Abstract

In this paper we develop tools to study families of non-selfadjoint operators $L(\varphi), \varphi \in P$, characterized by the property that the spectrum of $L(\varphi)$ is (partially) simple. As a case study we consider the Zakharov-Shabat operators $L(\varphi)$ appearing in the Lax pair of the focusing NLS on the circle. The main result says that the set of potentials $\varphi$ of Sobolev class $H^N, N \geq 0$, so that all small eigenvalues of $L(\varphi)$ are simple, is path connected and dense.

1 Introduction

In this paper we develop tools to study families of non-selfadjoint operators $L(\varphi)$, depending on a parameter $\varphi$. To fix ideas assume that the parameter space $P$ is a subset of some real Hilbert space and for any $\varphi \in P$, $L(\varphi)$ has a discrete spectrum. Ideally, the spectrum $\text{spec} L(\varphi)$ is simple for any $\varphi \in P$, i.e. any eigenvalue of $L(\varphi)$ has algebraic multiplicity one, and $\text{spec} L(\varphi)$ can then be represented, under appropriate regularity assumptions on the parameter dependence of $L(\varphi), \varphi \in P$, by a family of eigenvalues $(\lambda_j(\varphi))_{j \in J}$ with $\lambda_j : P \to \mathbb{C}$ being real-analytic for any $j \in J$. However, typically, such a situation does not hold and one is interested in tools to estimate the size of the subset

$$P' = \{ \varphi \in P \mid \text{spec} L(\varphi) \text{ simple} \}.$$ 

In particular, it is of interest to know if $P'$ is open, dense, or connected. As an illustration we recall the classical theorem of Neumann and Wigner [18] saying that within the space $P$ of all real symmetric $n \times n$ matrices, $n \geq 2$, the ones with multiple eigenvalues form an algebraic variety of codimensions two. In particular, the set $P'$ of all real symmetric $n \times n$
matrices with simple spectrum is path-wise connected and dense. See also [2, 3], [15], [6], [4] for related results.

As a case study we consider in this paper the Zakharov-Shabat operators (ZS) appearing in the Lax pair of the defocussing nonlinear Schrödinger equation (dNLS) and the focusing one (fNLS). These operators are differential operators of first order of the form \((x \in \mathbb{R}, \partial_x = \partial/\partial x)\)

\[
L(\varphi) = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix}
\]

(1.1)

where the potential \(\varphi = (\varphi_1, \varphi_2)\) is in \(L^2_c = L^2(\mathbb{T}, \mathbb{C}) \times L^2(\mathbb{T}, \mathbb{C})\) and \(\mathbb{T} = \mathbb{R}/\mathbb{Z}\). In the case of dNLS, \(\varphi\) is in the subspace \(L^2_r\) whereas in the case of fNLS, \(\varphi\) is in \(iL^2_r\). Here \(L^2_r \subseteq L^2_c\) denotes the real subspace.

For \(\varphi \in L^2_c\) arbitrary we denote by \(\text{spec}_p L(\varphi)\) the spectrum of the operator \(L = L(\varphi)\) with domain

\[
\text{dom}_p L(\varphi) = \{F \in H^1_{\text{loc}} \times H^1_{\text{loc}} | F(1) = \pm F(0)\}.
\]

As \(L(\varphi)\) has a compact resolvent \(\text{spec}_p L(\varphi)\) is discrete and each of its eigenvalues has finite algebraic multiplicity. It is referred to as the periodic spectrum of \(L(\varphi)\) and its eigenvalues as periodic eigenvalues of \(L(\varphi)\) – or by a slight abuse of terminology as periodic eigenvalues of \(\varphi\). First let us state the following rough estimate of the periodic eigenvalues of \(L(\varphi)\) – cf. Lemma 1 in [16] as well as Theorem 4.1 and Theorem 4.2 in [17]. For the convenience of the reader it is proved in Section 2.

**Lemma 1.1.** [Counting Lemma] For each potential in \(L^2_c\) there exist a neighborhood \(W \subseteq L^2_c\) and an integer \(R \in \mathbb{Z}_{\geq 0}\) so that for any \(\varphi \in W\), when counted with their algebraic multiplicities, \(L(\varphi)\) has two periodic eigenvalues in each disk

\[
D_n = \{\lambda \in \mathbb{C} | |\lambda - n\pi| < \pi/4\}
\]

with \(|n| > R\) and \(4R + 2\) eigenvalues in the disk

\[
B_R = \{\lambda \in \mathbb{C} | |\lambda| < R\pi + \pi/4\}.
\]

There are no other periodic eigenvalues.

The Counting Lemma shows that given a potential \(\varphi\) in \(L^2_c\), for any \(|n| > R\) with \(R\) sufficiently large, the periodic eigenvalues of \(L(\varphi)\) come in pairs, located in the disjoint disks \(D_n\). In case they are equal, one gets an eigenvalue of geometric and algebraic multiplicity two (cf. Section 2). For \(\varphi\) in \(L^2_r\) or \(iL^2_r\) one can say more. Let us first consider the case \(\varphi \in L^2_r\). Then \(L(\varphi)\) is self-adjoint and hence \(\text{spec}_p L(\varphi)\) real. It is well known that when
listed with their algebraic multiplicities, the periodic eigenvalues are given by two doubly infinite real sequences, \((\lambda_n^-)_{n \in \mathbb{Z}}\) and \((\lambda_n^+)_{n \in \mathbb{Z}}\) satisfying \(\lambda_n^+ = n\pi + \ell_n^2\) and
\[
\cdots < \lambda_n^- < \lambda_{n+1}^- \leq \lambda_n^+ < \lambda_{n+1}^+ < \cdots .
\]

– see e.g. [9] for a proof. In particular, \(L(\varphi)\) has a multiple eigenvalue iff there exists \(n \in \mathbb{Z}\) with \(\lambda_n^- = \lambda_n^+\). The set \(Z_n\) of potentials in \(L^2_r\) with \(\lambda_n^- = \lambda_n^+\) is a real-analytic submanifold of codimension two. Hence, for any \(N \in \mathbb{Z}_{\geq 0}\) the set \(L^2_r \setminus \bigcup_{|n|\leq N} Z_n\) is open, dense, and connected in \(L^2_r\). Furthermore, \(\bigcup_{n \in \mathbb{Z}} Z_n\) is dense in \(L^2_r\).

For \(\varphi \in iL^2_r\), the periodic spectrum of \(L(\varphi)\) is more complicated. If \(\varphi \neq 0\), \(L(\varphi)\) is not selfadjoint and hence its periodic spectrum is not necessarily real. Moreover, besides the asymptotic properties provided by the Counting Lemma, the spectrum has a symmetry. For any \(\lambda \in \text{spec}_p L(\varphi)\), its complex conjugate \(\overline{\lambda}\) is also a periodic eigenvalue and its algebraic and geometric multiplicities are the same as the ones of \(\lambda\) (cf. Section 2). In addition any real eigenvalue has geometric multiplicity two and its algebraic multiplicity is even. No further constraints are known for the \(4R+2\) periodic eigenvalues in the disk \(B_R\), given by the Counting Lemma. It turns out that some of the feature of \(\text{spec}_p L(\varphi)\) are still comparable to the ones in the case where the potential is in \(L^2_r\). To describe them we introduce the following notion.

**Definition 1.** We say that a potential \(\varphi \in iL^2_r\) is standard, if any real periodic eigenvalue of \(L(\varphi)\) has algebraic multiplicity two and any periodic eigenvalue in \(\mathbb{C} \setminus \mathbb{R}\) is simple.

Denote by \(S_p\) the set of all standard potentials in \(iL^2_r\). Due to the Counting Lemma, the property of being a standard potential involves only the \(4R+2\) eigenvalues in \(B_R\). One can show in a straightforward way that \(S_p\) is open in \(iL^2_r\) and contains the zero potential. To state our main result we need to introduce some additional notation. For any \(N \in \mathbb{Z}_{\geq 0}\), let \(H^N_c = H^N(T, \mathbb{C}) \times H^N(T, \mathbb{C})\) and \(iH^N_r = H^N_c \cap iL^2_r\) where \(H^N(T, \mathbb{C})\) denotes the Sobolev space of functions \(f : T \to \mathbb{C}\) with distributional derivatives up to order \(N\) in \(L^2(T, \mathbb{C})\). Note that \(H^0(T, \mathbb{C}) = L^2(T, \mathbb{C})\) and \(H^0_c = L^2_c\).

**Theorem 1.2.** For any \(N \in \mathbb{Z}_{\geq 0}\), \(S_p \cap iH^N_r\) is path-wise connected.

**Remark 1.3.** Concerning the proof of this theorem let us first point out that in contrast to papers such as [2], the Hilbert spaces \(iH^N_r, N \geq 0\), considered in Theorem 1.2 are real. Therefore one can not apply the standard arguments used to prove that the complement of a proper algebraic variety in a complex Hilbert space is path-wise connected.

We begin by analyzing potentials with a multiple eigenvalue \(\lambda\). It turns out that the case where the geometric multiplicity of \(\lambda\) is equal to 1 and the one where it is 2 have to be treated differently. In Section 3 we show by general arguments that for any given \(\psi \in iL^2_r\) with a periodic eigenvalue \(\lambda_\psi\) of \(L(\psi)\) of geometric multiplicity one and algebraic
multiplicity $m_p \geq 2$ there is a neighborhood $W$ of $\psi \in iL^2_r$ so that the set of potentials in $W$, having a periodic eigenvalue near $\lambda_\psi$ of algebraic multiplicity $m_p$ and geometric multiplicity one, is contained in a real-analytic submanifold of codimension two (Theorem 3.2). A corresponding result is proved for a potential $\psi$ in $iL^2_r$ admitting a periodic eigenvalue of geometric multiplicity two (Theorem 3.3). The proof of Theorem 3.2 and Theorem 3.3 are based on a Theorem formulated in general terms and proved in Appendix A, providing a class of functionals which can be used to construct submanifolds with the properties stated in Theorem 3.2 and Theorem 3.3. In Section 2 we describe the set-up used throughout the paper and in Appendix B we illustrate our results for the constant potentials and show an auxiliary result needed in the proof of Theorem 3.2.

It follows from the proof of Theorem 1.2 that for any $N \in \mathbb{Z}_{\geq 0}$, $\mathcal{S}_p \cap iH^N_r$ is dense in $iH^N_r$. However, there is a much easier way to prove this density result and it turns out that a stronger result holds. First we need to introduce some more notation. For $\varphi \in L^2_c$, denote by $\text{spec}_D L(\varphi)$ the Dirichlet spectrum of the operator $L(\varphi)$, i.e., the spectrum of the operator $L(\varphi)$ with the domain

$$\text{dom}_D L(\varphi) = \{ f = (f_1, f_2) \in H^1([0, 1], \mathbb{C})^2 \mid f_1(0) = f_2(0), \ f_1(1) = f_2(1) \}. \quad (1.4)$$

The Dirichlet spectrum is discrete and each eigenvalue has finite algebraic multiplicity. Let

$$\mathcal{S}_D := \{ \varphi \in iL^2_r \mid \text{spec}_D L(\varphi) \text{ is simple} \}. \quad (1.5)$$

**Theorem 1.4.** For any $N \in \mathbb{Z}_{\geq 0}$, $\mathcal{S}_p \cap iH^N_r$ and $\mathcal{S}_D \cap iH^N_r$ are open and dense in $iH^N_r$. To prove the statement of Theorem 1.4 concerning density, we locally reduce the problem to one for matrices and then use the discriminant to conclude the theorem.

The basis for the study of the geometry of the phase space of fNLS are the spectral properties of $L(\varphi)$. Such an analysis was initiated in [1] and later in more detail, taken up in [16]. However, much remains to be discovered – see also [5]. In a forthcoming paper we will use Theorem 1.2 to construct action and angle coordinates for the fNLS in a neighborhood of a standard potential $\varphi \in iL^2_r$.

## 2 Set-up

In this section we introduce some more notations, recall several known results needed in the sequel and establish some auxiliary results. We consider the ZS operator $L(\varphi)$, defined by (1.1), for $\varphi = (\varphi_1, \varphi_2)$ in $L^2_c$. For any $\lambda \in \mathbb{C}$, let $M = M(x, \lambda, \varphi)$ be the fundamental $2 \times 2$ matrix of the equation

$$L(\varphi)M = \lambda M$$
satisfying the initial condition \( M(0, \lambda, \varphi) = \text{Id}_{2 \times 2}, \)

\[
M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}.
\]

Further, we denote by \( M_1, M_2 \) the first, respectively second column of \( M \). The fundamental solution \( M(x, \lambda, \varphi) \) is a continuous function on \( \mathbb{R} \times \mathbb{C} \times L^2_c \) and for any given \( x \in \mathbb{R} \), it is analytic in \( \lambda, \varphi \) on \( \mathbb{C} \times L^2_c \) — see e.g. Section 1 in [9]. Moreover, the proof of Theorem 1.1 in [9] shows that the following stronger statement holds.

**Lemma 2.1.** The fundamental matrix \( M \) defines an analytic map

\[
M : \mathbb{C} \times L^2_c \to C([0, 2]), \quad (\lambda, \varphi) \mapsto M(\cdot, \lambda, \varphi).
\]

For \( \varphi = 0 \), the fundamental solution \( E_\lambda(x) := M(x, \lambda, 0) \) is given by the diagonal matrix \( \text{diag}(e^{-i\lambda x}, e^{i\lambda x}) \). In the sequel we denote by \( (\cdot) : \) the derivative with respect to \( \lambda \).

**Symmetry:** The ZS operator has various symmetries — see e.g. [7]. In this paper, the following one is used frequently. For any function \( f : \mathbb{R} \to \mathbb{C}^2 \) with components \( f_1, f_2 \) introduce the functions \( \hat{f}, \check{f} : \mathbb{R} \to \mathbb{C}^2 \), given by

\[
\hat{f} = (-\bar{f}_2, \bar{f}_1) \quad \text{and} \quad \check{f} = -(\bar{f}_2, \bar{f}_1).
\]

Note that for any \( \varphi \in L^2_c \), one has \( \varphi = \check{\varphi} \) iff \( \varphi \in iL^2_r \) and that \( \Im f := \hat{f} \) is an anti-involution, \( \Im^2 f = -f \).

**Lemma 2.2.** Assume that \( \varphi \in L^2_c, \lambda \in \mathbb{C}, \) and \( f \) in \( H^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^2) \) solves \( (L(\varphi) - \lambda)^n f = 0 \) for some \( n \in \mathbb{Z}_{\geq 1} \). Then

\[
(L(\check{\varphi}) - \bar{\lambda})^n \check{f} = 0.
\]

**Proof.** Introduce the matrices

\[
P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

A direct computation shows that \( PR = J, \ PR = -(PR)^{-1}, \ P^2 = \text{Id}, \) and \( R^2 = \text{Id} \). As \( L(\varphi) = iR\partial_x + \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix} \) it then follows that

\[
PR(L(\bar{\varphi}) - \bar{\lambda})(PR)^{-1} = L(\check{\varphi}) - \bar{\lambda}
\]

and hence

\[
PR(L(\varphi) - \lambda)^n(PR)^{-1} = (L(\check{\varphi}) - \bar{\lambda})^n.
\]

As \( \check{f} = PR\check{f} \) one then concludes for any \( f \in H^1_{\text{loc}}(\mathbb{R}, \mathbb{C}) \) satisfying \( (L(\varphi) - \lambda)^n f = 0 \) that \( (L(\check{\varphi}) - \bar{\lambda})^n \check{f} = 0 \) as claimed. \( \square \)
Periodic spectrum: By the definition of the fundamental solution $M$, any solution $f$ of the equation $L(\varphi)f = \lambda f$ is given by $f(x) = M(x, \lambda)f(0)$. Hence, a complex number $\lambda$ is a periodic eigenvalue of $L(\varphi)$ iff there exists a non zero solution of $L(\varphi)f = \lambda f$ with

$$f(1) = M(1, \lambda)f(0) = \pm f(0).$$

It means that 1 or $-1$ is an eigenvalue of the Floquet matrix $M(1, \lambda)$. Denote by $\Delta(\lambda) \equiv \Delta(\lambda, \varphi)$ the discriminant of $L(\varphi)$, $\Delta(\lambda, \varphi) := m_1(1, \lambda, \varphi) + m_4(1, \lambda, \varphi)$, i.e., the trace of the fundamental matrix $M$, evaluated at $x = 1$. In view of the Wronskian identity, $\det M(1, \lambda) = 1$, it then follows that $\lambda$ is a periodic eigenvalue of $L(\varphi)$ iff $\Delta(\lambda) = \pm 2$. For later reference we record the following

Proposition 2.3. For any $\varphi \in L^2_c$, the periodic spectrum of $L(\varphi)$ coincides as a set with the zero set of the function

$$\chi_p(\lambda) \equiv \chi_p(\lambda, \varphi) = \Delta^2(\lambda, \varphi) - 4.$$

The discriminant $\Delta$ and hence the characteristic function $\chi_p$ are analytic on $\mathbb{C} \times L^2_c$.

Actually, more is true. We will see below that for any periodic eigenvalue $\lambda_\varphi$ of $L(\varphi)$, the algebraic multiplicity of $\lambda_\varphi$ coincides with the multiplicity of $\lambda_\varphi$ as a root of $\chi_p(\cdot, \varphi)$. Recall that the algebraic multiplicity of a periodic eigenvalue $\lambda$ of $L(\varphi), \varphi \in L^2_c$, equals the dimension of the root space $R_\lambda(\varphi)$, defined as the following subspace of $\text{dom}_p L(\varphi)$,

$$R_\lambda(\varphi) = \{ f \in \text{dom}_p L(\varphi) \mid \exists n \in \mathbb{N} \forall 1 \leq k \leq n, L(\varphi)^k f \in \text{dom}_p L(\varphi), (\lambda - L(\varphi))^n f = 0 \}.$$ 

First we give the following rough localization of the roots of $\chi_p$ – see Section 6 in [9]. Recall that the disks $D_n$ and $B_R$ have been introduced in (1.2) respectively (1.3).

Lemma 2.4. For each potential in $L^2_c$ there exist a neighborhood $W$ in $L^2_c$ and $R \in \mathbb{Z}_{\geq 0}$ such that for any $\varphi \in W$ the entire function $\chi_p(\cdot, \varphi)$ has exactly two roots in each disk $D_n$ with $|n| > R$, and $4R + 2$ roots in the disk $B_R$, counted with their multiplicities. There are no other roots.

Lemma 2.4 leads to the following corollary. To formulate it, denote by $\|\varphi\|_1$ the norm of $\varphi \in H^1_c$,

$$\|\varphi\|_1 := (\|\varphi\|^2 + \|\partial_x \varphi\|^2)^{1/2}.$$

Corollary 2.5. For any $\rho > 0$ there exists $R \equiv R_\rho \geq 1$ so that for any $\varphi \in H^1_c$ with $\|\varphi\|_1 \leq \rho$, the entire function $\chi_p(\cdot, \varphi)$ has exactly two roots in each disk $D_n$ with $|n| > R$ and exactly $4R + 2$ roots in the disk $B_R$, counted with their multiplicities. There are no other roots.
Proof. For any $\varphi \in H^1_c$, let $W_\varphi := W$ and $R_\varphi := R$ be as in the statement of Lemma 2.4. By Rellich’s theorem, $H^1_c$ is compactly embedded in $L^2_c$. Hence there exist finitely many potentials $(\varphi^j)_{j \in J}$ in $H^1_c$ with $\|\varphi^j\|_1 \leq \rho$ so that $(W_{\varphi^j})_{j \in J}$ covers the closed ball of radius $\rho$ in $H^1_c$ centered at 0. Then $R \equiv R_\rho := \max_{j \in J} R_{\varphi^j}$ has the claimed properties.

We now prove that the algebraic multiplicity of a periodic eigenvalue equals its multiplicity as a root of the characteristic function $\chi_p$. First we note that by functional calculus, the algebraic multiplicity $m_p(\lambda) \equiv m_p(\lambda, \varphi)$ of a periodic eigenvalue $\lambda$ of $L(\varphi)$ with $\varphi \in L^2_c$ is equal to the dimension of the subspace of $\text{dom}_p L(\varphi)$, given by the image of the Riesz projector $\Pi_\lambda(\varphi)$,

$$\Pi_\lambda(\varphi) = \frac{1}{2\pi i} \int_{\partial B(\lambda)} (z - L_p(\varphi))^{-1} \, dz,$$

where $L_p(\varphi)$ denotes the operator $L(\varphi)$ with domain $\text{dom}_p L(\varphi)$, $B(\lambda)$ denotes the open disk centered at $\lambda$ with sufficiently small radius so that $B(\lambda) \cap \text{spec}_p L(\varphi) = \{\lambda\}$, and the circle $\partial B(\lambda)$ is counterclockwise oriented. By Proposition 2.3, $\lambda$ is a root of $\chi_p(\cdot, \varphi)$. Denote by $m_r(\lambda)$ the multiplicity of $\lambda$ as a root of $\chi_p(\cdot, \varphi)$.

Lemma 2.6. For any periodic eigenvalue $\lambda$ of $L(\varphi)$ with $\varphi \in L^2_c$, $m_r(\lambda) = m_p(\lambda)$.

Proof. First, note that a direct computation shows that the statement of the Lemma holds for the zero potential $\varphi = 0$. A simple perturbation argument involving Proposition 2.3, Lemma 2.4, the argument principle, and the properties of the Riesz projector (see the arguments below), then shows that the Lemma continues to hold in an open neighborhood of zero in $L^2_c$.

Now, consider the general case. Take $\varphi \in L^2_c$. As $\{s\varphi | 0 \leq s \leq 1\}$ is compact in $L^2_c$ there exist a connected open neighborhood $W$ of the line segment $[0, \varphi]$ in $L^2_c$ so that the integer $R \geq 1$ of Lemma 2.4 can be chosen independently of $\psi \in W$. First consider the periodic eigenvalues in $B_R$. For $\psi \in W$ denote by $\Pi_R(\psi)$ the Riesz projector

$$\Pi_R(\psi) = \frac{1}{2\pi i} \int_{\partial B_R} (z - L_p(\psi))^{-1} \, dz.$$

Note that by functional calculus

$$\text{Image} \Pi_R(\psi) = \bigoplus_{\lambda \in \text{spec}_p L(\psi)} R_\lambda(\psi)$$

(2.1)

where $R_\lambda(\varphi)$ is the root space corresponding to $\lambda$. Moreover, standard arguments show that $W \to \mathcal{L}(L^2_c, L^2_c)$, $\psi \mapsto \Pi_R(\psi)$, is analytic. In particular, by the general properties of the projection operators the dimension of $\text{Image} \Pi_R(\psi)$ is independent on $\psi \in W$ (see [10], Chapter III, §3). Consider the operator,

$$A(\psi) = \frac{1}{2\pi i} \int_{\partial B_R} z(z - L_p(\psi))^{-1} \, dz.$$

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One easily sees that $W \to \mathcal{L}(L_c^2, L_c^2)$, $\psi \mapsto A(\psi)$, is analytic. By functional calculus $L_p(\psi)|_{\text{Image } \Pi_R(\psi)} = A(\psi)|_{\text{Image } \Pi_R(\psi)}$, and hence
\[
\det \left( \lambda - L_p(\psi)|_{\text{Image } \Pi_R(\psi)} \right) = \det \left( \lambda - A(\psi)|_{\text{Image } \Pi_R(\psi)} \right).
\]
Hence the polynomial
\[
Q(\lambda, \psi) := \det \left( \lambda - L_p(\psi)|_{\text{Image } \Pi_R(\psi)} \right)
\]
is well defined, analytic in $\mathbb{C} \times W$, and has leading coefficient one. By Proposition 2.4, the roots of $Q(\cdot, \psi)$ are precisely the periodic eigenvalues of $L(\psi)$ in $B_R$ counted with their multiplicities. On the other hand, define
\[
P(\lambda, \psi) := \prod_{|j| \leq R} (\lambda - \lambda^+_j)(\lambda - \lambda^-_j).
\]
Note that $P(\lambda, \psi)$ is a polynomial in $\lambda$ of degree $4R + 2$ with leading coefficient 1. By the argument principle and the last statement of Proposition 2.3, $P(\lambda, \psi)$ is analytic in $\mathbb{C} \times W$. Hence, the coefficients of $Q(\cdot, \psi)$ and $P(\cdot, \psi)$ are analytic on $W$. As $Q(\cdot, \psi) = P(\cdot, \psi)$ in an open neighborhood of zero in $L_c^2$ we get by analyticity that
\[
Q(\cdot, \psi) = P(\cdot, \psi)
\]
for any $\psi \in W$. In particular, the Lemma holds also for any $\lambda \in B_R \cap \text{spec}_p L(\psi)$. The same argument shows that the statement of the Lemma holds also for any $\lambda \in D_n \cap \text{spec}_p L(\psi)$, $|n| > R$.

Now we are ready to prove Lemma 1.1 stated in the introduction.

**Proof of Lemma 1.1.** By Lemma 2.6, for any $\varphi \in L_c^2$, the roots of $\chi_p(\cdot, \varphi)$ coincide with the eigenvalues of $L_p(\varphi)$, together with the corresponding multiplicities. Lemma 1.1 thus follows from Lemma 2.4.

For potentials $\varphi \in iL_c^2$, the results discussed so far lead to a convenient description of the periodic spectrum of $L(\varphi)$. To state it we introduce the following order of $\mathbb{C}$. We say that two complex numbers $a, b$ are lexicographically ordered, $a \preceq b$, if $\text{Re}(a) < \text{Re}(b)$ or $\text{Re}(a) = \text{Re}(b)$ and $\text{Im}(a) \leq \text{Im}(b)$.

**Proposition 2.7.** For any $\varphi \in iL_c^2$, any real periodic eigenvalue of $L(\varphi)$ has geometric multiplicity two and even algebraic multiplicity. For any periodic eigenvalue $\lambda$ of $L(\varphi)$ in $\mathbb{C} \setminus \mathbb{R}$, its complex conjugate $\bar{\lambda}$ is also a periodic eigenvalue of $L(\varphi)$ and has the same algebraic and geometric multiplicity as $\lambda$. It then follows that the periodic eigenvalues of $L(\varphi)$, when counted with their algebraic multiplicities, are given by two doubly infinite sequences $(\lambda^+_n)_{n \in \mathbb{Z}}$ and $(\lambda^-_n)_{n \in \mathbb{Z}}$ where $\lambda^-_n = \overline{\lambda^+_n}$ and $\text{Im}(\lambda^+_n) \geq 0$ for any $n \in \mathbb{Z}$ so that $(\lambda^+_n)_{n \in \mathbb{Z}}$ is lexicographically ordered.
Proof. It follows from Lemma \([2.2]\) that for any \(\lambda \in \text{spec}_{p} L(\varphi)\), its complex conjugate \(\bar{\lambda}\) is in \(\text{spec}_{p} L(\varphi)\) as well and that \(\lambda\) and \(\bar{\lambda}\) have the same geometric and the same algebraic multiplicities. In addition, it follows from Lemma \([2.2]\) that the geometric multiplicity of each real periodic eigenvalue is two. It remains to show that any real periodic eigenvalue of \(L(\varphi)\) has even algebraic multiplicity. Denote by \(R_{\lambda}(\varphi)\) the root space of \(L(\varphi) - \lambda\). By Lemma \([2.2]\), if \(f = \hat{f}\) is a \(\mathbb{R}\)-linear anti-involution, leaving the finite dimensional vector space invariant. Hence \(f\) defines a complex structure on \(R_{\lambda}(\varphi)\) and hence \(\dim_{\mathbb{R}} R_{\lambda}(\varphi)\) is even. This means that the algebraic multiplicity of \(\lambda\) is even. \(\square\)

Finally, we state the following well-known result on the asymptotics of the roots of \(\chi_{p}\) – see e.g. Section 6 in \([9]\).

**Proposition 2.8.** For any \(\varphi \in L_{c}^{2}\), the set of roots of \(\chi_{p}(\cdot, \varphi)\), listed with multiplicities, consists of a sequence of pairs \(\lambda_{n}^{-}(\varphi), \lambda_{n}^{+}(\varphi), n \in \mathbb{Z}\), of complex numbers satisfying

\[
\lambda_{n}^{\pm}(\varphi) = n\pi + \ell_{n}^{2}
\]

locally uniformly in \(\varphi\), i.e., the sequences \((\lambda_{n}^{\pm}(\varphi) - n\pi)_{n \in \mathbb{Z}}\) are locally bounded in \(\ell^{2}(\mathbb{Z}, \mathbb{C})\).

**Discriminant:** Denote by \(\hat{\Delta}\) the partial derivative of the discriminant \(\Delta(\lambda, \varphi)\) with respect to \(\lambda\). Then \(\hat{\Delta}(\lambda, \varphi)\) is analytic on \(\mathbb{C} \times L_{c}^{2}\) as well. The following properties of \(\Delta\) and \(\hat{\Delta}\) are well known – see e.g. Section 6 in \([9]\) as well as Proposition \([2.7]\) above. To state them, introduce

\[
\pi_{n} := n\pi \text{ for } n \in \mathbb{Z} \setminus \{0\} \quad \text{and} \quad \pi_{0} := 1.
\]

**Proposition 2.9.** Let \(\varphi\) be an arbitrary element in \(L_{c}^{2}\).

(i) The function \(\lambda \mapsto \Delta^{2}(\lambda, \varphi) - 4\) is entire and admits the product representation

\[
\Delta^{2}(\lambda, \varphi) - 4 = -4 \prod_{n \in \mathbb{Z}} \left( \frac{(\lambda_{n}^{+}(\varphi) - \lambda)(\lambda_{n}^{-}(\varphi) - \lambda)}{\pi_{n}} \right).
\]

(ii) The function \(\lambda \mapsto \hat{\Delta}(\lambda, \varphi)\) is entire and has countably many roots. They can be listed when counted with their order in such a way that they are lexicographically ordered and satisfy the asymptotic estimates

\[
\hat{\lambda}_{n} = n\pi + \ell_{n}^{2},
\]

locally uniformly in \(\varphi\). In addition, \(\hat{\Delta}(\lambda, \varphi)\) admits the product representation

\[
\hat{\Delta}(\lambda, \varphi) = 2 \prod_{n \in \mathbb{Z}} \frac{\hat{\lambda}_{n} - \lambda}{\pi_{n}}.
\]
(iii) For any $\varphi \in iL^2_r$ and $\lambda \in \mathbb{C}$,
\[ \Delta(\bar{\lambda}, \varphi) = \bar{\Delta}(\lambda, \varphi) \quad \text{and} \quad \bar{\Delta}(\bar{\lambda}, \varphi) = \bar{\Delta}(\lambda, \varphi). \]
In particular, the zero set of $\bar{\Delta}(\cdot, \varphi)$ is invariant under complex conjugation. In view of the asymptotics stated in (ii), for $n$ sufficiently large, $\lambda_n$ is real.

**Dirichlet spectrum:** Recall from the introduction that for $\varphi \in L^2_c$ we denote by $\text{spec}_D(L(\varphi))$ the Dirichlet spectrum of the operator $L(\varphi)$, i.e., the spectrum of the operator $L(\varphi)$ considered with domain $(1.4)$. As $L(\varphi)$, when viewed as an operator with domain $\text{dom}_D(L)$ has compact resolvent the Dirichlet spectrum is discrete. For any $\lambda \in \mathbb{C}$ and $\varphi \in L^2_c$, denote
\[ \hat{M} := \begin{pmatrix} \hat{m}_1 & \hat{m}_2 \\ \hat{m}_3 & \hat{m}_4 \end{pmatrix} = M(1, \lambda, \varphi). \]
Similarly as in the periodic case, one can show that the operator $L(\varphi)$ with domain $\text{dom}_D(L)$ admits the entire function
\[ \chi_D(\lambda, \varphi) := \frac{\hat{m}_4 + \hat{m}_3 - \hat{m}_2 - \hat{m}_1}{2i} \]
as a characteristic function and that the following results hold.

**Lemma 2.10.** For an arbitrary potential in $L^2_c$ there exist a neighborhood $W$ in $L^2_c$ and an integer $R \geq 1$ so that when counted with their algebraic multiplicity, for any $\varphi \in W$, there is exactly one Dirichlet eigenvalue in each disk
\[ D_n := \{ \lambda \in \mathbb{C} \mid |\lambda - n\pi| < \pi/4 \} \quad |n| > R, \]
and there are exactly $2R + 1$ Dirichlet eigenvalues in the disk
\[ B_R := \{ \lambda \in \mathbb{C} \mid |\lambda| < R\pi + \pi/4 \}. \]
There are no other Dirichlet eigenvalues.

**Proposition 2.11.** (i) For any $\varphi \in L^2_r$, the Dirichlet eigenvalues $(\mu_n(\varphi))_{n \in \mathbb{Z}}$ of $L(\varphi)$ can be listed with their algebraic multiplicities in such a way that they are lexicographically ordered and satisfy the asymptotic estimates
\[ \mu_n(\varphi) = n\pi + \ell^2_n, \]
locally uniformly in $\varphi$. Moreover, $\chi_D(\lambda, \varphi)$ admits the product representation
\[ \chi_D(\lambda, \varphi) = -\prod_{n \in \mathbb{Z}} \frac{\mu_n - \lambda}{\pi_n}. \]
(ii) For $\varphi \in L^2_r$, the Dirichlet eigenvalues are real and for any $n \in \mathbb{Z}$
\[ \lambda_n^-(\varphi) \leq \mu_n(\varphi) \leq \lambda_n^+(\varphi). \]
By Lemma 2.10 for $|n|$ sufficiently large, the Dirichlet eigenvalue $\mu_n$ is simple. Moreover one has

**Lemma 2.12.** (i) If for a given potential $\varphi \in L^2_c$, $\lambda$ is a periodic eigenvalue of $L(\varphi)$ of geometric multiplicity 2, then $\lambda$ is a Dirichlet eigenvalue of $L(\varphi)$. (ii) If for a given potential $\varphi \in iL^2_t$, $\lambda$ is a real periodic eigenvalue of $L(\varphi)$ then it is of geometric multiplicity two and hence also a Dirichlet eigenvalue of $L(\varphi)$.

**Proof.** (i) If $\lambda$ is a periodic eigenvalue of $L(\varphi)$ of geometric multiplicity two, then $M_1, M_2$ and hence $M_1 + M_2$ satisfy periodic or anti-periodic boundary conditions. As $m_1(0) + m_2(0) = 1 = m_3(0) + m_4(0)$ it then follows that $\lambda$ is a Dirichlet eigenvalue. (ii) follows from (i) and Proposition 2.7. \qed

$L^2$-gradients: Let $F: V \to \mathbb{C}$ be an analytic function on an open set $V$ in $L^2_c$. The $L^2$-gradient $\partial F$ of $F$ at $\psi \in V$ is an element in $L^2_c$ such that for any $h \in L^2_c$

$$d_{\psi}F(h) = \langle \partial F, h \rangle_{r}$$

where $d_{\psi}F$ denotes the differential of $F$ at $\psi$ and

$$\langle \partial F, h \rangle_{r} := \int_0^1 ((\partial_1 F)(x)h_1(x) + (\partial_2 F)(x)h_2(x)) \, dx.$$

Let $\lambda_{\varphi}$ be a periodic eigenvalue of $L(\varphi), \varphi \in L^2_c$, of geometric multiplicity one. Then $\hat{M}(\lambda_{\varphi}, \varphi) \neq \pm \text{Id}_{2 \times 2}$, and hence $\hat{m}_2(\lambda_{\varphi}, \varphi)$ or $\hat{m}_3(\lambda_{\varphi}, \varphi)$ is not equal to zero. The proof of the following lemma can be found e.g. in Section 4 in [9] (cf. Lemma 2 in [16]). To state it introduce the * product, $(f_1, f_2) * (g_1, g_2) := (f_2g_2, f_1g_1)$.

**Lemma 2.13.** Under the conditions listed above and if in addition $\hat{m}_2(\lambda_{\varphi}, \varphi) \neq 0$ one has

$$i\partial \Delta = \hat{m}_2 f * f$$

where $f$ is the eigenfunction of $\lambda_{\varphi}$ normalized so that

$$f(x) = M(x, \lambda_{\varphi}, \varphi) \begin{pmatrix} 1 \\ \zeta \end{pmatrix} \quad \text{with} \quad \zeta := (\xi - \hat{m}_1)/\hat{m}_2,$$

where $\xi \in \{ \pm 1 \}$ is the eigenvalue of $\hat{M}(\lambda_{\varphi}, \varphi)$. Similarly, if $\hat{m}_3(\lambda_{\varphi}, \varphi) \neq 0$, then at $(\lambda_{\varphi}, \varphi)$

$$i\partial \Delta = -\hat{m}_3 f * f$$

where

$$f(x) = M(x, \lambda_{\varphi}, \varphi) \begin{pmatrix} \zeta \\ 1 \end{pmatrix} \quad \text{with} \quad \zeta := (\xi - \hat{m}_4)/\hat{m}_3.$$
Actually, the formulas for $\partial \Delta$ above can be obtained from the following formula for $\partial \hat{M}$ (see Section 3 in [9]).

**Lemma 2.14.** The $L^2$-gradient of the Floquet matrix $\hat{M} \equiv M(1, \lambda, \varphi)$ is given by

$$
\iota \partial \hat{M} = \begin{pmatrix}
-\hat{m}_1 M_1 * M_2 + \hat{m}_2 M_1 * M_1 & -\hat{m}_1 M_2 * M_2 + \hat{m}_2 M_1 * M_2 \\
-\hat{m}_3 M_1 * M_2 + \hat{m}_4 M_1 * M_1 & -\hat{m}_3 M_2 * M_2 + \hat{m}_4 M_1 * M_2
\end{pmatrix}
$$

(2.2)

where $M_1$ and $M_2$ are the two column vectors of $M$ and the elements of the matrix in parentheses are column vectors.

### 3 Proof of Theorem 1.2

The aim of this section is to prove Theorem 1.2 saying that $S_p \cap iH^N_r$ is path connected for any $N \in \mathbb{Z}_{\geq 0}$. First we need to analyze multiple periodic eigenvalues of $L(\varphi)$ locally in $iL^2_r$. Recall that the characteristic functions for the Dirichlet and the periodic spectrum of $L(\varphi)$, $\varphi \in L^2_c$, denoted by $\chi_D$ and $\chi_p$ respectively, are given by

$$
2i\chi_D(\lambda, \varphi) = (\hat{m}_4 + \hat{m}_3 - \hat{m}_2 - \hat{m}_1)|_{\lambda, \varphi}, \quad \chi_p(\lambda, \varphi) = ((\hat{m}_1 + \hat{m}_4)^2 - 4)|_{\lambda, \varphi}.
$$

Assume that $\lambda \in \mathbb{C}$ is a periodic eigenvalue of $L(\varphi)$ of geometric multiplicity two,

$$
m_g(\lambda, \varphi) = 2 \quad \text{1}
$$

By Lemma 2.12 it then follows that $\lambda$ is at the same time a Dirichlet eigenvalue, i.e.,

$$
\chi_D(\lambda) = 0, \quad \chi_p(\lambda) = 0, \quad \text{and} \quad \partial_\lambda \chi_p(\lambda) = 0.
$$

One can easily see that the following more general statement holds.

**Lemma 3.1.** Let $\varphi \in L^2_c$ and let $\lambda$ be a periodic eigenvalue of $L(\varphi)$. Then $m_g(\lambda, \varphi) = 2$ iff $\hat{M}(\lambda) \equiv M(1, \lambda)$ is diagonalizable or, equivalently, $\hat{M}(\lambda) \in \{\pm \text{Id}_{2 \times 2}\}$.

A periodic eigenvalue of geometric multiplicity two, $m_g(\lambda, \varphi) = 2$, is said to be non-degenerate if the algebraic multiplicity of $\lambda$, when viewed as a periodic eigenvalue of $L(\varphi)$, is two, $m_p(\lambda, \varphi) = 2$, and degenerate otherwise. Note that for the zero potential, any periodic eigenvalue is of geometric multiplicity two and non-degenerate. More generally, by Lemma 2.12(ii) any real periodic eigenvalue of $L(\varphi)$ with $\varphi \in iL^2_r$ is of geometric multiplicity two. It might be degenerate – see Corollary 6.7(iii) in Appendix B. Furthermore note that a non-degenerate periodic eigenvalue of $L(\varphi)$, $\varphi \in iL^2_r$, of geometric multiplicity two is not

\[\text{In what follows, } m_g(\lambda) \equiv m_g(\lambda, \varphi) \text{ denotes the geometric multiplicity of } \lambda \in \mathbb{C} \text{ as a periodic eigenvalue of } L(\varphi).\]
necessarily a simple Dirichlet eigenvalue. Indeed, by Corollary 6.7 for the constant potential \( \varphi_a = (a, -\bar{a}), a \in \mathbb{C}, \) and \( n \in \mathbb{Z}_{\geq 1} \) with \( 0 < n\pi < |a| \), the points \( \pm i \frac{\sqrt{|a|^2 - n^2 \pi^2}}{2} \) are non-degenerate periodic eigenvalues of geometric multiplicity two. As \( i \text{Im}(a) \) is a Dirichlet eigenvalue of \( L(\varphi_a) \), the phase of \( a \) can be chosen so that \( \text{Im}(a) \) equals \( \frac{\sqrt{|a|^2 - n^2 \pi^2}}{2} \) – see Corollary 6.8. For such an \( a, i \frac{\sqrt{|a|^2 - n^2 \pi^2}}{2} \) is a Dirichlet eigenvalue of algebraic multiplicity two.

The first result concerns potentials \( \psi \in iL^2_c \) with the property that \( L(\psi) \) admits a periodic eigenvalue \( \lambda_\psi \) with \( s_p(\lambda_\psi) \geq 2 \) and \( s_g(\lambda_\psi) = 1 \). In this case it is convenient to distinguish between a periodic eigenvalue in the proper sense, characterized by \( \Delta(\lambda_\psi, \psi) = 2 \) and an anti-periodic eigenvalue, characterized by \( \Delta(\lambda_\psi, \psi) = -2 \). The corresponding characteristic functions are

\[
\chi^\pm_p(\lambda, \psi) = \Delta(\lambda, \psi) \mp 2.
\]

Note that \( \chi_p(\lambda, \psi) = \chi^+_p(\lambda, \psi) \chi^-_p(\lambda, \psi) \). Finally denote by \( D^\varepsilon(\lambda_\psi) \subseteq \mathbb{C} \) the open disk of radius \( \varepsilon > 0 \) centered at \( \lambda_\psi \).

**Theorem 3.2.** Assume that for \( \psi \in iH^N_c, N \geq 0, \lambda_\psi \) is a periodic eigenvalue of \( L(\psi) \) in the proper sense [alternatively, anti-periodic eigenvalue of \( L(\psi) \)] of algebraic multiplicity \( m \geq 2 \), and geometric multiplicity one. Then for any \( \varepsilon > 0 \) sufficiently small there exists an open neighborhood \( \mathcal{V} \subseteq iH^N_c \) of \( \psi \) such that the set

\[
X := \{ \varphi \in \mathcal{V} | \exists \lambda \in D^\varepsilon(\lambda_\psi) \text{ with } s_p(\lambda, \varphi) = m \text{ and } s_g(\lambda, \varphi) = 1 \}
\]

is contained in a real-analytic submanifold \( \mathcal{Y} \) of \( iH^N_c \) of (real) codimension two, which is closed in \( \mathcal{V} \). In addition, \( \mathcal{V} \) can be chosen so that for any \( \varphi \in \mathcal{V} \), all periodic eigenvalues of \( L(\varphi) \) in \( D^\varepsilon(\lambda_\psi) \) have geometric multiplicity one.

**Proof.** First assume that \( N = 0 \). As the cases where \( \lambda_\psi \) is a periodic eigenvalue in the proper sense and where it is an anti-periodic eigenvalue can be treated in the same way we concentrate on the first case only. First we remark that due to Proposition 5.1 one has \( \text{Im}(\lambda_\psi) \neq 0 \). By the first part of Theorem 5.1 applied to the characteristic function \( \chi^+_p(\lambda, \varphi) = \Delta(\lambda, \varphi) - 2 \), for any \( \varepsilon > 0 \) sufficiently small there exists an open neighborhood \( \mathcal{V} \subseteq iL^2_c \) of \( \psi \) so that for any \( \varphi \in \mathcal{V} \), \( L(\varphi) \) has \( m \) periodic eigenvalues \( \lambda^1(\varphi), \ldots, \lambda^m(\varphi) \), listed with their algebraic multiplicities, in the the open disk \( D^\varepsilon = D^\varepsilon(\lambda_\psi) \) and none on the boundary \( \partial D^\varepsilon \). By the characterization of the geometric multiplicity of Lemma 3.1 \( s_g(\lambda_\psi) = 1 \) implies that either \( \hat{m}_2(\lambda_\psi, \psi) \neq 0 \) or \( \hat{m}_3(\lambda_\psi, \psi) \neq 0 \). Hence by shrinking \( \mathcal{V} \) and \( \varepsilon > 0 \) if necessary it follows that \( s_g(\lambda^k(\varphi)) = 1 \) for any \( 1 \leq k \leq m \) and \( \varphi \in \mathcal{V} \).

In order to apply Theorem 5.1(i) we look for an analytic function \( F : \mathbb{C} \times L^2_c \to \mathbb{C} \) so that \( X \) – after shrinking \( \mathcal{V} \), if necessary – is contained in the zero set of

\[
F_{\chi^+_p} : \mathcal{V} \to \mathbb{C}, \quad \varphi \mapsto \sum_{j=1}^m F(\lambda^j(\varphi), \varphi).
\]
For any \( q \geq 1 \), take \( F_q(\lambda) = (\lambda - \lambda_\psi)^q \). By Theorem 5.1(ii) applied to the pair \((F_q, \chi^+_p)\) one concludes that

\[
E_q : \mathcal{V} \to \mathbb{C}, \quad \varphi \mapsto \sum_{j=1}^m (\lambda^j(\varphi) - \lambda_\psi)^q
\]
is analytic.\footnote{A function \( F : \mathcal{V} \to \mathbb{C}, \mathcal{V} \subseteq iL^2 \), is called analytic if it is the restriction to \( \mathcal{V} = \mathcal{V}_c \cap iL^2 \) of an analytic function \( F : \mathcal{V}_c \to \mathbb{C} \) where \( \mathcal{V}_c \) is an open set in \( L^2 \).}

Note that for any \( \varphi \in X \),

\[
E_m(\varphi) = m(\lambda_\varphi - \lambda_\psi)^m \quad \text{and} \quad E_1(\varphi) = m(\lambda_\varphi - \lambda_\psi)
\]

where for \( \varphi \in X \), \( \lambda_\varphi \) denotes the unique periodic eigenvalue of \( L(\varphi) \) in \( \mathcal{D}((\lambda_\psi)) \). To obtain a functional which vanishes on \( X \) we set

\[
G : \mathcal{V} \to \mathbb{C}, \quad \varphi \mapsto m^{m-1} E_m(\varphi) - E_1(\varphi)^m.
\]

Note that \( G \) is analytic and

\[
\big| G \big|_X = 0. \tag{3.1}
\]

As \( \partial F_m = 0 \) and as \( F_m(\lambda) \) has a zero of order \( m \) at \( \lambda = \lambda_\psi \) one concludes from Theorem 5.1(ii) that at \((\lambda, \varphi) = (\lambda_\psi, \psi)\),

\[
\partial E_m = a\partial \Delta, \quad a \neq 0.
\]

As \( m \geq 2 \) and \( E_1(\psi) = 0 \) it follows that

\[
\partial(E_1(\varphi)^m) \bigg|_{\varphi=\psi} = mE_1(\varphi)^{m-1}\partial E_1 \bigg|_{\varphi=\psi} = 0
\]

and hence at \( \varphi = \psi \)

\[
\partial G = a\partial \Delta, \quad a \neq 0.
\]

It remains to show that near \( \psi \) the zero set of \( G \) is a real-analytic submanifold of codimension two. Clearly

\[
G_R : \mathcal{V} \to \mathbb{R}, \varphi \mapsto \text{Re} \, G(\varphi) \quad \text{and} \quad G_I : \mathcal{V} \to \mathbb{R}, \varphi \mapsto \text{Im} \, G(\varphi) \tag{3.2}
\]

are two real-analytic functionals. In view of the implicit function theorem it then remains to show that the differentials \( d_\psi G_R \) and \( d_\psi G_I \) as elements in \( \mathcal{L}(iL^2_\mathbb{R}, \mathbb{R}) \) are \( \mathbb{R} \)-linearly independent. Recall that by assumption, \( \lambda_\psi \) has geometric multiplicity one. Then \( \hat{m}_2(\lambda_\psi, \psi) \) or \( \hat{m}_3(\lambda_\psi, \psi) \) is not equal to zero. Assume for simplicity that \( \hat{m}_2(\lambda_\psi, \psi) \neq 0 \). The case when \( \hat{m}_3(\lambda_\psi, \psi) \neq 0 \) is treated in the same way. It follows from Lemma 2.13 that at \( (\lambda_\psi, \psi) \)

\[
i\partial \Delta = \hat{m}_2 f \star f
\]
where $f$ is the appropriately normalized 1-periodic eigenfunction of $L(\psi)$ corresponding to $\lambda_\psi$. Summarizing the computations above, one has in the case where $\hat{m}_2(\lambda_\psi) \neq 0$

$$\partial G = -ia \cdot \hat{m}_2(\lambda_\psi) f \ast f, \quad a \neq 0.$$  \hfill (3.3)

In view of Lemma 5.3 (iii) it is to show that the $\mathbb{R}$-linear functionals in $iL^2_\ast$

$$\ell_R(h) := \text{Re}((f \ast f, h)_\tau) \quad \text{and} \quad \ell_I(h) := \text{Im}((f \ast f, h)_\tau), \quad h \in iL^2_\ast,$$

are $\mathbb{R}$-linearly independent at $\psi$ (see the discussion before Lemma 5.3 in Appendix A). By Lemma 5.3(iv) we know that $\ell_R$ and $\ell_I$ are $\mathbb{R}$-linearly dependent iff there exists $c \in \mathbb{C}\{0\}$ so that

$$cf \ast cf + \hat{c}f \ast cf = 0.$$  \hfill (3.4)

Assume that (3.4) holds for some $c \neq 0$. It is convenient to introduce $g := cf$ and $s := \hat{g} = (-\bar{g}_2, \bar{g}_1)$. Then equation (3.4) reads

$$(g_1^2, g_2^2) = (s_1^2, s_2^2).$$  \hfill (3.5)

By Lemma 2.2 $s$ satisfies

$$L(\psi)s = \bar{\lambda}_\psi s.$$  

Hence,

$$ig'_1 + \psi_1 g_2 = \lambda_\psi g_1 \quad \text{and} \quad is'_1 + \psi_1 s_2 = \bar{\lambda}_\psi s_1$$  \hfill (3.6)

$$-ig'_2 + \psi_2 g_1 = \lambda_\psi g_2 \quad \text{and} \quad -is'_2 + \psi_2 s_1 = \bar{\lambda}_\psi s_2.$$  \hfill (3.7)

As $g = (g_1, g_2) \in H^1_{\text{loc}}(\mathbb{R}, \mathbb{C}^2)$ is a non-zero 1-periodic solution of $L(\psi)g = \lambda_\psi g$ we conclude that $g(x) \neq 0$ for any $x \in \mathbb{R}$. This and the periodicity of $g$ imply that there are only four possible cases:

**Case 1:** There exists a non-empty finite interval $(a, b) \subseteq \mathbb{R}$ such that $\forall x \in (a, b)$

$$g_1(x)g_2(x) \neq 0, \quad g_1(a)g_2(a) = 0, \quad \text{and} \quad g_1(b)g_2(b) = 0;$$

**Case 2:** There exists a non-empty finite interval $(a, b) \subseteq \mathbb{R}$ such that $\forall x \in (a, b)$

$$g_1(x) = 0 \quad \text{and} \quad g_2(x) \neq 0;$$

**Case 3:** There exists a non-empty finite interval $(a, b) \subseteq \mathbb{R}$ such that $\forall x \in (a, b)$

$$g_2(x) = 0 \quad \text{and} \quad g_1(x) \neq 0;$$

**Case 4:** $\forall x \in \mathbb{R}$

$$g_1(x)g_2(x) \neq 0.$$
First, assume that Case 1 holds. It follows from (3.5) that on \((a, b)\),
\[
g_1 = \sigma_1 s_1 \quad \text{and} \quad g_2 = \sigma_2 s_2
\]
(3.8)
where \(\sigma_1, \sigma_2 \in \{\pm 1\}\). If \(\sigma_1 = \sigma_2\) one obtains from (3.6) that \(\text{Im}(\lambda_\psi)g_1 = 0\) on \((a, b)\). As \(\text{Im}(\lambda_\psi) \neq 0\) we see that \(g_1 = 0\) on \((a, b)\). This contradicts one of the assumptions in Case 1. Now, assume that \((\sigma_1, \sigma_2) = (1, -1)\). Summing up the two equations in (3.6) we get that \(ig_1' = \text{Re}(\lambda_\psi)g_1\) on \((a, b)\), or \(g_1(x) = \eta_1 e^{-i \text{Re}(\lambda_\psi)x}\), with constant \(\eta_1 \neq 0\). Similarly, one gets from (3.7) that \(g_2(x) = \eta_2 e^{i \text{Re}(\lambda_\psi)x}\), \(\eta_2 \neq 0\). This implies that \(g_1(x)g_2(x) = \eta_1 \eta_2 \neq 0\) on \((a, b)\). By continuity, \(g_1(a)g_2(a) \neq 0\), which contradicts again one of the assumptions in Case 1. The case \((\sigma_1, \sigma_2) = (-1, 1)\) is treated in the same way. Hence, Case 1 does not occur.

Now, assume that Case 2 holds. Then, it follows from (3.5) that on \((a, b)\)
\[
g_1 = s_1 = 0 \quad \text{and} \quad g_2 = s_2
\]
where \(s \in \{\pm 1\}\). This together with (3.7) implies that \(\text{Im}(\lambda_\psi)g_2 = 0\) on \((a, b)\). As \(\text{Im}(\lambda_\psi) \neq 0\) we see that \(g_2 = 0\) on \((a, b)\). This contradicts one of the conditions in Case 2. In the same way one treats Case 3.

Finally, consider Case 4. Arguing as in Case 1 we see that (3.8) holds and the only possible cases are \((\sigma_1, \sigma_2) = (1, -1)\) and \((\sigma_1, \sigma_2) = (-1, 1)\). If \((\sigma_1, \sigma_2) = (1, -1)\) one concludes from (3.6) that
\[
ig_1' = \text{Re}(\lambda_\psi)g_1 \quad \text{and} \quad \psi_1g_2 = i \text{Im}(\lambda_\psi)g_1
\]
and from (3.7) that
\[
\psi_2g_1 = i \text{Im}(\lambda_\psi)g_2 \quad \text{and} \quad -ig_2' = \text{Re}(\lambda_\psi)g_2.
\]
Hence
\[
g_1(x) = \eta_1 e^{-i \text{Re}(\lambda_\psi)x}, \quad g_2(x) = \eta_2 e^{i \text{Re}(\lambda_\psi)x}
\]
with \(\eta_1, \eta_2 \in \mathbb{C}\setminus\{0\}\). Solving (3.9)–(3.10) for \(\psi_1, \psi_2\) one then gets
\[
\psi_1 = i \text{Im}(\lambda_\psi) \frac{g_1}{g_2} = i \text{Im}(\lambda_\psi) \frac{\eta_1}{\eta_2} e^{-2i \text{Re}(\lambda_\psi)x}
\]
and
\[
\psi_2 = i \text{Im}(\lambda_\psi) \frac{g_2}{g_1} = i \text{Im}(\lambda_\psi) \frac{\eta_2}{\eta_1} e^{2i \text{Re}(\lambda_\psi)x}.
\]
As \(\psi \in iL^2_p\) and thus \(\overline{\psi}_1 = -\psi_2\) one has \(\eta_1/\eta_2 = e^{i\alpha}\) with \(\alpha \in \mathbb{R}\), and as \(\psi\) is 1-periodic it follows that \(\text{Re}(\lambda_\psi) = k\pi\) for some \(k \in \mathbb{Z}\). Hence
\[
\psi_1(x) = i \text{Im}(\lambda_\psi)e^{i\alpha}e^{-2k\pi ix} \quad \text{and} \quad \lambda_\psi = k\pi + i \text{Im}(\lambda_\psi).
\]
By Lemma 6.6 $\lambda_\psi = k\pi + i\text{Im}(\lambda_\psi)$ has algebraic multiplicity one. This contradicts the assumption $m_\nu(\lambda_\psi) = m \geq 2$. The case $(\sigma_1, \sigma_2) = (-1, 1)$ is treated in the same way as the case $(\sigma_1, \sigma_2) = (1, -1)$. Altogether we have shown that $\ell_R$ and $\ell_I$, and hence the differentials $d_\psi G_R$ and $d_\psi G_I$, are $\mathbb{R}$-linearly independent. As $G |_X = 0$, the claimed statement concerning $X$ then follows from the implicit function theorem.

Finally, assume that $N \geq 1$. Take $\psi \in iH^N_r$ and note that the restrictions $G_R |_{V \cap iH^N_r}$ and $G_I |_{V \cap iH^N_r}$ of the functionals \([3.2]\) considered above are real-analytic. Moreover, for any $h \in iH^N_r$,

$$d_\psi(G_R |_{V \cap iH^N_r})(h) = \langle \partial G_R, h \rangle_r \quad \text{and} \quad d_\psi(G_I |_{V \cap iH^N_r})(h) = \langle \partial G_I, h \rangle_r.$$

Assume that there exist $\alpha, \beta \in \mathbb{R}$, $(\alpha, \beta) \neq 0$, such that for any $h \in iH^N_r$, $\alpha \langle \partial G_R, h \rangle_r + \beta \langle \partial G_I, h \rangle_r = 0$. As $iH^N_r$ is dense in $iL^2_r$ we see by a continuity argument that the last equality holds also for any $h \in iL^2_r$. As this contradicts to the result obtained in the case $N = 0$ we get that the differentials $d_\psi(G_R |_{V \cap iH^N_r})$ and $d_\psi(G_I |_{V \cap iH^N_r})$ are $\mathbb{R}$-linearly independent in $L(iH^N_r, \mathbb{R})$. Then, arguing as in the case $N = 0$ we complete the proof of Theorem 3.2. \(\square\)

The second result deals with potentials $\psi \in iL^2_r$ with the property that $L(\psi)$ admits a periodic eigenvalue $\lambda_\psi$ with $m_g(\lambda_\psi) = 2$. In this case, $\hat{M}(\lambda_\psi) \in \{\pm \text{Id}_{2 \times 2}\}$ and hence $\lambda_\psi$ is at the same time a Dirichlet eigenvalue. Denote by $m_D(\lambda_\psi)$ the algebraic multiplicity of $\lambda_\psi$ as Dirichlet eigenvalue.

**Theorem 3.3.** Assume that for $\psi$ in $iH^N_r$, $N \geq 0$, $\lambda_\psi$ is a periodic eigenvalue of $L(\psi)$ in the proper sense [alternatively, anti-periodic eigenvalue of $L(\psi)$] with $m_g(\lambda_\psi) = 2$ and $m_D(\lambda_\psi) = m \geq 1$. Then for any $\varepsilon > 0$ sufficiently small there exists an open neighborhood $\mathcal{V} \subseteq iH^N_r$ of $\psi$ such that the set

$$X := \{ \varphi \in \mathcal{V} | \exists \lambda \in D^\varepsilon(\lambda_\psi) \text{ with } m_g(\lambda, \varphi) = 2, m_D(\lambda, \varphi) = m \}$$

is contained in a real-analytic submanifold $Y$ in $iH^N_r$ of (real) codimension two, which is closed in $\mathcal{V}$.

**Proof.** As the case $N \in \mathbb{Z}_{\geq 1}$ is treated in the same way as $N = 0$ – see the proof of Theorem 3.2 above – we concentrate on the latter case only. Similarly, as the cases where $\lambda_\psi$ is a periodic eigenvalue in the proper sense and where it is an anti-periodic eigenvalue can be treated in the same way we concentrate on the first case only. It turns out that we have to distinguish between two different cases. We begin with the case where $m = m_D(\lambda_\psi) \geq 2$.

**Case 1:** $m \geq 2$. By Theorem 5.1(i), applied to the characteristic function $\chi_D(\lambda, \varphi) = \frac{1}{2} (\tilde{m}_2 + \tilde{m}_3 - \tilde{m}_1 - \tilde{m}_4) |_{\lambda, \varphi}$, for any $\varepsilon > 0$ sufficiently small there exists an open neighborhood $\mathcal{V} \subseteq iL^2_r$ of $\psi$ so that for any $\varphi \in \mathcal{V}$, $L(\varphi)$ has $m$ Dirichlet eigenvalues $\mu^1(\varphi), \ldots, \mu^m(\varphi)$.
listed with their algebraic multiplicities, in the open disk \( D^\varepsilon \equiv D^\varepsilon(\lambda_\psi) \) and none on the boundary \( \partial D^\varepsilon \). Similarly as in the proof of Theorem 3.2, introduce the functional
\[
G : \mathcal{V} \to \mathbb{C}, \quad \varphi \mapsto m^{m-1}E_m(\varphi) - E_1(\varphi)^m
\]
where here, for any \( q \geq 1 \),
\[
E_q : \mathcal{V} \to \mathbb{C}, \quad \varphi \mapsto \sum_{j=1}^{m} (\mu_j(\varphi) - \lambda_\psi)^q.
\]
By Theorem 5.1(ii), applied to \( F_q(\lambda) := (\lambda - \lambda_\psi)^q \) and \( \chi_D(\lambda, \varphi) \), one concludes that \( E_q \) is analytic for any \( q \geq 1 \). Note that for any \( \varphi \in X \),
\[
E_m(\varphi) = m(\mu_\varphi - \lambda_\psi)^m \quad \text{and} \quad E_1(\varphi) = m(\mu_\varphi - \lambda_\psi)
\]
where for \( \varphi \in X, \mu_\varphi \) denotes the unique Dirichlet eigenvalue of \( L(\varphi) \) in \( D^\varepsilon \). It then follows that
\[
G \big|_X = 0.
\]
As \( \partial F_m = 0 \) and as \( F_m(\lambda) \) has a zero of order \( m \) at \( \lambda = \lambda_\psi \) one concludes from Theorem 5.1(ii) and (3.11) that at \( (\lambda, \varphi) = (\lambda_\psi, \psi) \),
\[
\partial G = a \partial \chi_D, \quad a \neq 0.
\]
By Lemma 2.14, the \( L^2 \)-gradient of the Floquet matrix \( \dot{M} \equiv M(1, \lambda, \varphi) \) is given by
\[
i\partial \dot{M} = \begin{pmatrix}
-\dot{m}_1 M_1 \ast M_2 + \dot{m}_2 M_1 \ast M_1 & -\dot{m}_1 M_2 \ast M_2 + \dot{m}_2 M_1 \ast M_2 \\
-\dot{m}_3 M_1 \ast M_2 + \dot{m}_4 M_1 \ast M_1 & -\dot{m}_3 M_2 \ast M_2 + \dot{m}_4 M_1 \ast M_2
\end{pmatrix}
\]
where \( M_1 \) and \( M_2 \) are the two column vectors of \( M \) and the elements of the matrix in parentheses are column vectors. Thus
\[
2 \partial \chi_D = i \partial \dot{m}_1 + i \partial \dot{m}_2 - i \partial \dot{m}_3 - i \partial \dot{m}_4 = (\dot{m}_2 - \dot{m}_4) M_1 \ast M_1 + (\dot{m}_3 - \dot{m}_1) M_2 \ast M_2 + (\dot{m}_2 + \dot{m}_3 - \dot{m}_1 - \dot{m}_4) M_1 \ast M_2.
\]
As at \( (\lambda, \varphi) = (\lambda_\psi, \psi) \), \( \dot{M} = Id_{2\times2} \) one gets
\[
2 \partial \chi_D = -M_1 \ast M_1 - M_2 \ast M_2 - 2M_1 \ast M_2.
\]
By Lemma 2.1 \( M(\cdot, \lambda_\psi, \psi) \in C([0, 2]) \). In particular, it can be evaluated at \( x = 0 \). One thus obtains
\[
2 \partial \chi_D(0, \lambda_\psi, \psi) = -\begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]
In view of (3.12),
\[ \partial G \bigg|_{x=0} = -\frac{a}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a \neq 0. \] (3.16)

In addition to \( G \) we need to introduce a second functional, denoted by \( H \),

\[ H : \mathcal{V} \to \mathbb{C}, \quad \varphi \mapsto \sum_{j=1}^{m} \hat{m}_2(\mu^j(\varphi), \varphi). \]

Note that \( H \bigg|_{X} = 0. \) By Lemma 2.1, \( \hat{m}_2 : \mathbb{C} \times L^2 \to \mathbb{C} \), \( (\lambda, \varphi) \mapsto m_2(1, \lambda, \varphi) \) is analytic. By Theorem 5.1 applied to \( (F, \chi) = (\hat{m}_2, \chi_D) \) it follows that \( H \) is analytic and that at \( (\lambda_\psi, \psi) \)

\[ \partial H = m \partial \hat{m}_2 + \sum_{j=1}^{m} a_j \partial^{m-j}_{\lambda} \partial \chi_D. \] (3.17)

Let us first discuss the term \( m \partial \hat{m}_2 \) in more detail. By (3.13) one has

\[ i \partial \hat{m}_2 = -\hat{m}_1 M_2 * M_2 + \hat{m}_2 M_1 * M_2. \]

As \( \hat{M} = \text{Id}_{2 \times 2} \) at \( (\lambda_\psi, \psi) \) one then gets

\[ i \partial \hat{m}_2 \bigg|_{x=0} = -\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad m \partial \hat{m}_2 \bigg|_{x=0} = \text{im} \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \] (3.18)

Next, let us turn to the second term of the right hand side of formula (3.17). It follows from (3.14) and Lemma 2.1 that

\[ \mathbb{C} \to C([0,2]), \quad \lambda \mapsto \partial \chi_D(\cdot, \lambda, \psi) \]

is analytic. This implies that

\[ \partial^k_{\lambda} \partial \chi_D(\cdot, \lambda, \psi) \bigg|_{x=0} = \partial^k_{\lambda} (\partial \chi_D(0, \lambda, \psi)). \]

For any \( \lambda \in \mathbb{C} \)

\[ 2\partial \chi_D(0, \lambda, \psi) = (\hat{m}_2 - \hat{m}_4) \begin{pmatrix} 0 \\ 1 \end{pmatrix} + (\hat{m}_3 - \hat{m}_1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\hat{m}_3 - \hat{m}_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + (\hat{m}_1 + \hat{m}_2 - \hat{m}_3 - \hat{m}_4) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \]
or
\[ 2\partial\chi_D(0, \lambda, \psi) = (\hat{m}_3 - \hat{m}_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} - 2i\chi_D(\lambda, \psi) \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \] (3.19)

As by assumption, \( \partial_k \chi_D(\lambda, \psi) = 0 \) for any \( 0 \leq k \leq m - 1 \), it then follows from formula (3.19) that
\[ 2\partial^{m-j}_\lambda (\partial\chi_D(0, \lambda, \psi)) \bigg|_{\lambda = \lambda_\psi} = \partial^{m-j}_\lambda (\hat{m}_3 - \hat{m}_1) \bigg|_{\lambda = \lambda_\psi} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]
for any \( 1 \leq j \leq m \). When combined with (3.17) and (3.18) one has
\[ \partial H \bigg|_{x=0} = im \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \kappa \begin{pmatrix} 1 \\ 1 \end{pmatrix} \] (3.20)
for some \( \kappa \in \mathbb{C} \).

Following the notation introduced in Appendix A, denote by \( \ell = \ell^G : iL^2_r \to \mathbb{C} \) the \( \mathbb{R} \)-linear functional induced by \( \partial G = (\partial_1 G, \partial_2 G) \)
\[ \ell^G(h) := \langle \partial G, h \rangle_r = \int_0^1 (\partial_1 Gh_1 + \partial_2 Gh_2) dx \]
and let
\[ \ell^G_R(h) := \text{Re}(\langle \partial G, h \rangle_r) \quad \text{and} \quad \ell^G_I(h) := \text{Im}(\langle \partial G, h \rangle_r). \]

According to (5.4), one has for \( h \in iL^2_r \)
\[ \partial_s \bigg|_{s=0} \text{Re} G(\varphi + sh) = \ell^G_R(h) = \left\langle \frac{\partial G + \hat{\partial} G}{2}, h \right\rangle_r \]
and similarly
\[ \partial_s \bigg|_{s=0} \text{Im} G(\varphi + sh) = \ell^G_I(h) = \left\langle \frac{\partial G - \hat{\partial} G}{2i}, h \right\rangle_r \]
where we recall that for \( f = (f_1, f_2) \in L^2_{c^*}, \hat{f} \) is given by \( \hat{f} = -\langle f_2, f_1 \rangle \). By formula (3.16), for \( \varphi = \psi \),
\[ \frac{1}{2}(\partial G + \hat{\partial} G) \bigg|_{x=0} = \frac{1}{4}(\alpha - a) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -\frac{i}{2} \text{Im}(a) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \] (3.21)
and
\[ \frac{1}{2i}(\partial G - \hat{\partial} G) \bigg|_{x=0} = \frac{1}{4i}(a + \bar{a}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{i}{2} \text{Re}(a) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \] (3.22)
where \( a \neq 0 \). Similarly, we define for \( H \)
\[ \ell^H_R(h) = \left\langle \frac{\partial H + \hat{\partial} H}{2}, h \right\rangle_r \quad \text{and} \quad \ell^H_I(h) = \left\langle \frac{\partial H - \hat{\partial} H}{2i}, h \right\rangle_r. \]
By formula (3.20), at $\varphi = \psi$,

$$\frac{1}{2}(\partial H + \partial \hat{H}) \bigg|_{x=0} = i\left(\frac{m}{2} + \text{Im}(\kappa)\right) \binom{1}{1}$$

(3.23)

and

$$\frac{1}{2i}(\partial H - \partial \hat{H}) \bigg|_{x=0} = \frac{m}{2} \binom{1}{-1} - i\text{Re}(\kappa) \binom{1}{1}.$$  (3.24)

In view of the identities (3.21) - (3.24) introduce

$$F_1 : V \to \mathbb{R}, \quad \varphi \mapsto \text{Im} H(\varphi)$$

and

$$F_2 : V \to \mathbb{R}, \quad \varphi \mapsto \begin{cases} \text{Re} G(\varphi), & \text{Im}(\alpha) \neq 0 \\ \text{Im} G(\varphi), & \text{Im}(\alpha) = 0 \end{cases}.$$  

As $\alpha \neq 0$, $\text{Im}(\alpha) = 0$ implies that $\text{Re}(\alpha) \neq 0$ and hence according to (3.22), $\frac{1}{2i}(\partial G - \partial \hat{G}) \bigg|_{x=0} = -\frac{i}{2} \text{Re}(\alpha) \binom{1}{1} \neq 0$. Now define

$$Y = \{ \varphi \in V \mid F_1(\varphi) = 0, F_2(\varphi) = 0 \}.$$  

By construction, $G \big|_{X} = 0$, $H \big|_{X} = 0$ and hence $X \subseteq Y$. By (3.21), (3.22), and (3.24), $\partial F_1$ and $\partial F_2$ are $\mathbb{R}$-linearly independent at $\varphi = \psi$. By the implicit function theorem, it then follows that after shrinking $\mathcal{V}$, if necessary, $X$ is contained in a real-analytic submanifold of $i\mathcal{L}^2$ of codimension two. Hence the claimed result for $X$ is established in Case 1.

**Case 2:** $m = m_D(\lambda_\psi) = 1$ & $m_g(\lambda_\psi) = 2$. By Theorem 5.1(i), applied to the characteristic function $\chi_D$, for any $\varepsilon > 0$ sufficiently small there exists an open neighborhood $\mathcal{V} \subseteq i\mathcal{L}^2$ of $\psi$ so that for any $\varphi \in \mathcal{V}$, $L(\varphi)$ has precisely one Dirichlet eigenvalue, denoted by $\mu(\varphi)$ in the open disk $D^\varepsilon = \mathcal{D}^\varepsilon(\lambda_\psi)$ and none on the boundary $\partial D^\varepsilon$. As $\mu(\varphi)$ is simple, it follows from the inverse function theorem that the mapping $\mu : \mathcal{V} \to \mathbb{C}$ is analytic. In view of Lemma 3.1,

$$X \subseteq \{ \varphi \in \mathcal{V} \mid \hat{m}_2(\mu(\varphi), \varphi) = \hat{m}_3(\mu(\varphi), \varphi) = 0 \}.$$  

(3.25)

Consider the functionals,

$$H_1 : \mathcal{V} \to \mathbb{C}, \quad \varphi \mapsto \hat{m}_2(\mu(\varphi), \varphi)$$

and

$$H_2 : \mathcal{V} \to \mathbb{C}, \quad \varphi \mapsto \hat{m}_3(\mu(\varphi), \varphi).$$

In view of Lemma 2.1 $H_1$ and $H_2$ are analytic, and by (3.25),

$$H_1|_X = H_2|_X = 0.$$  

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Next, we will compute the $L^2$-gradients of $H_1$ and $H_2$ at $\varphi = \psi$. By the chain rule, we have that at $\varphi = \psi$

$$\partial H_1 = \partial \lambda \dot{m}_2(\lambda \psi, \psi) \partial \mu + \partial \ddot{m}_2(\lambda \psi, \psi)$$

(3.26)

and

$$\partial H_2 = \partial \lambda \dot{m}_3(\lambda \psi, \psi) \partial \mu + \partial \ddot{m}_3(\lambda \psi, \psi).$$

(3.27)

Using the identity $\chi_D(\mu(\varphi), \varphi) = 0$ for $\varphi \in V$ and that $\dot{\chi}_D(\mu(\varphi), \varphi) \neq 0$ by the assumed simplicity of $\mu(\varphi)$ we obtain that

$$\partial \mu = -\frac{1}{\dot{\chi}_D} \partial \chi_D$$

where $\partial \chi_D = \partial \chi_D(\lambda \psi, \psi)$ and $\dot{\chi}_D = \dot{\chi}_D(\lambda \psi, \psi)$. By (3.15),

$$-2 \partial \chi_D = M_1 \ast M_1 + M_2 \ast M_2 + 2M_1 \ast M_2,$$

and hence,

$$\partial \mu = \frac{1}{2\dot{\chi}_D} (M_1 \ast M_1 + M_2 \ast M_2 + 2M_1 \ast M_2).$$

(3.28)

In particular,

$$\partial \mu|_{x=0} = \frac{1}{2\dot{\chi}_D} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$  

(3.29)

By Lemma [2.14]

$$\partial \dot{m}_2 = i\dot{m}_1 M_2 \ast M_2 - iM_2 \ast M_1$$

(3.30)

and

$$\partial \dot{m}_3 = i\dot{m}_3 M_1 \ast M_2 - iM_4 M_1 \ast M_1.$$  

(3.31)

As at $(\lambda, \varphi) = (\lambda \psi, \psi)$, $\dot{M} = Id_{2 \times 2}$ we get that

$$\partial \dot{m}_2|_{x=0} = i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

(3.32)

and

$$\partial \dot{m}_3|_{x=0} = -i \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$  

(3.33)

Combining (3.26)-(3.33) we then obtain at $\varphi = \psi$

$$\partial H_1 = \kappa_1 (M_1 \ast M_1 + M_2 \ast M_2 + 2M_1 \ast M_2) + iM_2 \ast M_2$$

(3.34)

$$\partial H_2 = \kappa_2 (M_1 \ast M_1 + M_2 \ast M_2 + 2M_1 \ast M_2) - iM_1 \ast M_1$$

(3.35)
and

\[ \partial H_1|_{x=0} = \kappa_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

\[ \partial H_2|_{x=0} = \kappa_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} - i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

where

\[ \kappa_1 := \frac{\partial \lambda m_2}{2 \dot{\chi}_D} \quad \text{and} \quad \kappa_2 := \frac{\partial \lambda m_3}{2 \dot{\chi}_D}. \]  

(3.36)

Arguing as in Case 1 we compute

\[ \frac{1}{2} (\partial H_1 + \partial \tilde{H}_1)|_{x=0} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - i \text{Re}(\kappa_1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]  

(3.37)

\[ \frac{1}{2i} (\partial H_1 - \partial \tilde{H}_1)|_{x=0} = \frac{1}{2i} \begin{pmatrix} 1 \\ -1 \end{pmatrix} - i \text{Re}(\kappa_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]  

(3.38)

and

\[ \frac{1}{2} (\partial H_2 + \partial \tilde{H}_2)|_{x=0} = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \text{Re}(\kappa_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]  

(3.39)

\[ \frac{1}{2i} (\partial H_2 - \partial \tilde{H}_2)|_{x=0} = \frac{1}{2i} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \text{Re}(\kappa_2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]  

(3.40)

For any analytic function \( F : \mathcal{V} \rightarrow \mathbb{C} \) denote for simplicity

\[ \partial_R F := \frac{1}{2} (\partial F + \partial \tilde{F}) \quad \text{and} \quad \partial_I F := \frac{1}{2i} (\partial F - \partial \tilde{F}). \]

We now show that

\[ \text{rank}_R \{ \partial_R H_1, \partial_I H_1, \partial_R H_2, \partial_I H_2 \} \geq 2. \]  

(3.41)

Assume on the contrary that the rank above is one. Then it follows from (3.37)-(3.40) that

\[ \kappa_1 = a - \frac{i}{2} \quad \text{and} \quad \kappa_2 = a + \frac{i}{2}, \quad \text{where} \quad a \in \mathbb{R}. \]  

(3.42)

This together with (3.34) and (3.35) imply that at \( \varphi = \psi \)

\[ \partial H_1 - \partial H_2 = -2i M_1 * M_2. \]  

(3.43)

As the rank in (3.41) is assumed to be one, we get from Lemma 3.4 below that

\[ M_1 * M_2 \equiv 0. \]

It means that at \( \varphi = \psi \), for any \( 0 \leq x \leq 1 \)

\[ m_1(x, \lambda_{\psi}) m_2(x, \lambda_{\psi}) = 0 \text{ and } m_3(x, \lambda_{\psi}) m_4(x, \lambda_{\psi}) = 0. \]  

(3.44)
Multiplying the first row of $L(\psi)M_1 = \lambda\psi M_1$ by $m_4$ and the second by $m_2$ yields

$$im'_4 m_4 = \lambda\psi m_1 m_4 \quad \text{and} \quad -im'_3 m_2 = \lambda\psi m_3 m_2.$$ 

Taking the difference of the two equations and using the Wronskian identity one then gets

$$i(m'_4 m_4 + m'_3 m_2) = \lambda\psi.$$ \hfill (3.45)

In the same way one gets, after multiplying the first row of $L(\varphi)M_2 = \lambda\psi M_2$ by $m_3$ and the second by $m_1$

$$im'_2 m_3 = \lambda\psi m_2 m_3 \quad \text{and} \quad -im'_4 m_1 = \lambda\psi m_4 m_1$$

leading to

$$i(m'_2 m_3 + m'_4 m_1) = -\lambda\psi.$$ \hfill (3.46)

Adding (3.45) and (3.46) one obtains

$$\partial_x (m_1 m_4 + m_2 m_3) = 0$$

or, in view of the Wronskian identity,

$$\partial_x (m_1 m_4) = 0.$$ 

As $m_1 m_4 \big|_{x=0} = 1$ one therefore has

$$m_1(x, \lambda\psi) m_4(x, \lambda\psi) = 1 \quad \forall \ 0 \leq x \leq 1.$$ 

This combined with (3.44) leads to

$$m_2(x, \lambda\psi) = 0 \quad \text{and} \quad m_3(x, \lambda\psi) = 0 \quad \forall \ 0 \leq x \leq 1.$$ 

Multiplying the first row of $L(\psi)M_2 = \lambda\psi M_2$ by $m_1$ and using that $m_2 = 0$ yields

$$0 = \psi_1 m_4 m_1 = \psi_1.$$ 

As $\psi$ is in $iL^2_r$ one has $\psi_2 = -\overline{\psi_1}$ and hence

$$\psi = 0.$$ 

A simple computation (cf. Lemma 6.4) shows that

$$\hat{M}(\lambda, \psi) \big|_{\psi=0} = \left( \begin{array}{cc} e^{-i\lambda} & 0 \\
0 & e^{i\lambda} \end{array} \right).$$

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Hence, 
\[ \hat{m}_2(\lambda, 0) = \hat{m}_3(\lambda, 0) \equiv 0 \quad \text{and} \quad \chi_D(\lambda, 0) = \sin \lambda \]
and by (3.36) 
\[ \kappa_1 = \kappa_2 = 0. \]
This contradicts (3.42). Therefore, (3.41) holds. By the implicit function theorem it then follows that, after shrinking \( \mathcal{V} \) if necessary, \( X \) is contained in a real-analytic submanifold in \( iL^2_r \) of codimension two.

**Lemma 3.4.** If \( M_1 \ast M_2 \neq 0 \) then 
\[ [\partial_I H_1 \text{ and } \partial_R (H_1 - H_2)] \text{ or } [\partial_I H_1 \text{ and } \partial_I (H_1 - H_2)] \]
are \( \mathbb{R} \)-linearly independent.

**Proof.** As \( M_1 \ast M_2 \neq 0 \) then in view of (3.43) \( \partial_R (H_1 - H_2) \neq 0 \) or \( \partial_I (H_1 - H_2) \neq 0 \). Assume for example that \( \partial_I (H_1 - H_2) \neq 0 \). Assume that 
\[ \alpha \partial_I H_1 + \beta \partial_I (H_1 - H_2) = 0 \]
where \( (\alpha, \beta) \neq 0 \), \( \alpha, \beta \in \mathbb{R} \). Restricting the equality above at \( x = 0 \) and using that by (3.37)-(3.40) and (3.42), \( \partial_I (H_1 - H_2) \big|_{x=0} = 0 \) and \( \partial_I H_1 \big|_{x=0} \neq 0 \), we obtain that \( \alpha = 0 \). Hence, \( \beta \partial_I H_1 \equiv 0 \). As \( \partial_I (H_1 - H_1) \neq 0 \) we see that \( \beta = 0 \). This shows that \( \partial_I H_1 \) and \( \partial_I (H_1 - H_2) \) are \( \mathbb{R} \)-linearly independent. The case \( \partial_R (H_1 - H_2) \neq 0 \) is considered in the same way.

Theorem 3.2 and Theorem 3.3 are now used to prove Theorem 1.2 stated in the introduction.

**Proof of Theorem 1.2.** As the case \( N \in \mathbb{Z} \geq 1 \) is treated in the same way as \( N = 0 \) we concentrate on the latter case only. Let \( \zeta, \xi \) with \( \zeta \neq \xi \) be arbitrary elements in \( \mathcal{S}_p \). It is to show that there exists a continuous path \( \gamma^* : [0, 1] \to \mathcal{S}_p \) with \( \gamma^*(0) = \zeta \) and \( \gamma^*(1) = \xi \). The path \( \gamma^* \) will be constructed by deforming the straight line \( \ell \), parametrized by
\[ \gamma^0 : [0, 1] \to iL^2_r, t \mapsto (1 - t)\zeta + t\xi. \]
First let us observe that as the straight line \( \ell \) is compact, Lemma 1.1 implies that there exist a tubular neighborhood \( \mathcal{U}_\ell \) of \( \ell \),
\[ \mathcal{U}_\ell := \{ \varphi \in iL^2_r | \text{dist}(\varphi, \ell) < \delta \} \]
for some \( \delta > 0 \) and an integer \( R > 0 \) so that for any \( \varphi \in \mathcal{U}_\ell \), the eigenvalues \( \lambda_n^\pm \) and \( \lambda_n^{-} = \lambda_n^{+} \) of \( L(\varphi) \) with \( |n| > R \) are in the disk \( D_n \) whereas the \( 4R + 2 \) remaining eigenvalues
\[ \lambda_n^\pm, |n| \leq R, \] are contained in \( B_R \). In addition, in view of Lemma 2.10, we can ensure that

for any \( \varphi \in U_\ell \) and for any \( |n| > R, \mu_n \in D_n \), and the remaining \( 2R+1 \) Dirichlet eigenvalues \( \mu_n \in B_R, |n| < R \). The path \( \gamma^0 \) will be deformed within \( U_\ell \). Note that for any \( |n| > R \),
either \( \lambda_n^+ \) and \( \lambda_n^- \) are both simple periodic eigenvalues or \( \lambda_n^\pm \) is a real periodic eigenvalue with \( m_g(\lambda_n^+) = 2 \) and \( m_p(\lambda_n^+) = 2 \). Hence to verify that a potential \( \varphi \in U_\ell \) is standard it suffices to study the eigenvalues \( \lambda_n^\pm \) with \( |n| \leq R \).

As by Theorem 1.4 \( S_p \) is open and the endpoints \( \zeta, \xi \) of \( \gamma^0 \) are assumed to be in \( S_p \) there exist open balls \( \mathcal{V}_\zeta, \mathcal{V}_\xi \) such that \( \mathcal{B}_0, \mathcal{B}_1 \) are contained in \( \mathcal{B}_{0,1} \).

In a first step we apply Proposition 3.5, based on Theorem 3.3, to show that there exists a path \( \gamma^1 : [0, 1] \to U_\ell \) with \( \gamma^1(0) \in \mathcal{V}_\zeta \) and \( \gamma^1(1) \in \mathcal{V}_\xi \) so that for any \( \varphi \) on \( \gamma^1 \), no periodic eigenvalue \( \lambda_n^\pm \) with \( |n| \leq R \) has geometric multiplicity two. Note that the path \( \gamma_\zeta : [0, 1] \to \mathcal{V}_\zeta, [\gamma_\zeta : [0, 1] \to \mathcal{V}_\xi] \), connecting \( \gamma_\xi(0) = \zeta, [\gamma_\xi(0) = \xi] \), with \( \gamma_\xi(1) = \gamma^1(0) [\gamma_\xi(1) = \gamma^1(1)] \) by a straight line is in \( S_p \cap U_\ell \).

Then we apply Proposition 3.7, based on Theorem 3.2, to show that \( \gamma^1 \) can be deformed within \( U_\ell \) to a path \( \gamma^2 \) with the same end points as \( \gamma^1 \) so that for any \( \varphi \) on \( \gamma^2 \), all its periodic eigenvalues \( \lambda_n^\pm \) with \( |n| \leq R \) are simple. In particular, \( \gamma^2 \) is contained in \( S_p \). The path \( \gamma^* \) is then defined by concatenating \( \gamma_\xi, \gamma^2, \) and \( \gamma_\xi^{-1} \), i.e., \( \gamma^* = \gamma_\xi^{-1} \circ \gamma^2 \circ \gamma_\xi \).

To describe our construction of \( \gamma^1 \) in more detail we first introduce some more notation. Recall that for any Dirichlet eigenvalue \( \mu \) of \( L(\varphi) \) with \( \varphi \in L^2_c, m_D(\mu) = m_D(\mu, \varphi) \) denotes its algebraic multiplicity. It is convenient to set \( m_D(\mu) = m_D(\mu, \varphi) = 0 \) for any \( \mu \) in \( \mathbb{C} \) which is not a Dirichlet eigenvalue of \( L(\varphi) \). Note that for any \( \varphi \in U_\ell \) one has

\[ m_D(\lambda_n^\pm(\varphi)) \leq 2R + 1 \quad \forall |n| \leq R. \]

Furthermore introduce for any \( \varphi \in U_\ell \)

\[ M^D_\varphi := \max\{m_D(\lambda_n^\pm(\varphi)) : |n| \leq R; m_g(\lambda_n^\pm(\varphi)) = 2\}. \]

We point out that

\[ 0 \leq M^D_\varphi \leq 2R + 1 \quad \forall \varphi \in U_\ell. \]

Finally, for any continuous path \( \gamma : [0, 1] \to U_\ell \) set

\[ M^D_\gamma := \max\{M^D_{\gamma(t)} : 0 \leq t \leq 1\}. \]

Note that \( M^D_\varphi = 0 \) implies that for any \( \varphi \in \gamma \) there is no periodic eigenvalue \( \lambda_n^\pm \) with \( |n| \leq R \) and \( m_g(\lambda_n^\pm(\varphi)) = 2 \). If \( M^D_\varphi = 0 \), choose \( \gamma^1 \) to be \( \gamma^0 \). On the other hand, if \( M^D_\varphi > 0 \), then Proposition 3.5 says that there exists a continuous path \( \gamma^0 : [0, 1] \to U_\ell \) connecting \( \mathcal{V}_\zeta \) with \( \mathcal{V}_\xi \) so that \( M^D_{\gamma^0} < M^D_\varphi \). In particular, \( \gamma^0(0) \in \mathcal{V}_\zeta \) and \( \gamma^0(1) \in \mathcal{V}_\xi \). This procedure is iterated till we get a continuous path \( \gamma^1 : [0, 1] \to U_\ell \) connecting \( \mathcal{V}_\zeta \) with \( \mathcal{V}_\xi \) so that \( M^D_{\gamma^1} = 0 \).
To deform $\gamma^1$ to $\gamma^2$ we have to deal with potentials $\varphi$ with multiple periodic eigenvalues $\lambda_n^\pm(\varphi)$ of geometric multiplicity one, i.e., $m_p(\lambda_n^\pm(\varphi)) \geq 2$ and $m_g(\lambda_n^\pm(\varphi)) = 1$. To this end introduce for any $\varphi \in U_\ell$

$$M_\varphi^p = \max\{m_p(\lambda_n^\pm(\varphi)) \mid |n| \leq R\}.$$ 

As $1 \leq m_p(\lambda_n^\pm(\varphi)) \leq 4R + 2$ for any $\varphi$ in $U_\ell$ it follows that $1 \leq M_\varphi^p \leq 4R + 2$. Moreover, by construction $M_\gamma^p(0) = 1$ and $M_\gamma^p(1) = 1$.

Finally, for a continuous path $\gamma : [0, 1] \to U_\ell$ define

$$M_\gamma^p := \max\{M_\gamma^p(t) \mid 0 \leq t \leq 1\}.$$ 

We now deform the path $\gamma^1$. If $M_\gamma^p = 1$, then $\gamma^1$ is already a path in $S_p$ and we set $\gamma^2 := \gamma^1$. On the other hand, if $M_\gamma^p \geq 2$, Proposition 3.7 implies that there exists a continuous path $\tilde{\gamma}^1 : [0, 1] \to U_\ell$ from $\gamma^1(0)$ to $\gamma^1(1)$ so that $M_{\tilde{\gamma}^1}^p < M_\gamma^p$ and $M_{\tilde{\gamma}^1}^D = 0$. This procedure is iterated till we get a continuous path $\gamma^2 : [0, 1] \to U_\ell$ from $\gamma^1(0)$ to $\gamma^1(1)$ so that $M_{\gamma^2}^p = 1$. Then $\gamma^2$ is a path inside $S_p$ connecting $\gamma^1(0)$ with $\gamma^1(1)$. 

It remains to prove the two propositions used in the proof of Theorem 1.2.

**Proposition 3.5.** Let $\gamma : [0, 1] \to U_\ell$ be a continuous path with standard potentials as end points, i.e., $\zeta := \gamma(0)$, $\xi := \gamma(1) \in S_p$, and $\zeta \neq \xi$. Denote by $V_\zeta, V_\xi$ open disjoint balls in $S_p \cap U_\ell$ centered at $\zeta$, respectively $\xi$. If $M_\gamma^D > 0$ then there exists a continuous path $\tilde{\gamma} : [0, 1] \to U_\ell$ with $\tilde{\gamma}(0) \in V_\zeta, \tilde{\gamma}(1) \in V_\xi$ and $M_{\tilde{\gamma}}^D < M_\gamma^D$.

**Proof.** For any $\varphi \in \gamma$, denote by

$$\lambda^1(\varphi), \ldots, \lambda^K(\varphi), \ K \equiv K_\varphi \in \mathbb{Z}_{\geq 0}$$ 

the list of different periodic eigenvalues of $L(\varphi)$ inside $B_R$ with $m_g(\lambda^k(\varphi)) = 2$ for any $1 \leq k \leq K$. If $M_\varphi^D < M_\gamma^D$, then choose an open ball $W_\varphi \subseteq U_\ell$ centered at $\varphi$ so that for any $\psi \in W_\varphi$,

$$M_\psi^D \leq M_\varphi^D \ (< M_\gamma^D). \quad (3.47)$$

The existence of such neighborhood follows easily from the second statement of Theorem 3.2 and Theorem 5.1 (i) applied with $\chi = \chi_D$. On the other hand, if $M_\varphi^D = M_\gamma^D$, let

$$I \equiv I_\varphi := \{1 \leq j \leq K \mid m_D(\lambda^j(\varphi)) = M_\gamma^D\}.$$
By Theorem 3.3 applied to \((\lambda(\varphi), \varphi)\) for any \(j \in I\), there exists a path connected neighborhood \(\mathcal{W}_\varphi\) of \(\varphi\) in \(\mathcal{U}_t\) and a union \(Z_\varphi = \bigcup_{j \in I} Z_\varphi^j\) of submanifolds \(Z_\varphi^j\) of codimension two which are closed in \(\mathcal{W}_\varphi\) so that

\[
\{ \psi \in \mathcal{W}_\varphi \mid M_\psi^D = M_\gamma^D \} \subseteq Z_\varphi. \tag{3.48}
\]

By shrinking \(\mathcal{W}_\varphi\), if necessary, we further can assume that

\[
M_\psi^D \leq M_\varphi^D \quad (= M_\gamma^D) \quad \forall \psi \in \mathcal{W}_\varphi. \tag{3.49}
\]

In addition, if \(\varphi\) is either \(\zeta\) or \(\xi\) we assume that \(\mathcal{W}_\varphi \subseteq \mathcal{V}_\zeta\) or \(\mathcal{W}_\varphi \subseteq \mathcal{V}_\xi\) respectively. As \(\{\gamma(t) \mid 0 \leq t \leq 1\}\) is compact the cover \((\mathcal{W}_\varphi)_{\varphi \in \gamma([0,1])}\) admits a finite subcover \((\mathcal{W}_\varphi)_{\varphi \in \Lambda}\) where \(\Lambda \subseteq \gamma([0,1])\) is finite and contains \(\zeta\) and \(\xi\). We claim that there exists a sequence \(\mathcal{W}_i \equiv \mathcal{W}_{\varphi_i}, 1 \leq i \leq N \equiv N_\gamma\) with \(\varphi_i \in \Lambda\) so that \(\mathcal{W}_{i-1} \cap \mathcal{W}_i \neq \emptyset\) for any \(2 \leq j \leq N_\gamma\) and \(\varphi_1 = \zeta\) or \(\varphi_N = \xi\). Indeed, choose \(\varphi_1 = \zeta\) to begin with. Then \(\gamma^{-1}(\mathcal{W}_1)\) is an open subset of \([0,1]\). As \(\mathcal{V}_\zeta \cap \mathcal{V}_\xi = \emptyset\) it follows that

\[
t_1 := \sup \gamma^{-1}(\mathcal{W}_1) < 1.
\]

As \((\mathcal{W}_\varphi)_{\varphi \in \Lambda}\) covers \(\gamma([0,1])\) there exists \(\varphi_2 \in \Lambda\) with \(\gamma(t_1) \in \mathcal{W}_2\). As \(\mathcal{W}_2\) is open and \(\gamma : [0,1] \to \mathcal{U}_t\) is continuous it then follows that \(\mathcal{W}_1 \cap \mathcal{W}_2 \neq \emptyset\). Continuing in this way one obtains the sequence \(\mathcal{W}_i\) with \(\varphi_i \in \Lambda, 1 \leq i \leq N \equiv N_\gamma\) so that \(\varphi_1 = \zeta, \varphi_N = \xi, \) and \(\mathcal{W}_{i-1} \cap \mathcal{W}_i \neq \emptyset\) for any \(2 \leq j \leq N\). As for any \(1 \leq i \leq N\), \(Z_i \equiv Z_{\varphi_i}\) is a finite union of submanifolds of codimension two it then follows that for any \(2 \leq i \leq N\), \((\mathcal{W}_{i-1} \cap \mathcal{W}_i) \setminus (Z_{i-1} \cup Z_i) \neq \emptyset\).

For any \(2 \leq i \leq N\), choose \(\eta_i \in (\mathcal{W}_{i-1} \cap \mathcal{W}_i) \setminus (Z_{i-1} \cup Z_i)\). By (3.47)-(3.49) one concludes that for any \(2 \leq i \leq N\), \(M_{\eta_i}^D < M_\gamma^D\) and

\[
\eta_i, \eta_{i+1} \in \mathcal{W}_i \setminus Z_i.
\]

As \(Z_i\) is a finite union of submanifolds of codimension two which are closed in \(\mathcal{W}_i\), Lemma 3.6 stated below applies repeatedly. Hence for any \(2 \leq i \leq N-1\), there exists a continuous path \(\gamma_i : [0,1] \to \mathcal{W}_i \setminus Z_i\) such that \(\gamma_i(0) = \eta_i\) and \(\gamma_i(1) = \eta_{i+1}\). By (3.47)-(3.49) one has \(M_{\gamma_i}^D < M_\gamma^D\). As \(\eta_2 \in Z_1 \subseteq \mathcal{V}_\zeta\) and \(\eta_N \in Z_N \subseteq \mathcal{V}_\xi\) it then follows that the concatenation \(\tilde{\gamma}\) of \(\gamma_2, \ldots, \gamma_{N-1}\) is a continuous curve \(\tilde{\gamma} : [0,1] \to \mathcal{U}_t\) with the properties listed in Proposition 3.5.

Let us now state and prove the lemma referred to in the proof of Proposition 3.5.

**Lemma 3.6.** Let \(\mathcal{U}\) be an open, path connected set in a Hilbert space \(E\) and let \(Z \subseteq \mathcal{U}\) be a closed smooth submanifold of codimension two. Then \(\mathcal{U} \setminus Z\) is open and path connected.

**Proof.** This lemma is well known. In fact, it can be proved following the line of arguments used in Proposition 3.5 and by taking into account the following special case, where \(Z\) is a linear subspace of \(E\) of codimension two and hence \(E \setminus Z\) is obviously path connected.

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Proposition 3.7. Let $\gamma : [0, 1] \to U_\ell$ be a continuous path with standard potentials as end points so that $M^D_\gamma = 0$. If $M^p_\gamma \geq 2$, then there exists a continuous path $\tilde{\gamma} : [0, 1] \to U_\ell$ with the same end points as $\gamma$, $M^D_{\tilde{\gamma}} = 0$, and $M^p_{\tilde{\gamma}} < M^p_\gamma$.

Proof. The assumption $M^D_{\gamma} = 0$ implies that $M^D_{\gamma(0)} = 0$ and $M^D_{\gamma(1)} = 0$. As $\gamma(0)$ and $\gamma(1)$ are both standard potentials one concludes from Proposition 2.7 that all their periodic eigenvalues $\lambda_n^\pm$ with $|n| \leq R$ are simple. In particular, one has $M^p_{\gamma(0)} = 1$ and $M^p_{\gamma(1)} = 1$.

For any $\varphi$ denote by

$$\lambda^1(\varphi), \ldots, \lambda^K(\varphi), \quad K \equiv K_\varphi \in \mathbb{Z}_{\geq 0}$$

the list of different multiple periodic eigenvalues of $L(\varphi)$ inside $B_R$. By assumption, $m_g(\lambda^i(\varphi)) = 1$ for $1 \leq i \leq K$. If $M^p_\varphi < M^p_\psi$, by the second statement of Theorem 3.2 and Theorem 5.1 (i) applied with $\chi = \chi_\varphi$ and $\chi = \chi_D$ there exists a neighborhood $W_\varphi \subseteq U_\ell$ of $\varphi$ so that for any $\psi \in W_\varphi$

$$M^p_\psi \leq M^p_\varphi (< M^p_\psi) \text{ and } M^D_\psi = 0.$$  

On the other hand, if $M^p_\varphi = M^p_\psi$, let

$$I \equiv I_\varphi := \{1 \leq j \leq K \mid m_p(\lambda^j(\varphi)) = M^p_\varphi\}.$$  

By Theorem 3.2 applied to $(\varphi, \lambda^j(\varphi))$ for any $j \in I$, there exists an open ball $W_\varphi$ in $U_\ell$, centered at $\varphi$ and a union $Z_\varphi = \cup_{j \in I_\varphi} Z^j_\varphi$ of submanifolds $Z^j_\varphi$ of codimension two which are closed in $W_\varphi$ so that

$$\{\psi \in W_\varphi \mid M^p_\psi = M^p_\varphi\} \subseteq Z_\varphi.$$  

By shrinking $W_\varphi$, if necessary, we further can assume that for any $\psi \in W_\varphi$

$$M^p_\psi \leq M^p_\varphi \text{ and } M^D_\psi = 0.$$  

Then argue as in the proof of Proposition 3.5 to conclude that there is a continuous path $\tilde{\gamma} : [0, 1] \to U_\ell$ with the same end points as $\gamma$, $M^D_{\tilde{\gamma}} = 0$, and $M^p_{\tilde{\gamma}} < M^p_\gamma$. $\square$

It turns out that the proof of Theorem 1.2 actually leads to the following additional result. We say that a potential $\varphi \in L^2_\ell$ is $R$-simple, $R \in \mathbb{Z}_{\geq -1}$, if $\mu_n, \lambda_n^\pm \in D_n$ for any $|n| > R$, $\mu_n, \lambda_n^\pm \in B_R$ for any $|n| \leq R$, and the eigenvalues $(\lambda_n^+, \lambda_n^-)_{|n| \leq R}$ are all simple. Note that the zero potential is $(-1)$-simple. Denote by $T^R$ the set of $R$-simple potentials in $L^2_\ell$ and by $T$ the set of potentials $\varphi \in L^2_\ell$ so that $\text{spec}_p L(\varphi)$ is simple.

Inspecting the proof of Theorem 1.2 one sees that at the same time, the following result has been proved.

Corollary 3.8. For any $N \in \mathbb{Z}_{\geq 0}$ and for any $\zeta, \xi \in T \cap iH^N_\ell$ there exists $R \in \mathbb{Z}_{\geq -1}$ such that $\zeta$ and $\xi$ are path connected in $T^R \cap iH^N_\ell$.  

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Proof. First, consider the case $N = 0$. Take $\zeta, \xi \in T \cap iL^2_r$ and denote by $\ell$ the straight line connecting $\zeta$ with $\xi$ in $iL^2_r$. By the counting lemmas and the fact that $\ell$ is compact, there exists $R \in \mathbb{Z}_{\geq -1}$ so that for any $\varphi \in \ell$, $\mu_n, \lambda_n^+ \in D_n$ for any $|n| > R$, and $\mu_n, \lambda_n^- \in B_R$, for any $|n| \leq R$. With this choice of $R$, one can follow the arguments of the proof of Theorem 1.2 to conclude that there exists a path $\gamma$ in $T \cap iL^2_r$ connecting $\zeta$ and $\xi$. The case $N \in \mathbb{Z}_{\geq 1}$ is treated similarly.

For later applications it is useful to consider also a stronger version of the notion of $R$-simple potentials. For any $\psi \in iL^2_r$ denote by $\text{Iso}_0(\psi)$ the connected component containing $\psi$ of the isospectral set

$$\text{Iso}(\psi) := \{ \varphi \in iL^2_r \mid \text{spec}_p L(\varphi) = \text{spec}_p L(\psi) \}$$

We say that a potential $\psi \in iL^2_r$ is uniformly $R$-simple, $R \in \mathbb{Z}_{\geq -1}$, if $\text{Iso}_0(\psi) \subseteq T^R$. In other words, we require that for any $\varphi \in \text{Iso}_0(\psi)$, the periodic and the Dirichlet eigenvalues of $L(\varphi)$ are contained in $B_R \cup \bigcup_{|n| > R} D_n$ and, when counted with their algebraic multiplicities, satisfy the following conditions:

(S1) $\#(D_n \cap \text{spec}_p L(\varphi)) = 2$, $\#(D_n \cap \text{spec}_D L(\varphi))) = 1 \forall |n| > R$;

(S2) $\#(B_R \cap \text{spec}_p L(\varphi)) = 4R + 2$, $\#(B_R \cap \text{spec}_D L(\varphi)) = 2R + 1$;

(S3) The ball $B_R$ contains only simple periodic eigenvalues.

We denote the set of uniformly $R$-simple potentials by $U^R$ and let $U^* := \bigcup_{R \geq -1} U^R$. In view of the Counting Lemmas (see Lemma 1.1 and Lemma 2.10) and the compactness of $\text{Iso}_0(\psi)$ (see Lemma 3.11 below) it follows that for any $N \in \mathbb{Z}_{\geq 0}$,

$$T \cap iH^N_r \subseteq U^* \cap iH^N_r. \tag{3.50}$$

**Proposition 3.9.** For any $N \in \mathbb{Z}_{\geq 0}$, the set $T \cap iH^N_r$ is dense in $iH^N_r$. As a consequence, $U^* \cap iH^N_r$ is dense in $iH^N_r$.

**Proof.** Let $N \in \mathbb{Z}_{\geq 0}$ and let $\psi \in iH^N_r$. In view of the Counting Lemmas there exists $R \in \mathbb{Z}_{\geq -1}$ and an open neighborhood $U(\psi)$ of $\psi$ in $iH^N_r$ such that any $\varphi \in U(\psi)$ satisfies conditions (S1) and (S2). It follows from Theorem 3.2 and Theorem 3.3 that $T^R \cap U(\psi)$ is open and dense in $U(\psi)$. In view of Proposition 2.7 Lemma 2.12 and Theorem 3.3 for any $|n| > R$, the set

$$Z_n := \{ \varphi \in U(\psi) \mid \lambda^-_n(\varphi) = \lambda^+_n(\varphi) \} \subseteq U(\psi)$$

---

The periodic eigenvalues are counted with their algebraic multiplicities.
is contained in a submanifold in \( U(\psi) \) of (real) codimension two. Hence, for any \( |n| > R \), \( Z_n \) is closed and nowhere dense in \( T^R \cap U(\psi) \), and by the Baire theorem the set

\[
T \cap U(\psi) = \left( T^R \cap U(\psi) \right) \setminus \bigcup_{|n| > R} Z_n
\]
is dense in \( U(\psi) \). This completes the proof of the first statement of Proposition 3.9. The second statement then follows from (3.50). \( \square \)

We finish this appendix by showing that \( U^* \cap iH^N_r, N \geq 1 \), contains a subset which is open in \( iH^N_r \). Here we use that for \( N \geq 1 \), the Counting Lemmas for the periodic and the Dirichlet spectrum of \( L(\varphi) \) with \( \varphi \) in \( iH^N_r \) hold uniformly on any bounded subset of \( iH^N_r \).

**Proposition 3.10.** \( \forall N \in \mathbb{Z}_{\geq 1}, U^* \cap iH^N_r \) contains a subset which is open and dense in \( iH^N_r \).

We first need to make some preliminary considerations. It follows from [9] Theorem 13.4 and 13.5 that the quantities \( J_1(\varphi) := \int_0^1 |\varphi|^2 \, dx \) and \( J_2(\varphi) := \int_0^1 (|\partial_x \varphi|^2 - |\varphi|^4) \, dx \) are spectral invariants of the periodic spectrum of \( L(\varphi) \) for \( \varphi \in iH^1_r \). By the generalized Gagliardo-Nirenberg inequality there exist absolute constants \( C_1, C_2 > 0 \) so that for any \( u \in L^2(\mathbb{T}, \mathbb{C}) \) and \( \varepsilon > 0 \)

\[
\|u\|_{L^4} \leq C_1 \|\partial_x u\|^{1/4} \|u\|^{3/4} + C_2 \|u\| \leq C_1 (\varepsilon^2 \|\partial_x u\|^{1/2} + \|u\|^{3/2}/\varepsilon^2) + C_2 \|u\|
\]

– see [19] Theorem 1] with \( n = 1 \) (for the case of a circle instead of an interval), \( j = 0, p = 4, m = 1, r = 2, q = 2, \) and \( a = 1/4 \). By taking \( \varepsilon^2 = 1/(3C_1) \) in the inequality above one obtains

\[
\|u\|_{L^4} \leq \frac{1}{3} \|\partial_x u\|^2 + 3^7 C_1^8 \|u\|^6 + 3^3 C_2^4 \|u\|^4.
\]

Writing \( \int_0^1 |\partial_x \varphi|^2 \, dx = J_2 + \int_0^1 |\varphi|^4 \, dx \) it then follows that \( \frac{2}{3} \int_0^1 |\partial_x \varphi|^2 \, dx \leq J_2 + C(J_1^3 + J_1^2) \) where \( C = \max(3^7 C_1^8, 3^3 C_2^4) \). We thus have proved that any \( \varphi \in \text{Iso}_0(\psi) \cap iH^1_r \), can be bounded by

\[
\|\varphi\|_{H^1}^2 \leq (3J_2 + 3C(J_1^3 + J_1^2) + J_1) \|\psi\| \leq 3\|\psi_1\|_{H^1}^2 + 3C(\|\psi\|^6 + \|\psi_1\|^2).
\]

where \( \|\psi_1\|^2_{H^1} := \|\partial_x \psi_1\|^2 + \|\psi_1\|^2 \). This implies that for any \( \rho > 0 \) there exists a positive constant \( C_\rho > 0 \) such that for any \( \psi \in B^1_\rho \),

\[
\text{Iso}_0(\psi) \cap iH^1_r \subseteq B^1_\rho,
\]

where \( B^N_\rho := \{ \varphi \in iH^N_r \mid \|\varphi\|_{H^N_r} < \rho \} \).
Proof of Proposition 3.10. Take $N \in \mathbb{Z}_{\geq 1}$, $\rho > 0$, and consider the set

$$I^N_\rho := \bigcup_{\varphi \in B^N_0} \text{Iso}_0(\varphi) \cap iH^N_r.$$ 

In view of (3.51) we have the following sequence of continuous embedding

$$I^N_\rho \subseteq I^1_\rho \subseteq B^1_C \subseteq iL^2_r$$

where the last embedding is compact. Hence, $I^N_\rho$ is a precompact set in $iL^2_r$ and, by the Counting Lemma, there exists $R \equiv R_\rho \geq 0$ such that any $\varphi \in I^N_\rho$ satisfies conditions (S1) and (S2). This implies that $T^R \cap B^N_\rho \subseteq U^* \cap iH^N_r$. Hence,

$$T \cap B^N_\rho \subseteq \mathcal{P}_\rho \subseteq U^* \cap iH^N_r,$$  

(3.52)

where $\mathcal{P}_\rho$ denotes the open subset $T^R \cap B^N_\rho$ of $iH^N_r$. By Proposition 3.9, the set $T \cap B^N_\rho$ is dense in $B^N_\rho$. This together with the first inclusion in (3.52) implies that $\mathcal{P}_\rho$ is open and dense in $iH^N_r$ and thus is a subset of $U^* \cap iH^N_r$ with the claimed properties. 

We complete this section by proving the following result used above to establish (3.50).

Lemma 3.11. For any $\psi$ in $iL^2_r$, $\text{Iso}(\psi)$ is compact.

Proof. Let $(\varphi_n)_{n \geq 1}$ be a sequence in $\text{Iso}(\psi)$. By [9, Theorem 13.4] the $L^2$-norm is a spectral invariant of the periodic spectrum of $L(\varphi)$ for $\varphi \in iL^2_r$. Hence, $\|\varphi_n\| = \|\psi\|$, for any $n \geq 1$, and therefore there exist $\varphi \in iL^2_r$ and a weakly convergent subsequence, which we again denote by $(\varphi_n)_{n \geq 1}$, such that $\varphi_n \rightharpoonup \varphi$ as $n \to \infty$. By [9, Theorem 4.1], for any $\lambda \in \mathbb{C}$ the map $L^2_r \to \mathbb{C}$, $\varphi \mapsto \Delta(\lambda, \varphi)$, is compact and hence $\Delta(\lambda, \psi) = \lim_{n \to \infty} \Delta(\lambda, \varphi_n) = \Delta(\lambda, \varphi)$. By the definition of $\text{Iso}(\psi)$ it then follows that $\varphi \in \text{Iso}(\psi)$, implying that $\|\varphi\| = \|\psi\|$. As a consequence $\varphi_n \to \varphi$ in $iL^2_r$. This shows that $\text{Iso}(\psi)$ is a compact subset of $iL^2_r$. 

4 Proof of Theorem 1.4

The aim of this section is to prove Theorem 1.4.

Proof of Theorem 1.4. We begin by showing that $S_p \cap iH^N_r$ and $S_D \cap iH^N_r$ are open. First consider the case $S_p \cap iH^0_r = S_p$. For $\psi \in S_p$ arbitrary choose $R \in \mathbb{Z}_{\geq 0}$ as in Lemma 1.1 and let $\varepsilon > 0$ be smaller than twice the distance between any two different periodic eigenvalues of $L(\psi)$ in $B_R$. By Lemma 1.1, Proposition 2.3 and Theorem 5.1(1), there exists an open
neighborhood $W_{\psi}$ of $\psi$ in $L^2_c$ so that for any $\varphi \in W_{\psi}$, \(#(D_n \cap \text{spec}_p L(\varphi)) = 2 \) and \(#(B_R \cap \text{spec}_p L(\varphi)) = 4R + 2 \) and for any eigenvalue $\lambda \in B_R \cap \text{spec}_p L(\psi)$, 
\[ \#(D^\varepsilon(\lambda) \cap \text{spec}_p L(\varphi)) = \#(D^\varepsilon(\lambda) \cap \text{spec}_p L(\psi)) \]

where $D^\varepsilon(\lambda) \subseteq \mathbb{C}$ is the open disk of radius $\varepsilon$ centered at $\lambda$. By the definition of $S_p$, one then concludes that $W_{\psi} \cap iL^2_c \subseteq S_p$. The openness of $S_p \cap iH^N_r (N \geq 1)$ and $S_D \cap iH^N_r (N \geq 0)$ is proved in a similar fashion. Next we show that $\lambda$ vanish for $\partial_r$ neighborhood $\partial$. Recall that the discriminant is the resultant of $Q$ and $\chi$. It is given by
\[ (\lambda, \psi) \subseteq B \]
\[ \\lambda = \det \left( \begin{array}{cccccc} a_0 & \cdots & a_{4R} & a_{4R+1} & a_{4R+2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_0 & a_1 & a_2 & a_3 & \cdots & a_{4R+2} \\ b_0 & \cdots & b_{4R} & b_{4R+1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & b_0 & b_1 & b_2 & \cdots & b_{4R+1} \end{array} \right) \]

Denote by $B^1_{\rho}$ the open ball of radius $\rho > 0$ in $H^1_c$, centered at 0. Choose $R = R_\rho \geq 1$ as in Corollary 2.5 and introduce
\[ S_{p,R} := \{ \varphi \in S_p | \lambda_n^\pm \text{ simple} \forall |n| \leq R \} \]

We claim that $S_{p,R} \cap B^1_{\rho}$ is dense in $B^1_{\rho} \cap iL^2_c$. For any $\varphi \in B^1_{\rho}$ and $R$ as above introduce
\[ Q_{p,R}(\lambda) := \prod_{|k| \leq R} (\lambda_k^+ - \lambda)(\lambda_k^- - \lambda). \]

Then $Q_{p,R}$ is a polynomial in $\lambda$ of degree $4R + 2$ with coefficients depending analytically in $\varphi$ on $B^1_{\rho}$. Indeed, any coefficient of the polynomial $Q_{p,R}$ is a symmetric polynomial in $\lambda_k^\pm$, $|k| \leq R$, and hence can be written as a polynomial in
\[ s_n := \sum_{|k| \leq R} (\lambda_k^+)^n + (\lambda_k^-)^n, \quad 0 \leq n \leq 4R + 2. \]

To see that each $s_n$ is analytic on $B^1_{\rho}$, note that by the argument principle, $s_n$ is given by
\[ s_n = \frac{1}{2\pi i} \int_{|\lambda|=\pi(R+1/4)} \lambda^n \frac{\dot{\chi}_p(\lambda)}{\chi_p(\lambda)} d\lambda \]

and hence analytic as $\chi_p(\lambda, \varphi)$ and $\dot{\chi}_p(\lambda, \varphi)$ are analytic on $\mathbb{C} \times L^2_c$ and $\chi_p(\lambda, \varphi)$ does not vanish for $\varphi \in B^1_{\rho}$ and $\lambda$ in $\{ \lambda \in \mathbb{C} | |\lambda| = R\pi + \pi/4 \}$. Denote by $\mathcal{D}_{p,R}$ the discriminant of the polynomial $Q_{p,R}$. Recall that the discriminant is the resultant of $Q_{p,R}$ and its derivative $\partial_\lambda Q_{p,R}(\lambda)$. It is given by
\[ \mathcal{D}_{p,R} = \det \left( \begin{array}{cccccc} a_0 & \cdots & a_{4R} & a_{4R+1} & a_{4R+2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_0 & a_1 & a_2 & a_3 & \cdots & a_{4R+2} \\ b_0 & \cdots & b_{4R} & b_{4R+1} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & b_0 & b_1 & b_2 & \cdots & b_{4R+1} \end{array} \right) \]
where \((a_0, a_1, ..., a_{4R+2})\) is the coefficient vector of \(Q_{p,R}\) and repeated \(4R+1\) times whereas \((b_0, b_1, ..., b_{4R+1})\) is the one of \(\dot{Q}_{p,R}\), and repeated \(4R+2\) times. Note that \(D_{p,R}\) is an analytic function on \(B^1_p\). Furthermore, it has the property that it vanishes at an element \(\varphi \in B^1_p\) iff \(Q_{p,R}(\cdot, \varphi)\) has at least one multiple zero – see e.g. [14]. In particular, we have

\[
\{ \varphi \in B^1_p \cap iH^1_\rho | D_{p,R}(\varphi) \neq 0 \} = S_{p,R} \cap B^1_p.
\]

To show that \(S_{p,R} \cap B^1_p\) is dense in \(B^1_p \cap iH^1_\rho\), it thus suffices to show that \(S_{p,R} \cap B^1_p \neq \emptyset\) and that \(D_{p,R}\) is real valued on \(B^1_p \cap iH^1_\rho\). To see that \(D_{p,R}\) is real valued on \(B^1_p \cap iH^1_\rho\), note that by Proposition 2.7 for any \(\varphi \in B^1_p \cap iH^1_\rho\), \(\lambda^+(\varphi) = \lambda^-(\varphi)\). Hence by the definition of \(\dot{Q}_{p,R}\), its coefficients are real valued and \(D_{p,R}\) is therefore real valued on \(B^1_p \cap iH^1_\rho\). Finally, to see that \(S_{p,R} \cap B^1_p \neq \emptyset\) we use that elements in \(iH^1_\rho\) near 0 can be represented by Birkhoff coordinates. Indeed, by Theorem 1.1 of [11] and formula (3.8) of [11], for any element \(\varphi\) of \(iH^1_\rho\) near 0 with Birkhoff coordinates \((x_k, y_k)_{k \in \mathbb{Z}}\) satisfying for any given \(k \in \mathbb{Z}\), \(x_k^2 + y_k^2 \neq 0\), one has \(\lambda_k^+ \neq \lambda_k^-\). Thus by Theorem 1.1 of [11], any sequence \((x_k, y_k)_{k \in \mathbb{Z}}\) with values in \(i\mathbb{R} \times \mathbb{R}\) and \(\sum_{k \in \mathbb{Z}} (1 + |k|)^2 (|x_k|^2 + |y_k|^2)\) sufficiently small so that \(x_k^2 + y_k^2 \neq 0\) for any \(|k| \leq R\) is an element in \(S_{p,R} \cap B^1_p\). In the same way one shows that \(S \cap iH^N\) is dense in \(iH^N\) for any \(N \in \mathbb{Z}_{\geq 1}\).

The corresponding density result for \(S_D\) is proved in a similar fashion. As \(iH^1_\rho\) is dense in \(iL^2_\rho\), it suffices to show that \(S_D \cap iH^1_\rho\) is dense in \(iH^1_\rho\). For any \(\rho > 0\) choose \(R \equiv R_\rho \geq 1\) so that for any \(\varphi \in B^1_p\) the statement of Lemma 2.10 holds. Introduce

\[
S_{D,R} := \{ \varphi \in iL^2_\rho | \mu_n \text{ simple and } \mu_n \neq \tilde{\mu}_n \forall |n| \leq R \}.
\]

We claim that for any \(\rho > 0\), \(S_{D,R} \cap B^1_p\) is dense in \(B^1_p \cap iH^1_\rho\). Arguing as above one reduces in a first step the proof of the density of \(S_{D,R} \cap B^1_p\) in \(B^1_p \cap iH^1_\rho\) to the proof of \(S_{D,R} \cap B^1_p \neq \emptyset\). Here

\[
Q_{D,R} := \prod_{|k| \leq R} (\mu_k - \lambda)(\tilde{\mu}_k - \lambda)
\]

plays the role of \(Q_{p,R}\). Next we use again that by Theorem 1.1 in [11], any sequence \((x_k, y_k)_{k \in \mathbb{Z}}\) with values in \(i\mathbb{R} \times \mathbb{R}\) and \(\sum_{k \in \mathbb{Z}} (1 + |k|)^2 (|x_k|^2 + |y_k|^2)\) sufficiently small, represents an element \(\varphi \in B^1_p\) close to 0. Together with Proposition 4.1 in [11] it follows that if \(x_k \neq 0\) and \(y_k = 0\), then \(\lambda_k^+ \neq \lambda_k^-\) and \(\mu_k \in \{\lambda_k^+, \lambda_k^-\}\). Hence \(\mu_k \notin \mathbb{R}\) and \(\mu_k \neq \tilde{\mu}_k\). In addition, for \(\sum_{k \in \mathbb{Z}} (1 + |k|)^2 (|x_k|^2 + |y_k|^2)\) sufficiently small, \(\mu_k, \lambda_k^\pm \in D_k\) for any \(k \in \mathbb{Z}\). Hence, any sequence \((x_k, y_k)_{k \in \mathbb{Z}}\) with values in \(i\mathbb{R} \times \mathbb{R}\) and \(\sum_{k \in \mathbb{Z}} (1 + |k|)^2 (|x_k|^2 + |y_k|^2)\) sufficiently small, so that \(y_k = 0\) for any \(|k| \leq R\), represents an element in \(S_{D,R} \cap iH^1_\rho\). The case \(N \in \mathbb{Z}_{\geq 1}\) is treated in a similar way.

Inspecting the proof of Theorem 1.4 one sees that actually the following result for the set of potentials \(S^*\) introduced at the end of Section 3 has been proved.

**Corollary 4.1.** For any \(N \in \mathbb{Z}_{\geq 0}\), \(S^* \cap iH^N_r\) is dense in \(iH^N_r\).
5 Appendix A: $L^2$-gradients of averaging functions

In this appendix, in a quite general set-up, we state and prove a theorem on multiple roots of characteristic functions applied in the proofs of Theorem 3.2 and Theorem 3.3.

**Theorem 5.1.** Let $F, \chi : \mathbb{C} \times L^2_c \to \mathbb{C}$ be analytic maps and $\psi$ an arbitrary but fixed element in $L^2_c$. Assume that at $z_{\psi} \in \mathbb{C}$, $\chi(\cdot, \psi)$ has a zero of order $m \geq 1$. Then the following statements hold:

(i) For any $\varepsilon > 0$ sufficiently small there exists an open neighborhood $V \subseteq L^2_c$ of $\psi$ such that for any $\phi \in V$, $\chi(\cdot, \phi)$ has exactly $m$ roots $z_1(\phi), \ldots, z_m(\phi)$, listed with their multiplicities, in the open disk $D^\varepsilon \equiv D^\varepsilon(z_{\psi}) := \{ \lambda \in \mathbb{C} \mid |\lambda - z_{\psi}| < \varepsilon \}$ and no roots on the boundary $\partial D^\varepsilon$ of $D^\varepsilon$.

(ii) The functional $F_\chi : V \to \mathbb{C}$, defined by

$$F_\chi(\phi) := \sum_{j=1}^m F(z_j(\phi), \phi),$$

is analytic and at $(\phi, \lambda) = (\psi, z_{\psi})$

$$\partial F_\chi = m \partial F + \sum_{j=0}^m a_j \partial^{m-j}_\lambda \partial \chi$$

where $a_j \in \mathbb{C}$, $0 \leq j \leq m$, and $\partial$ denotes the $L^2$-gradient with respect to $\phi$ and $\partial \lambda$ denotes the derivative with respect to $\lambda$. If $F(\cdot, \psi)$ has a zero of order $k \geq 1$ at $z_{\psi}$, then $a_0 = \ldots = a_{k-1} = 0$; if $k = m$, then

$$a_m = -\frac{1}{m!} \partial^m_\lambda \left( F(\lambda, \psi) \frac{(\lambda - z_{\psi})^{m+1} \partial \chi(\lambda, \psi)}{\chi(\lambda, \psi)^2} \right) \bigg|_{\lambda = z_{\psi}} \neq 0.$$

**Remark 5.2.** As $\chi : \mathbb{C} \times L^2_c \to \mathbb{C}$ is analytic it follows that for any $\phi \in L^2_c$, $\partial \chi : \mathbb{C} \to L^2_c$, $\lambda \mapsto \partial \chi|_{(\phi, \lambda)}$ is analytic and so is $\partial^k \chi$ for any $k \geq 1$.

**Proof.** (i) By the analyticity of $\chi(\cdot, \psi)$ there exists $\varepsilon > 0$ so that $\chi(\cdot, \psi)$ does not vanish on $D_\varepsilon \setminus \{ z_{\psi} \}$. By the analyticity of $\chi$ it then follows that there exists a neighborhood $V$ of $\psi$ in $L^2_c$ so that for any $\phi \in V$, $\chi(\cdot, \phi)$ does not vanish in a small tubular neighborhood of $\partial D^\varepsilon$ in $\mathbb{C}$. It then follows by the argument principle that for any $\phi \in V$, $\chi(\cdot, \phi)$ has precisely $m$ zeros in $D^\varepsilon$, when counted with their multiplicities.

(ii) Again by the argument principle, for any $\phi \in V$ one has

$$F_\chi(\phi) = \frac{1}{2\pi i} \int_{\partial D^\varepsilon} F(\lambda, \phi) \frac{\dot{\chi}(\lambda, \phi)}{\chi(\lambda, \phi)} d\lambda \quad (5.1)$$
where $\dot{\chi} = \partial_{\lambda} \chi$. Note that the integrand in (5.1) is analytic on $\partial D^c \times \mathcal{V}$, whence $F_\chi$ is analytic on $\mathcal{V}$. To compute its $L^2$-gradient $\partial F_\chi$ it is convenient to introduce for $g = (g_1, g_2), h = (h_1, h_2) \in L^2_c$,
\[
\langle g, h \rangle_r = \int_0^1 (g_1 h_1 + g_2 h_2) \ dx.
\]
Then, by the definition of the $L^2$-gradient, one has at $\varphi = \psi$,
\[
\langle \partial F_\chi, h \rangle_r = \frac{d}{ds} \bigg|_{s=0} F_\chi(\psi + sh) = \frac{1}{2\pi i} \int_{\partial D^c} \frac{d}{ds} \bigg|_{s=0} \left( F(\lambda, \psi + sh) \frac{\dot{\chi}(\lambda, \psi + sh)}{\chi(\lambda, \psi + sh)} \right) \ d\lambda.
\]
By the product rule one gets at $\varphi = \psi$
\[
\langle \partial F_\chi, h \rangle_r = \frac{1}{2\pi i} \int_{\partial D^c} \left[ \langle \partial F, h \rangle_r \dot{\chi} + F \cdot \left( \frac{1}{\chi} \langle \partial \dot{\chi}, h \rangle_r - \frac{1}{\chi^2} \langle \partial \chi, h \rangle_r \dot{\chi} \right) \right] \ d\lambda.
\]
Hence $\partial F_\chi$ is given by
\[
\frac{1}{2\pi i} \int_{\partial D^c} \left( \frac{\dot{\chi}}{\chi} \partial F + \frac{1}{(\lambda - z_\psi)^m} \frac{\lambda - z_\psi)^m F}{\chi} (\partial \chi) - \frac{1}{(\lambda - z_\psi)^{m+1}} \frac{\lambda - z_\psi)^{m+1} F}{\chi^2} \dot{\chi} (\partial \chi) \right) \ d\lambda.
\]
(5.2)
Here we used that $\partial \dot{\chi} = (\partial \chi)'$ and that $\partial F, \partial \chi : C \to L^2_c$ are analytic and hence in particular, the maps $\partial F, \partial \chi, (\partial \chi)' : C \to L^2_c$ are continuous. Furthermore, as by assumption, $\chi(\cdot, \psi)$ has a zero of order $m$ at $\lambda = z_\psi$, $\frac{(\lambda - z_\psi)^m F}{\chi} (\partial \chi)$ and $\frac{(\lambda - z_\psi)^{m+1} F}{\chi^2} \dot{\chi} (\partial \chi)$ are both analytic functions on $\overline{D^c}$ with values in $L^2_c$. Hence by the argument principle, at $\varphi = \psi$,
\[
\frac{1}{2\pi i} \int_{\partial D^c} \frac{\dot{\chi}}{\chi} \partial F d\lambda = m \partial F \bigg|_{\lambda = z_\psi}
\]
and by Cauchy’s integral formula,
\[
\frac{1}{2\pi i} \int_{\partial D^c} \frac{1}{(\lambda - z_\psi)^m} \frac{\lambda - z_\psi)^m F}{\chi} (\partial \chi) \ d\lambda = \frac{1}{(m-1)!} \partial^m_{\lambda} \bigg|_{\lambda = z_\psi} \left( \frac{(\lambda - z_\psi)^m F}{\chi} (\partial \chi) \right)
\]
and
\[
\frac{1}{2\pi i} \int_{\partial D^c} \frac{1}{(\lambda - z_\psi)^{m+1}} \frac{\lambda - z_\psi)^{m+1} F}{\chi^2} \dot{\chi} (\partial \chi) \ d\lambda = \frac{1}{m!} \partial^m_{\lambda} \bigg|_{\lambda = z_\psi} \left( \frac{(\lambda - z_\psi)^{m+1} F}{\chi^2} \dot{\chi} (\partial \chi) \right).
\]
Thus $\partial F\chi$ at $\varphi = \psi$ is given by
\[
\partial F\chi = m \partial F\bigg|_{\lambda = z\psi} \frac{1}{(m-1)!} \partial^{m-1}_{\lambda} \left( \frac{(\lambda - z\psi)^m F\chi}{\chi} \right)_{\lambda = z\psi}.
\]
The claimed formula for $\partial F\chi$ at $\varphi = \psi$ then follows from the Leibniz rule. If $F(\cdot, \psi)$ has a zero of order $k \geq 1$ at $\lambda = z\psi$, then at $(\varphi, \lambda) = (\psi, z\psi)$
\[
\partial F\chi = m \partial F + \sum_{j=k}^{m} a_j \partial^{m-j}_{\chi} \partial\chi,
\]
i.e., $a_j = 0$ for $0 \leq j \leq k - 1$. If $k = m$, then
\[
\partial F\chi = m \partial F + a_m \partial\chi
\]
where in this case
\[
a_m = - \frac{1}{m!} \partial^m_{\lambda} \left( F(\lambda, \psi) \frac{(\lambda - z\psi)^{m+1}\chi(\lambda, \psi)}{\chi(\lambda, \psi)^2} \right)_{\lambda = z\psi} \neq 0.
\]

Finally we record a few simple facts from linear algebra, also needed in Section 3. Consider $f = (f_1, f_2)$ in $L^2_{\mathbb{R}}$ and denote by $\ell \equiv \ell_f$ the $\mathbb{R}$-linear functional on the $\mathbb{R}$-vector space $iL^2_{\mathbb{R}}$ induced by $f$,
\[
\ell : iL^2_{\mathbb{R}} \to \mathbb{C}, h \mapsto \langle f, h \rangle_r,
\]
where
\[
\langle f, h \rangle_r = \int_0^1 (f_1 h_1 + f_2 h_2) dx.
\]
Write $\ell(h)$ as $\ell_R(h) + i\ell_I(h)$ where $\ell_R \equiv \ell_{f,R}$ and $\ell_I \equiv \ell_{f,I}$ are the elements in the dual $L(iL^2_{\mathbb{R}}, \mathbb{R})$ of $iL^2_{\mathbb{R}}$ given by
\[
\ell_R(h) = \Re(\langle f, h \rangle_r) \quad \text{and} \quad \ell_I(h) = \Im(\langle f, h \rangle_r).
\]
They can be expressed in terms of $f$ and $\hat{f} = -(\overline{f_2}, \overline{f_1})$ as follows
\[
\ell_R(h) = \langle f + \hat{\overline{f}}, h \rangle_r \quad \text{and} \quad \ell_I(h) = \langle f - \hat{\overline{f}}, h \rangle_r.
\]
As the subspace $iL^2_{\mathbb{R}} \subseteq L^2_{\mathbb{C}}$ is the subset of all elements $\varphi \in L^2_{\mathbb{C}}$ satisfying $\varphi = \hat{\varphi}$ it follows that $\frac{f + \hat{\overline{f}}}{2}$ and $\frac{f - \hat{\overline{f}}}{2i}$ are in $iL^2_{\mathbb{R}}$.

Using that $f \mapsto \hat{f}$ is an involution and that for any $c \in \mathbb{C}$, $(\overline{cf}) = \overline{c}\hat{f}$, the following lemma can be proved in a straightforward way.
Lemma 5.3. (i) $\ell_R, \ell_I$ are $\mathbb{R}$-linearly independent iff $f + \hat{f}, f - \hat{f}$ are $\mathbb{R}$-linearly independent.
(ii) $\ell_R, \ell_I$ are $\mathbb{R}$-linearly independent iff $f + \hat{f}, f - \hat{f}$ are $\mathbb{C}$-linearly independent.
(iii) For any $\lambda \in \mathbb{C} \setminus \{0\}$, $\ell_{f,R}, \ell_{f,I}$ are $\mathbb{R}$-linearly independent iff $\ell_{\lambda f,R}, \ell_{\lambda f,I}$ are $\mathbb{R}$-linearly independent.
(iv) $\ell_{f,R}, \ell_{f,I}$ are $\mathbb{R}$-linearly dependent iff there exists $\lambda \in \mathbb{C} \setminus \{0\}$ so that
\[
\frac{\lambda f + (\lambda f)}{2} = 0.
\]

6 Appendix B: Examples

In this section we consider potentials in $iL^2_r$ of the form $(a \in \mathbb{C}, k \in \mathbb{Z})$
\[
\varphi_{a,k}(x) = (ae^{2\pi ikx}, -ae^{-2\pi ikx}).
\] (6.1)

Most of the results presented in this section can be found in [16]. We include them for the convenience of the reader. First we show that we can easily relate various spectra of $L(\varphi_{a,k})$ with the corresponding ones for $k = 0$. More generally, for an arbitrary potential $\varphi \in L^2_c$, various spectra of $L(\varphi_1 e^{2\pi ikx}, \varphi_2 e^{-2\pi ikx})$ are related to the corresponding spectra of $(\varphi_1, \varphi_2)$ by the following lemma which can be verified in a straightforward way.

Lemma 6.1. Assume that $f = (f_1, f_2)$ is a solution of $L(\varphi)f = \lambda f$ where $\varphi \in L^2_c$ is arbitrary. Then $(f_1 e^{i\pi kx}, f_2 e^{-i\pi kx})$ is a solution of
\[
L(\varphi_1 e^{2\pi ikx}, \varphi_2 e^{-2\pi ikx})g = (\lambda - k\pi)g.
\]

Corollary 6.2. For any $\varphi \in L^2_c$ and $k \in \mathbb{Z}$, the fundamental solution
\[
\tilde{M}(x, \lambda) \equiv M(x, \lambda, (\varphi_1 e^{2\pi ikx}, \varphi_2 e^{-2\pi ikx}))
\]
of $L(\varphi_1 e^{2\pi ikx}, \varphi_2 e^{-2\pi ikx})$ is related to the fundamental solution $M(x, \lambda)$ of $L(\varphi_1, \varphi_2)$ by
\[
\tilde{M} = \text{diag}(e^{i\pi kx}, e^{-i\pi kx}) \cdot M(x, \lambda + k\pi).
\]

Corollary 6.2 yields the following application.

Proposition 6.3. For any $\varphi = (\varphi_1, \varphi_2) \in L^2_c$ and any $k \in \mathbb{Z}$,
\[
\text{spec}_p(L(\varphi_1 e^{2\pi ikx}, \varphi_2 e^{-2\pi ikx})) = \text{spec}_p(L(\varphi_1, \varphi_2)) - k\pi
\]
and
\[
\text{spec}_D(L(\varphi_1 e^{2\pi ikx}, \varphi_2 e^{-2\pi ikx})) = \text{spec}_D(L(\varphi_1, \varphi_2)) - k\pi
\]
(with multiplicities).
Proof. Recall that the characteristic functions \( \chi_p \) and \( \chi_D \) are given by

\[
\chi_p(\lambda) = (\hat{m}_3(\lambda) + \hat{m}_4(\lambda))^2 - 4
\]

\[
2i\chi_D(\lambda) = \hat{m}_4(\lambda) + \hat{m}_3(\lambda) - \hat{m}_2(\lambda) - \hat{m}_1(\lambda).
\]

By Corollary 6.2,

\[
\chi_p(\lambda, (\varphi_1 e^{2\pi ikx}, \varphi_2 e^{-2\pi ikx})) = \chi_p(\lambda + k\pi, \varphi)
\]

and

\[
\chi_D(\lambda, (\varphi_1 e^{2\pi ikx}, \varphi_2 e^{-2\pi ikx})) = (-1)^k \chi_D(\lambda + k\pi, \varphi).
\]

As spec\(_p\)(\(L(\varphi_1 e^{2\pi ikx}, \varphi_2 e^{-2\pi ikx})\)) and spec\(_D\)(\(L(\varphi_1 e^{2\pi ikx}, \varphi_2 e^{-2\pi ikx})\)) are the zero sets (with multiplicities) of \(\chi_p(\lambda, (\varphi_1 e^{2\pi ikx}, \varphi_2 e^{-2\pi ikx}))\) respectively \(\chi_D(\lambda, (\varphi_1 e^{2\pi ikx}, \varphi_2 e^{-2\pi ikx}))\), the claimed identities follow.

In view of Proposition 6.3, instead of the potentials \(\varphi_{a,k}\) defined by (6.1), it suffices to consider the case \(k = 0\),

\[
\varphi_a \equiv \varphi_{a,0} = (a, -\bar{a}), \quad a \in \mathbb{C}.
\]

In a straightforward way one verifies the following

**Lemma 6.4.** For any \(a \in \mathbb{C}\),

\[
M(x, \lambda, \varphi_a) = \begin{pmatrix}
\cos(\kappa x) - i\lambda \frac{\sin(\kappa x)}{\kappa} & ia \frac{\sin(\kappa x)}{\kappa} \\
-i\bar{a} \frac{\sin(\kappa x)}{\kappa} & \cos(\kappa x) + i\lambda \frac{\sin(\kappa x)}{\kappa}
\end{pmatrix}
\]

where

\[
\kappa \equiv \kappa(\lambda, a) = \sqrt{\lambda^2 + |a|^2}
\]

**Remark 6.5.** Note that \(\kappa\) depends only on the modulus \(|a|\) of \(a\) and that the right hand side of (6.2) does not depend on the choice of the sign of the root \(\sqrt{\lambda^2 + |a|^2}\) as cosine is an even function whereas sine is odd. Furthermore, the right hand side of (6.2) is well defined at \(\kappa = 0\) as \(\frac{\sin(\kappa x)}{\kappa} = x + O(\kappa^2)\).

**Periodic spectrum of \(L(\varphi_a)\):** By Lemma 6.4 one has \(\Delta(\lambda, \varphi_a) = 2\cos \kappa(\lambda)\) and hence the characteristic function of \(L(\varphi_a)\), considered on the interval \([0, 2]\) with periodic boundary conditions, is given by

\[
\chi_p(\lambda, \varphi_a) = \Delta^2(\lambda, \varphi_a) - 4 = -4\sin^2(\kappa(\lambda)).
\]

The periodic eigenvalues of \(L(\varphi_a)\) are thus given by the \(\lambda\)’s satisfying \(\kappa(\lambda) = n\pi\) for some \(n \in \mathbb{Z}\), or

\[
\lambda^2 + |a|^2 = n^2 \pi^2.
\]
The monodromy matrix $M$ for such a $\lambda$ is given by

$$M = \begin{pmatrix} (-1)^n & 0 \\ 0 & (-1)^n \end{pmatrix}$$

(6.6)

when $n \neq 0$ and by

$$M = \begin{pmatrix} 1 - i\lambda & ia \\ ia & 1 + i\lambda \end{pmatrix} = \begin{pmatrix} 1 \pm |a| & ia \\ ia & 1 \mp |a| \end{pmatrix}$$

(6.7)

when $n = 0$. It is convenient to list the periodic eigenvalues not in lexicographic ordering, but rather use the integer $n \in \mathbb{Z}$ in (6.5) as an index. When listed in this way, we denote the periodic eigenvalues by $\hat{\lambda}^\pm_n$, $n \in \mathbb{Z}$, which are defined as follows. For any $n \in \mathbb{Z}$ with $|n\pi| > |a|$ denote

$$\hat{\lambda}^+_n = \hat{\lambda}^-_n = \text{sgn}(n) \cdot \sqrt{n^2 \pi^2 - |a|^2}.$$

In view of (6.6), $\hat{\lambda}^\pm_n$ defined above is a periodic eigenvalue of $L(\varphi_a)$ of geometric multiplicity two. Using (6.3) and (6.4) one easily sees that $\hat{\lambda}^+_n$ has algebraic multiplicity is two. Further, for any $n \in \mathbb{Z}$ with $0 < |n\pi| < |a|$ denote

$$\hat{\lambda}^+_n = \hat{\lambda}^-_n = \text{sgn}(n) \cdot i \sqrt{|a|^2 - n^2 \pi^2}.$$

Again, in view of (6.6), for $n \in \mathbb{Z}$ with $0 < |n\pi| < |a|$, $\hat{\lambda}^+_n$ is a periodic eigenvalue of $L(\varphi_a)$ of geometric multiplicity two and, by (6.3) and (6.4), its algebraic multiplicity is two.

Next note that for $n = 0$, one has $\hat{\lambda}^+_0 = \hat{\lambda}^-_0 = \pm i|a|$. In view of (6.7), for $a \neq 0$ the geometric multiplicity of $\hat{\lambda}^+_0$ as well as of $\hat{\lambda}^-_0$ equals one. In view of (6.3) and (6.4) the algebraic multiplicity of $\hat{\lambda}^+_0$ and the one of $\hat{\lambda}^-_0$ is one. For $a \neq 0$, the eigenfunctions corresponding to $\hat{\lambda}^+_0$ and $\hat{\lambda}^-_0$ are the constant vectors $(a, ia|a|)$ resp. $(a, -i|a|)$. We then obtain the following result, used in the proof of Theorem 3.2.

**Lemma 6.6.** For any $k \in \mathbb{Z}$, consider the potential $\varphi_{a,-k} = (ae^{-2i\pi kx}, -ae^{2i\pi kx})$. Then $\hat{\lambda}^\pm_0 = k\pi \pm i|a|$ are periodic eigenvalues of $L(\varphi_{a,-k})$ of algebraic multiplicity one.

In the special case where $|a| = n_a\pi$ for some $n_a \in \mathbb{Z}_{>0}$ set $\hat{\lambda}^\pm_{n_a} = 0$. The above computations yield

**Corollary 6.7.**

(i) For $a \in \mathbb{C}$, $\varphi_a$ is a standard potential iff $|a| < \pi$.

(ii) For $a \in \mathbb{C}$, any multiple periodic eigenvalue $\lambda$ of $L(\varphi_a)$ satisfies $m_p(\lambda) = 2$ and $m_g(\lambda) = 2$ iff $|a| > \pi$ and $|a| \neq \pi \mathbb{Z}$.

(iii) If $a \in \mathbb{C} \setminus \{0\}$ satisfies $|a| \in \pi \mathbb{Z}$, then 0 is a periodic eigenvalue of $L(\varphi_a)$ of algebraic multiplicity four.
Isospectral set $\text{Iso}_0(\varphi_a)$: Denote by $\text{Iso}_0(\varphi_a)$ the connected component containing $\varphi_a$ of the set $\text{Iso}(\varphi_a)$ of all potentials $\varphi \in iL^2$ with $\text{spec}_L(\varphi) = \text{spec}_L(\varphi_a)$. By the computations above one sees that
\[
\{ |a|e^{i\alpha} | \alpha \in \mathbb{R} \} \subseteq \text{Iso}_0(\varphi_a).
\]
For $|a|$ sufficiently small, $\varphi_a$ is in the domain of the Birkhoff map introduced in Theorem 1.1 in [11]. As the $L^2$-norm is a spectral invariance it then follows that, for $|a|$ sufficiently small, all of $\text{Iso}(\varphi_a)$ is contained in this domain. According to the computations above $\varphi_a$ is a 1-gap potential. It then follows from Theorem 1.1 in [11] and its proof that $\text{Iso}(\varphi_a)$ is homeomorphic to a circle. As a consequence
\[
\text{Iso}(\varphi_a) = \text{Iso}_0(\varphi_a) = \{ |a|e^{i\alpha} | \alpha \in \mathbb{R} \}.
\]
Most likely the latter identities remain true for any $|a| < \pi$, but we have not verified this. For $|a| > \pi$, Li and McLaughlin observed that $\text{Iso}_0(\varphi_a)$ is larger than $\{ |a|e^{i\alpha} | \alpha \in \mathbb{R} \}$. Indeed, let $\pi < |a| < 2\pi$. Then $\lambda_{\pm 1} = \pm i \sqrt{|a|^2 - \pi^2}$ are periodic eigenvalues of geometric multiplicity two. In subsection 4.3 of [16], using Bäcklund transformation techniques, formulas of solutions of fNLS are presented which evolve on $\text{Iso}_0(\varphi_a)$ and depend explicitly on $x$. They are parametrized by the punctured complex plane $\mathbb{C}^* := \{ e^{\rho}e^{i\beta} \}$ with coordinates $(\rho, \beta) \in \mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z}$, whereas the angle variable $\alpha$ in $\{ |a|e^{i\alpha} | \alpha \in \mathbb{R} \}$ is proportional to the time $t$. As $t \to \pm \infty$ these solutions approach the $x$ independent solutions evolving on $\{ |a|e^{i\alpha} | \alpha \in \mathbb{R} \}$. Due to the trace formulas ([16], Section 2.4), on the orbits of these solutions, the periodic eigenvalues $\lambda_{\pm 1}$ have geometric multiplicity one.

Dirichlet spectrum of $L(\varphi_a)$: By Lemma 6.4, the characteristic function of the Dirichlet spectrum of $L(\varphi_a)$ is given by
\[
\chi_D(\lambda, \varphi_a) = \frac{\sin \kappa}{\kappa} \left( \lambda + \frac{\bar{a} - a}{2} \right).
\] (6.8)
The Dirichlet eigenvalues of $L(\varphi_a)$ are thus given by the $\lambda$’s satisfying
\[
\kappa(\lambda) = n\pi
\] (6.9)
for some $n \in \mathbb{Z} \setminus \{0\}$ or
\[
\lambda + \frac{\bar{a} - a}{2} = 0.
\] (6.10)
Note that by the definition of the Dirichlet boundary conditions, any Dirichlet eigenvalue is of geometric multiplicity one. It is convenient to list the Dirichlet eigenvalues not in lexicographic ordering, but rather use the integer $n$ in (6.9) as an index. When listed in this way, we denote the Dirichlet eigenvalues by $\hat{\mu}_n$, $n \in \mathbb{Z}$, which are defined as follows. For all $n \in \mathbb{Z}$ with $|n\pi| > |a|$ denote
\[
\hat{\mu}_n = \text{sgn}(n) \cdot \sqrt{\frac{n^2\pi^2 - |a|^2}{n^2\pi^2 - |a|^2}}.
\]
From (6.8) it follows that $\hat{\mu}_n$ has algebraic multiplicity one. For all $n \in \mathbb{Z}$ with $0 < |n\pi| < |a|$ denote

$$\hat{\mu}_n = \text{sgn}(n) \cdot i \sqrt{|a|^2 - n^2\pi^2}.$$ 

By the same arguments as in the case $|n\pi| > |a|$, the algebraic multiplicity of $\hat{\mu}_n$ is equal to one iff $\hat{\mu}_n + \frac{\bar{a} - a}{2} \neq 0$ and two otherwise. For $n = 0$ denote

$$\hat{\mu}_0 = i \text{Im}(a).$$

Again, by the same arguments, $\hat{\mu}_0$ has algebraic multiplicity equal to one if

$$\text{Im}(a) \neq 0 \text{ and } \text{Im}(a) \neq \pm \sqrt{|a|^2 - n^2\pi^2} \forall 0 < |n\pi| < |a| \text{ or } [\text{Im}(a) = 0 \text{ and } |a| \not\in \pi\mathbb{Z}_0]$$

or two if

$$\text{Im}(a) \in \left\{ \pm \sqrt{|a|^2 - n^2\pi^2} \mid 0 < |n\pi| < |a| \right\}$$

or three if

$$\text{Im}(a) = 0 \quad \text{and} \quad |a| \in \pi\mathbb{Z}_{>0}.$$ 

In the special case where $|a| = n_a\pi$ for some $n_a \in \mathbb{Z}_{>0}$ one has $\hat{\mu}_{n_a} = \hat{\mu}_{-n_a} = 0$. The algebraic multiplicity of $\hat{\mu}_{n_a}$ is two ($\text{Im}(a) \neq 0$) or three ($\text{Im}(a) = 0$). These computations lead to the following

**Corollary 6.8.** Let $a \in \mathbb{C}$. Then

(i) If $|a| < \pi$, then the Dirichlet spectrum of $L(\varphi_a)$ is simple.

(ii) If $|a| \not\in \pi\mathbb{Z}_{>0}$, then the only possible multiple Dirichlet eigenvalue is $i \text{Im}(a)$. It is at most of algebraic multiplicity two.

(iii) If $|a| \in \pi\mathbb{Z}_{>0}$, then 0 is a Dirichlet eigenvalue of algebraic multiplicity two or three.

(iv) For any $0 < n\pi < |a|$ or $n\pi > |a|$, $\hat{\mu}_n$ is a periodic eigenvalue of geometric and algebraic multiplicity two whereas for $|a| = n\pi \in \pi\mathbb{Z}_{>0}$, $\hat{\mu}_n = 0$ is a periodic eigenvalue of algebraic multiplicity 4.

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