WONDERFUL VARIETIES: A GEOMETRICAL REALIZATION

S. CUPIT-FOUTOU

Abstract. We give a geometrical realization of wonderful varieties by means of a suitable class of invariant Hilbert schemes. Consequently, we prove Luna’s conjecture asserting that wonderful varieties are classified by combinatorial invariants, called spherical systems.

CONTENTS

Introduction 2
1. Wonderful varieties 6
  1.1. Combinatorial invariants 6
  1.2. Total coordinate ring 7
  1.3. Spherical varieties 8
2. Spherical systems 8
  2.1. Set of weights associated to a spherical system 9
  2.2. Relations with wonderful varieties 10
3. Invariant Hilbert schemes 11
  3.1. Definition 11
  3.2. Toric action 11
  3.3. Tangent space 12
  3.4. Invariant infinitesimal deformations 12
  3.5. Obstruction space 12
4. Geometrical construction of wonderful varieties 15
  4.1. Invariant Hilbert scheme attached to a spherical system 15
  4.2. The wonderful variety attached to a spherical system 17
Appendix A. Computations of cohomology groups 19
  A.1. Properties of the weights attached to a spherical system 20
  A.2. Auxiliary lemmas 21
  A.3. Computations for degree 0 22
  A.4. Computations for degree 1 24
Appendix B. Injectivity 31

I was partially supported by the DFG Schwerpunktprogramm 1388 - Darstellungstheorie.
INTRODUCTION

Wonderful varieties are complex algebraic varieties which encompass De Concini-Procesi compactifications of symmetric spaces studied in [DP] and flag varieties: they are projective, smooth, equipped with an action of a connected reductive algebraic group $G$, spherical (they contain a dense orbit for a Borel subgroup $B$ of $G$),... The unique closed $G$-orbit, the $B$-weights of the function field of a wonderful $G$-variety as well as its $B$-stable but not $G$-stable prime divisors are invariants of special interest; they are/yield some combinatorial invariants.

After Wasserman completed the classification of rank 2 wonderful varieties ([W]), Luna highlighted in [Lu3] some properties enjoyed by such combinatorial invariants and took them as axioms to set up the definition of spherical systems. Luna’s conjecture asserts that there corresponds a unique wonderful variety to any spherical system. This conjecture has been positively answered in case the group $G$ is of simply laced type ([Lu3, BP, Bra]) and for some other peculiar spherical systems ([Lu4, BC2]). The approach followed therein is that initiated by Luna in [Lu3], it is Lie theoretical and involves case-by-case considerations: for a given spherical system of some group $G$, a subgroup $H$ of $G$ is exhibited and thereafter $G/H$ is proved to admit a wonderful compactification with the desired spherical system. A full answer to this conjecture is known only for the uniqueness part of this problem: thanks to Losev’s work ([Lo]), the uniqueness holds true.

Besides the combinatorial invariants, let us mention the total coordinate ring (known also as the Cox ring) of a wonderful variety, an algebro-geometric invariant introduced by Brion in [Bri4]. The structure of this ring gives insight to the combinatorial invariants. Moreover, as shown in loc. cit., this ring is factorial and finitely generated; the spectrum of this ring is the total space of a flat family of (normal) spherical varieties. As stated in [Lu4], some of these families provide a nice geometric interpretation of Luna’s parametrization (obtained therein) of the isomorphism classes of the so-called model spaces of some group $G$ by the orbits of a ”model” wonderful variety.

As shown in [AB], families of closed $G$-schemes of a given finite dimensional $G$-module are parameterized by quasi-projective schemes, the invariant Hilbert schemes; a special feature of some of these schemes is that they allow to prove several results concerning spherical varieties. Connections with invariant Hilbert schemes and wonderful varieties
were clearly established in [J, BC1]; the invariant Hilbert schemes considered there are proved to be affine and the corresponding universal families turn out to be the families occurring in Brion’s work. These results were obtained by means of the already known classification of wonderful varieties of rank 1 (resp. of strict wonderful varieties) ([A] and also [Bri1], resp. [BC1]).

The problem of determining classification-free these invariant Hilbert schemes then arises naturally (see also [Lu4] for related questions). The main goal of this paper is to solve this problem; we study a suitable class of invariant Hilbert schemes by deformation theoretic methods in order to construct wonderful varieties. As a consequence, we shall obtain a positive answer to Luna’s conjecture.

The first two sections concern wonderful varieties (and more generally spherical varieties) and spherical systems. In the first section, we make some brief recalls on wonderful varieties and some of their invariants; we state Luna’s definition of spherical systems in the next section. We then attach to any spherical system of some group $G$ a set of characters $\lambda_D = (\omega_D, \chi_D)$ indexed by a finite set $\Delta$. The $\omega_D$’s are dominant weights of $G$ defined after [F] and the $\chi_D$’s are characters of some well-determined diagonalizable group $C$. The characters $\omega_D$ (resp. $\chi_D$) encapsulate the first (resp. the third) datum of the spherical system under consideration. We end up the second section by recalling how wonderful varieties and spherical systems are related and by stating explicitly Luna’s conjecture. Further, we shall give a geometrical interpretation of the characters $\lambda_D$.

The third section is devoted to definitions and results concerning invariant Hilbert schemes; after [AB], we give a characterization of their tangent spaces and define a toric action on these schemes. We introduce also the functor of invariant deformations and give an interpretation of its obstruction space.

In the fourth section, we settle our main results. Many geometrical properties of wonderful varieties can be read off spherical systems and vice versa ([Lu3]). This provides a natural dictionary which in turn allows many reductions to prove Luna’s conjecture. In particular, it suffices to consider the so-called spherically closed spherical systems (Section 6 in loc. cit.). Given a spherically closed spherical system $\mathcal{S}$ of some group $G$, we consider the $G := G \times C^\circ$-module

$$V = \oplus_\Delta V(\lambda_D)^*$$

where the weights $\lambda_D$ stand for the characters associated to $\mathcal{S}$ and $V(\lambda_D)^*$ for the dual of the irreducible $G$-module associated to $\lambda_D$. We then study the corresponding invariant Hilbert scheme $\text{Hilb}(\mathcal{S})$. This
invariant Hilbert scheme contains in particular as closed point the $G$-orbit closure within $V$

$$X_0(S) = \overline{G.v^*_\lambda D} \quad \text{with} \quad v^*_\lambda D = \sum_{D \in \Delta} v^*_\lambda D$$

where $v^*_\lambda D$ denotes a highest weight vector of $V(\lambda D)^*$; Hilb($S$) parameterizes the $G$-subschemes of $V$ whose coordinate ring is isomorphic as a $G$-module to that of $X_0(S)$.

In case of spherical systems with emptyset as third datum, the invariant Hilbert schemes we are studying fall in the setting considered in [J, BC2]; see Remark 22 for details.

We then describe the tangent space at $X_0(S)$ of Hilb($S$). We shall show that the second datum of the given spherical system $S$ (the so-called set of spherical roots of $S$) is encoded in this tangent space. More specifically, generalizing the arguments developed in [BC1], we obtain

**Theorem 1.** (Corollary 24) The tangent space at $X_0(S)$ of Hilb($S$) is a multiplicity free module for an adjoint torus of $G$; its weights are the opposites of the spherical roots of $S$.

Afterwards, we show that the invariant Hilbert scheme is smooth. This is achieved essentially by proving that the obstruction space of the functor of invariant deformations of $X_0(S)$ is trivial (Theorem 25). As a consequence, we obtain the following statement.

**Theorem 2.** (Theorem 26) The scheme Hilb($S$) is an affine toric variety for an adjoint torus of $G$; its weights are the opposites of the spherical roots of $S$. More specifically, it is an affine space $\mathbb{A}^r$.

Consider the universal family of Hilb($S$)

$$\mathcal{X}^{\text{univ}} \xrightarrow{\pi} \mathbb{A}^r$$

and let $\mathcal{X}^{\text{univ}}$ be the open subset of $\mathcal{X}^{\text{univ}}$ defined as follows

$$\mathcal{X}^{\text{univ}} = \{ x \in \mathcal{X}^{\text{univ}} : G.x \text{ is open in } \pi^{-1}(x) \}.$$ 

There is an action of the algebraic torus $G_m^\Delta = GL(V)^G$ on $\mathcal{X}^{\text{univ}}$. This toric action stabilizes the set $\mathcal{X}^{\text{univ}}$; see Section 5.2.

**Theorem 3.** (Theorem 29) Let

$$X(S) = \mathcal{X}^{\text{univ}}/G_m^\Delta.$$ 

Then $X(S)$ is a wonderful $G$-variety whose spherical system is the given $S$. Further its total coordinate ring is the coordinate ring of $\mathcal{X}^{\text{univ}}$. 
Combining the above result with Luna’s reduction procedure, we conclude by proving that Luna’s conjecture is true (Corollary 30). The appendix can be read independently to the text body. In Appendix A, we prove the following theorem

**Theorem 4.** Let

\[
\Sigma(\Delta) = \Sigma \cup \{ \alpha + \alpha' : \alpha, \alpha' \text{ adjacent simple roots in } \Sigma \}.
\]

Then (for \(i = 0, 1\))

\[
H^i(G_{v_\lambda}, V/\mathfrak{g}.v_\lambda) \cong \bigoplus_{\ell(w) = i, \gamma \in \Sigma(\Delta)} k_{w^*}\gamma \text{ as } T_{\text{ad}}\text{-modules.}
\]

If \(w\) is of length 1, the one dimensional \(T_{\text{ad}}\text{-module} k_{w^*}\gamma\) is spanned by \(X^*_\alpha \otimes [wv_\gamma]\) where \(w\) is the simple reflection associated to \(\alpha\) with \(\alpha \in S \setminus S^p\) and \(\gamma + \alpha \in \mathbb{Z}\Delta\).

This theorem is used to characterize the tangent space at \(X_0(S)\) of \(\text{Hilb}(S)\) as well as the obstruction space of the functor of invariant infinitesimal deformations of \(X_0(S)\). In Appendix B, we prove that the latter is in fact trivial.

**Acknowledgments.** I am very grateful to Michel Brion for the interest he demonstrated for my work, through his help and advice. I thank also Dominique Luna for enlightening exchanges and Peter Littelmann for the support he provides me.

**Notation.** The ground field \(k\) is the field of complex numbers. Throughout this paper, \(G\) denotes a connected reductive algebraic group. We fix a Borel subgroup \(B\) of \(G\) and \(T \subset B\) a maximal torus; the unipotent radical of \(B\) is denoted \(U\). The choice of \(B\) defines the set of simple roots \(S\) of \(G\) as well as the set \(\Lambda^+\) of dominant weights. Recall that the latter parametrizes the simple \(G\)-modules; we denote as usual by \(V(\lambda)\) the simple \(G\)-module associated to \(\lambda \in \Lambda^+\). The dual module \(V(\lambda)^*\) is isomorphic to \(V(-w_0(\lambda))\), \(w_0\) being the longest element of the Weyl group of \((G, T)\). We label the simple roots as in Bourbaki ([Bo]).

The character group of any group \(H\) is denoted \(\Xi(H)\); note that \(\Xi(B)\) and \(\Xi(T)\) are naturally identified. For any \(\mu \in T, e^\mu\) refers to the corresponding regular function in \(k[T]\). Given \(V\) any \(H\)-module and a weight \(\mu \in \Xi(H)\) of \(V\), \(V_\mu\) stands for the \(\mu\)-weight space of \(V\) and \(V(\mu)\) for the isotypical component of type \(V(\mu)\) (\(\mu \in \Lambda^+\)).
1. Wonderful varieties

Throughout this section, $G$ is assumed to be semisimple and simply connected. We recalled freely below notions and results concerning wonderful varieties; see [Lu3] and the references given therein.

**Definition 5.** A smooth complete algebraic variety equipped with an action of $G$ is said to be wonderful (of rank $r$) if

(i) it contains a dense $G$-orbit whose complementary is a finite union of smooth prime divisors $D_i$ ($i = 1, ..., r$) with normal crossings;

(ii) its $G$-orbit closures are given by the partial intersections of the boundary divisors $D_i$.

Wonderful varieties are spherical (see [Lu1]). Furthermore, a $G$-variety is wonderful if and only if it is smooth, spherical, toroidal and contains a unique closed orbit of $G$ (see [Lu2]).

Denote $D_{X}$ the set of $B$-stable but not $G$-stable divisors of a wonderful variety $X$. As usual, we call $D_{X}$ the set of colors of $X$; this set yields a basis of the Picard group $\text{Pic}(X)$ of $X$.

Let $H$ be the stabilizer of a point in the open $G$-orbit of $X$. Choose $H$ such that $BH$ is open in $G$ and let $p : G \to G/H$ be the natural projection. For any $D \in D_{X}$, $p^{-1}(D)$ is a $B \times H$-stable divisor hence has an equation $f_{D}$ in $k[G]$. The weights $(\omega_{D}, \chi_{D})$ of the $f_{D}$'s generate freely the abelian group $\Xi(B) \times \Xi(B \cap H) \Xi(H)$; see Lemma 3.2.1 in [Bri4].

Denote $H^{\sharp}$ the intersection of the kernels of all characters of $H$; $H/H^{\sharp}$ is thus a diagonalizable subgroup whose character group is $\Xi(H)$. Note that $G/H^{\sharp}$ is a quasi-affine variety which is spherical under the natural action of $G \times H/H^{\sharp}$.

1.1. Combinatorial invariants. Retain the notation set up for a wonderful $G$-variety $X$. After Luna, we attach to $X$ three combinatorial invariants as follows.

The (unique) closed $G$-orbit $Y$ of $X$ yields the first invariant, denoted $S^{p}_{X}$: this is the set of simple roots of $G$ associated to the parabolic subgroup $P_{X}$ containing $B$ and such that $Y \cong G/P_{X}$.

The second invariant is the set $\Sigma_{X}$ of spherical roots of $X$ defined as the following set $\{\sigma_{1}, ..., \sigma_{r}\}$ of characters of $T$. Let $X_{B}$ be the complementary in $X$ of the union of the colors of $X$; it is isomorphic to an affine space. The character $\sigma_{i}$ is the opposite of the $B$-weight of an equation $f_{i}$ (uniquely determined up to a non-zero scalar) in $k[X_{B}]$ of $X_{B} \cap D_{i}$. 
The third invariant is an integral matrix $A_X$. Let us index now the boundary divisors $D_i$ according to the spherical roots of $X$, we have

$$[D_\sigma] = \sum_{D \in D_X} a_{\sigma,D}[D] \quad \text{in Pic}X \quad \text{for all } \sigma \text{ in } \Sigma_X.$$ 

Equivalently, regarding the equations $f_i$ as $B$-weight vectors in the function field of $X$, we get

$$(\sigma, 0) = \sum_{D \in D_X} a_{\sigma,D}(\omega_D, \chi_D).$$

The matrix $A_X$ is then the matrix with coefficients the $a_{\alpha,D}$’s for $\alpha$ a simple root in $\Sigma_X$. In particular, when there is no simple root in $\Sigma_X$, the third invariant is taken to be the emptyset.

1.2. **Total coordinate ring.** Let $X$ be a wonderful $G$-variety and $D = D_X$ be its set of colors.

The following definition and results of this subsection are freely gathered from Section 3 in [Bri4].

Set

$$\tilde{G} = G \times \mathbb{G}_m^D$$

with $\mathbb{G}_m^D$ being the torus with character group $\mathbb{Z}^D \cong \text{Pic}(X)$.

Define the total coordinate ring of $X$

$$R(X) = \oplus_{(n_D) \in \mathbb{Z}^D} H^0(X, \mathcal{O}_X(\sum_{D \in D} n_D D)).$$

This is a $\mathbb{Z}^D$-graded finitely generated $k-$algebra. Further

$$\tilde{X} := \text{Spec} R(X)$$

is a spherical factorial $\tilde{G}$-variety.

The canonical sections of the boundary divisors of $X$ from a regular sequence in $R(X)$ and generate freely the ring of invariants $R(X)^G$.

The principal fibers of the quotient morphism

$$\pi : \tilde{X} \to \text{Spec}(R(X)^G)$$

are isomorphic to the spherical $G \times H/H^\sharp$-variety $\text{Spec} (k[G/H^\sharp])$.

**Theorem 6.** There is an isomorphism of $G \times T$-algebras

$$R(X) \cong \oplus_{\lambda} k[G/K|_{\lambda}]e^\mu$$

where the sum runs over the dominant weights $\lambda$ of $G$ and characters $\mu$ of $T$ such that $\lambda - \mu$ is a positive sum of spherical roots of $X$; the left-hand side is a subalgebra of $k[G \times T]$. 

1.3. Spherical varieties. Some of the invariants associated to a wonderful variety can also be assigned to any spherical variety $X$ for the action of any connected reductive group $G$. In particular, one defines similarly the set of colours of a spherical variety.

The set of spherical roots of $X$ is defined as follows. Let $H$ be a generic stabilizer of $X$. The set $\Sigma_X$ of spherical roots of $X$ is defined as the set of spherical roots of the rank 1 wonderful $G$-varieties which can be realized as $G$-subvarieties of embeddings of $G/H$; see [Lu3].

If in addition $X$ is affine, there is another characterization of $\Sigma_X$. First, let us recall that the algebra $k[X]$ of regular functions is multiplicity-free as a $G$-module, that is every simple $G$-module occurs in $k[X]$ at most once. Denote $\Xi(X)$ the weight monoid of $X$, that is the submonoid of $\Lambda^+$ given by the highest weights of $k[X]$. Let $\lambda, \mu$ and $\nu$ be in $\Xi(X)$ and such that $V(\mu) \subset V(\lambda)V(\mu)$ - the product being taken in $k[X]$. The closure of the cone generated by the $\lambda + \mu - \nu$ in $\Xi(T) \otimes_{\mathbb{Z}} \mathbb{R}$ is freely generated by a basis of a certain root system - the set $\Sigma_X$; see Theorem 1.3 in [K].

2. Spherical systems

**Definition 7 ([W], [Lu3]).** A spherical root is one of the following characters of $T$: $\alpha_1$ and $2\alpha_1$ of type $A_1$; $\alpha_1 + \alpha'_1$ of type $A_1 \times A_1$; $\alpha_1 + 2\alpha_2 + \alpha_3$ of type $A_3$; $\alpha_1 + \ldots + \alpha_n$ of type $A_n$, $n \geq 2$; $\alpha_1 + 2\alpha_2 + 3\alpha_3$ of type $B_3$; $\alpha_1 + \ldots + \alpha_n$ and $2(\alpha_1 + \ldots + \alpha_n)$ of type $B_n$, $n \geq 2$; $\alpha_1 + 2\alpha_2 + \ldots + 2\alpha_{n-1} + \alpha_n$ of type $C_n$; $2\alpha_1 + \ldots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$ of type $D_n$, $n \geq 4$; $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ of type $F_4$; $\alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2$, and $4\alpha_1 + 2\alpha_2$ of type $G_2$.

This list is that of spherical roots of rank 1 wonderful varieties; see loc. cit.

The set of spherical roots of $G$, denoted by $\Sigma(G)$, is the set of spherical roots of rank 1 wonderful $G$-varieties.

In the following, $\text{Supp}\beta$ denotes the support of any $\beta = \sum n_\alpha \alpha$ (where $\alpha \in S$), that is the set of simple roots $\alpha$ such that $n_\alpha$ is non-zero.

**Definition 8 ([BL], 1.1.6).** Let $S^p$ be a subset of $S$ and $\sigma$ be a spherical root of $G$. The couple $(S^p, \sigma)$ is said to be compatible if

$$S^{pp}(\sigma) \subset S^p \subset S^p(\sigma)$$

where $S^p(\sigma)$ is the set of simple roots orthogonal to $\sigma$ and $S^{pp}(\sigma)$ is one of the following sets

- $S^p(\sigma) \cap \text{Supp}\sigma \setminus \{\alpha_r\}$ if $\sigma = \alpha_1 + \ldots + \alpha_r$ with $\text{Supp}\sigma$ of type $B_r$. 
- $S^p(\sigma) \cap \text{Supp}\sigma \setminus \{\alpha_1\}$ if $\text{Supp}\sigma$ is of type $C_r$,
- $S^p(\sigma) \cap \text{Supp}\sigma$ otherwise.

**Definition 9.** Let $S^p$ be a set of simple roots of $G$, $\Sigma$ a set of spherical roots of $G$ and $A$ a matrix with integer coefficients $a^\pm_{\alpha,\sigma}$. The triple $(S^p, \Sigma, A)$ is called a spherical system of $G$ if

(A1) $A = (a^\pm_{\alpha,\sigma})$ with $\alpha \in \Sigma \cap S$ and $\sigma \in \Sigma$.
(A2) $a^\pm_{\alpha,\sigma} \leq 1$ for every $\sigma \in \Sigma$. Further if $a^-_{\alpha,\sigma} = 1$ then $\sigma \in \Sigma \cap S$.
(A3) $a^+_{\alpha,\sigma} + a^-_{\alpha,\sigma} = (\alpha^\vee, \sigma)$ for any $\sigma \in \Sigma$.
(S1) $(\alpha^\vee, \sigma) \in 2\mathbb{Z}_{\leq 0}$ for all $\sigma \in \Sigma \setminus \{2\alpha\}$ and all $\alpha \in S$ such that $2\alpha \in \Sigma$.
(S2) $(\alpha^\vee, \sigma) = (\beta^\vee, \sigma)$ for all $\sigma \in \Sigma$ and all $\alpha, \beta \in S$ which are mutually orthogonal and such that $\alpha + \beta \in \Sigma$.
(S) The couple $(S^p, \sigma)$ is compatible for any $\sigma \in \Sigma$.

**Definition 10.** A spherical system is called spherically closed if none of its spherical roots $\sigma \in \Sigma \setminus S$ is such that $(S^p, 2\sigma)$ is compatible.

**Example 11.** A spherical system whose set of spherical roots contains the root $\alpha_1 + \ldots + \alpha_n$ of type $B_n$ is not spherically closed whenever its set $S^p$ contains the roots $\alpha_2, \ldots, \alpha_n$.

2.1. **Set of weights associated to a spherical system.** The purpose of this subsection is to attach to any spherical system $S = (S^p, \Sigma, A)$ of $G$, a set of linearly independent characters $(\omega_D, \chi_D)$ of $T \times C$ for some group $C$.

2.1.1. These characters are indexed by a finite set $\Delta$, the set of colors of $S$. The set $\Delta$ is defined as follows (see [Lu2]). Set

$$S^a = 2S \cap \Sigma \quad \text{and} \quad S^b = S \setminus (S^p \cup (S \cap \Sigma) \cup S^a).$$

If $\alpha$ and $\beta$ are orthogonal simple roots whose sum is in $\Sigma$, write $\alpha \sim \beta$. Define now

$$\Delta := \{D_\alpha^\pm : \alpha \in S \cap \Sigma\} \cup \{D_\alpha : \alpha \in S^a \cup (S^b / \sim)\}.$$

2.1.2. Let $\omega_\alpha$ denote the fundamental weight associated to the simple root $\alpha$.

Given $D \in \Delta$, we define (after [E])

$$\omega_D = \begin{cases} 
\sum_{\beta} \omega_\beta & \text{if } D = D_\alpha^+ \text{ and } a^+_{\alpha,\beta} = 1 \\
\sum_{\beta} \omega_\beta & \text{if } D = D_\alpha^- \text{ and } a^-_{\alpha,\beta} = 1 \\
2\omega_\alpha & \text{if } D = D_\alpha \text{ with } 2\alpha \in \Sigma \\
\omega_\alpha + \omega_\beta & \text{if } D = D_\alpha \text{ with } \alpha \in S^b \text{ and } \alpha \sim \beta \\
\omega_\alpha & \text{otherwise}
\end{cases}.$$
Note that these weights may not be pairwise distinct: among the first two defined types of dominant weights, some may occur twice—but not more.

2.1.3. We now introduce some additional characters $\chi_D$ indexed by $\Delta$ (see also [Lu2] and Lemma 3.2.1 in [Bri4]).

Given $D$ in $\Delta$ and $\sigma$ in $\Sigma$, define (after [Lu2])

$$a_{D,\sigma} = \begin{cases} a_{\alpha,\sigma}^+ & \text{if } D = D^\alpha_+ \\ a_{\alpha,\sigma}^- & \text{if } D = D^\alpha_- \\ \frac{1}{2}(\sigma, \alpha^\vee) & \text{if } D = D_\alpha \text{ with } \alpha \in S^a \\ (\sigma, \alpha^\vee) & \text{otherwise} \end{cases}$$

Let $G^r_m$ be the torus whose character group is spanned freely by the set $\Sigma$ of spherical roots and $G^\Delta_m$ be the torus with character group $\mathbb{Z}\Delta$. Consider the morphism

$$G^\Delta_m \to G^r_m : (t_D)_{D \in \Delta} \mapsto \left( \prod_{D \in \Delta} t_{D,\sigma}^{a_{D,\sigma}} \right)_{\sigma \in \Sigma}.$$

Let $C$ be its kernel; it is a diagonalizable group.

Define the character $\chi_D$ as the restriction to $C$ of the $D$-component character

$$\varepsilon_D : (t_D)_{D \in \Delta} \mapsto t_D.$$

Lemma 12. The $(\omega_D, \chi_D)$’s generate freely a subgroup of the character group of $T \times C$. Further they satisfy the following equalities

$$(\sigma, 0) = \sum_{D \in \Delta} a_{D,\sigma}(\omega_D, \chi_D) \quad \text{for all } \sigma \in \Sigma.$$

Proof. The lemma follows essentially from the definition of the characters $(\omega_D, \chi_D)$ along with the compatibility condition (S) of spherical systems. \hfill \qed

The set of characters $(\omega_D, \chi_D)$ will be referred in the remainder as the set of weights associated to $S$.

2.2. Relations with wonderful varieties. Let $X$ be a wonderful $G$-variety. Recall the notation set up in Section 1.

Theorem 13 ([Lu3]). Suppose $G$ is of adjoint type, i.e. the center of $G$ is trivial. The triple $(S^p_X, \Sigma_X, A_X)$ associated to a wonderful $G$-variety $X$ is a spherical system of $G$.

Conjecture 14 ([Lu2]). Suppose $G$ is of adjoint type. There corresponds to any spherical system of $G$ a unique (up to $G$-isomorphism) wonderful $G$-variety.
The set of $B \times H$-weights of the $f_D$’s ($D \in \mathcal{D}_X$) is the set associated to the spherical system $\mathcal{S}_X$ of $X$; see Lemma 3.2.1 and its proof in [Bri4]. Further, the diagonalizable subgroup $C$ attached to $\mathcal{S}_X$ is the group $H/H^\bullet$.

3. Invariant Hilbert schemes

3.1. Definition. The definitions and results stated in this section are taken from [AB] except that they are formulated in a more general setting in loc. cit.

Let $\lambda_1, \ldots, \lambda_s$ be linearly independent dominant weights of $G$. Denote by $\underline{\lambda}$ the corresponding s-tuple and by $\Gamma$ the submonoid of $\Lambda^+$ generated by the $\lambda_i$’s. Set

$$V := V(\lambda_1) \oplus \ldots \oplus V(\lambda_s).$$

Definition 15. Given a scheme $S$ equipped with the trivial action of $G$, a family of closed $G$-subschemes $\mathcal{X}$ of $V$ over $S$ is a closed $G$-subscheme of $V \times S$ such that

1. the projection $\pi : \mathcal{X} \to S$ is $G$-invariant;
2. the sheaf $\mathfrak{F}_\lambda := (\pi_* \mathcal{O}_X)_\lambda^U$ of $\mathcal{O}_S$-modules is invertible for every $\lambda \in \Gamma$.

With the preceding notation, the sheaf $\pi_* \mathcal{O}_X$ is isomorphic (as an $\mathcal{O}_S \times G$-module) to $\bigoplus_{\lambda \in \Gamma} \mathfrak{F}_\lambda \otimes V(\lambda)^*$. See Lemma 1.2. in loc. cit.

Theorem 16 (Theorem 1.7/Corollary 1.17 in [AB]). The functor which assigns to any scheme $S$ (endowed with the trivial action of $G$) the set of families of closed $G$-subschemes of $V$ over $S$ is representable by a connected affine, the invariant Hilbert scheme $\text{Hilb}_G^\underline{\lambda}$.

In particular, $\text{Hilb}_G^\underline{\lambda}$ contains as closed point the horospherical $G$-variety $X_0 := X_0(\underline{\lambda})$ given by the $G$-orbit closure within $V$ of

$$v_\underline{\lambda} = v_{\lambda_1} + \ldots + v_{\lambda_s}$$

where $v_{\lambda_i}$ stands for a highest weight vector in $V$ of weight $\lambda_i$.

3.2. Toric action. We recall briefly how the action of the adjoint torus $T_{\text{ad}} := T/Z(G)$ on $\text{Hilb}_G^\underline{\lambda}$ is defined; see Section 2.1 in [AB] for details.

Consider a family $\pi : \mathcal{X} \to S$ of closed $G$-subschemes of $V$. Let the center $Z(G)$ of $G$ act on $\mathcal{X} \times T$ by $z.(x, t) = (z.x, z^{-1}t)$. Then

$$\tilde{\mathcal{X}} = (\mathcal{X} \times T)/Z(G)$$

is a scheme equipped with an action of $G$ and the morphism $\pi \times \text{id} : \mathcal{X} \times T \to S \times T$ descends to a morphism

$$\tilde{\pi} : \tilde{\mathcal{X}} \to (S \times T)/Z(G) = S \times T_{\text{ad}}.$$
Moreover, we have an isomorphism of \(G - \mathcal{O}_S[T_{\text{ad}}]\)-modules

\[
\tilde{\pi}_* \mathcal{O}_{\tilde{X}} = \bigoplus_{\lambda \in \Lambda^+} (\pi_* \mathcal{O}_X)(\lambda)e^\lambda \otimes_k k[T_{\text{ad}}].
\]

Letting \(T_{\text{ad}}\) act on \(V(\lambda)\) by

\[
t.v = w_0(\lambda)(t)t^{-1}v,
\]

one gets a \(\tilde{G} := (G \times T)/Z(G)\)-module structure on \(V\). Further, for some suitable action of \(\tilde{G}\) on \(V \times T_{\text{ad}}\), the corresponding \(\tilde{G}\)-module \(V \times T_{\text{ad}}\) becomes isomorphic to \((V \times T)/Z(G)\); and \(\tilde{\pi} : \tilde{X} \to T_{\text{ad}} \times S\) is then a family of closed \(G\)-subscheme of \(V \times T_{\text{ad}}\). Applying this construction to the universal family \(\pi : X_{\text{univ}} \to \text{Hilb}_G^\lambda\), one obtains a morphism of schemes

\[
a : T_{\text{ad}} \times \text{Hilb}_\Delta^G \to \text{Hilb}_\Delta^G.
\]

**Theorem 17** (Proposition 2.1/ Theorem 2.7 in [AB]). The morphism \(a\) defines an action of \(T_{\text{ad}}\) on \(\text{Hilb}_\Delta^G\). Furthermore \(X_0\) is the unique closed orbit (hence fixed point) of \(T_{\text{ad}}\) in \(\text{Hilb}_\Delta^G\).

### 3.3. Tangent space.

Let \(I \subseteq \text{Sym}(V^*)\) be the ideal of \(X_0 \subseteq V\) and \(B_0\) be the coordinate ring of \(X_0\). Denote by \(G^\circ\) the identity component of the isotropy group \(G_{v_\lambda}\) and by \(g_{v_\lambda}\) the isotropy Lie algebra of \(v_\lambda\).

**Proposition 18** ([AB]).

(i) The tangent space \(T_{X_0} \text{Hilb}_\Delta^G\) of \(\text{Hilb}_\Delta^G\) at \(X_0\) is canonically isomorphic to \(\text{Hom}_{B_0}(I/I^2, B_0)^{G^\circ}\).

(ii) There is an injection of \(T_{X_0} \text{Hilb}_\Delta^G\) into \((V/ g_{v_\lambda})^{G^\circ}\).

(iii) If \(X_0\) is normal and the boundary \(X_0 \setminus G.v_\lambda\) is of codimension at least 2 then \(T_{X_0} \text{Hilb}_\Delta^G\) is isomorphic to \((V/ g_{v_\lambda})^{G^\circ}\).

### 3.4. Invariant infinitesimal deformations.

For local studies purposes (e.g. smoothness of \(\text{Hilb}_\Delta^G\)), we shall need to consider the functor \(\text{Def}_{X_0}^G\) of invariant infinitesimal deformations of \(X_0\):

\[
\text{Def}_{X_0}^G(A) = \text{Hom}(\text{Spec}A, \text{Hilb}_\Delta^G)[X_0]
\]

where \(A\) is any local artinian \(k\)-algebra. By Theorem 16 the functor \(\text{Def}_{X_0}^G\) is representable by the completion \(\hat{\mathcal{O}}_{\text{Hilb}_\Delta^G}[X_0]\).

### 3.5. Obstruction space.

First let us recall (see [S]) the definition of the obstruction space of a covariant functor \(\mathcal{F}\) defined on the category of local artinian \(k\)-algebras.

Let \(\Lambda\) be a local noetherian \(k\)-algebra with residue field \(k\). Denote \(\mathcal{A}_\Lambda\) the category of local artinian \(\Lambda\)-algebras with residue field \(k\) and \(\text{Ex}_\Lambda(A, k)\) the module of isomorphism classes of \(\Lambda\)-extensions of \(A\) by \(k\).
Definition 19. A $k$-vector space $v(F)$ is called an obstruction space for the functor $F$ if for every object $A$ of $A$ and every $\xi \in F(A)$, there exists a $k$-linear map

$$\xi_v : \text{Ex}_A(A, k) \to v(F)$$

with the following property:

$\ker(\xi_v)$ consists of the isomorphism classes of extensions $(\tilde{A}, \varphi)$ such that $\xi \in \text{Im}[F(\tilde{A}) \to F(A)]$.

If the obstruction space $v(F)$ is trivial then the functor $F$ is smooth.

In order to give a characterization of the obstruction space of $G_X^0$, we shall recall the definition of the second cotangent module $T^2_{B_0}$ of $B_0$; see for instance Section 3.1.2 in [S].

Take a presentation of the ideal $I$ of $X_0$

$$0 \to R \to F \to I \to 0$$

where $F$ is a finitely generated free $\text{Sym}(V^*)$-module. Let $f_1, \ldots, f_n$ generate the ideal $I$ so that $F = \text{Sym}(V^*)^n$.

Consider $K \subset R$ the module of trivial relations: $K$ is generated by the relations $r_{i,j} = (a_1, \ldots, a_n) \in F^n$ with $a_j = -f_i$, $a_j = f_i$ and $a_i = 0$ otherwise; $i, j = 1, \ldots, n$. We thus get the exact sequence of $B_0$-modules

$$R/K \to F \otimes B_0 \to I/I^2 \to 0.$$  

Apply $\text{Hom}_{B_0}(-, B_0)$ to the last exact sequence then the second cotangent module $T^2_{B_0}$ of $B_0$ is defined by the exact sequence

$$\text{Hom}_{B_0}(I/I^2, B_0) \to \text{Hom}_{B_0}(F \otimes B_0, B_0) \to \text{Hom}_{B_0}(R/K, B_0) \to T^2_{B_0} \to 0.$$  

The second cotangent module is independent of the presentation of $I$.

For any $1 \leq i \neq j \leq s$, let

$$m_{i,j} : V(\lambda_i)^* \otimes V(\lambda_j)^* \to V(\lambda_i + \lambda_j)^*$$

be the Cartan product and

$$K_{i,j} = \ker m_{i,j}.$$  

In the following, we take the following presentation of $I/I^2$ as $B_0 - G$-modules

(1)  

$$R/K \to \oplus_{i,j} B_0 \otimes K_{i,j} \to I/I^2 \to 0.$$  

Proposition 20. (i) The $G$-invariant space $(T^2_{B_0})^G$ of the second cotangent module of $X_0$ is an obstruction space for $\text{Def}^G_{X_0}$.  

(ii) There is an injection of \((T_{B_0}^2)^G\) into the kernel of the map

\[
H^1(\mathfrak{g}_{v\Delta}, V/\mathfrak{g}_{v\Delta})^{G_v\Delta/G_v\Delta} \rightarrow \oplus_{1 \leq i,j \leq s} H^1(\mathfrak{g}_{v\Delta}, K^{*}_{i,j})^{G_v\Delta/G_v\Delta}
\]

induced by the map of \(\mathfrak{g}_{v\Delta}\)-modules from \(V/\mathfrak{g}_{v\Delta}\) to \(\oplus_{i,j} K^{*}_{i,j}\)

\[
[v] \mapsto \sum_{i,j} [v^i \otimes v_{\lambda^j}] \quad \text{where } v^i \text{ denotes the projection of } v \in V \text{ onto } V(\lambda_i).
\]

(iii) If \(X_0\) is normal and the boundary \(X_0 \setminus G.v\Delta\) is of codimension at least 2 then the above injection is an isomorphism.

Proof. The proof of the first assertion is essentially the same as that of Proposition 3.1.12 in [S]. For convenience, we reproduce some of its main steps.

Let \(A\) be a local artinian \(k\)-algebra. Denote by \(\text{Ex}(A, k)\) the module of isomorphism classes of extensions of \(A\) by \(k\).

Let \(\xi \in \text{Def}_G^{X_0}(A)\). We shall assign to \(\xi\) a \(k\)-linear map \(\hat{\xi} : \text{Ex}(A, k) \rightarrow (T_{B_0}^2)^G\) satisfying the property of the definition of an obstruction space.

Let \([(\tilde{A}, \varphi)] \in \text{Ex}(A, k)\) be represented by

\[
0 \longrightarrow k\varepsilon \longrightarrow \tilde{A} \xrightarrow{\varphi} A \longrightarrow 0
\]

where \(\varepsilon^2 = 0\).

The given element \(\xi\) can be regarded as a morphism \(X \rightarrow \text{Spec}A\) where \(X = \text{Spec}B\) with \(B = \text{Sym}(V^*) \otimes_k A/J\) and \(J\) a \(G\)-stable ideal of \(\text{Sym}(V^*) \otimes_k B\). Note that \(B\) is flat over \(A\) and \(\text{Sym}(V^*) \otimes_k A/(J, \varepsilon) = B_0\). Further \(J\) is generated by elements \(F_1, \ldots, F_n\) such that \(f_i - F_i\) is in the maximal ideal of \(A\). Thanks to the flatness of \(B\) over \(A\), for every \(\underline{r} = (r_1, \ldots, r_n) \in B\), there exist \(R_1, \ldots, R_n\) in \(\text{Sym}(V^*) \otimes_k A\) such that \(r_i = R_i\) modulo the maximal ideal of \(A\) and \(\sum_i R_i F_i = 0\).

Send

\[
(\underline{r}) \mapsto \sum_i \tilde{R}_i \tilde{F}_i
\]

where \(\tilde{F}_i\) (resp. \(\tilde{R}_i\)) is a lifting of \(f_i\) (resp. \(R_i\)) through \(\varphi\) for \(i = 1, \ldots, n\). One thus checks easily that this yields an element \(\hat{\xi}\) of the cokernel of \((T_{B_0}^2)^G\) and that \(\hat{\xi}\) satisfies our requirement.

Let \(N_\Delta\) be the normal sheaf of \(G.v\Delta\); note that \(K^{*}_{i,j} \otimes \mathcal{O}_{G.v\Delta}\) is the dual sheaf \(\text{Hom}_{\mathcal{O}_{X_0}}(K_{i,j} \otimes \mathcal{O}_{X_0}, \mathcal{O}_{X_0})\). The second cotangent module \(T_{X_0}^2\) being supported on the singular locus of \(X_0\), that is contained in
We have:

\[ H^0(X_0, \oplus_{i,j} K^*_{i,j} \otimes \mathcal{O}_{X_0}) \longrightarrow T^2_{B_0} \longrightarrow 0 \]

\[ H^0(G.v_{\lambda}, \oplus_{i,j} K^*_{i,j} \otimes \mathcal{O}_{G.v_{\lambda}}) \longrightarrow H^1(G.v_{\lambda}, \mathcal{N}_{\lambda}) \longrightarrow H^1(G.v_{\lambda}, \oplus_{i,j} K^*_{i,j} \otimes \mathcal{O}_{G.v_{\lambda}}) \]

The normal sheaf \( \mathcal{N}_{\lambda} \) being the \( G \)-linearized sheaf on \( G/G.v_{\lambda} \) associated to the \( G.v_{\lambda} \)-module \( V/g.v_{\lambda} \), we have:

\[ H^1(G.v_{\lambda}, \mathcal{N}_{\lambda}) = H^1(G.v_{\lambda}, V/g.v_{\lambda}) \]

and similarly

\[ H^1(G.v_{\lambda}, \oplus_{i,j} K^*_{i,j} \otimes \mathcal{O}_{G.v_{\lambda}}) = \bigoplus_{1 \leq i,j \leq s} H^1(G.v_{\lambda}, K^*_{i,j}). \]

From [H], we know that

\[ H^1(G.v_{\lambda}, V/g.v_{\lambda}) \simeq H^1(g.v_{\lambda}, V/g.v_{\lambda})^{G.v_{\lambda}/G.v_{\lambda}}. \]

and

\[ H^1(G.v_{\lambda}, K^*_{i,j} \otimes \mathcal{O}_{G.v_{\lambda}}) \simeq H^1(g.v_{\lambda}, K^*_{i,j} \otimes \mathcal{O}_{G.v_{\lambda}})^{G.v_{\lambda}/G.v_{\lambda}}. \]

The second assertion thus follows.

Note that the sheaf \( \oplus_{i,j} K^*_{i,j} \otimes \mathcal{O}_{X_0} \) is reflexive. Therefore, in case \( X_0 \) is normal and the codimension of \( X_0 \setminus G.v_{\lambda} \) is at least 2, the vertical injection of the above diagram becomes an isomorphism whence the last assertion of the proposition.

\[ \square \]

**Corollary 21.** If \( (T^2_{B_0})^G \) is trivial then \( \text{Hilb}_{\lambda} \) is smooth at \( X_0 \).

**Proof.** As recalled, if \( (T^2_{B_0})^G \) is trivial then the functor of invariant deformations of \( X_0 \) is smooth; the latter being represented by \( \hat{\mathcal{O}}_{\text{Hilb}_{\lambda}[X_0]} \), the assertion follows. \( \square \)

### 4. Geometrical construction of wonderful varieties

#### 4.1. Invariant Hilbert scheme attached to a spherical system

Throughout this section, \( S = (S^p, \Sigma, A) \) denotes a spherically closed spherical system of \( G \).

Recall the definition of the diagonalizable subgroup \( C \) as well as the set \( \Delta \) of weights \( \lambda_D = (\omega_D, \chi_D) \) of \( G \times C \) - both canonically associated to \( S \). Set

\[ G = G \times C^0 \quad \text{and} \quad V = \bigoplus_{D \in \Delta} V(\lambda_D)^*. \]
In order to keep track of the data $\mathcal{S}$, let us denote the invariant Hilbert scheme $\text{Hilb}_{\Delta}^G$ rather by $\text{Hilb}(\mathcal{S})$.

**Remark 22.** When the third datum of a spherical system is empty (i.e. none of the simple roots is a spherical root) then the monoid $\mathbb{N}_{\Delta}$ is saturated: the dominant weights in the integral span of $\mathbb{N}_{\Delta}$ are themselves elements of $\mathbb{N}_{\Delta}$. The invariant Hilbert scheme associated to the group $G$ itself and to $V$ as a $G$-module is the one studied in [J, BC1]. Further, it maps naturally to $\text{Hilb}(\mathcal{S})$.

We denote in the following $\ell(w)$ the length of an element $w$ in the Weyl group (relative to $T$) of $G$. In particular $\ell(w) = 0$ if and only if $w$ is the identity in $W$.

**Theorem 23.** Let

$$\Sigma(\Delta) = \Sigma \cup \{\alpha + \alpha' : \alpha, \alpha' \text{ adjacent simple roots in } \Sigma\}. $$

Then (for $i = 0, 1$)

$$H^i(G_{v_\Delta}, V/\mathfrak{g}.v_\Delta) \approx \bigoplus_{\ell(w)=i, \gamma \in \Sigma(\Delta)} k_{w\ast \gamma} \text{ as } T_{\text{ad}}\text{-modules.}$$

If $w$ is of length 1, the one dimensional $T_{\text{ad}}\text{-module } k_{w\ast \gamma} \text{ is spanned by } X_{\alpha}^* \otimes [wv_\gamma] \text{ where } w \text{ is the simple reflection associated to } \alpha \text{ with } \alpha \in S \setminus S^p \text{ and } \gamma + \alpha \in \mathbb{Z}_{\Delta}.$

**Proof.** See Appendix A. □

**Corollary 24.** The tangent space at $X_0$ of the invariant Hilbert scheme $\text{Hilb}(\mathcal{S})$ is a multiplicity free $T_{\text{ad}}\text{-module}; its } T_{\text{ad}}\text{-weights are the opposites of the spherical roots of } \mathcal{S}.

**Proof.** The corollary follows essentially from the characterization of the $G$-orbit closures of codimension 1 in $X_0$ along with the preceding theorem. □

**Corollary 25.** The obstruction space for the functor $\text{Def}_{X_0}^G$ of invariant infinitesimal deformations of $X_0$ is trivial.

**Proof.** We will prove the injectivity of the map displayed in the second assertion of Proposition 20. This is achieved in Appendix B. □

**Theorem 26.** The scheme $\text{Hilb}(\mathcal{S})$ is an affine toric variety for the adjoint torus of $G$; its weights are the opposites of the spherical roots of $\mathcal{S}$. More specifically, it is an affine space.

**Proof.** Recall that $X_0$ is the single fixed point of $T_{\text{ad}}$ in $\text{Hilb}(\mathcal{S})$. Corollaries 21 and 25 imply that $\text{Hilb}(\mathcal{S})$ is smooth. Thanks to Lemma 12
WONDERFUL VARIETIES: A GEOMETRICAL REALIZATION 17

and Theorem 17 the scheme $\text{Hilb}(\mathcal{S})$ is affine, connected hence irreducible and in turn a toric $T_{\text{ad}}$-variety which has to be an affine space. □

4.2. The wonderful variety attached to a spherical system. Denote $r$ the number of spherical roots of $\mathcal{S}$ and let $X_1$ be the closed point of $\text{Hilb}(\mathcal{S})$ whose $T_{\text{ad}}$-orbit is dense. In virtue of the definition of the weights $\lambda_D$, $X_1$ regarded as a $G$-subvariety, is spherical.

Consider the universal family of $\text{Hilb}(\mathcal{S})$

$$\mathcal{X}^{\text{univ}} \xrightarrow{\pi} \mathbb{A}^r.$$ Then the coordinate ring $R(\mathcal{S})$ of $\mathcal{X}^{\text{univ}} \subset V \times \mathbb{A}^r$ is isomorphic as a $G \times T_{\text{ad}}$-algebra to

$$R(\mathcal{S}) = \oplus_{\lambda \in \mathbb{N}\Delta} k[X_1]^\lambda \otimes k[e^\sigma : \sigma \in \Sigma];$$

see the recalls made in Section 3.2. Here $e^\lambda$ with $\lambda \in \mathbb{N}\Delta$ stands for the character $e^\omega$ in $k[T]$ whenever $\lambda = (\omega, \chi)$.

The $G \times T$-variety $\mathcal{X}^{\text{univ}}$ is thus spherical.

**Proposition 27.** (i) The set of spherical roots of the $G \times T$-variety $\mathcal{X}^{\text{univ}}$ coincides with the set $\Sigma$.

(ii) The colors of $\mathcal{X}^{\text{univ}}$ are indexed by the set $\Delta$.

**Proof.** By Theorem 1.3 in [K] along with Proposition 2.13 in [AB] and Theorem 23, the spherical roots of $\mathcal{X}^{\text{univ}}$ are exactly the elements of the given set $\Sigma$.

The set colors of an affine spherical variety is entirely determined by the spherical roots and the weight monoid of this variety; see Lemma 10.1 in [Ca] and Subsection in [Lu3], or Theorem 1.2 and its proof in [Lo], for a proof of this fact and how to determine the colors. More specifically, the colors of $\mathcal{X}^{\text{univ}}$ are given by equations $f_D$, $D \in \Delta$, which are $B \times C^\circ \times T$-weight vectors of weight $(\lambda_D, \omega_D)$; the second assertion follows. □

Let $\mathcal{X}^{\text{univ}}$ be the open subset of $\mathcal{X}^{\text{univ}}$ defined as follows

$$\mathcal{X}^{\text{univ}} = \{x \in \mathcal{X}^{\text{univ}} : G.x \text{ is open in } \pi^{-1}\pi(x)\}.$$ The closed subset $\mathcal{X} \setminus \mathcal{X}^{\text{univ}}$ is of codimension at least 2.

**Lemma 28.** (i) The elements of $\mathcal{X}^{\text{univ}}$ are the elements of $\mathcal{X}^{\text{univ}} \subset V \times \mathbb{A}^r$ which project non-trivially onto each simple $G$-submodule $V(\lambda_D)^*$ of $V$.

(ii) We have: $\mathcal{X}^{\text{univ}} = G.(\mathcal{X}^{\text{univ}} \setminus \cup_{\Delta} D)$. 
Proof. The first assertion of the lemma follows readily from the definition of $\text{Hilb}(S)$.

Let $f_D$ be the equation in $R(S)$ of a color $D \in \Delta$. From the comment at the end of the proof of the previous statement, $f_D$ can be identified to the weight vector $v_{\lambda_D} e^{\lambda_D} \otimes 1$ in $R(S)$. Take $y = (v, s) \in V \times \mathbb{A}^r$. If $f_D(y) \neq 0$ then clearly $v_{\lambda_D}(v) \neq 0$. Conversely, if $y$ projects non-trivially onto each $V(\lambda_D)^*$ then there exists $g \in G$ such that the support of $g.v$ contains all the dominant weights $\lambda_D^*$ and in turn $v_{\lambda_D}(g.v) \neq 0$ for each $D \in \Delta$. The second assertion of the lemma follows.

Recall that the dominant weights $\lambda_D$ are linearly independent (see Lemma 12). Let $\mathbb{G}_m^\Delta$ be the torus $GL(V)^G$. Then the $T_{ad} \times C$-action on $V$ via $t \mapsto (\lambda_D(t))$ yields naturally a $\mathbb{G}_m^\Delta$-action on $V$: the componentwise multiplication. Note that the open set $\mathcal{X}^{\text{univ}}$ is $\mathbb{G}_m^\Delta$-stable.

**Theorem 29.** The $G$-variety $X(S) = \mathcal{X}^{\text{univ}}/\mathbb{G}_m^\Delta$ is wonderful. Further its spherical system is $S$ and its total coordinate ring is $R(S)$.

Proof. From the first assertion of Lemma 28, we have a morphism

$$\mathcal{X}^{\text{univ}} \xrightarrow{\varphi} \bigoplus_{D \in \Delta} V(\lambda_D)^* \setminus \{0\}.$$

From our previous observations, we know that the color $D$ is contained in the inverse image of $\varphi$ of the hyperplane ($f_D = 0$). The $G \times \mathbb{G}_m^\Delta$-variety $\mathcal{X}^{\text{univ}}$ is thus toroidal and so is $\mathcal{X}^{\text{univ}}/\mathbb{G}_m^\Delta$ as a $G$-variety.

Furthermore, together with the second assertion of Lemma 28, it follows that there exists an affine toric $T \times \mathbb{G}_m^\Delta$-variety $W$ such that $\mathcal{X}^{\text{univ}}$ is $P$-equivariantly isomorphic to $P^u \times W$ where $P$ is the parabolic subgroup of $G$ stabilizing the colors $D$ in $\Delta$.

Considering the universal family $\pi$ of $\text{Hilb}(S)$, we get that the quotient $W/T$ is isomorphic to $\mathbb{A}^r$ and in turn that the $G \times \mathbb{G}_m^\Delta$-stable prime divisors of $\mathcal{X}^{\text{univ}}$ are principal; their equations are given by the $e^\sigma$'s with $\sigma \in \Sigma$.

Finally the divisor class group of $\mathcal{X}^{\text{univ}}$ being generated by the colors and the $G \times \mathbb{G}_m^\Delta$-stable prime divisors of $\mathcal{X}^{\text{univ}}$, we get that $\mathcal{X}^{\text{univ}}$ is factorial and so is $W$. The affine variety $W$ being toroidal and factorial, it is smooth and so is $\mathcal{X}^{\text{univ}}/\mathbb{G}_m^\Delta$.

The $G$-variety $\mathcal{X}^{\text{univ}}/\mathbb{G}_m^\Delta$ is thus complete, spherical, toroidal and smooth $G$-variety and has a unique closed $G$-orbit, namely $X_0/\mathbb{G}_m^\Delta$. Equivalently, $\mathcal{X}^{\text{univ}}/\mathbb{G}_m^\Delta$ is $G$-wonderful.

The two last assertions of the theorem follow from Proposition 27 as well as the definition of the spherical system of $X$ recalled in Section 4 and that of $R(X)$ (when forgetting the $C^\circ$-module structure of $R(S)$).
Corollary 30. Luna’s conjecture is true: to any spherical system $S$ of an adjoint reductive algebraic group $G$, there corresponds a unique (up to $G$-isomorphism) wonderful $G$-variety which has $S$ as spherical system.

Proof. It suffices to consider spherical systems which are spherically closed; see Section 6 in [Lu3]. The existence part is given by the previous theorem. The proof of the uniqueness is conducted as follows.

Let $X$ be a wonderful $G$-variety, $D_X$ be its set of colors and $\omega_D$ be the $B$-weight of a color $D$. The variety $X$ coincides with the normalization (in the function field of $X$) of some $G$-orbit closure within $\prod_{\Delta} \mathbb{P}(V(\omega_D)^*)$; see Section 4.4 in [Br14]. Further, $H^0(X, \mathcal{O}_X(D))$ is either isomorphic as a $G$-module to $V(\omega_D)$ or $V(\omega_D) \oplus V(0)$ (see [Br12]).

Consider the corresponding finite morphism

$$X \longrightarrow \prod_{D \in D_X} \mathbb{P}(H^0(X, \mathcal{O}_X(D))^*) .$$

Let $\hat{X}$ be the affine multicone over $X$ with respect to this morphism, i.e.

$$\hat{X} = \text{Spec} R(X)$$

where $R(X)$ is the total coordinate ring of $X$.

Thanks to Theorem 26 together with the recalls made in Subsection 1.2, the quotient morphism $\pi : \hat{X} \to \text{Spec}(R(X)^G)$ is the universal family of the invariant Hilbert scheme $\text{Hilb}(S_X)$, $S_X$ being the spherical system of $X$.

This implies that if $X$ and $X'$ are wonderful $G$-varieties which have the same spherical system then $K$ and $K'$ are isomorphic; recall the definition and the properties of the subgroup $K$ (and $K'$) from Section 1. And, since $H/K$ and $H'/K'$ are isomorphic so are the subgroups $H$ and $H'$. In turn, $X$ and $X'$ are isomorphic thanks to the uniqueness of the wonderful compactification. \hfill $\square$

Appendix A. Computations of cohomology groups

As in the text body, $G$ is a simply connected semisimple algebraic group, $T$ an algebraic torus contained in a Borel subgroup $B$ of $G$. The associated set of simple roots of $G$ (resp. of dominant weights) is denoted $\Delta$ (resp. $\Delta^+$). Let $Z(G)$ be the center of $G$, the adjoint torus $T/Z(G)$ is denoted $T_{ad}$.

Let $\mathfrak{g} = \oplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ be the decomposition of the Lie algebra of $G$ into root spaces. We denote by $X_\alpha$ a non-zero vector of the root space $\mathfrak{g}_\alpha$. The Lie algebra of the Levi subgroup containing $T$ associated to a subset $S'$ of $S$ is denoted by $\mathfrak{g}(S')$. 

Given a finite dimensional $G$-module, a weight vector (resp. a weight space) in $V$ of weight $\mu$ ($\mu$ a character of $T$) is denoted $v_{\mu}$ (resp. $V_{\mu}$). Let

$$V = \oplus_{i=1}^{s} V(\lambda_i)$$

be the decomposition of $V$ into irreducible $G$-modules. In particular, every $\lambda_i$ ($i = 1, \ldots, r$) is an highest weight of $V$. We set

$$v_{\Delta} = v_{\lambda_1} + \ldots + v_{\lambda_s}.$$

We consider in the following the normalized action of the adjoint torus $T_{\text{ad}}$ of $G$ on $V$, that is the action of $T_{\text{ad}}$ naturally induced by setting

$$t.v = (\lambda_i)(t)t^{-1}.v \quad \text{if} \quad v \in V(\lambda_i).$$

In the following, we fix a spherical system $S = (S^p, \Sigma, A)$ and we denote (by abuse of notation) $\Delta$ also the set of weights $\lambda_D = (\omega_D, \chi_D)$ ($D \in \Delta$) associated to $S$. Let $G = G \times C^\circ$. Unless otherwise stated, $V$ is a $G$-module whose highest weights are the $\lambda_D$.

A.1. Properties of the weights attached to a spherical system.

**Lemma 31.** The dominants weights $\omega_D$ ($D \in \Delta$) associated to a spherical system satisfies the following properties

1. $(\lambda, \alpha^\vee) \leq 2$ for all $\lambda \in \Delta$ and all $\alpha$ in $S$;
2. if $(\lambda, \alpha^\vee) = 2$ then $\lambda = 2\omega_{\alpha}$ and $(\lambda', \alpha) = 0$ for all $\lambda' \neq \lambda \in \Delta$;
3. if $(\lambda, \alpha) (\lambda, \alpha') \neq 0$ then there exists $\lambda' \in \Delta$ such that $(\lambda', \alpha) (\lambda', \alpha') = 0$;
4. if $(\lambda, \alpha) (\lambda', \alpha) \neq 0$ then there exists $\alpha' \in S$ such that $(\lambda, \alpha') \neq 0$ and $(\lambda', \alpha') = 0$.

*Proof.* The lemma follows readily from the definition of $\Delta$ along with the axioms defining the spherical systems. \qed

**Lemma 32.** Let $\alpha$, $\alpha'$, $\delta$ and $\delta'$ be pairwise simple roots not in $S^p$. Suppose that $(\alpha, \alpha')(\delta, \delta') \neq 0$ and $(\delta, \alpha) = 0$. If $\Delta$ contains more than two elements then one of its elements is orthogonal to $\alpha + \alpha'$.

*Proof.* Suppose first that $\alpha$ and $\alpha'$ are not in $\Sigma$. If $(\lambda, \alpha + \alpha') \neq 0$ for all $\lambda \in \Gamma$ then $S \setminus S^p$ consists of the four given simple roots and in turn $\Delta$ is of cardinality 2 – a contradiction.

To be definite, let $\alpha$ be in $\Sigma$. We proceed again by contradiction. We thus obtain that $\delta$ and $\delta'$ are also in $\Sigma$. One thus concludes by applying Axiom A3. \qed
A.2. **Auxiliary lemmas.** For convenience, we shall recall the following statements from [BC1].

**Lemma 33.** ([BC1] Proposition 3.4) Let $V$ be a finite dimensional $G$-module of $G$. Suppose the highest weights of $V$ are linearly independent and generate a monoid $\Gamma$ such that $Z\Gamma \cap \Lambda^+ = \Gamma$. Let $\gamma$ be a $T_{ad}$-weight vector of $(V/\mathfrak{g}.\mathfrak{v}_\lambda)^{G_{\Delta}}$. If $\delta$ is a simple root in the support of $\gamma$ such that $\gamma - \delta$ is not a root then $(\gamma, \delta) \geq 0$. Further if $\gamma$ and $\delta$ are orthogonal then $\delta$ is orthogonal to all the $\lambda_i$’s.

**Remark 34.** We will generalize the above statement in Proposition 37.

**Lemma 35.** ([BC1]) Keep the assumptions of the preceding lemma.

(i) The $T_{ad}$-module $(V/\mathfrak{g}.\mathfrak{v}_\lambda)^{G_{\Delta}}$ is multiplicity-free and its $T_{ad}$-weights are non-loose spherical roots of $G$.

(ii) Let $[v]$ be a $T_{ad}$-weight vector of $(V/\mathfrak{g}.\mathfrak{v}_\lambda)^{G_{\Delta}}$. Then one of the representatives $v \in V$ of $[v]$ can be taken as follows

$$[v] \in (V(\lambda)/\mathfrak{g}.\mathfrak{v}_\lambda)^{G_{\Delta}} \quad \text{or} \quad v = X_{\gamma}v_{\lambda}$$

where $\lambda$ is one of the given dominant weights $\lambda_i$. The second case occurs only when $(V(\lambda)/\mathfrak{g}.\mathfrak{v}_\lambda)^{G_{\Delta}}$ is trivial.

**Lemma 36.** Let $\gamma$ be a spherical root of $G$. Suppose $\gamma$ is neither a loose spherical root nor a simple root of $G$ and consider $S^p$ such that $(S^p, \gamma)$ is compatible. Then $\gamma$ is a $T_{ad}$-weight vector of $(V/\mathfrak{g}.\mathfrak{v}_\lambda)^{G_{\Delta}}$ where $V$ is the $G$-module whose highest weights are the dominant weights $\omega_D$ associated to the spherical system $(S^p, \{\gamma\}, \emptyset)$.

**Proof.** The lemma is essentially due to Proposition in [J]. \(\square\)

The following proposition is the announced generalization of Lemma 33.

**Proposition 37.** Take a $T_{ad}$-weight vector $[v_\gamma] \in V/\mathfrak{g}.\mathfrak{v}_\Delta$ of weight $\gamma$. Let $\alpha, \delta$ be orthogonal simple roots with $\delta$ in the support of $\gamma$. Suppose that $\gamma - \delta$ is not a root and that

$$(X_\beta v_\gamma) = 0 \quad \text{for all } \beta \neq \alpha.$$

Then $(\gamma, \delta)$ is positive. Moreover, if $\gamma$ is orthogonal to $\delta$ then so are the dominant weights $\lambda$ in $\Delta$.

**Proof.** Remark that $\gamma$ is a sum of two positive roots. Since $\gamma - \delta$ is not a root, there exists a simple root, say $\delta'$, adjacent to $\delta$ which lies in the support of $\gamma$.

We proceed by contradiction: Suppose $(\gamma, \delta) \leq 0$ and there exists a dominant weight $\lambda$ which is not orthogonal to $\delta$.

Let $\beta$ be a simple root of $G$ such that $(\gamma, \beta) > 0$. 

Assume first that $\beta$ is distinct to $\alpha$. Then by assumption $X_{\beta}v_\gamma \in g.v_\lambda$. Since $(\lambda - \gamma, \beta) \leq 0$ for all $\lambda \in \Delta$, we can assume that $v_\gamma \in \oplus_\lambda V(\lambda)$ with $\lambda$ non-orthogonal to $\beta$.

If $X_{\nu}v_\gamma$ is not trivial for some simple root $\nu$ distinct to $\alpha$ then $\nu \neq \delta$ and $(\lambda, \delta)(\lambda, \beta) \neq 0$. Thanks to the third item of Lemma 31, $\delta + \beta$ is of type $A_1 \times A_1$; the roots $\delta$, $\beta$ and $\delta'$ are pairwise distinct and in turn $\delta'$ is not in $S^\circ$. As a consequence, $X_{\nu}v_\gamma$ has to be trivial for every simple root $\nu$ distinct to $\alpha$. Moreover, since $\gamma$ is assumed not to be a multiple of $\alpha$, there exists a simple root $\alpha'$ adjacent to $\alpha$ such that $X_{\alpha + \alpha'}v_\gamma$ is not trivial. In particular, $\alpha$ lies in the support of $\gamma$ and the latter is of cardinality at least 4. Finally $X_{\alpha + \alpha'}v_\gamma \in g.v_\lambda \setminus \{0\}$ yields a contradiction.

Assume now that $(\gamma, \beta) \leq 0$ for every simple root $\beta$ distinct to $\alpha$. If $X_{\beta}v_\gamma$ is not trivial for such a simple root $\beta$ then $\gamma - \beta$ is a root and so is $\gamma$. Furthermore, $(\gamma, \alpha + \alpha') > 0$ for some simple root $\alpha'$ adjacent to $\alpha$. Similarly as before, we can assume that $v_\gamma \in \oplus_\lambda V(\lambda)$ for $\lambda$ non-orthogonal to $\alpha + \alpha'$. It follows that if $X_{\nu}v_\gamma$ is not trivial for some simple root $\nu$ distinct to $\alpha$ then $(\nu, \delta)(\nu, \alpha + \alpha') \neq 0$. If $X_{\nu}v_\gamma$ is trivial for every simple root $\nu$ distinct to $\alpha$ then as before $X_{\alpha + \alpha'}v_\gamma \neq 0$ and in turn $\gamma - \alpha - \alpha'$ is a root whence $(\gamma, \alpha + \alpha') > 0$ and we end up with the same conclusion: $(\nu, \delta)(\nu, \alpha + \alpha') \neq 0$.

If the roots $\alpha$, $\alpha'$, $\delta$ and $\delta'$ are distinct, we get a contradiction by means of Lemma 32. We are thus left with $\gamma$ of support consisting only of the roots $\alpha$, $\delta$ and $\delta'$ (hence $\alpha' = \delta'$). Easy computations yield the desired contradiction. □

A.3. Computations for degree $0$. We denote $\Sigma(\Delta)$ the set of $T_{\text{ad}}$-weights of $(V/ g.v_\Delta)^{G_{\Delta}}$.

**Proposition 38.**

$$\Sigma \subset \Sigma(\Delta) \subset \Sigma \cup \{\alpha + \alpha' : (\alpha, \alpha') \neq 0 \text{ and } \alpha, \alpha' \in S \cap \Sigma\}.$$  

Furthermore, the $T_{\text{ad}}$-module $(V/ g.v_\Delta)^{G_{\Delta}}$ is multiplicity-free.

**Remark 39.** This proposition does not hold in general while forgetting the requirement of being spherically closed for the spherical system. Indeed if $\gamma$ is a loose spherical root of $S$ then $2\gamma$ belongs to $\Sigma(\Delta)$ but $\gamma$ itself does not; one may consider $\gamma$ as in Example 11.

**Proof.** In case of characters of $T$ whose support does not contain any simple root in $\Sigma$, the proposition follows essentially from the auxiliary lemmas recalled above. Any spherical root whose support contains a spherical root $\alpha$ is either $\alpha$ itself or of shape $\alpha + \alpha'$ where $\alpha$ is orthogonal
to \( \alpha + \alpha' \) and \( \alpha' \) is a simple root adjacent to \( \alpha \) but not in \( \Sigma \). Both \( \alpha \) and \( \alpha + \alpha' \) are obviously in \( \Sigma(\Delta) \) when they are spherical roots of \( S \).

The rest of the proof follows from Proposition 40 below. □

**Proposition 40.** Suppose \( S \) is a spherically closed spherical system of \( G \). Let \( \gamma \) be a \( T_{\text{ad}} \)-weight in \( \Sigma(\Delta) \) whose support contains a simple root \( \alpha \) in \( \Sigma \). Then \( \gamma \) is a root and it is equal to \( \alpha \) or \( \alpha + \alpha' \) with \( \alpha' \) a simple root. In the latter case, either \( \gamma \) or \( \alpha' \) belongs to \( \Sigma \).

**Proof.** Note that \((\alpha + \alpha',0)\) is in the integral span of the \( \lambda_D \)'s if and only if either \( \alpha \) and \( \alpha' \) are both in \( \Sigma \) or \( \alpha + \alpha' \) is in \( \Sigma \). The latter assertion of the proposition follows.

In order to prove the first assertion of the proposition, we shall proceed as follows. Note first that \( \alpha \) is clearly in \( \Sigma(\Delta) \). When \( \gamma \) is not equal to \( \alpha \), we shall prove in the following lemmas that \( \gamma, \gamma - \alpha \) and \( \gamma - \alpha' \) are roots (\( \alpha' \) being one of the simple roots adjacent to \( \alpha \)). □

**Lemma 41.** The character \( \gamma - \alpha \) is a root.

**Proof.** Let us proceed by contradiction: suppose \( \gamma - \alpha \) is not a root. Since \( X_\alpha v_\gamma \) has to lie in \( g.v_\Delta \), the vector \( X_\alpha v_\gamma \) is trivial in \( V \). Moreover, by Proposition 37, \((\gamma, \alpha)\) is strictly positive. Consequently, the representative \( v_\gamma \) thus lies in \( V(\lambda_\alpha^+) \oplus V(\lambda_\alpha^-) \) where \( \lambda_\alpha^+ \) and \( \lambda_\alpha^- \) are the dominant weights among the \( \omega_D \)'s which are not orthogonal to \( \alpha \). Since the vector \( v_\gamma \) can not be dominant, there exists a simple root \( \delta \) in the support of \( \gamma \) such that the vector \( X_\delta v_\gamma \) is not trivial in \( V \). It follows that the weight \( \gamma - \delta \) is a root. Thanks to Lemma 31, \( \gamma \) is thus equal to \( \alpha + \delta \) hence \( \gamma \) has to be a root: a contradiction with \( \gamma - \alpha \) non-being a root and \((\gamma, \alpha) > 0 \).

**Lemma 42.** If the weight \( \gamma \) is not a root then the vector \( X_\alpha v_\gamma \) is not trivial in \( V \).

**Proof.** Thanks to Lemma 41, \((\gamma - \alpha, \alpha^\vee)\) is positive hence \((\gamma, \alpha)\) is strictly positive. By the same arguments as those used in the proof of Lemma 41, we get a contradiction whenever \( X_\alpha v_\gamma \) is trivial in \( V \).

**Lemma 43.** The supports of \( \alpha \) and \( \gamma - \alpha \) are not orthogonal.

**Proof.** Let us proceed by contradiction. Then the weight vector \( v_\gamma \) can be written as \( X_{-\alpha} v_{\gamma - \alpha} \) where \( v_{\gamma - \alpha} \) is a weight vector of \( T_{\text{ad}} \)-weight \( \gamma - \alpha \). In particular, \( X_\alpha v_\gamma \) is not trivial in \( V \). By means of Lemma 31, we get a contradiction.
Lemma 44. There exists a simple root \( \alpha' \) adjacent to \( \alpha \) such that the character \( \gamma - \alpha' \) is a root. In particular, \( \alpha' \) lies in the support of \( \gamma \).

Proof. Note first that by the previous lemma, the support of \( \gamma \) contains a simple root \( \alpha' \) adjacent to \( \alpha \). One of the dominant weights \( \omega_D \) is thus non-orthogonal to \( \alpha' \) since so is \( \alpha \). Let us proceed by contradiction: suppose \( \gamma - \alpha' \) is not a root. Then \( X_{\alpha'}v \) is trivial in \( V \) and \( (\gamma, \alpha') \) is strictly positive by Proposition 37. It follows that \( \gamma \) is not a root.

Thanks to the previous lemmas, \( \alpha' \) is not a weight in \( \Sigma(\Delta) \) hence it is not a spherical root in \( \Sigma \). Recalling that \( \gamma \) is in the integral span of the weights \( \omega_D \), we get that \( (\omega_D - \gamma, \alpha') \) is negative for every \( D \). But since \( X_{\alpha'}v \) is trivial in \( V \), the representative of \( v_\gamma \) can be taken in the module \( V(\omega_D) \) associated to the (single) dominant weight which is not orthogonal to \( \alpha' \). Since \( X_{\alpha'}v \) is not trivial in \( V \) by Lemma 42, the support of \( \gamma - \alpha \) does not contain the root \( \alpha \). Together with the fact that \( \alpha' \) belongs to the support of \( \gamma \), we get that \( (\gamma - \alpha, \alpha) \) is strictly negative. It follows that \( \gamma \) is a root since so is \( \gamma - \alpha \) by Lemma 41: a contradiction. \( \square \)

Lemma 45. The weight \( \gamma \) is a root.

Proof. We first claim: Let \( \alpha \) and \( \alpha' \) be non-orthogonal pairwise simple roots and \( \delta \) an arbitrary root. If \( \delta + \alpha \) is not a root then neither is \( \delta + \alpha - \alpha' \). Apply the claim to \( \delta := \gamma - \alpha \) which is a root as previously proved. We get that if \( \gamma \) is not a root then neither is \( \gamma - \alpha' \) for any simple root \( \alpha' \) adjacent to \( \alpha \). This yields a contradiction with Lemma 44. \( \square \)

A.4. Computations for degree 1. Recall that the set \( \Sigma(\Delta) \) is defined as the set of \( T_{\text{ad}} \)-weights of \( (V/\mathfrak{g}.v_\lambda)^{G_{v_\lambda}} \). Further, let \( S^p \) be the set of simple roots of \( G \) orthogonal to \( \sum_\Delta \lambda_D \).

Theorem 46. \( H^1(\mathbf{G}_{v_\Delta}, V/\mathfrak{g}.v_\lambda) \approx \bigoplus_{\alpha \in S \setminus S^p, \gamma \in \Sigma(\Delta)} k_{s_\alpha^* \gamma} \) as \( T_{\text{ad}} \)-modules.

The one dimensional \( T_{\text{ad}} \)-module \( k_{s_\alpha^* \gamma} \) is spanned by \( X_{\alpha}^* \otimes [s_\alpha v_\gamma] \).

The proof of this theorem requires the following proposition.

Proposition 47. Let \( \varphi \) be a non-zero \( T_{\text{ad}} \)-weight vector in \( H^1(\mathfrak{g}_{v_\Delta}, V/\mathfrak{g}.v_\lambda) \). Then \( X_\beta \varphi(X_\alpha) = 0 \) for every simple root \( \alpha \) and every root \( \beta \neq \alpha \) of the isotropy Lie algebra \( \mathfrak{g}_{v_\lambda} \).

Remark 48. The vanishing condition fulfilled by the vector \( \varphi(X_\alpha) \) is that stated in Proposition 37.
Proof of Theorem 46. The proof follows essentially from Proposition 47 together with the isomorphism

\[ H^1(G_{\nu\lambda}, V/\mathfrak{g}.v_\lambda) \approx (H^1(\mathfrak{g}_{\nu\lambda}, V/\mathfrak{g}.v_\lambda))^{G_{\nu\lambda}/G_{\nu\lambda}}. \]

Let \( \gamma \) be the \( T_{ad} \)-weight of \( \varphi(X_\alpha) \) and denote by \( v_\gamma \) a representative of \( \varphi(X_\alpha) \) in \( V/\mathfrak{g}.v_\lambda \).

We shall thus assume in the remainder of the proof that \( \alpha \) does belong to the support of the \( T_{ad} \)-weight \( \gamma \) and that \( v_\gamma \) is not equal to \( X_r^{-\alpha}v_\lambda \) - in which case the proposition is obvious. We shall proceed along the type of the support of \( \gamma \). Let us work out a few cases in detail. The main ingredients of the proof are Proposition 37 and 47 along with the properties enjoyed by the dominant weights in \( \Delta \) (see section A.1). As a consequence of Proposition 47 the weight \( \gamma \) can be written as a sum of two positive roots, say \( \beta_1 \) and \( \beta_2 \).

Consider first the case where the supports of the roots \( \beta_1 \) and \( \beta_2 \) are orthogonal. Thanks to Proposition 37, the roots \( \beta_1 \) and \( \beta_2 \) have to be simple. In virtue of Lemma 31, there is a single dominant weight, say \( \lambda \), which is neither orthogonal to \( \beta_1 \) nor to \( \beta_2 \). Thanks to Proposition 47, \( \gamma \in \Sigma(\Delta) \).

Suppose now that the support of \( \gamma \) is of type \( A_n \). If \( \gamma \) is not a root, Proposition 37 and 47 yield: \( \gamma = \alpha_{i-1} + 2\alpha_i + \alpha_{i+1} \) with \( \alpha = \alpha_i \) and all the dominant weights \( \lambda_k \) are orthogonal to both \( \alpha_{i-1} \) and \( \alpha_{i+1} \). Clearly, we thus have: \([v_\gamma] \in (V/\mathfrak{g}.v_\lambda)^{G_{\nu\lambda}}\). If \( \gamma \) is now a root then one gets: \( \gamma = \alpha_i + \ldots + \alpha_j \) and \( \alpha = \alpha_i \) (or \( \alpha_j \)) by Proposition 37. Since \((\gamma, \alpha_j) > 0\), applying the above remark to \( \alpha_j \), we get that either \( \gamma \) or \( \gamma - \alpha \) belongs to \( \Sigma(\Delta) \).

In case of type \( B_n \), we obtain similarly as before that \( \gamma = \alpha_i + \ldots + \alpha_n \) or \( \gamma = 2(\alpha_i + \ldots + \alpha_n) \) whenever \( \alpha \) lies in the support of \( \gamma \). Note that the same arguments as for the case of a root \( \gamma \) of type \( A \) can be applied. Suppose thus \( \gamma \) is the weight \( 2(\alpha_i + \ldots + \alpha_n) \). Then by the above remark along with Proposition 37 and 47 we get: \( \alpha = \alpha_i \). Moreover all simple roots, except \( \alpha_i \) and \( \alpha_{i+1} \), in the support of \( \gamma \) are orthogonal to the weights in \( \Delta \). Remark that \( \alpha_{i+1} \) can not be orthogonal to \( S^p \) by Whitehead lemma. From Lemma 31 we deduce that the fundamental weight attached to \( \alpha_i \) (resp. \( \alpha_{i+1} \)) is the unique weight in \( \Delta \) non-orthogonal to \( \alpha_i \) (resp. \( \alpha_{i+1} \)). It follows that \( \gamma - 2\alpha \) is a weight in \( \Sigma(\Delta) \).

The other types can be conducted similarly.

A.4.1. Proof of Proposition 47.
Theorem 49 (Kostant).

\[ H^1(g_{v_\lambda}, V(\lambda)) = \bigoplus_{\alpha} \mathbb{C}X^*_\alpha \otimes v_{s_\alpha \lambda}, \text{ as } T_{\text{ad}}-\text{modules} \]

where \( v_{s_\alpha \lambda} \) is a non-zero weight vector in \( V(\lambda) \) of weight \( s_\alpha \lambda \), \( \alpha \) a simple root non-orthogonal to \( \lambda \) and \( s_\alpha \) being the simple reflection associated to \( \alpha \).

Let \( \varphi \) be a non-zero \( T_{\text{ad}} \)-weight vector in \( H^1(g_{v_\lambda}, V/\mathfrak{g}v_\lambda) \). Then there exist a simple root \( \alpha \) and a \( T_{\text{ad}} \)-weight vector \([v_\gamma]\) in \( V/\mathfrak{g}v_\lambda \) such that \( \varphi(X_\alpha) = [v_\gamma] \neq 0 \) and one can write \( \varphi \) as

\[ \varphi = \sum_{\beta + \nu = \alpha + \gamma} X^*_\beta \otimes [v_\nu] \]

Consider the short exact sequence of \( g_{v_\Delta} \)-modules

\[ 0 \to \mathfrak{g}v_\Delta \to V \to V/\mathfrak{g}v_\Delta \to 0 \]

and the associated long exact sequence in cohomology.

In order to prove Proposition 47 we shall study separately the following situations regarding the simple roots \( \beta \) which lie in the support of \( \gamma \): \((\gamma, \beta) < 0\); when \( \gamma - \beta \) is a root, we work out first the case when \( \alpha \) and \( \beta \) are orthogonal and then the case where they are not; finally, we consider the roots \( \beta \) such that \( \gamma - \beta \) is not a root and \((\gamma, \beta) \geq 0\).

Before dealing with different situations, note that the following lemma holds in general.

Lemma 50. Assume that for any weight \( \phi(\alpha, \beta) \) distinct to \( \alpha \) in the integral span of \( \alpha \) and \( \beta \), \( \gamma - \phi(\alpha, \beta) \) is not a root. Then \( \varphi \) maps trivially in \( H^2(g_{v_\Delta}, \mathfrak{g}v_\Delta) \).

A.4.2. \((\gamma, \beta) < 0\). Throughout this section, \( \beta \) lies in the support of \( \gamma \) and is such that \((\gamma, \beta) < 0\).

Lemma 51. The weight \( \gamma - \beta \) is not a root except if \( \gamma = 3\alpha_1 + 2\alpha_2 \) is a root of type \( G_2 \).

Proof. Note that if \( \gamma - \beta \) was a root then \((\gamma, \beta) \) would be positive - a contradiction except if \( \gamma \) is as in the lemma. \( \square \)

Corollary 52. If \( \alpha \) and \( \beta \) are orthogonal then \( X_\beta \varphi(X_\alpha) \) is trivial.

Lemma 53. Suppose \( \gamma - \alpha - \beta \) is a root.

(1) The weight \( \gamma - \alpha \) is a root.

(2) \((\beta, \alpha^\vee) = -1\).

(3) The weight \( \gamma \) is one of the following:

(i) \( \gamma = \beta + 3\alpha + 2(\alpha^+ + \ldots + \alpha_{n-1} + \alpha_n) \) in type \( C_n \);
(ii) $\gamma = \ldots + \alpha^- + 2\alpha + \beta$ with $\beta = \alpha_n$ in type $B_n$;
(iii) $\gamma = \ldots + \beta^- + 2\beta + 2\alpha$ with $\alpha = \alpha_n$ in type $C_n$;
(iv) $\gamma = \beta + 2\alpha$ with $\alpha = \alpha_n$ in type $C_n$.

**Proof.** The first assertion follows from the inequality: $(\gamma - \alpha - \beta, \beta) < 0$
- except possibly in type $G_2$.

The weight $\gamma - \alpha$ being a root, we have $(\gamma - \alpha, \beta') \geq 0$ and in turn
$0 \leq (\gamma - \alpha, \beta') < 2$, i.e. $(\gamma - \alpha, \beta') = 0$ or $1$. The lemma follows readily. \[\square\]

**Lemma 54.** Let $\gamma - 2\alpha - \beta$ and $2\alpha + \beta$ be roots. Then the root
$\gamma - 2\alpha - \beta$ is one of the following: $\ldots + \beta^- + \ldots + \beta^- + 2\alpha$ of type
$B_n$; $\ldots + \alpha^- + \alpha$ of type $C_n$; $1242, 1121, 1122$ in type $F_4$; $\alpha_1$ or $3\alpha_1 + \alpha_2$
in type $G_2$.

**Proof.** One review all possible roots $\gamma - 2\alpha - \beta$ satisfying all our requirements
(in particular, $\gamma$ is a sum of two positive roots, $(\gamma, \beta) < 0$). \[\square\]

**Lemma 55.** If $\gamma = \beta + 2\alpha$ then $2\alpha$ is a spherical root in $\Sigma$. Further
any $T_{ad}$-weightvector $X^*_\alpha \otimes v_\gamma$ is not fixed by $G_{v_\lambda}/(G_{v_\lambda})^\circ$.

Proof of Proposition 47. In virtue of Lemma 55, $\varphi$ can not be of weight
$\beta + 3\alpha$. We thus assume in the remainder that $\gamma$ is distinct to $\beta + 2\alpha$.

Thanks to Lemma 54, we can assume there exists $\phi(\alpha, \beta)$ in the integral span of $\alpha$ and $\beta$ such that $\gamma - \phi(\alpha, \beta)$ is a root; otherwise the proposition is already proved. Considering the long exact sequence of
cohomology associated to (\ref{charank}), $\phi(\alpha, \beta)$ is either a root or a sum of shape
$-\alpha + \delta + \delta'$ with $\delta$ and $\delta'$ being positive roots in $\Phi(\alpha, \beta)$.

Note that $(\gamma - a\alpha - b\beta, \beta') = 0$ for any positive integers $a$ and $b$ with
$a \leq b$. Along with Lemma 51 (not $G_2$ type with $\gamma \in \Phi !$), it follows that
the weight $\phi(\alpha, \beta)$ has to be $\alpha + \beta$ or $2\alpha + \beta$; the latter weight occurs only in case $(\beta, \alpha') = -2$.

Suppose first that $\gamma - \alpha - \beta$ is a root. A glance at the weight of $\varphi(X_\beta)$
(see Lemma 53) shows that this vector is trivial: this weight should be equal to $\gamma + \alpha - \beta$ and should fulfill the required property. Further, since $2\alpha + \beta$ is not a root, $\varphi([X_\alpha, X_{\alpha + \beta}]) = X_\alpha X_\beta \varphi(X_\alpha) - X_{\alpha + \beta} \varphi(X_\alpha)$
is trivial. We shall prove that there is a representative of $\varphi(X_\alpha)$ such that
$\varphi([X_\alpha, X_{\alpha + \beta}])$ is trivial in $V$.

Suppose $X_\beta \varphi(X_\alpha)$ is not trivial. Then considering again the weights
$\gamma$ listed in Lemma 53, we see that a representative $v_{\gamma - \beta}$ of $X_\beta \varphi(X_\alpha)$
can be taken to be in $V(\lambda)_{\lambda - \gamma + \beta}$ where $\lambda$ is not orthogonal to $\alpha$. The
support of $\gamma - \beta$ contains a simple root $\alpha'$ adjacent to $\alpha$ such that
$(\gamma, \alpha')$ and $(\alpha, \alpha')$ differ. From Lemma 51, we deduce that whatever
$X_\beta \varphi(X_\alpha)$ is, the vector $X_\alpha v_{\gamma - \beta}$ does not lie in $\mathfrak{g}.v_\lambda \setminus \{0\}$. 

Consequently, if $X_\beta X_\alpha v_\gamma$ is not trivial (in $V$) then the $\lambda$-component of $X_{\alpha+\beta}v_\gamma$ equals up to a scalar to $X_{-\gamma+\alpha+\beta}v_\lambda$ for $\lambda$ orthogonal to $\alpha$. Since $(\gamma, \alpha + \beta) > 0$, there exists a representative of $\varphi(X_\alpha)$ whose $\lambda$-component is trivial for every dominant weight $\lambda$ orthogonal to $\alpha + \beta$. Note that such a dominant weight $\lambda$ exists under the assumption that $\gamma$ is distinct to $\beta + 2\alpha$.

Let now $\gamma - \alpha - \beta$ not be a root. As mentioned above, the weight $\gamma - 2\alpha - \beta$ has to be a root and so has $2\alpha + \beta$. We proceed similarly as above while considering instead $[v] = X_\alpha \varphi(X_{2\alpha+\beta}) - X_{2\alpha+\beta}\varphi(X_\alpha)$ which is obviously trivial because of the cocycle property.

Assume $\varphi(X_{2\alpha+\beta})$ is not trivial. A glance at its weight $\gamma' = \gamma - \alpha - \beta$ (see Lemma 54) shows that the representative $v_{\gamma'}$ of $\varphi(X_\alpha)$ in $\oplus_\lambda V(\lambda)_{\lambda - \gamma'}$ projects trivially onto $V(\lambda)$ if $\lambda$ is orthogonal to $\alpha$. For such a $v_{\gamma'}$, $X_\alpha v_{\gamma'}$ does not lie in $g\cdot \mathfrak{g}_\alpha \setminus \{0\}$. If $X_\alpha v_{\gamma'} - X_{2\alpha+\beta}v_\gamma$ is not trivial in $V$ for some representative $v_\gamma$ of $\varphi(X_\alpha)$ then any $\lambda$-component of $X_{2\alpha+\beta}v_\gamma$ has to be non-trivial whenever $\lambda$ is orthogonal to $\alpha$. Further, $X_{2\alpha+\beta}v_\gamma^\lambda = X_{-\gamma + 2\alpha + \beta}v_\lambda$. Since $(\lambda - \gamma, 2\alpha + \beta) < 0$ for $\lambda$ orthogonal to both $\alpha$ and $\beta$ (existence!), there exists a representative of $\varphi(X_\alpha)$ whose $\lambda$-components are trivial for $\lambda$ orthogonal to $\alpha$ and $\beta$. It follows that the corresponding representative of $[v]$ is trivial in $V$ whence the proposition.

A.4.3. $\gamma - \beta$ is a root and $(\alpha, \beta) = 0$. First observe that $X_\beta \varphi(X_\alpha)$ is trivial whenever so is $\varphi(X_\beta)$ (thanks to the cocycle property). We shall thus suppose in the remainder of this section that $\varphi(X_\beta)$ is not trivial; let $\gamma'$ be its $T_{\text{ad}}$-weight. Recall that $\gamma' = \gamma + \alpha - \beta$.

A.4.4. $(\gamma, \alpha) \leq 0$.

**Lemma 56.** Assume there exists a simple root $\delta$ in the support of $\gamma$ such that $\varphi$ restricted onto the Lie algebra spanned by $\alpha$ and $\delta$ is harmonic. Then $\gamma$ is orthogonal to $\alpha$ and there exists a representative $v_\gamma$ of $\varphi(X_\alpha)$ such that the $\lambda$-component of $v_\gamma$ is trivial for $\lambda$ non-orthogonal to $\alpha$.

Let $v_{\gamma'}$ be a representative of $\varphi(X_\beta)$ in $\oplus V(\lambda)_{\lambda - \gamma'}$. Suppose the assumptions of the lemma right above are satisfied. If $X_\beta v_\gamma - X_\alpha v_\gamma^\lambda$ is not trivial (in $V$) then neither is the $\lambda$-component of $X_\alpha v_\gamma^\lambda$ for any dominant weight $\lambda$ non-orthogonal to $\alpha$. More precisely, we have

$$X_\alpha v_\gamma^\lambda = X_{-\gamma + \beta}v_\lambda \quad \text{for } (\lambda, \alpha) \neq 0.$$

We thus claim that there exists a dominant weight $\lambda'$ in $\Delta$ which is not orthogonal to $\alpha$ but orthogonal to $\beta$. This is due essentially to the fact that $\alpha + \beta$ is not a spherical root in $\Sigma$. Since $(\gamma', \alpha) > 0$, there
exists a representative $v_{\gamma'}$ whose $\lambda'$-component is trivial. The lemma follows in the case under consideration.

Assume now that we are not in the setting of the lemma right above. Since the simple roots $\alpha$ and $\beta$ are assumed to be orthogonal and both lie in the support of $\gamma$, at least one of the adjacent roots, say $\beta^-$ should belong also to the support of $\gamma$.

Claim 57. If $\gamma - \beta$ and $\gamma - \beta^-$ are roots then so is $\gamma$.

Assume first that $\gamma$ is not a root then thanks to this first claim, neither $\gamma - \beta^-$ nor $\gamma - \alpha$ is a root. The latter is due to the fact that $(\gamma - \alpha, \alpha)$ is strictly negative (recall that $(\gamma, \alpha) \leq 0$ by assumption).

Claim 58. If $\gamma - \alpha - \beta^-$ is a root then $(\beta^-, \alpha') = -2$. Further $\gamma = \beta + 2\beta^- + \alpha$.

The second claim is due to the inequality $(\gamma - \alpha - \beta^-, \alpha) \geq 0$ which holds since $\gamma - \beta^-$ is not a root.

Assume thus now that $\gamma$ is a root. Recall that $(\gamma, \alpha) \leq 0$, $\gamma - \beta$ is a root and $\gamma$ does not satisfy the conditions of the above lemma. One obtains a few roots $\gamma$ and can conclude as before.

A.4.5. $(\gamma, \alpha) > 0$. If $(\gamma', \beta)$ is negative then as previously proved, $X_\alpha \varphi(X_\beta)$ is trivial and so is $X_\beta \varphi(X_\alpha)$ (by cocyclicity). Let us thus assume that $(\gamma', \beta)$ is strictly positive i.e. $(\gamma - \beta, \beta) > 0$ since $(\alpha, \beta) = 0$.

The weight $\gamma - \beta$ being a root, we have either $(\gamma - \beta, \beta') = 1$ or 2 whenever not of type $G_2$. One can thus list the very few possible roots $\gamma - \beta$. Let us work out explicitly the type $C_n$: we have either $\gamma - \beta = \alpha + \ldots + 2(\beta + \ldots) + \alpha_n$ or $\gamma - \beta = \beta + \ldots + 2(\alpha + \ldots) + \alpha_n$.

Consider the first weight $\gamma$. Note that $v_\alpha^\lambda$ is not trivial only if $\lambda \in \Delta$ is not orthogonal to $\beta$ and distinct to the fundamental weight associated to $\beta$.

Claim 59. There exists a dominant weight $\lambda$ in $\Delta$ which is orthogonal to both $\alpha$ and $\beta$ but not $\gamma$.

It follows that $X_\alpha v_\gamma^\lambda$ equals (up to a non-trivial scalar) $X_{-\gamma + \beta} v_\lambda$. Recall that $(\gamma', \beta) > 0$; we thus conclude as before.

Consider now the second possible weight $\gamma$ of type $C_n$. Then $\gamma' = \gamma + \alpha + \beta = 2\alpha + \ldots + 2(\beta + \ldots) + \alpha_n$ and $v_\gamma^\lambda \neq 0$ only if $(\lambda, \alpha) \neq 0$.

We conclude as previously.

A.4.6. $\gamma - \beta$ is a root with $(\alpha, \beta) \neq 0$.

Lemma 60. Under the assumptions of the subsection, we have $(\gamma, \alpha) \geq 0$ unless $\gamma = \alpha_{n-2} + \alpha_{n-1} + \alpha_n$ of type $C_n$. 

Proof. Let us proceed by contradiction.

Suppose first that \((\gamma - \beta, \alpha^\vee)\) is strictly positive. Then \((\beta, \alpha^\vee)\) equals 
\(-2\) or \(-3\) and \(\gamma - \beta - \alpha\) has to be a root. In type \(\mathcal{B}_n\), the simple root \(\alpha_n\) 
has to be \(\alpha\) itself and \((\gamma - \beta, \alpha)\) being strictly positive, it has to be equal 
to \(2\) and in turn \((\gamma, \alpha) = 0\) -whence a contradiction. Similarly, in type \(\mathcal{C}_n\), we get as possibilities for \(\gamma\) the weights \(\gamma_1 = \alpha_i + \ldots + 2\alpha_{n-1} + \alpha_n\) 
with \(i < n - 1\) and \(\beta = \alpha_n\) and \(\gamma_2 = \ldots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n\). Note that \(\gamma_1 - \phi(\alpha, \alpha_i)\) is not a root for any weight in the \(\mathbb{Z}\)-span of \(\alpha\) and \(\alpha_i\). 
Together with Lemma 50, this yields a contradiction with \((\gamma, \alpha)\) being 
strictly negative. And similarly, we are left with \(\gamma = \alpha_{n-2} + \alpha_{n-1} + \alpha_n\).
We handle the type \(\mathcal{F}_4\) by analogous arguments.

Suppose now that \((\gamma - \beta, \alpha)\) is negative. Assume further the sup-
port of \(\gamma\) contains a simple root \(\delta\) which is orthogonal to \(\alpha\). Then by 
Proposition 37, the weight \(\gamma - \delta\) has to be a root.

Claim 61. The weight \(\gamma\) is a root and \(\gamma = \beta + \delta\).

Indeed, if \(\alpha\) belongs to the support of \(\gamma\) then \(\gamma\) is not a root and 
neither is \(\gamma - \delta\). Therefore the simple root \(\alpha\) does not belong to the 
support of \(\gamma\) and the claim follows from Lemma 1.8.

Finally assume that there is no root orthogonal to \(\alpha\) in the support 
of \(\gamma\). It follows that \(\alpha\) does belong to the support of \(\gamma\) whenever \(\gamma\) is 
distinct to \(2\beta\).

Claim 62. The weight \(\gamma - \beta\) is one of the roots \(\alpha + \beta\), \(2\beta + \alpha\), \(2\alpha^- + \alpha\) 
or \(\alpha^- + \alpha + \beta\).

To obtain the above claim, we list the possible roots \(\gamma - \beta\) such that 
there is no simple root \(\delta\) orthogonal to \(\alpha\) in the support of \(\gamma\).

\[\square\]

A.4.7. \(\gamma - \beta\) is not a root and \((\gamma, \beta) \geq 0\). Remark that if \(X_\beta \varphi(X_\alpha)\) 
is not trivial then (by Lemma 50), the simple roots \(\alpha\) and \(\beta\) are not 
orthogonal and in turn, one of the weights \(\gamma - \alpha - \beta\) and \(\gamma - 2\alpha - \beta\) 
has to be a root. Note that the latter may occur only in case \(2\alpha + \beta\) is 
a root.

Suppose first that \(\gamma - \alpha - \beta\) is a root.

Claim 63. \(\varphi(X_\beta)\) is trivial.
If \((\beta, \alpha^\vee) = -1\) then \((\gamma, \alpha) > 0\) and \(0 = \varphi([X_\alpha, X_{\alpha+\beta}] = X_\beta X_\alpha \varphi(X_\alpha).\)

If there exists a representative \(v_\gamma\) such that \(X_\beta X_\alpha v_\gamma\) is trivial (in \(V\)) then the lemma is proved. Let thus \(X_\beta X_\alpha v_\gamma\) be non-trivial then it equals to \(X_{-\gamma+\alpha+\beta} v_\Delta\). Note that whenever \(\lambda\) is orthogonal to \(\beta\) (exists!!!), we have \((\lambda - \gamma + \alpha, \beta) < 0\) in the case under study. It follows that \(X_\alpha v_\gamma = X_{-\beta} X_{-\gamma+\alpha+\beta} v_\lambda\) for \(\lambda\) orthogonal to \(\beta\) (recall that \(X_\alpha v_\gamma\) may be assumed to be non-trivial otherwise the lemma is already proved).

**Claim 64.** There exists a dominant weight in \(\Delta\) which is orthogonal to \(\alpha + \beta\).

Since \((\gamma, \alpha) > 0\), we can deduce the existence of a representative \(v_\gamma\) whose \(\lambda\)-components are trivial when \(\lambda\) is orthogonal to \(\alpha + \beta\). The lemma follows by the same arguments as before.

Suppose now that \((\beta, \alpha^\vee) = -2\) (and \(\gamma - 2\alpha - \beta\) may be a root). The possible weights can be explicitly listed.

**APPENDIX B. Injectivity**

**Proposition 65.** The map

\[ f : H^1(\mathfrak{g}_{v_\lambda}, V/\mathfrak{g}.v_\lambda) \rightarrow H^1(\mathfrak{g}_{v_\lambda}, S^2V/V(2\Delta)) \]

is injective.

**Proof.** Let \(\varphi\) be a \(T_{ad}\)-weight vector in \(H^1(\mathfrak{g}_{v_\lambda}, V/\mathfrak{g}.v_\lambda)\). By Proposition \([46]\) there exist \(\alpha\) simple and \(\gamma \in \Sigma(\Delta)\) such that \(\varphi = X_\alpha^* \otimes [v_{s_\alpha\gamma}]\).

It is thus enough to prove that there exists \(v_{s_\alpha\gamma} \cdot v_{\lambda_i}\) non-trivial in \(S^2 V/V(2\Delta)\) for which there is no \(v \in S^2 V/V(2\Delta)\) such that \(v_{s_\alpha\gamma} \cdot v_{\lambda_i} = X_\alpha v\) in \(S^2 V/V(2\Delta)\).

Note that this assertion holds whenever

\[(4) \quad X_{-\alpha}(v_{s_\alpha\gamma} \cdot v_{\lambda_i}) = 0 \quad \text{in} \quad S^2 V/V(2\Delta)\]

or

\[(5) \quad X_\alpha^a v_{s_\alpha\gamma} \neq 0 \quad \text{in} \quad V \quad \text{for} \quad a = (\lambda_i, \alpha^\vee).\]

Let us consider first \(X_{-\alpha}(v_{s_\alpha\gamma} \cdot v_{\lambda_i})\). Note that by definition of \(r\), we have: \(X_{-\alpha} v_{s_\alpha\gamma} = 0\) in \(V\). We thus have

\[X_{-\alpha}(v_{s_\alpha\gamma} \cdot v_{\lambda_i}) = v_{s_\alpha\gamma} \cdot X_{-\alpha} v_{\lambda_i}.\]

When \(\lambda_i\) is orthogonal to \(\alpha\), assertion \((4)\) to be proved is thus clear.

Suppose that \(\lambda_i\) is not orthogonal to \(\alpha\). If \((\lambda_i, \gamma) \neq 0\), we have \((\gamma, \alpha) \geq 0\) by Proposition \([37]\). We shall prove assertion \((5)\) considering the cases where \((\lambda_i, \gamma) = 0\) and \((\lambda_i, \gamma) \neq 0\) separately; this is done in the next lemmas. \(\square\)
Lemma 66. Let $\varphi(X_\alpha) = X^\gamma_\alpha v_{\lambda_j}$ for some $\lambda_j$. Then Assertion (3) holds with $\lambda_i = \lambda_j$.

Proof. Note that $r = (\lambda_j, \alpha^\vee)$. Hence if $r > 1$ then $X^r_\alpha v_{\lambda_j}, v_{\lambda_j} \neq 0$ and Assertion (5) is clear. If $r = 1$ then there exists $\lambda_i \neq \lambda_j$ which is not orthogonal to $\alpha$. Thanks to Lemma 31 $(\lambda_i, \alpha^\vee) = 1$, Assertion (5) follows with that chosen $\lambda_i$. \hfill $\square$

Lemma 67. Let $v_{s_{\alpha^\vee}} \cdot v_{\lambda_i} \neq 0$ in $S^2 V/V(2\Delta)$ with $\gamma \in \Sigma(\Delta)$. If $\lambda_i$ is orthogonal to $\gamma$ then Assertion (2) holds.

Proof. Note first that the support of $\gamma$ does not contain $\alpha$. Indeed $\lambda_i$ being orthogonal to $\gamma$ it can not be orthogonal to $\alpha$ otherwise $v_{s_{\alpha^\vee}} v_{\lambda_i}$ will be 0. Hence $(\gamma, \alpha^\vee) \leq 0$ and further $X_\alpha v_\gamma = 0$ in $V$. It follows in turn that $v_{s_{\alpha^\vee}} = X^\gamma_\alpha v_\gamma$ with $r = (\lambda - \gamma, \alpha^\vee)$ and $v_\gamma \in V(\lambda)$. Further, since $v_{s_{\alpha^\vee}} v_{\lambda_i} \neq 0$ and $(\lambda_i, \gamma) = 0$, we have: $v_{s_{\alpha^\vee}} \neq v_\gamma$.

The weight $\lambda$ being non-orthogonal to $\gamma$, it is different from $\lambda_i$.

Assume that $(\gamma, \alpha) = 0$ then since $\alpha$ does not belong to the support of $\gamma$, it has to be orthogonal to every simple root lying in the support of $\gamma$. Let $\delta$ be a simple root $\delta$ in the support of $\gamma$ such that $X_\delta v_\gamma \neq 0$ in $V$. Considering $X_\delta \varphi(X_\alpha)$ along with $(\lambda, \alpha)$, one ends up with a contradiction.

We deduce that $(\gamma, \alpha) < 0$. Thanks to Proposition 37, $\lambda_i$ is the single weight among the $\lambda_j$’s which is not orthogonal to $\alpha$. Recall that $\gamma$ belongs to $Z\Delta$. It follows that $r = (\lambda - \gamma, \alpha^\vee) \geq (\lambda_i, \alpha^\vee)$ whence the lemma. \hfill $\square$

Lemma 68. Suppose $(\gamma, \alpha^\vee) > 1$ with $\gamma \in \Sigma(\Delta)$. Then Assertion (5) holds with $\lambda_i$ such that $v_\gamma \in V(\lambda_i)$.

Proof. If $\gamma = 2\alpha$, we fall in the case of Lemma 66.

By Definition 7 the weight $\gamma$ under consideration is such that $(\gamma, \alpha^\vee) = 2$. Further $\gamma - \delta$ is a root with $\delta$ a simple root if and only if $\delta$ equals $\alpha$. Hence $X_\delta v_\gamma = 0$ in $V$ for all simple roots $\delta$ distinct to $\alpha$ and a fortiori $X_\alpha v_\gamma \neq 0$ in $V$. Moreover $\gamma - 2\alpha$ not being a root, we have: $X_\alpha^2 v_\gamma = 0$ in $V$. Since $(\lambda_i, \alpha^\vee)$ equals 1 or 2, we have respectively $v_{s_{\alpha^\vee}}\gamma$ equals $v_\gamma$ or $X_\alpha v_\gamma$. The lemma follows readily. \hfill $\square$

Lemma 69. Let $\gamma \in \Sigma(\Delta)$ and $(\gamma, \alpha^\vee) = 1$ for some simple root $\alpha$. Then Assertion (3) holds.

Proof. Note that $\gamma - \alpha$ is a root; see Table 7. By Proposition 37 there exists a unique $\lambda$ non-orthogonal to $\alpha$ and $(\lambda, \alpha^\vee) = 1$. Further $v_\gamma$ can be chosen in $V(\lambda)$. If $X_\alpha v_\gamma = 0$ in $V$, it follows from [1] and Lemma 35 that $[v_\gamma] = [X_\gamma v_\lambda] = [X_\gamma v_{\lambda_j}]$ for some $\lambda_j \neq \lambda$ and such
that \((\lambda_j, \gamma - \alpha) \neq 0\). In particular \(X_\alpha v_\gamma \neq 0\) in \(V\) for \(v_\gamma = X_{-\gamma} v_{\lambda_j}\).

Then \(v_\gamma\) can be chosen such that \(v_\gamma \in V(\lambda_k)\) with \(X_\alpha v_\gamma \neq 0\) in \(V\) and \(\lambda_k = \lambda\) or \(\lambda_j\) as above. Assertion (4) thus holds with \(\lambda_i = \lambda\).

**Lemma 70.** Let \(\alpha\) be a simple root not in \(S^p\). Suppose \((\gamma, \alpha) = 0\) then Assertion (4) or Assertion (5) holds.

*Proof.* First assume that \(\alpha\) does not belong to the support of \(\gamma\). Then \(\alpha\) is orthogonal to every simple root in the support of \(\gamma\). Let \(v_\gamma \in V(\lambda)\).

It follows that \(v_{s_\alpha \gamma} = v_\gamma\) in \(V\) if and only if \((\lambda, \alpha) = 0\). If \(v_\gamma, v_\lambda \neq 0\) then Assertion (4) holds whenever \((\lambda, \alpha) = 0\). If \(v_\gamma, v_\lambda = 0\) then \(v_\gamma = X_{-\gamma} v_\lambda\) and there exists \(\lambda_j \neq \lambda\) such that and \(0 \neq v_\gamma \cdot v_{\lambda_j} = v_\lambda \cdot X_{-\gamma} v_{\lambda_j}\) whence Assertion (4) whenever \((\lambda, \alpha) = 0\). Let now \((\lambda, \alpha) \neq 0\). Note that \(X_\alpha v_\gamma = 0\) in \(V\) since \(\alpha\) does not belong to the support of \(\gamma\). Then \(v_{s_\alpha \gamma} = X^{r_\alpha} v_\gamma\) with \(r = (\lambda - \gamma, \alpha^\vee) = (\lambda, \alpha^\vee)\). Further \(v_{s_\alpha \gamma} \notin g/v_\lambda\); Assertion (5) thus holds with \(\lambda_i = \lambda\).

Assume now that \(\alpha\) lies in the support of \(\gamma\). Then by Lemma 33 \(\gamma - \alpha\) has to be a root; the type \(F_4\) is easily ruled out. More precisely \(\gamma\) is a root of type \(B_n\) or \(C_n\). Further in type \(B_n\), we can choose \(v_\gamma = X_{-\gamma} v_\lambda\) whereas \(v_\gamma \in V(\lambda)\) \(g/v_\lambda\) in type \(C_n\) along with \((\lambda, \alpha) = 0\) in both cases. In the first situation, \(v_{s_\alpha \gamma} = X_{-\gamma - \alpha} v_\lambda\) and there exists \(\lambda_i \neq \lambda\) non-orthogonal to \(\gamma\). In type \(B_n\), we then have \(0 \neq v_{s_\alpha \gamma} \cdot v_{\lambda_i} = X_{-\gamma - \alpha} v_{\lambda_i} \cdot v_\lambda\) whence Assertion (4). In type \(C_n\), Assertion (4) holds with \(\lambda_i = \lambda\).

If \(\gamma = \alpha\) then clearly there exists \(\lambda_D\) such that \(v_\gamma \cdot v_{\lambda_D}\) is not trivial and annihilated by \(X_{-\alpha}\) since \(X_{-\alpha} v_\gamma = 0\) in \(V\) and \((\lambda_D, \alpha^\vee) = 1\).

**References**

[A] Akhiezer, D., *Equivariant completions of homogeneous algebraic varieties by homogeneous divisors*, Ann. Global Anal. Geom. 1 (1983), no. 1, 49–78.

[AB] Alexeev, V. and Brion, M., *Moduli of affine schemes with reductive group action*, J. Algebraic Geom., 14 (2005), no. 1, 83–117.

[Bo] Bourbaki, N., *Éléments de mathématique. Groupes et Algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: Systèmes de racines*. Actualités Scientifiques et Industrielles, No. 1337 Hermann, Paris, 1968.

[Bra] Bravi, P., *Wonderful varieties of type E*, Represent. Theory 11 (2007), 174–191.

[BC1] Bravi, P. and Cupit-Foutou S., *Equivariant deformations of the affine multicone over a flag variety*, Adv. Math. 217 (2008), 2800–2821.

[BC2] Bravi, P. and Cupit-Foutou S., *Classification of strict wonderful varieties*, preprint arXiv:math.AG/08062263.

[BL] Bravi, P. and Luna D., *An introduction to wonderful varieties with many examples of type F_4*, preprint arXiv:math.0812.2340.

[BP] Bravi, P. and Pezzini, G., *Wonderful varieties of type D*, Represent. Theory 9 (2005), 578–637.
[Bri1] Brion, M., *On spherical varieties of rank one*, CMS Conf. Proc. **10** (1989), 31–41.

[Bri2] Brion, M., *Groupe de Picard*, Duke Math. Journal, Volume **58**, Number 2 (1989), 397–424.

[Bri3] Brion, M., *Variétés sphériques*, Notes de la session de la S. M. F. “Opérations hamiltoniennes et opérations de groupes algébriques”, Grenoble, 1997, 1–60.

[Bri4] Brion, M., *The total coordinate ring of a wonderful variety*, J. Algebra **313** (2007), 61–99.

[Ca] Camus, R., *Variétés sphériques affines lisses*, Ph. D. thesis, Grenoble, 2001; available at www-fourier.ujf-grenoble.fr.

[DP] De Concini, C. and Procesi, C., *Complete symmetric varieties*, Invariant theory (Montecatini, 1982), Lecture Notes in Math., 996, Springer, Berlin, 1983, 1–44.

[F] Foschi, A., *Variétés maginifiques et polytopes moment*, PhD thesis, Institut Fourier, Universit J. Fourier, Grenoble, 1998.

[H] Hoschild, G., *Cohomology of algebraic linear groups*, Illinois J. Math, **5** (1961), 492–519.

[J] Jansou, S., *Déformations invariantes des cônes de vecteurs primitifs*, Math. Ann. **338** (2007), 627–647.

[K] Knop, F., *The Luna-Vust theory of spherical embeddings*, Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989), 225–249, Manoj Prakashan, Madras, 1991.

[K] Knop, F., *Automorphisms, root systems, and compactifications of homogeneous varieties*, J. Amer. Math. Soc. **9** (1996), 153–174.

[Lo] Losev, I.V., *Uniqueness property for spherical homogeneous spaces*, Duke Math. Journal, Volume **147**, Number 2 (2009), 315–343.

[Lu1] Luna, D., *Toute variété magnifique est sphérique*, Transform. Groups **1** (1996), 249–258.

[Lu2] Luna, D., *Grosses cellules pour les variétés maginifiques*, in Algebraic Groups and Lie Groups, ed. by G. I. Lehrer, Australian Math. Soc. Lecture, Series **9** (1997), 267–280.

[Lu3] Luna, D., *Variétés sphériques de type A*, Inst. Hautes Études Sci. Publ. Math. **94** (2001), 161–226.

[Lu4] Luna, D., *La variété magnifique modèles*, Journal of Algebra **313**, (2007), 292–319.

[S] Sernesi, E., Deformation of algebraic schemes, Grundlehren der mathematischen Wissenschaften **334**, Springer-Verlag, Berlin, 2006.

[W] Wasserman, B., *Wonderful varieties of rank two*, Transform. Groups **1** (1996), 375–403.

Mathematisches Institut, Universität zu Köln, Weyertal Str. 86–90, 50931 Köln, Germany

E-mail address: scupit@mi.uni-koeln.de