LOCALIZATION AND DELOCALIZATION OF EIGENMODES OF HARMONIC OSCILLATORS

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Abstract. We characterize quantum limits and semi-classical measures corresponding to sequences of eigenfunctions for systems of coupled quantum harmonic oscillators with arbitrary frequencies. The structure of the set of semi-classical measures turns out to depend strongly on the arithmetic relations between frequencies of each decoupled oscillator. In particular, we show that as soon as these frequencies are not rational multiples of a fixed fundamental frequency, the set of semi-classical measures is not convex and therefore, infinitely many measures that are invariant under the classical harmonic oscillator are not semi-classical measures.

1. Introduction

Understanding the distribution of high-frequency eigenfunctions of elliptic operators, in particular the presence of scarring (concentration on subsets of zero Lebesgue measure), has been the subject of intensive study in the past fifty years. In this note, we discuss this problem for eigenfunctions of Schr"odinger operators acting on $L^2(\mathbb{R}^d)$ of the form:

$$\hat{P} := -\frac{1}{2} \Delta_x + \frac{1}{2} Q,$$

with $Q$ a positive definite quadratic form on $\mathbb{R}^d$.

Since $\hat{P}$ has compact resolvent in $L^2(\mathbb{R}^d)$, its spectrum is discrete and unbounded. Moreover, the non-compactness of $\mathbb{R}^d$ implies that the $L^2(\mathbb{R}^d)$-mass of any sequence of normalized eigenfunctions with eigenvalues tending to infinity will escape any bounded set, both in position and momentum variables. In order to observe eventual concentration-like behavior on the eigenfunctions, it is natural to rescale the problem. Consider the unitary operators:

$$T_h : L^2(\mathbb{R}^d) \ni \psi \mapsto \frac{1}{h^{d/4}} \psi \left( \cdot \sqrt{\frac{1}{h}} \right) \in L^2(\mathbb{R}^d), \quad h > 0;$$

then

$$\hat{P}_h := -\frac{1}{2} h^2 \Delta_x + \frac{1}{2} Q = hT_h \hat{P} T_h^*.$$

A further unitary conjugation, which essentially amounts to diagonalizing the quadratic form $Q$, transforms $\hat{P}_h$ into a semi-classical quantum harmonic oscillator:

$$\hat{H}_h := \frac{1}{2} \sum_{j=1}^d \omega_j \left( -h^2 \partial^2_{x_j} + x_j^2 \right),$$

where $\omega_1^2 \leq \ldots \leq \omega_d^2$ are the eigenvalues of $Q$ and the frequencies are chosen to satisfy $\omega_j > 0$ for $j = 1, \ldots, d$. In particular, the spectrum of this operator satisfies:

$$\text{Sp}(\hat{H}_h) = h \text{Sp}(\hat{P}).$$

The vector

$$\omega := (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d_+$$
is called the vector of frequencies; its arithmetic properties will play an important role in
the sequel.

The above considerations lead to naturally consider the asymptotic properties of eigen-
functions in the semi-classical limit, that is, letting the eigenvalue $\lambda_n$ tend to infinity
and the semi-classical parameter $\hbar_n$ tend to zero while keeping the energy $\hbar_n\lambda_n$ constant.

More explicitly, we will consider sequences of eigenfunctions $(\Psi_{\hbar})$ satisfying

$$\hat{H}_\hbar \Psi_{\hbar} = \lambda_{\hbar} \Psi_{\hbar}, \quad \|\Psi_{\hbar}\|_{L^2(\mathbb{R}^d)} = 1,$$

$(3)$

where $\lambda_{\hbar} \to 1$ as $\hbar \to 0^+$. In particular, given a sequence $(\Psi_{\hbar})$ of solutions to $(3)$, we are interested in investigating
the structure of the accumulation points of the sequence of probability densities $(|\Psi_{\hbar}|^2 dx)$
in the weak-$\star$ topology of Radon measures on $\mathbb{R}^d$. These accumulation points are again
probability measures in $\mathbb{R}^d$ and are sometimes called quantum limits.

The existence of quantum limits that are singular with respect to the Lebesgue measure
is known as eigenfunction scarring; understanding which systems exhibit this type of
behavior is a notoriously difficult problem.

Quantum limits can be defined for general elliptic operators with compact resolvent, for
instance a Schrödinger operator on a compact manifold or a bounded domain of Euclidean
space. In this setting, they have been completely characterized in relatively few cases:
the Laplacian on spheres [22], and more generally on compact rank-one symmetric spaces
[24] or space forms [7]; and the Laplacian on the two dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ [21].

Spheres and their relatives exhibit the strongest form of scarring: any uniform measure
supported on a geodesic is a quantum limit. The situation is completely different for
tori: quantum limits are always absolutely continuous (although concentration in the
momentum variable is possible, this will be discussed below).

Other settings that are relatively well-understood include the Laplacian on Zoll mani-
folds [25] (manifolds whose geodesics are closed), where scarring takes place, but usually in
a weaker form: there are uncountable families of geodesics onto which no eigenfunctions
scar. The Laplacian on manifolds with non-degenerate completely integrable geodesic
flow has also been studied [2, 31, 33]; scarring cannot take place on regions that can be
parametrized by action-angle coordinates. A similar phenomenology can be found for the
Dirchlet Laplacian on planar domains with completely integrable billiard flow: rational
polygonal domains [20, 26] or the Euclidean disk [3].

Not much is known in the case the Laplacian on a manifold whose geodesic flow is
a small perturbation of a completely integrable system (KAM systems); however, there
have been some interesting recent developments in that direction [6, 17, 18], where some
form scarring has been proved. On the other side of the dynamics lie manifolds of negative
curvature: characterizing the set of Quantum Limits in this case is part of the Quantum
Unique Ergodicity conjecture [28]: the conjecture implies that the only quantum limit is
the Riemannian volume. The literature is vast in this setting; see, among many others,
[1, 4, 8, 9, 14, 19, 23, 29, 34].

As all the preceding examples indicate, a central role in this problem is played by a
Hamiltonian dynamical system (the geodesic flow) acting on phase-space (the cotangent
bundle of a manifold). Here, the relevant object is the classical harmonic oscillator:

$$H(x, \xi) = \frac{1}{2} \sum_{j=1}^{d} \omega_j (\xi_j^2 + x_j^2), \quad (x, \xi) \in T^*\mathbb{R}^d = \mathbb{R}^d \times \mathbb{R}^d,$$

$(4)$

whose induced Hamiltonian flow will be denoted by $\phi^{H}_t$. It is therefore natural to lift
to phase-space $T^*\mathbb{R}^d = \mathbb{R}^d_\xi \times \mathbb{R}^d_\xi$ all objects of interest, since this is where the classical
Hamiltonian flow is defined. Here we will lift the density $|\psi|^2$ of a function $\psi \in L^2(\mathbb{R}^d)$
through its associated Wigner function $W^h_{\psi} \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$, which is defined by:

$$W^h_{\psi}(x, \xi) := \int_{\mathbb{R}^d} \psi \left( x - \frac{hv}{2} \right) \psi \left( x + \frac{hv}{2} \right) e^{i\xi \cdot v} \frac{dv}{(2\pi)^d}.$$ 

That this is actually a lift follows from this useful property: for every $C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ that is bounded,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} a(x, \xi) W^h_{\psi}(x, \xi) \, dx \, d\xi = \langle \psi, \text{Op}_h(a)\psi \rangle_{L^2(\mathbb{R}^d)},$$

where $\text{Op}_h(a)$ denotes the semiclassical Weyl quantization of the symbol $a$ (see [13, 15, 35], for instance) and $\langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d)}$ is the scalar product on $L^2(\mathbb{R}^d)$. When $a$ does not depend on the $\xi$ variable, the operator $\text{Op}_h(a)$ is the multiplication operator by $a$, and therefore:

$$\int_{\mathbb{R}^d} W^h_{\psi}(:, \xi) \, d\xi = |\psi|^2. \tag{5}$$

If $(\Psi_h)$ solves (3) then $(W^h_{\Psi_h})$ is bounded in the space of tempered distributions $S'(\mathbb{R}^d \times \mathbb{R}^d)$. Moreover, every accumulation point of $(W^h_{\Psi_h})$ belongs to the set $\mathcal{M}(H)$ of Radon probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ that are concentrated on the level set $H^{-1}(1)$ and that are invariant by the Hamiltonian flow $\phi^H_t$. See for instance [16] for proofs of these facts.

We will denote by $\mathcal{M}(\hat{H}_h)$ the set of all accumulation points obtained in this way, as the sequence $(\Psi_h)$ varies among those satisfying (3). Its elements are called semi-classical measures.

The lift property (3) still holds after taking limits. Therefore, the set of quantum limits is obtained from $\mathcal{M}(\hat{H}_h)$ by projecting semi-classical measures onto the $x$-variable.

In order to state our result, we recall some basic facts on the dynamics of the harmonic oscillator. Set

$$H_j(x, \xi) = \frac{1}{2} (\xi_j^2 + x_j^2), \quad j \in \{1, \ldots, d\}.$$ 

We can write $H$ as a function of $H_1, \ldots, H_d$ by putting

$$H = L_\omega(H_1, \ldots, H_d), \tag{6}$$

where $L_\omega : \mathbb{R}^4_d \to \mathbb{R}$ is the linear form defined by $L_\omega(E) = \omega \cdot E$. Since $\{H_j, H_k\} = 0$ for every $j, k \in \{1, \ldots, d\}$, the Hamiltonian flow of $H$ can be written as

$$\phi^H_t(z) = \phi^H_{\omega_d} \circ \cdots \circ \phi^H_{\omega_1}(z), \quad z = (x, \xi) \in \mathbb{R}^{2d},$$

where $\phi^H_{\omega_j}$ denotes the flow of $H_j$. These flows are totally explicit, they act as a rotation of angle $t$ on the plane $(x_j, \xi_j)$. If one identifies points $(x_j, \xi_j)$ in this plane to the complex numbers $z_j := x_j + i\xi_j$, then $\phi^H_t(z)$ acts on this plane as $e^{-it}z_j$ and fixes the points in its orthogonal complement.

These flows are $2\pi$-periodic, therefore Konecker’s theorem shows that the orbit of $\phi^H_t$ of any point $z_0 \in \mathcal{X}$, where

$$\mathcal{X} := \{ z \in H^{-1}(1) : H_j(z) > 0, \quad j = 1, \ldots, d \},$$

is dense in a torus of dimension

$$d_\omega := \dim S_\omega,$$

where

$$S_\omega = \langle \omega_1, \ldots, \omega_d \rangle_Q$$

is the linear subspace of $\mathbb{R}^d$, viewed as a vector space over the rationals, spanned by the frequencies.
Remark 1. When $d_\omega = d$ then orbits corresponding to points $z_0 \in \mathcal{X}$ are dense in a $d$-dimensional Lagrangian torus. When $d_\omega = 1$, the flow $\phi^H_t$ is periodic of period

$$T_\omega := \frac{2\pi k_\omega}{\omega_1},$$

where $k_\omega$ is the least positive integer such that $k_\omega \omega_j/\omega_1 \in \mathbb{Z}$ for every $j = 1, \ldots, d$.

The Hamiltonian of the harmonic oscillator admits a decomposition that is particularly useful to our purposes. Let $\{I_n\}_{n=1}^{d_\omega}$ denote the partition of the set of indexes $\{1, \ldots, d\}$ induced by the equivalence relation $i \sim j$ if and only if $\omega_i = q\omega_j$ for some $q \in \mathbb{Q}$. Write for every $n = 1, \ldots, d_\omega$,

$$H_n := \sum_{j \in I_n} \omega_j H_j,$$

so that $H = \sum_{n=1}^{d_\omega} H_n$. The flows $\phi^H_t$ of $H_n$ with $n = 1, \ldots, d_\omega$ are periodic, since $\dim(\{\omega_j : j \in I_n\})_Q = 1$. Moreover, those measures invariant by the flow of he harmonic oscillator are precisely those simultaneously invariant by all $\phi^H_t$.

$$\mu \in \mathcal{M}(H) \iff (\phi^H_t)^* \mu = \mu, \quad \forall t \in \mathbb{R}, \ n = 1, \ldots, d_\omega. \tag{8}$$

Our main result characterizes the set of semi-classical measures associated to eigenstates of the Hamiltonian $\widehat{H}_h$. It turns out that it consists precisely of those invariant measures that supported on the intersection of the level sets of the Hamiltonians $H_n$, $n = 1, \ldots, d_\omega$.

**Theorem 1.** A probably measure $\mu$ on $\mathbb{R}^d \times \mathbb{R}^d$ satisfies $\mu \in \mathcal{M}(\widehat{H}_h)$ if and only if there exists $E \in [0,1]^{d_\omega}$ such that $\sum_{j=1}^{d_\omega} E_j = 1$,

$$\text{supp} \mu \subseteq \mathcal{H}_1^{-1}(E_1) \cap \cdots \cap \mathcal{H}_{d_\omega}^{-1}(E_{d_\omega}),$$

and

$$(\phi^H_t)^* \mu = \mu, \quad \forall t \in \mathbb{R}, \ n = 1, \ldots, d_\omega.$$ 

Note that, since the commuting Hamiltonian vector fields $X_{H_n}$ of $H_n$ are linearly independent when restricted to $\mathcal{X}$, Theorem 1 implies that the restriction of the measures in $\mathcal{M}(\widehat{H}_h)$ to $\mathcal{X}$ are smoother as $d_\omega$ increases.

Remark 2. In particular, Theorem 1 implies that $\mathcal{M}(\widehat{H}_h)$ is not a convex set, unless $d_\omega = 1$. In other words:

$$\mathcal{M}(\widehat{H}_h) = \mathcal{M}(H) \iff \phi^H_t \text{ is periodic}.$$ 

The first step in the proof of Theorem 1 follows from an argument on the propagation of wave-packets for periodic harmonic oscillators which is inspired in [12] and already yields the proof of Theorem 1 in the periodic case (see Lemma 1 in Section 2). The particular case of an isotropic harmonic oscillator, i.e. with $\omega = (1, \ldots, 1)$, was addressed in previous works [5, 27, 30], although those proofs are not easy to adapt to the general case. In Section 3 we show how to conclude the proof of the theorem using the decomposition (7).

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2. SCARRING FOR PERIODIC HARMONIC OSCILLATORS

We first prove a particular case of our result: when the Hamiltonian flow $\phi^H_t$ is periodic, one can find, for any orbit, sequences of eigenfunctions that concentrate on that orbit. Recall that the periodicity of the flow is equivalent to the condition $d_\omega = 1$ and the period $T_\omega$, which only depends on $\omega$, is defined in Remark 1.

The following result precisies and extends [12] and [27, Proposition 5] (see also [5], [11] and [30] for related works).

Lemma 1. Suppose that $\phi^H_t$ is periodic. Then for every $E_0 \geq 0$ there exist $h_0 > 0$ and a sequence $(\lambda_h)_{0<\omega\leq h_0}$ with $\lambda_h \to E_0$ as $h \to 0^+$ satisfying the following. For every $z_0 \in H^{-1}(E_0)$ there exists $(\Psi_h)_{0<\omega\leq h_0}$ such that

$$\hat{H}_h \Psi_h = \lambda_h \Psi_h, \quad \|\Psi_h\|_{L^2(\mathbb{R}^d)} = 1,$$

and, for every $a \in C_c^\infty(\mathbb{R}^{2d})$,

$$\langle \Psi_h, \text{Op}_h(a)\Psi_h \rangle_{L^2(\mathbb{R}^d)} = \frac{1}{T_\omega} \int_0^{T_\omega} a(\phi^H_t(z_0))dt + O(h^{1/2}), \quad h \to 0^+.$$

Proof. As it is well-known, the spectrum of $\hat{H}_h$ on $L^2(\mathbb{R}^d)$ is:

$$\text{Sp}(\hat{H}_h) = \left\{ \lambda_{k,h} = \mathcal{L}_\omega(hk) + \frac{\hbar|\omega|^2}{2} : k \in \mathbb{Z}^d \right\},$$

where $|\omega|^2 := \omega_1 + \ldots + \omega_d$. The fact that the flow $\phi^H_t$ is periodic implies that the spacing between consecutive (and large enough) eigenvalues of $\hat{H}_h$ is constant and that their corresponding multiplicities is very high. We claim that

$$\lambda_{h,N} := h \left( \frac{2\pi}{T_\omega} N + \frac{|\omega|^2}{2} \right), \quad N \in \mathbb{N}, N \geq N_0,$$

are precisely the eigenvalues of $\hat{H}_h$ that are greater or equal to $\lambda_{h,N_0}$. To see this, recall (see Remark 1) that $d_\omega = 1$ implies that there exists a least positive integer $k_\omega$ such that

$$\frac{k_\omega}{\omega_1} =: n_\omega \in \mathbb{Z}^d_+,$$

(the period of $\phi^H_t$ is then $T_\omega = 2\pi k_\omega/\omega_1$). The components of $n_\omega$ are positive integers $n_{\omega,1} = k_\omega \leq n_{\omega,2} \leq \ldots \leq n_{\omega,d}$ that are relatively prime. Therefore, there exists $N_0 \in \mathbb{N}$ such that, for every integer $N \geq N_0$, there exists $k \in \mathbb{Z}^d$ such that $k \cdot n_\omega = N$ (the smallest $N_0$ with this property is called the Frobenius number of the family $\{n_{\omega,j}\}_{j=1}^d$).

The claim then follows from the general form of the spectrum of the quantum harmonic oscillator [10].

With this in mind, we set:

$$N(h,E_0) := \left[ \frac{T_\omega}{2\pi} \left( \frac{E_0}{\hbar} - \frac{|\omega|^2}{2} \right) \right], \quad \text{for } E_0 > 0, \quad N(h,0) := 0,$$

and

$$\lambda_h := \lambda_{h,N(h,E_0)} = h \left( \frac{2\pi}{T_\omega} N(h,E_0) + \frac{|\omega|^2}{2} \right), \quad \text{which satisfies } |E_0 - \lambda_h| < \frac{2\pi}{T_\omega} \hbar.$$

We now write

$$\Psi^h_0(x) := \frac{1}{(\pi \hbar)^{d/4}} e^{-\frac{|x|^2}{2\hbar}},$$

and, provided $z_0 = (x_0,\xi_0)$, define the coherent state

$$\Psi^h_{z_0}(x) := e^{-\frac{i\xi_0 \cdot x}{\hbar}} e^{\frac{i\xi_0 \cdot x}{\hbar}} \Psi^h_0(x - x_0).$$
As in [12], consider its time-average:

\begin{equation}
\langle \Psi^h_{z_0} \rangle := \frac{1}{T_\omega} \int_{0}^{T_\omega} e^{i\frac{2\pi}{T_\omega} H_h \frac{\hbar}{T_\omega} z_0} dt.
\end{equation}

Since \( e^{-i\frac{\hbar}{T_\omega} H_h} = e^{-i\frac{\hbar}{T_\omega} |z_0|^2} \text{Id} \), it follows that \( \langle \Psi^h_{z_0} \rangle = \Psi^h_{z_0} \) is the ground state of \( \hat{H}_h \); more generally

\( \hat{H}_h \langle \Psi^h_{z_0} \rangle = \lambda_h \langle \Psi^h_{z_0} \rangle \).

On the other hand, it is known (see for instance [22, Sect. 23.4.2] or [10, Chpt. 3]) that

\begin{equation}
\langle \Psi^h_{z_0} \rangle := \frac{1}{T_\omega} \int_{0}^{T_\omega} e^{i\frac{2\pi N(h,E_0)}{T_\omega} H_h \frac{\hbar}{T_\omega} z_0} dt.
\end{equation}

We next give a stationary phase-type argument that is well suited to the particular structure of this integral.

We start by noting that the inner product in (16), is the cross-Wigner distribution of two phase-space translates of the same Gaussian \( \Psi^0_0 \) acting on the test function \( a \). Define:

\( \varphi(t, s, z) := -\frac{\sigma(H^0(z_0), H^0(z_0))}{2} + \sigma(H^0(z_0), H^0(z_0), z), \)

where \( \sigma(\cdot, \cdot) \) is the canonical symplectic form in \( \mathbb{R}^{2d} \). A direct computation then shows:

\begin{equation}
\langle \langle \Psi^h_{z_0}, \Psi^h_{z_0} \rangle \rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} a(z) e^{\frac{i\sigma(t-s,z)}{\hbar}} W^h_{\Psi^h_0}(z - \frac{\phi^h_t(z_0) + \phi^h_s(z_0)}{2}) dz,
\end{equation}

where \( W^h_{\Psi^h_0} \), the Wigner distribution of \( \Psi^h_0 \), is:

\( W^h_{\Psi^h_0}(z) = \frac{1}{\hbar^d} G \left( \frac{z}{\sqrt{\hbar}} \right), \quad G(z) := \frac{1}{\pi^d} e^{-|z|^2}. \)

Inserting this into (16) gives:

\begin{equation}
\int_{0,T_\omega} \int_{\mathbb{R}^{2d}} e^{i\frac{2\pi N(h,E_0)}{T_\omega} H_h \frac{\hbar}{T_\omega} z_0} a \left( \frac{\phi^h_t(z_0) + \phi^h_s(z_0)}{2} + \sqrt{\hbar}z \right) G(z) dz ds dt \frac{ds dt}{(T_\omega)^2}.
\end{equation}

In order to simplify this expression, note first that, since the flow \( \phi^H_t \) is linear and Hamiltonian:

\begin{equation}
\frac{\sigma(\phi^H_t(z_0), \phi^H_s(z_0))}{2} = \frac{\sigma(\phi^H_s(z_0), \phi^H_t(z_0))}{2} = -H(z_0)(t-s) + (t-s)^3g(t-s),
\end{equation}

where \( g \) is analytic. Second, by [12], we have

\begin{equation}
m_h := \frac{2\pi N(h,E_0)}{T_\omega} - \frac{E_0}{\hbar} = \frac{\lambda_h - E_0}{\hbar} - \frac{|\omega|_1}{2}, \quad |m_h| \leq \frac{2\pi}{T_\omega} + \frac{|\omega|_1}{2}.
\end{equation}
Taking into account that the integrand of (18) is a $T_0\mathbb{Z}^2$-periodic function with respect to $(t, s)$, after performing a change of variables, and using (19), (20) we conclude that (18) is equal to:

$$\int_{[0,T_0]} \int_{\mathbb{R}^{2d}} e^{it\text{ms}} e^{i\frac{t^3g(t)}{\hbar}} e^{i\left(\phi^H_{t+\frac{s}{\hbar}}(z) - \phi^H_s(z)\right)} a\left(\frac{\phi^H_{t+\frac{s}{\hbar}}(z) + \phi^H_s(z)}{2}\right) + \sqrt{\hbar}G(z)dz dt = O(\hbar) =$$

Now we use that:

$$\sigma\left(\phi^H_{t+\frac{s}{\hbar}}(z) - \phi^H_s(z), z\right) = -t\hbar\phi^H_s(z) + t^2 r(t, s) \cdot z,$$

where $r$ is analytic in both arguments, and that:

$$e^{it^2 r(t, s) \sqrt{\hbar}a}\left(\frac{\phi^H_{t+\frac{s}{\hbar}}(z) + \phi^H_s(z)}{2} + \sqrt{\hbar}\right) = a\left(\frac{\phi^H_{t+\frac{s}{\hbar}}(z) + \phi^H_s(z)}{2}\right) + O(\sqrt{\hbar})R_h,$$

with a remainder such that $R_0G$ is uniformly bounded in $\mathcal{S}(\mathbb{R}^{2d})$ with respect to $\hbar, t, s$. Then (21) can be transformed in:

$$\sqrt{\hbar}\int_{0}^{T_0} \int_{\mathbb{R}^{2d}} e^{i\sqrt{h}(tm_n + t^3g(t))} e^{-it\hbar\phi^H_s(z)} a\left(\frac{\phi^H_{t+\frac{s}{\hbar}}(z) + \phi^H_s(z)}{2}\right) G(z)dz dt + O(h) =$$

$$= \frac{1}{T_\omega} \int_{0}^{T_0} \sqrt{\hbar}\int_{0}^{T_\omega} e^{i\sqrt{h}(tm_n + t^3g(t))} a\left(\frac{\phi^H_{t+\frac{s}{\hbar}}(z) + \phi^H_s(z)}{2}\right) G(\hbar\phi^H_s(z))\frac{dt}{T_\omega}dz =$$

$$= \int_{0}^{T_\omega} e^{i\sqrt{h}(tm_n + t^3g(t))} a\left(\frac{\phi^H_{t+\frac{s}{\hbar}}(z) + \phi^H_s(z)}{2}\right) G(\hbar\phi^H_s(z))\frac{dt}{T_\omega}dz =$$

Inserting this identity back in (22), we conclude that, taking $a = 1$:

$$c_h := \frac{T_\omega|dH_{z_0}|}{\sqrt{\pi\hbar}||\langle \Psi^h_{z_0} \rangle||^2_{L^2(\mathbb{R})}} \longrightarrow 1, \quad \hbar \rightarrow 0^+,$$

and that

$$\Psi^h := \left(\frac{T_\omega|dH_{z_0}|}{c_h\sqrt{\pi\hbar}}\right)^{1/2} \langle \Psi^h_{z_0} \rangle$$

are the desired normalized eigenstates satisfying (9).

\[\square\]

3. PROOF OF THEOREM 1

We start with the case $d_\omega = 1$. This is a direct consequence of Lemma 11 and a density-orthogonality argument (see for instance [24]) that we recall here for the sake of completeness. The Krein-Milman theorem ensures that the convex hull of the set of orbit measures in $H^{-1}(1)$ is dense in $\mathcal{M}(H)$. Therefore, it suffices to show that any measure $\mu$ that is a convex combination of distinct orbit measures $\delta_{\gamma_1}, \ldots, \delta_{\gamma_r}$ belongs to $\mathcal{M}(\widehat{H}_h)$. 

\[\square\]
Lemma \[1\] shows the existence of sequences of normalized eigenfunctions \((\psi_h^j)\), \(j = 1, \ldots, r\) corresponding to the same sequence of eigenvalues \((\lambda_h)\) such that:

\[
W_h^{k, j} \xrightarrow{n} \delta_{\eta_j}, \quad h \to 0^+, \quad j = 1, \ldots, r.
\]

Suppose that \(\mu = \sum_{j=1}^r \alpha_j \delta_{\eta_j}\) with \(\alpha_j \in (0, 1)\); since the measures \(\delta_{\eta_j}\) are mutually singular, it follows (see for instance \[16\] Proposition 3.3) that

\[
\lim_{h \to 0^+} \langle \psi_h^j, \text{Op}_h(a)\psi_h^k \rangle_{L^2(\mathbb{R}^d)} = 0, \quad j \neq k.
\]

This suffices to show that the sequence of eigenfunctions \(\psi_h := \sum_{j=1}^r \sqrt{\alpha_j} \psi_h^j\) has \(\mu\) as a semiclassical measure. Even though the sequence \((\psi_h)\) is not normalized, its norm tends to one so one concludes that \(\mu \in \mathcal{M}(\hat{H}_h)\).

Let us turn now to the case \(d_\omega > 1\). Using the decomposition

\[
\hat{H}_h = \sum_{j=1}^{d_\omega} \text{Op}_h(\mathcal{H}_n),
\]

where the periodic Hamiltonians \(\mathcal{H}_n\) were defined in \([7]\). Since, by construction, the families \(\{\omega_j : j \in I_n\}_{n \in \{1, \ldots, d_\omega\}}\) are linearly independent over \(\mathbb{Q}\), it follows from \([10]\) that given any \(\lambda_h \in \text{Sp} \hat{H}_h\) there exist unique \(\lambda_h^n \in \text{Sp} (\text{Op}_h(\mathcal{H}_n))\), \(n = 1, \ldots, d_\omega\), such that:

\[
\lambda_h = \lambda_h^1 + \ldots + \lambda_h^{d_\omega},
\]

and the corresponding eigenfunctions \(\psi_h\) satisfy:

\[
\text{Op}_h(\mathcal{H}_n)\psi_h = \lambda^n_h \psi_h. \tag{24}
\]

Suppose now that \((\psi_h)\) is a sequence of normalized eigenfunctions with eigenvalues \((\lambda_h)\) converging to one and such that \(W_h^{k, j} \xrightarrow{n} \mu \in \mathcal{M}(\hat{H}_h)\) as \(h \to 0^+\). Then there exists a subsequence \((h_k)\), converging to zero, such that:

\[
\forall n = 1, \ldots, d_\omega, \quad \lim_{k \to \infty} \lambda_{h_k}^n = E_n \in [0, 1], \quad \text{and} \quad \sum_{n=1}^{d_\omega} E_n = 1.
\]

Identity \[(24)\] then ensures that:

\[
(\mathcal{H}_n - E_n)\mu = 0, \quad X_{\mathcal{H}_n}\mu = 0, \quad n = 1, \ldots, d_\omega,
\]

where \(X_{\mathcal{H}_n}\) denotes the Hamiltonian vector field of \(\mathcal{H}_n\). Therefore,

\[
\mu \in \mathcal{M}_{\{E_1, \ldots, E_{d_\omega}\}}(\mathcal{H}) := \{\mu \in \mathcal{P}(H^{-1}(1)) : \text{supp} \mu \subseteq \mathcal{H}_n^{-1}(E_n), \ X_{\mathcal{H}_n}\mu = 0, \forall n = 1, \ldots, d_\omega\}.
\]

Conversely, let us show that for any choice of \(E = (E_1, \ldots, E_{d_\omega}) \in [0, 1]^{d_\omega}\) such that \(|E|_1 = 1\) one has that \(\mathcal{M}_E(\mathcal{H})\) is contained in \(\mathcal{M}(\hat{H}_h)\). To this aim, note that up to permutation of the coordinates, we can assume that

\[
1 \leq n < m \leq d_\omega \implies i < j, \quad \forall i \in I_n, \forall j \in I_m.
\]

Each flow \(\phi_t^{\mathcal{H}_n}\) can be identified in a natural way to a flow acting on \(\mathbb{R}^r_n\), with \(r_n := \#I_n\). Using again the Krein-Milman theorem, we deduce that the convex hull of the set of uniform multi-orbit measures, \(i.e.\) measures of the form:

\[
\delta_{\gamma_1} \otimes \ldots \otimes \delta_{\gamma_{d_\omega}}, \quad \text{where} \ \gamma_n \subseteq \mathcal{H}_n^{-1}(E_n) \text{ is an orbit of } \phi_t^{\mathcal{H}_n},
\]

is dense in \(\mathcal{M}_E(\mathcal{H})\). Note also that each operator \(\text{Op}_h(\mathcal{H}_n)\) acts naturally on \(L^2(\mathbb{R}^r_n)\); the spectrum on this space coincides with that on \(L^2(\mathbb{R}^d)\). Therefore, we can apply Lemma \[1\] to each operator \(\text{Op}_h(\mathcal{H}_n)\) and energy \(E_n\) to ensure the existence of a sequence of
eigenvalues $\lambda_n^\hbar \to E_n$ as $\hbar \to 0^+$ and normalized eigenfunctions $\psi_n^\hbar \in L^2(\mathbb{R}^n)$ such that,
\[
\lim_{\hbar \to 0^+} \langle \psi_n^\hbar, \text{Op}_\hbar(a)\psi_n^\hbar \rangle_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^{2n}} a(x, \xi) \delta_{\gamma_n}(dx, d\xi).
\]
Let now $\psi_\hbar := \psi_1^\hbar \otimes \cdots \otimes \psi_d^\hbar$. This is a normalized eigenfunction of $\hat{H}_\hbar$ with eigenvalue $\lambda_\hbar = \sum_{n=1}^{d\omega} \lambda_n^\hbar$. By construction, it follows that:
\[
W_\psi^\hbar \nrightarrow \delta_{\gamma_1} \otimes \cdots \otimes \delta_{\gamma_d}, \quad \lambda_\hbar \to 1, \quad \text{as } \hbar \to 0^+.
\]
Since the sequences $(\lambda_n^\hbar)$ only depend on the energies $E_n$ and not on the particular orbits $\gamma_n \subseteq H_n^{-1}(E_n)$ we have chosen, we can argue as we did in the case $d\omega = 1$ to conclude that any convex combination of measures of the form (25) can be obtained as the semiclassical measure of a normalized sequence of eigenfunctions. This concludes the proof of the theorem.

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