EFFECTIVE BOUNDS FOR QUASI-INTEGRAL POINTS IN ORBITS OF RATIONAL MAPS

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Abstract. We prove the existence of bounds for the number quasi-integral points in orbits of semigroups of rational maps under some conditions, exhibiting formulas for this.

1. Introduction

Let $K$ be a number field and $x = [x_0, ..., x_N] \in \mathbb{P}^N(K)$, the naive logarithmic height $h(P)$ is given by

$$ \sum_{v \in M_K} \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]} \log(\max_i |x_i|_v), $$

where, $M_K$ is the set of places of $K$, $M_K^\infty$ is the set of archimedean (infinite) places of $K$, $M_K^0$ is the set of nonarchimedean (finite) places of $K$, and for each $v \in M_K$, $|.|_v$ denotes the corresponding absolute value on $K$ whose restriction to $\mathbb{Q}$ gives the usual $v$-adic absolute value on $\mathbb{Q}$. Also, we write $K_v$ for the completion of $K$ with respect to $|.|$, and we let $\mathbb{C}_v$ denote the completion of an algebraic closure of $K_v$.

To simplify notation, we let $d_v = [K_v : \mathbb{Q}_v]/[K : \mathbb{Q}]$. Let $\mathcal{F} = \{\phi_1, ..., \phi_k\} \subset K(z)$ be a dynamical system of rational functions of degree at least 2, let $P \in K$, and let $O_{\mathcal{F}}(P) = \{\phi_{i_n} \circ ... \circ \phi_{i_1}(P)|n \in \mathbb{N}, i_j = 1, ..., k\}$ denote the forward orbit of $P$ under $\mathcal{F}$. When $k = 1$ and $\phi^2 \notin k[z]$, Hsia and Silverman proved [2] that the number of quasi-$(S, \epsilon)$-integral points in the orbit of a point $P$ with infinite orbit is bounded by a constant depending only on $\phi_1, \hat{h}_{\phi_1}(P), \epsilon, S$, and $[K : \mathbb{Q}]$. In this paper we generalize this bound for cases of dynamical systems with several rational functions, obtaining some consequences. In section 2 and 3 we remind important facts about height functions, distance and dynamics on the projective line. In section 4 we state a quantitative version of Roth’s theorem and some facts about the index of ramification. The main results are then in section 5.

2. Canonical Heights

Initially, We recall some theorems on height functions.

**Theorem 2.1.** (Weil’s height Machine [8]) There is a way to attach to any projective variety $X$ over $\mathbb{Q}$ and any line bundle $L$ on $X$ a function

$$ h_{X,L} : X(\bar{\mathbb{Q}}) \rightarrow \mathbb{R} $$

with the following properties:

(i) $h_{X,M\otimes M} = h_{X,M} + h_{X,M} + O(1)$ for any line bundles $L$ and $M$ on $X$, where $O(1)$ is a bounded function on $X(\overline{\mathbb{Q}})$

(ii) If $X = \mathbb{P}^N$ and $L = \mathcal{O}_{\mathbb{P}^N}(1)$, then $h_{\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(1)} = h + O(1)$.

(iii) If $f : Y \rightarrow X$ is a morphism of projective varieties and $L$ is a line bundle on $Y$ then $h_{X,L} = \bar{f}^* h_{Y,L}$

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Theorem 2.2 (7). Assume $L$ is ample. Let $h_{X,L}$ be a height function corresponding to $L$.

(1) (Northcott’s finiteness theorem) For any real number $c$ and positive integer $D$, the set
\[ \{ x \in X(\overline{\mathbb{Q}}) | |Q(x) : \mathbb{Q}| \leq D, h_{X,L} \leq c \} \]
is finite.

(2) (positivity) There is a constant $c'$ such that $h_{X,L}(x) \geq c'$ for all $x \in X(\overline{\mathbb{Q}})$.

Given a projective variety $X$ over a number field $K$ and $L$ a line bundle on $X$, a height function $h_{X,L}$ corresponding to $L$ is fixed. Let $H$ be the set of all morphisms $f : X \to X$ over $K$ such that $f^*L \cong L^\otimes d_f$ for some integer $d_f \geq 2$. For $f \in H$, we set
\[ c(f) := \sup_{x \in X(K)} \frac{1}{d_f} h_L(f(x)) - h_L(x). \]
For $f = (f_i)_{i=1}^\infty$ a sequence with $f_i \in H$, i.e., $f \in \prod_{i=1}^\infty H$, we set
\[ c(f) := \sup_{i \geq 1} c(f_i) \in \mathbb{R} \cup \{ +\infty \}. \]
When $c(f) < +\infty$, the sequence is said to be bounded. The property of being bounded is independent of the choice of height functions corresponding to $L$. Let $\mathcal{B}$ be the set of all bounded sequences in $H$, and for $c > 0$, $\mathcal{B}_c := \{ f = (f_i)_{i=1}^\infty \in \mathcal{B} | c(f) \leq c \}$.

It is easy to see that if $H$ is a finite set of self-maps on a projective space, then any sequence of maps arising from $H$ belongs to $\mathcal{B}_c$ for some $c$. In fact, if $H = \{ g_1, \ldots, g_k \}$, and we set $J = \{ 1, \ldots, k \}, W := \prod_{i=1}^\infty J$, and $f_w := (g_{w_i})_{i=1}^\infty$ for $w = (w_i) \in W$. If $c := \max \{ c(g_1), \ldots, c(g_k) \}$, then $\{ f_w | w \in W \} \subset \mathcal{B}_c$.

We also let $S : \prod_{i=1}^\infty H \to \prod_{i=1}^\infty H$ be the shift map which sends $f = (f_i)_{i=1}^\infty$ to $S(f) = (f_{i+1})_{i=1}^\infty$. Then $S$ maps $\mathcal{B}$ into $\mathcal{B}$ and $\mathcal{B}_c$ into $\mathcal{B}_c$ for any $c$.

For $f = (f_i)_{i=1}^\infty \in \prod_{i=1}^\infty H$ and $x \in X(\overline{K})$, denoting $f^{(n)} := f_n(f_{n-1}(\ldots(f_1(x))))$, the set $\{ x, f^{(1)}(x), f^{(2)}(x), f^{(3)}(x), \ldots \} = \{ x, f_1(x), f_2(f_1(x)), f_3(f_2(f_1(x))), \ldots \}$ is called the forward orbit of $x$ under $f$, denoted by $O_f(x)$. The point $x$ is said to be $f$-preperiodic if $O_f(x)$ is finite. If $f = f_1 = f_2 = \ldots$, then the forward orbit is the forward orbit under $f$ in the usual sense.

Theorem 2.3 (Kawaguchi 3, theorem 2.3). Let $X$ be a projective variety over a number field $K$, and $L$ a line bundle on $X$. Let $h_L$ be a height function corresponding to $L$.

(1) There is a unique way to attach to each bounded sequence $f = (f_i)_{i=1}^\infty \in \mathcal{B}$ a canonical height function
\[ \hat{h}_{L,f} : X(\overline{K}) \to \mathbb{R} \]
such that
\begin{enumerate}
\item $\sup_{x \in X(\overline{K})} |\hat{h}_{L,f}(x) - h_L(x)| \leq 2c(f)$.
\item $\hat{h}_{L,S^x(f)} \circ f_1 = d_{f_1} \hat{h}_{L,f}$. In particular, $\hat{h}_{L,S^x(f)} \circ f_n \circ \ldots \circ f_1 = d_{f_n} \ldots d_{f_1} \hat{h}_{L,f}$.
\end{enumerate}

(2) Assume $L$ is ample. Then $\hat{h}_{L,f}$ satisfies the following properties:
\begin{enumerate}
\item $\hat{h}_{L,f}(x) \geq 0$ for all $x \in X(\overline{K})$.
\item $\hat{h}_{L,f}(x) = 0$ if and only if $x$ is $f$-preperiodic. We call $\hat{h}_{L,f}$ a canonical height function (normalized) for $f$. 
\end{enumerate}
Corollary 2.4 (Kawaguchi 3, cor 2.4). Assume $L$ is ample.

(1) Let $c$ be a nonnegative number, and $D$ a positive integer. Then the set
\[
\bigcup_{f \in B_c} \{ x \in X(\bar{K}) | [K(x) : K] \leq D, x \text{ is } f\text{-preperiodic} \}
\]
is finite.

(2) Let $H = \{g_1, \ldots, g_k\}$, and we set $J = \{1, \ldots, k\}$, $W := \prod_{i=1}^{\infty} J$, and $f_w := (g_{w_i})_{i=1}^{\infty}$ for $w = (w_i) \in W$. Then for any positive integer $D$, the set
\[
\{ x \in X(\bar{K}) | [K(x) : K] \leq D, x \text{ is } f_w\text{-preperiodic for some } w \in W \}
\]
is finite.

Considering conditions as above, namely, $X$ is a projective variety over a number field $K$, $L$ is a line bundle on $X$, $H = \{g_1, \ldots, g_k\}$, $J = \{1, \ldots, k\}$, $W := \prod_{i=1}^{\infty} J$, and $f_w := (g_{w_i})_{i=1}^{\infty}$ for $w = (w_i) \in W$, let $\hat{h}_{L,f_w}$ be the canonical height function for $f_w$.

Since $g_j^*L \cong L^{\otimes d_j}$, we have $g_1^*L \otimes \ldots \otimes g_k^*L \cong L^{\otimes (d_1 + \ldots + d_k)}$. Thus $(X, g_1, \ldots, g_k)$ becomes a particular case of what we call a dynamical eigensystem for $L$ of degree $d_1 + \ldots + d_k$. Then Kawaguchi also proved [3, thm 1.2.1] that there exists the canonical height function
\[
\hat{h}_{L,\{g_1, \ldots, g_k\}} : X(\bar{K}) \to \mathbb{R}
\]
for $(X, g_1, \ldots, g_k)$ characterized by the following two properties: (i) $\hat{h}_{L,\{g_1, \ldots, g_k\}} = h_L + O(1)$; and (ii) $\sum_{j=1}^{k} \hat{h}_{L,\{g_1, \ldots, g_k\}} \circ g_j = (d_{g_1} + \ldots + d_{g_k}) \hat{h}_{L,\{g_1, \ldots, g_k\}}$.

Proposition 2.5 (Kawaguchi 3, prop 3.1). Given $J$ the discrete topology, and let $\nu$ be the measure on $J$ that assigns mass $\frac{d_{g_j}}{d_{g_1} + \ldots + d_{g_k}}$ to $j \in J$. Let $\mu := \prod_{i=1}^{\infty} \nu$ be the product measure on $W$. Then we have, for $x \in X(\bar{K})$,
\[
\hat{h}_{L,\{g_1, \ldots, g_k\}}(x) = \int_W \hat{h}_{L,f_w}(x)d\mu(w).
\]
In particular, $|\hat{h}_{L,\{g_1, \ldots, g_k\}}(x) - h_L(x)| \leq 4c$ for all $x \in X(\bar{K})$, where $c = \max\{c(g_1), \ldots, c(g_k)\}$.

3. Distance and dynamics on the projective line

For each $v \in M_K$, we let $\rho_v$ denote the chordal metric defined on $\mathbb{P}^1(C_v)$, where we recall that for $[x_1, y_1], [x_2, y_2] \in \mathbb{P}^1(C_v)$,
\[
\rho_v([x_1, y_1], [x_2, y_2]) = \begin{cases} 
|y_2 - x_2y_1|_v & \text{if } v \in M^\infty_K, \\
\sqrt{|x_1|^2 + |y_1|^2} \sqrt{|x_2|^2 + |y_2|^2} - |x_1y_2 - x_2y_1|_v & \text{if } v \in M^0_K.
\end{cases}
\]

Definition 3.1. The logarithmic chordal metric function
\[
\lambda_v : \mathbb{P}^1(C_v) \times \mathbb{P}^1(C_v) \to \mathbb{R} \cup \{\infty\}
\]
is defined by
\[
\lambda_v([x_1, y_1], [x_2, y_2]) = -\log \rho_v([x_1, y_1], [x_2, y_2]).
\]
It is a matter of fact that $\lambda_v$ is a particular choice of an arithmetic distance function as defined by Silverman, which is a local height function $\lambda_{\mathbb{P}^1 \times \mathbb{P}^1, \Delta}$, where $\Delta$ is the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$. The logarithmic chordal metric and the usual metric can relate in the following way.
Lemma 3.2 (2, lemma 3). Let \( v \in M_K \) and let \( \lambda_v \) be the logarithmic chordal metric on \( \mathbb{P}^1(\mathbb{C}_v) \). Define \( l_v = 2 \) if \( v \) is archimedean, and \( l_v = 1 \) if \( v \) is nonarchimedean. Then for \( x, y \in \mathbb{C}_v \) the inequality \( \lambda_v(x, y) > \lambda_v(y, \infty) + \log l_v \) implies
\[
\lambda_v(y, \infty) \leq \lambda_v(x, y) + \log |x - y|_v \leq 2\lambda_v(x, y) + \log l_v.
\]

Now, let \( \mathcal{F} = \{ \phi_1, \ldots, \phi_k \} \) such that each \( \phi_j : \mathbb{P}^1 \to \mathbb{P}^1 \) is a rational map of degree \( d_j \geq 2 \) defined over a number field \( K \). We set again \( J = \{1, \ldots, k\} \), \( W = \prod_{i=1}^k J_i \), and \( \Phi_w := (\phi_{w_1})_{j=1}^\infty \) for \( w = (w_j)_{j=1}^\infty \in W \). In this situation we let \( \Phi_{w_i}^{(n)} = \phi_{w_n} \circ \cdots \circ \phi_{w_1} \)
with \( \Phi_{w_i}^{(0)} = \text{Id} \), and also \( \mathcal{F}_n := \{ \Phi_{w_i}^{(n)} | w \in W \} \). For a \( P \in \mathbb{P}^1 \), the \( \mathcal{F} \)-orbit of \( P \) is defined as
\[
\mathcal{O}_F(P) = \{ \phi(P) | \phi \in \bigcup_{n \geq 1} \mathcal{F}_n \} = \{ \Phi_{w_i}^{(n)}(P) | n \geq 0, w \in W \} = \bigcup_{w \in W} \mathcal{O}_{\Phi_w}(P).
\]
The point \( P \) is called preperiodic for \( \mathcal{F} \) if \( \mathcal{O}_F(P) \) is finite.

We identify \( K \cup \{ \infty \} = \mathbb{P}^1 \) by fixing an affine coordinate \( z \) on \( \mathbb{P}^1 \), so \( \alpha \in K \) is equal to \([ \alpha, 1 ] \in \mathbb{P}^1(\mathbb{K}) \), and the point at infinity is \([1, 0] \). With respect to the affine coordinate, we identify rational maps with rational functions in \( K(z) \).

Remember for \( P = [x_0, x_1] \in \mathbb{P}_1(K) \) the height of \( P \) is
\[
h(P) = \sum_{v \in M_K} d_v \log(\max\{|x_1|_v, |x_1|_v\}).
\]
And using the definition of \( \lambda_v \), we see that
\[
h(P) = \sum_{v \in M_K} d_v \lambda_v(P, \infty) + O(1).
\]
For a polynomial \( f = \sum a_i z^i \) and an absolute value \( v \in M_K \), we define
\[
|f|_v = \max\{|a_i|_v\} \quad \text{and} \quad h(f) = \sum_{v \in M_K} d_v \log |f|_v.
\]
Given a rational function \( \phi(z) = f(z)/g(z) \in K(z) \) of degree \( d \) written in normalized form, let us say, \( f(z) = \sum_{i \leq d} a_i z^i \), \( g(z) = \sum_{i \leq d} b_i z^i \) with \( a_i \) and \( b_i \) different from zero, and \( f \) and \( g \) relatively prime in \( K[z] \). For \( v \in M_K \), we set \( |\phi|_v = \max\{|f|_v, |g|_v\} \), and then the height of \( \phi \) is defined by
\[
h(\phi) := \sum_{v \in M_K} d_v \log |\phi|_v.
\]
For \( \mathcal{F} = \{ \phi_1, \ldots, \phi_k \} \) set of rational maps, we define \( h(\mathcal{F}) := \max \{ h(\phi) \} \).

Also, for any \( \phi(z), \psi(z) \) rational functions in \( K(z) \), it is a fact that
\[
h(\phi \circ \psi) \leq h(\phi) + (\deg \phi) h(\psi) + (\deg \phi)(\deg \psi) \log 8.
\]
Using this one can conclude the following preliminary estimate:

Proposition 3.3. Let \( \mathcal{F} = \{ \phi_1, \ldots, \phi_k \} \) be a finite set of rational functions with \( \deg \phi_i = d_i \geq 2 \), and \( d := \max_i d_i \). Then for all \( n \geq 1 \) and \( \phi \in \mathcal{F}_n \), we have
\[
h(\phi) \leq \left( \frac{d^n - 1}{d - 1} \right) h(\mathcal{F}) + d^n \left( \frac{d^{n-1} - 1}{d - 1} \right) \log 8.
\]

Proof. For \( n = 1 \) the result is easily true. We assume the it is true for \( n \). Let \( \phi = \phi_{n+1} \circ \cdots \circ \phi_1 \in \mathcal{F}_{n+1} \). Then by the previous proposition and the induction hypothesis
\[
h(\phi) \leq h(\phi_{n+1} \circ \cdots \circ \phi_2) + d^n h(\phi_1) + d^{n+1} \log 8
\]
\[
\leq \frac{d^n - 1}{d - 1} h(\mathcal{F}) + d^n \left( \frac{d^{n-1} - 1}{d - 1} \right) \log 8 + d^n h(\mathcal{F}) + d^{n+1} \log 8
\]
\[
\leq \left( \frac{d^{n+1} - 1}{d - 1} \right) h(\mathcal{F}) + d^n \left( \frac{d^n - 1}{d - 1} \right) \log 8.
\]
For a rational map \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \) of degree \( d \geq 2 \) defined \( K \), some well known facts for any \( P \in \mathbb{P}^1(\bar{K}) \) are that

(a) \( |h(\phi(P)) - dh(P)| \leq c_1 h(\phi) + c_2 \).

(b) \( \hat{h}_\phi(P) = \lim_n h(\phi^{(n)}(P))/d^n \).

(c) \( |\hat{h}_\phi(P) - h(P)| \leq c_3 h(\phi) + c_4 \).

Where \( c_1, c_2, c_3 \) and \( c_4 \) above depend only on \( d \).

Gathering these facts with Theorem 2.3 and Proposition 2.5, omitting \( L \) on the notation, we derive the following:

**Proposition 3.4.** Let \( \mathcal{F} = \{ \phi_1, \ldots, \phi_k \} \) such that each \( \phi_j : \mathbb{P}^1 \to \mathbb{P}^1 \) is a rational function of degree \( d_j \geq 2 \) defined over a number field \( K \). We set again \( J = \{1, \ldots, k\}, W = \prod_{i=1}^\infty J \), and \( \Phi_w := (\phi_{w_j})_{j=1}^\infty \) for \( w = (w_j)_{j=1}^\infty \in W \). Thus, there are constants \( c_1, c_2, c_3 \) and \( c_4 \) depending only on the degrees \( d_1, \ldots, d_k \) such that

(i) \( |\hat{h}_{\Phi_w}(P) - h(P)| \leq c_1 h(\mathcal{F}) + c_3 \)

(ii) \( |\hat{h}_{\Phi_w}(P) - h_{\Phi_w}(P)| \leq c_3 h(\mathcal{F}) + c_4 \)

For any \( P \) where the corresponding orbits are well defined, and any \( w = (w_j)_{j=1}^\infty \in W \).

4. A distance estimate and a quantitative version of Roth’s Theorem

We will state two known results that will be needed to prove our main theorems.

The first one is a result due to Silverman that gives explicit estimates for the dependence on local heights of points and function. Let us recall that, for a rational function \( f(z) \) and \( P \neq \infty \), the ramification index of \( f \) at \( P \) is defined as the order of \( P \) as a zero of the rational function \( f(z) - f(P) \), i.e.,

\[ e_P(f) = \text{ord}_P(f(z) - f(P)). \]

If \( P = \infty \), we change coordinates through a linear fractional transformation \( L \), sending \( P \) to \( \beta \neq \infty \), and define \( e_P(f) = e_\beta(L^{-1} \circ f \circ L) \), and it will not depend on the choice of \( L \).

The result is as following.

**Proposition 4.1** (Hsia and Silverman 2, prop 7). Let \( \phi \in K(z) \) be a nontrivial rational function, let \( S \subset M_K \) be a finite set of absolute values on \( K \), each extended in some way to \( \bar{K} \), and let \( A, P \in \mathbb{P}^1(\bar{K}) \). Then

\[ \sum_{v \in S} \max_{A' \in \Phi_{v^{-1}(A)}} e_{A'}(\psi) d_v \lambda_v(P, A') \geq \sum_{v \in S} d_v \lambda_v(\psi(P), A) - O(h(A) + h(\psi) + 1), \]

where the implied constant depends only on the degree of the map \( \psi \).

The second result is the following quantitative version of Roth’s theorem.

**Theorem 4.2** (2, Th 10). Let \( S \) be a finite subset of \( M_K \) that contains all infinite places. We assume that each place in \( S \) is extended to \( \bar{K} \) in some fashion. Set the following notation.

- \( s \) is the cardinality of \( S \).
- \( \Upsilon \) a finite \( G_{K/\mathbb{Q}} \)-invariant subset of \( \bar{K} \).
- \( \beta \) a map \( S \to \Upsilon \).
- \( \mu > 2 \).
- \( M \geq 0 \).

There are constants \( r_1 \) and \( r_2 \), depending only on \([K : \mathbb{Q}], \#\Upsilon \) and \( \mu \), such that there are at most \( 4^sr_1 \) elements \( x \in K \) satisfying both of the following conditions:
\( (1) \sum_{v \in S} d_v \log^+ |x - \beta_v|_v^{-1} \geq \mu h(x) - M. \)

\( (2) \ h(x) \geq r_2 \max_{v \in S} \{h(\beta_v), M, 1\}. \)

To end this section let again be \( F = \{\phi_1, \ldots, \phi_k\} \) such that each \( \phi_j : \mathbb{P}^1 \to \mathbb{P}^1 \) is a rational function of degree \( d_j \geq 2 \) defined over a number field \( K \). We set again \( J = \{1, \ldots, k\}, W = \prod_{i=1}^{\infty} J, \) and \( \Phi_w := (\phi_{w_j})_{j=1}^\infty \) for \( w = (w_j)_{j=1}^\infty \in W \) and \( \Phi^{(0)}_w = \phi_{w_1} \circ \cdots \circ \phi_{w_j} \) with \( \Phi^{(0)}_w = \text{Id} \).

We fix \( w \) and denote \( \Phi := \Phi_w, \Phi^n := \Phi^{(n)}_w \) by simplicity. Now, let let \( P \in \mathbb{P}^1(K) \) be a point whose \( \Phi \)-orbit does not have any periodic points within it, namely, that \( \Phi^n(P) \neq \Phi^m(P) \) for all \( n \neq m \). Then using well known facts such as the multiplicativity of the ramification index, and the formula

\[ \sum_P e_P(f) = \deg f - 2 \]

for rational functions \( f \), we can compute that

\[ e_P(\Phi^n) = e_P(\phi_{w_1} \circ \cdots \circ \phi_{w_j}) = e_P(\phi_{w_1}) e_{\phi_{w_2}}(\phi_{w_2}) \cdots e_{\phi_{w_m-1}}(\phi_{w_m})(\phi_{w_m}) = e_P(\phi_{w_1}) e_{\phi_{w_2}}(\phi_{w_2}) \cdots e_{\phi_{w_m}}(\phi_{w_m}) = e_1 e_2 \cdots e_m, \]

where we make

\[ e_i := e_{\phi_{w_i-1}}(\phi_{w_i}) = e_{\phi_{w_i}}(\phi_{w_i}). \]

Therefore

\[ e_P(\Phi^n) = e_1 e_2 \cdots e_m \leq \left( \frac{e_1 + \cdots + e_m}{m} \right)^m = \left( \frac{(e_1 - 1) + \cdots + (e_m - 1)}{m} + 1 \right)^m \]

\[ \leq \left( \frac{\sum_{i \leq k} (2d_i - 2)}{m} \right)^m \leq e_{\sum_{i \leq k} (2d_i - 2)} = M = M \left( \frac{1}{d_1} d_1 \right)^m. \]

Hence, generalizing a result for just one function, we have just easily proved that

**Lemma 4.3.** Under the conditions above with \( P \) wandering point for \( \Phi \), whose orbit does not repeat points, then there exist two positive constants \( \kappa_1 > 0 \) and \( 0 < \kappa_2 < 1 \) depending only on the degrees of the functions of the dynamical system \( F = \{\phi_1, \ldots, \phi_k\} \) such that

\[ e_P(\Phi^m) \leq \kappa_1(\kappa_2 d_1)^m \text{ for any } m \geq 0. \]

It is possible, under some conditions on the system \( F \), to prove this kind of result for any \( P \), replacing \( d_1 \) by \( \deg \Phi^m \) namely below.

**Lemma 4.4.** Under the conditions above, supposing that the orbit of \( P \) does not repeat points or otherwise that no point in the orbit of \( P \) is totally ramified for any \( \phi_j \) in \( F \), then there exist two positive constants \( \kappa_1 > 0 \) and \( 0 < \kappa_2 < 1 \) depending only on the degrees of the functions of the dynamical system \( F = \{\phi_1, \ldots, \phi_k\} \) such that

\[ e_P(\Phi^m) \leq \kappa_1 \kappa_2^m \deg \Phi^m \text{ for any } m \geq 0. \]

**Proof.** Previous lemma deals with the situation without repeated points on orbits, so we prove the second situation. Again, making \( e_i := e_{\phi_{w_{i-1}}}(...(\phi_{w_1})(\phi_{w_i})) = e_{\phi_{w_i}}(\phi_{w_i}) \), the ramification hypothesis implies that \( e_i \leq d_i - 1 \) for each \( i \). Therefore

\[ e_P(\Phi^m) = e_1 e_2 \cdots e_m \leq \prod_{j \leq m} (d_i - 1) \leq \prod_{j \leq m} (1 - \frac{1}{d_i}) \prod_{j \leq m} d_i \]

\[ \leq \left( 1 - \frac{1}{\max_i d_i} \right)^m \prod_{j \leq m} d_i = \left( 1 - \frac{1}{\max_i d_i} \right)^m \deg \Phi^m, \]
which is as desired with $\kappa_1 = 1, \kappa_2 = \left(1 - \frac{1}{\max_i d_i}\right)$.

\[
\square
\]

5. A Bound for the Number of Quasiintegral Points in an Orbit

In this section, we show explicit bounds for the number of $S$-integral points in a given orbit of a wandering point for a dynamical system of rational functions extending previous work by Hsia and Silverman. In general, orbits of these kind have always only a finite number of $S$-integers, as it was showed by J.Silverman, which we recall below.

**Theorem 5.1** (Silverman 6, Th 2.1). Let $K$ be a number field, let $R_S$ be a rink of $S$-integers of $K$, and let $\phi_1, ..., \phi_k : \mathbb{P}^1 \to \mathbb{P}^1$ be rational maps of degree at least two defined over $K$. Let $\mathcal{F}$ be the monoid of maps (the dynamical system) generated by the $\phi_i$’s under composition, and for any $P \in \mathbb{P}^1$, let $\mathcal{O}_P$ be the orbit of $P$ under $\mathcal{F}$. Assume that none map in the monoid is totally ramified in its fixed points. Then for any function $z \in \mathbb{P}^1$, the set

\[
\{Q | Q \in \mathcal{O}_P(P) \text{ and } z(Q) \in R_S\}
\]

is finite.

The next quantitative theorem generalizes a theorem of Hsia and Silverman

**Theorem 5.2.** Let $\mathcal{F} = \{\phi_1, \phi_2, ..., \phi_k\} \subset K(z)$ be a dynamical system of rational maps of respective degrees $2 \leq d_1 \leq d_2 \leq ... \leq d_k$. We consider sequences $\Phi = (\phi_i)^{\infty}_{j=1}$ of functions in $\mathcal{F}$, with $\Phi^n = \phi_{i_n} \circ ... \circ \phi_{i_1} \in \mathcal{F}$, and $P \in \mathbb{P}^1(K)$. Fix $A \in \mathbb{P}^1(K)$ such that none two points in the $\Phi$-orbit of $A$ coincide, or otherwise the none point in its orbit is totally ramified for any map in $\mathcal{F}$. For any finite set of places $S \subset M_K$ and any constant $1 \geq \epsilon > 0$, define a set of nonnegative integers by

\[
\Gamma_{\Phi,S}(A, P, \epsilon) := \{n \geq 0; \sum_{v \in S} d_v \lambda_v(\Phi^n(P), A) \geq \epsilon h_{\Phi^n}(\Phi^n(P))\}.
\]

(a) There exist constants

\[
\gamma_1 = \gamma_2(d_1, ..., d_k, \epsilon, [K : \mathbb{Q}]) \quad \text{and} \quad \gamma_2 = \gamma_2(d_1, ..., d_k, \epsilon, [K : \mathbb{Q}])
\]

such that

\[
\# \{n \in \Gamma_{\Phi,S}(A, P, \epsilon); n > \gamma_1 + \log_{d_1} \left(\frac{\hat{h}_\mathcal{F}(A) + h(\mathcal{F})}{\hat{h}_\Phi(P)}\right)\} \leq 4^S \gamma_2.
\]

In particular, there is a constant $\gamma_3(d_1, ..., d_k, \epsilon, [K : \mathbb{Q}])$ such that

\[
\# \Gamma_{\Phi,S}(A, P, \epsilon) \leq 4^S \gamma_3 + \log_{d_1} \left(\frac{\hat{h}_\mathcal{F}(A) + h(\mathcal{F})}{\hat{h}_\Phi(P)}\right)
\]

(c) There is a constant $\gamma_3(K, S, \mathcal{F}, A, \epsilon)$ that is independent of $P$ and of the sequence $\Phi$ chosen from $\mathcal{F}$ such that

\[
\max \Gamma_{\Phi,S}(A, P, \epsilon) \leq \gamma_4.
\]

Proof. For simplicity, we write $\Gamma_S(\epsilon)$ instead of $\Gamma_{\Phi,S}(A, P, \epsilon)$. Taking $\kappa_1$ and $\kappa_2 < 1$ the constants from lemma 4.3 and 4.4, we choose $m \geq 1$ minimal such that $\kappa_2^m \leq \epsilon/5\kappa_1$. Then $\kappa_1$, $\kappa_2$ and $m$ depend only on $d_1, ..., d_k$ and on $\epsilon$.

If $n \leq m$ for all $n \in \Gamma_S(\epsilon)$, then

\[
\# \Gamma_S(\epsilon) \leq m \leq \frac{\log(5\kappa_1) + \log(\epsilon^{-1})}{\log(\kappa_2^{-1})} + 1,
\]
which is in the desired form. If there is an $n \in \Gamma_S(\epsilon)$ such that $n > m$, we fix $n$ for instance. Then by definition of $\Gamma_S(\epsilon)$ we have

$$e h_{\Phi^n}(\Phi^n(P)) \leq \sum_{v \in S} d_v \lambda_v(\Phi^n(P), A).$$

We can write $\Phi^n = \psi \circ \Phi^{n-m}$ for $\psi = \phi_{n-m} \circ \ldots \circ \phi_{n-m+1} \in \mathcal{F}_m$.

For our chosen $m$, we denote

$$e_m := \max_{A' \in \psi^{-1}(A)} e_{A'(\psi)},$$

and we notice by our choice of $m$ that

$$e_m \leq \kappa_1 \kappa_2^m \deg \psi \leq \epsilon \deg \psi / 5.$$

Therefore, proposition 4.1 yields, for $Q \in \mathbb{P}^1(K)$ and $\psi \in \mathcal{F}_m$, that

$$\sum_{v \in S} d_v \lambda_v(\psi(Q), A) - O(h(A) + h(\psi) + 1) \leq e_m \sum_{v \in S} \max_{A' \in \psi^{-1}(A)} d_v \lambda_v(Q, A').$$

Gathering the last two inequalities with $Q := \Phi^{n-m}(P)$ implies that

$$e h_{\Phi^n}(\Phi^n(P)) \leq e_m \sum_{v \in S} \max_{A' \in \psi^{-1}(A)} d_v \lambda_v(\Phi^{n-m}(P), A') + O(h(A) + h(\mathcal{F}_m) + 1),$$

where the big $O$ depends only on the degree of the functions in $\mathcal{F}_m$, and so on $d_1, \ldots, d_k$ and on $\epsilon$.

For each $v \in S$, we choose $A'_v \in \psi^{-1}(A)$ such that

$$\lambda_v(\Phi^{n-m}(P), A'_v) = \max_{A' \in \psi^{-1}(A)} \lambda_v(\Phi^{n-m}(P), A'),$$

so that

$$e h_{\Phi^n}(\Phi^n(P)) \leq e_m \sum_{v \in S} d_v \lambda_v(\Phi^{n-m}(P), A'_v) + O(h(A) + h(\mathcal{F}_m) + 1).$$

For instance, we can assume that $z(A') \neq \infty$ for all $A' \in \psi^{-1}(A), \Psi \in \mathcal{F}_m$. If this is not the case, we use $z$ for some of the $A'$ and $z^{-1}$ for the others.

Let $S' \subset S$ be the set of places in $S$ defined by

$$S' = \{ v \in S; \lambda_v(\Phi^{n-m}(P), A'_v) > \lambda_v(A'_v, \infty) + \log l_v \},$$

where again $l_v = 2$ for $v$ archimedean and $l_v = 1$ otherwise.

Set $S'' := S - S'$. Applying lemma 3.2 to the places in $S''$ and using the definition of $S''$ we find that

$$e h_{\Phi^n}(\Phi^n(P)) \leq e_m (\sum_{v \in S'} + \sum_{v \in S''}) d_v \lambda_v(\Phi^{n-m}(P), A'_v) + O(h(A) + h(\mathcal{F}_m) + 1)$$

$$\leq e_m \sum_{v \in S'} d_v (2 \lambda_v(A'_v, \infty) - \log |z(\Phi^{n-m}(P)) - z(A'_v)| + \log l_v)$$

$$+ e_m \sum_{v \in S''} d_v (\lambda_v(A'_v, \infty) + \log l_v) + O(h(A) + h(\mathcal{F}_m) + 1)$$

$$\leq e_m \sum_{v \in S'} d_v \log |z(\Phi^{n-m}(P)) - z(A'_v)|^{-1}$$

$$+ e_m \sum_{v \in S} d_v (2 \lambda_v(A'_v, \infty) + \log l_v) + O(h(A) + h(\mathcal{F}_m) + 1).$$

Now using theorem 2.3 and proposition 3.4 it can be checked that
\[
\sum_{v \in S} d_v \lambda_v(A'_v, \infty) \leq \sum_{A' \in \psi^{-1}(A)} \sum_{v \in S} d_v \lambda_v(A', \infty) \leq \sum_{A' \in \psi^{-1}(A)} h(A')
\]
\[
\leq \sum_{A' \in \psi^{-1}(A)} \hat{h}_{S^{n-m}(\Phi)}(A') + O(h(F) + 1)
\]
\[
= \sum_{A' \in \psi^{-1}(A)} (\deg \psi)^{-1} \hat{h}_{S^{n}(S^{n-m}(\Phi))}(\psi(A')) + O(h(F) + 1)
\]
\[
\leq \sum_{A' \in \psi^{-1}(A)} (\deg \psi)^{-1} \hat{h}_{S^{n}(\Phi)}(A) + O(h(F) + 1)
\]
\[
\leq \hat{h}_F(A) + O(h(F) + 1).
\]

The constants depend only on \(m\) and \(d_1, \ldots, d_k\).

Further, from the definition of \(l_v\), we have
\[
\sum_{v \in S} d_v \log l_v \leq \log 2.
\]

Also, from proposition 3.3 it follows that \(h(F_m) = O(h(F) + 1)\).

All the inequalities above together imply that
\[
\epsilon(\hat{h}_{S^n}(\Phi^n(P))) \leq e_m(\sum_{v \in S'} d_v \log |z(\Phi^{n-m}(P)) - z(A'_v)|^{-1}) + O(\hat{h}_F(A) + h(F) + 1).
\]

Let us set some definitions in order to apply Roth’s theorem. We define
\[
\Upsilon = \{z(A'); A' \in \psi^{-1}(A)\} \subset \bar{K},
\]
which is \(G_{K/K}\)-invariant and \(#\Upsilon \leq d_k^n\). We define the map \(\beta : S' \to \Upsilon\) by \(\beta_v := A'_v\) and analyze the points \(x = \Phi^{n-m}(P)\) for \(n \in \Gamma_S(\epsilon)\). Applying theorem 3.2 for the set of places \(S', M = 0\) and \(\mu = 5/2\), yields that there exist constants \(r_1, r_2\) depending only on \([K : \mathbb{Q}], d_1, \ldots, d_k\) and \(\epsilon\) such that the set of \(n \in \Gamma_S(\epsilon)\) with \(n > m\) can be written as a union
\[
\{n \in \Gamma_S(\epsilon); n > m\} = T_1 \cup T_2 \cup T_3
\]
such that
\[
\#T_1 \leq 4\#S' r_1,
\]
\[
T_2 = \{n > m; \sum_{v \in S'} d_v \log |z(\Phi^{n-m}(P)) - z(A'_v)|^{-1} \leq \frac{5}{2} h(\Phi^{n-m}(P))\},
\]
\[
T_3 = \{n > m; h(\Phi^{n-m}(P)) \leq r_2 \max_{v \in S'} \{h(A'_v, 1)\}\}.
\]

We already have a bound for the size of \(T_1\). For \(T_3\), we use again theorem 2.3 and proposition 3.4 to compute
\[
h(A'_v) \leq \hat{h}_{S^{n-m}(\Phi)}(A'_v) + c_3 h(F) + c_4 = (\deg \psi)^{-1} \hat{h}_{S^{n}(S^{n-m}(\Phi))}(A) + c_3 h(F) + c_4
\]
\[
= (\deg \psi)^{-1} \hat{h}_{S^{n}(\Phi)}(A) + c_3 h(F) + c_4
\]
\[ h(\Phi^{n-m}(P)) \geq \hat{h}_{S_{n-m}^{\circ}}(\Phi^{n-m}(P)) - c_3 h(\mathcal{F}) - c_4 = \deg(\Phi^{n-m})\hat{h}_\Phi(P) - c_3 h(\mathcal{F}) - c_4. \]

Hence

\[ T_3 \subset \{ n > m; d_1^{n-m}\hat{h}_\Phi(P) \leq c_5 \hat{h}_\mathcal{F}(A) + c_3 h(\mathcal{F}) + c_4 \}, \]

so every \( n \in T_3 \) satisfies

\[ n \leq m + \log_{d_1}^+ \left( \frac{c_5 \hat{h}_\mathcal{F}(A) + c_6 h(\mathcal{F}) + c_7}{\hat{h}_\Phi(P)} \right) \leq c_8 + \log_{d_1}^+ \left( \frac{\hat{h}_\mathcal{F}(A) + h(\mathcal{F})}{\hat{h}_\Phi(P)} \right). \]

Finally, we consider the set \( T_2 \). Again using theorem 2.3 and proposition 3.4 we derive

\[ \hat{h}(\Phi^{n-m}(P)) \leq \hat{h}_{S_{n-m}^{\circ}}(\Phi^{n-m}(P)) + c_3 h(\mathcal{F}) + c_4 \]

\[ = \deg(\Phi^{n-m})\hat{h}_\Phi(P) + c_3 h(\mathcal{F}) + c_4, \]

and then, for \( n \in T_2 \), using that \( e_m \leq \epsilon \deg \psi/5 \)

\[ \epsilon \hat{h}_{S_{\circ}^{\circ}}(\Phi^n(P)) = \epsilon \deg(\Phi^n)\hat{h}_\Phi(P) \]

\[ \leq e_m \left( \sum_{v \in S'} d_v \log |z(\Phi^{n-m}(P)) - z(A')|^{-1} \right) + c_9 (\hat{h}_\mathcal{F}(A) + h(\mathcal{F}) + 1) \]

\[ \leq \left( \frac{\epsilon \deg \psi}{5} \right) 2 \deg(\Phi^{n-m})\hat{h}_\Phi(P) + c_{10} (\hat{h}_\mathcal{F}(A) + h(\mathcal{F}) + 1) \]

\[ = \frac{\epsilon}{2} \deg(\Phi^n)\hat{h}_\Phi(P) + c_{10} (\hat{h}_\mathcal{F}(A) + h(\mathcal{F}) + 1). \]

Thus

\[ \frac{\epsilon}{2} \deg(\Phi^n)\hat{h}_\Phi(P) \leq c_{10} (\hat{h}_\mathcal{F}(A) + h(\mathcal{F}) + 1), \]

which implies that

\[ \frac{\epsilon}{2} d_1^n \hat{h}_\Phi(P) \leq c_{10} (\hat{h}_\mathcal{F}(A) + h(\mathcal{F}) + 1) \]

, equivalent to

\[ n \leq c_{11} + \log_{d_1}^+ \left( \frac{\hat{h}_\mathcal{F}(A) + h(\mathcal{F})}{\hat{h}_\Phi(P)} \right). \]

We observe that the set \( \Upsilon \) does not depend on the point, so the largest element in \( T_1 \) is bounded independently of \( P \). We also note that the quantity

\[ \hat{h}_{\min}(\mathcal{F},K) := \inf \{ \hat{h}_\Phi(P); \Phi \text{ a sequence generated by } \mathcal{F}, P \in \mathbb{P}^1(K) \text{ wandering for } \Phi \} \]

is strictly positive. Namely, from proposition 3.4, we know that

\[ \hat{h}_\Phi(P) \leq \hat{h}_\mathcal{F}(P) + O(h(\mathcal{F})) \text{ for any } \Phi \text{ generated by } \mathcal{F}, \text{ and any } P. \]

So if \( P_0 \) is a \( \Phi \)-wandering point, \( J = \{ 1, \ldots, k \} \) and \( W = \prod_{i=1}^\infty J \), then \( \hat{h}_\mathcal{F}(P) > 0 \) and

\[ \hat{h}_{\min}(\mathcal{F},K) := \inf \{ \hat{h}_{\Phi_w}(P); w \in W, P \in \mathbb{P}^1(K) \text{ and } 0 < \hat{h}_{\Phi_w}(P) \leq \hat{h}_\mathcal{F}(P) + O(h(\mathcal{F})) \}, \]
for this last set is finite by the Northcott property for \( \hat{h}_F \), so the infimum is taken over a finite set of positive numbers.

Therefore, \( \max(T_1 \cup T_2 \cup T_3) \) can be bounded independently of \( P \) and the choice of the sequence \( \Phi \) generated by the semigroup \( F \).

□

**Corollary 5.3.** Let \( K \) be a number field, \( S \subset M_K \) a finite set of places that includes all archimedean places, let \( R_S \) be the ring of \( S \)-integers of \( K \), and let \( 2 \leq d_1 \leq \ldots \leq d_k \).

There is a constant \( \gamma = \gamma(d_1, \ldots, d_k, [K : \mathbb{Q}]) \) such that for all dynamical systems \( F = \{\phi_1, \ldots, \phi_k\} \subset K(z) \) of rational maps of respective degrees \( d_1, \ldots, d_k \) that are not totally ramified at the \( F \)-orbit of \( \infty \) or that the \( F \)-orbit of \( \infty \) has no repeated points, and for any sequence \( \Phi \) of maps from \( F \) and all points \( P \in \mathbb{P}^1(K) \), the number of \( S \)-integers in the \( \Phi \)-orbit of \( P \) is bounded by

\[
\#\{n \geq 1; z(\Phi^n(P)) \in R_S\} \leq 4^{#S} \gamma + \log_{d_1} \left( \frac{h(F)}{\hat{h}_\Phi(P)} \right).
\]

**Proof.** An element \( \alpha \in K \) is in \( R_S \) if and only if \( |\alpha|_v \leq 1 \) for all \( v \notin S \), or equivalently, if and only if

\[
h(\alpha) = \sum_{v \in S} d_v \log \max\{|\alpha|_v, 1\}.
\]

Another fact is that

\[
\log \max\{|\alpha|_v, 1\} \leq \lambda_v(\alpha, \infty).
\]

This implies for \( \alpha \in R_S \) that \( h(\alpha) \leq \sum_{v \in S} d_v \lambda_v(\alpha, \infty) \).

Let \( n \geq 1 \) satisfy \( z(\Phi^n(P)) \in R_S \). Then

\[
h(\Phi^n(P)) \leq c.
\]

Proposition 2.4 and theorem 1.3 tell us that

\[
h(\Phi^n(P)) \geq \hat{h}_{S^c(\Phi)}(\Phi^n(P)) - c_3 h(F) - c_4 = \deg(\Phi^n) \hat{h}_\Phi(P) - c_3 h(F) - c_4,
\]

which implies that

\[
\deg(\Phi^n) \hat{h}_\Phi(P) - c_3 h(F) - c_4 \leq \sum_{v \in S} d_v \lambda_v(\Phi^n(P), \infty).
\]

The rest of the proof is divided into two cases: First one, when

\[
\deg(\Phi^n) \hat{h}_\Phi(P) \leq 2C_3 h(F) + 2c_4.
\]

In this case, \( d_1^n \hat{h}_\Phi(P) \leq 2C_3 h(F) + 2c_4 \), and then

\[
n \leq \log_{d_1}^+ \left( \frac{2C_3 h(F) + 2c_4}{\hat{h}_\Phi(P)} \right).
\]

In the second case , \( \deg(\Phi^n) \hat{h}_\Phi(P) \geq 2C_3 h(F) + 2c_4 \). Therefore

\[
\sum_{v \in S} d_v \lambda_v(\Phi^n(P), \infty) \geq \frac{1}{2} \deg(\Phi^n) \hat{h}_\Phi(P) = \frac{1}{2} h_{S^c(\Phi)}(\Phi^n(P)).
\]

Now the previous theorem with \( \epsilon = 1/2 \), \( A = \infty \) (\( \infty \) is not totally ramified for any map of the system) tells us that \( n \) is at most

\[
4^{#S} \gamma_3 + \log_{d_1}^+ \left( \frac{h(F) + \hat{h}_\Phi(\infty)}{\hat{h}_\Phi(P)} \right),
\]
for $\gamma_3$ depending only on $[K : \mathbb{Q}], d_1, \ldots, d_k$. Both bounds are on the desired form since $h_\Phi(\infty) << h(\infty) = 0$.

**Proposition 5.4.** Under the conditions and notations of the proof of theorem 5.2, $\max T_1$ is bounded by a constant that depends only on $A, d_1, \ldots, d_k, K, S$ and $\epsilon$, independent on which sequence of maps in the initial dynamical system was chosen. In particular, there exists $\gamma_2$ depending only on $A, d_1, \ldots, d_k, K, S$ such that

$$\max T_1 \leq \gamma_2 + \log_{d_1}^+ \left( \frac{h_F(A) + h(F)}{h_{\min,F,K}(P)} \right).$$

**Proof.** According to the proof of theorem B from [1], page 131, line 6, for the algebraic numbers $x$ approximating $\alpha$ satisfying Roth's theorem hypothesis, there exists a finite number (depending on the constants given by Roth's theorem 4.2) of $\beta_i$'s approximating $\alpha$ that depend only on $\alpha$ and on the parameters of theorem 4.2 such that

$$\log(4H(x)) \leq \frac{4\eta n}{\eta} \left( \frac{1}{\eta} \log(4H(\alpha)) + \log(4 \max_i H(\beta_i)) \right).$$

This implies that $h(x) \leq C(h(\alpha) + \max_i h(\beta_i))$, where $C$ depends only on the parameters of theorem 4.2. Translating this for the notation of our set $T_1$, as in the proof of theorem 5.2, we have that

$$h(\Phi^{n-m}(P)) \leq C(\max_{v \in S, \psi} h(A'_v) + \max_{i,v,\psi} h(\beta_{i,v}))$$

$$= O(\hat{h}_F(A) + h(F) + \max_{i,v,\psi} h(\beta_{i,v})) = O(\hat{h}_F(A) + h(F)) + \gamma,$$

for each $n \in T_1$ where $\gamma$ depends only on $A, d_1, \ldots, d_k, K, S$ and $\epsilon$ by our previous choice of $m$.

We have then that

$$d_1^{n-m} \hat{h}_\Phi(P) \leq \deg(\Phi^{n-m}) \hat{h}_\Phi(P) = \hat{h}_{S^{n-m} \Phi}(\Phi^{n-m}(P))$$

$$= h(\Phi^{n-m}(P)) + O(1) \leq O(\hat{h}_F(A) + h(F)) + \gamma_1,$$

for each $n \in T_1$, where $\gamma_1$ depends only on $A, d_1, \ldots, d_k, K, S$ and $\epsilon$.

Therefore,

$$\max T_1 \leq m + \log_{d_1}^+ \left( \frac{O(\hat{h}_F(A) + h(F)) + \gamma_1}{h_{\min,F,K}(P)} \right) \leq \gamma_2 + \log_{d_1}^+ \left( \frac{\hat{h}_F(A) + h(F)}{h_{\min,F,K}(P)} \right),$$

where $\gamma_2([\beta_{i,v})_{i,v,\psi})$ depends only on $A, d_1, \ldots, d_k, K, S$ and $\epsilon$, concluding the proof.

**Corollary 5.5.** Under the conditions of corollary 5.3, there is a constant $\gamma = \gamma(S, d_1, \ldots, d_k, [K : \mathbb{Q}])$ such that for all dynamical systems $F = \{\phi_1, \ldots, \phi_k\} \subset K(z)$ of rational maps of respective degrees $d_1, \ldots, d_k$ that are not totally ramified at the $F$-orbit of $\infty$ or that the $F$-orbit of $\infty$ has no repeated points, and all points $P \in \mathbb{P}^1(K)$, the number of $S$-integers in the $F$-orbit of $P$ is bounded by

$$\#\{Q \in O_F(P); z(Q) \in R_S\} \leq \frac{k^M - 1}{k - 1},$$

where $M = [\gamma + \log_{d_1}^+ (\frac{h(F)}{h_{\min,F,K}(P)})] + 1$.

**Proof.** If $Q \in O_F(P), z(Q) \in R_S$, then there exists a sequence $\Phi$ of maps from $F$, and a $n \geq 1$, such that $Q = \Phi^n(P)$ and $z(\Phi^n(P)) \in R_S$. By theorem 5.2, corollary 5.3 and proposition 5.4, there exists a $\gamma$ on the conditions stated in this theorem such that
And for each $m$, there are at most $k^m$ maps inside $F_m$, and therefore at most $k^m S$-integer points on the set $\{f(P) | f \in F_m\}$. The result follows from the identity $1 + k + \ldots + k^n = \frac{k^{n+1} - 1}{k - 1}$.

\[ n \leq \gamma + \log^+ \left( \frac{h(F)}{h_{\text{min}}^{F,K}(P)} \right). \]

Remark 5.6. In the particular case of a system of polynomial maps, that are non-special, the number of points whose orbit has repeated points is finite, due to Theorem 1.7 of [Ostafe, Young, 5], and therefore only for a finite number of points $A$ the hypothesis of theorem 5.2 will not be satisfied.

Remark 5.7. Theorem 5.2 delivers, in particular, under its conditions for sequences $\Phi$ of rational functions in a given system over a certain number field and $P, A$ rational numbers, an explicit upper bound for

\[ \#\{n \geq 1; \frac{1}{\Phi^n(P) - A} \text{ is quasi-}(S, \epsilon)\text{-integral }\}, \]

and this does not depend on which $\Phi$ was chosen from the initial dynamical system $F$.

Corollary 5.8. Under the hypothesis of theorem 5.2,

\[ \lim_{n \to \infty} \frac{\lambda_v(\Phi^n(P), A)}{\deg(\Phi^n)} = 0 \text{ for every } v \in M_K. \]

Proof. Applying theorem 5.2 for the set of places that contains just the place $v$, we conclude that for every natural $n$ big enough, it will be true that

\[ \frac{\lambda_v(\Phi^n(P))}{\deg(\Phi^n)} \leq \epsilon, \frac{\hat{h}_\Phi(P)}{d_v}. \]

Choosing $\epsilon$ sufficiently small, the result is proven.

Note that, due to theorem 5.2, the convergence above has an uniformity for the semigroup of maps, in the sense that the big natural $n$ does not depend on the $\Phi$ chosen in the semigroup generated by the initial dynamical system, so that actually the stronger fact

\[ \lim_{n \to \infty} \left( \sup_{\Phi \text{ seq. of } F} \frac{\lambda_v(\Phi^n(P), A)}{\deg(\Phi^n)} \right) = 0 \text{ for every } v \in M_K \]

is also true.

Corollary 5.9. Suppose that a dynamical system $=\{\phi_1, \ldots, \phi_k\} \subset \mathbb{Q}(z)$ of rational functions of degree at least 2 satisfies the hypothesis of theorem 5.2 with $P = \alpha \in \mathbb{Q}$ for $A = 0$ and $A = \infty$, and let $\Phi$ be a sequence of functions of $F$ such that $\#O_\Phi(\alpha) = \infty$. Write

\[ \Phi^n(\alpha) = \frac{a_n(t)}{b_n(t)} \in \mathbb{Q} \text{ as a fraction in lowest terms}. \]

Then

\[ \lim_{n \to \infty} \frac{\log |a_n(\alpha)|}{\log |b_n(\alpha)|} = 1. \]

Proof. From previous corollary, for $v$ the place at infinity, it is true that
\[
\lim_{n \to \infty} \frac{\lambda_v(\Phi^n(\alpha), 0)}{\deg(\Phi^n)} = \lim_{n \to \infty} \frac{\lambda_v(\Phi^n(\alpha), \infty)}{\deg(\Phi^n)} = 0.
\]

Working out similarly as in the proof of previous corollary, using proposition 3.4 (i), it is true that
\[
\lim_{n \to \infty} \frac{\lambda_v(\Phi^n(\alpha), 0)}{h(\Phi^n(\alpha))} = \lim_{n \to \infty} \frac{\lambda_v(\Phi^n(\alpha), \infty)}{h(\Phi^n(\alpha))} = 0.
\]

On the other hand, if \( t = \frac{a}{b} \in \mathbb{Q} \) written in lowest terms, since \( \max\{|a|, |b|\} \leq \sqrt{|a|^2 + |b|^2} \), then \( h(t) = \log \max\{|a|, |b|\} \) and
\[
\lambda_v(t, \infty) = \lambda_v([a, b], [1, 0]) = \log \left( \frac{\sqrt{|a|^2 + |b|^2}}{|b|} \right) = -\log |b| + \log(\sqrt{|a|^2 + |b|^2}) \geq -\log |b| + h(t).
\]

And in the same way
\[
\lambda_v(t, 0) = \lambda_v([a, b], [0, 1]) = \log \left( \frac{\sqrt{|a|^2 + |b|^2}}{|a|} \right) = -\log |a| + \log(\sqrt{|a|^2 + |b|^2}) \geq -\log |a| + h(t).
\]

Gathering these facts, and recalling that \( \Phi^n(\alpha) = \frac{a_n(t)}{b_n(t)} \) yields
\[
\lim_{n \to \infty} \frac{-\log |b_n(\alpha)| + h(\Phi^n(\alpha))}{h(\Phi^n(\alpha))} = \lim_{n \to \infty} \frac{-\log |a_n(\alpha)| + h(\Phi^n(\alpha))}{h(\Phi^n(\alpha))} = 0,
\]
and thus
\[
\lim_{n \to \infty} \frac{\log |b_n(\alpha)|}{\log \max\{|a_n(\alpha)|, |B_n(\alpha)|\}} = \lim_{n \to \infty} \frac{\log |a_n(\alpha)|}{\log \max\{|a_n(\alpha)|, |B_n(\alpha)|\}} = 1.
\]

This implies that
\[
\lim_{n \to \infty} \frac{\log |a_n(\alpha)|}{\log |b_n(\alpha)|} = 1.
\]

\[\square\]

**Remark 5.10.** Again, from theorem 5.2, for a given \( \alpha \in \mathbb{Q} \), the last result does not depend on \( \Phi \), in the sense that for every sequence \( \Phi \) of functions in the tree of functions determined by the initial dynamical system \( F \), the correspondent quotient sequences \( \frac{\log |a_n(\alpha)|}{\log |b_n(\alpha)|} \) converge to 1 as \( n \) goes to \( \infty \) with the same speed.

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