Relativistic effects due to gravimagnetic moment of a rotating body

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We compute exact Hamiltonian (and corresponding Dirac brackets) for spinning particle with gravimagnetic moment $\kappa$ in an arbitrary gravitational background. $\kappa = 0$ corresponds to the Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations. $\kappa = 1$ leads to modified MPTD equations with reasonable behavior in the ultrarelativistic limit. So we study the modified equations in the leading post-Newtonian approximation. Rotating body with unit gravimagnetic moment has qualitatively different behavior as compared with MPTD body: A) If a number of gyroscopes with various rotation axes are freely traveling together, the angles between the axes change with time. B) For specific binary systems, gravimagnetic moment gives a contribution to frame-dragging effect with the magnitude, that turns out to be comparable with that of Schiff frame dragging.

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I. INTRODUCTION

Rotating body in general relativity is usually described on the base of manifestly generally covariant Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations, that prescribe the dynamics of both trajectory and spin of the body in an external gravitational field [6]. Starting from the pioneer works, these equations were considered as a Hamiltonian-type system. In the recent work [7], we realized this idea by constructing the minimal interaction with gravity in the vector model of spinning particle, and showed that this indeed leads to MPTD equations in the Hamiltonian formalism (see also below). This allowed us to study ultra relativistic limit in exact equations for trajectory of MPTD particle in the laboratory time. Using the Landau-Lifshitz $(1 + 3)$-decomposition [8] we observed that, unlike a geodesic equation, the MPTD equations lead to the expression for three-acceleration which contains divergent terms as $\nu \to c$. Fast test particles are now under intensive investigation [10–14], and represent an important tool in the study, for example, of near horizon geometry of black holes [15–23]. So, it would be interesting to find a generalization of MPTD equations with improved behavior in ultra relativistic regime. This can be achieved, if we add a non-minimal spin-gravity interaction through gravimagnetic moment [24]. In the theory with unit gravimagnetic moment, both acceleration and spin torque have reasonable behavior in ultra relativistic limit. In the present work we study the modified equations in the regime of small velocities in the leading post-Newtonian approximation. In Schwarzschild and Kerr space-times, the modified equations imply a number of qualitatively new effects, that could be used to test experimentally, whether a rotating body in general relativity has null or unit gravimagnetic moment.

The work is organized as follows. In Sect. II we shortly describe Lagrangian and Hamiltonian formulations of vector model of spinning particle and compute Dirac brackets of the theory in an arbitrary gravitational background. In the formulation with use of Dirac brackets, the complete Hamiltonian acquires a simple and expected form, while an approximate $\frac{1}{c^2}$ Hamiltonian, further obtained in Sect. IV strongly resembles that of spinning particle in electromagnetic background. This is in correspondence with the known analogy between gravity and electromagnetism [25–28]. In Sect. III we introduce non-minimal spin-gravity interaction through the gravimagnetic moment and obtain the corresponding equations of motion. We show that constants of motion due to isometries of space-time for the MPTD and the modified equations are the same. In section IV we compute the leading post-Newtonian corrections to the trajectory and spin of our particle with unit gravimagnetic moment, and present the corresponding effective Hamiltonian in $\frac{1}{c^2}$-approximation. The non-minimal interaction implies extra contributions into both trajectory and spin, as compared with MPTD equations in the same approximation. A number of effects due to non-minimal interaction are discussed in Sect. V.

Notation. Our variables are taken in arbitrary parametrization $\tau$, then $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$. The square brackets mean antisymmetrization, $\omega[^{\mu}[\nu]] = \omega^{\mu}{}_{\nu} - \omega^{\nu}{}_{\mu}$. For the four-dimensional quantities we suppress the contracted indexes and use the notation $\dot{x}^\mu G_{\mu\nu} \dot{x}^\nu = \dot{x}\dot{G}\dot{x}$, $N^\nu \dot{x}^\nu = (\dot{N})^\nu$, $\omega^2 = g_{\mu\nu} \omega^\mu \omega^\nu$, $\mu, \nu = 0, 1, 2, 3$. Notation for the scalar functions constructed from second-rank tensors are $\Theta S = \Theta^{\mu\nu} S_{\mu\nu}$, $S^2 = S^\mu S^\nu S_{\mu\nu}$. When we work in four-dimensional Minkowski space with coordinates $x^\mu = (x^0 = ct, x^i)$, we use the metric $\eta_{\mu\nu} = (-,+,+,+)$, then $\dot{x}\omega = \dot{x}^i \omega_i = -\dot{x}^0 \omega^0 + \dot{x}^i \omega^i$ and so on. Suppressing the indexes of three-dimensional quantities.
ties, we use bold letters, $v^i\gamma_{ij}a^j = v\gamma a$, $v^i G_{ij} v^j = v G v$, $i, j = 1, 2, 3$, and so on.

The covariant derivative is $\nabla \omega^\mu = \frac{\partial \omega^\mu}{\partial \tau} + \Gamma^\mu_{\alpha\beta} \omega^\alpha \omega^\beta$. The tensor of Riemann curvature is $R^\sigma_{\lambda\mu\nu} = \partial^\sigma_{\mu} \Gamma^\lambda_{\nu\sigma} - \partial^\sigma_{\nu} \Gamma^\lambda_{\mu\sigma} - \partial^\lambda_{\mu} \Gamma^\sigma_{\nu\sigma} + \partial^\lambda_{\nu} \Gamma^\sigma_{\mu\sigma}$.

The most convenient for analyzing the dynamics of the phase-space variables, the Hamiltonian formalism is thermore, one of the basic observables is constructed from dipole electric moment \cite{31}.

where $S^\mu$ represents the spin-tensor of the particle. We decompose the vectors $\omega^\mu$ and $\pi^\mu$ into their own planes \cite{30}. Being affected by the local transformation, these vectors do not represent observable quantities. But their combination, $S^{\mu\nu} = 2(\omega^\mu \pi^\nu - \omega^\nu \pi^\mu)$, is an invariant quantity, which represents the spin-tensor of the particle. We decompose the spin-tensor as follows:

$$S^{\mu\nu} = 2(\omega^\mu \pi^\nu - \omega^\nu \pi^\mu) = (S^{\alpha\beta} D^\lambda, S_{ij} = 2\epsilon_{ijk} S_k) \quad (2)$$

where $S$ is three-dimensional spin-vector, and $D_\lambda$ is dipole electric moment \cite{31}.

Since we deal with a local-invariant theory and, furthermore, one of the basic observables is constructed from the phase-space variables, the Hamiltonian formalism is the most convenient for analyzing the dynamics of the theory. So, we first obtain the Hamiltonian equations of motion, and next, excluding momenta, we arrive at the Lagrangian equations for the physical-vector variables $x$ and $S$.

Conjugate momenta for $x^\mu$ and $\omega^\mu$ are $p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}$ and $\pi^\mu = \frac{\partial L}{\partial \dot{\omega}^\mu}$, respectively. Due to the presence of $\dot{\omega}^\mu$ in $\nabla \omega^\mu$, the conjugated momentum $p_\mu$ does not transform as a vector, so it is convenient to define the canonical momentum

$$P_\mu \equiv p_\mu - \Gamma^\beta_{\alpha\mu} \omega^\alpha \pi^\beta, \quad (3)$$

which transforms as a vector under general-coordinate transformations. The full set of phase-space coordinates consists of the pairs $x^\mu, p_\mu$ and $\omega^\mu, \pi^\mu$. They fulfill the fundamental Poisson brackets \{ $x^\mu, p_\nu$ \} = $\delta^\mu_\nu$, \{ $\omega^\mu, \pi_\nu$ \} = $-\Gamma^\mu_{\alpha\nu} \pi_\alpha$, \{ $P_\mu, \pi_\nu$ \} = $\delta^\mu_\nu$,

$$\{ P_\mu, \omega^\nu \} = \Gamma^\nu_{\mu\alpha} \omega^\alpha, \quad \{ P_\mu, \pi_\nu \} = -\Gamma^\alpha_{\mu\nu} \pi_\alpha, \quad \{ P_\mu, \omega^2 \} = \{ P_\mu, \pi^2 \} = \{ P_\mu, \omega \pi \} = 0. \quad (4)$$

For the quantities $x^\mu$, $P^\mu$ and $\omega^{\mu\nu}$, the basic Poisson brackets imply the typical relations used by people for spinning particles in Hamiltonian formalism

$$\{ x^\mu, P_\nu \} = \delta^\mu_\nu, \quad \{ P_\mu, P_\nu \} = -\frac{1}{4} R_{\mu\nu\alpha\beta} S^{\alpha\beta}, \quad \{ P_\mu, \omega^{\nu\beta} \} = \Gamma^\nu_{\mu\alpha} S^{\alpha\beta} - \Gamma^\beta_{\nu\alpha} S^{\alpha\mu}, \quad \{ S^{\mu\nu}, \omega^{\beta\gamma} \} = 2 (g^{\mu\alpha} S^{\nu\beta} - g^{\nu\beta} S^{\mu\alpha} - g^{\mu\beta} S^{\nu\alpha} + g^{\nu\alpha} S^{\mu\beta}). \quad (5)$$

Applying the Dirac-Bergman procedure for a singular system to the theory \cite{4}, we arrive at the Hamiltonian

$$H = \frac{\lambda_1}{2} \left[ T_1 + 4a(\pi \theta P) T_3 - 4a(\omega \theta P) T_4 + T_5 \right] + \lambda_2 T_6 \quad (6)$$

composed of the constraints

$$T_1 \equiv P^2 + m^2 c^2 = 0, \quad T_2 \equiv \omega \pi = 0, \quad T_3 \equiv P \omega = 0, \quad T_4 \equiv P \pi = 0, \quad T_5 \equiv \pi^2 - \frac{\lambda_1}{2} = 0. \quad (7)$$

In the expression for $H$ we have denoted

$$\theta_{\mu\nu} \equiv R_{\alpha\beta\mu\nu} S^{\alpha\beta}, \quad a \equiv \frac{2}{16m^2 c^2 + \theta S}. \quad (9)$$

The antisymmetric tensor $\theta_{\mu\nu}$ turns out to be gravitational analogy of the electromagnetic field strength $F_{\mu\nu}$, see below. $T_1, \ldots, T_4$ appear as the primary constraints in the course of Dirac-Bergman procedure. $T_5$ is the only secondary constraint of the theory, and $\lambda_1, \lambda_2$ are the Lagrangian multipliers associated to $T_1$ and $T_2$. Poisson brackets of the constraints are summarized in Table \cite{1}. The Table implies that $T_3$ and $T_4$ represent a pair of second-class constraints, while $T_2$, $T_5$ and the combination $T_1 + 4a(\pi \theta P) T_3 - 4a(\omega \theta P) T_4$ are the first-class constraints. So the Hamiltonian \cite{6} consist of the first-class constraints.

Taking into account that each second-class constraint rules out one phase-space variable, whereas each first-class constraint rules out two variables, we have the right number of spin degrees of freedom, $8 - (2 + 4) = 2$. The meaning of the constraints becomes clear if we consider...
their effect over the spin tensor. The second-class constraints $T_3 = 0$ and $T_4 = 0$ imply the spin supplementary condition

$$S^\mu{}\nu P_\nu = 0, \quad (10)$$

while the first-class constraints $T_2$ and $T_3$ fix the value of square of the spin tensor

$$S^\mu{}\nu S_\mu{}\nu = 8\alpha. \quad (11)$$

The equations (10) and (11) imply that only two components of spin-tensor are independent, as it should be for an elementary spin one-half particle.

We could use Poisson brackets to obtain the Hamiltonian equations, $\dot{z} = \{z, H\}$, for the variables of physical sector $z = (x, P, S)$. But in this case we are forced to work with rather inconvenient Hamiltonian (6). Instead, we construct the Dirac bracket associated with second-class constraints $T_3$ and $T_4$. It is convenient to denote $\{T_3, T_4\} = -\frac{1}{8\alpha}$, where $\Delta = \frac{1}{16\mu^2 - (\pi\theta)^2}$, then $\Delta \approx a$ on the surface of mass-shell constraint $T_1 = 0$. The Dirac bracket reads

$$\{A, B\}_D = \{A, B\} - 8\Delta \{\{A, T_3\}\{T_4, B\} - \{A, T_4\}\{T_3, B\}\}. \quad (12)$$

By construction, the Dirac bracket of any variable with the constraints vanishes, so $T_3$ and $T_4$ can be omitted from the Hamiltonian. The first-class constraints $T_2$ and $T_3$ can be omitted as well, since brackets of the variables $x, P$ and $S$ with them vanish on the constraint surface. In the result we arrive at a simple Hamiltonian

$$H_0 = \frac{\lambda_1}{2} \left( P^2 + m^2 c^2 \right), \quad (13)$$

which looks like that of a free point particle. All the information on spin and interaction is encoded now in the Dirac bracket. In particular, equations of motion are obtained according the rule $\dot{z} = \{z, H_0\}_D$.

Poisson brackets of our variables with $T_3$ and $T_4$ are

$$\{x^\mu, T_3\} = \omega^\mu, \quad \{x^\mu, T_4\} = \pi^\mu, \quad \{P_\alpha, T_3\} = -\frac{1}{2} \theta_{\alpha\beta} \omega^\beta + \Gamma_{\alpha\beta\lambda} P_\lambda \omega^\beta, \quad \{P_\alpha, T_4\} = -\frac{1}{2} \theta_{\alpha\beta} \pi^\beta + \Gamma_{\alpha\beta\lambda} P_\lambda \pi^\beta, \quad \{S^\mu{}\nu, T_3\} = 2 P^\mu \omega^\nu + \Gamma_{\alpha\beta\gamma\lambda} S^\mu{}\nu S^\alpha{}\beta, \quad \{S^\mu{}\nu, T_4\} = 2 P^\mu \pi^\nu + \Gamma_{\alpha\beta\gamma\lambda} S^\mu{}\nu S^\alpha{}\beta. \quad (14)$$

Using these expressions in (12), we obtain manifest form of the Dirac brackets

$$\{x^\mu, x^\nu\}_D = 4\Delta S^\mu{}\nu, \quad \{P_\mu, P_\nu\}_D = -\frac{1}{4} \theta_{\mu\nu} + 4\Delta (\Gamma P)_\mu{}\nu S^\alpha{}\beta (\Gamma P)_{\beta\nu} - \frac{\Delta}{8} \left( \theta_{\mu\alpha} S^\alpha{}\beta \left[ \theta_{\beta\nu} + 4(\Gamma P)_{\beta\nu} - (\mu \leftrightarrow \nu) \right] \right), \quad (15)$$

$$\{x^\mu, P_\alpha\}_D = \delta_\mu^\alpha + \Delta S^\mu{}\beta \left[ \theta_{\beta\alpha} + 4(\Gamma P)_{\beta\alpha} \right], \quad \{x^\mu, S^\alpha{}\beta\}_D = -8\Delta \left[ S_{[\mu} P^{[\beta]} - \frac{1}{2} S_{[\mu} \Gamma^{\alpha\beta\lambda} S^\alpha{}\beta \right], \quad \{P_\alpha, S^\mu{}\nu\}_D = -\Gamma_{\mu\alpha} S^\mu{}\nu, \quad \{S^\mu{}\nu, S^\alpha{}\beta\}_D = 2 S^\mu{}\nu P^\alpha S^\beta{}\gamma - 2 S^\mu{}\nu S^\alpha{}\beta P^\gamma + 2 S^\mu{}\nu S^\alpha{}\beta P^\gamma + 2 S^\mu{}\nu S^\alpha{}\beta P^\gamma. \quad (16)$$

Their right hand sides do not contain explicitly the variables $\omega$ and $\pi$, so the brackets form a closed algebra for the set $(x, P, S)$.

The Dirac brackets remain different from the Poisson brackets even in the limit of a free theory, $g_{\mu\nu} \to \eta_{\mu\nu}$. In particular, in the sector of canonical variables $x$ and $p$ we have

$$\{x^\mu, x^\nu\}_D = -\frac{S^\mu{}\nu}{2p}, \quad \{x^\mu, P^\nu\}_D = -\eta_{\mu\nu}, \quad \{P^\mu, P^\nu\}_D = 0. \quad (16)$$

Hence, account of spin leads to deformation of the phase-space symplectic structure: the position variables of relativistic spinning particle obey the noncommutative bracket, with the noncommutativity parameter being proportional to the spin-tensor. This must be taken into account in construction of quantum mechanics of a spinning particle. In particular, for an electron in electromagnetic field, the spin-induced noncommutativity explains the famous one-half factor in the Pauli equation without appeal to the Thomas precession, Dirac
equation or to the Foldy-Wouthuysen transformation, see [34]. Besides, for a spinning body in gravitational field, the spin-induced noncommutativity clarifies the discrepancy in expressions for three-acceleration obtained by different methods, see [35].

Using the Dirac brackets with the Hamiltonian [13], we obtain equations of motion

\[ \dot{x}^\mu = \{ x^\mu, H_0 \}_D = \lambda_1 \left[ P^\mu + a S^{\alpha\beta} \theta_{\beta\gamma} P^\gamma \right], \]

\[ \dot{P}_\mu = \{ P_\mu, H_0 \}_D = \left( -\frac{1}{4} \theta_{\mu\nu} + (\Gamma P)_{\mu\nu} \right) \lambda_1 \left[ P^\nu + a S^{\alpha\beta} \theta_{\beta\gamma} P^\gamma \right] - \frac{1}{4} \theta_{\mu\nu} \dot{x}^\nu + \Gamma^\alpha_{\mu\nu} P^\alpha \dot{x}^\nu, \]

\[ \dot{S}^{\mu\nu} = \{ S^{\mu\nu}, H_0 \}_D = (2P^\mu \delta^{\nu}_\alpha - \Gamma^\mu_{\alpha\sigma} S^{\sigma\nu}) \lambda_1 \left[ P^\alpha + a S^{\alpha\beta} \theta_{\beta\gamma} P^\gamma \right] - \frac{1}{4} \theta_{\mu\nu} \dot{x}^\nu. \]

They can be rewritten in a manifestly general-covariant form as follows:

\[ \dot{x}^\mu = \lambda_1 \left( \delta^\mu_\nu + a S^{\alpha\beta} \theta_{\beta\gamma} \right) P^\nu, \]

\[ \nabla P_\mu = -\frac{1}{4} R_{\mu\nu\alpha\beta} S^{\alpha\beta} \dot{x}^\nu \equiv \frac{1}{4} \theta_{\mu\nu} \dot{x}^\nu, \]

\[ \nabla S^{\mu\nu} = 2(P^\mu \dot{x}^\nu - P^\nu \dot{x}^\mu). \]

Some relevant comments are in order.

1. **Comparison with MPTD equations.** Despite the fact that the vector model has been initially constructed as a theory of an elementary particle of spin one-half, it turns out to be suitable to describe a rotating body in general relativity in the pole-dipole approximation [16]. Indeed, the equations (19) and (20) coincide with Dixon equations of the body (our spin is twice of that of Dixon), while our constraint (10) is just the Tulczyjew spin supplementary condition. Besides, the Hamiltonian equation (13) can be identified with the velocity-momentum relation, implied by MPTD-equations, see [24] for a detailed comparison. The only difference is that values of momentum and spin are conserved quantities of MPTD equations, while in the vector model they are fixed by constraints. In summary [24] to study the class of trajectories of a body with \( \sqrt{-g} = k \) and \( S^2 = \beta \), we can use our spinning particle with \( m = \frac{k}{2} \) and \( \alpha = \frac{\beta}{2} \).

2. **Ultra relativistic limit.** Using the Landau-Lifshitz 1 + 3-decomposition [8], we showed in [24] that MPTD equations yield a paradoxical behavior in ultra relativistic limit: three-dimensional acceleration of the particle grows with its speed, and diverges as \( |v| \to c \). In the next section, we improve this by adding a non-minimal spin-gravity interaction through the gravimagnetic moment.

3. **Analogy between gravitation and electromagnetism.** Many people mentioned remarkable analogies between gravitation and electromagnetism in various circumstances [14, 22, 28]. Here we observe an analogy, comparing (18)-(20) with equations of motion of spinning particle (with null gyromagnetic ratio) [20] in electromagnetic field with the strength \( F_{\mu\nu} \)

\[ \dot{x}^\mu = \lambda_1 (\delta^\mu_\nu + a S^{\alpha\beta} F_{\beta\gamma}) P^\nu, \]

\[ \text{where} \quad a = \frac{-2e}{4m^2 c^3 - e(F S)}, \]

\[ \dot{P}_\mu = \frac{e}{c} F_{\mu\nu} \dot{x}^\nu, \]

\[ \dot{S}^{\mu\nu} = 2P^{[\mu} \dot{x}^{\nu]} \].

(17) The system just turns into another if we identify \( \theta_{\mu\nu} \equiv R_{\mu\nu\rho\sigma} S^{\rho\sigma} \sim F_{\mu\nu} \), and set \( e = -\frac{c}{a} \). That is a curvature influences trajectory of a spinning particle in the same way as an electromagnetic field with the strength \( \theta_{\mu\nu} \).

We now use this analogy to construct a non-minimal spin-gravity interaction.

### III. Rotating Body with Gravimagnetic Moment

The Hamiltonian [13] is a combination of constraints, so the Hamiltonian formulation of our model is completely determined by the set of constraints [7, 8], and by the expression [3] for canonical momentum \( P^\mu \) through the conjugated momentum \( p^\mu \). We observe that algebraic properties of the constraints do not change, if we replace the mass-shell constraint \( T_1 = P^2 + m^2 c^2 \) by \( \bar{T}_1 = P^2 + f(x, P, S) + m^2 c^2 \), where \( f(x^\mu, P^\nu, S^{\mu\nu}) \) is an arbitrary scalar function. Indeed, in the modified theory \( T_3 \) and \( T_4 \) remain the second-class constraints, while \( T_2 \) and the combination \( \bar{T}_1 - \{ T_3, T_4 \}^{-1} \{ \bar{T}_1, T_4 \} T_3 + \{ T_3, T_4 \}^{-1} \{ \bar{T}_1, T_3 \} T_4 \), form a set of first-class constraints. If we confine ourselves to the linear in curvature and quadratic in spin approximation, the only scalar function \( f \), which can be constructed from the quantities at our disposal is \( \frac{\kappa}{16} R_{\mu\rho\sigma\beta} S^{\mu\rho} S^{\sigma\beta} \equiv \kappa R_{\mu\rho\sigma\beta} \omega^\mu \omega^\rho \omega^\sigma \omega^\beta \), where \( \kappa \) is a dimensionless parameter. The resulting constraint

\[ \bar{T}_1 = P^2 + \frac{\kappa}{16} \theta(S) + m^2 c^2 = 0, \]

is similar to the Hamiltonian \( \frac{\lambda_1}{2} (P^2 - \frac{e}{2} (FS) + m^2 c^2) \) of a spinning particle interacting with electromagnetic field through the gyromagnetic ratio \( g \), see [20]. In view of this similarity, the interaction constant \( \kappa \) is called gravimagnetic moment [13, 16], and we expect that non-minimally interacting theory with the Hamiltonian (24) could be consistent generalization of MPTD equations.

The consistency has been confirmed in [24], where we presented the Lagrangian action of a spinning particle that implies the constraints (24) and (8) in Hamiltonian formalism.

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1. While the variational problem dictates [37] the equation [10], in the multipole approach there is a freedom in the choice of a spin supplementary condition, related with the freedom in the choice of a representative point \( x^\mu \) describing position of the body [3, 4, 6]. Different conditions lead to the same results in \( \omega^\mu \approx 0 \) approximation, see [4, 33, 36].
Poisson brackets of the constraints $\tilde{T}_1$, $T_3$ and $T_4$ read

$$\{\tilde{T}_1, T_3\} = \frac{1}{2}(1 - \kappa)(\omega \theta P) + \kappa \omega^\sigma (\nabla_\sigma R_{\mu \nu \alpha \beta}) \omega^\mu \pi^\alpha \pi^\beta. \quad (25)$$

$$\{\tilde{T}_1, T_4\} = \frac{1}{2}(1 - \kappa)(\pi \theta P) + \kappa \pi^\alpha (\nabla_\alpha R_{\mu \nu \alpha \beta}) \omega^\mu \pi^\alpha \omega^\beta. \quad (26)$$

$$\{T_3, T_4\} = P^2 - \frac{1}{16}(\theta S) \approx -8a, \quad \text{where} \quad a = \frac{16m^2c^2 + (\kappa + 1)(\theta S)}{2}. \quad (27)$$

These expressions must be substituted in place of terms $\frac{1}{2}(\omega \theta P), \frac{1}{2}(\pi \theta P)$ and $a$ in the Table [1]. The Dirac brackets [18], being constructed with help of $T_3$ and $T_4$, remain valid in the modified theory. Our new Hamiltonian is $H = \frac{1}{2}H_0 + \frac{1}{2}H_\kappa$, with $H_0$ from (13) and $H_\kappa = \frac{8}{16}(\theta S)$. Hence, to obtain the manifest form of equations of motion $\dot{z} = \{z, H_0\} + \{z, H_\kappa\}$, we need only to compute the brackets $\{z, H_\kappa\}_D$. They are

$$\{x^\mu, H_\kappa\}_D = -\lambda_1 \kappa \tilde{T}_0 S^{\alpha \beta} \sigma_\alpha \sigma_\beta, \quad \frac{1}{8}S^{\mu \nu}(\nabla_\nu R_{\alpha \beta \sigma \lambda})S^{\alpha \beta} S^{\sigma \lambda}, \quad (28)$$

$$\{P_\mu, H_\kappa\}_D = -\lambda \kappa \tilde{T}_0 \theta_{\mu \alpha} \{x^\alpha, H_\kappa\}_D + \Gamma^\beta_{\mu \alpha} P_\beta \{x^\alpha, H_\kappa\}_D \quad \frac{\kappa_1}{32}(\nabla_\mu R_{\alpha \beta \sigma \lambda})S^{\alpha \beta} S^{\sigma \lambda}, \quad (29)$$

$$\{S^{\mu \nu}, H_\kappa\}_D = \frac{\kappa_1}{32}(\Gamma^\mu_{\alpha \beta} S^{\nu \alpha} + 2P^{\mu \nu}) \{x^\beta, H_\kappa\}_D. \quad (30)$$

Adding them to the equations $\dot{z} = \{z, H_0\}_D$ given in (18)-(20), we arrive at the dynamical equations

$$\dot{x}^\mu = \lambda_1 [\tilde{T}_0 + a(\kappa - 1)S^{\mu \alpha} \theta_\alpha] P^\nu + \frac{\kappa_1}{8}(\nabla_\nu R_{\alpha \beta \sigma \lambda})S^{\alpha \beta} S^{\sigma \lambda}, \quad (31)$$

$$\dot{P}_\mu = -\frac{\kappa_1}{8}(\nabla_\mu R_{\alpha \beta \sigma \lambda})S^{\alpha \beta} S^{\sigma \lambda} \theta_\lambda, \quad (32)$$

$$\dot{S}^{\mu \nu} = 2P^{\mu \nu} - \frac{1}{4}\kappa_1 \theta^{\mu \alpha} S^{\nu \alpha}. \quad (33)$$

Together with the constraints (19), (21), and (24), they give complete system of Hamiltonian equations of spinning particle with gravimagnetic moment $\kappa$. As it should be, our equations reduce to MPTD equations (18)-(20) when $\kappa = 0$. Comparing the two systems, we see that the non-minimal interaction yields quadratic and cubic in spin corrections to MPTD equations.

The equations (31)–(33) are greatly simplified for a particle with unit gravimagnetic moment, $\kappa = 1$ (gravimagnetic particle). It has a qualitatively different behavior as compared with MPTD particle. First, gravimagnetic particle has an expected behavior in the ultra relativistic limit $\|v\| \to c$, while the longitudinal acceleration vanishes in the limit. Second, at low velocities, taking $\kappa = 1$ and keeping only the terms which may give a contribution in the leading post-Newton approximation, $\sim \frac{1}{c^2}$, we obtain from (31)–(33) the approximate equations

$$\dot{x}^\mu = \lambda_1 P^\mu, \quad \nabla P^\mu = -\frac{1}{32}(\nabla_\mu R_{\alpha \beta \sigma \lambda})S^{\alpha \beta} S^{\sigma \lambda}, \quad (34)$$

$$\nabla S^{\mu \nu} = \frac{1}{4}\theta^{\mu \alpha} S^{\nu \alpha}. \quad (35)$$

while MPTD equations ($\kappa = 0$) in the same approximation read

$$\dot{x}^\mu = \lambda_1 P^\mu, \quad \nabla P^\mu = -\frac{1}{32}(\nabla_\mu R_{\alpha \beta \sigma \lambda})S^{\alpha \beta} S^{\sigma \lambda}, \quad (36)$$

In Sect. [18] we compute $\frac{1}{c^2}$ corrections due to the extra-terms appeared in (34).

Conserved charges. In curved space which possesses some isometry, MPTD equations admit a constant of motion (see, for example, [7])

$$J^{(\xi)} = P^\mu \theta_\mu - \frac{1}{4}S^{\mu \nu} \nabla_\nu \xi_\mu, \quad (37)$$

where $\xi_\mu$ is Killing vector which generates the isometry, i.e., $\nabla_\nu \xi_\mu + \nabla_\nu \xi_\mu = 0$. Let us show that $J^{(\xi)}$ remains a constant of motion when the gravimagnetic interaction is included. Using (32) and (33), we obtain by direct calculation

$$J^{(\xi)} = \frac{\kappa_1}{8}[S^{\alpha \beta} R^{\mu \nu \alpha \beta} S^{\sigma \nu} \nabla_\sigma \xi_\mu - \frac{1}{4}S^{\alpha \beta} S^{\sigma \lambda} \xi_\mu \nabla_\nu R_{\mu \alpha \beta \lambda}] . \quad (38)$$

Using the Bianchi identities we find the relation

$$S^{\alpha \beta} S^{\sigma \nu} \xi_\mu \nabla_\nu R_{\mu \alpha \beta \lambda} = 2S^{\alpha \beta} S^{\sigma \nu} \xi_\mu \nabla_\nu R_{\mu \alpha \beta \lambda}. \quad (39)$$

Derivative of a curvature tensor is related with derivative of a Killing vector by the formula $\xi_\mu \nabla_\sigma R_{\mu \alpha \beta \lambda} = \xi_\mu \nabla_\sigma R_{\mu \alpha \beta \lambda} = R_{\mu \alpha \beta \lambda} \nabla_\nu \xi_\mu + R_{\mu \alpha \beta} \nabla_\nu \xi_\mu - \nabla_\nu \xi_\mu$. Contracting twice with the spin tensor we obtain

$$S^{\alpha \beta} S^{\sigma \nu} \xi_\mu \nabla_\nu R_{\mu \alpha \beta \lambda} = 2S^{\alpha \beta} R^{\mu \nu \alpha \beta} S^{\sigma \nu} \nabla_\nu \xi_\mu. \quad (40)$$

Using this expression in (38), we obtain

$$S^{\alpha \beta} S^{\sigma \nu} \xi_\mu \nabla_\nu R_{\mu \alpha \beta \lambda} = 4S^{\alpha \beta} R^{\mu \nu \alpha \beta} S^{\sigma \nu} \nabla_\nu \xi_\mu. \quad (41)$$

This implies that the right hand side of (37) vanishes, so $J^{(\xi)} = 0$. Thus, the quantity (40) represents a constant of motion of a spinning particle with gravimagnetic moment.

Lagrangian System of equations of motion. Since we are interested in the influence of non-minimal spin-gravity interaction on trajectory and spin of the particle, we eliminate the momenta $P^\mu$ and the auxiliary variable $\lambda_\xi$ from the equations (31)–(33), obtaining their Lagrangian form. In the equation (61), which relates velocity and momentum, appeared the matrix

$$T^\alpha_\nu = \delta^\alpha_\nu - (\kappa - 1)aS^{\alpha \sigma} \theta_\sigma. \quad (42)$$
Using the identity $(S\theta S)^{\mu\nu} = -\frac{1}{2}(S^{\alpha\beta}\theta_{\alpha\beta})S^{\mu\nu}$, we find inverse of the matrix $T$

$$T^\alpha_\nu \equiv \delta^\alpha_\nu + (\kappa - 1)bS^{\alpha\sigma}\theta_{\sigma\nu}, \quad b = \frac{1}{8m^2c^2 + \kappa(S\theta)} \tag{41}$$

Using (41), we solve (31) with respect to $P^\mu$. Using the resulting expression in the constraint (24), we obtain

$$\lambda_1 = \frac{c^2}{m^2c^2 + \kappa(S\theta)}, \quad m^2 \equiv m^2 + \frac{e}{16\pi\sigma}(S\theta) - \kappa^2Z^2$$

is the radiation mass in gravitational field. By $Z^\mu$ we have denoted the vector, which vanishes in spaces with covariantly-constant curvature, $Z^\mu = \frac{b}{k}S^{\alpha\nu}(\nabla_\alpha R_{\beta\alpha\beta})S^{\nu\beta}S^{\mu\nu}$. Besides, in the expression for $\lambda_1$ a kind of effective metric $G$ induced by spin-gravity interaction along the world-line, $G_{\mu\nu} = T^\alpha_\mu g_{\alpha\beta}T^\beta_\nu$. Only for the gravimagnetic particle ($\kappa = 1$), the effective metric reduces to the original one. Using (31) and (41), we obtain expression for momentum in terms of velocity

$$P^\mu = \frac{m_r c}{\sqrt{-xGx}} \tilde{T}^{\mu}_\nu \dot{x}^\nu - \kappa c Z^\mu. \tag{42}$$

We substitute this $P^\mu$ into (32) and (33), arriving at the Lagrangian equations of our spinning particle with gravimagnetic moment $\kappa$

$$\nabla \left[ \frac{m_r c}{\sqrt{-xGx}} \tilde{T}^{\mu}_\nu \dot{x}^\nu \right] = -\frac{1}{4c} \theta^{\mu\nu} \dot{x}^\nu + \kappa \frac{c^2}{32m_r^2c^2} \nabla^\mu (S\theta)$$

$$+ \kappa \nabla Z^\mu, \tag{43}$$

$$\nabla S^{\mu\nu} = -\frac{\kappa c^2}{4m_r c} (\theta S)^{[\mu\nu]} - \frac{2m_r c(\kappa - 1)b}{\sqrt{-xGx}} \tilde{\xi}^{[\mu\nu]} (S\theta \dot{x})^{\nu}$$

$$+ 2\kappa c\tilde{z}^{[\mu\nu]} Z^\nu. \tag{44}$$

**IV. LEADING POST-NEWTONIAN CORRECTIONS DUE TO UNIT GRAVIMAGNETIC MOMENT**

Taking $\kappa = 1$ in (33) and (41), we obtain equations of our gravimagnetic body

$$\nabla \left[ \frac{m_r c}{\sqrt{-xGx}} \tilde{T}^{\mu}_\nu \dot{x}^\nu \right] = -\frac{1}{4c} \theta^{\mu\nu} \dot{x}^\nu - \frac{\sqrt{-xGx}}{32m_r^2c^2} \nabla^\mu (S\theta)$$

$$+ \nabla Z^\mu, \tag{45}$$

$$\nabla S^{\mu\nu} = -\frac{\sqrt{-xGx}}{4m_r c} (\theta S)^{[\mu\nu]} + 2\kappa c\tilde{z}^{[\mu\nu]} Z^\nu. \tag{46}$$

To test these equations, we compute the leading relativistic corrections due to unit gravimagnetic moment to the trajectory and precession of a gyroscope, orbiting around a rotating spherical body of mass $M$ and angular momentum $J$. To this aim, we write equations of motion implied by (15) and (16) for the three-dimensional position $x^i(t)$ and for the spin-vector

$$S = \frac{1}{2} \left( S^{23}, S^{31}, S^{12} \right), \quad \text{or} \quad S_i(t) = \frac{1}{2} \epsilon_{ijk} S^{jk}(t),$$

$$S^{ij} = 2\epsilon^{ijk} S_k,$$  \tag{47}

as functions of the coordinate time $t = \frac{x^i}{c}$. Due to the reparametrization invariance, the desired equations are obtained by setting $\tau = t$ in (15) and (16). We consider separately the trajectory and the spin.

**Trajectory.** We denote $x^\mu \equiv \frac{dx^\mu}{dt} = (c, v)$, so $\sqrt{-xGx} = \sqrt{-xG\dot{x}x} = \sqrt{-c^2g_{00} - 2c^2g_{0i}v^i - g_{ij}v^i v^j}$. The temporal and spatial parts of Eq. (45) read

$$\frac{d}{dt} \left[ \frac{m_r c}{\sqrt{-xGx}} \right] + \frac{m_r c}{\sqrt{-xGx}} \dot{\theta}^{\mu\nu} \dot{x}^\nu$$

$$= \frac{1}{4c^2} \theta^{\mu\nu} \dot{x}^\nu - \frac{\sqrt{-xGx}}{32m_r^2c^2} \nabla^0 (S\theta) + \frac{1}{c} \nabla_i Z^0,$$

$$\frac{d^2 x^i}{dt^2} + \Gamma^{\mu_{\nu\rho}}_{\mu\nu\rho} v^\nu v^\rho + \frac{v^i}{c^2} \nabla^0 (S\theta) + \frac{1}{m_r c} \frac{d}{dt} \left[ \frac{m_r c}{\sqrt{-xGx}} \right]$$

$$= \frac{\sqrt{-xGx}}{4m_r c} \theta^{\mu\nu} v^\nu v^\rho + \frac{v^i}{32m_r^2c^2} \nabla^0 (S\theta) + \frac{\sqrt{-xGx}}{m_r c} \nabla_i Z^0. \tag{48}$$

Now we assume a non relativistic motion, $\frac{x}{c} << 1$, and expand all quantities in (48) in series with respect to $\frac{1}{c}$. Typical metric of stationary spaces has the series of the form

$$g_{00} = -1 + 2g_{00} + 4g_{00} + \ldots$$

$$g_{ij} = \delta_{ij} + 2g_{ij} + 4g_{ij} + \ldots$$

$$g_{00} = 3g_{00} + 5g_{00} + \ldots,$$

where $\eta_{\mu\nu}$ denotes the term in $g_{\mu\nu}$ of order $1/c^0$. As a consequence, the series of connection, curvature and its covariant derivative starts from $\frac{1}{c^0}$ or from higher order. In some details, we have

$$\Gamma^{\mu}_{\nu\alpha} = 2\Gamma^{\mu}_{\nu\alpha} + 4\Gamma^{\mu}_{\nu\alpha} + \ldots$$

for $\Gamma_{00}^0$, $\Gamma_{i0}^0$, $\Gamma_{00}^0$, (50)

$$\Gamma^{\mu}_{\nu\alpha} = 3\Gamma^{\mu}_{\nu\alpha} + 5\Gamma^{\mu}_{\nu\alpha} + \ldots$$

for $\Gamma_{0m}^0$, $\Gamma_{00}^0$, $\Gamma_{0m}^0$, (51)

$$R^{\mu}_{\nu\alpha\beta} =$$

$$2R^{\mu}_{\nu\alpha\beta} + 4R^{\mu}_{\nu\alpha\beta} + \ldots$$

for $R_{mn0}$, $R_{0mn}$, $R_{i0m}$, $R_{jmn}$, (52)

$$R^{\mu}_{\nu\alpha\beta} =$$

$$3R^{\mu}_{\nu\alpha\beta} + 5R^{\mu}_{\nu\alpha\beta} + \ldots$$

for $R_{ijmn}$, $R_{0m0}$, $R_{i0m}$, $R_{jm0}$, (53)

$^2$ We point out that the analogous matrix present in MPTD equations can not be explicitly inverted in the multipole approach.
Besides, for various quantities which appear in equations \(^{(45)}\) and \(^{(46)}\), we have the estimations

\[
\sqrt{-vgv} \sim c + \frac{1}{c} + \ldots, \quad -vgv \sim c^2 + 1 + \frac{1}{c^2} + \ldots,
\]

\[
m^2_r \sim m^2 + \frac{1}{c^4} + \ldots, \quad \theta_{\mu\nu} \sim \frac{1}{c^2} + \ldots,
\]

\[
b \sim \frac{1}{c^2} + \ldots, \quad Z^\mu \sim \frac{1}{c^5} + \ldots.
\]

(54)

At last, the spin supplementary condition implies

\[
S^{0i} = \frac{1}{c} S^{ij} v^j + \ldots.
\]

(55)

Keeping only the terms which may contribute up to order \(\frac{1}{c}\) in the equation \(^{(45)}\), we obtain

\[
\frac{d^2x^i}{dt^2} = -\Gamma^i_{\mu\nu} v^\mu v^\nu + \frac{v^i}{c^2} \Gamma^i_{\mu\nu} v^\mu v^\nu + \frac{1}{4m^2} \left( \theta^i (\theta^j v^j) - \frac{1}{32m^2} \nabla^i (S \theta) \right). \quad (56)
\]

The terms on right-hand side of this equation are conveniently grouped according to their origin

\[
\frac{d^2x^i}{dt^2} = a^i_T + a^i_R + a^i_{\nabla R}.
\]

(57)

Here \(a^i_T\) is the contribution due to connection, \(a^i_R\) comes from interaction between spin and space-time curvature, and \(a^i_{\nabla R}\) is the contribution which involves derivatives of the Riemann tensor. Using \(^{(56)}\)-\(^{(58)}\) we obtain

\[
a^i_T = -\Gamma^i_{\alpha\beta} v^\alpha v^\beta + \frac{1}{4m^2} \Gamma^0_{\alpha\beta} v^\alpha v^\beta
\]

\[
= -c^2 2i_0^i - 2\Gamma^0_{mn} v^m v^n + 2i^i 2\Gamma^0_{mn} v^m - c^2 4i_0^i + ci^i 3\Gamma^0_{\mu0} - 2c^2 3\Gamma^0_{\mu0}, \quad (58)
\]

\[
a^i_R = \frac{1}{4m^2} \left[ (\theta^i (\theta^j v^j) - \frac{1}{32m^2} \nabla^i (S \theta) \right] + \frac{1}{4m^2} \left[ v^i \theta^0_0 - c\theta^0_0 - \theta^i_j v^j \right] - \frac{1}{32m^2} \nabla^i (S \theta).
\]

(59)

As a concrete example of an external gravitational field, we take a stationary, asymptotically flat metric in the post-Newtonian approximation up to order \(\frac{1}{c}\). With this metric, the equations \(^{(58)}\)-\(^{(60)}\) are

\[
\mathbf{a}_T = \frac{MG}{r^2} \hat{r} + \frac{4GM}{c^2 r^2} (\hat{r} \cdot \mathbf{v}) \mathbf{v} - \frac{GM}{c^2 r^2} v^2 \hat{r} + \frac{4GM^2}{c^2 r^3} \hat{r} + 2 \frac{G}{c^2} \left[ \frac{3 (J \cdot \hat{r}) \hat{r} - J}{r^3} \right] \times \mathbf{v},
\]

(62)

\[
\mathbf{a}_R = \frac{3GM}{mc^2 r^3} [(\hat{r} \times \mathbf{v}) (\hat{r} \cdot \mathbf{S}) + (\mathbf{S} \cdot (\hat{r} \times \mathbf{v}))]
\]

\[
- \frac{1}{m} \nabla \left[ \frac{G}{c^2} \left( \frac{3 (J \cdot \hat{r}) \hat{r} - J}{r^3} \right) \cdot \mathbf{S} \right],
\]

(63)

\[
\mathbf{a}_{\nabla R} = \frac{1}{2m} \nabla \left[ \frac{G}{c^2} \left( \frac{M}{m} \left( \frac{3 (S \cdot \hat{r}) \hat{r} - S}{r^3} \right) \cdot \mathbf{S} \right) \right].
\]

(64)

We denote by \(\hat{r}\) the unit vector in the direction of \(r\).

**Spin torque.** Setting \(\kappa = 1\) and \(t = \frac{x^0}{c}\) in the spatial part of Eq. \(^{(46)}\), this reads

\[
\frac{dS^{ij}}{dt} = -\Gamma^i_{\alpha\beta} v^\alpha S^{\beta j} - \Gamma^i_{\mu\alpha} v^\mu S^{\alpha j} + \frac{\sqrt{-vgv}}{4m v^c} \delta^i_{\nu 0} S^{0j} + 2 \alpha i^i Z^j.
\]

(65)

For the spin-vector \(^{(47)}\), this equation implies

\[
\frac{dS^{i}}{dt} = -\frac{1}{c} \epsilon^{ijk} \Gamma^j_{\mu\nu} v^\mu v^k \mathbf{S}^{\nu k} - \frac{\sqrt{-vgv}}{8m v^c} \epsilon^{ijk} \frac{\theta^j_0 S^{\nu i}}{v^j} + \epsilon c^2 \epsilon^{ijk} Z^k.
\]

(66)

Taking into account the equations \(^{(50)}\)-\(^{(54)}\), we keep only the terms which may contribute up to order \(\frac{1}{c}\)

\[
\frac{dS^{i}}{dt} = -\frac{1}{c} \epsilon^{ijk} \frac{\theta^{i0} S^{0k} + v^n \Gamma^j_{nk} S^{nk} + c \Gamma^j_{0n} S^{nk}}{8m v^c} \epsilon^{ijk} \frac{\theta^j_0 S^{\nu i}}{v^j} + \epsilon c^2 \epsilon^{ijk} Z^k + \frac{1}{2m} \epsilon_{mnkl} 2 \Gamma_{kn} S^{kl} S^i.
\]

(67)

The total torque on right hand side of this equation can be conveniently grouped as follows:

\[
\frac{dS}{dt} = \tau_v + \tau_J + \tau_R,
\]

(68)

where \(\tau_v\) contains the velocity-dependent terms, \(\tau_J\) depends on inner angular momentum of central body, and

\[
\tau_R = -3 \frac{GM}{mc^2 r^3} [(\mathbf{v} \times \mathbf{S}) - 2 \hat{r} (\hat{r} \cdot (\mathbf{v} \times \mathbf{S})) - (\hat{r} \cdot \mathbf{v}) (\hat{r} \times \mathbf{S})].
\]

The first two terms in \(a_R\) can be written also as follows:

\[
-3 \frac{GM}{mc^2 r^3} [(\mathbf{v} \times \mathbf{S}) - 2 \hat{r} (\hat{r} \cdot (\mathbf{v} \times \mathbf{S})) - (\hat{r} \cdot \mathbf{v}) (\hat{r} \times \mathbf{S})].
\]
\( \tau_R \) is due to spin-curvature interaction. Computing these terms for the metric (61), we obtained

\[
\tau_v = \frac{GM}{c^2 r^2} \left( 2(S \cdot \hat{r})v + (\hat{r} \cdot v)S - (S \cdot v)\hat{r} \right),
\]

(69)

\[
\tau_J = \frac{G}{c^2} \left[ \frac{3(J \cdot \hat{r})\hat{r} - J}{r^3} \right] \times S,
\]

(70)

\[
\tau_R = \frac{G}{c^2} \left( \frac{M}{r^3} \right) \left[ \frac{3(S \cdot \hat{r})\hat{r}}{r^3} \right] \times S.
\]

(71)

Magnitude of the torque (68) does not represent directly measurable quantity. Indeed, evolution of the gyroscope axis is observed in the frame co-moving with the gyroscope, so the measurable quantity is \( \frac{ds}{dr} \), where \( S' \) are components of spin-vector in the rest frame of gyroscope, and \( s \) is its proper time. Magnitudes of the two torques do not coincide, since \( S \) is not a covariant object. According to the classical work of Schiff [38], we can present \( S' \) through \( S \), and then use the resulting relation to compute \( \frac{ds}{dr} \) through \( \frac{ds}{dr} \) given in (68). The procedure is as follows. First, we use the tetrad formalism, presenting original metric along an infinitesimal arc of the gyroscope trajectory as \( g_{\mu\nu} = \epsilon_{\mu\nu}^{\rho} \eta_{AB} \). Let \( e_{\rho}^A \) is inverse matrix of \( e^A_\rho \). Applying a general-coordinate transformation \( x^\nu \rightarrow x^A \) with the transition functions \( \frac{dx^A}{dx^\nu} = e^A_\nu \), the metric acquires the Lorentz form, \( \eta_{AB} = \epsilon_{\mu\nu}^{AB} \epsilon^\mu_A \epsilon^\nu_B \).

So the transformed spin-tensor, \( S^CD \equiv \epsilon_C^{\mu} \epsilon_D^{\nu} S_{\mu\nu} \), represents spin of gyroscope in a free-falling frame. Second, we apply the Lorentz boost \( \Lambda^A_A(v) \), where \( v \) is velocity of gyroscope, to make the frame co-moving with gyroscope. This gives the spin-tensor \( S^{CD} = \Lambda^C_A \Lambda^D_B \epsilon^{AB}_\mu \epsilon_{\rho}^\mu S_{\rho\nu} \).

Then three-dimensional spin (77) in the co-moving frame can be presented through the quantities given in original coordinates as follows:

\[
S'_i = \frac{1}{4} \epsilon_{ijk} \Lambda^k_A \Lambda^j_B e_B^e e^C_{\mu} S_{\mu\nu}.
\]

(72)

Since our metric is diagonal in \( \frac{1}{c^2} \)-approximation, the tetrad field is diagonal as well, and reads \( \epsilon^0_0 = 1 - \frac{GM}{c^2 r} \), \( \epsilon^i_i = 1 + \frac{GM}{c^2 r} \), \( i = 1, 2, 3 \), again to \( \frac{1}{c^2} \)-order. The Lorentz boost is given by the matrix with components \( \Lambda^0_0 = \gamma \), \( \Lambda^0_i = \gamma v^i / c \), \( \Lambda^i_0 = -\gamma v^i / c \), \( \Lambda^i_j = \delta^i_j + \gamma^2 - 1 v^i v_j \), where \( \gamma = (1 - v^2 / c^2)^{-1/2} \). Using these expressions in Eq. (72), we write it in \( \frac{1}{c^2} \)-approximation

\[
S' = S + \frac{2GM}{c^2 r} S - \frac{1}{2c^2} \left[ v^2 S - (v \cdot S)v \right].
\]

(73)

To compute derivative \( dS' / ds \) of this expression, we note that the difference between \( ds \) and \( dt \) can be neglected, being of order \( \frac{1}{c^2} \), so we can replace \( ds \) on \( dt \) on the right hand side of (73). For \( dS'/ds \) we use its expression (77) in the leading approximation, \( dS' = -\frac{MG}{c^2 r^2} \hat{r} \). The result is

\[
\frac{dS'}{ds} = \frac{dS}{dt} - \frac{GM}{c^2 r^2} \left[ (\hat{r} \cdot v)S + \frac{1}{2}(\hat{r} \cdot S)v + \frac{1}{2}(v \cdot S)\hat{r} \right].
\]

(74)

We substitute (68) into (74), and then replace \( S \) on \( S' \) in the resulting expression, since according to (73), \( S \) differs from \( S' \) only by terms of order \( \frac{1}{c^2} \). The final result for total torque in the rest frame of gyroscope is

\[
\frac{dS'}{ds} = \tau_v + \tau_J + \tau_R,
\]

(75)

where

\[
\tau_v' = \frac{3GM}{2c^2 r^2} \hat{r} \times v \times S',
\]

(76)

while \( \tau_J' \) and \( \tau_R' \) are given by (70) and (71), where \( S \) must be replaced on \( S' \).

Comments. 1. Curiously enough, spin torque in original coordinates, being averaged over a revolution along an almost closed orbit, almost coincides with instantaneous torque in the co-moving frame. This has been observed by direct computation of the mean value of \( \langle \frac{ds}{dr} \rangle \), see [42].

The same result is implied by Eq. (74): \( \langle \frac{ds}{dr} \rangle - \langle \frac{ds}{dr} \rangle \sim \frac{\tau_v}{GM} \sim \frac{1}{c^2} \), and since \( \langle \frac{ds}{dr} \rangle \approx \frac{ds}{dr} \), we have \( \langle \frac{ds}{dr} \rangle \approx \frac{GM}{c^2 r^2} \).

2. Spin-tensor subject to the condition \( S^{\mu\nu} P_{\nu} = 0 \) can be used to construct four-dimensional Pauli-Lubanski vector

\[
s_{\mu} = \frac{\sqrt{-det g_{\mu\nu}}}{\sqrt{-P^2}} \epsilon_{\mu\alpha\beta\gamma} P_{\alpha} S^{\beta\gamma}, \quad \epsilon_{0123} = -1(77)
\]

In a free theory, where \( P^{\alpha} \) does not depend on \( S^{\beta\gamma} \), this equation can be inverted, so \( \epsilon_{\alpha\beta\gamma} \) and \( s \) are mathematically equivalent. Hence spatial components \( s \) could be equally used to describe spin of a gyroscope [40]. In \( \frac{1}{c^2} \)-approximation we have \( P^\alpha = m\hat{v}^\alpha \), and it implies \( s = S \) in the rest frame of gyroscope. Under general-coordinate transformations, \( S \) transforms as spatial part of a tensor, while \( s \) transforms as a part of four-vector. So the two spins differ in all frames except the rest frame. Let us find the relation between them in \( \frac{1}{c^2} \)-approximation. Using the approximate equalities \( (-v^2)^{-1/2} = \frac{1}{c} \left( 1 + \frac{v^2}{c^2} + \frac{GM}{c^2 r} \right) \) and \( -\sqrt{-det g_{\mu\nu}} = 1 + \frac{2GM}{c^2 r} \) together with Eqs. (72), (54) and (68), we obtain for spatial part of (77)

\[
s = \frac{1}{\gamma} S + \frac{1}{c^2} (v \cdot S)v + \frac{3GM}{c^2 r} S.
\]

(78)

Computing derivative of this equality and using (68)- (77), we arrive at the following expression for variation rate of \( s \):

\[
\frac{ds}{dt} = \frac{GM}{c^2 r^2} \left[ (s \cdot \hat{r})v - (v \cdot \hat{r})s - 2(s \cdot v)\hat{r} \right] + \tau_J + \tau_R.
\]

(79)

The first term coincides with that of Weinberg [40].

Post-Newtonian Hamiltonian. Let us obtain an effective Hamiltonian, which yields the equations (57) and (68) in \( \frac{1}{c^2} \)-approximation. According to the procedure described in [33], complete Hamiltonian for dynamical variables as functions of the coordinate time \( t \) is
\[ H = -c p_0, \text{ where } p_0 \text{ is a solution to the mass-shell constraint (24) with } P_\mu \text{ given in (3).} \]

Solving the constraint, we obtain
\[
H = \frac{c}{\sqrt{-g^{00}}} \sqrt{(mc)^2 + g^{ij} P_i P_j + \frac{1}{16} (\theta S)} - c \pi_{\mu} \Gamma^\mu_{\nu \omega} \omega^\nu + \frac{cg^{00}}{g^{00}} P_0. \tag{80}
\]

where \( \gamma_{ij} = g^{ij} - \frac{\delta^{00}}{g^{00}} \). After tedious computations, this gives the following expression up to \( \frac{1}{r^2} \)-order:
\[
H = mc^2 + \frac{1}{2m} \left[ p + \frac{m}{c} \left( \frac{2G}{c} [J \times \frac{r}{r^3}] + \frac{2M}{m c} [S \times \frac{r}{r^3}] \right) \right]^2
- \frac{1}{8m^3 c^2} \left( \frac{2G}{c} [J \times \frac{r}{r^3}] + \frac{2M}{m c} [S \times \frac{r}{r^3}] \right) \cdot [\bigtriangledown \times [J \times \frac{r}{r^3}] + \frac{2}{m c} S \times \frac{r}{r^3}] \cdot S. \tag{81}
\]

Together with the Dirac brackets \( (15) \), also taken in \( \frac{1}{r^2} \)-approximation, this gives Hamiltonian equations of motion. Excluding from them the momentum \( p \), we arrive at the Lagrangian equations \( (57) \) and \( (68) \).

To write the Hamiltonian in a more convenient form, we introduce vector potential \( A_{ij} = -c^{-1} g_{0i} \) for the gravitomagnetic field \( B_J \), produced by rotation of central body (we use the conventional factor \( 2G/c \)), different from that of Wald \( (26) \). In the result, our \( B_J = 4B_{Wald} \)
\[
A_J = \frac{2G}{c} [J \times \frac{r}{r^3}],
\]
then \( B_J = \left[ \bigtriangledown \times A_J \right] = \frac{2G}{c} \frac{3J \hat{r} \times J - J}{r^3} \).

Then Eq. \( (81) \) prompts to introduce also the vector potential \( A_S \) of fictitious gravitomagnetic field \( B_S \) due to rotation of a gyroscope
\[
A_S = \frac{2M}{m c} [S \times \frac{r}{r^3}],
\]
then \( B_S = \left[ \bigtriangledown \times A_S \right] = \frac{2G}{m c} \frac{3S \hat{r} \times S - S}{r^3}, \tag{83}
\]
as well as the extended momentum
\[
\Pi \equiv p + \frac{m}{c} (A_J + 2A_S). \tag{84}
\]

With these notation, the Hamiltonian \( (81) \) becomes similar to that of spinning particle in a magnetic field
\[
H = mc^2 + \frac{1}{2m} \Pi^2 - \frac{(\Pi^2)^2}{8m^3 c^2} - \frac{3GM}{2m c^2 r^2} \Pi^2
- \frac{mGM}{r} + \frac{m(MG)^2}{2c^2 r^2} + \frac{1}{16} \left( B_J + B_S \right) \cdot S \tag{85}
\]
\[
= \frac{c}{\sqrt{-g^{00}}} \sqrt{(mc)^2 + g^{ij} \Pi^i \Pi^j + \frac{1}{2c} \left( B_J + B_S \right) \cdot S}. \tag{86}
\]

Note that the Hamiltonian \( \sqrt{g^{00}} \sqrt{(mc)^2 + g^{ij} p_i p_j} \) corresponds to the usual Lagrangian \( L = -mc \sqrt{-g_{\mu \nu} \dot{x}^\mu \dot{x}^\nu} \) describing a particle propagating in the Schwarzschild metric \( g_{\mu \nu} \). So, the approximate Hamiltonian \( (86) \) can be thought as describing a gyroscope orbiting in the field of Schwarzschild space-time and interacting with the gravitomagnetic field.

Effective Hamiltonian for MPTD equations turns out to be less symmetric: it is obtained from \( (86) \) excluding the term \( \frac{1}{c^2} (B_J \cdot S) \), while keeping the potential \( A_S \) in \( (84) \). Hence the only effect of non-minimal interaction is the deformation of gravitomagnetic field of central body according to the rule
\[
B_J \rightarrow B_J + B_S. \tag{87}
\]

V. DISCUSSION

Starting from a variational problem, we have studied relativistic spinning particle with non-minimal spin-gravity interaction through the gravimagnetic moment \( \kappa \). Hamiltonian equations for an arbitrary \( \kappa \) are presented in \( (31) - (33) \). When \( \kappa = 0 \), our variational problem yields MPTD equations \( (19) \) and \( (20) \), accompanied by the momentum-velocity relation \( (18) \) and by the expected constraints \( (7), (10) \) and \( (11) \). When \( \kappa = 1 \), the MPTD equations are modified by extra terms, see Eqs. \( (31) \) and \( (35) \) above.

We have computed, in the coordinate-time parametrization \( t = \frac{r^2}{\sqrt{\eta}} \), the acceleration \( (88) \) and the spin torque \( (69) - (71) \) of our gravimagnetic particle in the field of a rotating central body \( (61) \) in the leading post-Newtonian approximation. We also obtained the approximate Hamiltonian \( (86) \), which implies these expressions in the Hamiltonian formulation with use of Dirac brackets. As it should be expected, the expressions \( (62), (63) \) and \( (69), (70) \) coincide with those of known from analysis of MPTD equations \( (39), (40), (43), (44), (45) \). The new terms due to the non-minimal interaction are \( (64) \) and \( (74) \). Using the notation \( (52) \) and \( (83) \), the total acceleration of spinning particle in \( \frac{1}{r^2} \)-approximation reads
\[
a = -\frac{MG}{c^2 r^2} (\hat{r} \times \mathbf{v}) - \frac{GM}{c^2 r^2} \hat{r}^2
+ \frac{4G^2 M^2}{c^2 r^3} \hat{r}
+ \frac{1}{c} \left( B_J + B_S \right) \times \mathbf{v} + \frac{GM}{mc^2 r^3} [S \times \mathbf{v} + 3(S \cdot (\hat{r} \times \mathbf{v})) \hat{r}]
- \frac{1}{2mc} \nabla \left( \left| \left| B_J + B_S \right| \right| \cdot S \right). \tag{89}
\]

The first term in \( (88) \) represents the standard limit of Newtonian gravity and implies an elliptical orbit. The next three terms represent an acceleration in the orbital plane and are responsible for the precession of perihelia \( (40), (45), (46) \). The term \( \frac{1}{c} B_J \times \mathbf{v} \) represents the
\[4\text{ We recall (44) that vector potential, produced by a localized current distribution } J(x') \text{ in electrodynamics is determined, in the leading order, by the vector of magnetic moment } \mu = \frac{1}{c} \int [\mathbf{J} \times J(x')] d^3x \text{ as follows: } \mathbf{A} = [\mu \times \mathbf{x}], \text{ and the corresponding magnetic field is } \mathbf{B} = [\nabla \times \mathbf{A}] = \frac{3(\mu \times \mathbf{r}) \times \mu}{r^3}.\]
acceleration due to Lense-Thirring rotation of central body, while the remaining terms in (89) and (90) describe the influence of the gyroscopes spin on its trajectory. The first term in (89) has been computed by Lense and Thirring [48, 50], the remaining terms in (89) have been discussed in [17, 24, 35]. The gravitational dipole-dipole force \( \frac{1}{2mc} \nabla (B_J \cdot S) \) has been computed by Wald [20]. The new contribution due to non-minimal interaction, \( \frac{1}{2mc} \nabla (B_S \cdot S) \), is similar to the Wald term. The acceleration (89) comes from second term of effective Hamiltonian (85), while (90) comes from the last term.

The geodetic precession (70) comes from second term of effective Hamiltonian (85), while the frame-dragging precession (71) is produced by the term \( \frac{1}{c^2} (B_J \cdot S) \). So they are the same for both gravimagnetic and MPTD particle. They have been first computed by Schiff [38], and measured during Stanford Gravity Probe B experiment [53, 54]. The term (71) is due to non-minimal interaction, and appears only for gravimagnetic interaction.

Comparing the expressions (70) and (71), we conclude that precession of spin \( S \) due to non-minimal interaction is equivalent to that of caused by rotation of central body with the momentum \( J = \frac{m}{c} S \).

Effective Hamiltonian for the case of non-rotating central body (Schwarzschild metric) is obtained from (86) by setting \( A_J = B_J = 0 \). We conclude that, due to the term \( \frac{1}{c^2} B_S \cdot S \), the spin of gravimagnetic particle will experience frame-dragging effect (71) even in the field of a non-rotating central body.

In a co-moving frame, gravimagnetic particle experiences the precession \( \frac{dS}{dt} = [\Omega \times S] \) with angular velocity

\[
\Omega = \frac{3GM}{2c^2r^2} [\hat{r} \times v] + \frac{1}{2c} B_J + \frac{1}{c} B_S, \tag{91}
\]

which depends on gyroscopes spin \( S \). Hence, two gyroscopes with different magnitudes and directions of spin will precess around different rotation axes. Then the angle between their own rotation axes will change with time in Schwarzschild or Kerr space-time. Since the variation of the angle can be measured with high precision, this effect could be used to find out whether a rotating body has unit or null gravimagnetic moment.

To estimate the relative magnitude of spin torques due to \( B_J \) and \( B_S \), we represent them in terms of angular velocities. Assuming that both bodies are spinning spheres of uniform density, we write \( J = I_1 \omega_1 \) and \( S = I_2 \omega_2 \), where \( \omega_1 \) is angular velocity and \( I_1 = (2/5)I_1 \) is moment of inertia. Then the last two terms in (91) read

\[
\Omega_{fd} = \frac{2Gm_1r_1^2}{5c^2r^3} \left[ 3 \left( \omega_1 + \rho^2 \omega_2 \right) \hat{r} - (\omega_1 + \rho^2 \omega_2) \right], \tag{92}
\]

where \( \rho \equiv (r_2/r_1) \). Note that \( \Omega_{fd} \) does not depend on mass of the test particle. The ratio \( \rho^2 \equiv (r_2/r_1)^2 \) is extremely small for the case of Gravity Probe B experiment, so the MPTD and gravimagnetic bodies are indistinguishable in this experiment. For a system like Sun-Mercury \( \rho^2 \sim 10^{-5} \). For a system like Sun-Jupiter \( \rho^2 \sim 10^{-2} \). The new effect could be relevant to the analysis of binary pulsars with massive companions, where the geodetic spin precession has been observed [55–57]. Besides, the two torques could have a comparable magnitudes in a binary system with stars of the same size (so \( \rho = 1 \)), but one of them much heavier than the other (neutron star or white dwarf). Then our approximation of a central field is reasonable and, according to Eq. (92), the frame-dragging effect due to gravimagnetic moment becomes comparable with the Schiff frame-dragging effect.

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