A Laplacian on the Full Shift Space

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Abstract

In this paper, we consider the one-sided shift space on finitely many symbols and extend the theory of what is known as rough analysis. We define difference operators on an increasing sequence of subsets of the shift space that would eventually render the Laplacian on the space of real-valued continuous functions on the shift space. We then define the Green’s function and the Green’s operator that come in handy to solve the analogue to the Dirichlet boundary value problem on the shift space.

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1 Introduction

Symbolic dynamics is a relatively new and popular branch of dynamical systems. It is considered as an effective tool in the study of general dynamical systems. The original dynamical system is discretised by equipartitioning the phase space into finitely many subsets, each represented by a different symbol. The trajectory of a point is then observed by tracking the symbols, corresponding to the sets in the partition the point visits, at a given time. This process generates an infinite sequence over finitely many symbols, determined by the partition. The space of all such sequences obtained for a particular labeling of the partition, is known as the symbolic space. Each sequence in the space is a symbolic trajectory corresponding to the dynamical trajectory of a point in the original system.

The first successful attempt to apply the techniques of symbolic dynamics was made by Hadamard in 1898, to investigate geodesic flows on negatively curved surfaces, in [13]. Forty years later, in [24], the term symbolic dynamics was formally proposed by Morse and Hedlund. This foundational work marked the beginning of the study of symbolic dynamics, in its own right. Morse and Hedlund in [24] analysed the dynamics on the symbolic space in its independent abstract dynamical setting. Having assigned a metric to the symbolic space, they established that the space is perfect, compact...
and totally disconnected. They further studied the transitivity and recurrence properties of the dynamics on the space, which are fundamental to any kind of dynamical systems. In 1940, Shannon employed these spaces to model the data and information channels in the theory of communication, in [23]. Since then, the branch has found a wide range of applications in ergodic theory, complex dynamics, topological dynamics, number theory, information theory etc. Interested readers may refer to [5, 6, 21, 23] for a detailed literature on the study of symbolic dynamics and its various applications.

In this work, we focus on studying the symbolic space on its own merit. It is well known among the dynamicists that the symbolic space can be used to model many naturally occurring non-smooth objects like fractals. The Sierpiński gasket is a popular model of a fractal, that one obtains through an iterated function scheme. Investigations into the analytical study of the Sierpiński gasket through the probabilistic approach were made by Goldstein [11], Kusuoka [22] and Barlow and Perkins [4], where the authors constructed a Laplacian on the Sierpiński gasket as a Brownian motion. Kigami formulated a more direct Laplacian on the Sierpiński gasket in [16] and later generalized it for a class of post critically finite self-similar sets in [17]. We first summarize the method here.

Let us begin by constructing the one sided full shift space on $N > 1$ symbols. For the symbol set, $S := \{1, 2, \cdots, N\}$, consider the space of one sided sequences as,

$$
\Sigma^+_N := S^\mathbb{N} = \{x = (x_1 x_2 \cdots) : x_i \in S\}.
$$

The shift operator $\sigma : \Sigma^+_N \rightarrow \Sigma^+_N$ given by $\sigma((x_1 x_2 x_3 \cdots)) = (x_2 x_3 \cdots)$ is an $N$-to-1 continuous transformation. The inverse branches of $\sigma$ are given by, $\sigma_l : \Sigma^+_N \rightarrow \Sigma^+_N$ for $l \in S$, which are defined as, $\sigma_l(x_1 x_2 \cdots) = (lx_1 x_2 \cdots)$. The pair $(\Sigma^+_N, \sigma)$ is known as the one sided full shift space.

Let $V$ be a set. Let $\ell(V) := \{u | u : V \rightarrow \mathbb{R}\}$ be the set of all real valued functions on $V$ and $\mathcal{C}(V)$ denote the set of all real valued continuous functions on $V$. If $V$ is a finite set, then the standard inner product on $\ell(V)$, denoted by $\langle u, v \rangle$, is defined as,

$$
\langle u, v \rangle := \sum_{p \in V} u(p)v(p).
$$

For any $q \in V$, its characteristic function is defined as,

$$
\chi_q(p) = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{otherwise}. \end{cases}
$$

Let $(K, S, \{F_l\}_{l \in S})$ be a self-similar structure, where $K$ is a compact metrizable topological space, $F_l : K \rightarrow K$ is a continuous injection for each $l \in S$ and there exists a continuous surjection $\pi : \Sigma^+_N \rightarrow K$ such that $F_l \circ \pi = \pi \circ \sigma_l$ for each $l \in S$. $K$ is called a post critically finite set (p.c.f., for short) if the set $P = \bigcup_{n \geq 1} \sigma^n\left(\bigcup_{l \neq j} F_l(K) \cap F_j(K)\right)$ is finite. Kigami in [19] constructed a compatible sequence $(V_m, H_m)$ of resistance networks (see [9] for probabilistic study of concepts of electrical networks) on $K$, where for each $m \geq 0$, $V_m$ is a finite set with $V_m \subset V_{m+1} \subset K$ and $H_m : \ell(V_m) \rightarrow \ell(V_m)$ is a non-positive definite symmetric linear operator known as the Laplacian. Each $H_m$ induces a natural Dirichlet form $\mathcal{E}_{H_m}$ on $\ell(V_m)$. Moreover, the set $V_* = \bigcup_{m \geq 0} V_m$ is dense in $K$. Interested readers may refer to [17] [13] [20] for definitions and fundamental properties of the Dirichlet forms and difference operators. For the compatibility of the sequence $\{(V_m, H_m)\}_{m \geq 0}$, the following must hold.

$$
\mathcal{E}_{H_m}(u, u) = \min \{ \mathcal{E}_{H_{m+1}}(v, v) : v \in \ell(V_{m+1}), v|_{V_m} = u \} \quad \text{for all} \quad m \geq 0. \tag{1.1}
$$
The ‘energy’ or ‘resistance form’ on $\ell(V_*)$ is then defined as,

$$E(u, v) = \lim_{m \to \infty} E_{H_m}(u|v_m, v|v_m), \quad \text{for } u, v \in \ell(V_*),$$

whenever the limit is finite. In the study of analysis on the general framework of resistance networks, the set $V_*$, which is merely a countable set, does not have any topology. To overcome this difficulty, Kigami in [18, 19] defined and studied a metric on the resistance networks, popularly known as effective resistance, given by,

$$R(p, q) := \left( \min \{E(u, u) : u(p) = 1, u(q) = 0\} \right)^{-1}, \quad \text{for } p, q \in V_*.$$ 

If $(\Omega, R)$ is the completion of $(V_*, R)$, then we have a Laplacian on $\Omega$ associated to the quadratic form $E$. The important fact to note here is, $\Omega$ can be identified with the original space $K$, if and only if $(\Omega, R)$ is bounded, see [20]. In case of p.c.f. self-similar set $K$, this metric $R$ is compatible with the original metric on $K$. Therefore, the Laplacian on $K$ can be directly defined as the renormalized limit of the difference operators $H_m$. The effective resistance plays a crucial role in the theory of Laplacians and Dirichlet forms, see [20] for a comprehensive study on the topic.

Further fundamental properties of this Laplacian on the Sierpiński gasket, like the Dirichlet and Neumann problems of the Poisson equation, complete Dirichlet spectrum, heat and wave equations were examined by Kigami [10], Shima [27], Fukushima and Shima [10], Dalrymple, Strichartz and Vinson [7] etc. Also similar boundary value problems, spectral properties of the Laplacian and various physical phenomena like heat and wave propagation on different and more complex rough spaces have been studied extensively over the last few decades by several mathematicians that include Teplyaev, Alonso-Ruiz, Freiberg, Kesseböhmer [1, 2, 12, 14, 15].

In [8], Denker et al., considered the abstract setting of the shift space $\Sigma^+_{\mathbb{N}}$ independent of its relation with fractals. Rather than constructing a nested sequence of finite sets and difference operators on them as described above, they define ‘thin’ equivalence relations on $\Sigma^+_{\mathbb{N}}$ and construct Dirichlet forms on the associated quotient spaces of $\Sigma^+_{\mathbb{N}}$. This Dirichlet form gives rise to the Laplace operator on such quotient spaces. The authors also prove that Kigami’s Laplacian on the Sierpiński gasket can be derived as a special case to this theory.

In the present paper, we consider the same abstract setting of the shift space. We extract a nested sequence of finite subsets $V_m$ of $\Sigma^+_{\mathbb{N}}$ by exploiting the dynamical aspects and the peculiar topology of the symbolic space, and define equivalence relations on each of these subsets as described in section (2). We then define the difference operators in section (3) and corresponding Dirichlet forms in section (4), on each of these sets. The set $V_*$ is unbounded with respect to the analogous effective resistance metric obtained using these Dirichlet forms in this setting, as will be proved in section (5), after introducing concepts like energy and energy minimizers. This establishes that the effective resistance is insufficient in obtaining a Laplacian on the full space $\Sigma^+_{\mathbb{N}}$. Therefore we resort to the standard metric existing on the shift space, which will be defined in the following section and derive a Laplacian on $\Sigma^+_{\mathbb{N}}$ as a renormalised limit of the difference operators in section (6).

The second part of the paper focuses on solving a problem analogous to the Dirichlet boundary value problem, as stated below, through the standard concepts of the Green’s function in section (7), the Green’s operator in section (8).

**Theorem 1.1** Let $C(\Sigma^+_{\mathbb{N}})$ denote the Banach space of real-valued continuous functions defined on $\Sigma^+_{\mathbb{N}}$ and $V_0$ be the set of fixed points of $\sigma$. For any $f \in C(\Sigma^+_{\mathbb{N}})$ and $\zeta \in \ell(V_0)$, there exists a function $u \in C(\Sigma^+_{\mathbb{N}})$ in the domain of the Laplacian that satisfies

$$\Delta u = f \quad \text{subject to } u|_{V_0} = \zeta.$$
We conclude the paper by giving a complete solution to the differential equation on this totally disconnected space in section (9). We aim to investigate the relation between the energy and the Laplacian on $\Sigma_N^+$ in subsequent papers.

2 $m$-relations in the full shift space

The full shift space $\Sigma_N^+$ introduced in the introduction is equipped with a natural metric defined by,

$$d(x, y) := \frac{1}{2^\rho(x,y)},$$

where $\rho(x, y) := \min\{i : x_i \neq y_i\}$ with $\rho(x, x) := \infty$.

This metric $d$ generates a product topology on $\Sigma_N^+$, where the symbol set $S$ is considered to have a discrete topology. In fact, for any $0 < \theta < 1$, the metric $d_\theta(x, y) := \theta^{\rho(x,y)}$ generates the same product topology. Unless otherwise mentioned, we always use the metric $d = d_\theta$ with $\theta = \frac{1}{2}$. In this topology, one may observe that the open sets can be written as a countable union of cylinder sets, which are themselves both closed and open. Thus, the cylinder sets form a basis for the topology on $\Sigma_N^+$. By a cylinder set of length $m$, we mean the set denoted by

$$[p_1 \cdots p_m] := \{x \in \Sigma_N^+ : x_1 = p_1, \cdots, x_m = p_m\},$$

where we fix the initial $m$ co-ordinates. We wish to draw the attention of the readers to the position of the cylinder sets, which can occur anywhere in general. However, we necessitate the cylinder sets to be placed at the initial co-ordinates, as defined. The reason for the same becomes clear during the course of this section. Under the topology defined, $\Sigma_N^+$ is a totally disconnected, compact, perfect metric space on which $\sigma$ is a non-invertible, continuous surjection, that has $N$ local inverse branches for any point $x$.

These cylinder sets form a semi-algebra that would, in turn generate the Borel sigma-algebra on $\Sigma_N^+$. The equidistributed Bernoulli measure $\mu$ on a cylinder set of length $m$ is defined as,

$$\mu([p_1 \cdots p_m]) := \frac{1}{N^m}. \quad (2.1)$$

Moreover, the shift space $\Sigma_N^+$ has a self similar structure. Recall the inverse branches $\sigma_l : \Sigma_N^+ \rightarrow [l]$ for each $l \in S$ as defined in the previous section. Each branch $\sigma_l$ maps an element $x$ in $\Sigma_N^+$ to its preimage (under $\sigma$) that has the symbol $l$ in its first position. $\sigma_l$ is a contractive similarity with the contraction ratio being $\frac{1}{2}$. In fact, for distinct $l \in S$, we note that the sets $\sigma_l(\Sigma_N^+)$ are mutually disjoint and satisfy, $\Sigma_N^+ = \bigcup_{l \in S} \sigma_l(\Sigma_N^+)$. 

We can also generalise the above structure of self-similarity, as follows. Fix $m > 0$ and consider any finite word $w$ of length $|w| = m$ i.e., $w = (w_1 w_2 \cdots w_m)$. One can then define the map $\sigma_w := \sigma_{w_1} \circ \sigma_{w_2} \circ \cdots \circ \sigma_{w_m} : \Sigma_N^+ \rightarrow [w_1 w_2 \cdots w_m]$ which concatenates the finite word $w$ as a prefix to the elements of $\Sigma_N^+$. Again, we can write the shift space as a disjoint union given by

$$\Sigma_N^+ = \bigcup_{\{w : |w| = m\}} \sigma_w(\Sigma_N^+).$$

We now understand the shift space $\Sigma_N^+$ as the limit of an increasing sequence of finite subsets of $\Sigma_N^+$. For $l \in S$, denote the point $(ll \cdots) \in \Sigma_N^+$ by $(l)$. This is a fixed point of $\sigma$ and of the corresponding map $\sigma_l$. Let $V_0$ denote the set of all fixed points of $\sigma$, namely,

$$V_0 := \{(1), (2), \cdots, (N)\}.$$
For $m \geq 1$, we define the sets $\{V_m\}_{m \geq 1}$ inductively as $V_m := \bigcup_{l \in S} \sigma_l(V_{m-1})$. Note that $V_m$ is the set of all $m$-th order pre-images of points in $V_0$, with cardinality $N^{m+1}$. Further, the sequence $\{V_m\}$ is increasing. Since $V_m = \bigcup_{l \in S} \sigma_l(V_0)$, any point $p$ in $V_m$ is of the form $p = (p_1 \cdots p_m p_{m+1} p_{m+2} \cdots)$. We denote this point by $(p_1 \cdots p_m \hat{p}_{m+1})$. In particular, for a point $p \in V_n \setminus V_{n-1}$, we have $p_m \neq p_{m+1}$.

Define $V_* := \bigcup_{m \geq 0} V_m$. Then, $V_*$ is a dense subset of $\Sigma_N^+$ in the standard topology, i.e., for any $x = (x_1 x_2 \cdots) \in \Sigma_N^+$, the sequence of points

$$\{(\hat{x}_1) \in V_0; (x_1 \hat{x}_2) \in V_1; \cdots; (x_1 x_2 \cdots x_m \hat{x}_{m+1}) \in V_m; \cdots\}$$

converges to $x$. We note that there could be different sequences that approach $x$ in the limit; we have provided only an example of one such to establish density. On each $V_m$, we define an equivalence relation to characterize the closest points to a given point in $V_m$.

**Definition 2.1** Any two points $p = (p_1 p_2 \cdots p_m \hat{p}_{m+1})$ and $q = (q_1 q_2 \cdots q_m \hat{q}_{m+1})$ in $V_m$ are said to be $m$-related, denoted by $p \sim_m q$, if $p_i = q_i$ for $1 \leq i \leq m$.

The definition entails that any two points in $V_0$ are 0-related. The $m$-related points $p, q \in V_m$ are obtained by the action of the same $\sigma_w$ on $V_0$. Any two points in $V_m$ are separated by a distance of at least $\frac{1}{2m+1}$. The $m$-relation, $\sim_m$ is an equivalence relation on $V_m$. The equivalence class of $p$ in $V_m$ is the set of all $m$-related points of $p$ in $V_m$. We denote it by

$$[p_1 \cdots p_m]_{V_m} := \{(p_1 p_2 \cdots p_m \hat{p}) : l \in S\} = [p_1 \cdots p_m] \cap V_m.$$

**Remark 2.2** The $m$-relation is clearly reflexive (being an equivalence relation). Therefore, when we say two points are $m$-related, we will only focus on distinct points being $m$-related. We adopt this as a convention, since reflexivity does not play any role in our analysis.

**Remark 2.3** Let $p = (p_1 p_2 \cdots p_m \hat{p}_{m+1}) \in V_m$. Among all the points in $V_m$ other than $p$, those that are $m$-related to $p$ are the closest to $p$ at a distance $\frac{1}{2m+1}$. We define these points to be the immediate neighbours of $p$ in $V_m$. We call the set of these immediate neighbours, as the deleted neighbourhood of $p$ in $V_m$, denoted by $U_{p,m}$. Observe that there are precisely $N-1$ immediate neighbours for any $p \in V_m$. Let us denote these neighbours by $q^1, q^2, \ldots, q^{N-1}$.

For example, when $N = 3$, the points $(1\hat{2})$, $(1\hat{3})$ and $(1\hat{1})$ in $V_1$ are 1-related to each other. Thus, $U_{(1\hat{2}),1} = \{(1\hat{3}), (1\hat{1})\}$. Similarly, the points $(3\hat{1})$, $(3\hat{2})$ and $(3\hat{3})$ in $V_2$ are 2-related to each other. Thus, $U_{(3\hat{2}),2} = \{(3\hat{1}), (3\hat{2})\}$.

**Remark 2.4** Consider $p \in V_m \setminus V_{m-1}$. Among the $N-1$ neighbours of $p$ accommodated in $U_{p,m}$, we note that one of them denoted by $q^{N-1} = (p_1 p_2 \cdots p_{m-1} \hat{p}_m)$ comes from $V_{m-1}$. The rest of the neighbours; $N-2$ of them are collected in the set

$$U_{p,m} := \{(p_1 p_2 \cdots p_m \hat{p}) : l \neq p_m \text{ and } l \neq p_{m+1}\} = \{q^1, q^2, \ldots, q^{N-2}\} \subset V_m \setminus V_{m-1}.$$
One can easily verify this for the example when \( N = 3 \).

We make one more interesting observation in the next remark which enables us to track any point in \( V_m \) from any point in \( V_0 \), through a chain of \( k \)-related points in the intermediary stages \( V_k \). Making use of this, we can connect any two points in \( V_m \) by a chain of \( k \)-related points from \( V_k \), from each stage \( 0 \leq k \leq m \).

**Remark 2.5** For \( p \in V_m \backslash V_{m-1} \), choose \( n_1, n_2, \ldots, n_d \in \mathbb{N} \) such that \( 1 \leq n_1 < n_2 < \cdots < n_d = m \), which are the only coordinates of \( p \) satisfying \( p_{n_i} \neq p_{n_i+1} \). Construct the points

\[
 r^0 = (\dot{p}_1) \in V_0; \quad r^{ni} = (p_1 p_2 \cdots p_{n_i} \dot{p}_{n_i+1}) \in V_n \setminus V_{n-1} \quad \text{and} \quad r^{nd} = p.
\]

Observe that, any \((\dot{i}) \in V_0 \) can be connected to \( p \in V_m \backslash V_{m-1} \) by means of this chain of distinct points, \( r^0, r^{n_1}, \ldots, r^{n_d} \), in the sense that,

\[
 (\dot{i}) \sim_0 r^0 \quad \text{or} \quad (\dot{i}) = r^0 \quad \text{and} \quad r^{n_i-1} \sim_n r^{n_i} \quad \text{implying} \quad r^{n_i-1} \in U_{r^{n_i}, n_i} \quad \text{for} \ 1 \leq i \leq d.
\]

Due to the peculiar topology on the space \( \Sigma^+_N \), the standard notion of topological boundary becomes irrelevant. Nevertheless, as the method of construction of \( \Sigma^+_N \) begins from \( V_0 \), we define the set \( V_0 \) as the boundary of \( \Sigma^+_N \).

### 3 Difference operators

In this section, we inductively define a difference operator \( H_m \) on \( \ell(V_m) \), which gives the total difference between the functional values at a point and its neighbouring points in \( V_m \). We also provide an easier approach to the operator so defined, by writing its matrix representation, however after defining an order amongst the finitely many points in \( V_m \). Whenever a point \( p \) appears before \( q \) in the ordering of \( V_m \), we denote it by \( p \prec q \). Then a matrix for \( H_m \) is arranged in such a way that, whenever \( p \prec q \), the row or column corresponding to the point \( p \) appears to the top or to the left of the row or column respectively, corresponding to the point \( q \). Let \( (H_m)_{pq} \) denote the entry in the matrix \( H_m \) corresponding to row \( p \) and column \( q \). The action of \( H_m \) on \( u \in \ell(V_m) \) at a point \( p \in V_m \) is given by,

\[
 H_m u(p) = \sum_{q \in V_m} (H_m)_{pq} u(q).
\]

For \( l \in S \), consider the point \( (\dot{l}) \in V_0 \). Observe that all other points in \( V_0 \) are at an equal distance of \( \frac{1}{2} \) from \( (\dot{l}) \). We then define a difference operator \( H_0 \) on \( \ell(V_0) \) as,

\[
 H_0 u(\dot{l}) := \sum_{(\dot{k}) \in V_0} \left( u(\dot{k}) - u(\dot{l}) \right) = - (N-1) u(\dot{l}) + \sum_{\substack{k \in S \\cap \cap \dot{l}}} u(\dot{k}). \tag{3.1}
\]

For \( l, k \in S \), we define \( (\dot{k}) \prec (\dot{l}) \) if and only if \( k \prec l \). Then \( V_0 \) can be written in an ascending order of elements as,

\[
 V_0 = \left\{ (1) \prec (2) \prec \cdots \prec (N) \right\}.
\]
We write $H_0$ as a matrix of order $N$ given by

$$(H_0)_{pq} = \begin{cases} 
1 & \text{if } q \in U_{p,0} \\
-(N-1) & \text{if } p = q,
\end{cases}$$

where $(H_0)_{pq}$ denotes the entry of the matrix $H_0$ corresponding to the row for $p \in V_0$ and the column for $q \in V_0$. Thus,

$$H_0 = \begin{pmatrix}
-(N-1) & 1 & 1 & \cdots & 1 \\
1 & -(N-1) & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & -(N-1)
\end{pmatrix}_{N \times N}.$$

Making use of this matrix representation of $H_0$, we observe that for $p \in V_0$, we have

$$H_0 u(p) = \sum_{q \in V_0} (H_0)_{pq} u(q). \quad (3.2)$$

It is now an easy observation, that the two expressions for $H_0$ as in equation (3.1) and (3.2) are the same. Having defined a difference operator, namely $H_0$ on $\ell(V_0)$, we now adopt the same philosophy for defining a difference operator $H_1$ on $\ell(V_1)$.

Let $u \in \ell(V_1)$ and consider the points $p = (p_1 \, p_2) \in V_1 \setminus V_0$ and $l \in V_0$. We define the action of $H_1$ on $V_1 \setminus V_0$ and $V_0$ separately as,

$$H_1 u(p) := \sum_{q \in U_{p,1}} (u(q) - u(p)) = -(N-1) u(p) + \sum_{q \in U_{p,1}} u(q), \quad (3.3)$$

$$H_1 u(l) := -2(N-1) u(l) + \sum_{k \in S} u(k) + \sum_{q \in V_1 \setminus V_0} \sum_{q_1 = l} u(q) \quad \quad (3.4)$$

$$= H_0 u(l) - (N-1) u(l) + \sum_{q \in U_{l,1}} u(q).$$

By construction, $V_1$ contains all the points in $V_0$ and its immediate predecessors under $\sigma$. We order the elements of $V_1$ as follows: The elements of $V_0$ appear first in $V_1$, with the prescribed order therein. That is, we define $p \prec q$ for any $p \in V_0$ and $q \in V_1 \setminus V_0$. We now define the order on $V_1 \setminus V_0$. For $l \in V_0$ and $i, j \in S$ with $i \neq l$ and $j \neq l$, we define $\sigma_i(l) \prec \sigma_j(l)$ if and only if $i < j$. And for distinct $(k), (l) \in V_0$ and any $i, j \in S$, we define $\sigma_i(k) \prec \sigma_j(l)$ if and only if $(k) \prec (l)$. Thus the set $V_1$ can be arranged in an ascending order of its points as,

$$V_1 = \left\{ (1) \prec \cdots \prec (\hat{N}) \prec (2\hat{1}) \prec \cdots \prec (\hat{N}1) \prec (1\hat{2}) \prec \cdots \prec (\hat{N}2) \prec \cdots \right\}.$$

We now obtain a matrix representation of the operator $H_1$. We expect $H_1$ to be a square matrix of order $N^2$, the cardinality of $V_1$. For this purpose, we split $H_1$ into 4 parts as follows:

$$H_1 = \begin{pmatrix}
T_1 & J_1^T \\
J_1 & X_1
\end{pmatrix},$$

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where
\[ T_1 : \ell(V_0) \rightarrow \ell(V_0) \text{ is of order } N, \]
\[ J_1 : \ell(V_0) \rightarrow \ell(V_1 \setminus V_0) \text{ is of order } (N^2 - N) \times N, \]
\[ X_1 : \ell(V_1 \setminus V_0) \rightarrow \ell(V_1 \setminus V_0) \text{ is of order } N^2 - N. \]

Making use of equations (3.3) and (3.4), we compute the entries in the relevant matrices \( T_1 \), \( X_1 \) and \( J_1 \) as
\[
(T_1)_{pq} = \begin{cases} -2(N - 1) & \text{if } p = q, \\ 1 & \text{otherwise.} \end{cases}
\]
\[
(X_1)_{pq} = \begin{cases} -(N - 1) & \text{if } q \in U_p, \\ 1 & \text{if } q \in U_p, \\ 0 & \text{otherwise.} \end{cases}
\]
\[
(J_1)_{pq} = \begin{cases} 1 & \text{if } q \in U_p, \\ 0 & \text{otherwise.} \end{cases}
\]

One can observe that the matrices \( T_1 \), \( J_1 \) and \( X_1 \) satisfy the following relation:
\[ T_1 = H_0 + J_1^T X_1^{-1} J_1. \]

We proceed inductively to define the appropriate difference operator \( H_m \) on \( u \in \ell(V_m) \). If \( p \in V_m \), then \( p \in V_n \setminus V_{n-1} \) for some \( 1 \leq n \leq m \), or \( p \in V_0 \) (choose \( n = 0 \) in that case). Then,
\[
H_m u(p) := -(m - n + 1) (N - 1) u(p) + \sum_{i=n}^m \sum_{q \in U_{p,i}} u(q) \tag{3.5}
\]
\[
= H_{m-1} u(p) + \left[ -(N - 1) u(p) + \sum_{q \in U_{p,m}} u(q) \right].
\]

In particular, if \( p \in V_m \setminus V_{m-1} \), then substituting \( n = m \) in (3.5) we obtain,
\[
H_m u(p) := -(N - 1) u(p) + \sum_{q \in U_{p,m}} u(q). \tag{3.6}
\]

To obtain the matrix representation of \( H_m \), we order the points in \( V_m \). Recall that any point \( p \in V_m \) looks like \( p = (p_1 p_2 \cdots p_m \hat{p}_{m+1}) \) with \( p_m = \hat{p}_{m+1} \) if \( p \in V_{m-1} \), and \( p_m \neq \hat{p}_{m+1} \) if \( p \in V_m \setminus V_{m-1} \). The order of points in \( V_{m-1} \) is retained as it is in \( V_m \). Moreover \( V_{m-1} \) appears first in the ordering of \( V_m \), that is, if \( p \in V_{m-1} \) and \( q \in V_m \setminus V_{m-1} \), then \( p < q \). Recall that \( V_m = \bigcup_{i \in S} \sigma_i(V_{m-1}) \). Now, for any \( p, q \in V_{m-1} \), we define \( \sigma_i(p) \prec \sigma_j(p) \) if and only if \( i < j \) and for any \( i, j \in S \), define \( \sigma_i(p) \prec \sigma_j(q) \) if and only if \( p < q \). In summary, the points in \( V_m \) can be listed in their ascending order as,
\[
V_m = \left\{ (1) \prec \cdots \prec (N) \prec \underbrace{(21) \prec \cdots \prec (N1)}_{V_0} \prec \cdots \prec (N1) \prec \underbrace{(N-1N)}_{V_0 \setminus V_1} \prec \cdots \prec (N-1N) \right\}
\]
\[
\underbrace{\cdots \prec \cdots \prec \cdots \prec \cdots}_{V_1 \setminus V_0, V_2 \setminus V_1, \ldots, V_{m-1} \setminus V_{m-2}}
\]
A Laplacian on $\Sigma_N^+$

\[
\left\{ \begin{array}{c}
1 \cdots 1 \\
N^m_1 \\
V_m \setminus V_{m-1}
\end{array} \right. \chi_1 \cdots \chi_{m-1} (N \cdots N N - 1 \tilde{N})
\]

The matrix representation for the difference operator $H_m$ (of order $N^{m+1}$) defined on $\ell(V_m)$ is split into four parts, analogous to what we did for $H_1$.

\[
H_m = \begin{pmatrix} T_m & J_m^T \\ J_m & X_m \end{pmatrix}
\]

where

\[
T_m : \ell(V_{m-1}) \rightarrow \ell(V_{m-1}) \text{ is of order } N^m,
\]

\[
J_m : \ell(V_{m-1}) \rightarrow \ell(V_m \setminus V_{m-1}) \text{ is of order } (N^{m+1} - N^m) \times N^m,
\]

\[
X_m : \ell(V_m \setminus V_{m-1}) \rightarrow \ell(V_m \setminus V_{m-1}) \text{ is of order } N^{m+1} - N^m.
\]

The entries in each of these submatrices are obtained using the definition of $H_m$, as given in equation (3.5).

\[
(T_m)_{pq} = \begin{cases} 
-(m-n+1)(N-1) & \text{if } p = q \in V_n \setminus V_{n-1} \text{ (n < m), or } V_0 \text{ (n = 0)}, \\
1 & \text{if } p \in V_0(n = 0) \text{ or } p \in V_n \setminus V_{n-1} \text{ with } q \in U_{p,i} \text{ for some } n \leq i < m, \\
0 & \text{otherwise.}
\end{cases}
\]

\[
(X_m)_{pq} = \begin{cases} 
-(N-1) & \text{if } p = q, \\
1 & \text{if } q \in U_{p,m}, \\
0 & \text{otherwise.}
\end{cases}
\]

\[
(J_m)_{pq} = \begin{cases} 
1 & \text{if } q \in U_{p,m}, \\
0 & \text{otherwise.}
\end{cases}
\]

It is now easy to verify that these submatrices satisfy the relation

\[
T_m = H_{m-1} + J_m^T X_{m-1} J_m.
\]

**Remark 3.1** For any $m \geq 0$, $(H_m)_{pq} = 1$ if and only if $p \in V_i$ and $q \in U_{p,i}$, for some $0 \leq i \leq m$.

Every difference operator $H_m$ satisfies the properties enlisted in the following lemma.

**Lemma 3.2**

1. $H_m$ is a symmetric matrix with the row sum being zero for every row.

2. The non-diagonal entries in $H_m$ are non-negative; in particular either 1 or 0.

3. $H_m$ is non-positive definite with rank $N^{m+1} - 1$.

4. The function $u \in \ell(V_m)$ is constant, if and only if $H_m u = 0$.

**Proof:** The first two properties directly follow from the construction of the difference operator $H_m$. The remaining two can be easily proved by reducing the matrix to its row echelon form. 

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4 Dirichlet forms on $V_m$

A non-positive definite symmetric linear operator on any finite set $V$ satisfying the properties (2) and (4) mentioned in lemma (3.2) gives rise to a Dirichlet form, a fundamental notion in the analysis on finite sets. Concerned readers may refer to [19] for more details. A Dirichlet form on $V$ is a non-negative definite symmetric bilinear form satisfying,

1. $\mathcal{E}(u, u) = 0$ if and only if $u$ is constant on $V$

2. for any $u \in \ell(V)$, $\mathcal{E}(u, u) \geq \mathcal{E}(\bar{u}, \bar{u})$ where $\bar{u}$ is defined by

$$\bar{u}(p) := \begin{cases} 
1 & \text{if } u(p) \geq 1, \\
u(p) & \text{if } 0 < u(p) < 1, \\
0 & \text{if } u(p) \leq 0.
\end{cases} \quad (4.1)$$

Now, returning to our setting of the shift space, the symmetric difference operator $H_m$ defined on $\ell(V_m)$ in section (3) naturally induces a symmetric bilinear form on $\ell(V_m)$. We denote it by $\mathcal{E}_{H_m}$ and is given by,

$$\mathcal{E}_{H_m}(u, v) := -\langle u, H_m v \rangle = -\sum_{p \in V_m} u(p) H_m v(p). \quad (4.2)$$

If $v = u$, for simplicity we denote $\mathcal{E}_{H_m}(u, u)$ by $\mathcal{E}_{H_m}(u)$. We verify that $\mathcal{E}_{H_m}$ defined in such a way is a Dirichlet form on $\ell(V_m)$, once we observe the following:

**Proposition 4.1** For $u, v \in \ell(V_m)$,

$$\mathcal{E}_{H_m}(u, v) = \frac{1}{2} \sum_{p,q \in V_m} (H_m)_{pq} (u(p) - u(q)) (v(p) - v(q)).$$

**Proof:** Consider,

$$\frac{1}{2} \sum_{p,q \in V_m} (H_m)_{pq} (u(p) - u(q)) (v(p) - v(q))$$

$$= \sum_{p \in V_m} u(p) v(p) \sum_{q \in V_m} (H_m)_{pq} + \sum_{q \in V_m} u(q) v(q) \sum_{p \in V_m} (H_m)_{pq}$$

$$- \sum_{p \in V_m} u(p) \sum_{q \in V_m} (H_m)_{pq} v(q) - \sum_{q \in V_m} u(q) \sum_{p \in V_m} (H_m)_{pq} v(p)$$

Since the row sum and column sum of $H_m$ are zero, the first two terms in the expression on the right side above vanish. Also by the definition of $H_m$, we have $H_m v(p) = \sum_{q \in V_m} (H_m)_{pq} v(q)$. So by reversing the roles of $p$ and $q$ in the last term of above expression on the right side, we obtain,

$$\frac{1}{2} \sum_{p,q \in V_m} (H_m)_{pq} (u(p) - u(q)) (v(p) - v(q)) = \sum_{p \in V_m} u(p) H_m v(p) = \mathcal{E}_{H_m}(u, v)$$

The following corollary follows when $v = u$, in the above proposition.

**Corollary 4.2** For any $u \in \ell(V_m)$,

$$\mathcal{E}_{H_m}(u) = \frac{1}{2} \sum_{p,q \in V_m} (H_m)_{pq} (u(p) - u(q))^2. \quad (4.3)$$
Theorem 4.3 The bilinear form \( \mathcal{E}_{H_m} \) defined in equation (4.2) is a Dirichlet form on \( V_m \).

**Proof:** Consider the alternate expression for \( \mathcal{E}_{H_m}(u) \) obtained in the corollary (4.2). Note that on the right hand side of equation (4.2), the terms corresponding to the points \( p, q \in V_m \) such that \( p = q \) or \((H_m)_{pq} = 0 \) contribute nothing to the sum. Among the remaining terms that contribute to the sum, the points \( p, q \) are such that \( p \neq q \) with \((H_m)_{pq} = 1 \). Thus, all the terms in the sum are non-negative and we have,

\[
\mathcal{E}_{H_m}(u) \geq 0, \text{ for all } u \in \ell(V_m).
\]

\( \mathcal{E}_{H_m}(u) = 0 \) if and only if \((H_m)_{pq} (u(p) - u(q))^2 = 0 \) for all \( p, q \in V_m \), since every individual term in the sum for \( \mathcal{E}_{H_m}(u) \) is non-negative. From the definition of \( H_m \), it follows that whenever \((H_m)_{pq} = 1 \), that there exists some \( k \in \mathbb{N} \) such that \( 0 \leq k \leq m \) and \( q \in U_{p,k} \). Then we obtain \((H_m)_{pq} (u(p) - u(q))^2 = 0 \) if and only if \( u(q) = u(p) \) whenever \( q \in U_{p,k} \) for some \( 0 \leq k \leq m \). In other words, \( u \) assumes a constant value for any two \( k \)-related points in \( V_m \). In particular, for any \( p, q \in V_0 \), \( q \in U_{p,0} \) holds, and thus the function \( u \) is constant on \( V_0 \). Recall that, any point in \( V_m \) can be connected to a point in \( V_0 \) by a chain of related points at intermediary steps, as described in remark (2.5). Therefore we obtain that \( \mathcal{E}_{H_m}(u) = 0 \) if and only if \( u \) is a constant function on \( V_m \).

For a function \( u \in \ell(V_m) \), construct a function \( \bar{u} \in \ell(V_m) \) as defined in (4.1). Consider,

\[
\mathcal{E}_{H_m}(\bar{u}) = \frac{1}{2} \sum_{p,q \in V_m} (H_m)_{pq} (\bar{u}(p) - \bar{u}(q))^2
\]

\[
= \frac{1}{2} \left[ \sum_{p,q \in V_m} (H_m)_{pq} (u(p) - u(q))^2 + \sum_{p \in V_m} (H_m)_{pq} (1 - u(q))^2 \right.
\]

\[
+ \sum_{p \in V_m} (H_m)_{pq} (u(p) - 1)^2 + \sum_{p \in V_m} (H_m)_{pq} (u(q))^2 \right]
\]

\[
+ \sum_{p \in V_m} (H_m)_{pq} (u(p))^2 \]

\[
\leq \frac{1}{2} \sum_{p,q \in V_m} (H_m)_{pq} (u(p) - u(q))^2
\]

\[
= \mathcal{E}_{H_m}(u)
\]

For any real valued function (not necessarily continuous) \( u \) on \( \Sigma_N^+ \), denote its restriction to \( V_m \) by \( u|_{V_m} \). Clearly \( u|_{V_m} \in \ell(V_m) \). The sequence \( \{\mathcal{E}_{H_m}(u|_{V_m})\}_{m \geq 0} \) is non-decreasing, as, for any \( m \geq 0 \), by the definition of \( H_m \), we have,

\[
\mathcal{E}_{H_{m+1}}(u|_{V_{m+1}}) = \frac{1}{2} \sum_{i=0}^{m+1} \sum_{p \in V_i} \sum_{q \in U_{p,i}} (u(p) - u(q))^2
\]
\[
= \mathcal{E}_{H_m}(u|_{V_m}) + \frac{1}{2} \sum_{p \in V_{m+1}} \sum_{q \in \mathcal{U}_{p, m+1}} \left( u(p) - u(q) \right)^2
(4.4)
\]

The following theorem states that the sequence \( \{(V_m, H_m)\}_{m \geq 0} \) that we have constructed, is compatible in the sense of equation (1.1).

**Theorem 4.4** Any \( u_m \in \ell(V_m) \) can be uniquely extended to a function \( u_{m+1} \in \ell(V_{m+1}) \) preserving the respective Dirichlet forms in the sense that,

\[
\mathcal{E}_{H_{m+1}}(u_{m+1}) = \mathcal{E}_{H_m}(u_m) = \min \{ \mathcal{E}_{H_{m+1}}(v) \mid v \in \ell(V_{m+1}), v|_{V_m} = u_m \}.
\]

**Proof:** Assuming such an extension \( u_{m+1} \) of \( u_m \) exists, let us explicitly construct the same. Since \( u_{m+1} \) should satisfy \( \mathcal{E}_{H_{m+1}}(u_{m+1}) = \mathcal{E}_{H_m}(u_m) \), from equation (4.4), we get

\[
\sum_{p \in V_{m+1}} \sum_{q \in \mathcal{U}_{p, m+1}} \left( u(p) - u(q) \right)^2 = 0,
\]

which holds if and only if for any \( p \in V_{m+1} \),

\[
u_{m+1}(q) = u_{m+1}(p) \quad \text{for all} \quad q \in \mathcal{U}_{p, m+1}.
\]

Thus, this extension is unique and it takes constant values on the equivalence classes \([p_1 p_2 \cdots p_{m+1}]_{V_{m+1}} \). Recall from remark (2.3), that the deleted neighbourhood of any \( p \in V_{m+1} \setminus V_m \) is given by,

\[
\mathcal{U}_{p, m+1} = \{ q^1, q^2, \cdots, q^{N-1} \} \subset [p_1 p_2 \cdots p_{m+1}]_{V_{m+1}},
\]

with only one immediate neighbour \( q^{N-1} \in V_m \). Therefore the extension \( u_{m+1} \) on \( V_{m+1} \setminus V_m \) is uniquely determined by \( u_m \) as

\[
u_{m+1}(p) = u_{m+1}(q^1) = \cdots = u_{m+1}(q^{N-2}) = u_m(q^{N-1}).
\]

\[ \bullet \]

5 **Energy**

The sequence of the Dirichlet forms corresponding to the compatible sequence of difference operators, as constructed in the last section, gives rise to a non-negative definite symmetric bilinear form in the limiting sense. We call this form as energy.

**Definition 5.1** The energy of \( u \) is defined as,

\[
\mathcal{E}(u) = \lim_{m \to \infty} \mathcal{E}_{H_m}(u_m),
\]

where \( +\infty \) is a possible limiting value. Further, we define the domain of energy as,

\[
dom \mathcal{E} := \{ u \in \mathcal{C}(\Sigma^+_N) : \mathcal{E}(u) < \infty \}.
\]
Since $E(u)$ is a series in which each summand is non-negative, $E(u) = 0$ if and only if $u$ is constant.

We now see that the energy is a Markovian form. Consider a function $\phi : \mathbb{R} \rightarrow \mathbb{R}$,

$$\phi(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ t & \text{if } t \in [0, 1] \\ 1 & \text{if } t \geq 1. \end{cases}$$

Then, for any $u \in \text{dom } E$, it follows directly from the definition of the energy $E$, that $\phi \circ u \in \text{dom } E$, with $E(\phi \circ u) \leq E(u)$, proving the Markovian property.

Similar to the way the energy is defined as the limit of finite Dirichlet forms, we define Laplacian as a renormalised limit of the finite difference operators, in the next section. At this juncture, we carefully restrict ourselves from calling this energy to be a Dirichlet form associated with the Laplacian (that we will soon define). Although the finite difference operators and finite Dirichlet forms are associated in a certain way, this relation may not get carried over in exactly the same way in the limit. But we can still expect some relation to exist between the energy and the Laplacian defined in the next section. However, this is not the aim of this paper and we will look into this matter in detail in a subsequent paper, [28].

Consider the function $u_{m+1} \in \ell(V_{m+1})$ constructed in the proof of theorem (4.4). Extend the same to a function $u \in C(\Sigma^+_N)$ by fixing it to be constant on the cylinder sets of length $m + 1$ in $\Sigma^+_N$. These constants are determined by the values of $u$ at the points in $V_m$. That is,

$$u(x) = u_m(p_1 p_2 \cdots p_m \hat{p}_{m+1}) \quad \text{whenever } x \in [p_1 p_2 \cdots p_m p_{m+1}].$$

Then for all $n \geq m + 1$, $E_{H_n}(u|_{V_n}) = E_{H_m}(u_m)$. Due to the compatibility of the difference operators as proved in theorem (4.4), this particular extension has the least energy among all the extensions of $u_m$,

$$E(u) = \min \{ E(v) : v \in \text{dom } E, v|_{V_m} = u_m \}.$$

Therefore, we call such an extension of a function as the energy minimizer extension. In general we can define such functions as follows.

**Definition 5.2** A real valued function $h$ on $\Sigma^+_N$ is called as an energy minimizer, if for some $n \geq 0$,

$$h(x) = h(p) \quad \text{whenever } x \in [p_1 p_2 \cdots p_n p_{n+1}],$$

where $p = (p_1 p_2 \cdots p_n p_{n+1}) \in V_n$.

The energy minimizer extension of a function in $\ell(V_m)$ takes constant values on cylinder sets of length $m + 1$. For instance, if $p = (p_1 p_2 \cdots p_m \hat{p}_{m+1}) \in V_m$ and $\chi_p \in \ell(V_m)$ is its characteristic function, then its energy minimizer extension is given by $\chi^m_p : \Sigma^+_N \rightarrow \mathbb{R}$ as,

$$\chi^m_p = \begin{cases} 1 & \text{on } [p_1 p_2 \cdots p_{m+1}] \\ 0 & \text{elsewhere}. \end{cases} \quad (5.1)$$

These are the simple functions in $C(\Sigma^+_N)$ which will play a crucial role in the study of Laplacian that we will define in the following section. Having defined the energy, let us look at the effective
Let us construct a function replaced by \( V \) points in the equivalence classes generated by the point \( a \). We denote the points in each of these equivalence classes as \( \{a_i \} \). This is the longest chain of \( \leq m \) points in the equivalence classes above generated by \( a \) and \( b \), \( \delta \) points in each of these equivalence classes as, \( \{a_1 \cdots a_m\} \) with \( a_m^{-1} \in V_{m-1} \setminus V_{m-2} \). Similarly, \( \{a_1 \cdots a_{m-1}\} \) with \( a_{m-1}^{-1} \in V_{m-2} \setminus V_{m-3} \), and so on. Finally, \( \{a_1\} \) with \( a_1^{-1} = (\hat{a}_1) \in V_0 \). The points in the equivalence classes generated by the point \( b \) are denoted in the same manner with \( a \) replaced by \( b \) in the above notation.

Let us construct a function \( u \in \ell(V_m) \) as follows. Set \( u(a) = 1 \) and \( u(b) = 0 \). Let \( \delta_1, \delta_2 > 0 \) satisfying

\[
\delta_i < \frac{1}{2m(N-1)} \text{ for } i = 1, 2 \quad \text{and} \quad \delta_1^2 + \delta_2^2 < \frac{6(m^2 - m - 1)}{(m^4 + m)(N-1)^2(2N^2 - N)}.
\]

At the points in the equivalence classes above generated by \( a \), set the values of \( u \) in the decreasing manner with a difference of \( \delta_1 \) as

\[
u(a_1) = 1 - \delta_1, \quad u(a_2) = 1 - 2\delta_1, \quad \cdots, \quad u(a_{m-1}) = 1 - (N - 1)\delta_1, \quad \cdots, \quad u((\hat{a}_1)) = 1 - m(N - 1)\delta_1.
\]

Similarly, fix the values of \( u \) at points in the equivalence classes generated by \( b \), each increasing by the quantity \( \delta_2 \). At all the remaining points of \( V_m \), \( u \) is set to take the constant value \( u(\hat{a}_1) - \frac{\delta_1 + \delta_2}{2} \).

On careful observation, we note that only the points in the equivalence classes above and all the points in \( V_0 \) contribute in determining \( \mathcal{E}_{H_m}(u) \) given by equation (4.3). The terms involving all other points vanish as the function \( u \) is set to take the same constant value. Upon substituting for these values of \( u \) in the expression of \( \mathcal{E}_{H_m}(u) \) in equation (4.3), we obtain,

\[
\mathcal{E}_{H_m}(u) < \frac{1}{m + 1}.
\]

Let \( h(u) \) denote the energy minimizer taking constant values on cylinder sets of length \( m + 1 \), obtained by extending \( u \). Clearly, \( (h(u))(a) = 1 \) and \( (h(u))(b) = 0 \) and \( \mathcal{E}(h(u)) = \mathcal{E}_{H_m}(u) \). Then,

\[
\left[ \min \{\mathcal{E}(u) : u \in \text{dom} \mathcal{E}, u(a) = 1, u(b) = 0 \} \right] \leq \mathcal{E}(h(u)) < \frac{1}{m + 1}.
\]
This implies that the effective resistance between a and b is, \( R(a,b) > m + 1 \), thus proving our claim. As already discussed in the introduction, due to the unboundedness, the completion of \( V \), with respect to \( R \) will only be a proper subset of \( \Sigma_N^+ \). Thus the Laplacian on \( \Sigma_N^+ \) cannot be obtained in the topology generated by the effective resistance. Despite this fact, in the next section we prove that the Laplacian of a continuous function in the standard topology induced by the metric \( d \), can be defined, as a scalar limit of the difference operators \( H_m \).

6 The Laplacian

We begin this section by considering the discrete approximation of the Laplacian of a twice differentiable \( i.e., C^2 \) function on \( \mathbb{R} \). If \( u : \mathbb{R} \rightarrow \mathbb{R} \) is a \( C^2 \) function, then we write,

\[
\Delta u(x) = \lim_{h \to 0} \frac{1}{h^2} [u(x+h) + u(x-h) - 2u(x)].
\]

(6.1)

Observe that the right hand side in the definition of \( H_m \) as in equation (3.6) is analogous to the quantity inside the bracket on right hand side of the equation (6.1). Hence, it makes sense to use this difference operator \( H_m \) normalized appropriately and define the Laplacian on \( \Sigma_N^+ \).

Now we use the density arguments to extend the operator on the full shift space, \( \Sigma_N^+ \). The Laplacian of a function \( u \) in \( C(\Sigma_N^+) \) can now be defined as the limit of \( H_m(u|_{V_m}) \) with some proper scaling as given below.

Definition 6.1 For the equidistributed Bernoulli measure \( \mu \) defined in equation (2.1), define the set

\[
D_\mu := \left\{ u \in C(\Sigma_N^+) : \exists f \in C(\Sigma_N^+) \text{ satisfying} \right. \\
\left. \lim_{m \to \infty} \max_{p \in V_m \setminus V_{m-1}} \left| \frac{H_m u(p)}{\mu([p_1 p_2 \cdots p_{m+1}])} - f(p) \right| = 0 \right\}.
\]

(6.2)

Then, for \( u \in D_\mu \), we write \( f = \Delta_\mu u \). We call the operator \( \Delta_\mu \) as the Laplacian on \( C(\Sigma_N^+) \) and the set \( D_\mu \) is referred to as the domain of the Laplacian.

A function \( h \) on \( \Sigma_N^+ \) is called a harmonic function, if \( \Delta h = 0 \). A natural question that may arise in the readers’ minds now, is whether the domain of the Laplacian \( D_\mu \) is vacuous. Let \( h \in C(\Sigma_N^+) \) be an energy minimizer, as defined in the previous section. There exists \( n \geq 0 \) such that \( h \) is constant on the cylinder sets of length \( n + 1 \). Consider the functions \( h_m \in \ell(V_m) \) given by \( h_m = h|_{V_m} \). Then for any \( m \geq n + 1 \), \( h_m(p) - h_m(q) = 0 \) whenever \( q \in U_{p,m} \) and we have,

\[
N^{m+1} \sum_{q \in U_{p,m}} (h_m(q) - h_m(p^m)) = 0 \quad \text{for any} \quad p^m \in V_m \setminus V_{m-1}.
\]

Therefore, \( \Delta h = 0 \), which implies that every energy minimizer belongs to \( D_\mu \) and is in fact a harmonic function. Also, \( \mathcal{E}(h) = \mathcal{E}_{H_m}(h_n) \) for some \( n \geq 0 \).

The convergence required in the definition of the Laplacian above, is relatively stronger to determine a function \( f \) directly for a given \( u \in D_\mu \). In the following theorem, we derive the pointwise formulation of the Laplacian which will come in handy in most of the calculations throughout. For convenience of notations, we lose the subscript \( \mu \) for the Laplacian, since we will only consider the equidistributed Bernoulli measure \( \mu \).
Theorem 6.2 Let \( u \in D_\mu \) and \( \Delta u = f \). For any \( x \in \Sigma^+_N \), there exists a sequence of points \( \{p^m\}_{m \geq 1} \) with \( p^m \in V_m \setminus V_{m-1} \) such that

\[
f(x) = \Delta u(x) = \lim_{m \to \infty} N^{m+1} H_m u(p^m).
\]

(6.3)

Proof: Any point \( x \in \Sigma^+_N \) looks like \( x = (x_1 x_2 \cdots) \). For \( m \geq 1 \), consider the sequence of points \( \{p^m = (x_1 x_2 \cdots x_m \hat{l})\}_{m \geq 1} \), where \( \hat{l} \in S \) is chosen such that \( \hat{l} \neq x_m \). Such a selection of \( \hat{l} \) guarantees that \( p^m \in V_m \setminus V_{m-1} \) for all \( m \geq 1 \). This sequence converges to the point \( x \) in the metric \( d \). Thus, it directly follows from the definition of the Laplacian of \( u \) as given in (6.2) that,

\[
\lim_{m \to \infty} |N^{m+1} H_m u(p^m) - f(p^m)| = 0.
\]

Then, by a simple use of triangle inequality we obtain the pointwise expression for the Laplacian as stated in equation (6.3).

The domain of the Laplacian and the domain of the energy both are dense linear subspaces of the space of continuous functions on \( \Sigma^+_N \) as proved in the following theorem.

Theorem 6.3

\[ D_\mu \subset \text{dom} \ E \subset C(\Sigma^+_N). \]

Moreover, both the inclusions are dense.

Before proving this theorem, we observe the following important fact.

Lemma 6.4 Any \( u \in C(\Sigma^+_N) \) is uniformly approximated by a sequence of harmonic functions.

Proof: Let \( u \in C(\Sigma^+_N) \). For each \( m \geq 0 \), define the functions \( u_m \in C(\Sigma^+_N) \) as,

\[
u_m := \sum_{p \in V_m} u(p) \chi_p^m, \tag{6.4}
\]

where \( \chi_p^m \) is as defined in equation (5.1). Note that each \( u_m \) is a harmonic function, being an energy minimizer. Since \( u \) is uniformly continuous and \( \Sigma^+_N \) is compact, the sequence \( \{u_m\}_{m \geq 0} \) uniformly converges to \( u \). Thus, every continuous function is approximated by a sequence of harmonic functions and the convergence is uniform.

Proof: (Proof of Theorem (6.3)) The inclusion \( \text{dom} \ E \subset C(\Sigma^+_N) \) follows directly from the definition of \( \text{dom} \ E \). For the first inclusion, consider \( u \in D_\mu \) and \( f = \Delta u \). By theorem (6.2), for any \( p \in V_m \setminus V_{m-1} \) we have,

\[
\lim_{m \to \infty} N^{m+1} \sum_{q \in U_{p,m}} |u(q) - u(p)| < \infty.
\]

Since \( u \) is continuous, there exist positive constants \( C > 0 \) and \( M \in \mathbb{N} \), such that for all \( m \geq M \),

\[
|u(q) - u(p)| \leq \frac{C}{N^{m+1}} \quad \text{whenever} \quad d(q,p) \leq \frac{1}{2m+1}. \tag{6.5}
\]

Observe that the energy of the function \( u \) can be written as,

\[
E(u) = E_{H_M}(u_M) + \lim_{n \to \infty} \sum_{i=1}^{n} \sum_{p \in V_i} \sum_{q \in U_{p,i}} (u(q) - u(p))^2.
\]
Clearly for \( q \in U_{p, i} \), with \( i \geq M + 1 \), we have, \( d(q, p) = \frac{1}{2^{i+1}} \leq \frac{1}{2^{M+1}} \). Thus by equation (6.5), we find a bound for \( \mathcal{E}(u) \) as,

\[
\mathcal{E}(u) \leq \mathcal{E}_{H_M}(u_M) + \frac{C^2(N-1)}{2} \lim_{n \to \infty} \sum_{i=M+1}^{n} \sum_{p \in V_i} \frac{1}{(N+i+1)^2}
\]

\[
= \mathcal{E}_{H_M}(u_M) + \frac{C^2(N-1)}{2} \lim_{n \to \infty} \sum_{i=M+1}^{n} \frac{1}{N+i+1}
\]

\[
< \mathcal{E}_{H_M}(u_M) + \frac{C^2(N-1)}{2} \sum_{i=1}^{\infty} \frac{1}{N+i+1}
\]

\[
< \infty.
\]

Therefore, \( D_{\mu} \subset \text{dom} \mathcal{E} \).

Lemma (6.4) states that the harmonic functions are dense in the set \( C(\Sigma_N^+) \). Since the harmonic functions are the members of \( D_{\mu} \), the inclusions in the statement of the theorem are dense. •

In the next part of this paper we proceed towards establishing the existence and uniqueness of a solution to the Dirichlet boundary value problem as stated in theorem (1.1). We follow the standard approach to obtain the Green’s function and the Green’s operator which produces the required solution.

7 Green’s function

In this section, we define the Green’s function on \( \Sigma_N^+ \). Recall that the matrix \( X_m \), as defined in equation (3.7) is symmetric and invertible. All the diagonal entries of \( X_m \) are \(-(N - 1)\). Among the non-diagonal entries, the row corresponding to any point \( p \in V_m \setminus V_{m-1} \) contains 1 at \( N - 2 \) places which correspond to the \( N - 2 \) neighbours of \( p \) in \( U_{p, m} \). The remaining entries in the matrix \( X_m \) are all 0.

**Lemma 7.1** Consider the matrix \( G_m : \ell(V_m \setminus V_{m-1}) \longrightarrow \ell(V_m \setminus V_{m-1}) \) defined by

\[
(G_m)_{pq} = \begin{cases} 
\frac{2}{N} & \text{if } q = p, \\
\frac{1}{N} & \text{if } (X_m)_{pq} = 1, \\
0 & \text{otherwise}.
\end{cases}
\]  

(7.1)

Then, \( X_m^{-1} = -G_m \).

**Proof:** Let us first calculate the diagonal entries of \( (X_m) \times (-G_m) \). For any \( p \in V_m \setminus V_{m-1} \) we get,

\[
((X_m) (-G_m))_{pp} = \sum_{r \in V_m \setminus V_{m-1}} [(X_m)_{pr} (-G_m)_{rp}]
\]

\[
= (X_m)_{pp} (-G_m)_{pp} + \sum_{r \in U_{p, m}} [(X_m)_{pr} (-G_m)_{rp}]
\]

\[
= -(N-1) \frac{2}{N} + (N-2) \frac{-1}{N}
\]

\[
= 1.
\]
For the non-diagonal entries, consider any two distinct points \( p, q \in V_m \setminus V_{m-1} \). Then we have,

\[
((X_m)(-G_m))_{pq} = (X_m)_{pp}(-G_m)_{pq} + (X_m)_{pq}(-G_m)_{qq} + \sum_{\substack{r \in V_m \setminus V_{m-1} \\ r \neq q, p}} [(X_m)_{pr}(-G_m)_{rq}]
\]

\[
= -(N-1)(-G_m)_{pq} + (X_m)_{pq}\left(\frac{-2}{N}\right) + \sum_{\substack{r \in U_{p, m} \\ r \neq q}} (-G_m)_{rq}. \quad (7.2)
\]

If \( q \in U_{p, m} \) then \((X_m)_{pq} = 1\), \((G_m)_{pq} = (G_m)_{rq} = \frac{1}{N}\) whenever \( r \in U_{p, m} \) with \( r \neq q \). Substituting these values in equation \((7.2)\), we get

\[
((X_m)(-G_m))_{pq} = -(N-1)\left(\frac{-1}{N}\right) + \frac{-2}{N} + (N-3)\left(\frac{-1}{N}\right) = 0.
\]

If \( q \notin U_{p, m} \) then \((X_m)_{pq} = (G_m)_{pq} = 0\). Now consider the third term in equation \((7.2)\). For \( r \in U_{p, m} \), we know that \( r \sim_m p \) with \( r \neq q \). Since the \( m \)-relation \( \sim_m \) is transitive, we have \( r \notin U_{q, m} \) and \((G_m)_{rq} = 0\). Therefore in this case too, we obtain

\[
[(X_m)(-G_m)]_{pq} = 0.
\]

\[\bullet\]

**Definition 7.2** Let \( G_m \) be the matrix as given in \((7.1)\). We define the Green’s function \( g : \Sigma_N^+ \times \Sigma_N^+ \rightarrow \mathbb{R} \cup \{\infty\} \) as,

\[
g(x, y) = \begin{cases} 
\frac{\rho(x,y)-1}{m} \sum_{m=1}^{\rho(x,y)-1} \sum_{r, s \in V_m \setminus V_{m-1}} (G_m)_{rs} \chi_r^m(x) \chi_s^m(y) & \text{if } \rho(x,y) > 1, \\
0 & \text{if } \rho(x,y) = 1.
\end{cases} \quad (7.3)
\]

Here \( \rho(x,y) \) is the first instance where \( x \) and \( y \) disagree, as defined in section \((2)\).

**Lemma 7.3** For any \( x, y \in \Sigma_N^+ \) with \( \rho(x,y) > 1 \),

\[
0 \leq g(x,y) \leq \frac{2\rho(x,y) - 3}{N}.
\]

**Proof:** Let \( x = (x_1 x_2 \cdots), y = (y_1 y_2 \cdots) \in \Sigma_N^+ \) such that \( \rho(x,y) \geq 2 \). Since all the entries of the matrix \( G_m \) are non-negative, it is clear that \( g(x,y) \geq 0 \). We know that there exist unique points \( r^m = (x_1 x_2 \cdots x_m \hat{x}_{m+1}) \) and \( s^m = (y_1 y_2 \cdots y_m \hat{y}_{m+1}) \) in \( V_m \) such that \( \chi_r^m(x) = 1 \) and \( \chi_s^m(y) = 1 \).

Suppose now that the point \( x \) and \( y \) are such that \( x_m \neq x_{m+1} \) and \( y_m \neq y_{m+1} \) for all \( 1 \leq m \leq \rho(x,y) - 1 \). Since \( x_m = y_m \) for all \( 1 \leq m \leq \rho(x,y) - 1 \), we have,

\[
r^m = s^m \quad \text{for all } \quad 1 \leq m \leq \rho(x,y) - 2
\]

\[
r^m \neq s^m \quad \text{for } \quad m = \rho(x,y) - 1.
\]

Thus,

\[
(G_m)_{r^ms^m} = \begin{cases} 
\frac{2}{N} & \text{for all } \quad 1 \leq m \leq \rho(x,y) - 2 \\
\frac{1}{N} & \text{when } m = \rho(x,y) - 1.
\end{cases}
\]
Substituting these values in equation (7.3), we get,
\[ g(x, y) = \frac{2}{N} (\rho(x, y) - 2) + \frac{1}{N} = \frac{2 \rho(x, y) - 3}{N}. \]
Suppose \( x \) and \( y \) are such that for some \( 1 \leq m \leq \rho(x, y) - 1 \), either \( x_m = x_{m+1} \) or \( y_m = y_{m+1} \). In this case either \( r^m \notin V_m \setminus V_{m-1} \) or \( s^m \notin V_m \setminus V_{m-1} \) respectively. In the definition of the Green’s function as in equation (7.3), the terms corresponding to such values of \( m \) do not contribute to the sum. Hence, we have the bound
\[ g(x, y) \leq \frac{2 \rho(x, y) - 3}{N}. \]

**Theorem 7.4** The Green’s function satisfies the following properties:

1. For any \( p \in V_m \setminus V_{m-1} \) and \( y \in \Sigma_N^+ \), \( g(p, y) < \infty \).
2. For \( x \in \Sigma_N^+ \setminus V_s \), \( g(x, x) = \infty \).
3. The Green’s function is continuous \((\mu \times \mu)\)-almost everywhere.

**Proof:**

1. Let \( p \in V_m \setminus V_{m-1} \) and \( m > M \). Then for any \( r \in V_m \setminus V_{m-1} \), we have \( \chi^m_r(p) = 0 \). So the sum in the equation (7.3) reduces to
\[ g(p, y) = \sum_{m=1}^{M} \sum_{r,s \in V_m \setminus V_{m-1}} (G_m)_{rs} \chi^m_r(p) \chi^m_s(y) < \infty. \tag{7.4} \]

2. If \( x \in \Sigma_N^+ \setminus V_s \), then \( \rho(x, x) = \infty \) and thus \( g(x, x) = \infty \).

3. Since \( V_s \) is a countable set, \( \mu(V_s) = 0 \). We prove the continuity of the Green’s function on the set \( \{x \in \Sigma_N^+ \times \Sigma_N^+ : x \in V_s\} \). Let \( (x, y) \in \Sigma_N^+ \setminus \Sigma_N^+ \) and \( (x^n, y^n) \) be a sequence converging to \((x, y)\). Then \( \rho(x^n, y^n) \rightarrow \rho(x, y) \) as \( n \rightarrow \infty \).

If \( x = y \in \Sigma_N^+ \setminus V_s \) then \( g(x, y) = \infty \) and \( g(x^n, y^n) \rightarrow \infty \) as \( \rho(x^n, y^n) \rightarrow \infty \).

If \( x, y \in \Sigma_N^+ \) such that \( x \neq y \), then \( \rho(x, y) < \infty \) and there exists some \( M_0 \in \mathbb{N} \) such that \( \rho(x^n, y^n) = \rho(x, y) \) for all \( n \geq M_0 \). Thus we obtain,
\[
g(x^n, y^n) = \sum_{m=1}^{\rho(x^n, y^n)-1} \sum_{r,s \in V_m \setminus V_{m-1}} (G_m)_{rs} \chi^m_r(x^n) \chi^m_s(y^n) = \sum_{m=1}^{\rho(x, y)-1} \sum_{r,s \in V_m \setminus V_{m-1}} (G_m)_{rs} \chi^m_r(x) \chi^m_s(y) \quad \text{for all } n \geq M_0.
\]

Thus, \( g(x^n, y^n) \rightarrow g(x, y) \) as \( n \rightarrow \infty \) in both the cases, proving the almost everywhere continuity of \( g \).

\[ \bullet \]
8 Green’s operator

In this section we define the Green’s operator and study some of its properties.

Definition 8.1 Let $L^1(\Sigma_N^+)$ be the space of $\mu$-integrable functions on $\Sigma_N^+$. We define the Green’s operator on $L^1(\Sigma_N^+)$ as an integral operator whose kernel is the Green’s function as,

\[ G_\mu f(x) := \int_{\Sigma_N^+ \setminus \{x\}} g(x, y) f(y) \, d\mu(y) \quad \text{for } f \in L^1(\Sigma_N^+). \]

As we proved in the last section, for any $x \in \Sigma_N^+ \setminus V_\ast$, $g(x, x) = \infty$. Since the points have no mass, we remove the point $x$ from the domain of the integration in the definition above.

Theorem 8.2 Let $f \in L^1(\Sigma_N^+)$. The Green’s operator satisfies the following:

1. $G_\mu f \in L^1(\Sigma_N^+)$. 
2. If $f \in C(\Sigma_N^+)$, then $G_\mu f \in C(\Sigma_N^+)$. 
3. $G_\mu f|_{V_0} = 0$.

Proof:

1. Fix $M \in \mathbb{N}$. Observe that,

\[ |G_\mu f(x)| \leq \int_{\Sigma_N^+ \setminus \{x\}} |g(x, y) f(y)| \, d\mu(y) + \int_{\{x\}} |g(x, y) f(y)| \, d\mu(y). \]

For any $y \in \Sigma_N^+ \setminus [x_1 \cdots x_M]$, we have $\rho(x, y) \leq M$ and thus by lemma (7.3) we have $|g(x, y)| \leq \frac{2M - 3}{N}$. Since $f \in L^1(\Sigma_N^+)$, the first integral above can be bounded by,

\[ \int_{\Sigma_N^+ \setminus [x_1 \cdots x_M]} |g(x, y) f(y)| \, d\mu(y) \leq \frac{2M - 3}{N} \int_{\Sigma_N^+ \setminus [x_1 \cdots x_M]} |f(y)| \, d\mu(y) \leq \frac{2M - 3}{N} \|f\|_{L^1} < \infty. \]

Let us now look at the second integral. For $i \geq 1$, consider the sets

\[ C_i := \bigcup_{y_{M+i} \in S \atop y_{M+i} \neq x_{M+i}} [x_1 \cdots x_{M+i-1} y_{M+i}]. \]

Note that $\{C_i : i \geq 1\}$ forms a partition of $[x_1 \cdots x_M] \setminus \{x\}$. As $y_{M+i}$ can be any of the $N - 1$ symbols from $S$ other than $x_{M+i}$, $\mu(C_i) = \frac{N-1}{N^M}$. Since $\rho(x, y) = M + i$ for any $y \in C_i$, again by lemma (7.3) we have $|g(x, y)| \leq \frac{2(M + i) - 3}{N}$. Thus we obtain a bound for the second integral as,

\[ \int_{[x_1 \cdots x_M] \setminus \{x\}} |g(x, y) f(y)| \, d\mu(y) \leq \|f\|_{L^1} \left[ \sum_{i \geq 1} \int_{C_i} \frac{(2(M + i) - 3)}{N} \, d\mu(y) \right]. \]
Let \( \rho \in C(\Sigma_N^+) \). Then \( f \) is bounded and \( \| f \|_\infty = \sup_{x \in \Sigma_N^+} |f(x)| < \infty \). We are interested to prove that \( G_\mu f \) is continuous at any point \( x = (x_1 x_2 \cdots) \in \Sigma_N^+ \). Let us take a sequence of points \( \{x^n\}_{n \in \mathbb{N}} \) such that \( x^n \to x \) in \( \Sigma_N^+ \), and \( x^n \neq x \) for any \( n \geq 1 \). Therefore for every \( n \geq 1 \), there exists \( M \in \mathbb{N} \) (which depends on \( n \)) such that \( d(x^n, x) = \frac{1}{2^M n} \) or equivalently, \( x^n \in [x_1 \cdots x_M] \) and \( \rho(x^n, x) = M + 1 \). Clearly, as \( n \to \infty \), \( x^n \to x \) and \( M \to \infty \). Consider,

\[
|G_\mu f(x) - G_\mu f(x^n)| \leq \int_{\Sigma_N^+ \setminus \{x, x^n\}} |g(x,y) - g(x^n,y)||f(y)| \, d\mu(y). \tag{8.1}
\]

We now analyse the term \( |g(x,y) - g(x^n,y)| \) for all possible combinations of \( \rho(x,y) \) and \( \rho(x^n,y) \). First, if \( \rho(x,y) = 1 \) then \( \rho(x^n,y) = 1 \) and thus by the definition of the Green’s function, \( g(x,y) = g(x^n,y) = 0 \). If \( 1 < \rho(x,y) \leq M \) then \( \rho(x^n,y) = \rho(x,y) = \rho \) (say). Therefore,

\[
g(x,y) - g(x^n,y) = \sum_{m=1}^{\rho - 1} \sum_{r,s \in V_m \setminus V_{m-1}} [(G_m)_{rs} \chi^m_r(x) \chi^m_s(y) - (G_m)_{rs} \chi^m_r(x^n) \chi^m_s(y)].
\]

In this case, since \( \rho \leq M \) and \( x \) and \( x^n \) agree on the initial \( M \) places, we have for every \( m \leq \rho - 1 \), \( \chi^m_r(x) = 1 \) if and only if \( \chi^m_r(x^n) = 1 \). Therefore all the terms in this sum get cancelled and we get \( g(x,y) - g(x^n,y) = 0 \). In short, for any \( y \notin [x_1 \cdots x_M] \), \( |g(x,y) - g(x^n,y)| = 0 \). Thus the integration in equation (8.1) reduces to the following integration on the set \([x_1 \cdots x_M] \setminus \{x, x^n\}\),

\[
|G_\mu f(x) - G_\mu f(x^n)| \leq \int_{[x_1 \cdots x_M] \setminus \{x, x^n\}} |g(x,y) - g(x^n,y)||f(y)| \, d\mu(y). \tag{8.2}
\]

Set \( A := [x_1 \cdots x_M] \setminus \{x^n, x\} \). For any \( y \in A \), \( y \neq x^n \) and \( y \neq x \). This implies, \( \rho(x,y) \) and \( \rho(x^n,y) \) are finite and \( |\rho(x,y) - \rho(x^n,y)| = k < \infty \), for some \( k \in \mathbb{N} \). Let us define the sets

\[ A_k := \{y \in A : |\rho(x,y) - \rho(x^n,y)| = k\} \cdot \]

Clearly, all \( A_k \)s are mutually disjoint and \( A = \bigcup_{k \geq 0} A_k \).

Let us first consider \( y \in A_0 \).

**Case I:** Suppose \( \rho(x,y) = \rho(x^n,y) = M + 1 \). Then \( y \in [x_1 \cdots x_M] \) such that \( y_{M+1} \neq x_{M+1} \) and \( y_{M+1} \neq x_{M+1}^n \).

**Case II:** Suppose \( y \in A \) such that \( \rho(x,y) \geq M + 2 \). Then \( y_{M+1} = x_{M+1} \) which implies \( y_{M+1} \neq x_{M+1}^n \) and \( \rho(x^n,y) = M + 1 \). For such a choice of \( y \), \( |\rho(x,y) - \rho(x^n,y)| = 1 \), thus
y ⤷ A₀. Note that the roles of x and xⁿ are interchangeable, so by the same argument, any y ∈ A satisfying ρ(xⁿ, y) ≥ M + 2 cannot belong to A₀.

This tells us that any y ∈ A₀ satisfies the condition in case I. The measure of A₀ is then obtained as,

\[ \mu( A₀ ) = \frac{1}{N^M} \frac{N - 2}{N}. \]

Now, for y ∈ A₀, consider,

\[ |g(x, y) - g(xⁿ, y)| = \sum_{m=1}^{M} \sum_{r,s \in Vₘ \setminus Vₘ₋₁} (Gₘ)_{rs} \chiₗₘ(x) \chiₗₘ(y) \]

\[ - \sum_{m=1}^{M} \sum_{r,s \in Vₘ \setminus Vₘ₋₁} (Gₘ)_{rs} \chiₗₘ(xⁿ) \chiₗₘ(y). \]

Here, x, xⁿ and y agree on first M coordinates. Thus, all the terms in the above sum get cancelled for 1 ≤ m ≤ M - 1, and we are left with the terms only corresponding to m = M.

\[ |g(x, y) - g(xⁿ, y)| = \sum_{r,s \in Vₘ \setminus Vₘ₋₁} (Gₘ)_{rs} \chiₗₘ(x) \chiₗₘ(y) \]

\[ - \sum_{r,s \in Vₘ} (Gₘ)_{rs} \chiₗₘ(xⁿ) \chiₗₘ(y). \]

In the term corresponding to x, \( \chiₗₘ(x) = 1 \) and \( \chiₗₘ(y) = 1 \) if and only if \( r = (x₁ \cdots xₘ \tilde{x}_{M+1}) \) and \( s = (x₁ \cdots xₘ \tilde{y}_{M+1}) \). Here \( r \neq s \), and depending on whether \( r \) and \( s \) belong to \( Vₘ \setminus Vₘ₋₁ \) or not, the only possible values of corresponding \((Gₘ)_{rs}\) are \( \frac{1}{N} \) or 0. Further, the minimum value that the term \( \sum_{r,s \in Vₘ} (Gₘ)_{rs} \chiₗₘ(xⁿ) \chiₗₘ(y) \) can take, is 0. Therefore,

\[ |g(x, y) - g(xⁿ, y)| \leq \frac{1}{N} \text{ for all } y ∈ A₀. \]

Let us now fix \( k ≥ 1 \) and consider \( y ∈ A_k \).

**Case III:** Suppose \( ρ(xⁿ, y) = M + 1 + k \). Then \( y ∈ [x₁ \cdots xₘ] \) such that \( y_{M+1} \neq x_{M+1} \) and \( ρ(x, y) = M + 1 \).

**Case IV:** Similar to the case III, another possible choice for y is when \( ρ(x, y) = M + 1 + k \) and \( ρ(xⁿ, y) = M + 1 \).

**Case V:** Suppose \( y \in A \) such that \( ρ(x, y) \neq M + 1 + k \) and \( ρ(xⁿ, y) \neq M + 1 + k \). Then \( ρ(xⁿ, y) = M + 1 \) which results in \( |ρ(x, y) - ρ(xⁿ, y)| \neq k \) and \( y \notin A_k \). Same argument holds if \( ρ(xⁿ, y) \neq M + 1 + k \) and \( ρ(xⁿ, y) \neq M + 1 + k \).

This establishes that any \( y \in A_k \) will satisfy the condition of either case III or case IV. Therefore the measure of \( A_k \) can be given by,

\[ \mu(A_k) = \frac{2}{N^{M+k}} \frac{N - 1}{N}. \]
Let \( y \in A_k \) which belongs to case IV, that is, \( \rho(x, y) = M + 1 + k \) and \( \rho(x^n, y) = M + 1 \). We get the same result for case III as well, so it is enough to work with case IV. Consider,

\[
|g(x, y) - g(x^n, y)| = \sum_{m=1}^{M+k} \sum_{r,s \in V_m \backslash V_{m-1}} (G_m)_{rs} \chi_r^m(x) \chi_s^m(y) - \sum_{m=1}^{M} \sum_{r,s \in V_m \backslash V_{m-1}} (G_m)_{rs} \chi_r^m(x^n) \chi_s^m(y).
\]

As discussed before, all the terms corresponding to \( 1 \leq m \leq M - 1 \) in the above expression get cancelled and the above sum reduces to,

\[
|g(x, y) - g(x^n, y)| = \sum_{m=M}^{M+k} \sum_{r,s \in V_m \backslash V_{m-1}} (G_m)_{rs} \chi_r^m(x) \chi_s^m(y) - \sum_{r,s \in V_M} (G_M)_{rs} \chi_r^M(x^n) \chi_s^M(y).
\]

Consider the term corresponding to \( x \) in the above sum. Observe that for \( M \leq m \leq M+k-1 \), \( \chi_r^m(x) = 1 \) and \( \chi_s^m(y) = 1 \) if and only if \( r = s = (x_1 \cdots x_{m+1}) \in V_m \backslash V_{m-1} \). Therefore \( (G_m)_{rs} = \frac{2}{N} \) or 0. For \( m = M + k \), \( \chi_r^{M+k}(x) = 1 \) and \( \chi_s^{M+k}(y) = 1 \) if and only if \( r = (x_1 \cdots x_{M+k} x_{M+k+1}) \) and \( s = (x_1 \cdots x_{M+k} y_{M+k+1}) \). Clearly \( r \neq s \) and depending on whether \( r \) and \( s \) belong to \( V_{M+k} \backslash V_{M+k-1} \) or not, we get \( (G_m)_{rs} = \frac{1}{N} \) or 0. The minimum value of the term in the above sum corresponding to \( x^n \) is 0. Substituting for these values we get,

\[
|g(x, y) - g(x^n, y)| \leq \frac{2k}{N} + \frac{1}{N} < \frac{2(k+1)}{N} \quad \text{for all} \quad y \in A_k.
\]

It is easy to verify that \( \mu(A) = \sum_{k \geq 0} \mu(A_k) \). Let us now evaluate the required integration from the equation \((8.2)\).

\[
|G_\mu f(x) - G_\mu f(x^n)| \leq \int_A \left| g(x, y) - g(x^n, y) \right| |f(y)| \, d\mu(y)
\]

\[
= \sum_{k \geq 0} \int_{A_k \backslash \{x, x^n\}} \left| g(x, y) - g(x^n, y) \right| |f(y)| \, d\mu(y)
\]

\[
< \frac{1}{N} \|f\|_\infty \left( \frac{N - 2}{NM+1} \right) + \sum_{k \geq 1} \frac{2(k+1)}{N} \|f\|_\infty \left( \frac{2(N - 1)}{NM+1+k} \right)
\]

\[
= \frac{1}{NM+2} \|f\|_\infty \left( N - 2 + 4(N - 1) \sum_{k \geq 1} \frac{k+1}{Nk} \right)
\]

\[
= \frac{C}{NM+2},
\]

where the ratio test guarantees the convergence of the series \( \sum_{k \geq 1} \frac{k+1}{Nk} \), and the constant \( C \) depends only on \( f \) and \( N \). Therefore, as \( n \to \infty \), \( M \to \infty \) and we can conclude that \( |G_\mu f(x) - G_\mu f(x^n)| \to 0 \), proving the continuity of \( G_\mu f \).
We only need to prove \( g((\tilde{i}), y) = 0 \) for any \( y \in \Sigma_N^+ \). Thus we have \( G_\mu f|_{V_0} = 0 \).

\[ \bullet \]

9 The solution to the BVP in theorem (1.1)

The objective of this section is to find the solutions to the analogous Dirichlet boundary value problem on the full one-sided shift space \( \Sigma_N^+ \). We begin with the following two lemmas.

**Lemma 9.1** For any \( n \geq 1 \) and \( p \in V_n \setminus V_{n-1} \),

\[
H_n G_\mu f(p) = -\int_{\Sigma_N^+} \chi_p^n f \, d\mu.
\]

**Proof:** Let \( q^1, q^2, \ldots, q^{N-1} \in U_{p,n} \) be as defined in section (2). Using the definition of \( H_m \) as in equation (3.6) and the definition of Green’s operator we obtain,

\[
H_n G_\mu f(p) = -(N-1) G_\mu f(p) + G_\mu f(q^1) + G_\mu f(q^2) + \cdots + G_\mu f(q^{N-1})
\]

\[
= \int_{\Sigma_N^+ \setminus \{p, q^1, \ldots, q^{N-1}\}} \left[ -(N-1) g(p, y) + g(q^1, y) + \cdots + g(q^{N-1}, y) \right] f(y) \, d\mu(y).
\]

We only need to prove

\[-(N-1) g(p, y) + g(q^1, y) + \cdots + g(q^{N-1}, y) = -\chi_p^n(y). \]

We know that \( p, q^1, \ldots, q^{N-2} \in V_n \setminus V_{n-1} \) and \( q^{N-1} \in V_{n-1} \). Therefore by equation (7.4) in the proof of Theorem (7.4) we have,

\[
-(N-1) g(p, y) + g(q^1, y) + \cdots + g(q^{N-1}, y)
\]

\[
= -(N-1) \left[ \sum_{m=1}^{n} \sum_{r,s \in V_m \setminus V_{m-1}} (G_m)_{rs} \chi_r^m(p) \chi_s^m(y) \right]
\]

\[
+ \sum_{m=1}^{n} \sum_{r,s \in V_m \setminus V_{m-1}} (G_m)_{rs} \chi_r^m(q^1) \chi_s^m(y)
\]

\[
+ \cdots + \sum_{m=1}^{n} \sum_{r,s \in V_m \setminus V_{m-1}} (G_m)_{rs} \chi_r^m(q^{N-2}) \chi_s^m(y)
\]

\[
+ \sum_{m=1}^{n-1} \sum_{r,s \in V_m \setminus V_{m-1}} (G_m)_{rs} \chi_r^m(q^{N-1}) \chi_s^m(y)
\]

\[
= -(N-1) \sum_{r,s \in V_n \setminus V_{n-1}} (G_n)_{rs} \chi_r^n(p) \chi_s^n(y)
\]

\[
+ \sum_{r,s \in V_n \setminus V_{n-1}} (G_n)_{rs} \chi_r^n(q^1) \chi_s^n(y) \quad (9.1)
\]
Here note that, for $1 \leq m \leq n - 1$, all the terms get cancelled and only the terms with $m = n$ remain. We now examine when this term will be nonzero. In the first term in equation (9.1) above, we get, 

After substituting the values of $(G_n)_{rs}$ for the corresponding choices of $r$ and $s$ in each term in equation (9.1) above, we get,

$$
-(N - 1) g(p, y) + g(q^1, y) + \cdots + g(q^{N-1}, y)
$$

$$
= -(N - 1) \left[ \frac{2}{N} \chi^n_p(y) + \frac{1}{N} \chi^n_q(y) + \cdots + \frac{1}{N} \chi^n_{q^{N-2}}(y) \right]
$$

$$
+ \cdots +
$$

$$
+ \frac{1}{N} \chi^n_p(y) + \frac{1}{N} \chi^n_q(y) + \cdots + \frac{2}{N} \chi^n_{q^{N-2}}(y)
$$

$$
= \chi^n_p(y) \left[ -\frac{2(N - 1)}{N} + \frac{1}{N} + \cdots + \frac{1}{N} \right]
$$

$$
+ \chi^n_q(y) \left[ -\frac{(N - 1)}{N} + \frac{2}{N} + \frac{1}{N} + \cdots + \frac{1}{N} \right]
$$

$$
+ \cdots +
$$

$$
+ \chi^n_{q^{N-2}}(y) \left[ -\frac{(N - 1)}{N} + \frac{1}{N} + \cdots + \frac{1}{N} + \frac{2}{N} \right]
$$

$$
= -\chi^n_p(y).
$$

\[ \bullet \]

**Lemma 9.2** For any $f \in C(\Sigma_N^+)$, we have $G_\mu f \in D_\mu$ and $\Delta (G_\mu f) = -f$.

**Proof:** Let $m \geq 1$ and $p \in V_m \setminus V_{m-1}$. With the help of Lemma (9.1), we get,

$$
\left| N^{m+1} H_m G_\mu f(p) + f(p) \right|
$$

$$
= -\int_{\Sigma_N^+} N^{m+1} \chi^n_p(y) f(y) d\mu(y) + \int_{\Sigma_N^+} N^{m+1} \chi^n_p(y) f(p) d\mu(y)
$$

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\[ \leq \int_{[p_1, p_2 \cdots p_{m+1}]} N^{m+1} |f(p) - f(y)| \, d\mu(y) \]

where \( \epsilon_m := \sup_{y \in [p_1, p_2 \cdots p_{m+1}]} |f(p) - f(y)| \).

Since \( f \) is uniformly continuous, \( \epsilon_m \to 0 \) as \( m \to \infty \). Therefore, \( \Delta (G_\mu f) = -f \).

Based on the ideas developed in this paper, we now restate our main theorem (1.1) and prove the same to conclude this work.

Theorem 9.3 For any \( f \in C(\Sigma_N^+) \) and \( \zeta \in \ell(V_0) \), there exists a continuous function \( u \in D_\mu \) such that the following holds:

\[ \Delta u = f, \quad u|_{V_0} = \zeta. \] (9.2)

This solution is unique up to the harmonic functions taking value 0 on the boundary \( V_0 \).

Proof: Define a function \( u: \Sigma_N^+ \to \mathbb{R} \) as,

\[ u := \sum_{p \in V_0} \zeta(p) \chi_p^m - G_\mu f. \]

Observe that \( \sum_{p \in V_0} \zeta(p) \chi_p^m \) is a harmonic function and thus its Laplacian is 0. Due to lemma (9.2) and the linearity of the Laplacian \( \Delta \), we obtain \( G_\mu f \in D_\mu \) and thus \( u \in D_\mu \) with

\[ \Delta u = -\Delta (G_\mu f) = f. \]

Since \( (G_\mu f)|_{V_0} = 0 \), clearly this choice of \( u \) satisfies \( u|_{V_0} = \zeta \). Further, if \( h: \Sigma_N^+ \to \mathbb{R} \) is any harmonic function satisfying \( h|_{V_0} = 0 \), then it is trivial to see that the function \( u + h \) is a solution to (9.2).

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