ON SUBLAPLACIANS OF SUB-RIEMANNIAN MANIFOLDS

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Abstract. In this note we address a notion of sublaplacians of sub-Riemannian manifolds. In particular for fat sub-Riemannian manifolds we answered the sublaplacian question proposed by R. Montgomery in [21, p.142]

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1. Introduction

In the literature the sub-Riemannian analogue of Riemannian Laplacians is Hörmander sum of squares of vector fields, see [17, 21, 24, 33, 27, 15, 31, 32, 13, 25, 26] and references therein. Let $M$ be a smooth manifold of dimension $m$ endowed with a smooth distribution (horizontal bundle) $\Sigma$ of dimension $k$ with $k < m$. If we a priori equip $\Sigma$ with an inner product $g_c$ (sub-Riemannian metric), we call $(M, \Sigma, g_c)$ a sub-Riemannian manifold with the sub-Riemannian structure $(\Sigma, g_c)$. If $\Sigma$ is integrable, it is just the Riemannian geometry. We will assume $\Sigma$ is not integrable. A piecewise smooth curve $\gamma(t), t \in [a, b]$ in $M$ is horizontal if $\dot{\gamma}(t) \in \Sigma_{\gamma(t)}$ a.e. $t \in [a, b]$. The length $\ell(\gamma)$ of the horizontal curve $\gamma(t), t \in [a, b]$ is the integral $\int_a^b g_c(\dot{\gamma}(t), \dot{\gamma}(t))dt$. Denote by $\Sigma_i$ the set of all vector fields spanned by all commutators of order $\leq i$ of vector fields in $\Sigma$ and let $\Sigma_i(p)$ be the subspace of evaluations at $p$ of all vector fields in $\Sigma_i$. We call $\Sigma$ satisfies the Chow or Hörmander condition if for any $p \in M$, there exists an integer $r(p)$ such that $\Sigma_r(p) = T_p M$ (the least such $r$ is called the degree of $\Sigma$ at $p$). If moreover $\Sigma_i$ is of constant dimension for all $i \leq r$, $\Sigma$ and also $(M, \Sigma, g_c)$ are called regular. If $M$ is connected and $\Sigma$ satisfies the Hörmander condition, the Chow connectivity theorem asserts that there exists at least one piecewise smooth horizontal curve connecting two given points (see [6, 1, 21]), and thus $(\Sigma, g_c)$ yields a metric (called Carnot-Carathéodory distance) $d_{cc}$ by letting $d_{cc}(p, q)$ as the infimum among the lengths of all horizontal curves joining $p$ to $q$. Let $\{X_1, \cdots, X_k\}$ be an orthonormal basis of $\Sigma$. The Hörmander operator is $\Box = \sum_{i=1}^k X_i^2 + X_0$ where $X_0 \in \Gamma(\Sigma)$ is a horizontal vector field. It is easy to see that the operator $\Box$ in general depends on the choice of orthonormal bases. Thus $\Box$ is not intrinsic to the sub-Riemannian structure $(\Sigma, g_c)$. Recall that the Riemannian Laplacian on a Riemannian manifold is a Riemannian invariant. Montgomery in [21, p.142] proposed the question whether there exists a canonical sublaplacian in the sub-Riemannian case. As observed by [21], this question is equivalent to the existence of a canonical measure $\mu$: the canonical sublaplacian $\Box$ and $\mu$...
should satisfy
\[- \int_M (\hat{\Delta} f) gd\mu = \int_M g_c(\nabla^H f, \nabla^H g)d\mu\]
for any smooth functions \(f, g\) with compact support, where \(\nabla^H f\) is the horizontal gradient of \(f\): \(g_c(\nabla^H f, X) = X f\) for any horizontal vector field \(X \in \Gamma(\Sigma)\).

The importance of the study of sublaplacians on sub-Riemannian manifolds lies in the conjectured close relationship between spectral asymptotics of sublaplacians and sub-Riemannian geodesics (in particular singular curves), see [21, 22] and references therein. This note is devoted to a rudimental study of sublaplacians. We will give a notion of sublaplacians for general sub-Riemannian manifolds, or the sublaplacian for fat sub-Riemannian manifolds. We recall that a sub-Riemannian manifold \((M, \Sigma, g_c)\) is fat if \(\Sigma\) is strong-bracket generating, that is, for each \(p \in M\) and each nonzero horizontal vector \(v \in \Sigma_p\) we have
\[\Sigma_p + [V, \Sigma]_p = T_p M\]
where \(V\) is any horizontal extension of \(v\). This notion is defined using the truncated connection (called horizontal connection). Given a complement \(\Sigma'\) of \(\Sigma\): \(TM = \Sigma \bigoplus \Sigma'\), then with respect to this decomposition there exists a unique horizontal connection \(D\) on \(\Sigma\) such that for any \(X, Y, Z \in \Gamma(M)\)
\begin{align*}
(1) & \quad X g_c(Y, Z) = g_c(D_X Y, Z) + g_c(Y, D_X Z) \\
(2) & \quad D_X Y - D_Y X = [X, Y]^H,
\end{align*}
where \([X, Y]^H\) denotes the projection on \(\Sigma\) of \([X, Y]\), see [29, 13] for details. We define

\[\Delta^H := \text{div}^H \circ \nabla^H\]

where the horizontal divergence operator is defined as \(\text{div}^H X := \sum_{i=1}^k g_c(D_X X_i, X_i)\) for \(X \in \Gamma(\Sigma)\) and an orthonormal basis \(\{X_i\}_{i=1}^k\). The operator \(\Delta^H\) depends only on \((\Sigma, g_c)\) and the decomposition. In many cases such as nilpotent groups, contact manifolds, principal bundles with connections, and Riemannian submersions, there exists a ‘natural’ decomposition of \(TM\). In particular when the sub-Riemannian metric \(g_c\) is the projection on \(\Sigma\) of a Riemannian metric \(g\), \(TM\) can be orthogonally decomposed as \(TM = \Sigma \bigoplus \Sigma'\). Conversely, given a decomposition, we always can extend \(g_c\) to a Riemannian metric \(g\) such that the decomposition is orthogonal. Note that such extensions are not unique. Let \(g\) be any such extension. Then for any \(X, Y \in \Gamma(\Sigma)\)

\[D_X Y = \mathcal{P}(\nabla_X Y)\]

where \(\nabla\) is the Riemannian connection of \(g\) and \(\mathcal{P}\) denotes the projection on \(\Sigma\), see [29].

As stated above the defined \(\Delta^H\) depends on the splitting of the tangent bundle. This makes the problem delicate. We remark that for some cases such as nilpotent groups with grading Lie algebra and contact Riemannian manifolds, there is a canonical notion of the sublaplacian. Recall that the sub-Riemannian geometry of \((M, \Sigma, g_c)\), i.e., the geometry of \((M, d_{gc})\), depends only on the sub-Riemannian structure \((\Sigma, g_c)\), not on complements of \(\Sigma\) or extensions of \(g_c\). Our motivation to study sublaplacians or weakly convex function on sub-Riemannian manifolds is to extract information about sub-Riemannian geometry as much as possible by exploring functions or invariants defined intrinsically. What should a canonical complement of \(\Sigma\), and then a canonical orthogonal extension of \(g_c\) be? For
instance, in our opinion it is desirable (under some conditions imposed on $\Sigma$ and topology of $M$) to find a complement of (regular) $\Sigma$ and then to select an extension $g$ of $g_c$ such that the Riemannian measure of $g$ is just the Hausdorff $Q$-measure of $d_{cc}$, where $Q = \sum_{i=1}^{r} f(\dim(\Sigma_i) - \dim(\Sigma_{i-1}))/2$ is the Hausdorff dimension of $(M, d_{cc})$. The following statement is our starting point.

**Theorem 1.1.** Let $(M, \Sigma, g_c)$ be a sub-Riemannian manifold. Then there exists a complement $\Sigma'$ of $\Sigma$, $TM = \Sigma \oplus \Sigma'$, such that for this decomposition there is an orthogonal extension $g$ of $g_c$ and an orthonormal basis $\{T_1', \cdots, T_{m-k}'\}$ of $\Sigma'$ satisfying

$$\mathcal{P}(\nabla_{T_{\beta}}T_{\beta}') = 0, \quad \beta = 1, \cdots, m-k$$

where $\nabla$ is the Riemannian connection of $g$. Moreover, for any local frame of $TM$, $\{X_1, \cdots, X_k, T_1, \cdots, T_{m-k}\}$, where $\{X_1, \cdots, X_k\}$ is a basis of $\Gamma(\Sigma)$, the matrix $[C_{ij}^\beta]$ is invertible for $\beta = 1, \cdots, m-k$, where $C_{ij}^\beta$ are the coefficients satisfying

$$[X_i, X_j] = \sum_{a=1}^{k} C_{ij}^a X_a + \sum_{\beta=1}^{m-k} C_{ij}^\beta T_\beta,$$

then such complement is unique.

In general, the condition in Theorem 1.1 guaranteeing the uniqueness of the complement is very strong. For most cases it is impossible. In fact this condition is just the strong-bracket generating condition for $\Sigma$, see Proposition 2.1. We recall that contact structures of contact manifolds are fat.

The following theorem motivates our definition of sublaplacians, see Definition 2.6.

**Theorem 1.2.** Let $(M, \Sigma, g_c)$ be a sub-Riemannian manifold. Assume $TM = \Sigma \oplus \Sigma'$ be a splitting and $g$ an orthogonal extension of $g_c$. Denote by $d\nu$ the Riemannian measure of $g$ and by $H^\perp$ the mean curvature of $\Sigma'$. Let $u$ be a positive, smooth function on $M$. Set $d\mu = u d\nu$. Then

$$- \int_M (\Delta^H e) f d\mu = \int_M g_c(\nabla^H e, \nabla^H f) d\mu$$

holds for any $e, f \in C^\infty_0(M)$ if and only if $\ln u$ is the horizontal potential of $H^\perp$, i.e.,

$$\nabla^H(\ln u) = H^\perp.$$  \hfill (1.3)

Here the mean curvature $H^\perp$ of $\Sigma'$ is defined as

$$H^\perp := \sum_{\beta=1}^{m-k} \mathcal{P}(\nabla_{T_{\beta}}T_{\beta}') = \sum_{\beta=1}^{m-k} \sum_{i=1}^{k} g(\nabla_{T_{\beta}}T_{\beta}, X_i) X_i,$$

where $\{X_i\}_{i=1}^k, \{T_{\beta}\}_{\beta=1}^{m-k}$ are orthonormal bases of $\Sigma, \Sigma'$ respectively. We note that for given $\Sigma'$ and $g$, equation (1.3) is not soluble in general. But by Theorem 1.1 we always can choose an orthogonal extension of $g_c$ such that (1.3) is soluble for $H^\perp = 0$, if $(M, \Sigma, g_c)$ satisfies the assumption in Theorem 1.1.

The paper is organized as follows. In the next section after proving Theorem 1.1, 1.2, we give the definition of sublaplacians. Several canonical examples are given. It turns out that our definition is compatible with the canonical sublaplacians in the literature. At
the end of Section 2, a Hopf type theorem is proven for closed sub-Riemannian manifolds. The last section is devoted to the closed eigenvalue problem on compact sub-Riemannian manifolds.

2. PROOFS, EXAMPLES, AND BASIC PROPERTIES OF SUBLAPLACIANS

**Proof of Theorem 1.1** Let \( \{X_1, \cdots, X_k, T_1, \cdots, T_{m-k}\} \) be any local frame of \( TM \), where \( \{X_1, \cdots, X_k\} \) is an orthonormal basis of \( \Gamma(\Sigma) \). Extend \( g_c \) to a Riemannian metric \( \bar{g} \) such that \( \{T_\beta\}_{\beta=1}^{m-k} \) is orthonormal. Denote by \( \bar{\nabla} \) the Riemannian connection of \( \bar{g} \). Then for \( \beta = 1, \cdots, m-k \),

\[
\bar{F}_\beta := \mathcal{P}(\bar{\nabla}_{T_\beta} T_\beta) = \sum_{i=1}^{k} \bar{g}(T_\beta, [X_i, T_\beta])X_i.
\]

Let \( \Sigma' \) be any complement of \( \Sigma \) and \( \{\bar{T}_\beta\}_{\beta=1}^{m-k} \) be a local basis of \( \Sigma' \). Then

\[
\bar{T}_\beta = \sum_{i=1}^{k} \bar{A}_i^\beta X_i + \sum_{\alpha=1}^{m-k} B_{\beta}^\alpha T_\alpha
\]

for smooth functions \( \bar{A}_i^\beta \) and \( B_{\beta}^\alpha \). Since \( \{\bar{T}_\beta\}_{\beta=1}^{m-k} \) is a basis of \( \Sigma' \), the matrix \( [B_{\beta}^\alpha] \) is invertible. Denote by \( [K_{\beta}^\alpha] \) the inverse matrix of \( [B_{\beta}^\alpha] \). Setting

\[
T'_\beta := \sum_{\alpha=1}^{m-k} K_{\beta}^\alpha T_\alpha
\]

i.e.,

\[
T'_\beta = \sum_{i=1}^{k} A_i^\beta X_i + T'_\beta, \quad \text{where} \quad A_i^\beta = \sum_{\alpha=1}^{m-k} K_{\beta}^\alpha \bar{A}_i^\alpha,
\]

then \( \{T'_\beta\}_{\beta=1}^{m-k} \) is also a basis of \( \Sigma' \). Now extend \( g_c \) to a Riemannian metric \( g \) such that \( \{T'_1, \cdots, T'_{m-k}\} \) is orthonormal with respect to \( g \). Noting that

\[
[X_i, T'_\beta] = [X_i, \sum_{j=1}^{k} A_j^\beta X_j + T_\beta]
\]

\[
= \sum_{j=1}^{k} A_j^\beta [X_i, X_j] + (X_i A_j^\beta)X_j + [X_i, T_\beta],
\]
we have for \( \beta = 1, \cdots, m - k \)

\[
\mathcal{P}(\nabla_{T_{\beta}'} T_{\beta}') = \sum_{i=1}^{k} g(T_{\beta}', [X_i, T_{\beta}']) X_i
\]

\[
= \sum_{i=1}^{k} g \left( T_{\beta}', \sum_{j=1}^{k} A_{\beta}^j [X_i, X_j] + (X_i A_{\beta}^j)X_j + [X_i, T_{\beta}] \right) X_i
\]

\[
= \sum_{i=1}^{k} g \left( T_{\beta}', \sum_{\alpha=1}^{m-k} \left\{ \sum_{j=1}^{k} A_{\beta}^j g([X_i, X_j], T_{\alpha}) + g([X_i, T_{\beta}], T_{\alpha}) \right\} T_{\alpha}^i \right) X_i
\]

\[
= \sum_{i=1}^{k} \left( \sum_{j=1}^{k} A_{\beta}^j g([X_i, X_j], T_{\beta}) + \tilde{g}([X_i, T_{\beta}], T_{\beta}) \right) X_i
\]

Thus \( \mathcal{P}(\nabla_{T_{\beta}'} T_{\beta}') = 0 \) if and only if

\[
\sum_{j=1}^{k} A_{\beta}^j \tilde{g}([X_i, X_j], T_{\beta}) = -\tilde{g}(\bar{F}_\beta, X_i) \quad \text{(2.1)}
\]

for any \( i = 1, \cdots, k \). The first part is from elementary knowledge of linear algebra. Since \( C_{ij}^j = \bar{g}([X_i, X_j], T_{\beta}) \), the uniqueness follows from (2.1) and the assumption that the matrix \( [C_{ij}^j] \) is invertible for any \( \beta \). \( \square \)

**Proposition 2.1.** Let \( \Sigma \) be a distribution of \( M \). Then \( \Sigma \) is strong-bracket generating if and only if for any local frame of \( TM, \{X_1, \cdots, X_k, T_1, \cdots, T_{m-k}\} \), where \( \{X_1, \cdots, X_k\} \) is a basis of \( \Gamma(\Sigma) \), the matrix \( [C_{ij}^j] \) is invertible for \( \beta = 1, \cdots, m - k \), where \( C_{ij}^j \) are the coefficients satisfying (1.2).

**Proof.** Denote by \( \Sigma^\perp \) be the set of all sections in \( T^*M \) annihilating \( \Sigma \). By the Cartan formula

\[
d\omega(X, Y) = X(\lambda(Y)) - Y(\lambda(X)) - \lambda([X, Y]),
\]

it is easy to verify that \( \Sigma \) is strong-bracket generating if and only if \( d\omega: \Gamma(\Sigma) \times \Gamma(\Sigma) \to \mathbb{R} \) is nondegenerate for any \( \omega \in \Sigma^\perp \), see e.g. [21, p.70]. Now assume \( \Sigma \) be strong-bracket generating. For a given frame \( \{X_1, \cdots, X_k, T_1, \cdots, T_{m-k}\} \) of \( TM \), for \( \beta = 1, \cdots, m - k \) we choose \( \omega^\beta \in \Sigma^\perp \) such that \( \omega^\beta(T_{\alpha}) = \delta^\beta_\alpha \). Then the nondegeneracy of \( \omega^\beta \) implies the matrix \( [C_{ij}^j] = -[\lambda^\beta([X_i, X_j])] \) is invertible.

Conversely if \( 0 \neq \omega \in \Sigma^\perp \), then we can choose a frame \( \{X_1, \cdots, X_k, T_1, \cdots, T_{m-k}\} \) such that \( \{X_1, \cdots, X_k\} \) is a basis of \( \Gamma(\Sigma) \), \( \omega(T_1) = 1 \) and \( \omega(T_\beta) = 0 \) for \( \beta \neq 1 \). The nondegeneracy of \( [C_{ij}^j] \) implies the nondegeneracy of \( \omega \) on \( \Gamma(\Sigma) \). \( \square \)

If we write out \( \Delta^H \) in terms of horizontal vector fields, we see that \( \Delta^H \) is a Hörmander operator. In fact, we have

**Lemma 2.2.** Let \( \{X_i\}_{i=1}^{k} \) be any orthonormal basis of \( \Sigma \). Then \( \Delta^H = \sum_{i=1}^{k} (X_i^2 - D_X, X_i) \).

Thus by [17] \( \Delta^H \) is hypoelliptic if \( M \) is connected and \( \Sigma \) satisfies the Hörmander condition. We will use the following technical lemma.
Lemma 2.3. Let \( M, \Sigma, \Sigma', g, g_c \) be as in Theorem 1.2. For \( \epsilon > 0 \), let \( g^\epsilon \) be the Riemannian metric \( g^\epsilon = g_c \bigoplus \epsilon^2 g' \) where \( g' := g|_{\Sigma'} \). Denote by \( \Delta^\epsilon \) the Riemannian Laplacian of \( g^\epsilon \). Then

\[
\lim_{\epsilon \to +\infty} -\Delta^\epsilon = -\Delta^H + H^\perp
\]

Proof. Denote by \( \nabla^\epsilon \) the Riemannian connection of \( g^\epsilon \). Assume \( \{X_1, \ldots, X_k, T_1, \ldots, T_{m-k}\} \) an orthonormal basis with respect to \( g \). Then \( \{X_1, \ldots, X_k, \frac{1}{\epsilon} T_1, \ldots, \frac{1}{\epsilon} T_{m-k}\} \) is an orthonormal basis with respect to \( g^\epsilon \). For any smooth function \( f \), by definition and Lemma 2.2 we get

\[
\Delta^\epsilon f = \sum_{i=1}^{k} (X_i^2 - \nabla_{X_i} X_i) f + \frac{1}{\epsilon^2} \sum_{\beta=1}^{m-k} \left( T_\beta^2 - \nabla_{T_\beta} T_\beta \right) f
\]

\[
= \Delta^H f + \sum_{i=1}^{k} B(X_i, X_i) f + \frac{1}{\epsilon^2} \sum_{\beta=1}^{m-k} \left( T_\beta^2 - \nabla_{T_\beta} T_\beta \right) f,
\]

where

\[
B(X_i, X_i) = \sum_{\beta=1}^{m-k} g^\epsilon \left( \nabla_{X_i} X_i, \frac{1}{\epsilon} T_\beta \right) \frac{1}{\epsilon} T_\beta = \frac{1}{\epsilon^2} \sum_{\beta=1}^{m-k} g_c(X_i, [T_\beta, X_i]^H) T_\beta
\]

and

\[
\nabla_{T_\beta} T_\beta = (\nabla_{T_\beta} T_\beta)^H + (\nabla_{T_\beta} T_\beta)^\perp
\]

\[
= \sum_{i=1}^{k} g^\epsilon(\nabla_{T_\beta} T_\beta, X_i) X_i + \sum_{s=1}^{m-k} g^\epsilon \left( \nabla_{T_\beta} T_\beta, \frac{1}{\epsilon} T_s \right) \frac{1}{\epsilon} T_s
\]

\[
= \epsilon^2 \sum_{i=1}^{k} g(T_\beta, [X_i, T_\beta]) X_i + \frac{1}{\epsilon^2} \sum_{s=1}^{m-k} g^\epsilon \left( \nabla_{T_\beta} T_\beta, T_s \right) T_s
\]

\[
= \epsilon^2 H^\perp + \sum_{s=1}^{m-k} g(T_\beta, [T_\beta, T_\beta]) T_s.
\]

Remark 2.4. (1), Some authors called \( \bar{\Delta}^H := \Delta^H - H^\perp \) sublaplacian. \([11, 13]\). If \( H^\perp \neq 0 \), \( \Delta^H \) explicitly depends on \( g' \). (2), the penalty metric \( g^\epsilon \) is very useful in sub-Riemannian geometry. The reason is that when \( \Sigma \) satisfies the Hörmander condition and \( M \) is connected, \( (M, d^\epsilon) \) \((d^\epsilon \) is the Riemannian distance corresponding to \( g^\epsilon \)) converges to \( (M, d_{cc}) \) in the sense of Hausdorff-Gromov, e.g. \([20, 14, 21]\).

Proof of Theorem 1.2 Let \( g^\epsilon \) as in Lemma 2.3. Denote by \( d(\text{vol})^\epsilon \) the Riemannian volume element of \( g^\epsilon \). It is easy to show that

\[
d(\text{vol})^\epsilon = e^{m-k} d\text{vol}.
\]
Let $e, f$ be any smooth (or Sobolev) functions with compact support. We abuse the notation to denote by $\nabla^e f$ the Riemannian gradient of $f$ with respect to $g^e$. Noting that

$$\nabla^e f = \nabla^H f + \frac{1}{\epsilon^2} \sum_{\beta=1}^{m-k} (T_\beta f) T_\beta$$

and hence

$$g^e(\nabla^e f, \nabla^e f) = g_c(\nabla^H e, \nabla^H f) + \frac{1}{\epsilon^2} \sum_{\beta=1}^{m-k} (T_\beta e)(T_\beta f),$$

from (2.2) and the Green formula

$$\int_M (-\Delta^e f) d(\text{vol})^e = \int_M g^e(\nabla^e e, \nabla^e f) d(\text{vol})^e$$

we derive

$$\int_M (-\Delta^e f) d\text{vol} = \int_M \left\{ g_c(\nabla^H e, \nabla^H f) + \frac{1}{\epsilon^2} \sum_{\beta=1}^{m-k} (T_\beta e)(T_\beta f) \right\} d\text{vol}. \tag{2.3}$$

Taking the limit $\epsilon \to +\infty$ in (2.3), we by Lemma 2.3 induce

$$\int_M ((-\Delta^H + H^\perp) e) f d\text{vol} = \int_M g_c(\nabla^H e, \nabla^H f) d\text{vol}. \tag{2.4}$$

Putting $f = u\bar{f}$ ($\bar{f} \in C^\infty_0$) in (2.4), we get

$$-\int_M \Delta^H e \bar{f} u d\text{vol} + \int_M (H^\perp e) \bar{f} u d\text{vol} = \int_M g_c(\nabla^H e, \nabla^H f u) d\text{vol} + \int_M g_c(\nabla^H e, \nabla^H \bar{f}) u d\text{vol}.$$

Thus

$$-\int_M (\Delta^H e) \bar{f} d\mu = \int_M g_c(\nabla^H e, \nabla^H f u) d\mu, \tag{2.5}$$

holds if and only if

$$\int_M (H^\perp e) \bar{f} u d\text{vol} = \int_M g_c(\nabla^H e, \nabla^H \bar{f}) u d\text{vol}.$$

By the arbitrariness of $\bar{f}$ in the last equation, we deduce that (2.5) holds for any $f, \bar{f} \in C^\infty_0(M)$ if and only

$$(H^\perp e) u = g_c(\nabla^H e, \nabla^H \bar{f}) u$$

for any $e \in C^\infty_0(M)$, that is,

$$H^\perp e = g_c(\nabla^H e, \nabla^H (\ln u)).$$

Since $H^\perp$ is a horizontal vector field and hence $H^\perp e = g_c(\nabla^H e, H^\perp)$, Theorem 1.2 follows. □

**Corollary 2.5.** Let $(M, \Sigma, g_c)$ be a sub-Riemannian manifold. Then there exists a complement $\Sigma'$ of $\Sigma$ such that we can extend $g_c$ to some Riemannian metric $g$ and $\Delta^H$ is a symmetric operator on $C^\infty_0(M)$: for any $e, f \in C^\infty_0(M)$

$$\int_M (-\Delta^H e) d\text{vol} = \int_M g_c(\nabla^H e, \nabla^H f) d\text{vol} = \int_M e(-\Delta^H f) d\text{vol}$$
where $\text{dvol}$ is the Riemannian measure of $g$.

**Definition 2.6.** Let $(M, \Sigma, g_c)$ be a sub-Riemannian manifold. Fix a complement of $\Sigma$ such that (1.1) holds for some extension $g$ of $g_c$ and for some orthonormal basis of $g$. We define $\Delta^H$ (with respect to the splitting $TM = \Sigma \oplus \Sigma'$) as a sublaplacian of $(M, \Sigma, g_c)$. When such complement is unique (see Theorem 1.1), we call $\Delta^H$ the sublaplacian of $(M, \Sigma, g_c)$.

One of the reasons we define $\Delta^H$ (not $\Delta^L$) as a (the) sublaplacian is that $\Delta^H$ is compatible with several notions such as the horizontal Hessian and weakly convex functions on sub-Riemannian manifolds, see [28]. As already pointed out in the introduction, the sublaplacian in Definition 2.6 is defined for few cases. Theorem 1.1 and Proposition 2.1 tell us that the sublaplacian is well defined for fat sub-Riemannian manifolds. This makes the case more interesting because fat sub-Riemannian manifolds are proven to admit no singular sub-Riemannian geodesics.

**Example 2.7** (Carnot groups, [8, 23]). A Carnot group (or a stratified group) $G$ is a connected, simply connected Lie group whose Lie algebra $G$ admits the grading $G = V_1 \oplus \cdots \oplus V_l$, with $[V_1, V_i] = V_{i+1}$, for any $1 \leq i \leq l - 1$ and $[V_1, V_l] = 0$ (the integer $l$ is called the step of $G$). Let $\{e_1, \ldots, e_m\}$ be a basis of $G$ with $m = \sum_{i=1}^l \dim(V_i)$. Let $X_i(x) = (L_x)_* e_i$ for $i = 1, \ldots, k := \dim(V_1)$ where $(L_x)_*$ is the differential of the left translation $L_x(x') = xx'$ and let $T_\alpha(x) = (L_x)_* e_{i+k}$ for $\alpha = 1, \ldots, m - k$. We call the system of left-invariant vector fields $\Sigma := V_1 = \text{span}\{X_1, \ldots, X_k\}$ the horizontal bundle of $G$. If we equip $\Sigma$ an inner product $g_c$ such that $\{X_1, \ldots, X_k\}$ is an orthonormal basis of $\Sigma$, $(G, \Sigma, g_c)$ is a sub-Riemannian manifold satisfying the Hörmander condition which is guaranteed by the grading of its Lie algebra. The role played by Carnot groups in sub-Riemannian geometry is similar that by Euclidean spaces in Riemannian geometry, [20]. Thus sub-Riemannian manifolds are also called Carnot manifolds. Fix a Carnot group $G$. Because of the grading condition of its Lie algebra, by choosing the natural splitting of $TG$ and a system of left-invariant vector fields $\{X_1, \ldots, X_k\}$ as an orthonormal basis, we easily deduce that the horizontal connection $D$ has the following simple form

$$D_XY = \sum_{i=1}^k X(Y^i)X_i, \quad \text{for any} \quad X, Y = \sum_{i=1}^k Y^i X_i \in \Gamma(\Sigma)$$

and hence

$$\Delta^H = \sum_{i=1}^k X_i^2.$$ 

$\Delta^H$ coincides with the sublaplacian of Carnot groups studied in the literature, see [24, 18, 14] and references therein. It is clear that $\Delta^H$ is a symmetric operator on $C^\infty_0(G) \hookrightarrow L^2(G, g)$, where $g$ is an extension of $g_c$ such that $\{X_1, \ldots, X_k, T_1, \ldots, T_{m-k}\}$ is orthonormal.

**Example 2.8** (contact Riemannian manifolds, [24, 3]). Let $M$ be a real $2n + 1$-dimensional smooth manifold. An almost contact Riemannian structure $(\varphi, \xi, \eta, g)$ on $M$ consists of a $(1,1)$-tensor field $\varphi$, a vector field $\xi$, a 1-form $\eta$, and a Riemannian metric $g$ such that

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \varphi \xi = 0, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

where $I$ is the identity tensor field.
for any \(X, Y \in \Gamma(M)\). It is a contact Riemannian structure if it satisfies \(\Omega = d\eta\) (the contact condition) where \(\Omega(X, Y) = g(X, \varphi Y)\). Set \(\Sigma = \ker(\eta)\). Then \((M, \Sigma, g|_{\Sigma})\) is a sub-Riemannian structure and \(\Sigma\) has a natural complement. Let \(\nabla\) be the Levi-Civita connection of \(g\). Then the horizontal connection \(D\) is
\[
D_XY = \nabla_XY - \eta(\nabla_XY)\xi, \quad X, Y \in \Gamma(\Sigma).
\]
Note that the Levi form
\[
L_\eta(X, Y) = -d\eta(X, \varphi Y) = \eta([X, \varphi Y]) \quad X, Y \in \Gamma(\Sigma)
\]
is nondegenerate. This in particular implies that \(\Sigma\) satisfies the Hörmander condition.

The (generalized) Tanaka-Webster connection \(\nabla^*\) on \((M, \varphi, \xi, \eta, g)\) is
\[
\nabla^*_X Y = \nabla_X Y + \eta(X)\varphi(Y) - \eta(Y)\nabla_X \xi + [(\nabla_X \eta)Y]_\xi.
\]
Denote by \(\mathcal{D}\) the complexification of \(\Sigma\), i.e., \(\mathcal{D} = \mathcal{D}' \oplus \overline{\mathcal{D}'}\) where \(\mathcal{D}' = \{X - i\varphi X, X \in \Gamma(\Sigma)\}\) and \(\overline{\mathcal{D}'}\) is the conjugate of \(\mathcal{D}'\). Set \(h := \frac{1}{2}L\xi\varphi\). Then the pair \((M, \mathcal{D})\) is a (strongly pseudo-convex) CR manifold, (i.e., \([\mathcal{D}', \overline{\mathcal{D}'}] \subset \mathcal{D}'\) and the Levi form \(L_\eta\) is positive definite), if and only if the contact Riemannian manifold \((M, \varphi, \xi, \eta, g)\) satisfies
\[
(\nabla_\mathcal{D} \varphi) = g(X + hX, Y)\xi - \eta(Y)(X + hX).
\]
Let \((M, \mathcal{D})\) be a strongly pseudo-convex CR manifold. Denote by \(D^*\) the restriction on \(\Sigma\) of the \(\nabla^*\), and extend \(D^*\) to the complexified bundle \(\mathcal{D}\). Note that \(D^*\) is just the Webster connection, \([34]\). Since for \(X, Y \in \Gamma(\Sigma)\), \(\eta(X) = \eta(Y) = 0\), \((\nabla_\mathcal{D} \eta)Y = X(\eta(Y)) - \eta(\nabla_X Y)\), we have \(D^* = D\). Here \(D\) is also extended to \(\mathcal{D}\). Let \(\{X_\alpha\}_{\alpha=1}^n\) be an orthonormal complex basis (with respect to the extended metric \(g\)) of \(\mathcal{D}'\). Then \(\{X_\alpha = \varphi(X_\alpha)\}_{\alpha=1}^n\) is an orthonormal complex basis of \(\overline{\mathcal{D}'}\). For a smooth function \(f\) on \(M\), the sublaplacian studied by \([15, 19]\) is
\[
\Delta f = \sum_{\alpha=1}^n f_{\alpha\overline{\alpha}} + f_{\alpha\alpha},
\]
where
\[
f_\alpha = X_\alpha f, \quad f_{\alpha\alpha} = X_{\alpha\alpha} f, \quad f_{\alpha\beta} = X_{\alpha\beta} f - \sum_{\gamma=1}^n \Gamma^\gamma_{\alpha\beta} f_{\gamma}, \quad f_{\alpha\overline{\alpha}} = X_{\alpha\overline{\alpha}} f - \sum_{\gamma=1}^n \Gamma^\gamma_{\alpha\overline{\alpha}} f_{\gamma},
\]
and \(\Gamma^\gamma_{\alpha\beta} = g(D^*_{X_\alpha} X_\beta, X_\gamma), \quad \Gamma^\gamma_{\alpha\overline{\alpha}} = g(D^*_{X_\alpha} X_\beta, X_\overline{\gamma})\). Set \(Y_\alpha = \frac{1}{\sqrt{2}}(X_\alpha + \overline{X_\alpha}), \quad Y_\overline{\alpha} = \frac{1}{\sqrt{2}}(X_\alpha - \overline{X_\alpha})\). Then \(\{Y_1, \cdots, Y_n, \overline{Y}_1, \cdots, \overline{Y}_n\}\) is an orthonormal basis of \(\Sigma\). Now by direct computation we get from Lemma \(2.2\)
\[
\Delta = \sum_{\alpha=1}^n (Y_\overline{\alpha}^2 + Y_\alpha^2 - D_{Y_\alpha} Y_\alpha - D_{\overline{Y}_\alpha} \overline{Y}_\alpha) = \Delta^H
\]
since \(D^* = D\). Thus for strongly pseudo-convex pseudo-Hermitian manifolds our definition for sublaplacians coincides with the canonical one. Because \(\nabla_\xi \xi = 0\) and the Levi form \(L_\eta\) is nondegenerate, by Corollary \(2.3\), \(\Delta^H\) is a symmetric operator on \(C^0_c(M) \hookrightarrow L^2(G, g)\).
Example 2.9 (Riemannian submersions with minimal fibres, [12]). Let \((M, g)\) and \((B, g')\) be smooth Riemannian manifolds. A smooth map \(\pi : M \to B\) is a submersion if \(\pi_* : T_q M \to T_{\pi(q)} B\) is a surjective linear map for each \(q \in M\). The vertical space at \(q\) is the tangent space of the fibre \(\pi^{-1}(\pi(q)) : V_q = \ker(\pi_*)\). The collection of vertical spaces is the vertical distribution \(V \subset TM\). Let \(\Sigma\) be the orthogonal complement of \(V\). \(M\) with the structure \((\Sigma, g_c = g|_\Sigma)\) is a sub-Riemannian manifold. If \(\pi_* : \Sigma_q \to T_{\pi(q)} B\) is linear isometry for any \(q \in M\), \(\pi\) is called a Riemannian submersion. The Riemannian submersion \(\pi\) is a harmonic map between \((M, g)\) and \((B, g')\) if and only if each fibre of \(\pi\) is a minimal surface, e.g. [12]. If \(\pi\) is a Riemannian submersion with minimal fibres, by Theorem 1.2 we can define a sublaplacian \(\Delta^H\) on \((M, \Sigma, g_c)\) such that \(\Delta^H\) is a symmetric operator on \(C^\infty_0(M) \hookrightarrow L^2(M, g)\).

The above examples show that our notion of sublaplacians covers the canonical ones in the literature.

Lemma 2.10 (divergence theorem). Let \((M, \Sigma, g_c)\) be a sub-Riemannian manifold. Let \(g\) be the orthogonal extension of \(g_c\) as in Theorem 1.1. Then for any horizontal vector field \(X \in \Gamma(\Sigma)\)

\[
\text{div}^H X = \text{div} X
\]  

where \(\text{div}\) is the Riemannian divergence of \(g\). Thus if \(M\) is moreover compact with boundary (possibly empty), we have for any horizontal vector field \(X\)

\[
\int_M \text{div}^H X \text{dvol} = \int_{\partial M} g(X, \nu) \text{ds}
\]

where \(\text{dvol}\) is the Riemannian measure of \(g\), \(\nu\) is the normal vector field of the boundary \(\partial M\), and \(\text{ds}\) is the area measure on \(\partial M\) induced by \(g\).

**Proof.** Choose \(\{X_1, \cdots, X_k, T'_1, \cdots, T'_{m-k}\}\) as an orthonormal basis of \(g\), such that (1.1) holds. Since the (horizontal) divergence is independent of the choice of orthonormal bases,

\[
\text{div} X = \sum_{i=1}^k g(D_{X_i} X, X_i) + \sum_{\alpha=1}^{m-k} g(T'_{\alpha} X, T'_{\alpha})
\]

\[
= \sum_{i=1}^k g_c(D_{X_i} X, X_i) + \sum_{\alpha=1}^{m-k} \{T'_{\alpha} g(X, T'_{\alpha}) - g(X, \nabla_{T'_{\alpha}} T'_{\alpha})\}
\]

\[
= \text{div}^H
\]

where we used (1.1) and the assumption that \(X\) is horizontal. (2.7) is from (2.6) and the classical divergence theorem. \(\square\)

**Theorem 2.11.** Let \(\Delta^H\) be a (the) sublaplacian of \((M, \Sigma, g_c)\) in the sense of Definition 2.6. If \(\Sigma\) satisfies the Hörmander condition and \(M\) is a closed, connected manifold, then any horizontal-harmonic function \(u\), i.e. \(u\) satisfies

\[
\Delta^H u = 0,
\]

is constant.
**Proof.** Note that
\[
\Delta^H u^2 = 2\text{div}^H (u \nabla^H u) = 2g_c(\nabla^H u, \nabla^H u) + 2u\Delta^H u \tag{2.8}
\]
If \( u \) is horizontal-harmonic, by (2.6) and (2.8) we get
\[
\text{div}(u \nabla^H u) = g_c(\nabla^H u, \nabla^H u)
\]
where \( \text{div} \) is the Riemannian divergence of some extension \( g \) of \( g_c \) as in Theorem 1.1. Integrating the last formula, by the green formula in the Riemannian case we induce
\[
\int_M g_c(\nabla^H u, \nabla^H u) \, d\text{vol} = 0
\]
since by assumption \( M \) is closed. Thus \( \nabla^H u = 0 \), that is, \( u \) is constant along horizontal curves. The statement follows because \( \Sigma \) satisfies the Hörmander condition and \( M \) is connected, by the Chow theorem \([6]\) any two points can be connected by a piecewisely smooth horizontal curve.

\[\blacksquare\]

### 3. Eigenvalues of sublaplacians of compact sub-Riemannian manifolds

In this section we always assume \((M, \Sigma, g_c)\) is a **compact and regular** sub-Riemannian manifold with smooth (possibly empty) boundary. Let \( \Delta^H \) be a (the) sublaplacian of \((M, \Sigma, g_c)\) in the sense of Definition 2.6 and \( g \) be an orthogonal extension of \( g_c \) with respect to the given decomposition as in Definition 2.6. The goal of this section is to study the eigenvalue problem of \( \Delta^H \). First we give the definition of horizontal Sobolev functions on \((M, \Sigma, g_c)\).

**Definition 3.1.** A function \( f \) in \( L^2(M) \) is called a horizontal Sobolev function if there exists a horizontal vector field \( Y \) belonging to \( L^2(M) \) such that the following
\[
\int_M g_c(Y, X) \, d\text{vol} = -\int_M f \text{div}^H X \, d\text{vol}
\]
holds for any horizontal vector field \( X \) with compact support on \( M \). \( Y \) denoted by \( \nabla^H f \) is called the weakly horizontal derivative of \( f \). The set of all horizontal Sobolev functions is denoted by \( W^{1,2}_e(M) \).

Here we call a horizontal vector field is in \( L^2(M) \) if its coefficients are in \( L^2(M) \). From (2.7) the above definition is well-defined. We denote by \( H^{1,2}(M)(H^{0,2}_0(M)) \) the completed space of \( C^\infty(M)(C^\infty_0(M)) \) functions with respect to the norm
\[
||f||_{H^{1,2}(M)} = \left( \int_M |f|^2 + g_c(\nabla^H f, \nabla^H f) \, d\text{vol} \right)^{\frac{1}{2}}.
\]

**Lemma 3.2.**

1. \( H^{1,2}(M) = W^{1,2}_e(M) \);
2. the embedding \( W^{1,2}_e(M) \hookrightarrow L^2(M) \) is compact;
3. If \( f \in W^{1,2}_e(M) \) and \( \Delta^H f = \lambda f \) for some \( \lambda \in \mathbb{R} \), then \( f \) must be smooth.
Proof. Since $M$ is compact, by choosing a smooth partition of unity subordinate to a finite cover of $M$, the first two statements are reduced to a local chart case. Let $\phi : U \subset M \to V = \phi(U) \subset \mathbb{R}^m$ be a coordinate chart. Since $\Sigma$ is regular, $\phi_*(\Sigma|_U)$ is also a regular distribution on $V$. Then $(V, \phi_*(\Sigma|_U), g')$ is a regular sub-Riemannian manifold, where $g'$ is the standard Euclidean metric. Now any function in $H^{1,2}(U)$ is pushed forward by $\phi$ to a horizontal weighted Sobolev function in the sense of [10]. Now the first two statements follow from the corresponding results proven in [10, 9]. The third is standard since $\Delta^H$ is a hypoelliptic operator.

Theorem 3.3. Let $M$ be without boundary. Consider the following closed eigenvalue problem

$$-\Delta^H f = \lambda f. \tag{3.1}$$

That is, we are looking for all numbers $\lambda$ for which there exists a nontrivial smooth solution satisfying (3.1). Then

1. The set of eigenvalues consists of an infinite sequence $0 \leq \lambda_1 < \lambda_2 < \lambda_3 \cdots \uparrow +\infty$
2. Each eigenvalue $\lambda_i$ has finite multiplicity and the eigenspaces corresponding to different eigenvalues are $L^2(M)$-orthogonal.
3. The direct sum of the eigenspaces $E(\lambda_i)$, $i = 1, \cdots$ is dense in $L^2(M)$.
4. Let $\Delta^\epsilon$ be as in Lemma 2.3. For each $\epsilon$, denote by $\lambda_i(\epsilon)$ be the $i$-th (counting the multiplicity) eigenvalue of the eigenvalue problem

$$\Delta^\epsilon f = \lambda(\epsilon)f.$$

Then

$$\lim_{\epsilon \to +\infty} \lambda_i(\epsilon) = \lambda_i$$

where $\lambda_i$ is renumbered counting the multiplicity.

Proof. By Corollary 2.5, $-\Delta^H$ is a positive and symmetric operator on $C^\infty(M)$ which is dense in $W^{1,2}(M)$. Thus by the first statement of Lemma 3.2, $-\Delta^H$ can be extended to a closed, positive self-adjoint operator on $W^{1,2}(M)$, which implies that the spectrum of $-\Delta^H$ is contained in $\mathbb{R}_+$. It follows from the compactness of the embedding $W^{1,2}(M) \hookrightarrow L^2(M)$ that the resolvent $(\Delta^H - \lambda)^{-1}$ is a compact operator in $L^2(M)$. The first three statements follows from the classical results on the spectral theory of compact operators and from the third claim of Lemma 3.2, see e.g. [7].

Fukaya in [11] proved the fourth statement for $\Delta^H - H^\perp$. Since by our choice of the orthogonal extension $H = 0$, the statement follows.

Remark 3.4. (1) The first three claims can also be proven by a variational argument minimizing the Rayleigh quotient

$$\frac{\int_M g_c(\nabla^H f, \nabla^H f) d\text{vol}}{\int_M f^2 d\text{vol}}.$$

(2) For complete sub-Riemannian manifolds, following the lines of [27] it is possible to develop a theory of heat semi-group of $\Delta^H$. 
REFERENCES

[1] A. Bellaiche and J. Risler (eds.), *Sub-Riemannian geometry*. Progress in Mathematics, 144, Birkhauser Verlag, Basel, 1996.

[2] J.-M. Bismut, *Large Deviations and the Malliavin calculus*. Birkhäuser, 1984.

[3] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, Vol. 203, Birkhauser, 2002.

[4] A. Bonfiglioli, F. Uguzzoni, Families of diffeomorphic sub-Laplacians and free Carnot groups, Forum Math. Vol. 16 (2004), 403-415.

[5] L. Capogna, D. Danielli, N. Garofalo, An embedding theorem and the Harnack inequality for nonlinear subelliptic equations. Comm. PDE. 18 (1993), 1765-1794.

[6] W. L. Chow, über Systeme non linearen partiellen Differentialgleichungen erster Ordnung. Math. Ann. 117 (1939), 98-105.

[7] N. Dunford, Jacob T. Schwartz, *Linear Operators, Part 2, Spectral Theory, Self Adjoint Operators in Hilbert Space*, New York, John Wiley & Sons, 1988.

[8] G. B. Folland, E. M. Stein, *Hardy spaces on homogeneous groups*. Princeton Univ. Press, Princeton, New Jersey, 1982.

[9] B. Franchi, G. Lu, Richard L. Wheeden, The representation formula and weighted Poincaré inequalities for Hörmander’s vector fields, Ann. Inst. Fourier (Grenoble), 45 (1995), 577-604.

[10] B. Franchi, R. P. Serapioni, F. Serra Cassano, Approximation and imbedding theorems for weighted Sobolev spaces associated with Lipschitz continuous vector fields, Bollettino della Unione matematica italiana. B, Vol. 11 (1997), p.83-117.

[11] Kenji Fukaya, Collapsing of Riemannian manifolds and eigenvalues of Laplace operator. Invent. Math., Vol. 87 (1987), 517-547.

[12] M. Falcitelli, S. Lanus and Anna M. Pastore, *Riemannian submersions and related topics*. World Scientific, Hong Kong, 2004.

[13] Zhong Ge, Collapsing Riemannian metrics to Carnot-Carathéodory metrics and Laplacians to Sub-Laplacians. Can. J. Math., Vol. 45, 537-553.

[14] V. Gershkovich, Sub-Riemannian manifolds as limits of Riemannian manifolds, Russian J. Math. Phys., Vol. 4 (1996), 151-172.

[15] V. Gershkovich and A. Vershik, Nonholonomic metrics and nilpotent analysis. J. Geom. Phys. Vol. 5 (1988), 407-452.

[16] A. Greenleaf, The first eigenvalue of a sub-Laplacian on a pseudo-Hermitian manifold, Comm. Partial Differential Equations, Vol. 10 (1985), 191-217.

[17] L. Hörmander, Hypoelliptic second order differential equations, Acta Math., Vol. 199 (1967), 147-171.

[18] D. Jerison, The Dirichlet problem for the Kohn Laplacian on the Heisenberg group. I,II, J. Functional Anal. Vol. 43 (1981), 97-142,224-257.

[19] Song-Ying Li and Hing-Sun Luk, The sharp lower bound for the first positive eigenvalue of a sub-laplacian on a pseudo-hermitian manifold, Proc. of AMS. Vol. 132 (2003), 789-798.

[20] J. Mitchell, On Carnot-Carathéodory metrics. J. Diff. Geom., 21 (1985), 35-45.

[21] R. Montgomery, *A Tour of Subriemannian Geometry, Their Geodesics and Applications*, Mathematical Surveys and Monographs, vol. 91, 2002.

[22] R. Montgomery, A survey of singular curves in subRiemannian geometry, J.Control and Dyn. Sys., Vol. 1 (1995), 49-90.

[23] P. Pansu, Métriques de CC et quasiisométries des espaces symétriques de rang un, Ann. of Math., 119 (1989), 1-60.

[24] L.P.Rothschild and E. Stein, Hypoelliptic differential operators and nilpotent groups. Acta Math., Vol. 137 (1976), 247-320.

[25] M. Rumin, Sub-Riemannian limit of the differential form spectrum of contact manifolds. Geom. Funct. Anal., Vol. 10 (2000), 407-452.
[26] M. Rumin, An introduction to spectral and differential geometry in Carnot-Carathéodory spaces. Rend. Circ. Mat. Palermo, Serie II, Suppl. Vol. 75 (2005), 139-196.
[27] Robert S. Strichartz, Sub-Riemannian geometry, J. Differential Geometry, Vol. 24 (1986), 221-263.
[28] K.-H. Tan, A notion of convex functions of sub-Riemannian manifolds, 2006, submitted.
[29] K.-H. Tan, X.-P. Yang, On some sub-Riemannian objects of hypersurfaces of sub-Riemannian manifolds, Bull. Austral. Math. Soc., Vol. 70 (2004), 177-198.
[30] S. Tanno, Variational problems on contact Riemannian manifolds, Trans. Amer. Math. Soc., Vol. 314 (1989), 349-379.
[31] M. Taylor, Some aspects of differential geometry associated with hypoelliptic second order operators. Pac. J. Math. Vol. 136 (1989), 355-378.
[32] M. Taylor, Off diagonal asymptotics of hypoelliptic diffusion equations and singular Riemannian geometry. Pac. J. Math. Vol. 136 (1989), 379-399.
[33] N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon, Analysis and geometry on groups. Cambridge University Press, New York, 1992.
[34] S. Webster, Pseudo-Hermitian structures on a real hypersurface, J. Differential Geom., Vol. 13 (1978), 25-41.

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