How Lock-free Data Structures Perform in Dynamic Environments: Models and Analyses

Aras Atalar†, Paul Renaud-Goud* and Philippas Tsigas†
† Chalmers University of Technology
* Institut de Recherche en Informatique de Toulouse

CONTENTS

I Introduction 3
II Related Work 4
III Preliminaries 5
   III-A System Settings ........................................... 6
   III-B Execution Description ...................................... 6
   III-C Our Approaches ............................................. 7
      III-C1 Average-based Analysis ................................. 8
      III-C2 Constructive Method ................................... 8
IV Average-based Approach 9
   IV-A Contended System ............................................ 9
      IV-A1 Expected Completion time ................................ 9
      IV-A2 Expected Slack Time ...................................... 10
      IV-A3 Expected Success Period ................................. 11
   IV-B Non-contended System ..................................... 11
   IV-C Unified Solving ............................................. 11
V Constructive Approach 13
   V-A Process .......................................................... 13
   V-B Expansion ...................................................... 14
   V-C Formalization .................................................. 16
      V-C1 Transition Matrix ......................................... 17
      V-C2 Stationary Distribution .................................. 20
      V-C3 Slack time and Throughput ............................... 21
      V-C4 Number of Failed Retries ................................. 22
VI Experiments 23
   VI-A Setting .......................................................... 23
   VI-B Basic Data Structures ...................................... 24
      VI-B1 Synthetic Tests ......................................... 24
      VI-B2 Treiber’s Stack .......................................... 27
| Section  | Title                                                                 | Page |
|----------|----------------------------------------------------------------------|------|
| VI-C     | Towards Advanced Data Structure Designs                              | 28   |
| VI-C1    | Expected Expansion for the Advanced Data Structures                 | 28   |
| VI-C2    | Expected Slack Time for the Advanced Data Structures                | 30   |
| VI-C3    | Enqueue on Michael-Scott Queue                                      | 30   |
| VI-C4    | Deque                                                                | 31   |
| VI-D     | Applications                                                         | 31   |
| VI-D1    | Back-off Optimizations                                               | 31   |
| VI-D2    | Memory Management Optimization                                       | 33   |
| VII      | Conclusion                                                           | 36   |
| References|                                                                     | 37   |
Abstract

In this paper we present two analytical frameworks for calculating the performance of lock-free data structures. Lock-free data structures are based on retry loops and are called by application-specific routines. In contrast to previous work, we consider in this paper lock-free data structures in dynamic environments. The size of each of the retry loops, and the size of the application routines invoked in between, are not constant but may change dynamically. The new frameworks follow two different approaches. The first framework, the simplest one, is based on queuing theory. It introduces an average-based approach that facilitates a more coarse-grained analysis, with the benefit of being ignorant of size distributions. Because of this independence from the distribution nature it covers a set of complicated designs. The second approach, instantiated with an exponential distribution for the size of the application routines, uses Markov chains, and is tighter because it constructs stochastically the execution, step by step.

Both frameworks provide a performance estimate which is close to what we observe in practice. We have validated our analysis on (i) several fundamental lock-free data structures such as stacks, queues, dequeues and counters, some of them employing helping mechanisms, and (ii) synthetic tests covering a wide range of possible lock-free designs. We show the applicability of our results by introducing new back-off mechanisms, tested in application contexts, and by designing an efficient memory management scheme that typical lock-free algorithms can utilize.
I. Introduction

During the last two decades, lock-free data structures have received a lot of attention in the literature, and have been accepted in industrial applications, e.g. in the Intel’s Threading Building Blocks Framework [2], the Java concurrency package [3] and the Microsoft .NET Framework [4]. Lock-free implementations provide indeed a way out of several limitations of their lock-based counterparts, in robustness, availability and programming flexibility. Last but not least, the advent of multi-core processors has pushed lock-freedom on top of the toolbox for achieving scalable synchronization.

Naturally, the development of lock-free data structures was accompanied by studies on the performance of such data structures, in order to characterize their scalability. Having no guarantee on the execution time of an individual operation, the time complexity analyses of lock-free algorithms have turned towards amortized analyses. The so-called amortized analyses are thus interested in the worst-case behavior over a sequence of operations, which can be seen as a worst-case bound on the average time per operation. In order to cover various contention environments, the time complexity of the algorithms is often parametrized by different contention measures, such as point [9], interval [5] or step [10] contention. Nonetheless these investigations are targeting worst-case asymptotic behaviors. There is a lack of analytical results in the literature capable of describing the execution of lock-free algorithms on top of a hardware platform, and providing predictions that are close to what is observed in practice. Asymptotic bounds are particularly useful to rank different algorithms, since they rely on a strong theoretical background, but the presence of potentially high constants might produce misleading results. Yet, an absolute prediction of the performance can be of great importance by constituting the first step for further optimizations.

The common measure of performance for data structures is throughput, defined as the number of operations on the data structure per unit of time. To this end, this performance measure is usually obtained by considering an algorithm that strings together a pure sequence of calls to an operation on the data structure. However, when used in a more realistic context, the calls to the operations are mixed with application-specific code (that we call here parallel work). For instance, in a work-stealing environment designed with deques, a thread basically runs one of the following actions: pushing a new-generated task in its deque, popping a task from a deque or executing a task. The modifications on the deques are thus interleaved with deque-independent work. There exist some papers that consider in their experiments local computations between calls to operations during their respective evaluations, but the amount of local computations follows a given distribution varying from paper to paper, e.g. constant [19], uniform [13], exponential [21].

In this work, we derive a general approach for unknown distributions of the size of the application-specific code, as well as a tighter method when it follows an exponential distribution.

As for modeling the data structure itself, we use as a basis the universal construction described by Herlihy in [14], where it is shown that any abstract data type can get such a lock-free implementation, which relies on one retry loop. Moreover, we have particularly focused our experiments on data structures that present a low level of disjoint-access parallelism [15] (stack, queue, shared counter, deque). Coming back to amortized analyses, the time complexity of an operation is often expressed as a contention-free time complexity added with a contention overhead. In this paper, we want to model and analyze the impact of contention. Loosely speaking, the data structures that exhibit low level of disjoint-access parallelism have lightweight operations (i.e. low contention-free complexity) and they are prone to high contention overheads.
In contrast, the data structures that present high level of disjoint-access parallelism, or that
employ contention alleviation techniques, provide heavyweight operations (i.e. high contention-
free complexity) and behave differently, compared to the previous ones, under contention. Our
analyses examine this trade-off and then facilitate conscious decisions in the data structures
design and use.

We propose two different approaches that analyze the performance of such data structures.
On the one hand, we derive an average-based approach invoking queuing theory, which provides
the throughput of a lock-free algorithm without any knowledge about the distribution of the
parallel work. This approach is flexible but allows only a coarse-grained analysis, and hence
a partial knowledge of the contention that stresses the data structure. On the other hand, we
exhibit a detailed picture of the execution of the algorithm when the parallel work is instantiated
with an exponential distribution, through a second complementary approach. We prove that the
multi-threaded execution follows a Markovian process and a Markov chain analysis allows us
to pursue and reconstruct the execution, and to compute a more accurate throughput.

We finally show several ways to use our analyses and we evaluate the validity of our ideas by
experimental results. Those two analysis approaches give a good understanding of the phenomena
that drive the performance of a lock-free data structure, at a high-level for the average-based
approach, and at a detailed level for the constructive method. Moreover, our results provide a
common framework to compare different implementations of a data structure, in a fair manner.
We also emphasize that there exist several concrete paths to apply our analyses. To this end, based
on the knowledge about the application at hand, we implement two back-off strategies. We show
the applicability of these strategies by tuning a Delaunay triangulation application [12] and a
streaming pipeline component which is fed with trade exchange workloads [1]. These experiments
reveal the validity of our analyses in the application domain, under non-synthetic workloads and
diverse access patterns. We confirm the benefits of our theoretical results by designing a new
adaptive memory management mechanism for lock-free data structures in dynamic environments
which surpasses the traditional scheme and which is such that the loss in performance, when
compared to a static data structure without memory management, is largely leveraged. This
memory management mechanism is based on the analyses presented in this paper.

The rest of the paper is organized as follows: we start by presenting related work in Section II,
then we define the algorithm and the platform that we consider, together with concepts that are
common to our both approaches in Section III. The average-based approach is described in
Section IV, while the constructive analysis is exposed in Section V, and both methods are
evaluated in the experiment part that is presented in Section VI.

II. RELATED WORK

Approaches that are based on Markov chains and queueing theory, are commonly employed
to analyze the performance of parallel programs in concurrent environments. Yu et al. [22]
have provided an analytical model to estimate the mean transaction completion time for the
transactional memory systems. They make use of a continuous-time Markov chain queuing
model to analyze the execution of transactions, in which they formulate the state transition
rates by considering the arrival rate, the service time for the transactions together with other
parameters such as conflict rate that statistically quantifies the spatial (intersecting data set)
and temporal (overlapped time) aspects of conflicts. In [6], Al-Bahra has mentioned Little’s
Law as an appropriate tool to understand the effects of contention on serialized resources for
synchronization paradigms.
Closer to our work, Alistarh et al. [7] have studied the same class of lock-free data structures that we consider in this paper. They show initially that the lock-free algorithms are statistically wait-free and going further they exhibit upper bounds on the performance. Their analysis is done in terms of scheduler steps, in a system where only one thread can be scheduled (and can then run) at each step. If compared with execution time, this is particularly appropriate to a system where the instructions of the threads cannot be done in parallel (e.g. multi-threaded program on a multi-core processor with only writes on the same cache line of the shared memory). In our paper, the execution is evaluated in terms of processor cycles, strongly related to the execution time. In addition, the “parallel work” and the “critical work” can be done in parallel. Also, they bound the asymptotic expected system latency (with a big O, when the number of threads tends to infinity), while in our paper we estimate the throughput (close to the inverse of system latency) for any number of threads.

Comparing to our previous work: In [8], we illustrate the performance impacting factors and the model we use to cover a subset of lock-free structures that we consider in this paper. In the former paper, the analysis is built upon properties that arise only when the sizes of the critical work and the parallel work are constant. There, we show that the execution is not memoryless due to the natural synchrony provided by the retry loops; at the end of the line, we prove that the execution is cyclic and use this property to bound the rate of failed retries.

Here, we provide two new approaches which serve different purposes. In the first approach, we relax the assumptions regarding the critical work and parallel work parameters, that we respectively use to model the data structure operations and the application specific code from which the data structure operations are called. The first approach relies on the expected values of the size of the critical work and the parallel work. This allows us to cover, compared to our previous analysis, more advanced lock-free data structure operations, see Section VI-C. Also, we can analyze the data structures running on a larger variety of application specific environments, thanks to the relaxed assumption on the size of the parallel work. The second approach provides a tight analysis when the parallel work follows an exponential distribution. We can observe a significant decrease in the performance when the parallel work is initiated with exponential distribution in comparison to the cases where the parallel work is constant as in our previous work, see Section VI-B1. The tight analyses, in our previous work and the second approach presented in this paper, reveal for the first time an analytical understanding of this phenomenon.

This paper is complementary to the previous work, not only because of the difference in the analysis tools, the extensive set of data structures and the application specific environments that it considers but also because they together exhibit the impact of the size distributions of the parallel work on the performance of lock-free data structures.

### III. Preliminaries

We describe in this section the structure of the algorithm that is covered by our model. We explain how to analyze the execution of an instance of such an algorithm when executed by several threads, by slicing this execution into a sequence of adjacent success periods, where a success period is an interval of time during which exactly one operation returns. Each of the success periods is further split into two by the first access to the data structure in the considered retry loop. This execution pattern reflects fundamental phases of both analyses, whose first steps and general direction are outlined at the end of the section.
A. System Settings

All threads call Procedure AbstractAlgorithm (see Figure 1) when they are spawned. So each thread follows a simple though expressive pattern: a sequence of calls to an operation on the data structure, interleaved with some parallel work during which the thread does not try to modify the data structure. For instance, it can represent a work-stealing algorithm, as described in the introduction.

The algorithm is decomposed in two main sections: the parallel section, represented on line 2, and the retry loop (which represents one operation on the shared data structure) from line 3 to line 6. A retry starts at line 4 and ends at line 6. The outer loop that goes from line 1 to line 6 is designated as the work loop.

In each retry, a thread tries to modify the data structure and does not exit the retry loop until it has successfully modified the data structure. It firstly reads the access point AP of the data structure, then, according to the value that has been read, and possibly to other previous computations that occurred in the past, the thread prepares, during the critical work, the new desired value as an access point of the data structure. Finally, it atomically tries to perform the change through a call to the CAS primitive. If it succeeds, i.e. if the access point has not been changed by another thread between the first Read and the CAS, then it goes to the next parallel section, otherwise it repeats the process. The retry loop is composed of at least one retry (and the first iteration of the retry loop is strictly speaking not a retry, but a try).

We denote by \( cc \) the execution time of a CAS when the executing thread does not own the cache line in exclusive mode, in a setting where all threads share a last level cache. Typically, there exists a thread that touches the data between two requests of the same thread, therefore this cost is paid at every occurrence of a CAS. As for the Reads, \( rc \) holds for the execution time of a cache miss. When a thread executes a failed CAS, it immediately reads the same cache line (at the beginning of the next retry), so the cache line is not missing, and the execution time of the Read is considered as null. However, when the thread comes back from the parallel section, a cache miss is paid. To conclude with the parameters related to the platform, we dispose of \( P \) cores, where the CAS (resp. the Read) latency is identical for all cores, i.e. \( cc \) (resp. \( rc \)) is constant.

The algorithm is parametrized by two execution times. In the general case, the execution time of an occurrence of the parallel section (application-specific section) is a random variable that follows an unknown probability distribution. In the same way, the execution time of the critical work (specific to a data structure) can vary while following an unknown probability distribution. The only provided information is the mean value of those two execution times: \( cw \) for the critical work, and \( pw \) for the parallel work. These values will be given in units of work, where 1 u.o.w. = 50 cycles.

B. Execution Description

It has been underlined in [8] that there are two main conflicts that degrade the performance of the data structures which do not offer a great degree of disjoint-access parallelism: logical and hardware conflicts.

Logical conflicts occur when there are more than one thread in the retry loop at a given time (happens typically when the number of threads is high or when the parallel section is small). At any time, considering only the threads that are in the retry loop, there is indeed at most one thread whose retry will be successful (i.e. whose ending CAS will succeed), which implies the execution of more retries for the failing threads. In addition, after a thread executes successfully
Procedure AbstractAlgorithm

1 while ! done do
2   Parallel_Work();
3   while ! success do
4     current ← Read(AP);
5     new ← Critical_Work(current);
6     success ← CAS(AP, current, new);

Figure 1: Thread procedure

its final CAS, the other threads of the retry loop have first to finish their current retry before starting a potentially successful retry, since they are not informed yet that their current retry is doomed to failure. This creates some “holes” in the execution where all threads are executing useless work.

The threads will also experience hardware conflicts: if several threads are requesting for the same data, so that they can operate a CAS on it, a single thread will be satisfied. All the other threads will have to wait until the current CAS is finished, and give a new try when this CAS is done. While waiting for the ownership of the cache line, the requesting threads cannot perform any useful work. This waiting time is referred to as expansion.

We now refine the description of the execution of the algorithm. The timeline is initially decomposed into a sequence of success periods that will define the throughput. A success period is an interval of time of the execution that (i) starts after a successful CAS, (ii) contains a single successful CAS, (iii) finishes after this successful CAS. As explained in the previous subsection, to be successful in its retry, a thread has first to access the data structure, then modify it locally, and finally execute a CAS, while no other thread performs changes on the data structure. That is why each success period is further cut into two main phases (see Figure 2). During the first phase, whose duration is called the slack time, no thread is accessing the data structure. The second phase, characterized by the completion time, starts with the first access to the data structure (by any thread). Note that this Access could be either a Read (if the concerned thread just exited the parallel section) or a failed CAS (if the thread was already in the retry loop). The next successful CAS will come at least after cw (one thread has to traverse the critical work anyway), that is why we split the latter phase into: cw, then expansion, and finally a successful CAS.

C. Our Approaches

In this work, we propose two different approaches to compute the throughput of a lock-free algorithm, which we name as average-based and constructive. The average-based approach relies
on queuing theory and is focused on the average behavior of the algorithm: the throughput is obtained through the computation of the expectation of the success period at a random time. As for the constructive approach, it describes precisely the instants of accesses and modifications to the data structure in each success period: in this way, we are able to deconstruct and reconstruct the execution, according to observed events. The constructive approach leads to a more accurate prediction at the expense of requiring more information about the algorithm: the distribution functions of the critical and parallel works have indeed to be instantiated.

In both cases, we partition the domain space into different levels of contention (or modes); these partitions are independent across approaches, even if we expect similarities, but in each case, cover the whole domain space (all values of critical work, parallel work and number of threads).

1) **Average-based Analysis:** We distinguish two main modes in which the algorithm can run: contended and non-contended. In the non-contended mode, \( P_{rl} \) when the parallel work is large or the number of threads is low, concurrent operations are not likely to collide. So every retry loop will count a single retry, and atomic primitives will not delay each other. In the contended mode, any operation is likely to experience unsuccessful retries before succeeding (logical conflicts), and a retry will last longer than in the non-contended mode because of the collision of atomic primitives (hardware conflicts).

Once all the parameters are given, the analysis is centered around the calculation of a single variable \( P_{rl} \), which represents the expectation of the number of threads inside the retry loop at a random instant. Based on this variable, we are able to express the expected expansion \( \tau(P_{rl}) \) at a random time. As a next step, we show how this expansion can be used to estimate the expected slack time \( \overline{sl}(P_{rl}) \) and the expected completion time \( \overline{cl}(P_{rl}) \), and at the end, the expected time of a success period \( \overline{sp}(P_{rl}) \).

2) **Constructive Method:** The previous average-based reasoning is founded on expected values at a random time, while in the constructive approach, we study each success period individually, based on the number of threads at the beginning of the considered success period. So we are able to exhibit more clearly the instants of occurrences of the different accesses and modifications to the data structure, and thus to predict the throughput more accurately.

We rely on the same set of values used in the average-based approach, but these values are now associated with a given success period. Thus the number of threads inside the retry loop \( P_{rl} \), as well as the slack time and the completion time are evaluated at the beginning of each success period. We denote these times in the same way as in the first approach, but remove the bar on top since these values are not expectations any more.

The different contention modes do not characterize here the steady-state of the data structure as in the previous approach but are associated with the current success period. Accordingly, the contention can oscillate through different modes in the course of the execution. First, a success period is not contended when \( P_{rl} = 0 \), i.e. when there is no thread in the retry loop after a successful CAS. In this case, the first thread that exits the parallel section will be successful, and the Access of the sequence will be a Read. Second, the contention of a success period is high when at any time during the success period, there exists a thread that is executing a CAS. In other words, at the end of each CAS, there is at least one thread that is waiting for the cache line to operate a CAS on it. This implies that the first access of the success period is a CAS and occurs immediately after the preceding successful CAS: the slack time is null. Third, the mid-contention mode takes place when \( P_{rl} > 0 \), while at the same time, there are not enough requesting threads to fill the whole success period with CAS’s (which implies a non-null slack
time). Since these requesting threads have synchronized in the previous success period, CAS’s do not collide in the current success period, and because of that, the expansion is null.

IV. AVERAGE-BASED APPROACH

We propose in this section our coarse-grained analysis to predict the performance of lock-free data structures. Our approach utilizes fundamental queuing theory techniques, describing the average behavior of the algorithm. In turn, we need only a minimal knowledge about the algorithm: the mean execution time values $cw$ and $pw$. As explained in Section III-C1, the system runs in one of the two possible modes: either contended or uncontended.

A. Contended System

We first consider a system that is contended. When the system is contended, we use Little’s law to obtain, at a random time, the expectation of the success period, which is the interval of time between the last and the next successful CAS’s (see Figure 2).

The stable system that we observe is the parallel section: threads are entering it (after exiting a successful retry loop) at an average rate, stay inside, then leave (while entering a new retry loop). The average number of threads inside the parallel section is $P_{ps} = P - P_{rl}$, each thread stays for an average duration of $pw$, and in average, one thread is exiting the retry loop every success period $\bar{sp}(P_{rl})$, by definition of the success period. According to Little’s law [16], we have:

$$P_{ps} = pw \times \frac{1}{\bar{sp}(P_{rl})}, \text{ i.e.}$$

$$\frac{1}{pw} \times \bar{sp}(P_{rl}) = \frac{1}{P - P_{rl}}.$$

As explained in Section III-B, we further decompose a success period into two parts, separated by the first access to the data structure after a successful CAS. We can then write the average success period as the sum of: (i) the expected time before some thread starts its Access (the slack time), and (ii) the expected completion time. We compute these two expectations independently and gather them into the success period thanks to:

$$\bar{sp}(P_{rl}) = \bar{st}(P_{rl}) + \bar{ct}(P_{rl}).$$

When the data structure is contended, a thread is likely to be successful after some failed retries. Therefore a thread that is successful was already in the retry loop when the previous successful CAS occurred. This implies that the Access to the data structure will be due to a failed CAS, instead of a Read. The time before a thread starts its Access is then the time before a thread finishes its current critical work since there is a thread currently executing a CAS.

1) Expected Completion time: Since the data structure is contended, numerous threads are inside the retry loop, and, due to hardware conflicts, a retry can experience expansion: the more threads inside the retry loop, the longer time between a CAS request and the actual execution of this CAS. The expectation of the completion time can be written as:

$$\bar{ct}(P_{rl}) = cc + cw + \bar{e}(P_{rl}) + cc,$$

where $\bar{e}(P_{rl})$ is the expectation of expansion when there are $P_{rl}$ threads inside the retry loop, in expectation. This expansion can be computed in the same way as in [8], through the following differential equation:
by assuming that the expansion starts as soon as strictly more than 1 thread are in the retry loop, in expectation.

2) Expected Slack Time: Concerning the slack time, we consider that, at any time, the threads that are running the retry loop have the same probability to be anywhere in their current retry. However, when a thread is currently executing a CAS, the other threads cannot execute as well a CAS. The other threads are thus in their critical work or expansion. For every thread, the time before accessing the data structure is then uniformly distributed between 0 and \( cw + \bar{e}(P_{rt}) \).

According to Lemma 1, we conclude that
\[
\bar{s}(P_{rt}) = \frac{(cw + \bar{e}(P_{rt}))(P_{rt} + 1)}{P_{rt}}.
\]

Lemma 1. Let an integer \( n \), a real positive number \( a \), and \( n \) independent random variables \( X_1, X_2, \ldots, X_n \), uniformly distributed within \([0, a] \). Let then \( X \) be the random variable defined by: \( X = \min_{i \in [1, n]} X_i \). The expectation of \( X \) is:
\[
E(X) = \frac{a}{n + 1}.
\]

Proof: Let a positive real number \( x \) be such that \( x < a \). We have
\[
P(X > x) = P(\forall i : X_i > x) = \prod_{i=1}^{n} P(X_i > x) = \left(\frac{a-x}{a}\right)^n.
\]

Therefore, the probability distribution of \( X \) is given by:
\[
t \mapsto \frac{n}{a} \left(\frac{a-x}{a}\right)^{n-1},
\]
and its expectation is computed through
\[
E(X) = \frac{n}{a} \int_{0}^{a} x \times \left(\frac{a-x}{a}\right)^{n-1} \, dx = \frac{n}{a} \int_{0}^{a} (a-u) \times \left(\frac{u}{a}\right)^{n-1} \, du = \frac{n}{a} a^n \int_{0}^{a} (a-u) \times u^{n-1} \, du = \frac{n}{a} a^n \left( a \times \frac{a^n}{n} - \frac{a^{n+1}}{n+1} \right) = \frac{a}{n + 1}.
\]
3) Expected Success Period: We just have to combine Equations 2, 3, and 4 to obtain the general expression of the expected success period under contention:

\[ sp \left( P_{rl} \right) = \left( 1 + \frac{1}{P_{rl} + 1} \right) \left( cw + \bar{e} \left( P_{rl} \right) \right) + 2 cc, \]

which leads, according to Equation 1, to

\[ \frac{1}{pw} \times \left( \frac{P_{rl} + 2}{P_{rl} + 1} \left( cw + \bar{e} \left( P_{rl} \right) \right) + 2 cc \right) = \frac{1}{P - P_{rl}}. \]  

(5)

B. Non-contended System

When the system is not contended, logical conflicts are not likely to happen, hence each thread succeeds in its retry loop at its first retry. A fortiori, no hardware conflict occurs. Each thread still performs one success every work loop, and the success period is given by

\[ sp \left( P_{rl} \right) = \frac{pw + rc + cw + cc}{P}. \]  

(6)

Moreover, a thread spends in average \( rc + cw + cc \) units of time in the retry loop within each work loop. As this holds for every thread, we can obtain the following expression for the total average number of threads inside the retry loop:

\[ P_{rl} = \frac{rc + cw + cc}{pw + rc + cw + cc} \times P = \frac{rc + cw + cc}{sp \left( P_{rl} \right)} \]  

(7)

Equation 6 also gives \( rc + cw + cc = P \times sp \left( P_{rl} \right) - pw \), hence, thanks to Equation 7,

\[ P_{rl} = \frac{P \times sp \left( P_{rl} \right) - pw}{sp \left( P_{rl} \right)}, \text{ i.e. } \frac{sp \left( P_{rl} \right)}{pw} = \frac{1}{P - P_{rl}}, \]  

(8)

where \( sp \left( P_{rl} \right) = \frac{rc + cw + cc}{P_{rl}} \).

C. Unified Solving

It remains to decide whenever the data structure is under contention or not, and to find the corresponding solution. Concerning the frontier between contended and non-contended system, we can remark that Equations 5 and 8 are equivalent if and only if

\[ \frac{rc + cw + cc}{P_{rl}} = \frac{P_{rl} + 2}{P_{rl} + 1} \left( cw + \bar{e} \left( P_{rl} \right) \right) + 2 cc, \]  

(9)

which leads to Lemma 2.

Lemma 2. The system switches from being non-contended to being contended at \( P_{rl} = P_{rl}^{(0)} \), where

\[ P_{rl}^{(0)} = \frac{-(cc + cw - rc) + \sqrt{(cc + cw - rc)^2 + 4(rc + cw + cc)(cw + 2cc)}}{2(cw + 2cc)}. \]

Proof: We show that:

- \( P_{rl}^{(0)} \) is the unique positive solution of Equation 9 if the expansion is set to 0,
- \( P_{rl}^{(0)} \leq 1, \)
- there is no solution of Equation 9 with a non-null expansion.

If the expansion is set to 0, then Equation 9 can be turned into the second order equation

\[ P_{rl}^2 (cw + 2cc) + P_{rl} (cw + cc - rc) - (rc + cw + cc) = 0, \]

that has a single positive solution: \( P_{rl}^{(0)} \).
While instantiating the binomial with $\bar{P}_{rl} = 1$, we obtain $cw + 2(\bar{c}c - \bar{r}c)$, which is not negative, since $\bar{c}c \geq \bar{r}c$ in all the architectures that we are aware of. As the second order equation has also a negative solution, and $cw + 2\bar{c}c$ is positive, we have that $1 \geq \bar{P}_{rl}^{(0)}$. This implies that $\bar{P}_{rl}^{(0)}$ is a solution of the former Equation 9: the expansion is indeed a non-decreasing function, thus $0 \leq \bar{e} \left( \bar{P}_{rl}^{(0)} \right) \leq \bar{e} \left( 1 \right) = 0$. Still we could have other solutions with a non-null expansion.

However, Equation 9 can be rewritten as:

$$rc + cw + \bar{c}c = \frac{\bar{P}_{rl} + 2}{\bar{P}_{rl} + 1} \times \bar{P}_{rl} \times \left( cw + \bar{e} \left( \bar{P}_{rl} \right) \right) + 2\bar{c}c.$$  \hspace{1cm} (10)

The left-hand side of Equation 10 is constant, while the right-hand side is increasing, which discards any other solution, hence the lemma.

Thanks to Lemma 2, we can unify the success period as:

$$\bar{s}p \left( \bar{P}_{rl} \right) = \begin{cases} \frac{(rc + cw + \bar{c}c)}{\bar{P}_{rl}} & \text{if } \bar{P}_{rl} \leq \bar{P}_{rl}^{(0)} \\ \left( cw + \bar{e} \left( \bar{P}_{rl} \right) \right) \times \frac{\bar{P}_{rl} + 2}{\bar{P}_{rl} + 1} + 2\bar{c}c & \text{otherwise.} \end{cases}$$

The unified success period obeys to the following equation

$$\bar{s}p \left( \bar{P}_{rl} \right) = \frac{pw}{P - \bar{P}_{rl}}.$$  \hspace{1cm} (11)

We show in the following theorem how to compute the throughput estimate; the proof manipulates equations in order to be able to use the fixed-point Knaster-Tarski theorem.

**Theorem 1.** The throughput can be obtained iteratively through a fixed-point search, as $T = \left( \bar{s}p \left( \lim_{n \to +\infty} u_n \right) \right)^{-1}$, where

$$\begin{cases} u_0 = \frac{rc + cw + \bar{c}c}{pw + rc + cw + \bar{c}c} \times P \\ u_{n+1} = \frac{u_n \bar{s}p(u_n)}{pw + u_n \bar{s}p(u_n)} \times P \quad \text{for all } n \geq 0. \end{cases}$$

**Proof:** Let us note $f_1 \left( \bar{P}_{rl} \right) = \bar{s}p \left( \bar{P}_{rl} \right) \times \bar{P}_{rl}$ and $f_2 \left( \bar{P}_{rl} \right) = pw \times \bar{P}_{rl} / (P - \bar{P}_{rl})$; then Equation 11 is equivalent to $f_1 \left( \bar{P}_{rl} \right) = f_2 \left( \bar{P}_{rl} \right)$, and we have some properties on $f_1$ and $f_2$.

Firstly, since $x \mapsto x(x + 2)/(x + 1)$ is non-decreasing on $[0, +\infty]$, as well as the expected expansion, we know that $f_1$ is a non-decreasing function. Secondly, $f_2$ is increasing on $[0, P]$, and is bijective from $[0, P]$ to $[0, +\infty]$. We can thus rewrite Equation 11 as:

$$\bar{P}_{rl} = f_2^{-1} \left( f_1 \left( \bar{P}_{rl} \right) \right).$$  \hspace{1cm} (12)

Moreover, $f_2^{-1} \circ f_1$ is a non-decreasing function, as a composition of two non-decreasing functions. Thirdly, $f_2^{-1}$ can be obtained through $x = f_2 \left( f_2^{-1} \left( x \right) \right) = pw \times f_2^{-1} \left( x \right) \times (P - f_2^{-1} \left( x \right))$, which leads to

$$f_2^{-1} \left( x \right) = \frac{x}{pw + x} P.$$  

In addition, we know by construction that if $\bar{P}_{rl} > \bar{P}_{rl}^{(0)}$, then

$$\left( cw + \bar{e} \left( \bar{P}_{rl} \right) \right) \times \frac{\bar{P}_{rl} + 2}{\bar{P}_{rl} + 1} + 2\bar{c}c \geq \frac{rc + cw + \bar{c}c}{\bar{P}_{rl}}.$$  \hspace{1cm} (13)

Indeed, on the one hand,

$$\lim_{\bar{P}_{rl} \to +\infty} \frac{rc + cw + \bar{c}c}{\bar{P}_{rl}} = +\infty,$$

and on the other hand, $(cw + \bar{e}(\bar{P}_{rl})) \times (\bar{P}_{rl} + 2)/(\bar{P}_{rl} + 1) + 2\bar{c}c$ remains bounded. According to Lemma 2, those two functions cross only once, hence Equation 13.

Since $\bar{s}p \left( \bar{P}_{rl} \right) = (rc + cw + \bar{c}c) / \bar{P}_{rl}$ if $\bar{P}_{rl} \leq \bar{P}_{rl}^{(0)}$, we have $\bar{s}p \left( \bar{P}_{rl} \right) \geq (rc + cw + \bar{c}c) / \bar{P}_{rl}$.
for any \( P_{rt} \), and then
\[
f_1 \left( P_{rt} \right) \geq rc + cw + cc.
\]

Let then
\[
P_{rt}^{(i)} = \frac{rc + cw + cc}{pw + rc + cw + cc} P.
\]
We have seen that \( f_2^{-1} \circ f_1 \) is a non-decreasing function, hence
\[
f_2^{-1} \left( f_1 \left( P_{rt}^{(i)} \right) \right) \geq f_2^{-1} \left( rc + cw + cc \right)
\]
\[
\geq \frac{rc + cw + cc}{pw + rc + cw + cc} \times P
\]
\[
f_2^{-1} \left( f_1 \left( P_{rt}^{(i)} \right) \right) \geq P_{rt}^{(i)}.
\]
Since \( f_2^{-1} \) is bounded, Equation 12 admits a solution.

We are interested in the solution whose \( P_{rt} \) is minimal since it corresponds to the first attained solution when the expansion grows, starting from 0. The current theorem comes then from the application of the Knaster-Tarski theorem.

V. CONSTRUCTIVE APPROACH

In this section, we instantiate the probability distribution of the parallel work with an exponential distribution. We have therefore a better knowledge of the behavior of the algorithm, particularly in medium contention cases, which allows us to follow a fine-grained approach that studies individually each successful operation together with every CAS occurrence. We provide an elegant and efficient solution that relies on a Markov chain analysis.

A. Process

We have seen in Section III-C2 that we split the contention domain into three modes: no contention, medium contention or high contention. The main idea is to start from a configuration with a given number of threads \( P_{rt} \) just after a successful CAS, and describe what will happen until the next successful CAS: what will be the mode of the next success period, and even more precisely, which will be the number of threads at the beginning of the next success period.

As a basis, we consider the execution that would occur without any other thread exiting the parallel section (then entering the retry loop); we call this execution the internal execution. This execution follows the success period pattern described in Figure 2 (with an infinite slack time if the system is not contended). On top of this basic success period, we inject the threads that can exit the parallel section, which has a double impact. On the one hand, they increase the number of threads inside the retry loop for the next success period. On the other hand, if the first thread that exits the parallel section starts its retry during the slack time of the success period of the internal execution, then this thread will succeed its Access, which is a Read, and will shrink the actual slack time of the current success period.

According to the distribution probability of the arrival of the new threads, we can compute the probability for the next success period to start with any number of threads. The expression of this stochastic sequence of success periods in terms of Markov chains results in the throughput estimate.
B. Expansion

The expansion, as before, represents the additional time in the execution time of a retry, due to the serialization of atomic primitives. However, in contrary to Section IV-A1, we compute here this additional time in the current success period, according to the number of threads $P_{rl}$ inside the retry loop at the beginning of the success period. The expansion only appears when the success period is highly contended, i.e. when we can find a continuous sequence of CAS’s all through the success period.

The expansion is highly correlated with the way the cache coherence protocol handles the exchange of cache lines between threads. We rely on the experiments of the research report associated with [7], which show that if several threads request for the same cache line in order to operate a CAS, while another thread is currently executing a CAS, they all have an equal probability to obtain the cache line when the current CAS is over.

We draw an illustrative example in Figure 3. The green CAS’s are successful while the red CAS’s fail. To lighten the picture, we hide what happened for the threads before they experience a failed CAS. The horizontal dash lines represent the time where a thread wants to access the data in order to operate a CAS but has to wait because another thread owns the data in exclusive mode. We can observe in this example that the first thread that accesses the data structure is not the thread whose operation returns.

We are given that $P_{rl}$ threads are inside the retry loop at the end of the previous successful CAS, and we only consider those threads. When such a thread executes a CAS for the first time, this CAS is unsuccessful. The thread was in the retry loop when the successful CAS has been executed, so it has read a value that is not up-to-date anymore. However, this failed CAS will bring the current version of the value (to compare-and-swap) to the thread, a value that will be up-to-date until a successful CAS occurs.

So we have firstly a sequence of failed CAS’s until the first thread that operated its CAS within the current success period finishes its critical work. At this point, there exists a thread that is
executing a CAS. When this CAS is finished, some threads compete to obtain the cache line. We have two bags of competing threads: in the first bag, the thread that just ended its critical work is alone, while in the second bag, there are all the threads that were in the retry loop at the beginning of the success period, and did not operate a CAS yet. The other, non-competing, threads are running their critical work and do not yet want to access the data.

As described before, every thread has the same probability to become the next owner of the cache line. If a thread from the first bag is drawn, then the CAS will be successful and the success period ends. Otherwise, the CAS is a failure, and we iterate at the end of this failed CAS. However, the thread that just failed its CAS is now executing its critical work, and does not request for a new CAS until this work has been done, thus it is not anymore in the second bag. In addition, the thread that had executed its CAS after the thread of the first bag is now back from its critical work and falls into the first bag. The process iterates until a thread is drawn from the first bag.

As a remark, note that we do not consider threads that are not in the retry loop at the beginning of the success period since even if they come back from the parallel section during the success period, their Read will be delayed and their CAS is likely to occur after the end of the success period.

Theorem 2 gives the explicit formula for the expansion, based on the previous explanations.

**Theorem 2.** The expected time between the end of the critical work of the first thread that operates a CAS in the success period and the beginning of a successful CAS is given by:

$$ e(P_{rl}) = \left\lceil \frac{cw}{cc} \right\rceil cc - cw + \sum_{i=1}^{P_{com}} \frac{i(i-1)(P_{com}-1)!}{(P_{com})^i} \times \frac{cc}{(P_{com}-i)!} $$

where $$ P_{com} = P_{rl} - \left\lceil \frac{cw}{cc} \right\rceil + 1. $$

**Proof:** Let us set the timeline so that at the beginning of the success period, i.e. just after a successful CAS, we are at $$ t = 0. $$ Firstly, a success cannot start before $$ t = t_0, $$ where $$ t_0 = cc + \left\lceil \frac{cw}{cc} \right\rceil cc. $$ The quickest thread indeed starts a failed CAS at $$ t = 0 $$ and comes back from critical work at $$ t = cc + cw. $$ It has then to wait for the current CAS to finish before being able to obtain the cache line. At $$ t = t_0, $$ $$ P_{rl} - t_0/cc + 1 $$ threads are competing for the data. Among them, 1 thread will lead to a successful CAS, while the $$ P_{rl} - t_0/cc $$ other threads will end up with a failed CAS. If a failed CAS occurs, then at $$ t = t_0 + cc, $$ the same number of threads compete, but now there is one more potential success and one less potential failure. In the worst case, it will continue until all competing threads will lead to a successful CAS.

Let $$ P_{com} = P_{rl} - t_0/cc + 1 $$ the number of threads that are competing at each round, and let, for all $$ i \in [1, P_{com}], $$ $$ p_i = i/P_{com} $$ the probability to draw a thread that will execute a successful CAS.

The expected number of failed CAS’s that occurs after the first thread comes back is then given by

$$ E(F) = p_1 \times 0 + (1-p_1)p_2 \times 1 + \cdots + (1-p_1)(1-p_2) \times \cdots \times (1-p_{P_{com}-1}) \times p_{P_{com}} \times (P_{com}-1). $$
More formally,

\[
E(F) = \sum_{i=1}^{P_{com}} i \prod_{j=1}^{i-1} (1 - p_j) p_i \times (i - 1)
\]

\[
= \sum_{i=1}^{P_{com}} i \prod_{j=1}^{i-1} \left(1 - \frac{j}{P_{com}}\right) \times (i - 1)
\]

\[
= \sum_{i=1}^{P_{com}} \frac{1}{(P_{com})^i} \prod_{j=1}^{i-1} (P_{com} - j)i(i - 1)
\]

\[
E(F) = \sum_{i=1}^{P_{com}} i(i - 1) \frac{(P_{com} - 1)!}{(P_{com} - i)!}
\]

C. Formalization

The parallel work follows an exponential distribution, whose mean is \(pw\). More precisely, if a thread starts a parallel section at the instant \(t_1\), the probability distribution of the execution time of the parallel section is

\[
t \mapsto \lambda e^{-\lambda(t-t_1)} \mathbb{1}_{[t_1, +\infty)}(t), \text{ where } \lambda = \frac{1}{pw}.
\]

This probability distribution is memoryless, which implies that the threads that are executing their parallel section cannot be differentiated: at a given instant, the probability distribution of the remaining execution time is the same for all threads in the parallel section, regardless of when the parallel section began. For all threads, it is defined by:

\[
t \mapsto \lambda e^{-\lambda t}, \text{ where } \lambda = \frac{1}{pw}.
\]

For the behavior in the retry loop, we rely on the same approximation as in the previous section, i.e. when a successful thread exits its retry loop, the remaining execution time of the retry of every other thread that is still in the retry loop is uniformly distributed between 0 and the execution time of a whole retry. We have seen that the expectation of this remaining time is the size of the execution time of a retry divided by the number of threads inside the retry loop plus one. Here, we assume that a thread will start a retry at this time. This implies another kind of memoryless property: the behavior of a thread that is in the retry loop does not depend on the moment that it entered its retry loop.

To tackle the problem of estimating the throughput of such a system, we use an approach based on Markov chains. We study the behavior of the system over time, step by step: a state of the Markov chain represents the state of the system when the current success period began (i.e. just after a successful CAS) and (thus) the system changes state at the end of every successful CAS. According to the current state, we are able to compute the probability to reach any other state at the beginning of the next success period. In addition, the two memoryless properties render the description of a state easy to achieve: the number of threads inside the retry loop when the current success begins, indeed fully characterizes the system.

We recall that \(P_{rt}\) is the number of threads inside the retry loop when the success period begins. The Markov chain is strongly connected with \(P_{rt}\), since it is composed of \(P\) states.
$S_0, S_1, \ldots, S_{P-1}$, where, for all $i \in [0, P - 1]$, the success period is in state $S_i$ iff $P_{rt} = i$. For all $(i, j) \in [0, P - 1]^2$, $P (S_i \rightarrow S_j)$ denotes the probability that a success characterized by $S_j$ follows a success in state $S_i$. $st (S_i \rightarrow S_j)$ denotes the slack time that passed while the system has gone from state $S_i$ to state $S_j$. This slack time can be expressed based on the slack time $st(i)$ of the internal execution, i.e. the execution that involves only the $i$ threads of the retry loop and ignores the other threads (see Section V-A). Recall that we consider that the slack time of the internal execution with $0$ thread is infinite, since no thread will access the data structure. In the same way, we denote by $ct(i)$ the completion time of the internal execution, hence $ct(i) = cc + cw + e(i) + cc$.

We have seen that the level of contention (mode) is determined by $P_{rt}$, hence the interval $[0, P - 1]$ can be partitioned into

$$[0, P - 1] = \mathcal{I}_{noc} \cup \mathcal{I}_{mid} \cup \mathcal{I}_{hi},$$

where the partitions correspond to the different contention levels. So, by definition, $\mathcal{I}_{noc} = \{0\}$, and for all $i \in \mathcal{I}_{noc} \cup \mathcal{I}_{mid}$, $e(i) = 0$ (see Section III-C2).

The success period is highly-contended, i.e. we have a continuous sequence of CAS’s in the success period, if the sum of the execution time of all the CAS’s that need to be operated exceeds the critical work. Hence $\mathcal{I}_{hi} = [i_{hi}, P - 1]$, where

$$i_{hi} = \min\{i \in [1, P - 1] \mid i \times cc > cw\}.$$

In addition, as the sequence of CAS’s is continuous when the contention is high, the slack time is null when the success period is highly contended, i.e., for all $i \in \mathcal{I}_{hi}$, $st(i) = 0$, and a fortiori, $st(S_i \rightarrow S_j) = 0$.

Otherwise, the success period is in medium contention, hence $\mathcal{I}_{mid} = [1, i_{hi} - 1]$. Moreover, if $i \in \mathcal{I}_{mid}$, $st(i) > 0$, and $e(i) = 0$, because the CAS’s synchronized during the previous success period and will not collide any more in the current success period.

1) Transition Matrix: We consider here that the system is in a given state, and we compute the probability that the system will next reach any other state. Without loss of generality, we can choose the origin of time such that the current success period begins at $t = 0$.

Let us first look at the core cases, i.e. let $i \in \mathcal{I}_{mid} \cup \mathcal{I}_{hi}$ and $k \in [0, P - i - 1]$; we assume that the system is currently in state $S_i$, and we are interested in the probability that the system will switch to $S_{i+k}$ at the end of the current state. In other words, we want to find the probability that, given that the current success period started when $i$ threads were in the retry loop, the next success period will begin while $i + k$ threads are in the retry loop.

As the successful thread will exit the retry loop at the end of the current success period, there is at least one thread that enters the retry loop during the current success period. Two non-overlapping events can then occur (see Figure 4): either the first thread exiting the parallel section starts within $[0, st(i)]$, i.e. in the slack time of the internal execution, and this event is written $E_{ext}$, or the first thread entering the retry loop starts after $t = st(i)$, and this event is denoted by $E_{int}$. Therefore, we have $P (S_i \rightarrow S_{i+k}) = P (E_{ext}) + P (E_{int})$.

First note that $E_{ext}$ cannot happen when the success period is highly contended; in this case, the slack time is indeed null, and we conclude $P (E_{ext}) = 0$. In addition, we have seen in Section V-B that external threads, i.e. threads that are in the parallel section at the beginning of the success period, do not participate to the game of expansion, so they cannot be successful. Under high-contention, $E_{int}$ happens, and the successful CAS that ends the success period is operated by an internal thread, i.e. a thread that was already in the retry loop when the success period began.

Under medium contention, $E_{ext}$ can occur. In this case, an external thread accesses the data
structure before any internal thread does. We have also seen that the expansion is null in medium contention level, thus the external thread will execute its critical work, and especially its CAS without being delayed; this implies that the first external thread that accesses the data structure will end the current success period with the end of its CAS. If however \( E_{\text{int}} \) occurs, an internal thread succeeds, but is not necessarily the first thread that accessed the data structure during the success period.

The two possible events are pictured in Figure 4, where the blue arrows represent the threads that exit the parallel section. Recall, we aim at computing the probability to start the next success period with \( i + k \) threads inside the retry loop. We formalize the idea drawn in the figure by using \( X_{[a,b]} \), which is defined as a random variable indicating the number of threads exiting the parallel section during the time interval \([a,b]\). The probability of having \( E_{\text{int}} \) is then given by

\[
P(E_{\text{int}}) = P\left( X_{[0,\text{st}(i)]} = 0 \mid P_{rl} = i \text{ at } t = 0^+ \right) \times P\left( X_{[\text{st}(i),\text{st}(i)+\text{ct}(i)]} = k + 1 \mid P_{rl} = i \text{ at } t = \text{st}(i)^+ \right)
\]

Concerning \( E_{\text{ext}} \), we know that if \( i \in \mathcal{T}_\text{hi} \), then \( P(E_{\text{ext}}) = 0 \). Otherwise, if we denote by \( t_3 \) the starting time of the first thread that exits the parallel section, we obtain

\[
P(E_{\text{ext}}) = P\left( X_{[0,\text{st}(i)]} > 0 \mid P_{rl} = i \text{ at } t = 0^+ \right) \times P\left( X_{[t_3,t_3+r_c+c_w+c_c]} = k \mid P_{rl} = i + 1 \text{ at } t = t_3^+ \right)
\]

To simplify the reasoning, and given that the costs of \( \text{Read} \) and \( \text{CAS} \) are approximately the same, we approximate \( t_3 + r_c + c_w + c_c \) with \( t_3 + c_c + c_w + c_c \), leading to

\[
P(E_{\text{ext}}) = P\left( X_{[0,\text{st}(i)]} > 0 \mid P_{rl} = i \text{ at } t = 0^+ \right) \times P\left( X_{[t_3,t_3+c(t+1)]} = k \mid P_{rl} = i + 1 \text{ at } t = t_3^+ \right)
\]

According to the exponential distribution, given a thread that is in the parallel section at \( t = a \),
the probability to exit the parallel section within \([a, b]\) is:
\[
\int_a^b \lambda e^{-\lambda(t-a)} \, dt = \int_0^{b-a} \lambda e^{-\lambda u} \, du.
\]
It is also the probability, given a thread that is in the parallel section at \(t = 0\), to exit the retry loop within \([a, b-a]\). This implies:
\[
P(E_{\text{int}}) = \mathbb{P}(X_{[0,\text{st}(i)]} = 0 \mid P_{rl} = i \text{ at } t = 0^+) \times \mathbb{P}(X_{[0,\text{ct}(i)]} = k + 1 \mid P_{rl} = i \text{ at } t = 0^+)
\]
and
\[
P(E_{\text{ext}}) = \mathbb{P}(X_{[0,\text{st}(i)]} > 0 \mid P_{rl} = i \text{ at } t = 0^+) \times \mathbb{P}(X_{[0,\text{ct}(i)]} = k \mid P_{rl} = i + 1 \text{ at } t = 0^+).
\]
To lighten the notations, let us define
\[
\begin{cases}
a_{i,k} = \mathbb{P}(X_{[0,\text{ct}(i)]} = k \mid P_{rl} = i \text{ at } t = 0) \\
b_i = \mathbb{P}(X_{[0,\text{st}(i)]} = 0 \mid P_{rl} = i \text{ at } t = ct(i))^+.
\end{cases}
\tag{14}
\]
In addition, given a thread that is in the parallel section at \(t = 0\), the probability to exit the parallel section within \([0, b-a]\) is \(\int_0^{b-a} \lambda e^{-\lambda u} \, du\). By counting the number of threads that need to exit the parallel section, we obtain:
\[
\begin{cases}
a_{i,k} = \binom{P-1}{k} (1 - e^{-\lambda ct(i)})^k (e^{-\lambda ct(i)})^{P-i-k} \\
b_i = (\exp(-\lambda sl(t(i)))^{P-i}.
\end{cases}
\tag{15}
\]
Altogether, we have that
\[
P(S_i \rightarrow S_{i+k}) = b_i \times a_{i,k+1} + (1 - b_i) \times a_{i+1,k}.
\]

The situation is slightly different if \(k = -1\); in this case, no thread should exit the parallel section during the slack time and no thread should exit during the retry of the first thread that accessed the data structure during the success period neither. This shows that
\[
P(S_i \rightarrow S_{i-1}) = b_i \times a_{i,0}.
\]

When the success period is not contended, \(i.e.\) if \(i = 0\), the slack time of the execution that ignores external threads can be seen as infinite, hence we can define \(b_0 = 0\) (the probability that a thread exits its parallel section during an infinite interval of time is 1). As for the \(a_{i,k}\)’s, they can be defined in the same way as earlier.

We have obtained the full transition matrix \((M_{i,j})_{(i,j) \in [0,P-1]^2}\), which is a triangular matrix, augmented with a subdiagonal:
\[
\begin{align*}
M_{i,i+k} &= b_i a_{i,k+1} + (1 - b_i) a_{i+1,k} & \text{if } k \in [0, P - i - 1] \\
M_{i,i-1} &= b_i \times a_{i,0} & \text{if } i > 0 \\
M_{i,i} &\quad = 0 & \text{otherwise}
\end{align*}
\]

**Lemma 3.** \(M\) is a right stochastic matrix.

**Proof:** First note that, by definition of \(a_{i,k}\), for all \(i \in [0, P - 1]\),
\[
\sum_{k=0}^{P-i} a_{i,k} = 1.
\]
If \(i\) threads are indeed inside the retry loop at \(t = 0\), then, within \([0, \text{st}(i)]\), at least 0 thread, and at most \(P - i\) threads (inclusive) will exit their parallel section.
We have first
\[ \sum_{j=0}^{P-1} M_{0,j} = \sum_{k=0}^{P-1} a_{0+1,k} = 1. \]

In the same way, for all \( i \in [1, P-1] \),
\[ \sum_{j=0}^{P-1} M_{i,j} = \sum_{k=-1}^{P-1-i} M_{i,i+k} \]
\[ = b_i \times a_{i,0} + \sum_{k=0}^{P-1-i} b_i a_{i,k+1} + (1 - b_i) a_{i+1,k} \]
\[ = b_i \times \sum_{k=-1}^{P-1-i} a_{i,k+1} + (1 - b_i) \sum_{k=0}^{P-1-i} a_{i+1,k} \]
\[ \sum_{j=0}^{P-1} M_{i,j} = 1. \]

\[ \]  

**Lemma 4.** The transition matrix has a unique stationary distribution, which is the unique left eigenvector of the transition matrix with eigenvalue 1 and sum of its elements equal to 1.

**Proof:** Note that the Markov chain is irreducible and aperiodic. Let \( X \geq P - 1 \), \( i \in [0, P - 1] \) and \( j \in [i, P - 1] \).
\[ \mathbb{P}(S_j \rightarrow S_i \text{ in } X \text{ steps}) \geq \mathbb{P}(S_j \rightarrow S_{j-1} \rightarrow \cdots \rightarrow S_i) \]
\[ \times \mathbb{P}(S_i \rightarrow S_i)^{X-(j-i)} \]
\[ \mathbb{P}(S_j \rightarrow S_i \text{ in } X \text{ steps}) > 0 \]
As
\[ \mathbb{P}(S_i \rightarrow S_j \text{ in } X \text{ steps}) \geq \mathbb{P}(S_i \rightarrow S_j) > 0, \]
the Markov chain is irreducible. Since \( S_1 \) is clearly aperiodic, and the chain is irreducible, the chain is aperiodic as well.

This implies that the Markov chain has a unique stationary distribution, which is the unique left eigenvector of the transition matrix with eigenvalue 1 and sum of its elements equal to 1.

2) **Stationary Distribution:**

**Theorem 3.** Given the transition matrix, the stationary distribution can be found in \((P+1)P-1\) operations.

**Proof:** As the Markov chain is irreducible, the stationary distribution does not contend any zero. The space of the left eigenvectors with unit eigenvalue is uni-dimensional; therefore, for any \( v_0 \), there exists a vector \( v = (v_0 \ v_1 \ldots \ v_{P-1}) \), such that \( v \) spans this space.

Let \( v_0 \) a real number; necessarily, \( v \) fulfills \( v \cdot M = v \), hence for all \( i \in [0, P-2] \)
\[ \sum_{k=0}^{i+1} v_k M_{k,i} = v_i, \]
which leads to, for all \( i \in [0, P - 2] \):
\[
v_{i+1} = \frac{1}{M_{i+1,i}} \left( (1 - M_{i,i})v_i - \sum_{k=0}^{i-1} v_k M_{k,i} \right).
\]
So we obtain the \( v_1, \ldots, v_{P-1} \) iteratively (we know that \( M_{i+1,i} = b_{i+1} \times a_{i+1,0} \), which is not null), with \( 2 \times i + 1 \) operations needed to compute \( v_{i+1} \).

The elements of the stationary distribution should sum to one, so we start from any \( v_0 \), compute the whole vector, and then normalize each element by their sum, hence the theorem.

3) Slack time and Throughput: In order to compute the final throughput, we have to compute the expectation of the slack time, when the system goes from state \( S_i \) to any other state, that we note \( \mathbb{E} (st (S_i \rightarrow S_\star)) \). Also, we will be able to exhibit a vector \( s = (s_0, s_1, \ldots, s_{P-1}) \) of expected success period, where \( s_i \) is the expectation of the execution time of the success period if \( i \) threads are in the retry loop when the success period begins:
\[
\begin{cases}
  s_i = \mathbb{E} (st (S_i \rightarrow S_\star)) + cc + cw + e (i) + cc & \text{if } i \notin I_{noc} \\
  s_i = \mathbb{E} (st (S_i \rightarrow S_\star)) + rc + cw + cc & \text{otherwise}.
\end{cases}
\]

Finally, the expected throughput (inverse of the success period) is calculated through
\[
T = \frac{1}{v \cdot s},
\]
where \( v \) is the stationary distribution of the Markov chain.

We know already that if \( i \in I_{hi} \), then \( \mathbb{E} (st (S_i \rightarrow S_{i+k})) = 0 \).

In the other extreme case, i.e. if \( i \in I_{noc} \), we rely on the following lemma.

**Lemma 5.** Let an integer \( n \), a real number \( \lambda \), and \( n \) independent random variables \( X_1, X_2, \ldots, X_n \), following an exponential distribution of mean \( \lambda^{-1} \). Let then \( X \) be the random variable defined by: \( X = \min_{i \in [1,n]} X_i \). The expectation of \( X \) is:
\[
\mathbb{E} (X) = \frac{1}{\lambda n}.
\]

**Proof:** We have
\[
\mathbb{P} (X > x) = \mathbb{P} (\forall i : X_i > x) = \prod_{i=1}^{n} \mathbb{P} (X_i > x) = \left( \int_{x}^{+\infty} \lambda e^{-\lambda t} \right)^n = \left( \int_{x}^{+\infty} e^{-\lambda t} \right)^n = e^{-\lambda nx}
\]
Therefore, the probability distribution of \( X \) is given by:
\[
t \mapsto \lambda ne^{-\lambda nt},
\]
and its expectation is computed through
\[
E(X) = \int_0^{+\infty} \lambda n t e^{-\lambda nt} dt
\]
\[
= \left[ e^{-\lambda nt} \right]_0^{+\infty} + \int_0^{+\infty} e^{-\lambda nt} dt
\]
\[
= \left[ \frac{1}{\lambda n} e^{-\lambda nt} \right]_0^{+\infty}
\]
\[
E(X) = \frac{1}{\lambda n}
\]

This proves that
\[
E(st(S_0 \rightarrow S_i)) = \frac{pw}{P}.
\]

Let now \( i \in I_{\text{mid}} \), and \( k \in [-1, P - i - 1] \); we are interested in \( E(st(S_i \rightarrow S_{i+k})) \). The slack time is less immediate, and we use the following reasoning. First note that the probability distribution of the first thread exiting the parallel section is given by \( t \mapsto \lambda(P - i)e^{-\lambda(P - i)t} \). If this thread comes back during \([0, st(i)]\), the time that passed since the beginning of the success period is the slack time, otherwise, it is \( st(i) \).

\[
E(st(S_i \rightarrow S_s)) = \int_0^{st(i)} \lambda(P - i)e^{-\lambda(P - i)t} dt + \int_{st(i)}^{+\infty} \lambda(P - i)e^{-\lambda(P - i)t} st(i) dt
\]
\[
= \left[ e^{-\lambda(P - i)t} \right]_0^{st(i)} + \left[ \frac{1}{\lambda(P - i)} e^{-\lambda(P - i)t} \right]_0^{st(i)} + st(i) \left[ e^{-\lambda(P - i)t} \right]_{+\infty}
\]
\[
E(st(S_i \rightarrow S_s)) = -st(i) e^{-\lambda(P - i)st(i)} + \frac{1 - e^{-\lambda(P - i)st(i)}}{\lambda(P - i)} + st(i) \left( e^{-\lambda(P - i)st(i)} \right)
\]

We conclude that
\[
E(st(S_i \rightarrow S_s)) = \frac{1 - e^{-\lambda(P - i)st(i)}}{P - i} pw.
\]

Putting all together, we obtain
\[
\begin{cases}
E(st(S_i \rightarrow S_s)) = \frac{1 - e^{-\lambda(P - i)st(i)}}{P - i} pw & \text{if } i \in I_{\text{noc}} \cup I_{\text{mid}} \\
E(st(S_i \rightarrow S_s)) = 0 & \text{if } i \in I_{\text{hi}}.
\end{cases}
\]

4) Number of Failed Retries: Another metric to estimate the quality of the model is the number of failed retries per successful retry. We compute it by counting the number of failed retries within the current success period, where a retry is billed to a given success period if its failed CAS occurs during this success period. We denote by \( E(f_i) \) the expected number of failed CAS during a success period that begins with \( i \) threads, where \( i \in [0, P - 1] \).

If the success period is not contended, \textit{i.e.} if \( i \in I_{\text{noc}} \), no failure will occur since the first CAS of the success period will be a success; hence \( E(f_i) = 0 = i \).

If the success period is mid-contended, \textit{i.e.} if \( i \in I_{\text{mid}} \), every thread that is in the retry loop in the beginning of the success period will execute at least one CAS during this success period, and exactly two if the thread is the successful one. We know indeed that, even if a thread exits its parallel section during the slack time, and is then successful, the failed CAS’s will occur before the thread entering the retry loop executes its successful CAS. As any thread that exits
its parallel section during the success period either is successful at its first CAS, or does not operate the CAS during the success period, we conclude that: $E(f_i) = i$.

If the success period is highly contended, \textit{i.e.} if $i \in \mathcal{I}_{hi}$, then we know that we have an uninterrupted sequence of failed CAS’s, from the beginning of the success period to the last ending successful CAS. The expected number of failed CAS’s is then directly related to the expected duration of the success period. Recalling that the expansion is given in Theorem 2, we obtain:

$$E(f_i) = 1 + \frac{cw + e(i)}{cc}.$$ 

VI. EXPERIMENTS

To validate our analysis results, we use two main types of lock-free algorithms. In the first place, we consider a set of basic algorithms where operations can be completed with a single successful CAS. This set of algorithms includes: (i) synthetic designs, that cover the design space of possible lock-free data structures; (ii) several fundamental designs of data structure operations such as lock-free stacks [20] (\texttt{Pop, Push}), queues [19] (\texttt{Dequeue}), counters [11] (\texttt{Increment, Decrement}). As a second step, we consider more advanced lock-free operations that involve helping mechanisms, and show how to use our analysis in this context. Finally, in order to highlight the benefits of the analysis framework, we show how it can be applied to i) determine a beneficial back-off strategy and ii) optimize the memory management scheme used by a data structure, in the context of an application.

We also give insights about the strengths of our two approaches. On the one hand, the constructive approach exhibits better predictions due to the tight estimation of the failing retries. On the other hand, the average-based approach is applicable to a broader spectrum of algorithmic designs as it leaves room to abstract complicated algorithmic designs.

A. Setting

We have conducted experiments on an Intel ccNUMA workstation system. The system is composed of two sockets equipped with Intel Xeon E5-2687W v2 CPUs with frequency band 1.2-3.4 GHz. The physical cores have private L1, L2 caches and they share an L3 cache, which is 25 MB. In a socket, the ring interconnect provides L3 cache accesses and core-to-core communication. Due to the bi-directionality of the ring interconnect, uncontented latencies for intra-socket communication between cores do not show significant variability. Our model assumes uniformity in the CAS and Read latencies on the shared cache line. Thus, threads are pinned to a single socket to minimize non-uniformity in Read and CAS latencies. In the experiments, we vary the number of threads between 4 and 8 since the maximum number of threads that can be used in the experiments are bounded by the number of physical cores that reside in one socket. We show the experimental results with 8 threads.

In all figures, the y-axis shows both the throughput values, \textit{i.e.} number of operations completed per second, and the ratio of failing to successful retries (multiplied by $10^6$, for readability), while the mean of the exponentially distributed parallel work $pw$ is represented on the x-axis. The number of failures per success in the average-based approach is computed as $\frac{P}{pw + rc + cw + cc}$ (a thread can succeed only once in each work loop).
B. Basic Data Structures

Here, we consider lock-free operations that can be completed with a single successful CAS, and provide predictions using both the average-based and the constructive approach together with the theoretical upper bound.

1) Synthetic Tests: We first evaluate our models using a set of synthetic tests that have been constructed to abstract different possible design patterns of lock-free data structures (value

![Figure 5: Synthetic program with exponentially distributed parallel work](image-url)
of \( cw \) and different application contexts (value of \( pw \)). The critical work is either constant, or follows a Poisson distribution; in Figure 5, its mean value \( cw \) is indicated at the top of the graphs.

A steep decrease in throughput, as \( pw \) gets low, can be observed for the cases with low \( cw \), that mainly originates due to expansion. When \( cw \) is high, performance continues to increase when \( pw \) decreases, though slightly. The expansion is indeed low but the slack time, which appears as a more dominant factor, decreases as the number of threads inside the retry loop

Figure 6: Synthetic program with parallel work following Poisson
increases.

When looking into the differences between the constructive and the average-based approach: the average-based approach estimations come out to be less accurate for mid-contention cases as it only differentiates between contended and non-contended modes. In addition, it fails to capture the failing retries when measured throughput starts to deviate from the theoretical upper bound, as $pw$ gets lower. In contrast, the constructive approach provides high accuracy in all metrics for almost every case.

![Figure 7: Synthetic program with Constant parallel work](image-url)
We have also run the same synthetic tests with a parallel work that follows a Poisson distribution (Figure 6) or is constant (Figure 7), in order to observe the impact of the distribution nature of the parallel work. Compared to the exponential distribution, a better throughput is achieved with a Poisson distribution on the parallel work. The throughput becomes even better with a constant parallel work, since the slack time is minimized due to the synchronization between the threads, as explained in [8].

![Figure 8: Treiber’s Stack](image)

2) Treiber’s Stack: The lock-free stack by Treiber [20] is a fundamental data structure that provides Pop and Push operations. To Pop an element, the top pointer is read and the next pointer of the initial element is obtained. The latter pointer will be the new value of the CAS that linearizes the operation. So, accessing the next pointer of the topmost element represents $cw$ as it takes place between the Read and the CAS. We initialize the stack by pushing elements with or without a stride from a contiguous chunk of memory. By this way, we are able to introduce both costly or not costly cache misses. We also vary the number of elements popped at the same time to obtain different $cw$; the results, with different $cw$ values are illustrated in Figure 8.
C. Towards Advanced Data Structure Designs

Advanced lock-free operations generally require multiple pointer updates that cannot be done with a single CAS. One way to design such operations, in a lock-free manner, is to use helping mechanisms: an inconsistency will be fixed eventually by some thread. Here we consider two data structures that apply immediate helping, the queue from [19] and the deque designed in [17]. In the queue experiment (Figure 9), we run the Enqueue operation on the queue with and without memory management; in the deque experiment, each thread is dedicated to an end of the deque (equally distributed), while we vary the proportion of push operations (colors in Figure 10).

Here, we consider data structures that apply immediate helping, where threads help for the completion of a recently linearized operation until the data structure comes into a stable state in which a new operation can be linearized. The crucial observation is that the data structure goes through multiple stages in a round robin fashion. The first stage is the one where the operation is linearized. The remaining ones are the stages in which other threads, that execute another operation, might help for the completion of the linearized operation, before attempting to linearize their own operations. Thus, the success period (ignoring the slack time) can be seen as the sum of the execution time of these stages, each ending with a CAS that updates a pointer. The CAS in the first stage might be expanded by the threads that are competing for the linearization of their operation, and consequent CAS’s might be expanded by the helper threads, which are still trying to help an already completed operation. Also, there might be slack time before the start of the first stage as the other stages will start immediately due to the thread that has completed the previous stage.

Although it is hard to stochastically reconstruct the executions with Markov chains, our average-based approach provides the flexibility required to estimate the performance by plugging the expected success period, given the number of threads inside the retry loop, into the Little’s Law. As the impacting factors are similar, we estimate the success period in the same vein as in Section IV; with a minor adaptation of the expansion formula and by slightly adapting the slack time estimation based on the same arguments.

1) Expected Expansion for the Advanced Data Structures: Consider an operation such that, the success period (ignoring the slack time) is composed of $S$ stages (denoted by $Stage_1, \ldots, Stage_S$) where each stage represents a step towards the completion of the operation. Let $CAS_i$ denote the CAS operation at the end of the $Stage_i$. From a system-wide perspective, $\{CAS_1, \ldots, CAS_S\}$ is the set of CAS’s that have to be successfully and consecutively executed to complete an operation, assuming all threads are executing the same operation. This design enforces that $CAS_i$ can be successful only if the last successful CAS is a $CAS_{i-1}$. And, $CAS_1$ can be successful only if the last successful CAS is a $CAS_S$. In other words, another operation can not linearize before the completion of the linearized but incomplete operation.

Now, let $e_i$ denote the expected expansion of $CAS_i$. If the data structure is in the stable state (i.e. is in $Stage_1$, where a new operation can be linearized), then we have to consider the probability, for all threads except one, to expand the successful $CAS_1$ which linearizes the operation. After the linearization, this operation will be completed in the remaining stages where again the successful CAS’s at the end of the stages are subject to the same expansion possibility by the threads in the retry loop, as they might be still trying to help for the completion of the previously completed operation.

Similar to the [8], our assumption here is that any thread that is in the retry loop, can launch $CAS_i$, with probability $h$, that might expand the successful $CAS_i$. We consider, the starting point of a failing $CAS_i$ is a random variable which is distributed uniformly within the retry loop, which
is composed of expanded stages of the operation. This is because an obsolete thread can launch a $CAS_i$, regardless of the stage in which the data structure is in (equally, regardless of the last successful $CAS$). Due to the uniformity assumption, the expansion for the successful $CAS$’s in all stages, would be equal. Similar to the [8], we estimate the expansion $e_i$ by considering the impact of a thread that is added to the retry loop. Let the cost function $delay_i$ provide the amount of delay that the additional thread introduces, depending on the point where the starting point of its $CAS_i$ hits. By using these cost functions, we can formulate the total expansion increase that each new thread introduces and derive the differential equation below to calculate the expected total expansion in a success period, where $\bar{e}(P_{rl}) = \sum_{i=1}^{S} e_i(P_{rl})$. Note that, we assume that the expansion starts as soon as strictly more than 1 thread are in the retry loop, in expectation.

**Lemma 6.** The expansion of a $CAS$ operation is the solution of the following system of equations, where $rlw = \sum_{i=1}^{S} rlw_i = \sum_{i=1}^{S} (rc_i + cw_i + cc_i)$:

$$
\begin{align*}
  \bar{v}'(P_{rl}) &= cc \times \frac{S \times \frac{cc}{2} + \bar{e}(P_{rl})}{rlw + \bar{e}(P_{rl})}, \\
  v(P_{rl}^{(0)}) &= 0,
\end{align*}
$$

where $P_{rl}^{(0)}$ is the point where expansion begins.

**Proof:**

We compute $\bar{e}(P_{rl} + h)$, where $h \leq 1$, by assuming that there are already $P_{rl}$ threads in the retry loop, and that a new thread attempts to $CAS$ during the retry, within a probability $h$. For simplicity, we denote $a_i = (\sum_{j=1}^{i-1} rlw_j + e_j(P_{rl})) + rc_i + cw_i$.

$$
\begin{align*}
  \bar{e}(P_{rl} + h) &= \bar{e}(P_{rl}) + h \sum_{i=1}^{S} \int_{0}^{rlw_i} \frac{delay_i(t_i)}{rlw_i} dt_i \\
  &= \bar{e}(P_{rl}) + h \sum_{i=1}^{S} \left( \int_{0}^{a_i} \frac{delay_i(t_i)}{rlw_i} dt_i + \int_{a_i}^{a_i + \bar{e}(P_{rl})} \frac{delay_i(t_i)}{rlw_i} dt_i \right) \\
  &= \bar{e}(P_{rl}) + h \sum_{i=1}^{S} \left( \int_{a_i}^{a_i + \bar{e}(P_{rl})} \frac{delay_i(t_i)}{rlw_i} dt_i + \int_{a_i + \bar{e}(P_{rl})}^{rlw_i} \frac{delay_i(t_i)}{rlw_i} dt_i \right) \\
  &= \bar{e}(P_{rl}) + h \sum_{i=1}^{S} \left( \int_{a_i}^{a_i + \bar{e}(P_{rl})} \frac{delay_i(t_i)}{rlw_i} dt_i + \int_{a_i + \bar{e}(P_{rl})}^{rlw_i} \frac{delay_i(t_i)}{rlw_i} dt_i \right)
\end{align*}
$$

This leads to

$$
\frac{\bar{e}(P_{rl} + h) - \bar{e}(P_{rl})}{h} = \frac{S \times \frac{cc}{2} + \bar{e}(P_{rl}) \times cc}{rlw_i}.
$$

When making $h$ tend to 0, we finally obtain

$$
\bar{e}'(P_{rl}) = cc \times \frac{S \times \frac{cc}{2} + \bar{e}(P_{rl})}{rlw + \bar{e}(P_{rl})}.
$$

In addition, if a set $S_k$ of $CAS$’s are operating on the same variable $var_k$, then $CAS_i \in S_k$ can be expanded by the $CAS_j \in S_k$. In this case, we can obtain $\bar{e}_k(P_{rl})$ by using the reasoning above. The calculation simply ends up as follows: Consider the problem as if no $CAS$ shares a variable
and denote expansion in Stage_i with $e_i^{(old)}$. Then, $e_k(P_{ri}) = \sum_{CAS_i \in S_k} e_i^{(old)}$.

2) Expected Slack Time for the Advanced Data Structures: We assume here the slack time can only occur after the completion of an operation (i.e., before stage 1), as the other stages are expected to start immediately due to the thread that completes the previous stage. Similar to Section IV-A2, we consider that, at any time, the threads that are running the retry loop have the same probability to be anywhere in their current retry. Thus, a thread can be in any stage just after the successful CAS that completes the operation. So, we need to consider the thread which is closest to the end of its current stage when the operation is completed. We denote the execution time of the expanded retry loop with $r_{lw}^{(*)}$ and the number of stages with $S$. For a thread executing Stage_i when the operation completes, the time before accessing the data structure is then uniformly distributed between 0 and $r_{lw}^{(*)}$.

Here, we take another assumption and consider all stages can be completed in the same amount of time (i.e., for all (i, j) in $\{1, \ldots, S\}^2$, $r_{lw}^{(*)} = r_{lw}^{(*)} = r_{lw}^{(*)}/S$). This assumption does not diverge much from the reality and provides a reasonable approximation. With these assumption and using Lemma 1, we conclude that:

$$\overline{st}_i(P_{ri}) = \frac{r_{lw}^{(*)}}{S \times (P_{ri} + 1)}.$$  \hspace{1cm} (16)

3) Enqueue on Michael-Scott Queue: As a first step, we consider the Enqueue operation of the MS queue to validate our approach. This operation requires two pointer updates leading to two stages, each ending with a CAS. The first stage, that linearizes the operation, updates the next pointer of the last element to the newly enqueued element. In the next and last stage, the queue’s head pointer is updated to point to the recently enqueued element, which could be done by a helping thread, that brings the data structure into a stable state. Here, we determine the $cw$ by subtracting the $rc$ and $cc$ from the non-contended cost of Enqueue operation.

![Figure 9: Enqueue on MS Queue](image)

We estimate the expansion in the success period as described above and throughput as explained in Section IV. The results for the Enqueue experiments where all threads execute Enqueue are presented in Figure 9.
4) **Deque**: We consider the deque designed in [17]. PushLeft and PushRight (resp. PopLeft and PopRight) operations are exactly the same, except that they operate on the different ends of the deque. The status flags, which depict the state of the deque, and the pointers to the leftmost element and the rightmost element are together kept in a single double-word variable, so-called **Anchor**, which could be modified by a double-word **CAS** atomically.

A **PopLeft** operation linearizes and even completes in one stage that ends with a double-word **CAS** that just sets the left pointer of the anchor to the second element from left.

A **PushLeft** operation takes three stages to complete. In the first stage, the operation is linearized by setting the left pointer of the **Anchor** to the new element and at the same time changing the status flags to “left unstable”, to indicate the status of the incomplete but linearized **PushLeft** operation. In the second stage, the left pointer of the leftmost element is redirected to the recently pushed element. In the third stage, a **CAS** is executed on **Anchor** to bring the deque status flags into “stable state”. Every operation can help an incomplete **PushLeft** or **PushRight** until the deque comes into the stable state; in this state, the other operations can attempt to linearize anew.

As noticed, the first and the third stage execute a **CAS** on the same variable (**Anchor**) so it is possible to delay the third stage of the success period by executing a **CAS** in the first stage. This implies that the expansion in stage one should also be considered when the delay in the third stage is considered, and the other way around. This can be done by summing expansion estimates of the stages that run the **CAS** on the same variable and using this expansion value in all these stages. Again, it just requires simple modifications in the expansion formula by keeping assumptions unchanged.

We first run pop-only and push-only experiments where dedicated threads operate on both ends of the deque, in a half-half manner. We provide predictions by plugging the slightly modified expansion estimate, as explained above, into the average-based approach. Then, we take one step further and mix the operations, assigning the threads inequally among push and pop operations. And, we obtain estimates for them by simply taking the weighted average (depending on the number of threads running each operation) of the success period of pop-only and push-only experiments, with the corresponding **pw** value.

In Figure 10, results are illustrated; they are satisfactory for the push-only and pop-only cases. For the mixed-case experiments, the results are mixed: our analysis follows the trend and becomes less accurate when the **pw** gets lower, as experimental curves tend toward push-only success period. This, presumably, happens because the first stage of a **PushLeft** (or **PushRight**) operation is shorter than the first stage of a **PopLeft** (or **PopRight**) operation. This brings indeed an advantage to push operations, under contention: they have higher chances to linearize before pop operations after the data structure comes into the stable state. It also provides an interesting observation which highlights the lock-free nature of operations: it is improbable to complete a pop operation if numerous threads try to push, due to the difference of work inside the first stage of their retry loop.

### D. Applications

1) **Back-off Optimizations**: When the parallel work is known, we can deduce from our analysis a simple and efficient back-off strategy: as we are able to estimate the value for which the throughput is maximum, we just have to back-off for the time difference between the peak **pw** and the actual **pw**. In Figure 12, we compare, on a synthetic workload, this constant back-off strategy against widely known strategies, namely exponential and linear, where the
back-off amount increases exponentially or linearly after each failing retry loop starting from a
115 cycles step size. In Figure 11, we apply our constant back-off on a Delaunay triangulation application [12], provided with several workloads. The application uses a stack in two phases, whose first phase pushes elements on top of the stack without delay. We are able to estimate a corresponding back-off time, and we plot the results by normalizing the execution time of our back-offed implementation with the execution time of the initial implementation.

A measure or an estimate of $pw$ is not always available (and could change over time, see next section), therefore we propose also an adaptive strategy: we incorporate in the data structure a monitoring routine that tracks the number of failed retries, employing a sliding window. As our analysis computes an estimate of the number of failed retries as a function of $pw$, we are able to estimate the current $pw$, and hence the corresponding back-off time like previously.

We test our adaptive back-off mechanism on a workload originated from [1], where global operators of exchanges for financial markets gather data of trades with a microsecond accuracy. We assume that the data comes from several streams, each of them being associated with a thread. All threads enqueue the elements that they receive in a concurrent queue, so that they can be later aggregated. We extract from the original data a trade stream distribution that we use to generate similar streams that reach the same thread; varying the number of streams to the same thread leads to different workloads. The results, represented as the normalized throughput (compared to the initial throughput) of trades that are enqueued when the adaptive back-off is used, are plotted in Figure 11. For any number of threads, the queue is not contended on workload $s3$, hence our improvement is either small or slightly negative. On the contrary, the workload $s50$ contends the queue and we achieve very significant improvement.

![Figure 12: Back-off Tuning on Treiber’s Stack](image)

2) Memory Management Optimization: Memory Management (MM) is an inseparable part of dynamic concurrent data structures. In contrary to lock-based implementations, a node that has been removed from a lock-free data structure can still be accessed by other threads, e.g. if they
have been delayed. Collective decisions are thus required in order to reclaim a node in a safe manner. A well-known solution to deal with this problem is the hazard pointers technique [18].

A traditional design to implement this technique works as follows. Each thread $T_i$ maintains two lists of nodes: $\mathcal{N}_i$ contains the nodes that $T_i$ is currently accessing, and $\mathcal{D}_i$ stores the nodes that have been removed from the data structure by $T_i$. Once a threshold on the size of $\mathcal{D}_i$ is reached, $T_i$ calls a routine that: (i) collects the nodes that are accessed by any other thread, i.e. $\mathcal{N}_j$ for $j \neq i$ (collection phase), and (ii) for each element in $\mathcal{D}_i$, checks whether someone is accessing the element, i.e. whether it belongs to $\bigcup_{j \neq i} \mathcal{N}_j$, and if not, reclaims it (reclamation phase).

The main goal of our adaptive MM scheme is to distribute this extra-work in a way that the loss in performance is largely leveraged, knowing that additional work can be an advantage under high-contention (see previous section). The optimization is based on two main modifications. First, the granularity has to be finer, since the additional quantum that the back-off mechanism uses, has to be rather small (hundreds of cycles for a queue). Second, we need to track the contention level on the data structure in order to be able to inject the work at a proper execution point.

**Fine-grain Memory Management Scheme:** We divide the routine (and further the phases) of the traditional MM mechanism into quanta (equally-sized chunks). One quantum of the collection phase is the collection of the list of one thread, while three nodes are reclaimed during one quantum of the reclamation phase. The traditional MM scheme was parameterized by a threshold based on the number of the removed nodes; the fine-grain MM scheme is parameterized by the number of quanta that are executed at each call.

![Figure 13: Performance of memory management mechanisms](image-url)
We apply different MM schemes on the \texttt{Dequeue} operation of the Michael-Scott queue, and plot the results in Figure 13. We initialize the queue with enough elements. Threads execute \texttt{Dequeue}, which returns an element, then call the MM scheme. On the left side, we compare a pure queue (without MM), a queue with the traditional MM (complete reclamation once in a while) and a queue with fine-grain MM (according to the numbers of quanta that are executed at each call). Note that the performance of the traditional MM is also subject to the tuning of the threshold parameter. We have tested and kept only the best parameter on the studied domain. First, unsurprisingly, we can observe that the pure queue outperforms the others as its $cw$ is lower (no need to maintain the list of nodes that a thread is accessing). Second, as the fine-grain MM is called after each completed \texttt{Dequeue}, adding a constant work, the MM can be seen as a part of the parallel work. We highlight this idea on the second experiment (on the right side). We first measure the work done in a quantum. It follows that, for each value of the
granularity parameter, we are able to estimate the effective parallel work as the sum of the initial $pw$ and the work added by the fine-grain MM. Finally, we run the queue with the fine-grain MM, and plot the measured throughput, according to the effective parallel work, together with our two approaches instantiated with the effective $pw$. The graph shows the validity of the model estimations for all values of the granularity parameter.

**Adaptive Memory Management Scheme:** We build the adaptive MM scheme on top of the fine-grain MM mechanism by adding a monitoring routine that tracks the number of failed retry loops, employing a sliding windows. Given a granularity parameter and a number of failed retry loops, we are able to estimate the parallel work and the throughput, hence we can decide a change in the granularity parameter to reach the peak performance. Note that one can avoid memory explosion by specifying a threshold like the traditional implementation in case the application provides a durable low contention; in the worst case, it performs like the traditional MM.

Numerous scientific applications are built upon a pattern of alternating phases, that are communication- or computation-intensive. If the application involves data structures, it is expected that the rate of the modifications to the data structures is high in the data-oriented phases, and conversely. These phases could be clearly separated, but the application can also move gradually between phases. The rate of modification to a data structure will anyway oscillate periodically between two extreme values. We place ourselves in this context, and evaluate the two MMs accordingly. The parallel work still follows an exponential distribution of mean $pw$, but $pw$ varies in a sinusoidal manner with time, in order to emulate the numerical phases. More precisely, $pw$ is a step approximation of a sine function. Thus, two additional parameters rule the experiment: the period of the oscillating function represents the length of the phases, and the number of steps within a period depicts how continuous are the phase changes.

In Figure 14, we compare our approach with the traditional implementation for different periods of the sine function, on the Dequeue of the Michael-Scott queue [19]. The adaptive MM, that relies on the analysis presented in this paper, outperforms the traditional MM because it provides an advantage both under low contention due to the costless (since delayed) invocation of the MM and under high contention due to the back-off effect.

**VII. Conclusion**

In this paper we have presented two analyses for calculating the performance of lock-free data structures in dynamic environments. The first analysis has its roots in queuing theory, and gives the flexibility to cover a large spectrum of configurations. The second analysis makes use of Markov chains to exhibit a stochastic execution; it gives better results, but it is restricted to simpler data structures and exponentially distributed parallel work. We have evaluated the quality of the prediction on basic data structures like stacks, as well as more advanced data structures like optimized queues and deques. Our results can be directly used by algorithmicians to gain a better understanding of the performance behavior of different designs, and by experimentalists to rank implementations within a fair framework. We have also shown how to use our results to tune applications using lock-free codes. These tuning methods include: (i) the calculation of simple and efficient back-off strategies whose applicability is illustrated in application contexts; (ii) a new adaptative memory management mechanism that acclimates to a changing environment.

The main differences between the data structures of this paper and linked lists, skip lists and trees occur when the size of the data structure grows. With large sizes, the performance is dominated by the traversal cost that is ruled by the cache parameters. The reduction in the size of the data structure decreases the traversal cost which in turn increases the probability of
encountering an on-going CAS operation that delays the threads which traverse the link. The expansion, which can additionally be supported unfavorably by helping mechanisms, appears then as the main performance degrading factor. While the analysis becomes easier for high degrees of parallelism (large data structure size), being able to describe the behavior of lock-free data structures as the degree of parallelism changes constitutes the main challenge of our future work.

REFERENCES

[1] Daily trades from 2015-08-05. http://www.nyxdataln/Data-Products/Daily-TAQ1#155. Accessed: 2016-05-05.

[2] Intel’s threading building blocks framework. https://www.threadingbuildingblocks.org/. Accessed: 2016-01-20.

[3] Java concurrency package. https://docs.oracle.com/javase/7/docs/api/java/util/concurrent/package-summary.html. Accessed: 2016-01-20.

[4] Microsoft .net framework. http://www.microsoft.com/net. Accessed: 2016-01-20.

[5] Yehuda Afek, Gideon Stupp, and Dan Touitou. Long lived adaptive splitter and applications. Distributed Computing, 15(2):67–86, 2002. URL: http://dx.doi.org/10.1007/s004446010060.

[6] Samy Al-Bahra. Nonblocking algorithms and scalable multicore programming. Communications of the ACM, 56(7):50–61, 2013. URL: http://doi.acm.org/10.1145/2483852.2483866.

[7] Dan Alistarh, Keren Censor-Hillel, and Nir Shavit. Are lock-free concurrent algorithms practically wait-free? In Symposium on Theory of Computing (STOC), pages 714–723. ACM, June 2014.

[8] Aras Atalar, Paul Renaud-Goud, and Philippas Tsigas. Analyzing the performance of lock-free data structures: A conflict-based model. In International Symposium on Distributed Computing (DISC), pages 341–355, 2015.

[9] Hagit Attiya and Arie Founen. Algorithms adapting to point contention. Journal of the ACM (JACM), 50(4):444–468, 2003. URL: http://doi.acm.org/10.1145/792538.792541.

[10] Hagit Attiya, Rachid Guerraoui, and Petr Kouznetsov. Computing with reads and writes in the absence of step contention. In International Symposium on Distributed Computing (DISC), pages 122–136, 2005. URL: http://dx.doi.org/10.1007/11561927_11.

[11] Dave Dice, Yossi Lev, and Mark Moir. Scalable statistics counters. In Proceedings of the ACM Symposium on Parallelism in Algorithms and Architectures (SPAA), pages 43–52. ACM, July 2013.

[12] Tanmay Gangwani, Adam Morrison, and Josep Torrellas. CASPAR: breaking serialization in lock-free multicore synchronization. In Proceedings of the Twenty-First International Conference on Architectural Support for Programming Languages and Operating Systems, ASPLOS ‘16, Atlanta, GA, USA, April 2-6, 2016, pages 789–804, 2016.

[13] Danny Hendler, Nir Shavit, and Lena Yerushalmi. A scalable lock-free stack algorithm. Journal of Parallel and Distributed Computing (JPDC), 70(1):1–12, 2010.

[14] Maurice Herlihy. Wait-free synchronization. ACM Transactions on Programming Languages and Systems (TOPLAS), 13(1):124–149, 1991.

[15] Amos Israeli and Lihu Rappoport. Disjoint-access-parallel implementations of strong shared memory primitives. In Proceedings of the ACM Symposium on Principles of Distributed Computing (PoDC), pages 151–160, 1994.

[16] John D. C. Little. A proof for the queuing formula: L= λ w. Operations research, 9(3):383–387, 1961.

[17] Maged M. Michael. Cas-based lock-free algorithm for shared dequeues. In Euro-Par, pages 651–660, 2003. URL: http://dx.doi.org/10.1007/3-540-45209-6_92.

[18] Maged M. Michael. Hazard pointers: Safe memory reclamation for lock-free objects. IEEE Transactions on Parallel and Distributed Systems (TPDS), 15(8), August 2004.

[19] Maged M. Michael and Michael L. Scott. Simple, fast, and practical non-blocking and blocking concurrent queue algorithms. In Proceedings of the ACM Symposium on Principles of Distributed Computing (PoDC), pages 267–275. ACM, May 1996.

[20] R. Kent Treiber. Systems programming: Coping with parallelism. International Business Machines Incorporated, Thomas J. Watson Research Center, 1986.

[21] J. D. Valois. Implementing Lock-Free Queues. In Proceedings of International Conference on Parallel and Distributed Systems (ICPADS), pages 64–69, December 1994.

[22] Xiao Yu, Zhengyu He, and Bo Hong. A queuing model-based approach for the analysis of transactional memory systems. Concurrency and Computation: Practice and Experience (CCPE), 25(6):808–825, 2013.