Constructing sparse Davenport-Schinzel sequences
by hypergraph edge coloring

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Abstract

A sequence is called \( r \)-sparse if every contiguous subsequence of length \( r \) has no repeated letters. A \( DS(n, s) \)-sequence is a 2-sparse sequence with \( n \) distinct letters that avoids alternations of length \( s + 2 \). Pettie and Wellman (2018) asked whether there exist \( r \)-sparse \( DS(n, s) \)-sequences of length \( \Omega(sn^2) \) for \( s \geq n \) and \( r > 2 \), which would generalize a result of Roselle and Stanton (1971) for the case \( r = 2 \).

We construct \( r \)-sparse \( DS(n, s) \)-sequences of length \( \Omega(sn^2) \) for \( s \geq n \) and \( r > 2 \). Our construction uses linear hypergraph edge-coloring bounds. We also use the construction to generalize a result of Pettie and Wellman by proving that if \( s = \Omega(n^{1/t}(t-1)!)) \), then there are \( r \)-sparse \( DS(n, s) \)-sequences of length \( \Omega(n^2s/(t-1)!)) \) for all \( r \geq 2 \). In addition, we find related results about the lengths of sequences avoiding \((r, s)\)-formations.

1 Introduction

A Davenport-Schinzel sequence of order \( s \) is a sequence with no adjacent same letters that avoids alternations of length \( s + 2 \) \([3]\). A \( DS(n, s) \)-sequence is a Davenport-Schinzel sequence of order \( s \) with \( n \) distinct letters. The function \( \lambda_s(n) \) is defined as the maximum possible length of a \( DS(n, s) \)-sequence. Davenport-Schinzel sequences have a variety of applications and connections to other problems, including upper bounds on the maximum complexity of lower envelopes of sets of polynomials of bounded degree \([3]\), the maximum complexity of faces in arrangements of arcs \([20]\), the maximum number of edges in certain \( k \)-quasiplanar graphs \([3][7]\), extremal functions of forbidden 0–1 matrices \([11][15]\), and extremal functions of tuples stabbing interval chains \([6]\).
Most research on Davenport-Schinzel sequences has focused on when \( s \) is fixed. It is easy to see that \( \lambda_1(n) = n \) and \( \lambda_2(n) = 2n - 1 \) \cite{3}. Nivasch \cite{14} and Pettie \cite{16} proved that \( \lambda_3(n) = 2n\alpha(n) + O(n) \), while Agarwal, Sharir, and Shor \cite{1} proved that \( \lambda_4(n) = \Theta(n2^{o(n)}) \). Pettie \cite{16} and Nivasch \cite{14} proved that \( \lambda_5(n) = \theta(n^2\alpha(2\alpha(n))) \) and \( \lambda_s(n) = n2^{(1-o(1))\alpha(n)/t!} \) for all \( s \geq 6 \), where \( t = \lfloor \frac{n^2}{2} \rfloor \).

A more general upper bound from Davenport and Schinzel \cite{3} \cite{12} shows that \( \lambda_s(n) \leq s \left( \frac{n}{2} \right) + 1 \), even when \( s \) is not fixed. Roselle and Stanton \cite{19} constructed a family of sequences to prove that if \( s \geq n \), then \( \lambda_s(n) = \Theta(sn^2) \). For the case of \( s = n \), the coefficient of \( n^3 \) in their lower bound is \( 1/3 \), and it is an open problem \cite{18} to determine what is the actual coefficient between \( 1/3 \) and \( 1/2 \). Pettie and Wellman \cite{18} proved several bounds for when \( s \) is not fixed but smaller than linear in \( n \), including that if \( s = \Omega(n^{1/(t - 1)!}) \), then \( \lambda_s(n) \) is between \( \Omega(n^2s/(t - 1)!) \) and \( O(n^2s) \).

Call a sequence \( r \)-sparse if every contiguous subsequence of length \( r \) has all letters distinct. Let \( \lambda_s(n, r) \) be the maximum possible length of an \( r \)-sparse \( DS(n, s) \)-sequence. Klazar proved for fixed \( s, r, t \) that \( \lambda_s(n, r) = \Theta(\lambda_s(n, t)) \) for all \( t \geq r \geq 2 \) \cite{13}, but the proof does not work when \( s \) is not fixed. Pettie and Wellman \cite{18} asked whether Roselle and Stanton’s \( \Omega(sn^2) \) bound can be generalized to \( r \)-sparse \( DS(n, s) \)-sequences.

In this note, we construct \( r \)-sparse \( DS(n, s) \)-sequences of length \( \Omega(sn^2) \) for \( s \geq n \), where the constant in the bound depends on \( r \). Our construction uses Kahn’s asymptotic bound on linear hypergraph edge-coloring \cite{10}. As a corollary, we obtain that if \( s = \Omega(n^{1/(t - 1)!}) \), then there are \( r \)-sparse \( DS(n, s) \)-sequences of length \( \Omega(n^2s/(t - 1)! \) for all \( r \geq 2 \).

We also prove related results about \( (r, s) \)-formations. An \( (r, s) \)-formation is a concatenation of \( s \) permutations of \( r \) distinct letters. The function \( F_{r,s}(n) \) is defined as the maximum possible length of an \( r \)-sparse sequence with \( n \) distinct letters that avoids all \( (r, s) \)-formations. Similarly we define the function \( F_{r,s,q}(n) \) to be the maximum possible length of a \( q \)-sparse sequence with \( n \) distinct letters that avoids all \( (r, s) \)-formations.

Nivasch \cite{14} and Pettie \cite{17} found tight bounds on \( F_{r,s}(n) \) for all fixed \( r, s > 0 \), which are mostly on the same order as the bounds for \( \lambda_{s-1}(n) \). Upper bounds on \( (r, s) \)-formations have been used to find tight bounds on the extremal functions of several families of forbidden sequences \cite{13} \cite{8}, including a family of sequences used to bound the maximum number of edges in \( k \)-quasiplanar graphs in which every pair of edges intersect at most a constant number of times \cite{7} \cite{5}.

We show that \( F_{2,s,r}(n) = \theta(sn^2) \) for all \( s \geq n \) using the same family of sequences that we constructed for \( \lambda_s(n, r) \). The upper bound is from
We also prove that $F_{r,s,r}(n) = \theta(sn^r)$ for all $s \geq n^{r-1}$, where the constant in the bound again depends on $r$. Using the previous bound as the initial case, we generalize the construction of $r$-sparse $DS(n,s)$-sequences of length $\Omega(sn^2)$ to prove that $F_{r,s,q}(n) = \theta(sn^r)$ for all $s \geq n^{r-1}$ and $q \geq r$, where the constant in the bound depends on $r$ and $q$.

2 Iterated hypergraph edge coloring

A hypergraph is called linear if every pair of distinct edges have intersection size at most 1. A proper edge-coloring of a hypergraph $H$ is a coloring of the edges of $H$ so that no intersecting distinct edges have the same color.

Erdös, Faber, and Lovász conjectured that any linear hypergraph on $n$ vertices has a proper edge-coloring with at most $n$ colors. The conjecture was originally stated for vertex colorings of graphs [4], but Hindman observed that both versions were equivalent and proved the conjecture for $n \leq 10$ [9].

Chang and Lawler [2] proved an upper bound of $\lceil \frac{3n}{2} - 2 \rceil$, before Kahn proved the asymptotic upper bound of $n + o(n)$ [10].

Our construction of $r$-sparse $DS(n,s)$-sequences of length $\Omega(sn^2)$ is inductive. The initial case for $r = 2$ is a family of sequences that looks similar to the sequences in [19]. For the inductive step, we turn each sequence into a linear hypergraph, color its edges with the minimal possible number of colors, and then add those colors into the sequence as letters to increase the sparsity by 1. The next lemma is part of what makes the induction work.

**Lemma 2.1.** Let $H = (V,E)$ be a linear hypergraph with $n$ vertices and let $f : E \rightarrow C$ be a proper-edge coloring of $H$, where $C$ denotes the set of colors. If $H'$ is the hypergraph obtained from $H$ with vertex set equal to $V \cup C$ and with edge set consisting of every edge of the form $e \cup \{f(e)\}$ for $e \in E$, then $H'$ is also linear.

**Proof.** Let $x$ and $y$ be any two distinct edges of $H'$. Then $x = e \cup \{f(e)\}$ and $y = d \cup \{f(d)\}$ for some edges $d, e \in H$. If $f(e) = f(d)$, then $e \cap d = \emptyset$ since $f$ is a proper edge-coloring of $H$. Otherwise if $f(e) \neq f(d)$, then $|x \cap y| \leq 1$ since $H$ is linear. \[\blacksquare\]

In the next theorem for the case of $r = 3$, we could use Vizing’s Theorem instead of Kahn’s upper bound for the proper edge-coloring of $H$, since $H$ is just a graph when $r - 1 = 2$. This would eliminate the $o(x)$ from the proof.

**Theorem 2.2.** Fix integer $r \geq 2$ and real number $0 < c \leq 1$. Then $\lambda_s(n,r) = \theta(sn^2)$ for all $s \geq cn$, where the constant in the bound depends on $c$ and $r$. 

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Proof. Since the upper bound was already proved in [3] [12], it suffices to prove that \( \lambda_s(n, r) = \Omega(sn^2) \) for all \( s \geq cn \), where the constant in the bound depends on \( c \) and \( r \).

Roselle and Stanton already proved the result for \( r = 2 \) [19]. Our construction is inductive. Rather than using Roselle and Stanton’s construction for the initial case of \( r = 2 \), we use a slightly simpler construction with a worse constant in the \( \Omega(sn^2) \) bound.

Let \( x, t > 0 \) be two parameters that will be chosen at the end of the proof in terms of \( n, s, c \), and \( r \). Define \( S_2(x, t) \) to be the sequence obtained by starting with the empty sequence and then adding \( t \) copies of the subsequence \( ij \) for each \( 1 \leq i \leq x \) and \( i+1 \leq j \leq x \), for \( (i, j) \) in lexicographic order. For each \( i, j \) with \( 1 \leq i \leq x \) and \( i+1 \leq j \leq x \), we call the consecutive copies of the subsequence \( ij \) in \( S_2(x, t) \) a block. We call the set of adjacent blocks with \( i = a \) the block-row \( B_a \). There are a total of \( x - 1 \) block-rows, and each block-row has fewer than \( x \) blocks.

First observe that \( S_2(x, t) \) has length \( 2t(\binom{x}{2}) \). Moreover for any pair of distinct letters \( i, j \) in \( S_2(x, t) \), the length of an alternation on the letters \( i, j \) is less than \( 2x + 2t \) since the block-row containing the block on letters \( i, j \) contains an alternation on \( i, j \) of length at most \( 2t + 1 \), and all other block-rows contain alternations on \( i, j \) of length at most 2. Note also that \( S_2(x, t) \) is 2-sparse, but not 3-sparse, and \( S_2(x, t) \) has \( x \) distinct letters.

For every \( r \geq 3 \), we will construct \( S_r(x, t) \) so that \( S_r(x, t) \) has length \( rt(\binom{x}{2}) \) and any pair of distinct letters \( i, j \) in \( S_r(x, t) \) have alternation length less than \( 2x + 2t \). In addition, \( S_r(x, t) \) will be \( r \)-sparse but not \((r + 1)\)-sparse, and \( S_r(x, t) \) will have at most \( 2^{r-2}x + o(x) \) distinct letters.

Like \( S_2(x, t) \), the sequences \( S_r(x, t) \) for \( r \geq 3 \) also have blocks, where each block consists of a sequence of \( r \) distinct letters repeated \( t \) times. In order to construct \( S_r(x, t) \) from \( S_{r-1}(x, t) \), we treat each block in \( S_{r-1}(x, t) \) as an edge in a \((r - 1)\)-uniform hypergraph.

Specifically, \( H = (V, E) \) is the \((r - 1)\)-uniform hypergraph with vertex set equal to the letters of \( S_{r-1}(x, t) \) and edge set \( E \) with \( e \in E \) if and only if there is a block in \( S_{r-1}(x, t) \) on the letters \( e \). Note that in the case that \( r - 1 = 2 \), \( H \) is a graph by construction, so \( H \) is also a linear hypergraph for \( r - 1 = 2 \).

Suppose for inductive hypothesis that \( H \) is a linear hypergraph. Then by the theorem of Kahn [10], there exists a proper edge-coloring \( f \) of \( H \) with at most \( 2^{r-3}x + o(x) + o(2^{r-3}x + o(x)) = 2^{r-3}x + o(x) \) colors.

For each edge \( e \in E \), insert the color \( f(e) \) after each of the \( t \) occurrences in \( S_{r-1}(x, t) \) of the \( r - 1 \) letters in \( e \). The resulting sequence \( S_r(x, t) \) is \( r \)-sparse but not \((r + 1)\)-sparse. It has length \( rt(\binom{x}{2}) \) and at most \( 2^{r-3}x + \)
\[ o(x) + 2^{r-3}x + o(x) = 2^{r-2}x + o(x) \] distinct letters.

As for alternations, we note that any pair \( i, j \) of letters in \( S_r(x,t) \) that were also in \( S_{r-1}(x,t) \) have alternation length less than \( 2x + 2t \) by inductive hypothesis. If \( i \) and \( j \) are both letters that are new to \( S_r(x,t) \), then \( i \) and \( j \) make an alternation of length less than \( 2x \). If \( i \) was in \( S_{r-1}(x,t) \) but \( j \) was not, then there are two cases.

If \( j \) appears in no block with \( i \), then \( i \) and \( j \) make an alternation of length less than \( 2x \). If \( i \) and \( j \) appear in a single block together, then \( i \) and \( j \) make an alternation of length less than \( 2x + 2t \). Note that \( i \) and \( j \) cannot occur in two blocks together since \( f \) is a proper edge-coloring.

We have one last step for the induction. Let \( H' \) be the \( r \)-uniform hypergraph with vertex set equal to the letters of \( S_r(x,t) \) and edge set \( E \) with \( e \in E \) if and only if there is a block in \( S_r(x,t) \) on the letters \( e \). \( H \) was a linear hypergraph, so \( H' \) is also a linear hypergraph by Lemma 2.1. This completes the induction.

The last part of the proof is choosing \( x \) and \( t \) in terms of \( n \) and \( s \). Since \( S_r(x,t) \) is an \( r \)-sparse sequence avoiding alternations of length \( 2x + 2t \) with at most \( 2^{r-2}x + o(x) \) distinct letters and length \( rt(x^2) \), choosing e.g. \( x = \frac{cn}{4x2^{r-2}} \) and \( t = s/4 \) suffices to give the bound of \( \lambda_s(n,r) = \Omega(sn^2) \) for all \( s \geq cn \), where the constant in the bound depends on \( c \) and \( r \).

Note also that even if \( S_r(x,t) \) has fewer than \( n \) distinct letters, we can add a sequence of new distinct letters at the end of \( S_r(x,t) \) to increase the number of distinct letters to \( n \) without increasing the maximum alternation length.

One of the constructions in Pettie and Wellman’s paper \[18\] is an inductive construction that uses Roselle and Stanton’s construction as its initial case. The construction in Theorem 2.2 can be substituted for Roselle and Stanton’s construction in Pettie and Wellman’s proof to generalize the result in \[18\].

**Corollary 2.3.** If \( s = \Omega(n^{1/4}(t-1)!)) \), \( \lambda_s(n,r) \) is between \( \Omega(n^2s/(t-1)!)) \) and \( O(n^2s) \) for all \( r \geq 2 \).

**Proof.** Pettie and Wellman proved the case \( r = 2 \) in Theorem 4.1 of their paper \[18\]. The initial case of their construction \( S_1(s,q) \) uses the Roselle-Stanton construction \( RS(s,q) \) for \( q \) a prime power and \( s \geq q \). For \( r \geq 3 \), their construction and the analysis in their proof also work if we replace the Roselle-Stanton construction in their initial case with our construction in Theorem 2.2 using \( c = 1 \) as the bound for \( s \geq cn \).

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3 Long \((r, s)\)-formations

We obtain the next result just from the construction in the last section and previously known upper bounds.

**Theorem 3.1.** Fix integer \(r \geq 2\) and real number \(0 < c \leq 1\). \(F_{2,s,r}(n) = \theta(sn^2)\) for all \(s \geq cn\), where the constant in the lower bound depends on \(c\) and \(r\).

**Proof.** The upper bound follows from Klazar’s bound \(F_{2,s}(n) \leq sn^2\) for all \(n \geq 2\) [12, 14]. The lower bound follows from our construction in Theorem 2.2. 

Klazar actually proved the more general result that \(F_{r,s}(n) \leq sn^r\) for all \(n \geq r\) [12, 14]. In the next theorem, we show for \(s\) sufficiently large that this bound is tight up to a factor that depends only on \(r\) (and not on \(s\) or \(n\)).

**Theorem 3.2.** Fix integer \(r \geq 2\) and real number \(0 < c \leq 1\). Then \(F_{r,s,r}(n) = \theta(sn^r)\) for all \(s \geq cn^{r-1}\), where the constant in the lower bound depends on \(c\) and \(r\).

**Proof.** It suffices to prove that \(F_{r,s,r}(n) = \Omega(sn^r)\) for all \(s \geq cn^{r-1}\), where the constant in the bound depends on \(c\) and \(r\). We will use a family of sequences similar to the one used in Theorem 2.2.

Let \(x, t > 0\) be two parameters that will be chosen at the end of the proof in terms of \(n, s, c,\) and \(r\). Define \(T_{r}(x, t)\) to be the sequence obtained by starting with the empty sequence and then adding \(t\) copies of the subsequence \(i_1i_2\ldots i_r\) for each \(1 \leq i_1 \leq x\) and \(i_1 + 1 \leq i_2 \leq x\) and \(i_{r-1} + 1 \leq i_r \leq x\), for \((i_1, i_2, \ldots, i_r)\) in lexicographic order.

We call the consecutive copies of the subsequence \(i_1i_2\ldots i_r\) in \(T_{r}(x, t)\) a **block**. We call the set of adjacent blocks with \(i_1 = a_1, i_2 = a_2, \ldots, i_{r-1} = a_{r-1}\) the **block-row** \(B_{a_1,a_2,\ldots,a_{r-1}}\). Note that each block-row contains fewer than \(x\) blocks, and there are a total of \(\binom{x-1}{r-1}\) block-rows.

First observe that \(T_{r}(x, t)\) has length \(rt \binom{x}{r}\), and that \(T_{r}(x, t)\) is \(r\)-sparse. Next we explain why the formations on \(r\) letters have length less than \(2(\binom{x-1}{r-1}) + t + 1\). Let \(a_1 < a_2 < \cdots < a_r\) be arbitrary distinct letters in \(T_{r}(x, t)\).

Note that we can find a longest formation on the letters \(a_1, a_2, \ldots, a_r\) by searching greedily from left to right in \(T_{r}(x, t)\). Suppose that we go through the block-rows from beginning to end, and we mark block-rows greedily on
the letters wherever the formation length increases by 1 (in other words, on the last letter of each permutation of the formation that we find greedily).

Then every block-row not equal to \(B_{a_1, a_2, \ldots, a_{r-1}}\) increases the length of the formation on letters \(a_1, a_2, \ldots, a_r\) by at most \(2\). Block-row \(B_{a_1, a_2, \ldots, a_{r-1}}\) increases the length of the formation on letters \(a_1, a_2, \ldots, a_r\) by at most \(t + 1\), so all \(r\)-tuples of letters in \(T_r(x, t)\) have formation length less than \(2\left(\frac{x-1}{r-1}\right) + t + 1\).

Since \(T_r(x, t)\) is an \(r\)-sparse sequence avoiding formations of length 2 \(\left(\frac{x-1}{r-1}\right) + t + 1\) with \(x\) distinct letters and length \(rt\left(\binom{x}{r}\right)\), choosing \(x = \frac{cn}{4}\) and \(t = s/2 - 1\) suffices to give the bound of \(F_{r,s,r}(n) = \Omega(sn^r)\) for all \(s \geq cn^{r-1}\), where the constant in the bound depends on \(c\) and \(r\).

The lemma below generalizes the first part of Chang and Lawler’s argument for their upper bound of \(\lceil 3n/2 - 2 \rceil\) on proper edge-coloring for linear hypergraphs. We use this lemma in place of Kahn’s theorem in the main result of this section, which parallels the proof of Theorem 2.2.

**Lemma 3.3.** Suppose that \(H\) is a \(k\)-uniform hypergraph in which every pair of edges have intersection size at most \(r\) for \(2 \leq r < k\). Then it is possible to color the edges of \(H\) with \(krn/r!\) colors so that no pair of edges with intersection size \(r\) receive the same color.

**Proof.** We color the edges of \(H\) in an arbitrary order. Assume that we next color an edge \(e\). Since every pair of edges in \(H\) have intersection size at most \(r\), there are at most \(\left\lfloor \frac{n-k}{k-r} \right\rfloor\) edges already assigned colors that meet \(e\) at each of the \(\binom{k}{r}\) size-\(r\) subsets of vertices that are contained in \(e\). Thus there will be an unused color for \(e\) if \(\left(\binom{k}{r}\right) \frac{n-k}{k-r} < k^rn/r!\), which holds for all \(2 \leq r < k\), so we color \(e\) with any unused color.

The construction for the theorem below generalizes the construction in the proof of Theorem 2.2. Also, the initial case of the construction uses the construction in Theorem 3.2.

**Theorem 3.4.** Fix integers \(q \geq r \geq 2\) and real number \(0 < c \leq 1\). Then \(F_{r,s,q}(n) = \theta(sn^r)\) for all \(s \geq cn^{r-1}\), where the constant in the lower bound depends on \(c\), \(r\), and \(q\).

**Proof.** The upper bound was already proved in [12, 14], so it suffices to prove that \(F_{r,s,q}(n) = \Omega(sn^r)\) for all \(s \geq cn^{r-1}\), where the constant in the lower bound depends on \(c\), \(r\), and \(q\).

In the last section, we proved that the theorem is true for \(r = 2\) (Theorem 2.2) and also for \(q = r\) (Theorem 3.2). As in Theorem 2.2, our construction
for this theorem is inductive. For the initial case of \( q = r \), we set \( T_{r,r}(x, t) = T_r(x, t) \), where \( T_r(x, t) \) is the same sequence defined in Theorem 3.2.

For every \( q \geq r + 1 \), we will construct \( T_{r,q}(x, t) \) so that \( T_{r,q}(x, t) \) has length \( qt(r\choose x) \) and any \( r \)-tuple of distinct letters \( a_1, a_2, \ldots, a_r \) in \( T_{r,q}(x, t) \) has intersection size less than \( 2(r-1\choose x-1) + t + 1 \). In addition, \( T_{r,q}(x, t) \) will be \( q \)-sparse but not \((q + 1)\)-sparse, and \( T_{r,q}(x, t) \) will have at most \((q!)^{r-1}x \) distinct letters.

Like \( T_{r,r}(x, t) \), the sequences \( T_{r,q}(x, t) \) for \( q \geq r + 1 \) also have blocks, where each block consists of a sequence of \( q \) distinct letters repeated \( t \) times. In order to construct \( T_{r,q}(x, t) \) from \( T_{r,q-1}(x, t) \), we treat each block in \( T_{r,q-1}(x, t) \) as an edge in a \((q-1)\)-uniform hypergraph.

Similarly to Theorem 2.2, \( H = (V, E) \) is the \((q-1)\)-uniform hypergraph with vertex set equal to the letters of \( T_{r,q-1}(x, t) \) and edge set \( E \) with \( e \in E \) if and only if there is a block in \( T_{r,q-1}(x, t) \) on the letters \( e \).

Suppose for inductive hypothesis that \( H \) is a hypergraph in which every pair of edges have intersection size at most \( r - 1 \). Then by Lemma 3.3, it is possible to color the edges of \( H \) with some coloring \( f \) using \((q-1)!^{r-1}((q-1)!)^{r-1}x/(r-1)! \) colors so that no pair of edges with intersection size \( r - 1 \) receive the same color.

For each edge \( e \in E \), insert the color \( f(e) \) after each of the \( t \) occurrences in \( T_{r,q-1}(x, t) \) of the \( q - 1 \) letters in \( e \). The resulting sequence \( T_{r,q}(x, t) \) is \( q \)-sparse but not \((q + 1)\)-sparse. It has length \( qt(r\choose x) \) and at most \( x + (q - 1)!^{r-1}((q-1)!)^{r-1}x/(r-1)! \leq (q!)^{r-1}x \) distinct letters.

For formations, we consider arbitrary distinct letters \( a_1 < a_2 < \cdots < a_r \) in \( T_{r,q}(x, t) \). Note that if all of the letters were also in \( T_{r,q-1}(x, t) \), then they have formation length less than \( 2(r-1\choose x-1) + t + 1 \) by inductive hypothesis. If two of the letters are both new to \( T_{r,q}(x, t) \), then the maximum possible formation length on \( a_1, a_2, \ldots, a_r \) is at most \( 2(r-1\choose x-1) \). If there is a single letter \( a_i \) that was not in \( T_{r,q-1}(x, t) \), then there are two cases.

If there is some \( a_j \) that appears in no block with \( a_i \), then \( a_1, a_2, \ldots, a_r \) make a formation of length at most \( 2(x-1\choose r-1) \). If all of the letters \( a_1, a_2, \ldots, a_r \) appear in a single block together, then their maximum formation length is less than \( 2(r-1\choose x-1) + t + 1 \), since each block-row not containing that block contributes at most \( 2 \) to the formation length. Note that the letters \( a_1, a_2, \ldots, a_r \) cannot all occur together in two blocks by the definition of the coloring \( f \).

We have one last step for the induction. Let \( H' \) be the \( q \)-uniform hypergraph with vertex set equal to the letters of \( T_{r,q}(x, t) \) and edge set \( E \) with \( e \in E \) if and only if there is a block in \( T_{r,q}(x, t) \) on the letters \( e \). \( H' \) was a hypergraph in which every pair of edges have intersection size at most \( r - 1 \),
so $H'$ is also a hypergraph in which every pair of edges have intersection size at most $r - 1$ by the definition of the coloring $f$. This completes the induction.

The last part of the proof is choosing $x$ and $t$ in terms of $n$ and $s$. Since $T_{r,q}(x,t)$ is a $q$-sparse sequence avoiding formations of length $2\binom{x-1}{r-1} + t + 1$ with $(q!)^{r-1}x$ distinct letters and length $qt\binom{x}{r}$, choosing e.g. $x = \frac{cn}{\binom{q}{r}}$ and $t = s/2 - 1$ suffices to give the bound of $F_{r,s,q}(n) = \Omega(sn^r)$ for all $s \geq cn^{r-1}$, where the constant in the lower bound depends on $c$, $r$, and $q$.

As in Theorem 2.2, note also that even if $T_{r,q}(x,t)$ has fewer than $n$ distinct letters, we can add a sequence of new distinct letters at the end of $T_{r,q}(x,t)$ to increase the number of distinct letters to $n$ without increasing the maximum formation length.

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