Quantum phase transitions and collapse of the Mott gap in the $d = 1 + \epsilon$ dimensional half-filled Hubbard model

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(March 24, 2022)

We study the low-energy asymptotics of the half-filled Hubbard model with a circular Fermi surface in $d = 1 + \epsilon$ continuous dimensions, based on the one-loop renormalization-group (RG) method. Peculiarity of the $d = 1 + \epsilon$ dimensions is incorporated through the mathematical structure of the elementary particle-particle (PP) and particle-hole (PH) loops: infrared logarithmic singularity of the PH loop is smeared for $\epsilon > 0$. The RG flows indicate that a quantum phase transition (QPT) from a metallic phase to the Mott insulator phase occurs at a finite on-site Coulomb repulsion $U$ for $\epsilon > 0$. We also discuss effects of randomness.

I. INTRODUCTION

Correlation-driven metal-insulator transition (MIT) in the Hubbard model has been a basic problem in condensed matter physics.\cite{1} Central to this issue is the problem of quantum fluctuation controlled by the on-site Coulomb repulsion $U$, the band width $W$, and the carrier concentration $n$. In the half-filled ($n = 1$) case, the zero temperature band width control MIT occurs at a critical ratio of $\bar{U} = U/W$, $\bar{U}_c$. The exact solution of the Hubbard model is available only in $d = 1$ dimension, where the half-filled ground state is always an insulator with a finite charge excitation gap (Mott gap) at $\bar{U} > \bar{U}_c = 0$.\cite{2} In $d = 2$, it is believed that there exists a finite $\bar{U}_c$ except in the case of the perfect nesting where $\bar{U}_c = 0$.\cite{3} In the case of the $d = 2$ half-filled Hubbard model with the nearest neighbor and the second nearest neighbor hoppings, a Monte Carlo simulation,\cite{4} a Hartree-Fock approximation, and the Gutzwiller approximation\cite{5} all indicate that $\bar{U}_c$ is finite. In $d = \infty$, the dynamical mean field approach implies that the quasiparticle spectral weight in the vicinity of the Fermi surface vanishes continuously as $U$ approaches a critical value $\bar{U}_c \sim 1$ from below.\cite{6} The filling control MIT, which occurs as the carrier concentration approaches half filling, has also been extensively studied.\cite{7}

In the $d = 1$ Hubbard model at half filling, low-energy asymptotics is also well understood in terms of the renormalization-group (RG) flow of two-particle scattering strengths, $g_1$, $g_2$, and $g_3$, which correspond to the backward, forward and $2k_F$-umklapp scatterings, respectively.\cite{8,9} The RG scheme in $d = 1$ is based on infrared logarithmic singularities of elementary particle-particle (PP) and particle-hole (PH) loops which have the same magnitude and opposite signs. In this context, the source of the Mott gap is the umklapp scattering which becomes a relevant perturbation for $U > 0$. The RG flow also indicates that the charge stiffness reaches zero during the renormalization process and the system becomes an insulator. Thus, the RG-based scenario is in perfect agreement with the exact solution.

Dimensionality effects on the $d = 1$ Mott insulator phase were phenomenologically treated by cutting off either of the PP or the PH loop below some characteristic temperature.\cite{10,11} In the case of weakly-coupled chains, dimensional crossovers caused by an interchain hopping $t_\perp$ have been studied by treating $t_\perp$ perturbatively and assuming that the scaling procedure in the one-dimensional regime at high energy scales ($\omega \gg t_\perp$) remains valid down to the crossover energy scales.\cite{12,13} Recently, the filling control MIT in coupled Hubbard chains with infinitely large coordination numbers was also studied.\cite{14} However, these attempts have not clarified the dimensionality effects on the Mott gap, because feedback effects of the interchain processes on the Mott gap have been missing.

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The RG method is straightforwardly extended to the case of \( d = 1 + \epsilon \) \((0 < \epsilon \ll 1)\) dimensions. In this case, only the PP loop remains logarithmically singular, while the PH loop is smeared for \( \epsilon > 0 \). By taking this fact into account from the beginning of renormalization processes, the feedback effects on the Mott gap may be incorporated. In this paper, by using the one-loop RG method, we examine dimensionality effects on the RG flow of the umklapp process and discuss possible QPTs in the Hubbard model with a circular Fermi surface in \( d = 1 + \epsilon \) dimensions.

If randomness exists, a QPT between the Mott insulator and a randomness-driven Anderson insulator might arise. In the case of \( d = 1 \) at half filling, based on the RG method, Fujimoto and Kawakami \cite{FujimotoKawakami2015} found that sufficiently strong random forward scattering destroys the Mott gap. Recently, Ohtsuka \textit{et al.} \cite{Ohtsuka2020} studied the half-filled Hubbard model containing site randomness by using the quantum Monte Carlo technique and found that the strong randomness destroys the Mott gap. In this paper, we also discuss the QPTs in the half-filled random Hubbard model in \( d = 1 + \epsilon \) dimensions.

This paper is organized as follows: In Sec. II we introduce the \( g \)-ology effective action, derive the one-loop RG equations and discuss possible QPTs in the absence of randomness. The effects of randomness are discussed in Sec. III, followed by concluding remarks in Sec. IV.

II. HALF-FILLED HUBBARD MODEL IN \( D = 1 + \epsilon \) DIMENSIONS

In this section, we study interplay of electron correlation and dimensionality effects in the half-filled Hubbard model with a circular Fermi surface in \( d = 1 + \epsilon \) dimensions.

A. Effective action

We start with the effective action,

\[
S_{\text{Hubbard}} = \sum_{\sigma} \int_{-\infty}^{\infty} \frac{d\varepsilon}{2\pi} \int \frac{d^d k}{(2\pi)^d} G^{-1}(k, i\varepsilon)c_{\sigma}^*(K)c_{\sigma}(K) - \frac{\pi v_F}{2} \prod_{\sigma, \sigma'} \int_{-\infty}^{\infty} \frac{d\varepsilon_1}{2\pi} \int \frac{d^d k_1}{(2\pi)^d} \delta(\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)\delta(k_4 + k_3 - k_2 - k_1 - G) g_{K_1, K_2, K_3}^{\sigma, \sigma'} c_{\sigma}^*(K_4)c_{\sigma'}(K_3)c_{\sigma'}(K_2)c_{\sigma}(K_1),
\]

(1)

where \( c_{\sigma}^*(K) \) and \( c_{\sigma}(K) \) are the Grassmann variables representing the electron with the spin \( \sigma \) and \( K = (k, \varepsilon) \) with \( k \) and \( \varepsilon \) being a \( d \)-dimensional momentum and a Fermion thermal frequency, respectively. The non-interacting one-particle propagator is given by

\[
G^{-1}(k, i\varepsilon) = i\varepsilon - \xi(k).
\]

(2)

The one-particle dispersion is \( \xi(k) = (k^2 - k_F^2)/2m \) and the Fermi surface is a \( d \)-dimensional sphere \(| k | = k_F \). Since we consider only the energy scale which is much smaller than the Fermi energy, we linearize one-particle dispersion as

\[
\xi(k) = v_F(|k| - k_F),
\]

(3)

where \( v_F \) is the Fermi velocity. The band width cut off \( E_0 \) is introduced and the one-particle processes are restricted to \(-E_0 \leq \xi(k) \leq E_0\). In the two-particle scattering part of \( S_{\text{Hubbard}}, \ |G| = 0 \) and \(|G| = 4k_F \) for the normal and the umklapp processes, respectively. It is reasonable to assume that the two-particle scattering processes which enter the RG equations in \( d = 1 + \epsilon \) are, as in the case of \( d = 1 \), the backward scattering with large momentum transfer \(|k_3 - k_2| \sim 2k_F\), the forward scattering with small momentum transfer \(|k_3 - k_2| \sim 0\), and the \( 2k_F \)-umklapp scattering characterized by \(|k_4 + k_3 - k_2 - k_1| \sim 4k_F \). We represent the corresponding scattering vertices in Figs. 1(a)-(c). Dimensionless scattering strengths for these processes are denoted by \( g_1 \), \( g_2 \), and \( g_3 \).
respectively. Unrenormalized scattering strengths are related to the on-site Coulomb repulsion, $U$, as

$$g_{1:0} = g_{2:0} = g_{3:0} = U/\pi v_F \equiv \bar{U}. \quad (4)$$

### B. The elementary particle-particle and particle-hole loops

The peculiarity of $d = 1 + \epsilon$ dimensions is incorporated only through the integration measure, $\int d^d\mathbf{k}/(2\pi)^d$. For our purpose here, it is sufficient to integrate over $k = |\mathbf{k}|$ and the angle $\theta$ spanned by $\mathbf{k}$ and another fixed momentum. Then we can use

$$\int \frac{d^d\mathbf{k}}{(2\pi)^d} (\cdots) = S_{d-1} \int k^{d-1} dk \int_0^\pi d\theta (\sin \theta)^{d-2} (\cdots), \quad (5)$$

where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the $d$-dimensional unit sphere.

As is well known, in any dimension, the real part of the elementary PP loop [Fig. 1(d)] at the zero total momentum, $\Delta_0(\omega) = \int d^d\mathbf{k} \frac{\Theta(\xi_{-\mathbf{k}}) - \Theta(-\xi_{\mathbf{k}})}{(2\pi)^d \omega - \xi_{-\mathbf{k}} - \xi_{\mathbf{k}} + i0^+}$, with $\Theta(x)$ being the step function, exhibits an infrared logarithmic singularity of the form $\frac{1}{\epsilon}$. However, in $d = 1$, we obtain

$$\Re \Delta_0(\omega) \sim -\frac{1}{2\pi v_F} \log(\omega/E_0), \quad (7)$$

In $d = 1 + \epsilon$, it takes the form

$$\Re \Delta_0(\omega) \sim -\frac{1}{2\pi v_F} \log(\omega/E_0), \quad (8)$$

which exactly reproduces the result in $d = 1$.

On the other hand, the real part of the elementary PH loop at $2k_F$ momentum transfer [Fig. 1(e)],

$$\Pi_{2k_F}(\omega) = \int d^d\mathbf{k} \frac{\Theta(\xi_{\mathbf{k}+Q}) - \Theta(\xi_{\mathbf{k}})}{(2\pi)^d \omega - \xi_{\mathbf{k}+Q} - \xi_{\mathbf{k}} + i0^+}, \quad (9)$$

with $|Q| = 2k_F$, unlike the case of $d = 1$, no longer exhibits an infrared singularity for $d > 1$ and in $d = 1 + \epsilon$, takes the form

$$\Re \Pi_{2k_F}(\omega) \sim -\frac{1}{2\pi v_F} \left[ \frac{\tilde{\omega}^\epsilon/2}{\epsilon/2} + C_\epsilon \right], \quad (10)$$

where $\tilde{\omega} = \omega/2v_F k_F$ and $C_\epsilon$ is a constant independent of $\omega$. Although this form has already been suggested in Ref. [16], we confirm, in the appendix, that the $\omega$-dependent term in (10) is uniquely determined.

### C. One-loop renormalization

One-loop renormalization of the scattering strengths comes from the vertex correction diagrams represented in Figs. 2(a) and 2(b) for the normal $[g_1, g_2]$ and umklapp $[g_3]$ processes, respectively. The renormalized scattering strength, $g'_1$, $g'_2$, and $g'_3$, are thus given by
\[ g'_1 = g_1 + g_1 g_2 \ln \frac{\omega}{E_0} - (g_2 - g_1)g_1 \pi(\omega), \]  
\[ g'_2 = g_2 + \frac{1}{2}(g_1^2 + g_2^2) \ln \frac{\omega}{E_0} - \frac{1}{2}(g_2^2 + g_3^2)\pi(\omega), \]  
\[ g'_3 = g_3 + g_3(g_1 - 2g_2)\pi(\omega), \]  
where \( \pi(\omega) = 2\pi v_F \mathcal{R} \Pi_{2k_F}(\omega) \). Fig. 2(a-1) contains the PP loop and all other diagrams contain the PH loop. In particular, renormalization of the umklapp process comes from the PH loop only. By differentiating equations (11)-(13) with respect to the scaling parameter \( l = \ln(\omega/E_0) \), we obtain the RG equations

\[ \frac{dg_1}{dl} = -g_1 g_2 + (g_2 - g_1)g_1 \lambda_l, \]  
\[ \frac{dg_2}{dl} = -(g_1^2 + g_2^2)/2 + (g_2^2 + g_3^2)\lambda_l/2, \]  
\[ \frac{dg_3}{dl} = -g_3(g_1 - 2g_2)\lambda_l. \]  

The PH loop gives rise to the smooth cutoff\[ ]

\[ \lambda_l = \left| \frac{\partial}{\partial l} \pi(\omega) \right| = \left( \frac{E_0}{2k_Fv_F} \right)^{\epsilon/2} \exp[-\epsilon l/2] \sim \exp[-\epsilon l/2], \]  
where the ratio of the two cut off energy scales, \( E_0/2k_Fv_F \), is of the order of unity. In the absence of the umklapp process, the RG equations obtained here reproduces those in Ref. [16].

**D. Renormalization-group flow and QPT**

In the \( d = 1 \) half-filled Hubbard model, the charge degrees of freedom are governed by the combination of \( (g_3, G) = (g_1 - 2g_2) \) with the flow lines \((G - \text{const.})^2 - g_3^2 = \text{const}\). For any finite \( U > 0 \), they are scaled to \( (g_3^2 = \infty, G^2 = -\infty) \), which implies the Mott gap opens due to the relevant umklapp scattering. This RG flow also indicates that the charge stiffness \( K_\rho = \sqrt{(1 + G)/(1 - G)} \) reaches zero during the renormalization process and the system becomes an insulator. To complement this scenario, it is useful to map the charge sector of the \( d = 1 \) half-filled Hubbard model onto the \((1+1)\)-dimensional sine-Gordon model by using the bosonization technique. The sine-Gordon action is of the form\[ ]

\[ S_{SG} = \int d^2r \left\{ \frac{u_\rho^2}{2} (\nabla r \Psi_\rho(r))^2 - \frac{2g_3}{(\pi\alpha)^2} \cos[\sqrt{8\pi K_\rho} \Psi_\rho(r)] \right\}, \]  
where \( r = (x, \tau) \) represents \((1+1)\)-dimensional space-time coordinates, \( \Psi_\rho(r) \) represents a charge boson (holon) field, \( u_\rho \) is the holon velocity, and \( \alpha \) is a short distance cut off. By applying the RG method directory to the sine-Gordon model\[ ] we obtain the RG equations

\[ \frac{dg_3}{dl} = 2(1 - K_\rho)g_3, \]  
\[ \frac{dK_\rho}{dl} = -\frac{8\pi^2}{\Xi_l} g_3^3 K_\rho^3, \]  
where \( a = \int_0^1 d\rho \rho^3 J_0(\rho) \) and \( \Xi_l = \Xi_0 e^{-l} \) is the space-time cutoff. We thus see that \( g_3 \) becomes relevant for the initial condition \( K_{\rho,0} < 1 \) [corresponding to \( U > 0 \) for the Hubbard model] and then, accordingly, the charge stiffness \( K_\rho \) is scaled to zero.

Now we consider the case of \( d = 1 + \epsilon \) dimensions. In Figs. 3(a) and (b) are shown the RG flows for \( U = U/\pi v_F = 0.08 \) and 0.04, respectively, in \( d = 1.1 \). In Fig. 3(c), we show the RG flows in
terms of $g_3$ and $G = g_1 - 2g_2$ for various $\tilde{U}$ in $d = 1.1$. It is found that there exists a critical value of $\tilde{U}$, $\tilde{U}_c = 0.0588$. For $\tilde{U} > \tilde{U}_c$, the RG flows exhibit runaway trajectories toward $(g_1^* = \infty, G^* = -\infty)$ [shaded region in Fig. 3(c)], which implies the Mott gap opens at the low-energy limit just as in the case of $d = 1$. The initial values of $g_3$ and $G$ at $l = 0$ correspond to the points on the line, $G = -g_3$, represented by a broken line in Fig. 3(c). In the $d = 1$ half-filled Hubbard model, $g_3$ and $G$ flow along this line (denoted by “1D Hubbard”) for any $\tilde{U} > 0$.

On the other hand, the RG flows approach the fixed points, $(g_3^* = \text{const.}, G^* = 0)$ for $\tilde{U} < \tilde{U}_c$. Marginal behavior of $g_3$ is in accordance with $G = g_1 - 2g_2 \to 0$ [see equation (17)] as $l \to \infty$. The smooth cutoff, $\lambda_l = \exp[-c\ell/2]$, in eq. (17) causes suppression of $g_3$ during the renormalization process. This suppression becomes more conspicuous for larger $\epsilon$. The fixed point $G^* = 0$, corresponding to the non-interacting value of the charge stiffness, $K^*_c = 1$, implies that the Mott gap collapses and the system becomes metallic at the low-energy limit. Thus a QPT from the metallic phase to the Mott insulator phase may occur at $\tilde{U} = \tilde{U}_c$.

Within the RG-based scheme, it remains debatable how the marginal behavior of $g_3$ for $\tilde{U} > \tilde{U}_c$ modifies the ground state property. Regarding this point, recently the density matrix renormalization group (DMRG) method was applied to three Hubbard chains coupled via the interchain one-particle hopping $t_{\perp}$. As a result, it was found that the Mott gap decreases as $t_{\perp}$ increases. This numerical result strongly supports the RG-based view given here.

We here give qualitative discussion on the magnitude of the Mott gap. There is no tractable method to quantitatively obtain the magnitude of the Mott gap in $d > 1$. However, the magnitude of the Mott gap is qualitatively given by the energy scale, $\omega_\text{gap} = E_0 e^{-K^*_c \infty}$, at which the umklapp scattering strength exceeds unity, $g_3 = 1$ [see Fig. 3(a)]. To see how $\omega_\text{gap}$ reproduces the Mott gap, we compare the $\tilde{U}$-dependence of the exact Mott gap in $d = 1$ with that of $\omega_\text{gap}$ in $d = 1$. There is arbitrariness in specification of the linearized bandwidth $E_0$. In Fig. 4, we show the case for $E_0 = 0.4v_F$, where $\omega_\text{gap}$ reproduces $\Delta_{\text{exact}}(\tilde{U})$ well at least for a weak $\tilde{U}$ where the weak coupling RG scheme is valid.

In Fig. 5, we show a low-energy asymptotic phase diagram. We also show $\omega_\text{gap}/E_0$ as a function of $d$ and $\tilde{U}$. For a fixed dimension, the low-energy asymptotic phase corresponds to a metal and the Mott insulator phases for $\tilde{U} < \tilde{U}_c$ and $\tilde{U} > \tilde{U}_c$, respectively. The critical value, $\tilde{U}_c$, increases as $d$ increases, suggesting a larger $\tilde{U}$ is required for the Mott gap to open as the dimension increases. Accordingly, for a fixed $\tilde{U}$, $\omega_\text{gap}/E_0$ decreases with increasing dimensions and disappears at some critical dimension, $d_c$ [for example, $d_c = 1.345$ for $\tilde{U} = 0.2$].

Apparently $\omega_\text{gap}/E_0$ grows continuously at $\tilde{U} = \tilde{U}_c$ for $d < 1.4$. However, the transition is always discontinuous by the following reason. As is seen from Fig. 3(c), $g_3$ exceeds unity during the renormalization process and gives a finite value of $\omega_\text{gap}/E_0$ even for $\tilde{U} < \tilde{U}_c$. Nevertheless, $\omega_\text{gap}/E_0$ has no physical interpretation for $\tilde{U} < \tilde{U}_c$, since the charge stiffness is scaled to the non-interacting value. Therefore, $\omega_\text{gap}/E_0$ grows discontinuously at $\tilde{U} = \tilde{U}_c$. The magnitude of discontinuity at the transition point becomes more conspicuous as $d$ increases. At present, it is not clear whether the discontinuity is an artifact of the one-loop RG method. To elaborate on this point would require the two-loop RG analysis, which is too involved a subject to be treated here.

We here comment on the relevance of the present study to the $d = 2$ lattice Hubbard model. In this paper, we have studied only the case with a circular Fermi surface. Accordingly, our results are not smoothly connected to the $d = 2$ square lattice Hubbard model. In particular, in the case of the perfect nesting, the van-Hove singularity at $(\pm \pi, \pm \pi)$ points gives rise to the “log-square” singularity of the elementary PH loop at the momentum transfer $Q = (\pi, \pi)$. Then the Hartree-Fock solution gives the gap $\Delta \sim t e^{-2\pi\sqrt{\nu}/U}$, which indicates that the ground state of the system is always an insulator for a finite $U$. This conclusion has been strongly supported by numerical studies. However possibility of the MIT at a finite $U$ in the half-filled lattice Hubbard model with a Fermi surface of various geometries has been open to question. In the case of the $d = 2$ half-filled Hubbard model with the nearest neighbor and the second nearest neighbor hopping integrals ($t$ and $t'$, respectively), a Monte Carlo simulation provides a Hartree-Fock approximation and the
Gutzwiller approximation including the antiferromagnetism\(^\text{[3]}\) give finite critical values, \(U/t = 2.5, 2.064, \) and \(3.902,\) respectively, for \(t'/t = 0.2.\) These results are consistent with the present findings that, in the case of a circular Fermi surface, the MIT occurs at a finite \(U\) for \(\epsilon > 0.\)

### III. EFFECTS OF RANDOMNESS

In this section, we study interplay of electron correlation, randomness and dimensionality effects in the \(d = 1 + \epsilon\) dimensional random Hubbard model at half filling\(^\text{[25]}\).

#### A. Effective action for quenched randomness

The action for the scattering processes by the random potentials,

\[
S_{\text{random}} = -\sum_{\sigma} \left( \int d^d x \int d\tau v(x) c^\dagger_{\sigma}(x, \tau) c_{\sigma}(x, \tau) \right)
\]

is added to the Hubbard action \([1]\). Here, \(c_{\sigma}(x, \tau) = \sum_k e^{ik \cdot x} c_{\sigma}(k, \tau)\) with \(\tau\) being an imaginary time and \(v(x) = \sum_q e^{iq \cdot x} v(q)\) is a random potential at the position \(x.\) We assume that the random scattering processes which enter the RG equations in \(d = 1 + \epsilon\) are, as in the case of \(d = 1,\)\(^\text{[26]}\) characterized by real and complex random fields \(\eta(x)\) and \(\xi(x)\) for the forward scattering with small momentum transfer \(|q| \sim 0\) and the backward scattering with large momentum transfer \(|q| \sim 2k_F\), respectively, due to the random potential. The random potentials are assumed to be governed by Gaussian distributions,

\[
P_\eta \propto \exp \left[ -D_\eta \int d^d x \eta(x)^2 \right],
\]

\[
P_\xi \propto \exp \left[ -D_\xi \int d^d x \xi(x) \xi^*(x) \right],
\]

which lead to

\[
\langle \eta(x) \eta(y) \rangle_{\text{random}} = \frac{D_\eta}{2} \delta(x - y),
\]

\[
\langle \xi(x) \xi^*(y) \rangle_{\text{random}} = \frac{D_\xi}{2} \delta(x - y),
\]

where \(D_\eta = (\pi N_F \tau_\eta)^{-1}\) and \(D_\xi = (\pi N_F \tau_\xi)^{-1}\) with \(\tau_\eta, \xi\) and \(N_F\) being the elastic-scattering mean free time and the one-particle density of states, respectively.

We consider the quenched randomness where averaging the free energy is accomplished by means of the replica trick which is based on the identity

\[
\ln Z = \lim_{N \to 0} \frac{Z^N - 1}{N}.
\]

We introduce \(N\) identical replicas of the system labeled by the index \(\alpha.\) Then, by using the path-integral representation of the partition function, we have

\[
Z^N = \int \prod_{\alpha=1}^N Dc^\alpha \cdot Dc^\alpha \exp \left[ \sum_{\alpha=1}^N S^\alpha \right],
\]
where \( S^{\alpha} = S^{\alpha}_{\text{Hubbard}} + S^{\alpha}_{\text{random}} \) is the total action and \( D \) symbolizes the measure over the fermionic Grassmann variables \( c^{\alpha*} \) and \( c^{\alpha} \) depending on a replica index \( \alpha \). The replica trick consists of performing the Gaussian ensemble average \( \langle Z^{N} \rangle_{\text{random}} \) for integer \( N \), continuing the result analytically to real \( N \), and taking the limit \( N \to 0 \). We thus obtain

\[
\langle Z^{N} \rangle_{\text{random}} = \int dq P_{q} \int d\xi d\xi^{*} \int \prod_{\alpha=1}^{N} Dc^{\alpha*} Dc^{\alpha} \exp \left[ \sum_{\alpha=1}^{N} S^{\alpha} \right] = \int \prod_{\alpha=1}^{N} Dc^{\alpha*} Dc^{\alpha} \exp [\sum_{\alpha=1}^{N} \tilde{S}^{\alpha}],
\]

where the random scattering parts contained in \( \tilde{S}^{\alpha} \) are given by

\[
\frac{D_{\xi}}{4} \sum_{\beta=1}^{\sigma} \sum \int d\tau_{1} \int d\tau_{2} \int \frac{d^{d}k}{(2\pi)^{d}} \int \frac{d^{d}k'}{(2\pi)^{d}} c^{\alpha*}_{\sigma}(k, \tau_{1}) c^{\alpha}_{\sigma}(k, \tau_{1}) c^{\beta*}_{\sigma}(k', \tau_{2}) c^{\beta}_{\sigma}(k', \tau_{2})
\]

\[
+ \frac{D_{\xi}}{4} \sum_{\beta=1}^{\sigma} \sum \int d\tau_{1} \int d\tau_{2} \int \frac{d^{d}k}{(2\pi)^{d}} \int \frac{d^{d}k'}{(2\pi)^{d}} c^{\alpha*}_{\sigma}(k + Q, \tau_{1}) c^{\alpha}_{\sigma}(k, \tau_{1}) c^{\beta*}_{\sigma}(k' - Q, \tau_{2}) c^{\beta}_{\sigma}(k', \tau_{2}),
\]

where \( |Q| \sim 2k_{F} \).

We here change imaginary time variables \( \tau_{1} \) and \( \tau_{2} \) into \( \Delta \tau = \tau_{1} - \tau_{2} \) and \( \tau = (\tau_{1} + \tau_{2})/2 \). In the integration over \( \Delta \tau \), we introduce a short distance cutoff \( \Lambda \) and keep only the region, \( v_{F} | \Delta \tau | \leq \Lambda \), which couples the two-particle scattering processes and contribute to the RG equations. Then, the random forward and backward scattering parts are written as

\[
\tilde{S}^{\alpha}_{\text{random}} \sim \frac{D_{\xi} \Lambda}{2v_{F}^{2}} \sum_{\sigma, \sigma'} \sum_{\beta=1}^{\sigma} \int d\tau_{1} \int d\tau_{2} \int \frac{d^{d}k}{(2\pi)^{d}} \int \frac{d^{d}k'}{(2\pi)^{d}} \prod_{i=1}^{A} \int_{-\infty}^{\infty} d\varepsilon_{i} \delta(\varepsilon_{4} + \varepsilon_{3} - \varepsilon_{2} - \varepsilon_{1})
\]

\[
\times c^{\alpha*}_{\sigma}(k, \varepsilon_{4}) c^{\alpha}_{\sigma}(k', \varepsilon_{3}) c^{\beta*}_{\sigma}(k', \varepsilon_{2}) c^{\beta}_{\sigma}(k, \varepsilon_{1})
\]

\[
- \frac{D_{\xi} \Lambda}{2v_{F}^{2}} \sum_{\sigma, \sigma'} \sum_{\beta=1}^{\sigma} \int d\tau_{1} \int d\tau_{2} \int \frac{d^{d}k}{(2\pi)^{d}} \int \frac{d^{d}k'}{(2\pi)^{d}} \prod_{i=1}^{A} \int_{-\infty}^{\infty} d\varepsilon_{i} \delta(\varepsilon_{4} + \varepsilon_{3} - \varepsilon_{2} - \varepsilon_{1})
\]

\[
\times c^{\alpha*}_{\sigma}(k + Q, \varepsilon_{4}) c^{\alpha}_{\sigma}(k', \varepsilon_{3}) c^{\beta*}_{\sigma}(k', \varepsilon_{2}) c^{\beta}_{\sigma}(k, \varepsilon_{1}),
\]

where \( c_{\sigma}^{\alpha}(x, \varepsilon) = T^{1/2} \sum_{k} e^{i(kx - \varepsilon\tau)} c_{\sigma}^{\alpha}(k, \varepsilon) \). The actions for the random scatterings inside the same replica \( [\beta = \alpha] \) are absorbed into the two-particle backward and forward scatterings by introducing

\[
\hat{g}_{1} = g_{1} - \hat{D}_{\xi},
\]

\[
\hat{g}_{2} = g_{2} - \hat{D}_{\eta},
\]

where \( \hat{D}_{\xi} = D_{\xi} \Lambda / \pi v_{F}^{2} \) and \( \hat{D}_{\eta} = D_{\eta} \Lambda / \pi v_{F}^{2} \). Now, in addition to the two-particle scattering vertices [Figs. 1(a)-(c)], there appear inter-replica vertices as shown in Figs. 6(a) and (b).

### B. One-loop renormalization

We obtain the vertex correction diagrams for \( \hat{g}_{1} \), \( \hat{g}_{2} \) and \( \hat{g}_{3} \) merely by replacing \( g_{1} \) and \( g_{2} \) in Fig. 2 with \( \hat{g}_{1} \) and \( \hat{g}_{2} \), respectively. However, we must avoid counting the diagram as shown in Fig. 6(c) which apparently renormalizes \( \hat{g}_{1} \), but vanishes in the replica limit, \( N \to 0 \), since summation over the replica indices \( \gamma = 1, 2, ..., N \) of the inner loop yields \( N \). Keeping this point in mind, we obtain the renormalized scattering strength, \( \hat{g}_{1} \), \( \hat{g}_{2} \), and \( \hat{g}_{3} \), which are analogous to equations (11)-(13) and given by

\[
\hat{g}_{1} = \hat{g}_{1} + \hat{g}_{2} \ln \frac{\omega}{E_{0}} - \left[ (\hat{g}_{2} - \hat{g}_{1}) \hat{g}_{1} + \hat{D}_{\xi}^{2} \right] \pi(\omega),
\]

\[
\hat{g}_{2} = \hat{g}_{2} + \frac{1}{2} (\hat{g}_{1}^{2} + \hat{g}_{2}^{2}) \ln \frac{\omega}{E_{0}} - \frac{1}{2} (\hat{g}_{2}^{2} + \hat{g}_{1}^{2}) \pi(\omega),
\]

\[
\hat{g}_{3} = g_{3} + g_{3} (\hat{g}_{1} - 2\hat{g}_{2}) \pi(\omega).
\]
Renormalization of the inter-replica vertices for \( \beta \neq \alpha \) in Figs. 6(a) and (b) comes from the vertex correction diagrams as shown in Figs. 7(a) and (b), respectively. We must avoid counting the diagram Fig. 7(b-4) which vanishes in the replica limit. We obtain

\[
\begin{align*}
D'_\eta &= \tilde{D}_\eta - \frac{1}{2} \tilde{D}_\eta^2 \ln \frac{\omega}{E_0}, \\
D'_\xi &= \tilde{D}_\xi - \tilde{D}_\eta \tilde{D}_\xi \ln \frac{\omega}{E_0} + [(2\tilde{g}_1 - \tilde{g}_2)\tilde{D}_\xi + 2\tilde{D}_\xi^2] \pi(\omega).
\end{align*}
\]

When we differentiate (35)-(39) with respect to the scaling parameter, \( l = \ln(E_0/\omega) \), the length-scale, \( \Lambda \), must also be scaled in accordance with the change of the energy scale, \( \omega \), as

\[
\frac{d\Lambda}{\Lambda} + \frac{d\omega}{\omega} = 0.
\]

Thus we obtain the RG equations

\[
\begin{align*}
\frac{d\tilde{g}_1}{dl} &= -\tilde{D}_\xi - \tilde{g}_1\tilde{g}_2 + [(\tilde{g}_2 - \tilde{g}_1)\tilde{g}_1 + \tilde{D}_\xi^2]\lambda_l, \\
\frac{d\tilde{g}_2}{dl} &= -\tilde{D}_\eta - (\tilde{g}_1^2 + \tilde{g}_2^2)/2 + (\tilde{g}_2^2 + \tilde{g}_1^2)\lambda_l/2, \\
\frac{d\tilde{g}_3}{dl} &= -\tilde{g}_3(\tilde{g}_1 - 2\tilde{g}_2)\lambda_l, \\
\frac{d\tilde{D}_\eta}{dl} &= \tilde{D}_\eta + \tilde{D}_\xi^2/2, \\
\frac{d\tilde{D}_\xi}{dl} &= \tilde{D}_\xi + \tilde{D}_\eta \tilde{D}_\xi - [(2\tilde{g}_1 - \tilde{g}_2)\tilde{D}_\xi + 2\tilde{D}_\xi^2]\lambda_l.
\end{align*}
\]

C. Renormalization-group flow and QPT

1. In the absence of the random backward scattering

First, we consider the case where the random forward scattering is present (\( \tilde{D}_{\eta,0} \neq 0 \)), but the random backward scattering is absent (\( \tilde{D}_{\xi,0} = 0 \)), where \( \tilde{D}_{\eta,0} \) and \( \tilde{D}_{\xi,0} \) are initial strengths of the random forward and backward scatterings, respectively. In this case, the RG flows indicate that the QPT occurs from a metallic fixed point, \( (\tilde{D}_\eta^* = \infty, g_3^* = 0) \), to the Mott insulator fixed point, \( (\tilde{D}_\xi^* = \infty, g_3^* = \infty) \), as \( \tilde{U} \) increases. Typical flows are shown in Figs. 8(a-1) and (a-2) for \( \tilde{U} = 0.1 \) and 0.4, respectively, in the case of \( d = 1.1 \) and \( \tilde{D}_{\eta,0} = 0.02 \), where the critical value of \( \tilde{U} \) is \( \tilde{U}_c \sim 0.330 \). This behavior in \( d = 1 + \epsilon \) qualitatively reproduces the case of \( d = 1 \) where sufficiently strong random forward scattering destroys the Mott gap.

In Fig. 9(a) is shown a low-energy asymptotic phase diagram, where we also show energy scales of the Mott gap, \( \omega_{\mathrm{gap}}/E_0 \), introduced in the previous section. We see that both the random forward scattering and the raising dimensionality tend to destroy the Mott gap and consequently widen the metallic region as compared with the pure case [the phase boundary in the pure case is shown by the gray solid line].

2. Effects of the random backward scattering

Next, we consider the case where both the random forward and backward scatterings are present: \( \tilde{D}_{\eta,0} \neq 0 \) and \( \tilde{D}_{\xi,0} \neq 0 \). The random backward scattering makes it possible for the Anderson localization to occur. In this case, there occurs a transition from the Anderson insulator fixed point, \( (\tilde{D}_\xi^* = \infty, \tilde{D}_\eta^* = \infty, g_3^* = 0) \), to the Mott insulator fixed point, \( (\tilde{D}_\xi^* = \infty, \tilde{D}_\eta^* = \infty, g_3^* = \infty) \), as \( \tilde{U} \) increases. Typical flows are shown in Figs. 8(b-1) and (b-3) for \( \tilde{U} = 0.1 \) and 0.4, respectively.
here introduce the scale $l_{\text{loc}}$ at which $\tilde{D}_\xi$ reaches unity [see Fig. 8(b-1)]. Then, $\omega_{\text{loc}} = E_0 e^{-l_{\text{loc}}}$ gives a qualitative energy scale around which a crossover to the Anderson insulator occurs. In the flows of the type of Fig. 8(b-3), $g_3$ always dominates $\tilde{D}_\xi$ and reaches unity at the scale $l = l_{\text{gap}} < l_{\text{loc}}$, indicating that the Mott gap formation overwhelms the Anderson localization.

We also find flows toward $(\tilde{D}_\xi^* = 0, D_n^* = \infty, g_3^* = 0)$, as shown in Fig. 8(b-2). This type of flows is found only for $0 < d < 1.575$ in the narrow region of $\tilde{U}$ in between the regions corresponding to Figs. 8(b-1) and 8(b-3). In these cases, however, $\tilde{D}_\xi$ exceeds unity at some scaling parameter $l_{\text{loc}}$ during the renormalization, which indicates that the perturbative treatment breaks down and the localization occurs around the energy scale specified by $l = l_{\text{loc}}$. Thus we interpret that the ground state corresponding to this fixed point is the Anderson insulator. It is beyond the RG-based scheme to settle this ambiguity and we do not go into the details on this issue here.

In Fig. 9(b) is shown a low-energy asymptotic phase diagram, where we also show $\omega_{\text{loc}}/E_0$ and $\omega_{\text{gap}}/E_0$. As compared with Fig. 9(a), the phase boundary remains nearly unchanged, but the metallic phase in Fig. 9(a) is replaced with the Anderson insulator phase due to the random backward scattering.

The present results indicate that the QPT from the Anderson to the Mott insulators occurs in both $d = 1$ and $d > 1$. Recently, Ohtsuka and Hatsugai studied the half-filled Hubbard model containing site randomness by using the Monte Carlo method. They found that the QPTs from an incompressible (Mott) to a compressible (Anderson) insulators occur in all the cases of $d = 1, 2, 3$. This numerical result is consistent with the RG-based views given here.

IV. CONCLUDING REMARKS

In this paper, based on the one-loop RG flows, we have studied QPTs in the half-filled Hubbard model with a circular Fermi surface in $d = 1 + \epsilon$ continuous dimensions. Peculiarity of the $d = 1 + \epsilon$ dimensions was incorporated only through the mathematical structure of the elementary PP and PH loops: infrared logarithmic singularity of the PH loop is smeared. We have studied the following three cases:

(1) In the absence of randomness: The QPT from the metallic phase to the Mott insulator phase occurs at a finite $\tilde{U}$ for $\epsilon > 0$.

(2) In the case where the random forward scattering is present, but the random backward scattering is absent: Both random forward scattering and raising dimensionality tend to destroy the Mott gap. Consequently, $\tilde{U}_c$ becomes finite for $\epsilon > 0$ and the metallic region becomes wider as compared with the pure case (1).

(3) In the case where both the random forward and backward scatterings are present: The phase boundary remains nearly unchanged as compared with the case (2), but the metallic phase in the case (2) is replaced with the Anderson insulator phase due to the random backward scattering.

In the present study, the ground state properties were conjectured based solely on the one-loop RG flows. At present, numerical studies are in progress to complement the views given here.

ACKNOWLEDGMENTS

The author acknowledges Professor K. Ueda for directing his attention to Ref. [16]. He also thanks Professors K. Yonemitsu and Y. Hatsugai for discussions and useful information. He was supported by a Grant-in-Aid for Encouragement of Young Scientists from the Ministry of Education, Science, Sports and Culture, Japan.

APPENDIX A: DERIVATION OF (10)

Our purpose here is to show that the $\tilde{\omega}$-dependence of $\Re \Pi_{2k_F}(\omega)$ is uniquely determined as $[11]$. We start with equation (9). The imaginary counterpart is given by...
\[ \Im \Pi_{2k_F}(\omega) = -\pi \frac{S_{d-1}}{(2\pi)^d} \int k^{d-1} dk \int_0^\pi d\theta (\sin \theta)^{d-2} \Theta(-\xi_k)\Theta(\xi_k+Q)\delta(\omega - \xi_k+Q + \xi_k), \quad (A1) \]

for \(0 < \omega < 2v_Fk_F\), \(\Im \Pi_{2k_F}(\omega) = 0\) for \(2v_Fk_F < \omega\) and satisfies \(\Im \Pi_{2k_F}(-\omega) = -\Im \Pi_{2k_F}(\omega)\). The delta function in the integrand of \(A1\) is rewritten as \(\delta(\omega - \xi_k+q + \xi_k) = \frac{1}{v_F k_F} \left(\frac{1}{2} + \frac{\omega}{\xi_k} \right) \delta(t - t_0)\), where we introduced \(\tilde{\omega} = \omega/2v_Fk_F\), \(\tilde{k} = k/k_F\), \(t = \cos \theta\) and \(t_0 = \tilde{\omega} - (1 - \tilde{\omega}^2)/\tilde{k}\). So we have

\[ \Im \Pi_{2k_F}(\omega) = -\pi \frac{S_{d-1} k_F^{d-1}}{(2\pi)^d v_F} \int_{1-\tilde{\omega}}^1 dk \frac{1}{2} + \frac{\tilde{\omega}}{\tilde{k}} \tilde{k}^{d-1} \left[1 - (\tilde{\omega} - \frac{1 - \tilde{\omega}^2}{\tilde{k}})^2 \right]^{\frac{d-3}{2}}, \quad (A2) \]

where \(u = \tilde{k} + \tilde{\omega}\). For small \(\tilde{\omega}\), since the region of integration is limited to the vicinity of \(u = 1\), it is reasonable to replace the integrand, \((u^2 - \tilde{\omega}^2)(u^2 - 1)^{\frac{d-1}{2}}\), with \(2^{d-1}(u - 1)^{\frac{d-1}{2}}\) and we have

\[ \Im \Pi_{2k_F}(\omega) \sim -\pi \frac{S_{d-1} k_F^{d-1}}{(2\pi)^d v_F} \int_{1-\tilde{\omega}}^1 du (u^2 - 1)^{\frac{d-1}{2}}, \quad (A3) \]

which corresponds to a special case of equation (7.12) in Ref. [17] for \(|q| = 2k_F\). In \(d = 1 + \epsilon\), noting \(S_{d-1} \sim \epsilon\), we obtain

\[ \Im \Pi_{2k_F}(\omega) \sim -\frac{1}{4v_F} (1 - \tilde{\omega})^{-1 + \epsilon/2} \tilde{\omega}^{\epsilon/2} \quad (A4) \]

The real counterpart is obtained by Kramers-Kronig transformation,

\[ \Re \Pi_{2k_F}(\omega) = \frac{2}{\pi} \int_0^1 d\tilde{\omega}' \tilde{\omega}' \Im \Pi_{2k_F}(\omega'), \quad (A5) \]

where the symbol \(\mathcal{P}\) denotes Cauchy principal value integral [Note that \(0 < \tilde{\omega} < 1\)]. Since the prefactor, \((1 - \tilde{\omega})^{-1 + \epsilon/2}\), on the r.h.s of \(A4\) has already appeared in the exact expression, \(A2\), and the remaining part of the integral, \(A3\), shows no singularity at \(\tilde{\omega} = 1\), the expression for small \(\tilde{\omega}\), \(A4\), holds analytical property of \(\Im \Pi_{2k_F}(\omega)\) correctly even for \(\tilde{\omega} \sim 1\). Thus it is reasonable to use the expression \(A4\) in \(A3\) and we have

\[ \Re \Pi_{2k_F}(\omega) \sim -\frac{1}{2\pi v_F} \mathcal{P} \int_0^1 d\tilde{\omega}' \tilde{\omega}' (\tilde{\omega}'^{1+\epsilon/2}/(\tilde{\omega}'^2 - \tilde{\omega}^2)(1 - \tilde{\omega}'^2)^{1-\epsilon/2}), \quad (A6) \]

which is to be evaluated for small \(\tilde{\omega}\) and \(\epsilon\).

To evaluate \(A6\), let

\[ f(z) = \frac{z}{(z^2 - \tilde{\omega}^2)(z^2 - 1)} [z(z^2 - 1)]^{\epsilon/2}, \quad (A7) \]

and consider the integral, \(\oint_C f(z)dz\), along the contour as depicted in Fig. 10. \(f(z)\) has poles at \(z = \pm \tilde{\omega}\) and branch points at \(z = 0, \infty\) and \(z = \pm 1\). We choose branch cuts in the region \(\{\Re z < -1\} \cup \{0 < \Re z\}\). In the limit as the large circle recedes to infinity, it gives no contribution. The residue at the pole \(z = -\tilde{\omega}\) gives

\[ \oint_C f(z)dz = -\pi i e^{\pi i \epsilon/2} \frac{\tilde{\omega}^{\epsilon/2}}{(1 - \tilde{\omega}^2)^{1-\epsilon/2}} \sim -\pi i \tilde{\omega}^{\epsilon/2}. \quad (A8) \]

The remainder of the contour is deformed into an integral enclosing the cuts and encircling the pole \(z = \tilde{\omega}\) and the branch points at \(z = 0\) and \(z = \pm 1\). The points \(z = 0, \pm 1\) give no contribution. The integrals encircling the pole \(z = \tilde{\omega}\) and the remainder along the real axis give
\[
\oint_C f(z)dz = \frac{\pi i}{2}(e^{\gamma i\epsilon} + e^{\gamma^* i\epsilon}) \frac{\tilde{\omega}^{\epsilon/2}}{(1 - \tilde{\omega}^2)^{1-\epsilon/2}} + 2\pi v_F(e^{\gamma i\epsilon} - e^{\gamma^* i\epsilon}) \Pi_{2k_F}(\omega) + (e^{\gamma i\epsilon} - e^{\gamma^* i\epsilon}) \int_1^\infty \frac{x[x(x^2 - 1)]^{\epsilon/2}}{(x^2 - \tilde{\omega}^2)(x^2 - 1)} dx.
\]

Here the integrals in the second line are evaluated for small \(\tilde{\omega}\) and \(\epsilon\) as

\[
\int_1^\infty \frac{x[x(x^2 - 1)]^{\epsilon/2}}{(x^2 - \tilde{\omega}^2)(x^2 - 1)} dx = -\int_{-\infty}^{-1} \frac{x[-x(x^2 - 1)]^{\epsilon/2}}{(x^2 - \tilde{\omega}^2)(x^2 - 1)} dx
\]

\[
\sim \int_1^\infty [x(x^2 - 1)]^{-1+\epsilon/2} dx = \frac{\Gamma(1-3\epsilon/4)\Gamma(\epsilon/2)}{2\Gamma(1-\epsilon/2)} \sim 1/\epsilon,
\]

where the integral in the second line converges for \(0 < \epsilon < 4/3\). We thus obtain, for small \(\tilde{\omega}\) and \(\epsilon\),

\[
\oint_C f(z)dz = \pi i\tilde{\omega}^{\epsilon/2} - 2\pi v_F i\epsilon \Pi_{2k_F}(\omega) + 2\pi i - \pi i. \quad \text{(A9)}
\]

Therefore we obtain \(\Pi_{2k_F}(\omega)\) of the form which is correct up to the leading order of \(\tilde{\omega}\),

\[
\Pi_{2k_F}(\omega) \sim \frac{1}{2\pi v_F} \left[ \tilde{\omega}^{\epsilon/2} - C_\epsilon \right], \quad \text{(A10)}
\]

which is (10). Although above manipulation gives \(C_\epsilon = 1/\epsilon\), it seems feasible to choose \(C_\epsilon = -2/\epsilon\) as suggested in Ref. [16] to reproduce correctly the limit form of \(\Pi_{2k_F}(\omega) = \frac{1}{2\pi v_F} \log \tilde{\omega}\) at \(\epsilon = 0\). This discrepancy may arise, because to evaluate (A5) we used the expression (A4) which holds analytical property of \(3\Pi_{2k_F}(\omega)\) correctly but misses contribution from \(\tilde{\omega} \sim 1\). We do not go into the details here, since an explicit form of \(C_\epsilon\) does not enter the RG equations [13-17] and (11)-(45).

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FIG. 1. Two-particle scattering vertices for the (a) backward, (b) forward, and (c) 2$k_F$-umklapp scattering processes, and the elementary (d) particle-particle (PP) and (e) particle-hole (PH) loops.

FIG. 2. Vertex correction diagrams for the (a) normal $[g_1, g_2]$ and (b) umklapp $[g_3]$ processes. White and black circles represent the normal and umklapp scatterings, respectively.

FIG. 3. RG flows in the case of $d = 1.1$ with (a) $U = 0.08$ and (b) $0.04$. (c) The RG trajectories in terms of $g_3$ and $G = g_1 - 2g_2$ in $d = 1.1$. A critical value is $\tilde{U}_c = 0.0588$. In the $d = 1$ half-filled Hubbard model, $g_3$ and $G$ flow along the broken line denoted by “1D Hubbard”.

FIG. 4. $U$-dependence of $\Delta_{\text{exact}}(\tilde{U})$ and $\omega_{\text{gap}} = E_0 e^{-l_{\text{gap}}}$ in $d = 1$.

FIG. 5. A low-energy asymptotic phase diagram of the $d = 1 + \epsilon$ dimensional Hubbard model at half filling. We also show energy scales of the Mott gap, $\omega_{\text{gap}}/E_0$, as a function of $d$ and $\tilde{U}$.

FIG. 6. Inter-replica vertices originating from the random (a) forward and (b) backward scatterings. (c) The diagram which is proportional to the number of replicas, $N$, and vanishes in the replica limit, $N \to 0$.

FIG. 7. The vertex correction diagrams for the inter-replica (a) forward $[\tilde{D}_{\eta}]$ and (b) backward $[\tilde{D}_{\xi}]$ processes. We must avoid counting the diagram (b-4) which vanishes in the replica limit.

FIG. 8. (a) RG flows of $\tilde{D}_{\eta}$ and $g_3$ for (a-1) $\tilde{U} = 0.1$ and (a-2) $\tilde{U} = 0.4$ in the case of $d = 1.1$, $\tilde{D}_{\eta,0} = 0.02$ and $\tilde{D}_{\xi,0} = 0$. (b) RG flows of $\tilde{D}_{\eta}$, $\tilde{D}_{\xi}$ and $g_3$ for (b-1) $\tilde{U} = 0.1$, (b-2) $\tilde{U} = 0.2$ and (b-3) $\tilde{U} = 0.4$ in the case of $d = 1.1$, $\tilde{D}_{\eta,0} = 0.02$ and $\tilde{D}_{\xi,0} = 0.08$.

FIG. 9. Low-energy asymptotic phase diagrams of the $d = 1 + \epsilon$ dimensional random Hubbard model at half filling in the cases where (a) the random forward scattering is present, but the random backward scattering is absent, and (b) both the random forward and backward scatterings are present. In (a), the phase boundary in the pure case [see Fig. 5] is shown by the gray solid line. We also show $\omega_{\text{loc}}/E_0$ and $\omega_{\text{gap}}/E_0$, as a function of $d$ and $\tilde{U}$.

FIG. 10. Contour to evaluate the integral (A6).
Fig. 1 J. Kishine

Fig. 2 J. Kishine

Fig. 3 J. Kishine

Fig. 4 J. Kishine

Fig. 5 J. Kishine
