A RENORMALIZATION GROUP STUDY OF HELIMAGNETS IN $D = 2 + \epsilon$ DIMENSIONS

P. Azaria*, B. Delamotte†, F. Delduc‡ and T. Jolicoeur§

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Abstract

The non linear sigma model $O(N) \otimes O(2)/O(N - 2) \otimes O(2)$ describing the phase transition of N-components helimagnets is built and studied up to two loop order in $D = 2 + \epsilon$ dimensions. It is shown that a stable fixed point exists as soon as $N$ is greater than 3 (or equal) in the neighborhood of two dimensions. The critical exponents $\nu$ and $\eta$ are obtained. In the $N = 3$ case, the symmetry of the system is dynamically enlarged at the fixed point from $O(3) \otimes O(2)/O(2)$ to $O(3) \otimes O(3)/O(3) \sim O(4)/O(3)$. We show that the order parameter for Heisenberg helimagnets involves a tensor representation of $O(4)$ and we verify it explicitly at one loop order on the value of the exponents. We show that for large $N$ and in the neighborhood of two dimensions this nonlinear sigma model describes the same critical theory as the Landau-Ginzburg linear theory. As a consequence, the critical behaviour evolves smoothly between $D = 2$ and $D = 4$ dimensions in this limit.

However taking into account the old results from the $D = 4 - \epsilon$ expansion of the linear theory, we show that most likely the nature of the transition must change between $D = 2$ and $D = 4$ dimensions for small enough $N$ (including $N = 3$). The simplest possibility is that there exists a dividing line $N_c(D)$ in the plane $(N, D)$ separating a first-order region containing the Heisenberg point at $D = 4$ and a second-order region containing the whole $D = 2$ axis. We conclude that the phase transition of Heisenberg helimagnets in dimension 3 is either first order or second order with $O(4)$ exponents involving a tensor representation or tricritical with mean field exponents.

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1 Introduction

In the recent past, the critical behavior of frustrated spin systems has been the subject of intensive theoretical, numerical and experimental studies (see [?][?] and references therein and [?]). However, there is still no definite conclusion about the nature of the phase transition that occurs in these systems. One of the most striking features of frustrated models is their non trivial ground state which for continuous spin models is in general a canted ground state. Well known examples are incommensurate helimagnets and triangular antiferromagnets among others. As a consequence of the non collinear ordering the $O(3)$ spin rotation group is completely broken in the low temperature phase so that the relevant order parameter is a rotation matrix instead of a vector as in ferromagnetic-like models. One may thus wonder if canted spin models belong to a new universality class. Up to now, no definite answer is known. Experiments on rare earth helimagnets such as Ho, Tb, or Dy for example, do not show any clear evidence for a universal critical behaviour.

From a theoretical point of view, early renormalization group (RG) studies by Garel and Pfeuty[?] found no stable fixed point in the neighborhood of $D = 4$ by studying the Landau-Ginzburg theory of a commensurate helimagnet. This is an example of a fluctuation-induced first order transition. However, early Monte-Carlo studies of several canted models [?][?] found evidence for a continuous transition in three dimensions. If taken seriously these numerical results are in contradiction with the $4 - \epsilon$ prediction. Of course, it is notoriously difficult to discriminate between first and second order phase transitions in Monte-Carlo studies (the two-dimensional five-states Potts model is a well-known weird case for example). Strictly speaking, one cannot exclude the existence of a stable fixed point which manifests itself at a finite distance from $D = 4$, unreachable in an $\epsilon$ expansion around $D = 4$. If this happens to be true, then standard perturbative methods are of no use to study the critical behavior of canted spin models.

There is an alternative perturbative approach to this problem which is the low temperature expansion of the non-linear sigma ($NL\sigma$) model suited to the symmetry breaking scheme of these canted models. In this paper we focus on a simple commensurate helimagnet which is the triangular antiferromagnet with $N$-component classical spins. By stacking triangular planes, this magnet exists in all integer dimensions ($D \geq 2$).

In the case of the triangular antiferromagnet (AFT) with Heisenberg classical spins i.e. $N = 3$, the massless modes live in the homogeneous space $G/H = O(3) \otimes O(2)/O(2)$. Some results from the $D = 2 + \epsilon$ expansion of a $NL\sigma$ model based on this coset $G/H$ have been recently reported[?][?]. It has been found that up to two loop a stable fixed point which is the $N = 4$ Wilson-Fisher fixed point shows up in the vicinity of $D = 2$. Thus no new universality class is required in the case of canted spin models. One meets the general phenomenon of increased symmetry at a critical point since at this point the model is $O(3) \otimes O(3) = O(4)$ instead of $O(3) \otimes O(2)$ symmetric.

In this paper, we extend our previous analysis to the case with $N \geq 3$ components. We build up the relevant nonlinear sigma model and analyze its RG properties by standard field-theoretic techniques.

If one believes that both the $\epsilon = 4 - D$ and $\epsilon = D - 2$ perturbative results can be extended to non-zero $\epsilon$, in the neighborhood of $D = 2$ and $D = 4$, as it is the case for the $O(N)$ models, the simplest hypothesis which agrees with both $\epsilon$ expansions is
the following: there is a tricritical surface separating the basin of attraction of the $O(4)$ fixed point found near $D = 2$ from a first order runaway region found in the vicinity of $D = 4$. The phase transition of canted magnets is thus either first order or second order with $O(4)$ or tricritical exponents (i.e. mean-field in $D = 3$). This hypothesis has been previously proposed in ref.\cite{7}. Recent extensive Monte-Carlo studies \cite{8} performed directly in $D=3$ point towards a second-order transition with $O(4)$ exponents, a fact that was missed by previous lower-statistics studies. This means that presumably the critical surface for $N = 3$ lies between $D=3$ and $D=4$.

The manifold $G/H$ is topologically equivalent to $O(3)$ but as metric spaces they are different. The RG properties of the corresponding non-linear sigma model are a priori sensitive to the metric properties. However the study of purely topological properties can be performed directly on $O(3)$ as in ref.\cite{9}. The study of defects reveals the presence of $Z_2$ vortices that are probably liberated in the high-temperature phase of the strictly two-dimensional AFT model. In this work we will ignore global aspects and concentrate on configurations with zero vorticity, leaving for the future the study of the defects on the phase transition.

In this paper, we present the detailed renormalization group study of canted spin systems in $D = 2 + \epsilon$. In section I we show how the effective continuum action is obtained from a lattice Hamiltonian with Heisenberg spins. In section II the group theoretical construction of the non-linear sigma (NL$\sigma$) model is presented. In section III the two loop recursion relations as well as the Callan-Symanzik $\gamma$-function are given. The critical exponents $\nu$ and $\eta$ are calculated. Special attention is given to the nature of the order parameter which is shown to belong to the tensor representation of $O(4)$. In section IV known results from both $4 - \epsilon$ and $1/N$ expansions are recalled for convenience. They are discussed and compared with the $2 + \epsilon$ results in section V. Our conclusions are contained in section VI.

## 2 Continuum limit and effective action

### 2.1 General analysis

The effective action that describes the long distance behavior of a lattice model is obtained by taking the continuum limit of the microscopic Hamiltonian:

$$H = - \sum_{ij} J_{ij} S_i \cdot S_j.$$  

(1)

In this equation the vectors $S_i$ are classical Heisenberg spins with fixed unit length. In a ferromagnetic system, this continuum limit is achieved by letting the spins $S$ fluctuate around their common expectation value. Relative fluctuations between neighboring spins are assumed to be smooth enough so that we may replace $S_i \cdot S_j$ by $(\nabla S(x))^2$.

When the interaction distribution $\{J_{ij}\}$ leads to a canted ground state the continuum limit is less obvious since neighboring spins do not fluctuate around the same mean expectation value. To overcome this difficulty, one has to consider the magnetic cell with $n$ sublattices $(S^1, \ldots, S^n)$ as the basis of a new superlattice where the continuum limit is taken. Practically, this procedure depends on the detailed microscopic model: lattice
symmetry, ground state structure and interaction parameters. We shall however give qualitative arguments valid for many canted models.

Let us define in each elementary cell an orthonormal basis \( \{ e_a(x) \} \):

\[
e_a(x), e_b(x) = \delta_{ab}; \quad a = 1, 2, 3,
\]

where \( x \) is a superlattice index. We may parametrize our \( n \) sublattice spins \( S^\alpha(x), \alpha = 1, \ldots, n \) as:

\[
S^\alpha(x) = \sum_a C^\alpha_a(x) e_a(x).
\]

In the ground state, all the \( S^\alpha \) are in general not independent. There is in fact a maximum of three of them which are independent. Equivalently, there is a minimum of \( n - 3 \) linear combinations of the \( S^\alpha(x) \) which have zero expectation value in the ground state. Such combinations cannot be part of an order parameter. They correspond to relative motions of the spins within each unit cell. They are massive modes with short range correlations and are thus irrelevant to the critical behavior. We ignore them by imposing the constraints that \( locally \), i.e., within each unit cell, the spins are in the ground state configuration. We call this requirement “local rigidity”. Thus, up to an appropriate field redefinition, the order parameter of canted magnets will be the orthonormal basis \( \{ e_a(x) \} \) defined on each site of the superlattice. As a consequence canted magnets are equivalent in the critical region to a system of interacting solid rigid bodies. The continuum effective action \( S_1 \) may be obtained through the standard gradient expansion of the \( e_a(x) \) as in ferromagnets:

\[
S_1 = \frac{1}{2} \int D^D x \left( \sum_{a=1}^{3} p_a (\nabla e_a)^2 \right),
\]

where the ground state is given by the minimization equations:

\[
\nabla e_a(x) = 0, \\
e_a(x) = e^0_a.
\]

The \( p_a, a = 1, 2, 3 \) are coupling constants which depend on the particular lattice model we started with. The partition function \( Z \) is:

\[
Z = \int D e_{1,2,3}(x) \left( \prod_{ab} \delta(e_a(x), e_b(x) - \delta_{ab}) \right) e^{-S_1/T}.
\]

When two couplings \( p_a \) vanish, one recovers, integrating over the corresponding \( e_a \), the action of the standard non linear sigma model \( O(3)/O(2) \) corresponding to collinear ferro or anti-ferromagnets. In all the other cases, among the nine fields \( e^a_0(x) \), taking into account the constraints (2), one sees that there are three independent fluctuating fields corresponding to the three Goldstone modes, or spin waves, resulting from the breakdown of the \( O(3) \) group. Each one corresponds to infinitesimal rotations around each of the \( e_a(x) \)'s. The couplings \( p_a \) are the associated stiffness constants which depend on the detailed microscopic model. They are deeply connected to the symmetry properties of the lattice Hamiltonian as we shall see.
In addition to the usual $O(3)$ rotational invariance, the symmetry group $G$ of the Hamiltonian contains, in general, a discrete group $\mathcal{T}$ of transformations $\{T^s\}$ mixing together the sublattices $S^a$, or equivalently the $e_a$. These transformations may belong to the space group of the lattice, as in triangular antiferromagnets, but may be some more complicated objects, such as “gauge” transformations as in the Villain lattice\[?\]. The order parameter $\{e_a\}, a = 1, 2, 3$ thus transforms under $G$ as:

$$e^i_a \rightarrow \sum_j U_{ij} e^j_a ; \quad U \in O(3),$$

$$e_a \rightarrow \sum_b (T^s)_{ab} e_b ; \quad T^s \in \mathcal{T}.$$  \hspace{1cm} (7)

The requirement that the action $S_1$ should be invariant under the group $\mathcal{T}$ implies several relations between the $p_a$’s. In general, the $e_a$'s span reducible representations of the group $\mathcal{T}$. Depending on the number of these representations, some of the coupling constants $p_a$ may be equal. If there are three irreducible representations of dimension 1, all the $p_a$ are different. If there is one representation of dimension 2 and one of dimension 1, as it is the case in the triangular lattice where $\mathcal{T}$ is $C_{3v}$, two coupling constants are equal: $p_1 = p_2$.

In this case, since the action is quadratic in the fields, the invariance under the discrete group $\mathcal{T}$ is enlarged to a continuous invariance group $O(2)$ generated by:

$$ (e_1, e_2, e_3) \rightarrow (e_1, e_2, e_3) \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$ (e_1, e_2, e_3) \rightarrow (e_1, e_2, e_3) \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

The action $S_1$ is thus $O(3) \otimes O(2)$ invariant in this case.

Finally, if there is only one representation of dimension 3, as it is the case in lattices with tetragonal symmetry, $p_1 = p_2 = p_3$ and $S_1$ is invariant under $G = O(3) \otimes O(3)$. To summarize, depending on the symmetry group of the lattice, $S_1$ can be symmetric under $O(3) \otimes O(p)$ with $p = 1, 2$ or 3.

Since any rotation matrix $R \in SO(3)$ is given by an orthonormal set of three vectors, we can gather the $e_a$’s into a rotation matrix $R$:

$$ R(x) = (e_1(x) e_2(x) e_3(x)), $$

and the action $S_1$ can be written into a different, but equivalent, form:

$$ S_2 = \frac{1}{2} \int d^D x \; Tr \left( P(\partial R^{-1})(\partial R) \right), $$

where $P$ is the diagonal matrix: $P = diag(p_1, p_2, p_3)$ and $R \in SO(3)$.

Using $R$, the symmetry operations on the $e_a$ can be written in a compact form. The action $S_2$ is invariant under left $O(3)$ transformations $R \rightarrow UR$, $U \in O(3)$ and right
transformations: \( R \to RV \) where \( V \) belongs to the \( O(p) \) group which commutes with matrix \( P \). We find again that depending on the value of \( p \), \( S_2 \) is invariant under the group \( G = O(3) \otimes O(p); p = 1, 2, 3 \). The right \( O(p) \) invariance reflects the original discrete symmetry of the microscopic Hamiltonian since it mixes the \( e_a \) while the left \( O(3) \) is the usual rotational symmetry. The discrete symmetry group of the Hamiltonian acts, in the continuum limit, as the \( O(p) \) group. This is an artefact of the continuum limit and has no dynamical consequences since the number of Goldstone modes is given by the breaking of the \( O(3) \) spin rotation group only. Indeed the number \( n \) of Goldstone modes resulting from the symmetry breaking \( G \to H \), where \( H \) is the subgroup of \( G \) which leaves the ground state invariant, is equal to the number of broken generators of \( G \): \( n = \text{dim} \, \text{Lie}(G) - \text{dim} \, \text{Lie}(H) \). In our case, the isotropy subgroup \( H \) consists of the transformations of \( G \) leaving the orthonormal basis \( e_0 \) invariant or equivalently using Eq.(5) and the above \( U \) and \( V \) transformations:

\[
H : R_0 \to \hat{V} R_0 V = R_0, \tag{12}
\]

where \( \hat{V} \in O(p) \subset O(3) \) and is determined once \( V \) is chosen (see Eq.(20) for an explicit expression of \( V \) in a particular example). This particular subgroup \( H \) is called the diagonal group \( O(p)_{\text{diag}} \) of a subgroup \( O(p) \) in \( O(3) \) times the \( O(p) \) of \( G \) acting on the right. The symmetry breaking patterns described by action (11) and equivalently by action (12) are therefore \( G/H = O(3) \otimes O(p)/O(p)_{\text{diag}} \) with \( p = 1, 2, 3 \) depending on matrix \( P \). These are all the possible symmetry breaking schemes that can undergo a frustrated Heisenberg spin model. In any case, there are three Goldstone modes. Before ending this section, let us emphasize that in the non linear sigma model, the dynamical properties depend only on the geometry of the coset space \( G/H \) and not on the field parametrization. Contrary to the Landau-Ginzburg model, the vanishing dimension of the order parameter in two dimensions allows infinitely many different parametrizations of the theory, differing in (possibly non linear) field redefinitions. The choice of one particular parametrization may be useful in discussing symmetry properties or renormalization group equations but does not change the physics. In particular we have seen that the actions \( S_1 \) and \( S_2 \) are equivalent. There is another equivalent form of \( S_1 \) and \( S_2 \) that we shall use for the discussion of the \( O(3) \otimes O(2)/O(2) \) case:

\[
S_3 = \frac{1}{2} \int d^p x \left( g_1 \left( \nabla e_1^2 + \nabla e_2^2 \right) + g_2 \left( e_1 \nabla e_2 - e_2 \nabla e_1 \right)^2 \right), \tag{13}
\]

with:

\[
e_a \cdot e_b = \delta_{ab}, \quad g_2 = -\frac{1}{2} p_2, \quad g_1 = p_1 + p_2. \tag{14}
\]

The latter expression of the action is obtained from (11) by integrating over the constraint \( e_3 = e_1 \wedge e_2 \) and the relations among the coupling constants are derived in Appendix B.

### 2.2 The particular case of the antiferromagnetic triangular lattice

We shall consider the antiferromagnetic triangular lattice with \( N \)-component spins, \( N \geq 3 \), as an example. Let us start by the case \( N = 3 \). The symmetry group of the system
is the product of the rotation group $O(3)$ acting on the spin components times the space group of the triangular lattice. The Hamiltonian density must then be $G = O(3) \otimes C_{3v}$ invariant. The three spins $S_i, i = 1, 2, 3$ of the elementary plaquette are co-planar in the ground state with the well-known 120 degrees structure. Then, we expect that only two vectors of the triad $(e_1, e_2, e_3)$ are necessary for the decomposition of the $S_i$.

i) $\Sigma = S_1 + S_2 + S_3$ spans the trivial representation of $C_{3v}$. This linear combination of the spins has a vanishing expectation value at $T = 0$. It cannot be an order parameter and corresponds to massive modes.

ii) The two vectors:

$$
\begin{pmatrix}
e_1 \\
e_2
\end{pmatrix} \propto
\begin{pmatrix}
-\frac{\sqrt{3}+1}{2}S_1 + \frac{\sqrt{3}-1}{2}S_2 + S_3 \\
\frac{\sqrt{3}-1}{2}S_1 - \frac{\sqrt{3}+1}{2}S_2 + S_3
\end{pmatrix}
$$

span the two dimensional representation of $C_{3v}$. They have a non vanishing expectation value in the low temperature phase and can thus be taken as an appropriate order parameter. The “local rigidity” constraint of section (2.1) means here:

$$
\Sigma(x) =< \Sigma(x) > = 0.
$$

This allows fluctuations of the spins between cells but not within the cells. This constraint is consistent with the symmetry since $\Sigma$ is a scalar for $C_{3v}$. Once the constraint is imposed, it is straightforward to show that $e_1(x)$ and $e_2(x)$ are orthonormal by use of $S_i^2 = 1$. The action for the triangular lattice is then:

$$
S_1 = \frac{1}{2} \int d^Dx \left( p_1 \left( (\nabla e_1(x))^2 + (\nabla e_2(x))^2 \right) \right).
$$

Dombre and Read have obtained, from a direct microscopic derivation [?]:

$$
p_1 = \frac{\sqrt{3} J}{4 T}.
$$

Note that the original $C_{3v}$ invariance has been enlarged in Eq.(14) to a $O(2)$ group given by $\otimes$: $G = O(3) \otimes O(2)$. We will see that this action is not stable under renormalization. The most general renormalizable action which is compatible with the symmetry $O(3) \otimes O(2)$ is given by equation (13). Its general form is stable under renormalization so that we shall work only with it in the following.

It is easy to generalize the above action to the case where the fields $e_a(x)$ have $N > 3$ components. The symmetry group $G$ is in this case $O(N) \otimes O(2)$. The ground states are given by eq.(5). Let us choose one of them, for example:

$$
(e_1^{(0)}, e_2^{(0)}) = \begin{pmatrix}
0 \\
\cdot \\
1 \\
0 \\
\cdot \\
0 \\
1
\end{pmatrix}.
$$

The unbroken symmetry group $H$ of the low-temperature phase is then the set of matrices leaving this configuration invariant:
This group $H$ consists in two subgroups: a $O(N - 2)$ and a diagonal $O(2)$. Therefore the action $S_1$ describes the symmetry breaking pattern $G \rightarrow H = O(N) \otimes O(2) \rightarrow O(N - 2) \otimes O(2)_{\text{diag}}$. 

## 3 Group theoretical construction of the non linear sigma model

In the last section, we have derived three equivalent forms of the relevant action for Heisenberg frustrated magnets $S_1, S_2, S_3$ (eq.(4,11,13)). Each of them has its own interests and shortcomings. Action $S_1$ is closely related to the microscopic Hamiltonian while action $S_2$ is suited to the discussion of the symmetry properties. Finally action $S_3$ offers a possible large $N, N \geq 3$, generalization which we shall discuss in detail. The particular form of the action is irrelevant near $D = 2$ since the RG properties depend only on the geometry of the manifold $G/H$. These intrinsic properties will be formulated in the language of group theory which provides an abstract but powerful framework.

### 3.1 The $O(N)/O(N - 1)$ partition function

The partition function of the $O(N)/O(N - 1)$ model is $[?, ?]$:

$$Z = \int DS \delta (S^2(x) - 1) \exp \left( -\frac{1}{2T} \int d^Dx (\partial S)^2 \right).$$

The functional delta selects the configurations of $S(x)$ with unit length. We can take advantage of this delta to integrate out one degree of freedom in $S(x)$. Let us choose $u, u^2 = 1$, collinear to the magnetization and write $S(x)$ as:

$$S(x) = \sigma(x)u + \pi(x) ; \quad \pi(x) \perp u ; \quad \sigma^2 + \pi^2 = 1.$$  \hfill (22)

After integrating out $\sigma(x)$, $Z$ can be rewritten as:

$$Z = \int_{|\pi| \leq 1} D\pi \exp \left( -\frac{1}{2T} \int d^Dx \left( (\partial \pi)^2 + (\partial \sqrt{1 - \pi^2})^2 \right) \right).$$  \hfill (23)

The low temperature $T$ perturbative calculation of (23) starts from small fluctuations around the ground state: $< S >= u$. They correspond to the excitations of the $\pi$-field and are the usual spin waves. The $\pi$’s consist of the $N - 1$ Goldstone modes coming from the breaking of $O(N)$ down to the rotation group $O(N - 1)$ that leaves the ground state
invariant, i.e. the $O(N-1)$ around the $u$-direction. Once the symmetry breaking pattern $O(N) \rightarrow O(N-1)$ is given the NL$\sigma$ model is entirely determined up to the coupling constant which in this case is the temperature.

We present now a matrix formulation of the $O(N)/O(N-1)$ model. Let us choose a ground state $S^0 = u$. We can write the $S(x)$ field as:

$$S(x) = R(x)S^0,$$

(24)

where $R(x)$ is the $O(N)$ matrix sending $S^0$ onto $S(x)$. The partition function $Z$ can be rewritten as:

$$Z = \int DR \exp\left(-\frac{1}{2T} \int d^P x Tr(K(\partial R^{-1}\partial R))\right),$$

(25)

where

$$K_{\alpha\beta} = S^O_\alpha S^O_\beta.$$  

(26)

In this last equation the indices are those of vectors $(\alpha, \beta) = 1,..,N$ and $R \in O(N)$. The relationship between $S(x)$ and $R(x)$ is not bi-univoque since for any rotation matrix $h(x)$ leaving $S^0$ invariant:

$$hS^0 = S^0,$$

(27)

one has:

$$R(x)h(x)S^0 = R(x)S^0 = S(x).$$

(28)

As a consequence, the action is locally (i.e. gauge) right invariant under the transformation:

$$R^h(x) = R(x)h(x), \ h \in H = O(N-1).$$

(29)

Some degrees of freedom in $R \in O(N)$ are thus unphysical. To obtain a bi-univoque representation in terms of matrices we have to choose one unique element in each equivalence class $R^h$, that is to fix the gauge. The set of these equivalence classes is the set of $O(N)$ rotations up to a $O(N-1)$ rotation: it is $O(N)/O(N-1)$. We can easily find one element per equivalence class in terms of the physical $\pi$-field. Let us write $R(x)$ and $h(x)$ as:

$$R(x) = \begin{pmatrix} A & V \\ iV' & B \end{pmatrix}, \quad h(x) = \begin{pmatrix} h' & 0 \\ 0 & 1 \end{pmatrix}, \quad h' \in O(N-1).$$

The matrix $A$ is $(N-1) \times (N-1)$, $V$ and $V'$ are a $N-1$-component vectors and $B$ is a scalar. We use relation (29) to eliminate as many degrees of freedom in $R(x)$ as possible. It is convenient to choose:

$$h'(x) = A^{-1}\sqrt{A^\dagger A}.$$  

(30)

This leads to exactly one element per class given by:

$$L = \begin{pmatrix} \sqrt{1-N^{-1}}V V' \sqrt{1-N^{-1}}V' \\ -iV \sqrt{1-N^{-1}}V' \sqrt{1-N^{-1}}V' \end{pmatrix}.$$  

(31)

We identify $V$ by applying $L$ to $S^0$, eq.(24):

$$\begin{pmatrix} \pi \\ \sigma \end{pmatrix} = L \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

(32)
The element $L$ can thus be written entirely in terms of the $\pi$ fields:

$$L(\pi^i) = \left( \sqrt{1 - \frac{\pi^i}{\pi}} \begin{pmatrix} \pi \\ \pi \sigma \end{pmatrix} \right).$$

(33)

The set of $L$-matrices is such that to any $\pi$ corresponds a unique $L(\pi) \in O(N)/O(N-1)$. The quantity $(L^{-1}\partial L)$ belongs to the Lie algebra $Lie(G)$ of $G = O(N)$ and we have:

$$(L^{-1}\partial L) = (L^{-1}\partial L)_{G-H} + (L^{-1}\partial L)_H$$

(34)

where $(L^{-1}\partial L)_H$ is in $Lie(H)$. The partition function $Z$ can be finally written as:

$$Z = \int D\pi \ e^{-\frac{1}{\lambda}S},$$

(35)

$$S = -\frac{1}{2} \int d^4x \ Tr([((L^{-1}\partial L)_{G-H})]^2).$$

(36)

We have used the fact that $K$ is a projector: $K(L^{-1}\partial L)_H = 0$. The partition function in Eq.(25) is globally $G$-invariant and locally (i.e. gauge) $H$-invariant. Once a gauge choice is made (as in (30,32)) no $H$-transformations are allowed in (35,36) and the $G$-transformations are in general not compatible with the gauge choice, i.e. they do not preserve the form of matrices $L$. This means that a $G$-transformation must be accompanied by a $H$-gauge-restoring-transformation:

$$L(\pi') = gL(\pi)h(g, \pi).$$

(37)

Thus, $G$ is non linearly realized on the $\pi$-fields. This is completely different from the Landau-Ginzburg model where $G$ is linearly realized on the $\pi_i$ fields.

Equation (35) is the general expression for the partition function of a NL$\sigma$ model defined on a coset space $G/H$. This coset space can be viewed as a metric manifold so that it is convenient to formulate the theory in the language of differential geometry.

Since $L^{-1}\partial L$ belongs to $Lie(G)$, eq.(34) can be rewritten as:

$$L^{-1}\partial_\mu L = e^I_\mu T_I + \omega^a_\mu T_a,$$

(38)

where the $T_a$'s are the generators of $Lie(H)$ while the $T_I$'s are generators in $Lie(G) - Lie(H)$. $e^I_\mu$ and $\omega^a_\mu$ are respectively the vielbein and the connection in the tangent space of $G/H$. Under (37) they transform as:

$$\begin{cases}
    e^I_\mu T_I = h^{-1}(x)(e^I_\mu T_I)h(x), \\
    \omega^a_\mu T_a = h^{-1}(x)(\omega^a_\mu T_a)h(x) + h^{-1}(x)\partial_\mu h(x).
\end{cases}$$

(39)

The $T_I$'s span a representation of $H$ since:

$$[T_a, T_I] = f_{aIJ}T_J.$$  

(40)

As a consequence, the $e^I_\mu$'s span a linear representation of $H$. Using (38), action $S$ in eq.(36) can be written as:
\[ S = \frac{1}{2} \int d^2x \, e^I_\mu e^J_\mu \eta_{IJ}, \]  
(41)

where \( \eta_{IJ} \) is the tangent space metric given by:

\[ \eta_{IJ} = -Tr(K T_I T_J), \]  
(42)

with \( K \) the projector on \( G - H \). In the \( O(N)/O(N-1) \) case it is given by eq.(26). In this case, the \( N-1 \) generators \( T_I \)'s of \( \text{Lie}(O(N)) - \text{Lie}(O(N-1)) \) span the vector representation of \( O(N-1) \) so that there is only one coupling constant: \( \eta_{IJ} = \eta \delta_{IJ} \). However, in general, \( \eta_{IJ} \) is a diagonal matrix with several different couplings. The number of these couplings is the number of quadratic invariants under transformation (40) constructed with the \( e^I_\mu \)'s. For a symmetric space there is only one such invariant. This is the case of \( O(N)/O(N-1) \) for example. For a non-symmetric homogeneous space such as \( O(N) \otimes O(2)/O(N-2) \otimes O(2) \) this number is larger than one. The formula \( e^I_\mu = e^I_\mu \partial_\mu \pi^i \) leads to the more conventional form eq.(23) of the action of the NL\( \sigma \) model defined on a coset space viewed as a metric space equipped with the metric \( g_{ij}(\pi) = e^I_\mu e^J_\mu \eta_{IJ} \):

\[ Z = \int_{|\pi| \leq 1} D\pi \, \exp \left( -\frac{1}{2T} \int d^2x \, g_{ij}(\pi) \partial_i \pi^i \partial_j \pi^j \right). \]  
(43)

Eq.(41) and eq.(43) provide alternative descriptions of NL\( \sigma \) models defined on a coset space \( G/H \) in terms of purely local geometrical quantities of the manifold \( G/H \) such as for example the metric, Riemann and Ricci tensors. It is equivalent to work either on the manifold itself (43) or in the tangent space (41). For practical calculations, it is extremely convenient to use the tangent space formulation we have discussed above. It can be shown that the geometrical quantities such as the Riemann tensor depend only in tangent space on the Lie algebras \( \text{Lie}(G) \) and \( \text{Lie}(H) \). More precisely they depend on the structure constants defined by the following commutation rules:

\[ [T_a, T_I] = f_{aI}^J T_J, \]  
\[ [T_a, T_b] = f_{ab}^c T_c, \]  
\[ [T_I, T_J] = f_{IJ}^K T_K + f_{IJ}^a T_a, \]  
(44)

where \( T_a \in \text{Lie}(H) \) and \( T_I \in \text{Lie}(G) - \text{Lie}(H) \).

### 3.2 The \( O(N) \otimes O(2)/O(N-2) \otimes O(2)_{\text{diag}} \) partition function

In the case of the \( O(N) \otimes O(2)/O(N-2) \otimes O(2)_{\text{diag}} \) model, the order parameter is the set of \( N \)-component vectors \( (e_1, e_2) \) and the action is given by action \( S_3 \) eq.(13). Let us define the order parameter as the rectangular matrix:

\[ \Phi = (e_1, e_2). \]  
(45)

The \( O(N-2) \otimes O(2) \) transformations can be written:

\[ '\Phi = ' r(x) \, ' \Phi \, ' R(x), \]  
(46)
where \( R \in O(N - 2) \) and \( r \in O(2) \). The ground state Eq. (19) is invariant under the transformations:

\[
\hat{t} \Phi^0 = h_1(x) \Phi^0 H(x),
\]

with \( h_1 \in O(2) \) and

\[
H(x) = \begin{pmatrix} h_2(x) & 0 \\ 0 & h_1^{-1}(x) \end{pmatrix}, \quad h_2 \in O(N - 2).
\]

Thus, the matrices \( r(x) \) and \( R(x) \) are defined up to the following local transformations:

\[
\begin{cases}
  r(x) &\to r(x) \, h_1(x) \\
  R(x) &\to R(x) \, H(x)
\end{cases}
\]

(49)

In the low temperature phase we can rewrite \( \Phi \) in terms of the \( 2N - 3 \) Goldstone modes:

\[
\Phi(x) = \begin{pmatrix} \pi(x) \\ \omega(x) \sqrt{1 - t \pi \pi} \end{pmatrix},
\]

where \( \pi \) is a \((N - 2) \times 2\) matrix and \( \omega(x) \in O(2) \). The \( \pi_i^\alpha, i = 1, \ldots, N, \alpha = 1, 2 \) transform as two independent vectors under \( O(N - 2) \) and as a vector under \( O(2) \). \( \omega(x) \sqrt{1 - t \pi \pi} \) represents one extra degree of freedom which is scalar under both \( O(N - 2) \) and \( O(2) \). One can use the gauge freedom (49) to go from a general element \( R(x) \otimes r(x) \) of \( O(N) \otimes O(2) \) to the unique element in the same gauge orbit \( L \otimes \mathbf{1}_2 \):

\[
L(\pi(x), \omega(x)) = \begin{pmatrix} \sqrt{1 - \pi \pi} & \pi \\ -\omega(x) \pi & \omega(x) \sqrt{1 - t \pi \pi} \end{pmatrix}.
\]

(51)

The matrix \( L \) thus parametrizes the coset space \( O(N) \otimes O(2) / O(N - 2) \otimes O(2) \). Note that this matrix is the same as for the coset space \( O(N) / O(N - 2) \).

In fact this is not accidental and we will use the following property to simplify our study: very generally the coset spaces \( G \otimes X / H \otimes X_{\text{diag}} \) (where \( X \) is the maximal subgroup of \( G \) commuting with \( H \)) and \( G/H \) are topologically equivalent. We can thus work directly with the coset \( G/H \) keeping in mind that we search for an action which has \( G \otimes X \) as symmetry (isometry) group.

The vielbein of \( G/H \) defined as in eq. (38) decompose into two irreducible representations under the action of \( H \otimes X \). \( X \) itself spans the adjoint representation of \( X \) and is a scalar under \( H \). \( G - H - X \) is irreducible because \( H \otimes X \) is maximal in \( G \), stated otherwise \( G/H \otimes X \) is a symmetric space. Thus, the two projected matrices \((L^{-1} \partial L)_{G-H-X} \) and \((L^{-1} \partial L)_{|X} \) transform independently under the right action of the \( H \otimes X \) group so that there are two independent couplings \( \eta_1 \) and \( \eta_2 \). We are thus led to the action:

\[
S = -\frac{1}{2} \int d^D x \left( \eta_1 \text{tr}(L^{-1} \partial L)_{|G-H-X}^2 + \eta_2 \text{tr}(L^{-1} \partial L)_{|X}^2 \right).
\]

(52)

Denoting by \( I \) the indices of \( \text{Lie}(G) - \text{Lie}(H) \), and among them by \( \alpha \) the indices of \( \text{Lie}(X) \), this action may be rewritten as:

\[
S = \frac{1}{2} \int d^D x \, e^J_{\mu} e^J_{\nu} \eta_{IJ}.
\]

(53)
where the tangent space metric $\eta_{IJ}$ is given by:

$$
\eta_{IJ} = -\eta_1 \text{tr}(T^IT^J) - (\eta_2 - \eta_1)\delta_{I\alpha}\delta_{J\beta}\text{tr}(T^\alpha T^\beta).
$$

We recall that in our case $T_I \in \text{Lie}(O(N)) - \text{Lie}(O(N - 2))$, $T_\alpha \in \text{Lie}(O(2))$ and $T_\beta \in \text{Lie}(O(N - 2))$ and that the corresponding algebra is given in (14). Action (52) is completely equivalent to the action $S_3$ we have obtained from the continuum limit (13). We prove it in appendix B and derive the relations between the couplings $g_1, g_2$ entering in $S_3$ and $\eta_1, \eta_2$: $\eta_1 = g_1/2$; $\eta_2 = g_1 + 2g_2$.

4 Renormalization of the $NL\sigma$ model in $D = 2 + \epsilon$

4.1 General case

The renormalizability in $D = 2 + \epsilon$ of $NL\sigma$ models defined on coset spaces $G/H$ was studied by D.H. Friedan[?]. The $\beta$ function gives the evolution of the metric $g_{ij}(\pi)$ with the scale:

$$
\frac{\partial g_{ij}}{\partial l} = \beta_{ij}.
$$

At two loop order it is given by the following expression:

$$
\beta_{ij}(g) = -\epsilon g_{ij} + R_{ij} + \frac{1}{2}TR_{ipqr}R_{jpqr} + O(T^2).
$$

where: $R_{ij}$ and $R_{ipqr}$ are the Ricci and the Riemann tensors of the manifold $G/H$ equipped with the metric $g_{ij}$.

In principle, it is enough to compute $R_{ij}$ and $R_{ijkl}$ from the metric $g_{ij}$ to obtain these recursion relations. In practice, these calculations are tedious and some formal algebraic work has to be done first. The trick is to get rid in the calculation of any dependence on the coordinates $\pi^i$ by going from the manifold itself to its tangent space, eq.(41). The crucial advantage is that in tangent space, the Riemann and Ricci tensors are functions only of the structure constants $f^k_{ij}$ of $\text{Lie}(G)$ and that the tangent space metric $\eta_{IJ}$ is constant, see eq.(54), and involves only the coupling constants. In the vielbein basis, eq.(55,56) becomes:

$$
\frac{\partial \eta_{IJ}}{\partial l} = \beta_{IJ},
$$

$$
\beta_{IJ}(\eta) = -\epsilon \eta_{IJ} + R_{IJ} + \frac{1}{2}TR_{IPQR}R_{JPQR} + O(T^2).
$$

The matrix $\eta_{IJ}$ is given in eq.(74) and the Riemann tensor in tangent space can be expressed as:

$$
R_{IJKL} = f_{IJ}^a f_{aKL} + \frac{1}{2}f_{IJ}^M (f_{MKL} + f_{LMK} - f_{KLM})
$$

$$
+ \frac{1}{4} (f_{IKM} + f_{MIK} - f_{KMI}) (f_{JL}^M + f_{LJ}^M - f^M_{LL})
$$

$$
- \frac{1}{4} (f_{JKM} + f_{MKJ} - f_{KJM}) (f_{LI}^M + f_{LJ}^M - f^M_{LI}).
$$
The indices $a$ and $\{I, J, \ldots\}$ refer to $H$ and $G - H$ respectively. $G - H$ indices are raised and lowered by means of $\eta^{IJ}$ and $\eta_{IJ}$ and repeated indices are summed over. In NL$\sigma$ models the $\beta$ function and its derivatives allows to compute the fixed point and the critical exponent $\nu$. Note that since the $\beta$ function is a tensor, it does not depend on a particular choice of coordinates. As a consequence, the mere existence of a fixed point as well as the value of the exponent $\nu$ do not depend on the representation spanned by the order parameter. The other renormalization group function which is needed to give a complete description of the critical behavior is the Callan-Symanzik $\gamma$-function. This function is determined by the field renormalization $Z$:

$$\gamma = -\frac{\partial \log Z}{\partial l}.$$  \hfill (60)

From this function follows the anomalous dimension $\eta$:

$$\eta = \gamma(\eta_1^*, \eta_2^*) - \epsilon$$  \hfill (61)

where $\eta_1^*, \eta_2^*$ are the fixed point values of the coupling constants.

The factor $Z$ is given at one loop order by the Laplace-Beltrami operator acting on the coordinate $\pi^i$. It can be shown that this is nothing but $g^{ij}\Gamma_{ij}^k$ where $\Gamma_{ij}^k$ is the Christoffel connection on the metric manifold $G/H$:

$$Z\pi^k = \pi^k + \frac{1}{\epsilon}g^{ij}\Gamma_{ij}^k.$$  \hfill (62)

Once again, it is simpler to compute $g^{ij}\Gamma_{ij}^k$ by working in tangent space. We find for any coset $G \otimes X/H \otimes X$ where $X$ is the subgroup of $G$ that commutes with $H$:

$$g^{ij}\Gamma_{ij}^k = -\frac{1}{\eta_1} \left( \sum_A T_AT_A \right) \pi^k + \eta_2 - \eta_2 \frac{\eta_2 - \eta_1}{\eta_1 \eta_2} \left( \sum_\alpha T_\alpha T_\alpha \right) \pi^k,$$  \hfill (63)

where $\{T_A\}$ and $\{T_\alpha\}$ are generators of $G$ and $X$ and where $\eta_1, \eta_2$ are defined in equation (52). $\sum_A T_AT_A$ and $\sum_\alpha T_\alpha T_\alpha$ are Casimir operators of $G$ and $X$. In general, a choice of coordinates is not stable under renormalization. Equations (62) and (63) show that a good coordinate system which renormalizes multiplicatively consists in the $\pi$ fields together with the massive $\sigma$ modes. They build up a linear representation of $G \otimes X$ such that the $\pi$’s are an eigenvector of the Casimir operators with an eigenvalue that depends on the representation. Therefore, the $\gamma$-function and thus the critical exponent $\eta$ depends on the representation $r$ of $G \otimes X$ spanned by the order parameter. In our case, the Casimir operators have to be taken in the vector representation of both the $O(N)$ and the $O(2)$ groups. Their values are therefore respectively $N - 1$ and 1. To summarize, in NL$\sigma$ models the existence of a fixed point depends only on the symmetry breaking pattern $G/H$ and not on the representation spanned by the order parameter. However, the universality class is completely determined once the representation $r$ of $G$ spanned by the observable is known. This scheme is completely different from what happens in the $4 - \epsilon$ expansion where even the $\beta$ function, and thus the mere existence of a fixed point, does depend on the representation of $G$ spanned by the order parameter. In the following, we apply these results to the $O(N) \otimes O(2)/O(N - 2) \otimes O(2)$ models. For reasons that will soon become clear, we shall distinguish between the $N = 3$ and $N > 3$ cases.
4.2 Results for \( N > 3 \)

Using Eq.(58) we obtain the following two loop recursion relations valid for any \( N \geq 3 \):

\[
\begin{align*}
\frac{\partial \eta_1}{\partial l} &= -\epsilon \eta_1 + N - 2 - \frac{1}{2} \frac{n_2}{\eta_1} + \frac{3}{8} \frac{N - 4}{\eta_1^2} + 3 \left( 1 - \frac{N}{2} \right) \frac{n_2}{\eta_1^2} \\
&
+ (3N - 8) \frac{1}{\eta_1} \\
\frac{\partial \eta_2}{\partial l} &= -\epsilon \eta_2 + \frac{N - 2}{2} \left( \frac{n_2}{\eta_1} \right)^2 + \frac{N - 2}{8} \frac{n_2}{\eta_1^2}
\end{align*}
\]

(64)

Defining \( T_{1,2} = 1/\eta_{1,2} \), we find that, apart from the trivial zero temperature line of fixed points: \( T_1 = T_2 = 0 \) with \( T_1/T_2 \) arbitrary there is one non trivial fixed point \( C_{NL} \) with coordinates:

\[
\begin{align*}
T_1^* &= \frac{N - 1}{(N - 2)^2} \left( \epsilon - \frac{1}{2} \frac{13N^2 - 10N + 4}{(N - 2)^3} \epsilon^2 \right) + O(\epsilon^3) \\
T_2^* &= \frac{1}{2} \frac{(N - 1)^2}{(N - 2)^3} \left( \epsilon - \frac{1}{2} \frac{15N^2 - 16N + 4}{(N - 2)^3} \epsilon^2 \right) + O(\epsilon^3)
\end{align*}
\]

(65)

This fixed point has one direction of instability so that our model undergoes an ordinary second order phase transition with critical exponent \( \nu \):

\[
\nu^{-1} = \epsilon + \frac{1}{2} \frac{16N^3 - 27N^2 + 32N - 12}{(N - 2)^3(2N - 3)} \epsilon^2 + O(\epsilon^3)
\]

(66)

In order to complete our discussion, we have to specify the representation \( r \) of \( O(N) \otimes O(2) \) spanned by the observable of the physical system under study. We are interested in the AFT model with \( N \)-component spins. In this case, the order parameter transforms under the vector representation of both \( O(N) \) and \( O(2) \), see Eq.(46). At one loop, it follows from Eq.(54) and Eq.(53) that the anomalous dimension \( \eta \) is:

\[
\eta = \frac{3N^2 - 10N + 9}{2(N - 2)^3} \epsilon + O(\epsilon^2)
\]

(67)

4.3 Results for \( N = 3 \)

Although both the recursion relations and the values of the exponents given in the preceding section are still valid in the \( N = 3 \) case the symmetry properties are less obvious. In this case, we can take advantage of the different equivalent parametrizations of the action we have derived in section 2. The convenient parametrization is given by Eq.(11):

\[
S_2 = \frac{1}{2} \int d^Dx \, Tr \left( P(\partial R^{-1})(\partial R) \right),
\]

(68)

where \( P \) is the diagonal matrix: \( P = diag(p_1, p_2, p_3) \) and \( R \in SO(3) \). In the \( O(3) \otimes O(2)/O(2) \) case we have \( p_1 = p_2 \neq p_3 \). The relationship between the \( p_i \)'s and the tangent
space couplings $\eta_{i,j}$ is given in Appendix B. Using Eq.(14) we can deduce the two loop recursion relations for the couplings $p_i$. At the fixed point we find $p^*_1 = p^*_2 = p^*_3$ and thus $P^* \propto 1$. It follows from the discussion given in section 2 that the action $S_2$ is $O(3) \otimes O(3)$ symmetric at the fixed point: the symmetry has been dynamically enlarged at the fixed point. Since $O(3) \otimes O(3)/O(3) \sim O(4)/O(3)$ the critical behavior of the $O(3) \otimes O(2)/O(2)$ NLσ model is given by that of the $O(4)/O(3)$ NLσ model. It is a new result to find such a $O(4)$ symmetry for a Heisenberg system. We stress that it is not trivial to identify such a symmetry using a different parametrization such as the one given in action $S_3$ (see Eq.(13)). In this case, the $O(4)$ symmetry is non-linearly realized on the fields $e_1, e_2$. The critical exponents $\nu$ and $\eta$ are given by Eqs.(16,17) with $N = 3$. Although the critical exponent $\nu$ is identical to that of the $N = 4$ vector model, it is not so simple to get $\eta$. The order parameter is a matrix $R(x) = (e_1(x), e_2(x), e_3(x))$ (Eq.(11)) and spans the tensor representation of $O(4)$. This point was previously missed in ref.[2]. As a consequence, the exponent $\eta$ of the Heisenberg AFT model is the anomalous dimension of a composite operator of the $N = 4$ vector model. To see this, we need the relationship between the $O(3)$ matrix $R$ and a $O(4)$ unit vector. It can be shown that to any unit 4-component vector:

$$\Psi = (\Psi_0, \Psi_i) \ ; \ \Psi_0^2 + \sum_i \Psi_i^2 = 1$$

there exists a matrix $R$ of $O(3)$ with components:

$$R_{ij} = 2(\Psi_i \Psi_j - \frac{1}{4} \delta_{ij}) + 2 \epsilon_{ijk} \Psi_k \Psi_k + 2(\Psi_0^2 - \frac{1}{4}) \delta_{ij}$$

Therefore, the expectation values of the vectors $< e_i(x) >, i = 1, 3$ are obtained from those of the bilinear forms $< (\Psi_i \Psi_j - \frac{1}{4} \delta_{ij}) >$.

We thus find no new universality class for Heisenberg canted models but instead the general phenomenon of increased symmetry at the fixed point. These models belong to the standard $N = 4$ Wilson-Fisher universality class. In dimension $D = 3$, the exponent $\nu$ is very accurately known [2]: $\nu = 0.74$. However, the anomalous dimension of the composite operator $(\Psi_i \Psi_j - \frac{1}{4} \delta_{ij})$ is only known at the two loop order in $\epsilon = 4 - D$. Let us finally emphasize that the phenomenon of increased symmetry at the fixed point is particular to the $N = 3$ case in the $O(N) \otimes O(2)/O(N - 2) \otimes O(2)$ NLσ models. For any $N > 3$, the phase transition belongs indeed to a universality class different from $O(N)$ but as one reaches the physical $N = 3$ case one falls in the well known $O(4)$ one. This conclude our analysis of the NLσ models associated to canted magnets.

The well-known $\epsilon = 4 - D$ expansion starting from the upper critical dimension of the appropriate Landau-Ginzburg-Wilson (LGW) action has been applied to helimagnets more than ten years ago by Garel and Pfeuty[2] and Bailin et al.[2]. More recently, renewed interest on this subject has been drawn by Kawamura[2]. We shall, in the next section, present the results obtained from this expansion.

5 The linear theory and the $\epsilon = 4 - D$ expansion

The LGW action can be obtained in the same spirit as the NLσ. Once the symmetry breaking pattern is known, in our case $O(N) \otimes O(2)/O(N - 2) \otimes O(2)$, all we have to
do is to find the most general action which is $O(N) \otimes O(2)$ symmetric and which ground state is $O(N - 2) \otimes O(2)$ invariant. Among all possible actions, one has to select those which are renormalizable in $D = 4$, i.e., to keep only terms up to order 4 in the fields and to order 2 in their derivatives. The LGW action does not possess the invariance under reparametrization of the NL$\sigma$ model. Moreover, only linear transformations of $O(N) \otimes O(2)$ are allowed since non-linear transformations involve higher powers of the fields and their derivatives than allowed by renormalizability. For this reason the whole LGW or Linear theory depends explicitly on the representation of $O(N) \otimes O(2)$ spanned by the physical order parameter. This fact have dramatic consequences on the renormalizability of LGW theories as compared to their corresponding NL$\sigma$ models. In order to build the LGW action, we shall start from the NL$\sigma$ model. In the particular case of canted models, we have to choose the parametrization which spans a linear representation of $O(N) \otimes O(2)$. In this case the partition function $Z$ is given by:

$$Z = \int \mathcal{D}\mathbf{e}_1 \mathcal{D}\mathbf{e}_2 \, \delta(\mathbf{e}_1, \mathbf{e}_2) \, \delta(\mathbf{e}_1^2 - 1) \, \delta(\mathbf{e}_2^2 - 1) e^{-S_3}, \quad (71)$$

$$S_3 = \frac{1}{2} \int d^Dx \left( g_1 \left( \nabla \mathbf{e}_1^2 + \nabla \mathbf{e}_2^3 \right) + g_2 \left( \mathbf{e}_1 \nabla \mathbf{e}_2 - \mathbf{e}_2 \nabla \mathbf{e}_1 \right)^2 \right). \quad (72)$$

The $\mathbf{e}_i$ are $N$-component vectors. The LGW action is now obtained in a standard way by relaxing the constraints in Eq. (71) and use of a potential:

$$V(\mathbf{e}_1, \mathbf{e}_2) = \frac{1}{2} m^2 (\mathbf{e}_1^2 + \mathbf{e}_2^2) + u_1 (\mathbf{e}_1^2 + \mathbf{e}_2^2)^2 + u_2 (\mathbf{e}_1 \cdot \mathbf{e}_2)^2 \quad (73)$$

The LGW action for canted magnets reads now:

$$S_{LWG} = \frac{1}{2} \int d^Dx \left( \frac{1}{2} \left( \nabla \mathbf{e}_1^2 + \nabla \mathbf{e}_2^2 \right) + V(\mathbf{e}_1, \mathbf{e}_2) \right) \quad (74)$$

We have rescaled the fields in order to obtain the standard normalization for the gradient term and have omitted the current term $(\mathbf{e}_1 \nabla \mathbf{e}_2 - \mathbf{e}_2 \nabla \mathbf{e}_1)^2$ since it is not renormalizable. Note that it is precisely this term which allowed the NL$\sigma$ action $S_3$ to be $O(3) \otimes O(3)/O(3)$ symmetric at the fixed point when $N = 3$.

The two loop recursion relations for the couplings $u_1$ and $u_2$ were first obtained by Bailin et al.[?] and Garel and Pfenty[?]. We shall here only summarize their results. Let us comment the RG flow:

i) there is a critical value of $N$ depending on the dimension: $N_c(D) = 21.8 - 23.4\epsilon + O(\epsilon^2)$, under which there is no fixed point. In this case the transition is expected to be first order. Let us emphasize that for $\epsilon = 1$ the second term in the $\epsilon$-expansion of $N_c(\epsilon)$ is not a small perturbation of the first one since it is $-23.4$. This is the signal that a precise determination of $N_c(D)$ needs some control of the $\epsilon$-expansion which, as it stands, can not be used directly for $\epsilon$ of order 1.

ii) for $N > N_c(D)$ there are still two different regions in the portion of the $(u_1, u_2)$ parameter space where the potential is stable: see fig. 1. One is the second order region. It lies above the line $L$ joining the origin to an unstable fixed point (called $C_-$) and is the basin of attraction of the stable fixed point, called $C_L$. The Heisenberg fixed point $H$ is unstable towards $C_L$. The other region lies between the line $L$ and the stability line
$S$ of the potential: $u_2 = -2u_1$. It is a region of runaway behaviour and is expected to correspond to a first order region. We note that, as $N$ tends to infinity, $L$ tends to $S$, as expected.

6 Interpolating between $D = 2 + \epsilon$ and $D = 4 - \epsilon$.

There is clearly a mismatch between RG results obtained in $D = 4 - \epsilon$ dimensions from the LGW model and in $D = 2 + \epsilon$ dimensions from the $O(N) \otimes O(2)/O(N-2) \otimes O(2)$ sigma model. Even though these NL\sigma models predict for any $N$ a continuous transition in the neighborhood of dimension 2, there are models with either $N < N_c$ or which do not belong to the basin of attraction of $C_L$ for which the transition is expected to be of first order at least near $D = 4$. This is very different from the ferromagnetic case where both the $O(N)/O(N-1)$ NL\sigma model and the LGW model predict the same critical behaviour. However, when $N > N_c(D)$, there exists a domain in the coupling constants space $M$ where a second order transition is predicted in both models. These domains are respectively the basins of attraction of $C_{NL}$ in $D = 2 + \epsilon$ and of $C_L$ in $D = 4 - \epsilon$. The natural question is whether or not these two fixed points are the same in a given dimension $D$ between 2 and 4. The 1/$N$ expansion allows to answer this question, at least for $N$ large enough, since this expansion is non perturbative in the dimension $D$. The critical exponent $\nu$ has been calculated to the lowest non trivial order in a 1/$N$ expansion of the Landau-Ginzburg action \cite{74}:

\[ \nu_{1/N}(D) = \frac{1}{D-2} \left( 1 - \frac{1}{ND} 12(D-1)S_D \right). \]  
\[ S_D = \frac{\sin \left( \frac{\pi}{2} (D-2) \right) \Gamma(D-1)}{2\pi (\Gamma(D/2))^2}. \]

By expanding eq.\((75)\) to lowest non trivial order in $\epsilon$, $\epsilon = 4 - D$ or $\epsilon = D - 2$, we find that $\nu_{1/N}(D)$ coincides with $\nu_{4-\epsilon}(N)$ and $\nu_{2+\epsilon}(N)$ to lowest order in $1/N$. The same type of expansion can be done on the other exponents with the same results.

We may thus conclude as in the ferromagnetic case that, when the fixed point exists near $D = 2$ and near $D = 4$, we can follow it smoothly from $D = 4 - \epsilon$ down to $D = 2 + \epsilon$. Therefore, in the whole space $E = \{(M) = \text{coupling constants, } D, N\}$ there should exist a domain $Z$ where the transition is of second order and which is governed by a unique fixed point $C_L(N, D) = C_{NL}(N, D)$. In the complementary of $Z$ the transition is expected to be of first order. On the boundary $\Gamma$ of these two domains, the transition should be tricritical in the simplest hypothesis.

The situation can be summarized in the plane $(N, D)$ of number of components of the model and dimension. The $4 - \epsilon$ findings have shown that there is a universal curve $N_c(D)$ separating a first-order region and a second-order region. If one believes that the $2 + \epsilon$ results survive perturbation theory then the neighborhood of $D = 2$ belongs to the second-order region for all $N \geq 3$. As a consequence, the line $N_c(D)$ intersects the $N = 3$ axis somewhere between $D = 2$ and $D = 4$. This defines thus a critical dimension $D_c$ that we do not expect to be a simple number. Making the hypothesis that the RG calculations have captured all the relevant fixed points then there are two possibilities:
i) The critical value is between $D = 3$ and $D = 4$. This implies that the physical case $N = 3, D = 3$ undergoes an $O(4)$ transition as shown in (4.3). This case is favored by present numerical studies [?].

ii) The critical value is between $D = 2$ and $D = 3$. Then the physical case is governed by a fluctuation-induced first-order transition. It cannot be excluded that $D_c = 2$ in which case the perturbative analysis of the nonlinear sigma model is always irrelevant.

We note in addition that there is an additional possibility namely $D_c = 3$. We do not see any reason why this would be realized since this looks like an artificial fine-tuning. In this case one may speculate that a tricritical mean-field like behaviour is seen for $N = 3, D = 3$.

These alternatives are consistent with all the RG results and do not require additional fixed points not seen in perturbation theory. In this picture the stable fixed point seen in $4 - \epsilon$ for $N \geq N_c$ can be followed smoothly by the large-N limit till $D = 2$ and then identified with the conventional $O(4)$ fixed point via the $2 + \epsilon$ calculation of (4.2-3) for $N = 3$.

7 Conclusion

We have shown in this article that the nonlinear sigma model provides a new approach to the analysis of the critical behavior of frustrated systems. The double expansion in $T$ and in $\epsilon = D - 2$ of the $O(N) \otimes O(2)/O(N - 2) \otimes O(2)$ NL$\sigma$ model has been performed and a fixed point has been found in $D = 2 + \epsilon$ for any $N$, which turns out to have a remarkable $O(4)$ symmetry for three component spins. Since for $N \leq 21$ the transition is expected to be first order near $D = 4$, we conjecture that for any $N \in [3, 21]$, the nature of the phase transition changes between 2 and 4 dimensions and is tricritical at the border of the second and first order region.

In the simplest hypothesis the $(N, D)$-plane is divided in two region: a first-order region containing $N = 3$ and $D = 4$ and a second-order region containing the $D = 2$ line (for any $N$), the whole large-N line (for all $D$) and also in the neighborhood of $D = 4$ the $N \geq 21.8$ points. In between lies a universal line $N_c(D)$ whose $4 - \epsilon$ expression was already known. This universal line intersects the $N=3$ axis for some unknown critical dimension $D_c$. If $3 < D_c < 4$ then the physical point $N=3, D=3$ is second order in the $O(4)$ universality class and its exponent $\eta$ is that of a tensor representation. If $2 < D_c < 3$ the physical point is first-order. It may happen that $D_c = 3$ (although we see no reason why) in which case one could see tricritical behaviour. This phase diagram is in agreement with all known RG results. To decide the fate of the physical point requires clearly additional techniques. Present Monte-Carlo results[?] favor $2 < D_c < 3$ although more work is needed. Direct RG calculations in $D=3$ may also help to confirm the phase diagram[?].

We note that in the generic case $O(N) \otimes O(2)/O(N - 2) \otimes O(2), N > 3$, the symmetry is not enlarged at the fixed point. In this respect $N = 3$ is exceptional: for other values of $N$, the fixed point does not belong to the $O(N)$ Wilson-Fisher family. It would be interesting of course to investigate the fate of the XY $N=2$ case since known helimagnets have significant anisotropies that lead to a non-Heisenberg behaviour.
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APPENDIX A

We have seen in section 6 that for \( N = 3 \) the LGW and NL\( \sigma \) models do not predict the same critical behaviour. No fixed point is found in \( D = 4 - \epsilon \) dimensions from the LGW action while the NL\( \sigma \) model predicts a fixed point in \( D = 2 + \epsilon \) with a \( O(3) \otimes O(3) \sim O(4) \) symmetry at this point. Actually, though these results are perturbative, it is easy to see that a \( O(4) \)-symmetric fixed point can not be obtained with the LGW action (74) since no value of \((u_1, u_2)\) makes this action \( O(4) \)-symmetric. The reason is that the rectangular matrix \((\phi_1, \phi_2)\) represents 6 degrees of freedom on which acts \( O(3) \) on the right and \( O(2) \) on the left and that this \( O(2) \) can not be enlarged to \( O(3) \) with only these 2 fields. This \( O(3) \) symmetry can be realized only with at least 3 fields: \((\phi_1, \phi_2, \phi_3)\).

This would correspond to the 9-dimensional representation of \( O(4) \). To check whether the discrepancy between the two models can be eliminated by allowing the LGW model to reach the \( O(3) \otimes O(3) \) symmetry, we have built and studied the most general action invariant under \( O(3) \otimes O(2) \) and involving 3 fields:

\[
H = \frac{1}{2} (\partial \phi_1)^2 + \frac{1}{2} (\partial \phi_2)^2 + \frac{1}{2} (\partial \phi_3)^2 + \frac{1}{2} u_1 (\phi_1^2 + \phi_2^2)^2 + \frac{1}{2} u_2 (\phi_1^2 \phi_2^2 - (\phi_1 \phi_2)^2) + \frac{1}{4} u' (\phi_3^2)^2 + \frac{1}{2} u_4 \phi_3^2 (\phi_1^2 + \phi_2^2) - \frac{1}{2} u_3 ((\phi_1 \phi_3)^2 + (\phi_2 \phi_3)^2)
\]

(\( \phi_1, \phi_2 \) is a doublet of \( O(2) \) and \( \phi_3 \) a singlet. \( H \) is \( O(3) \otimes O(3) \) invariant when:

\[
u_2 = u_3 ; \quad u_1 + u_2 = u_4 ; \quad u_1 = u'\]

A one-loop RG calculation shows that no attractive fixed point exists in \( D = 4 - \epsilon \) in this model. This is the proof that the root of the problem is not only a question of symmetry but also a problem of dimension, field content and renormalizability. More precisely, if we set \( u_2, u_4 \) and \( u' \) to the values eq.(78) which make hamiltonian eq.(77) \( O(3) \otimes O(3) \)-symmetric, we find that the remaining symmetry in the broken phase is \( O(3)_{diag} \). Then, the NL\( \sigma \) model associated with this symmetry breaking scheme is \( O(4)/O(3) \). This NL\( \sigma \) model is unique since it depends only on the Goldstone modes, and then only on the Lie algebras of \( O(4) \) and \( O(3) \) and not on the representations of these groups. On the other hand, there are as many associated LGW models as there are representations of \( O(4) \) (or at least as there are actions built with representations of \( O(4) \) that can be broken down to \( O(3) \)). Surprisingly, the LGW action built with the 4-component vector representation of \( O(4) \) admits a fixed point in \( 4 - \epsilon \) dimensions (the usual Heisenberg fixed point) and that built with the 9-dimensional tensor representation \((\phi_1, \phi_2, \phi_3)\) admits no such fixed point, as we have seen above. Since in these two LGW models, the symmetry breaking scheme is the same and then the Goldstone modes are the same, it means that the difference between these models lies in the massive modes. It is not clear up to now whether these modes can indeed be physically relevant for the critical behaviour. At least perturbatively and near 2 dimensions, the NL\( \sigma \) model does not take care of these modes. In our case, we have to deal with the vector and tensor representations of \( O(4) \) which both allow to replace the constraints of the NL\( \sigma \) model by potentials and which lead to two different results. The relevance of the massive modes in the critical behaviour is then directly related to the way one chooses to go from the microscopic Hamiltonian to the different continuous actions, linear or non linear.
APPENDIX B

We give in this appendix the relationship between different parametrizations of the sigma model.

From tangent space to the constraints (for any $N$):

We parametrize the matrix $L$ in Eq.(51) as:

$$L = (A \phi_1 \phi_2)$$ (79)

where $A$ is a rectangular $N \times (N - 2)$ matrix, $\phi_1$ and $\phi_2$ are $N$-component vectors. Since $L$ is in $O(N)$, they must satisfy:

$$t^A A = 1_{N-2}, \quad t^A \phi_1 = 0, \quad t^A \phi_2 = 0, \quad \phi_1^2 = \phi_2^2 = 1, \quad \phi_1 \phi_2 = 0$$ (80)

$$A^t A + \phi_1 \phi_1 + \phi_2 \phi_2 = 1_N$$ (81)

We are interested in the action of the NL$\sigma$ model so that we have to compute $L^{-1} \partial L$:

$$L^{-1} \partial L = \begin{pmatrix}
0 & t^A \phi_1 & t^A \phi_2 \\
-t^A \phi_1 & 0 & \phi_1 \phi_2 \\
t^A \phi_2 & -\phi_1 \phi_2 & 0
\end{pmatrix}$$ (82)

The projection of $L^{-1} \partial L$ onto $\text{Lie}(G') - \text{Lie}(H')$ leads to two sets of vielbein that are not mixed under the $H$-transformations:

$$(L^{-1} \partial L)_{|G'-H'-X} = \begin{pmatrix}
0 & t^A \phi_1 & t^A \phi_2 \\
-t^A \phi_1 & 0 & 0 \\
t^A \phi_2 & 0 & 0
\end{pmatrix}$$ (83)

and

$$(L^{-1} \partial L)_{|X} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \phi_1 \phi_2 \\
0 & -\phi_1 \phi_2 & 0
\end{pmatrix}$$ (84)

The total action is the sum of the traces of the squares of these two matrices weighted by $\eta_1$ and $\eta_2$. These coefficients are by definition those coming from the tangent space metric $\eta_{IJ}$.

$$\eta_1 \text{Tr} \left[ (L^{-1} \partial L)_{|G'-H'-X}^2 \right] = -2 \eta_1 \left( (\phi_1^2 + (\phi_2)^2) + 4 \eta_1 (\phi_1 \phi_2)^2 \right)$$

$$\eta_2 \text{Tr} \left[ (L^{-1} \partial L)_{|X}^2 \right] = -2 \eta_2 (\phi_1 \phi_2)^2$$ (85)

The coupling constants $\eta_1, \eta_2$ are now easily related to $g_1, g_2$ defined in (13):

$$\begin{cases}
\eta_1 = g_1/2 \\
\eta_2 = g_1 + 2g_2
\end{cases}$$ (86)

From the P-matrix to the constraints ($N = 3$)

We now compute directly $\text{Tr}(P(R^{-1} \partial R)^2)$ to obtain explicitly the action (11). Let us start with $R \in O(3)$, parametrized as follows:

$$R = (e_1 \ e_2 \ e_1 \wedge e_2)$$ (87)
where $e_1$ and $e_2$ are two 3-component vectors such that $e_1^2 = e_2^2 = 1$ and $e_1 \cdot e_2 = 0$. The diagonal part of $(R^{-1} \partial R)^2$ is:

$$
(R^{-1} \partial R)^2_{\text{diag}} = \begin{pmatrix}
-(\partial e_1)^2 & - (\partial e_2)^2 \\
-(\partial e_1)^2 - (\partial e_1)^2 + 2(\partial e_1 \cdot e_2)^2
\end{pmatrix}
$$

so that:

$$
Tr \left( P(R^{-1} \partial R)^2 \right) = -(p_1 + p_2) \left( (\partial e_1)^2 + (\partial e_2)^2 \right) + \frac{p_2}{2} (\partial e_1 \cdot e_2 - \partial e_2 \cdot e_1)^2
$$

We obtain the relations between $(p_1, p_2)$ and $(g_1, g_2)$:

$$
\begin{cases}
p_1 &= g_1 + 2g_2, \\
p_2 &= -2g_2.
\end{cases}
$$
FIGURE CAPTIONS

Figure 1:
The renormalization group flow in the neighborhood of four dimensions for the Landau-Ginzburg model in the case $N \geq N_C(D)$. On the $u_1$ axis one finds the conventional O(2N) Wilson-Fisher fixed point which is unstable towards the fixed point $C_L$. By following a smooth path in the (N, D) plane we find that $C_L$ is the O(4) fixed point when N=3 in the neighborhood of D=2.