Fourier Coefficients of Beurling Functions and a Class of Mellin Transform Formally Determined by its Values on the Even Integers

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Abstract. It is a well-known fact that Riemann Hypothesis will follow if the function identically equal to \(-1\) can be arbitrarily approximated in the norm \(\|\cdot\|\) of \(L^2([0, 1], dx)\) by functions of the form \(f(x) = \sum_{k=1}^{n} a_k \rho \left( \frac{\theta_k}{x} \right)\), where \(\rho(x) := x - [x]\), and \(a_k \in \mathbb{C}, 0 < \theta_k \leq 1\) satisfies \(\sum_{k=1}^{n} a_k \theta_k = 0\). Parsevall Identity \(\|f(x) + 1\| = \sum_{n \in \mathbb{Z}} |c(n)|^2\) is a possible tool to compute or estimate this norm. In this note we give an expression for the Fourier coefficients \(c(n)\) of \(f + 1\), when \(f\) is a function defined as above. As an application, we derive an expression for \(M_f(s) := \int_{0}^{1} (f(x) + 1) x^{s-1} dx\) as a series that only depends on \(M_f(2k), k \in \mathbb{N}\). We remark that the Fourier coefficients \(c(n)\) depend on \(M_f(2k)\) which, for a function \(f\) defined as above, can be expressed also in terms of the \(a_k\)'s and \(\theta_k\)'s. Therefore, a better control on these parameters will allow to estimate \(M_f(2k)\) and therefore eventually to handle \(\|f + 1\|\) via our expression for the Fourier coefficients and Parsevall Identity.

1 Introduction

Denote as \([x]\) the “integer part of \(x\)”, i.e. the greatest integer less than or equal to \(x\) and define the “fractional part” function by \(\rho(x) = x - [x]\). When \(f\) is a function of the form

\[ f_N(x) := \sum_{k=1}^{N} a_k \rho \left( \frac{\theta_k}{x} \right), \tag{1} \]

where \(N \in \mathbb{N}, a_k \in \mathbb{C}\) and \(0 < \theta_k \leq 1\) for all \(k \in \mathbb{N}\), an elementary computation shows

\[
\int_{0}^{1} (f_N(x) + 1) x^{s-1} dx = \sum_{k=1}^{N} a_k \theta_k \frac{x}{s-1} + \frac{1}{s} \left( 1 - \zeta(s) \sum_{k=1}^{N} a_k \theta_k^s \right); \tag{2}
\]

for \(\sigma > 0\), where, as usual, we denote \(s = \sigma + it\). See, for instance, [2, p. 253] for a proof. We will assume that function \(f_N\) satisfies also the additional condition

\[
\sum_{k=1}^{N} a_k \theta_k = 0. \tag{3}
\]
Identity (2) is the starting point of a theorem by Beurling. See [1, p. 252] for a proof and further references. As in the proof of (the easy half of) Beurling’s Theorem, application of Schwarz inequality to the left side of (2) allows to show that a sufficient condition for Riemann Hypothesis is that \( \| f(x) + 1 \| \) be done arbitrarily small for a convenient choice of \( a_k \) and \( \theta_k \), where \( \| . \| \) denotes the norm in \( L^2([0, 1], dx) \).

Just for reference, we will call a function \( f_N \) as in (1) as an approximation or Beurling function, and the sequence \( \{ f_N \}_{N \in \mathbb{N}} \) is called an approximation sequence. We remark that approximation sequences do not necessarily converge to \( -1 \) in \( L^2([0, 1], dx) \); see the excellent work [2] on this topic. For abuse of notation and language, when \( f_N \) is a Beurling function we will call \( f_N + 1 \) also a Beurling function.

A method to compute \( \| f_N + 1 \| := \left( \int_0^1 | f_N(t) + 1 |^2 dt \right)^{1/2} \) would be to use not this definition but the Parsevall Identity \( \| f_N + 1 \|^2 = \sum_{n \in \mathbb{Z}} | c(N, n) |^2 \). In the Sec. 2 we give an expression for the Fourier coefficients \( c(N, n) \) of the Beurling function \( f_N + 1 \).

In this note, unless explicit statement on contrary, we assume condition (3) on a Beurling function. At a certain point we will assume also that \( \theta_k = 1/b_k \) where \( b_k \in \mathbb{N} \) and \( | a_k | \leq 1 \). This restriction on the \( \theta_k \)'s is not serious at all after Theorem 1.1 in [3]. On the other hand, the restriction on the \( a_k \)'s include some of the so-called natural approximations considered in [2].

# 2 The Fourier Coefficients for a Beurling function

For convenience we define \( F_N := f_N(x) + 1 \). We extend \( F_N \) to an odd function in \([-1, 1]\). Therefore, \( \int_{-1}^1 F_N(x) \cos(n \pi x) dx = 0 \), and

\[
c(N, n) := \int_{-1}^1 F_N(x) \sin(n \pi x) dx = 2 \int_0^1 F_N(x) \sin(n \pi x) dx
\]

\[
= 2 \left[ \int_0^1 \sin(n \pi x) dx + \sum_{k=1}^N a_k \int_0^1 \rho \left( \frac{\theta_k}{x} \right) \sin(n \pi x) dx \right]
\]

\[
= 2 \left[ -\frac{\cos(n \pi x)}{n \pi} \bigg|_0^1 + \sum_{k=1}^N a_k \left( \sum_{j=1}^\infty \int_{\theta_k x}^{\theta_k x + \pi \theta_k} \rho \left( \frac{\theta_k}{x} \right) \sin(n \pi x) dx + \int_{\theta_k x}^{\theta_k x + \pi \theta_k} \rho \left( \frac{\theta_k}{x} \right) \sin(n \pi x) dx \right) \right]
\]
\[
= 2 \left[ \frac{1}{n\pi} (1 - \cos(n\pi)) \right] \\
+ \sum_{k=1}^{N} a_k \left( \sum_{j=1}^{\infty} \theta_k \int_{\frac{\theta_k}{j+1}}^{\frac{\theta_k}{j}} \frac{\sin(n\pi x)}{x} \, dx - \int_{\frac{\theta_k}{j+1}}^{\frac{\theta_k}{j}} \sin(n\pi x) \, dx + \theta_k \int_{\theta_k}^{1} \frac{\sin(n\pi x)}{x} \, dx \right) \\
= \frac{2}{n\pi} (1 - \cos(n\pi)) + 2 \sum_{k=1}^{N} a_k \left( \theta_k \int_{0}^{1} \frac{\sin(n\pi x)}{x} \, dx - \sum_{j=1}^{\infty} \int_{\frac{\theta_k}{j+1}}^{\frac{\theta_k}{j}} \sin(n\pi x) \, dx \right) \\
= \frac{2}{n\pi} (1 - \cos(n\pi)) + 2 \left( \sum_{k=1}^{N} a_k \theta_k \right) \left( \int_{0}^{1} \frac{\sin(n\pi x)}{x} \, dx \right) - 2 \sum_{k=1}^{N} a_k \sum_{j=1}^{\infty} j \frac{\cos(n\pi x)}{n\pi} \left|_{\frac{\theta_k}{j+1}}^{\frac{\theta_k}{j}} \right| \\
\]

Denote the first term in last line by \(A_1 := \frac{2}{n\pi} (1 - \cos(n\pi))\), and note that the second term vanishes because \(3\), hence

\[
c(N, n) = A_1 + \frac{2}{n\pi} \sum_{k=1}^{N} a_k \sum_{j=1}^{\infty} j \left[ \cos \left( \frac{n\pi \theta_k}{j} \right) - \cos \left( \frac{n\pi \theta_k}{j+1} \right) \right]. \tag{4}
\]

Now, if \(L\) is any natural number, replacing in (4) the expression for the \(L\)-th Taylor approximation for \(\cos x\) given by

\[
\cos x = \sum_{l=0}^{L} (-1)^l \frac{x^{2l}}{(2l)!} + \frac{1}{L!} \int_{0}^{x} \cos^{(L+1)}(t) (x-t)^L \, dt, \tag{5}
\]

we get

\[
c(N, n) \\
= A_1 + \frac{2}{n\pi} \sum_{k=1}^{N} a_k \sum_{j=1}^{\infty} j \left( \sum_{l=0}^{L} (-1)^l \frac{(n\pi \theta_k)^{2l}}{(2l)!} \left[ \frac{1}{j^{2l}} - \frac{1}{(j+1)^{2l}} \right] + R(L, j, n, k) \right) \\
= A_1 + \frac{2}{n\pi} \sum_{k=1}^{N} a_k \left( \sum_{l=1}^{L} (-1)^l \frac{(n\pi \theta_k)^{2l}}{(2l)!} \sum_{j=1}^{\infty} j \left[ \frac{1}{j^{2l}} - \frac{1}{(j+1)^{2l}} \right] + \sum_{j=1}^{\infty} j R(L, j, n, k) \right) \tag{6}.
\]
where

\[ R(L, j, n, k) := \frac{1}{L!} \int_0^{n\pi\theta_k} \cos^{(L+1)}(t) \left( \frac{n\pi\theta_k}{j} - t \right)^L dt \]

\[ - \frac{1}{L!} \int_0^{n\pi\theta_{k+1}} \cos^{(L+1)}(t) \left( \frac{n\pi\theta_k}{j+1} - t \right)^L dt. \]  (7)

Now, observe that

\[ \infty \sum_{j=1}^\infty j \left[ \frac{1}{j^{2l}} - \frac{1}{(j+1)^{2l}} \right] = \lim_{J \to \infty} J \sum_{j=1}^J \left[ \frac{j}{j^{2l}} - \frac{j+1}{(j+1)^{2l}} \right] \]

\[ = \lim_{J \to \infty} \left( \sum_{j=1}^J \left[ \frac{j}{j^{2l}} - \frac{j+1}{(j+1)^{2l}} \right] + \sum_{j=1}^J \frac{1}{(j+1)^{2l}} \right) \]

\[ = \lim_{J \to \infty} \left( 1 - \frac{J+1}{(J+1)^{2l}} + \sum_{j=2}^{J+1} \frac{1}{j^{2l}} \right) = 1 + \sum_{j=2}^\infty \frac{1}{j^{2l}} = \zeta(2l); \]  (8)

Substituting (8) in (6) and denoting \( R(L, n, k) := \sum_{j=1}^\infty j R(L, j, n, k) \), we get

\[ c(N, n) = A_1 + \frac{2}{n\pi} \sum_{k=1}^N a_k \left( \sum_{l=1}^L (-1)^l \frac{(n\pi\theta_k)^{2l}}{(2l)!} \zeta(2l) + R(L, n, k) \right) \]

\[ = A_1 + \frac{2}{n\pi} \sum_{k=1}^L (-1)^l \frac{(n\pi)^{2l}}{(2l)!} \zeta(2l) \left( \sum_{k=1}^N a_k \frac{\theta_k^{2l}}{k!} \right) + \sum_{k=1}^N a_k R(L, n, k). \]  (9)

If we denote \( M_g(s) := \int_0^1 g(x) x^{s-1} dx \), then by (2), under condition (3), we have

\[ \sum_{k=1}^N a_k \theta_k^{2l} = \frac{1}{\zeta(2l)} (1 - (2l) M_{FN}(2l)). \]  (10)

Substituting now (10) in (9) we get
\[ c(N, n) = A_1 + \frac{2}{n\pi} \sum_{l=1}^{L} (-1)^l \frac{(n\pi)^{2l}}{(2l)!} \]

\[ - \frac{2}{n\pi} \sum_{l=1}^{L} (-1)^l \frac{(n\pi)^{2l}}{(2l)!} (2l) M_{F_N}(2l) + \sum_{k=1}^{N} a_k R(L, n, k) \]

\[ = A_1 + \frac{2}{n\pi} \sum_{l=1}^{L} (-1)^l \frac{(n\pi)^{2l}}{(2l)!} \]

\[ + 2 \sum_{l=1}^{L} (-1)^{l-1} \frac{(n\pi)^{2l-1}}{(2l-1)!} M_{F_N}(2l) + \sum_{k=1}^{N} a_k R(L, n, k). \quad (11) \]

This is an exact expression valid for all \( L \in \mathbb{N} \) but a better expression arises in the limit \( L \to \infty \). In this case, the first term cancels the second and is not difficult to prove the following

**Lemma 1** For each \( k \in \mathbb{N} \) assume \(|a_k| \leq 1\) and \( \theta_k = 1/b_k \) with \( b_k \in \mathbb{N} \). Then, for any fixed \( N \) and \( n \) in \( \mathbb{N} \) is

\[ \lim_{L \to \infty} \sum_{k=1}^{N} a_k R(L, n, k) = 0. \]

(The proof will be given in Sec. 4). Therefore, the final expression for the Fourier coefficient is

\[ c(N, n) = 2 \sum_{l=1}^{\infty} (-1)^{l-1} \frac{(n\pi)^{2l-1}}{(2l-1)!} M_{F_N}(2l). \quad (12) \]

### 3 An Expression for the Mellin Transform

Directly from (12) we have

\[ M_{F_N}(s) = \int_{0}^{1} F_N(x) x^{s-1} dx = \int_{0}^{1} \left( \sum_{n=1}^{\infty} c(N, n) \sin(n\pi x) \right) x^{s-1} dx \]

\[ = \sum_{n=1}^{\infty} \left( \int_{0}^{1} \sin(n\pi x) x^{s-1} dx \right) 2 \sum_{l=1}^{\infty} (-1)^{l-1} \frac{(n\pi)^{2l-1}}{(2l-1)!} M_{F_N}(2l). \quad (13) \]
The last expression gives the value of $M_{F_N}(s)$ in term of its values in the even integers $M_{F_N}(2l)$.

Note also that under the hypothesis of Lemma 1 we have $\left| \sum_{k=1}^{N} a_k \frac{\theta_k^{2l}}{k} \right| \leq \zeta(2l)$, and this combined with (10), or (2), gives an estimation on the Mellin transform

$$\left| M_{F_N}(2l) \right| \leq \frac{1 + \zeta^2(2l)}{2l}. \quad (14)$$

As in the derivation of relation (18) in Sec. 4, a better control on the $a_k$’s and $\theta_k$’s will allows to control $M_{F_N}(2l)$ via (10), and therefore to control $\|f_N(x) + 1\|$ via (12) and Parsevall Identity.

## 4 Proof of Lemma 1

For sake of brevity, denote $I(j) := (1/L!) \int_0^{\frac{n \pi \theta_k}{j}} \cos^{(L+1)}(t) \left( \frac{n \pi \theta_k}{j} - t \right)^L$. From (7) we have

$$R(L, n, k) := \sum_{j=1}^{\infty} j R(L, j, n, k) = \lim_{J \to \infty} \sum_{j=1}^{J} j I(j) - (j + 1 - 1) I(j + 1)$$

$$= \lim_{J \to \infty} \left( \sum_{j=1}^{J} [j I(j) - (j + 1) I(j + 1)] + \sum_{j=1}^{J} I(j + 1) \right)$$

$$= \lim_{J \to \infty} \left( I(1) - (J + 1) I(J + 1) + \sum_{j=1}^{J} I(j + 1) \right)$$

$$= \lim_{J \to \infty} \left( \sum_{j=1}^{J+1} I(j) - (J + 1) I(J + 1) \right). \quad (15)$$

Using the elementary estimative

$$|I(j)| \leq \frac{1}{L!} \left( \frac{n \pi \theta_k}{j} - t \right)^{L+1} \left| \frac{n \pi \theta_k}{j} \right|^{\frac{n \pi \theta_k}{j}} \left| \frac{n \pi \theta_k}{j} \right|^L \leq \frac{1}{j^{L+1}} \frac{(n \pi \theta_k)^{L+1}}{(L+1)!}, \quad (16)$$

for any $L \in \mathbb{N}$ we have

$$|R(L, n, k)| \leq \lim_{J \to \infty} \frac{(n \pi \theta_k)^{L+1}}{(L+1)!} \left( \sum_{j=1}^{J+1} \frac{1}{j^{L+1}} + \frac{1}{(J + 1)^L} \right) = \frac{(n \pi \theta_k)^{L+1}}{(L+1)!} \zeta(L + 1). \quad (17)$$
Therefore, for any \( N \in \mathbb{N} \),

\[
\left| \sum_{k=1}^{N} a_k R(L, n, k) \right| \leq \sum_{k=1}^{N} |R(L, n, k)| \leq \frac{(n\pi)^{L+1}}{(L+1)!} \zeta(L+1) \sum_{k=1}^{N} \theta_k^{L+1}
\]

\[
= \frac{(n\pi)^{L+1}}{(L+1)!} \zeta(L+1) \sum_{k=1}^{N} \frac{1}{\theta_k^{L+1}} \leq \frac{(n\pi)^{L+1}}{(L+1)!} \zeta^2(L+1). \quad (18)
\]

Now the assertion of Lemma follows observing that \( \zeta(L+1) \) remains close to 1 for large \( L \) and the first factor in the right side of (18) goes to zero because is the \((L+1)\)-th term of the (convergent) series for \( \exp(n\pi) \).

### References

[1] Donoghue W. F. *Distributions and Fourier Transforms*, Academic Press, New York, 1969.

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