(\Theta(W_1), \Theta(W_1^-))$ is not necessarily a Gromoll-Meyer pair of $N_2$ of $f_2$ (because $\Theta$ is only a homeomorphism). Moreover, for $c = f_1|_{N_1}$ and $\epsilon > 0$ it is clear that

$$(W_1,W_1^-) \subset (f_1^{-1}[c-\epsilon, c+\epsilon], f_1^{-1}(c-\epsilon))$$

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implies \((\Theta(W_1), \Theta(W_1^-)) \subset (f_2^{-1}[c - \epsilon, c + \epsilon], f_2^{-1}(c - \epsilon))\).

**Proof.** Take a small open neighborhood \(U\) of \(N_1\) so that \(U \subset W_1\). Since \(\Theta(\{f_1 \leq c\} \cap U) = \{f_2 \leq c\} \cap U\) and \(\Theta((\{f_1 \leq c\} \setminus N_1) \cap U) = (\{f_2 \leq c\} \setminus N_2) \cap \Theta(U)\), we have isomorphisms

\[
\Theta_* : H_*(W_1, W_1^-; \mathbb{K}) \rightarrow H_*(\Theta(W_1), \Theta(W_1^-); \mathbb{K}),
\]

\[
\Theta_* : H_*(\{f_1 \leq c\} \cap U, ((f_1 \leq c) \cap N_1) \cap U; \mathbb{K}) \rightarrow
\]

\[
H_*(\{f_2 \leq c\} \cap \Theta(U), (\{f_2 \leq c\} \setminus N_2) \cap \Theta(U); \mathbb{K})
\]

\[
= C_*(f_2, N_2; \mathbb{K}).
\]

By (4.1) and (4.2), \(H_*(W_1, W_1^-; \mathbb{K}) \cong H_*(\{f_1 \leq c\} \cap U, ((f_1 \leq c) \cap N_1) \cap U; \mathbb{K})\). The desired conclusion is obtained. \(\Box\).

It is this result that we may often treat \((\Theta(W_1), \Theta(W_1^-))\) as a Gromoll-Meyer pair without special statements. For conveniences we call it a **topological Gromoll-Meyer** of \(f_2\) at \(N_2\). The usual Gromoll-Meyer pair can be viewed the special case of it. Moreover, if \(\Gamma : \mathcal{M}_2 \rightarrow \mathcal{M}_3\) is a \(C^4\)-diffeomorphism onto another \(C^2\) Hilbert-Riemannian manifold \(\mathcal{M}_3\), then \((\Gamma \circ \Theta(W_1), \Gamma \circ \Theta(W_1^-))\) is also a topological Gromoll-Meyer pair of \(f_3 = f_2 \circ \Gamma^{-1}\) at \(N_3 = \Gamma(N_2)\). (4.2) and Lemma 4.1 show that the topological Gromoll-Meyer may be used to give an equivalent definition of the critical group.

To understand the Note at the end of proof of Theorem 5.1 of [Ch] pp. 44 we add a lemma, which is need in this paper.

**Lemma 4.2** Let \(H_i\) be Hilbert spaces with origins \(\theta_i\), \(i = 1, 2, 3\). For \(\varepsilon > 0\) let \(f \in C^2(B_\varepsilon(\theta_1) \times B_\varepsilon(\theta_2) \times B_\varepsilon(\theta_3), \mathbb{R})\). Assume that \(d_3f(x_1, \theta_2, \theta_3) = 0\) for \(x_1 \in B_\varepsilon(\theta_1)\) and that \(d_3^2f(\theta_1, \theta_2, \theta_3) : H_3 \rightarrow H_3\) is a Banach space isomorphism. Then there exist a small \(0 < \delta \ll \varepsilon\) and \(C^1\)-map \(h : B_\delta(\theta_1) \times B_\delta(\theta_2) \rightarrow H_3\) such that

(i) \(d_3f(x_1, x_2, h(x_1, x_2)) = \theta_3\) for all \((x_1, x_2) \in B_\delta(\theta_1) \times B_\delta(\theta_2)\),

(ii) \(g : B_\delta(\theta_1) \times B_\delta(\theta_2) \rightarrow \mathbb{R}, (x_1, x_2) \mapsto g(x_1, x_2) = f(x_1, x_2, h(x_1, x_2))\) is \(C^2\).

**Proof.** Applying the implicit function theorem to the map \(d_3f : B_\varepsilon(\theta_1) \times B_\varepsilon(\theta_2) \times B_\varepsilon(\theta_3) \rightarrow H_3\) we get a \(0 < \delta \ll \varepsilon\) and a \(C^1\)-map \(h : B_\delta(\theta_1) \times B_\delta(\theta_2) \rightarrow H_3\) such that \(h(\theta_1, \theta_2) = \theta_3\) and

\[
d_3f(x_1, x_2, h(x_1, x_2)) = 0 \forall (x_1, x_2) \in B_\delta(\theta_1) \times B_\delta(\theta_2).
\]

Set \(g(x_1, x_2) = f(x_1, x_2, h(x_1, x_2)).\) Then

\[
dg(x_1, x_2) = d_{(1, 2)}f(x_1, x_2, h(x_1, x_2)) + d_3f(x_1, x_2, h(x_1, x_2)) \circ d_{(x_1, x_2)}h(x_1, x_2)
\]

\[
= d_{(1, 2)}f(x_1, x_2, h(x_1, x_2))
\]

because \(d_3f(x_1, x_2, h(x_1, x_2)) = 0\), where \(d_{(1, 2)}\) denotes the differential for the first two variables of \(f\). Hence

\[
d^2g(x_1, x_2) = d_{(1, 2)}^2f(x_1, x_2, h(x_1, x_2)) + d_3d_{(1, 2)}f(x_1, x_2, h(x_1, x_2)) \circ d_{(x_1, x_2)}h(x_1, x_2).
\]
The desired claims are proved. □

4.1. The arguments in this section are following Section 3 in [Lo2]. However, since our arguments are on a Hilbert manifold, rather than Hilbert space, some new techniques are needed. The precise proofs are also given for reader’s convenience. In this subsection we always assume: $M$ is $C^3$-smooth, $L$ is $C^2$-smooth and satisfies (L1)-(L3).

**Lemma 4.3** Let $\gamma \in H_\tau(\alpha)$ be an isolated critical point of the functional $L_\tau$ on $H_\tau(\alpha)$ such that $\gamma^k$ is an isolated critical point of the functional $L_{k\tau}$ in $H_{k\tau}(\alpha^k)$ for some $k \in \mathbb{N}$. Suppose that $\gamma^kT^*M \to S_\tau$ is trivial. Then there exist Gromoll-Meyer pairs $(W(\gamma), W(\gamma)^-)$ of $L_\tau$ at $\gamma$ and $(W(\gamma^k), W(\gamma^k)^-)$ of $L_{k\tau}$ at $\gamma^k$ such that

$$
(\psi^k(W(\gamma)), \psi^k(W(\gamma)^-)) \subset (W(\gamma^k), W(\gamma^k)^-).
$$

**Proof.** For each $j \in \mathbb{N}$, let

$$
\phi_{j\tau}: W^{1,2}(S_{j\tau}, B_\rho^n(0)) \to H_{j\tau}(\alpha^j) \quad \text{and} \quad \tilde{\phi}_{j\tau} = L_{j\tau} \circ \phi_{j\tau}
$$

as in (3.8) and (3.10). They satisfy (3.9), i.e., $\phi_{j\tau} \circ \tilde{\psi}^j = \psi^j \circ \phi_{j\tau} \quad \forall j \in \mathbb{N}$, where $\psi^j: H_{\tau}(\alpha) \to H_{j\tau}(\alpha^j)$ and $\tilde{\psi}^j: W^{1,2}(S_{\tau}, \mathbb{R}^n) \to W^{1,2}(S_{j\tau}, \mathbb{R}^n)$ are the iteration maps. Let $\tilde{\gamma} = (\phi_{j\tau})^{-1}(\gamma)$. Then $\phi_{j\tau}(\tilde{\gamma}^j) = \gamma^j$ for any $j \in \mathbb{N}$.

Let $\| \cdot \|$ and $\| \cdot \|$ denote the norms in $W^{1,2}(S_{\tau}, \mathbb{R}^n)$ and $W^{1,2}(S_{k\tau}, \mathbb{R}^n)$ respectively. By the construction on page 49 of [Ch], we set

$$
\tilde{W}(\tilde{\gamma}) := L_{\tau}^{-1}[c - \epsilon, c + \epsilon] \cap \{ x \in W^{1,2}(S_{\tau}, \mathbb{R}^n) \mid \lambda L_{\tau}(x) + \| x \|^2 \leq \mu \},
$$

$$
\tilde{W}(\tilde{\gamma})^- := L_{\tau}^{-1}[c - \epsilon] \cap \{ x \in W^{1,2}(S_{\tau}, \mathbb{R}^n) \mid \lambda L_{\tau}(x) + \| x \|^2 \leq \mu \},
$$

$$
\tilde{W}(\tilde{\gamma})^k := L_{k\tau}^{-1}[k\epsilon - k\epsilon, k\epsilon + k\epsilon] \cap \{ y \in W^{1,2}(S_{k\tau}, \mathbb{R}^n) \mid \lambda L_{k\tau}(y) + \| y \|^2 \leq k\mu \},
$$

$$
\tilde{W}(\tilde{\gamma})^- k := L_{k\tau}^{-1}[k\epsilon - k\epsilon] \cap \{ y \in W^{1,2}(S_{k\tau}, \mathbb{R}^n) \mid \lambda L_{k\tau}(y) + \| y \|^2 \leq k\mu \},
$$

where positive numbers $\lambda, \mu, \epsilon$ and $k\epsilon, k\mu, k\epsilon$ are such that the conditions as in (5.13)-(5.15) on page 49 of [Ch] hold. Then $(\tilde{W}(\gamma), \tilde{W}(\gamma)^-)$ and $(\tilde{W}(\gamma^k), \tilde{W}(\gamma^k)^-)$ are Gromoll-Meyer pairs of $L_{j\tau}$ at $\tilde{\gamma}$ and of $L_{k\tau}$ at $\tilde{\gamma}^k$, and

$$
(\psi^k(\tilde{W}(\tilde{\gamma})), \psi^k(\tilde{W}(\tilde{\gamma})^-)) \subset (\tilde{W}(\gamma^k), \tilde{W}(\gamma^k)^-).
$$

Define

$$
(W(\gamma, W(\gamma)^-) := (\phi_{\tau}(\tilde{W}(\tilde{\gamma})), \phi_{\tau}(\tilde{W}(\tilde{\gamma})^-)),
$$

$$
(W(\gamma^k, W(\gamma^k)^-) := (\phi_{k\tau}(\tilde{W}(\tilde{\gamma}^k)), \phi_{k\tau}(\tilde{W}(\tilde{\gamma}^k)^-)).
$$

Since $\phi_{k\tau} \circ \psi^k = \psi^k \circ \phi_{\tau}$, (4.3) follows from (4.5). □

When $\gamma$ and $\gamma^k$ are isolated, according to the definition of critical groups in (4.4) it is easy to see that the iteration map $\psi^k: H_{\tau}(\alpha) \to H_{k\tau}(\alpha^k)$ induces homomorphisms

$$
(\psi^k)_* : C_*(L_{\tau}, \gamma; \mathbb{K}) \to C_*(L_{k\tau}, \gamma^k; \mathbb{K}).
$$

Lemma 4.3 shows that the homomorphisms are still well-defined when the critical groups $C_*(L_{\tau}, \gamma; \mathbb{K})$ and $C_*(L_{k\tau}, \gamma^k; \mathbb{K})$ are defined by (4.2). Later similar cases are always understood in this way. Our purpose is to prove:
Theorem 4.4 Let \( \gamma \in H_\tau(\alpha) \) be an isolated critical point of the functional \( L_\tau \) on \( H_\tau(\alpha) \) such that \( \gamma^* TM \to S_\tau \) is trivial. Suppose that for some \( k \in \mathbb{N} \) the iteration \( \gamma^k \) is also an isolated critical point of the functional \( L_{k\tau} \) in \( H_{k\tau}(\alpha^k) \), and

\[
m_{k\tau}(\gamma^k) = m_\tau(\gamma) \quad \text{and} \quad m^0_{k\tau}(\gamma^k) = m^0_\tau(\gamma).
\]

Then for \( c = L_\tau(\gamma) \) and any \( \epsilon > 0 \) there exist topological Gromoll-Meyer pairs of \( L_\tau \) at \( \gamma \in H_\tau(\alpha) \) and of \( L_{k\tau} \) at \( \gamma^k \in H_{k\tau}(\alpha^k) \),

\[
(W_\tau, W^-_\tau) \subset ((L_\tau)^{-1}[c - \epsilon, c + \epsilon], (L_\tau)^{-1}(c - \epsilon)) \quad \text{and} \quad (W_{k\tau}, W^-_{k\tau}) \subset ((L_{k\tau})^{-1}[kc - k\epsilon, kc + k\epsilon], (L_{k\tau})^{-1}(kc - k\epsilon)),
\]

such that

\[
(\psi^k(W_\tau), \psi^k(W^-_\tau)) \subset (W_{k\tau}, W^-_{k\tau})
\]

and that the homomorphism

\[
(\psi^k)_* : C_*(L_\tau, \gamma; \mathbb{K}) := H_*(W_\tau, W^-_\tau; \mathbb{K}) \to C_*(L_{k\tau}, \gamma^k; \mathbb{K}) := H_*(W_{k\tau}, W^-_{k\tau}; \mathbb{K})
\]

is an isomorphism. Specially, \( (\psi^1)_* = id \), and \( (\psi^k)_* \circ (\psi^l)_* = (\psi^{kl})_* \) if the iterations \( \gamma^l \) and \( \gamma^{kl} \) are also isolated, and

\[
\left\{ m_{k\tau}(\gamma^{kl}) = m_\tau(\gamma^l) = m_\tau(\gamma), \quad m^0_{k\tau}(\gamma^{kl}) = m^0_\tau(\gamma^l) = m^0_\tau(\gamma) \right\}.
\]

When \( M = \mathbb{R}^n \), this theorem was proved by [Lo2, Th.3.7]. We shall reduce the proof of Theorem 4.4 to that case.

Using the chart in (4.4) let \( \tilde{\gamma} = (\phi_\tau)^{-1}(\gamma) \). Then \( \tilde{\gamma}^j = (\phi_\tau)^{-1}(\gamma^j) \) for each \( j \in \mathbb{N} \). Then \( \tilde{\gamma}^j \) are isolated critical points of \( \tilde{L}_\tau = L_\tau \circ \phi_\tau \) in \( W^{1,2}(S_\tau, \mathbb{R}^n) \), \( j = 1, k, l, kl \). Moreover, \( m_{\tilde{\tau}}(\tilde{\gamma}^j) = m_\tau(\tilde{\gamma}) \) and \( m^0_{\tilde{\tau}}(\tilde{\gamma}^j) = m^0_\tau(\tilde{\gamma}) \) for \( j = k, l, kl \). Let \( (\tilde{W}(\tilde{\gamma}), \tilde{W}(\tilde{\gamma})^-) \) and \( (\tilde{W}(\tilde{\gamma}^k), \tilde{W}(\tilde{\gamma}^k)^-) \) be Gromoll-Meyer pairs of \( \tilde{L}_\tau \) at \( \tilde{\gamma} \) and of \( \tilde{L}_{k\tau} \) at \( \tilde{\gamma}^k \), satisfying (4.5). Define

\[
C_*(\tilde{L}_\tau, \tilde{\gamma}; \mathbb{K}) := H_*(\tilde{W}(\tilde{\gamma}), \tilde{W}(\tilde{\gamma})^-; \mathbb{K}), \quad C_*(\tilde{L}_\tau, \gamma; \mathbb{K}) := H_*(W(\gamma), W(\gamma)^-; \mathbb{K}),
\]

\[
C_*(\tilde{L}_{k\tau}, \tilde{\gamma}^k; \mathbb{K}) := H_*(\tilde{W}(\tilde{\gamma}^k), \tilde{W}(\tilde{\gamma}^k)^-; \mathbb{K}), \quad C_*(\tilde{L}_{k\tau}, \gamma^k; \mathbb{K}) := H_*(W(\gamma^k), W(\gamma^k)^-; \mathbb{K}).
\]

Since \( \phi_{k\tau} \circ \tilde{\psi}^k = \psi^k \circ \phi_\tau \), we have \( (\phi_{k\tau})_* \circ (\tilde{\psi}^k)_* = (\psi^k)_* \circ (\phi_\tau)_* \). Clearly,

\[
(\phi_\tau)_* : C_*(\tilde{L}_\tau, \tilde{\gamma}; \mathbb{K}) \to C_*(L_\tau, \gamma; \mathbb{K}) \quad \text{and} \quad (\phi_{k\tau})_* : C_*(\tilde{L}_{k\tau}, \tilde{\gamma}^k; \mathbb{K}) \to C_*(L_{k\tau}, \gamma^k; \mathbb{K})
\]

are isomorphisms. Hence we only need to prove that

\[
(\tilde{\psi}^k)_* : C_*(\tilde{L}_\tau, \tilde{\gamma}; \mathbb{K}) \to C_*(\tilde{L}_{k\tau}, \tilde{\gamma}^k; \mathbb{K})
\]

(4.11)
is an isomorphism which maps generators to the generators. This is exactly one proved by [Lo2, Th.3.7]. Theorem 3.7 in [Lo2] also gives that \((\tilde{\psi}^1)_* = id\) and \((\tilde{\psi}^k)_* \circ (\tilde{\psi}^l)_* = (\tilde{\psi}^{kl})_*\). So other conclusions follow immediately.

For later conveniences we outline the arguments therein. Let

\[
W^{1,2}(S_{kr}, \mathbb{R}^n) = M^0(\tilde{\gamma}_k) \oplus M(\tilde{\gamma}_k)^- \oplus M(\tilde{\gamma})^+
\]

be the orthogonal decomposition of the space \(W^{1,2}(S_{kr}, \mathbb{R}^n)\) according to the null, negative, and positive definiteness of the quadratic form \(\tilde{L}_{kr}''(\tilde{\gamma}_k)\). The generalized Morse lemma ([Ch, Th.5.1, pp. 44]) yields a homeomorphism \(\tilde{\Theta}_{kr}\) from some open neighborhood \(\tilde{U}_{kr}\) of \(0\) in \(W^{1,2}(S_{kr}, \mathbb{R}^n)\) to \(\tilde{\Theta}_{kr}(\tilde{U}_{kr}) \subset W^{1,2}(S_{kr}, \mathbb{R}^n)\) with \(\tilde{\Theta}_{kr}(0) = \tilde{\gamma}_k\), and a map \(\tilde{h}_{kr} \in C^1(\tilde{U}_{kr} \cap M(\tilde{\gamma}_k)^0, M(\tilde{\gamma}_k)^-1)\) such that

\[
\tilde{L}_{kr}(\tilde{\Theta}_{kr}(\eta + \xi)) = \tilde{L}_{kr}(\tilde{\gamma}_k + \eta + \tilde{h}_{kr}(\eta)) + \frac{1}{2}(\tilde{L}_{kr}'(\tilde{\gamma}_k)\xi, \xi) \equiv \tilde{\alpha}_{kr}(\eta) + \tilde{\beta}_{kr}(\xi) \tag{12.41}
\]

for any \(\eta + \xi \in \tilde{U}_{kr} \cap (M(\tilde{\gamma}_k)^0 \oplus M(\tilde{\gamma}_k)^-1)\). (Note: \(\tilde{\beta}_{kr}\) is \(C^\infty\), \(\tilde{\alpha}_{kr}\) is \(C^2\) as noted at the end of proof of Theorem 5.1 on the page 44 of [Ch]. Carefully checking the beginning proof therein one can easily derive this from Lemma 12.2. It is easy to prove that

\[
\tilde{\psi}^k(\tilde{L}_{kr}'(x)) = \tilde{L}_{kr}'(\tilde{\psi}^k(x)) \quad \text{and} \quad \tilde{\psi}^k(\tilde{L}_{kr}''(x)\xi) = \tilde{L}_{kr}''(\tilde{\psi}^k(x))\tilde{\psi}^k(\xi) \tag{12.13}
\]

for any \(\tau, k \in \mathbb{N}, x \in W^{1,2}(S_{kr}, B^0_{kr}(0))\) and \(\xi \in W^{1,2}(S_{kr}, \mathbb{R}^n)\), and that

\[
\tilde{\alpha}_{kr}(\tilde{\psi}^k(\eta)) = k\tilde{\alpha}(\eta) \quad \text{and} \quad \tilde{\beta}_{kr}(\tilde{\psi}^k(\xi)) = k\tilde{\beta}(\xi) \tag{12.14}
\]

for any \(\eta \in \tilde{U}_{kr} \cap M^0(\tilde{\gamma})\) and \(\xi \in \tilde{U}_{kr} \cap M^+(\tilde{\gamma})\).

**Lemma 4.5** ([Lo2, Lem. 3.2, 3.3]) The iteration map \(\tilde{\psi}^k : M^*(\tilde{\gamma}) \to M^*(\tilde{\gamma}_k)\) for \(* = 0, -, +\) is linear, continuous and injective. If \(m_{kr}^-(\tilde{\gamma}_k) = m_\tau^-\), the map \(\tilde{\psi}^k : M^-(\tilde{\gamma}) \to M^-(\tilde{\gamma}_k)\) is a linear diffeomorphism. If \(m_{kr}^0(\tilde{\gamma}_k) = m_\tau^0\), then the map \(\tilde{\psi}^k : M^0(\tilde{\gamma}) \to M^0(\tilde{\gamma}_k)\) is a linear diffeomorphism, and \(\tilde{U}_{kr}\), the homeomorphism \(\tilde{\Theta}_{kr}\) and map \(\tilde{h}_{kr} \in C^1(\tilde{U}_{kr} \cap M(\tilde{\gamma}_k)^0, M(\tilde{\gamma}_k)^-1)\) are chosen to satisfy:

\[
\tilde{U}_{kr} \cap \tilde{\psi}^k(W^{1,2}(S_{kr}, \mathbb{R}^n)) = \tilde{\psi}^k(\tilde{U}_\tau), \tag{15.15}
\]

\[
\tilde{\Theta}_{kr} \circ \tilde{\psi}^k = \tilde{\psi}^k \circ \tilde{\Theta}_\tau : \tilde{U}_\tau \to \tilde{\Theta}_\tau(\tilde{U}_\tau \cap M^0(\tilde{\gamma}_k)), \tag{16.16}
\]

\[
\tilde{h}_{kr}(\tilde{\psi}^k(\eta)) = \tilde{\psi}^k(\tilde{h}_\tau(\eta)) \quad \forall \eta \in \tilde{U}_\tau \cap M(\tilde{\gamma}). \tag{17.17}
\]

Let \((W_0, W_0^-)\) and \((W_1, W_1^-)\) be Gromoll-Meyer pairs of \(\tilde{\alpha}_\tau\) and \(\tilde{\beta}_\tau\) at their origins respectively. By [Lo2, Prop.3.5. 2^o], \((\tilde{\psi}^k(W_0), \tilde{\psi}^k(W_0^-))\) is a Gromoll-Meyer pair of \(\tilde{\alpha}_{kr}\) at the origin. The Gromoll-Meyer pair \((W_1, W_1^-)\) can also be chosen to satisfy

\[
(\tilde{\psi}^k(W_1), \tilde{\psi}^k(W_1^-)) \subset (V, V^-) \tag{18.18}
\]
for some Gromoll-Meyer pair \((V, V^-)\) of \(\beta_{kr}\) at the origin. By [Ch] Lem.5.1. pp.51

\[
(W_0 \times W_1, (W_0 \times W^-_1) \cup (W^-_0 \times W_1)), \tag{4.19}
\]
\[
\left(\tilde{\psi}^k(W_0) \times V, (\tilde{\psi}^k(W_0) \times V^-) \cup (\tilde{\psi}^k(W^-_0) \times V)\right). \tag{4.20}
\]

are Gromoll-Meyer pairs of \(\tilde{\alpha}_r + \tilde{\beta}_r\) and \(\tilde{\alpha}_{kr} + \tilde{\beta}_{kr}\) at their origins respectively, and also satisfy

\[
\left(\tilde{\psi}^k(W_0 \times W_1), \tilde{\psi}^k((W_0 \times W^-_1) \cup (W^-_0 \times W_1))\right) \subset \left(\tilde{\psi}^k(W_0) \times V, (\tilde{\psi}^k(W_0) \times V^-) \cup (\tilde{\psi}^k(W^-_0) \times V)\right). \tag{4.21}
\]

Note that

\[
(\tilde{W}_r, \tilde{W}^-_r) := \tilde{\Theta}_r(W_0 \times W_1, (W_0 \times W^-_1) \cup (W^-_0 \times W_1)), \quad \tag{4.22}
\]
\[
(\tilde{W}_{kr}, \tilde{W}^-_{kr}) := \tilde{\Theta}_{kr}(\tilde{\psi}^k(W_0) \times V, (\tilde{\psi}^k(W_0) \times V^-) \cup (\tilde{\psi}^k(W^-_0) \times V)). \tag{4.23}
\]

are topological Gromoll-Meyer pairs of \(\tilde{\mathcal{L}}_r\) at \(\tilde{\gamma}\) and \(\tilde{\mathcal{L}}_{kr}\) at \(\tilde{\gamma}^k\) respectively. Let

\[
C_*(\tilde{\alpha}_r + \tilde{\beta}_r, 0; \mathbb{K}) := H_*(W_0 \times W_1, (W_0 \times W^-_1) \cup (W^-_0 \times W_1); \mathbb{K}),
\]
\[
C_*(\tilde{\mathcal{L}}_r, 0; \mathbb{K}) := H_*(\tilde{W}_r, \tilde{W}^-_r; \mathbb{K}),
\]
\[
C_*(\tilde{\alpha}_{kr} + \tilde{\beta}_{kr}, 0; \mathbb{K}) := H_*\left(\tilde{\psi}^k(W_0) \times V, (\tilde{\psi}^k(W_0) \times V^-) \cup (\tilde{\psi}^k(W^-_0) \times V); \mathbb{K}\right),
\]
\[
C_*(\tilde{\mathcal{L}}_{kr}, 0; \mathbb{K}) := H_*\left(\tilde{W}_{kr}, \tilde{W}^-_{kr}; \mathbb{K}\right).
\]

We have the isomorphisms on critical modules,

\[
(\tilde{\Theta}_r)_* : C_*(\tilde{\alpha}_r + \tilde{\beta}_r, 0; \mathbb{K}) \cong C_*(\tilde{\mathcal{L}}_r, \tilde{\gamma}; \mathbb{K}),
\]
\[
(\tilde{\Theta}_{kr})_* : C_*(\tilde{\alpha}_{kr} + \tilde{\beta}_{kr}, 0; \mathbb{K}) \cong C_*(\tilde{\mathcal{L}}_{kr}, \tilde{\gamma}^k; \mathbb{K}).
\]

By (4.21) we have a homomorphism

\[
(\tilde{\psi}^k)_* : C_*(\tilde{\alpha}_r + \tilde{\beta}_r, 0; \mathbb{K}) \rightarrow C_*(\tilde{\alpha}_{kr} + \tilde{\beta}_{kr}, 0; \mathbb{K}). \tag{4.24}
\]

Moreover, (4.16) and (4.21) show that

\[
(\tilde{\psi}^k(\tilde{W}_r), \tilde{\psi}^k(\tilde{W}^-_r)) \subset (\tilde{W}_{kr}, \tilde{W}^-_{kr}) \tag{4.25}
\]

and therefore the homomorphism

\[
(\tilde{\psi}^k)_* : C_*(\tilde{\mathcal{L}}_r, 0; \mathbb{K}) \rightarrow C_*(\tilde{\mathcal{L}}_{kr}, 0; \mathbb{K})
\]

satisfy

\[
(\tilde{\psi}^k)_* \circ (\tilde{\Theta}_r)_* = (\tilde{\Theta}_{kr})_* \circ (\tilde{\psi}^k)_*. \tag{4.26}
\]

Hence the problem is reduced to prove:
Lemma 4.6 The Gromoll-Meyer pairs $(W_1, W_1^-)$ and $(V, V^-)$ in (4.18) can be chosen such that

$$(\tilde{\psi}^k)_* : C_*(\tilde{\alpha}_r + \tilde{\beta}_r, 0; \mathbb{K}) \to C_*(\tilde{\alpha}_{kr} + \tilde{\beta}_{kr}, 0; \mathbb{K}) \quad (4.27)$$

is an isomorphism.

Proof. For $j = 1, k$, decompose $\xi \in M(\tilde{\gamma}_j)^- = M(\tilde{\gamma}_j)^- \oplus M(\tilde{\gamma}_j)^+$ into $\xi = \xi^- + \xi^+$ and write

$$\tilde{\beta}_{j\tau}(\xi) = \tilde{\beta}_{j\tau}(\xi^-) + \tilde{\beta}_{j\tau}(\xi^+) = \tilde{\beta}_{j\tau}^-(\xi^-) + \tilde{\beta}_{j\tau}^+(\xi^+).$$

Then $\tilde{\beta}_{j\tau}^-$ and $\tilde{\beta}_{j\tau}^+$ are negative and positive definite quadratic forms on $M(\tilde{\gamma}_j)^-$ and $M(\tilde{\gamma}_j)^+$ with Morse indexes $m^-(\tilde{\gamma}_j)$ and $0$ respectively, $j = 1, k$. The (4.12)-(4.14) imply

$$\tilde{\beta}_{kr}^- (\tilde{\psi}^k(\xi^-)) = k\tilde{\beta}_{\tau}^- (\xi^-) \quad \text{and} \quad \tilde{\beta}_{kr}^+ (\tilde{\psi}^k(\xi^+)) = k\tilde{\beta}_{\tau}^+ (\xi^+)$$

for any $\xi^- \in M^-(\tilde{\gamma})$ and $\xi^+ \in M^+(\tilde{\gamma})$. Since $m_{kr}^-(\tilde{\gamma}^k) = m_{kr}^-(\tilde{\gamma})$, by Lemma 4.5, the map $\tilde{\psi}^k : M^-(\tilde{\gamma}) \to M^-(\tilde{\gamma}^k)$ is a linear diffeomorphism. Let $(W_{11}, W_{11}^-)$ be a Gromoll-Meyer pair of $\tilde{\beta}_{\tau}^-$ at the origin. Then

$$(\tilde{\psi}^k(W_{11}), \tilde{\psi}^k(W_{11}^-)) \quad (4.28)$$

is a Gromoll-Meyer pair of $\tilde{\beta}_{kr}^-$ at the origin. For $\delta > 0$ sufficiently small, set

$$W_{12} := \{\xi^+ \in M(\tilde{\gamma})^+ | \|\xi^+\|_\tau \leq \delta \},$$
$$W_{12}^- := \{\xi^+ \in M(\tilde{\gamma})^+ | \|\xi^+\|_\tau = \delta \},$$
$$V_{12} := \{\xi^+ \in M(\tilde{\gamma}^k)^+ | \|\xi^+\|_{kr} \leq \sqrt{k}\delta \},$$
$$V_{12}^- := \{\xi^+ \in M(\tilde{\gamma}^k)^+ | \|\xi^+\|_{kr} = \sqrt{k}\delta \}.$$

It is easily checked that $(W_{12}, W_{12}^-)$ and $(V_{12}, V_{12}^-)$ are Gromoll-Meyer pairs of $\tilde{\beta}_{kr}^+$ and $\tilde{\beta}_{kr}^+$ at their origins respectively, and that

$$(\tilde{\psi}^k(W_{12}), \tilde{\psi}^k(W_{12}^-)) \subset (V_{12}, V_{12}^-). \quad (4.29)$$

By [Ch] Lem.5.1. pp.51, we may take

$$(W_1, W_1^-) := (W_{11} \times W_{12}, (W_{11} \times W_{12}) \cup (W_{11}^- \times W_{12})), \quad (4.30)$$
$$(V, V^-) := \left(\tilde{\psi}^k(W_{11}) \times V_{12}, (\tilde{\psi}^k(W_{11}) \times V_{12}) \cup (\tilde{\psi}^k(W_{11}^-) \times V_{12})\right). \quad (4.31)$$

Then $(W_0 \times W_1, (W_0 \times W_1^-) \cup (W_0^- \times W_1))$ becomes $(W, W^-)$, and

$$C_*(\tilde{\alpha}_r + \tilde{\beta}_r, 0; \mathbb{K}) = H_*(W, W^-; \mathbb{K}), \quad (4.32)$$

where $W := W_0 \times W_{11} \times W_{12}$ and

$$W^- := (W_0 \times (W_{11} \times W_{12}) \cup (W_{11} \times W_{12})) \cup (W_0^- \times W_{11} \times W_{12}). \quad (4.33)$$

Moreover, $(\tilde{\psi}^k(W_0) \times V, (\tilde{\psi}^k(W_0) \times V^-) \cup (\tilde{\psi}^k(W_0^-) \times V))$ becomes $(U, U^-)$, and

$$C_*(\tilde{\alpha}_{kr} + \tilde{\beta}_{kr}, 0; \mathbb{K}) = H_*(U, U^-; \mathbb{K}), \quad (4.34)$$

29
where \( U = \tilde{\psi}^k(W_0) \times \tilde{\psi}^k(W_{12}) \times V_{12} \) and

\[
U^- = (\tilde{\psi}^k(W_0) \times (\tilde{\psi}^k(W_{11}) \times V_{12}^-) \cup (\tilde{\psi}^k(W_{11}^-) \times V_{12})) \\
\cup (\tilde{\psi}^k(W_0^-) \times \tilde{\psi}^k(W_{11}) \times V_{12}). \tag{4.35}
\]

Note that \( \tilde{\psi}^k(W) = \tilde{\psi}^k(W_0) \times \tilde{\psi}^k(W_{11}) \times \tilde{\psi}^k(W_{12}) \) and

\[
\tilde{\psi}^k(W_-) = (\tilde{\psi}^k(W_0) \times (\tilde{\psi}^k(W_{11}) \times \tilde{\psi}^k(W_{12}^-)) \cup (\tilde{\psi}^k(W_{11}^-) \times \tilde{\psi}^k(W_{12}))) \\
\cup (\tilde{\psi}^k(W_0^-) \times \tilde{\psi}^k(W_{11}) \times \tilde{\psi}^k(W_{12})). \tag{4.36}
\]

Since \( \tilde{\psi}^k : M^+(\bar{\gamma}) \to M^+(\bar{\gamma}^k) \) is a linear, continuous and injection, by (1.29) and the constructions of \((V_{12}, V_{12}^-)\) and \((W_{12}, W_{12}^-)\) it is readily checked that \( (\tilde{\psi}^k(W_{12}), \tilde{\psi}^k(W_{12}^-)) \) is a deformation retract of \((V_{12}, V_{12}^-)\). It follows that

\[
(\tilde{\psi}^k(W), \tilde{\psi}^k(W_-)) \subset (U, U^-)
\]

is a deformation retract of \((U, U^-)\). Hence

\[
(\tilde{\psi}^k)_* : H_*(W, W^-; \mathbb{K}) \to H_*(U, U^-; \mathbb{K})
\]

and therefore, by (4.32) and (4.34), the homomorphism \((\tilde{\psi}^k)_*\) in (4.27) is an isomorphism.

We may also prove the conclusion as follows. By the arguments at the middle of [Ch pp. 51] we can use Kähler formula to arrive

\[
C_*(\bar{\alpha}_r + \bar{\beta}_r, 0; \mathbb{K}) = H_*(W_0, W_0^-; \mathbb{K}) \otimes \\
H_*(W_{11}, W_{11}^-; \mathbb{K}) \otimes H_*(W_{12}, W_{12}^-; \mathbb{K}), \tag{4.37}
\]

\[
C_*(\bar{\alpha}_k r + \bar{\beta}_k r, 0; \mathbb{K}) = H_*(\tilde{\psi}^k(W_0), \tilde{\psi}^k(W_0^-); \mathbb{K}) \otimes \\
H_* (\tilde{\psi}^k(W_{11}), \tilde{\psi}^k(W_{11}^-); \mathbb{K}) \otimes H_*(V_{12}, V_{12}^-; \mathbb{K}). \tag{4.38}
\]

Now \( m_{k r}^- (\bar{\gamma}^k) = m_{r}^- (\bar{\gamma}) \) and \( m_{k r}^0 (\bar{\gamma}^k) = m_{r}^0 (\bar{\gamma}) \) imply that

\[
(\tilde{\psi}^k)_* : H_*(W_0, W_0^-; \mathbb{K}) \to H_*(\tilde{\psi}^k(W_0), \tilde{\psi}^k(W_0^-); \mathbb{K}), \\
(\tilde{\psi}^k)_* : H_*(W_{11}, W_{11}^-; \mathbb{K}) \to H_*(\tilde{\psi}^k(W_{11}), \tilde{\psi}^k(W_{11}^-); \mathbb{K})
\]

are isomorphisms. Since \( (\tilde{\psi}^k(W_{12}), \tilde{\psi}^k(W_{12}^-)) \) is a deformation retract of \((V_{12}, V_{12}^-)\) as above, it follows that

\[
(\tilde{\psi}^k)_* : H_* (\tilde{\psi}^k(W_{12}), \tilde{\psi}^k(W_{12}^-); \mathbb{K}) \to H_*(V_{12}, V_{12}^-; \mathbb{K}) \tag{4.39}
\]

is an isomorphism. By (4.37) and (4.38) we get the proof of Lemma 4.6 \( \square \)

For \((\hat{W}_r, \hat{W}_r^-)\) in (1.22) and \((\hat{W}_{k r}, \hat{W}_{k r}^-)\) in (1.23), where the Gromoll-Meyer pairs \((W_1, W_1^-)\) and \((V, V^-)\) in (1.18) are also required to satisfy Lemma 4.6 Set

\[
(W_r, W_r^-) := (\phi_r(\hat{W}_r), \phi_r(\hat{W}_r^-)) \quad \text{and} \quad (W_{k r}, W_{k r}^-) := (\phi_{k r}(\hat{W}_{k r}), \phi_{k r}(\hat{W}_{k r}^-)).
\]
Since $\phi_{kr} \circ \tilde{\psi}^k = \psi^k \circ \phi_r$, by (4.25) we have $(\psi^k(W_\tau), \psi^k(W_\tau^-)) \subset (W_{kr}, W_{kr}^-)$ and that the homomorphism

$$(\psi^k)_* : H_*(W_\tau, W_\tau^-; \mathbb{K}) \to H_*(W_{kr}, W_{kr}^-; \mathbb{K})$$

is an isomorphism. Consequently, $(W_\tau, W_\tau^-)$ and $(W_{kr}, W_{kr}^-)$ are desired topological Gromoll-Meyer pairs.

The other conclusions are also easily proved. So Theorem 4.4 holds. □

4.2. In this subsection we always assume: $M$ is $C^3$-smooth, $L$ is $C^2$-smooth and satisfies (L1)-(L4). Let $\gamma \in EH_\tau$ be an isolated critical point of the functional $\mathcal{L}^E_{kr}$ on $EH_\tau$, and

$$\phi^E_{kr} : EW^{1,2}(S_{kr}, B^*_\rho(0)) \to EH_{kr} \quad \text{and} \quad \tilde{\mathcal{L}}^E_{kr} = \mathcal{L}_{kr} \circ \phi^E_{kr} \quad (4.40)$$

be as in (3.29) and (3.32) for each $k \in \mathbb{N}$. They satisfy (3.30), i.e. $\phi^E_{kr} \circ \tilde{\psi}^k = \psi^k \circ \phi^E_{kr}$ for any $k \in \mathbb{N}$, where $\psi^k : EH_\tau \to EH_{kr}$ and

$$\tilde{\psi}^k : EW^{1,2}(S_\tau, \mathbb{R}^n) \to EW^{1,2}(S_{kr}, \mathbb{R}^n)$$

are the iteration maps. Let $\tilde{\gamma} = (\phi^E_{kr})^{-1}(\gamma)$ and thus $\phi^E_{kr}(\tilde{\gamma}^k) = \gamma^k$ for any $k \in \mathbb{N}$. Suppose that $\gamma^k$ and therefore $\tilde{\gamma}^k$ are also isolated. Denote by

$$C_q(\tilde{\mathcal{L}}^E_{kr}, \tilde{\gamma}^k; \mathbb{K}) = H_q(W(\tilde{\gamma}^k)_E, W(\tilde{\gamma}^k)_E^-; \mathbb{K})$$

the critical module of $\tilde{\mathcal{L}}^E_{kr}$ at $\tilde{\gamma}^k$ via the relative singular homology with coefficients in $\mathbb{K}$, where $(W(\tilde{\gamma}^k)_E, W(\tilde{\gamma}^k)_E^-)$ is a Gromoll-Meyer pair via some pseudo-gradient vector field of $\tilde{\mathcal{L}}^E_{kr}$ near $\tilde{\gamma}^k$ in $EW^{1,2}(S_{kr}, \mathbb{R}^n)$. Let

$$EW^{1,2}(S_{kr}, \mathbb{R}^n) = M^0(\tilde{\gamma}^k)_E \oplus M(\tilde{\gamma}^k)_E^- \oplus M(\tilde{\gamma})_E^1$$

be the orthogonal decomposition of the space $EW^{1,2}(S_{kr}, \mathbb{R}^n)$ according to the null, negative, and positive definiteness of the quadratic form $(\tilde{\mathcal{L}}^E_{kr})''(\tilde{\gamma})$. As above we can use the generalized Morse lemma to get a homeomorphism $\tilde{\Theta}^E_{kr}$ from some open neighborhood $\tilde{U}_{kr}$ of 0 in $EW^{1,2}(S_{kr}, \mathbb{R}^n)$ to $\tilde{\mathcal{L}}^E_{kr}(\tilde{U}_{kr})^E \subset EW^{1,2}(S_{kr}, \mathbb{R}^n)$ with $\tilde{\Theta}^E_{kr}(0) = \tilde{\gamma}^k$, and a map $\tilde{h}^E_{kr} \in C^1(\tilde{U}_{kr} \cap M(\tilde{\gamma}^k)_E^0; M(\tilde{\gamma}^k)_E^-)$ and $\tilde{h}^E_{kr} \in C^\infty$ and $C^2$ as noted below (4.12). Then $\tilde{\Theta}^E_{kr}$ induces isomorphisms on critical modules,

$$(\tilde{\Theta}^E_{kr})_* : C_*(\tilde{\mathcal{L}}^E_{kr}, \tilde{\gamma}^k; 0; \mathbb{K}) \iff C_*(\tilde{\mathcal{L}}^E_{kr}, \tilde{\gamma}^k; \mathbb{K}). \quad (4.41)$$

Note that

$$(W(\gamma^k)_E, W^-((\gamma^k)_E)) = (\phi^E_{kr}(W(\tilde{\gamma}^k)_E), \phi^E_{kr}(W^-(\tilde{\gamma}^k)_E)) \quad (4.42)$$
is a Gromoll-Meyer pair of $\mathcal{L}_E^{k_1}$ at $\gamma^k$. Define the critical modules
\[ C_*(\mathcal{L}_E^{k_1}, \gamma^k; \mathbb{K}) := H_*(W(\gamma^k)_E, W^-(\gamma^k)_E; \mathbb{K}). \] (4.43)

Then corresponding to Theorem 4.4, we have the following generalization of [LuW2, Lemma 4.1].

**Theorem 4.7** Let $\gamma \in EH_\tau$ be an isolated critical point of the functional $\mathcal{L}_E^{k_1}$ on $EH_\tau$. If the iteration $\gamma^k$ is also isolated for some $k \in \mathbb{N}$, and
\[ m_{1, k_1}^-(\gamma^k) = m_{1, \tau}^-(\gamma) \quad \text{and} \quad m_{0, k_1}^0(\gamma^k) = m_{0, \tau}^0(\gamma), \]
then for $c = \mathcal{L}_E^{k_1}(\gamma)$ and any $\epsilon > 0$ there exist topological Gromoll-Meyer pairs of $\mathcal{L}_E^{k_1}$ at $\gamma \in EH_\tau$ and of $\mathcal{L}_E^{k_1}$ at $\gamma^k \in EH_\tau$,
\[ (W_\tau, W_\tau^-) \subseteq ((\mathcal{L}_E^{k_1})^{-1}(c - \epsilon, c + \epsilon), (\mathcal{L}_E^{k_1})^{-1}(c - \epsilon)) \quad \text{and} \quad (W_{k_1 \tau}, W_{k_1 \tau}^-) \subseteq ((\mathcal{L}_E^{k_1})^{-1}(kc - kc, kc + kc), (\mathcal{L}_E^{k_1})^{-1}(kc - kc)), \]
such that
\[ (\psi^k(W_\tau), \psi^k(W_\tau^-)) \subset (W_{k_1 \tau}, W_{k_1 \tau}^-) \] (4.44)
and that the iteration map $\psi^k : EH_\tau \to EH_\tau$ induces isomorphisms
\[ (\psi^k)_* : C_*(\mathcal{L}_E^{k_1}, \gamma; \mathbb{K}) := H_*(W_\tau, W_\tau^-; \mathbb{K}) \]
\[ \to C_*(\mathcal{L}_E^{k_1}, \gamma^k; \mathbb{K}) := H_*(W_{k_1 \tau}, W_{k_1 \tau}^-; \mathbb{K}). \] (4.45)
Specially, $(\psi^1)_* = id$, and $(\psi^1)_* \circ (\psi^k)_* = (\psi^{k1})_*$ if the iterations $\gamma^l$ and $\gamma^{k1}$ are also isolated, and
\[ m_{1, k_1 \tau}^-(\gamma^{k1}) = m_{1, \tau}^-(\gamma^l) = m_{1, \tau}^-(\gamma), \]
\[ m_{0, k_1 \tau}^0(\gamma^{k1}) = m_{0, \tau}^0(\gamma^l) = m_{0, \tau}^0(\gamma). \]

### 4.3. Let us consider the case $L$ is independent $t$. In this subsection we always assume: $M$ is $C^5$-smooth, $L$ is $C^4$-smooth and satisfies (L1)-(L3). The goal is to generalize [LoLu Th.2.5] to the present general case. However, unlike the last two cases we cannot choose a local coordinate chart around a critical orbit. For $\tau > 0$, let $S_\tau := \mathbb{R} / \tau \mathbb{Z} = \{[s]_\tau \mid [s]_\tau = s + \tau \mathbb{Z}, s \in \mathbb{R}\}$ and the functional $\mathcal{L}_\tau : H_\tau(\alpha) \to \mathbb{R}$ be still defined by (1.14). By [K1 Chp.2, §2.2], there exist equivariant and also isometric operations of $S_\tau$-action on $H_\tau(\alpha)$ and $TH_\tau(\alpha)$:
\[ [s]_\tau \cdot \gamma(t) = \gamma(s + t), \quad \forall [s]_\tau \in S_\tau, \gamma \in H_\tau(\alpha), \]
\[ [s]_\tau \cdot \xi(t) = \xi(s + t), \quad \forall [s]_\tau \in S_\tau, \xi \in T\gamma H_\tau(\alpha). \] (4.46)
which are continuous, but not differentiable. Clearly, $\mathcal{L}_\tau$ is invariant under this action. Since under our assumptions each critical point $\gamma$ of $\mathcal{L}_\tau$ is $C^4$-smooth, by [GM12 p. 499], the orbit $S_\tau \cdot \gamma$ is a $C^3$-submanifold in $H_\tau(\alpha)$. It is easily checked that $S_\tau \cdot \gamma$ is a $C^3$-smooth critical submanifold of $\mathcal{L}_\tau$. Seemingly, the theory of [Wa] cannot be applied to this case because the action of $S_\tau$ is only continuous. However, as pointed
out in the second paragraph of [GM2, pp. 500] this theory still hold since critical orbits are smooth and $S_\tau$ acts by isometries.

For any $k \in \mathbb{N}$, there is a natural $k$-fold cover $\varphi_k$ from $S_{k\tau}$ to $S_\tau$ defined by

$$
\varphi_k : [s]_{k\tau} \mapsto [s]_{\tau}.
$$

(4.47)

It is easy to check that the $S_\tau$-action on $H_\tau(\alpha)$, the $S_{k\tau}$-action on $H_{k\tau}(\alpha^k)$, and the $k$-th iteration map $\psi^k$ defined above [5.9] satisfy:

$$
\begin{align*}
([s]_{\tau} \cdot \gamma)^k &= [s]_{k\tau} \cdot \gamma^k, \\
\mathcal{L}_{k\tau}([s]_{k\tau} \cdot \gamma^k) &= k\mathcal{L}_\tau([s]_{\tau} \cdot \gamma) = k\mathcal{L}_\tau(\gamma)
\end{align*}
$$

(4.48)

for all $\gamma \in H_\tau(\alpha)$, $k \in \mathbb{N}$, and $s \in \mathbb{R}$.

Let $\gamma \in H_\tau(\alpha)$ be a non-constant critical point of $\mathcal{L}_\tau$ with minimal period $\tau/m$ for some $m \in \mathbb{N}$. Denote by $\mathcal{O} = S_\tau \cdot \gamma = S_{\tau/m} \cdot \gamma$. It is a 1-dimensional $\mathcal{C}^3$-submanifold diffeomorphic to the circle. Let $c = \mathcal{L}_\tau|_{\mathcal{O}}$. Assume that $\mathcal{O}$ is isolated. We may take a neighborhood $U$ of $\mathcal{O}$ such that $K(\mathcal{L}_\tau) \cap U = \mathcal{O}$. By [1.1] we have critical group $C_*(\mathcal{L}_\tau, \mathcal{O}; \mathbb{K})$ of $\mathcal{L}_\tau$ at $\mathcal{O}$. For every $s \in [0, \tau/m]$ the tangent space $T_{s \cdot \gamma}(S_\tau \cdot \gamma)$ is $\mathbb{R}(s \cdot \gamma)^\perp$, and the fiber $N(\mathcal{O})_{s \cdot \gamma}$ at $s \cdot \gamma$ of the normal bundle $N(\mathcal{O})$ of $\mathcal{O}$ is a subspace of codimension 1 which is orthogonal to $(s \cdot \gamma)^\perp$ in $T_{s \cdot \gamma}H_\tau(\alpha)$, i.e.

$$
N(\mathcal{O})_{s \cdot \gamma} = \{\xi \in T_{s \cdot \gamma}H_\tau(\alpha) \mid \langle\xi, (s \cdot \gamma)^\perp\rangle_1 = 0\}.
$$

Since $H_\tau(\alpha)$ is $\mathcal{C}^4$-smooth and $\mathcal{O}$ is a $\mathcal{C}^3$-smooth submanifold, $N(\mathcal{O})$ is $\mathcal{C}^2$-smooth manifold. Notice that $N(\mathcal{O})$ is invariant under the $S_\tau$-actions in [4.20] and each $[s]_{\tau}$ gives an isometric bundle map

$$
N(\mathcal{O}) \to N(\mathcal{O}), (z, v) \mapsto ([s]_{\tau} \cdot z, [s]_{\tau} \cdot v).
$$

(4.49)

Under the present case it is easily checked that $\mathcal{L}_\tau$ satisfies the Assumption 7.1 on the page 71 of [Ch], that is, there exists $\epsilon > 0$ such that

$$
\sigma(\mathcal{L}_\tau''(x)) \cap (\{-\epsilon, \epsilon\} \setminus \{0\}) = \emptyset, \quad \dim \ker(\mathcal{L}_\tau''(x)) = \text{constant}
$$

(4.50)

for any $x \in \mathcal{O}$. Then Lemma 7.4 of [Ch, pp. 71] gives the orthogonal $\mathcal{C}^2$-smooth bundle decomposition

$$
N(\mathcal{O}) = N(\mathcal{O})^+ \oplus N(\mathcal{O})^- \oplus N(\mathcal{O})^0, \quad N(\mathcal{O})^* = P_* N(\mathcal{O})
$$

(4.51)

for $* = +, -, 0$. Here $P_* : N(\mathcal{O}) \to N(\mathcal{O})^*$, $* = +, 0, -$, are orthogonal bundle projections. Each $N(\mathcal{O})^*$ is a $\mathcal{C}^2$-smooth submanifold. It is not hard to check that $\mathcal{L}_\tau'$ and $\mathcal{L}_\tau''$ satisfy

$$
\mathcal{L}_\tau'([s]_{\tau} \cdot x) = [s]_{\tau} \cdot \mathcal{L}_\tau'(x) \quad \text{and} \quad \mathcal{L}_\tau''([s]_{\tau} \cdot x)([s]_{\tau} \cdot \xi) = [s]_{\tau} \cdot (\mathcal{L}_\tau''(x)(\xi))
$$

for all $x \in H_\tau(\alpha)$, $\xi \in T_x H_\tau(\alpha)$ and $[s]_{\tau} \in S_\tau$. It follows that the bundle map (4.49) preserves the decomposition (4.51). In particular, we obtain

$$
\langle \text{rank } N(\mathcal{O})^-, \text{rank } N(\mathcal{O})^0 \rangle = \langle m^-_\tau(x), m^0_\tau(x) - 1 \rangle \quad \forall x \in \mathcal{O},
$$

[1] This is the reason that we require higher smoothness of $M$ and $L$. 
where $m^-_r(x)$ and $m^0_r(x)$ are Morse index and nullity of $L_r$ at $x$ respectively. Define
\[(m^-_r(O), m^0_r(O)) := (\text{rank} N(O)^-, \text{rank} N(O)^0).\] (4.52)
Then
\[(m^-_r(O), m^0_r(O)) = (m^-_r(x), m^0_r(x) - 1) \quad \forall x \in O.\] (4.53)
For a single point critical orbit $O = \{\gamma\}$, i.e., $\gamma$ is constant, we define
\[(m^-_r(O), m^0_r(O)) := (m^-_r(\gamma), m^0_r(\gamma)).\] (4.54)

Note that for sufficiently small $\varepsilon > 0$ the set
\[N(O)(\varepsilon) := \{(y, v) \in N(O) | y \in O, \|v\|_1 < \varepsilon\}\]
is contained in an open neighborhood of the zero section of the tangent bundle $TH_r(\alpha)$. By [Kl] Th.1.3.7, pp. 20 we have a $C^2$-embedding from $N(O)(\varepsilon)$ to an open neighborhood of the diagonal of $H_r(\alpha) \times H_r(\alpha)$,
\[N(O)(\varepsilon) \rightarrow H_r(\alpha) \times H_r(\alpha), \quad (y, v) \mapsto (y, \exp y v),\]
where $\exp$ is the exponential map of the chosen Riemannian metric on $M$ and $(\exp_y v)(t) = \exp_{y(t)} v(t) \forall t \in \mathbb{R}$. This yields a $C^2$ diffeomorphism from $N(O)(\varepsilon)$ to an open neighborhood $Q_\varepsilon(O)$ of $O$,
\[\Psi_r : N(O)(\varepsilon) \rightarrow Q_\varepsilon(O), \quad \Psi_r(y, v)(t) = \exp_{y(t)} v(t) \forall t \in \mathbb{R},\] (4.55)
(Note that it is not the exponential map of the Levi-Civita connection derived the Riemannian metric $\langle \cdot, \cdot \rangle$ on $H_r(\alpha)$.) Clearly,
\[\Psi_r(y, 0) = y \forall y \in O \quad \text{and} \quad \Psi_r([s]_r \cdot y, [s]_r v) = [s]_r \cdot \Psi_r(y, v)\] (4.56)
for any $(y, v) \in N(O)(\varepsilon)$ and $[s]_r \in S_r$. It follows that $Q_\varepsilon(O)$ is a $S_r$-invariant neighborhood of $O$, and that $\Psi_r$ is $S_r$-equivariant. We also require $\varepsilon > 0$ so small that $Q_\varepsilon(O)$ contains no other critical orbit besides $O$, and that $\Psi_r(\{y\} \times N(O)(\varepsilon))$ and $O$ have a unique intersection point $y$ (after identifying $O$ with the zero section $N(O)(\varepsilon)$). Then $L_r \circ \Psi_r|_{N(O)(\varepsilon)}$ possesses $y$ as an isolated critical point. Checking the proofs of Theorem 7.3 and Corollary 7.1 in [Ch] pp. 72, and replacing $f \circ \exp|_{\xi}$ and $\exp_x \phi_x$ therein by $L_r \circ \Psi_r|_{N(O)(\varepsilon)}$ and $\Psi_r|_{N(O)(\varepsilon)} \circ \phi_x$ for $x \in O$, one easily gets:

**Lemma 4.8** For sufficiently small $0 < \varepsilon < \varepsilon$, there exist a $S_r$-equivariant homeomorphism $\Phi_r$ from $N(O)(\varepsilon)$ to a $S_r$-invariant open neighborhood $\Omega_\varepsilon(O) \subset Q_\varepsilon(O)$ of $O$, and a $C^1$-map $h_r : N(O)(\varepsilon) \rightarrow N(O)^+(\varepsilon) \oplus N(O)^-(\varepsilon)$ such that
\[L_r \circ \Phi_r(y, v) = \frac{1}{2} (\|P_+(y)v\|_1^2 - \|P_-(y)v\|_1^2) + L_r \circ \Psi_r((y, P_0(y)v) + h_r(P_0(y)v))\]
for $(y, v) \in N(O)(\varepsilon)$, where $P_*$ is as in [4.10].
Let $N(O)^+ = N(O)^+ + N(O)^-$ and write $v = v^0 + v^\perp$. Set
\begin{align*}
\Xi_r(y,v^\perp) &= \frac{1}{2} \left( \|P_+(y)v\|_1^2 - \|P_-(y)v\|_1^2 \right), \\
\Upsilon_r(y,v^0) &= \mathcal{L}_r \circ \Psi_r((y, P_0(y)v) + h_r(P_0(y)v))
\end{align*}
for $(y,v) \in N(O)(\epsilon)$. Then define $F_r : N(O)(\epsilon) \to \mathbb{R}$ by
\begin{equation}
F_r(y,v) = \mathcal{L}_r \circ \Phi_r(y,v) = \Upsilon_r(y,v^0) + \Xi_r(y,v^\perp)
\end{equation}
for all $(y,v) \in N(O)(\epsilon)$. (Note: Though we require the higher smoothness of $M$ and $L$ we do not know whether or not $\mathcal{L}_r$ has higher smoothness than order two unlike the special $L$ considered in [Lo2]. Hence from [Ch7.3, pp. 72] we can only get that $\Phi_r$ is a homeomorphism. However, $N(O)(\epsilon)$ is a $C^2$-bundle and therefore both $\Xi_r$ and $\Upsilon_r$ are $C^2$.)

By the local trivialization of $N(O)(\epsilon)$ the final claim can be derived from Lemma 4.7 and the proofs of [ChTh.5.1, pp. 44] and [ChTh.7.3, pp. 72].) Clearly, both $\Upsilon_r$ and $\Xi_r$ are also $S_r$-invariant, and have the unique critical orbit $O$ in $N(O)^+(\epsilon)$ and $N(O)^0(\epsilon)$ respectively. Since $F_r$ is $C^2$-smooth, we can follow [Wa] to construct a Gromoll-Meyer pair of $O$ as a critical submanifold of $F_r$ on $N(O)(\epsilon)$,
\begin{equation}
(W(O), W(O)^-).
\end{equation}
(Note that different from [Wa] the present $S_r$-action on $N(O)(\epsilon)$ is only continuous; but the arguments there can still be carried out due to the special property of our $S_r$-action in (4.20) and the definition of $F_r$.) In the present case, for any $y \in O$, $F_r|_{N(O)(\epsilon)}$ has a unique critical point $y$ in $N(O)(\epsilon)$ (the fibre of disk bundle $N(O)(\epsilon)$ at $y$), and
\begin{equation}
(W(O), y, W(O)^-) := (W(O) \cap N(O)_{y}(\epsilon), W(O)^- \cap N(O)_{y}(\epsilon))
\end{equation}
is a Gromoll-Meyer pair of $F_r|_{N(O)(\epsilon)}$ at its isolated critical point $y$ satisfying
\begin{equation}
(W(O)_{[s \cdot y]}, W(O)^-_{[s \cdot y]}) = ([s] \cdot W(O)_y, [s] \cdot W(O)_y)
\end{equation}
for any $[s] \in S_r$ and $y \in O$. Clearly,
\begin{equation}
(\hat{W(O)}, \hat{W(O)^-}) := (\Phi_r(W(O)), \Phi_r(W(O)^-))
\end{equation}
is a topological Gromoll-Meyer pair of $\mathcal{L}_r$ at $O$, which is also $S_r$-invariant. Define
\begin{align*}
C_s(\mathcal{L}_r, O; \mathbb{K}) &:= H_s(\hat{W(O)}, \hat{W(O)^-}; \mathbb{K}), \\
C_s(F_r, O; \mathbb{K}) &:= H_s(W(O), W(O)^-; \mathbb{K})
\end{align*}
via the relative singular homology. $\Phi_r$ induces an obvious isomorphism
\begin{equation}
(\Phi_r)_* : C_s(\mathcal{L}_r, O; \mathbb{K}) \cong C_s(F_r, O; \mathbb{K}).
\end{equation}

\footnote{The requirements of the higher smoothness of $M$ and $L$ is used to assure this.}
Since the normal bundle $N(O)$ is differentiably trivial, it follows from [Wad (2.13), (2.14)] (cf. the also shifting theorem in [GM1] and [Ch]) that for any $q \in \{0\} \cup \mathbb{N},$

$$C_q(F_\tau, O; \mathbb{K}) \cong \oplus_{j=0}^q \left[ C_{q-j} \left( F_\tau \big|_{N(O)^q(t)}, y; \mathbb{K} \right) \otimes H_j(S_{\tau}; \mathbb{K}) \right]$$

$$\cong \oplus_{j=0}^q \left[ C_{q-j-m_{\tau}}(O) \left( F_\tau \big|_{N(O)^q(t)}, y; \mathbb{K} \right) \otimes H_j(S_{\tau}; \mathbb{K}) \right]$$

$$\cong C_{q-1-m_{\tau}}(O) \left( F_\tau \big|_{N(O)^q(t)}, y; \mathbb{K} \right) \quad \forall y \in O.$$  

Here $C_{q-1-m_{\tau}}(O) \left( F_\tau \big|_{N(O)^q(t)}, y; \mathbb{K} \right)$ is independent of the choice of $y \in O = S_{\tau} \cdot \gamma.$

Taking $y = \gamma$ we obtain

$$C_* \left( L_\tau, S_{\tau} \cdot \gamma; \mathbb{K} \right) \cong C_{*-1-m_{\tau}}(S_{\tau} \cdot \gamma) \left( F_\tau \big|_{N(S_{\tau} \cdot \gamma)^q(t)}, \gamma; \mathbb{K} \right). \quad (4.67)$$

Suppose that $\psi^k(O) = S_{k\tau} \cdot \gamma^k$ is also an isolated critical orbit of the functional $L_{k\tau}$ on $H_{k\tau}(\alpha^k)$ for some $k \in \mathbb{N}.$ Our purpose is to study the relations between critical groups $C_* \left( L_\tau, O; \mathbb{K} \right)$ and $C_* \left( L_{k\tau}, \psi^k(O); \mathbb{K} \right).$

Let $N(S_{k\tau} \cdot \gamma^k)$ be the normal bundle of $S_{k\tau} \cdot \gamma^k$ in $H_{k\tau}(\alpha^k)$ and

$$N(S_{k\tau} \cdot \gamma^k)(\varepsilon) = \{ (y, v) \in N(S_{k\tau} \cdot \gamma^k) \mid y \in S_{k\tau} \cdot \gamma^k, \|v\|_1 < \varepsilon \}.$$  

Corresponding to (4.51) there exist natural orthogonal bundle decompositions

$$N(\psi^k(O)) = N(\psi^k(O))^+ \oplus N(\psi^k(O))^- \oplus N(\psi^k(O))^0,$$

$$N(\psi^k(O))((\varepsilon) = N(\psi^k(O))^+(\varepsilon) \oplus N(\psi^k(O))^-((\varepsilon) \oplus N(\psi^k(O))^0((\varepsilon), \quad (4.69)$$

where $N(\psi^k(O))^+((\varepsilon) = N(\psi^k(O))(\varepsilon) \cap N(\psi^k(O))^* \text{ for } * = +, -, 0.$

It is not hard to check that

$$\psi^k(N(O)((\varepsilon) \subset N(S_{k\tau} \cdot \gamma^k)(\sqrt{\varepsilon}) \quad \text{and} \quad \psi^k(N(O)^*((\varepsilon) \subset N(S_{k\tau} \cdot \gamma^k)^* \quad (4.70)$$

for $* = +, 0, -.$ By shrinking $\varepsilon > 0$ we have also a $C^2$-smooth $S_{\tau}$-equivariant diffeomorphism from $N(S_{k\tau} \cdot \gamma^k)(\sqrt{\varepsilon})$ to a $S_{k\tau}$-invariant open neighborhood $Q_{\sqrt{\varepsilon}}(S_{k\tau} \cdot \gamma^k)$ of $S_{k\tau} \cdot \gamma^k,$

$$\Psi_{k\tau} : N(S_{k\tau} \cdot \gamma^k)(\sqrt{\varepsilon}) \to Q_{\sqrt{\varepsilon}}(S_{k\tau} \cdot \gamma^k), \quad (4.71)$$

$$\Psi_{k\tau}(y, v)(t) = \exp_y(t, v(t) \forall t \in \mathbb{R}.$$  

With the same arguments as above Lemma 1.3 by furthermore shrinking $0 < \varepsilon < \varepsilon,$ there exist a $S_{k\tau}$-equivariant homeomorphism $\Phi_{k\tau}$ from from $N(\psi^k(O))(\sqrt{\varepsilon})$ to a $S_{k\tau}$-invariant open neighborhood $\Omega_{\sqrt{\varepsilon}}(\psi^k(O)) \subset Q_{\sqrt{\varepsilon}}(\psi^k(O))$ of $\psi^k(O),$ and a $C^1$-map

$$h_{k\tau} : N(\psi^k(O))^0(\sqrt{\varepsilon}) \to N(\psi^k(O))^+(\sqrt{\varepsilon}) \oplus N(\psi^k(O))^-((\sqrt{\varepsilon})$$

such that

$$L_{k\tau} \circ \Phi_{k\tau}(y, v) = \Psi_{k\tau}(y, v^0) + \Xi_{k\tau}(y, v^1) \quad (4.72)$$
Indeed, the key in the proof of [Lo2, Lem. 3.3] is that the maps \( h \) uniquely determined by the implicit function theorem as shown in the proof of the Generalized Morse lemma [Ch, pp. 44]. It follows from (4.78) that

\[
\Xi_{k\tau}(y, v^+) = \frac{1}{2} \left( \|v^+\|^2 - \|v^-\|^2 \right), \\
\Upsilon_{k\tau}(y, v^0) = L_{\tau} \circ \Psi_{\tau}(y, v^0) + h(y, v^0) 
\]

have the similar properties to (4.59). As in [4.55] we define a \( S_{k\tau} \)-invariant, \( C^2 \)-smooth function \( F_{k\tau} : N(\psi^k(\mathcal{O}))(\sqrt{k\epsilon}) \to \mathbb{R} \) by

\[
F_{k\tau}(y, v) = L_{k\tau}(\Phi_{k\tau}(y, v)) = \Upsilon_{k\tau}(y, v^0) + \Xi_{k\tau}(y, v^+). 
\]

It has the unique critical orbit \( \psi^k(\mathcal{O}) \) in \( N(\psi^k(\mathcal{O}))(\sqrt{k\epsilon}) \). Note that (4.75) and (4.77) imply

\[
\Psi_{k\tau} \circ \psi^k = \psi^k \circ \Psi_{\tau}. 
\]

As in [LoLu] Prop. 2.3], we can suitably modify the proof of [Lo2, Lem. 3.3] to get:

**Lemma 4.9** Suppose that \( m^0_{k\tau}(\psi^k(\mathcal{O})) = m^0_{\tau}(\mathcal{O}) \). Then:

(i) The maps \( h_{\tau} \) and \( h_{k\tau} \) satisfy

\[
h_{k\tau}(\psi^k(p)) = \psi^k(h_{\tau}(p)), \quad \forall p = (y, v) \in N(\mathcal{O})^0(\epsilon). 
\]

(ii) The homeomorphisms \( \Phi_{\tau} \) and \( \Phi_{k\tau} \) satisfy

\[
\Phi_{k\tau} \circ \psi^k = \psi^k \circ \Phi_{\tau} 
\]

as maps from \( N(\mathcal{O})(\epsilon) \) to \( H_{k\tau}(\alpha^k) \).

(iii) For \( q \in N(\mathcal{O})^0(\epsilon), p \in N(\mathcal{O})^+(\epsilon) \), there hold

\[
\Upsilon_{k\tau}(\psi^k(q)) = k\Upsilon_{\tau}(q), \quad \Xi_{k\tau}(\psi^k(p)) = k\Xi_{\tau}(p). 
\]

Indeed, the key in the proof of [Lo2] Lem. 3.3] is that the maps \( h_{\tau} \) and \( h_{k\tau} \) are uniquely determined by the implicit function theorem as showed in the proof of the Generalized Morse lemma [Ch] pp. 44]. It follows from (4.78) that

\[
F_{k\tau} \circ \psi^k = kF_{\tau}. 
\]

By the construction of the Gromoll-Meyer pair in [Wa] we can construct such a pair of \( F_{k\tau} \) at \( \psi^k(\mathcal{O}) \) on \( N(\psi^k(\mathcal{O}))(\sqrt{k\epsilon}) \), \( (W(\psi^k(\mathcal{O})), W(\psi^k(\mathcal{O}))) \) such that

\[
(\psi^k(W(\mathcal{O})), \psi^k(W(\mathcal{O}))^-) \subset (W(\psi^k(\mathcal{O})), W(\psi^k(\mathcal{O})))^- 
\]

for the pair \( (W(\mathcal{O}), W(\mathcal{O}))^- \) in (4.60). Set

\[
(\tilde{W}(\psi^k(\mathcal{O})), \tilde{W}(\psi^k(\mathcal{O})))^- := (\Phi_{k\tau}(W(\psi^k(\mathcal{O}))), \Phi_{k\tau}(W(\psi^k(\mathcal{O})))^-), 
\]

which is a topological Gromoll-Meyer pair, and

\[
C_*(L_{k\tau}, \psi^k(\mathcal{O}); \mathbb{K}) := H_*(\tilde{W}(\psi^k(\mathcal{O})), \tilde{W}(\psi^k(\mathcal{O})))^-; \mathbb{K}), \\
C_*(F_{k\tau}, \psi^k(\mathcal{O}); \mathbb{K}) := H_*(W(\psi^k(\mathcal{O})), W(\psi^k(\mathcal{O})))^-; \mathbb{K}). 
\]

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It follows from (4.77) and (4.80) that
\[
\left( \psi^k(\tilde{W}(O)), \psi^k(\tilde{W}(O^\perp)) \right) \subset \left( \tilde{W}(\psi^k(O)), \tilde{W}(\psi^k(O)) \right) \quad (4.84)
\]
and that \( \psi^k \) induces homomorphisms
\[
(\psi^k)_* : C_*(L_T, O; K) \to C_*(L_{kT}, \psi^k(O); K), \quad (4.85)
\]
\[
(\psi^k)_* : C_*(F_T, O; K) \to C_*(F_{kT}, \psi^k(O); K) \quad (4.86)
\]
satisfying
\[
(\psi^k)_* \circ (\Phi_T)_* = (\Phi_{kT})_* \circ (\psi^k)_* \quad (4.87)
\]
because of (4.77). By (4.66) and the isomorphism
\[
(\Phi_{kT})_* : C_*(L_{kT}, \psi^k(O); K) \cong C_*(F_{kT}, \psi^k(O); K) \quad (4.88)
\]
we only need to prove:

**Lemma 4.10** The Gromoll-Meyer pairs in (4.80) can be chosen so that the homomorphism in (4.86) is an isomorphism provided that
\[
m_{kT}^{-}(\psi^k(O)) = m_T^{-}(O) \quad \text{and} \quad m_{kT}^{0}(\psi^k(O)) = m_T^{0}(O). \quad (4.89)
\]

**Proof.** By (4.8), (4.72) and (4.74) we have
\[
\begin{align*}
C_*(F_T, O; K) &= C_*(\Upsilon_T + \Xi_T, O; K), \\
C_*(F_{kT}, \psi^k(O); K) &= C_*(\Upsilon_{kT} + \Xi_{kT}, \psi^k(O); K)
\end{align*}
\quad (4.90)
\]
We shall imitate the proof of Lemma 4.6 to prove that the homomorphism
\[
(\psi^k)_* : C_*(\Upsilon_T + \Xi_T, O; K) \to C_*(\Upsilon_{kT} + \Xi_{kT}, \psi^k(O); K) \quad (4.91)
\]
is an isomorphism.

Let \((W_0(O), \tilde{W}_0(O))\) be a Gromoll-Meyer pair of \(\Upsilon_T\) at \(O \subset N(O)^0(\epsilon)\). Since (4.89) implies that \(\psi^k : N(O)^0(\epsilon) \to N(\psi^k(O))^0(\sqrt{k}\epsilon)\) is a bundle isomorphism. Hence
\[
\left( \psi^k(W_0(O)), \psi^k(\tilde{W}_0(O)) \right)
\]
is a Gromoll-Meyer pair of \(\Upsilon_{kT}\) at \(\psi^k(O) \subset N(\psi^k(O))^0(\sqrt{k}\epsilon)\). For \(j = 1, k\) let us write \(N(\psi^j(O)) = N(\psi^j(O))^+ \oplus N(\psi^j(O))^\perp\) and
\[
N(\psi^j(O))^{\perp}(\sqrt{j}\epsilon) = N(\psi^j(O))^+(\sqrt{j}\epsilon) \oplus N(\psi^j(O))^-(\sqrt{j}\epsilon),
\]
\[
\Xi_{jT}(y, v^+) = \Xi_{jT}^{\perp}(y, v^+) + \Xi_{jT}(y, v^-), \quad v^+ = v^+ + v^-.
\]
By (4.78), for \(p \in N(O)^{\perp}(\epsilon)\), there hold
\[
\Xi_{kT}^{\pm}(\psi^k(p)) = k\Xi_{T}^{\pm}(p). \quad (4.92)
\]
Let \((W_{11}(O), \tilde{W}_{11}(O))\) be a Gromoll-Meyer pair of \(\Xi_T\) at \(O \subset N(O)^{-}(\epsilon)\). Then
\[
(\psi^k(W_{11}(O)), \psi^k(\tilde{W}_{11}(O))) \quad (4.93)
\]
is a Gromoll-Meyer pair of $\Xi^+$ at $\psi^k(O) \subset N(\psi^k(O))^-(\sqrt{k}\epsilon)$ because (4.89) implies that $\psi^k : N(O)^-(\epsilon) \to N(\psi^k(O))^-(\sqrt{k}\epsilon)$ is a bundle isomorphism. For $0 < \delta \ll \epsilon$, set

$$W_{12} := \{(y, v) \in N(O)^+(\epsilon) \mid \|v\|_r \leq \delta\},$$
$$W_{12}^- := \{(y, v) \in N(O)^+(\epsilon) \mid \|v\|_r = \delta\},$$
$$V_{12} := \{(y, v) \in N(\psi^k(O))^+(\epsilon) \mid \|v\|_{k\tau} \leq \sqrt{k}\delta\},$$
$$V_{12}^- := \{(y, v) \in N(\psi^k(O))^+(\epsilon) \mid \|v\|_{k\tau} = \sqrt{k}\delta\}.$$  

Then $(W_{12}, W_{12}^-)$ (resp. $(V_{12}, V_{12}^-)$) is a Gromoll-Meyer pair of $\Xi^+$ (resp. $\Xi^+_{k\tau}$) at $O \subset N(O)^+(\epsilon)$ (resp. $\psi^k(O) \subset N(\psi^k(O))^+(\sqrt{k}\epsilon)$), and that

$$\left(\psi^k(W_{12}), \psi^k(W_{12}^-)\right) \subset (V_{12}, V_{12}^-).$$  

By Lemma 5.1 on the page 51 of [Ch], we may take

$$W_1(O) := W_{11}(O) \oplus W_{12},$$
$$W_{11}^- (O) := (W_{11}(O) \oplus W_{12}^-) \cup (W_{11}^-(O) \oplus W_{12}),$$
$$V := \psi^k(W_{11}(O)) \oplus V_{12},$$
$$V := (\psi^k(W_{11}(O)) \oplus V_{12}^-) \cup (\psi^k(W_{11}^-(O)) \oplus V_{12})$$

and get a Gromoll-Meyer pair of $\Upsilon_\tau + \Xi_\tau$ at $O \subset N(O)(\epsilon)$, $(W(O), W(O)^-)$, where

$$W(O) := W_0(O) \oplus W_{11}(O) \oplus W_{12},$$
$$W^-(O) := (W_0(O) \oplus [W_{11}(O) \oplus W_{12}^-] \cup (W_{11}^-(O) \oplus W_{12}))$$
$$\cup (W_0^-(O) \oplus W_{11}(O) \oplus W_{12}).$$

Therefore

$$C_*(\Upsilon_{\tau} + \Xi_{\tau}, 0; \mathbb{K}) = H_*(W(O), W^-(O); \mathbb{K}).$$

Similarly, we have a Gromoll-Meyer pair of $\Upsilon_{k\tau} + \Xi_{k\tau}$ at $\psi^k(O) \subset N(\psi^k(O))(\sqrt{k}\epsilon)$, $(W(\psi^k(O)), W(\psi^k(O))^-)$, where

$$W(\psi^k(O)) := \psi^k(W_0(O)) \oplus \psi^k(W_{11}(O)) \oplus V_{12},$$
$$W^-(\psi^k(O)) := (\psi^k(W_0(O)) \oplus [(\psi^k(W_{11}(O)) \oplus V_{12}^-) \cup (\psi^k(W_{11}^-(O)) \oplus V_{12})]$$
$$\cup (\psi^k(W_0^-(O)) \oplus \psi^k(W_{11}(O)) \oplus V_{12}).$$

It follows that

$$C_*(\Upsilon_{k\tau} + \Xi_{k\tau}, \psi^k(O); \mathbb{K}) = H_*(W(\psi^k(O)), W^-(\psi^k(O)); \mathbb{K}).$$

Note that $\psi^k(W(O)) = \psi^k(W_0(O)) \oplus \psi^k(W_{11}(O)) \oplus \psi^k(W_{12})$ and

$$\psi^k(W^-(O)) = (\psi^k(W_0(O)) \oplus (\psi^k(W_{11}(O)) \oplus \psi^k(W_{12}^-))$$
$$\cup (\psi^k(W_{11}(O)) \oplus \psi^k(W_{12}))$$
$$\cup (\psi^k(W_0^-(O)) \oplus \psi^k(W_{11}(O)) \oplus \psi^k(W_{12})).$$
Since \( \psi^k : N^+(\mathcal{O}) \rightarrow N^+(\psi^k(\mathcal{O})) \) is a continuous bundle injection, by (4.94) and

the constructions of \((V_{12}, V_{12}')\) and \((W_{12}, W_{12}')\) above (4.94) it is readily checked that

\((\psi^k(W_{12}), \psi^k(W_{12}'))\) is a deformation retract of \((V_{12}, V_{12}')\). It follows that

\((\psi^k(W(\mathcal{O})), \psi^k(W^-(\mathcal{O}))) \subset (W(\psi^k(\mathcal{O})), W^-(\psi^k(\mathcal{O})))\)

is a deformation retract of \((W(\psi^k(\mathcal{O})), W^-(\psi^k(\mathcal{O})))\). Hence

\((\psi^k)_* : H_*(W(\mathcal{O}), W^-(\mathcal{O}); \mathbb{K}) \rightarrow H_*(W(\psi^k(\mathcal{O})), W^-(\psi^k(\mathcal{O})); \mathbb{K})\)

is an isomorphism. Therefore, by (4.97) and (4.98), the homomorphism \((\psi^k)_* \) in (4.91) is an isomorphism. Lemma 4.10 is proved. \(\Box\).

When \( \gamma \) is constant, i.e. \( \mathcal{O} = S_{\tau'} \cdot \gamma \) is an isolated critical point, this case has been proved in Theorem 4.4. Combing this with Lemma 4.10 and (4.66) and (4.88) we get

**Theorem 4.11** For an isolated critical submanifold \( \mathcal{O} = S_{\tau'} \cdot \gamma \) of \( \mathcal{L}_\tau \) in \( H_{\tau}(\alpha) \), suppose that for some \( k \in \mathbb{N} \) the critical submanifold \( \psi^k(\mathcal{O}) = S_{\tau^k} \cdot \gamma^k \) of \( \mathcal{L}_{\tau^k} \) in \( H_{\tau^k}(\alpha^k) \) is also isolated, i.e. (4.89) is satisfied, i.e. \( m^k_{\tau}(S_{\tau^k} \cdot \gamma^k) = m^k_{\tau}(S_{\tau} \cdot \gamma) \)

and \( m^0_{\tau^k}(S_{\tau^k} \cdot \gamma^k) = m^0_{\tau^k}(S_{\tau} \cdot \gamma) \). Then for \( c = \mathcal{L}_{\tau^k}(\mathcal{O}) \) and small \( \epsilon > 0 \) there exist topological Gromoll-Meyer pairs of \( \mathcal{L}_{\tau} \) at \( \mathcal{O} \subset H_{\tau^k}(\alpha) \) and of \( \mathcal{L}_{\tau^k} \) at \( \psi^k(\mathcal{O}) \subset H_{\tau^k}(\alpha^k) \)

\((\tilde{W}(\mathcal{O}), \tilde{W}(\mathcal{O})^{-}) \subset ((\mathcal{L}_{\tau})^{-1} |c - \epsilon, c + \epsilon|, (\mathcal{L}_{\tau})^{-1} |c - \epsilon|) \) and

\((\tilde{W}(\psi^k(\mathcal{O})), \tilde{W}(\psi^k(\mathcal{O})^{-}) \subset ((\mathcal{L}_{\tau^k})^{-1} |kc - k\epsilon, kc + k\epsilon|, (\mathcal{L}_{\tau^k})^{-1} |kc - k\epsilon|),\)

such that

\((\psi^k(\tilde{W}(\mathcal{O})), \psi^k(\tilde{W}(\mathcal{O})^{-})) \subset (\tilde{W}(\psi^k(\mathcal{O})), \tilde{W}(\psi^k(\mathcal{O})^{-}))\)

and that the iteration map \( \psi^k : H_{\tau}(\alpha) \rightarrow H_{\tau^k}(\alpha^k) \) induces an isomorphism:

\[ \psi^k_* : C_*(\mathcal{L}_\tau, \mathcal{O}; \mathbb{K}) := H_*(\tilde{W}(\mathcal{O}), \tilde{W}(\mathcal{O})^{-}; \mathbb{K}) \]

\[ \rightarrow C_*(\mathcal{L}_{\tau^k}, \psi^k(\mathcal{O}); \mathbb{K}) := H_*(\tilde{W}(\psi^k(\mathcal{O})), \tilde{W}(\psi^k(\mathcal{O})^{-}); \mathbb{K}) \].

**Lemma 4.12** Suppose that \( C_0(\mathcal{L}_\tau, \mathcal{O}; \mathbb{K}) \neq 0 \) for \( \mathcal{O} = S_{\tau} \cdot \gamma \). Then

\[ q - 2n \leq q - 1 - m^0_{\tau}(\mathcal{O}) \leq m^-_{\tau}(\mathcal{O}) \leq q - 1 \] (4.100)

if \( \mathcal{O} \) is not a single point critical orbit, i.e. \( \gamma \) is not constant, and

\[ q - 2n \leq q - m^0_{\tau}(\mathcal{O}) \leq m^-_{\tau}(\mathcal{O}) \leq q \] (4.101)

otherwise.

**Proof.** If \( \gamma \) is not a constant solution, it follows from (4.66) and (4.67) that

\[ C_{q-1-\dim m^-_{\tau}(\mathcal{O})}(F_{\tau}|_{N(\mathcal{O})^0(\gamma)}, \gamma; \mathbb{K}) \cong C_0(\mathcal{L}_\tau, \mathcal{O}; \mathbb{K}) \neq 0. \] (4.102)

Since \( \gamma \) is an isolated critical point of \( F_{\tau}|_{N(\mathcal{O})^0(\gamma)} \) in \( N(\mathcal{O})^0(\gamma) \) and \( N(\mathcal{O})^0(\gamma) \) has dimension \( m^0_{\tau}(\mathcal{O}) \), we get

\[ 0 \leq q - 1 - m^-_{\tau}(\mathcal{O}) \leq \dim N(\mathcal{O})^0(\gamma) = m^0_{\tau}(\mathcal{O}). \] (4.103)
By (4.53), \( m^0_\tau(O) = m^0_\tau(\gamma) - 1 \leq 2n - 1 \). (4.100) easily follows from this and (4.103).

If \( \gamma \) is a constant solution, i.e. \( O = \{ \gamma \} \), using the isomorphisms above (4.11) and (4.24) we derive
\[
C_q(\tilde{\alpha}_\tau + \tilde{\beta}_\tau, 0; \mathbb{K}) \cong C_q(\tilde{L}_\tau, \tilde{\gamma}; \mathbb{K}) \neq 0, \quad \text{where} \quad \tilde{\gamma} = (\phi_\tau)^{-1}(\gamma).
\]

On the other hand, (5.11) and the shifting theorem ([GM1] and [Ch, pp.50]) imply
\[
C_q(\alpha_\tau + \beta_\tau, 0; \mathbb{K}) \cong C_q(-m_\tau(\gamma)\tilde{\alpha}_\tau, 0; \mathbb{K}).
\]

Since \( \tilde{\alpha}_\tau \) is defined on a manifold of dimension \( m^0_\tau(\gamma) \leq 2n \), (4.101) follow immediately.

\[\square\]

**Lemma 4.13** Suppose that \( C_q(L_\tau, O; \mathbb{K}) \neq 0 \) for \( O = S_\tau \cdot \gamma \). If either \( O \) is not a single point critical orbit and \( q > 1 \), or \( O \) is a single point critical orbit and \( q > 0 \), then each point in \( O \) is non-minimal saddle point.

**Proof.** When \( O \) is a single point critical orbit and \( q > 0 \), the conclusion follows from [Ch, Ex.1, pp.33]. Now assume that \( O \) is not a single point critical orbit and \( q > 1 \). For any \( y \in O \), by (4.66) and the formula above (4.67) we have
\[
0 \neq C_q(F_\tau, O; \mathbb{K}) \cong \bigoplus_{j=0}^q \left[ C_{q-j} \left( F_\tau \bigg|_{N(O)_y(\epsilon)}; y; \mathbb{K} \right) \otimes H_j(S_\tau; \mathbb{K}) \right]
\]
\[
\cong C_{q-1} \left( F_\tau \bigg|_{N(O)_y(\epsilon)}; y; \mathbb{K} \right).
\]

Since \( y \) is an isolated critical point of \( F_\tau \bigg|_{N(O)_y(\epsilon)} \) and \( q - 1 > 0 \), we derive from [Ch, Ex.1, pp.33] that \( y \) is a non-minimal saddle point of \( F_\tau \bigg|_{N(O)_y(\epsilon)} \). This implies that \( y \) is a non-minimal saddle point of \( L_\tau \) on the submanifold \( \Psi_\tau(N(O)_y(\epsilon)) \subset H_\tau(\alpha) \) (and therefore on \( H_\tau(\alpha) \)). \( \square \)

## 5 Proof of Theorem 1.1

### 5.1. Proof of (i).

For any \( \tau \in \mathbb{N} \), let \( H_\tau(\alpha^k) \) denote the Hilbert manifold of \( W^{1,2} \)-loops \( \gamma : \mathbb{R}/\tau \mathbb{Z} \to M \) representing \( \alpha^k \). Since \( H_\tau(C(\mathbb{R}/\tau \mathbb{Z}, M; \alpha^k); \mathbb{K}) = H_\tau(C(\mathbb{R}/\mathbb{Z}, M; \alpha^k); \mathbb{K}) \) and the inclusion \( H_\tau(\alpha^k) \hookrightarrow C(\mathbb{R}/\tau \mathbb{Z}, M; \alpha^k) \) is a homotopy equivalence,

\[
\text{rank} H_\tau(\alpha^k); \mathbb{K} \neq 0 \quad \forall \tau, k \in \mathbb{N}.
\] (5.1)

By [Be] the functional \( L_\tau \) on the Hilbert manifold \( H_\tau(\alpha^k) \) is \( C^2 \)-smooth, bounded below, satisfies the Palais-Smale condition, and all critical points of it have finite Morse indexes and nullities. In particular, the critical set \( K(L_\tau, \alpha^k) \) of \( L_\tau \) on \( H_\tau(\alpha^k) \) is nonempty because \( L_\tau \) can attain the minimal value on \( H_\tau(\alpha^k) \). Clearly, for any \( \tau, k \in \mathbb{N} \) we may assume that each critical point of \( L_\tau \) on \( H_\tau(\alpha^k) \) is isolated. By contradiction we make:
Assumption $F(\alpha)$: (i) For any given integer $k > 0$, the system (1.6) only possesses finitely many distinct, $k$-periodic solutions representing $\alpha^k$, (ii) there exists an integer $k_0 > 1$ such that for each integer $k > k_0$, any $k$-periodic solution $\gamma$ of the system (1.6) representing $\alpha^k$ must be an iteration of some $l$-periodic solution $\gamma$ of the system (1.6) representing $\alpha^l$ with $l \leq k_0$ and $k = ls$ for some $s \in \mathbb{N}$.

Under this assumption we have integer periodic solutions $\tilde{\gamma}_i$ of the system (1.6) of period $\tau_i \leq k_0$ and representing $\alpha^{\tau_i}$, $i = 1, \cdots, p$, such that for each integer $k > k_0$ any integer $k$-periodic solution $\gamma$ of the system (1.6) representing $\alpha^k$ must be an iteration of some $\tilde{\gamma}_i$, i.e. $\gamma = \tilde{\gamma}_i^l$ for some $l \in \mathbb{N}$ with $l \tau_i = k$. Set $\tau := k_0!$ (the factorial of $k_0$) and $\gamma_i = \tilde{\gamma}_i^{\tau_i}$, $i = 1, \cdots, p$. Then each $\gamma_i$ is a $\tau$-periodic solution of the system (1.6) representing $\alpha^{\tau_i}$. We conclude

**Claim 5.1** For any $k \in \mathbb{N}$, it holds that

$$K(L_{k\tau}, \alpha^{k\tau}) = \{ \gamma_j^{k} \mid 1 \leq j \leq p \}. \quad (5.2)$$

**Proof.** Let $\gamma \in K(L_{k\tau}, \alpha^{k\tau})$. Since $k\tau > k_0$, by (ii) in Assumption $F(\alpha)$ we have $\gamma = \tilde{\gamma}_i^l$ for some $l \in \mathbb{N}$ with $l \tau_i = k$. Hence $\gamma = \tilde{\gamma}_i^l = (\tilde{\gamma}_i)^{k\tau_l} = (\tilde{\gamma}_i^{\tau_i})^k = k\gamma_i^k$. $\square$

Since $M$ is not assumed to be orientable, it is possible that the pullback bundle $\gamma_j^*TM \to \mathbb{R}/\tau \mathbb{Z}$ is not trivial. However, each 2-fold iteration $\gamma_j^2, (\gamma_j^2)^*TM \to \mathbb{R}/2\tau \mathbb{Z}$ is always trivial. Note that (5.2) implies

$$K(L_{2k\tau}, \alpha^{2k\tau}) = \{ (\gamma_j^2)^k = \gamma_j^{2k} \mid 1 \leq j \leq p \}. \quad (5.3)$$

Hence replacing $\{\gamma_1 \cdots \gamma_p\}$ by $\{\gamma_1^2 \cdots \gamma_p^2\}$ we may assume:

$$\gamma_j^*TM \to \mathbb{R}/\tau \mathbb{Z}, \quad j = 1, \cdots, p, \quad \text{are all trivial.} \quad (5.4)$$

**Lemma 5.2** For each $k \in \mathbb{N}$ there exists $\gamma_k' \in K(L_{k\tau}, \alpha^{k\tau})$ such that

$$C_r(L_{k\tau}, \gamma_k'; \mathbb{K}) \neq 0 \quad \text{and} \quad r - 2n \leq r - m_{k\tau}^0(\gamma_k') \leq m_{k\tau}^- (\gamma_k') \leq r.$$  

**Proof.** Let $c_1 < \cdots < c_l$ be all critical values of $L_{\tau}, l \leq p$. Then $kc_1 < \cdots < kc_l$ are all critical values of $L_{k\tau}, k = 1, 2, \cdots$. In particular, $\inf L_{k\tau} = kc_1$ because $L_{k\tau}$ is bounded below and satisfies the Palais-Smale condition.

By (5.1), rank$H_r(H_{k\tau}(\alpha^{k\tau}); \mathbb{K}) \geq m$ for some $m \in \mathbb{N}$. Recall that a subset of an abelian group is defined to be **linearly independent** if it satisfies the usual condition with integer coefficients, cf. [Ma pp. 87]. Take linearly independent elements of $H_r(H_{k\tau}(\alpha^{k\tau}); \mathbb{K}), \beta_1, \cdots, \beta_m$, and singular cycles $Z_1, \cdots, Z_m$ of $H_{k\tau}(\alpha^{k\tau})$ which represent them. Let $S$ be a compact set containing the supports of $Z_1, \cdots, Z_m$. Then $S \subset (L_{k\tau})_b := \{ L_{k\tau} \leq b \}$ for a sufficiently large regular value $b > kc_1$. Note that $Z_1, \cdots, Z_m$ are also singular cycles of $(L_{k\tau})_b$, and that non-trivial $\mathbb{K}$-linear combination of them cannot be homologous to zero in $(L_{k\tau})_b$ (otherwise the same combination is homologous to zero in $H_{k\tau}(\alpha^{k\tau})$.) Hence we get

$$\text{rank}H_r((L_{k\tau})_b; \mathbb{K}) \geq m > 0.$$
Lemma 5.3 Without Assumption $F(\alpha)$, let $\gamma$ be an isolated critical point of $\mathcal{L}_T$ in $H_T(\alpha^t)$ such that $\gamma^*TM \to S_T$ is trivial. For every integer $q \geq n + 1$, let $k(q, \gamma) = 1$ if $\tilde{m}_k^-(\gamma) = 0$, and $k(q, \gamma) = a_{q+n}^{+n} - \tilde{m}_k^-(\gamma)$ if $\tilde{m}_k^-(\gamma) \neq 0$. Assume that $\gamma^k$ is also an isolated critical point of $\mathcal{L}_k$ for some integer $k > k(q, \gamma)$. Then

$$C_q(\mathcal{L}_k, \gamma^k; \mathbb{K}) = 0.$$  
(5.6)

Proof. Let $\phi_{k^T} : W^{1,2}(\mathbb{S}_r, B_p^m(0)) \to H_{k^T}(\alpha^{k^T})$ be a coordinate chart on $H_{k^T}(\alpha^{k^T})$ around $\gamma^k$ as in (3.3). Set $\hat{\gamma} = (\phi_{k^T})^{-1}(\gamma)$. Then $\hat{\gamma}^k = (\phi_{k^T})^{-1}(\gamma^k)$ and $m_k^-($

Take the regular values of $\mathcal{L}_k^T$, $a_0 < a_1 < \cdots < a_l = b$ such that $k \in (a_{i-1}, a_i)$, $i = 1, \cdots, l$. By Theorem 4.2 of [Ch] pp. 23,

$$H_r((\mathcal{L}_k^T)_{a_i}, (\mathcal{L}_k^T)_{a_{i-1}}; \mathbb{K}) \cong \bigoplus_{\mathcal{L}_k^T(z) = k \alpha_i, d\mathcal{L}_k^T(z) = 0} C_r(\mathcal{L}_k^T, z; \mathbb{K}).$$  
(5.5)

Since each critical point has finite Morse index, it follows from the generalized Morse lemma that each group $C_r(\mathcal{L}_k^T, z; \mathbb{K})$ has finite rank, and therefore that

$$\text{rank } H_r((\mathcal{L}_k^T)_{a_i}, (\mathcal{L}_k^T)_{a_{i-1}}; \mathbb{K}) < +\infty, \ i = 1, \cdots, l.$$  

By the arguments on the page 38 of [Ch] and the fact (b) on the page 87 of [Ma], for a triple $Z \subset Y \subset X$ of topological spaces it holds that

$$\text{rank } H_q(X, Z; \mathbb{K}) \leq \text{rank } H_q(X, Y; \mathbb{K}) + \text{rank } H_q(Y, Z; \mathbb{K})$$

if these three numbers are finite. It follows that

$$0 < m \leq \text{rank } H_r((\mathcal{L}_k^T)_{b_1}; \mathbb{K})$$
$$= \text{rank } H_r((\mathcal{L}_k^T)_{a_1}, (\mathcal{L}_k^T)_{a_0}; \mathbb{K})$$
$$\leq \sum_{i=1}^m \text{rank } H_r((\mathcal{L}_k^T)_{a_i}, (\mathcal{L}_k^T)_{a_{i-1}}; \mathbb{K}) < +\infty.$$  

Hence $\text{rank } H_r((\mathcal{L}_k^T)_{a_i}, (\mathcal{L}_k^T)_{a_{i-1}}; \mathbb{K}) \geq 1$ for some $i$. By (5.5), we get a $\gamma'_k \in \mathcal{K}(\mathcal{L}_k^T, \alpha^{k^T})$ such that $\text{rank } C_r(\mathcal{L}_k^T, \gamma'_k; \mathbb{K}) \neq 0$ and thus $C_r(\mathcal{L}_k^T, \gamma'_k; \mathbb{K}) \neq 0$. Noting (5.4), we can use the isomorphism above (4.11) to derive

$$C_r(\tilde{\mathcal{L}}_k^T, \gamma'_k; \mathbb{K}) \neq 0,$$  

where $\tilde{\gamma}'_k = (\phi_{k^T})^{-1}(\gamma'_k).$

Replacing $\tilde{\gamma}^k$ in (4.12) by $\tilde{\gamma}'_k$, and using the isomorphism above (4.21), (3.11) and the shifting theorem ([GM] and [Ch] pp.50]) we get

$$C_{r-m_{k^T}(\gamma'_k)}(\alpha_{k^T}, 0; \mathbb{K}) \cong C_r(\alpha_{k^T} + \beta_{k^T}, 0; \mathbb{K}) \cong C_r(\tilde{\mathcal{L}}_k^T, \gamma'_k; \mathbb{K}) \neq 0.$$  

Since $\tilde{\alpha}^{k^T}$ is defined on a manifold of dimension $m_{k^T}(\gamma'_k) \leq 2n$, the desired inequalities follow immediately. $\square$

Lemma 5.3

Since $\tilde{\gamma}$ is defined on a manifold of dimension $m_{k^T}(\gamma'_k) \leq 2n$, the desired inequalities follow immediately. $\square$
\( m^- (\gamma), m^0_\tau (\tilde{\gamma}) = m^0_\tau (\gamma) \) and \( m^-_{\bar{\tau}} (\tilde{\gamma}) = m^-_{\bar{\tau}} (\gamma) \) and \( m^0_{\bar{\tau}} (\tilde{\gamma}) = m^0_{\bar{\tau}} (\gamma) \). As in the
\[ \text{proof of Lemma 5.2} \] by the isomorphisms above \( (4.11) \) and \( (4.24) \) we have
\[
\begin{align*}
C_q (L_{\bar{\tau}}, \gamma^k; K) & \cong C_r (\hat{L}_{\bar{\tau}}, \tilde{\gamma}^k; K) \\
& \cong C_q (\hat{\alpha}_{\bar{\tau}} + \tilde{\beta}_{\bar{\tau}}, 0; K) \\
& \cong C_{q - m^-_{\bar{\tau}} (\gamma)} (\hat{\alpha}_{\bar{\tau}}, 0; K).
\end{align*}
\]
Here \( \hat{\alpha}_{\bar{\tau}} \) is defined on a manifold of dimension \( m^0_{\bar{\tau}} (\gamma) \leq 2n \).

If \( \hat{\tilde{m}}^- (\gamma) = 0 \), by \( (3.2) \) (or \( (3.18) \)) we have \( 0 \leq m^-_{\bar{\tau}} (\gamma^k) \leq n - m^0_{\bar{\tau}} (\gamma^k) \). Hence
\[ q - m^-_{\bar{\tau}} (\gamma^k) = q - (n - m^0_{\bar{\tau}} (\gamma^k)) \leq 1 + m^0_{\bar{\tau}} (\gamma^k). \]
This gives \( C_{q - m^-_{\bar{\tau}} (\gamma)} (\hat{\alpha}_{\bar{\tau}}, 0; K) = 0 \).

If \( \hat{\tilde{m}}^- (\gamma) > 0 \), by \( (3.2) \) (or \( (3.18) \)) we have \( k\hat{\tilde{m}}^- (\gamma) - n \leq m^-_{\bar{\tau}} (\gamma^k) \) and thus
\[
q - m^-_{\bar{\tau}} (\gamma^k) \leq q - (k\hat{\tilde{m}}^- (\gamma) - n) = q + n - k\hat{\tilde{m}}^- (\gamma) < 0
\]
if \( k > \frac{q + n}{\hat{\tilde{m}}^- (\gamma)} \). This also leads to \( C_{q - m^-_{\bar{\tau}} (\gamma)} (\hat{\alpha}_{\bar{\tau}}, 0; K) = 0 \). \( \square \)

So we immediately get the following generalization of Lemma 4.2 in [Lo2].

**Corollary 5.4** Under Assumption \( F(\alpha) \), for every integer \( q \geq n + 1 \) there exists a
constant \( k_0 (q) > 0 \) such that for every integer \( k \geq k_0 (q) \) there holds
\[ C_q (L_{\bar{\tau}}, y; K) = 0 \quad \forall y \in \mathcal{K} (L_{\bar{\tau}}, \alpha_{\bar{\tau}}). \]
Here \( k_0 (q) = 1 \) if \( \hat{\tilde{m}}^- (\gamma_j) = 0 \) for all \( 1 \leq j \leq p \), and
\[ k_0 (q) = 1 + \max \left\{ \left\lfloor \frac{q + n}{\hat{\tilde{m}}^- (\gamma_j)} \right\rfloor \mid \hat{\tilde{m}}^- (\gamma_j) \neq 0, 1 \leq j \leq p \right\} \]
otherwise. \( \left\lfloor \frac{s}{\gamma} \right\rfloor \) denotes the largest integer less than or equal to \( s \).

Indeed, by \( (5.2) \) we may assume \( y = \gamma^k_j \) for some \( 1 \leq j \leq p \). Then Lemma 5.3
yields the desired conclusion.

Clearly, if \( r \geq n + 1 \) then Lemma 5.2 and Lemma 5.3 immediately give a
contradiction. Theorem 1.1(i) is proved in this case.

In the following we consider the case \( r = n \).

Under Assumption \( F(\alpha) \) we apply Lemma 5.2 to all \( k \in \{ 2^m \mid m \in \{ 0 \} \cup \mathbb{N} \} \) to get an infinite subsequence \( Q \) of \( \{ 2^m \mid m \in \{ 0 \} \cup \mathbb{N} \} \), some \( l \in \mathbb{N} \), and \( \gamma \in \{ \gamma_1, \ldots, \gamma_p \} \) such that \( C_n (L_{\bar{\tau}}, \gamma^k; \mathbb{Z}) \neq 0 \), \( m^-_{\bar{\tau}} (\gamma^k_l) = m^-_{\bar{\tau}} (\gamma^k) \), and \( m^0_{\bar{\tau}} (\gamma^k_l) = m^0_{\bar{\tau}} (\gamma^k) \) for any \( k \in Q \). In order to save notations we always assume \( l = 1 \) in the following. That is, we have \( \gamma^k \in \mathcal{K} (L_{\bar{\tau}}, \alpha_{\bar{\tau}}) \) with
\[
\begin{align*}
C_n (L_{\bar{\tau}}, \gamma^k; K) & \neq 0, \\
m^-_{\bar{\tau}} (\gamma^k) & = m^-_{\bar{\tau}} (\gamma), \\
m^0_{\bar{\tau}} (\gamma^k) & = m^0_{\bar{\tau}} (\gamma)
\end{align*}
\]
\[ (5.7) \]
for any $k \in Q$. By Corollary 5.4 there exists $k_0 > 0$ such that for any $\gamma \in \{\gamma_1, \cdots, \gamma_p\}$,

$$C_{n+1}(\mathcal{L}_{kr}, \gamma^k; \mathbb{K}) = 0 \quad \forall k \in Q(k_0) := \{k \in Q \mid k \geq k_0\}. \quad (5.8)$$

To avoid the finite energy homology introduced and used in [Lo2] we need to improve the proof and conclusions of Theorem 4.3 in [Lo2]. Let $c = \mathcal{L}_r(\gamma)$. Take $\epsilon > 0$ sufficiently small so that for each $k \in \mathbb{N}$ the interval $[k(c - 3\epsilon), k(c + 3\epsilon)]$ contains an unique critical value $kc$ of $\mathcal{L}_{kr}$ on $H_{kr}(\alpha^{kr})$, i.e.

$$\mathcal{L}_{kr}(\mathcal{K}(\mathcal{L}_{kr}, \alpha^{kr})) \cap [k(c - 3\epsilon), k(c + 3\epsilon)] = \{kc\}.$$ By Theorem 4.4 for each integer $k \in Q$ we may choose topological Gromoll-Meyer pairs of $\mathcal{L}_r$ at $\gamma$ and $\mathcal{L}_{kr}$ at $\gamma^k$, $(W(\gamma), W(\gamma)^{-})$ and $(W(\gamma^k), W(\gamma^k)^{-})$, such that

$$(W(\gamma), W(\gamma)^{-}) \subset ((\mathcal{L}_r)^{-1}([c - 2\epsilon, c + 2\epsilon]), (\mathcal{L}_r)^{-1}(c - 2\epsilon)), \quad (5.9)$$

$$(W(\gamma^k), W(\gamma^k)^{-}) \subset ((\mathcal{L}_{kr})^{-1}([kc - 2k\epsilon, kc + 2k\epsilon]), (\mathcal{L}_{kr})^{-1}(kc - 2k\epsilon)), \quad (5.10)$$

$$(\psi^k(W(\gamma)), \psi^k(W(\gamma)^{-})) \subset (W(\gamma^k), W(\gamma^k)^{-})(5.11)$$

and that the iteration map $\psi^k : H_r(\alpha) \rightarrow H_{kr}(\alpha^k)$ induces isomorphisms

$$(\psi^k)_* : C_*(\mathcal{L}_r, \gamma; \mathbb{K}) = H_*(W(\gamma), W(\gamma)^{-}; \mathbb{K})$$

$$\rightarrow C_*(\mathcal{L}_{kr}, \gamma^k; \mathbb{K}) = H_*(W(\gamma^k), W(\gamma^k)^{-}; \mathbb{K}).$$

For $j = 1, k$, denote by the inclusions

$$h^j_1 : (W(\gamma^j), W(\gamma^j)^{-}) \hookrightarrow ((\mathcal{L}_{jr})_{j(c+2\epsilon)}, (\mathcal{L}_{jr})_{j(c-2\epsilon)}),$$

$$h^j_2 : ((\mathcal{L}_{jr})_{j(c+2\epsilon)}, (\mathcal{L}_{jr})_{j(c-2\epsilon)}) \hookrightarrow ((\mathcal{L}_{jr})_{j(c+2\epsilon)}, (\mathcal{L}_{jr})_{j(c-\epsilon)}^0),$$

$$h^j_3 : ((\mathcal{L}_{jr})_{j(c+2\epsilon)}, (\mathcal{L}_{jr})_{j(c-\epsilon)}^0) \hookrightarrow (H_{jr}, (\mathcal{L}_{jr})_{j(c-\epsilon)}^0).$$

Hereafter $B^0$ denote the interior of $B$ without special statements. The arguments above [Lo2] Th.4.3 show that

$$(h^j_3 \circ h^j_2)_* : H_*(W(\gamma^j), W(\gamma^j)^{-}; \mathbb{K}) \rightarrow H_*((\mathcal{L}_{jr})_{j(c+2\epsilon)}, (\mathcal{L}_{jr})_{j(c-\epsilon)}^0; \mathbb{K}),$$

$$(h^j_2)_* : H_*((\mathcal{L}_{jr})_{j(c+2\epsilon)}, (\mathcal{L}_{jr})_{j(c-\epsilon)}^0; \mathbb{K}) \rightarrow H_*((H_{jr}, (\mathcal{L}_{jr})_{j(c-\epsilon)}^0; \mathbb{K})$$

are monomorphisms on homology modules. For $j = 1, k$, we have also inclusions

$$I^j : (W(\gamma^j), W(\gamma^j)^{-}) \hookrightarrow ((\mathcal{L}_{jr})^{-1}([jc - 2j\epsilon, jc + 2j\epsilon]), (\mathcal{L}_{jr})^{-1}(jc - 2j\epsilon)),$$

$$J^j : ((\mathcal{L}_{jr})^{-1}([jc - 2j\epsilon, jc + 2j\epsilon]), (\mathcal{L}_{jr})^{-1}(jc - 2j\epsilon)) \hookrightarrow (H_{jr}, (\mathcal{L}_{jr})_{j(c-j\epsilon)}^0).$$

It is clear that

$$J^j \circ I^j = h^j_3 \circ h^j_2 \circ h^j_1, \quad j = 1, k. \quad (5.12)$$

By (5.11), we have also

$$\psi^k \circ I^1 = I^k \circ \psi^k$$

as maps from $(W(\gamma), W(\gamma)^{-})$ to $((\mathcal{L}_{kr})^{-1}([kc - 2k\epsilon, kc + 2k\epsilon]), (\mathcal{L}_{kr})^{-1}(kc - 2k\epsilon))$. So we get the following result, which is a slightly strengthened version of [Lo2] Th. 4.3 in the case $M = T^n$. 

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Proposition 5.5  Under Assumption $F(\alpha)$, there exist a periodic solution $\gamma$ of (1.6) of integer period $\tau$ and representing $\alpha$, a large integer $k_0 > 0$, an infinite integer set $Q$ containing 1, and a small $\epsilon > 0$ having properties: For any $k \in Q(k_0) := \{k \in Q \mid k \geq k_0\}$ there exist topological Gromoll-Meyer pairs $(W(\gamma), W(\gamma)^-)$ and $(W(\gamma^k), W(\gamma^k^-))$ satisfying (5.9)-(5.11) such that for the inclusion

$$j_k = h_3 \circ h_2 \circ h_1 : (W(\gamma^k), W(\gamma^k^-)) \to (H_{k^\tau}(\alpha^{k^\tau}), (\mathcal{L}_{k^\tau})^\circ_{k(c-\epsilon)})$$

the following diagram holds:

$$0 \neq C_n(\mathcal{L}_{\tau}, \gamma; \mathbb{K}) \xrightarrow{(\psi^k)} C_n(\mathcal{L}_{k^\tau}, \gamma^k; \mathbb{K}) \xrightarrow{(j_k)} H_n(H_{k^\tau}(\alpha^{k^\tau}), (\mathcal{L}_{k^\tau})^\circ_{k(c-\epsilon)}; \mathbb{K}) \equiv \mathcal{H}_k, \quad (5.13)$$

where $c = \mathcal{L}_{\tau}(\gamma)$, $(\psi^k)_*$ is an isomorphism, and $(j_k)_*$ is a monomorphism among the singular homology modules. In particular, if $\omega$ is a generator of $C_n(\mathcal{L}_{\tau}, \gamma; \mathbb{K}) = H_n(W(\gamma), W(\gamma)^-; \mathbb{K})$, then

$$(j_k)_* \circ (\psi^k)_*(\omega) \neq 0 \quad \text{in} \mathcal{H}_k, \quad (5.14)$$

$$(j_k)_* \circ (\psi^k)_*(\omega) = (J_k)_* \circ (I_k)_* \circ (\psi^k)_*(\omega) = (J_k)_* \circ (\psi^k)_* \circ (I)_*(\omega) \quad \text{in} \mathcal{H}_k. \quad (5.15)$$

It is (5.15) that helps us avoiding to use the finite energy homology.

The notion of a $C^1$-smooth singular simplex in Hilbert manifolds can be defined as on page 252 of [Ma].

Proposition 5.6  For $\tau \in \mathbb{N}$, $c \in \mathbb{R}$, $\epsilon > 0$, $q \geq 0$, and a $C^1$-smooth $q$-simplex

$$\eta : (\Delta_q, \partial \Delta_q) \to (H_{\tau}(\alpha^\tau), (\mathcal{L}_{\tau})^\circ_{k(c-\epsilon)})$$

there exists an integer $k(\eta) > 0$ such that for every integer $k \geq k(\eta)$, the $q$-simplex

$$\eta^k \equiv \psi^k(\eta) : (\Delta_q, \partial \Delta_q) \to (H_{k^\tau}(\alpha^{k^\tau}), (\mathcal{L}_{k^\tau})^\circ_{k(c-\epsilon)})$$

is homotopic to a singular $q$-simplex

$$\eta_k : (\Delta_q, \partial \Delta_q) \to ((\mathcal{L}_{k^\tau})^\circ_{k(c-\epsilon)}, (\mathcal{L}_{k^\tau})^\circ_{k(c-\epsilon)}) \quad (5.16)$$

with $\eta^k = \eta_k$ on $\partial \Delta_q$ and the homotopy fixes $\eta^k|_{\partial \Delta_q}$.

This is an analogue of [BK Th.1], firstly proved by Y. Long [Lo2 Prop. 5.1] in the case $M = T^n$. Proposition 5.1 in [Lo2] actually gave stronger conclusions under weaker assumptions: If the $q$-simplex $\eta$ above is only a finite energy one ($C^1$-smooth simplex must be of finite energy), then the simplex $\eta^k$ is finite energy homotopic to a finite energy $q$-simplex $\eta_k$. Hence Proposition 5.6 can be derived with the same reason as in [Lo2 Prop. 5.1] as long as we generalize an inequality as done in Lemma A.4 of Appendix. But we also give necessary details for the reader’s convenience.
Proof of Proposition 5.6 Recall that for paths $\sigma : [a_1, a_2] \to M$ and $\delta : [b_1, b_2] \to M$ with $\sigma(a_2) = \delta(b_1)$ one often define new paths $\sigma^{-1} : [a_1, a_2] \to M$ by $\sigma^{-1}(t) := \sigma(a_2 + a_1 - t)$ and $\sigma \ast \delta : [a_1, a_2 + b_2 - b_1] \to M$ by $\sigma \ast \delta|_{a_1, a_2} = \sigma$ and 
\[
\sigma \ast \delta(t) := \delta(t - a_2 + b_1) \quad \text{for} \quad t \in [a_2, a_2 + b_2 - b_1].
\]

Given a $C^1$-path $\rho : [a, b] \to H_\tau(\alpha^\tau)$ and an integer $k \geq 3$ we want to construct a path $\rho_k : [a, b] \to H_{k\tau}(\alpha^{k\tau})$ such that 
\[
\rho_k(a) = \psi^k(\rho(a)) \quad \text{and} \quad \rho_k(b) = \psi^k(\rho(b)).
\]
Define the initial point curve $\beta_\rho$ of $\rho$ by 
\[
[a, b] \to M, \ s \mapsto \beta_\rho(s) = \rho(s)(0).
\]
It is $C^1$-smooth. Following [Lo2] pp. 460 and [BK] pp. 381, for $0 \leq s \leq (b - a)/k$ and $1 \leq j \leq k - 2$ define 
\[
\begin{align*}
\rho_k(a + s) &= \rho(a)^{k-1} \ast (\beta_\rho|_{a,a+ks}) \ast \rho(a + ks)^{1} \ast (\beta_\rho|_{a,a+ks})^{-1}, \\
\rho_k(a + j(b - a)/k + s) &= \rho(a)^{k-j} \ast (\beta_\rho|_{a,a+ks}) \ast \rho(a + ks)^{1} \ast (\beta_\rho|_{a,a+ks})^{-1} \ast (\beta_\rho)^{j} \ast (\beta_\rho)^{j-1}, \\
\rho_k((b - a)/k + s) &= \rho(a + ks)^{1} \ast (\beta_\rho|_{a,a+ks}) \ast \rho(b)^{k-1} \ast (\beta_\rho|_{a,a+ks})^{-1}.
\end{align*}
\]
These are piecewise $C^1$-smooth loops in $M$ representing $\alpha^k$, and 
\[
\bar{\rho}(a) = \rho(a)^{k-1} \quad \text{and} \quad \bar{\rho}(b) = \rho(b) \ast \beta_\rho \ast (\rho(b)^{k-1}) \ast (\beta_\rho)^{j}.
\]
For each $u \in [a, b]$, reparametrising the loop $\bar{\rho}_k(u)$ on $\mathbb{R}/k\tau$ as in [Lo2] pp.461] we get a piecewise $C^1$-smooth loop $\rho_k(u) \in H_{k\tau}(\alpha^{k\tau})$ and therefore a piecewise $C^1$-smooth path $\rho_k : [a, b] \to H_{k\tau}(\alpha^{k\tau})$ with $\rho_k(a) = \psi^k(\rho(a)) = \rho(a)^{k}$ and $\rho_k(b) = \psi^k(\rho(b)) = \rho(b)^{k}$.

Replacing all the terms of powers of $\rho(a)$ and $\rho(b)$ by the constant point paths in the definition of $\bar{\rho}_k$ above, we get a piecewise $C^1$-smooth path $\beta_{\rho_k} : [a, b] \to H_\tau(\alpha)$. For $s \in [a, b]$ and $j = [k(s - a)/(b - a)]$, by the arguments of [Lo2] pp. 461,
\[
L_{k\tau}(\rho_k(s)) = (k - j - 1)L_\tau(\rho(a)) + jL_\tau(\rho(b)) + L_\tau(\beta_{\rho,k}(s)) \leq (k - 1)M_0(\rho) + M_1(\rho) + 2M_2(\rho), \quad (5.17)
\]
where $M_0(\rho) = \max\{L_\tau(\rho(a)), L_\tau(\rho(b))\}$, $M_1(\rho) = \max_{a \leq s \leq b} |L_\tau(\rho(s))|$ and 
\[
M_2(\rho) = \int_a^b |L(s, \beta_\rho(s), \dot{\beta}_\rho(s))|ds. \quad (5.18)
\]
Note that $(L_3)$ implies 
\[
|L(t, q, v)| \leq C(1 + \|v\|^2) \quad \forall (t, q, v) \in \mathbb{R} \times TM \quad (5.19)
\]
for some constant $C > 0$. Therefore it follows from Lemma A.4 that 
\[
M_2(\rho) = \int_a^b |L(s, \beta_\rho(s), \dot{\beta}_\rho(s))|ds \leq (b - a)C + C \int_a^b |\dot{\beta}_\rho(s)|^2 ds \leq (b - a)C + \frac{1 + \tau}{2\tau}Cc(\rho).
\]
This and \((5.17)\) yield
\[
\lim_{k \to +\infty} \sup_{a \leq s < b} \frac{1}{k} \mathcal{L}_{k\tau}(\rho_k(s)) \leq M_0(\rho).
\] (5.20)

Next replacing \([L02, \text{Lem.2.3}]\) by Lemma A.4, and almost repeating the reminder arguments of the proof of \([L02, \text{Prop.5.1}]\), we can complete the proof of Proposition \(5.6\). □

**Lemma 5.7 (BK Lem.1)** Let \((X, A)\) be a pair of topological spaces and \(\beta\) a singular relative \(p\)-cycle of \((X, A)\). Let \(\Sigma\) denote the set of singular simplices of \(\beta\) together with all their faces. Suppose to every \(\sigma \in \Sigma, \sigma : \Delta^q \to X, 0 \leq q \leq p\), there is assigned a map \(P(\sigma) : \Delta^q \times [0,1] \to X\) such that

(i) \(P(\sigma)(z,0) = \sigma(z)\) for \(z \in \Delta^q\),

(ii) \(P(\sigma)(z, t) = \sigma(z)\) if \(\sigma(\Delta^q) \subset A\),

(iii) \(P(\sigma)(\Delta^q \times \{1\}) \subset A\),

(iv) \(P(\sigma)(e^q_i \times \text{id}) = P(\sigma \circ e^q_i)\) for \(0 \leq i \leq q\).

Then the homology class \([\beta] \in H_p(X, A)\) vanishes.

For the class \(\omega\) in \((5.15)\), by the definition of \(I_1\) above \((5.12)\) we have
\[
(I_1)_*(\omega) \in H_n((L_r)^{-1}([c-2\epsilon, c+2\epsilon]), (L_r)^{-1}(c-2\epsilon); \mathbb{K}).
\] (5.21)

Since both \((L_r)^{-1}([c-2\epsilon, c+2\epsilon])\) and \((L_r)^{-1}(c-2\epsilon)\) are at least \(C^2\)-smooth Hilbert manifolds, we can choose a \(C^1\)-smooth cycle representative \(\sigma\) of the class \((I_1)_*(\omega)\). Denote by \(\Sigma(\sigma)\) the set of all simplexes together with all their faces contained in \(\sigma\). By \([Ch]\,\text{Ex.1, pp.33}]\) each \(\gamma^k\) in \((5.17)\) is a non-minimal saddle point of \(L_{k\tau}\) on \(H_{k\tau}(\alpha^{k\tau})\). As in the proof of \([L02, \text{Prop.5.2}]\) we can use Proposition 5.6 and Lemma A.4 to get the corresponding result without using the finite energy homology.

**Proposition 5.8** There exists a sufficiently large integer \(k(\sigma) \geq k_0\) such that for every integer \(k \in Q(k(\sigma))\) and for every \(\mu \in \Sigma(\sigma)\) with \(\mu : \Delta_r \to H_{r}(\alpha^{r})\) and \(0 \leq r \leq n\), there exists a homotopy \(P(\psi^k(\mu)) : \Delta_r \times [0,1] \to H_{k\tau}(\alpha^{k\tau})\) such that the properties (i) to (iv) in Lemma 5.7 hold for \((X, A) = (H_{k\tau}(\alpha^{k\tau}), (L_{k\tau})^\epsilon_{(c-\epsilon)})\).

It follows that the homology class \((J_k)_* \circ (\psi^k)_* \circ (I_1)_* (\omega) \in H_k\) vanishes. By \((5.15)\), \((j_{k\tau})_* \circ (\psi^k)_* (\omega) = 0\) in \(H_k\). This contradicts to \((5.14)\). Therefore Assumption \(F(\alpha)\) can not hold. Theorem \((5.1)\) is proved.

5.2. Proof of (ii). Since the inclusion \(E_\tau \to C(\mathbb{R}/\mathbb{Z}, M)\) is a homotopy equivalence, and therefore \(\text{rank} H_r(E_r; \mathbb{K}) \neq 0\) for all \(r \in \mathbb{N}\). Consider the functional \(L_{k\tau}\) on \(E_{k\tau}\). It has still a nonempty critical point set. Replace Assumption \(F(\alpha)\) by

**Assumption F:** (i) For any given integer \(k > 0\), the system \((1.0)\) only possesses finitely many distinct, \(k\)-periodic solutions, (ii) there exists an integer \(k_0 > 1\) such that for each integer \(k > k_0\), any \(k\)-periodic solution \(\gamma\) of the system \((1.0)\) must be an iteration of some \(l\)-periodic solution \(\gamma\) of the system \((1.0)\) with \(l \leq k_0\) and \(k = 1s\) for some \(s \in \mathbb{N}\).

Then slightly modifying the proof of (i) above one can complete the proof. □
6 Proof of Theorem 1.4

The proof is similar to that of Theorem 1.1. We only give the main points. Identifying \( \mathbb{R}/\tau \mathbb{Z} = [-\frac{\tau}{2}, \frac{\tau}{2}] \), let

\[
C(\mathbb{R}/\tau \mathbb{Z}, M) := \{ x \in C(\mathbb{R}/\tau \mathbb{Z}, M) \mid x(-t) = x(t) - \tau/2 \leq t \leq \tau/2 \}.
\]

We have a contraction from \( C(\mathbb{R}/\tau \mathbb{Z}, M) \) to the subset of constant loops in \( C(\mathbb{R}/\tau \mathbb{Z}, M) \) which is identified with \( M \):

\[
[0,1] \times C(\mathbb{R}/\tau \mathbb{Z}, M) \to C(\mathbb{R}/\tau \mathbb{Z}, M), \ (s, x) \mapsto x_s,
\]

where \( x_s(t) = x(st) \) for \( -\tau/2 \leq t \leq \tau/2 \). Since the inclusion \( C(\mathbb{R}/\tau \mathbb{Z}, M) \hookrightarrow EH_\tau \) is also a homotopy equivalence, we get

\[
H_n(EH_\tau; \mathbb{Z}_2) = H_n(C(\mathbb{R}/\tau \mathbb{Z}, M)_e; \mathbb{Z}_2) = H_n(M; \mathbb{Z}_2) \neq 0 \quad (6.1)
\]

for any \( \tau > 0 \). Note that \( L^E_\tau \) can always attain the minimal value on \( EH_\tau \) and therefore has a nonempty critical set \( \mathcal{K}(L^E_\tau) \). Under the conditions (L1)-(L4) we replace the Assumption F(\( \alpha \)) in §5 by

Assumption FE: (i) For any given integer \( k > 0 \), the system (1.6) possesses only finitely many distinct reversible \( k\tau \)-periodic solutions, (ii) there exists an integer \( k_0 > 1 \) such that for each integer \( k > k_0 \), any reversible \( k\tau \)-periodic solution \( \gamma \) of the system (1.6) is an iteration of some reversible \( l\tau \)-periodic solution \( \tilde{\gamma} \) of the system (1.6) with \( l \leq k_0 \) and \( k = ls \) for some \( s \in \mathbb{N} \).

Under this assumption, as the arguments below Assumption F(\( \alpha \)) we may get an integer \( \tau \in \mathbb{N} \) and finitely many reversible \( \tau \)-periodic solutions of the system (1.6), \( \gamma_1 \cdots \gamma_p \), such that for any \( k \in \mathbb{N} \) every reversible \( k\tau \)-periodic solution of the system (1.6) has form \( \gamma_j^k \) for some \( 1 \leq j \leq p \). Namely,

\[
\mathcal{K}(L^E_{k\tau}) = \{ \gamma_j^k \mid 1 \leq j \leq p \}.
\]

(6.2)

Using the same proof as one of Lemma 5.2 we may obtain:

Lemma 6.1 Under Assumption FE, for each \( k \in \mathbb{N} \) there exists a critical point \( \gamma'_k \) of \( L^E_{k\tau} \) such that

\[
C_n(L^E_{k\tau}, \gamma'_k; \mathbb{Z}_2) \neq 0 \quad \text{and} \quad -n \leq n - m^0_{1,k\tau}(\gamma'_k) \leq m^-_{1,k\tau}(\gamma'_k) \leq n.
\]

(6.3)

Let \( k_0 = 1 \) if \( \tilde{m}_{1,\tau}(\gamma_j) = 0 \) for all \( 1 \leq j \leq p \), and

\[
k_0 = 1 + \max \left\{ \left[ \frac{3n + 2}{2\tilde{m}_{1,\tau}(\gamma_j)} \right] \mid \tilde{m}_{1,\tau}(\gamma_j) \neq 0, 1 \leq j \leq p \right\}
\]

otherwise. Corresponding with Corollary 5.4 we have the following generalization of [LW2, Lem.4.4].
Lemma 6.2 Under Assumption FE, for any integer number $k \geq k_0$, every isolated critical point $z$ of $L^E_{kr}$ has the trivial $(n+1)$-th critical module, i.e.

$$C_{n+1}(L^E_{kr}, z; \mathbb{K}) = 0.$$  

Proof. Using the chart $\phi^E_{kr}$ in (3.29) let $\tilde{z} = (\phi^E_{kr})^{-1}(z)$. We only need to prove

$$C_{n+1}(\tilde{L}^E_{kr}, \tilde{z}; \mathbb{K}) = 0 \ \forall k \geq k_0. \quad (6.4)$$

Let $z = \gamma^k_j$ and thus $\tilde{z} = \tilde{\gamma}^k_j$ with $\tilde{\gamma}_j = (\beta^E_{kr})^{-1}(\gamma_j)$. By (4.41), it follows from Shifting theorem ([Ch, p.50, Th. 5.4]) and the Künneth formula that

$$C_{n+1}(\tilde{L}^E_{kr}, \tilde{z}; \mathbb{K}) \cong C_{n+1}(\alpha^E_{kr} + \beta^E_{kr}, 0; \mathbb{K})$$

$$\cong C_{n+1-m_1,kr}(\gamma^k_j)(\alpha^E_{kr}, 0; G) \otimes C_{m_1,kr}(\gamma^k_j)(\beta^E_{kr}, 0; \mathbb{K})$$

$$\cong C_{n+1-m_1,kr}(\gamma^k_j)(\alpha^E_{kr}, 0; \mathbb{K}) \otimes \mathbb{K}$$

$$\cong C_{n+1-m_1,kr}(\gamma^k_j)(\alpha^E_{kr}, 0; \mathbb{K})$$

because $0$ is a nondegenerate critical point of quadratic function $\beta^E_{kr}$. If (6.4) does not hold, we get that $0 \leq n + 1 - m_{1,kr}(\gamma^k_j) \leq m^0_{1,kr}(\gamma^k_j)$ because $\tilde{\gamma}^k_j$ is defined on a manifold of dimension $m^0_{1,kr}(\gamma^k_j)$. Note that

$$m_{1,kr}(\gamma^k_j) = m^0_{1,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j) = m^0_{1,kr}(\tilde{\gamma}^k_j),$$

$$m^0_{1,kr}(\gamma^k_j) = m^0_{1,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j) = m^0_{1,kr}(\tilde{\gamma}^k_j).$$

We have

$$m^0_{1,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j) \leq n + 1 \leq m^0_{1,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j) + m^0_{1,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j) \quad (6.5)$$

or $$m^0_{1,kr}(\gamma^k_j) \leq n + 1 \leq m^0_{1,kr}(\gamma^k_j) + m^0_{1,kr}(\gamma^k_j) \quad (6.6)$$

for any $k \in \mathbb{N}$. By (2.24)

$$m^0_{1,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j) - 2n \leq m^0_{2,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j) \leq m^0_{2,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j) \quad \forall k \in \mathbb{N}.$$  

Hence it follows from this, (3.26) and (6.5) that

$$2km^0_{1,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j) - n \leq m^0_{2,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j) + m^0_{1,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j)$$

$$\leq 2m^0_{1,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j) \leq 2n + 2.$$  

Therefore, when $\tilde{m}^0_{1,kr}(\gamma^k_j) = \tilde{m}^0_{1,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j) > 0$, $k \leq \left\lceil \frac{3n + 2}{2m^0_{1,kr}(\gamma^k_j)} \right\rceil$, which contradicts to $k \geq k_0$.

When $\tilde{m}^0_{1,kr}(\gamma^k_j) = \tilde{m}^0_{1,kr}(\tilde{L}^E_{kr}, \tilde{\gamma}^k_j) = 0$, (3.22) and (6.6) also give a contradiction. The desired (6.4) is proved. □

Now as the arguments below Corollary 5.1 under Assumption FE we may use Lemma 6.1 to get an infinite subsequence $Q$ of $\{2^m \mid m \in \{0\} \cup \mathbb{N}\}$ and an $\gamma \in \{\gamma_1, \cdots, \gamma_p\}$ such that

$$C_n(L^E_{kr}, \gamma^k; \mathbb{Z}_2) \neq 0,$$

$$m^0_{1,kr}(\gamma^k) = m^0_{1,kr}(\gamma), \quad m^0_{1,kr}(\gamma^k) = m^0_{1,kr}(\gamma) \quad \text{(6.7)}$$
for any $k \in Q$. By Lemma 6.2 for any $x \in \{\gamma_1, \ldots, \gamma_p\}$ we have also
\[
C_{n+1}(L_{kr}^E, x^k; \mathbb{K}) = 0 \quad \forall k \in Q(k_0) := \{k \in Q \mid k \geq k_0\}.
\] (6.8)

Then from Proposition 5.1 to the end of §5.1 we only need to make suitable replacements for some notations such as $H_jr(\alpha^j)$, $L_jr$ by $EH_jr$, $L_j^r$ for $j = 1, k$, and so on, and can complete the proof of Theorem 1.4.

7 Proof of Theorem 1.6

7.1. Proof of (i). Denote by $\mathcal{KO}(\mathcal{L}_r, \alpha^k)$ the set of critical orbits of $\mathcal{L}_r$ on $H_r(\alpha^k)$. It is always nonempty because $\mathcal{L}_r$ can attain the minimal value on $H_r(\alpha^k)$. Clearly, we may assume that each critical orbit of $\mathcal{L}_r$ on $H_r(\alpha^k)$ is isolated for any $k \in \mathbb{N}$. As in §5.1, by contradiction we assume:

Assumption $\text{FT}(\alpha)$: (i) For any given integer $k > 0$, the system (1.6) only possesses finitely many distinct, $k\tau$-periodic solution orbit towers based on $k\tau$-periodic solutions of (1.6) representing $\alpha^k$, (ii) there exists an integer $k_0 > 1$ such that for each integer $k > k_0$, any $k\tau$-periodic solution $\tilde{\gamma}$ of the system (1.6) representing $\alpha^k$ must be an iteration of some $l\tau$-periodic solution $\gamma$ of the system (1.6) representing $\alpha^l$ with $l \leq k_0$ and $k = lq$ for some $q \in \mathbb{N}$.

Under this assumption, there only exist finitely many periodic solution orbit towers $\{s \cdot \tilde{\gamma}_k^i\}_{k \in \mathbb{N}}, \ldots, \{s \cdot \tilde{\gamma}_k^p\}_{k \in \mathbb{N}}$ of the system (1.6) such that
- $\tilde{\gamma}_k^i$ has period $k\tau \leq k_0\tau$ and represents $\alpha^{ku}$ for some $k_i \in \mathbb{N}$, $i = 1, \ldots, p$;
- for each integer $k > k_0$ any $k\tau$-periodic solution $\gamma$ of the system (1.6) representing $\alpha^k$ must be an iteration of some $s \cdot \tilde{\gamma}_i$, i.e. $\gamma = (s \cdot \tilde{\gamma}_i)^l = s \cdot \tilde{\gamma}_i^l$ for some $s \in \mathbb{R}$ and $l \in \mathbb{N}$ with $l k_i = k$.

Set $m := k_0!$ (the factorial of $k_0$) and $\gamma_i = \tilde{\gamma}_i^{m/k_i}, i = 1, \ldots, p$. Then each $\gamma_i$ is a $m\tau$-periodic solution of the system (1.6) representing $\alpha^m$. We conclude

Claim 7.1 For any $k \in \mathbb{N}$, it holds that
\[
\mathcal{KO}(\mathcal{L}_{km\tau}, \alpha^{km}) = \{S_{mk\tau} \cdot \gamma_j^k \mid 1 \leq j \leq p\}.
\]

Proof. Let $\gamma \in \mathcal{K}(\mathcal{L}_{km\tau}, \alpha^{km})$. Since $km > k_0$, then $\gamma = (s \cdot \tilde{\gamma}_i)^l$ for some $s \in \mathbb{R}$ and $l \in \mathbb{N}$ with $l k_i = km$. Hence $\gamma = s \cdot \tilde{\gamma}_i^l = s \cdot (\tilde{\gamma}_i)^{km/k_i} = s \cdot (\tilde{\gamma}_i^{m/k_i})^k = s \cdot \gamma_i^k$. □

Hence replacing $\tau$ by $m\tau$ we may assume $m = 1$ below, i.e.
\[
\mathcal{KO}(\mathcal{L}_{k\tau}, \alpha^k) = \{S_{k\tau} \cdot \gamma_j^k \mid 1 \leq j \leq p\} \quad \forall k \in \mathbb{N}.
\] (7.1)

As in §5.1 we can also assume: $\gamma_j^s T \mathbb{R} \to \mathbb{R}/\tau \mathbb{Z}, j = 1, \ldots, p$, are all trivial.

Lemma 7.2 For each $k \in \mathbb{N}$ there exists $\mathcal{O}_k \subseteq \mathcal{KO}(\mathcal{L}_{k\tau}, \alpha^k)$ such that
\[
C_r(\mathcal{L}_{k\tau}, \mathcal{O}_k; \mathbb{K}) \neq 0.
\]

Moreover, $r - 2n \leq r - 1 - m_{k\tau}^0(\mathcal{O}_k) \leq m_{k\tau}^-(\mathcal{O}_k) \leq r - 1$ if $\mathcal{O}_k$ is not a single point critical orbit, and $r - 2n \leq r - m_{k\tau}^0(\mathcal{O}_k) \leq m_{k\tau}^-(\mathcal{O}_k) \leq r$ otherwise.
Proof. By Lemma 4.12, we only need to prove the first claim. The proof is similar to that of Lemma 5.2. Let $kc_1 < \cdots < kc_l$ be all critical values of $L_{kr}$, $l \leq p$, and $\inf L_{kr} = kc_1$, $k = 1, 2, \cdots$. As in the proof of Lemma 5.2, we have a large regular value $b$ of $L_{kr}$ such that rank $H_r((L_{kr})_b; \mathbb{K}) > 0$. Take the regular values of $L_{kr}$, $a_0 < a_1 < \cdots < a_l = b$ such that $kc_i \in (a_{i-1}, a_i)$, $i = 1, \cdots, l$. Noting (7.3), by Theorem 2.1 of [W a] or the proof of Lemma 4 of [GM2, pp. 502], we get

$$H_r((L_{kr})_{a_i}, (L_{kr})_{a_{i-1}}; \mathbb{K}) \cong \bigoplus_{L_{kr}(\gamma_j) = kc_i} C_r(L_{kr}, S_{kr} \cdot \gamma_j^k; \mathbb{K}).$$

Since each critical point has finite Morse index, (4.67) implies that each critical group $C_r(L_{kr}, S_{kr} \cdot \gamma_j^k; \mathbb{K})$ has finite rank. Almost repeating the proof of Lemma 5.2, we get some $S_{kr} \cdot \gamma_j^k$ in $KO(L_{kr}, \alpha^k)$ such that rank $C_r(L_{kr}, S_{kr} \cdot \gamma_j^k; \mathbb{K}) > 0$ and thus rank $C_r(L_{kr}, S_{kr} \cdot \gamma_j^k; \mathbb{K}) = 0$. □

Corresponding to Corollary 5.4, we have

**Lemma 7.3** Under Assumption FT($\alpha$), for every integer $q \geq n + 1$ there exists a constant $k_0(q) > 0$ such that

$$C_q(L_{kr}, O_k; \mathbb{K}) = 0$$

for every integer $k \geq k_0(q)$ and $O_k \in KO(L_{kr}, \alpha^k)$. Here $k_0(q) = 1$ if $\hat{m}_r(\gamma_j) = 0$ for all $1 \leq j \leq p$, and

$$k_0(q) = 1 + \max \left\{ \left[ \frac{q + n}{\hat{m}_r(\gamma_j)} \right] \mid \hat{m}_r(\gamma_j) \neq 0, 1 \leq j \leq p \right\}$$

otherwise.

**Proof.** Let $O_k = S_{kr} \cdot \gamma_j^k$. If $\gamma_j$ is constant, by the proof of Lemma 5.3, we have

$$C_q(L_{kr}, O_k; \mathbb{K}) = C_q(L_{kr}, \gamma_j^k; \mathbb{K}) = 0$$

for any $k > k(q, \gamma_j)$, where $k(q, \gamma_j) = 1$ if $\hat{m}_r(\gamma_j) = 0$, and $k(q, \gamma_j) = \frac{q + n}{\hat{m}_r(\gamma_j)}$ if $\hat{m}_r(\gamma_j) \neq 0$.

Suppose that $\gamma_j$ is not a constant solution. If $C_q(L_{kr}, O_k; \mathbb{K}) \neq 0$, Lemma 4.12 yields

$$m_{kr}(S_{kr} \cdot \gamma_j^k) \leq q - 1 \leq m_{kr}(S_{kr} \cdot \gamma_j^k) + m_{kr}^0(S_{kr} \cdot \gamma_j^k).$$

By (4.53), this becomes

$$m_{kr}(\gamma_j^k) \leq q - 1 \leq m_{kr}(\gamma_j^k) + m_{kr}(\gamma_j^k) - 1. \quad (7.2)$$

If $\hat{m}_r(\gamma_j) > 0$, it follows from (7.3) and (5.2) that

$$k\hat{m}_r(\gamma_j) - n \leq m_{kr}(\gamma_j^k) \leq q - 1$$

and therefore $k \leq \frac{q + n - 1}{\hat{m}_r(\gamma_j)}$. This contradicts to $k \geq k_0(q)$. If $\hat{m}_r(\gamma_j) = 0$, by (3.2)

$$0 \leq m_{kr}(\gamma_j^k) \leq n - m_{kr}(\gamma_j^k) \quad \forall k \in \mathbb{N}.$$
It follows that

\[ m_{kτ}^−(S_{kτ} \cdot \gamma_j^k) + m_{kτ}^0(S_{kτ} \cdot \gamma_j^k) = m_{kτ}^−(\gamma_j^k) + m_{kτ}^0(\gamma_j^k) - 1 \leq n - 1. \]

Since \( q \geq n + 1 \), (7.2) implies that \( m_{kτ}^−(S_{kτ} \cdot \gamma_j^k) + m_{kτ}^0(S_{kτ} \cdot \gamma_j^k) \geq n \). This also gives a contradiction. Lemma 7.3 is proved. □

Clearly, Lemma 7.2 and Lemma 7.3 imply Theorem 1.1(i) in the case \( r \geq n + 1 \).

In the following we consider the case \( r = n \).

Under Assumption \( FT(α) \) we apply Lemma 7.2 to all \( k \in \{2^m \mid m \in \{0\} \cup \mathbb{N}\} \) to get an infinite subsequence \( Q \) of \( \{2^m \mid m \in \{0\} \cup \mathbb{N}\} \), some \( l \in \mathbb{N} \) and an \( γ \in \{γ_1, \cdots, γ_p\} \) such that \( C_n(L_{kτ}, S_{kτ} \cdot γ_k^l; \mathbb{K}) \neq 0 \), \( m_{kτ}^−(S_{kτ} \cdot γ_k^l) = m_{τ}(S_γ \cdot γ) \) and \( m_{kτ}^0(S_{kτ} \cdot γ_k^l) = m_{τ}^0(S_γ \cdot γ) \) for any \( k \in Q \). As before we always assume \( l = 1 \) in the following. Then we have

\[
\begin{align*}
C_n(L_{kτ}, S_{kτ} \cdot γ^k; \mathbb{K}) & \neq 0 \quad \text{and} \\
m_{kτ}^−(S_{kτ} \cdot γ^k) = m_{τ}^−(S_γ \cdot γ), & \quad m_{kτ}^0(S_{kτ} \cdot γ^k) = m_{τ}^0(S_γ \cdot γ) \end{align*}
\] (7.4)

for any \( k \in Q \). By Lemma 7.3 there exists \( k_0 > 0 \) such that for any \( γ \in \{γ_1, \cdots, γ_p\}, C_{n+1}(L_{kτ}, S_{kτ} \cdot γ_k^l; \mathbb{K}) = 0 \quad \forall k \in Q(k_0) := \{k \in Q \mid k \geq k_0\}. \) (7.5)

Denote by \( O = S_γ \cdot γ \), and by \( c = L_γ(γ) = L_τ(O) \). Under Assumption \( FT(α) \), as in §5.1 let us take \( ν > 0 \) sufficiently small so that for each \( k \in \mathbb{N} \) the interval \([k(c - 3ν), k(c + 3ν)]\) contains a unique critical value \( kc \) of \( L_{kτ} \) on \( H_{kτ}(α_k) \), i.e.

\[ L_{kτ}(\mathcal{K}O(L_{kτ}, α_k)) \cap [k(c - 3ν), k(c + 3ν)] = \{kc\}. \]

For any \( k \in Q \), by Theorem 4.11 we may choose a topological Gromoll-Meyer pair of \( L_τ \) at \( O \subset H_{τ}(α) \), \((\tilde{W}(O), \tilde{W}(O)^-)\) satisfying

\[ ((L_τ)^{-1}([c - 2ν, c + 2ν]), (L_τ)^{-1}(c - 2ν)), \] (7.6)

and a topological Gromoll-Meyer pair of \( L_{kτ} \) at \( ψ_k^k(O) \subset H_{kτ}(α_k) \),

\[ (\tilde{W}(ψ_k^k(O)), \tilde{W}(ψ_k^k(O))^-) \]

such that

\[
\begin{align*}
(ψ_k^k(\tilde{W}(O)), ψ_k^k(\tilde{W}(O)^-)) & \subset (\tilde{W}(ψ_k^k(O)), \tilde{W}(ψ_k^k(O))^-) \quad \text{and} \\
(\tilde{W}(ψ_k^k(O)), \tilde{W}(ψ_k^k(O))^-) & \subset \quad ((L_{kτ})^{-1}([kc - 2kν, kc + 2kν]), (L_{kτ})^{-1}(kc - 2kν)) \end{align*}
\] (7.7)

and that the iteration map \( ψ_k^k : H_{τ}(α) \rightarrow H_{kτ}(α_k) \) induces an isomorphism:

\[
ψ_k^k : C_*(L_τ, O; \mathbb{K}) := H_*(\tilde{W}(O), \tilde{W}(O)^-; \mathbb{K}) \rightarrow C_*(L_{kτ}, ψ_k^k(O); \mathbb{K}) := H_*(\tilde{W}(ψ_k^k(O)), \tilde{W}(ψ_k^k(O))^-) \mathbb{K}).
\]
Identifying $\psi(\mathcal{O}) = \mathcal{O}$, for $j = 1, k$, denote by the inclusions
\[ h_j^i : (\hat{W}(\psi^j(\mathcal{O})), \hat{W}(\psi^j(\mathcal{O}))^{-}) \hookrightarrow (L_{j\tau}j_{c+2\nu}, (L_{j\tau}j_{c-2\nu})^{-}), \]
\[ h_2^j : ((L_{j\tau})j_{c+2\nu}, (L_{j\tau})j_{c-2\nu}) \hookrightarrow ((L_{j\tau})j_{c+2\nu}, (L_{j\tau})j_{c-2\nu})^{-}, \]
\[ h_3^j : ((L_{j\tau})j_{c+2\nu}, (L_{j\tau})j_{c-2\nu}) \hookrightarrow (H_{j\tau}, (L_{j\tau})j_{c-\nu})^{-}. \]

As in §5.1 we have monomorphisms on homology modules,
\[ (h_j^i \circ h_j^1)_* : H_*((\hat{W}(\psi^j(\mathcal{O})), \hat{W}(\psi^j(\mathcal{O}))^{-}); \mathbb{K}) \rightarrow H_*((L_{j\tau})j_{c+2\nu}, (L_{j\tau})j_{c-2\nu})^{-}; \mathbb{K}), \]
\[ (h_j^i)_* : H_*((L_{j\tau})j_{c+2\nu}, (L_{j\tau})j_{c-\nu})^{-}; \mathbb{K}) \rightarrow H_*((H_{j\tau}, (L_{j\tau})j_{c-\nu})^{-}; \mathbb{K}). \]

Moreover, the inclusions
\[ I_j : (\hat{W}(\psi^j(\mathcal{O})), \hat{W}(\psi^j(\mathcal{O}))^{-}) \hookrightarrow ((L_{j\tau})j_{c-2\nu}, (L_{j\tau})j_{c+2\nu})^{-}, \]
\[ J_j : ((L_{j\tau})j_{c-2\nu}, (L_{j\tau})j_{c+2\nu})^{-} \hookrightarrow (H_{j\tau}, (L_{j\tau})j_{c-\nu})^{-} \]
satisfy
\[ J_j \circ I_j = h_j^i \circ h_j^1 \circ h_j^i, \quad j = 1, k. \quad (7.9) \]

By (7.10), we have also
\[ \psi^k \circ I_1 = I_k \circ \psi^k \]
as maps from $(\hat{W}(\mathcal{O}), \hat{W}(\mathcal{O})^{-})$ to $((L_{k\tau})^{-1}(kc-2k\nu), (L_{k\tau})^{-1}(kc-2k\nu))^{-}$. These yield the following corresponding result with Proposition 5.5.

**Proposition 7.4** Under Assumption FT(\alpha), there exist a $\tau$-periodic solution $\gamma$ of (7.1) representing $\alpha$, a large integer $k_0 > 0$, an infinite integer set $Q$ containing 1, and an angle $\epsilon > 0$ having properties: For the orbit $\mathcal{O} = S_{\tau} \cdot \gamma$ and any $k \in Q(k_0) := \{k \in Q | k \geq k_0\}$ there exist topological Gromoll-Meyer pairs $(\hat{W}(\mathcal{O}), \hat{W}(\mathcal{O})^{-})$ and $(\hat{W}(\psi^k(\mathcal{O})), \hat{W}(\psi^k(\mathcal{O}))^{-})$ satisfying (7.8) such that for the inclusion
\[ j_{k\tau} = h_k^1 \circ h_k^1 \circ h_k^1 : (\hat{W}(\psi^k(\mathcal{O})), \hat{W}(\psi^k(\mathcal{O}))^{-}) \rightarrow (H_{k\tau}(\alpha^k), (L_{k\tau})j_{c-\nu})^{-} \]
the following diagram holds:
\[ 0 \neq C_n(L_{\tau}, \mathcal{O}; \mathbb{K}) \xrightarrow{\psi^k} C_n(L_{k\tau}, \psi^k(\mathcal{O}); \mathbb{K}) \xrightarrow{(j_{k\tau})_*} H_n(H_{k\tau}(\alpha^k), (L_{k\tau})j_{c-\nu})^{-}; \mathbb{K}) \equiv \mathcal{H}_k, \quad (7.11) \]
where $e = L_{\tau}(\gamma)$, $\psi^k_* \circ \psi^k$ is an isomorphism, and $(j_{k\tau})_*$ is a monomorphism among the singular homology modules. In particular, if $\omega$ is a generator of $C_n(L_{\tau}, \mathcal{O}; \mathbb{K}) = H_n(\hat{W}(\mathcal{O}), \hat{W}(\mathcal{O})^{-}; \mathbb{K})$, then
\[ (j_{k\tau})_* \circ (\psi^k)_* (\omega) \neq 0 \quad \text{in } \mathcal{H}_k, \quad (7.12) \]
\[ (j_{k\tau})_* \circ (\psi^k)_* (\omega) = (J_k)_* \circ (I_k)_* \circ (\psi^k)_* (\omega) \]
\[ = (J_k)_* \circ (\psi^k)_* \circ (I_1)_* (\omega) \quad \text{in } \mathcal{H}_k. \quad (7.13) \]

Now we can slightly modify the arguments from Proposition 5.6 to Proposition 5.8 to complete the proof of (i). The only place which should be noted is that for $\psi^k(\mathcal{O})$ in (7.11) Lemma 4.13 implies each point $y \in \psi^k(\mathcal{O})$ to be a non-minimum saddle point of $L_{k\tau}$ on $H_{k\tau}(\alpha^k)$ in the case $\dim M = n > 1$.

**7.2. Proof of (ii)** can be completed by the similar arguments as in §5.2.
8 Questions and remarks

For a $C^3$-smooth compact $n$-dimensional manifold $M$ without boundary, and a $C^2$-smooth map $H : \mathbb{R} \times T^* M \rightarrow \mathbb{R}$ satisfying the conditions (H1)-(H5), we have shown in $1^\circ$ of Theorem [112] that the Poincaré map $\Psi^H$ has infinitely many distinct periodic points sitting in the zero section $0_T M$ of $T^* M$. Notice that the condition (H5) can be expressed as: $H(t, x) = H(-t, \tau_0(x)) \forall (t, x) \in \mathbb{R} \times M$, where $\tau_0 : T^* M \rightarrow T^* M$, $(q, p) \mapsto (q, -p)$, is the standard anti-symplectic involution. So it is natural to consider the following question: Let $(P, \omega, \tau)$ be a real symplectic manifold with an anti-symplectic involution $\tau$ on $(P, \omega)$, i.e. $\tau^* \omega = -\omega$ and $\tau^2 = id_P$. A smooth time dependent Hamiltonian function $H : \mathbb{R} \times P \rightarrow \mathbb{R}$, $(t, x) \mapsto H(t, x) = H_t(x)$ is said to be $1$-periodic in time and symmetric if it satisfies

$$H_t(x) = H_{t+1}(x) \quad \text{and} \quad H(t, x) = H(-t, \tau(x)) \forall (t, x) \in \mathbb{R} \times P.$$ 

In this case, the Hamiltonian vector fields $X_{H_t}$ satisfies $X_{H_{t+1}}(x) = X_{H_t}(x) = -d\tau(\tau(x))X_{H_{t-1}}(\tau(x))$ for all $(t, x) \in \mathbb{R} \times P$. If the global flow of

$$\dot{x}(t) = X_{H_t}(x(t))$$

(8.1) exists, denoted by $\Psi^H_t$, then it is obvious that

$$\Psi^H_{t+1} = \Psi^H_t \circ \Psi^H_1 \forall t \in \mathbb{R}, \quad \Psi^H_1 \circ \tau = \tau \circ (\Psi^H_1)^{-1}.$$ 

So each $\tau$-invariant $k$-periodic point $x_0$, i.e. $\tau(x_0) = x_0$, of $\Psi^H = \Psi^H_1$ with $k \in \mathbb{N}$ yields a $k$-periodic contractible solution $x(t) = \Psi^H_t(x_0)$ of (8.1) satisfying $x(-t) = \tau(x(t))$ for all $t \in \mathbb{R}$. Such a solution is called $\tau$-reversible. By [Vi1, p.4] the fixed point set $L := \text{Fix}(\tau)$ of $\tau$ is either empty or a Lagrange submanifold. It is natural to ask the following more general question of the Conley conjecture.

**Question 8.1** Suppose that $L$ is nonempty and compact, and that $(P, \omega)$ satisfies some good condition (e.g. geometrically bounded for some $J \in \mathbb{R} \mathcal{J}(P, \omega) := \{J \in \mathcal{J}(P, \omega) \mid J \circ d\tau = -d\tau \circ J\}$ and Riemannian metric $\mu$ on $P$). Has the system (8.1) infinitely many distinct $\tau$-reversible contractible periodic solutions of integer periods? Furthermore, if the flow $\Psi^H_t$ exists globally, has the Poincaré map $\Psi^H = \Psi^H_1$ infinitely many distinct periodic points sitting in $L$?

Let $\mathcal{P}_0(H, \tau)$ denote the set of all contractible $\tau$-reversible $1$-periodic solutions of (8.1). Since the Conley conjecture came from the Arnold conjecture, Question 8.1 naturally suggests the following more general versions of the Arnold conjectures.

**Question 8.2** Under the assumptions of Question 8.1, $\#\mathcal{P}_0(H, \tau) \geq \text{Cuplength}_\mathbb{F}(L)$ for $\mathbb{F} = \mathbb{Z}, \mathbb{Z}_2$? Moreover, if some nondegenerate assumptions for elements of $\mathcal{P}_0(H, \tau)$ are satisfied, $\#\mathcal{P}_0(H, \tau) \geq \sum_{k=0}^{\frac{\dim L}{2}} b_k(L, \mathbb{F})$?
This question is closely related to the Arnold-Givental conjecture, cf. [Lu1]. In order to study it we try to construct a real Floer homology \( \text{FH}^*(P, \omega, \tau, H) \) with \( P_0(H, \tau) \) under some nondegenerate assumptions for elements of \( P_0(H, \tau) \), which is expected to be isomorphic to \( H^*(M) \). Moreover, if \( L \in C^2(\mathbb{R}/\mathbb{Z} \times TM) \) satisfies (L1)-(L4) and the functional \( L(\gamma) = \int_0^1 L(t, \gamma(t), \dot{\gamma}(t))dt \) on \( EH_1 \) has only nondegenerate critical points, then one can, as in [AbSc, §2.2], construct a Morse complex \( CM_*(L) \) whose homology is isomorphic to \( H^*(M) \) as well. As in [Vi3, SaWe, AbSc], it is also natural to construct an isomorphism between \( HF_*(T^*M, \omega_{\text{can}}, \tau_0, H) \) and \( H(CM_*(L)) \) and to study different product operations in them.

The author believes that the techniques developed in this paper are useful for one to generalize the results of multiple periodic solutions of some Lagrangian and Hamiltonian systems on the Euclidean space to manifolds.

9 Appendix

A.1. Proof of Proposition A. The first claim is a direct consequence of the following (9.4). As to the second, since for each \( t \in \mathbb{R} \) the functions \( L_t = L(t, \cdot) \) and \( H_t = H(t, \cdot) \) are Fenchel transformations of each other, we only need to prove that (H2)-(H3) can be satisfied under the assumptions (L2)-(L3). For conveniences we omit the time variable \( t \). In any local coordinates \((q_1, \cdots, q_n, v_1, \cdots, v_n)\), we write \((q,v) = (q_1, \cdots, q_n, v_1, \cdots, v_n)\). By definition of \( H \) we have

\[
H(q, \frac{\partial L}{\partial v}(q,v)) = -L(q,v) + \sum_{j=1}^n \frac{\partial L}{\partial v_j}(q,v)v_j. \tag{9.1}
\]

Differentiating both sides with respect to the variable \( v_i \) we get

\[
\sum_{j=1}^n \frac{\partial H}{\partial p_j}(q, \frac{\partial L}{\partial v}(q,v)) \frac{\partial^2 L}{\partial v_i \partial v_j}(q,v) = \sum_{j=1}^n v_j \frac{\partial^2 L}{\partial v_i \partial v_j}(q,v).
\]

Since the matrix \( \left[ \frac{\partial^2 L}{\partial v_i \partial v_j}(q,v) \right] \) is invertible, it follows that

\[
\frac{\partial H}{\partial p_j}(q, \frac{\partial L}{\partial v}(q,v)) = v_j. \tag{9.2}
\]

Let \( p = \frac{\partial L}{\partial v}(q,v) \). Differentiating both sides of (9.1) with respect to the variable \( q_i \) and using (9.2) we obtain

\[
\sum_{j=1}^n v_j \frac{\partial^2 L}{\partial q_i \partial v_j}(q,v) - \frac{\partial L}{\partial q_i}(q,v) = \frac{\partial H}{\partial q_i}(q, \frac{\partial L}{\partial v}(q,v)) + \sum_{j=1}^n \frac{\partial H}{\partial p_j}(q, \frac{\partial L}{\partial v}(q,v)) \frac{\partial^2 L}{\partial q_i \partial v_j}(q,v).
\]

\[
= \frac{\partial H}{\partial q_i}(q, \frac{\partial L}{\partial v}(q,v)) + \sum_{j=1}^n v_j \frac{\partial^2 L}{\partial q_i \partial v_j}(q,v)
\]
and hence

\[ \frac{\partial H}{\partial q_i}(q, \frac{\partial L}{\partial v}(q, v)) = -\frac{\partial L}{\partial q_i}(q, v). \]  \quad (9.3)

Differentiating both sides of (9.2) with respect to the variable \( v_i \) yields

\[ \sum_{k=1}^{n} \frac{\partial^2 H}{\partial p_j \partial p_k} (q, p) \frac{\partial^2 L}{\partial v_i \partial v_k} (q, v) = \delta_{ij}, \ i.e. \]

\[ \left[ \frac{\partial^2 H}{\partial p_i \partial p_j} (q, p) \right] = \left[ \frac{\partial^2 L}{\partial v_i \partial v_j} (q, v) \right]^{-1}. \]  \quad (9.4)

Differentiating both sides of (9.2) with respect to the variable \( q_i \), and both sides of (9.3) with respect to the variable \( q_j \) respectively, we arrive at

\[ \frac{\partial^2 H}{\partial q_i \partial q_j} (q, \frac{\partial L}{\partial v}(q, v)) + \sum_{k=1}^{n} \frac{\partial^2 H}{\partial q_i \partial p_k} (q, \frac{\partial L}{\partial v}(q, v)) \frac{\partial^2 L}{\partial v_k \partial q_j} (q, v) = 0, \]

\[ \frac{\partial^2 H}{\partial q_i \partial q_j} (q, \frac{\partial L}{\partial v}(q, v)) + \sum_{k=1}^{n} \frac{\partial^2 H}{\partial q_j \partial p_k} (q, \frac{\partial L}{\partial v}(q, v)) \frac{\partial^2 L}{\partial v_k \partial q_i} (q, v) = -\frac{\partial^2 L}{\partial q_i \partial q_j} (q, v), \]

or their equivalent expressions of matrices,

\[ \left[ \frac{\partial^2 H}{\partial q_i \partial q_j} (q, \frac{\partial L}{\partial v}(q, v)) \right] + \left[ \frac{\partial^2 H}{\partial q_i \partial p_j} (q, \frac{\partial L}{\partial v}(q, v)) \right] \left[ \frac{\partial^2 L}{\partial v_i \partial q_j} (q, v) \right] = 0, \]

\[ \left[ \frac{\partial^2 H}{\partial q_i \partial q_j} (q, \frac{\partial L}{\partial v}(q, v)) \right] + \left[ \frac{\partial^2 H}{\partial q_j \partial p_i} (q, \frac{\partial L}{\partial v}(q, v)) \right]^t \left[ \frac{\partial^2 L}{\partial v_i \partial q_j} (q, v) \right] = -\left[ \frac{\partial^2 L}{\partial q_i \partial q_j} (q, v) \right]. \]

It follows from these that

\[ \left[ \frac{\partial^2 L}{\partial q_i \partial q_j} (q, v) \right] \]

\[ = \left[ \frac{\partial^2 H}{\partial q_i \partial q_j} (q, \frac{\partial L}{\partial v}(q, v)) \right]^t \left[ \frac{\partial^2 H}{\partial p_i \partial p_j} (q, \frac{\partial L}{\partial v}(q, v)) \right]^{-1} \left[ \frac{\partial^2 H}{\partial p_i \partial q_j} (q, \frac{\partial L}{\partial v}(q, v)) \right] \]

\[ - \left[ \frac{\partial^2 H}{\partial q_i \partial q_j} (q, \frac{\partial L}{\partial v}(q, v)) \right]. \]  \quad (9.5)

Finally, differentiating both sides of (9.3) with respect to the variable \( v_j \) we get

\[ \frac{\partial^2 L}{\partial q_i \partial v_j} (q, v) = -\sum_{k=1}^{n} \frac{\partial^2 H}{\partial q_i \partial p_k} (q, \frac{\partial L}{\partial v}(q, v)) \frac{\partial^2 L}{\partial v_k \partial v_j} (q, v), \ i.e. \]

\[ \left[ \frac{\partial^2 L}{\partial q_i \partial v_j} (q, v) \right] = -\left[ \frac{\partial^2 H}{\partial q_i \partial p_j} (q, \frac{\partial L}{\partial v}(q, v)) \right] \left[ \frac{\partial^2 L}{\partial v_i \partial v_j} (q, v) \right] \]

\[ = -\left[ \frac{\partial^2 H}{\partial q_i \partial p_j} (q, \frac{\partial L}{\partial v}(q, v)) \right] \left[ \frac{\partial^2 L}{\partial p_i \partial p_j} (q, \frac{\partial L}{\partial v}(q, v)) \right]^{-1}. \]  \quad (9.6)

Here the final equality is due to (9.4). Since \( p = \frac{\partial L}{\partial v}(q, v) \) and \( v = \frac{\partial H}{\partial p}(q, p) \), the desired conclusions will follow from (9.4)-(9.6). Indeed, by (9.4) it is easily seen that (L2) is equivalent to
(H2') \[ \sum_{ij} \frac{\partial^2 H_{ij}}{\partial p_i \partial p_j} (t, q, p) u_i u_j \leq \frac{1}{c} |u|^2 \quad \forall u = (u_1, \ldots, u_n) \in \mathbb{R}^n. \]

Moreover, the three inequalities in (L3) have respectively the following equivalent versions in terms of matrix norms:

\[
\left| \frac{\partial^2 L}{\partial q_i \partial q_j} (t, q, v) \right| \leq C(1 + |v|^2), \quad \left| \frac{\partial^2 L}{\partial q_i \partial v_j} (t, q, v) \right| \leq C(1 + |v|) \]

and

\[
\left| \frac{\partial^2 L}{\partial v_i \partial v_j} (t, q, v) \right| \leq C. \]

Then (L3) is equivalent to the following

(H3') \[
\left[ \frac{\partial^2 H}{\partial p_i \partial p_j} (t, q, p) \right]^t \left[ \frac{\partial^2 H}{\partial p_i \partial p_j} (t, q, p) \right]^{-1} \left[ \frac{\partial^2 H}{\partial p_i \partial p_j} (t, q, p) \right] - \left[ \frac{\partial^2 H}{\partial p_i \partial p_j} (t, q, p) \right] \leq C \left( 1 + \left| \frac{\partial H}{\partial p} (t, q, p) \right|^2 \right),
\]

\[
\left| \left[ \frac{\partial^2 H}{\partial p_i \partial p_j} (t, q, p) \right]^{-1} \right| \leq C.
\]

Here \( \frac{\partial H}{\partial p} (t, q, p) = (\frac{\partial H}{\partial p_1} (t, q, p), \ldots, \frac{\partial H}{\partial p_n} (t, q, p)) \), and \( |A| \) denotes the standard norm of matrix \( A \in \mathbb{R}^{n \times n} \), i.e. \( |A| = (\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2)^{1/2} \) if \( A = (a_{ij}) \).

Note that \( |A| = \sup_{|x|=1} |(Ax, x)_{\mathbb{R}^n}| \) for any symmetric matrix \( A \in \mathbb{R}^{n \times n} \), and \( |A| = \sup_{|x|=1} (Ax, x)_{\mathbb{R}^n} \) if \( A \) is also positive definite, where \((\cdot, \cdot)_{\mathbb{R}^n}\) is the standard inner product in \( \mathbb{R}^n \). As usual, for two symmetric positive matrices \( A, B \in \mathbb{R}^{n \times n} \), by “\( A \leq B \)” we mean that \( (Ax, x)_{\mathbb{R}^n} \leq (Bx, x)_{\mathbb{R}^n} \) for any \( x \in \mathbb{R}^n \). Then it is easily proved that

\[
\left| \left[ \frac{\partial^2 H}{\partial p_i \partial p_j} (t, q, p) \right]^{-1} \right| \leq C \iff \left| \frac{\partial^2 H}{\partial p_i \partial p_j} (t, q, p) \right| \geq \frac{1}{C} I_n. \quad (9.7)
\]

This and (H2') yield \( \frac{1}{c} I_n \leq \left[ \frac{\partial^2 H}{\partial p_i \partial p_j} (t, q, p) \right] \leq \frac{1}{c} I_n. \)

**Lemma A.1.** For a matrix \( B \in \mathbb{R}^{n \times n} \) and symmetric matrices \( A, B \in \mathbb{R}^{n \times n} \), suppose that there exist constants \( 0 < c < C \) and \( \alpha \geq 0 \) such that

(i) \( \frac{1}{c} I_n \leq A \leq \frac{1}{c} I_n \),

(ii) \( |BA^{-1}| \leq C(1 + \alpha) \),

(iii) \( |B^t A^{-1} B - E| \leq C(1 + \alpha^2) \).

Then it holds that

\[
|B| \leq \frac{C}{c} (1 + \alpha) \quad \text{and} \quad |E| \leq \left( \frac{2C^3}{c^2} + C \right)(1 + \alpha^2). \quad (9.8)
\]

Conversely, if (i) and (9.8) are satisfied, then

\[
|BA^{-1}| \leq \frac{C^2}{c} (1 + \alpha) \quad \text{and} \quad |B^t A^{-1} B - E| \leq \left( \frac{4C^3}{c^2} + C \right)(1 + \alpha^2). \quad (9.9)
\]
Proof. By (i), $|A| \leq \frac{1}{c}$ and $|A^{-1}| \leq C$. Hence

$$|B| = |BA^{-1}A| \leq |BA^{-1}| |A| \leq \frac{C}{c}(1 + \alpha),$$

$$|E| = |B^t A^{-1} B - E - B^t A^{-1} B| \leq |B^t A^{-1} B - E| + |B^t A^{-1} B|$$

$$\leq C(1 + \alpha^2) + |B|^2 |A^{-1}| \leq C(1 + \alpha^2) + \frac{C^3}{c^2}(1 + \alpha)^2$$

$$\leq C(1 + \alpha^2) + \frac{2C^3}{c^2}(1 + \alpha^2)$$

$$\leq (C + \frac{2C^3}{c^2})(1 + \alpha^2).$$

(9.8) is proved. The “conversely” part is easily proved as well. □

By this lemma we get immediately:

Proposition A.2. In any local coordinates $(q_1, \cdots, q_n)$, the conditions (L2)-(L3) are equivalent to the fact that there exist constants $0 < C_1 < C_2$, depending on the local coordinates, such that

$$C_1 I_n \leq \left[ \frac{\partial^2 H}{\partial q_i \partial p_j}(t, q, p) \right] \leq C_2 I_n,$$

$$\left[ \frac{\partial^2 H}{\partial q_i \partial p_j}(t, q, p) \right] \leq C_2 \left( 1 + \left| \frac{\partial H}{\partial p}(t, q, p) \right| \right),$$

$$\left[ \frac{\partial^2 H}{\partial q_i \partial q_j}(t, q, p) \right] \leq C_2 \left( 1 + \left| \frac{\partial H}{\partial p}(t, q, p) \right|^2 \right).$$

For each $(t, q) \in \mathbb{R}/\mathbb{Z} \times M$, since the function $T_q^* M \to \mathbb{R}$, $p \mapsto H(t, q, p)$ is strictly convex, it has a unique minimal point $\bar{p} = \bar{p}(t, q)$. In particular, $D_p H(t, q, \bar{p}) = 0$. Recall that the diffeomorphism $L_H$ in (1.3) is the inverse of $L_L$ in (1.5), and that $L(t, q, v) = \langle p(t, q, v), v \rangle - H(t, q, p(t, q, v))$, where $p = p(t, q, v)$ is a unique point determined by the equality $v = D_p H(t, q, p)$. It follows that

$$\{(t, q, \bar{p}(t, q)) \in \mathbb{R}/\mathbb{Z} \times T^* M \mid (t, q) \in \mathbb{R}/\mathbb{Z} \times M\} = \mathcal{L}_H(\mathbb{R}/\mathbb{Z} \times 0_{TM})$$

is a compact subset. So in any local coordinates $(q_1, \cdots, q_n)$, there exists a constant $C_3 > 0$, depending on the local coordinates, such that the expression of $\bar{p} = \bar{p}(t, q)$ in the local coordinate $(q_1, \cdots, q_n)$, denoted by $\bar{p} = (\bar{p}_1, \cdots, \bar{p}_n)$, satisfies

$$|\bar{p}| = |(\bar{p}_1, \cdots, \bar{p}_n)| \leq C_3. \quad (9.10)$$

By the mean value theorem we have $0 < \theta = \theta(t, q, p) < 1$ such that

$$\left| \frac{\partial H}{\partial p}(t, q, p) \right| = \left| \frac{\partial H}{\partial p}(t, q, p) - \frac{\partial H}{\partial p}(t, q, \bar{p}) \right|$$

$$= \left| \left[ \frac{\partial^2 H}{\partial q_i \partial q_j}(t, q, \theta p + (1 - \theta)\bar{p}) \right] (p - \bar{p}) \right|.\]
Since the first inequality in Proposition A.2 implies
\[ C_1|u| \leq \left| \frac{\partial^2 H}{\partial p_i \partial p_j} (t, q, p) \right| \leq C_2|u| \quad \forall u = (u_1, \ldots, u_n)^t \in \mathbb{R}^n, \]
using (9.10) and the inequality \( ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2 \forall \varepsilon > 0 \) we easily get
\[ C_1|p| - C_1C_3 \leq C_1|p - \bar{p}| \leq \left| \frac{\partial H}{\partial p} (t, q, p) \right| \leq C_2|p - \bar{p}| \leq C_2|p| + C_2C_3, \]
\[ \frac{C_2^2}{2}|p|^2 - 2C_1^2C_3 \leq \left| \frac{\partial H}{\partial p} (t, q, p) \right|^2 \leq 2C_2^2|p|^2 + 2C_2^2C_3. \]

These two inequalities and Proposition A.2 lead to: In any local coordinates \((q_1, \ldots, q_n)\), the conditions (L2)-(L3) are equivalent to the fact that there exist constants \(0 < c < C\), depending on the local coordinates, such that
\[ cL_n \leq \left| \frac{\partial^2 H}{\partial p_i \partial p_j} (t, q, p) \right| \leq CI_n \quad \text{and} \]
\[ \left| \frac{\partial^2 H}{\partial q_i \partial p_j} (t, q, p) \right| \leq C(1 + |p|), \quad \left| \frac{\partial^2 H}{\partial q_i \partial q_j} (t, q, p) \right| \leq C(1 + |p|^2). \]

Proposition A is proved. □

**A.2. An inequality for \(C^1\)-simplex in \(C^1\) Riemannian-Hilbert manifolds.**

For every integer \(q \geq 0\) we denote by \(\Delta_q\) the standard closed \(q\)-dimensional simplex in \(\mathbb{R}^q\) with vertices \(e_0 = 0, e_1, \ldots, e_q\), i.e. \(\Delta_0 = \{0\}\) and
\[ \Delta_q := \{(t_1, \ldots, t_q) \in \mathbb{R}_{\geq 0}^q | t_1 + \cdots + t_q \leq 1\} \]
with \(q \geq 1\). For \(1 \leq i \leq q\) denote by \(F_i^q : \Delta_{q-1} \to \Delta_q\) the \(i\)-th face. Let \(e(s) = (s, \ldots, s) \in \mathbb{R}^q\) with \(s \in [0, 1]\), \(\bar{e} = e(1/(q + 1))\), and \(L\) be the straight line passing through \(e(0)\) and \(\bar{e}\) successively in \(\mathbb{R}^q\), i.e. \(L = \{s\bar{e} | s \in \mathbb{R}\}\). Then we have an orthogonal subspace decomposition
\[ \mathbb{R}^q = V_{q-1} \times L, \]
and each \(w \in \Delta_q\) may be uniquely written as \(w = (v, s_0) \in [V_{q-1} \times L] \cap \Delta_q\). Denote by \(l(v)\) the intersection segment of \(\Delta_q\) with the straight line passing through \(w\) and parallel to \(L\), i.e. \(l(v) = \{w + s\bar{e} \in \Delta_q | s \in \mathbb{R}\} = \{(v, s) | s_1 \leq s \leq s_2\}\) for some \(s_1 \leq s_0\) and \(s_2 \geq s_0\). Clearly, each \(l(v)\) has length no more than \(\sqrt{q}/2\).

Let \((\mathcal{M}, \langle , \rangle)\) be a \(C^1\) Riemannian-Hilbert manifold and \(\|\cdot\|\) be the induced Finsler metric. For \(\phi \in C(\Delta_q, \mathcal{M})\) and each \(w = (v, s_0) \in [V_{q-1} \times L] \cap \Delta_q\) we define
\[ \tilde{\phi}_q : l(v) \to \mathcal{M}, s \mapsto \phi(v, s). \]
If \(\phi \in C^1(\Delta_q, \mathcal{M})\), i.e. \(\phi\) can be extended into a \(C^1\)-map from some open neighborhood of \(\Delta_q\) in \(\mathbb{R}^q\) to \(\mathcal{M}\), then there exists a constant \(c = c(\phi) > 0\) such that
\[ \left\| \frac{\partial}{\partial s} \phi(v, s) \right\|^2 \leq c(\phi), \quad \forall (v, s) \in \Delta_q. \]
So for any \((v, s) \in \triangle \gamma\) we get

\[
\int_{l(v)} \left\| \frac{d}{ds} \tilde{\phi}_v(s) \right\|^2 ds \leq c(\phi) \text{Length}(l(v)) \leq \frac{\sqrt{q}}{2} c(\phi). \quad (9.11)
\]

Now consider the case \(M = E_\tau = W^{1,2}(S_\tau, M)\) with the Riemannian metric given by (1.13). Using the local coordinate chart in (3.3) it is easy to prove

**Lemma A.3.** For each \(t \in S_\tau\) the evaluation map

\[
\mathbf{EV}_t : W^{1,2}(S_\tau, M) \to M, \gamma \mapsto \gamma(t)
\]

is continuous and maps \(W^{1,2}\)-curves in \(E_\tau\) to \(W^{1,2}\)-curves in \(M\).

**Proof.** We only need to prove the case \(M = \mathbb{R}^n\). Let \([a, b] \to \gamma(s)\) be a \(W^{1,2}\)-curve in \(W^{1,2}(S_\tau, \mathbb{R}^n)\). Then \(\xi(s) := \frac{d}{ds} \gamma(s)\) is a \(W^{1,2}\)-vector field along \(\gamma(s)\). Since \(T_{\gamma(s)} W^{1,2}(S_\tau, \mathbb{R}^n) = W^{1,2}(S_\tau, \mathbb{R}^n)\), \(\xi(s) \in W^{1,2}(S_\tau, \mathbb{R}^n)\) and

\[
\lim_{\epsilon \to 0} \left\| \frac{\gamma(s + \epsilon) - \gamma(s)}{\epsilon} - \xi(s) \right\|_{W^{1,2}(S_\tau, \mathbb{R}^n)} = 0.
\]

Carefully checking the proof of Proposition 1.2.1(ii) in [Kl, pp. 9] one easily derives

\[
\|\eta\|_{C^0} \leq \sqrt{\frac{1 + \tau}{\tau}} \|\eta\|_{W^{1,2}} \quad \forall \eta \in W^{1,2}(S_\tau, \mathbb{R}^n). \quad (9.12)
\]

Hence we get

\[
\lim_{\epsilon \to 0} \left\| \frac{\gamma(s + \epsilon)(t) - \gamma(s)(t)}{\epsilon} - \xi(s)(t) \right\|_{\mathbb{R}^n} = 0
\]

uniformly in \(t\). This means that \([a, b] \to M, s \to \mathbf{EV}_t(\gamma(s))\), is differentiable and

\[
\frac{d}{ds} \mathbf{EV}_t(\gamma(s)) = \xi(s)(t) \quad \text{at each} \quad s \in [a, b]. \quad (9.13)
\]

Fix a \(\epsilon > 0\) such that

\[
\left\| \frac{\gamma(s + \epsilon) - \gamma(s)}{\epsilon} - \xi(s) \right\|_{W^{1,2}(S_\tau, \mathbb{R}^n)} \leq \sqrt{\frac{1 + \tau}{\tau}}.
\]

By (9.12) we get

\[
\left\| \frac{\gamma(s + \epsilon)(t) - \gamma(s)(t)}{\epsilon} - \xi(s)(t) \right\|_{\mathbb{R}^n} \leq 1 \quad \forall t \in \mathbb{R}.
\]

It follows that for any \(s \in [a, b]\),

\[
\|\xi(s)(t)\|_{\mathbb{R}^n}^2 \leq 2 \left[ \left\| \frac{\gamma(s + \epsilon)(t) - \gamma(s)(t)}{\epsilon} - \xi(s)(t) \right\|_{\mathbb{R}^n}^2 + \left\| \frac{\gamma(s + \epsilon)(t) - \gamma(s)(t)}{\epsilon} \right\|_{\mathbb{R}^n}^2 \right] \leq 2 \left[ 1 + \frac{1}{\epsilon} \|\gamma(s + \epsilon) - \gamma(s)(t)\|_{\mathbb{R}^n}^2 \right] \leq 2 \left[ 1 + \frac{1 + \tau}{\tau \epsilon^2} \|\gamma(s + \epsilon) - \gamma(s)\|_{W^{1,2}(S_\tau, \mathbb{R}^n)}^2 \right].
\]
Here the final inequality is due to \([9.12]\). Hence \(\int_a^b \|\xi(s)(t)\|_{\mathbb{R}^n}^2 ds < +\infty\), and thus \(\int_a^b \|d\sigma^t E\mathbf{V}_t(\gamma(s))\|_{\mathbb{R}^n}^2 ds < +\infty\) because of \([9.13]\). □

For a singular simplex \(\sigma\) from \(\triangle_q\) to \(E_\tau\) and every \(w = (v, s_0) \in \triangle_q\), define curves \(\tilde{\sigma}_v^t : l(v) \to M, s \mapsto E\mathbf{V}_t(\tilde{\sigma}_v(s)) = \tilde{\sigma}_v(s)(t)\)

\(\text{(9.14)}\)

for each \(t \in S_\tau\). The curve \(\tilde{\sigma}_v^t\) is called the initial point curve. Suppose that \(\sigma \in C^1(\triangle_q, E_\tau)\). Then \(\tilde{\sigma}_v \in C^1(l(v), E_\tau)\), and by \([9.11]\) there exists a positive constant \(c(\sigma)\) such that

\[\int_{l(v)} \left\| \frac{d\tilde{\sigma}_v(s)}{ds} \right\|_{W^{1,2}(\tilde{\sigma}_v(s)^*TM)}^2 ds \leq \frac{\sqrt{q}}{2} c(\sigma) \quad \text{(9.15)}\]

for any \((v, s) \in \triangle_q\), where \(\frac{d\tilde{\sigma}_v(s)}{ds} \in T_{\tilde{\sigma}_v(s)}E_\tau = W^{1,2}(\tilde{\sigma}_v(s)^*TM)\). Specially, by Lemma A.3 we get each \(\tilde{\sigma}_v^t \in W^{1,2}(l(v), M)\) for any \(t\). As in the proof of Proposition 1.2.1(ii) in [Kl] pp. 9] one can easily derive that

\[\|\xi\|_{C^0(\gamma^*TM)} \leq \sqrt{\frac{1 + \tau}{\tau}} \|\xi\|_{W^{1,2}(\gamma^*TM)}\]

for any \(\gamma \in W^{1,2}(S_\tau, M)\) and \(\xi \in W^{1,2}(\gamma^*TM)\). Applying to \(\gamma = \tilde{\sigma}_v(s)\) and \(\xi = \frac{d\tilde{\sigma}_v(s)}{ds}\) we get

\[\left\| \frac{d}{ds} \tilde{\sigma}_v(s) \right\|_{C^0(\tilde{\sigma}_v(s)^*TM)} \leq \frac{1 + \tau}{\tau} \left\| \frac{d}{ds} \tilde{\sigma}_v(s) \right\|_{W^{1,2}(\tilde{\sigma}_v(s)^*TM)} \cdot \quad \text{(9.16)}\]

Moreover, it follows from \([9.13]\) and \([9.14]\) that

\[\left( \frac{d}{ds} \tilde{\sigma}_v(s) \right)(t) = \frac{d}{ds}(\tilde{\sigma}_v(s)(t)) \in T_{\tilde{\sigma}_v(s)(t)}M\]

for all \(s \in [a, b]\) and \(t \in S_\tau\). Hence for any \(t \in S_\tau\), we can derive from \([9.13]\) that

\[\left\| \frac{d}{ds} \tilde{\sigma}_v^t(s) \right\|_{T_{\tilde{\sigma}_v(s)(t)}M}^2 = \left\| \left( \frac{d}{ds} \tilde{\sigma}_v(s) \right)(t) \right\|_{T_{\tilde{\sigma}_v(s)(t)}M}^2 \leq \left( \max_{t \in S_\tau} \left\| \left( \frac{d}{ds} \tilde{\sigma}_v(s) \right)(t) \right\|_{T_{\tilde{\sigma}_v(s)(t)}M} \right)^2 \leq \frac{1 + \tau}{\tau} \left\| \frac{d}{ds} \tilde{\sigma}_v(s) \right\|_{C^0(\tilde{\sigma}_v(s)^*TM)}^2 \leq \frac{1 + \tau}{\tau} \left\| \frac{d}{ds} \tilde{\sigma}_v(s) \right\|_{W^{1,2}(\tilde{\sigma}_v(s)^*TM)}^2 \cdot \]

This and \([9.15]\) together give the following generalization of [Lo2 Lem. 2.3].

**Lemma A.4.** If \(\sigma \in C^1(\triangle_q, E_\tau)\), for every \(w = (v, s_0) \in \triangle_q\), it holds that

\[\int_{l(v)} \left\| \frac{d}{ds} \tilde{\sigma}_v^0(s) \right\|_{T_{\tilde{\sigma}_v(s)^*M}}^2 ds \leq \frac{(1 + \tau)\sqrt{q}}{2\tau} c(\sigma).\]
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