Design of Robust Positively Invariant Set with Associated Controller for the Multi-rotor UAV

Zhinan Li¹, Weihua Li¹ and Peng Wang*¹

¹ Information and Navigation College, Air Force Engineering University, Xi'an, ShanXi, 710003, China
* Corresponding author’s e-mail: blueking1985@hotmail.com

Abstract. In this paper, a nonlinear multi-rotor unmanned aerial vehicle (UAV) system with bounded disturbance and input saturation is considered and we design a robust positively invariant (RPI) set for the trajectory tracking problem of the multi-rotor UAV. A nonlinear controller with disturbances offset is designed firstly to ensure the stability of tracking, and the RPI set is constructed based on linear matrix inequalities (LMIs). For the tracking error within the designed RPI set, the designed controller can always guarantee that the tracking error stays in the RPI set forever and eventually converges to zero. The effectiveness of the proposed results are confirmed by the numerical simulation.

1. Introduction
In recent years, the researches about the multi-rotor UAV have been widely studied. Because of its small size, strong maneuverability, vertical take-off and landing and so on, the UAV has been used in industry, agriculture, military and civil fields[1-4]. As is universally known, a multi-rotor UAV system is a nonlinear, strongly coupled and underactuated system with multi-input and multi-output, whose control has become the key technology affecting its practical application. The trajectory tracking control is a basic problem in the flight control of the UAV and has received more and more attention. At present, for the trajectory tracking control problem of multi-rotor UAV, many scholars have proposed a variety of effective solutions, such as sliding mode control[3], predictive control[5], backstepping control[6], PID control[7].

It is also necessary to consider robust property when designing the controller because UAV is affected by many external disturbances in the actual flight. Based on the idea of cascade theory, the complete closed-loop system is divided into two reduced-order subsystems and a coupling term in [8]. Furthermore, the disturbances compensation controller is designed by backstepping control based on the disturbances estimation obtained by the observer to avoid the interference. In [6], the trajectory tracking problem of the UAV is studied by backstepping control according to the cascade control theory. The disturbances observer is designed reasonably through the coupling of attitude and position error, thus the disturbances is suppressed better. In this paper, aiming at the trajectory tracking control problem of multi-rotor UAV with external bounded disturbances, a disturbances cancellation term is added to the design of nonlinear feedback controller to realize the more stable control of UAV trajectory tracking.

Due to the limitation of the motor power and propeller speed, the state and input (such as thrust force and torque) of multi-rotor UAV in the control process are restricted. Therefore, the guarantee of input constraints should be considered in the construction of controller. The existing research often restricts the controller through saturation function, such as reference [9], through adding saturation function to the finite time backstepping control strategy, the adverse effects of input saturation on the convergence...
speed of the system are inhibited to a certain extent. In addition, model predictive control (MPC), as an advanced control strategy of moving optimization and moving implementation, is perfectly suitable for dealing with constrained control problems. However, the stability of (robust) predictive control needs to be based on (robust) positively invariant sets. In this paper, based on the designed nonlinear feedback controller, a robust positive definite invariant set is constructed for trajectory tracking control of multi-rotor UAV with external disturbances using LMIs.

2. Problem statement

The multi-rotor UAV, shown in figure 1 (taking quadrotor UAV as an example), is driven by the rotation of its propellers and each of them produces the thrust force. Consider a multi-rotor UAV with perturbed input, which is formulated by the following kinematic and dynamic models as

\[
\begin{align*}
\dot{p} &= v, \\
\dot{v} &= \frac{R(q)T e_z}{m} - g e_z + d_F, \\
\dot{q}_0 &= -\frac{1}{2} q^T \omega, \\
\dot{q}_r &= \frac{1}{2} (q_0 I + [q_r]_e) \omega, \\
J \omega &= -[\omega], J \omega + \tau + d_t,
\end{align*}
\]

where \( p = [p_x, p_y, p_z]^T \), \( v = [v_x, v_y, v_z]^T \), \( q = [q_0, q_1, q_2, q_3]^T \) and \( \omega = [\omega_x, \omega_y, \omega_z]^T \) represent the position, velocity, attitude and angular velocity of the UAV, respectively; \( R(q) = |q_0|^2 I_{3x3} + 2q_0 [q_0]^T + 2q_0 [q_1, q_2, q_3]^T \) represents the rotation operator corresponding to the quaternion \( q \); \( T \) and \( \tau = [\tau_x, \tau_y, \tau_z]^T \) represent the thrust force and torque provided by the UAV propellers; the unit vector \( e_z = [0, 0, 1]^T \) represents the \( z \) direction in the earth coordinate; \( m \) and \( g \) represent the mass of the UAV and the gravitational acceleration; \( I \) and \( J = \text{diag} \{ J_x, J_y, J_z \} \) represent the unit matrix and the symmetric positive definite constant inertia matrix of the UAV, respectively; \( d_F = [d_{F1}, d_{F2}, d_{F3}]^T \) and \( d_t = [d_{t1}, d_{t2}, d_{t3}]^T \) indicate external disturbances acting on thrust force and torque, respectively; \( [\cdot]_e \) denotes the skew-symmetric matrix.

In general, the disturbances and control input are bounded, which are required to satisfy

\[
\begin{align*}
d_F \in \mathbb{D}_F & \triangleq \{ d_F \ | d_{F1} \leq d_{F1}^{\max}, |d_{F2}| \leq d_{F2}^{\max}, |d_{F3}| \leq d_{F3}^{\max} \}, \\
d_t \in \mathbb{D}_t & \triangleq \{ d_t \ | d_{t1} \leq d_{t1}^{\max}, |d_{t2}| \leq d_{t2}^{\max}, |d_{t3}| \leq d_{t3}^{\max} \}, \\
u = [T \tau]^T & \in \mathbb{U} \triangleq \{ u \ | |T| \leq T^{\max}, |\tau_x| \leq \tau_{x,\max}, |\tau_y| \leq \tau_{y,\max}, |\tau_z| \leq \tau_{z,\max} \},
\end{align*}
\]
\( d_{\text{max}}, T_{\text{max}} \) and \( r_{(\text{max})} \) represent the known upper bound of disturbances component, thrust force and torque, respectively.

In order to facilitate the design and description of the RPI set, the following definitions are given in this paper.

**Definition**: A robust positively invariant set \( S \) is such that, for any \( z_{(\text{r})}^{(i)}(t) \in S, d_{(\text{r})}(r) \in \mathbb{R}_{+} \) and \( r > 1 \),

\[
\kappa^{(r)}(z^{(i)}_{(\text{r})}(r), \bullet) \in U,
\]

where \( \kappa^{(r)}(z^{(i)}_{(\text{r})}(r), \bullet) \) is feedback controller to be designed.

The control objective is to drive the perturbed UAV to track a pre-specified reference trajectory, which is described by \( p_d, v_d, \dot{v}_d, \dot{q}_d, \omega_d \) and \( \dot{\omega}_d \). We define the tracking error state \( p_e = p_d - p \), \( v_e = v_d - v \), \( q_e = q^* - q_d = [q_{e\theta} \ q_{e\phi} \ q_{e\psi}]^T \) and \( \omega_e = R(q_e) \omega_d - \omega \) represent the tracking errors of UAV, where \( q^* \) represents the adjoint of the quaternion. Then the differential dynamics of the tracking error can be derived as outer loop control system (7) and inner loop control system:

\[
\dot{z}_e \equiv \begin{bmatrix} \dot{p}_e \\ \dot{v}_e \end{bmatrix} = \begin{bmatrix} v_e \\ \frac{\tau}{m} - \frac{1}{2} q_{e\phi} \omega_e \end{bmatrix},
\]

\[
\dot{z}_e^i = \begin{bmatrix} \dot{q}_{e\theta} \\ \dot{q}_{e\phi} \\ \dot{\omega}_e \end{bmatrix} = \begin{bmatrix} \frac{1}{2} (q_e I + [q_{e\theta}]_\times) \omega_e \\ \frac{1}{2} (q_e I + [q_{e\phi}]_\times) \omega_e \\ \frac{1}{2} (q_e I + [q_{e\psi}]_\times) \omega_e \end{bmatrix},
\]

3. Development of a robust positively invariant set with associated controller

3.1. Development of a robust positively invariant set with associated controller for inner loop

3.1.1. Controller design for inner loop. Define a new state as \( \tilde{x} = \omega_e + \mu q_{e\psi} \), and select the Lyapunov candidate as \( V_2 = \frac{1}{2} \dot{x}^T \dot{x} \). Take the derivative of \( V_2 \), then we get

\[
\dot{V}_2 = \dot{x}^T \left( \ddot{x} + \mu \dot{q}_{e\psi} \right)
\]

\[
= \dot{x}^T \left( \ddot{x} + \mu \dot{q}_{e\psi} \right)
= \ddot{x}^T \{ R(q_e) \dot{\omega}_e + R(q_e) \dot{\omega}_e - J^{-1}(\tau + d_t) + J^{-1}\{ R(q_e) \dot{\omega}_e - \omega_e \} J \{ R(q_e) \dot{\omega}_e - \omega_e \} + \frac{1}{2} \mu (q_{e\psi} I + [q_{e\theta}]_\times) \omega_e \}.
\]

Design the virtual control function as

\[
\tau = -K \ddot{x} + d_{l_{\text{max}}} \text{sign}(\dot{x}) + \frac{1}{2} \mu J(q_e I + [q_{e\theta}]_\times) \omega_e + \{ R(q_e) \dot{\omega}_e - \omega_e \} J \{ R(q_e) \dot{\omega}_e - \omega_e \} + J \{ \dot{R}(q_e) \dot{\omega}_e + R(q_e) \dot{\omega}_e \}.
\]

(10)

Where \( K \leq 0, d_{l_{\text{max}}} = \text{diag}(d_{l_{\text{max}}}^1, d_{l_{\text{max}}}^2, d_{l_{\text{max}}}^3) \). Substitute (10) into (9),

\[
\dot{V}_2 = \ddot{x}^T J^{-1}(K \ddot{x} - d_t - d_{l_{\text{max}}} \text{sign}(\dot{x}))
= \ddot{x}^T J^{-1}(K \ddot{x} - \dot{x}^T J^{-1}(d_t + d_{l_{\text{max}}} \text{sign}(\dot{x}))
\leq 0.
\]

After finite time we get \( \dot{x} = \omega_e + \mu q_{e\psi} \equiv 0 \), which can be rewritten as \( \omega_e = -\mu q_{e\psi} \). According to (8), we get

\[
\omega_e = \omega_e - \mu q_{e\psi} \equiv 0,
\]
\[
\begin{align*}
\dot{q}_{e0} &= \frac{1}{2} \mu q_{e0}^T q_{e0}, \\
\dot{q}_{e} &= -\frac{1}{2} \mu q_{e} q_{e}^T.
\end{align*}
\]

Select the Lyapunov candidate as
\[
V_s = \frac{1}{2} q_{e0} q_{e0} + \frac{1}{2} (1 - q_{e0})^2\]

Take the derivative of \( V_s \), then we get \( \dot{V}_s = -\frac{1}{2} \mu q_{e0}^T q_{e0} \leq 0 \), i.e., \( \lim_{t \to \infty} z_s^e = [1 \ 0 \ 0]^T \). We can conclude that the attitude and angular velocity tracking error converge to zero in finite time.

### 3.1.2. Development of a robust positively invariant set for inner loop.

Substitute \( (10) \) into \( (8) \):
\[
\begin{align*}
\dot{q}_{e0} &= -\frac{1}{2} q_{e0} q_{e0}, \\
\dot{q}_{e} &= \frac{1}{2} \left( q_{e0} I + [q_{e0}]_n \right) q_{e}, \\
\dot{q}_{e} &= -\frac{1}{2} \mu \left( q_{e0} I + [q_{e0}]_n \right) q_{e} + J^T K \dot{x} - J^T \left( d_t + d_{t}^{\max} \text{sign}(\dot{x}) \right).
\end{align*}
\]

Let \( x = \left[ q_{e0}^T \ q_{e}^T \right]^T \), then we get \( \dot{x} = \left[ q_{e0}^T \ q_{e}^T \right]^T = [A(z_r') + A_s F_t] x + B_z \), where
\[
A(z_r') = \begin{bmatrix}
0 & q_{e0} I + [q_{e0}]_n \\
\mu [\omega_r]_n & -\mu q_{e0} I
\end{bmatrix}, \quad A_s = \begin{bmatrix}
0 & J^T \\
\mu K & K
\end{bmatrix}, \quad B_z = \begin{bmatrix}
0 \\
0 -(d_t + d_{t}^{\max} \text{sign}(\dot{x}))^T J^{-1}
\end{bmatrix}.
\]

The third and above power of the nonlinear parts in \( (10) \) are constrained to quadratic terms related to the specified trajectory parameters through the norm of the unit quaternion:
\[
|r_s| \leq |C x| + \lambda_i - F_t x, \quad s = 1, 2, 3
\]

where \( C_i = [c_{i1} \ c_{i2} \ c_{i3} \ c_{i4} \ c_{i5} \ c_{i6}] \) and \( \lambda_i \) are the known constant. Consider the input saturation \( |r_s| \leq \tau_{s\max}, \quad |r_i| \leq \tau_{s\max}, \quad \dot{r}_s \leq \tau_{s\max} \), then we get
\[
\begin{align*}
\|C x| + \lambda_i - F_t x| &\leq \eta C P_- z_r'^T + \lambda_i + \eta \sqrt{F_t P_{t}^{-1} F_t^T} \\
&\leq \eta C P_- z_r'^T + \lambda_i + \eta^2 + F_t P_{t}^{-1} F_t^T \\
&\leq \tau_{s\max}, \quad s = 1, 2, 3
\end{align*}
\]

On the premise of ensuring the asymptotic stability of the inner loop control system, the RPI set for inner loop can be constructed as \( S' = \{ x | x^T P_2 x \leq \eta^2 \} \), where \( P_2 > 0 \) and \( \eta > 0 \). The parameters \( F_t = [F_{t1}^T \ F_{t2}^T \ F_{t3}^T]^T \) and \( \eta \) can be obtained by solving the following LMIs:\n
\[
\begin{align*}
\max_{F_t, \ \eta, \ \zeta} \eta \quad & \\
\text{s.t.} \quad & A(z_r')^T P_2 + P_2 A(z_r') + A_s^T F_t^T P_2 + P_2 A_s F_t \leq 0, \\
& \begin{bmatrix} -\eta & z_r'^T \\ z_r'^T & -\eta P_2^{-1} \end{bmatrix} \leq 0, \\
& \begin{bmatrix} \lambda_s - \tau_{s\max} & \eta \\ \eta & -(C P_{t}^{-1} C_s^{-1} + 1)^{-1} \end{bmatrix} \begin{bmatrix} F_t \\ F_{t2} \\ F_{t3} \\ K \end{bmatrix} \leq 0, \quad \forall s = 1, 2, 3, \\
& K \leq 0.
\end{align*}
\]
3.2. Development of a robust positively invariant set with associated controller for outer loop
The outer loop control system (7) can be rewritten as
\[ \dot{z}_e^o = A_t z_e^o + B_t u_t - B_d s_e, \]  
(18)
where \( A_t = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} \), \( B_t = \begin{bmatrix} 0 \\ I \end{bmatrix} \) and \( u_t = \dot{v}_d - \frac{R(q)}{m} + ge_z \). Select the suitable Lyapunov candidate as \( V_t = z_e^o P_t z_e^o \) and we can design the virtual control function easily as
\[ u_t = F_t z_e^o - d_f \max \text{sign}(B_t P_t z_e^o), \]  
(19)
where \( d_f \max = \text{diag}(d_{f1}^\max, d_{f2}^\max, d_{f3}^\max) \).

Finally we will get \( V_t \leq 0 \) and the outer loop control system (7) is stable. Furthermore, the actual control input (i.e., thrust force) of the outer loop control system can be designed as
\[ T = \kappa(\dot{z}_e^o, \dot{v}_d) = m \left\| F_t z_e^o - d_f \max \text{sign}(B_t P_t z_e^o) - \dot{v}_d - ge_z \right\|. \]  
(20)
Using ideas and steps of RPI set design for linear disturbances systems in [13], the RPI set can be constructed as
\[ S^o = \{ z_e^o | z_e^o P_t z_e^o \leq \alpha \}, \]  
(21)
where \( P_t, F_t \) and \( \alpha \) can be obtained by solving the following LMIs(12):
\[
\max_{w, Y, z_e^o, \alpha} \alpha \\
\text{s.t.} \\
WA_t^T + A_t W + Y^T B_t^T + B_t Y \leq 0, \\
\begin{bmatrix}
-\alpha & z_e^o \\
z_e^o & -W
\end{bmatrix} \leq 0, \\
\begin{bmatrix}
\beta & Y \\
Y^T & -W
\end{bmatrix} \leq 0,
\]  
(22)
where the \( P_t = W^{-1} > 0, F_t = YW^{-1} \) and \( \beta = \alpha - 2 \left( \frac{T_{\max}}{m} - \left\| d_f \max \text{sign}(B_t P_t z_e^o) \right\| - \left\| s_e \right\| - \left\| ge_z \right\| \right) \).

4. Simulation
In this paper, the Matlab is used to simulate and the specific parameters are selected as follows: \( m = 3 \text{ kg}, g = 9.8 \text{ m/s}^2, J = \text{diag}(0.039, 0.039, 0.12) \text{ kg m}^2 \). The specified trajectory is formulated as \( p_s = [2.5 \pi \sin (\frac{t}{2\pi}), 2.5 \pi \cos (\frac{t}{2\pi}), -0.5 t^2] \text{ m} \). \( \cup \triangle \{ u \left\| u \right\| \leq T_{\max}, 0 \leq r_s \leq r_{\max}, \left\| r_s \right\| \leq \left\| r \right\| \leq \left\| r \right\| \leq \left\| r \right\| \leq r_{\max} \} \) is the input saturation constraint, where \( T_{\max} = 36 \text{ N}, r_{\max} = 1 \text{ N m}, \forall s = 1, 2, 3 \). The disturbances are generated by \( d_s = [0.4 \sin (t), 0.2 \sin (t) + 0.2, 0.4 \cos (t)] \text{ m/s}^2 \) and \( d_e = [0.49 \sin (t), 0.49 \sin (t) + 0.49 \cos (t)] \text{ m/s}^2 \).

Solving the optimization problem (22) and (17), we get the parameters about the RPI sets for the outer loop and inner loop: \( P_s = \begin{bmatrix} 0.0004I & 0.0052I & 0.0052I \\ 0.0052I & 0.2972I \end{bmatrix} \), \( \alpha = 21.5361 \) and \( \eta^2 = 1.0420 \). Select the \( P_s, \mu \) as \( P_s = \begin{bmatrix} 1.5I \\ I \end{bmatrix} \) and \( \mu = 1 \) offline. Select the boundary points of the RPI sets as the initial tracking error states: \( q_s (0) = [0.7071 0.5 0.4 0.3]^T, \omega_s (0) = [0.1 0.1 0.1]^T \text{ rad/s}, \) \( p_s (0) = [32 44 23]^T \text{ m}, v_s (0) = [5.8 3.5 2.6]^T \text{ m/s} \). We apply the controllers (20) and (10) to the
outer and inner control systems (7) and (8), respectively, and the simulation results are shown in the figure 2–9.

Figure 2, figure 3, figure 4 and figure 5 show the attitude error, angular velocity error, position error and velocity error curve of multi-rotor UAV trajectory tracking control, respectively. It can be seen that the state of outer loop control system converges to zero gradually with the rapid convergence of the state of inner loop control system to the equilibrium point, which verifies the stability of inner and outer loop control system under the action of controller designed in this paper.

![Figure 2. Attitude tracking error](image)

![Figure 3. Angular velocity tracking error](image)

![Figure 4. Position tracking error](image)

![Figure 5. Velocity tracking error](image)

Figure 6, and figure 7 show the curves of $z_c^o P_1 z_c^o$ and $z_c^o P_2 z_c^o$, respectively. We can see that both the $z_c^o P_1 z_c^o \leq \alpha$ and $z_c^o P_2 z_c^o \leq \eta^2$ are guaranteed all the time during the control, which means the tracking error states are always in the RPI set under the controllers we designed.
5. Conclusion

In this paper, a robust positively invariant set with associated controller is studied for the trajectory tracking problem of multi-rotor unmanned aerial vehicle, whose control input is subject to the saturation constraint and the bounded disturbances. The theoretical and simulation results show that when the tracking error state of UAV is in the designed RPI set, the error state is always in the invariant set under the action of the controller designed in this paper. In addition, because of the cancellation of external disturbances in the control design, the controller in this paper can realize a more stable control of trajectory tracking.

References

[1] B. Kalantar, S. B. Mansor, A. A. Halin, H. Z. M. Shafrri, and M. Zand, (2017) Multiple Moving Object Detection From UAV Videos Using Trajectories of Matched Regional Adjacency Graphs, IEEE Transactions on Geoscience & Remote Sensing, 99:1-16.
[2] E. Camci, D. R. Kripalani, L. Ma, E. Kayacan, and M. A. Khanesar, (2018) An aerial robot for rice farm quality inspection with type–2 fuzzy neural networks tuned by particle swarm optimization–sliding mode control hybrid algorithm, Swarm and Evolutionary Computation, 41:1-8.

[3] R. Bhola, N. H. Krishna, K. N. Ramesh, J. Senthilnath, and G. Anand, (2018) Detection of the power lines in UAV remote sensed images using spectral-spatial methods, Journal of Environmental Management, 206:1233-1242.

[4] J. Wang and G. Liu, (2019) Saturated control design of a quadrotor with heterogeneous comprehensive learning particle swarm optimization, Swarm and Evolutionary Computation, 46: 84–96.

[5] P. N. Chikasha and C. Dube, (2017) Adaptive Model Predictive Control of a Quadrotor, IFAC PapersOnLine, 50:157–162.

[6] R. Wang and J. Liu, (2018) Trajectory Tracking Control of a 6-DOF Quadrotor UAV with Input Saturation Via Backstepping, Journal of the Franklin Institute, 355:3288–3309.

[7] A. A. Najm and I. K. Ibraheem, (2019) Nonlinear PID controller design for a 6-DOF UAV quadrotor system, Engineering Science and Technology, an International Journal. doi: https://doi.org/10.1016/j.jestch.2019.02.005

[8] L. Wang and H. Jia, (2014) The Trajectory Tracking Problem of Quadrotor UAV: Global Stability Analysis and Control Design Based on the Cascade Theory, Asian Journal of Control, 16: 574-588.

[9] C. Fu, Y. Tian, H. Huang, L. Zhang, and C. Peng, (2018) Finite-time trajectory tracking control for a 12-rotor unmanned aerial vehicle with input saturation,” ISA Transactions, 81: 52-62.

[10] P. Wang, X. Feng, and W. Li, (2016) Design of robust positively invariant set for nonholonomic vehicle, presented at the 2016 Chinese Control and Decision Conference (CCDC) Yinchuan, China.

[11] R. Zhang, Q. Quan, and K. Y. Cai, (2011) Attitude control of a quadrotor aircraft subject to a class of time-varying disturbance, Iet Control Theory & Applications, 5:1140-1146.

[12] M. Kothare, V. Balkrishnan, and M. Morari, (1995) Robust constrained model predictive control using linear matrix inequalities, in American Control Conference.

[13] P. Wang and B. Ding, (2014) A Synthesis Approach of Distributed Model Predictive Control for Homogeneous Multi-Agent System with Collision Avoidance, International Journal of Control, 87:52-63.