HYPONORMAL TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACES

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ABSTRACT. We consider the Toeplitz operator with symbol $z^n + C|z|^s$ acting on certain weighted Bergman spaces and determine for what values of the constant $C$ this operator is hyponormal. The condition is presented in terms of the norm of an explicit block Jacobi matrix.

Keywords: Hyponormal operator, Toeplitz Operator, Weighted Bergman Space, Block Jacobi Matrix

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1. Introduction

1.1. Weighted Bergman Spaces. Let $\mu$ be a probability measure on the interval $[0, 1]$ with $1 \in \text{supp}(\mu)$ and $\mu(\{1\}) = 0$. Using $\mu$, define the measure $\nu$ on the open unit disk $\mathbb{D}$ by $d\nu(re^{i\theta}) = d\mu(r) \times \frac{dr}{2\pi}$. Let $A^2_\nu(\mathbb{D})$ denote the weighted Bergman space of the unit disk defined by

$$A^2_\nu(\mathbb{D}) = \left\{ f : \int_\mathbb{D} |f(z)|^2 d\nu(z) < \infty, \ f \text{ is analytic in } \mathbb{D} \right\}$$

We equip $A^2_\nu(\mathbb{D})$ with the inner product

$$\langle f, g \rangle_\nu = \int_\mathbb{D} f(z)\overline{g(z)} \ d\nu(z).$$

Notice that the rotation invariance of the measure $\nu$ means the monomials $\{z^n\}_{n=0}^\infty$ are an orthogonal set in $L^2(\mathbb{D}, d\nu)$ and $A^2_\nu(\mathbb{D})$. It is a standard fact that $A^2_\nu(\mathbb{D})$ is a reproducing kernel Hilbert space. Let us define the set $\{\gamma_t\}_{t \in [0, \infty)}$ by

$$\gamma_t := \int_\mathbb{D} |z|^t d\nu(z) = \int_{[0,1]} x^t d\mu(x).$$

Since $1 \in \text{supp}(\mu)$, the sequence $\{\gamma_n\}_{n \in \mathbb{N}}$ decays subexponentially as $n \to \infty$, meaning for all $t > 0$ it holds that $\gamma_{m+t}/\gamma_m \to 1$ as $m \to \infty$. Since $\mu(\{1\}) = 0$, we know $\gamma_t$ approaches 0 as $t \to \infty$. With this notation it is true that

$$A^2_\nu(\mathbb{D}) = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n : \sum_{n=0}^{\infty} |a_n|^2 \gamma_{2n} < \infty \right\}$$

and the inner product becomes

$$\left\langle \sum_{n=0}^{\infty} a_n z^n, \sum_{n=0}^{\infty} b_n z^n \right\rangle_\nu = \sum_{n=0}^{\infty} a_n \overline{b_n} \gamma_{2n}.$$

Of particular interest is the case when

$$d\nu(z) = (\beta + 1)(1 - |z|^2)^\beta dA$$
for some \( \beta \in (-1, \infty) \), where \( dA \) is normalized area measure on \( \mathbb{D} \) (see \([8, 9, 10, 11]\)). Notice that when \( \beta = 0 \), the space \( \mathcal{A}_\nu(\mathbb{D}) \) is just the usual Bergman space of the unit disk.

A bounded operator \( T \) acting on a Hilbert space is said to be hyponormal if \( [T^*, T] \geq 0 \), where \( T^* \) denotes the adjoint of \( T \). The motivation for studying such operators comes from Putnam’s inequality (see \([13, \text{Theorem 1}]\)), which says that hyponormal operators satisfy

\[
\| [T^*, T] \| \leq \frac{|\sigma(T)|_2}{\pi}
\]

where \( \sigma(T) \) is the spectrum of \( T \) and \( | \cdot |_2 \) denotes the two-dimensional area.

If \( \varphi \in L^\infty(\mathbb{D}) \), then we define the operator \( T_\varphi : \mathcal{A}_\nu^2(\mathbb{D}) \to \mathcal{A}_\nu^2(\mathbb{D}) \) with symbol \( \varphi \) by

\[
T_\varphi(f) = P_\nu(\varphi f),
\]

where \( P_\nu \) denotes the orthogonal projection to \( \mathcal{A}_\nu^2(\mathbb{D}) \) in \( L^2(\mathbb{D}, d\nu) \). There is an extensive literature aimed at characterizing those symbols \( \varphi \) for which the corresponding operator \( T_\varphi \) is hyponormal, much of which focuses on the special case of the classical Bergman space of the unit disk (see \([1, 3, 5, 6, 7, 8, 9, 10, 11, 13, 16]\)). The specific symbol we will focus on is \( \varphi(z) = z^n + C|z|^s \), where \( n \in \mathbb{N} \), \( C \in \mathbb{C} \), and \( s \in (0, \infty) \). The case \( n = 1 \) and \( s = 2 \) in the classical Bergman space was considered in \([5]\) while a broader range of \( n \) and \( s \) was previously considered in \([16]\), where it was shown that hyponormality of \( T_\varphi \) acting on the classical Bergman space implies \( |C| \leq \frac{2}{n} \) and the converse holds if \( s \geq 2n \). Theorem \( 2.1 \) below will complete that result by providing necessary and sufficient conditions on the constant \( C \) for \( T_\varphi \) acting on any \( \mathcal{A}_\nu^2(\mathbb{D}) \) to be hyponormal. As a result, we will recover the aforementioned result from \([16]\). Our condition is stated in terms of the norm of a certain self-adjoint operator that happens to be a block Jacobi matrix.

1.2. **Block Jacobi Matrices.** Block Jacobi matrices are matrices of the form

\[
\mathcal{M} = \begin{pmatrix}
B_1 & A_1 & 0 & \cdots & \cdots \\
A_1^* & B_2 & A_2 & \cdots & \cdots \\
0 & A_2^* & B_3 & A_3 & \cdots \\
& \vdots & \ddots & \ddots & \ddots
\end{pmatrix}
\]

where each \( A_m \) and \( B_m \) is a \( k \times k \) matrix for some fixed \( k \in \mathbb{N} \) with \( B_m = B_m^* \) and \( \det(A_m) \neq 0 \) for all \( m \in \mathbb{N} \). An extensive introduction to the theory and applications of these operators can be found in \([4]\), so we will only mention the facts that are directly relevant to our investigation.

In the context of our problem, a block Jacobi matrix is a bounded self-adjoint operator from \( \ell^2(\mathbb{N}_0) \) to \( \ell^2(\mathbb{N}_0) \) (where \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \)) so its spectrum is a compact subset of the real line. While the spectrum of such an operator is in general difficult to compute, one can easily verify that if \( B_m \equiv 0 \) and \( A_m \equiv I_{k \times k} \) for all \( m \in \mathbb{N} \), then the spectrum of the corresponding block Jacobi matrix is \([-2, 2]\). In particular, the norm of this operator is 2 in this case.

2. **Main Result**

With this basic knowledge of block Jacobi matrices in hand, we can now state our main result. Suppose \( \nu \) is as in Section \([1, 11]\), \( s \in (0, \infty) \), and \( n \in \mathbb{N} \) have been fixed. Define the
block Jacobi matrix $\mathcal{J}(\nu)$ by

$$\mathcal{J}(\nu)_{n+k,k} = \mathcal{J}(\nu)_{k,n+k} = \begin{cases} \frac{\gamma_{2k+2n}\gamma_{2k+s}}{\sqrt[2]{\gamma_{2k+2n}^2 + \gamma_{2k+s}}} & k = 0, \ldots, n-1 \\ \frac{\gamma_{2k+2n}\gamma_{2k+s}}{\sqrt[2]{\gamma_{2k+2n}^2 - \gamma_{2k+s}}} & k \geq n \end{cases}$$

and all other entries of $\mathcal{J}(\nu)$ are equal to 0. Note that the log-convexity of $\{\gamma_t\}_{t>0}$ (see [12, Theorem 1.3.4]) implies that the quantities under the square roots in the entries of $\mathcal{J}(\nu)$ are all non-negative. However, if we write

$$\gamma_{2k+4n}\gamma_{2k} - \gamma_{2k+2n}^2 = \int_{[0,1]^2} (x^{2k+4n}y^{2k+2n} - x^{2k+2n}y^{2k+2n})d\mu(x)d\mu(y)$$

$$= \int_{[0,1]^2} (xy)^{2k}(x^{2n} - y^{2n})d\mu(x)d\mu(y)$$

and then symmetrize to obtain

$$\gamma_{2k+4n}\gamma_{2k} - \gamma_{2k+2n}^2 = \frac{1}{2} \int_{[0,1]^2} (xy)^{2k}(x^{2n} - y^{2n})^2d\mu(x)d\mu(y) > 0,$$

we see that the quantities under the square roots in the entries of $\mathcal{J}(\nu)$ are all strictly positive. Similar reasoning shows the numerators of those expressions are also strictly positive.

Our main result can be stated as follows.

**Theorem 2.1.** Suppose $C \in \mathbb{C}$, $s \in (0, \infty)$, and $n \in \mathbb{N}$. The operator $T_{z^n + C|z|^s}$ acting on $A_2^\nu(D)$ is hyponormal if and only if $|C| \leq \|\mathcal{J}(\nu)\|^{-1}$.

The above discussion verifies that $\mathcal{J}(\nu)$ is a block Jacobi matrix. Our next task is to prove the following result, which has clear implications for the application of Theorem 2.1.

**Theorem 2.2.** The spectrum of the operator $\mathcal{J}(\nu)$ consists of the interval $[-s/n, s/n]$ and an at most countable set of isolated points whose only accumulation points are among $\{\pm s/n\}$. In particular, the operator $\mathcal{J}(\nu)$ is bounded.

The proof will require the following elementary lemma.

**Lemma 2.3.** It holds that

$$\lim_{k \to \infty} \left( \int_{[0,1]^2} (xy)^k(x^{2n} - y^{2n})^2d\mu(x)d\mu(y) \right)^{1/k} = 1$$

**Proof.** It is clear that

$$\limsup_{k \to \infty} \left( \int_{[0,1]^2} (xy)^k(x^{2n} - y^{2n})^2d\mu(x)d\mu(y) \right)^{1/k} \leq 1$$
Also notice that if $\delta \in (0, 1)$ is fixed, then
\[
\liminf_{k \to \infty} \left( \int_{[0,1]^2} (xy)^k(x^{2n} - y^{2n})^2 d\mu(x)d\mu(y) \right)^{1/k}
\geq \liminf_{k \to \infty} \left( \int_{[1-\delta,1]^2} (xy)^k(x^{2n} - y^{2n})^2 d\mu(x)d\mu(y) \right)^{1/k}
\geq (1 - \delta)^2 \liminf_{k \to \infty} \left( \int_{[1-\delta,1]^2} (x^{2n} - y^{2n})^2 d\mu(x)d\mu(y) \right)^{1/k}
= (1 - \delta)^2
\]
Sending $\delta \to 0$ proves the lemma.

**Proof of Theorem 2.2.** We will show that
\[
\lim_{k \to \infty} \mathcal{J}(\nu)_{n+k,k} = \frac{s}{2n}
\]
This will show that $\mathcal{J}(\nu)$ is a compact perturbation of the block Jacobi matrix $\frac{1}{2n}(\mathcal{L}^n + \mathcal{R}^n)$, where $\mathcal{L}$ is the left shift and $\mathcal{R}$ is the right shift on $\ell^2(\mathbb{N}_0)$. The operator $\frac{1}{2n}(\mathcal{L}^n + \mathcal{R}^n)$ has spectrum equal to $[-s/n, s/n]$, from which the desired conclusion follows.

Recall that for any $\eta > 0$ it holds that
\[
\lim_{t \to \infty} \frac{\gamma_{t+\eta}}{\gamma_t} = 1.
\]
This implies
\[
\mathcal{J}(\nu)_{n+k,k} = (1 + o(1)) \frac{\gamma_{2k+2n+\gamma}2k - \gamma_{2k+2n}\gamma_{2k+s}}{\sqrt{\gamma_{2k+2n}\gamma_{2k-2n} - \gamma_{2k}^2}} \sqrt{\gamma_{2k+4n}\gamma_{2k} - \gamma_{2k+2n}^2}
\]
as $k \to \infty$. Now we write
\[
\gamma_{2k+2n+s} \gamma_{2k} - \gamma_{2k+2n} \gamma_{2k+s} = \int_{[0,1]^2} (x^{2k+2n+s} y^k - x^{2k+2n} y^{2k+s}) d\mu(x)d\mu(y)
= \int_{[0,1]^2} (xy)^k x^{2n} (x^s - y^s) d\mu(x)d\mu(y)
\]
Interchanging the roles of $x$ and $y$ and adding these expressions, we find
\[
\gamma_{2k+2n+s} \gamma_{2k} - \gamma_{2k+2n} \gamma_{2k+s} = \frac{1}{2} \int_{[0,1]^2} (xy)^k (x^{2n} - y^{2n}) (x^s - y^s) d\mu(x)d\mu(y)
\]
Using similar reasoning on the expressions in the denominator of (1), we can rewrite the leading term of (1) as
\[
\frac{\int_{[0,1]^2} (xy)^{k+2n} (x^2 - y^2)^2 d\mu(x)d\mu(y)}{\sqrt{\int_{[0,1]^2} (xy)^{k-2n} (x^{2n} - y^{2n})^2 d\mu(x)d\mu(y)} \left( \int_{[0,1]^2} (xy)^{2k} (x^{2n} - y^{2n})^2 d\mu(x)d\mu(y) \right)}
\]
We can bound (2) from above by
\[
\frac{\int_{[0,1]^2} (xy)^{2k} (x^{2n} - y^{2n}) (x^s - y^s) d\mu(x)d\mu(y)}{\int_{[0,1]^2} (xy)^{2k} (x^{2n} - y^{2n})^2 d\mu(x)d\mu(y)}
\]
and from below by
\[
\frac{\int_{[0,1]^2}(xy)^{2k}(x^{2n} - y^{2n})(x^s - y^s)d\mu(x)d\mu(y)}{\int_{[0,1]^2}(xy)^{2k-2n}(x^{2n} - y^{2n})^2d\mu(x)d\mu(y)}
\] (4)

Lemma 2.3 implies that the denominators in (3) and (4) decay subexponentially as \( k \to \infty \), and hence we obtain the same asymptotic behavior as \( k \to \infty \) if we replace each integral in the numerators of (3) and (4) by the integral over \([1 - \epsilon, 1]^2\) for some \( \epsilon > 0 \).

Now, fix some \( \epsilon \in (0, 1) \) and define
\[
g(\epsilon) := \max_{1-\epsilon \leq t \leq 1} t^{s-2n} = \begin{cases} 
(1-\epsilon)^{s-2n} & \text{if } s < 2n \\
1 & \text{if } s \geq 2n.
\end{cases}
\]

If \( Z > \frac{sg(\epsilon)}{2n} \), then \( Z \mu^{2n} - u^s \) is an increasing function of \( u \) on \([1 - \epsilon, 1]\). Thus, when \( Z > \frac{sg(\epsilon)}{2n} \) and \( 1 \geq x \geq y \geq 1 - \epsilon \) it holds that \((x^s - y^s) < Z(x^{2n} - y^{2n})\). It follows from (3) that for such a \( Z \) we have
\[
\limsup_{k \to \infty} J(\nu)_{n+k,k} \leq \limsup_{k \to \infty} \frac{Z \int_{[1-\epsilon,1]^2}(xy)^{2k}(x^{2n} - y^{2n})^2d\mu(x)d\mu(y)}{\int_{[0,1]^2}(xy)^{2k}(x^{2n} - y^{2n})^2d\mu(x)d\mu(y)} = Z,
\]
where we used the fact that the denominator of the expression in (3) decays subexponentially as \( k \to \infty \). Since \( Z > \frac{sg(\epsilon)}{2n} \) was arbitrary, we conclude that
\[
\limsup_{k \to \infty} J(\nu)_{n+k,k} \leq \frac{sg(\epsilon)}{2n}.
\]
Taking \( \epsilon \to 0 \), we obtain the desired upper bound.

By applying similar reasoning, we see that if \( h(\epsilon) := \min_{1-\epsilon \leq t \leq 1} t^{s-2n} \) and \( Z' < \frac{sh(\epsilon)}{2n} \), then
\[
\liminf_{k \to \infty} J(\nu)_{n+k,k} \geq \liminf_{k \to \infty} \frac{Z' \int_{[1-\epsilon,1]^2}(xy)^{2k}(x^{2n} - y^{2n})^2d\mu(x)d\mu(y)}{\int_{[0,1]^2}(xy)^{2k}(x^{2n} - y^{2n})^2d\mu(x)d\mu(y)}
\]
\[
\geq \liminf_{k \to \infty} \frac{Z'(1-\epsilon)^{4n} \int_{[1-\epsilon,1]^2}(xy)^{2k-2n}(x^{2n} - y^{2n})^2d\mu(x)d\mu(y)}{\int_{[0,1]^2}(xy)^{2k-2n}(x^{2n} - y^{2n})^2d\mu(x)d\mu(y)}
\]
\[
= Z'(1-\epsilon)^{4n}.
\]
Since \( Z' < \frac{sh(\epsilon)}{2n} \) was arbitrary, we conclude that
\[
\liminf_{k \to \infty} J(\nu)_{n+k,k} \geq \frac{sh(\epsilon)(1-\epsilon)^{4n}}{2n}.
\]
Taking \( \epsilon \to 0 \), we obtain the desired lower bound. \(\square\)

As an example of Proposition 2.2, consider the case when \( \mu = 2rdr \). In this case the measure \( \nu \) is normalized area measure on \( \mathbb{D} \) and \( \gamma_n = 2(t+2)^{-1} \) so \( J(dA) \) is
\[
J(dA)_{n+k,k} = J(dA)_{k,n+k} = \begin{cases} 
\frac{s\sqrt{(k+n+1)(k+2n+1)}}{2(k+1+s/2)(k+n+1+s/2)} & k = 0, \ldots, n-1 \\
\frac{s(k+1)\sqrt{(k+n+1)(k+2n+1)}}{2n(k+1+s/2)(k+n+1+s/2)} & k \geq n.
\end{cases}
\]
We see that the conclusion of Theorem 2.2 holds true for this matrix. Furthermore, one can quickly verify by hand that if \( s \geq 2n \), then each non-zero entry of \( J(dA) \) is less than.
This implies that when \( s \geq 2n \) the spectrum of \( J(dA) \) is precisely \([-s/n, s/n]\) and so Theorem 2.1 implies [16, Theorem 2].

3. Proof of Theorem 2.1

Throughout this section, let us suppose that \( n \in \mathbb{N} \), \( s \in (0, \infty) \), and \( \nu \) as in Section 1.1 are fixed. As in [5, 16], we will use the formula

\[
\langle (T + S)^*, T + S \rangle u, u \rangle = \langle Tu, Tu \rangle - \langle T^*u, T^*u \rangle + 2\text{Re} \langle Tu, Su \rangle - \langle S^*u, S^*u \rangle \tag{5}
\]

with \( T = T_{z^n} \) and \( S = T_{C[z^n]} \). The first step in our proof will be the following adaptation of [16] Lemma 1] to weighted Bergman spaces.

**Lemma 3.1.** If \( k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and \( t \in (0, \infty) \), then

\[
P_{\nu}(z^k | z^t) = \frac{\gamma_{2k+t} z^k}{\gamma_{2k}}, \quad \text{and} \quad P_{\nu}(z^{k+1} | z^t) = 0.
\]

**Proof.** A calculation shows that

\[
\left\langle z^q, \frac{\gamma_{2k+t}}{\gamma_{2k}} z^k \right\rangle = \left\langle z^q, z^k | z^t \right\rangle
\]

for every \( q \in \mathbb{N}_0 \), so the first claim follows from the fact that polynomials are dense in \( \mathcal{A}_\nu^2(D) \). The second claim follows from a similar calculation. \( \square \)

With Lemma 3.1 in hand, we consider \( u = \sum_{k=0}^{\infty} u_k z^k \in \mathcal{A}_\nu^2(D) \) and calculate

\[
\langle T_{z^n} u, T_{z^n} u \rangle = \sum_{k=0}^{\infty} |u_k|^2 \gamma_{2k+2n}
\]

\[
\langle T_{z^n} u, T_{z^n} u \rangle = \sum_{k=n}^{\infty} \frac{\gamma_{2k}}{\gamma_{2k-2n}} |u_k|^2
\]

\[
\text{Re}[\langle T_{z^n} u, T_{C[z^n]} u \rangle - \langle T_{z^n} u, T_{C[z^n]} u \rangle] = \sum_{k=0}^{\infty} \text{Re}[u_k \bar{u}_{k+n} C] \left( \gamma_{2k+2n+} \frac{\gamma_{2k+2n} \gamma_{2k+2n} - \gamma_{2k+2n} \gamma_{2k+2n}}{\gamma_{2k}} \right),
\]

Notice that \( \gamma_{2k+2n+} \frac{\gamma_{2k+2n} \gamma_{2k+2n} - \gamma_{2k+2n} \gamma_{2k+2n}}{\gamma_{2k}} \) is the numerator of the entry \( J(\nu)_{n+k,k} \) and hence the discussion before Theorem 2.1 implies each of these terms is strictly positive. That discussion also implies \( \gamma_{2k+2n+} \frac{\gamma_{2k+2n}}{\gamma_{2k+2n}} \) when \( k \geq n \). Thus we may reason as in [16] and conclude that \( T_{z^n+C[z^n]} \) is hyponormal if and only if

\[
|C| \leq \inf \left\{ \sum_{k=0}^{n-1} u_k^2 \gamma_{2k+2n} + \sum_{k=n}^{\infty} u_k^2 \left( \gamma_{2k+2n} - \frac{\gamma_{2k}}{\gamma_{2k-2n}} \right) \right\} \quad \text{and} \quad 2 \sum_{k=0}^{\infty} u_k u_{k+n} \left( \gamma_{2k+2n+} - \frac{\gamma_{2k+2n} \gamma_{2k+2n}}{\gamma_{2k}} \right), \tag{6}
\]
where the infimum is taken over all non-negative sequences \( \{u_k\} \) satisfying \( \sum_{k=0}^{\infty} |u_k|^2 \gamma_{2k} < \infty \) that are not the zero sequence. For convenience, we will restate the condition \( \mathcal{G} \) as

\[
\kappa := \sup \left\{ \frac{2 \sum_{k=0}^{\infty} u_k u_{k+n} \left( \frac{\gamma_{2k+2n+s} - \frac{\gamma_{2k} \gamma_{2k+s}}{\gamma_{2k}}}{\gamma_{2k}} \right)}{\sum_{k=0}^{n-1} u_k^2 \gamma_{2k+2n} + \sum_{k=n}^{\infty} u_k^2 \left( \frac{\gamma_{2k+2n} - \frac{\gamma_{2k}^2}{\gamma_{2k-2n}}}{\gamma_{2k-2n}} \right)} \right\} \leq \frac{1}{|\mathcal{C}|} \tag{7}
\]

Now we make the substitution

\[
v_j = \begin{cases} \frac{u_j \sqrt{\gamma_{2k+2n}}}{\gamma_{2k+2n}} & j \leq n - 1 \\ \frac{u_j \sqrt{\gamma_{2k+2n} - \frac{\gamma_{2k}^2}{\gamma_{2k-2n}}}}{\gamma_{2k-2n}} & j \geq n \end{cases}
\]

in (7) (we used the discussion before Theorem 2.1 here). This gives

\[
\kappa = \sup \left\{ \frac{2 \left( \sum_{k=0}^{n-1} v_k v_{k+n} \left( \frac{\gamma_{2k+2n+s} - \frac{\gamma_{2k} \gamma_{2k+s}}{\gamma_{2k}}}{\gamma_{2k}} \right) + \sum_{k=n}^{\infty} v_k v_{k+n} \left( \frac{\gamma_{2k+2n+s} - \frac{\gamma_{2k} \gamma_{2k+s}}{\gamma_{2k}}}{\gamma_{2k}} \right) \right)}{\sum_{k=0}^{\infty} v_k^2} \right\} \tag{8}
\]

and we take the supremum over all non-negative \( \{v_k\}_{k=0}^{\infty} \) that satisfy

\[
0 < \sum_{k=0}^{\infty} v_k^2 \frac{\gamma_{2k}}{\gamma_{2k+2n} - \frac{\gamma_{2k}^2}{\gamma_{2k-2n}}} < \infty.
\]

Notice that our assumptions on \( \nu \) imply

\[
\lim_{k \to \infty} \left[ \frac{\gamma_{2k}}{\gamma_{2k+2n} - \frac{\gamma_{2k}^2}{\gamma_{2k-2n}}} \right] = \infty,
\]

so in particular

\[
\ell^2_\gamma(N_0) := \left\{ v_k \right\}_{k=0}^{\infty} : \sum_{k=0}^{\infty} |v_k|^2 \frac{\gamma_{2k}}{\gamma_{2k+2n} - \frac{\gamma_{2k}^2}{\gamma_{2k-2n}}} < \infty \right\} \subseteq \ell^2(N_0)
\]

and in fact \( \ell^2_\gamma(N_0) \) is dense in \( \ell^2(N_0) \) in the \( \ell^2(N_0) \)-metric because \( \ell^2(N_0) \) contains all finite sequences. Therefore, by the scale invariance of the expression in (5), we may define \( \kappa \) by taking the supremum only over those non-negative sequences \( \{v_k\}_{k=0}^{\infty} \in \ell^2_\gamma(N_0) \) such that \( \sum v_k^2 = 1 \).

Since \( \|\mathcal{J}(\nu)\| < \infty \) (by Theorem 2.2) and hence self-adjoint, we can use the well-known formula

\[
\|\mathcal{J}(\nu)\| = \sup_{\|x\|=1} \langle x, \mathcal{J}(\nu)x \rangle_{\ell^2(N_0)} \tag{9}
\]

(see [14] page 216), where the supremum is taken over all \( x \in \ell^2(N_0) \) with norm 1 in this space. By the density property we just mentioned, it suffices to take the supremum over all
$x \in \ell^2_0(N_0)$ with norm 1 in $\ell^2(N_0)$. Since all the entries of $J(\nu)$ are positive real numbers, we recognize that the right-hand side of (9) is equal to the right-hand side of (8). We conclude that $\|J(\nu)\| = \kappa$ as desired.

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