AN OPTIMAL ANISOTROPIC POINCARÉ INEQUALITY FOR CONVEX DOMAINS

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Abstract. In this paper, we prove a sharp lower bound of the first (nonzero) eigenvalue of Finsler-Laplacian with the Neumann boundary condition. Equivalently, we prove an optimal anisotropic Poincaré inequality for convex domains, which generalizes the result of Payne-Weinberger [16]. A lower bound of the first (nonzero) eigenvalue of Finsler-Laplacian with the Dirichlet boundary condition is also proved.

Keywords. Finsler-Laplacian, first eigenvalue, gradient estimate, optimal Poincaré inequality.

1. Introduction and main results

In this paper we are interested in studying the eigenvalues of the Finsler-Laplacian $Q$, which is a natural generalization of the ordinary Laplacian $\Delta$. We say that $F$ is a norm on $\mathbb{R}^n$, if $F : \mathbb{R}^n \to [0, +\infty)$ is a convex function of class $C^1(\mathbb{R}^n \setminus \{0\})$, which is even and positively 1-homogeneous, i.e.

$$F(t\xi) = |t|F(\xi)$$

for any $t \in \mathbb{R}$, $\xi \in \mathbb{R}^n$,

and

$$F(\xi) > 0$$

for any $\xi \neq 0$.

A typical norm on $\mathbb{R}^n$ is $F(\xi) = (\sum_{i=1}^{n} |\xi_i|^q)^{1/q}$ for $q \in (1, \infty)$. The Finsler-Laplacian on $(\mathbb{R}^n, F)$ is defined by

$$Qu := \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (F(\nabla u) F_{\xi_i} (\nabla u)) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( \frac{1}{2} F^2 \right) (\nabla u),$$

where $F_{\xi_i} = \frac{\partial F}{\partial \xi_i}$. When $F(\xi) = |\xi| = (\sum_{i=1}^{n} |\xi_i|^2)^{1/2}$, the Finsler-Laplacian $Q = \Delta$, the usual Laplacian.

The Finsler-Laplacian has been studied by many mathematicians, both in the context of Finsler geometry (see e.g. [1, 7, 13, 18]) and quasilinear PDE (see e.g. [2, 4, 8, 20, 21, 22]). Especially, many problems related to the first eigenvalue of Finsler-Laplacian have been already considered in [3, 7, 10, 14, 21]. In this paper we investigate the estimates of the first eigenvalue of the Finsler-Laplacian.

Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ and $\nu$ be the outward normal of its boundary $\partial \Omega$. The first eigenvalue $\lambda_1$ of Finsler-Laplacian $Q$ is defined by the smallest positive constant such that there exists a nonconstant function $u$ satisfying

$$-Qu = \lambda_1 u \quad \text{in } \Omega$$

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with the Dirichlet boundary condition
\[ u = 0 \text{ on } \partial \Omega \]
or the Neumann boundary condition
\[ \langle F \xi (\nabla u), \nu \rangle = 0 \text{ on } \partial \Omega. \]

We call \( \lambda_1 \) the \textit{first Dirichlet eigenvalue} (the \textit{first Neumann eigenvalue} resp.) and denote it by \( \lambda_1^D \) (by \( \lambda_1^N \) resp.). Here \( \langle F \xi (\nabla u), \nu \rangle = \sum_{i=1}^{n} F_{\xi_i} (\nabla u) \nu^i \) and \( \nu = (\nu^1, \ldots, \nu^n) \).

\( \langle F_{\xi} (\nabla u), \nu \rangle = \frac{\partial u}{\partial \nu}. \) is a natural Neumann boundary condition for the Finsler Laplacian. When \( F(\xi) = |\xi| \),

\[ \langle F_{\xi} (\nabla u), \nu \rangle = \frac{\partial u}{\partial \nu}. \]

The first Dirichlet (Neumann, resp.) eigenvalue can be formulated as a variational problem by
\[ \lambda_1^D (\Omega) = \inf \left\{ \frac{\int_{\Omega} F^2(\nabla u) dx}{\int_{\Omega} u^2 dx} \middle| 0 \neq u \in W_0^{1,2}(\Omega) \right\}. \]

\[ \lambda_1^N (\Omega) = \inf \left\{ \frac{\int_{\Omega} F^2(\nabla u) dx}{\int_{\Omega} u^2 dx} \middle| 0 \neq u \in W^{1,2}(\Omega), \int_{\Omega} u dx = 0 \right\}. \]

Therefore obtaining a sharp estimate of first eigenvalue is equivalent to obtaining the best constant in Poincaré type inequalities.

We remark that equation (2) should be understood in a weak sense, i.e.
\[ \int_{\Omega} \sum_{i=1}^{n} \frac{\partial}{\partial \xi_i} \left( \frac{1}{2} F^2 \right) (\nabla u) \varphi_i dx = \int_{\Omega} \lambda_1 u \varphi dx \text{ for any } \varphi \in C_0^\infty(\Omega). \]

Finding a lower bound for the first eigenvalue is always an interesting problem. In [4, 7], the authors proved the Faber-Krahn type inequality for the first Dirichlet eigenvalue of the Finsler-Laplacian. A Cheeger type estimate for the first eigenvalue of the Finsler-Laplacian involving isoperimetric constant was also obtained there. In this paper, we are interested in the Payne-Weinberger type sharp estimate [16] of the first eigenvalue in terms of some geometric quantity, such as the diameter with respect to \( F \).

Before stating our main result, we need to introduce some concepts and definitions. We say that \( \partial \Omega \) is \textit{weakly convex} if the second fundamental form of \( \partial \Omega \) is nonnegative definite. We say that \( \partial \Omega \) is \textit{F-mean convex} if the F-mean curvature \( H_F \) is nonnegative. For the definition of F-mean curvature, see section 2.

There is another convex function \( F^0 \) related to \( F \), which is defined to be the support function of \( K := \{ x \in \mathbb{R}^n : F(x) < 1 \} \), namely
\[ F^0(x) := \sup_{\xi \in K} \langle x, \xi \rangle. \]

It is easy to verify that \( F^0 : \mathbb{R}^n \rightarrow [0, +\infty) \) is also a convex, even, 1-positively homogeneous function of class \( C^1 (\mathbb{R}^n \setminus \{0\}) \). Actually \( F^0 \) is dual to \( F \) (see for instance [2]) in the sense that
\[ F^0(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F(\xi)} \text{ and } F(x) = \sup_{\xi \neq 0} \frac{\langle x, \xi \rangle}{F^0(\xi)}. \]

Hence the Cauchy-Schwarz inequality holds in the sense that
\[ \langle \xi, \eta \rangle_{\mathbb{R}^n} \leq F(\xi) F^0(\eta). \]
We call \( W_r(x_0) := \{ x \in \mathbb{R}^n | F^0(x - x_0) \leq r \} \) a Wulff ball of radius \( r \) with center at \( x_0 \).
We say \( \gamma : [0, 1] \to \Omega \) a minimal geodesic from \( x_1 \) to \( x_2 \) if
\[
d_F(x_1, x_2) := \int_0^1 F^0(\dot{\gamma}(t))dt = \inf \int_0^1 F^0(\dot{\tilde{\gamma}}(t))dt,
\]
where the infimum takes on all \( C^1 \) curves \( \tilde{\gamma}(t) \) in \( \Omega \) from \( x_1 \) to \( x_2 \). In fact \( \gamma \) is a straight line and \( d_F(x_1, x_2) = F^0(x_2 - x_1) \). We call \( d_F(x_1, x_2) \) the \( F \)-distance between \( x_1 \) and \( x_2 \).

Now we can define the \textit{diameter} \( d_F \) of \( \Omega \) with respect to the norm \( F \) on \( \mathbb{R}^n \) as
\[
d_F := \sup_{x_1, x_2 \in \Omega} d_F(x_1, x_2).
\]
In the same spirit we define the \textit{inscribed radius} \( i_F \) of \( \Omega \) with respect to the norm \( F \) on \( \mathbb{R}^n \) as the radius of the biggest Wulff ball that can be enclosed in \( \Omega \).

Our main result of this paper is

**Theorem 1.1.** Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^n \) and \( F \in C^1(\mathbb{R}^n \setminus \{0\}) \) be a norm on \( \mathbb{R}^n \). Let \( \lambda_1^N \) be the first Neumann eigenvalue of the Finsler-Laplacian (1). Assume that \( \partial \Omega \) is weakly convex. Then \( \lambda_1^N \) satisfies
\[
\lambda_1^N \geq \frac{\pi^2}{d_F^2}.
\]
Moreover, equality in (8) holds if and only if \( \Omega \) is a segment in \( \mathbb{R} \).

Estimate (8) for the Neumann boundary problem is optimal. This is in fact a generalization of the classical result of Payne-Weinberger in [16] on an optimal estimate of the first Neumann eigenvalue of the ordinary Laplacian. See also [3]. There are many interesting generalizations. Here we just mention its generalization to Riemannian manifolds, since we will use the methods developed there. It should be also interesting to ask if the methods of [16] and [3] work to reprove our result, since there are lots of motivations in computational mathematics.

For a smooth compact \( n \)-dimensional Riemannian manifold \( (M, g) \) with nonnegative Ricci curvature and diameter \( d \), possibly with boundary, the first Neumann eigenvalue \( \lambda_1 \) of Laplace operator \( \Delta \) is defined to be the smallest positive constant such that there is a nonconstant function \( u \) satisfying
\[
-\Delta u = \lambda_1 u \text{ in } M,
\]
with
\[
\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial M,
\]
if \( \partial M \) is not empty, where \( \nu \) denotes the outward normal of \( \partial M \). A fundamental work of Li [12], Li-Yau [13], Zhong-Yang [24] gives us the following optimal estimate
\[
\lambda_1 \geq \frac{\pi^2}{d^2},
\]
where \( d \) is the diameter of \( M \) with respect to \( g \). Li-Yau [13] derived a gradient estimate for the eigenfunction \( u \) and proved that \( \lambda_1 \geq \frac{\pi^2}{4d^2} \) and Li [12] used another auxiliary function to obtain a better estimate \( \lambda_1 \geq \frac{\pi^2}{2d^2} \). Finally, Zhong-Yang [24] was able to use a more
precise auxiliary function to get the sharp estimate $\lambda_1 \geq \frac{\pi^2}{4F}$, which is optimal in the sense that the lower bound is achieved by a circle or a segment. Recently Hang-Wang [9] proved that equality in (9) holds if and only if $M$ is a circle or a segment. Very recently these results were generalized to the $p$-Laplacian in [23] and to the Laplacian on Alexandrov spaces in [17].

For the Dirichlet problem we have

**Theorem 1.2.** Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n$ and $F \in C^1(\mathbb{R}^n \setminus \{0\})$ be a norm on $\mathbb{R}^n$. Assume that $\lambda^D_1$ are the first Dirichlet eigenvalue of the Finsler-Laplacian (1). Assume further that $\partial \Omega$ is $F$-mean convex. Then $\lambda^D_1$ satisfies

$$\lambda^D_1 \geq \frac{\pi^2}{4F}. \tag{10}$$

Estimate (10) is by no mean optimal.

Our idea to prove the result on the Dirichlet eigenvalue is based on the gradient estimate technique for eigenfunctions of Li-Yau [12, 13]. This idea also works for the first Neumann eigenvalue to get a rough estimate, say $\lambda^N_1 \geq \frac{\pi^2}{4F}$. However, for getting the sharp estimate of the first Neumann eigenvalue (8), the method of Zhong-Yang seems hard to apply. Instead, we adopt the technique based on gradient comparison with a one dimensional model function, which was developed by Kr"oger [11] and improved by Chen-Wang [6] and Bakry-Qian [5]. Surprisingly, we find that the one dimensional model coincides with that for the Laplacian case. In fact, this must be the case because when we consider $F$ in $\mathbb{R}$, it can only be $F(x) = c|x|$ with $c > 0$, a multiple of the standard Euclidean norm. In order to get the gradient comparison theorem, we need a Bochner type formula (13), a Kato type inequality (14) and a refined inequality (15), which was referred to as the “extended Curvature-Dimension inequality” in the context of Bakry-Qian [5]. Interestingly, the proof of these inequalities sounds more “naturally” than the proof of their counterpart for the usual Laplace operator. These inequalities may have their own interest. Another difficulty we encounter is to handle the boundary maximum due to the different representation of the Neumann boundary condition (4). We find a suitable vector field $V$ (see its explicit construction in Section 3) to avoid this difficulty. With the gradient comparison theorem, we are able to follow step by step the work of Bakry-Qian [5] to get the sharp estimate. The proof for the rigidity part of Theorem 1.1 follows closely the work of Hang-Wang [9]. Here we need pay more attention on the points with vanishing $|\nabla u|$.

A natural question arises whether one can generalize Theorem 1.1 to manifolds? The Finsler-Laplacian with the norm $F$ has not a direct generalization to Riemannian manifolds. However, it has a (natural) generalization to Finsler manifolds. In fact, $\mathbb{R}^n$ with $F$ can be viewed as a special Finsler manifold. On a general Finsler manifold, there is a generalized Finsler-Laplacian, see for instance [7, 14, 18]. A Lichnerowicz type result for the first eigenvalue of this Laplacian was obtained in [14] under a condition on some kind of new Ricci curvature $\text{Ric}_N, N \in [n, \infty]$. A Li-Yau-Zhong-Yang type sharp estimate, i.e., a generalization of Theorem 1.1 for this generalized Laplacian on Finsler manifolds would be a challenging problem. We will study this problem in a forthcoming paper.
The paper is organized as follows. In Section 2 we give some preliminary results on 1-homogeneous convex functions and the F-mean curvature and prove useful inequalities. In Section 3 we prove the sharp estimate for the first Neumann eigenvalue and classify the equality case. We handle the first Dirichlet eigenvalue in Section 4.

2. Preliminary

Without of loss generality, we may assume that \( F \in C^3(\mathbb{R}^n \setminus \{0\}) \) and \( F \) is a strongly convex norm on \( \mathbb{R}^n \), i.e. \( F \) satisfies

\[
\text{Hess}(F^2) \text{ is positive definite in } \mathbb{R}^n \setminus \{0\}.
\]

In fact, for any norm \( F \in C^1(\mathbb{R}^n \setminus \{0\}) \), there exists a sequence \( F_\varepsilon \in C^3(\mathbb{R}^n \setminus \{0\}) \) such that the strongly convex norm \( \tilde{F_\varepsilon} := \sqrt{F_\varepsilon^2 + \varepsilon|x|^2} \) converges to \( F \) uniformly in \( C^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\}) \), then the corresponding first eigenvalue \( (\lambda_1)_\varepsilon \) of Finsler-Laplacian with respect to \( \tilde{F_\varepsilon} \), converges to \( \lambda_1 \) as well. Here \( |\cdot| \) denotes the Euclidean norm.

Therefore, in the following sections, we assume that \( F \in C^3(\mathbb{R}^n \setminus \{0\}) \) and \( F \) is a strongly convex norm on \( \mathbb{R}^n \). Thus \( F \) is degenerate elliptic among \( \Omega \) and uniformly elliptic in \( \Omega \setminus C \), where \( C := \{x \in \Omega | \nabla u(x) = 0\} \) denotes the set of degenerate points. The standard regularity theory for degenerate elliptic equation (see e.g. [4, 19]) implies that \( u \in C^{1,\alpha}(\Omega) \cap C^{2,\alpha}(\Omega \setminus C) \).

The following property is an obvious consequence of 1-homogeneity of \( F \).

**Proposition 2.1.** Let \( F : \mathbb{R}^n \to [0, +\infty) \) be a 1-homogeneous function, then the following holds:

(i) \( \sum_{i=1}^n F_{\xi_i}(\xi)\xi_i = F(\xi) \);

(ii) \( \sum_{j=1}^n F_{\xi_i\xi_j}(\xi)\xi_j = 0 \), for any \( i = 1, 2, \ldots, n \);

For simplicity, from now on we will follow the summation convention and frequently use the notations \( F = F(\nabla u) \), \( F = F_\xi(\nabla u) \), \( u_i = \frac{\partial u}{\partial x_i} \), \( u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j} \) and so on. Denote

\[
a_{ij}(\nabla u)(x) := \frac{\partial^2}{\partial \xi_i \partial \xi_j} \left( \frac{1}{2} F^2 \right) (\nabla u(x)) = (F_i F_j + F F_{ij})(\nabla u(x)),
\]

\[
a_{ijk}(\nabla u)(x) := \frac{\partial^3}{\partial \xi_i \partial \xi_j \partial \xi_k} \left( \frac{1}{2} F^2 \right) (\nabla u(x)).
\]

In the following we shall write it simply by \( a_{ij} \) and \( a_{ijk} \) if no confusion appears. With these notations, we can rewrite the Finsler-Laplacian \( (\Pi) \) as

\[
Q u = a_{ij} u_{ij}.
\]

For the function \( \frac{1}{2} F^2(\nabla u) \) we have a Bochner type formula.

**Lemma 2.1** (Bochner Formula). At a point where \( \nabla u \neq 0 \), we have

\[
a_{ij} \left( \frac{1}{2} F^2(\nabla u) \right)_{ij} = a_{ij} a_{kl} u_{ik} u_{jl} + (Q u)_k \frac{\partial}{\partial \xi_k} \left( \frac{1}{2} F^2 \right) (\nabla u) - a_{ijl} \frac{\partial}{\partial x_l} \left( \frac{1}{2} F^2(\nabla u) \right) u_{ij}.
\]

\[\square\]
Lemma 2.2 (Kato inequality). At a point where \( \nabla u \neq 0 \), we have
\[
 a_{ij}a_{kl}u_{ik}u_{jl} \geq a_{ij}F_kF_lu_{ik}u_{jl}. \tag{14} \]

Proof. It is clear that
\[
a_{ij}a_{kl}u_{ik}u_{jl} - a_{ij}F_kF_lu_{ik}u_{jl} = a_{ij}F_kF_lu_{ik}u_{jl} = F(F_kF_lu_{ik}u_{jl} + F^2F_ku_{ik}u_{jl}).
\]
Since \((F_{ij})\) is positive definite, we know the first term
\[
 F(F_kF_lu_{ik}u_{jl} + F_kF_lu_{ik}u_{jl}) \geq 0.
\]
The second term \( F_kF_lu_{ik}u_{jl} \) is nonnegative as well. Indeed, we can write the matrix
\( (F_{kl})_{k,l} = O^T\Lambda O \) for some orthogonal matrix \( O \) and diagonal matrix \( \Lambda = \text{diag}(\mu_1, \mu_2, \cdots, \mu_n) \) with \( \mu_i \geq 0 \) for any \( i = 1, 2, \cdots, n \). Set \( U = (u_{ij})_{i,j} \) and \( \tilde{U} = OUO^T = (\tilde{u}_{ij})_{i,j} \). Then we have
\[
 F_{ij}F_{kl}u_{ij}u_{ki} = tr(O^T\Lambda O^T\Lambda OU) = tr(\Lambda O^T\Lambda O^T)
 = tr(\Lambda \tilde{U} \Lambda \tilde{U}) = \mu_i\mu_j\tilde{u}_{ij}^2 \geq 0,
\]
and hence the proof of (14).

When \( F(\xi) = |\xi| \), (14) is the usual Kato inequality
\[
|\nabla^2 u|^2 \geq |\nabla |\nabla u||^2.
\]
The following inequality is crucial to apply the gradient comparison argument in the next Section.

Lemma 2.3. At a point where \( \nabla u \neq 0 \), we have
\[
a_{ij}a_{kl}u_{ik}u_{jl} \geq \frac{(a_{ij}u_{ij})^2}{n} + \frac{n}{n-1} \left( \frac{a_{ij}u_{ij}}{n} - F_lF_ju_{ij} \right)^2 \tag{15}
\]
Proof. Let

\[ A = F_i F_j u_{ij} \quad \text{and} \quad B = F F_{ij} u_{ij} . \]

The right hand side of [15] equals to

\[ \frac{(A + B)^2}{n} + \frac{n}{n - 1} \left( \frac{B - n - 1}{n} A \right)^2 = A^2 + \frac{1}{n - 1} B^2 . \]

The left hand side of [15] is

\[ A^2 + 2 F F_{ij} F_{kl} u_{ik} u_{jl} + F^2 F_{ij} F_{kl} u_{ik} u_{jl} . \]

Since \((F_{ij})\) is semi-positively definite, we know

\[ F F_{ij} F_{kl} u_{ik} u_{jl} = F F_{kl} (F_{ij} u_{ik}) (F_{jl} u_{ij}) \geq 0 . \]

Using the same notations in the proof of Lemma 2.2, we have

\[ F^2 F_{ij} F_{kl} u_{ik} u_{jl} = F^2 \mu_i \mu_j \bar{u}_{ij}^2 + F^2 \sum_{i \neq k} \mu_i \mu_k \bar{u}_{ik}^2 \geq F^2 \mu_i^2 \bar{u}_{ii}^2 , \]

\[ B = F F_{ij} u_{ij} = tr(O^T \Lambda O U) = tr(\Lambda O U O^T) = \mu_i \bar{u}_{ii} . \]

We claim that \((F_{ij})\) is a matrix of rank \(n - 1\), in other words, one of \(\mu_i\) is zero. Firstly, \(F_{ij} u_{ij} = 0\). Secondly, for any nonzero \(V \perp F_{ij} (\nabla u), F_{ij} V^i V^j = a_{ij} V^i V^j > 0\). The claim follows easily. Thus the Hölder inequality gives

\[ F^2 \mu_i^2 \bar{u}_{ii}^2 \geq \frac{1}{n - 1} F^2 (\mu_i \bar{u}_{ii})^2 = \frac{1}{n - 1} B^2 . \]

Altogether we complete the proof of the Lemma. \(\square\)

When \(F(\xi) = |\xi|\), then [15] is

\[ |\nabla^2 u|^2 \geq \frac{(\Delta u)^2}{n} + \frac{n}{n - 1} \left( \frac{\Delta u}{n} - \frac{u_{ij} u_{ij}}{|\nabla u|^2} \right)^2 . \]

We now recall the definition of \(F\)-mean curvature. Let \(\Omega \subset \mathbb{R}^n\) be a smooth bounded domain, whose boundary \(\partial \Omega\) is a \((n - 1)\)-dimensional, oriented, compact submanifold without boundary in \(\mathbb{R}^n\). We denote by \(\nu\) and \(d\sigma\) the outward normal of \(\partial \Omega\) and area element respectively. Let \(\{e_\alpha\}_{\alpha = 1}^{n-1}\) be a basis of the tangent space \(T_p(\partial \Omega)\) and \(g_{\alpha \beta} = g(e_\alpha, e_\beta)\) and \(h_{\alpha \beta}\) be the first and second fundamental form respectively. \(\partial \Omega\) is called weakly convex, if \(h_{\alpha \beta}\) is nonnegative definite. Moreover let \((g^{\alpha \beta})\) be the inverse matrix of \((g_{\alpha \beta})\) and \(\nabla\) the covariant derivative in \(\mathbb{R}^n\). The \(F\)-second fundamental form \(h^F_{\alpha \beta}\) and \(F\)-mean curvature \(H_F\) are defined by

\[ h^F_{\alpha \beta} := \langle F_{\xi \xi} \circ \nabla e_\alpha, e_\beta \rangle \]

and

\[ H_F = \sum_{\alpha, \beta = 1}^{n-1} g^{\alpha \beta} h^F_{\alpha \beta} \]

respectively. \(\vec{H}_F = -H_F \nu\) are called \(F\)-mean curvature vector (it is easy to check that all definitions are independent of the choice of coordinate). \(\partial \Omega\) is called weakly \(F\)-convex (\(F\)-mean convex, resp.) if \((h^F_{\alpha \beta})\) is nonnegative definite \((H_F \geq 0\) resp.). It is well known
Lemma 2.4 ([20], Theorem 3). Let \( u \) be a \( C^2 \) function with a regular level set \( S_t := \{ x \in \Omega \mid u(x) = t \} \). Let \( H_F(S_t) \) be the F-mean curvature of the level set \( S_t \). We then have

\[
Q u(x) = -FH_F(S_t) + F_i F_j u_{ij} = -FH_F(S_t) + \frac{\partial^2 u}{\partial \nu^2_F}
\]

for \( x \) with \( u(x) = t \), where \( \nu_F := F_\xi(\nu) = -F_\xi(\nabla u) \).

We point out that we have used the inward normal in [20] and there is an sign error in the formula (5) there. Hence the term \( FH_F(S_t) \) in the formula (9) there should be read as \( -FH_F(S_t) \).

3. Sharp estimate of the first Neumann eigenvalue

It is well-known that the existence of Neumann first eigenfunction can be obtained from the direct method in the calculus of variations. We note that the first Neumann eigenfunction must change sign, for its average vanishes.

In this section we first prove the following gradient comparison theorem, which is the most crucial part for the proof of the sharp estimate. For simplicity, we write \( \lambda_1 \) instead of \( \lambda_1^F \) through this section.

**Theorem 3.1.** Let \( \Omega, u, \lambda_1 \) be as in Theorem 1.1. Let \( v \) be a solution of the 1-D model problem on some interval \( (a,b) \):

\[
v'' - Tv' = -\lambda_1 v, \quad v'(a) = v'(b) = 0, \quad v' > 0,
\]

with \( T(t) = -\frac{n-1}{t} \) or 0. Assume that \( \min u, \max u \subset [\min v, \max v] \), then

\[
F(\nabla u)(x) \leq v'(v^{-1}(u(x))).
\]

**Proof.** First, since \( \int u = 0 \), we know that \( \min u < 0 \) while \( \max u > 0 \). We may assume that \( \min u, \max u \subset (\min v, \max v) \) by multiplying \( u \) by a constant \( 0 < c < 1 \). If we prove the result for this \( u \), then letting \( c \to 1 \) we have (17).

Under the condition \( \min u, \max u \subset (\min v, \max v) \), \( v^{-1} \) is smooth on a neighborhood \( U \) of \( \min u, \max u \).

Consider \( P := \psi(u)(\frac{1}{2}F(\nabla u)^2 - \phi(u)) \), where \( \psi, \phi \in C^\infty(U) \) are two positive smooth functions to be determined later. We first assume that \( P \) attains its maximum at \( x_0 \in \Omega \), and then we will consider the case that \( x_0 \in \partial \Omega \). If \( \nabla u(x_0) = 0 \), \( P \leq 0 \) is obvious. Hence
we assume $\nabla u(x_0) \neq 0$. From now on we compute at $x_0$. As in Section 2, we use the notation $\text{(11)}$. Since $x_0$ is the maximum of $P$, we have that

$\begin{align*}
(18) & \quad P_i(x_0) = 0,
(19) & \quad a_{ij}(x_0)P_{ij}(x_0) \leq 0.
\end{align*}$

Equality in $\text{(18)}$ gives

$\begin{align*}
& \frac{\partial}{\partial x_i} \left( \frac{1}{2} F^2(\nabla u) - \phi(u) \right) = -\frac{\psi(u)}{\psi^2}P, \quad F_iF_ju_{ij} = \phi' - \frac{\psi'}{\psi^2}P.
\end{align*}$

Then we compute $a_{ij}P_{ij}$.

$\begin{align*}
a_{ij}P_{ij} &= \frac{P}{\psi}a_{ij}(\psi(u))_{ij} + \psi a_{ij} \frac{\partial}{\partial x_i} \left( \frac{1}{2} F^2(\nabla u) - (\phi(u)) \right) \\
&\quad + 2a_{ij}(\psi(u))_{i} \frac{\partial}{\partial x_j} \left( \frac{1}{2} F^2(\nabla u) - \phi(u) \right).
\end{align*}$

It is easy to see from Proposition 2.1 that

$\begin{align*}
(21) & \quad \frac{\partial}{\partial x_i} \left( \frac{1}{2} F^2 \right) (\nabla u)_{ii} = F^2(\nabla u), \quad a_{ij}u_iu_j = F^2(\nabla u), \quad a_{ijk}u_k = 0.
\end{align*}$

By using $\text{(20)}$, $\text{(21)}$, the Bochner formula $\text{(13)}$ and eigenvalue equation $\text{(2)}$, we get

$\begin{align*}
a_{ij}P_{ij} &= (-\lambda_1 u \frac{\psi'}{\psi} + F^2 \frac{\psi''}{\psi^2} - 2F^2 \frac{\psi'^2}{\psi^2})P \\
&\quad + \psi(a_{ij}a_{kl}u_{ik}u_{jl} - \lambda_1 F^2) + \psi(\lambda_1 u \phi' - F^2 \phi'').
\end{align*}$

Applying Lemma 2.3 to $\text{(22)}$, replacing $F^2$ by $2\frac{F^2}{\psi} + \phi$ and using $\text{(20)}$, $\text{(2)}$, $\text{(19)}$, we deduce

$\begin{align*}
0 \geq a_{ij}P_{ij} &\geq (-\lambda_1 u \frac{\psi'}{\psi} + F^2 \frac{\psi''}{\psi^2} - 2F^2 \frac{\psi'^2}{\psi^2})P + \psi(\lambda_1 u \phi' - F^2 \phi'') \\
&\quad + \psi \left( \frac{(a_{ij}u_{ij})^2}{n} + \frac{n}{n-1} \left( a_{ij}u_{ij} - F_iF_ju_{ij} \right)^2 - \lambda_1 F^2 \right) \\
&\quad = \frac{1}{\psi} \left[ 2 \frac{\psi''}{\psi} - \left( 4 - \frac{n}{n-1} \right) \frac{\psi'^2}{\psi^2} \right] P^2 \\
&\quad + \left[ 2\phi' \left( \frac{\psi''}{\psi} - \frac{2\psi'^2}{\psi^2} \right) - \frac{n+1}{n-1} \phi' \lambda_1 u - \frac{2n}{n-1} \frac{\psi'}{\psi} \phi' - 2\lambda_1 - 2\phi'' \right] P \\
&\quad + \psi \left[ \frac{1}{n-1} \lambda_1^2 u^2 + \frac{n}{n-1} \frac{\psi}{\psi} \phi' + \frac{n}{n-1} \phi'^2 - 2\lambda_1 \phi - 2\phi'' \right]
\end{align*}$

:= a_1P^2 + a_2P + a_3.

We are lucky to observe that the coefficients $a_i, i = 1, 2, 3$, coincide with those appearing in the ordinary Laplacian case (see e.g. [5], Lemma 1). The next step is to choose suitable positive functions $\psi$ and $\phi$ such that $a_1, a_2 > 0$ everywhere and $a_3 = 0$, which had already be done in [5]. For completeness, we sketch the main idea here.
Choose \( \phi(u) = \frac{1}{2}v'(v^{-1}(u))^2 \), where \( v \) is a solution of 1-D problem [16]. One can compute that
\[
\phi'(u) = v''(v^{-1}(u)), \quad \phi''(u) = \frac{v'''}{v'}(v^{-1}(u)).
\]
Setting \( t = v^{-1}(u) \) and \( u = v(t) \) we have
\[
a_3(t) = \frac{1}{n-1}\lambda_1^2 v^2 + \frac{n+1}{n-1} \lambda_1 vv'' + \frac{n}{n-1} v''^2 - \lambda_1 v'^2 - v'v''
\]
\[
= -v'(v'' - T v' + \lambda_1 v') + \frac{1}{n-1}(v'' - T v' + \lambda_1 v)(nv'' + T v' + \lambda_1 v) = 0.
\]
Here we have used that \( T \) satisfies \( T = \frac{r_2}{n-1} \). For \( a_1, a_2 \), we introduce
\[
X(t) = \lambda_1 \frac{v(t)}{v'(t)}, \quad \psi(u) = \exp\left(\int h(v(t))\right), \quad f(t) = -h(v(t))v'(t).
\]
With these notations, we have
\[
f' = -h' v'^2 + f(T - X),
\]
\[
v' |_{v=1}^2 a_1 \psi = 2f(T - X) - \frac{n-2}{n-1} f'^2 - 2f' := 2(Q_1(f) - f'),
\]
\[
a_2 = f(-\frac{3n-1}{n-1}T - 2X) - 2T(X - \frac{n-1}{n-1} T - X) - f' - f' := Q_2(f) - f'.
\]
We may now use Corollary 3 in [5], which says that there exists a bounded function \( f \) on \([\min u, \max u] \subset (\min v, \max v)\) such that \( f' < \min\{Q_1(f), Q_2(f)\}\).
In view of [23], we know that by our choice of \( \psi \) and \( \phi \), \( P(x_0) \leq 0 \), and hence \( P(x) \leq 0 \) for every \( x \in \Omega \), which leads to [17].

Now we consider the case \( x_0 \in \partial \Omega \). Suppose that \( P \) attains its maximum at \( x_0 \in \partial \Omega \). Consider a new vector field \( V(x) = (V_i(x))_{i=1}^n \) defined on \( \partial \Omega \) by
\[
V^i(x) = \sum_{j=1}^n a_{ij}(\nabla u(x))v^j(x).
\]
Thanks to the positivity of \( a_{ij} \), \( V(x) \) must point outward. Hence \( \partial P \partial \nu(x_0) \geq 0 \).
On the other hand, we see from the Neumann boundary condition and homogeneity of \( F \) that
\[
\frac{\partial u}{\partial \nu}(x_0) = u_i a_{ij}(\nabla u(x))v^j = FF_jv^j = 0.
\]
Thus we have
\[
0 \leq \frac{\partial P}{\partial \nu}(x_0) = \psi F u_i a_{ij} v^k.
\]
Choose now local coordinate \( \{e_i\}_{i=1,\ldots,n} \) around \( x_0 \) such that \( e_n = \nu \) and \( \{e_\alpha\}_{\alpha=1,\ldots,n-1} \) is the orthonormal basis of tangent space of \( \partial \Omega \). Denote by \( h_{\alpha \beta} \) the second fundamental form of \( \partial \Omega \). By the assumption that \( \partial \Omega \) is weakly convex, we know the matrix \( (h_{\alpha \beta}) \geq 0 \).

The Neumann boundary condition implies
\[
F_i v^i(x_0) = F_{n}(x_0) = 0.
\]
By taking tangential derivative of (25), we have
\[ D_{\epsilon_\beta} \left( \sum_{i=1}^{n} F_i \nu^i \right)(x_0) = 0, \]
for any \( \beta = 1, \cdots, n-1 \). Computing \( D_{\epsilon_\beta} \left( \sum_{i=1}^{n} F_i \nu^i \right)(x_0) \) explicitly, we have
\[
0 = D_{\epsilon_\beta} \left( \sum_{i=1}^{n} F_i \nu^i \right)(x_0) = \sum_{i,j=1}^{n} F_{ij} u_{j\beta} \nu^i + \sum_{i=1}^{n} F_i \nu^i_{\beta} \\
= \sum_{i,j=1}^{n} F_{ij} u_{j\beta} \nu^i + \sum_{i=1}^{n-1} \sum_{\gamma=1}^{n} F_i h_{\beta\gamma} e^i_\gamma \tag{26}
\]
In the last equality we have used \( \nu_n = 1 \) and \( \nu_\beta = 0 \) for \( \beta = 1, \cdots, n-1 \) in the chosen coordinate.

Combining (24), (25) and (26), we obtain
\[
0 \leq \frac{\partial P}{\partial V}(x_0) = \sum_{i,j,k=1}^{n} \psi F_i u_{ij} a_{jk} \nu^k = \psi F \sum_{\alpha=1}^{n-1} \sum_{j=1}^{n} F_{\alpha} u_{\alpha j} a_{jn} \\
= \psi F \sum_{\alpha=1}^{n-1} \sum_{j=1}^{n} F_{\alpha} u_{\alpha j} F_{jn} = -\psi F \sum_{\alpha,\gamma=1}^{n-1} F_{\alpha} F_{\gamma} h_{\alpha\gamma} \leq 0.
\]

Therefore we obtain that \( \frac{\partial P}{\partial V}(x_0) = 0 \). Since the tangent derivatives of \( P \) also vanishes, we have \( \nabla P(x_0) = 0 \). It’s also the case that (19) holds. Thus the previous proof for an interior maximum also works in this case. This finishes the proof of Theorem 3.1. \( \square \)

Following the idea of [5], besides the gradient comparison with the 1-D models, in order to prove the sharp estimate on the first eigenvalue of the Finsler-Laplacian, we need to study many properties of the 1-D models, such as the difference \( \delta(a) = b(a) - a \) as a function of \( a \in [0, +\infty] \), where \( b(a) \) is the first number that \( v'(b(a)) = 0 \) (Note that \( v' > 0 \) in \( (a, b(a)) \)). As we already saw in Theorem 3.1 the 1-D model (16) appears the same as that in the Laplacian case. Therefore, we can use directly the results of [5] on the properties of the 1-D models. Here we use some simpler statement from [23].

We define \( \delta(a) \) as a function of \( a \in [0, +\infty] \) as follows. On one hand, we denote \( \delta(\infty) = \frac{x}{\sqrt{\lambda_1}} \). This number comes from the 1-D model (16) with \( T = 0 \). In fact, it is easy to see that solutions of the 1-D model (16) with \( T = 0 \) can be explicitly written as
\[ v(t) = \sin \sqrt{\lambda_1} t \]
up to dilations. Hence in this case, \( b(a) - a = \frac{x}{\sqrt{\lambda_1}} \) for any \( a \in \mathbb{R} \). On the other hand, we denote \( \delta(a) = b(a) - a \) as a function of \( a \in [0, +\infty] \) relative to the 1-D model (16) with \( T = -\frac{a-1}{x} \).

The following property of \( \delta(a) \) was proved in [5, 23].
Lemma 3.1 ([5] or [23], Th. 5.3, Cor. 5.4). The function $\delta(a) : [0, \infty) \to \mathbb{R}^+$ is a continuous function such that

$$\delta(a) > \frac{\pi}{\sqrt{\lambda_1}},$$

$$\delta(\infty) = \frac{\pi}{\sqrt{\lambda_1}}.$$

$m(a) := v(b(a)) < 1$, $\lim_{a \to \infty} m(a) = 1$ and $m(a) = 1$ if and only if $a = \infty$.

In order to prove the main result, we also need the following comparison theorem on the maximum values of eigenfunctions. This theorem is obtained as a consequence of a standard property of the volume of small balls with respect to some invariant measure (see [5], Section 6).

Lemma 3.2. Let $\Omega, u, \lambda_1$ be as in Theorem 1.1. Let $v$ be a solution of the 1-D model problem on some interval $(0, \infty)$:

$$v'' = -\frac{n-1}{t} v' - \lambda_1 v, \quad v(0) = -1, \quad v'(0) = 0.$$

Let $b$ be the first number after 0 with $v'(b) = 0$ and denote $m = v(b)$. Then $\max u \geq m$.

The proof of Lemma 3.2 is similar to that of [5], Th. 11. The essential part is the gradient comparison Theorem 3.1. We omit it here.

Now we are in position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $u$ be an eigenfunction with eigenvalue $\lambda_1$. Since $\int u = 0$, we may assume $\min u = -1$ and $0 \leq k = \max u \leq 1$. Given a solution $v$ to (16), denote $m(a) = v(b(a))$ with $b(a)$ the first number with $v'(b(a)) = 0$ after $a$.

Lemma 3.1 and 3.2 imply that for any eigenfunction $u$, there exists a solution $v$ to (16) such that $\min v = \min u = -1$ and $\max v = \max u = k \leq 1$.

We now get the expected estimate by using Theorem 3.1. Choosing $x_1, x_2 \in \overline{\Omega}$ with $u(x_1) = \min u = -1, u(x_2) = \max u = k$ and $\gamma(t) : [0,1] \to \overline{\Omega}$ the minimal geodesic from $x_1$ to $x_2$. Consider the subset $I$ of $[0,1]$ such that $\frac{d}{dt} u(\gamma(t)) \geq 0$. By the gradient comparison estimate (17) and Lemma 3.1 we have

$$d_F \geq \int_0^1 F^0(\dot{\gamma}(t))dt \geq \int_I F^0(\dot{\gamma}(t))dt \geq \int_0^1 \frac{1}{F(\nabla u)}(\nabla u, \dot{\gamma}(t))dt = \int_{-1}^k \frac{1}{F(\nabla u)}du \geq \int_{-1}^k \frac{1}{v'(v^{-1}(u))}du = \int_{a}^{b(a)} dt = \delta(a) \geq \frac{\pi}{\sqrt{\lambda_1}},$$

which leads to

$$\lambda_1 \geq \frac{\pi^2}{d_F^2}.$$
Assume that $\lambda_1 = \frac{a^2}{4F}$. It can be easily seen from the proof of Theorem 4.1 that $a = \infty$, which leads to $\max u = \max v = 1$ by Lemma 3.1. We will prove that $\Omega$ is in fact a segment in $\mathbb{R}$. We divide the proof into several steps.

**Step 1:** $S := \{ x \in \overline{\Omega}: u(x) = \pm 1 \} \subset \partial \Omega$.

Let $\mathcal{P} = F(\nabla u)^2 + \lambda_1 u^2$. After a simple calculation by using Bochner formula (13) and Kato inequality (14), we obtain

$$ \frac{1}{2} a_{ij} P_{ij} = a_{ij} a_{kl} u_{ik} u_{jl} - \frac{1}{2} a_{ij} u_{ij} P_l - \lambda_1^2 u^2 $$

$$ \geq a_{ij} F_k F_l u_{ik} u_{jl} - \frac{1}{2} a_{ij} u_{ij} P_l - \lambda_1^2 u^2 $$

$$ = - \frac{1}{2} a_{ij} u_{ij} P_l + \frac{1}{4F^2} (a_{ij} P_i P_j - 4\lambda_1 u u_i P_l) \text{ on } \Omega \setminus C. $$

Namely,

$$ (27) \quad \frac{1}{2} a_{ij} P_{ij} + b_i P_i \geq 0 \text{ on } \Omega \setminus C $$

for some $b_i \in C^0(\Omega)$. If $\mathcal{P}$ attains its maximum on $x_0 \in \partial \Omega$, then arguing as in Theorem 3.1, we have that $\nabla P(x_0) = 0$. However, from the Hopf Theorem, $\nabla P(x_0) \neq 0$, a contradiction. Hence $\mathcal{P}$ attains its maximum at $C$, and therefore,

$$ (28) \quad \mathcal{P} \leq \lambda_1. $$

Take any two points $x_1, x_2 \in S$ with $u(x_1) = -1, u(x_2) = 1$. Let

$$ \gamma(t) = \left( 1 - \frac{t}{F^0(x_2 - x_1)} \right) x_1 + \frac{t}{F^0(x_2 - x_1)} x_2 : [0, l] \rightarrow \overline{\Omega} $$

be the straight line from $x_1$ to $x_2$, where $l := F^0(x_2 - x_1)$ is the distance from $x_1$ to $x_2$ with respect to $F$. Denote $f(t) := u(\gamma(t))$. It is easy to see $F^0(\dot{\gamma}(t)) = 1$. It follows from (28) and Cauchy-Schwarz inequality (7) that

$$ (29) \quad |f'(t)| = |\nabla u(\gamma(t)) \cdot \dot{\gamma}(t)| \leq F(\nabla u)(\gamma(t)) \leq \sqrt{\lambda_1(1 - f(t)^2)}. $$

Here we have used the Cauchy-Schwarz inequality (7) again. Hence

$$ d_F \geq l \geq \int_{[0 \leq t \leq l, f'(t) > 0]} dt \geq \int_0^l \frac{f'(t)}{\sqrt{\lambda_1(1 - f(t)^2)}} dt $$

$$ = \frac{1}{\sqrt{\lambda_1}} \int_{-1}^1 \frac{1}{\sqrt{1 - x^2}} dx = \frac{\pi}{\sqrt{\lambda_1}}. $$

Since $d_F = \frac{\pi}{\sqrt{\lambda_1}}$, we must have $d_F = l$, which means $S \subset \partial \Omega$.

**Step 2:** $\mathcal{P} = F^2(\nabla u) + \lambda_1 u^2 \equiv \lambda_1$ in $\overline{\Omega}$, hence $S \equiv C$.

Indeed, from Step 1, we know that $\Omega^* := \overline{\Omega}\setminus S$ is connected. Let $E := \{ x \in \Omega^* : \mathcal{P} = \lambda_1 \}$. It is clear that $E$ is closed. In view of (27), thanks to the strong maximum principle we know that $E$ is also open. We now show that $E$ is nonempty. Indeed, from the fact that all inequalities in (29) and (30) are equality, we obtain $f(t) = u(\gamma(t)) = -\cos \sqrt{\lambda_1} t$ for $t \in (0, l)$. Hence

$$ \mathcal{P}(\gamma(t)) = f'(t)^2 + \lambda_1 f(t)^2 = \lambda_1. $$
Thus $E$ is nonempty, open, closed in $\Omega^*$. Therefore, we obtain $\mathcal{P} \equiv \lambda_1$ in $\Omega$ (for $x \in S$, $\mathcal{P} = \lambda_1$ is obvious).

**Step 3:** Define $X = \frac{\nabla u}{F(\nabla u)}$ in $\Omega^*$ and $X^*$ the cotangent vector given by $X^*(Y) = \langle X, Y \rangle$ for any tangent vector $Y$. Then in $\Omega^*$, we claim that

$$D^2 u = -\lambda_1 u X^* \otimes X^*, \quad (31)$$

and moreover $X = \overrightarrow{c}$ for some constant vector $\overrightarrow{c}$.

First, taking derivative of $F^2(\nabla u) + \lambda_1 u^2 \equiv \lambda_1$ gives

$$F_i F_j u_{ij} = -\lambda_1 u. \quad (32)$$

On the other hand, since $\mathcal{P} \equiv \lambda_1$, the proof of (27) leads to

$$a_{ij} a_{kl} u_{ik} u_{jl} = \lambda_1^2 u^2 = (F_i F_j u_{ij})^2. \quad (33)$$

(33) in fact gives that

$$F_i F_j u_{ij} F_{kl} u_{ik} u_{jl} = 0. \quad (34)$$

Set $X^\perp := \{ V \in \mathbb{R}^n | V \perp X \}$. $X^\perp$ is an $(n - 1)$-dim vector subspace. Note that $(F_{ij})$ is exactly matrix of rank $n - 1$ (see the proof of Lemma 2.3) and $F_{ij} X^j = 0$. It follows from this fact and (34) that

$$u_{ij} V^i V^j = 0 \text{ for any } V \in X^\perp. \quad (35)$$

(32) and (35) imply (31), which in turn implies

$$u_{ij} = \frac{-\lambda_1 u u_i u_j}{F^2(\nabla u)}. \quad (36)$$

By differentiating $X$, we obtain from (36) that

$$\nabla_i X^j = \frac{u_{ij}}{F(\nabla u)} - \frac{u_j}{F^2(\nabla u)} F_k u_{ki} = 0. \quad \text{Thus } X = \overrightarrow{c} \text{ in } \Omega^*. \quad (37)$$

**Step 4:** The maximum point and the minimum point are unique.

We already knew that $f(t) = u(\gamma(t)) = -\cos \sqrt{\lambda_1} t$ and $\nabla u(\gamma(t)) \neq 0$ for $t \in (0, l)$. Hence $u$ is $C^2$ along $\gamma(t)$ for $t \in (0, l)$ and it follows that

$$D^2 u (\dot{\gamma}(t), \dot{\gamma}(t)) \bigg|_{\gamma(t)} = \lambda_1 \cos t \text{ for any } t \in (0, l). \quad (37)$$

On the other hand, we deduce from (31) that

$$D^2 u (\dot{\gamma}(t), \dot{\gamma}(t)) \bigg|_{\gamma(t)} = -\lambda_1 u(\gamma(t)) \langle X, \dot{\gamma}(t) \rangle^2. \quad (38)$$

Combining (37) and (38), taking $t \to 0$, we get

$$|\langle X, \dot{\gamma}(t) \rangle| = 1 = F(X) F^0(\dot{\gamma}(t)), \quad \text{for any } t \in (0, l). \quad (39)$$
which means equality in Cauchy-Schwarz inequality \( \square \) holds. Hence \( X = \pm F_{\xi}^0(\dot{\gamma}(t)) \). Noting that \( \dot{\gamma}(t) = \frac{x_2 - x_1}{F^0(x_2 - x_1)} \), we have
\[
X = F_{\xi}^0(x_2 - x_1).
\]

Suppose there is some point \( x_3 \) with \( u(x_3) = 1 \), using the same argument, we obtain \( X = F_{\xi}^0(x_3 - x_1) \). In view of \( F^0(x_3 - x_1) = F^0(x_2 - x_1) \), we conclude \( x_3 = x_2 \). Therefore, there is only one maximum point as well as one minimum point.

**Step 5:** \( \Omega \) is a segment in \( \mathbb{R} \).
Suppose \( \Omega \subset \mathbb{R}^n \) for \( n \geq 2 \). We see from Step 4 that for most of points of \( \partial \Omega \), \( \nabla u \neq 0 \), and at these points \( X = \frac{\nabla u}{|\nabla u|} \) lies in the tangent spaces due to the Neumann boundary condition, which is impossible because \( X \) is a constant vector, a contradiction. We complete the proof. \( \square \)

### 4. Estimate of the first Dirichlet eigenvalue

As in Section 3 for simplicity, we write \( \lambda_1 \) instead of \( \lambda_1^D \) through this section.

It is well-known that the existence of first Dirichlet eigenfunction can be easily proved by using the direct method in the calculus of variations. Moreover, by the assumption that \( F \) is even, the first Dirichlet eigenfunction \( u \) does not change sign (see [4], Th. 3.1).

We may assume \( u \) is non-negative. By multiplying \( u \) by a constant, we can also assume that \( \sup_{\Omega} u = 1 \) and \( \inf_{\Omega} u = 0 \) without loss of generality.

For any \( \alpha, \beta \in \mathbb{R} \) with \( \alpha > 0, \beta^2 > \sup(\alpha + u)^2 \), consider function
\[
P(x) = \frac{F^2(\nabla u)}{2(\beta^2 - (\alpha + u)^2)}.
\]

Suppose that \( P(x) \) attains its maximum at \( x_0 \in \partial \Omega \).

With the assumption that \( \Omega \) is \( F \)-mean convex, we first exclude the possibility \( x_0 \in \partial \Omega \) with \( \nabla u(x_0) \neq 0 \). Indeed, suppose we have \( x_0 \in \partial \Omega \) with \( \nabla u(x_0) \neq 0 \). Define \( \nu_F := F_{\xi}(\nu) \) on \( \partial \Omega = \{ x \in \Omega | u(x) = 0 \} \). In view of \( \langle \nu_F, \nu \rangle = F(\nu) > 0 \), \( \nu_F \) must point outward. From the Dirichlet boundary condition, we know \( \nu = -\frac{\nabla u}{|\nabla u|} \) for \( \nabla u \neq 0 \). Hence \( \nu_F = -F_{\xi}(\nabla u) \).

Since \( P \) attains maximum at \( x_0 \), we have
\[
0 \leq \frac{\partial P}{\partial \nu_F}(x_0) = \frac{FF_{\xi}u_{ij}\nu_F^j}{\beta^2 - (\alpha + u)^2} + F^2 \frac{\alpha \frac{\partial u}{\partial \nu_F}}{(\beta^2 - (\alpha + u)^2)^2}
\]

Hence
\[
-\frac{\partial^2 u}{\partial \nu_F^2} + \frac{F\alpha \frac{\partial u}{\partial \nu_F}}{\beta^2 - \alpha^2} \geq 0.
\]

Note that \( \frac{\partial u}{\partial \nu_F} = -F(\nabla u) \). Since \( \partial \Omega \) itself is a level set of \( u \), we can apply Lemma [2.4] to obtain
\[
\frac{\partial^2 u}{\partial \nu_F^2} = Qu + FH_F.
\]
In view of $Qu(x_0) = -\lambda_1 u(x_0) = 0$, we obtain that
\[-FH_F - F^2 \frac{\alpha}{\beta^2 - \alpha^2} \geq 0.\]
This contradicts the fact that $H_F(\partial \Omega) \geq 0$.

On the other hand, if $\nabla u(x_0) = 0$, then $F(\nabla u)(x_0) = 0$ and $P(x_0) = 0$ which implies $F(\nabla u) = 0$, i.e., $u$ is constant, a contradiction.

Therefore we may assume $x_0 \in \Omega$ and $\nabla u(x_0) \neq 0$. Since $a_{ij}$ is positively definite on $\overline{\Omega} \setminus \mathcal{C}$, where $\mathcal{C} := \{x|\nabla u(x) = 0\}$, it follows from the maximum principle that
\[
P_i(x_0) = 0,
\]
\[
a_{ij}(x_0)P_{ij}(x_0) \leq 0.
\]
From now on we will compute at the point $x_0$. Equality (39) gives
\[
\frac{\partial}{\partial x_i} \left( \frac{1}{2} F^2(\nabla u) \right) = - \frac{F^2(\nabla u)(\alpha + u)u_i}{\beta^2 - (\alpha + u)^2},
\]
Then we compute $a_{ij}(x_0)P_{ij}(x_0)$.
\[
a_{ij}(x_0)P_{ij}(x_0) = \frac{1}{\beta^2 - (\alpha + u)^2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{1}{2} F^2(\nabla u) \right) + 2a_{ij} \frac{\partial}{\partial x_i} \left( \frac{1}{2} F^2(\nabla u) \right) \frac{\partial}{\partial x_j} \left( \frac{1}{\beta^2 - (\alpha + u)^2} \right) + a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \left( \frac{1}{\beta^2 - (\alpha + u)^2} \right) \frac{1}{2} F^2(\nabla u),
\]
\[
= I + II + III.
\]

By using (11), (21), Bochner formula (13) and equation (2), we obtain
\[
I = \frac{1}{\beta^2 - (\alpha + u)^2} \left[ a_{ij} a_{kl} u_{ik} u_{jl} - \lambda_1 F^2 \right],
\]
\[
II = - \frac{4F^4(\alpha + u)^2}{(\beta^2 - (\alpha + u)^2)^3},
\]
\[
III = \frac{F^4}{(\beta^2 - (\alpha + u)^2)^2} + \frac{4F^4(\alpha + u)^2}{(\beta^2 - (\alpha + u)^2)^3} - \frac{\lambda_1 F^2 u(\alpha + u)}{(\beta^2 - (\alpha + u)^2)^2}.
\]

We now apply Lemma 2.2 to (42) and obtain
\[
a_{ij} a_{kl} u_{ik} u_{jl} \geq a_{ij} F_k F_l u_{ik} u_{jl}
\]
\[
= \frac{1}{F^2} a_{ij} \frac{\partial}{\partial x_i} \left( \frac{1}{2} F^2(\nabla u) \right) \frac{\partial}{\partial x_j} \left( \frac{1}{2} F^2(\nabla u) \right)
\]
\[
= \frac{F^4(\alpha + u)^2}{(\beta^2 - (\alpha + u)^2)^2}.
\]

Here we have used (11) and (21) again in the last equality. Therefore, we have
\[
I \geq \frac{F^4(\alpha + u)^2}{(\beta^2 - (\alpha + u)^2)^3} - \frac{\lambda_1 F^2}{(\beta^2 - (\alpha + u)^2)^2}.
\]
Combining (40), (43), (44) and (45), we obtain
\[0 \geq a_{ij} p_{ij} \geq F^4 \beta^2 \frac{\lambda_1 F^2}{(\beta^2 - (\alpha + u)^2)^3} - \frac{\lambda_1 F^2 u(\alpha + u)}{(\beta^2 - (\alpha + u)^2)^2} - \lambda_1 F^2 (\alpha + u).\]

It follows that
\[F^2(\nabla u)\beta^2 (\alpha + 1)^2 (x_0) \leq \lambda_1 \beta^2 (\alpha + 1)^2 (x_0),\]

(46)\]

Noting that \(\sup_{\Omega} u = 1\) we choose \(\alpha > 0\) and \(\beta = \alpha + 1\). Then estimate (46) becomes
\[F^2(\nabla u)\beta^2 (\alpha + 1)^2 (x_0) \leq \lambda_1 \beta^2 (\alpha + 1)^2 (x_0) \leq \lambda_1.\]

Hence we conclude, for any \(x \in \Omega\),
\[F^2(\nabla u)\beta^2 (\alpha + 1)^2 (x_0) \leq \lambda_1.\]

(47)\]

Choose \(x_1 \in \Omega\) with \(u(x_1) = \sup u = 1\) and \(x_2 \in \partial \Omega\) with \(d_F(x_1, x_2) = d_F(x_1, \partial \Omega) \leq i_F\) and \(\gamma(t) : [0, 1] \to \Omega\) the minimal geodesic connected \(x_1\) with \(x_2\). Using the gradient estimates (47), we have
\[\pi^2 - \arcsin\left(\alpha \frac{\alpha}{\alpha + 1}\right) = \int_0^1 \frac{1}{\sqrt{(\alpha + 1)^2 - (\alpha + u)^2}} du \leq \sqrt{\lambda_1} \int_0^1 \frac{1}{F(\nabla u)} du \leq \sqrt{\lambda_1} \int_0^1 \frac{1}{F(\nabla u(\gamma(t)))} (\nabla u(\gamma(t)), \gamma(t)) dt \leq \sqrt{\lambda_1} \int_0^1 F(\gamma(t)) dt \leq \sqrt{\lambda_1} i_F.\]

Here we have used the Cauchy-Schwarz inequality (7). Letting \(\alpha \to 0\), we obtain
\[\lambda_1 \geq \frac{\pi^2}{4i_F}.\]

Thus we finish the proof of Theorem 1.2.

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