A FIXED POINT THEOREM FOR CLOSED-GRAPHED
DECOMPOSABLE-VALUED CORRESPONDENCES

IDIONE MENEGHEL AND RABEE TOURKY

ABSTRACT. We prove a fixed point theorem for closed-graphed, decomposable-valued correspondences whose domain and range is a decomposable set of functions from an atomless measure space to a topological space. One consequence is an improvement of the fixed point theorem in Cellina, Colombo, and Fonda [3].

1. Introduction

Let \((S, \Sigma, \mu)\) be an atomless measure space and let \(L\) be a Banach space. Denote by \(L_1(\mu, L)\) the Banach space of all \(L\)-valued Bochner \(\mu\)-integrable functions (equivalence classes) on \(S\). A set \(F \subseteq L_1(\mu, L)\) is decomposable if for every \(f, g \in F\) and each \(E \in \Sigma\) the decomposition \(\chi_{E}g + \chi_{S \setminus E}f\) is in \(F\). A set \(X \subseteq L_1(\mu, L)\) is \(\mu\)-uniformly compact if for each \(\varepsilon > 0\) there is \(E \in \Sigma\) with \(\mu(E) < \varepsilon\) such that
\[
\{\chi_{S \setminus E}f : f \in X\}
\]
is compact when endowed with the metric \(\delta(f, g) = \|f - g\|_{\infty} \wedge 1\) and there is an integrable function \(g \in L_1(\mu, L)\) satisfying \(\|g(s)\| \geq \|f(s)\|\) almost surely for all \(f \in X\). The following is one consequence of the theorem in this paper. It illustrates the relation between the work here and the literature on fixed point theory for decomposable sets.

Corollary 1.1. Let \(F\) be a nonempty closed decomposable set in \(L_1(\mu, L)\) and let \(B : F \rightarrow F\) be a decomposable-valued correspondence with a closed graph. If there is a nonempty \(\mu\)-uniformly compact set \(X \subseteq F\) satisfying \(X \cap B(f) \neq \emptyset\) for each \(f \in X\), then \(B\) has a fixed point.

A related result is the fixed point theorem for continuous functions in Cellina [2], and its generalization in Fryszkowski [7]. There \(B\) is a continuous function that maps \(F\) into a norm compact, not necessarily decomposable, set \(X\). Cellina, Colombo and Fonda’s [3] proved the existence of approximate continuous selections for decomposable-valued upper hemicontinuous correspondences. One application that appears in [3] of this selection result is a fixed point theorem for decomposable-valued correspondences. However, this corollary required that the range of the correspondence be norm compact and decomposable. It was latter shown by Cellina and Mariconda [4] that decomposable compact sets in \(L_1(\mu, L)\) contain at most one element.

Date: May 7, 2014.

We wish to thank Bill Johnson, Ben de Pagter, and Tony Wickstead for informative discussions on the local continuity properties of sublattices of Banach lattices.
Let \((S, \Sigma, \mu)\) be an atomless probability space and let \(T\) be a nonempty topological space. Let \(L(S, T)\) be the set of all functions, not necessarily measurable, from \(S\) to \(T\). Endow \(L(S, T)\) with the topology of pointwise convergence.

For any \(E \in \Sigma\) and any pair of functions \(f, g \in L(S, T)\) let \(g_E f \in L(S, T)\) be defined as follows:

\[
g_E f(s) = \begin{cases} g(s) & \text{if } s \in E, \\ f(s) & \text{if } s \in S \setminus E, \end{cases}
\]

for all \(s \in S\).

A subset \(F \subseteq L(S, T)\) is decomposable if for each \(E \in \Sigma\) and \(f, g \in F\) we have \(g_E f \in F\). A set-valued (possibly empty valued) mapping \(B : F \rightarrow F\) is a decomposable mapping if its domain \(F\) and values \(B(f)\), for all \(f \in F\), are decomposable subsets of \(L(S, T)\). A decomposable mapping \(B\) is \(\mu\)-sequentially closed graphed if the following hold:

1. If \(\mu(E) = 0\) and \(g \in B(f)\), then \(h_E g \in B(f)\) and \(g \in B(h_E f)\) for all \(h \in F\).
2. \(F\) is sequentially closed in \(L(S, T)\).
3. \(B\) has a sequentially closed graph in \(F \times F\).

A fixed point of \(B\) is a function \(f \in F\) satisfying \(f \in B(f)\).

**Definition 2.1.** A subset \(Y \subseteq L(S, T)\) has the \(\mu\)-fixed point property if every decomposable \(\mu\)-sequentially closed-graphed mapping \(B : F \rightarrow F\) that satisfies the following:

- (a) \(Y \cap F \neq \emptyset\).
- (b) \(Y \cap B(f) \neq \emptyset\) for all \(f \in Y \cap F\).

has a fixed point in \(Y \cap F\).

A subset \(X\) of \(L(S, T)\) is metrizable if it is a metrizable topological space when endowed with the topology of pointwise convergence. Sets (of measurable functions) that are compact and metrizable in the topology of pointwise convergence are extensively studied in [10, 11].

**Theorem 2.2.** Each compact and metrizable set \(X \subseteq L(S, T)\) is a subset of a sequentially compact set \(Y \subseteq L(S, T)\) that has the \(\mu\)-fixed point property.

We have the following immediate corollary.

**Corollary 2.3.** Let \(B : F \rightarrow F\) be a decomposable \(\mu\)-sequentially closed-graphed mapping. If for a nonempty compact and metrizable \(X \subseteq F\) we have \(X \cap B(f) \neq \emptyset\) for each \(f \in F\), then \(B\) has a fixed point.

3. **Proof of Theorem 2.2**

Let \(C \subseteq [0,1]\) be the Cantor ternary set. Let \(L([0,1], C)\) be the set of all functions from \([0,1]\) to \(C\). A function \(\theta : L([0,1], C) \rightarrow L(S, T)\) is sequentially pointwise continuous if for any sequence \(f_n \in L([0,1], C)\) converging pointwise to \(f \in L([0,1], C)\), the sequence \(\theta(f_n)\) converges pointwise to \(\theta(f)\) in \(L(S, T)\). Extending the notion of decomposability to subsets of \(L([0,1], C)\), we say that
$D \subseteq L([0,1], C)$ is decomposable if for any pair of functions $f, g \in D$ and Borel set $E \subseteq [0,1]$ we have $g_E f \in D$.

**Lemma 3.1.** There is a function $\theta: L([0,1], C) \to L(S, T)$ satisfying the following:

1. $\theta$ is sequentially pointwise continuous.
2. $\theta$ maps the constant functions in $L([0,1], C)$ onto $X$.
3. If $D \subseteq L(S, T)$ is decomposable, then $\theta^{-1}(D)$ is decomposable.
4. If $f, g \in L([0,1], C)$ differ on exactly a countable set of points in $[0,1]$, then $\theta(f)$ and $\theta(g)$ differ on a $\mu$-zero measure set.

**Proof.** Order the members of $\Sigma$ as follows $E \leq E'$ if either $E = E'$ or $E \subseteq E'$ and $\mu(E) < \mu(E')$. Consider a maximal chain $\{E_\lambda\}$ of this ordering containing $S$ and the empty set. Because $\mu$ is atomless $E_\lambda \mapsto \mu(E_\lambda)$ is a one to one mapping from $\{E_\lambda\}$ to $[0,1]$. So we can reindex the maximal chain by means of the identity $\lambda = \mu(E_\lambda)$. Let $\mathbb{Q}$ be the set of rational numbers in $[0,1]$ and for each $s \in S$ let

$$r(s) = \inf\{\lambda \in \mathbb{Q}: s \in E_\lambda\} = \sup\{\lambda \in \mathbb{Q}: s \notin E_\lambda\}.$$ 

This is a measurable function satisfying $\mu(r^{-1}(E)) = 0$ for any zero measure Borel subset of $[0,1]$.

Because $X$ is compact and metrizable, the Hausdorff-Alexandroff Theorem says that there is a continuous function $\psi$ mapping $C$ onto $X$.

Define the $\theta: L([0,1], C) \to L(S, T)$ as follows:

$$\theta(f)(s) = \psi(f(r(s)))(s),$$

for all $f \in L([0,1], C)$ and $s \in S$. Note that

$$\theta(g_E f) = \theta(g)_{r^{-1}(E)} \theta(f)$$

for any $f, g \in L([0,1], C)$ and $E \subseteq [0,1]$.

We prove that $\theta$ has the required properties:

1. If $f_n \in L([0,1], C)$ is a sequence that converges pointwise to $f \in L([0,1], C)$, then for any $\alpha \in [0,1]$ the sequence $\psi(f_n(\alpha))$ converges to $f(\alpha)$ in $X$. Thus, for all $s \in S$ the sequence $\psi(f_n(\alpha))(s)$ converges in $T$. This tells us that $\theta(f_n)$ converges pointwise to $\theta(f)$ in $L(S, T)$.
2. If $f(\alpha) = c$ for all $\alpha \in [0,1]$, then $\theta(f)(s) = \psi(c)(s)$ for all $s$.
3. Let $D \subseteq L(S, T)$ be a decomposable set. If $f, g \in \theta^{-1}(D)$ and $E$ is a Borel subset of $[0,1]$, then $r^{-1}(E)$ is in $\Sigma$ and $\theta(g_E f) = \theta(g)_{r^{-1}(E)} \theta(f) \in D$. Thus, $g_E f \in \theta^{-1}(D)$.
4. If $f, g \in L([0,1], C)$ and $g = h_E f$ for some zero measure Borel set $E \subseteq [0,1]$, then $\theta(g) = \theta(h_E f) = \theta(h)_{r^{-1}(E)} \theta(f)$ and $r^{-1}(E) = 0$.

$\square$

Fix a function $\theta: L([0,1], C) \to L(S, T)$ satisfying the properties in Lemma 3.1. Let $\mathcal{M}$ be the set of monotone functions from $[0,1]$ to $C$. This is a sequentially compact set in the topology of pointwise convergence. Let $Y = \theta(\mathcal{M})$, which is also a sequentially compact subset of $L(S, T)$, because $\theta$ is sequentially continuous. The set $Y$ contains $X$ because $\mathcal{M}$ contains the constant functions, and $\theta$ maps the constant functions onto $X$. We want to show that $Y$ has the $\mu$-fixed point property.

Fix a set valued mapping $B: F \to F$ that is decomposable, $\mu$-sequentially upper hemicontinuous, and that satisfies the following:

(a) $Y \cap F \neq \emptyset$. 

(b) $Y \cap B(f) \neq \emptyset$ for all $f \in Y \cap F$.

We need to show that $B$ has a fixed point in $Y$.

Let $\mathcal{F} = \theta^{-1}(F)$, which is a subset of $L([0,1],C)$, and note that it is decomposable and sequentially closed, because of properties (3) and (1), respectively, of Lemma 3.1. For each $f \in \mathcal{F}$ let

$$P(f) = \theta^{-1}(B(\theta(f))).$$

We record the following properties of the mapping $P: \mathcal{F} \to \mathcal{F}$.

**Lemma 3.2.** The following hold true:

1. $\mathcal{F}$ is sequentially closed, decomposable, and $\mathcal{M} \cap \mathcal{F}$ is non-empty.
2. For each $f \in \mathcal{F}$, the set $P(f)$ is sequentially closed and decomposable.
3. $P$ has a sequentially closed graph in $\mathcal{F} \times \mathcal{F}$.
4. If $E \subseteq [0,1]$ is countable, then $g \in P(f)$ implies that $h_E g \in P(f)$ and $g \in P(h_E f)$ for all $h \in \mathcal{F}$.
5. For any $f \in \mathcal{M} \cap \mathcal{F}$, the set $P(f) \cap \mathcal{M}$ is nonempty.
6. If $f$ is a fixed point of $P$, then $\theta(f)$ is a fixed point of $B$.

**Proof.** (1) and (2) are consequences of (1) and (3) of Lemma 3.1. (3) is a consequence of (1) of Lemma 3.2. (4) follows from (4) of Lemma 3.1. (5) holds because $B(f) \cap Y$ is not empty for any $f \in Y \cap F$. Finally, (6) holds because if $f \in P(f)$, then $\theta(f) \in B(\theta(f))$. \qed

So our task now is to show that $P$ has a fixed point in $\mathcal{M}$.

Let $\mathcal{Z} = \mathcal{F} \cap \mathcal{M}$, which is sequentially closed and nonempty by (1) of Lemma 3.2. Define the mapping $Q: \mathcal{Z} \to \mathcal{Z}$ by letting

$$Q(f) = P(f) \cap \mathcal{M},$$

for all $f \in \mathcal{Z}$. This is a nonempty valued correspondence with sequentially closed graph in $\mathcal{Z} \times \mathcal{Z}$, because of (2) of Lemma 3.2. For any $f \in \mathcal{M}$ let $f^{-}$ be the right continuous version of $f$; setting $f^{-}(1) = 1$ for all $f \in \mathcal{M}$. Recall that $f^{-}$ differs from $f$ over a countable subset of $[0,1]$. Also, if $g^{-} = f^{-}$, then $g$ differs from $f$ on a countable subset of $[0,1]$. In particular, $Q(f) = Q(g)$ and if $f \in Q(h)$, then $g \in Q(h)$. This is, as a result of property (1) of the definition of $\mu$-sequentially graphed mappings and (4) of Lemma 3.1.

For $G \subseteq \mathcal{M}$, we write $G^{-}$ for the set $\{f^{-}: f \in G\}$. For each $f \in \mathcal{Z}$ choose an arbitrary $g \in \mathcal{Z}$ satisfying $g^{-} = f$ and let

$$\hat{Q}(f) = Q(f)^{-}.$$

The mapping $\hat{Q}: \mathcal{Z}^{-} \to \mathcal{Z}^{-}$ is nonempty valued, because $Q$ is nonempty valued. Further, if $f$ is a fixed point of $\hat{Q}$, then any $g \in \mathcal{Z}$ satisfying $g^{-} = f$ is a fixed point of $Q$, and the required fixed point of $P$. So we are done if we show that $\hat{Q}$ has a fixed point.

Endow $\mathcal{M}$ with the pseudometric

$$\delta(f,g) = \int_{0}^{1} |f(a) - g(a)| \, da.$$

Notice that $(\mathcal{M}^{-}, \delta)$ and $(\mathcal{Z}^{-}, \delta)$ are a compact metric spaces. Furthermore, $\hat{Q}$ has a $\delta$-closed graph in $\mathcal{Z}^{-} \times \mathcal{Z}^{-}$. 


Order the set $\mathcal{M}^\prec$ of right-continuous monotone functions by means of the pointwise ordering whereby $f \geq g$ if $f(\alpha) \geq g(\alpha)$ for all $\alpha \in [0,1]$. The set $(\mathcal{M}^\prec, \delta)$ is a $\delta$-compact topological meet semilattice using the terminology in [8].

Let $\Gamma$ be the set of all nonempty closed subsets of $(\mathcal{M}^\prec, \delta)$ endowed with the metric induced by Hausdorff distances. For any $U \subseteq \mathcal{M}^\prec$ we write $\inf U$ for the pointwise inf of the set of functions in $U$. This is a monotone right continuous function in $\mathcal{M}^\prec$ and the infimum of the set $U$ in the lattice $\mathcal{M}^\prec$. Notice that $\inf U = \inf \overline{U}$, where $\overline{U}$ is the closure of $U$ in $(\mathcal{M}^\prec, \delta)$. This is because if $f_n$ is a sequence in $U$ that $\delta$-converges to $f$, then it pointwise converges to some $g \in \mathcal{M}$ satisfying $g^\prec = f$. But $g(a) \leq f(a)$ for all $a \in [0,1]$. We will now show that the function $U \mapsto \inf U$ from $\Gamma$ to $(\mathcal{M}^\prec, \delta)$ is continuous.

**Lemma 3.3.** If a sequence $U_n \in \Gamma$ converges to $U \in \Gamma$, then $\inf U_n$ converges to $\inf U$ in $(\mathcal{M}^\prec, \delta)$.

**Proof.** Let $f = \inf U$. For each $n$ let $f_n = \inf U_n$. All of these are in $\mathcal{M}^\prec$. Let $f^*$ be an accumulation point in $(\mathcal{M}^\prec, \delta)$ of $f_n$, by moving to a subsequence we shall suppose that $f_n$ converges to $f^* \in \mathcal{M}^\prec$. We want to show that $f^* = f$.

For each $n$ let $V_n = \cup_{m \geq n} U_m$. Let $g_n = \inf V_n$ for each $n$, and recall that $g_n \in \mathcal{M}^\prec$. The sequence $g_n$ is increasing pointwise, so let $g$ be sup{$g_n$} (taking the pointwise supremum), which is in $\mathcal{M}$ but not necessarily right continuous. Now $U$ is in the closure of $V_n$ for each $n$. Thus, $g_n = \inf (V_n \cup U) \leq \inf U = f$ for all $n$. In particular, $g(a) \leq f(a)$ for all $a \in [0,1]$.

Let $\tilde{f}$ be the left continuous version of $f$, setting $\tilde{f}(0) = 0$. Suppose by way of contradiction that for some $a \in [0,1]$ we have $g(a) < d_2 < d_1 < \tilde{f}(a)$. There is $\gamma > 0$ such that $f(a - \gamma) > d_1$. Pick $n$ large enough such that $\delta(h, U) < (d_1 - d_2)\gamma$ for all $h \in V_n$. Pick $h \in V_n$ satisfying $h(a) < d_2$ and $h' \in U$ satisfying $\delta(h, h') < (d_1 - d_2)\gamma$. But $d_1 < \tilde{f}(b) \leq f(b) \leq h'(b)$ for all $a - \gamma \leq b$. Thus, $\delta(h, h') \geq (d_1 - d_2)\gamma$. This is impossible. We conclude that $g(a) \leq \tilde{f}(a)$ for all $a$. Thus, $g^\prec = f$ and $g_n$ converges to $f$.

Now note that $f_n \geq g_n$ for all $n$. For each $a$, for every $\epsilon > 0$, and $n$ there is $m \geq n$ and $h \in U_m$ such that $|h(a) - g_n(a)| < \epsilon$. But $h(a) \geq f_m(a) \geq g_n(a)$. Thus, $f^* = f$. \hfill \Box

The result of Wojdyslawski [15] (cf. [5]) tells us that when endowed with the metric induced by Hausdorff distances, the family of all nonempty closed subsets of a Peano continuum is an absolute retract. We employ this and the previous lemma to establish the next result.

**Lemma 3.4.** If $\mathcal{G} \subseteq L([0,1], C)$ is decomposable and $\mathcal{G} \cap \mathcal{M}$ is nonempty and sequentially closed, then $(\mathcal{G} \cap \mathcal{M})^\prec$ is a compact absolute retract.

**Proof.** The set $(\mathcal{G} \cap \mathcal{M})^\prec$ is nonempty and compact. If $f, g \in \mathcal{G} \cap \mathcal{M}$, then $f \wedge g$ is monotone and differs from $f, g$ on Borel sets. Thus, $f \wedge g$ is in $\mathcal{G} \cap \mathcal{M}$. Noting that $(f \wedge g)^\prec = f^\prec \wedge g^\prec$, we see that $(\mathcal{G} \cap \mathcal{M})^\prec$ is a sub-semilattice of $\mathcal{M}^\prec$. We show that it is locally connected, and thus a Peano continuum.

First, notice that if $f, g \in \mathcal{G} \cap \mathcal{M}$ and $f \geq g$, then $g_{[0,\alpha]} f \in \mathcal{G} \cap \mathcal{M}$ for all $\alpha \in [0,1]$. Thus, if $f, g \in (\mathcal{G} \cap \mathcal{M})^\prec$, then $g_{[0,\alpha]} f \in (\mathcal{G} \cap \mathcal{M})^\prec \cap \mathcal{M}$ for all $\alpha \in [0,1]$.

If $U_n$ is a neighborhood base in $(\mathcal{G} \cap \mathcal{M})^\prec \cap \mathcal{M}$ of $f \in (\mathcal{G} \cap \mathcal{M})^\prec$, then $\inf U_n$ converges to $f$ by Lemma 3.3. Thus, $V_n = \{\inf \{U_n\}, h\} : h \in U_n\},$ where $\{g, h\} = \{h' \in (\mathcal{G} \cap \mathcal{M})^\prec : g \leq h' \leq h\}$, is a neighborhood base at $f$. Let $g = \inf U_n$ and
Let $X \subseteq F$ be the $\mu$-essentially compact subset of $L_1(\mu, L)$ and fix an integrable function $g \in L_1(\mu, L)$ satisfying $\|g(s)\| \geq \|f(s)\|$ almost surely for all $f \in X$. Let $\tilde{F}$ be the functions $f \in F$ satisfying $\|f(s)\| \leq \|g(s)\|$ almost surely. The restriction $\tilde{B}: \tilde{F} \to \tilde{F}$ of $B$ satisfies the conditions of Corollary \ref{corol}.

For each subset $Z$ of $L_1(\mu, L)$ we write $|Z|$ for all the measurable functions in each of equivalence classes of $Z$. For any $f \in \tilde{F}$ let $P(f) = |B(f)|$. Notice that $P$ is a decomposable $\mu$-sequentially closed-graphed mapping. Corollary \ref{corol} is a consequence of Corollary \ref{corol} and the following result.

**Lemma 4.1.** If $X$ is a $\mu$-essentially compact subset of $L_1(\mu, L)$, then $\inf X$ contains a subset that is compact and metrizable in the topology of pointwise convergence.

**Proof.** Let $E_n$ be a decreasing sequence in $\Sigma$ satisfying $\mu(E_n) = \frac{1}{n}$ such that

$$\chi_{\Omega \setminus E_n}, f : f \in X,$$

is compact for the metric $\|f - g\|_{\infty} \wedge 1$.

Let $[f_n]$ be a sequence of equivalence classes in $X$ whose $L_1$-accumulation points comprise every function in $X$. Pick a version $f_n$ from each equivalent class $[f_n]$. There exists $S'$ satisfying $\mu(S') = 1$ such that for any $\ell, m, n$ and $s \in S' \setminus E_m$ we have

$$\|\chi_{S' \setminus E_n}, f_\ell - \chi_{S' \setminus E_m}, f_n\|_{\infty} \geq \|f_\ell(s) - f_n(s)\|.$$ 

Now if $f_{n'}$ is a $L_1$-converging subsequence sequence, then $f_n$ converges pointwise on $S'$. Furthermore, if two subsequence $f_{n'}$ and $f_{n''}$ converge in $L_1$ to $[f]$, then they converge pointwise on $S'$ to the same version of $[f]$. This implies that there is a measurable set $S'' \subseteq S'$ of full measure such that if $f_{n'}$ converges to $f_n$ for a fixed $n$, then it converges pointwise on $S''$ to $f_n$.

So for each $[f] \in X$ we can pick a version $f$ that is zero on the complement of $S''$ and equal to the pointwise limit on $S''$ of any $L_1$-convergent to $f$ subsequence of $f_n$. We see that the collection $X$ of these versions is sequentially compact, and metrizable by means of the $L_1$-metric. □

**References**

[1] S. Athey, Single crossing properties and the existence of pure strategy equilibria in games of incomplete information, *Econometrica* 69 (2001), 861–889.

[2] A. Cellina, A fixed point theorem for subsets of $L^p$, in *Multifunctions and Integrands* (G. Salinetti, ed.), Springer Berlin Heidelberg, 1984, pp. 129–137.
A. Cellina, G. Colombo, and A. Fonda, Approximate selections and fixed points for upper semicontinuous maps with decomposable values, *Proceedings of the American Mathematical Society* **98** (1986), 663–666.

A. Cellina and C. Mariconda, Kuratowski’s index of a decomposable set, *Bulletin of the Polish Academy of Sciences Mathematics* **37** (1989), 679–685.

D. Curtis and R. Schori, Hyperspaces of peano continua are hubert cubes, *Fundamenta Mathematicae* **101** (1978), 19–38.

S. Eilenberg and D. Montgomery, Fixed point theorems for multi-valued transformations, *American Journal of Mathematics* **68** (1946), 214–222.

A. Fryszkowski, *Fixed point theory for decomposable sets*, second ed., Kluwer Academic Publishers, 2004.

G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. W. Mislove, and D. S. Scott, *A compendium of continuous lattices*, 611, Springer Heidelberg, 1980.

S. Grant, I. Meneghel, and R. Tourky, *Savage games*, Mimeo, 2013.

A. Ionescu Tulcea, On pointwise convergence, compactness and equicontinuity in the lifting topology. I, *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete* **26** (1973), 197–205.

A. Ionescu Tulcea, On pointwise convergence, compactness, and equicontinuity. II, *Advances in Mathematics* **12** (1974), 171–177.

D. McAdams, Isotone equilibrium in games of incomplete information, *Econometrica* **71** (2003), 1191–1214.

I. Meneghel and R. Tourky, *A new approach to the existence of equilibrium in Bayesian games*, Mimeo, 2013.

P. J. Reny, On the existence of monotone pure strategy equilibria in Bayesian games, *Econometrica* **79** (2011), 499–553.

M. Wojdyslawski, Retractes absolus et hyperespaces des continus, *Fundamenta Mathematicae* **32** (1939), 184–192.

School of Economics, The University of Queensland, Brisbane St Lucia QLD 4072, Australia.

School of Economics, The University of Queensland, Brisbane St Lucia QLD 4072, Australia.