NON-INTEGRATED DEFECT RELATIONS FOR THE GAUSS MAP OF A COMPLETE MINIMAL SURFACE WITH FINITE TOTAL CURVATURE IN $\mathbb{R}^m$

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ABSTRACT. In this article, we give the non-integrated defect relations for the Gauss map of a complete minimal surface with finite total curvature in $\mathbb{R}^m$. This is a continuation of previous work of Ha-Trao [J. Math. Anal. Appl., 430 (2015), 76-84.], which we extend here to targets of higher dimension.

1. Introduction

In 1988, H. Fujimoto [6] proved Nirenberg’s conjecture that if $M$ is a complete non-flat minimal surface in $\mathbb{R}^3$, then its Gauss map can omit at most 4 points, and the bound is sharp. After that, he [8] also extended that result for minimal surfaces in $\mathbb{R}^m$. He proved that the Gauss map of a non-flat complete minimal surface in $\mathbb{R}^m$ can omit at most $m(m + 1)/2$ hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ located in general position. He also gave an example to show that the number $m(m + 1)/2$ is the best possible when $m$ is odd. Beside that, many mathematicians also studied the value distribution of the Gauss map of a minimal surface with finite total curvature and got many good results (see Fang [4] and Ru [19] for examples). On the other hand, Mo-Osserman [18] (1990), Mo [17] (1994) and Ha-Phuong-Thoan [13] recently showed the relations between the value distribution of the Gauss map and the total curvature of a complete minimal surface. Related to the value distribution of the Gauss map of a complete minimal surface and the value distribution of the Gauss map of a complete minimal surface with finite total curvature, many results were given (see [12], [2], [3], [16] and [15] for examples).

On the other hand, Fujimoto [7, 8, 9] improved the previous results on the value distribution theory of the Gauss map of a complete minimal surface by introducing the modified defect relations for the Gauss map of a complete minimal surface.

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which have analogy to the defect relations given by R. Nevanlinna in his value distribution theory. The author and Trão [14] recently improved the Fujimoto’s results in the case the Gauss map of a complete minimal surface with finite total curvature in \(\mathbb{R}^3, \mathbb{R}^4\) by studying the non-integrated defect relations for the Gauss map. In this article, we would like to be continuous to study the non-integrated defect relations for the Gauss map of a complete minimal surface with finite total curvature in \(\mathbb{R}^m\). These are the strict improvements of all previous results of Fujimoto on the modified defect relations for the Gauss map of a complete minimal surface with finite total curvature in \(\mathbb{R}^m\). Thus, they also are improvements of previous results on ramifications for the Gauss map of a complete minimal surface with finite total curvature in \(\mathbb{R}^m\).

2. Statements of the main results

Let \(M\) be an open Riemann surface and \(f\) a nonconstant holomorphic map of \(M\) into \(\mathbb{P}^k(\mathbb{C})\). Assume that \(f\) has reduced representation \(f = (f_0 : \cdots : f_k)\). Set \(|f| = (|f_0|^2 + \cdots + |f_k|^2)^{1/2}\) and, for each a hyperplane \(H : \bar{a}_0w_0 + \cdots + \bar{a}_kw_k = 0\) in \(\mathbb{P}^k(\mathbb{C})\) with \(|a_0|^2 + \cdots + |a_k|^2 = 1\), we definition \(f(H) := \bar{a}_0f_0 + \cdots + \bar{a}_kf_k\).

**Definition 1.** We definition the \(S\)-defect of \(H\) for \(f\) by

\[
\delta^S_f(H) := 1 - \inf\{\eta \geq 0; \eta \text{ satisfies condition } (*)_S\}.
\]

Here, condition \((*)_S\) means that there exists a \([-\infty, \infty)\)-valued continuous subharmonic function \(u (\neq -\infty)\) on \(M\) satisfying the following conditions:

(C1) \(e^u \leq ||f||^\eta\),

(C2) for each \(\xi \in f^{-1}(H)\), there exists the limit

\[
\lim_{z \to \xi} (u(z) - \min(u_{f(H)}(\xi), k) \log |z - \xi|) \in [-\infty, \infty),
\]

where \(z\) is a holomorphic local coordinate around \(\xi\).

**Remark 1.** We always have that \(\eta = 1\) satisfies condition \((*)_S\) when \(u = \log |f(H)|\).

**Definition 2.** We definition the \(H\)-defect of \(H\) for \(f\) by

\[
\delta^H_f(H) := 1 - \inf\{\eta \geq 0; \eta \text{ satisfies condition } (*)_H\}.
\]
Here, condition \((*)_H\) means that there exists a \([-\infty, \infty)\)–valued continuous sub-harmonic function \(u\) on \(M\) which is harmonic on \(M - f^{-1}(H)\) and satisfies the conditions (C1) and (C2).

**Definition 3.** We definition the \(O\)–defect of \(H\) for \(f\) by
\[
\delta^O_f(H) := 1 - \inf \{ \frac{1}{n}; \ f(H) \text{ has no zero of order less than } n \}.
\]

**Remark 2.** We always have \(0 \leq \delta^O_f(H) \leq \delta^H_f(H) \leq \delta^S_f(H) \leq 1\).

Moreover, Fujimoto [5, page 672] also gave the reasons why he calls \(\delta^S_f(H)\) the non-integrated defect by showing a relation between the non-integrated defect and the defect (as in Nevanlinna theory) of a nonconstant holomorphic map of \(\Delta_R\) into \(\mathbb{P}^k(\mathbb{C})\).

**Definition 4.** One says that \(f\) is ramified over a hyperplane \(H\) in \(\mathbb{P}^k(\mathbb{C})\) with multiplicity at least \(e\) if all the zeros of the function \((f, H)\) have orders at least \(e\). If the image of \(f\) omits \(H\), one will say that \(f\) is ramified over \(H\) with multiplicity \(\infty\).

**Remark 3.** If \(f\) is ramified over a hyperplane \(H\) in \(\mathbb{P}^k(\mathbb{C})\) with multiplicity at least \(n\), then \(\delta^S_j(H) \geq \delta^H_j(H) \geq \delta^O_j(H) \geq 1 - \frac{1}{n}\). In particular, if \(f^{-1}(H) = \emptyset\), then \(\delta^O_j(H) = 1\).

In this article, we would like to study the \(S\)–defect relations for the Gauss maps of minimal surfaces with finite total curvature in \(\mathbb{R}^m\) which generalize the previous results of Ha-Tran in [14] to targets of higher dimension. In particular, we prove the following.

**Main theorem.** Let \(M\) be a non-flat complete minimal surface with finite total curvature in \(\mathbb{R}^m\) and its Gauss map \(G\). Let \(H_1, \ldots, H_q\) be hyperplanes in \(\mathbb{P}^{m-1}(\mathbb{C})\) located in \(N\)-subgeneral position \((q > 2N - k + 1, N \geq m - 1)\). Assume that \(G\) is \(k\)–non-degenerate (that is \(G(M)\) is contained in a \(k\)–dimensional linear subspace in \(\mathbb{P}^{m-1}(\mathbb{C})\), but none of lower dimension), \(1 \leq k \leq m - 1\), then
\[
\sum_{j=1}^q \delta^S_j(H_j) \leq (k + 1)(N - \frac{k}{2}) + (N + 1).
\]

**Remark 4.** For the case of the Gauss maps of minimal surfaces with finite total curvature, we can show that the Main theorem improved strictly Theorem 1.2 in
Remark 5. It is well known that the image of the (generalized) Gauss map $g : M \to \mathbb{P}^{m-1}(\mathbb{C})$ is contained in the hyperquadric $Q_{m-2}(\mathbb{C}) \subset \mathbb{P}^{m-1}(\mathbb{C})$, and that $Q_1(\mathbb{C})$ is biholomorphic to $\mathbb{P}^1(\mathbb{C})$ and that $Q_2(\mathbb{C})$ is biholomorphic to $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. So the results Ha-Trao in ([14]) which only treat the cases $m = 3$ and $m = 4$ are better than a result which holds for any $m \geq 3$ can be if restricted to the special cases $m = 3, 4$. The easiest way to see the difference is to observe that 6 lines in $\mathbb{P}^2(\mathbb{C})$ in general position may have only 4 points of intersection with the quadric $Q_1(\mathbb{C}) \subset \mathbb{P}^2(\mathbb{C})$.

3. Preliminaries and auxiliary lemmas

In this section, we recall some auxiliary lemmas in [9, 10, 11]. Let $M$ be an open Riemann surface and $ds^2$ a pseudo-metric on $M$, namely, a metric on $M$ with isolated singularities which is locally written as $ds^2 = \lambda^2 |dz|^2$ in terms of a nonnegative real-value function $\lambda$ with mild singularities and a holomorphic local coordinate $z$. We definition the divisor of $ds^2$ by $\nu_{ds} := \nu_\lambda$ for each local expression $ds^2 = \lambda^2 |dz|^2$, which is globally well-definitiond on $M$. We say that $ds^2$ is a continuous pseudo-metric if $\nu_{ds} \geq 0$ everywhere.

**Definition 5.** (see [9]) We definition the Ricci form of $ds^2$ by

$$\text{Ric}_{ds^2} := -dd^c \log \lambda^2$$

for each local expression $ds^2 = \lambda^2 |dz|^2$.

In some cases, a $(1, 1)$–form $\Omega$ on $M$ is regarded as a current on $M$ by defining $\Omega(\varphi) := \int_M \varphi \Omega$ for each $\varphi \in \mathcal{D}$, where $\mathcal{D}$ denotes the space of all $C^\infty$ differentiable functions on $M$ with compact supports.

**Definition 6.** (see [9]) We say that a continuous pseudo-metric $ds^2$ has strictly negative curvature on $M$ if there is a positive constant $C$ such that

$$-\text{Ric}_{ds^2} \geq C \cdot \Omega_{ds^2},$$

where $\Omega_{ds^2}$ denotes the area form for $ds^2$, namely,

$$\Omega_{ds^2} := \lambda^2(\sqrt{-1}/2)dz \wedge d\bar{z}$$
for each local expression $ds^2 = \lambda^2 |dz|^2$.

As is well-known, if the universal covering surface of $M$ is biholomorphic with the unit disc in $\mathbb{C}$, then $M$ has the complete conformal metric with constant curvature $-1$ which is called the Poincaré metric of $M$ and denoted by $d\sigma_M^2$.

Let $f$ be a linearly non-degenerate holomorphic map of $M$ into $\mathbb{P}^k(\mathbb{C})$. Take a reduced representation $f = (f_0 : \cdots : f_k)$. Then $F := (f_0, \cdots, f_k) : M \to \mathbb{C}^{k+1} \setminus \{0\}$ is a holomorphic map with $\mathbb{P}(F) = f$. Consider the holomorphic map

$$F_p = (F_p)_z := F^{(0)} \wedge F^{(1)} \wedge \cdots \wedge F^{(p)} : M \to \wedge^{p+1}\mathbb{C}^{k+1}$$

for $0 \leq p \leq k$, where $F^{(0)} := F = (f_0, \cdots, f_k)$ and $F^{(l)} := (F^{(l)})_z = (f_0^{(l)}, \cdots, f_k^{(l)})$ for each $l = 0, 1, \cdots, k$, and where the $l$-th derivatives $f^{(l)}_i = (f_i^{(l)})_z$, $i = 0, \ldots, k$, are taken with respect to $z$. (Here and for the rest of this paper the index $|_z$ means that the corresponding term is definition) by using differentiation with respect to the variable $z$, and in order to keep notations simple, we usually drop this index if no confusion is possible). The norm of $F_p$ is given by

$$|F_p| := \left( \sum_{0 \leq i_0 < \cdots < i_p \leq k} |W(f_{i_0}, \cdots, f_{i_p})|^2 \right)^{1/2},$$

where $W(f_{i_0}, \cdots, f_{i_p}) = W_z(f_{i_0}, \cdots, f_{i_p})$ denotes the Wronskian of $f_{i_0}, \cdots, f_{i_p}$ with respect to $z$.

**Proposition 1.** ([11, Proposition 2.1.6]).

For two holomorphic local coordinates $z$ and $\xi$ and a holomorphic function $h : M \to \mathbb{C}$, the following holds:

a) $W_\xi(f_0, \cdots, f_p) = W_z(f_0, \cdots, f_p) \cdot \left( \frac{dz}{d\xi} \right)^{p(p+1)/2}$.

b) $W_z(hf_0, \cdots, hf_p) = W_z(f_0, \cdots, f_p) \cdot (h)^{p+1}$.

**Proposition 2.** ([11, Proposition 2.1.7]).

For holomorphic functions $f_0, \cdots, f_p : M \to \mathbb{C}$ the following conditions are equivalent:

(i) $f_0, \cdots, f_p$ are linearly dependent over $\mathbb{C}$.

(ii) $W_z(f_0, \cdots, f_p) \equiv 0$ for some (or all) holomorphic local coordinate $z$.

We now take a hyperplane $H$ in $\mathbb{P}^k(\mathbb{C})$ given by

$$H : \tau_0 \omega_0 + \cdots + \tau_k \omega_k = 0,$$
with \( \sum_{i=0}^{k} |c_i|^2 = 1 \). We set
\[
F_0(H) := F(H) := \overline{c}_0 f_0 + \cdots + \overline{c}_k f_k
\]
and
\[
|F_p(H)| = |(F_p)_z(H)| := \left( \sum_{0 \leq i_1 < \cdots < i_p \leq k} \left| \sum_{l \neq i_1, \ldots, i_p} \overline{c}_l W(f_{i_1}, f_{i_2}, \ldots, f_{i_p}) \right|^2 \right)^{\frac{1}{2}},
\]
for \( 1 \leq p \leq k \). We note that by using Proposition 1, \( |(F_p)_z(H)| \) is multiplied by a factor \( \frac{d}{d|e^{p+1}/2} \) if we choose another holomorphic local coordinate \( \xi \), and it is multiplied by \( |h|^{p+1} \) if we choose another reduced representation \( f = (h f_0 : \cdots : h f_k) \) with a nowhere zero holomorphic function \( h \). Finally, for \( 0 \leq p \leq k \), set the \( p \)-th contact function of \( f \) for \( H \) to be \( \phi_p(H) := \frac{|F_p(H)|^2}{|F_p|^2} = \frac{|(F_p)_z(H)|^2}{|(F_p)_z|^2} \).

We next consider \( q \) hyperplanes \( H_1, \ldots, H_q \) in \( \mathbb{P}^k(\mathbb{C}) \) given by
\[
H_j : \langle \omega, A_j \rangle \equiv \overline{c}_{j0} \omega_0 + \cdots + \overline{c}_{jk} \omega_k \quad (1 \leq j \leq q)
\]
where \( A_j := (c_{j0}, \ldots, c_{jk}) \) with \( \sum_{i=0}^{k} |c_{ji}|^2 = 1 \).

Assume now \( N \geq k \) and \( q \geq N + 1 \). For \( R \subseteq Q := \{1, 2, \ldots, q\} \), denote by \( d(R) \) the dimension of the vector subspace of \( \mathbb{C}^{k+1} \) generated by \( \{A_j; j \in R\} \).

The hyperplanes \( H_1, \ldots, H_q \) are said to be in \( N \)-subgeneral position if \( d(R) = k + 1 \) for all \( R \subseteq Q \) with \( \sharp(R) \geq N + 1 \), where \( \sharp(A) \) means the number of elements of a set \( A \). In the particular case \( N = k \), these are said to be in general position.

**Theorem 3.** ([11, Theorem 2.4.11]) For given hyperplanes \( H_1, \ldots, H_q \) (\( q > 2N - k + 1 \)) in \( \mathbb{P}^k(\mathbb{C}) \) located in \( N \)-subgeneral position, there are some rational numbers \( \omega(1), \ldots, \omega(q) \) and \( \theta \) satisfying the following conditions:

(i) \( 0 < \omega(j) \leq \theta \leq 1 \) \( (1 \leq j \leq q) \),

(ii) \( \sum_{j=1}^{q} \omega(j) = k + 1 + \theta(q - 2N + k - 1) \),

(iii) \( \frac{k+1}{2N-k+1} \leq \theta \leq \frac{k+1}{N+1} \),

(iv) If \( R \subset Q \) and \( 0 < \sharp(R) \leq n + 1 \), then \( \sum_{j \in R} \omega(j) \leq d(R) \).

Constants \( \omega(j) \) \( (1 \leq j \leq q) \) and \( \theta \) with the properties of Theorem 3 are called Nochka weights and a Nochka constant for \( H_1, \ldots, H_q \) respectively.

We need the three following results of Fujimoto combining the previously introduced concept of contact functions with Nochka weights:
Theorem 4. ([11, Theorem 2.5.3]) Let $H_1, \cdots, H_q$ be hyperplanes in $\mathbb{P}^k(\mathbb{C})$ located in $N-$subgeneral position and let $\omega(j)$ $(1 \leq j \leq q)$ and $\theta$ be Nochka weights and a Nochka constant for these hyperplanes. For every $\epsilon > 0$ there exist some positive numbers $\delta(> 1)$ and $C$, depending only on $\epsilon$ and $H_j$, $1 \leq j \leq q$, such that

$$dd^c \log \frac{\Pi_{p=0}^{k-1} |F_p|^{2\epsilon}}{\Pi_{1 \leq j \leq q, 0 \leq p \leq k-1} \log^{2\omega(j)}(\delta/\phi_p(H_j))} \geq C \left( \frac{|F_0|^{2\theta(q-2N+k-1)} |F_k|^{2}}{\Pi_{j=1}^{q} (|F(H_j)|^{2} \Pi_{p=0}^{k-1} \log^{2}(\delta/\phi_p(H_j)))^{\omega(j)}} \right)^{2/(\kappa + 1)} dd^c |z|^2.$$  \hspace{1cm} (3.1)

Proposition 5. ([11, Proposition 2.5.7]) Set $\sigma_p = p(p+1)/2$ for $0 \leq p \leq k$ and $\tau_k = \sum_{p=0}^{k} \sigma_p$. Then,

$$dd^c \log(|F_0|^2 |F_1|^2 \cdots |F_{k-1}|^2) \geq \frac{\tau_k}{\sigma_k} \left( \frac{|F_0|^2 |F_1|^2 \cdots |F_k|^2}{|F_0|^{2\sigma_{k+1}}} \right)^{1/\tau_k} dd^c |z|^2.$$ \hspace{1cm} (3.2)

Proposition 6. ([11, Lemma 3.2.13]) Let $f$ be a non-degenerate holomorphic map of a domain in $\mathbb{C}$ into $\mathbb{P}^k(\mathbb{C})$ with reduced representation $f = (f_0 : \cdots : f_k)$ and let $H_1, \cdots, H_q$ be hyperplanes located in $N$-subgeneral position $(q > 2N - k + 1)$ with Nochka weights $\omega(1), \cdots, \omega(q)$ respectively. Then,

$$\nu_\phi + \sum_{j=1}^{q} \omega(j) \cdot \min(\nu(f,H_j), k) \geq 0,$$

where $\phi = \frac{|F_k|}{\Pi_{j=1}^{q} |F(H_j)|^{\omega(j)}}$.

Lemma 7. (Generalized Schwarz’s Lemma [1]) Let $v$ be a non-negative real-valued continuous subharmonic function on $\Delta_R$. If $v$ satisfies the inequality $\Delta \log v \geq v^2$ in the sense of distribution, then

$$v(z) \leq \frac{2R}{R^2 - |z|^2}.$$  

4. The proof of the Main Theorem

Proof. For the convenience of the reader, we first recall some notations on the Gauss map of minimal surfaces in $\mathbb{R}^m$. Let $M$ be a complete immersed minimal surface in $\mathbb{R}^m$. Take an immersion $x = (x_0, \ldots, x_{m-1}) : M \to \mathbb{R}^m$. Then $M$ has the structure of a Riemann surface and any local isothermal coordinate $(x, y)$ of $M$
gives a local holomorphic coordinate $z = x + \sqrt{-1}y$. The generalized Gauss map of $x$ is defined to be

$$g : M \to \mathbb{P}^{m-1}(\mathbb{C}), g = \mathbb{P}\left(\frac{x}{z}\right) = \left(\frac{\partial x_0}{\partial z}, \cdots, \frac{\partial x_{m-1}}{\partial z}\right).$$

Since $x : M \to \mathbb{R}^m$ is immersed,

$$G = G_z := (g_0, \ldots, g_{m-1}) = ((g_0)_z, \ldots, (g_{m-1})_z) = \left(\frac{\partial x_0}{\partial z}, \cdots, \frac{\partial x_{m-1}}{\partial z}\right)$$

is a (local) reduced representation of $g$, and since for another local holomorphic coordinate $\xi$ on $M$ we have $G_\xi = G_z \cdot \left(\frac{dz}{d\xi}\right)$, $g$ is well defined (independently of the (local) holomorphic coordinate). Moreover, if $ds^2$ is the metric on $M$ induced by the standard metric on $\mathbb{R}^m$, we have

$$ds^2 = 2|G_z|^2|dz|^2. \quad (4.3)$$

Finally since $M$ is minimal, $g$ is a holomorphic map.

Since by hypothesis of the Main theorem, $g$ is $k$-non-degenerate ($1 \leq k \leq m-1$) without loss of generality, we may assume that $g(M) \subset \mathbb{P}^k(\mathbb{C})$; then

$$g : M \to \mathbb{P}^k(\mathbb{C}), g = \mathbb{P}\left(\frac{x}{z}\right) = \left(\frac{\partial x_0}{\partial z}, \cdots, \frac{\partial x_k}{\partial z}\right)$$

is linearly non-degenerate in $\mathbb{P}^k(\mathbb{C})$ (so in particular $g$ is not constant) and the other facts mentioned above still hold.

Let $H_j(j = 1, \ldots, q)$ be $q(\geq N + 1)$ hyperplanes in $\mathbb{P}^{m-1}(\mathbb{C})$ in $N$-subgeneral position ($N \geq m-1 \geq k$). Then $H_j \cap \mathbb{P}^k(\mathbb{C})(j = 1, \ldots, q)$ are $q$ hyperplanes in $\mathbb{P}^k(\mathbb{C})$ in $N$-subgeneral position. Let each $H_j \cap \mathbb{P}^k(\mathbb{C})$ be represented as

$$H_j \cap \mathbb{P}^k(\mathbb{C}) : \bar{c}_{j0}\omega_0 + \cdots + \bar{c}_{jk}\omega_k = 0$$

with $\sum_{i=0}^k |c_{ji}|^2 = 1$.

Set

$$G(H_j) = G_z(H_j) := \bar{c}_{j0}g_0 + \cdots + \bar{c}_{jk}g_k.$$ 

We will now, for each contact function $\phi_p(H_j)$ for each of our hyperplanes $H_j$, choose one of the components of the numerator $|(G_z)_p(z(H_j))|$ which is not identically zero: More precisely, for each $j, p (1 \leq j \leq q, 1 \leq p \leq k)$, we can choose $i_1, \ldots, i_p$ with $0 \leq i_1 < \cdots < i_p \leq k$ such that

$$\psi(G)_{jp} = (\psi(G_z)_{jp})_z := \sum_{l \neq i_1, \ldots, i_p} \bar{c}_{jl}W_z(g_l, g_{i_1}, \cdots, g_{i_p}) \neq 0,$$
(indeed, otherwise, we have \( \sum_{l \neq i_1, \ldots, i_p} \bar{\tau}_{jl}W(g_l, g_{i_1}, \ldots, g_{i_p}) = 0 \) for all \( i_1, \ldots, i_p \), so \( W(\sum_{l \neq i_1, \ldots, i_p} \bar{\tau}_{jl}g_l, g_{i_1}, \ldots, g_{i_p}) = 0 \) for all \( i_1, \ldots, i_p \), which contradicts the non-degeneracy of \( g \) in \( \mathbb{P}^k(\mathbb{C}) \). Alternatively we simply can observe that in our situation none of the contact functions vanishes identically).

Now we prove the Main theorem in four steps:

**Step 1:** We may assume that
\[
\sum_{j=1}^{q} \delta_{S}^{S}(H_j) > (k + 1)(N - \frac{k}{2}) + (N + 1),
\]
otherwise our Main theorem is already proved. By definition, there exist constants \( \eta_j \geq 0 (1 \leq j \leq q) \) such that \( q - \sum_{j=1}^{q} \eta_j > (k + 1)(N - \frac{k}{2}) + (N + 1) \) and continuous subharmonic functions \( u_j (1 \leq j \leq q) \) on \( M \) satisfies conditions (C1) and (C2).

Thus
\[
\sum_{j=1}^{q} (1 - \eta_j) - 2N + k - 1 > \frac{(2N - k + 1)k}{2} > 0,
\]
and this implies in particular
\[
q > 2N - k + 1 \geq N + 1 \geq k + 1.
\]

By Theorem 3, we have
\[
(q - 2N + k - 1)\theta = \sum_{j=1}^{q} \omega(j) - k - 1, \quad \theta \geq \omega(j) > 0 \text{ and } \theta \geq \frac{k + 1}{2N - k + 1},
\]
so
\[
2\left( \sum_{j=1}^{q} \omega(j)(1 - \eta_j) - k - 1 \right) = 2\left( \sum_{j=1}^{q} \omega(j) - k - 1 - \sum_{j=1}^{q} \omega(j)\eta_j \right)
\]
\[
= 2(q - 2N + k - 1)\theta - 2\sum_{j=1}^{q} \omega(j)\eta_j
\]
\[
\geq 2(q - 2N + k - 1)\theta - 2\sum_{j=1}^{q} \theta\eta_j
\]
\[
= 2\theta \left( \sum_{j=1}^{q} (1 - \eta_j) - 2N + k - 1 \right)
\]
\[
\geq 2\left( k + 1 \right) \left( \sum_{j=1}^{q} (1 - \eta_j) - 2N + k - 1 \right) \frac{2N - k + 1}{2N - k + 1}.
\]
Thus, we now can conclude with (4.5) that
\[ 2 \left( \sum_{j=1}^{q} \omega(j)(1 - \eta_j) - k - 1 \right) > k(k + 1) \]
\[ \Rightarrow \sum_{j=1}^{q} \omega(j)(1 - \eta_j) - k - 1 - \frac{k(k + 1)}{2} > 0. \quad (4.7) \]

We set \( \gamma := \sum_{j=1}^{q} \omega(j)(1 - \eta_j) - k - 1 \).
Then, by (4.7), we get \( \gamma > \sigma_k \Rightarrow \frac{\gamma}{\sigma_k} > 1 \).

So we can choose a positive real number \( \epsilon \) such that \( \frac{\gamma - \epsilon \sigma_{k+1}}{\sigma_k + \epsilon \tau_k} > 1 \).

**Step 2:**

We set
\[ \lambda_z := \left( \frac{|G_z|^{\gamma - \epsilon \sigma_{k+1}} e^{\sum_{j=1}^{q} \omega(j) u_j} |(G_k)_z| \prod_{p=0}^{k} |(G_p)_z|^\epsilon}{\prod_{j=1}^{q} (|G(H_j)| \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j)))^\omega(j)} \right)^{\frac{1}{\sigma_k + \epsilon \tau_k}}, \]
and definition the pseudometric \( d\tau_z^2 := \lambda_z^2 |dz|^2 \). Using Proposition 1, we can see that
\[ d\tau_z := \left( \frac{|G_z|^{\gamma - \epsilon \sigma_{k+1}} e^{\sum_{j=1}^{q} \omega(j) u_j} |(G_k)_z| \prod_{p=0}^{k} |(G_p)_z|^\epsilon}{\prod_{j=1}^{q} (|G(H_j)| \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j)))^\omega(j)} \right)^{\frac{1}{\sigma_k + \epsilon \tau_k}} |d\xi| \]
\[ = \left( \frac{|G_z|^{\gamma - \epsilon \sigma_{k+1}} e^{\sum_{j=1}^{q} \omega(j) u_j} |(G_k)_z| \prod_{p=0}^{k} |(G_p)_z|^\epsilon |d\xi|}{\prod_{j=1}^{q} (|G(H_j)| \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j)))^\omega(j)} \right)^{\frac{1}{\sigma_k + \epsilon \tau_k}} |dz| \]
\[ = d\tau_z. \]

Thus \( d\tau_z^2 \) is independent of the choice of the local coordinate \( z \). We will denote \( d\tau_z^2 \) by \( d\tau^2 \) for convenience. So \( d\tau^2 \) is well-definition on \( M \) and
\[ d\tau^2 := \left( \frac{|G_z|^{\gamma - \epsilon \sigma_{k+1}} e^{\sum_{j=1}^{q} \omega(j) u_j} |G_k| \prod_{p=0}^{k} |G_p|^\epsilon}{\prod_{j=1}^{q} (|G(H_j)| \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j)))^\omega(j)} \right)^{\frac{2}{\sigma_k + \epsilon \tau_k}} |dz|^2 := \lambda^2 |dz|^2. \]

**Step 3:** We will show that \( d\tau^2 \) is continuous and has strictly negative curvature on \( M \) in this step.

Indeed, it is easy to see that \( d\tau \) is continuous at every point \( z_0 \) with \( \prod_{j=1}^{q} G(H_j)(z_0) \neq 0 \). Now we take a point \( z_0 \) such that \( \prod_{j=1}^{q} G(H_j)(z_0) = 0 \). Hence, it follows from
(C2) that we get
\[
\lim_{z \to z_0} e^{u_j(z)} |z - z_0|^{-\min(\nu_{G(H_j)}(z_0), k)} = \lim_{z \to z_0} e^{u_j(z)} - \min(\nu_{G(H_j)}(z_0), k) \log |z - z_0| < \infty, \forall 1 \leq j \leq q.
\]

Thus, combining this with Proposition 6, we get
\[
\nu_{d\tau}(z_0) \geq \frac{1}{\sigma_k + \epsilon \tau_k} \left( \nu_{G_k}(z_0) - \sum_{j=1}^{q} \omega(j) \nu_{G(H_j)}(z_0) + \sum_{j=1}^{q} \omega(j) \min\{\nu_{G(H_j)}(z_0), k\} \right)
+ \sum_{j=1}^{q} \omega(j) e^{u_j(z_0)} |z - z_0|^{-\min(\nu_{G(H_j)}(z_0), k)}(z_0)
\geq 0.
\]

This concludes the proof that \( d\tau \) is continuous on \( M \).

On the other hand, by using Proposition 5, Theorem 4 and noting that \( dd^c \log |G_k| = 0, dd^c \log e^{u_j} \geq 0(1 \leq j \leq q) \), we have
\[

\begin{align*}
dd^c \log \lambda &\geq \frac{\gamma - \epsilon \sigma_{k+1}}{\sigma_k + \epsilon \tau_k} \dd^c \log |G| + \frac{\epsilon}{4(\sigma_k + \epsilon \tau_k)} \dd^c \log (|G_0| |G| |G_1| |G_2| \cdots |G_{k-1}|)
+ \frac{1}{2(\sigma_k + \epsilon \tau_k)} \dd^c \log \prod_{j=1}^{q} \frac{|G_j|^{2\omega(j)}}{\prod_{j=1}^{q} \log^2(\delta/\phi_p(H_j))}
\geq \frac{\epsilon}{4(\sigma_k + \epsilon \tau_k)} \frac{\tau_k}{\sigma_k} \left( \frac{|G_0|^2 |G_1|^2 \cdots |G_k|^2}{|G_0|^{2\sigma_{k+1}}} \right)^{1/\tau_k} \dd^c |z|^2
+ C_0 \left( \frac{|G_0|^{2\theta(q-2N+k-1)} |G_k|^2}{\prod_{j=1}^{q} (|G(H_j)|^2 \prod_{p=0}^{k-1} \log^2(\delta/\phi_p(H_j)))^{\omega(j)}} \right)^{\frac{1}{1+\tau_k}} \dd^c |z|^2
\geq \min\left\{ \frac{1}{4\sigma_k(\sigma_k + \epsilon \tau_k)} \cdot C_0 \left( \frac{|G_0|^2 |G_1|^2 \cdots |G_k|^2}{|G_0|^{2\sigma_{k+1}}} \right)^{1/\tau_k}
+ \sigma_k \left( \frac{|G_0|^{2\theta(q-2N+k-1)} |G_k|^2}{\prod_{j=1}^{q} (|G(H_j)|^2 \prod_{p=0}^{k-1} \log^2(\delta/\phi_p(H_j)))^{\omega(j)}} \right)^{1/\tau_k} \right\} \dd^c |z|^2
\end{align*}
\]

where \( C_0 \) is the positive constant. So, by using the basic inequality
\[
\alpha A + \beta B \geq (\alpha + \beta) A^{\alpha/\alpha + \beta} B^{\beta/\alpha + \beta} \text{ for all } \alpha, \beta, A, B > 0,
\]
we can find a positive constant $C_1$ satisfying the following

$$dd^c \log \lambda \geq C_1 \left( \frac{|G|^2(q-2N+k-1)-1-\sigma_{k+1} |G_k|, \prod_{p=0}^k |G_p|^\epsilon}{\prod_{j=1}^q (|G(H_j)| \cdot \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j))) \omega(j)} \right) \frac{2}{|G|^2} dd^c |z|^2$$

$$= C_1 \left( \frac{|G|^2 \sum_{j=1}^q |G_j|, \prod_{p=0}^k |G_p|^\epsilon \cdot \prod_{j=1}^q |G|_{e^{u_j}} \omega(j)}{\prod_{j=1}^q (|G(H_j)| \cdot \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j))) \omega(j)} \right) \frac{2}{|G|^2} dd^c |z|^2.$$ (by Theorem 3)

On the other hand,

$$\left( \frac{|G|_{e^{u_j}}}{e^{u_j}} \right) \omega(j) \geq 1 \text{ for all } j = 1, \ldots, q,$$

so we get

$$dd^c \log \lambda \geq C_1 \left( \frac{|G|^2 \sum_{j=1}^q |G_j|, \prod_{p=0}^k |G_p|^\epsilon \cdot \prod_{j=1}^q |G|_{e^{u_j}} \omega(j)}{\prod_{j=1}^q (|G(H_j)| \cdot \prod_{p=0}^{k-1} \log(\delta/\phi_p(H_j))) \omega(j)} \right) \frac{2}{|G|^2} dd^c |z|^2$$

$$= C_1 \chi^2 dd^c |z|^2.$$ (Step 4: Set $\Omega_g = dd^c \log |G|^2$.)

Now, by using the classical Nevanlinna theory [11, Theorem 3.3.15] and remarks of Fujimoto [5, Proposition 4.7] we get $\sum_{j=1}^q \delta_g(H_j) \leq 2N - k + 1$ if the universal covering surface of $M$ is biholomorphic to $\mathbb{C}$. This is a contradiction with (4.4). Thus, we only need consider the case that the universal covering surface of $M$ is biholomorphic to the unit disc. By Lemma 7, there exists a positive constant $C_0$ such that

$$d\tau^2 \leq C_0 d\sigma_M^2,$$

where $d\sigma_M^2$ denotes the Poincaré metric on $M$. Now, it follows from the assumption $M$ has finite total curvature in $\mathbb{R}^m$ that $M$ is bi-holomorphic with a compact Riemann surface $\overline{M}$ with finitely many points $a_i$'s removed. For each $a_i$, we take a neighborhood $U_i$ of $a_i$ which is biholomorphic to $\Delta^* = \{z; 0 < |z| < 1\}$, where $z(a_i) = 0$. The Poincaré metric on domain $\Delta^*$ is
given by
\[ d\sigma^2_{\Delta'} = \frac{4|dz|^2}{|z|^2 \log^2 |z|^2}. \]

By using the distance decreasing property of the Poincaré metric, we have
\[ d\tau^2 \leq C_l \frac{|dz|^2}{|z|^2 \log^2 |z|^2} \]
with some \( C_l > 0 \). This implies that, for a neighborhood \( U_i^* \) of \( a_i \) which is relatively compact in \( U_i \), we have
\[ \int_{U_i^*} \Omega_{d\tau^2} < +\infty. \]

Since \( \overline{M} \) is compact, we have
\[ \int_M \Omega_{d\tau^2} \leq \int_{M \cup \bigcup_i U_i^*} \Omega_{d\tau^2} + \sum_i \int_{U_i^*} \Omega_{d\tau^2} < +\infty. \tag{4.8} \]

On the other hand, we have
\[ d\Sigma \log \lambda \geq \left( \frac{\gamma - \epsilon \sigma_{k+1}}{\sigma_k + \epsilon \tau_k} \right) d\Sigma \log |G| + \frac{1}{(\sigma_k + \epsilon \tau_k)} d\Sigma \log \frac{\prod_{p=0}^{k-1} |G_p|^\epsilon}{\prod_{j=1}^q \prod_{p=0}^{k-1} \log^{2\omega(j)}(\delta/\phi_p(H_j))} \]
\[ \geq \left( \frac{\gamma - \epsilon \sigma_{k+1}}{\sigma_k + \epsilon \tau_k} \right) d\Sigma \log |G|, \text{ by Theorem 4.} \]

This implies
\[ d\Sigma \log \lambda^2 \geq \left( \frac{\gamma - \epsilon \sigma_{k+1}}{\sigma_k + \epsilon \tau_k} \right) \Omega_g. \]

Thus, we now can find a subharmonic function \( v \) such that
\[ \lambda^2 |dz|^2 = e^v |G|^2 \left( \frac{\gamma - \epsilon \sigma_{k+1}}{\sigma_k + \epsilon \tau_k} \right)^2 |dz|^2 \]
\[ = e^{v+\left( \frac{\gamma - \epsilon \sigma_{k+1}}{\sigma_k + \epsilon \tau_k} - 1 \right) \log |G|^2} |G|^2 |dz|^2 \]
\[ = e^w ds^2. \]

So \( d\tau^2 = e^w ds^2 \), where \( w \) is a subharmonic function. Here, we can apply the result of Yau in [20] to see
\[ \int_M e^w \Omega_{ds^2} = +\infty, \]
because of the completeness of \( M \) with respect to the metric \( ds^2 \). This contradicts the assertion \((4.8)\). The proof of the Main theorem is completed. \( \square \)
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REFERENCES

[1] L. V. Ahlfors, *An extension of Schwarz’s lemma*, Trans. Amer. Math. Soc., 43 (1938), 359-364.
[2] G. Dethloff and P. H. Ha, *Ramification of the Gauss map of complete minimal surfaces in \( \mathbb{R}^3 \) and \( \mathbb{R}^4 \) on annular ends*, Ann. Fac. Sci. Toulouse Math., 23 (2014), no. 4, 829-846.
[3] G. Dethloff, P. H. Ha and P. D. Thoan, *Ramification of the Gauss map of complete minimal surfaces in \( \mathbb{R}^m \) on annular ends*, Colloq. Math., 142 (2016), 149-167.
[4] Y. Fang, *On the Gauss map of complete minimal surfaces with finite total curvature*, Indiana Univ. Math. J., 42 (1993), 1389-1411.
[5] H. Fujimoto, *Value distribution of the Gauss maps of complete minimal surfaces in \( \mathbb{R}^m \)*, J. Math. Soc. Japan, 35 (1983), no. 4, 663-681.
[6] H. Fujimoto, *On the number of exceptional values of the Gauss maps of minimal surfaces*, J. Math. Soc. Japan, 40 (1988), 235-247.
[7] H. Fujimoto, *Modified defect relations for the Gauss map of minimal surfaces*, J. Differential Geom., 29 (1989), 245-262.
[8] H. Fujimoto, *Modified defect relations for the Gauss map of minimal surfaces II*, J. Differential Geom., 31 (1990), 365-385.
[9] H. Fujimoto, *Modified defect relations for the Gauss map of minimal surfaces III*, Nagoya Math. J., 124 (1991), 13-40.
[10] H. Fujimoto, *Unicity theorems for the Gauss maps of complete minimal surfaces II*, Kodai Math. J., 16 (1993), 335-354.
[11] H. Fujimoto, *Value Distribution Theory of the Gauss map of Minimal Surfaces in \( \mathbb{R}^m \)*, Aspect of Math., Vol. E21, Vieweg, Wiesbaden (1993).
[12] P. H. Ha, *An estimate for the Gaussian curvature of minimal surfaces in \( \mathbb{R}^m \) whose Gauss map is ramified over a set of hyperplanes*, Differential Geom. Appl., 32 (2014), 130-138.
[13] P. H. Ha, L. B. Phuong and P. D. Thoan, *Ramification of the Gauss map and the total curvature of a complete minimal surface*, Topol. Appl., 199 (2016), 32-48.
[14] P. H. Ha and N. V. Trao, *Non-integrated defect relations for the Gauss map of complete minimal surfaces with finite total curvature*, J. Math. Anal. Appl., 430 (2015), 76-84.
[15] L. Jin and M. Ru, *Algebraic curves and the Gauss map of algebraic minimal surfaces*, Differential Geom. Appl., 25 (2007), 701-712.
[16] Y. Kawakami, R. Kobayashi, and R. Miyaoka, *The Gauss map of pseudo-algebraic minimal surfaces*, Forum Math., 20 (2008), 1055-1069.
[17] X. Mo, *Value distribution of the Gauss map and the total curvature of complete minimal surface in \( \mathbb{R}^m \)*, Pacific. J. Math., 163 (1994), 159-174.
[18] X. Mo and R. Osserman, *On the Gauss map and total curvature of complete minimal surfaces and an extension of Fujimoto’s theorem*, J. Differential Geom., 31 (1990), 343-355.

[19] M. Ru, *On the Gauss map of minimal surfaces with finite total curvature*, Bull. Austral. Math. Soc., 44 (1991), 225-232.

[20] S. T. Yau, *Some function-theoretic properties of complete Riemannian manifolds and their applications to geometry*, Indiana Univ. Math. J., 25 (1976), 659-670.

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