COMPOSITION OPERATORS ON HAAGERUP $L^p$-SPACES

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Abstract. Building on the ideas in [Lab] we indicate how the concept of a composition operator may be extended to the context of Haagerup $L^p$-spaces.

1. Introduction

Classically a (generalised) composition operator $C$ is a bounded linear operator $C : L^p(X_1, \Sigma_1, m_1) \to L^q(X_2, \Sigma_2, m_2)$ which in a canonical way is induced by a non-singular measurable transformation $T : Y \subset X_2 \to X_1$ from a measurable subset $Y$ of $X_2$ into $X_1$ in the sense that $C(f)(t) = f \circ T(t)$ if $t \in Y$ and $C(f)(t) = 0$ otherwise. In the setting of standard Borel spaces, up to sets of measure zero, such non-singular measurable transformations are in 1-1 correspondence with $\ast$-homomorphisms $L^\infty(X_1, \Sigma_1, m_1) \to L^\infty(X_2, \Sigma_2, m_2)$. (See for example the discussion in section 2.1 of [SM].) So in the noncommutative world the study of composition operators on $L^p$-spaces translates to a description and study of those Jordan $\ast$-morphisms $J : M_1 \to M_2$ which in some canonical sense induce a bounded operator $C_J : L^p(M_1) \to L^q(M_2)$, where $L^p(M_1)$ and $L^q(M_2)$ are the corresponding noncommutative spaces (The definitions we use will be given in the next section). Now even in the commutative setting the case $p < q$ tends to be pathological (see [TY, Corollary, Lemma 1.5]). In the noncommutative setting one has a negative result of Junge and Sherman [JS, Corollary 2.7]. Thus we will focus on the case where $\infty \geq p \geq q \geq 1$.

At the outset of any self-respecting theory of composition operators two questions need to be answered: Firstly the question of which point transformations actually induce composition operators, and secondly the question of how in the class of all bounded linear maps from $L^p$ to $L^q$ we may recognise those that come from point transformations. In our noncommutative endeavour this translates to firstly identifying those Jordan $\ast$-morphisms $J : M_1 \to M_2$ that canonically induce bounded maps $C_J : L^p(M_1) \to L^q(M_2)$, and secondly describing those bounded maps between noncommutative $L^p$-spaces that come from Jordan $\ast$-morphisms. In section 3 we will indicate how the classical process for constructing composition operators on $L^p$-spaces may be extended to the setting of von Neumann algebras as well as indicating a possible answer to the above two questions.

We tried to make the exposition accessible to both specialists in operator algebras, and also specialists dealing with composition operators on classical function spaces. This means that in many places we explain more than is strictly necessary,
especially for specialists in operator algebras. However we do this consciously for
the sake of reaching a larger audience.

We would like to thank David Sherman, who directed our attention to the paper
of Junge and Sherman [55], and to the fact that their Theorem 2.5 on the gen-
eral form of the (right) $\mathcal{M}$-module homomorphisms of noncommutative $L^p$
spaces implies our change of weight result (see Step II in Section 3). It turned out that
after a slight modification we were able to prove their theorem using our method,
at least in the case when $1 \leq q \leq p \leq \infty$. We decided to show the proof to the
reader, as it differs substantially from the proof of Junge and Sherman in that it
uses essentially only duality arguments.

2. PREREQUISITES

Throughout this paper we will assume that $\mathcal{M}_1$ and $\mathcal{M}_2$ are von Neumann alge-
bras with faithful normal semifinite (fns for short) weights $\varphi_1$ and $\varphi_2$ respectively.
For a von Neumann algebra $\mathcal{M}$ with an fns weight $\varphi$, the crossed product of $\mathcal{M}$
with the modular action induced by $\varphi$ will be denoted by $\mathcal{M} \rtimes_\varphi \mathbb{R}$ and the
canonical trace on $\mathcal{M} \rtimes_\varphi \mathbb{R}$ by $\tau$. The Haagerup $L^p$ space constructed by means of the action
of $\varphi$ is denoted by $L^p_\varphi(\mathcal{M})$. Now let $h = \frac{d^\circ}{\tau}$ where $\varphi$ is the dual weight on
the crossed product. Then $h$ is a closed densely defined positive non-singular operator
affiliated with the crossed product. In general, $h$ is not $\tau$-measurable, so it has to
be manipulated with caution.

Define, for $q \in [2, \infty[$,

\[ n^{(q)}_\varphi = n^{(q)} := \{ a \in \mathcal{M} : ah^{1/q} \text{ is closable and } [ah^{1/q}] \in L^p_\varphi(\mathcal{M}) \} \]

(here $[\cdot]$ is used to denote the minimal closure of a given closable operator). For $p \in
[1, \infty[$, denote by $m^{(p)}$ the linear span of elements of the form $b^*a$ with $a, b \in \mathcal{N}^{(2p)}$.
Then $n_\varphi = n^{(2)}$, where $n_\varphi = \{ a \in \mathcal{M} : \varphi(a^*a) < \infty \}$, and $m_\varphi = m^{(1)} \subset m^{(p)}$ for
each $p > 1$, where $m_\varphi$ is linearly spanned by positive elements $a$ from the algebra
satisfying $\varphi(a) < \infty$. The linear extension to $m^{(p)}$ of the map

\[ a \mapsto h^{1/(2p)}a^{1/2} : [a^{1/2}h^{1/(2p)}] : m^{(p)}_\varphi \rightarrow L^p_\varphi(\mathcal{M}) \]

is denoted by $i^{(p)}$, and the image of $a$ under the mapping by $h^{1/(2p)}ah^{1/(2p)}$.
Other than that, we use the following convention: whenever a formula consists of (pre)measurable operators only, their juxtaposition denotes their strong product;
otherwise, it denotes the usual operator product, and we use square brackets
for the closure of a closable operator. Sometimes we add parentheses to avoid ambi-
guity. For example, if $h$ is not measurable, but $a, b$ and $h^{1/p}b$ are, we write $a(h^{1/p}b)$
to denote the strong product of $a$ and $h^{1/p}b$.

Let now $X_0$ denote the completion of $m_\varphi$ equipped with the norm $\|a\|_0$ equal to
the maximum of $\|a\|$ and $\|i^{(1)}(a)\|_1$. The mappings $i^{(p)}$ can be extended to bounded
maps from $X_0$ into $L^p_\varphi(\mathcal{M})$. Denote by $\kappa_p, 1 < p \leq \infty$, the Banach space adjoint of
$i^{(p^*)}$, where $p^*$ is the conjugate index of $p$. Define additionally $\kappa_1$: if $h_\psi$ is an element
of $L^1_\varphi(\mathcal{M})$ corresponding to the functional $\psi \in \mathcal{M}_\varphi$ (i.e. $h_\psi = \frac{d\psi}{\tau}$), then $\kappa_1(h_\psi)$ is
an element of $X^*_\varphi$ which maps $a \in m_\varphi$ onto $\varphi(a)$. Then $\kappa_p$ maps $L^p_\varphi(\mathcal{M})$ boundedly
into $X^*_\varphi$. If we denote by $X_1$ the closure of $\kappa_\infty(L^\infty_\varphi(\mathcal{M})) + \kappa_1(L^1_\varphi(\mathcal{M}))$ in $X^*_\varphi$ and
by $L^p(\mathcal{M}, \varphi)$ the image $\kappa_p(L^p_\varphi(\mathcal{M}))$ equipped with the norm $\|a\|_p^\varphi = \|(\kappa_p(\varphi))^{-1}a\|_p$, then
$L^p(\mathcal{M}, \varphi) = C_{1/p}(X_0, X_1)$, where $C_{0, \theta}, 0 \leq \theta \leq 1$ is the $\theta$'s interpolation functor.
for the complex interpolation method of Calderon. The spaces $L^p(M, \varphi)$ are the Terp interpolation spaces. (For a precise explanation of the interpolation method the reader is directed to Terp’s paper [Tp2].)

The theory is simpler if $\varphi$ is a state. Then we may define the embeddings $\kappa_p(\varphi) : L^p_f(M) \rightarrow L^1_f(M) : a \mapsto h^{1/(3p')} a h^{1/(2p')}$. The Kosaki interpolation spaces (Kos) then correspond to the spaces $L^p(M, \varphi) = \kappa_p(\varphi)(L^p_f(M))$ equipped with the norm $\|a\|_{p}^p = \|(\kappa_p(\varphi))^{-1}a\|_p$. In this setting the derivative $h$ may also be used to define embeddings $M \rightarrow L^p_f(M) : a \mapsto h^{(1-c)/p} ah^{c/p} (0 \leq c \leq 1)$ of $M$ into $L^p_f(M)$ (GLL). For these embeddings the case $c = \frac{1}{2}$ has the added advantage of being positivity preserving, and so for this distinguished case we will employ the notation $i(p)$ for the associated embedding.

As we have seen above, the Terp interpolation spaces are defined only for the situation when $c = 1/2$. The interested reader can find a further generalization of the interpolation for the weight case, so as to incorporate the cases when $c \neq 1/2$, in [I]. In settings where several algebras or weights are involved we will employ suitable subscripts to distinguish these cases.

In the sequel, by the term Jordan *-morphism we understand a map from a $C^*$-algebra into another $C^*$-algebra which preserves adjoints and squares of elements.

3. Defining generalised composition operators

Let $(X_i, \Sigma_i, m_i) (i = 1, 2)$ be standard Borel spaces and let $T : Y \subset X_2 \rightarrow X_1$ be a given non-singular measurable transformation from a measurable subset $Y$ of $X_2$ into $X_1$. Then for $\infty > p \geq q \geq 1$ the formula $C_T(f)(t) = f \circ T(t)$ if $t \in Y$ and $C_T(f)(t) = 0$ otherwise, induces a bounded linear operator $C_T : L^p(X_1, \Sigma_1, m_1) \rightarrow L^q(X_2, \Sigma_2, m_2)$ if and only if $m_2 \circ T^{-1}$ is absolutely continuous with respect to $m_1$ and $\frac{dm_2 \circ T^{-1}}{dm_1}$ belongs to $L^r(X_1, \Sigma_1, m_1)$ where $r = \frac{p}{p-q}$. So we see that when it comes to the formal existence of a (generalised) composition operator in the case where $1 \leq q < \infty$, some form of absolute continuity is crucial. (Boundedness of the composition operator is in turn conditioned by the behaviour of the associated Radon-Nikodym derivative.) We will see that even in the noncommutative world it is precisely some form of absolute continuity that once again enables us to formally introduce the concept of a (generalised) composition operator.

Let $M$ be a von Neumann algebra with $fns$ weight $\varphi$ and let $h = \frac{d\varphi}{dt}$. Then the span of the set

$$\{h^{1/(2p')} e h^{1/(2p')} | e \in M \text{ a projection}, \varphi_1(e) < \infty\}$$

is known to be norm dense in $L^p_f(M)$ if $1 \leq p < \infty$. We may think of this span as representing the simple functions in $L^p_f(M)$. In the context of classical $L^p$ spaces on standard Borel measure spaces a bounded linear operator from $L^p$ to $L^q$ is known to be a (generalised) composition operator precisely if it takes characteristic functions in $L^p$ to characteristic functions in $L^q$. (See for example [Lab].) Now let $h_i = \frac{d\varphi_i}{dt}$, and let $J : M_1 \rightarrow M_2$ be a normal Jordan *-morphism satisfying the condition that for any projection $e \in M_1$ with $\varphi_1(e) < \infty$, we always have that $\varphi_2(J(e)) < \infty$. In such a case the formal process $h_1^{1/(2p')} ah_1^{1/(2p')} \rightarrow h_2^{1/(2q')} ah_2^{1/(2q'')} (a \in M)$ is at least densely defined on $L^p_f(M_1)$. If indeed the process extends to a bounded map $C_J : L^p_f(M_1) \rightarrow L^q_f(M_2)$, then by analogy with the classical context mentioned above, we may think of $C_J$ as a (generalised) composition operator induced by $J$. 
We proceed to indicate that the condition regarding the Jordan *-morphism's action on projections with finite weight may be interpreted as a type of local absolute continuity. Thus the proposed definition of composition operators compares well with the classical setting in that here too some form of absolute continuity of \( \varphi_2 \circ J \) with respect to \( \varphi_1 \) is a prerequisite for the existence of a composition operator.

We start with a simple generalisation of a well known fact regarding absolute continuity of finite measures. First, we give the following definitions.

**Definition 3.1.** Let \( \varphi_0, \varphi_1 \) be weights on a von Neumann algebra \( \mathcal{M} \).

1. We say that \( \varphi_0 \) is \( \epsilon, \delta \) absolutely continuous with respect to \( \varphi_1 \) if, for every \( \epsilon > 0 \) we can then find a \( \delta > 0 \) so that for any projection \( e \in \mathcal{M} \) with \( \varphi_1(e) < \delta \) we will have that \( \varphi_0(e) < \epsilon \). For a projection \( e \in \mathcal{M} \), the weight \( \varphi_0 \) is called \( \epsilon, \delta \) absolutely continuous with respect to \( \varphi_1 \) on \( e \) if the restriction of \( \varphi_0 \) to the von Neumann algebra \( e \mathcal{M} e \) is \( \epsilon, \delta \) absolutely continuous with respect to the restriction of \( \varphi_1 \) to \( e \mathcal{M} e \).

2. We say that \( \varphi_0 \) is locally absolutely continuous with respect to \( \varphi_1 \) if, for each projection \( e \in \mathcal{M} \), \( \varphi_1(e) < \infty \) implies \( \varphi_0(e) < \infty \). If this is the case, we write \( \varphi_0 \ll_{\text{loc}} \varphi_1 \).

We are going to show that (under very mild conditions) local absolute continuity is, in fact, absolute continuity on each projection of finite weight, so that the name is well chosen. In the sequel, we assume that the weight \( \varphi_1 \) is semifinite. Although this assumption is not really needed, it makes statements of the results slightly easier, and is exactly what we need in practice. Moreover, if \( \varphi_1 \) is not semifinite, there exists a greatest projection \( e \) such that \( \varphi_1 \) is semifinite when restricted to \( e \mathcal{M} e \); it is enough to take for \( e \) the unit of the von Neumann algebra generated by projections of finite \( \varphi_1 \)-weight. Thus, we can always restrict our attention to an algebra on which the weight in question is semifinite.

**Lemma 3.2.** Let \( \varphi_1, \varphi_0 \) be normal states on a von Neumann algebra \( \mathcal{M} \) with \( \varphi_1 \) also faithful. Then \( \varphi_0 \) is \( \epsilon, \delta \) absolutely continuous with respect to \( \varphi_1 \).

**Proof.** Note that the sets \( \{ x \in (\mathcal{M})_1 : \varphi_1(x^*x) < \epsilon \} \), with \( \epsilon > 0 \), form a basis of neighbourhoods of zero for the strong topology on the unit ball of \( \mathcal{M} \). Hence, the conclusion follows from the strong continuity of \( \varphi_0 \) on the ball. \( \square \)

The next two lemmas collect various facts belonging to the mathematical folklore.

**Lemma 3.3.** Let \( \mathcal{M} \) be a von Neumann algebra with no minimal projections. Then any maximal abelian von Neumann subalgebra \( \mathcal{M}_0 \) of \( \mathcal{M} \) also has no minimal projections \([\text{GJL}]\). If \( \mathcal{M} \) admits a faithful normal state \( \varphi \), then the algebra \( \mathcal{M}_0 \) corresponds to a classical \( L^\infty(\Omega, \Sigma, \mu_\varphi) \), where \( (\Omega, \Sigma, \mu_\varphi) \) is a nonatomic probability space and the measure \( \mu_\varphi \) is defined by \( \mu_\varphi(E) = \varphi(\chi_E) \) for each \( E \in \Sigma \).

**Proof.** The first statement was noted in \([\text{GJL}]\). The second follows from the fact that any commutative von Neumann subalgebra \( \mathcal{M}_0 \) will correspond to some \( L^\infty(\Omega, \Sigma, \nu) \). In particular given a faithful normal state \( \varphi \) on \( \mathcal{M} \), it is an exercise to show that the restriction of \( \varphi \) to \( \mathcal{M}_0 = L^\infty(\Omega, \Sigma, \nu) \) defines a probability measure \( \mu_\varphi = \mu \) on \( (\Omega, \Sigma) \) (with the same sets of measure zero as \( \nu \)) by means of the prescription \( \mu(E) = \varphi(\chi_E) E \in \Sigma \). Replacing \( \nu \) by \( \mu \) if necessary, all that remains is to note that the subalgebra \( \mathcal{M}_0 = L^\infty(\Omega, \Sigma, \mu) \) has no minimal projections precisely when \( (\Omega, \Sigma, \mu) \) is nonatomic. \( \square \)
Lemma 3.4. Let \((\Omega, \Sigma, \mu)\) be a nonatomic probability space and let \(\nu\) be a measure on \((\Omega, \Sigma)\) which is \(\epsilon\)-\(\delta\) absolutely continuous with respect to \(\mu\). Then \(\nu\) is a finite measure.

Proof. Let \(\epsilon\) be given and select \(\delta\) so that for any \(E \in \Sigma\) with \(\mu(E) < \delta\) we will have that \(\nu(E) < \epsilon\). We show that we may write \(\Omega\) as the union of a finite collection \(E_1, E_2, \ldots, E_n\) of disjoint sets in \(\Sigma\) with \(\mu(E_k) < \delta\) for each \(1 \leq k \leq n\). It then trivially follows that \(\nu(\Omega) = \sum_{k=1}^{n} \nu(E_k) < n \epsilon < \infty\) as required. To see that such a partitioning of \(\Omega\) is indeed possible let \(n \in \mathbb{N}\) be given such that \(\frac{1}{n} < \delta\), and use Zorn’s lemma to find a maximal set \(E_1 \in \Sigma\) with \(\mu(E_1) \leq \frac{1}{n}\). Now given any \(E \in \Sigma\) with \(\mu(E) < \frac{1}{n}\) we can then use the nonatomicity of \((\Omega, \Sigma, \mu)\) to find a larger set \(F \in \Sigma\) with \(\mu(E) < \mu(F) < \frac{1}{n}\). Hence the maximality of \(E_1\) ensures that \(\mu(E_1) = \frac{1}{n}\). To complete the proof we may now continue inductively by finding a measurable subset \(E_2\) of \(\Omega - E_1\) such that \(\mu(E_2) = \frac{1}{n} \mu(\Omega - E_1) = \frac{1}{n}\), and so on.

Theorem 3.5. Let \(\mathcal{M}\) be an arbitrary von Neumann algebra, \(\varphi_1\) be a faithful normal semifinite weight on \(\mathcal{M}\) and \(\varphi_0\) a normal weight on \(\mathcal{M}\), semifinite on its atomic part. The following conditions are equivalent:

1. \(\varphi_0\) is locally absolutely continuous with respect to \(\varphi_1\);
2. \(\varphi_0\) is \(\epsilon\)-\(\delta\) absolutely continuous with respect to \(\varphi_1\) on each projection \(e \in \mathcal{M}\) with \(\varphi_1(e) < \infty\).

Proof. That local absolute continuity implies \(\epsilon\)-\(\delta\) absolute continuity on each projection \(e \in \mathcal{M}\) with \(\varphi_1(e) < \infty\) follows immediately from Lemma 3.2. For the reverse implication, we fix \(\epsilon > 0\) and take the corresponding \(\delta\) from the definition of the \(\epsilon\)-\(\delta\) absolute continuity. One notes first that if the algebra \(\mathcal{M}\) is a direct sum of a finite number of summands, it is enough to prove the implication on each summand separately. Thus, it is enough to consider the following four cases:

1. The algebra \(\mathcal{M}\) is non-atomic (i.e. it has no minimal projections). Let \(e\) be a projection such that \(\varphi_1(e) < \infty\). Now since \(e\) belongs to a maximal abelian subalgebra, say \(\mathcal{M}_0\), of \(e\mathcal{M}e\), it suffices to prove that if \(\varphi_0\) restricts to a normal weight on \(\mathcal{M}_0\) which is \((\epsilon - \delta\) absolutely continuous with respect to the action of \(\varphi_1\) on \(\mathcal{M}_0\), then \(\varphi_0|_{\mathcal{M}_0}\) is a finite weight on \(\mathcal{M}_0\). Without loss of generality we may of course normalise the action of \(\varphi_1\) on \(\mathcal{M}_0\). Then by Lemma 3.3 the algebra \(\mathcal{M}_0\) corresponds to a classical \(L^\infty(\Omega, \Sigma, \mu)\), where \((\Omega, \Sigma, \mu)\) is a nonatomic probability space and the measure \(\mu\) is defined by \(\mu(E) = \varphi_1(\chi_E)\) for each \(E \in \Sigma\). In a similar fashion the weight \(\varphi_0\) also defines a measure \(\nu\) on \((\Omega, \Sigma)\) by means of the formula \(\nu(E) = \varphi_0(\chi_E)\) for each \(E \in \Sigma\). We may then directly conclude from Lemma 3.4 that \(\varphi_0(e) = \varphi_0(\chi_{\Omega}) = \nu(\Omega) < \infty\) as required.
2. The algebra \(\mathcal{M}\) is a factor of type \(I_\infty\) (where \(\infty\) stands for any infinite cardinal). Let \(e\) be a projection such that \(\varphi_1(e) = 1\). Note that \(\varphi_0\) is finite on any minimal projection of \(\mathcal{M}\), by semifiniteness. Hence we may assume that \(e\) is (properly) infinite. Write \(e\) in the form \(e = \sum_{k=1}^{n} e_k\), where the projections \(e_k\) are all equivalent to \(e\). Choose \(n\) so that \(\frac{1}{n} < \delta\). Then, for some \(k\), \(\varphi_1(e_k) < 1/n\). Since \(\varphi_1(e_k) \leq \varphi_1(e - e_k)\) and \(e_k \sim e - e_k\), there is a projection \(f_1\) in \(\mathcal{M}\) such that \(f_1 \leq e\) and \(\varphi_1(f_1) = 1/n\) (see, for example,
(3) Assume now that $\mathcal{M}$ is finite and atomic. Then $\mathcal{M}$ is of the form $\sum_{i \in I} M_i$, where each $M_i$ is a factor of type $I_n$, with $n_i < \infty$. As before, since $\varphi_0$ is semifinite on $\mathcal{M}$, we may assume that the index set $I$ is infinite. Let $e$ be a projection such that $\varphi_0(e) < \infty$. Then $e$ is of the form $\sum_{i \in I} e_i$ and there exists a finite subset $J$ of $I$ such that $\varphi_0(\sum_{i \in I \setminus J} e_i) < \delta$. Hence $\varphi_0(e) = \sum_{i \in J} \varphi_0(e_i) + \varphi_0(\sum_{i \in I \setminus J} e_i) < \infty$, by the $\epsilon$-$\delta$ condition and the semifiniteness of $\varphi_0$.

(4) Assume finally that $\mathcal{M}$ is an infinite direct sum of type $I_\infty$ factors. We obtain the result as in (3), from the $\epsilon$-$\delta$ condition and (2).

Note that we did not use the assumption that $\varphi_0$ is normal in the proof of the reverse implication. □

**Remark 3.6.** Let $\mathcal{M}$ be a von Neumann algebra with two normal weights $\varphi_0$ and $\varphi_1$, with $\varphi_1$ also semifinite and faithful. Now if $\varphi_0$ was locally absolutely continuous with respect to $\varphi_1$, then $\varphi_0$ would also be semifinite! To see this all we need to notice is that the linear span of all projections $e \in \mathcal{M}$ with $\varphi_1(e) < \infty$ is $\sigma$-weakly dense in $\mathcal{M}$.

In the sequel, whenever we deal with a von Neumann algebra $\mathcal{M}$ with a fixed weight $\varphi$, we shall write $\mathcal{M}^{(0)}$ for the span of the set $\{e|e \in \mathcal{M} \text{ a projection}, \varphi(e) < \infty\}$. The weight used to define $\mathcal{M}^{(0)}$ will always be clear from the context.

We are now ready to formally define the concept of a composition operator on Haagerup $L^p$-spaces.

**Definition 3.7.** Let $J : \mathcal{M}_1 \to \mathcal{M}_2$ be a normal Jordan $*$-morphism, let $h_i = \frac{d\varphi_i}{d\tau_i}$.

Given $1 \leq q \leq p < \infty$, we say that $J$ induces a generalised composition operator (or just a composition operator) if $J(1) = 1l$ from $L^p_{\varphi_1}(\mathcal{M}_1)$ into $L^q_{\varphi_2}(\mathcal{M}_2)$

- if $\varphi_2 \circ J$ is locally absolutely continuous with respect to $\varphi_1$,
- and if the process $h_1^{1/(2p)} a h_1^{1/(2p)} \to h_2^{1/(2q)} J(a) h_2^{1/(2q)} (a \in \mathcal{M}_1^{(0)})$ is continuous.

The above process then extends uniquely to a bounded map $C_J : L^p_{\varphi_1}(\mathcal{M}_1) \to L^q_{\varphi_2}(\mathcal{M}_2)$, which we shall call the (generalised) composition operator induced by $J$ from $L^p_{\varphi_1}(\mathcal{M}_1)$ into $L^q_{\varphi_2}(\mathcal{M}_2)$. (Here we used the fact that $1^{(p)}(\mathcal{M}_1^{(0)})$ is norm dense in $L^p_{\varphi_1}(\mathcal{M}_1)$.)

**Remark 3.8.** By analogy with the above definition we may say that $J$ induces a generalised composition operator from $\mathcal{M}_1 = L^\infty_{\varphi_1}(\mathcal{M}_1)$ into $L^q_{\varphi_2}(\mathcal{M}_2)$ $(1 \leq q < \infty)$ if the map $C_J : a \to h_2^{1/(2q)} J(a) h_2^{1/(2q)} (a \in \mathcal{M})$ is well-defined and continuous from $\mathcal{M}_1$ into $L^q_{\varphi_2}(\mathcal{M}_2)$. For this map to be well-defined we at least need $J(1)h_2^{1/(2q)}$ to be closable with closure an element of $L^q_{\varphi_2}(\mathcal{M}_2)$ (see (GL2): 2.7 and 2.8).

Conversely if $J(1)h_2^{1/(2q)}$ is closable with closure an element of $L^q_{\varphi_2}(\mathcal{M}_2)$, then the above map is well-defined and continuous. In fact, for for any $a \in \mathcal{M}_1$ we will have that $J(a)J(1) \in n_{\varphi_2}^{(2q)}$ and hence that $\|J(1)h_2^{1/(2q)}a^{*}[J(a)J(1)h_2^{1/(2q)}] = [h_2^{1/(2q)}J(a)J(1)]h_2^{1/(2q)}J(1)J(a)h_2^{1/(2q)}a^{*}\| \leq \|h_2^{1/(2q)}J(a)J(1)\| \leq \|a\|_{\infty} h_2^{1/(2q)}J(1)J(a)h_2^{1/(2q)}a^{*}\| \leq \|a\|_{\infty} h_2^{1/(2q)}J(1)J(a)h_2^{1/(2q)}a^{*}\|$. 

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**[GP], Proposition 1.1.** The rest follows the lines of the proof of Lemma 3.4.
From this it follows that
\[ \|h_2^{1/(2q)} J(a) h_2^{1/(2q)}\|_q \leq \|a\|_\infty \|h_2^{1/(2q)} J(\|h_2^{1/(2q)}\|_q \|a\|_\infty). \]
for each \(a \in M_1\) (or rather \(\|C_J(a)\|_q \leq \|C_J(\|a\|_\infty).\) This clearly suffices to force continuity of the induced map.

4. Identifying and describing composition operators

Having introduced the concept of a composition operator on Haagerup \(L^p\)-spaces, we now focus on the two-fold task of firstly finding a way to identify those operators that actually are composition operators, and secondly describing those Jordan morphisms between von Neumann algebras that do indeed induce composition operators on the associated \(L^p\)-spaces.

Operators on Haagerup \(L^p\)-spaces that come from Jordan \(*\)-morphisms.

We noted earlier that on classical \(L^p\) spaces of standard Borel measure spaces, bounded linear operators from \(L^p\) to \(L^q\) are (generalised) composition operators precisely when they take characteristic functions in \(L^p\) to characteristic functions in \(L^q\). (See for example [Lab].) The primary result of this section shows that a similar structure pertains even in the noncommutative context. In this regard we remind the reader that given a von Neumann algebra \(M\) equipped with a faithful normal semifinite weight \(\varphi\), the role that is classically played by characteristic functions \(\varphi(h)\) will here be played by elements of the form \(h^{1/(2p)} \varphi \ h^{1/(2q)}\) where \(e\) is a self-adjoint projection in \(M\) with finite weight, and \(h = \frac{d\varphi}{d\tau}\). Thus by analogy with the classical setting, it is natural to try and describe composition operators in terms of their action on elements of the above form.

Definition 4.1. Let \(1 \leq p, q \leq \infty\) and let \(M_i\) \((i = 1, 2)\) be von Neumann algebras equipped with faithful normal semifinite weights \(\varphi_i\). We say that a bounded linear operator \(S : L^p_{\varphi_i}(M_1) \to L^q_{\varphi_2}(M_2)\) preserves characteristic functions if for any projection \(e \in M_1\) (with \(\varphi_1(e) < \infty\) if \(p < \infty\)) there exists a unique projection \(\tilde{e} \in M_2\) (with \(\varphi_2(\tilde{e}) < \infty\) if \(q < \infty\)) such that \(S(h_1^{1/(2p)} \varphi_1 h_1^{1/(2p)}) = h_2^{1/(2q)} \tilde{e} h_2^{1/(2q)}\) (where \(h_i = \frac{d\varphi_i}{d\tau}\)).

Theorem 4.2. Let \(1 \leq p, q \leq \infty\) and let \(M_i\) \((i = 1, 2)\) be von Neumann algebras equipped with faithful normal semifinite weights \(\varphi_i\). Let \(C(M_1)\) denote the \(C^*\)-subalgebra of \(M_1\) generated by \(M_1^{(0)}\). Let \(S : L^p_{\varphi_1}(M_1) \to L^q_{\varphi_2}(M_2)\) be a bounded linear operator which preserves characteristic functions.

If \(p = \infty\) then for some Jordan morphism \(J : M_1 \to M_2\), \(S\) is precisely of the form \(a \mapsto h_2^{1/(2q)} J(a) h_2^{1/(2q)}\) where \(a \in M_1\).

If \(p < \infty\), there exists a (not necessarily normal) Jordan \(*\)-morphism \(J : C(M_1) \to M_2\) such that \(S\) appears as the continuous extension of the map \(h_1^{1/(2p)} a h_1^{1/(2p)} \to h_2^{1/(2q)} J(a) h_2^{1/(2q)}\) where \(a \in M_1^{(0)}\). In this case \(J\) will be normal precisely when it satisfies the requirement that if mutually orthogonal projections \(e_1, \ldots, e_n\) in \(M_1^{(0)}\), sets of mutually orthogonal projections \(\{f_i^{(k)}\}\) \((k = 0, 1, \ldots, m)\) in \(M_1^{(0)}\), and positive scalars
\( \lambda_1, \lambda_2, \ldots, \lambda_n \) and \( \mu_1, \mu_2, \ldots, \mu_m \) are such that

\[
\sum_{p=1}^{n} \lambda_p \epsilon_p \leq \sum_{k=1}^{m} \mu_k \left( \sum_{i \in I} f_i^{(k)} \right),
\]

it then follows that

\[
\sum_{p=1}^{n} \lambda_p J(e_p) \leq \sum_{k=1}^{m} \mu_k \left( \sum_{i \in I} J(f_i^{(k)}) \right).
\]

If in fact \( \varphi_1, \varphi_2 \) are states and \( p < \infty \), then \( J \) is necessarily normal (and of course defined on all of \( \mathcal{M}_1 \)).

Note that in the above we do not require that \( q \leq p \).

Next let \( 1 \leq p < \infty \). In the commutative case normality of \( J \) will then still be automatic even if \( \varphi_1, \varphi_2 \) are not states. \( \) (See [Lab 4.3 & 4.15(ii)]) However, although we have no proof for this as yet, we suspect that in the noncommutative setting \( \sigma \)-finiteness is essential to obtain automatic normality of \( J \).

**Proof.** The proofs for the cases \( p = \infty \) and \( p < \infty \) are similar, and hence we prove only the latter case. Let \( S : L_{p_q}^p(\mathcal{M}_1) \to L_{q_2}^q(\mathcal{M}_2) \) be a bounded linear operator which preserves characteristic functions in the sense described above.

Firstly note that by hypothesis \( S \) will map all elements of the form \( h_1^{1/(2p)} a h_1^{1/(2p)} \) where \( a \in \mathcal{M}_1^{(0)} \) onto elements of the form \( h_2^{1/(2q)} \tilde{a} h_2^{1/(2q)} \) where \( \tilde{a} \in \mathcal{M}_2^{(0)} \). So if for some \( a \in \mathcal{M}_1^{(0)} \) we have \( S(h_1^{1/(2p)} a h_1^{1/(2p)}) = h_2^{1/(2q)} \tilde{a} h_2^{1/(2q)} \), we set \( J(a) = \tilde{a} \). The linearity of \( S \) and the injectivity of \( \vartheta \) ensures that \( J : \mathcal{M}_1^{(0)} \to \mathcal{M}_2 \) is well-defined and linear.

Notice that if \( e \) and \( f \) are mutually orthogonal projections in \( \mathcal{M}_1^{(0)} \), then by construction each of \( J(e) \), \( J(f) \) and \( J(e + f) = J(e) + J(f) \) is also a projection. However the latter can only hold if in fact \( J(e) \perp J(f) \). It therefore follows that \( J \) preserves the orthogonality of projections in \( \mathcal{M}_1^{(0)} \). But then \( J \) will also preserve the order of projections.

Now let \( a \in \mathcal{M}_1^{(0)} \) be given with \( a = a^* \). Since \( a \) is in \( \mathcal{M}_1^{(0)} \), we surely have \( \varphi_1(\text{supp}(a)) \) \( \) < \( \infty \). For the sake of simplicity write \( e = \text{supp}(a) \). Then by passing to Riemann sums of spectral projections of \( a \), we can find a sequence

\[
b_n = \sum_{k=1}^{m_n} \mu_k^{(n)} e_k^{(n)} \in \mathcal{M}_1^{(0)},
\]

converging uniformly to \( a \) such that for each fixed \( n \in \mathbb{N} \):

- the projections \( \{e_k^{(n)} \mid 1 \leq k \leq m_n \} \) are mutually orthogonal and satisfy \( 0 \leq e_k^{(n)} \leq e \);
- \( -\|a\| \leq \mu_k^{(n)} \leq \|a\|, 1 \leq k \leq m_n \). Then of course \( -\|a\| e \leq b_n \leq \|a\| e. \)

Since \( J \) preserves both the order and orthogonality of projections, it is clear from the above facts that

\[
-\|a\| J(e) \leq J(b_n) \leq \|a\| J(e)
\]

and hence that

\[
-\|a\| h_2^{1/(2q)} J(e) h_2^{1/(2q)} \leq h_2^{1/(2q)} J(b_n) h_2^{1/(2q)} \leq \|a\| h_2^{1/(2q)} J(e) h_2^{1/(2q)}
\]
for each \( n \). Since \( \ell_1^{1/(2p)} \) is measurable and \( \text{supp}(b_n) \leq 1 \), the uniform convergence of the \( b_n \)'s to \( a \) ensures that \( h_1^{1/(2p)} b_n h_1^{1/(2p)} \to h_1^{1/(2p)} a h_1^{1/(2p)} \). The continuity of \( S \) then yields

\[
h_2^{1/(2q)} J(b_n) h_2^{1/(2q)} = S(h_1^{1/(2p)} b_n h_1^{1/(2p)}) \to S(h_1^{1/(2p)} a h_1^{1/(2p)}) = h_2^{1/(2q)} J(a) h_2^{1/(2q)}
\]

in \( L^q(M_2) \). Together these two facts force

\[
-\|a\| h_2^{1/(2q)} J(e) h_2^{1/(2q)} \leq h_2^{1/(2q)} J(a) h_2^{1/(2q)} \leq \|a\| h_2^{1/(2q)} J(e) h_2^{1/(2q)}.
\]

Let \( b \in \mathfrak{m}_{\varphi_2} \) be given. Since \( \varphi^q(b) \in L_+^q(M_2) \), we have

\[
-\|a\| \text{tr}(i^{(q^*)}(b) i^{(q)}(J(e))) \leq \text{tr}(i^{(q^*)}(b) i^{(q)}(J(a))) \leq \|a\| \text{tr}(i^{(q^*)}(b) i^{(q)}(J(e)))
\]

or equivalently

\[
-\|a\| \text{tr}(i^{(1)}(b)(J(e))) \leq \text{tr}(i^{(1)}(b)(J(a))) \leq \|a\| \text{tr}(i^{(1)}(b)(J(e))).
\]

On applying [BL2] Proposition 2.11(b), it now follows that \(-\|a\| J(e) \leq J(a) \leq \|a\| J(e)\), and hence that \( \|J(a)\| \leq \|a\| \). Thus \( J \) is bounded. By continuity we may then extend \( J \) to the uniform closure of \( M_1^{(0)} \). This closure is however exactly \( \mathcal{C}(M_1) \). To see this note that if \( b = b^* \) is in the dense \( \ast \)-subalgebra of \( \mathcal{C}(M_1) \) generated by finite algebraic combinations of elements of \( M_1^{(0)} \), then \( \varphi_1(\text{supp}(b)) < \infty \), and hence as before by passing to Riemann sums we may write \( b \) as a norm limit of terms of the form \( d_n = \sum_{k=1}^{n} \mu_k f_k(n) \in M_1^{(0)} \) where the \( f_k(n) \)'s are mutually orthogonal. Then \( b = \lim_n d_n \in M_1^{(0)} \). Thus \( \mathcal{C}(M_1) \subset M_1^{(0)} \). The converse inclusion is clear. Now with \( b \) as above, notice that also \( J(b^2) = \lim_n J(d_n^2) = \lim_n J(\sum_{k=1}^{n} \mu_k f_k(n)^2) = \lim_n (\sum_{k=1}^{n} \mu_k f_k(n))^2 = \lim_n (J(d_n)^2) = J(b^2) \). Thus \( J \) preserves squares of self-adjoint elements on \( \mathcal{C}(M_1) \), and hence must be a Jordan \( \ast \)-morphism.

The claim about the normal extension of \( J \) to all of \( M_1 \) may be proved by a similar argument as was employed in the proof of the implications \((iii) \Rightarrow (iv) \Rightarrow (ii)\) in [Lab] 4.4. The only change that needs to be made is that wherever semifiniteness of \( M_1 \) was used in [Lab] to select a finite subprojection \( e \), we should here use the semifiniteness of \( \varphi_1 \) to select a subprojection \( e \) with \( \varphi_1(e) < \infty \).

It remains to show that \( J \) is normal when \( \varphi_1, \varphi_2 \) are states and \( p < \infty \). Since \( \varphi_1 \) is a state, it is clear that in this case \( J \) is defined on all of \( M_1 \). So suppose that \( p < \infty \), and let \( \{ e_\mu \} \) be a set of mutually orthogonal projections in \( M_1 \). If we can show that \( J(\sum_\mu e_\mu) = \sum_\mu J(e_\mu) \), \( J \) will be normal by [Lab] 4.3. Now

\[
e = \sum_\mu e_\mu
\]

is of course a projection in \( M_1 \) with convergence of the series taking place in the \( \sigma \)-strong topology (and hence also the weak* topology) of \( M_1 \). But then

\[
h_1^{1/(2p)} e h_1^{1/(2p)} = \sum_\mu h_1^{1/(2p)} e_\mu h_1^{1/(2p)}
\]

with convergence taking place in the weak topology of \( L^p(M_1) \). To see this note that if \( a_\lambda \to a \) in the weak* topology on \( M_1 \), then for any \( b \in L^p_\ast(M_1) \) we will have \( \text{tr}(h_1^{1/(2p)} a_\lambda h_1^{1/(2p)} b) = \text{tr}(a_\lambda (h_1^{1/(2p)} b h_1^{1/(2p)})) \to \text{tr}(a (h_1^{1/(2p)} b h_1^{1/(2p)})) = \text{tr}(h_1^{1/(2p)} a h_1^{1/(2p)} b) \) (since then \( h_1^{1/(2p)} b h_1^{1/(2p)} \in L^1_\varphi(M_1) \)).
Since $S$ is norm continuous, it is also weak-weak continuous. Thus we may conclude from the above that
\[
    h_2^{1/(2q)} J(e) h_2^{1/(2q)} = \sum_{\mu} h_2^{1/(2q)} J(e_{\mu}) h_2^{1/(2q)}
\]
with convergence taking place in the weak topology on $L^q_{\varphi_2}(\mathcal{M}_2)$. But since $\{J(e_{\mu})\}_\mu$ is a set of mutually orthogonal projections in $\mathcal{M}_2$, it follows that
\[
    f = \sum_{\mu} J(e_{\mu})
\]
is a projection in $\mathcal{M}_2$ with convergence taking place in the weak* topology on $\mathcal{M}_2$. Now if $q = \infty$, uniqueness of limits will then force $J(\sum_{\mu} e_{\mu}) = J(e) = f = \sum_{\mu} J(e_{\mu})$. If however $q < \infty$, we may argue as before to conclude that
\[
    \sum_{\mu} h_2^{1/(2q)} J(e_{\mu}) h_2^{1/(2q)} = \sum_{\mu} h_2^{1/(2q)} J(e_{\mu}) h_2^{1/(2q)}
\]
with convergence taking place in the weak topology on $L^q_{\varphi_2}(\mathcal{M}_2)$. Once again uniqueness of limits will then force $h_2^{1/(2q)} J(e) h_2^{1/(2q)} = h_2^{1/(2q)} f h_2^{1/(2q)}$. Since $h_2$ is an injective positive element of $L^q_{\varphi_2}(\mathcal{M}_2)$, this is enough to ensure that $J(\sum_{\mu} e_{\mu}) = J(e) = f = \sum_{\mu} J(e_{\mu})$ as required. \qed

**Jordan *-morphisms that induce operators on Haagerup $L^p$-spaces.** The main focus of this subsection is to try and describe those Jordan *-morphisms which allow for the construction of a (generalised) composition operator on a given pair of $L^p$-spaces. Although we do not succeed in giving a completely general description, we do manage to describe a large class of morphisms from which we may construct such operators. We will assume throughout that $\mathcal{M}_i$ ($i = 1, 2$) are von Neumann algebras equipped with faithful normal semifinite weights $\varphi_i$, and that $J : \mathcal{M}_1 \to \mathcal{M}_2$ is a normal Jordan *-morphism. Moreover, $\mathcal{B}$ is the von Neumann algebra generated by $J(\mathcal{M}_1)$ and $\varphi_B$ denotes the restriction of $\varphi_2$ to $\mathcal{B}$. Note that the unit of $\mathcal{B}$ is $J(1)$.

It turns out that the construction of composition operators from such a Jordan *-morphism may be broken up into five distinct steps. To avoid any pathologies associated with this process, we will for the remainder of this section consistently assume that $\varphi_2 \circ J$ is locally absolutely continuous with respect to $\varphi_1$. To gain some clarity regarding the processes involved, we first take some time to review the classical situation.

**Preamble to the construction of composition operators.** Let $(X_i, \Sigma_i, m_i)$ ($i = 1, 2$) be measure spaces and let $T : Y \subset X_2 \to X_1$ be a given non-singular measurable transformation from a measurable subset $Y$ of $X_2$ into $X_1$. For any $q$ we may then regard $L^q(Y, m_2)$ as a subspace of $L^q(X_2, m_2)$ by simply assigning the value 0 on $X_2 \setminus Y$ to each element of $L^q(Y, m_2)$. If the process $f \to f \circ T$ directly yields a bounded linear operator from $L^p(X_1, m_1)$ to $L^q(Y, m_2) \subset L^q(X_2, m_2)$, we call the resultant operator a generalised composition operator from $L^p(X_1, m_1)$ to $L^q(X_2, m_2)$ and denote it by $C_T$. If in fact $Y = X_2$, we simply call $C_T$ a composition operator.

Notice that we may use $T$ to define a new measure $m_2 \circ T^{-1}$ on $X_1$. With this new measure in place one should now be very careful about what one calls a “composition operator”. For example the map $L^q(X_1, m_2 \circ T^{-1}) \to L^q(X_2, m_2)$ defined by $f \to f \circ T$ is a very nice map (in fact an isometry), but it is not a
composition operator from $L^p(X_1, m_1)$ to $L^q(X_2, m_2)$ in the true sense of the word. Part of the problem is that the measure on the domain space is wrong.

Now if we do have a bounded map of the form $C_T : L^p(X_1, m_1) \to L^q(Y, m_2) \subset L^q(X_2, m_2) : f \mapsto f \circ T$, the construction of such a map may be broken up into five subprocesses. In the following let $Z \subset \Sigma_1$ be the support of $m_2 \circ T^{-1}$ in $X_1$, let $\Sigma_2^Y = \{ E \in \Sigma_2 | E \subset Y \}$, and let $\Sigma_T$ be the $\sigma$-subalgebra of $\Sigma_Y$ generated by sets of the form $T^{-1}(E)$ where $E \in \Sigma_1$. Our composition operator is then made up of the following processes:

(I) Restricting to the support of $m_2 \circ T^{-1}$: $L^p(X_1, m_1) \to L^p(Z, m_1|_Z) : f \mapsto f|_Z$

(II) Changing weights: $L^p(Z, m_1|_Z) \to L^q(Z, m_2 \circ T^{-1}) : f \mapsto f$

(III) Isometric equivalence of spaces: $L^q(Z, \Sigma_1^Z, m_2 \circ T^{-1}) \to L^q(Y, \Sigma_T, m_2) : f \mapsto f \circ T$ (Here $\Sigma_1^Z = \{ E \in \Sigma_1 | E \subset Z \}$.)

(IV) Refining the $\sigma$-algebra: $L^q(Y, \Sigma_T, m_2) \to L^q(Y, \Sigma_2^Y, m_2) : f \mapsto f$

(V) Canonical embedding: $L^q(Y, \Sigma_1^Y, m_2) \to L^q(X_2, \Sigma_2, m_2) : f \mapsto j(f)$ where $j(f) = f$ on $Y$ and $j(f) = 0$ on $X_2 \setminus Y$.

Notice that the map in step (V) will be the identity whenever $X_2 \setminus Y$ is a set of measure zero. Now for the combination of these five processes to yield a composition operator, we must careful about HOW we change weights. Suppose by way of example that $m_1$ and $m_2 \circ T^{-1}$ have the same sets of measure zero and that $\frac{d m_2}{d m_2 \circ T^{-1}}$ exists. Then the map $f \mapsto f \left( \frac{d m_2}{d m_2 \circ T^{-1}} \right)^{1/p}$ will certainly yield an isometry from $L^p(X_1, m_1)$ to $L^p(X_1, m_2 \circ T^{-1})$, but using this to change weights will not in general yield a composition operator. In the following we give some indication of how one may construct “composition operators” on noncommutative $L^p$-spaces associated with von Neumann algebras, by successively extending each of these processes to the noncommutative context. Thus given von Neumann algebras $M_i (i = 1, 2)$ the basic idea is to classify and study those Jordan $\ast$-morphisms $J : M_1 \to M_2$ that canonically induce bounded linear operators $L^p(M_1) \to L^q(M_2)$ along the lines suggested above. We proceed to look at noncommutative versions of each of the above steps.

**Step (I): Reducing matters to the case where $J$ is injective.** Notice that $\varphi_2 \circ J$ defines a semifinite normal weight on $M_1$. So the noncommutative analogue of the first step would be to pass from $(M_1, \varphi_1)$ to $(e M_1 e, e \varphi_1 e)$, where $e$ is the support projection of $\varphi_2 \circ J$, in a way that allows us to compare the associated $L^p$-spaces. The object of this exercise is basically to reduce matters to the case where $\varphi_2 \circ J$ is also faithful. We point out that no real information is lost in making such a reduction since it follows from $J(1) = J(e) \varphi_1(e)$ that $J(a) = J(1 a \varphi_1) = J(1) J(a) J(1) = J(e) J(a) J(e) = J(e a c e)$ for each $a \in M_1$. It turns out that such a reduction is always possible. We start with two easy lemmas concerning facts generally known, which we chose to prove here for completeness.

Note that the algebra generated by $J(e M_1 e)$ is the same as the algebra generated by $J(M)$, that is $B$.

Assume now that we have a von Neumann algebra $M$ acting in a Hilbert space $H$, with a fns weight $\varphi$. If $e$ is a projection from $M$, we denote by $\varphi_\ast$ the restriction of $\varphi$ to $e M_1 e$. Furthermore, we denote by $\tau_\ast$ the canonical trace on the crossed product $(e M_1 e) \ltimes \sigma_{\varphi_\ast} \wp$. Finally, we put $\hat{e} := \pi_\varphi(e)$. 

Lemma 4.3. If the projection e belongs to the subalgebra $M_\varphi$ of fixed points for the modular group of $M$ with respect to an fns weight $\varphi$ (in particular, when $e$ is central), then $L^p_{\varphi_e}(eMe)$ consists of operators from $eL^p_\varphi(M)e$ restricted to $L^2(\mathbb{R},eH)$. Moreover, $\frac{d\varphi_e}{dt}$ commutes with $\hat{e}$ and $\frac{d\varphi_e}{dr_e}$ may be identified with the restriction of $\frac{d\varphi_e}{dt}$ to $L^2(\mathbb{R},eH)$.

Proof. It is clear that the weight $\varphi_e$ is faithful, normal and semifinite, and the modular group for the pair $(eMe, \varphi_e)$ is the restriction to $eMe$ of the modular group for $(M, \varphi)$. Similarly, one checks easily that $\hat{e}$ projects $L^2(\mathbb{R}, H)$ onto $L^2(\mathbb{R}, eH)$. Consequently, the operators $\pi_{\varphi_e}(eae)$ with $a \in M$ are just $\hat{e}\pi_\varphi(a)\hat{e}$ restricted to $L^2(\mathbb{R}, eH)$. Similarly, $\lambda_{\varphi_e}(s)$ is just $\lambda_\varphi(s)$ restricted to $L^2(\mathbb{R}, eH)$. Hence $(eMe)\ltimes_{\varphi_e} \mathbb{R} = \hat{e}(M\ltimes_{\varphi} \mathbb{R})\hat{e}$, where the von Neumann algebra on the right hand side of the equation acts on $L^2(\mathbb{R}, eH)$. Now, if $(\theta_s)$ is the dual action on $M\ltimes_{\varphi} \mathbb{R}$, then it restricts to the dual action on $(eMe)\ltimes_{\varphi_e} \mathbb{R}$, and $\theta_s(\hat{e}x\hat{e}) = \exp(-s/p)\hat{e}x\hat{e}$ for each $x \in L^p_\varphi(M)$, which implies the required equality. The final claim follows from noting that the shift operators $\lambda_{\varphi_e}(s)$ commute with $\hat{e}$, and that $\frac{d\varphi_e}{dr_e}$ (resp. $\frac{d\varphi_e}{dt}$) is the (positive) generator of the unitary group $\lambda_{\varphi_e}(s), s \in \mathbb{R}$ (resp. $\lambda_{\varphi_e}(s), s \in \mathbb{R}$).

Remark 4.4. The lemma shows that there is a natural embedding of $L^p_{\varphi_e}(eMe)$ into $L^p_{\varphi_e}(M)$, namely $x \mapsto \hat{e}xe$, and that the image of $L^p_{\varphi_e}(eMe)$ under the embedding is exactly $\hat{e}L^p(M)e$. In the sequel we stick to the usual convention of identifying $e$ with $\hat{e}$ and $L^p_{\varphi_e}(eMe)$ with $eL^p_{\varphi_e}(M)e$.

Lemma 4.5. The support of $\varphi_2 \circ J$ is central.

Proof. Let $z$ be a central projection in $B$ such that $a \mapsto zJ(a)$ is a *-homomorphism and $a \mapsto (J(\mathbb{I}) - z)J(a)$ is a *-antihomomorphism. If $J(a) = 0$ for some $a \in M$, then $J(ab) = zJ(a)J(b) + (J(\mathbb{I}) - z)J(b)J(a) = 0$ and similarly $J(ba) = 0$. Hence the kernel of $J$ is a two-sided, $\sigma$-weakly closed because of $J$’s normality. Thus there exists a central projection $e$ such that $\ker(J) = eM$ (see [Tak, Proposition II.3.12]). Now, it follows from Remark 4.6 that $\varphi_2 \circ J$ is semifinite, which shows that its support must be equal to $\mathbb{I} - e$.

The above results show that the reduction to the support of $\varphi_2 \circ J$ is, in fact, multiplication by a central projection. Since for any pair $(M, \varphi)$, central projections are automatically fixed points of the modular group of $M$ induced by $\varphi$ (in fact they are even central in $M\ltimes_{\varphi} \mathbb{R}$), in the light of Lemma 4.3 this reduction is particularly simple.

Step (II): Changing weights. Let $J$ be as before and let $e$ be the support projection of $\varphi_2 \circ J = \varphi_J$. Our primary interest in step (II) is to describe the situation in which we may pass from $L^p_{\varphi_1}(eM_1e)$ to $L^p_{\varphi_J}(eM_1e)$ (where $1 \leq q \leq p < \infty$) by means of a change of weights. In this regard notice that since by assumption $\varphi_2 \circ J \ll_{loc} \varphi_1$, $\varphi_2 \circ J$ is necessarily semifinite. Given that we are only really interested in the action of $\varphi_1$ and $\varphi_J$ on $eM_1e$, we may assume for the sake of argument that $\varphi_2 \circ J$ is faithful. As was noted in the preamble, care should be taken in exactly how we change weights, if we are to end up with a composition operator. So in particular in the noncommutative world we can not just willy nilly apply (II.37 & II.38) and leave it at that. To gain some insight into what is required we take some time to consider the semifinite case. So suppose that $M_i$ $(i = 1, 2)$ are equipped
with fins traces $\tau_1$ and $\tau_2$ respectively. From (Lab) we see that if $J$ is in fact $\sigma$-weakly continuous (as we are assuming here), then roughly speaking it will induce a projection preserving bounded linear map from $L^p(M_1, \tau_1)$ to $L^q(M_2, \tau_2)$ if and only if $f_J = \frac{d\tau_J}{d\tau_1}$ exists as an element of $L^r(M_1, \tau_1)$ (where $r = \frac{p}{p-q}$) and $\tau_2 \circ J(a) = \tau_1(f_{j_a}^{1/2} a f_{j_a}^{1/2})$ for each $a \in M_1$.

For any $a \in L_p(M_1, \tau_1) \cap M_1$ we then have
\[
\|J(a)\|_q = (\tau_2(J(|a|_q^q)))^{1/q} = (\tau_2(J(|a|_q^q)))^{1/q} \leq \|J\|_q^{1/q} \|a\|_p
\]
(In the above $|a|_q$ denotes the so-called $q$-th symmetric modulus discussed in [Lab].)

Here the first line corresponds to the isometric embedding of $L^p(M_1, \tau_1) \cap M_1$ into $L^q(M_2, \tau_2)$, and the next two to the passage from $L^p(M_1, \tau_1)$ to $L^q(M_1, \tau_2 \circ J)$ by means of a change of weights. So we see that it is the derivative $f_J$ that not only enables us to pass from $L^p(M_1, \tau_1)$ to $L^q(M_1, \tau_2 \circ J)$ by means of the identity
\[
\tau_2 \circ J(\cdot) = \tau_1(f_{j_a}^{1/2} \cdot f_{j_a}^{1/2}),
\]
but also conditions the boundedness of the induced map.

Passing to the general case the assumption that $J$ is normal ensures that $\varphi_J = \varphi_2 \circ J$ is normal, in addition to being faithful and semifinite. So for the sake of clarity we may assume for now that $M_1 \times_{\sigma, 1} \mathbb{R} = M_1 \times_{\sigma, 2} \mathbb{R}$ [Tp1 II.37 & II.38]. Now let $\text{tr}_1$ and $\text{tr}_J$ be the canonical trace functionals associated with $L_{\varphi_1}^q(M_1)$ and $L_{\varphi_J}^q(M_1)$ respectively, and let $h_1 = \frac{d\tau_1}{d\tau_2}$ and $h_J = \frac{d\tau_J}{d\tau_1}$. In a simplistic world we would then by analogy with the semifinite case hope to achieve the change of weights by means of some positive element $f_J \in (M_1 \times_{\sigma, 1} \mathbb{R})$ for which $\text{tr}_J(\cdot) = \text{tr}_1(f_{j_a}^{1/2} \cdot f_{j_a}^{1/2})$. However this is too much to hope for in general, as the type III case is rather more exotic than the semifinite case. This makes for a type III theory of “composition operators” which shows some interesting variations to the semifinite theory. If the weights $\varphi_1$ and $\varphi_J$ actually commute, then by [Tak Corollary VIII.3.6] there indeed does exist some $v \geq 0$ affiliated to $(M_1)_{\varphi_1}$, such that $\varphi_J(\cdot) = \varphi_1(v^{1/2} \cdot v^{1/2})$.

Although the above is already reminiscent of the equality in the semifinite setting, it would be more useful to translate this to a statement concerning $\text{tr}_1$ and $\text{tr}_J$.

Now by mimicking the argument of [Cl2 Proposition 2.13] we may show that $\varphi_1(\sigma_{i/2}(b) \sigma_{-i/2}(c) b^*) = \text{tr}_1(h^{(p)}(c) b^*)$ $b \in m_\infty$, $c \in n^*$.

Arguing formally, the fact that $v$ is affiliated to $(M_1)_{\varphi_1}$ then seems to suggest that in the case of commuting weights we will have
\[
\text{tr}_J(h_{j_a}^{1/2} \cdot h_{j_a}^{1/2}) = \text{tr}_1(v^{1/2} h_{j_a}^{1/2} \cdot h_{j_a}^{1/2} v^{1/2}),
\]
or in other words
\[
\text{tr}_J(i^{(1)}_{j_a} \cdot \cdot) = \text{tr}_1(h^{(p)}(\cdot) d^*)
\]
d where $d = v^{1/2} h_{j_a}^{1/2}$. If now $d \in L^{2p^*}(M_1)$, we could use Hölder’s inequality to show that then the process $i^{(p)}(a) = h_{j_a}^{1/2} \tau_1(a) h_{j_a}^{1/2} \tau_1(a) h_{j_a}^{1/2} \tau_1(a) h_{j_a}^{1/2} = i^{(1)}(a)$, $a \in M_1^{(0)}$ extends to a well defined bounded map $L^p_{\varphi_1}(M_1) \rightarrow L^1_{\varphi_J}(M_1)$. At
least for the the case $q = 1$ the resultant map then seems to represent a means of
passing from $i_1^p(M_1) \subset L^p_p(M_1)$ to $i_j^{(1)}(M_1) \subset L^1_j(M_1)$ by means of a “change
of weights” in a way that is categorically more in line with what is required for
the construction of composition operators. Admittedly this “change of weights” is
dependent on the manner in which $M_1$ is embedded in $L_\mu$, but this fact seems to be
a challenge inherent in the type III theory.

It remains to develop a suitable strategy for dealing with the case $L^p_{\varphi_1}(M_1) \rightarrow
L^p_{\varphi_2}(M_1)$ where $1 < q \leq \infty$. Formally one may consider something like
\( i_1^p(a) = h^{1/(2p)}(a)h^{1/(2p)} \rightarrow h^{1/(2q)}(a)h^{1/(2q)} = i_j^q(a) \) ($a \in M_1^{(0)}$). We deal with
the situation by first considering change of weights mapping acting in one specific
crossed product (say, the one given by $\varphi_1$), and then by applying the natural
isometry $\gamma$ (described in detail in [TP]: II.37 & II.38) that identifies this crossed
product with the one given by the other weight ($\varphi_2$ in our case). The following
proposition deals with the change of weights:

**Proposition 4.6.** Let $\mathcal{M}$ be a von Neumann algebra with two fns weights $\varphi$ and
$\varphi_0$ with $\varphi_0 \ll_{loc} \varphi$. Let $h = \frac{dx}{x^r}$ and $k = \frac{dx}{x^s}$. Also let $1 \leq q \leq p \leq \infty$.

Then the following statements are equivalent:

1. The embedding $h^{1/(2p)}a^1/(2p) \rightarrow k^{1/(2q)}a^{1/(2q)}$ ($a \in \mathcal{M}^{(0)}$) extends to a
continuous map $T : L^p_{\varphi}(\mathcal{M}) \rightarrow L^q_{\varphi_0}(\mathcal{M})$;

2. for $r$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, there exists some $d \in L^2_{\varphi}(\mathcal{M}) \subset \mathcal{M} \otimes_\sigma \mathbb{R}$, such
that $f[|dh^{1/(2p)}|^2] = f[k^{1/q}f$ for any projection $f \in \mathcal{M}^{(0)}$;

3. for each pair $1 \leq q_0 \leq p_0 \leq \infty$ with $(p_0,q_0) \geq (p,q)$ (by the lexicographic ordering) and with $p_0/q_0 = p/q$, the embedding $h^{1/(2p_0)}a^{1/(2p_0)} \rightarrow k^{1/(2q_0)}a^{1/(2q_0)}$ ($a \in \mathcal{M}^{(0)}$) extends to a continuous linear map $T^{(p_0,q_0)} : L^p_{\varphi}(\mathcal{M}) \rightarrow L^q_{\varphi_0}(\mathcal{M})$.

In our construction of composition operators the operator $d$ above will then fulfill the
role played by $f_{\varphi,\varphi_0}^p$ in the semifinite setting - see the preceding discussion.

**Definition 4.7.** Let $\mathcal{M}$ be a von Neumann algebra with an fns weight $\varphi$ and let $\varphi_0$ be a
normal weight with support projection $e$ belonging to the fixed point algebra
of $\varphi$, and with $\varphi_0 \ll_{loc} \varphi$. Let $h$ and $k$ be as in the preceding discussion. Note
that our assumptions imply that $k^{1/(2q)}a^{1/(2q)} = k^{1/(2q)}eaeak^{1/(2q)}$ is well-defined
for any ($a \in \mathcal{M}^{(0)}$).

Given $1 \leq p,q \leq \infty$, we say that $\mathcal{M}$ admits of a bounded change of weights
from $\varphi$ to $\varphi_0$ for the pair $(p,q)$, if the embedding $h^{1/(2p)}a^{1/(2p)} \rightarrow k^{1/(2q)}a^{1/(2q)}$
($a \in \mathcal{M}^{(0)}$) extends to a continuous linear map $T : L^p_{\varphi}(\mathcal{M}) \rightarrow L^q_{\varphi_0}(\mathcal{M})$.

Given $1 \leq r < \infty$, we say that $\mathcal{M}$ admits of a bounded change of weights scale
from $\varphi$ to $\varphi_0$ for the ratio $r$ if for each pair $1 \leq q \leq p < \infty$ with $r = p/q$, the embedding
$h^{1/(2p)}a^{1/(2p)} \rightarrow k^{1/(2q)}a^{1/(2q)}$ extends to a continuous map $T^{(p,q)} : L^p_{\varphi}(\mathcal{M}) \rightarrow L^q_{\varphi_0}(\mathcal{M})$.

Notice that the support of $k$ is just $e$. Thus in the above definition, the maps
$T,T^{(p,q)}$ actually maps into $eL^q_{\varphi_0}(\mathcal{M})e$. On canonically identifying $eL^q_{\varphi_0}(\mathcal{M})e$
with $L^q_{\varphi_0}(e\mathcal{M}e)$, we may therefore equivalently speak of a bounded change of weights
from $(\mathcal{M},\varphi)$ to $(e\mathcal{M}e,\varphi_0)$ for the pair $(p,q)$, etc.

The proposition will be an easy consequence of the following, more general,
themorem (see [JS] Theorem 2.5) and the comments in the introduction).
Theorem 4.8. Let $\mathcal{M}$ be a von Neumann algebra with an fnns weight $\varphi$, and let $1 \leq q \leq p \leq \infty$ and $T \in \text{Hom}(L^p(\mathcal{M}), L^q(\mathcal{M}))$ (Thus $T$ is a bounded linear map from $L^p$ to $L^q$ which is a homomorphism with respect to the right-module action of $\mathcal{M}$ on $L^p$.) Then there exists $c \in L^q(\mathcal{M})$ (where $\frac{1}{q} = \frac{1}{p} + \frac{1}{r}$) such that $T(a) = ca$ for all $a \in L^p(\mathcal{M})$.

We first show how Proposition 4.6 can be deduced from the above theorem.

Proof of Proposition 4.6: (1) $\Rightarrow$ (2): The implication clearly holds if $p = \infty$, and hence we may assume that $p < \infty$.

Assume that

$$h^{1/(2p)}ah^{1/(2p)} \to k^{1/(2q)}ak^{1/(2q)} \quad a \in \mathcal{M}^{(0)}$$

extends to a continuous map $T$ from $L^p(\mathcal{M})$ to $L^q(\mathcal{M})$. Given any $a \in \mathcal{M}^{(0)}$, the spectral resolution for selfadjoint operators ensures that we may find a sequence of Riemann sums of the form

$$\sum_{i=1}^{n} \lambda_i e_i$$

with each $e_i$ a projection majorized by $s_\lambda(a)$ (where $s_\lambda(a)$ is the right support of $a$), and $e_i$'s mutually orthogonal, which converges uniformly to $|a|^2$ in the compression $s_\lambda(a)\mathcal{M}s_\lambda(a)$. Since $\varphi(s_\lambda(a)) < \infty$, we have that $h^{1/(2p)}s_\lambda(a) \in L^2(\mathcal{M})$. Hence an application of Hölder's inequality reveals that the terms $h^{1/(2p)}s_\lambda(a)(\sum_{i=1}^{n} \lambda_i e_i)s_\lambda(a)h^{1/(2p)} = h^{1/(2p)}(\sum_{i=1}^{n} \lambda_i e_i)h^{1/(2p)}$ must converge to $h^{1/(2p)}|a|^2h^{1/(2p)}$. Since $\varphi_0 \ll_{\text{loc}} \varphi$, we of course also have $\varphi_0(s_\lambda(a)) < \infty$, and hence essentially the same argument shows that the terms $k^{1/(2q)}(\sum_{i=1}^{n} \lambda_i e_i)k^{1/(2q)}$ must converge to $k^{1/(2q)}|a|^2k^{1/(2q)}$.

Thus for any $a \in \mathcal{M}^{(0)}$ the continuity of $T$ ensures that it will map the term $h^{1/(2p)}|a|^2h^{1/(2p)}$ onto the term $k^{1/(2q)}|a|^2k^{1/(2q)}$. From this observation it now follows that

$$|||ak^{1/(2q)}||2q = ||h^{1/(2p)}|a|^2h^{1/(2p)}||2q = ||T(h^{1/(2p)}|a|^2h^{1/(2p)})||2p \leq ||T||^2||h^{1/(2p)}|a|^2h^{1/(2p)}||2p = ||T||^2||h^{1/(2p)}|a|^2h^{1/(2p)}||2p.$$}

Thus the formal map $[ah^{1/(2p)}] \to [ak^{1/(2q)}]$ ($a \in \mathcal{M}^{(0)}$) extends continuously to a map $T_0 : L^p(\mathcal{M}) \to L^q(\mathcal{M})$. This map is a homomorphism with respect to the left module action of $\mathcal{M}$ on $L^p(\mathcal{M})$. Thus an application of the left version of Theorem 4.6 now establishes (2).

(2) $\Rightarrow$ (1): For the converse note that if an element $d$ of the form described in (2) exists, then given any $a \in \mathcal{M}^{(0)}$, we may select a partial isometry $u$ so that $ud^{1/2}(h^{1/(2p)})f = k^{1/(2q)}f$ where $f = s_\lambda(a)$ (here $s_\lambda(a)$ is the left and right supports of $a$). A simple application of Hölder's inequality then reveals that

$$||k^{1/(2q)}ak^{1/(2q)}||q = ||udh^{1/(2p)}ah^{1/(2p)}d^*u^*||q \leq ||ud||_{(2q)} ||h^{1/(2p)}ah^{1/(2p)}||_{(2q)} ||d^*u^*||_{(2q)} \leq ||d||^2 ||h^{1/(2p)}ah^{1/(2p)}||_{p}.$$}

Since this holds for each $a \in \mathcal{M}^{(0)}$, the embedding $h^{1/(2p)}ah^{1/(2q)} \to k^{1/(2q)}ah^{1/(2q)}$ ($a \in \mathcal{M}^{(0)}$) therefore clearly extends to a continuous map $\tilde{T} : L^q(\mathcal{M}) \to L^q(\mathcal{M})$.

(1) $\Rightarrow$ (3): Here $T^{(p,q)}$ is nothing but the unique operator for which $T^{(p,q)} \circ i^{(p)}(\mathcal{M}) = i^{(q)} \circ id_{\mathcal{M}^{(0)}}$. Now let $T^{(p,q)}$ be the unique bounded operator on the Terp interpolation space $L^p(\mathcal{M}, \varphi)$ such that $T^{(p,q)} \circ \kappa^{(p)} = \kappa^{(q)} \circ T^{(p,q)}$. It is clear that $T^{(p,q)}|\kappa^{(p)}(\mathcal{M}^{(0)}) = T^{(\infty,\infty)}|\kappa^{(q)}(\mathcal{M}^{(0)})$ where $T^{(\infty,\infty)} = id_{\mathcal{M}}$. By the
reiteration property of the complex interpolation method ([BeL]: Theorem 4.6.1), $T^{[p_0,q_0]}|_{K_0^p}(\mathcal{M}(\mathcal{O})) = T^{[p,q]}|_{K_0^p}(\mathcal{M}(\mathcal{O}))$ is bounded, which implies the boundedness of $T^{[p_0,q_0]}$.

The implication (3) $\Rightarrow$ (1) is entirely trivial, and hence the result follows. \hfill $\square$

We begin the proof of Theorem 4.8 with two lemmas.

**Lemma 4.9.** Let $\mathcal{M}$ be a von Neumann algebra with an fns weight $\varphi$ and let $1 \leq r < \infty$ be given. For any $0 < t, s < \infty$ satisfying $\frac{1}{s} = \frac{1}{r} + \frac{1}{t}$ and any $b \in \mathcal{L}_r^1(\mathcal{M})$, we have

$$\|b\|_r = \sup\{|bg|_s : g \in \mathcal{L}_r^1(\mathcal{M}), \|g\|_t \leq 1\}.$$  

If $1 \leq t, s < \infty$, the formula also holds for the case $r = \infty$.

**Proof.** The statement obviously holds if $b = 0$. If $b \neq 0$ we may normalise and assume that $\|b\|_r = 1$. Hölder’s inequality then ensures that

$$1 \geq \sup\{|bg|_s : g \in \mathcal{L}_r^1(\mathcal{M}), \|g\|_t \leq 1\}.$$  

To see that we get equality when $1 \leq r < \infty$, consider the element of $\mathcal{L}_r^1(\mathcal{M})$ defined by $g_0 = |b|^{r/t}$. Then $\|g_0\|_t = (\|b\|_r)^{r/t} = 1$ with $\|bg_0\|_s = \text{tr}(\|bg_0\|_r^{1/s}) = \text{tr}(\|b\|^{r/t})^{1/s} = \text{tr}(\|b\|^{1/s}) = 1$ as required.

Finally let $1 \leq t = s < \infty$ and $r = \infty$. For the case $1 = t$ this formula is known. Hence let $1 < t < \infty$. Given any $0 < \varepsilon < 1$, we may use $L^p$ duality to select $f \in \mathcal{L}_s^1(\mathcal{M})$ with $1 - \varepsilon < \text{tr}(bf) \leq 1$ and $\|f\|_s = 1$. Let $u|f|$ be the polar decomposition of $f$ and set $g_0 = u|f|^{r/t} \in \mathcal{L}_r^1(\mathcal{M})$. Since $\|\text{tr}(|f|^{1/(s+t')}\|_{s+t'}) = 1$, it therefore follows from Hölder’s inequality that $1 - \varepsilon < \text{tr}(bf) = \text{tr}(bg_0|f|^{1/(r+t')}) \leq \|bg_0\|_s$. Now by construction $\|bg_0\|_s = \text{tr}(|f|^{1/(s+t')})^{1/s} = \text{tr}(|f|)^{1/s} = 1$. Hence $1 - \varepsilon \leq \sup\{|bg|_s : g \in \mathcal{L}_r^1(\mathcal{M}), \|g\|_t \leq 1\}$. From these considerations it is clear that $1 = \sup\{|bg|_s : g \in \mathcal{L}_r^1(\mathcal{M}), \|g\|_t \leq 1\}$ as required. \hfill $\square$

**Lemma 4.10.** Let $\mathcal{M}$ be a von Neumann algebra equipped with an fns weight $\varphi$, and let $h = \frac{d\varphi}{d\tau}$. Let $2 \leq q \leq p < \infty$ and let $T \in \text{Hom}(\mathcal{L}_r^q(\mathcal{M}), \mathcal{L}_s^q(\mathcal{M}))$ (Thus $T$ is a bounded linear map from $L^p$ to $L^q$ which is a homomorphism with respect to the right-module action of $\mathcal{M}$ on $L^p$.) Let $e \in \mathcal{M}$ be a projection in $\mathcal{M}$ with $\varphi(e) < \infty$, and let $d_e = T(eh^{1/p})$. Then the following holds:

1. For any $a \in \mathcal{M}$, we have that $T([eh^{1/p}]a) = d_e a$.
2. For any $g \in \mathcal{L}_r^q(\mathcal{M})$ we have $gd_e = T^*(g)[eh^{1/p}]$.
3. The formal map $h^{1/p} \mapsto d_e b$ defined for all $b \in \{a \in \mathcal{L}_r^q(\mathcal{M}) : \varphi(s_t(a)) < \infty\}$, extends continuously and uniquely to a linear map $\mathcal{L}_r^q(\mathcal{M}) \mapsto \mathcal{L}_r^q(\mathcal{M})$ where $1 \leq v \leq \infty$ is such that $\frac{1}{v} = \frac{1}{p} + \frac{1}{q}$.

**Proof.** Let $T$ be a bounded linear map from $\mathcal{L}_r^q(\mathcal{M})$ to $\mathcal{L}_s^q(\mathcal{M})$. By continuity and the density of $h^{1/p}\mathcal{M}(0)$ in $\mathcal{L}_r^q(\mathcal{M})$, it is not difficult to see that (1) follows directly from the requirement that $T \in \text{Hom}(\mathcal{L}_r^q(\mathcal{M}), \mathcal{L}_s^q(\mathcal{M}))$. It therefore remains to demonstrate the validity of (2) and (3).

Next consider claim (2). For any $a \in \mathcal{M}$ and $g$ as in the hypothesis, we have

$$\text{tr}(gd_e a) = \text{tr}(g(d_e a)) = \text{tr}(gT([eh^{1/p}]a)) = \text{tr}(T^*(g)[eh^{1/p}]a).$$
It follows from this equality that \( gd_{e} = T^{*}(g)[eh^{1/p}] \).

Finally consider claim (3). Given \( b \in \{ a \in L_{p}^{\infty}(\mathcal{M}) : \varphi(s_{l}(a)) < \infty \} \), it follows from claim (2) that we will then have

\[
gd_{e}b = T^{*}(g)[eh^{1/p}]b, \quad g \in L_{p}^{\infty}(\mathcal{M}).
\]

Since \( h^{1/p}b = h^{1/p}s_{l}(b) \in L_{p}^{\infty}(\mathcal{M}) \cdot L_{p}^{\infty}(\mathcal{M}) \subset L_{p}^{\infty}(\mathcal{M}) \), we may apply Lemma 4.9 (with \( t = q^{*}, r = 1 \)) and Hölder’s inequality to get

\[
\|d_{e}b\|_{1} = \sup\{\|gd_{e}b\|_{*} : g \in L^{q^{*}}(\mathcal{M}), \|g\| \leq 1\}
\leq \sup\{\|T^{*}(g)[eh^{1/p}]b\|_{*} : g \in L^{q^{*}}(\mathcal{M}), \|g\| \leq 1\}
\leq \|T\| \cdot \|eh^{1/p}b\|_{v}.\]

Now, since \( s_{l}(b) \in (n(2))^{*} \), the operator \( h^{1/p}s_{l}(b) \) is premeasurable (in fact, even measurable). Hence, \( h^{1/p}b = (h^{1/p}s_{l}(b))b \), being a product of two premeasurable operators, is also premeasurable. Since \( eh^{1/p}b \subset [eh^{1/p}]b \) and \( eh^{1/p}b \subset e[h^{1/p}b] \), the rigidity of measurable operators yields \( [eh^{1/p}]b = e[h^{1/p}b] \). Thus,

\[
\|T\| \cdot \|eh^{1/p}b\|_{v} \leq \|T\| \cdot \|h^{1/p}b\|_{v},
\]

as required. (Here we made use of the fact that \( \frac{1}{q} = 1 + \frac{1}{q^{*}} = 1 - \frac{1}{p} + \frac{1}{q^{*}} = \frac{1}{p} + \frac{1}{q^{*}} \).) The last part of the claim now follows from the density of \( \{ h^{1/p}a : a \in L_{p}^{\infty}(\mathcal{M}), \varphi(s_{l}(a)) < \infty \} \) in \( L_{p}^{\infty}(\mathcal{M}) \).

\[\square\]

**Proof of Theorem 4.8**. As noted in [JS], the implication clearly holds if \( p = \infty \), and hence we may assume that \( p < \infty \). Suppose for the sake of argument that \( p > q \). Notice that the above assumptions in turn ensure that \( 1 < r < \infty \), and hence that \( L^{r} \) is reflexive.

First assume that \( 2 \leq p < \infty \). Let \( e \) be a projection in \( \mathcal{M} \) with \( \varphi(e) < \infty \). As noted in the preceding lemma, the restriction of \( T \) to \( eL_{p}^{\infty}(\mathcal{M}) \) is a continuous extension of the formal map \([eh^{1/p}]a \mapsto d_{e}a \) (\( a \in \mathcal{M} \)) where \( d_{e} = T([eh^{1/p}]) \). Lemma 4.10 now additionally informs us that the map \([h^{1/p}]b \mapsto d_{e}b \) (\( b \in \{ a \in L_{p}^{\infty}(\mathcal{M}) : \varphi(s_{l}(a)) < \infty \} \)) is a continuous map from a dense subspace of \( L_{p}^{\infty}(\mathcal{M}) \) into \( L_{p}^{\infty}(\mathcal{M}) \) where \( v \) is such that \( \frac{1}{q} = \frac{1}{p} + \frac{1}{q^{*}} \). We may therefore compose this map with the trace functional \( \text{tr} \) on \( L_{p}^{\infty}(\mathcal{M}) \) to get a densely defined continuous linear functional \([h^{1/p}]b \mapsto \text{tr}(d_{e}b) \) (\( b \in \{ a \in L_{p}^{\infty}(\mathcal{M}) : \varphi(s_{l}(a)) < \infty \} \)) on \( L_{p}^{\infty}(\mathcal{M}) \). Thus by \( L^{p} \) duality there must exist \( c_{e} \in L_{p}^{\infty}(\mathcal{M}) = L_{p}^{\infty}(\mathcal{M}) \) with

\[
\text{tr}(c_{e}[h^{1/p}]b) = \text{tr}(d_{e}b)
\]

for all \( b \) with \( \varphi_{1}(s_{l}(b)) < \infty \). It is clear that the \( b \) in the formula above can be replaced with \( ba \), where \( a \in \mathcal{M} \), and that \([h^{1/p}ba] = [h^{1/p}b]a \). This implies \( c_{e}[h^{1/p}]b = c_{e}[h^{1/p}b] = d_{e}b \), so that \( c_{e}[h^{1/p}ah^{1/q}] = d_{e}ah^{1/q} \) for each \( a \in \mathcal{M}^{(0)} \). Consequently, \( c_{e}h^{1/p}ah^{1/q} \subset d_{e}ah^{1/q} \) and the invertibility of \( h \) yields \( c_{e}h^{1/p}a \subset d_{e}a \). Again, by rigidity \( c_{e}(h^{1/p}a) = d_{e}a = T([eh^{1/p}]a) \) for all \( a \in \mathcal{M}^{(0)} \).

Now let \( \{ e_{\lambda} : \lambda \in \Lambda \} \) be a mutually orthogonal family of projections with \( \varphi(e_{\lambda}) < \infty \) for each \( \lambda \), and \( \sum_{\lambda \in \Lambda} e_{\lambda} = 1 \). Let \( a_{0} \) be a fixed element of \( \mathcal{M}^{(0)} \). For
any finite subset \( F \) of \( \Lambda \) we have by linearity that

\[
(\sum_{\lambda \in F} c_{\lambda})(h^{1/p}a_0) = T((\sum_{\lambda \in F} e_{\lambda})h^{1/p}a_0).
\]

The net of terms of the form \( \sum_{\lambda \in F} e_{\lambda} \) converges to \( \mathbb{1} \) in the weak* topology, and hence the net \( \{\sum_{\lambda \in F} e_{\lambda}(h^{1/p}a_0)\}_F \) (where \( F \) ranges over the finite subsets of \( \Lambda \)) will converge weakly to \( h^{1/p}a_0 \). Thus \( T((\sum_{\lambda \in F} e_{\lambda}h^{1/p}a_0) \to T(h^{1/p}a_0) \) weakly.

If now we combine the density of \( h^{1/p}\mathcal{M}(0) \) in \( L^p_\infty(\mathcal{M}) \) with the previous centered equation, we get that

\[
\|\sum_{\lambda \in F} c_{\lambda}\|_r = \sup\{\|((\sum_{\lambda \in F} c_{\lambda})(h^{1/p}a))\|_q : a \in \mathcal{M}(0), \|h^{1/p}a\|_p \leq 1\}
\]

\[
= \sup\{\|T((\sum_{\lambda \in F} e_{\lambda})h^{1/p}a)\|_q : a \in \mathcal{M}(0), \|h^{1/p}a\|_p \leq 1\}
\]

\[
\leq ||T||, 
\]

since again \( [e]h^{1/p}a = e[h^{1/p}a] \) for \( e, a \in \mathcal{M}(0) \). Therefore by the weak compactness of the unit ball of \( L^r \) we may select a subnet of terms of the form \( \sum_{\lambda \in F} c_{\lambda} \in L^r \) (where \( F \subseteq \Lambda \) is finite) converging to some \( c \in L^r \). (In the case \( p = q \) we would have \( r = \infty \). Hence we could then use weak* compactness instead of weak compactness.)

By now taking limits it follows that

\[
c(h^{1/p}a_0) = \lim_{F} (\sum_{\lambda \in F} c_{\lambda})(h^{1/p}a_0) = \lim_{F} T((\sum_{\lambda \in F} e_{\lambda})(h^{1/p}a_0)) = T(h^{1/p}a_0).
\]

Since \( a_0 \) was an arbitrary element of \( \mathcal{M}(0) \), we may now finally appeal to the density of \( h^{1/p}\mathcal{M}(0) \) in \( L^p \), to conclude that as required

\[
Cb = T(b) \quad \text{for all} \quad b \in L^p.
\]

Notice that everything we have done so far is entirely symmetrical, and hence we may similarly prove that if \( 2 \leq p \leq \infty \), then all left \( \mathcal{M} \)-module homomorphisms from \( L^p(\mathcal{M}) \) to \( L^q(\mathcal{M}) \) are right multiplication operators induced by some \( c \in L^r(\mathcal{M}) \).

Now suppose that \( 1 \leq p < 2 \). It is an exercise to show that \( T : L^p(\mathcal{M}) \to L^q(\mathcal{M}) \) is a right \( \mathcal{M} \)-module homomorphism if and only if \( T^* : L^q(\mathcal{M}) \to L^p(\mathcal{M}) \) is a left \( \mathcal{M} \)-module homomorphism, and that \( T \) is a left multiplication operator induced by some element \( c \in L^r(\mathcal{M}) \) if and only if \( T^* \) is a right multiplication operator induced by the same element \( c \) (notice that here \( \frac{1}{q^*} = \frac{1}{q'} + \frac{1}{r} \)). Notice for example that if for some \( a \in \mathcal{M} \) we have that \( T(b)a = T(ba) \) for every \( b \in L^p(\mathcal{M}) \), we will then have that \( tr(T^*(ax)b) = T(axT(b)) = tr(xT(b)a) = tr(aT(x)b) \) for every \( b \in L^p(\mathcal{M}) \) and every \( x \in L^{q^*}(\mathcal{M}) \). Thus we then clearly have that \( T^*(ax) = aT^*(x) \) for every \( x \in L^{q^*}(\mathcal{M}) \). Therefore since \( 1 \leq p < 2 \) forces \( 1 \leq q < 2 \) (or equivalently \( 2 < q^* \leq \infty \)), the present case clearly follows by duality from the case \( 2 \leq p \leq \infty \).

**Step (III) : Applying the Jordan morphism.** We start with the simplest case when \( \mathcal{B} = \mathcal{M}_2 \) and \( J : \mathcal{M}_1 \to \mathcal{M}_2 \) is a Jordan \( + \)-isomorphism of \( \mathcal{M}_1 \) onto \( \mathcal{M}_2 \). The challenge is then to find a natural canonical way of isometrically identifying \( L^p_{\varphi_J}(\mathcal{M}_1) \) (where \( \varphi_J = \varphi_\varphi \circ J \)) with \( L^p_{\varphi_J}(\mathcal{M}_2) \). In a sequence of papers ([W1] - [W2]) Keiichi Watanabe developed just such a construction. (See for example §3 of [W2] and
the discussion preceding 3.1 of [W3].) All we need to do is to apply Watanabe’s construction to $J^{-1}$ to get the following

**Lemma 4.11.** Let $J$ be a bijective Jordan *-isomorphism. Then $J$ canonically extends to a Jordan *-isomorphism $\tilde{J}$ from $(\mathcal{M}_1 \times_{\varphi_1} \mathbb{R})$ onto $(\mathcal{M}_2 \times_{\varphi_2} \mathbb{R})$ which canonically identifies $h_J = \frac{d\varphi}{dz}$ with $h_2 = \frac{d\varphi_2}{dz}$, and isometrically identifies $L^p_{\varphi_1}(\mathcal{M}_1)$ with $L^p_{\varphi_2}(\mathcal{M}_2)$. (Here $\varphi_J = \varphi_2 \circ J$.)

(Note that in the computation in the middle of p 275 of [W1] it is shown that $\tilde{J}$ takes the shift map $\lambda_z$ onto $\lambda_z^2$. This fact together with the continuity of $J$ in the topology of convergence in measure, now ensures that speaking loosely $h_J$ maps onto $h_2$ with respect to this identification.)

We now move on to the more general case when the image of $\mathcal{M}_1$ under $J$ is not necessarily a von Neumann algebra. Let $e$ be the support projection of $\varphi_J$. We remind the reader that $e$ belongs to the center of $\mathcal{M}_1$. Let $z$ be a central projection in $\mathcal{B}$ such that $zJ$ is a *-homomorphism and $(1 - z)J$ is a *-antihomomorphism. Since the kernels of $zJ$ and $(1 - z)J$ are both two-sided ideals in $\mathcal{M}_1$, there exist central projections $e_z$ and $e_{1 - z}$ in $\mathcal{M}_1$ such that $\ker(zJ) = \mathcal{M}_1(1 - e_z)$ and $\ker((1 - z)J) = \mathcal{M}_1(1 - e_{1 - z})$. Note that $e_z$ is the support of $\varphi_z = \varphi_2 \circ zJ$ and $e_{1 - z}$ is the support of $\varphi_{1 - z} = \varphi_2 \circ (1 - z)J$. Note also that $zJ(\mathcal{M}_1 e_z) = \mathbb{B} z$ and $(1 - z)J(\mathcal{M}_1 e_{1 - z}) = \mathbb{B}(1 - z)$. This follows easily from the fact that the smallest von Neumann algebra containing $J(\mathcal{M}_1)$ must also contain the projection $z$, and by then realizing that the direct sum of $zJ(\mathcal{M}_1 e_z)$ and $(1 - z)J(\mathcal{M}_1 e_{1 - z})$ is a von Neumann algebra contained in $\mathcal{B}$, and containing both $z$ and $J(\mathcal{M}_1) = J(\mathcal{M}_1 e)$ (obviously $e = e_z \vee e_{1 - z}$). Therefore $\mathcal{B} = zJ(\mathcal{M}_1 e_z) \oplus (1 - z)J(\mathcal{M}_1 e_{1 - z})$; thus $zJ$ restricted to $\mathcal{M}_1 e_z$ is a *-isomorphism of $\mathcal{M}_1 e_z$ onto $\mathbb{B} z$ and $(1 - z)J$ restricted to $\mathcal{M}_1 e_{1 - z}$ is a *-antihomomorphism of $\mathcal{M}_1 e_{1 - z}$ onto $\mathbb{B}(1 - z)$, and Lemma 4.11 shows that the spaces $L^q_{\varphi_z}(\mathcal{M}_1 e_z \times \mathcal{M}_1 e_{1 - z})$ and $L^q_{\varphi_{1 - z}}(\mathbb{B} z \oplus \mathbb{B}(1 - z)) = L^q_{\varphi_2}(\mathbb{B})$ are isometric. The ‘direct product’ notation for the first space is used remind the reader that $e_z$ and $e_{1 - z}$ are not, in general, orthogonal to each other. We denote the isometry mentioned above by $W^J$. With reference to Lemma 4.11 it is clear that this isometry is constructed from the action of $(zJ, (1 - z)J)$ on $(\mathcal{M}_1 e_z \times \mathcal{M}_1 e_{1 - z})$. Setting $h_z = \frac{d\varphi_z}{dz}$ and $h_{1 - z} = \frac{d\varphi_{1 - z}}{dz}$, it is therefore an exercise to see that $W^J$ will map elements of the form $(h_z^{1/2q} \pi_z(a)h_z^{1/2q}, h_{1 - z}^{1/2q} \pi_{1 - z}(b)h_{1 - z}^{1/2q})$ (where $a, b \in \mathcal{M}(0)$), onto $h_z^{1/2q} \pi_B(zJ(a) + (1 - z)J(b))h_{1 - z}^{1/2q}$.

**Step (IV) and (V): Passing from $L^q_{\varphi_1}(\mathcal{B})$ to $L^q_{\varphi_2}(\mathcal{M}_2)$.** Let us remind the reader that $\mathcal{B}$ is the von Neumann subalgebra of $\mathcal{M}_2$ generated by $J(\mathcal{M}_1)$. We shall need the following results:

**Proposition 4.12.** Let $(\mathcal{M}_i, \varphi_i), i = 1, 2$ be von Neumann algebras with fins weights, and let $j : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ be a positive map satisfying $\varphi_2 \circ j \leq C \varphi_1$ for some $C > 0$. Denote by $j^{(p)}$, $1 \leq p < \infty$ the maps defined on a dense subspace of $L^p_{\varphi_1}(\mathcal{M}_1)$ by $h_1^{(1/2p)} a h_1^{(1/2p)} \mapsto j^{(1/2p)} a h_1^{(1/2p)} j^{(1/2p)} h_1^{(1/2p)}$, where $h_i$ are the densities of the dual weights of $\varphi_i$ with respect to canonical traces on the corresponding crossed products. Then all the $j^{(p)}$‘s extend to bounded linear operators from $L^p_{\varphi_1}(\mathcal{M}_1)$ into $L^p_{\varphi_2}(\mathcal{M}_2)$.

The proposition was proved for finite weights by Junge and Xu [JX, Theorem 5.1], where the norms of the mappings are also calculated. The proof is based on
Haagerup’s lemma [Haa, Lemma 1.1] (see also [Tak, Lemma VII.1.9]). The lemma shows essentially that for a self-adjoint element \(a\) of \(m_\varphi\),
\[
\|h_1^{1/2} a h_1^{1/2}\| = \inf\{\varphi(b) + \varphi(c) : a = b - c, b, c \in m_\varphi^+\}.
\]
Note that the assumed inequality gives boundedness of our mappings on positive elements, and Haagerup’s lemma allows us to extend the bound to self-adjoint elements. That this implies boundedness of the mappings on arbitrary elements is trivial. Since the lemma is true for weights, the proposition is also true for weights, essentially without changes.

**Lemma 4.13.** Let \(M_1\) and \(\varphi_i\) \((i = 1, 2)\) be as before and let \(J : M_1 \to M_2\) be a normal Jordan \(*\)-isomorphism. If \(\varphi_2 \circ J \ll_{loc} \varphi_1\), then \(\varphi_2\) is semifinite on \(B\).

**Proof.** It is enough to show that for any projection \(e \in B\) there exists a projection \(f \in B\) such that \(f \leq e\) and \(\varphi_2(f) < \infty\). Let \(z\) be the central projection in \(B\) such that \(a \mapsto zJ(a)\) is a \(*\)-homomorphism and \(a \mapsto (J(\|) - z)J(a)\) is a \(*\)-antihomomorphism. Note that both \(zJ(M_1)\) and \((J(\|) - z)J(M_1)\) are von Neumann algebras, as the (anti)homomorphic images of a von Neumann algebra, with both the homomorphism and the antihomomorphism normal (see [Tak, Proposition III.3.12]). As noted at the close of the discussion pertaining to Step (III), the direct sum of these two von Neumann algebras is precisely \(B\). Thus there exist projections \(e_1\) and \(e_2\) in \(M_1\) such that \(e = zJ(e_1) + (J(\|) - z)J(e_2)\). Choose now projections \(f_1, f_2\) in \(M_1\) so that \(f_i \leq e_i\) and \(\varphi_1(f_i) < \infty\) for \(i = 1, 2\). Put \(f = zJ(f_1) + (J(\|) - z)J(f_2)\). Then \(f\) is a projection in \(B\), \(f \leq e\) and \(\varphi_2(f) < \infty\).

Let us apply the proposition to the natural embedding \(j\) of the von Neumann algebra \(B\) with weight \(\varphi_2\) restricted to the algebra, into the algebra \(M_2\). The inequality required for the lemma is clearly satisfied with constant 1.

**Remark 4.14.** The maps \(j^{(p)}\) are especially simple if the algebra \(B\) is invariant under the modular group for the couple \((M_2, \varphi_2)\). Then the space \(L^p_{\varphi_2}J(B)\) can be treated as a subspace of \(L^p_{\varphi_2}M_2\) and \(j^{(p)}\) is the natural embedding. To see this, we can mimic the proof of Lemma 4.3. In fact up to this canonical embedding, the maps \(i^{(p)} \circ J\) will in this case yield essentially identical terms on \(M_1^{(0)}\) for both \(M_2\) and \(B\). Thus in dealing with composition operators we may then freely replace \(M_2\) with \(B\). To see this note that in this case \(J(\|)\) (the unit of \(B\)) will be a fixed point of the modular group generated by \(\varphi_2\). In this regard observe that the identity \(\sigma^2_t(J(\|)a) = \sigma^2_t(a)\) for all \(a \in B\) and all \(t \in \mathbb{R}\), ensures that \(\sigma^2_t(J(\|))\) is an identity for \(\sigma^2_t(B) = B\), and hence that \(\sigma^2_t(J(\|)) = J(\|)\) for all \(t \in \mathbb{R}\). By Lemma 4.3 the density of \(\varphi_2\) restricted to \(J(\|)M_2J(\|)\) may then be identified with \(J(\|)h_2 = h_2J(\|)\) (where as before \(h = \frac{A}{A + B}\)). In addition by [Tak IX,4.2] there exists a faithful normal conditional expectation \(E : J(\|)M_2J(\|) \to B\) such that \(\varphi_2 \circ E = \varphi_2\) on \(J(\|)M_2J(\|)\). Hence [G 4.8] assures us that the density of the restriction of \(\varphi_2\) to \(B\), may be identified with that of the restriction to \(J(\|)M_2J(\|)\), described above. Thus up to canonical inclusion we have \(i^{(p)} \circ J(a) = h_2^{1/2p} J(a) h_2^{1/2p} (a \in M_1^{(0)})\) for both \(M_2\) and \(B\).

**The main result.** With a description of steps (I) – (V) now finally behind us, we are ready to give a description of a large class of Jordan \(*\)-morphisms which do yield composition operators.
Lemma 4.15. Let $M_i$ ($i = 1, 2$) be von Neumann algebras equipped with faithful normal semifinite weights $\phi_i$, and let $J : M_1 \to M_2$ be a normal $\ast$-(anti)homomorphism. (In this case $B = J(M_1).)$ Denote the support projection of $\phi_2 \circ J = \phi_J$ by $e$. Suppose that $\sigma_t^{\phi_J}(B) = B$ for each $t \in \mathbb{R}$. Then for each $1 \leq q \leq p < \infty$, $J$ canonically induces a composition operator from $L^p_{\phi_J}(M_1)$ to $L^q_{\phi_J}(M_2)$ if and only if firstly $\phi_J \ll \text{loc} \phi_1$, and secondly $M_1$ admits of a bounded change of weights from $\phi_1$ to $\phi_J$ for the pair $(p, q)$.

Proof. By steps (IV) and (V), the assumption that $\sigma_t^{\phi_J}(B) = B$ for each $t \in \mathbb{R}$, enables us to reduce to the case where $J$ is surjective (see Remark 4.14). The rest of the proof is then essentially contained in step (I), step (III), and Proposition 4.6. \hfill \Box

Before actually extracting our main theorem from the above lemma, we need one final technical observation regarding commuting weights. The result is surely reflected in the literature somewhere, but we have been unable to find a reference, and hence elect to prove the relevant lemmas in full.

Given two densely defined closed operators affiliated to some von Neumann algebra $M$, we say that such operators commute if they are affiliated to a common abelian von Neumann subalgebra of $M$ (or equivalently if their spectral projections commute).

Lemma 4.16. Let $M$ be a von Neumann algebra equipped with faithful normal semifinite weight $\phi$, and let $d \in M$. Then $d$ commutes with $h_\phi = \frac{\partial}{\partial \phi}$ if and only if $d \in M^\psi$, the centralizer of $\phi$ in $M$.

Proof. For a faithful normal semifinite weight $\psi$ on $M$ and a positive self-adjoint densely defined operator $h$ affiliated with $M^\psi$, the weight $\psi_h$ is defined as in [Tak, Lemma VIII.2.8].

We can assume that the crossed product is built using the weight $\phi$. By definition of the Radon-Nikodym derivative, $\tilde{\phi} = \tau_{h_\phi}$. Thus by formula (11) in chapter II of [Tak] and [Tak, Lemma VIII.2.10],

$$\sigma_t^{\phi}(d) = \sigma_t^{\tilde{\phi}}(d) = h_\phi^t d h_\phi^{-it}.$$  

From this it clearly follows that $d \in M^\psi$ if and only if $h_\phi^t d h_\phi^{-it} = d$. Since the latter equality holds if and only if $d$ and $h_\phi$ commute (see [RS, Theorem 1.VIII.13]), we are done. \hfill \Box

Lemma 4.17. Let $\phi$, $\psi$ be faithful normal semifinite weights on $M$. Then $\phi$ and $\psi$ commute (in the sense of satisfying the conditions in [Tak, Corollary VIII.3.6]) if and only if the densities $h_\phi$ and $h_\psi$ commute.

Proof. If $\phi$ and $\psi$ commute, then there exists a nonsingular positive self-adjoint densely defined operator $d$ affiliated with the algebra $M^\psi$ such that $\psi = \phi_d$. Using formula (12) from chapter II of [Tak], [Str 4.8] and the chain rule for the Connes cocycle derivative, we conclude that

$$h_\psi^t = (D\tilde{\phi}_d : D\tau)_t = (D\tilde{\phi}_d : D\tilde{\phi})_t (D\tilde{\phi} : D\tau)_t = (D\tilde{\phi}_d : D\tilde{\phi}) h_\phi^t = h_\phi^t h_\psi^t.$$  

Since $d$ is affiliated with the algebra $M^\psi$, $d^it$ must commute with $h_\phi^it$ by the previous lemma. Hence,

$$h_\psi^t h_\phi^it = d^it h_\phi^it h_\psi^t = h_\phi^it d^it h_\phi^it = h_\phi^it h_\psi^t,$$

which means, again by [RS, Theorem 1.VIII.13], that $h_\phi$ and $h_\psi$ commute.
Conversely, assume that \( h_\varphi \) and \( h_\psi \) commute. Let \( \mathcal{A} \) be the abelian von Neumann algebra generated by the two operators. Since \( h_\varphi^{-1} \) is a densely defined positive self-adjoint operator affiliated with \( \mathcal{A} \), we can put \( d = h_\psi \cdot h_\varphi^{-1} \) (see [KR], Theorem 5.6.15 (iii)). Obviously, \( d \) commutes with both \( h_\varphi \) and \( h_\psi \), and, for each \( t \in \mathbb{R} \), \( d^it = h_\psi^i h_\varphi^{-it} \) (for if \( \mathcal{A} \) is identified with the algebra of continuous functions on an extremely disconnected compact Hausdorff space \( X \), then \( f^it = f_\psi^i f_\varphi^{-it} \) for functions \( f_\psi \) and \( f_\varphi \) corresponding to the operators \( d, h_\psi \) and \( h_\varphi \), respectively). Moreover,
\[
\sigma^\varphi_s(d^it) = h_\psi^is h_\varphi^{-is} = d^it,
\]
so that \( d \) is affiliated with \( \mathcal{M}^\varphi \). Hence (using formula (11) from chapter II of [Tp1], [Str 4.8] and the chain rule for the Connes cocycle derivative),
\[
(D\psi : D\varphi)_t = (D\tilde\psi : D\tilde\varphi)_t = (D\tilde\psi : D\tau)_t(D\tilde\varphi : D\tau)_t^t = h_\psi^it h_\varphi^{-it} \in \mathcal{M}^\varphi,
\]
which guarantees, by the Pedersen-Takesaki theorem (see [Str 4.10(iii)]) that \( \varphi \) and \( \psi \) commute.

**Definition 4.18.** We say that two normal semifinite weights on \( \mathcal{M} \) commute if the support projections of the weights commute, and the restrictions of the weights to the product of support projections also commute.

**Theorem 4.19.** Let \( \mathcal{M}_i \ (i = 1, 2) \) be as before, and let \( 1 \leq r < \infty \) and \( 1 \leq q \leq p \leq \infty \) be given. Further, let \( J : \mathcal{M}_1 \to \mathcal{M}_2 \) be a normal Jordan \(*\)-morphism. Finally let \( z \) be a central projection in \( B \) for which \( zJ \) and \( (1 - z)J \) are respectively \(*\)-homomorphism and a \(*\)-antihomomorphism. As in Step (III) we write \( e_z, e_{1-z} \) for the central support projections of \( zJ \) and \( (1 - z)J \), and set \( \varphi_z = \varphi_2 \circ zJ, \varphi_{1-z} = \varphi_2 \circ (1 - z)J \).

(a) For each pair \( (p, q) \) with \( p/q = r \), \( J \) canonically induces a composition operator from \( L^p_{\varphi_1}(\mathcal{M}_1) \) to \( L^q_{\varphi_2}(\mathcal{M}_2) \) if and only if \( \varphi_J \ll \varphi_1 \) and \( \mathcal{M}_1 \) admits of a bounded change of weights scale from \( \varphi_1 \) to \( \varphi_J \) for the ratio \( r \).

(b) Consider the following statements:

(1) For \( 1 \leq q \leq p \leq \infty \), \( J \) canonically induces a composition operator from \( L^p_{\varphi_1}(\mathcal{M}_1) \) to \( L^q_{\varphi_2}(\mathcal{M}_2) \).

(2) \( \varphi_J \ll_{\text{loc}} \varphi_1 \) and \( \mathcal{M} \times \mathcal{M} \) admits of a bounded change of weights from \( (\varphi_1, \varphi) \) to \( (\varphi_z, \varphi_{1-z}) \) for the pair \( (p, q) \).

(3) \( \varphi_J \ll_{\text{loc}} \varphi_1 \) and \( \mathcal{M} \) admits of a bounded change of weights from \( \varphi_1 \) to \( \varphi_J \) for the pair \( (p, q) \).

In general \( b(3) \Rightarrow b(2) \Rightarrow b(1) \). If \( \sigma^\varphi_t(B) = B \) for each \( t \in \mathbb{R} \), then statements \( b(1) \) and \( b(2) \) are equivalent. If the weights \( \varphi_z \) and \( \varphi_{1-z} \) commute, \( b(2) \) and \( b(3) \) are equivalent. If \( B = J(\mathcal{M}_1) \), \( z \) can be chosen so that \( \varphi_z \) and \( \varphi_{1-z} \) commute.

**Proof.** Throughout the proof we will let \( B \) and \( e \) be as before. As noted in steps (I) and (II), the centrality of \( e \) enables us to assume that \( \varphi_2 \circ J \) is faithful (ie. that \( e = \mathbb{1} \)). Again for the sake of simplicity we will now suppress the technicalities inherent in [Tp1 II.37 & II.38], and identify the crossed products of \( \mathcal{M}_1 \) with \( \varphi_1 \), and \( \mathcal{M}_1 \) with \( \varphi_J \).

(a): To see the **only if** part, assume that for each pair \( (p, q) \) with \( p/q = r \), \( J \) canonically induces a composition operator from \( L^p_{\varphi_1}(\mathcal{M}_1) \) to \( L^q_{\varphi_2}(\mathcal{M}_2) \). Then for
q = 1, p = r, the map \( h_1^{1/(2r)}a h_1^{1/(2r)} \mapsto h_2^{1/2} J(a) h_2^{1/2} \) extends to a bounded map \( C_J : L^r_\varphi(M_1) \to L^1_\varphi(M_2) \). (Here we have as before that \( h_1 = \frac{d\varphi_1}{dt_1} \).

On composing the operator \( C_J \) with the trace functional \( tr_2 \) on \( L^r_\varphi(M_2) \), we obtain a positive bounded linear functional \( tr_2 \circ C_J : L^r_\varphi(M_1) \to \mathbb{C} \). Hence by \( L^p \) duality there exists \( b \in L^\infty_r(M_1) \) with \( tr_1(bc) = tr_2(C_J(c)) \) for each \( c \in L^r_\varphi(M_1) \). Thus

\[
tr_1(b(h_1^{1/(2r)}a h_1^{1/(2r)})) = tr_2(C_J(h_1^{1/(2r)}a h_1^{1/(2r)})) = tr_2(h_2^{1/2} J(a) h_2^{1/2})
\]

for each \( a \in M_1^{(0)} \). But with \( e \) as in the hypothesis and \( k = \frac{d\varphi_2 \circ J}{dt_1} \), we have by [GL2] 2.13(a) that

\[
tr_2(h_2^{1/2} J(a) h_2^{1/2}) = \varphi_2(J(a)) = tr_1(k^{1/2} ak^{1/2}) \quad \text{for all } a \in M_1^{(0)}.
\]

But then

\[
tr_1(b(h_1^{1/(2r)} a h_1^{1/(2r)})) = tr_1(k^{1/2} ak^{1/2}) \quad \text{for all } a \in M_1^{(0)}.
\]

This suffices to force \([f h_1^{1/(2r)}]b(h_1^{1/(2r)} f) = f k f \) for any projection \( f \) with \( \varphi_1(f) < \infty \). The claim follows.

For the \( if \) part suppose that \( \varphi_J \ll \text{loc} \varphi_1 \) and \( M_1 \) admits a bounded change of weights scale from \((M_1, \varphi_1) \) to \((e M_1 e, \varphi_J) \) for the ratio \( r \). A perusal of steps (I) to (III) will reveal that this is enough to ensure that for each \( 1 \leq p, q \leq \infty \) with \( \frac{1}{p} + \frac{1}{q} = r \), \( J \) induces a (generalised) composition operator from \( L^p_\varphi(M_1) \) to \( L^q_{\varphi_1}([B]) \) (for further details see the proof of (b) below). An application of Proposition 4.1.2 to the injection \( B \to M_2 \) now completes the proof.

(b) To facilitate the task of reading the proof we write below the explicit decomposition of the composition operator \( C_J \) for the normal Jordan *-morphism \( J \).

\[
h_1^{1/(2p)} \pi_1(a) h_1^{1/(2p)} \mapsto h_1^{1/(2p)} \pi_1(e a e) h_1^{1/(2p)}
\]

\[
\mapsto k^{1/(2q)} \pi_1(e a e) k^{1/(2q)} \mapsto h_J^{1/(2q)} \pi_J(e a e) h_J^{1/(2q)}
\]

\[
\mapsto (h_2^{1/(2q)} \pi_J(e a e) h_2^{1/(2q)}, h_2^{1/(2q)} \pi_{\varpi - J}(e a e) h_2^{1/(2q)})
\]

\[
\mapsto h_B^{1/(2q)} \pi_B(J(a)) h_B^{1/(2q)} \mapsto h_2^{1/(2q)} \pi_2(J(a)) h_2^{1/(2q)}
\]

In the above scheme \( h_z = \frac{\Xi z}{\delta_1} \) and \( h_{\varpi - z} = \frac{\Xi \varpi - z}{\delta_1} \). In addition to the simplifying assumptions made at the start of the proof, we will in \( b \) also use [T1, II.37] to identify the crossed product of \( M_1 e_z \) and \( \varphi_z \), with \( e_z(M_1 \rtimes_{\varphi_J} \mathbb{R}) \). Similarly the crossed product of \( M_1 e_{\varpi - z} \) and \( \varphi_{\varpi - z} \), is identified with \( e_{\varpi - z}(M_1 \rtimes_{\varphi_J} \mathbb{R}) \). All of these simplifying assumptions have the effect of identifying \( k \) with \( h_J \), and of forcing \( h_J = h_z + h_{\varpi - z} \). (This last equality is a simple consequence of the fact that \( \varphi_J = \varphi_z + \varphi_{\varpi - z} \).)

**b(3) \Rightarrow b(2):** Suppose that \( \varphi_J \ll \text{loc} \varphi_1 \). Since \( \varphi_J \geq \varphi_z, \varphi_{\varpi - z} \), the continuity of the maps

\[
h_J^{1/(2q)} a h_J^{1/(2q)} \mapsto h_z^{1/(2q)} a e_z h_z^{1/(2q)}
\]

and

\[
h_J^{1/(2q)} a h_J^{1/(2q)} \mapsto h_{\varpi - z}^{1/(2q)} a e_{\varpi - z} h_{\varpi - z}^{1/(2q)}
\]
(where \(a \in \mathcal{M}_1^{(0)}\)), is an easy consequence of Proposition 4.14. If therefore \(b(3)\) holds, we merely need to compose the above maps with the given bounded change of weights from \(\varphi_1\) to \(\varphi_J\) for the pair \((p, q)\), to see that

\[
(h_1^{1/(2p)} a h_1^{1/(2p)}, h_1^{1/(2p)} b h_1^{1/(2p)}) \mapsto (h_z^{1/(2q)} a e_z h_z^{1/(2q)}, h_{\|z\|_2}^{1/(2q)} b e_{\|z\|_2} h_{\|z\|_2}^{1/(2q)})
\]

(where \(a, b \in \mathcal{M}_1^{(0)}\)) is continuous.

**b(2) \(\implies\) b(1):** Suppose that \(b(2)\) holds. We show that the hypothesis of \(b(2)\) is strong enough to ensure that \(J\) induces a composition operator from \(L^p_{\varphi_1}(\mathcal{M}_1)\) to \(L^q_{\varphi_2}(\mathcal{B})\). The conclusion will then follow from applying Proposition 4.14 to the inclusion \(\mathcal{B} \to \mathcal{M}_2\). In the remainder of the proof of this implication, we may therefore assume that \(\mathcal{B} = \mathcal{M}_2\). From Lemma 4.3 and the discussion following Lemma 4.5, it is clear that \(L^p_{\varphi_2}(\mathcal{M}_2) = z L^p_{\varphi_2}(\mathcal{M}_2) \oplus (1 - z) L^p_{\varphi_2}(\mathcal{M}_2) = L^p_{\varphi_2}(z \mathcal{M}_2) \oplus L^p_{\varphi_2}((1 - z) \mathcal{M}_2)\). Taking into account the action of the isometry described in the discussion following Lemma 4.11, it is clear that \(J\) will induce the required composition operator from \(L^p_{\varphi_1}(\mathcal{M}_1)\) to \(L^q_{\varphi_2}(\mathcal{M}_2)\), if the map

\[
h_1^{1/(2p)} a h_1^{1/(2p)} \mapsto (h_z^{1/(2q)} a e_z h_z^{1/(2q)}, h_{\|z\|_2}^{1/(2q)} a e_{\|z\|_2} h_{\|z\|_2}^{1/(2q)})
\]

(where \(a \in \mathcal{M}_1^{(0)}\)) extends to a bounded linear map from \(L^p_{\varphi_1}(\mathcal{M}_1)\) to \(L^q_{(\varphi_1, \varphi_{\|z\|_2})}(\mathcal{M}_1 e_z \times \mathcal{M}_1 e_{\|z\|_2})\). But this map is just the given bounded change of weights described in \(b(2)\), composed with the bounded injection

\[
L^p_{\varphi_1}(\mathcal{M}_1) \to L^p_{(\varphi_1, \varphi_{\|z\|_2})}(\mathcal{M}_1 \times \mathcal{M}_1) : a \mapsto (a, a)
\]

Hence the claim follows.

**b(1) \(\implies\) b(2):** Suppose that \(\varphi_J \leq_{\text{loc}} \varphi_1\). Assume that \(\sigma^{q, z}(\mathcal{B}) = \mathcal{B}\) for each \(t \in \mathbb{R}\).

Steps (IV) and (V) ensure that we may then assume \(\mathcal{M}_2 = \mathcal{B}\) (see Remark 4.14). Under this assumption we may therefore select a central projection \(z \in \mathcal{M}_2\) so that \(z J\) is a \(*\)-homomorphism onto \(z \mathcal{M}_2\) and \((1 - z) J\) a \(*\)-antihomomorphism onto \((1 - z) \mathcal{M}_2\).

Now suppose that \(b(1)\) holds. From the action of the isometry described in the discussion following Lemma 4.11, it is clear that \(b(1)\) is exactly equivalent to the continuity of the map

\[
h_1^{1/(2p)} a h_1^{1/(2p)} \mapsto (h_z^{1/(2q)} a e_z h_z^{1/(2q)}, h_{\|z\|_2}^{1/(2q)} a e_{\|z\|_2} h_{\|z\|_2}^{1/(2q)})
\]

(where \(a \in \mathcal{M}_1^{(0)}\)). Thus each of \(h_1^{1/(2p)} a h_1^{1/(2p)} \mapsto h_z^{1/(2q)} a e_z h_z^{1/(2q)}\) and \(h_1^{1/(2p)} a h_1^{1/(2p)} \mapsto h_{\|z\|_2}^{1/(2q)} a e_{\|z\|_2} h_{\|z\|_2}^{1/(2q)}\) are separately continuous, which in turn is sufficient to force the continuity of

\[
(h_1^{1/(2p)} a h_1^{1/(2p)}, h_1^{1/(2p)} b h_1^{1/(2p)}) \mapsto (h_z^{1/(2q)} a e_z h_z^{1/(2q)}, h_{\|z\|_2}^{1/(2q)} b e_{\|z\|_2} h_{\|z\|_2}^{1/(2q)})
\]

(where \(a, b \in \mathcal{M}_1^{(0)}\)) as required.

**b(2) \(\implies\) b(3):** Suppose that \(b(2)\) holds, and assume that \(\varphi_z, \varphi_{\|z\|_2}\) commute. By the lemma, this has the effect of ensuring that \(h_z\) and \(h_{\|z\|_2}\) are commuting affiliated operators. On composing the map

\[
h_1^{1/(2p)} a h_1^{1/(2p)} \mapsto (h_1^{1/(2p)} a h_1^{1/(2p)}, h_1^{1/(2p)} a h_1^{1/(2p)}) \quad a \in \mathcal{M}_1^{(0)}
\]

with the given change of weights, we obtain the continuity of the map

\[
h_1^{1/(2p)} a h_1^{1/(2p)} \mapsto (h_z^{1/(2q)} a e_z h_z^{1/(2q)}, h_{\|z\|_2}^{1/(2q)} a e_{\|z\|_2} h_{\|z\|_2}^{1/(2q)}) \quad a \in \mathcal{M}_1^{(0)}
\]
By continuity this map will also map terms of the form $h_1^{1/(2p)} |a|^2 h_1^{1/(2p)}$ (where $a \in \mathcal{M}_1^{(0)}$), onto the terms $(h_z^{1/(2p)} |a|^2 e_z h_z^{1/(2p)}, h_{\|e_{\|z}}^1 |a|^2 e_{\|z} h_{\|e_{\|z}}^1)$. Since this convergence takes place in the compression $\|e_{\|z} - \|z\| - z$ and simi-
larly for such operators (see [KR, §5.6]), that the terms

$$h_z^{1/(2p)} |a|^2 e_z h_z^{1/(2p)}, h_{\|e_{\|z}}^1 |a|^2 e_{\|z} h_{\|e_{\|z}}^1)$$

will converge to $(h_z^{1/(2p)} |a|^2 e_z h_z^{1/(2p)}, h_{\|e_{\|z}}^1 |a|^2 e_{\|z} h_{\|e_{\|z}}^1)$. Consequently

$$1 \leq M \geq 0 \text{ so that }$$

$$(||h_z^{1/(2p)}||^{2q} + ||h_{\|e_{\|z}}^1||^{2q})^{1/q} = ||h_z^{1/(2p)} |a|^2 h_z^{1/(2p)}, h_{\|e_{\|z}}^1 |a|^2 h_{\|e_{\|z}}^1)\|^{1/q} \leq M ||h_z^{1/(2p)} |a|^2 h_z^{1/(2p)}\|^{1/q} = M ||h_z^{1/(2p)}||^{2q}.$$

To conclude the proof we need only show that there exists some $K > 0$ with

$$||a h_j^{1/(2q)}|| \leq K (||a h_z^{1/(2q)}||^{2q} + ||a h_{\|e_{\|z}}^1||^{2q})^{1/(2q)}$$

for all $a \in \mathcal{M}_1^{(0)}$, and apply Lemma 4.10. This fact is palpably clear if $q = \infty$, and hence we will assume $1 \leq q < \infty$. For such a $q$ it is a simple matter to show that $(r+s)^{1/q} \leq r^{1/q} + s^{1/q}$ for any $r, s \in \mathbb{R}^+$. Thus since $h_z$ and $h_{\|e_{\|z}}$ are commuting positive affiliated operators with $h_j = h_z + h_{\|e_{\|z}}$, it follows from the Borel functional calculus for such operators (see [KR, §5.6]), that $h_j^{1/q} \leq h_z^{1/q} + h_{\|e_{\|z}}^{1/q}$. Given any $a \in \mathcal{M}_1^{(0)}$,

this in turn has the effect of ensuring that $[h_j^{1/(2p)} a]^{*2} \leq [h_z^{1/(2p)} a]^{*2} + [h_{\|e_{\|z}}^{1/(2p)} a]^{*2}$. Consequently

$$||h_j^{1/(2p)} a]^{*2}||_q \leq ||h_z^{1/(2p)} a]^{*2} + [h_{\|e_{\|z}}^{1/(2p)} a]^{*2}||_q \leq ||h_z^{1/(2p)} a]^{*2}||_q + ||h_{\|e_{\|z}}^{1/(2p)} a]^{*2}||_q.$$
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