LOGARITHMIC BOUNDS FOR TRANSLATION-INVARIANT EQUATIONS IN SQUARES

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Abstract. We show that the equation \( \lambda_1 n_1^2 + \cdots + \lambda_s n_s^2 = 0 \) admits non-trivial solutions in any subset of \([N]\) of density \((\log N)^{-c}\), provided that \(s \geq 7\) and the coefficients \(\lambda_i \in \mathbb{Z} \setminus \{0\}\) sum to zero and satisfy certain sign conditions. This improves upon previous known density bounds of the form \((\log \log N)^{-c}\).

1. Introduction

We are interested in quantitative quadratic analogues of Roth’s theorem [14], which asserts the existence of non-trivial three-term arithmetic progressions in subsets of \([N]\) of density at least \((\log \log N)^{-1}\). It was shown in the late 1980s by Heath-Brown [10] and Szemerédi [22] that better bounds of the form \((\log N)^{-c}\) hold for this theorem. A well-studied nonlinear analogue is Sarközy’s theorem [15], which establishes the existence of square differences in sets of density \((\log N)^{-1/3+\alpha(1)}\), however it is less directly relevant to our work, since we consider equations with all variables lying in a dense set.

Roth’s method and subsequent improvements generalize readily to all translation-invariant equations, that is to say all equations of the form

\[
\lambda_1 x_1 + \cdots + \lambda_s x_s = 0,
\]

where \(\lambda_i \in \mathbb{Z} \setminus \{0\}, \lambda_1 + \cdots + \lambda_s = 0\) and \(x_1, \ldots, x_s\) are integer variables. From an additive number-theoretic point of view, it is natural to ask whether (1.1) is solvable in a dense subset of an arithmetic set instead, such as that of the primes or the squares. In the case of primes, the analogue of Roth’s theorem was shown to hold by Green [7], who for that purpose developed the now-standard transference principle, and used restriction estimates for primes in arithmetic progressions akin to those of Bourgain [1] for primes. By contrast, the case of squares can be handled directly, although rather uneconomically, by invoking the formidable bounds of Gowers [5] for Szemerédi’s theorem. Indeed, let \(x_i = n_i^2\) in (1.1) and consider the larger system

\[
\begin{align*}
\lambda_1 n_1^2 + \cdots + \lambda_s n_s^2 &= 0, \\
\lambda_1 n_1^2 + \cdots + \lambda_s n_s^2 &= 0.
\end{align*}
\]

in variables \(n_1, \ldots, n_s\), which has the crucial property of being invariant under translation and dilation. We call a solution \((n_1, \ldots, n_s)\) to the above non-trivial when the \(n_i\) are all
distinct. Assuming the existence of a non-trivial integer solution \((m_1, \ldots, m_s)\) to (1.2), Gowers’ argument [5] enables one to locate a pattern of the form \((x + dm_1, \ldots, x + dm_s)\) with \(x, d \in \mathbb{N}\) in any subset of \([N]\) of density at least \(C_s (\log \log N)^{-c_s}\) with \(c_s = 2^{-2^{s+9}}\), and this pattern again satisfies (1.2).

However, for large \(s\), one can expect the system (1.2) to be governed by Fourier analysis rather than higher-order uniformity, and the quantitative bounds to be superior. The work of Smith [21], who initiated the study of the system (1.2) in dense sets, and that of Keil [12] progressively confirmed both of these expectations. These two authors pursued a density increment strategy made possible by the translation and dilation invariance of the system, where at each step the set either contains the expected number of solutions given by the classical circle method, or correlates with a small arithmetic progression. Smith relied on quadratic Fourier analysis and on the Hardy-Littlewood circle method to show that the system (1.2) admits non-trivial solutions in sets of density as low as \((\log \log N)^{-8\cdot10^{-7}}\), for \(s \geq 9\) and certain conditions on the coefficients. Keil later simplified Smith’s approach to use only linear Fourier analysis, and reduced the necessary number of variables to 7, respectively by using appropriately the technique of linearization of a quadratic from Gowers’ work [5] and by incorporating (as well as giving a new proof of) certain restriction estimates of Bourgain [2] for lattice sets.

**Theorem 1** (Keil). Assume that \(s \geq 7\) and \(\lambda_1, \ldots, \lambda_s \in \mathbb{Z} \setminus \{0\}\) are such that \(\sum_{i=1}^s \lambda_i = 0\), and that at least two of these coefficients are positive and at least two are negative. There exists a positive constant \(C\) depending on \(s\) and \(\lambda_1, \ldots, \lambda_s\) such that the following holds. Suppose that \(A\) is a subset of \([N]\) of density

\[
\delta \geq C (\log \log N)^{-1/15}.
\]

Then \(A\) contains a non-trivial solution to the system of equations (1.2).

The translation-invariance and sign conditions on the coefficients, which in particular force \(s \geq 4\), are necessary for the theorem to hold, as explained by Keil [12]. The constraint \(s \geq 7\), on the other hand, is due to limitations in the classical circle method. The drawback of Theorem 1 is that the density bound is still doubly logarithmic, in contrast with the linear case where logarithmic bounds are available. Confirming an expectation of Smith [21, p. 276], we remove this discrepancy.

**Theorem 2.** Assume that \(s \geq 7\) and \(\lambda_1, \ldots, \lambda_s \in \mathbb{Z} \setminus \{0\}\) are as in Theorem 1. There exists a positive constant \(c\) depending on \(s\) and \(\lambda_1, \ldots, \lambda_s\) such that the following holds. Suppose that \(A\) is a subset of \([N]\) of density

\[
\delta \geq 2 (\log N)^{-c}.
\]

Then \(A\) contains a non-trivial solution to the system of equations (1.2).
Our main input is to adapt to the problem at hand the energy-increment strategy of Heath-Brown [10] and Szemerédi [22], by which one collects several large frequencies of the Fourier transform to obtain a more efficient density increment. In fact we use the framework from Green’s exposition [6] of this technique, where the discrete Fourier transform is used to simplify combinatorial arguments. Our proof also relies on the circle method analysis of Smith [21] and Keil [12], to which we make no further contribution, and again on the estimates of Bourgain [2] for exponential sums of the form \( \sum_n a_n e(\alpha n + \beta n^2) \). In our situation however, we now fully exploit Bourgain’s \( L^p - L^2 \) estimate, and we could not content ourselves with an \( L^p - L^\infty \) bound. A new ingredient of our approach is the use of simultaneous linearization of quadratics, a technique developed by Green and Tao [8] in the context of deriving efficient bounds for Szemerédi’s theorem for progressions of length four. Our setting is different than in the higher-order situation and we work only with arithmetic progressions, but we have to be careful with how this process interacts with the arithmetic energy increment strategy.

The dependency of the exponent \( c \) on the coefficients \( \lambda_i \) in Theorem 2 is an unescapable feature of our argument, and even in the case where \( s = 7 \) and \( \lambda_i = \pm 1 \) (say) we do not expect it to produce numerically very efficient constants. This seems to be an intrinsic limitation in the original method of Heath-Brown [10] and Szemerédi [22], and obtaining competitive exponents would likely require an adaptation of the machinery of density increment on Bohr sets developed by Bourgain [3]. We note finally that Theorem 2 would follow as a corollary if logarithmic bounds for Szemerédi’s theorem for progressions of length \( s \) were to be established, and in fact there is ongoing work in that direction by Green and Tao for \( s = 4 \), but the cases \( s \geq 5 \) seem far from accessible at present.

In the converse direction, an argument of Shapira [20] shows that for most choices of \( (\lambda_i) \), there exist Behrend-type sets of density \( e^{-c\sqrt{\log N}} \) containing no solutions to the first equation in (1.2), although this does not answer the question of solving the squares equation alone. In the special case where \( s = 3 \) and \( (\lambda_i) = (1, 1, -2) \), which is currently out of reach of analytic methods, a construction of Gyarmati and Ruzsa [9] gives a set of density \( (\log \log N)^{-1/2} \) without solutions to the second equation in (1.2). For small values of \( s \), we do not have a good heuristic for what the true density bound in Theorem 2 should be, but for large values it is tempting to conjecture Behrend-type bounds, in light of recent developments in Roth’s theorem in many variables [18, 19].

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2. Notation

In this preliminary section we introduce our basic notation and our normalization
conventions. We write $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ for the torus, equipped with the distance $\| \cdot \| = d(\cdot, \mathbb{Z})$. We write $X \sim Y$ when $X, Y$ are positive reals such that $Y \leq X \leq 2Y$. We also let $[N]$ denote the interval $\{1, \ldots, N\}$ when $N$ is an integer. We let $c$ and $C$ denote positive
constants whose values may change from line to line, and when we want to temporarily
fix those values we use subscripts $c_1, c_2$, and so on.

Given integers $N_1, \ldots, N_d \geq 1$ and a function $F : \mathbb{Z}_{N_1} \times \cdots \times \mathbb{Z}_{N_d} \to \mathbb{C}$, we define the $L^p$ norm of $F$ for $p \geq 1$ by
\[
\|F\|_p = \left( \mathbb{E}_{n_1 \in \mathbb{Z}_{N_1}, \ldots, n_d \in \mathbb{Z}_{N_d}} |F(n_1, \ldots, n_d)|^p \right)^{1/p},
\]
For a function $f : \mathbb{Z} \to \mathbb{C}$ and an integer $N \geq 1$, we define
\[
\|f\|_{L^p(N)} = \left( \mathbb{E}_{n \in [N]} |f(n)|^p \right)^{1/p}.
\]
We also use the standard $\ell^p$ and $L^p$ norms respectively on $\mathbb{Z}^d$ and on $\mathbb{T}^d$, and we write $\|f\|_p = \|f\|_{\ell^p(\mathbb{Z}^d)}$ for functions on $\mathbb{Z}^d$ and $\|f\|_p = \|f\|_{L^p(\mathbb{T}^d)}$ for functions on $\mathbb{T}^d$. We occasionally use the Fourier transform of a function $g : \mathbb{Z}^d \to \mathbb{C}$ with finite support, which is then defined by $\hat{g}(\gamma) = \sum_{n \in \mathbb{Z}^d} g(n) e(n \cdot \gamma)$ for $\gamma \in \mathbb{T}^d$.

3. Outline and organization of the paper

Our argument is modeled after the original energy increment strategy of Heath-
Brown [10] and Szemerédi [22], which we briefly recall, following the exposition of
Green [6]. In the case of three-term arithmetic progressions, the counting operator
acting on functions $f_i : \mathbb{Z} \to \mathbb{C}$ with support in $[N]$ takes the form
\[
M^{-2} \sum_{n_1, n_2, n_3 \in \mathbb{Z}} f_1(n_1) f_2(n_2) f_3(n_3) = \sum_{r \in \mathbb{Z}_M} \hat{f}_1(r) \hat{f}_2(r) \hat{f}_3(-2r),
\]
where $M = 2N$ and we have defined the Fourier transform $\hat{f} : \mathbb{Z}_M \to \mathbb{C}$ by
\[
(3.1) \quad \hat{f}(r) = \mathbb{E}_{n \in [M]} f(n) e\left(\frac{nr}{M}\right).
\]
When $A$ is a subset of $[N]$ of density $\delta$ containing no (non-trivial) three-term arithmetic
progressions, the usual multilinear expansion process coupled with Hölder’s inequality
shows that $\|\hat{f}_A\|_3 \gg \delta$, where $f_A = 1_A - \delta 1_{[N]}$. Via a clever “energy pigeonholing”
argument, one can then extract a larger restricted moment of lower order (2 in this
case), that is to say one may find an integer $1 \leq R \ll \delta^{-C}$ and distinct frequencies
\( r_1, \ldots, r_R \in \mathbb{Z}_M \) such that

\[
(3.2) \quad R^c \ll \sum_{i=1}^{R} \left| \frac{\hat{f}_A(r_i)}{\delta} \right|^2.
\]

The next step is to observe that the phases \( x \mapsto e(r_i x / M) \) are simultaneously approximately constant on a progression \( P \) of length \( \sim N^{c/R} \) by Dirichlet's theorem on simultaneous linear recurrence [17]. This can be used in the expansion (3.1) to show that \( \hat{f}_A(r_i) = \hat{f}_A \ast \mu_P(r_i) + O(N^{-c/R}) \) for all \( i \in [R] \), an error term which is strong enough to replace \( f_A \) by its convolution with \( \mu_P \) in (3.2) at little to no cost. Completing the sum in (3.2) and using Plancherel's identity, it follows that \( \|f_A \ast \mu_P\|_{L^2(N)}^2 \gg c R^c \). By unfolding \( f_A = 1_A - \delta 1_{[N]} \), performing a couple regularity computations and using an \( L^\infty - L^1 \) bound\(^1\), one is quickly led to a density increment of the form

\[
\delta \leftarrow (1 + c R^c) \cdot \delta,
\]

\[
N \leftarrow N^{c/R},
\]

upon passing to the smaller arithmetic progression and rescaling. Iterating these bounds, one obtains a logarithmic density bound in Roth’s theorem, and the reason behind this success is the efficient lower bound in (3.2), which crucially does not lose any power of \( \delta \) and even adds a gain factor \( R \).

In the quadratic case then, the counting operator associated to the system (1.2) is

\[
M^{-(s-3)} \sum_{n_1, \ldots, n_s \in \mathbb{Z}_M : (1.2)} f_1(n_1) \ldots f_s(n_s) = \sum_{z \in \mathbb{Z}_M \times \mathbb{Z}_M^2} S_{f_1}(\lambda_1 z) \ldots S_{f_s}(\lambda_s z)
\]

where \( s \geq 7 \) and \( S_f : \mathbb{Z}_M \times \mathbb{Z}_M^2 \to \mathbb{C} \) is defined at \( z = (x, y) \) by

\[
(3.3) \quad S_f(z) = \mathbb{E}_{n \in \mathbb{Z}[M]} f(n) e\left(\frac{x n}{M} + \frac{y n^2}{M^2}\right).
\]

This last expression is best interpreted as a quadratic version of the usual Fourier transform. By the discrete version of a restriction estimate of Bourgain [2], we know that \( S_f \) has bounded moments of order \( > 6 \) for bounded \( f \), while the 6-th moment for \( f = 1_{[N]} \) is already unbounded due to arithmetic considerations [13]. This suggests the following modification to the previous strategy: Starting with a set with fewer solutions to (1.2) than expected from the circle method, we now extract a large restricted sum

\[
(3.4) \quad R^c \ll \sum_{i=1}^{R} |S_{f_A / \delta}(z_i)|^r,
\]

\(^1\)The argument in Green’s note [6] is in fact even simpler in that it involves the function \( 1_A \) instead of \( f_A \), and thus does not require the said regularity computations. However it requires an additional Fejer-smoothing operation in the counting step, which we could not reproduce here: see the remarks at the end of the article.
with $6 < r < s$; this is done in Sections 7 and 8. This time we simultaneously linearize the corresponding quadratic phases $n \mapsto e(\alpha_i n + \beta_i n^2)$, taking our inspiration from Green and Tao [8] and relying in particular on their version of Schmidt’s result on simultaneous quadratic recurrence [16]. This lets us replace $f_A$ in (3.4) by a smoothed version $\tilde{f}_A$, which corresponds roughly to an average of convolutions of $f_A$ with smaller progressions of size $\sim N^{c/R^3}$. Completing the sum in (3.4), and applying Bourgain’s $\ell^r - L^2$ restriction bound [2] in place of Plancherel’s identity, we deduce that

$$R^c \ll \|S_{\tilde{f}_A/\delta}\|_r \ll \|\tilde{f}_A/\delta\|_{L^2(M)},$$

and from there the rest of the argument proceeds much as in the linear case. The linearization and the $L^2$ density increment steps are the most technical parts of this work and are found in Section 9. The main density increment lemma is proved in Section 10, but the final density increment iteration is carried out prior to all of these steps, in Section 4. Section 11 contains technical remarks on possible simplifications or extensions of the argument.

4. The density increment iteration

In this section we deduce Theorem 2 from the following density increment statement, whose proof will occupy the rest of this article. Although formally this is the last logical step of our argument, it is convenient to dispense with it at the beginning of the article, so that we may fix a scale $N$ at the outset. The iteration process below is essentially that carried out in Green’s note [6].

**Proposition 4.1.** Suppose that $s \geq 7$ and $\lambda_1, \ldots, \lambda_s \in \mathbb{Z} \setminus \{0\}$ are as in Theorem 1. There exist positive constants $c_1, c_2, c_3$, $D$ depending on $s$ and $\lambda_1, \ldots, \lambda_s$ such that the following holds. Suppose that $A$ is a subset of $[N]$ of density $\delta$ containing no non-trivial solutions to (1.2), and that $N \geq e^{(2/\delta)^D}$. Then there exists an arithmetic progression $Q$ contained in $[N]$ and an integer $R \geq 1$ such that

$$|A \cap Q| / |Q| \geq (1 + c_1 R^{c_2}) \cdot \delta, \quad |Q| \geq N^{c_3/R^3}.$$

**Proof of Theorem 2.** We consider a subset $A$ of $[N]$ of density $\delta$, and we construct iteratively a sequence of subsets $A_i$ of intervals $[N_i]$ of density $\delta_i$, initializing at $(A_0, N_0) = (A, N)$. As long as $\delta_i (\log N_i)^{1/D} \geq 2$, we run the following algorithm: If $A_i$ contains no non-trivial solutions to (1.2), we let $Q_{i+1} = u_{i+1} + q_{i+1} [N_{i+1}]$ be the arithmetic progression given by Proposition 4.1, and we define $A_{i+1}$ by $A_i \cap Q_{i+1} = u_{i+1} + q_{i+1} A_{i+1}$. Thus there
exists $R_i \geq 1$ such that
\[
\delta_{i+1} \geq (1 + c_1 R_i^{c_2}) \cdot \delta_i,
\]
\[
N_{i+1} \geq N_i^{c_3/R_i^3}.
\]

In particular, $\delta_{i+1} \geq (1 + c_1) \cdot \delta_i$ and the algorithm carries on for a finite number of steps, since all densities are bounded by 1. We now show that there exists a positive constant $\kappa = \kappa(c_1, c_2, c_3, D) \leq 1/D$ such that if $\delta_i (\log N_i)^\kappa \geq 2$, then $\delta_i (\log N_i)^\kappa \geq 2$ for every $i$ for which $A_i$ is defined. Indeed by (4.1), it follows that
\[
\frac{\delta_{i+1}(\log N_{i+1})^\kappa}{\delta_i(\log N_i)^\kappa} \geq 1 + c_1 R_i^{c_2}\left(\frac{R_i^3}{c_3}\right)^\kappa =: f(R_i).
\]

Assuming that $\kappa \leq c_2/6$, we clearly have $f(R) \geq 1$ for $R \geq C(c_1, c_2, c_3)$ large enough, and for $R \leq C(c_1, c_2, c_3)$ we can also guarantee that $f(R) \geq 1$ by choosing $\kappa \leq c(c_1, c_2, c_3)$ small enough.

Assume now that $\delta \geq 2(\log N)^{-\kappa}$, so that the condition $\delta_i (\log N_i)^{1/D} \geq 2$ is always met. Since the iteration cannot go on indefinitely, some set $A_i$ necessarily contains a non-trivial solution to (1.2), and so does $A$ by translation and dilation invariance.

\[\square\]

5. Analytic preparation

Our goal is now to prove Proposition 4.1. For the rest of the article, we thus fix an integer $s \geq 7$ and coefficients $\lambda_1, \ldots, \lambda_s \in \mathbb{Z} \setminus \{0\}$ such that $\lambda_1 + \cdots + \lambda_s = 0$, and such that at least two of the $\lambda_i$ are positive and at least two are negative. We also fix an integer $N \geq 1$, but at this point we do not impose size conditions on it, and we introduce those as our argument progresses. The same applies to the subset $A$ of $[N]$ of density $\delta$ which will be introduced later on, although we fix the notation $f_A = 1_A - \delta 1_{[N]}$ here.

More importantly, from this point onwards, we let all further implicit or explicit constants depend on $s$ and on $\lambda_1, \ldots, \lambda_s$. While it would be possible in theory to track down all of these dependencies, it would require a sizeable effort on our part, which we do not think worthwhile in light of the asymptotic nature of our results. Furthermore, such a process would almost certainly not allow us to eliminate the dependency of the logarithmic exponent in Theorem 2 on $s$ and on $\lambda_1, \ldots, \lambda_s$, given the shape of the density-increment statement in the previous section.

Throughout the article we embed the interval $[N]$ in a larger interval $[M]$ for Fourier analytic purposes, where $M$ is prime number of magnitude $M \sim 2(|\lambda_1| + \cdots + |\lambda_s|) \cdot N$ chosen via Bertrand’s postulate. Therefore, we have $(M, \lambda_1 \ldots \lambda_s) = 1$ and for integers
n_1, \ldots, n_s \in [N], the system of equations (1.2) is equivalent to
\[ \lambda_1 n_1 + \cdots + \lambda_s n_s \equiv 0 \pmod{M}, \]
\[ \lambda_1 n_1^2 + \cdots + \lambda_s n_s^2 \equiv 0 \pmod{M^2}. \]

We use both the discrete and continuous (quadratic) Fourier transforms in this article: the discrete transform is more convenient for combinatorics, while the continuous one is more suitable for number theory. Precisely, given a function \( f : \mathbb{Z} \to \mathbb{C} \) we define
\[ S_f : \mathbb{Z}_M \times \mathbb{Z}_{M^2} \to \mathbb{C} \]
and
\[ V_f : \mathbb{T}^2 \to \mathbb{C} \]
by
\[ S_f(x, y) = E_{n \in [M]} f(n) e\left( \frac{xn}{M} + \frac{yn^2}{M^2} \right) \]
and
\[ V_f(\alpha, \beta) = \sum_{n \in \mathbb{Z}} f(n) e(\alpha n + \beta n^2). \]

In practice we usually work with functions \( f : [N] \to \mathbb{C} \), which we view throughout the article as functions on \( \mathbb{Z} \) or \( [M] \) with support in \([N]\). We also define the normalized counting operator acting on functions \( f_1, \ldots, f_s : [N] \to \mathbb{C} \) by
\[ T(f_1, \ldots, f_s) = M^{-(s-3)} \sum_{n_1, \ldots, n_s \in \mathbb{Z}} f_1(n_1) \cdots f_s(n_s). \]

From the previous congruence considerations and by discrete Fourier inversion, we deduce the following harmonic expression for the counting operator.

**Proposition 5.1.** For functions \( f_1, \ldots, f_s : [N] \to \mathbb{C} \), we have
\[ T(f_1, \ldots, f_s) = \sum_{\mathbf{z} \in \mathbb{Z}_M \times \mathbb{Z}_{M^2}} S_{f_1}(\lambda_1 \mathbf{z}) \cdots S_{f_s}(\lambda_s \mathbf{z}). \]

6. Discrete restriction estimates

In this short section, we translate to the discrete setting a restriction estimate of Bourgain [2], which will prove extremely useful in the sequel. We start with (the \( d \)-dimensional version of) a very useful lemma due to Marcinkiewicz and Zygmund, and instrumental in Green’s proof of Roth’s theorem in the primes [7].

**Proposition 6.1.** Suppose that \( M_1, \ldots, M_d \geq 1 \) are integers and let \( p \geq 1 \). We have, for every function \( f : \mathbb{Z}^d \to \mathbb{C} \) with support in \([M_1] \times \cdots \times [M_d]\),
\[ \sum_{r_1 \in [M_1]} \cdots \sum_{r_d \in [M_d]} |f\left( \frac{r_1}{M_1}, \ldots, \frac{r_d}{M_d} \right)|^p \ll M_1 \cdots M_d \int_{\mathbb{T}^d} |\hat{f}(\theta_1, \ldots, \theta_d)|^p d\theta_1 \cdots d\theta_d, \]
where the implicit constant depends on \( d \) and \( p \) only.

**Proof.** Define the usual triangular functions \( \Delta_M : \mathbb{Z} \to \mathbb{C} \) by \( \Delta_M(m) = (1 - \frac{|m|}{M})^{+} \) for \( 1 \leq i \leq d \), and consider the tensor product \( \Delta_M = \Delta_{M_1} \otimes \cdots \otimes \Delta_{M_d} \). We have
\(\Delta_{M_i}(m_i) + 1 \leq 2\Delta_{2M_i}(m_i)\) for all \(m_i \in [1, M_i]\) for all \(i\) (with equality in fact), and by taking products over \(i \in [d]\) we can deduce that \(g_M := 2^d\Delta_{2M} - \Delta_M\) is at least 1 on \([M_1] \times \ldots \times [M_d]\). The proof of [7, Lemma 6.5] now generalizes straightforwardly to the \(d\)-dimensional setting upon replacing the function \(g\) from there by \(g_M\).

The key restriction estimate we need is the following, established by Bourgain [2] in the continuous setting.

**Proposition 6.2** (Bourgain). Let \(p > 6\). We have, uniformly in functions \(f : [M] \to \mathbb{C}\),

\[
\|Sf\|_p \ll_p \|f\|_{L^2(M)}.
\]

**Proof.** Apply Proposition 6.1 to the function \(g(n, m) = f(n)1(m = n^2)\) supported on \([M] \times [M^2]\). Recalling (5.1) and (5.2), this yields

\[
\|Sf\|_p^p = M^{-p}\sum_{x \in \mathbb{Z}_M} \sum_{y \in \mathbb{Z}_{M^2}} |\hat{g}\left(\frac{x}{M}, \frac{y}{M^2}\right)|^p
\]

\[
\ll_p M^{-p} \cdot M \cdot M^2 \int_{T^2} |\hat{g}(\alpha, \beta)|^p d\alpha d\beta
\]

\[
= M^{3-p}\|Vf\|_p^p.
\]

The proposition then follows from Bourgain’s restriction estimate [2, (3.115)] (see [2, (1.7), (3.1)] for the notation used there) and renormalizing.

We note that using Keil’s estimate [12, Theorem 2.1] instead, we could obtain an analogue of Proposition 6.2 with \(\|f\|_\infty\) in place of \(\|f\|_2\), however this would not suffice for our argument in its present form.

7. From non-uniformity to large energy

In this section we proceed with the first step of our energy increment strategy, which consists in converting the physical-space information that a set contains few solutions to (1.2) into an exploitable harmonic information: that the quadratic Fourier transform of this set has a large \(s\)-th moment. This part of our argument is very similar to the work of Keil [12] and Smith [21], and in particular we borrow highly non-trivial number-theoretic estimates from there; one minor technical difference (at this point) is that we work with the discrete Fourier transform.

We first need to bound the number of trivial solutions to (1.2), and for that purpose we import two supplementary estimates on even moments of exponential sums from the litterature: [2, Proposition 2.1] and [12, Lemma 5.1]. Both of these concern the exponential sum \(V(\alpha, \beta) = \sum_{n \in [N]} e(\alpha n + \beta n^2)\) viewed as a function \(V : T^2 \to \mathbb{C}\), and can be established by relatively elementary divisor considerations.
**Proposition 7.1.** We have
\[ \|V\|_4^4 \ll N^2, \quad \text{and} \quad \|V\|_6^6 \ll N^3 \log N. \]

We can now bound the number of trivial solutions easily, assuming that \( s \geq 7 \).

**Proposition 7.2.** Suppose that \( A \subset [N] \) contains no non-trivial solutions to (1.2). Then
\[ T(1_A, \ldots, 1_A) \ll \frac{\log N}{N}. \]

**Proof.** Consider distinct indices \( i, j \in [s] \). The number of solutions \( (n_1, \ldots, n_s) \in [N]^s \) to (1.2) with \( n_i = n_j \) is bounded by the number of solutions \( (m_1, \ldots, m_r) \in [N]^r \) to a new system of the form
\[
\begin{align*}
\mu_1 m_1 + \cdots + \mu_r m_r &= 0, \\
\mu_1 m_1^2 + \cdots + \mu_r m_r^2 &= 0,
\end{align*}
\]
where \( \mu_i \in \mathbb{Z} \setminus \{0\} \), and \( r = s - 2 \) or \( r = s - 1 \) according to whether \( \lambda_i + \lambda_j = 0 \) or not.

By the continuous circle method and Hölder’s inequality, the number of such solutions is at most
\[ \left| \int_{\mathbb{T}^2} V(\mu_1 \gamma) \cdots V(\mu_r \gamma) d\gamma \right| \leq \|V\|_r^r, \]
where we have used the 1-periodicity of \( V \). When \( r \geq 6 \) we have, by the second bound in Proposition 7.1,
\[ \|V\|_r^r \leq \|V\|_\infty^{r-6} \|V\|_6^6 \ll N^{r-3} \log N \ll N^{s-4} \log N. \]

Since \( s \geq 7 \), the only other possible case is \( s = 7, r = 5 \) for which the first bound in Proposition 7.1 provides a similar (and even stronger) conclusion. Summing finally over all \( \binom{s}{2} \) indices \( i, j \) and recalling the normalizing factor in (5.3), this concludes the proof. \( \square \)

We also need an estimate on the number of solutions to the system (1.2) in the complete integer interval \([N]\). Luckily for us, this is provided by the delicate circle method analysis of Keil [12, Proposition 7.1] and Smith [21].

**Proposition 7.3** (Smith, Keil). For \( N \geq C \), we have
\[ T(1_{[N]}, \ldots, 1_{[N]}) \gg 1. \]

It is now easy to derive the conclusion of large Fourier energy in the non-uniform case, via the usual multilinear expansion process and Hölder’s inequality.
Proposition 7.4 (Non-uniformity implies large energy). Suppose that $A$ is a subset of $[N]$ of density $\delta$ containing no non-trivial solutions to (1.2), and that $N \geq C\delta^{-2s}$. Then
\[ 1 \ll \|S_{f_A/\delta}\|_s. \]

Proof. We start from the bound of Proposition 7.2, and we expand $1_A = f_A + 1_{[N]}$ by multilinearity to obtain
\[ O(N^{-1/2}) = T(1_A, \ldots, 1_A) = \delta^s T(1_{[N]}, \ldots, 1_{[N]}) + \sum T(*, \ldots, f_A, \ldots, *), \]
where the sum has $2^s - 1$ terms and the stars denote functions equal to $f_A$ or $1_{[N]}$. By Proposition 7.3, we know that $T(1_{[N]}, \ldots, 1_{[N]}) \gg 1$, and from our assumption on $N$ and the pigeonhole principle we may obtain a bound of the form
\[ \delta^s \ll |T(f_1, \ldots, f_s)|, \]
where $\ell \geq 1$ of the functions $f_i$ are equal to $f_A$ and $s - \ell$ of them are equal to $1_{[N]}$. By Proposition 5.1 and Hölder’s inequality, and since $M$ is coprime to the $\lambda_i$, it follows that
\[ \delta^s \ll \|S_{f_A}\|_s^\ell \cdot \delta^{s - \ell} \|S_{1_{[N]}}\|_s^{s - \ell} \ll \delta^{s - \ell} \|S_{f_A}\|_s^\ell \]
where we have used the restriction bound of Proposition 6.2 in the last inequality. The inequality above is easily transformed into the desired result. \qed

8. Obtaining a large restricted energy

We now have at our disposal an $\ell^s$ estimate of the form $\sum \ell^s |S_{f_A/\delta}(z_i)|^s \gg 1$. In the spirit of the Heath-Brown-Szemerédi energy-increment strategy [10, 22], our next move is to extract a larger restricted moment $\sum_{i=1}^R |S_{f_A/\delta}(z_i)|^r \gg R^c$ with $6 < r < s$, where $R$ is a certain “gain” parameter of manageable size. This is made possible by the following innocent-looking combinatorial lemma, which is implicitly present in the exposition of Green [6, Lemma 4].

Lemma 8.1 (Energy pigeonholing). Let $X \geq 1$ and $0 < r < s$ be parameters. Suppose that $(a_k)_{k \geq 1}$ is a finite non-increasing sequence of non-negative real numbers such that
\[ \sum_k a_k^s \gg 1 \quad \text{and} \quad \sum_k a_k^r \leq X. \]

Then there exists $1 \leq R \ll_{r,s} X^{s/(s-r)}$ such that
\[ \sum_{k \leq R} a_k^r \gg_{r,s} R^{(s-r)/2s}. \]
Proof. Since \((a_k)\) is non-increasing, we have a Markov-type bound
\[
j a_j^r \leq \sum_{k \leq j} a_k^r \leq X,
\]
that is \(a_j \leq X^{1/r} j^{-1/r}\) for all \(j\). Since \(s > r\) we have therefore, for any \(Y \geq 1\),
\[
\sum_{k > Y} a_k^s \leq X^{s/r} \sum_{k > Y} k^{-s/r} \ll_{r,s} X^{s/r} Y^{-(s-r)/r}.
\]
Recalling (8.1) and choosing \(Y = CX^{s/(s-r)}\) with \(C = C(r, s)\) large enough, we have (8.2)
\[
\sum_{k \leq Y} a_k^s \gg 1.
\]

Let \(\theta, \eta > 0\), and assume for contradiction that \(a_k \leq \theta k^{-(1+\eta)/s}\) for all \(1 \leq k \leq Y\). Then
\[
\sum_{k \leq Y} a_k^s \leq \theta \sum_{k \leq Y} k^{-1-\eta} \ll_{\eta} \theta.
\]
Choosing \(\theta = c(\eta)\) small enough, this contradicts (8.2) and therefore we have found
\(1 \leq R \leq Y\) such that \(a_R \gg \eta R^{-(1+\eta)/s}\). Hence, by monotonicity again,
\[
\sum_{k < R} a_k^r \geq Ra_R^r \gg \eta R^{1-(1+\eta)r/s}.
\]
Choosing \(\eta = (s-r)/2r\), we obtain the desired conclusion. \(\square\)

**Corollary 8.2.** Suppose that \(A\) is a subset of \([N]\) of density \(\delta\) containing no non-trivial solutions to (1.2), and that \(N \geq C\delta^{-2s}\). Then there exists \(1 \leq R \leq \delta^{-C}\) and distinct frequencies \(z_1, \ldots, z_R \in \mathbb{Z}_M \times \mathbb{Z}_M^2\) such that
\[
\sum_{i=1}^R |S_{f_A/\delta}(z_i)|^{6.1} \gg R^c.
\]

**Proof.** We know that \(\sum_{z} |S_{f_A/\delta}(z)|^s \gg 1\) by Proposition 7.4, and that \(\sum_{z} |S_{f_A/\delta}(z)|^r \ll_{r} \delta^{-r/2}\) for every \(r > 6\) by Proposition 6.2. We fix \(r = 6.1 < 7 \leq s\) for definiteness. Ordering the absolute values \(|S_{f_A/\delta}(z)|\) for \(z \in \mathbb{Z}_M \times \mathbb{Z}_M^2\) by size, and applying Lemma 8.1, we obtain the desired conclusion. \(\square\)

### 9. Linearization and density increment

In this section we carry out the most technical part of our argument, by which we turn the previous large restricted energy into a density increment. We have previously isolated a collection of quadratic harmonics \(|S_{f_A/\delta}(z)|\) which is large in an \(\ell^{6.1}\) sense, and the next three steps consist in replacing the function \(f_A\) in each of these harmonics by its convolution \(\tilde{f}_A\) with smaller progressions, extracting a large second moment of
\( \tilde{f}_A \) via a restriction bound, and carrying out the classical \( L^2 \) density increment strategy. To alleviate technical statements, we fix an integer \( R \) and a collection of frequencies \( z_1, \ldots, z_R \) throughout this section, and we define quadratic polynomials \( \phi_i(n) = \alpha_i n + \beta_i n^2 \), where \( z_i = (x_i, y_i) \) and \( \alpha_i = x_i/M, \beta_i = y_i/M \).

Recalling (5.1) we thus have, for every function \( h : \mathbb{Z} \to \mathbb{C} \) and \( i \in [R] \),

\[
S_h(z_i) = \mathbb{E}_{n \in [M]} h(n) e(\phi_i(n)).
\]

The only non-trivial fact from diophantine approximation that we require is a version by Green and Tao [8, Proposition A.2] of Schmidt’s result on simultaneous quadratic recurrence, where the dependency of the error in \( d \) is made explicit.

**Proposition 9.1** (Schmidt, Green-Tao). Let \( \theta_1, \ldots, \theta_d \in \mathbb{T} \) be real numbers modulo 1, and let \( X \geq 1 \) be an integer. Then there exists an integer \( 1 \leq q \leq X \) such that

\[
\|q^2 \theta_i\| \ll dX^{-c/d^2} \quad \text{uniformly in } 1 \leq i \leq d.
\]

We note here that for the purpose of proving Theorem 2, we could work equally well with the slightly weaker estimate of Croot, Lyall and Rice [4, Theorem 1], which has the advantage of having a completely elementary proof. Yet we stick with Proposition 9.1, only because the error term there is more concise to state. Our first step, then, is to find a collection of progressions on which the quadratics phases in (9.1) are nearly constant.

In this we follow Green and Tao [8], using however the language of translates rather than that of partitions.

**Proposition 9.2** (Simultaneous linearization of quadratics). Let \( \varepsilon, \delta \in (0, 1] \) be parameters, and suppose that \( N \geq (2/\varepsilon \delta)^{CR^3} \). Then there exist integers \( q, (r_n)_{n \in \mathbb{Z}}, U, V \) of size

\[
U \sim N^{c/R^2} \quad \quad V \sim U^{c/R^2} \\
q \leq N^{1/4} \quad \quad r_n \leq U^{1/4}
\]

such that for \( P = q[U], Q_n = q r_n[V] \), we have

\[
\|\phi_i(n + m + k) - \phi_i(n + m)\| \leq \varepsilon \delta
\]

for all \( n \in [N], m \in P, k \in Q_n \) and \( i \in [R] \).

**Proof.** We consider at first arbitrary integers \( n, q, a, b, \) and we write

\[
\phi_i(n + qa) - \phi_i(n + qb) = \alpha_i [(n + qa) - (n + qb)] + \beta_i [(n + qa)^2 - (n + qb)^2] = (a^2 - b^2)q^2 \beta_i + (a - b) \gamma_i, n.
\]

(9.2)
where $\gamma_{i,n} = q(\alpha_i + 2n\beta_i)$. Via Proposition 9.1, we now choose $1 \leq q \leq N^{1/4}$ such that $\|q^2\beta_i\| \ll RN^{-c_0/R^2}$ for all $i \in [R]$. We also pick an integer $U \sim N^{c_0/4R^2}$ and we let $a = x + r_n y$ and $b = x$, where $x \in [U]$ and $y \in Z$ is arbitrary, and where $1 \leq r_n \leq U^{1/4}$ are chosen so that $\|r_n \gamma_{i,n}\| \leq U^{-c_1/R}$ for all $i \in [R]$ and $n \in Z$, via Dirichlet’s theorem on simultaneous linear recurrence [17, Theorem II.1A]. If we now pick another integer $V \sim U^{c_1/2R}$ and insist that $y, z \in [V]$, it follows from (9.2) that
\[
\|\phi_i(n + qa) - \phi_i(n + qb)\| \leq |a^2 - b^2| \cdot \|q^2\beta_i\| + |y - z| \cdot |r_n \gamma_{i,n}|
\leq RU^2 N^{-c_0/R^2} + VU^{-c_1/R}
\leq RN^{-c_0/R^2} + N^{-c_2/R^3}
\]
with $c_2 = c_0 c_1/8$, and the error is less than $\varepsilon \delta$ for $N \geq (2/\varepsilon \delta)^{CR^3}$.

We chose the sizes of the parameters in the previous proposition so that $P \subset [N^{1/2}]$ and $r_n[V] \subset [U^{1/2}]$. Before proceeding further, we recall two standard averaging techniques, which we use implicitly throughout the section.

**Proposition 9.3 (Regularity calculus).** Let $\varepsilon \in (0, 1]$ be a parameter and let $f : Z \to \mathbb{C}$ be a function. Consider two arithmetic progressions $P = q[N]$ and $P' \subset q[N']$, where $q, N, N' \geq 1$. Then
\[
\text{[Shifting]} \quad \mathbb{E}_{n \in P} f(n + n') = \mathbb{E}_{n \in P} f(n) + O\left(\frac{N'}{N} \|f\|_\infty\right) \quad \forall n' \in P',
\]
\[
\text{[Sub-averaging]} \quad \mathbb{E}_{n \in P, n' \in P'} f(n + n') = \mathbb{E}_{n \in P} f(n) + O\left(\frac{N'}{N} \|f\|_\infty\right).
\]

The next proposition allows us to approximate the quadratic Fourier transform of a function by the transform of an additively smoothed version of itself, at several frequencies at once. Since we have the function $f_A/\delta$ in mind we assume an $L^\infty$ bound of the form $\delta^{-1}$ below. We also adopt a handy notation: for complex numbers $X, Y$ we write $X \approx \varepsilon Y$ when $|X - Y| \ll \varepsilon$.

**Proposition 9.4 (Additive smoothing).** Let $\varepsilon, \delta \in (0, 1]$ be parameters, and assume that $N \geq (2/\varepsilon \delta)^{CR^3}$ and $P, (Q_n)_{n \in Z}$ are as in Proposition 9.2. Given a function $h : [N] \to \mathbb{C}$, define $\tilde{h} : [N] \to \mathbb{C}$ by
\[
\tilde{h}(n) = \mathbb{E}_{m \in P, k \in Q_{n-m}} h(n + k).
\]
Provided that $\|h\|_\infty \leq \delta^{-1}$, we then have
\[
\mathbb{E}_{n \in [N]} h(n) e(\phi_i(n)) \approx \varepsilon \mathbb{E}_{n \in [N]} \tilde{h}(n) e(\phi_i(n)) \quad \text{for all } i \in [R],
\]
(9.5)
\[
\mathbb{E}_{n \in [N]} h(n) \approx \varepsilon \mathbb{E}_{n \in [N]} \tilde{h}(n).
\]
Proof. Sub-averaging twice, using the bound $N \geq (\varepsilon \delta)^{-CR^2}$, we derive

$$I := \mathbb{E}_{n \in [N]} h(n) e(\phi_i(n))$$
$$\approx \varepsilon \mathbb{E}_{n \in [N]} \mathbb{E}_{m \in P, k \in Q_n} h(n + m + k) e(\phi_i(n + m + k)).$$

By Proposition 9.2, we thus have

$$I \approx \varepsilon \mathbb{E}_{n \in [N], m \in P, k \in Q_n} h(n + m + k) e(\phi_i(n + m))$$
$$= \mathbb{E}_{m \in P} \mathbb{E}_{n \in [N]} \left[ \mathbb{E}_{k \in Q_n} h((n + m) + k) e(\phi_i(n + m)) \right].$$

By shifting in $n$ at fixed $m$, using the bound $N \geq (\varepsilon \delta)^{-2}$, we obtain

$$I \approx \varepsilon \mathbb{E}_{n \in [N]} \mathbb{E}_{k \in Q_n, n-m} h(n + k) e(\phi_i(n))$$

After reordering variables, we have $I \approx \varepsilon \mathbb{E}_{n \in [N]} \tilde{h}(n) e(\phi_i(n))$ as expected, and an identical computation gives an analog approximation where $e(\phi_i(n))$ is replaced by 1.

We can now perform the promised transformation of the function $f_A/\delta$ in Corollary 8.2 into its smoothed version, which is then guaranteed to have a large second moment via the $\ell^4 - L^2$ restriction estimate of Section 6.

**Proposition 9.5.** Let $\varepsilon, \delta \in (0, 1]$ be parameters, and assume that $N \geq (2/\varepsilon \delta)^{CR^3}$ and $P, (Q_n)_{n \in \mathbb{Z}}$ are as in Proposition 9.2. Define $h \mapsto \tilde{h}$ as in (9.3) Suppose also that $A$ is a subset of $[N]$ of density $\delta$ such that

$$R^c \ll \sum_{i=1}^R |S_{f_A/\delta}(z_i)|^{6.1}. \tag{9.6}$$

Provided that $\varepsilon \leq c/R$, we then have

$$R^c \ll \left\| \frac{\tilde{f}_A}{\delta} \right\|_{L^2([N])}^2.$$

**Proof.** Since $M \sim CN$ and the functions under consideration are supported in $[N]$, we can pass from averages over $[M]$ to averages over $[N]$ and back, losing only a constant factor in the process. We rescale the averages (9.1) to $[N]$ and apply (9.4) to $h = f_A/\delta$, so that by the inequality $|x + y|^{6.1} \ll |x|^{6.1} + |y|^{6.1}$ we have

$$cR^c - CR^c\varepsilon^{6.1} \ll \sum_{i=1}^R \left| \mathbb{E}_{n \in [N]} \frac{\tilde{f}_A}{\delta}(n) e(\phi_i(n)) \right|^{6.1}.$$

Assuming that $\varepsilon \leq c/R$ with $c$ small enough (say), the left-hand side is $\gg 1$. Rescaling the average on the right-hand side to $[M]$ and completing the sum, we can apply the restriction estimate from Proposition 6.2 and rescale back to $[N]$ to finish the proof. □
We know from the last proposition that the balanced function of our set, averaged over a family of small progressions, has a large $L^2$ norm. One can pass from this information to a density increment by a standard argument combining regularity computations and an $L^\infty - L^1$ bound, much as in the Bohr set setting of Roth’s theorem [11, Proposition 2]. At this stage we rescale norms to the interval $[N]$ on which our functions naturally live.

**Proposition 9.6** ($L^2$ density increment). Let $\varepsilon, \delta \in (0, 1]$ and $\nu > 0$ be parameters, and suppose that $N \geq (2/\varepsilon \delta)^{CR^3}$ and $P, (Q_n)_{n \in \mathbb{Z}}$ are as in Proposition 9.2, $h \mapsto \tilde{h}$ is defined by (9.3), and $A$ is a subset of $[N]$ of density $\delta$ such that

$$\left\| \frac{f_A}{\delta} \right\|^2_{L^2(N)} \geq \nu.$$  

If $\varepsilon \leq c \nu \delta$, there exists an arithmetic progression $Q$ contained in $[N]$ such that

$$|A \cap Q|/|Q| \geq (1 + \nu/2) \cdot \delta,$$

$$|Q| \geq N^{c/R^3}.$$  

**Proof.** We write $\|h\|_2 = \|h\|_{L^2(N)}$ for functions $h : [N] \to \mathbb{C}$ for conciseness. Before engaging in computations notice that, by (9.3) and considering the lengths of the arithmetic progressions $(Q_n)_{n \in \mathbb{Z}}$, we have, for every $n \in [N],$

$$\tilde{1}_{[N]}(n) = \begin{cases} 1 & \text{if } n \in [1, N - N^{1/2}], \\ \in [0, 1] & \text{always}. \end{cases}$$

(9.7)

Therefore $\tilde{1}_{[N]}$ acts as a mollified indicator function of $[N]$. Recalling next that $f_A = 1_A - \delta \tilde{1}_{[N]}$, we can unfold the $L^2$ norm in the proposition to obtain

$$\nu \leq \left\| \frac{1_A}{\delta} \right\|^2_2 - 2\left< \frac{1_A}{\delta}, \tilde{1}_{[N]} \right> + \|\tilde{1}_{[N]}\|_2^2.$$  

(9.8)

A quick computation using (9.7) and (9.5) yields

$$\left< \frac{1_A}{\delta}, \tilde{1}_{[N]} \right> \approx_\varepsilon 1, \quad \|\tilde{1}_{[N]}\|_2^2 \approx_\varepsilon 1.$$  

(9.9)

We want to avoid certain overwrapping scenarios near the edges of our interval, and to do so we introduce the auxiliary set $E = [1, N - N^{1/2}]$. Writing $\mu_A = 1_A/\delta$ and using the bound $N \geq \varepsilon^{-2} \delta^{-4}$, we then have

$$\left\| \frac{1_A}{\delta} \right\|^2_2 = \mathbb{E}_{n \in [N]} 1_E(n) \tilde{\mu}_A(n)^2 + O(\varepsilon)$$

$$\leq \|1_E \tilde{\mu}_A\|_\infty \|\tilde{\mu}_A\|_1 + O(\varepsilon).$$
Applying (9.5) to \( h = \mu_A \), we deduce that
\[
\frac{1}{\delta} \left\| \frac{1_A}{\delta} \right\|_2^2 \leq \| 1_E \widehat{\mu_A} \|_\infty \| \mu_A \|_1 + O(\varepsilon / \delta)
\]
(9.10)
\[= \| 1_E \widehat{\mu_A} \|_\infty + O(\varepsilon / \delta).\]
Inserting (9.9) and (9.10) in (9.8), we obtain
\[\nu \leq \| 1_E \widehat{\mu_A} \|_\infty - 1 + O(\varepsilon / \delta).\]
Assuming that \( \varepsilon \leq c \nu \delta \), it follows that there exists \( n \in E \) such that
\[1 + \nu / 2 \leq \mathbb{E}_{m \in P} \mathbb{E}_{k \in Q_{n-m}} \mu_A(n + k).\]
By the pigeonhole principle, we may therefore find \( m \in P \) such that
\[(1 + \nu / 2) \cdot \delta \leq \mathbb{E}_{k \in Q_{n-m}} 1_A(n + k) = \frac{|A \cap (n + Q_{n-m})|}{|n + Q_{n-m}|}.
\]
Given our choice of \( n \in E \) and the bounds on the size of parameters in Proposition 9.2, we are guaranteed that \( n + Q_{n-m} \) is contained in \([N]\), and this concludes the proof. \( \square \)

10. Assembling all the pieces

In this section we finish the proof of Proposition 4.1, gathering the main statements of the two previous sections and assigning explicit values to parameters. As explained in Section 4, this completes the proof of Theorem 2.

Proof of Proposition 4.1. By Corollary 8.2, we can find a parameter \( 1 \leq R \ll \delta^{-C} \) and distinct frequencies \( z_1, \ldots, z_R \) such that
\[R^c \ll \sum_{i=1}^{R} |S_{fA/\delta}(z_i)|^{6.1}.\]
We assume that these frequencies are those that we started with at the beginning of Section 9, and we fix \( \varepsilon = c \delta / R \). Proposition 4.1 then follows at once from Propositions 9.5 and 9.6. \( \square \)

11. Final remarks

We make a few last technical comments on our method in this section. Consider a subset \( A \) of \([N]\) of density \( \delta \) containing no non-trivial solutions to (1.2). If we could draw a conclusion of the form \( R^c \ll \sum_{i=1}^{R} |S_{1_A/\delta}(z_i)|^s \) for distinct non-zero frequencies \( z_1, \ldots, z_R \), then we could use Keil’s [12] restriction estimate \( \| S_f \|_s \ll \| f \|_\infty \) to obtain a slight simplification in our argument. Indeed we could then deduce a density increment from the lower bound \( \| 1_A/\delta \|_\infty \geq 1 + c R^c \) (with the notation of Section 9), without resorting to any regularity computations. However we do not see how to do this at
present, and the information \( \|\hat{f}_A/\delta\|_\infty \gg R^c \) is hard to exploit by itself: it provides either a density increment or a density decrement, without the means to decide between the two.

Finally, we close this article with a prediction inspired by a remark of Smith [21, p. 276]. We remark that the density increment strategy employed here can be adapted to handle any translation/dilation-invariant system of equations of the form
\[
\lambda_1 n_1 + \cdots + \lambda_s n_s = 0
\]
\[
(11.1)
\]
\[
\lambda_1 n_1^k + \cdots + \lambda_s n_s^k = 0,
\]
where \( k \geq 1 \) and \( \lambda_1 + \cdots + \lambda_s = 0 \), under two additional conditions: that the classical circle method succeeds in bounding by below the number of solutions to (11.1) in \([N]\), and that the discrete exponential sums
\[
W_f(x_1, \ldots, x_k) = \mathbb{E}_{n \in [M]} f(n) e\left(\frac{x_1 n}{M} + \cdots + \frac{x_k n^k}{M^k}\right)
\]
again satisfy a restriction estimate of the form
\[
\|W_f\|_s \ll_s \|f\|_{L^2(M)}.
\]
It may well be that both these conditions are met for \( s \) large enough with respect to \( k \).

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