Approximate lattices and Meyer sets in nilpotent Lie groups

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Abstract

We show that uniform approximate lattices in nilpotent Lie groups are subsets of model sets. This extends Y.Meyer’s theorem about quasicrystals in Euclidean spaces.

We derive from this structure theorem a characterisation of connected, simply connected, nilpotent Lie groups containing approximate lattices as those groups whose Lie algebra have structure constants lying in \( \mathbb{Q} \).
1 Outline

A uniform approximate lattice in a Lie group $G$ is an approximate subgroup which is both discrete and relatively dense. Approximate lattices were defined by M.Björklund and T.Hartnick in \cite{BH16}. An important example of approximate lattices is given by so-called cut-and-project schemes and model sets (also defined in \cite{BH16}), which are projection of certain well chosen subsets of lattices in product groups $G \times H$. We refer the reader to Section 2 below for precise definitions.

In 1972, Y.Meyer proved, albeit in a different language, that uniform approximate lattices in locally compact abelian groups are relatively dense subsets of model sets, see \cite[Thm 3.2]{Mey72}, \cite[Thm 2]{Sch73}. Moreover, M.Björklund and T.Hartnick asked, in \cite{BH16}, the following related question:

**Question.** Are all uniform approximate lattices of locally compact second countable groups subsets of model sets?

The main purpose of this note is to answer affirmatively in the case of nilpotent Lie groups.

The main theorem is as follows.

**Theorem 1.1.** Let $\Lambda \subset G$ be a uniform approximate lattice in a connected, simply connected, nilpotent Lie group. Then there exists a unique connected, simply connected, nilpotent Lie group $H$ such that:

1. $\langle \Lambda \rangle$ is isomorphic as an abstract group to a lattice $\Gamma$ in $G \times H$;
2. There is a compact neighbourhood $W_0$ of $e$ in $H$ such that $\Lambda \subset \pi_G(G \times W_0)$.

Here $\langle \Lambda \rangle$ denotes the subgroup generated by $\Lambda$ and $\pi_G$ the projection to the first factor.

As a corollary of Theorem 1.1 we get a “Meyer type” theorem for nilpotent Lie groups.

**Corollary 1.2.** Let $\Lambda \subset G$ be a uniform approximate lattice in a connected nilpotent Lie group. Then there exists a simply connected nilpotent Lie group $H$ such that $\Lambda$ is a Meyer set given by a cut-and-project scheme $(G, H, \Gamma)$.

In order to prove Theorem 1.1 we first prove a rigidity result for morphisms defined over uniform approximate lattices. In the case of lattices in nilpotent Lie groups, Theorem 1.3 reduces to Malcev’s rigidity theorem \cite[Chap 2]{Rag72}.

**Theorem 1.3.** Let $\Lambda \subset G$ be a uniform approximate lattice in a connected, simply connected, nilpotent Lie group and $\Gamma := \langle \Lambda \rangle$ the subgroup generated by $\Lambda$. Let $f : \Gamma \to N$ be an abstract group homomorphism from $\Gamma$ to a connected, simply connected, nilpotent Lie group. Then there are unique group homomorphisms $\tilde{f} : G \to N$ and $\rho : \Gamma \to N$ such that:

1. $\tilde{f}$ is a continuous group homomorphism;
(ii) \( f|_\Gamma = \tilde{f}|_{\Gamma} \cdot \rho; \)

(iii) The image of \( \rho \) lies in the centralizer \( C(\text{Im}(\tilde{f})) \) of \( \text{Im}(\tilde{f}) \);

(iv) \( \rho|_{\Lambda} \) is bounded.

This is done by considering the logarithm of a uniform approximate lattice, and showing it is a uniform approximate lattice in the Lie algebra seen as a locally compact abelian group. Indeed, the abelian case of the previous theorem is easier to prove, fairly elementary and can be used to deduce the general case.

Once Theorem 1.1 is proved we will be able to show the following

**Theorem 1.4.** A connected, simply connected, nilpotent Lie group contains a uniform approximate lattice if and only if its Lie algebra has a basis with structure constants in \( \mathbb{Q} \).

To prove this last result, we mainly use structure theorems about Lie algebras and their decomposition into indecomposable ideals. Thus, the tools are fairly different from those used in previous proofs.

## 2 Basic definitions

### 2.1 Uniform Approximate Lattices

Although approximate lattices can be defined in greater generality, we will focus on uniform approximate lattices.

**Definition.** A subset \( \Lambda \) of group \( G \) is called an approximate subgroup if it is symmetric, \( e \in \Lambda \) and there is a finite subset \( F \subset G \) such that \( \Lambda \cdot \Lambda \subset F \cdot \Lambda \). The group \( G \) is the ambient group of the approximate subgroup \( \Lambda \).

A uniform approximate lattice is an approximate subgroup satisfying further topological conditions.

**Definition.** A subset \( X \) of a locally compact group is:

- A relatively dense set if there is a compact set \( K \subset G \) such that \( \Lambda \cdot K = G \)
- A uniformly discrete set if there is a compact neighbourhood \( K \) of the identity element in \( G \) such that \( \forall g \in G, |g \cdot K \cap \Lambda| \leq 1 \);
- A Delone set if it is both relatively dense and uniformly discrete.

**Remark 2.1.** Equivalently, \( X \) is uniformly discrete if and only if \( e \) is not an accumulation point in \( X^{-1} \cdot X \).

We are now able to define uniform approximate lattices.

**Definition.** A subset \( \Lambda \) of a locally compact group \( G \) is a uniform approximate lattice if:
i) \( \Lambda \) is an approximate subgroup;

ii) \( \Lambda \) is a Delone set.

**Example 2.2.** 1. All uniform lattices are uniform approximate lattices.

2. According to a result of de Bruijn, the set of vertices of the Penrose rhombus tiling \( P3 \) is a uniform approximate lattice in the plane, see [dB81b] and [dB81a].

3. If \( \gamma \in \mathbb{Q} \) is a Pisot number, set

\[
X = \left\{ \sum_{i \in I} \gamma^i \mid I \subset \mathbb{N}, |I| < +\infty \right\}
\]

then \( Y := X \cup (-X) \) is a uniform approximate lattice in \( \mathbb{R} \), see [Mey72, 8.2] .

The following statement gives a handy characterisation of uniform approximate lattices.

**Proposition 2.3** (M. Björklund and T. Hartnick, [BH16]). Let \( \Lambda \) be a relatively dense, symmetric, containing the identity element subset of a locally compact group \( G \). Then the following statements are equivalent:

(i) \( \Lambda \) is a uniform approximate lattice;

(ii) \( \Lambda^k := \{ \lambda_1 \cdots \lambda_k \mid \lambda_1, \ldots, \lambda_k \in \Lambda \} \) is discrete for all \( k \in \mathbb{N} \):

(iii) \( \Lambda^6 \) is discrete;

(iv) For all compact \( K \subset G \), the set \( K \cap \Lambda^3 \) is finite; \( \Lambda^3 \) is then said to be locally finite.

**Remark 2.4.** Without a relative density assumption, only the implications (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (iv) hold. Indeed, for every positive integer \( k \) there is a symmetric set \( X \subset \mathbb{R} \) such that \( 0 \in X \), and \( X + \ldots + X \) is locally finite whereas \( \underbrace{X + \ldots + X}_{k+1-\text{terms}} \) is not.

Set \( Y \) as \( \{ k^n + \frac{1}{k^n} \mid n \geq 1 \} \) and \( X := Y \cup (-Y) \cup \{ 0 \} \) a quick computation shows that a \( k \) terms sum of elements that belong to \( X \) is equal to zero or greater than \( k - 1 \) in absolute value, while \( \frac{1}{k^n} - \frac{1}{k^n} \in \underbrace{X + \ldots + X}_{k+1-\text{terms}} \) for all \( n \geq 1 \).

**Corollary 2.5.** Let \( \Lambda \subset G \) be a uniform approximate lattice in a locally compact group. A symmetric subset containing the identity of \( \Lambda \) is a uniform approximate lattice if and only if it is relatively dense.

For convenience, we mention yet another result proved in [BH16].

**Proposition 2.6.** All approximate lattices (as defined [BH16]) in a nilpotent, locally compact, second countable group are uniform approximate lattices.

From now on, we will use “approximate lattices” instead of “uniform approximate lattices”.
2.2 Model sets and Meyer sets

The following scheme is the main tool to build uniform approximate lattices.

**Definition.** A cut-and-project scheme is a triple $(G, H, \Gamma)$ such that $G, H$ are locally compact groups, $\Gamma < G \times H$ is a lattice and when restricted to $\Gamma$ the projection $\pi_G$ on $G$ is injective and the projection $\pi_H$ on $H$ has dense image.

Hence, we can define $\tau := \pi_H \circ \pi_G^{-1}$ which is sometimes called the star map.

Given a cut-and-project scheme $(G, H, \Gamma)$ we are able to build a whole family of uniform approximate lattices in $G$. Indeed, pick a compact neighbourhood $W_0$ of the identity element in $H$ and define:

$$P_0(G, H, \Gamma, W_0) := \pi_G((G \times W_0) \cap \Gamma) = \tau^{-1}(W_0).$$

**Proposition 2.7** (M.Björklund and T.Hartnick, [BH16]). Denote $P_0(G, H, \Gamma, W_0)$ by $P_0$. Then:

1. $P_0$ and $P_0^{-1}P_0$ are uniformly discrete;
2. There is a finite subset $F \subset G$ such that $P_0^2 \subset F \cdot P_0$. Thus, if $W_0$ is symmetric and contains the identity element, $P_0$ is an approximate subgroup.
3. If $\Gamma$ is a uniform lattice, then $P_0$ is relatively dense.
4. If $P_0$ is relatively dense, then $\Gamma$ is a uniform lattice.

**Definition.** For a cut-and-project scheme $(G, H, \Gamma)$ and a compact neighbourhood of the identity $W_0 \subset H$, $P_0(G, H, \Gamma, W_0)$ is called a model set. Moreover, any relatively dense subset of a model set is called a Meyer set.

**Remark 2.8.** According to Proposition 2.7 model sets are approximate lattices whenever $W_0$ is symmetric and $\Gamma$ is uniform.

Conversely, Y.Meyer proved the following statement.

**Theorem.** [Mey72, Thm 3.2] In compactly generated locally compact abelian groups, all approximate lattices are Meyer sets.

3 Nilpotent Lie groups

In this section we will show Theorem 1.3 and Theorem 1.1.

Recall that we consider lattices in connected, simply connected, nilpotent Lie groups. Let $G$ be such a group, and denote by $\mathfrak{g}$ its Lie algebra. The exponential map $\exp : \mathfrak{g} \to G$ is then a diffeomorphism, denote by $\log$ its
inverse. The group structure on \( G \) and the Lie algebra structure on \( \mathfrak{g} \) are linked by the Baker-Campbell-F ormula:

\[
\log(\exp X \exp Y) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left( X^{r_1} Y^{s_1} X^{r_2} Y^{s_2} \cdots X^{r_n} Y^{s_n} \right)
\]

where the sum is performed over all nonegative values of \( s_i \) and \( r_i \) and:

\[
[X^{r_1} Y^{s_1} \cdots X^{r_n} Y^{s_n}] := [X, [X, \cdots [X, [Y, Y, \cdots Y] \cdots]].
\]

### 3.1 Logarithms of approximate lattices

Thanks to the Baker-Campbell-Hausdorff formula we will be able to express the sum and Lie bracket of two elements of the Lie algebra \( \mathfrak{g} \) using only the logarithm map and products.

**Definition.** Let \( n \in \mathbb{N} \) and \( w \in F_n \), an element of the free group of rank \( n \) with \( S = \{ s_1, \ldots, s_n \} \) a set of generators. For any group \( G \) and elements \( g_1, \ldots, g_n \in G \) we denote by \( w(x_1, \ldots, x_n) \) the image of \( w \) by the only group homomorphism \( f : F_n \rightarrow G \) such that \( s_i \mapsto x_i \), for \( i \in \{ 1, \ldots, n \} \). This is called a word in \( n \) letters.

As a consequence of this lemma, we get the following statement.

**Lemma 3.1.** For \( c \in \mathbb{N} \), there exist words in two letters \( w_c, w'_c \) and natural numbers \( m_c, m'_c \), depending only on \( c \), such that for all connected, simply connected nilpotent Lie groups of nilpotent class \( \leq c \), and \( x, y \in N \) we have:

\[
m_c(\log(x) + \log(y)) = \log(w_{c}(x, y)) \quad \text{and} \quad m'_c([\log(x), \log(y)]) = \log(w'_{c}(x, y))
\]

where \( \log : N \rightarrow \mathfrak{n} \) is the inverse of the exponential map, with source \( N \) and target \( N \)'s Lie algebra \( \mathfrak{n} \).

A direct consequence of this lemma is the following.

**Corollary 3.2.** Let \( X \subset N \) be a subset of a connected, simply connected, nilpotent Lie group. Then there are integers \( n, n', m, m' \in \mathbb{N} \) such that

\[
\log(X) + \log(X) \subset \frac{1}{m} \log(X^n) \quad \text{and} \quad [\log(X), \log(X)] \subset \frac{1}{m'} \log(X^{n'}).
\]

In particular, as \( \log \) is a homeomorphism, if \( X^n \) is discrete for all \( n \in \mathbb{N} \) then \( \log(X^n) \) is discrete and so \( \log(X) + \cdots + \log(X) \) and \( \log(X) + [\log(X), \log(X)] \) are discrete, according to Corollary 3.2.

Now let us show a result about relatively dense sets in nilpotent groups.
Lemma 3.3. Let $X \subset N$ be a relatively dense subset of a connected, simply connected, nilpotent Lie group. Then there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the set $\log(X^n)$ is relatively dense in $\mathfrak{g}$. Moreover, $n_0$ depends only on the nilpotency class of $N$.

Proof. Let us prove this by induction on the nilpotency class $c$. If $c = 1$ log is also a group homomorphism, and we are done.

Assume the lemma to be true for $c$ and let $N$ be a connected, simply connected, nilpotent Lie group with nilpotency class $c+1$. Denote by $p$ the canonical projection $N \to N/Z(N)$ where $Z(N)$ is the center of $N$ and by $\pi$ the canonical projection $n \to n/z(n)$ where $z(n)$ is the center of $n$ as a Lie algebra. Then $p(X)$ is relatively dense in $N/Z(N)$ and by induction hypothesis there is $n \in \mathbb{N}$ such that $\log(p(X)^n) = \pi(\log(X))$ is relatively dense too. Thus, there is a compact subset $K \subset N$ such that $n = \log(X^n) + K + z(n)$.

Let $L \subset N$ be a compact subset such that $\Lambda \cdot L = N$, then for any $z \in z(n)$ there are $x \in X$ and $b \in L$ such that $\exp(z) = xb$, thus $x$ and $b$ commute and $z = \log(x) + \log(b)$. Hence, $z(n) \subset \log(X) + \log(L)$ and $\log(X^n) + K + \log(X^n) + \log(L) = n$. According to Corollary 3.2 there are $m', n' \in \mathbb{N}$ such that $\log(X^n) + \log(X^n) \subset \frac{1}{m'} \log(X^n)$, so $\log(X^n) + m'(K + L) = N$.

As a consequence of the lemmas stated above we obtain:

Proposition 3.4. Let $\Lambda \subset N$ be an approximate lattice in a connected, simply connected, nilpotent Lie group. Then, there is a natural number $n_0$, depending only on the nilpotency class of $N$, such that $\log(\Lambda^{n_0})$ and $\log(\Lambda^{n_0}) + [\log(\Lambda^{n_0}), \log(\Lambda^{n_0})]$ is an approximate lattice in the Lie algebra $\mathfrak{n}$ endowed with its additive group structure.

Proof. For $n$ large enough $\log(\Lambda^n)$ is relatively dense according to Lemma 3.3. As a consequence of Corollary 3.2 $\log(\Lambda^n) + \cdots + \log(\Lambda^n)$ is discrete. So $\log(\Lambda^n)$ is an approximate lattice according to 2.3. According to Corollary 3.2 again, $\log(\Lambda^{n_0}) + [\log(\Lambda^{n_0}), \log(\Lambda^{n_0})]$ is also an approximate lattice.

3.2 Rigidity

Now, let us turn to the proof of Theorem 1.3. Recall its statement.

Theorem. Let $\Lambda \subset G$ be an approximate lattice in a connected, simply connected, nilpotent Lie group and $\Gamma := \langle \Lambda \rangle$ the subgroup generated by $\Lambda$. Let $f : \Gamma \to N$ be an abstract group homomorphism from $\Gamma$ to a connected, simply connected, nilpotent Lie group. Then there are unique group homomorphisms $\tilde{f} : G \to N$ and $\rho : \Gamma \to N$ such that:

(i) $\tilde{f}$ is a Lie group homomorphism;

(ii) $f|_{\Gamma} = \tilde{f}|_{\Gamma} \cdot \rho$;

(iii) The image of $\rho$ lies in the centralizer $C(\text{Im}(\tilde{f}))$ of $\text{Im}(\tilde{f})$;
(iv) $\rho_{\mathbf{1}_A}$ is bounded.

We will need an abelian version of this lemma.

**Lemma 3.5.** Let $\Lambda \subset \mathbb{R}^n$ be an approximate lattice and note $\Gamma := \langle \Lambda \rangle$. Let $\phi : \Gamma \to \mathbb{R}^m$ be a group homomorphism. Then there is a continuous homomorphism $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}^m$ such that $(\tilde{\phi} - \phi)|_{\Lambda}$ is bounded. The homomorphism $\tilde{\phi}$ is called the harmonization of $\phi$.

A proof of this lemma can be found in [Moo97, Lemma 8.5]

**Proof of existence.** Let us start with a lemma.

**Lemma 3.6.** Denote by $A$ the Lie ring generated by $\log(\Lambda)$. Then $\log \circ f \circ \exp$ extends to a Lie ring homomorphism $\hat{\phi} : A \to \mathfrak{n}$.

**Proof.** According to Corollary 3.2, $\log(\Gamma) \subset A \subset \text{span}_\mathbb{Q}(\log(\Lambda))$, where $\text{span}_\mathbb{Q}(\log(\Lambda))$ denotes the $\mathbb{Q}$-linear span of $\log(\Lambda)$.

Assume $G$ and $N$ are of nilpotency class less or equal to $c$ and $w_{c,m}$ as in Lemma 3.1. Define by induction the word $w_{c,n}$:

$$w_{c,n+1}(x_1, \ldots, x_{n+1}) = w_c(w_{c,n}(x_1, \ldots, x_n), w_{m,n}^{n+1}).$$

Then, for $g_1, \ldots, g_n \in G$, we have

$$\log(w_{c,n}(g_1, \ldots, g_n)) = m_n^c(\log(g_1) + \cdots + \log(g_n)).$$

Thus for other elements $g'_1, \ldots, g'_n \in G$, $\log(g_1) + \cdots + \log(g_n) = \log(g'_1) + \cdots + \log(g'_n)$ if and only if $w_{c,n}(g_1, \ldots, g_n) = w_{c,n}(g'_1, \ldots, g'_n)$.

And the same is true for elements of $N$. Therefore, for $\lambda_1, \ldots, \lambda_n, \lambda'_1, \ldots, \lambda'_n \in G$ such that $\log(\lambda_1) + \cdots + \log(\lambda_n) = \log(\lambda'_1) + \cdots + \log(\lambda'_n)$ we have $w_{c,n}(\lambda_1, \ldots, \lambda_n) = w_{c,n}(\lambda'_1, \ldots, \lambda'_n)$ and $w_{c,n}(f(\lambda_1), \ldots, f(\lambda_n)) = w_{c,n}(f(\lambda'_1), \ldots, f(\lambda'_n))$ since $f$ is a group homomorphism. Hence, let $\hat{\phi}$ denote $\log \circ f \circ \exp$,

$$\hat{\phi}(\log(\lambda_1)) + \cdots + \hat{\phi}(\log(\lambda_n)) = \phi(\log(\lambda'_1)) + \cdots + \phi(\log(\lambda'_n))$$

So we can extend $\log \circ f \circ \exp$ to a group homomorphism $\text{span}_\mathbb{Q}(\log(\Lambda)) \to \mathfrak{n}$, again denoted by $\phi$.

It remains to prove that the restriction to $A$ is a Lie ring homomorphism. We already know it is a group homomorphism. Let $w'_c$ and $m'_c$ be as in Lemma 3.1 then for $x_1, x_2 \in A$ there is a natural number $m$ such that $mx_1, mx_2 \in \log(\Lambda)$. Let $g_1, g_2 \in G$ be such that $\log(g_1) = mx_1$, then

$$\hat{\phi}([x_1, x_2]) = \frac{1}{m'_c m^2} \phi(m'_c [\log(g_1), \log(g_2)])$$

$$= \frac{1}{m'_c m^2} \phi((w'_c(g_1, g_2)))$$

$$= \frac{1}{m'_c m^2} \log(w'_c(f(g_1), f(g_2)))$$

$$= \frac{1}{m^2} [\log(f(g_1)), \log(f(g_2))] = [\phi(x_1), \phi(x_2)].$$

□
According to Lemma 3.3 up to considering $\Lambda^k$ for $k$ large enough, we can that $\log(\Lambda)$ and $\log(\Lambda) + \log(\Lambda)$ are approximate lattices. Denote by $\phi$ the Lie ring homomorphism given by Lemma 3.6 and $\tilde{\phi}$ the one given by Lemma 3.5 applied to $\phi$ and $\log(\Lambda) + \log(\Lambda)$.

Now, let us show that $\tilde{\phi}$ is a Lie algebra homomorphism. Let

$$B := \langle \phi - \tilde{\phi} \rangle (\log(\Lambda) + \log(\Lambda))$$

(1)

$x_1, x_2 \in \log(\Lambda)$ and $N_1, N_2$ be two norms on $g$ and $n$ respectively. There are $b \geq 0$ and $C \geq 0$ such that $B \subset B_{N_2}(0, b)$ and for all $y_2 \in n, y_1 \in B$:

$$N_2([y_1, y_2]) \leq CN_2(y_2)$$

Therefore,

$$N_2([x_1, x_2] - [\tilde{\phi}(x_1), \tilde{\phi}(x_2)]) \leq b + N_2(\phi([x_1, x_2] - [\tilde{\phi}(x_1), \tilde{\phi}(x_2)]) - [\phi(x_1) - \phi(x_1), \phi(x_2)] - [\tilde{\phi}(x_1), \tilde{\phi}(x_2) - \phi(x_2)] - [\tilde{\phi}(x_1) - \phi(x_1), \tilde{\phi}(x_2) - \phi(x_2)])$$

$$\leq b + C(N_2(\phi(x_1)) + N_2(\phi(x_2))) + Cb$$

As $\log(\Lambda)$ is an approximate lattice in $g$ and $\tilde{\phi}([y_1, y_2]) - [\tilde{\phi}(y_1), \tilde{\phi}(y_2)] = O(N_2(\phi(x_1)) + N_2(\phi(x_2)))$, the bilinear form $\langle y_1, y_2 \rangle \mapsto \tilde{\phi}([y_1, y_2]) - [\tilde{\phi}(y_1), \tilde{\phi}(y_2)]$ is null. Hence, $\tilde{\phi}$ is a Lie algebra homomorphism.

Finally, $\text{Im}(\tilde{\phi})$ and $\text{Im}(\phi - \tilde{\phi})$ commute in $n$. Indeed, choose $x_1, x_2 \in \log(\Lambda)$ and let $r$ denote $\phi - \tilde{\phi}$ then:

$$r([x_1, x_2]) = \phi([x_1, x_2]) - \tilde{\phi}([x_1, x_2])$$

$$= [\phi(x_1), \phi(x_2)] - [\tilde{\phi}(x_1), \tilde{\phi}(x_2)] + [\tilde{\phi}(x_1), \phi(x_2)] - [\tilde{\phi}(x_1), \tilde{\phi}(x_2)]$$

$$= [r(x_1), \phi(x_2)] + [\tilde{\phi}(x_1), r(x_2)]$$

So $[\tilde{\phi}(x_1), r(x_2)] = r([x_1, x_2]) - [r(x_1), \phi(x_2)]$. As a consequence, for a given $x_2 \in \mathcal{L}$, $x_1 \mapsto [\tilde{\phi}(x_1), r(x_2)]$ is linear form, bounded on $\log(\Lambda)$, thus bounded on $g$ as $\mathcal{L}$ is relatively dense. Then, $x \mapsto [\tilde{\phi}(x), r(x_2)] = 0$ for all $x_2$. Therefore, $\text{Im}(r)$ and $\text{Im}(\tilde{\phi})$ commute.

It remains only to check that $\tilde{f} := \exp \circ \tilde{\phi}$ and $\rho := \exp \circ r$ satisfy all the conditions of Theorem 3.3

\begin{proof}

\textbf{Proof of uniqueness.}

\textbf{Lemma 3.7.} Let $f, g : G \to N$ be Lie group homomorphisms. If $f(\cdot)g(\cdot)^{-1} \mid_\Lambda$ is bounded, then $f = g$.

Pick some compact subset $K \subset G$ such that $K\Lambda = G$. Then, for all $x \in G$ there are $\lambda \in \Lambda$ and $b \in K$ such that $x = b\lambda$, so $f(x)g(x)^{-1} = f(b)f(\lambda)g(\lambda)^{-1}g(b)^{-1}$ and $x \mapsto f(x)g(x)^{-1}$ is bounded.

We induct on $n$, the dimension of $N$ as a Lie group. If $n = 1$ the claim is true. Now, assume that the induction hypothesis is true for any $k < n$. Define $\pi :
\[ N \to N/Z(N) \text{ the canonical projection, then } \pi \circ f = \pi \circ g, \text{ so } g(x)^{-1}f(x) \in Z(N) \text{ for all } x \in G. \]\[ \text{Thus, } x \mapsto g(x)^{-1}f(x) \text{ is a group homomorphism. In addition, it has bounded image and target } Z(N) \simeq \mathbb{R}^m. \text{ As a conclusion, } x \mapsto g(x)^{-1}f(x) \text{ is the trivial homomorphism.} \]

Note that if \( \Lambda \) is a uniform lattice then Theorem 1.3 becomes as follows.

**Theorem 3.8.** Let \( \Lambda \subset G \) be a uniform lattice in a connected, simply connected, nilpotent Lie group, and \( f : \Lambda \to N \) a group homomorphism with target a connected, simply connected, nilpotent Lie group. Then, \( f \) extends to a unique Lie group homomorphism \( \tilde{f} : G \to N. \)

This is the well known Malcev rigidity Lemma, see [Mal49] and [Rag72].

### 3.3 Proof of Theorem 1.1

**Proof.** The subgroup \( \langle \Lambda \rangle \) generated by \( \Lambda \) is finitely generated, nilpotent and torsion-free, according to a Malcev’s Theorem it is isomorphic as an abstract group to \( \Gamma < N \) a lattice in a connected, simply connected, nilpotent Lie group, see [Rag72]. \( N \) is called the Malcev completion of \( \Gamma \). Define \( \Gamma_H := \langle \Lambda \rangle \) and \( i : \Gamma_H \to \Gamma \) an isomorphism.

According to Malcev rigidity (Theorem 3.8) there is a unique morphism, denoted by \( p \), extending \( i^{-1} \) to a Lie group homomorphism \( p : N \to G \). The morphism \( p \) has connected, co-compact image in \( N \), so \( p \) is surjective.

Now, let \( \tilde{i} \) and \( \rho \) given by Theorem 1.3 applied to \( i \).

First of all, let us prove that \( \tilde{i} \) is a section of \( p \). Indeed, \( p \circ \tilde{i} \) and \( \text{Id}_G \) satisfy the conditions of Lemma 3.7, so \( p \circ \tilde{i} = \text{Id}_G \). In particular, \( \text{Im}(\rho) \subset \ker(p) \).

Define \( H := \ker(p) \) and let us show that \( N = H \times \tilde{i}(G) \). The group homomorphism \( \rho \circ \tilde{i}^{-1} \) extends to a unique morphism \( \tilde{\rho} : N \to N \) according to Theorem 3.8. Moreover, as \( \text{Im}(\rho) \subset C(\text{Im}(\tilde{i})) \) the inclusion \( \text{Im}(\tilde{\rho}) \subset C(\text{Im}(\tilde{i})) \) holds too and so does \( \text{Im}(\tilde{\rho}) \subset H \).

Now, the map

\[
N \to N \\
n \mapsto \tilde{\rho}(n) \tilde{\phi}(n)
\]

is a group homomorphism that induces the identity on \( \Gamma \). So for all \( n \in N \), \( \tilde{\rho}(n) \tilde{\phi}(n) = n \) according to Theorem 3.8.

Now, we see that \( \text{Im}(\tilde{\rho}) = H \) and \( N = H \times \tilde{i}(G) \). Furthermore, \( \pi_H = \tilde{\rho} \) and \( \pi_{\tilde{i}(G)} = \tilde{\phi} \).

Finally, we see that any compact neighbourhood \( W_0 \) of \( e \) containing \( \rho(\Lambda) \) works.

\[\square\]

### 3.4 From Theorem 1.1 to Corollary 1.2

Recall the statement of Corollary 1.2.
Corollary. Let $\Lambda \subset G$ be an approximate lattice in a compactly generated, nilpotent Lie group $G$. Then there exists a simply connected nilpotent Lie group $H$ such that $\Lambda$ is a Meyer set given by a cut-and-project scheme $(G, H, \Gamma)$.

The proof relies on two general facts.

The first one is about how approximate lattices and model sets behave with respect to coverings.

Proposition 3.9. Let $p : G \to B$ be a covering of locally compact groups, and $\Lambda \subset B$ an approximate lattice. Define $\Lambda_G = p^{-1}(\Lambda)$. Then $\Lambda_G$ is an approximate lattice in $G$.

In addition, if $\Lambda_G$ is a Meyer set with respect to a cut-and-project scheme $(G, H, \Gamma)$ such that $H$ has no compact subgroup, then $\Lambda$ is a Meyer set with respect to the cut-and-project scheme $(B, H, \Gamma/\ker(p) \times \{e\})$.

Proof. 1. For all $n \in \mathbb{N}, \Lambda^n$ is discrete. As $p$ is a local homeomorphism $\Lambda^n_G$ is also discrete.

It remains only to prove that $\Lambda_G$ is relatively dense. Let $K \subset B$ be a compact set such that $\Lambda K = B$, as $p$ is a covering and $G$ is locally compact there is a compact subset $L \subset G$ such that $K \subset p(L)$. Now, for all $x \in G$ there is $\lambda \in \Lambda$ and $k \in K$ such that $p(x) = \lambda k$. Let $l \in L$ be such that $p(l) = k$, then $p(\lambda^{-1}l) = \lambda \in \Lambda$ so $\Lambda_G L = G$.

2. Pick some compact set $W \subset G$ such that $\Lambda_G \subset \pi_G(G \times W)$. As $\ker(p) \subset \Lambda$ we have $\pi(\ker(p)) \subset W$, so $\pi(\ker(p))$ is compact. By assumption, $\pi_H(\ker(p)) = \{e\}$ which implies $\ker(p \times \text{Id}_H) = \ker(p) \times \{e\} \subset \Gamma$. Finally, according to the first part of the proof we have in particular that $\Gamma' = p \times \text{Id}_H(\Gamma)$ is a uniform lattice in its ambient group if and only if $\Gamma$ is one in $G \times H$.

The second one is about simply connected nilpotent Lie groups.

Proposition 3.10. Let $G$ denote a simply connected nilpotent Lie group and assume that $G/G^0$ is finitely generated (equivalently, $G$ is compactly generated). Then there are a connected, simply connected, nilpotent Lie group $\tilde{G}$ and a Lie group embedding $i : G \to \tilde{G}$ such that $\tilde{G}/G$ is compact.

A proof can be found in [Rag72, Thm 2.21].

According to these two facts, it is clear that the connected, simply connected case implies the general case.

As an easy consequence we show the following corollary.

Corollary 3.11. Let $\Lambda \subset G$ be an approximate lattice in a nilpotent Lie group and $p : G \to G/[G; G]$ the canonical projection. Then,

1. $p(\Lambda)$ is an approximate lattice in $G/[G, G]$.

2. for some integer $n$, $[\Lambda^n, \Lambda^n]$ is an approximate lattice in $[G, G]$.
Proof. According to Corollary 1.2 there are a nilpotent Lie group \( H \), a lattice \( \Gamma < G \times H \) and a compact neighbourhood \( W_0 \) of \( e \) in \( H \).

1. There is an integer \( n \in \mathbb{N} \) such that \( \pi_G(((G \times W_0) \cap \Gamma) \subset \Lambda^n \), where \( \pi_G \) is the projection to the first factor. Moreover, \( \Gamma \cap [G, G] \times [H, H] \) is a lattice in \( [G, G] \times [H, H] \). Hence, \( [\Lambda^n, \Lambda^n] \) is relatively dense in \( [G, G] \times [H, H] \) because
   \[
   \pi_G([G, G] \times [W_0, W_0]) \cap (\Gamma \cap ([G, G] \times [H, H])) \subset [\Lambda^n, \Lambda^n]
   \]
Finally, \( [\Lambda^n, \Lambda^n] \subset \Lambda^{4n} \) so \( [\Lambda^n, \Lambda^n]^m \) is discrete for all \( m \in \mathbb{N} \). Hence, \( [\Lambda^n, \Lambda^n] \) is an approximate lattice.

2. As \( \Lambda \) is relatively dense, so is \( p(\Lambda) \). Moreover, define \( q : H \to H/[H, H] \) the canonical projection,
   \[
p(\Lambda) \subset \pi_{G/[G, G]}([G/[G, G] \times q(W_0)] \cap (p \times q)(\Gamma))
   \]
where \( \pi_{G/[G, G]} \) is the projection to the first factor. As \( p \times q(\Gamma) \) is also a lattice in \( G/[G, G] \times H/[H, H] \), \( p(\Lambda)^n \) is discrete for all \( n \in \mathbb{N} \). Therefore, \( p(\Lambda) \) is an approximate lattice.

\( \square \)

Remark 3.12. Furthermore, one can see that part 1 of the previous corollary implies that \( \Lambda^2 \cap [G, G] \) is an approximate lattice.

4 Criterion for existence

In [Mal49], Malcev proved a criterion for the existence of uniform lattices in connected, simply connected nilpotent Lie groups. The main consequence of the structure Theorem 1.1 is a criterion for the existence of approximate lattices, analogous to Malcev Theorem.

**Definition.** Let \( g \) be a Lie algebra over a field \( K \), and \( (e_i) \) a basis of \( g \) as a \( K \)-vector space. The coordinates in the basis \( (e_i) \) of the elements \( ([e_i, e_j])_{i,j} \) are called structure constants.

Malcev’s theorem goes as follows.

**Theorem** (Malcev, 1949, [Mal49]). Let \( G \) be a connected, simply connected, nilpotent Lie group and \( g \) its Lie algebra. There is a uniform lattice in \( G \) if and only if \( g \) has a basis with rational structure constants.

Our statement is similar.

**Theorem 4.1.** Let \( G \) be a connected, simply connected, nilpotent Lie group and \( g \) its Lie algebra. There is an approximate lattice in \( G \) if and only if \( g \) has a basis with structure constants algebraic over \( \mathbb{Q} \).
4.1 Necessity

If \( G \) contains an approximate lattice, then there is a connected, simply connected, nilpotent Lie group \( H \) such that \( G \times H \) contains a uniform lattice, according to Theorem 1.1. Thus, \( g \oplus h \) has a basis with rational structure constants. We will now prove that a direct factor of such a group is defined over \( \mathbb{Q} \).

In order to prove the necessity part, we will need to work with Lie algebras over varying ground fields.

**Definition.** Let \( K \hookrightarrow L \) be a field extension and \( g \) a \( K \)-Lie algebra. Then the \( L \)-vector space \( g \otimes L \) can be endowed with a \( L \)-Lie algebra structure by extending linearly the Lie bracket of \( g \). \( g(L) \) denotes the \( L \)-Lie algebra obtained this way.

**Remark 4.2.** This operation is well behaved, for instance we have \((g \oplus h)(L) = g(L) \oplus h(L)\) and for \( L \hookrightarrow M \) we have \((g(L))(M) = g(M)\). Moreover, a \( L \)-Lie algebra \( g \) has a basis with structure constants lying in a subfield \( K \subset L \) if and only if there is a \( K \)-Lie algebra \( h \) such that \( h(L) \simeq g \). We will say that a Lie algebra \( g \) can be defined over \( K \) if it admits a basis with structure constants in \( K \).

According to the previous remark, if \( G \) contains an approximate lattice then there is a nilpotent \( \mathbb{R} \)-Lie algebra \( h \) such that \( g \oplus h \) can be defined over \( \mathbb{Q} \). To show that such a Lie algebra can be defined over \( \mathbb{Q} \) we will need further results on Lie algebras.

**Definition.** Let \( g \) be a non-trivial \( K \)-Lie algebra, then \( g \) is indecomposable if there are no non-trivial \( K \)-Lie algebras \( g_1, g_2 \) such that \( g_1 \oplus g_2 \simeq g \).

**Proposition 4.3.** Let \( g \) be a \( K \)-Lie algebra, then there are indecomposable \( K \)-Lie algebras \( I_1, \ldots, I_r \) such that:

\[
g \simeq I_1 \oplus \cdots \oplus I_r
\]

Moreover, if \( J_1, \ldots, J_s \) are other indecomposable \( K \)-Lie algebras such that \( g \simeq J_1 \oplus \cdots \oplus J_s \) then \( s = r \) and there is a bijection \( \sigma : \{1, \ldots, r\} \to \{1, \ldots, s\} \) such that \( \forall i, I_i \simeq J_{\sigma(i)} \).

This is Krull-Schmidt theorem for Lie algebras, for a proof see [Kra15, Cor 3.3.3].

**Definition.** Let \( g \) be a \( K \)-Lie algebra. It is said absolutely indecomposable if \( g(L) \) is indecomposable for every field extension \( K \hookrightarrow L \).

**Proposition 4.4.** A \( K \)-Lie algebra \( g \) is absolutely indecomposable if and only if \( g(\overline{K}) \) is indecomposable, where \( \overline{K} \) is the algebraic closure of \( K \).

**Proof.** Let \( (e_i)_{1 \leq i \leq n} \) be a \( K \)-basis of \( g \) and \( K \hookrightarrow L \) a field extension. Now, \( (L) \) is not indecomposable if and only if there are two matrices \( A, B \in M_{n,n}(L) \)
such that:

\[
\begin{aligned}
A + B &= \text{Id}_{g(L)} \\
det(A) &= det(B) = 0 \\
A^2 &= A \quad \text{et} \quad B^2 = B \\
A([e_i, e_j]) &= [Ae_i, Ae_j], \forall i, j \in \{1, \ldots, n\} \\
B([e_i, e_j]) &= [Ae_i, Be_j], \forall i, j \in \{1, \ldots, n\}
\end{aligned}
\]

As a consequence, according to the Nullstellensatz, if \( g(\mathbb{K}) \) is indecomposable, so is \( g(L) \).

\[\square\]

**Proof of 4.1, necessity.** Let \( G \) be a connected, simply connected, nilpotent Lie group containing an approximate lattice and \( g \) its Lie algebra. As explained above there is a Lie algebra \( h \) such that \( g \oplus h \) is defined over \( \mathbb{Q} \). Now let \( \mathfrak{k} \) denote a \( \mathbb{Q} \)-algebra such that \( \mathfrak{k}(\mathbb{R}) \cong g \oplus h \) and \( \mathfrak{k}_1, \ldots, \mathfrak{k}_r \) be indecomposable ideals such that \( \mathfrak{k}(\mathbb{Q}) = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r \). According to [4.4] there is a set \( I \subset \{1, \ldots, r\} \) such that \( g(\mathbb{C}) \cong \bigoplus_{i \in I} \mathfrak{k}_i \). Therefore, the \( \mathbb{C} \)-Lie \( g(\mathbb{C}) \) is defined over both \( \mathbb{Q} \) and \( \mathbb{R} \).

As a consequence, its minimal field of definition (see [Spr10] for the existence of a minimal field of definition) is a subfield of \( \mathbb{Q} \cap \mathbb{R} \) and \( g \) is defined over \( \mathbb{Q} \).

\[\square\]

### 4.2 Sufficiency

**Proof of 4.1, sufficiency.** Let \( G \) be such a connected, simply connected, nilpotent Lie group. Then \( G \) can be seen as the group of real points of an algebraic group \( \mathbb{G} \) defined over a number field \( K \). Let \( \text{Res}_{K/\mathbb{Q}} G \) denote the Weil restriction of \( \mathbb{G} \) (see [Spr10]), then the group of real points \( \text{Res}_{K/\mathbb{Q}} G(\mathbb{R}) \) is a connected, simply connected, nilpotent Lie group, defined over \( \mathbb{Q} \) and isomorphic to a product \( G \times H \). As it is defined over \( \mathbb{Q} \), it contains a uniform lattice \( \Gamma < G \times H \) according to Malcev’s theorem.

Now let \( L \) be the Zariski closure of \( \Gamma \cap H \). As \( \Gamma \cap H \) is normal in \( \Gamma \), \( L \) is normal in \( H \). The image of \( \Gamma \) in \( G \times H/L \) is still a uniform lattice so we can assume that \( \Gamma \cap H = \{e\} \). Then \( (G, \overline{\Gamma}, \Gamma) \) is a cut-and-project scheme, so \( G \) contains an approximate lattice.

\[\square\]

**Remark 4.5.** This part of Theorem 4.1 was already mentioned in [BH16].

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