TOPOLOGICAL COMPLEXITY OF THE WORK MAP

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Abstract. We introduce the topological complexity of the work map associated to a robot system. In broad terms, this measures the complexity of any algorithm controlling, not just the motion of the configuration space of the given system, but the task for which the system has been designed. From a purely topological point of view, this is a homotopy invariant of a map which generalizes the classical topological complexity of a space.

INTRODUCTION

The theory of topological complexity was initiated by Michael Farber [5, 6] and it has become one of most active fields in the area of applied topology during last decade. In broad terms, this theory measures the complexity of any algorithm controlling the motion planning of a given configuration space. One can also regard this invariant as the minimum number of navigational instabilities of such a motion planning.

However, in many situations in robotics, more important than controlling the motion of a given configuration space, it is designing the resulting motion on the corresponding workspace. We briefly support this assertion by looking at two different key examples:

In robotics, see for instance [4, Chap. 1], a robot manipulator or robot arm consists of multiple rigid segments (sub-arms, links) where successive, neighboring links are connected by joints of different kind (rotational, translational, polar, cylindrical). The base of a robot manipulator is the end of the first link which is fixed to a point through a given joint. The end effector is the device (screw driver, welding device,...) at the end of the last link of the robotic arm, designed to interact with the environment performing the proposed task. Finally, The workspace of a robot arm is defined as the set of points that can be reached by the end effector.

From the topological point of view, the configuration space of a robot arm, i.e., the set of all possible states of such a manipulator, was first...
modeled in [12], see also [17] or the modern reference [7] for a complete treatment. Less attention has been given to the topological study of the workspace as this is not an (even diffeomorphic) invariant of the configuration space. For instance, consider two robot manipulators $C$ and $D$, each of which consisting of two links of lengths $\ell_1, \ell_2$ with $\ell_1 > \ell_2$ and both with one degree of freedom. In other words, both $C$ and $D$ are diffeomorphic to $S^1 \times S^1$.

However, if we assume $C$ to be “planar”, then its workspace $W_C$ is an annulus of radius $\ell_1 - \ell_2$ [7, §1.2]. On the other hand, if in $D$, the circle generated by the end effector link is “transversal” to the plane containing the circle generated by the based link, then $W_D$ is a torus. This is discussed in detail in Example 2.6.

Nevertheless, both the configuration space $C$ and the workspace $W$ of a given robot arm are connected by the continuous work map

$$f : C \rightarrow W$$

which assigns to each state of the configuration space the position of the end effector at that state.

This map is a crucial object for implementing algorithms controlling the task performed by the robot manipulator. Indeed, the input of such an algorithm are pairs $(a, b) \in W \times W$ of points of the workspace, that is, pairs of possible positions of the end effector. The output for such a pair ought to be a curve in the configuration space $\alpha \in C^I$ such that $f(\alpha(0)) = a$ and $f(\alpha(1)) = b$, where $C^I$ is the space of curves in $C$ (that is the space of continuous maps $I = [0, 1] \rightarrow C$).

One may argue that a motion planner of the configuration space $C$ produces such an algorithm by composing with the work map. However, the efficiency of such an algorithm might not be optimal as, for instance, the work map is not injective in general and therefore, many states of the configuration space may give the same position of the end effector.

The second example to which the above can be applied is the following: according to [1], a multi-robot system consists of two or more robots executing a task requiring collaboration among them. Assume that such a multi-robot system is formed by $n$ autonomous mobile robots running in a space $X$ without colliding. The configuration space of such a system is the standard $n$-th configuration space

$$F(X, n) = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

with the subspace topology of the $n$-fold Cartesian product of $X$. On the other hand, the task requiring the collaboration of the robots can
be described as a continuous work map
\[ f: F(X, n) \rightarrow Y, \]
depending on the locations of the robots, and with values on the workspace \( Y \) which is often described as a subspace of some Euclidean space \( \mathbb{R}^N \). Hence, an algorithm controlling the task performed by the multi-robot system can be described, as before, in terms of the work map.

These examples motivate the main purpose of this article which is to propose the notion of topological complexity of a (work) map as a generalization of the topological complexity of a given configuration space.

**Definition 0.1.** Given a continuous map \( f: X \rightarrow Y \), the topological complexity of \( f \), \( \text{TC}(f) \), is the least integer \( n \leq \infty \) such that \( X \times X \) can be covered by \( n + 1 \) open sets \( \{U_i\}_{i=0}^n \) on each of which there is a continuous map \( \sigma_i: U_i \rightarrow X \) satisfying \( f \pi \sigma_i \simeq f|_{U_i} \).

Here, \( \pi: X^I \rightarrow X \times X \) is the path fibration, \( \pi(\alpha) = (\alpha(0), \alpha(1)) \).

We first introduce and study in Section 1, for a given pair of maps \( E \xrightarrow{p} B \xrightarrow{g} X \), the \( g \)-sectional category of \( p \), \( \text{secat}_g(p) \), a generalization of the Svarc genus or sectional category. Then, the topological complexity of a map \( f: X \rightarrow Y \) can be thought of as \( \text{secat}_f(X \times f(\pi)) \). As such, we show in Section 2 that \( \text{TC}(f) \) is an invariant of the homotopy type of \( f \) and is 0 if and only if \( f \) is inessential. Moreover, \( \text{TC}(X) = \text{TC}(\text{id}_X) \). We also provide upper and lower bounds for the topological complexity of a map, see propositions 2.2 and 2.4:
\[
\max\{\text{cat}(f), \text{nil} \ker \cup |f \times f|_\ast \} \leq \text{TC}(f) \leq \min\{\text{TC}(X), \text{cat}(f \times f)\}.
\]
This indicates in particular that the complexity of an algorithm controlling the task performed by a system is in general smaller than the one controlling just the motion of the system. We finish the section with several examples.

In Section 3 we give a characterization of the topological complexity of the rationalization \( fQ \) of a given map \( f \) between simply connected spaces in terms of its Sullivan models. This is highly computable in algebraic terms and becomes a lower bound of the topological complexity of \( f \) as \( \text{TC}(fQ) \leq \text{TC} f \). As an application, we show that the topological complexity of a formal map always coincides with its cohomological lower bound.

Finally, we would like to stress that our purpose is not being exhaustive in the study of the the topological complexity of a map, but just laying the groundwork for its further development and presenting the general behaviour of this new invariant.
1. $f$-Sectional Category

In what follows, and unless explicitly stated otherwise, a topological space will always be pointed, path-connected, and of the homotopy type of a CW-complex. Continuous maps are assumed to preserve base points.

We recall the definition of the most classical Lusternik-Schnirelmann invariants. The category of a space $X$, $\text{cat}(X)$, is the least $n \leq \infty$ such that $X$ can be covered by $n + 1$ open sets contractible within $X$. On the other hand, the category of a map $f$, $\text{cat}(f)$, is the least $n \leq \infty$ such that the domain can be covered by $n + 1$ open sets on each of which the restriction of $f$ is homotopically trivial. Finally, given a map $p: E \to B$, the sectional category of $p$, denoted by $\text{secat}(p)$, is the least $n \leq \infty$ for which $B$ can be covered by $n + 1$ open sets on each of which there is a local homotopy section of $p$. Here we extend this invariant.

**Definition 1.1.** Let $E \xrightarrow{p} B \xrightarrow{f} X$ be two continuous maps. An open set $U \subset B$ is $f$-categorical if there is a map $s: U \to E$ such that $fps \simeq f|_U$. We call $s$ an $f$-section. The $f$-sectional category of $p$, $\text{secat}_f(p)$, is the least $n \leq \infty$ for which $B$ admits a covering of $n + 1$ $f$-categorical open sets.

Obviously, if $f \simeq g$ and $p \simeq q$, then $\text{secat}_f(p) = \text{secat}_g(q)$. Also, observe that $\text{secat}(p) = \text{secat}_{\text{id}_B}(p)$.

Recall that $p: E \to B$ is said to be dominated by $p': E' \to B'$ if there is a (homotopy) commutative diagram,

\[
\begin{array}{ccc}
E & \xrightarrow{i} & E' \xrightarrow{r} E \\
\downarrow{p} & & \downarrow{p'} \\
B & \xrightarrow{j} & B' \xrightarrow{t} B
\end{array}
\]

such that $ri \simeq \text{id}_E$ and $tj \simeq \text{id}_B$.

**Lemma 1.2.** Let $p: E \to B$ be dominated by $p': E' \to B'$ and let $f: B \to X$ be any map. Then, $\text{secat}_f(p') \leq \text{secat}_f(p)$.

**Proof.** Let $U \subset B$ be $f$-categorical and let $s: U \to E$ with $fps \simeq f|_U$. Let $V = t^{-1}(U)$ and $s' = ist_{|_V}: V \to E'$. Then, $ftp's' = ftp'ist \simeq ftjpst \simeq fps \simeq ft$, that is, $V$ is $ft$-categorical for $p'$. \qed

We write $p \sim p'$ is there is a homotopy commutative square,

\[
\begin{array}{ccc}
E & \xrightarrow{g} & E' \\
\downarrow{p} & & \downarrow{p'} \\
B & \xrightarrow{h} & B'
\end{array}
\]
in which the horizontal arrows are homotopy equivalences. An immediate consequence of the lemma above is:

**Proposition 1.3.** Let \( p \sim p' \) and \( f: B' \to X \). then \( \text{secat}_{hf}(p) = \text{secat}_{f}(p') \).

The following summarize how the classical bounds for sectional category (see for instance [3, §9]) have to be modified for this new invariant.

**Proposition 1.4.** For any \( p: E \to B \) and any map \( f: B \to X \),
\[
\text{secat}_f(p) \leq \min\{\text{secat}(p), \text{cat}(f)\}.
\]
Moreover, if \( E \) is contractible, then \( \text{secat}_f(p) = \text{cat}(f) \).

**Proof.** The inequality \( \text{secat}_f(p) \leq \text{secat}(p) \) is obvious. Also, if \( f|_U \simeq * \) the constant map \( *: U \to B \) is an \( f \)-section. This proves the other inequality. Finally, if \( U \subset B \) is \( f \)-categorical and \( E \) is contractible, \( f|_U \) is homotopically trivial. This shows that \( \text{cat}(f) \leq \text{secat}_f(p) \) whenever \( E \) is contractible, which proves the last assertion. \( \square \)

In particular, as \( \text{cat}(f) \leq \min\{\text{cat}(B), \text{cat}(X)\} \), we obtain:

**Corollary 1.5.** \( \text{secat}_f(p) \leq \min\{\text{cat}(B), \text{cat}(X)\} \). In particular, if either \( B \) or \( X \) is a co-H-space, then \( \text{secat}_f(p) \leq 1 \).

An interesting feature of this invariant is the following.

**Proposition 1.6.** Let \( p: E \to B \) be the pullback of a fibration \( p': E' \to B' \) along \( f: B \to B' \). If \( E' \) is contractible, then,
\[
\text{secat}_f(p) = \text{cat}(f).
\]

**Proof.** Let \( U \subset B \) be an \( f \)-categorical open set and \( s: U \to E \) an \( f \)-section. As \( E' \) is contractible then \( fps \simeq f|_U \simeq * \). This proves that \( \text{cat}(f) \leq \text{secat}_f(p) \). The other inequality is given by Proposition 1.4 \( \square \)

The following immediate consequences are examples of this situation:

**Corollary 1.7.** Let \( p: E \to B \) a principal fibration (resp. \( G \)-bundle) classified by a map \( f: B \to K(\pi, n) \) (resp. \( f: B \to BG \)). Then,
\[
\text{secat}_f(p) = \text{cat}(f).
\]

We now set the lower cohomological lower bound of the \( f \)-sectional category. Recall that the **nilpotency index** of a ring \( R \), \( \text{nil } R \), is the biggest \( n \geq \infty \) such that \( R^n \neq 0 \).

**Proposition 1.8.** \( \text{secat}_f(p) \geq \text{nil } \text{ker } p^*|_{\text{lin } f^*} \).

Here \((-)^* \) denotes the morphism induced in reduced cohomology over any fixed ring.
Proof. Assume $\text{secat}_f(p) = n$ and let $k_i: U_i \hookrightarrow B$, $i = 1, \ldots, n + 1$, be $f$-categorical open sets covering $B$ with $f$-sections $s_i$. Consider the long exact sequence

$$\cdots \to H^*(B, U_i) \xrightarrow{q_i^*} H^*(B) \xrightarrow{k_i^*} H^*(U_i) \to \cdots$$

induced by the pair $(B, U_i)$ and let $\gamma_i = f^*(\alpha_1), \ldots, \gamma_{n+1} = f^*(\alpha_{n+1}) \in \ker p^* \cap \text{Im } f^*$. Then,

$$k_i^*(\gamma_i) = k_i^* f^*(\alpha_i) = k_i^* s_i^* p^* f^*(\alpha_i) = 0,$$

and thus $\gamma_i \in \ker k_i^* = \text{Im } q_i^*$. Write $\gamma_i = q_i^*(\tau_i)$, $\tau_i \in H^*(B, U_i)$. To finish observe that $\tau_1 \cup \cdots \cup \tau_{n+1} \in H^*(B, B) = 0$ and, denoting $q: B \to (B, B)$, we have $\gamma_1 \cup \cdots \cup \gamma_{n+1} = q^*(\tau_1 \cup \cdots \tau_{n+1}) = 0$. 

\[\square\]

Observe that $\text{nil ker } p^*|_{\text{im } f^*}$ is in general smaller than $\text{nil ker } (fp)^*$ which is the classical cohomological lower bound of $\text{secat}(fp)$.

Next we give the “Ganea and Whitehead characterizations” of the $f$-sectional category. For the first, we follow the classical approach of [13], improved in [11, §2], with the suitable modifications.

Recall that given a fibration $q: Z \to Y$, the $n$-fold join of $q$ is the space $*^n Z$ inductively defined as follows: $*^0 Z = Z$; $*^n Z = Z \times_Y Z$ is the double mapping cylinder of the projections of $Z \times Y$ over $Z$,

$$E \ast_Y E = ((E \times_Y E) \times I \ast I \ast I) / (x, y, 0) \sim x, (x, y, 1) \sim y.$$

Finally, $*^n Z = (*^{n-1}_Y Z) \ast_Y Z$. The $n$-fold join fibration is the fibration $*^n q: *^n Z \to Y$ inductively defined by $*^0 q = q; *^1 q = q \ast_Y q$ where $(q \ast_Y q)[x, y, t] = q(x) = q(y)$; and $*^n q = (*^{n-1}_Y q) \ast_Y q$.

Now, factor a given composition $E \xrightarrow{p} B \xrightarrow{j} X$ as $qj$ where $j: E \xrightarrow{\sim} Z$ is a homotopy equivalence and $q: Z \to Y$ is a fibration. Then, we have:

**Proposition 1.9.** $\text{secat}_f(p)$ is the least integer $n$ for which there exists $\sigma: B \to Z$ such that $(*^n_Y q)\sigma = f$.

**Proof.** Assume $\text{secat}_f(p) = n$. By induction on $m$, with $0 \leq m \leq n$ we show the existence of an open covering \{\(U_i\)\}_{i=0}^{n-m} of $B$ for which:

There exists a map $\sigma_0: U \to *^n_Y E$ such that $(*^n_Y q)\sigma_0 = f|_{U_0}$.

There are maps $\sigma_i: U_i \to Z$, for $i = 1, \ldots, n-m$, such that $q\sigma_i = f|_{U_i}$.

For $m = 0$ choose a covering \{\(U_i\)\}_{i=0}^{n-m} of $B$ and $f$-sections $\tau_i: U_i \to E$ of $p$. As $q$ is a fibration we may replace the maps $j\tau_i$ by $\sigma_i: U_i \to Z$ so that $q\sigma_i = f|_{U_i}$.

Let $m < n$ and $U = \{U_i\}_{i=0}^{n-m}$ an open covering of $B$ with maps $\sigma_0: U \to *^n_Y E, \sigma_i: U_i \to Z$, for $i = 1, \ldots, n-m$, satisfying the induction hypothesis. Choose refinements of $U$, \{\(V_i\)\}_{i=0}^{n-m}, \{W_i\}_{i=0}^{n-m}$ with

$$V_i \subset V_i \subset W_i \subset W_i \subset U_i,$$

and consider the disjoint closed subspaces $A_0 = V_0 \cap (B - W_1), A_1 = V_1 \cap (B - W_0)$. Observe that $A_0 \cap A_1 \cap C = V_0 \cup V_1$ being $C =$
\( \overline{W}_0 \cap \overline{W}_1 \cap (\overline{V}_0 \cup \overline{V}_1) \). By the Urysohn Lemma choose a continuous function \( h: B \to I \) such that \( h(A_0) = 0 \) and \( h(A_1) = 1 \) and define \( \sigma: \overline{V}_0 \cup \overline{V}_1 \to *^{n+1}_Y Z \),
\[
\sigma(x) = \begin{cases} 
[\sigma_0(x)], & x \in A_0, \\
[\sigma_1(x)], & x \in A_1, \\
[\sigma_0(x), \sigma_1(x), h(x)], & x \in C.
\end{cases}
\]

To finish, consider the open covering of \( B \) given by \( \{V_0 \cap V_1, U_i\}_{i=2}^{n-m} \) and the maps \( \sigma: V_0 \cup V_1 \to *^n_Y Z \), \( \sigma_i: U_i \to Z \), \( i = 2, \ldots, m - n \).

Conversely, assume that \( \sigma: B \to Z \) is such that \( (*^n_Y q)\sigma = f \). By [30] the sectional category of the fibration \( q_n \) given by the pullback diagram
\[
\begin{array}{ccc}
(*^n_Y Z) \times_Y Z & \xrightarrow{p} & Z \\
\downarrow{q_n} & & \downarrow{q} \\
*^n_Y Z & \xrightarrow{*^n_Y q} & Y
\end{array}
\]
is bounded above by \( n \). Hence we may choose an open covering \( \{U_i\}_{i=0}^n \) of \( *^n_Y Z \) and local sections \( \gamma_i: V_i \to (*^n_Y Z) \times_Y Z \) of \( q_n \). For each \( i = 0, \ldots, n \) consider \( U_i = \sigma^{-1}(V_i) \) and \( \tau_i = \pi \gamma_i \sigma; U_i \to Z \). Then \( \{U_i\}_{i=0}^n \) is an open covering of \( B \) and the composition \( \sigma_i = k\tau_i; U_i \to E \), with \( k \) a homotopy inverse of \( j \), is a local \( f \)-section of \( p \).

As \( \text{secat}_f(p) \) is a homotopy invariant it is useful to have a less rigid analogue of the above result. For it, and abusing of notation, we denote by \( E*^n_Y E \) the (homotopy) join, i.e., the homotopy pushout of the homotopy pullback of \( E \downarrow Y \leftarrow E \). More generally, define \( *^n_Y E \) and \( *^n_Y E = (**^n_Y E) *^n_Y E \). By the weak universal property of the homotopy pushout we also get maps, \( *^n_Y fp: *^n_Y E \to Y \). Observe that \( *^n_Y E \simeq *^n_Y Z \) and the Proposition above readily implies:

**Theorem 1.10.** \( \text{secat}_f(p) \) is the least integer \( n \) for which there exists \( \sigma: B \to E \) such that \( (*^n_Y fp)\sigma \simeq f \).

**Remark 1.11.** Note that for each \( n \geq 0 \), there are maps \( \omega_n: E \to *^n_Y E \) inductively defined by the construction of the homotopy join, making commutative the following diagram,
\[
\begin{array}{ccc}
E & \xrightarrow{\omega_n} & *^n_Y E \\
\downarrow{p} & & \downarrow{*^n_Y fp} \\
B & \xrightarrow{f} & Y
\end{array}
\]

For the Whitehead characterization of \( \text{secat}_f(p) \) factor \( fp \) as \( qj \) where \( j: E \to Z \) is a cofibration and \( q: Z \to Y \) is a homotopy equivalence. Recall that the \( n \)th fat wedge of \( j \) is defined as
\[
T^n(j) = \{(x_0, \ldots, x_n) \in Z^{n+1} \mid \text{such that } x_k \in \text{Im } j \text{ for some } k\}.
\]
Observe that this space can also be characterized as the polyhedral product

$$T^n(j) = (Z, E)^{S^n}$$

of the pair $(Z, E)$ and the simplicial complex $S^n$ regarded as the border of the standard $n$-simplex.

Consider the composition

$$h_n: T^n(j) \hookrightarrow Z^{n+1} \xrightarrow{\sim} Y^{n+1}$$

and observe [8, Thm. 3.3.2], [16] that there is a homotopy pullback of the form

$$\begin{array}{ccc}
\ast^n_{\gamma} E & \xrightarrow{\eta} & T^n(j) \\
\downarrow \ast^n_{\gamma} f_{fp} & & \downarrow h_n \\
Y & \xrightarrow{\Delta^n} & Y^{n+1}
\end{array}$$

where $\Delta^n$ is the $n$-diagonal map. Then:

**Theorem 1.12.** $\text{secat}_f(p)$ is the least integer $n$ for which the map $\Delta^n f: B \to Y^{\times n+1}$ homotopy factors through the $n$th fat wedge $T^n(j)$,

$$\begin{array}{ccc}
T^n(j) & \xrightarrow{T^n(j)} & B \\
\downarrow h_n & & \downarrow \Delta^n f \\
Y^{n+1} & \xrightarrow{\sigma} & Y^{n+1}
\end{array}$$

**Proof.** If $\text{secat}_f(p) = n$ then, via Theorem 1.10, there exists a map $\sigma: B \to \ast^n_{\gamma} E$ such that $(\ast^n_{\gamma} f_{fp})\sigma \simeq f$. Hence, the map $\eta\sigma: B \to T^n(j)$ is the dotted lifting in the above diagram.

Conversely, given $\xi: B \to T^n(j)$ such that $h_n \xi \simeq f\Delta$, the weak universal property of the homotopy pullback produces a map $\sigma: B \to \ast^n_{\gamma} E$ such that $\eta\sigma \simeq \xi$ and $(\ast^n_{\gamma} f_{fp})\sigma \simeq f$. \qed

2. **Topological complexity of a map**

**Definition 2.1.** Let $f: X \to Y$ be a continuous map. The *topological complexity of $f$* is defined as

$$\text{TC}(f) = \text{secat}_{f \times f}(p)$$

where $\pi: X^I \to X \times X$ is the path fibration, $\pi(\alpha) = (\alpha(0), \alpha(1))$. In view of the commutative diagram

$$\begin{array}{ccc}
X & \xrightarrow{\sim} & X^I \\
\downarrow \Delta & & \downarrow \pi \\
X \times X & &
\end{array}$$

where $c(x)$ is the constant path in $x$, Proposition 1.3 implies that

$$\text{TC}(f) = \text{secat}_{f \times f}(\Delta).$$
Note that $TC(X) = TC(id_X)$.

**Proposition 2.2.** For any map $f: X \to Y$,
\[
cat(f) \leq TC(f) \leq \min\{TC(X), cat(f \times f)\}.
\]

**Proof.** Proposition [1.4] proves the second inequality. For the first, fix $x_0 \in X$, suppose $s: U \to X$ is a $f \times f$-categorical section of $\pi$, and let $V = \{x \in X, (x_0, x) \in U\}$. Consider the map $k: V \to U$, $k(x) = (x_0, x)$ and denote by $q_2: X \times X \to X$ the projection over the second factor. Then,
\[
q_2(f \times f)\pi sk \simeq f_{|V}: V \to Y.
\]
Hence, in view of the commutative diagram
\[
\begin{array}{ccc}
X^I & \xrightarrow{f^I} & Y^I \\
\downarrow \pi & & \downarrow \pi \\
X \times X & \xrightarrow{f \times f} & Y \times Y,
\end{array}
\]
it follows that
\[
q_2(f \times f)\pi sk = q_2 \pi f^I sk.
\]
However, observe that the image of the map $f^I sk$ lies in the contractible subspace of $Y^I$ of paths starting at $f(x_0)$. Thus,
\[
f_{|V} \simeq q_2(f \times f)\pi sk \simeq *.
\]
Finally, observe that if $\{U_i\}_{i \in J}$ is an open covering with each $U_i$ as above, then the corresponding family $\{V_i\}_{i \in J}$ is an open covering of $X$. This proves the first inequality. $\square$

Taking into account that in general, $cat(f \times g) \leq cat(f) + cat(g)$, we immediately deduce:

**Corollary 2.3.** $TC(f) = 0$ if and only if $f \simeq *$. $\square$

On the other hand, Proposition [1.8] readily implies:

**Proposition 2.4.** For any map $f: X \to Y$,
\[
\text{nil ker} \cup_{(f \times f)^*} \mu \leq TC(f).
\]

$\square$

**Example 2.5.** (1) Obviously $TC(f) = 0$ for any $f: S^n \to S^m$, $n < m$. For $f: S^n \to S^n$, $TC(f)$ is also zero if $\deg f = 0$. If $\deg f > 0$, then
\[
TC(f) = TC(S^n) = \begin{cases} 
1, & \text{if } n \text{ is odd,} \\
2, & \text{if } n \text{ is even,}
\end{cases}
\]
Indeed, by Proposition [2.2] if $n$ is odd, $TC(f) \leq TC(S^n) = 1$. On the other hand, by Corollary [2.3], $TC(f)$ cannot be zero as $f$ is essential.
If \( n \) is even, again by Proposition 2.2, \( \text{TC}(f) \leq \text{TC}(S^n) = 2 \). On the other hand, choose a non zero class \( \alpha \in H^n(S^n) \) which is also in the image of \( f^* \) and observe that \( \gamma = \alpha \otimes 1 - 1 \otimes \alpha \in \ker \cup \) and \( \gamma^2 \neq 0 \). Hence, by Proposition 2.4, \( \text{TC}(f) \geq 2 \).

(2) Let \( f : \mathbb{C}P^n \to \mathbb{C}P^m \) with \( 1 \leq n \leq m \). These maps are also classified by the degree. If \( \deg f = 0 \), then \( \text{TC}(f) = 0 \). If \( \deg f > 0 \), then choose again a non zero class \( \alpha \in H^2(\mathbb{C}P^n) \) which is also in the image of \( f^* \) and observe that \( \gamma = \alpha \otimes 1 - 1 \otimes \alpha \in \ker \cup \) and \( \gamma^2 \neq 0 \). Hence, by Proposition 2.4, \( \text{TC}(f) \geq 2n \). On the other hand, by Proposition 2.2, \( \text{TC}(f) \leq \text{TC}(\mathbb{C}P^n) = 2 \). Hence, \( \text{TC}(f) = 2n \).

(3) For any map \( f : X \to Y \) into a co-\( H \)-space, \( \text{TC}(f) \leq 2 \). Indeed, co-\( H \)-spaces have category 1 and therefore, by Proposition 2.2, \( \text{TC}(f) \leq \text{cat}(f \times f) \leq 2 \text{ cat } f \leq 2 \text{ cat } Y = 2 \).

**Example 2.6** (2-link robot arm). Let \( T = S^1 \times S^1 \) be the configuration space of a robot manipulator of two links of lengths \( \ell_1 > \ell_2 \) and both with one degree of freedom. Then, the work map may have two different homotopy types:

Assume that, at each state of the configuration space, the plane defined by the circle generated by the end effector link contains the tangent vector of the circle generated by the base link. Then, the workspace is a strip homotopy equivalent to \( S^1 \) and the work map is homotopic to the projection \( f : T \to S^1 \) over the first factor.

We show that in this case \( \text{TC}(f) = 1 \). Indeed, as \( \text{TC}(S^1) = 1 \) let \( \sigma_i : V_i \to (S^1)^I \) be local sections of the path fibration \( \pi, i = 1, 2 \), with \( V_1 \cup V_2 = S^1 \). Let \( U_i = f^{-1}(V_i) \) and choose a section \( s : S^1 \to C \) of the projection \( f \). Define \( \alpha_i : U_i \to T^I, \alpha_i = s^i \sigma_i(f \times f) \), for \( i = 1, 2 \). Obviously \( (f \times f) \pi \alpha_i \simeq (f \times f)_{|U_i} \).

In the remaining cases, the workspace is a torus and the work map is homotopic to the identity \( \text{id}_T \). Hence \( \text{TC}(f) = \text{TC}(S^1 \times S^1) = 2 \).

### 3. Rational topological complexity of a map

As rational homotopy theory is particularly fond of algorithms which permit explicit computations we give in this section a characterization of the topological complexity of a map in the rational homotopy category. For it, we will be using known results in rational homotopy for which the excellent reference [9] is now standard. Here we simply
present a brief summary of some basic facts. Any space $X$ considered within this section is simply connected and of the homotopy type of a CW-complex of finite type. Its rationalization $X_Q$ is a rational space (its homotopy groups are rational vector spaces) together with a map $X \to X_Q$ inducing isomorphisms in rational homotopy. On the other hand, to any space $X$ there corresponds, in a contravariant way, a rational commutative differential graded algebra (cdga henceforth) of the form, $(\Lambda V, d)$ which algebraically models the rational homotopy type of the space $X$, or equivalently, the homotopy type of its rationalization $X_Q$. By $\Lambda V$ we mean the free commutative algebra generated by the graded vector space $V = \bigoplus_{p \geq 2} V^p$, i.e., $\Lambda V = TV/I$ where $TV$ denotes the tensor algebra over $V$ and $I$ is the ideal generated by $v \otimes w - (-1)^{|w||v|} w \otimes v$, $\forall v, w \in V$, homogeneous elements of degrees $|v|$ and $|w|$ respectively. Moreover, there exists a well ordered basis of $V$ consisting of homogeneous elements $\{v_i\}_{i \in I}$ such that, for each $i$, the differential $dv_i$ is a polynomial on the generators $\{v_j\}_{j < i}$. The cdga $(\Lambda V, d)$, or simply $\Lambda V$ when the differential is implicitly considered, is called a Sullivan model of $X$. In general, a model of $X$ is any cdga connected by quasi-isomorphisms to a Sullivan model of $X$. This correspondence yields an equivalence between the homotopy categories of simply connected rational spaces of finite type and that of cdga’s of finite type and also simply connected.

For the rest of the section fix a map $f : X \to Y$ and let $\psi : (\Lambda V, d) \to (\Lambda W, d)$ be a surjective Sullivan model of its rationalization $f_Q$. denote by $K$ the kernel of the composition

$$
\Lambda V \otimes \Lambda V \xrightarrow{\psi \otimes \psi} \Lambda W \otimes \Lambda W \xrightarrow{\mu} \Lambda W
$$

where $\mu$ denotes the multiplication. Then, we prove:

**Theorem 3.1.** $\text{TC}(f_Q)$ is the least $n$ for which $\psi \otimes \psi$ factors up to homotopy through $(\Lambda V \otimes \Lambda V)/K^{n+1}$.

In other words, $\text{TC}(f_Q)$ is the least $n$ for which, decomposing the projection $\Lambda V \otimes \Lambda V \to (\Lambda V \otimes \Lambda V)/K^{n+1}$ as the composition

$$
\Lambda V \otimes \Lambda V \xrightarrow{\psi \otimes \psi} \Lambda W \otimes \Lambda W \xrightarrow{\mu} \Lambda W
$$

of a cdga cofibration (i.e., a Sullivan extension) and a quasi-isomorphism, there exists a cdga morphism $\rho : \Lambda V \otimes \Lambda V \otimes \Lambda U \to \Lambda W \otimes \Lambda W$ such that, the following diagram commutes,

$$
\Lambda V \otimes \Lambda V \xrightarrow{\psi \otimes \psi} \Lambda W \otimes \Lambda W \xrightarrow{\rho} \Lambda V \otimes \Lambda V \otimes \Lambda U.
$$
Remark 3.2. (i) Standard arguments on classical localization, taking into account that rationalization commutes with homotopy fibration and cofibration sequences, let us assert that

\[ \text{TC}(f_Q) \leq \text{TC}(f). \]

Hence, Theorem 3.1 produces an algebraic lower bound for the target topological complexity.

(ii) Observe that in the case \( f = \text{id}_X \), we have \( \psi = \text{id}_{\Lambda V} \) and Theorem 3.1 recovers the characterization of \( \text{TC}(X_Q) \) in [2, Thm. 2].

(iii) For completeness we recall how to obtain a surjective model of a given map. Let \( A \to B \) any cdga model of \( f_Q \). Choose a surjective cdga morphism \( \alpha: \Lambda R \to B \) and extend the model above to \( \gamma: A \otimes \Lambda R \otimes \Lambda dR \to B \) by \( \gamma(w) = \alpha(w), \gamma(dw) = d\alpha(w), w \in R. \) Then, the inclusion \( A \to A \otimes \Lambda R \otimes \Lambda dR \) is a quasi-isomorphism and thus, \( \gamma \) is a surjective model of \( f_Q \).

In the proof of Theorem 3.1 we will use the following results:

On the one hand, recall that \( \text{TC}(f) = \text{secat}_{f \times f}(\Delta) \) and let \( h_n: T^n(j) \to (Y \times Y)^{n+1} \) be the associated map [2] in this particular case. Then:

**Lemma 3.3.** [8, Thm. 8.4.1], [10, Thm. 1] A model of \( h_n \) is given by the projection

\[ (\Lambda V \otimes \Lambda V)^{\otimes n+1} \longrightarrow (\Lambda V \otimes \Lambda V)^{\otimes n+1}/K^{\otimes n+1}. \]

\[ \square \]

On the other hand, consider the diagram of Remark 1.11 in our particular case,

\[ \begin{array}{ccc}
X & \xrightarrow{I^n} & X^{Y \times Y} \\
\Delta & & \downarrow r^n_{f \times f}(\Delta) \\
X \times X & \xrightarrow{f \times f} & Y \times Y.
\end{array} \] (3)

Then, a straightforward adaptation to our context of the proof of [2, Prop. 7], based mainly on [2, Lemma 5], gives the following.

**Lemma 3.4.** There is a model of diagram (3)

\[ \begin{array}{ccc}
\Lambda V \otimes \Lambda V & \xrightarrow{k_n} & C_n \\
\psi \otimes \psi & & q_n \\
\Lambda W \otimes \Lambda W & \xrightarrow{\mu} & \Lambda W
\end{array} \]

for which \( (\ker q_n)^{n+1} = 0. \) \[ \square \]
Proof of Theorem 3.1. Consider the commutative diagram induced by the nth multiplication \( \mu_n \):

\[
\begin{array}{ccc}
(\Lambda V \otimes \Lambda V) \otimes n+1 & \xrightarrow{\mu_n} & \Lambda V \otimes \Lambda V \\
\downarrow & & \downarrow \\
(\Lambda V \otimes \Lambda V) \otimes n+1 / K \otimes n+1 & \xrightarrow{\pi_n} & (\Lambda V \otimes \Lambda V) / K^{n+1},
\end{array}
\]

and assume that \( \psi \otimes \psi : \Lambda V \otimes \Lambda V \to \Lambda W \otimes \Lambda W \) factors up to homotopy through \( (\Lambda V \otimes \Lambda V) / K^{n+1} \), for some \( n \).

Then, the composition,

\[
(\Lambda V \otimes \Lambda V) \otimes n+1 \xrightarrow{\mu_n} \Lambda V \otimes \Lambda V \xrightarrow{\psi \otimes \psi} \Lambda W \otimes \Lambda W
\]

factors up to homotopy through the projection

\[
(\Lambda V \otimes \Lambda V) \otimes n+1 \to (\Lambda V \otimes \Lambda V) \otimes n+1 / K \otimes n+1.
\]

But by Theorem 1.12 and Lemma 3.3, \( \text{TC}(f_Q) \) is precisely the least \( n \) for which this occurs. Hence \( \text{TC}(f_Q) \leq n \).

Conversely, assume that \( \text{TC}(f_Q) = n \) and observe that, by Theorem 1.10 and considering the model in Lemma 3.4, \( \text{TC}(f_Q) \) is precisely the least \( n \) for which \( \psi \otimes \psi \) factors up to homotopy through \( k_n \). But \( k_n(K^{n+1}) \subset (\ker q_n)^{n+1} = 0 \). Hence, \( k_n \) factors through \( (\Lambda V \otimes \Lambda V) / K^{n+1} \) and therefore, \( \psi \otimes \psi \) factors up to homotopy through \( (\Lambda V \otimes \Lambda V) / K^{n+1} \).

As an application we show that the topological complexity of certain rational work maps are given in purely cohomological terms.

Recall that a map \( f : X \to Y \) is formal if there is a commutative diagram

\[
\begin{array}{ccc}
(\Lambda V, d) & \xrightarrow{\psi} & (\Lambda W, d) \\
\alpha \downarrow & \simeq & \downarrow \beta \\
H^*(\Lambda V, d) & \xrightarrow{\psi} & H^*(\Lambda W, d),
\end{array}
\]

in which \( \psi \) is a model of \( f \). Note that, in particular, \( X \) and \( Y \) are formal spaces, that is, their rational homotopy type depends only on its rational cohomology. We show that for such maps, the cohomological bound of Proposition 2.4 is always reached. This generalizes [15, Thm. 1.2], see also [14, Cor. 2.2].

**Theorem 3.5.** For any formal map \( f \),

\[
\text{TC}(f_Q) = \text{nil ker } \cup_{(f \times f)^*},
\]

Here, cohomology is considered with rational coefficients.

**Proof.** In view of Proposition 2.4 it is enough to show that

\[
\text{TC}(f_Q) \leq \text{nil ker } \cup_{(f \times f)^*}.
\]
The diagram (4), in which we may assume that $\psi$ is a surjective model of $f$, produces a commutative diagram,

$$\begin{array}{ccc}
\Lambda V \otimes \Lambda V & \xrightarrow{\psi \otimes \psi} & \Lambda W \otimes \Lambda W \\
\cong & \uparrow & \cong \\
H^*(\Lambda V) \otimes H^*(\Lambda V) & \xrightarrow{\phi^* \otimes \phi^*} & H^*(\Lambda W) \otimes H^*(\Lambda W) \\
\end{array}$$

This induces, for any $n \geq 1$, another commutative diagram

$$\begin{array}{ccc}
\Lambda V \otimes \Lambda V & \xrightarrow{(\beta \otimes \beta)(\psi \otimes \psi)} & H^*(\Lambda W) \otimes H^*(\Lambda W) \\
\downarrow & & \downarrow \\
(\Lambda V \otimes \Lambda V)/K^n & \longrightarrow & (H^*(\Lambda W) \otimes H^*(\Lambda W))/L^n \\
\end{array}$$

where $K = \ker \mu(\psi \otimes \psi)$ and $L = \ker \mu^* |_{\text{Im}(f \times f)^*}$, which is in turn identified with $\ker \cup_{\text{Im}(f \times f)^*}$. Now, assume nil $\ker \cup_{\text{Im}(f \times f)^*} = n$. Hence $L^{n+1} = 0$ and, for $n + 1$, the above diagram becomes

$$\begin{array}{ccc}
\Lambda V \otimes \Lambda V & \xrightarrow{(\beta \otimes \beta)(\psi \otimes \psi)} & H^*(\Lambda W) \otimes H^*(\Lambda W) \\
\downarrow & & \downarrow \\
(\Lambda V \otimes \Lambda V)/K^{n+1} & \longrightarrow & (H^*(\Lambda W) \otimes H^*(\Lambda W))/L^{n+1} \\
\end{array}$$

As $\beta \otimes \beta$ is a quasi-isomorphism, this shows that $\psi \otimes \psi$ factors up to homotopy through $(\Lambda V \otimes \Lambda V)/K^{n+1}$. By Theorem 3.1 the result follows.

\begin{proof}
\end{proof}

\textit{Example} 3.6. Let $f$ be any of the maps considered in Example 2.5. Such a map is formal and therefore,

$$\text{TC}(f_Q) = \text{TC}(f) = \text{nil} \ \ker \cup_{\text{Im}(f \times f)^*}.$$ 

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