AN ALMOST ZOLL AFFINE SURFACE

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Abstract. An affine surface is said to be an affine Zoll surface if all affine geodesics close smoothly. It is said to be an affine almost Zoll surface if thru any point, every affine geodesic but one closes smoothly (the exceptional geodesic is said to be alienated as it does not return). We exhibit an affine structure on the cylinder which is almost Zoll. This structure is geodesically complete, affine Killing complete, and affine symmetric.

Key words: affine manifold, Zoll surface
Subject class 53C21

1. Introduction

A Riemannian manifold is said to be a Zoll manifold the geodesics are all simple closed curves of the same length. Zoll [21] showed that the sphere $S^2$ admits many such metrics in addition to the round one. Later authors called these surfaces auf Wiedersehensflächen or “until we meet again” surfaces. In Spanish, this becomes te veo de nuevo superficie.

The only compact surfaces which admit Zoll metrics are the sphere $S^2$ and real projective space $\mathbb{RP}^2$. Green [10] showed that the only Zoll metric on $\mathbb{RP}^2$ is the standard homogeneous metric, up to isometry and rescaling; this was later extended by Pries [18] to show that if all the geodesics of a metric on $\mathbb{RP}^2$ are closed, then the metric is the standard homogeneous metric, up to isometry and rescaling. One can consider analogous questions in the Lorentzian setting – see, for example, the work of Mounoud and Suhr [15]. Zoll surfaces have been used in many contexts; see, for example, the work on Balacheff et al. [3] concerning a geodesic length conjecture. Zoll surfaces also have been considered in the orbifold context; see, for example, Lange [12]. We refer to Besse [4] for a discussion of more of the history of the subject than is available here and also to the references cited above.

An affine surface $M$ is a pair $(M, \nabla)$ where $M$ a 2-dimensional manifold and $\nabla$ is a torsion free connection on the tangent bundle of $M$. A diffeomorphism $\Phi$ from $(M_1, \nabla_1)$ to $(M_2, \nabla_2)$ is said to be affine if $\Phi$ intertwines $\nabla_1$ and $\nabla_2$. An affine surface $M$ is said to be homogeneous if the group of affine diffeomorphisms acts transitively on $M$. A vector field $X$ on $M$ is said to be an affine Killing vector field if the (locally defined) flows of $X$ are (locally defined) affine diffeomorphisms of $M$ or, equivalently by Kobayashi and Nomizu [11] the Lie derivative $L_X \nabla = 0$. The Lie bracket makes the set $\mathcal{K}(M)$ of affine Killing vector fields into a Lie algebra.

Let $R(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ be the curvature operator. The Ricci tensor is given by $\rho(X,Y) := \text{Tr}(Z \rightarrow R(Z,Y)Z)$. Because the Ricci tensor $\rho$ need not be symmetric in the affine context, one introduces the symmetric Ricci tensor $\rho_s(X,Y) := \frac{1}{4} \{\rho(X,Y) + \rho(Y,X)\}$. We say two affine structures $\nabla$ and $\tilde{\nabla}$ are projectively equivalent if there exists a smooth 1-form $\omega$ so that one may express $\tilde{\nabla}_XY = \nabla_XY + \omega(X)Y + \omega(Y)X$. Two affine structures are projectively equivalent if and only if their unparametrized geodesics coincide; see for example, Schouten [19]: thus it is natural to pass to projective structures. Note that projective equivalence does not preserve geodesic completeness since the parametrized
Theorem 2.1. Let surface. Riemannian extension to study Walker metrics on the cotangent bundle of an affine further discussion of the quasi-Einstein equation and its use through the modified to Gilkey and Valle-Regueiro [8] for the proof of the following result as well as a tions of the quasi-Einstein equation. We refer to Brozos-Vázquez et al. [6] and There is a close relationship between strong projective equivalence and the solu-

2. The quasi-Einstein equation

The solutions to the quasi-Einstein equation is a fundamental invariant in the theory of affine surfaces. We define the Hessian by setting:

$$\mathcal{H}\phi = (\partial_x \partial_x \phi - \Gamma_{ij}^k \partial_x \phi) dx^i \otimes dx^j \in \mathbb{S}^2(M).$$

Let $\mathcal{Q}(\mathcal{M})$ be the solution space of the quasi-Einstein equation:

$$\mathcal{Q}(\mathcal{M}) := \{ \phi \in C^\infty(M) : \mathcal{H}\phi + \phi p_s = 0 \}.$$ 

There is a close relationship between strong projective equivalence and the solutions of the quasi-Einstein equation. We refer to Brozos-Vázquez et al. [6] and to Gilkey and Valle-Regueiro [8] for the proof of the following result as well as a further discussion of the quasi-Einstein equation and its use through the modified Riemannian extension to study Walker metrics on the cotangent bundle of an affine surface.

Theorem 2.1. Let $\mathcal{M} = (M, \nabla)$ be a simply connected affine surface.

1. $\dim \{\mathcal{Q}(\mathcal{M})\} \leq 3$.
2. $\dim \{\mathcal{Q}(\mathcal{M})\} = 3$ if and only if $\mathcal{M}$ is strongly projectively flat.
3. $\mathcal{Q}(\Phi \mathcal{M}) = e^\phi \mathcal{Q}(\mathcal{M})$.
4. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two strongly projectively flat affine surfaces and let $\Phi$ be a diffeomorphism from $\mathcal{M}_1$ to $\mathcal{M}_2$. If $\Phi^* \mathcal{Q}(\mathcal{M}_2) = \mathcal{Q}(\mathcal{M}_1)$, then $\Phi$ is an affine morphism from $\mathcal{M}_1$ to $\mathcal{M}_2$.

3. The affine structures $\mathcal{M}(c)$

The following family of surfaces was introduced by D’Ascanio et al. [2] in their study of geodesically complete homogeneous affine surfaces. Let $\mathcal{M}(c) := (\mathbb{R}^2, \nabla)$ where the only (possibly) non-zero Christoffel symbols of $\nabla$ are $\Gamma_{22}^{1} = (1 + c^2)c^1$
and $\Gamma_{22}^2 = 2c$. Replacing $x^2$ by $-x^2$ interchanges the roles of $\mathcal{M}(c)$ and $\mathcal{M}(-c)$ so we shall assume $c \geq 0$ henceforth. A direct computation shows

$$\rho = (1 + c^2)dx^2 \otimes dx^2. \quad (3.a)$$

An affine connection is said to be symmetric if and only if $\nabla \rho = 0$; one verifies that $\mathcal{M}(c)$ is affine symmetric if and only if $c = 0$.

**Theorem 3.1.** Let $\mathcal{G}(c)$ be the set of all smooth maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ of the form:

$$T(\alpha, \beta, \gamma, \delta)(x^1, x^2) := (e^\alpha x^1 + \beta e^{\alpha + \delta} \cos(x^2) + \gamma e^{\alpha + \delta} \sin(x^2), x^2 + \delta).$$

Then $\mathcal{G}(c)$ is a 4-dimensional Lie group which is diffeomorphic to $\mathbb{R}^4$ which acts transitively on $\mathbb{R}^2$ so $\mathcal{M}(c)$ is a homogeneous space. The Lie algebra of $\mathcal{G}(c)$ is

$$\mathfrak{g}(c) := \text{Span}\{x^1 \partial_{x^1}, e^{\alpha + \delta} \cos(x^2) \partial_{x^1}, e^{\alpha + \delta} \sin(x^2) \partial_{x^1}, \partial_{x^2}\}.$$ 

The vector fields of $\mathfrak{g}(c)$ are all complete and $\mathfrak{g}(c) = \mathcal{K}(\mathcal{M}(c))$ is the set of affine Killing vector fields of $\mathcal{M}(c)$.

**Proof.** We follow the discussion in Gilkey et al. [9]. A direct computation shows that $\{e^{\alpha x^1} \cos(x^2), e^{\alpha x^2} \sin(x^2), x^1\} \subset \mathcal{Q}(\mathcal{M}(c))$. Consequently, by Theorem 2.1, $\mathcal{M}(c)$ is strongly projectively flat and

$$\mathcal{Q}(\mathcal{M}(c)) = \text{Span}\{e^{\alpha x^1} \cos(x^2), e^{\alpha x^2} \sin(x^2), x^1\}. \quad (3.b)$$

It is then immediate that $T(\alpha, \beta, \gamma, \delta)^* \mathcal{Q}(\mathcal{M}(c)) = \mathcal{Q}(\mathcal{M}(c))$ so $T(\alpha, \beta, \gamma, \delta)$ is a diffeomorphism of $\mathbb{R}^2$ preserving the affine structure. We show that $\mathcal{G}(c)$ is a Lie group with Lie algebra $\mathfrak{g}(c)$ by computing:

$$T(\alpha, \beta, \gamma, \delta) \circ T(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}) = T(e^{\alpha + \tilde{\gamma}} e^{\alpha \tilde{\gamma} + \beta e^{\beta \delta} \cos(\tilde{\delta}) + \gamma e^{\beta \delta} \sin(\tilde{\delta}), e^{\alpha \tilde{\gamma} - \beta e^{\beta \delta} \sin(\tilde{\delta}) + \gamma e^{\beta \delta} \cos(\tilde{\delta}), \tilde{\delta} + \delta})$$

$$T(\alpha, \beta, \gamma, \delta)^{-1} = T(-\alpha, -e^{-\alpha - \beta \delta} (\beta \cos(\delta) - \gamma \sin(\delta)), -e^{-\alpha - \beta \delta} (\beta \sin(\delta) + \gamma \cos(\delta)), -\delta). \quad \square$$

Following the notation of Opozda [10], an affine structure on $\mathbb{R}^2$ is said to be Type $A$ if the Christoffel symbols are constant; work of Arias-Marco and Kovalevski [1] and of Vanzurova [20] show that if the Ricci tensor of such a structure is non-zero, then it is not metrizable, i.e. it does not arise as the Levi-Civita connection of a pseudo-Riemannian metric. Let $\mathcal{M}_0(c)$ be the affine structure on $\mathbb{R}^2$ where the only (possibly) non-zero Christoffel symbols are $\Gamma_{11}^1 = 1$, $\Gamma_{22}^1 = 1 + c^2$, and $\Gamma_{22}^2 = 2c$. This is a Type $A$ geometric structure on $\mathbb{R}^2$ in the notation of Opozda [10]; it is linearly equivalent to the surfaces $\mathcal{M}_0^c$ of [5] but is in a slightly more convenient form for our purposes. Let $(u^1, u^2)$ be coordinates on $\mathbb{R}^2$. A direct computation shows

$$\mathcal{Q}(\mathcal{M}_0(c)) = \text{Span}\{e^{cu^2} \cos(u^2), e^{cu^2} \sin(u^2), e^{u^1}\}.$$ 

Let $\Phi(u^1, u^2) = (e^{u^1}, u^2)$. We then have $\Phi^* \mathcal{Q}(\mathcal{M}(c)) = \mathcal{M}_0(c)$ so $\Phi$ is an affine embedding of $\mathcal{M}_0(c)$ in $\mathcal{M}(c)$. We have $\rho = (1 + c^2)dx^2 \otimes dx^2$. Results of Brozos-Vázquez et al. [5] show that if $\mathcal{M}$ is a Type $A$ structure on $\mathbb{R}^2$ and if $\text{Rank}(\rho) = 1$, then $\dim(\mathcal{R}(\mathcal{M}(c))) = 4$. Consequently, the Lie algebra of Killing vector fields $\mathcal{K}(\mathcal{M})$ is 4-dimensional and is the Lie algebra of the Lie group $\mathcal{M}(c)$. Consequently, every affine Killing vector field on $\mathcal{M}(c)$ is complete. \square

Patera et al. [17] classified the 4-dimensional Lie algebras. We follow their notation. Let $A_{4,12}$ be the 4-dimensional Lie algebra with bracket relations $[e_1, e_3] = e_1$, $[e_2, e_3] = e_2$, $[e_1, e_4] = -e_2$, $[e_2, e_4] = e_1$. Results of [5] show $\mathcal{R}(\mathcal{M}_0(c))$ is isomorphic to $A_{4,12}$ and, consequently, $\mathcal{R}(\mathcal{M}(c))$ is isomorphic to $A_{4,12}$ as well.
Theorem 3.2.

(1) The geometries $\mathcal{M}(c)$ and $\mathcal{M}(\bar{c})$ are not locally affine equivalent for $c \neq \bar{c}$.
(2) The geometries $\mathcal{M}(c)$ and $\mathcal{M}(\bar{c})$ are not strongly projectively equivalent for $c \neq \bar{c}$.
(3) Let $\phi(x^1, x^2) = (e^{cx^1} x^1, x^2)$ define a diffeomorphism of $\mathbb{R}^2$. Then $\phi^*\mathcal{M}(c)$ is strongly projectively equivalent to $\mathcal{M}(0)$.

Proof. Let $\alpha(c) := \nabla_p(\partial_{x^1}, \partial_{x^2})^2 \nabla_p(\partial_{x^1}, \partial_{x^2})^3$. Results of [5] show that $\alpha(c)$ is an affine invariant in this setting. One computes $\alpha(\mathcal{M}(c)) = \frac{4c^2}{1 + 2c^2}$. This shows that $\mathcal{M}(c)$ is locally affine equivalent to $\mathcal{M}(\bar{c})$ if and only if $c = \bar{c}$; these geometries are all distinct. Suppose $c \neq \bar{c}$. There is no function $h$ so that $Q(\mathcal{M}(c)) = e^h Q(\mathcal{M}(\bar{c}))$. Consequently, by Theorem 2.1 these geometries are not strongly projectively equivalent. However, we verify that $\phi^*Q(\mathcal{M}(c)) = e^{cx^2}Q(\mathcal{M}(0))$. Consequently, by Theorem 2.1 $\phi^*\mathcal{M}(c)$ is strongly projectively equivalent to $\mathcal{M}(0)$. □ □

The geometry $\mathcal{M}(0)$ is geodesically complete; the geometries $\mathcal{M}(c)$ for $c > 0$ are geodesically incomplete. The following result follows from a direct computation.

Theorem 3.3.

(1) The curves $\sigma_{u,v,a,b}(t) = \begin{cases} (u \cos(bt) + \frac{a}{b} \sin(bt), bt + v) & \text{if } b \neq 0 \\ (at + u, 0) & \text{if } b = 0 \end{cases}$ are geodesics for the geometry $\mathcal{M}(0)$ with initial position $(u,v)$ and initial velocity $(a,b)$. $\mathcal{M}(0)$ is geodesically complete.
(2) For $c > 0$, let $\tau(t) = \log(1 + 2bct)$. Then the curves $\sigma_{u,v,a,b}(t) = \begin{cases} \left( \frac{\sqrt{1+2bct} \left( bu \cos(\frac{\tau(t)}{2c}) + (a - bcu) \sin(\frac{\tau(t)}{2c}) \right)}{b} , v + \frac{\tau(t)}{2c} \right) & \text{if } b \neq 0 \\ \left( at + u, 0 \right) & \text{if } b = 0 \end{cases}$ are the geodesics of $\mathcal{M}(c)$ with initial position $(u,v)$ and initial velocity $(a,b)$. $\mathcal{M}(c)$ is geodesically incomplete since these curves are only defined for the parameter range $1 + 2bct > 0$.

Proof. One verifies directly the curves $\sigma_{u,v,a,b}$ are geodesics with the given initial conditions. Since they are defined for all time if $c = 0$, $\mathcal{M}(0)$ is geodesically complete. On the other hand, they fail to be defined when $1 + 2bct = 0$ and thus $\mathcal{M}(c)$ is geodesically incomplete for $c > 0$. □

We give below a picture of the geodesic structure at the origin $(u,v) = (0,0)$; we emphasize, it makes no difference since the geometry is homogeneous. The geodesics fall into 2 families; those with $b > 0$ all meet at the point $(0, \pi)$; those with $b < 0$ all meet at the point $(0, -\pi)$, and those with $b = 0$ lie along the $x^1$ axis. We also present similar pictures for the geodesics that start at $(u,v) = (1,0)$ and at $(u,v) = (2,0)$.

Figure 1. Geodesic structure
Let \(\phi(x^1, x^2) = (e^{c_{x^2}}x^1, x^2)\). By Theorem 2.1, \(\phi^*\mathcal{M}(c)\) is strongly projectively equivalent to \(\mathcal{M}(0)\). Thus modulo the diffeomorphism \(\phi\), the picture of the geodesic structure for \(\mathcal{M}(c)\) is the same as that of \(\mathcal{M}(0)\).

### 3.1. An affine geometry on the cylinder.

Let \(\Phi(x^1, x^2) = (x^1, x^2 + 2\pi)\) generate a fixed point free action of \(\mathbb{Z}\) on \(\mathbb{R}^2\); this corresponds to taking \(\alpha = 0\), \(\beta = 0\), \(\gamma = 0\), and \(\delta = \pi\) in Theorem 3.1. We divide by this action to define an affine structure on the cylinder \(C := (\mathbb{R} \times S^1, \nabla)\). We verify immediately that if \(c = 0\), then

\[
T(\alpha, \beta, \gamma, \delta)\Phi_k(x^1, x^2) = (e^c x^1 + \beta \cos(x^2 + 2k\pi) + \gamma \sin(x^2 + 2k\pi), x^2 + 2k\pi) = (e^c x^1 + \beta \cos(x^2) + \gamma \sin(x^2), x^2 + 2k\pi)
\]

so \(\Phi\) is in the center of the group \(\mathcal{G}(0)\) and hence \(\mathcal{G}(0)\) extends to a transitive affine action on \(C\); this is a homogeneous geometry. All the geodesics with a nontrivial vertical component close smoothly and where the horizontal geodesic is the alienated geodesic. This is the desired quasi Zoll affine geometry. The exponential map is surjective on \(\mathbb{R}^2\); it is not surjective on \(\mathbb{R}^2\). It is globally affine homogeneous, affine geodesically complete, and affine Killing complete.

### 3.2. An affine geometry on the Möbius strip.

Let \(\Psi(x^1, x^2) := (-x^1, x^2 + \pi)\); this generates a fixed point free action of \(\mathbb{Z}\) on \(\mathbb{R}^2\). Let \(\mathcal{L} := \mathbb{R}^2/\Psi(\mathbb{Z})\) be the quotient; this is the Möbius strip. In a purely formal sense, this corresponds to taking \(\alpha = \pi \sqrt{-1}, \beta = \gamma = 0\), and \(\delta = \pi\) in Theorem 3.1. We compute

\[
T(\alpha, \beta, \gamma, \delta)\Psi(x^1, x^2) = (-e^c x^1 + \beta \cos(x^2 + \pi) + \gamma \sin(x^2 + \pi), x^2 + \pi) = (-e^c x^1 - \beta \cos(x^2) - \gamma \sin(x^2), x^2 + \pi) = \Psi T(\alpha, \beta, \gamma, \delta)(x^1, x^2)
\]

Thus this is a homogeneous affine structure as well and we have a double cover \(\mathbb{Z}_2 \to C \to \mathcal{L}\) on which \(\mathcal{G}(0)\) acts equivariantly. With this structure, the Möbius strip is affine homogeneous, geodesically complete, affine complete, and almost Zoll.

Let \(\tau(t) = \log(1 + 2bct)\). Then

\[
\sigma_{u,v,a,b}(t) = \left\{ \begin{array}{ll}
\frac{1}{1 + 2bct} \left( bu \cos\left(\frac{\tau(t)}{2c}\right) + (a - bcu) \sin\left(\frac{\tau(t)}{2c}\right) \right) & \text{if } b \neq 0 \\
(at + u, 0) & \text{if } b = 0
\end{array} \right.
\]

This geometry is geodesically incomplete; it is only defined for the parameter range \(1 + 2bct > 0\). Still, all geodesics thru the origin either focus vertically above or below the \(x^1\)-axis or are horizontal and the general pattern is the same. A geodesic \(\sigma\) is an alienated geodesic if and only if \(\rho(\sigma, \delta) = 0\). Dividing by \(\mathbb{Z}\) yields an affine quasi Zoll geometry. This geometry is locally homogeneous but not globally homogeneous since \(\Phi\) is not in the center of \(\mathcal{G}(c)\) for \(c \neq 0\) owing to the presence of the exponential factor \(e^{c x^2}\).

Let \(T(\alpha, \delta)(x^1, x^2) = (\alpha x^1, x^2 + \delta)\) define an affine action of \((\mathbb{R} \setminus \{0\}) \times \mathbb{R}\) on \(\mathbb{R}^2\). This action commutes with \(\Psi\); there are 2 \(\mathcal{C}\) orbits – the horizontal axis and everything else. Thus this geometry is “almost” affine homogeneous; the complement of the alienated geodesic thru \((0,0)\) is homogeneous as is the alienated geodesic thru \((0,0)\). We also obtain an almost Zoll geometry on the Möbius strip.

### 3.3. Speed.

We have \(\rho = (1 + c^2)dx^2 \otimes dx^2\). We use \(\rho\) to define a positive semi-definite inner product and let \(\rho(X, X)\) be the “speed”. We suppose \(c \neq 0\). We compute

\[
\rho(\sigma_{u,v,a,b}, \sigma_{u,v,a,b}) = \left\{ \begin{array}{ll}
(1 + c^2)\frac{2b}{(1 + 2bct)^2} & \text{if } b \neq 0 \\
0 & \text{if } b = 0
\end{array} \right.
\]
Thus the alienated geodesics are the null geodesics in the geometry $M(c)$. Although the remaining geodesics all return to the basepoint in the cylinder, the speed increases to $\infty$ as $t \to -\frac{1}{bc}$; the return to the basepoint occurs more and more rapidly.

3.4. The projective tangent bundle. We digress briefly to relate this example to the results of LeBrun and Mason [13]. If $M$ is a smooth manifold, let $\mathbb{P}(M)$ be the projective tangent bundle. If $\nabla$ is an affine structure on $M$, the tangent of lifted geodesics defines a natural foliation on $\mathbb{P}(M)$; the affine structure is said to be *tame* if this structure on $\mathbb{P}(M)$ is locally trivial. LeBrun and Mason showed (see Lemma 2.7 and Lemma 2.8)

**Lemma 3.4.** Suppose that $M$ is an affine tame Zoll manifold.

(1) The universal cover of $M$ with the induced affine structure is tame Zoll.

(2) $M$ is compact and any two points of $M$ can be joined by an affine geodesic.

(3) $M$ has finite fundamental group.

Our examples show that these results fail for almost Zoll structures. Let $\mathbb{P}$ be an almost Zoll surface. Associating to any point of $\mathbb{P}$ the tangent to the alienated geodesic through that point defines a natural section to $\mathbb{P}(M)$; let $\tilde{\mathbb{P}}$ be the complement of this section. We adopt the notation of Section 3.1 to define $C$; the alienated geodesics are the horizontal geodesics; $\sigma_{u,v,a,b}$ is not alienated if and only if $b \neq 0$. Since we are working projectively, we may set $b = 1$. Let $\chi(r, s, t) := \sigma_{r,0,s,1}(t)$ parametrize $\tilde{\mathbb{P}}(C)$; this identifies $\tilde{\mathbb{P}}(C)$ with $\mathbb{R} \times \mathbb{R} \times S^1$ since, of course, $\sigma_{r,0,s,1}(t)$ is periodic with period $2\pi$ in $t$. This shows the foliation of $\mathbb{P}(TM)$ by lifted geodesics is a trivial circle bundle over $\mathbb{R}^2$ and hence is tame. Similarly, if we lift to the universal cover, the foliation of $\mathbb{P}(T\mathbb{R}^2) - S(\mathbb{R}^2)$ by lifted geodesics is a trivial $\mathbb{R}$ bundle over $\mathbb{R}^2$ and hence tame. Clearly, however, the affine structure on $\mathbb{R}^2$ is no longer almost Zoll and we can not join any two points of $\mathbb{R}^2$ by geodesics. And the cylinder does not have finite fundamental group. On the other hand, the cylinder is the oriented double cover of the Möbius strip.

3.5. Global topology. As noted above, the tangent to the alienated geodesic thru any point of an almost Zoll manifold is a section to $\mathbb{P}(TM)$. Consequently, if $M$ is compact, then the Euler-Poincare characteristic of $M$ vanishes. Thus, in particular, the only compact surfaces which could potentially admit an almost Zoll structure are the torus and the Klein bottle. The example we have constructed passes to the cylinder and the Möbius strip; it does not pass to the torus or the Klein bottle. We do not know if these admit an almost Zoll structure but we suspect the answer is no.

4. Effect of the fundamental group

Up to affine equivalence, there is a unique surface with a $Z_3$ symmetry. Let $T \in \text{GL}(2, \mathbb{R})$ satisfy $T^3 = \text{id}$ and $T \neq \text{id}$. Let $e_2 := Te_1$ and set $Te_2 = ae_1 + be_2$ so that

\[
T = \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \quad \text{so} \quad T^3 = \begin{pmatrix} ab & a^2 + b^2a \\ b^2 + a & ab + (b^2 + a)b \end{pmatrix}.
\]

The equation $T^3 = \text{id}$ forces $a = b = -1$ and we have

\[
T = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

\[
T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad T\begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
\]
We suppose \( Q = \text{Span}\{e^x, e^x, e^{-x^1} - e^x\} \). Then \( T^*Q = Q \). Our ansatz tells us this is realized by

\[
\Gamma \left( \frac{b^2 + a - 1, b^2 - b, ba, ba, a^2 - a, b + a^2 - 1}{b + a - 1} \right)
\]

\[
= \Gamma \left( \frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3} \right).
\]

We let \( M := (\mathbb{R}^2 - \{0\}) / \mathbb{Z}_3 \); topologically, this is the cylinder. We give this the inherited affine structure to define \( M \). We then have \( \mathcal{R}(M) = \{0\} \) and \( Q(M) \) is 1-dimensional and may be identified with \( e^x + e^x + e^{-x^1} - e^x \). This is the type \( A \) surface with \( \text{Rank}\{\rho\} = 2 \) with a \( \mathbb{Z}_3 \) symmetry. The Ricci tensor takes the form

\[
\rho = \frac{1}{3} \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}.
\]

This is the cusp point in the moduli space of negative definite Ricci tensors.

Research partially supported by Project MTM2016-75897-P (AEI/FEDER, Spain).

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