SYMMETRIC RECOLLEMENTS INDUCED BY BIMODULE EXTENSIONS

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Abstract. Inspired by the work of Jørgensen [J], we define a (upper-, lower-) symmetric recollements; and give a one-one correspondence between the equivalent classes of the upper-symmetric recollements and one of the lower-symmetric recollements, of a triangulated category. Let \( \Lambda = \left( \begin{array}{cc} A & M \\ 0 & B \end{array} \right) \) with bimodule \( A_M = B_M \). We construct an upper-symmetric abelian category recollement of \( \Lambda\text{-}\text{mod} \); and a symmetric triangulated category recollement of \( \Lambda\text{-}\text{Gproj} \) if \( A \) and \( B \) are Gorenstein and \( A_M \) and \( M_B \) are projective.

Key words and phrases. abelian category, triangulated category, symmetric recollement, Gorenstein-projective modules

Introduction

A triangulated category recollement, introduced by A. A. Beilinson, J. Bernstein, and P. Deligne [BBD], and an abelian category recollement, formulated by V. Franjou and T. Pirashvili [FV], play an important role in algebraic geometry and in representation theory ([MV], [CPS], [K], [M]).

Recently, P. Jørgensen [J] observed that if a triangulated category \( C \) has a Serre functor, then a triangulated category recollement of \( C \) relative to \( C' \) and \( C'' \) can be interchanged in two ways to triangulated category recollements of \( C \) relative to \( C'' \) and \( C' \). Inspired by [J] we define in Section 2 a (upper-, lower-) symmetric recollement; and prove that there is a one-one correspondence between the equivalent classes of the upper-symmetric triangulated category recollements of \( C \) relative to \( C' \) and \( C'' \), and the ones of the lower-symmetric triangulated category recollements of \( C \) relative to \( C'' \) and \( C' \). Let \( A \) and \( B \) be Artin algebras, \( M \) an \( A\text{-}B \)-bimodule, and \( \Lambda = \left( \begin{array}{cc} A & M \\ 0 & B \end{array} \right) \) the upper triangular matrix algebra. We construct an upper-symmetric abelian category recollement of \( \Lambda\text{-}\text{mod} \), the category of finitely generated \( \Lambda \)-modules.

An important feature of Gorenstein-projective modules is that the category \( A\text{-Gproj} \) of Gorenstein-projective \( A \)-modules is a Frobenius category, and hence the stable category \( A\text{-Gproj} \) is a triangulated category ([Hap]). Iyama-Kato-Miyachi ([IKM], Theorem 3.8) prove that if \( A \) is a Gorenstein algebra, then \( T_2(A)\text{-Gproj} \) admits a triangulated category recollement, where \( T_2(A) = \left( \begin{array}{cc} A & A \\ 0 & A \end{array} \right) \). In Section 3, if \( A \) and \( B \) are Gorenstein algebras and \( A_M \) and \( M_B \) are projective, we extend this result by asserting that \( \Lambda\text{-Gproj} \) admits a symmetric triangulated category recollement, and by explicitly writing out the involved functors.

Supported by the NSF of China (10725104), and STCSM (09XD1402500).
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1. An equivalent definition of triangulated category recollements

1.1. Recall the following

**Definition 1.1.** (1) ([BBD]) Let $C'$, $C$ and $C''$ be triangulated categories. The diagram

\[
\begin{array}{ccc}
C' & \xrightarrow{i^*} & C & \xrightarrow{j_!} & C'' \\
\downarrow{i_*} & & \downarrow{j_*} & & \\
C & \xrightarrow{j^*} & C''
\end{array}
\]

(1.1)

of exact functors is a triangulated category recollement of $C$ relative to $C'$ and $C''$, if the following conditions are satisfied:

(R1) $(i^*, i_*)$, $(i_*, i^!)$, $(j_!, j^*)$, and $(j^*, j_*)$ are adjoint pairs;

(R2) $i_*$, $j_!$ and $j_*$ are fully faithful;

(R3) $j^*i_* = 0$;

(R4) For each object $X \in C$, the counits and units give rise to distinguished triangles:

\[
\begin{align*}
& j_!j^*X \xrightarrow{\epsilon_X} X \xrightarrow{\eta_X} i_*i^*X \quad \text{and} \quad i_*i^!X \xrightarrow{\omega_X} X \xrightarrow{\kappa_X} j_*j^*X.
\end{align*}
\]

(2) ([FV]) Let $C'$, $C$ and $C''$ be abelian categories. The diagram (1.1) of additive functors is an abelian category recollement of $C$ relative to $C'$ and $C''$, if (R1), (R2) and (R5) are satisfied, where

(R5) $\text{Im} i_* = \text{Ker} j^*$.

**Remark 1.2.** (1) Let (1.1) be an abelian category recollement. If all the involved functors are exact, then one can prove that there is an equivalence $C \cong C' \times C''$ of categories. This explains why Franjou-Pirashvili [FV] did not require the exactness of the involved functors in Definition 1.1(2).

(2) For any adjoint pair $(F, G)$, it is well-known that $F$ is fully faithful if and only if the unit $\eta: \text{Id} \to GF$ is a natural isomorphism, and $G$ is fully faithful if and only if the counit $\epsilon : FG \to \text{Id}$ is a natural isomorphism; and that if $F$ is fully faithful then $G\epsilon_X$ is an isomorphism for each object $X$, and if $G$ is fully faithful then $F\eta_Y$ is an isomorphism for each object $Y$.

(3) In any triangulated or abelian category recollement, under the condition (R1), the condition (R2) is equivalent to the condition (R2'): the units $\text{Id}_{C'} \to i^!i_*$ and $\text{Id}_{C''} \to j^*j_!$, and the counits $i^*i_* \to \text{Id}_{C'}$ and $j^*j_* \to \text{Id}_{C''}$, are natural isomorphisms.

(4) In an abelian category recollement one has $i^*j_! = 0$ and $i^!j_* = 0$; and in a triangulated category recollement one has $\text{Im} i_* = \text{Ker} j^*$, $\text{Im} j^! = \text{Ker} i^*$ and $\text{Im} j_* = \text{Ker} i^!$.

(5) In any abelian category recollement (1.1), the counits and units give rise to exact sequences of natural transformations $j_!j^* \to \text{Id}_{C'} \to i_*i^* \to 0$ and $0 \to i_*i^! \to \text{Id}_{C''} \to j_*j^*$; and if $C', C$, and $C''$ have enough projective objects, then $i^*$ is exact if and only if $i^!j_! = 0$; and dually, if $C', C$, and $C''$ have enough injective objects, then $i^!$ is exact if and only if $i^*j_* = 0$. See [FV].
1.2. We will need the following equivalent definition of a triangulated category recollement, which possibly makes the construction of a triangulated category recollement easier.

**Lemma 1.3.** Let (1.1) be a diagram of exact functors of triangulated categories. Then it is a triangulated category recollement if and only if the conditions (R1), (R2) and (R5) are satisfied.

**Proof.** This seems to be well-known, however we did not find an exact reference. For the convenience of the reader we include a proof.

We only need to prove the sufficiency. Embedding the counit morphism $\epsilon_x$ into a distinguished triangle $j_! j^* X \xrightarrow{\epsilon_x} X \xrightarrow{h} Z \rightarrow$. Applying $j^*$ we get a distinguished triangle $j^* j_! j^* X \xrightarrow{j^* \epsilon_x} j^* X \xrightarrow{j^* h} j^* Z \rightarrow$. Since $j^* \epsilon_x$ is an isomorphism by Remark 1.2(2), we have $j^* Z = 0$. By $\text{Im} i_! = \text{Ker} j^*$ we have $Z = i_! Z'$. Applying $i^*$ to the distinguished triangle $j_! j^* X \xrightarrow{\epsilon_x} X \xrightarrow{h} i_! Z' \rightarrow$, by $i^* j_! = 0$ we know that $i^* h : i^* X \rightarrow i^* i_! Z'$ is an isomorphism. Since the counit morphism $i^* i_! Z' \xrightarrow{\epsilon_x} Z'$ is an isomorphism, we have isomorphism $i_! ((i^* h)^{-1}) i_! (\epsilon_x^{-1}) : i_! Z' \rightarrow i_! i^* X$, and hence we get a distinguished triangle of the form $j_! j^* X \xrightarrow{\epsilon_x} X \xrightarrow{f} i_! i^* X \rightarrow$ with $f = i_! ((i^* h)^{-1}) i_! (\epsilon_x^{-1}) h$, which also means $\text{Im} j_! = \text{Ker} i^*$. Since $i^* h$ is an isomorphism, $i^* f$ is an isomorphism.

In order to complete the first distinguished triangle in (R4), we need to show that $f$ can be chosen to be the unit morphism. Embedding the unit morphism $\eta_X$ into a distinguished triangle $Y \rightarrow X \xrightarrow{\eta_X} i_! i^* X \rightarrow$. By the similar argument (but this time we use $\text{Im} j_! = \text{Ker} i^*$) we get a distinguished triangle of the form $j_! j^* X \xrightarrow{g} X \xrightarrow{\eta_X} i_! i^* X \rightarrow$. By the following commutative diagram given by the adjoint pair $(i^*, i_!)$

\[
\begin{array}{ccc}
\text{Hom}_C(i^* i_! X, i^* X) & \xrightarrow{\sim} & \text{Hom}_C(i_! i^* X, i_! i^* X) \\
\downarrow{(i^* f, -)} & & \downarrow{(f, -)} \\
\text{Hom}_C(i^* X, i^* X) & \xrightarrow{\sim} & \text{Hom}_C(X, i_! i^* X)
\end{array}
\]

we see that $\text{Hom}_C(f, i_! i^* X)$ is also an isomorphism, and hence there is $u \in \text{Hom}_C(i_! i^* X, i_! i^* X)$ such that $uf = \eta_X$. Since $(i^*, i_!)$ is an adjoint pair and $i_!$ is fully faithful, it follows that $i^* \eta_X$ is an isomorphism. Replacing $f$ by $\eta_X$ we get $v \in \text{Hom}_C(i_! i^* X, i_! i^* X)$ such that $v \eta_X = f$. Thus we have morphisms of distinguished triangles

\[
\begin{array}{ccc}
j_! j^* X \xrightarrow{\epsilon_X} X & \xrightarrow{f} & i_! i^* X \\
\downarrow{\eta_X} & = & \downarrow{uv} \\
j_! j^* X \xrightarrow{\epsilon_X} X & \xrightarrow{f} & i_! i^* X
\end{array}
\]

and

\[
\begin{array}{ccc}
j_! j^* X \xrightarrow{g} X & \xrightarrow{\eta_X} & i_! i^* X \\
\downarrow{\eta_X} & = & \downarrow{uv} \\
j_! j^* X \xrightarrow{g} X & \xrightarrow{\eta_X} & i_! i^* X
\end{array}
\]
So $uv$ and $vu$, and hence $u$ and $v$, are isomorphisms. By the isomorphism of triangles

\[
\begin{array}{ccc}
  j^* W & \xrightarrow{\varepsilon} & X \\
  \downarrow{=} & \phantom{=} & \downarrow{=} \\
  j^* W & \xrightarrow{\varepsilon} & X
\end{array}
\]

we see that $j^* X \xrightarrow{\varepsilon} X \xrightarrow{\eta} i_* i^* X$ is a distinguished triangle.

In order to obtain the second distinguished triangle, we embed the unit morphism $\zeta_X$ into a distinguished triangle $W \xrightarrow{v} X \xrightarrow{\eta} j_* j^* X \to$. Applying $j^*$ we get a distinguished triangle $j^* W \xrightarrow{j^* w} j^* X \xrightarrow{j^* \zeta_X} j^* j_* j^* X \to$. Since $j^* \zeta_X$ is an isomorphism by Remark 1.2(2), we have $j^* W = 0$. By $\text{Im} i_* = \text{Ker} j^*$ we have $W = i_* X'$. Applying $i^!$ to the distinguished triangle $i_* X' \xrightarrow{\eta_i} X' \xrightarrow{\zeta} j_* j^* X \to$ and by $i^! j_* = 0$ we know that $i^! w : i^! i_* X' \to i^! X$ is an isomorphism. Using the unit isomorphism $X' \to i^! i_* X'$, we get a distinguished triangle of the form $i_* i^! X' \xrightarrow{\eta_i} X' \xrightarrow{\zeta} j_* j^* X \to$ with $i^! a$ an isomorphism. It follows that $\text{Im} j_* = \text{Ker} j^!$.

Now since $\text{Im} j_* = \text{Ker} j^!$ and $\text{Im} i_* = \text{Ker} j^*$, it follows that we can replace $(i^*, i_*)$ by $(j^*, j_*)$, and replace $(j_!*, j_!)$ by $(i_!, i_!)$, in the distinguished triangle $j^* j_* X \xrightarrow{\varepsilon} X \xrightarrow{\eta} i_* i^* X \to$. In this way we get the second distinguished triangle $i_* i^! X' \xrightarrow{\varepsilon} X \xrightarrow{\zeta} j_* j^* X \to$.

\[\blacksquare\]

2. Upper-symmetric recollements

2.1. Given a recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$, one usually can not expect a recollement of $\mathcal{C}$ relative to $\mathcal{C}''$ and $\mathcal{C}'$. Inspired by [J] we define

**Definition 2.1.** ([J]) A triangulated category recollement

\[
\begin{array}{ccc}
  \mathcal{C}' & \xrightarrow{i^*} & \mathcal{C} \\
  \phantom{=} & \phantom{=} & \phantom{=} \\
  \mathcal{C} & \xrightarrow{j^*} & \mathcal{C}''
\end{array}
\]

(2.1)

of $\mathcal{C}$ is upper-symmetric, if there are exact functors $j^*$ and $i^*$ such that

\[
\begin{array}{ccc}
  \mathcal{C}'' & \xrightarrow{j^*} & \mathcal{C} \\
  \phantom{=} & \phantom{=} & \phantom{=} \\
  \mathcal{C} & \xrightarrow{i^*} & \mathcal{C}'
\end{array}
\]

(2.2)

is a recollement; and it is lower-symmetric, if there are exact functors $j^!$ and $i^!$ such that

\[
\begin{array}{ccc}
  \mathcal{C}'' & \xrightarrow{j^!} & \mathcal{C} \\
  \phantom{=} & \phantom{=} & \phantom{=} \\
  \mathcal{C} & \xrightarrow{i^!} & \mathcal{C}'
\end{array}
\]

(2.3)

is a recollement. A recollement is symmetric if it is upper- and lower-symmetric.

Similarly, we have a (upper-, lower-) symmetric abelian category recollement, and note that in abelian situations, all the involved functors, in particular $j^*$, $i^*$, $j^!$ and $i^!$, are only required to be additive functors, not required to be exact.
Let $k$ be a field. P. Jørgensen [J] observed that if a Hom-finite $k$-linear triangulated category $\mathcal{C}$ has a Serre functor, then any recollement of $\mathcal{C}$ is symmetric: his proof does not use any triangulated structure of $\mathcal{C}$ and hence also works for a Hom-finite $k$-linear abelian category having a Serre functor. For a similar notion of symmetric recollements of unbounded derived categories we refer to S. König [K], and also Chen-Lin [CL].

2.2. Given two triangulated or abelian category recollements

$$
\begin{array}{ccc}
C' & \xrightarrow{i^*} & \mathcal{C} & \xrightarrow{j^*} & \mathcal{C}'' \\
\downarrow{i_!} & & \downarrow{j_!} & & \downarrow{j_!}
\end{array}
$$

and

$$
\begin{array}{ccc}
C' & \xrightarrow{i'^*} & \mathcal{D} & \xrightarrow{j'^*} & \mathcal{D}'' \\
\downarrow{i'_!} & & \downarrow{j'_!} & & \downarrow{j'_!}
\end{array}
$$

if there is an exact functor $f: \mathcal{C} \to \mathcal{D}$ such that there are natural isomorphisms

$$
i^* \cong i^*_D f, \quad f i_* \cong i^*_D, \quad i^! \cong i'^*_D f, \quad f j_! \cong j^!_D, \quad j^* \cong j'^*_D f, \quad f j_* \cong j^*_D,$$

then we call $f$ a comparison functor. Two (triangulated or abelian category) recollements are equivalent if there is a comparison functor $f$ which is an equivalence of categories. According to Parshall-Scott [PS, Theorem 2.5], a comparison functor between triangulated category recollements is an equivalence of categories. However, Franjou-Pirashvili [FV] pointed out that this is not necessarily the case for abelian category recollements.

2.3. In this subsection we only consider triangulated category recollements. If (2.1) is an upper-symmetric recollement, then we call (2.2) a upper-symmetric version of (2.1); and if (2.1) is an lower-symmetric recollement, then we call (2.3) a lower-symmetric version of (2.1).

**Lemma 2.2.** (1) Any two upper-symmetric versions of a upper-symmetric recollement are equivalent.

(1') Any two lower-symmetric versions of a lower-symmetric recollement are equivalent.

(2) Equivalent upper-symmetric recollements have equivalent upper-symmetric versions.

(2') Equivalent lower-symmetric recollements have equivalent lower-symmetric versions.

**Proof.** (1) Let (2.2) and

$$
\begin{array}{ccc}
\mathcal{C}'' & \xrightarrow{j^*} & \mathcal{C} & \xrightarrow{i^*} & \mathcal{C}' \\
\downarrow{j_!} & & \downarrow{i_!} & & \downarrow{i_!}
\end{array}
$$

be two upper-symmetric versions of a upper-symmetric recollement (2.1). Then $j_! i_! = 0$. In fact, for $Y \in \mathcal{C}'$ we have

$$
\text{Hom}_{\mathcal{C}''}(j_! i_! Y, j_! i_! Y) \cong \text{Hom}_{\mathcal{C}}(j_! j_! i_! Y, i_! Y) \cong \text{Hom}_{\mathcal{C}''}(i_! j_! j_! i_! Y, Y) = 0.
$$

For $X \in \mathcal{C}$, by (2.2) and (R4) we have distinguished triangle $j_! j_! X \xrightarrow{\eta} X \xrightarrow{\eta} i_! i_! X \to$. Applying exact functor $j_!$ and using the unit $\text{Id}_{\mathcal{C}''} \to j_! j_!$, we have

$$
j_! X \cong j_! j_! i_! X \cong j_! X.
$$
which means that \( j \alpha \) is naturally isomorphic to \( j \gamma \). Similarly one can prove that \( i \gamma \alpha \) is naturally isomorphic to \( i \gamma \). Thus \( \text{Id}_C \) is an equivalence between (2.2) and (2.4). This proves (1).

(1′) can be similarly proved.

(2) Given two equivalent upper-symmetric recollements

\[
\begin{array}{ccccccc}
\mathcal{C}' & \xrightarrow{i'} & \mathcal{C} & \xrightarrow{j} & \mathcal{C}'' & \xrightarrow{i''} & \mathcal{C}' \\
\mathcal{C}' & \xrightarrow{i'} & \mathcal{D} & \xrightarrow{j'} & \mathcal{C}'' & \xrightarrow{i''} & \mathcal{C}'
\end{array}
\]

with comparison functor \( f \), let (2.2) as an upper-symmetric version of the first recollement. By Lemma 1.3 we know that

\[
\begin{array}{ccccccc}
\mathcal{C}'' & \xrightarrow{j''} & \mathcal{D} & \xrightarrow{i''} & \mathcal{C}'
\end{array}
\]

is a triangulated category recollement, and that \( f \) is an equivalence between (2.2) and (2.5). Note that (2.5) is an upper-symmetric version of the second given upper-symmetric recollement, and hence the assertion follows from (1).

(2′) can be similarly proved.

Let \( \mathcal{C}', \mathcal{C}, \mathcal{C}'' \) be triangulated categories. Denote by \( \text{USR}(\mathcal{C}', \mathcal{C}, \mathcal{C}'') \) the class of equivalence classes of the upper-symmetric recollements of triangulated category \( \mathcal{C} \) relative to \( \mathcal{C}' \) and \( \mathcal{C}'' \); and denote by \( \text{LSR}(\mathcal{C}', \mathcal{C}, \mathcal{C}'') \) the class of the lower-symmetric recollements of triangulated category \( \mathcal{C} \) relative to \( \mathcal{C}' \) and \( \mathcal{C}'' \).

**Theorem 2.3.** There is a one-one correspondence between \( \text{USR}(\mathcal{C}', \mathcal{C}, \mathcal{C}'') \) and \( \text{LSR}(\mathcal{C}'', \mathcal{C}, \mathcal{C}') \).

**Proof.** Given an upper-symmetric recollement (2.1), observe that an upper-symmetric version (2.2) of (2.1) is lower-symmetric: in fact, (2.1) could be a lower-symmetric version of (2.2). Similarly, a lower-symmetric recollement could be an upper-symmetric version of a lower-symmetric version of itself. Thus by Lemma 2.2 we get a one-one correspondence between \( \text{USR}(\mathcal{C}', \mathcal{C}, \mathcal{C}'') \) and \( \text{LSR}(\mathcal{C}'', \mathcal{C}, \mathcal{C}') \). □

2.4. We consider Artin algebras over a fixed commutative artinian ring, and finitely generated modules. Let \( A \) and \( B \) be Artin algebras, and \( M \) an \( A-B \)-bimodule. Then \( \Lambda = ( \begin{array}{c} \Lambda \\ \frac{A}{B} \end{array} ) \) is an Artin algebra with multiplication given by the one of matrices. Denoted by \( A \)-mod the category of finitely generated left \( A \)-modules. A left \( \Lambda \)-module is identified with a triple \( ( X, Y, f ) \), or simply \( ( X ) \) if \( \phi \) is clear, where \( X \in A \)-mod, \( Y \in B \)-mod, and \( \phi : M \otimes_B Y \rightarrow X \) is an \( A \)-map. A \( \Lambda \)-map \( ( X ) \rightarrow ( Y ) \) is identified with a pair \( ( f, g ) \), where \( f \in \text{Hom}_A(X, X') \), \( g \in \text{Hom}_B(Y, Y') \), such that \( \phi f ( \text{Id} \otimes g) = f \phi \). The indecomposable projective \( \Lambda \)-modules are exactly \( ( \begin{array}{c} \Lambda \\ X \end{array} ) \) and \( ( \frac{M}{Q} \otimes_B P \otimes Q, \text{id} ) \), where \( P \) runs over indecomposable projective \( A \)-modules, and \( Q \) runs over indecomposable projective \( B \)-modules. See [ARS], p.73.

For any \( A \)-module \( X \) and \( B \)-module \( Y \), denote by \( \alpha_{X,Y} \) the adjoint isomorphism

\[
\alpha_{X,Y} : \text{Hom}_A(M \otimes_B Y, X) \rightarrow \text{Hom}_B(Y, \text{Hom}_A(M, X))
\]
given by
\[ \alpha_{X,Y}(\phi)(y)(m) = \phi(m \otimes y), \forall \phi \in \text{Hom}_A(M \otimes_B Y, X), \, y \in Y, \, m \in M. \]

Put \( \psi_X \) to be \( \alpha_{X,\text{Hom}(M,X)}^{-1}(\text{Id}_{\text{Hom}(M,X)}) \). Thus \( \psi_X : M \otimes_B \text{Hom}_A(M,X) \to X \) is given by \( m \otimes f \mapsto f(m) \).

**Theorem 2.4.** Let \( A \) and \( B \) be Artin algebras, \( A_M \) an \( A-B \)-bimodule, and \( \Lambda = (\begin{smallmatrix} A & M \\ M & B \end{smallmatrix}) \). Then we have an upper-symmetric (but non lower-symmetric) abelian category recollement

\[
\begin{array}{ccc}
A\text{-mod} & \xrightarrow{i^*} & \Lambda\text{-mod} & \xleftarrow{i^!} \& B\text{-mod} \\
\xrightarrow{j_*} & \Lambda\text{-mod} & \xleftarrow{j^*} \& B\text{-mod}
\end{array}
\]

(2.6)

where

- \( i^* \) is given by \( \left( \begin{smallmatrix} X \\ \phi \end{smallmatrix} \right)_\phi \mapsto \text{Coker}\phi; \) \( i_* \) is given by \( X \mapsto \left( \begin{smallmatrix} X \\ 0 \end{smallmatrix} \right); \) \( i^! \) is given by \( \left( \begin{smallmatrix} X \\ \phi \end{smallmatrix} \right)_\phi \mapsto X; \)

- \( j_i \) is given by \( Y \mapsto \left( \begin{smallmatrix} M \otimes Y \\ \phi \end{smallmatrix} \right) \text{Id}; \) \( j^* \) is given by \( \left( \begin{smallmatrix} X \\ \phi \end{smallmatrix} \right)_\phi \mapsto Y; \) \( j_* \) is given by \( Y \mapsto \left( \begin{smallmatrix} Y \\ \phi \end{smallmatrix} \right); \)

- \( j_! \) is given by \( \left( \begin{smallmatrix} X \\ \phi \end{smallmatrix} \right)_\phi \mapsto \text{Ker}\alpha_{X,Y}(\phi); \) and \( i_! \) is given by \( X \mapsto \left( \begin{smallmatrix} X \\ \phi \end{smallmatrix} \right)_{\psi_X}. \)

**Proof.** By construction \( i_*, \ j_! \) and \( j_* \) are fully faithful; \( \text{Im} i_* = \text{Ker} j^* \), and \( \text{Im} j_* = \text{Ker} i^! \). For \( \left( \begin{smallmatrix} X \\ \phi \end{smallmatrix} \right) \in \Lambda\text{-mod}, \ X' \in A\text{-mod}, \) and \( Y' \in B\text{-mod}, \) we have the following isomorphisms of abelian groups, which are natural in both positions

\[
\text{Hom}_A(\text{Coker}\phi, X') \cong \text{Hom}_A(\left( \begin{smallmatrix} X \\ \phi \end{smallmatrix} \right), \left( \begin{smallmatrix} X' \\ 0 \end{smallmatrix} \right)) \tag{2.7}
\]

given by \( f \mapsto \left( \begin{smallmatrix} f \pi \\ 0 \end{smallmatrix} \right) \), where \( \pi : X \to \text{Coker}\phi \) is the canonical \( A \)-map;

\[
\text{Hom}_A(\left( \begin{smallmatrix} X' \\ 0 \end{smallmatrix} \right), \left( \begin{smallmatrix} X \\ \phi \end{smallmatrix} \right)) \cong \text{Hom}_A(X', X); \tag{2.8}
\]

\[
\text{Hom}_A\left( \left( \begin{smallmatrix} M \otimes Y' \\ \phi \end{smallmatrix} \right) \text{Id}, \left( \begin{smallmatrix} X \\ \phi \end{smallmatrix} \right) \right) \cong \text{Hom}_B(Y', Y) \tag{2.9}
\]

given by \( \left( \phi(\text{Id} \otimes g) \right) \mapsto g; \) and

\[
\text{Hom}_B(Y, Y') \cong \text{Hom}_A(\left( \begin{smallmatrix} X \\ \phi \end{smallmatrix} \right), \left( \begin{smallmatrix} Y' \\ 0 \end{smallmatrix} \right)).
\]

Thus \( (i^*, i_*), \ (i_*, i'^!), \ (j_i, j^*), \) and \( (j_!, j_*) \) are adjoint pairs, and hence (2.6) is a recollement. It is not lower-symmetric since \( \text{Im} j_! \neq \text{Ker} i^! \).

In order to see that it is upper-symmetric, it remains to prove that \( (j_!, j^!) \) and \( (i', i^!) \) are adjoint pairs, and that \( i^! \) is fully faithful. For \( g \in \text{Hom}_B(Y, Y') \) and \( \left( \begin{smallmatrix} X' \\ \phi \end{smallmatrix} \right) \in \Lambda\text{-mod}, \) we have

\[
\left( \begin{smallmatrix} 0 \\ g \end{smallmatrix} \right) \in \text{Hom}_A(\left( \begin{smallmatrix} 0 \\ X' \end{smallmatrix} \right), \left( \begin{smallmatrix} X \\ \phi \end{smallmatrix} \right)) \iff \phi'(\text{Id} \otimes g) = 0 \iff \phi'(m \otimes g(y)) = 0, \forall \ y \in Y, \forall \ m \in M
\]
\[
\iff \alpha_{X',Y'}(\phi')(g(y)) = 0, \forall \ y \in Y \iff g(Y) \subseteq \text{Ker} \alpha_{X',Y'}(\phi') \iff g \in \text{Hom}_B(Y, \text{Ker} \alpha_{X',Y'}(\phi')).
\]

It follows that \( \left( \begin{smallmatrix} 0 \\ g \end{smallmatrix} \right) \mapsto g \) gives an isomorphism \( \text{Hom}_A(\left( \begin{smallmatrix} 0 \\ X' \end{smallmatrix} \right), \left( \begin{smallmatrix} X \\ \phi \end{smallmatrix} \right)) \to \text{Hom}_B(Y, \text{Ker} \alpha_{X',Y'}(\phi')) \) of abelian groups, which is natural in both positions, i.e., \( (j_!, j^!) \) is an adjoint pair. Let \( \left( \begin{smallmatrix} f \\ g \end{smallmatrix} \right) \in \text{Hom}_A(\left( \begin{smallmatrix} X' \\ \phi \end{smallmatrix} \right), \left( \begin{smallmatrix} X \\ \phi \end{smallmatrix} \right)_{\psi_X}). \) By \( \psi_X(\text{Id} \otimes g) = f \phi \) we have

\[
\alpha_{X',Y'}(f \phi)(y)(m) = f \phi(m \otimes y) = \psi_X(\text{Id} \otimes g)(m \otimes y)
\]
\[
= \psi_X(m \otimes g(y)) = g(y)(m), \forall \ y \in Y, \forall \ m \in M,
\]
which means \( g = \alpha_{X',\psi}(f\phi) \). Thus \( f \mapsto (\alpha_{X',\psi}(f\phi)) \) gives an isomorphism

\[
\text{Hom}_A(X, X') \rightarrow \text{Hom}_A((\psi_X)^{-1}((X)'_{\phi})),
\]

of abelian groups, which is natural in both positions, i.e., \((i^! , i^?)\) is an adjoint pair. Since \( \alpha_{X',\text{Hom}(M,X)}(f\psi_X) = \text{Hom}_A(M,f) \), this isomorphism also shows that \( i^? \) is fully faithful. This completes the proof.

By Theorem 2.4 we have

**Corollary 2.5.** Let \( A \) be a Gorenstein algebra, and \( T_2(A) = (\begin{array}{cc} A & A \\ 0 & A \end{array}) \). Then we have an upper-symmetric (but non lower-symmetric) abelian category recollement

\[
\begin{array}{ccc}
A\text{-mod} & \xrightarrow{i_*} & \text{T}_2(A)\text{-mod} & \xleftarrow{i^!} & A\text{-mod} \\
\end{array}
\]

**Remark 2.6.** As we see from (2.6) and its upper symmetric version, in an abelian category recollement, the following statement may not be true:

1. \( \text{Im} j_! = \text{Ker} i^*; \ \text{Im} j_* = \text{Ker} i^!; \)

2. The counits and units give rise to exact sequences of natural transformations:

\[
0 \rightarrow j_! j^* \rightarrow \text{Id}_C \rightarrow i_* i^* \rightarrow 0 \quad \text{and} \quad 0 \rightarrow i_* i^! \rightarrow \text{Id}_C \rightarrow j_* j^* \rightarrow 0.
\]

3. \( i^! j_! = 0; \text{ and } i^* j_* = 0. \)

In triangulated situations, (1) and the corresponding version of (2) always hold; but (3) is also not true in general.

### 3. Symmetric recollements induced by Gorenstein-projective modules

#### 3.1. Let \( A \) be an Artin algebra. An \( A \)-module \( G \) is \emph{Gorenstein-projective}, if there is an exact sequence \( \cdots \rightarrow P^{-1} \rightarrow P^0 \xrightarrow{d^0} P^1 \rightarrow \cdots \) of projective \( A \)-modules, which stays exact under \( \text{Hom}_A(-, A) \), and such that \( G \cong \text{Ker} d^0 \). Let \( A\text{-Gproj} \) be the full subcategory of \( A\text{-mod} \) consisting of the Gorenstein-projective modules. Then \( A\text{-Gproj} \subseteq A\text{-proj} \), where \( A\text{-proj} = \{ X \in A\text{-mod} | \text{Ext}_i^A(X,A) = 0, \forall i \geq 1 \}; \) and \( \text{Hom}_A(-, A) \) induces a duality \( A\text{-Gproj} \cong A\text{-proj} \) with a quasi-inverse \( \text{Hom}_A(-, A) \) ([B], Proposition 3.4). An important feature is that \( A\text{-Gproj} \) is a Frobenius category with projective-injective objects being projective \( A \)-modules, and hence the stable category \( A\text{-Gproj} \) modulo projective \( A \)-modules is a triangulated category ([Hap]).

An Artin algebra \( A \) is \emph{Gorenstein}, if \( \text{inj.dim } A < \infty \) and \( \text{inj.dim } A \) \( < \infty \). We have the following well-known fact (E. Enochs - O. Jenda [EJ], Corollary 11.5.3).

**Lemma 3.1.** Let \( A \) be a Gorenstein algebra. Then

1. If \( P^\bullet \) is an exact sequence of projective left (resp. right) \( A \)-modules, then \( \text{Hom}_A(P^\bullet, A) \) is again an exact sequence of projective right (resp. left) \( A \)-modules.
(2) A module $G$ is Gorenstein-projective if and only if there is an exact sequence $0 \to G \to P^0 \to P^1 \to \cdots$ with each $P^i$ projective.

(3) $A\text{-Gproj} = \perp A$.

**Proof.** For convenience we include an alternating proof.

1. Let $0 \to K \to I_0 \to I_1 \to 0$ be an exact sequence with $I_0, I_1$ injective modules. Then $0 \to \text{Hom}_A(P^*, K) \to \text{Hom}_A(P^*, I_0) \to \text{Hom}_A(P^*, I_1) \to 0$ is an exact sequence of complexes. Since $\text{Hom}_A(P^*, I_i)$ ($i = 0,1$) are exact, it follows that $\text{Hom}_A(P^*, K)$ is exact. Repeating this process, by $\text{inj.dim }_A A < \infty$ we deduce that $\text{Hom}_A(P^*, A)$ is exact.

2. This follows from definition and (1).

3. Let $G \in \perp A$. Applying $\text{Hom}_A(-, A)$ to a projective resolution of $G$ we get an exact sequence. By (2) this means that $\text{Hom}_A(G, A)$ is a Gorenstein-projective right $A$-module, and hence $G$ is Gorenstein-projective by the duality $\text{Hom}_A(-, _A A): A\text{-Gproj} \cong A^{\text{op}}\text{-Gproj}$.

We need the following description of Gorenstein-projective $\Lambda$-modules.

**Proposition 3.2.** Let $A$ and $B$ be Gorenstein algebras, $M$ an $A$-$B$-bimodule such that $AM$ and $MB$ are projective, and $\Lambda = (A \ M B)$. Then $(\begin{smallmatrix} A & M \\ \phi & B \end{smallmatrix})$ is a Gorenstein-projective $\Lambda$-module if and only if $\phi: M \otimes Y \to X$ is monic, $X$ and $\text{Coker } \phi$ are Gorenstein-projective $\Lambda$-modules, and $Y$ is a Gorenstein-projective $B$-module. In this case $M \otimes Y$ is a Gorenstein-projective $A$-module.

**Proof.** If $(\begin{smallmatrix} A & M \\ \phi & B \end{smallmatrix})$ is a Gorenstein-projective $\Lambda$-module, then there is an exact sequence

$$0 \to (\begin{smallmatrix} A & M \\ \phi & B \end{smallmatrix}) \to \left(\begin{smallmatrix} P_{0} \oplus (M \otimes Q_{0}) \\ Q_0 \end{smallmatrix}\right)_{\phi_{0}} \to \left(\begin{smallmatrix} P_{1} \oplus (M \otimes Q_{1}) \\ Q_1 \end{smallmatrix}\right)_{\phi_{1}} \to \cdots$$

(3.1)

where $P_i$ and $Q_i$ are respectively projective $A$- and $B$-modules, $i \geq 0$, i.e., we have exact sequences

$$0 \to X \to P_{0} \oplus (M \otimes Q_{0}) \to P_{1} \oplus (M \otimes Q_{1}) \to \cdots$$

(3.2)

and

$$0 \to Y \to Q_{0} \to Q_{1} \to \cdots$$

(3.3)

such that the following diagram commutes

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M \otimes_{B} Y & \longrightarrow & M \otimes_{B} Q_{0} & \longrightarrow & M \otimes_{B} Q_{1} & \longrightarrow & \cdots \\
\downarrow{\phi} & & \downarrow{(\phi_{0})} & & \downarrow{(\phi_{1})} & & & & \\
0 & \longrightarrow & X & \longrightarrow & P_{0} \oplus (M \otimes Q_{0}) & \longrightarrow & P_{1} \oplus (M \otimes Q_{1}) & \longrightarrow & \cdots
\end{array}$$

(3.4)

By Lemma 3.1(2) $Y$ is Gorenstein-projective. Since $AM$ and $BM$ are projective, it follows that $M \otimes Q_i$ are projective $A$-modules, and hence $X$ is Gorenstein-projective by Lemma 3.1(2). Since $M_B$ is projective, by (3.3) the upper row of (3.4) is exact, and hence $M \otimes Y$ is Gorenstein-projective and $\phi$ is monic. By (3.4) we get exact sequence $0 \to \text{Coker } \phi \to P_{0} \to P_{1} \to \cdots$, thus $\text{Coker } \phi$ is Gorenstein-projective by Lemma 3.1(2).
Conversely, we have exact sequence (3.3) with \(Q_i\) being projective \(B\)-modules. Since \(M_B\) is projective and \(\text{Coker } \phi\) is Gorenstein-projective, we get the following exact sequences

\[
0 \to M \otimes Y \to M \otimes Q_0 \to M \otimes Q_1 \to \cdots
\]

\[
0 \to \text{Coker } \phi \to P_0 \to P_1 \to \cdots
\]

with \(P_i\) projective. Since \(M \otimes Q_i\) \((i \geq 0)\) are projective \(A\)-modules and projective \(A\)-modules are injective objects in \(\text{A-Gproj}\), it follows from the exact sequence \(0 \to M \otimes Y \to X \to \text{Coker } \phi \to 0\) and a version of Horseshoe Lemma that there is an exact sequence (3.2) such that the diagram (3.4) commutes. This means that (3.1) is exact. Since \(\Lambda\) is also Gorenstein (see e.g. [C], Theorem 3.3), it follows from Lemma 3.1(2) that \((\frac{X}{\lambda})_\phi\) is a Gorenstein-projective \(\Lambda\)-module.

3.2. The main result of this section is as follows.

**Theorem 3.3.** Let \(A\) and \(B\) be Gorenstein algebras, \(M\) an \(A\)-\(B\)-bimodule such that \(\Lambda M\) and \(M_B\) are projective, and \(\Lambda = (\Lambda^A M_B)\). Then we have a triangulated category recollement

\[
\begin{array}{ccc}
\text{A-Gproj} & \xrightarrow{i^*} & \Lambda \text{-proj} & \xleftarrow{i_*} & \text{B-Gproj} \\
\end{array}
\]

Moreover, if \(A\) and \(B\) are in additional finite-dimensional algebras over a field, then it is a symmetric recollement.

3.3. Before giving a proof, we construct all the functors in Theorem 3.3. If a \(\Lambda\)-map \((\frac{X}{\lambda})_\phi \to (\frac{X'}{\lambda'})_\phi\) factors through a projective \(\Lambda\)-module \((\frac{P}{\sigma}) \oplus (\frac{M \otimes Q}{\phi})\), then it is easy to see that the induced \(\Lambda\)-map \(\text{Coker } \phi \to \text{Coker } \phi\) factors through \(P\). By Proposition 3.2 this implies that the functor \(\Lambda \text{-proj} \to \text{A-Gproj}\) given by \((\frac{X}{\lambda})_\phi \mapsto \text{Coker } \phi\) induces a functor \(i^* : \text{A-Gproj} \to \text{A-Gproj}\).

By Proposition 3.2 there is a unique functor \(i_* : \text{A-Gproj} \to \Lambda \text{-proj} \) given by \(X \mapsto (\frac{X}{\lambda})_\phi\), which is fully faithful.

If a \(\Lambda\)-map \((\frac{f}{g}) : (\frac{X}{\lambda})_\phi \to (\frac{X'}{\lambda'})_\phi\) factors through a projective \(\Lambda\)-module \((\frac{P}{\sigma}) \oplus (\frac{M \otimes Q}{\phi})\), then \(f : X \to X'\) factors through a projective \(A\)-module \(P \oplus (M \otimes \phi)\). By Proposition 3.2 this implies that there is a unique functor \(i' : \text{A-Gproj} \to \text{A-Gproj}\) given by \((\frac{X}{\lambda})_\phi \mapsto X\).

By Proposition 3.2 there is a unique functor \(j^* : \text{B-Gproj} \to \Lambda \text{-proj}\) given by \((\frac{X}{\lambda})_\phi \mapsto Y\).

Let \(\text{B-Y}\) be a Gorenstein-projective module. Since \(M_B\) is projective, by Lemma 3.1(2) \(M \otimes Y\) is a Gorenstein-projective \(\Lambda\)-module. By Proposition 3.2 there is a unique functor \(j_* : \text{B-Gproj} \to \text{A-Gproj}\) given by \(Y \mapsto (\frac{M \otimes Y}{\lambda'})_\phi\), which is fully faithful.

**Lemma 3.4.** Let \(A, B, M,\) and \(\Lambda\) be as in Theorem 3.3. Then there exists a unique fully faithful functor \(j_* : \text{B-Gproj} \to \Lambda \text{-proj}\) given by \(Y \mapsto (\frac{Y}{\lambda'})_\phi\), where \(P\) is a projective \(A\)-module such that there is an exact sequence \(0 \to M \otimes Y \to P \to \text{Coker } \sigma \to 0\) with \(\text{Coker } \sigma \in \text{A-Gproj}\).

**Proof.** Let \(\text{B-Y}\) be Gorenstein-projective. Then \(M \otimes Y\) is Gorenstein-projective, and hence there is an exact sequence \(0 \to M \otimes Y \to P \to \text{Coker } \sigma \to 0\) with \(P\) projective and \(\text{Coker } \sigma \in \text{A-Gproj}\). Let \(g : Y \to Y'\) be a \(B\)-map with \(Y, Y' \in \text{B-Gproj}\), and \(P'\) a projective \(A\)-module such that...
0 \to M \otimes Y' \xrightarrow{\sigma'} P' \to \text{Coker}\sigma' \to 0 \text{ is exact with } \text{Coker}\sigma' \in A\text{-proj}. \text{ Since projective } A\text{-modules are injective objects in } A\text{-proj}, \text{ it follows that there is a commutative diagram}

\[
\begin{array}{ccccccccc}
0 & \to & M \otimes Y & \xrightarrow{\sigma} & P & \xrightarrow{\pi} & \text{Coker}\sigma & \to & 0 \\
\downarrow{1 \otimes g} & & \downarrow{f} & & \downarrow{} & & \downarrow{} & & \\
0 & \to & M \otimes Y' & \xrightarrow{\sigma'} & P' & \to & \text{Coker}\sigma' & \to & 0.
\end{array}
\]

Taking \( g = \text{Id} \) we see \( (\sigma')_\sigma \cong (\sigma')_{\sigma'} \) in \( A\text{-proj} \). If we have another map \( f' : P \to P' \) such that \( f' \sigma = \sigma'(1 \otimes g) \), then \( f - f' \) factors through \( \text{Coker}\sigma \). Since \( \text{Coker}\sigma \in A\text{-proj} \), we have a monomorphism \( \tilde{\sigma} : \text{Coker}\sigma \to \hat{P} \) with \( \hat{P} \) projective. Then we easily see that \( (\sigma)_{\sigma} - (\sigma')_{\sigma} \) factors through projective \( \Lambda\) module \( (\tilde{P})_0 \), and hence \( (\sigma)_{\sigma} = (\sigma')_{\sigma} \). Thus we get a unique functor \( j_* : B\text{-proj} \to A\text{-proj} \) given by \( Y \mapsto (\sigma')_\sigma \) and \( g \mapsto (\sigma)_{\sigma} \).

Assume that \( g : Y \to Y' \) factors through a projective module \( bQ \) with \( g = g_2g_1 \). Since \( M \otimes Q \) is projective and hence an injective object in \( A\text{-proj} \), there is an \( A\)-map \( \alpha : P \to M \otimes Q \) such that \( 1 \otimes g_1 = \alpha \sigma \). Since \( (f - \sigma'(1 \otimes g_2)\alpha) \sigma = 0 \), there is an \( A\)-map \( \tilde{f} : \text{Coker}\sigma \to P' \) such that \( \tilde{f} \pi = f - \sigma'(1 \otimes g_2)\alpha \). Let \( \tilde{\sigma} : \text{Coker}\sigma \to \hat{P} \) be a monomorphism with \( \hat{P} \) projective. Then we get an \( A\)-map \( \beta : \hat{P} \to P' \) such that \( \tilde{f} = \beta \tilde{\sigma} \). Thus \( (\sigma)_{\sigma} \) factors through projective \( \Lambda\) module \( (M \otimes Q)_0 \oplus (\tilde{P})_0 \) with \( (\sigma)_{\sigma} = \left( \begin{array}{c} (\sigma(1 \otimes g_1), \beta) \\ (\tilde{\sigma} \pi) \end{array} \right) \). Therefore \( j_* : B\text{-proj} \to A\text{-proj} \) induces a functor \( B\text{-proj} \to A\text{-proj} \), again denoted by \( j_* \), which is given by \( Y \mapsto (\sigma')_\sigma \) and \( g \mapsto (\sigma)_{\sigma} \).

By the above argument we know that \( j_* \) is full. If \( (\sigma)_{\sigma} \) factors through projective \( \Lambda\) module \( (M \otimes Q)_0 \oplus (\tilde{P})_0 \), then \( g \) factors through projective module \( bQ \). Thus \( j_* \) is faithful. \( \blacksquare \)

3.4. Let \( \mathcal{A} \) be a Frobenius category and \( \underline{\mathcal{A}} \) the corresponding stable category. Then \( \underline{\mathcal{A}} \) is a triangulated category with shift functor \([1] \) given by \( X[1] = \text{Coker}(X \to I(X)) \), where \( I(X) \) is a projective-injective object of \( \mathcal{A} \); each exact sequence \( 0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0 \) in \( \mathcal{A} \) gives rise to a distinguished triangle \( X \xrightarrow{u} Y \xrightarrow{v} Z \to \text{in } \underline{\mathcal{A}} \), and each distinguished triangle in \( \underline{\mathcal{A}} \) is of this form up to an isomorphism. See D. Happel [H], Chapter 1, Section 2. It follows that we have

**Lemma 3.5.** All the functors \( i^* \), \( i_* \), \( i! \), \( j^* \), \( j_* \), \( j! \) constructed above are exact functors; and \( i_* \), \( j_* \) are full.

3.5. **Proof of Theorem 3.3.** By construction \( \text{Ker} j^* = \{(\tilde{X})_\phi \in A\text{-proj} | bQ \text{ is projective} \} \).

By Proposition 3.2 there is an exact sequence \( 0 \to M \otimes Q \xrightarrow{\phi} X \to \text{Coker}\phi \to 0 \) in \( A\text{-proj} \). Since \( M \otimes Q \) is a projective \( A\) module, and hence an injective object in \( A\text{-proj} \), it follows that \( \phi \) splits and then \( (\tilde{X})_\phi \cong (M \otimes Q)_0 \oplus (X'_0) \) in \( A\text{-proj} \). Thus \( \text{Im} i_* = \text{Ker} j^* \).

In the following \( (\tilde{X})_\phi \in A\text{-proj} \), \( X' \in A\text{-proj} \), and \( Y' \in B\text{-proj} \).

It is easy to see that a \( A\)-map \( (\tilde{f}) : (\tilde{X})_\phi \to (X'_0) \) factors through a projective \( A\)-module if and only if the induced \( A\)-map \( \text{Coker}\phi \to X' \) factors through a projective \( A\)-module. This implies that the isomorphism (2.7) induces the following isomorphism, which are natural in both positions

\[ \text{Hom}_{A\text{-proj}}((\tilde{X})_\phi, (X'_0)) \cong \text{Hom}_{A\text{-proj}}(\text{Coker}\phi, X') \],

where \( \text{Hom}_{A\text{-proj}} \) denotes the morphisms in the category of \( A\)-modules.
i.e., \((i^*, i_*)\) is an adjoint pair.

It is easy to see that a \(\Lambda\)-map \(\left( \begin{array}{c} f \\ 0 \end{array} \right) : (X'_0) \to (X)_{\phi}\) factors through a projective \(\Lambda\)-module if and only if \(f : X' \to X\) factors through a projective \(A\)-module. This implies that the isomorphism (2.8) induces the following isomorphism, which are natural in both positions

\[
\Hom_{\text{-proj}}(X'_0, (X)_{\phi}) \cong \Hom_{\text{-proj}}(X', X),
\]

i.e., \((i^*, i_*)\) is an adjoint pair.

Note that \(M \otimes Q\) is a projective \(A\)-module for any projective \(B\)-module \(Q\). It is easy to see that a \(\Lambda\)-map \(\left( \begin{array}{c} \phi(\text{Id}_{\Lambda} \otimes g) \\ g \end{array} \right) : \left( \begin{array}{c} M \otimes Y' \\ Y \end{array} \right)_{\phi} \to \left( \begin{array}{c} X \\ Y \end{array} \right)_{\phi}\) factors through a projective \(\Lambda\)-module if and only if \(g : Y' \to Y\) factors through a projective \(B\)-module. This implies that the isomorphism (2.9) induces the following isomorphism, which are natural in both positions

\[
\Hom_{\text{-proj}}(M \otimes Y', (X)_{\phi}) \cong \Hom_{\text{-proj}}(Y', Y),
\]

i.e., \((j^*, j_*)\) is an adjoint pair.

Let \(\left( \begin{array}{c} g \\ \sigma \end{array} \right) : \left( \begin{array}{c} X \\ Y \end{array} \right)_{\phi} \to \left( \begin{array}{c} Y' \\ X' \end{array} \right)_{\phi}\) be a \(\Lambda\)-map, \(0 \to M \otimes Y' \to P' \to \text{Coker}\sigma \to 0\) an exact sequence with \(P'\) projective and \(\text{Coker}\sigma \in \text{proj}\). In the proof of Lemma 3.4 we know that \(\left( \begin{array}{c} g \\ \sigma \end{array} \right)\) factors through a projective \(\Lambda\)-module if and only if \(g : Y \to Y'\) factors through a projective \(B\)-module. This implies that the map \(g \mapsto \left( \begin{array}{c} g \\ \sigma \end{array} \right)\) gives rise to the following isomorphism, which is natural in both positions

\[
\Hom_{\text{-proj}}(\left( \begin{array}{c} X \\ Y \end{array} \right)_{\phi}, (X')_{\phi}) \cong \Hom_{\text{-proj}}(Y, Y'),
\]

i.e., \((j^*, j_*)\) is an adjoint pair. Now the first assertion follows from Lemmas 3.5 and 1.3.

Assume that \(A\) and \(B\) are in additional finite-dimensional algebras over a field \(k\). Note that \(\Lambda\text{-proj}\) is a resolving subcategory of \(\Lambda\text{-mod}\) (see e.g. Theorem 2.5 in [Hol]). Since \(\Lambda\) is a Gorenstein algebra, it is well-known that \(\Lambda\text{-proj}\) contravariantly finite in \(\Lambda\text{-mod}\) (see Theorem 11.5.1 in [EJ], where the result is stated for arbitrary \(\Lambda\)-modules, but the proof holds also for finitely generated modules. See also Theorem 2.10 in [Hol]). Then by Corollary 0.3 of H. Krause and O. Solberg [KS], which asserts that a resolving contravariantly finite subcategory in \(\Lambda\text{-mod}\) is also covariantly finite in \(\Lambda\text{-mod}\), \(\Lambda\text{-proj}\) is functorially finite in \(\Lambda\text{-mod}\), and hence \(\Lambda\text{-proj}\) has Auslander-Reiten sequences, by Theorem 2.4 of M. Auslander and S. O. Smalø [AS]. Since each distinguished triangle in the stable category \(A\) of a Frobenius category \(A\) is induced by an exact sequence in \(A\), \(\Lambda\text{-proj}\) has Auslander-Reiten triangles. By assumption \(\Lambda\) is finite-dimensional \(k\)-algebra, thus \(\Lambda\text{-proj}\) is a Hom-finite \(k\)-linear Krull-Schmidt category, and hence by Theorem 1.2.4 of I. Reiten and M. Van den Bergh [RV] \(\Lambda\text{-proj}\) has a Serre functor. Now the second assertion follows from Theorem 7 of P. Jørgensen [J], which claims that any recollement of a triangulated category with a Serre functor is symmetric. 

3.6. By Theorem 3.3 we have

**Corollary 3.6.** Let \(A\) be a Gorenstein algebra, and \(T_2(A) = \left( \begin{array}{c} A \\ 0 \end{array} \right)\). Then we have a recollement of triangulated categories
\[ A\text{-Gproj} \quad \mathbb{T}_{2}(A)\text{-Gproj} \quad A\text{-Gproj}; \]

and it is symmetric if \(A\) and \(B\) are finite-dimensional algebras over a field.

For the first part of Corollary 3.6 see also Theorem 3.8 in [IKM].

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