Partitioning a graph into defensive $k$-alliances

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Abstract

A defensive $k$-alliance in a graph is a set $S$ of vertices with the property that every vertex in $S$ has at least $k$ more neighbors in $S$ than it has outside of $S$. A defensive $k$-alliance $S$ is called global if it forms a dominating set. In this paper we study the problem of partitioning the vertex set of a graph into (global) defensive $k$-alliances. The (global) defensive $k$-alliance partition number of a graph $\Gamma = (V,E)$, $(\psi_{gd}^k(\Gamma))$, is defined to be the maximum number of sets in a partition of $V$ such that each set is a (global) defensive $k$-alliance. We obtain tight bounds on $\psi_{gd}^k(\Gamma)$ and $\psi_{gd}^{gd}(\Gamma)$ in terms of several parameters of the graph including the order, size, maximum and minimum degree, the algebraic connectivity and the isoperimetric number. Moreover, we study the close relationships that exist among partitions of $\Gamma_1 \times \Gamma_2$ into (global) defensive $(k_1 + k_2)$-alliances and partitions of $\Gamma_i$ into (global) defensive $k_i$-alliances, $i \in \{1, 2\}$.
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1 Introduction

Since (defensive, offensive and dual) alliances in graph were first introduced by P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi [10], several authors have studied their mathematical properties [1–3, 6–8, 13–19, 21–23]. We are interested in a generalization of defensive alliances, called $k$-alliances, introduced by K. H. Shafique and R. D. Dutton in [17, 18]. We focus our attention in the problem of partitioning the vertex set of a graph into defensive $k$-alliances. This problem has been previously studied by K. H. Shafique and R. D. Dutton [19, 20] and the particular case $k = -1$ has been studied by L. Eroh and R. Gera [4, 5] and by T. W. Haynes and J. A. Lachniet [9].

We begin by stating the terminology used. Throughout this article, $\Gamma = (V, E)$ denotes a simple graph of order $|V| = n$ and size $|E| = m$. We denote two adjacent vertices $u$ and $v$ by $u \sim v$, the degree of a vertex $v \in V$ by $\delta(v)$, the minimum degree by $\delta$ and the maximum degree by $\Delta$.

For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors $v$ has in $X$: $N_X(v) := \{u \in X : u \sim v\}$, and the degree of $v$ in $X$ will be denoted by $\delta_X(v) = |N_X(v)|$. The subgraph induced by $S \subseteq V$ will be denoted by $\langle S \rangle$ and the complement of the set $S$ in $V$ will be denoted by $\bar{S}$.

A nonempty set $S \subseteq V$ is a defensive $k$-alliance in $\Gamma = (V, E)$, $k \in \{-\Delta, \ldots, \Delta\}$, if for every $v \in S$,

$$\delta_S(v) \geq \delta_S(v) + k. \quad (1)$$

Notice that (1) is equivalent to

$$\delta(v) \geq 2\delta_S(v) + k.$$

For example, if $k > 1$, the star graph $K_{1,t}$ has no defensive $k$-alliances and every set composed by two adjacent vertices in a cubic graph is a defensive $(-1)$-alliance. For graphs having defensive $k$-alliances, the defensive $k$-alliance number of $\Gamma$, denoted by $a_d^k(\Gamma)$, is defined as the minimum cardinality of a defensive $k$-alliance in $\Gamma$. Notice that

$$a_d^{k+1}(\Gamma) \geq a_d^k(\Gamma).$$
For the study of the mathematical properties of $a^d_k(\Gamma)$ we cite [15].

A set $S \subset V$ is a dominating set in $\Gamma = (V,E)$ if for every vertex $u \in S$, $\delta_S(u) > 0$ (every vertex in $S$ is adjacent to at least one vertex in $S$). The domination number of $\Gamma$, denoted by $\gamma(\Gamma)$, is the minimum cardinality of a dominating set in $\Gamma$.

A defensive $k$-alliance $S$ is called global if it forms a dominating set. For graphs having global defensive $k$-alliances, the global defensive $k$-alliance number of $\Gamma$, denoted by $\gamma^d_k(\Gamma)$, is the minimum cardinality of a global defensive $k$-alliance in $\Gamma$. Clearly,

$$\gamma^d_{k+1}(\Gamma) \geq \gamma^d_k(\Gamma) \geq \gamma(\Gamma) \quad \text{and} \quad \gamma^d_k(\Gamma) \geq a^d_k(\Gamma).$$

For the study of the mathematical properties of $\gamma^d_k(\Gamma)$ we cite [16].

The (global) defensive $k$-alliance partition number of $\Gamma$, $(\psi^g_d(k)(\Gamma))$, $k \in \{-\Delta, \ldots, \delta\}$, is defined to be the maximum number of sets in a partition of $V(\Gamma)$ such that each set is a (global) defensive $k$-alliance. Extreme cases are $\psi^g(\Gamma) = n$, where each set composed of one vertex is a defensive $(-\Delta)$-alliance, and $\psi^g(\Gamma) = 1$ for the case of a connected $\delta$-regular graph where $V(\Gamma)$ is the only defensive $\delta$-alliance. A graph $\Gamma$ is partitionable into (global) defensive $k$-alliances if $(\psi^g_d(\Gamma) \geq 2) \psi^g_d(\Gamma) \geq 2$. Hereafter we will say that $\Pi_r(\Gamma) = \{V_1, V_2, \ldots, V_r\}$ is a partition of $\Gamma$ into $r$ (global) defensive $k$-alliances.

Notice that if every vertex of $\Gamma$ has even degree and $k$ is odd, $k = 2l - 1$, then every (global) defensive $(2l - 1)$-alliance in $\Gamma$ is a (global) defensive $(2l)$-alliance and vice versa. Hence, in such a case, $a^d_{2l-1}(\Gamma) = a^d_{2l}(\Gamma)$, $\Gamma^d_{2l-1}(\Gamma) = \gamma^d_{2l}(\Gamma)$, and $\psi^d_{2l-1}(\Gamma) = \psi^d_{2l}(\Gamma)$.

Analogously, if every vertex of $\Gamma$ has odd degree and $k$ is even, $k = 2l$, then every defensive $(2l)$-alliance in $\Gamma$ is a defensive $(2l + 1)$-alliance and vice versa. Hence, in such a case, $a^d_{2l}(\Gamma) = a^d_{2l+1}(\Gamma)$, $\gamma^d_{2l}(\Gamma) = \gamma^d_{2l+1}(\Gamma)$, $\psi^d_{2l}(\Gamma) = \psi^d_{2l+1}(\Gamma)$, and $\psi^d_{2l}(\Gamma) = \psi^d_{2l+1}(\Gamma)$.

### 2 Partitioning a graph into defensive $k$-alliances

**Example 1.** Let $k$ and $r$ be integers such that $r > 1$ and $r + k > 0$ and let $\mathcal{H}$ be a family of graphs whose vertex set is $V = \bigcup_{i=1}^r V_i$ where, for every $V_i$, $\langle V_i \rangle \cong K_{r+k}$ and $\delta_{V_j}(v) = 1$, for every $v \in V_i$ and $j \neq i$. Notice that $\{V_1, V_2, \ldots, V_r\}$ is a partition of the graphs belonging to $\mathcal{H}$ into $r$ global defensive $k$-alliances. A particular family of graphs included in $\mathcal{H}$ is $K_{r+k} \times K_r$. 


Hereafter, \( \mathcal{H} \) will denote the family of graphs defined in the above example.

From the following relation between the defensive \( k \)-alliance number, \( a^d_k(\Gamma) \), and \( \psi^d_k(\Gamma) \) we obtain that lower bounds on \( a^d_k(\Gamma) \) lead to upper bounds on \( \psi^d_k(\Gamma) \):

\[
a^d_k(\Gamma) \psi^d_k(\Gamma) \leq n. \tag{2}
\]

For instance, it was shown in [15] that

\[
a^d_k(\Gamma) \geq \left\lceil \frac{\delta + k + 2}{2} \right\rceil. \tag{3}
\]

An example of equality in the above bound is provided by the graphs belonging to the family \( \mathcal{H} \), for which we obtain \( a^d_k(\Gamma) = r + k \).

By (2) and (3) we obtain the following bound,

\[
\psi^d_k(\Gamma) \leq \begin{cases} 
\left\lfloor \frac{2n}{\delta + k + 2} \right\rfloor, & \delta + k \text{ even} \\
\left\lfloor \frac{2n}{\delta + k + 3} \right\rfloor, & \delta + k \text{ odd}.
\end{cases}
\]

This bound gives the exact value of \( \psi^d_k(\Gamma) \), for instance, for every \( \Gamma \in \mathcal{H} \), where \( \psi^d_k(\Gamma) = r \), and in the following cases: \( \psi^d_{-1}(K_4 \times C_4) = 5 \), \( \psi^d_0(K_3 \times C_4) = \psi^d_{-1}(K_2 \times C_4) = 4 \) and \( \psi^d_1(K_2 \times C_4) = 2 \).

Analogously, for global alliances we have

\[
\gamma^d_k(\Gamma) \psi^gd_k(\Gamma) \leq n. \tag{4}
\]

One example of bounds on \( \gamma^d_k(\Gamma) \) is the following, obtained in [16],

\[
\gamma^d_k(\Gamma) \geq \left\lceil \frac{n}{\frac{\Delta - k}{2}} + 1 \right\rceil. \tag{5}
\]

For the graphs in \( \mathcal{H} \), the above bound gives the exact value \( \gamma^d_k(\Gamma) = r + k \). Thus, the bound obtained by combining (4) and (5),

\[
\psi^gd_k(\Gamma) \leq \left\lfloor \frac{\Delta - k}{2} \right\rfloor + 1,
\]

leads to the exact value of \( \psi^gd_k(\Gamma) = r \) for every \( \Gamma \in \mathcal{H} \). Even so, this bound can be improved.
Theorem 2. For every graph \(\Gamma\) partitionable into global defensive \(k\)-alliances,

\[
\psi_k^{gd}(\Gamma) \leq \left\lfloor \sqrt{\frac{k^2}{2} + 4n - k} \right\rfloor,
\]
\[
\psi_k(\Gamma) \leq \left\lfloor \frac{\delta - k + 2}{2} \right\rfloor.
\]

Proof. Since, every \(V_i \in \Pi_r(\Gamma)\) is a dominating set, we have that for every \(v \in V_i\), \(\delta_{V_i}(v) \geq r - 1\). Thus, the bounds are obtained as follow.

(i) \(|V_i| - 1 \geq \delta_{V_i}(v) \geq \delta_{V_i}(v) + k \geq r - 1 + k\), so \(n = \sum_{i=1}^{r} |V_i| \geq r(r + k)\). By solving the inequality \(r^2 + kr - n \leq 0\) we obtain the result.

(ii) Taking \(v \in V_i\) as a vertex of minimum degree we obtain the result from \(\delta = \delta(v) \geq 2\delta_{V_i}(v) + k \geq 2(r - 1) + k\).

Remark 3. For every \(k \in \{1 - \delta, \ldots, \delta\}\), if \(\psi_k^{gd}(\Gamma) \geq 2\), then

\[
\gamma_k^{d}(\Gamma) + \psi_k^{gd}(\Gamma) \leq \frac{n + 4}{2}.
\]

Proof. By (4) we have \(\gamma_k^{d}(\Gamma) + \psi_k^{gd}(\Gamma) \leq \frac{n + \left(\psi_k^{gd}(\Gamma)\right)^2}{\psi_k^{gd}(\Gamma)}\). On the other hand, if \(k \in \{1 - \delta, \ldots, \delta\}\), then \(\gamma_k^{d}(\Gamma) \geq 2\). Moreover, if \(\psi_k^{gd}(\Gamma) \geq 2\), then \(\gamma_k^{d}(\Gamma) \leq \frac{n}{2}\). So, \(2 \leq \psi_k^{gd}(\Gamma) \leq \frac{n}{\gamma_k^{d}(\Gamma)} \leq \frac{n}{2}\). As a consequence, the result is obtained as follow,

\[
\max_{2 \leq r \leq n} \left\{ \frac{n + x^2}{x} \right\} = \max \left\{ \frac{n + 4}{2}, \frac{n + \left(\gamma_k^{d}(\Gamma)\right)^2}{\gamma_k^{d}(\Gamma)} \right\} = \frac{n + 4}{2}.
\]

Example of equality in above bound is \(\gamma_{d-1}(C_4 \times K_2) + \psi_{d-1}^{gd}(C_4 \times K_2) = 6\).

Theorem 4. Let \(C_{(r,k)}^{gd}(\Gamma)\) be the minimum number of edges having its endpoints in different sets of a partition of \(\Gamma\) into \(r \geq 2\) global defensive \(k\)-alliances. Then
(i) \( C_{(r,k)}^{gd}(\Gamma) \geq \frac{1}{2} r(r-1) \gamma_{k}^{d}(\Gamma) \),

(ii) \( C_{(r,k)}^{gd}(\Gamma) \geq \frac{1}{2} r(r-1)(r+k) \),

(iii) \( C_{(r,k)}^{gd}(\Gamma) \leq \frac{2m-nk}{4} \).

(iv) \( C_{(r,k)}^{gd}(\Gamma) = \frac{1}{2} r(r-1) \gamma_{k}^{d}(\Gamma) = \frac{1}{2} r(r-1)(r+k) = \frac{2m-nk}{4} \) if and only if \( \Gamma \in \mathcal{H} \).

Proof. Let \( x = \min_{V_i \in \Pi_r(\Gamma)} |V_i| \). From the fact that every set of \( \Pi_r(\Gamma) \) is a dominating set, we obtain that the number of edges adjacent to \( v \in V_i \) with one endpoint in \( \cup_{j=i+1}^r V_j \) is bounded by \( \sum_{j=i+1}^r \delta_{V_j}(v) \geq r-i \). Therefore,

\[
C_{(r,k)}^{gd}(\Gamma) \geq \sum_{i=1}^{r-1} (r-i)|V_i| \geq x \sum_{i=1}^{r-1} (r-i) = \frac{x}{2} r(r-1). \tag{6}
\]

Since every \( V_i \in \Pi_r(\Gamma) \) is a global defensive \( k \)-alliance, we have \( x \geq r+k \) and \( x \geq \gamma_{k}^{d}(\Gamma) \), as a consequence, (i) and (ii) follow.

Proof of (iii). In order to obtain the upper bound we note that the number of edges in \( \Gamma \) with one endpoint in \( V_i \) and the other endpoint in \( V_j \) is \( C(V_i, V_j) = \sum_{v \in V_i} \delta_{V_j}(v) = \sum_{v \in V_j} \delta_{V_i}(v) \). Hence,

\[
2m = \sum_{i=1}^r \sum_{v \in V_i} \delta(v) \geq 2 \sum_{i=1}^r \sum_{v \in V_i} \delta_{V_j}(v) + k \sum_{i=1}^r |V_i| \\
= 2 \sum_{i=1}^r \sum_{v \in V_i} \sum_{j=1,j \neq i}^r \delta_{V_j}(v) + kn \\
= 2 \sum_{i=1}^r \sum_{j=1,j \neq i}^r \delta_{V_j}(v) + kn \\
= 2 \sum_{i=1}^r \sum_{j=1,j \neq i}^r C(V_i, V_j) + nk \\
= 4C_{(r,k)}^{gd}(\Gamma) + nk.
\]

Proof of (iv). \((\Rightarrow)\) If for some \( V_i \in \Pi_r(\Gamma) \) there exists \( v \in V_i \) such that \( \delta_{V_i}(v) > \delta_{V_i}(v) + k \), then, by analogy to the proof of (iii) we obtain
Therefore, if \( C_{(r,k)}^{gd}(\Gamma) = \frac{2m-nk}{4} \), then for every \( V_i \in \Pi_r(\Gamma) \), and for every \( v \in V_i \), we have
\[
\delta_{V_i}(v) = \delta_{V_i'}(v) + k. \tag{7}
\]
Moreover, if for some \( V_i \in \Pi_r(\Gamma) \) there exists \( v \in V_i \) such that \( \sum_{j \neq i} \delta_{V_i}(v) > r - 1 \), then, by analogy to the proof of (i) and (ii) we obtain \( C_{(r,k)}^{gd}(\Gamma) > \frac{1}{2}r(r-1)\gamma^d_k(\Gamma) \) and \( C_{(r,k)}^{gd}(\Gamma) > \frac{1}{2}r(r-1)(r+k) \). Therefore, if \( C_{(r,k)}^{gd}(\Gamma) = \frac{1}{2}r(r-1)\gamma^d_k(\Gamma) = \frac{1}{2}r(r-1)(r+k) \), then for every \( V_i \in \Pi_r(\Gamma) \), and for every \( v \in V_i \), we have
\[
\delta_{V_i}(v) = \sum_{j \neq i} \delta_{V_i}(v) = r - 1. \tag{8}
\]
So, by (7) and (8) we obtain that for every \( V_i \in \Pi_r(\Gamma) \), \( \langle V_i \rangle \) is regular of degree \( r + k - 1 \). Thus, \( \Gamma \) is a regular graph of degree \( 2(r-1) + k \) and, by \( \frac{1}{2}r(r-1)\gamma^d_k(\Gamma) = \frac{1}{2}r(r-1)(r+k) = \frac{2m-nk}{4} \) we have \( n(\Gamma) = r(r+k) \) and \( \gamma^d_k(\Gamma) = r + k \). Hence, \( |V_i| = r + k \), so \( \langle V_i \rangle \cong K_{r+k} \). Moreover, as every \( V_j \in \Pi_r(\Gamma) \) is a dominating set, by (8) we have \( \delta_{V_j}(v) = 1 \), for every \( v \in V_i \), \( i \neq j \). Therefore, \( \Gamma \in \mathcal{H} \). (\( \Leftarrow \)) The result is immediate.

By (6) and Theorem 4 (iii) we obtain the following result.

**Corollary 5.** For every graph \( \Gamma \) partitionable into \( r \) global defensive \( k \)-alliances of equal cardinality, \( r \leq \frac{2(m+n)-kn}{2n} \).

A family of graphs that achieve equality for Corollary 5 is the family \( \mathcal{H} \) defined in Example 1.

By Theorem 4 and (3) we obtain the following two necessary conditions for the existence of a partition of a graph into \( r \) global defensive \( k \)-alliances.

**Corollary 6.** If for a graph \( \Gamma \), \( k > \frac{2m-r(r-1)(\delta+2)}{n+r(r-1)} \) or \( k > \frac{2(m-r^2(r-1))}{n+2r(r-1)} \), the \( \Gamma \) cannot be partitioned into \( r \) global defensive \( k \)-alliances.

By the above corollary we conclude, for instance, that the 3-cube graph cannot be partitioned into \( r > 2 \) global defensive \( k \)-alliances.

**Remark 7.** The size of the subgraph induced by a set belonging to a partition of \( \Gamma \) into \( r \) global defensive \( k \)-alliances is bounded below by \( \frac{1}{2}\gamma^d_k(\Gamma)(r+k-1) \).
Proof. The result follows from the fact that for every 
\[ V_i \in \Pi_r(\Gamma), \sum_{v \in V_i} \delta_{V_i}(v) \geq \]
\[ ((r - 1) + k)|V_i| \geq (r - 1 + k)\gamma_k(\Gamma). \]

The above bound is tight as we can check by taking \( \Gamma \in \mathcal{H} \).

2.1 Isoperimetric number, bisection and \( k \)-alliances
The isoperimetric number of \( \Gamma \) is defined as
\[ i(\Gamma) := \min_{S \subset \mathcal{V}(\Gamma) : |S| \leq \frac{n}{2}} \left\{ \frac{\sum_{v \in S} \delta_S(v)}{|S|} \right\}. \]

As a consequence of Theorem 4 (iii) we obtain the following result.

**Corollary 8.** If there exists a partition \( \Pi_r \) of \( \Gamma \) into \( r \geq 2 \) global defensive \( k \)-alliances such that, for every \( V_i \in \Pi_r, |V_i| \leq \frac{n}{2} \), then
\[ i(\Gamma) \leq \frac{2m - nk}{2n}. \]

**Proof.** For every \( V_i \in \Pi_r \) we have
\[ |V_i|i(\Gamma) \leq \sum_{v \in V_i} \delta_{V_i}(v) = \sum_{v \in V_i} \sum_{j=1}^{r} \delta_{V_j}(v). \]

Hence,
\[ n i(\Gamma) = i(\Gamma) \sum_{i=1}^{r} |V_i| \leq \sum_{i=1}^{r} \sum_{v \in V_i} \sum_{j=1, j \neq i}^{r} \delta_{V_j}(v) = 2C_{(r,k)}^{gd}(\Gamma) \leq \frac{2m - nk}{2}. \]

Example of equality in above bound is the graph \( \Gamma = C_3 \times C_3 \) for \( k = 0 \). That is, \( C_3 \times C_3 \) can be partitioned into \( r = 3 \) global defensive 0-alliances of cardinality 3, moreover, \( i(C_3 \times C_3) = 2 \). Other example is the 3-cube graph \( \Gamma = C_4 \times K_2 \), for \( k = 1 \). In this case each copy of the cycle \( C_4 \) is a global defensive 1-alliance and \( i(C_4 \times K_2) = 1 \).

Notice that if \( i(\Gamma) > \frac{2m - nk}{2m} \), then \( \Gamma \) cannot be partitioned into \( r \geq 2 \) global defensive \( k \)-alliances with the condition that the cardinality of every set in the partition is at most \( \frac{n}{2} \). One example of this is the graph \( \Gamma = C_3 \times C_3 \) for \( k \geq 1 \).
Theorem 9. For any graph $\Gamma$,

(i) if $\Gamma$ is partitionable into global defensive $k$-alliances, then

$$\psi_{gd}^k(\Gamma) \leq \Delta + 1 - i(\Gamma) - k,$$

(ii) if $\Gamma$ is partitionable into defensive $k$-alliances, then

$$a_d^k(\Gamma) \geq i(\Gamma) + k + 1.$$

Proof.

(i) Let $\Pi_r(\Gamma)$ be a partition of $\Gamma$ into $r \geq 2$ global defensive $k$-alliances. Then, there exists $V_i \in \Pi_r(\Gamma)$ such that $|V_i| \leq \frac{n}{r}$. Hence, $|V_i|i(\Gamma) \leq \sum_{v \in V_i} \delta_{V_i}(v) \leq \sum_{v \in V_i} (\delta(v) - r + 1) \leq |V_i|((\Delta - r + 1) - k)$. Thus, $r \leq \Delta + 1 - i(\Gamma) - k$.

(ii) If $\psi_{gd}^k(\Gamma) \geq 2$, then there exists a defensive $k$-alliance $S$ such that $|S| \leq \frac{n}{r}$. Therefore, $|S|i(\Gamma) \leq \sum_{v \in S} \delta_S(v) \leq \sum_{v \in S} (\delta_S(v) - k) \leq |S|(|S| - 1) - k|S|$. Thus, the result follows.

The following relation between the algebraic connectivity and the isoperimetric number of a graph was shown by Mohar in [12]: $i(\Gamma) \geq \frac{\mu}{2}$.

Corollary 10. For any graph $\Gamma$,

(i) if $\Gamma$ is partitionable into global defensive $k$-alliances, then

$$\psi_{gd}^k(\Gamma) \leq \left\lfloor \Delta + 1 - \frac{\mu}{2} - k \right\rfloor,$$

(ii) if $\Gamma$ is partitionable into defensive $k$-alliances, then

$$a_d^k(\Gamma) \geq \left\lceil \frac{\mu + 2(k + 1)}{2} \right\rceil.$$
Example of equality in above bounds is the graph \( \Gamma = C_3 \times C_3 \) for \( k = 0 \), in this case \( \mu = 3 \).

From above corollary, we emphasize that if \( \mu > 2(\Delta - 1 - k) \), then \( \Gamma \) cannot be partitioned into global defensive \( k \)-alliances. For instance, we conclude that \( \Gamma = C_3 \times C_3 \) cannot be partitioned into global defensive \( k \)-alliances for \( k > 1 \). Moreover, by Corollary 10 (ii) we conclude, if \( a^d_k(\Gamma) < \left\lceil \frac{\mu + 2(k+1)}{2} \right\rceil \), then \( \Gamma \) cannot be partitioned into defensive \( k \)-alliances.

A bisection of \( \Gamma \) is a 2-partition \( \{X, Y\} \) of the vertex set \( V(\Gamma) \) in which \( |X| = |Y| \) or \( |X| = |Y| + 1 \). The bisection problem is to find a bisection for which \( \sum_{v \in X} \delta_Y(v) \) is as small as possible. The bipartition width, \( bw(\Gamma) \), is defined as

\[
bw(\Gamma) := \min_{X \subset V(\Gamma), |X| = \left\lfloor \frac{n}{2} \right\rfloor} \left\{ \sum_{v \in X} \delta_X(v) \right\}.
\]

It was shown by Merris [11] and Mohar [12] that

\[
bw(\Gamma) \geq \begin{cases} 
\left\lfloor \frac{nu}{4} \right\rfloor & \text{if } n \text{ is even;} \\
\left\lfloor \frac{(n^2-1)\mu}{4n} \right\rfloor & \text{if } n \text{ is odd.}
\end{cases}
\]

We are interested in the bisection of a graph into global defensive \( k \)-alliances, i.e., the bisection \( \{X, Y\} \) of \( V \) such that \( X \) and \( Y \) are global defensive \( k \)-alliances. An example of bisection into global defensive \( k \)-alliances is obtained for the family of hypercube graphs \( Q_{t+1} = Q_t \times K_2 \), by taking \( \{X, Y\} \) such that \( \langle X \rangle \cong Q_t \cong \langle Y \rangle \).

By Theorem 4 (iii) and the above bound we obtain the following result.

**Corollary 11.** If \( \left\lfloor \frac{2m-nk}{4} \right\rfloor < \left\lfloor \frac{nu}{4} \right\rfloor \), for \( n \) even, or \( \left\lfloor \frac{2m-nk}{4} \right\rfloor < \left\lfloor \frac{(n^2-1)\mu}{4n} \right\rfloor \), for \( n \) odd, then \( \Gamma \) cannot be bisectioned into global defensive \( k \)-alliances.

For example, according to Corollary 11 we can conclude that, for \( k > 0 \), the graph \( C_3 \times C_3 \) cannot be bisectioned into global defensive \( k \)-alliances.

## 3 Partitioning \( \Gamma_1 \times \Gamma_2 \) into (global) defensive \( k \)-alliances

In Subsection 3.1 we will discuss the close relationships that exist among \( \psi^d_{k_1+k_2}(\Gamma_1 \times \Gamma_2) \) and \( \psi^d_{k_i}(\Gamma_i) \), \( i \in \{1, 2\} \). Obviously, we begin with the study
of the relationship between \( a_{k_1+k_2}^d(\Gamma_1 \times \Gamma_2) \) and \( a_{k_i}^d(\Gamma_i), \ i \in \{1,2\} \). The case of global alliances will be studied in Subsection 3.2.

### 3.1 Partitioning \( \Gamma_1 \times \Gamma_2 \) into defensive \( k \)-alliances

**Theorem 12.** For any graphs \( \Gamma_1 \) and \( \Gamma_2 \),

(i) if \( \Gamma_i \) contains a defensive \( k_i \)-alliance, \( i \in \{1,2\} \), then \( \Gamma_1 \times \Gamma_2 \) contains a defensive \((k_1+k_2)\)-alliance and

\[
a_{k_1+k_2}^d(\Gamma_1 \times \Gamma_2) \leq a_{k_1}^d(\Gamma_1)a_{k_2}^d(\Gamma_2),
\]

(ii) if there exists a partition of \( \Gamma_i \) into defensive \( k_i \)-alliances, \( i \in \{1,2\} \), then there exists a partition of \( \Gamma_1 \times \Gamma_2 \) into defensive \((k_1+k_2)\)-alliances and

\[
\psi_{k_1+k_2}^d(\Gamma_1 \times \Gamma_2) \geq \psi_{k_1}^d(\Gamma_1)\psi_{k_2}^d(\Gamma_2).
\]

**Proof.** Let \( S_i \) be a defensive \( k_i \)-alliance in \( \Gamma_i, \ i \in \{1,2\} \), and let \( X = S_1 \times S_2 \). Then for every \( x = (u,v) \in X \),

\[
\delta_X(x) = \delta_{S_1}(u) + \delta_{S_2}(v) \\
\geq (\delta_{S_1}(u) + k_1) + (\delta_{S_2}(v) + k_2) \\
= \delta_X(x) + k_1 + k_2.
\]

Thus, \( X \) is a defensive \((k_1+k_2)\)-alliance in \( \Gamma_1 \times \Gamma_2 \) and, as a consequence, (i) follows. Moreover, we conclude that every partition

\[
\Pi_{r_i}(\Gamma_i) = \{S_{1}^{(i)}, S_{2}^{(i)}, ..., S_{r_i}^{(i)}\}
\]

of \( \Gamma_i \) into \( r_i \) defensive \( k_i \)-alliances induces a partition of \( \Gamma_1 \times \Gamma_2 \) into \( r_1r_2 \) defensive \((k_1+k_2)\)-alliances:

\[
\Pi_{r_1r_2}(\Gamma_1 \times \Gamma_2) = \left\{ \begin{array}{c}
S_1^{(1)} \times S_1^{(2)} \quad \cdots \quad S_1^{(1)} \times S_{r_2}^{(2)} \\
S_2^{(1)} \times S_1^{(2)} \quad \cdots \quad S_2^{(1)} \times S_{r_2}^{(2)} \\
\vdots \quad \vdots \quad \vdots \\
S_{r_1}^{(1)} \times S_1^{(2)} \quad \cdots \quad S_{r_1}^{(1)} \times S_{r_2}^{(2)}
\end{array} \right\}.
\]

Therefore, (ii) follows. \( \square \)
In the particular case of the Petersen graph, \( P \), and the 3-cube graph, \( Q_3 \), we have \( a_{d-2}(P \times Q_3) = 4 = a_{d-1}(P)a_{d-1}(Q_3) \), \( \psi_{d-2}(P \times Q_3) = 20 = \psi_{d-1}(P)\psi_{d-1}(Q_3) \) and \( 16 = a_{d}^2(P \times Q_3) < a_{1}^d(P)a_{1}^d(Q_3) = 20, \) \( 5 = \psi_{d}^2(P \times Q_3) > \psi_{1}^d(P)\psi_{1}^d(Q_3) = 4. \)

An example where we cannot apply Theorem 12 (i) is the book graph \( \Gamma_1 \times \Gamma_2 = K_{1,4} \times K_2 \), for \( k_1 = 2 \) and \( k_2 = 0 \); the star graph \( \Gamma_1 = K_{1,4} \) does not contain defensive 2-alliances, although \( \Gamma_1 \times \Gamma_2 \) contains some of them and \( a^d_2(\Gamma_1 \times \Gamma_2) = 8. \)

We note that from Theorem 12 we obtain \( a_{2k}^d(\Gamma_1 \times \Gamma_2) \leq a_k^d(\Gamma_1)a_k^d(\Gamma_2) \) and \( \psi_{2k}^d(\Gamma_1 \times \Gamma_2) \geq \psi_k^d(\Gamma_1)\psi_k^d(\Gamma_2) \). Another interesting consequence of Theorem 12 is the following.

**Corollary 13.** Let \( \Gamma_1 \) and \( \Gamma_2 \) be two graphs of order \( n_1 \) and \( n_2 \) and maximum degree \( \Delta_1 \) and \( \Delta_2 \), respectively. Let \( s \in \mathbb{Z} \) such that \( \max\{\Delta_1, \Delta_2\} \leq s \leq \Delta_1 + \Delta_2 + k. \) Then

(i) \( a_{d-s}^d(\Gamma_1 \times \Gamma_2) \leq \min\{a_k^d(\Gamma_1), a_k^d(\Gamma_2)\}, \)

(ii) \( \psi_{d-s}^d(\Gamma_1 \times \Gamma_2) \geq \max\{n_2\psi_k^d(\Gamma_1), n_1\psi_k^d(\Gamma_2)\}. \)

As example of equalities we take \( \Gamma_1 = P, \Gamma_2 = Q_3, k = 1 \) and \( s = 3. \) In such a case, \( 4 = a_{d-2}(P \times Q_3) = \min\{a_1^d(P), a_1^d(Q_3)\} = \min\{5, 4\} \) and \( 20 = \psi_{d-2}(P \times Q_3) = \max\{8\psi_1^d(P), 10\psi_1^d(Q_3)\} = \max\{16, 20\}. \)

### 3.2 Partitioning \( \Gamma_1 \times \Gamma_2 \) into global defensive \( k \)-alliances

**Theorem 14.** Let \( \Pi_{r_i}(\Gamma_i) \) be a partition of a graph \( \Gamma_i \), of order \( n_i \), into \( r_i \geq 1 \) global defensive \( k_i \)-alliances, \( i \in \{1, 2\}, r_1 \leq r_2. \) Let \( x_i = \min_{X \in \Pi_{r_i}(\Gamma_i)} \{|X|\}. \) Then,

(i) \( \gamma_{d-k_1+k_2}^d(\Gamma_1 \times \Gamma_2) \leq \min \{x_1n_2, x_2n_1\}, \)

(ii) \( \psi_{d-k_1+k_2}^{gd}(\Gamma_1 \times \Gamma_2) \geq \max \left\{\psi_{k_1}^{gd}(\Gamma_1), \psi_{k_2}^{gd}(\Gamma_2)\right\}. \)

**Proof.** From the procedure showed in the proof of Theorem 12 we obtain that for every \( S_j^{(1)} \in \Pi_{r_1}(\Gamma_1) \) and every \( S_i^{(2)} \in \Pi_{r_2}(\Gamma_2) \), the sets \( M_j = S_j^{(1)} \times V_2 \) and \( N_i = V_1 \times S_i^{(2)} \) are defensive \((k_1 + k_2)\)-alliances in \( \Gamma_1 \times \Gamma_2 \). Moreover \( M_j \) and \( N_i \) are dominating sets in \( \Gamma_1 \times \Gamma_2 \). Thus, by taking \( S_j^{(1)} \) and \( S_i^{(2)} \) of
cardinality $x_1$ and $x_2$, respectively, we obtain $|M_j| = x_1n_2$ and $|N_i| = x_2n_1$, so (i) follows. Moreover, as $\{M_1, ..., M_{r_1}\}$ and $\{N_1, ..., N_{r_2}\}$ are partitions of $\Gamma_1 \times \Gamma_2$ into global defensive $(k_1 + k_2)$-alliances, (ii) follows.

**Corollary 15.** If $\Gamma_i$ is a graph of order $n_i$ such that $\psi_{gd}^{k_i}(\Gamma_i) \geq 1$, $i \in \{1, 2\}$, then

$$\gamma_{k_1+k_2}^{d}(\Gamma_1 \times \Gamma_2) \leq \frac{n_1n_2}{\max_{i \in \{1,2\}} \{\psi_{gd}^{k_i}(\Gamma_i)\}}.$$

**Theorem 16.** If $\Gamma_1$ contains a global defensive $k_1$-alliance, then for every $k_2 \in \{-\Delta_2, ..., \delta_2\}$, $\Gamma_1 \times \Gamma_2$ contains a global defensive $(k_1 + k_2)$-alliance and $\gamma_{k_1+k_2}^{d}(\Gamma_1 \times \Gamma_2) \leq \gamma_{k_1}^{d}(\Gamma_1)n_2$.

**Proof.** Following a similar procedure used in the proof of Theorem 14 (i) we deduce the result.

For the graph $\Gamma_1 \times \Gamma_2 = C_4 \times Q_3$, by taking $k_1 = 0$ and $k_2 = 1$, we obtain equalities in Theorem 14, Corollary 15 and Theorem 16.

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