On the Hopf algebra structure of the AdS/CFT S-matrix

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Abstract

We formulate the Hopf algebra underlying the $\mathfrak{su}(2|2)$ worldsheet S-matrix of the $AdS_5 \times S^5$ string in the AdS/CFT correspondence. For this we extend the previous construction in the $\mathfrak{su}(1|2)$ subsector due to Janik to the full algebra by specifying the action of the coproduct and the antipode on the remaining generators. The nontriviality of the coproduct is determined by length-changing effects and results in an unusual central braiding. As an application we explicitly determine the antiparticle representation by means of the established antipode.
1 Introduction

Integrable structures play a very important role in the most recent developments concerning the AdS/CFT correspondence. After the discovery of one-loop integrability of the planar dilatation operator in the scalar sector of four-dimensional $\mathcal{N}=4$ SYM theory [1], important progress has been made in uncovering integrable structures in both the gauge and the string theory side of the correspondence. On the gauge theory side integrability could be shown to persist to higher orders in perturbation theory as well as to the full set of field excitations [2]. Analogously, integrability has been established at the classical level of string theory [3] and first steps towards quantum integrability have been undertaken [4]. As a result of this work, the S-matrix of the problem has been fully determined up to an overall scalar factor, which seems to be not constrained completely by symmetry arguments and whose form should interpolate between the known expressions obtained perturbatively on both sides of the correspondence.

The S-matrix of AdS/CFT was shown to possess a centrally extended $\su(2|2) \oplus \su(2|2)$ symmetry by Beisert [5], where the central extensions are related to length-changing effects of an underlying dynamic spin chain. It is completely fixed up to the abovementioned scalar factor, and can be written in a manifestly $\su(1|2)$ symmetric form, in terms of a combination of projectors into irreducible representations of the tensor product $\su(1|2) \otimes \su(1|2)$. On the other hand, the undetermined phase factor contains an exponential of quadratic terms in the conserved charges of the theory, with coefficient functions depending on the ‘t Hooft coupling [6]. First orders in the expansion of these coefficients can be determined comparing with perturbation theory [7]. Remarkable progress in constraining the scalar factor was made by Janik, who advocated an underlying Hopf algebra structure from which he derived an equation analog to the crossing symmetry condition of relativistic S-matrices [8]. In order to derive this equation he made use of the antipode acting on $\su(1|2)$ generators, where one can avoid length-changing effects. The resulting conditions, together with unitarity, put strong constraints on the form of the scalar factor. This equation was checked against the known perturbative expansions and agreement was found [9] (see also [10]). Unfortunately, these equations do not constrain the overall factor completely, and new insights, in particular on understanding its analytic structure, are needed [11]. In this note we take a step further and establish the Hopf algebra structure underlying the full $\su(2|2)$ symmetry algebra. This can be done by simply analyzing the condition of invariance of the $\su(1|2)$ S-matrix under the remaining dynamic $\su(2|2)$ generators. The effect of length-changing operators produces momentum dependent phase factors which can be mathematically represented through a nontrivial bialgebra coproduct, which we determine. We derive the antipode and verify the whole set of Hopf algebra axioms. By using the derived antipode we are able to fully determine the antiparticle representation directly from the Hopf algebra structure.

The most important step beyond this would be to determine the universal R-matrix of the problem. The requirement of the quasi-triangularity condition would lead to an expression of the universal R-matrix which would solve the crossing relation directly at the algebraic level, without making use of representation-dependent coefficients. This could be a very powerful and clean way of fixing the structure of the overall factor, and might teach us a lot about the origin of this symmetry from the string theory perspective.
2 Deriving the Coproduct

The relevant centrally extended \( su(2|2) \) commutation relations read \[5\]

\[
[\mathcal{R}_a^\beta, \mathcal{J}^c] = \delta_c^b \mathcal{J}_a^b - \frac{1}{2} \delta_c^a \mathcal{J}_a^c,
\]

\[
[\mathcal{L}_\alpha^\beta, \mathcal{J}^c] = \delta_c^\beta \mathcal{L}_\alpha^\beta - \frac{1}{2} \delta_c^\alpha \mathcal{L}_\alpha^\beta,
\]

\[
\{\Omega_a^\alpha, \Omega_b^\beta\} = \delta_a^b \Omega_a^\alpha + \delta_b^a \Omega_a^\beta + \delta_a^b \delta_b^\mu \mathcal{C},
\]

\[
\{\Omega_a^\alpha, \mathcal{S}_b^\beta\} = \epsilon_{\alpha \beta} \epsilon_{ab} \mathfrak{P},
\]

\[
\{\mathcal{S}_a^\alpha, \mathcal{S}_b^\beta\} = \epsilon_{\alpha \beta} \epsilon_{ab} \mathfrak{R}.
\]

(1)

Here \( \mathcal{J} \) represents any generator with the appropriate index, and the \( \mathcal{R} \)'s and \( \mathcal{L} \)'s close on two copies of \( su(2) \) respectively among themselves. The central charges are \( \mathcal{C}, \mathfrak{P} \) and \( \mathfrak{K} \).

Working in the \( su(1|2) \) language, the S-matrix can be determined as a combination of projectors into the irreducible representations of the tensor product \( su(1|2) \otimes su(1|2) \), weighted with representation-dependent coefficients. Invariance of the S-matrix under all \( su(1|2) \) generators amounts in fact to impose a trivial coproduct condition. More specifically, let us indicate collectively with \( \Xi \) any of the \( su(1|2) \) generators \( \mathcal{R}_1^1, \mathcal{R}_2^2 = -\mathcal{R}_1^1, \mathcal{L}_\alpha^\beta, \mathcal{L}_1^1, \mathcal{L}_2^2, \mathcal{S}_1^1 \) and \( \mathcal{S}_1^2 \), together with the central charge \( \mathcal{C} \). The ambient space we set is the universal enveloping algebra \( U(su(2|2)) \) containing generators of the Lie algebra and all their products. Together with the (undeformed) defining commutation relations, and the standard notion of unit, which define the multiplication structure, we will endow this space with a coalgebra structure by specifying the coproduct and the counit, making it a bialgebra. Finally, the antipode will determine the Hopf algebra structure on \( U(su(2|2)) \).

Invariance of the S-matrix under \( \Xi \) amounts to the following condition \[5\]

\[
[\Xi_1 + \Xi_2, S_{12}] = 0.
\]

(2)

This can be rewritten in terms of an R-matrix

\[
\mathcal{S} = \sigma \circ \mathcal{R},
\]

(3)

where \( \sigma(A \otimes B) = (-1)^{deg(A)deg(B)} B \otimes A \), as a coproduct relation

\[
\Delta^\text{op}(\Xi) \mathcal{R} = \mathcal{R} \Delta(\Xi),
\]

(4)

with the coproduct \( \Delta \) defined as

\[
\Delta(\Xi) = (\Xi \otimes \text{id} + \text{id} \otimes \Xi)
\]

(5)

and \( \Delta^\text{op} = \sigma \circ \Delta \). One recalls that \( \mathcal{S} : V_1 \otimes V_2 \to V_2 \otimes V_1 \), \( V_i \) being modules for \( su(2|2) \) representations. In order to get eq. (2) from eqs. (1) and (3) one has simply to project the two factors of the tensor product in the abstract algebra onto some definite representations 1 and 2. We recall that a bialgebra with the property that the opposite coproduct \( \Delta^\text{op} \) is equal to the coproduct \( \Delta \) is referred to as “co-commutative”.

An R-matrix of the form

\[
\mathcal{R}_{12} = \sum_i S_i P_i,
\]

(6)
where $P_i$ are the projectors into the irreducible representations of $su(1|2) \otimes su(1|2)$, and $S_i$ are arbitrary coefficients, solves eq. (2). There are three such projectors, whose expression in terms of the quadratic $su(1|2)$ Casimir can be found in [3]. The coefficients $S_i$ are then fixed by requiring invariance under the remaining generators of $su(2|2)$ which are not in the $su(1|2)$ subalgebra\(^1\), namely $R_2^1$, $R_1^2$, $Q_2^1$, $Q_2^2$, $S_2^1$, and $S_2^2$, together with the central charges $P$ and $K$. These generators are called “dynamic” and we collectively denote them by $D$. All these generators change the length of the spin chain when acting on all magnons. In order to close their action on states of a same chain, one can use the basic relation (2.13) of [5] to move all length-changing operators to the right of all excitations, and exploit the limit of having an infinite chain. This produces the appearance of braiding factors. For example, invariance under $\Omega^1_2$ leads to eq. (34) of [8],

\[
(e^{-ip_1} [\tilde{\Omega}^1_2]_2 \otimes id_1 + id_2 \otimes [\tilde{\Omega}^1_2]_1)S = S(e^{-ip_2} [\tilde{\Omega}^1_2]_1 \otimes id_2 + id_1 \otimes [\tilde{\Omega}^1_2]_2),
\]

where the subscript indicates the representations, and the tilded version of the operator means the same action as the untilded one but disregarding length-changing effects, which are taken into account by the braiding factors. In this case one defines (cfr. [8])

\[
[\tilde{\Omega}^1_2] = a|\psi^1\rangle\langle\chi| - b|\phi\rangle\langle\psi^2|. 
\]

Now note that one can rewrite eq. (7) in terms of a deformed coproduct

\[
\Delta^\psi(\tilde{\Omega}^1_2) = R \Delta(\tilde{\Omega}^1_2),
\]

with the coproduct $\Delta$ defined as

\[
\Delta(\tilde{\Omega}^1_2) = (\tilde{\Omega}^1_2 \otimes e^{-ip} + id \otimes \tilde{\Omega}^1_2).
\]

We have lifted the coproduct relation to a representation-independent level, where we understand now $e^{-ip}$ as a central element of the (universal enveloping algebra of the) $su(2|2)$ algebra. One could make use of the natural choice

\[
e^{-ip} = 1 + \frac{1}{\alpha}P,
\]

imposed by physical requirement, where $\alpha$ is a parameter related to the coupling constant [5], and another similar relation for

\[
e^{ip} = 1 + \frac{1}{\beta}K.
\]

One can immediately realize that the coproduct (10) is not co-commutative. The existence of an element $R$ of the tensor algebra $U(su(2|2)) \otimes U(su(2|2))$ such that (10) holds makes the bialgebra “quasi-cocommutative”. The presence of $e^{-ip}$ is connected to the fact that the dynamic generator $\Omega^1_2$ adds one $Z$ field to the chain. Let us spell out the coproducts one obtains in an analog way for all the other dynamic generators:

\(^1\)In a quite different context, this is however similar to the standard Jimbo-equation procedure for determining the R-matrix for quantum affine algebras, see for instance [12].
\[ \Delta(\tilde{Q}^2_1) = (\tilde{Q}^2_1 \otimes e^{-ip} + id \otimes \tilde{Q}^2_1), \]
\[ \Delta(\tilde{S}^2_{21}) = (\tilde{S}^2_{21} \otimes e^{ip} + id \otimes \tilde{S}^2_{21}), \]
\[ \Delta(\tilde{S}^2_{22}) = (\tilde{S}^2_{22} \otimes e^{ip} + id \otimes \tilde{S}^2_{22}), \]
\[ \Delta(\tilde{R}^1_{12}) = (\tilde{R}^1_{12} \otimes e^{-ip} + id \otimes \tilde{R}^1_{12}), \]
\[ \Delta(\tilde{R}^2_{12}) = (\tilde{R}^2_{12} \otimes e^{ip} + id \otimes \tilde{R}^2_{12}). \]  

One can notice the conjugate braiding \( e^{ip} \) for the generators subtracting one \( Z \) field from the chain.

The central charges \( \mathfrak{P} \) and \( \mathfrak{K} \) also add length-changing operators to all states. Therefore, their coproduct should also be deformed in the following fashion.

\[ \Delta(\tilde{\mathfrak{P}}) = (\tilde{\mathfrak{P}} \otimes e^{-ip} + id \otimes \tilde{\mathfrak{P}}), \]
\[ \Delta(\tilde{\mathfrak{K}}) = (\tilde{\mathfrak{K}} \otimes e^{ip} + id \otimes \tilde{\mathfrak{K}}). \]  

Making use of the relations (11) and (12) one realizes that

\[ \Delta_{op}(\tilde{\mathfrak{P}}) = \Delta(\tilde{\mathfrak{P}}) \quad \Delta_{op}(\tilde{\mathfrak{K}}) = \Delta(\tilde{\mathfrak{K}}). \]  

Together with their centrality, this makes the coproduct relation with the R-matrix automatically satisfied \( \Delta_{op}(\tilde{\mathfrak{P}}) R = \Delta(\tilde{\mathfrak{P}}) R = R \Delta(\tilde{\mathfrak{P}}) \) and identically for \( \tilde{\mathfrak{K}} \). Conversely, the requirement of co-commutativity for the central elements at the abstract algebraic level automatically enforces the quartic constraint \[5, 8\].

3 The deformed Hopf Algebra structure

We have seen in the previous section that, in order to implement the length-changing effect at the algebraic level, one can deform the universal enveloping algebra of the symmetry algebra. We do it in such a way that the ordinary commutation relations remain unchanged, therefore only the coalgebra structure gets modified. This produces a well defined bialgebra which we equip with an antipode, making it a Hopf algebra. We would like now check the Hopf algebra axioms for our construction. The reader is referred to [13] for standard textbooks on Hopf algebras.

3.1 The Coproduct

Since one has no length-changing effects on the \( su(1|2) \) sector, one defines the coproduct to be the trivial one:

\[ \Delta(J) = 1 \otimes J + J \otimes 1 \quad \forall J \in su(1|2). \]  

For \( D \in su(2|2)/su(1|2) \) we need to introduce a braiding \( B(D) \) to account for those length-changing effects (tildes on generators are implicitly understood):

\[ \Delta(D) = 1 \otimes D + D \otimes B(D). \]
In the following we check that this indeed gives a consistent Hopf algebra structure, provided that we also modify the antipode $S$, and that there are consistency relations between the different braiding factors $B(D)$. Looking at the commutation relations (11) of $\mathfrak{su}(2|2)$, and naming the sets $\mathcal{J} = \mathfrak{su}(1|2)$ and $\mathcal{D} = \mathfrak{su}(2|2)/\mathfrak{su}(1|2)$ we see that

$$[\mathcal{J}, \mathcal{J}] \subseteq \mathcal{J}, \quad (18)$$

$$[\mathcal{J}, \mathcal{D}] \subseteq \mathcal{D}, \quad (19)$$

$$[\mathcal{D}, \mathcal{D}] \subseteq \mathcal{J}. \quad (20)$$

Now let $J \in \mathfrak{su}(1|2)$, $D \in \mathfrak{su}(2|2)/\mathfrak{su}(1|2)$, then $W := [J, D] \in \mathfrak{su}(2|2)/\mathfrak{su}(1|2)$. The coproduct is required to be a homomorphism, that is, it has to respect the commutation relations $^2$:

$$\Delta W = \Delta([J, D]) = [\Delta(J), \Delta(D)]. \quad (21)$$

This equality holds iff

$$B(D) = B(W). \quad (22)$$

Similarly, if $D_1, D_2 \in \mathfrak{su}(2|2)/\mathfrak{su}(1|2)$, we again demand the equality

$$\Delta([D_1, D_2]) = [\Delta D_1, \Delta D_2]. \quad (23)$$

Thus we conclude that

$$B(D_1) = B(D_2)^{-1} \quad (24)$$

whenever the commutator does not vanish. In particular, we expect the braiding to be an invertible element. The commutation relations then imply

$$B(\Omega_1^2) = B(\Omega_2^2), \quad (25)$$

$$B(\Phi_1^1) = B(\Phi_2^1), \quad (26)$$

$$B(\Omega_2^a) = B(\Phi_1^a), \quad (27)$$

$$B(\Omega_1^b) = B^{-1}(\Phi_2^b), \quad (28)$$

$$B(\Omega_2^b) = B^{-1}(\Phi_2^b). \quad (29)$$

Our result is that there can be just one braiding factor $B \equiv B(\Omega_1^1)$ and its inverse, consistently with the analysis of the previous section.

Now let us derive the braiding of the central charges $\Phi, \Phi$ and $\mathcal{C}$, which we can read off from the relations

$$\{\Omega_\alpha^a, \Omega_\beta^b\} = \epsilon_\alpha^\beta \epsilon_{ab} \Phi, \quad (30)$$

$$\{\Phi_\alpha^a, \Phi_\beta^b\} = \epsilon_\alpha^\beta \epsilon^{ab} \Phi, \quad (31)$$

$$\frac{1}{2} ([\Omega_1^1, \Phi_1^1] - [\Omega_2^2, \Phi_2^2]) = \mathcal{C}. \quad (32)$$

We get

$$B(\Phi) = B(\Omega_2^2), \quad (33)$$

$$B(\Phi) = B(\Phi_2^2). \quad (34)$$

$^2[A, B]$ denotes the usual supercommutator: $[A, B] := AB - (-1)^{deg(A)deg(B)}BA$, and we recall that $(A \otimes B)(C \otimes D) = (-1)^{deg(B)deg(C)}AC \otimes BD$. 

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while $\mathcal{C}$ remains unbraided. This is again consistent with what we argued before.

The action of the coproduct on the braiding $B$ is determined from the co-associativity condition

$$\Delta \otimes \text{id}(\Delta(A)) = (\text{id} \otimes \Delta)(\Delta(A)),$$

which has to be satisfied by every coalgebra. We get

$$\Delta(B) = B \otimes B,$$

which means that our braiding is grouplike. Again, this is consistent with the physical requirement $[11], [12]$.

### 3.2 The Antipode and the Counit

The antipode has to obey the equation

$$\mu(S \otimes \text{id})\Delta(A) = \mu(\text{id} \otimes S)\Delta(A) = i \circ \epsilon(A).$$

Here $\epsilon$ denotes the counit, which is given by $\epsilon(A) = 0$ $\forall A \in \mathfrak{su}(2|2) \ltimes \mathbb{R}^2$, $i$ is the unit and $\mu$ the multiplication. We recall that if a bialgebra has an antipode, then this is unique. For $J \in \mathfrak{su}(1|2)$ the antipode is the trivial one:

$$S(J) = -J.$$ (38)

If $D \in \mathfrak{su}(2|2)/\mathfrak{su}(1|2)$, we expect the braiding to appear in the antipode. Indeed,

$$\mu(S \otimes \text{id})\Delta(D) = S(D)B(D) + D = 0$$

gives

$$S(D) = -DB^{-1}(D).$$

Furthermore, the action of $S$ on the braiding itself is

$$S(B(D)) = B^{-1}(D).$$

Using the defining coalgebra relation between the coproduct and the counit

$$(\text{id} \otimes \epsilon)\Delta(A) = A = (\epsilon \otimes \text{id})\Delta(A)$$

we see that

$$\epsilon(B) = 1.$$ (43)

### 3.3 Charge Conjugation

For the representations we use the same convention as Beisert [5]. The represented generators $\pi(A)$ act on the 4-dimensional graded vector space spanned by $|\phi\rangle$, $|\chi\rangle$, $|\psi^1\rangle$, $|\psi^2\rangle$, and the representations are labelled by the numbers $a, b, c, d$, with the constraint $ad - bc = 1$. For the $\mathfrak{su}(1|2)$ subalgebra we have

$\begin{align*}
\mathcal{Q}_1^\alpha &= a |\psi^\alpha\rangle \langle \phi| + b \epsilon^{\alpha\beta}\langle \chi| \langle \psi^\beta| \\
\mathcal{G}_1^\alpha &= c \epsilon_{\alpha\beta} |\psi^\beta\rangle \langle \chi| + d |\phi\rangle \langle \psi^\alpha| \\
\mathcal{R}_1 &= \frac{1}{2} (|\phi\rangle \langle \phi| - |\chi\rangle \langle \chi|) \\
\mathcal{L}_1^\alpha &= |\psi^\alpha\rangle \langle \psi^\beta| - \frac{1}{2} \delta^\alpha_\beta |\psi^\gamma\rangle \langle \psi^\gamma| 
\end{align*}$

It is understood that $A \equiv \pi(A)$

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For the generators in $\mathfrak{su}(2|2)/\mathfrak{su}(1|2)$ we have
\begin{align}
\Omega_2^\alpha &= a |\phi^\alpharangle \langle \chi| - b e^{\alpha \beta} |\phi \psi^\betarangle | \tag{48} \\
\mathfrak{S}_2^\alpha &= -c \epsilon_{\alpha \beta} |\psi^\beta \rangle \langle \phi| + d |\chi \rangle \langle \psi^\alpha| \\
\mathfrak{R}_1^\alpha &= |\phi \rangle \langle \chi| \\
\mathfrak{R}_2^\alpha &= |\chi \rangle \langle \phi|.
\end{align}

Finally, for the centre we have
\begin{align}
\mathfrak{C} &= \frac{1}{2}(ad + cb), \\
\mathfrak{P} &= ab, \\
\mathfrak{R} &= cd.
\end{align}

In this section we derive the charge conjugation $C$, which has to fulfill $\mathfrak{S}$
\[ \pi(S(A)) = C^{-1} \pi(A)^{st} C. \]  

We will adopt the notation of $\mathfrak{S}$, and we will show how it is possible to get all the parameters $\bar{a}, \bar{b}, \bar{c}, \bar{d}$, which parametrise $\bar{\pi}$, directly from the knowledge of the full Hopf algebra. We start with the representation of the charge conjugation matrix determined by Janik $\mathfrak{S}$
\[ C = |\chi \rangle \langle \phi| \frac{a_1 b_1}{ab} + |\phi \rangle \langle \chi| - \frac{b_1}{a_1} |\psi^1 \rangle \langle \psi^1| + \frac{b_1}{a_1} |\psi^2 \rangle \langle \psi^2|. \]  

Then the inverse is given by
\[ C^{-1} = |\phi \rangle \langle \chi| \frac{\bar{a} \bar{b}}{a_1 b_1} + |\chi \rangle \langle \phi| - \frac{\bar{a}}{a_1} |\psi^1 \rangle \langle \psi^1| + \frac{\bar{a}}{a_1} |\psi^2 \rangle \langle \psi^2|. \]  

For $J \in \mathfrak{su}(1|2)$ the antipode is the trivial one and we get the equations
\[ C \pi(J)^{-1} = -\pi(J)^{st}, \]  

whilst for $D \in \mathfrak{su}(2|2)/\mathfrak{su}(1|2)$ we have the relation
\[ C \pi(D)^{-1} = -B(D) \pi(D)^{st}. \]  

We can therefore determine the parameters of $\bar{\pi}$ in terms of the parameters of $\pi$:
\begin{align}
C \mathfrak{S}_1^1 C^{-1} &= -\bar{\pi}^{st}(\mathfrak{S}_1^1) \quad \Rightarrow \quad \bar{d} = -\frac{c_1 b_1}{a}, \quad \bar{c} = -\frac{d_1 a_1}{b} \\
C \Omega_2^1 C^{-1} &= -\pi(B(\Omega_2^1))^{st}(\Omega_2^1) \quad \Rightarrow \quad \pi(B(\Omega_2^1)) = -\frac{a_1 b_1}{ab} \\
C \mathfrak{S}_2^2 C^{-1} &= -\bar{\pi}^{st}(\mathfrak{S}_2^2) \pi(B(\mathfrak{S}_2^2)) \quad \Rightarrow \quad \pi(B(\mathfrak{S}_2^2)) = \frac{d_1 \bar{a}}{b_1 c} = \frac{c_1 \bar{b}}{da_1}.
\end{align}

With the condition
\[ a_1 b_1 = \alpha(e^{-i\varphi} - 1), \]  

which stems from the requirement of having total zero momentum for physical states, and with $\pi(B(\mathfrak{R}_2^2)) = e^{i\varphi}$, we finally obtain
\[ e^{i\varphi} = \frac{\alpha}{\alpha + a_1 b_1} = -\frac{\bar{a} \bar{b}}{a_1 b_1} \quad \Rightarrow \quad \bar{b} = -\frac{a_1 b_1 \alpha}{\bar{a} \alpha + a_1 b_1}. \]  

This indeed coincides with eq. (58) of Janik $\mathfrak{S}$, who could determine $\bar{b}$ only after imposing the crossing conditions $\mathfrak{S}$. 

7
4 Conclusions

In this paper we have shown how it is possible to extend the Hopf algebra structure discovered by Janik for the subsector $su(1|2)$ to the full $su(2|2)$ algebra, by determining the action of the coproduct and of the antipode on the remaining generators. The result is a nontrivial action via a central element of the Hopf algebra. This construction appears to be novel from the mathematical point of view and we obtain a different structure to the one familiar from quantum groups. However, this algebra most probably resides at the core of the problem of determining additional symmetries and constraints for the S-matrix of the AdS/CFT correspondence. We have verified the Hopf algebra axioms for this structure, and we have used it to determine the antiparticle representation directly from the algebra.

The next step would be to construct the correspondent universal R-matrix. The conditions of “quasi-triangularity”

\[(\Delta \otimes id)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23},\]
\[(id \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}\]  \hspace{1cm} (63)

imply that $\mathcal{R}$ satisfies the Yang-Baxter equation, and, in the presence of an antipode, the crossing conditions

\[(S \otimes id)(\mathcal{R}) = \mathcal{R}^{-1},\]
\[(id \otimes S^{-1})(\mathcal{R}) = \mathcal{R}^{-1}\]  \hspace{1cm} (64)

which in turn imply $(S \otimes S)\mathcal{R} = \mathcal{R}$. The constraint (63) is reminiscent of the “bootstrap” principle for a relativistic S-matrix [14], and is in fact its translation in Hopf-algebraic terms (see e.g. [15]). The fact that the S-matrix of AdS/CFT satisfies Yang-Baxter and crossing relations is a hint that it might also satisfy the quasi-triangularity condition.

Being expressed in a representation independent way purely in terms of the abstract algebra generators, the universal R-matrix may give us significant help in determining the overall scalar factor in a clean way directly in terms of the generators of the universal enveloping algebra. One is likely to gain also a better understanding of the origin of the phase factor and of the whole Hopf algebra structure from the point of view of the string theory sigma-model. We plan to investigate these issues in a future publication.

Note added: While we were finalizing this note, we received notice of the publication of similar results by Gómez and Hernández in hep-th/0608029 [16]. The problem of having a non symmetric coproduct on the center, as mentioned in their paper, is avoided by imposing the physical requirements [11], [12] at the algebraic level.

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References

[1] J. A. Minahan and K. Zarembo, “The Bethe-ansatz for N = 4 super Yang-Mills,” JHEP 0303 (2003) 013 [arXiv:hep-th/0212208].

[2] N. Beisert, C. Kristjansen and M. Staudacher, “The dilatation operator of N = 4 super Yang-Mills theory,” Nucl. Phys. B 664 (2003) 131 [arXiv:hep-th/0303060].
N. Beisert, “The complete one-loop dilatation operator of N = 4 super Yang-Mills theory,” Nucl. Phys. B 676 (2004) 3 [arXiv:hep-th/0307015].
N. Beisert and M. Staudacher, “The N = 4 SYM integrable super spin chain,” Nucl. Phys. B 670 (2003) 439 [arXiv:hep-th/0307042].
N. Beisert, “The su(2|3) dynamic spin chain,” Nucl. Phys. B 682 (2004) 487 [arXiv:hep-th/0310252].
D. Serban and M. Staudacher, “Planar N = 4 gauge theory and the Inozemtsev long range spin chain,” JHEP 0406 (2004) 001 [arXiv:hep-th/0401057].
N. Beisert, V. Dippel and M. Staudacher, “A novel long range spin chain and planar N = 4 super Yang-Mills,” JHEP 0407 (2004) 075 [arXiv:hep-th/0405001].
N. Beisert, “The dilatation operator of N = 4 super Yang-Mills theory and integrability,” Phys. Rept. 405 (2005) 1 [arXiv:hep-th/0407277].
M. Staudacher, “The factorized S-matrix of CFT/AdS,” JHEP 0505 (2005) 054 [arXiv:hep-th/0412188].
N. Beisert and M. Staudacher, “Long-range PSU(2,2|4) Bethe ansaetze for gauge theory and strings,” Nucl. Phys. B 727 (2005) 1 [arXiv:hep-th/0504190].

[3] S. Frolov and A. A. Tseytlin, “Multi-spin string solutions in AdS(5) x S**5,” Nucl. Phys. B 668 (2003) 77 [arXiv:hep-th/0304255].
I. Bena, J. Polchinski and R. Roiban, “Hidden symmetries of the AdS(5) x S**5 superstring,” Phys. Rev. D 69 (2004) 046002 [arXiv:hep-th/0305116].
G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, “Spinning strings in AdS(5) x S**5 and integrable systems,” Nucl. Phys. B 671 (2003) 3 [arXiv:hep-th/0307191].
G. Arutyunov, J. Russo and A. A. Tseytlin, “Spinning strings in AdS(5) x S**5: New integrable system relations,” Phys. Rev. D 69 (2004) 086009 [arXiv:hep-th/0311004].
A. A. Tseytlin, “Spinning strings and AdS/CFT duality,” arXiv:hep-th/0311139.
V. A. Kazakov, A. Marshakov, J. A. Minahan and K. Zarembo, “Classical / quantum integrability in AdS/CFT,” JHEP 0405 (2004) 024 [arXiv:hep-th/0402207].
G. Arutyunov and S. Frolov, “Integrable Hamiltonian for classical strings on AdS(5) x S**5,” JHEP 0502 (2005) 059 [arXiv:hep-th/0411089].
L. F. Alday, G. Arutyunov and A. A. Tseytlin, “On integrability of classical superstrings in AdS(5) x S**5,” JHEP 0507 (2005) 002 [arXiv:hep-th/0502240].

[4] C. G. Callan, H. K. Lee, T. McLoughlin, J. H. Schwarz, I. J. Swanson and X. Wu, “Quantizing string theory in AdS(5) x S**5: Beyond the pp-wave,” Nucl. Phys. B 673 (2003) 3 [arXiv:hep-th/0307032].
C. G. Callan, T. McLoughlin and I. J. Swanson, “Holography beyond the Penrose limit,” Nucl. Phys. B 694 (2004) 115 [arXiv:hep-th/0404007].

C. G. Callan, T. McLoughlin and I. J. Swanson, “Higher impurity AdS/CFT correspondence in the near-BMN limit,” Nucl. Phys. B 700 (2004) 271 [arXiv:hep-th/0405153].

S. Frolov, J. Plefka and M. Zamaklar, “The AdS(5) x S**5 superstring in light-cone gauge and its Bethe equations,” arXiv:hep-th/0603008.

T. Klose and K. Zarembo, “Bethe ansatz in stringy sigma models,” J. Stat. Mech. 0605 (2006) P006 [arXiv:hep-th/0603039].

[5] N. Beisert, “The su(2|2) dynamic S-matrix,” arXiv:hep-th/0511082.

[6] N. Beisert and T. Klose, “Long-range gl(n) integrable spin chains and plane-wave matrix theory,” J. Stat. Mech. 0607 (2006) P006 [arXiv:hep-th/0510124].

[7] G. Arutyunov, S. Frolov and M. Staudacher, “Bethe ansatz for quantum strings,” JHEP 0410 (2004) 016 [arXiv:hep-th/0406256].

S. Schäfer-Nameki, M. Zamaklar and K. Zarembo, “Quantum corrections to spinning strings in AdS(5) x S**5 and Bethe ansatz: A comparative study,” JHEP 0509 (2005) 051 [arXiv:hep-th/0507189].

S. Schäfer-Nameki and M. Zamaklar, “Stringy sums and corrections to the quantum string Bethe ansatz,” JHEP 0510 (2005) 044 [arXiv:hep-th/0509096].

N. Beisert and A. A. Tseytlin, “On quantum corrections to spinning strings and Bethe equations,” Phys. Lett. B 629 (2005) 102 [arXiv:hep-th/0509084].

R. Hernandez and E. Lopez, “Quantum corrections to the string Bethe ansatz,” JHEP 0607 (2006) 004 [arXiv:hep-th/0603204].

L. Freyhult and C. Kristjansen, “A universality test of the quantum string Bethe ansatz,” Phys. Lett. B 638 (2006) 258 [arXiv:hep-th/0604069].

[8] R. A. Janik, “The AdS(5) x S**5 superstring worldsheet S-matrix and crossing symmetry,” Phys. Rev. D 73 (2006) 086006 [arXiv:hep-th/0603038].

[9] G. Arutyunov and S. Frolov, “On AdS(5) x S**5 string S-matrix,” Phys. Lett. B 639 (2006) 378 [arXiv:hep-th/0604043].

[10] N. Beisert, “On the scattering phase for AdS(5) x S**5 strings,” arXiv:hep-th/0606214

[11] R. A. Janik, Talk at the Workshop on "Integrability in Gauge and String Theory", Max-Planck Institute for Gravitational Physics (Albert-Einstein Institute), Potsdam, 24-28 July 2006.

[12] W. Delius, M. D. Gould and Y. Z. Zhang, “On the construction of trigonometric solutions of the Yang-Baxter equation,” Nucl. Phys. B 432 (1994) 377 [arXiv:hep-th/9405030].

A. J. Bracken, M. D. Gould, Y. Z. Zhang and G. W. Delius, “Solutions of the quantum Yang-Baxter equation with extra nonadditive parameters,” J. Phys. A 27 (1994) 6551 [arXiv:hep-th/9405138].
[13] V. Chari and A. Pressley, “A guide to quantum groups,” Cambridge, UK: Univ. Pr. (1994)
C. Kassel, “Quantum groups,” New York, USA: Springer (1995) (Graduate text in mathematics, 155)
E. Abe, “Hopf algebras,” Cambridge, UK: Univ. Pr. (1977)

[14] A. B. Zamolodchikov and A. B. Zamolodchikov, “Factorized S-Matrices In Two Dimensions As The Exact Solutions Of Certain Relativistic Quantum Field Models,” Annals Phys. 120 (1979) 253.
A. B. Zamolodchikov, “Integrable Field Theory From Conformal Field Theory,” Adv. Stud. Pure Math. 19 (1989) 641.

[15] G. W. Delius, “Exact S matrices with affine quantum group symmetry,” Nucl. Phys. B 451 (1995) 445 [arXiv:hep-th/9503079].

[16] C. Gomez and R. Hernandez, “The magnon kinematics of the AdS/CFT correspondence,” arXiv:hep-th/0608029.