High-Dimensional Inference for Cluster-Based Graphical Models

Carson Eisenach ∗ Florentina Bunea † Yang Ning‡ Claudiu Dinicu§

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Abstract

Motivated by modern applications in which one constructs graphical models based on a very large number of features, this paper introduces a new class of cluster-based graphical models. Unlike standard graphical models, variable clustering is applied as an initial step for reducing the dimension of the feature space. We employ model assisted clustering, in which the clusters contain features that are similar to the same unobserved latent variable. Two different cluster-based Gaussian graphical models are considered: the latent variable graph, corresponding to the graphical model associated with the unobserved latent variables, and the cluster-average graph, corresponding to the vector of features averaged over clusters. We derive estimates tailored to these graphs, with the goal of pattern recovery under false discovery rate (FDR) control. Our study reveals that likelihood based inference for the latent graph is analytically intractable, and we develop alternative estimation and inference strategies. We replace the likelihood of the data by appropriate empirical risk functions that allow for valid inference in both graphical models under study. Our main results are Berry-Esseen central limit theorems for the proposed estimators, which are proved under weaker assumptions than those employed in the existing literature on Gaussian graphical model inference. As a corollary of the main results, we make explicit the implications of the asymptotic approximations on graph recovery under FDR control, and show when it can be controlled asymptotically. Our analysis takes into account the uncertainty induced by the initial clustering step. We find that in the model-assisted clustering framework, the errors induced by clustering are asymptotically ignorable in the follow-up analysis, under no further restrictions on the parameter space for which inference is valid. The theoretical properties of the proposed procedures are verified on simulated data and an fMRI data analysis.

Keyword: Berry-Esseen bound, Graphical model, High-dimensional inference, Clustering, False discovery rate

∗Department of Operations Research and Financial Engineering, Princeton University, Princeton, NJ 08544, USA; e-mail: eisenach@princeton.edu
†Department of Statistical Science, Cornell University, Ithaca, NY 14850, USA; e-mail: fb238@cornell.edu
‡Department of Statistical Science, Cornell University, Ithaca, NY 14850, USA; e-mail: yn265@cornell.edu
§Department of Statistical Science, Cornell University, Ithaca, NY 14850, USA; e-mail: cd535@cornell.edu
1 Introduction

High dimensional graphical models have become increasingly popular, over the last several decades, for understanding independence and conditional independence relationships among components of high dimensional random vectors. The challenges posed by the estimation and statistical analysis of a graphical model with many more nodes than the number of observations led to renewed interest in these models, and to a very large body of literature, including Meinshausen and Bühlmann (2006); Yuan and Lin (2007); Friedman et al. (2008); Verzelen (2008); Lam and Fan (2009); Rothman et al. (2008); Peng et al. (2009); Ravikumar et al. (2011a); Yuan (2010); Cai et al. (2011); Sun and Zhang (2012); Liu et al. (2012); Xue and Zou (2012); Ning and Liu (2013); Jankova and van de Geer (2014); Cai et al. (2016); Janková and van de Geer (2017); Fan et al. (2017), to give only an incomplete list. However, in practice, when the dimension is very large and the sample size is small, the dependency among variables may become very weak, if it exists, and difficult to detect without auxiliary information. Moreover, when the number of variables is in the tens of thousands, it is difficult to form an opinion of their overall dependency structure, even at the visual level, from a graph estimated via a graphical model.

A solution to these issues is to employ an initial dimension reduction procedure on the very high dimensional vector. For example, in neuroscience applications, a typical functional magnetic resonance image (fMRI) consists of blood-oxygen-level-dependent (BOLD) activities measured at 200,000+ voxels of the brain, over a period of time. Instead of analyzing voxel-level data directly, scientists routinely cluster voxels into several regions of interest (ROI) with homogeneous functions based on the domain knowledge, and then carry out the analysis at the ROI-level. In this example, the group structure of variables may boost the dependency signals, in the context of graphical models. Similar pre-processing steps are used in other application domains, such as genomics, finance and economics.

Motivated by a rich set of applications, we consider variable clustering as the initial dimension reduction step applied to the observed vector \( X = (X_1, \ldots, X_d) \in \mathbb{R}^d \). To the best of our knowledge, very little is known about the effect of clustering on down-stream analysis and, consequently, on the induced graphical models. Our contribution is the provision of a framework that allows for such an analysis. We introduce cluster-based graphical models and show how they can be estimated, under FDR control.

The model is built on the assumption that the observed variables \( X = (X_1, \ldots, X_d) \in \mathbb{R}^d \) can be partitioned into \( K \) unknown clusters \( G^* = \{G_1^*, \ldots, G_K^*\} \) such that variables in the same cluster share the same behavior. Following the intuition behind the widely-used \( K \)-means type procedures, we define a population-level cluster as a group of variables that are noise corrupted versions of a hub-like variable, which is not directly observable, and treated as a latent factor. Formally, we assume there exists a latent random vector \( Z \in \mathbb{R}^K \), with mean zero and covariance matrix \( \text{Cov}(Z) = C^* \), such that

\[
X = AZ + E,
\]

for a zero mean error vector \( E \) with independent entries. The entries of the \( d \times K \) matrix \( A \) are given by \( A_{jk} = I\{j \in G_k^*\} \). A cluster of variables consist in those components of \( X \) with indices in the
same $G_k^*$. We denote $\text{Cov}(E) = \Gamma^*$, a diagonal matrix with entries $\Gamma^*_{jj} = \gamma^* j$ for any $1 \leq j \leq d$. We also assume that the mean-zero noise $E$ is independent of $Z$. The clusters are uniquely defined by model (1.1) provided that the smallest cluster contains at least two variables and that $C^*$ is strictly positive definite, as shown in Bunea et al. (2018), and this result holds irrespective of distributional assumptions. To keep the presentation focused, in this work we assume that $Z \sim N(0, C^*)$ and $E \sim N(0, \Gamma^*)$, which implies $X \sim N(0, \Sigma^*)$ with $\Sigma^* = AC^*A^T + \Gamma^*$. In this context we consider two related, but different, graphical models:

(i) The latent variable graph, associated with the sparsity pattern of the precision matrix

$$\Theta^* := C'^{-1}$$

of the Gaussian vector $Z \in \mathbb{R}^K$. The latent variable graph encodes conditional independencies (CI’s) among the unobservable latent variables $Z$.

(ii) The cluster-average graph, associated with the sparsity pattern of the precision matrix

$$\Omega^* := S'^{-1},$$

where $S^*$ is the covariance matrix of $\bar{X} \in \mathbb{R}^K$, and $\bar{X} = (\bar{X}_1, \ldots, \bar{X}_K)$ is the within cluster average given by $\bar{X}_k = \frac{1}{|G_k|} \sum_{i \in G_k} X_i$. The cluster-average graph encodes CI’s among averages of observable random variables. In particular, we have

$$S^* = C^* + \bar{\Gamma}^*,$$

where $\bar{\Gamma}^* = \text{diag}(\bar{\gamma}_1^*, \ldots, \bar{\gamma}_K^*)$ with $\bar{\gamma}_k^* = \frac{1}{|G_k|} \sum_{j \in G_k} \gamma^*_j$.

Although both graphs correspond to vectors of dimension $K$, possibly much lower than $d$, they are in general different, as the sparsity patterns of $\Theta^*$ and $\Omega^*$ will typically differ, and have different interpretations. It would be therefore misleading to use one as a proxy for the other when drawing conclusions in practice. For instance, in the neuroscience example, if we interpret each latent variable as the function of a ROI, the latent variable graph encodes the CI relationships between functions, which may be one question of scientific interest. The relationship between $\Theta^*$ and $\Omega^*$ shows that this question will not be typically answered by studying the cluster-average graph, although that may be tempting to do. On the other hand, the cluster-average graph may be of independent interest, as it encodes CI’s relations among the average signals within each ROI.

Since the two cluster-based graphical models introduced above can both be of interest in a large array of applications, we provide inferential tools for each of them in this work. We assume that we observe $n$ i.i.d. copies $X_1, \ldots, X_n$ of $X$. The focus of our work is on post-clustering and post-regularization inference for these two sparse precision matrices, with the ultimate goal of sparsity pattern recovery under FDR control.

Inference for the entries of a Gaussian precision matrix has received a large amount of attention in the past few years, most notably post-regularization inference, for instance Ren et al. (2013); Zhang and Zhang (2014); Jankova and van de Geer (2014); Gu et al. (2015); Barber and Kolar (2015); Janková and van de Geer (2017); Javanmard and Montanari (2013); van de Geer et al.
Ning and Liu (2017); Cai et al. (2017); Neykov et al. (2018). These works generalize to the high-dimensional setup the classical ideas of one-step estimation (Bickel, 1975), by first constructing a sparse estimator of the precision matrix, via regularization, and then building de-sparsified updates that are asymptotically normal. The effect of the initial regularization step is controlled in the second step, and inference after regularization becomes valid.

Whereas we will also consider a similar estimation strategy, we differ from the existing literature in important ways. In our work, we add another layer of data-dependent dimension reduction, via clustering, and provide a framework within which the variability induced by clustering can be controlled. Even after controlling for the clustering variability, we note that the existing procedures for estimation and, especially, post-regularization inference in Gaussian graphical models are not immediately applicable to our problem for the following reasons: (1) They are developed for variables that can be observed directly. From this perspective, they could, in principle, be applied to the cluster-average graph, but are not directly extendable to the latent graph; (2) To the best of our knowledge, all existing methods for precision matrix inference require the largest eigenvalue of the corresponding covariance matrix to be upper bounded by a constant. Such an assumption implies, in turn, that the Euclidean norm of each row of the covariance matrix is bounded, which reduces significantly the parameter space for which inference is valid. The assumption holds, for instance, when the number of variables is bounded, or when the entries of each row are appropriately small.

To overcome these limitations, we take a different approach in this work, that allows us to lift unpleasant technical conditions associated with other procedures, while maintaining the validity of inference for both the latent and the average graph. We summarize our contributions below.

1. Post clustering inference does not impose additional restrictions on the parameter space. We discuss, in Section 2.2, clustering methods tailored to model (1.1), where the number of clusters $K$ is unknown and is allowed to grow with $n$. Using the results of Bunea et al. (2018), these methods yield a partition $\hat{G} = G^*$, with high probability, provided that $\lambda_{\min}(C^*) > c$, for a small positive quantity $c$ made precise in Section 2.2. A lower bound on the smallest eigenvalue of the covariance matrix is the minimal condition under which inference in any general graphical model can be performed. Therefore, consistent clustering via model (1.1) does not require a further reduction of the parameter space for which the more standard post-regularized inference can be developed. Moreover, as Section 4 shows, asymptotic inference based on the estimated clusters reduces to asymptotic inference relative to the true clusters, $G^*$, without any need for data splitting. This fact is in contrast with the phenomenon encountered in post-model selection inference, for instance in variable selection in linear regression (Lockhart et al., 2014; Lee et al., 2013; Taylor et al., 2014). In that case, reducing inference to the consistently selected set of variables can only be justified over a reduced part of the parameter space (Bunea, 2004).

2. Methods for estimation tailored to high dimensional inference in cluster-based graphical models. We develop a new estimation strategy tailored to our final goal, that of constructing approximately Gaussian estimators for the entries of the precision matrices $\Theta^*$ and $\Omega^*$ given in (1.2) and (1.3) above. Although we work under the assumption that the data is
Gaussian, likelihood based estimators may be unsatisfactory, as their analysis may require stringent assumptions, as explained above, or may become analytically intractable, as argued in Section 3.3, for the latent graph. We do propose a method that mimics very closely the principles underlying the construction of an efficient score function for estimation in the presence of high dimensional nuisance parameters, see for instance Bickel et al. (1993), but we do not base it on the corresponding likelihood-derived quantities. We explain the underlying principles in Section 3.1.

3. Berry-Esseen-type bounds for Gaussian approximations and FDR control. Our goal is to estimate the cluster-average and latent graphs, respectively, and to provide guarantees on the recovery error. Existing literature focuses on constructing approximate confidence intervals for one or a finite number of entries of a CI graph, or known linear functionals of such entries (Ren et al., 2013; Zhang and Zhang, 2014; Jankova and van de Geer, 2014; Gu et al., 2015; Barber and Kolar, 2015; Janková and van de Geer, 2017; Javanmard and Montanari, 2013; van de Geer et al., 2013; Ning and Liu, 2017; Cai et al., 2017; Neykov et al., 2018). In all these cases, deriving the asymptotic limiting distribution of appropriate test statistics suffices. Our focus here is on the estimation of the sparsity pattern, which can be equivalently viewed as a multiple-testing problem. It is well known that the exact sparsity pattern can be recovered, with high probability, only if the entries of each precision matrix are above the minimax optimal noise level $O(\sqrt{\log d/n})$ (Ravikumar et al., 2011a; Meinshausen and Bühlmann, 2006). Since our aim is inference on the sparsity pattern without further restrictions on the parameter space, the next best type of error that we can control is the False Discovery Rate (FDR) (Benjamini and Hochberg, 1995). For this, we first derive Gaussian approximations for the distribution of our estimates. We establish Berry-Esseen type bounds on the difference between the cumulative distribution function of our estimators and that of a standard Gaussian random variable that are valid for each $K,d$ and $n$, and are presented in Theorems 4.3 and 4.4, respectively. In Section 4.3 we use these results for pattern recovery under FDR control, and explain the effect of the asymptotic approximations on this quantity.

The paper is organized as follows. Section 2 below contains a brief summary of existing results on model-assisted clustering, via model (1.1). Section 3 describes the estimation procedures for the latent variable graph and the cluster-average graph, respectively. In Section 4 we establish Berry-Esseen type central limit theorems for the estimators derived in Section 3, and provide bounds on the FDR associated with each graphical model under study, respectively. Section 5 gives numerical results using both simulated and real data sets.

2 Background

2.1 Notation

The following notation is adopted throughout this paper. Let $d$ denote the ambient dimension, $n$ the sample size, $K$ the number of clusters and $m$ the minimum cluster size. The matrix $C^*$ denotes the population covariance of the latent vector $Z$. Likewise, the matrices $\Gamma^*$, $\Sigma^*$, $\Theta^*$, $S^*$ and $\Omega^*$ denote population-level quantities.

For $v = (v_1, ..., v_d)^T \in \mathbb{R}^d$, and $1 \leq q \leq \infty$, we define $\|v\|_q = (\sum_{i=1}^d |v_i|^q)^{1/q}$, $\|v\|_0 = |\text{supp}(v)|$. 


where \( \text{supp}(v) = \{j : v_j \neq 0\} \) and \(|A|\) is the cardinality of a set \(A\). Denote \(\|v\|_\infty = \max_{1 \leq i \leq d} |v_i|\) and \(v \otimes 2 = vv^T\). Assume that \(v\) can be partitioned as \(v = (v_1, v_2)\). Let \(\nabla f(v)\) denote the gradient of the function \(f(v)\), and \(\nabla_1 f(v) = \partial f(v)/\partial v_1\). Similarly, let \(\nabla^2 f(v)\) denote the Laplacian of the function \(f(v)\) and \(\nabla_{1,2}^2 f(v) = \partial^2 f(v)/\partial v_1 \partial v_2\). For a \(d \times d\) matrix \(M = [M_{jk}]\), let \(\|M\|_{\text{max}} = \max_{j,k} |M_{jk}|\), \(\|M\|_1 = \sum_{j,k} |M_{jk}|\), and \(\|M\|_\infty = \max_k \sum_j |M_{jk}|\). If the matrix \(M\) is symmetric, then \(\lambda_{\min}(M)\) and \(\lambda_{\max}(M)\) are the minimal and maximal eigenvalues of \(M\). Let \([d] = \{1, 2, \ldots, d\}\). For any \(j \in [d]\), we denote the \(j\)th row and \(j\)th column of \(M\) as \(M_j\) and \(M_{j,j}\), respectively. Similarly, let \(M_{-j,-k}\) be the sub-matrix of \(M\) with the \(j\)th row and \(k\)th column removed. Define \(M \otimes 2 = M \otimes M\).

The notation \(S\) refers to the set of all real, symmetric \(d \times d\) matrices. Likewise, \(S^+_{d \times d} \subset S_{d \times d}\) is the positive semi-definite cone. We use \(\otimes\) and \(\circ\) to denote the Kronecker and Hadamard product of two matrices, respectively. Let \(e_j\) denote the vector of all zeros except for a one in the \(j\)th position. The vector \(1\) is the vector of all ones. \(a \vee b = \max(a, b)\).

### 2.2 Model Assisted Variable Clustering

In this section we review existing results on variable clustering that will be used throughout this paper. If model (1.1) is used as a background for defining clusters of variables, then Bunea et al. (2018) showed that these clusters are uniquely defined, up to label switching, as soon as \(m =: \min_{1 \leq k \leq K} |G_k^*| \geq 2\) and the components of the latent vector \(Z\) are different a.s.:

\[
\Delta(C^*) =: \min_{j < k} \mathbb{E}(Z_j - Z_k)^2 > 0.
\]

Since

\[
\Delta(C^*) = \min_{j < k} (e_j - e_k)^T C^* (e_j - e_k) \geq 2 \lambda_{\min}(C^*),
\]

the clusters are uniquely defined as soon as \(\lambda_{\min}(C^*) > 0\), which is the minimal condition under which one can study properties of the inverse of \(C^*\). Moreover, Bunea et al. (2018) developed two algorithms, PECOK and COD, that are shown to recover the clusters exactly, from \(n\) i.i.d. copies \(X_1, \ldots, X_n\) of \(X\), as soon as

\[
\lambda_{\min}(C^*) \geq c,
\]

for a positive quantity \(c\) that approaches 0 as \(n\) grows. Specifically, for the COD procedure,

\[
c = O\left(\|\Sigma^*\|_{\text{max}} \sqrt{\log(d \vee n) / n}\right).
\]

On the other hand, for the PECOK procedure

\[
c = O\left(\|\Gamma^*\|_{\text{max}} \sqrt{K \log(d \vee n) / mn}\right),
\]

which can be much smaller when one has a few, balanced, clusters.

These values of \(c\) are further shown to be minimax or near-minimax optimal for cluster recovery. We refer to Theorems 3 and 4 in Bunea et al. (2018) for the precise expressions and details. Under these minimal conditions on \(\lambda_{\min}(C^*)\), the exact recovery of the clusters holds with probability larger than \(1 - 1/(d \vee n)\). This will allow us to show, in Section 4, that inference in cluster-based graphical models is not hampered by the clustering step.
For completeness, we outline the PECOK algorithm below, which consists in a convex relaxation of the $K$-means algorithm, further tailored to estimation of clusters $G^* = \{G_1^*, \ldots, G_K^*\}$ defined via the interpretable model (1.1). The PECOK algorithm consists in the following three steps:

1. Compute an estimator $\hat{\Gamma}$ of the matrix $\Gamma^*$.

2. Solve the semi-definite program (SDP)

$$\hat{B} = \arg\max_{B \in D} \langle \hat{\Sigma} - \hat{\Gamma}, B \rangle,$$

(2.1)

where $\hat{\Sigma}$ is the sample covariance matrix and

$$D := \left\{ B \in \mathbb{R}^{d \times d} : \begin{array}{l} B \succeq 0 \text{ (symmetric and positive semidefinite)} \\ \sum_a B_{ab} = 1, \forall b \\ B_{ab} \geq 0, \forall a, b \\ \text{tr}(B) = K \end{array} \right\}. \quad (2.2)$$

3. Compute $\hat{G}$ by applying a clustering algorithm on the rows (or equivalently columns) of $\hat{B}$.

The construction of an accurate estimator $\hat{\Gamma}$ of $\Gamma^*$, before the cluster structure is known, is a crucial step for guaranteeing the statistical optimality of the PECOK estimator. Its construction is given in Bunea et al. (2018), and included in Appendix F, for the convenience of the reader.

We will employ an efficient algorithm for solving (2.1). Standard black-box SDP solvers, for a fixed precision, exhibit $O(d^7)$ running time on (2.1), which is prohibitively expensive. Eisenach and Liu (2017) recently introduced the FORCE algorithm, which requires worst case $O(d^6K^{-2})$ time to solve the SDP, and in practice often performs the clustering rapidly.

The key idea behind the FORCE algorithm is that an optimal solution to (2.1) can be attained by first transforming (2.1) into an eigenvalue problem, and then using a first-order method. Iterations of the first-order method are interleaved with a dual step to round the current iterate to an integer solution of the clustering problem, and then searches for an optimality certificate. By using knowledge of both the primal and the dual SDPs, FORCE is able to find the solution much faster than a standard SDP solver. We refer to Eisenach and Liu (2017) for the detailed algorithm.

3 Estimation of Cluster-based Graphical Models

In this section, we propose a unified estimation approach, that utilizes similar loss functions for estimation and inference in the cluster-average and the latent variable graphs. We first describe our general principle, and then apply it to the two graphical models, respectively.

3.1 One-step Estimators for High-Dimensional Inference

Assume that we observe $n$ i.i.d. realizations $X_1, \ldots, X_n$ of $X \in \mathbb{R}^d$. Let $Q(\beta, X)$ denote a known function of $\beta$ and $X$, where $\beta$ is a $q$-dimensional unknown parameter that parametrizes the distribution of $X$. We define the target parameter $\beta^*$ as

$$\beta^* = \arg\min \mathbb{E}(Q(\beta, X)).$$
Let us partition $\beta$ as $\beta = (\theta, \gamma)$, where $\theta \in \mathbb{R}$ is a univariate parameter of interest, and $\gamma \in \mathbb{R}^{q-1}$ is a nuisance parameter. Our goal is to construct a $n^{1/2}$-consistent and asymptotically normal estimator for $\theta$ in high-dimensional models with $q = \dim(\beta) \gg n$. In this case, the dimension of the nuisance parameter $\gamma$ is large, which makes the inference on $\theta$ challenging. We start from the empirical risk function over $n$ observations defined as

$$Q_n(\beta) = \frac{1}{n} \sum_{i=1}^{n} Q(\beta, X_i).$$ (3.1)

One standard instance of $Q_n$ is the negative log-likelihood function of the data. In this work, we will conduct inference based on an alternative loss function, as the analysis of the log-likelihood may require unpleasant technical conditions that we would like to avoid, as discussed in Sections 4.1. However, we mimic as much as possible the likelihood principles, in order to aid the understanding of the construction below. For these reasons we will refer to $Q_n(\beta)$ as the negative pseudo-likelihood function. For now, we will assume that $Q_n(\beta)$ is given, and a detailed discussion of its respective choice for inference in the latent variable graph and the cluster-average graph will be given in the following two subsections. The pseudo-information matrix for one observation is defined as

$$I = \mathbb{E}(\nabla^2 Q(\beta^*, X_i)),$$

relative to the partition of $\beta = (\theta, \gamma)$. When $Q_n$ is the negative log-likelihood function, and the dimension of the parameter is independent of $n$, then $h(\theta; \gamma)$ given by (3.3) is called the efficient score function for $\theta$, and classical theory shows that it admits solutions that are consistent, asymptotically normal and attain the information bound given by the reciprocal of (3.4) (Van der Vaart, 1998; Bickel et al., 1993).

With these goals in mind, we similarly define the corresponding pseudo-score function for estimating $\theta$ in the presence of the nuisance parameter $\gamma$ as

$$h(\theta; \gamma) = \nabla_1 Q_n(\beta) - I_{12} I_{22}^{-1} \nabla_2 Q_n(\beta) = \frac{1}{n} \sum_{i=1}^{n} \left( \nabla_1 Q(\beta, X_i) - I_{12} I_{22}^{-1} \nabla_2 Q(\beta, X_i) \right),$$ (3.3)

and define the pseudo information of $\theta$, in the presence of the nuisance parameter $\gamma$, as

$$I_{1|2} = I_{11} - I_{12} I_{22}^{-1} I_{21}.$$ (3.4)

When the dimension of $\gamma$ is fixed, one can easily estimate $I_{12}$ and $I_{22}$ in (3.3) by their sample versions $\tilde{I}_{12}$ and $\tilde{I}_{22}$. However, such simple procedure fails when the dimension of $\gamma$ is greater than the sample size, as $\tilde{I}_{22}$ is rank deficient. To overcome this difficulty, rather than estimating $I_{12}$ and $I_{22}$ separately, we directly estimate

$$w^T = I_{12} I_{22}^{-1}$$ (3.5)

by

$$\hat{w} = \text{argmin} \|w\|_1, \quad \text{s.t.} \quad \|\nabla_1^2 Q_n(\hat{\beta}) - w^T \nabla_2^2 Q_n(\hat{\beta})\|_\infty \leq \lambda,$$ (3.6)

8
where $\lambda$ is a non-negative tuning parameter, and $\hat{\beta} = (\hat{\theta}, \hat{\gamma})$ is an initial estimator, which is usually defined case by case, for a given model. Then, we can plug $\hat{\omega}$ and $\hat{\gamma}$ into the pseudo-score function, which gives

$$
\hat{h}(\theta, \hat{\gamma}) = \nabla_1 Q_n(\theta, \hat{\gamma}) - \hat{\omega}^T \nabla_2 Q_n(\theta, \hat{\gamma}).
$$

(3.7)

Following the Z-estimation principle (Van der Vaart, 1998; Bickel et al., 1993), one could define the final estimator of $\theta$ as the solution of the pseudo-score function $\hat{h}(\theta, \hat{\gamma})$. However, in many examples, the pseudo-score function $\hat{h}(\theta, \hat{\gamma})$ may have multiple solutions and it becomes unclear which root serves as a consistent estimator; see Small et al. (2000) for further discussion in the general estimating function context. To bypass this issue, we consider the following simple one-step estimation approach. Given the initial estimator $\hat{\theta}$ from the partition of $\hat{\beta}$, we perform a Newton-Raphson update based on the pseudo-score function $\hat{h}(\theta, \hat{\gamma})$, to obtain $\bar{\theta}$, which is classically referred to as a one-step estimator by Bickel (1975). Specifically, we construct

$$
\bar{\theta} = \hat{\theta} - \widehat{I}_{1/2}^{-1} \hat{h}(\hat{\theta}, \hat{\gamma}),
$$

(3.8)

where $\widehat{I}_{1/2}$ is an estimator of the partial information matrix $I_{1/2}$. In Sections 3.2 and 3.3 below we show that, under appropriate conditions, the one-step estimator $\bar{\theta}$ constructed relative to the empirical risk functions $Q_n$ defined in (3.11) and (3.26), respectively, satisfies

$$
n^{1/2}(\bar{\theta} - \theta^*) = -\widehat{I}_{1/2}^{-1} n^{1/2} h(\beta^*) + o_p(1).
$$

(3.9)

By applying the central limit theory to $h(\beta^*)$, we can establish the asymptotic normality of $\bar{\theta}$ in Theorems 4.3 and 4.4. When $Q_n(\beta)$ is the negative log-likelihood of the data, this approach has been successfully used in Ning and Liu (2017) and, moreover, the estimator $\bar{\theta}$ is asymptotically equivalent to the de-biased estimator in Zhang and Zhang (2014); van de Geer et al. (2013). As will be explained in the following subsections, the analysis based on the log-likelihood becomes intractable for the latent graphical model and requires stringent technical conditions for the cluster-average graphical model. To overcome this difficulty, we employ the pseudo score functions relative to the empirical risk functions $Q_n(\beta)$ defined in (3.11) and (3.26). The resulting one-step estimator still attains the information bound established in the literature, and more importantly requires weaker technical assumptions than the existing methods. Moreover, in addition to (3.9), we also derive explicitly the speed at which the normal approximation is attained.

### 3.2 Estimation of the Cluster-Average Graph

Recall that we assume $Z \sim N(0, C^*)$ and $E \sim N(0, \Gamma^*)$, which implies $X \sim N(0, \Sigma^*)$ with $\Sigma^* = AC^*A^T + \Gamma^*$. The within-cluster average $\bar{X} = (\bar{X}_1, \ldots, \bar{X}_K) \in \mathbb{R}^K$ is given by $\bar{X}_k =: \frac{1}{|G^*_k|} \sum_{i \in G^*_k} X_i$, corresponding to the population level clusters. Because $X \sim N(0, \Sigma^*)$, we can verify that $\bar{X} \sim N(0, S^*)$, where

$$
S^* = C^* + \Gamma^*,
$$

(3.10)

and $\Gamma^* = \text{diag}(\gamma^*_1, \ldots, \gamma^*_K)$ with $\gamma^*_k = \frac{1}{|G^*_k|} \sum_{j \in G^*_k} \gamma^*_j$. Recall that the precision matrix of $\bar{X}$ is

$$
\Omega^* = S^{-1} = (C^* + \Gamma^*)^{-1}.
$$
In this section we give the construction of the estimators of the cluster-average graph corresponding to \( \bar{X} \). Specifically, we use the generic strategy outlined in the previous section in order to construct \( n^{1/2} \)-consistent and asymptotically normal estimators for each component \( \Omega^*_t \) of the precision matrix \( \Omega^* \), for \( 1 \leq t < k \leq K \). For the estimation of each entry, the remaining \( K(K+1)/2 - 1 \) parameters in \( \Omega^* \) are treated as nuisance parameters.

Since we observe \( n \) i.i.d. samples of \( X \in \mathbb{R}^p \), if the clusters and their number were known, then we implicitly observe \( n \) i.i.d. samples of \( \bar{X} \in \mathbb{R}^K \). To explain our method, we first assume that clustering is given, and then explain how to lift this assumption.

Following our general principle, we would naturally tend to choose the negative log-likelihood function of the cluster-averages \( (\bar{X}_1, ..., \bar{X}_n) \) as the empirical risk function \( Q_n(\beta) \) in (3.1). Along this line, Jankova and van de Geer (2014) proposed the de-biased estimator for Gaussian graphical models. However, the inference requires the irrepresentable condition (Ravikumar et al., 2011b) on \( S^* \), which can be restrictive. The alternative methods proposed by Ren et al. (2013); Janková and van de Geer (2017) imposed the condition that the largest eigenvalue of \( S^* \) is bounded. These technical conditions on \( S^* \) are difficult to justify and can be avoided by using our approach. We propose to estimate each sparse row of \( \Omega^* \) as explained below.

Let \( \bar{S} = n^{-1} \sum_{i=1}^n \bar{X}_i \bar{X}_i^T \) denote the sample covariance matrix of \( \bar{X}_i \). When \( K \) is small, the maximum likelihood estimator of \( \Omega^* \) is \( \bar{S}^{-1} \), which can be viewed as the solution of the following equation \( \bar{S} \Omega^* - I_K = 0 \). Thus, in the low dimensional setting, this equation defines the maximum likelihood estimator. Since we are only interested in \( \Omega^*_t \), we can extract the \( k \)th column from the left hand side of the above equation, and use it as the pseudo-score function \( U_n(\Omega^*_k) = \bar{S} \Omega^*_k - e_k \).

To apply the inference strategy in Section 3.1, we need to construct a valid empirical risk function \( Q_n(\Omega^*_k) \) such that \( \nabla Q_n(\Omega^*_k) = U_n(\Omega^*_k) \).

Simple algebra shows that a possible choice is

\[
Q_n(\Omega^*_k) = \frac{1}{2} \Omega^T_k \bar{S} \Omega^*_k - e_k^T \Omega^*_k = \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{2} \Omega^T_k \bar{X}_i \bar{X}_i^T \Omega^*_k - e_k^T \Omega^*_k \right),
\]

which we view in the sequel as the empirical risk corresponding to the population level risk

\[
\mathbb{E} Q(\Omega^*_k, \bar{X}) = \frac{1}{2} \Omega^T_k \bar{S} \Omega^*_k - e_k^T \Omega^*_k,
\]

based on the loss function

\[
Q(\Omega^*_k, \bar{X}) = \frac{1}{2} \Omega^T_k \bar{X} \bar{X}^T \Omega^*_k - e_k^T \Omega^*_k.
\]

Since

\[
\nabla \mathbb{E} Q(\Omega^*_k, \bar{X}) = S^* \Omega^*_k - e_k = 0
\]

and

\[
\nabla^2 \mathbb{E} Q(\Omega^*_k, \bar{X}) = S^*,
\]

then the theoretical risk \( \mathbb{E} Q(\Omega^*_k, \bar{X}) \) has the rows \( \Omega^*_k \) of the target precision matrix \( \Omega^* \) as the unique minimizers, as desired, provided that \( S^* \) is positive definite, an assumption we make in Section 4.1.
We note that the choice of the empirical risk $Q_n(\cdot)$ and that of the corresponding pseudo-score $U_n(\cdot)$ is not unique. We chose the particular form (3.11) because it is quadratic in $\Omega_{k,k}$, which greatly simplifies the theoretical analysis and leads to weaker technical assumptions. Moreover, the property (3.14) is the same as that of the score function corresponding to the negative log-likelihood function, supporting our terminology.

We use the general strategy presented in Section 3.1 to construct estimators that employ the empirical risk $Q_n(\cdot)$ defined by (3.11) above. We first recall that $Q_n(\cdot)$ depends on the unknown cluster structure $G^*$ via $\tilde{X}_i$. We note that in general the estimated group $\tilde{G}_k$ may differ from $G^*_k$ by a label permutation. For notational simplicity, we ignore this label permutation issue and treat $\tilde{G}_k$ as an estimate of $G^*_k$ (rather than $G^*_j$ for some $j \neq k$). To define the estimator of $\Omega^*_{t,k}$, we first replace $\tilde{X}_i$ by $\hat{X}_i$ and denote $\hat{S} = n^{-1} \sum_{i=1}^n \hat{X}_i\hat{X}_i^T$, where $\hat{X}_{ik} = \frac{1}{|\tilde{G}_k|} \sum_{j \in \tilde{G}_k} X_{ij}$.

Let $(t,k)$ be arbitrary, fixed. Replacing $\tilde{S}$ by $\hat{S}$ in $Q_n(\cdot)$, we follow Section 3.1 to define the pseudo-score function

$$h(\Omega_{k}) = v_t^T (\hat{S} \Omega_k - e_k),$$

(3.16)

where $v_t^*$ is a $K$-dimensional vector with $(v_t^*)_t = 1$ and $(v_t^*)_{-t} = -w_t^*$ with $w_t^* = (S^*_{-t,-t})^{-1}s^*_{-t,t}$ consistent with the definition in (3.5) above. To make inference based on $h(\Omega_{k})$, we need to further estimate $w_t^*$ and $\Omega^*_{k,k}$. Following (3.6), an estimate of $w_t^*$ is given by

$$\hat{w}_t = \arg\min \|w\|_1, \text{ s.t } \|\hat{S}_{k,-t} - w^T \hat{S}_{-t,-t}\|_\infty \leq \lambda',$$

(3.17)

where $\lambda'$ is a tuning parameter. Then we can define $\hat{v}_t$ accordingly, and

$$h(\Omega_{k}) = \hat{v}_t^T (\hat{S} \Omega_k - e_k).$$

(3.18)

Recall that the construction of the one-step estimator (3.8) requires an initial estimator of $\Omega^*_k$. To be concrete, we consider the following initial estimator of $\Omega^*_k$,

$$\hat{\Omega}_k = \arg\min \|\beta\|_1, \text{ s.t } \|\hat{S} \beta - e_k\|_{\max} \leq \lambda,$$

(3.19)

where $\lambda$ is a tuning parameter. This estimator has the same form as the CLIME estimator for the $k$-th column of $\Omega$ (Cai et al., 2011). However, unlike the CLIME estimator which requires $\lambda \asymp \|\Omega^*_k\|_1 \sqrt{\log K/n}$, in our Theorem 4.3 we assume $\lambda = C \sqrt{\log (K \lor n)/n}$, where $C$ only depends on the minimum eigenvalue of $C^*$ and the largest diagonal entries of $C^*$ and $\Gamma^*$ which are assumed bounded by constants in Assumptions 4.1 and 4.2. With this choice of $\lambda$, we show in Lemma A.8 in Appendix A that

$$\|\hat{\Omega}_k - \Omega^*_k\|_1 \lesssim s_1 \sqrt{\frac{\log (K \lor n)}{n}},$$

(3.20)

with high probability, where $s_1$ is the sparsity level of $\Omega^*_k$. As a comparison, Theorem 6 in Cai et al. (2011) only implies $\|\hat{\Omega}_k - \Omega^*_k\|_1 \lesssim s_1 \|\Omega^*_k\|_1^2 \sqrt{\frac{\log K}{n}}$. For many sparse matrices, the $\ell_1$ norm of a column, $\|\Omega^*_k\|_1$, can grow to infinity with $K$ or $s_1$, and thus (3.20) gives a faster rate. This is possible to obtain under the Gaussian assumption on $X$, which can be easily relaxed to sub-Gaussian. In this case, and when $\lambda_{\min}(C^*) > c$, Lemma A.7 is instrumental in showing that the extra $\|\Omega^*_k\|_1^2$
factor in the rate of the original CLIME estimator can be avoided, whereas if only the marginal components of \( \mathbf{X} \) are assumed to be sub-Gaussian, as in Cai et al. (2011), it may be unavoidable, without further conditions on \( \Omega^* \).

Based on the block matrix inverse formula, we can show that the partial pseudo information matrix reduces to \( I_{1|2} = 1/\Omega^*_{t,t} \). Finally, the one-step estimator is defined as

\[
\hat{\Omega}_{t,k} = \hat{\Omega}_{t,k} - \hat{h}(\Omega_{k})\hat{\Omega}_{t,t},
\]

in accordance with \((3.8)\). In Section 4, we will show that under mild regularity conditions \( n^{1/2}(\hat{\Omega}_{t,k} - \Omega^*_{t,k}) \sim N(0, s_{tk}^2) \), where \( s_{tk}^2 = \Omega^*_{t,k} + \Omega^*_t \Omega^*_k \). Let \( \hat{s}_{tk}^2 = \hat{\Omega}_{t,k}^2 + \hat{\Omega}_{t,t}\hat{\Omega}_{k,k} \) be a consistent estimator of the asymptotic variance. Then, a \((1 - \alpha) \times 100\%\) confidence interval for \( \Omega_{t,k} \) is

\[
[\hat{\Omega}_{t,k} - z_{1-\alpha/2}\hat{s}_{tk}/n^{1/2}, \hat{\Omega}_{t,k} + z_{1-\alpha/2}\hat{s}_{tk}/n^{1/2}],
\]

where \( z_\alpha \) is the \( \alpha \)-quantile of a standard normal distribution. Equivalently, we can use the scaled test statistics \( \hat{\Omega}_{t,k} \) to construct a test for \( H_0 : \Omega_{t,k}^* = 0 \) versus \( H_1 : \Omega_{t,k}^* \neq 0 \) with \( \alpha \) significance level. Namely, the null hypothesis is rejected if and only if the above \((1 - \alpha) \times 100\%\) confidence interval does not contain 0. We will employ such tests in Section 4.

### 3.3 Latent Variable Graph

Recall that the structure of the latent variable graph is encoded by the sparsity pattern of \( \Theta^* = C^{*-1} \), which is generally different from the cluster-average group as \( C^{*-1} \) and \( \Omega^* = (C^* + \Gamma^*)^{-1} \) may have different sparsity patterns. In this section, we focus on the inference on the component \( \Theta_{t,k}^* \), for some \( 1 \leq t < k \leq K \). Similar to the cluster-average graph, we first discuss the likelihood approach. The negative log-likelihood corresponding to model \((1.1)\) indexed by the parameter \((\Theta, \Gamma)\) is

\[
\ell(\Theta, \Gamma) = -\text{tr}(\hat{\Sigma}(\Theta A^{-1}A^T + \Gamma)^{-1}) + \log \det((\hat{\Sigma}(\Theta A^{-1}A^T + \Gamma)^{-1})^{-1}),
\]

where \( \hat{\Sigma} = n^{-1}\sum_{i=1}^n \mathbf{X}_i \mathbf{X}_i^T \). After tedious algebra, we show that the Fisher information matrix for \((\Theta, \Gamma)\) is given by

\[
I = \begin{bmatrix}
(M^* A^T \Gamma^{*-1} \Sigma^* \Gamma^{*-1} A \mathbf{M}^*)^{\otimes 2} & (M^* A^T \Gamma^{*-1} F^* T)^{\otimes 2} D_d \\
D_d^T (F^* \Sigma^* \Gamma^{*-1} A \mathbf{M}^*)^{\otimes 2} & D_d^T (F^* \Sigma^* \Gamma^{*-1} F^* T)^{\otimes 2} D_d
\end{bmatrix},
\]

where \( D_d = (I_d \otimes I_d^T) \odot (I_d^T \otimes I_d), \mathbf{M}^* = (\Theta^* + A^T \Gamma^{*-1} A)^{-1} \) and \( F^* = I_d - \mathbf{A} \mathbf{M}^* A^T \Gamma^{*-1}. \) As seen in Section 3.1, the inference based on the likelihood or equivalently efficient score function \((3.3)\) requires the estimation of \( I_{12} L_{22}^{-1} \) which, given the complicated structure of the information matrix \((3.22)\), becomes analytically intractable.

A solution to this problem is inference based on an empirical risk function similar to \((3.11)\), but tailored to the latent variable graph. With a slight abuse of notation, and reasoning as in \((3.14)\) and \((3.15)\), we notice that, for each \( k \),

\[
\mathbb{E} Q(\Theta_k, \mathbf{X}) = \frac{1}{2} \Theta_k^T C^* \Theta_k - c_k^T \Theta_k,
\]

(3.23)
has the target $\Theta^*_k$ as a unique minimizer, where the loss function $Q(\Theta_k, X)$ is defined as

$$Q(\Theta_k, X) = \frac{1}{2} \Theta_k^T \mathbf{C} \Theta_k - \mathbf{e}_k^T \Theta_k,$$  \hfill (3.24)

and the matrix $\mathbf{C} := (\hat{C}_{jk})_{j,k}$ has entries

$$\hat{C}_{jk} = \frac{1}{|G_j||G_k|} \sum_{a \in G_j^*, b \in G_k^*} (X_a X_b - \hat{\Gamma}_{ab}),$$ \hfill (3.25)

and $\hat{\Gamma}_{ab} = 0$ if $a \neq b$ and $\hat{\Gamma}_{aa} = X_a X_a - \frac{1}{|G_j^*|} \sum_{a \in G_j^*, a \neq j} X_a X_a$. Since $\mathbb{E}(\hat{\mathbf{C}}) = \mathbf{C}^*$, the risk relative to the loss function in (3.24) is indeed (3.23), and the empirical risk is

$$Q_n(\Theta_k) = \frac{1}{n} \sum_{i=1}^n \left( \frac{1}{2} \Theta_k^T \hat{\mathbf{C}}^{(i)} \Theta_k - \mathbf{e}_k^T \Theta_k \right),$$ \hfill (3.26)

where $\hat{\mathbf{C}}^{(i)}$ is obtained by replacing $X$ in $\hat{C}_{jk}$ by $X_i$. Similar to the cluster-average graph, $Q_n(\Theta_k)$ also depends on the unknown cluster structure. We estimate $G_k^*$ by $\hat{G}_k$, and define $\hat{\Gamma} = (\hat{\Gamma}_{ab})$, where $\hat{\Gamma}_{ab} = 0$ if $a \neq b$ and $\hat{\Gamma}_{aa} = \frac{1}{n} \sum_{i=1}^n (X_{ia} X_{ia} - \frac{1}{|G_i^*|} \sum_{a \in \hat{G}_j, a \neq j} X_{ia} X_{ij})$, and $\hat{\mathbf{C}} = (\hat{C}_{jk})$ where $\hat{C}_{jk} = \frac{1}{n} \sum_{i=1}^n \frac{1}{|G_j||G_k|} \sum_{a \in \hat{G}_j, b \in \hat{G}_k} (X_{ia} X_{ib} - \hat{\Gamma}_{ab})$. We replace $\hat{\mathbf{C}}$ by $\hat{\mathbf{C}}$ in (3.26) above and follow exactly the strategy of Section 3.2, with $\hat{S}$ replaced by $\hat{\mathbf{C}}$, to construct the corresponding pseudo-score function $\hat{h}(\Theta_k)$, similarly to (3.18), and the initial estimator $\hat{\Theta}_k$, similarly to (3.19). We combine these quantities, following the general strategy (3.8), as above, to obtain the final one-step estimator of $\Theta^*_{t,k}$, defined as

$$\tilde{\Theta}_{t,k} = \hat{\Theta}_{t,k} - \hat{h}(\Theta_k) \hat{\Theta}_{t,t},$$ \hfill (3.27)

after observing that, in this case, $I_{1|2} = 1/\Theta^*_{t,t}$.

Although the form of this estimator is similar to (3.21), derived for the cluster-average graph, the study of the asymptotic normality of $\tilde{\Theta}_{t,k}$ reveals that its asymptotic variance is much more involved, and will be discussed in detail in Section 4.2.2.

### 4 Main Theoretical Results

#### 4.1 Assumptions

In this section we state the two assumptions under which all our results are proved.

**Assumption 4.1.** The covariance matrix $\mathbf{C}^*$ of $Z$ satisfies: $c_1 \leq \lambda_{\min}(\mathbf{C}^*)$ and $\max_{i} C^*_{i,t} \leq c_2$, for some absolute constants $c_1, c_2 > 0$.

**Assumption 4.2.** The matrix $\mathbf{\Gamma}^*$ satisfies: $\max_{1 \leq i \leq d} \gamma^*_i \leq c_3$ for some absolute constant $c_3 > 0$, where $\gamma^*_i$ are the entries of the diagonal matrix $\mathbf{\Gamma}^*$.  

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Assumptions 4.1 and 4.2 are minimal conditions for inference on precision matrices. Furthermore, they imply the conditions needed for clustering consistency derived in Bunea et al. (2018) and discussed in Section 2.2, for \( n \) large enough. The latter only require that \( \lambda_{\min}(C^*) \) is bounded from below by a sequence that converges to zero, as soon as \( \|\Sigma^*\|_{\text{max}} \) and \( \|\Gamma^*\|_{\text{max}} \) are bounded. This is strengthened by our assumptions. In particular, a constant lower bound on \( C^* \) is standard in any inference on graphical models and needed to show the asymptotic normality of the estimator introduced above (Ren et al., 2013; Jankova and van de Geer, 2014; Janková and van de Geer, 2017).

4.2 Asymptotic Normality via Berry-Esseen-type Bounds

4.2.1 Results for the Cluster-Average Graph

In the section, we show that the estimators \( \tilde{\Omega}_{t,k} \) given by (3.21) are asymptotically normal, for all \( t < k \). We define the sparsity of the cluster-average graph as

\[
\max_{1 \leq j \leq K} \sum_{k=1}^{K} \mathbb{I}(\Omega_{j,k}^* \neq 0) \leq s_1.
\]

Recall that the estimators (3.17) and (3.19) depend on the tuning parameters \( \lambda \) and \( \lambda' \). In the following theorem, we choose \( \lambda \asymp \lambda' \asymp \sqrt{\log(K \lor n)/n} \). For notational simplicity, we use \( C \) to denote a generic constant, the value of which may change from line to line.

**Theorem 4.3.** If Assumptions 4.1 and 4.2 hold, we have

\[
\max_{1 \leq t < k \leq K} \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \frac{n^{1/2}(\tilde{\Omega}_{t,k} - \Omega_{t,k}^*)}{\hat{s}_{tk}} \leq x \right) - \Phi(x) \right| \leq \frac{C s_1 \log(K \lor n)}{n^{3/2}} + \frac{C}{(K \lor n)^3} (4.1)
\]

where \( \hat{s}_{tk}^2 = \tilde{\Omega}_{t,k}^2 + \tilde{\Omega}_{t,t}^* \tilde{\Omega}_{k,k}^* \) and \( C \) is a positive constant.

Theorem 4.3, proved in Appendix A, gives the rate of the normal approximation of the distribution of the scaled and centered entries \( \tilde{\Omega}_{t,k} \). The right hand side in (4.1) is non-asymptotic and is valid for each \( K, n \) and \( d \). Its first, small, term is the price to pay for having first used the data for clustering, and it is dominated by the other two terms. From this perspective, the clustering step is the least taxing, as long as we can ensure its consistency, which in turn can be guaranteed under the minimal assumptions 4.1 and 4.2 already needed for the remaining steps.

The second, and dominant, term regards the normal approximation of the distribution of

\[
n^{1/2}(\tilde{\Omega}_{t,k} - \Omega_{t,k}^*). (4.2)
\]

Specifically, as an intermediate step, Proposition A.2 in Appendix A shows that the difference between the c.d.f. of (4.2), scaled by \( s_{tk} = \sqrt{\Omega_{t,k}^2 + \Omega_{t,t}^* \Omega_{k,k}^*} \), and that of a standard Gaussian random variable is bounded by \( s_{1} \log(K \lor n)/n^{3/2} \). Therefore, asymptotic normality holds as soon as this quantity converges to zero, which agrees with the weakest sparsity conditions for Gaussian graphical model inference in the literature (Ren et al., 2013; Janková and van de Geer, 2017). In addition,
the asymptotic variance $s_{tk}^2$ agrees with the minimum variance bound in Gaussian graphical models (Janková and van de Geer, 2017). Thus, inference based on the empirical risk function (3.11) does not lead to any asymptotic efficiency loss. Unlike the previous works, we do not require the bounded operator norm condition, $\lambda_{\max}(S^*) \leq C$. This condition is avoided in our analysis by using a more convenient empirical risk function (3.11), as opposed to the log-likelihood in Jankova and van de Geer (2014), and a CLIME-type initial estimator (3.19) satisfying (3.20), as opposed to the node-wise Lasso estimator in Janková and van de Geer (2017).

The last term in the normal approximation is $O(\frac{1}{(K\lor n)^\delta})$ which is dominated by the second one, and is associated with the replacement of the theoretical variance $s_{tk}^2$ by the estimate $\hat{s}_{tk}^2$.

Finally, we note that the powers of the first and the third term in the right hand side of (4.1) can be replaced by $2 + \delta$, for any $\delta > 0$, and a change in this power also changes the associated constant $C$ in the term $Cs_0^2 \log(K\lor n)n^{1/2}$, which we need for valid FDR control, we need $K^2/(K\lor n)^{2+\delta} = o(1)$, which holds for any $\delta > 0$. For simplicity, we choose $\delta = 1$ which gives the power 3.

### 4.2.2 Results for the Latent Variable Graph

In this section we show that the estimators $\tilde{\Theta}_{tk}$ given by (3.27) are asymptotically normal, for all $t < k$. We define the sparsity of the latent graph as $s_0 \in \mathbb{N}$ such that

$$\max_{1 \leq j \leq K} \sum_{k=1}^{K} \mathbb{I}(\Theta_{j,k}^* \neq 0) \leq s_0.$$ 

Inference for the estimator $\tilde{\Theta}_{tk}$ follows the general approach outlined in Section 3.1. We prove in Proposition B.1 in Appendix B that

$$n^{1/2}(\tilde{\Theta}_{tk} - \Theta_{tk}^*) = \frac{1}{n^{1/2}} \sum_{i=1}^{n} \Theta_{t,t}^* \mathbf{v}_i^T (\tilde{\mathbf{C}}^{(i)} \tilde{\Theta}_{ik}^* - e_k) + o_p(1),$$

where $\mathbf{v}_i^*$ is a $K$-dimensional vector with $(\mathbf{v}_i^*)_t = 1$ and $(\mathbf{v}_i^*)_{-t} = -w_i^*$ with $w_i^* = (\mathbf{C}_{-t,-t}^*)^{-1}\mathbf{C}_{-t,t}^*$, and $\tilde{\mathbf{C}}^{(i)}$ is defined in (3.25). The terms of the sum in display (4.3) are mean zero random variables, and their variance is

$$\sigma_{tk}^2 = \mathbb{E}(\Theta_{t,t}^* \mathbf{v}_i^T (\tilde{\mathbf{C}}^{(i)} \tilde{\Theta}_{ik}^* - e_k))^2,$$

which does not have an explicit closed form, unlike the asymptotic variance of the estimates of the entries of $\Omega^*$ above. However, we show in Proposition B.7 in Appendix B that $\sigma_{tk}^2$ admits an approximation that is easy to estimate:

$$\left| \sigma_{tk}^2 - \left[ (\Theta_{t,k}^*)^2 + \Theta_{k,k}^* \Theta_{t,t}^* \right] \right| \leq \frac{s_0}{m},$$

where $m = \min_{1 \leq k \leq K} |G_k^*|$. Guided by this approximation, we estimate $\sigma_{tk}^2$ by

$$\tilde{\sigma}_{tk}^2 = \tilde{\Theta}_{t,k}^2 + \tilde{\Theta}_{k,k} \tilde{\Theta}_{t,t}.$$
When all clusters have the equal size, we obtain $K = d/m$. Thus the $O(\frac{m}{d})$ terms can be ignored asymptotically in the sense that $\frac{s_0}{m} = \frac{s_0K}{d} \leq \frac{K^2}{d} = o(1)$, when the clusters are approximately balanced, and their number satisfies $K^2 = o(d)$. This is a reasonable assumption in most applied clustering problems. We note that the estimator $\hat{\sigma}_{tk}^2$ may be inconsistent when the size of some clusters is too small. However, we recall that our ultimate goal is to use these estimators for recovering the sparsity pattern of $\Omega^*$ under FDR control. To evaluate the sensitivity of our overall procedure to the size of the smallest cluster, we conduct simulation studies in Section 5. The results shows that the proposed method works well as soon as $m > 4$.

The following theorem gives the Berry–Esseen normal approximation bound for the estimators of the entries of the precision matrix corresponding to the latent variable graph.

**Theorem 4.4.** If Assumptions 4.1 and 4.2 hold, then

$$\max_{1 \leq t < k \leq K} \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \frac{n^{1/2}(\hat{\Theta}_{t,k} - \Theta^*_{t,k})}{\hat{\sigma}_{tk}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{(d \vee n)^3} + \frac{C}{(K \vee n)^3} + \frac{C s_0 \log(K \vee n)}{n^{1/2}} + \frac{Cs_0}{m},$$

(4.4)

where $C$ is a positive constant.

Compared to the average graph, the Berry-Esseen bound in (4.4) contains an additional $O(s_0m)$ term, stemming from the approximation of the analytically intractable asymptotic variance by an estimable quantity. The proof is deferred to Appendix B.

### 4.3 Post-clustering FDR Control

Given the edge-wise inferential results for the cluster-average and latent variable graphs established above, we explain in this section how to combine them to control graph-wise inferential uncertainty. We view the problem of recovering the sparsity pattern of a precision matrix as one of multiple hypotheses testing, by setting:

$$H_{0;tk} : \Omega^*_{t,k} = 0 \quad \text{vs.} \quad H_{1;tk} : \Omega^*_{t,k} \neq 0 \quad \text{for all } 1 \leq t < k \leq K,$$

for the cluster-average graph, and

$$H'_{0;tk} : \Theta^*_{t,k} = 0 \quad \text{vs.} \quad H'_{1;tk} : \Theta^*_{t,k} \neq 0 \quad \text{for all } 1 \leq t < k \leq K.$$

(4.5)

(4.6)

for the latent variable graph. In the following, we describe our procedure for the estimation of the cluster-average graph. The same procedure applies to the latent variable graph.

Define the set of true null hypotheses, $H_0 := \{(t,k) : 1 \leq t < k \leq K, \text{ such that } \Omega^*_{t,k} = 0\}$, as the set of indices $(t,k)$ for which there is no edge between the nodes $t$ and $k$. To control the error incurred by multiple testing, we focus on the false discovery rate (FDR), which is the average number of Type I errors relative to the total number of discoveries (Benjamini and Hochberg, 1995). Recall that $\hat{\Omega}_{t,k}$ is a consistent and asymptotically normal estimator of $\Omega_{t,k}$. We consider the natural test statistic $\hat{\tilde{W}}_{t,k} = n^{1/2} \hat{\Omega}_{t,k} / \tilde{s}_{tk}$ for $H_{0;tk}$, where $\tilde{s}_{tk}^2 = \hat{\sigma}_{tk}^2 + \hat{\Omega}_{t,t} \hat{\Omega}_{k,k}$. Given a cutoff $\tau > 0$, the total number of discoveries is

$$R_\tau := \sum_{1 \leq t < k \leq K} \mathbb{I}[|\hat{\tilde{W}}_{t,k}| > \tau].$$

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Similarly, the number of false positives or false discoveries is given by
\[ V_\tau := \sum_{(t,k) \in H_0} \mathbb{I}[|\tilde{W}_{t,k}| > \tau]. \]

The FDR is formally defined as the expected ratio of \( V_\tau \) over \( R_\tau \),
\[ \text{FDR}(\tau) := E \left[ \frac{V_\tau}{R_\tau} \mathbb{I}[R_\tau > 0] \right], \]
where the indictor function is included to remove the trivial case \( R_\tau = 0 \).

Our goal is to find a data-dependent cutoff \( \tau \) such that \( \text{FDR}(\tau) \leq \alpha + o(1) \) for any given \( 0 < \alpha < 1 \). This is the best one can hope for when, as in our case, the distribution of the test statistics \( \tilde{W}_{t,k} \) is only available asymptotically. The Berry-Esseen type bounds derived in Theorem 4.3 and 4.4 above allow us to quantify precisely the price that needs to be paid for the asymptotic approximation, and become instrumental for understanding asymptotic FDR control.

In addition, the test statistics \( \tilde{W}_{t,k} \) for different hypotheses are dependent. To allow for the dependence, instead of the standard B-H procedure (Benjamini and Hochberg, 1995), we consider the more flexible B-Y procedure by Benjamini and Yekutieli (2001). The resulting FDR procedure is as follows: reject all hypotheses such that \( |\tilde{W}_{t,k}| \geq \hat{\tau} \), where
\[ \hat{\tau} := \min \left\{ \tau > 0 : \tau \geq \Phi^{-1} \left( 1 - \frac{\alpha R_\tau}{2N_{\text{BY}}|H|} \right) \right\} \quad \text{and} \quad N_{\text{BY}} = \sum_{i=1}^{|H|} \frac{1}{i}, \quad (4.7) \]
where \( |H| = K(K - 1)/2 \) is the total number of hypotheses. Our next result shows when the FDR based on our test statistics is guaranteed to be no greater than \( \alpha \), asymptotically.

**Proposition 4.5.**

1. Assume that the conditions in Theorem 4.3 hold. For any \( 0 < \alpha < 1 \), we have
\[ \text{FDR}(\hat{\tau}) \leq \alpha + 2|H_0|b_n, \quad (4.8) \]
where \( b_n = \frac{C}{d\sqrt{n}} + \frac{C}{(K\sqrt{n})^3} + \frac{C_{s_1} \log(K \vee n)}{n^{1/2}} \), and \( |H_0| \) is the number of true null hypotheses in (4.5).

2. Assume that the conditions in Theorem 4.4 hold. If we define the test statistic as \( \tilde{W}_{t,k} = n^{1/2} \tilde{\Theta}_{t,k}/\tilde{\sigma}_k \), and \( \hat{\tau} \) as in (4.7), we have
\[ \text{FDR}(\hat{\tau}) \leq \alpha + 2|H_0'|c_n, \quad (4.9) \]
where \( c_n = \frac{C}{d\sqrt{n}} + \frac{C}{(K\sqrt{n})^3} + \frac{C_{s_0} \log(K \vee n)}{n^{1/2}} + \frac{C_{s_0}}{m} \), and \( |H_0'| \) is the number of true null hypotheses in (4.6).

Thus, this theorem implies that our method can control the FDR asymptotically, in the sense that \( \text{FDR}(\hat{\tau}) \leq \alpha + o_p(1) \), provided that \( s_1|H_0| \log(K \vee n) = o(n^{1/2}) \), for the average graph, and \( s_0|H_0'| \log(K \vee n) = o(n^{1/2}) \) for the latent graph. Gaussian graphical model estimation under FDR
control was recently studied by Liu (2013). Their approach is based on the following Cramer-type moderate deviation result using our terminology,

$$\max_{(t,k) \in \mathcal{H}_0} \sup_{0 \leq t \leq 2\sqrt{\log K}} \left| \frac{P(\hat{T}_{t,k} \geq t) - \frac{1}{2} - 2\Phi(t)}{\sqrt{2\log K}} \right| = o(1),$$

(4.10)

where $\hat{T}_{t,k}$ is the test statistic they proposed for estimation of the Gaussian graphical model structure. The above result (4.10) controls the relative error of the Gaussian approximation within the moderate deviation regime $[0, 2\sqrt{\log K}]$, whereas our result is based on the control of the absolute error via the Berry-Esseen-type Gaussian approximation. One of the main advantages of their result is that the number of clusters is allowed to be $K = o(n^r)$, where $r$ is a constant that can be greater than 1. However, to prove (4.10), they required that the number of strong signals tends to infinity, that is $|\{(t, k) : \Omega_{t,k} \geq C \sqrt{\log K/n}\}| \to \infty$, which reduces significantly the parameter space for which inference is valid. In contrast, the aim of this work is the study of pattern recovery under no conditions on the signal strength of the entries of the target precision matrices, as in practice it is difficult to assess whether these conditions are met.

The overall message conveyed by Proposition 4.5 is that, in the absence of any signal strength assumptions, cluster-based graphical models can still be recovered, under FDR control, provided that the number of clusters $K$ is not very high relative to $n$, and provided that the clusters are not very small. This further stresses the importance of an initial dimension reduction step in high-dimensional graphical model estimation. For instance, results similar to those of Theorem 4.3 can be derived along the same lines for the estimation of the sparsity pattern of $\Sigma^{-1}$, for a generic, unstructured, covariance matrix of $X$, where one replaces $K$ by $d$ throughout, and $s_1$ is replaced by $s$, the number of non-zero entries in the $d \times d$ matrix $\Sigma^{-1}$. Then, the analogue of (4.9) of Proposition 4.5 shows that FDR control in generic graphical models, based on asymptotic approximations of $p$-values, cannot generally be guaranteed if $d > n$. Our work shows that extra structural assumptions, for instance those motivated by clustering, do alleviate this problem. The simulation study presented in the next section provides further support to our findings.

5 Numerical Results

This section contains simulations and a real data analysis that illustrate the finite sample performance of the inferential procedures developed in the previous sections for the latent variable graph and cluster-average graph, respectively.

5.1 Synthetic Datasets

In this subsection, we demonstrate the effectiveness of the FDR control procedures on synthetic datasets. We consider two settings $(n, d) = (800, 400)$ and $(500, 1000)$, and in each setting we vary the value of $K$ and $m$. The error variable $E$ is sampled from the multivariate normal distribution with covariance $\Gamma^*$ whose entries $\gamma_i^*$ are generated from $U[0.25, 0.5]$. Recall that the latent variable $Z$ follows from $Z \sim N(0, C^*)$. We consider three different models to generate the graph structure
of \( Z \). Once the graph structure is determined, the corresponding adjacency matrix \( W \) is found, and the precision matrix \( \Theta^* = C^{*-1} \) is taken as \( \Theta^* = cW + (|\lambda_{\min}(W)| + 0.2)I \), where \( c = 0.3 \) when \( d = 400 \) and \( c = 0.5 \) when \( d = 1000 \). Finally, we assign the cluster labels for all variables so that all clusters have approximately equal size, which gives us the matrix \( A \). Given \( A, Z \) and \( E \), we can generate \( X \) according to the model (1.1).

We consider the following three generating models for the graph structure of \( Z \):

- **Scale-Free Graph** – The Scale-Free model is a generative model for network data, whose degree distribution follows a power law. To be concrete, we generate the graph one node at a time, starting with a 2 node chain. For nodes \( 3, \ldots, K \), node \( t \) is added and one edge is added between \( t \) and one of the \( t-1 \) previous nodes. Denoting by \( k_i \) the current degree of node \( i \) in the graph, the probability that node \( t \) and node \( i \) are connected is \( p_i = k_i / (\sum_i k_i) \). The number of edges in the resulting graph is always \( K \).

- **Hub Graph** – The \( K \) nodes of the graph are partitioned evenly into groups of size \( N \). Within each group, one node is selected to be the group hub, and an edge is added between it and the remainder of its group. \( N \) is either 5 or 6 depending upon the choice of \( K \). The number of edges in the graph is \( K(N-1)/N \), so for \( K = 100 \) with \( N = 5 \), the number of edges in the resulting graph is 80.

- **Band3 Graph** – This model generates a graph with a Toeplitz adjacency matrix. There is an edge between node \( i \) and node \( j \) if and only if \( |i-j| \leq B \), where we set \( B = 3 \) in this scenario. In general, the number of edges in a band graph with \( K \) nodes is given by \( BK - \frac{3}{2}B^2 + \frac{5}{2}B \). So, for \( K = 100 \) and \( B = 3 \), the number of edges in the graph is 294.

Recall that \( \bar{X} \sim N(0, S^*) \), where \( S^* \) is defined in (3.10). To determine the structure of the average graph, we numerically compute \( S^{*-1} \) and threshold the matrix at \( 10^{-8} \).

We examine the empirical FDR of our procedures on some synthetic datasets. The following protocol is followed in all the experiments:

1. Generate the graph structure of \( Z \) as specified above.
2. Simulate \( n \) observations from our model (1.1).
3. Estimate the cluster partition \( \hat{G} \). For computational convenience, we apply the FORCE algorithm (Eisenach and Liu, 2017) when \( d = 400 \) and the COD algorithm (Bunea et al., 2018) when \( d = 1000 \).
4. Construct the test statistic \( \tilde{W}_{l,k} \) defined in Section 4.3. The regularization parameters \( \lambda \) and \( \lambda' \) are chosen by 5-fold cross validation.
5. Find the FDR cutoff (4.7) at level \( \alpha \), where we consider three cases \( \alpha = 0.05, 0.1, 0.2 \).

The simulation is repeated 100 times. To compare with our Benjamini-Yekutieli based FDR procedure, we also report the empirical FDR based on the more classical Benjamini-Hochberg
procedure. That is, we apply the same procedures 1-4, but in step 5 we replace the FDR cutoff in (4.7) with the Benjamini-Hochberg (B-H) cutoff, i.e.,

$$\hat{\tau}_{BH} := \min \left\{ \tau > 0 : \tau \geq \Phi^{-1} \left( 1 - \frac{\alpha R_{\tau}}{2|\mathcal{H}|} \right) \right\}.$$  

Table 1 compares the empirical FDR based on our method with the B-H procedure under different $m, K$ settings when $d = 400$. When $m$ is relatively large (e.g., $m = 20$), both methods can control FDR on average, although our method is relatively more conservative. As expected, the FDR control problem becomes more challenging for large $K$ and small $m$. In this case the graph contains more nodes and each cluster contains fewer variables. We observe that when $m = 5$ our method can still control FDR reasonably well but the B-H method produces empirical FDR far beyond the nominal level, especially for the hub graph. The inferior performance of the B-H procedure is due to the fact that the dependence among the test statistics is not accounted in the B-H method. Finally, we examine the empirical power of the FDR procedure under each scenario, which is defined as

$$\text{Average} \left[ \sum_{(t,k) \in \mathcal{H}_1} \frac{I[\hat{W}_{t,k} \geq \hat{\tau}]}{|\mathcal{H}_1|} \right],$$

where $\mathcal{H}_1$ is the set of alternative hypothesis. Table 2 gives the empirical power of our FDR procedure, and the B-H procedure when $d = 400$. It shows that our procedure and the B-H procedure have very high power in all scenarios. The same phenomenon is observed when $d$ is large, i.e., $d = 1000$; see Tables 3 and 4. In summary, the proposed procedure can identify most signals in the graph with well controlled FDR.

| $K$ | $m$ | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub |
|-----|-----|-----------|-------|-----|-----------|-------|-----|-----------|-------|-----|
| 80  | 5   | 1.16%     | 1.24% | 5.99%| 2.01%     | 2.64% | 7.60%| 4.01%     | 4.73% | 10.33%|
| 66  | 6   | 0.93%     | 1.03% | 1.08%| 1.99%     | 1.98% | 1.96%| 3.73%     | 3.68% | 4.00%|
| 50  | 8   | 1.16%     | 0.99% | 1.09%| 2.20%     | 1.77% | 1.98%| 3.90%     | 3.49% | 3.73%|
| 20  | 20  | 0.99%     | 0.88% | 1.30%| 1.81%     | 1.66% | 2.26%| 3.75%     | 3.38% | 4.71%|

| $K$ | $m$ | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub |
|-----|-----|-----------|-------|-----|-----------|-------|-----|-----------|-------|-----|
| 80  | 5   | 1.15%     | 1.40% | 6.29%| 2.14%     | 2.67% | 7.89%| 4.00%     | 4.70% | 10.43%|
| 66  | 6   | 0.91%     | 1.04% | 1.04%| 1.85%     | 2.00% | 2.14%| 3.70%     | 3.71% | 3.83%|
| 50  | 8   | 1.16%     | 1.00% | 1.11%| 2.22%     | 1.80% | 1.94%| 3.98%     | 3.49% | 3.61%|
| 20  | 20  | 0.94%     | 0.88% | 1.23%| 1.81%     | 1.68% | 2.26%| 3.65%     | 3.38% | 4.71%|

Table 1: Averaged empirical FDR for synthetic data experiments with $d = 400$ and $n = 800$. 

| $K$ | $m$ | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub |
|-----|-----|-----------|-------|-----|-----------|-------|-----|-----------|-------|-----|
| 80  | 5   | 8.23%     | 9.24% | 15.60%| 15.01%   | 16.39%| 24.00%| 28.16%   | 28.97%| 38.58%|
| 66  | 6   | 7.18%     | 7.38% | 7.31% | 14.12%   | 14.01%| 13.78%| 26.56%   | 25.84%| 27.28%|
| 50  | 8   | 6.94%     | 6.75% | 6.69% | 13.23%   | 12.73%| 12.74%| 25.20%   | 23.62%| 26.05%|
| 20  | 20  | 5.43%     | 4.54% | 6.38% | 11.09%   | 8.47% | 10.71%| 21.46%   | 17.15%| 20.75%|

| $K$ | $m$ | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub |
|-----|-----|-----------|-------|-----|-----------|-------|-----|-----------|-------|-----|
| 80  | 5   | 8.42%     | 9.26% | 15.77%| 15.25%   | 16.43%| 24.15%| 28.03%   | 29.02%| 38.57%|
| 66  | 6   | 7.21%     | 7.37% | 7.45% | 13.99%   | 13.92%| 13.84%| 26.19%   | 25.77%| 27.17%|
| 50  | 8   | 6.82%     | 6.78% | 6.67% | 13.23%   | 12.67%| 12.89%| 25.10%   | 23.61%| 25.91%|
| 20  | 20  | 5.38%     | 4.51% | 6.49% | 11.01%   | 8.52% | 10.66%| 21.58%   | 17.09%| 20.71%|
\[ \alpha = 0.05 \]

| \( K \) x \( m \) | \( \alpha = 0.05 \) | \( \alpha = 0.1 \) | \( \alpha = 0.2 \) |
|---|---|---|---|
| | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub |
| 80 x 5 | 88.22% | 99.99% | 99.00% | 91.10% | 100.00% | 99.04% | 93.38% | 100.00% | 99.11% |
| 66 x 6 | 93.62% | 100% | 100% | 95.42% | 100% | 100% | 96.86% | 100% | 100% |
| 50 x 8 | 97.18% | 100% | 100% | 97.90% | 100% | 100% | 98.47% | 100% | 100% |
| 20 x 20 | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% | 100% |

Table 2: Averaged FDR power for synthetic data experiments with \( d = 400 \) and \( n = 800 \).

### 5.2 fMRI Dataset

Power et al. (2011) finds that the human brain can be divided into regions of interest (ROIs) which can be further organized into functional networks. We apply our inferential procedures to these regions of interest in publicly available resting-state fMRI data from the Neuro-bureau pre-processed repository (Bellec et al., 2015). Specifically, we use the data from patient 1018959, session 1 in the KKI dataset. This fMRI data was pre-processed using the Athena pipeline and mapped to T1 MNI152 coordinate space. We choose this dataset to make our experiments easily reproducible, as the data are available pre-processed using standard alignment and denoising procedures.

Using the T1 MNI152 coordinates, we extract the 264 ROIs identified in Power et al. (2011), which gives us \( d = 264 \) mean activities across \( n = 148 \) time periods. We apply the FORCE algorithm to cluster the 264 ROIs, and obtain an estimate of \( K = 53 \) and the corresponding clusters. Using the FDR control procedures with \( \alpha = 0.01 \), we obtain the networks shown in Figures 1 and 2. For clarification purpose, in these two figures, we only display the nodes and connections corresponding to the 10 largest groups. The groups are colored according to the functional network the majority of their nodes belong to as given in Power et al. (2011). In the latent graph, the nodes \( Z_{5}, Z_{29}, Z_{30}, \) and \( Z_{32} \) are highly connected. This finding is consistent with Power et al. (2011) where the authors note that the graph of the observable variables is highly connected within a functional group. By contrast, the average graph shows completely different patterns. For instance, \( \bar{X}_{5}, \bar{X}_{29}, \bar{X}_{30}, \) and \( \bar{X}_{32} \) which belong to the same functional group are not connected in the cluster-averages graph. If one uses the cluster-averages graph to interpret the dependence structure of the functional network, the scientific results could be misleading.
Figure 1: Recovered latent graph structure between 10 largest clusters in fMRI data with FDR level $\alpha = 0.01$ colored according to their functions.

Figure 2: Recovered cluster-averages graph structure between 10 largest clusters in fMRI data with FDR level $\alpha = 0.01$. 
| K   | m   | α = 0.05 |   | α = 0.1 |   | α = 0.2 |   |
|-----|-----|-----------|---|----------|---|----------|---|
|     |     | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub |
| 80  | 12  | 1.40%     | 1.52% | 1.78% | 2.78% | 2.81% | 3.29% | 5.71% | 5.09% | 6.30% |
| 66  | 15  | 1.23%     | 1.29% | 1.31% | 2.62% | 2.39% | 2.69% | 5.08% | 4.33% | 4.84% |
| 50  | 20  | 1.05%     | 0.94% | 1.26% | 2.27% | 1.85% | 2.79% | 4.28% | 3.65% | 4.58% |
| 20  | 50  | 1.04%     | 1.03% | 1.05% | 1.81% | 1.96% | 2.20% | 3.46% | 3.26% | 4.19% |
| 80  | 12  | 1.55%     | 1.57% | 1.78% | 2.78% | 2.80% | 3.26% | 5.57% | 5.13% | 6.13% |
| 66  | 15  | 1.30%     | 1.30% | 1.24% | 2.64% | 2.35% | 2.64% | 5.07% | 4.34% | 4.89% |
| 50  | 20  | 1.09%     | 0.93% | 1.26% | 2.32% | 1.89% | 2.75% | 4.29% | 3.64% | 4.51% |
| 20  | 50  | 1.09%     | 1.01% | 1.05% | 1.91% | 1.96% | 2.14% | 3.50% | 3.28% | 4.25% |
| 80  | 12  | 10.99%    | 10.04% | 12.00% | 20.04% | 18.18% | 21.47% | 35.50% | 31.46% | 37.24% |
| 66  | 15  | 9.28%     | 8.20% | 9.14% | 16.88% | 14.94% | 16.82% | 31.53% | 27.75% | 31.81% |
| 50  | 20  | 7.73%     | 6.91% | 7.96% | 14.74% | 13.13% | 15.33% | 27.03% | 23.92% | 28.72% |
| 20  | 50  | 5.00%     | 4.85% | 6.38% | 11.01% | 9.38% | 11.60% | 21.49% | 17.00% | 23.15% |
| 80  | 12  | 10.99%    | 10.00% | 12.00% | 20.04% | 18.20% | 21.38% | 35.56% | 31.48% | 37.10% |
| 66  | 15  | 9.28%     | 8.24% | 8.99% | 17.03% | 14.87% | 16.87% | 31.49% | 27.82% | 31.63% |
| 50  | 20  | 7.60%     | 6.93% | 7.94% | 14.61% | 13.13% | 15.40% | 27.12% | 23.94% | 28.71% |
| 20  | 50  | 4.95%     | 4.91% | 6.38% | 10.97% | 9.35% | 11.60% | 21.36% | 16.94% | 23.41% |

Table 3: Averaged empirical FDR for synthetic data experiments with \(d = 1000\) and \(n = 500\).

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\[ \alpha = 0.05 \]
\[ \alpha = 0.1 \]
\[ \alpha = 0.2 \]

| K | m | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub |
|---|---|-----------|-------|-----|-----------|-------|-----|-----------|-------|-----|
| 80 | 12 | 71.42% | 99.95% | 100.00% | 77.14% | 99.97% | 100.00% | 82.29% | 99.98% | 100.00% |
| 66 | 15 | 82.98% | 99.98% | 100.00% | 86.32% | 99.99% | 100.00% | 90.02% | 100.00% | 100.00% |
| 50 | 20 | 90.45% | 99.99% | 100.00% | 93.04% | 99.99% | 100.00% | 94.86% | 100.00% | 100.00% |
| 20 | 50 | 99.95% | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% |

| K | m | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub | Scalefree | Band3 | Hub |
|---|---|-----------|-------|-----|-----------|-------|-----|-----------|-------|-----|
| 80 | 12 | 71.54% | 99.95% | 100.00% | 77.13% | 99.97% | 100.00% | 82.25% | 99.98% | 100.00% |
| 66 | 15 | 82.74% | 99.98% | 100.00% | 86.26% | 99.99% | 100.00% | 90.08% | 100.00% | 100.00% |
| 50 | 20 | 90.45% | 99.99% | 100.00% | 93.00% | 99.99% | 100.00% | 94.76% | 100.00% | 100.00% |
| 20 | 50 | 99.95% | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% | 100.00% |

Table 4: Averaged FDR power for synthetic data experiments with \( d = 1000 \) and \( n = 500 \).
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A Proofs Regarding Estimation in the Cluster-Average Graph

In this section we provide the proofs of the results needed for establishing the asymptotic normality of the estimators of the entries of $\Omega^*$. The results below make use of the fact that consistent clustering is possible, under our assumptions, as stated below.

**Lemma A.1.** Let $\mathcal{E} = \{\hat{G} = G^*\}$, for $\hat{G}$ estimated by either the COD or the PECOK algorithm of Bunea et al. (2018). Then, under Assumptions 4.1 and 4.2, we have

$$\mathbb{P}(\mathcal{E}) \geq 1 - \frac{C}{(d \vee n)^3}.$$

The conclusion of this Lemma is proved in Theorem 3, for the COD algorithm, and Theorem 4, for the PECOK algorithm, of Bunea et al. (2018).

Lemma A.1 allows us to replace $\hat{G}$ by $G^*$ in all the results below, while incurring a small error, of order $O\left(\frac{C}{(d \vee n)^3}\right)$, which will be shown to be dominated by other error bounds.

## A.1 Main Proofs for the Cluster-Average Graph Estimators

**Proof of Theorem 4.3.** The proof relies crucially on Proposition A.2, stated and proved after the end of this proof. Assuming that this result has been obtained, the proof of Theorem 4.3 follows the standard steps explained below.

Denote $\mathcal{E}' = \{\max_{1 \leq t < k \leq K} |\hat{s}_{tk}/s_{tk} - 1| \leq r\}$, where $r = C\sqrt{\frac{s_1 \log(K \vee n)}{n}}$, and $\mathcal{E}'$ is the complement of the event $\mathcal{E}'$. We first consider the bound

$$\mathbb{P}\left(\frac{n^{1/2}(\hat{\Omega}_{t,k} - \Omega^*_{t,k})}{\hat{s}_{tk}} \leq x\right) - \Phi(x) \leq \mathbb{P}\left(\frac{n^{1/2}(\hat{\Omega}_{t,k} - \Omega^*_{t,k})}{\hat{s}_{tk}} \leq x, \mathcal{E}', \mathcal{E}\right) - \Phi(x) + \mathbb{P}(\mathcal{E}') + \mathbb{P}(\mathcal{E}).$$

Proposition A.3 below implies $\mathbb{P}(\mathcal{E}') \leq C(K \vee n)^{-3}$ and Lemma A.1 above implies $\mathbb{P}(\mathcal{E}) \leq C(d \vee n)^{-3}$ for some constant $C$. In addition, for $x \geq 0$,

$$\mathbb{P}\left(\frac{n^{1/2}(\hat{\Omega}_{t,k} - \Omega^*_{t,k})}{s_{tk}} \leq x, \mathcal{E}', \mathcal{E}\right) - \Phi(x) \leq \mathbb{P}\left(\frac{n^{1/2}(\hat{\Omega}_{t,k} - \Omega^*_{t,k})}{s_{tk}} \leq x(1 + r), \mathcal{E}\right) - \Phi(x)$$

$$= \left\{\mathbb{P}\left(\frac{n^{1/2}(\hat{\Omega}_{t,k} - \Omega^*_{t,k})}{s_{tk}} \leq x(1 + r), \mathcal{E}\right) - \Phi(x(1 + r))\right\} + \left\{\Phi(x(1 + r)) - \Phi(x)\right\}. \quad (A.1)$$

For the first term, Proposition A.2 implies

$$\max_{1 \leq t < k \leq K} \sup_{x \in \mathbb{R}} \left|\mathbb{P}\left(\frac{n^{1/2}(\hat{\Omega}_{t,k} - \Omega^*_{t,k})}{s_{tk}} \leq x(1 + r), \mathcal{E}\right) - \Phi(x(1 + r))\right| \lesssim \frac{s_1 \log(K \vee n)}{n^{1/2}}.$$

By the mean value theorem, the second term $\Phi(x(1 + r)) - \Phi(x) = \phi(x(1 + tr))x r$, for some $t \in [0, 1]$. It is easily seen that $\sup_{x \in \mathbb{R}} \sup_{t \in [0, 1]} |\phi(x(1 + tr))x| \leq C$ for some constant $C$. Plugging it into
(A.1), we obtain
\[
\max_{1 \leq t < k \leq K} \sup_{x \in \mathbb{R}} \left\{ \mathbb{P}\left( \frac{n^{1/2}(\tilde{\Omega}_{t,k} - \Omega^*_t)}{s_{tk}} \leq x \frac{s_{tk}}{s_{tk}}, \mathcal{E}', \mathcal{E} \right) - \Phi(x) \right\} \lesssim \frac{s_1 \log(K \vee n)}{n^{1/2}} + r \lesssim \frac{s_1 \log(K \vee n)}{n^{1/2}}.
\]
(A.2)

When \( x < 0 \), similar to (A.1), the bound is
\[
\mathbb{P}\left( \frac{n^{1/2}(\tilde{\Omega}_{t,k} - \Omega^*_t)}{s_{tk}} \leq x \frac{s_{tk}}{s_{tk}}, \mathcal{E}', \mathcal{E} \right) - \Phi(x) \\
\leq \left\{ \mathbb{P}\left( \frac{n^{1/2}(\tilde{\Omega}_{t,k} - \Omega^*_t)}{s_{tk}} \leq x(1-r), \mathcal{E} \right) - \Phi(x(1-r)) \right\} + \left\{ \Phi(x(1-r)) - \Phi(x) \right\}.
\]
Thus (A.2) holds for \( x < 0 \) as well. Combining these results, we obtain
\[
\max_{1 \leq t < k \leq K} \sup_{x \in \mathbb{R}} \left\{ \mathbb{P}\left( \frac{n^{1/2}(\tilde{\Omega}_{t,k} - \Omega^*_t)}{s_{tk}} \leq x \right) - \Phi(x) \right\} \lesssim \frac{s_1 \log(K \vee n)}{n^{1/2}} + \frac{1}{(K \vee n)^2} + \frac{1}{(d \vee n)^2}.
\]
Following the similar argument, we can also derive
\[
\max_{1 \leq t < k \leq K} \sup_{x \in \mathbb{R}} \left\{ \Phi(x) - \mathbb{P}\left( \frac{n^{1/2}(\tilde{\Omega}_{t,k} - \Omega^*_t)}{s_{tk}} \leq x \right) \right\} \lesssim \frac{s_1 \log(K \vee n)}{n^{1/2}} + \frac{1}{(K \vee n)^2} + \frac{1}{(d \vee n)^2}.
\]
This completes the proof.

Proof. The proof is done in two steps. In Step 1, we show that intersected with the event \( \mathcal{E} \),
\[
n^{1/2}|(\tilde{\Omega}_{t,k} - \Omega^*_t)/s_{tk} + \Omega^*_t h(\Omega^*_t)/s_{tk}| \leq \frac{s_1 \log(K \vee n)}{n^{1/2}},
\]
(A.3)
with probability at least \( 1 - C/(K \vee n)^3 \) and then use Lemma A.9 to obtain the result. To prove (A.3), we decompose it as
\[
n^{1/2}|(\tilde{\Omega}_{t,k} - \Omega^*_t) + \Omega^*_t h(\Omega^*_t)| \\
= n^{1/2}|(\tilde{\Omega}_{t,k} - \Omega^*_t) - \tilde{\Omega}_{t,k} h(\Omega_k) + \Omega^*_t h(\Omega^*_t)| \\
\leq n^{1/2}|(\tilde{\Omega}_{t,k} - \Omega^*_t) - \Omega^*_t h(\Omega_k) - h(\Omega^*_t)| + n^{1/2}|\Omega^*_t h(\Omega_k) - h(\Omega^*_t)\Omega_k| + n^{1/2}|(\tilde{\Omega}_{t,k} - \Omega^*_t) h(\Omega_k)|.
\]
In the following, we study these three terms separately. Recall that \( h(\Omega_k) = v_t^T(\hat{\Omega}_k - e_k) \), and \( \hat{h}(\Omega_k) = \hat{v}_t^T(\hat{\Omega}_k - e_k) \), where \( \hat{v}_t \) is a \( K \)-dimensional vector with \( (\hat{v}_t)_t = 1 \) and \( (\hat{v}_t)_t = -w_t^* \) with \( w_t^* = (S_{t,t}^*-1)S_{t,t}^* \). Term I.1 reduces to

\[
|I.1| = n^{1/2}|(\hat{\Omega}_{t,k} - \Omega_{t,k}^*) - \Omega_{t,t}^*v_t^T\hat{\Omega}_k - \Omega_k^*| \\
\leq n^{1/2}|(\hat{\Omega}_{t,k} - \Omega_{t,k}^*)(1 - \Omega_{t,t}^*(\hat{S}_{t,t} - w_t^*\hat{S}_{t,t}))| + n^{1/2}\Omega_{t,t}^*(\hat{S}_{t,t} - w_t^*\hat{S}_{t,t})(\hat{\Omega}_{t,k} - \Omega_{t,k}^*). \tag{A.4}
\]

Note that \( 1/\Omega_{t,t}^* = S_{t,t}^* - w_t^*\hat{S}_{t,t} \). The term in (A.4) can be bounded by

\[
n^{1/2}|(\hat{\Omega}_{t,k} - \Omega_{t,k}^*)|\Omega_{t,t}^*(\hat{S}_{t,t} - S_{t,t}^*) + n^{1/2}|(\hat{\Omega}_{t,k} - \Omega_{t,k}^*)\Omega_{t,t}^*(w_t^T(\hat{S}_{t,t} - S_{t,t}^*))| \\
\leq n^{1/2}\Omega_{t,t}^*(\hat{S}_{t,t} - S_{t,t}^*)\max(\|S - S^*\|_\text{max}, \|w_t^T(\hat{S}_{t,t} - S_{t,t}^*)\|_\infty) \\
\leq C_s \log(K \vee n) \frac{n^{1/2}}{n^{1/2}}.
\]

with probability at least \( 1 - (K \vee n)^{-3} \), by \( \lambda_{\text{max}}(\Omega^*) \leq C \) and the concentration and error bound results in Lemmas A.6, A.7, A.8. The term in (A.5) can be bounded by

\[
n^{1/2}\Omega_{t,t}^*(\hat{S}_{t,t} - w_t^T\hat{S}_{t,t})(\hat{\Omega}_{t,k} - \Omega_k^*) \leq C_s \log(K \vee n) \frac{n^{1/2}}{n^{1/2}},
\]

with probability at least \( 1 - (K \vee n)^{-3} \) again by Lemmas A.7, A.8. Thus, \( |I.1| \leq \frac{s_1 \log(K \vee n)}{n^{1/2}} \) with probability at least \( 1 - (K \vee n)^{-3} \).

For term I.2, we have

\[
|I.2| = n^{1/2}\Omega_{t,t}^*|v_t^* - v^*(\hat{\Omega}_k - e_k)| \\
\leq n^{1/2}\Omega_{t,t}^*\|v_t^* - v^*\|_1\|\hat{\Omega}_k - e_k\|_\infty \leq \frac{C_s \log(K \vee n)}{n^{1/2}},
\]

with probability at least \( 1 - (K \vee n)^{-3} \) by Lemma A.8 and the constraint of the CLIME-type estimator.

To control term I.3, first we observe that

\[
|\hat{h}(\hat{\Omega}_k)| = |v_t^*\hat{\Omega}_k - e_k| \\
\leq |v_t^*\hat{\Omega}_k - e_k| + |v_t^*\hat{\Omega}_k - \Omega_k^*| + |v_t^* - v^*(\hat{\Omega}_k - e_k)| \\
\leq \|v_t^*\|\|\hat{\Omega}_k - e_k\|_\infty + \|\hat{\Omega}_{t,t} - w_t^*\hat{S}_{t,t}\|_\infty\|\hat{\Omega}_k - \Omega_k^*\|_1 + \|\hat{\Omega}_k - e_k\|_\infty \|\hat{\Omega}_k - e_k\|_\infty.
\]

As shown in the proof of Lemma A.9, \( \|v_t^*\|_1 \leq s_1^{1/2} \|v_t^*\|_2 \leq C s_1^{1/2} \). The rest of the bounds on the above terms follows easily from Lemmas A.7, A.8. Thus, we have \( |\hat{h}(\hat{\Omega}_k)| \leq C(s_1 \log(K \vee n)/n)^{1/2} \) with high probability. Since

\[
|\hat{\Omega}_{t,t} - \Omega_{t,t}^*| \leq C(s_1 \log(K \vee n)/n)^{1/2},
\]

by Lemma A.8, we obtain that \( |I.3| \leq \frac{s_1 \log(K \vee n)}{n^{1/2}} \) with probability at least \( 1 - (K \vee n)^{-3} \). It is easily seen that \( \Omega_{t,t}^* \geq \frac{s_1}{s_{t,t}} \geq C > 0 \), see Remark A.4. This implies that \( s_{t,t}^2 = \Omega_{t,t}^* + \Omega_{t,t}^2 \) is lower bounded by a positive constant. The proof of (A.3) is complete.
In step 2, we need to verify that

$$\max_{1 \leq t < k \leq K} \sup_{x \in \mathbb{R}} \left| P \left( \frac{n^{1/2} \Omega_{t,k}^* h(\Omega_{t,k}^*)}{s_{t,k}} \leq x \right) - \Phi(x) \right| \leq \frac{C}{(K \vee n)^3} + \frac{1}{n^{1/2}},$$

which has been done in Lemma A.9. Thus, combining with result (A.3), we can use the same simple union bound in the proof of Theorem 4.3 to obtain the desired result. \qed

**Proposition A.3** (Convergence Rate of the Variance Estimator). Under the same conditions as in Theorem 4.3, we have

$$\max_{1 \leq t < k \leq K} \left| s_{t,k}^2 - s_{t,k}^* \right| \leq C \sqrt{\frac{s_1 \log(K \vee n)}{n}}, \quad \max_{1 \leq t < k \leq K} \left| \frac{s_{t,k}}{s_{t,k}} - 1 \right| \leq C \sqrt{\frac{s_1 \log(K \vee n)}{n}},$$

with probability at least $1 - (K \vee n)^{-3}$.

**Proof.** By Lemma A.8, under the event $\hat{G} = G^*$, we have

$$\max_{1 \leq t < k \leq K} \left| \hat{\Omega}_{t,k} - \Omega_{t,k}^* \right| \leq \max_{1 \leq k \leq K} \| \hat{\Omega}_{k} - \Omega_{k}^* \|_2 \leq C_1 \sqrt{\frac{s_1 \log(K \vee n)}{n}},$$

with probability at least $1 - \frac{C_2}{(K \vee n)^3}$. Under this event,

$$\max_{1 \leq t < k \leq K} \left| s_{t,k}^2 - s_{t,k}^* \right| = \max_{1 \leq t < k \leq K} \left| \hat{\Omega}_{t,k}^2 + \hat{\Omega}_{t,t} \hat{\Omega}_{k,k} - (\Omega_{t,k}^* + \Omega_{t,t}^* \Omega_{k,k}^*) \right|
\leq \max_{1 \leq t < k \leq K} \left| (\hat{\Omega}_{t,k} - \Omega_{t,k}^*) (\hat{\Omega}_{t,k} + \Omega_{t,k}^*) + \hat{\Omega}_{t,t} \hat{\Omega}_{k,k} - \Omega_{k,k}^* \right|
\leq C_1 \sqrt{\frac{s_1 \log(K \vee n)}{n}} (4 \| \Omega^* \|_{\max} + \delta) \leq C \sqrt{\frac{s_1 \log(K \vee n)}{n}},$$

for some constant $\delta > 0$ since $\| \Omega^* \|_{\max} \leq \lambda_{\max}(\Omega^*) \leq C$. It is easily seen that $\Omega_{t,t}^* \geq \frac{1}{s_{t,t}^*} \geq C > 0$, see Remark A.4. This implies that $s_{t,k}^2 = \Omega_{t,t}^* \Omega_{k,k}^* + \Omega_{t,k}^* \Omega_{k,k}^*$ is lower bounded by a positive constant. Thus,

$$\max_{1 \leq t < k \leq K} \left| \frac{s_{t,k}}{s_{t,k}} - 1 \right| \leq \max_{1 \leq t < k \leq K} \left| \frac{s_{t,k}^2 - s_{t,k}^*}{s_{t,k}(s_{t,k} + s_{t,k})} \right| \leq C \sqrt{\frac{s_1 \log(K \vee n)}{n}}.$$

\qed

### A.2 Key Lemmas for Estimators of the Cluster-Average Graph

**Remark A.4.** While Assumptions 4.1 and 4.2 are made for $C^*$, they do imply that $c_1 \leq \lambda_{\min}(S^*)$ and $\max_i S_{i,i}^* \leq c_2 + c_3$ holds for $S^*$. Furthermore Lemma D.3 implies the same restricted eigenvalue condition on $\lambda_{\min}(S^*)$ as on $C^*$.

**Remark A.5.** In the following proof, we always assume the event $E = \{ \hat{G} = G^* \}$ holds. Using a similar argument to that used in the proof of Theorem 4.3, the following bounds will hold with probability at least $1 - \frac{C}{(K \vee n)^3}$. 

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Lemma A.6 (Consistency of \(\hat{S}\)). If Assumptions 4.1 and 4.2 hold, then with probability greater than \(1 - \frac{C}{(K \land n)^3}\),
\[
\|\hat{S} - S^*\|_{\text{max}} \leq C \sqrt{\frac{\log(K \land n)}{n}}
\]
for some constant \(C\) dependent only on \(c_1, c_2, \) and \(c_3\) from Assumptions 4.1 and 4.2.

Proof of Lemma A.6. The proof follows from the proof of Theorem 1 in Cai et al. (2011).

Lemma A.7 (Concentration of Gradient and Hessian). If Assumptions 4.1 and 4.2 hold, then with probability greater than \(1 - \frac{C}{(K \land n)^3}\), we have that

\[
\begin{align*}
(a) \quad & \max_{1 \leq k \leq K} \|\hat{S}\Omega^*_k - e_k\|_{\infty} \leq C_1 \sqrt{\frac{\log(K \land n)}{n}}, \\
(b) \quad & \max_{1 \leq k \leq K} \|\hat{S}_{t,-t} - w_t^T \hat{S}_{t,-t}\|_{\infty} \leq C_2 \sqrt{\frac{\log(K \land n)}{n}}, \\
(c) \quad & \max_{1 \leq k \leq K} \|w_t^T (\hat{S}_{t,-t} - S^*_{t,-t})\|_{\infty} \leq C_2 \sqrt{\frac{\log(K \land n)}{n}},
\end{align*}
\]

for some constants \(C_1\) and \(C_2\) dependent only on \(c_1, c_2, \) and \(c_3\) from Assumptions 4.1 and 4.2.

Proof of Lemma A.7. We start from the decomposition
\[
\hat{S} - S^* = (A^{*T}A^*)^{-1} A^* T (\hat{\Sigma} - \Sigma^*) A^* (A^{*T}A^*)^{-1}.
\]

Denoting by \(B^* = A^{*T}A^*\), we can write
\[
(A^{*T}A^*)^{-1} A^* T (\hat{\Sigma} - \Sigma^*) A^* (A^{*T}A^*)^{-1} = \frac{1}{n} \sum_{i=1}^{n} B^* \left\{ (A^* Z_i + E_i)(A^* Z_i + E_i)^T - A^* C^* A^{*T} - \Gamma^* \right\} A^* B^*^{-1}
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} Z_i Z_i^T - C^* + \frac{1}{n} \sum_{i=1}^{n} Z_i E_i^T A^* B^*^{-1} + \frac{1}{n} \sum_{i=1}^{n} B^* A^{*T} E_i Z_i^T \\
+ \frac{1}{n} \sum_{i=1}^{n} B^* A^{*T} E_i E_i^T - \Gamma^* A^* B^*^{-1}.
\]

Note that \(\hat{S}\Omega^*_k - e_k = (\hat{S} - S^*)\Omega^*_k\), and \(\|\Omega^*_k\|_2 \leq \lambda_{\text{max}} (\Omega^*) \leq \lambda_{\text{min}} (S^*)^{-1}\). As seen in Remark A.4, we can bound the smallest eigenvalue of \(S^*\) from below by \(\lambda_{\text{min}} (C^*)\). We therefore obtain that \(\|\Omega^*_k\|_2 \leq c_1 + c_3\). By using the triangle inequality, we can apply Lemma C.1, Lemma C.2 and Lemma C.3 to bound (A.6). Combining these results, we obtain
\[
\max_{1 \leq k \leq K} \|\hat{S}\Omega^*_k - e_k\|_{\infty} \leq C_1 \sqrt{\frac{\log(K \land n)}{n}}
\]
with probability at least \(1 - \frac{C}{(K \land n)^3}\), concluding the proof of claim (a).
For the remaining two claims, we can rewrite
\[
\mathbf{w}_t^* = (S^*_{t,t} - S^*_{t,t}^T (S^*_{t,t} - S^*_{t,t})^{-1} S^*_{t,t}) \mathbf{\Omega}^*_{t,t,t} = \frac{1}{\Omega^*_{t,t,t}} \mathbf{\Omega}^*_{t,t,t}
\]
by the block matrix inverse formula. Using Lemma D.4, it follows that \(||\mathbf{w}_t^*||_2 \leq \lambda_{\min} (\mathbf{\Omega}^*) m_{\max} S^*_{t,t}.\)
Then we can see that
\[
\max_{1 \leq k \leq K} ||\mathbf{\hat{S}}_{t,t} - \mathbf{w}_t^* \mathbf{\hat{S}}_{t,t, t}||_\infty = \max_{1 \leq k \leq K} ||(\mathbf{\hat{S}}_{t,t} - S^*_{t,t}) - \mathbf{w}_t^* (\mathbf{\hat{S}}_{t,t} - S^*_{t,t})||_\infty
\leq \max_{1 \leq k \leq K} ||(\mathbf{\hat{S}}_{t,t} - S^*_{t,t})|| + \max_{i \neq t} |\mathbf{w}_t^* (\mathbf{\hat{S}}_{t,i,t} - S^*_{t,i,t})|.
\]

By using Lemma A.6, (i) is bounded with high probability. Similar to the the proof of Theorem 1 in Cai et al. (2011), we can show that (ii) \( \leq C_2 \sqrt{\frac{\log(K \vee n)}{n}} \), with probability at least \(1 - \frac{C}{(K \vee n)^2} \).
Part (c) is the same as the term (ii), concluding the proof.

**Lemma A.8** (Consistency of Initial Estimator). If Assumptions 4.1 and 4.2 hold, then,
\[
\begin{align*}
(a) \ & \max_{1 \leq k \leq K} ||\mathbf{\hat{\Omega}}_k - \mathbf{\Omega}^*_{k,t}||_1 \leq C_1 s_1 \sqrt{\frac{\log(K \vee n)}{n}}, \quad \max_{1 \leq k \leq K} ||\mathbf{\hat{\Omega}}_k - \mathbf{\Omega}^*_{k,t}||_2 \leq C_1 \sqrt{\frac{s_1 \log(K \vee n)}{n}} , \\
(b) \ & \max_{1 \leq k \leq K} ||\mathbf{\hat{v}}_t - \mathbf{v}_t^*||_1 \leq C_2 s_1 \sqrt{\frac{\log(K \vee n)}{n}}, \quad \text{and} \\
(c) \ & \max_{1 \leq k \leq K} |(\mathbf{\hat{\Omega}}_k - \mathbf{\Omega}^*_{k,t})^T \mathbf{\hat{S}} (\mathbf{\hat{\Omega}}_k - \mathbf{\Omega}^*_{k,t})| \leq C_3 \frac{s_1 \log(K \vee n)}{n},
\end{align*}
\]
with probability at least \(1 - \frac{C_c}{(K \vee n)^2} \). \(C_1, C_2, C_3, C_4 \) and \(C_4 \) are constants, dependent only upon \(c_0, c_1, c_2, \) and \(c_3 \) from Assumptions 4.1 and 4.2.

**Proof of Lemma A.8.** The proof follows from the same argument as in the proof of Lemma B.5 with Lemma B.4 replaced by Lemma A.7.

**Lemma A.9** (CLT for the Pseudo-Score Function). Recall that \(s^2_{t,k} = \mathbf{\Omega}^*_{t,k,t} + \mathbf{\Omega}^*_{t,k,t} \mathbf{\Omega}^*_{t,k,t} \). Let \(F_n\) denote the CDF of \(n^{1/2} \mathbf{v}_t^T (\mathbf{\hat{\Theta}}_k - e_k)/(s_{t,k} / \mathbf{\Theta}_{t,k}^*)\). If Assumptions 4.1 and 4.2 hold, then
\[
\max_{1 \leq t < k \leq K} \sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| \leq C(n^{-1/2} + (d \vee n)^{-3}),
\]
where \(C \) is a constant dependent only upon \(c_0, c_1, \) and \(c_2.\)

**Proof of Lemma A.9.** We have the similar bound that \(F_n(x) - \Phi(x) \leq \mathbf{\bar{F}}_n(x) - \Phi(x) + \mathbb{P}(\mathcal{E})\), where \(\mathcal{E}\) is the event \(\mathbf{\bar{G}} = G^*\), and \(\mathbf{\bar{F}}_n(x)\) is the CDF of \(n^{1/2} \sum_{i=1}^n \mathbf{v}_t^T (\mathbf{\bar{X}}_i \mathbf{\bar{X}}^T_i \mathbf{\bar{\Theta}}_k^* - e_k)/(s_{t,k} / \mathbf{\Theta}_{t,k}^*)\). Lemma A.1 implies that \(\mathbb{P}(\mathcal{E}) \leq (d \vee n)^{-3}\). We can similarly verify the Lyapunov condition to control \(\mathbf{\bar{F}}_n(x) - \Phi(x)\). Finally, \(\mathbb{E}[\mathbf{v}_t^T (\mathbf{\bar{X}}_i \mathbf{\bar{X}}^T_i \mathbf{\bar{\Theta}}_k^* - e_k)^2] = (s_{t,k} / \mathbf{\Theta}_{t,k}^*)^2\) by applying the Isserlis’ theorem \(\text{Var}(\mathbf{v}_1^T \mathbf{\bar{X}} \mathbf{\bar{X}}^T \mathbf{v}_2) = (\mathbf{v}_1^T \mathbf{\bar{S}}^* \mathbf{v}_1)(\mathbf{v}_2^T \mathbf{\bar{S}}^* \mathbf{v}_2) + (\mathbf{v}_1^T \mathbf{\bar{S}}^* \mathbf{v}_2)^2\), for any vector \(\mathbf{v}_1\) and \(\mathbf{v}_2.\)
Proof of Proposition 4.5. The proof follows verbatim that of Theorem 8.5 in Giraud (2014), with the exception of the fact that exact \( p \)-values are replaced by approximate \( p \)-values, including the rate of approximation. We included the full proof for the convenience of the reader. By the definition of the FDR,

\[
\text{FDR}(\hat{\tau}) = \mathbb{E}\left[ \frac{\sum_{(t,k) \in \mathcal{H}_0} \mathbb{I}[|\hat{W}_{t,k}| > \hat{\tau}][R_{\hat{\tau}} > 0]}{R_{\hat{\tau}}} \right] = \sum_{(t,k) \in \mathcal{H}_0} \mathbb{E}\left[ \frac{\mathbb{I}[|\hat{W}_{t,k}| > \hat{\tau}][R_{\hat{\tau}} > 0]}{R_{\hat{\tau}}} \right]. \tag{A.7}
\]

To handle the \( R_{\hat{\tau}} \) in the denominator, we use the identity \( 1 = \sum_{i=R_{\hat{\tau}}}^{\infty} \frac{R_{\hat{\tau}}}{i(i+1)} \). This implies

\[
\frac{1}{R_{\hat{\tau}}} = \sum_{i=R_{\hat{\tau}}}^{\infty} \frac{1}{i(i+1)} = \sum_{i=1}^{\infty} \frac{\mathbb{I}[i \geq R_{\hat{\tau}}]}{i(i+1)}.
\]

Plugging this into (A.7) and bringing the expectation inside the summation gives that

\[
\text{FDR}(\hat{\tau}) = \sum_{(t,k) \in \mathcal{H}_0} \sum_{i=1}^{\infty} \frac{1}{i(i+1)} \mathbb{E}\left[ \mathbb{I}[|\hat{W}_{t,k}| > \hat{\tau}][R_{\hat{\tau}} > 0][i \geq R_{\hat{\tau}}] \right] \leq \sum_{(t,k) \in \mathcal{H}_0} \sum_{i=1}^{\infty} \frac{1}{i(i+1)} \mathbb{E}\left[ |\hat{W}_{t,k}| > \Phi^{-1}\left(1 - \frac{\alpha R_{\hat{\tau}}}{2N_{BY}|\mathcal{H}|} \right) \mathbb{I}[R_{\hat{\tau}} > 0][i \geq R_{\hat{\tau}}] \right]
\]

\[
\leq \sum_{(t,k) \in \mathcal{H}_0} \sum_{i=1}^{\infty} \frac{1}{i(i+1)} \mathbb{E}\left[ |\hat{W}_{t,k}| > \Phi^{-1}\left(1 - \frac{\alpha(i \wedge |\mathcal{H}|)}{2N_{BY}|\mathcal{H}|} \right) \mathbb{I}[R_{\hat{\tau}} > 0][i \geq R_{\hat{\tau}}] \right], \tag{A.8}
\]

where the second line follows from the definition of the FDR cutoff and the last inequality holds since \( R_{\hat{\tau}} \leq (i \wedge |\mathcal{H}|) \). The Berry-Esseen bound in Theorem 4.3 implies that

\[
\mathbb{P}(|\hat{W}_{t,k}| > \Phi^{-1}\left(1 - \frac{\alpha(i \wedge |\mathcal{H}|)}{2N_{BY}|\mathcal{H}|} \right)) \leq \frac{\alpha(i \wedge |\mathcal{H}|)}{N_{BY}|\mathcal{H}|} + 2b_n.
\]

Thus, it follows that

\[
\text{FDR}(\hat{\tau}) \leq \sum_{(t,k) \in \mathcal{H}_0} \sum_{i=1}^{\infty} \frac{1}{i(i+1)} \left( \frac{\alpha(i \wedge |\mathcal{H}|)}{N_{BY}|\mathcal{H}|} + 2b_n \right) = \alpha \frac{\mathcal{H}_0}{|\mathcal{H}|} \left( \sum_{i=1}^{\mathcal{H}_0} \frac{i}{i(i+1)N_{BY}} + \sum_{i=\mathcal{H}_0+1}^{\infty} \frac{|\mathcal{H}|}{i(i+1)N_{BY}} \right) + 2|\mathcal{H}_0|b_n
\]

\[
= \alpha \frac{\mathcal{H}_0}{|\mathcal{H}|} \left( \sum_{i=1}^{\mathcal{H}_0} \frac{1}{i+1} \frac{1}{N_{BY}} + \frac{|\mathcal{H}|}{|\mathcal{H}|+1} \frac{1}{N_{BY}} \right) + 2|\mathcal{H}_0|b_n
\]

\[
\leq \alpha + 2|\mathcal{H}_0|b_n,
\]

where the last step follows from \( \frac{|\mathcal{H}_0|}{|\mathcal{H}|} \leq 1 \) and the definition of \( N_{BY} \). This completes the proof.  \( \square \)
B Proofs Regarding Estimation in the Latent Variable Graph

This section contains the proofs that establish the asymptotic normality of the estimator of $\Theta^*$.

B.1 Main Proofs for Estimators of the Latent Variable Graph

Proof of Theorem 4.4. The proof follows exactly the line of the proof of Theorem 4.3, but invokes Proposition B.2, instead of Proposition A.3, and Proposition B.1, instead of Proposition A.2, as one needs to establish different intermediate results, specifically tailored to estimation of the latent graph.

Proposition B.1 (Asymptotic Normality of $\tilde{\Theta}_{t,k}$). Under the same conditions as in Theorem 4.4, we get

$$\max_{1 \leq t < k \leq K} \sup_{x \in \mathbb{R}} \left| \mathbb{P}\left( \frac{n^{1/2}(\tilde{\Theta}_{t,k} - \Theta^*_{t,k})}{\sigma_{t,k}} < x, \mathcal{E} \right) - \Phi(x) \right| \leq \frac{C}{(K \lor n)^3} + \frac{Cs_0 \log(K \lor n)}{n^{1/2}}.$$ 

Proof. The proof follows all the steps of Proposition A.2, with Lemma A.6 replaced by Lemma B.3, A.7 by B.4, A.8 by B.5 and A.9 by B.6.

Proposition B.2 (Convergence Rate of the Variance Estimator). Under the same conditions as in Theorem 4.4, we get

$$\max_{1 \leq t < k \leq K} |\sigma_{t,k}^2 - \hat{\sigma}_{t,k}^2| \leq C \sqrt{\frac{s_0 \log(K \lor n)}{n}} + \frac{Cs_0}{m}, \quad \max_{1 \leq t < k \leq K} |\hat{\sigma}_{t,k} - 1| \leq C \sqrt{\frac{s_0 \log(K \lor n)}{n}} + \frac{Cs_0}{m},$$

with probability at least $1 - (K \lor n)^{-3}$.

Proof. Similar to the proof of Proposition A.3, we can prove that

$$\max_{1 \leq t < k \leq K} |\hat{\sigma}_{t,k}^2 - (\Theta_{t,k}^* + \Theta_{t,t}^* \Theta_{k,k}^*)| \leq C \sqrt{\frac{s_0 \log(K \lor n)}{n}},$$

with probability at least $1 - (K \lor n)^{-3}$. Then, by Lemma B.7, we obtain

$$\max_{1 \leq t < k \leq K} |\hat{\sigma}_{t,k}^2 - \sigma_{t,k}^2| \leq C \sqrt{\frac{s_0 \log(K \lor n)}{n}} + \frac{Cs_0}{m}.$$ 

The second statement can be similarly derived.

B.2 Key Lemmas for Estimators of the Latent Graph

Lemma B.3 (Consistency of $\hat{C}$). If Assumptions 4.1 and 4.2 hold, then with probability greater than $1 - \frac{C}{(K \lor n)^3}$,

$$\|\hat{C} - C^*\|_{\max} \leq C \sqrt{\frac{\log(K \lor n)}{n}},$$

for some constant $C$ dependent only on $c_1$, $c_2$, and $c_3$ from Assumptions 4.1 and 4.2.
Proof of Lemma B.3. Let $\mathcal{E}$ denote the event that $\hat{G} = G^*$. Under the event $\mathcal{E}$, we have
\[
\hat{C} - C^* = (A^{*T}A^*)^{-1}A^{*T}(\hat{\Sigma} - \Sigma^*)A^*(A^{*T}A^*)^{-1} - (A^{*T}A^*)^{-1}A^{*T}(\hat{\Gamma} - \Gamma^*)A^*(A^{*T}A^*)^{-1}. \tag{B.1}
\]
We can write, for $i \in G_k^*$,
\[
(\hat{\Gamma} - \Gamma^*)_{i,i} = \hat{\gamma}_i - \gamma_i^* = \hat{\Sigma}_{i,i} - \frac{1}{|G_k^*| - 1} \sum_{j \in G_k^* \setminus i} \hat{\Sigma}_{i,j} - \gamma_i^* = \hat{\Sigma}_{i,i} - \frac{1}{|G_k^*| - 1} \sum_{j \in G_k^* \setminus i} \hat{\Sigma}_{i,j} - \Sigma_{i,i}^* + \frac{1}{|G_k^*| - 1} \sum_{j \in G_k^* \setminus i} \Sigma_{i,j}^* = \hat{\Sigma}_{i,i} - \Sigma_{i,i}^* - \frac{1}{|G_k^*| - 1} \sum_{j \in G_k^* \setminus i} \left[ \hat{\Sigma}_{i,j} - \Sigma_{i,j}^* \right],
\]
which implies that $||\hat{\Sigma} - \Sigma^*||_{\text{max}} \leq 2||\hat{\Sigma} - \Sigma^*||_{\text{max}}$. Therefore from Lemma D.1 we see that
\[
||(A^{*T}A^*)^{-1}A^{*T}(\hat{\Sigma} - \Sigma^*)A^*(A^{*T}A^*)^{-1}||_{\text{max}} \leq \frac{2}{m}||(A^{*T}A^*)^{-1}A^{*T}(\hat{\Sigma} - \Sigma^*)A^*(A^{*T}A^*)^{-1}||_{\text{max}},
\]
demonstrating that it will suffice to bound the first term in (B.1), which we now do.

Let $B^* = A^{*T}A^*$. For the first term in (B.1), we have
\[
(A^{*T}A^*)^{-1}A^{*T}(\hat{\Sigma} - \Sigma^*)A^*(A^{*T}A^*)^{-1} = \frac{1}{n} \sum_{i=1}^{n} B^{*\text{-}1}A^{*T} \left\{ \left( A^*Z_i + E_i \right) (A^*Z_i + E_i)^T - A^*C^*A^{*T} - \Gamma^* \right\} A^*B^{*\text{-}1}
\]
\[
= \frac{1}{n} \sum_{i=1}^{n} Z_iZ_i^T - C^* + \frac{1}{n} \sum_{i=1}^{n} Z_iE_i^T A^{*\text{-}1} A^{*T} E_i Z_i^T + \frac{1}{n} \sum_{i=1}^{n} B^{*\text{-}1}A^{*T} E_i E_i^T A^{*\text{-}1} A^{*T} E_i Z_i^T
\]
\[
+ \frac{1}{n} \sum_{i=1}^{n} B^{*\text{-}1}A^{*T} (E_i E_i^T - \Gamma^*) A^*B^{*\text{-}1}. \tag{B.2}
\]
Using the triangle inequality, we can apply Lemma C.1, Lemma C.2 and Lemma C.3 to bound B.2. Combining these results with that $\mathbb{P}(\mathcal{E}) \geq 1 - c_0/(d \lor n)^3$, we obtain
\[
||\hat{C} - C^*||_{\text{max}} \leq C \sqrt{\frac{\log(K \lor n)}{n}}
\]
with probability at least $1 - \frac{C}{(K \lor n)^3}$ for some constant $C$ dependent only on $c_1$, $c_2$, and $c_3$ from Assumptions 4.1 and 4.2.

Lemma B.4 (Concentration of Gradient and Hessian of the Loss Function). If Assumptions 4.1 and 4.2 hold, then with probability greater than $1 - \frac{C}{(K \lor n)^3}$, we have that
\[
(a) \max_{1 \leq k \leq K} ||\hat{C} \Theta_k^* - e_k||_\infty \leq C_1 \sqrt{\frac{\log(K \lor n)}{n}},
\]
(b) \( \max_{1 \leq k \leq K} \| \tilde{C}_{t,-t} - w_t^T \tilde{C}_{t,-t} \|_\infty \leq C_2 \sqrt{\frac{\log(K \sqrt{n})}{n}} \), and

(c) \( \max_{1 \leq k \leq K} \| w_t^T (\tilde{C}_{t,-t} - C_{t,-t}^*) \|_\infty \leq C_2 \sqrt{\frac{\log(K \sqrt{n})}{n}} \),

for absolute constants \( C_1 \) and \( C_2 \) dependent only on \( c_1, c_2, \) and \( c_3 \) from Assumptions 4.1 and 4.2.

Proof of Lemma B.4. Let \( \mathcal{E} \) denote the event that \( \tilde{G} = G^* \). Note that \( \tilde{C} \Theta^*_k - e_k = (\tilde{C} - C^*) \Theta^*_k \). Under \( \mathcal{E} \), the decomposition (B.2), we can similarly show that

\[
(A^* T A^*)^{-1} A^* T (\hat{\Sigma} - \Sigma^*) A^* (A^* T A^*)^{-1} \Theta^*_k = 1 \sum_{i=1}^n Z_i Z_i^T \Theta^*_k - C^* \Theta^*_k + 1 \sum_{i=1}^n Z_i E_i^T A^* B_{t,i}^{-1} \Theta^*_k + 1 \sum_{i=1}^n B_{t,i}^{-1} A^* T E_i Z_i^T \Theta^*_k + 1 \sum_{i=1}^n B_{t,i}^{-1} A^* T (E_i E_i^T - \Gamma^*) A^* B_{t,i}^{-1} \Theta^*_k. 
\]

(B.3)

As in the proof of Lemma B.3, we have that

\[
||(A^* T A^*)^{-1} A^* T (\hat{\Sigma} - \Sigma^*) A^* (A^* T A^*)^{-1} \Theta^*_k||_\infty \leq \frac{2}{m}||(A^* T A^*)^{-1} A^* T (\hat{\Sigma} - \Sigma^*) A^* (A^* T A^*)^{-1} \Theta^*_k||_\infty,
\]

demonstrating that again it will suffice to bound the first term in B.3.

Note that \( ||\Theta^*_k||_2 \leq \lambda_{\max}(\Theta^*) \leq c_1^{-1} \). Therefore, by using the triangle inequality, we can apply Lemma C.1, Lemma C.2 and Lemma C.3 to bound the first term in B.3. Combining these results and \( P(\mathcal{E}) \geq 1 - C/(d \sqrt{n})^3 \), we obtain

\[
\max_{1 \leq k \leq K} \| \tilde{C} \Theta^*_k - e_k \|_\infty \leq C_1 \sqrt{\frac{\log(K \sqrt{n})}{n}},
\]

with probability at least \( 1 - C/(K \sqrt{n})^3 \) for some constant \( C_1 \) dependent only on \( c_1, c_2, \) and \( c_3 \) from Assumptions 4.1 and 4.2.

For the remaining two claims, we can rewrite

\[
0 = (C_{t,t}^* - C_{t,-t}^* (C_{t,-t}^*)^{-1} C_{t,-t}^*) \Theta^*_{t,-t} = \frac{1}{\Theta^*_{t,t}} \Theta^*_{t,-t},
\]

by the block matrix inverse formula. Using Lemma D.4, it follows that \( ||w^*_t||_2 \leq \lambda_{\max}(\Theta^*) \max_t C_{t,t}^* \). Then we see that

\[
\max_{1 \leq k \leq K} ||\tilde{C}_{t,-t} - w^*_t T \tilde{C}_{t,-t}||_1 = \max_{1 \leq k \leq K} ||(\tilde{C}_{t,-t} - C_{t,-t}^*) - w^*_t T (\tilde{C}_{t,-t} - C_{t,-t}^*)||_1 \leq \max_{1 \leq k \leq K} ||(\tilde{C}_{t,-t} - C_{t,-t}^*)|| + \max_{i \neq t} ||w^*_t T (\tilde{C}_{t,-t} - C_{t,-t}^*)||
\]

Clearly, using Lemma B.3, (i) is bounded with high probability. Likewise, Lemma B.3 demonstrates that \( \tilde{C}_{t,-i} - C_{t,-i}^* \) is a sub-exponential random vector with parameters dependent only on \( c_1, c_2, \)
and $c_3$ from Assumptions 4.1 and 4.2. Thus $w_t^T(\hat{C}_{-t,i} - C^*_{-t,i})$ is sub-exponential and because $||w_t^*||_2 \leq \lambda_{\text{max}}(\hat{\Theta}) \max_t C^*_{t,t}$, we obtain that

$$\max_{1 \leq k \leq K} \| \hat{C}_{t,-t} - w_t^T \hat{C}_{t,-t} \|_\infty \leq C_2 \sqrt{\frac{\log(K \vee n)}{n}}$$

with probability at least $1 - \frac{C}{(K \vee n)^q}$ for some constant $C_2$. $C_2$ is dependent only on $c_1$, $c_2$, and $c_3$ from Assumptions 4.1 and 4.2. The final result is bounded by the previous one, concluding the proof.

Lemma B.5 (Consistency of Initial Estimators). If Assumptions 4.1 and 4.2 hold, then

(a) $\max_{1 \leq k \leq K} \| \hat{\Theta}_{t,k} - \Theta^*_{t,k} \|_1 \leq C_1 s_0 \sqrt{\frac{\log(K \vee n)}{n}}$, $\max_{1 \leq k \leq K} \| \hat{\Theta}_{t,k} - \Theta^*_{t,k} \|_2 \leq C_1 \sqrt{\frac{s_0 \log(K \vee n)}{n}}$,

(b) $\max_{1 \leq k \leq K} \| \hat{v}_t - v^*_t \|_1 \leq C_2 s_0 \sqrt{\frac{\log(K \vee n)}{n}}$, and

(c) $\max_{1 \leq k \leq K} \| (\hat{v}_t - v^*_t)^T \hat{C}(\hat{\Theta}_{t,k} - \Theta^*_{t,k}) \| \leq C_3 \sqrt{\frac{s_0 \log(K \vee n)}{n}}$,

with probability at least $1 - \frac{C_4}{(K \vee n)^q}$. $C_1$, $C_2$, $C_3$, $C_4$ are constants, dependent only upon $c_0$, $c_1$, $c_2$, and $c_3$ from Assumptions 4.1 and 4.2.

Proof of Lemma B.5. Below, the constants $C_a$, $C'_a$, $C_b$, $C'_b$, $C_c$ and $C'_c$ will depend only upon $c_0$, $c_1$, $c_2$, and $c_3$ from Assumptions 4.1 and 4.2.

We first prove part (b). The proof of part (a) is similar. Let $\hat{\Delta} = \hat{w}_t - w^*_t$, noting that we can consider $w_t$ instead of $v_t$ as the $t^{th}$ entries in both the estimated and true value are 1. By $S$ denote the support of $w^*_t$. $w^*_t$ is $s_0$-sparse because $w^*_t$ is a multiple of $\Theta^*_{-t,k}$, which we know to be $s_0$-sparse.

By Lemma B.4, there exists $C_b$ such that for $\lambda \geq C_b \sqrt{\frac{\log(K \vee n)}{n}}$, $w^*_t$ is feasible for (3.17) with probability at least $1 - \frac{C_4}{(K \vee n)^q}$. Assuming $w^*_t$ is feasible, then it follows by definition that $||w^*_t||_S \geq ||\hat{w}_t||_S + ||\hat{w}_t||_S - 1$. This in turn implies by the triangle inequality that $||\hat{\Delta}_S||_1 \geq ||\hat{\Delta}_S||_1$. Letting $\lambda = C_b \sqrt{\frac{\log(K \vee n)}{n}}$, it follows from the triangle inequality that

$$||\hat{C}_{-t,-t}\hat{\Delta}||_\infty \leq ||\hat{w}_t^T \hat{C}_{-t,-t} - C_{t,-t}||_\infty + ||w^*_t T \hat{C}_{-t,-t} - C_{t,-t}||_\infty \leq 2C_b \sqrt{\frac{\log(K \vee n)}{n}}.$$ 

In addition, note that $||\hat{\Delta}||_1 \leq 2||\hat{\Delta}_S||_1 \leq 2\sqrt{s_0}||\hat{\Delta}_S||_2 \leq 2\sqrt{s_0}||\hat{\Delta}||_2$. Therefore combining with the above, this gives

$$\hat{\Delta}^T \hat{C}_{-t,-t}\Delta \leq ||\hat{\Delta}||_1||\hat{\Delta}_{-t}||_\infty \leq 2C \sqrt{\frac{\log(K \vee n)}{n}} ||\hat{\Delta}||_1 \leq 4C_b \sqrt{\frac{s_0 \log(K \vee n)}{n}} ||\hat{\Delta}||_2.$$ 

From Lemma D.3, $\hat{\Delta}^T \hat{C}_{-t,-t}\Delta \geq \frac{4C_1}{3} ||\hat{\Delta}||_2^2$ with probability at least $1 - \frac{C''_4}{(K \vee n)^q}$. Therefore

$$||\hat{\Delta}||_2 \leq \frac{16C_1}{3} \sqrt{\frac{s_0 \log(K \vee n)}{n}} \quad \text{and} \quad ||\hat{\Delta}||_1 \leq \frac{32C_1 s_0 C_1}{3} \sqrt{\frac{\log(K \vee n)}{n}},$$
with \(\lambda\) quantities are bounded above by constants per Assumptions 4.1 and 4.2. Thus, \(\max v\)

Using Lemma D.4, it follows that from Lemmas C.1 - C.3 we see that the entries in \(Q\) are bounded by \(C\). Let \(\tilde{\Theta}_k\) defined in (3.25). Let \(F_n\) denote the CDF of \(n^{-1/2}v_t^T(\bar{C}(\theta)_k^* - e_k)/\sigma_k\). If Assumptions 4.1 and 4.2 hold, then we have

\[
\max_{1 \leq t < k \leq K} \sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| \leq C (n^{-1/2} + (d \vee n)^{-3}).
\]

where \(C\) is a constant dependent only upon \(c_0, c_1,\) and \(c_2\).

**Proof of Lemma B.6.** Denote by \(\mathcal{E}\) the event that \(\hat{G} = G^*\). We have

\[
F_n(x) - \Phi(x) \leq \tilde{F}_n(x) - \Phi(x) + \mathbb{P}(\mathcal{E}),
\]

where \(\tilde{F}_n(x)\) is the CDF of \(n^{-1/2} \sum_{i=1}^n v_i^T(\bar{C}(i\theta)_k^* - e_k)/\sigma_k\). To control \(\tilde{F}_n(x) - \Phi(x)\), we now verify the Lyapunov condition. As in the proof of Lemma B.4, we can write

\[
v_t^T(\bar{C}(i\theta)_k^* - e_k) = v_t^T(\bar{C}_k^* - C^*)\theta_k^*.
\]

From Lemmas C.1 - C.3 we see that the entries in \(Q_i\) are sub-exponential with parameters \(\alpha = C_1\) and \(\nu = C_2\) which depend only upon \(\lambda_{\max}(\theta^*)\), \(\max_k \gamma_k^*\), and \(\max_t C_{k,t}^*\).

Recall the definition of \(w_t^*\): \((v_i^*)_t = 1\) and \((v_i^*)_{t-\rho} = -(C_{k,t-\rho})^{-1}C_{k,t}\). By the block matrix inverse formula, we can rewrite

\[
w_t^* = (C_{k,t}^* - C_{k,t}^* C_{k,t}^* C_{k,t}^* - C_{k,t}^* C_{k,t}^* - C_{k,t}^*)^{-1}C_{k,t}^* C_{k,t}^* - C_{k,t}^* = \frac{1}{C_{k,t}^* C_{k,t}^* - C_{k,t}^*}.\]

Using Lemma D.4, it follows that \(\|w_t^*\|_2 \leq \lambda_{\max}(\theta^*)\max_t C_{k,t}^* + 1\). From Lemma E.3 and the above, \(v_t^T Q_i\) is sub-exponential with parameters \(\alpha = C_1\) and \(\nu = \|v_i^*\|_2 C_2 \leq (\lambda_{\max}(\theta^*)\max_t C_{k,t}^* + 1) C_2\). Therefore, \(v_t^T(\bar{C}(i\theta)_k^* - e_k)\) has third moments bounded above by some constant \(\rho\) that depends only upon \(\lambda_{\max}(\theta^*)\), \(\max_k \gamma_k^*\), and \(\max_t C_{k,t}^*\). All three quantities are bounded above by constants per Assumptions 4.1 and 4.2. Thus, \(\max_{1 \leq t < k \leq K} \sup_{x \in \mathbb{R}} (\tilde{F}_n(x) - \Phi(x)) \leq C n^{-1/2}\) by the classical Berry-Esseen Theorem, and therefore

\[
\max_{1 \leq t < k \leq K} \sup_{x \in \mathbb{R}} (F_n(x) - \Phi(x)) \leq C (n^{-1/2} + (d \vee n)^{-3}).
\]
Similarly, it can be shown that $\sup_{x \in \mathbb{R}} (\Phi(x) - F_n(x)) \leq C(n^{-1/2} + (d \vee n)^{-3})$. This completes the proof. \hfill \square

**Lemma B.7** (Approximation for Asymptotic Variance). Under Assumptions 4.1 and 4.2, we have that

$$\sigma_{ik}^2 = \Theta_{i,k}^{*2} + \Theta_{i,t}^{*} \Theta_{k,t}^{*} + \Delta, \quad \text{(B.5)}$$

where $|\Delta| \leq \frac{C_m}{m}$ and $C$ is a constant dependent only upon $c_1$, $c_2$ and $c_3$.

**Proof.** Recall that $\sigma_{ik}^2 = \mathbb{E}(\Theta_{i,t}^{*} v_i^{*T} (\mathbf{C}^{(i)} \Theta_k^{*} - e_k))^2$. Using the identity $\text{vec}(M_1 M_2 M_3) = (M_3^T \otimes M_1)^T \text{vec}(M_2)(M_3^T \otimes M_1)$, we have

$$\sigma_{ik}^2 = (\Theta_{i,t}^{*})^2(\Theta_k^{*} \otimes v_i^{*})^T \mathbb{E} \left[ \text{vec}(\mathbf{C}^{(i)}) \text{vec}(\mathbf{C}^{(i)})^T \right] (\Theta_k^{*} \otimes v_i^{*}) \quad \text{(B.6)}$$

**Computing the expectation:** After some straightforward, albeit lengthy, algebra we can show that

$$\mathbb{E} \left[ \text{vec}(\mathbf{C}^{(i)}) \text{vec}(\mathbf{C}^{(i)})^T \right] = M_1 + M_2 + M_3,$$

where $M_1 := C^* \otimes C^*$ and $M_2 := [C^*_{j,i} C_{i}^{*T}]_{ij}$. The matrices $M_1$ and $M_2$ contribute to the first two terms in (B.5). The term $M_3 := \mathbb{E} \left[ \text{vec}(\mathbf{C}^{(i)}) \text{vec}(\mathbf{C}^{(i)})^T \right] - M_1 - M_2$, however, is unique to the latent graph and contributes the higher order term $\Delta$ in (B.5).

**Evaluating the first order terms:** By (B.6), we have

$$\sigma_{ik}^2 = (\Theta_{i,t}^{*})^2(\Theta_k^{*} \otimes v_i^{*})^T (M_1 + M_2)(\Theta_k^{*} \otimes v_i^{*}) + \Delta$$

with $\Delta$ defined as

$$\Delta := (\Theta_{i,t}^{*})^2(\Theta_k^{*} \otimes v_i^{*})M_3(\Theta_k^{*} \otimes v_i^{*}) \quad \text{(B.7)}$$

Next, we observe that

$$(\Theta_k^{*} \otimes v_i^{*})^T M_1(\Theta_k^{*} \otimes v_i^{*}) = \Theta_k^{*T} C^* \Theta_k^{*} \otimes v_i^{*T} C^* v_i^{*}$$

$$= \frac{\Theta_{k,t}^{*2}}{\Theta_{i,t}^{*2}},$$

where we used that $v_i^{*T} C^* v_i^{*} = (\Theta_{i,t}^{*})^{-1}$. Similarly, we can find that

$$(\Theta_k^{*} \otimes v_i^{*})^T M_2(\Theta_k^{*} \otimes v_i^{*}) = \frac{\Theta_{k,t}^{*2}}{\Theta_{i,t}^{*2}}.$$ 

**Bounding the higher order terms:** What remains is to bound the magnitude of the term $\Delta$ in (B.7). Lengthy algebra yields:

$$|(\Theta_k^{*} \otimes v_i^{*})^T M_3(\Theta_k^{*} \otimes v_i^{*})| \leq \frac{C'}{m} (\Theta_k^{*} \otimes v_i^{*})^T (M_4 + M_5 + M_6 + M_7)(\Theta_k^{*} \otimes v_i^{*}),$$

where $C'$ depends only on $c_1$, $c_2$ and $c_3$. Here, $M_4 = I \otimes (11^T)$. For $l = 5, 6, 7$ the matrices $M_l$ are defined block-wise by

$$M_{5;ij} := \begin{cases} 1 e_i^T & \text{if } i \neq j \\ 0 & \text{o/w} \end{cases} \quad \text{and} \quad M_{6;ij} := \begin{cases} e_j^T & \text{if } i \neq j \\ 0 & \text{o/w} \end{cases} \quad \text{and} \quad M_{7;ij} := \begin{cases} I & \text{if } i \neq j \\ 0 & \text{o/w} \end{cases}.$$
Further lengthy algebra gives:

\[ |(\Theta_k^* \otimes v_t^*)^T M_4(\Theta_k^* \otimes v_t^*)| \leq s_0 \frac{2}{c_1^2} (1 + \frac{c_2^2}{c_1^2}), \]

\[ |(\Theta_k^* \otimes v_t^*)^T M_5(\Theta_k^* \otimes v_t^*)| \leq s_0 \frac{2\sqrt{2}}{c_1^2} (1 + \frac{c_2^2}{c_1^2}), \]

\[ |(\Theta_k^* \otimes v_t^*)^T M_6(\Theta_k^* \otimes v_t^*)| \leq \frac{2\sqrt{2}(c_1^2 + c_2^2)}{c_1^4} \sqrt{1 + \frac{c_2^2}{c_1^2}}, \]

and

\[ |(\Theta_k^* \otimes v_t^*)^T M_7(\Theta_k^* \otimes v_t^*)| \leq \frac{2(c_1^2 + c_2^2)}{c_1^4}. \]

Plugging these bounds into the expression for \( \Delta \) in (B.7), we obtain

\[ |\Delta| \leq \frac{Cs_0}{m}, \]

concluding the proof.

\[ \square \]

C Concentration Results

The lemmas below provide important results regarding the concentration properties of some of the estimators \( \hat{C} \) and variables \( Z \).

**Lemma C.1.**

(a) \( Z_iZ_i^T \) consists of entries which are sub-exponential with parameters \( \alpha = 4 \max_t(C_{t,t}^*)^2 \) and \( \nu = 2\sqrt{2} \max_t(C_{t,t}^*)^2 \).

(b) \( \mathbb{P}\left( \| \frac{1}{n} \sum_{i=1}^n Z_iZ_i^T - C^* \|_{\text{max}} \geq C \max_t(C_{t,t}^*)^2 \sqrt{\log(K \vee n)} \right) \leq \frac{2}{(K \vee n)^3} \)

(c) \( Z_i\Theta_k^* \) consists of entries which are sub-exponential with parameters \( \alpha = 4 \max_t(C_{t,t}^*)^2 \) and \( \nu = 2\sqrt{2} \|\Theta_k^*\|_2 \max_t(C_{t,t}^*)^2 \), and

(d) \( \mathbb{P}\left( \| \frac{1}{n} \sum_{i=1}^n Z_i\Theta_k^* - C^*\Theta_k^* \|_{\text{max}} \geq C \|\Theta_k^*\|_2 \max_t(C_{t,t}^*)^2 \sqrt{\log(K \vee n)} \right) \leq \frac{2}{(K \vee n)^3} \),

where \( C = 4\sqrt{3} \).

**Proof.** From Lemma E.1, each element in the matrices \( Z_iZ_i^T \) are sub-exponential with parameters \( \alpha = 4 \max_t(C_{t,t}^*)^2 \) and \( \nu = 2\sqrt{2} \max_t(C_{t,t}^*)^2 \). Therefore by Corollary E.3, the entries in \( \frac{1}{n} \sum_{i=1}^n Z_iZ_i^T \) are sub-exponential with parameters \( \alpha = \frac{1}{n} \max_t(C_{t,t}^*)^2 \) and \( \nu = \frac{2\sqrt{2}}{\sqrt{n}} \max_t(C_{t,t}^*)^2 \). Therefore by the tail bound for sub-exponential random variables, we see that

\[ \mathbb{P}\left( \left( \frac{1}{n} \sum_{i=1}^n Z_iZ_i^T - C^* \right)_s \geq D_1 \sqrt{\frac{\log(K \vee n)}{n}} \right) \]
Therefore, as 
where $C$ vector where the $k$
Proof. Let $\alpha$ parameters
Lemma E.1 gives that $(Y, 2 \max_t (C_{t,t}^*)^2$
further, $\alpha > 0$. Observe that for $n$ sufficiently large, $D_1 \sqrt{\log(K \vee n)/n} \leq 2 \max_t (C_{t,t}^*)^2$, and thus we need only consider this case. Choose $D_1 \geq 4 \sqrt{3} \max_t (C_{t,t}^*)^2$. Then it is clear that

$$
P \left( \left\| \frac{1}{n} \sum_{i=1}^{n} Z_i^T A^* - C^* \right\|_{s,t} \geq D_1 \sqrt{\log(K \vee n)/n} \right) \leq \frac{1}{(K \vee n)^5}. $$

By applying the union bound across all entries in the matrix, we get the desired result that

$$
P \left( \left\| \frac{1}{n} \sum_{i=1}^{n} Z_i^T A^* - C^* \right\|_{\max} \geq D_1 \sqrt{\log(K \vee n)/n} \right) \leq \frac{2}{(K \vee n)^5}, $$

concluding the proof of parts (a) and (b). The proof of (c) and (d) are very similar and omitted. 

Lemma C.2.

(a) $(Z_i E_i^T A^* B^{*-1})_{s,t}$ is sub-exponential with parameters $\alpha = \sqrt{2} \max (\sigma_s^2, C_{t,t}^*)$ and $\nu = \sqrt{2} \max (\sigma_s^2, C_{t,t}^*)$, 

(b) $P \left( \left\| \frac{1}{n} \sum_{i=1}^{n} Z_i E_i^T A^* B^{*-1} \right\|_{\max} \geq C \max (\max_s \sigma_s^2, \max_t C_{t,t}^*) \sqrt{\log(K \vee n)/n} \right) \leq \frac{2}{(K \vee n)^5}$

(c) $(Z_i E_i^T A^* B^{*-1} \Theta_k^*)_{s,t}$ is sub-exponential with parameters $\alpha = \sqrt{2} \max (\sigma_s^2, C_{t,t}^*)$ and $\nu = \sqrt{2} \max (\sigma_s^2, C_{t,t}^*)$, and

(d) $P \left( \left\| \frac{1}{n} \sum_{i=1}^{n} Z_i E_i^T A^* B^{*-1} \Theta_k^* \right\|_{\infty} \geq C \max (\max_s \sigma_s^2, \max_t C_{t,t}^*) \max (\sigma_s^2, C_{t,t}^*) \mid \Theta_k^* \mid \sqrt{\log(K \vee n)/n} \right) \leq \frac{2}{(K \vee n)^{\gamma}}$,

where $C = 2 \sqrt{3}$ and $\sigma_s^2 = \frac{1}{|G_k^*|} \sum_{i \in G_k^*} \gamma_i$.

Proof. Let $M = \sum_{i=1}^{n} Z_i E_i^T A^* B^{*-1}$. From Lemma D.1, $Y_1 = B^{*-1} A^T E_1$ is a $K$-dimensional vector where the $k^{th}$ entry is given by

$$(Y_1)_k = \frac{1}{|G_k^*|} \sum_{i \in G_k^*} (E_1)_i.$$  

Because the errors are all independent mean zero Gaussian random variables, $(Y_1)_s \sim N(0, \sigma_s^2)$. Therefore, as $Y_1$ is independent of $Z_1$ by definition, $E((Y_1)_s(Z_1)_t) = E((Y_1)_s)E((Z_1)_t) = 0$. Further, Lemma E.1 gives that $(Y_1)_s(Z_1)_t$ is sub-exponential with parameters $\alpha = \nu = \sqrt{2} \max (\sigma_s^2, C_{t,t}^*)$.

Using the independence of the samples, Corollary E.3 gives that $M_{s,t}$ is sub-exponential with parameters $\alpha = \sqrt{2} \max (\sigma_s^2, C_{t,t}^*)$ and $\nu = \sqrt{2n} \max (\sigma_s^2, C_{t,t}^*)$. Then, Corollary E.5 gives that for arbitrary choice of $D_1 > 0$,

$$P \left( \frac{1}{n} M_{s,t} \geq D_1 \sqrt{\log(K \vee n)/n} \right)$$

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Then by the union bound we can obtain
\[ \exp \left( -\frac{(\log(K \lor n))D_1^2}{4 \max(\sigma_i^2, C_{t,t}^*)} \right) \quad \text{if } 0 \leq D_1 \sqrt{\frac{\log(K \lor n)}{n}} \leq \sqrt{2} \max(\sigma_s^2, C_{t,t}^*) \]
\[ \exp \left( \frac{-D_1 \sqrt{n \log(K \lor n)}}{2 \max(\sigma_i^2, C_{t,t}^*)} \right) \quad \text{if } D_1 \sqrt{\frac{\log(K \lor n)}{n}} > \sqrt{2} \max(\sigma_s^2, C_{t,t}^*) . \]

Observe that for \( n \) sufficiently large, \( D_1 \sqrt{\log K/n} \leq \sqrt{2} \max(\sigma_s^2, C_{t,t}^*) \). If we choose \( D_1 \geq 2\sqrt{3} \max(\sigma_s^2, C_{t,t}^*) \), then we obtain that for \( n \) sufficiently large,
\[
\mathbb{P}\left( \frac{1}{n} M_{s,t} \geq D_1 \sqrt{\frac{\log(K \lor n)}{n}} \right) \leq \frac{1}{(K \lor n)^3}.
\]

Then by the union bound we can obtain
\[
\mathbb{P}\left( \left\| \frac{1}{n} \sum_{i=1}^{n} Z_i E_i^T A^* B^{*-1} \right\|_{\max} \geq D_1 \sqrt{\frac{\log(K \lor n)}{n}} \right) \leq \frac{2}{(K \lor n)^3}
\]
for \( D_1 \geq 2\sqrt{3} \max(\sigma_s^2, \max, C_{t,t}^*) \), concluding the proof of parts (a) and (b). The proof of (c) and (d) are very similar and omitted.

**Lemma C.3.** Recall that \( m = \min_k |G_k^*| \), then

(a) \( (B^{*-1} A^{*T}(E_i E_i^T - \Gamma^*) A^* B^{*-1})_{t,k} \) is sub-exponential with parameters
\[
\alpha_{t,k} = \frac{\sqrt{2}}{|G_k^*||G_k^*|} \max_{i \in G_k^*} \gamma_i^* \quad \text{and} \quad \nu_{t,k} = \frac{2}{|G_k^*||G_k^*|} \max_{i \in G_k^*} \gamma_i^* ;
\]

(b) \( \mathbb{P}\left( \left\| \frac{1}{n} \sum_{i=1}^{n} B^{*-1} A^{*T}(E_i E_i^T - \Gamma^*) A^* B^{*-1} \right\|_{\max} \geq C \max_k \gamma_k \sqrt{\frac{\log(K \lor n)}{nm^2}} \right) \leq \frac{2}{(K \lor n)^3} \)

(c) \( (B^{*-1} A^{*T}(E_i E_i^T - \Gamma^*) A^* B^{*-1} \Theta_k^*)_{t,k} \) is sub-exponential with parameters
\[
\alpha_t = \frac{\sqrt{3}}{m^2} \max_k \gamma_k^* \quad \text{and} \quad \nu_t = \sqrt{\frac{2}{m^2}} \| \Theta_k^* \|_2 \max_k \gamma_k^* ;
\]

(d) \( \mathbb{P}\left( \left\| \frac{1}{n} \sum_{i=1}^{n} B^{*-1} A^{*T}(E_i E_i^T - \Gamma^*) A^* B^{*-1} \Theta_k^* \right\|_{\max} \geq C \| \Theta_k^* \|_2 \max_k \gamma_k \sqrt{\frac{\log(K \lor n)}{nm^2}} \right) \leq \frac{2}{(K \lor n)^3} \),

where \( C = 2\sqrt{3} \).

**Proof.** We bound the sum entrywise
\[
\left( \frac{1}{n} \sum_{i=1}^{n} B^{*-1} A^{*T}(E_i E_i^T - \Gamma^*) A^* B^{*-1} \right)_{t,k}.
\]

First, from Lemma D.2, Corollary E.2 and Corollary E.3, we have that \( (B^{*-1} A^{*T} E_1 E_1^T A^* B^{*-1})_{t,k} \) is sub-exponential with parameters
\[
\alpha_{t,k} = \frac{\sqrt{2}}{|G_k^*||G_k^*|} \max_{i \in G_k^*} \gamma_i^* \quad \text{and} \quad \nu_{t,k} = \sqrt{\frac{2}{|G_k^*||G_k^*|}} \max_{i \in G_k^*} \gamma_i^* ;
\]

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Therefore \( \frac{1}{n} M_{t,k} \), defined by \( M := \sum_{i=1}^{n} B^{-1} A^T E_i E_i^T A B^{-1} \), is sub-exponential with parameters
\[
\alpha = \frac{\sqrt{2}}{n|G_k^*||G_1^*|} \max_{i \in G_1^* \cup G_k^*} \gamma_i^* \quad \text{and} \quad \nu = \frac{2}{n|G_k^*||G_1^*|} \max_{i \in G_1^* \cup G_k^*} \gamma_i^*.
\]
Denote
\[
\mu_{t,k} = \begin{cases} \frac{1}{|G_1^*|} \sum_{p \in G_1^*} \gamma_p^* & \text{if } t = k \\ 0 & \text{otherwise} \end{cases}
\]
and \( N = \max_{i \in G_1^* \cup G_k^*} \gamma_i^* \). Then from Lemma E.4 we obtain
\[
\mathbb{P} \left( \frac{1}{n} M_{t,k} - \mu_{t,k} \geq D_1 \sqrt{\frac{\log(K \lor n)}{n|G_k^*||G_1^*|}} \right) \leq \exp \left( \frac{(\log(K \lor n)) D_1^2}{4n^2} \right) \exp \left( -D_1 \sqrt{\frac{n\log(K \lor n)}{\nu}} \right)
\]
if \( 0 \leq D_1 \sqrt{\frac{\log(K \lor n)}{n|G_k^*||G_1^*|}} \leq \sqrt{2} N \).

Observe that for \( n \) sufficiently large, \( D_1 \sqrt{\frac{\log(K \lor n)}{n|G_k^*||G_1^*|}} \leq \sqrt{2} N \). If we choose \( D_1 \geq 2\sqrt{3} N \), then we obtain that for \( n \) sufficiently large,
\[
\mathbb{P} \left( \frac{1}{n} M_{t,k} - \mu_{t,k} \geq D_1 \sqrt{\frac{\log(K \lor n)}{n|G_k^*||G_1^*|}} \right) \leq \frac{1}{(K \lor n)^5}.
\]
Therefore by taking the union bound, lower bounding \( n|G_k^*||G_1^*| \) by \( nm^2 \) and choosing \( D_1 \geq 2\sqrt{3} \max_k \gamma_k^* \),
\[
\mathbb{P} \left( \frac{1}{n} \sum_{i=1}^{n} B^{-1} A^T (E_i E_i^T - \mathbf{I}^*) A B^{-1} \right)_{max} \geq D_1 \sqrt{\frac{\log(K \lor n)}{nm^2}} \leq \frac{2}{(K \lor n)^3},
\]
concluding the proof of parts (a) and (b). The proof of (c) and (d) are very similar and omitted.

\[\square\]

### D Auxiliary Technical lemmas

**Lemma D.1.** For \( 1 \leq k \leq K \), denote \( m_k = |G_k^*| \). Then the matrix \( (A^* A^T)^{-1} A^T \) is a \( K \times d \) dimensional matrix given as
\[
[(A^* A^T)^{-1} A^T]_{k,i} = \begin{cases} \frac{1}{m_k} & \text{if } i \in G_k^* \\ 0 & \text{otherwise} \end{cases}
\]

**Proof.** First, we must calculate \( B^{-1} A^T \). For \( 1 \leq k \leq K \), denote \( m_k = |G_k^*| \) and let \( e_k \) be a unit vector in \( \mathbb{R}^K \) with 1 on the \( k \) position and 0 otherwise. Without loss of generality, we permute the rows of \( A^* \) such that for any \( 1 \leq k \leq K \) \( A_{j}^* = e_k \), for \( \sum_{i=1}^{k-1} m_i + 1 \leq j < \sum_{i=1}^{k} m_i + 1 \) – that is rows are ordered according to ascending group index. Here, for notational simplicity, we let \( m_0 = 0 \). Thus, \( A^T A = \text{diag}(m_1, ..., m_K) \) and the result follows immediately. \[\square\]
Lemma D.2. The matrices \( B^{-1}A^T (E_i E_i^T) A^* B^{-1} \) and \( B^{-1}A^T (\Gamma^*) A^* B^{-1} \) are given by

\[
(B^{-1}A^T (E_i E_i^T) A^* B^{-1})_{t,k} = \frac{1}{|G_t^* G_k^*|} \sum_{p \in G_t^*} \sum_{q \in G_k^*} E_{i,p} E_{i,q}
\]

and

\[
(B^{-1}A^T (\Gamma^*) A^* B^{-1})_{t,k} = \begin{cases} 
\frac{1}{|G_t^*|} \sum_{p \in G_t^*} \gamma_p^* & \text{if } t = k \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. The result can be obtained by a straightforward computation.

Lemma D.3 (Restricted Eigenvalue Condition for \( \hat{C} \)). If Assumptions 4.1 and 4.2 hold, then the matrix \( \hat{C} \) satisfies with probability at least \( 1 - \frac{C}{(K \lor n)^3} \),

\[
\kappa \leq \min \left\{ \frac{v^T \hat{C} v}{\|v\|_2^2} : v \in \mathbb{R}^K \setminus \{0\}, \|v_{\bar{S}}\|_1 \leq 3\|v_S\|_1 \right\}, \text{ and }
\]

\[
\kappa \leq \min \left\{ \frac{v^T \hat{C} v}{\|v\|_2^2} : v \in \mathbb{R}^K \setminus \{0\}, \|v_{\bar{S}'}\|_1 \leq 3\|v_S'\|_1 \right\},
\]

where \( \kappa \geq \frac{3}{4c_1} > 0 \).

Proof. We begin by proving the first claim. By Lemma B.3, we have that \( \|\hat{C} - C^*\|_{\max} \leq C_1 \sqrt{\frac{\log(K \lor n)}{n}} \) with high probability. Therefore, for \( K \) sufficiently large and for any \( v \in \mathbb{R}^K \setminus \{0\} \),

\[
\frac{v^T \hat{C} v}{\|v\|_2^2} \geq \frac{3}{4} \frac{v^T C^* v}{\|v\|_2^2}.
\]

The proof is then done for \( \kappa = \frac{3}{4c_1} \) as we assume the minimum eigenvalue of \( C^* \) is bounded below by \( c_0^{-1} \). The proof of the second claim is identical because \( C^* \) is positive semidefinite, and it is well known that the minimum eigenvalue of any principal submatrix \( C^*_{-t,-t} \) is bounded below by \( \lambda_{\min}(C^*) \geq c_0^{-1} \).

Lemma D.4. Let \( M \) be a \( n \times n \) positive definite matrix and denote its inverse by \( L \). Then, for all \( i = 1, \ldots, n \)

\[
M_{i,i} L_{i,i} \geq 1.
\]

Proof. By the block matrix inverse formula, it follows that

\[
M_{i,i}^{-1} = L_{i,i} - L_{i,i}^{-1} L_{i,i}^{-1} L_{i,i}. \tag{D.1}
\]

Because \( M \) is positive definite, so is \( L \). Recall that a matrix is positive definite if and only if all its principal minors are also positive definite. Therefore, \( L_{-i,-i} \) is positive definite, as is \( L_{-i,-i}^{-1} \). Therefore, \( L_{i,i} L_{-i,-i}^{-1} L_{i,i} \geq 0 \) and (D.1) becomes \( M_{i,i}^{-1} \leq L_{i,i} \). Lastly, if a matrix is positive definite, all its diagonal elements must be nonnegative, giving that \( M_{i,i} L_{i,i} \geq 1 \) as desired.
E Basic Tail Bounds for Random Variables

This section collects some basic tail probability results for random variables. The proof is standard and omitted.

**Lemma E.1.** Let \( Y = (Y_1, Y_2) \) be a jointly Gaussian random vector with covariance matrix \( C \). Then \( Y_1Y_2 \) is sub-exponential with parameters \( \alpha = 4\lambda_{\max}(C_Y) \) and \( \nu = 2\sqrt{2}\lambda_{\max}(C_Y) \).

**Corollary E.2.** Let \( Y_1 \sim N(0, \sigma_1^2) \) and \( Y_2 \sim N(0, \sigma_2^2) \) where \( \sigma_1^2 \geq \sigma_2^2 \). Then \( Y_1Y_2 \) is sub-exponential with parameters \( \alpha = \sqrt{2}\sigma_1^2 \) and \( \nu = \sqrt{2}\sigma_2^2 \).

**Corollary E.3.** Consider \( \sum_{i=1}^n X_i \) where \( X_i \) are centered, independent sub-exponential random variables. Then \( Y = \sum_{i=1}^n X_i \) is sub-exponential with parameters \( \alpha = \max_i \alpha_i \) and \( \nu = \sqrt{\sum_{i=1}^n \nu_i^2} \).

**Lemma E.4** (Tail Bound for Sub-Exponential Random Variables). Let \( X \) be a sub-exponential random variable with mean \( \mu \) and parameters \( \alpha \) and \( \nu \). Then

\[
\mathbb{P}(X - \mu \geq t) \leq \begin{cases} 
\exp(-\frac{t^2}{2\nu^2}) & \text{for } 0 \leq t \leq \frac{\nu^2}{\alpha} \\
\exp(-\frac{t^2}{2\nu^2}) & \text{for } t > \frac{\nu^2}{\alpha}.
\end{cases}
\]

**Corollary E.5.** Consider \( Y = \sum_{i=1}^n X_i \), where \( X_i \) are centered, independent sub-exponential random variables. Let \( \alpha = \max_i \alpha_i \) and \( \nu = \sqrt{\sum_{i=1}^n \nu_i^2} \). Then,

\[
\mathbb{P}\left(\frac{1}{n} \sum_{i=1}^n X_i \geq t\right) \leq \begin{cases} 
\exp(-\frac{nt^2}{2\nu^2/n}) & \text{for } 0 \leq t \leq \frac{\nu^2}{\alpha n} \\
\exp(-\frac{nt^2}{2\nu^2/n}) & \text{for } t > \frac{\nu^2}{\alpha n}.
\end{cases}
\]

F Construction of a pre-clustering estimator of \( \Gamma \)

We include in this section the construction of the pre-clustering estimator of \( \Gamma \) needed as an input of the PECOK algorithm of Section 2.2 above. For any \( a, b \in [d] \), define

\[
V(a, b) := \max_{c,d \in [p] \setminus \{a,b\}} \left| \left( \hat{\Sigma}_{ac} - \hat{\Sigma}_{ad} \right) - \left( \hat{\Sigma}_{bc} - \hat{\Sigma}_{bd} \right) \right|, \quad (F.1)
\]

with the convention \( 0/0 = 0 \). Guided by the block structure of \( \Sigma \), we define

\[
b_1(a) := \arg\min_{b \in [p]\setminus\{a\}} V(a, b) \quad \text{and} \quad b_2(a) := \arg\min_{b \in [p]\setminus\{a, b_1(a)\}} V(a, b),
\]

to be two elements "close" to \( a \), that is two indices \( b_1 = b_1(a) \) and \( b_2 = b_2(a) \) such that the empirical covariance difference \( \hat{\Sigma}_{bc} - \hat{\Sigma}_{bd} \), \( i = 1, 2 \), is most similar to \( \hat{\Sigma}_{ac} - \hat{\Sigma}_{ad} \) for all variables \( c \) and \( d \) not equal to \( a \) or \( b_i \), \( i = 1, 2 \). It is expected that \( b_1(a) \) and \( b_2(a) \) either belong to the same group as \( a \), or belong to some "close" groups. Then, our estimator \( \hat{\Gamma} \) is a diagonal matrix, defined by

\[
\hat{\Gamma}_{aa} = \hat{\Sigma}_{aa} + \hat{\Sigma}_{b_1(a)b_2(a)} - \hat{\Sigma}_{ab_1(a)} - \hat{\Sigma}_{ab_2(a)}, \quad \text{for } a = 1, \ldots, d. \quad (F.2)
\]
Intuitively, $\widetilde{\Gamma}_{aa}$ should be close to $\Sigma_{aa} + \Sigma_{b_1(a)b_2(a)} - \Sigma_{ab_1(a)} - \Sigma_{ab_2(a)}$, which is equal to $\Gamma_{aa}$ in the favorable event where both $b_1(a)$ and $b_2(a)$ belong to the same group as $a$.

In general, $b_1(a)$ and $b_2(a)$ cannot be guaranteed to belong to the same group as $a$. Nevertheless, these two surrogates $b_1(a)$ and $b_2(a)$ are close enough to $a$ so that $\|\widetilde{\Gamma} - \Gamma\|_{\max} \lesssim |\Gamma|_{\max} \sqrt{\log d/n}$. This last fact and the above construction are shown in Bunea et al. (2018).