Geometry of Generalized Coherent States : Some Calculations of Chern Characters

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Abstract

This is a continuation of the preceding paper (hep–ph/0108219).

First of all we make a brief review of generalized coherent states based on Lie algebra su(1,1) and prove that the resolution of unity can be obtained by the curvature form of some bundle.

Next for a set of generalized coherent states we define a universal bundle over the infinite–dimensional Grassmann manifold and construct the pull–back bundle making use of a projector from the parameter space to this Grassmann one. We mainly study Chern characters of these bundles.

Although the Chern characters in the infinite–dimensional case are in general not easy to calculate, we can perform them for the special cases. In this paper we report our calculations and propose some problems.

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1 Introduction

Coherent states or generalized coherent states play very important role in quantum physics, in particular, quantum optics, see [1] and its references or [2]. They also play an important one in mathematical physics. See the book [3]. For example, they are very useful in performing stationary phase approximations to path integral, see [4], [5] and [6].

This is a continuation of the preceding paper [7]. Namely, since we have studied coherent states from the geometric point of view, we continue to study generalized coherent states from the same point of view. For a set of generalized coherent states we can define a projector from the manifold consisting of parameters of them to infinite–dimensional Grassmann manifold. Making use of this we can calculate several geometric quantities, see for example [8].

In this paper we mainly focus on Chern characters because they play an very important role in global geometry. But their calculations are not so easy. Our calculations are only $m = 1, 2$ (see the section 3). Even the case $m = 2$ the calculations are very complicated. We must leave the case $m = 3$ to the readers with high powers.

The hidden aim of this paper and [7] is to apply the results to Quantum Information Theory (QIT) including Quantum Computation (QC) · · · a geometric construction of QIT. As for QC or QIT see [9], [10] and [11] for general introduction. We are in particular interested in Holonomic Quantum Computation, see [12]–[16]. We are also interested in Homodyne Tomography [17], [18] or Quantum Cryptgraphy [19], [20]. In sequel papers we will discuss these points.

By the way it seems to the author that our calculations suggest some deep relation to recent non–commutative differential geometry or non–commutative field theory, see [21] or [22]. But this is beyond the scope of this paper.

The author expects strongly that young mathematical physicists or information theorists will enter to this fruitful field.
2 Coherent States and Generalized Coherent Ones

2.1 Coherent States

We make a brief review of some basic properties of coherent operators within our necessity, [4] and [3].

Let \( a(a^\dagger) \) be the annihilation (creation) operator of the harmonic oscillator. If we set \( N \equiv a^\dagger a \) (number operator), then

\[
\begin{align*}
[N, a^\dagger] &= a^\dagger, \\
[N, a] &= -a, \\
[a^\dagger, a] &= -1.
\end{align*}
\]

(1)

Let \( \mathcal{H} \) be a Fock space generated by \( a \) and \( a^\dagger \), and \( \{ |n\rangle | n \in \mathbb{N} \cup \{0\} \} \) be its basis. The actions of \( a \) and \( a^\dagger \) on \( \mathcal{H} \) are given by

\[
\begin{align*}
a|n\rangle &= \sqrt{n}|n-1\rangle, \\
a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle, \\
N|n\rangle &= n|n\rangle
\end{align*}
\]

(2) where \( |0\rangle \) is a normalized vacuum (\( a|0\rangle = 0 \) and \( \langle 0|0\rangle = 1 \)). From (2) state \( |n\rangle \) for \( n \geq 1 \) are given by

\[
|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle.
\]

(3)

These states satisfy the orthogonality and completeness conditions

\[
\langle m|n\rangle = \delta_{mn}, \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1.
\]

(4)

Let us state coherent states. For the normalized state \( |z\rangle \in \mathcal{H} \) for \( z \in \mathbb{C} \) the following three conditions are equivalent :

\[
\begin{align*}
(i) \quad &a|z\rangle = z|z\rangle \quad \text{and} \quad \langle z|z\rangle = 1 \\
(ii) \quad &|z\rangle = e^{-|z|^2/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle = e^{-|z|^2/2} e^{za^\dagger} |0\rangle \\
(iii) \quad &|z\rangle = e^{z a^\dagger - \bar{z} a} |0\rangle.
\end{align*}
\]

(5) (6) (7)

In the process from (5) to (6) we use the famous elementary Baker-Campbell-Hausdorff formula

\[
e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B
\]

(8)
whenever \([A, [A, B]] = [B, [A, B]] = 0\), see \([1]\) or \([3]\). This is the key formula.

**Definition** The state \(|z\rangle\) that satisfies one of (i) or (ii) or (iii) above is called the coherent state.

The important feature of coherent states is the following partition (resolution) of unity.

\[
\int_{\mathbb{C}} \frac{|d^2 z|}{\pi} |z\rangle\langle z| = \sum_{n=0}^{\infty} |n\rangle\langle n| = 1 ,
\]

where we have put \([d^2 z] = d(\text{Re} z)d(\text{Im} z)\) for simplicity. We note that

\[
\langle z|w\rangle = e^{-\frac{1}{2}|z|^2 - \frac{1}{2}|w|^2 + \bar{z}w} \Rightarrow |\langle z|w\rangle| = e^{-\frac{1}{2}|z-w|^2}, \quad \langle w|z\rangle = \overline{\langle z|w\rangle},
\]

so \(|\langle z|w\rangle| < 1\) if \(z \neq w\) and \(|\langle z|w\rangle| \ll 1\) if \(z\) and \(w\) are separated enough. We will use this fact in the following.

Since the operator

\[
U(z) = e^{za^{\dagger} - \bar{z}a} \quad \text{for} \quad z \in \mathbb{C}
\]

is unitary, we call this a (unitary) coherent operator. For these operators the following properties are crucial. For \(z, w \in \mathbb{C}\)

\[
U(z)U(w) = e^{z\bar{w} - \bar{z}w} \quad U(w)U(z),
\]

\[
U(z + w) = e^{-\frac{1}{2}(z\bar{w} - \bar{z}w)} \quad U(z)U(w).
\]

### 2.2 Generalized Coherent States

Next we make a brief review of some basic properties of generalized coherent operators based on \(su(1, 1)\), see \([4]\) or \([3]\).

We consider a spin \(K\) \((> 0)\) representation of \(su(1, 1) \subset sl(2, \mathbb{C})\) and set its generators \(\{K_+, \bar{K}_-, K_3\} \quad ((K_+)^\dagger = K_-)\),

\[
[K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3.
\]

We note that this (unitary) representation is necessarily infinite dimensional. The Fock space on which \(\{K_+, \bar{K}_-, K_3\}\) act is \(\mathcal{H}_K \equiv \{|K, n\rangle|n \in \mathbb{N} \cup \{0\}\}\) and whose actions are

\[
K_+|K, n\rangle = \sqrt{(n+1)(2K+n)}|K, n+1\rangle,
\]
\[ K_-|K, n\rangle = \sqrt{n(2K + n - 1)}|K, n - 1\rangle, \]
\[ K_3|K, n\rangle = (K + n)|K, n\rangle, \]  
(15)

where \(|K, 0\rangle\) is a normalized vacuum (\(K_-|K, 0\rangle = 0\) and \(\langle K, 0|K, 0\rangle = 1\)). We have written \(|K, 0\rangle\) instead of \(|0\rangle\) to emphasize the spin \(K\) representation, see [1]. We also denote by \(1_K\) the unit operator on \(H_K\). From (15), states \(|K, n\rangle\) are given by

\[ |K, n\rangle = \frac{(K_+)^n}{\sqrt{n!(2K)_n}}|K, 0\rangle, \]  
(16)

where \((a)_n\) is the Pochammer’s notation \((a)_n \equiv a(a + 1) \cdots (a + n - 1)\). These states satisfy the orthogonality and completeness conditions

\[ \langle K, m|K, n\rangle = \delta_{mn}, \sum_{n=0}^{\infty} |K, n\rangle\langle K, n| = 1_K. \]  
(17)

Now let us consider a generalized version of coherent states:

**Definition**  
We call a state

\[ |w\rangle \equiv e^{wK_+ - \bar{w}K_-}|K, 0\rangle \quad \text{for} \quad w \in \mathbb{C}. \]  
(18)

the generalized coherent state (or the coherent state of Perelomov’s type based on \(su(1, 1)\) in our terminology, [24]).

We note that this is the extension of (7) not (5), see [3]. For this the following disentangling formula is well–known:

\[ e^{wK_+ - \bar{w}K_-} = e^{\zeta K_+}e^{\log(1-|\zeta|^2)K_3}e^{-\bar{\zeta}K_-} \quad \text{or} \]
\[ = e^{-\bar{\zeta}K_-}e^{-\log(1-|\zeta|^2)K_3}e^{\zeta K_+}. \]  
(19)

where

\[ \zeta = \zeta(w) \equiv \frac{wtanh(|w|)}{|w|} \quad (\implies |\zeta| < 1). \]  
(20)

This is the key formula for generalized coherent operators. Therefore from (15)

\[ |w\rangle = (1 - |\zeta|^2)^K e^{\zeta K_+}|K, 0\rangle \equiv |\zeta\rangle. \]  
(21)


This corresponds to the right hand side of (6). Moreover since
\[ e^{\zeta K_{+}}|K,0\rangle = \sum_{n=0}^{\infty} \frac{\zeta^n}{n!} K_{+}^n |K,0\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{(2K)_n}{n!}} \frac{\zeta^n K_{+}^n}{(2K)_n n!} |K,0\rangle = \sum_{n=0}^{\infty} \sqrt{\frac{(2K)_n}{n!}} \zeta^n |K,n\rangle \]
we have
\[ |w\rangle = (1 - |\zeta|^2) \sum_{n=0}^{\infty} \sqrt{\frac{(2K)_n}{n!}} \zeta^n |K,n\rangle \equiv |\zeta\rangle. \quad (22) \]
This corresponds to the left hand side of (6). Then the partition of unity corresponding to (9) is
\[ \int_{C} \frac{2K - 1}{\pi} \frac{\tanh(|w|)[d^2w]}{1 - \tanh^2(|w|)} |w\rangle\langle w| = \int_{D} \frac{2K - 1}{\pi} \frac{[d^2\zeta]}{1 - |\zeta|^2} |\zeta\rangle\langle \zeta| = \sum_{n=0}^{\infty} |K,n\rangle\langle K,n| = 1_K, \quad (23) \]
where \( C \to D : w \mapsto \zeta = \zeta(w) \) and \( D \) is the Poincare disk in \( C \), see (23).

Here let us construct the spin \( K \)–representations making use of Schwinger’s boson method.
First we set
\[ K_{+} \equiv \frac{1}{2} (a^\dagger)^2 , \quad K_{-} \equiv \frac{1}{2} a^2 , \quad K_{3} \equiv \frac{1}{2} (a^\dagger a + \frac{1}{2}) , \quad (24) \]
then we have
\[ [K_{3},K_{+}] = K_{+} , \quad [K_{3},K_{-}] = -K_{-} , \quad [K_{+},K_{-}] = -2K_{3} . \quad (25) \]
That is, the set \( \{K_{+},K_{-},K_{3}\} \) gives a unitary representation of \( su(1,1) \) with spin \( K = 1/4 \) and \( 3/4 \). Now we also call an operator
\[ S(w) = e^{\frac{w}{2}(a^\dagger a^\dagger - a^\dagger a)} \quad \text{for} \quad w \in \mathbb{C} \quad (26) \]
the squeezed operator.

Next we consider the system of two-harmonic oscillators. If we set
\[ a_1 = a \otimes 1, \quad a_1^\dagger = a^\dagger \otimes 1; \quad a_2 = 1 \otimes a, \quad a_2^\dagger = 1 \otimes a^\dagger , \quad (27) \]
then it is easy to see
\[ [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0, \quad [a_i, a_j^\dagger] = \delta_{ij}, \quad i, j = 1, 2. \quad (28) \]
We also denote by \( N_i = a_i^\dagger a_i \) number operators.

Now we can construct representation of Lie algebras \( su(2) \) and \( su(1,1) \) making use of Schwinger’s boson method, see [4], [5]. Namely if we set

\[
\begin{align*}
\text{su}(2): & \quad J_+ = a_1^\dagger a_2, \quad J_- = a_2^\dagger a_1, \quad J_3 = \frac{1}{2} (a_1^\dagger a_1 - a_2^\dagger a_2), \\
\text{su}(1,1): & \quad K_+ = a_1^\dagger a_2^\dagger, \quad K_- = a_2 a_1, \quad K_3 = \frac{1}{2} (a_1^\dagger a_1 + a_2^\dagger a_2 + 1),
\end{align*}
\]

then we have

\[
\begin{align*}
\text{su}(2): & \quad [J_3, J_+] = J_+, \quad [J_3, J_-] = -J_-, \quad [J_+, J_-] = 2J_3, \\
\text{su}(1,1): & \quad [K_3, K_+] = K_+, \quad [K_3, K_-] = -K_-, \quad [K_+, K_-] = -2K_3.
\end{align*}
\]

In the following we define (unitary) generalized coherent operators based on Lie algebras \( su(2) \) and \( su(1,1) \).

**Definition** We set

\[
\begin{align*}
\text{su}(2): & \quad W(w) = e^{wJ_+ - \bar{\omega}J_-} \quad \text{for} \quad w \in \mathbb{C}, \\
\text{su}(1,1): & \quad V(w) = e^{wK_+ - \bar{\omega}K_-} \quad \text{for} \quad w \in \mathbb{C}.
\end{align*}
\]

For the latter convenience let us list well-known disentangling formulas once more. We have

\[
\begin{align*}
\text{su}(2): & \quad W(w) = e^{\eta J_+} e^{\log(1 + |\eta|^2)} J_3 e^{-\eta J_-}, \quad \text{where} \quad \eta = \frac{w \tan(|w|)}{|w|}, \\
\text{su}(1,1): & \quad V(w) = e^{\zeta K_+} e^{\log(1 - |\zeta|^2)} K_3 e^{-\zeta K_-}, \quad \text{where} \quad \zeta = \frac{w \tanh(|w|)}{|w|}.
\end{align*}
\]

As for a generalization of these formulas see [25]. Before ending this section let us ask a question.

What is a relation between (34) and (26) of generalized coherent operators based on \( su(1,1) \) ?

The answer is given by Paris [18]:

**Formula** We have

\[
W(-\frac{\pi}{4})S_1(w)S_2(-w)W(-\frac{\pi}{4})^{-1} = V(w),
\]

\( S_1 \) and \( S_2 \) are unitary operators.
where $S_j(w) = (24)$ with $a_j$ instead of $a$.
Namely, $V(w)$ is given by “rotating” the product $S_1(w)S_2(-w)$ by $W(-\frac{\pi}{4})$. This is an interesting relation.

### 2.3 Some Formulas on Generalized Coherent States

We make some preliminaries for the following section. For that we list some useful formulas on generalized coherent states. Since the proofs are not so difficult, we leave them to the readers.

**Formulas** For $w_1, w_2$ we have

(i) $\langle w_1 | w_2 \rangle = \left\{ \frac{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)}{(1 - \bar{\zeta}_1 \zeta_2)^2} \right\}^K , \quad (38)$

(ii) $\langle w_1 | K_+ | w_2 \rangle = \langle w_1 | w_2 \rangle \frac{2K\bar{\zeta}_1}{1 - \bar{\zeta}_1 \zeta_2} , \quad (39)$

(iii) $\langle w_1 | K_- | w_2 \rangle = \langle w_1 | w_2 \rangle \frac{2K\zeta_2}{1 - \bar{\zeta}_1 \zeta_2} , \quad (40)$

(iv) $\langle w_1 | K_- K_+ | w_2 \rangle = \langle w_1 | w_2 \rangle \frac{2K + 4K^2\bar{\zeta}_1 \zeta_2}{(1 - \bar{\zeta}_1 \zeta_2)^2} . \quad (41)$

where

$$\zeta_j = \frac{w_j \tanh(|w_j|)}{|w_j|} \quad \text{for} \quad j = 1, 2. \quad (42)$$

When $w_1 = w_2 \equiv w$, then $\langle w | w \rangle = 1$, so we have

$$\langle w | K_+ | w \rangle = \frac{2K\bar{\zeta}}{1 - |\zeta|^2} , \quad \langle w | K_- | w \rangle = \frac{2K\zeta}{1 - |\zeta|^2} , \quad (43)$$

$$\langle w | K_- K_+ | w \rangle = \frac{2K + 4K^2|\zeta|^2}{(1 - |\zeta|^2)^2} . \quad (44)$$

Here let us make a comment. From (38)

$$|\langle w_1 | w_2 \rangle|^2 = \left\{ \frac{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)}{|1 - \bar{\zeta}_1 \zeta_2|^2} \right\}^{2K} , \quad (38)$$
so we want to know the property of

\[ \frac{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)}{|1 - \zeta_1 \zeta_2|^2} \].

It is easy to see that

\[ 1 - \frac{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)}{|1 - \zeta_1 \zeta_2|^2} = \frac{|\zeta_1 - \zeta_2|^2}{|1 - \zeta_1 \zeta_2|^2} \geq 0 \quad (45) \]

and (45) = 0 if and only if (iff) \( \zeta_1 = \zeta_2 \). Therefore

\[ |\langle w_1 | w_2 \rangle|^2 = \left\{ \frac{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)}{|1 - \zeta_1 \zeta_2|^2} \right\}^{2K} \leq 1 \quad (46) \]

because \( 2K > 1 \) (\( 2K - 1 > 0 \) from (23)). Of course

\[ |\langle w_1 | w_2 \rangle| = 1 \iff \zeta_1 = \zeta_2 \iff w_1 = w_2. \quad (47) \]

by (42).

### 2.4 A Supplement on Generalized Coherent States

Before ending this section let us make a brief comment on generalized coherent states (18). The coherent states \(|z\rangle\) has been defined by (5) : \(a|z\rangle = z|z\rangle\). Why do we define the generalized coherent states \(|w\rangle\) as \(K_-|w\rangle = w|w\rangle\) because \(K_-\) is an annihilation operator corresponding to \(a\) ? First let us try to calculate \(K_-|w\rangle\) making use of (21).

\[ K_-|w\rangle = (1 - |\zeta|^2)^K K_- e^{\zeta K_+}|K, 0\rangle = (1 - |\zeta|^2)^K e^{\zeta K_+} e^{-\zeta K_+} K_- e^{\zeta K_+}|K, 0\rangle. \]

Here it is easy to see

\[ e^{-\zeta K_+} K_- e^{\zeta K_+} = \sum_{n=0}^{\infty} \frac{1}{n!} [\zeta K_+, [-\zeta K_+, [\zeta K_+, \cdots [\zeta K_+, K_- \cdots ]]] = K_- + 2\zeta K_3 + \zeta^2 K_+ , \]

from the relations (14), so that

\[ K_-|w\rangle = (1 - |\zeta|^2)^K e^{\zeta K_+} (K_- + 2\zeta K_3 + \zeta^2 K_+)|K, 0\rangle \]

\[ = 2\zeta K(1 - |\zeta|^2)^K e^{\zeta K_+}|K, 0\rangle + \zeta^2 K_+(1 - |\zeta|^2)^K e^{\zeta K_+}|K, 0\rangle \]

\[ = (2K\zeta + \zeta^2 K_+)|w\rangle \quad (48) \]
because $K_-|K,0\rangle = 0$. Namely we have the equation

$$(K_- - \zeta^2 K_+)|w\rangle = 2K\zeta|w\rangle, \quad \text{where} \quad \zeta = \frac{wtanh(|w|)}{|w|},$$

(49)

or more symmetrically

$$(\zeta^{-1}K_- - \zeta K_+)|w\rangle = 2K|w\rangle, \quad \text{where} \quad \zeta = \frac{wtanh(|w|)}{|w|}.$$  

(50)

This equation is completely different from (3).

A comment is in order. The states $||w\rangle\rangle$ ($w \in \mathbb{C}$) defined by

$$K_-||w\rangle\rangle = w||w\rangle\rangle$$

(51)

are called the Barut–Girardello coherent states, [26]. The solution is given by

$$||w\rangle\rangle = \sum_{n=0}^{\infty} \frac{w^n}{\sqrt{n!(2K)_n}}|K,n\rangle$$

(52)

up to the normalization factor. Compare this with (22).

Their states have several interesting structures, but we don’t consider them in this paper. See [27], [28] and [29] as to further developments and applications.

### 3 Universal Bundles and Chern Characters

We make a brief review of some basic properties of pull–backed ones of universal bundles over the infinite–dimensional Grassmann manifolds and Chern characters, see [8].

Let $\mathcal{H}$ be a separable Hilbert space over $\mathbb{C}$. For $m \in \mathbb{N}$, we set

$$St_m(\mathcal{H}) \equiv \{V = (v_1, \cdots, v_m) \in \mathcal{H} \times \cdots \times \mathcal{H} \mid V^\dagger V \in GL(m; \mathbb{C})\}.$$  

(53)

This is called a (universal) Stiefel manifold. Note that the unitary group $U(m)$ acts on $St_m(\mathcal{H})$ from the right:

$$St_m(\mathcal{H}) \times U(m) \longrightarrow St_m(\mathcal{H}) : (V, a) \mapsto Va.$$  

(54)
Next we define a (universal) Grassmann manifold

\[ Gr_m(\mathcal{H}) \equiv \left\{ X \in M(\mathcal{H}) \mid X^2 = X, X^\dagger = X \text{ and } \text{tr}X = m \right\} , \tag{55} \]

where \( M(\mathcal{H}) \) denotes a space of all bounded linear operators on \( \mathcal{H} \). Then we have a projection

\[ \pi : St_m(\mathcal{H}) \longrightarrow Gr_m(\mathcal{H}) , \quad \pi(V) \equiv V(V^\dagger V)^{-1}V^\dagger , \tag{56} \]

compatible with the action (54) \( (\pi(Va) = Va\{a^{-1}(V^\dagger V)^{-1}a\}(Va)^\dagger = \pi(V)) \).

Now the set

\[ \{ U(m), St_m(\mathcal{H}), \pi, Gr_m(\mathcal{H}) \} , \tag{57} \]

is called a (universal) principal \( U(m) \) bundle, see \[8\] and \[11\]. We set

\[ E_m(\mathcal{H}) \equiv \{(X, v) \in Gr_m(\mathcal{H}) \times \mathcal{H} \mid Xv = v\} . \tag{58} \]

Then we have also a projection

\[ \pi : E_m(\mathcal{H}) \longrightarrow Gr_m(\mathcal{H}) , \quad \pi((X, v)) \equiv X . \tag{59} \]

The set

\[ \{ \mathcal{C}^m, E_m(\mathcal{H}), \pi, Gr_m(\mathcal{H}) \} , \tag{60} \]

is called a (universal) \( m \)-th vector bundle. This vector bundle is one associated with the principal \( U(m) \) bundle (57).

Next let \( \mathcal{M} \) be a finite or infinite dimensional differentiable manifold and the map

\[ P : \mathcal{M} \longrightarrow Gr_m(\mathcal{H}) \tag{61} \]

be given (called a projector). Using this \( P \) we can make the bundles (57) and (58) pullback over \( \mathcal{M} : \)

\[ \{ U(m), St, \pi_{St}, \mathcal{M} \} \equiv P^* \{ U(m), St_m(\mathcal{H}), \pi, Gr_m(\mathcal{H}) \} , \tag{62} \]

\[ \{ \mathcal{C}^m, \tilde{E}, \pi_{\tilde{E}}, \mathcal{M} \} \equiv P^* \{ \mathcal{C}^m, E_m(\mathcal{H}), \pi, Gr_m(\mathcal{H}) \} , \tag{63} \]
see [8]. (63) is of course a vector bundle associated with (62).

For this bundle the (global) curvature (2–) form \( \Omega \) is given by

\[
\Omega = PdP \wedge dP
\]

making use of (61), where \( d \) is the usual differential form on \( \Omega \). For the bundles Chern characters play an essential role in several geometric properties. In this case Chern characters are given by

\[
\Omega, \; \Omega^2, \; \cdots, \; \Omega^{m/2}; \quad \Omega^2 = \Omega \wedge \Omega, \; \text{etc,}
\]

where we have assumed that \( m = \dim \mathcal{M} \) is even. In this paper we don’t take the trace of (63), so it may be better to call them densities for Chern characters.

To calculate these quantities in infinite–dimensional cases is not so easy. In the next section let us calculate these ones in the special cases.

We now define our projectors for the latter aim. In the following \( \mathcal{H} = \mathcal{H}_K \). For \( w_1, w_2, \cdots, w_m \in \mathbb{C} \) we consider a set of generalized coherent states \( \{|w_1\}, |w_2\>, \cdots, |w_m\rangle \) and set

\[
V_m(\mathbf{w}) = (|w_1\rangle, |w_2\rangle, \cdots, |w_m\rangle) \equiv V_m
\]

where \( \mathbf{w} = (w_1, w_2, \cdots, w_m) \). Since \( V_m \dagger V_m = (\langle w_i|w_j \rangle) \in M(m, \mathbb{C}) \), we define

\[
\mathcal{D}_m \equiv \{ \mathbf{w} \in \mathbb{C}^m \mid \det(V_m \dagger V_m) \neq 0 \}.
\]

For example \( V_1 \dagger V_1 = 1 \) for \( m = 1 \), and for \( m = 2 \)

\[
\det(V_2 \dagger V_2) = \begin{vmatrix} 1 & a \\ \bar{a} & 1 \end{vmatrix} = 1 - |a|^2 \geq 0
\]
where \( a = \langle w_1 | w_2 \rangle \). So from (47) we have
\[
\mathcal{D}_1 = \{ w \in \mathbb{C} \mid \text{no conditions} \} = \mathbb{C}, \tag{68}
\]
\[
\mathcal{D}_2 = \{(w_1, w_2) \in \mathbb{C}^2 \mid w_1 \neq w_2 \}. \tag{69}
\]
For \( \mathcal{D}_m \ (m \geq 3) \) it is not easy for us to give a simple condition like (69). At any rate
\( V_m \in St_m(\mathcal{H}) \) for \( w \in \mathcal{D}_m \). Now let us define our projector \( P \) as follows:
\[
P : \mathcal{D}_m \longrightarrow Gr_m(\mathcal{H}) , \quad P(w) = V_m(V_m^\dagger V_m)^{-1}V_m^\dagger. \tag{70}
\]
In the following we set \( V = V_m \) for simplicity. Let us calculate (64). Since
\[
dP = V(V^\dagger V)^{-1}dV^\dagger \{1 - V(V^\dagger V)^{-1}V^\dagger\} \{1 - V(V^\dagger V)^{-1}V^\dagger\}dV(V^\dagger V)^{-1}V^\dagger \tag{71}
\]
where \( d = \sum_{j=1}^{m} \left( dw_j \frac{\partial}{\partial w_j} + d\bar{w}_j \frac{\partial}{\partial \bar{w}_j} \right) \), we have
\[
P dP = V(V^\dagger V)^{-1}dV^\dagger \{1 - V(V^\dagger V)^{-1}V^\dagger\}
\]
after some calculation. Therefore we obtain
\[
P dP \wedge dP = V(V^\dagger V)^{-1} [dV^\dagger \{1 - V(V^\dagger V)^{-1}V^\dagger\}dV](V^\dagger V)^{-1}V^\dagger. \tag{72}
\]
Our main calculation is \( dV^\dagger \{1 - V(V^\dagger V)^{-1}V^\dagger\}dV \), which is rewritten as
\[
dV^\dagger \{1 - V(V^\dagger V)^{-1}V^\dagger\}dV = [\{1 - V(V^\dagger V)^{-1}V^\dagger\}dV]^\dagger [\{1 - V(V^\dagger V)^{-1}V^\dagger\}dV] \tag{73}
\]
since \( Q \equiv 1 - V(V^\dagger V)^{-1}V^\dagger \) is also a projector (\( Q^2 = Q \) and \( Q^\dagger = Q \)). Therefore the first step for us is to calculate the term
\[
\{1 - V(V^\dagger V)^{-1}V^\dagger\}dV. \tag{74}
\]
Let us summarize our process of calculations:

1–st step \( \{1 - V(V^\dagger V)^{-1}V^\dagger\}dV \cdots (74) \),

2–nd step \( dV^\dagger \{1 - V(V^\dagger V)^{-1}V^\dagger\}dV \cdots (73) \),

3–rd step \( V(V^\dagger V)^{-1}[dV^\dagger \{1 - V(V^\dagger V)^{-1}V^\dagger\}dV](V^\dagger V)^{-1}V^\dagger \cdots (72) \).
4 Calculations of Chern Characters

In this section we calculate the Chern characters only for the cases \( m = 1 \) and \( m = 2 \). Even for \( m = 2 \) the calculation is complicated enough. For \( m \geq 3 \) calculations may become miserable.

4.1 \( M=1 \)

In this case \( \langle w|w \rangle = 1 \), so our projector is very simple to be

\[
P(w) = |w\rangle\langle w|.
\]

(75)

In this case the calculation of curvature is relatively simple. From (72) we have

\[
PdP \wedge dP = |w\rangle\langle d\langle w|(1_K - |w\rangle\langle w|)d|w\rangle\rangle Channel \langle w|d\rangle = |w\rangle\langle w|\{d\langle w|(1_K - |w\rangle\langle w|)d|w\rangle\},
\]

(76)

where \( d = dw\frac{\partial}{\partial w} + d\bar{w}\frac{\partial}{\partial \bar{w}} \). Since \( |w\rangle = (1 - |\zeta|^2)^K \exp(\zeta K_+)|K,0 \rangle \) by (21),

\[
d|w\rangle = \left\{d\zeta K_+ + Kd\log(1 - |\zeta|^2)\right\}|w\rangle
\]

(77)

by some calculation, so that

\[
(1_K - |w\rangle\langle w|)d|w\rangle = (1_K - |w\rangle\langle w|)K_+|w\rangle d\zeta = (K_+ - \langle w|K_+|w\rangle)|w\rangle d\zeta
\]

(78)

because \( (1_K - |w\rangle\langle w|)|w\rangle = 0 \). Similarly we have

\[
d\langle w|(1_K - |w\rangle\langle w|) = \langle w| \left( K_- - \frac{2K\zeta}{1 - |\zeta|^2} \right) d\zeta
\]

(79)

Now we are in a position to determine the curvature form (70).}

\[
d\langle w|(1_K - |w\rangle\langle w|)d|w\rangle = \langle w| \left( K_- - \frac{2K\zeta}{1 - |\zeta|^2} \right) d\zeta \wedge d\zeta
\]

\[
= \left\{\langle w|K_-K_+|w\rangle - \frac{2K\zeta}{1 - |\zeta|^2}\langle w|K_-|w\rangle - \frac{2K\zeta}{1 - |\zeta|^2}\langle w|K_+|w\rangle + \frac{4K^2|\zeta|^2}{(1 - |\zeta|^2)^2}\right\} d\zeta \wedge d\zeta
\]

\[
= \frac{2K}{(1 - |\zeta|^2)^2} d\zeta \wedge d\zeta
\]

(80)
after some algebra with (43) and (44). Therefore
\[
\Omega = PdP \wedge dP = |w\rangle\langle w| \frac{2Kd\bar{\zeta} \wedge d\zeta}{(1 - |\zeta|^2)^2}.
\] (81)

From this result we know
\[
\frac{\Omega}{2\pi i} = \frac{2K}{\pi} \frac{d\zeta_1 \wedge d\zeta_2}{(1 - |\zeta|^2)^2} |w\rangle\langle w| = \frac{2K}{\pi} \frac{d\zeta_1 \wedge d\zeta_2}{(1 - |\zeta|^2)^2} |\zeta\rangle\langle \zeta|
\]
by (21) when \( \zeta = \zeta_1 + \sqrt{-1}\zeta_2 \). If we define a constant
\[
C_K = \frac{2K - 1}{2K},
\] (82)
then we have
\[
C_K \frac{\Omega}{2\pi i} = \frac{2K - 1}{\pi} \frac{d\zeta_1 \wedge d\zeta_2}{(1 - |\zeta|^2)^2} |\zeta\rangle\langle \zeta|.
\] (83)

This gives the resolution of unity in (23). But the situation is a bit different from [7] in which the constant corresponding to \( C_K \) is just one.

**Problem** What is a (deep) meaning of \( C_K \) ?

### 4.2 \( M=2 \) · · · Main Result

First of all let us determine the projector. Since \( V = (|w_1\rangle, |w_2\rangle) \) we have easily
\[
P(w_1, w_2) = (|w_1\rangle, |w_2\rangle) \left( \begin{array}{cc} 1 & \langle w_1|w_2\rangle \\ \langle w_2|w_1\rangle & 1 \end{array} \right)^{-1} \left( \begin{array}{c} \langle w_1| \\ \langle w_2| \end{array} \right)
\]
\[
= \frac{1}{1 - |\langle w_1|w_2\rangle|^2} (|w_1\rangle\langle w_1| - \langle w_2|w_1\rangle |w_2\rangle \langle w_1| - \langle w_1|w_2\rangle |w_1\rangle \langle w_2| + |w_2\rangle \langle w_2|).
\] (84)

Let us calculate (74) : Since
\[
dV = (d|w_1\rangle, d|w_2\rangle)
\]
\[
= \left( \left\{ d\zeta_1 K_+ + Kd\log(1 - |\zeta_1|^2) \right\} |w_1\rangle, \left\{ d\zeta_2 K_+ + Kd\log(1 - |\zeta_2|^2) \right\} |w_2\rangle \right)
\]
by (74) where \( d = \sum_{j=1}^{2} \left( dw_j \frac{\partial}{\partial w_j} + d\bar{w}_j \frac{\partial}{\partial \bar{w}_j} \right) \), the long but straightforward calculation leads to

\[
\{1_K - V(V\dagger V)^{-1}V\dagger\}dV = (G_1, G_2) \quad (85)
\]

where

\[
G_1 = \left\{ \left( K_+ - \frac{2K\bar{\zeta}_2}{1 - \zeta_2\zeta_1} \right) - \frac{2K}{1 - |\langle w_1|w_2\rangle|^2} \frac{\bar{\zeta}_1 - \bar{\zeta}_2}{(1 - |\zeta_1|^2)(1 - \zeta_2\zeta_1)} \right\} d\zeta_1|w_1\rangle
d\bar{\zeta}_1|w_2\rangle, \\
+ \frac{2K\langle w_2|w_1\rangle}{1 - |\langle w_1|w_2\rangle|^2} \frac{\bar{\zeta}_1 - \bar{\zeta}_2}{(1 - |\zeta_1|^2)(1 - \zeta_2\zeta_1)} d\zeta_1|w_2\rangle, \\
G_2 = \frac{2K\langle w_1|w_2\rangle}{1 - |\langle w_1|w_2\rangle|^2} \frac{\bar{\zeta}_2 - \bar{\zeta}_1}{(1 - \zeta_1\zeta_2)(1 - |\zeta_2|^2)} d\zeta_2|w_1\rangle
+ \left\{ \left( K_+ - \frac{2K\bar{\zeta}_1}{1 - \zeta_1\zeta_2} \right) - \frac{2K}{1 - |\langle w_1|w_2\rangle|^2} \frac{\bar{\zeta}_1 - \bar{\zeta}_2}{(1 - |\zeta_2|^2)(1 - \zeta_1\zeta_2)} \right\} d\zeta_2|w_2\rangle. \quad (86)
\]

Therefore by (73)

\[
dV\dagger\{1_K - V(V\dagger V)^{-1}V\dagger\}dV = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \quad (87)
\]

where

\[
F_{11} = \frac{2K}{(1 - |\zeta_1|^2)^2} \left\{ 1 - \frac{|\langle w_1|w_2\rangle|^2}{1 - |\langle w_1|w_2\rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{|1 - \bar{\zeta}_1\zeta_2|^2} \right\} d\bar{\zeta}_1 \wedge d\zeta_1, \\
F_{12} = \frac{2K\langle w_1|w_2\rangle}{1 - \zeta_1\zeta_2} \left\{ 1 - \frac{1}{1 - |\langle w_1|w_2\rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)} \right\} d\bar{\zeta}_1 \wedge d\zeta_2, \\
F_{21} = \frac{2K\langle w_2|w_1\rangle}{1 - \zeta_2\zeta_1} \left\{ 1 - \frac{1}{1 - |\langle w_1|w_2\rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{(1 - |\zeta_2|^2)(1 - |\zeta_1|^2)} \right\} d\bar{\zeta}_2 \wedge d\zeta_1, \\
F_{22} = \frac{2K}{(1 - |\zeta_2|^2)^2} \left\{ 1 - \frac{|\langle w_1|w_2\rangle|^2}{1 - |\langle w_1|w_2\rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{|1 - \bar{\zeta}_2\zeta_1|^2} \right\} d\bar{\zeta}_2 \wedge d\zeta_2. \quad (88)
\]

Now we are in a position to determine the curvature form (72). Since

\[
V(V\dagger V)^{-1} = \frac{1}{1 - |\langle w_1|w_2\rangle|^2}(\langle w_1\rangle - \langle w_2|w_1\rangle|w_2\rangle, |w_2\rangle - \langle w_1|w_2\rangle|w_1\rangle),
\]
\[(V^\dagger V)^{-1} V^\dagger = \frac{1}{1 - |\langle w_1 | w_2 \rangle|^2} \left( \langle w_1 | - \langle w_2 | \langle w_1 | w_2 \rangle \rangle \langle w_2 | - \langle w_1 | \langle w_2 | w_1 \rangle \rangle \right)\]

we obtain after some algebra

\[\Omega = PdP \wedge dP\]
\[= \frac{1}{(1 - |\langle w_1 | w_2 \rangle|^2)^2} \left( |\langle w_1 | L_1 - \langle w_2 | w_1 \rangle | w_2 \rangle | \langle w_1 | L_2 - \langle w_1 | w_2 \rangle | w_1 \rangle | w_2 \rangle L_3 + |w_2 \rangle \langle w_2 | L_4 \right),\]

where

\[L_1 = \frac{2K}{(1 - |\zeta_1|^2)^2} \left\{ 1 - \frac{|\langle w_1 | w_2 \rangle|^2}{1 - |\langle w_1 | w_2 \rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{|1 - \zeta_1 \zeta_2|^2} \right\} d\zeta_1 \wedge d\zeta_1 \]
\[- \frac{2K|\langle w_1 | w_2 \rangle|^2}{(1 - \zeta_2 \zeta_1)^2} \left\{ 1 - \frac{1}{1 - |\langle w_1 | w_2 \rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)} \right\} d\zeta_2 \wedge d\zeta_1 \]
\[- \frac{2K|\langle w_1 | w_2 \rangle|^2}{(1 - \zeta_1 \zeta_2)^2} \left\{ 1 - \frac{1}{1 - |\langle w_1 | w_2 \rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)} \right\} d\zeta_1 \wedge d\zeta_2 \]
\[+ \frac{2K|\langle w_1 | w_2 \rangle|^2}{(1 - |\zeta_2|^2)^2} \left\{ 1 - \frac{|\langle w_1 | w_2 \rangle|^2}{1 - |\langle w_1 | w_2 \rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{|1 - \zeta_1 \zeta_2|^2} \right\} d\zeta_2 \wedge d\zeta_2 ,\]

\[L_2 = \frac{2K}{(1 - |\zeta_1|^2)^2} \left\{ 1 - \frac{|\langle w_1 | w_2 \rangle|^2}{1 - |\langle w_1 | w_2 \rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{|1 - \zeta_1 \zeta_2|^2} \right\} d\zeta_1 \wedge d\zeta_1 \]
\[- \frac{2K}{(1 - \zeta_2 \zeta_1)^2} \left\{ 1 - \frac{1}{1 - |\langle w_1 | w_2 \rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)} \right\} d\zeta_2 \wedge d\zeta_1 \]
\[- \frac{2K}{(1 - \zeta_1 \zeta_2)^2} \left\{ 1 - \frac{1}{1 - |\langle w_1 | w_2 \rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)} \right\} d\zeta_1 \wedge d\zeta_2 \]
\[+ \frac{2K}{(1 - |\zeta_2|^2)^2} \left\{ 1 - \frac{|\langle w_1 | w_2 \rangle|^2}{1 - |\langle w_1 | w_2 \rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{|1 - \zeta_1 \zeta_2|^2} \right\} d\zeta_2 \wedge d\zeta_2 ,\]

\[L_3 = \frac{2K}{(1 - |\zeta_1|^2)^2} \left\{ 1 - \frac{|\langle w_1 | w_2 \rangle|^2}{1 - |\langle w_1 | w_2 \rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{|1 - \zeta_1 \zeta_2|^2} \right\} d\zeta_1 \wedge d\zeta_1 \]
\[- \frac{2K}{(1 - \zeta_2 \zeta_1)^2} \left\{ 1 - \frac{1}{1 - |\langle w_1 | w_2 \rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)} \right\} d\zeta_2 \wedge d\zeta_1 \]
\[- \frac{2K}{(1 - \zeta_1 \zeta_2)^2} \left\{ 1 - \frac{1}{1 - |\langle w_1 | w_2 \rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)} \right\} d\zeta_1 \wedge d\zeta_2 \]
\[+ \frac{2K}{(1 - |\zeta_2|^2)^2} \left\{ 1 - \frac{|\langle w_1 | w_2 \rangle|^2}{1 - |\langle w_1 | w_2 \rangle|^2} \frac{2K|\zeta_1 - \zeta_2|^2}{|1 - \zeta_1 \zeta_2|^2} \right\} d\zeta_2 \wedge d\zeta_2 ,\]
This is our main result. Next let us calculate $\Omega^2$ ($\Omega^k = 0$ for $k \geq 3$): From (89) we obtain after long calculation

$$\Omega^2 = \frac{1}{(1 - |\langle w_1 | w_2 \rangle|^2)^4} \left( |\langle w_1 | M_1 - \langle w_2 | w_1 \rangle | w_2 \rangle \langle w_1 | M_2 - \langle w_1 | w_2 \rangle | w_1 \rangle \langle w_2 | M_3 + |w_2 \rangle \langle w_2 | M_4 \rangle \right) \times d\bar{\zeta}_1 \wedge d\zeta_1 \wedge d\bar{\zeta}_2 \wedge d\zeta_2 ,$$

(91)

where

$$M_1 = \frac{4K^2 |\langle w_1 | w_2 \rangle|^2}{(1 - |\zeta_1|^2)^2(1 - |\zeta_2|^2)^2|1 - \bar{\zeta}_1 \zeta_2|^4} \left[ 2|1 - \bar{\zeta}_1 \zeta_2|^4(1 - |\langle w_1 | w_2 \rangle|^2) - (1 - |\zeta_1|^2)^2(1 - |\zeta_2|^2)^2(1 - |\langle w_1 | w_2 \rangle|^4) \right]$$

$$-2 \left\{ 2|1 - \bar{\zeta}_1 \zeta_2|^2|\langle w_1 | w_2 \rangle|^2 - (1 - |\zeta_1|^2)(1 - |\zeta_2|^2)(1 + |\langle w_1 | w_2 \rangle|^2) \right\} 2K|\zeta_1 - \zeta_2|^2$$

$$-(1 + 2|\langle w_1 | w_2 \rangle|^2)4K^2|\zeta_1 - \zeta_2|^4 \right],$$

$$M_2 = \frac{4K^2}{(1 - |\zeta_1|^2)^2(1 - |\zeta_2|^2)^2|1 - \bar{\zeta}_1 \zeta_2|^4} \left[ |1 - \bar{\zeta}_1 \zeta_2|^4(1 - |\langle w_1 | w_2 \rangle|^4) - 2(1 - |\zeta_1|^2)^2(1 - |\zeta_2|^2)^2|\langle w_1 | w_2 \rangle|^2(1 - |\langle w_1 | w_2 \rangle|^2) \right]$$

$$-2 \left\{ |1 - \bar{\zeta}_1 \zeta_2|^2|\langle w_1 | w_2 \rangle|^2(1 + |\langle w_1 | w_2 \rangle|^2) - 2(1 - |\zeta_1|^2)(1 - |\zeta_2|^2)|\langle w_1 | w_2 \rangle|^2 \right\} 2K|\zeta_1 - \zeta_2|^2$$

$$-|\langle w_1 | w_2 \rangle|^2(2 + |\langle w_1 | w_2 \rangle|^2)4K^2|\zeta_1 - \zeta_2|^4 \right],$$

$$M_3 = M_2, \text{ and } M_4 = M_1 .$$

(92)
This is a second main result in this paper.

We have calculated the Chern characters for $m = 2$. Since our results are in a certain sense “raw” (remember that we have not taken the trace), one can freely “cook” them. We leave it to the readers.

### 4.3 Problems

Before concluding this section let us propose problems.

**Problem 3** For the case of $m = 3$ perform the similar calculations!

We want to calculate them up to this case. Therefore let us give an explicit form to the projector:

$$P(w_1, w_2, w_3) = (|w_1>, |w_2>, |w_3>) \begin{pmatrix} 1 & \langle w_1|w_2 \rangle & \langle w_1|w_3 \rangle \\ \langle w_2|w_1 \rangle & 1 & \langle w_2|w_3 \rangle \\ \langle w_3|w_1 \rangle & \langle w_3|w_2 \rangle & 1 \end{pmatrix}^{-1} \begin{pmatrix} \langle w_1| \\ \langle w_2| \\ \langle w_3| \end{pmatrix}$$

$$= \frac{1}{\det M} \left[ \{1 - |\langle w_2|w_3 \rangle|^2\}|w_1\rangle\langle w_1| - \{\langle w_1|w_2 \rangle - \langle w_1|w_3 \rangle\langle w_3|w_2 \rangle\}|w_1\rangle\langle w_2| - \{\langle w_1|w_3 \rangle - \langle w_2|w_1 \rangle\langle w_1|w_3 \rangle\}|w_2\rangle\langle w_3| + \{1 - |\langle w_1|w_3 \rangle|^2\}|w_2\rangle\langle w_2| - \{\langle w_2|w_3 \rangle - \langle w_2|w_1 \rangle\langle w_1|w_3 \rangle\}|w_3\rangle\langle w_3| \right. $$

$$+ \left. \{\langle w_3|w_1 \rangle - \langle w_3|w_2 \rangle\langle w_2|w_1 \rangle\}|w_3\rangle\langle w_1| - \{\langle w_3|w_2 \rangle - \langle w_3|w_1 \rangle\langle w_1|w_2 \rangle\}|w_3\rangle\langle w_2| + \{1 - |\langle w_1|w_2 \rangle|^2\}|w_3\rangle\langle w_3| \right], \tag{93}$$

where

$$\det M = 1 - |\langle w_1|w_2 \rangle|^2 - |\langle w_2|w_3 \rangle|^2 - |\langle w_1|w_3 \rangle|^2$$

$$+ \langle w_1|w_2 \rangle\langle w_2|w_3 \rangle\langle w_3|w_1 \rangle + \langle w_1|w_3 \rangle\langle w_3|w_2 \rangle\langle w_2|w_1 \rangle.$$

Perform the calculations of $\Omega$, $\Omega^2$ and $\Omega^3$. Moreover

**Problem $\infty$** For the general case perform the similar calculations (if possible).
It seems to the author that the calculations in the general case are very hard.

5 Further Problem

In this paper we treated generalized coherent states based on $su(1,1)$ (34)

$$|w\rangle = \exp(wK_+ - \bar{w}K_-)|K,0\rangle \quad \text{for} \quad w \in \mathbb{C}$$

and calculated the Chern characters for $m = 1, 2$.

But we didn’t treat generalized coherent states based on $su(2)$ (33)

$$|v\rangle = \exp(vJ_+ - \bar{v}J_-)|J,0\rangle \quad \text{for} \quad v \in \mathbb{C} \subset \mathbb{C}P^1.$$ 

See [4] or [3]. In this case the parameter space is $S^2 \cong \mathbb{C}P^1$ for $m = 1$ (a compact manifold). To calculate the Chern characters for $m = 1, 2$ is very interesting problem.

We leave them to keen graduate students.

It seems to the author that this calculation will be deeply related to the recent non–commutative field theory, see for example [21] or [22] and their references.

6 Discussion

We have calculated Chern characters for pull–back bundles on $D_1$ and $D_2$ which are based on generalized coherent states and suggested a relation to some non–commutative field theory (differential geometry).

On the other hand one of aims of this paper and [7] is to apply the results to Quantum Information Theory including Quantum Computation or Quantum Cryptography. The study is under progress. In the forthcoming paper we would like to discuss this point.


References

[1] J. R. Klauder and Bo-S. Skagerstam (Eds) : Coherent States, World Scientific, Singapore, 1985.

[2] L. Mandel and E. Wolf : Optical Coherence and Quantum Optics, Cambridge University Press, 1995.

[3] A. Perelomov : Generalized Coherent States and Their Applications, Springer–Verlag, 1986.

[4] K. Funahashi, T. Kashiwa, S. Sakoda and K. Fujii : Coherent states, path integral, and semiclassical approximation, J. Math. Phys., 36(1995), 3232.

[5] K. Funahashi, T. Kashiwa, S. Sakoda and K. Fujii : Exactness in the Wentzel-Kramers-Brillouin approximation for some homogeneous spaces, J. Math. Phys., 36(1995), 4590.

[6] K. Fujii, T. Kashiwa, S. Sakoda : Coherent states over Grassmann manifolds and the WKB exactness in path integral, J. Math. Phys., 37(1996), 567.

[7] K. Fujii : Geometry of Coherent States : Some Examples of Calculations of Chern Characters, hep–ph/0108219.

[8] M. Nakahara : Geometry, Topology and Physics, IOP Publishing Ltd, 1990.

[9] H-K. Lo, S. Popescu and T. Spiller (eds) : Introduction to Quantum Computation and Information, 1998, World Scientific.

[10] A. Hosoya : Lectures on Quantum Computation (in Japanese), 1999, Science Company (in Japan).

[11] K. Fujii : Introduction to Grassmann Manifolds and Quantum Computation, quant-ph/0103011.
[12] K. Fujii : Note on Coherent States and Adiabatic Connections, Curvatures, J. Math. Phys., 41(2000), 4406, quant-ph/9910069.

[13] K. Fujii : Mathematical Foundations of Holonomic Quantum Computer, to appear in Rept. Math. Phys., quant-ph/0004102.

[14] K. Fujii : More on Optical Holonomic Quantum Computer, quant-ph/0005129.

[15] K. Fujii : Mathematical Foundations of Holonomic Quantum Computer II, quant-ph/0101102.

[16] K. Fujii : From Geometry to Quantum Computation, quant-ph/0107128.

[17] G. M. D’Ariano, L. Maccone and M. G. A. Paris : Quorum of observables for universal quantum estimation, quant-ph/0006006.

[18] M. G. A. Paris : Entanglement and visibility at the output of a Mach–Zehnder interferometer, quant-ph/9811078.

[19] K. Banaszek : Optical receiver for quantum cryptography with two coherent states, quant-ph/9901067.

[20] K. Banaszek and K. Wodkiewicz : Direct Probing of Quantum Phase Space by Photon Counting, atom–ph/9603003.

[21] M. Spradlin and A. Volovich : Noncommutative solitons on Kahler manifolds, hep-th/0106180.

[22] A. P. Balachandran, B. P. Dolan, J. Lee, X. Martin and D. O’Conner : Fuzzy Complex Projective Spaces and their Star–products, hep-th/0107099.

[23] K. Fujii : Basic Properties of Coherent and Generalized Coherent Operators Revisited, Mod. Phys. Lett. A, 16(2001), 1277, quant-ph/0009012.

[24] K. Fujii : Note on Extended Coherent Operators and Some Basic Properties, quant-ph/0009116.
[25] K. Fujii and T. Suzuki: A Universal Disentangling Formula for Coherent States of Perelomov’s Type, hep-th/9907049.

[26] A. O. Barut and L. Girardello: New “coherent” states associated with noncompact groups, Commun. Math. Phys., 21(1971), 222.

[27] K. Fujii and K. Funahashi: Extension of the Barut–Girardello coherent state and path integral J. Math. Phys., 38(1997), 4422, quant-ph/9708011.

[28] K. Fujii and K. Funahashi: Extension of the Barut–Girardello coherent state and path integral II, quant-ph/9708041.

[29] D. A. Trifonov: Barut–Girardello coherent states for u(p,q) and sp(N,R) and their macroscopic superpositions, J. Phys. A, 31(1998), 5673, quant-ph/9711066.