Levitation of Current Carrying States in the Lattice Model for the Integer Quantum Hall Effect

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The disorder driven quantum Hall to insulator transition is investigated for a two-dimensional lattice model. The Hall conductivity and the localization length are calculated numerically near the transition. For uncorrelated and weakly correlated disorder potentials the current carrying states are annihilated by the negative Chern states originating from the band center. In the presence of correlated disorder potentials with correlation length larger than approximately half the lattice constant the floating up of the critical states in energy without merging is observed. This behavior is similar to the levitation scenario proposed for the continuum model.

Soon after the discovery of the integer quantum Hall effect (QHE) the fate of the current carrying states had been of much concern. At zero temperature and magnetic field $B = 0$ a disordered two-dimensional electron gas should behave as an Anderson insulator. Therefore the question arose, how the transition from the QHE state to the insulating phase takes place when $B$ goes to zero. Alternatively, for fixed magnetic field one can also consider the increase of the disorder and the broadening of the Landau bands. In the presence of disorder, both Khmelnitzkii [1] and Laughlin [2] argued that the current carrying states that reside at the center of the Landau bands cannot disappear discontinuously as $B \to 0$, but rather float up in energy above the Fermi level. These general quasi-classical considerations which are based on scaling theory and supported by numerical results [3] temporarily settled the case. Furthermore, the floating up of the critical energies to infinity became an essential cornerstone in constructing a global phase diagram (GPD) [4] of the QHE. The levitation picture also implies that in the disorder–magnetic field plane only such transitions between QHE-states are possible at which the change of the quantized Hall conductivity, $\Delta \sigma_{xy}$, is $\pm e^2/h$. Therefore, the final transition to the insulator must occur always from the lowest ($\nu = 1$) Hall plateau.

Recently, the appealing levitation scenario has been questioned both experimentally and theoretically and the debate about the fate of the current carrying states and the appearance of the phase diagram has been restarted [5,6]. One reason for this development is probably the fact that despite considerable efforts [7] the microscopic origin of the levitation remains still unclear. Although transitions to the insulator from $\nu = 1$, as expected from the GPD, and the floating up of the current carrying states have been observed in some experiments [5,6,7], direct transitions from higher Hall plateaus ($\nu > 2$) which according to the GPD should not be possible have also been reported [4,8,9]. On the numerical side, the observation of direct QHE to insulator transitions with $\Delta \sigma_{xy} > e^2/h$ in tight-binding lattice models and the absence of floating states [4,8,9] have led these authors to conclude that due to lattice effects which are always present in real systems the conventional floating up model which neglects the periodic background potential is inappropriate for explaining the experimental results. In addition, a new universality class has been assigned to direct transitions from the QHE-liquid to the insulating phase where at the critical point the conductivities are found to be equal, $\sigma_{xy}^{\nu} = \sigma_{xy}^{\nu - 1}$ [10,11].

In fact, the studies presented in [4,8,9] showed that with increasing disorder the critical states do not float up in energy, but instead, the states with opposite Chern numbers from the center of the tight-binding band move downwards and finally annihilate the current carrying states below the Fermi energy. In lattice systems the extended states can be characterized by a topological nonzero Chern integer [12,16]. Even with next nearest neighbor hopping, which breaks the particle-hole symmetry, no floating up of states across the Landau gap, but only the moving outward of states with opposite Chern numbers, away from the center, has been found [20]. This holds also for general magnetic fields commensurate with the lattice size, which are given by a rational number of flux quanta per plaquette, $B = p/q \cdot h/(ea^2)$ with $p \neq 1$, that lead to Chern numbers for sub-bands which are $\neq \pm 1$ [20]. Here $p$ and $q$ are coprime integers, and $a$ is the lattice constant.

Although successful in explaining the experimental observation of direct transitions to the insulator from higher Hall plateaus, the lattice model investigated in [4,8,9] possesses a serious drawback. If magnetic field and lattice effects are weak as in the experimental situation in GaAs heterostructures an effective mass approximation should be appropriate and allow for the floating up results of the continuum model. In the latter, states with negative Chern numbers are absent so that the proposed annihilation cannot occur.

The aim of our paper is to resolve this controversy and to reconcile the predictions for the continuum model with the results of the lattice model. The key for achieving this goal is to refrain from unphysical realizations of the disorder potentials usually considered in this context. We will show that if the assumption of independent random site energies is abandoned and long range correlations of the disorder potentials are allowed as it is the case in the experiments [21], the floating up of critical states across the Landau gap is also seen in the lattice model.

We consider a single-band tight-binding Hamiltonian describing the properties of non-interacting particles in a disor-
dered two-dimensional system in the presence of a perpendicular magnetic field,

$$H = \sum_k w_k c_k^\dagger c_k + \sum_{\langle kl\rangle} V(e^{ib_{kl}} c_k^\dagger c_l + e^{-ib_{kl}} c_l^\dagger c_k),$$

(1)

where $c_k$ is the fermionic operator on lattice site $k$. The magnetic field enters the transfer terms connecting nearest neighbors via the phases which in the Landau gauge read $b_{kl} = \pm 2\pi (p/q) (\vec{r}_k \cdot \vec{e}_y) / a$, if $\vec{r}_l = \vec{r}_k \pm a \vec{e}_x$, and $b_{kl} = 0$ else, where $\vec{r}_k$ is the position of site $k$ and $\vec{e}_x$, $\vec{e}_y$ are unit vectors pointing in the $x$ and $y$ directions. $W$ is the disorder strength of the correlated random potentials $w_k \in [-W,W]$. We choose $V = 1$ and $p/q = 1/8$ so that the tight-binding band splits into $8$ sub-bands.

The behaviour of the current carrying states is investigated through a numerical calculation of the localization length $\lambda_M$

$$\lambda_M^{-1}(E,W) = -\lim_{L \to \infty} \frac{1}{2L} \ln \text{Tr}[G^+_{LL}],$$

(2)

and the Hall conductivity

$$\sigma_{xy}(E,W;M) = -\lim_{\varepsilon \to 0^+} \lim_{L \to \infty} \frac{e^2}{h} \frac{2}{LM} \times \text{Tr}\left\{ \sum_{n} i\varepsilon (G_{nn}^+ - G_{nn}^-) x_n y_n - 2 \sum_{n,n'} \varepsilon^2 G_{nn'}^+ y_n G_{n'n}^- x_n \right\}.$$

(3)

The $G_{m,n}^\pm$ are $M \times M$ sub-matrices of the advanced and retarded Green functions, $G^\pm = (E - H \pm i\varepsilon)^{-1}$, acting in the subspace of the columns $m$, $n$ on the lattice. The $x_n$ and $y_n$ are $M \times M$ matrices with elements $(x_n)_{ij} = n\delta_{ij}$ and $(y_n)_{ij} = j\delta_{ij}$. Periodic boundary conditions are applied in the $y$-direction for the calculation of $\lambda_M$ while $\sigma_{xy}$ is obtained for Dirichlet boundary conditions. The length (x-direction) and width (y-direction) are denoted by $L$ and $M$, respectively, and $\varepsilon$ takes care of the proper thermodynamic limit.

The task is accomplished by means of a recursive Green function method developed previously [24,23]. This procedure is well suited for calculating very long disordered 2d systems of finite width without getting into trouble with storage-space problems. Therefore, also the correlated on-site disorder potentials $w_k$ have to be generated iteratively site by site. This is implemented in analogy to the successive creation of Ising spin chains. In this case the algorithm can be based on the Gibbs representation of Markov random fields [24]. The correlated random number $\mu_{x_k,y_k}$ at a given site $k$ with coordinates $x_k,y_k$ is calculated with the help of two neighboring numbers already determined, $\mu_{x_k,y_k} = (\mu_{x_k-a,y_k} + \mu_{x_k+a,y_k-a}) / 2 + 2 \varepsilon^{-C} \gamma$. The uncorrelated random numbers $\gamma$ are drawn from an interval $[-1,1]$ with constant probability density. If $|\mu_{x_k,y_k}| > 1$, the value is reflected back into the unit interval by $\mu_{x_k,y_k} \to -2 - \mu_{x_k,y_k}$. The correlation strength is tuned via the correlation parameter $C$, where $C = 0$ corresponds to the uncorrelated case. The correlated random numbers $\mu$ generated in this way are multiplied by the disorder strength $W$ which results in the set of disorder potentials $w_k \in [-W,W]$ used in Eq. (1).

An example is shown in Fig. 1 where gray-scale plots for arrays ($128 \times 128$) of correlated disorder potentials with $C = 0.0, 0.5, 1.0,$ and $1.5$ are shown in comparison with the uncorrelated case $C = 0.0$ (upper left). White spots correspond to disorder potentials $w_k \simeq W$ whereas black ones to $w_k \simeq -W$. To better characterize the disorder potentials, we have calculated the correlation function $K(\rho) = \langle w_k w_l \rangle_{\rho=|\vec{r}_k-\vec{r}_l|}$ averaged over all pairs of disorder potentials at sites which are a given distance $|\vec{r}_k - \vec{r}_l|$ apart. We find an exponential relation, $K(\rho) \sim \exp(-\rho/\eta(C))$, where the spatial decay of the correlations is governed by the correlation length $\eta(C)$ which for example amounts to $\eta(C)/a = 0.8, 1.1,$ and $1.9$ for correlation parameters, $C = 0.5, 0.7$, and $1.0$, respectively. Thus, for $C < 0.7$ the correlation length of the disorder potentials is less than the magnetic length, $l_B/a = 1.1284$ for $p/q = 1/8$, while it is larger for $C \geq 0.7$. Our data follow nicely the relation $\eta(C) = 1/2 |\ln(\tanh(C))|^{-1}$ in analogy to Ising spin chains.

First, to trace the QHE to insulator transition we calculate the Hall conductivity $\sigma_{xy}(W)$ at energy $E/V = -1.5$, which corresponds to a filling factor $\nu \approx 2$, as a function of disorder $W$. The result for various correlation parameters is shown in Fig. 2. Smooth transitions from $\sigma_{xy}$ for correlated disorder potentials $C = 0.5, 1.0,$ and $1.5$ are shown in comparison with the uncorrelated case $C = 0.0$ (upper left). White spots correspond to disorder potentials $w_k \simeq W$ whereas black ones to $w_k \simeq -W$. Similar direct transitions have been reported also for larger filling factors in the case of uncorrelated disorder potentials [14,17,18]. This observation and the assignment of the critical point to the disorder where $\sigma_{xy} = 0.5\nu e^2/h = \sigma_{xx}$ have led the authors of [13] to conclude that for uncorrelated
disorder potentials this transition corresponds to a new universality class. Although the transition region gets broader with increasing correlations, no indication of a Hall plateau at \( \sigma_{xy}/(e^2/h) = 1 \) can be observed at first sight. From this point of view our results of the Hall conductivity for correlated disorder potentials seem qualitatively to be the same as for the uncorrelated case.

![FIG. 2. The Hall conductivity versus disorder at \( E = -1.5 \) (filling factor \( \nu \approx 2 \)). The correlation parameter of the disorder potentials are \( C = 0 \) (○), \( C = 0.5 \) (□), \( C = 1.0 \) (△), and \( C = 1.5 \) (◇). The system length is \( L/a = 5 \cdot 10^2 \) and the width is \( M/a = 128 \) (\( C = 0.0 \)), \( M/a = 160 \) (\( C = 0.5, 1.0 \)), and \( M/a = 256 \) (\( C = 1.5 \)).](image)

Next, to find out the reason for the broadening of the transition region in the correlated disorder case, we have calculated the disorder dependence of the localization length. The position of the current carrying states can be extracted from the finite size scaling of the normalized localization length \( \lambda_M(E,W)/M \). With increasing system width \( M \), \( \lambda_M/M \) goes to zero for localized states while it converges to a finite value for critical states. The results for \( \nu = 2 \) and \( C = 1.0 \) are shown in Fig. 3 where the averaged \( \lambda_M/M \) is displayed versus disorder for different system widths in the range \( 32 \leq M/a \leq 160 \). The averaging is performed over 3 to 5 realizations of the correlated disorder potentials and, for the calculation of \( \lambda_M \), \( L \) is always larger than \( 5 \cdot 10^5 \) lattice spacings \( a \). With increasing system width \( M \) two distinct peaks, which correspond to two separate divergences of the localization length at about \( W_c^{(1)} \approx 1.8 \) and \( W_c^{(2)} \approx 3.0 \), emerge out of the broad decay of the Hall conductivity shown in Fig. 2.[4]

In fact, a closer look at the Hall conductivity curves of Fig. 2 suggests that there is, hardly noticeable, a tiny shoulder near \( \sigma_{xy} = e^2/h \) for correlation parameters \( C = 1.0 \) and \( C = 1.5 \). Taking the derivative \( d\sigma_{xy}(W)/dW \) approximately by \( |\Delta \sigma_{xy}(W)/\Delta W| \), one sees that the peak corresponding to maximal steepness is not situated at \( \sigma_{xy} = e^2/h \), which should be the case for a direct transition, but instead near \( \sigma_{xy}/(e^2/h) \approx 1.5 \), and a second somewhat weaker hump at about 0.5 can also be detected. This clearly means that there is no new single direct transition, but two distinct usual transitions which, however, are not easy to discern in finite size studies. This difficulty of resolving the quantized Hall steps applies also to the experiments in addition to temperature and inelastic scattering effects which may mask the transitions.

![FIG. 3. The normalized localization length versus disorder at filling factor \( \nu = 2 \). The correlation parameter is \( C = 1.0 \) and the system sizes are \( M/a = 32 \) (○), \( M/a = 64 \) (□), \( M/a = 96 \) (◇), and \( M/a = 160 \) (△).](image)

To see the details of the QH to insulator transitions more clearly the energy and disorder dependence of \( \lambda_M(E,W)/M \) has been calculated in the filling factor range \( 0 \lesssim \nu \lesssim 2.5 \). Depending on the value of the correlation parameter we find the surprising result that with increasing disorder the current carrying states are either annihilated by the anti-Chern states or do float up in energy as seen in the continuum model. The essence of our extensive numerical calculations is represented in Fig. 3 which shows the location of the two lowest critical states within the disorder–energy plane which has been extracted from the finite size scaling of \( \lambda_M/M \).

We start with looking at the results for the correlation parameter \( C = 0.2 \). With increasing disorder \( W \) the downward movement of the anti-Chern states (depicted by ○) coming from the band center can be tracked. The last current carrying state, i.e. that from the lowest Landau band (marked by ○) which moves slightly to smaller energies, disappears at a disorder of about \( W/V \approx 2.5 \) when both Chern and anti-Chern states touch. This behavior is similar to the uncorrelated case (\( C = 0 \), Chern and anti-Chern states are labeled by □ and ▼, respectively) where the last current carrying state vanishes at about \( W/V \approx 2.85 \), also shown in Fig. 2, but hardly to discern from the \( C = 0.2 \) data in case of the Chern state.

On the other hand, the floating of the energy \( E_c \) of the critical states across the Landau levels without merging is seen for both \( C = 0.5 \) and \( C = 1.0 \). Here, the states move to higher energies when the disorder is increased. For a given energy that corresponds to a filling factor in the range \( 1.5 < \nu < 2.5 \), the separation between the two critical disorders, which mark the current carrying states floating up in energy, is enlarged by enhancing the long range correlations of the disorder. The floating up is slower, i.e. a smaller shift in energy is caused
by the same increase in disorder, for larger correlation length. For $C = 0.5$, the states are closer and float up in energy within a small range of disorder values. These data are near a special correlation strength $C_s(B)$ with $\eta \approx \alpha / 2$ when the upward floating critical states and downward moving anti-Chern states meet halfway. In that situation the critical disorder $W_c(E)$ is constant which implies that only an infinitesimal change in disorder is needed for the the lowest critical state to float up and for the anti-Chern state to float down.

In conclusion, we have shown that floating up of current carrying states with increasing disorder can take place also in the lattice model, if correlation in the random disorder potentials is considered. In this case, no single direct transition from higher Hall plateaus to the insulator is seen. Instead, a one by one transition is observed as it is known from the continuum model. These usual transitions may be, however, difficult to report. The merging was not observed in our calculations.

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