TWO COLOURS QCD AT NONZERO CHEMICAL POTENTIAL

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We identify the Goldstone modes appropriate to the low and high density phases of SU(2) lattice gauge theory with staggered fermions. We present hybrid Monte Carlo simulation results for susceptibilities on a $6^4$ lattice at $\beta = 1.5, m = 0.05$. We specify how to implement a diquark source term in a lattice simulation and present first measurements of the lattice diquark condensate as a function of the diquark source.

The symmetry breaking pattern of SU(2) lattice gauge theory with fermions in the fundamental representation of the group is different from that of the continuum model where the symmetry breaking pattern for fermions in a pseudoreal representation is $SU(2N_f) \to Sp(2N_f)\). To identify the symmetries appropriate to the lattice model with staggered fermions we start with the kinetic term in the lattice action for a gauged isospinor doublet of staggered fermions. For clarity we consider $N = 1$ flavors.

\begin{align}
S_{\text{kin}} &= \sum_{x, \nu = 1,3} \frac{\eta_\nu(x)}{2} \left[ \bar{\chi}(x)U_\nu(x)\chi(x + \hat{\nu}) - \bar{\chi}(x)U^\dagger_\nu(x - \hat{\nu})\chi(x - \hat{\nu}) \right] \\
&\quad + \sum_x \frac{\eta_t(x)}{2} \left[ \bar{\chi}(x)e^{\mu_0}U_t(x)\chi(x + \hat{t}) - \bar{\chi}(x)e^{-\mu_0}U^\dagger_t(x - \hat{t})\chi(x - \hat{t}) \right].
\end{align}

(1)

Defining:

\begin{align}
\bar{X}_e &= (\bar{\chi}_e, -\chi^t_e \tau_2) : \ X_o = \left( \begin{array}{c} \chi_o \\ -\tau_2 \chi^t_o \end{array} \right)
\end{align}

(2)

and using $\eta_\mu(x \pm \hat{\mu}) = \eta_\mu(x)$ and $\tau_2 U_\mu \tau_2 = U'^*_\mu$ (where $\tau_2$ is a Pauli matrix) the kinetic term can be expressed in the following form:

\begin{align}
S_{\text{kin}} &= \sum_{x, \epsilon, \nu = 1,3} \frac{\eta_\nu(x)}{2} \left[ \bar{X}_e(x)U_\nu(x)X_o(x + \hat{\nu}) - \bar{X}_e(x)U^\dagger_\nu(x - \hat{\nu})X_o(x - \hat{\nu}) \right] \\
&\quad + \sum_{x, \epsilon, \nu = 1,3} \frac{\eta_t(x)}{2} \left[ \bar{X}_e(x) \left( \begin{array}{cc} e^{\mu_0} & 0 \\ 0 & e^{-\mu_0} \end{array} \right) U_t(x)X_o(x + \hat{t}) - \\
&\quad \bar{X}_e(x) \left( \begin{array}{cc} e^{-\mu_0} & 0 \\ 0 & e^{\mu_0} \end{array} \right) U^\dagger_t(x - \hat{t})X_o(x - \hat{t}) \right].
\end{align}

(3)
There are two manifest global $U(1)$ symmetries in the original action (1):

$$
\chi \mapsto e^{i\alpha} \chi \quad \bar{\chi} \mapsto \bar{\chi} e^{-i\alpha} \quad \bar{\chi} \mapsto \bar{\chi} e^{i\alpha} (4)
$$

where $\varepsilon(x) = (-1)^{x_1+x_2+x_3+x_4}$ The first invariance corresponds to baryon number conservation; the second which holds only in the chiral limit ($m \to 0$) corresponds to conservation of axial charge. Both of these are subsumed in a larger $U(2)$ symmetry which only holds for $m = \mu = 0$:

$$
X_o \mapsto V X_o \quad \bar{X} \mapsto \bar{X} V^\dagger \quad V \in U(2). (5)
$$

In a continuum approach the axial anomaly would reduce the symmetry to $SU(2)$.

For $\mu \neq 0$ the full $U(2)$ lattice symmetry is reduced to $U(1)_V \otimes U(1)_A$ and is further reduced to $U(1)_V$ for $\mu \neq 0$, $m \neq 0$. It has been proposed that $U(1)_V$ is likely to be broken spontaneously by the formation of a diquark condensate. At $\mu = 0$ the model displays spontaneous chiral symmetry breaking, with a chiral condensate of same form as the mass term $\sum_x \bar{\chi}(x)\chi(x)$.

$$
\bar{\chi} \chi = \frac{1}{2} \left[ \bar{X} e \left( \begin{array}{c} 1 \\ 1 \\
\end{array} \right) \tau_2 X e^\dagger + X e^\dagger \left( \begin{array}{c} 1 \\ 1 \\
\end{array} \right) \tau_2 X \right]. (6)
$$

This chiral condensate breaks the global symmetry (5). The residual symmetry is generated by the subgroup which leaves $\left( \begin{array}{cc} 0 & 1 \\
1 & 0 \\
\end{array} \right)$ invariant of which the most general element is $\left( \begin{array}{cc} e^{i\alpha} & \\
& e^{-i\alpha} \end{array} \right)$, which generates a $U(1)$. Thus we identify the pattern of chiral symmetry breaking appropriate to the lattice model with $N = 1$ as $U(2) \to U(1)$. Now our hybrid Monte Carlo algorithm simulates with $\det M \dagger M$ and since $\det M \dagger = \det M^* = \det \tau_2 M \tau_2 = \det M$, this describes two identical staggered fermion flavours resulting in pattern of symmetry breaking $U(4) \to O(4)$, with 10 associated Goldstone modes.

At high density, we postulate that a large Fermi surface will promote the formation of a diquark condensate. In principle many diquark wavefunctions can be written down therefore it is a dynamical question as to which condensate actually forms. We can deduce the “maximally attractive channel” for the diquark condensate by insisting that the condensate wavefunction is anti-symmetric under exchange of fields and by assuming the condensate is gauge invariant, invariant under lattice parity and as local as possible in the $\chi$ fields. A local condensate $qq_2$ which satisfies these conditions is:

$$
qq_2 = \frac{1}{2} \left[ \chi^\dagger(x) \tau_2 \chi(x) + \bar{\chi}(x) \tau_2 \bar{\chi}^\dagger(x) \right]. (7)
$$
In terms of the $X, \bar{X}$ fields the **diquark condensate** is written

$$qq_2 = \frac{1}{2} \left[ \bar{X}_e \begin{pmatrix} 1 & \tau_2 \bar{X}_e^\text{tr} + X_\nu^\text{tr} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \tau_2 X_\nu \right]. \quad (8)$$

The diquark condensate (8) can be obtained from the chiral condensate (6) by an explicit global $U(2)$ rotation:

$$V = i \sqrt{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \quad (9)$$

Therefore in the limit $m = 0, \mu = 0$ the pattern of symmetry breaking remains $U(2) \to U(1)$: as $\mu$ increases the condensate simply rotates from a chiral $\bar{q}q$ to a diquark $qq$.

The full set of Goldstone modes (again for the simplest case $N = 1$) can be found by considering infinitesimal rotations of either the chiral condensate (6) in the low density phase or the diquark condensate (8) in the high density phase by $V_\delta = 1 + i \delta \lambda$, with $\lambda$ one of the $U(2)$ generators $\{1, \tau_i\}$, and identifying the mode as the coefficient of $O(\delta)$. In the chiral limit and for $\mu = 0$ we expect 3 massless Goldstone modes (since $\text{dim } U(2) = 4$). Since the full $U(2)$ symmetry of the action is reduced for $\mu \neq 0$ only two of the four generators correspond to Goldstone modes i.e. the generators appropriate to $U(1)_V$ and $U(1)_A$ which are $\tau_3$ and 1 respectively. In the low density phase the rotation generated by $1$ gives the pion: $\bar{\chi} \varepsilon \chi$, while the rotation generated by $\tau_3$ leaves the condensate invariant. Rotation of the diquark condensate by 1 gives a pseudoscalar diquark: $\chi^\text{tr} \tau_2 \varepsilon \chi + \bar{\chi} \bar{\tau}_2 \bar{\varepsilon} \bar{\chi}^\text{tr}$, which is pseudo-Goldstone for $\mu \neq 0$ and rotation by $\tau_3$ gives a scalar diquark: $\chi^\text{tr} \tau_2 \chi - \bar{\chi} \bar{\tau}_2 \bar{\chi}^\text{tr}$, which remains an exact Goldstone mode even for $m \neq 0$. The fact that the low and high density regimes are characterised by different numbers of massless modes indicates that they should be separated by a true phase transition at some $\mu = \mu_c$. In general we expect $\mu_c = m_{b}/N_c$, where $m_{b}$ is the mass of the lightest baryon and $N_c$ is the number of colours. The $SU(2)$ theory is a special case because we expect $\mu_c = m_{\pi}/2$ i.e. there is a Goldstone baryon in the theory due to the $SU(2)$ symmetry which relates quark to anti-quark. In this respect the model is crucially different from QCD where we expect $\mu_c = m_{\text{proton}}/3$ yet find $\mu_c = m_{\pi}/2$ due to pathologies in the hybrid Monte Carlo simulations for $\mu \neq 0$.

We can formally introduce a diquark source term into the fermionic sector of the partition function as follows:

$$Z_{\text{term.}} = \int d\bar{\psi}d\psi \exp \left( \bar{\psi} M \psi + \bar{\psi} J \psi + \bar{\psi} \bar{J} \bar{\psi} \right) \quad (10)$$
where $M$ is the fermion matrix while $J$ and $\bar{J}$ are diquark and anti-diquark source terms respectively. It is necessary to recast this equation in order to perform the integration over the fermion fields. To do so we define a new 2 component Grassmann field: $\phi \equiv \begin{pmatrix} \bar{\psi} \\ \psi \end{pmatrix}$ and then reconstruct $Z_{\text{ferm}}$ in terms of a “double matrix”

$$Z_{\text{ferm}} = \int d\phi \bar{\psi} \psi \left( \begin{array}{cc} \frac{\bar{J}}{M} & \frac{M}{\bar{J}} \\ \frac{M}{\bar{J}} & \frac{\bar{J}}{M} \end{array} \right) \left( \begin{array}{c} \bar{\psi} \\ \psi \end{array} \right) = \left\{ \text{det} \left( \begin{array}{cc} \frac{\bar{J}}{M} & \frac{M}{\bar{J}} \\ \frac{M}{\bar{J}} & \frac{\bar{J}}{M} \end{array} \right) \right\}^{1/2} \quad (11)$$

where we have used $\int d\phi \exp(\phi A\phi) = \sqrt{\text{det} A} \equiv \text{Pfaffian}(A)$. Note that we require an antisymmetric matrix therefore we chose $J = \bar{J} = j\tau_2$ with $j$ a real number.

Making use of the property $\ln \text{det} A = Tr \ln A$, on the double matrix leads to the lattice diquark condensate:

$$\langle \psi \bar{\psi} \rangle = \lim_{j \to 0} Tr \begin{pmatrix} \frac{j\tau_2}{M} & \frac{M}{j\tau_2} \\ \frac{M}{j\tau_2} & \frac{j\tau_2}{M} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \tau_2 \end{pmatrix} \right\} \quad (12)$$

In Fig. 1 we show the diquark condensate as a function of $j$ and $\mu$. Polynomial fits to these curves were used to obtain the $j \to 0$ limit plotted alongside the standard observables in Fig. 2. The sharp rise in the diquark condensate at $\mu \simeq 0.4$ coincides with the downward jump in the average plaquette while the chiral condensate decreases smoothly as $\mu$ is increased.

Measurements of the susceptibilities (Figs. 3,4) are strongly suggestive of a phase transition at $\mu_c = 0.4$. This is consistent with the conclusion from the observables in Fig. 2. We shall refer to $\mu < m_\pi/2 \simeq 0.4$ as the low density phase and $\mu > m_\pi/2$ as the high density phase. We find, as
expected, that at $\mu = 0$ the scalar diquark is degenerate with the pion while the pseudoscalar diquark is degenerate with the scalar meson. The pseudoscalar diquark which is heavy in the low density phase becomes light in the high density phase whereas the pion which is a pseudo-Goldstone in the low density phase becomes heavy in dense (chirally symmetric) phase. As seen in Fig. 4 susceptibilities of scalar and pseudoscalar diquarks become very large in the dense phase. In the dense phase we observe that the scalar diquark and the pseudoscalar diquark are light states while the connected contributions to the scalar imply that it is comparatively heavy. In our symmetries analysis we identified the scalar diquark as an exactly massless Goldstone mode in the dense phase while we expected the pseudoscalar diquark to be a pseudo-Goldstone mode for $m \neq 0$. The scalar meson is not expected to be a Goldstone mode in the high density phase. Although the pion and scalar meson appear to be approximately degenerate in the high density phase this degeneracy may be broken when the disconnected contributions to the scalar are taken into account.

References

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