FLRW metric $f(R)$ cosmology with a perfect fluid by generating integrals of motion

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Abstract

In the context of metric $f(R)$ gravity, we consider a FLRW space-time, filled with a perfect fluid described by a barotropic equation of state ($p = \gamma \rho$). We give the equivalent mini-superspace description and use the reparametrization invariance of the resulting Lagrangian to work in the equivalent constant potential description. At that point, we restrict our analysis to those models for which the ensuing scaled mini-superspace is maximally symmetric. Those models exhibit the maximum number of autonomous integrals of motion linear in the momenta, which are constructed by the Killing vectors of the respective mini-supermetric. The integrals of motion are used to analytically solve the equations of the corresponding models. Finally, a brief description of the properties of the resulting Hubble parameters is given.

1 Introduction

General relativity, since its formulation, led to many experimental predictions. In the field of cosmology it provided us with the Standard Model, which can be considered quite successful with respect to a series of observational facts. Of course there exist some discrepancies as well, such as flatness and/or horizon problems together with the observed accelerated expansion of the universe at present times. In order to address these issues, many deviations from the gravitational theory of General Relativity have been proposed. The most widespread among them is the theory of $f(R)$ gravity, where the scalar curvature $R$ in the usual Einstein Hilbert action is replaced by a non linear function of itself. The method by which the field equations of motion are obtained defines different kinds of theories: a) metric, b) affine and c) metric-affine $f(R)$ gravity (for an overview see [1] and references therein).
In cosmology, there exists an increasing interest in the study of analytical solutions for a variety of $f(R)$ models (\cite{2}, \cite{3}, \cite{4}, \cite{5}) particularly by the use of Noether symmetries (\cite{6}, \cite{7}, \cite{8}, \cite{9}), even at the quantum regime \cite{10}. There also have been proposed criteria for testing whether particular models are physically accepted or not (\cite{11}, \cite{12}), although these are not always dogmatically imposed. Nevertheless, and in spite of the physical significance, one can not overlook the mathematical interest in the procedure of obtaining analytical solutions, in particular wherever symmetry conditions are imposed. In this paper, we adopt the metric $f(R)$ gravity point of view, i.e. we consider the set of equations of motion that is obtained by varying the action with respect to the spacetime metric $g_{\mu\nu}$. We consider an FLRW spacetime and supply the model with a perfect fluid satisfying the barotropic equation $p = \gamma \rho$, where $p$ is the pressure, $\rho$ the energy density and $\gamma$ a constant. Step by step, we construct the equivalent mini-superspace model having as configuration variables the scalar curvature $R$ and the scale factor $a$. The Euler-Lagrange equations for this system are seen to be equivalent to the field equations of the model under consideration and, additionally, consistent with the definition of the scalar curvature. By using the reparameterization invariance of the theory, we scale the lapse function in order to work in the constant potential parameterization. At this point the Killing vector fields of the scaled mini-superspace metric are used to define autonomous integrals of motion linear in the momenta. We choose to study those particular $f(R)$ models for which the ensuing mini-superspace exhibits the maximum number of symmetries, meaning that the mini-superspace is maximally symmetric. The corresponding integrals of motion are used in order to completely integrate the equations of motion and find the general analytical solution in each case.

The paper is organized as follows: In section 2 we give a brief outline of the theory regarding the existence and derivation of linear integrals of motion for singular Lagrangians. In section 3 we derive the equivalent mini-superspace model. In section 4 we examine every case for which the corresponding scaled mini-superspace metric is maximally symmetric and acquire the analytical solutions for these models. In section 5 we examine some of the properties of the Hubble parameters that are obtained through the previous mentioned solutions. Finally in the discussion we sum up our results together with some concluding remarks.

2 Preliminary remarks

A general relativistic action is of the form

$$S = \int d^4x \sqrt{-g} \mathcal{L}$$

(2.1)

with $\mathcal{L}$ being the Lagrangian density of the system and $\sqrt{-g} d^4x$ the infinitesimal space-time volume element. Variation with respect to the space-time metric $g_{\mu\nu}$ leads to the equations of motion for the gravitational field

$$E_{\mu\nu} = T_{\mu\nu}$$

(2.2)
with $E_{\mu\nu}$ and $T_{\mu\nu}$ the respective contributions from pure geometry (assuming minimal coupling) and $T_{\mu\nu}$ matter and/or cosmological constant (if present).

The imposition of certain symmetries in cosmology can lead to a simplification of the above problem and its reduction to a mechanical one. Whenever this is the case one is led to a reduced action

$$S_{\text{red}} = \int dt L,$$

(2.3)

where $t$ is the dynamical variable of the system and $L$ a singular Lagrangian of the general form

$$L = \frac{1}{2N} G_{\alpha\beta}(q) \dot{q}^{\alpha} \dot{q}^{\beta} - NV(q),$$

(2.4)

with $G_{\alpha\beta}$ ($\det G_{\alpha\beta} \neq 0$) being the mini-superspace metric, $N$ the lapse function of the base manifold, $q^{\alpha}$’s the configuration space variables and $\dot{q}^{\alpha} := \frac{dq^{\alpha}}{dt}$ their velocities with respect to the dynamical variable $t$. By adopting the usual definition of the momenta $p_{\alpha} := \frac{\partial L}{\partial \dot{q}^{\alpha}}$, $p_N := \frac{\partial L}{\partial \dot{N}}$ and following Dirac’s algorithm ([13], [14], [15]) one is led to the Hamiltonian

$$H_C = NH + u^N p_N = N \left( \frac{1}{2} G^{\alpha\beta} p_{\alpha} p_{\beta} + V \right) + u^N p_N$$

(2.5)

where $u^N$ is an arbitrary function. According to the theory, there exist two first class constraints

$$p_N \approx 0, \quad H \approx 0$$

(2.6)

that define the restricted phase space.

As shown in [16] and even more generally in [17], for singular systems described by Lagrangians of the form (2.3), every conformal vector field of the mini-supermetric can be used to define integrals of motion linear in the momenta. In brief, let $\xi$ be a vector field over the configuration space spanned by the $q$’s, then one can define in phase space the quantity

$$Q = \xi^{\alpha} p_{\alpha},$$

(2.7)

where $\xi^{\alpha}$ are the components of $\xi$. If now,

$$\mathcal{L}_{\xi} G_{\alpha\beta} = \omega(q) G_{\alpha\beta}$$

(2.8)

holds, we distinguish two possibilities:

- Apart from (2.8), the relation

$$\mathcal{L}_{\xi} V = -\omega(q) V$$

(2.9)

also holds, then $Q$ is itself an autonomous integral of motion, because

$$\frac{dQ}{dt} = \{Q, H_C\} = \omega N H \approx 0.$$

(2.10)
• The Lie derivative of the potential with respect to $\xi$ has a different conformal factor, i.e.
\[ L_\xi V = -\sigma(q)V \]  
with $\sigma \neq -\omega$. Then there can be defined a rheonomic integral of motion
\[ I = Q + \int N(\omega(q(t)) + \sigma(q(t)))V dt \]
with $q(t)$ being the trajectories obtained by solving the Euler - Lagrange equations. It is easy to check, that
\[ \frac{dI}{dt} = \{Q, H_C\} + N(\omega + \sigma)V = \frac{N\omega}{2}G_{\alpha\beta}p_\alpha p_\beta + N\omega V = \omega NH \approx 0. \]

However, one can argue, that these rheonomic integrals are not useful, for the purpose of integrating the equations of motion, since their solutions $q(t)$ need to be known a priori. Later on we will see that it is possible to overcome this difficulty, for one such quantity.

We can exploit the reparametrization invariance that is exhibited by theories with Lagrangians of the form of (2.4) to make the previous results even clearer. A scaling transformation of the lapse function $N = \frac{n}{V}$ leads to an equivalent Lagrangian
\[ L = \frac{1}{2n}G_{\alpha\beta}(q)\dot{q}^\alpha \dot{q}^\beta - n \]
and consequently to a Hamiltonian
\[ H_C = nH + u^\alpha p_\alpha = n\left(\frac{1}{2}G_{\alpha\beta}p_\alpha p_\beta + 1\right) + u^\alpha p_\alpha \]
with $G_{\alpha\beta} = VG_{\alpha\beta}$. The autonomous integrals of the system, $Q = \xi^\alpha p_\alpha$, are now constructed by Killing fields of the scaled supermetric ($L_\xi G_{\alpha\beta} = 0$) and their evolution is strictly zero, not just weakly, i.e. $\{Q, H_C\} = 0$. The rheonomic integrals of motion assume the form
\[ I = Q + \int n\omega(q)dt \]
with $\omega \neq 0$ defined by $L_\xi G_{\alpha\beta} = \omega G_{\alpha\beta}$. It is now clear from the above relation that, one of these rheonomic integrals can be utilized prior to the explicit knowledge of $q(t)$, in two ways: (a) by adoption of a particular gauge choice $n \propto \frac{1}{\omega(q)}$ or (b) if $\xi$ is a homothecy of the scaled supermetric $G_{\alpha\beta}$, i.e. $\omega = \text{const.}$

It is obvious that, in our analysis, not fixing the time gauge by selecting a particular lapse function (prior to the derivation of the equations of motion) is imperative. In the appendix of [16] we have proved that the gauge fixing of the lapse in Lagrangian (2.4)
may lead to a loss of some existing symmetries (ξ’s related to integrals of motion). Since Lagrangians of the form
\[ L = \frac{1}{2} G_{\alpha\beta} \dot{q}^\alpha \dot{q}^\beta - V, \] (2.17)
that correspond to the gauge \( N = 1 \), are commonly used in the literature of mini-superspace cosmology instead of (2.4), let us explain briefly why this is the case: Every physical theory, with finite degrees of freedom, and its properties are described by an action principle similar to (2.3). Lagrangians (2.4) and (2.17) correspond to different physical theories. If the dimension of \( G_{\alpha\beta} \) is \( d \), then (2.4) corresponds to a (singular) system of \( d+1 \) degrees of freedom, while (2.17) refers to a (regular) system with \( d \) degrees of freedom. The first is invariant under transformations of the form \( t \rightarrow v = h^{-1}(t) \); these transformations are the remnant of the general coordinate covariance of the base manifold metric components, i.e., the covariance under an arbitrary change in the time variable \( N(t)^2 dt^2 \rightarrow N(u)^2 \left( \frac{dh}{dv} \right)^2 dv^2 = \tilde{N}^2(v)dv^2 \). The second system is just a Newtonian-like system i.e. its action is invariant under only rigid time translations, \( t \rightarrow u = t + \varepsilon \), where \( \varepsilon \) is constant. This happens because, in the latter case, there is no constraint equation of motion \( \frac{dN}{dt} = 0 \). As a consequence, the resulting Hamiltonian is constant but not necessarily equal to zero destroying the reparameterization invariance of the theory, which in itself is imperative for many of the existing symmetries. Moreover, in order for someone to acquire the correct solution space, the condition of the Hamiltonian being zero must be demanded as an ad hoc condition (not already incorporated in the action principle).

For all the above mentioned reasons, we never gauge fix the lapse at the Lagrangian level, prior to the derivation of the symmetries and the equations of motion. Any gauge fixing, where necessary - for the sake of simplifying the equations - is imposed strictly after the derivation of symmetries and never inside the Lagrangian function.

3 FLRW f(R) cosmology with perfect fluid

We assume a Lagrangian density \( \mathcal{L} \) consisting of the gravitational part \( \mathcal{L}_g = f(R) \) plus a possible contribution of matter \( \mathcal{L}_m \)
\[ S = \int d^4 x \sqrt{-g} \left( f(R) + 2\mathcal{L}_m \right). \] (3.1)
Variation with respect to the base manifold metric \( g_{\mu\nu} \) leads to the known set of field equations for the metric \( f(R) \) gravity
\[ f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + g_{\mu\nu} f'(R) \delta^{\lambda}_{\mu} - f'(R) \delta_{\mu\nu} = T_{\mu\nu}, \] (3.2)
where \( f' = \frac{df}{dR} \), “;” stands for the covariant derivative with respect to \( g_{\mu\nu} \) and \( T_{\mu\nu} := \frac{\delta L_m}{\delta g_{\mu\nu}} \) is the energy momentum tensor, which is bound to satisfy
\[ T^{\mu\nu} = 0. \] (3.3)
By considering an FLRW space-time
\[ ds^2 = -N(t)^2 dt^2 + \frac{a(t)^2}{1 - kr^2} dr^2 + a(t)^2 (r^2 d\theta^2 + r^2 \sin \theta d\phi^2) \] (3.4)
and when one integrates out the redundant degrees of freedom \((r, \theta, \phi)\), the contribution of \(\sqrt{-g} f(R)\) in (3.1) is just \(Na^3 f(R)\). The scalar curvature is
\[ R = \frac{6 \left( -a \dot{a} \dot{N} + N (a \ddot{a} + \dot{a}^2) + k N^3 \right)}{a^2 N^3}, \] (3.5)
where the dot denotes differentiation with respect to the time coordinate \(t\). In order for (3.5) to be reproduced as an equation of motion in the corresponding minisuperspace model, we add it as a constraint in the ensuing Lagrangian and then properly fix the Lagrange multiplier (18, 19). In short, we assume a Lagrangian for the gravitational part that is of the form
\[ L_{g_1} = Na^3 f(R) + \lambda(t) \left( R - \frac{6 \left( -a \dot{a} \dot{N} + N (a \ddot{a} + \dot{a}^2) + k N^3 \right)}{a^2 N^3} \right). \] (3.6)
The equation of motion with respect to \(R\) is just
\[ \frac{\partial L_{g_1}}{\partial R} = 0 \Rightarrow \lambda(t) = Na^3 f'(R). \] (3.7)
If we insert this result into (3.6) and eliminate the acceleration term \(\ddot{a}\) by subtracting a total derivative of time, we are led to the following Lagrangian for the gravitational part
\[ L_g = -\frac{6}{N} \left( a f'(R) \dot{a}^2 + a^2 f''(R) \ddot{a} \right) + Na \left( a^2 (f(R) - R f'(R)) + 6 k f'(R) \right). \] (3.8)

For the matter part we consider a perfect fluid described by a barotropic equation of state \(p(t) = \gamma \rho(t)\), where \(p\) is the pressure, \(\rho\) the energy density and \(\gamma\) a constant. The energy momentum tensor is
\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu + pg_{\mu\nu}, \] (3.9)
where \(u^\mu\) is the 4-velocity of the fluid with \(u^\mu u_\mu = -1\). The equation derived by the conservation of energy (3.3) is
\[ 3(1 + \gamma) \rho \dot{a} + a \dot{\rho} = 0 \] (3.10)
with the well known solution
\[ \rho = ma^{-3(1+\gamma)}, \] (3.11)
where \(m\) is the constant of integration. The Lagrangian density for the perfect fluid is consisted solely of the energy density \(\rho\), i.e. \(\mathcal{L}_m = -\rho\). By considering (3.11) we
make the assumption that the matter contribution in the Lagrangian for the equivalent mechanical system is

$$L_m = -mN a^{-3\gamma}. \quad (3.12)$$

So the full Lagrangian under consideration becomes

$$L = L_g + 2L_m = -\frac{6}{N}\left(af'(R)\dot{a}^2 + a^3f''(R)\dot{a}\dot{R}\right)
+ Na\left(a^2\left(f(R) - Rf'(R)\right) + 6kf'(R) - 2ma^{-(1+3\gamma)}\right). \quad (3.13)$$

It is an easy task to check that the Euler-Lagrange equations of (3.13)

$$\frac{\partial L}{\partial N} = 0 \quad (3.14a)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{a}}\right) - \frac{\partial L}{\partial a} = 0 \quad (3.14b)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) - \frac{\partial L}{\partial R} = 0 \quad (3.14c)$$

are equivalent to the field equations (3.2), with (3.14c) revealing as it’s solution equation (3.5), which is given by the definition of $\dot{R}$. As a result, we state that Lagrangian (3.13) is valid, hence we can proceed with the mini-superspace analysis.

At this point, we choose to exploit the reparametrization invariance exhibited by Lagrangian (3.13) through a re-scaling of the lapse function ($N \to n$) as follows

$$N = -\frac{na^{3\gamma}}{6ka^{3\gamma+1}f'(R) - Ra^{3(\gamma+1)}f'(R) + a^{3(\gamma+1)}f(R) - 2m}. \quad (3.15)$$

We have already stated that this corresponds to a valid transformation ($t \to h(t)$) for a relativistic theory. Under the above transformation the Lagrangian assumes the form

$$L = \frac{A(a,R)}{n}a^2 + \frac{B(a,R)}{n}\dot{a}\dot{R} - n \quad (3.16)$$

with

$$A(a,R) = 6a f'(R)\left(6ka f'(R) + a^3\left(f(R) - Rf'(R)\right) - 2ma^{-3\gamma}\right) \quad (3.17a)$$
$$B(a,R) = 6a^{2-3\gamma}f''(R)\left(6ka^{3\gamma+1}f'(R) + a^{3(\gamma+1)}\left(f(R) - Rf'(R)\right) - 2m\right). \quad (3.17b)$$

As already mentioned in the previous section, it is in this parametrization of the constant potential, that the Killing fields of the scaled mini-supermetric

$$G_{\alpha\beta} = \begin{pmatrix} 2A & B \\ B & 0 \end{pmatrix} \quad (3.18)$$

generate autonomous integrals of motion in phase space (for simplicity we avoid the bar symbolism that we used in the previous section for this mini-supermetric).
The scaled Hamiltonian reads

\[ H_C = \frac{n}{2} G^{\alpha \beta} p_\alpha p_\beta + n + u^n p_n \]  

(3.19)

where \( \alpha, \beta = a, R \) with

\[ p_a = \frac{\partial L}{\partial \dot{a}}, \quad p_R = \frac{\partial L}{\partial \dot{R}}. \]  

(3.20)

The constraint space is defined by the two first class constraints

\[ p_n \approx 0 \]  

(3.21a)

\[ H = \frac{1}{2} G^{\alpha \beta} p_\alpha p_\beta + 1 \approx 0 \]  

(3.21b)

and each Killing vector field of (3.18), \( \mathcal{L}_{\xi} G_{\alpha \beta} = 0 \), defines the autonomous integral of motion \( Q_J = \xi^\alpha J p_\alpha \), because \( \{ Q_J, H_C \} = 0 \); while a homothecy of \( G_{\alpha \beta} \) \( (\mathcal{L}_h G_{\alpha \beta} = G_{\alpha \beta}) \) leads to the rheonomic integral \( I_h = Q_h + \int ndt = \xi^\alpha h p_\alpha + \int ndt \). Thus in each case we can define the following set of equations

\[ Q_J = \kappa_J \quad \text{and} \quad I_h = \kappa_h, \]  

(3.22)

where \( \kappa_J, \kappa_h \) are constants with \( J \) counting the number of possibly existing Killing fields.

## 4 Specific models

The number of Killing fields of (3.18) is upper bounded by its dimension. As it is known for a metric of dimension \( d \), the maximum number of possible isometries is \( \frac{d(d+1)}{2} \). Thus, in our case, there can be at most three Killing fields of \( G_{\mu \nu} \) and, accordingly, three autonomous integrals of motion linear in the momenta. Their number depends on the choice of \( f(R), k \) and \( \gamma \) since they are the only free parameters inside the scaled mini-supermetric. In other words, it is their choice that determines the geometry of mini-superspace.

We choose to focus our analysis on those models that correspond to a maximally symmetric superspace, i.e. \( G_{\alpha \beta} \) admits the maximum number of Killing vectors. Of course, this does not constitute a physical argument, it is a rather cynical choice so as to have in our disposal as many first integrals there is possible, in order to integrate the equations of motion. Our mini-superspace is two dimensional, hence, for maximal symmetry, the space needs to be either flat or of constant non zero scalar curvature \( \mathcal{R} \).

One can easily check, that the scalar curvature corresponding to \( G_{\alpha \beta} \) becomes a non zero constant only for a linear function \( f(R) = c_1 R + c_2 \) and \( k = 0 \) (driving also \( \gamma \) to particular values). Hence, restricting ourselves to all other cases of \( f(R) \) we need to investigate those models with a flat mini-superspace. As a starting point we distinguish two major cases, \( k = 0 \) and \( k \neq 0 \).
4.1 Spatially flat models \((k = 0)\)

By rejecting the possibility for \(f(R)\) being a linear function of \(R\), the mini-superspace is flat, i.e. \(\mathcal{R}_{\alpha\beta\rho\tau} = 0\) iff

\[
a^{3(\gamma+1)}(f(R)(Rf''(R) + f'(R)) - Rf'(R)^2) - 2m(f'(R) - 8f''(R)) = 0. \tag{4.1}
\]

Now, we have to proceed to a further separation of possible cases, according to what happens with \(a^{3(\gamma+1)}\) and its coefficient.

4.1.1 Case \(\gamma = -1\)

In this case the matter contribution in the Lagrangian corresponds to that of a cosmological constant, since \(L_m = -Na^3m = -\sqrt{-g}m\), with \(m\) being this constant. Under this selection for \(\gamma\), equation (4.1) holds for

\[
f(R) = 2m + \lambda R^\mu, \tag{4.2}
\]

with \(\lambda, \mu\) constants \((\mu \neq 0, 1)\). This choice results in the elimination of the cosmological constant \(m\) in the full Lagrangian (3.13). Henceforth, the corresponding model becomes equivalent to a vacuum model with \(f(R) = \lambda R^\mu\). Thus, we can state that, a cosmological constant cannot produce (at least not with an FLRW line element) a maximally symmetric mini-superspace. If one uses the latter as a requirement, it results in a necessary annihilation of \(m\) from the action itself. For reasons that will become evident later on in the analysis we have to distinguish several subcases.

- \(\mu \neq \frac{1}{2}\) and \(\mu \neq \frac{5}{4}\)

The three Killing vectors of the scaled mini-supermetric are

\[
\begin{align*}
    \xi_1 &= a\partial_a + \frac{6R}{1 - 2\mu}\partial_R, \\
    \xi_2 &= a^{\frac{4-3\mu}{\mu}}\partial_a - \frac{a^{\frac{5-4\mu}{\mu}}R}{\mu - 1}\partial_R, \\
    \xi_3 &= a^{\frac{1-2\mu}{\mu}}R^{2(1-\mu)}\partial_R,
\end{align*}
\tag{4.3}
\]

where from the form of \(\xi_1\) it becomes evident why we excluded the value \(\mu = \frac{1}{2}\). Additionally, there exists a homothetic vector \((\mathcal{L}_{\xi_h}G_{\alpha\beta} = G_{\alpha\beta})\)

\[
\xi_h = \frac{a}{6}\partial_a. \tag{4.4}
\]

Thus, we can define three autonomous integrals of motion linear in the momenta

\[
Q_J = \xi_J^\alpha p_\alpha, \quad J = 1, 2, 3, \quad \alpha = a, R \tag{4.5}
\]

and a rheonomic integral due to the homothetic vector field

\[
I_h = Q_h + \int n(t)dt = \xi_h^\alpha p_\alpha + \int n(t)dt, \quad \alpha = a, R. \tag{4.6}
\]
It can easily be verified that $\dot{Q}_J = \{Q_J, H_C\} = 0$ for $J = 1, 2, 3$ and $\dot{I}_h = \{Q_h, H_C\} + n(t) \approx 0$, thus on the solution space, relations

$$Q_J = \kappa_J$$

$$Q_h + \int n(t) dt = \kappa_h$$

with $\kappa_J, \kappa_h$ being constants hold. By substituting in (4.7a) and (4.7b) the momenta with respect to the configuration space variables and the velocities from (3.20), we are led to the following set of equations:

$$\kappa_1(1 - 2\mu)nR^2 - 6\lambda^2(\mu - 1)\mu a^5 R^{2\mu} \left( (2\mu^2 - 3\mu + 1) a\dot{R} - 2(\mu - 2)R\dot{a} \right) = 0 \quad (4.8a)$$

$$6\lambda^2(\mu - 1)\mu a^{2\mu - 1} R^{2\mu} \left( \dot{R} + (\mu - 1)a\dot{R} \right) + \kappa_2 nR^2 = 0 \quad (4.8b)$$

$$6\lambda^2(\mu - 1)^2 \mu a^{\frac{3\mu - 4}{\mu - 1}} \dot{a} + \kappa_3 n = 0 \quad (4.8c)$$

$$nR^2 \left( \int ndt - \kappa_h \right) - \lambda^2(\mu - 1)\mu a^5 R^{2\mu} (2R\dot{a} + (\mu - 1)a\dot{R}) = 0. \quad (4.8d)$$

By solving equation (4.8c) with respect to $n$ we get

$$n = -\frac{6\lambda^2(\mu - 1)^2 \mu a^{\frac{3\mu - 4}{\mu - 1}} \dot{a}}{\kappa_3}. \quad (4.9)$$

Two remarks are in order: (a) $\kappa_3$ cannot be zero, for then $a(t)$ would be constant, $R(t)$ zero and the resulting space-time would correspond to a trivial solution with $\gamma = -1$ and $f(R) = 2m$, (b) the fact that we solved (4.8c) with respect to $n$ does not constitute a gauge choice. This is because relations (4.8) are valid in any time gauge (as long as the field equations hold), no matter what is the particular functional form of $n$, $a$ or $R$.

By using (4.9) in the remaining equations, the gauge freedom passes over to one of the last two degrees of freedom, $a$ and $R$. So, we can choose either of them as an explicit function of time. A rather convenient choice is

$$a(t) = tR^{1-\mu}, \quad (4.10)$$

for then, equation (4.8a) can be integrated to give

$$R(t) = \frac{e^{\sigma \frac{1}{t^{\mu - 1}}} \left( \kappa_1(2\mu - 1)(\mu - 1)t^{\frac{1}{\mu - 1}} + \kappa_3(4\mu - 5)t^{\frac{2\mu - 4}{4\mu - 5}} \right)}{1} \quad (4.11)$$

where $\sigma$ is an integration constant. From (4.11) we can see why we also had to exclude the case $\mu = \frac{5}{4}$. Solutions (4.9), (4.10) and (4.11) together with (3.15) satisfy the Euler - Lagrange equations (3.14b) and (3.14c) for this model, while the quadratic constraint equation (3.14a) yields a relation between constants

$$\kappa_3 = 6\lambda^2 \mu(\mu - 1)^3(2\mu - 1)e^{(5 - 4\mu)\sigma}. \quad (4.12)$$
Relations (4.9), (4.10), (4.11) and (4.12) constitute the solution of the given system. If we substitute them into (4.8b) and (4.8d) we can evaluate the value of two more constants, \( \kappa_2 = \frac{e^{(4\mu - 5)s}}{24\mu^2 - 3\mu + 1} \) and \( \kappa_h = \frac{\kappa_1(1 - 2\mu)}{6(4\mu - 5)} \). The ensuing line element is (3.4) \((k = 0)\) with a lapse function

\[
N = 6\lambda \mu (\mu - 1)(2\mu - 1)e^{-2(\mu - 1)s}t\left(\frac{2\mu - 3}{(\mu - 1)(4\mu - 5)}\right) \left( \kappa_1(2\mu - 1)(\mu - 1)t^{\frac{1}{\mu - 1}} + \kappa_3(4\mu - 5)t^{\frac{2}{4\mu - 5}} \right)
\]

and a scale factor given by (4.10) with the substitution of (4.11).

- \( \mu = \frac{1}{2} \)

This time the three Killing vectors of the supermetric are

\[
\xi_1 = a\partial_a + 6R \ln \left( \frac{a^2}{R} \right) \partial_R, \quad \xi_2 = \frac{1}{a^5} \partial_a - \frac{2R}{a^8} \partial_R, \quad \xi_3 = R \partial_R.
\]

The latter two are the same as the \( \xi_2 \) and \( \xi_3 \) from (4.3) when one sets \( \mu = \frac{1}{2} \), but the first one cannot be obtained this way. Apart from these, there also exists a homothetic vector which is identical to (4.4). Thus, it is possible to define in the same way the linear integrals of motion (4.5), (4.6) and consequently the respective four equations (4.7a) and (4.7b) that yield following the system:

\[
3\lambda^2 a^5 \left( 2R\dot{a} \left( 3 \log \left( \frac{a^2}{R} \right) - 2 \right) + aR \right) + 4\kappa_1 nR = 0
\]

\[
\frac{3\lambda^2 \dot{a}}{2an} - \kappa_2 = \frac{3\lambda^2 \dot{R}}{4nR} = 0
\]

\[
\frac{3\lambda^2 a^5 \dot{\dot{a}}}{4n} - \kappa_3 = 0
\]

\[
\int ndt + \frac{\lambda^2 a^5 (4R\dot{a} - a\dot{R})}{8nR} - 8\kappa_h nR = 0.
\]

Here the situation is quite simpler than before, because the system can be solved algebraically for two of the three degrees of freedom and their derivatives, leaving unuttered the gauge freedom (for additional examples see [17] and [20]). We choose to solve system (4.15) with respect to \( fn \, dr, \, n, \, R \) and \( \dot{R} \). As a result we get

\[
\int ndt = -\frac{1}{6}\kappa_2 a^6 + \kappa_h + \frac{\kappa_3}{3}
\]

\[
n = -\frac{3\lambda^2 a^5 \dot{a}}{4\kappa_3}
\]

\[
R = a^2 e^{-\frac{-\kappa_2 a^6 + \kappa_3 + 2\kappa_4}{6\kappa_3}}
\]

\[
\dot{R} = a\dot{a} \left( \kappa_2 a^6 + 2\kappa_3 \right) e^{-\frac{-\kappa_2 a^6 + \kappa_3 + 2\kappa_4}{6\kappa_3}}.
\]
The consistency condition $\dot{R} = \frac{dR}{dt}$ is identically satisfied, while $n = \frac{d}{dt} \int n dt$ leads to a relation between constants

$$\kappa_2 = \frac{3 \lambda^2}{4 \kappa_3}. \quad (4.17)$$

This value of $\kappa_2$ is a realization of the constraint equation $H \approx 0$ (see [17] and [20]).

Expressions (4.16b), (4.16c) and (4.17) (with the use of (3.15)) are solutions of the Euler - Lagrange equations (3.14a), (3.14b) and (3.14c) for this model. Henceforth, they also solve the field equations (3.2) under the specific requirements we have made. The corresponding line element is

$$ds^2 = -\left(\frac{3 \lambda a(t) \dot{a}(t) e^{\frac{4 \kappa_3 (\kappa_1 + 2 \kappa_3) - 3 \lambda^2 a(t)^6}{48 \kappa_3}}}{2 \kappa_3}\right)^2 dt^2 + a(t)^2 (dr^2 + r^2 d\theta^2 + r^2 \sin \theta d\phi^2) \quad (4.18)$$

with $a(t)$ being an arbitrary function, since we did not need to make use of the gauge freedom for the integration of system (4.15).

- $\mu = \frac{5}{4}$

This value of the exponent leads to the following three Killing vectors in the configuration space:

$$\xi_1 = a \partial_a - 4 R \partial_R, \quad \xi_2 = a \ln(a) \partial_a - \frac{2}{3} R (1 + 6 \ln(a)) \partial_R, \quad \xi_3 = \frac{1}{a^6 4 \kappa_3} \partial_R, \quad (4.19)$$

while the homothetic vector remains the same as in the previous cases.

In the usual way we produce the ensuing set of equations (3.22) from the definition of the first integrals

$$15 \lambda^2 a^5 \sqrt{R} \left(4 \dot{a} + a \dot{R}\right) + 32 \kappa_1 n = 0 \quad (4.20a)$$

$$5 \lambda^2 a^5 \sqrt{R} \left(2 \dot{a} (6 \ln(a) - 1) + 3 a \ln(a) \dot{R}\right) + 32 \kappa_2 n = 0 \quad (4.20b)$$

$$-\frac{15 \lambda^2 \dot{a}}{32 a n} - \kappa_3 = 0 \quad (4.20c)$$

$$5 \lambda^2 a^5 \sqrt{R} \left(8 \dot{R} a + a \dot{R}\right) + 64 n \left(\kappa_h - \int n dt\right) = 0. \quad (4.20d)$$
Again the system can be solved algebraically with respect to $\int ndr, n, R$ and $\dot{R}$ and doing so we get

$$\int ndt = -\kappa_1 \ln(a) + \kappa_h - \frac{\kappa_1}{6} + \kappa_2 \quad (4.21a)$$

$$n = -\frac{15\lambda a \dot{a}}{32\kappa_3 a} \quad (4.21b)$$

$$R = \frac{\left(\frac{3}{2}\right)^{2/3} (\kappa_1 \ln(a) - \kappa_2)^{2/3}}{\kappa_3^{2/3} a^4} \quad (4.21c)$$

$$\dot{R} = \frac{\left(\frac{2}{3}\right)^{1/3} \dot{a} (-6\kappa_1 \ln(a) + \kappa_1 + 6\kappa_2)}{\kappa_3^{2/3} a^5 (\kappa_1 \ln(a) - \kappa_2)^{1/3}}. \quad (4.21d)$$

Once more, the consistency condition $\dot{R} = \frac{dR}{dt}$ is identically satisfied, while $n = \frac{d}{dt} \int ndt$ leads to the relation

$$\kappa_3 = 15\lambda^2 \frac{32\kappa_1}{32\kappa_1}. \quad (4.22)$$

As is expected, (4.21b), (4.21c) and (4.22) together with (3.15) solve the field equations for this particular model. The resulting line element is

$$ds^2 = -\left(\frac{5^{5/6} \kappa_1^{1/6} \lambda^{2/3} a(t) \dot{a}(t)}{2^{4/3}(\kappa_1 \ln(a(t)) - \kappa_2)^{5/6}}\right)^2 dt^2 + a(t)^2 (dr^2 + r^2 d\theta^2 + r^2 \sin \theta d\phi^2) \quad (4.23)$$

with $a(t)$ remaining an arbitrary function of time $t$.

All the solutions obtained in this section (4.18 for $\mu = 1/2$, 4.23 for $\mu = 5/4$ and the set (4.10), (4.11) and (4.13) for all the other values of $\mu$), that actually correspond to the vacuum case with $f(R) = \lambda R^\mu$, have also been obtained by different means in [3].

4.1.2 Case $\gamma \neq -1, \gamma \neq \pm\frac{2}{3}$ and $\gamma \neq -\frac{4}{3}$

The reason why we exclude the values $\pm\frac{2}{3}$ and $-\frac{4}{3}$ for $\gamma$ will become evident later on in the analysis. In order for (4.11) to be zero, one has to eliminate the coefficients of different powers of $a$. When $\gamma \neq -1$, at first we set

$$f(R) = \lambda R^\mu, \quad (4.24)$$

for the cancelation of the coefficient of $a^{3(\gamma + 1)}$. Then the requirement for the full vanishing of (4.11) leads to a relation between $\mu$ and $\gamma$, i.e. a correspondence between gravity and equation of state,

$$\mu = \frac{3(\gamma + 1)}{3\gamma + 2}. \quad (4.25)$$

As we can see the value $\gamma = -\frac{2}{3}$ would lead to a non flat mini-superspace which would not be maximally symmetric and thus out of our scope.
Under the assumptions \(\text{(4.24) and (4.25)}\) the mini-supermetric exhibits three Killing vectors

\[\xi_1 = a\partial_a - \left(\frac{6(3\gamma + 2)(\gamma - 1)(3\gamma + 2)(3\gamma + 4)mR - \lambda a^{3(\gamma + 1)}R^{\frac{3\gamma + 6}{\gamma + 3}})}{(3\gamma + 4)\left(\lambda a^{3(\gamma + 1)}R^{\frac{3(\gamma + 1)}{\gamma + 3}} + (6\gamma + 4)m\right)}\right)\partial_R\]  
(4.26a)

\[\xi_2 = a^{3\gamma - 1}\partial_a - (3\gamma + 2)Ra^{3\gamma - 2}\partial_R\]  
(4.26b)

\[\xi_3 = \left(\frac{5(3\gamma + 2)R^{\frac{\gamma + 4}{\gamma + 3}}}{a(3\gamma + 4)(\lambda a^{3(\gamma + 1)}R^{\frac{3(\gamma + 1)}{\gamma + 3}} + (6\gamma + 4)m)}\right)\partial_R,\]  
(4.26c)

and a homothetic Killing vector

\[\xi_h = \frac{a}{2 - 3\gamma}\partial_a - \frac{2 + 3\gamma}{2 - 3\gamma}R\partial_R.\]  
(4.27)

From the form of \(\text{(4.26) and (4.27)}\) it becomes clear that cases \(\gamma = \frac{2}{3}\) and \(\gamma = -\frac{4}{3}\) should be treated separately as special cases. For now, we proceed by excluding these particular values.

By the use of \(\text{(4.5), (4.6) and the definition of the momenta, we can} \) obtain four equations involving \(a, \dot{a}, R, \dot{R}, \int ndt\) and \(n.\) Unfortunately, due to the arbitrariness of \(\gamma, \) it is not possible to solve those equations algebraically (i.e. without making a gauge choice). The simplest equation is the one defined by \(\xi_3\) which yields

\[90(\gamma + 1)\lambda a^{1 - 3\gamma}\dot{a} \over (3\gamma + 2)(3\gamma + 4)n + \kappa_3 = 0.\]  
(4.28)

If we solve the latter with respect to \(n\) we get

\[n = -\frac{90(\gamma + 1)\lambda a^{1 - 3\gamma}\dot{a}}{(3\gamma + 2)^2(3\gamma + 4)\kappa_3},\]  
(4.29)

with \(\kappa_3 \neq 0\) for the same reason that we mentioned in the first subsection. As we stated previously \(\text{(4.29) is not a gauge choice, this freedom is} \) transported to the degrees of freedom \(a\) or \(R.\) As gauge we choose

\[a = tR^{-\frac{1}{2\gamma + 3}}.\]  
(4.30)

By substitution of \(\text{(4.29) and (4.30) in the equation defined by} Q_1 = \kappa_1 \) we get a first order ODE for \(R(t),\) which upon integration yields

\[R = \frac{\sigma t^{3\gamma + 2}}{(5\kappa_1 - (3\gamma - 2)\kappa_3 t (2(9\gamma^2 + 18\gamma + 8)m + \lambda t^{3\gamma + 3})^{2(3\gamma + 1)}}},\]  
(4.31)
with $\sigma$ being a constant. Relations (4.29), (4.30), (4.31) and by the use of (3.15) solve the spatial equations of motion (3.14b) and (3.14c), while the quadratic constraint (3.14a) gives rise to a relation between constants

$$\kappa_3 = \frac{450(\gamma + 1)\lambda \sigma^{\frac{3\gamma - 2}{3\gamma + 2}}}{(3\gamma + 2)^3(3\gamma + 4)}.$$  

(4.32)

The other equations for the integrals of motion, lead also to relations regarding the respective constants. The equation defined by $Q_2^2 = \kappa_2^2$ gives $\kappa_2 = \frac{1}{3}\sigma^{\frac{3\gamma - 2}{3\gamma + 2}}$, while $Q_h + \int n(t)dt = \kappa_h$ leads to $\kappa_h = 0$.

From (3.15) we can evaluate the lapse function in line element (3.4) ($k = 0$) to be

$$N = -\frac{90(\gamma + 1)\lambda \sigma^{-\frac{3\gamma - 2}{3\gamma + 2}} (5\kappa_1 - (3\gamma - 2)\kappa_3 t (2(9\gamma^2 + 18\gamma + 8)m + \lambda t^{3\gamma + 3})))^{\frac{2\gamma}{2\gamma - 3\gamma}}}{3\gamma + 2}.$$  

(4.33)

with $\kappa_3$ given by (4.32) and $\kappa_1$, $\sigma$ remaining arbitrary constants. The scale factor $a(t)$ is given by (4.30) with substitution of (4.31). This solution was also attained in [3] with the same exceptions $\gamma \neq \pm \frac{2}{3}$ and $\gamma \neq -\frac{4}{3}$.

Since we managed to solve the field equations for a general $\gamma$ (apart from four specific values), we can use (4.33), (4.30), (4.31) and (4.32) to evaluate solutions for specific equations of state (of course we must bear in mind that $\mu$ is also fixed through (4.25)).

For a pressureless matter $\gamma = 0 \Rightarrow \mu = \frac{3}{2}$ we have

$$N = -45\frac{\lambda}{\sigma^2} a = \left(\frac{5\kappa_1 + 32\kappa_3 m t + 2\kappa_3 \lambda t^4}{\sigma}\right)^{1/2}, \quad \kappa_3 = \frac{225\lambda}{6\sigma},$$  

(4.34)

for radiation $\gamma = \frac{1}{3} \Rightarrow \mu = \frac{4}{3}$ we get

$$N = -\frac{40\lambda (5\kappa_1 + 30\kappa_3 m t + \kappa_3 \lambda t^5)}{\sigma^{2/3}}, \quad a = \frac{5\kappa_1 + 30\kappa_3 m t + \kappa_3 \lambda t^5}{\sigma^{1/3}}, \quad \kappa_3 = \frac{40\lambda}{3\sigma^{1/3}},$$  

(4.35)

and finally for stiff matter $\gamma = 1 \Rightarrow \mu = \frac{6}{5}$ the result is

$$N = -\frac{36\lambda}{\sigma^{2/5} (5\kappa_1 - \kappa_3 t (70m + \lambda t^6))^{3}}, \quad a = \frac{1}{\sigma^{1/5} (5\kappa_1 - \kappa_3 t (70m + \lambda t^6))^{1/5}}, \quad \kappa_3 = \frac{36\lambda\sigma^{1/5}}{7},$$  

(4.36)

4.1.3 Case $\gamma = \frac{2}{3}$

This is a special case of a model with $f(R)$ given by (4.24). Under this specific choice for $\gamma$, and through (4.25), we are led to $\mu = \frac{5}{4}$. The resulting mini-superspace is of course
(since we chose so) flat and its metric has the following three Killing vectors

\[\xi_1 = a\partial_a - 4R\partial_R\]  
\[\xi_2 = a\ln(a)\partial_a - \frac{2R(a^5\lambda R^{5/4} + 6\ln(a)(a^5\lambda R^{5/4} + 8m) + 48m)}{3(a^5\lambda R^{5/4} + 8m)}\partial_R\]  
\[\xi_3 = \frac{10R^{5/4}}{3a^6\lambda R^{5/4} + 24am}\partial_R,\]  

along with the homothetic vector

\[\xi_h = \frac{2a^5\lambda R^{9/4} + 96mR}{3a^5\lambda R^{9/4} + 24m}\partial_R\]

The equations corresponding to those four quantities that are constants of motion cannot be given in closed form by algebraically solving for two of the degrees of freedom and their derivatives. Thus, we are obligated to proceed by a suitable gauge fixing. The same procedure is applied, first we solve equation \(Q_3 = \kappa_3\) with respect to the scaled lapse \(n\)

\[n = \frac{25\lambda}{16\kappa_3 a}\]  
then we adopt the gauge choice

\[a = tR^{-1/4}\]  
and equation \(Q_1 = \kappa_1\) can be integrated to give

\[R = \sigma t^4 e^{-\frac{4\kappa_3 t}{6\kappa_1}(48m + \lambda t^5)}\]  

Relations (4.39), (4.40) and (4.41) solve the spatial field equations of the model under consideration. The quadratic constraint reveals a relation between constants

\[\kappa_3 = \frac{25\lambda}{16\kappa_1}\]  
which completes the solution of the equations of motion. The constant values of the other integrals can be evaluated to be

\[\kappa_2 = -\kappa_h = -\frac{1}{4}\kappa_1\ln(\sigma)\]  
Finally, the lapse in line element (3.14) (with \(k = 0\)) can be calculated through (3.15), the latter yields

\[N = \frac{5\lambda e^{\frac{\kappa_1 t}{8\kappa_1}}}{2\kappa_1 \sqrt{\sigma}}\]  
and of course the scale factor is given by (4.40). To the best of our knowledge this solution has not been previously presented in the relevant literature.
4.1.4 Case $\gamma = -\frac{4}{3}$

One can proceed exactly as in the previous subcase. This time $\mu = \frac{1}{2}$, the minisupermetric has the three Killing vectors

$$\xi_1 = a \partial_a + \left( -\frac{12\lambda R^{3/2} \ln(a) + 56amR + 6\lambda R^{3/2} \ln(R)}{4am - \lambda \sqrt{R}} \right) \partial_R$$

(4.45a)

$$\xi_2 = a^{-5} \partial_a + \frac{2R}{a^6} \partial_R$$

(4.45b)

$$\xi_3 = \frac{R^{3/2}}{4am - \lambda \sqrt{R}} \partial_R,$$

(4.45c)

and the homothetic vector

$$\xi_h = \frac{a}{6} \partial_a + \frac{R}{3} \partial_R.$$  (4.46)

By following the gauge fixing approach we arrive at the solution

$$n = \frac{3\lambda a^5 \dot{a}}{4\kappa_3}$$

(4.47a)

$$a = t\sqrt{R}$$

(4.47b)

$$R = \frac{(\sigma \kappa_2 - 48\kappa_3 mt + 12\kappa_3 \lambda \ln(t))^{1/3}}{\kappa_2^{1/3} t^2}$$

(4.47c)

$$\kappa_2 = -\frac{3\lambda}{4\kappa_1}.$$  (4.47d)

We note that, from the equations defined by the integrals, we chose to solve $Q_3 = \kappa_3$ and $Q_2 = \kappa_2$. The other two equations give the relations

$$\kappa_1 = \kappa_2 \sigma + 2\kappa_3 \lambda, \quad \kappa_h = 0.$$  (4.48)

Equation (3.15) yields the lapse function of the space time line element, which is

$$N = \frac{4\kappa_3 \lambda^{2/3}}{(\lambda \sigma + 64\kappa_3^2 mt - 16\kappa_3^2 \lambda \ln(t))^{2/3}}.$$  (4.49)

Again the solution here obtained is, to the best of our knowledge, new.

4.2 Non vanishing $k$

For a non vanishing $k$, the mini-superspace is flat if

$$6a^2 f''(R) \left( -f''(R) \left( 2m \left( 18\gamma k - a^2(3\gamma + 2)R \right) - a^3(\gamma + 1) f(R) \left( a^2 R - 18k \right) \right) 
+ a^3(\gamma + 1) \left( - \left( a^2 R - 6k \right) \right) f'(R)^2 + a^2 f'(R) \left( a^3(\gamma + 1) f(R) - 2m \right) \right) = 0.$$  (4.50)
We ignore the case of \( f(R) \) being a linear function of \( R \), and by taking the coefficient of \( a^{7+3\gamma} \) we see that again \( f(R) \) must be of the form \( f(R) = \lambda R^{\mu} \). In order to proceed we consider the coefficient of \( a^{5+3\gamma} \) that leads to the relation

\[
\mu \left( 2\mu^2 - 5\mu + 3 \right) = 0. \tag{4.51}
\]

We again do not take into account the solutions \( \mu = 0 \) and \( \mu = 1 \), since they are trivial for the theory, so we are left with just \( \mu = \frac{3}{2} \). Under these assumptions, equation (4.50) leads finally to \( \gamma = 0 \). We note here, that the specific choices \( \gamma = -\frac{7}{3} \) and \( \gamma = -\frac{5}{3} \), which could change the coefficient arrangements in (4.50), lead to trivial choices for \( \mu \) and \( k \), i.e. \( \mu = 0,1 \) and \( k = 0 \). To summarize, the requirement that we must have a maximally symmetric mini-superspace in the \( k \neq 0 \) case leaves, as the only option, a pressureless perfect fluid and a (modified) gravitation theory that is described by

\[
f(R) = \lambda R^{3/2}. \tag{4.52}
\]

The three Killing vectors of the mini-supermetric for the model under consideration are

\[
\begin{align*}
\xi_1 &= \frac{a^2 + 1}{2a} \partial_a - \frac{R \left( a\lambda \sqrt{R} (a^2 (3a^2 + 2) R - 36 (2a^2 + 1) k) + 8 (3a^2 + 1) m \right)}{2a^2 \left( a\lambda \sqrt{R} (a^2 R - 18 k) + 4m \right)} \partial_R, \\
\xi_2 &= \frac{a^2 - 1}{2a} \partial_a + \frac{a\lambda R^{3/2} (36 (2a^2 - 1) k + (2 - 3a^2) a^2 R) + 8 (1 - 3a^2) mR}{2a^2 \left( a\lambda \sqrt{R} (a^2 R - 18 k) + 4m \right)} \partial_R, \\
\xi_3 &= \frac{\sqrt{R}}{a \left( a\lambda \sqrt{R} (a^2 R - 18 k) + 4m \right)} \partial_R.
\end{align*}
\]

Additionally, there exists a homothetic vector

\[
\xi_h = \frac{a}{2} \partial_a - R \partial_R. \tag{4.54}
\]

The system of equations that corresponds to the constant integrals of motion that are constructed by (4.53) and (4.54) cannot be solved algebraically, thus we follow the procedure that entails a convenient gauge choice. By solving equation \( Q_3 = \kappa_3 \) with respect to \( n \) we get

\[
n = -\frac{9\lambda a\dot{a}}{4\kappa_3}. \tag{4.55}
\]

Substitution of (4.55) into \( Q_1 = \kappa_1 \) and under the gauge choice

\[
a = t R^{-1/2}, \tag{4.56}
\]

yields a first order ODE for \( R(t) \) which can be integrated to give

\[
R = \frac{t^2}{\sigma (2\kappa_1 - 36\kappa_3 k\lambda t^2 + 16\kappa_3 m t + \kappa_3 \lambda t^4) - 1},
\]
with \( \sigma \) being the constant of integration.

The spatial field equations of this model are satisfied by (4.55) (with the use also of (3.15)), (4.56) and (4.57). The quadratic constraint equation as in the previous cases leads to a relation between constants

\[
\kappa_3 = \frac{9 \lambda \sigma}{4}
\]

and that completes the solution of the model. Now, it is an easy task to also compute the values of the other two constants of motion

\[
\kappa_2 = \frac{\kappa_1 \sigma}{\sigma} - 1, \quad \kappa_h = 0.
\]

The resulting lapse in line element (3.4) is (relation (3.15))

\[
N = -9 \lambda \sigma
\]

and the scale factor is given by (4.56) under the substitution of (4.57). This solution is also new - in its generality - to the best of our knowledge. It is interesting to note that setting \( m = 0 \) in the line element one obtains the vacuum solution \((T_{\mu \nu} = 0)\). It is also noteworthy that, the same solution holds for \( k = 0 \) \((m \neq 0)\); this latter case was investigated in [21] where the corresponding solution was expressed in a different set of variables.

5 The Hubble parameter

The definition of the Hubble parameter in an arbitrary time gauge is

\[
H(t) := \frac{1}{a(t) N(t)} \frac{d a(t)}{d t}.
\]

With the use of (5.1) it is an easy task to evaluate this function for all the solutions obtained in the previous sections. However if one needs to derive useful conclusions by comparison to observational data, function \( H(t) \) needs to be expressed in a time coordinate that exhibits a constant gauge (usually \( N = 1 \)). In order to do this, one has to apply the transformation \( t \rightarrow \tau = \int N dt \), which in most cases is practically impossible, since a closed form for the function \( t(\tau) \) is not always attainable.

In what follows we evaluate \( H(t) \) for all the derived solutions and wherever possible we give in closed form the transformation that links \( H(t) \) to \( H(\tau) \). At first, we consider the cases were \( k = 0 \):

- \( \gamma = -1, \mu \neq \frac{1}{2} \) and \( \mu \neq \frac{5}{4} \)

  The desired function is

  \[
  H_\mu(t) = \frac{\kappa_3 e^{2(\mu - 1) \sigma t^{\frac{\mu^2 - 3 \mu - 2}{4 \mu^2 - 2 \mu + 5}}} (\kappa_1 (2 \mu^2 - 3 \mu + 1) t^{1 - \mu} + \kappa_3 (4 \mu - 5) t^{2 \mu})^{-\frac{2(\mu - 1)}{4 \mu - 5}}}{6 \lambda (\mu - 1) \mu}
  \]

  (5.2)
with $\kappa_3 = 6 \lambda^2 \mu (\mu - 1)^3 (2\mu - 1)e^{(5 - 4\mu)\sigma}$ being the only fixed constant. The behavior of function (5.2) is highly dependent on $\mu$. For example, if we set $\lambda = \sigma = \kappa_1 = 1$, then for $\mu = \frac{3}{2}$

$$H_{3/2} = \frac{2et^3}{9t^4 + 4e}, \quad (5.3)$$

while for $\mu = 2$

$$H_2 = \frac{3\sqrt{3}e^{8/3}}{(t(36t^3 + e^3))^{2/3}}. \quad (5.4)$$

It is easy to check that $H_{3/2}$ is an increasing function of $t$ until the maxima exhibited at $t = \sqrt{\frac{2e}{\sqrt{3}}}$, from then it just decreases until at $t \to +\infty$, $H_{3/2} \to 0$. On the other hand $H_2$ is an increasing function that becomes upper bounded at infinity $\lim_{t \to +\infty} H_2 = \frac{6e^6}{\sqrt[3]{2}}$. By choosing even bigger values of $\mu$, functions that diverge at infinity can be obtained.

As is evident from (4.13) one can not obtain a closed form solution $t(\tau)$ from $\tau(t) = \int N(t)dt$ for arbitrary values of $\mu$. We only mention that in the case where $\mu = 3/2$ the corresponding relation is just $t(\tau) = e^{6\tau}$. This linear relation between $t$ and $\tau$ means that $H_{3/2}(t)$ and $H_{3/2}(\tau)$ exhibit the same functional behavior.

- $\gamma = -1$ and $\mu = \frac{5}{4}$

In this case we have obtained the solution space in an arbitrary gauge. But, for the analysis of the Hubble parameter, we have to choose some functional form for $a(t)$. The simplest admissible one (so that the lapse does not become zero) is $a(t) = t$, then the corresponding Hubble parameter becomes

$$H_{5/4} = \frac{2\sqrt{2}(\kappa_1 \log(t) - \kappa_2)^{5/6}}{5^{5/6}\kappa_1^{1/6}\lambda^{2/3}t^2}. \quad (5.5)$$

It can be seen that $H_{5/4}$ is an increasing function until the maxima at $t = e^{\frac{5\kappa_1 + 12\kappa_2}{12\kappa_1}}$, then it decreases and when $t \to +\infty$, $H_{5/4}$ converges to zero. In this case, a closed form expression for $t(\tau)$ cannot be found, since $\tau(t) = \int N dt$ is expressed in terms of the incomplete Gamma function.

- $\gamma = -1$ and $\mu = \frac{1}{2}$

As in the previous case we choose $a(t) = t$. The Hubble parameter is

$$H_{1/2} = -\frac{2\kappa_3 e^{-4\kappa_1(\kappa_1 + 2\kappa_2) - 3\lambda^2t^6}}{3\lambda t^2}. \quad (5.6)$$

This is an even function that diverges ($H_{1/2} \to -\infty$) for both $t \to 0$ and $t \to +\infty$, its maxima are at $t = \pm^{2/3}\frac{\sqrt{2}}{\sqrt[3]{3}}$. Again the transformation $t \to \tau(t)$ cannot be applied.
• $\gamma \neq -1, \pm \frac{2}{3}$ and $-\frac{4}{3}$

As we have seen, in this case $\mu = \frac{3(\gamma + 1)}{2(\gamma + 2)}$, so we expect $\gamma$ to play a dominant role in the Hubble parameter, with the latter being

$$H_{\mu,\gamma} = -(3\gamma + 2)(3\gamma + 4)\kappa_3 \sigma^{\frac{2}{3\gamma + 2}} \left( (6\gamma + 4)m + \lambda t^{3\gamma + 3} \right) x \left( 5\kappa_1 - (3\gamma - 2)\kappa_3 t (2(9\gamma^2 + 18\gamma + 8)m + \lambda t^{3\gamma + 3}) \right) \frac{\sigma^{\frac{2}{3\gamma + 2}}}{90(\gamma + 1)\lambda}$$

(5.7)

where $\kappa_3 = \frac{450(\gamma + 1)\lambda \sigma^{\frac{2}{3\gamma + 2}}}{(3\gamma + 2)^2(3\gamma + 4)}$.

In the case of stiff matter, $\gamma = 1$ and $\mu = \frac{6}{5}$, function (5.7) becomes

$$H_{6/5,1} = \sigma \left( -(10m + \lambda t^6) \right) \left( \frac{5}{2} \kappa_1 \sigma^{1/5} - \frac{36}{7} \lambda t \left( 70m + \lambda t^6 \right) \right)^2,$$

(5.8)

which converges at $t = 0$ to $H_{6/5,1}(0) = -250\kappa_1^2 \sigma^{2/5}$, while for $t \to \pm \infty$ diverges to $\pm \infty$ depending on the sign of $\lambda$. A positive lambda leads to $\lim_{t \to \pm \infty} H_{6/5,1} = -\infty$.

For dust, where $\gamma = 0$ and $\mu = \frac{3}{2}$, we get

$$H_{3/2,0} = -\frac{2\sigma (4m + \lambda t^3)}{4\kappa_1 \sigma + 720\lambda mt + 45\lambda^2 t^4}.$$ 

(5.9)

At zero, $H_{3/2,0}(0) = -\frac{2m}{k_1}$, while at infinity $\lim_{t \to \pm \infty} H_{3/2,0} = 0$.

Finally, if we consider radiation $\gamma = \frac{1}{3}$, $\mu = \frac{4}{3}$ the function

$$H_{4/3,1/3} = -\frac{3\sigma (6m + \lambda t^4)}{5(3\kappa_1 (3^\frac{1}{2}\sigma + 240\lambda mt + 8\lambda^2 t^5)^\frac{1}{2})}$$

(5.10)

is obtained. This time $H_{4/3,1/3}(0) = -\frac{2m \sqrt{\sigma}}{5\kappa_1}$ and again $\lim_{t \to \pm \infty} H_{4/3,1/3} = 0$.

Of course, the arbitrariness of $\gamma$ makes the derivation of a general transformation $t(t)$ impossible. In the particular subcase $\gamma = 0$, the proper time is $\tau = \frac{45\lambda}{\sigma}$ thus, $H_{3/2,0}(\tau)$ behaves exactly like $H_{3/2,0}(t)$.

• $\gamma = \frac{2}{3}$

This means that $\mu = \frac{5}{4}$ and it can easily be seen that

$$H_{5/4,2/3} = -\frac{4\kappa_3 \sqrt{\sigma} (8m + \lambda t^5) e^{\frac{5\lambda (48m + \lambda t^5)}{8m t^2}}}{25\lambda},$$

(5.11)

where $\kappa_3 = \frac{25\lambda}{16\kappa_1}$. The value of $H_{5/4,2/3}$ at zero is $-\frac{2m \sqrt{\sigma}}{5\kappa_1}$ and when $t$ goes to plus or minus infinity the function converges to zero.
• $\gamma = -\frac{4}{3}$

The corresponding value of $\mu$ is $\frac{1}{2}$ and the Hubble parameter becomes

$$H_{1/2,-4/3} = \frac{2\kappa_3 (4mt - \lambda)}{3\lambda^{2/3} t \sqrt{\lambda \sigma + 64\kappa_3^2 mt - 16\kappa_3^2 \lambda \log(t)}}.$$  (5.12)

As it can easily be seen, $H_{1/2,-4/3}$ is defined for positive $t$ and specifically for those values that satisfy the condition

$$\lambda \sigma + 64\kappa_3^2 mt - 16\kappa_3^2 \lambda \log(t) > 0.$$ 

However, what can easily be deduced is, that for $t \to \pm\infty$, $H_{1/2,-4/3}$ becomes zero.

Finally, we are left with the case $k \neq 0$. Under our assumptions, the sole case that emerged leads to $\mu = \frac{3}{2}$, $\gamma = 0$ and

$$kH_{3/2,0} = \frac{2\kappa_3 (\lambda (t^2 - 18k) + 4m)}{9\lambda (2\kappa_1 \sigma - 36\kappa_3 k \lambda \sigma t^2 + 16\kappa_3 m \sigma t + \kappa_3 \lambda \sigma t^4 - 1)}.$$  (5.13)

with $\kappa_3 = \frac{9\lambda \sigma}{4}$. Since the lapse function is $N = -9\lambda \sigma$, $kH_{3/2,0}(\tau)$ can be obtained by a constant scaling of time $t = -\frac{\tau}{9\lambda \sigma}$. Irrespectively of the choice of $k$, the value of $kH_{3/2,0}(\tau)$ at $\tau = 0$ is $-\frac{8\mu \sigma}{8\kappa_1 \sigma - 4}$ and for $\tau \to +\infty$ the function converges to zero. This of course does not mean that for different $k$, $kH_{3/2,0}$ exhibits the same behavior. One can easily see, for example that $k = +1$ and $k = -1$ correspond to quite different functional behaviors, since that change of sign inside the polynomial of the denominator, changes the number and the nature of possible extrema throughout the domain of the function.

6 Conclusion

Throughout this paper we exploit the reparametrization invariance of singular systems, with a Lagrangian of the form (2.3). Autonomous linear integrals of motion in phase space, are generated by the Killing vector fields of the scaled super metric $G_{\alpha\beta} = VG_{\alpha\beta}$, while its proper conformal Killing vectors give rise to integrals with an explicit time dependance.

This scheme is implemented in the context of an FLRW space-time in $f(R)$ gravity minimally coupled with a perfect fluid governed by a barotropic equation of state. As a first step, we write the equivalent mechanical system (equivalent in the sense that the field equations are satisfied whenever the Euler - Lagrange equations hold). Then we require that the reduced system describes a maximally symmetric mini-superspace. This imposes conditions upon $f(R)$, as well as on any other characteristic parameters that enter the Lagrangian. Since the mini-supermetric is two dimensional there exist three Killing fields. Thus, the three autonomous integrals of motion together with one of the infinite rheonomic, are used to completely integrate the equations of motion. It is interesting to observe that the latter are not used anywhere in the analysis, since
the number of the integrals define enough independent relations to completely solve the
system (even algebraically in some cases). Some known solutions are obtained, together
with some others that, to our knowledge, are new to the literature (subsections 4.1.3
and the general case in section 4.2 with $k, m \neq 0$).

Finally, we conclude with a brief investigation of the resulting Hubble parameters
$H(t)$. It is a common fact in the search of analytical solutions, that sometimes a time
gauge different than $N = 1$ is a more convenient choice to be employed so as to integrate
the field equations. The same holds in our case as well, so most of the solutions are
expressed in time coordinates where the lapse function is not constant. If one wants to
express the Hubble parameter with respect to the proper time defined by $\tau = \int N(t)dt$,
the transformation $t = t(\tau)$ must be found. As we observe from the particular examples,
this is a rather difficult task, even impossible in some cases. However, the gauge freedom
of the lapse function is a too powerful tool, to senselessly restrict it prior to writing
down the equations of motion. Let alone that any integral of motion that is constant
modulo the constraint equation $H_C \approx 0$, would be impossible to be recovered if $N = 1$
was naively imposed inside the Lagrangian.

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