Bosonic interactions with a domain wall

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Abstract We consider here the interaction of scalar bosons with a topological domain wall. Not only is there a continuum of scattering states, but there is also an interesting “quasi-discretuum” of positive energy bosonic bound states, describing bosons entrapped within the wall’s core. The full spectrum of the scattering and bound state energies and eigenstates is obtainable from a Schrödinger-type of equation with a Pöschl-Teller potential. We also consider the presence of a boson gas within the wall and high energy boson emission.

Keywords domain wall · topological soliton · interacting scalar fields

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1 Introduction

A simple planar topological domain wall emerges as a solitonic solution of a scalar field equation possessing a quartic potential with a broken $Z_2$ symmetry (see, e.g., [1]-[5]). In this familiar type of model the domain wall scalar field $\chi$ interpolates between the disconnected vacuum states located by $\chi = \pm \eta$, and the energy density of the wall is localized near the wall’s core, where $\chi \to 0$. Other particles, both fermions and bosons, can interact with the wall in various ways (see [1]-[4] for example). In particular, the scattering of scalar bosons from such a wall has been examined in [6]. We re-examine this type of scalar field interaction with more of a focus upon bound states of a complex scalar field $\phi$ interacting with the domain wall field $\chi$.

Other models of interacting scalar fields have demonstrated how it is possible for entrapped bosons to stabilize or metastabilize nontopological solitons formed from domain wall bubbles (see, e.g., [5],[7]). However here, our focus, at least partly, is on the mathematical description of a set of exact, analytical eigenstates and energy spectra associated with $\phi$ boson scattering states and $\phi$ boson bound states. Using a simple model for the interacting scalars, and a simple ansatz for the field $\phi$, we obtain a Schrödinger-type of equation with a sech$^2$ Pöschl-Teller potential, which has known solutions. These solutions can be incorporated to obtain a full description of the $\phi$ scattering and bound states. Certain values of the potential strength allow for reflectionless scattering. However, the bound state spectrum is especially interesting in that there is a discretuum of boson energies associated with motion transverse to the wall, but for each discrete energy there is a continuum as-
sociated with unconstrained motion parallel to the wall.

2 Interacting scalar fields

Our model of interacting scalar fields is basically the same as that presented in [6]. (See also [7] for a similar type of model with quartic self interactions present.) The lagrangian for our system of interacting scalars consisting of a real scalar $\chi$ and a complex scalar $\phi$ is taken to be

$$\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + \partial_{\mu} \phi^{*} \partial^{\mu} \phi - V(\chi, |\phi|)$$

(1)

where the potential is

$$V = \frac{1}{4} \lambda (\chi^2 - \eta^2)^2 + f(\chi^2 - \eta^2) \phi^{*} \phi + m^2 \phi^{*} \phi$$

(2)

The parameters $\lambda$, $f$, $\eta$, and $m$ are all taken to be positive real-valued constants. The vacuum states of the theory are given by $\chi = \pm \eta$ and $\phi = 0$. The $\chi$ particle and $\phi$ particle masses are given, respectively, by $m_{\chi} = \sqrt{2 \lambda \eta}$ and $m_{\phi} = m$. The field equations for the system that follow from $\mathcal{L}$ are

$$\Box \phi + \left[ \lambda (\chi^2 - \eta^2)^2 + f(\chi^2 - \eta^2) \right] \phi = 0$$

(3a)

$$\Box \chi = 0$$

(3b)

where $\Box = \partial_{\mu} \partial^{\mu} - \nabla^2$. This system admits the exact solution set $\phi = 0$ and

$$\chi(x) = \pm \eta \tanh \bar{x} = \pm \eta \tanh (\Omega x),$$

$$\Omega = \eta \sqrt{\frac{2}{\lambda}}$$

(4)

where we have defined the dimensionless variable $\bar{x} = \Omega x$, where $\Omega$ is the inverse of the wall’s width $w$. The solution for $\chi$ is that describing a static domain wall (for, say, $+\eta$) or anti-wall (for, say, $-\eta$) centered on the $y$-$z$ plane at $x = 0$ (see, e.g., [1], [2], [3]). The width $w$ of the wall is

$$w = \Omega^{-1} = \frac{\eta}{\sqrt{\lambda}} = 2m_{\chi}^{-1}$$

(5)

We next consider the behavior of the $\phi$ field in the domain wall background, as considered in previous works (e.g., [2], [3], [4]). We note that the $\chi$-$\phi$ coupling term is proportional to $f(\chi^2 - \eta^2) = -f\eta^2 \text{sech}^2 \bar{x}$. Using this, we rewrite the $\phi$ field equation (3a) as

$$\partial_{\mu}^2 \phi - \nabla^2 \phi + m^2 \phi + U(x) \phi = 0$$

(6)

with

$$U(x) = -f\eta^2 \text{sech}^2 \bar{x}$$

(7)

Adopting the plane wave ansatz [6]

$$\phi(x, y, z, t) = \psi(x) \exp[-i(\omega t - \kappa_y y - \kappa_z z)]$$

(8)

with $\omega$, $\kappa_y$, $\kappa_z$ being real-valued parameters, (6) reduces to

$$\frac{d^2 \psi}{dx^2} - U(x) \psi + [\omega^2 - (\kappa^2 + m^2)]\psi = 0$$

(9)

where we define $\kappa = \kappa_y^2 + \kappa_z^2$. The ansatz describes a $\phi$ boson freely propagating in the $y$ and $z$ directions, but subject to a force in the $x$ direction, and having total energy $\omega$. In the limit $U \to 0$ we have a free boson in three dimensions with the usual dispersion relation $\omega^2 = p^2 + m^2$ where $p = (p_x, p_y, p_z)$ is the relativistic momentum and $\omega$ is the energy. Let us define

$$K^2 \equiv \omega^2 - (\kappa^2 + m^2)$$

(10)

The quantity $\sqrt{\kappa^2 + m^2}$ can be considered to be the portion of the boson energy associated with the two dimensional translational motion in the $y$ and $z$ directions. With (7) and (9), (10) becomes

$$\frac{d^2 \psi}{dx^2} + \left[ K^2 + \omega^2 \text{sech}^2 \bar{x} \right] \psi = 0$$

(11)

This is just a Schrödinger equation for the eigenfunction $\psi$ with a Pöschl-Teller potential (see, e.g., [8], [9] and references therein)

$$U(x) = -\Omega^2 \nu(\nu + 1) \text{sech}^2 \bar{x}$$

(12)

where $\nu$ is a real-valued positive constant, not necessarily an integer. Comparison of (7) and (12) implies that

$$\nu(\nu + 1) = \frac{f\eta^2}{\Omega^2} = \frac{2f}{\lambda}$$

(13)

where [5] has been used. The depth of the potential well depends upon $\nu$, or equivalently, $\sqrt{\nu}/\Omega$. 

Exact, analytical sets of scattering eigenstates and bound eigenstates and the associated spectral values are known, and are elucidated clearly by Lekner in [9], whose results we adopt here. In particular, [9] studies the solutions of the Schrödinger equation

$$\frac{d^2\psi}{dx^2} + [K^2 + \Omega^2 \nu(\nu + 1) \text{sech}^2 x] \psi = 0$$

(14)

which coincides with (11) using the identification (13). Following [9], we define an energy parameter $\mathcal{E} = K^2/2M$ which would correspond to the energy of a nonrelativistic particle of mass $M$ described by an eigenfunction $\psi$ of (14). We can use the spectral values of $K^2$ to obtain the $\phi$ boson energy spectral values for $\omega$ by using (10). Scattering states for $\psi$, and hence scattering states for the $\phi$ bosons, form a continuum with $K^2 > 0$, and bound states of $\psi$ are described by $K^2 = 2M\mathcal{E} < 0$, and by the relation (10) will describe the states of $\phi$ bosons trapped in the domain wall with energy $\omega(\kappa) < \sqrt{\kappa^2 + m^2}$.

3 Scattering states

The scattering states are the positive energy states with $K^2 > 0$. By (10) these are states with $\omega^2 > m^2 + \kappa^2$ and bosons with this energy can exist outside the domain wall and are not confined by it, but, in general, scatter from it. The eigenstates are given by two independent solutions of (14) forming even and odd functions of $x$ and are described by hypergeometric functions (see, e.g., [9] and references therein)

$$\psi_+(x) = (\cosh x)^{\nu+1} F (\alpha; \beta; \frac{1}{2} - \sin^2 x)$$

(15a)

$$\psi_-(x) = (\cosh x)^{\nu+1} \sinh x \times F (\alpha + \frac{1}{2}; \beta + \frac{1}{2} - \sin^2 x)$$

(15b)

where

$$\alpha = \frac{1}{2} (\nu + 1 + i\frac{\kappa}{m}),$$

$$\beta = \frac{1}{2} (\nu + 1 - i\frac{\kappa}{m})$$

(16)

and a representation for $F$ is given by

$$F(\alpha; \beta; \gamma; \zeta; \xi) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \times$$

$$\times \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + n)\Gamma(\beta + n)}{\Gamma(\gamma + n)} \frac{\xi^n}{n!}$$

(17)

(We ignore the possible scattering of $\chi$ particles from the $\chi$ wall, as the reflection coefficient is zero for this case [6].) Now, the scattering of $\phi$ bosons from the domain wall becomes reflectionless, for any energy, for the case that $\nu = \text{positive integer}$ [9]. This can also be seen from the analysis of boson scattering from a domain wall [9], where there is a similar $\chi$-$\phi$ interaction, which by using the same plane wave ansatz, leads to the same Schrödinger equation, provided that we identify the coupling constant $\lambda$ of [6] with our coupling $f$ [10]. The quoted reflection coefficient is

$$R = \frac{\cos^2 \theta}{\sinh^2 \varphi + \cos^2 \theta}$$

(18)

where (in our notation, with our parameters) $\varphi = \sqrt{2/\lambda} (\pi \kappa/\eta)$ and $\theta = (\pi/2) \sqrt{1 + 8f/\lambda}$. Using (13) we have $\sqrt{1 + 8f/\lambda} = \sqrt{4\nu(\nu + 1) + 1}$. The reflection coefficient vanishes for $\nu = \text{positive integer}$, where $\cos \theta = 0$.

4 Bound states

Bound states of the Schrödinger equation (14) occur when $K^2 = 2M\mathcal{E} < 0$. In this case the energy spectrum for $\mathcal{E}$ is given by [9]

$$\mathcal{E}_n = -\frac{\Omega^2 (\nu - n)^2}{2M}, \quad n = 0, 1, 2, ..., [\nu]$$

(19)

where $[\nu]$ is the integer part of $\nu$ ($\nu$ need not be integer). These negative energy eigenstates can be obtained from (15) and (16) by replacing $K$ by $Q = iK$ [9]. The identification $K^2_n = 2M\mathcal{E}_n$ gives $\omega_n^2 = (\kappa^2 + m^2)$ or

$$\omega_n^2(\kappa) = -\Omega^2 (\nu - n)^2 + (\kappa^2 + m^2)$$

(20)

The energy spectrum given by the set $\{\omega_n(\kappa)\}$ depends upon the discrete set of integers $n$, as well as upon the continuous parameter $\kappa \geq 0$. We might refer to this as a “quasi-discretuum”. We note the following:

$$\omega_n^2(\kappa) \geq -\Omega^2 (\nu - n)^2 + m^2$$

(21a)

$$\min[\omega_n^2(\kappa)] \geq 0$$

(21b)
since $\omega_n(\kappa)$ is assumed to be real-valued. (Otherwise the $\phi$ particles are not stable, as is assumed to be the case.) The above, however, requires that $\nu - n \leq m/\Omega$, i.e.,

$$n \geq \nu - m/\Omega \quad \text{and} \quad n \geq 0$$

(22)

Therefore, there exists a minimal value for the integer $n$ with $n_{\text{min}} = 0$ if $\nu \leq m/\Omega$ and $n_{\text{min}} = \text{smallest integer} \geq \nu - m/\Omega$ for $\nu \geq m/\Omega$ (a “ceiling function” value).

We now envision a set $\{\omega_n(\kappa)\}$ with energies given in [20] with $n = n_{\text{min}}, n_{\text{min}} + 1, n_{\text{min}} + 2, \ldots, [\nu]$ and $\kappa \in [0, \infty)$. The minimal value of $\omega_n(\kappa)$, i.e., $\min[\omega_n(\kappa)]$, increases with increasing $n$, so that the different $n$ discrete energy levels overlap. Note that for $\nu = \text{positive integer}$ then for $n = \nu$ we have $\min[\omega_n(\kappa)] = m$, and we have a barely free $\phi$ boson at rest.

Consider now a gas of $\phi$ bosons at a temperature $T = \beta^{-1}$. The gas particles collide and interact, and we expect that the ensemble average boson momentum to be zero, i.e., $\langle p \rangle = 0$, but we have $\langle \kappa \rangle \geq 0$. For a gas of $\phi$ bosons with a large ensemble of particle energies, we now abandon the monochromatic bosonic beam ansatz of [5] in favor of a quantum statistical description. We take the chemical potential $\mu$ of the boson gas to be negligible ($\beta\mu \ll 1$), and using boson gas statistics we have that the number of particles with energy $\omega$ is given by $N(\omega) = (e^{\beta\omega} - 1)^{-1}$. Now, the ideal boson gas can allow sufficiently high energy particles to escape the wall to $|x| = \infty$ with a minimal energy $\omega_\infty = m$. Therefore gas particles within the wall with average energies $\omega \geq m$ have the possibility of eventually escaping the wall. If the gas temperature is $T \ll m$, i.e., $\beta m \gg 1$, then the number of particles $N(\omega \geq m)$ is small and the rate of boson leakage is low. However, for high temperature $T \gg m$, i.e., $\beta m \ll 1$, $N(\omega \geq m)$ is large and the rate of $\phi$ boson leakage from the wall is high. The total number density of relativistic $\phi$ particles is $\frac{4}{3}$ $n \sim T^3$, and the total energy density of relativistic particles is $\frac{4}{3} \rho_\phi \sim T^4$. So for a planar domain wall inhabited by $\phi$ bosons, we expect escaping bosons to result in a drop of $n$ and $\rho_\phi$, and consequently a drop in $T$. A drop in temperature $T$ then reduces the rate of $\phi$ boson emission from the wall.

The model considered here is presented at a classical level, i.e., the domain wall solution follows from the tree-level potential $\phi$ for $\phi = 0$. These tree-level results can be used for an analysis of one-loop quantum corrections, which will result in a shift of the surface tension of the domain wall. (See, for example [11] and [12]).

5 Summary

A tree-level model describing two interacting scalar fields has been studied, wherein one of the scalar fields ($\chi$) exhibits a $Z_2$ discrete symmetry. Consequently, the field equations admit a solution set where the real field $\chi$ forms a topological planar domain wall of width $w = \Omega^{-1}$ centered on the $y$-$z$ plane, while the second field ($\phi$) assumes its vacuum value of $\phi = 0$. We then consider excitations of the $\phi$ field, i.e., $\phi$ boson particles, in the domain wall background. Upon adopting a modulated plane wave type ansatz with $\phi = \psi(x) \exp[-i(\omega t - \kappa_\rho y - \kappa_z z)]$, with $\omega$ and $\kappa = (\kappa_\rho, \kappa_z)$ being real-valued parameters, a Schrödinger equation with a Pöschl-Teller potential $U(x) = -f\eta^2\sech^2\Omega x$ is obtained for the eigenfunction $\psi(x)$. The eigenvalue $K^2$ for the Schrödinger equation can be written in terms of boson energy $\omega$ and momentum parameter $\kappa = |\kappa|$.

Solutions describing scattering states of the $\phi$ bosons from the domain wall are presented. A comparison made with the work in [6] verifies that the reflection coefficient $R$ vanishes for positive integers $\nu$ where we have the identification $\nu(\nu + 1) = \Omega^{-2}f\eta^2 = 2f/\lambda$.

Next, the bound state energy spectrum for $\phi$ bosons trapped within the domain wall is found. This spectrum is described by a discrete set of nonnegative integers $n$, along with a set formed by the continuous parameter $\kappa$. We have therefore referred to this spectrum formed by the set $\{\omega_n(\kappa)\}$ as a “quasi-discretuum”, since it is described by a discrete integer $n$, as well as by a continuous parameter $\kappa$.  


Since there can be many $\phi$ bosons in each discrete energy level, then one can consider the existence of a $\phi$ boson gas at some temperature $T = 1/\beta$ trapped within the domain wall. We suggest, in a way similar to that along the lines of [7], that sufficiently energetic particles with $\omega \geq m$ can escape the wall. The leakage rate, however, depends upon the gas temperature $T$, with a low rate of $\phi$ boson emission for $T \ll m$ and a high rate of emission at $T \gg m$.

References

1. A. Vilenkin, Phys. Rep. 121, 263 (1985)
2. A. Vilenkin and E.P.S. Shellard, Cosmic Strings and Other Topological Defects (Cambridge University Press, 1994)
3. See, for example, R. Rajaraman, Solitons and Instantons (North-Holland Publishing Co., 1982)
4. See, for example, E.W. Kolb and M.S. Turner, The Early Universe (Addison-Wesley, 1990)
5. J.A. Frieman, G.B. Gelmini, M. Gleiser, E.W. Kolb, “Solitogenesis: Primordial Origin of Non-topological Solitons”, Phys. Rev. Lett. 60 (1988) 2101
6. See, for example, section 13.4 of Ref. [2].
7. J.R. Morris, “Stability of a class of neutral vacuum bubbles”, Phys. Rev. D87 (2013) 8, 085022 (6 pp.) {e-Print: arXiv:1304.4560 [hep-th]}
8. L. D. Landau and E. M. Lifshitz, Quantum Mechanics, (Pergamon, Oxford, 1965 , 2nd ed.), Secs. 23 and 25.
9. J. Lekner, “Reflectionless eigenstates of the sech$^2$ potential”, Am. J. Phys. 75, 1151-1157 (2007)
10. In [6] the interaction between a wall field and a boson field in the wall background is written in the form $\mathcal{L}_{\text{int}} = -\frac{1}{4}\phi^2 \chi^2$, where, in [6], $\phi$ represents the wall field and $\chi$ represents the interacting boson field. With the change of variables $\lambda \to f$, $\phi \to \chi$, and $\frac{1}{2} \chi^2 \to \phi^* \phi$, then $\mathcal{L}_{\text{int}}$ of [6] maps into that of our Eq. (2) and the equations for a complex $\chi$ in [6] maps into those of our Eqs. (9) - (10).