Moufang symmetry II.
Moufang-Mal’tsev pairs and triality

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Abstract

A concept of the Moufang-Mal’tsev pair is elaborated. This concept is based on the
generalized Maurer-Cartan equations of a local analytic Moufang loop. Triality can be seen
as a fundamental property of such pairs. Based on triality, the Yamagutian is constructed.
Properties of the Yamagutian are studied.

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1 Introduction

In [1], the generalized Maurer-Cartan equations for a local analytic Moufang loop were found.
In this paper, we elaborate a concept of the Moufang-Mal’tsev pair which is based on the
generalized Maurer-Cartan equations. Such a concept may also be inferred from the theory of
alternative algebras [2, 3]. Triality can be seen as a fundamental property of such pairs. Based
on triality, the Yamagutian is constructed. Properties of the Yamagutian are studied.

2 Moufang-Mal’tsev pairs

Let $\mathcal{M}$ be an anti-commutative algebra and let $\mathcal{L}$ be a Lie algebra. Throughout this paper we
assume that both algebras have the same base field $\mathbb{F}$ ($\cong \mathbb{R}$ or $\mathbb{C}$; as a matter of fact, only
$\text{char} \mathbb{F} \neq 2, 3$ is essential). Denote by $(S, T)$ a pair of the linear maps $S, T : \mathcal{M} \to \mathcal{L}$.

Definition 2.1 (Moufang-Mal’tsev pair). We call $(S, T)$ a Moufang-Mal’tsev pair if

\begin{align}
[S_x, S_y] &= S_{[x,y]} - 2[S_x, T_y] \quad (2.1a) \\
[T_x, T_y] &= T_{[y,x]} - 2[T_x, S_y] \quad (2.1b)
\end{align}

for all $x, y$ in $\mathcal{M}$.

We call (2.1a, b) the Moufang-Mal’tsev relations. Note that the same brackets $\lbrack, \rbrack$ are used
to denote multiplications in $\mathcal{M}$ and $\mathcal{L}$.

Proposition 2.2. Let $(S, T)$ a Moufang-Mal’tsev pair. Then

$[S_x, T_y] = [T_x, S_y]$ for all $x, y$ in $\mathcal{M}$.

Proof. Use anti-commutativity in $\mathcal{L}$ and $\mathcal{M}$. □

Corollary 2.3 (minimality conditions). The Moufang-Mal’tsev conditions read

\[2[S_x, T_y] = S_{[x,y]} - [S_x, S_y] = T_{[y,x]} - [T_x, T_y] = 2[T_x, S_y]\]

Remark 2.4. Deviations of $S$ and $T$ from algebra maps may be considered as "minimal" for
the Moufang-Mal’tsev pair.
3 Triality

Proposition 3.1. Let \((S, T)\) be a Moufang-Malt’sev pair. Then \((-T, -S)\) is a Moufang-Malt’sev pair as well.

Proof. By using anti-commutativity in \(M\), rewrite the Moufang-Malt’sev conditions of \((S, T)\) as follows:

\[
\begin{align*}
-T_x, -T_y &= -T_{[x, y]} - 2T_x, -S_y \\
-S_x, -S_y &= -S_{[y, x]} - 2S_x, -T_y
\end{align*}
\]

Consider a triple \((S, T, P)\) of the linear maps \(S, T, P : M \to L\), such that

\[S + T + P = 0\]

With a given triple \((S, T, P)\) we can associate the pairs

\[(S, T), (-T, -S), (T, P), (-P, -T), (P, S), (-S, -P)\]

Lemma 3.2 (triality). Let some pair from \((3.1)\) be a Moufang-Malt’sev pair. Then all other pairs from \((3.1)\) are the Moufang-Malt’sev pairs as well.

Proof. Assume that \((S, T)\) is a Moufang-Malt’sev pair. Then we know from above Proposition 3.1 that \((-T, -S)\) is a Moufang-Malt’sev pair. The required minimality conditions for \((T, P)\) and \((P, S)\) read, respectively,

\[
\begin{align*}
2[T_x, P_y] &= T_{[x, y]} - [T_x, T_y] = P_{[y, x]} - [P_x, P_y] = 2[P_x, T_y] \\
2[P_x, S_y] &= P_{[x, y]} - [P_x, P_y] = S_{[y, x]} - [S_x, S_y] = 2[S_x, P_y]
\end{align*}
\]

As an example, calculate

\[
-2[S_x, P_y] = 2[P_x + T_x, P_y] = 2[P_x, P_y] + 2[P_x, T_y] = [P_x, P_y] + [P_x, T_y] + 2[P_x, T_y] = [P_x, P_y] + [S_x + T_x, S_y + T_y] - 2[S_x, T_y] - 2[T_x, T_y] = [P_x, P_y] + [S_x, T_y] - 2[T_x, T_y] = [P_x, P_y] + S_{[x, y]} - 2[S_x, T_y] - T_{[y, x]} + 2[T_x, S_y] = P_x, T_y - P_{[x, y]}
\]

All other required equalities in \((3.1)\) can be verified in the same way. The result tells us that \((T, P)\) and \((P, S)\) are the Moufang-Malt’sev pairs, which in turn implies that \((-P, -T)\) and \((-S, -P)\) must be the Moufang-Malt’sev pairs as well.

It follows from Lemma 3.2 that the Moufang-Malt’sev pair \((S, T)\) is invariant under the substitutions

\[
\begin{align*}
\sigma &\doteq (S \to T \to P \to S) \\
\tau &\doteq (S \to -T \to S)(P \to -P) \\
\sigma^2 &\doteq (S \to P \to T \to S) \\
\sigma \circ \tau &\doteq (S \to -P \to S)(T \to -T) \\
\sigma^2 \circ \tau &\doteq (T \to -P \to T)(S \to -S)
\end{align*}
\]

which we call the triality substitutions. So it is natural to proclaim:
Theorem 3.3 (principle of triality). All algebraic consequences of the Moufang-Mal’tsev conditions are triality invariant.

Such a symmetry we call triality. It suggests that we should try to handle the Moufang-Mal’tsev pairs in triality symmetric manner.

In particular, by using the triality conjugation,

\[
\begin{align*}
P^+ &= S - T = P + 2S = -P - 2T \\
S^+ &= T - P = S + 2T = -S - 2P \\
T^+ &= P - S = T + 2P = -T - 2S
\end{align*}
\]

with the evident property

\[S^+ + T^+ + P^+ = 0\]

one can rewrite the Moufang-Mal’tsev conditions as follows:

\[
\begin{align*}
[S_x, S^+_x] &= [S^+_x, S_y] = S_{[x,y]} \\
[T_x, T^+_x] &= [T^+_x, T_y] = T_{[x,y]} \\
[P_x, P^+_x] &= [P^+_x, P_y] = P_{[x,y]}
\end{align*}
\]

Note that

\[
\begin{align*}
3P &= T^+ - S^+ = P^+ + 2T^+ = -P^+ - 2S^+ \\
3S &= P^+ - T^+ = S^+ + 2P^+ = -S^+ - 2T^+ \\
3T &= S^+ - P^+ = T^+ + 2S^+ = -T^+ - 2P^+
\end{align*}
\]

which means that the triality conjugation is invertible.

4 Yamagutian

We introduced the triple \((S, T, P)\) via the triality symmetric identity (3.1). Following triality, it is natural to search for other but nontrivial triality invariant combinations of the maps from the triple \((S, T, P)\).

Definition 4.1 (Yamagutian [4]). The Yamagutian of \((S, T)\) is the skew-symmetric bilinear map \(Y : M \otimes M \to L\) defined (cf (3.1)) by

\[
6Y(x; y) = [S_x, S_y] + [T_x, T_y] + [P_x, P_y] = -Y(y; x)
\]

We can see the evident but important

Proposition 4.2. The Yamagutian \(Y\) is triality invariant.

By triality symmetry, the Yamagutian \(Y\) can be redefined in several useful ways. In particular,

\[
\begin{align*}
6Y(x; y) &= 3[S_x, S_y] + S^+_{[x,y]} \\
&= 3[T_x, T_y] + T^+_{[x,y]} \\
&= 3[P_x, P_y] - P^+_{[x,y]}
\end{align*}
\]
and one can also verify that

\[ 6Y(x; y) = 2P^+_{[x,y]} - 6[S_x, T_y] \]
\[ = 2S^+_{[x,y]} - 6[T_x, P_y] \]
\[ = 2T^+_{[x,y]} - 6[P_x, S_y] \]

Later we shall need the

**Proposition 4.3.** Let \((S, T)\) be a Moufang-Mal’tsev pair. Then

\[ 6Y(x; y) = [S^+_{x,y}, S^+_{x,y}] + S^+_{[x,y]} \] \hfill (4.1a)
\[ = [T^+_{x,y}, T^+_{x,y}] + T^+_{[x,y]} \] \hfill (4.1b)
\[ = [P^+_{x,y}, P^+_{x,y}] + P^+_{[x,y]} \] \hfill (4.1c)

for all \(x, y\) in \(M\).

**Proof.** Due to triality, check only the last formula (4.1c):

\[
[P^+_{x,y}] = [S_x - T_x, S_y - T_y] \\
= [S_x, S_y] - 2[S_x, T_y] + [T_x, T_y] \\
= \frac{1}{3} Y(x; y) + \frac{1}{3} S^+_{[x,y]} + \frac{1}{3} Y(x; y) - \frac{2}{3} T^+_{[x,y]} + \frac{1}{3} Y(x; y) + \frac{1}{3} T^+_{[x,y]} \\
= Y(x; y) - P^+_{[x,y]} \qed
\]

**Remark 4.4.** Formulae (4.1a-c) tell us that the Yamagutian \(Y\) measures the deviation of \(S^+, T^+\) and \(P^+\) from the anti-algebra maps.

**Corollary 4.5.** We have

\[ 18Y(x, y) = [S^+_{x,y}, S^+_{x,y}] + [T^+_{x,y}, T^+_{x,y}] + [P^+_{x,y}, P^+_{y}] \]

**Theorem 4.6.** Let \((S, T)\) be a Moufang-Mal’tsev pair. Then

\[
[S_x, S_y] = 2Y(x; y) + \frac{1}{3} S_{[x,y]} + \frac{2}{3} T_{[x,y]} \]
\[
[S_x, T_y] = -Y(x; y) + \frac{1}{3} S_{[x,y]} - \frac{1}{3} T_{[x,y]} \]
\[
[T_x, T_y] = 2Y(x; y) - \frac{2}{3} S_{[x,y]} - \frac{1}{3} T_{[x,y]} \]

**Proof.** Evident. \qed

**Corollary 4.7.** By triality, we have

\[
[T_x, T_y] = 2Y(x; y) + \frac{1}{3} T_{[x,y]} + \frac{2}{3} P_{[x,y]} \]
\[
[T_x, P_y] = -Y(x; y) + \frac{1}{3} T_{[x,y]} - \frac{1}{3} P_{[x,y]} \]
\[
[P_x, P_y] = 2Y(x; y) - \frac{2}{3} T_{[x,y]} - \frac{1}{3} P_{[x,y]} \]
and

\[
\begin{align*}
[P_x, P_y] &= 2Y(x; y) + \frac{1}{3}P_{[x,y]} + \frac{2}{3}S_{[x,y]} \\
[P_x, S_y] &= -Y(x; y) + \frac{1}{3}P_{[x,y]} - \frac{1}{3}S_{[x,y]} \\
[S_x, S_y] &= 2Y(x; y) - \frac{2}{3}P_{[x,y]} - \frac{1}{3}S_{[x,y]}
\end{align*}
\]

**Proposition 4.8.** Let \((S, T)\) be a Moufang-Mal’tsev pair. Then

\[
\begin{align*}
6[&Y(x; y), S_z] = 3[[S_x, S_y], S_z] - S_{[[x,y],z]} \\
6[&Y(x; y), T_z] = 3[[T_x, T_y], T_z] - T_{[[x,y],z]} \\
6[&Y(x; y), P_z] = 3[[P_x, P_y], P_z] - P_{[[x,y],z]}
\end{align*}
\]

for all \(x, y, z\) in \(M\).

**Proof.** Due to triality, only the first identity must be checked:

\[
\begin{align*}
6[&Y(x; y), S_z] &= 3[[S_x, S_y], S_z] - S_{[[x,y],z]} \\
&= 3[[S_x, S_y], S_z] - [S^+_{[x,y]}, S_z] \\
&= 3[[S_x, S_y], S_z] - S_{[[x,y],z]} \quad \blacksquare
\end{align*}
\]

**Corollary 4.9.** Adding formulae (4.2a–c) we obtain (cf. (4.7)) the triality symmetric identity

\[
[[S_x, S_y], S_z] + [[T_x, T_y], T_z] + [[P_x, P_y], P_z] = 0
\]

**Corollary 4.10.** In (4.2a–c) make twice cyclic permutation of \(x, y, z\) and add the resulting equalities with the original ones. Then we obtain

\[
\begin{align*}
6[&Y(x; y), S_z] + 6[&Y(y; z), S_x] + 6[&Y(z; x), S_y] = S_{J(x,y,z)} \\
6[&Y(x; y), T_z] + 6[&Y(y; z), T_x] + 6[&Y(z; x), T_y] = T_{J(x,y,z)} \\
6[&Y(x; y), P_z] + 6[&Y(y; z), P_x] + 6[&Y(z; x), P_y] = P_{J(x,y,z)}
\end{align*}
\]

**Proposition 4.11.** Let \((S, T)\) be a Moufang-Mal’tsev pair. Then

\[
\begin{align*}
[S^+_{[x,y]}, S^+_{z,x}] + [S^+_{[y,z]}, S^+_{x,y}] + [S^+_{[z,x]}, S^+_{y,z}] &= P^+_{J(x,y,z)} \\
[T^+_{[x,y]}, T^+_{z,x}] + [T^+_{[y,z]}, T^+_{x,y}] + [T^+_{[z,x]}, T^+_{y,z}] &= T^+_{J(x,y,z)} \\
[P^+_{[x,y]}, P^+_{z,x}] + [P^+_{[y,z]}, P^+_{x,y}] + [P^+_{[z,x]}, P^+_{y,z}] &= P^+_{J(x,y,z)}
\end{align*}
\]

for all \(x, y, z\) in \(M\).

**Proof.** Subtracting (4.2b) from (4.2a), we obtain

\[
6[&Y(x; y), P^+_z] = 3[[S_x, S_y], S_z] - 3[[T_x, T_y], T_z] - P^+_{[[x,y],z]}
\]

On the other hand, using (4.2c), we have

\[
6[&Y(x; y), P^+_z] = [[P^+_x, P^+_y], P^+_z] + [P^+_x, P^+_z]
\]
and so we obtain
\[ [P_{[x,y]}^+, P_z^+] = -P_{[x,y],z}^+ + 3[[S_x, S_y], S_z] - 3[[T_x, T_y], T_z] - [[P_x^+, P_y^+], P_z^+] \]

Now make twicely cyclic permutation of \( x, y, z \) and add the resulting equalities with the original one. Then, using Jacobi conditions (in \( L \)) and the definition of \( J \) on the right hand-side of the resulting equality we obtain (4.3c). The remaining identities (4.3a,b) are evident from triality.

\[ \text{Lemma 4.12. Let } (S, T) \text{ be a Moufang-Mal’tsev pair. Then} \]
\[ Y([x, y]; z) + Y([y, z]; x) + Y([z, x]; y) = 0, \quad \forall x, y, z \in M \] (4.4)

\[ \text{Proof. Use (4.1c) to obtain} \]
\[ 6Y([x, y]; z) = [P_{[x,y]}^+, P_z^+] + P_{[x,y],z}^+ \]

Make here twicely the cyclic permutation of \( x, y, z \) and add the resulting equalities with the original one. Then use (4.3b) and the definition of \( J \) to obtain the desired identity (4.4).

\[ \begin{align*}
\end{align*} \]

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