Coded Caching with Demand Privacy: Constructions for Lower Subpacketization and Generalizations

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Abstract

Coded caching is a technique where we utilize multi-casting opportunities to reduce rate in cached networks. One limitation of coded caching schemes is that they reveal the demands of all users to their peers. In this work, we consider coded caching schemes that assure privacy for user demands. We focus on reducing subpacketization in such schemes. For the 2-user, 2-file case, we propose a new linear demand-private scheme with the lowest possible subpacketization. This is done by presenting the scheme explicitly and proving impossibility results under lower subpacketization. We then propose new construction schemes for placement delivery arrays. This includes direct as well as lifting constructions. Coded caching schemes based on these can achieve lower subpacketization. A new notion of privacy with security is introduced which combines demand privacy and content security and schemes to achieve the same are proposed. Additionally, when only partial privacy is required, we show that subpacketization can be significantly reduced when there are a large number of files.

I. INTRODUCTION

Data traffic has been growing rapidly in recent years with content delivery, especially that of multimedia files, contributing a significant part. One important aspect of such traffic is its temporal variation. Network usage during peak demand times could be much higher than the demand in off-peak hours. Caching is a way to alleviate network congestion during peak hours by prefetching popular content nearer to the user during off-peak hours. Depending on the limitations on memory, a part of these files would be prefetched and once the user makes a demand, the rest of the requested file will be transmitted. Early literature on caching focused on cache placement/replacement policies [1], caching architectures [4], [18], [20], web request models [3] etc.

Maddah-Ali and Niesen had shown in their seminal paper that coding can achieve significant gain over uncoded caching by making use of multicast opportunities [15]. Coded caching achieves an additional global caching gain, which is proportional to the number of users. Their scheme is shown to be order optimal with an information-theoretic lower bound on the number of files needed to be transmitted (known as rate). This scheme is later improved in [36] and is optimal for uncoded prefetching. Though the exact lower bound on peak rate is still an open problem several works had investigated this and came up with tighter bounds [8], [23], [31], [32], [37]. The problem has been studied in several settings like decentralized caching [16], non-uniform demands [19], multiple levels of cache [13] to name a few. Most of the schemes in these works involve storing the prefetched parts of files in uncoded form. Coded prefetching is investigated in [5], [9], [28], where linear combinations of subfiles are stored in caches. In a few regimes this approach can improve the rate-memory trade-off over uncoded prefetching.

Yan et al. developed a structure called placement delivery arrays that could model both the placement and delivery schemes in a single array [34]. Graphical models for caching have been investigated in [24], [26], [35]. Schemes can also be derived using combinatorial designs and linear block codes [27]. A limitation with the original centralized scheme was the high subpacketization of files [25]. In the original scheme due to [15], the number of subfiles a file is split into, increases exponentially with the number of users. These combinatorial models have helped in developing schemes that have lower subpacketization but with a small penalty on rate [6], [26].

One area of particular interest is security and privacy in coded caching. In typical coded caching schemes, other users involved in the multicast or eavesdroppers might get to know the identity of the file a particular user demanded and its contents. Furthermore, users will be able to partially access files which they have not demanded. This is in part due to the cache that contains contents of files not requested by them and also because, during delivery, they may be able to decode packets not meant for them. Sengupta et al. [22] proposed a method for preventing information leakage to an external wiretapper with the use of cryptographic keys. Vaishakh et al. [21] had recently shown that the contents of a file could be revealed only to the user/users who requested it, using secret sharing techniques.

One aspect that has not been investigated much is the privacy of the user requests in the specific context of coded caching, while it has been studied in closely related areas like index coding [14] and private information retrieval (PIR) [7]. As we were preparing this manuscript, we became aware of work due to Wan and Caire [30] who take a different approach for user request privacy from ours. Another paper by Kamath [11] also addressed the problem of demand privacy and their approach is similar to the one in this work. We point out the specific differences in our results when compared to those from [30] and [11] below.

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In this work, we explore methods to obtain privacy of each user’s requests from the other users in coded caching keeping subpacketization constraints as an important parameter.

Our specific contributions are as follows:
i) We focus on the 2-user, 2-file case in detail and provide an achievable multicast transmission rate versus cache storage curve under a demand privacy constraint.

ii) For the 2-user, 2-file case with cache storage of 1 file, we show an explicit demand-private scheme achieving a multicast transmission rate of 2/3 with a subpacketization of 3. This scheme cannot be obtained using the general scheme proposed in [11], which, in fact, requires a subpacketization of 6.

iii) For the 2-user, 2-file case, we prove some impossibility results on subpacketization of 2 and uncoded cache storage for linear coded caching with demand privacy. These are some of the first negative results in this new area.

iv) We propose some general construction schemes for placement delivery arrays.

v) We propose the notion of privacy with security and show how demand privacy schemes can be combined with content security schemes to achieve this.

vi) We show how demand privacy can be incorporated into other settings including the hierarchical system of caches.

vii) Finally, we propose a general $K$-user, $N$-file partially demand-private scheme and a general scheme for non-uniform demand privacy that provides a trade-off between the level of privacy and reduction in subpacketization.

The rest of the paper is organized as follows. In Section II, we describe the system setup and the problem statement. In Section III, we provide demand-private schemes and an achievable rate vs cache memory curve for the case of two users and two files. We prove certain impossibility results with respect to packetization and coded prefetching. In Section III-D, we describe the general scheme for constructing demand-private coded caching schemes from non-private coded caching schemes from [11], and provide specific instances of the construction from PDAs resulting in lesser subpacketization. We also introduce the notion of partially private schemes and show how to construct a partially private scheme. We conclude with a brief discussion on scope for future work in Section VI.

II. PROBLEM STATEMENT

A. System setup

Assume that we have a server with $N$ files. Each file is assumed to be of $F$ bits and the $i$-th file is denoted $W_i$. The server is connected to $K$ users via a multicast link. Each user has a cache of size $MF$ bits. The cache contents of the $i$-th user are denoted $Z_i$. The system setup is shown in Fig. 1. The cache system works in two phases. In the first phase, called the placement phase, the cache of each user is populated with content by the server. In addition, the server sends metadata or header information $\Theta(Z_i)$ about how the cache content was derived from the files to User $i$. The header information is assumed to be small in size when compared to the file size but crucial for decoding purposes. Note that during the placement phase the server is unaware of the files demanded by the users. We assume that the transmission of cache content and header takes place over a private link between the server and each user.

In the second phase, called the delivery phase, each user requests the server for one of the files from the set of $N$ files. The demand of the $i$-th user is denoted $D_i$, where $D_i \in [N] = \{0, 1, \ldots, N - 1\}$. The demands of all the users 0 to $K - 1$ is denoted by the demand vector $D = (D_0, D_1, \ldots, D_{K-1})$. We assume that the $D_i$ are all i.i.d. random variables uniformly distributed over $[N]$ and that the demands are sent over a private link between the user and the server. Based on the demands, the server multicasts $\ell$ packets, typically of the same size. The entire multicast transmission from the server is denoted $X^D$ for a demand vector $D$. It consists of $RF$ bits. The transmission $X^D$ depends on the cache $Z_i$ and the demands $D_i$. The quantity $R$ is called the rate of transmission. In addition to $X^D$, some additional metadata or header information about the

![Fig. 1. Caching system with $K$ users connected to a server with $N$ files over a broadcast link. Server transmits $X^D$ for a demand vector $D = (D_0, \ldots, D_{K-1})$. User $k$ uses the server transmission $X^D$ and its cache content $Z_k$ to recover the file it has requested, $W_{D_k}$.

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transmission is typically multicast in coded caching schemes. This metadata, denoted \( \Theta(X^D) \), is usually small compared to the file size and provides critical information for decoding by the users.

The main requirement in a coded caching scheme is that User \( i \) should be able to decode the file \( W_{D_i} \), using \( Z_i, \Theta(Z_i), X^D \) and \( \Theta(X^D) \). In other words, we require

\[
H(W_{D_i} | Z_i, \Theta(Z_i), X^D, \Theta(X^D)) = 0. \tag{1}
\]

We denote a coded caching scheme with \( K \) users, \( N \) files, local cache size \( M \), and rate \( R \) as a \((K; N; M, R)\) coded caching scheme, or as a \((K, N)\) scheme in short.

### B. Demand privacy in coded caching

We will introduce the notion of demand privacy in coded caching with a simple example. Consider the \((2, 2)\) coded caching scheme due to Maddah-Ali and Niesen [15]. The two files, denoted \( A, B \), are sub-packetized into two pieces each. The pieces of \( A \) and \( B \) are denoted \( A_0, A_1 \) and \( B_0, B_1 \), respectively. The scheme is illustrated in Fig. 2.

![Non-private scheme](image)

Fig. 2. Non-private scheme from [15] for \( N = 2 \) files, \( K = 2 \) users and demand vector, \( D = (D_0, D_1) \).

Suppose that the demand is \((A, A)\). This results in the transmission \( A_1 \oplus A_0 \). To recover the files, each user must know what linear combination of subfiles has been transmitted. So, we will suppose that the server sends the linear combination information as header along with the transmission. It is easy to see that each user can recover the missing portion of the file demanded by them. However, the scheme has the unfortunate side effect of revealing the demands of each user to the other parties. From the header and scheme details, it is clear to User 0 that User 1 demanded the file \( A \) and vice versa.

If the transmission is \( A_1 \oplus B_j \), then the \( i \)-th user can infer that the \( j \)-th user has requested \( A \) based on the linear combination header information. In general, users can use the combined information of their cache, demands and header data from the server to learn about another user’s demands.

Based on the preceding discussion, to achieve demand privacy in a coded caching scheme, we impose the following additional condition for all demand vectors \( D \):

\[
I(D_i, Z_i, \Theta(Z_i), X^D, \Theta(X^D); D_j) = 0, \quad i \neq j. \tag{2}
\]

In other words, we require that the \( i \)-th user is completely uncertain about what the \( j \)-th user demands, given all information available to User \( i \) in the coded caching scheme. It can be shown that the standard Maddah-Ali-Niesen scheme [15] does not satisfy the demand privacy condition in Eq. (2).

### C. Subpacketization in coded caching

Another parameter of interest in coded caching is subpacketization. It is the number of parts a file will be split into, for the purpose of coding. For instance, it is 2 in the example given in Fig. 2. In the standard Maddah-Ali-Niesen non-private scheme [15], the subpacketization grows exponentially with \( K \). This limits the utility of the scheme in practical scenarios where there may be a large number of users. Hence, reducing subpacketization is important in non-private and, consequently, in demand-private coded caching schemes.

### III. \((K = 2, N = 2)\) CODED CACHING WITH DEMAND PRIVACY

We will first consider the case when there are two files and two users. A complete characterization of the \( M \) vs \( R \) region in the case of two files/users was one of the starting points of the area of coded caching. Therefore, it is important to fully characterize the same region with demand privacy. We have made some partial progress towards this problem.

First, we will show the design of a linear \((2, 2; 1/2/3)\) coded caching scheme with demand privacy having a subpacketization (number of parts into which each file is divided) of 3. We will show later that this scheme provides better subpacketization than a known lifting construction.
Fig. 3. Private scheme for $N = 2$ files, $K = 2$ users. Caches assigned to each user is highlighted. The server transmits $X^D = \{A_2, B_2\}$ for demand vector, $D = (A, A)$.

A. $(M = 1, R = 2/3)$ scheme with subpacketization 3

The two files, $A$ and $B$, are divided into 3 parts $A_i$, $i = 0, 1, 2$ and $B_i$, $i = 0, 1, 2$. Table I summarizes the entire scheme.

| Notation | Possible cache contents |
|----------|-------------------------|
| $Z_{00}$ | $A_0 \oplus A_1, B_0 \oplus B_1, A_2 \oplus B_1$ |
| $Z_{01}$ | $A_0 \oplus A_1, B_0 \oplus B_1, A_1 \oplus B_2$ |
| $Z_{10}$ | $A_0 \oplus A_2, B_0 \oplus B_2, A_1 \oplus B_2$ |
| $Z_{11}$ | $A_0 \oplus A_2, B_0 \oplus B_2, A_2 \oplus B_1$ |

| $D_0D_1$ | $X^{D_0D_1}$ (at User $i$) |
|----------|-----------------------------|
| $AA$     | $A_0, B_0$ |
| $AB$     | $A_1, B_1$ |
| $BA$     | $A_2, B_2$ |
| $BB$     | $A_0 \oplus A_1 \oplus A_2, B_0 \oplus B_1 \oplus B_2$ |

Table II also specifies the transmission for a given choice of cache assignment and demand pair. For instance, if cache $Z_{01}$ is assigned to User 0 and $Z_{10}$ is assigned to User 1 and the demand pair is $A, A$, the transmission is $(A_2, B_2)$.

Table III is the set of recoverable files under each possible cache content for a given transmission. For the same transmission, each user is able to recover either file $A$ or file $B$ with the two possible cache contents. Since the actual cache content is private, we readily see that this scheme satisfies the demand privacy condition in Eq. (2).

B. Dual private schemes

We show that a $(2, 2; M = M_1, R = R_1)$ scheme with demand privacy can be converted into a $(2, 2; M = R_1, R = M_1)$ demand-private scheme and this results in symmetric $R$ vs $M$ capacity bounds for the $(2, 2)$ case.
One can observe that the roles of caches and transmissions can be interchanged in the symmetric file recovery matrix in Table III. Hence, from the scheme given in Table I, we can arrive at a scheme given in Table IV with rate $R = 1$ for $M = 2/3$. We call this scheme the dual of the original scheme. Our next result generalizes the above for all $(2,2)$ private schemes that use one of two caches uniformly at random.

**Lemma 1** (Duality of transmissions and caches). Suppose that there exists a $(2,2; M = M_1, R = R_1)$ private scheme where the user places one of two possible cache contents uniformly at random. Then, there exists a $(2,2; M = R_1, R = M_1)$ private scheme.

**Proof.** Consider a $(2,2; M, R)$ private scheme constructed with users having two options to populate their caches. Let $\{Z_{00}, Z_{01}\}$ be the set of two cache options for User 0 and $\{Z_{10}, Z_{11}\}$ be the set of two cache options for User 1. Let $X_{D_1}^D$ be the transmission corresponding to the user demands $D = (D_1, D_2)$ and cache $Z_{i0}$ at User $i$. The sets $Z = \{Z_{00}, Z_{01}, Z_{10}, Z_{11}\}$ and $X = \{X^{AA}, X^{AB}, X^{BA}, X^{BB}\}$ are able to recover files $A$ and $B$ as given in Table III. Let the size of $Z_i$ be $R_iF$ bits and that of $X_{W_0W_i}^D$ be $M_iF$ bits. We can interchange the role of these caches and transmissions. Let $\{X^{AB}, X^{BA}\}$ be the set of two cache options for User 0 and $\{X^{BB}, X^{AA}\}$ be the set of two cache options for User 1. Then if $Z_{11}$ is transmitted and the Users 0 and 1 are assigned $\{X^{AB}\}$ and $\{X^{BB}\}$ as their caches, both can recover file $A$. Instead if User 1 had $X^{BA}$ in its cache, the users would have recovered $B$ and $A$, respectively. This way of interchangeability between caches and transmissions gives rise to a new scheme for 2 users and 2 files, where the cache size is $M_1$ bits and transmission size is $R_1$ bits. \qed

A consequence of the above duality is that the achievable trade-off between memory and rate for $(2,2)$ private schemes is symmetric about the line $M = R$.

**Lemma 2** (Time sharing with file splitting). Given two achievable $(M, R)$ pairs for a $(2,2)$ private scheme, all values of $(M, R)$ along the line joining these points are achievable.

**Proof.** Consider $0 \leq \alpha \leq 1$. Split the file $A$ into two parts $A_\alpha$ and $A_{\bar{\alpha}}$ of size $\alpha F$ bits and $(1-\alpha)F$ bits respectively. Similarly, split $B$ into $B_\alpha$ and $B_{\bar{\alpha}}$. Denote the two achievable private caching schemes as $(2,2; M, R)$ and $(2,2; M', R')$ respectively. We can use the $(2,2; M, R)$ scheme for sharing $A_\alpha$ and $B_\alpha$ and the $(2,2; M', R')$ scheme for sharing $A_{\bar{\alpha}}$ and $B_{\bar{\alpha}}$. The overall scheme shares $A$ and $B$ with effective cache size $(\alpha M + (1-\alpha)M')F$ bits and transmission $(\alpha R + (1-\alpha)R')F$ bits giving a $(2,2; \alpha M + (1-\alpha)M', \alpha R + (1-\alpha)R')$ private scheme. \qed

Note that the time sharing scheme in Lemma 2 has a subpacketization that is equal to the sum of the two schemes used for time sharing. Using Lemma 2 and Lemma 1, we can plot the upper bounds for the achievable $(M, R)$ pair for $(2,2)$ private schemes. The plot is symmetric about the line $M = R$ as can be seen in Fig. 4.

### C. Towards lower bounds and optimal subpacketization

Kamath et. al. had extended our results on $(2,2)$ private schemes by providing two more achievable $(M, R)$ pairs and a tight lower bound [12]. We present here a basic impossibility result involving subpacketization and coding of cache contents which cannot be captured by a lower bound on rate. For subpacketization of 3, the scheme in Section III-A uses coded cache contents,
which is not typical in the non-private setting. In the setting considered here for demand privacy, we have the following result on coding in cache contents.

**Theorem 3 (Necessity of coded prefetching).** Consider \( N = 2, K = 2 \) with subpacketization of 3 and \( M = 1 \). If the cache contents are not allowed to be coded (i.e. linear combinations of two or more file parts are not allowed to be stored in cache), a rate \( R = 2/3 \) cannot be achieved with demand privacy when using a linear scheme.

**Proof.** A proof is given in Appendix A.

\[ \square \]

**D. General Scheme**

Here we describe the general scheme from [11] that provides the design of a demand-private coded caching scheme from non-private schemes.

**Theorem 4 (Existence of private schemes [11]).** If there exists a \((KN, N; M, R)\) coded caching scheme, then there exists a private \((K, N; M, R)\) scheme.

Using the Maddah-Ali-Niesen scheme from [15] as the non-private scheme in Theorem 4, we obtain the following:

**Corollary 5.** There exists a demand private \((K, N; M, R)\) scheme for integer values of \( KM \), where the rate

\[
R = \begin{cases} 
\frac{K(N-M)}{(1+KM)} & \text{if } M \geq \frac{K-1}{K} \\
N-M & \text{if } M < \frac{K-1}{K}
\end{cases}
\]  

Using the scheme proposed by Maddah Ali and Niesen [15, Theorem 1], for integer values of \( KM \) we can construct a \((NK, N; M, R)\) non-private scheme. If \( KM \geq K-1 \), then \( R = \frac{K(N-M)}{(1+KM)} \). If \( 1 \leq KM < K-1 \), then \( R = N-M \). We can map each user in the demand-private scheme to \( N \) virtual users in the non-private scheme and extend the demand vector of the private \((K, N; M, R)\) scheme to the non-private scheme using

\[
d'_j = d_k + j - (k-1)N \mod N \text{ for } (k-1)N \leq j < kN,
\]

where \( 0 \leq k < K \). Then, the \((NK, N; M, R)\) scheme from [15] gives \( N \) possible cache contents that could be stored in each user and a transmission for each demand from which each user can recover their files. The cache memory and rate required in the private scheme will be the same as that in the non-private scheme. More details can be found in [11]. Note that there is no coding gain when \( KM < K-1 \).

**E. Case of two files, two users**

For the \( N = 2, K = 2 \) case considered earlier, the \( M = 1, R = 2/3 \) construction presented in Section III-A is not derived from a non-private scheme but constructed directly. In fact, a construction from the Maddah-Ali-Niesen scheme using Theorem 4 results in a subpacketization of 6, when compared to the subpacketization of 3 needed for the scheme in Section III-A. This shows that direct construction has the benefits of improved subpacketization. In general, the subpacketization increases exponentially with the number of users in the Maddah-Ali-Niesen scheme. When such a non-private scheme is used to construct a private scheme using Theorem 4, the subpacketization grows exponentially with \( NK \). Hence reducing subpacketization becomes a major concern in demand private schemes.
IV. Low subpacketization schemes from PDAs

A general framework for non-private centralized coded caching schemes was proposed in [34] using placement delivery arrays (PDAs). Some of them improve upon the Maddah-Ali-Niesen scheme [15] in subpacketization or other parameters. In this section, at first, we briefly review the concept of PDAs. Secondly, we propose new constructions of PDAs for low subpacketization. We begin with the construction of 2-regular PDAs for a range of parameters. Then we discuss some lifting-type constructions that can achieve higher coding gains. We also present the details of a partial Latin square based construction. Finally, we demonstrate how they can be used to construct low subpacketization schemes for both non-private and private coded caching schemes.

For positive integers $K, f, Z$ and $S$, a $(K, f, Z, S)$ placement delivery array is a $f \times K$ matrix $P = [p_{j,k}]$ with $j \in [f], k \in [K]$ containing either a “*” or integers from $\{0, 1, \ldots, S-1\}$ in each cell such that they satisfy the following conditions [34].

C1. The symbol “*” appears $Z$ times in each column.

C2. Each integer $s \in [S]$ occurs at least once in the array.

C3. If two distinct entries $p_{j_1,k_1}$ and $p_{j_2,k_2}$ are the same integer $s \in [S]$, then

a) $j_1 \neq j_2$ and $k_1 \neq k_2$, i.e., they lie in distinct rows and distinct columns

b) $p_{j_1,k_1} = p_{j_2,k_2} = \neq s$, i.e., the other two corners in the rectangle formed by $(j_1, k_1)$ and $(j_2, k_2)$ must be “*”.

Note that the parameters of a PDA are invariant under row and columns permutations. Existence of coded caching scheme corresponding to a PDA was proved by Yan et al. [34].

**Theorem 6** (Coded caching schemes from PDAs [34]). For a given $(K, f, Z, S)$ PDA, $P = [p_{j,k}]_{f \times K}$, there exists a corresponding $(K, N; M, R)$ caching system with subpacketization $f$, $M/N = Z/f$ and $R = S/f$.

In the caching scheme, each file $W_i$ is split into $f$ subfiles $W_{i,j}$, $i \in [N], j \in [f]$. Each row of the PDA corresponds to a subfile label and if $p_{j,k} = \neq s$, then User $k$ caches the $j$-th subfile of every file. So the cache $Z_k$ of User $k$ is given by

$$Z_k = \{W_{i,j} : \forall i \in [N], p_{j,k} = \neq s\}$$

(5)

For the demand vector $D = \{D_k : k \in [K]\}$, the server transmits

$$X^D = \left\{\bigoplus_{j \in [f], k \in [K], p_{j,k} = s} W_{D_k,j} : s \in [S]\right\}.$$  

(6)

In the transmission corresponding to $s$, User $k_0$ has all the subfiles in its cache except $W_{D_{k_0,j_0}}$, where $j_0$ is such that $p_{j_0,k_0} = s$. This is because, if $p_{j,k} = s$ for $k \neq k_0$, then $j \neq j_0$ and $p_{j,k} = \neq s$ by C3. So, User $k_0$ recovers $W_{D_{k_0,j}}$ whenever $p_{j,k} \neq \neq s$. In this manner, each user can recover the entire file demanded using cache contents and transmissions.

A placement delivery array $P$ is said to be a $g$-regular $(K, f, Z, S)$ PDA, $g$-$(K, f, Z, S)$ PDA or $g$-PDA for short, if each integer in $[S]$ appears $g$ times in $P$. The constant $g$, called the coding gain [34] (or global caching gain in [15]), indicates the number of users that recover a subfile from a single transmission. The rate of the coded caching scheme obtained from a $g$-regular PDA is

$$R = \frac{S(f/f)}{F} = \frac{K(f-Z)}{fg},$$  

(7)

where we have used $Sg = K(f-Z)$ = number of integer cells in the PDA.

In some circumstances PDAs are equivalent to combinatorial objects called partial Latin squares [24]. A partial Latin square of size $n$ is an $n \times n$ array filled with different symbols such that every symbol appears in every row and column at most once and some cells are allowed to be empty. A partial Latin square $P$ is said to have the Blackburn property if whenever two distinct cells $P_{j_1,k_1}$ and $P_{j_2,k_2}$ are occupied by the same symbol, the opposite corners $P_{j_1,k_2}$ and $P_{j_2,k_1}$ are empty. An $n \times n$ partial Latin square with $S$ symbols satisfying Blackburn property and $Z$ empty cells in every column is an $(n, n, Z, S)$-PDA [24]. Conversely, a $(K, f, Z, S)$-PDA is a partial Latin square with Blackburn property when $f = K \geq S$.

A. Constructions for PDAs

We focus on $(n, n, Z, S)$-PDAs that are $g$-regular. The corresponding $(K, N; M, R)$ coded caching scheme can have arbitrary number of files $N$, $K = n$ users, $M = (NZ/n)$, and the rate $R = (n-Z)/g$ (using Eq. (7)). We present multiple constructions for PDAs, and they are organized into three types.

1) $2$-regular constructions: We begin with a direct construction of a $2$-regular PDA.

**Lemma 7.** For an integer $n$, there exists a $2$-regular $(n, n, 1, n(n-1)/2)$ PDA.

**Proof.** Take an $n \times n$ array. Set all diagonal (or anti-diagonal) entries to “*”. There are $S = n(n-1)/2$ cells below the diagonal. Fill them with the integers from 0 to $S-1$. Symmetrically fill the cells above the diagonal. □
Examples with \( n = 4 \) (diagonal) and \( n = 5 \) (anti-diagonal) are shown in Table V. The PDAs constructed using Lemma 7 is similar in structure to the PDA corresponding to Maddah-Ali-Niesen schemes [19] for \( t = \frac{KM}{N} = 1 \).

So far, in an \((n, n, Z, S)\) PDA, we have considered \( S \) to be a positive integer and used the integers \( \{0, 1, \ldots, S-1\} \) to populate the array. We will now generalise and consider populating the array with a set of distinct integers \( S \). Such PDAs are denoted either \((n, n, Z, S)\) or \((n, n, Z, |S|)\) PDAs.

In an \((n, n, Z, S)\) PDA, \(|S|\) determines the rate of the corresponding coded caching scheme. In the simple construction of Lemma 7, \(|S| = n(n-1)/2\). We now provide methods to modify a PDA \( P \) from Lemma 7 to lower \(|S|\), while retaining regularity of 2. The basic idea is to replace integers in \( P \) with \( \ast \) without affecting regularity.

For this purpose, given a 2-regular \((n, n, Z, S)\) PDA \( P \) with rows/columns indexed from 0 to \( n-1 \) and \( s \in S \) at locations \((i_1(s), j_1(s))\) and \((i_2(s), j_2(s))\), we associate a graph \( G(P) = (V, E) \) with vertex set \( V = \{0, 1, \ldots, n-1\} \) representing the columns of \( P \) and edge set \( E = \{(j_2(s), j_1(s)) : s \in S\} \). We refer the reader to [10] for definitions and basic results in graph theory. The edge \( e(s) = (j_2(s), j_1(s)) \in E \) is labelled with the triple \((i_1(s), i_2(s), s)\). For a symmetric PDA \( P \) (such as the one from Lemma 7), since \( i_2(s) = j_1(s) \) and \( j_2(s) = i_1(s) \), we have \( e(s) = (i_1(s), j_1(s)) \) and the edge label is shortened to \( s \).

**Lemma 8.** Let \( P \) be the 2-regular \((n, n, 1, n(n-1)/2)\) PDA from Lemma 7. Then, the associated graph \( G(P) \) is equal to \( K_n \), the complete graph on \( n \) vertices.

**Proof.** By symmetry, the first column of \( P \) is equal to the first row, if diagonal is set to \( \ast \), or the last row, if anti-diagonal is set to \( \ast \). So, in \( G(P) \), vertex 1 is connected to all other vertices \( \geq 2 \).

Now, delete the first column and its corresponding first or last symmetric row, and see that vertex 2 is connected to vertices \( \geq 3 \). Proceed iteratively to complete the proof.

A spanning subgraph has the same vertex set as the original graph and a subset of its edges with no isolated vertices. A graph or subgraph is said to be \( r \)-regular if every vertex has degree equal to \( r \). An \( r \)-regular spanning subgraph is very useful for modifying PDAs as shown in the following lemma.

**Lemma 9.** Consider a 2-regular \((n, n, Z, S)\) PDA \( P \) and its associated graph \( G(P) \). Suppose \( G(P) \) has an \( r \)-regular spanning subgraph with its \( nr/2 \) edges being \( \{e(s) : s \in S_r\} \), where \( S_r \subset S \). The array obtained by setting \( s \in S_r \) as \( \ast \) in \( P \) is a 2-regular \((n, n, Z + r, S_r \setminus S_{\ast})\) PDA.

**Proof.** Since the subgraph is spanning and \( r \)-regular, exactly \( r \) integers in \( S_r \) are present in each column of \( P \). So, setting the integers in \( S_r \) to \( \ast \) results in the modified PDA as claimed.

To find regular spanning subgraphs, the notions of 1-factors and 1-factorization are useful [29]. A matching in a graph is a set of non-intersecting (or parallel) edges. A matching is said to be a 1-factor if it covers all vertices. A 1-factor is clearly a 1-regular spanning subgraph.

A complete graph \( K_n \), for \( n \) even, has multiple 1-factors each with \( n/2 \) edges [29]. A 1-factorization of \( K_n \), \( n \) even, is a partition of its \( n(n-1)/2 \) edges into \( n-1 \) edge-disjoint 1-factors. It is well known that 1-factorizations exist for \( K_n \) when \( n \) is even [17]. The union of \( r \) different 1-factors in a 1-factorization is clearly an \( r \)-regular spanning subgraph of \( K_n \), which can be used in Lemma 9 as follows.

**Lemma 10.** Let \( P \) be the 2-regular \((n, n, 1, n(n-1)/2)\) PDA from Lemma 7 with associated graph \( K_n \), \( n \) even. Let \( \{M_1, M_2, \ldots, M_{n-1}\} \) be a 1-factorization of \( K_n \) with the \( i \)-th 1-factor \( M_i = \{e(s_{ij}) : j \in \{1, 2, \ldots, n/2\}\} \). For \( z \in \{1, 2, \ldots, n-2\} \), let \( P_z \) be the array obtained by setting \( s_{ij} \) to \( \ast \) in \( P \) for \( 1 \leq i \leq z \) and \( j \in \{1, 2, \ldots, n/2\} \). Then, \( P_z \) is a 2-regular \((n, n, z+1, n(n-z-1)/2)\) PDA.

**Proof.** Since \( M_1 \cup M_2 \cup \cdots \cup M_z \) is a \( z \)-regular spanning subgraph of \( K_n \), the result follows by the use of Lemma 9.

For \( n = 4 \), consider the \((4, 4, 1, 6)\) PDA in Table V. A 1-factorization for the associated graph is \( M_1 = \{e(0), e(5)\}, M_2 = \{e(1), e(4)\}, M_3 = \{e(2), e(3)\} \). The modified PDAs obtained using this 1-factorization in Lemma 10 are easy to write down.
For \( n \) odd, there are no 1-factors in \( K_n \), and the smallest regular spanning subgraph is a Hamiltonian cycle, which is a cycle with \( n \) edges passing through all \( n \) vertices [10]. It is well-known that \( K_n \) for \( n \) odd can be decomposed into \((n-1)/2\) edge-disjoint Hamiltonian cycles [2], [29]. This decomposition leads to the following lemma.

**Lemma 11.** Let \( P \) be the 2-regular \((n, n, 1, n(n-1)/2)\) PDA from Lemma 7 with associated graph \( K_n \), \( n \) odd. Let \( \{H_1, H_2, \ldots, H_{(n-1)/2}\} \) be a set of edge-disjoint Hamiltonian cycles of \( K_n \) with \( H_i = \{e(s_{ij}) : j \in \{1, 2, \ldots, n\}\} \). For \( z \in \{1, 2, \ldots, (n-3)/2\} \), let \( P_z \) be the array obtained by setting \( s_{ij} \) to \( * \) in \( P \) for \( 1 \leq i \leq z \) and \( j \in \{1, 2, \ldots, n\} \). Then, \( P_z \) is a 2-regular \((n, n, 2z+1, n(n-2z-1)/2)\) PDA.

**Proof.** Since \( H_1 \cup H_2 \cdots \cup H_{l} \) is a 2-regular spanning subgraph of \( K_n \), the result follows by the use of Lemma 9.

For \( n = 5 \), consider the \((5, 5, 1, 10)\) PDA in Table V. A Hamiltonian cycle decomposition for the associated graph is \( H_1 = \{e(0), e(1), e(5), e(6), e(7)\} \), \( H_2 = \{e(2), e(8), e(9), e(4), e(3)\} \). The modified PDAs obtained using this decomposition in Lemma 11 are easy to write down.

We will conclude this section on modifying PDAs by providing explicit constructions of 1-factors and regular spanning subgraphs for the graph \( G \) associated with different types of PDAs \( P \). Let \( P_1, n \) be the \((n-1)/2\) PDA created using Lemma 7 for \( n \) even (anti-diagonal) with \( p_{i,j} = * \) if \( i + j = n - 1 \). The integers are placed row-wise above the anti-diagonal as in the \( n = 6 \) example in Table V. Define the \( l \)-th diagonal \((0 \leq l \leq n-1)\) of a PDA \( P \) to be the cells \((i, j)\) such that \( j = (i + l) \mod n \). The \( l \)-th diagonal is called even if \( l \) is even, and odd otherwise. The following two properties are easy to establish for \( P_1, n \).

1) The \( * \)'s in \( P_1, n \) occur only in the odd diagonals.
2) Every even diagonal has \( n/2 \) distinct integers with each integer occurring twice and no \( * \).

So, we see that the \( n/2 \) even diagonals are edge-disjoint 1-factors of \( G(P_1, n) \).

All the odd diagonals of \( P_1, n \) together form an \((n/2 - 1)\)-regular spanning subgraph. Using this spanning subgraph in Lemma 9, we obtain a PDA, which we call \( P_2, n \). It can be shown that \( P_2, n \) is a 2-regular \((n, n/2, n^2/4)\) PDA. Further, the even diagonals of \( P_2, n \) continue to be 1-factors of \( G(P_2, n) \).

**Example IV.1.** Constructions using \( P_1, n \) and \( P_2, n \) for \( n = 6 \) are demonstrated in Fig 5.

![Example IV.1. Constructions using P1,n and P2,n for n = 6 are demonstrated in Fig 5.](image)

Next, we consider the case of \( P_1, n \) when \( n \) is odd and anti-diagonal elements are set as \( * \). In this case, for \( 0 \leq j \leq (n-3)/2 \), it can be shown that the locations \((j, j), (n-1-j, n-1-j), (2j+1)-th diagonal and (n-2j-1)-th diagonal in \( P_1, n \) form a Hamiltonian cycle in \( G(P_1, n) \). This provides the necessary Hamiltonian cycle decomposition.

**Example IV.2.** For \( n = 7 \), a 2-(7, 7, 3, 14) PDA is constructed from a 2-(7, 7, 1, 21) PDA as shown below in Fig. 6 using the Hamiltonian cycle (highlighted locations) with \( j = 2 \).
2) Lifting or protograph-type constructions: The constructions in Lemmas 10 and 11 create PDAs with the same size and same coding gain as of the original PDA. To increase coding gain and obtain $g$-regular PDAs for $g > 2$, we employ the idea of lifting or protograph construction. Similar to the popular notion of protograph or lifted LDPC codes, we start with a base PDA, and replace each entry with another PDA. An important difference when lifting PDAs is that we have to ensure that the Blackburn property is preserved during the lifting.

The component PDAs used in the lifting construction are as follows.

1) $I_n(t)$ denotes the $(n, n, n - 1, 1)$ PDA with the integer $t$ on the main diagonal and $\ast$ in all other cells. $\hat{I}_n(t)$ denotes the $(n, n, n - 1, 1)$ PDA with the integer $t$ on the main anti-diagonal and $\ast$ in all other cells. For example,

\[
I_3(1) = \begin{pmatrix} 1 & \ast & \ast \\ \ast & 1 & \ast \\ \ast & \ast & 1 \end{pmatrix}, \quad \hat{I}_3(0) = \begin{pmatrix} \ast & \ast & 0 \\ \ast & 0 & \ast \\ 0 & \ast & \ast \end{pmatrix}.
\]

2) $G_n(S)$ and $H_n(S)$ denote the $(n, n, 1, S)$ PDAs obtained using Lemma 7 with anti-diagonal and diagonal cells set as $\ast$, respectively. The set $S$ is ordered and contains $n(n-1)/2$ integers, which are arranged row-wise above the anti-diagonal in $G_n(S)$ or below the diagonal in $H_n(S)$. For example,

\[
G_3(4, 6, 5) = \begin{pmatrix} 4 & 6 & \ast \\ 5 & 6 & \ast \\ \ast & \ast & 5 \end{pmatrix}, \quad H_3(13, 12, 11) = \begin{pmatrix} \ast & 13 & 12 \\ 13 & \ast & 11 \\ 12 & 11 & \ast \end{pmatrix}.
\]

In the first lifting method, we start with a base PDA and replace $\ast$’s with an all-$\ast$ array, and replace integers with PDAs that contain disjoint sets of integers.

Lemma 12. Let $P_b$ be a $(K, f, Z, S_b)$ PDA. Let an array $P$ be defined as follows:

1) Each $\ast$ in $P_b$ is replaced by an $a \times b$ all-$\ast$ array.

2) Each integer $i \in S_b$ is replaced by an $(a, b, e, S_i)$ PDA, where $S_i$, $S_j$ are disjoint for $i \neq j$.

Then, $P$ is a $(K a, f b, Z a + (f - Z) e, S)$ PDA, where $S = \bigcup_{i \in S_b} S_i$.

Proof. Clearly, $P$ is an $f b \times K a$ array. Each column of $P_b$ has $Z \ast$’s and $f - Z$ integers. So, each column of $P$ has $Z a + (f - Z) e \ast$’s.

Since each column of $P_b$ has distinct integers and distinct integers are replaced with PDAs containing disjoint sets of integers, each column of $P$ has distinct integers as well.

Finally, we need to verify the Blackburn property for $P$. Let $p_{i,j}$ and $q_{i,j}$ denote the $(i,j)$-th elements of $P_b$ and $P$, respectively. Let $i/b$ denote the quotient when $i$ is divided by $b$, and let $j/a$ be defined similarly. If $q_{i_{1,j_{1}}} = q_{i_{2,j_{2}}} = s$ in the lifted PDA $P$, we necessarily have that $p_{i_{1}/b,j_{1}/a} = p_{i_{2}/b,j_{2}/a}$ in the base PDA $P_b$ because different integers are expanded to PDAs containing disjoint sets of integers. So, we have $p_{i_{1}/b,j_{2}/a} = p_{i_{2}/b,j_{1}/a} = \ast$ by the Blackburn property. Since an $\ast$ is replaced by an all-$\ast$ array, $q_{i_{1,j_{2}}} = q_{i_{2,j_{1}}} = \ast$ in $P$ and the Blackburn property is satisfied.

Example IV.3. Two PDAs obtained using Lemma 12 are shown in Table VI. The first one is with $P_b = \begin{pmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{pmatrix}$ and integer $i$ replaced with $I_3(i)$ resulting in a 3-regular $(9, 9, 6, 9)$ PDA. The second one uses $P_b = \begin{pmatrix} \ast & 0 & 1 \\ 0 & \ast & 2 \\ 1 & 2 & \ast \end{pmatrix}$ with 0 replaced
by \( G_3(0, 1, 2), 1 \) replaced by \( H_3(3, 4, 5) \) and 2 replaced by \( G_3(6, 7, 8) \) resulting in a 4-regular \((9, 9, 5, 9)\) PDA.

The second lifting method we propose is for 2-regular base PDAs. In this method, a \( * \) in the base PDA will be replaced by the PDA \( I_4(t) \), which has integers on the diagonal. So, the two occurrences of every integer in the base PDA have to be replaced so that the Blackburn property is not violated. The following definition characterizes non-violation of Blackburn property during lifting.

Two \( n \times n \) PDAs \( P_1 \) and \( P_2 \) are said to be Blackburn-compatible with respect to (w.r.t.) a third \( n \times n \) PDA \( P_0 = [p_{ij}^{(0)}] \) if, whenever \( p_{ij}^{(1)} = p_{ij}^{(2)} \neq * \), we have \( p_{ij}^{(0)} = \hat{p}_{ij}^{(0)} = * \). Firstly, we see that any two PDAs are Blackburn-compatible w.r.t. the trivial all-\( * \) PDA, and this enables the lifting method of Lemma 12. Next, the following two PDAs \( A_3(x) \) and \( A_3'(x) \) are examples of PDAs that are Blackburn-compatible w.r.t. \( I_3(t) \):

\[
A_3(x) = \begin{pmatrix}
    x & x + 2 & x + 1 \\
    x + 2 & * & x \\
    x + 3 & x & *
\end{pmatrix}, \quad A_3'(x) = \begin{pmatrix}
    x & x + 1 & * \\
    x + 3 & * & x + 1 \\
    * & x + 3 & x + 2
\end{pmatrix}.
\]

Note that there are 4 distinct integers occurring in \( A_3(x) \) and \( A_3'(x) \). The following two PDAs \( A_4(x) \) and \( A_4'(x) \) with 6 distinct integers are Blackburn-compatible w.r.t. \( I_4(t) \):

\[
A_4(x) = \begin{pmatrix}
    x + 2 & x + 4 & x + 1 & * \\
    x + 5 & x + 3 & * & x + 1 \\
    x & * & x + 3 & x + 4 \\
    * & x & x + 5 & x + 2
\end{pmatrix}, \quad A_4'(x) = \begin{pmatrix}
    x + 3 & x + 4 & x + 1 & * \\
    x + 5 & x + 2 & * & x + 1 \\
    x & * & x + 2 & x + 4 \\
    * & x & x + 5 & x + 3
\end{pmatrix}.
\]

The following two PDAs with 4 distinct integers are Blackburn-compatible w.r.t. \( I_4(t) \):

\[
C_4(x) = \begin{pmatrix}
    * & x + 2 & x + 1 & * \\
    x + 3 & * & * & x + 1 \\
    x & * & x + 2 & * \\
    * & x & x + 3 & *
\end{pmatrix}, \quad C_4'(x) = \begin{pmatrix}
    * & x + 2 & x + 1 & * \\
    x + 3 & * & * & x + 1 \\
    x & * & x + 2 & * \\
    * & x & x + 3 & *
\end{pmatrix}.
\]

Other such examples can be readily constructed for larger-sized PDAs as well.

Lifting of 2-regular PDAs using Blackburn-compatibility of PDAs is described in the next lemma.

**Lemma 13.** Let \( P_b \) be a 2-regular \((K, f, Z, S_b)\) PDA. Let an array \( P \) be defined as follows:

1. There are \( KZ \) \( * \)'s in \( P_b \). The \( t \)-th \( * \) is replaced by \( I_n(t), t \in [KZ] \).
2. The two occurrences of an integer \( i \in S_b \) are replaced by two \((n, n, e, S_i)\) PDAs that are Blackburn-compatible w.r.t. \( I_n(t), \) where \( S_i = [KZ] \cap (\bigcup_{s \in S_b} S_s) \).

Then, \( P \) is a \((Kn, fn, Z(n-1)+(f-Z)e, S)\) PDA, where \( S = [KZ] \bigcup (\bigcup_{i \in S_b} S_i) \).

**Proof.** The parameters of \( P \) are established easily. No row or column has repeating integers by the disjointedness of the sets. The Blackburn-compatibility ensures that \( P \) satisfies the Blackburn property after lifting.

Using 2-regular PDAs obtained from Lemmas 7 and 10 as base PDAs and lifting using the Blackburn-compatible \( 3 \times 3 \) PDAs in (8) and \( 4 \times 4 \) PDAs in (9), we obtain the following corollary.

**Corollary 14.** (a) For \( n \) even, there exist \( c \)-regular \((cn, cn, n + (c - 2)z, nz + (c - 1)n(n - z))\) PDAs for \( c = 3, 4 \) and \( z \in \{1, 2, \ldots, n\} \).

(b) For \( n \) even, there exist \( 4 \)-regular \((4n, 4n, n + 2z + 1, nz + 3n(n - z) - n)\) PDA for \( z \in \{1, 2, \ldots, n - 1\} \).
Proof. (a) The construction is that of Lemma 13 with the 2-regular \((n, n, z, n(n - z)/2)\) base PDA from Lemma 7 for \(z = 1\) and Lemma 10 for \(2 \leq z \leq n - 1\). For \(z = n\), the all-\(\ast\) PDA is used as the base PDA.

To replace an integer \(i\) occurring twice in the base PDA, the \(c \times c\) PDAs \(A_c(x_i)\) and \(A'_c(x_i)\) from (8) for \(c = 3\) and (9) for \(c = 4\) are used with \(x_i = nz + 2(c - 1)i\), which ensures disjointedness for different values of \(i\).

Finally, the \(c\)-regularity of the lifted PDA follows because each integer in \([nz]\) occurs \(c\) times in \(I_c(t)\) and each integer in \(A_c(x_i)\) and \(A'_c(x_i)\) occurs \(c\) times as well.

(b) Once again, the construction is that of Lemma 13 with the 2-regular \((n, n, z, n(n - z)/2)\) base PDA from Lemma 7 for \(z = 1\) and Lemma 10 for \(2 \leq z \leq n - 1\).

Select a 1-factor of the base PDA (say, an even diagonal) and replace an integer \(i\) occurring twice in the 1-factor with \(C_4(x_i)\) and \(C'_4(x_i)\), where \(x_i = nz + 6i\). Any other integer \(j\) occurring twice outside of the chosen 1-factor is replaced with \(A_4(x_j)\) and \(A'_4(x_j)\).

Since \(C_4(x)\) and \(C'_4(x)\) contain \(2\) \(\ast\)’s per column, the number of \(\ast\)’s in the lifted PDA increases by 1 when compared to Part (a). Since the number of distinct integers in \(C_4(x)\) and \(C'_4(x)\) is 4, which is lesser by 2, when compared to 6, the number of distinct integers in the lifted PDA decreases by \(2 \times n/2 = n\) when compared to Part (a).

Note that the second lifting construction improves the coding gain from 2 to \(c\) (3 or 4). A \((12, 12, 5, 21)\) PDA constructed from a \((3, 3, 1, 3)\) PDA using Corollary 14 (Part (a)) is given below.

### (3, 3, 1, 3) PDA
\[
\begin{array}{ccc}
* & 0 & 1 \\
0 & * & 2 \\
1 & 2 & *
\end{array}
\]

### (12, 12, 5, 21) PDA
\[
\begin{array}{cccc}
I_4(0) & A_4(3) & A_4(9) \\
A'_4(3) & I_4(1) & A_4(15) \\
A'_4(9) & A'_4(15) & I_4(2)
\end{array}
\]

Recasting the parameters of the PDAs obtained using Corollary 14 (Parts (a) and (b)), we obtain the following:

**Corollary 15.** For \(c \in \{3, 4\}\) and integer \(n\) divisible by \(2c\), there exists \(c\)-regular \((n, n, z, \frac{n(n-2z)}{c})\) PDA for integers \(z \in \{\frac{n}{c} + 1, \frac{n}{c} + 2, \ldots, \frac{(c-1)n}{c}\}\).

**Proof.** In Corollary 14 (Part (a)), we set \(n' = cn\) and \(z' = n + (c - 2)z\) and simplify. In Corollary 14 (Part (a)), we set \(n' = 4n\) and \(z' = n + 2z + 1\) and simplify. \(\square\)

**Example IV.4.** A 4-regular \((24, 24, 11, 78)\) PDA from Corollary 15 is illustrated in Fig. 7. The base PDA is \((6, 6, 2, 12)\). The 1-factor replaced with \(C_4(x)\) and \(C'_4(x)\) has been highlighted.
Corollary 15 clearly brings out the coding gain of 3- and 4-regular lifted PDAs when compared to the 2-regular PDAs of the same size from Lemma 10.

Finally, to show the versatility of the lifting construction, we provide a construction for 8-regular PDAs. The following two $8 \times 8$ PDAs are Blackburn-compatible w.r.t. $I_{\hat{S}}(t)$ as can be directly verified.

$$A_8(x) = \left( \begin{array}{cccccccc} x & * & * & * & x+5 & x+6 & x+3 & * \\ * & x & * & * & x+7 & x+4 & * & x+3 \\ * & * & x & * & x+2 & * & x+4 & x+6 \\ * & * & * & x & * & x+2 & x+7 & x+5 \\ x+4 & x+6 & x+3 & * & x+1 & * & * & * \\ x+7 & x+5 & * & x+3 & * & x+1 & * & * \\ x+2 & * & x+5 & x+6 & * & * & x+1 & * \\ * & x+2 & x+7 & x+4 & * & * & * & x+1 \end{array} \right)$$ \hspace{1cm} (11a)

$$A'_8(x) = \left( \begin{array}{cccccccc} x & * & * & * & x+4 & x+7 & x+2 & * \\ * & x & * & * & x+6 & x+5 & * & x+2 \\ * & * & x & * & x+3 & * & x+5 & x+7 \\ * & * & * & x & * & x+3 & x+6 & x+4 \\ x+5 & x+7 & x+2 & * & x+1 & * & * & * \\ x+6 & x+4 & * & x+2 & * & x+1 & * & * \\ x+3 & * & x+4 & x+7 & * & * & x+1 & * \\ * & x+3 & x+6 & x+5 & * & * & * & x+1 \end{array} \right)$$ \hspace{1cm} (11b)

**Corollary 16.** Given a 2-regular $(K, f, Z, S)$ PDA $P$, there exists an 8-regular $(8K, 8f, 4f + 3Z, 8|S| + KZ)$ PDA.

**Proof.** The construction is that of Lemma 13 with the given 2-regular base PDA $P$, * replaced by $I_{\hat{S}}(t)$, $t \in [KZ]$, and integers replaced by the $8 \times 8$ PDAs $A_8(x_i)$ from (11a) and $A'_8(x_i)$ from (11b) with $x_i = KZ + 8i$. Since $I_{\hat{S}}(t)$ has 7 *'s per column and $A_8(x_i), A'_8(x_i)$ have 4 *'s per column, the number of * in each column of the lifted PDA is $(Z+7)(f-Z)/4 = 4f + 3Z$. The other parameter values are easy to establish.

3) **Partial Latin square construction:** The next construction is based on Wanless’ construction of partial latin squares with Blackburn property [33]. This is an algebraic construction and can provide various different values for regularity determined by the existence of certain constrained sets.

Explicit constructions of PDAs and coded caching schemes using Wanless’ construction of partial Latin squares have not appeared in the literature. So, we provide some details. Let $Z_n$ denote the ring of integers $\{0, 1, \ldots, n-1\}$ modulo $n$. A subset $S \subseteq Z_n$, $n$: odd, satisfies the law of excluded middle (LEM) property if

$$\frac{x+y}{2} \notin S, \forall x, y \in S, x \neq y.$$ \hspace{1cm} (12)
Lemma 17 (Wanless construction). Suppose $S$ is a subset of $\mathbb{Z}_n$, $n$: odd, satisfying LEM. There exists an $|S|$-regular $(n, n, n - |S|, n)$ PDA.

Proof. Let $P = [p_{j,k}]$, $j,k \in \mathbb{Z}_n$, be defined as

$$p_{j,k} = \begin{cases} \frac{1}{2}(j + k) & \text{if } k - j \in S, \\ \ast & \text{otherwise.} \end{cases}$$

We claim that $P$ is a $|S|$-regular $(n, n, n - |S|, n)$ PDA.

For $\mathbb{Z}_n$ with odd $n$, $\frac{1}{2}(j + k) \equiv \frac{1}{2}(j + k_2) \mod n$ only when $k_1 \equiv k_2 \mod n$. This ensures no duplication in rows. Similarly, there is no duplication in columns.

For verifying that $P$ has Blackburn property, assume that cells $(j_1, k_1)$ and $(j_2, k_2)$ are such that $k_1 - j_1, k_2 - j_2 \in S$ and

$$\frac{1}{2}(j_1 + k_1) = \frac{1}{2}(j_2 + k_2) \mod n.$$

Because of LEM, $\frac{1}{2}(k_1 - j_1 + k_2 - j_2) \notin S$. So, we have

$$k_1 - j_2 = \frac{1}{2}(k_1 - j_1 + k_2 - j_2) + \frac{1}{2}(j_1 + k_1) - \frac{1}{2}(j_2 + k_2) \notin S$$

implying that the cell $(j_2, k_1)$ contains an $\ast$. Similarly, we can show that the cell $(j_1, k_2)$ contains an $\ast$.

The number of $\ast$s in each column is readily seen to be $n - |S|$. For every $i \in \mathbb{Z}_n$, $s \in S$, $i = \frac{1}{2}(j + k)$ and $s = k - j$ has a unique solution for $(j, k)$. So, every element of $\mathbb{Z}_n$ occurs $|S|$ times in the PDA $P$.

A list of maximal subsets satisfying LEM is given in Table VII. The maximal subsets need not be unique. As an example, a PDA constructed using Lemma 17 for $S = \{0, 1, 3, 4\} \in \mathbb{Z}_0$ is given in Table VIII.

| TABLE VII |
| Maximal subsets of $\mathbb{Z}_n$ satisfying LEM for odd integers from $\{3, \ldots, 25\}$. |
|---|---|---|
| $n$ | A maximal subset of $\mathbb{Z}_n$ with LEM | Size |
| 3 | {0, 1} | 2 |
| 5 | {0, 1} | 2 |
| 7 | {0, 1, 3} | 3 |
| 9 | {0, 1, 3, 4} | 4 |
| 11 | {0, 1, 3, 4} | 4 |
| 13 | {0, 1, 3, 4} | 4 |
| 15 | {0, 1, 3, 4} | 4 |
| 17 | {0, 1, 3, 7, 8} | 5 |
| 19 | {0, 1, 3, 12, 14, 15} | 6 |
| 21 | {0, 1, 3, 4, 9, 10} | 6 |
| 23 | {0, 1, 3, 4, 9, 10} | 6 |
| 25 | {0, 1, 3, 4, 9, 10, 12} | 7 |

| TABLE VIII |
| A $(9, 9, 5, 9)$ PDA created using Wanless’ construction (Lemma 17) for $\{0, 1, 3, 4\} \in \mathbb{Z}_0$. |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 6 | * | 7 | 3 | * | * | * | * | * |
| * | 2 | 7 | * | 8 | 4 | * | * | * | * |
| * | * | 3 | 8 | * | 0 | 5 | * | * | * |
| * | * | * | 4 | 0 | * | 1 | 6 | * | * |
| * | * | * | * | 5 | 1 | * | 2 | 7 | * |
| 8 | * | * | * | * | 6 | 2 | * | 3 | | |
| 4 | 0 | * | * | * | 7 | 3 | * | | |
| * | 5 | 1 | * | * | * | 8 | 4 | | |
| 5 | * | 6 | 2 | * | * | * | 0 | | |

The parameters of all the coded caching schemes derived from the PDAs that we have presented above are summarized in the following theorem.

Theorem 18. There exist $(K, N; M, R)$ coded caching schemes for the parameters tabulated below.
| Construction | Common Parameters | PDA | Caching Scheme | Constraints |
|--------------|------------------|-----|---------------|-------------|
| Lemma 7      | $n$ $n$ $2$ $1$ $\frac{n(n-1)}{2}$ $\frac{1}{n}$ $\frac{n-1}{2}$ |     |               | $n$ is even $z \in \{1, \ldots, n - 1\}$ |
| Lemma 10     | $n$ $n$ $2$ $z$ $\frac{n(n-z)}{2}$ $\frac{z}{n}$ $\frac{n-z}{2}$ |     |               | $n$ is odd $i \in \{0, \ldots, \frac{n^2-3}{2}\}$ |
| Lemma 11     | $n$ $n$ $2$ $2i + 1$ $\frac{n(n-1-2i)}{2}$ $\frac{2i+1}{n}$ $\frac{n-1-2i}{2}$ |     |               | $c \in \{3, 4\}, 2c|n$ $z \in \left\{\frac{n}{2} + 1, \frac{n}{2} + 2, \ldots, \frac{(c-1)n}{c}\right\}$ |
| Corollary 15 | $n$ $n$ $c$ $z$ $\frac{n(n-z)}{c}$ $\frac{z}{n}$ $\frac{n-z}{c}$ |     |               | $S \subseteq \mathbb{Z}_n$ satisfies LEM $|S|=q$, $i \in \{1, \ldots, q\}$ |
| Lemma 17     | $n$ $n$ $i$ $n-i$ $\frac{n-i}{n}$ $1$ |     |               | $\frac{M}{N}$ |

A comparison of these coded caching schemes with the Maddah-Ali-Niesen scheme for $K = N = 24$ is shown in Fig. 8. The PDA-based constructions are for $K = 24$ using Corollary 15, which is possible for $c = 3, 4$ since both 6 and 8 divides 24. The PDA-based schemes from Lemma 10 and from Corollary 15 are close to the Maddah-Ali-Niesen (MN) scheme at different parts of the rate versus memory trade-off curve.

![Fig. 8. Comparison of memory-rate tradeoff between the PDA-based schemes and Maddah-Ali-Niesen (MN) scheme. Rate ($R$) and subpacketization ($f$) are plotted against normalized cache memory ($\frac{M}{N}$) for $K = 24$.](image)

The comparison of subpacketization clearly shows the advantage in the PDA-based schemes over the MN scheme. For instance, when $K = 24$, $\frac{M}{N} = 11$ and $N \geq 24$, the $(24, 24, 11, 78)$ PDA in Example IV.4 gives a scheme with $R = 3.25$ and $f = 24$. For these parameters, the matching Maddah-Ali-Niesen scheme has $R = 1.083$ with $f = 2496144$ and the uncoded caching has $R = 13$. Since the subpacketization in Maddah-Ali-Niesen scheme could become very large in some situations, the PDA-based schemes become practical even if they have a lower rate.

### B. Demand private schemes from PDAs

We have seen that there are several construction schemes for PDAs which reduce the subpacketization in coded-caching schemes when compared to the Maddah-Ali-Niesen scheme, although with a penalty on rate. We can construct demand-private schemes from the non-private schemes constructed from PDAs using Theorem 4 and achieve a significantly lower subpacketization.

**Corollary 19** (Private schemes from PDAs). If there exists a $(NK, f, Z, S)$ placement delivery array, we can obtain a private $(K, N; \frac{NZ}{f}, \frac{S}{f})$ coded caching scheme, for any $N$.

**Proof.** Given a $(NK, f, Z, S)$ placement delivery array, there exists a non-private $(NK, N; \frac{NZ}{f}, \frac{S}{f})$ (see [34] for details). From this we can obtain the private $(K, N; \frac{NZ}{f}, \frac{S}{f})$ scheme using Theorem 4. \qed

Since the number of "virtual" users become $NK$ in the demand-private scheme, reducing subpacketization is even more significant in the context of demand privacy.

To illustrate the construction of demand-private schemes, we show a couple of complete examples.
Example IV.5. We now present an example of a private scheme with $N = 2$, $K = 3$, derived from a PDA. Consider the PDA from [34, Eq. (7)] corresponding to 6 users and 4 subfiles.

$$P = \begin{pmatrix}
  * & 1 & * & 2 & * & 0 \\
  0 & * & 3 & 1 & * \\
  * & 3 & 0 & * & 2 & * \\
  2 & * & 1 & * & * & 3
\end{pmatrix} \quad (15)$$

We assume that each file $W_i$ is split into $f = 4$ subfiles which are denoted as $W_{i,j}$, where $0 \leq j < f$. In the non-private scheme, the cache contents of the $i$-th user are given below.

$$Z'_0 = \{W_{i,0}, W_{i,2} : i \in [0, 2]\}$$
$$Z'_1 = \{W_{i,1}, W_{i,3} : i \in [0, 2]\}$$
$$Z'_2 = \{W_{i,0}, W_{i,1} : i \in [0, 2]\}$$
$$Z'_3 = \{W_{i,2}, W_{i,3} : i \in [0, 2]\}$$
$$Z'_4 = \{W_{i,0}, W_{i,1} : i \in [0, 2]\}$$

The transmission for demand vector $D' = (D'_0, \ldots, D'_5)$ is

$$X^{D'} = \left\{ \begin{array}{c}
W_{D'_0,1} \oplus W_{D'_2,2} \oplus W_{D'_0,0} \\
W_{D'_1,0} \oplus W_{D'_2,3} \oplus W_{D'_1,1} \\
W_{D'_2,3} \oplus W_{D'_4,0} \oplus W_{D'_2,2} \\
W_{D'_1,2} \oplus W_{D'_4,1} \oplus W_{D'_2,3}
\end{array} \right\}. \quad (16)$$

For $N = 2$ files, $A$ and $B$, we can create a private $(3, 2; 1, 1)$ scheme as shown in Fig. 9.

Fig. 9. A $(3, 2; 1, 1)$ private scheme for $D = (A, A, B)$ from a $(6, 2; 1, 1)$ non-private scheme from the PDA given in (15).

Example IV.6. Given in Table VIII is a $9 \times 9$ PDA constructed using Lemma 17, from which we can create a demand-private $(3, 3; \frac{3}{5}, 1)$ coded-caching scheme as shown in Figure 10.
V. Generalizations: Beyond Demand Privacy

The caching system we have considered so far has a single tier of users with uniform cache size, uniform file size and the full privacy for all users regarding the user demands. In practical applications the setup and requirements could vary. We consider a few such scenarios and analyse how the existing results could be applied there.

A. Partial privacy

In Section IV, we saw that there exist a trade-off between subpacketization and rate. There exists a similar trade-off between subpacketization and privacy as well. The scheme modified from the non-private scheme can have less subpacketization if full privacy is not needed. For instance, suppose that 2-file privacy suffices. That is, at the end of the multicast transmission, every user has an ambiguity of one of two files about any other user’s demand. This would require lesser subpacketization.

For example, if \( N = 10 \) and \( K = 2 \), then a private scheme created using Corollary 5 would require the number of users in the non-private scheme to be \( K' = NK = 20 \). With \( M = 5 \), such a scheme would require a subpacketization \( f = \binom{K'}{M} = \binom{20}{10} = 184756 \). For 2-file privacy, we need to provide only two options to populate the cache content of a user. Hence, we can use a \((2N, K)\) non-private scheme to arrive at an \((N, K)\) partially private scheme where any user’s demand is possibly one of two files to another user. In this setup, \( K' = 2K = 4 \), and we can use a subpacketization as low as \( f = \binom{4}{2} = 6 \). In Fig. 11, we show a partially private \((2, 4; 2, 2/3)\) scheme from a \((4, 4; 2, 2/3)\) non-private scheme providing an ambiguity of two files. These schemes are important particularly when we have large number of files compared to users.

B. Non-uniform privacy

A general setting of interest is when different users need different levels of privacy. As we have seen earlier, we need to provide \( \alpha \) options to populate the cache content of a user when that user needs \( \alpha \)-file privacy. So the User \( k \) in a \((K, N; M, R)\) coded caching scheme with non-uniform privacy is assured of \( \alpha_k \)-file privacy for \( \alpha_k \in \{1, \ldots, N\} \). We can design such a scheme using a non-private \((\kappa, N; M, R)\) coded caching where

\[
\kappa = \sum_{k=0}^{K-1} \alpha_k. \tag{17}
\]
Theorem 20. If there exists a $(\kappa, N; M, R)$ coded caching scheme such that $\kappa = \sum_{k=0}^{K-1} \alpha_k$, then there exists a $(K, N; M, R)$ caching scheme such that User $k$ has $\alpha_k$-privacy.

Proof. \hfill \Box

Example V.1. A $(3, 2; 1, 2/3)$ caching system with non-uniform privacy obtained from a $(4, 2; 1, 2/3)$ non-private scheme is shown in Fig. 12. User 0 is assured of full privacy since it has two option to populate its cache. But User 1 and User 2 got no privacy.
C. Privacy with security

Vaishakh et al. had proposed a coded caching scheme that ensured that the contents of a file remain secret to the users unless they had requested it from the server [21]. We term this as content security in coded caching. We define a coded caching scheme is private and secure when we have both demand privacy and content security.

The content private scheme proposed in [21] is actually three schemes for three regimes of cache memory \( M \). Two of these makes use of secret sharing. They define an \((m,n)\)-secret sharing scheme wherein for each file \( n \) shares are created, each with \( \frac{F}{n-m} \) bits, such that \( m \) shares do not reveal any information about the file, but \( n \) shares completely reveals the file. The three regimes and their respective schemes are detailed below.

1. \( M = 1 \): The cache memory should be able to store at least one file for ensuring content security in a multicast. When \( M = 1 \), we store cryptographic keys, \( T_k \) of \( F \) bits in each cache. i.e.,
   
   \[
   Z_k = T_k
   \]

   During delivery phase, server sends
   
   \[
   X^D = \{W_{D_k} \oplus T_k, \forall k \in [K]\}
   \]

   where \( D_k \) is the demand of User \( k \). Here \( R = K \).

2. \( M = N(K-1) \): Here a \((K-1,K)\) secret sharing scheme is used. \( K \) shares of each file \( W_i \) are created, each of size \( F \) bits. For \( j \in [K] \), \( j \)-th share of the file \( W_j \) is denoted by \( S^j_i \). For each file \( W_i \), User \( k \) stores all its shares except \( S^k_i \).
   
   \[
   Z_k = \{S^j_i, j \in [K], j \neq k\}
   \]

   During delivery the server transmits
   
   \[
   X^D = \oplus_{k \in [K]} S^k_i
   \]

   resulting in \( R = 1 \).

3. \( 1 < M < N(K-1) \) and \( M = \frac{Nt}{K-t} + 1 \) for \( t \in \{0,1,\ldots,K-2\} \): Here a \( \binom{K-1}{t-1}, \binom{K}{t} \) secret sharing scheme is used to create \( \binom{K}{t} \) shares. Size of each share is given by
   
   \[
   F_s = \frac{F}{\binom{K}{t}-\binom{K-1}{t-1}} = \frac{Ft}{(K-t)\binom{K-1}{t-1}}
   \]

   For each file \( W_i \), we denote its shares by \( D_t \equiv \{S^L_i : L \subset [K], |L|=t\} \). The content of cache of User \( k \) is given by
   
   \[
   Z_k = \{S^L_i : i \in [N], L \subset [K], |L|=t, k \in L\}
   \]

   For each subset \( V \subset [K] \) of users of size \( |V|=t+1 \), an independently and uniformly generated key \( T_V \) of size \( F_s \) bits is cached at each user \( k \in V \). During the delivery phase the server transmits, for each such that
   
   \[
   X^D = \{T_V \oplus_{k \in V} S^V\setminus\{k\} : V \subset [K], |V|=t+1\}
   \]

   Rate \( R = \frac{K}{1+t} \) where \( t = \frac{(M-1)K}{(N+M-1)} \).

For any other value of \( M \), we can employ time-sharing between the above schemes such that the convex hull of these \((M,R)\) pairs are achievable.

**Theorem 21.** The following \((M,R)\) pairs and their convex hull is achievable for a private and secure \((K,N)\) scheme.

- \( M = 1, R = K \)
- \( M = N(K-1), R = 1 \)
- \( M = \frac{Nt}{K-t} + 1, R = \frac{NK}{1+t} \) for \( t \in \{0,1,\ldots,NK-2\} \)

**Proof.** The content private scheme for \( M = 1 \) provides demand privacy as well. Hence the first \((M,R)\) pair is achievable. For the other two regimes, considering the content secure schemes, Scheme 2 and Scheme 3, as the non-private schemes in Theorem 4, we can arrive at the \((M,R)\) pairs listed in Theorem with privacy and security. With time sharing, their convex hull also becomes achievable.

\[ \square \]

A comparison of the rate-memory tradeoff for these schemes is given in Figure 13.
D. Privacy in hierarchical coded caching

Karamchandani et al. devised a scheme to implement coded caching in hierarchical topology [13]. Here the server is connected to $K_1$ mirrors. A mirror is an intermediate node in a hierarchical topology which represents base stations, internet service providers etc. Each mirror connected to $K_2$ users through multicast links. There could be a limitation in cache memory with users as well as the mirrors. Each mirror is having cache memory of $M_1F$ bits while that of each user is $M_2F$ bits.

This setting also suffers from user requests being revealed to other users and mirrors. We know how to achieve demand privacy in a single tier caching system. Using a similar method, we can obtain private schemes for the hierarchical topology from non-private hierarchical coded caching schemes. We denote a hierarchical coded caching scheme with $K_1$ mirrors each with $K_2$ users, $N$ files, mirror cache size $M_1$, user cache size $M_2$, rate between server and mirrors $R_1$ and rate between a mirror and its users $R_2$ as a $(K_1, K_2, N; M_1, M_2, R_1, R_2)$ coded caching scheme. If a hierarchical scheme has demand privacy, we call it a private hierarchical coded caching scheme.

We will use the caching and delivery functions from a $(K_1, NK_2, N; M_1, M_2, R_1, R_2)$ non-private hierarchical coded caching scheme due to [13] to construct a $(K_1, K_2, N; M_1, M_2, R_1, R_2)$ private hierarchical scheme. The construction of $(K_1, K_2, N; M_1, M_2, R_1, R_2)$ private hierarchical scheme is as follows. The scheme is a combination of two achievable schemes, denoted as scheme A and scheme B. Scheme A, comprises of two single layer coded caching schemes, one between the server and the mirrors and the second between each mirror and its users. Scheme B is a single layer coded caching scheme between the server and all the users, where the storage at mirrors is ignored and mirrors only forward the part of transmission useful for its users. Two parameters $\alpha$ and $\beta$ dictate what fraction of each file and user memory respectively will be allocated to these two schemes. For an achievable rate $r(K, M, N)$ for single layer coded caching with $K$ users, $N$ files and cache memory $M$, the expressions in Eq. (22) gives the achievable rate pair.

Note that each expression has two components, one from scheme A and the other from scheme B. For the three different regimes of the memory sizes $M_1$ and $M_2$,

I) $M_1 + M_2NK_2 \geq N$ and $0 < M_1 \leq N/4$

II) $M_1 + M_2NK_2 < N$

III) $M_1 + M_2NK_2 \geq N$ and $N/4 < M_1 \leq N$

we will set the values of $\alpha$ and $\beta$ as given below

$$\langle \alpha^*, \beta^* \rangle = \begin{cases} 
\left( \frac{M_1}{N}, \frac{M_2}{N} \right) & \text{in regime I} \\
\left( \frac{M_1}{M_1+M_2NK_2}, 0 \right) & \text{in regime II} \\
\left( \frac{M_1}{N}, \frac{1}{4} \right) & \text{in regime III}
\end{cases}$$

Then we have the following result on the achievable rate of private hierarchical coded caching schemes.

Fig. 13. Comparing rate-memory tradeoff for different levels of privacy when $N = K = 12$
Theorem 22. The following rates $R_1$ and $R_2$ are achievable for a private $(K_1, K_2, N; M_1, M_2, R_1, R_2)$ hierarchical scheme.

$$
R_1(\alpha, \beta) = \alpha N K_2 \cdot r \left( K_1, \frac{M_1}{\alpha}, N \right) + (1 - \alpha) \cdot r \left( N K_1 K_2, \frac{(1 - \beta) M_2}{1 - \alpha}, N \right)
$$

$$
R_2(\alpha, \beta) = \beta M_2 \cdot r \left( N K_2, \frac{M_2}{\alpha}, N \right) + (1 - \alpha) \cdot r \left( N K_2, \frac{(1 - \beta) M_2}{1 - \alpha}, N \right)
$$

(19)

where $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $r(K, M, N)$ is the optimal rate for single layer caching scheme with $K$ users, $N$ files and each user with a cache memory $M$.

Proof. The proof uses the same idea for the single tier case. Since for the $(K_1, NK_2, N; M_1, M_2, R_1, R_2)$ non-private hierarchical scheme, rates $R_1 = R_1(\alpha, \beta)$ and $R_2 = R_2(\alpha, \beta)$ are achievable, it is also achievable for the $(K_1, K_2, N; M_1, M_2, R_1, R_2)$ private hierarchical scheme made based on it. The demand privacy is derived from the fact that for each option of a user’s cache, each of the $N$ files could be recovered from the transmission. Since the cache assignment is private, demand also remains private.

We now show an example of a $(2, 2, 4; 1, 0.665, 1.991)$ private hierarchical coded caching scheme as shown in Fig. 14. These parameters corresponds to regime III. Using (18), we set $\alpha = 1/2$ and $\beta = 1/4$. So half of the files will be transmitted through scheme A and the other half through scheme B. In scheme A, since the mirrors could store all the files, $R_{1A} = 0$. For the transmission between mirror and users, the available memory is $M_{2A} = \frac{1}{4}$ resulting in a transmission of $R_{2A}F = 1.375F$ bits. The available memory for scheme B is $M_{2B} = \frac{3}{4}$ resulting in transmission rates $R_{1B} = 0.6648$ and $R_{2B} = 0.616$.

VI. CONCLUSION

We have investigated the problem of demand privacy in systems employing coded caching techniques with a focus on minimizing subpacketization. For the 2-user, 2-file case, we provided a new construction with a subpacketization of 3. Additionally, we proved that the subpacketization of 3 is indeed minimal for a linear code for the 2-user, 2-file case. We were able to demonstrate that coded prefetching has an edge over uncoded prefetching in this scenario. We had proposed several construction schemes for PDAs that provides linear subpacketization for the caching schemes built from them. Also,
we proposed partially private caching schemes and showed how to construct such private schemes with less subpacketization in the general $K$-user, $N$-file case.

A lower bound on the rate and optimal scheme remains an open problem in demand private coded caching. The relationship between subpacketization with other parameters also need to be well-characterised.

**APPENDIX**

Here we show that without coded prefetching we cannot obtain a private $(2, 2; 1, 2/3)$ scheme with three subfiles and thereby prove Lemma 3.

Informally, the proof is organized as follows. First, we show that without coded prefetching the subfiles must be cached in an uncoded form i.e. without linear combinations. This restricts the possibilities for the caches. Furthermore any given cache restricts the possibilities for the other user’s cache. Demand privacy is possible only if the set of caches consistent with a user allow the reconstruction of both the files for any demand. We show that is not possible and hence a linear private $(2, 2; 1, 2/3)$ scheme with subpacketization of three subfiles does not exist.

### A. Permissible caches without coded prefetching

Without coded prefetching, the subfiles can only be replicated in the cache. With three subfiles, $M = 1$ implies that each user can store 3 subfiles. $R = 2/3$ implies that there are two independent subfile combinations in the transmission. If all the subfiles in a cache belongs to a file, that user cannot recover the other file from a transmission of rate $R = 2/3$. So a cache should contain two subfiles of one file and one subfile of the other file. Let the two files be $A$ and $B$. Without loss of generality, let us assume the cache of first user, $Z_0$ contains two subfiles of file $A$ and one subfile of $B$.

$$Z_0 = \{A_0, A_1, B_2\}. \tag{20}$$

Let the cache of User 1 be

$$Z_1 = \{G_0, G_1, G_2\}, \tag{21}$$

where $G_i \in \{A_0, A_1, A_2, B_0, B_1, B_2\}$.

**Lemma 23.** If $Z_0 = \{A_0, A_1, B_2\}$, then the permissible cache for $Z_1$ must be one of the following.

$$Z_1 = \{G_0, G_1, A_2 \mid G_0, G_1 \in \{B_0, B_1, B_2\}\} \tag{22a}$$

or

$$Z_1 = \left\{\begin{array}{c} G_0, G_1, A_2 \\ G_0 \in \{A_0, A_1\} \\ G_1 \in \{B_0, B_1, B_2\} \end{array}\right\}. \tag{22b}$$

**Proof.** Consider the transmission

$$X^{BA} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2 + \beta_0 B_0 + \beta_1 B_1 + \beta_2 B_2 \\ \gamma_0 A_0 + \gamma_1 A_1 + \gamma_2 A_2 + \delta_0 B_0 + \delta_1 B_1 + \delta_2 B_2 \end{bmatrix} \tag{23}$$

User 0 can use its cache contents to eliminate three variables from the system of linear equations in Eq. (23). The reduced/equivalent equations for User 0 is

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \alpha_2 A_2 + \beta_0 B_0 + \beta_1 B_1 \\ \gamma_2 A_2 + \delta_0 B_0 + \delta_1 B_1 \end{bmatrix}$$

Since User 0 does not have access to $A_2$, for recovering $B_0$ and $B_1$ we need

$$\text{rk} \left(\begin{bmatrix} \beta_0 \\ \beta_1 \\ \delta_0 \\ \delta_1 \end{bmatrix}\right) = 2 \text{ and } \begin{bmatrix} \alpha_2 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{24}$$

The transmission $X^{BA}$ cannot involve $A_2$. So $A_2$ must be in $Z_1$ for it to recover file $A$ from $X^{BA}$.

$$G_2 = A_2$$

All the subfiles in $Z_1$ cannot be that of file $A$. So, we have two cases for the possible values of $\{G_0, G_1\}$ based on the associated files. Either both of them are subfiles of $B$ as in Eq. (22a) or one of them is subfile of $A$ and the other is of $B$ as in Eq.(22b).

$\square$
B. Two subfiles of file B in $Z_1$

In this section, we will show that if the cache of User 1 is of the form given in Eq. (22a), then the scheme is not private.

**Lemma 24.** If $Z_0 = \{A_0, A_1, B_2\}$, and $Z_1 = \{G_0, G_1, A_2\}$ and $G_i \in \{B_0, B_1, B_2\}$, then demand privacy is not satisfied.

**Proof.** Let $G_0, G_1 \in \{B_0, B_1, B_2\}$. The reduced equations corresponding to $X^{BA}$ for User 1 will be

$$
\begin{bmatrix}
  u'' \\
  v''
\end{bmatrix} =
\begin{bmatrix}
  \alpha_0 A_0 + \alpha_1 A_1 + \beta_0 B_0 + \beta_1 B_1 + \beta_2 B_2 \\
  \gamma_0 A_0 + \gamma_1 A_1 + \delta_0 B_0 + \delta_1 B_1 + \delta_2 B_2
\end{bmatrix}
$$

(25)

For User 1 being able to obtain subfiles $A_0$ and $A_1$ from the transmission, we need

$$
\text{rk} \left( \begin{bmatrix} \alpha_0 & \alpha_1 \\ \gamma_0 & \gamma_1 \end{bmatrix} \right) = 2, \text{ and }
\begin{bmatrix} \beta_2 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

(26a)

(26b)

Due to Eq. (24), $Z_2$ must contain the subfiles $B_0$ and $B_1$, for eliminating those variables from $X^{BA}$ and recover the subfiles of $A$.

$$
\{G_0, G_1\} = \{B_0, B_1\}
$$

(27)

Now consider the transmission for $D = (A, A)$.

$$
X^{AA} = \begin{bmatrix} u \\ v \end{bmatrix} =
\begin{bmatrix}
  \alpha_0' A_0 + \alpha_1' A_1 + \beta_0' B_0 + \beta_1' B_1 + \beta_2' B_2 \\
  \gamma_0' A_0 + \gamma_1' A_1 + \delta_0' B_0 + \delta_1' B_1 + \delta_2' B_2
\end{bmatrix}
$$

(28)

For User 1, reduced equations are

$$
\begin{bmatrix}
  u'' \\
  v''
\end{bmatrix} =
\begin{bmatrix}
  \alpha_0 A_0 + \alpha_1 A_1 + \beta_2 B_2 \\
  \gamma_0 A_0 + \gamma_1 A_1 + \delta_2 B_2
\end{bmatrix}
$$

(29a)

(29b)

For demand privacy we require the existence of some cache $Z'_1$ which can recover file $B$ from $X^{AA}$. Since $X^{AA}$ doesn’t involve $B_2$, it must be present in $Z'_1$.

$$
Z'_1 = \{H_0, B_2, A_2\}
$$

For no value of $H_0 \in \{B_0, B_1, A_0, A_1\}$, it can recover both $B_0$ and $B_1$ (or file $B$ completely) from $X^{AA}$ due to Eq. (29a). Thus, if $Z_0$ has two subfiles of $A$, $Z_1$ cannot contain two subfiles of $B$ as given in Eq. (22a).

C. Two subfiles of file A in $Z_1$

If the cache of User 1 is of the form given in Eq. (22b), then we can restrict the cache even further as the following lemma shows.

**Lemma 25.** If $Z_0 = \{A_0, A_1, B_2\}$, then the permissible cache for $Z_1$ must be of the form $Z_1 = \{G_0, G_1, A_2\}$, where $G_0, G_1$ are distinct and $G_0 \in \{A_0, A_1\}$ and $G_1 \in \{B_0, B_1\}$.

**Proof.** We need to show $G_1 \neq B_2$. Since $Z_1$ already contains $A_2$, it can have either $A_0$ or $A_1$, both of which are in $Z_0$. Without loss of generality, let $G_0 = A_1$. Assume $G_1 = B_2$. Then $Z_1 = \{B_2, A_1, A_2\}$ Consider the transmission

$$
X^{AB} = \begin{bmatrix} u \\ v \end{bmatrix} =
\begin{bmatrix}
  \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2 + \beta_0 B_0 + \beta_1 B_1 + \beta_2 B_2 \\
  \gamma_0 A_0 + \gamma_1 A_1 + \gamma_2 A_2 + \delta_0 B_0 + \delta_1 B_1 + \delta_2 B_2
\end{bmatrix}
$$

(30)

For User 0, these equations reduces to

$$
\begin{bmatrix}
  u' \\
  v'
\end{bmatrix} =
\begin{bmatrix}
  \alpha_2 A_2 + \beta_0 B_0 + \beta_1 B_1 \\
  \gamma_2 A_2 + \delta_0 B_0 + \delta_1 B_1
\end{bmatrix}
$$

(31)
Since User 0 have no access to $B_0$ and $B_1$, for recovering $A_2$, we need
\[
\text{rk} \left( \begin{bmatrix} \beta_0 & \beta_1 \\ \delta_0 & \delta_1 \end{bmatrix} \right) \leq 1 \quad \text{and} \quad \begin{bmatrix} \alpha_2 \\ \gamma_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  \hspace{1cm} (32a)
\[
\begin{bmatrix} \alpha_0 \\ \gamma_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  \hspace{1cm} (32b)

For User 1 the equations from $X^{AB}$ reduces to
\[
\begin{bmatrix} u'' \\ v'' \end{bmatrix} = \begin{bmatrix} \alpha_0 A_0 + \beta_0 B_0 + \beta_1 B_1 \\ \gamma_0 A_0 + \delta_0 B_0 + \delta_1 B_1 \end{bmatrix}
\]  \hspace{1cm} (33)

For User 1 recovering $B_0$ and $B_1$, it requires
\[
\text{rk} \left( \begin{bmatrix} \beta_0 & \beta_1 \\ \delta_0 & \delta_1 \end{bmatrix} \right) = 2 \quad \text{and} \quad \begin{bmatrix} \alpha_0 \\ \gamma_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  \hspace{1cm} (34a)
\[
\begin{bmatrix} \alpha_2 \\ \gamma_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  \hspace{1cm} (34b)

Equations (34a) and (32a) are contradictory, Hence $G_1 \neq B_2$. So $G_1 \in \{B_0, B_1\}$.

By Lemma 25, there are four possible choices for $Z_1$ as given below.
\[
Z_a = \{A_1, A_2, B_0\} \quad \hspace{1cm} (35a)
\]
\[
Z_b = \{A_1, A_2, B_1\} \quad \hspace{1cm} (35b)
\]
\[
Z_c = \{A_0, A_2, B_0\} \quad \hspace{1cm} (35c)
\]
\[
Z_d = \{A_0, A_2, B_1\} \quad \hspace{1cm} (35d)
\]

**Lemma 26.** If $Z_0 = \{A_0, A_1, B_2\}$ and $Z_1 \in \{Z_a, Z_b, Z_c, Z_d\}$, then demand privacy is not possible.

**Proof.** It suffices to show demand privacy is not possible for $Z_1 = Z_a$, since we can arrive at the other cache combinations by relabeling.

Suppose the cache $Z_a = \{A_1, A_2, B_0\}$ is assigned to the User 1. Then, by arguments similar to Lemmas 23, 24 and 25, the User 1 is aware that the cache of User 0 must have two subfiles of $A$, with one being $A_0$ and the subfile of $B$ in $Z_0$ is not $B_0$. The four possible caches for $Z_0$ consistent with $Z_1 = Z_a$ are given below.
\[
Z_e = \{A_0, A_1, B_1\} \quad \hspace{1cm} (36a)
\]
\[
Z_f = \{A_0, A_1, B_2\} \quad \hspace{1cm} (36b)
\]
\[
Z_g = \{A_0, A_2, B_1\} \quad \hspace{1cm} (36c)
\]
\[
Z_h = \{A_0, A_2, B_2\} \quad \hspace{1cm} (36d)
\]

Note that $Z_0 = Z_f$. For demand privacy we need the caches consistent with $Z_0$ to be able to recover both files and vice versa.

Consider the transmission
\[
X^{AB} = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \alpha_0 A_0 + \alpha_1 A_1 + \alpha_2 A_2 + \beta_0 B_0 + \beta_1 B_1 + \beta_2 B_2 \\ \gamma_0 A_0 + \gamma_1 A_1 + \gamma_2 A_2 + \delta_0 B_0 + \delta_1 B_1 + \delta_2 B_2 \end{bmatrix}
\]  \hspace{1cm} (37)

For the User 0 with cache $Z_0 = \{A_0, A_1, B_2\}$ the transmission $X^{AB}$ reduces to the following set of equations after eliminating the subfiles which are already present in $Z_0$.
\[
\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} \alpha_2 A_2 + \beta_0 B_0 + \beta_1 B_1 \\ \gamma_2 A_2 + \delta_0 B_0 + \delta_1 B_1 \end{bmatrix}
\]  \hspace{1cm} (38)

For User 0 whose demand is $A$ already has $A_0$ and $A_1$. Only $A_2$ needs to be recovered from $X^{AB}$. This is possible only if the following conditions are satisfied.
\[
\text{rk} \left( \begin{bmatrix} \beta_0 & \beta_1 \\ \delta_0 & \delta_1 \end{bmatrix} \right) \leq 1 \quad \text{and} \quad \begin{bmatrix} \alpha_2 \\ \gamma_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  \hspace{1cm} (39a)
\[
\begin{bmatrix} \alpha_0 \\ \gamma_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]  \hspace{1cm} (39b)
Similarly, for the User 1, whose cache is $Z_1 = Z_a$, the transmission $X^{AB}$ reduces to
\[
\begin{bmatrix}
  u_1 \\
v_1
\end{bmatrix} = \begin{bmatrix}
  \alpha_0 A_0 + \beta_1 B_1 + \beta_2 B_2 \\
  \gamma_0 A_0 + \delta_1 B_1 + \delta_2 B_2
\end{bmatrix}
\] (40)

For User 1 to recover $B_0$ and $B_1$, the following conditions must be satisfied.
\[
\text{rk} \left( \begin{bmatrix}
  \beta_1 \\
  \delta_1 \\
\end{bmatrix} \begin{bmatrix}
  \beta_2 \\
  \delta_2 \end{bmatrix} \right) = 2 \quad \text{and} \quad \begin{bmatrix}
  \alpha_0 \\
  \gamma_0
\end{bmatrix} = \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\] (41a)
\[
\begin{bmatrix}
  \alpha_0 A_0 + \beta_0 B_0 + \beta_2 B_2 \\
  \gamma_0 A_0 + \delta_0 B_0 + \delta_2 B_2
\end{bmatrix}
\] (41b)

The reduced equations for $Z_b$ are
\[
\begin{bmatrix}
  u_b \\
v_b
\end{bmatrix} = \begin{bmatrix}
  \alpha_0 A_0 + \beta_0 B_0 + \beta_2 B_2 \\
  \gamma_0 A_0 + \delta_0 B_0 + \delta_2 B_2
\end{bmatrix}
\] (42)

The reduced equations for $Z_c$ are
\[
\begin{bmatrix}
  u_c \\
v_c
\end{bmatrix} = \begin{bmatrix}
  \alpha_1 A_1 + \beta_1 B_1 + \beta_2 B_2 \\
  \gamma_1 A_1 + \delta_1 B_1 + \delta_2 B_2
\end{bmatrix}
\] (43)

The reduced equations for $Z_d = Z_g$ are
\[
\begin{bmatrix}
  u_d \\
v_d
\end{bmatrix} = \begin{bmatrix}
  \alpha_1 A_1 + \beta_0 B_0 + \beta_2 B_2 \\
  \gamma_1 A_1 + \delta_0 B_0 + \delta_2 B_2
\end{bmatrix}
\] (44)

The reduced equations for $Z_e$ are
\[
\begin{bmatrix}
  u_e \\
v_e
\end{bmatrix} = \begin{bmatrix}
  \alpha_2 A_2 + \beta_0 B_0 + \beta_2 B_2 \\
  \gamma_2 A_2 + \delta_0 B_0 + \delta_2 B_2
\end{bmatrix}
\] (45)

The reduced equations for $Z_h$ are
\[
\begin{bmatrix}
  u_h \\
v_h
\end{bmatrix} = \begin{bmatrix}
  \alpha_1 A_1 + \beta_0 B_0 + \beta_1 B_1 \\
  \gamma_1 A_1 + \delta_0 B_0 + \delta_1 B_1
\end{bmatrix}
\] (46)

From the above constraints and the reduced equations for all users we can infer the following.

1) Due to Eq. (41b), $Z_0$ cannot recover file $A$ since it has no access to $A_0$.

2) Due to Eq. (41a), and since $Z_c$ has no access to $B_1$ and $B_2$, it cannot eliminate them from the transmission to recover file $A$.

3) Due to Eq. (39b), $Z_c$ cannot recover file $B$.

4) Due to Eq. (39a), $Z_h$ cannot recover file $B$.

Hence, from the four possible caches for $Z_1$, the only cache that might be able to recover file $A$ and might achieve privacy for User 1 is $Z_d$. But since there are only five equations from the cache and transmissions, it is impossible for $Z_d$ to recover all the six subfiles $A_0, A_1, A_2, B_0, B_1, B_2$ and thus recover file $B$ also. That means, no possible cache for User 0 consistent with $Z_1$ is able to recover file $B$ from $X^{AB}$. This results in no demand privacy for User 0.

On the other hand, if $X^{AB}$ is such that $Z_d$ can recover file $B$, then it results in no privacy for User 1.

Note that any consistent set of caches for User 0 and User 1 can be obtained by permuting subfile labels of file $A$ and file $B$ (permutation $\pi_A$ to relabel $A$ and $\pi_B$ to relabel $B$). Applying the same relabeling, the above proof will hold true for them as well. This concludes the proof of Lemma 3.

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