Modified Momentum Euler Equation for Water Wave Modeling

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Abstract—In this research, weighted total acceleration for a function \( f(x,z,t) \) was formulated. This total acceleration equation was done at the Euler momentum equation. Then, the Euler momentum equation was done together with free surface boundary condition equation to formulate water wave constant at the solution of Laplace equation. The velocity potential of the solution of Laplace equation actually consists of two components that were used in this research.

Keywords—weighted total acceleration, convective acceleration, complete velocity potential.

I. INTRODUCTION

Momentum equation is an important basic equation in mathematic modeling of hydrodynamics, including water wave modeling. Momentum equation commonly used in water wave modeling is Euler momentum equation. There is a constraint in this equation, i.e. Euler momentum equation has no hydrodynamic force in the horizontal direction or convective acceleration has a value of zero when velocity potential is substituted to the term. To overcome this problem, weighted total acceleration equation was formulated where there are two weighted coefficients, i.e. at the time differential term and at the differential term of vertical-zdirection.

Laplace equation solution consists of two velocity potential components (Dean (1991)). However, only one component that has been used. Equations from water wave constant, i.e. wave number \( k \) and wave constant \( G \) can be formulated using only one velocity potential component, but the value is determined by both the two velocity components. In this research, the water wave surface equation is formulated using the two velocity potential components, then the condition of the water wave surface that has been produced is studied.

II. WEIGHTED TOTAL ACCELERATION

Hutahaean (2019a) formulated weighted total acceleration in a function \( f = f(x,z,t) \), \( x \) is horizontal axis and \( t \) is time, using Taylor series. The formulation of weighted total acceleration in a function \( f = f(x,z,t) \), \( z \) is vertical axis, is done using similar method, therefore the formulation of weighting total acceleration in \( f = f(x,z,t) \) will be preceded by reviewing the formulation of weighted total acceleration in \( f = f(x,t) \) to obtain a clearer description.

2.1. Weighted Total Acceleration for the function \( f = f(x,t) \)

The changes in the value of a function in a function \( f = f(x,t) \) for a very small \( \delta x \) and \( \delta t \) using Taylor series only until the second derivative is,

\[
\begin{align*}
f(x + \delta x, t + \delta t) &= f(x, t) + \delta x \frac{af}{dx} + \delta t \frac{af}{dt} + \delta x^2 \frac{a^2f}{dx^2} + \delta x \delta t \frac{a^2f}{dx dt} + \frac{\delta t^2}{2} \frac{a^2f}{dt^2} + \delta x \delta t \delta x \frac{a^2f}{dx dt dx} + \delta t \delta t \delta x \frac{a^2f}{dt dt dx} \end{align*}
\]

By working on the argument of Courant (1928) that in order to obtain a good result on horizontal velocity \( u = \frac{dx}{dt} \), then weighting coefficient \( \gamma \), is done which is a positive number, in time differential in Taylor series.

\[
f(x + \delta x, t + \gamma \delta t) = f(x, t) + \delta x \frac{af}{dx} + \gamma \delta t \frac{af}{dt} + \frac{\delta x^2}{2} \frac{a^2f}{dx^2} + \gamma \delta t \delta x \frac{a^2f}{dx dt} + \frac{\gamma^2 \delta t^2}{2} \frac{a^2f}{dt^2} + \delta x \delta t \delta x \frac{a^2f}{dx dt dx} + \delta t \delta t \delta x \frac{a^2f}{dt dt dx} \]

At the limit \( \delta x, \delta t \) close to zero the following equation is obtained,

\[
\frac{df}{dt} = u \frac{af}{dx} + \gamma \frac{af}{dt} \quad \text{or} \quad \frac{df}{dt} = \gamma \frac{af}{dt} + u \frac{af}{dx} \quad \text{.....(2)}
\]

This equation is weighted total derivative equation or weighted total acceleration for the function of \( f = f(x,t) \) where \( \gamma \) is weighting coefficient.

The method of calculating weighting coefficient \( \gamma \) will be formulated using Taylor series (1). The second derivative term can be omitted if,

\[
\left| \frac{\delta x^2 \frac{a^2f}{dx^2} + \gamma \delta t \delta x \frac{a^2f}{dx dt} + \frac{\gamma^2 \delta t^2}{2} \frac{a^2f}{dt^2} + \delta x \delta t \delta x \frac{a^2f}{dx dt dx} + \delta t \delta t \delta x \frac{a^2f}{dt dt dx}}{\delta x \delta t \delta x \frac{a^2f}{dx dt dx} + \delta t \delta t \delta x \frac{a^2f}{dt dt dx}} \right| \leq \epsilon \quad \text{.....(3)}
\]
Then it was defined $\delta x = C_\delta t = \frac{L}{T} \delta t = \frac{2\pi}{k} \delta t$, where $C$ is wave celerity, $k$ is wave number $T$ is wave period, $\sigma = \frac{2\pi}{T}$ is angular frequency. $\delta x$ is substituted with $\frac{\sigma}{k} \delta t$, and the following equation is obtained, 

$$\left| \frac{a^2 f}{\delta x^2} + \gamma_2 f + \frac{\gamma_2^2 f}{2} \right| \leq \varepsilon \quad \ldots \ldots (4)$$

The completion of this equation requires a function form of $f = f(x,t)$. And the following sinusoidal function form will be used, $f(x,t) = \cos kx \cos \sigma t \quad \ldots \ldots (5)$

This equation is wave surface equation of the linear wave theory. The derivative of the function is as follows:

| Table 1: Derivative Equation of (5) |
|------------------------------------|
| $\frac{df}{dx} = -k \sin kx \cos \sigma t$ | $\frac{a^2 f}{\delta x^2} = -k^2 \cos kx \cos \sigma t$ |
| $\frac{df}{dt} = -\sigma \cos kx \sin \sigma t$ | $\frac{\partial^2 f}{\partial t \partial x} = k\sigma \sin kx \sin \sigma t$ |
| $\frac{df}{dt} = -\sigma^2 \cos kx \cos \sigma t$ |

Using the condition of $\cos kx = \sin kx = \cos \sigma t = \sin \sigma t$, the elements of sinusoidal function will cancel out each other as a result of a division. Substitute the derivative equations to (4), the following equation is obtained

$$\left| \frac{1}{2} - \gamma + \frac{\gamma^2}{4} \right| \leq \frac{\varepsilon}{\sigma \delta t}$$

The numerator $(1 + \gamma)$ is a positive number, then the equation can be written as,

$$\frac{1}{2} - \gamma + \frac{\gamma^2}{4} \leq \frac{\varepsilon}{\sigma \delta t} (1 + \gamma)$$

If equals ( = ) relation is used, then

$$\frac{1}{2} - \gamma + \frac{\gamma^2}{4} = \frac{\varepsilon}{\sigma \delta t} (1 + \gamma) \quad \ldots \ldots (5)$$

Considering that $\gamma$ is a positive number, the right side of the equation is a positive number. Therefore, the left side of the equation is also a positive number. The calculation of the value $\gamma$ can be done by releasing the sign $\mid \mid$ in the left side of the equation, i.e. using equation (5).

The calculation of the value with (5) requires an input $\delta t$. The value of $\delta t$ is obtained from the function $f = f(t)$. The approximation of Taylor series for the function is, 

$$f(t + \delta t) = f(t) + \delta t \frac{df}{dt} + \frac{\delta t^2}{2} \frac{d^2 f}{dt^2}$$

In order to be able to use $\delta t$ only until the first derivative, then $\left| \frac{\delta t}{T} \frac{d^2 f}{dt^2} \right| \leq \varepsilon$ or $\left| \frac{\delta t}{T} \frac{d^2 f}{dt^2} \right| \leq \varepsilon$. For the function, $f(t) = \cos \sigma t$; $\frac{df}{dt} = -\sigma \sin \sigma t$; $\frac{d^2 f}{dt^2} = -\sigma^2 \cos \sigma t$, and it is done in a $\cos \sigma t = \sin \sigma t$ condition, and

$$\frac{\delta t}{\sigma} \frac{d^2 f}{dt^2} \leq \varepsilon \or \delta t = \frac{2\pi}{\sigma} \ldots \ldots \ldots (6)$$

is obtained. Substitution of (6) to (5) obtains

$$\gamma = 3 \quad \ldots \ldots \ldots (7)$$

It is obtained that $\gamma$ has a constant value, i.e. independent of wave period or the level of accuracy $\varepsilon$.

2.2 Weighted Total Acceleration for the function $f = f(x,z,t)$

To obtain weighted total acceleration equation in a function $f = f(x,z,t)$, the similar method will be done as in the function $f = f(x,t)$, where,

$$f(x + \delta x, z + \gamma_2 \delta z, t + \gamma_3 \delta t) = f(x,t) + \delta x \frac{df}{dx} + \gamma_2 \delta z \frac{df}{dz} + \gamma_3 \delta t \frac{df}{dt} + \gamma_2 \delta x \frac{d^2 f}{dx^2} + \gamma_2 \delta z \frac{d^2 f}{dz^2} + \gamma_3 \delta t \frac{d^2 f}{dt^2} + \gamma_2 \gamma_3 \frac{d^2 f}{dxz} + \gamma_2 \gamma_3 \frac{d^2 f}{dzt} + \gamma_2 \gamma_3 \frac{d^2 f}{dtz} \quad \ldots \ldots \ldots (8)$$

In (8), for $\delta z = \delta x$, it is meant that $\gamma_2 \delta z \frac{df}{dz} = \delta z \left( \gamma_2 \frac{df}{dz} \right)$, therefore in a change of $z$ for $\delta z$, the value of the first derivative function against $z$ is $\left( \gamma_2 \frac{df}{dz} \right)$, and so also $\gamma_2 \delta z \frac{df}{dxz}$. In the previous section, the value of $\delta x \delta z \delta t$ is $\delta x = C_\delta t = \frac{L}{T} \delta t = \frac{2\pi \delta t}{k}$ which means $\frac{d^2 f}{dx \delta z}$. As in the previous section, the value of $\delta x \delta z \delta t$ is $\delta x = C_\delta t = \frac{L}{T} \delta t = \frac{2\pi \delta t}{k}$.

Then, a function $f(x,z,t)$ is reviewed with the following form.

$$f(x, z, t) = \cos kx \cos kx (k + z) \cos \sigma \ldots \ldots (9)$$

$$A_\delta z = 0, \quad c_1 = \cos (kh) \quad \text{and} \quad c_2 = \sinh (kh)$$

are defined and done in the deep water where $tankh = 1$ with the value of $kh = 2.0\pi$. Then $c_1 = \cos (2.0\pi) = c_2 = \sin (2\pi)$ and (8) is done in a condition of $c_1 \cos kx = \sin kx = \cos \sigma t = \sin \sigma t$, then the sinusoidal function cancelled out each other. The derivative equations (9) can be written in the forms shown in Table 2.

| Table 2: Differential of (9) |
|-----------------------------|
| $\frac{df}{dx} = -kc_1$ | $\frac{d^2 f}{dx^2} = -k^2c_1$ |
| $\frac{df}{dt} = c_{1+1}$ | $\frac{d^2 f}{dt^2} = \sigma c_{1+1}$ |
\[
\begin{align*}
\frac{\partial f}{\partial z} &= kc_2 \\
\frac{\partial^2 f}{\partial x \partial z} &= -k^2 c_2 \\
\frac{\partial^2 f}{\partial \tau \partial z} &= -\sigma kc_2 \\
\frac{\partial f}{\partial \tau} &= -\sigma c_1 \\
\frac{\partial^2 f}{\partial \tau^2} &= k^2 c_1 \\
\frac{\partial^2 f}{\partial \tau \partial z} &= -\sigma^2 c_1 
\end{align*}
\]

To simplify the writing, the followings are defined

\[
A = \frac{\delta^2 x \delta^2 y}{2} + \gamma_0 \delta z \delta x \frac{\partial^2 f}{\partial x \partial z} + \frac{\gamma_0 \delta z \delta x}{2} \frac{\partial^2 f}{\partial x \partial z} + \frac{\gamma \delta x \delta y}{2} \frac{\partial^2 f}{\partial y \partial z} + \frac{\gamma \delta y \delta z}{2} \frac{\partial^2 f}{\partial y \partial z} + \frac{\delta y \delta z}{2} \frac{\partial^2 f}{\partial y \partial z} \quad (6)
\]

\[
B = \delta x \frac{\partial f}{\partial x} + \gamma_0 \delta z \frac{\partial f}{\partial x} + \gamma_0 \delta z \frac{\partial f}{\partial \tau} \quad (7)
\]

In order for (8) to be able to be used with only the first derivative, then

\[
|\frac{\partial f}{\partial \tau}| \leq \epsilon \quad \text{..............(9)}
\]

The substitution of differential equations in (2) to (9) will obtain,

\[
\begin{align*}
-\frac{y^2}{2} c_1 + y - c_1 - \frac{\epsilon}{\delta t} (-y c_1 - c_1) \\
-\left( y + 1 + \frac{\epsilon}{\delta t} \right) c_2 y_2 + \frac{c_1}{2} y_2 = 0
\end{align*}
\]

Substituting from (6), \( \delta t = \frac{2c}{\sigma} \)

\[
-\frac{y^2}{2} c_1 + y - c_1 - \frac{1}{2} \left(-y c_1 - c_1\right) - \left(y + 1 + \frac{1}{2}\right) c_2 y_2 + \frac{c_1}{2} y_2 = 0 \quad \text{..............(10)}
\]

With (10), \( y_0 \) can be calculated where \( y \) is a known function from (9). With an input \( y = 3, \ y_0 = 1,630 \) is obtained for the velocity potential equation that consists of two potential velocities, i.e.

\[
\begin{align*}
\varphi(x, z, t) &= A \cos k x (Ce^{kz} + De^{-kz}) \sin \sigma t + B \sin k x (Ce^{kz} + De^{-kz}) \sin \sigma t \quad \text{.........(12)}
\end{align*}
\]

The constants of \( A, B, C, \) and \( D \) will be determined using (14) where water particle velocity at the vertical- \( z \)-direction is

\[
w = \frac{-\partial \varphi}{\partial z} = -(A + B) k \cos k x \quad (Ce^{kz} - De^{-kz}) \sin \sigma t
\]

Substitute equation for \( w \) to the kinematic bottom boundary condition equation

\[
\begin{align*}
\Phi(x, z, t) &= (A + B) \cos k x \quad (De^{kz}e^{kz} + De^{-kz}) \sin \sigma t
\end{align*}
\]

or

\[
\begin{align*}
\Phi(x, z, t) &= (A + B) De^{kz} \cos k x \quad e^{k(h+z)} + e^{-k(h+z)} \sin \sigma t
\end{align*}
\]

A new constant is defined

\[
G_A = (A + B) De^{kz} \quad \text{..............(16)}
\]

\[
\Phi(x, z, t) = G_A \cos k x (e^{k(h+z)} + e^{-k(h+z)}) \sin \sigma t
\]

\[
\Phi(x, z, t) = (A + B) De^{kz} \cos k x (Ce^{kz} + De^{-kz}) \sin \sigma t
\]

There are four constants that should be determined, i.e. \( A, B, C, D \). Hutahaean (2019b) has shown that the two equations have similar constant value, or in other words there is only one constant value in velocity potential total (12). However, in the next section it will be proven again with another method that (12) has one constant value.

Equation (12) can be written as,

\[
\varphi(x, z, t) = (A \cos k x + B \sin k x) (Ce^{kz} + De^{-kz}) \sin \sigma t \quad \text{..............(13)}
\]

At a condition of \( \cos k x = \sin k x \), (13) can be written as

\[
\varphi(x, z, t) = (A + B) \cos k x \quad (Ce^{kz} + De^{-kz}) \sin \sigma t \quad \text{..............(14)}
\]

or

\[
\varphi(x, z, t) = (A + B) \sin k x \quad (Ce^{kz} + De^{-kz}) \sin \sigma t \quad \text{..............(15)}
\]

The constants of \( A, B, C, \) and \( D \) will be formulated using (14) and (15), where it will be proven that either using (14) or (15) similar constant will be obtained. The formulation is done by doing kinematic bottom boundary condition on flat bottom, as was done by Dean (1991).

a. Alternative I

The constants \( A, B, C, \) and \( D \) will be determined using (14) where water particle velocity at the vertical- \( z \)-direction is

\[
w = \frac{-\partial \varphi}{\partial z} = -(A + B) k \cos k x \quad (Ce^{kz} - De^{-kz}) \sin \sigma t
\]

Substitute equation for \( w \) to the kinematic bottom boundary condition equation

\[
\Phi(x, z, t) = (A + B) \cos k x \quad (De^{kz}e^{kz} + De^{-kz}) \sin \sigma t
\]

or

\[
\Phi(x, z, t) = (A + B) De^{kz} \cos k x \quad e^{k(h+z)} + e^{-k(h+z)} \sin \sigma t
\]

A new constant is defined

\[
G_A = (A + B) De^{kz} \quad \text{..............(16)}
\]

\[
\Phi(x, z, t) = G_A \cos k x (e^{k(h+z)} + e^{-k(h+z)}) \sin \sigma t
\]

There are four constants that should be determined, i.e. \( A, B, C, D \). Hutahaean (2019b) has shown that the two equations have similar constant value, or in other words there is only one constant value in velocity potential total (12). However, in the next section it will be proven again with another method that (12) has one constant value.
\[ -(A + B)k \sin k x (Ce^{-kh} - De^{kh}) \sin \sigma t = 0 \]

The equation is divided by \(- (A + B)k \sin k x (Ce^{-kh} - De^{kh})\)

\[ \Phi(x, z, t) = (A + B) \sin k x \]

\[ (De^{2kh} e^{kz} - De^{-kz}) \sin \sigma t \]

or

\[ \Phi(x, z, t) = (A + B) De^{kh} \sin k x \left( e^{kh+z} + e^{-kh+z} \right) \sin \sigma t \]

A new constant is defined

\[ G_B = (A + B) De^{kh} \]

...(18)

\[ \Phi(x, z, t) = G_B \cos k x \]

\[ (e^{kh+z} + e^{-kh+z}) \sin \sigma t \]...(19)

From (16) and (18) obtained that \( G_A = G_B = G \), so it is proven that in (1) there is only one wave constant \( \mathcal{G} \), then (7) becomes

\[ \Phi(x, z, t) = G (\cos k x + \sin kx) \]

\[ (e^{kh+z} + e^{-kh+z}) \sin \sigma t \]...(20)

The hyperbolic function is, \( e^{kh+z} + e^{-kh+z} = 2 \cos kh (h + z) \)

...(13) becomes

\[ \Phi(x, z, t) = 2 G (\cos k x + \sin kx) \cos kh (h + z) \sin \sigma t \]

Defined \( G = 2G \)

\[ \Phi(x, z, t) = G (\cos k x + \sin kx) \cos kh (h + z) \sin \sigma t \]....(21)

A complete velocity potential equation is obtained with the form as in (21). In that equation, there are still two wave constants where the form should be known, i.e. wave number \( k \) and wave constant \( G \). Considering that the values of wave number \( k \) and wave constant \( G \) is similar along the wave curve, then the calculation of the two parameters will be done at the point of characteristic where \( \cos kx = \sin kx \), this condition...(21) becomes

\[ \Phi(x, z, t) = 2 G \cos kx \cos kh (h + z) \sin \sigma t \]....(26)

The particle velocity in horizontal-x-direction is,

\[ u = - \frac{\partial \Phi}{\partial x} = 2 G \sin kx \cos kh (h + z) \sin \sigma t \]....(27)

The particle velocity in vertical-z-direction is,

\[ w = - \frac{\partial \Phi}{\partial z} = - 2 G \cos kx \sin h (h + z) \sin \sigma t \]....(28)

IV. Application of Weighted Total Acceleration on Euler Momentum Equation

From (28), the total derivative for horizontal x-direction velocity is,

\[ \frac{Du}{dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \gamma w \frac{\partial u}{\partial z} \]

With this total derivative equation, the Euler momentum equation in horizontal-x-direction becomes,

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \gamma w \frac{\partial u}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial x} \]

By doing the characteristic of irrotational flow, \( \frac{\partial u}{\partial x} \) obtained,

\[ \gamma \frac{\partial u}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} \left( u^2 + \gamma w^2 \right) = - \frac{1}{\rho} \frac{\partial p}{\partial x} \]...(29)

Total derivative equation for vertical velocity in axis-z direction.

\[ \frac{Dw}{dt} = \gamma \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + \gamma u \frac{\partial w}{\partial z} \]

The Euler momentum equation in vertical-z-direction becomes,

\[ \gamma \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + \gamma u \frac{\partial w}{\partial z} = - \frac{1}{\rho} \frac{\partial p}{\partial z} \]

The execution of irrotational flow characteristic, \( \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} \)

\[ \gamma \frac{\partial w}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} \left( u^2 + \gamma w^2 \right) = - \frac{1}{\rho} \frac{\partial p}{\partial z} \]...(30)

(29) and (30) are modified Euler momentum equations, where there are time weighting coefficient \( \gamma \) and weighting coefficient vertical z-direction of weighting coefficient, i.e. \( \gamma z \). Using (30) pressure equation will be formulated where (30) is written as an equation for pressure.

\[ - \frac{1}{\rho} \frac{\partial p}{\partial z} = \gamma \frac{\partial w}{\partial t} + \frac{1}{2} \frac{\partial}{\partial x} \left( u^2 + \gamma w^2 \right) + g \]

This equation is multiplied by \( dz \) and integrated against vertical-z axis.

\[ \rho = \gamma \int_z^\eta \frac{\partial w}{\partial t} dz + \frac{1}{2} \left( u^2_z + \gamma w^2_z \right) + g \left( \eta - z \right) \]

Differentiated against horizontal-x axis

\[ \frac{\partial p}{\partial x} = \gamma \frac{\partial}{\partial x} \int_x^z \frac{\partial w}{\partial t} dz + \frac{1}{2} \frac{\partial}{\partial x} \left( u^2_z + \gamma w^2_z \right) + g \int_x^\eta \frac{\partial}{\partial z} dz \]

Substituted to (29)

\[ \gamma \frac{\partial u}{\partial t} + \gamma \frac{\partial}{\partial x} \int_x^z \frac{\partial w}{\partial t} dz + \frac{1}{2} \frac{\partial}{\partial x} \left( u^2_z + \gamma w^2_z \right) = - \frac{1}{\rho} \frac{\partial p}{\partial x} \]...(31)

The completion of \( \frac{\partial}{\partial x} \int_x^z \frac{\partial w}{\partial t} dz \) is done using velocity potential (21), where the particle velocity in horizontal direction is in equation (27), and the particle velocity in vertical-direction (28). From (28) the following is obtained.

\[ \frac{\partial w}{\partial t} = -2 G k \cos kx \sinh (h + z) \cos \sigma t \]

This equation is integrated against time \( t \),

\[ \int_x^\eta \frac{\partial w}{\partial t} dz = -2 G k \cos kx \]

\[ (\cosh k (h + \eta) - \cosh k (h + z)) \cos \sigma t \]

Then, it is differentiated against horizontal-x axis.

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Equation (27) is differentiated against time $t$, $\frac{du}{dt} = 2Gk\sin kxcosh(h + z)cos\sigma$.

This equation is a surface momentum equation that will be used in the calculation of Gandk.

V. THE FORMULATION OF AN EQUATION FOR THE CALCULATION OF $G$ AND $k$

As has been mentioned in the previous section that the calculation of Gandk is done in the point of characteristic where $coskx = sinkx$. Therefore, (27) is used as the particle velocity in horizontal $x$ direction and (28) is particle velocity equation in vertical $z$ direction.

5.1 Wave number conservation

In the formulation of an equation for the calculation of Gandk, in the following subsection, the wave number conservation equation will be done. The equation comes from the principle of variable separation at the completion of Laplace equation, i.e. that velocity potential is considered as a multiplication of three functions, i.e. $\Phi(x, z, t) = X(x)Z(z)T(t)$ where $X(x)$ is just a function-$x$, $Z(z)$ is just a function-$z$ and $T(t)$ is just a function-$t$.

In this case $Z(z) = coshz(h + z)$. As just function-$z$, then

$$\frac{dz}{dx}(z) = 0 .$$

$$\frac{d}{dx}(sinkz(h + z)) = sinkz(h + z) - \frac{dk(h + z)}{dx} = 0$$

For $sinkz(h + z)$ is not equal to zero, then

$$\frac{dk(h + z)}{dx} = 0 \ldots (33)$$

This equation (33) is called wave number conservation equation. This means that all area of calculation has similar values for the function $tanhk(h + z)$, $cothk(h + z)$ dan $sinkh(h + z)$. As deep water, it can be defined as water depth where $tanhk(h + \eta) = 1$, which $\eta(x, t)$ is the water surface elevation against still water level. Bearing in mind that the wave number conservation equation or law, in the entire domain applies $tanhk(h + \eta) = 1 \ldots (34)$

In this research, the following is used $k(h + \eta) = 2.0\pi \ldots (35)$

Where $tanhk(2.0\pi)$ = 0.999993.
5.2 The formulation of wave amplitude function
The weighted total acceleration equation (2), done at the water wave surface equation \( \eta = \eta(x, t) \), obtained \( \frac{\partial \eta}{\partial t} = \sqrt{\frac{g}{\alpha}} + u \frac{\partial \eta}{\partial x} \). The original kinematic free surface boundary condition (KFSBC) equation is, \( w_\eta = \frac{\partial \eta}{\partial t} + u \frac{\partial \eta}{\partial x} \). By comparing the two equations, then the KFSBC equation should be in the form of \( w_\eta = \sqrt{\frac{g}{\alpha}} + u \frac{\partial \eta}{\partial x} \), or \( \frac{\partial \eta}{\partial t} = w_\eta - u \frac{\partial \eta}{\partial x} \) ...

Substitutefrom (27) andfrom (28) and done at \( z = \eta \),
\[
\frac{\partial \eta}{\partial t} = -2Gk\sinh(h + \eta)\cos kx\sin t
\]
Water wave surface equation was obtained by integrating (43) against time \( t \). The right side of the equation is a non-linear function against time \( t \) of which the integration completion is difficult. However, there is an argument that can simplify the integration (43) completion. First bearing in mind (36), i.e. \( \cosh(h + \eta) = \cosh(2.0\pi) = \) constant. Then, (43) is written as,
\[
\frac{\partial \eta}{\partial t} = -2Gk \left( \cosh(x)\sinh(h + \eta) + \sinh(x)\cosh(h + \eta)\frac{\partial \eta}{\partial x} \right) \sin t
\]
In (44) the one that is the function of time \( t \) is only the element \( \sin t \). In addition, as a periodical function against time \( t \), the element \( -2Gk \left( \cosh(x)\sinh(h + \eta) + \sinh(x)\cosh(h + \eta)\frac{\partial \eta}{\partial x} \right) \) should be a constant number against time \( t \). Thus, the integration (44) against time \( t \), is sufficient by integrating only the \( \sin t \) element, obtained
\[
\eta(x, t) = \frac{2Gk}{\gamma_\sigma} \left( \cosh(x)\sinh(h + \eta) + \sinh(x)\cosh(h + \eta)\frac{\partial \eta}{\partial x} \right) \cos t
\]
At the characteristics point, (45) can be written as
\[
\eta(x, t) = \frac{2Gk}{\gamma_\sigma} \left( \sinh(h + \eta) + \cosh(h + \eta)\frac{\partial \eta}{\partial x} \right) \cos kx \cos t
\]
The form \( \cos kx \) was selected because it has been determined that the velocity potential component that was used is \( \cos kx \) component itself is defined
\[
A = \frac{2Gk}{\gamma_\sigma} \left( \sinh(h + \eta) + \cosh(h + \eta)\frac{\partial \eta}{\partial x} \right)
\]
Then (46) becomes
\[
\eta(x, t) = Acoskxcos\sigma
\]
At the characteristic point, then \( \eta = \frac{A}{2} \), wave amplitude function equation,
\[
A = \frac{2Gk}{\gamma_\sigma} \left( \sinh(h + \frac{A}{2}) - \cosh(h + \frac{A}{2})kA \right)
\]
From (36) where \( \sinh(h + \frac{A}{2}) = \cosh(h + \frac{A}{2}) \) the wave amplitude function equation becomes,
\[
A = \frac{2Gk}{\gamma_\sigma} \cosh(h + \frac{A}{2}) \left( 1 - \frac{kA}{2} \right)
\]
5.3 Equation for the calculation of \( k \) and \( G \)
Substitute (47) to (41) at the characteristic point
\[
\gamma_\sigma Gk \cosh k \left( h + \frac{A}{2} \right) + \frac{1}{2} (1 - \gamma_\sigma) Gk^2 k^2 \cosh^2 k \left( h + \frac{A}{2} \right) = \frac{g kA}{2}
\]
Substitute wave amplitude function,
\[
\gamma_\sigma Gk \cosh k \left( h + \frac{A}{2} \right) + \frac{1}{2} (1 - \gamma_\sigma) Gk^2 k^2 \cosh^2 k \left( h + \frac{A}{2} \right) = \frac{g kA}{2}
\]
The equation is divided by \( \frac{Gk}{\gamma_\sigma} \cosh k \left( h + \frac{A}{2} \right) \),
\[
\gamma_\sigma^2 + \frac{1}{2} (1 - \gamma_\sigma) Gk^2 \cosh k \left( h + \frac{A}{2} \right) = \frac{g k(1 - kA)}{2}
\]
Wave amplitude equation is written as an equation for \( G \), i.e.
\[
G = \frac{\gamma_\sigma}{2k\cosh k \left( h + \frac{A}{2} \right) \left( 1 - \frac{kA}{2} \right)}
\]
and substitute it to the last equation,
\[
\gamma_\sigma^2 + \frac{1}{2} (1 - \gamma_\sigma) GkA = g k(1 - kA)^2
\]
The calculation of the value \( k \) with this equation using Newton-Rhapson method requires initial estimation of \( k \) for the initial value of the iteration. The initial value of \( k \) can be obtained by ignoring convective acceleration, then (50) becomes
\[
\gamma_\sigma^2 + \frac{1}{2} \left( 1 - \frac{kA}{2} \right) + \frac{1}{4} (1 - \gamma_\sigma) kA = g k(1 - kA)^2
\]
This equation is the quadratic equation of wave number \( k \) that can be easily completed. The use of (51) maximum value of wave amplitude \( A \) in a wave period in deep water is obtained, i.e. if the determinant \( D \) from (51) has a value of zero.
\[
A_{\text{max}} = \frac{g}{2\gamma_\sigma^2} \]
The value of \( G \) can be calculated using (49).

VI. THE FORMULATION OF WATER WAVE SURFACE EQUATION.
Water wave surface equation is formulated using a complete velocity potential equation, i.e. equation (21).
By using (21), particle velocity in horizontal-x direction and particle velocity in vertical-z direction are consecutively,

\[ u = Gk (\sin kx - \cos kx) \cosh k(h + z) \sin \sigma t \]
\[ w = -Gk (\cos kx + \sin kx) \sinh k(h + z) \sin \sigma t \]

The two particle velocity equations are done at \( z = \eta \) and substituted to equation KFSBC (42),

\[ \frac{d\eta}{dt} = -Gk (\cos kx + \sin kx) \sinh (h + \eta) \sin \sigma t \]
\[ -Gk(\sin kx - \cos kx) \cosh (h + \eta) \sin \sigma t \frac{d\eta}{dx} \]

\[ \ldots (53) \]

As in the previous section, the water wave surface equation is obtained by integrating (53) against time, where the integration is sufficient to be done only at the \( \sin \sigma \) telement,

\[ \eta(x, t) = \frac{Gk}{\gamma'} (\cos kx + \sin kx) \sinh (h + \eta) \cos \sigma t \]
\[ + \frac{Gk}{\gamma'} (\sin kx - \cos kx) \cosh (h + \eta) \cos \sigma t \frac{d\eta}{dx} \]

In the deep water the equation can be written as,

\[ \eta(x, t) = c_0 \left((c_2 + c_1) + (c_1 - c_2) \frac{d\eta}{dx} \right) c_3 \ldots (54) \]

where, to simplify the writing \( c_0 = \frac{Gk}{\gamma'} \cosh (h + \eta), \ c_1 = \sin kx, \ c_2 = \cos kx \) dan \( c_3 = \cos \sigma t \) are defined. Equation (54) is differentiated against horizontal-x axis

\[ \frac{d\eta}{dx} = c_0 k \left((-c_1 + c_2) + (c_2 + c_1) \frac{d\eta}{dx} \right) c_3 \ldots \ldots (55) \]

Equation (54) is water wave surface equation that is used to calculate water surface elevation where \( \frac{d\eta}{dx} \) in (54) is calculated using (55). \( \eta \) in \( c_0 = \frac{Gk}{\gamma'} \cosh (h + \eta) \) is calculated using the equation,

\[ \eta(x, t) = A(\cos kx + \sin kx) \cos \sigma t \ldots (56) \]

Whereas \( \frac{d\eta}{dx} \) in (55) it is calculated with,

\[ \frac{d\eta}{dx} = Ak(-\sin kx + \cos kx) \cos \sigma t \ldots (57) \]

### VII. THE RESULTS OF THE EQUATION.

#### 7.1 The characteristic of water wave surface.

In the calculations that will be done in this section, the value of \( \gamma = 3.0 \) and \( \eta_{\text{max}} = 1.630 \) are used and the calculation is done in the deep water. Deep water depth \( h_{\text{0is}} \) obtained with the following equation

\[ h_0 = \frac{1}{k} \left(2.0\pi - \frac{kA}{2}\right) \ldots (58) \]

Where \( A \) is calculated using (52).

### Table 3: The result of calculation of wave parameter and other characteristic

| T (sec) | H (m) | L (m) | \( H/L \) | \( H/A \) | \( \eta_{\text{max}}/H \) |
|---------|-------|-------|-----------|-----------|------------------|
| 6       | 1,409 | 5,026 | 0.28      | 2.865     | 0.851            |
| 7       | 1,918 | 6,842 | 0.28      | 2.865     | 0.851            |
| 8       | 2,506 | 8,936 | 0.28      | 2.865     | 0.851            |
| 9       | 3,171 | 11,309| 0.28      | 2.865     | 0.851            |
| 10      | 3,915 | 13,962| 0.28      | 2.865     | 0.851            |
| 11      | 4,737 | 16,894| 0.28      | 2.865     | 0.851            |
| 12      | 5,638 | 20,105| 0.28      | 2.865     | 0.851            |
| 13      | 6,617 | 23,595| 0.28      | 2.865     | 0.851            |
| 14      | 7,674 | 27,365| 0.28      | 2.865     | 0.851            |
| 15      | 8,81  | 31,413| 0.28      | 2.865     | 0.851            |

Using water wave surface equation, the elevation of wave crest \( \eta_{\text{max}} \) and the elevation of wave trough \( \eta_{\text{min}} \) are calculated. The wave height is \( H = \eta_{\text{max}} - \eta_{\text{min}} \), whereas Wilson (1963) criteria is \( \eta_{\text{max}}/H \). Table (3) presented the result of the calculations of wave height, wavelength, wave steepness, and the comparison of wave height \( H \) and wave amplitude \( A \).

Wave steepness \( \frac{H}{L} = 0.280 \), where considering the calculation used maximum wave Amplitude \( A \) that was calculated using (52), then wave steepness is critical wave steepness.

### Table 4: Types of wave, according to Wilson criteria (1963)

| Wave Type          | \( \eta_{\text{max}}/H \) |
|--------------------|---------------------------|
| Airy waves         | < 0.505                   |
| Stoke's waves      | < 0.635                   |
| Cnoidal waves      | 0.635 < \( \eta_{\text{max}}/H \) < 1 |
| Solitary waves     | = 1                       |

The critical wave steepness is bigger than the criteria of Michell (1893) i.e. \( \frac{H}{L} = 0.142 \). The comparison between wave height and wave amplitude \( \frac{H}{A} = 2.865 \) which is bigger than 2. Therefore, there will be wave steepness between wave height and wave amplitude is \( H = 2A \) cannot be used. The obtained Wilson parameter is \( \frac{\eta_{\text{max}}}{H} = 0.851 \). Based on Wilson criteria (1963), Table (4), the value of the parameter shows that the wave profile has a cnoidal wave type, with wave profile presented in Fig.1 and Fig.2 for wave period \( T = 8 \) sec.
using the model, where the input in the model is wave period $T$ and wave amplitude calculated using (52), so that the wave height that is obtained is the wave height maximum $H_{\text{max}}$ in the related wave period. Table (5) shows that for $\gamma = 3$, the obtained $T_{\text{Wiegel}}$ is almost similar with the $T$ that is a wave period to calculate $H_{\text{max}}$ with the model. Whereas in $\gamma = 2.97102$, it can be said that the obtained $T_{\text{Wiegel}}$ is equal with $T$. The result of this calculation concludes that the values of $\gamma$, $\gamma_x$, and equations formulated in this research are in line with the result of Wiegel research (1949-1964) which is the result of an observation.

### VIII. CONCLUSION

If the characteristic of ideal fluid i.e. irrotational flow is done at Euler momentum equation, and the velocity potential as the product of Laplace equation solution is substituted, the hydrodynamic force or convective equation in the horizontal direction becomes zero. This problem can be solved using weighted total acceleration where there is weighting coefficient at the differential term against vertical-$z$ axis and the resulted model producewave height that corresponds to Wiegel equation. Another finding that should be noticed is that the value of wave height is not twice the value of wave amplitude.

A further research needed is formulating shoaling and breaking model by doing the weighted total acceleration equation, because there are many researches result in the laboratory on breaker height that are stated in the form of Breaker Height Index equation, so that the shoaling-breaking model and its various basic theories can easily be calibrated.

### REFERENCES

[1] Dean, R.G., Dalrymple, R.A. (1991). Water wave mechanics for engineers and scientists. Advance Series on Ocean Engineering.2. Singapore: World Scientific. ISBN 978-981-02-0420-4. OCLC 229072424.

[2] Hutahaean , S. (2019a). Application of Weighted Total Acceleration Equation on Wavelength Calculation. International Journal of Advance Engineering Research and Science (IJAERS). Vol-6, Issue-2, Feb-2019. ISSN-2349-6495(P)/2456-1908(O).

https://dx.doi.org/10.22161/ijaers.6.2.31

[3] Hutahaean , S. (2019b). Water Wave Modeling Using Complete Solution of Laplace Equation. International Journal of Advance Engineering Research and Science (IJAERS). Vol-6, Issue-8, Aug-2019. ISSN-2349-6495(P)/2456-1908(O).

https://dx.doi.org/10.22161/ijaers.6.8.33

[4] Courant, R.,Friedrichs,K., Lewy, H. (1928). Uber die partiellen Differenzengleichungen der mathematischen Physik. Matematische Annalen (in German). 100 (1): 32-74.
Bibcode:1928. MatAn 100.32.C. doi:10.1007/BF01448839. JFM 54.0486.01 MR 1512478.

[5] Michell, J.H. (1893). On the highest wave in water: Philosophical Magazine, (5), vol. XXXVI, pp. 430-437.

[6] Wilson, B.W., (1963). Condition of Existence for Types of Tsunami waves, paper presented at XIII th General

[7] Wiegel, R.L. (1949). An Analysis of Data from Wave Recorders on the Pacific Coast of the United States, Trans. Am. Geophys. Union, Vol.30, pp.700-704.

[8] Wiegel, R.L. (1964). Oceanographical Engineering, Prentice-Hall, Englewoods Cliffs, N.J.