Geodetic Line Interpolation Algorithm with Absolute Precision Threshold Constraints

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Abstract. The existing geodetic line interpolation algorithm mainly has the following problems: first, the map projection property and the limitation of the scale size can not be separated from the chart carrier; the second is that the maximum interpolation interval needs to be given artificially, and the third is that the precision of the algorithm results can not be adjusted adaptively. Aimed at the above problems, in this paper, in this paper, the length of the earth spherop corresponding to the replacement of the geodetic line by the space linear line is calculated according to the curvature of the earth, and it was taken as the maximum interpolation interval so that a geodetic interpolation algorithm with absolute precision threshold constraint was proposed in this paper, too. The correctness of the algorithm and the precision of the results are verified experimentally by using the classical example, and the results show that the algorithm can not only adjust the maximum interpolation interval adaptively according to the absolute precision threshold, but also improve the absolute precision of the results, and also has wide applicability to the distance of geodetic line.

1. Introduction

In order to improve the accuracy of the results of maritime delimitation, the new technology of implementing the precise delimitation of the ocean on the earth spherop was identified in [1] so that the precision of the delimitation results got rid of the limitation of the projected nature and scale size of the chart. To visualize the delimitation results, the baseline of the territorial sea generated on the earth spherop needs to be drawn on to the chart plane according to the appropriate map projection and scale size. In [1], it is held that the geodetic line should be the ideal model of “straight line” in the baseline of the straight territorial sea, so the plotting of the baseline of the straight-line territorial sea is essentially the extension of the geodetic line.

For the plotting of geodetic lines, it is common practice to use the "straight-replace-curve" technique of plotting plane curves in computer cartography. The idea in [1] is to set the point interval of the plot to the minimum distance between the two points that can be discerned by the naked eye on the chart, to solve a series of dense points on the map by the geodesy theme solutions, and to be connected to achieve the precision plotting of the geodetic line. This idea was designed in [2] by the corresponding geodetic interpolation algorithm. The use of the interpolation method is based on a certain geodetic line distance, the same spacing to obtain a number of internal insertion points between
the two points of the geodetic line, and then connect the points to get the map accurately illustrated geodetic line[3]. If the geodetic coordinates of the interpolated points obtained by the algorithm in [3] are converted to space right-angle coordinates, and these interpolated points are connected one by one by space linear segments, the approximate expression of the geodetic line on the earth spherop can also be realized[4-6].

In [2-6], the geodetic interpolation algorithms all belong to the equal-lying-spacing geodetic line interpolation algorithm without constraints, and their basic idea is that the maximum interpolation interval is given by human before the algorithm begins. However, these algorithms still have the following shortcomings: firstly, the result accuracy of the algorithm is directly affected by the maximum interpolation interval given, the interpolation spacing can be set small enough to ensure the accuracy of the geodetic line plotting, but the purpose of doing so is to take into account the map projection nature and scale size of the chart, so it is a relative practice; the second is that the algorithm does not have a feedback link when executing. In other words, for different accuracy requirements, unless the maximum interpolation interval is set just right, or is set to the limit under the corresponding conditions just like the method in [2], it is difficult to make the algorithm only once to enable the results to meet the accuracy threshold.

For the practice of maritime delimitation, the ideal straight-line territorial sea baseline (geodetic line) plotting method should be independent of the map projection nature and scale size of the chart carrier, and the maximum interpolation interval in the geodetic interpolation algorithm should be adjusted adaptively rather than artificially pre-given for different accuracy requirements. To this end, a set of absolute precision threshold constraints of the geodetic line interpolation algorithm is proposed. The algorithm mainly solves the length of the earth spherop corresponding to the geodetic line by the space linear line according to the curvature of the earth, and takes it as the maximum interpolation interval, so as to realize the adaptive adjustment of the maximum interpolation interval and improve the conversion accuracy.

### 2. The basic principles of the algorithm

As shown in Figure 1(a), \( P(B_1, L_1), Q(B_2, L_2) \) are any two points on the earth spherop, and \( w \) is the geodetic line connecting points \( P, Q \) on the earth spherop, while \( w' \) is the space polyline with \( P, Q \) as the ends of the straight segment. Define \( h \) as the geodetic line arch height, which represents the maximum distance from any point on the geodetic line \( w \) to the space straight segment \( w' \) (see Figure 1(b)). Note that the length of the geodetic line \( w \) is \( S(w) \), which represents the distance of the geodetic line on the earth spherop, and the length of the space straight segment \( w' \) is recorded as \( d(w') \).

![Fig.1 The principle of "straight-replace-curve" for geodetic line plotting](image)

Due to the influence of the earth's flatness rate, the geodetic line attached to the earth spherop (except for the meridian and the equator) is actually a double curve, with both curvature and deflection. Therefore, the length of the geodetic line \( w \) is always greater than the length of the space line segment \( w' \), and the difference between them is \( \delta \), there exists

\[
\delta = S(w) - d(w') > 0
\]
Obviously, \( \delta \) is positively correlated with \( h \) and \( S(w) \), and ideally, when \( \delta \to 0 \) and \( h \to 0 \), the geodetic line \( w \) coincides with the straight segment \( w' \).

For the calculation of \( d(w') \), involving the conversion between the geodesic coordinates and the space right-angle coordinates, \((B_1, L_1, 0)\) and \((B_2, L_2, 0)\), the geodesy coordinates of points \( P \) and \( Q \) in order, can be substituted into formula (2) (in the formula, \( N \) is the radius of the ring, \( e \) is the first eccentricity) to find their respective space right-angle coordinates \((X_1, Y_1, Z_1)\), \((X_2, Y_2, Z_2)\):

\[
\begin{align*}
X &= (N + H) \cos B \cos L \\
Y &= (N + H) \cos B \sin L \\
Z &= N(1-e^2) + H \sin B
\end{align*}
\]  

(2)

So, the value of \( d(w') \) can be calculated in formula (3):

\[
d(w') = \sqrt{(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2}
\]  

(3)

Because the flatness of the earth spheroid is relatively small (about one-third of its value), engineering measurements are usually used in a small range (in this case, the geodetic line length \( S(w) \) is small) and the spherical approximation is usually replaced by the earth spheroid. Accordingly, the geodetic line \( w \) can be approximated as the large arc line \( w_c \) on the sphere, and the approximate value of the geodetic line length \( S(w) \) corresponds to the large arc length \( S_c(w) \).

When engineering measurements are made in a local range, \( \delta_c \), the difference between the length of the geodetic line \( S(w) \) and the large arc length \( S_c(w) \) can be calculated in formula (4):

\[
\delta_c = S_c(w) - S(w)
\]  

(4)

Moreover, if the earth spheroid pointcut radius is used instead of the spherical radius of the earth spheroid, the difference between them is less than 0.5 meters within 200 km\(^7\). According to the above analysis, when the length accuracy threshold \( \delta_{\Omega} \) of the geodetic line is known, the geodetic line can be divided and interpolated with half of the geodetic line length \( S(w) \) (\( \frac{S(w)}{2^n} \), \( n \) represents the number of interpolation and takes positive integers). Note that the total length of the geodetic line consisting of two ends \( P(T_0), Q(T_2) \) and interstitial \( T_i(i=1, 2, \ldots, 2^n-1) \) of the geodetic line is \( \sum_{i=0}^{2^n-1} d(w'_i) \), and calculate the difference between \( S(w) \) and \( \sum_{i=0}^{2^n-1} d(w'_i) \), which is recorded as \( \delta_n \):

\[
\delta_n = S(w) - \sum_{i=0}^{2^n-1} d(w'_i)
\]  

(5)

Compare the difference \( \delta_n \) and the length precision threshold \( \delta_{\Omega} \), if \( \delta_n \leq \delta_{\Omega} \), then stop the geodetic semi-divided interpolation.

On this basis, it is possible to further expand fine interpolation of the geodetic line. As shown in Figure 2(a), note that the length of small geodetic line between any adjacent two interpolated points (including the two ends of the earth line) \( T_i \) and \( T_{i+1} \) is \( S(w_i)(i=0, 1, \ldots, 2^n-1) \).
Because the value of $S(w_i)$ is small and has met the requirements of small-scale engineering measurement, then under the premise that the high-precision threshold $h_{i3}$ of the geodetic line arch height is known, $h_i (i = 0, 1, \ldots, 2^n - 1)$, the geodetic line arch height between any adjacent two interpolated points, can be calculated by replacing the earth spheroid with the sphere (that is, the length of the space straight segment $AB$ in Figure 2(b), which can be calculated by the triangular relationship between the right triangle $OT_B$). If $h_i > h_{i3}$, then use half of the length of the corresponding small section of the geodetic line, namely $\frac{S(w_i)}{2}$, to divide and interpolate. When the geodetic line arch height between any adjacent two interpolated points does not exceed the given threshold $h_{i3}$, it represents the end of the fine interpolation of the geodetic line. Because the length precision threshold $\delta_{i3}$ and arch height precision threshold $h_{i3}$ all belong to the absolute precision threshold, the above algorithm is called the absolute precision threshold constraint of the geodetic line interpolation algorithm.

2.1. Geodetic line semi-divided interpolation algorithm with length precision threshold constraint

Figure 3 is an effect diagram of the geodetic line semi-divided interpolation under different interpolation times. It can be seen by induction that when interpolating $n$ times, the interpolation distance is $\frac{S(w)}{2^n}$, and the interpolation error $\delta_n$ is $S(w) - d_{P_{r1}} - d_{P_{r\ldots2}} - \sum_{i=1}^{2^n} d_{T_{r_{i1}}}$ (see Figure 3(d)).

The following is a concrete step for the geodetic line semi-divided interpolation algorithm with length precision threshold constraints:

i. According to the practice of maritime delimitation, the length precision threshold for the plotting baseline (geodetic line) is determined as $\delta_{i3}$;

ii. The geodesy theme reverse solution is carried out on the known geodetic line starting point $P(B_1, L_1)$ and the end $Q(B_2, L_2)$, and the geodetic line distance $S(w)$ and the positive azimuth $A_{12}$ are solved.

iii. Interpolate the geodetic line $PQ$, the interpolation distance $\Delta S(n) = \frac{S(w)}{2^n}$ ( $n$ represents the number of interpolation times).
iv. Calculate the total length of the spatial polyline formed by \( P, Q \) and each interpolated point \( T_i \), i.e. 
\[
d_{pT_i} + \sum_{i=1}^{2^n-1} d_{T_{i-1}T_i} + d_{T_nQ},
\]
\( T_i \) \((i=1,2,\cdots,2^n-1)\).

v. Calculate the difference between the geodetic line length \( S(w) \) and the previous step calculation result, i.e. \( \delta_n \).

vi. Compare the size relationship between difference \( \delta_n \) and the length precision threshold \( \delta_\Omega \), if \( \delta_n > \delta_\Omega \), go to step iii, and if \( \delta_n \leq \delta_\Omega \), that is the end of interpolating.

2.2. Geodetic line fine interpolation algorithm with arch height precision threshold constraint

Figure 4 is the flow chart of the geodetic line fine interpolation algorithm with arch height precision threshold constraint, where Figure 4(a) is a scene zooming in a small segment of the geodetic line after the interpolation with the over-limit arch height. The figures from Figure 4(b) to Figure 4(d) are the semi-divided interpolation for the small segment of the geodetic line.

![Flow chart of the geodetic line fine interpolation algorithm with arch height precision threshold constraint](image)

The specific steps of the geodetic line fine interpolation algorithm for arch height precision threshold constraint are as follows:

i. According to the practice of maritime delimitation, the arch height precision threshold for the plotting baseline (geodetic line) is determined as \( h_\Omega \);

ii. According to the latitude and longitude coordinates \( (T_i) \) and \( (T_{i+1}) \) of adjacent interpolators \( T_i \) and \( T_{i+1} \) in order, through the geodesy theme reverse solution, this step can get the distance \( S(w) \) of the small segment of the geodetic line between the two points, and the positive azimuth \( A_{i(i+1)} \) of the small segment;

iii. Calculate the midpoint \( \left( \frac{B_i + B_{i+1}}{2}, \frac{L_i + L_{i+1}}{2} \right) \) in the longitude and latitude coordinates of \( T_i \) and \( T_{i+1} \);

iv. Based on the coordinate midpoint obtained by the previous step, and using the spherical instead of the earth sphero to calculate the geodetic line arch height \( h_i \) between adjacent interpolated points \( T_i \) and \( T_{i+1} \);

v. Compare the size relationship between arch height \( h_i \) and arch height precision threshold \( h_\Omega \), if \( h_i > h_\Omega \), then the interpolation distance \( \frac{S(w)}{2} \), azimuth \( A_{i(i+1)} \) and the latitude and longitude coordinates \( (B_i, L_i) \) of point \( T_i \) are expanded together to solve the midpoint of the geodetic line between \( T_i \) and \( T_{i+1} \), and to replace \( T_{i+1} \) and \( T_i \) with the obtained geodetic line midpoint, and go to step ii, if \( h_i \leq h_\Omega \), then that is the end of interpolation.
3. Samples comparison and precision verification

In order to verify the result precision of the geodetic line interpolation algorithm of absolute precision threshold constraint proposed in this paper, and to test the wide application of the algorithm to the geodetic line distance, the following comparison experiments are carried out by two classical samples.

3.1. Description of the experiment

The two samples used in the experiment are cited from [9], each of which takes the coordinates \((B_1, L_1), (B_2, L_2)\), which belong to the start and ending points of a geodetic line on the earth spheroid, as known conditions, and takes the length of the geodetic line calculated in the references as the theoretical value (see Table 1 and Table 2, respectively).

| Tab.1 Parameters of sample I |   |   |
|------------------------------|---|---|
| Spheroid type                |   |   |
| Krasovsky                    |   |   |
| \(B_1\)                      | 30°29’58.2043″ |
| \(L_1\)                      | 120°05’40.2184″ |
| Known conditions             |   |   |
| \(B_2\)                      | 30°24’05.8354″ |
| \(L_2\)                      | 119°49’23.3854″ |
| Geodetic line distance       |   | 28230.936m |

| Tab.2 Parameters of sample II|   |   |
|------------------------------|---|---|
| Spheroid type                |   |   |
| Krasovsky                    |   |   |
| \(B_1\)                      | 35°00’00.22″ |
| \(L_1\)                      | 90°00’00.11″ |
| Known conditions             |   |   |
| \(B_2\)                      | -30°29’20.96″ |
| \(L_2\)                      | 215°59’04.34″ |
| Geodetic line distance       |   | 15000000.1m |

For each sample, the following five interpolation algorithms are used to interpolate the known geodetic lines:

1. No interpolation algorithm: connect the geodetic line directly with the space straight polyline segment from start point to end point;
2. Fixed distance interpolation algorithm: equal-lying-spacing interpolation of known geodetic line with unconditional constraints on the given interpolation spacing;
3. Interpolation algorithm with length precision threshold constraint: corresponding to the algorithm introduced in section 2.1 of this paper;
4. Interpolation algorithm with arch height precision threshold constraint: corresponding to the algorithm introduced in section 2.2 of this paper;
5. Interpolation algorithm with comprehensive constraints: for the known geodetic line, the semi-divided interpolation of the length precision threshold constraint is carried out first, and then the fine interpolation of the arch height precision threshold constraint is carried out.

For each interpolation algorithm, the following two errors are calculated: first, the length error of the geodetic line is represented by the theoretical length of the geodetic line minus the total length of the space polyline connecting the start point, end point, and all of the interpolation points. The other is the arch height error, which represents the maximum value of the arch height of the small segment of the geodetic line between any adjacent interpolated points (including the start point and the end point of the geodetic line, see Table 3 and Table 4, respectively).
### Tab.3 Calculating deviation statistics of sample I (in meters)

| Interpolation algorithms | Algorithm parameters | Length error | Arch height error |
|--------------------------|----------------------|--------------|------------------|
|                          | Interpolation distance | Length threshold | Arch height threshold |
| ①                       | —                     | —             | 0.0228           |
| ②                       | 10000.0               | —             | 0.0069           |
| ③                       | —                     | 1.0           | 0.0055           |
| ④                       | —                     | 1.0           | 0.0012           |
| ⑤                       | —                     | 1.0           | 0.0008           |

### Tab.4 Calculating deviation statistics of sample II (in meters)

| Interpolation algorithms | Algorithm parameters | Length error | Arch height error |
|--------------------------|----------------------|--------------|------------------|
|                          | Interpolation distance | Length threshold | Arch height threshold |
| ①                       | —                     | —             | 3234544.4016     |
| ②                       | 10000.0               | —             | 1.3533           |
| ③                       | —                     | 1.0           | 0.6395           |
| ④                       | —                     | 1.0           | 0.0201           |
| ⑤                       | —                     | 1.0           | 0.0201           |

3.2. Results Analysis

For charting, when the Earth is considered to be a sphere, the difference between the arc length and the string length can be ignored within 78km of the distance. In other words, it is possible to treat it as a flat circle when the radius of the sphere circle is no more than 39km. Assuming that the linear territorial sea baseline (geodetic line) is plotted on the conventional chart, taking the maximum scale of 1:5000 as an example, in order to meet the error requirement of 0.1mm on the chart, the difference between the two points on the sphere should not be greater than 0.5m. As sample I, it can be seen from Table 3 that the length error of the five interpolation algorithms are all less than 0.5m, then the difference between the arc length and the string length, which are corresponding to any adjacent two interpolated points obtained by each algorithm, must be less than 0.5m. This is the reason that when plotting geodetic lines on the chart plane, it can be realized through connecting the interpolation points by polyline segments.

Theoretically, if the length error and arch height error of an algorithm are very small, the spatial polyline formed by the start point, the end point and the interpolation points of the geodetic line will be more closely to the actual geodetic line, which means the effect of the algorithm is better, and every small polyline segment obtained by the algorithm can maintain a high absolute accuracy. A comprehensive analysis of the results of the above two samples can be obtained from the following conclusions:

First, the effect of the geodetic line interpolation algorithm with arch height precision threshold constraint is obviously better than the geodetic line interpolation algorithm with length precision threshold constraint, which also reflects that the requirement of the arch height precision threshold is much stronger than that of the length precision threshold.

Second, when the distance of the geodetic line is very long (e.g. sample II), if the equal-lying-spacing interpolation algorithm without constraints is carried out, its length error can reach the meter level, while the geodetic line interpolation algorithm with arch height precision threshold constraint and the geodetic line interpolation algorithm with comprehensive constraints are both very good to control the length error in the centimeter level. This also shows that the geodetic line interpolation
algorithm with absolute precision threshold constraints has good universality for the distance of the geodetic line.

Third, compared with the geodetic line interpolation algorithm with length precision threshold constraint or the geodetic line interpolation algorithm with arch height precision threshold constraint, the geodetic line interpolation algorithm with comprehensive constraints is better, the absolute accuracy of the results is also higher.

4. Conclusions

In this paper, a set of geodetic line interpolation algorithms with absolute precision threshold constraints are designed to achieve high-precision plotting on the chart plane for the baseline of the straight-line territorial sea on the earth spheroid. All the calculation process in this algorithm is carried out on the earth spheroid so that the acquisition of the geodetic line interpolation is not limited by the map projection and scale size, which provides the theoretical support of the underlying technology for the high-precision exploration of maritime delimitation on the earth spheroid. The semi-divided interpolation algorithm of the geodetic line with length threshold constraint can complete the calculation of interpolation points more quickly, while the fine interpolation algorithm of the geodetic line with arch height threshold constraint achieves further increasing of interpolation point. Experiments show that the algorithm has a good universality to the distance of the geodetic line. Because interpolation intervals are adapted according to the absolute precision threshold, the algorithm not only has high precision, but also avoids the man-made experimental setting of the relevant parameters. At the same time, it should be pointed out that because the calculation volume of the algorithm is large and relatively complex, how to balance the relationship between its execution efficiency and computational precision is worth further study.

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References

[1] PENG Rencan, Xu Jian, SHEN Wenzhou. The Solutions to Some Key Problems of Accurately Delimitating Sea Area Boundary on the Ellipsoid[J]. Journal of Topography Academy, 2001, 18(3): 210-212.
[2] DONG Jian. Research and Realization of the High-precision Distance Isometric Law[D]. Dalian: Dalian Naval Academy, 2009.
[3] LIANG Deqing, Xu Jian, PENG Rencan. Application of the Geodetic Line Crowley Clairaut's Equation in Geodetic Line Plotting[J]. Hydrographic Surveying and Charting, 2001(2): 20-23.
[4] WANG Wei. Expression Method of Small-scale Surface of the Earth Object Based on a Large Area of the Ellipsoid[D]. Xi’an: Xi’an University of Science and Technology, 2012.
[5] TANG Hongtao, Wang Wei, Yang Yongchong, et al. An Algorithm of Expressing Geodesic on the Ellipsoid Surface[J]. Science of Surveying and Mapping, 2015, 40(04): 7-10.
[6] ZHANG Fengsheng, ZHUANG Jianming. A New Method for Expression of Geodesic Lines and DLG on the Reference Ellipsoid in AutoCAD[J]. Geomatics World, 2018, 25(05): 83-86.
[7] YOU Baoping. Surveying[M]. Dalian: Dalian Naval Academy, 2005.
[8] HUA Tang. Mathematical Foundations of Charts[M]. Maritime Assurance Department of the Naval Command of the Chinese People's Liberation Army, 1985.
[9] HUANG Jiwen. Ellipsoidal Geodesy[M]. Zhengzhou: Surveying And Mapping Academy of PLA, 1991.