FRW Universe in Hořava Gravity

Nairwita Mazumder* , Subenoy Chakraborty†.

1Department of Mathematics, Jadavpur University, Kolkata-32, India.

(Dated: March 9, 2010)

Recently, a field theoretic model for a UV complete theory of gravity has been proposed by Hořava. This theory is a non-relativistic renormalizable gravity theory which coincides with Einstein’s general relativity at large distances. Subsequently Lü et al have formulated the modified Friedmann equations and have presented a solution in vacuum. In the present work, we rewrite the modified FRW equations in the form of usual FRW equations in Einstein gravity and consequences has been analyzed. Also the thermodynamics of the FRW universe has been studied.

Key words: FRW Universe, Hořava Gravity, Thermodynamics.

PACS numbers: 98.80.Cq, 98.80.-k, 04.60.-m

I. INTRODUCTION

Recently, a renormalizable theory of gravity was proposed by Hořava [1,2]. As this theory follows Lifshitz-type anisotropic scaling so it is commonly known as Hořava-Lifshitz (HL) gravity. This theory of gravitation has four possible versions so far - with/without the detailed balance condition and with/without projectability condition. Among these version without detailed balance and with the the projectability condition is the most viable one.

Due to detailed balance condition the potential in the 4D Lorentzian action has a specific form in terms of a 3D Euclidean theory and it leads to obstacles from cosmological view point. The projectability condition on the other hand, is due to the foliation preserving diffeomorphism invariance - the fundamental symmetry of the theory. This new theory of gravity do not considered the space and time on an equal footing- only the general covariance is retained at large distance and coincides with general relativity. In fact, it is a non-relativistic renormalizable field theory model for gravity having UV completeness. Using parameterized form [2] of the four dimensional metric as

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt)$$ (1)

and using ADM formalism, the Einstein-Hilbert action has the form

$$S_{EH} = \frac{1}{16\pi G} \int d^4x \sqrt{g} N (k_{ij} k^{ij} - k^2 + R - 2\Lambda),$$ (2)

where $G$ is the Newton’s constant and $k_{ij}$, the extrinsic curvature of a space-like hyper surface with a fixed time has the expression

$$k_{ij} = \frac{1}{2N} [g_{ij} - (\nabla_i N_j)(\nabla_j N_i)]$$ (3)

Here a over dot denotes derivative with respect to ‘t’ and covariant derivatives are defined with respect to the spatial metric $g_{ij}$.

The action of the proposed non-relativistic generalized theory by Hořava [1] has the expression (known as Hořava-Lifshitz (HL) action)

$$S_{HL} = \int dt d^3x \sqrt{g} N (L_0 + L_1)$$ (4)

* nairwita15@gmail.com
† schakraborty@math.jdvu.ac.in
with

\[ L_0 = \left[ \frac{2}{k^2} (k_{ij} k^{ij} - \lambda k^2) + \frac{k^2 \mu^2 (\Lambda R - 3 \Lambda^2)}{8(1 - 3\lambda)} \right] \]

and

\[ L_1 = \left[ \frac{k^2 \mu^2 (1 - 4\lambda) R^2}{32(1 - 3\lambda)} - \frac{k^2 z_{ij} z^{ij}}{2\omega^4} \right] \tag{5} \]

Here

\[ z_{ij} = c_{ij} - \frac{\mu \omega^2}{2} R_{ij} \]

\[ c^{ij} = \epsilon^{ikl} \nabla_k (R_{jl} - \frac{1}{4} \delta^i_j) = \epsilon^{ikl} \nabla_k R_{jl} - \frac{1}{4} \epsilon^{i[k} \partial_{l]} R ; \tag{6} \]

is known as Cotten tensor and \( k^2, \lambda, \mu, \omega, \Lambda \) are constant parameters.

In the above HL-action (4) the first two terms are kinetic terms and the rest correspond to potential of the theory (in 'detailed balance' form). A comparative study with general relativity shows that the speed of light, Newton’s constant and the cosmological constant have the expressions

\[ c = \frac{k^2 \mu}{4} \sqrt{\frac{\Lambda}{1 - 3\lambda}} \quad \text{and} \quad G = \frac{k^2 c}{32\pi} = \frac{3}{2}\Lambda . \tag{7} \]

In the present theory [1] \( \lambda \) is a dynamical coupling constant, subject to quantum correction. For \( \lambda = 1 \) the first three terms in the action (4) are the usual one’s in Einstein’s general relativity. Also from the equation (7) the expression for the velocity of light shows that \( \Lambda \) should be negative if \( \lambda > \frac{1}{3} \). Note that the HL-action (4) remains real [3] under the analytic continuation [4]

\[ \mu \rightarrow i\mu, \quad \omega^2 \rightarrow -i\omega^2, \tag{8} \]

so that one may choose \( \Lambda \) to be positive for \( \lambda > \frac{1}{3} \). The cosmological implications of the HL-actions has been studied in [5-11].

Now variation of the HL-action with respect to \( N, N^i \) and \( g_{ij} \) gives the equations of motion

\[ \frac{2}{k^2} (k_{ij} k^{ij} - \lambda k^2) - \frac{k^2 \mu^2 (1 - 4\lambda) R^2}{32(1 - 3\lambda)} + \frac{k^2 z_{ij} z^{ij}}{2\omega^4} - \frac{k^2 \mu^2 (\Lambda R - 3 \Lambda^2)}{8(1 - 3\lambda)} = 0 \tag{9} \]

\[ \nabla_l (k^{ls} - \lambda k g^{ls}) = 0 \tag{10} \]

and

\[ \frac{2}{k^2} E^{(1)}_{ij} - \frac{2\lambda}{k^2} E^{(2)}_{ij} + \frac{k^2 \mu^2 \Lambda}{8(1 - 3\lambda)} E^{(3)}_{ij} + \frac{k^2 \mu^2 (1 - 4\lambda)}{32(1 - 3\lambda)} E^{(4)}_{ij} - \frac{\mu k^2}{4\omega^2} E^{(5)}_{ij} - \frac{k^2}{2\omega^4} E^{(6)}_{ij} = 0 \tag{11} \]

where the tensors \( E^{(\alpha)}_{ij} \) (\( \alpha = 1, 2, \ldots, 6 \)) are combination of \( k_{ij}, g_{ij}, N, N_i \) and their covariant derivatives with respect to the three dimensional metric and detailed expressions can be found in [3].

After Hořava [1] developed the new gravity theory, within two-three months Lü et al [3] have obtained static, spherically symmetric solutions in HL gravity and have shown asymptotically \( AdS_4 \) solution for \( \lambda = 1 \). Also they have formulated the modified FRW equations and have obtained vacuum solution for the isotropic model.

In the present work, we rearrange the modified Friedmann equation so that it can be written in the usual Friedmann equations and interprets the extra terms from the point of view of cosmology. Also thermodynamics of the FRW universe in HL theory will be investigated.
II. FRIEDMANN EQUATIONS IN HL-GRAVITY

Immediately, after the proposal for the new gravity theory by Hořava [1], Lü et al [4] give cosmological solutions for this theory. At first they have solved the equations of motion for spherically symmetric space-time model and have shown the correspondence with $AdS_4$ asymptotically for $\lambda = 1$. They have also obtained a solution, deviating slightly from the detailed balance by changing the lagrangian in (4) as

$$L = L_0 + (1 - \epsilon^2) L_1.$$  

They have found that for $\epsilon \neq 0$ the metric has a finite mass which diverges in the detailed-balance limit ($\epsilon = 0$). Further, they have written down the Friedmann equations in the new gravity theory and have solved these Friedmann equations for vacuum case. For $k = +1$, they have obtained a bounce in the solution.

For the Friedmann-Lemaître-Robertson-Walker model of the space-time with line element

$$ds^2 = -c^2 dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right],$$

($k = 1, 0, -1$ correspond to a closed, flat or open universe) the non-vanishing equations of motion are [4]

$$\left( \frac{\dot{a}}{a} \right)^2 = \frac{2c^4}{3(\lambda - 1)} \left[ \frac{\Lambda}{2} + \frac{8\pi G \rho}{3} - \frac{k}{a^2} + \frac{k^2}{2\Lambda a^4} \right]$$  

and

$$\left( \frac{\ddot{a}}{a} \right) = \frac{2c^4}{3(\lambda - 1)} \left[ \frac{\Lambda}{2} - \frac{4\pi G (\rho + 3p)}{3} - \frac{k^2}{2\Lambda a^4} \right]$$  

or equivalently we write

$$H^2 = \frac{2c^4}{3(\lambda - 1)} \left[ \frac{\Lambda}{2} + \frac{\kappa_4^2 \rho}{3} - \frac{k}{a^2} + \frac{k^2}{2\Lambda a^4} \right]$$  

and

$$\dot{H} = \frac{2c^4}{3(\lambda - 1)} \left[ -\kappa_4^2 (\rho + p) + \frac{k}{a^2} - \frac{k^2}{\Lambda a^4} \right]$$

where $H = \frac{\dot{a}}{a}$, usual Hubble parameter, $\kappa_4^2 = 8\pi G$ and $\rho, p$ are respectively the thermodynamic energy density and pressure of the fluid in the universe. Note that for $k = 0$, there is no contribution from the higher order derivative terms in the action. However, for $k \neq 0$, these higher derivative terms are significant for small volume (i.e. small $a$) and become insignificant for large $a$, where it agrees with general relativity.

Now, choosing $\frac{2c^4}{3(\lambda - 1)} = 1$ the above Friedmann equations can be written as

$$H^2 + \frac{k}{a^2} = \frac{\kappa_4^2}{3} (\rho + \rho_{HL})$$  

and

$$\dot{H} = \frac{k}{a^2} = -\frac{\kappa_4^2}{2} (\rho + p + \rho_{HL} + p_{HL})$$

where
\[
\rho_{HL} = \frac{3}{\kappa^4} \left( \frac{\Lambda}{2} + \frac{k^2}{2 \Lambda a^4} \right)
\]

and

\[
p_{HL} = \frac{1}{\kappa^4} \left( -\frac{3 \Lambda}{2} + \frac{k^2}{2 \Lambda a^4} \right)
\]

One may note that, the equations (16) and (17) are same as the usual Friedmann equations in Einstein gravity having two fluid system - one the usual fluid present in the universe and the other may be interpreted as the effect of the HL gravity. If we write \( \rho_t = \rho + \rho_{HL} \) and \( p_t = p + p_{HL} \), then from equations (16) and (17) the conservation equation will be

\[
\dot{\rho}_t + 3H(\rho_t + p_t) = 0
\]

Now if we assume the energy conservation for the ordinary matter i.e.

\[
\dot{\rho} + 3H(\rho + p) = 0
\]

then combining (20) and (21) and using the definition of \( \rho_t \) and \( p_t \) we have

\[
\rho_{HL} \dot{3} H(\rho_{HL} + p_{HL}) = 0
\]

Hence we may say that the apparent two fluid system are non-interacting. Therefore we may conclude that in the present cosmological setting gravity in HL theory may be considered as the Einstein gravity with two non-interacting fluid system.

We now study the induced fluid systems due to HL gravity. If the cosmological constant is positive then \( \rho_{HL} > 0 \) throughout the evolution while \( p_{HL} \) is initially positive and becomes negative at \( a^2 = \frac{|k|}{\sqrt{3} \Lambda} \). On the other hand, for negative cosmological constant \( \rho_{HL} \) is always negative while \( p_{HL} \) starts with negative value but becomes positive when \( a^2 = \frac{|k|}{\sqrt{3} |\Lambda|} \). However, for \( k = 0 \), it behaves as a cosmological constant. Finally, for large \( 'a' \) whatever be the choice of \( k \) the effect of HL gravity reduces to a cosmological constant i.e. HL-gravity becomes Einstein gravity with a cosmological constant at large \( 'a' \).

### III. THERMODYNAMICS OF FRW UNIVERSE IN HL-GRAVITY

It is well known in the literature that the laws of thermodynamics are valid for the universe bounded by the apparent horizon. This is true not only in Einstein gravity [12-14] but also in higher derivative Lovelock theory [15] of gravity. As the present HL gravity theory is shown to be the generalization of Einstein gravity by including an effective matter term to the original matter so it is expected that laws of thermodynamics will be valid on the apparent horizon. In this section we shall examine the validity of the generalized second law of thermodynamics assuming the first law of thermodynamics on the event horizon. Also the matter in the universe is chosen as the holographic dark energy. Form the principle of the holographic dark energy [16] model the matter density of the holographic dark energy component can be written as [16]

\[
\rho_D = 3c^2 R_E^{-2}
\]

where \( c \) is any arbitrary parameter. Now the form of the equations of motion are the following:

\[
H^2 + \frac{k}{a^2} = \frac{\kappa^2}{3}(\rho_D + \rho_{HL})
\]
\[ \dot{H} - \frac{k}{a^2} = -\frac{\kappa^2}{2}(\rho_D + p_D + \rho_{HL} + p_{HL}) \]  

(25)

Using the definition of event horizon

\[ R_E = a \int_a^\infty \frac{da}{Ha^2} = \frac{c}{(\sqrt{\Omega_D})H} \]  

(26)

where \( \Omega_D = \frac{\rho_D}{M_D} \) is the density parameter corresponding to dark energy. The equation of state for the dark energy can be written as

\[ \rho_D = \omega_D p_D \]  

(27)

with \( \omega_D \) is not necessarily a constant.

The amount of energy crossing the event horizon in time \( dt \) is given by the expression [15]

\[ -dE = 4\pi R_E^3 H(\rho_t + p_t)dt \]  

(28)

Thus assuming the validity of the first law of thermodynamics the time variation of the horizon entropy is given by

\[ \frac{dS_E}{dt} = \frac{4\pi R_E^3 H}{T_E}(\rho_t + p_t) \]  

(29)

where \( S_E \) and \( T_E \) are respectively the entropy and temperature of the event horizon.

To determine the time variation of the entropy of the matter inside the event horizon we use the Gibb’s equation [17]

\[ T_E dS_I = dE_I + p_I dV \]  

(30)

where \( S_I \) and \( E_I \) are the entropy and energy of the matter inside the event horizon. Note that due to thermodynamical equilibrium we choose the temperature of the matter distribution is same as that of the boundary surface (the event horizon).

Using

\[ E_I = \frac{4}{3} \pi R_E^3 \rho_t \quad \text{and} \quad V = \frac{4}{3} \pi R_E^3 \]

in the Gibb’s equation and with the help of equations of motion (24) and (25) we have

\[ dS_I = \frac{4\pi R_E^2}{T_E}(\rho_t + p_t)dR_E + \frac{HR_E^3}{T_E}(\dot{H} - \frac{k}{a^2})dt \]  

(31)

To obtain \( (dR_E) \) we start with the expression of \( R_E \) in equation (26) and using the conservation equation (20) for holographic dark energy, we obtain

\[ dR_E = \frac{3}{2} R_E H(1 + \omega_D)dt \]  

(32)

Hence the time variation of the matter entropy is given by (after some simplification)
\[
\frac{dS_I}{dt} = \frac{2\pi R_E^3}{T_E} H (\rho_t + p_t) (3\omega_D + 1) \tag{33}
\]

Thus combining equations (29) and (33) the resulting change of total entropy is given by

\[
\frac{d}{dt}(S_I + S_E) = \frac{6\pi R_E^3 H}{T_E} (\rho_t + p_t)(\omega_D + 1) \tag{34}
\]

which gives the same form as in Einstein gravity [18]. But here the restrictions are different from that of the Einstein gravity.

Now using the deceleration parameter \( q = -1 - \frac{\dot{H}}{H^2} \) the above expression can be written as

\[
\frac{d}{dt}(S_I + S_E) = \frac{12\pi R_E^3 H}{T_E} \left[ (1 + q)H^2 + \frac{k}{a^2} \right] (\omega_D + 1) \tag{35}
\]

Before going to examine the validity of the second law (generalized) of thermodynamics we first write the explicit form of \((\rho_t + p_t)\) as follows:

\[
\rho_t + p_t = \rho_D + \frac{3}{\kappa_4^2} \left( \frac{\Lambda}{2} + \frac{k^2}{2\Lambda a^4} \right) + p_D + \frac{1}{\kappa_4^2} \left( \frac{3\Lambda}{2} + \frac{k^2}{2\Lambda a^4} \right) = \rho_D (1 + \omega_D) + \frac{2k^2}{\kappa_4^2 \Lambda a^4} \tag{36}
\]

The conclusions are the following:

**case I:** \( \Lambda > 0 \)

If the holographic dark energy satisfies the weak energy condition then the generalized second law of thermodynamics will always be satisfied as in Einstein gravity. However, if the dark energy does not obey the weak energy condition then the result is distinct from Einstein gravity: at very early stages of the evolution of the universe, HL term (i.e. the \( \Lambda \) term) in equation (36) dominates and there is violation of the second law of thermodynamics. But at later epoch the HL term becomes insignificant, the second law of thermodynamics will again be valid.

**case II:** \( \Lambda < 0 \)

We see from (36) that \( \Lambda \) term (i.e. HL term) may have a significant contribution at very early stages of the evolution of the universe. This means that if the holographic dark energy satisfies the weak energy condition, \( \rho_t + p_t \) may not be positive at very early stages of the evolution of the universe. Thus second law of thermodynamics may not be satisfied at early stages of the evolution of the universe even if the dark energy satisfies the weak energy condition. However at later stages of the evolution when HL-term becomes insignificant, \( \rho_t + p_t \) becomes positive then generalized second law is obeyed.

Further, if the weak energy condition is not satisfied by the dark energy (phantom in nature) then second law of thermodynamics will always be satisfied. Hence we may conclude that HL term has a significant effect for the validity of the second law of thermodynamics compare to Einstein gravity particularly when dark energy violates weak energy condition.

For future work, it will be interesting to examine the validity of the first law of thermodynamics at the event horizon.

**Acknowledgement:**

This paper has been carried out during a visit to IUCAA, Pune, India. The authors are thankful to IUCAA for warm hospitality and facility of doing research works.
References:

[1] P. Hořava , *Phys. Rev. D* **79** 084008 (2009);

[2] R. L. Arnowitt , S. Deser and C.W. Misner , The Dynamics of General Relativity, ”gravitation: an introduction to current research , L. Witten ed.” (Wiley 1962), Chapter 7, pp 227-265, arXiv: [gr-qc/ 0405109](http://arxiv.org/abs/gr-qc/0405109)

[3] G. Calcagni , arXiv: 0904.0829 [hep-th];

[4] H. Lü , J. Mei , and C.N. Pope , arXiv: 0904.1595 [hep-th];

[5] E. Kiritsis and G. Kofinas , arXiv: 0904.1334 [hep-th];

[6] P. Hořava , *JHEP* **0903** 020 (2009);

[7] P. Hořava , arXiv: 0902.3657 [hep-th];

[8] T. Takahasi and J. Soda , arXiv: 0904.0554 [hep-th];

[9] J. Kluson , arXiv: 0904.1343 [hep-th];

[10] R.G. Cai , L.M. Cao and N. Ohta , arXiv: 0904.3670 [hep-th];

[11] R.G. Cai , L.M. Cao and N. Ohta , arXiv: 0905.0751 [hep-th];

[12] B. Wang, Y. Gong, E. Abdalla, *Phys. Rev. D* **74** 083520 (2006).

[13] Y. Gong , B. Wang , A. Wang , *JCAP* **0701** 024 (2007);

[14] M.R. Setare , *JCAP* **01** 023 (2007).

[15] R. G. Cai and S. P. Kim, *JHEP* **02** 050 (2005).

[16] M. Li , *Phys. Lett. B* **603** 01 (2004);

[17] G. Izquierdo and D. Pavon, *Phys. Lett. B* **633** 420 (2006).

[18] N. Mazumder and S. Chakraborty, Accepted for publication in *Gen. Rel. Grav.* in September doi:10.1007/s10714-009-0881-z.