ON ROZANOV’S THEOREM AND STRENGTHENED ASYMPTOTIC UNIFORM DISTRIBUTION

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ABSTRACT. For sums $S_n = \sum_{k=1}^{n} X_k$, $n \geq 1$ of independent random variables $X_k$ taking values in $\mathbb{Z}$ we prove, as a consequence of a more general result, that if (i) For some function $1 \leq \phi(t) \uparrow \infty$ as $t \to \infty$, and some constant $C$, we have for all $n$ and $\nu \in \mathbb{Z}$,

$$|B_n| P\{S_n = \nu\} - \frac{1}{\sqrt{2\pi}} e^{\frac{(\nu - Mn)^2}{2h^2}} \leq \frac{C}{\phi(B_n)},$$

then (ii) There exists a numerical constant $C_1$, such that for all $n$ such that $B_n \geq 6$, all $h \geq 2$, and $\mu = 0, 1, \ldots, h - 1$,

$$\left| P\{S_n \equiv \mu \pmod{h}\} - \frac{1}{h} \right| \leq \frac{1}{\sqrt{2\pi} B_n} + \frac{1 + 2C/h}{\phi(B_n)^{2/3}} + C_1 e^{-(1/16)\phi(B_n)^{3/2}}.$$

Assumption (i) holds if a local limit theorem in the usual form is applicable, and (ii) yields a strengthening of Rozanov’s necessary condition.

Assume in place of (i) that $\vartheta_j = \sum_{k \in \mathbb{Z}} P\{X_j = k\} \wedge P\{X_j = k+1\} > 0$, for each $j$ and that $\nu_n = \sum_{j=1}^{n} \vartheta_j \uparrow \infty$. We prove strengthened forms of the asymptotic uniform distribution property. (iii) Let $\alpha > \alpha' > 0, 0 < \varepsilon < 1$. Then for each $n$ such that

$$|x| \leq \frac{1}{2}\left(\frac{2\alpha \log(1-\varepsilon)\nu_n}{(1-\varepsilon)\nu_n}\right)^{1/2} \Rightarrow \frac{\sin x}{x} \geq (\alpha'/\alpha)^{1/2},$$

we have

$$\sup_{u \geq 0} \sup_{d \leq n\left(\frac{1-\varepsilon}{\nu_n}\right)^{1/2}} \left| P\{d|S_n + u\} - \frac{1}{d}\right| \leq 2 e^{-\frac{\pi^2}{4} \nu_n} + \left((1-\varepsilon)\nu_n\right)^{-\alpha'}.$$

(iv) Let $0 < \rho < 1$ and $0 < \varepsilon < 1$. The sharper uniform bound $2 e^{-\frac{\pi^2}{4} \nu_n} + e^{-(1-\varepsilon)\nu_n} \rho$ is also proved (for a corresponding $d$-region of divisors), for each $n$ such that

$$|x| \leq \frac{1}{2}\left(\frac{2}{((1-\varepsilon)\nu_n)^{1-\rho}}\right)^{1/2} \Rightarrow \frac{\sin x}{x} \geq \sqrt{1-\varepsilon}.$$

1. Local limit theorem and asymptotic uniform distribution.

Let $X = \{X_i, i \geq 1\}$ be a sequence of independent variables taking values in $\mathbb{Z}$, and let $S_n = \sum_{k=1}^{n} X_k$, for each $n$.

The sequence $X$ is said to be asymptotically uniformly distributed with respect to lattices of span $d$, in short a.u.d.$(d)$, if for $m = 0, 1, \ldots, d - 1$, we have

$$(1.1) \quad \lim_{n \to \infty} P\{S_n \equiv m \pmod{d}\} = \frac{1}{d}.$$
Equivalently for \( m = 0, 1, \ldots, d - 1 \), we have
\[
\lim_{n \to \infty} \mathbb{P}\{d|S_n - m\} = \frac{1}{d}.
\]
The sequence \( X \) is asymptotically uniformly distributed, in short a.u.d., if (1.1) holds true for any \( d \geq 2 \) and \( m = 0, 1, \ldots, d - 1 \).

Dvoretzky and Wolfowitz [3] proved the following characterization. Assume that \( X \) is composed with independent random variables taking only the values \( 0, 1, \ldots, h - 1 \).

In order that the partial sums \( \{S_n, n \geq 1\} \) be a.u.d.\((h)\), it is necessary and sufficient that
\[
\prod_{k=1}^{\infty} \left( \sum_{m=0}^{h-1} \mathbb{P}\{X_k = m\} e^{2\pi i r m} \right) = 0, \quad (r = 1, \ldots, h - 1).
\]
Equivalently,
\[
\prod_{k=1}^{\infty} \left( \mathbb{E} e^{2\pi i r X_k} \right) = \lim_{N \to \infty} \left( \mathbb{E} e^{2\pi i r S_N} \right) = 0, \quad (r = 1, \ldots, h - 1).
\]

This notion plays an important role in the study of the local limit theorem. Let us assume that the random variables \( X_k \) take values in a common lattice \( L(v_0, D) \), namely defined by the sequence \( v_k = v_0 + Dk, k \in \mathbb{Z}, v_0 \) and \( D > 0 \) being reals, and are square integrable, and let
\[
M_n = \mathbb{E} S_n, \quad B_n^2 = \text{Var}(S_n) \to \infty.
\]

We say that the local limit theorem (in the usual form) is applicable to \( X \) if
\[
\sup_{N=v_0n+Dk} \left| B_n \mathbb{P}\{S_n = N\} - \frac{D}{\sqrt{2\pi} B_n} e^{-\frac{(N-M_n)^2}{2B_n^2}} \right| = o(1), \quad n \to \infty.
\]
When the random variables \( X_i \) are identically distributed, (1.6) reduces to
\[
\sup_{N=v_0n+Dk} \left| \sigma \sqrt{n} \mathbb{P}\{S_n = N\} - \frac{D}{\sqrt{2\pi}} e^{-\frac{(N-n\mu)^2}{2n\sigma^2}} \right| = o(1),
\]
where \( \mu = \mathbb{E} X_1, \sigma^2 = \text{Var}(X_1) \). By Gnedenko’s Theorem [10], see also [20], p. 187, [24], Th. 1.4, (1.7) holds if and only if the span \( D \) is maximal (there are no other real numbers \( v_0' \) and \( D' > D \) for which \( \mathbb{P}\{X \in L(v_0', D')\} = 1 \)).

Note that the transformation
\[
X_j' = \frac{X_j - v_0}{D},
\]
allows one to reduce to the case \( v_0 = 0, D = 1 \).

**Remark 1.1.** Note that the series (in \( k \))
\[
\sum_{N=v_0n+Dk} \left( \mathbb{P}\{S_n = N\} - \frac{D}{\sqrt{2\pi} B_n} e^{-\frac{(N-M_n)^2}{2B_n^2}} \right),
\]
is obviously convergent, whereas nothing can be deduced concerning its order from the very
definition of the local limit theorem. Further by using Poisson summation formula the series
associated to the second summand verifies

\[(1.10) \sum_{N=\nu_0+n+Dk} \frac{D}{\sqrt{2\pi B_n}} e^{-\frac{(N-Mn)^2}{2B_n}} = \sum_{\ell \in \mathbb{Z}} e^{2\pi i \ell \frac{v_0}{D} - \frac{2\pi^2 \ell^2 D_k^2}{B_n}}, \]

and so is \(1 + O(D/B_n)\), whereas the one associated to the first is 1. Therefore

\[(1.11) \sum_{N=\nu_0+n+Dk} \left| P\{S_n = N\} - \frac{D}{\sqrt{2\pi B_n}} e^{-\frac{(N-Mn)^2}{2B_n}} \right| = O(D/B_n). \]

When a strong local limit theorem with convergence in variation holds we have the more
informative result

\[(1.12) \lim_{n \to \infty} \sum_{N=\nu_0+n+Dk} \left| P\{S_n = N\} - \frac{D}{\sqrt{2\pi B_n}} e^{-\frac{(N-Mn)^2}{2B_n}} \right| = 0. \]

The following result is well-known.

**Theorem 1.2 (Rozanov).** Let \(X = \{X_i, i \geq 1\}\) be a sequence of independent variables taking
values in \(\mathbb{Z}\), and let \(S_n = \sum_{k=1}^n X_k\), for each \(n\). The local limit theorem is applicable to \(X\)
only if \(X\) satisfies the a.u.d. property.

**Remark 1.3.** In Petrov [20], Lemma 1, p. 194, also in Rozanov's [23] Lemma 1, p. 261, Theorem 1.2 is stated under the assumption that a local limit theorem in the strong form holds, which is not necessary.

We will in fact prove the following stronger result providing an explicit link between the
local limit theorem and the a.u.d. property, through a quantitative estimate of the difference
\(P\{S_n \equiv m \pmod{h}\} - 1/h\).

**Theorem 1.4.** Let \(X = \{X_i, i \geq 1\}\) be a sequence of independent variables taking values
in \(\mathbb{Z}\), and let \(S_n = \sum_{k=1}^n X_k\), for each \(n\). Assume that for some function \(1 \leq \phi(t) \uparrow \infty\) as \(t \to \infty\), and some constant \(C\), we have for all \(n\)

\[(1.13) \sup_{m \in \mathbb{Z}} \left| B_n P\{S_n = m\} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(m-Mn)^2}{2B_n^2}} \right| \leq \frac{C}{\phi(B_n)}. \]

Then there exists a numerical constant \(C_1\), such that for all \(0 < \varepsilon \leq 1\), all \(n\) such that \(B_n \geq 6\), and all \(h \geq 2\),

\[\sup_{\mu=0,1,\ldots,h-1} \left| P\{S_n \equiv \mu \pmod{h}\} - \frac{1}{h} \right| \leq \frac{1}{\sqrt{2\pi B_n}} + \frac{2C}{h \sqrt{\varepsilon} \phi(B_n)} + P\left\{ \frac{|S_n - M_n|}{B_n} > \frac{1}{\sqrt{\varepsilon}} \right\} + C_1 e^{-1/(10\varepsilon)}. \]

**Remark 1.5.** It follows from the proof that \(C_1 = 2e\sqrt{\pi}\) is suitable.

Choosing \(\varepsilon = \phi(B_n)^{-2/3}\) and using Tchebycheff’s inequality, we get the following
Corollary 1.6. For all \( n \) such that \( B_n \geq 6 \), and all \( h \geq 2 \), we have

\[
\sup_{\mu=0,1,\ldots,h-1} \left| \mathbb{P}\{S_n \equiv \mu \pmod{h}\} - \frac{1}{h} \right| \leq H_n,
\]

with

\[
H_n = \frac{1}{\sqrt{2\pi} B_n} + \frac{1 + 2C/h}{\phi(B_n)^{2/3}} + C_1 e^{-(1/16)\phi(B_n)^{2/3}}.
\]

Theorem 1.4 contains Theorem 1.2 since by definition such a function \( \phi \) exists if the local limit theorem is applicable to \( X \). Further condition (1.13) implies that the local limit theorem is applicable to \( X \).

Remark 1.7. Examples of LLT’s with speed of convergence are given in Appendix.

Proof. By assumption,

\[
|B_n\mathbb{P}\{S_n = m\} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(m-M_n)^2}{2B_n}}} \leq \frac{C}{\phi(B_n)},
\]

for all \( m \) and \( n \). Let \( \varepsilon > 0 \). We have

\[
\left| \mathbb{P}\{S_n \equiv m \pmod{h}\} - \sum_{|k-M_n|\leq B_n/\sqrt{\pi}} \mathbb{P}\{S_n = k\} \right| \leq \mathbb{P}\left\{ \left| \frac{S_n-M_n}{B_n} \right| > \frac{1}{\varepsilon} \right\}.
\]

Letting \( z_n = |M_n| \), we have

\[
\sum_{|k-M_n|\leq B_n/\sqrt{\pi}} \mathbb{P}\{S_n = k\} = \frac{1}{\sqrt{2\pi} B_n} \sum_{|k-M_n|\leq B_n/\sqrt{\pi}} e^{-\frac{(k-M_n)^2}{2B_n} - \frac{1}{\varepsilon} \phi(B_n)} \leq \frac{C}{B_n \phi(B_n)} \sum_{|k-M_n|\leq B_n/\sqrt{\pi}} 1 \leq \frac{2C}{h \varepsilon} \phi(B_n).
\]

Now using the elementary inequality \((a+b)^2 \leq 2(a^2 + b^2)\) for reals \( a, b \), we have \( |Z-z_n| \leq \sqrt{2(|Z| + |z_n|)} \) and \( |Z-z_n|^2 \geq |Z|^2/2 - z_n^2 \). We can thus continue as follows

\[
\leq \sum_{|Z-z_n| > B_n/\sqrt{\pi}} e^{-\frac{(Z-z_n)^2}{2B_n} - \frac{1}{\varepsilon} \phi(B_n)} \leq e^{1/2B_n} \sum_{|Z-z_n| > B_n/\sqrt{\pi}} e^{-\frac{Z^2}{4B_n^2}}.
\]

Assume that \( B_n \geq \max(1/\sqrt{2}, 4\sqrt{2}) \), then \( B_n/2\sqrt{2} - 2 \geq B_n/(\sqrt{2}) - 1 \). In particular \( |Z| \geq 1 \) in the previous series, and so we have the estimates

\[
\leq 2e^{1/2B_n} \sum_{Z > (B_n/2\sqrt{2}) + 1} e^{-\frac{Z^2}{4B_n^2}} \leq 2e^{\int_{B_n/(2\sqrt{2})+1}^{\infty} e^{-\frac{t^2}{4B_n^2}} dt} \leq 2e^{\int_{B_n/(2\sqrt{2})}^{\infty} e^{-\frac{t^2}{4B_n^2}} dt}.
\]
\[
(t = \sqrt{2}B_n u) = 2\sqrt{2}eB_n \int_{1/4\sqrt{\pi}}^{\infty} e^{-\frac{u^2}{\pi}} du \\
\leq 2\sqrt{2}eB_n \sqrt{\frac{\pi}{2}} e^{-1/(16\varepsilon)} \\
= 2e\sqrt{\pi}B_n e^{-1/(16\varepsilon)},
\]

since \(e^{x^2/2} \int_x^{\infty} e^{-t^2/2} dt \leq \sqrt{\pi}, \) for any \(x \geq 0.\)

Therefore

\[
\left| \mathbb{P}\{S_n \equiv m \pmod{h}\} - \frac{1}{\sqrt{2\pi}B_n} \sum_{k \equiv m (h)} e^{-\frac{(k-M_n)^2}{2B_n^2}} \right| \\
\leq \mathbb{P}\left\{ \frac{|S_n - M_n|}{B_n} > \frac{1}{\sqrt{\varepsilon}} \right\} + \frac{2C}{h\sqrt{\varepsilon} \phi(B_n)} + C_1 e^{-1/(16\varepsilon)},
\]

with \(C_1 = 2e\sqrt{\pi}.\)

Recall Poisson summation formula: for \(x \in \mathbb{R}, \ 0 \leq \delta \leq 1,\)

\[
\sum_{\ell \in \mathbb{Z}} e^{-(\ell + \delta)^2 \pi x^{-1}} = x^{1/2} \sum_{\ell \in \mathbb{Z}} e^{2\pi\ell\delta - \ell^2 \pi x}.
\]

Write \(k = m + lh, \ M'_n = M_n - m,\)

\[
(\frac{k - M_n}{2B_n^2})^2 = \frac{(lh - M'_n)^2}{2B_n^2} = \frac{(l - [M'_n/h] + \{M'_n/h\})^2}{2B_n^2/h^2} = \frac{(\ell + \{M'_n/h\})^2}{2B_n^2/h^2},
\]

letting \(\ell = l - [M'_n/h].\)

By applying it with \(x = 2B_n^2 \pi/h^2, \ \delta = \{M'_n/h\},\) we get

\[
\sum_{k \equiv m (h)} e^{-\frac{(k-M_n)^2}{2B_n^2}} = \sum_{\ell \in \mathbb{Z}} e^{-\frac{(\ell-(M'_n/h))^2}{2B_n^2/h^2}} = \frac{\sqrt{2\pi}B_n}{h} \sum_{\ell \in \mathbb{Z}} e^{-2\pi\ell\{M'_n/h\} - 2\pi^2 B_n^2 \ell^2/h^2}.
\]

Whence

\[
\left| \frac{h}{\sqrt{2\pi}B_n} \sum_{k \equiv m (h)} e^{-\frac{(k-M_n)^2}{2B_n^2}} - 1 \right| \leq \sum_{|\ell| \geq 1} e^{-2\pi^2 B_n^2 \ell^2/h^2}.
\]

But for any positive real \(a,\)

\[
\sum_{H=1}^{\infty} e^{-a H^2} \leq \frac{\sqrt{\pi}}{2} \min(\frac{1}{\sqrt{a}}, a).
\]

Therefore with \(a = 2\pi^2 B_n^2/h^2,\)

\[
\left| \frac{h}{\sqrt{2\pi}B_n} \sum_{k \equiv m (h)} e^{-\frac{(k-M_n)^2}{2B_n^2}} - 1 \right| \leq \sqrt{\pi} \min(\frac{h}{\sqrt{2\pi}B_n}, \frac{h^2}{2\pi^2 B_n^2}) \leq \frac{h}{\sqrt{2\pi} B_n}.
\]

We have thus obtained the explicit bound

\[
\left| \frac{1}{\sqrt{2\pi}B_n} \sum_{k \equiv m (h)} e^{-\frac{(k-M_n)^2}{2B_n^2}} - \frac{1}{h} \right| \leq \frac{1}{\sqrt{2\pi} B_n}.
\]

By carrying it back to (1.16), we get for any \(\varepsilon > 0,\) all \(n\) such that \(B_n \geq \max(1/\sqrt{2}, 4\sqrt{2\varepsilon}),\)

and all \(h \geq 2,\)
This is fulfilled if we choose $0 < \varepsilon \leq 1$, and $n$ such that $B_n \geq 6$, whence the claimed estimate. \hfill \Box

2. Local limit theorem in the strong form

There are easy examples of sequences $X$ for which the fulfillment of the local limit theorem depends on the behavior of the first members of $X$. Hence it is reasonable to introduce the following definition due to Prohorov [21]. A local limit theorem in the strong form (or in a strengthened form) is said to be applicable to $X$, if a local limit theorem in the usual form is applicable to any subsequence extracted from $X$, which differs from $X$ only in a finite number of members.

This definition can be made a bit more convenient, see Gamkrelidze [7]. Let

$$S_{k,n} = X_{k+1} + \ldots + X_{k+n}, \quad A_{k,n} = E(S_{k,n}), \quad B_{k,n}^2 = \text{Var}(S_{k,n}).$$

The local limit theorem in the strong form holds if and only if

$$P\{S_{k,n} = m\} = \frac{D}{B_{k,n}\sqrt{2\pi}} e^{-\frac{(m-A_{k,n})^2}{2B_{k,n}^2}} + o\left(\frac{1}{B_{k,n}}\right),$$

uniformly in $m$ and every finite $k$, $k = 0, 1, 2, \ldots$, as $n \to \infty$ and $B_{k,n} \to \infty$.

Rozanov’s necessary condition states as follows.

**Theorem 2.1** ([23], Th. I). Let $X = \{X_j, j \geq 1\}$ be a sequence of independent, square integrable random variables taking values in $\mathbb{Z}$. Let $b_k^2 = \text{Var}(X_k), B_n^2 = b_1^2 + \ldots + b_n^2$. Assume that

$$B_n \to \infty \quad \text{as} \quad n \to \infty.$$  

The following condition is necessary for the applicability of a local limit theorem in the strong form to the sequence $X$,

$$\prod_{k=1}^{\infty} \left[ \max_{0 \leq m < h} P\{X_k \equiv m \pmod{h}\} \right] = 0 \quad \text{for any} \quad h \geq 2.$$  

Condition (2.4) is also sufficient in some important examples, in particular if $X_j$ have stable limit distribution, see Mitalasauskas [15]. We briefly indicate how Theorem 2.1 is proved. If the local limit theorem in the strong form is applicable to the sequence $X$, then

$$\sum_{k=1}^{\infty} P\{X_k \not\equiv 0 \pmod{h}\} = \infty, \quad \text{for any} \quad h \geq 2.$$  

Indeed, otherwise given $h \geq 2$, by the Borel–Cantelli lemma, on a set of measure greater than $3/4$, $X_k \equiv 0 \pmod{h}$ for all $k \geq k_0$, say. The new sequence $X'$ defined by $X'_k = 0$ if $k < k_0$, $X'_k = X_k$ unless, with partial sums $S'_n$, verifies $P\{S'_n \equiv 0 \pmod{h}\} > 3/4$ for all $n$ large.
enough, and this can be used to bring a contradiction with the fact that \( \mathbb{P}\{S_n' \equiv 0 \pmod{h}\} \) should converge to \( 1/h \).

The arithmetical quantity

\[
\max_{0 \leq m < h} \mathbb{P}\{X_k \equiv m \pmod{(h)}\}
\]

also appears in the study of local limit theorems with arithmetical sufficient conditions. The approaches used (Freiman, Moskvin and Yudin [5], Mitalauskas [16], Raudelyunas [22] and later Fomin [4], for instance) require the random variables to do not overly much concentrate in a particular residue class \( m \pmod{h} \) of \( \mathbb{Z} \), and impose arithmetical conditions of type: For all \( h \geq 2 \)

\[
\max_{0 \leq m < h} \mathbb{P}\{X_k \equiv m \pmod{(h)}\} \leq 1 - \alpha_k,
\]

for all \( k \), where \( \alpha_k \) is some specific sequence of reals decreasing to 0. In addition, one generally have that \( \sum_k \alpha_k = \infty \). Although the simple form of local limit theorem is here considered, for obvious reasons, condition (2.6) brings nothing more in this context.

As a consequence of the quantitative formulation of the a.u.d. property obtained in Theorem 1.4, we have the following result.

**Theorem 2.2.** Under the assumptions of Theorem 2.1, assume further that the local limit theorem is applicable to a sequence \( X \). Then

1. \( \limsup_{h \to \infty} \prod_{k=1}^{\infty} \max_{0 \leq m < h} \mathbb{P}\{X_k \equiv m \pmod{(h)}\} = 0. \)

2. There exists a function \( 1 \leq \phi(t) \to \infty \) as \( t \to \infty \), such that

\[
\sum_{k=1}^{n} \max_{0 \leq m < h} \mathbb{P}\{X_k \equiv m \pmod{(h)}\} \leq \phi(B_n)^2/3, \quad \text{and} \quad C, C_1 \text{ are absolute constants.}
\]

**Proof.** We purport a direct argument. Consider a sequence \( Y \) where \( Y_k = X_k - m_k \), \( m_k \) are integers, for all \( k \geq 1 \). Let \( h \geq 2 \) be fixed. Choose \( m_k \) so that

\[
\max_{0 \leq m < h} \mathbb{P}\{X_k \equiv m \pmod{(h)}\} = \mathbb{P}\{X_k \equiv m_k \pmod{(h)}\} = \mathbb{P}\{Y_k \equiv 0 \pmod{(h)}\},
\]

and let \( \mu_n = \sum_{k=1}^{n} m_k \). Note that \( \sum_{k=1}^{n} Y_k = S_n - \mu_n \), \( \text{Var}(\sum_{k=1}^{n} Y_k) = \text{Var}(S_n) = B_n^2 \).

As the local limit theorem is applicable to the sequence \( X \), condition (1.13) is satisfied for some function \( 1 \leq \phi(t) \to \infty \) as \( t \to \infty \), namely we have for all \( n \),

\[
\sup_{\nu \in \mathbb{Z}} B_n \mathbb{P}\{S_n = \nu\} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(\nu - \mu_n)^2}{2B_n^2}} \leq \frac{C}{\phi(B_n)}.
\]

Given \( n \), letting \( \nu = m + \mu_n \) and observing that \( \mathbb{P}\{\sum_{k=1}^{n} Y_k = m\} = \mathbb{P}\{S_n - \mu_n = m\} \), we get for \( m \in \mathbb{Z} \), \( n \geq 1 \),

\[
|B_n \mathbb{P}\{\sum_{k=1}^{n} Y_k = m\} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(m + \mu_n - \mu_n)^2}{2B_n^2}}| \leq \frac{C}{\phi(B_n)}.
\]
Thus $Y$ satisfies condition (1.13) with the same function $\phi(n)$.

Applying Remark 1.5 to the sequence $Y$, it follows that,

$$\prod_{k=1}^{n} \max_{0 \leq m < h} \mathbb{P}\{X_k \equiv m \pmod{h}\} = \prod_{k=1}^{n} \mathbb{P}\{Y_k \equiv 0 \pmod{h}\} \leq \mathbb{P}\left\{ \sum_{k=1}^{n} Y_k \equiv 0 \pmod{h}\right\} \leq \frac{1}{h} + H_n,$$

(2.7)

where $H_n$ has the form given in the statement, and $H_n \to 0$ as $n \to \infty$.

Letting $n$ tend to infinity in (2.7) implies,

$$\prod_{k=1}^{\infty} \max_{0 \leq m < h} \mathbb{P}\{X_k \equiv m \pmod{h}\} \leq \frac{1}{h}.$$

(2.8)

This being true for each $h$, $h \geq 2$, letting now $h$ tend to infinity in (2.8) yields,

$$\limsup_{h \to \infty} \prod_{k=1}^{\infty} \max_{0 \leq m < h} \mathbb{P}\{X_k \equiv m \pmod{h}\} = 0.$$

(2.9)

We also have by using the elementary inequality $\log(1 - x) \geq -x/(1 - x)$, $0 \leq x < 1$,

$$\prod_{k=1}^{n} \mathbb{P}\{Y_k \equiv m \pmod{h}\} = \prod_{k=1}^{n} \left(1 - \mathbb{P}\{Y_k \not\equiv m \pmod{h}\}\right) = e^{-\sum_{k=1}^{n} \log(1 - \mathbb{P}\{Y_k \not\equiv m \pmod{h}\})} \geq e^{-\sum_{k=1}^{n} \mathbb{P}\{Y_k \not\equiv m \pmod{h}\}/(1 - \mathbb{P}\{Y_k \not\equiv m \pmod{h}\})}.$$

Thus by Remark 1.5

$$\sum_{k=1}^{n} \frac{\max_{0 \leq m < h} \mathbb{P}\{X_k \equiv m \pmod{h}\}}{1 - \max_{0 \leq m < h} \mathbb{P}\{X_k \equiv m \pmod{h}\}} = \sum_{k=1}^{n} \frac{\mathbb{P}\{Y_k \not\equiv m \pmod{h}\}}{1 - \mathbb{P}\{Y_k \not\equiv m \pmod{h}\}} \geq -\log\left(\frac{1}{h} + H_n\right).$$

□

**Remark 2.3. (i)** Note that the bound used in (2.7) is very weak since

$$\prod_{k=1}^{n} \mathbb{P}\{Y_k \equiv m \pmod{h}\} = \mathbb{P}\{\forall J \subset [1, n], \sum_{k \in J} Y_k \equiv m \pmod{h}\}.$$

One can replace individuals $Y_k$ by sums over blocks according to any partition of $\{1, \ldots, n\}$.

(ii) Sets of multiples serve as good test sets for the applicability of the local limit theorem because addition is a closed operation. What can be derived when testing the applicability of the local limit theorem with other remarkable sets of integers (squarefree numbers, primes numbers, power numbers, geometric growing sequences, …) is unknown. Concerning the squarefree integers, namely having no squared prime factors, we note the bound

$$\left| 2^{-n} \sum_{j \text{ squarefree}} C_n^j - \frac{6}{\pi^2} \right| \leq C_1 e^{-C_2 (\log n)^{3/5}/(\log \log n)^{1/5}}.$$

(2.10)

We refer to [2].
3. Random sequences satisfying the a.u.d. property

It has some interest to relate the a.u.d. property for Bernoulli sums to the one of sets having Euler density, in this particular case here, arithmetic progressions. A subset \( A \) of \( \mathbb{N} \) is said to have Euler density \( \lambda \) with parameter \( \rho \) (in short \( E_\rho \) density \( \lambda \)) if

\[
\lim_{n \to \infty} \sum_{j \in A} C_n^j \rho^j (1 - \rho)^{n-j} = \lambda.
\]

By a result due to Diaconis and Stein, we have the following characterization.

**Theorem 3.1** ([2], Th. 1). For any \( A \subset \mathbb{N} \), and \( \rho \in [0,1[ \) the following assertions are equivalent:

(i) \( A \) has \( E_\rho \) density \( \lambda \),

(ii) \( \lim_{t \to \infty} e^{-t} \sum_{j \in A} t^j / j! = \lambda \),

(iii) for all \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \frac{\# \{ j \in A : n \leq j < n + \varepsilon \sqrt{n} \}}{\varepsilon \sqrt{n}} = \lambda.
\]

Applying (iii) with \( \rho = \frac{1}{2} \), to \( (3.1) \) \( A = \{ u + kd, \ k \geq 1 \} \), straightforwardly implies

**Lemma 3.2.** Let \( \mathcal{B}_n = \beta_1 + \ldots + \beta_n \), where \( \beta_i \) are i.i.d. Bernoulli random variables. Then \( \{ \mathcal{B}_n, n \geq 1 \} \) is a.u.d.\((d)\) for any \( d \geq 2 \).

Now consider the independent case and introduce the following characteristic. Let \( Y \) be a random variable with values in \( \mathbb{Z} \). Put

\[
\vartheta_Y = \sum_{k \in \mathbb{Z}} \mathbb{P}\{ Y = k \} \wedge \mathbb{P}\{ Y = k + 1 \},
\]

where \( a \wedge b = \min(a,b) \). Note that \( 0 \leq \vartheta_Y < 1 \).

**Theorem 3.3.** Let \( X = \{ X_j, j \geq 1 \} \) be a sequence of independent random variables taking values in \( \mathbb{Z} \). Assume that \( \vartheta_{X_j} > 0 \) for each \( j \). Further assume that the series \( \sum_{j=1}^{\infty} \vartheta_{X_j} \) diverges. Then \( X \) is a.u.d., the conclusion holds in particular if the \( X_j \) are i.i.d. and \( \vartheta_{X_1} > 0 \).

Note that no integrability condition is required, whereas square integrability is required in order that the local limit theorem be applicable. We prove in the next section that if the series \( \sum_{j=1}^{\infty} \vartheta_{X_j} \) diverges, much more is in fact true. Under the assumption made, each \( X_j \) admits a Bernoulli component. This is the principle of a coupling method (the Bernoulli part extraction) introduced by McDonald [14], Davis and McDonald [1] in the study of the local limit theorem. See Weber [28] for an application of this method to almost sure local limit theorem, and Giuliano and Weber [9] where this method is used to obtain approximate local limit theorems with effective rate.

Before passing to the proof, we briefly recall some facts and state an auxiliary Lemma. Let \( \mathcal{L}(v_0, D) \) be a lattice defined by the sequence \( v_k = v_0 + Dk, \ k \in \mathbb{Z}, \ v_0 \) and \( D > 0 \) being real numbers. Let \( X \) be a random variable such that \( \mathbb{P}\{ X \in \mathcal{L}(v_0, D) \} = 1 \), and assume that
\( \vartheta_X > 0 \). Let \( f(k) = \mathbb{P}\{X = \tau_k\}, k \in \mathbb{Z} \). Let also \( 0 < \vartheta \leq \vartheta_X \). Associate to \( \vartheta \) and \( X \) a sequence \( \{\tau_k, k \in \mathbb{Z}\} \) of non-negative reals such that
\[
(3.3) \quad \tau_{k+1} + \tau_k \leq 2f(k), \quad \sum_{k \in \mathbb{Z}} \tau_k = \vartheta.
\]

For instance \( \tau_k = \frac{\vartheta}{\vartheta_X} (f(k) \wedge f(k+1)) \) is suitable. Next define a pair of random variables \((V, \varepsilon)\) as follows:
\[
\begin{align*}
\mathbb{P}\{(V, \varepsilon) = (\tau_k, 1)\} &= \tau_k, \\
\mathbb{P}\{(V, \varepsilon) = (\tau_k, 0)\} &= f(k) - \frac{\tau_{k-1} + \tau_k}{2}.
\end{align*}
\]

(\forall k \in \mathbb{Z})

**Lemma 3.4.** Let \( L \) be a Bernoulli random variable which is independent of \((V, \varepsilon)\), and let \( Z = V + \varepsilon DL \). Then \( Z \overset{d}{=} X \).

**Proof of Theorem 3.3.** We apply Lemma 3.4 with \( D = 1 \) to each \( X_j \), and choose \( 0 < \vartheta_j \leq \vartheta_{X_j} \) so that the series \( \sum_{j=1}^{\infty} \vartheta_j \) diverges. One can associate to them a sequence of independent vectors \((V_j, \varepsilon_j, L_j), j = 1, \ldots, n\) such that
\[
(3.5) \quad \{V_j + \varepsilon_j L_j, j = 1, \ldots, n\} \overset{d}{=} \{X_j, j = 1, \ldots, n\}.
\]

Further the sequences \( \{(V_j, \varepsilon_j), j = 1, \ldots, n\} \) and \( \{L_j, j = 1, \ldots, n\} \) are independent. For each \( j = 1, \ldots, n \), the law of \((V_j, \varepsilon_j)\) is defined according to (3.4) with \( \vartheta = \vartheta_j \). And \( \{L_j, j = 1, \ldots, n\} \) is a sequence of independent Bernoulli random variables. Set
\[
(3.6) \quad W_n = \sum_{j=1}^{n} V_j, \quad M_n = \sum_{j=1}^{n} \varepsilon_j L_j, \quad B_n = \sum_{j=1}^{n} \varepsilon_j.
\]

Denoting again \( X_j = V_j + \varepsilon_j L_j, j \geq 1 \), we have
\[
(3.7) \quad \mathbb{P}\{d|S_n + u\} = \mathbb{E}(V, \varepsilon) \mathbb{P}_L\{d\left(\sum_{j=1}^{n} \varepsilon_j L_j + W_n\right) + u\}.
\]

As \( \sum_{j=1}^{n} \varepsilon_j L_j \overset{d}{=} \sum_{j=1}^{B_n} L_j \), we have
\[
\mathbb{P}_L\{d\left(\sum_{j=1}^{n} \varepsilon_j L_j + W_n\right) + u\} = \mathbb{P}_L\{d\sum_{j=1}^{B_n} L_j + (W_n + u)\}.
\]

In view of the dominated convergence theorem, it suffices to prove that for each \( d \geq 2 \),
\[
\mathbb{P}_L\{d\sum_{j=1}^{B_n} L_j + (W_n + u)\} \rightarrow \frac{1}{d},
\]

as \( n \rightarrow \infty, \mathbb{P}(V, \varepsilon) \) almost surely. But the set (compare with (3.1))
\[
A = \{(W_n + u) + kd, k \geq 1\},
\]

now depends on \( W_n \), thus on \( n \), which is complicating things. However we can write
\[
\chi\left(d\left|\sum_{j=1}^{B_n} L_j + (W_n + u)\right\right) = \frac{1}{d} \sum_{j=0}^{d-1} e^{2i\pi d(W_n + u)} e^{2i\pi \frac{d}{d} \sum_{j=1}^{B_n} L_j}.
\]
By integrating with respect to $\mathbb{P}_L$ we get,
\[
\mathbb{P}_L\left\{ d\sum_{j=1}^{B_n} L_j + (W_n + u) \right\} = \frac{1}{d} + \frac{1}{d}\sum_{j=1}^{d-1} e^{2i\pi \frac{x}{d}(W_n + u)} \left( \cos \frac{\pi j}{d} \right) B_n.
\]
By the assumption made, $B_n$ tends to infinity $\mathbb{P}_{(V,\varepsilon)}$ almost surely, ((8.3.5) in [29] for instance). Thus the latter sum tends to 0 as $n \to \infty$, $\mathbb{P}_{(V,\varepsilon)}$ almost surely. Therefore by the convergence argument invoked before, $\mathbb{P}\{d|S_n + u\}$ tends to $\frac{1}{d}$ as $n$ tends to infinity, for any $d \geq 2$ and $u \in \mathbb{N}$. Whence it follows that the sequence $\{S_n, n \geq 1\}$ is a.u.d.

4. Random sequences satisfying a strengthened a.u.d. property.

For Bernoulli sums, the a.u.d. property is only a rough aspect of the value distribution of divisors of $\mathbb{B}_n + u$, $u \geq 0$ integer. Much more is known.

Theorem 4.1 ([25], Th. 2.1). We have the uniform estimate
\[
\sup_{u \geq 0} \sup_{2 \leq d \leq n} \left| \mathbb{P}\{d|\mathbb{B}_n + u\} - \frac{1}{d} \sum_{0 \leq |j| < d} e^{2\pi i (2u+n) \frac{j}{d}} e^{-\frac{n\pi^2 j^2}{2d^2}} \right| = \mathcal{O}\left((\log n)^{5/2} n^{-3/2}\right).
\]

The special case $u = 0$ was proved in [30, Th. II]. Introduce the Theta function
\[
\Theta_u(d, n) = \sum_{\ell \in \mathbb{Z}} e^{i\pi (2u+n) \frac{\ell}{d}} e^{-\frac{n\pi^2 \ell^2}{2d^2}}.
\]

By Poisson summation formula
\[
\Theta_u(d, n) = \left(d\sqrt{\frac{2}{\pi n}}\right) \sum_{\ell \in \mathbb{Z}} e^{-\left(\ell + \left\{\frac{u+n/2}{d}\right\}\right)^2 \frac{2d^2}{n}}.
\]

As a consequence of Theorem 4.1, we get

Corollary 4.2. We have the uniform estimate
\[
\sup_{u \geq 0} \sup_{2 \leq d \leq n} \left| \mathbb{P}\{d|\mathbb{B}_n + u\} - \frac{\Theta_u(d, n)}{d} \right| \leq C (\log n)^{5/2} n^{-3/2}.
\]

Apart from this important but specific case, it seems that the speed of convergence in the limit (1.2) was not investigated, in particular when $d$ and $n$ are varying simultaneously.

Consider the independent case and assume as in Theorem 3.3 that $\nu_n = \sum_{j=1}^{n} \vartheta_j \uparrow \infty$. The speed of uniform convergence over regions (in $d$ and $n$) presents a singularity when $d$ is getting too close to $\sqrt{\nu_n}$. That quantity already appears in Davis and McDonald [1]. On the other hand when $d$ is not close to $\sqrt{\nu_n}$, in a sense that we shall make precise, we show that an explicit speed of convergence can be assigned, this under the sole divergence assumption of the series $\sum_{j=1}^{\infty} \vartheta_j$. So, for this important class of independent sequences, the well-known a.u.d. necessary condition turns up to be a particularly weak requirement. Further one can show by using Poisson summation formula that in the Bernoulli case, the local limit theorem implies a weaker speed of convergence than the one obtained in Theorem 4.1.

The speed of uniform convergence problem for all $d$ and $n$, $n \geq d \geq 2$, $n \to \infty$, is more complicated and one must restrict to the i.i.d. case. In place of the limiting term $1/d$ appears a more complicated Theta elliptic function. See [25]. For the independent case, the approach used becomes inoperant, due to appearance of integral products with interlaced integrants.
In fact, what will make possible to handle the independent case, is not just that $d$ and $\sqrt{\nu_n}$ are not too close, but also that in background, symmetries properties of the Bernoulli model permitted to effect the necessary calculations in the first quadrant and not in the half-circle. This point is crucial for getting the uniform speed of convergence in Theorem 4.1. This is explained in [25], see reduction Lemma 2.3. In short, when the Bernoulli extraction part applies, these symmetry properties allow one to get a speed of convergence. The proof in the Bernoulli case is transposable to other systems of random variables when such symmetries exist. This is not the case for the Hwang and Tsai model of the Dickman function [11], [8], neither for the Cramér model of primes [27].

We prove the following result.

**Theorem 4.3.** Assume that $D = 1$, $\vartheta X_j > 0$ for each $j$, and that the series $\sum_{j=1}^{\infty} \vartheta X_j$ diverges. Let $\alpha > \alpha' > 0$, $0 < \varepsilon < 1$. Then for each $n$ such that

$$|x| \leq \frac{1}{2} \sqrt{\frac{2 \alpha \log (1 - \varepsilon) \nu_n}{(1 - \varepsilon) \nu_n}} \Rightarrow \frac{\sin x}{x} \geq (\alpha'/\alpha)^{1/2},$$

recalling that $\nu_n = \sum_{j=1}^{n} \vartheta_j$, we have

$$\sup_{u \geq 0} \sup_{d < \pi \sqrt{\frac{1 - \varepsilon \nu_n}{2 \alpha \log (1 - \varepsilon) \nu_n}}} \left| \mathbb{P} \{ d | S_n + u \} - \frac{1}{d} \right| \leq 2 e^{-\frac{2}{2} \nu_n} + ((1 - \varepsilon) \nu_n)^{-\alpha'}.$$  

For the proof we use the following Lemma.

**Lemma 4.4 ([13], Theorem 2.3).** Let $X_1, \ldots, X_n$ be independent random variables, with $0 \leq X_k \leq 1$ for each $k$. Let $S_n = \sum_{k=1}^{n} X_k$ and $\mu = \mathbb{E} S_n$. Then for any $\varepsilon > 0$,

(a) $\mathbb{P} \{ S_n \geq (1 + \varepsilon) \mu \} \leq e^{-\frac{\varepsilon^2 \mu}{2(1 + \varepsilon/3)}}$.

(b) $\mathbb{P} \{ S_n \leq (1 - \varepsilon) \mu \} \leq e^{-\frac{2}{2} \mu}$.

We also need the following result.

**Proposition 4.5 ([25], Corollary 2.4).** (i) For each $\alpha > \alpha' > 0$ and $n$ such that $\tau_n \geq (\alpha'/\alpha)^{1/2}$, where

$$\tau_n = \frac{\sin \varphi_n/2}{\varphi_n/2}, \quad \varphi_n = \left(\frac{2 \alpha \log n}{n}\right)^{1/2},$$

we have

$$\sup_{u \geq 0} \sup_{d < \pi \sqrt{\frac{1 - \varepsilon \nu_n}{2 \alpha \log (1 - \varepsilon) \nu_n}}} \left| \mathbb{P} \{ d | S_n + u \} - \frac{1}{d} \right| \leq n^{-\alpha'}.$$

(ii) Let $0 < \rho < 1$. Let also $0 < \eta < 1$, and suppose $n$ sufficiently large so that $\tau_n \geq \sqrt{1 - \eta}$, where

$$\bar{\tau}_n = \frac{\sin \psi_n/2}{\psi_n/2}, \quad \psi_n = \left(\frac{2n^\rho}{n}\right)^{1/2}.$$  

Then,

$$\sup_{u \geq 0} \sup_{d < (\pi/\sqrt{2}) n (1 - \rho)/2} \left| \mathbb{P} \{ d | B_n + u \} - \frac{1}{d} \right| \leq e^{-(1 - \eta) n^\rho}.$$
Proof of Theorem 4.3. We use the Bernoulli part extraction displayed at Lemma 3.4, 3.5, 3.6 as well as the notation introduced. Let

\[ A_n = \{ B_n \leq (1 - \epsilon)\nu_n \}. \]

We deduce from Lemma 4.4 that \( \mathbb{P}\{ A_n \} \leq e^{-\frac{c_1}{\nu_n}} \) for all positive \( n \). We write

\[ \mathbb{P}\{ d | S_n \} - \frac{1}{d} = \mathbb{E}_{(V, \epsilon)} (\chi(A_n) + \chi(A_n^c)) \left( \mathbb{P}_L \{ d | (\sum_{j=1}^{n} \epsilon_j L_j + W_n) \} - \frac{1}{d} \right). \]

On the one hand,

\[ \mathbb{E}_{(V, \epsilon)} \chi(A_n) \big| \mathbb{P}_L \{ d | (\sum_{j=1}^{n} \epsilon_j L_j + W_n) \} - \frac{1}{d} \big| \leq 2 \mathbb{P}\{ A_n \} \leq 2e^{-\frac{c_1}{\nu_n}}. \]

So that

\[ \left| \mathbb{P}\{ d | S_n \} - \frac{1}{d} \right| \leq 2e^{-\frac{c_1}{\nu_n}} + \mathbb{E}_{(V, \epsilon)} \chi(A_n^c) \cdot \left| \mathbb{P}_L \{ d | (\sum_{j=1}^{n} \epsilon_j L_j + W_n) \} - \frac{1}{d} \right|. \]

Now on \( A_n^c, B_n \geq (1 - \epsilon)\nu_n \), and since \( \sqrt{x}\log x \) is increasing on \( [e, \infty) \), we have

\[ \sqrt{\frac{(1 - \epsilon)\nu_n}{2\alpha \log(1 - \epsilon)\nu_n}} \leq \sqrt{\frac{B_n}{2\alpha \log B_n}}. \]

Also

\[ \varphi_n = \sqrt{\frac{2\alpha \log B_n}{B_n}} \leq \sqrt{\frac{2\alpha \log(1 - \epsilon)\nu_n}{(1 - \epsilon)\nu_n}} \quad \text{and thus} \quad \frac{\sin \varphi_n}{\varphi_n} / 2 \geq \frac{(\alpha' / \alpha)^{1/2}}, \]

by the assumption made.

By applying Proposition 4.5 we have \( \mathbb{P}_{(V, \epsilon)} \) almost surely,

\[ \sup_{u \geq 0} \sup_{d < \pi \sqrt{\frac{(1 - \epsilon)\nu_n}{2\alpha \log(1 - \epsilon)\nu_n}}} \left| \mathbb{P}_L \{ d | (\sum_{j=1}^{B_n} L_j + W_n + u) \} - \frac{1}{d} \right| \leq B_n^{-\alpha'}. \]

Whence on \( A_n^c \),

\[ \sup_{u \geq 0} \sup_{d < \pi \sqrt{\frac{(1 - \epsilon)\nu_n}{2\alpha \log(1 - \epsilon)\nu_n}}} \left| \mathbb{P}_L \{ d | (\sum_{j=1}^{B_n} L_j + W_n + u) \} - \frac{1}{d} \right| \leq \sup_{u \geq 0} \sup_{d < \pi \sqrt{\frac{(1 - \epsilon)\nu_n}{2\alpha \log(1 - \epsilon)\nu_n}}} \left| \mathbb{P}_L \{ d | (\sum_{j=1}^{B_n} L_j + W_n + u) \} - \frac{1}{d} \right| \leq B_n^{-\alpha'} \leq (1 - \epsilon)\nu_n)^{-\alpha'}. \]

In view of (4.6) and (4.9), we get for all \( u \geq 0 \) and \( d < \pi \sqrt{\frac{(1 - \epsilon)\nu_n}{2\alpha \log(1 - \epsilon)\nu_n}} \),

\[ \left| \mathbb{P}\{ d | S_n + u \} - \frac{1}{d} \right| \leq 2e^{-\frac{c_2}{\nu_n}} + ((1 - \epsilon)\nu_n)^{-\alpha'} \mathbb{E}_{(V, \epsilon)} \chi(A_n^c) \]

\[ \leq 2e^{-\frac{c_2}{\nu_n}} + ((1 - \epsilon)\nu_n)^{-\alpha'}. \]

\( \Box \)
The next result shows a considerable variation of the speed of convergence when $d$ is less close to $\sqrt{\nu_n}$.

**Theorem 4.6.** Let $0 < \rho < 1$ and $0 < \varepsilon < 1$. Then for each $n$ such that

$$|x| \leq \frac{1}{2} \sqrt{\frac{2}{((1 - \varepsilon)\nu_n)^{1-\rho}}} \Rightarrow \frac{\sin x}{x} \geq \sqrt{1 - \varepsilon}$$

we have

$$\sup_{u \geq 0} \sup_{d < (\pi/\sqrt{2})B_n^{(1-\rho)/2}} \left| \mathbb{P}\{d|S_n + u\} - \frac{1}{d} \right| \leq 2e^{-\frac{c^2}{2}\nu_n} + e^{-(1-\varepsilon)\rho,\nu_n}.$$

**Proof.** The proof is similar. We operate with the same set $A_n$ as in (4.3), and use the decomposition (3.4). Let $0 < \rho < 1$ and $0 < \varepsilon < 1$.

By applying Proposition 4.5 with $\eta = \varepsilon$, we have $\mathbb{P}(V,\varepsilon)$ almost surely, for $n$ such that $\tilde{\tau}_n \geq \sqrt{1 - \varepsilon}$, where here

$$\tilde{\tau}_n = \frac{\sin \psi_n/2}{\psi_n/2} \quad \text{with} \quad \psi_n = \left(\frac{2B_n^\rho}{B_n}\right)^{1/2},$$

$$\sup_{u \geq 0} \sup_{d < (\pi/\sqrt{2})B_n^{(1-\rho)/2}} \left| \mathbb{P}_L\left\{ d \left( \sum_{j=1}^{B_n} L_j + W_n + u \right) \right\} - \frac{1}{d} \right| \leq e^{-(1-\varepsilon)B_n^\rho}.$$

By using corresponding estimates to (4.7), (4.8), namely that on $A_n^c$,

$$\psi_n = \left(\frac{2}{B_n^{(1-\rho)}}\right)^{1/2} \leq \left(\frac{2}{((1 - \varepsilon)\nu_n)^{1-\rho}}\right)^{1/2},$$

so that $\tilde{\tau}_n \geq \sqrt{1 - \varepsilon}$, we deduce that on $A_n^c$,

$$\sup_{u \geq 0} \sup_{d < (\pi/\sqrt{2})B_n^{(1-\rho)/2}} \left| \mathbb{P}_L\left\{ d \left( \sum_{j=1}^{B_n} L_j + W_n + u \right) \right\} - \frac{1}{d} \right| \leq \sup_{u \geq 0} \sup_{d < (\pi/\sqrt{2})B_n^{(1-\rho)/2}} \left| \mathbb{P}_L\left\{ d \left( \sum_{j=1}^{B_n} L_j + W_n + u \right) \right\} - \frac{1}{d} \right| \leq e^{-(1-\varepsilon)B_n^\rho}.$$

Therefore

$$\sup_{u \geq 0} \sup_{d < (\pi/\sqrt{2})B_n^{(1-\rho)/2}} \left| \mathbb{P}\{d|S_n + u\} - \frac{1}{d} \right| \leq 2e^{-\frac{c^2}{2}\nu_n} + \mathbb{E}(V,\varepsilon) \chi(A_n^c) e^{-(1-\varepsilon)B_n^\rho} \leq 2e^{-\frac{c^2}{2}\nu_n} + e^{-(1-\varepsilon)^{1+\rho}\nu_n^\rho}. \quad (4.11)$$

**Remark 4.7.** So far we only have considered necessary conditions for the validity of the local limit theorem, which are formulated in terms of a.u.d. property, as well as strengthenings of this property yielding effective speed of convergence bounds. It is important to mention in that context, that in 1984, Mukhin found a remarkable necessary and sufficient condition for the validity of the local limit theorem. Let $\{S_n, n \geq 1\}$ be a sequence of $\mathbb{Z}$-valued random variables such that an integral limit theorem holds: there exist $a_n \in \mathbb{R}$ and real $b_n \to \infty$ such that the sequence of distributions of $(S_n - a_n)/b_n$ converges weakly to an absolutely
continuous distribution $G$ with density $g(x)$, which is uniformly continuous in $\mathbb{R}$. The local limit theorem is valid if
\begin{equation}
\mathbb{P}\{S_n = m\} = B_n^{-1}g\left(\frac{m - A_n}{B_n}\right) + o(B_n^{-1}),
\end{equation}
uniformly in $m \in \mathbb{Z}$. Muhkin showed that the validity of the local limit theorem is equivalent to the existence of a sequence of integers $v_n = o(b_n)$ such that
\begin{equation}
\sup_m \left| \mathbb{P}\{S_n = m + v_n\} - \mathbb{P}\{S_n = m\} \right| = o\left(\frac{1}{b_n}\right),
\end{equation}
Revisiting the succinct proof given in [19], we however could only prove rigorously a weaker necessary and sufficient condition, with a significantly different formulation, namely that a necessary and sufficient condition for the local limit theorem in the usual form to hold is
\begin{equation}
\sup_{m,k \in \mathbb{Z}, |m-k| \leq \max\{1,\sqrt{n}\}} \left| \mathbb{P}\{S_n = m\} - \mathbb{P}\{S_n = k\} \right| = o\left(\frac{1}{b_n}\right),
\end{equation}
where
\begin{equation}
\varepsilon_n := \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left\{\frac{S_n - a_n}{b_n} < x\right\} - G(x) \right| \to 0,
\end{equation}
by the integral limit theorem. This is the object of the Note [26], with remarks and references on general relations of type (4.13) therein. Mukhin wrote at this regard in [19]: "... getting from here more general sufficient conditions turns out to be difficult in view of the lack of good criteria. Working with asymptotic equidistribution properties are more convenient in this respect".

**Appendix A. LLT’s with speed of convergence.**

Let $S_n = X_1 + \ldots + X_n$, $n \geq 1$, where $X_j$ are independent random variables such that $\mathbb{P}\{X_j \in \mathcal{L}(v_0, D)\} = 1$.

Assume first that the random variables $X_j$ are identically distributed. Then we have the following characterization result.

**Theorem A.1.** Let $F$ denote the distribution function of $X_1$.

(i) ([12], Theorem 4.5.3) In order that the property
\begin{equation}
\sup_{N=an+Dk} \left| \frac{\sigma \sqrt{n}}{D} \mathbb{P}\{S_n = N\} - \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(N-a\mu)^2}{2\sigma^2}} \right| = O(n^{-\alpha/2}),
\end{equation}
where $0 < \alpha < 1$, it is necessary and sufficient that the following conditions be satisfied:

1. $D$ is maximal,
2. $\int_{|x| \geq u} x^2 F(dx) = O(u^{-\alpha})$ as $u \to \infty$.

(ii) ([20] Theorem 6 p. 197) If $E|X_1|^3 < \infty$, then (A.1) holds with $\alpha = 1/2$.

Now consider the non-identically distributed case. Assume that (see (3.2))
\begin{equation}
\vartheta X_j > 0, \quad j = 1, \ldots, n.
\end{equation}
Let \( \nu_n = \sum_{j=1}^{n} \vartheta_j \). Let \( \psi : \mathbb{R} \to \mathbb{R}^+ \) be even, convex and such that \( \frac{\psi(x)}{x^2} \) and \( \frac{x^3}{\psi(x)} \) are non-decreasing on \( \mathbb{R}^+ \). We further assume that
\[
\mathbb{E} \psi(X_j) < \infty.
\]
(A.3)

Put
\[
L_n = \frac{\sum_{j=1}^{n} \mathbb{E} \psi(X_j)}{\psi(\sqrt{\text{Var}(S_n)})}.
\]

The following result is Corollary 1.7 in Giuliano-Weber in [9].

**Theorem A.2.** Assume that \( \frac{\log \nu_n}{\nu_n} \leq 1/14 \). Then, for all \( \kappa \in L(v_{0n}, D) \) such that
\[
\frac{(\kappa - E S_n)^2}{\text{Var}(S_n)} \leq \sqrt{\frac{7 \log \nu_n}{2 \nu_n}},
\]
we have
\[
\left| \mathbb{P}\{S_n = \kappa\} - De^{-\frac{(\kappa - E S_n)^2}{2 \text{Var}(S_n)}} \right| \leq C_3 \left\{ D \left( \frac{\log \nu_n}{\text{Var}(S_n) \nu_n} \right)^{1/2} + \frac{L_n + \nu_n^{-1}}{\sqrt{\nu_n}} \right\}.
\]
And \( C_3 = \max(C_2, 2^{3/2}C_E) \), \( C_E \) being an absolute constant arising from Berry-Esseen's inequality.

We pass to another speed of convergence result due to Mukhin. Consider the structural characteristic of a random variable \( X \), introduced and studied by Mukhin in [17] and [18] for instance,
\[
H(X, d) = \mathbb{E} \langle X^* d \rangle^2,
\]
where \( \langle \alpha \rangle \) denotes the distance from \( \alpha \) to the nearest integer, and \( X^* \) is a symmetrization of \( X \). Let \( \varphi_X \) be the characteristic function \( X \). The two-sided inequality
\[
1 - 2 \pi^2 H(X, \frac{t}{2\pi}) \leq |\varphi_X(t)| \leq 1 - 4H(X, \frac{t}{2\pi}),
\]
is established in the above references. See also Szewczak and Weber [24] for more.

The following is the one-dimensional version of Theorem 5 in [18], see also [24] and is stated without proof, however.

**Theorem A.3** (Mukhin). Let \( X_1, \ldots, X_n \) have zero mean and finite third moments. Let
\[
B_n^2 = \sum_{j=1}^{n} \mathbb{E} |X_j|^2, \quad H_n = \inf_{1/4 \leq d \leq 1/2} \sum_{j=1}^{n} H(X_j, d), \quad L_n = \frac{\sum_{j=1}^{n} \mathbb{E} |X_j|^3}{(B_n)^{3/2}}.
\]

Then
\[
\sup_{N = v_{0n} + Dk} \left| B_n \mathbb{P}\{S_n = N\} - \frac{D}{\sqrt{2\pi}} e^{-\frac{(N-M_n)^2}{2B_n^2}} \right| \leq C L_n \left( B_n/H_n \right).
\]
(A.5)
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