Fundamental study of subharmonic vibration of order 1/2 in automatic transmissions for cars

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Abstract. A torque converter is an element that transfers torque from the engine to the gear train in the automatic transmission of an automobile. The damper spring of the lock-up clutch in the torque converter is used to effectively absorb the torsional vibration caused by engine combustion. A damper with low stiffness reduces fluctuations in rotational speed but is difficult to use because of space limitations. In order to address this problem, the damper is designed using a piecewise-linear spring with three stiffness stages. However, the damper causes a nonlinear vibration referred to as a subharmonic vibration of order 1/2. In the subharmonic vibration, the frequency is half that of the vibrations from the engine. In order to clarify the mechanism of the subharmonic vibration, in the present study, experiments are conducted using the fundamental experimental apparatus of a single-degree-of-freedom system with two stiffness stages. In the experiments, countermeasures to reduce the subharmonic vibration by varying the conditions of the experiments are also performed. The results of the experiments are evaluated through numerical analysis using the shooting method. The experimental and analytical results were found to be in close agreement.

1. Introduction
In recent years, diesel and high-power engines have become widely used in automobiles. However, such engines generate strong torsional vibrations in the powertrain. In order to address this problem, the damper for the lock-up clutch must be designed to effectively absorb torsional vibrations. A damper with low stiffness reduces fluctuations in rotational speed caused by combustion in the engine combustion chamber. However, low-stiffness springs, which are applicable to a wide range of static torque, are difficult to use because of space limitations.

In order to address this problem, dampers have been designed using a piecewise-linear spring having three different stages of stiffness, so that the overall spring constant of the damper increases...
with the increase in the static torque. This type of spring can realize a wide range of restoring torque characteristics in a small space.

However, a nonlinear subharmonic vibration of order 1/2 occurs because of the nonlinearity of the piecewise-linear spring of the damper. In this vibration, the main frequency component is half the excitation frequency. The subharmonic vibration of order 1/2 occurs near a switching point in the piecewise-linear restoring torque. In previous studies, we analytically clarified the occurrence mechanism of the subharmonic vibration of order 1/2 in automatic transmissions of automobiles [1], [2]. A number of studies have examined nonlinear vibrations in mechanical systems that have piecewise-linear spring properties [3]-[10]. However, a fundamental study involving both experiments and numerical analyses to analyze the characteristics of the subharmonic vibration of order 1/2 has not yet been reported.

In the present study, we performed experiments and numerical analyses using a simple one-degree-of-freedom system with the nonlinearity of a piecewise-linear spring. The effects of the stiffness ratio, damping, the initial position of the switching points of the piecewise-linear spring, and the external force on the subharmonic vibration of order 1/2 were examined. The results of the experiments and the numerical analyses were found to be in good agreement.

2. Subharmonic vibration of order 1/2

Figure 1 shows a schematic diagram of a torque converter. The torque converter consists of a pump impeller, a turbine runner, and a stator. Since the torque converter transmits torque through the fluid, the rotational speed of the turbine runner is slower than that of the pump impeller, which causes an inefficiency associated with the torque converter. In order to overcome this disadvantage, a lock-up clutch, which connects the input and output sides, is used. When the clutch is locked, the torsional vibration due to engine combustion is transmitted directly to the gear train. In order to reduce the fluctuations in the speed of rotation, a set of torsional springs, referred to collectively as a damper, is placed between the piston and the output shaft.

A traveling test using an actual automobile was conducted in order to investigate the occurrence mechanism of the subharmonic vibration of order 1/2 (referred to hereinafter simply as the subharmonic vibration). Figure 2 shows a Campbell diagram for the turbine runner. The abscissa shows the engine speed, and the ordinate shows the vibration frequency of the fluctuation of the rotational speed. The color scale represents the vibration amplitude of the rotational speed. The engine frequency in figure 2 represents the component of the torsional vibration frequency due to engine combustion and increases with the increase in the engine speed. The strong vibration indicated by the dashed white oval is the subharmonic vibration. The vibration frequency is half the engine frequency.

Figure 3 shows the piecewise-linear spring restoring torque characteristics. In the actual automobile,
there are three stages of the spring constant \( (K_1, K_2, K_3) \). The subharmonic vibration occurred at around point B. Figure 4 shows the waveforms of the rotational speed fluctuation at the pump impeller and the turbine runner when the subharmonic vibration occurred. It is confirmed that the engine torque fluctuation does not include the \( 1/2 \) frequency component of the fundamental engine excitation frequency. The period of the turbine runner and the pump impeller is twice that of the engine excitation frequency. When subharmonic vibrations occur, the engine frequency is approximately twice that of the natural frequency of the second mode.

![Figure 3. Piecewise-linear spring restoring torque characteristics.](image1)

**Figure 4. Waveform of subharmonic vibration.**

### 3. Fundamental experiment

#### 3.1. Experimental setup

Figure 5 shows the experimental setup used to reproduce the subharmonic vibration. Even though the subharmonic vibration occurred as a torsional vibration, the experimental setup is designed by the translational vibration system to evaluate fundamental mechanism of the subharmonic vibration. A mass is supported by leaf springs A and B. The other sides of leaf springs A and B are connected to the base and the exciter, respectively. Leaf spring C is placed so as to provide the second stage of the spring restoring force characteristics. In the experiment, the piecewise-linear spring restoring force characteristics have two stages. For additional damping, sponges are attached around the base and leaf spring A.

![Figure 5. Experimental setup.](image2)

Figure 6 shows a schematic diagram of the experimental setup. In the figure, \( m \) is the mass, \( K \), \( k \), and \( K_p \) are the spring constants of leaf springs A, B, and C, respectively, and \( d \) is the initial position.
Table 1. Standard parameters of the experimental setup.

| Parameter | Value       |
|-----------|-------------|
| m         | 0.75 kg     |
| d         | 0 mm        |
| a         | 3 mm        |
| k         | 441 N/m     |
| K         | 928 N/m     |
| K_p       | 1369 N/m    |
| C         | 1.7 Ns/m    |

of spring C. The origin point \( d = 0 \) is the point at which leaf spring C and the mass just touch. When \( d \) is positive, \( d \) is the gap between leaf spring C and the mass. In contrast, when \( d \) is negative, the mass is preloaded by spring C. The damping of the system is assumed to be \( C \) generated by leaf spring A. Here, \( x \) is the displacement of the mass, \( u = a \cos \omega t \) is the forced displacement, \( a \) is the amplitude of the forced displacement, and \( \omega \) is the angular velocity of the exciter. Thus, this experimental setup has a two-stage piecewise-linear spring, as shown in figure 6. In this experimental setup, the spring coefficients \( k + K \) and \( k + K + K_p \) are simulated as spring coefficients of \( K_2 \) and \( K_3 \) of figure 3 in the actual automobile, respectively.

Table 1 shows the standard parameter of the experimental setup. The natural frequencies in the first and second stages of the spring are 6.8 Hz and 9.6 Hz, respectively. The amplitude of the forced displacement of the exciter is maintained constant in the entire excitation frequency range by adjusting the input voltage to the exciter. Considering the value of the damping ratio used in the calculation of actual automobile, the damping ratio in the first stage \( C / 2 \sqrt{m(k + K)} \) is set to become the same order as the value used in the calculation of actual automobile [1]. The stiffness ratio of the first to second stage of the spring, \( \gamma \), is defined as follows:

\[
\gamma = \frac{k + K + K_p}{k + K} \quad (1)
\]

Figure 7 shows the piecewise-linear spring restoring force characteristics for the standard condition shown in table 1. The stiffness ratio \( \gamma \) is set to 2.0, because \( \gamma = 2 \) is the boundary for the occurrence of subharmonic vibration in actual automobile [1].

![Figure 7. Piecewise-linear spring restoring force characteristics in the standard condition.](image-url)
3.2. Experimental results

3.2.1. Occurrence of the subharmonic vibration. Figure 8 shows the frequency response curve for the standard condition shown in Table 1. The abscissa indicates the excitation frequency of the exciter, and the ordinate indicates the peak-to-peak amplitude of displacement of the mass. The amplitude of the vibrations becomes large when the excitation frequency is approximately 16 Hz, even though the natural frequencies of the first and second stages are not around 16 Hz. The resonance frequency range is approximately twice the natural frequency. Figure 9 shows the frequency analysis at point P (in figure 8). Even though the excitation frequency is 15.7 Hz, the main vibration frequency component is half the excitation frequency. This is characteristic of the subharmonic vibration of order 1/2. Figure 10 shows a Campbell diagram of the curve shown in figure 8. The abscissa indicates the excitation frequency, and the ordinate indicates the vibration frequency. The color scale represents the amplitude of displacement (P-P). Strong vibration occurred in the area indicated by the dashed white oval, and the vibration frequency is half the excitation frequency. The tendency of the experimental results coincides with that of the experimental results obtained using an actual automobile.

3.2.2. Effect of the stiffness ratio on the occurrence of subharmonic vibration. The subharmonic vibration occurs because of the nonlinearity of the piecewise-linear spring. Then, the effect of the stiffness ratio on the subharmonic vibration is confirmed experimentally. The stiffness ratio is changed by adjusting the length of leaf spring C. Figure 11 shows the frequency response curves for various values of the stiffness ratio $\gamma$. The dashed line indicates the result for the standard condition. When the stiffness ratio becomes large, $\gamma = 3$, the amplitude of the subharmonic vibration also becomes
large. In contrast, when the stiffness ratio becomes small, \( \gamma = 1.5 \), the amplitude of the subharmonic vibration also becomes small. This is because the nonlinearity is increased by the larger stiffness ratio. In other words, the subharmonic vibration can be suppressed by using a smaller stiffness ratio. The frequency ranges of the subharmonic vibrations change according to the change in the stiffness.

3.2.3. **Effect of damping on the occurrence of subharmonic vibration.** As a countermeasure, the effect of the system damping on the subharmonic vibration is evaluated. The system damping is increased to \( C = 3.5 \text{ Ns/m} \) and \( 4.9 \text{ Ns/m} \). Figure 12 shows the frequency response curves for the three values of the damping considered herein. The amplitude of the subharmonic vibration is reduced by increasing the damping. Because of the increase in stiffness associated with the attachment of the sponges for additional damping, the peak frequency moves slightly to the higher-frequency side. Additional damping is effective for suppressing the subharmonic vibration.

3.2.4. **Effect of the initial position of the second spring on the occurrence of subharmonic vibration.** Figure 13 shows the frequency response curve for various initial positions of the second spring, \( C \). The subharmonic vibration occurs even when the initial position of the spring restoring force is not at the switching point. The frequency range of the subharmonic vibrations varies due to the change of the nonlinear natural frequency. In this experiment, only the condition of \( d = \pm 0.1 \text{ mm} \) was examined. As the nonlinearity exists only at the switching point, the subharmonic vibration will disappear if \( d \) is larger or smaller than in this experiment. This was confirmed by the numerical analysis.
3.2.5. **Effect of external force on the occurrence of subharmonic vibration.** In the nonlinear vibration system, it is important to confirm the effect of the external force on the subharmonic vibration. Figure 14 shows the frequency response curves for various external forces. The amplitude of the displacement of the forced displacement is varied as $a = 2.5$ mm and 2 mm. The experiment revealed that the maximum amplitude of the subharmonic vibration is reduced as the external force is decreased. However, the frequency range of the subharmonic vibrations is approximately the same.

4. **Numerical analysis**

4.1. **Analytical model**

In order to analyze the occurrence mechanism and the characteristics of the subharmonic vibration, numerical analyses were conducted using the mathematical model of figure 6. The equation of motion is written as

$$m\ddot{x} + C\dot{x} + f(x) = ka\cos\omega t$$  \hspace{1cm} (2)

where $f(x)$ is the spring restoring force characteristics and is given by

$$f(x) = \begin{cases} 
    x < d : f(x) = Kx + kx \\
    x \geq d : f(x) = Kx + K_p(x - d) + kx
\end{cases}$$  \hspace{1cm} (3)

Equations (2) and (3) are used to calculate the nonlinear vibration. The shooting method [11] was used to solve this equation of motion. In the process of the numerical integration used in the shooting method, the time at the switching point of the piecewise-linear spring between the time steps is calculated precisely using the Newton-Raphson method.

4.2. **Results of numerical analysis and comparison with experimental results**

4.2.1. **Occurrence of subharmonic vibration.** Figure 15 shows the frequency response curve for the standard parameters listed in table 1. The solid black line indicates the stable solution, and the dotted red line indicates the unstable solution. The boundaries of the unstable solution are approximately 14.5 Hz and 17.9 Hz. These points are classified as flip (period-doubling) bifurcations. Subharmonic vibration occurs between these flip bifurcations. The results of the numerical analysis, shown in figure 15, and the experimental results, shown in figure 8, coincide. Figure 16 shows the frequency analysis at point Q (in figure 15). The excitation frequency is 15.9 Hz. The amplitude of 8 Hz which is half of the excitation frequency, is larger than that of 15.9 Hz. This is characteristic of the subharmonic vibration.

![Figure 15. Frequency response curve for the standard condition.](image)

![Figure 16. Frequency analysis at point Q.](image)
4.2.2. *Effect of the stiffness ratio on the occurrence of subharmonic vibration.* Figure 17 shows the results of numerical analysis as the stiffness ratio $\gamma$ is varied. The subharmonic vibration becomes larger when the stiffness ratio is larger. The condition under which the subharmonic vibration occurred was in good agreement with the experimental results shown in figure 11. Figure 18 shows the frequency range of the flip bifurcations as the stiffness ratio is varied. Subharmonic vibration occurs in the shaded region. The dashed line indicates the standard condition. The subharmonic vibration was found to be suppressed as the stiffness ratio $\gamma$ approached 1. In actual automobiles, the boundary of the stiffness ratio for the occurrence of subharmonic vibrations is approximately 2.0 [1]. The reason for the difference is thought to be that the damping ratio of the vibration mode in actual automobiles is larger than that of the experimental setup.

![Figure 17](image1.png)  
**Figure 17.** Frequency response curve for various values of the stiffness ratio.

![Figure 18](image2.png)  
**Figure 18.** Existence region of subharmonic vibration for various values of the stiffness ratio.

4.2.3. *Effect of damping on the occurrence of subharmonic vibration.* Figure 19 shows the results of numerical analysis when the damping coefficient $C$ is varied as 3.5 Ns/m and 4.9 Ns/m. The tendency of the suppressive effect caused by the additional damping is similar to that indicated by the experimental results shown in figure 12. Figure 20 shows the variation in the range of frequencies of the flip bifurcations as the damping coefficient is varied. If the damping coefficient is greater than 9 Ns/m, the subharmonic vibration can be completely suppressed.

![Figure 19](image3.png)  
**Figure 19.** Frequency response curves for various values of damping.

![Figure 20](image4.png)  
**Figure 20.** Existence region of subharmonic vibration for various values of the damping coefficient.

4.2.4. *Effect of initial position of the second spring on the occurrence of subharmonic vibration.* Figure 21 shows the results of numerical analysis as the initial position of the second spring $d$ is
varied. In the condition of $d = \pm 0.1 \text{mm}$, the subharmonic vibration still exists. The results of the numerical analysis are in good agreement with the experimental results shown in figure 13. Figure 22 shows the frequency range of the flip bifurcations as the initial position of the second spring is varied. The region in which the subharmonic vibration occurs (shaded region) disappears when the initial position $d$ becomes too large or too small. This means that when the vibration region is far from the switching point, the subharmonic vibration does not occur. Figure 23 shows the frequency response curve for the case in which the initial position of the second spring is $d = 0.4 \text{mm}$. In figure 22, although the flip bifurcation disappeared, the stable region of the subharmonic vibration still remains as an island shape. In the frequency range of the subharmonic vibration, there are two stable solutions. In this case, even though subharmonic vibration did not occur in the experiment, subharmonic vibration can suddenly occur due to a large disturbance.

4.2.5. Effect of external force on the occurrence of subharmonic vibration. Figure 24 shows the results of numerical analysis as the external force is varied. The external force is varied by changing the amplitude of the forced displacement, $a$. The maximum amplitude of the subharmonic vibration decreases as the external force decreases, and the results of the numerical analysis were in good agreement with the experimental results shown in figure 14. Figure 25 shows the frequency range of the flip bifurcations as the external force is varied. The frequency range of the flip bifurcations was found not to change even when the external force is changed. This is characteristic of the nonlinear vibration of the piecewise-linear spring that has nonlinearity only at the switching point.
5. Conclusions
In order to clarify the mechanism of and countermeasures for the subharmonic vibration of order 1/2 in an automatic transmission powertrain, the present study considered a simple one-degree-of-freedom system with a piecewise-linear spring both experimentally and analytically. The following results were obtained:

(1) Experiments revealed that subharmonic vibration of order 1/2 occurs in a single-degree-of-freedom system with a piecewise-linear spring. The excitation frequency range of the subharmonic vibration of order 1/2 is approximately twice that of the natural frequency of the system.

(2) A smaller stiffness ratio, a larger additional damping, an initial position of the spring set further from the switching point, and a smaller external force can reduce the amplitude of the subharmonic vibration of order 1/2. By designing a damper to have a small stiffness ratio and large damping, subharmonic vibration can be suppressed completely.

(3) The analytical results were in good agreement with the experimental results.

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