Structure of elementary operators defining $m$-left invertible, $m$-selfadjoint and related classes of operators

B.P. Duggal, I.H. Kim

Dedicated to Professor Woo Young Lee on the occasion of his 65th birthday

Abstract

We use elementary, algebraic properties of left, right multiplication operators to prove some deep structural properties of left $m$-invertible, $m$-isometric, $m$-selfadjoint and other related classes of Banach space operators, often adding value to extant results.

1. Introduction

Let $B(X)$ (resp., $B(H)$) denote the algebra of operators on an infinite dimensional complex Banach Space $X$ (resp., Hilbert space $H$) into itself. For $A, B \in B(X)$, let $L_A$ and $R_B \in B(B(X))$ denote, respectively, the operators

$$L_A(X) = AX, \quad R_B(X) = XB$$

of left multiplication by $A$ and right multiplication by $B$. The elementary operators $\triangle_{A,B}$ and $\delta_{A,B} \in B(B(X))$ are then defined by

$$\triangle_{A,B}(X) = (L_AR_B - I)(X) = L_AR_B(X) - X = AXB - X$$

and

$$\delta_{A,B}(X) = (L_A - R_B)(X) = L_A(X) - R_B(X) = AX - XB.$$}

Let $d_{A,B} \in B(B(X))$ denote either of the operators $\triangle_{A,B}$ and $\delta_{A,B}$. Let $m \geq 1$ be some integer. The elementary operators

$$d_{A,B}^m(I) = d_{A,B}(d_{A,B}^{m-1}(I)) = \begin{cases} \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} A^{m-j}B^{m-j} = 0, \quad d = \triangle \\ \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} A^{m-j}B^j = 0, \quad d = \delta \end{cases}$$

AMS(MOS) subject classification (2010). Primary: Primary47A05, 47A55 Secondary47A80, 47A10.

Keywords: Banach space, Left/right multiplication operator, elementary operator, $m$-left invertible, $m$-isometric and $m$-selfadjoint operators, product of operators, perturbation by nilpotents, commuting operators.

The second named author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2019R1F1A1057574).
have been considered in the recent past by a large number of authors (see [2], [3], [4], [8], [9], [13], [14], [15], [16], [17], [19], [20] for further references). Operators satisfying $\Delta_{A,B}^m(I) = 0$ have been called left $m$-invertible ([6], [17], [19]): $m$-isometric operators $S \in B(H)$ satisfying $\Delta_{S^*S}^m(I) = 0$ and $(m,C)$-isometric operators $S \in B(H)$ satisfying $\Delta_{S^*SC^*C}^m(I) = 0$ (for some conjugation $C$ of $H$) are a couple of important example of left $m$-invertible operators. Again, $m$-selfadjoint operators $S \in B(H)$ satisfying $\delta_{S^*S}^m(I) = 0$ ([21], [23]) and $(m,C)$-symmetric operators $S \in B(H)$ satisfying $\delta_{S^*SC^*}^m(I) = 0$ ([13]) are a couple of important examples of operators $A, B$ satisfying $\delta_{A,B}^m(I) = 0$.

An array of results, amongst them that.

\[ d_{A,B}^m(I) = 0 \iff d_{A,B}^m(I) = 0 \text{ for all integers } n \geq m; \]
\[ d_{A,B}^m(I) = 0 \iff d_{A^n,B^n}^m(I) = 0 \text{ for all integers } n \geq 1, \]

conditions on $A_i, B_i \in B(\mathcal{A})$, $i = 1, 2$, ensuring

\[ d_{A_1,B_1}^m(I) = 0 = d_{A_2,B_2}^m(I) \implies d_{A_1,A_2,B_1,B_2}^{m+n-1}(I) = 0; \]

and conditions on $B$ and an $n$-nilpotent $N$ ensuring

\[ d_{A,B}^m(I) = 0 \implies d_{A,B+N}^{m+n-1}(I) = 0, \]

is available in extant literature. The authors of these papers have used a wide variety of arguments, amongst them combinatorial arguments, arithmetical progressions, (interpolation by) Lagrange polynomials/properties of operator roots of polynomials and the hereditary functional calculus (as developed in [1]).

This paper considers $d_{A,B}^m(I) = 0$ from the view point that

\[ d_{A,B}^m(I) = (L_AR_B - 1)^m(I) = \left( \sum_{j=0}^{m} (-1)^j \binom{m}{j} (L_AR_B)^{m-j} \right)(I) = 0, \quad d = \triangle; \]
\[ d_{A,B}^m(I) = (L_A - R_B)^m(I) = \left( \sum_{j=0}^{m} (-1)^j \binom{m}{j} L_A^{m-j}R_B^j \right)(I) = 0, \quad d = \delta \]

and, using little more them algebraic operations with left-right multiplication operators, provides a uniformly simple argument to prove (often, improved versions of) these results.

The plan of this paper is as follows: Preliminaries are dealt with in Section 2. Here we use elementary arguments to prove some basic properties which operators $A, B$ satisfying $d_{A,B}^m(I) = 0$ share with $m$-isometric operators. Amongst other results, we prove here that if $\Delta_{A,B}^m(I) = 0$ and $\Delta_{A,B}^{m-1}(I) \neq 0$, then

\[ \{ L_AR_B^s, L_A^{\pm 1}R_B^{\pm 1}\Delta_{A,B}(I), \cdots, L_A^{\pm m-1}R_B^{\pm m-1}\Delta_{A,B}^{m-1}(I) \} \]

is a linearly independent set. Sections 3, 4 and 5, the final three sections (before the section on references), deal with products, perturbations by commuting nilpotents and commuting $A,B$. Here we give elementary proofs for some results which have been proved for $m$-isometric, $(m,C)$-isometric (etc.) operators using a wide variety of often lengthy, occasionally involved, arguments.
2. Preliminaries

Henceforth, unless otherwise stated, $A$ and $B$ shall denote operators in $B(\mathcal{X})$. We say that the pair $(A, B) \in d^m(I)$ if either $\Delta_{A,B}^m(I) = 0$ or $\delta_{A,B}^m(I) = 0$. Since

$$
\Delta_{A,B}^{m+n}(I) = (L_A R_B - I)^n (\Delta_{A,B}^m(I)) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} L_A^{n-r} R_B^{-r} (\Delta_{A,B}^m(I))
$$

and

$$
\delta_{A,B}^{m+n}(I) = (L_A - R_B)^n (\delta_{A,B}^m(I)) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} L_A^{n-r} R_B^{-r} (\delta_{A,B}^m(I))
$$

for all integers $n \geq 1$,

$$
d_{A,B}^m(I) = 0 \iff d_{A,B}^t(I) = 0 \text{ for all integers } t \geq m.
$$

Again, given an integer $n \geq 1$, since

$$
\Delta_{A^n,B^n}^m(I) = (L_A^n R_{B^n} - I)^m(I) = (L_A^n R_{B^n} - I)^m(I)
$$

and

$$
\delta_{A^n,B^n}^m(I) = (L_A^n - R_{B^n})^m(I) = (L_A^n - R_{B^n})^m(I)
$$

for some scalars $\alpha_j$ and $\beta_j$,

$$(A, B) \in d^m(I) \iff (A^n, B^n) \in d^m(I) \text{ for all integers } n \geq 1.
$$

([10], [17], [19]).

The hypothesis $(A, B) \in \Delta^m(I)$ implies $B$ is left invertible (and $A$ is right invertible). Thus, if $(A, B) \in \Delta^m(I)$, $B$ has a dense range and $A$ is injective, then $A, B$ are invertible. Again, if $(A, B) \in \delta^m(I)$ and $B$ is invertible, then

$$
\delta_{A,B}^m(I) = 0 \implies \Delta_{A,B}^{m+1}(I) = 0 \implies A \text{ is right invertible.}
$$

Hence, if $B$ is invertible and $A$ is injective, then

$$
\delta_{A,B}^m(I) = 0 \implies A, B \text{ invertible.}$$
Furthermore, since
\[(L_{A^{-1}}R_{B^{-1}} - I)^m(I) = (-1)^m L_{A^{-m}} R_{B^{-m}} (L_A R_B - I)^m(I) = 0,\]
and
\[(L_{A^{-1}} - R_{B^{-1}})^m(I) = (-1)^m L_{A^{-m}} R_{B^{-m}} (L_A - R_B)^m(I) = 0,\]
we have \((A^{-1}, B^{-1}) \in d^m(I)\).

We say in the following that the pair of operators \((A, B)\) is strictly \(d^m(I)\), \((A, B) \in \text{strict-}d^m(I)\), if \((A, B) \in d^m(I)\) and \(d_{A,B}^{m-1}(I) ≠ 0\). The definition implies that if \((A, B) \in \text{strict-}d^m(I)\), then \(d_{A,B}^r(I) ≠ 0\) for all \(0 ≤ r ≤ m - 1\) and the set
\[\{d_{A,B}^r(I)\}_{r=0}^{m-1} = \{I, d_{A,B}(I), \cdots, d_{A,B}^{m-1}(I)\}\]
is linearly independent. To see this, assume (to the contrary) that there exist non-zero scalars \(a_r\) such that \(\sum_{r=0}^{m-1} a_r d_{A,B}^r(I) = 0\). Then

\[0 = d_{A,B}^{m-1} \left( \sum_{r=0}^{m-1} a_r d_{A,B}^r(I) \right) = a_0 d_{A,B}^{m-1}(I) \iff a_0 = 0,\]
\[0 = d_{A,B}^{m-2} \left( \sum_{r=1}^{m-1} a_r d_{A,B}^r(I) \right) = a_1 d_{A,B}^{m-1}(I) \iff a_1 = 0,\]
\[0 = a_{m-1} d_{A,B}^{m-1}(I) \iff a_{m-1} = 0.\]

This being a contradiction, our assertion follows. More is true in the case in which \(d = \Delta\).

**Lemma 2.1** If \((A, B) \in \text{strict-}d^m(I)\), then the sets \(\{L_A^{t+r} \Delta_{A,B}^r(I)\}_{r=0}^{m-1}\), \(\{R_B^{r-s-r} \Delta_{A,B}^r(I)\}_{r=0}^{m-1}\) and \(\{L_A^{t-r} R_B^{s-r} \Delta_{A,B}^r(I)\}_{r=0}^{m-1}\), where \(s, t ≥ m - 1\) are some integers, are linearly independent.

**Proof.** The proof in all cases being similar, we prove the linear independence of the set \(\{L_A^{t-r} R_B^{s-r} \Delta_{A,B}^r(I)\}_{r=0}^{m-1}\). Suppose to the contrary that there exist scalars \(a_r\) such that \(\sum_{r=0}^{m-1} a_r L_A^{t-r} R_B^{s-r} \Delta_{A,B}^r(I) = 0\). Assume without loss of generality that \(s = t + n\) for some integer \(n ≥ 0\). Then

\[0 = \Delta_{A,B}^{m-1} \left( \sum_{r=0}^{m-1} a_r L_A^{t-r} R_B^{s-r} \Delta_{A,B}^r(I) \right) \implies a_0 A^t \Delta_{A,B}^{m-1}(I) B^{t+n} = 0\]
\[\implies a_0 A^{t+n} \Delta_{A,B}^{m-1}(I) B^{t+n} = 0 \iff a_0 \Delta_{A,B}^{m-1}(I) = 0 \iff a_0 = 0;\]
\[0 = \Delta_{A,B}^{m-2} \left( \sum_{r=1}^{m-1} a_r L_A^{t-r} R_B^{s-r} \Delta_{A,B}^r(I) \right) \implies A^{t-1} R_B^{s-1} \Delta_{A,B}^{m-1}(I) = 0\]
\[\implies a_1 A^{t+n-1} \Delta_{A,B}^{m-1}(I) B^{t+n-1} = 0 \iff a_1 \Delta_{A,B}^{m-1}(I) = 0 \iff a_1 = 0;\]
\[\cdots\]
\[0 = a_{m-1} A^{t+m-1} \Delta_{A,B}^{m-1}(I) B^{s-m+1}\]
\[\implies a_{m-1} A^{t+n-m+1} \Delta_{A,B}^{m-1}(I) B^{t+n-m+1} = 0\]
\[\iff a_{m-1} \Delta_{A,B}^{m-1}(I) = 0 \iff a_{m-1} = 0.\]
This is a contradiction. \[\square\]

A similar result does not hold for \(d = \delta\), as the following example shows.

**Example 2.2** Let \(\mathcal{H} = \mathbb{C}^3\) (the 3-dimensional complex space) and let \(A \in B(\mathcal{H})\) be the operator

\[
A = \begin{pmatrix}
0 & a & 0 \\
0 & 0 & b \\
0 & 0 & 0
\end{pmatrix}, \text{a and } b \text{ distinct non-zero reals.}
\]

Let \(D\) be the conjugation \(D(x, y, z) = (\bar{x}, \bar{y}, \bar{z})\). Then

\[
A^3 = 0, \quad DAD = A, \quad \delta_{A^*, DAD}^3(I) = 0, \quad A^{*2} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
ab & 0 & 0
\end{pmatrix},
\]

\[
DA^2D = \begin{pmatrix}
0 & 0 & ab \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix} \quad \text{and} \quad \delta_{A^*, DAD}^1(I) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & a^2b^2
\end{pmatrix} \neq 0.
\]

Thus \((A^*, DAD) \in \text{strict-}\delta^5(I)\). However,

\[
L_{A^*, DAD}^3(I) = L_{A^*, DAD}^3(I) = 0,
\]

the set \(\{L_{A^*, DAD}^r(I)\}_{r=0}^4\) is not linearly independent and the set \(\{\delta_{A^*, DAD}^r(I)\}_{r=0}^4\) is linearly independent.

We shall require the following lemma in our deliberations.

**Lemma 2.3** If \(R_{A,B}^{n-1-i} \Delta_{A,B}^{m+i}(I) = 0\) for some integer \(n \geq 1\) and \(0 \leq i \leq n - 1\), then \((A, B) \in \Delta^m(I)\).

**Proof.** The hypothesis implies \(A^{n-1-i} \Delta_{A,B}^{m+i}(I)B^{n-1-i} = 0\) for all \(0 \leq i \leq n - 1\).

Since \(A\Delta_{A,B}^{r+1}(I)B = \Delta_{A,B}^{r+1}(I) + \Delta_{A,B}^r(I)\),

\[
A^{n-1-i} \Delta_{A,B}^{m+i}(I)B^{n-1-i} = A^{n-2-i} \Delta_{A,B}^{m+i+1}(I)B^{n-2-i} + A^{n-2-i} \Delta_{A,B}^{m+i}(I)B^{n-2-i},
\]

hence

\[
A^{n-2-i} \Delta_{A,B}^{m+i}(I)B^{n-2-i} = 0, \quad 0 \leq i \leq n - 2.
\]

Repeating the argument, another \((n - 2)\)-times, it follows that \(\Delta_{A,B}^m(I) = 0\). \[\square\]

Let \(\mathcal{X} \otimes \mathcal{X}\) denote the completion, endowed with a reasonable uniform cross-norm, of the algebraic tensor product of \(\mathcal{X}\) with itself. Let \(A \otimes B \in B(\mathcal{X} \otimes \mathcal{X})\) denote the tensor product operator defined by \(A\) and \(B\). The following lemma is well known (see [18], [22] for similar results).

**Lemma 2.4** If \(\sum_{i=1}^n A_i \otimes B_i = 0\) for some operators \(A_i, B_i \in B(\mathcal{X})\), \(1 \leq i \leq n\) and the set \(\{B_i\}_{1 \leq i \leq n}\) is linearly independent, then \(A_i = 0\) for all \(1 \leq i \leq n\).
3. Results: Products

We start in the following by considering products of commuting pairs of operators satisfying the \( d^m(I) \) property. Products of commuting \( m \)-isometric operators have been considered by Bermúdez et al (8) and [23]), of commuting left \( m \)-invertible operators by [19], of tensor products of left \( m \)-invertible operators by Duggal and Muller ([17]) and of commuting \( m \)-selfadjoint operators by Trieu Le ([23]). The following theorem employs an elementary algebraic argument to prove these results for operators satisfying the \( d^m(I) \) property. Let \([A,B] = AB - BA\) denote the commutator of \( A \) and \( B \).

**Theorem 3.1** Let \(A_i, B_i \in B(\mathcal{X}), i = 1, 2\), be such that \([A_1, A_2] = 0 = [B_1, B_2]\).

(i) If \((A_1, B_1) \in d^m(I)\) and \((A_2, B_2) \in d^n(I)\), then \( (A_1A_2, B_1B_2) \in d^{m+n-1}(I)\).

(ii) If \((A_1, B_1) \in \triangle^m(I)\) and \((A_2, B_2) \in \triangle^n(I)\), then \((A_1A_2, B_1B_2) \in \text{strict}\triangle^{m+n-1}(I)\) if and only if \((L_{A_2}R_{B_2})^{m-1} \triangle^{m-1}_{A_1, B_1}(I) \triangle^{n-1}_{A_2, B_2}(I) \neq 0\), equivalently, \(\triangle^{m-1}_{A_1, B_1}(I) \triangle^{n-1}_{A_2, B_2}(I) \neq 0\).

(iii) If \((A_1, B_1) \in \delta^m(I)\) and \((A_2, B_2) \in \delta^n(I)\), then \((A_1A_2, B_1B_2) \in \text{strict}\delta^{m+n-1}(I)\) if and only if \(R_{B_1}^{-1}L_{A_2}^{-1}L_{A_1}^{-1} \delta^{m-1}_{A_1, B_1}(I) \delta^{n-1}_{A_2, B_2}(I) = L_{A_1}^{-1}R_{B_2}^{-1} \delta^{m-1}_{A_1, B_1}(I) \delta^{n-1}_{A_2, B_2}(I) \neq 0\).

**Proof.** (i) The hypothesis \([A_1, A_2] = 0 = [B_1, B_2]\) implies \(L_{A_1}, L_{A_2}, R_{B_1}\) and \(R_{B_2}\) all (mutually) commute. Since

\[
(L_{A_1}L_{A_2}R_{B_1}R_{B_2} - I)^r = \sum_{j=0}^{r} \binom{r}{j} (L_{A_2}R_{B_2} - I)^r j (L_{A_1}R_{B_1} - I)^r j (L_{A_2}R_{B_2} - I)^j
\]

and

\[
\{ (L_{A_1}R_{B_1} - I)^r j (L_{A_2}R_{B_2} - I)^j \} (I) = \Delta^{r j}_{A_2, B_2}(I) \Delta^{r j}_{A_1, B_1}(I) = \Delta^{r j}_{A_1, B_1}(I) \Delta^{r j}_{A_2, B_2}(I)
\]

for all integers \(j \geq 1\), we have

\[
\Delta^{r j}_{A_1, A_2, B_1, B_2}(I) = \sum_{j=0}^{r} \binom{r}{j} \Delta^{r j}_{A_2, B_2}(I) \Delta^{r j}_{A_1, B_1}(I) \Delta^{r j}_{A_2, B_2}(I).
\]

Let \(r = m + n - 1\). Then \(\Delta^{r j}_{A_2, B_2}(I) = 0\) for all \(j \geq n\), and since

\[
j \leq n - 1 \implies m + n - 1 - j \geq m \implies \Delta^{m+n-1-j}_{A_1, B_1}(I) = 0,
\]

we have

\[
(A_1A_2, B_1B_2) \in \triangle^{m+n-1}(I).
\]

Considering now

\[
(L_{A_1}L_{A_2} - R_{B_1}R_{B_2})^{r} = \{L_{A_2}(L_{A_1} - R_{B_1}) + (L_{A_2} - R_{B_2})R_{B_1}\}^{r}
\]

\[
= \sum_{j=0}^{r} \binom{r}{j} L_{A_2}^{r-j} R_{B_1}^{j} (L_{A_1} - R_{B_1})^{r-j} (L_{A_2} - R_{B_2})^{j},
\]

we have

\[
\delta^{r j}_{A_1, A_2, B_1, B_2}(I) = \sum_{j=0}^{r} \binom{r}{j} L_{A_2}^{r-j} R_{B_1}^{j} \delta^{r-j}_{A_1, B_1}(I) \delta^{j}_{A_2, B_2}(I).
\]
Hence, letting \( r = m + n - 1 \) and recalling \( \delta_{A_1,B_1}^m (I) = 0 = \delta_{A_2,B_2}^n (I) \), we have

\[
\delta_{A_1,A_2,B_1,B_2}^{m+n-1} = \sum_{j=0}^{m+n-1} \binom{m+n-1}{j} R_{B_1}^j \delta_{A_1,B_1}^{m+n-j} (I) L_{A_2}^{m+n-j} \delta_{A_2,B_2}^{j} (I) = 0.
\]

(ii) If \( \Delta_{A_1,B_1}^m (I) = 0 = \Delta_{A_2,B_2}^n (I) \), then

\[
\Delta_{A_1,A_2,B_1,B_2}^{m+n-2} = \sum_{j=0}^{m+n-2} \binom{m+n-2}{j} (L_{A_2} R_{B_2})^{m+n-j} \Delta_{A_2,B_2}^{m+n-j} (I) \Delta_{A_1,B_1}^{j} (I)
\]

(since \( \Delta_{A_2,B_2}^{n} (I) = 0 \) implies \( A_2^{m} \Delta_{A_2,B_2}^{m-1} (I) B_2^{n} = \Delta_{A_2,B_2}^{n-1} (I) \)). Hence

\[
\Delta_{A_1,A_2,B_1,B_2}^{m+n-2} (I) \neq 0 \iff (L_{A_2} R_{B_2})^{m-1} \Delta_{A_2,B_2}^{n-1} (I) \Delta_{A_1,B_1}^{m-1} (I) \neq 0
\]

(equivalently, \( (A_1,B_1) \in \text{strict-} \Delta^m (I) \) and \( (A_2,B_2) \in \text{strict-} \Delta^n (I) \)).

(iii) Again, since \( \delta_{A_1,B_1}^m (I) = 0 = \delta_{A_2,B_2}^n (I) \),

\[
\delta_{A_1,A_2,B_1,B_2}^{m+n-2} = \sum_{j=0}^{m+n-2} \binom{m+n-2}{j} L_{A_2}^{m+n-j} R_{B_1}^j \delta_{A_1,B_1}^{m+n-j} (I) \delta_{A_2,B_2}^{j} (I)
\]

Evidently,

\[
\delta_{A_2,B_2}^n (I) = 0 \iff L_{A_2} \delta_{A_2,B_2}^{n-1} (I) = R_{B_2} \delta_{A_2,B_2}^{n-1} (I)
\]

\[
\iff L_{A_2}^{n-1} \delta_{A_2,B_2}^{n-1} (I) = R_{B_2}^{n-1} \delta_{A_2,B_2}^{n-1} (I)
\]

and

\[
\delta_{A_1,B_1}^m (I) = 0 \iff L_{A_1} \delta_{A_1,B_1}^{m-1} (I) = R_{B_1} \delta_{A_1,B_1}^{m-1} (I)
\]

\[
\iff R_{B_1}^{n-1} \delta_{A_1,B_1}^{m-1} (I) = L_{A_1}^{n-1} \delta_{A_1,B_1}^{m-1} (I).\]

Hence

\[
(A_1 A_2, B_1 B_2) \in \text{strict-} \delta^{m+n-1} (I) \iff L_{A_2}^{m-1} R_{B_1}^{n-1} \delta_{A_1,B_1}^{m-1} (I) \delta_{A_2,B_2}^{n-1} (I) \neq 0
\]

\[
\iff L_{A_1}^{n-1} R_{B_2}^{m-1} \delta_{A_1,B_1}^{m-1} (I) \delta_{A_2,B_2}^{n-1} (I) \neq 0.
\]

The condition that \( \delta_{A_1,B_1}^{m-1} (I) \neq 0 \neq \delta_{A_2,B_2}^{n-1} (I) \), although necessary for \( (A_1 A_2, B_1 B_2) \in \text{strict-} \delta^{m+n-1} (I) \) in Theorem 3.1(iii), is not sufficient.
Example 3.2 Let \( A \in \mathcal{B}(\mathcal{H}) \), \( \mathcal{H} = \mathbb{C}^3 \), be the operator of Example 2.2 and let the conjugation \( C \) of \( \mathcal{H} \) be defined by \( C(x, y, z) = (z, y, x) \). Then \( [C, D] = 0 \), \((A^*, CAC) \in \text{strict-} \delta^3(I), (A^*, DAD) \in \text{strict-} \delta^3(I), (A^2, CACDAD) \in \text{strict-} \delta^I(I) \) and

\[
A^2DA^*DA^2C(I)\delta^4_{A^*,DAD}(I) = 0
\]

(even though neither of the operators \( \delta^2_{A^*,CAC}(I) \) and \( \delta^4_{A^*,DAD}(I) \) is 0).

Translated to tensor products, Theorem 3.1 implies:

Proposition 3.3 Let \( A_i, B_i \in \mathcal{B}(\mathcal{X}), i = 1, 2 \).

(i) If \( (A_1, B_1) \in d^m(I) \) and \( (A_2, B_2) \in d^n(I) \), \( d = \Delta \) or \( \delta \) (exclusive 'or' as always), then \((A_1 \otimes A_2, B_1 \otimes B_2) \in d^{m+n-1}(I \otimes I)\).

(ii) Any two of the hypotheses \( (a) \) \( (A_1, B_1) \in \text{strict-} \Delta^m(I); (b) \) \( (A_2, B_2) \in \text{strict-} \Delta^n(I); (c) \) \( (A_1 \otimes A_2, B_1 \otimes B_2) \in \text{strict-} \Delta^{m+n-1}(I \otimes I) \) implies the other.

(iii) If \( d = \delta \) in part (i), then \((A_1 \otimes A_2, B_1 \otimes B_2) \in \text{strict-} \Delta^{m+n-1}(I \otimes I) \) if and only if

\[
(A_1^{-1} \otimes A_2^{-1})\delta_{A_1 \otimes I, I \otimes B_1}^{-1}(I \otimes I)\delta_{A_2 \otimes I, I \otimes B_2}^{-1}(I \otimes I) \neq 0.
\]

Proof. Define \( S_i, T_i \in \mathcal{B}(\mathcal{X} \otimes \mathcal{X}), i = 1, 2 \), by \( S_1 = (A_1 \otimes I), S_2 = (I \otimes A_2), T_1 = (B_1 \otimes I) \) and \( T_2 = (I \otimes B_2) \). Then \( [S_1, S_2] = 0 = [T_1, T_2], S_1S_2 = (A_1 \otimes A_2) \) and \( T_1T_2 = (B_1 \otimes B_2) \). Theorem 3.1 applies and we have:

(i) \( d^{m+n-1}_{S_iS_jT_k} = 0 \) implies \( d^{m+n-1}_{A_1 \otimes A_2, B_1 \otimes B_2}(I \otimes I) = 0 \).

(ii) Since \((A_1 \otimes A_2, B_1 \otimes B_2) \in \text{strict-} \Delta^{m+n-1}(I \otimes I) \) if and only if \( \Delta^{m-1}_{A_1 \otimes I, I \otimes B_1}(I \otimes I) \neq 0 \) \( \Delta^{n-1}_{I \otimes A_2, I \otimes B_2}(I \otimes I) \neq 0 \) \((A_1, B_1) \in \text{strict-} \Delta^m(I) \) if and only if \( \Delta^{m-1}_{A_1, B_1}(I) \neq 0 \) (equivalently, \( \Delta^{m-1}_{A_1 \otimes I, B_1 \otimes I}(I \otimes I) \neq 0 \)) and \((A_2, B_2) \in \text{strict-} \Delta^n(I) \) if and only if \( \Delta^{m-1}_{A_2, B_2}(I) \neq 0 \) (equivalently, \( \Delta^{n-1}_{I \otimes A_2, I \otimes B_2}(I \otimes I) \neq 0 \)), any two of (a), (b) and (c) imply the third.

The proof of (iii) being evident, the proof is complete. \( \Box \)

4. Results: Perturbation by commuting nilpotents

Perturbation by commuting nilpotents of \( m \)-isometric, \( m \)-selfadjoint and left \( m \)-invertible (etc.) operators has been considered by a number of authors in the recent past, amongst them Bermúdez et al (9), [11], Le [23] and Gu [19]). The following theorem provides a particularly elementary proof of the currently available results for pairs \((A, B) \in d^m(I)\).

Theorem 4.1 If \((A, B) \in d^m(I)\), and \( N \) is an \( n \)-nilpotent which commutes with \( B \), then \((A, B + N) \in d^{m+n-1}(I)\). Furthermore, \((A, B + N) \in \text{strict-} d^{m+n-1}(I) \) if and only if

\[
R_N^{-1}\delta_{A,B}^{-1}(I) \neq 0, \quad d = \delta
\]

and

\[
L_A^{-1}R_N^{-1}\Delta_{A,B}^{-1}(I) \neq 0, \quad d = \delta.
\]
Proof. A straightforward calculation shows that

\[
\Delta_{A,B+N}^{m+n-1}(I) = \{(L_A R_B - I) + L_A R_N\}^{m+n-1}(I)
\]

\[
= \left\{ \sum_{j=0}^{m+n-1} \left( \begin{array}{c} m+n-1 \\ j \end{array} \right) (L_A R_N)^j (L_A R_B - I)^{m+n-1-j} \right\}(I)
\]

\[
= \sum_{j=0}^{m+n-1} \left( \begin{array}{c} m+n-1 \\ j \end{array} \right) (L_A R_N)^j \Delta_{A,B}^{m+n-1-j}(I)
\]

and

\[
\delta_{A,B+N}^{m+n-1}(I) = \{(L_A - R_B) - R_N\}^{m+n-1}(I)
\]

\[
= \left\{ \sum_{j=0}^{m+n-1} (-1)^j \left( \begin{array}{c} m+n-1 \\ j \end{array} \right) R_N^j (L_A - R_B)^{m+n-1-j} \right\}(I)
\]

\[
= \sum_{j=0}^{m+n-1} (-1)^j \left( \begin{array}{c} m+n-1 \\ j \end{array} \right) R_N^j \delta_{A,B}^{m+n-1-j}(I).
\]

The operator \( N \) being \( n \)-nilpotent, \( N^j = 0 \) for all \( j \geq n \). For \( j \leq n-1 \) (equivalently, \( -j \geq -n+1 \)), \( m+n-1-j \geq m \). Hence \( (A,B+N) \in d^{m+n-1}(I) \).

Since \( (A,B+N) \in \text{strict-}d^{m+n-1}(I) \) if and only if \( d_{A,B+N}^{m+n-2}(I) \neq 0 \), it follows that \( (A,B+N) \in \text{strict-}d^{m+n-1}(I) \) if and only if

\[
\left( \begin{array}{c} m+n-1 \\ n-1 \end{array} \right) (L_A R_N)^{n-1} \Delta_{A,B}^{m-1}(I) \neq 0, \ d = \Delta
\]

and

\[
\left( \begin{array}{c} m+n-1 \\ n-1 \end{array} \right) R_N^{n-1} \delta_{A,B}^{m-1}(I) \neq 0, \ d = \delta.
\]

The proof is now evident. \( \square \)

Evidently, \( (A,B) \in d^m(I) \) implies \( (B^*, A^*) \in d^m(I^*) \), and if \( M \) is an \( n_1 \)-nilpotent which commutes with \( A \), then \( (B^*, A^* + M^*) \in d^{m+n_1-1}(I^*) \). Hence, Theorem 4.1 implies:

**Corollary 4.2** If \( (A,B) \in d^m(I) \), and \( N_i \ (i = 1, 2) \) are \( n_i \)-nilpotents such that \( [A,N_2] = 0 = [B,N_1] \), then \( (A+N_2, B+N_1) \in d^{m+n_1+n_2-2}(I) \).

If \( A, B \) and \( N_i \ (i = 1, 2) \) are the operators of Corollary 4.2, then

\[
\Delta_{A+N_2,B+N_1}^{r}(I) = \left\{ \sum_{j=0}^{r} \left( \begin{array}{c} r \\ j \end{array} \right) \Delta_{A,B+N_1}^{r-j}(L_{N_2} R_{B+N_1})^j \right\}(I)
\]

and

\[
\delta_{A+N_2,B+N_1}^{r}(I) = \left\{ \sum_{j=0}^{r} \left( \begin{array}{c} r \\ j \end{array} \right) \delta_{A,B+N_1}^{r-j}(L_{N_2})^j \right\}(I)
\]
and it follows from an argument similar to the one above that \((A + N_2, B + N_1) \in \text{strict-}d^{m+n_1+n_2-2}(I)\) if and only if
\[
\Delta_{A,B+N_1}^{m+n_1-2}(I)(L_{N_2}R_{B+N_1})^{n_2-1} \neq 0, \quad d = \Delta
\]
and
\[
\delta_{A,B+N_1}^{m+n_1-2}(I)L_{N_2}^{n_2-1} \neq 0, \quad d = \delta.
\]

In the particular case in which \(m = 1\), Theorem 4.1 implies that a necessary and sufficient condition for \((A, B + N) \in \text{strict-}\Delta^n(I)\) is that \(A^{n-1}N^{n-1}B^{n-1} = A^{n-1}B^{n-1}N^{n-1} = N^{n-1} \neq 0\). Hence:

**Corollary 4.3** If \((A, B) \in \triangle(I)\), and \(N\) is an \(n\)-nilpotent which commutes with \(B\), then \((A, B + N) \in \text{strict-}\Delta^n(I)\).

An \(m\)-isometric version of Corollary 4.3 was first proved by Bermúdez et al. ([9], Theorem 2.2). Observe that if \(d = \delta\) and \(m = 1\), then \(\delta_{A,B}(I) = 0\) if and only if \(A = B\) and then \((A, B + N) \in \text{strict-}\delta^n(I)\) in Theorem 4.1.

If \(m > 1\), then the hypothesis \((A, B) \in \text{strict-}d^m(I)\) is not enough to guarantee \((A, B + N) \in \text{strict-}d^{m+n-1}(I)\), as the following (slightly changed) example from [19, Example 7] shows.

**Example 4.4** Define \(A, B \in B(\mathcal{X} \oplus \mathcal{X})\) by \(A = I \oplus A_1\) and \(B = I \oplus B_1\), where \((A_1, B_1) \in \text{strict-}d^n(I)\). Let \(N_1 \in B(\mathcal{X})\) be an \(n\)-nilpotent operator, and define \(N \in B(\mathcal{X} \oplus \mathcal{X})\) by \(N = N_1 \oplus 0\). Then neither of the operators \(d_{A,B}^r(I), N^{n-1}\) and \(A^{n-1}\) is the 0 operator,

\[
d_{A,B+N}^r(I) = d_{I,N_1}^r(I) \oplus d_{A_1,B_1}^r(I),
\]

\[(I, I + N_1) \in \text{strict-}d^n(I), \quad (A_1, B_1) \in \text{strict-}\Delta^n(I), \quad \text{and}
\]

\[(A, B + N) \in \text{strict-}d^t(I), \quad t = \max\{n, m\} < m + n - 1.
\]

**Proposition 4.5** If \(A_1, B_1 \in B(\mathcal{X})\) are such that \([A_1, A_2] = 0 = [B_1, B_2], (A_1, B_1) \in \delta^m(I)\) and \((A_2, B_2) \in \delta^n(I)\), then \((A_1 + A_2, B_1 + B_2) \in \text{strict-}\delta^{m+n-1}(I)\) if and only if \(\delta_{A_1,B_1}^{m-1}(I)\delta_{A_2,B_2}^{n-1}(I) \neq 0\).

**Proof.** If we let \(S = (L_{A_1} + L_{A_2} - R_{B_1} - R_{B_2})(I) = \{(L_{A_1} - R_{B_1}) + (L_{A_2} - R_{B_2})\}(I)\), then

\[
S^{m+n-1} = \sum_{j=0}^{m+n-1} \left( m + n - 1 \atop j \right) \delta_{A_1,B_1}^{m+n-1-j}(I)\delta_{A_2,B_2}^j(I)
\]

and

\[
= \sum_{j=0}^{m+n-1} \left( m + n - 1 \atop j \right) \delta_{A_2,B_2}^j(I)\delta_{A_1,B_1}^{m+n-1-j}(I).
\]
Remark 4.6

If \( A \) isometric if

\[
S^{m+n-2} = \sum_{j=0}^{m+n-2} \binom{m+n-2}{j} \delta_{A_1,B_1}^{m+n-2-j}(I) \delta_{A_2,B_2}^j(I)
\]

it follows that \( S^{m+n-2} \neq 0 \) if and only if \( \delta_{A_1,B_1}^{m+1}(I) \delta_{A_2,B_2}^1(I) \neq 0 \). ☐

The case \( d = \delta \) of Theorem 4.1 is obtained from Proposition 4.5 upon choosing \( A_2 = 0 \) and \( B_2 = N \) (for then \( \delta_{0,B_2}^n(I) = 0 \) if and only if \( B_2^n = 0 \)). It is clear from Example 4.4 that the condition \( \delta_{A_1,B_1}^{n-1}(I) \neq 0 \neq \delta_{A_2,B_2}^{n-1}(I) \) in Proposition 4.5 though necessary is in no way sufficient for \( (A_1 + A_2, B_1 + B_2) \in \text{strict}-\delta^{m+n-1}(I) \). If we let \( B_i = S_i^* \) and \( A_i = S_i \), \( i = 1,2 \), for some operators \( S_i \in B(\mathcal{H}) \), and choose \( n = 1 \), then Proposition 4.5 says: \( S_1 + S_2 \) is \( m \)-selfadjoint ([23], Corollary 2.9). Furthermore, \( S_1 + S_2 \in \text{strict}-m \)-selfadjoint if and only if \( S_1 \in \text{strict}-m \)-selfadjoint.

Remark 4.6 A \( B(\mathcal{H}) \) is said to be an \( n \)-Jordan operator (or, a Jordan operator of order \( n \)) if \( A = S + N \) for some selfadjoint operator \( S \in B(\mathcal{H}) \) (i.e., for some \( S \in B(\mathcal{H}) \) satisfying \( \delta_{S^*S}(I) = 0 \)) and an \( n \)-nilpotent operator \( N \in B(\mathcal{H}) \) such that \( [S,N] = 0 \). Theorem 4.1 says that \( n \)-Jordan and satisfies \( \delta_{A^*,A}^{2n-1}(I) = 0^n \) ([24], Theorem 3.2).

Remark 4.7 Bayart ([7]) defines a Banach space operator \( S \in B(\mathcal{X}) \) to be \( m \)-isometric if \( \sum_{j=0}^m (-1)^j \binom{m}{j} \|S^{m-j}x\|^2 = 0 \) for all \( x \in \mathcal{X} \). Hilbert space \( m \)-isometric operators are left \( m \)-invertible operators, but a similar description does not fit the (generalized) Banach space definition. Arguments similar to those used to prove Theorems 3.1 and 4.1 do not extend to prove similar results for Banach space \( m \)-isometries. This in a way is to be expected: Whereas it is true that if \( A,B \in B(\mathcal{X}) \) commuting operators such that \( A \) is \( m \)-isometric and \( B \) is \( m \)-isometric then \( AB \) is \( (m+n-1) \)-isometric ([8], Theorem 3.3), the perturbation by a commuting \( n \)-nilpotent operator of an \( m \)-isometric (Banach space) operator may not be an \( (m+2n-2) \)-isometric operator ([11], Example 4.1). Observe that Bayart’s definition of \( m \)-isometries converts the problem of commuting products into a problem in arithmetic progressions.

Let \( N \in B(\mathcal{X}) \) be an \( n \)-nilpotent operator, and let \( S^r, T^r \in B(\mathcal{X} \bar{\otimes} \mathcal{X}) \), \( r \geq 1 \) some integer, be the operators

\[
S^r = \Delta_{A \otimes I + I \otimes N, B \otimes I}(I \otimes I) = \{(L_{A \otimes I} R_{B \otimes I} - I \otimes I) + L_{A \otimes I} R_{I \otimes N}\}^r (I \otimes I)
\]

\[
= \sum_{j=0}^r \binom{r}{j} (L_{I \otimes N} R_{B \otimes I})^j (L_{A \otimes I} R_{B \otimes I} - I \otimes I)^{r-j} (I \otimes I)
\]

\[
= \sum_{j=0}^r \binom{r}{j} \Delta_{A,B}^{r-j}(I) B^j \otimes N^j
\]
Proposition 4.8

Given operators $A, B, N \in B(\mathcal{X})$, any two of the following conditions implies the third.

(i) $(A, B) \in \text{strict-}d^n(I)$.

(ii) $N^n = 0$.

(iii) $(A \otimes I + I \otimes N, B \otimes I) \in \text{strict-}d^{m+n-1}(I \otimes I)$.

Proof. $(i) \land (ii) \implies (iii)$: We have already seen (above) that $(A \otimes I + I \otimes N, B \otimes I) \in d^{m+n-1}(I \otimes I)$. Since $\Delta_{A,B}^{m+n-1}(I)B^n = 0$ implies $A^n \Delta_{A,B}^{m-1}(I)B^n = \Delta_{A,B}^{m-1}(I) = 0$ (Recall: $\Delta_{A,B}^{m}(I) = A\Delta_{A,B}^{m-1}(I)B - \Delta_{A,B}^{m-1}(I) = 0$), and since $N^n \neq 0 \neq d^{m-1}(I)$, (1) holds.

$(i) \land (iii) \implies (ii)$: Hypothesis (i) implies

$$S^{m+n-1} = \sum_{j=n}^{m+n-1} \binom{m+n-1}{j} \Delta_{A,B}^{m+n-1-j}(I)B^j \otimes N^j$$

and

$$T^{m+n-1} = \sum_{j=n}^{m+n-1} \binom{m+n-1}{j} \delta_{A,B}^{m+n-1-j}(I) \otimes N^j.$$ 

Here the sets $\left\{ \Delta_{A,B}^{m+n-1-j}(I)B^j \right\}_{j=n}^{m+n-1}$ and $\left\{ \delta_{A,B}^{m+n-1-j}(I) \right\}_{j=n}^{m+n-1}$ are linearly independent. Hence we must have $N^j = 0$ for all $n \leq j \leq m + n - 1$.

$(ii) \land (iii) \implies (i)$: Hypotheses $(ii) \land (iii)$ imply

$$S^{m+n-1} = \sum_{j=0}^{n-1} \binom{m+n-1}{j} \Delta_{A,B}^{m+n-1-j}(I)B^j \otimes N^j = 0$$
and
\[ T^{m+n-1} = \sum_{j=0}^{n-1} \binom{m+n-1}{j} \delta_{A,B}^{-m+n-1-j}(I) \otimes N^j = 0. \]

The set \( \{N^j\}_{j=0}^{n-1} \) being linearly independent, we must have \( \Delta_{A,B}^{m+j}(I)B^j = 0 = \delta_{A,B}^{m+j}(I) \) for all \( 0 \leq j \leq n-1 \). Applying Lemma 2.1, it follows that \((A, B) \in d^m(I)\).

Again, since \(d^m(I)\) is left invertible, hence left invertible, and
\[
S^{m+n-2} = \binom{m+n-2}{n-1} \Delta_{A,B}^{m-1}(I)B^{n-1} \otimes N^{n-1},
\]
\[
T^{m+n-2} = \binom{m+n-2}{n-1} \delta_{A,B}^{m-1}(I) \otimes N^{n-1}
\]
and
\[
\Delta_{A,B}^{m-1}(I)B^{n-1} = 0 \implies \Delta_{A,B}^{m-1}(I) = 0,
\]
we must have \( \Delta_{A,B}^{m-1}(I) \neq 0 \neq \delta_{A,B}^{m-1}(I) \), i.e., \((A, B) \in \text{strict-}d^m(I)\).

\[ \square \]

5. Results: Commuting \( A \) and \( B \)

Pairs of operators \((A, B) \in d^m(I)\) for which \([A, B] = 0\) have a particular representation: There exists an \(m\)-nilpotent \(N\) satisfying \([B, N] = 0\) and such that \(A = B^{-1} + N\) if \(d = \Delta\) and \(A = B + N\) if \(d = \delta\). The following theorem is a generalization of ([19], Proposition 8.9).

**Theorem 5.1** (a) Given \(A, B \in B(\mathcal{X})\) such that \([A, B] = 0\), if:
(i) \((A, B) \in \Delta^m(I)\), then \(B\) is invertible and there exists an \(m\)-nilpotent operator \(N\) satisfying \([B, N] = 0\) and \(A = B^{-1} + N\);
(ii) \((A, B) \in \delta^m(I)\), then there exists an \(m\)-nilpotent operator \(N\) satisfying \([B, N] = 0\) and \(B = A + N\).

(b) If \(A, B \in B(\mathcal{X})\) satisfy
(iii) \(\Delta_{A,B}^2(I) = 0\), then \(B\) is invertible if and only if \([A, B] = 0\);
(iv) \(\delta_{A,B}^2(I) = 0\), then \([A, B] = 0\).

**Proof.** (i) The hypotheses \([A, B] = 0\) and \((A, B) \in \Delta^m(I)\) imply \(\Delta_{A,B}^m(I) = 0 = \Delta_{B,A}^m(I)\), and this in turn implies that both \(A\) and \(B\) are invertible. \((\Delta_{A,B}^m(I) = 0\) implies \(B\) is left \(m\)-invertible, hence left invertible, and \(A\) is right \(m\)-invertible, hence right invertible). The invertibility of \(B\) implies
\[
\Delta_{A,B}^m(I) = 0 \iff \sum_{j=0}^{m} (-1)^j \binom{m}{j} A^j B^{-m+j} = 0 \iff \delta_{A,B}^m(I) = 0.
\]

Define \(N \in B(\mathcal{X})\) by
\[
\delta_{A,B}^{-1} = L_A - R_{B^{-1}} = L_N.
\]

Then
\[
N^m = L_N^m(I) = \delta_{A,B}^{-1}(I) = 0
\]
and
\[
L_N(I) = \delta_{A,B}^{-1}(I) \iff A = B^{-1} + N.
\]
Evidently,
\[ I + NB = AB = BA = I + BN \iff [B, N] = 0. \]

(ii) If \( \delta_{A,B}^m(I) = 0 \), then define \( N \) by \( \delta_{A,B} = L_N \). We have
\[ A - B = \delta_{A,B}(I) = L_N(I) = N \iff A = B + N, \]
\[ N^m = L_N^m(I) = \delta_{A,B}^m(I) = 0 \]
and
\[ NB = AB - B^2 = BA - B^2 = B(A - B) = BN. \]

(iii) It being clear that if \([A, B] = 0\) and \( \Delta_{A,B}^2(I) = 0 \), then \( B \) is invertible. Then
\[ \Delta_{A,B}^2(I) = 0 \iff A^2 - 2AB^{-1} + B^{-2} = \delta_{A,B}^{-1}(I) = 0. \]
The operator \( \delta_{A,B}^{-1} = L_N \) satisfies \( \delta_{A,B}^{-2}(I) = L_N^2(I) = N^2 = 0 \). Since \( \delta_{A,B}^{-1}(I) = A - B^{-1} = N \), \( (A - B^{-1})^2 = A^2 - AB^{-1} - B^{-1}A + B^{-2} \). Hence (since \( \delta_{A,B}^{-2}(I) = A^2 - 2AB^{-1} + B^{-2} \))
\[ B^{-1}A = AB^{-1} \iff [A, B] = 0. \]

(iv) If \( \delta_{A,B}^2(I) = 0 \), then the operator \( \delta_{A,B} = L_N \) satisfies \( N = A - B \) and
\[ \delta_{A,B}^2(I) = N^2 = 0. \]
Thus
\[ \delta_{A,B}^2(I) = A^2 - 2AB + B^2 = N^2 = (A - B)^2 = A^2 - AB - BA + B^2 \iff [A, B] = 0. \]
\[ \Box \]

If \( X = \mathbb{C}^n \) is a finite dimensional Hilbert space, then \( (A, B) \in \Delta^m(I) \) implies \( B \) (also, \( A \)) is invertible. The conclusion of Theorem 5.1 (iii) does not extend to \( m \geq 3 \) even for \( A, B \in B(\mathbb{C}^n) \) (see [19], Example 10). Also, \( [A, B] = 0 \) in (iv) of Theorem 5.1 does not guarantee \( \delta_{A,B}^2(I) = 0 \): Consider, for example, the operators \( A \) and \( DAD \) of Example 2.2: \( [A, DAD] = 0 \) and \( \delta_{A,DAD}^2(I) \neq 0 \).

If \( \mathcal{B} \subset B(\mathcal{H}) \) is an algebraic operator, then \( B \) has a representation \( B = \bigoplus_{r=1}^m B|_{H_0(B - \lambda_i I)} \), where \( \sigma(B) = \{ \lambda_i : 1 \leq i \leq r \} \) and \( H_0(B - \lambda_i I) = \{ x \in \mathcal{H} : \lim_{n \to \infty} \| (B - \lambda_i I)^n x \| = 0 \} = (B - \lambda_i I)^{-p_i}(0) \) for some integer \( p_i > 0 \). (Indeed, the points \( \lambda_i \) are poles of the resolvent of \( B \) of some order \( p_i \).) Each \( B_i = B|_{H_0(B - \lambda_i I)} \) has a representation \( B_i = \lambda_i I_i + N_i \) for some nilpotent \( N_i \) (of order \( p_i \)) and \( B = \bigoplus_{i=1}^r \lambda_i I_i + N = B_0 + N \) for some normal operator \( B_0 \) and a nilpotent \( N \) such that \( [B_0, N] = 0 \). If we now assume (additionally) that \( B \) is \( m \)-isometric, then \( B \) is invertible, hence \( |\lambda_i| = 1 \) for all \( \lambda_i \in \sigma(B) \) ([7], Proposition 2.3). The operator \( B_0 \) being normal is (hence) unitary. Consequently, \((B_0^*, B_0) \in \Delta(I)\). Since \( B \) is \( m \)-isometric (by hypothesis), it follows from an application of Corollary 4.3 that the order \( n \) of nilpotency of \( N \) is given by \( n = m^2 + 1 \). Observe here that the invertibility of \( B \) forces \( m \) to be an odd integer ([7], Proposition 2.4). We have proved:

**Proposition 5.2** Let \( B \in B(\mathcal{H}) \) be an algebraic \( m \)-isometric operator. Then \( m \) is odd and \( B \) is the perturbation of a unitary operator by a commuting \( \frac{m+1}{2} \)-nilpotent operator.

Recall that operators \( B \in B(\mathbb{C}^t) \), \( \mathbb{C}^t \) the \( t \)-dimensional complex space, are algebraic. Proposition 5.2 was proved for operators \( B \in B(\mathbb{C}^t) \) by Bermúdez et al ([9], Theorem 2.7). That an \( m \)-isometric operator \( B \in B(\mathbb{C}^t) \) is the perturbation of a unitary by a commuting nilpotent was first observed by Agler et al ([5], Page 134).
References

[1] J. Agler, Subjordan operators, ProQuest LLC, Ann Arbor, MI(1980), Thesis(Ph.D.), Indiana University (MR2639769)

[2] J. Agler and M. Stankus, m-Isometric transformations of Hilbert space I, Integr. Equat. Oper. Theory 21(1995), 383-420.

[3] J. Agler and M. Stankus, m-Isometric transformations of Hilbert space II, Integr. Equat. Oper. Theory 23(1995), 1-48.

[4] J. Agler and M. Stankus, m-Isometric transformations of Hilbert space III, Integr. Equat. Oper. Theory 24(1996), 379-421.

[5] J. Agler, W. Helton and M. Stankus, Classification of hereditary matrices, Linear Alg. Appl. 274(1998), 125-160.

[6] O.A.M. Sid Ahmed, Some properties of m-isometries and m-invertible operators in Banach spaces, Acta Math. Sci. Ser. B English Ed. 32(2012), 520-530.

[7] F. Bayart, m-isometries on Banach Spaces, Math. Nachr. 284(2011), 2141-2147.

[8] T. Bermúdez, A. Martinón and J.N. Noda, Products of m-isometries, Lin. Alg. Appl. 408(2013) 80-86.

[9] T. Bermúdez, A. Martinón and J.N. Noda, An isometry plus a nilpotent operator is an m-isometry, Applications, J. Math. Anal Appl. 407(2013) 505-512.

[10] T. Bermúdez, C. D. Mendoza and A. Martinon, Powers of m-isometries, Studia Math. 208(2)(2012), 249-255.

[11] T. Bermúdez, A. Martinón, V. Müller and J.N. Noda, Perturbation of m-isometries by nilpotent operators, Abstract and Applied Analysis, Volume 2014, Article ID 745479(6pages).

[12] F. Botelho and J. Jamison, Isometric properties of elementary operators, Linear Alg. Appl. 432(2010), 357-365.

[13] M. Chō, E. Ko and J.E. Lee, Properties of m-complex symmetric operators, Stud. Univ. Babes-Bolyai Math. 62(2017), 233-248.

[14] B. P. Duggal, Tensor product of n-isometries, Linear Alg. Appl. 437(2012), 307-318.

[15] B.P. Duggal, Tensor product of n-isometries II, Functional Anal. Approx. and Computation 4:1(2012), 27-32.

[16] B.P. Duggal, Tensor product of n-isometries III, Functional Anal. Approx. and Computation 4:2(2012), 61-67.

[17] B.P. Duggal and V. Müller, Tensor product of left n-invertible operators, Studia Math. 215(2)(2013), 113-125.

[18] C. K. Fong AND A. R. Sourour, On the operator identity \( \sum A_rX B_r = 0 \), Canadian J. Math. XXXI(1979), 845-857.
[19] C. Gu, *Structure of left n-invertible operators and their applications*, Studia Math. **226** (2015), 189-211.

[20] C. Gu, *Elementary operators which are m-isometric*, Lin. Alg. Appl. **451** (2014), 49-64.

[21] J.W. Helton, *Infinite dimensional Jordan operators and Sturm-Liouville conjugate point theory*, Trans. Amer. Math. Soc. **170** (1972), 305-331.

[22] Hou Jinchuan, *On the tensor products of operators*, Acta Math. Sinica, New Series **9** (1993) 195-202.

[23] Trieu Le, *Algebraic properties of operator roots of polynomials*, J. Math. Anal. Appl. **421** (2015), 1238-1246.

[24] Scott A. McCllough and Leiba Rodman, *Hereditary classes of operators and matrices*, Amer. Math. Monthly **104** (1977), 415-430.

B.P. Duggal, 8 Redwood Grove, London W5 4SZ, England (U.K.).
e-mail: bpduggal@yahoo.co.uk

I. H. Kim, Department of Mathematics, Incheon National University, Incheon, 22012, Korea.
e-mail: ihkim@inu.ac.kr