On the mass function at the inner horizon of a regular black hole

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Abstract

Calculations of the inner mass function of the Hayward regular black hole with fluxes are reviewed and rederived. We present detailed calculations of the inner mass function in two forms of the Ori approach (the ingoing flux is continuous, the outgoing flux is modeled by a thin null shell) and compare them with calculations in Reissner-Nordström black hole. A formal reason of different results is discussed. The energy density of scalar perturbations propagating from the event horizon into the Hayward black hole measured by a free falling observer near the inner horizon is calculated.

1 Introduction

Black hole solutions of general relativity, Schwarzschild, Reissner-Nordström (RN), Kerr-Newman, have the central singularity at \( r = 0 \) which is considered as undesirable in models of astrophysical black holes. Regular black holes were proposed as configurations in which the central singularity is replaced by a non-singular core [1–7]. Regular black holes are static, spherically-symmetric and satisfy the weak energy condition.

In this note we discuss the Hayward black hole [3] which can be considered as a regularization of the Schwarzschild solution. Outside the event horizon both geometries have the same asymptotic form at \( r \to \infty \). The important difference between Schwarzschild and Hayward black holes is that Hayward black hole can have no, one double and two horizons. In the case of the Hayward black hole with two horizons its causal structure is similar to that of the RN and Kerr-Newman solutions. In these black holes the inner horizon is the Cauchy horizon, a null hypersurface beyond which predictability of the theory is lost.

In the process of collapse of a star and formation of a black hole an outgoing flux of radiation is produced which, after a partial reflection at the potential near the outer horizon, forms an influx of radiation propagating into black hole [8]. The influx partially reflects at the potential in the vicinity of the inner horizon and produces an outflux. The inner horizon is a surface of infinite blueshift, and in [9–16] and many subsequent papers it was shown that a freely falling observer near the Cauchy horizon of the RN and Kerr-Newman black holes will see an unbounded energy density of scalar, electromagnetic and gravitational fields. This was interpreted as an instability of the Cauchy horizon under external perturbations.

These properties have tended to an expectation that only the blueshifted ingoing flux (Price radiation tail [8]) near the inner horizon would result in an increase of the inner mass function. However, in paper [17] on the example of the RN black hole it was shown that only the blueshifted influx is insufficient for the unbounded increase of the inner mass function (mass inflation). It was shown that mass inflation appears only as a combined effect of the incoming and outgoing fluxes. The mass inflation produces an increase of curvature at the inner horizon, and instead of the Cauchy
horizon there appears a curvature singularity shielding the Cauchy horizon. In paper [17] the in- and outfluxes were modeled by the incoming and outgoing charged Vaidia solutions [18], which in turn were modeled by the thin null shells of lightlike particles crossing without interaction through each other. The space-time is divided by the crossing fluxes into four regions, the metric in each region is characterized by its mass. Mass inflation of the outgoing flux appears when the ingoing shell is near the Cauchy horizon. Solution for the inner mass function was obtained with the use of the Dray-'t Hooft-Redmond (DTR) [19, 20] relation between the masses in the metrics of four regions. Singularity at the Cauchy horizon was analytically discussed in [21–24] and in a number of papers cited therein.

In paper [25], in the RN black hole with fluxes, the problem of mass inflation was studied by modeling the outgoing radiation by a null thin shell, but considering the ingoing Price flux as continuous (the Ori approach). In this model, starting from the Einstein equations and using continuity of the ingoing flux through the shell, a relation was obtained connecting the masses of the metrics of space-times inside and outside the shell, which predicted mass inflation near the inner horizon [21, 25, 26].

Since the causal structure of the Hayward black hole is similar to that of the RN black hole, it is natural to discuss, if, in the Hayward black hole with fluxes, there also appears mass inflation. This question was discussed within the Ori approach and using the generalized DTR construction [27], in [6] for the “loop black hole” and in [28–30] for the Hayward black hole. Somewhat surprising result was that, contrary to the RN black hole, in the case of regular (loop, Hayward) black holes the Ori approach does not show the mass inflation. However the use of the (generalized) DTR relation in these models leads to mass inflation.

In this work we review previous calculations of the inner mass function and provide a detailed calculations of the inner mass function in two forms of the Ori approach. We compare our results with calculations in the RN case.

We show that, as in the RN case, in Hayward black hole with an incoming flux the energy density near the inner horizon measured by a free falling observer increases without a bound showing that both the Hayward and RN black holes are unstable under the external perturbations.

2 Inner mass function in the Hayward model

The metric of the Hayward black hole is [3]

\[ ds^2 = -f(r)dv^2 + 2dvdr + r^2d\Omega^2, \]  
(1)

where

\[ f(r) = 1 - \frac{M(r)}{r} = 1 - \frac{2mr^2}{2m^2 + r^3}. \]  
(2)

The metric can have no, one double, or two horizons. We discuss the case with two horizons. If the mass \( m \) is a function of retarded or advanced time \( v \), \( m = m(v) \), the metric takes a form

\[ ds^2 = -f(r,v)dv^2 - 2dvdr + r^2d\Omega^2 \]  
(3)

and there appears an additional component of the energy-momentum tensor \( T^r_v \).

In the case there are in- and outfluxes of energy, following the Ori approach [25], the outflux is modeled by a thin null shell \( \Sigma \). The shell divides the interior of the black hole into an outer \( V^+ \) and an inner \( V^- \) regions. In both parts \( V^\pm \) the metric is of the form (3) with different \( v^\pm \) and \( m^\pm \). The variable \( r \) can be introduced continuous through the shell [27, 31].
For a metric (2)-(3) the Einstein equations take a form [27, 31]
\[ \frac{\partial M_\pm}{\partial r} = -4\pi r^2 T^r_v, \quad \frac{\partial M_\pm}{\partial v_\pm} = 4\pi r^2 T^v_\pm. \] (4)
Continuity of the metric through the shell yields the equation
\[(f_+ dv_+ = f_- dv_-)|_\Sigma = 2dr. \] (5)
Below we write \( v_+ \equiv v, \ f_+(v_+) = f(v) \).
Continuity of the flux across the shell \( [T_{\mu\nu} n^\mu n^\nu] = 0, \ [A] = A_+ - A_-, \ n^\mu \) being a normal to the shell, is expressed as
\[ \frac{T_{v_+ v_+}}{f_+^2} = \frac{T_{v_- v_-}}{f_-^2} \bigg|_\Sigma. \] (6)
From Eq.(5) variable \( v_- \) is determined as a function of \( v \)
\[ \frac{dv_-(v)}{dv} = \frac{f(v)}{f_-(v)} \bigg|_\Sigma \] (7)
where \( \check{f}_-(v) = f_-(v_-) \). In the same way, \( \check{m}_-(v) = m_-(v_-) \). We have
\[ \frac{\partial \check{M}_-(v)}{\partial v} = \frac{\partial M_-(v_-)}{\partial v_-} \frac{f(v)}{f_-(v)} \bigg|_\Sigma. \] (8)
From Eqs.(6) and (8) we obtain
\[ \frac{1}{f(v)} \frac{\partial M_+}{\partial v} \bigg|_\Sigma = \frac{1}{\check{f}_-(v)} \frac{\partial \check{M}_-}{\partial v} \bigg|_\Sigma. \] (9)
Eq.(9) will be used to obtain the mass function \( \check{m}_-(v) \).

2.1 Mass function from continuity of flux across the shell

In the case of a black hole with the Price influx [8], the mass function in the region \( V_+ \) is \( m_+ = m_0 - \delta m_{pr} \), where \( m_0 \) is the mass without the Price flux, and \( \delta m_{pr}(v) = \beta/v^p \), where \( p \geq 12 \). We assume that \( m_0 \gg l \).

Without the Price flux, the horizons of the black hole are determined from the equation \( f(r, m_0) = 0 \), or, equivalently, from the equation
\[ r^3 - 2m_0(r^2 - l^2) = 0. \] (10)
The outer horizon is located at \( r_+ \simeq 2m_0 - l^2/2m_0 + \cdots \), the inner horizon is at
\[ r_- \simeq l \left( 1 + \frac{l}{4m_0} + \frac{5}{2} \left( \frac{l}{4m_0} \right)^2 + \cdots \right) \equiv l(1 + \eta + 5\eta^2/2 + \cdots). \] (11)
In the case with fluxes, the locations of the horizons are
\[ r_+(v) \simeq r_+ - \delta r_+(v), \quad r_-(v) \simeq r_- + \delta r_-(v). \] (12)
In the first order in $\delta m_{pr}$, $\delta r_-$ is determined from the equation

$$f_r(r_-, m_0)\delta r_--f_m(r_-, m_0)\delta m_{pr}=0,$$

which gives

$$\delta r_- = \delta m_{pr} f_m(r_-, m_0)/f_r(r_-, m_0). \tag{13}$$

The shell modeling the outflux is located at the radius $r_{shell} = r_s$, and in vicinity of the inner horizon we can write $r_s = r_- + y(v)$, $\dot{r}_- > y(v)$. The location of the shell is determined by the null geodesic equation (5)

$$2\dot{r}_s(v) = f(r_s, m_+ (v)) = f(r_- + y(v), m_0 - \delta m_{pr}(v)). \tag{14}$$

The dot denotes the derivative with respect to the variable $v$. In the first order in $y$ and $\delta m_{pr}$, we obtain

$$2\dot{y} = f_r(r_-, m_0)y - f_m(r_-, m_0)\delta m_{pr}, \tag{15}$$

where

$$f_r(m_0, r_-) = 2m_0 r_-\frac{r_-^3 - 4m_0^2}{(r_-^3 + 2m_0^2)^2} \approx -\frac{2}{l}(1 - 4\eta), \tag{16}$$

$$f_m(m_0, r_-) = -\frac{2r_-^3}{r_-^3 + 2m_0^2} \approx -\frac{l}{2m_0^2}(1 + \eta). \tag{17}$$

Solving (15), we have

$$y(v) = e^{-v|f_r|/2} \left(C - \int_v^0 dv e^{v|f_r|/2} \delta m_{pr} \frac{f_m}{2} \right). \tag{18}$$

In the limit $v \to \infty$, $y(v)$ approximately is

$$y(v) \approx \frac{l^2}{4m_0^2} \delta m_{pr}(1 + 5\eta) - \delta \dot{m}_{pr} \frac{l^3}{4m_0^3}(1 + 9\eta). \tag{19}$$

If the location of the shell is determined with respect to the $v$-dependent horizon $r_-(v)$, $r_s = r_-(v) + z(v)$, $z(v)$ is determined from the geodesic equation

$$2(\dot{z} + \delta \dot{r}_-) = f_r(r_-, m_0)(\delta r_- + z) - f_m(r_-, m_0)\delta m_{pr}, \tag{20}$$

where $\delta r = \delta m_{pr}(l^2/4m_0^4)(1 + 5\eta)$. Asymptotic solution of (20) is

$$z(v) = e^{-v|f_r|/2} \left(C - \int_v^0 dv e^{v|f_r|/2} \delta m_{pr} \frac{l^2}{4m_0^2}(1 + 5\eta) \right) \approx -\delta \dot{m}_{pr} \frac{l^3}{4m_0^3}(1 + 9\eta). \tag{21}$$

The terms proportional to $\delta m_{pl}$ have canceled. The location of the shell in both calculations is

$$r_s(v) \approx r_- + \frac{l^2}{4m_0^2} \delta m_{pr}(1 + 5\eta) - \frac{l^3}{4m_0^3} \delta \dot{m}(1 + 9\eta). \tag{22}$$

Using (9), let us find the inner mass function $\dot{\tilde{m}}_-(v)$. First, we calculate

$$R_+ = \frac{1}{f(v)} \frac{\partial M_+}{\partial v} = \frac{r_s^6 \dot{m}_+}{(r_s^3 + 2l^2m_+)(r_s^3 - 2m_+(r_s^2 - l^2))} = \frac{r_s^6 \dot{m}_+}{2r_s^2(2m_+l^2 + r_s^3)^2}. \tag{23}$$
Here we have used the geodesic equation of the shell written as $r^2_s - 2m_\pm (r_s^2 - l^2) = 2r_s (2m_\pm l^2 + r^3_s)$. Using (22) and retaining in (23) the leading in $\delta m_{pr}$ terms, we have

$$R_+ \simeq -\frac{r^6_s}{2l^6(1 + 9\eta)}. \quad (24)$$

The equation for the inner mass function $\tilde{m}_-$ is

$$R_- \equiv \frac{1}{f_-(v)} \frac{\partial \tilde{M}_-}{\partial v} = -\frac{r^6_s \ddot{\tilde{m}}_-}{-4m^2 l^2 (r^2_s - l^2) + 2\tilde{m}_- r^3_s (2l^2 - r^2_s) + r^6_s} = -\frac{r^6_s}{2l^6(1 + 9\eta)}. \quad (25)$$

Using the relations

$$r^2_s - l^2 \simeq l^2 [(1 + \eta + 5\eta^2/2 + \cdots)^2 - 1] \simeq \frac{l^3}{2m_0} (1 + 3\eta), \quad 2l^2 - r^2_s \simeq l^2 (1 - 2\eta), \quad (26)$$

we rewrite Eq.(25) as

$$\dot{\tilde{m}}_- = \frac{1}{1 + 9\eta} \left[ \tilde{m}^2 \frac{2m_0}{(1 + 3\eta)} - \frac{\tilde{m}_-}{l} (1 + \eta) - \frac{1}{2}(1 + 6\eta) \right], \quad (27)$$

and further is manipulated to

$$\dot{\tilde{m}}_- = \frac{1 - 6\eta}{m_0 l} \left[ \left( \tilde{m}_- - \frac{m_0}{2} (1 - 2\eta) \right)^2 - \frac{m^2_0}{4} (1 - 2\eta)^2 - \frac{m_0 l}{2}(1 + 3\eta) \right] \simeq \frac{1 - 6\eta}{m_0 l} \left[ \left( \tilde{m}_- - \frac{m_0}{2} (1 - 2\eta) \right)^2 - \left( \frac{m_0}{2} (1 + 2\eta) \right)^2 \right]. \quad (28)$$

Integrating Eq.(28), we have

$$\frac{1}{m_0(1 + 2\eta)} \ln \left| \frac{(\tilde{m}_- - m_0(1 - 2\eta)/2) - m_0(1 + 2\eta)/2}{(\tilde{m}_- - m_0(1 - 2\eta)/2) + m_0(1 + 2\eta)/2} \right| = C v/2l. \quad (29)$$

In the limit $v \rightarrow \infty$ we obtain

$$\tilde{m}_- = -2m_0 \eta = -\frac{l}{2}. \quad (30)$$

The negative value for $m_-$ was discussed in [6] for the loop black hole and in [29] for the Hayward black hole, and in particular it was noted that the inner mass function $m_-(v)$ is not a directly measurable quantity and the result may be an artifact of parametrization.

### 2.2 Mass function in double null coordinates

Let us calculate the inner mass function following the original Ori approach [25]. As above, the shell divides the interior of the black hole into two regions $V_\pm$ with different $v_\pm$ and $m_\pm$; $r$ is continuous across the shell. In double null coordinates the metric is $ds^2 = -2e^\nu dU dV + r^2 d\Omega$. The coordinate $U$ is set equal to zero at the shell. Since the shell is pressureless, it is possible to introduce an affine parameter $\lambda$ at both sides of the shell [27, 31]. Coordinate $V$ is equal to $\lambda$ along the shell. Location of the shell $r_s = r_- + y(v)$ (see(22)), is written as $r_s = R(\lambda) = r(V = \lambda, U = 0)$. It is supposed that the shell reaches the inner horizon at $v \rightarrow \infty$, or, equivalently, at $\lambda = 0$. In the following we write $v_+ = v$. 

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The geodesic equation of the shell is
\[ R'/v'_\pm = \frac{1}{2} f_\pm(m_\pm(v), R), \] (31)
where the prime denotes the derivative with respect to \( \lambda \). One introduces
\[ z_\pm = R/v'_\pm. \] (32)
Differentiating Eq.(32), \( z'_\pm = R'/v'_\pm - Rv''_\pm/v'_\pm^2 \), and using the geodesic equation for \( v(\lambda) \)
\[ v''_\pm + \frac{1}{2} f_\pm, r v'_\pm^2 = 0, \] (33)
we obtain an equation
\[ 2z'_\pm = f_\pm + Rf_\pm, r. \] (34)
At the (+) side of the shell Eq.(34) takes a form
\[ z'_+ \simeq -1 - 2 \left( \frac{l}{4m_0 + \delta m_{pr}/m_0 + \delta y/l} \right) \] (36)
Here we have set \( R(\lambda) = r_+ + y(R, v) \) and \( m_+ = m_0 - \delta m_{pr} \). Integrating (35), we obtain \( z_+ \simeq Z_+ - \lambda \). From (32) it follows that
\[ v_+ = \int^\lambda d\lambda \frac{R}{z_+} \simeq r_+ \ln \frac{1}{\lambda}, \] (37)
In (37) \( Z_+ \) was set to zero to have \( v_+ \to \infty \) for \( \lambda \to 0 \).

Differentiating (31) with respect to \( \lambda \), we have \[ v''_\pm = 2(R''_\pm f_\pm - R' f'_\pm)/f'_\pm. \] Substituting \( v'' \) in (33), we obtain an equation for \( f \) (See also [29])
\[ f(R) \frac{R''}{R'} = f'(R) - f, r(R) R'. \] (38)
In the case \( f = f_- \),
\[ f_-(\tilde{m}_-(v), R) = 1 - \frac{2\tilde{m}_-(v)R_2^2}{R^3 + 2\tilde{m}_-(v)l_2^2}, \]
transforming (38), we obtain
\[ \frac{R''}{2R'} f_-(\tilde{m}_-(v), R) = -\frac{\tilde{m}'_-(v)R_5^5}{(R^3 + 2\tilde{m}_-(v)l_2^2)^2}. \] (39)
Substituting \( R(v(\lambda)) = r_s \simeq r_+ + \delta m(v)l_2^2/4m_0^2 \), we have
\[ \frac{R''}{R'} = \frac{p(p + 1)v^{-p-2}v^2 - pv^{-p-1}v''}{-pv^{-p-1}v'} = -(p + 1) \frac{v'}{v} + \frac{v''}{v'} \simeq \frac{-1}{\lambda}, \]
Eq.(39) takes a form
\[ \tilde{m}'_- = -\frac{1}{2\lambda R^5} [4\tilde{m}_-^2 l_2^2 (R^2 - l^2) - 2\tilde{m}_- R^3 (2l^2 - R^2) - R^6]. \] (40)

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Noting that
\[ \dot{\tilde{m}} = \frac{d\tilde{m}(v)}{dv} \left( -\frac{r_+}{\lambda} \right), \]
we have
\[ \dot{m}_- = \frac{2l^2(r^2 - l^2)}{r_+^6} \left[ \dot{m}_- - 2\tilde{m}_- \frac{r^3(2l^2 - r^2)}{4l^2(r^2 - l^2)} - \frac{r_+^6}{4l^2(r^2 - l^2)} \right]. \]  
(41)

In Eq.(41) we recognize the structure of Eq.(25). With the use of (26), Eq.(41) takes the same form as (28).

3 Hayward versus RN black hole

Let us compare the Hayward black hole with the RN black hole with the Price flux.

The function \( f(v) \) in (1) is
\[ f(v) = 1 - \frac{2m(v)}{r} + \frac{e^2}{r^2}. \]  
(42)

We assume that \( m(v) \gg e^2 \). The mass of the black hole, \( m_+ \), is \( m_0 - \delta m_{pr}, m_0 \gg \delta m_{pr} \). Without the Price flux, the locations of the horizons are determined from the equation \( f(r, m_0) = 0 \). The outer and inner horizons are located at \( \tilde{r}_+ = m_0 + \sqrt{m_0^2 - e^2} \) and \( \tilde{r}_- = m_0 - \sqrt{m_0^2 - e^2} \). In the case with flux, locations of horizons are
\[ r_+ = \tilde{r}_+ \left( 1 - \frac{\delta m_{pr}}{\sqrt{m_0^2 - e^2}} \right), \quad r_- = \tilde{r}_- \left( 1 + \frac{\delta m_{pr}}{\sqrt{m_0^2 - e^2}} \right). \]  
(43)

The shell modeling the outflux is located in the vicinity of the inner horizon at \( r_s(v) = r_-(v) + y(v) \). Below we use the notations of the preceding section.

The location of the null shell is determined by the geodesic equation \( 2\dot{r}_s = f(r_s, m_+). \) Noting that \( f(\tilde{r}_-, m_0) = 0 \) and expanding the function \( f(r_s, m(v)) \) to the first order in \( y \) and \( \delta m_{pr} \), we obtain
\[ 2 \left( \dot{y} + \frac{\delta m_{pr}}{\kappa \tilde{r}_-} \right) = f_r(\tilde{r}_-, m_0) \left( y + \frac{\delta m_{pr}}{\kappa \tilde{r}_-} \right) - f_m(\tilde{r}_-, m_0) \delta m_{pr}, \]  
(44)

where
\[ r_s = \tilde{r}_+ + y + \frac{\delta m_{pr}}{\kappa \tilde{r}_-}; \quad \kappa = \frac{\sqrt{m_0^2 - e^2}}{\tilde{r}_-^2}; \quad f_r(\tilde{r}_-, m_0) = -2\kappa; \quad f_m(\tilde{r}_-, m_0) = -\frac{2}{\tilde{r}_-}. \]  
(45)

Substituting expressions (45) into (44), we rewrite Eq.(44) in a form
\[ \dot{y} + \kappa y \simeq -\frac{\dot{m}_{pr}}{\kappa \tilde{r}_-}. \]  
(46)

Note that as in (22) the terms proportional to \( \delta m_{pr} \) have canceled. Solution of Eq.(46) is
\[ y(v) = e^{-\kappa v} \left( C - \int^v dve^{+\kappa v} \frac{\dot{m}_{pr}}{\kappa \tilde{r}_-} \right). \]  
(47)

In the limit \( v \to \infty \) solution (47) is simplified to
\[ y(v) \simeq -\frac{\dot{m}_{pr}}{\tilde{r}_- \kappa^2}. \]
The null shell is located at
\[ r_s = \tilde{r}_- + \frac{\delta m_{pr}}{\tilde{r}_- \kappa} - \frac{\dot{\delta m}_{pr}}{\tilde{r}_- \kappa^2}. \] (48)

Let us find the inner mass function. With the use of relations (4)-(6) continuity of the flux across the shell is written as
\[ \frac{\dot{m}_-(v_-)}{f_-(\tilde{m}_-, v_-)} = \frac{\dot{m}_+(v_+)}{f(v_+)}. \] (49)

Using the geodesic equation to determine the location of the shell, \( f(v) = 2\tilde{r}_s \), and writing \( f_-(\tilde{m}_-, v_-) = f(v) + 2(m_+ - \tilde{m}_-)/r_s \), we obtain Eq.(49) in a form
\[ \frac{\dot{\tilde{m}}_-(v_-)}{2\tilde{r}_s + 2(m_+ - \tilde{m}_-)/r_s} = -\frac{\delta m_{pr}}{2\tilde{r}_s}, \] (50)

From (48), we have
\[ \frac{\delta m_{pr}}{2\tilde{r}_s} = \frac{\kappa \tilde{r}_-}{2} \left( 1 + \frac{p + 1}{v \kappa} \right). \] (51)

Neglecting in the l.h.s. of (50) the small term \( \dot{r}_s \), we obtain
\[ \dot{\tilde{m}}_- \simeq (\tilde{m}_- - m_0) \kappa \left( 1 + \frac{p + 1}{v \kappa} \right). \] (52)

Here we meet the crucial difference from the Hayward black hole: Eq.(52) is of the first order in \( \tilde{m}_- \) while (28) contains \( \tilde{m}_- \) quadratically. Solving (51), we obtain
\[ \tilde{m}_-(v) = e^{\kappa v + (p+1)\ln v} \left( C - \kappa m_0 \int v (1 + \frac{p + 1}{v \kappa}) e^{-\kappa v - (p+1)\ln v} \right) \simeq C e^{\kappa v (p+1)} - m_0. \] (53)

To finish the comparison of calculations of the inner mass in the RN and Hayward black holes, we calculate the inner mass in the RN black hole in the Ori approach as in Sect.2.2.

Eq.(34) for the (+) side, \( 2z_+ = f_+ + Rf_{+, r} \), gives
\[ 2z_+ = 1 - \frac{e^2}{r_s^2} \simeq 1 - \frac{e^2}{\tilde{r}_s^2} = -2\kappa \tilde{r}_-, \] (54)

(for definitions of \( \tilde{r}_- \) and \( \kappa \) see (43) and (45)) and we obtain
\[ v_+ \equiv v = \frac{1}{\kappa} \ln \frac{1}{\lambda}. \] (55)

Eq.(38) for the case of the RN black hole has a form
\[ \frac{-2\tilde{m}'_- R'}{R} = \frac{R''}{R'} f(R, \tilde{m}_-(v) \big|_{R=r_s}). \] (56)

Substituting in (56) the relations
\[ \frac{-2\tilde{m}_-'}{r_s} \simeq \frac{2\dot{\tilde{m}}_-}{\tilde{r}_- \kappa \lambda}, \quad \frac{R''}{R'} = \frac{r_s''}{r_s'} = -\frac{1}{\lambda} \left( 1 + \frac{p + 1}{\ln \lambda} \right) \]
and
\[ f(r_s, \tilde{m}_-) = \left( 2\tilde{r}_s + \frac{2(m_+ - \tilde{m}_-)}{r_s} \right), \]
we obtain the equation
\[ \dot{\tilde{m}}_- \simeq (\tilde{m}_- - m_0) \kappa \left( 1 + \frac{p + 1}{v \kappa} \right), \] (57)

which coincides with (52).
4 Instability of the inner horizon under external perturbations

In this section we consider propagation of scalar field in a neighborhood of the Cauchy horizon and show that the power-law tails entering the black hole as seen by a free falling observer diverge at the inner horizon.

The problem of external perturbations for the Hayward black hole is discussed in a similar way as in the case of the RN black hole [12–15, 32], because the causal structures of both metrics are similar.

To set the problem, we consider the Hayward metric

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2, \]

where

\[ f(r) = f_{\text{RN}} g(r) = \frac{(r_+ - r)(r - r_-)}{r^2} \frac{r^2(r - \tilde{r})}{r^2 + 2ml^2}. \] (58)

Here \( \tilde{r} = -r_-(m) \) is the negative root of the equation

\[ r^3 - 2m(r^2 - l^2) = 0 = \left( r^3 - (2m)(r^2 - l^2) \right). \]

g(r) is a bounded function without zeroes and poles at \( r > 0 \).

In the region \((r_+, r_-)\) one introduces the tortoise variable \( r_* = -\int dr/f(r) \)

\[ r_* = \int dr \frac{r^3 + 2ml^2}{(r_+ - r)(r - r_-)(r - \tilde{r})} = \]

\[ -r - A_1(r_+^3 + 2ml^2) \ln(r_+ - r) - A_2(r_-^3 + 2ml^2) \ln(r - r_-) - A_3(\tilde{r}^3 + 2ml^2) \ln(r - \tilde{r}) + \text{const}, \] (59)

where

\[ A_1 = \frac{1}{(r_+ - r_-)(r_+ - \tilde{r})}, \quad A_2 = \frac{1}{(r_+ - r_-)(\tilde{r} - r_-)}, \quad A_3 = \frac{1}{(r_+ - \tilde{r})(r_- - \tilde{r})}. \] (60)

Assuming as in Sect.2 that \( m \gg l \), so that \( r_+ \simeq 2m, r_- \simeq l, \tilde{r} \simeq -l \), and

\[ A_1 \simeq 1/(2m)^2, \quad A_2 \simeq -1/4ml, \quad A_3 \simeq 1/4ml, \]

we obtain

\[ r_* \simeq -r - 2m \ln(r_+ - r) + \frac{l}{2} \ln(r - r_-) - \frac{l}{2} \ln(r - \tilde{r}). \] (61)

In the limit \( r \to r_- \) we have

\[ r_* \simeq \frac{l}{2} \ln \frac{r - r_-}{l}. \] (62)

Defining the null coordinates \( v = -r_* + t, \ u = -r_* - t \), the left and right branches of the Cauchy horizon are the hypersurfaces \((r_-, u = \infty)\) and \((r_-, v = \infty)\). Propagation of the scalar field \( \Phi(x) \) is described by the wave equation \( \Phi_{;\mu;\nu} f^{\mu\nu} = 0 \), where \( f_{\mu\nu} \) are components of the metric (58). To solve the equation, the field \( \Phi(x) \) is expanded in spherical harmonics

\[ \Phi(x) = \int e^{-ik\theta} Y_{lm}(\theta, \varphi) H_{lm}(k) \frac{\varphi_{klm}(r)}{r} dk. \] (63)
The functions $\varphi(r_\ast)$ (below the indices $klm$ are omitted) satisfy the equation
\begin{equation}
\frac{d^2 \varphi(r_\ast)}{dr_\ast^2} + [k^2 - V_i(r_\ast)] \varphi(r_\ast) = 0, \tag{64}
\end{equation}
where the potential $V_i$ is
\begin{equation}
V_i(r_\ast) = -f(r) \left[ \frac{l(l+1)}{r^2} + \frac{1}{r} \frac{df(r)}{dr} \right].
\end{equation}
The function $H(k)$ in (63) is determined by the initial data $h(v)$ given at the branch $(r_+, u = -\infty)$ of the outer horizon
\begin{equation}
H(k) = \frac{1}{2\pi} \int e^{ikr} h(v) dv. \tag{65}
\end{equation}
The solutions of (64) $\varphi(r_\ast)$ which have the asymptotic form $e^{-ikt} \varphi(r_\ast) \sim e^{-ikv}$ at the $r_+$ horizon, at the $r_-$ horizon are $e^{-ikt} \varphi(r_\ast) \sim A(k)e^{-iku} + B(k)e^{iku} \quad r_\ast \to -\infty,$
where $A_{lm}^2 - B_{lm}^2 = 1.$

The ingoing field $\Phi(r_\ast, t)$ propagates inside the black hole and near $r_-$ is scattered in fluxes $X(v)$ and $Y(u)$
\begin{equation}
\Phi(r_\ast, t) \to X(v) + Y(u) = \int dk H(k)(A(k) - 1)e^{-iku} + \int dk H(k)B(k)e^{iku}. \tag{66}
\end{equation}
In the limit $v, u \to \infty$, the main contribution to $X(v)$ and $Y(u)$ comes from integration in a vicinity of $k = 0$ [12,13]. For the Price power-law tail $h(v) = \delta \theta(v - v_0)v^{-p}$ one obtains
\begin{align*}
X(v) &= \beta v^{-p}(A(0) - 1), \quad v \to \infty, \\
Y(u) &= \beta u^{-p}B(0), \quad u \to \infty. \tag{67}
\end{align*}
The fields $X(v)$ and $Y(u)$ are finite on the Cauchy horizon.

Let us find what energy density measures a free falling observer in a vicinity of the inner horizon. The velocity components of the radially falling observer are [14,32]
\begin{equation}
U^t = \frac{E}{f(r)}, \quad U^r = -\sqrt{E^2 - f(r)}, \tag{68}
\end{equation}
and $U^{r\ast} = \sqrt{E^2 - f(r)}/f(r)$. The flux seen by a free falling observer is
\begin{equation}
\mathcal{F} = U^t \Phi_t + U^r \Phi_r + U^{r\ast} \Phi_{r\ast} = \frac{E}{f(r)}(X_u - Y_u) + \frac{\sqrt{E^2 - f(r)}}{f(r)}(-X_v - Y_v) = \frac{X_v}{f(r)}(E - \sqrt{E^2 - f(r)}) - \frac{Y_u}{f(r)}(E + \sqrt{E^2 - f(r)}). \tag{69}
\end{equation}
In the limit $r \to r_-$, or $r_\ast \to -\infty,$ the function $f(r)$ vanishes:
\begin{equation}
f(r) \sim 2 \frac{r - r_-}{l} \sim \frac{e^{2r_\ast/l}}{2}. \nonumber
\end{equation}
If $E > 0$, the flux is
\begin{equation}
\mathcal{F} \sim -\frac{f(r)}{2E} X_v - \frac{2E}{f(r)} Y_u. \tag{70}
\end{equation}
At the branch \((r_-, u \to \infty)\) the first term is finite, and the second increases. If \(E < 0\), we obtain
\[
\mathcal{F} \simeq \frac{2|E|}{f(r)} X_v + \frac{f(r)}{2|E|} Y_{,u}.
\] (71)

At the branch \((r_-, v \to \infty)\) the first term increases, and the second is finite. The fluxes measured by a free falling observer are
\[
E > 0 : \quad \mathcal{F}|_{(r_-, u \to \infty)} = -2E \beta p u^{-p-1} B(0)e^{2u/l},
\] (72)
\[
E < 0 : \quad \mathcal{F}|_{(r_-, v \to \infty)} = 2E \beta p u^{-p-1}(A(0) - 1)e^{2v/l}.
\] (73)

It is seen that the observed fluxes exponentially diverge at the inner horizon.

5 Conclusions and discussion

In this work we studied the inner mass function in the Hayward model of regular black hole with fluxes and compared it with the RN black hole. We assumed that the mass of the black hole without fluxes, \(m_0\), is much larger than the core parameter \(l\). Assuming the validity of the classical treatment, we consider the core parameter \(l\) larger than the Planck length \(l_p\).

We calculated the inner mass via two methods, the first based on continuity of the flux across the shell and the second one using the original Ori approach [25] (to be precise, both methods are within the Ori approach, because in both methods the incoming flux was taken as the continuous Price flux, and the outgoing flux was modeled by a pressureless null shell). Both methods give a finite negative value for the inner mass function. The inner mass is not a directly measurable quantity, and in [6,29] it was suggested that this result is an artifact of the parameterization. However, a good parameterization is not known.

Formally, different behavior of the inner mass functions in RN and Hayward black holes is traced to a following. Schematically, in the RN case, the equation for the inner mass is
\[
\frac{dm}{dv} - cm = -\delta m_{pr},
\]
where \(c > 0\), leading to the exponential grows in \(v\). For the Hayward black hole, we have
\[
\frac{dm}{dv} = (m - c)^2 - a^2, \quad c, a > 0,
\]
which gives
\[
\left| \frac{(m - c) - a}{(m - c) + a} \right| = Ce^{2av}.
\]
In the limit \(v \to \infty\) the solution for \(m(v)\) is \(m = c - a\) which for the specific values of \(c\) and \(a\) gives \(m < 0\).

Another approach used in [6,17] to find the inner mass function, is based on the Dray–t’Hooft-Redmond (DTR) [19,20] relation. In this approach, the fluxes in the interior of the black hole are modeled as thin shells. The DTR formula provides a relation between the masses of the metrics of the ingoing and outgoing spherical shells in the regions between the shells before and after collision. In [17], in the case of the RN black hole, the DTR relation was derived from the system of the Einstein equations. For the loop and Hayward black holes one must use the generalized DTR formula [6,27] which does not require the use of the Einstein equations. It appears that in all cases of RN, Hayward
and loop black holes the generalized DTR relation shows the divergence of the mass function of the spacetime near the Cauchy horizon after the shells have crossed. It was noted that DTR relation being nonperturbative accounts for nonlocal and nonlinear effects [27].

However, the DTR approach is not directly comparable with the Ori approach and there lacks a clear connection between the DTR formula and the Ori-like approaches.

In [33] it was shown that in a system with crossing streams inside a black hole generated by accretion appears mass inflation. The streams propagate in the background of spherically-symmetric space-time and mass inflation appears as the 4-velocities of the streams increase at the approach to the inner horizon. Because of the Lorentz boost of 4-velocity of the observed flux with respect to the observing flux, the counter-streaming velocity of fluxes exponentiates along with the center-of-mass energy density of the streams, causing the increase of the interior mass. However, in paper [33] calculations were made with the specially constructed metric, and a concrete reformulation of the results of [33] to models with the metrics of the form (1) is a problem.

Because of the similar causal structure of the metrics of the Hayward and RN black holes, in both cases propagation of external perturbations is also similar. If perturbation is the Price flux, in both cases a free falling observer approaching the inner horizon measures an increasing energy flux. This property is interpreted as an instability of the inner horizon. However, this effect is not directly connected with the inner mass inflation.

Acknowledgments
I thank M. Smolyakov and I. Volobuev for discussion and valuable comments.

The research was carried out within the framework of the scientific program of the National Center for Physics and Mathematics, the project “Particle Physics and Cosmology”, and was partially supported by the Project 01201255504 of the Ministry of Science and Higher Education of Russian Federation.

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