ITERATED SHIMURA INTEGRALS

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Abstract. In this paper I continue the study of iterated integrals of modular forms and noncommutative modular symbols for \( \Gamma \subset SL(2, \mathbb{Z}) \) started in [Ma3]. Main new results involve a description of the iterated Shimura cohomology and the image of the iterated Shimura cocycle class inside it. The concluding section of the paper contains a concise review of the classical modular symbols for \( SL(2) \) and a discussion of open problems.

§0. Introduction

Let \( M \) be a linearly connected space, and \( G \) a group acting on it. Then \( G \) acts on the fundamental groupoid of \( M \) thus creating a situation where the well known formalism of cohomology of \( G \) with noncommutative coefficients applies.

If \( M \) is a differentiable manifold, Chen iterated integrals produce a representation of the fundamental groupoid so that we get relations between such integrals reflecting the action of \( G \).

In [Ma3] I have studied this situation for the case when \( M \) is the upper complex half plane partially completed by cusps, and the iterated integrals involve cusp forms (and eventually Eisenstein series). The questions asked and the form of answers I would like to get in this case were motivated by Drinfeld’s associators and the classical theory of ordinary integrals including the basics of Mellin transform and modular symbols.

Here I continue this study, stressing the Shimura approach to the \( SL(2) \)-modular symbols of arbitrary weight and attempting its iterated extension.

The paper is structured as follows. In §1 the notation and some background of noncommutative group cohomology is reviewed. In §2 the theory of the iterated Shimura cocycle is given. Finally, §3 sketches the classical theory of modular symbols and discusses open problems. The reader might prefer to read this section first, as a motivation for our attempt to produce its iterated version.
§1. Noncommutative cohomology
and abstract Shimura–Eichler relations

In this section I set notation and collect some general background facts.

1.1. Noncommutative group cohomology: a general formalism. Let $G$ be a group, and $N$ a group endowed with a left action of $G$ by automorphisms: $(g, n) \mapsto gn$. Generally, both $G$ and $N$ can be noncommutative, and the group laws are written multiplicatively.

The set of 1–cocycles is defined by

$$Z^1(G, N) := \{ u : G \to N \mid u(g_1g_2) = u(g_1)g_1u(g_2) \}.$$ 

It follows that $u(1_G) = 1_N$.

Two cocycles are cohomological, $u' \sim u$, iff there exists an $n \in N$ such that for all $g \in G$ we have $u'(g) = n^{-1}u(g) \cdot gn$. This is an equivalence relation, and by definition,

$$H^1(G, N) := Z^1(G, N)/\sim.$$

This is a set with a marked point: the class of the trivial cocycles $u_n(g) = n^{-1}gn$.

Assume now that $G$ is embedded into a larger group $G \subset H$, $[H : G] < \infty$. Denote by $N_H$ the induced noncommutative $H$–module: $N_H$ is the space of $G$–covariant maps $\phi : H \to N$, $\phi(gh) = g\phi(h)$ for all $g \in G$, $h \in H$, with pointwise multiplication and the left action of $H$

$$(h\phi)(h') := \phi(h'h).$$

The map $N_H \to N : \phi \mapsto \phi(1_G)$, is a group homomorphism, compatible with the action of $G$. Hence it induces a map of pointed sets $Z^1(H, N_H) \to Z^1(G, N)$. One easily checks that cohomological cocycles go to the cohomological ones so that we have an induced map $c : H^1(H, N_H) \to H^1(G, N)$.

1.1.1. Proposition (noncommutative Shapiro Lemma). The map $c$ is a bijection.

For a proof, see [PlRap], I.1.3. Here we only describe for future use a map $Z^1(G, N) \to Z^1(H, N_H)$ which sends equivalent cocycles to equivalent ones and induces the inverse map $c^{-1} : H^1(G, N) \to H^1(H, N_H)$.

To this end we will slightly modify notation: for a cocycle $u : G \to N$ and $g \in G$, we will now denote by $u_g \in N$ the former $u(g)$. We want to produce from
u a cocycle \( \tilde{u} \) whose value at \( h \in H \) will be denoted \( \tilde{u}_h \in N_H \). Thus, \( \tilde{u}_h \) is a \( G \)-covariant function \( H \to N \) whose value at \( h' \in H \) will be denoted \( \tilde{u}_h(h') \). A well defined prescription for obtaining this value, according to [PlRap], requires a choice of representatives of \( G \setminus H \) in \( H \): let \( H = \bigsqcup_i G h_i \), such that \( G \) is represented by \( 1_H \). Then we put, for any \( g, g' \in G \):

\[
\tilde{u}_{gh_i}(g'h_j) := g'u_{gji},
\]

where \( g_{ji} \in G \) is determined by

\[
h_jgh_i = g_{ji}h_k
\]

for some representative \( h_k \).

1.2. Cohomology of \( PSL(2, \mathbb{Z}) \). Let now \( G = PSL(2, \mathbb{Z}) \) and \( N \) a noncommutative \( G \)-module. It is known that \( PSL(2, \mathbb{Z}) \) is the free product of its two subgroups \( \mathbb{Z}_2 \) and \( \mathbb{Z}_3 \) generated respectively by

\[
\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}.
\]

\( PSL(2, \mathbb{Z}) \) acts transitively on \( \mathbb{P}^1(\mathbb{Q}) \), the set of cusps of upper complex half-plane. The stabilizer of \( \infty \) is a cyclic subgroup \( G_\infty \) generated by \( \sigma \tau \). Hence the stabilizer \( G_a \) of any cusp \( a \in \mathbb{P}^1(\mathbb{Q}) \) is generated by \( g^{-1}\sigma \tau g \) where \( ga = \infty \).

Below we will give a concise description of the set \( H^1(PSL(2, \mathbb{Z}), N) \) and its cuspidal subset \( H^1(PSL(2, \mathbb{Z}), N)_{\text{cusp}} \) consisting by definition of those cocycle classes that become trivial after restriction to any \( G_a \).

1.2.1. Proposition. (i) Restriction to \( (\sigma, \tau) \) of any cocycle in \( Z^1(PSL(2, \mathbb{Z}), N) \) belongs to the set

\[
\{ (X, Y) \in N \times N \mid X \cdot \sigma X = 1, Y \cdot \tau Y \cdot \tau^2 Y = 1 \}. \tag{1.2}
\]

(ii) Conversely, any element of the set (1.2) comes from a unique 1–cocycle so that we can and will identify these two sets. The cohomology relation between cocycles translates as

\[
(X, Y) \sim (n^{-1}X\sigma n, n^{-1}Y\tau n), \quad n \in N. \tag{1.3}
\]
(iii) The cuspidal part of the cohomology consists of classes of pairs of the form
\[ \{ (X,Y) | \exists n \in N, X \cdot \sigma Y = n^{-1} \cdot \sigma n \}. \] (1.4)

We will call (1.2) abstract (noncommutative) Shimura–Eichler relations.

This result must be well known to experts, but I will sketch a proof because I do not know a reference.

**Proof.** Equations (1.2) are a translation of the cocycle relations applied to \( \sigma^2 = 1 \) and \( \tau^3 = 1 \).

Each nonidentical element \( g \) of \( PSL(2, \mathbb{Z}) \) can be written uniquely as a product \( g = \sigma^a \tau^{b_1} \ldots \sigma^a \tau^{b_n} \) with \( n \geq 1 \), \( a_i = 0 \) or 1, \( b_i = 0 \) or 1 or 2, satisfying the condition that \( a_i \neq 0 \) for \( i > 1 \) and \( b_i \neq 0 \) for \( i < n \). Define the length \( l(g) \) of such a word as \( \sum_i (a_i + b_i) \). Identity has length zero.

Each \( g \neq 1 \) ends with either \( \sigma \), or \( \tau \) that is, can be represented as \( h \sigma \) or \( h \tau \) with \( l(h) < l(g) \). All proofs proceed by induction on the length and use the cocycle relations. Here are some details. Denote by \( Z \) the set (1.2) and by \( \rho : Z^1(G,N) \rightarrow Z \) the restriction map.

(A) \( \rho \) is injective.

Assume that \( X = u(\sigma), Y = u(\tau) \) where \( u \) is a cocycle. Then we know \( u \) on words of length \( \leq 1 \). If \( g = h \sigma \) with \( l(h) < l(g) \geq 2 \), then \( u(g) = u(h) \cdot hX \) so that \( u(g) \) is uniquely defined by induction. The case \( g = h \tau \) with \( l(h) < l(g) \) is treated similarly.

(B) \( \rho \) is surjective.

Take arbitrary \( (X,Y) \in Z \). Construct a (well defined) map \( u : G \rightarrow N \) such that \( u(1_G) = 1_N, u(\sigma) = X, u(\tau) = Y \) and \( u(g) = u(h) \cdot hX \) (resp. \( u(g) = u(h) \cdot hY \)) if \( g = h \sigma \) (resp. \( g = h \tau \)) and \( l(h) < l(g) \).

We have to check the cocycle relations (1.1): for arbitrary \( h, g \in G \), \( u(hg) = u(h) \cdot hu(g) \). We make induction on \( l(g) \).

Start of induction: \( l(g) = 1 \). Then \( g = \sigma \) or \( \tau \), and \( l(hg) \neq l(h) \).

If \( l(hg) > l(h) \), then \( hg \) ends with \( g \), and the cocycle relation holds by construction.

If \( l(hg) < l(h) \), then \( h \) ends with \( g \). There are two subcases to consider: (a) \( g = \sigma, h = h' \sigma, l(h') = l(h) - 1 \); (b) \( g = \tau, h = h' \tau^2, l(h') = l(h) - 2 \). By construction, in the first case \( u(h) = u(h') \cdot h'X \), in the second case \( u(h) = u(h' \tau \cdot \tau) = u(h' \tau) \cdot h' \cdot \tau Y = (u(h') \cdot h' \cdot \tau Y) = u(h') \cdot h' \cdot ((\tau Y)^{-1}) = u(h') \cdot (hY)^{-1} \).
The relations we want to prove, namely \( u(h\sigma) = u(h) \cdot hX \) in the case (a), and \( u(h\tau) = u(h) \cdot hY \) in the case (b), easily follow.

**Inductive step:** assuming cocycle relations checked for all \( g \) with \( l(g) \leq n - 1 \), check them for longer words \( g\sigma \) or \( g\tau \) with \( l(g) = n \).

In fact, applying the inductive assumption twice we get:

\[
u(hg\sigma) = u(hg) \cdot hu(g) \cdot hgu(\sigma) = u(h) \cdot h[u(g) \cdot gu(\sigma)].\]

On the other hand,

\[
u(h) \cdot hu(g\sigma) = u(h) \cdot hu(g) \cdot hu(gu(\sigma)).\]

One treats \( g\tau \) similarly. Thus \( \rho \) is a bijection.

(C) The equivalence relations between cocycles clearly restricts to (1.3) on \( Z \). On the other hand, if we start with a pair \((X, Y)\) and produce a cocycle \( u \), and then start with \((n^{-1}X\sigma n, n^{-1}Y\tau n)\) and produce another cocycle \( v \), then by induction one can check that \( v(g) = n^{-1}u(g) \cdot gn \) for any \( g \). We leave this calculation to the reader.

(D) Finally, (1.4) means exactly that our cohomology class becomes trivial on \( G_\infty \), and triviality on other stabilizers of cusps follows from this.

The description given above can be further cut down under some additional conditions.

1.2.2. **Proposition.** Assume that the equation \( n^2 = m \) has a unique solution \( n := m^{1/2} \) in \( N \) for any \( n \in N \). Then we have:

(i) Any cocycle is homologous to one taking value 1 at \( \sigma \). Hence \( H^1(\text{PSL}(2, \mathbb{Z}), N) \) can be identified with the following quotient set:

\[
\{ Y \in N \mid Y \cdot \tau Y \cdot \tau^2 Y = 1 \} / (Y \sim m^{-1}Y\tau m \text{ for some } m \in N^\sigma). \tag{1.5}
\]

(ii) The cuspidal part of the cohomology consists of classes of elements \( Y \) of the form \( Y = n^{-1}\sigma n \).

**Proof.** Obviously \((gm)^{1/2} = g(m^{1/2})\) for all \( g \in G, m \in N \).

Hence from \( X \cdot \sigma X = 1 \) it follows that \( n^{-1}X\sigma n = 1 \) where \( n = X^{1/2} \). Thus any cocycle is homologous to one with \( X = 1 \). On the subset of such cocycles, which we will identify with a part of \( N \) satisfying the second relation in (1.2), the homology relation (1.3) becomes (1.5), and the cuspidal relation (1.4) becomes (ii).
Whenever in \( N \) equations \( n^3 = m \) are uniquely solvable and \( N \) is commutative, one can similarly show that any cocycle (represented by) \((X, Y)\) is homologous to one with \( Y = 1 \): take \( n = Y^{2/3}(\tau Y)^{1/3} \) in (1.3). I was unable to check this in the noncommutative case. However, in 2.6 we will see that the iterated Shimura cohomology class for \( PSL(2, \mathbb{Z}) \) can anyway be represented by a cocycle with \( Y = 1 \). Hence the following description parallel to the one in 1.2.2 will be relevant;

1.2.3. Proposition. (i) The part of the set \( H^1(PSL(2, \mathbb{Z}), N) \) represented by cocycles taking value 1 on \( \tau \) can be identified with the following quotient set:

\[
\{ X \in N \mid X \cdot \sigma X = 1 \} / (X \sim n^{-1} X \sigma n \text{ for some } n \in N^\tau). \tag{1.6}
\]

(ii) The cuspidal part of the cohomology consists of classes of elements \( X \) of the form \( X = n^{-1} \cdot \sigma \tau n \).

§2. Iterated Shimura integrals

2.1. Forms of cusp modular type. Let \( \Gamma \) be a subgroup of finite index of \( SL(2, \mathbb{Z}) \), \( k \geq 2 \) an integer, \( S_k(\Gamma) \) the space of cusp forms of weight \( k \). Denote by \( Sh_k(\Gamma) \) the space of 1–forms on the complex upper half plane \( H \) of the form

\[
f(z)P(z, 1)dz \text{ where } f \in S_k(\Gamma), \text{ and } P = P(X, Y) \text{ runs over homogeneous polynomials of degree } k - 2 \text{ in two variables. Thus, the space } Sh_k(\Gamma) \text{ is spanned by 1–forms of cusp modular type with integral Mellin arguments in the critical strip in the terminology of [Ma3], Def. 2.1.1.}
\]

2.2. Action of \( GL^+(2, \mathbb{R}) \). The group of real matrices with a positive determinant \( GL^+(2, \mathbb{R}) \) acts on \( H \) by fractional linear transformations \( z \mapsto [g]z \). Let \( j(g, z) := cz + d \) where \((c, d)\) is the lower row of \( g \). Then we have, for any function \( f \) on \( H \):

\[
g^*[f(z)P(z, 1)dz] := f([g]z)[j(g, z)]^{-k}P(az + b, cz + d) \det g \, dz
\]

where \((a, b)\) is the upper row of \( g \). From the definition it is clear that the diagonal matrices act identically so that we have in fact an action of \( PGL^+(2, \mathbb{R}) \).

This can be rewritten in terms of the weight \( k \) action of \( GL^+(2, \mathbb{R}) \) upon functions on \( H \). Actually, in the literature one finds at least two different normalizations of such an action. They differ by a determinantal twist and therefore coincide on \( SL(2, \mathbb{R}) \). For example, in [He1], [He2] one finds

\[
f|[g]_k(z) := f([g]z)j(g, z)^{-k}(\det g)^{k-1}, \tag{2.2a}
\]
whereas in [Me2] and [Ma3] the action
\[ f \, [[g]]^k \,(z) := f([[g]z) \, j(g, z)^{-k} \,(\det g)^k/2 \quad (2.2b) \]
is used.

Comparing this with (2.1), we get
\[
g^*f(z) \, P(z, 1) \, dz = f([[g]]^k \,(z) \, P(az + b, cz + d) \,(\det g)^{2-k} \, dz
\]
\[
= f([[g]]^k \,(z) \, P(az + b, cz + d) \,(\det g)^{(2-k)}/2 \, dz.
\]

Since \( S_k(\Gamma) \) consists of holomorphic functions which are \( \Gamma \)-invariant with respect to the (coinciding) right actions (1.2a), (1.2b), the space \( Sh_k(\Gamma) \) is \( \Gamma \)-stable and can be viewed as a tensor product of the trivial representation on \( S_k(\Gamma) \) and the \((k - 2)\)-th symmetric power of the basic 2-dimensional representation: for \( g \in \Gamma \) we have
\[
g^*f(z) \, P(z, 1) \, dz = f(z) \, P(az + b, cz + d) \, dz. \quad (2.3)
\]

2.3. Space \( SH_k \) and formal series. In the following we choose and fix a group \( \Gamma \) as above and a finite family of pairwise distinct weights \( k = (k_i) \). Put \( Sh_k := \oplus_i Sh_{k_i} \). Denote by \( Sh_k^* \) the dual space to \( Sh_k \), together with the adjoint left action \( g^* \) of \( \Gamma \) on it, so that \( (g^*(\omega), \nu) = (\omega, g^*(\nu)) \) for all \( \omega \in Sh_k, \nu \in Sh_k^* \), and \( g \in \Gamma \).

We will consider the completed tensor algebra of \( Sh_k^* \) as a ring of formal series in a finite number of associative non-commutative variables. Using the conventions of [Ma3], we may and will choose a basis \( (\omega_v) \) of \( Sh_k^* \) indexed by a finite set \( V \), and the dual basis \( (A_v) \) of \( Sh_k^* \). Then \( \Gamma \) acts on the left by linear transformations \( g^* \) on \( (A_v) \) inducing automorphisms on the formal series ring \( C\langle\langle A_v \rangle\rangle \). This ring has a continuous comultiplication defined by \( \Delta(A_v) = A_v \otimes 1 + 1 \otimes A_v \).

Group-like elements \( F \) of \( C\langle\langle A_v \rangle\rangle \) are characterized by the property \( \Delta(F) = F \otimes F, F \equiv 1 \mod (A_v) \). As is well known, \( F \) is group-like if and only if \( \log F \) belongs to the completed free Lie algebra freely generated by \( (A_v) \) inside \( C\langle\langle A_v \rangle\rangle \).

We may extend the scalars \( C \) of \( C\langle\langle A_v \rangle\rangle \) to functions or 1-forms on \( H \). All scalars are assumed to commute with \( (A_v) \).

In particular, the \( C\langle\langle A_v \rangle\rangle \)-bimodule \( \Omega_H\langle\langle A_v \rangle\rangle \) contains a canonical element
\[
\Omega := \sum_v A_v \omega_v \in Sh_k^* \otimes Sh_k \quad (2.4)
\]
which does not depend on the initial choice of basis \((\omega_v)\).

2.4. Iterated Shimura cocycles. We will consider now the iterated Shimura integrals

\[
J^a_z(\Omega) := 1 + \sum_{n=1}^{\infty} \int_a \Omega(z_1) \int_a \Omega(z_2) \cdots \int_a \Omega(z_n)
\]

where \(a, z\) are points of \(\overline{H} := H \cup \mathbb{P}^1(\mathbb{Q})\). Such an integral is well defined and takes values in the group \(\Pi\) of group–like elements of \(C(\langle A_v \rangle)\). For more details, see [Ma3].

The group \(P\Gamma\) acts on \(\Pi\) as was described in 1.3.

The following result is a slightly more precise version of [Ma3], 2.6.1.

2.4.1. Theorem. (i) For any \(a \in \overline{H}\), the map \(P\Gamma \to \Pi: \gamma \mapsto J^a_{\gamma a}(\Omega)\) is a noncommutative 1–cocycle in \(Z^1(P\Gamma, \Pi)\).

(ii) The cohomology class of this cocycle in \(H^1(P\Gamma, \Pi)\) does not depend on the choice of the reference point \(a \in \overline{H}\).

(iii) This cohomology class belongs to the cuspidal subset \(H^1(P\Gamma, \Pi)_{\text{cusp}}\) consisting of those cohomology classes whose restrictions on the stabilizers of cusps in \(\Gamma\) are trivial.

The last statement which was not mentioned in [Ma3] can be checked as follows. Let \(\gamma\) belong to the stabilizer \(\Gamma_a\) of a point \(a \in \mathbb{P}^1(\mathbb{Q})\). Then if we take \(a\) for the reference point, the respective cocycle is identically 1 on \(\Gamma_a\) since \(J^a_{\gamma a} = J^a_a = 1\).

2.5. Reductions of the coefficient group. We will call the class \(\zeta \in H^1(P\Gamma, \Pi)\), represented by \(u_\gamma := J^a_{\gamma a}(\Omega)\) the Shimura class. The same name will be applied to its various incarnations obtained by changing \(\Pi\) or \(P\Gamma\) (and using Shapiro Lemma).

In this subsection, we will cut down \(\Pi\) and exhibit representatives of this class satisfying conditions stated in 1.2.2 and 1.2.3.

2.5.1. The continued fractions trick. The following result which we reproduce from [Ma3] drastically reduces the size of a subgroup of \(\Pi\) containing a representative of \(\zeta\).

Choose a set of representatives \(C\) of left cosets \(P\Gamma \setminus PSL_2(\mathbb{Z})\). The iterated integrals of the form \((J^{g(0)}_{g(i\infty)})^{\pm 1}\), \(g \in C\), will be called primitive ones. Notice that when \(g \notin \Gamma\) the space spanned by \((\omega_v)\) is not generally \(g^*\)–stable so that we cannot define \(g_*\).
2.5.2. Proposition. Each \( J^a_b(\Omega) \), \( a, b \in P^1(\mathbb{Q}) \), in particular components of any Shimura cocycle with a cuspidal initial point \( a \in P^1(\mathbb{Q}) \), can be expressed as a noncommutative monomial in \( \gamma(\Omega) \) where \( \gamma \) runs over \( \Gamma \) and \( J^a_b(\Omega) \) runs over primitive integrals.

Proof. In fact, it suffices to express in this way \( J^a_{i\infty} \) for \( a > 0 \) (we omit \( \Omega \) for brevity). Produce a sequence of matrices \( g_k \) from the consecutive convergents to \( a \):

\[
a = \frac{p_n}{q_n}, \frac{p_{n-1}}{q_{n-1}}, \ldots, \frac{p_0}{q_0} = \frac{p_0}{1}, \frac{p_{-1}}{q_{-1}} := \frac{1}{0},
\]

\[
g_k := \begin{pmatrix} p_k & (-1)^{k-1}p_{k-1} \\ q_k & (-1)^{k-1}q_{k-1} \end{pmatrix}, \quad k = 0, \ldots, n.
\]

We have \( g_k = g_k(a) \in SL_2(\mathbb{Z}) \). Put \( g_k = \gamma_k c_k \) where \( \gamma_k \in \Gamma \) and \( c_k \in C \) are two sequences of matrices depending on \( a \). Then

\[
J^a_{i\infty} = \prod_{k=n}^0 \gamma_k(\Omega)_{\gamma_k(0)}
\]

which ends the proof.

2.6. The case \( P\Gamma = PSL(2, \mathbb{Z}) \). In this case, one can directly apply Proposition 1.2.1 providing noncommutative Shimura-Eichler relations between iterated integrals. It can be also applied to the smaller coefficient group \( \Pi_0 \) generated by the \( g^a(\Omega) \) as in 2.5.3.

In \( \Pi \), any element has a unique square root. Hence one can apply Proposition 1.2.2 as well. Another way to produce a Shimura cocycle taking value 1 at \( \sigma \), without using square roots, is to choose \( a = i \) for the initial point, because \( \sigma i = i \). Then the components of the respective Shimura cocycles will be iterated integrals between points of complex multiplication by \( i \) rather than cusps, and the all-important \( \tau \)-component is simply \( \tau = J^i_{\tau i}(\Omega) \).

A similar trick is applicable to \( \tau \): its fixed point is \( \rho = e^{\pi i/3} \) and so we get \( Y = 1, X = J^\rho_{\sigma \rho}(\Omega) \).

Notice finally that

\[
J^\rho_{\sigma \rho}(\Omega) = J^\rho_i(\Omega) \cdot \sigma(J^\rho_i(\Omega))^{-1}
\]

because \( J^\rho_i = J^\rho_i J^\rho_{\sigma \rho} \) and \( J^\rho_{\sigma \rho} = \sigma(J^\rho_i) = \sigma(J^\rho_i)^{-1} \).
2.7. Application of the Shapiro Lemma. We can apply the Shapiro Lemma for $P\Gamma \subset P\text{SL}(2,\mathbb{Z})$ in order to be able to use Proposition 1.2.1 for arbitrary $\Gamma$. The $P\Gamma$–module $\Pi$ gets replaced by the module $\Pi_{P\Gamma}$ of $P\Gamma$–covariant maps $P\text{SL}(2,\mathbb{Z}) \to \Pi$. Formula (1.1) shows that the Shimura class is still represented by a cocycle whose components are iterated integrals between two cusps. Square roots still exist and are unique in $\Pi_{P\Gamma}$ so that Proposition 1.2.2 (i) is applicable as well. However, the description of the cuspidal subset becomes somewhat clumsier.

§3. Linear term of $J^z_a(\Omega)$ and classical modular symbols

The linear (in $(A_v)$) term of $J^z_a(\Omega)$ involves ordinary integrals of the form $\int_z f(z)z^{s-1}dz$, $f \in S_k(\Gamma)$, $s \in \mathbb{C}$ (Mellin transform) or $s = 1, \ldots, k-1$ (Shimura integrals).

In this section, I review some basic facts of the classical theory of such integrals and explain how they extend (or otherwise) to the iterated setting, following [Ma3].

3.1. Classical and iterated Mellin transforms. The classical Mellin transform of $f \in S_k(\Gamma)$ is

$$\Lambda(f; s) := \int_\infty^0 f(z)z^{s-1}dz.$$ 

Let $N > 0$ and assume that $\Gamma$ is normalized by

$$g = g_N := \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$ 

Then $[N^{-1/2}g_N]_k$ defines an involution on $S_k(\Gamma)$ (see (2.2a)). Let $f$ be an eigenform with eigenvalue $\varepsilon_f = \pm 1$ with respect to this involution. Then

$$\Lambda(f; s) = -\varepsilon_f e^{\pi is}N^{k/2-s}\Lambda(f; k-s).$$

3.1.1. The iterated extension. The iterated Mellin transform of a finite sequence of cusp forms $f_1, \ldots, f_k$ with respect to $\Gamma$ was defined in [Ma3] as follows. Put $\omega_j(z) := f_j(z)z^{s_j-1}dz$. Then

$$M(f_1, \ldots, f_k; s_1, \ldots, s_k) := I^0_{i\infty}(\omega_1, \ldots, \omega_k) =$$

$$= \int_\infty^0 \omega_1(z_1) \int_\infty^{z_1} \omega_2(z_2) \cdots \int_\infty^{z_{n-1}} \omega_n(z_n)$$
A neat functional equation however can be written not for these individual integrals but for their generating series. More precisely, let \( f_V = (f_v | v \in V) \) be a finite family of cusp forms with respect to \( \Gamma \), \( s_V = (s_v | v \in V) \) a finite family of complex numbers, \( \omega_V = (\omega_v) \), where \( \omega_v(z) := f_v(z) \frac{z^{s_v-1}}{s_v} \). The total Mellin transform of \( f_V \) is

\[
TM(f_V; s_V) := J_{i\infty}^0(\omega_V) = 1 + \sum_{n=1}^{\infty} \sum_{\substack{(v_1,\ldots,v_n) \in V^n}} A_{v_1} \cdots A_{v_n} M(f_{v_1}, \ldots, f_{v_n}; s_{v_1}, \ldots, s_{v_n})
\]

Let \( k_v \) be the weight of \( f_v(z) \), and \( k_V = (k_v) \). Then we have

\[
TM(f_V; s_V) = g_{N^*}(TM(f_V; k_V - s_V))^{-1}
\]

for an appropriate linear transformation \( g_{N^*} \) of formal variables \( A_v \).

3.2. Dirichlet series. It is well known that \( \Lambda(f; s) \) for general \( s \) can be represented by a product of a \( \Gamma \)-factor and a formal Dirichlet series convergent in a right half plane of \( s \):

\[
f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z} \quad \Rightarrow \quad \Lambda(f; s) = -\frac{\Gamma(s)}{(-2\pi i)^s} \sum_{n=1}^{\infty} \frac{a_n}{n^s}
\]

In [Ma3], §3, it was shown that the iterated Mellin transforms at integral points of the product of critical strips can be expressed as multiple Dirichlet series of a special form. We omit the precise statements here.

3.3. The problem of iterated Hecke operators. If \( \Gamma \) is a congruence subgroup, there is a well known classical correspondence between the cusp forms which are eigenfunctions for the Hecke algebra and their Mellin transforms admitting an Euler product. Moreover:

(i) Shimura integrals of such a form span over \( \overline{\mathbb{Q}} \) a linear space of dimension \( \leq 2 \).

This was proved in [Ma2] for \( \Gamma = SL(2,\mathbb{Z}) \), and in [Sh3] for arbitrary (non necessarily congruence) \( \Gamma \).

(ii) In the case of a congruence subgroup \( \Gamma \), Fourier coefficients of such forms are expressed by explicit formulas involving summation of some simple linear functionals over universal sets of matrices.
This was also proved in [Ma2] for $\Gamma = \text{SL}(2, \mathbb{Z})$, and extended in several papers to general congruence $\Gamma$. For an especially neat version, see Merel’s “universal Fourier expansion” in [Me2].

The problem of extending these results to the iterated case remains a major challenge. One obstacle is that correspondences (in particular, Hecke correspondences) do not act directly on the fundamental groupoid (as opposed to the cohomology) and hence do not act on the iterated integrals which provide homomorphisms of this groupoid.

However, a part of the theory which is used in (ii), that of the classical modular symbols, allows a partial iterated extension. We will give below a brief review of this theory.

3.4. Classical modular symbols. The space of modular symbols $M S_k(\Gamma)$, by definition, is essentially the space of linear functionals on $S_k(\Gamma)$ spanned by the Shimura integrals

$$f(z) \mapsto \int_{\alpha}^{\beta} f(z) z^{m-1} dz; \quad 1 \leq m \leq k - 1; \quad \alpha, \beta \in \mathbb{P}^1(\mathbb{Q}).$$

(but see more precise information below). Three descriptions of $M S_k(\Gamma)$ are known:

(i) Combinatorial (Shimura – Eichler – Manin): generators and relations.

(ii) Geometric (Shokurov): $M S_k(\Gamma)$ can be identified with a (part of) the middle homology of the Kuga–Sato variety $M^k$.

(iii) Cohomological (Shimura): The dual space to $M S_k(\Gamma)$ can be identified with the cuspidal group cohomology $H^1(\Gamma, W_{k-2})_{\text{cusp}}$, with coefficients in the $(k-2)$–th symmetric power of the basic representation of $\text{SL}(2)$.

The noncommutative cohomology sets that we have described in §2, are iterated extensions of this last description.

Here are some details.

3.5. Combinatorial modular symbols. In this description, $M S_k(\Gamma)$ appears as an explicit subquotient of the space $W_{k-2} \otimes \mathcal{C}$ where $W_{k-2}$ consists of polynomial forms $P(X, Y)$ of degree $k - 2$ of two variables, and $\mathcal{C}$ is the space of formal linear combinations of pairs of cusps $\{\alpha, \beta\} \in \mathbb{P}^1(\mathbb{Q})$. Coefficients of these linear combinations can be $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$, as in the theory of Hodge structure.

Each element of the form $P \otimes \{\alpha, \beta\}$ produces a linear functional

$$f \mapsto \int_{\alpha}^{\beta} f(z) P(z, 1) dz.$$
This is extended to the total $W_{k-2} \otimes C$ by linearity.

Denote by $C$ the quotient of $\overline{C}$ by the subspace generated by sums $\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\}$. Since $\int_\beta^\alpha + \int_\gamma^\beta + \int_\alpha^\gamma = 0$, our linear functional (Shimura integral) descends to $W_{k-2} \otimes C$. We will still denote by $P \otimes \{\alpha, \beta\}$ the class of this element in $C$.

The group $GL^+(2, \mathbb{Q})$ acts from the left upon $W_{k-2}$ by $(gP)(X, Y) := P(bX - dY, -cX + aY)$ (notation as in (2.1)), and upon $C$ by $g\{\alpha, \beta\} := \{g\alpha, g\beta\}$. Hence it acts on the tensor product. A change of variable formula then shows that the Shimura integral restricted to $S_k(\Gamma)$ vanishes on the subspace of $W_{k-2} \otimes C$ spanned by $P \otimes \{\alpha, \beta\} - gP \otimes \{g\alpha, g\beta\}$ for all $P \in W_{k-2}$, $g \in \Gamma$.

Denote by $\text{MS}_k(\Gamma)$ the quotient of $W_{k-2} \otimes C$ by the latter subspace.

The subspace of cuspidal modular symbols $\text{MS}_k(\Gamma)_{\text{cusp}}$ is defined by the following construction. Consider the space $B$ freely spanned by $P_1(Q)$. Define the space $B_k(\Gamma)$ as the quotient of $W_{k-2} \otimes B$ by the subspace generated by $P \otimes \{\alpha\} - gP \otimes \{g\alpha\}$ for all $g \in \Gamma$. There is a well defined boundary map $\text{MS}_k(\Gamma) \to B_k(\Gamma)$ induced by $P \otimes \{\alpha, \beta\} \mapsto P \otimes \{\alpha\} - P \otimes \{\beta\}$. Its kernel is denoted $\text{MS}_k(\Gamma)_{\text{cusp}}$.

By construction, any (real) modular symbol in $\text{MS}_k(\Gamma)_{\text{cusp}}$ defines a $C$–valued functional $\int$ on $S_k(\Gamma)$ and in fact even on $S_k(\Gamma) \oplus \overline{S}_k(\Gamma)$.

The first result of the theory is:

3.5.1. Theorem (Shimura). $\int$ is an isomorphism of $\text{MS}_k(\Gamma)_{\text{cusp}}$ with the dual space of $S_k(\Gamma) \oplus \overline{S}_k(\Gamma)$.

3.5.2. Remark. Probably, the most interesting recent result involving combinatorial modular symbols is Herremans’ combinatorial reformulation of Serre’s conjecture in [He1], [He2].

3.6. Geometric modular symbols. Let $\Gamma^{(k)}$ be the semidirect product $\Gamma \times (\mathbb{Z}^{k-2} \times \mathbb{Z}^{k-2})$ acting upon $H \times C^{k-2}$ via

$$(\gamma; n, m)(z, \zeta) := ([\gamma]z; j(\gamma, z)^{-1}(\zeta + zn + m))$$

where $n = (n_1, \ldots, n_{k-2})$, $m = (m_1, \ldots, m_{k-2})$, $\zeta = (\zeta_1, \ldots, \zeta_{k-2})$, and $nz = (n_1z, \ldots, n_{k-2}z)$.

If $f(z)$ is a $\Gamma$–invariant cusp form of weight $k$, then

$$f(z)dz \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_{k-2}$$
is a $\Gamma^{(k)}$–invariant holomorphic volume form on $H \times C^{k-2}$. Hence one can push it down to a Zariski open smooth subset of the quotient $\Gamma^{(k)} \setminus (H \times C^{k-2})$. An appropriate smooth compactification $M^{(k)}$ of this subset is called a Kuga–Sato variety, cf. [Sh1]–[Sh3].

Denote by $\omega_f$ the image of this form on $M^{(k)}$. Notice that it depends only on $f$, not on any Mellin argument. The latter can be accommodated in the structure of (relative) cycles in $M^{(k)}$, so that integrating $\omega_f$ over such cycles we obtain the respective Shimura integrals.

Concretely, let $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ be two cusps in $\mathcal{H}$ and let $p$ be a geodesic joining $\alpha$ to $\beta$. Fix $(n_i)$ and $(m_i)$ as above. Construct a cubic singular cell $p \times (0, 1)^{k-2} \to H \times C^{k-2}$: $(z, (t_i)) \mapsto (z, (t_i(zn_i + m_i)))$. Take the $S_{k-2}$–symmetrization of this cell and push down the result to the Kuga–Sato variety. We will get a relative (modular fibers of $M^{(k)}$ over cusps) cycle whose homology class is Shokurov’s higher modular symbol $\{\alpha, \beta; n, m\}_\Gamma$. One easily sees that

$$\int_\alpha^{\beta} f(z) \prod_{i=1}^{k-2} (nz_i + m_i) \, dz = \int_{\{\alpha, \beta; n, m\}_\Gamma} \omega_f.$$ 

The singular cube $(0, 1)^{k-2}$ may also be replaced by an evident singular simplex.

3.6.1. Theorem (Shokurov). (i) The map $f \mapsto \omega_f$ is an isomorphism $S_k(\Gamma) \to H^0(M^{(k)}, \Omega^{k-2}_{M^{(k)}})$.

(ii) The homology subspace spanned by Shokurov modular symbols with vanishing boundary is canonically isomorphic to the space of cuspidal combinatorial modular symbols.

3.6.2. Remark. I suggested in [Ma3] that it would be desirable to replace in this description Kuga–Sato varieties by moduli spaces of curves of genus 1 with marked points and a level structure. For $\Gamma = SL(2, \mathbb{Z})$, this was essentially accomplished in a recent paper [CF] by C. Consani and C. Faber. Namely, they proved that the Chow motive associated with $S_k(SL(2, \mathbb{Z}))$ (with coefficients in $\mathbb{Q}$) is cut off by the alternating projector from the motive of $\overline{M}_{1,k-2}$. Recall that the symmetric group $S_{k-2}$ renumbering marked points naturally acts on $\overline{M}_{1,k-2}$.

3.7. Cohomological modular symbols. In this description, the space dual to $MS_k(\Gamma)$ is identified with the group cohomology $H^1(\Gamma, W_{k-2})$.

A bridge between the geometric and the cohomological descriptions is furnished by the identification of $H^1(\Gamma, W_{k-2})_{\text{cusp}}$ with the cohomology of a local system on $M_{1,1}$, namely $H^1_!(M_{1,1}, \text{Sym}^{k-2} R^1 \pi_* \mathbb{Q})$. 
Our iterated version explained in §2 was an attempt to extend this version of modular symbols.

3.7.1. Remark. $SL(2)$–modular symbols (and their generalization to groups of higher rank) made their appearance also in the context of (relations between) multiple polylogarithms: see A. Goncharov’s papers [Go3], [Go4]. It is not clear (at least to me) how to connect this description with the former ones.

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