QCD Dirac Spectrum at Finite Chemical Potential:
Anomalous Effective Action, Berry Phase and Composite Fermions

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We show that the QCD Dirac spectrum at finite chemical potential using a matrix model in the spontaneously broken phase, is amenable to a generic 2-dimensional effective action. The eigenvalues form a droplet with strong screening and plasmon oscillation. The droplet is threaded by a magnetic vortex which is at the origin of a Berry phase. For quarks in the complex or Dirac representation, the anomalous transport in the droplet of eigenvalues bear some similarities with that in droplets of composite fermions at half filling suggesting that the latters maybe Dirac fermions.

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I. INTRODUCTION

QCD breaks spontaneously chiral symmetry with a wealth of evidence in hadronic processes at low energies [1]. First principle lattice simulations strongly support that [2]. The spontaneous breaking is characterized by a large accumulation of eigenvalues of the Dirac operator near zero-virtuality [3]. The zero virtuality regime is ergodic, and its neighborhood is diffusive [4].

The ergodic regime of the QCD Dirac spectrum is amenable to a chiral random matrix model [5]. In short, the model simplifies the Dirac spectrum to its zero-mode zone (ZMZ). The Dirac matrix is composed of hopping between N-zero modes and N-anti-zero modes because of chirality, which are Gaussian sampled by the maximum entropy principle. The model was initially suggested as a null dynamical limit of the random instanton model [6]. QCD at finite chemical potential $\mu$ is subtle on the lattice due to the sign problem [7]. A number of effective models have been proposed to describe the effects of matter in QCD with light quarks [1]. Chiral random matrix models offer a simple construct that retains some essentials of chiral symmetry both in vacuum and matter. For instance, in the chiral 1-matrix model finite $\mu$ is captured by a constant deformation of Gaussian matrix ensembles [8] [9]. In the chiral 2-matrix model the deformation with $\mu$ is also random [10] [11]. Chiral matrix models in matter were discussed by many [12] [13]. Recently both a universal shock analysis [14] and a hydrodynamical description of the Dirac spectra were suggested [15].

In the ergodic regime the 1- and 2-matrix models exhibit the same microscopic universality for vanishingly small $\mu^2$ in the large volume limit [13]. This limit corresponds to a weak non-hermitean deformation of the standard chiral matrix models and therefore preserves the underlying chiral symmetry of the coset space. It follows the general strictures of the epsilon-expansion in chiral power counting [16]. The microscopic universality for small $\mu^2$ leads to new Leutwyler-Smilga sum rules [17] for the eigenvalues of the QCD Dirac operator that put first principle constraints on the effective Lagrangian approach in matter as well QCD lattice simulations in the regime of small quark masses, small $\mu^2$ and large volumes.

In the first part of the paper, we will show that at finite $\mu$ the distribution of Dirac eigenvalues in the complex plane maps onto a 2-dimensional Coulomb gas whose effective action is constrained by Coulomb’s law, the conformal and gravitational anomalies in 2-dimensions. These contributions are generic and go beyond the specifics of matrix models at finite $\mu$. The mapping offers a physical framework through anomalies, for understanding first principle aspects of the QCD Dirac spectrum at finite $\mu$ with the hope of constraining further the effective Lagrangian approaches and current and future lattice simulations near the chiral point at finite $\mu$ and for large volumes. We will also show that the 2-dimensional Coulomb gas of eigenvalues exhibits both screening and a plasmon branch. The latter translates to a diffusive mode in the stochastic evolution in chiral matrix models [13]. The plasmon frequency sets an estimate for the relaxation time for the restoration/breaking of chiral symmetry at finite $\mu$, solely through the stochastic re-organization of the low-lying modes of the Dirac spectrum.

In the second part of the paper, we follow by noting that under adiabatic changes the complex eigenvalues at finite $\mu$, viewed as particles in the complex 2-plane, behave as fermions for $\beta = 2$ due to the emergence of a Berry phase of $\pi$. There is no Berry phase for $\beta = 1$ or when the quarks are in the real representation. We use this observation to note that a dynamical droplet of quark eigenvalues for $\beta = 2$ and finite $\mu$ share some similarities with composite fermions at half filling suggesting that the latters are Dirac particles. We show that the anomalous charged transport contributions from an induced Wess-Zumino-Witten term are consistent with constituents of charge $e = 1$ and spin $s = 1/2$. We use these observations to derive novel effects for the compos-
ite fermions at half filling.

This paper consists of the following new results: 1/ a generic anomalous effective action for the QCD Dirac eigenvalues as a droplet in the complex 2-plane of eigenvalues with scalar curvature; 2/ a plasmon dispersion law in the bulk of the droplet with an improved estimate for the relaxation time for the breaking/restoration of chiral symmetry in QCD at finite $\mu$; 3/ an identification of the eigenvalue droplet as a 2-dimensional Fermi liquid threaded by a magnetic vortex; 4/ a description of the anomalous transport on the Fermi surface through a Berry induced Wess-Zumino-Witten type term; 5/ a suggestion through geometry that composite fermions in the fractional quantum Hall effects at half filling are Dirac particles; 6/ two novel anomalous transport effects in composite fermions at half filling, caused by a rotation or a temperature gradient.

II. 2-MATRIX MODEL

The low lying eigenmodes of the QCD Dirac operator capture some aspects of the spontaneous breaking of chiral symmetry both in vacuum and in matter. Remarkably, their fluctuations follow by approximating the entries in the Dirac operator by purely random matrix elements which are chiral (paired spectrum) and fixed by time-reversal symmetry (Dyson ensembles). At finite $\mu$ the Dirac spectrum on the lattice is complex \cite{20}. The matrix models at finite $\mu$ \cite{8,10} capture this aspect of the lattice spectra and the nature of the chiral phase transition \cite{11,12,13}. For a 2-matrix model, the partition function is \cite{10,11}

$$Z[\beta, \mu, N_f, m_f] =$$

$$\int dA \, dB \, e^{-a N \text{Tr}(A^A A)} e^{-a N \text{Tr}(B^B B)}$$

$$\times \det \begin{pmatrix} -i m_f & A - i \mu B \\ A^\dagger - i \mu B^\dagger & -i m_f \end{pmatrix}^{N_f} \quad (1)$$

for equal quark masses $m_f$ in the complex representation. Here $A, B$ are $C(N+N) \times N$ valued. $\nu$ accounts for the difference between the number of zero modes and anti-zero modes. The vacuum Banks-Casher formula \cite{3} fixes the dimensionfull parameter $\sqrt{\alpha} = |q| q_0 / n$ in terms of the massless quark condensate and the density of zero modes $n = N/V_4$. Throughout, we will set the units using $\sqrt{\alpha} \rightarrow 1$. All canonical units are recovered by inspection.

The Dirac matrix in (1) has $\nu$ unpaired zero modes and $N$ paired eigenvalues $\pm iz_j$ in the massless limit. The paired eigenvalues delocalize and are represented by (1). The unpaired zero-modes decouple. In terms of the fixed eigenvalues and large but finite $N$, (1) reads \cite{10,11,11}.

$$Z[\beta, \mu, N_f, m_f] \approx \int \prod_{i=1}^{N} d^2 z_i \ (z_i^2 + m_f^2)^{N_f} e^{-S[\beta, \mu; z]}$$

(2)

The action is

$$S[\beta, \mu; z] = -\beta \sum_{i<j=1}^{N} \ln[z_i^2 - z_j^2] + \sum_{i=1}^{N} W(z_i) \quad (3)$$

with

$$W(z) = -\alpha \ln|z| + \frac{B}{2} w(z) \quad (4)$$

and the quasi-harmonic potential $w(z) = |z|^2 - \frac{\tau}{2}(z^2 + \bar{z}^2)$. Here $\alpha = \beta(\xi + 1) - 1$ and $\xi$ accounts for the difference between the number of zero modes and anti-zero modes in the ZMZ. We define

$$\frac{B}{N \beta} = \frac{1}{1-\tau^2} = \frac{1 + \mu^2}{2 \mu^2} = \frac{1}{l^2} \quad (5)$$

with $B = 1/l_B^2$ acting as a magnetic field with magnetic length $l_B$ as we will suggest below. Throughout $\beta = 2$ unless indicated otherwise. The 2-matrix model for $\beta = 1, 4$ \cite{21} is more subtle at finite $\mu$ \cite{11}.

In Fig. \[1\] we display the distribution of eigenvalues following from the 2-matrix model with $A$ and $B$ sampled from a Gaussian ensemble of $200 \times 200$ matrices with $\nu = 0$ and $\mu = 0.3$. The mean density in the droplet is $\rho_0 \approx \nu B / 2\pi$ with $\nu = 1/\beta$. The boundary curves follow from the analysis in \cite{12}. The domain is an ellipse

$$\frac{x^2}{a_+^2} + \frac{y^2}{a_-^2} = 1 \quad (6)$$

with semi-axes $a_+^2/2l^2 = 1 \pm \tau / 1 + \tau$. The ellipse remains un-split with area $A = \pi a_+ a_- = 2\pi l^2$ for all values.
of $\mu$. For the other quark representations with $\beta = 1,4$ the joint distribution in the 2-matrix model is more subtle [11]. Throughout, (2) will be assumed for $\beta = 2$, but most results extend to $\beta = 1, 2, 4$ for large $N$.

For completeness we recall that the 1-matrix model corresponds to setting $B = 1$. In this case, the eigenvalue distribution forms a connected droplet in the $z$-plane for $\mu < \mu_c$, and splits into two droplets symmetric about the real-axis for $\mu > \mu_c$, restoring chiral symmetry [8,9]. In the spontaneously broken phase, all droplets are connected and symmetric about the real-axis.

Throughout, (2) is to be understood in the large $N$ limit to allow for a course graining of the eigenvalue density. We note that in this limit, the Bessel kernel in [10] is expanded. While the 2-matrix model is exactly solvable in terms of the orthogonal polynomial method, our analysis of this model shows the emergence of a generic effective action for the complex eigenvalues of the Dirac spectrum at finite $\mu$. Therefore our analysis encompasses the weak non-hermiticity range of the model. Besides chiral symmetry through the pairing of the complex eigenvalues, the new guiding principles for the construct of this effective action are Coulomb law, the conformal and gravitational anomalies in 2-dimensions as we now detail.

III. STATIC EFFECTIVE ACTION

In this section we will re-write (2) in terms of the effective potential generated by the mean charged density, for a sufficiently dense ensemble of eigenvalues. This assumes that $N$ is large to allow for a smoothening of the eigenvalue density over distances larger than the inter-level spacing but much shorter than the size of the eigenvalue droplet. While the 2-matrix model is solvable [10,11], it is important to stress that its re-writing at large but fixed $N$, using an effective action unravels universal physical aspects of the eigenvalue droplet that are not restricted to the specifics of the model. We will be able to go beyond the strictures of chiral symmetry by making use of emergent anomalies when using the collective potential sourced by the smoothened eigenvalue density as we detail below.

For the sake of generality, we will assume that the eigenvalue space is curved with a measure $ds^2 = g_{zz}dzd\bar{z}$. The explicit form of the metric is not important for our general arguments. The curved eigenvalue space will allow for the unraveling of two generic contributions to the effective action through the emergence of conformal and gravitational anomalies, that make the effective action construct more general than the 2-matrix model. Also, for bulk quantities a fixed space curvature $R$ is conjugate to a fixed space curvature $R_4 \equiv 1/R$.

The re-writing of (2) will closely follow the effective action construction in [22] for the Laughlin states to which we refer for further details. In brief, on a curved eigenvalue manifold of volume $V_2$ with a metric $g_{zz}$ and large $N$, the ensemble described by (4) allows the change in the measure, from integrating over specific eigenvalues $z_i$ to integrating over the eigenvalue density $\rho(z)$. Specifically

$$\prod_{i=1}^{N} \sqrt{g} dz_i \to e^{\int dz \sqrt{g} \rho(z) \ln(\rho_{0}/\rho(z))} D\rho$$

with the induced Boltzmann entropy in the exponent [18]. The mean density $\rho_0$ will be made explicit below. Since the spectrum is chirally symmetric, we reset $(\rho(z) + \rho(-z))/2 \to \rho(z)$ for convenience. We will assume that the density and thus the potential are real, which effectively corresponds to a phase quenched approximation. This approximation while limited in bulk [8,9], still contains useful physical information at the edge of the Dirac spectrum [23].

Following [22], we substitute the collective potential $\varphi(z)$ to the collective density $\rho(z)$ through the Poisson equation

$$\nabla^2 \varphi(z) = -4\pi \left( \rho(z) - \frac{N}{V_2} \right)$$

This change of variable involves the conformal anomaly in 2-dimensional curved space [22,25]

$$D\rho \to e^{-\Gamma_2} \det(-\nabla^2) D\varphi$$

with the Liouville action

$$\Gamma_2 = \frac{1}{24\pi} \int dz \sqrt{g} \left( (\nabla \ln \rho)^2 + R(z) \ln \rho \right)$$

as we briefly detail in the Appendix. Here $R(z)$ is the scalar Ricci curvature on the curved 2-dimensional complex manifold of eigenvalues. The determinant in (9) induces a gravitational anomaly [22]. The final result for the effective action is

$$Z[\beta, \mu, N_f, m_f] \approx \left( \det(-\nabla^2) \right)^{\frac{1}{2}} \times \int D\varphi e^{-\frac{N_f}{2} \left( \varphi(im_f) + \varphi(-im_f) \right)} e^{-\Gamma[\beta; \varphi]}$$

with the effective quenched action $\Gamma = \Gamma_0 + \Gamma_0 + \Gamma_1 + \Gamma_2$ and

$$\Gamma_0 = \frac{1}{8\pi \nu} \int dz \sqrt{g} \left( (\nabla \varphi)^2 - R(z) \varphi - 4\nu B(z) \varphi \right)$$

$$\Gamma_1 = \frac{1}{\nu} \left( \nu - \frac{1}{2} \right) \int dz \sqrt{g} \rho \ln \rho$$

$$E_0 = \frac{N}{\nu V_2} \int dz dz' \sqrt{gg'} \ln |z - z'|^2 \left( \rho_0(z) - \frac{1}{2} \frac{N}{V_2} \right)$$

(12)
Here \( B(z) = B - \pi \alpha \delta(z) \) and \( \Gamma_1 = 0 \) for Dirac quarks with \( \nu = 1/2 \). \((11, 12)\) differs from the one in \([22]\) in three ways: 1/ Both \( p(z) \) and \( \varphi(z) \) are \( z \)-even because of chiral symmetry; 2/ \( B \) is of order \( N \); 3/ \( B(z) \) carries a magnetic vortex which will be exploited below. We note that \( \Gamma \) is real and of order \( N^2 \).

Although \((11, 12)\) was derived using the 2-matrix model, we observe that each of its contributions are generic and therefore not specific to the 2-matrix model. The exception is \( w(z) \) which is model specific and in this case quasi-harmonic. This suggests that \((11)\) is the effective partition function for QCD Dirac spectra at finite \( \mu \) in the spontaneously broken phase provided that \( w(z) \) in \((4)\) is extended to include non-quasi-harmonic potentials.

### IV. GROUND STATE

In this section we analyze the ground state properties following from \((2)\) in terms of the collective potential in \((11)\). We will explicit the solution to \( \delta \Gamma / \delta \varphi = 0 \) in the linearized approximation. The result is an expression for the mean charged density in the droplet \( \rho_0(z) \) without the contribution from the \( N \)-exponent in \((11)\). We will refer to this solution as the quenched saddle point which is not to be confused with the standard quenched approximation using in the litterature. We will use this result to analyze the mean contribution of the low-lying quark eigenmodes in the ZMZ to both the bulk energy and quark condensate at finite chemical potential and for fixed curvature \( R \approx 1/R_4 \). Again, because of the generic nature of \((11)\) as we noted earlier, we expect the results to reflect on the QCD eigenvalue spectrum at finite \( \mu \) beyond the 2-matrix model.

#### A. Leading

With this in mind, the quenched saddle point equation \( \delta \Gamma / \delta \varphi = 0 \) in the linearized approximation yields the mean density

\[
\rho_0(z) \approx \frac{\nu B}{2 \pi} + \frac{R}{8 \pi} - \frac{\nu \alpha}{2} \delta(z) \equiv \rho_0 - \frac{\nu \alpha}{2} \delta(z) \quad (13)
\]

For vanishingly small \( \mu^2 \), \((13)\) yields \( \pi \rho_0(z) \approx N/4 \mu^2 \) in agreement with the quenched asymptotic density in the weak non-hermiticity limit \([24]\). The quenched energy in the ZMZ is

\[
E_0 \approx -\ln Z[\beta, \mu, 0, 0] \approx -\frac{1}{2} \int_A \sqrt{g} \rho_0(z) W(z) \quad (14)
\]

with pertinent changes in \( W(z) \) in curved space. Using \([13]\) we obtain

\[
E_0 \approx -\frac{1}{8 \pi} \left( \nu B^2 + \frac{1}{4} B R \right) \int_A \sqrt{g} w(z) \quad (15)
\]

We note the quadratic form of the magnatic-like contribution, and the mixed curvature-magnetic contribution which is a Casimir effect. Recently, a similar mixed term between the gauge holonomy and the curvature in hyperbolic space was noted in the cosmological context \([25]\).

The quenched quark condensate in the ZMZ is \( \langle m_f = 0 \rangle \)

\[
\int_4 \langle \bar{q} q \rangle_0 \approx \left( \frac{\nu B}{2 \pi} + \frac{R}{8 \pi} \right) \int_A \sqrt{g} (\sqrt{2} \pi z \varphi(z, z'))_{z' = 0} (16)
\]

with \( \nabla^2 G(z, z') = \delta(z - z')/\sqrt{g} \) in curved eigenvalue space. In zeroth order in the curvature \( 2 \pi \varphi G(z, z') = \ln |z - z'| \) and \((16)\) vanishes

\[
V_4 \langle \bar{q} q \rangle_0 \approx \frac{i}{2} \left( \nu B + \frac{R}{4} \right) \int_A \frac{1}{z} = 0 \quad (17)
\]

as the last integral is zero when carried over the elliptic droplet \( \Omega \).

\[
\int_A \frac{1}{z} = \int_{a_+}^{a_+} dx 2 \text{sign}(x) \tan^{-1} \left( \frac{a_-(x)}{|x|} \right) = 0
\]

with \( a_-(x) = a_-(1 - x^2/a_+^2)^{1/2} \). This result is conform with the phase quenched limit, with the inside of the droplet breaking conformal symmetry with a condensation of mixed pairs made of a quark and a conjugate quark \([8,9]\).

#### B. Sub-leading

The unquenched but subleading results follow from the linearized saddle point in \((11)\) including the \( N \)-contribution. Specifically

\[
\frac{\delta \Gamma}{\delta \varphi(z)} = -N \left( 1 + \frac{N \nu}{16 \pi \rho(z) \nabla^2} \right) |z| \delta(z^2 + m_f^2) \quad (18)
\]

The \( N^2 \) contribution follows from the arguments presented in \([22]\) for the emergence of a conformal dimension associated to \( e^{-N \varphi/2} \) in \((11)\). Solving for \( \varphi \) in \((18)\) yields the unquenched density to order \( O(N_f N^0) \)

\[
\rho_f(z) \approx \rho_0(z) - \frac{N_f}{2} (\delta(z + im_f) + \delta(z - im_f)) \quad (19)
\]

The unquenched energy in the ZMZ in flat eigenvalue space is

\[
E_f \approx -\frac{1}{2} \int_A \rho_f(z) (W(z) - N_f \ln(z^2 + m_f^2)) \quad (20)
\]

which differs from \((14)\) to order \( O(N_f N) \) by
\[ \mathcal{E}_f - \mathcal{E}_0 \approx \frac{\nu N_f}{4} \left( \frac{B}{\pi} \int_{A} \ln(z^2 + m_f^2) + \sum_{\pm} W(\pm im_f) \right) \]

(21)

The unquenched chiral condensate to the same order follows from \( V_4 \langle \bar{q}q \rangle_f = \partial \mathcal{E}_f / \partial \rho_f \). We note that \( |\langle \bar{q}q \rangle_f| < |\langle \bar{q}q \rangle_0| \). Indeed, (21) yields to order \( N_f/N \)

\[ V_4 \left( \langle \bar{q}q \rangle_f - \langle \bar{q}q \rangle_0 \right) \rightarrow N_f m_f \frac{\nu B}{4} \left( \frac{a+\pi}{a_+ + a_- + (1+\tau)} \right) \]

(22)

The effect of the fermion determinant on the 2-dimensional droplet amounts to inserting two static charges of \( \nu N_f/2 \) at the mirror locations \( z = \pm im_f \). The charges are strongly screened as we now detail.

V. SCREENING AND PLASMONS

In this section we will explicit the screening nature of the droplet of eigenvalues viewed as a 2-dimensional 1-specie plasma with unit charges \( e = 1 \). This is one of the universal feature of the effective action (11) which goes beyond the 2-matrix model used. In particular, we will derive the static structure factor of the droplet using a small and longitudinal deformation of the droplet density. We will observe that the static pole structure emerging from the structure factor nicely extends time-like to the plasmon pole contribution we have derived recently using a hydrodynamical analysis [19], with an improved estimate for a relaxation time in QCD at finite \( \mu \).

A. Static Structure Factor

By rescaling \( \tilde{z} = \sqrt{N} z \) in the microscopic limit, both \( B \) and \( \rho_0 \) are of order 1 and the droplet fills out the entire 2-plane in flat space. Within the center of the droplet, the \( \varphi(\pm im_f) \)-insertions do not affect the density and its response. The electro-static properties of the droplet are captured by the structure factor

\[ S(k) = \frac{1}{N} \left\{ \int d z \ e^{i k \cdot z} \ \rho(z) \right\}^{\text{conn.}} \]

(23)

The longitudinal deformation \( \delta \rho(z) \approx -\rho_0 \nabla \cdot \vec{\varphi} \) yields \( S(k) \approx \rho_0 k^2 \left\{ \langle \hat{\varphi}(k) \rangle^2 \right\} \). It follows by expanding \( \Gamma \) in (11) to quadratic order in \( \delta \rho \). The result is

\[ S(\tilde{k}) \approx \frac{\tilde{k}^2 \omega_p^2}{\omega_p - \frac{\beta - 2}{4} \tilde{k}^2 + \frac{\beta}{48} \frac{k^4}{\omega_p}} \]

(24)

with \( \omega_p = B/N \) and the rescaled momentum \( \tilde{k} = k/\sqrt{N} \). For \( \beta = 2 \), the screening length follows from \( \tilde{k}^4 + 24 \omega_p^2 = 0 \) or \( l_S \approx t/\sqrt{N} \). Back to z-space

\[ S(z) = \int \frac{d^2 \tilde{k}}{4\pi^2} e^{-i \tilde{k} \cdot z} S(\tilde{k}) = \frac{3 \omega_p^3}{4} \frac{3 |z|^4 \omega_p^2}{32} \]

(25)

\( G(r) \) is Meijer G-function shown in Fig. 2 with a logarithmic core \( G(r \ll l_S) \approx -\ln r \), a hole of range \( l_S \) and an asymptotic tail \( G(r \gg l_S) \approx e^{-2\sqrt{2}r} \).

B. Relaxation Time

The pole in (24) is the static limit of the longitudinal plasmon mode in the droplet. Indeed, (24) together with the hydrodynamical plasmon analysis in [19] implies the non-linear plasmon dispersion relation for the time-dependent longitudinal modes \( \dot{\phi}(z) \rightarrow \dot{\phi}(t, z) \)

\[ \left( \partial_t^2 + \left( \omega_p + \frac{\beta - 2}{4} \nabla^2 + \frac{\beta}{48} \frac{k^4}{\omega_p} \right)^2 \right) \tilde{\phi}(t, z) \approx 0 \]

(26)

which confirms and extends the hydrodynamical result [19]

\[ \left( \partial_t^2 + \left( \omega_p + \frac{\beta - 2}{4} \nabla^2 \right)^2 \right) \tilde{\phi}(t, z) \approx 0 \]

(27)

FIG. 2: Radial structure factor \( G(r) \) versus \( r \).
Now consider a time-dependent deviation of the eigenvalue density through $\delta \rho(t, z) \approx -\rho_0 \mathbf{\nabla} \cdot \mathbf{\tilde{\phi}}(t, z)$ in the complex 2-plane. For large droplets, the hydrodynamical arguments presented in [19] show that the deformation relaxes through Euler equations with a large time asymptotics controlled by the plasmon branch [27], i.e. $\delta \rho(t \rightarrow \infty, z)/\rho_0 \approx e^{-2\omega_p t}$, with a relaxation time $T_R \approx 1/2\omega_p$. If the initial condition for the eigenvalue distribution $\rho(0, z)$ is chosen to describe a chirally symmetric state at finite $\mu$, then the time it takes for the distribution to relax to the spontaneously broken phase is again given by the plasmon branch. From [26] it follows that the relaxation time for spontaneously breaking/restoring chiral symmetry in QCD at finite chemical potential in droplets of finite sizes $\mathcal{A} = 2\pi l^2$ is now of order

$$T_R \approx \frac{1}{2 \left( \omega_p - \frac{\beta - 2}{4} + \frac{\beta}{48} \omega_p A^2 \right)}$$

(28)

with $\nabla^2 \approx -1/\mathcal{A}$ and $\omega_p = B/N = 2\pi \beta / \mathcal{A}$. Although [26, 28] follow from the model with $\beta = 2$, they nicely agree and extend the results in [19] following from the hydrodynamical arguments for the three Dyson ensembles $\beta = 1, 2, 4$.

VI. BERRY PHASE

In this section we will show that if the Dirac eigenvalues where to change adiabatically with some mathematical time, i.e. $z_i \rightarrow z_i(t)$, then the effective action will develop among other thinghs a geometrical contribution of the Wess-Zumino-Witten type. The origin of this term will be traced to a particular contribution in the eigenvalue measure which distinguishes between real $\beta = 1$ or complex $\beta = 2$ representations for the quarks, i.e. whether the underlying quarks are Majorana or Dirac particles. This suggests that an adiabatically deformed droplet of eigenvalues at finite $\mu$ maps onto a 2-dimensional fermionic system in a magnetic field.

With this in mind, the magnetic field induced by the 2-matrix model is $B(z) = \mathcal{B} - \pi \alpha \delta(z)$. The first contribution defines the mean density $\nu \mathcal{B}/2\pi$ in flat space. The second contribution is a magnetic vortex of strength $\alpha/2 (z \rightarrow \tilde{z})$

$$A^i_z = \frac{\alpha}{2} \frac{\epsilon_{ij} z_j}{|z|^2}$$

(29)

is multivalued and generates a Berry phase [29, 30]. For that, consider an adiabatic time-dependent change in the eigenvalue through $\tilde{z} \rightarrow \tilde{z}(t)$ and $\tilde{v}(t) = \tilde{z}(t)/|z(t)|$. In the presence of [29] an anomalous Berry contribution is generated for each eigenvalue change as

$$S_1 = \int \tilde{A}^i_z \cdot \tilde{v} dt = \frac{\alpha}{2} \int \frac{v dz}{|z|^2} = \frac{\alpha}{2} \int v dv$$

(30)

The form notation is subsumed. The line integral is over the mathematical time which is understood as a relaxation time for Dirac spectra in 1+4-dimensions [19]. Each of the eigenvalue when adiabatically rotated around the origin accumulates a phase $S_1 = \alpha \pi$. For Dirac quarks in the complex representation and particle-anti-particle symmetry, $\beta = 2$ and $\alpha = 1$. The phase accumulation is $\pi$.

To generalize (30) to all particles in the droplet we borrow from the arguments in [30], and covariantize $v(t)$ in (30) by using the embedding $v(t) \rightarrow v^\mu(t, z) \equiv (1, \tilde{v}(t, z))$ in 1+2 dimensions. Thus

$$S_B = \frac{1}{2} \alpha \rho_0 \mathcal{A} \int_1 v dv - \frac{1}{2} \alpha \rho_0 \int_1^2 v dv$$

(31)

which is a Wess-Zumino-Witten type term.

VII. FERMI SURFACE

The complex eigenvalues form a droplet in the complex 2-plane for all three Dyson ensembles. However, for $\beta = 2$ with the underlying quarks in the Dirac representation, adiabatically changing eigenvalues behave as moving fermions in 2-dimensions with a fixed magnetic field $B$ as defined in [5] with filling fraction $\nu = 1/\beta = 1/2$ since the mean droplet density is $\rho_0 = \nu B / 2\pi$. Note that for $\beta = 1$ with the underlying quarks in the real or Majorana representation, the Berry phase is zero with no identification to fermions. We now explore the consequences of this relationship to fermions in 2-dimensions.

A. Anomalous Fermi Surface

We identify the Fermi momentum of the droplet as $p_0 \rightarrow k_F^2/4\pi$ or $k_F \equiv \sqrt{2\nu B}$ for $\beta = 2$. The anomalous transport on the Fermi surface follows by setting $p^\mu = k_F v^\mu \rightarrow k_F (\nu_F, \tilde{v}(t, z))$ in (31) and gauging by minimal substitution. The Fermi velocity $v_F < 1$. Thus

$$S_B = \frac{\alpha}{2\pi} \int_1^{1+2} \sqrt{g} (p + eA + sw) d(p + eA + sw)$$

(32)

Here $A$ is a U(1) gauge field with $F = dA$ and $\omega$ a U(1) spin connection with $R = d\omega$. Similar anomalous effective actions for the quantum Hall effects were recently discussed in [22, 31].

The U(1) current $J = \delta S_B/\delta A$ stemming from (32) is anomalous

$$\frac{1}{\sqrt{g}} \int \sqrt{g} \left( J - \frac{\alpha e}{8\pi k_F v} \right) = \frac{\alpha e^2}{8\pi} F + \frac{\alpha e s}{8\pi} R$$

(33)

The $k_F = \sqrt{2\nu B}$ contribution in (33) is the analogue of the chiral vortical effect in Fermi surfaces threaded by a
Berry phase in higher dimensions \[30\]. The anomalous contribution to the density is

\[
\rho_B(z) = \frac{1}{e} \frac{\delta S_B}{\delta A_0(z)} = \frac{\alpha e}{4\pi} B + \frac{\alpha s}{4\pi} R \quad (34)
\]

A comparison of (34) with \(\rho_0\) in (13) suggests that \(e = 2/\alpha \beta\) and \(s = 1/2\alpha\). For \(\beta = 2\) and \(\alpha = 1\), we have \(e = 1\) and \(s = 1/2\). The droplet of eigenvalues maps onto a fermionic droplets in 2-dimensions with constituents of charge \(e = 1\) and spin \(s = 1/2\).

B. Relation to Composite Fermions

Recently, Son has argued that in a non-zero magnetic field the composite fermions of the fractional quantum Hall effect at half filling, have a finite density and live in a zero magnetic field \[29\]. He argued that they exhibit particle-hole symmetry and that they are Dirac particles. Their ground state is a Fermi liquid with a Berry phase of \(\pi\).

Our construction suggests that the anomalous transport of composite fermions in the fractional quantum Hall effect at half filling in \(1 + 2\) dimensions with a magnetic field, share similarities with a 2-matrix model of the ZMZ effect at half filling in \(1 + 2\) dimensions with a magnetic field. The suggestion that composite fermions at half filling behave as Dirac particles with a Wess-Zumino-Witten term of the type \[32\] is falsifiable as it leads to specific and measurable predictions as we now detail.

C. Rotating Fermi Surface

We recall that novel chiral vortical effects were noted in rotating Weyl Fermi liquids with a Berry phase in \(1 + 3\) dimensions \[30\]. We now show that similar effects are expected for composite fermions at filling fraction \(\nu = 1/2\) in \(1 + 2\) dimensions if they are Dirac particles. Throughout this sub-section \(B\) is understood as a real (residual) magnetic field.

Consider now that the \(2\)-dimensional fermions are in a rotating frame along the \(z\)-direction with velocity \(\Omega = \Omega \hat{z}\). Each fermion experiences centrifugation that is best captured by the gravito-electro-magnetic fields \(E_g = p^0 \hat{\Omega} \theta\) and \(\vec{B}_g = p^0 \hat{\Omega}\) with a metric \(g_{00} = 1 - \Omega^2 |z|^2 + 2\theta \) \[32\]. \(\theta\) acts as a gravitational-like potential. In the Fermi sea, the inertial 2-force on a quasi-particle of 3-momentum \(p^\mu \rightarrow (\epsilon, s k_F)\) is the Lorentz-like force \[30\]

\[
F_g = e \hat{\Omega} \theta + \vec{k}_F \times s \hat{\Omega} \quad (35)
\]

with the Coriolis force manifest. Here \(s = \pm\) for a particle or anti-particle (hole). The corresponding Berry induced mixed Chern-Simons term in the Fermi liquid droplet is constructed following the arguments in \[30\] to which we refer to for more details. Since \(\epsilon \alpha = 2/\beta = 2\nu\), the result in our case is

\[
S_\Omega = \frac{\nu}{4\pi} \left( \sum_{s=\pm} \int_0^\infty \frac{d(se)}{\epsilon} f(\epsilon, s) \right) \times 2 \int_{1+2} \left( A_0 \hat{s} \hat{\Omega} + (\vec{A} \times \hat{\nabla} \theta) \right) \quad (36)
\]

The Fermi distribution \(f(\epsilon, s) = 1/(1+e^{(\epsilon-s k_F)/T})\) for the composite fermions with particle-hole symmetry satisfies

\[
\left( \sum_{s=\pm} \int_0^\infty d(se) f(\epsilon, s) \right) = \mu_F = k_F = \sqrt{2\nu B} \quad (37)
\]

\[
\left( \sum_{s=\pm} \int_0^\infty d(\epsilon) f(\epsilon, s) \right) = T \ln \left( 2 + 2 \mathrm{ch} \left( \frac{k_F}{T} \right) \right) \quad (38)
\]

\[30\] yields an anomalous and \(\Omega\)-driven contribution to the composite fermion density

\[
\rho_\Omega(z) = \frac{\delta S_\Omega}{\delta A_0} = \frac{\nu}{2\pi} T \ln \left( 2 + 2 \mathrm{ch} \left( \frac{k_F}{T} \right) \right) \Omega \quad (39)
\]

The \(\theta\)-contribution in \[39\] can be exploited using an observation by Luttinger who noted that the effect of a temperature gradient can be balanced by a gravitational potential \[35\]. The response of the composite fermions in a Fermi disc to a small temperature gradient can be captured by \(\hat{\nabla} \theta = -\hat{\nabla} \ln T\). As a result, an anomalous and measurable stationary gradient flow develops

\[
J_{\Omega,i}(z) = \frac{\delta S_\Omega}{\delta A_i} = \frac{\nu \sqrt{2\nu B}}{2\pi} \epsilon_{ij} \hat{\nabla} \ln T \quad (40)
\]

The results of this sub-section rely only on the interpretation that composite fermions are Dirac particles with particle-antiparticle (hole) symmetry and the origin of the Wess-Zumino-Witten term \[32\], with \(B\) a real (residual) magnetic field as we noted earlier. Therefore they are more general than the correspondence we have developed.
VIII. CONCLUSIONS

The QCD Dirac spectrum at finite chemical potential following from a 2-matrix model at large $N$ maps onto a generic but anomalous effective theory in 2-dimensions. Both gauge and conformal anomalies play a role at half filling in 1+2 dimensions. Anomalous freedoms share similarities with a 2-matrix model of Dirac with both gauge and conformal anomalies in bulk, much like the fractional quantum Hall effect [22]. The emergent Liouville effective action (9-10) follows from the conformal anomaly [22, 28]. In this Appendix, we give a brief account of the derivation following [22]. The change from the collective density $\rho$ to the collective potential $\varphi$ follows formally from (8) as

$$\det(-\nabla^2) D\varphi = \left(\int D\eta D\bar{\eta} e^{\int dV_2 \bar{\eta} \nabla^2 \eta} \right) D\varphi$$

(41)

with $\eta, \bar{\eta}$ as Grassmanians. The regularized measure $D\eta D\bar{\eta} D\varphi$ depends implicitly on the collective density $\rho$ and thus the collective potential $\varphi$ through regularization [22] [28]. As a result, $\rho$ can be treated as a conformal factor of the mathematical metric $g_{z \bar{z}}$ on the eigenvalues, and removed from the measure by a conformal transformation of the coordinates

$$dV_2 \equiv \sqrt{g} dz \rightarrow dV_2/\rho$$

(42)

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X. APPENDIX: CONFORMAL ANOMALY

The emergence of the Liouville effective action in [9, 10] follows from the conformal anomaly [22, 28]. The result is (9-10) after using the central charges $c_\rho = +1$ and $c_\eta = c_{\bar{\eta}} = -1$.

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