Implications of non-extensivity on Gamow theory

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Relying on the quantum tunneling concept and the Maxwell-Boltzmann-Gibbs statistics, Gamow shows that the star burning process happens at temperatures comparable to a critical value, called the Gamow temperature (T) and less than the prediction of the classical framework. In order to highlight the role of the equipartition theorem in the Gamow argument, a thermal length scale is defined and thereafter, the effects of non-extensivity on the Gamow temperature have been investigated focusing on the Tsallis and Kaniadakis statistics. The results attest that while the Gamow temperature decreases in the framework of the Kaniadakis statistics, it can be bigger or smaller than T when the Tsallis statistics is employed.

I. INTRODUCTION

In the framework of the Maxwell-Boltzmann-Gibbs statistics, also known as the Gibbs statistics, consider a gas with temperature Tg, including particles with mass m and mean velocity v. In this manner, using the equipartition theorem, one finds

\[ \frac{1}{2}mv^2 = \frac{3}{2}K_BT_g. \]  (1)

Here, KB is the Boltzmann constant. For a pair of particles (called the i-th and j-th particles) with atomic numbers Zi and Zj, respectively, located at a distance r0 from each other, the kinetic energy lets particles overcome the Coulomb barrier meaning that nuclear fusion begins, and consequently, a star is born if \( \frac{3}{2}K_BT_g \geq U(r_0) = \frac{Z_iZ_j\epsilon^2}{4\pi\epsilon_0r_0} \) leading to [1]

\[ T_g \geq \frac{Z_iZ_j\epsilon^2}{6\pi\epsilon_0K_Br_0} \sim 1 \cdot 10^6 \frac{Z_iZ_j\epsilon^2}{r_0} \equiv T_0, \]  (2)

for the gas temperature. On the other hand, for the temperature of the gas, we also have [1]

\[ T \approx 4 \times 10^6 \left( \frac{M}{M_\odot} \right) \left( \frac{R_\odot}{R} \right), \]  (3)

in which M (M_\odot) and R (R_\odot) denote the mass and radius of the gas (Sun), respectively. As an example, consider the Sun for which we have T < T_0 meaning that the Sun should not burn [1]. Therefore, nuclear fusion cannot be launched in gasses whose temperature (T) is lower than T_0 (i.e., T < T_0) [1].

Thanks to the scorching Sun, the above argument becomes questionable. Indeed, Gamow was the first one who was able to find a proper answer by proposing a mechanism: quantum tunneling [1]. Based on this theory, if the particles become close to each other by their de Broglie wavelength (r_0 \approx \frac{\hbar}{p} \equiv r_\lambda), then they overcome the Coulomb barrier. In this manner, the corresponding de Broglie wavelength of particles can be calculated as

\[ \lambda = \frac{2\pi\epsilon_0\hbar^2}{mZ_iZ_je^2}, \]  (4)

where p = mv is considered. We have replaced r_0 with \( \lambda \equiv \frac{\hbar}{p} \) along with using

\[ \frac{p^2}{2m} = \frac{Z_iZ_je^2}{4\pi\epsilon_0\lambda}. \]  (5)

Now, with the help of \( \frac{3}{2}K_BT_g \geq U(r_0) \), one reaches [1]

\[ T_g \geq \frac{Z_iZ_je^2}{6\pi\epsilon_0K_B\lambda} \approx 9 \cdot 6 \times 10^6 \frac{Z_iZ_je^2}{r_0^2} \equiv T, \]  (6)

instead of Eq. (2) meaning that nuclear fusion can be started in gases whose temperature is comparable with T (the Gamow temperature) not T_0 [1]. Moreover, using \( \frac{3}{2}K_BT_g \geq U(r_0) \), one gets

\[ \lambda \geq \frac{Z_iZ_je^2}{6\pi\epsilon_0K_B\lambda} \equiv r_0^T. \]  (7)

Now, bearing the equal signs within Eqs. (1) and (3) in mind, we can finally deduce that the minimum requirement for quantum tunneling in a gas with temperature T_g \geq T is \( \lambda = r_0^T \). Hence, the equipartition theorem has a vital role i.e., if it changes then both Eq. (6) and r_0^T would change.

Although extensivity is the backbone of the Gibbs statistics, there are various arguments in favor of the non-extensivity, especially in relativistic systems and those that involve long-range interactions [2, 3]. The Tsallis and Kaniadakis (κ) statistics are two of the most famous and widely used generalized statistics [3, 4] which
propose generalized versions of the equipartition theorem \([7, 8]\). Motivated by various reasons such as the long-range nature of gravity, and the probable relationship between quantum aspects of gravity and the non-extensivity \([8, 11]\), these statistics have been employed leading to notable outcomes in i) describing dark energy \([8, 12]\), MOND theory \([13]\), ii) studying the Jeans instability \([14, 17]\), and also iii) stellar sciences \([15, 22]\).

Therefore, finding the Gamow temperature in the Tsallis and Kaniadakis statistics is an important task that also helps to get a better understanding of the non-extensivity, and correspondingly, the non-extensive aspects of gravity. To achieve this goal, we focus on the Tsallis and Kaniadakis statistics in the next section, and a summary will be presented at the end.

II. GENERALIZED STATISTICS AND THE GAMOW TEMPERATURE

A. Tsallis framework

The Tsallis entropy content of a statistical distribution with \(W\) states while the \(i\)-th state happens with probability \(P_i\) is defined as \([2]\)

\[
S_q^T = \frac{1}{1-q} \sum_{i=1}^{W} (P_i^q - P_i),
\]

where \(q\) is a free parameter calculated by other parts of physics or matching with experiments \([1, 3]\). The Gibbs entropy is recovered at \(q \to 1\); in fact, each sample has its own \(q\) \([1, 3]\). For a three dimensional particle, the ordinary thermal energy \((\frac{1}{2} K_B T)\) is modified as

\[
\frac{1}{2} m v^2 = \frac{3}{5 - 3q} K_B T_g,
\]

where \(0 \leq q < \frac{4}{3} \) \([3]\). Now, simple calculations lead to

\[
T_g \geq \frac{5 - 3q}{2} T \equiv T_q \Rightarrow 0 < T_q \leq 2 \cdot 5 \ T,
\]

\[
\lambda_q = \frac{2 \pi \varepsilon g h^2}{m Z_i Z_j e^2} = \lambda,
\]

in which the subscript \(q\) is used to distinguish the previous results with those of the Tsallis statistics. Moreover, solving \(\frac{5 - 3q}{2} K_B T_g = U(r_0)\), one reaches the Tsallis thermal length scale \(r_0^T = \left(\frac{5 - 3q}{2} Z_i Z_j e^2 \right) \frac{1}{2 \pi \varepsilon g h^2} \) that meets the condition \(r_0^T (q \to 1) \to r_0^T\). Therefore, quantum tunneling can happen at temperature comparable to \(T_q\), and in this manner, we should have at least \(r_0^T = \lambda_q\).

B. The \(\kappa\) statistics

In this framework, entropy is given by \([8]\)

\[
S_\kappa = \frac{1}{\kappa} \sum_{i=1}^{W} \left( \frac{P_i^{1+\kappa} - P_i^{1-\kappa}}{2\kappa} \right) = \frac{1}{\kappa} \left( \sum_{i=1}^{W} \frac{P_i^{1-\kappa} - P_i}{\kappa} + \sum_{i=1}^{W} \frac{P_i^{1+\kappa} - P_i}{\kappa} \right),
\]

leading to \([12]\)

\[
S_\kappa = \frac{S_T^{\kappa+1} + S_T^{1-\kappa}}{2},
\]

which clearly testifies that the Gibbs entropy is achieved for \(\kappa = 0 \) \([3]\). Indeed, \(\kappa\) is an unknown free parameter estimated by observations that varies from case to case \([8]\). Moreover, the equipartition theorem changes \([2, 9]\), and thus, Eq. (11) takes the form

\[
\frac{p^2}{2m} = \frac{3}{2} \kappa K_B T_g,
\]

in which

\[
\gamma_\kappa = \frac{(1 + \frac{\kappa}{2}) \Gamma(\frac{\kappa}{2} - \frac{1}{3}) \Gamma(\frac{\kappa}{2} + \frac{1}{3})}{2\kappa(1 + \frac{3\kappa}{4}) \Gamma(\frac{\kappa}{2} + \frac{2}{3}) \Gamma(\frac{\kappa}{2} - \frac{2}{3})},
\]

where \(0 \leq \kappa < \frac{2}{3}\) and \(\Gamma(n)\) denote the Gamma function \([3]\). Moreover, \(\gamma_\kappa\) diverges for \(\kappa = \frac{2}{3}\) and the ordinary equipartition theorem \((\frac{1}{2} K_B T)\), and thus Eq. (11) are recovered when \(\kappa = 0\) leading to \(\gamma_\kappa = 1 \) \([9]\).

Finally, it is a matter of calculation to find the Kaniadakis counterpart of Eq. (10) and the Kaniadakis length scale as

\[
T_g \geq \frac{T}{\gamma_\kappa} \equiv T_\kappa,
\]

\[
\lambda_\kappa = \frac{2 \pi \varepsilon g h^2}{m Z_i Z_j e^2} = \lambda,
\]

and

\[
r_{0\kappa}^T = \frac{r_T^0}{\gamma_\kappa},
\]

respectively. Since \(1 \leq \gamma_\kappa \) \([8]\), the conditions \(T_\kappa \leq T\) and \(r_{0\kappa}^T \leq r_T^0\) are obtained as the allowed intervals for \(T_\kappa\) and \(r_{0\kappa}^T\).
III. CONCLUSION

Reviewing the Gamow theory shows the role of equipartition theorem in more clarification via defining a thermal length scale \( r_T^0 \). It was deduced that the nuclear fusion would occur in a gas whose temperature is comparable to the Gamow temperature \( T \) if the minimum requirement \( \lambda = r_T^0 \) is satisfied. Moreover, equipped with the fact that generalized statistics modifies the equipartition theorem and motivated by their considerable achievements in various setups \[9–22\], we studied the effects of non-extensivity on the Gamow temperature. The results indicate that the Gamow temperature \( T \) decreases in the Kaniadakis statistics \( T_\kappa \leq T \), and in the Tsallis statistics, it can be smaller or bigger than \( T \) (i.e., \( 0 < T_q < 2 \cdot 5 \cdot T \)), depending on the value of \( q \). The same result applies to the corresponding thermal length scales.

Finally, if stars obey the Tsallis (Kaniadakis) statistics, then by using \( T = T_q (T = T_\kappa) \), one can find the value of \( q (\kappa) \) corresponding to each star. Therefore, the upper and lower bounds on the \( q (\kappa) \) parameter for nuclear fusion process occurring can be found in the coldest and hottest stars.

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