STABILITY OF THE SELFSIMILAR DYNAMICS OF A VORTEX FILAMENT

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90. Vortices above an inclined triangular wing. Lines of colored fluid in water show the symmetrical pair of vortices behind a thin wing of 15° semi-vertex angle at 20° angle of attack. The Reynolds number is 20,000 based on chord. Although the Mach number is very low, the flow field is practically conical over most of the wing, quantitatively being constant along rays from the apex. ONERA photograph, Welté 1963

91. Cross section of vortices on a triangular wing. Tiny air bubbles in water show the vortex pair for the flow above in a section at the trailing edge of the wing. ONERA photograph, Welté 1963
Euler equations

$u$: velocity field

$\omega = \text{curl } u = \nabla \wedge u$: vorticity

$\omega = \Gamma T ds \quad T = X_s$

$X = X(t, s)$ curve in $\mathbb{R}^3$ support of $\omega$

$\text{div } u = 0$

$$u(P) = \frac{\Gamma}{4\pi} \int_{-\infty}^{\infty} \frac{X(s) - P}{|X(s) - P|^3} \wedge T(s) ds$$

Examples: straight lines, vortex rings, helical vortices
BINORMAL FLOW (Vortex Filament Flow)

\[ X_t = X_s \land X_{ss} = cb \]

- \( X = X(t, s) \in \mathbb{R}^3 \)
- \( c = c(t, s) \) curvature
- \( b = b(t, s) \) binormal

Examples:

a) circle

b) straight line

c) helix

Remark.– \( X_s = T \) \( |T|^2 = \text{constant} \)
SELF SIMILAR solutions

\[ X(t, s) = \sqrt{t}G\left(\frac{s}{\sqrt{t}}\right) \quad T(t, s) = T\left(\frac{s}{\sqrt{t}}\right) \]

\[ T_t = T \wedge T_{ss} \]

Differentiating and making \( t = 1 \)

\[ -\frac{s}{2}T' = T \wedge T_{ss} \]

Frenet equations:

\[ T' = cn \]

\[ n' = -cT + \tau b \]

\[ b' = -\tau n \]

\[ -\frac{s}{2}cn = T \wedge (c'n - c^2T + c\tau b) \]

\[ c' = 0 \quad c = a \quad \tau = s/2 \]

Buttke’88
\[ G'(s) = T \quad \frac{1}{2} G - \frac{s}{2} G' = ab \]

\[
\left( \frac{G}{s} \right)' = \frac{G' s - G}{s^2} = -2a \frac{b}{s^2} \quad |b| = 1
\]

\[
\frac{G}{s} \rightarrow A^\pm \quad s \rightarrow \pm \infty
\]

- \( X(1) = G \) is regular
- \( X(t, s) = \sqrt{t} G' \left( s/\sqrt{t} \right) : \quad X(0, s) = \begin{cases} A^+ s & s > 0 \\ A^- s & s < 0 \end{cases} \)
- \( Q : A^+ = A^- ? \)

NO \quad \int_{-\infty}^{+\infty} e^{is^2 + i a \lg s} \frac{ds}{s}.
\[ A^+ = A^- ? \quad \theta = \text{angle} \ (A^+, A^-) \]

\[ G \quad c = a \quad \tau = s/2 \]

**Lemma.**— Let \( c \) and \( \tau \) be the curvature and the torsion of a curve in \( \mathbb{R}^3 \) with tangent \( T = (T_1, T_2, T_3) \). Then

\[ T_j = 1 - |\eta_j|^2 \quad j = 1, 2, 3 \]

for some \( \eta_j \) solution of

\[ \eta_j'' + \left( i\tau - \frac{c'}{c} \right) \eta_j' + \frac{c^2}{4} \eta_j = 0 \]
(With S. Gutierrez and J. Rivas)

There exist $A_a^\pm$, $E_a^\pm$

- $\sqrt{t}G_a\left(\frac{s}{\sqrt{t}}\right) = A_a^\pm s + O\left(\frac{\sqrt{t}}{s}\right)$

- $T_a\left(\frac{s}{\sqrt{t}}\right) = A_a^\pm + O\left(\frac{\sqrt{t}}{s}\right)$

- $(n_a - ib_a)\left(\frac{s}{\sqrt{t}}\right) = E_a^\pm e^{is^2/4t + ia^2 \log \frac{|s|}{\sqrt{t}}} + O\left(\frac{\sqrt{t}}{s}\right)$
SCHRÖDINGER EQUATION

Hasimoto transformation:

\[ \psi(t, s) = c(t, s)e^{i \int_0^s \tau(t, s')ds'} \]

\[ c = c(t, s) \quad \text{curvature} \]

\[ \tau = \tau(t, s) \quad \text{torsion} \]

\[ \partial_t \psi(t, s) = i \left( \partial_s^2 \psi \pm \frac{1}{2}(|\psi|^2 + A(t))\psi \right) \]

\[ \int_{-\infty}^{\infty} |\psi(t, s)|^2 ds = \int_{-\infty}^{\infty} |\psi(0, s)|^2 ds = \int_{-\infty}^{\infty} c^2(0, s) ds \]

In our case

\[ \psi(t, s) = \frac{a}{\sqrt{t}} e^{i \frac{s^2}{4t}} \quad , \quad \int_{-\infty}^{\infty} |\psi|^2 ds = +\infty. \]
STABILITY / INSTABILITY (V. Banica)

The equations are time reversible

\[
\begin{align*}
\psi(t) &= i \left( \psi_{ss} \pm \frac{1}{2} \left( |\psi|^2 - \frac{|a|^2}{t} \right) \psi \right) \\
\psi(1, s) &= ae^{i \frac{s^2}{4t}} + \epsilon_1(s)
\end{align*}
\]

**Q1** Solve (*) for \(0 < t < 1\) for reasonable (small) \(\epsilon_1\).

**Q2** \(\lim_{t \downarrow 0} \psi(t, s) = ?\)
Conformal transformation

$$\psi(t, s) = e^{is^2/4t} \frac{v}{\sqrt{it}} \left( \frac{1}{t}, \frac{s}{t} \right)$$

$$-v_t = i \left( v_{ss} \pm \frac{1}{2t} (|v|^2 - |a|^2) v \right)$$

**Particular solution**

$$\Phi_a = ae^{is^2/4} \quad v_a = a$$

$$v_1 = a + \epsilon$$

$$\psi_1 = (a + \epsilon)e^{is^2/4}$$
RENORMALIZED ENERGY

\[ E(t) = \frac{1}{2} \int |v_s(t)|^2 ds \mp \frac{1}{4t} \int (|v(t)|^2 - |a|^2)^2 ds \]

\[ \frac{d}{dt} E(t) = \pm \frac{1}{4t^2} \int (|v|^2 - |a|^2)^2 \]
Asymptotic profile (Ozawa, Hayashi–Naumkin)

\[
\begin{align*}
-\nu_t &= i \left( \nu_{xx} \pm \frac{1}{2t} (|\nu|^2 - |a|^2) \nu \right) \\
\nu(t, x) &= a + \epsilon(t, x)
\end{align*}
\]

Asymptotic profile (Ozawa, Hayashi–Naumkin)

\[
v_1(t, x) = a + e^{\mp i \frac{a^2}{2} \log t} e^{it} \Delta u_+(x)
\]

for any \( u_+ \).
\[ X_\gamma = \left\{ f : \| f \|_{L^2} + \left\| |\xi|^{2\gamma} \hat{f}(\xi) \right\|_{L^\infty(|\xi| \leq 1)} \right\} \]

**Theorem 1.**– Given \( 0 < \gamma < 1/4 \) and \( a > 0 \) there exists \( \delta > 0 \) such that if
\[ \| v(\cdot, 1) - a \|_{X_\gamma} < \delta \]
then \( \exists! v \) solution of (**) and \( u_+ \in X_{\gamma-} \) with

(i) \( v - v_1 \in C \left([1, \infty) : X_{\gamma-}\right) \cap \text{Strichartz} \)

(ii) \( \| v - v_1 \|_{L^2} = \mathcal{O}(t^{-1/4}). \)

Hence

**Q1** YES

**Q2** NO: \[
\psi(t, s) = \frac{e^{is^2/4t}}{\sqrt{it}} v \left( \frac{1}{t}, \frac{s}{t} \right) \]

\[
= \frac{a}{\sqrt{it}} e^{is^2/4t} + e^{-i \frac{a^2}{2} \lg t} \hat{u}_+(s) + o(t)
\]
\[ \hat{\epsilon}(t, 0) = \int_{-\infty}^{\infty} \epsilon(t, x)dx \]

\[ i\epsilon_t + \epsilon_{xx} = \pm \frac{1}{2t} (|\epsilon + a|^2 - a^2) (\epsilon + a) \]

\[ \frac{d}{dt} \int |\epsilon + a|^2 - a^2 = 0 \quad ; \quad \frac{1}{2} \int |\epsilon + a|^2 - a^2 = C_0 \]

\[ i\hat{\epsilon}(t, 0) = \pm aC_0 \lg t + \int_{1}^{t} \frac{1}{\tau}NL(\epsilon)d\tau \]

Hence \[ |\hat{\epsilon}(t, 0)| \geq \frac{a|C_0|}{2} \lg t. \]
Theorem 2.– Let $X_1(s)$ be the curve obtained from

$$\psi(1, s) = (a + \epsilon_1)e^{is^2/4}.$$ 

Then, there exist a unique $X(t, s)$ solution of the B.F. for $0 < t < 1$ with $X(1, s) = X_1(s)$ and a unique $X_0(s)$ such that

$$\sup_s |X(t, s) - X_0(s)| \leq Ca\sqrt{t}.$$
Moreover, assume \( s \in \mathbb{C}_1 \) is in \( L^2 \) then there exist 
\((T^\pm, E^\pm), (T_0(s), E_0(s)), \) and \((T_0(0^\pm), E_0(0^\pm))\) such that 

(i) \( T(t, s) = T^\pm + \mathcal{O} \left( \frac{1}{s} \right), \quad s \to \pm \infty, \)

(ii) \( n - ib = E^\pm e^{is^2/4t + ia^2 \log \frac{|s|}{\sqrt{t}}} + \mathcal{O} \left( \frac{1}{s} \right), \quad s \to \pm \infty, \)

(iii) For \( s \neq 0 \) \( \lim_{t \downarrow 0} (T(t, s), E(t, s)) = (T_0(s), E_0(s)) \)

(iv)

- \( \lim_{s \to 0^\pm} (T_0(s), E_0(s)) = (T_0(0^\pm), E_0(0^\pm)) \)
- \( (T_0'(s), E_0'(s)) \in L^2 \)
- \( \sin \frac{\theta}{2} = e^{-\frac{a^2}{2}} \)

and \( \theta \) the angle between \( T_0(0^+) \) and \(-T_0(0^-)\).
About the proof

\[ T_s = \text{Re} \overline{\psi} E \]

\[ E_s = -\psi T \]

\[ T_t = \text{Im} \overline{\psi}_s E \]

\[ T^+ - T(t, x) = \int_x^\infty T_s ds \]

\[ = \text{Re} \int_x^\infty \frac{e^{-is^2/4t}}{\sqrt{t}} (a + \epsilon) \left( \frac{1}{t}, \frac{s}{t} \right) E(t, s) ds \]

+ integration by parts

+ growth of \( \|J\epsilon\|_{L^2} \) \( J = x + 2it\partial_x \)
• For $T_t$, notice that the $\log t$ correction of the phase of $\psi_s$ and $E$ cancel each other.

• 

$$T_0(x) = T^+ + \text{Im } E^+ \int_x^\infty h(s)ds + \text{Re } \int_x^\infty h(s) \int_x^\infty h(s')T_0(s')ds'ds.$$ 

$$h(s) = \hat{u}_+ \left( \frac{s}{2} \right) \frac{i}{s^{ia^2}}.$$ 

• The rigidity result about the angle follows from a blow–up argument and a more precise result on the $\lim_{t \to 0^\pm} (T(t, s), E(t, s))$
Conclusions
**Theorem 3.** Assume in **Theorem 2** the extra hypothesis that \( \hat{u}_+ \in \dot{H}^{-2} \), then there exist \((T, n, b)(t, 0)\) such that

\[
| (T, \hat{n}, \hat{b}) - (T_a, n_a, b_a) | (t, 0) \leq O (\|v_1 - a\|)
\]

with

1. \( \hat{n} + i \hat{b} = e^{i\Phi/2} (n + ib) \)
2. \( \frac{a^2}{t} + \Phi' = 2 \left( \frac{c_{xx} - cT^2}{c} \right) (t, 0) + c^2 (t, 0) \)
Proof. Fefferman–Stein & Tsutsumi

$$\left\| \partial_x e^{it\partial_x^2} u_0 \right\|^2_{L^2} = \left( \int \int |\hat{u}_0(\xi)|^2 |\hat{u}_0(\eta)|^2 |\xi - \eta| d\xi d\eta \right)^{1/2}.$$ 

- Gustafson, Nakanishi, Tsai. (G-P.)
THANK YOU FOR YOUR ATTENTION