Black Remnants from T-Duality

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In this paper, we will analyze the effects of T-duality on thermodynamics of black holes. It will be demonstrated that such a string theoretical corrections can produce non-trivial modification for the temperature of black holes. This modification leads to the production of black remnants, and such black remnants can have important consequences for the black hole information paradox. This is because a small number of quanta can contain a large amount of information. We will also comment on various different physical consequences of these black hole remnants formed due to string theoretical corrections.

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I. INTRODUCTION

It is known that black holes have an entropy, and they evaporate by radiation Hawking radiation [1]. As the Hawking radiation is thermal, so it is expected that it may not contain any information. Now as the black hole evaporates, there is a possibility that the information would be lost inside a black hole. This consideration has led to the information loss paradox [2]. It is hoped that quantum gravitational modification to the physical of black holes could help resolve such a paradox. As string theory is one of the most well known approaches to quantum gravity, it is hoped that the physics of black holes in string theory can help resolve this paradox. It has been demonstrated that the entropy of black holes in string theory, agrees with the microscopic counting of the states of the brane system [3–6]. These results can also holds beyond the supergravity approximation.

It has also been demonstrated that the folded strings, which are spontaneously created behind the horizon of the $SL(2, R)/U(1)$ black hole, can violate the averaged null energy condition [7, 8]. The back-reaction from these folded string can prevent information from falling into the black holes. Furthermore, the black hole entropy can be related to the number of these folded strings. The black hole information paradox in string theory has also been studied using the diffeomorphism invariance for the Hilbert space [9]. However, usually the black hole information paradox is address in string theory is the framework of AdS/CFT correspondence [10–13]. This has also led to the development of the fuzzball proposal, in which the black hole is replaced by a fuzzball [14, 15]. As fuzzballs do not have a horizon, the black hole information paradox is resolved by replacing the horizon of a black hole by an object without horizon. However, it has been demonstrated that even though it is possible to reflect photons from the surface of a fuzzball, the of such probability of such a reflection is very small [16]. So, the fuzzballs and the classical black holes resemble each other, and fuzzballs can only be differentiated from classical black holes using some quantum effects.

It is also important to study the physics of black holes as it can have important cosmological consequences. This is because the physics of black holes in string theory has been used to understand production of primordial black hole [17, 18]. These primordial black hole can be produced from an amplification of the power spectrum of curvature perturbations. This can in turn be achieved by breaking the slow-roll conditions during inflation. As in the string axion models of inflation, non-perturbative effects can violate the slow-roll conditions, it is possible that such string theoretical effects can produce primordial black hole. It is also possible to produce such an amplification from the warp factor of D-brane with a sufficiently large step during the inflationary [18, 19]. The production of primordial black holes has also been studied in brane world model [20, 21]. In these brane world models, the physics of primordial black holes is corrected because of the modification in the Hawking temperature. Furthermore, in brane world models,
there is an increase the number of such primordial black holes in the radiation-dominated epoch. Thus, black holes in string theory can have important consequences for important physical phenomena like the production of primordial black hole. So, it is important to study the corrections to the physics of black hole from string theory.

The entropy of a black hole solutions in heterotic string theory compactified on a torus has also been studied, and it has been demonstrated that this solution is characterized by a charge vector \(24\). Furthermore, axion-dilaton black holes and Kaluza-Klein black holes are obtained for special values of the charge vector. It may be noted that the entropy of a Kaluza-Klein black hole has also been obtained using the Cardy formula \(24\). It is expected that the T-duality can lead to ultraviolet finiteness of string theory \(24, 25\), and this can in turn correct black holes solutions \(26, 28\).

Furthermore, as the double field theory is constructed using T-duality \(26, 30\), the black hole thermodynamics in also invariant under the duality of double field theory \(31\). As such non-perturbative corrections modify the short distance physics of black holes, they are expected modify the thermodynamics at the last stages of black holes significantly.

In this paper, we will analyze different aspects of thermodynamics of such a black hole. We will demonstrated that this modified thermodynamics of black holes produces black remnants. These black remnants can have important physical consequences. So, we will analyze those physical consequences in this paper.

II. BLACK HOLES MODIFIED FROM T-DUALITY

It is possible to obtain corrected black hole solution using T-duality \(26, 28, 31\). This is because the ultraviolet finiteness of string theory can modify the short distance physics of black holes. It may be noted that the ultraviolet finiteness of string occurs as the geometry in string theory cannot be probed below a certain limit \(22, 32\). This is because in perturbative string theory, string length scale is the smallest probe available \(34, 35\). Even though non-perturbative objects like D0-branes also occur in string theory, it has been argued that space-time cannot be probed below \(\alpha’ g_s^{1/3}\), even with such non-perturbative objects (where \(g_s\) is the string coupling constant) \(24, 25\). This behavior of string theory occurs due to T-duality. As due to T-duality, it is possible to show that the description of string theory below the length \(l_s = \alpha’\) is the same as its description above \(l_s = \alpha’\).

It may be noted that the bosonic string exists in 26 dimensions, so the physics of four dimensions can be obtained by compactifying 22 dimensions. However, the important consequences of such compactification can also be understood by analyzing compactification with one compact dimension. Now for a single compact dimension with radius \(R\), the boundary conditions for a string can be written as \(24\)

\[ X^y(\tau, \sigma + 2\pi) = X^y(\tau, \sigma) + 2\pi wR, \]  

where \(w\) is the winding number. The winding states for a string are topologically stable, and will exist if the internal manifold a non-contractible loop. Now the mass spectrum for such a system, can be written as \(24\)

\[ m^2 = \frac{1}{2\alpha’} \left(n^2 \frac{\alpha’}{R^2} + w^2 \frac{R^2}{\alpha’}\right) + \ldots, \]  

where \(n\) is the Kaluza-Klein excitation level. Now the important property of this spectrum is that if we exchange winding number \(w\) and Kaluza-Klein excitation level \(n\), this expression does not change. So, this expression is invariant under T-duality,

\[ w \rightarrow n, \quad R \rightarrow \frac{\alpha’}{R}. \]  

This is the main reason for the description of string theory below the length \(l_s = \alpha’\) being equivalent to its description above \(l_s = \alpha’\). As the double field theory is constructed using T-duality \(26, 30\), such a short distance behavior for double field theory has also been studied \(36\).

The T-duality for the effective path integral of strings propagating in compactified extra-dimensions has also been studied \(37, 38\). This has been done by using the string center of mass for analyzing the Green’s function for this system. So, for such a system, using the string center of mass five-momentum \(P_M = (p_\mu, p_4)\), propagation kernel can be written as

\[ K[x - y; T] = \sum_{n = -\infty}^{\infty} \int_{x(0) = x}^{x(T) = y} \int_{x(0) = 0}^{x(0) = n l_0} [Dz] [Dp] \left[Dx^4\right] [Dp_4] \times \]

\[ \exp \left[i \int_0^T d\tau \left(p_\mu \dot{x}^\mu + p_4 \dot{x}^4 - \frac{i}{2\mu_0} \left(p_\mu p_\mu + p_4 p_4\right)\right)\right] \]  

(4)
where $\mu_0$ is a dimensional parameter. The contribution from the compact dimension are integrated to obtain the effective four dimensional propagator. Here for a closed string (winding around the compact dimension) all the oscillator modes are neglected, and dynamics is expressed using its center of mass. Now as two points are connected in the four dimensional space-time, the string winds around the compactified dimension. So, the Green’s function is obtained by taking a double sum over $n$ and $w$ \[37, 38\]. It has been demonstrated that this Green’s function is invariant under T-duality, $w \rightarrow n, R \rightarrow \alpha^2/R$, with suitable identification of its parameters. In momentum space, this Green’s function can be expressed as \[37, 38\],

\[
G(k) = -\frac{2\pi R}{\sqrt{k^2}}K_1(2\pi R\sqrt{k^2}),
\]

where $K_1(k)$ is a modified Bessel function of second kind. In this system a minimum length ($l = 2\pi R$) is produced due to the compactified extra-dimension of radius $R$, and the system cannot be probed below this length. In the limit, $2\pi Rk^2 = lk^2 \rightarrow 0$, the usual Green’s function is obtained, $G(k) = -k^{-2}$. This is the limit, in which the string is approximated as a particle, and the stringy effects are totally neglected. This modified Green’s function occurs due to smearing of the matter density from a point-like source to a distribution, due to the extended nature of strings. This smearing of the matter density in turn modifies the geometry of a black hole. So, a corrected black hole solution is obtained by using the modified matter density, which produces this modified Green’s function. So, the metric for a static and spherically symmetric black hole gets modified by these string theoretical corrections as \[26\],

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2,
\]

where $d\Omega^2 = \sin^2\theta d\phi^2 + d\theta^2$, and

\[
f(r) = 1 - \frac{2Mr^2}{(r^2 + 4\pi^2 R^2)^{3/2}},
\]

where $M$ denotes the Komar mass. This is a non-pertubative modification of a static and spherically symmetric black hole, and occurs due to the non-perturbative purely stringy effects. So, it is possible to obtain stringy corrections to black holes from T-duality \[26–28, 31\]. As such non-perturbative corrections modify the short distance physics of black holes, they are expected to significantly modify the thermodynamics at the last stages of black holes. This can have important physical consequences.

### III. BLACK REMNANTS

We will analyze the formation of black remnants from the stringy corrected black hole solutions obtained using T-duality \[26–28, 31\]. Now, for such a corrected solution, and for a large mass ($M > 3\sqrt{\pi^2 R^2}$), there exists two horizons \[20\],

\[
r_+ \sim 2M - \frac{12\pi^2 R^2}{4M}, \quad r_- \sim \sqrt{2\pi R} \left(\frac{2\pi R}{M}\right)^{1/2},
\]

where higher order $\frac{R}{M}$ are neglected. It is clear that the outer horizon produces the Schwarzschild event horizon for the large mass. However, exact solution of $f(r) = 0$ obtained for $M < 3\sqrt{\pi^2 R^2}$, produces the following event horizon radius,

\[
r_+ = \sqrt{48M^4 + 16\alpha M^3 + (12M^2 - 36\pi^2 R^2)\alpha^2 - 432M^2\pi^2 R^2 + \alpha^4 + 24\pi RM^2\sqrt{324\pi^2 R^2 - 48M^2}} \frac{3\alpha}{3\alpha},
\]

where $\alpha$ is given by

\[
\alpha = (-108\pi M^2 R^2 + 8M^4 + 6M\pi R\sqrt{324\pi^2 R^2 - 48M^2})^{1/3}.
\]

It is clear that $R \rightarrow 0$, we obtain $r_+ = 2M$. However, as this is an imaginary solution, there is a lower bound for the black hole mass $M_e$, and only for $M > M_e = \frac{3\sqrt{\pi^2 R}}{2}$ the black hole horizon exists. Such minimum and maximum
mass of black hole already pointed out by the Ref. [39]

The Hawking temperature of the stringy corrected black hole at the outer horizon is given by

$$T = \frac{1}{4\pi r_+} \left(1 - \frac{12\pi^2 R^2}{r_+^2 + 4\pi^2 R^2}\right),$$

(11)

which is represented by the Fig. 2(a) in terms of the black hole mass. We should note that regions of interest for stringy T-dual black holes are for small mass or volume i.e within the interval [0, 1], however we consider wide variety of range in figures. It seems that $T \propto \sin \kappa$, when $M$ is small, and $T = c_1 \kappa$ when $M$ is large ($\kappa$ is surface gravity). In fact, from Fig. 2(a) suggests that temperature of black hole oscillates for mass $0 < M < 1$, i.e. increasing, decreasing, and even vanishing at certain point. Even though it becomes non-zero beyond this point, physically the black hole cannot shrink beyond this point. This is because when the temperature of the black hole becomes zero, it will stop radiating Hawking radiation, and form a black remnants. It is possible to calculate the mass of this black remnant.

![Fig. 1: The mass loss versus $M$ for $a = 1$.](image)

From the Fig. 2(a) we can see that there is a maximum mass ($M_{max}$), which is root of the following equation,

$$-32768M^{10} + 94208M^8(2\pi R)^2 - 26624M^6(2\pi R)^4 - 9984M^4(2\pi R)^6 + 4968M^2(2\pi R)^8 - 243(2\pi R)^{10} = 0.$$  

(12)

For the $2\pi R = 1$, we obtain $M_{max} \approx 1.5$.

Because of black hole evaporation, the black hole mass may decreased to $M_{min}$ where the black hole temperature is zero, hence the approximation value of this mass illustrated by violet circle in Fig. 2(a), but we can obtain exact value by using equations (4) and (11). It is exactly the point where entropy become positive (Fig. 2(b)). From the equation (11) we can obtain $T = 0$ if $r_+ = 2\sqrt{2\pi R}$. In that case,

$$M_{min} = M(T = 0) = \frac{3\sqrt{3}}{2\pi R}.$$  

(13)

This is the minimum mass of the black remnant from due to string theory corrections. The existence of such black remnants could solve the information paradox, and the information can be stored in such a black remnant, and this black remnant would not evaporate [40, 41]. It has been demonstrated that due to quantum effects information need not be additive, and so a small number of quanta can contain a large amount of information [42]. Thus, the last stages of the black holes, when they form a black remnants can contain the entire information of a black hole. There is a cosmological limit on the size and total information of a large black hole. So, taking the Lloyd’s estimate for the total information of the universe is around $10^{120}$ bits [43], and this is the maximum information that can be contained in a black hole. It has been demonstrated that all this information can be stored in a black remnant which is only 40 Planck masses. Thus, the formation of such a black remnant has to potential to resolve
the information loss paradox. Here we have demonstrated that such a resolution to the information loss paradox can occur due to non-perturbative modifications of black hole solutions motivated from T-duality. It is also interesting to note that such black remnants can also form primordial black hole, as these primordial black holes would evaporate to such black remnants. Such primordial black hole remnants can form a candidate for cold dark matter as they can form weakly interacting massive particles \[44, 45\]. Thus, the modification of black holes from T-duality can also be used to motivate the production of primordial black hole remnants, which can in turn be used as a candidate for cold dark matter.

It had been suggested that such black remnants should occur due to stringy theoretical effects \[46\]. This is because four dimensional magnetic black hole solutions of heterotic string theory can be constructed using \(SU(2)/Z(2Q + 2)\) WZW orbifold. It was observed that certain marginal operators can deform this theory to an asymptotically flat black hole. It was also argued using the renormalization group flow that the gravitational singularity can be resolved in this system, and black remnants can form in it. Thus, black hole remnants can form string theoretical effects. It has also been suggested that such black remnants can have some important implication for the detection of black holes at the LHC \[47, 48\], and this is because they will modify the minimum size of such black holes which could be produced at the LHC (which is the size of remnants) \[49, 50\]. In fact, this can be used as a reason for the non-existence of such black holes at LHC \[47, 48\]. Now, we have demonstrated here that such black remnants also form due to a modification of black hole solution by T-duality, and so the reason why we might not have observed the production of black holes at the LHC might be due to the existence of such black remnants. This can happen in a brane world model with one compactified dimension. In this case, the mass loss for a black hole can be obtained as \[51\],

\[
M_t \equiv \frac{dM}{dt} = -aAT^4,
\]

where \(a\) is a positive constant depend on the effective Stefan-Boltzmann constant and Planck mass. In the Fig. 2 we can see typical behavior of mass loss. Thus, the existence of a black remnant change this behavior.

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**IV. CORRECTED THERMODYNAMICS**

It is also important to analyze other thermodynamic quantities for such a black hole corrected from these string theory corrections. Since \(A = 4\pi r_+^2\) denotes the area of the horizon and \(V = \frac{4}{3}\pi r_+^3\) is the volume of the black hole,
the black hole entropy given by [26],
\[
S = \frac{4(M^4 - 5\pi^2 R^2 M^2 + \frac{9\pi^4 R^4}{4})}{M^2} \sqrt{1 + \frac{\pi^2 R^2}{(2M - \frac{12\pi^2 R^2}{4M})^2}} - 12\pi^3 R^2 \left( \sinh^{-1}\sqrt{2} - \frac{1}{\sinh^{-1}\left(\frac{2\pi R}{2M - \frac{12\pi^2 R^2}{4M}}\right)} \right). \quad (15)
\]

In the Fig. 2(b) we observe the behavior of entropy. The entropy is an increasing function of \(M\), which becomes equal to the Schwarzschild entropy for the large \(M\). This is expected as the string theoretical effects should only modify the short distance behavior of the black holes. So, in the infrared limit, the black hole resembles the usual black hole obtained from general relativity. In the Fig. 2(b), the entropy has a sharp minimum (\(M \approx 0.5\)) and than sudden increase as \(M\) further increases. This behavior is completely different from classical black hole thermodynamics, and suggests that \(S \propto A\) does not hold with \(0 < M < 1\). It suggests that entropy and area have a different relation in this limit, like \(S = c_1/(M - a)^n\), where \(a\) and \(n\) are positive numbers while \(c_1\) is an arbitrary constant. It is a power-law form of entropy admitted in the ultraviolet limit.

Helmholtz free energy for this stringy corrected black hole can obtain by using the following relation,
\[
F = -\int SdT. \quad (16)
\]

One can obtain analytic expression of the free energy in terms of the black hole volume. The expression is so large, however an approximate relation for the case of \(2\pi R = 1\) can be written as,
\[
F \approx -\frac{5\mu \left(100(7V^2 + 20) \tan^{-1}(2\nu) + (3V^2 + 7) \tanh^{-1}\left(\frac{100}{\nu}\right)\right)}{\nu(4V^2 + 10)}, \quad (17)
\]
where
\[
\mu = \sqrt{600V^2 + 1559}, \quad \nu = \sqrt{3600V^2 + 9354}. \quad (18)
\]

In the Fig. 3(a), we observe the exact behavior of the Helmholtz free energy in terms of \(V\). We can see there is a maximum for certain value of volume. Internal energy obtained using the following relation,
\[
E = F + ST. \quad (19)
\]

![Figure 3](image-url)
In the case of $2\pi R = 1$, one can obtain,

$$E \approx \frac{1}{6} \left( \frac{1}{3} (1 - 5V^{-\frac{2}{3}}) \xi - V^{-\frac{1}{3}} \right) \left( 1 - \frac{3}{\zeta^2} \right) - \frac{5\mu \left( 100(7V^2 + 20) \tan^{-1}(2\nu) + (3V^2 + 7) \tanh^{-1}(\frac{100}{\nu}) \right)}{\nu(4V^\frac{2}{3} + 10)},$$

(20)

where $\xi$ and $\zeta$ are given by

$$\xi = \sqrt{9V^\frac{2}{3} + 24},$$
$$\zeta = \sqrt{0.4V^\frac{2}{3} + 1}.$$  

(21)

In order to see exact behavior we give graphical analysis. In the Fig. 3 (b) behavior of the internal energy obtained. There is critical volume with special value of $l$, where internal energy is minimum. As $l = 2\pi R$, so by varying the value of the compactification radius the internal energy of this black hole solution can also change.

It is important to understand the behavior of specific heat for this system, and analyze the string theoretical corrections to the specific heat of this black hole. The black hole specific heat given by,

$$C = T \left( \frac{dS}{dT} \right).$$  

(22)

We calculate specific heat in terms of $M$ and find that there is the first order phase transition around $M_c$ [26]. It may be expressed for the $l = 2\pi R = 1$ as,

$$C \approx -\frac{4 \times 10^8}{\sqrt{\delta^2 - 1} (84M^2 - 3)^3 M (4 \times 10^3 M^8 - 10^4 M^6 + 8 \times 10^3 M^4 - 2 \times 10^4 M^2 + 81)},$$ 

(23)

where

$$\delta = \sqrt{1 + (2M - 0.75M^{-1})^{-2}},$$ 

(24)

and

$$M = M^{20} - 3.5M^{18} + 5.4M^{16} + 5M^{14} + 3M^{12} - 1.4M^{10} + \ldots$$

(25)
We plot behavior of the specific heat in terms of the black hole mass in the Fig. 4 and show that in the limit $R \to 0$ the black hole is completely unstable, while there is some stable regions when $R \neq 0$. These regions are corresponding to the lower mass of black hole. By increasing mass black hole transit to the unstable phase. Indeed by the Fig. 4 we show that there is a discontinuity within range $0 < M < 2$. The specific heat goes from initially positive to negative values showing a second order phase transition. It confirms that black holes with small mass and string correction are stable while classical large Schwarzschild like black holes are unstable. Furthermore, for small black hole masses $M = M_{\text{min}}$, the specific heat goes to zero at $C = 0$. This indicates the existence of remnants, as because when the heat capacity is zero, the black hole cannot exchange radiation with the surrounding space [52].

In order to investigate stability of the system we calculate Gibbs free energy ($G = H - TS$), its minimum represent stability, and its maximum represent unstable to stable phase transition (corresponding to asymptotic behavior of the Fig. 4). The Gibbs energy will always be decreasing for a system in equilibrium. From the Fig. 5 we can observe that the Schwarzschild black hole is unstable. However, the string theoretical corrections to the Schwarzschild black hole can make it stable for small black hole radius. This is an important change in the physical behavior of this system, which occurs due to short distance corrections coming from string theory.

\[\text{FIG. 5: Gibbs free energy in terms of } V.\]

V. CONCLUSION

In this work, we have analyzed the string theoretical corrections to a black hole, which occur from the finiteness of string theory. Such finiteness of string theory occurs due to T-duality, as according to T-duality, the behavior of string theory below string length scale is the same as its behavior above it. This finiteness modifies the short distance behavior of black holes, which in turn changes the physics of such black holes. We have analyzed the thermodynamics of such black holes. It has been demonstrated that at large distances the thermodynamic quantities coincide with the usual quantities, and they only have a different behavior at short distances. We also analyze the effect of compactification on the behavior of such a system. It was observed that such a string theoretical corrections produced non-trivial modification for the temperature of black holes. Such mini black holes in string theory may related to non-commutative space [53, 54]. This modification produced black remnants. Such remnants would modified the last stages of the evaporation of black holes, and thus can have important consequences for the black hole information paradox. This is because a small number of quanta can contain a large amount of information, and so black remnants can contain all the information of a black hole. These black remnants can also form for primordial black holes and form a candidate for cold dark matter. They can also have important consequences for the detection of black holes at the LHC. It may be noted that it is known that it is possible to use string theory models, in which black remnants
form. This is because a four dimensional magnetic black hole solutions of heterotic string theory has been constructed using $SU(2)/Z(2Q + 2)$ WZW orbifold, and it has also been demonstrated certain marginal operators can deform this theory to an asymptotically flat black hole. Furthermore, it has been demonstrated that black remnants can form in this system \[46\]. However, in this paper, it was demonstrated that such black remnants can also form in a modification to a black hole solution, which was motivated from T-duality. Finally, it was also demonstrated that such string theoretical corrections can change the behavior of this system, and make change it from an unstable to a stable system. This is done by investigating the critical points and phase transition for such a system.

It would be interesting to analyze such modification to the thermodynamics of various geometries that occur in string theory. This can be done by analyzing various solution to different supergravity equations, and then using analyzing the modifications to such solutions produced from this approach. It may be noted that such modifications are produced by non-perturbative effects, and cannot be produced by perturbative calculations. This is because the modified Green’s function does not diverge, but if it is expressed as a pertubative sum, each term would tend to diverge. So, to analyze such modifications of supergravity solution, it would be important to first obtain the usual black hole solution. Then a non-perturbative modification of such a solution would produce such effects. One of the important differences between this non-perturbative modification to a black hole solution, and the usual black hole solution is that this black hole solution is free from singularities. Now singularities are also important in cosmology, and it might be possible to remove the big bang singularity, by modification of different cosmological solutions, using this formalism. It would be interesting to analyze the constraints on such cosmological models from different cosmological data sets. It would be interesting to analyze add a cosmological constant term to this system, and then the analyze its $P−V$ criticality. It is expected that such string theory corrections would could lead to non-trivial modification to the $P−V$ criticality of this system.
