R-deformed Heisenberg algebra

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Abstract

It is shown that the deformed Heisenberg algebra involving the reflection operator $R$ ($R$-deformed Heisenberg algebra) has finite-dimensional representations which are equivalent to representations of paragrassmann algebra with a special differentiation operator. Guon-like form of the algebra, related to the generalized statistics, is found. Some applications of revealed representations of the $R$-deformed Heisenberg algebra are discussed in the context of $\text{OSp}(2|2)$ supersymmetry. It is shown that these representations can be employed for realizing (2+1)-dimensional supersymmetry. They give also a possibility to construct a universal spinor set of linear differential equations describing either fractional spin fields (anyons) or ordinary integer and half-integer spin fields in 2+1 dimensions.

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1 Introduction

The deformed Heisenberg algebra involving the reflection operator $R$ has found many interesting physical applications. It appeared naturally in the context of parafields $[1, 2]$, but earlier it was known in connection with some quantum mechanical systems $[3]$. Recently this algebra was used for investigating quantum mechanical $N$-body Calogero model $[4]$, for bosonization of supersymmetric quantum mechanics $[5, 6, 7]$ and describing anyons in (2+1) $[7, 8]$ and (1+1) dimensions $[9]$. In all the listed applications the infinite-dimensional unitary representations of the $R$-deformed Heisenberg algebra were used.

In the present paper it will be shown that this algebra has also finite dimensional representations which are equivalent to representations of some paragrassmann algebra $[10]$ with differentiation operator realized in a special form. We shall show that guon-like algebra $[11]$ can be constructed in a natural way proceeding from the $R$-deformed Heisenberg algebra. Such guon-like algebra can be related in some way to the $q$-deformed Heisenberg algebra $[12, 13]$ with deformation parameter $q$ being a primitive root of unity $[10]$. We shall discuss some applications of finite-dimensional representations of the $R$-deformed Heisenberg algebra. In particular, they will be used for realization of OSp(2|2) supersymmetry. The relationship of revealed representations to finite-dimensional representations of (2+1)-dimensional Lorentz group will be established. The latter will be employed for realizing (2+1)-dimensional supersymmetry. We shall also use them for constructing a universal spinor set of linear differential equations describing either fractional spin fields (anyons) or ordinary integer or half-integer spin fields in 2+1 dimensions.

2 Representations of $R$-deformed Heisenberg algebra

The $R$-deformed Heisenberg algebra is given by the generators $a^-$, $a^+$, 1 and by the reflection operator $R$ satisfying the (anti)commutation relations $[1]–[8]:$

\[ [a^-, a^+ + 1 = 1 + \nu R, \quad R^2 = 1, \quad \{a^\pm, R\} = 0, \quad (2.1) \]

and $[a^\pm, 1] = [R, 1] = 0$, where $\nu \in \mathbb{R}$ is a deformation parameter. The reflection operator $R$ is hermitian, whereas $a^+$ and $a^-$ will be considered as mutually conjugate operators with respect to appropriate scalar product. One introduces the vacuum state $|0\rangle$, $a^-|0\rangle = 0$, $\langle 0|0 \rangle = 1$, $R|0\rangle = |0\rangle$, and defines the states $|n\rangle = C_n(a^+)^n|0\rangle$ with some normalization constants $C_n$. Then, from the relation

\[ [a^-, (a^+)^n] = \left(n + \frac{1}{2}(1 - (-1)^n \nu R)\right)(a^+)^{n-1} \]

(2.2)

one concludes that algebra (2.1) has infinite-dimensional unitary representations when $\nu > -1$. In this case the states $|n\rangle$ with $C_n = (|n|!)^{-1/2}$, $|n|! = \Pi_{l=1}^{n}|l|\nu$, $|l\rangle = \frac{l}{2}(1 - (-1)^l)\nu$, form the complete orthonormal basis of Fock space representation, $\langle n|n' \rangle = \delta_{nn'}$. The reflection operator can be realized in terms of creation and annihilation operators via the number operator $[2, 8]$,

\[ N = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}(\nu + 1), \quad N|n\rangle = n|n\rangle, \]

(2.3)
On the other hand, one can consider $R$-deformed Heisenberg algebra (2.4) working in the Schrödinger representation, $\Psi = \Psi(x)$, with creation-annihilation operators realized in the usual form $a^\pm = \frac{1}{\sqrt{2}}(x \mp ip)$. Here the deformed momentum operator is $[p] = -i\frac{d}{dx} + \frac{\nu}{2}R$, and operator $R$ acts as $R\Psi(x) = \Psi(-x)$, and so, $R\Psi_\pm(x) = \pm\Psi_\pm(x)$, $\Psi(x) = \Psi(x)\pm\Psi(-x)$. This explains the name of operator $R$. One can note that if we write realization (2.4) in the Schrödinger representation just in the case of non-deformed ($\nu = 0$) Heisenberg algebra, we shall reveal a hidden nonlocal nature of the reflection operator, $R = \sin H_0$, $H_0 = \frac{1}{2}(x^2 - d^2/dx^2)$. Therefore, the reflection operator has a nature similar to the nonlocal nature of the Klein operator [1].

One can get the realization of the $R$-deformed Heisenberg algebra in terms of non-deformed algebra with creation-annihilation operators $b^\pm$ obeying the commutation relation $[b^-, b^+] = 1$. For the purpose, one represents the operators $a^\pm$ as $a^- = F(N_b)b^-$, $a^+ = (a^-)\dagger = b^+F(N_b)$ with $F = F\dagger$ being a function of the number operator $N_b = b^+b^-$. Let us substitute these expressions for $a^+$ and $a^-$ and $R = (-1)^{N_b}$ into the first relation from (2.4), and act on the complete set of orthonormal states $\langle n\rangle_b \equiv (n!)^{-1/2}(b^+)^n|0\rangle = |n\rangle$, $N_b|n\rangle_b = n|n\rangle_b$, where $|n\rangle$, $n = 0, 1, \ldots$, are the Fock space states of deformed algebra. As a result, we arrive at the sought for realization of deformed creation-annihilation operators in terms of non-deformed ones,

$$a^- = F(N_b)b^-, \quad a^+ = b^+F(N_b), \quad (2.5)$$

$$F(N_b) = \sqrt{1 + \frac{\nu}{2(N_b + 1)}(1 + (-1)^{N_b})}, \quad \nu > -1. \quad (2.6)$$

Function (2.6) takes zero values if we put $\nu = -(2p + 1)$, $p = 1, 2, \ldots$. This indicates [14] that at these special values of the deformation parameter algebra (2.7) has finite-dimensional representations. Then, using eq. (2.2), one finds that for $\nu = -(2p + 1)$, $p = 1, 2, \ldots$, the relation $\langle m|n\rangle = 0$, $|n\rangle \equiv (a^+)^m|0\rangle$, takes place for $n \geq 2p + 1$ and arbitrary $m$. This means, in turn, that the relations $(a^+)^{2p+1} = (a^-)^{2p+1} = 0$ are valid in this case. These latter relations specify finite-dimensional representations of the $R$-deformed Heisenberg algebra. Since in such representations for any $p = 1, 2, \ldots$, there are the states with negative norm (see eq. (2.2)), it means that these finite-dimensional Fock space representations are non-unitary.

### 3 $R$-paragrassmann algebra

Let us consider the revealed finite-dimensional representations in more detail. We have arrived at the nilpotent algebra

$$[a^-, a^+] = 1 - (2p + 1)R, \quad (a^\pm)^{2p+1} = 0, \quad p = 1, 2, \ldots, \quad (3.1)$$

$$\{a^\pm, R\} = 0, \quad R^2 = 1. \quad (3.2)$$

One can interpret $a^+$ as a paragrassmann variable $\theta$, $\theta^{2p+1} = 0$, and in this case $a^-$ can be considered as a differentiation operator [10]. Therefore, the algebra (3.1), (3.2) is a
A paragrassmann algebra of order $2p + 1$ with a special differentiation operator whose action can be defined by relation (2.2). We shall call it the R-paragrassmann algebra. Here, in addition to universal representation (2.3), (2.4), one has also the normal ordered representation for the operator $R$,

$$R = \sum_{n=0}^{2p} f_n a^{+n} a^{-n},$$

with finite recursive relations defining coefficients $f_n$,

$$2f_{n-1} + [n]_\nu f_n - (2p + 1) \sum_{i=0}^{[n/2]-1} f_{2i+1} f_{n-(2i+1)} = 0, \quad n = 1, \ldots, 2p,$$

where $f_0 = 1$ and $[n/2]$ is an integer part of $n/2$.

As a consequence of eqs. (3.1), (3.2), we have the relations $(1-R)a^{+2p} = (1-R)a^{-2p} = 0$. They are equivalent to the nilpotency conditions $a^{\pm (2p+1)} = 0$. Besides, here operators $a^{\pm}$ satisfy the relation

$$a^{+} a^{-2p} + a^{-} a^{+ (2p-1)} + \ldots + a^{-(2p-1)} a^{+} a^{-} + a^{-2p} a^{+} = 0$$

and corresponding conjugate relation. Relations of form (3.4) take place in parasupersymmetric quantum mechanics [17].

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As it has been mentioned above, finite-dimensional Fock space representations of $R$-deformed Heisenberg algebra (2.1) contain the states with negative norm. One may introduce the normalized states as $|n\rangle = |\langle(n|n)\rangle^{-1/2}\langle n|\rangle\rangle$. They define the metric operator $\eta = \eta^\dagger$, $\eta^2 = 1$, whose matrix elements are $||\eta||_{mn} = ||\langle m|n\rangle|| = \text{diag}(1, -1, -1, 1, 1, 1, 1, 1, \ldots, (1)^{p-1}, (1)^{p-1}, (1)^{p}, (1)^{p})$. With this metric operator, the indefinite scalar product is given by the relation $(\Psi_1, \Psi_2) = \langle\Psi_1|\eta\Psi_2\rangle = \Psi_{1n}^* \eta_{mn} \Psi_{2m}$, where $\Psi_n = \langle n|\Psi \rangle$. The operators $a^+$ and $a^-$ can be represented by the matrices $(a^+)_{mn} = A_n \delta_{m-1,n}$, $(a^-)_{mn} = B_m \delta_{m+1,n}$, with $A_{2k+1} = -B_{2k+1} = \sqrt{2}^k$, $k = 0, 1, \ldots, p-1$, $A_{2k} = B_{2k} = \sqrt{2^k}$, $k = 1, \ldots, p$. They satisfy the relation $(a^-)^\dagger = \eta a^+ \eta$, and, as a consequence, are mutually conjugate operators with respect to this scalar product, $(\Psi_1, a^- \Psi_2)^* = (\Psi_2, a^+ \Psi_1)$. The reflection operator has here the diagonal form $R = \text{diag}(+1, -1, +1, \ldots, -1, +1)$.

Below we shall reveal the ‘physical explanation’ of non-unitarity of finite-dimensional representations of the $R$-deformed Heisenberg algebra in which $a^+$ and $a^-$ are interpreted as mutually conjugate operators. On the other hand, one can define hermitian conjugate operators $f^+ = a^+$, $f^- = a^- R$, in terms of which $R$-paragrassmann algebra (3.1), (3.2) is rewritten equivalently as

$$\{f^+, f^-\} = (2p + 1) - R, \quad \{R, f^\pm\} = 0, \quad R^2 = 1, \quad (f^\pm)^{2p+1} = 0, \quad p = 1, 2, \ldots$$

With these operators one could work in a Hilbert space with positive definite scalar product $(\Psi_1, \Psi_2) = \Psi_{1n}^* \Psi_{2n}$, considering $a^+$ and $a^-$ as not basic operators. However, due to concrete physical applications to be considered in what follows, here we shall work in terms of operators $a^\pm$ using the corresponding indefinite scalar product. The described possibility of employing hermitian conjugate operators $f^+$ and $f^-$ will be discussed in last section.
4 Guons, fermions and $q$-deformed Heisenberg algebra

Let us suppose that $\nu \neq 1$, and define the operators $c^- = a^{-G^{-1/2}_\nu}(R)$, $c^+ = a^{G^{-1/2}_\nu}(R)^*$, $G_\nu(R) = |1 - \nu R|$, where for the moment we suppose that $R = (-1)^N$ with $N$ given by eq. (2.3). These operators anticommute with reflection operator, $\{R, c^\pm\} = 0$, and satisfy the commutation relation $c^- c^+ - G_\nu(R)G^{-1}_\nu(R) c^+ c^- = \text{sign}(1 + \nu R)$, where $\text{sign} x$ is +1 for $x > 0$ and −1 for $x < 0$. The operator $G_\nu(R)$ is reduced to $G_\nu(R) = 1 - \nu R$ for $-1 < \nu < 1$; for two other cases we have $G_\nu(R) = \nu - R$, $\nu > 1$, and $G_\nu(R) = R - (2p + 1)$, $\nu = -(2p + 1)$. As a result, commutation relation is represented in first case as

$$c^- c^+ - g_\nu c^+ c^- = 1,$$  
$$g_\nu = (1 - \nu)^R(1 + \nu)^{-R}, \quad -1 < \nu < 1,$$  
(4.1)

whereas in two other cases it is reduced to

$$c^- c^+ - g_\nu c^+ c^- = R,$$  
(4.2)

where $g_\nu = (\nu - 1)^R(1 + \nu)^{-R}$ for $\nu > 1$ and $g_\nu = p^R(1 + p)^{-R}$ for $\nu = -(2p + 1)$. In the case corresponding to finite-dimensional representations the final form (4.2) has been obtained via additional changing $R \to -R$. In all three cases operator-valued function $g_\nu$ satisfies the relation $g_\nu c^\pm = c^\pm g_\nu^{-1}$. The deformed algebra of form (4.1) was introduced in ref. [11] in the context of generalized statistics. The algebra (4.2) represents some modification of (4.1).

The corresponding number operator $N = N(c^+, c^-)$ is given by

$$N = -\frac{\alpha}{2} + \frac{1}{2} \sqrt{|1 - \nu^2| (2c^c - \beta)(2c^c - \beta) + 1},$$

where $\alpha = -\nu$, $\beta = 1$ in the case $-1 < \nu < 1$, and $\alpha = -\nu + \nu^2 - 1$, $\beta = |\nu|$ for two other cases. Implying in relations (4.1), (4.2) that $R = (-1)^N(c^+, c^-)$, one can represent them in a closed form containing only creation-annihilation operators $c^\pm$.

Let us take a limit $\nu \to \infty$ for the case $\nu > 1$ and $p \to \infty$ for $\nu = -(2p + 1)$ proceeding from relation (4.2). Both cases lead to the algebra

$$c^- c^+ - c^+ c^- = R,$$  
$$\{R, c^\pm\} = 0, \quad R^2 = 1.$$  
(4.3)

Considering the Fock space representation defined by relations $c^- |0\rangle = 0$, $R|0\rangle = |0\rangle$, $\langle 0|0\rangle = 1$, one gets the relations $\langle 1|1\rangle = 1$, $\langle 0|1\rangle = 0$, and $\langle m|n\rangle = 0$ for any $m \geq 2$ or $n \geq 2$, where $|n\rangle = (c^+)^n|0\rangle$. It means that the (anti)commutation relations (4.3) have two-dimensional irreducible representation, in which $(c^\pm)^2 = 0$. In this case the operator $R$ is realized as $R = 1 - 2c^+ c^-$, that reduces commutation relations (4.3) to the standard fermionic anticommutation relations, $c^+ c^- + c^+ c^- = 1$, $c^+ c^+ = c^- c^- = 0$. Therefore, fermionic algebra can be obtained from the guon-like form of the $R$-deformed Heisenberg algebra in the limit $|\nu| \to +\infty$.

The substitution of operators $c^\pm$ into bosonic realization (2.3), (2.4) gives for $\nu \to \infty$ the well known realization of fermionic operators in terms of bosonic operators $b^\pm$ [10, 13, 14]:

$$c^- = \frac{\Pi^+}{\sqrt{N + 1}} b^-, \quad c^+ = (c^-)^\dagger.$$  
(4.4)
Here $\Pi_+$ and supplementary operator $\Pi_-$, $\Pi_{\pm} = \frac{1}{2} (1 \pm R)$, are projector operators, $\Pi_{\pm}^2 = \Pi_{\pm}$, $\Pi_+ \Pi_- = 0$, $\Pi_+ + \Pi_- = 1$, and in eq. (4.4) we mean that $R = \pm (-1)^{N_b}$. Due to relations $\Pi_\pm b^- = b^- \Pi_{\mp}$, $\Pi_\pm b^+ = b^+ \Pi_{\mp}$, operators (4.4) satisfy the standard fermionic anticommutation relations.

This bosonization construction for fermions gives us a hint for realization of $R$-paragrassmann algebra in terms of non-deformed creation-annihilation operators $b^\pm$. Indeed, the operators $a^\pm$ satisfying algebra (3.1), (3.2) can be realized as follows:

$$
a^- = \varphi_p(N_b) F_p(N_b) b^-, \quad a^+ = b^+ F_p(N_b) \varphi_p(N_b), \quad (4.5)
$$

where instead of projector operator $\Pi_+ = \sin (\frac{\pi}{2}(N_b + 1))$, we have

$$
\varphi_p(N_b) = \frac{\sin \left( \frac{\pi}{2p+1} (1 + N_b) \right)}{\sin \left( \frac{\pi}{2p+1} (1 + N_p) \right)} \quad (4.6)
$$

with operator $N_p = N_p(N_b)$,

$$
N_p(N_b) = p + \frac{1}{2} - \left| N_b - \left( p + \frac{1}{2} + (2p + 1) \left\lfloor \frac{N_b}{2p+1} \right\rfloor \right) \right|. \quad (4.7)
$$

Here $[X]$ is an integer part of $X$, and operator $F_p(N_b)$ is given by

$$
F_p(N_b) = i^{N_b+1} \sqrt{1 - \frac{2p+1}{2(N_b + 1)} (1 + (-1)^{N_b})}. \quad (4.8)
$$

Operator $\varphi_p$ has the properties $|\varphi_p(n)| = 1$, $n \neq 2p \mod (2p + 1)$, $\varphi_p(2p + k(2p + 1)) = 0$, $k \in \mathbb{Z}$, and so, $\varphi_p(N_b) \varphi_p(N_b + 1) \ldots \varphi_p(N_b + 2p) = 0$. Due to the latter property and relations $G(N_b) b^\pm = b^\mp G(N_b - 1)$ being valid for any function $G(N_b)$, one concludes that operators $b^\pm$ satisfy relations $(a^\pm)^{2p+1} = 0$. They are odd operators, $Ra^\pm = -a^\mp R$, $R = (-1)^{N_b}$, and satisfy $R$-deformed commutation relations (2.3). Since the relation $(a^-)^i \eta = \eta a^+$ takes place with $\eta = \eta^i = (-1)^{(N_b + 1)/2}$, the operators $a^-$ and $a^+$ are mutually conjugate with respect to the indefinite scalar product $\langle \Psi_1, \Psi_2 \rangle = \langle \Psi_1 | \eta \Psi_2 \rangle$. The obtained bosonized representation corresponds to finite-dimensional matrix representation of the $R$-deformed Heisenberg algebra described in the previous section.

Special form of fermionic algebra (4.3) can be generalized into the algebra related to the $q$-deformed oscillator. To realize such a generalization, we note that since $R^2 = 1$, $R$ is a phase operator. Then commutation relations (4.3) ($R$-algebra) can be generalized into the $P$-algebra,

$$
[a, \bar{a}] = P, \quad (4.8)
$$

where $P$ is a phase operator with properties generalizing the corresponding properties of operator $R$,

$$
P^p = 1, \quad P a = qaP, \quad P \bar{a} = q^{-1}\bar{a}P, \quad q = e^{-i\frac{2\pi}{p}}, \quad p = 2, 3, \ldots. \quad (4.9)
$$

Using these relations, one finds that the operators $a^p$ and $\bar{a}^p$ commute with operators $a$, $\bar{a}$ and $P$. In an irreducible representation they are reduced to some constants. Assuming the
existence of the vacuum state $|0\rangle$, $a|0\rangle = 0$, we find that in Fock space representation of algebra (4.8), (4.9), there are the relations $a^p = a^p = 0$. Multiplying relation (4.8) from the left by the operator $P^{-1} = P^{p-1}$, we represent it in the form of Lie-admissible algebra (4.1), $aT\bar{a} - \bar{a}S\bar{a} = 1$, $T = q^{-1}P^{-1}$, $S = qP^{-1}$. Defining new creation-annihilation operators, $c = q^{-1/2}aP^{-1/2}$, $\bar{c} = q^{-1/2}P^{-1/2}\bar{a}$, one gets finally the $q$-deformed Heisenberg algebra $c\bar{c} - q\bar{c}c = 1$, with deformation parameter $q$ being the primitive root of unity.

5 OSp($2|2$) supersymmetry

The $R$-deformed Heisenberg algebra gives a possibility to realize OSp($2|2$) supersymmetry. As a result we can get unitary infinite-dimensional half-bounded representations of $sl(2, R)$, $sl(2, R) \subset osp(1|2) \subset osp(2|2)$, and its non-unitary finite-dimensional representations.

In terms of generators of algebra (2.1), the generators of $osp(2|2)$ superalgebra can be realized as follows. The even generators $J_0, J_\pm = J_1 \pm iJ_2$ and $\Delta$ are given by relations

$$J_0 = \frac{1}{4}\{a^-, a^+\}, \quad J_\pm = \frac{1}{2}(a^\pm)^2, \quad \Delta = \frac{1}{2}(R + \nu). \quad (5.1)$$

They satisfy $sl(2, R) \times u(1)$ algebra,

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_-, J_+] = 2J_0, \quad [\Delta, J_0] = [\Delta, J_\pm] = 0.$$ 

Odd generators $Q^\pm, S^\pm$ are represented as $Q^\pm = a^+\Pi_-, \quad Q^- = a^-\Pi_+, \quad S^\pm = a^+\Pi_+, \quad S^- = a^-\Pi_-$, where $\Pi_\pm$ are the projectors, $\Pi_\pm = \frac{1}{2}(1 \pm R)$. These operators satisfy the following anticommutation relations:

$$Q^{\pm2} = S^{\pm2} = 0, \quad \{S^+, Q^-\} = \{S^-, Q^+\} = 0, \quad (5.2)$$

$$\{Q^+, Q^-\} = 2J_0 + \Delta, \quad \{S^+, S^-\} = 2J_0 - \Delta, \quad \{S^+, Q^+\} = J_+, \quad \{S^-, Q^-\} = J_. \quad (5.3)$$

Nontrivial commutators between even and odd generators are

$$[J_+, Q^-] = -S^+, \quad [J_+, S^-] = -Q^+, \quad [J_-, Q^+] = S^-, \quad [J_-, S^+] = Q^-, \quad (5.4)$$

$$[J_0, Q^\pm] = \pm\frac{1}{2}Q^\pm, \quad [J_0, S^\pm] = \pm\frac{1}{2}S^\pm, \quad [\Delta, Q^\pm] = \pm Q^\pm, \quad [\Delta, S^\pm] = \pm S^\pm. \quad (5.5)$$

In the case $\nu > -1$, the generators $J_0, J_\pm$ give the direct sum of half-bounded infinite-dimensional unitary representations $D^+_\alpha$ and $D^-\alpha_\alpha$ of $sl(2, R)$, being representations of the so called discrete series. Here $\alpha_+ = \frac{1}{2}(1 + \nu)$, $0$, and $\alpha_- = \alpha_+ + \frac{1}{2}$, and these representations are realized on the subspaces spanned by $|2n\rangle$ and $|2n + 1\rangle$, $n = 0, 1, \ldots$, where the corresponding Casimir operator of $sl(2, R)$, $C = -J_0^2 + \frac{1}{2}(J_+, J_-)$, takes the values $C = -\alpha_+(\alpha_+ - 1)$ and $C = -\alpha_-.(\alpha_- - 1)$, and $J_0$ has the spectra $j_0 = \alpha_+ + n$ and $j_0 = \alpha_- + n$, respectively [4, 8].

In the case of the revealed finite-dimensional representations of the $R$-deformed Heisenberg algebra, one finds that the generators $J_0, J_\pm$ give a direct sum of two non-unitary $(p + 1)$- and $p$-dimensional irreducible representations characterized by the values of the Casimir operator $C = -j_{\pm}(j_{\pm} + 1)$ with $j_+ = p/2$ and $j_- = (p - 1)/2$. These representations are realized on the subspaces of even and odd states, $|m\rangle_+ = a^{\dagger 2m}|0\rangle$, $m = 0, 1, \ldots, p$, $|m\rangle_- = a^{\dagger 2m}|0\rangle$, $m = 0, 1, \ldots, p$. 

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fixed value

We have seen, these representations are the direct sum of finite-dimensional representations of (2+1)-dimensional Lorentz group. As it was noted in section 3, the appearance of indefinite scalar product in the case of finite-dimensional representations means that such representations are non-unitary. As we have seen, representations are the direct sum of finite-dimensional representations of (2+1)-dimensional Lorentz group, and since finite-dimensional representations of this group are non-unitary (see, e.g., ref. [17]), we have here a ‘physical explanation’ for non-unitarity of finite-dimensional representations of the $R$-deformed Heisenberg algebra with operators $a^+$ and $a^−$ to be mutually conjugate.

In the simplest cases given by $p = 1$ and $p = 2$, the corresponding metric operator in two-dimensional even ($p = 1$, $j_+ = 1/2$) and odd ($p = 2$, $j_− = 1/2$) subspaces coincides up to a c-number factor with the operator $J_0$ being restricted to the corresponding subspaces. As a result, the indefinite scalar product on these subspaces is the Dirac scalar product. In the case of 3-dimensional vector representations corresponding to $j_+ = 1$, $p = 2$ and $j_− = 1$, $p = 3$, the metric operator and generators $J_\mu$, $\mu = 0, 1, 2$, being restricted to the corresponding even and odd subspaces, can be reduced by appropriate unitary transformation to the standard form of the vector realization with $(J_\mu)^\nu_\lambda = -i\epsilon^{\nu_\mu_\lambda}$ and $\eta_{\mu\nu} = diag(-, +, +)$ [17].

As we have seen, the generators of $sl(2, R)$ algebra act reducibly in the cases of infinite-dimensional and finite-dimensional representations of algebra (2.1). On the other hand, these generators together with operators $a^\pm = Q^\pm + S^\pm$ give irreducible realization of $osp(1|2)$ generators with corresponding Casimir operator $\mathbb{C} = J_\mu J^\mu - \frac{1}{8}[a^−, a^+]$ taking the fixed value $C = \frac{1}{16}(1 − \nu^2)$.

In conclusion of this section we note that relations (5.2) and (5.5) mean that the pair of odd generators $Q^+$ and $Q^−$ together with even generator $H_+ = 2J_0 + \Delta$ form $s(2)$ superalgebra, $Q^{\pm 2} = 0$, $\{Q^+, Q^−\} = H_+$, $[Q^\pm, H_+] = 0$, whereas operators $S^+$ and $S^−$ are odd generators of $s(2)$ superalgebra with even generator $H_− = 2J_0 − \Delta$.

6 Outlook and concluding remarks

The constructed guon-like algebra of the form (4.1), (4.2) contains the operator-valued function $g_\nu$. But unlike the original guon algebra [11], here $g_\nu \neq 1$ and $[g_\nu, c^\pm] \neq 0$. The condition of the form $[g, c^\pm] = 0$ appeared in [11] from the requirement of micro causality under assumption that observables should be bilinear in fields or in creation-annihilation operators. On the other hand, it is known that in the field-theoretical anyonic constructions involving the Chern-Simons gauge field, there are observables (e.g., total angular momentum operator) which are not bilinear in creation-annihilation operators [18]. Moreover, the gauge-invariant fields carrying fractional spin and statistics themselves turn out to be nonlocal operators [19] being decomposable in some infinite series in degrees of creation-annihilation operators of the initial matter field. It seems that the guon-like algebra appeared here could find some applications in the theory of anyons.

The revealed finite-dimensional representations of the $R$-deformed Heisenberg algebra and their relationship to representations of (2+1)-dimensional Lorentz group can be used for the construction of universal minimal spinor set of linear differential equations describing,
on one hand, ordinary integer and half-integer spin fields and, on other hand, fractional spin fields in 2 + 1 dimensions. Moreover, it is natural to try to apply these representations for constructing (2 + 1)-dimensional supersymmetric field systems since, as it was shown, any (2p + 1)-dimensional representation of the \( R \)-deformed Heisenberg algebra carries the direct sum of spin-\( j \), \( j = p/2 \), and spin-(\( j - 1/2 \)) representations of (2 + 1)-dimensional Lorentz group. For the purpose, let us consider the simplest possible nontrivial case corresponding to the choice of 5-dimensional representation of the \( R \)-deformed Heisenberg algebra with \( \nu = -5 \) (\( p = 2 \)), and construct the operators

\[
D_\alpha = \left( \frac{1}{2} - R \right) \mathcal{P}_\alpha - \mathcal{J}_\alpha + \frac{1}{2}emL_\alpha, \quad \epsilon = +, -. 
\]

Here \( \alpha = 1, 2, m \) is a mass parameter, \( \mathcal{P}_\alpha = -i(\gamma^\mu \partial_\mu)_\alpha^\beta \mathcal{L}_\beta \), \( \partial_\mu = \partial/\partial x^\mu \), \( x^\mu \) are external space-time coordinates independent from \( a^\pm \); \( \gamma_\mu \) is the set of (2 + 1)-dimensional \( \gamma \)-matrices taken in the Majorana representation, \( (\gamma^0)_{\alpha \beta} = -(\sigma^2)_{\alpha \beta} \), \( (\gamma^1)_{\alpha \beta} = i(\sigma^1)_{\alpha \beta} \), \( (\gamma^2)_{\alpha \beta} = i(\sigma^3)_{\alpha \beta} \), \( \mathcal{L}_1 = \frac{1}{\sqrt{2}}(a^+ + a^-) \), \( \mathcal{L}_2 = \frac{i}{\sqrt{2}}(a^+ - a^-) \), and \( \mathcal{J}_\alpha = \mathcal{L}^\beta_{\mu \lambda} \partial^\mu J^\nu(\gamma^\nu)_{\alpha \beta} \) with \( J_\mu \) given by eq. (5.1). Operators \( \mathcal{L}_\alpha \), \( \mathcal{J}_\alpha \) and \( \mathcal{P}_\alpha \) are spinor operators with respect to the action of the total angular momentum vector operator, \( M_\mu = i\epsilon_{\mu \lambda \nu} x^\nu \partial^\lambda + J_\mu \). \([M_\mu, M_\nu] = -i\epsilon_{\mu \lambda \nu} M_\lambda \), \([M_\mu, \mathcal{L}_\alpha] = \frac{i}{2}(\gamma_\mu)_{\alpha \beta} \mathcal{L}_\beta \) etc., whereas the reflection operator \( R \) is a scalar, \([M_\mu, R] = 0 \). These properties of the operators are, in fact, the consequence of the \( osp(1|2) \) superalgebra generated by the operators \( J_\mu \) and \( a^\pm \), which has been discussed in the previous section. As a result, operator \( D_\alpha \) is (2 + 1)-dimensional translation-invariant spinor operator. One can consider the set of linear (in \( \partial_\mu \)) differential field equations

\[
D_\alpha \Psi(x) = 0 \tag{6.1}
\]

having in mind that \( \Psi(x) \) is a 5-component field, which with respect to (2 + 1)-dimensional Lorentz group is transformed as \( \Psi(x) \rightarrow \Psi'(x') = \exp(iM_\mu \omega^\mu)\Psi(x) \), where \( \omega^\mu \) are the transformation parameters. Therefore, eq. (6.1) is the covariant (spinor) set of (2 + 1)-dimensional field equations. One can find that the field \( \Psi(x) \) satisfying equations (6.1) is decomposed into the sum of fields \( \Psi_\pm = \Pi_\pm \Psi \), \( \Psi = \Psi_+ + \Psi_- \), carrying spins \( s_+ = -\epsilon \) and \( s_- = \frac{1}{2}s_+ \), respectively. Field \( \Psi_- \) is a 2-component Dirac field, whereas 3-component field \( \Psi_+ \) is, in fact, topologically massive Jackiw-Templeton-Deser-Schonfeld vector field [20, 17]. Both fields have the same mass \( m \), and, therefore, spinor set of equations (6.1) describes a supermultiplet of (2 + 1)-dimensional massive fields. The spinor supercharge operator generating the corresponding supertransformations is

\[
Q_\alpha = em\mathcal{L}_\alpha + R\mathcal{P}_\alpha. \tag{6.2}
\]

It anticommutes with \( D_\alpha \) on mass shell, i.e. on the surface of equations (6.1), \( \{Q_\alpha, D_\beta\} \approx 0 \), and satisfies the relations \( \{Q_\alpha, Q_\beta\} \approx -16em(\gamma_\mu \partial^\mu)_{\alpha \beta}, [\partial_\mu, Q_\alpha] = 0 \). Now, one can introduce a field \( \Phi(x) \) carrying arbitrary (fixed) infinite- or finite-dimensional representation of the \( R \)-deformed Heisenberg algebra, and consider another spinor set of equations,

\[
Q_\alpha \Phi(x) = 0. \tag{6.3}
\]

Here operator \( Q_\alpha \) (6.3) is generalized to the case of the corresponding representation. Solution of eq. (6.3) is decomposable into the trivial field \( \Psi_-(x) = \Pi_- \Phi(x) = 0 \) and field
\[ \Phi_+(x) = \Pi_+ \Phi(x) \]
carrying irreducible representation of the (2+1)-dimensional Poincaré group characterized by mass \( m \) and spin \( s_+ = \epsilon_1^+(1 + \nu) \). Therefore, spin of nontrivial field \( \Phi_+ \) is defined by the value of deformation parameter \( \nu \), and one concludes that the spinor set of equations (6.3) is the above-mentioned universal set of linear differential equations giving some link between fractional spin fields (anyons) in the case of choosing \( \nu > -1 \) [4, 8], and ordinary (2+1)-dimensional integer and half-integer spin fields in the case \( \nu = -(2p + 1) \). The described (2+1)-dimensional supersymmetry as well as the universal spinor set of linear differential field equations will be considered in detail elsewhere [21]. We only note here that the spinor sets of equations (6.1) and (6.3) are analogous to (3+1)-dimensional Dirac positive-energy equations [22] in the sense that they represent by themselves the covariant (spinor) sets of equations imposed on one multi-component field.

In conclusion we note that in terms of hermitian conjugate operators \( f^+ = a^+ \) and \( f^- = a - R \) satisfying anticommutation relations (3.5), the described finite-dimensional representations of the \( R \)-deformed Heisenberg algebra supply us with some special deformation of parafermionic algebra of order \( 2p \) with internal \( Z_2 \) grading structure [21]. Recently it was shown [23] that new variants of parasupersymmetry can be constructed with the help of finite-dimensional representations of the \( q \)-deformed Heisenberg algebra. It turns out that physical properties of such new variants can be different from the properties of the parasupersymmetry realized in terms of the standard parafermionic generators [15]. Therefore, the revealed finite-dimensional representations of the \( R \)-deformed Heisenberg algebra may also be interesting from the point of view of constructing parasupersymmetric systems. Perhaps, there the intrinsic \( Z_2 \)-grading structure of the corresponding deformed parafermionic algebra could find physically interesting consequences.

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