I. INTRODUCTION

In the conventional quark model\cite{1, 2}, hadrons generally have two kinds of structures: a meson consisting of a quark and an antiquark, and a baryon consisting of three quarks. However, quantum chromodynamics (QCD) allows the existence of hadrons different from the above two structures, such as the tetraquarks, hadronic molecules, pentaquarks, hybrids and so on\cite{3–6}.

A compact tetraquark is composed of a diquark and an antiquark, bounding by the color force among quarks and antiquarks. The light tetraquarks have been widely studied via different theoretical methods\cite{7–10}. For the heavy quark sector, the hidden-charm/bottom $Q\bar{Q}q\bar{q}$ tetraquarks have been extensively investigated to interpret some observed XYZ states in various methods, such as the constituent quark models\cite{11–13}, meson exchange and scattering methods\cite{14–16}, QCD sum rules\cite{17–20}, chromomagnetic interaction models\cite{21, 22}, etc. The doubly heavy tetraquark states $QQ\bar{q}\bar{q}$ have been studied to investigate the stability of tetraquarks\cite{23, 24}. In Ref. \cite{25–27}, the open-flavor heavy $bcq\bar{s}$ tetraquark states have also been investigated, the results suggest that their masses may lie below the corresponding two-meson thresholds. In addition, such tetraquarks cannot decay via the annihilation channels and thus they will be very stable with narrow widths.

Comparing to the above several tetraquark configurations, the fully open-flavor tetraquarks $bcq\bar{s}$ and $scq\bar{b}$ ($q = u, d$) are more exotic since they contain four valence quarks with totally different flavors. However, the studies of these tetraquarks have drawn much less interest to date. In Ref. \cite{21}, the authors studied the masses of $qsc\bar{b}$ and $qc\bar{s}$ tetraquarks by using the color-magnetic interaction with the flavor symmetry breaking corrections. Their results show that the masses of $qsc\bar{b}$ and $qc\bar{s}$ tetraquark states are about 7.1 GeV and 7.2 GeV, which are lower than the corresponding two-meson S-wave thresholds. In the heavy quark symmetry, the mass of $bcq\bar{s}$ tetraquark states with $J^P = 1^+$ was also evaluated to be around 7445 MeV \cite{28}, which is about 163 MeV above the $D\bar{B}_s^*$ threshold and thus allows such decay channel via strong interaction. The above conflicting results from different phenomenological models are inspiring more theoretical studies for the existence of these fully open-flavor tetraquark states. In this paper, we shall study the mass spectra of the fully open-flavor $bcq\bar{s}$ and $scq\bar{b}$ tetraquarks in the method of QCD sum rules \cite{29, 30}.

This paper is organized as follows. In Sec. II, we construct the interpolating tetraquark currents of the $bcq\bar{s}$ and $scq\bar{b}$ systems with $J^P = 0^+, 1^+$, respectively. In Sec. III, we evaluate the correlation functions and spectral densities for these interpolating currents. The spectral densities will listed in the appendix because of their complicated form. We extract the masses for these tetraquarks by performing the QCD sum rule analyses in Sec. IV. The last section is a brief summary.

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II. INTERPOLATING CURRENTS FOR THE $bc\bar{q}s$ AND $sc\bar{q}b$ TETRAQUARK SYSTEMS

In this section, we construct the interpolating currents for $bc\bar{q}s$ and $sc\bar{q}b$ tetraquarks with $J^P = 0^+, 1^+$. In general, there are five independent diquark fields, $q_a^T C \gamma_5 q_b$, $q_a^T C q_b$, $q_a^T C \gamma_\mu \gamma_5 q_b$, $q_a^T C \gamma_\mu q_b$, and $q_a^T C \sigma_{\mu\nu} q_b$, where $q$ stands for quark field, $a,b$ are the color indices, $C$ denotes the charge conjugate operator, and $T$ represents the transpose of the quark fields. The $q_a^T C \gamma_5 q_b$ and $q_a^T C \gamma_\mu q_b$ are S-wave operators while $q_a^T C q_b$ and $q_a^T C \gamma_5 q_b$ are P-wave operators. The $q_a^T C \sigma_{\mu\nu} q_b$ contains both S-wave and P-wave operators according to its different components. To study the lowest lying $bc\bar{q}s$ and $sc\bar{q}b$ tetraquark states, we use only S-wave diquarks and corresponding antidiquark fields to construct the tetraquark interpolating currents with quantum numbers $J^P = 0^+, 1^+$. For the $bc\bar{q}s$ system, the scalar currents with $J^P = 0^+$ can be written as

$$
\begin{align*}
J_1 &= b_a^T C \gamma_5 c_b \left( \bar{q}_a \gamma_5 C s_b^T + \bar{q}_b \gamma_5 C s_a^T \right), \\
J_2 &= b_a^T C \gamma_5 c_b \left( \bar{q}_a \gamma_5 C s_b^T - \bar{q}_b \gamma_5 C s_a^T \right), \\
J_3 &= b_a^T C \gamma_\mu c_b \left( \bar{q}_a \gamma_\mu C s_b^T + \bar{q}_b \gamma_\mu C s_a^T \right), \\
J_4 &= b_a^T C \gamma_\mu c_b \left( \bar{q}_a \gamma_\mu C s_b^T - \bar{q}_b \gamma_\mu C s_a^T \right),
\end{align*}
$$

in which $q$ is light quark field (up or down). The color structure for the currents $J_1$ and $J_3$ are symmetric $[\mathbf{6}_c]_{bc} \otimes [\mathbf{\bar{6}}_c]_{\bar{q} \bar{s}}$, while for the $J_2$ and $J_4$ are antisymmetric $[\mathbf{3}_c]_{bc} \otimes [\mathbf{\bar{3}}_c]_{\bar{q} \bar{s}}$. The axial-vector currents with $J^P = 1^+$ can be written as

$$
\begin{align*}
J_{1\mu} &= b_a^T C \gamma_\mu c_b \left( \bar{q}_a \gamma_\mu C s_b^T + \bar{q}_b \gamma_\mu C s_a^T \right), \\
J_{2\mu} &= b_a^T C \gamma_\mu c_b \left( \bar{q}_a \gamma_\mu C s_b^T - \bar{q}_b \gamma_\mu C s_a^T \right), \\
J_{3\mu} &= b_a^T C \gamma_\mu c_b \left( \bar{q}_a \gamma_\mu C s_b^T + \bar{q}_b \gamma_\mu C s_a^T \right), \\
J_{4\mu} &= b_a^T C \gamma_\mu c_b \left( \bar{q}_a \gamma_\mu C s_b^T - \bar{q}_b \gamma_\mu C s_a^T \right),
\end{align*}
$$

where the currents $J_{1\mu}$ and $J_{3\mu}$ are color symmetric while the $J_{2\mu}$ and $J_{4\mu}$ are color antisymmetric. For the $sc\bar{q}b$ system, the currents with $J^P = 0^+$ are

$$
\begin{align*}
\eta_1 &= s_a^T C \gamma_5 c_b \left( \bar{q}_a \gamma_5 C \bar{b}_b^T + \bar{q}_b \gamma_5 C \bar{b}_a^T \right), \\
\eta_2 &= s_a^T C \gamma_5 c_b \left( \bar{q}_a \gamma_5 C \bar{b}_b^T - \bar{q}_b \gamma_5 C \bar{b}_a^T \right), \\
\eta_3 &= s_a^T C \gamma_\mu c_b \left( \bar{q}_a \gamma_\mu C \bar{b}_b^T + \bar{q}_b \gamma_\mu C \bar{b}_a^T \right), \\
\eta_4 &= s_a^T C \gamma_\mu c_b \left( \bar{q}_a \gamma_\mu C \bar{b}_b^T - \bar{q}_b \gamma_\mu C \bar{b}_a^T \right),
\end{align*}
$$

where the currents $\eta_1$ and $\eta_3$ are color symmetric with $[\mathbf{6}_c]_{sc} \otimes [\mathbf{\bar{6}}_c]_{\bar{q} \bar{b}}$, while the $\eta_2$ and $\eta_4$ are color antisymmetric with $[\mathbf{3}_c]_{sc} \otimes [\mathbf{\bar{3}}_c]_{\bar{q} \bar{b}}$. The currents with $J^P = 1^+$ are

$$
\begin{align*}
\eta_{1\mu} &= s_a^T C \gamma_\mu c_b \left( \bar{q}_a \gamma_\mu C \bar{b}_b^T + \bar{q}_b \gamma_\mu C \bar{b}_a^T \right), \\
\eta_{2\mu} &= s_a^T C \gamma_\mu c_b \left( \bar{q}_a \gamma_\mu C \bar{b}_b^T - \bar{q}_b \gamma_\mu C \bar{b}_a^T \right), \\
\eta_{3\mu} &= s_a^T C \gamma_\mu c_b \left( \bar{q}_a \gamma_\mu C \bar{b}_b^T + \bar{q}_b \gamma_\mu C \bar{b}_a^T \right), \\
\eta_{4\mu} &= s_a^T C \gamma_\mu c_b \left( \bar{q}_a \gamma_\mu C \bar{b}_b^T - \bar{q}_b \gamma_\mu C \bar{b}_a^T \right),
\end{align*}
$$

in which the currents $\eta_{1\mu}$ and $\eta_{3\mu}$ are color symmetric while the $\eta_{2\mu}$ and $\eta_{4\mu}$ are color antisymmetric.

III. QCD SUM RULES

In this section, we investigate the two-point correlation functions of the above scalar and axial-vector interpolating currents. For the scalar currents, the correlation function can be written as

$$
\Pi(p^2) = i \int d^4x e^{ip\cdot x} \langle 0 | T [J(x) J^I(0)] | 0 \rangle,
$$

and for the axial-vector current

$$
\Pi_{\mu\nu}(p^2) = i \int d^4x e^{ip\cdot x} \langle 0 | T [J_\mu(x) J^I_\nu(0)] | 0 \rangle.
$$
The correlation function $\Pi_{\mu\nu}(p^2)$ in Eq. (6) can be expressed as

$$\Pi_{\mu\nu}(p^2) = \left(\frac{p_{\mu}p_{\nu}}{p^2} - g_{\mu\nu}\right)\Pi_0\left(p^2\right) + \frac{p_{\mu}p_{\nu}}{p^2}\Pi_1\left(p^2\right),$$

(7)

where $\Pi_0\left(p^2\right)$ and $\Pi_1\left(p^2\right)$ are the scalar and vector current polarization functions related to the spin-0 and spin-1 intermediate states, respectively.

At the hadron level, the correlation function can be written through the dispersion relation

$$\Pi\left(p^2\right) = \frac{(p^2)^N}{\pi} \int_{(m_q+m_c)^2}^{\infty} \frac{\text{Im}\Pi(s)}{s^N(s-p^2-i\epsilon)}\,ds + \sum_{n=0}^{N-1} b_n\left(p^2\right)^n,$$

(8)

in which $b_n$ is the subtraction constant. In QCD sum rules, the imaginary part of the correlation function is defined as the spectral function

$$\rho(s) = \frac{1}{\pi} \text{Im}\Pi(s) = f_H^2\delta(s-m_H^2) + \text{QCD continuum and higher states},$$

(9)

where the "pole plus continuum parametrization" is adopted. The parameters $f_H$ and $m_H$ are the coupling constant and mass of the lowest-lying hadronic resonance $H$ respectively

$$\langle 0|J|H\rangle = f_H,$$
$$\langle 0|J_\mu|H\rangle = f_H\epsilon_\mu$$

(10)

with the polarization vector $\epsilon_\mu$.

At the quark-gluon level, we can evaluate the correlation function $\Pi(p^2)$ and spectral density $\rho(s)$ using the method of operator product expansion (OPE). To calculate the Wilson coefficients, we use the light quark propagator in coordinate space and heavy quark propagator in momentum space

$$iS_q(x) = \frac{i\delta^{ab}}{2\pi^2x^4} \bar{x} + \frac{i}{32\pi^2} \frac{\lambda^a}{2} g_s G_{\mu\nu} \frac{1}{x^2} \left(\sigma^{\mu\nu} \bar{x} + \bar{x} \sigma^{\mu\nu}\right) - \frac{\delta^{ab}x^2}{12} \langle \bar{q}g_s \sigma \cdot Gq \rangle - \frac{m_q\delta^{ab}}{4\pi^2x^2},$$

$$iS_Q(p) = \frac{i\delta^{ab}}{p-m_Q} + \frac{i}{4} g_s \frac{\lambda^a}{2} \sigma^{\mu\nu} \frac{\bar{p} + m_Q}{12} + \frac{i\delta^{ab}}{12} \langle g_s G\sigma \cdot G\rangle m_Q \frac{p^2 + m_Q^2}{(p^2 - m_Q^2)^2},$$

(11)

where $q$ represents $u$, $d$ or $s$ quark and $Q$ represents $c$ or $b$ quark. The superscripts $a, b$ are color indices and $\bar{x} = x^\mu\gamma_\mu, \bar{p} = p^\mu\gamma_\mu$. In this work, we evaluate the Wilson coefficients up to dimension eight condensates at the leading order in $\alpha_s$. To improve the convergence of the OPE series and suppress the contributions from continuum and higher states region, one can perform the Borel transformation to the correlation function in both hadron and quark-gluon levels. The QCD sum rules are then established as

$$\mathcal{L}_k\left(s_0, M_H^2\right) = f_H^2 m_H^{2k} e^{-m_H^2/M_H^2} = \int_{(m_q+m_c)^2}^{s_0} ds e^{-s/M_H^2} \rho(s)s^k,$$

(12)

where $M_H$ is the Borel mass introduced via the Borel transformation and $s_0$ the continuum threshold. The lowest-lying hadron mass can be thus extracted via the following expression

$$m_H\left(s_0, M_H^2\right) = \sqrt{\frac{\mathcal{L}_1\left(s_0, M_H^2\right)}{\mathcal{L}_0\left(s_0, M_H^2\right)}},$$

(13)
IV. NUMERICAL ANALYSIS

In this section, we perform the QCD sum rule analyses for the $bc\bar{q} \bar{s}$ and $sc\bar{q} \bar{b}$ tetraquarks. We use the following values of quark masses and condensates\cite{27, 31–34}

\[
\begin{align*}
    m_u(2\text{GeV}) & = (2.2^{+0.5}_{-0.4})\text{MeV}, \\
    m_d(2\text{GeV}) & = (4.7^{+0.3}_{-0.3})\text{MeV}, \\
    m_s(2\text{GeV}) & = (3.5^{+0.5}_{-0.2})\text{MeV}, \\
    m_c(m_c) & = (1.275^{+0.025}_{-0.035})\text{GeV}, \\
    m_b(m_b) & = (4.18^{+0.04}_{-0.03})\text{GeV}, \\
    \langle \bar{q}q \rangle & = -(0.23 \pm 0.03)^3\text{GeV}^3, \\
    \langle \bar{q}g_\sigma \cdot Gq \rangle & = -M_0^2 \langle \bar{q}q \rangle, \\
    M_0^2 & = (0.8 \pm 0.2)\text{GeV}^2, \\
    \langle \bar{s}s \rangle / \langle \bar{q}q \rangle & = 0.8 \pm 0.1, \\
    \langle g_0^2GG \rangle & = (0.48 \pm 0.14)\text{GeV}^4,
\end{align*}
\]

in which the masses of $u, d, s$ are the current quark masses in the $\overline{MS}$ scheme at a scale $\mu = 2$ GeV. We consider the scale dependence of the charm and bottom quark masses at the leading order

\[
\begin{align*}
    m_c(\mu) & = \bar{m}_c \left( \frac{\alpha_s(\mu)}{\alpha_s(\bar{m}_c)} \right)^{12/25}, \\
    m_b(\mu) & = \bar{m}_b \left( \frac{\alpha_s(\mu)}{\alpha_s(\bar{m}_b)} \right)^{12/23},
\end{align*}
\]

where

\[
\alpha_s(\mu) = \frac{\alpha_s(M_\tau)}{1 + \frac{25\alpha_s(M_\tau)}{12 \pi} \log \left( \frac{\mu^2}{M_\tau^2} \right)}, \quad \alpha_s(M_\tau) = 0.33
\]

is determined by evolution from the $\tau$ mass using PDG values.

To obtain a stable sum rule, the working regions should also be determined, i.e, the continuum threshold $s_0$ and the Borel mass $M_B^2$. The threshold $s_0$ can be fixed by minimizing the variation of the hadronic masses $m_H$ with the Borel mass $M_B^2$. The Borel mass $M_B^2$ can be obtained by requiring the OPE convergence, which results in the lower bound of $M_B^2$, and a sufficient pole contribution, which results in the upper bound of $M_B^2$. Specific details of these procedures will be shown later. The pole contribution is defined as

\[
\text{PC} \left(s_0, M_B^2 \right) = \frac{\mathcal{L}_0 \left(s_0, M_B^2 \right)}{\mathcal{L}_0 \left(\infty, M_B^2 \right)},
\]

in which $\mathcal{L}_0$ is defined in Eq. (12).

A. $bc\bar{q} \bar{s}$ systems

We firstly perform the QCD sum rule analyses for $bc\bar{q} \bar{s}$ tetraquarks. The spectral densities for the interpolating currents in Eqs. (1)–(2) are calculated and listed in the appendix A. For any interpolating current in the $bc\bar{q} \bar{s}$ system, contributions from the quark condensate $\langle \bar{q}q \rangle$ and quark-gluon mixed condensate $\langle \bar{q}g_\sigma \cdot Gq \rangle$ are numerically small since they are proportional to the quark mass $m_q$ and $m_c$. The dominant nonperturbative contribution to the correlation function comes from the four-quark condensate $\langle \bar{q}q \rangle \langle \bar{s}s \rangle$. In Fig.1, we take the scalar interpolating current $J_1$ as an example to present the contributions to correlation function from the perturbative and various condensate terms. To extract the output parameters, the Borel mass $M_B^2$ should be large enough to guarantee the convergence of OPE series. Here, we require that the four-quark condensate contribution be less than one-fifth of the perturbative term. In Fig.1, we can see that the convergence of OPE series can be ensured while $M_B^2 \geq 5.4 \text{GeV}^2$. 

To get the upper bound of $M_B^2$, we need to fix the value of $s_0$ at first. As mentioned before, the output hadron mass $m_H$ should be irrelevant to $M_B^2$. In Fig. 2, we show the variations of $m_H$ with the threshold $s_0$ and Borel mass $M_B^2$. It is shown that the variation of $m_H$ with $M_B^2$ minimizes around $s_0 \sim 58$ GeV$^2$, which will result in the working region $56 \leq s_0 \leq 60$ GeV$^2$ for the scalar current $J_1$. Using this value of $s_0$, the upper bound of $M_B^2$ can be obtained by requiring the pole contribution be larger than 30%. Finally, the working region of the Borel parameter for the scalar current $J_1$ can be determined to be $5.4 \leq M_B^2 \leq 5.8$ GeV$^2$. We show the Borel curves in these regions in Fig. 2 and extract the hadron mass to be $m_H = 7.17 \pm 0.11$ GeV. The errors come from the continuum threshold $s_0$, condensates $\langle \bar{q}q \rangle$ and $\langle \bar{q}g_\sigma \cdot Gq \rangle$, the heavy quark masses $m_b$ and $m_c$. The errors from the Borel mass and the gluon condensate are small enough to be neglected.

For all other interpolating currents in Eqs. (1)–(2), we perform similar analyses and obtain the suitable working regions for the threshold $s_0$, Borel mass $M_B^2$, output hadron masses, pole contributions. We collect the numerical results in Table I for the scalar $bc\bar{q}\bar{s}$ tetraquarks and Table II for the axial-vector $bc\bar{q}\bar{s}$ tetraquarks. The last columns are the $S$-wave two-meson $\bar{B}_sD$ and $B_s^*D$ thresholds for these possible tetraquark states. It is shown that both the scalar and axial-vector $bc\bar{q}\bar{s}$ tetraquarks lie below the corresponding two-meson thresholds, implying their stabilities against the strong interaction.

B. $sc\bar{q}b$ systems

For the $sc\bar{q}b$ systems, we calculate and list the correlation functions and spectral densities in the appendix A for all interpolating currents in Eqs. (3)–(4). Comparing to the $bc\bar{q}\bar{s}$ system, the OPE series behaviors are very different as shown in Fig.3 (for the scalar current $\eta_1$), where the contributions from the quark condensate $\langle \bar{q}q \rangle$ and quark-gluon
Table I: The continuum threshold, Borel window, hadron mass and pole contribution for the $bc\bar{q}\bar{s}$ system with $J^P = 0^+$.  

| Current | $s_0$(GeV$^2$) | $M_B^2$(GeV$^2$) | $m_H$(GeV) | PC(%) | Two-meson threshold(GeV) |
|---------|---------------|----------------|------------|-------|-------------------------|
| $J_1$   | 58±2          | 5.4~ 5.8       | 7.17 ± 0.11 | 33.1  |                         |
| $J_2$   | 56±2          | 5.4~ 5.6       | 7.04 ± 0.13 | 31.7  | 7.24                    |
| $J_3$   | 57±2          | 5.4~ 5.6       | 7.12 ± 0.15 | 32.7  | ($\bar{B}_sD$)          |
| $J_4$   | 57±2          | 5.3~ 5.5       | 7.11 ± 0.20 | 32.4  |                         |

Table II: The continuum threshold, Borel window, hadron mass and pole contribution for the $bc\bar{q}\bar{s}$ system with $J^P = 1^+$.  

| Current | $s_0$(GeV$^2$) | $M_B^2$(GeV$^2$) | $m_H$(GeV) | PC(%) | Two-meson threshold(GeV) |
|---------|---------------|----------------|------------|-------|-------------------------|
| $J_{1\mu}$ | 58±2          | 5.4~ 5.8       | 7.19 ± 0.12 | 33.1  |                         |
| $J_{2\mu}$ | 57±2          | 5.3~ 5.7       | 7.10 ± 0.11 | 34.3  | 7.28                    |
| $J_{3\mu}$ | 57±2          | 5.4~ 5.7       | 7.10 ± 0.14 | 33.9  | ($B_{1\mu}$)           |
| $J_{4\mu}$ | 58±2          | 5.4~ 5.9       | 7.16 ± 0.13 | 33.7  |                         |

mixed condensate $\langle \bar{q}g_\sigma \cdot Gq \rangle$ are dominant while the contribution from the four-quark condensate $\langle \bar{q}q \rangle^2$ is relatively small. Such difference is due to the existence of $m_Q \langle \bar{q}q \rangle$ and $m_Q \langle \bar{q}g_\sigma \cdot Gq \rangle$ in the $sc\bar{q}\bar{b}$ system. These terms are proportional to the heavy quark mass but absent in the OPE series of the $bc\bar{q}\bar{s}$ system.

Figure 3: The OPE behavior for the current $\eta_1$ in the $J^P = 0^+$ $sc\bar{q}\bar{b}$ system.

For the interpolating current $\eta_1$ with $J^P = 0^+$, we perform the numerical analysis and find the suitable working regions for the continuum threshold and Borel parameter are $55 \leq s_0 \leq 59$ GeV$^2$ and $7.6 \leq M_B^2 \leq 7.9$ GeV$^2$, respectively. We show the variations of $m_H$ with threshold $s_0$ and Borel mass $M_B^2$ in Fig. 4. Accordingly, the hadron mass can be extracted in these parameter regions. For all interpolating currents in the $sc\bar{q}\bar{b}$ systems, we can only establish reliable mass sum rules for $\eta_1$, $\eta_3$, $\eta_4$, $\eta_{1\mu}$ and $\eta_{3\mu}$. We list the numerical results for the threshold $s_0$, Borel mass $M_B^2$, output masses, pole contributions and the two-meson thresholds in Table III for the scalar $sc\bar{q}\bar{b}$ system and Table IV for the axial-vector channel. In Table III, the extracted masses for the scalar $sc\bar{q}\bar{b}$ tetraquarks are slightly below the $B_sD$ threshold while higher than the $B_{1\mu}^+K$ threshold. However, the numerical results in Table IV show that the axial-vector $sc\bar{q}\bar{b}$ tetraquarks lie below both the $B_{1\mu}^+D$ and $B_{1\mu}^+K^*$ thresholds.
Table III: The continuum threshold, Borel window, mass and pole contribution for the \(sc\bar{q}b\) system with \(J^P = 0^+\).

| Current | \(s_0\)(GeV\(^2\)) | \(M_B^2\)(GeV\(^2\)) | \(m_H\)(GeV) | PC(%) | Two-meson threshold(GeV) |
|---------|-----------------|-----------------|--------------|-------|-------------------|
| \(\eta_1\) | 55±2           | 7.6~ 7.9        | 7.11 ± 0.11  | 10.7  | \(6.77(B_s^+K)\) |
| \(\eta_2\) | –              | –              | –            | –     | \(7.24(B_sD)\)    |
| \(\eta_3\) | 54±2           | 6.4~ 7.3        | 7.02 ± 0.12  | 12.7  |                   |
| \(\eta_4\) | 56±2           | 6.4~ 7.7        | 7.13 ± 0.12  | 14.6  |                   |

Table IV: The continuum threshold, Borel window, mass and pole contribution for the \(sc\bar{q}b\) system with \(J^P = 1^+\).

| Current | \(s_0\)(GeV\(^2\)) | \(M_B^2\)(GeV\(^2\)) | \(m_H\)(GeV) | PC(%) | Two-meson threshold(GeV) |
|---------|-----------------|-----------------|--------------|-------|-------------------|
| \(\eta_{1\mu}\) | 55±2           | 7.7~ 7.9        | 7.10 ± 0.12  | 11.2  | \(7.17(B_s^+K^)\) |
| \(\eta_{2\mu}\) | –              | –              | –            | –     | \(7.28(B_s^*D)\)  |
| \(\eta_{3\mu}\) | 54±2           | 7.6~ 7.8        | 7.04 ± 0.12  | 10.4  |                   |
| \(\eta_{4\mu}\) | –              | –              | –            | –     |                   |

V. CONCLUSION

We have investigated the mass spectra for the fully open-flavored \(bc\bar{s}\) and \(sc\bar{q}b\) tetraquark states in the framework of QCD sum rules. We construct the interpolating tetraquark currents with \(J^P = 0^+\) and \(1^+\) and calculate their two-point correlation functions and spectral densities up to dimension eight condensates at the leading order of \(\alpha_s\).

For the \(bc\bar{s}\) tetraquark states, we find that the quark condensate \(\langle \bar{q}q \rangle\) and quark-gluon mixed condensate \(\langle \bar{q}g, \sigma \cdot Gq \rangle\) are proportional to the light quark mass and thus numerically small. The dominant nonperturbative contribution to the correlation function comes from the four-quark condensate \(\langle \bar{q}q \rangle \langle \bar{s}s \rangle\). The OPE series are very different for the \(sc\bar{q}b\) tetraquark systems, where the quark condensate and quark-gluon mixed condensate provide more important contribution than the four-quark condensate. Such difference leads to distinct behavior of the mass sum rules between the \(bc\bar{s}\) and \(sc\bar{q}b\) tetraquark systems.

After the numerical analyses, we extract the masses around \(7.1 \sim 7.2\) GeV for both the scalar and axial-vector \(bc\bar{s}\) tetraquark states while \(7.0 \sim 7.1\) GeV for the \(sc\bar{q}b\) tetraquarks. These results show that the masses of the \(bc\bar{s}\) tetraquark states are below the \(B_s\)\(D\) and \(B_s^*D\) two-meson S-wave thresholds, which are consistent with the results from the color-magnetic interaction model [21]. For the axial-vector \(sc\bar{q}b\) tetraquarks, their masses are also lower than the two-meson thresholds of \(B_s^+K^*\) and \(B_s^*D\) modes. Such results indicate that the two-meson strong decay modes are kinematically forbidden for these possible tetraquark states. They can only decay via the weak interaction if they do exist. For the scalar \(sc\bar{q}b\) tetraquark state, their decay to the \(B_s\)\(D\) final states is also forbidden, but the \(B_s^+K\) decay mode is allowed due to their slightly higher masses. However, such decay will be difficult since the low
production rate of the $B_{c}^{+}$ meson. These stable teraquark states may be found in BelleII and LHCb in future.

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Appendix A: The spectral densities

In this appendix, we list the spectral densities evaluated for the $bc \bar{s}\bar{s}$ and $sc\bar{q}\bar{b}$ tetraquark systems with $J^{P} = 0^{+}$ and $1^{+}$. The spectral density includes the perturbative term, quark condensate, gluon condensate, quark-gluon mixed condensate, four-quark condensate and dimension eight condensate

$$
\rho(s) = \rho^{0}(s) + \rho^{3}(s) + \rho^{4}(s) + \rho^{5}(s) + \rho^{6}(s) + \rho^{8}(s),
$$

(A1)
in which the superscripts stand for the dimension of various condensates.
1. The spectral densities for the $bc\bar{q}\bar{s}$ tetraquarks

1. Spectral densities for $J_1$:

\[
\rho_{J_1}^{9}(s) = \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \, d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} \, d\beta \frac{(1 - \alpha - \beta)^2 (m_b^2 + m_c^2 - \alpha \beta s - 2m_b m_c)}{256 \pi^6 \alpha^3 \beta^3},
\]

\[
\rho_{J_1}^{3}(s) = \left[ m_q \langle q\bar{q} \rangle + m_s \langle s\bar{s} \rangle \right] \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \, d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} \, d\beta \frac{(m_b^2 + m_c^2 - \alpha \beta s - 2m_b m_c)}{16 \pi^4 \alpha \beta},
\]

\[- \left[ m_s \langle q\bar{q} \rangle + m_q \langle s\bar{s} \rangle \right] \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \, d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} \, d\beta \frac{(m_b^2 + m_c^2 - \alpha \beta s - 2m_b m_c)}{8 \pi^4 \alpha \beta},
\]

\[
\rho_{J_1}^{4}(s) = \langle g^2 GG \rangle \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \, d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} \, d\beta \frac{(1 - \alpha - \beta)^2}{1536 \pi^6} \left[ (2m_b^2 + 2m_c^2 - 3\alpha \beta s) \frac{m_b^2 + m_c^2}{\alpha \beta} \right]
\]

\[- \frac{m_b m_c}{\alpha \beta} \left( \frac{4m_b^2 + 3m_c^2 - 3\alpha \beta s}{\alpha^2} + \frac{3m_b^2 + 4m_c^2 - 3\alpha \beta s}{\beta^2} \right) \right],
\]

\[
\rho_{J_1}^{5}(s) = - \langle g^2 GG \rangle \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \, d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} \, d\beta \frac{(m_b^2 + m_c^2 - \alpha \beta s - 2m_b m_c)}{1024 \pi^4}
\]

\[+ \frac{(1 - \alpha - \beta)^2}{2\alpha^2 \beta^2} \left( m_b^2 + m_c^2 - 2\alpha \beta s - 2m_b m_c \right),
\]

\[
\rho_{J_1}^{6}(s) = - \left[ m_q \langle \bar{s}g_s \sigma \cdot Gs \rangle + m_s \langle qg_s \sigma \cdot Gq \rangle \right] \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \, d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} \, d\beta \frac{s - (m_b - m_c)}{64 \pi^4} \sqrt{1 + \frac{m_b^2 - m_c^2}{s} - \frac{4m_b^2}{s}},
\]

\[+ \left[ m_s \langle \bar{s}g_s \sigma \cdot Gs \rangle + m_q \langle qg_s \sigma \cdot Gq \rangle \right] \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} \, d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} \, d\beta \frac{s - (m_b - m_c)}{192 \pi^4} \sqrt{1 + \frac{m_b^2 - m_c^2}{s} - \frac{4m_b^2}{s}},
\]

\[
\rho_{J_1}^{7}(s) = \left[ \langle q\bar{q} \rangle \langle \bar{s}s \rangle \langle qg_s \sigma \cdot Gq \rangle \right] \int_{0}^{1} d\alpha \frac{1}{24 \pi^2} \left[ \frac{m_b m_c^3 + m_c \alpha + m_b^2 m_c^2 (1 - \alpha)}{(1 - \alpha)} \right]
\]

\[- \delta \left[ m_b^2 (1 - \alpha) + 2m_c^2 \alpha \left[ s - \frac{m_b^2 (1 - \alpha) + m_c^2 \alpha}{\alpha(1 - \alpha)} \right] - 2\alpha \left[ s - \frac{m_b^2 (1 - \alpha) + m_c^2 \alpha}{\alpha(1 - \alpha)} \right] \right],
\]

where

\[
\alpha_{\text{min}} = \frac{1}{2} \left[ 1 + \frac{m_b^2 - m_c^2}{s} - \sqrt{1 + \frac{m_b^2 - m_c^2}{s} - \frac{4m_b^2}{s}} \right],
\]

\[
\alpha_{\text{max}} = \frac{1}{2} \left[ 1 + \frac{m_b^2 - m_c^2}{s} + \sqrt{1 + \frac{m_b^2 - m_c^2}{s} - \frac{4m_b^2}{s}} \right],
\]

\[
\beta_{\text{min}} = \frac{\alpha m_c^2}{\alpha s - m_b}, \quad \beta_{\text{max}} = 1 - \alpha.
\]
\(\delta\) and \(H\) denote the Dirac delta and Heaviside theta function, respectively.

2. Spectral densities for \(J_2\):
\[
\rho^0_{J_2}(s) = \frac{1}{2} \rho^0_{J_1}(s), \quad \rho^3_{J_2}(s) = \frac{1}{2} \rho^3_{J_1}(s), \quad \rho^{4a}_{J_2}(s) = \frac{1}{2} \rho^{4a}_{J_1}(s), \quad \rho^{4b}_{J_2}(s) = -\rho^{4b}_{J_1}(s), \quad \rho^5_{J_2}(s) = \frac{1}{2} \rho^5_{J_1}(s), \quad \rho^6_{J_2}(s) = \frac{1}{2} \rho^6_{J_1}(s).
\]

3. Spectral densities for \(J_3\):
\[
\rho^0_{J_3}(s) = \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(1 - \alpha - \beta)^2(m_b^2 \beta + m_c^2 \alpha - \alpha \beta s)^3(m_b^2 \beta + m_c^2 \alpha - 3 \alpha \beta s - m_b m_c)}{64\pi^6 \alpha^3 \beta^3},
\]
\[
\rho^3_{J_3}(s) = \left[ m_q \langle \bar{q}q \rangle + m_s \langle \bar{s}s \rangle \right] \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(m_b^2 \beta + m_c^2 \alpha - \alpha \beta s)(2m_b^2 \beta + 2m_c^2 \alpha - 4 \alpha \beta s - m_b m_c)}{8\pi^4 \alpha \beta},
\]
\[
- \left[ m_e \langle \bar{q}q \rangle + m_q \langle \bar{s}s \rangle \right] \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(m_b^2 \beta + m_c^2 \alpha - \alpha \beta s)(m_b^2 \beta + m_c^2 \alpha - 2 \alpha \beta s - 2m_b m_c)}{4\pi^3 \alpha \beta},
\]
\[
\rho^{4a}_{J_3}(s) = \langle g_s^2 GG \rangle \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(1 - \alpha - \beta)^2}{384\pi^6} \left[ (2m_b^2 \beta + 2m_c^2 \alpha - 3 \alpha \beta s) \left( \frac{m_b^2}{\alpha^3} + \frac{m_c^2}{\beta^3} \right) \right. - \frac{m_b m_c}{2\alpha \beta} \left( \frac{4m_b^2 \beta + 3m_c^2 \alpha - 3 \alpha \beta s}{\alpha^2} + \frac{3m_b^2 \beta + 4m_c^2 \alpha - 3 \alpha \beta s}{\beta^2} \right)
\]
\[
\left. \right] \frac{(m_b^2 \beta + m_c^2 \alpha - \alpha \beta s)}{512\pi^6 \alpha \beta},
\]
\[
\rho^{4b}_{J_3}(s) = - \langle g_s^2 GG \rangle \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(m_b^2 \beta + m_c^2 \alpha - \alpha \beta s) m_b m_c}{32\pi^4 \alpha \beta},
\]
\[
\rho^5_{J_3}(s) = - \left[ m_q \langle \bar{s}g_s \sigma \cdot Gs \rangle + m_s \langle \bar{g}_s \sigma \cdot Gq \rangle \right] \int^s_{m_b} \frac{d \beta}{d \beta} \frac{\left( 1 + \frac{m_b^2}{s} \right)^2}{32\pi^4} - \frac{4m_b^2}{s},
\]
\[
+ \left[ m_s \langle \bar{s}g_s \sigma \cdot Gs \rangle + m_q \langle \bar{g}_s \sigma \cdot Gq \rangle \right] \frac{m_b}{6\pi^4} \int^s_{m_b} \frac{d \beta}{d \beta} \left( \frac{1 + \frac{m_b^2}{s}}{32\pi^4} - \frac{4m_b^2}{s} \right),
\]
\[
\rho^6_{J_3}(s) = \left[ \langle \bar{q}q \rangle \langle \bar{s}s \rangle \right] \frac{s - (m_b - m_c)^2 + 2m_b m_c}{6\pi^4} \sqrt{\left( 1 + \frac{m_b^2 - m_c^2}{s} \right)^2 - \frac{4m_b^2}{s}},
\]
\[
\rho^5_{J_3}(s) = \left[ \langle \bar{q}q \rangle \langle \bar{s}s \rangle \cdot Gs \right] \left[ \langle \bar{s}s \rangle \langle \bar{g}_s \sigma \cdot Gq \rangle \right] \int^1_0 \frac{d \alpha}{d \alpha} \frac{1}{12\pi^2} \left\{ \left[ \frac{2m_b m_c^3 + m_b^3 + m_c^3}{(1-\alpha)^2} \right] \right.
\]
\[
\times \left[ \right. \frac{m_b^2 (1 - \alpha) + m_c^2 \alpha}{\alpha (1 - \alpha)} \right] + \left[ \right. \frac{2m_b m_c^2 (1 - \alpha) + 2m_b^2 \alpha}{(1 - \alpha)} \right] \delta \left[ \right. \frac{m_c (1 - \alpha)}{\alpha (1 - \alpha)} \left. \right] - 2\alpha H \left[ \right. \frac{m_c^2 (1 - \alpha) + m_c^2 \alpha}{\alpha (1 - \alpha)} \right].
\]

4. Spectral densities for \(J_4\):
\[
\rho^0_{J_4}(s) = \frac{1}{2} \rho^0_{J_1}(s), \quad \rho^3_{J_4}(s) = \frac{1}{2} \rho^3_{J_1}(s), \quad \rho^{4a}_{J_4}(s) = \frac{1}{2} \rho^{4a}_{J_1}(s), \quad \rho^{4b}_{J_4}(s) = -\rho^{4b}_{J_1}(s), \quad \rho^5_{J_4}(s) = \frac{1}{2} \rho^5_{J_1}(s), \quad \rho^6_{J_4}(s) = \frac{1}{2} \rho^6_{J_1}(s).
\]

5. Spectral densities for \(J_1\mu\):
\[
\rho^0_{J_1\mu}(s) = \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(1 - \alpha - \beta)^2(m_b^2 \beta + m_c^2 \alpha - \alpha \beta s)^3(m_b^2 \beta + m_c^2 \alpha - 5 \alpha \beta s - 4m_b m_c)}{512\pi^6 \alpha^3 \beta^3},
\]
\[
\rho_{J_{1\mu}}^3(s) = \left[ m_q \langle \bar{q}q \rangle + m_s \langle \bar{s}s \rangle \right] \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(m_b^2 \beta + m_c^2 \alpha - n \beta s)(m_b^2 \beta + m_c^2 \alpha - 3 \alpha \beta s - 2m_b m_c)}{32\pi^4 \alpha \beta},
\]

\[
- \left[ m_s \langle \bar{q}q \rangle + m_q \langle \bar{s}s \rangle \right] \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(m_b^2 \beta + m_c^2 \alpha - n \beta s)(m_b^2 \beta + m_c^2 \alpha - 3 \alpha \beta s - 2m_b m_c)}{16\pi^4 \alpha \beta},
\]

\[
\rho_{J_{1\mu}}^{4a}(s) = (g_{2G}^2 G^2 \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(1 - \alpha - \beta)^2}{1536\pi^6} \left[ (m_b^2 \beta + m_c^2 \alpha - 2 \alpha \beta s) \left( \frac{m_b^2}{\alpha \beta} + \frac{m_c^2}{\beta^2} \right) \right],
\]

\[
- \frac{m_s m_c}{\alpha \beta} \left( 4m_b^2 \beta + 3m_c^2 \alpha - 3 \alpha \beta s + \frac{3m_b^2 \beta + 4m_c^2 \alpha - 3 \alpha \beta s}{\beta^2} \right),
\]

\[
\rho_{J_{1\mu}}^{4b}(s) = - (g_{2G}^2 G^2 \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(m_b^2 \beta + m_c^2 \alpha - 3 \alpha \beta s - 2m_b m_c)}{2048\pi^6} \left[ (m_b^2 \beta + m_c^2 \alpha - 3 \alpha \beta s - 2m_b m_c) \right],
\]

\[
- \frac{(1 - \alpha - \beta)^2}{6\alpha^2 \beta^2} \left( 3m_b^2 \beta + 3m_c^2 \alpha - 5 \alpha \beta s \right),
\]

\[
\rho_{J_{1\mu}}^{5a}(s) = - \left[ m_q \langle \bar{q}g_\sigma \cdot Gs \rangle + m_s \langle \bar{g}g_\sigma \cdot Gq \rangle \right] \left[ \frac{(m_b + m_c)^2 + 2s[s - (m_b - m_c)]}{192\pi^4 s} \sqrt{\frac{(1 + \frac{m_b^2 - m_c^2}{s})^2 - 4\frac{m_b^2}{s} \beta^2}{s}}
\]

\[
+ \left[ m_s \langle \bar{q}g_\sigma \cdot Gs \rangle + m_q \langle \bar{g}g_\sigma \cdot Gq \rangle \right] \left[ \frac{(m_b + m_c)^2 + 2s[s - (m_b - m_c)]}{576\pi^4 s} \sqrt{\frac{(1 + \frac{m_b^2 - m_c^2}{s})^2 - 4\frac{m_b^2}{s}}{s}} \right],
\]

\[
\rho_{J_{1\mu}}^{6}(s) = \left[ \langle \bar{q}q \rangle \langle \bar{s}s \rangle \right] \left[ \frac{(m_b + m_c)^2 + 2s[s - (m_b - m_c)]}{36\pi^2 s} \sqrt{\frac{(1 + \frac{m_b^2 - m_c^2}{s})^2 - 4\frac{m_b^2}{s}}{s}} \right],
\]

\[
\rho_{J_{1\mu}}^{7}(s) = \left[ \langle \bar{q}q \rangle \langle \bar{s}s \rangle \cdot Gs \right] \langle \bar{s}s \rangle \langle \bar{g}g_\sigma \cdot Gq \rangle \int_0^1 \frac{1}{24\pi^2} \left[ \left( \frac{m_b m_c^2 + m_c^4 \alpha + m_b^2 m_c^2 (1 - \alpha)}{(1 - \alpha)^2} \right) \right]
\]

\[
\times \delta \left[ s - \frac{m_b^2 (1 - \alpha) + m_c^2 \alpha}{\alpha (1 - \alpha)} \right] \left[ \frac{m_b^2 (1 - \alpha) + m_c^2 \alpha}{(1 - \alpha)} \beta \left[ s - \frac{m_b^2 (1 - \alpha) + m_c^2 \alpha}{\alpha (1 - \alpha)} \right] \right].
\]

6. Spectral densities for \( J_{2\mu} \):

\[
\rho_{J_{2\mu}}^{0}(s) = \frac{1}{2} \rho_{J_{1\mu}}^{0}(s), \quad \rho_{J_{2\mu}}^{3}(s) = \frac{1}{2} \rho_{J_{1\mu}}^{3}(s), \quad \rho_{J_{2\mu}}^{4a}(s) = \frac{1}{2} \rho_{J_{1\mu}}^{4a}(s), \quad \rho_{J_{2\mu}}^{4b}(s) = - \rho_{J_{1\mu}}^{4b}(s), \quad \rho_{J_{2\mu}}^{5}(s) = \frac{1}{2} \rho_{J_{1\mu}}^{5}(s), \quad \rho_{J_{2\mu}}^{6}(s) = \frac{1}{2} \rho_{J_{1\mu}}^{6}(s), \quad \rho_{J_{2\mu}}^{7}(s) = \frac{1}{2} \rho_{J_{1\mu}}^{7}(s).
\]

7. Spectral densities for \( J_{3\mu} \):

\[
\rho_{J_{3\mu}}^{0}(s) = \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(1 - \alpha - \beta)^2}{1536\pi^6 \alpha^3 \beta^3} \left[ 6(m_b^2 \beta + m_c^2 \alpha - 3 \alpha \beta s - 2m_b m_c) \right]
\]

\[
- (1 - \alpha - \beta)(3m_b^2 \beta + 3m_c^2 \alpha - 7 \alpha \beta s - 4m_b m_c),
\]

\[
\rho_{J_{3\mu}}^{3}(s) = \left[ m_q \langle \bar{q}q \rangle + m_s \langle \bar{s}s \rangle \right] \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(m_b^2 \beta + m_c^2 \alpha - 2 \alpha \beta s - m_b m_c)}{16\pi^4 \alpha \beta} \left[ (m_b^2 \beta + m_c^2 \alpha - 2 \alpha \beta s - m_b m_c) \right],
\]

\[
- \frac{1}{2} \left[ (1 - \alpha - \beta)(3m_b^2 \beta + 3m_c^2 \alpha - 5 \alpha \beta s - 2m_b m_c) \right]
\]

\[
- \left[ m_s \langle \bar{q}q \rangle + m_q \langle \bar{s}s \rangle \right] \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(m_b^2 \beta + m_c^2 \alpha - 2 \alpha \beta s - m_b m_c)}{8\pi^4 \alpha \beta} \left[ (m_b^2 \beta + m_c^2 \alpha - 2 \alpha \beta s - m_b m_c) \right].
\]
\[ \rho_{J_{\mu}}^{4a}(s) = (g_s^2 G G) \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(1 - \alpha - \beta)^2}{4608\pi^6} \left\{ 3(2m_c^2 \beta + 2m_c^2 \alpha - 3\alpha\beta s) - (1 - \alpha - \beta)(3m_c^2 \beta + 3m_c^2 \alpha - 4\alpha\beta s) \left[ \frac{m_b^2}{\alpha^3} + \frac{m_c^2}{\beta^3} \right] - m_b m_c (\alpha + \beta + 2) \left( \frac{4m_b^2 \beta + 3m_c^2 \alpha - 3\alpha\beta s}{\alpha^2} + \frac{3m_b^2 \beta + 4m_c^2 \alpha - 3\alpha\beta s}{\beta^2} \right) \right\}. \]

\[ \rho_{J_{\mu}}^{4b}(s) = (g_s^2 G G) \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(m_b^2 \beta + m_c^2 \alpha - \alpha\beta s)}{12288\pi^6} \left\{ \frac{2}{\alpha \beta} \left[ 2(m_b^2 \beta + m_c^2 \alpha - 2\alpha\beta s - m_b m_c) + (1 - \alpha - \beta)(3m_c^2 \beta + 3m_c^2 \alpha - 5\alpha\beta s - 2m_b m_c) \right] - (1 - \alpha - \beta)(3m_c^2 \beta + 3m_c^2 \alpha - 5\alpha\beta s - 4m_b m_c) \right\}. \]

\[ \rho_{J_{\mu}}^{5a}(s) = - m_q \langle \bar{s}g_s \sigma \cdot G s \rangle + m_s \langle \bar{q}g_s \sigma \cdot G q \rangle \left[ \frac{s - (m_b - m_c)}{64\pi^4} \sqrt{1 + \frac{m_b^2 - m_c^2}{s}} - \frac{4m_b^2}{s} \right] + m_s \langle \bar{s}g_s \sigma \cdot G s \rangle + m_q \langle \bar{q}g_s \sigma \cdot G q \rangle \left[ \frac{s - (m_b - m_c)}{192\pi^4} \sqrt{1 + \frac{m_b^2 - m_c^2}{s}} - \frac{4m_b^2}{s} \right]. \]

\[ \rho_{J_{\mu}}^{5b}(s) = \left[ m_s \langle \bar{s}g_s \sigma \cdot G s \rangle + m_q \langle \bar{q}g_s \sigma \cdot G q \rangle \right] \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(3m_c^2 \beta + 3m_c^2 \alpha - 4\alpha\beta s - m_b m_c)}{96\pi^4}. \]

\[ \rho_{J_{\mu}}^{5c}(s) = \left[ \langle \bar{q}q \rangle \langle \bar{s}s \rangle \right] \left[ \frac{s - (m_b - m_c)}{12\pi^2} \sqrt{1 + \frac{m_b^2 - m_c^2}{s}} - \frac{4m_b^2}{s} \right]. \]

\[ \rho_{J_{\mu}}^{8}(s) = \left[ \langle \bar{q}q \rangle \langle \bar{s}s \rangle \langle \bar{g}_s \sigma \cdot G s \rangle + \langle \bar{s}s \rangle \langle \bar{q}g_s \sigma \cdot G q \rangle \right] \int_{0}^{\alpha_{\text{max}}} d\alpha \frac{1}{24\pi^2} \left\{ \left[ \frac{m_b m_c^3 + m_b^4 \alpha + m_c^5 \alpha^2}{(1 - \alpha)^2} \right] \frac{(1 - \alpha - \beta)^2}{12 \pi^2} \left\{ \frac{m_b m_c^3 + m_b^4 \alpha + m_c^5 \alpha^2}{(1 - \alpha)^2} \right\} \right\} \times \delta' \left[ s - \frac{m_b^2(1 - \alpha) + m_c^2 \alpha}{\alpha(1 - \alpha)} \right] + m_b^2(1 - \alpha) + \frac{2m_b m_c^2 \alpha}{(1 - \alpha)} \delta' \left[ s - \frac{m_b^2(1 - \alpha) + m_c^2 \alpha}{\alpha(1 - \alpha)} \right] - 2\alpha H \left[ s - \frac{m_b^2(1 - \alpha) + m_c^2 \alpha}{\alpha(1 - \alpha)} \right]. \]

8. Spectral densities for $J_{4\mu}$:

\[ \rho_{J_{4\mu}}^{0}(s) = \frac{1}{2} \rho_{J_{\mu}}^{0}(s), \quad \rho_{J_{4\mu}}^{3}(s) = \frac{1}{2} \rho_{J_{\mu}}^{3}(s), \quad \rho_{J_{4\mu}}^{4a}(s) = \frac{1}{2} \rho_{J_{\mu}}^{4a}(s), \quad \rho_{J_{4\mu}}^{4b}(s) = - \rho_{J_{4\mu}}^{4b}(s), \quad \rho_{J_{4\mu}}^{5a}(s) = \frac{1}{2} \rho_{J_{\mu}}^{5a}(s), \quad \rho_{J_{4\mu}}^{5b}(s) = \frac{1}{2} \rho_{J_{\mu}}^{5b}(s), \quad \rho_{J_{4\mu}}^{5c}(s) = \frac{1}{2} \rho_{J_{\mu}}^{5c}(s). \]

2. **The spectral densities for $sc\bar{q}\bar{b}$ tetraquarks**

1. Spectral densities for $\eta_1$:

\[ \rho_{\eta_1}^{0}(s) = \int_{\alpha_{\text{min}}}^{\alpha_{\text{max}}} d\alpha \int_{\beta_{\text{min}}}^{\beta_{\text{max}}} d\beta \frac{(1 - \alpha - \beta)^2}{256\pi^6} \left\{ \frac{1}{\alpha^3 \beta^3} (m_b^2 \beta + m_c^2 \alpha - \alpha\beta s) \right\} \left\{ \frac{1}{\alpha^3 \beta^3} (m_b^2 \beta + m_c^2 \alpha - \alpha\beta s) \right\} \left\{ \frac{1}{\alpha^3 \beta^3} (m_b^2 \beta + m_c^2 \alpha - \alpha\beta s) \right\} \left\{ \frac{1}{\alpha^3 \beta^3} (m_b^2 \beta + m_c^2 \alpha - \alpha\beta s) \right\} \times \left\{ m_b^2 \beta + m_c^2 \alpha - 3\alpha\beta s \right\} - \left\{ \frac{2m_b m_q}{\alpha^3 \beta^3} + \frac{2m_c m_s}{\alpha^3 \beta^3} \right\} \left\{ (2m_b^2 \beta + 2m_c^2 \alpha - 5\alpha\beta s) \right\} \left\{ (2m_b^2 \beta + 2m_c^2 \alpha - 5\alpha\beta s) \right\} \left\{ (2m_b^2 \beta + 2m_c^2 \alpha - 5\alpha\beta s) \right\} \left\{ (2m_b^2 \beta + 2m_c^2 \alpha - 5\alpha\beta s) \right\} \right\}. \]
\[
\rho^3_{\eta}(s) = \left[ m_q \langle \bar{q}q \rangle + m_s \langle s\bar{s} \rangle \right] \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{m_q^2 + m_s^2 - \alpha \beta s}{16\pi^4}\right) \frac{(m_q^2 \beta + m_s^2 - 2\alpha \beta s)(m_q^2 \beta + m_s^2 - 2\alpha \beta s)}{16\pi^4 \alpha \beta} \\
+ \left[ m_s \langle \bar{q}q \rangle + m_q \langle s\bar{s} \rangle \right] \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{m_s^2 \beta + m_q^2}{8\pi^4}\right) \frac{(m_q^2 \beta + m_s^2 - 2\alpha \beta s)(m_q^2 \beta + m_s^2 - 2\alpha \beta s)}{8\pi^4 \alpha \beta} \\
- m_b \langle \bar{q}q \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{(1 - \alpha - \beta)(m_b^2 \beta + m_q^2 - 2\alpha \beta s)(m_b^2 \beta + m_q^2 - 2\alpha \beta s)}{8\pi^4 \alpha \beta^2} \right),
\]

\[
\rho^{4a}_{\eta}(s) = \left( g_s^2 \right) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{1 - \alpha - \beta}{1024\pi^6}\right) \frac{(1 - \alpha - \beta)(2m_b^2 \beta + 2m_s^2 - 3\alpha \beta s)}{1024\pi^6 \alpha \beta^2} \left( \frac{m_b^2}{\alpha^3} + \frac{m_s^2}{\beta^3} \right),
\]

\[
\rho^{4b}_{\eta}(s) = - \left( g_s^2 \right) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{1 - \alpha - \beta}{1024\pi^6 \alpha \beta^2}\right) \frac{(1 - \alpha - \beta)(2m_b^2 \beta + 2m_s^2 - 3\alpha \beta s)}{1024\pi^6 \alpha \beta^2} \left( \frac{m_b^2}{\alpha^3} + \frac{m_s^2}{\beta^3} \right),
\]

\[
\rho^{5a}_{\eta}(s) = m_b \langle \bar{q}q, \sigma \cdot Gq \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{m_b^2 \beta + m_s^2 - 3\alpha \beta s}{32\pi^4 \alpha \beta} \right) \frac{(m_b^2 \beta + m_s^2 - 2\alpha \beta s)}{32\pi^4 \beta} \\
+ m_c \langle \bar{s}g_s \cdot Gs \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{m_c \beta + m_s^2 - 3\alpha \beta s}{64\pi^4 \alpha \beta^2} \right) \frac{(m_c \beta + m_s^2 - 2\alpha \beta s)}{64\pi^4 \alpha \beta^2},
\]

\[
\rho^{5b}_{\eta}(s) = \langle \bar{q}g_s \cdot Gq \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{m_b \beta + m_s^2 - 3\alpha \beta s}{64\pi^4 \alpha \beta^2} \right) \frac{(m_b \beta + m_s^2 - 2\alpha \beta s)}{64\pi^4 \alpha \beta^2} \\
+ \langle \bar{s}g_s \cdot Gs \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{m_c \beta + m_s^2 - 3\alpha \beta s}{64\pi^4 \alpha \beta^2} \right) \frac{(m_c \beta + m_s^2 - 2\alpha \beta s)}{64\pi^4 \alpha \beta^2},
\]

\[
\rho^{6a}_{\eta}(s) = \left( \bar{q}q \langle \bar{q} \bar{q} \cdot Gq \rangle + \bar{s}s \langle \bar{s} \bar{s} \cdot Gs \rangle \right) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{m_b \beta + m_s^2 - 3\alpha \beta s}{192\pi^4 \alpha \beta} \right) \frac{(m_b \beta + m_s^2 - 2\alpha \beta s)}{192\pi^4 \alpha \beta} \\
- \left[ m_q \langle \bar{q}q, \sigma \cdot Gq \rangle + m_s \langle s\bar{s} \cdot Gs \rangle \right] \frac{m_b m_c s}{32\pi^4} \sqrt{\left( 1 + \frac{m_b^2 - m_s^2}{s} \right)^2 - \frac{4m_b^2}{s}},
\]

\[
\rho^{6b}_{\eta}(s) = \left( \frac{\bar{q}q}{12\pi^2} \right) \left[ m_b m_c + m_b m_s + m_c m_q - \frac{m_b m_c m_q (s - m_b^2 + m_s^2) + m_b m_s m_q (s + m_b^2 - m_s^2)}{(s - m_b^2 + m_s^2)^2 - 4m_b^2 s} \right] \\
\times \sqrt{\left( 1 + \frac{m_b^2 - m_s^2}{s} \right)^2 - \frac{4m_b^2}{s}},
\]

\[
\rho^{8a}_{\eta}(s) = \left( \bar{q}q \langle \bar{q} \bar{q} \cdot Gq \rangle + \langle s \bar{s} \rangle \langle s \bar{s} \cdot Gs \rangle \right) \int_{0}^{1} d\alpha \frac{1}{24\pi^2} \frac{m_b m_c^3}{(1 - \alpha)^2} \delta' \left[ s - \frac{m_b^2 (1 - \alpha) + m_s^2 s}{\alpha (1 - \alpha)} \right],
\]

\[
\rho^{8b}_{\eta}(s) = \langle \bar{q}q \langle \bar{q} \bar{q} \cdot Gq \rangle \rangle \int_{0}^{1} d\alpha \frac{m_b m_c^3}{48\pi^2 (1 - \alpha)} \delta \left[ s - \frac{m_b^2 (1 - \alpha) + m_s^2 s}{\alpha (1 - \alpha)} \right] \\
+ \langle \bar{s}s \rangle \langle s \bar{s} \cdot Gs \rangle \int_{0}^{1} d\alpha \frac{m_b m_c^3}{48\pi^2 \alpha} \delta \left[ s - \frac{m_b^2 (1 - \alpha) + m_s^2 s}{\alpha (1 - \alpha)} \right].
\]
2. Spectral densities for $\eta_2$:

\[ \rho^0_{\eta_2}(s) = \frac{1}{2} \rho^0_{\eta_2}(s), \quad \rho^3_{\eta_2}(s) = \frac{1}{2} \rho^3_{\eta_2}(s), \quad \rho^{4a}_{\eta_2}(s) = \frac{1}{2} \rho^{4a}_{\eta_2}(s), \quad \rho^{4b}_{\eta_2}(s) = -\rho^{4b}_{\eta_2}(s), \quad \rho^{5a}_{\eta_2}(s) = \frac{1}{2} \rho^{5a}_{\eta_2}(s), \quad \rho^{5b}_{\eta_2}(s) = -\rho^{5b}_{\eta_2}(s), \quad \rho^{6a}_{\eta_2}(s) = \frac{1}{2} \rho^{6a}_{\eta_2}(s), \quad \rho^{6b}_{\eta_2}(s) = \frac{1}{2} \rho^{6b}_{\eta_2}(s), \quad \rho^{6c}_{\eta_2}(s) = -\rho^{6c}_{\eta_2}(s) . \]

3. Spectral densities for $\eta_3$:

\[
\rho^0_{\eta_3}(s) = \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)^2(m^2_2 \beta + m^2_2 \alpha - \alpha \beta s)^2}{64\pi^6} \left\{ \frac{1}{\alpha^3 \beta^3} (m^2_2 \beta + m^2_2 \alpha - \alpha \beta s) \times \left( m^2_2 \beta + m^2_2 \alpha - 3\alpha \beta s \right) - \left( \frac{m^2_m}{\alpha^2 \beta^2} + \frac{m^2_m}{\alpha^2 \beta^2} \right) \left( 2m^2_2 \beta + 2m^2_2 \alpha - 5\alpha \beta s \right) \right\} ,
\]

\[
\rho^3_{\eta_3} = \left[ m_q(q \bar{q}) + m_s(s \bar{s}) \right] \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(m^2_2 \beta + m^2_2 \alpha - \alpha \beta s)(m^2_2 \beta + m^2_2 \alpha - 2\alpha \beta s)}{4\pi\alpha^3 \beta^3} + \left[ m_q(q \bar{q}) + m_q(s \bar{s}) \right] \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(m^2_2 \beta + m^2_2 \alpha - \alpha \beta s)(m^2_2 \beta + m^2_2 \alpha - 2\alpha \beta s)}{4\pi\alpha^3 \beta^3} - m_b(q \bar{q}) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)(m^2_2 \beta + m^2_2 \alpha - \alpha \beta s)(m^2_2 \beta + m^2_2 \alpha - 2\alpha \beta s)}{4\pi\alpha^3 \beta^3} - m_c(s \bar{s}) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)(m^2_2 \beta + m^2_2 \alpha - \alpha \beta s)(m^2_2 \beta + m^2_2 \alpha - 2\alpha \beta s)}{4\pi\alpha^3 \beta^3} ,
\]

\[
\rho^{4a}_{\eta_3} = (g_s^2 G G) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)^2(2m^2_2 \beta + 2m^2_2 \alpha - 3\alpha \beta s)}{384\pi^6} \left( \frac{m^2_2}{\alpha^3} + \frac{m^2_2}{\beta^3} \right) ,
\]

\[
\rho^{4b}_{\eta_3} = (g_s^2 G G) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)(m^2_2 \beta + m^2_2 \alpha - \alpha \beta s)(m^2_2 \beta + m^2_2 \alpha - 2\alpha \beta s)}{512\pi^4 \alpha \beta} \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) ,
\]

\[
\rho^{4c}_{\eta_3} = (g_s^2 G G) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(m^2_2 \beta + m^2_2 \alpha - \alpha \beta s)(m^2_2 \beta + m^2_2 \alpha - 2\alpha \beta s)}{512\pi^4 \alpha \beta} \left[ 1 + \frac{(1 - \alpha - \beta)^2}{2\alpha \beta} \right] ,
\]

\[
\rho^{5a}_{\eta_3} = m_b(qg_s \sigma \cdot Gq) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{2m^2_2 \beta + 2m^2_2 \alpha - 3\alpha \beta s}{16\pi^4 \alpha} + m_c(sg_s \sigma \cdot Gs) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{2m^2_2 \beta + 2m^2_2 \alpha - 3\alpha \beta s}{16\pi^4 \beta} ,
\]

\[
\rho^{5b}_{\eta_3} = \left[ m_q(qg_s \sigma \cdot Gq) + m_s(\bar{s}g_s \sigma \cdot Gs) \right] \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{m^2_2 m_c}{4\pi^4} \sqrt{\left( 1 + \frac{m^2_2 - m^2_c}{s} \right)^2 - \frac{4m^2_2}{s}} - \left[ m_q(\bar{s}g_s \sigma \cdot Gs) + m_s(qg_s \sigma \cdot Gq) \right] \frac{m^2 m_c}{8\pi^4} \sqrt{\left( 1 + \frac{m^2_2 - m^2_c}{s} \right)^2 - \frac{4m^2_2}{s}} ,
\]

\[
\rho^{6a}_{\eta_3} = \frac{(q \bar{q}) (s \bar{s})}{6\pi^2} \left[ 4m_b m_c + m_s m_q + m_c m_q - \frac{m^2_2 m_c m_q (s - m^2_2 + m^2_2)}{(s - m^2_2 + m^2_2)^2 - 4m^2_2 s} \right] \times \sqrt{\left( 1 + \frac{m^2_2 - m^2_c}{s} \right)^2 - \frac{4m^2_2}{s}} ,
\]

\[
\rho^{6c}_{\eta_3} = \frac{\langle q \bar{q} \rangle (s \bar{s})}{6\pi^2} \left[ 4m_b m_c + m_s m_q + m_c m_q - \frac{m^2_2 m_c m_q (s - m^2_2 + m^2_2)}{(s - m^2_2 + m^2_2)^2 - 4m^2_2 s} \right] \int_0^1 d\alpha \frac{m^3 m^3_c}{6\pi^2 (1 - \alpha)^2} \delta^3 \left( s - \frac{m^2_2 (1 - \alpha) + m^2_2 \alpha}{(1 - \alpha)} \right) .
\]
4. Spectral densities for $\eta_4$:

$$
\rho^0_{\eta_4}(s) = \frac{1}{2} \rho^0_{\eta_4}(s), \quad \rho^3_{\eta_4}(s) = \frac{1}{2} \rho^3_{\eta_4}(s), \quad \rho^4_{\eta_4}(s) = \frac{1}{2} \rho^4_{\eta_4}(s), \quad \rho^{2b}_{\eta_4}(s) = -\rho^{2b}_{\eta_4}(s), \quad \rho^{5c}_{\eta_4}(s) = -\rho^{5c}_{\eta_4}(s), \quad \rho^{5a}_{\eta_4}(s) = \frac{1}{2} \rho^{5a}_{\eta_4}(s), \quad \rho^{5h}_{\eta_4}(s) = \frac{1}{2} \rho^{5h}_{\eta_4}(s), \quad \rho^{6a}_{\eta_4}(s) = \frac{1}{2} \rho^{6a}_{\eta_4}(s).
$$

5. Spectral densities for $\eta_{1\mu}$:

$$
\rho^0_{\eta_{1\mu}}(s) = \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta (1 - \alpha - \beta)^2 \left( m^2_{\nu} \beta + m^2_{\nu} \alpha - \alpha \beta s \right)^2 \cdot \frac{1}{128 \pi^4 \alpha^2 \beta^2} \left( 6 \left( m^2_{\nu} \beta + m^2_{\nu} \alpha - \alpha \beta s \right) \right) 
$$

$$
\times \left( m^2_{\nu} \beta + m^2_{\nu} \alpha - 3 \alpha \beta s \right) \cdot \left( 1 - \alpha - \beta \right) \left( m^2_{\nu} \beta + m^2_{\nu} \alpha - \alpha \beta s \right) \left( 3 m^2_{\nu} \beta + 3 m^2_{\nu} \alpha - 7 \alpha \beta s \right) 
$$

$$
- \left[ \frac{2 m^2_{\nu} \beta + 2 m^2_{\nu} \alpha - 5 \alpha \beta s \right) m_b m_q + \left( m^2_{\nu} \beta + m^2_{\nu} \alpha - 4 \alpha \beta s \right) m_c m_s \right]\}
$$

$$
\rho^3_{\eta_{1\mu}}(s) = \left[ m_q \langle \bar{q} q \rangle + m_s \langle \bar{s} s \rangle \right] \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{m^2_{\nu} \beta + m^2_{\nu} \alpha - \alpha \beta s}{32 \pi^4 \alpha \beta} \right) \left( 2 m^2_{\nu} \beta + m^2_{\nu} \alpha - 2 \alpha \beta s \right) 
$$

$$
- \left( 1 - \alpha - \beta \right) \left( 3 m^2_{\nu} \beta + 3 m^2_{\nu} \alpha - 5 \alpha \beta s \right) \right] 
$$

$$
+ \left[ m_s \langle \bar{q} q \rangle + m_q \langle \bar{s} s \rangle \right] \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{m^2_{\nu} \beta + m^2_{\nu} \alpha - \alpha \beta s}{8 \pi^4 \alpha \beta} \right) \left( m_b m_q \right) 
$$

$$
- m_b \langle \bar{q} q \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{1 - \alpha - \beta \right) \left( m^2_{\nu} \beta + m^2_{\nu} \alpha - \alpha \beta s \right) \left( m^2_{\nu} \beta + m^2_{\nu} \alpha - 2 \alpha \beta s \right) 
$$

$$
- m_c \langle \bar{s} s \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left( \frac{1 - \alpha - \beta \right) \left( m^2_{\nu} \beta + m^2_{\nu} \alpha - \alpha \beta s \right) \left( m^2_{\nu} \beta + m^2_{\nu} \alpha - 3 \alpha \beta s \right) 
$$

$$
\rho^{4a}_{\eta_{1\mu}}(s) = (g^2_{\nu} GG) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)^2}{4608 \pi^6 \alpha \beta} \left[ 3 \left( 2 m^2_{\nu} \beta + 2 m^2_{\nu} \alpha - 3 \alpha \beta s \right) 
$$

$$
- \left( 1 - \alpha - \beta \right) \left( 3 m^2_{\nu} \beta + 3 m^2_{\nu} \alpha - 4 \alpha \beta s \right) \right] \left( \frac{m^2_{\nu} \beta}{\alpha^2} + \frac{m^2_{\nu} \alpha}{\beta^3} \right),
$$

$$
\rho^{5a}_{\eta_{1\mu}}(s) = (g_{\nu} g_{\nu} \cdot G q) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left[ \frac{m_b (2 m^2_{\nu} \beta + 2 m^2_{\nu} \alpha - 3 \alpha \beta s)}{32 \pi^4 \alpha} + \frac{m_q (3 m^2_{\nu} \beta + 3 m^2_{\nu} \alpha - 4 \alpha \beta s)}{96 \pi^4} \right] 
$$

$$
+ (g_{\nu} g_{\nu} \cdot G s) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left[ \frac{m_c (2 m^2_{\nu} \beta + 2 m^2_{\nu} \alpha - 2 \alpha \beta s)}{32 \pi^4 \beta} + \frac{m_s (3 m^2_{\nu} \beta + 3 m^2_{\nu} \alpha - 4 \alpha \beta s)}{96 \pi^4} \right],
$$

$$
\rho^{5b}_{\eta_{1\mu}}(s) = (g_{\nu} g_{\nu} \cdot G s) \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left[ \frac{m_c (1 - \alpha - \beta \right) \left( m^2_{\nu} \beta + m^2_{\nu} \alpha - 2 \alpha \beta s \right) - m_b m_q}{64 \pi^4 \beta^2} \right],
$$

$$
\rho^{5c}_{\eta_{1\mu}}(s) = \left[ m_q \langle \bar{q} g_{\nu} \cdot G q \rangle + m_s \langle \bar{s} g_{\nu} \cdot G s \rangle \right] \frac{s - m^2_{\nu} - m^2_{\gamma}}{192 \pi^4 \sqrt{\left( \frac{m^2_{\nu} - m^2_{\gamma}}{s} \right)^2 - \frac{4 m^2_{\nu}}{s}}} - \left[ m_q \langle \bar{q} g_{\nu} \cdot G s \rangle + m_s \langle \bar{s} g_{\nu} \cdot G q \rangle \right] \frac{m_b m_q}{32 \pi^4 \sqrt{\left( \frac{m^2_{\nu} - m^2_{\gamma}}{s} \right)^2 - \frac{4 m^2_{\nu}}{s}}},
$$

where $\alpha_{\min}$ and $\beta_{\min}$ are defined as

$$
\alpha_{\min} = \frac{1}{2} \left[ 1 - \left( \frac{m^2_{\nu} \beta + m^2_{\nu} \alpha - \alpha \beta s}{\alpha \beta s} \right)^2 \right], \quad \beta_{\min} = \frac{1}{2} \left( \frac{m^2_{\nu} \beta + m^2_{\nu} \alpha - \alpha \beta s}{\alpha \beta s} \right)^2.
$$
\[ \rho^{6}_{\eta_{\mu
u}}(s) = \frac{\langle \bar{q}q \rangle \langle \bar{s}s \rangle}{12\pi^2} \left[ 2m_b m_c + m_b m_s + m_c m_q - \frac{m_b^2 m_c m_q}{(s - m_b^2 + m_c^2)^2 - 4m_b^2 s} \right] \]

\[ - \frac{m_c m_q}{2s} \left( \frac{m_b^2 - m_c^2}{s} \right) \sqrt{\left( 1 + \frac{m_b^2 - m_c^2}{s} \right)^2 - \frac{4m_b^2}{s}} , \]

\[ \rho^{8a}_{\eta_{\mu
u}}(s) = \left[ \langle \bar{q}q \rangle \langle \bar{g}_s \sigma \cdot Gs \rangle + \langle \bar{s}s \rangle \langle \bar{q}q \sigma \cdot Gq \rangle \right] \int_0^1 d\alpha \frac{1}{24\pi^2} \frac{m_b m_c}{(1 - \alpha)^2} \delta' \left[ s - \frac{m_b^2(1 - \alpha) + m_c^2\alpha}{\alpha(1 - \alpha)} \right] , \]

\[ \rho^{8b}_{\eta_{\mu
u}}(s) = \langle \bar{q}q \rangle \langle \bar{g}_s \sigma \cdot Gs \rangle \int_0^1 d\alpha \frac{m_b m_c}{48\pi^2(1 - \alpha)} \delta \left[ s - \frac{m_b^2(1 - \alpha) + m_c^2\alpha}{\alpha(1 - \alpha)} \right] . \]

6. Spectral densities for \( \eta_{2\mu} \):

\[ \rho^{0}_{\eta_{2\mu}}(s) = \frac{1}{2} \rho^{a}_{\eta_{2\mu}}(s) , \quad \rho^{3}_{\eta_{2\mu}}(s) = \frac{1}{2} \rho^{b}_{\eta_{2\mu}}(s) , \quad \rho^{4a}_{\eta_{2\mu}}(s) = \frac{1}{2} \rho^{4b}_{\eta_{2\mu}}(s) , \quad \rho^{4b}_{\eta_{2\mu}}(s) = -\rho^{4a}_{\eta_{2\mu}}(s) , \quad \rho^{5a}_{\eta_{2\mu}}(s) = \frac{1}{2} \rho^{5a}_{\eta_{2\mu}}(s) , \quad \rho^{5b}_{\eta_{2\mu}}(s) = -\rho^{5a}_{\eta_{2\mu}}(s) . \]

7. Spectral densities for \( \eta_{3\mu} \):

\[ \rho^{0}_{\eta_{3\mu}}(s) = \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)^2}{128\pi^6\alpha^2\beta^2} \left\{ \frac{1}{12\alpha\beta} \left[ 6(m_b^2 + m_c^2\alpha - \alpha\beta s) \right. \right. \]

\[ \times \left. \left. (m_b^2 + m_c^2\alpha - 3\alpha\beta s) - (1 - \alpha - \beta)(m_b^2 + m_c^2\alpha - \alpha\beta s)(3m_b^2 + 3m_c^2 - 7\alpha\beta s) \right] \right. \]

\[ \left. - \left[ (m_b^2 + m_c^2\alpha - 4\alpha\beta s) m_b m_q + (2m_b^2 + 2m_c^2 - 5\alpha\beta s) m_c m_s \right] \alpha \right\} , \]

\[ \rho^{3}_{\eta_{3\mu}}(s) = \left[ m_q \langle \bar{q}q \rangle + m_s \langle \bar{s}s \rangle \right] \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(m_b^2 + m_c^2\alpha - \alpha\beta s)}{32\pi^4\alpha\beta} \left[ 2(m_b^2 + m_c^2\alpha - 2\alpha\beta s) \right] \]

\[ - (1 - \alpha - \beta)(3m_b^2 + 3m_c^2\alpha - 5\alpha\beta s) \right] \]

\[ + \left[ m_s \langle \bar{q}q \rangle + m_c \langle \bar{s}s \rangle \right] \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(m_b^2 + m_c^2\alpha - \alpha\beta s)m_b m_c}{8\pi^4\alpha\beta} \]

\[ - m_b \langle \bar{q}q \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)(m_b^2 + m_c^2\alpha - \alpha\beta s)(m_b^2 + m_c^2\alpha - 3\alpha\beta s)}{16\pi^4\alpha^2\beta} \]

\[ - m_c \langle \bar{s}s \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)(m_b^2 + m_c^2\alpha - \alpha\beta s)(m_b^2 + m_c^2\alpha - 2\alpha\beta s)}{8\pi^4\alpha^2\beta} , \]

\[ \rho^{4a}_{\eta_{3\mu}}(s) = \langle g_s^2 GG \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)^2}{4608\pi^6} \left[ 3(2m_b^2 + 2m_c^2\alpha - 3\alpha\beta s) \right] \]

\[ - (1 - \alpha - \beta)(3m_b^2 + 3m_c^2\alpha - 4\alpha\beta s) \left( \frac{m_b^2}{\alpha^3} + \frac{m_c^2}{\beta^3} \right) , \]

\[ \rho^{4b}_{\eta_{3\mu}}(s) = - \langle g_s^2 GG \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \frac{(1 - \alpha - \beta)(m_b^2 + m_c^2\alpha - \alpha\beta s)}{4096\pi^6\alpha\beta} \left[ 4(3\beta - \alpha)(m_b^2 + m_c^2\alpha - 2\alpha\beta s) \right. \]

\[ - (\alpha + 3\beta)(1 - \alpha - \beta)(3m_b^2 + 3m_c^2\alpha - 5\alpha\beta s) \right] \frac{1}{3\alpha\beta} \right) , \]

\[ \rho^{5a}_{\eta_{3\mu}}(s) = \langle \bar{g}_s \sigma \cdot Gq \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left[ \frac{m_b(m_b^2 + m_c^2\alpha - 2\alpha\beta s)}{32\pi^4\alpha} + \frac{m_q(3m_b^2 + 3m_c^2\alpha - 4\alpha\beta s)}{96\pi^4} \right] \]
\[
\begin{align*}
\rho^{5b}_{\eta_\mu}(s) &= \langle \bar{q}g_s \sigma \cdot Gq \rangle \int_{\alpha_{\min}}^{\alpha_{\max}} d\alpha \int_{\beta_{\min}}^{\beta_{\max}} d\beta \left[ \frac{m_b(1-\alpha-\beta)(m_b^2\beta + m_b^2\alpha - 2\alpha\beta s)}{64\pi^4\alpha^2} - \frac{m_b m_c m_s}{64\pi^4\alpha} \right], \\
\rho^{5c}_{\eta_\mu}(s) &= \left[ m_q \langle \bar{q}g_s \sigma \cdot Gq \rangle + m_s \langle \bar{s}g_s \sigma \cdot Gs \rangle \right] \sqrt{\frac{s-m_b^2-m_c^2}{192\pi^4} \left[ 1 + \frac{m_b^2-m_c^2}{s} \right]^2 - \frac{4m_b^2}{s}}, \\
\rho^{6}_{\eta_\mu}(s) &= \frac{\langle \bar{q}q \rangle \langle \bar{s}s \rangle}{12\pi^2} \left[ 2m_b m_c + m_b m_s + m_c m_q - \frac{m_b^2 m_c m_q(s-m_b^2+m_c^2)+m_b m_c^2 m_s(s+m_b^2-m_c^2)}{(s-m_b^2+m_c^2)^2-4m_c^2 s} ight. \\
&\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\left. - \frac{m_b m_s(s-m_b^2+m_c^2)}{2s} \right] \sqrt{1 + \frac{m_b^2-m_c^2}{s}} \left[ 1 + \frac{m_b^2-m_c^2}{s} \right]^2 - \frac{4m_b^2}{s}, \\
\rho^{8a}_{\eta_\mu}(s) &= \left[ \langle \bar{q}q \rangle \langle \bar{s}g_s \sigma \cdot Gs \rangle + \langle \bar{s}s \rangle \langle \bar{q}g_s \sigma \cdot Gq \rangle \right] \int_0^1 d\alpha \frac{1}{24\pi^2} \frac{m_b m_c^2}{(1-\alpha)^2} \delta \left[ s - \frac{m_b^2(1-\alpha)+m_c^2\alpha}{\alpha(1-\alpha)} \right], \\
\rho^{8b}_{\eta_\mu}(s) &= \langle \bar{s}s \rangle \langle \bar{q}g_s \sigma \cdot Gq \rangle \int_0^1 d\alpha \frac{m_b m_c}{48\pi^2\alpha} \delta \left[ s - \frac{m_b^2(1-\alpha)+m_c^2\alpha}{\alpha(1-\alpha)} \right].
\end{align*}
\]

8. Spectral densities for \(\eta_{4\mu}\):
\[
\rho^0_{\eta_{4\mu}}(s) = \frac{1}{2} \rho^0_{\eta_{3\mu}}(s), \quad \rho^3_{\eta_{4\mu}}(s) = \frac{1}{2} \rho^3_{\eta_{3\mu}}(s), \quad \rho^{4a}_{\eta_{4\mu}}(s) = \frac{1}{2} \rho^{4a}_{\eta_{3\mu}}(s), \quad \rho^{4b}_{\eta_{4\mu}}(s) = -\rho^{4b}_{\eta_{3\mu}}(s), \quad \rho^{5a}_{\eta_{4\mu}}(s) = \frac{1}{2} \rho^{5a}_{\eta_{3\mu}}(s), \quad \rho^{5b}_{\eta_{4\mu}}(s) = -\rho^{5b}_{\eta_{3\mu}}(s), \quad \rho^{6}_{\eta_{4\mu}}(s) = \frac{1}{2} \rho^{6}_{\eta_{3\mu}}(s), \quad \rho^{6}_{\eta_{4\mu}}(s) = \frac{1}{2} \rho^{6}_{\eta_{3\mu}}(s), \quad \rho^{8a}_{\eta_{4\mu}}(s) = \frac{1}{2} \rho^{8a}_{\eta_{3\mu}}(s), \quad \rho^{8b}_{\eta_{4\mu}}(s) = -\rho^{8b}_{\eta_{3\mu}}(s).
\]