Super-Simple \((v, 5, 2)\) Directed Designs and Their Smallest Defining Sets with Application in LDPC Codes

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Abstract
In this paper, we show that for all \(v \equiv 0, 1 \pmod{5}\) and \(v \geq 15\), there exists a super-simple \((v, 5, 2)\) directed design. Moreover, for these parameters, there exists a super-simple \((v, 5, 2)\) directed design such that its smallest defining sets contain at least half of its blocks. Also, we show that these designs are useful in constructing parity-check matrices of LDPC codes.

Keywords
Super-simple directed design · Trade · LDPC codes · Tanner graph

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1 Introduction and Preliminaries
A group divisible design (or GDD) is a triple \((X, G, B)\) which satisfies the following properties:

1. \(G\) is a partition of the set \(X\) into subsets called groups;
2. \(B\) is a set of subsets of \(X\) called blocks such that a group and a block intersect in at most one point;
3. Each pair of points from distinct groups occurs in exactly \(\lambda\) blocks.

The group type of GDD is the multiset \(\{\vert G \vert : G \in G\}\). We use the notation \(g_1^{u_1}g_2^{u_2} \cdots g_n^{u_n}\) to denote \(u_i\) occurrences of \(g_i\) for \(1 \leq i \leq n\) in the multiset. A GDD with block sizes from a set of positive integers \(K\) is called a \((K, \lambda)\)-GDD. When \(K = \{k\}\) (resp. \(\lambda = 1\)), we simply call it a \((k, \lambda)\)-GDD (resp. \(k\)-GDD). When \(\lambda = 1\),
we simply write $K$-GDD. A $(K, \lambda)$-GDD with group type 1 is called a pairwise balanced design and denoted by PBD$(v, K, \lambda)$. A $(k, \lambda)$-GDD with group type 1 is called a balanced incomplete block design, denoted by $(v, k, \lambda)$-BIBD.

Some generalizations have been introduced for the concept of designs. Gronau and Mullin [17], for the first time, introduced super-simple block designs. A super-simple $(v, k, \lambda)$ design is a block design such that any two blocks of it intersect in at most two points. A simple block design is a block design which has no repeated blocks. The existence of super-simple $(v, 4, \lambda)$ designs has been characterized for $2 \leq \lambda \leq 9$; see Refs. [4–8, 10, 17, 25, 28]. Also, the existence of super-simple $(v, 5, \lambda)$ designs has been characterized for $2 \leq \lambda \leq 5$; see Refs. [9, 11, 12, 14].

A directed group divisible design $(K, \lambda)$-DGDD is a group divisible design in which every block is ordered and each ordered pair formed from distinct elements of different groups occurs in exactly $\lambda$ blocks. A $(k, \lambda)$-DGDD with group type 1 is called a directed balanced incomplete block design and denoted by $(v, k, \lambda)$-DBIBD or $(v, k, \lambda)$-DD. A $(K, \lambda)$-DGDD is super-simple if its underlying $(K, 2\lambda)$-GDD is super-simple. A transversal design TD$(k, \lambda, n)$ is a $(k, \lambda)$-GDD of group type $n^k$.

Lemma 1.1 [1]

1. A TD$(q + 1, q)$ exists, consequently, a TD$(k, q)$ exists for any positive integer $k (k \leq q + 1)$, where $q$ is a prime power.
2. A TD$(7, n)$ exists for all $n \geq 63$.

Definition 1.2 [22] Let $0 < t \leq k \leq v$ and $\lambda > 0$ be integers. A $t - (v, k, \lambda)$ directed design (or simply a $t - (v, k, \lambda)$-DBIBD) is a pair $(V, B)$, where $V$ is a set of $v$ elements, called points, and $B$ is a collection of ordered $k$-tuples of distinct elements of $V$, called blocks, with the property that every ordered $t$-tuple of distinct elements of $V$ occurs in exactly $\lambda$ blocks (as a subsequence). We say that a $t$-tuple appears in a $k$-tuple if its components appear in that $k$-tuple with the same order.

A super-simple $(v, k, \lambda)$ directed design is a directed design for which any two blocks intersect in at most two points. A simple directed block design is a directed block design having no repeated blocks.

A set of blocks which is a subset of a unique $(v, k, \lambda)$-DBIBD is said to be a defining set of the directed design. A defining set is minimal if no proper subset of it is a defining set. A smallest defining set is a defining set with the smallest cardinality. A $(v, k, t)$ directed trade of volume $s$ consists of two disjoint collections $T_1$ and $T_2$ each of $s$ ordered $k$-tuples of a $v$-set $X$, called blocks, such that every ordered $t$-tuple of distinct elements of $X$ is covered by exactly the same number of blocks of $T_1$ as of $T_2$. Such a directed trade is usually denoted by $T = T_1 - T_2$. In a $(v, k, t)$ directed trade, both collections of blocks cover the same set of elements. This set of elements is called the foundation of the trade. In [22], it has been shown that the minimum volume of a $(v, k, t)$ directed trade is $2\lfloor \frac{t}{2} \rfloor$ and that directed trades of minimum volume and minimum foundation exist. In this paper, we use a special type of a directed trade which is defined as follows.

Definition 1.3 [23] Let $T = T_1 - T_2$ be a $(v, k, 2)$ directed trade of volume $s$ with blocks $b_0, b_1, \ldots, b_{s-1}$ such that each pair of consecutive blocks of $T_1$ ($b_i, b_{i+1}$
The corresponding graph of trades in Example 1.5

\[ (1, 2, 3) \ (4, 3, 2) \]

\[ (2, 1, 4) \ (3, 4, 1) \]

\( i = 0, 1, \cdots, s - 1 \text{ (mod } s) \) is a trade of volume 2. Such a trade is called a cyclical trade (we denote a cyclical trade of volume \( s \) by \( CT_s \)).

If \( D = (V, B) \) is a directed design with \( T_1 \subset B \), we say that \( D \) contains the directed trade \( T \). Defining sets for directed designs are strongly related to trades. This relation is illustrated by the following result.

**Example 1.4** Consider \( v = 8, G = \{ [0, 6], [1, 3], [2, 4], [5, 7] \} \) and \( B = \{ (1, 7, 2, 0), (7, 1, 4, 6), (0, 4, 1, 5), (4, 0, 3, 7), \)
\((5, 3, 0, 2), (3, 5, 6, 4), (2, 6, 5, 1), (6, 2, 7, 3) \} \); this is a super-simple 4-DGDD, and contains 12 trades of volume 2. The trades of volume 4 of this design are \( T_1 = \{ (1, 7, 2, 0), (5, 3, 0, 2), (6, 2, 7, 3), (4, 0, 3, 7) \} \) and \( T_2 = \{ (7, 1, 4, 6), (0, 4, 1, 5), \)
\((2, 6, 5, 1), (3, 5, 6, 4) \} \); \( T = T_1 - T_2 \) is an \( (8, 4, 2) \) directed trade of volume 4. In this example, \( CT_4 = \{ (1, 7, 2, 0), (6, 2, 7, 3), (4, 0, 3, 7), (5, 3, 0, 2) \} \) is a cyclical trade of volume 4.

**Example 1.5** On the set, \( V = \{ 1, \ldots, 4 \} \) given a 2 - (4, 3, 1)-DBIBD with blocks \( B = \{ (1, 2, 3), (2, 1, 4), (3, 4, 1), (4, 3, 2) \} \). As shown in Fig. 1, each edge between two vertices (blocks) represents a trade of volume two and the corresponding trades to the edges are shown in the following table.

| \( T_1 \) | \( T_2 \) |
|-----------|-----------|
| \( (1, 2, 3) \) | \( (2, 1, 3) \) |
| \( (2, 1, 4) \) | \( (1, 2, 4) \) |

| \( T_1 \) | \( T_2 \) |
|-----------|-----------|
| \( (1, 2, 3) \) | \( (2, 1, 4) \) |
| \( (4, 3, 2) \) | \( (4, 2, 3) \) |

| \( T_1 \) | \( T_2 \) |
|-----------|-----------|
| \( (1, 3, 2) \) | \( (2, 4, 1) \) |
| \( (3, 4, 1) \) | \( (3, 1, 4) \) |

**Proposition 1.6** \[20\] Let \( D = (V, B) \) be a \( (v, k, \lambda) \)-DBIBD and let \( S \subset B \). Then, \( S \) is a defining set of \( D \) if and only if \( S \) contains a block of every \( (v, k, 2) \) directed trade \( T = T_1 - T_2 \) such that \( T \) is contained in \( D \).

Each defining set of a \( (v, k, \lambda) \)-DBIBD \( D \) contains at least one block from each trade in \( D \). In particular, if \( D \) contains \( m \) mutually disjoint directed trades, then the smallest defining set of \( D \) must contain at least \( m \) blocks. If a directed design \( D \)
contains a cyclical trade of volume \( s \), then each defining set of \( D \) must contain at least \( \lceil \frac{s}{2} \rceil \) blocks of \( T_1 \).

Some results have been obtained on \((v, k, \lambda)\)-DBIBDs for special \( k \) and \( \lambda \) and their defining sets. For example, in Ref. [20], it is proved that if \( D \) is a \((v, 3, 1)\)-DBIBD, then a defining set of \( D \) has at least \( \frac{v}{2} \) blocks. In Ref. [16], it has been shown that for each admissible value of \( v \), there exists a simple \((v, 3, 1)\)-DBIBD whose smallest defining sets have at least a half of the blocks. In Ref. [23], it has been shown that the necessary and sufficient condition for the existence of a super-simple \((v, 4, 1)\)-DBIBD is \( v \equiv 1 \pmod{3} \) and for these values of \( v \) except for \( v = 7 \), there exists a super-simple \((v, 4, 1)\)-DBIBD whose smallest defining sets have at least half of the blocks. Also, in Ref. [24], the authors proved that for all \( v \equiv 1, 5 \pmod{10} \) except for \( v = 5, 15 \), there exists a super-simple \((v, 5, 1)\)-DBIBD such that its smallest defining sets have at least half of the blocks. In Ref. [3], the authors showed that for all \( v \equiv 1 \pmod{3} \), there exists a super-simple \((v, 4, 2)\)-DBIBD whose smallest defining sets have at least half of the blocks.

In this paper, we prove that the necessary and sufficient condition for the existence of a super-simple \((v, 5, 2)\)-DBIBD is \( v \equiv 0, 1 \pmod{5} \) (\( v \geq 15 \)), and for these values of \( v \), there exists a super-simple \((v, 5, 2)\)-DBIBD whose smallest defining sets have at least half of the blocks. We introduce the following quantity:

\[
d = \frac{\text{the total number of blocks in a smallest defining set in } D}{\text{the total number of blocks in } D}
\]

and we show for all admissible values of \( v \), \( d \geq \frac{1}{2} \).

In the last section, we use the idea from Ref. [2] to construct parity-check matrices of LDPC codes by \((v, 5, 2)\) directed designs.

### 2 Recursive Constructions

For some values of \( v \), the existence of a super-simple \((v, 5, 2)\)-DBIBD will be proved by recursive constructions presented in this section for later use.

**Construction 2.1** (Weighting) [3] Let \((X, G, B)\) be a super-simple DGDD with index \( \lambda_1 \) and with \( d \geq \frac{1}{2} \). Let \( w : X \to \mathbb{Z}^+ \cup \{0\} \) be a weight function on \( X \), where \( \mathbb{Z}^+ \) is the set of positive integers. Suppose that for each block \( B \in B \), there exists a super-simple \((k, \lambda_2)\)-DGDD of type \( \{w(x) : x \in B\} \) with \( d \geq \frac{1}{2} \). Then, there exists a super-simple \((k, \lambda_1 \lambda_2)\)-DGDD of type \( \{\sum_{x \in G_i} w(x) : G_i \in G\} \) with \( d \geq \frac{1}{2} \).

**Construction 2.2** [3] If there exist a super-simple \((k, \lambda)\)-DGDD of type \( g_1^{u_1} \cdots g_t^{u_t} \) with \( d \geq \frac{1}{2} \) and a super-simple \((g_i + \eta, k, \lambda)\)DBIBD for each \( i \) (\( 1 \leq i \leq t \)) with \( d \geq \frac{1}{2} \), then there exists a super-simple \((\sum_{i=1}^t g_i u_i + \eta, k, \lambda)\)DBIBD with \( d \geq \frac{1}{2} \), where \( \eta = 0 \) or \( 1 \).
3 Direct Construction

In this section, we construct some super-simple \((v, 5, 2)\)-DBIBDs for some small admissible values of \(v\) and some super-simple directed group divisible designs by direct construction. Moreover, for these values of \(v\), we show that the parameter \(d\) for the constructed designs is at least \(\frac{1}{2}\).

In what follows, we use the notation \(+d\ (\text{mod}\ v)\), which denotes that all elements of the base blocks should be developed cyclically by adding \(d\ (\text{mod}\ v)\) to them, while the infinite point \(\infty\), if it occurs in the base blocks, is always fixed. We usually omit \(+d\) when \(d = 1\). Let \([a, b]_{5}^{0,1}\) be the set of positive integers \(v \equiv 0, 1\ (\text{mod}\ 5)\) and \(a \leq v \leq b\).

**Lemma 3.1** There exists a super-simple \((v, 5, 2)\)-DBIBD, for all \(v \in [15, 86]_{5}^{0,1} \cup \{95, 110, 111, 115, 116, 130, 131\}\), whose smallest defining sets have at least half of the blocks.

**Proof** For \(v = 15\) and \(G = Z_{14} \cup \{\infty\}\), the following base blocks by \(+2\ (\text{mod}\ 14)\) form a super-simple \((15, 5, 2)\)-DBIBD:

\[
(1,0,2,3,8) \quad (0,3,13,11,9) \quad (0,1,4,10,9) \\
(0,7,11,4,2) \quad (1,0,\infty,5,7) \quad (13,2,\infty,0,10)
\]

This design contains 42 blocks and each of the three columns has 7 disjoint directed trades of volume 2. Since every defining set for this design must contain one 5-tuple of each directed trade in each of column, it follows that each defining set contains at least \(7 \times 3 = 21\) blocks. Therefore, \(d \geq \frac{1}{2}\).

For \(v = 25\) and \(G = Z_{24} \cup \{\infty\}\), the following base blocks by \(+1\ (\text{mod}\ 24)\) form a super-simple \((25, 5, 2)\)-DBIBD:

\[
(0,5,1,7,15) \quad (22,0,5,21,11) \quad (12,0,1,10,4) \\
(2,0,\infty,17,21) \quad (13,6,1,0,9)
\]

There are 120 blocks in a super-simple \((25, 5, 2)\)-DBIBD. The first two columns have 48 disjoint directed trades of volume 2, and the last column is a cyclical trade of volume 24. Since each defining set for this super-simple directed design must contain at least one 5-tuple of each directed trade in the first two columns and twelve 5-tuples of cyclical trade in the last column, we deduce that each defining set must contain at least \(48 + 12 = 60\) blocks. Therefore, for this super-simple \((25, 5, 2)\)-DBIBD, the inequality \(d \geq \frac{1}{2}\) is satisfied.

For \(v \in [16, 36]_{5}^{0,1}\) except \(v = 15, 25\), the results are summarized in the following table.
The above table has five columns. The first column contains the values of \(v\) and the second one contains the base blocks. The third column shows how to develop the elements of the base blocks. The last two columns contain the number of blocks of corresponding design and the least possible value of \(d\), respectively. For the remaining values of \(v\), their associated super-simple directed designs are presented in Appendix.

\[\square\]

**Lemma 3.2** There exists a super-simple \((5, 2)\)-DGDD of type 5\(^5\) with \(d \geq \frac{1}{2}\).
| \( v \) | Base blocks | \( \text{mod} \) | \( b_v \) | \( d \geq \) |
|---|---|---|---|---|
| 16 | \((3,0,1,8,6)\) | \((1,7,14,2,11)\) | \((2,5,0,1,4)\) | +2 \( \text{mod} \) 16 | 48 | \( \frac{3 \times 8}{48} \) |
| \( \geq 16 \) | \( (7,0,3,11,5) \) | \( (0,2,12,7,8) \) | \( (1,0,10,7,9) \) | | | |
| 20 | \((0,4,3,9,16)\) | \((9,0,1,8,14)\) | \((11,2,4,8,0)\) | \( \text{mod} \) 19 | 76 | \( \frac{2 \times 19}{76} \) |
| \( (5,0,\infty, 7,8) \) | \((8,2,4,0,16)\) | \((0,19,18,7,10)\) | \((0,9,14,4,15)\) | | | |
| 21 | \((3,0,6,8,7)\) | \((0,4,3,9,16)\) | \((9,0,1,18,14)\) | \( \text{mod} \) 21 | 84 | \( \frac{2 \times 21}{84} \) |
| \( \geq 20 \) | \( (5,13,0,7,22)\) | \((0,19,18,7,10)\) | \((1,0,10,7,9)\) | | | |
| 26 | \((0,11,20,19)\) | \((3,0,19,7,25)\) | \((8,0,13,14,24)\) | \( \text{mod} \) 26 | 130 | \( \frac{2 \times 26 + 13}{130} \) |
| \( (16,0,11,20,19) \) | \((0,12,1,24,3)\) | \((0,9,14,4,15)\) | \((0,9,14,4,15)\) | | | |
| 30 | \((0,3,16,21,23)\) | \((2,20,10,0,25)\) | \((0,27,4,10,11)\) | \( \text{mod} \) 29 | 174 | \( \frac{3 \times 29}{174} \) |
| \( (1,0,15,2,26) \) | \((9,0,4,12,26)\) | \((20,8,\infty, 1,0)\) | \((20,8,\infty, 1,0)\) | | | |
| 31 | \((3,7,0,15,1)\) | \((9,0,12,26)\) | \((27,0,3,19,2)\) | \( \text{mod} \) 31 | 186 | \( \frac{3 \times 31}{186} \) |
| \( (18,0,27,16,26) \) | \((0,19,9,6,26)\) | \((27,0,3,19,2)\) | \((27,0,3,19,2)\) | | | |
| 35 | \((6,7,0,30,12)\) | \((0,24,18,10,31)\) | \((0,4,6,5,21)\) | \( (23,0,32,8,25)\) | \( \text{mod} \) 34 | 238 | \( \frac{3 \times 34 + 17}{238} \) |
| \( (0,32,11,29,20) \) | \((5,8,12,20,0)\) | \((1,15,\infty, 4,0)\) | \((1,15,\infty, 4,0)\) | | | |
| 36 | \((1,0,6,9,21)\) | \((3,13,0,2,27)\) | \((13,22,0,18,26)\) | \( (2,17,9,30,0)\) | \( \text{mod} \) 36 | 252 | \( \frac{2 \times 36 + 36 + 2 \times 18}{252} \) |
Proof Let $X = Z_{25}$ and let $G = \{ [i, 5+i, 10+i, 15+i, 20+i] | 0 \leq i \leq 4 \}$. Here are the base blocks. These blocks are developed by $+5 \pmod{25}$.

This directed group divisible design has 100 blocks, containing 10 disjoint directed trades of volume 2 in each of the five columns. Because each defining set for this design must contain one block of each directed trades, each defining set contains at least 50 blocks. Therefore, $d \geq \frac{1}{2}$.

Lemma 3.3 For each $t$ $6 \leq t \leq 10$, there exists a super-simple $(5, 2)$-DGDD of type $5^t$ with $d \geq \frac{1}{2}$.

Proof Let the point set be $Z_{5t}$ and let the group set be $\{ [i, i+t, i+2t, i+3t, i+4t] | 0 \leq i \leq t-1 \}$. The required base blocks are listed below. All the base blocks are developed by mod $5t$.

4 Main Theorem

In this section, we try to find super-simple $(v, 5, 2)$-DBIBDs for some admissible values of $v$ by recursive constructions presented in Sect. 2 and using super-simple DGDDs obtained in Sect. 3.

Lemma 4.1 There exists a super-simple $(v, 5, 2)$-DBIBD for each $v \in \{ 20i + \eta | 5 \leq i \leq 9, \eta = 0, 1 \}$ with $d \geq \frac{1}{2}$.

Proof Using a super-simple $(5, 2)$-DGDD of type $5^t$ for $5 \leq t \leq 9$ with $d \geq \frac{1}{2}$ obtained in Lemmas 3.2 and 3.3 and applying Construction 2.1 with a TD$(5, 4)$ as an input design coming from Lemma 1.1, we obtain a super-simple $(5, 2)$-DGDD of type $(20)^t$ with $d \geq \frac{1}{2}$. On the other hand, by Lemma 3.1, there exists a super-simple $(20+\eta, 5, 2)$-DBIBD. Therefore, by Construction 2.2, we obtain a super-simple $(20t + \eta, 5, 2)$-DBIBD with $d \geq \frac{1}{2}$, where $\eta = 0$ or 1.

Lemma 4.2 There is a super-simple $(v, 5, 2)$-DBIBD for each $v \in \{ 125, 126, 145, 146, 150, 151 \}$ with $d \geq \frac{1}{2}$. 
| $b_t$ | $d$ | $t$ | $b_x$ | $d$ | $t$ | $b_o$ | $d$ | $t$ | $b_i$ | $d$ | $t$ | $b_f$ | $d$ | $t$ |
|-------|-----|-----|-------|-----|-----|-------|-----|-----|-------|-----|-----|-------|-----|-----|
| 450   | 30  | 150 | 30    | 150 | 2     | 30    | 150 | 2     | 30    | 150 | 2     | 30    | 150 | 2     |
| 450   | 30  | 150 | 2     | 30    | 150 | 2     | 30    | 150 | 2     | 30    | 150 | 2     | 30    | 150 | 2     |
| 450   | 30  | 150 | 2     | 30    | 150 | 2     | 30    | 150 | 2     | 30    | 150 | 2     | 30    | 150 | 2     |
| 450   | 30  | 150 | 2     | 30    | 150 | 2     | 30    | 150 | 2     | 30    | 150 | 2     | 30    | 150 | 2     |

**Base blocks**

| $< p$ | $t_q$ |
|-------|-------|
| 1     | 1     |
| $t$ | Base blocks | $b_t$ | $d \geq$ |
|-----|-------------|-------|--------|
| 6   | (32,15,0,83,1) | (16,56,15,0,43) | (85,2,88,81,0) | (0,77,1,57,82) | 1350 | $\frac{7 \times 90 + 45}{1350}$ |
|     | (0,22,68,85,33) | (40,87,37,59,0) | (26,51,55,64,0) | (63,55,0,76,23) |     |         |
|     | (44,0,41,7,15) | (0,31,47,51,70) | (0,45,73,71,44) | (0,80,10,55,69) |     |         |
|     | (74,25,69,16,0) | (0,71,43,33,10) | (38,0,15,17,49) |                 |     |         |
| 7   | (0,31,57,75,76) | (0,50,55,102,72) | (27,40,93,0,99) | (0,66,74,82,93) | 1890 | $\frac{9 \times 105}{1890}$ |
|     | (0,11,85,10,100) | (15,73,82,0,95) | (0,3,55,88,79) | (81,0,62,101,64) |     |         |
|     | (0,34,81,73,99) | (0,82,31,102,37) | (19,760,44,1) | (5,0,53,94,69) |     |         |
|     | (76,22,0,58,68) | (0,51,92,54,94) | (45,0,43,81,93) | (0,69,92,96,101) |     |         |
|     | (22,26,60,0,59) |                 |                 |                 |     |         |
|     | (0,46,71,44,61) |                 |                 |                 |     |         |
| 9   | (25,55,60,0,85) | (12,49,52,80,0) | (0,42,10,48) | (22,7,0,51,110) | 3240 | $\frac{12 \times 135}{3240}$ |
|     | (23,0,105,65,76) | (38,6,0,107,22) | (8,420,0,96) | (23,0,84,71,58) |     |         |
|     | (46,0,7,116,85) | (3,20,0,13,46) | (88,28,69,35,0) | (33,52,0,120,91) |     |         |
|     | (70,0,28,59,49) | (0,11,32,134,66) | (84,57,16,0) | (6,0,49,92,26) |     |         |
|     | (0,24,98,104,25) | (46,123,0,125,61) | (102,44,85,0,14) | (1,2,0,5,24) |     |         |
|     | (41,3,56,0,70) | (98,0,74,23,87) | (97,35,92,14,0) | (5,0,56,98,118) |     |         |
Proof We delete \(5 - a\) points from the last group of a TD(6, 5) coming from Lemma 1.1 to obtain a \(\{5, 6\}\)-GDD of type \(5^a a^1\). Applying Construction 2.1 and using a super-simple \((5, 2)\)-DGDD of group type \(5^5\) and \(5^6\) with \(d \geq \frac{1}{2}\) from Lemmas 3.2 and 3.3, we get a super-simple \((5, 2)\)-DGDD of type \((25)^5(5a)^1\) with \(d \geq \frac{1}{2}\). Since by Lemma 3.1 there exist a super-simple \((25 + \eta, 5, 2)\)-DBIBD and a super-simple \((5a + \eta, 5, 2)\)-DBIBD for \(a \in \{0, 4, 5\}\) and \(\eta = 0, 1\), we get a super-simple \((125 + 5a + \eta, 5, 2)\)-DBIBD with \(d \geq \frac{1}{2}\) by Construction 2.2.

Lemma 4.3 There is a super-simple \((v, 5, 2)\)-DBIBD for \(v \in \{155, 156\}\) with \(d \geq \frac{1}{2}\).

Proof Starting from a \(5\)-GDD of type \(3^{71}\) (given by Lemma 4.3 in Ref. [12]) and applying Construction 2.1, using a super-simple \((5, 2)\)-DGDD of type \(5^5\) as an input design, we get a super-simple \((5, 2)\)-DGDD of type \((15)^8(35)^1\) with \(d \geq \frac{1}{2}\). Since by Lemma 3.1 there exists a super-simple \((15 + \eta, 5, 2)\)-DBIBD and a super-simple \((175 + \eta, 5, 2)\)-DBIBD, we get a super-simple \((155 + \eta, 5, 2)\)-DBIBD with \(d \geq \frac{1}{2}\) by Construction 2.2, where \(\eta = 0\) or 1.

Lemma 4.4 There is a super-simple \((v, 5, 2)\)-DBIBD for each \(v \in \{170, 171, 175, 176, 185, 186\}\) with \(d \geq \frac{1}{2}\).

Proof Beginning with a \(\{5, 6, 7, 8\}\)-GDD of type \(4^{761}\) (given in Lemma 4.4 in Ref. [12]) and applying Construction 2.1 using a super-simple \((5, 2)\)-DGDD of types \(5^5\), \(5^6\), \(5^7\) with \(d \geq \frac{1}{2}\) given in Lemmas 3.2 and 3.3, we get a super-simple \((5, 2)\)-DGDD of type \((20)^7(30)^1\) with \(d \geq \frac{1}{2}\). Since by Lemma 3.1 there exist a super-simple \((20 + \eta, 5, 2)\)-DBIBD and a super-simple \((30 + \eta, 5, 2)\)-DBIBD with \(d \geq \frac{1}{2}\), we get a super-simple \((170 + \eta, 5, 2)\)-DBIBD with \(d \geq \frac{1}{2}\) by Construction 2.2, where \(\eta = 0\) or 1.

Beginning with a \(TD(5, 7)\) coming from Lemma 1.1 and applying Construction 2.1 using a super-simple \((5, 2)\)-DGDD of type \(5^5\) with \(d \geq \frac{1}{2}\) coming from Lemma 3.2, we get a super-simple \((5, 2)\)-DGDD of type \((35)^5\) with \(d \geq \frac{1}{2}\). Since by Lemma 3.1 there exists a super-simple \((35 + \eta, 5, 2)\)-DBIBD, we have a super-simple \((175 + \eta, 5, 2)\)-DBIBD with \(d \geq \frac{1}{2}\) by Construction 2.2, where \(\eta = 0\) or 1.

Starting from a \(\{5, 6, 7\}\)-GDD of type \(5^{671}\) (given in Lemma 4.4 in Ref. [12]) and applying Construction 2.1 using a super-simple \((5, 2)\)-DGDD of types \(5^5\), \(5^6\) and \(5^7\) with \(d \geq \frac{1}{2}\) given in Lemmas 3.2 and 3.3, we get a super-simple \((5, 2)\)-DGDD of type \((25)^6(35)^1\) with \(d \geq \frac{1}{2}\). Since by Lemma 3.1 there exist a super-simple \((25 + \eta, 5, 2)\)-DBIBD and a super-simple \((35 + \eta, 5, 2)\)-DBIBD, we get a super-simple \((185 + \eta, 5, 2)\)-DBIBD with \(d \geq \frac{1}{2}\) by Construction 2.2, where \(\eta = 0\) or 1.

Lemma 4.5 There exists a super-simple \((v, 5, 2)\)-DBIBD for any \(v \in \{90, 91, 105, 106, 135, 136\}\) with \(d \geq \frac{1}{2}\).

Proof By Lemma 3.4, there is a super-simple \((5, 2)\)-DGDD of type \((15)^t\) with \(d \geq \frac{1}{2}\) for \(t \in \{6, 7, 9\}\). Since by Lemma 3.1 there exist a super-simple \((15 + \eta, 5, 2)\)-DBIBD with \(d \geq \frac{1}{2}\) for \(\eta = 0, 1\), Construction 2.2 results in a super-simple \((15t + \eta, 5, 2)\)-DBIBD with \(d \geq \frac{1}{2}\), where \(\eta = 0\) or 1.
Lemma 4.6 There exists a super-simple $(96, 5, 2)$-DBIBD with $d \geq \frac{1}{2}$.

Proof A super-simple $(5, 2)$-DGDD of group type $4^6$ is listed as follows. Let $X = Z_{24}$ and $G = \{(i, i + 6, 12 + i, 18 + i) | 0 \leq i \leq 5\}$. Below are the required base blocks. All the base blocks are developed by mod 24.

\[(0, 2, 1, 4, 11), (1, 0, 5, 22, 15), (13, 2, 0, 16, 21), (0, 1, 20, 9, 16)\]

This super-simple DGDD has 96 blocks, each of two columns has 24 disjoint directed trades of volume 2. Therefore each defining set for this super-simple DGDD contains at least $24 \times 2 = 48$ blocks. So $d \geq \frac{1}{2}$.

By making use of this DGDD and applying Construction 2.1 with a $TD(5, 4)$ from Lemma 1.1, we get a super-simple $(5, 2)$-DGDD of type $(16)^6$ with $d \geq \frac{1}{2}$. Since by Lemma 3.1 there exist a super-simple $(16, 5, 2)$-DBIBD with $d \geq \frac{1}{2}$, Construction 2.2 leads to super-simple $(96, 5, 2)$-DBIBD with $d \geq \frac{1}{2}$.

Lemma 4.7 There is a super-simple $(v, 5, 2)$-DBIBD for any $v \in \{165, 166\}$ with $d \geq \frac{1}{2}$.

Proof Let the point set be $X = Z_{33}$ and the group set be $G = \{(i, i + 11, i + 22) | 0 \leq i \leq 10\}$. Below are the required base blocks. All the base blocks are developed by mod 33.

\[(6, 2, 0, 3, 27) - (10, 0, 26, 2, 19), (1, 0, 4, 6, 5), (9, 15, 19, 0, 29), (1, 13, 20, 0, 8), (2, 0, 15, 30, 5)\]

This super-simple DGDD has 198 blocks, each of three columns has 33 disjoint directed trades of volume 2. Hence each defining set for this super-simple DGDD contains at least $33 \times 3 = 99$ blocks. So $d \geq \frac{1}{2}$.

Starting from this DGDD and applying Construction 2.1 with a $TD(5, 5)$ coming from Lemma 1.1, we get a super-simple $(5, 2)$-DGDD of type $(15)^{11}$ with $d \geq \frac{1}{2}$. Because by Lemma 3.1 there exists a super-simple $(15 + \eta, 5, 2)$-DBIBD, Construction 2.2 leads to super-simple $(165 + \eta, 5, 2)$-DBIBD with $d \geq \frac{1}{2}$, where $\eta = 0$ or $1$.

Lemma 4.8 Suppose that $5 \leq k \leq 10$ is an integer. Let $N(m) \geq k - 2$, and let $M = \{5m, 5a_1, \ldots, 5a_r\}$, where $a_i \in [3, m] \cup \{0\}$, $1 \leq i \leq r$. If there is a super-simple $(l + \eta, 5, 2)$-DBIBD with $d \geq \frac{1}{2}$, then there exists a super-simple $(25m + 5 \sum_{i=1}^r a_i + \eta, 5, 2)$-DBIBD with $d \geq \frac{1}{2}$, where $\eta = 0$ or $1$.

Proof By Lemma 4.8 in Ref. [12], there exists a $(5, 6, \ldots, k)$-GDD of type $m^5(a_1)^1(a_2)^1 \cdots (a_r)^1$. Beginning with this GDD and applying Construction 2.1 using a super-simple $(5, 2)$-DGDDs of type $5^t$ for $t \in \{5, 6, \ldots, k\}$ with $d \geq \frac{1}{2}$ coming from Lemmas 3.2 and 3.3, we get a super-simple $(5, 2)$-DGDD of type $(5m)^5(5a_1)^1(5a_2)^1 \cdots (5a_r)^1$ with $d \geq \frac{1}{2}$. Because there exists a super-simple $(u + \eta, 5, 2)$DBIBD for any $u \in M$, by Construction 2.2 we have a super-simple $(25m + 5 \sum_{i=1}^r a_i + \eta, 5, 2)$-DBIBD with $d \geq \frac{1}{2}$, where $\eta = 0$ or $1$.

Lemma 4.9 There exists a super-simple $(v, 5, 2)$-DBIBD for any $v \in [190, 1591]^{0.1}_{5}$ with $d \geq \frac{1}{2}$.
Proof Applying Lemma 4.8 with parameters in the following table, we obtain a super-simple \((v, 5, 2)\)-DBIBD for every \(v \in [190, 1591]_5^0, 1\). All required \(TD(k, m)\) exist by Lemma 1.1.

We are now in a position to present the main theorem of this section.

**Theorem 4.10** For all \(v \equiv 0, 1 \pmod{5}\) and \(v \geq 15\), there exist a super-simple \((v, 5, 2)\)-DBIBD with \(d \geq \frac{1}{2}\).

**Proof** The proof is by induction on \(v\). By the above lemmas, the result is true for \(v \in [15, 1591]_5^0, 1\). Therefore, we assume that \(v \geq 1595\). We can write \(v = 25m + 5(a_1 + a_2) + \eta\), where \(m \geq 63\), \(\eta = 0, 1\), \(\{a_1, a_2\} \subset [3, m] \cup \{0\}\) and \(a_1 + a_2 \in [3, 2m]\). By the induction hypothesis, there exists a super-simple \((5m + \eta, 5, 2)\)-DBIBD and a super-simple \((5a_i + \eta, 5, 2)\)-DBIBD, for \(i = 1, 2\) with \(d \geq \frac{1}{2}\). Since \(N(m) \geq 5\), we deduce that there exist a super-simple \((v, 5, 2)\)-DBIBD by Lemma 4.8.

5 Constructing Some LDPC Codes Using Super-Simple \((v, 5, 2)\)-DBIBDs

A low-density parity-check (LDPC) code was first proposed by Gallager in 1960 in his dissertation. Then, these codes were ignored for about 36 years and then rediscovered by Mackey [15, 19].

An \(m \times n\) sparse matrix is one that many of its entries are equal to zero. An LDPC code is a linear code with a sparse parity check matrix. A binary code with a parity-check matrix in which the number of 1’s in each column is \(d_c\) and the number of 1’s in each row is \(d_r\), is called a regular (or bi-regular) LDPC code. Otherwise it is an irregular LDPC code. Tanner generalized the LDPC codes and introduced a graphical representation of them, called Tanner graph [26]. It is a bipartite graph \(G = (L \cup R, E)\), whose adjacency matrix is the parity-check matrix of the code where columns are identified with the left nodes (\(L\)) of the bipartite graph (the message nodes), rows with the right nodes (\(R\)) of the graph (the check nodes) and \(E\) is the set of edges. The girth of the code is the length of shortest cycle in the Tanner graph \(G\) denoted by \(g\).

Reducing the parity-check matrix of an LDPC code to standard form to allow systematic coding may render it nonsparse, leading to inefficient encoding. Better encoding techniques are given in Ref. [21]. Some approaches to construct LDPC codes are algebraic-based, protograph-based, and convolutional LDPC codes. Numerous techniques for the construction of LDPC codes have been proposed. These include the original codes of Gallager [15], MacKay codes [19], irregular degree sequence codes, and codes based on combinatorial structures such as finite geometries and designs [13, 18, 27].

In this section, using the super-simple \((v, 5, 2)\) directed designs, we obtain parity-check matrices of trade-based LDPC codes. We also use a graph of trade, in which its vertices are associated with the blocks of a super-simple directed design and each edge between two vertices (blocks) represents a trade of volume two, and then provide an example to show this method in an irregular LDPC code with girth 8.
\[ v = 25m + 5 \sum_{i=1}^{r} a_i + \eta \]

|       |   |   |                     |   |
|-------|---|---|---------------------|---|
|       |   |   | \[\sum_{i=1}^{k-5} a_i\] |   |
| [190, 281]^{0,1}_5 | 7 | 8 | [3, 21] | 0, 1 |
| [285, 451]^{0,1}_5 | 9 | 10 | [12, 45] | 0, 1 |
| [455, 651]^{0,1}_5 | 13 | 10 | [26, 65] | 0, 1 |
| [655, 1251]^{0,1}_5 | 25 | 10 | [6, 125] | 0, 1 |
| [1255, 1591]^{0,1}_2 | 36 | 10 | [71, 138] | 0, 1 |
Let $V = \{0, 1, \ldots, v - 1\}$ be a $v$-set, corresponding to a super-simple \((v, 5, 2)\)-DBIBD with $n$ blocks $b_1, b_2, \ldots, b_n$. We construct a \((\frac{v}{2}) \times n\) matrix $A$ with columns indices $b_1, b_2, \ldots, b_n$ and rows indices $(x_i, x_j)$, where $x_i < x_j$ and $x_i, x_j \in \{0, 1, \ldots, v - 1\}$, as follows:

$$A_{(x_i, x_j)l} = \begin{cases} 
1 & \text{if } (x_i, x_j) \in b_l \text{ which } b_l \text{ appears in a trade}, \\
0 & \text{o.w.}
\end{cases}$$

The matrix denoted by $C$, that obtained by removing all zero rows and columns of $A$ is considered as the parity-check matrix of trade-based LDPC code if the number of rows of $C$ is less than or equal to the number of its columns. Otherwise, $C^\perp$ will be the parity-check matrix of the code. This parity-check matrix is the adjacency matrix of the Tanner graph. In each column and row the number of 1s is at most $\lambda(k - 1)$ and $2\lambda$, respectively. Because any pair $(x_i, x_j)$, where $x_i < x_j$, occurs in $\lambda$ blocks and each block of length $k$ contains $k - 1$ pairs $(x_i, x_j)$s with $x_i < x_j$, which may appear in $\lambda$ trades.

The main theorem in previous section shows that for all $v \equiv 0, 1 \pmod{5}$ and $v \geq 15$, there exists a super-simple \((v, 5, 2)\)-DBIBD whose smallest defining sets have at least half of the blocks. Because of this property and Proposition 1, we can make parity-check matrices of LDPC codes where the number of rows and columns is greater than or equal to the number of blocks. Thus, the length of such LDPC code is at least the number of blocks.

In the super-simple \((v, 5, 2)\)-DBIBD, we have the following possibilities for its directed trades since $\lambda = 2$.

1. Two blocks have a common pair in which the first block contains $(x_i, x_j)$ and the second one contains $(x_j, x_i)$. In this case, the directed trade is of the form

   $\begin{array}{|c|c|}
   \hline
   T_1 & T_2 \\
   \hline
   b_1 : (x_i, x_j, \ldots) & (x_j, x_i, \ldots) \\
   b_2 : (x_j, x_i, \ldots) & (x_i, x_j, \ldots) \\
   \hline
   \end{array}$

2. Three blocks have a common pair in which two blocks contain $(x_i, x_j)$ and the third one contains $(x_j, x_i)$, or vice versa. In this case, we have two directed trades of volume 2 of the form

   $\begin{array}{|c|c|}
   \hline
   T_1 & T_2 \\
   \hline
   b_1 : (x_i, x_j, \ldots) & (x_j, x_i, \ldots) \\
   b_3 : (x_j, x_i, \ldots) & (x_i, x_j, \ldots) \\
   \hline
   \end{array}$

   $\begin{array}{|c|c|}
   \hline
   \hline
   T_1 & T_2 \\
   \hline
   b_2 : (x_i, x_j, \ldots) & (x_j, x_i, \ldots) \\
   b_3 : (x_j, x_i, \ldots) & (x_i, x_j, \ldots) \\
   \hline
   \end{array}$

   The third block containing $(x_j, x_i)$ is common in two trades.

3. Four blocks have a common pair in which two blocks contain $(x_i, x_j)$ and two other ones contain $(x_j, x_i)$. In this situation we have four directed trades of volume 2 as shown below:
The next proposition describes the relation between the smallest volume of cyclical trade and the girth of trade-based LDPC code, which is used to construct the codes.

**Proposition 5.1** [2] A super-simple directed design has a cyclical trade of volume $s$ if and only if the Tanner graph of the corresponding trade-based LDPC code has $2s$-cycles.

**Proposition 5.2** The girth of an LDPC code constructed using super-simple $(v, 5, 2)$-DBIBD is at least 6.

**Proof** From Proposition 5.1, we conclude that the girth of the Tanner graph of the corresponding trade-based LDPC code constructed by using super-simple $(v, 5, 2)$-DBIBD must be even. By the concept of super-simple directed designs, every two blocks intersect in at most two points or an $(x_i, x_j)$. Therefore, the Tanner graph is 4-cycle free. In other words, there is no cycle in its Tanner graph of the following type:

\[ b_i : (x_i, x_j, \ldots), (x_j, x_i, \ldots) \]

\[ b_j : (x_j, x_i, \ldots), (x_i, x_j, \ldots) \]

If we have cyclical trade of volume 3, then the girth of the Tanner graph of its trade-based LDPC code must be 6, otherwise the code has girth at least 8. \hfill \Box

Combining proposed technique to construct a parity-check matrix for an LDPC code based on the concept of trades in super-simple directed designs that explained above, Propositions 5.1 and 5.2, we have the following result.

**Theorem 5.3** The existence of a super-simple $(v, 5, 2)$-DBIBD whose smallest defining set has at least half of the blocks can deduce an LDPC code with the girth at least 6 whose length is at least the number of blocks.

For a better understanding of the above, we give the following example.

**Example 5.4** Here are the blocks of a super-simple $(15, 5, 2)$DBIBD in Lemma 3.1:

\[ B_1 = (1, 0, 2, 3, 8) \quad B_{15} = (0, 1, 4, 10, 9) \quad B_{29} = (1, 0, \infty, 5, 7) \]
\[ B_2 = (3, 2, 4, 5, 10) \quad B_{16} = (3, 2, 6, 12, 11) \quad B_{30} = (2, 3, \infty, 7, 9) \]
Fig. 2 The normal graph corresponding to super-simple \((15, 5, 2)\)-DBIBD based on directed trades

\[
\begin{align*}
B_3 &= (5, 4, 6, 7, 12) & B_{17} &= (5, 4, 8, 0, 13) & B_{31} &= (4, 5, \infty, 9, 11) \\
B_4 &= (7, 6, 8, 9, 0) & B_{18} &= (7, 6, 10, 2, 1) & B_{32} &= (6, 7, \infty, 11, 13) \\
B_5 &= (9, 8, 10, 11, 2) & B_{19} &= (9, 8, 12, 4, 3) & B_{33} &= (8, 9, \infty, 13, 1) \\
B_6 &= (11, 10, 12, 13, 4) & B_{20} &= (11, 10, 0, 6, 5) & B_{34} &= (10, 11, \infty, 1, 3) \\
B_7 &= (13, 12, 0, 1, 6) & B_{21} &= (13, 12, 2, 8, 7) & B_{35} &= (12, 13, \infty, 3, 5) \\
B_8 &= (0, 3, 13, 11, 9) & B_{22} &= (0, 7, 11, 4, 2) & B_{36} &= (13, 2, \infty, 0, 10) \\
B_9 &= (2, 5, 1, 13, 11) & B_{23} &= (2, 9, 13, 6, 4) & B_{37} &= (1, 4, \infty, 2, 12) \\
B_{10} &= (4, 7, 3, 1, 13) & B_{24} &= (4, 11, 1, 8, 6) & B_{38} &= (3, 6, \infty, 4, 0) \\
B_{11} &= (6, 9, 5, 3, 1) & B_{25} &= (6, 13, 3, 10, 8) & B_{39} &= (5, 8, \infty, 6, 2) \\
B_{12} &= (8, 11, 7, 5, 3) & B_{26} &= (8, 1, 5, 12, 10) & B_{40} &= (7, 10, \infty, 8, 4) \\
B_{13} &= (10, 13, 9, 7, 5) & B_{27} &= (10, 3, 7, 0, 12) & B_{41} &= (9, 12, \infty, 10, 6) \\
B_{14} &= (12, 1, 11, 9, 7) & B_{28} &= (12, 5, 9, 2, 0) & B_{42} &= (11, 0, \infty, 12, 8) \\
\end{align*}
\]
Fig. 3 The Tanner graph of an irregular LDPC code of girth 8 corresponding to the super-simple $(15, 5, 2)$-DBIBD

Fig. 4 A normal graph (4-cycle) and its corresponding (8-cycle)
Using this super-simple directed design, we construct its corresponding matrix \( C \) having the same 42 rows and columns, and also, its normal graph on 42 vertices (see Fig. 2). As shown in Fig. 2, each block appears in \( \{2, 3, 4, 5\} \) trades of volume 2 (we show each \( B_i \) by \( i, i \in \{1, 2, \ldots, 42\} \) in Fig. 2). The graph is triangle-free which shows that there is no cyclical trades of volume 3. In the normal graph, the smallest cycle is of size 4 which corresponds to a \( CT_4 \). Therefore, \( C \) is the parity-check matrix of an irregular LDPC code with \( d_c = \{1, 2, 3, 4\} \) 1s in columns and \( d_r = \{2, 3, 4\} \) 1s in rows. Figure 3 corresponds to the tanner graph of irregular LDPC code of super-simple \((15, 5, 2)\)-DBIBD whose girth of it is 8. This girth that shown in Fig. 4 can be \( g = \{B_2, (4, 5), B_{31}, (9, 11), B_{14}, (7, 9), B_{30}, (2, 3), B_2\} \).

The same can be done for the other existing super-simple \((v, 5, 2)\)-DBIBD.

Appendix
| $v$ | Base blocks | $b_v$ | $d$ |
|-----|-------------|-------|-----|
| 40  | (7,8,18,5,0) (0.38,22,12,30) | (0.11,13,27,36) (0.28,∞, 4,33) | (0.14,17,2,22) (6,0,1,38,18) | (0.20,24,30,6) (0.7,26,35,3) | mod 39 | 312 | $\frac{4 \times 39}{312}$ |
| 41  | (0.4,1.11,29) (36,39,28,7,0) | (6,8,27,0,32) (1.19,0,25,15) | (0,11,7,10,23) (0.8,38,36,29) | (0.39,20,6,15) (0.23,1,27,17) | mod 41 | 328 | $\frac{4 \times 41}{328}$ |
| 45  | (0.1,12,1,2,7) (0,4,10,23,12) | (0.40,9,26,43) (0,41,24,5,36) | (0,18,38,32,39) (3.9,∞, 0,32) | (0,3,1,29,19) (0.16,43,14,36) | mod 44 | 396 | $\frac{4 \times 44+22}{396}$ |
| 46  | (14,1,7,0,10) (0,7,2,16,33) | (12,0,24,26,32) (10,0,15,37,28) | (31,0,28,25,29) (17,0,25,35,38) | (0.5,24,22,45) (0.20,1,36,31) | mod 46 | 414 | $\frac{4 \times 46+23}{414}$ |
| 50  | (0,15,8,47,48) (0,32,41,11,45) | (5,45,0,19,42) (0,30,45,47,16) | (1,23,0,3,31) (0,38,35,41,10) | (0.5,25,12,36) (20,0,27,33,21) | mod 49 | 490 | $\frac{5 \times 49}{490}$ |
| 51  | (40,17,0,22,29) (5,0,17,27,43) | (17,27,30,32,0) (6,42,0,32,44) | (0.33,20,39,50) (0,4,31,41,8) | (5,27,35,0,42) (28,2,42,16,0) | mod 51 | 510 | $\frac{5 \times 51}{510}$ |
| 55  | (15,0,18,47,31) (24,30,0,50,47) | (6,8,0,28,31) (17,0,6,33,52) | (14,7,22,0,40) (0,20,31,45,30) | (6,0,21,42,44) (5,11,∞,0,24) | mod 54 | 594 | $\frac{5 \times 54+27}{594}$ |
| 56  | (4,29,0,10,48) (9,11,25,0,34) | (0,40,21,39,41) (4,2,32,0,36) | (0,23,15,20,28) (0,51,11,44,26) | (0.7,29,6,46) (6,20,13,32,0) | mod 56 | 616 | $\frac{2 \times 56+4 \times 56}{616}$ |
| \( \nu \) | Base blocks | \( b_\nu \) | \( d \) |
|---|---|---|---|
| 60 | \((4,1,0,10,52)\) | \((2,37,5,0,41)\) | \((0,0,0,14,23)\) | \text{mod 59} | 708 | \( \frac{6 \times 59}{708} \) |
| 60 | \((3,0,1,17,34)\) | \((0,3,8,21,32)\) | \((0,5,16,42,6)\) | \text{mod 59} | 708 | \( \frac{6 \times 59}{708} \) |
| 60 | \((8,0,2,20,45)\) | \((25,10,0,32,12)\) | \((16,0,4,40,31)\) | \text{mod 61} | 732 | \( \frac{6 \times 61}{732} \) |
| 60 | \((0,41,10,40,48)\) | \((5,24,0,20,50)\) | \(\infty\) | 708 | \( \frac{6 \times 59}{708} \) |
| 61 | \((3,0,55,1,21)\) | \((0,2,6,49,42)\) | \((4,0,12,23,37)\) | \text{mod 61} | 732 | \( \frac{6 \times 61}{732} \) |
| 61 | \((3,0,9,2,43)\) | \((6,0,18,4,25)\) | \((12,0,36,8,50)\) | \text{mod 61} | 732 | \( \frac{6 \times 61}{732} \) |
| 61 | \((32,0,35,1,52)\) | \((0,11,24,16,39)\) | \(0,16,48,31,26\) | \text{mod 61} | 732 | \( \frac{6 \times 61}{732} \) |
| 61 | \((35,0,3,44,34)\) | \((0,48,22,32,17)\) | \(0,16,48,31,26\) | \text{mod 61} | 732 | \( \frac{6 \times 61}{732} \) |
| 65 | \((10,25,0,23,39)\) | \((23,20,0,22,48)\) | \((0,23,19,53,51)\) | \text{mod 64} | 832 | \( \frac{6 \times 64 + 32}{832} \) |
| 65 | \((0,41,\infty,31,52)\) | \((0,10,22,27,1)\) | \((0,46,25,19,49)\) | \text{mod 64} | 832 | \( \frac{6 \times 64 + 32}{832} \) |
| 65 | \((0,74,15,16,56)\) | \((4,49,44,0,56)\) | \((28,14,0,32,61)\) | \text{mod 64} | 832 | \( \frac{6 \times 64 + 32}{832} \) |
| 65 | \((0,5,7,0,53)\) | \((9,56,0,10,13)\) | \(0,16,48,31,26\) | \text{mod 64} | 832 | \( \frac{6 \times 64 + 32}{832} \) |
| 66 | \((17,37,36,0,65)\) | \((15,27,5,0,24)\) | \((22,4,0,20,27)\) | \text{mod 66} | 858 | \( \frac{6 \times 66 + 33}{858} \) |
| 66 | \((20,0,31,62,24)\) | \((6,29,63,0,42)\) | \((5,0,35,57,9)\) | \text{mod 66} | 858 | \( \frac{6 \times 66 + 33}{858} \) |
| 66 | \((34,0,6,32,59)\) | \((45,0,58,39,47)\) | \((32,16,0,33,54)\) | \text{mod 66} | 858 | \( \frac{6 \times 66 + 33}{858} \) |
| 66 | \((0,3,14,15,55)\) | \((29,0,43,35,45)\) | \(0,16,48,31,26\) | \text{mod 66} | 858 | \( \frac{6 \times 66 + 33}{858} \) |
| 70 | \((0,26,33,52,64)\) | \((5,59,0,25,43)\) | \((18,37,47,0,65)\) | \text{mod 69} | 966 | \( \frac{7 \times 69}{966} \) |
| 70 | \((9,23,38,0,62)\) | \((32,0,44,23,57)\) | \((29,33,0,9,50)\) | \text{mod 69} | 966 | \( \frac{7 \times 69}{966} \) |
| 70 | \((0,9,11,59,51)\) | \((22,52,0,55,49)\) | \((0,16,13,21,43)\) | \text{mod 69} | 966 | \( \frac{7 \times 69}{966} \) |
| 70 | \((4,27,0,20,28)\) | \((6,12,0,4,58)\) | \((1,0,15,56,59)\) | \text{mod 69} | 966 | \( \frac{7 \times 69}{966} \) |
| 71 | \((8,48,41,35,0)\) | \((20,0,67,49,52)\) | \((0,66,41,1,58)\) | \text{mod 71} | 994 | \( \frac{7 \times 71}{994} \) |
| 71 | \((0,7,4,27,42)\) | \((47,0,69,19,37)\) | \((1,6,0,31,14)\) | \text{mod 71} | 994 | \( \frac{7 \times 71}{994} \) |
| 71 | \((43,0,55,34,45)\) | \((34,0,62,50,60)\) | \((19,0,43,53,21)\) | \text{mod 71} | 994 | \( \frac{7 \times 71}{994} \) |
| 71 | \((62,0,16,17,5)\) | \((16,25,0,70,11)\) | \((0,67,15,47,18)\) | \text{mod 71} | 994 | \( \frac{7 \times 71}{994} \) |
| $v$ | Base blocks | $b_v$ | $d$ |
|-----|-------------|-------|-----|
| 75  | (0,30,19,25,47) (0,44,36,51,57) (0,30,39,10,65) (32,21,64,0,44) | 1110 | $\frac{7 \times 74 + 37}{1110}$ |
|     | (5,1,0,73,57) (37,0,4,33,55) (24,0,72,31,65) (0,4,\infty,66,68) |      |     |
|     | (18,36,0,20,47) (0,36,21,60,61) (0,32,1,49,46) (0,3,14,38,64) |      |     |
|     | (40,55,0,8,60) (12,25,28,65,0) |      |     |
| 76  | (36,17,58,0,71) (0,36,53,20,55) (0,9,1,67,22) (29,37,52,0,73) | 1140 | $\frac{7 \times 76 + 38}{1140}$ |
|     | (5,58,7,0,65) (0,38,63,37,48) (0,3,4,52,23) (11,45,0,73,56) |      |     |
|     | (0,5,51,55,67) (16,4,7,0,52) (10,0,54,6,52) (0,30,17,26,33) |      |     |
|     | (15,0,27,42,56) (27,0,8,37,51) (42,0,74,30,68) |      |     |
| 80  | (21,38,20,0,28) (37,70,0,20,68) (0,38,\infty,64,10) (0,66,41,53,71) | 1264 | $\frac{8 \times 79}{1264}$ |
|     | (32,8,33,0,11) (40,0,61,39,63) (16,4,7,0,52) (48,45,61,56,0) |      |     |
|     | (0,50,44,51,40) (0,2,44,17,76) (0,37,53,67,72) (0,35,62,12,21) |      |     |
|     | (19,0,43,65,13) (65,23,29,0,33) (56,40,69,70) |      |     |
| 81  | (49,0,47,26,30) (0,41,15,49,31) (52,0,17,60,78) (9,18,0,45,57) | 1296 | $\frac{8 \times 81}{1296}$ |
|     | (27,38,0,65,79) (0,50,7,75,78) (14,0,20,68,69) (28,48,0,51,70) |      |     |
|     | (7,31,66,68,0) (8,0,74,17,63) (0,33,45,67,73) (57,0,59,80,76) |      |     |
|     | (0,16,58,21,52) (37,42,0,74,46) (9,34,19,40) |      |     |
| 85  | (4,7,60,0,82) (0,10,45,15,75) (18,52,57,0,55) (5,0,\infty,19,40) | 1428 | $\frac{8 \times 84+42}{1428}$ |
|     | (48,0,68,25,11) (17,35,0,48,73) (1,22,0,72,65) (75,0,76,4,83) |      |     |
|     | (25,41,0,57,58) (15,0,41,67,74) (4,0,78,28,48) (0,49,72,51,57) |      |     |
|     | (0,18,68,30,47) (56,38,0,53,42) (29,0,60,38,40) (3,26,48,40,0) |      |     |
| 86  | (68,41,72,69,0) (13,64,0,12,67) (30,78,24,0,44) (16,0,42,65,19) | 1462 | $\frac{8 \times 86+43}{1462}$ |
|     | (0,59,84,31,33) (10,0,72,70,82) (0,36,28,78,65) (0,8,69,82,75) |      |     |
|     | (0,63,43,7,78) (0,23,41,50,47) (0,39,1,17,54) (0,20,77,66,71) |      |     |
|     | (0,22,77,21,79) (56,38,0,53,42) (0,29,71,39,6) |      |     |
| 95  | (3,77,0,1,36) (49,53,0,77,84) (3,68,86,52,0) (0,6,\infty,20,72) | 1598 | $\frac{8 \times 94+47}{1598}$ |
|     | (40,1,0,13,4) (21,73,0,81,38) (12,28,62,0,13) (55,0,26,88,37) |      |     |
|     | (0,80,82,64,69) (79,0,89,14,10) (19,58,0,63,25) (0,31,88,16,40) |      |     |
|     | (48,0,92,67,59) (46,7,0,48,75) (38,72,1,8,0) (23,0,3,50,43) |      |     |
| $v$  | Base blocks                      | $b_v$ | $d$    |
|------|----------------------------------|-------|--------|
| 110  | $(0,101,30,34,43)$               |       |        |
|      | $(0,18,26,40,98)$               |       |        |
|      | $(0,100,73,78,85)$              |       |        |
|      | $(23,48,63,0,104)$              |       |        |
|      | $(0,74,71,108,61)$              |       |        |
|      | $(0,36,50,85,32)$               |       |        |
|      | $(8,0,80,85,104)$               |       |        |
|      | $(2,19,50,41,0)$                |       |        |
|      | $(0,3,20,107,86)$               |       |        |
|      | $(53,36,56,0,80)$               |       |        |
|      | $(11,0,93,79,33)$               |       |        |
|      | $(0,47,49,58,72)$               |       |        |
| 111  | $(0,106,11,1,109)$              |       |        |
|      | $(18,50,106,87,0)$              |       |        |
|      | $(0,102,12,79,82)$              |       |        |
|      | $(51,101,34,0,30)$              |       |        |
|      | $(0,100,24,51,29)$              |       |        |
|      | $(92,0,9,37,75)$                |       |        |
| 115  | $(0,106,11,1,109)$              |       |        |
|      | $(18,50,106,87,0)$              |       |        |
|      | $(0,102,12,79,82)$              |       |        |
|      | $(51,101,34,0,30)$              |       |        |
|      | $(0,100,24,51,29)$              |       |        |
|      | $(92,0,9,37,75)$                |       |        |
| 116  | $(1,94,67,88,0)$                |       |        |
|      | $(8,0,42,65,11)$                |       |        |
|      | $(0,62,55,99,100)$              |       |        |
|      | $(0,64,71,18,74)$               |       |        |

\( \mod 109 \times 11 \times 109 \) 2398

\( \mod 111 \times 11 \) 2442

\( \mod 114 \times 114 + 57 \) 2622

\( \mod 116 \times (6 + 2 + 4) \times 116 \) 2668

\( (6 + 2 + 4) \times 116 \) 2668
| $v$    | Base blocks                                                                 | $b_v$ | $d$  |
|-------|------------------------------------------------------------------------------|-------|------|
| 130   | $(26,67,0,16,97)$ $(101,36,0,95,79)$ $(0,54,57,9,108)$ $(3,0,112,7,0,69)$ $(1,32,33,0,116)$ $(10,0,101,7,20)$ $(95,0,21,40,120)$ $(8,93,0,108,52)$ |       |      |
|       | $(100,18,0,19,91)$ $(0,82,11,21,34)$ $(0,83,114,11,123)$ $(0,124,41,101,7,4)$ $(0,26,50,94,53)$ $(23,11,0,85,127)$ $(11,0,105,42,122)$ $(8,26,\infty,51,0)$ |       |      |
|       | $(71,12,0,75,127)$ $(0,12,49,110,63)$ $(11,0,105,111,71)$ $(0,24,32,46,122)$ $(0,39,68,77,3,5)$ $(0,15,105,42,122)$ $(0,54,57,59,108)$ $(8,69,39,75,92)$ |       |      |
|       | $(0,45,18,82,110)$ $(57,122,79,0,120)$ $(8,26,\infty,51,0)$ $(40,52,36,0,56)$ $(0,69,39,75,92)$ $(0,30,5,38,114)$ $(0,124,41,101,7,4)$ $(131,3406)$ |       |      |
|       | $(13 \times 131)$ $3354$                                                  |       |      |
| 131   | $(70,53,11,69,0)$ $(16,0,72,10,27)$ $(72,62,45,56,0)$ $(0,25,107,23,116)$ $(129,23,47,0,38)$ $(123,21,92,0,57)$ $(0,104,65,110,75)$ |       |      |
|       | $(0,118,2,34,9)$ $(32,20,0,13,54)$ $(0,40,108,26,64)$ $(101,50,0,46,83)$ $(127,76,46,0,94)$ $(0,124,13,112,90,0)$ $(128,80,52,0,85)$ |       |      |
|       | $(68,0,4,18,105)$ $(26,49,0,117,93)$ $(104,125,39,0,29)$ $(22,9,0,106,7)$ $(0,124,13,112,90,0)$ $(128,80,52,0,85)$ |       |      |
|       | $(8,36,5,0,79)$ $(55,0,52,98,103)$ $(78,119,58,77,0)$ $(0,28,36,88,31)$ $(128,80,52,0,85)$ |       |      |
|       | $(13 \times 131)$ $3406$                                                  |       |      |
|       | $3406$                                                                      |       |      |
References

1. Abel, R.J.R., Brouwer, A.E., Colbourn, C.J., Dinitz, J.H.: Mutually orthogonal Latin squares. In: Colbourn, C.J., Dinitz, J.H. (eds.) C.R.C. Handbook of Combinatorial Designs, pp. 111–142. CRC Press, Boca Raton (1996)
2. Amirzade, F., Panario, D., Sadeghi, M. R.: Trade-Based LDPC Codes (2021). arXiv:2107.07466
3. Boostan, M., Goralizadeh, S., Soltankhah, N.: Super-simple \((v, 4, 2)\) directed designs and a lower bound for the minimum size of their defining set. Discrete Appl. Math. 201, 14–23 (2016)
4. Cao, H., Chen, K., Wei, R.: Super-simple balanced incomplete block designs with block size 4 and index 5. Discrete Math. 309, 2808–2814 (2009)
5. Chen, G., Chen, K., Zhang, Y., Jiang, N.: Super-simple group divisible designs with block size 4 and index \(\lambda = 7, 8\). Discrete Math. 344(12), 112592 (2021)
6. Chen, K.: On the existence of super-simple \((v, 4, 3)\)-BIBDs. J. Comb. Math. Comb. Comput. 17, 149–159 (1995)
7. Chen, K.: On the existence of super-simple \((v, 4, 4)\)-BIBDs. J. Stat. Plan. Inference 51, 339–350 (1996)
8. Chen, K., Cao, Z., Wei, R.: Super-simple balanced incomplete block designs with block size 4 and index 6. J. Stat. Plan. Inference 133, 537–554 (2005)
9. Chen, K., Chen, G., Li, W., Wei, R.: Super-simple balanced incomplete block designs with block size 5 and index 3. Discrete Appl. Math. 161, 2396–2404 (2013)
10. Chen, K., Sun, Y.G., Zhang, Y.: Super-simple balanced incomplete block designs with block size 4 and index 8. Util. Math. 91, 213–229 (2013)
11. Chen, K., Wei, R.: Super-simple \((v, 5, 5)\) designs. Des. Codes Crypt. 39, 173–187 (2006)
12. Chen, K., Wei, R.: Super-simple \((v, 5, 4)\) designs. Discrete Appl. Math. 155, 904–913 (2007)
13. Dengsheng, L., Qiang, L., Shaoqian, L.: Construction of nonsystematic low-density parity-check codes based on symmetric balanced incomplete block designs. J. Electron. 25(4), 445–449 (2008)
14. Dietrich, H., Gronau, H.O.F., Kreher, D., Ling, A.: Super-simple \((v, 5, 2)\) designs. Discrete Appl. Math. 138, 65–77 (2004)
15. Gallager, R.G.: Low-Density Parity-Check Codes. MIT Press, Cambridge (1963)
16. Grannell, M.J., Griggs, T.S., Quinn, K.A.S.: Smallest defining sets of directed triple systems. Discrete Math. 309, 4810–4818 (2009)
17. Gronau, H.O.F., Mullin, R.C.: On super-simple \(2 - (v, 4, \lambda)\) designs. J. Comb. Math. Comb. Comput. 11, 113–121 (1992)
18. Gruner, A., Huber, M.: Low-density parity-check codes from transversal designs with improved stopping set distributions. IEEE Trans. Commun. 61(6), 2190–2200 (2013)
19. MacKay, D.: Good error-correcting codes based on very sparse matrices. IEEE Trans. Inf. Theory 45, 399–431 (1999)
20. Mahmoodian, E.S., Soltankhah, N., Street, A.P.: On defining sets of directed designs. Australas. J. Comb. 19, 179–190 (1999)
21. Richardson, T.J., Urbanke, R.L.: Efficient encoding of low-density parity-check codes. IEEE Trans. Inf. Theory 47, 638–656 (2001)
22. Soltankhah, N.: On directed trades. Australas. J. Comb. 11, 59–66 (1995)
23. Soltankhah, N., Amirzade, F.: Smallest defining sets of super-simple \(2 - (v, 4, 1)\) directed designs. Util. Math. 96, 331–344 (2015)
24. Soltankhah, N., Amirzade, F.: Super-simple \(2 - (v, 5, 1)\) directed designs and their smallest defining sets. Australas. J. Comb. 54, 85–106 (2012)
25. Sun, X.: Super-simple BIBDs with block size 4 and index 7. Discrete Math. 343(12), 112089 (2020)
26. Tanner, M.R.: A recursive approach to low complexity codes. IEEE Trans. Inf. Theory 27, 533–547 (1981)
27. Vasic, B., Milenkovic, O.: Combinatorial constructions of low-density parity-check codes for iterative decoding. IEEE Trans. Inf. Theory 50(6), 1156–1176 (2004)
28. Zhang, Y., Chen, K., Sun, Y.: Super-simple balanced incomplete block designs with block size 4 and index 9. J. Stat. Plan. Inference 139, 3612–3624 (2009)

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