Research Article
Symmetry Classification and Solutions for the Third-Order of Kudryashov–Sinelshchikov Equation

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This paper studies nonlinear wave propagation in a bubble-liquid mixture based on the third-order Kudryashov–Sinelshchikov (KS) equation. Symmetry is classified and reduced through the symmetry group method. Through the generalized conditional symmetry method, this equation is classified, and the generalized conditional symmetry considered are second, third, fourth, and fifth-order, respectively. Meanwhile, the classified equations are transformed into a system of ordinary differential equations and solved.

1. Introduction

The nonlinear evolution equation (NLEE) is essential to describe wave propagation in a bubble-liquid mixture, a typical nonlinear medium. The NLEE includes the Korteweg–de Vries (KdV) [1], the Burgers–Korteweg–de Vries equations. It is of significance to investigate the wave propagation in a bubble-liquid mixture.

Obtaining the solutions of the NLEE is vital due to the physical information and insights into the problems that the NLEE can provide [2–9]. In order to solve the NLEE, several methods were proposed, including Inverse Scattering transformation [10], symmetry approach [11–13], and Darboux transformation (DT) [14]. Among them, the symmetry group algorithm has been the most widely used in obtaining the exact solutions of the NLEE. The nonlinear partial differential equation was introduced in [1], which is a vanish dissipation defined as follows:

\[ u_t + auu_x + u_{xxx} - (uu_{xx})_x - \beta u_x u_{xx} = 0, \]  

where \( u \) represents a density that models viscosity and heat transfer. \( \alpha \) and \( \beta \) are parameters. Equation (1) is a generalized equation of the KdV and BKdV equations, and was called the Kudryashov–Sinelshchikov (KS) equation. This equation was studied for various methods [15–29]. In [19], the characteristics of a bubble-liquid mixture, such as peaked solitons, were studied. Also, the KS symmetry, optimal system, and solutions of the following KS equation were discussed,

\[ u_t + auu_x + bu_{xxx} + c(uu_{xx})_x + du_x u_{xx} = 0. \]  

This paper aims to study the following type of KS equation,

\[ u_t = F(u)u_x + u_{xxx} + [G(u)u_{xx}]_x + u_x u_{xx}, \]  

where \( F(u) \) and \( G(u) \) indicate arbitrary functions. By taking some appropriate forms for \( F(u) \) and \( G(u) \), equation (3) is simplified to the KdV equation [24].

\[ u_t + 6uu_x + u_{xxx} = 0, \]  

it was widely used to model various mechanical engineering and physical phenomena.

The rest of this paper is structured as follows. Section 2 presents the symmetry classification for the system (1), admitting the presence of GCS. Section 3 conducts the GCS method to reduce the symmetry and find exact solutions to the classified equations. Lastly, Section 4 is the conclusion.

2. Symmetry Classification

The symmetry group methods, especially the Lie point symmetry method, are effectively used to obtain the exact solution of partial differential equations. However, there are many problems that allow us to use symmetry groups. Based on the Lie point symmetry, many other methods were proposed to obtain the exact solutions, such as the
Theorem 1. Equation (3) admits the second-order GCSs in the form
\[
Q = \eta \frac{\partial}{\partial u} \equiv \left[ u_2 - \sum_{i=0}^{1} a_i u_i \right] \frac{\partial}{\partial u} \quad \text{and}
\]
\[
\eta = u_{xx} - a_0 u,
\]
if and only if the equation is equivalent to one of the followings:

(i) \[
u_t = \left( G_0 u a_0 - a_0 G(u) - a_0 u + f_1 \right) u_x + u_{xxxx} + \left[ (G_0^2 u^2 + G_0 u + G_0) u_{xxx} \right]_x + u_x u_{xxx}
\]

(ii) \[
u_t = \left( u f_1 + f_2 \right) u_x + u_{xxxx} + \left[ (G_0^2 u^2 + G_0 u + G_0) u_{xxx} \right]_x + u_x u_{xxx}
\]

(iii) \[
u_t = \left( u f_1 + f_2 \right) u_x + u_{xxxx} + \left[ \frac{1}{2} \left( 2G_0 a_0^2 - a_0^2 - u f_1 \right) \right]_x + u_x u_{xxx}
\]

(iv) \[
u_t = \left( -3G_0 u^2 a_0 - G_0 u a_0 - a_0 u + f_2 \right) u_x + u_{xxxx} + \left[ (G_0^2 u^2 + G_0 u + G_0) u_{xxx} \right]_x + u_x u_{xxx}
\]

(v) \[
u_t = \left( \frac{4}{27} \left( 2G_0 a_0^2 + f_2 \right) \right) u_x + u_{xxxx} + \left[ \left( G_0 - \frac{2}{3} \right) u_{xxx} \right] + u_x u_{xxx}
\]

Proof: According to the definition of the GCS method and the computation procedure, we obtain the following equation:
\[
F_1 u_x^4 + F_2 u_x^3 + F_3 u_x^2 + F_4 u + F_0 = 0,
\]
where
\[
F_1 = a_1 G_{uuu},
F_2 = a_0 u G_{uuu} + 5a_0^2 G_{uu} + 3a_0 G_{uu} + F_{uu},
F_3 = 8a_0 a_1 u G_{uu} + 4a_1^2 G_{u} + 7a_0 a_1 G_{u}
+ 2a_1^2 + 2a_1 F_u + 3a_0 a_1,
F_4 = 3a_0^2 u G_{uu},
F_0 = 3a_0^3 a_1 G_{u} + 3a_0^2 a_1.
\]

We solve the system of differential equations \( F_0 = 0, F_1 = 0, F_2 = 0, F_3 = 0, \) and \( F_4 = 0. \) When \( F_4 = 0, \) we obtain two cases, \( a_1 = 0 \) or \( G_{uuu} = 0. \)

Case 1. When \( a_1 = 0, \) we can show \( F(u) = G_0 u a_0 - a_0 G(u) - au + f_1. \)

Case 2. When \( G_{uu} = 0, \) we can obtain \( G(u) = G_1 u^2 + G_3 u + G_0, \) \( F(u) = -5G_1 u^2 a_0^2 - 3G_1 u^2 a_0 + u f_1 + f_2. \) By substituting \( F(u), G(u) \) into \( F_i (i = 0, 1, 2, 3, 4), \) so the classification results are shown above.

Theorem 2. The equation admits the third-order GCSs in the form
\[
Q = \eta \frac{\partial}{\partial u} \equiv \left[ u_3 - \sum_{i=0}^{2} a_i u_i \right] \frac{\partial}{\partial u}
\]
\[
u_0 = u.
\]

if and only if the equation is equivalent to one of the followings:

(i) \[
u_t = f_2 u_x + u_{xxx} + \left[ (G_1 u + G_2) u_{xxx} \right] + u_x u_{xxx}
\]

(ii) \[
u_t = \left[ f_2 - (a_1 + 2a_0 G_1) u \right] u_x + u_{xxx}
\]
\[
\left[ (G_1 u + G_3) u_{xxx} \right] + u_x u_{xxx}
\]

(iii) \[
u_t = \left( f_2 + a_0^2 u \right) u_x + u_{xxx} + \left[ (G_2 - u) u_{xxx} \right] + u_x u_{xxx}
\]
\[
\left( a_1 u_x + 2a_0 G_1 \right) u_x
\]

Proof: ...
(iv) 
\[ u_t = \left( f_2 + \frac{4}{27} a_2 u \right) u_x + u_{xxx} \]
\[ + \left[ \left( G_2 - \frac{2}{3} u \right) u_{xx} \right] + u_x u_{xx} \text{ and} \]
\[ \eta = u_{xxx} - a_2 u_{xx} + \frac{2}{9} a_2^2 u_x. \] 

\[ \frac{d\phi_1}{dt} = f_1 \phi_1^2 \text{ and} \]
\[ \frac{d\phi_2}{dt} = \phi_1 (f_1 \phi_2 + f_2), \] 
and the solutions are
\[ \phi_1 = \frac{1}{-f_1 t + c_1} \] 
\[ \phi_2 = \frac{f_2 t + c_2}{-f_1 t + c_1}. \] 

Theorem 3. The equation admits the fourth- and fifth-order GCSs, if and only if the equation satisfies one of the followings

(i)
\[ u_t = \left( f_2 + f_1 u \right) u_x + u_{xxx} \]
\[ + \left[ \left( G_2 + G_1 u \right) u_{xx} \right] + u_x u_{xx} \text{ and} \]
\[ \eta = u_{xxx}, \] 

(ii)
\[ u_t = \left( f_2 + a_2 u \right) u_x + u_{xxx} \]
\[ + \left[ \left( G_2 - u \right) u_{xx} \right] + u_x u_{xx} \text{ and} \]
\[ \eta = u_{xxx} - a_2 u_{xx}, \] 

(iii)
\[ u_t = \left( f_2 + \frac{4}{15} a_2 u \right) u_x + u_{xxx} \]
\[ + \left[ \left( G_2 - \frac{2}{3} u \right) u_{xx} \right] + u_x u_{xx} \text{ and} \]
\[ \eta = u_{xxx} - a_2 u_{xx} + \frac{4}{25} a_2^2 u_x; \] 

(iv)
\[ u_t = \left( f_2 + f_1 u \right) u_x + u_{xxx} \]
\[ + \left[ \left( G_2 - \frac{2}{3} u \right) u_{xx} \right] + u_x u_{xx} \text{ and} \]
\[ \eta = u_{xxx} - \frac{15}{4} f_1 u_{xxx} + \frac{9}{4} f_1 a_2 u_x. \] 

3. Symmetry Reduction and Solution

Example 1. Symmetry reduction of equation (7a), which admits symmetry operator (7b).

We obtain the solutions form by integrating \( u_{xx} = 0 \),
\[ u_t = \phi_1 (t) x + \phi_2 (t). \] 

In the following, solution (22) is inserted into equation (7a), yielding the system of ordinary differential equations as follows:

\[ \frac{d\phi_1}{dt} = f_1 \phi_1^2 \text{ and} \]
\[ \frac{d\phi_2}{dt} = \phi_1 (f_1 \phi_2 + f_2), \] 
and the solutions are
\[ \phi_1 = \frac{1}{-f_1 t + c_1} \] 
\[ \phi_2 = \frac{f_2 t + c_2}{-f_1 t + c_1}. \] 

Example 2. Symmetry reduction of equation (8a), which admits operator (8b).

We obtain the solutions form by integrating \( u_{xx} = a_1 u_x = 0 \),
\[ u = \phi_1 (t) + \phi_2 (t) e^{ax}. \] 

Equation (8a) can be reduced to the following system of ordinary differential equations:
\[ \frac{d\phi_1}{dt} = 0 \text{ and} \]
\[ \frac{d\phi_2}{dt} = -\frac{1}{2} a_1 \left( a_1^2 - f_1 \right) \phi_1 \phi_2 + a_1 \left( G_3 a_1^2 + a_1^2 + f_2 \right) \phi_2, \] 
the solutions of (27) are
\[ \phi_1 = c_1 \text{ and} \]
\[ \phi_2 = c_2 e^{-\frac{1}{2} a_1 \left( c_1 a_1^2 - 2G_3 a_1^2 - c_1 f_1 - 2a_1^2 - 2f_2 \right) t}. \] 

So we can obtain the solution of (8a) \( u = c_1 + c_2 e^{-\frac{1}{2} a_1 \left( c_1 a_1^2 - 2G_3 a_1^2 - c_1 f_1 - 2a_1^2 - 2f_2 \right) t + a_1 x} \). 
When \( f_2 = 0 \) and \( G_3 = 0 \), the special solution is \( u = c_1 + c_2 e^{-\frac{1}{2} a_1 \left( c_1 a_1^2 - c_1 f_1 - 2a_1^2 \right) t + a_1 x} \).

Example 3. Symmetry reduction of equation (16a), which admits operator (16b).

By integrating \( u_{xxx} - a_2 u_{xx} = 0 \), the solutions can be obtained as follows:
\[ u = \phi_1 (t) + \phi_2 (t) x + \phi_3 (t) e^{a_2 x}. \] 

Accordingly, we can reduce the equation (16a) to the system of ODEs,
\[ \frac{d\phi_1}{dt} = \phi_1 \phi_2 a_1^2 + \phi_2 f_2, \]
\[ \frac{d\phi_2}{dt} = \phi_2 a_2^2, \text{ and} \]
\[ \frac{d\phi_3}{dt} = \phi_3 \phi_2 a_2^2 + \phi_2 a_2 (G_2 a_2^2 + a_2^2 + f_2), \]
the solutions of above equations are

\[ \phi_1 (t) = \frac{f_2 t + C_3}{-a_2^2 t + C_1}, \]
\[ \phi_2 (t) = \frac{1}{-a_2^2 t + C_1}, \text{ and} \]
\[ \phi_3 (t) = \frac{C_2 e^{a_2 f (G_2 a_2^2 t^2 + f_2 t)}}{-a_2^2 t + C_1}. \]  

By inserting \( \phi_1 (t), \phi_2 (t) \) into (29), we get the solution of (16a).

4. Conclusion

In this paper, using the symmetry group theory and maple software, we investigated the symmetry classification, symmetry reduction, and solutions of the Kudryashov–Sinelshchikov equation for the third-order. We also obtain the polynomial and exponential solutions of the classical KS equation and enrich the results of KS equations. We will study the other type of KS equations and extend the application scope of the symmetry group method in the future.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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