De Branges spaces and Krein’s theory of entire operators

Luis O. Silva
Departamento de Física Matemática
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas
Universidad Nacional Autónoma de México
C.P. 04510, México D.F.
silva@iimas.unam.mx

Julio H. Toloza*
CONICET
Centro de Investigación en Informática para la Ingeniería
Universidad Tecnológica Nacional – Facultad Regional Córdoba
Maestro López esq. Cruz Roja Argentina
X5016ZAA Córdoba, Argentina
jtoloza@scdt.frc.utn.edu.ar

Abstract

This work presents a contemporary treatment of Krein’s entire operators with deficiency indices (1,1) and de Branges’ Hilbert spaces of entire functions. Each of these theories played a central role in the research of both renown mathematicians. Remarkably, entire operators and de Branges spaces are intimately connected and the interplay between them has had an impact in both spectral theory and the theory of functions. This work exhibits the interrelation between Krein’s and de Branges’ theories by means of a functional model and discusses recent developments, giving illustrations of the main objects and applications to the spectral theory of difference and differential operators.

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1 Introduction

In a series of papers [Kre44a, Kre44b, Kre44c] M. G. Krein formulated the foundations of the theory of entire operators that systematized some abstracted essential facts shared by various, seemingly unrelated, classical problems of mathematical analysis such as the moment problem, the continuation of positive definite functions, and the theory of spiral curves in Hilbert spaces. This unifying approach eventually allowed to tackle other problems in various fields.
and revealed interesting connections between them. Krein’s main motivation for constructing the theory of entire operators seems to have been the classical moment problem since he considered the works on the matter to be the germ of the theory (see [Kre44b] and [GG97, Appendix 3]). Entire operators were present (not always explicitly) in a large part of Krein’s mathematical research and they occupied a prominent position in his panoramic lectures at the jubilee session of the Moscow Mathematical Society (1964) [GG97, Appendix 3] and the International Congress of Mathematicians (1966) [Kre68].

Krein’s theory of entire operators combines methods of operator theory, particularly spectral theory, and the theory of analytical functions, particularly entire functions. This combination has produced an interplay of ideas between these two fields that has been very fruitful in both areas. Here, it is pertinent to mention that Krein developed new results and posed new problems in the theory of functions because of his investigations related with entire operators.

The connection of operator theory and the theory of functions mentioned above arises from the modeling of a symmetric operator of a certain class as the operator of multiplication by the independent variable in a certain functional space. This key part of the theory of entire operators was called by Krein the representation theory of symmetric operators, but it is actually a functional model (see Section 4). Functional models for various classes of operators have been studied by various authors throughout the history of operator theory. The best known functional model is the so-called canonical form of a simple selfadjoint operator [AG93, Section 69]. This model is obtained via the spectral theorem. Other instances are functional models for contractions [SNF70] and dissipative operators [Pav75a, Pav75b] (see also [Nab76, Nab77]). It is worth remarking that Krein’s approach to the construction of functional models for symmetric operators was further developed and generalized by Strauss in his theory of functional models for closed linear operators [Str98, Str00, Str01].

For any entire operator with deficiency indices (1,1), Krein’s functional model yields a Hilbert space of scalar entire functions. Krein noticed that this space had very distinctive properties [GG97, Appendix 3] and studied some of them in the course of his research on the theory of entire operators. The Hilbert spaces of entire functions corresponding to entire operators were the first instances of the spaces that were later introduced and studied by L. de Branges who was not aware of Krein’s results. The works by de Branges on the theory of Hilbert spaces of entire functions [dB59, dB60, dB61a, dB61b, dB61c, dB62] (which were later compiled in the book [dB68]), were considered very deep and
far reaching by Krein [GG97, Appendix 3].

The theory of Hilbert spaces of entire functions has played a central role in de Branges research work. This theory is an important ingredient in his celebrated proof of the Bieberbach conjecture [dB85]. Noteworthily, de Branges theory has been applied to various aspects of spectral theory of differential operators [Dym70, Eck12, Rem02, ST13c].

Krein not only studied entire operators with deficiency indices $(1, 1)$. He also incursioned into the investigation of entire operators with finite and infinite deficiency indices. Krein’s functional model in the case of arbitrary finite and equal deficiency indices yields to Hilbert spaces of vector entire functions. Coincidentally, de Branges studied also spaces of vector entire functions [dBR66], however these spaces are no longer, strictly speaking, de Branges spaces and complications arise when dealing with the parallels between the theory of these spaces and Krein’s entire operators. For this reason, since this review paper deals with the relations between de Branges spaces and Krein’s entire operators, the discussion is restricted to the case of deficiency indices $(1, 1)$.

This work is not exhaustive, many things were deliberately left out in order to keep the material neat, handy, and user-friendly. The aim of this review paper is to introduce the reader to this theory which has multiple ramifications and is interconnected with many objects in analysis and analytic function theory.

2 On a class of symmetric operators

This section introduces the class of symmetric operators relevant to this work and recollects material on operator and spectral theories that will be used in the course of the exposition. A more refined classification of symmetric operators, containing the main object of the present article, is given by the end of the section.

2.1 Closed symmetric operators and their selfadjoint extensions

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, the latter assumed antilinear in its first argument. To any linear operator $T$ acting within $\mathcal{H}$, there corresponds a linear subset

$$\left\{ \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} : \phi \in \text{dom}(T), \; \psi = T\phi \right\}$$
which is called its graph. In this section it is useful to identify an operator with its graph. By this approach an operator is a particular case of a linear relation which is a linear subset of $\mathcal{H} \oplus \mathcal{H}$. In this work, all operators and relations are linear.

For any relation $T$, one has

$$
\text{ker}(T) := \left\{ \phi \in \mathcal{H} : \begin{pmatrix} \phi \\ 0 \end{pmatrix} \in T \right\}, \quad \text{mul}(T) := \left\{ \phi \in \mathcal{H} : \begin{pmatrix} 0 \\ \phi \end{pmatrix} \in T \right\}, \quad \text{dom}(T) := \left\{ \phi \in \mathcal{H} : \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in T \right\}, \quad \text{ran}(T) := \left\{ \psi \in \mathcal{H} : \begin{pmatrix} \phi \\ \psi \end{pmatrix} \in T \right\}.
$$

(1)

The relation $T$ is an operator if and only if $\text{mul}(T) = \{0\}$. A relation is closed if it is a closed set with respect to the norm in $\mathcal{H} \oplus \mathcal{H}$, that is, a closed relation is a subspace of $\mathcal{H} \oplus \mathcal{H}$. Thus, an operator is closed if and only if its graph is a subspace of $\mathcal{H} \oplus \mathcal{H}$.

For any operator $T$, its adjoint $T^*$ is defined by

$$
T^* := \left\{ \begin{pmatrix} \eta \\ \omega \end{pmatrix} \in \mathcal{H} \oplus \mathcal{H} : \langle \eta, T\phi \rangle = \langle \omega, \phi \rangle \text{ for all } \phi \in \text{dom}(T) \right\},
$$

(2)

where $T^*$ is an operator whenever $\[2\]$ is the graph of an operator and a multi-valued relation otherwise. It is straightforward to verify that $T^*$ is an operator if and only if $\text{dom}(T)$ is dense in $\mathcal{H}$ [BS87, Lemma 3, Section 1, Chapter 3].

Let $A$ be a linear closed operator which is symmetric, that is, $A \subset A^*$ (as subsets of $\mathcal{H} \oplus \mathcal{H}$). It is also assumed that the deficiency indices

$$
n_+(A) := \dim[\mathcal{H} \ominus \text{ran}(A - zI)], \quad z \in \mathbb{C}^+, \\
n_-(A) := \dim[\mathcal{H} \ominus \text{ran}(A - zI)], \quad z \in \mathbb{C}^{-}.
$$

are such that

$$
n_+(A) = n_-(A) = 1.
$$

(3)

Since the operator $A$ is closed and symmetric, $\text{ran}(A - zI)$ is closed whenever $\text{im}(z) \neq 0$. Thus, for any nonreal $z$ the Hilbert space $\mathcal{H}$ admits the decomposition into subspaces

$$
\mathcal{H} = \text{ran}(A - zI) \oplus \ker(A^* - \overline{z}I)
$$

(4)

(see [BS87, Theorem 5, Section 3, Chapter 3] for the case when $\overline{\text{dom}(A)} = \mathcal{H}$)
and [Arc61, Proposition 3.31] for the general case). Now, in view of (4), the assumption on the deficiency indices (3) implies that

\[ \dim \ker (A^* - zI) = 1 \quad \text{for all} \quad z \in \mathbb{C} \setminus \mathbb{R}. \]  

(5)

According to (1) and (2), one has

\[ \text{mul}(A^*) = \{ \omega \in H : \langle \omega, \psi \rangle = 0 \text{ for all } \psi \in \text{dom}(A) \}. \]

Therefore \( \text{mul}(A^*) = \text{dom}(A)^\perp \).

Besides the symmetric operator \( A \), this work deals with its canonical selfadjoint extensions. A canonical selfadjoint extension of a given symmetric operator is a selfadjoint extension within the original space \( H \). In other words, a canonical selfadjoint extension \( A_\gamma \) of \( A \) satisfies

\[ A \subset A_\gamma = A_\gamma^* \subset A^*, \quad \text{as subsets of } H \oplus H. \]

Since the restriction of an operator is an operator, one obviously has that all canonical selfadjoint extensions of \( A \) are operators whenever \( \text{dom}(A) = H \). If \( A \) is nondensely defined, this is no longer true. However, under the condition imposed on the deficiency indices (3), the situation is not quite dissimilar.

**Theorem 2.1.** Let \( A \) be a closed, nondensely defined, symmetric operator in a Hilbert space. If (3) holds, then:

(i) The codimension of \( \text{dom}(A) \) equals one.

(ii) All except one of the canonical selfadjoint extensions of \( A \) are operators.

A proof of this theorem follows from [HdS97, Section 1, Lemma 2.2 and Theorem 2.4] (see also [HdS97, Proposition 5.4] and the comment below it). This work deals only with canonical selfadjoint extensions.

The spectral properties of the selfadjoint extensions of \( A \) are essential in this work and, in view of (ii) above, the reader is reminded that the spectrum of a closed linear relation \( T \) in \( H \), denoted \( \text{spec}(T) \) is the complement of the set of all \( z \in \mathbb{C} \) such that \((T - zI)^{-1}\) is a bounded operator defined on all \( H \). Moreover, \( \text{spec}(T) \subset \mathbb{R} \) when \( T \) is a selfadjoint linear relation [DdS74, Theorem 3.20].
2.2 Generalized Cayley transform

A closed symmetric operator $A$ with equal finite deficiency indices has always canonical selfadjoint extensions, with some of them being proper linear relations if $\text{dom}(A)$ is not dense in $H$ (Theorem 2.1 describes this fact when the both deficiency indices are equal to one). In any case, the resolvent of a given canonical selfadjoint extension, say $A_\gamma$, is always an operator. Given such an extension of $A$, define

$$V_\gamma(w,z) := (A_\gamma - wI)(A_\gamma - zI)^{-1} = I + (z - w)(A_\gamma - zI)^{-1},$$

(6)

for $w \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \text{spec}(A_\gamma)$. This operator is the generalized Cayley transform of $A_\gamma$. Unlike the (regular) Cayley transform of a selfadjoint operator, $V_\gamma(w,z)$ is not unitary for arbitrary values of $w$ and $z$ where it is defined. This operator, however, has various relevant properties. Indeed, directly from the first resolvent identity [BS87, Equation 12, Section 7, Chapter 3] (which also holds when $A_\gamma$ is a relation), one verifies that for any $v, w, z \in \mathbb{C} \setminus \text{spec}(A_\gamma)$

$$V_\gamma(w,z) = V_\gamma(z,w)^{-1}, \quad V_\gamma(w,z)V_\gamma(z,v) = V_\gamma(w,v).$$

(7)

Also, it is straightforward to establish that

$$V_\gamma(w,z)^* = V_\gamma(w,z).$$

(8)

By means of the first identity in (7) and (8), the following simple assertion is proven.

**Theorem 2.2.** For any choice of a canonical selfadjoint extension $A_\gamma$ of a closed symmetric operator $A$, the operator $V_\gamma(w,z)$ maps $\ker(A^* - wI)$ injectively onto $\ker(A^* - zI)$.

Define the function

$$\psi(z) := V_\gamma(w_0,z)\psi_{w_0},$$

(9)

for given $\psi_{w_0} \in \ker(A^* - w_0I)$ with $w_0 \in \mathbb{C}$. It follows from Theorem 2.2 that $\psi(z)$ is in $\ker(A^* - zI)$. Clearly, $\psi(z)$ is an analytic function in $\mathbb{C} \setminus \text{spec}(A_\gamma)$ because of the analytic properties of the resolvent. Obviously, $\psi(w_0) = \psi_{w_0}$. Moreover, as a consequence of the second identity in (7), one has

$$\psi(z) = V_\gamma(v,z)\psi(v),$$

(10)
for any pair \( z, v \in \mathbb{C} \setminus \text{spec}(A_\gamma) \).

Note that, in this subsection, (3) is not relevant. All assertions on the properties of the generalized Cayley transform only require the existence of a self-adjoint extension of \( A \), i.e., the equality of the deficiency indices.

### 2.3 Complete nonselfadjointness

**Definition 2.3.** A closed symmetric nonselfadjoint operator is said to be completely nonselfadjoint if it is not a nontrivial orthogonal sum of a symmetric and a selfadjoint operators.

Since an invariant subspace of a symmetric operator is a subspace reducing that operator [BS87, Theorem 4.6.1], a symmetric operator \( A \) is completely nonselfadjoint when there is not a nontrivial invariant subspace of \( A \) on which \( A \) is selfadjoint.

With the help of Theorem 2.2 it can be proven that

\[
\bigcap_{z \in \mathbb{C} \setminus \mathbb{R}} \text{ran}(A - zI)
\]

is the maximal invariant subspace in which \( A \) is selfadjoint. Hence, a necessary and sufficient condition for the symmetric operator \( A \) to be completely nonselfadjoint is

\[
\bigcap_{z \in \mathbb{C} \setminus \mathbb{R}} \text{ran}(A - zI) = \{0\} \quad (11)
\]

(see [LT77, Proposition 1.1] for the general case and [GG97, Theorem 1.2.1] for the densely defined case). Note that, due to (4), the condition (11) is equivalent to

\[
\overline{\text{span}_{z \in \mathbb{C} \setminus \mathbb{R}} \{\ker(A^* - zI)\}} = \mathcal{H} \quad (12)
\]

Complete nonselfadjointness plays an important role in this work’s further considerations. Here, some of the distinctive features that a closed symmetric operator has when it is completely nonselfadjoint are briefly discussed. Consider the function \( \psi(z) \) given by (9) and take a sequence \( \{z_k\}_{k=1}^\infty \) with elements in \( \mathbb{C} \setminus \mathbb{R} \) having accumulation points in the upper and lower half-planes. Suppose that there is \( \eta \in \mathcal{H} \) such that \( \langle \eta, \psi(z_k) \rangle = 0 \) for all \( k \in \mathbb{N} \). This implies that \( \langle \eta, \psi(z) \rangle = 0 \) for \( z \in \mathbb{C} \setminus \mathbb{R} \) because of the analyticity of the function \( z \mapsto \langle \eta, \psi(z) \rangle \). Therefore, by (12), one concludes that \( \eta = 0 \). Thus, completely nonselfadjoint, closed symmetric operators can exist only in a separable Hilbert
space. From now on, the reader should assume that $\mathcal{H}$ is separable.

As in the previous subsection, the condition (3) was so far not assumed in the ongoing discussion. However, for the next property related to complete nonselfadjointness, it is required that (3) holds. First, some definitions:

**Definition 2.4.** A mapping $J$ of $\mathcal{H}$ onto itself such that, for any $\phi, \psi \in \mathcal{H}$ and $a, b \in \mathbb{C}$,

$$J(a\phi + b\psi) = \overline{a}J\phi + \overline{b}J\psi, \quad J^2 = I, \quad \text{and} \quad \langle J\psi, J\phi \rangle = \langle \psi, \phi \rangle,$$

is called an involution.

**Definition 2.5.** An involution $J$ is said to commute with a selfadjoint relation $T$ if

$$J(T - zI)^{-1}\phi = (T - \overline{z}I)^{-1}J\phi,$$

for every $\phi \in \mathcal{H}$ and $z \in \mathbb{C} \setminus \mathbb{R}$. If $T$ is moreover an operator this is equivalent to the usual notion of commutativity, that is,

$$J \text{ dom}(T) \subseteq \text{ dom}(T), \quad JT\phi = TJ\phi,$$

for every $\phi \in \text{ dom}(T)$.

**Theorem 2.6.** Let $A$ be a completely nonselfadjoint, closed, symmetric operator with deficiency indices $n_+(A) = n_-(A) = 1$. Then there exists an involution $J$ that commutes with all its canonical selfadjoint extensions.

The proof of this assertion is constructive. As was already shown, the complete nonselfadjointness of $A$ implies that the sequence $\{\psi(z_k)\}_{k=1}^{\infty}$, used in the paragraph following (12), is total. Define

$$J \left( \sum_{k=1}^{N} c_k \psi(z_k) \right) := \sum_{k=1}^{N} \overline{c_n} \psi(\overline{z_k}),$$

for some $N \in \mathbb{N}$. Then $J$ is extended to the whole space and has the property

$$J\psi(z) = \psi(\overline{z}), \quad \text{for all } z \in \mathbb{C} \setminus \text{spec}(A_{\gamma}).$$

Using the properties (7) and (8) of the generalized Cayley transform, it is shown that $J$ is an involution and commutes with $A_{\gamma}$. Finally, due to (3), a generalization of Krein’s resolvent formula (see [HdS97, Theorem 3.2]) implies the result (some details of this proof can be found in [ST13a, Proposition 2.3]).
The following assertion (cf. [ST13a, Proposition 2.11]) is related to the previous one and, again, its proof relies on the assumption (3).

**Theorem 2.7.** Let $A$ be a completely nonselfadjoint, closed, symmetric operator with deficiency indices $n_+(A) = n_-(A) = 1$, and $J$ be an involution that commutes with a canonical selfadjoint extension $A_\gamma$ of $A$ (hence it commutes with all canonical selfadjoint extensions). For every $v \in \text{spec}(A_\gamma)$, there exists $\psi_v \in \ker(A^* - vI)$ such that $J\psi_v = \psi_v$.

**2.4 Regularity**

**Definition 2.8.** A closed operator $T$ is regular if for every $z \in \mathbb{C}$ there exists $c_z > 0$ such that
\[
\| (T - zI)\phi \| \geq c_z \| \phi \| ,
\]
for all $\phi \in \text{dom}(T)$.

In other words, $T$ is regular when every point of the complex plane is a point of regular type.

**Remark 2.9.** It is easy to see that a regular, closed symmetric operator is necessarily completely nonselfadjoint as regularity implies that the spectral kernel is empty and, therefore, the operator cannot have selfadjoint parts. On the other hand, there are completely nonselfadjoint operators that are not regular.

Since the residual spectrum of a regular operator $T$ fills up the whole complex plane, it follows from [BS87, Section 7.3 Chapter 3] that every complex number is an eigenvalue of $T^*$.

Consider again the case of a closed symmetric operator $A$ with equal deficiency indices and assume that $A$ is regular. If $A_\gamma$ is any canonical selfadjoint extension of $A$, then, since $A_\gamma$ is a restriction of $A^*$, it follows from [BS87, Section 7.4, Chapter 3] that $\text{spec}(A_\gamma) = \text{spec}_{pp}(A_\gamma)$, that is, every element of the spectrum is an eigenvalue.

The following theorem is well known for the case when the operator is densely defined. The proof in the general case can be found in [ST13a, Proposition 2.4].

**Theorem 2.10.** Let $A$ be a regular, closed, symmetric operator such that (3) holds. The following assertions are true:

(i) The spectrum of every canonical selfadjoint extension of $A$ consists solely of isolated eigenvalues of multiplicity one.
(ii) Every real number is part of the spectrum of one, and only one, canonical selfadjoint extension of $A$.

(iii) The spectra of the canonical selfadjoint extensions of $A$ are pairwise interlaced.

Note that (i) above implies that every selfadjoint extension of $A$ is a simple operator [AG93, Section 69].

## 2.5 The classes $S(\mathcal{H})$ and $E_n(\mathcal{H})$

In this subsection, the main classes of operators considered in this work are introduced.

**Definition 2.11.** The class $S(\mathcal{H})$ is the set of all regular, closed symmetric operators with both deficiency indices equal 1, that is,

$$S(\mathcal{H}) := \{ A \text{ is a regular, closed symmetric operator} : n_+(A) = n_-(A) = 1 \}.$$

By Remark 2.9, all operators in $S(\mathcal{H})$ are completely nonselfadjoint. Furthermore, for any element of $S(\mathcal{H})$, Theorem 2.10 holds, and, by Theorem 2.6, one can construct an involution that commutes with all its selfadjoint extensions.

**Definition 2.12.** An operator $A \in S(\mathcal{H})$ is said to belong to the class $E_n(\mathcal{H})$, $n \in \mathbb{N} \cup \{0\} = \mathbb{Z}^+$, if there exist $n + 1$ vectors $\mu_0, \ldots, \mu_n \in \mathcal{H}$ such that

$$\mathcal{H} = \text{ran}(A - zI) \oplus \text{span}\{\mu_0 + z\mu_1 + \cdots + z^n\mu_n\}, \text{ for all } z \in \mathbb{C}. \quad (14)$$

The class $E_0(\mathcal{H})$ admits a further breaking up into the following subclasses.

**Definition 2.13.** An operator $A \in E_0(\mathcal{H})$ is in $E_{-n}(\mathcal{H})$, $n \in \mathbb{N}$, if there exists a vector $\mu_{-n} \in \text{dom}(A^n)$ such that

$$\mathcal{H} = \text{ran}(A - zI) \oplus \text{span}\{\mu_{-n}\}, \text{ for all } z \in \mathbb{C}.$$

Thus, there is an operator class $E_n(\mathcal{H})$ for any $n \in \mathbb{Z}$. Moreover, one has the following chain of inclusions

$$\cdots \subset E_{-1}(\mathcal{H}) \subset E_0(\mathcal{H}) \subset E_1(\mathcal{H}) \subset \cdots \subset S(\mathcal{H}).$$
The following notation will be used

\[ E_{-\infty}(\mathcal{H}) := \bigcap_{n \in \mathbb{Z}} E_n(\mathcal{H}). \]  

(15)

It turns out that the class \( E_{-\infty}(\mathcal{H}) \) is the class of nonselfadjoint Jacobi operators (see §5.1). Also it is easy to see that \([ST13a, Example 3.13]\)

\[ \bigcup_{n \in \mathbb{Z}^+} E_n(\mathcal{H}) \subsetneq S(\mathcal{H}). \]

An operator of the class \( E_n(\mathcal{H}), n \in \mathbb{Z} \), will be henceforth called \( n \)-entire. The classes \( E_0(\mathcal{H}) \) and \( E_1(\mathcal{H}) \) correspond to those defined originally by Krein; this point will be elucidated later in §4.4.

3 On de Branges Hilbert spaces

Most of the elementary albeit profound aspects of the theory of de Branges Hilbert spaces were introduced by L. de Branges himself in [dB59, dB60, dB61a, dB61b, dB61c, dB62], and later compiled and given some further development in [dB68]. An introductory and more amenable exposition of this theory, intended toward its application to the spectral analysis of Sturm-Liouville operators, can be found in [Dym70]. Another introductory presentation is found in [DM76, Chapter 6]. In passing, it is worth mentioning that the de Branges Hilbert space theory has been generalized to Pontryagin spaces of entire functions, an ambitious task being carried out by M. Kaltenbäck and H. Woracek in a series of papers [KW99a, KW99b, KW03, KW06, KW10, KW11, Wor11].

3.1 Definition and elementary properties

There are two essentially different ways of defining a de Branges Hilbert space (dB space from now on). The one introduced next is of axiomatic nature.

**Definition 3.1.** A Hilbert space of entire functions \( \mathcal{B} \) is an (axiomatic) dB space if and only if, for every function \( f(z) \) in \( \mathcal{B} \), the following conditions holds:

(A1) For every \( w \in \mathbb{C} \), the linear functional \( f(\cdot) \mapsto f(w) \) is continuous;

(A2) for every non-real zero \( w \) of \( f(z) \), the function \( f(z)(z-\overline{w})(z-w)^{-1} \) belongs to \( \mathcal{B} \) and has the same norm as \( f(z) \);
(A3) the function \( f^\#(z) := \overline{f(z)} \) also belongs to \( \mathcal{B} \) and has the same norm as \( f(z) \).

By Riesz lemma, condition (A1) is equivalent to saying that \( \mathcal{B} \) has a reproducing kernel, that is, there exists a function \( k : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) such that, for every \( w \in \mathbb{C} \), the function \( z \mapsto k(z, w) \) belongs to \( \mathcal{B} \) and has the property

\[
\langle k(\cdot, w), f(\cdot) \rangle_{\mathcal{B}} = f(w) \quad \text{for all } f(z) \in \mathcal{B};
\]

here \( \langle \cdot, \cdot \rangle_{\mathcal{B}} \) denotes the inner product in \( \mathcal{B} \) (assumed linear in the second argument). Moreover,

\[
k(w, w) = \langle k(\cdot, w), k(\cdot, w) \rangle_{\mathcal{B}} \geq 0
\]

where, as a consequence of (A2), the positivity is strict for every non-real \( w \) as long as \( \mathcal{B} \) contains a nonzero element [dB59, Lemma 1]. Note that

\[
k(z, w) = \langle k(\cdot, z), k(\cdot, w) \rangle_{\mathcal{B}},
\]

therefore \( k(w, z) = \overline{k(z, w)} \). Furthermore, (A3) implies that \( \overline{k(z, w)} \in \mathcal{B} \) for every \( w \in \mathbb{C} \) from which it can be shown that \( \overline{k(z, w)} = k(z, w) \) [dB59, Lemma 1]. Since by the previous discussion one obtains \( k(w, z) = k(z, w) \), it follows that \( k(z, w) \) is antientire with respect to the second argument (it is obviously entire with respect to the first one).

The other way of defining a dB space is constructive and requires two ingredients. The first one is the Hardy space

\[
\mathcal{H}_2(\mathbb{C}^+) := \left\{ f(z) \text{ holomorphic in } \mathbb{C}^+ : \sup_{y>0} \int_\mathbb{R} |f(x + iy)|^2 \, dx < \infty \right\},
\]

where \( \mathbb{C}^+ := \{ z = x + iy : y > 0 \} \). The second ingredient is an Hermite-Biehler function (HB function for short), that is, an entire function \( e(z) \) such that \( |e(z)| > |e(\overline{z})| \) for all \( z \in \mathbb{C}^+ \).

**Definition 3.2.** The (canonical) dB space associated with an HB function \( e(z) \) is the linear manifold

\[
\mathcal{B}(e) := \left\{ f(z) \text{ entire : } \frac{f(z)}{e(z)}, \frac{f^\#(z)}{e(z)} \in \mathcal{H}_2(\mathbb{C}^+) \right\},
\]
equipped with the inner product

\[ \langle f(x), g(x) \rangle_{\mathcal{B}(e)} := \int_{\mathbb{R}} \frac{f(x)g(x)}{|e(x)|^2} dx. \]

Thus defined, \( \mathcal{B}(e) \) is a Hilbert space [dB68, Theorem 21], indeed, it can trivially be identified with a subspace of \( L^2(\mathbb{R}, |e(x)|^{-2} dx) \).

The set \( \mathcal{B}(e) \) can be characterized as

\[ \mathcal{B}(e) = \left\{ f(z) \text{ entire} : \int_{\mathbb{R}} \frac{|f(x)|^2}{|e(x)|^2} dx < \infty \text{ and } \frac{|f(z)|}{|e(z)|} \leq \frac{c_f}{\sqrt{\text{im}(z)}}, \frac{|f^\#(z)|}{|e(z)|} \leq \frac{c_f}{\sqrt{\text{im}(z)}} \text{ for all } z \in \mathbb{C}^+ \right\} \] (16)

[Rem02, Proposition 2.1] so one can alternatively define \( \mathcal{B}(e) \) by (16).

Definitions 3.1 and 3.2 are equivalent (as expected) in the following sense [dB68, Problem 50 and Theorem 23]; see also [DM76, Section 6.1].

**Theorem 3.3.** Let \( \mathcal{B} \) be an axiomatic dB space that contains a nonzero element. Then there exists an HB function \( e(z) \) such that \( \mathcal{B} = \mathcal{B}(e) \) isometrically. Conversely, for every HB function \( e(z) \), the associated canonical dB space \( \mathcal{B}(e) \) satisfies (A1), (A2) and (A3).

Given an HB function \( e(z) \), the reproducing kernel of \( \mathcal{B}(e) \) can be written as [dB68, Theorem 19]

\[ k(z, w) = \begin{cases} 
\frac{e^\#(z)e(\overline{w}) - e(z)e^\#(\overline{w})}{2\pi i(z - \overline{w})}, & w \neq \overline{z}, \\
\frac{1}{2\pi i} \left[ e^\#(z)e(z) - e'(z)e^\#(z) \right], & w = \overline{z}.
\end{cases} \]

On the other hand, for a given dB space \( \mathcal{B} \) an HB function that makes Theorem 3.3 hold is

\[ e(z) = i \sqrt{\frac{\pi}{\text{im}(w_0)k(w_0, w_0)}}(z - \overline{w_0})k(z, w_0), \]

where \( w_0 \) is some fixed number in \( \mathbb{C}^+ \) [dB59, Lemma 4]. Note that a given dB space \( \mathcal{B} \) is not associated to a unique HB function, as it is apparent from the previous formula. However, the different HB functions that give rise to the same dB space are all related in the precise form asserted below [dB60, Theorem 1]. The following statement makes use of the customary decomposition
\[ e(z) = a(z) - ib(z), \]

where

\[ a(z) := \frac{e(z) + e^\#(z)}{2}, \quad b(z) := \frac{e(z) - e^\#(z)}{2}. \]

Notice that these newly introduced entire functions are real in the sense that they satisfy the identity \( f^\#(z) = f(z) \).

**Theorem 3.4.** Suppose \( M \) is a \( 2 \times 2 \) real matrix such that \( \det M = 1 \). Let \( e(z) = a(z) - ib(z) \) be an HB function. Define \( e_M(z) := a_M(z) - ib_M(z) \), where

\[
\begin{bmatrix}
a_M(z) \\ b_M(z)
\end{bmatrix} = M \begin{bmatrix}
a(z) \\ b(z)
\end{bmatrix}.
\]

Then \( e_M(z) \) is an HB function and \( B(e_M) = B(e) \) isometrically. Conversely, if \( \tilde{e}(z) \) is an HB function such that \( B(\tilde{e}) = B(e) \) isometrically, then \( \tilde{e}(z) = e_M(z) \) for some \( 2 \times 2 \) real matrix \( M \).

Orthogonal sets in a dB space can be constructed by means of phase functions [dB68, Theorem 22]. A phase function associated with an HB function \( e(z) \) is a real, monotonically increasing continuous (indeed differentiable) function \( \phi(x) \) such that \( e(x) \exp[i\phi(x)] \in \mathbb{R} \) for all \( x \in \mathbb{R} \) [dB68, Problem 48].

**Theorem 3.5.** Given \( B = B(e) \), let \( \phi(x) \) be a phase function associated with \( e(z) \). Then, for every \( \alpha \in \mathbb{R} \), the following assertions hold true:

(i) \( K_\alpha := \left\{ k(z, t_n) : t_n \in \mathbb{R} \text{ such that } \phi(t_n) = \alpha \mod \pi \right\} \) is an orthogonal set in \( B \);

(ii) \( \text{span} K_\alpha \neq B(e) \) if and only if \( e^{i\alpha}e(z) - e^{-i\alpha}e^\#(z) \in B \);

(iii) if \( e^{i\alpha}e(z) - e^{-i\alpha}e^\#(z) \notin B \) then

\[
\| f(\cdot) \|_B^2 = \sum_n \left| \frac{f(t_n)}{e(t_n)} \right|^2 \frac{\pi}{\phi'(t_n)},
\]

for every \( f(\cdot) \in B \).

In connection with the last theorem it is worth mentioning that, associated with every orthogonal set \( K_\alpha \) of a dB space \( B \) (and assuming \( \text{span} K_\alpha = B \)), one has the sampling formula

\[
f(z) = \sum_n \frac{k(z, t_n)}{k(t_n, t_n)} f(t_n) = \sum_n \frac{g(z)}{(z - t_n)g'(t_n)} f(t_n), \quad f(z) \in B,
\]
where the latter expression has the form of a standard Lagrange interpolation formula; here \( g(z) = \frac{1}{2\pi i} \left[ e^\#(z)e(t_n) - e(z)e^\#(t_n) \right] \).

### 3.2 Spaces of associated functions

**Definition 3.6.** An entire function \( h(z) \) is said to be associated to a dB space \( B \) if

\[
\frac{f(z)h(w) - f(w)h(z)}{z - w} \in B,
\]

for every \( f(z) \in B \) and \( w \in \mathbb{C} \) such that \( f(w) \neq 0 \). The set of all functions associated to \( B \) is denoted by \( \text{assoc} B \).

Clearly, \( \text{assoc} B \) is a linear manifold which can also be constructed in terms of \( B \) itself \[KW99a, \text{Lemma 4.5}\],

\[
\text{assoc} B = B + zB.
\]

Another insightful characterization of \( \text{assoc} B \) is given by \[dB68, \text{Theorem 25}\], which can be formulated as follows \[KW05, \text{p. 236}\].

**Theorem 3.7.** Let \( e(z) \) be an HB function such that \( B = B(e) \). Then

\[
\text{assoc} B = \left\{ f(z) \text{ entire} : \frac{f(z)}{(z + i)e(z)} - \frac{f^\#(z)}{(z + i)e^\#(z)} \in \mathcal{H}_2(\mathbb{C}^+) \right\}.
\]

Note that the characterization above implies that \( \text{assoc} B(e) \) itself turns out to be a dB space, since \((z + i)e(z)\) is an HB function. Furthermore, it is also clear that \( e(z) \in \text{assoc} B(e) \setminus B(e) \).

Within \( \text{assoc} B(e) \) there is a distinguished family of functions, defined as

\[
s_\beta(z) := \frac{i}{2} \left[ e^{i\beta} e(z) - e^{-i\beta} e^\#(z) \right], \quad \beta \in [0, \pi).
\]

Generically \( s_\beta(z) \in \text{assoc} B(e) \setminus B(e) \). More precisely \[dB59, \text{Lemma 7}\]:

**Lemma 3.8.** Assume \( B(e) \) contains a nonzero element. Then at most one of the functions \( s_\beta(z) \) belongs to \( B(e) \).

A special role is played by the zeros of the functions \( s_\beta(z) \).

**Lemma 3.9.** Let \( e(z) \) be an HB function having no real zeros. Assume furthermore that \( B(e) \) contains a nonzero element. Then, for every \( \beta \in [0, \pi) \), the
zeros of \( s_\beta(z) \) are all simple and real. Moreover, the zeros of any two functions \( s_\beta(z) \) and \( s_\gamma(z) \), with \( \beta \neq \gamma \), are interlaced.

The notion of functions associated to a dB space has been generalized in [LW02, Wor11]:

**Definition 3.10.** Given \( n \in \mathbb{Z} \), the set of \( n \)-associated functions of a dB space \( B \) is

\[
\text{assoc}_n B := \begin{cases} 
B + \mathcal{B} + \cdots + z^n \mathcal{B}, & n \geq 0, \\
\text{dom}(S'|n|), & n < 0.
\end{cases}
\]

These linear sets also become dB spaces when equipped with suitable inner products; see [LW02, Corollary 3.4] and [Wor11, Example 2.7].

An important result concerns the existence of a real zero-free function \( n \)-associated to a dB space; see [LW02, Theorem 5.1], [Wor11, Theorem 3.2], and [ST13b, Theorem 2.7] for further details.

**Theorem 3.11.** Suppose \( e(x) \neq 0 \) for all \( x \in \mathbb{R} \) and \( e(0) = (\sin \gamma)^{-1} \) for some fixed \( \gamma \in (0, \pi) \). Furthermore assume that \( \dim B(e) = \infty \). Let \( \{x_j\}_{j \in \mathbb{N}} \) be the sequence of zeros of the function \( s_\gamma(z) \). Also, let \( \{x^+_j\}_{n \in \mathbb{N}} \) and \( \{x^-_j\}_{n \in \mathbb{N}} \) be the sequences of positive, respectively negative, zeros of \( s_\gamma(z) \), arranged according to increasing modulus. Then a zero-free, real entire function belongs to \( \text{assoc}_n B(e) \) if and only if the following conditions hold true:

1. **(C1)** The limit \( \lim_{r \to \infty} \sum_{0 < |x_j| \leq r} \frac{1}{x_j} \) exists.
2. **(C2)** \( \lim_{j \to \infty} \frac{j}{x^+_j} = -\lim_{j \to \infty} \frac{j}{x^-_j} < \infty. \)
3. **(C3)** Assuming that \( \{b_j\}_{n \in \mathbb{N}} \) are the zeros of \( s_\beta(z) \), define

\[
h_\beta(z) := \begin{cases} 
\lim_{r \to \infty} \prod_{|b_j| \leq r} \left( 1 - \frac{z}{b_j} \right) & \text{if } 0 \text{ is not a root of } s_\beta(z), \\
z \lim_{r \to \infty} \prod_{0 < |b_j| \leq r} \left( 1 - \frac{z}{b_j} \right) & \text{otherwise}.
\end{cases}
\]

The series \( \sum_{x_j \neq 0} \left| \frac{1}{x_j^n h_0(x_j) h'_\gamma(x_j)} \right| \) is convergent.
Remark 3.12. In the previous theorem, the assumption that a real zero-free function exists in $\mathcal{B}$ can be weakened to just requiring the existence of a zero-free function (not necessarily real) in $\mathcal{B}$. This follows from the fact that if a zero-free function is in $\mathcal{B}$ then a real zero-free function is also in $\mathcal{B}$ [ST13a, Remark 2.8].

3.3 The operator of multiplication by the independent variable

Given a dB space $\mathcal{B}$, the operator of multiplication by the independent variable is defined by

$$\text{dom}(S) = \{ f(z) \in \mathcal{B} : zf(z) \in \mathcal{B} \}; \quad (Sf)(z) = zf(z), \quad f(z) \in \text{dom}(S).$$

The basic properties of this operator are summarized next.

Theorem 3.13. Let $S$ be the operator defined as above in a dB space $\mathcal{B}$. Then:

(i) $S$ is closed, symmetric, completely nonselfadjoint, and has deficiency indices $(1, 1)$;

(ii) $S$ is real with respect to the involution $\#$, that is, $f^\#(z) \in \text{dom}(S)$ whenever $f(z) \in \text{dom}(z)$ and $(Sf)^\#(z) = zf^\#(z)$;

(iii) $\text{dom}(S)$ is dense in $\mathcal{B}$ if and only if $s_\beta(z) \notin \mathcal{B}$ for all $\beta \in (0, \pi]$;

(iv) if otherwise $s_\gamma(z) \in \mathcal{B}$ for some (necessarily unique) $\gamma \in [0, \pi)$, then $\text{dom}(S)^\perp = \text{span}\{s_\gamma(z)\}$.

Assume moreover that $\mathcal{B} = \mathcal{B}(e)$ isometrically for some HB function $e(z)$ having no real zeros. Then:

(v) $S$ is regular;

(vi) $\text{ran}(S - wI)^\perp = \text{span}\{k(z, w)\}$, for every $w \in \mathbb{C}$.

In view of (iii) and (iv) above, all the canonical selfadjoint extensions of $S$, with at most the exception of one, are operators. In any case their resolvents can be described as follows [KW99a, Propositions 4.6 and 6.1]:

Theorem 3.14. The canonical selfadjoint extensions of $S$ have a bijective correspondence with the associated functions $s_\beta(z)$, $\beta \in [0, \pi)$. This correspondence is given by

$$\left(S_\beta - wI\right)^{-1} f(z) := \frac{f(z) - \frac{s_\beta(z)}{s_\beta(w)} f(w)}{z - w}, \quad (17)$$
for every \( f(z) \in B(e) \) and \( w \in \mathbb{C} \setminus \{ \text{zeros of } s_\beta(z) \} \). Moreover, \( S_\beta \) is a proper linear relation if and only if \( s_\beta(z) \in B(e) \), in which case \( \text{mul } S_\beta = \text{span}\{s_\beta(z)\} \).

Whenever \( s_\beta(z) \notin B(e) \) the resolvent description (17) is equivalent to the following one,

\[
\text{dom}(S_\beta) := \left\{ g(z) = \frac{s_\beta(w)f(z) - s_\beta(z)f(w)}{z-w}, f(z) \in B(e), w \in \mathbb{C} \right\},
\]

\[
(S_\beta g)(z) := zg(z) + f(w)s_\beta(z).
\]

This later form can be obtained from the standard von Neumann theory when \( S \) has dense domain in \( B(e) \) [ST10, p. 367]. In this case the adjoint operator can likewise be specified in terms of the function \( e(z) \). Namely,

\[
\text{dom}(S^*) := \left\{ g(z) = \frac{f(z) - \frac{e(z)}{e(w)}f(w)}{z-w} + \frac{h(z) - \frac{e^\#(z)}{e^\#(w)}h(w)}{z-w}, f(z), h(z) \in B(e), w : \text{im } w > 0 \right\},
\]

\[
(S^* g)(z) := zg(z) + \frac{e(z)}{e(w)}f(w) + \frac{e^\#(z)}{e^\#(w)}h(w).
\]

When \( \text{dom}(S) \) is not dense in \( B(e) \), the canonical selfadjoint operator extensions of \( S \) can be written also as a family of rank-one perturbations [ST13c, Theorem 1.1 and 1.2]:

**Theorem 3.15.** Assume \( s_0(z) \in B \). Then the set of canonical selfadjoint operator extensions of \( S \) is given by

\[
\text{dom}(S_\beta) = \text{dom}(S_{\pi/2}), \quad S_\beta = S_{\pi/2} - \frac{\cot \beta}{\pi} \langle s_0(\cdot), \cdot \rangle_B s_\beta(z),
\]

for \( \beta \in (0, \pi) \). Moreover, for every \( \beta \in (0, \pi) \), \( s_0(z) \) is a generating vector for the operator \( S_\beta \).

From (18) it is clear that the eigenvectors of the selfadjoint operator extensions are also related to the functions \( s_\beta(z) \). Up to a normalization they are given by \( \frac{s_\beta(z)}{x_\beta^\#} \), where \( x_\beta^\# \in \text{spec}(S_\beta) \) is the associated eigenvalue (hence \( s_\beta(x_\beta^\#) = 0 \)).
3.4 Isometric inclusion of dB spaces

A subspace \( L \) of a dB space \( B \) is a dB subspace if itself is a dB space with respect to the inner product inherited from \( B \).

A distinctive structural property of dB spaces states that the set of dB subspaces of a given dB space is totally ordered by inclusion [dB68, Theorem 35]. Below is a simplified version of this assertion, taken from [DM76, Section 6.5]. In what follows a dB space \( B \) is called regular (or short, according to [DM76]) if \( 1 \in \text{assoc } B \).

**Theorem 3.16.** Assume \( B_1, B_2 \) and \( B_3 \) are regular dB spaces such that \( B_1 \) and \( B_2 \) are both isometrically contained in \( B_3 \). Then either \( B_1 \subset B_2 \) or \( B_2 \subset B_1 \).

Recall that the mean type of a function \( f(z) \) (of bounded type on \( \mathbb{C}^+ \); for a definition of this see [KW05, Section 2]) is the number

\[
\text{mt } f := \limsup_{y \to +\infty} \frac{1}{y} \log |f(iy)|.
\]

Note that, for an HB function \( e(z) \), the mean type of \( e^\#(z)/e(z) \) is always nonpositive (this follows from the very definition of an HB function).

The following is an abridged version of [KW05, Theorem 2.7]. Notice that here no condition of regularity is assumed.

**Theorem 3.17.** Given a dB space \( \mathcal{B}(e) \), define \( \tau_e = \text{mt } \frac{e^\#}{e} \) and \( e_\tau(z) = e(z)e^{i\tau z} \).

Then:

(i) \( e_\tau(z) \) is an HB function if and only if \( \tau \geq \tau_e \);

(ii) for each \( \tau \in [\tau_e, 0] \), \( \mathcal{B}(e_\tau) \) is a dB subspace of \( \mathcal{B}(e) \);

(iii) the chain of all dB subspaces \( \mathcal{B}(\tilde{e}) \) of \( \mathcal{B}(e) \), where \( \tilde{e}(z) \) has the same real zeros as \( e(z) \) (counting multiplicities), is given by \( \{ \mathcal{B}(e_\tau) : \tau \in [\tau_e, 0] \} \).

3.5 Examples

A rather comprehensive discussion of special classes of dB spaces is found in [dB68 Chapter 3]. Here, some more or less ubiquitous examples are presented.
Spaces of polynomials. Consider the linear set
\[ \mathcal{P}_n := \{ \text{polynomials of degree } < n \}. \]

The set \( \mathcal{P}_n \) can be turned out into a dB space by choosing an HB function from \( \text{assoc } \mathcal{P}_n \setminus \mathcal{P}_n = \mathcal{P}_{n+1} \setminus \mathcal{P}_n \) (here \( \text{assoc } \mathcal{P}_n = \mathcal{P}_n + z\mathcal{P}_n \)). That is (allowing some abuse of notation), \( \mathcal{P}_n = \mathcal{B}(p_n) \) where \( p_n(z) \) is a polynomial of degree \( n \) with all its roots in \( \mathbb{C}^- \). Clearly, \( \mathcal{P}_n \) has finite dimension, \( \dim \mathcal{P}_n = n \). Conversely, if \( \mathcal{B} = \mathcal{B}(e) \) is a dB space of dimension \( \dim \mathcal{B} = m < \infty \) then there exists a zero-free, real entire function \( g(z) \) such that \( e(z) = g(z)p_m(z) \), where \( p_m(z) \) is a polynomial of degree \( m \) whose roots are in \( \mathbb{C}^- \) [dB68, Problem 88]. In other words, \( \mathcal{B} = g(z)\mathcal{P}_m \) as sets.

The operator of multiplication \( S \) is not densely defined in \( \mathcal{P}_n \), in fact, \( \text{dom}(S) = \mathcal{P}_{n-1} \). Also, \( \text{assoc}_k \mathcal{P}_n = \mathcal{P}_{n+k} \). Finally, one verifies that the chain of dB subspaces of \( \mathcal{P}_n \) is \( \{ \mathcal{P}_m : m \leq n \} \).

It is worth mentioning at this point that a subject of great interest in the theory of dB spaces concerns the role of polynomials. Contrary to a somewhat naive intuition may suggest, an infinite dimensional dB space may have no polynomials at all, it may contain polynomials only up to certain finite degree, or even if all of them are included in the space they may not be a dense subset. In connection with the latter case, note that if the polynomials are contained in a dB space \( \mathcal{B} \) (assumed infinite dimensional), then they are all in the domain of the operator \( S \). Thus a trivial necessary condition for the set of polynomials to be dense in \( \mathcal{B} \) is that \( S \) be densely defined. Also note that if \( 1 \in \text{assoc } \mathcal{B} \setminus \mathcal{B} \), then no polynomial belongs to \( \mathcal{B} \); conspicuous examples are the Paley-Wiener spaces introduced below. There is a good deal of research done on this topic, see for instance [Bar06] (and references therein).

Paley-Wiener spaces. Arguable, they are the prototypical, as well as motivational, examples of dB spaces, as attested by de Branges himself [dB68]. The Paley-Wiener space, parametrized by \( a > 0 \), is defined as
\[ \mathcal{PW}_a := \left\{ f(z) \text{ entire : } \int_{\mathbb{R}} |f(x)|^2 dx < \infty \text{ and } |f(z)| \leq c f e^{a|z|} \right\}, \]
equipped with the inner product of $L^2(\mathbb{R})$. It can be shown that

$$\mathcal{PW}_a = \left\{ f(z) \text{ entire : } \int_{\mathbb{R}} |f(x)|^2 \, dx < \infty \text{ and } |f(z)| \leq c_f e^{a |\text{Im}(z)|/|\text{Im}(z)|^{1/2}} \right\}$$

(see for instance [DM76, Chapter 6]). In view of (16) one obtains $\mathcal{PW}_a = \mathcal{B}(e^{-ia z})$; note that $e^{-ia z}$ is an HB function as long as $a \geq 0$. The Paley-Wiener theorem states that every function in $\mathcal{PW}_a$ is the analytic continuation (to the entire complex plane) of the Fourier transform of a function in $L^2(-a, a)$. That is,

$$\mathcal{PW}_a = \left\{ f(z) \text{ entire : } f(z) = \int_{-a}^{a} e^{-ix z} \varphi(x) \, dx, \quad \varphi(x) \in L^2(-a, a) \right\}.$$

No polynomial belongs to $\mathcal{PW}_a$. However, it is easy to verify that if $p_n(z)$ is a polynomial of degree $n \geq 0$, then $p_n(z) \in \text{assoc}_{n+1} \mathcal{PW}_a \setminus \text{assoc}_n \mathcal{PW}_a$. Also, the chain of dB subspaces of $\mathcal{PW}_a$ is $\{ \mathcal{PW}_b : b \in (0, a) \}$.

Paley-Wiener spaces have many more distinctive properties; further details are accounted for in [dB68, Chapter 2].

**dB spaces associated to Bessel functions.** This kind of dB spaces appears in connection with the radial Hamiltonian operator of a quantum free particle in spherical coordinates [ST13b, Section 3]; see also §5.3 below.

Given $l \in \mathbb{Z}^+$ and $b > 0$, define

$$\mathcal{G}_b^l := \left\{ f(z) \text{ entire : } f(z) = f(-z), \int_0^b x^{2l+2} |f(x)|^2 \, dx < \infty, \right\}$$

$$\left\{ |z^{l+1} f(z)| \leq c_f e^{b |\text{Im}(z)|} \text{ for all } z \in \mathbb{C} \right\}$$

equipped with the inner product of $L^2(\mathbb{R}^+; x^{2l+2} \, dx)$. By a theorem due to Griffith [Gri55] (see also [Zem66]), which is to some extend the analog of the Paley-Wiener theorem but involving the Hankel transform, one has

$$\mathcal{G}_b^l = \left\{ f(z) \text{ entire : } z^{l+1} f(z) = \int_0^b \sqrt{zx} J_{l+\frac{1}{2}}(zx) \varphi(x) \, dx, \quad \varphi(x) \in L^2(0, b) \right\};$$

here $J_m(w)$ denotes the Bessel function of order $m$. In order to verify that $\mathcal{G}_b^l$ is a dB space, define

$$\xi_l(z, x) := z^{-(l+1)} \sqrt{zx} J_{l+\frac{1}{2}}(zx).$$
On one hand, in terms of this function, it holds true that
\[ G_b^l = \left\{ f(z) \text{ entire} : f(z) = \int_0^b \xi_l(z,x) \varphi(x) dx, \quad \varphi(x) \in L^2(0,b) \right\}. \]

On the other hand, \( \xi_l(z,x) \) is the \( L^2(0,b) \) fundamental solution of the differential equation
\[-\psi''(x) + \frac{l(l+1)}{x^2} \psi(x) = z^2 \psi(x)\]
with suitable boundary conditions at \( x = 0 \) (for details, see \[ST13b, Section 3\]). An argument involving the Lagrange identity (see \[Eck12, Theorem 3.2\]) shows that
\[ G_b^l = B(e_b^l), \]
where
\[ e_b^l(z) := \xi_l(z,b) + i \xi'_l(z,b) \]
(the prime denotes derivative with respect to the second argument).

These dB spaces do not contain polynomials. Moreover, \( 1 \in \text{assoc}_{n_l+1} G_b^l \setminus \text{assoc}_n, G_b^l \), where \( n_l := \left\lfloor \frac{l}{2} + \frac{3}{4} \right\rfloor \) (the standard notation for the floor function has been used here).

4  **A functional model for operators in \( S(H) \)**

A functional model for a given operator \( A \) in a Hilbert space \( H \) is a unitary map of \( H \) onto a Hilbert space \( \hat{H} \) of functions with certain analytical properties, such that the operator \( A \) is transformed into the operator of multiplication by the independent variable in \( \hat{H} \).

This section describes a functional model for operators in \( S(H) \) which is suitable for studying the classes \( E_n(H), n \in \mathbb{Z} \). This functional model, which was developed in \[ST10, ST13a, ST13d\], stems from Krein’s theory of representation of symmetric operators developed in his original work \[Kre44b, Theorems 2 and 3\] (cf. \[GG97, Section 1.2\]), but differs from it in a crucial way as will be explained below (see Remark \[4.7\]). The functional model presented here can be viewed as a particular realization (with some modifications) of the general theory developed by Strauss \[Str98, Str00, Str01\] and it is different from (and simpler than) an equivalent functional model introduced in \[Mar11\].
4.1 The functional space

Fix an operator $A \in S(\mathcal{H})$ and let $J$ be an involution that commutes with the selfadjoint extensions of $A$. Consider a function $\xi_A : \mathbb{C} \to \mathcal{H}$ such that

(P1) $\xi_A(z)$ is zero-free and entire,

(P2) $\xi_A(z) \in \ker(A^* - zI)$ for all $z \in \mathbb{C}$, and

(P3) $J\xi_A(z) = \xi_A(z) \overline{z}$ for every $z \in \mathbb{C}$.

Since, for an operator $A \in S(\mathcal{H})$, one has that $\dim \ker(A^* - zI) = 1$ for all $z \in \mathbb{C}$, the following assertion clearly holds true (see \cite[Proposition 2.12 and Remark 2.13]{ST13a}).

**Lemma 4.1.** If $\xi_A^{(1)} : \mathbb{C} \to \mathcal{H}$ and $\xi_A^{(2)} : \mathbb{C} \to \mathcal{H}$ are two functions satisfying (P1), (P2), and (P3), then there exists a zero-free real entire function $g(z)$ such that $\xi_A^{(1)}(z) = g(z)\xi_A^{(2)}(z)$.

The function $\xi_A(z)$, which is crucial for the functional model described below, can be constructed as follows. Pick a canonical selfadjoint extension $A_\gamma$ of $A \in S(\mathcal{H})$ and let $h_\gamma(z)$ be a real entire function whose zero set (counting multiplicities) equals $\text{spec}(A_\gamma)$ (hence, by (i) of Theorem 2.10, the zeros of $h_\gamma(z)$ are simple). On the basis of the analytical properties of the generalized Cayley transform (see §2.2 and Theorem 2.2) it is straightforward to verify that if one sets

$$\xi_A(z) = h_\gamma(z)V_\gamma(w, z)\psi_w,$$

where $\psi_w$ is in $\ker(A^* - wI)$ and $V_\gamma(w, z)$ is given by (6), then (19) will satisfy (P1) and (P2). Moreover, either by defining the involution as in Theorem 2.6 or by choosing $w$ and $\psi_w$ as in Theorem 2.7, the function (19) also satisfies (P3). This follows either from the proof of Theorem 2.6 or from the proof of Theorem 2.7 (see \cite[Proposition 2.11]{ST13a}). Note that, since $h_\gamma(z)$ is defined up to a multiplying zero-free real entire function, Lemma 4.1 implies that (19) does not depend on the choice of the selfadjoint extension $A_\gamma$ nor on $w$ and, furthermore, every function $\xi_A : \mathbb{C} \to \mathcal{H}$ can be written as in (19).

Fix $A \in S(\mathcal{H})$ and any function $\xi_A : \mathbb{C} \to \mathcal{H}$ satisfying (P1), (P2), and (P3). Then define

$$(\Phi_A \varphi)(z) := \langle \xi_A(z), \varphi \rangle, \quad \varphi \in \mathcal{H}.$$  

Due to (P1), $\Phi_A$ maps $\mathcal{H}$ onto a certain linear manifold $\Phi_A \mathcal{H}$ of entire functions. The notation $\mathcal{H} = \Phi_A \mathcal{H}$ will be used when it is no need of referring to $A$. Note
that if one fixes \( z \in \mathbb{C} \) and allows \( \varphi \) to run over \( \mathcal{H} \), the inner product in (20) becomes a bounded linear functional whose kernel is \( \text{ran}(A - zI) \). Hence, the complete nonselfadjointness condition (11) and the analyticity of the functions in \( \hat{\mathcal{H}} \) imply that \( \Phi_A \) is injective. A generic element of \( \hat{\mathcal{H}} \) will be denoted by \( \hat{\varphi}(z) \), as a reminder of the fact that it is the image under \( \Phi_A \) of a unique element \( \varphi \in \mathcal{H} \). Clearly, the linear space \( \hat{\mathcal{H}} \) is turned into a Hilbert space by defining
\[
\langle \hat{\eta}(\cdot), \hat{\varphi}(\cdot) \rangle := \langle \eta, \varphi \rangle ,
\]
and \( \Phi_A \) is an isometry from \( \mathcal{H} \) onto \( \hat{\mathcal{H}} \).

### 4.2 Properties of functional space

The properties of the isometry \( \Phi_A \) and the space of functions \( \hat{\mathcal{H}} \) previously defined are discussed here.

The following assertion (see [ST13a]) follows from the properties of the function \( \xi_A(z) \) and the fact that
\[
k(z, w) := \langle \xi_A(z), \xi_A(w) \rangle
\]
is a reproducing kernel in \( \hat{\mathcal{H}} \) (cf. [Str01, Proposition 1]).

**Theorem 4.2.** Let \( \Phi_A \) be defined by (20). For any operator \( A \in \mathcal{S}(\mathcal{H}) \), the space of functions \( \hat{\mathcal{H}} = \Phi_A \mathcal{H} \) is a de Branges space.

On the other hand, the isometry \( \Phi_A \) transforms \( A \) as expected:

**Theorem 4.3.** Fix an operator \( A \in \mathcal{S}(\mathcal{H}) \). Let \( J \) be the involution that appears in (P2) and \( S \) be the operator of multiplication by the independent variable in the dB space \( \hat{\mathcal{H}} = \Phi_A \mathcal{H} \). Then, the following holds:

(i) \( S = \Phi_A A \Phi_A^{-1} \) and \( \text{dom}(S) = \Phi_A \text{dom}(A) \).

(ii) \( \# = \Phi_A J \Phi_A^{-1} \).

(iii) If \( A_\gamma \) is a canonical selfadjoint extension of \( A \), then \( \Phi_A A_\gamma \Phi_A^{-1} \) is a canonical selfadjoint extension of \( S \).

### 4.3 Functional spaces for \( \mathcal{E}_n(\mathcal{H}) \)

As already shown, to every operator \( A \in \mathcal{S}(\mathcal{H}) \) there corresponds a dB space such that \( A \) is unitarily equivalent to the operator \( S \) of multiplication by the
independent variable in that dB space. On the other hand, by (i) and (v) of Theorem 3.13 the operator of multiplication in every dB space $\mathcal{B}$ is an element of $\mathcal{S}(\mathcal{B})$. The following assertion gives a characterization of the dB spaces that correspond to operators in $\mathcal{E}_n(\mathcal{H})$.

**Theorem 4.4.** Let $\Phi_A$ be defined by (27), with $A \in \mathcal{S}(\mathcal{H})$, and $\hat{\mathcal{H}} = \Phi_A \mathcal{H}$. For any $n \in \mathbb{Z}$, the operator $A$ is in $\mathcal{E}_n(\mathcal{H})$ if and only if $\text{assoc}_n(\hat{\mathcal{H}})$ contains a zero-free entire function.

This theorem follows directly from Definitions 2.12 and 2.13 and the properties of the $\xi_A(z)$, taking into account (4) (cf. [ST13a, Proposition 3.1]).

In view of Theorem 3.11 and Remark 3.12, one has the following criterion for an operator to be in $\mathcal{E}_n(\mathcal{H})$ (cf. [ST13a, Proposition 3.7]).

**Theorem 4.5.** Let $A_1, A_2$ be two canonical selfadjoint extensions of $A \in \mathcal{S}(\mathcal{H})$. For any $n \in \mathbb{Z}$, the operator $A$ is in $\mathcal{E}_n(\mathcal{H})$ if and only if the sequences $\text{spec}(A_1)$ and $\text{spec}(A_2)$ comply with the conditions (C1), (C2), and (C3) of Theorem 3.11.

There are other results concerning the properties of the operator classes $\mathcal{E}_n(\mathcal{H})$, $n \in \mathbb{Z}$, which are obtained by means of the functional model given in this section. For instance, [ST13a, Proposition 3.11] states that for the definition of the class $\mathcal{E}_n(\mathcal{H})$, with $n \in \mathbb{Z}^+$, it is sufficient to require that (14) holds for all $z \in \mathbb{C}$ with the exception of a finite set. A more involved assertion stemming from the functional model is the following one due to Strauss [Str01, Propositions 9 and 10].

**Theorem 4.6.** Let $A$ be an operator in $\mathcal{S}(\mathcal{H})$.

(i) $A \in \mathcal{E}_1(\mathcal{H}) \setminus \mathcal{E}_0(\mathcal{H})$ if and only if there is an extension $B \supset A$ with empty spectrum (the resolvent set is the whole complex plane).

(ii) $A \in \mathcal{E}_0(\mathcal{H})$ if and only if $A^{-1}$ has a quasinilpotent extension $K$ (that is, $\text{spec}(K) = \{0\}$) such that $\text{dom}(K) = \mathcal{H}$ and $\text{ran}(K) = \text{dom}(A)$.

### 4.4 Krein’s entire operators

This section concludes with an elaboration of the relation between the notions of operators entire and entire in the generalized sense introduced by Krein, and the classes $\mathcal{E}_n(\mathcal{H})$ of $n$-entire operators.
According to Krein’s terminology [Kre44b, Section 2], a vector \( \mu \in \mathcal{H} \) is said to be a \textit{gauge} for a densely defined operator \( A \in \mathcal{S}(\mathcal{H}) \) whenever

\[
\mathcal{H} = \text{ran}(A - zI) + \text{span}\{\mu\}
\]

for some complex number \( z = z_0 \). Given a gauge, the set

\[
\{ z \in \mathbb{C} : (21) \text{ fails to hold} \}
\]

has no finite accumulation points and, therefore, its cardinality is at most infinite countably. Furthermore, depending on the choice of the gauge \( \mu \), the set (22) could be placed inside \( \mathbb{R} \) [ST10, Lemma 2.1] or be contained outside \( \mathbb{R} \) (see [Kre44c, Theorem 8c] or [ST10, Theorem 2.2]). Krein calls a gauge entire if the set (22) turns out to be empty and, in this case, the operator \( A \) is entire [Kre44a, Section 1] (see also [GG97, Chapter 2, Section 5]). Thus, by comparing Definition 2.12 with (21), one concludes that the densely defined operators in \( \mathcal{E}_0(\mathcal{H}) \) correspond to Krein’s class of entire operators.

For an entire operator \( A \) with \textit{real} entire gauge \( \mu \), Krein defined the mapping

\[
\varphi \mapsto \hat{\varphi}(z) := \frac{\langle V_\gamma(w, \zbar) \psi_w, \varphi \rangle}{\langle V_\gamma(w, \zbar) \psi_w, \mu \rangle}, \quad \varphi \in \mathcal{H},
\]

where \( w \in \mathbb{C} \) and \( \psi_w \) are suitable chosen. Comparing with the functional model outlined in this section, one sees that (23) corresponds to a specialized choice of the function \( h_\gamma(z) \) in (19), namely,

\[
h_\gamma(z) = \frac{1}{\langle V_\gamma(w, \zbar) \psi_w, \mu \rangle}.
\]

Furthermore, due to the coincidence of the models in this case, Krein’s assertion that the existence of an entire gauge implies the existence of a real entire gauge [Kre44c, Theorem 8] is a simple consequence of Remark 3.12.

**Remark 4.7.** In fact, Krein considered the mapping (23) not only for densely defined operators in \( \mathcal{E}_0(\mathcal{H}) \) but for all densely defined operators in \( \mathcal{S}(\mathcal{H}) \), where \( \mu \) is then an appropriately chosen element of \( \mathcal{H} \) [GG97, Chapter 1, Section 2]. Thus, Krein’s functional space is a dB space if and only if \( A \) is a densely defined operator in \( \mathcal{E}_0(\mathcal{H}) \) and \( \mu \) is an entire gauge.

In addition to the entire operators, Krein considered the so-called entire operators in the generalized sense [Kre68, Section 4], which were later studied
in [Shm71] and [TS77, Chapter 6]. To the end of defining these operators, first note that (4) and Definition 2.12 imply that

\[ A \in E_0(H) \] whenever

\[ \langle \xi_A(z), \mu \rangle \neq 0 \quad \text{for all} \quad z \in \mathbb{C}, \tag{24} \]

and some fixed \( \mu \in H \) (which, as already pointed out, can also be assumed real). Now, take a densely defined operator \( A \in \mathcal{S}(H) \) and define

\[ H_+ := \text{dom}(A^*) \], equipped with graph norm.

Then \( H_+ \) is a Hilbert space. Its dual is

\[ H_- := \{ \text{anti-linear functionals } H_+ \text{-continuous on } H_+ \}. \]

Clearly, one has \( H_+ \subset H \subset H_- \) (for details on triplets of this kind —the so-called Gelfand triplets— refer to [Ber68]).

With this setup, \( A \) is said to be entire in the generalized sense if there exists \( \mu \in H_- \setminus H \) such that, for all \( z \in \mathbb{C} \), one has [24] with the inner product replaced by the duality bracket between \( H_+ \) and \( H_- \). [ST10, Section 5]. Note that this definition makes sense because \( \xi_A(z) \in H_+ \) for all \( z \in \mathbb{C} \). Moreover, one can prove the following [ST10, Proposition 5.1],

\[ \text{assoc}_1 \hat{H} = \{ \hat{\eta}(z) \text{ entire} : \hat{\eta}(z) = \langle \xi_A(z), \eta \rangle \text{ for some } \eta \in H_- \}. \]

In view of (24), \( A \in \mathcal{S}(H) \) is then entire in the generalized sense as long as \( \text{assoc}_1 \hat{H} \) contains a zero-free, entire function (which can also be chosen real). Recalling Definition 3.10, this amounts to saying that there are vectors \( \mu_0, \mu_1 \in H \) such that

\[ H = \text{ran}(A - zI)^\perp + \text{span}\{ \mu_0 + z\mu_1 \}, \tag{25} \]

for every \( z \in \mathbb{C} \). All in all, a densely defined operator \( A \in \mathcal{S}(H) \) is entire in the generalized sense of Krein if and only if it belongs to the class \( E_1(H) \).

The use of triplet of spaces, to define the notion of entire operators in the generalized sense, can be replicated to a certain extent for \( n \)-entire operators. In [ST13a, Section 4], given \( A \in \mathcal{S}(H) \) densely defined and \( n \in \mathbb{N} \), a Gelfand triplet \( H_{+n} \subset H \subset H_{-n} \) is constructed in such a way that \( H_{-n} \cong \text{assoc}_n \hat{H} \) and \( \xi_A(z) \in H_{+n} \) for all \( z \in \mathbb{C} \). Then \( A \) is \( n \)-entire if and only if there exists \( \mu \in H_{-n} \) such that

\[ \langle \xi_A(z), \mu \rangle_n \neq 0 \]
for all $z \in \mathbb{C}$; here $\langle \cdot , \cdot \rangle_n$ denotes the duality bracket between $H_{+n}$ and $H_{-n}$.

Unlike the construction due to Krein, the one sketched above is a bit convoluted, rendering this alternative definition more difficult to use.

5 Applications

Most of the classical applications of the theory of entire operators are discussed in detail in Gorbachuk's monography [GG97, Chapter 3]; the first example below is probably the one most frequently used as an illustration of an entire operator. In fact, it was the first example (see [Kre44a, Section 4 A]) and, as mentioned in the Introduction, it is the germ of the theory. For the definition of the operator, the exposition of this first example follows [GG97, Chapter 3 Section 1] and [Sim98, Section 1]. The second example is also classical but presented in a somewhat novel approach. Finally, the last example illustrates a non-trivial class of Schrödinger operators that are $n$-entire with $n \in \mathbb{N}$ fixed but arbitrary (pedantically, one should say that these Schrödinger operators are selfadjoint extensions of some operator in $\mathcal{E}_n(\mathcal{H})$).

5.1 The Hamburger moment problem

Recall the formulation of this classical problem: Given a sequence of real numbers $\{s_n\}_{n=0}^\infty$, one is interested in finding conditions for the existence of a positive measure $m(x)$ such that

$$s_n = \int_{-\infty}^{\infty} x^n dm(x), \quad n \in \mathbb{N} \cup \{0\}.$$  \hfill (26)

Assuming the existence of such a measure, one may ask whether it is unique. If it is not unique, one is interested in describing all the solutions to problem (26). The moment problem is said to be determinate when it has only one solution and indeterminate otherwise. A very complete treatment of this classical problem can be found in [Akh65].

As it is well known, a necessary and sufficient condition for the existence of a solution of (26) is that

$$\sum_{j,k=0}^{n} s_{k+j} z_j z_k \geq 0,$$  \hfill (27)

for every $n \in \mathbb{N} \cup \{0\}$ and arbitrary numbers $z_j \in \mathbb{C}$ [Sim98, Proposition 1.3].
Under the condition that (27) holds, one considers the set $\mathcal{L}$ of all polynomials in $\mathbb{R}$ with complex coefficients,

$$p(x) = \sum_{k=0}^{n} z_k x^k, \quad z_k \in \mathbb{C}, \quad n \in \mathbb{N} \cup \{0\},$$

equipped with the sesquilinear form

$$\langle p, q \rangle := \sum_{j=0}^{n} \sum_{k=0}^{m} s_{j+k} \overline{x_j} z_k.$$

Then one obtains a Hilbert space $\mathcal{H}$ as the completion of $\mathcal{L}/\mathcal{L}_0$ under (28), where

$$\mathcal{L}_0 := \{ p(x) \in \mathcal{L} : \langle p, p \rangle = 0 \}.$$

In $\mathcal{H}$ one defines the operator $A$, with domain $\mathcal{L}/\mathcal{L}_0$, as the lifting of the operator defined by the mapping $p(x) \mapsto xp(x)$ with domain $\mathcal{L}$. This operator is symmetric and real with respect to (lifting to $\mathcal{L}/\mathcal{L}_0$ of) the usual complex conjugation in $\mathcal{L}$.

**Theorem 5.1.** Assume (27). Let $A$ be the operator defined as above. Then, either

(i) $A$ is essentially selfadjoint, in which case the problem (26) has a unique solution; or

(ii) the closure of $A$ has deficiency indices $(1, 1)$, in which case the solution of (26) is not unique.

It turns out that there is an orthonormal basis $\{P_{k-1}(x)\}_{k=1}^{\infty}$ in $\mathcal{H}$ such that $A$ has a Jacobi matrix as its matrix representation with respect to it (see [AG93, Section 47] for a discussion on the matrix representation of unbounded symmetric operators). Note that one could have taken a Jacobi matrix as the starting point for defining the operator $A$ (see [Akh65, Chapter 4]).

The element $P_k(x)$ of the basis mentioned above is a polynomial of degree $k$ and it is known as the $k$-th orthogonal polynomial of the first kind associated with the Jacobi matrix. It happens that $P_0(x) \equiv 1$.

One has the following assertion [GG97, Chapter 3, Theorem 1.2].

**Theorem 5.2.** If (ii) of Theorem 5.1 takes place, then $P_0(x)$ is an entire gauge and, therefore, $A \in \mathcal{E}_0(\mathcal{H})$.  

29
As a matter of fact $A$ is in $E_{-\infty}(\mathcal{H})$. Indeed, as it is straightforward to verify, $P_0(x)$ is in the domain of $A^n$ for any $n \in \mathbb{N}$.

5.2 The linear momentum operator

In $\mathcal{H} = L^2[-a,a]$, $0 < a < +\infty$, consider the operator

$$\text{dom}(A) = \{ \varphi(x) \in AC[-a,a] : \varphi(a) = 0 = \varphi(-a) \}, \quad A := i \frac{d}{dx}.$$ 

Clearly, $A$ is closed and symmetric. Moreover,

$$\text{dom}(A^*) = AC[-a,a], \quad A^* = i \frac{d}{dx},$$

from which it is straightforward to verify that the deficiency indices of $A$ are $(1,1)$. The canonical selfadjoint extensions of $A$ can be parametrized as

$$\text{dom}(A_\gamma) = \{ \varphi(x) \in AC[-a,a] : \varphi(a) = e^{-i2\gamma}\varphi(-a) \}, \quad A_\gamma = i \frac{d}{dx},$$

for $\gamma \in [0,\pi)$. These selfadjoint extensions correspond to different realizations of the linear momentum operator within the interval $[-a,a]$. By a straightforward calculation,

$$\text{spec}(A_\gamma) = \left\{ \frac{\gamma + k\pi}{a} : k \in \mathbb{Z} \right\}. \quad (29)$$

Clearly, the spectra are interlaced and their union equals $\mathbb{R}$ so it follows that $A$ is regular, hence completely nonselfadjoint.

This operator can be shown to be entire in the generalized sense (that is, 1-entire) by methods of directing functionals. This is the classical approach discussed for instance in [GG97]. An alternative method for showing that $A$ is in $E_1(\mathcal{H})$ is used here. This method resorts directly to Definition 2.12 and may be generalized to other differential operators. Another example treated in a similar manner is found in [ST13a, Example 3.4].

Define $\xi(x,z) := e^{-izx}$, $x \in [-a,a]$, $z \in \mathbb{C}$. This zero-free entire function belongs to $\ker(A^* - zI)$ for all $z \in \mathbb{C}$. For proving that $A$ is 1-entire it suffices to find $\mu_0(x), \mu_1(x) \in L^2[-a,a]$ such that

$$\int_{-a}^{a} e^{-igx} \mu_0(x)dx + y \int_{-a}^{a} e^{-igx} \mu_1(x)dx = 1 \quad (30)$$
for all $y \in \mathbb{R}$ (and then use analytic continuation to the whole complex plane). The search will be guided by formally taking the inverse Fourier transform of \textbf{(30)} and switching without much questioning the order of integration, obtaining in that way the differential equation

$$
\mu_0(x) - i\mu_1'(x) = \delta(x),
$$

where $\delta(x)$ is the Dirac’s distribution. This equation suggests to set

$$
\mu_0(x) = \frac{1}{2a} \chi_{[-a,a]}(x) \quad (31)
$$

$$
\mu_1(x) = -i\frac{a}{2a} \chi_{[-a,0]}(x) + i\frac{a}{2a} \chi_{[0,a]}(x), \quad (32)
$$

where $\chi_S(x)$ denotes the characteristic function of the set $S$. A simple computation shows that indeed \textbf{(31)} and \textbf{(32)} satisfy \textbf{(30)} thus $A$ is 1-entire as asserted.

This operator is associated with a Paley-Wiener space. Indeed, it is apparent that

$$
\mathcal{PW}_a = \left\{ \hat{\varphi}(z) = \int_{-a}^{a} \xi(x,z)\varphi(x)dx : \varphi(x) \in L^2[-a,a] \right\};
$$

notice that here one has an instance of application of the abstract functional model discussed in this work. Also, notice that this implies the sharper statement $A \in \mathcal{E}_1(\mathcal{H}) \setminus \mathcal{E}_0(\mathcal{H})$.

### 5.3 Spectral analysis of radial Schrödinger operators

Consider the selfadjoint operators that arise from the differential expression

$$
\tau := -\frac{d^2}{dx^2} + \frac{l(l+1)}{x^2} + q(x), \quad x \in (0,1), \quad l \geq -\frac{1}{2},
$$

along with separated selfadjoint boundary conditions. These operators describe the radial part of the Schrödinger operator for a particle confined to a ball of finite radius, when the potential is spherically symmetric.

The potential function $q(x)$ is assumed real such that $\tilde{q}(x) \in L_1(0,1)$, where

$$
\tilde{q}(x) := \begin{cases} 
  xq(x) & l > -\frac{1}{2}, \\
  x(1 - \log x)q(x) & l = -\frac{1}{2}.
\end{cases}
$$

Under this hypothesis, it is shown in [KST10, Theorem 2.4] that $\tau$ is regular.
at \( x = 1 \), and the limit point case (resp. limit circle case) at \( x = 0 \) for \( l \geq 1/2 \) (resp. \( l \in [-1/2, 1/2] \)). If \( \tau \) is in the limit circle case at \( x = 0 \), it is usual to add the boundary condition

\[
\lim_{x \to 0^+} x^l \left[ (l+1)\varphi(x) - x\varphi'(x) \right] = 0.
\] (33)

Other boundary conditions can serve as well. A comprehensive investigation of them can be found in [BG85].

With this setup \( \tau \) gives rise to a family of selfadjoint operators \( H_\beta \), for \( \beta \in [0, \pi) \), associated to the boundary conditions \( \varphi(1) \cos \beta = \varphi'(1) \sin \beta \). These operators are the canonical selfadjoint extensions of a certain closed, regular, symmetric operator \( H \), having deficiency indices \((1, 1)\). In [ST13b, Theorem 4.3 and Corollary 4.4] it is shown the following.

**Theorem 5.3.** Let \( l \geq -\frac{1}{2} \) and assume that \( \tilde{q}(x) \) belongs to \( L_p(0, 1) \), with \( p > 2 \). Then,

(i) the operator \( H \) is \( n \)-entire if and only if \( n > \frac{l}{2} + \frac{3}{4} \). In that case,

(ii) the spectra of two canonical selfadjoint extensions \( H_{\beta_1}, H_{\beta_2} \) of \( H \) satisfy conditions \((C1), (C2)\) and \((C3)\).

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