Classical $r$-matrices via semidualisation

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Abstract
We study the interplay between double cross sum decompositions of a given Lie algebra and classical $r$-matrices for its semidual. For a class of Lie algebras which can be obtained by a process of generalised complexification we derive an expression for classical $r$-matrices of the semidual Lie bialgebra in terms of the data which determines the decomposition of the original Lie algebra. Applied to the local isometry Lie algebras arising in three-dimensional gravity, decomposition and semidualisation yields the main class of non-trivial $r$-matrices for the Euclidean and Poincaré group in three dimensions. In addition, the construction links the $r$-matrices with the Bianchi classification of three dimensional real Lie algebras.

1 Introduction
In classical geometry and physics, the geometry of a homogeneous space and its isometry group mutually determine each other. Any deformation of either the geometry into a non-commutative version or of the isometry group into a Hopf algebra should preserve as much of this interplay as possible. One way of achieving this, at least at the algebraic level, is to combine the space and its isometry into one algebraic object. In physics terms, such an algebra should combine position coordinates for space with generators of rotations and translations, i.e., with angular momenta and momenta. 

Moments and positions are in duality and rotations act on both. This is manifest in three important subalgebras: the Heisenberg algebra generated by positions and momenta, the isometry algebra generated by angular momenta and momenta and finally the algebra of angular momenta and positions, which one may interpret as an isometry algebra of momentum space. In these general terms, semiduality is a bijection between the isometry algebras of space and momentum space.
A mathematically precise version of these ideas was formulated by Majid in terms of Hopf algebras \([1]\) or, infinitesimally, Lie bialgebras \([2]\), and is summarised in the textbook \([3]\). Majid’s approach is in turn inspired by Born’s proposal of a reciprocity between momenta and positions \([4]\). Here we will use Majid’s framework, working at the level of Lie bialgebras. Thus, non-commutativity of a space is parametrised by Lie brackets of position coordinates and its curvature by their co-commutator. In the dual Lie bialgebra, commutators and co-commutators are swapped so that curvature of space is also captured in the commutators of momenta; similarly curvature of momentum space is captured in the co-commutators of momenta or the commutators of positions.

Curved, homogenous geometries and the associated Lie brackets of isometry algebras are a standard topic in classical geometry and physics. Curved, homogeneous momentum spaces and the associated co-commutators of momenta are less familiar but play a central role in 3d (quantum) gravity (see \([5]\) for a review), in so-called \(\kappa\)-deformations of the Poincaré Lie algebra \([6, 7, 8]\) and more recently in the discussion about relative locality \([9]\). In this paper we exploit our familiarity with curvature and non-commutativity on one hand to enhance our understanding of co-commutativity on the other. We consider a family of Lie algebras, thinking of them as rotation-position algebras or infinitesimal isometries of momentum space, and systematically compute their semidual bialgebras. The semiduals have non-trivial co-commutators which are co-boundary, i.e., given by a classical \(r\)-matrix. We derive a formula for the \(r\)-matrix in terms of the map which characterises the split of the original Lie algebra into rotation generators and positions.

Specialising to three dimensions, the semiduals of our family of Lie bialgebras have the Lie brackets of the 3d Euclidean or Poincaré groups. Lie bialgebra structures on these Lie algebras are necessarily co-boundary and the possible \(r\)-matrices were classified a while ago by Stachura in \([10]\). Denoting angular momenta by \(J_a\) and momenta by \(P_a\), the list found by Stachura includes \(r\)-matrices which only contain rotation or Lorentz generators (these are just the standard \(r\)-matrices for \(sl(2, \mathbb{R})\)), trivial solutions which only contain the commuting momentum generators, and then a more interesting and complicated list of solutions of the mixed form \(r = R^b_a P_a \wedge J_b\). Our approach via semiduality reproduces all these mixed solutions and relates them to the Lie brackets of the original algebra. In particular, we are able to relate these \(r\)-matrices to the Bianchi classification of three-dimensional Lie algebras.

The work reported here is closely related to our previous paper \([11]\) in which we studied the semidual Hopf algebras of the universal enveloping algebras of the isometry Lie algebras arising in 3d gravity. Here we generalise and extend the results of \([11]\) at the infinitesimal, Lie bialgebra level. However, in comparing the current paper with \([11]\) and also with previous work on semiduality in 3d quantum gravity \([12, 13]\) the reader should be aware of a possible confusion between different interpretations of semiduality. Here we interpret semiduality essentially as the exchange of position and momentum generators, as outlined above. In \([13, 11]\), by contrast, pairs of semidual Hopf algebras are both thought of as ‘rotation-momentum’ algebras, but semiduality exchanges the regimes of 3d quantum gravity where they play the role of quantum isometry groups.

The paper is organised as follows. In Sect. 2 we introduce the family of Lie algebras whose direct sum decomposition we would like to study. They are necessarily even-
dimensional and obtained from a given real Lie algebra \( \mathfrak{g} \) by a process of generalised complexification. We parametrise decompositions of Lie algebras in the original family as double cross sums with \( \mathfrak{g} \) as one of the factors in terms of a map \( F : \mathfrak{g} \rightarrow \mathfrak{g} \) and derive a quadratic Lie-algebraic relation for \( F \) which ensures that it does indeed define a double cross sum. The main result of this section is that such maps also characterise classical \( r \)-matrices of the semidual Lie bi-algebra, and that the quadratic relation obtained as factorisation condition is equivalent to the modified classical Yang-Baxter equation for the semidual Lie bi-algebra. In Sect. 3 we specialise to the case where \( \mathfrak{g} \) is the Lie algebra of either the rotation group in Euclidean 3-space or the Lorentz group in three spacetime dimensions. The resulting family of complexified Lie algebras are the local isometry groups of 3d gravity \([5]\), but here we think of them as the Lie algebras of rotations or Lorentz boosts together with position coordinates, so that, in keeping with the general philosophy explained above, their semiduals, which have the Lie algebra structure of the 3d Euclidean or Poincaré Lie algebra, can be interpreted as spacetime symmetries. Using the isomorphism between \( \mathfrak{g} \wedge \mathfrak{g} \) and \( \mathfrak{g} \) in this case, we reformulate the quadratic Lie-algebraic relation for \( F \) as a quadratic equation for the linear map \( F \), and, using rotational or Lorentz symmetry, project out three equations which have to be satisfied independently. Solving these equations in Sect. 4 we recover a family of \( r \)-matrices for the Euclidean and Poincaré Lie algebras first found by Stachura \([10]\), and show that they are in one-to-one correspondence with the types in the Bianchi classification of three-dimensional Lie algebras bar the Heisenberg algebra. Sect. 5 contains a summary of our results in tabular form and a discussion.

2 A general theorem on double cross sums and \( r \)-matrices for their semidual

2.1 Direct sum decompositions of complexified Lie algebras

Consider an arbitrary \( n \)-dimensional real Lie algebra \( \mathfrak{g} \) with generators \( \{J_a\}_{a=1,...,n} \) and brackets
\[
[J_a, J_b] = f^{c}_{ab} J_c. \tag{2.1}
\]
Here and in the following we use the Einstein summation convention. Picking a real number \( \lambda \), one can associate to this Lie algebra the following \( 2n \)-dimensional real Lie algebra \( \mathfrak{g}_\lambda \) with generators \( \{J_a\} \), additional generators \( Q_a, a = 1, \ldots, n \), and brackets of the Cartan form
\[
[J_a, J_b] = f^{c}_{ab} J_c, \quad [Q_a, J_b] = f^{c}_{ab} Q_c, \quad [Q_a, Q_b] = \lambda f^{c}_{ab} J_c. \tag{2.2}
\]
For \( \lambda = -1 \) this is the usual complexification \( \mathfrak{g} \otimes \mathbb{C} \), for \( \lambda = 0 \) it is the semidirect product \( \mathfrak{g} \ltimes \mathbb{R}^n \) and for \( \lambda = 1 \) it is isomorphic to \( \mathfrak{g} \oplus \mathfrak{g} \). One can unify these three cases by considering the Lie algebra \( \mathfrak{g}_\lambda \) as a generalised complexification with a formal parameter \( \theta \) which satisfies \( \theta^2 = -\lambda \) and the identification \( Q_a = \theta J_a \), see \([14]\) and \([15]\). We will not emphasise this viewpoint in the following, but will make use of the linear map \( \theta \) defined via
\[
\theta : \mathfrak{g}_\lambda \mapsto \mathfrak{g}_\lambda, \quad J_a \mapsto Q_a, \quad Q_a \mapsto \lambda J_a \quad \text{for} \quad a = 0, \ldots, n. \tag{2.3}
\]
Lie algebras like (2.2) arise in physics, particularly in three spacetime dimensions. In that context, the generators $J_a$ are rotation- or Lorentz generators, the $Q_a$ are usually interpreted as components of momentum, i.e., as generators of spacetime translations, and the constant $\lambda$ is related to the cosmological constant. In the current context we also think of the $J_a$ as generalisations of rotation or Lorentz generators. However, when we apply semiduality we switch from the generators $Q_a$ to generators of the dual vector space, which we then interpret as momentum space. From that point of view, the generators $Q_a$ should be thought of as position coordinates, and $\lambda$ as a real constant related to the curvature of momentum space $^{13,11}$

We are interested in all decompositions of $g_\lambda$ as double cross sum $g_\lambda = g \rtimes m$ of $g$ and a second $n$-dimensional Lie algebra $m \subset g_\lambda$. We can assume without loss of generality that the generators of $m$ are of the form

$$Q'_a = Q_a + F^b_a J_b, \quad a = 1, \ldots, n,$$

for a real $n \times n$ matrix $F^b_a$. As explained in $^3$, the most general form of the brackets in such a double cross sum is

$$[J_a, J_b] = f_{ab}^c J_c, \quad [Q'_a, J_b] = f_{ab}^c Q'_c + L_{ab}^c J_c, \quad [Q'_a, Q'_b] = g_{ab}^c Q'_c. \quad \text{(2.5)}$$

If the matrix elements $F^b_a$ can be found so that the generators $Q'_a$ close under Lie brackets then we can express the structure constants $L_{ab}^c$ and $g_{ab}^c$ in terms of $F$ and $f_{ab}^c$ as follows:

$$g_{ab}^c = f_{ad}^c F^d_b + F^d_a f_{db}^c, \quad L_{ab}^c = F^d_a f_{db}^c - F^c_d f_{ab}^d. \quad \text{(2.6)}$$

To determine the condition on the generators (2.4) to form a Lie subalgebra we think of $F^b_a$ as the matrix relative to the basis $\{J_a\}_{a=1,\ldots,n}$ of a linear map

$$F : g \to g, \quad X = X^a J_a \mapsto F^b_a X^a J_b, \quad \text{(2.7)}$$

Then we define the map

$$I : g \to g_\lambda, \quad X \mapsto \theta X + F(X), \quad \text{(2.8)}$$

so that the sought-after generators in (2.4) are

$$Q'_a = I(J_a), \quad a = 1, \ldots, n. \quad \text{(2.9)}$$

Using this notation, we show the following.

**Lemma 2.1** The condition on $F : g \to g$ for the generators (2.9) to form a Lie subalgebra of $g_\lambda$ is

$$[F(X), F(Y)] - F([X, F(Y)] + [F(X), Y]) = -\lambda [X, Y] \quad \forall X, Y \in g. \quad \text{(2.10)}$$
Proof In terms of the map (2.8) we need to find the condition under which the commutator of the two elements of the form \(I(X)\) and \(I(Y)\) lies in the image of \(I\). One computes
\[
[I(X), I(Y)] = \lambda [X, Y] + [F(X), F(Y)] + \theta ([X, F(Y)] + [F(X), Y]).
\] (2.11)
For this to be of the form \(I(Z) = \theta Z + F(Z)\) we require
\[
F([X, F(Y)] + [F(X), Y]) = \lambda [X, Y] + [F(X), F(Y)],
\] (2.12)
as claimed. □

Note that, when the map \(F\) solving (2.10) can be found, then the map \(I\) is a bijection \(g \rightarrow m\).

2.2 Semidual Lie bialgebras and their classical r-matrices

The process of semidualisation is defined for Lie bialgebras which are double cross sums, see [3] for a systematic treatment. We are going to apply it to the Lie bialgebras \(g \bowtie m\) with the Lie algebra structure already discussed and trivial co-commutator. In the semidual Lie bialgebra, the Lie algebra \(m\) is replaced by its dual Lie bialgebra \(m^*\). In our case, \(m\) has trivial co-commutators so that the Lie algebra structure of \(m^*\) is abelian. Following the notation of [3], we denote the semidual Lie bialgebra of \(g \bowtie m\) by \(m^* \bowtie g\). In order to give its commutators and co-commutators, we introduce generators \(P^a, a = 1, \ldots, n\) of \(m^*\) which are dual to the generators \(Q'_a\) of \(m\), i.e., they satisfy
\[
P^a(Q'_b) = \delta^a_b. \tag{2.13}
\]
Then, in terms of the generators \(\{J_a, P^b\}_{a,b=1,\ldots,n}\) of the semidual Lie bialgebra, the commutators are
\[
[J_a, J_b] = f_{ab}^c J_c, \quad [J_a, P^b] = -f_{ac}^b P^c, \quad [P^a, P^b] = 0, \tag{2.14}
\]
or
\[
[J_a, J_b] = f_{ab}^c J_c, \quad [P^a, J_b] = f_{bc}^a P^c, \quad [P^a, P^b] = 0.
\]
Applying the general formulae in Sect. 8.3 of [3], the co-commutators in our case come out as
\[
\delta(P^a) = g_{cb}^a P^c \otimes P^b, \quad \delta(J_a) = L_{ba}^c (J_c \otimes P^b - P^b \otimes J_c). \tag{2.15}
\]

Theorem 2.2 For each double cross sum decomposition of the Lie algebra \(g_\lambda\) in the form \(g \bowtie m\) and with generators of \(m\) given in terms of the map \(F : g \rightarrow g\) by (2.9), the semidual Lie bialgebra \(m^* \bowtie g\) has a co-commutator which is co-boundary, with classical r-matrix
\[
r = P^a \wedge F(J_a) = F^b_a P^a \wedge J_b. \tag{2.16}
\]
Proof  Consider the following ansatz for an $r$-matrix of $\mathfrak{m}^* \rightarrow \mathfrak{g}$

$$r = R^b_a P^a \wedge J_b.$$  \hfill (2.17)

For the associated co-commutators one finds

$$\delta(P^a) = (-R^d_b f^a_{dc} + R^d_c f^a_{db}) P^c \otimes P^b$$
$$\delta(J_a) = (-R^d_b f^c_{ad} + R^c_d f^d_{ab}) (J_c \otimes P^b - P^b \otimes J_c)$$  \hfill (2.18)

so that we obtain the relation

$$g^c_{ab} = R^d_b f^c_{ad} + R^d_c f^c_{db}, \quad L^c_{ab} = R^d_a f^c_{db} - R^c_d f^d_{ab}.$$  \hfill (2.19)

Comparing with the expressions for $g^c_{ab}$ and $L^c_{ab}$ given in (2.6) we conclude that we can reproduce them by choosing

$$R^a_b = F^a_b.$$  \hfill (2.20)

Finally, we determine a condition for (2.17) to satisfy the modified classical Yang-Baxter equation

$$[[r, r]] + \lambda \Omega = 0,$$  \hfill (2.21)

where, for now, $\lambda$ is an arbitrary real constant and $\Omega$ is the invariant element

$$\Omega = f^c_{ab} (P^a \otimes P^b \otimes J_c - P^a \otimes J_c \otimes P^b + J_c \otimes P^a \otimes P^b).$$  \hfill (2.22)

Inserting (2.17) into

$$[[r, r]] = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$$  \hfill (2.23)

yields three terms,

$$\left( R^b_a R^c_d f^d_{be} - R^b_e R^c_d f^d_{bd} + R^b_e R^d_a f^d_{be} \right) P^c \otimes P^a \otimes J_c$$  \hfill (2.24)

and similar terms proportional to $P^a \otimes J_c \otimes P^b$, and $J_c \otimes P^a \otimes P^b$. We deduce that the modified Yang-Baxter equation (2.21) is equivalent to

$$R^b_a R^c_d f^d_{be} - R^b_e R^c_d f^d_{bd} + R^b_e R^d_a f^d_{be} + \lambda f^c_{ea} = 0.$$  \hfill (2.25)

However, this is simply the matrix form of the equation (2.10) for the map $F$. Thus, substituting (2.20) into (2.17) we indeed obtain a solution of the modified classical Yang-Baxter equation. □

3  Application to the isometry Lie algebras of 3d gravity

3.1  Special properties in three dimensions

We now turn to solutions of the factorisation condition for isometry Lie algebras arising in 3d gravity. In the following $\mathfrak{g}$ stands for either $so(3)$ or $so(2, 1)$, with generators $J_a$, $a = 0, 1, 2$. This range of indices is unconventional in the Euclidean setting but well-adapted to the more intricate Lorentzian situation. We write $\eta_{ab} = \eta^{ab}$ for either the
Euclidean metric $\text{diag}(1,1,1)$ or the Lorentzian metric $\text{diag}(1,-1,-1)$, and use it lower or raise indices. The Lie brackets of $\mathfrak{g}$ are then
\[ [J_a, J_b] = \epsilon_{abc} J^c, \] (3.1)
where we adopt the convention $\epsilon_{012} = \epsilon^{012} = 1$. The invariant inner product
\[ \langle J_a, J_b \rangle = \eta_{ab} \] (3.2)
on $\mathfrak{g}$ plays an important role in our analysis. In the Lorentzian case, it is sometimes convenient to work with normalised raising and lowering operators
\[ N = \frac{1}{\sqrt{2}} (J_0 + J_2), \quad \tilde{N} = \frac{1}{\sqrt{2}} (J_0 - J_2), \] (3.3)
whose names are chosen to reflect the fact that they are null (or lightlike) with respect to (3.2):
\[ \langle N, N \rangle = \langle \tilde{N}, \tilde{N} \rangle = 0, \quad \langle N, \tilde{N} \rangle = 1. \] (3.4)
Their commutators are
\[ [\tilde{N}, N] = J_1, \quad [J_1, N] = N, \quad [J_1, \tilde{N}] = -\tilde{N}. \] (3.5)

The Lie algebra $\mathfrak{g}_\lambda$ is that of the Poincaré, de Sitter or anti-de Sitter group or their Euclidean analogues in three dimensions, with brackets
\[ [J_a, J_b] = \epsilon_{abc} J^c, \quad [Q_a, J_b] = \epsilon_{abc} Q^c, \quad [Q_a, Q_b] = \lambda \epsilon_{abc} J^c. \] (3.6)

As in the general case, we are looking for basis change
\[ Q'_a = Q_a + F^b_a J_b, \] (3.7)
so that $\{J_0, J_1, J_2, Q'_0, Q'_1, Q'_2\}$ is a basis and $\{Q'_0, Q'_1, Q'_2\}$ closes under Lie brackets, i.e.,
\[ [J_a, J_b] = \epsilon_{ab}^c J_c, \quad [Q'_a, J_b] = \epsilon_{ab}^c Q'_c + L_{ab}^c J_c, \quad [Q'_a, Q'_b] = g_{ab}^c Q'_c. \] (3.8)

In this case, the semidual Lie algebra (2.14) has the brackets
\[ [J_a, J_b] = \epsilon_{ab}^c J_c, \quad [J_a, P^b] = \epsilon_{ab}^c P^c, \quad [P^a, P^b] = 0, \] (3.9)
which are the brackets of the Euclidean Lie algebra in the case of Euclidean signature and those of the Poincaré Lie algebra in the case of Lorentzian signature. Every decomposition of (3.6) according to (3.7) will therefore lead to a co-boundary Lie bialgebra structure on those Lie algebras.

In order to determine all possible solutions of the condition (2.10), we will be making use of the invariant inner product (3.2) on $\mathfrak{g}$. We write $F^t : \mathfrak{g} \to \mathfrak{g}$ for the transpose of a map $F : \mathfrak{g} \to \mathfrak{g}$ relative to $\langle \ , \ \rangle$, i.e., for the map which satisfies
\[ \langle F^t(X), Y \rangle = \langle X, F(Y) \rangle \quad \forall X, Y \in \mathfrak{g}. \] (3.10)
We also need the fundamental identity
\[ \epsilon_{abc}^{efg} = \delta_b^e \delta_c^f - \delta_c^e \delta_b^f, \tag{3.11} \]
which holds in both the Euclidean and Lorentzian context. It implies
\[ \epsilon_{abc}^{afg} = \delta_f^b \delta_g^c - \delta_g^b \delta_f^c, \tag{3.12} \]
which, in turn, is equivalent to
\[ [X, [Y, Z]] = \langle X, Z \rangle Y - \langle X, Y \rangle Z, \quad \forall X, Y, Z \in g. \tag{3.13} \]
This can be used to prove the useful result
\[ \langle [X, Y], V \rangle \langle V, Z \rangle + \langle [Y, Z], V \rangle \langle V, X \rangle + \langle [Z, X], V \rangle \langle V, Y \rangle = \langle V, V \rangle \langle [X, Y], Z \rangle \tag{3.14} \]
for any \( X, Y, Z, V \in g \), see [16] for details and a related identity.

3.2 Reformulating the factorisation condition

The Lie bracket or, equivalently, the epsilon tensor provide an identification of \( g \wedge g \) with \( g \). It follows that for any linear map \( F : g \to g \), the assignment
\[ (X, Y) \in g \wedge g \mapsto [F(X), F(Y)] \in g \tag{3.15} \]
defines a linear map \( g \to g \). Our factorisation condition (2.10) allows for a convenient ‘dual’ formulation in terms of this map, which turns out to be the adjugate of \( F \):

**Lemma 3.1** For every linear map \( F : g \to g \), there is a uniquely determined linear map
\[ F^{\text{adj}} : g \to g, \tag{3.16} \]
which satisfies
\[ \langle F^{\text{adj}}(Z), [X, Y] \rangle = \langle Z, [F(X), F(Y)] \rangle \quad \forall X, Y, Z \in g. \tag{3.17} \]
It is given by
\[ F^{\text{adj}} = F^2 - \text{tr}(F) F + \frac{1}{2} \left( \text{tr}(F) \right)^2 - \text{tr}(F^2) \text{id}, \tag{3.18} \]
which is the adjugate of \( F \).

**Proof** Inserting the basis elements \( J_a, J_b, J_c \) for \( X, Y, Z \) in (3.17), one deduces the matrix relation
\[ (F^{\text{adj}})^f e \epsilon_{fab} = \epsilon_{ecd} F^e_a F^d_b. \tag{3.19} \]
Multiplying with \( \epsilon^{ab} \), summing over \( a, b \) and repeatedly applying (3.11) one arrives at the formula (3.18). To see why (3.18) gives the adjugate of \( F \) (usually defined as the transpose of the matrix of co-factors) note that for any \( n \times n \) matrix \( A \), the characteristic polynomial \( p_A(t) = \det(A - t \text{id}) \) has the constant term \((-1)^n \det A\) so that
\[ q_A(t) = \frac{p_A(t) - (-1)^n \det A}{t} \tag{3.20} \]
is a polynomial of degree $n - 1$ which satisfies

$$q_A(A)A = A q_A(A) = (-1)^{n-1} \det(A) \text{id}, \quad (3.21)$$

by the Cayley-Hamilton Theorem. It is proved in [17] that, in fact,

$$A^{\text{adj}} = q_A(A). \quad (3.22)$$

The expression $(3.18)$ is easily seen to be $q_F(F)$ for $n = 3$ so that the solution of $(3.17)$ is indeed the adjugate of $F$ as claimed. □

Note that for $R \in SO(3)$ or $R \in SO(2,1)$, the invariance of the epsilon tensor implies $R^{\text{adj}} = R^{-1}$. Inserting this into $(3.18)$ is simply the Cayley-Hamilton theorem for the $3 \times 3$ matrix $R$.

**Lemma 3.2** In the case $\mathfrak{g} = \text{so}(3) \text{ or } \mathfrak{g} = \text{so}(2,1)$, the factorisation condition $(2.10)$ for the linear map $F : \mathfrak{g} \rightarrow \mathfrak{g}$ is equivalent to the quadratic relation

$$(F - \text{tr}F \text{id})(F + F^t) + \frac{1}{2} \left((\text{tr}F)^2 - \text{tr}(F^2)\right) \text{id} = -\lambda \text{id} \quad (3.23)$$

**Proof** The factorisation condition $(2.10)$ can equivalently be written as

$$\langle Z, [F(X), F(Y)] - \langle F'(Z), [X, F(Y)] + \langle F(X), Y] \rangle \rangle + \langle [F(X), Y], [X, F(Y)] \rangle = 0 \quad \forall X, Y, Z \in \mathfrak{g}. \quad (3.24)$$

Now observe that

$$[X, F(Y)] + [F(X), Y] = [(\text{id} + F)(X), (\text{id} + F)(Y)] - [F(X), F(Y)] - [X, Y], \quad (3.25)$$

and apply $(3.18)$ both to $F$ and $\text{id} + F$ to deduce $(3.23)$. □

### 3.3 Exploiting rotational invariance

In order to analyse the (equivalent) conditions $(2.10)$ and $(3.23)$ further, we split $F$ into symmetric and antisymmetric parts with respect to $\langle \ , \ \rangle$. The antisymmetric part can be written as the adjoint action of a general element $V \in \mathfrak{g}$ by virtue of the identity

$$\langle [X, Y], Z \rangle + \langle Y, [X, Z] \rangle = 0 \quad \forall X, Y, Z \in \mathfrak{g}. \quad (3.26)$$

Thus we write the map $F$ as

$$F = S + \text{ad}_V. \quad (3.27)$$

for $V \in \mathfrak{g}$ and $S : \mathfrak{g} \rightarrow \mathfrak{g}$ satisfying

$$\langle S(X), Y \rangle = \langle X, S(Y) \rangle \quad \text{for all } X, Y \in \mathfrak{g}. \quad (3.28)$$

Inserting the split $(3.27)$ for $F$ and simplifying using $(3.14)$ and standard identities we deduce

$$\langle Z, [S(X), S(Y)] \rangle - \langle X, [S(Y), S(Z)] \rangle - \langle Y, [S(Z), S(X)] \rangle + \langle 2[V, S(Z)] + (\lambda + \langle V, V \rangle)Z, [X, Y] \rangle = 0 \quad \forall X, Y, Z \in \mathfrak{g}. \quad (3.29)$$
which is equivalent to
\[
[S(X), S(Y)] - S([X, S(Y)] + [S(X), Y])
+ 2((V, X)S(Y) - (V, Y)S(X)) = -(\lambda + \langle V, V \rangle)[X, Y] \quad \forall X, Y \in \mathfrak{g}.
\] (3.30)

This turns out to be a very useful formulation of the factorisation condition for the Lie algebras (3.6).

We can derive an equivalent condition to (3.30) by inserting the split (3.27) into (3.23) and using \( F + F^t = 2S \). A short calculation gives
\[
2S^2 - 2\text{tr}(S)S + \frac{1}{2}((\text{tr}(S))^2 - \text{tr}(S^2))\text{id} + 2\text{ad}_V S = -(\lambda + \langle V, V \rangle)\text{id}.
\] (3.31)

This can also be derived directly from (3.30) using the methods of Sect. 3.2. The equation (3.31) is invariant under \( SO(3) \) or \( SO(2, 1) \) conjugation, and this can be used to split it into irreducible components under this action, namely into symmetric traceless, antisymmetric and scalar matrices. Noting that the commutator \([\text{ad}_V, S]\) is symmetric but that the anticommutator \(\{\text{ad}_V, S\} = \text{ad}_V S + S \text{ad}_V\) is antisymmetric we write
\[
2\text{ad}_V S = [\text{ad}_V, S] + \{\text{ad}_V, S\}
\] (3.32)
and deduce three equations. For the scalar part we have
\[
\frac{1}{6}((\text{tr}S)^2 - \text{tr}(S^2)) = \lambda + \langle V, V \rangle,
\] (3.33)
for the vector part we find
\[
\{\text{ad}_V, S\} = 0
\] (3.34)
and for the symmetric, traceless part we deduce
\[
[\text{ad}_V, S] + 2\left(S^2 - \frac{1}{3}\text{tr}(S^2)\text{id}\right) - 2\left(\text{tr}S S - \frac{1}{3}(\text{tr}S)^2\text{id}\right) = 0.
\] (3.35)

The equations (3.33), (3.34) and (3.35) are derived in a different form and by a different method in [10], where they are used for determining the most general \( r \)-matrix for the Poincaré and Euclidean Lie algebras. Adding (3.34) and (3.35) we deduce the relation
\[
\text{ad}_V S + \left(S^2 - \frac{1}{3}\text{tr}(S^2)\text{id}\right) - \left(\text{tr}S S - \frac{1}{3}(\text{tr}S)^2\text{id}\right) = 0.
\] (3.36)

This has a various useful consequences, including the following lemma which is also implicit in some of the manipulations in the appendix of [10].

**Lemma 3.3** If the map \( F : \mathfrak{g} \rightarrow \mathfrak{g} \) solves the factorisation condition (3.23) and \( S \) and \( \text{ad}_V \) are its symmetric and antisymmetric part as in (3.27), then the following hold:

1. The antisymmetric part \( \text{ad}_V \) is the zero map on any subspace of \( \mathfrak{h} \subset \mathfrak{g} \) where the restriction \( S|_\mathfrak{h} \) is invertible. In particular, if \( S \) is invertible then \( V \) vanishes.
(2) If \( \text{dim ker } S = 1 \) then: \( \text{ker } S \) is not a null space \( \Rightarrow V = 0 \).

In the following, the contrapositive of result (2) will be particularly useful: if \( \text{ker } S \) is one-dimensional and \( V \neq 0 \) then \( \text{ker } S \) is necessarily a null space.

**Proof**

(1) On any subspace \( \mathfrak{h} \) where \( S_\mathfrak{h} \) is invertible we can multiply (3.36) by the (symmetric) map \( S_\mathfrak{h}^{-1} \) to obtain an expression for the antisymmetric map \( \text{ad}_V \) in terms of a symmetric map. Thus \( (\text{ad}_V)|_\mathfrak{h} = 0 \).

(2) It follows from (3.34) that \( \text{ker } S \) is invariant under the action of \( \text{ad}_V \). If \( \text{ker } S \) is one-dimensional then any basis vector \( X \) of it is necessarily an eigenvector of \( \text{ad}_V \), i.e., \([V, X] = \mu X\). Since \( \langle [V, X], X \rangle = 0 \), we deduce that \( \mu = 0 \) if \( X \) is not null. However, in that case, \( S \) restricted to the orthogonal complement of \( X \) is invertible, so we know from part (1) that \( \text{ad}_V \) vanishes on the orthogonal complement of \( X \). Since we already have \( \text{ad}_V(X) = 0 \) we deduce that \( V = 0 \). \( \square \)

4 Solving the factorisation condition

4.1 Solutions of definite symmetry

Before we systematically study solution of the equation (3.30) in the next sections, we note some special cases which can be found by inspection. Setting the symmetric part \( S \) to zero, the condition (3.30) reduces to a condition on the Lie algebra element \( V \). Expanding

\[
V = -v^a J_a, \tag{4.1}
\]

where the sign is chosen to match conventions in [11], we have

\[
\langle V, V \rangle = v^a v_a = -\lambda. \tag{4.2}
\]

The resulting bialgebra structure on the semidual Lie algebra has the \( r \)-matrix

\[
r_\kappa = v^c \epsilon^b_{ac} P^a \wedge J_b, \tag{4.3}
\]

which is the familiar 3d \( \kappa \)-Poincaré structure with deformation parameter \( v = (v^0, v^1, v^2) \), which may be timelike, spacelike or timelike, depending on \( \lambda \) [18, 19, 20]. The resulting Lie algebra structure of \( \mathfrak{m} \) is a semidirect sum \( \mathbb{R} \ltimes \mathbb{R}^2 \), with \( \mathbb{R} \) acting on \( \mathbb{R}^2 \) by scaling. We will recover this solution as part of a more general family in the next section.

Similarly, setting the antisymmetric part \( \text{ad}_V \) of the map \( F \) to zero, we deduce from equations (3.35) and (3.33) that

\[
S^2 - \text{tr}(S) S = -2\lambda \text{id} \quad \text{for} \quad V = 0 \tag{4.4}
\]

At the same time, we know from the Cayley Hamilton theorem that

\[
S^3 - \text{tr}(S) S^2 + \frac{1}{2} \left( (\text{tr}(S))^2 - \text{tr}(S^2) \right) S - \text{det}(S) \text{id} = 0. \tag{4.5}
\]

Thus, from (3.33) with \( V = 0 \)

\[
(S^2 - \text{tr}(S) S + 3\lambda \text{id}) S - \text{det}(S) \text{id} = 0. \tag{4.6}
\]
Combining with (4.4) we deduce

\[ \lambda S = \det S \text{id} \quad \text{for } V = 0. \quad (4.7) \]

It follows that for \( \lambda > 0 \) any purely symmetric solutions is diagonal and given by

\[ F = \sqrt{\lambda} \text{id.} \quad (4.8) \]

The resulting classical \( r \)-matrix

\[ r_{\text{double}} = \sqrt{\lambda} P^a \wedge J_a \quad (4.9) \]

is that of the classical double of \( so(3) \) or \( so(2, 1) \). The associated Lie brackets on \( \mathfrak{m} \) are

\[ [Q'_a, Q'_b] = 2\sqrt{\lambda} \epsilon_{abc} Q'_c, \quad (4.10) \]

i.e., \( \mathfrak{m} \) is \( so(3) \) or \( so(2, 1) \) in this case.

When \( \lambda = 0 \) we have another symmetric solution that can be found by inspection (but which we will recover systematically later). For any vector \( m \), the liner map with matrix

\[ F^b_a = m^b m_a. \quad (4.11) \]

trivially satisfies (3.30) (for \( \lambda = 0 \), so that

\[ r_{m} = m_a m^b P^a \wedge J_b \quad (4.12) \]

is a classical \( r \)-matrix for the Euclidean or the Poincaré Lie algebras. The Lie algebra structure of \( \mathfrak{m} \) turns out to be a semidirect sum \( \mathbb{R} \ltimes \mathbb{R}^2 \) with \( \mathbb{R} \) acting on \( \mathbb{R}^2 \) by rotations, Lorentz boosts or the nilpotent matrix

\[ M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (4.13) \]

depending on whether \( \mathfrak{m} \) is timelike, spacelike or lightlike. We will study this solution, too, as part of a larger family in the next sections and give more details there.

4.2 The standard case

In the simplest case, the symmetric part \( S \) has a diagonalising orthonormal basis \( K_a \), \( a = 0, 1, 2 \), of \( \mathfrak{g} \) with real eigenvalues \( \lambda_a, a = 0, 1, 2 \). The diagonalisation is always possible in the Euclidean case but cannot always be achieved in the Lorentzian case. We will consider the other, non-standard cases below. Since the diagonalising basis is related to the original basis \( J_a \), \( a = 0, 1, 2 \) by an orthogonal (i.e., \( SO(3) \) or \( SO(2, 1) \)) transformation, we know that the commutators are still

\[ [K_a, K_b] = \epsilon_{abc} K^c. \quad (4.14) \]

The map \( S \) in (3.27) then takes the from

\[ S = \sum_{a=0}^{2} \lambda_a K_a \langle K_a, \cdot \rangle. \quad (4.15) \]
In order to find all solutions of this form, we consider two cases.

(I) $\dim \ker S \leq 1$. In that case, it follows from Lemma 3.3 that $V = 0$: if $S$ is invertible this is a consequence of part (1) while for a one-dimensional kernel of $S$ the vanishing of $S$ follows from part (2) since $\ker S$ cannot be a null space in the diagonalisable case under consideration. With $V = 0$ we insert the diagonal form of $S$ into (4.14) to find

$$
\begin{align*}
\lambda_0(\lambda_1 + \lambda_2) &= 2\lambda, \\
\lambda_1(\lambda_0 + \lambda_2) &= 2\lambda, \\
\lambda_2(\lambda_0 + \lambda_1) &= 2\lambda.
\end{align*}
$$

(4.16)

Taking pairwise differences we deduce

$$\lambda_0 = \lambda_1 = \lambda_2 = \sqrt{\lambda}.
$$

(4.17)

This is the solution already found by inspection. Since the eigenvalues $\lambda_a$ are assumed to be real, this is only a valid and non-trivial solution if $\lambda > 0$.

(II) $\dim \ker S \geq 2$. In that case, $S^2 = \text{tr}(S)S$, so that the scalar equation (3.33) gives the condition

$$\langle V, V \rangle = -\lambda
$$

(4.18)

for $V$, and the matrix condition (3.35) reduces to

$$[\text{ad}_V, S] = 0.
$$

(4.19)

If $S = 0$ then this imposes no further restriction of $V$ and we recover the purely anti-symmetric solution $F = \text{ad}_V$ found by inspection in the previous section. However, if $\dim \ker S = 2$ so that $S = \lambda_a K_a\langle K_a, \cdot \rangle$ for some fixed $a$ and $\lambda_a \neq 0$, we see that we can now choose $V$ to be a multiple of $K_a$, with the multiple chosen so that the normalisation condition (4.2) holds. When $\lambda \neq 0$, the resulting solution can be written

$$F = \beta V \langle V, \cdot \rangle + \text{ad}_V,
$$

(4.20)

with $\beta \in \mathbb{R}$ arbitrary and $V$ satisfying (4.18). When $\lambda = 0$, the requirement that $S = \lambda_a K_a\langle K_a, \cdot \rangle$ (no sum) and $\text{ad}_V$ commute enforces $V = 0$. In that case we recover the solution (4.11), at least for space- or timelike vectors $m$. The case of light-like $m$ represents a non-diagonalisable map $S$ and will appear in Sect. 4.3.

In the remainder of this subsection we study the solution (4.20) in more detail. With the convention (4.1), the generators of the subalgebra $m$ in this case are

$$Q'_a = Q_a + \epsilon_{abc}v^b J^c + \beta(v^b J_b) v_a,
$$

(4.21)

where $\beta$ is an arbitrary (real) parameter. Using 3-vector notation $v = (v^0, v^1, v^2)$ etc. we can also write

$$Q' = Q + v \times J + \beta(v \cdot J) v.
$$

(4.22)

Then, with $v \cdot w = v^a w_a$ etc., the brackets are

$$
\begin{align*}
[p \cdot Q', q \cdot Q'] &= -v \times (p \times q) \cdot (Q' - \beta v \times Q') \\
&= (v \cdot p)(q \cdot Q') - (v \cdot q)(p \cdot Q') - \beta(p \times q) \cdot v \cdot (v \cdot Q') - \lambda \beta(p \times q) \cdot Q'.
\end{align*}
$$

(4.23)
as well as
\[ [p \cdot J, q \cdot Q'] = p \times q \cdot Q + p \times (q \times v) \cdot J + \beta(q \cdot v) p \times v \cdot J, \] (4.24)
which can be written in terms of the generators \(Q'_a\) as
\[ [p \cdot J, q \cdot Q'] = p \times q \cdot Q' + (v \times p - \beta v \times (v \times p)) \cdot q \times J. \]
\[ = p \times q \cdot Q' + (v \cdot q)(p \cdot J) - (p \cdot q)(v \cdot J) - \beta (p \cdot v)(v \times q \cdot J) - \lambda \beta (p \times q \cdot J). \] (4.25)

To identify the resulting Lie algebra structure of \(m\), we consider the various cases.

### 4.2.1 The Euclidean case with \(\lambda < 0\)

We pick \(v = \sqrt{-\lambda}(1, 0, 0)\), so that (4.22) gives
\[ Q'_0 = Q_0 - \beta \lambda J_0, \quad Q'_1 = Q_1 - \sqrt{-\lambda} J_2, \quad Q'_2 = Q_2 + \sqrt{-\lambda} J_1. \] (4.26)

Then (4.23) yields the following brackets for \(m\):
\[ [Q'_1, Q'_2] = 0, \]
\[ [Q'_0, Q'_1] = \sqrt{-\lambda} Q'_1 - \beta \lambda Q'_2, \]
\[ [Q'_0, Q'_2] = \sqrt{-\lambda} Q'_2 + \beta \lambda Q'_1. \] (4.27)

Thus \(Q'_1\) and \(Q'_2\) span a commutative subalgebra, and \(Q'_0\) acts on this by infinitesimal scaling and rotation. In terms of the Bianchi classification of three-dimensional real Lie algebras, this is type VII.

The commutators involving \(J_0\) and \(Q'_b\) are
\[ [J_0, Q'_1] = Q'_2, \quad [J_0, Q'_2] = -Q'_1, \]
\[ [J_1, Q'_0] = \sqrt{-\lambda} J_1 + \beta \lambda J_2 - Q'_2, \quad [J_2, Q'_0] = \sqrt{-\lambda} J_2 - \beta \lambda J_1 + Q'_1, \]
\[ [J_1, Q'_2] = Q'_0 - \beta \lambda J_0, \quad [J_2, Q'_1] = -Q'_0 + \beta \lambda J_0, \]
\[ [J_0, Q'_0] = 0, \quad [J_1, Q'_1] = [J_2, Q'_2] = \sqrt{-\lambda} J_0, \] (4.28)

and again allow for a geometric interpretation. The action of \(J_0\) on the vector \(Q'_1\) is infinitesimal rotation around the \(a\)-th axis. The action of \(Q'_0\) on the \(J_1J_2\)-plane is infinitesimal scaling and rotation, as on the \(Q'_1Q'_2\)-plane above. The action of both \(Q'_1\) and \(Q'_2\) on the \(J_1J_2\)-plane is to map it onto \(J_0\), which is consistent with the fact that \(Q'_1\) and \(Q'_2\) commute.

### 4.2.2 Lorentzian case with \(\lambda < 0\)

If \(v\) is the timelike vector \(v = \sqrt{-\lambda}(1, 0, 0)\), then the generators (4.21) of \(m\) are
\[ Q'_0 = Q_0 - \beta \lambda J_0, \quad Q'_1 = Q_1 + \sqrt{-\lambda} J_2, \quad Q'_2 = Q_2 - \sqrt{-\lambda} J_1, \] (4.29)
so that from (4.23) we obtain
\[
\begin{align*}
[Q'_0, Q'_2] &= 0, \\
[Q'_0, Q'_1] &= \sqrt{-\lambda} Q'_1 + \beta \lambda Q'_2, \\
[Q'_0, Q'_2] &= -\sqrt{-\lambda} Q'_0 - \beta \lambda Q'_0.
\end{align*}
\] (4.30)

Again we have a two-dimensional abelian algebra spanned by \( Q'_1, Q'_2 \), with \( Q'_0 \) acting by infinitesimal rotation and scaling. The Bianchi type is again VII. Computing the mixed commutators is straightforward and leads to a geometric interpretation analogous to the Euclidean case, but we omit the details here.

### 4.2.3 Lorentzian case with \( \lambda > 0 \)

With the spacelike vector \( v = \sqrt{\lambda}(0, 1, 0) \), we have from (4.21) that
\[
\begin{align*}
Q'_0 &= Q_0 - \sqrt{\lambda} J_2, \\
Q'_1 &= Q_1 - \beta \lambda J_1, \\
Q'_2 &= Q_2 - \sqrt{\lambda} J_0,
\end{align*}
\] (4.31)

and therefore (4.23) gives
\[
\begin{align*}
[Q'_0, Q'_2] &= 0, \\
[Q'_1, Q'_0] &= -\sqrt{-\lambda} Q'_0 - \beta \lambda Q'_2, \\
[Q'_1, Q'_2] &= -\sqrt{-\lambda} Q'_0 - \beta \lambda Q'_2.
\end{align*}
\] (4.32)

It is illuminating to write the commutators in this case in terms of the generators \( N, \bar{N}, J_1 \) defined in (3.3) and the corresponding generators
\[
\begin{align*}
Q_N &= \frac{1}{\sqrt{2}} (Q_0 + Q_2), \\
Q_{\bar{N}} &= \frac{1}{\sqrt{2}} (Q_0 - Q_2).
\end{align*}
\] (4.33)

Then
\[
\begin{align*}
Q'_N &= Q_N - \sqrt{\lambda} N, \\
Q'_{\bar{N}} &= Q_{\bar{N}} + \sqrt{\lambda} \bar{N}, \\
Q'_1 &= Q_1 - \beta \lambda J_1,
\end{align*}
\] (4.34)

and therefore (4.32) gives
\[
\begin{align*}
[Q'_N, Q'_{\bar{N}}] &= 0, \\
[Q'_1, Q'_N] &= -(\beta \lambda + \sqrt{\lambda}) Q'_N, \\
[Q'_1, Q'_{\bar{N}}] &= (\beta \lambda - \sqrt{\lambda}) Q'_{\bar{N}}.
\end{align*}
\] (4.35)

This is again a semidirect sum \( \mathbb{R} \ltimes \mathbb{R}^2 \) and Bianchi type VI: \( Q'_N \) and \( Q'_{\bar{N}} \) span a two-dimensional abelian algebra, and \( Q'_1 \) acts on them with a matrix that has (generically distinct) real eigenvalues.

### 4.3 The non-standard case

In the Lorentzian case, maps \( S : \mathfrak{g} \rightarrow \mathfrak{g} \) which are symmetric with respect to the non-degenerate symmetric form \( \langle \cdot, \cdot \rangle \) cannot always be diagonalised. We now consider the different normal forms (which are discussed, for example, in the textbook \[21\]). In each case, the solution of the factorisation condition (3.30) or, equivalently, Eq. (3.31) uses Lemma 3.3 and standard arguments from linear algebra; some proceed along similar lines to the discussion in the appendix of \[10\].
4.3.1 Small Jordan Block

In this case the symmetric map $S$ has a lightlike and a spacelike eigenvector and can be brought into the form

$$S = a(N\langle \tilde{N}, \cdot \rangle + \tilde{N}\langle N, \cdot \rangle) + bN\langle N, \cdot \rangle - cJ_1\langle J_1, \cdot \rangle, \quad b \neq 0,$$

so that

$$S(N) = aN, \quad S(\tilde{N}) = bN + a\tilde{N}, \quad S(J_1) = cJ_1.$$  \hspace{1cm} (4.36)

It is easy to check that, if $a \neq 0$ and $c \neq 0$ (so that $S$ is invertible) we have $V = 0$ by Lemma 3.3 and hence the diagonal solution (4.8). If $a \neq 0$ then $S$ has a one-dimensional kernel which is not null, so by Lemma 3.3 we again deduce $V = 0$ and obtain a solution of the form (4.11). If $a = 0$ but $c \neq 0$ we have, from (3.33), that $\langle V, V \rangle = -\lambda$. The requirement that $ad_V$ leave the kernel of $S$ invariant implies that $V$ is spacelike or lightlike, i.e., $\lambda \geq 0$. When $\lambda > 0$, we can use (3.36) to deduce that $c = \sqrt{\lambda}$. Renaming $b$ as $\beta$ we have the new solution

$$F = \beta N\langle N, \cdot \rangle - \sqrt{\lambda}J_1\langle J_1, \cdot \rangle + \sqrt{\lambda} \text{ad}_{J_1}, \quad \lambda > 0, \quad \beta \in \mathbb{R},$$

so that

$$Q'_N = Q_N + \sqrt{\lambda}N, \quad Q'_\tilde{N} = Q_\tilde{N} + \beta N - \sqrt{\lambda}\tilde{N}, \quad Q'_1 = Q_1 + \sqrt{\lambda}J_1.$$ \hspace{1cm} (4.37)

The commutators of $m$ are

$$[Q'_N, Q'_N] = 0, \quad [Q'_1, Q'_N] = 2\sqrt{\lambda}Q'_N,$$

$$Q'_1, \beta Q'_N - 2\sqrt{\lambda}Q'_\tilde{N} = 0.$$ \hspace{1cm} (4.40)

This is Bianchi type III, which is isomorphic to the direct sum of the one dimensional Lie algebra $\mathbb{R}$ (generated here by $\beta Q'_N - 2\sqrt{\lambda}Q'_\tilde{N}$) and the non-abelian two-dimensional Lie algebra $L(2)$ (generated here by $Q'_N, Q'_\tilde{N}$).

When $\lambda = 0$, we again use (3.36) to obtain the solution

$$F = \beta N\langle N, \cdot \rangle + \text{ad}_N,$$

for an arbitrary $\beta \in \mathbb{R}$. This may be viewed as a limit as $\lambda \to 0$ of (4.38) and is also a lightlike version of (4.20). The generators of $m$ are

$$Q'_N = Q_N, \quad Q'_\tilde{N} = Q_\tilde{N} - J_1 + \beta N, \quad Q'_1 = Q_1 - N,$$

and the commutators of $m$ are

$$[Q'_N, Q'_1] = 0, \quad [Q'_N, Q'_\tilde{N}] = -Q'_N,$$

$$[Q'_\tilde{N}, Q'_1] = -Q'_1 - \beta Q'_\tilde{N}.$$ \hspace{1cm} (4.43)

which are again those of a semidirect sum $\mathbb{R} \ltimes \mathbb{R}^2$. When $\beta = 0$, this is Bianchi type V, with $-Q'_\tilde{N}$ acting on the span of $Q'_N, Q'_1$ via the identity. When $\beta \neq 0$, it is Bianchi type IV, with $-\frac{1}{\beta}Q'_\tilde{N}$ acting on the span of $Q'_N, Q'_1$ via $\text{id}/\beta + M$, where $M$ is the nilpotent linear map with matrix (4.13).
### 4.3.2 Large Jordan Block

The symmetric map $S$ only has a single, lightlike eigenvector and can be brought into the form

$$S = a(N\langle \tilde{N}, \cdot \rangle + \tilde{N}\langle N, \cdot \rangle - J_1\langle J_1, \cdot \rangle) + b(N\langle J_1, \cdot \rangle + J_1\langle N, \cdot \rangle), \quad b \neq 0$$

so that

$$S(N) = aN, \quad S(\tilde{N}) = a\tilde{N} + bJ_1, \quad S(J_1) = -bN + aJ_1.$$  \hspace{1cm} (4.45)

Again we require $a = 0$ for a non-vanishing $V$. Then $\ker S$ is one-dimensional and spanned by the lightlike generator $N$; this space is invariant under $\text{ad}_V$ provided $V$ is a linear combination of $N$ and $J_1$. However, since $\text{tr} S = \text{tr} S^2 = 0$ we can use (3.36) to deduce that $\text{ad}_V S + S^2 = 0$, which in turn implies that $V = -bN$. Thus we are in the case $\lambda = 0$ and, after re-naming $b = \beta/2$, we have the solution

$$F = \frac{\beta}{2} (N\langle J_1, \cdot \rangle + J_1\langle N, \cdot \rangle - \text{ad}_N),$$

which can be written more compactly as

$$F = \beta J_1\langle N, \cdot \rangle.$$  \hspace{1cm} (4.47)

The resulting basis of $\mathfrak{m}$ is

$$Q'_N = P_N, \quad Q'_{\tilde{N}} = Q'_{\tilde{N}} + \beta J_1, \quad Q'_1 = Q_1,$$  \hspace{1cm} (4.48)

and we obtain the commutators

$$[Q'_1, Q'_N] = 0, \quad [Q'_1, Q'_{\tilde{N}}] = 0, \quad [Q'_{\tilde{N}}, Q'_N] = \beta Q'_N.$$  \hspace{1cm} (4.49)

This is again a direct sum of $\mathbb{R}$ (spanned by $Q'_1$) and the two-dimensional non-abelian Lie algebra, i.e., Bianchi type III.

The remaining normal form of a symmetric map $S : \mathfrak{g} \to \mathfrak{g}$ is

$$S = a(N\langle \tilde{N}, \cdot \rangle + \tilde{N}\langle N, \cdot \rangle) + b(\tilde{N}\langle \tilde{N}, \cdot \rangle - N\langle N, \cdot \rangle) - cJ_1\langle J_1, \cdot \rangle$$  \hspace{1cm} (4.50)

which has the form of a ‘rotation’ in the span of $\{N, \tilde{N}\}$:

$$S(N) = aN + b\tilde{N}, \quad S(\tilde{N}) = -bN + a\tilde{N}, \quad S(J_1) = cJ_1.$$  \hspace{1cm} (4.51)

In this case we do not obtain a new solution: the cases $\dim \ker S \leq 1$ can be dealt with as above. However, $\dim \ker S = 2$ requires $a = b = 0$, in which case we recover a solution in the family (4.20).
5 Summary and Discussion

We can unify a large family of solutions of the factorisation condition (3.23) in the form

\[ F = \beta V \langle V, \cdot \rangle + \alpha \text{ad}_V, \quad \beta \in \mathbb{R}, \quad \alpha \in \{0, 1\}, \quad \alpha \langle V, V \rangle = -\lambda. \quad (5.1) \]

This reduces to the purely symmetric solutions (4.11) when \( \alpha = 0 \) and to the family (4.20), including the lightlike case (4.41), when \( \alpha = 1 \). This factorisation map gives rise to the \( r \)-matrices

\[ \tilde{r}_\kappa = (\beta \epsilon^a v_b + \alpha v^c \epsilon^b_{\text{ac}}) P^a \wedge J_b, \quad (5.2) \]

which generalise the familiar \( \kappa \)-Poincaré \( r \)-matrix (4.3); we therefore call them generalised \( \kappa \)-Poincaré solutions.

Figure 1: The \( \mathbb{R} \)-actions in the semidirect sums \( \mathbb{R} \ltimes \mathbb{R}^2 \) and their degenerate limit \( \mathbb{R} \oplus L(2) \); explanation in the main text

The Lie algebra structure of \( \mathfrak{m} \) arising from the family (5.1) is of the semidirect sum form \( \mathbb{R} \ltimes \mathbb{R}^2 \) for all values of the parameters. The \( \mathbb{R} \)-action on \( \mathbb{R}^2 \) has two components, corresponding to the parameters \( \alpha \) and \( \beta \) being non-zero. The \( \alpha \)-component (for \( \alpha = 1 \) and \( \beta = 0 \)) is simple overall scaling, shown in the bottom left of Fig. 1 and associated with the standard \( \kappa \)-Poincaré algebra. The \( \beta \)-component (for \( \beta \neq 0 \) and \( \alpha = 0 \)) is more interesting and depends on the value of \( \lambda \). Its matrix is proportional to one of the \( 2 \times 2 \) matrices \( \rho(\theta) \) representing the relation \( \theta^2 = \lambda \):

\[ \rho(\theta) = \begin{cases} 
\begin{pmatrix} 0 & \sqrt{-\lambda} \\ -\sqrt{-\lambda} & 0 \end{pmatrix} & \text{if } \lambda < 0 \\
\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{if } \lambda = 0 \\
\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & -\sqrt{\lambda} \end{pmatrix} & \text{if } \lambda > 0 
\end{cases} \quad (5.3) \]
The flows generated by exponentiating these matrices are shown from left to right in the top row of Fig. 1.

In addition to the family \( (5.1) \) we found the solution \( F = \sqrt{\lambda} \text{id} \), which gives rise to the \( r \)-matrix of the classical double \( (4.9) \) and \( m = so(3) \) or \( m = so(2,1) \). Finally, we have the exceptional, Jordan-type solutions \( (4.38) \) and \( (4.47) \). With the notation \( P_N = (P_0 + P_2)/\sqrt{2} \), the \( r \)-matrix associated to the factorisation map \( (4.38) \) is

\[
    r_{SJ} = \beta P_N \wedge N + \sqrt{\lambda}(P_1 \wedge J_1 + \epsilon^{b}_{\ a1} P^a \wedge J_b), \quad (5.4)
\]

while the \( r \)-matrix associated to the solution \( (4.47) \) is simply

\[
    r_{LJ} = \beta P_N \wedge J_1. \quad (5.5)
\]

The associated Lie algebra structure of \( m \) is \( \mathbb{R} \oplus L(2) \), which may be viewed as a degenerate case of the semidirect sums \( \mathbb{R} \ltimes \mathbb{R}^2 \) with a diagonalisable \( \mathbb{R} \)-action but one eigenvalue vanishing. The eigenvector for the non-vanishing eigenvalue is always the lightlike vector \( N \), but the eigenvector for the zero eigenvalue varies between the two cases and as a function of the parameter \( \beta \). The flow obtained by exponentiating such matrices is shown for a generic case in the bottom right of Fig. 1.

| \( F \) | \( r \)-matrix | \( m \) | Bianchi type |
|--------|----------------|-------|-------------|
| 0 \( \sqrt{\lambda} \text{id} \) | \( r_{\text{double}} \) \( r_\kappa \) \( r_{SJ} \) and \( r_{LJ} \) | \( \mathbb{R}^3 \) \( so(2,1) \) or \( so(3) \) \( \mathbb{R} \oplus L(2) \) | I VIII or IX IV-VII III |

Table 1: Factorisations and associated \( r \)-matrices

Table 1 summarises our results and discussion of the factorisation of the Lie algebras \( (3.6) \) and the associated \( r \)-matrices of the semiduals \( (3.9) \). As reviewed in the Introduction, semidualisation gives all \( r \)-matrices of the form \( R^b_a P^a \wedge J_b \) in this case. It is interesting that for each type of \( r \)-matrices there is an associated Bianchi type for the Lie algebra \( m \). All Bianchi types except type II (the Heisenberg algebra) arise in this way.

Viewing the generators \( Q^a \) of the original algebra \( (3.8) \) as spacetime coordinates and the dual generators \( P_a \) in \( (3.9) \) as momenta, the summary in Table 1 may thus also be viewed as a list of non-commutative spaces and associated non-co-commutative momentum.
spaces, parametrised by the relevant $r$-matrices. We thus obtain the unified picture of non-commutative geometries and isometry algebras promised in the Introduction.

There are several questions and topics for further research which follow from the results reported here. Mathematically, one would like to understand more generally when semidualising families of Lie algebras with a given double cross sum decomposition provides an effective way of finding $r$-matrices for the semidual Lie bialgebra. It would also be interesting to apply the same technique to families of Lie bialgebras with non-trivial co-commutators, so that the semidual has a more complicated Lie algebra structure than the semidirect sums found here.

In the physics literature, the standard $\kappa$-Poincaré (with a timelike deformation parameter) and the classical double Lie bialgebras in Table 1 have been much studied, the latter (but not the former - see [10]) being related to 3d quantum gravity. It would be interesting to see if the generalised $\kappa$-Poincaré $r$-matrices (5.2) with $\beta \neq 0$ and the Jordan-type $r$-matrices (5.4) and (5.5) also have applications in real or toy models of mathematical physics.

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