A Fast Certificate for Power System Small-Signal Stability

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Abstract—Swing equations are an integral part of a large class of power system dynamical models used in rotor angle stability assessment. Despite intensive studies, some fundamental properties of lossy swing equations are still not fully understood. In this paper, we develop a sufficient condition for certifying the stability of equilibrium points (EPs) of these equations, and illustrate the effects of damping, inertia, and network topology on the stability properties of such EPs. The proposed certificate is suitable for real-time monitoring and fast stability assessment, as it is purely algebraic and can be evaluated in a parallel manner. Moreover, we provide a novel approach to quantitatively measure the degree of stability in power grids using the proposed certificate. Extensive computational experiments are conducted, demonstrating the practicality and effectiveness of the proposal.

I. INTRODUCTION

Power system stability has been an important topic in power engineering for many years. There has been continuing advancement in the understanding of the stability issues of the system. In the recent decade, the proliferation of renewable energy resources has added new dimensions to the problem. The uncertainty and volatility of these resources have brought about significant stochastic transitions from one operating point to another [1], thereby making the system more prone to instability.

Owing to the complexity and high dimensionality of power systems, several CIGRE and IEEE Task Forces have classified power system stability into appropriate categories with the aim of facilitating the assessment of the problem [2]. In each category, a set of simplifying assumptions are made and an appropriate system model with a reasonable level of details is adopted. One of the most fundamental models used in several categories of stability (especially rotor angle stability) is the swing equation model. This model describes the nonlinear relation between the power output and voltage angles of synchronous generators and can be used to analyze the short term dynamical behaviour of the system.

The application of swing equations is not restricted to the characterization of interconnected synchronous machines. They can also be used to model the behavior of inverter-based resources, which can be controlled to emulate the behavior of synchronous machines [3]. Despite such a wide range of applications, some basic questions on the equilibrium points (EPs) of swing equations are not fully understood. In particular,

(i) Under what conditions an EP of swing equations with nontrivial transfer conductance is asymptotically stable?

(ii) What is the relation between the network structure of a power system and the stability of the EPs of swing equations?

Such challenging questions have perplexed many researchers over the years, and some parts of the puzzle have been solved. For instance, the EPs of swing equations with zero transfer conductance (the so-called lossless model) have been studied in the 1980s (see e.g. Chiang et al. [4] and Zaborszky et al. [5]). They assume that there is a unique stable EP and a finite number of unstable EPs in any $2\pi$ interval of generator angle coordinate. It is shown that the stability boundary of a stable EP consists of the stable manifolds of all the EPs (and/or closed orbits) on the stability boundary. Moreover, various methods in the broad category of the so-called direct methods have been developed to estimate the region of attraction of EPs [6], [7]. These methods not only avoid expensive time-domain integration of swing equations, but also provide a quantitative measure of the degree of stability. Unfortunately, the existing methods are mostly limited to lossless systems and require a significant computational effort. More recently, the authors in [8] have alleviated some of these drawbacks.

The characteristics of swing equations with nontrivial transfer conductance (the so-called lossy model) are more challenging to analyze. This is partly due to the fact that there is no global energy function for such systems [9], and therefore, some main approaches (e.g., the energy function method) to investigate these equations cannot be directly applied. Nonetheless, several approaches are devised over the years. For instance, reference [10] computes numerical energy functions to deal with the effects of transfer conductances on the system behavior. In [11], the authors extend the lossy swing equation model by considering the dynamics of the excitation system and ensure the asymptotic stability of the operating points by designing a nonlinear feedback control for the generator excitation field. In [12], the local stability of swing equations with nontrivial transfer conductance is examined by linearization and conditions for stability of EPs are established. It is found that undamped swing equations can be stable only under very special circumstances. Another set of literature that address similar questions are the recent studies of the synchronization of Kuramoto oscillators that are applicable to the stability analysis of lossy swing equations with strongly overdamped generators [13]. Furthermore, exploring question (ii), the
recent work [14] statistically studies the impact of topology of the network on transient stability.

In this paper, we aim to address questions (i) and (ii), and provide a rigorous analysis of the stability of EPs in lossy swing equation models. There are two main contributions in the present paper.

- We characterize the relationship between the Jacobian of swing equations and the underlying graph of power grids. Specifically, we associate a weighted graph with the swing equation model and then mathematically describe the relationship between the spectrum of the graph Laplacian and the spectrum of the swing equation Jacobian.
- We develop a sufficient condition under which the EPs of lossy swing equations are stable. In addition to providing new insights into the theory of stability, the derived conditions are easy to check, use only local information, and are suitable for real-time monitoring and fast stability assessment. The proposed stability certificate can be interpreted as enforcing an upper bound on the matrix norm of the Laplacian of the underlying graph of the system. We show that the aforementioned upper bound is proportional to the square of damping and inverse of inertia at each node of the power grid. These results provide new insights into the way the damping and inertia at each node of the system would affect the stability of EPs. We also illustrate how the proposed condition provides a quantitative measure of the degree of stability in power systems.

The rest of our paper is organized as follows. Section II provides a brief background on dynamical systems and swing equations. In Section III, the swing equation model is linearized and the linkage between the Jacobian of swing equations and the underlying graph of the power grid is established. Section IV is devoted to the main results on the stability of the swing equation EPs. Section V further illustrates the developed analytical results through numerical examples, and finally, the paper concludes with Section VI.

II. BACKGROUND

A. Notations

We use \( \mathbb{C}_- \) to denote the set of complex numbers with negative real part, and \( \mathbb{C}_0 \) to denote the set of complex numbers with zero real part. \( j = \sqrt{-1} \) is the imaginary unit, which should not be confused with the subscript \( j \) that is used as an index. The spectrum of a matrix \( A \in \mathbb{R}^{n \times n} \) is denoted by \( \sigma(A) \).

B. Autonomous Ordinary Differential Equations

Suppose \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a smooth vector field, where the term smooth here means continuously differentiable. An autonomous ordinary differential equation (ODE) is an equation of the form

\[
\dot{x} = f(x),
\]

where the dot denotes differentiation with respect to the independent variable \( t \) (here a measure of time), and the dependent variable \( x \) is a vector of state variables. If \( f(x_0) = 0 \) for some \( x_0 \in \mathbb{R}^n \), then \( x_0 \) is called an EP. Let us define the function \( \phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) as follows: For any \( x \in \mathbb{R}^n \), let \( t \mapsto \phi(t,x) \) be the solution of the ODE (1), that is, \( \frac{d\phi}{dt}(t,x) = f(\phi(t,x)), \forall t \in \mathbb{R} \). Moreover, \( \phi(0,x) = x \). Now, an EP \( x_0 \) of the ODE (1) is

- stable (in the sense of Lyapunov) if for each \( \epsilon > 0 \), there exits a number \( \xi > 0 \) such that \( ||\phi(t,x) - x_0|| < \epsilon, \forall t \geq 0 \) whenever \( ||x - x_0|| < \xi \);
- unstable if it not stable;
- asymptotically stable if it is stable and \( \xi \) can be chosen such that \( \lim_{t \to \infty} ||\phi(t,x) - x_0|| = 0 \) whenever \( ||x - x_0|| < \xi \).

Recall that if \( x_0 \) is an EP for the ODE (1) and if all eigenvalues of the linear transformation \( \nabla f(x_0) \) have negative real parts, then \( x_0 \) is asymptotically stable.

C. Multi-Machine Swing Equations

Consider a power system with the set of generators \( \mathcal{N} = \{1, \ldots, n\}, n \in \mathbb{N} \). Based on the classical small-signal stability assumptions [15], the mathematical model for a power system is described by the following system of nonlinear autonomous ODEs, aka swing equations:

\[
\begin{align*}
\dot{\delta}_i(t) &= \omega_i(t), \quad \forall i \in \mathcal{N}, \\
\frac{M_i}{\omega_s} \dot{\omega}_i(t) + \frac{D_i}{\omega_s} \omega_i(t) &= P_{m_i} - P_{e_i}(\delta(t)), \quad \forall i \in \mathcal{N},
\end{align*}
\]

where for each generator \( i \in \mathcal{N} \), \( P_{m_i} \) and \( P_{e_i} \) are respectively the mechanical and electrical power in per unit, \( M_i \) is the inertia constant in seconds, \( D_i \) is the unitless damping coefficient, \( \omega_s \) is the synchronous angular velocity in electrical radians per seconds, \( t \) is the time in seconds, \( \delta_i(t) \) is the rotor angle in radians, and \( \omega_i(t) \) is the deviation of the rotor angular velocity from the synchronous velocity in electrical radians per seconds. Henceforth we do not explicitly write the dependence of the state variables \( \delta \) and \( \omega \) on time \( t \). The electrical power \( P_{e_i} \) in (2b) is given by:

\[
P_{e_i}(\delta) = \sum_{j=1}^{n} V_i V_j Y_{ij} \cos(\theta_{ij} - \delta_i - \delta_j),
\]

where \( V_i \) is the terminal voltage magnitude of generator \( i \), and \( Y_{ij} \cos(\theta_{ij}) \) is the entry of the reduced admittance matrix.

Definition 1 (flow function): The smooth function \( P_e : \mathbb{R}^n \to \mathbb{R}^n \) given by \( \delta \mapsto P_e(\delta) \) in (3) is called the flow function. Since the flow function is smooth, there exists a unique solution to the swing equation (2). The flow function is invariant to the translation of \( \delta \mapsto \delta + \alpha 1 \), where \( \alpha \in \mathbb{R} \) and \( 1 \in \mathbb{R}^n \) is the vector of all ones, i.e., \( P_e(\delta + \alpha 1) = P_e(\delta) \). A common way to deal with this situation is to define a reference bus and refer all other bus angles to it. This is equivalent to projecting the original state space onto a lower dimensional space.
III. LINEARIZATION AND SPECTRUM OF JACOBIAN

A. Linearization

The Jacobian of the vector field in (2) is given by

$$J := \begin{bmatrix} 0 & I \\ -M^{-1}L & -M^{-1}D \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \quad (4)$$

where $I \in \mathbb{R}^{n \times n}$ is the identity matrix, $M = \frac{1}{\omega} \text{diag}(M_1, \ldots, M_n)$, and $D = \frac{1}{\omega} \text{diag}(D_1, \ldots, D_n)$. Moreover, $L \in \mathbb{R}^{n \times n}$ is the Jacobian of the flow function with the diagonal entries:

$$\frac{\partial P_{ei}}{\partial \delta_i} = \sum_{j \neq i} V_i V_j Y_{ij} \sin (\theta_{ij} - \delta_i + \delta_j), \forall i \in \mathcal{N},$$

and off-diagonal entries

$$\frac{\partial P_{ei}}{\partial \delta_j} = -V_i V_j Y_{ij} \sin (\theta_{ij} - \delta_i + \delta_j), \forall i, j \in \mathcal{N}, i \neq j.$$

In the following subsection, we study the role of matrix $L$ in the spectrum of the Jacobian matrix $J$.

B. Spectral Relationship Between Matrices $J$ and $L$

We establish the spectral relationship between $J$ and $L$ via a singularity constraint. Let us first define the concept of a quadratic matrix pencil $[16]$. Consider $n \times n$ real matrices $Q_0$, $Q_1$, and $Q_2$. A quadratic matrix pencil is a matrix-valued function $P : \mathbb{C} \to \mathbb{R}^{n \times n}$ given by $\lambda \mapsto P(\lambda)$ such that $P(\lambda) = \lambda^2 Q_2 + \lambda Q_1 + Q_0$.

Lemma 1: $\lambda$ is an eigenvalue of $J$ if and only if the quadratic matrix pencil $P(\lambda) := \lambda^2 M + \lambda D + L$ is singular.

Proof: Let $\lambda$ be an eigenvalue of $J$ and $[v_1, v_2]$ be a corresponding eigenvector. Then

$$\begin{bmatrix} 0 & I \\ -M^{-1}L & -M^{-1}D \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \quad (5)$$

Thus, $v_2 = \lambda v_1$ and $M^{-1}L + (M^{-1}D + \lambda I)v_1 = 0$, which implies

$$(L + \lambda D + \lambda^2 M)v_1 = 0. \quad (6)$$

Since the eigenvector $v$ is nonzero, we have $v_1 \neq 0$ (otherwise $v_2 = \lambda \times 0 = 0 \implies v = 0$). Equation (6) implies that the matrix pencil $P(\lambda) = \lambda^2 M + \lambda D + L$ is singular. Conversely, suppose there exists $\lambda \in \mathbb{C}$ such that $P(\lambda) = \lambda^2 M + \lambda D + L$ is singular. Choose a nonzero $v_1 \in \ker(P(\lambda))$ and let $v_2 := \lambda v_1$. Accordingly, the characteristic equation (5) holds, and consequently, $\lambda$ is an eigenvalue of $J$.

Lemma 1 illustrates the role of matrix $L$ in the stability of the EPs. We note that [19, Proposition 5.14] presents similar results under symmetry assumptions, while matrices $L$, $M$, and $D$ here in Lemma 1 are not necessarily symmetric. Next, we look more closely at the spectrum of $L$.

C. Graph Induced by $L$ and Its Spectral Properties

The Jacobian $L$ of the flow function encodes the graph structure of the power network. To see this, we can define a weighted directed graph $G = (\mathcal{N}, \mathcal{A}, W)$ where each node $i \in \mathcal{N}$ corresponds to a generator and each directed arc $(i, j) \in \mathcal{A}$ corresponds to the entry $(i, j) \in \mathcal{A}$ of the Jacobian matrix $L$. We further define a weight for each arc $(i, j) \in \mathcal{A}$:

$$w_{ij} = V_i V_j Y_{ij} \sin (\varphi_{ij}), \quad \forall (i, j) \in \mathcal{A}, \quad (7)$$

where $\varphi_{ij} := \theta_{ij} - \delta_i + \delta_j$. With the above definitions, we can see that the Jacobian matrix $L$ of the flow function in (4) is indeed the Laplacian of the directed graph $G$ defined as $L = D^+ (G) - A(G)$, where $D^+ (G)$ is a diagonal matrix with the $i$-th diagonal entry being the sum of all the weights of the out-going arcs incident to node $i$, and $A(G)$ is the adjacency matrix of $G$.

In general, the arc weights $w_{ij}$ can be positive or negative, and matrix $L$ is not necessarily symmetric. In practice, however, $w_{ij}$ varies in a small positive range. Fig. 1 illustrates the histogram of the angle $\varphi_{ij}$ for all $(i, j)$ in different reduced IEEE standard test cases.

![Figure 1: Histogram of the distribution of $\varphi_{ij}$ for all $(i, j)$ in different reduced IEEE standard test cases.](image)

In the stability of the EPs, we have $\partial P_{ei}/\partial \delta_i \geq 0, \forall i \in \mathcal{N}$ and $\partial P_{ei}/\partial \delta_j \leq 0, \forall (i, j) \in \mathcal{A}, i \neq j$. Since $L$ has zero row sum, we have $L \mathbf{1} = 0 \implies 0 \in \sigma(L)$. Furthermore, the sum of
the absolute values of the nondiagonal entries in the $i$-th row of $L$ is equal to $|L|_{ij}$, that is

$$[L]_{ii} = \sum_{j \neq i} |[L]_{ij}| \quad \forall i \in \mathcal{N}. \quad (8)$$

Let $\mathbb{D}([L]_{ii})$ be a closed disc centered at $[L]_{ii}$ with radius $|L|_{ii}$. According to the Gershgorin circle theorem, every eigenvalue of $L$ lies within at least one of the Gershgorin discs $\mathbb{D}([L]_{ii})$, which are located on the right half plane. This shows (iii). The equivalence of (iii) with (i) and (ii) is a fundamental property of M-matrices [18].

We will use property (iii) of matrix $L$ shown in the above proposition later to prove our main result in the next section.

IV. SUFFICIENT CONDITION FOR THE STABILITY OF SWING EQUATIONS: A FAST CERTIFICATE

In this section, we present our main result on the stability of the swing equation EPs.

**Theorem 1:** Let $[\delta^*, \omega^*] \in \Omega$ be an EP of swing equations (2). Suppose all generators have positive damping coefficient and inertia, and the underlying undirected graph of the power grid is connected. If condition

$$\sum_{j \neq i} V_i V_j Y_{ij} \sin(\theta_{ij} - \delta_i^* + \delta_j^*) \leq \frac{D_i^2}{2M_i}, \forall i \in \mathcal{N} \quad \text{(C)}$$

holds, then the EP is asymptotically stable.

The proof of Theorem 1 is given in Appendix I.

**Remark 1:** Condition (C) provides a practical and efficient way to certify the small-signal stability of the EPs. The left-hand side of condition (C) is closely related to the reactive power output of a generator. Note that at an EP the reactive power injected from bus $i$ into the network is $Q_i = -\sum_{j=1}^n V_i V_j Y_{ij} \sin(\theta_{ij} - \delta_i^* + \delta_j^*)$. Intuitively, when a generator is supplying more reactive power, the left-hand side of condition (C) decreases, and this helps make condition (C) satisfied.

It is worth mentioning that in [20], small-signal stability of lossless swing equations is studied. It is shown that if $[\delta^*, \omega^*] \in \Omega$ is an EP, then the EP is locally asymptotically stable. Theorem 1 is a generalization of such results to lossy swing equations. Contrary to the lossless case, we will show in the next section that an EP in lossy networks could be unstable even if it belongs to the set $\Omega$.

V. SIMULATION RESULTS

In this section, we test the practicality of the assumptions on which Theorem 1 is based. We also show how conservative condition (C) is, and how it can be used not only as a fast stability certificate, but also as a quantitative measure of the degree of stability.

Table 1 provides the details of testing Theorem 1 and condition (C) on different IEEE standard test systems [17]. All these systems have a connected underlying graph and nonzero transfer conductances. The second column of Table 1 shows the domain of $\varphi_{ij}$ in these test cases. Recall that $\varphi_{ij} = \theta_{ij} - \delta_i + \delta_j$ is the argument of the sin function, and having $\varphi_{ij} \in (0, \pi)$ ensures that an EP $[\delta^*, \omega^*]$ belongs to the set $\Omega$. As can be seen, this property holds in all test cases of Table 1, and therefore, the assumptions of Theorem 1 hold in a wide variety of practical power systems.

Next, let us define

$$S_i := \sum_{j \neq i} V_i V_j Y_{ij} \sin(\theta_{ij} - \delta_i^* + \delta_j^*) - \frac{D_i^2}{2M_i},$$

and recall that according to condition (C) in Theorem 1, if $S_i \leq 0, \forall i \in \mathcal{N}$, then the EP of swing equations is asymptotically stable. The third column of Table 1 provides the domain of $S_i$, i.e., $[\min S_i, \max S_i]$. Accordingly, $S_i \leq 0$ holds for all test cases, except the IEEE 89-bus system. Note that the corresponding EPs in these systems are all stable. While the evaluation of condition (C) confirms the stability of EPs in all other cases, it gives an inconclusive answer in the IEEE 89-bus case. However, here we show how condition (C) can be used as a quantitative measure of the degree of stability. The positive values of $S_i$ in the IEEE 89-bus system pertain to the bus numbers 6233, 6798, 7960, and 9239, indicating that the stability of the system can be improved by making $S_i$ negative in these buses via appropriate corrective actions. Exploring the structure of the system reveals that each of these buses is connected to the rest of the grid through a line with a relatively small resistance. As a corrective action, we change these resistances as follows: $r(659, 9239) = 6 \times 10^{-5} \rightarrow 0.5 \times 10^{-3}$, $r(659, 7960) = 6 \times 10^{-5} \rightarrow 1 \times 10^{-3}$, $r(659, 6233) = 6 \times 10^{-5} \rightarrow 2 \times 10^{-3}$, and $r(659, 6798) = 7 \times 10^{-5} \rightarrow 1.5 \times 10^{-3}$, where all the values are in p.u. With this corrective action (which can be implemented through flexible AC transmission system (FACTS) devices), we will have $S_i \leq 0, \forall i \in \mathcal{N}$ and condition (C) will hold true, certifying the stability of the system (see the test case IEEE 89-bus mod. in Table 1).

Fig. 2 depicts the spectrum of $J$ in the IEEE 89-bus system before and after implementing the corrective actions. As can be seen, the magnitude of the imaginary parts of the eigenvalues in $\sigma(J)$ is reduced, and their real parts are mainly moved towards $-\infty$, thereby making the modified system less oscillatory. Evidently, condition (C) increased the stability margins of the system. Finally, $\lambda_2 \in \sigma(J)$ denotes the closest nonzero eigenvalue of $J$ to the imaginary axis, and the fourth column of Table 1 depicts this value in different cases. Note that the proposed stability certificate can be fully parallelized, thereby making it even more reliable and resilient for real-time applications.

| Test case     | $\text{Dom}(\varphi_{ij})$ | $\text{Dom}(S_i)$ | $|\mathbb{N}(\lambda_2)|$ |
|---------------|-----------------------------|-------------------|--------------------------|
| IEEE 9-bus    | [0.48, 0.52]                | $[-0.79, -0.22]$  | 3.18                     |
| IEEE 14-bus   | [0.43, 0.66]                | $[-5.08, -0.03]$  | 2.17                     |
| IEEE 30-bus   | [0.36, 0.66]                | $[-12.26, -0.51]$ | 0.75                     |
| IEEE 39-bus   | [0.37, 0.62]                | $[-7.73, -0.12]$  | 4.96                     |
| IEEE 89-bus   | [0.45, 0.59]                | $[-143.75, 1166.9]$ | 4.15                 |
| IEEE 89-bus mod. | [0.25, 0.97]          | $[-280.19, -0.49]$ | 4.14                 |
| IEEE 113-bus  | [0.42, 0.66]                | $[-241.75, -0.21]$ | 0.11                    |
| IEEE 900-bus  | [0.59, 0.72]                | $[-260.90, -3.04]$ | 0.16                    |
TABLE II: Dynamic parameters and converged load flow data of the 3-bus test system.

| i | $M_i$ [sec.] | $D_i$ | $P_m$ [p.u.] | $V_i$ [p.u.] | $\delta_i$ [rad] | $S_i$ |
|---|-------------|------|-------------|-------------|----------------|--------|
| 1 | 6.1         | 1.5  | 0.89        | 0.9         | −0.30         | 6.98   |
| 2 | 10          | 1    | 15.06       | 0.9         | 0.36          | 12.73  |
| 3 | 4.5         | 1.8  | 2.53        | 0.913       | −0.12         | 8.91   |

Next, we provide an example of an unstable EP and show how enforcing condition (C) will make the EP stable. Consider the 3-bus system in Fig. 3 whose dynamic parameters and converged load flow data are provided in Table II. As can be observed from the last column of Table II, we have $S_i \geq 0, \forall i \in \mathcal{N}$, i.e., condition (C) is violated in all buses of this system, indicating that the system does not have sufficient stability margins. The instability of this EP can be verified through eigenvalue analysis and time domain simulation, as depicted in Fig. 4. In order to achieve stability, the power system operator can enforce condition (C) either by moving the current EP to a new point (e.g., through adding constraint (C) to the optimal power flow problem) or by making the current EP stable through adjusting the right-hand side of condition (C). Particularly, the latter is possible if we have inverter-based resources where the inertia $M_i$ and damping $D_i$ are adjustable parameters of their controllers. In this case, by setting $M = \text{diag}(0.9, 0.9, 0.9)$ and $D = \text{diag}(4.5, 4.9, 4.8)$, we would have $S_1 = −4.08, S_2 = −0.55,$ and $S_3 = −3.52,$ thereby certifying the stability of the system.

We conclude our numerical experiments by further illustrating the effect of condition (C) on the spectrum of matrix $J$. We have varied the operating point and parameters (inertia and damping) of the IEEE 9-bus system, and for each operating point or parameter value we have recorded $\lambda_2$ as well as $\min_i S_i$. Fig. 5 shows the relationship between $\lambda_2$ and $\min_i S_i$ as the system operating point and parameters change. Accordingly, a smaller $\min_i S_i$ yields a farther $\lambda_2$ from the imaginary axis.

VI. CONCLUSIONS AND OUTLOOK

This paper is aimed at finding a computationally efficient way to certify the stability of power system EPs. We have shown if the matrix norm of the Laplacian of the underlying graph is upper bounded by a specific value, then the EP is stable. The aforementioned upper bound is proportional to the square of damping and inverse of inertia at each node of the power grid. This fact also sheds light on the interplay of inertia, damping, and graph of the system, and provides profound insights into how power system should be designed and operated to be stable. A worthwhile direction for future research would be extending condition (C) as a function of network connectivity measure.

APPENDIX I

PROOF OF THEOREM 1

We complete the proof in three steps:

Step 1: First, we show that the zero eigenvalue of $J$ is simple. According to Proposition 1, if $[\delta^*, \omega^*] \in \Omega$, the Jacobian matrix $L$ is a singular M-matrix, and consequently, it has at least one zero eigenvalue. Consider the weighted directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{A}, W)$ constructed in the beginning of Section III-C. If $[\delta^*, \omega^*] \in \Omega$, the arc weights $w_{ij}$ are positive for all arcs $(i, j) \in \mathcal{A}$. Moreover, there are two arcs $(i, j)$ and $(j, i)$ between nodes $i$ and $j$ if and only if the two nodes are connected in the underlying undirected graph of the power grid. Therefore, if the underlying undirected graph of the power grid is connected, then the directed graph $\mathcal{G}$ is strongly connected. Now, we need the following lemma from graph theory to complete the proof: consider a weighted directed graph $\mathcal{G}$ with positive weights. If $\mathcal{G}$ is strongly connected, then the zero eigenvalue of its Laplacian is simple (see [19] and references therein). Note that the geometric multiplicity of the zero eigenvalue in $\sigma(J)$ and $\sigma(L)$ are equal.

Step 2: Next, we show all the nonzero real eigenvalues of $J$ are negative. Let $\lambda \in \mathbb{R}$ be an eigenvalue of $J$, then according to Lemma 1,

$$\det \left( L + \lambda D + \lambda^2 M \right) = 0.$$  \hspace{1cm} (9)

Consider the Gershgorin disk $D_i$, centered at $c_i := [L]_{ii} + \lambda D_i + \lambda^2 M_i$ with radius $r_i := |[L]_{ii} + \sum_{j \neq i} |[L]_{ij}|$, Fig. 4: Instability of the EP in the 3-bus test system. (a) There exist two eigenvalues with positive real part. (b) Starting from a neighborhood of the EP, the trajectories become unbounded.
According to our assumption in condition (C), we have $(2[J]_{kk}M_k - D_k^2) \leq 0$, thus the left-hand side of the inequality (13) is nonpositive. If $\alpha \geq 0$ and $\beta \neq 0$, the right-hand side of (13) would be positive, which is the desired contradiction. The idea used in this part of the proof was inspired by Skar [12]. Note that the simple zero eigenvalue of the Jacobian matrix $J$ stems from the translational invariance of the flow function (3). As mentioned earlier, we can eliminate this eigenvalue by choosing a reference bus and refer all other bus angles to it. Therefore, the set of EPs $\{\delta + \alpha \lambda : \alpha \in \mathbb{R}\}$ will collapse into one EP. Such an EP will be asymptotically stable.

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