Modern Approaches to Cosmological Singularities

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Abstract. We review recent work and present new examples about the character of singularities in globally and regularly hyperbolic, isotropic universes. These include recent singular relativistic models, tachyonic and phantom universes as well as inflationary cosmologies.

1. Introduction

An important question eventually arising in every study of the global geometric and physical properties of the universe is that of deciding whether or not the resulting model is geodesically complete. Geodesic completeness is associated with an infinite proper time interval of existence of privileged observers and implies that such a universe will exist forever. Its negation, geodesic incompleteness or the existence of future and/or past singularities of a spacetime, is often connected to an ‘end of time’ for the whole universe modeled by the spacetime in question. By now there exist simple singular cosmological models of all sorts, not incompatible with recent observations, in which an all-encompassing singularity features as such a catastrophic event.

It is well known that in general relativity there are a number of rigorous theorems predicting the existence of spacetime singularities in the form of geodesic incompleteness under certain geometric and topological conditions (see, e.g., [1] for a recent review). These conditions can be interpreted as restrictions on the physical matter content as well as plausibility assumptions on the causal structure of the spacetime in question. Because all these assumptions are not unreasonable, the singularity theorems predicting the existence of spacetime singularities in cosmology and gravitational collapse have become standard ingredients of the current cosmological theory.

However, such existence results cannot offer any clue about the generic nature of the singularities they predict. In addition, there are new completeness theorems (cf. [2]) which say that under equally general geometric assumptions, generic spacetimes are future (or past) geodesically complete. Among the chief hypotheses of the completeness theorems, except the usual causality ones also present in the singularity theorems, is the assumption that the space slice does not ‘vibrate’ too much as it moves forward (or backward) in time, and also the assumption that space does not curve itself too much in spacetime. It may well be that cosmological models in a generic sense are only mildly singular or even complete. It remains thus a basic open problem to decide about the character of the cosmological singularities predicted by the singularity theorems.

In preparing to tackle such basic open questions more information is required, and one feels that perhaps different techniques are needed which the singularity theorems cannot provide. We
are therefore faced with the following problem: Suppose we have a spacetime which is known to have a singularity. How can we unravel its basic characteristics and find criteria classifying different singular spacetimes? What methods are to be used in such pursuits? Indeed, how are we to make a start into the ‘zoology’ of cosmological singularities? Such questions apply with equal interest to different classes of cosmological models (classified according to symmetry), as well as to the general case.

In this paper, we review recent work on this subject contained mainly in Refs. [2]-[4] about the character of cosmological singularities. We present a basic theorem providing necessary conditions for singularities and give new examples to illustrate this result. These examples are constructed using relativistic cosmological models, phantom cosmologies and inflationary models.

2. Completeness and the character of singularities

Although we focus below exclusively on isotropic models, it is instructive to begin our analysis by taking a more general stance. Consider a slice space (cf. [3] for this terminology), that is a spacetime \((\mathcal{V}, g)\) with \(\mathcal{V} = \mathcal{M} \times I\), \(I = (t_0, \infty)\), where \(\mathcal{M}\) is a smooth manifold of dimension \(n\) and \((n+1)g\) a Lorentzian metric which in the usual \(n+1\) splitting, reads

\[
(n+1)g \equiv -N^2(\theta^0)^2 + g_{ij} \theta^i \theta^j, \quad \theta^0 = dt, \quad \theta^i = dx^i + \beta^i dt. \tag{1}
\]

Here \(N = N(t, x^i)\) is the lapse function, \(\beta^i(t, x^i)\) the shift function and the spatial slices \(\mathcal{M}_t (= \mathcal{M} \times \{t\})\) are spacelike submanifolds endowed with the time-dependent spatial metric \(g_t \equiv g_{ij}dx^i dx^j\). We assume that \((\mathcal{V}, g)\) is globally hyperbolic and so time-oriented by increasing \(t\). \(^1\) We also assume that \((\mathcal{V}, g)\) is regularly hyperbolic meaning that the lapse, shift and spatial metric are uniformly bounded. It is known that a regularly hyperbolic spacetime is globally hyperbolic if and only if each slice is a complete Riemannian manifold, cf. [3].

A Friedmann universe is a sliced space with \(N = 1\), \(\beta = 0\) and the spatial metric is described by a single function of the (proper) time, the expansion scale factor \(a(t)\). Thus the metric has the form \(ds^2 = -dt^2 + a^2(t)d\sigma^2\) with \(d\sigma^2\) denoting the time-independent slice metric of constant curvature \(k\), and the Hubble expansion rate is proportional to the extrinsic curvature of the slices, \(|K|_g^2 = 3(\dot{a}/a)^2 = 3H^2\). Using the completeness theorems of [2] we arrive at the following result for isotropic cosmologies.

Theorem 2.1 (Completeness of Friedmann universes) Every globally hyperbolic, regularly hyperbolic Friedmann solution such that for each finite \(t_1\) the Hubble expansion rate \(H(t)\) is bounded by a function of \(t\) which is integrable on \([t_1, +\infty)\), is future timelike and null geodesically complete.

We can further use this result to arrive at a characterization of the different singularities that may arise in isotropic universes. Consider a singular, globally and regularly hyperbolic (scale factor assumed bounded only below in this case) Friedmann universe. Then according to Theorem 2.1, there is a finite time \(t_1\) for which \(H\) fails to be integrable on the proper time interval \([t_1, \infty)\). In turn, this non-integrability of the expansion rate \(H\) can be implemented in different ways and we arrive at the following result for the types of future singularities that can occur in isotropic universes (see [4]).

Theorem 2.2 (Character of future singularities) Necessary conditions for the existence of future singularities in globally hyperbolic, regularly hyperbolic Friedmann universes are:

**S1** For each finite \(t\), \(H\) is non-integrable on \([t_1, t]\), or

\(^1\) We choose \(I = (t_0, \infty)\) because we study the future singularity behaviour of an expanding universe with a singularity in the past, for instance at \(t = 0 < t_0\). However, since \(t\) is just a coordinate, our study could apply as well to any interval \(I \subset \mathbb{R}\).
\textbf{S2} \textit{H blows up in a finite time, or}

\textbf{S3} \textit{H is defined and integrable (that is bounded, finite) for only a finite proper time interval.}

The character of the singularities in this theorem is expected to be in a sense somewhat milder than standard all-encompassing big-crunch type ones predicted by the Hawking-Penrose singularity theorems. For instance, those satisfying condition S1 may correspond to ‘sudden’ singularities located at the right end (say \(t_s\)) at which \(H\) is defined and finite but the left limit, \(\lim_{\tau\to t_1^-} H(\tau)\), may fail to exist, thus making \(H\) non-integrable on \([t_1, t_s]\), for any finite \(t_s\) (which is of course arbitrary but fixed from the start). We shall see examples of this behaviour in the next Section. Condition S2 leads to what is called here a blow-up singularity corresponding to a future singularity characterized by a blow-up in the Hubble parameter\(^2\). Condition S3 may also lead to a singularity but for this to be a genuine species (in the sense of geodesic incompleteness) one needs to demonstrate that the metric is non-extendible to a larger interval.

3. \textbf{General relativistic isotropic models}

We now consider an example of a cosmological model with a future singularity which fulfills precisely our condition S1 of the previous Section. This model, as given by Barrow in his definite recent works [5, 6], is described by the most general solution of the Friedmann equations for a perfect fluid source with equation of state \(p = w\rho\) in a local neighborhood of the singularity which is located at the time \(t = t_s\) ahead:

\[
a(t) = 1 + \left(\frac{t}{t_s}\right)^q (a_s - 1) + \tau^n\Psi(\tau), \quad \tau = t_s - t. \tag{2}
\]

Here we take \(1 < n < 2, 0 < q < 1, a(t_s) = a_s\) and \(\Psi(\tau)\) is the so-called \textit{logarithmic psi-series} which is assumed to be convergent, tending to zero as \(\tau \to 0\). Barrow shows in [5, 6] that the form (2) exists as a smooth solution only on the interval \((0, t_s)\). Also \(a_s\) and \(H_s \equiv H(t_s)\) are finite at the right end but \(\dot{a}\) blows up as \(t \to 0\) making \(H\) continuous only on \((0, t_s)\). In addition, \(a(0)\) is finite and we can extend \(H\) and define it to be finite also at \(0\), \(H(0) = H_0\), so that \(H\) is defined on \([0, t_s]\). However, since \(\lim_{\tau\to 0^+} H(t) = \pm\infty\), we conclude that this model universe implements exactly Condition S1 of the previous Section and thus \(H\) is non-integrable on \([0, t_s]\), \(t_s\) arbitrary.

This then provides an example of the so-called big rip singularity characterized by the fact that as \(t \to t_s\) one obtains \(\ddot{a} \to -\infty\). Then using the field equation we see that this is really a divergence in the pressure, \(p \to \infty\). In particular, we cannot have in this universe a family of privileged observers each having an infinite proper time and finite \(H\). A further calculation shows that the product \(E_{\alpha\beta}E^{\alpha\beta}\), \(E_{\alpha\beta}\) being the Einstein tensor, is unbounded at \(t_s\). Hence we find that this spacetime is geodesically incomplete.

4. \textbf{Tachyonic cosmologies}

Tachyons, phantoms, Chaplygin gases and quintessence represent unobserved and unknown, tensile, negative energy and/or pressure density substances, violating some or all of the usual energy conditions, whose purpose is to cause cosmic acceleration and drive the late phases of the evolution of the universe (see [7, 8] and references therein). Are such universes generically singular or complete? Such scalar sources usually have the unpleasant property of super-luminal sound speed at low density (an exception to this rule is given in [9]) and the counter-intuitive property of the sound speed going to zero at large density.

\(^2\) Note that S1 is not implied by S2 for if \(H\) blows up at some finite time \(t_s\) after \(t_1\), then it may still be integrable on \([t_1, t], t_1 < t < t_s\).
Despite these negative features, one can easily construct complete as well as singular models (see [4] for more examples of this sort). Consider a Friedmann universe filled with a generalized Chaplygin gas with equation of state given by [10]

\[ p = -\rho^{-\alpha}[C + (\rho^{1+\alpha} - C)^{\alpha/(1+\alpha)}], \]  

where \( C = A/(1 + w) - 1 \) and subject to the condition \( 1 + \alpha = 1/(1 + w) \). The scale factor is given by the form

\[ a(t) = (C_1 e^{-C_3 \tau} + C_2 e^{C_3 \tau})^{2/3}, \]  

where \( C_1, C_2 \) and \( C_3 \) are constants. Therefore we find that in the asymptotic limits \( \tau \to 0 \) and \( \tau \to \infty \), \( H \) tends to suitable constants, that is it remains finite on \([t_0, \infty)\) and the model is geodesically complete.

Completeness, however, is really a property sensitive to the equation of state assumed in any particular model. If instead one assumes that the dark energy component satisfies at late times a general equation of state of the form [11]

\[ p = -\rho - f(\rho), \quad f(\rho) = A\rho^\alpha, \]  

with \( A \) and \( \alpha \) being real parameters, then setting \( \bar{A} = A\rho_0^{\alpha - 1} > 0 \) one finds that for \( \alpha \in (1/2, 1) \) the Hubble expansion rate becomes

\[ H = C \left( \frac{1 + 3\bar{A}(1 - \alpha) \ln \frac{a_d}{a_0}}{\bar{A}(2\alpha - 1)} + \frac{3}{2} \bar{A}(1 - 2\alpha)C(t - t_d) \right)^{\frac{1}{1 - 2\alpha}}, \]  

where \( t_d \) is the time when dark energy dominance commences. Therefore at the time \( t_f \), where

\[ t_f = t_d + \frac{2}{3\bar{A}(2\alpha - 1)C} \left( 1 + 3\bar{A}(1 - \alpha) \ln \frac{a_d}{a_0} \right)^{(1-2\alpha)/(2(1-\alpha))}, \]  

\( H \) blows up. This is clearly a type-S2 singularity according to Theorem 2.2. In contrast, when \( \alpha < 1/2 \) we see from (7) that \( H \) is always finite and therefore the model is geodesically complete.

It is clear that in the field of phantom cosmology more work is needed to decide on the issue of future singularities.

5. Inflationary models

Inflation continuously produces thermalized regions coexisting with ones still in an inflationary phase and so the universe, assumed to consist generically of regions of both types, must be future geodesically complete (‘future eternal’ according to the inflationistic terminology, cf. [12, 13]). It is an intriguing question for every inflationary model whether or not it is eternal in the past direction. Assume for the moment that it is not past eternal, so the universe is geodesically incomplete to the past. Then from Theorem 2.2 one expects that \( H \) will be non-integrable in some way, e.g., according to one of the conditions \( S1 - S3 \). Indeed, in Ref. [14], it was shown that inflation is past singular in the sense of \( S3 \). Suppose we are in a flat Friedmann universe and that \( H \) is integrable on a finite interval \([t_i, t_f]\). We assume that the mean of \( H(\lambda) \) on \([t_i, t_f]\), \( H_{av} \), satisfies the averaged expansion condition [14]

\[ H_{av} > 0, \]  

along some null geodesic with affine parameter \( \lambda \). Then one finds that

\[ 0 < (\lambda(t_f) - \lambda(t_i)) H_{av} < \infty, \]
which means that the affine parameter of this past-directed null geodesic must take values only in
a finite interval. This in turn says that this geodesic is incomplete. A similar proof is obtained for
the case of a timelike geodesic. Observe that condition (9) holds if and only if the hypotheses in
Theorem 2.1 are valid for a finite interval of time only, thus leading to incompleteness according
to Theorem 2.2, condition S3. A similar bound for the Hubble parameter is obtained in [14] for
the general case, and one therefore concludes that such a model must be geodesically incomplete.

As expected from Theorem 2.1, a relaxation of the requirement that $H$ be finite for only a
finite amount of proper time leads, to singularity-free inflationary models evading the previously
encountered singularity behaviour. Such models have been preliminary considered in [4].

Space does not allow us to consider the singularity problem in many interesting recent
quantum cosmological models. We leave such matters to a future publication.

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