Lorentz invariance and the zero-point stress-energy tensor

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Abstract: Some 65 years ago (1951) Wolfgang Pauli noted that the net zero-point energy density could be set to zero by a carefully fine-tuned cancellation between bosons and fermions. In the current article I will argue in a slightly different direction: The zero-point energy density is only one component of the zero-point stress energy tensor, and it is this tensor quantity that is in many ways the more fundamental object of interest. I shall demonstrate that Lorentz invariance of the zero-point stress energy tensor implies finiteness of the zero-point stress energy tensor, and vice versa. Under certain circumstances, (in particular, but not limited to, the finite QFTs), Pauli’s cancellation mechanism will survive the introduction of particle interactions. I shall then relate the discussion to BSM physics, to the cosmological constant, and to Sakharov-style induced gravity.

Keywords: Lorentz invariance; zero-point stress–energy; zero-point energy density; zero-point pressure.

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1 Introduction

In his ETH lectures of 1951, (transcribed and translated into English in 1971), Wolfgang Pauli noted that the net zero-point energy density could be set to zero by imposing a carefully fine-tuned cancellation between bosons and fermions [1]. In more modern notation he observed that for relativistic QFTs on a Minkowski background one has:

\[ \rho_{zpe} = \sum_n \left\{ (-1)^{2S_n} g_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \hbar \omega_n(k) \right\}. \] (1.1)

This integrates the zero-point energy \( \pm \frac{1}{2} \hbar \omega(k) \) over all modes (all three-momenta), counting boson contributions as positive and fermion contributions as negative. The degeneracy factor \( g \) includes a spin factor \( g = 2S + 1 \) for massive particles, whereas the spin factor is \( g = 2 \) for massless particles. The degeneracy factor \( g \) also includes an additional factor of 2 when particle and antiparticle are distinct, and an additional factor of 3 due to colour. (So for example, \( g = 2 \) for the photon, \( g = 4 \) for the electron, and \( g = 12 \) for quarks.) Finally one sums over all particle species indexed by \( n \). It is the physical relevance of this sum over the entire particle physics spectrum that is Pauli’s key insight. (In related discussion in reference [2] the \( (-1)^{2S} \) has been absorbed into the degeneracy factor \( g \).) Explicitly introducing particle masses one sees that

\[ \rho_{zpe} = \sum_n \left\{ (-1)^{2S_n} g_n \frac{1}{2} \hbar \int \frac{d^3k}{(2\pi)^3} \sqrt{m_n^2 + k^2} \right\}. \] (1.2)

The key integral is

\[
\int_0^K d^3k \sqrt{m^2 + k^2} = 4\pi \int_0^K dk \ k^2 \sqrt{m^2 + k^2} \\
= \pi \left\{ K(m^2 + K^2)^{3/2} - \frac{1}{2} m^2 K \sqrt{m^2 + K^2} \\
- \frac{1}{2} m^4 \ln \left( \frac{K + \sqrt{m^2 + K^2}}{m} \right) \right\} \\
= \pi \left\{ K^4 + m^2 K^2 + \frac{m^4}{8} - \frac{1}{2} m^4 \ln(2K/m) \right\} + O \left( \frac{1}{K^2} \right). \] (1.3)

(There is an inconsequential and non-propagating typo in Pauli’s corresponding formula in reference [1]...) Using this integral, Pauli then observed that the total zero-point energy density vanishes if and only if we first impose the three polynomial-in-mass conditions

\[ \sum_n (-1)^{2S_n} g_n = 0; \quad \sum_n (-1)^{2S_n} g_n m_n^2 = 0; \quad \sum_n (-1)^{2S_n} g_n m_n^4 = 0; \] (1.4)
and then supplement this with a fourth logarithmic-in-mass condition
\[
\sum_n (-1)^{2S_n} g_n m_n^4 \ln(m_n^2/\mu_*) = 0.
\] (1.5)

(This fourth, logarithmic-in-mass condition, is actually completely independent of the arbitrary-but-fixed parameter $\mu_*$, due to one already having imposed the third polynomial-in-mass condition.) This enforced vanishing of the zero-point energy density certainly requires an extremely delicate fine-tuning of the particle physics spectrum.

Let us now modify Pauli’s discussion and take it in a rather different direction — the zero-point energy density is only one component of the zero-point stress-energy tensor; and we shall soon see that this zero-point stress-energy tensor is of more fundamental importance than the zero-point energy density considered in isolation. Specifically, we shall demonstrate that Lorentz invariance of the zero-point stress energy tensor implies finiteness of the zero-point stress energy tensor, and vice versa.

2 The zero-point stress-energy tensor

The on-shell zero-point stress-energy tensor is simply
\[
(T_{zpe})^{ab} = \sum_n \left\{ (-1)^{2S_n} g_n \int \frac{d^3k}{2\omega_n(k) (2\pi)^3} \hbar k^a_n k^b_n \right\}. \tag{2.1}
\]

Note that we are now integrating a tensor product of 4-momenta $k^a = (\omega(k); k^i) = (\sqrt{m^2+k^2}; k^i)$ over Lorentz invariant phase space $d^3k/(2\omega)$. This zero-point stress-energy tensor is certainly Lorentz covariant; but making sure it is Lorentz invariant (ie, independent of the inertial frame chosen to do the calculation) is more subtle. Being more explicit about this
\[
(T_{zpe})^{ab} = \sum_n \left\{ (-1)^{2S_n} g_n \int \frac{d^3k}{2\omega_n(k) (2\pi)^3} \hbar \left[ \frac{\omega_n(k)^2}{\omega_n(k) k^i} \frac{\omega_n(k) k^j}{k^i k^j} \right]^{ab} \right\}. \tag{2.3}
\]

Rotational invariance is enough to bring this into the form
\[
(T_{zpe})^{ab} = \sum_n \left\{ (-1)^{2S_n} g_n \int \frac{d^3k}{2\omega_n(k) (2\pi)^3} \hbar \left[ \frac{\omega_n(k)^2}{0} \frac{0}{\frac{1}{3} k^2 \delta^{ij}} \right]^{ab} \right\}. \tag{2.4}
\]
That is, based solely on rotational invariance,

$$ (T_{zpe})^{ab} = \begin{bmatrix} \rho_{zpe} & 0 \\ 0 & p_{zpe} \delta^{ij} \end{bmatrix}. $$

(2.5)

Here the formula for $\rho_{zpe}$ is exactly the same as in Pauli’s calculation, equation (1.1), while the zero-point pressure $p_{zpe}$ is seen to be

$$ p_{zpe} = \sum_n \left\{ (-1)^{2S_n} g_n \frac{\hbar}{2} \int \frac{d^3k}{\sqrt{m^2_n + k^2}} \left( \frac{k^2}{(2\pi)^3} \right) \right\}. $$

(2.6)

In references [3–5] one encounters rather similar formulae for zero-point energy density and zero-point pressure:

$$ \rho_{zpe} = \pm \frac{\hbar}{2} \int \frac{d^3k}{(2\pi)^3} \sqrt{m^2_n + k^2}; \quad p_{zpe} = \pm \frac{\hbar}{6} \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{\sqrt{m^2 + k^2}}; $$

(2.7)

but without any weighted sum over particle species. See also equation (8) of reference [6]. A somewhat similar formula for the zero-point stress-energy tensor is given in equation (3) of reference [7], and equation (146) of reference [8], but initially without any weighted sum over particle species, and then subsequently regulated by Pauli–Villars ghost terms, (rather than using Pauli’s weighted sum over physical particle species). The physical framework considered those articles is somewhat different from that considered in the current article.

## 3 Lorentz invariance implies finiteness

If the zero-point stress energy tensor is to be Lorentz invariant, then we must demand $\rho_{zpe} = -p_{zpe}$, or equivalently $\rho_{zpe} + p_{zpe} = 0$. But then we have

$$ \rho_{zpe} + p_{zpe} = \sum_n \left\{ (-1)^{2S_n} g_n \frac{\hbar}{2} \int \frac{d^3k}{\sqrt{m^2_n + k^2}} \left( \frac{\omega_n(k)^2 + k^2/3}{(2\pi)^3} \right) \right\} = 0. $$

(3.1)

That is

$$ \rho_{zpe} + p_{zpe} = \sum_n \left\{ (-1)^{2S_n} g_n \frac{\hbar}{2} \int \frac{d^3k}{\sqrt{m^2_n + k^2}} \left( \frac{m_n^2 + 4k^2/3}{(2\pi)^3} \right) \right\} = 0. $$

(3.2)
Now observe that
\[
\int_0^K \frac{d^3k}{\sqrt{m^2 + k^2}} \left( m^2 + \frac{4}{3} k^2 \right) = 4\pi \int_0^K \frac{dk}{\sqrt{m^2 + k^2}} \left( k^2 m^2 + \frac{4}{3} k^4 \right) \\
= \frac{4\pi}{3} K^3 \sqrt{K^2 + m^2} \\
= \frac{\pi}{6} \left( 8K^4 + 4m^2K^2 - m^4 \right) + \mathcal{O}\left( \frac{1}{K^2} \right). \quad (3.3)
\]

Consequently, the zero-point stress-energy tensor is Lorentz invariant if and only if Pauli’s three polynomial-in-mass constraints of equation (1.4) are satisfied. (The logarithmic-in-mass constraint of equation (1.5) need not be satisfied, and in fact the finite value of \( \rho_{zpe} \) will be seen to be proportional to the extent to which this logarithmic-in-mass condition is violated.) If these three polynomial-in-mass constraints are satisfied then one has
\[
(T_{zpe})^{ab} = -\rho_{zpe} \eta^{ab} = p_{zpe} \eta^{ab}, \quad (3.4)
\]
and, returning to Pauli’s analysis for \( \rho_{zpe} \), one now sees:
\[
\rho_{zpe} = -p_{zpe} = \frac{\hbar}{64\pi^2} \sum_n (-1)^{2s_n} g_n m_n^4 \ln(m_n^2/\mu^2). \quad (3.5)
\]

Thus Lorentz invariance of the zero-point stress-energy tensor implies finiteness of the zero-point stress-energy tensor. Pauli’s sum over all particle species is essential to deriving this result.

(For a somewhat related formula see equation (6) of reference [7], or equation (150) of reference [8]; but these expressions are regulated by Pauli–Villars ghost terms. There is no sum over the physical particle spectrum.)

## 4 Finiteness implies Lorentz invariance

Working in the other direction, if we assume finiteness of the zero-point stress-energy tensor, then in particular \( \rho_{zpe} \) must be finite.

- Pauli’s original argument, when applied to finiteness of the zero-point energy density, (rather than forcing the zero-point energy density to be zero), merely requires that the three polynomial-in-mass constraints of equation (1.4) be enforced. Once the third polynomial-in-mass constraint is enforced the logarithmic cutoff dependence is eliminated. (See also reference [9].)

- But once these three polynomial-in-mass constraints are enforced, the argument of the previous section immediately implies Lorentz invariance.
• As a first consistency check, from Lorentz invariance we see $p_{zpe} = -\rho_{zpe}$, whence the zero point pressure must also be finite.

• As a second consistency check, note that the key integral appearing in $p_{zpe}$ is

$$\int_0^K \frac{d^3k}{\sqrt{m^2 + k^2}} k^2 = 4\pi \int_0^K \frac{dk}{\sqrt{m^2 + k^2}} k^4$$

$$= \pi \left\{ K^3 \sqrt{K^2 + m^2} - \frac{3}{2} m^2 K \sqrt{K^2 + m^2} 
+ \frac{3}{2} m^4 \ln \left( \frac{K + \sqrt{m^2 + K^2}}{m} \right) \right\}$$

$$= \pi \left\{ K^4 - \frac{7}{8} m^4 + \frac{3}{2} m^4 \ln \left( \frac{2K}{m} \right) \right\} + \mathcal{O}\left( \frac{1}{K^2} \right). \quad (4.1)$$

So demanding that $p_{zpe}$ be finite again merely requires that the three polynomial-in-mass constraints of equation (1.4) be enforced. (Once the third polynomial-in-mass constraint is enforced the logarithmic cutoff dependence is eliminated.)

• So finiteness of both $\rho_{zpe}$ and $p_{zpe}$ are controlled by exactly the same conditions: The three polynomial-in-mass Pauli constraints of equation (1.4).

Overall, we see that demanding finiteness of the zero-point stress-energy tensor implies Lorentz invariance of the zero-point stress-energy tensor, and we again have the explicit formula of equation (3.5). Pauli’s sum over all particle species is again essential to deriving this result.

5 Summary of Pauli’s sum rules

If Pauli’s three polynomial-in-mass constraints hold

$$\sum_n (-1)^{2s_n} g_n = 0; \quad \sum_n (-1)^{2s_n} g_n m_n^2 = 0; \quad \sum_n (-1)^{2s_n} g_n m_n^4 = 0; \quad (5.1)$$

then the zero-point energy density and zero-point pressure are both finite, and also equal-but-opposite:

$$\rho_{zpe} = -p_{zpe} = \frac{\hbar}{64\pi^2} \sum_n (-1)^{2s_n} g_n m_n^4 \ln(m_n^2/\mu^2); \quad (5.2)$$

and the zero-point stress-energy tensor is Lorentz invariant. Conversely, the Lorentz invariance of the zero-point stress-energy tensor implies both finiteness and Pauli’s three polynomial-in-mass constraints.
6 Supersymmetry: Neither necessary nor sufficient

It is perhaps worthwhile emphasizing the role that supersymmetry does not play in the considerations of this article.

- Supersymmetry is not necessary in order to set up and understand any of the preceding analysis. Specifically, note that Pauli’s 1951 lectures pre-date even the earliest versions of supersymmetry by some 20 years [12–17]. Certainly Pauli’s sum rules were known to some of the originators of supersymmetry, so the sum rules did have a historical input into the foundations of supersymmetry, but they are logically orthogonal thereto.

- While unbroken supersymmetry automatically satisfies all of Pauli’s constraints, unbroken supersymmetry is also in violent conflict with empirical reality.

- Broken supersymmetry, (either spontaneously broken or explicitly broken), need not (and often does not) satisfy the second and third ($m^2$ or $m^4$) Pauli constraints. (The first Pauli constraint, since it just counts bosonic and fermionic degrees of freedom, will generally survive supersymmetry breaking.)

- The finite QFTs developed in the mid 1980’s automatically satisfy all of Pauli’s sum rules, with the most general finite QFTs being manifestly non-supersymmetric, in the sense that they are based on supersymmetric theories that are softly but explicitly broken; the supersymmetry is, at best, a book-keeping device [18–23].

- The quasi-finite QFTs of reference [19] require some explicit clarification: In that specific reference “finiteness” refers only to the scattering amplitudes, the authors are explicitly excluding the zero-point contributions (the vacuum bubbles) from consideration. For that reason they can avoid imposing the third Pauli sum rule, but the third Pauli sum rule must be reinstated if the vacuum bubbles are to be rendered finite.

- If desired one can rewrite the sum over the particle spectrum in Pauli’s constraints as a “supertrace” [25],

$$\sum_{n} (-1)^{2S_{n}} g_{n} X_{n} = \text{Str}[X], \quad (6.1)$$

but this is merely a book-keeping device, it is not an appeal to supersymmetry.

That is: Supersymmetry, or lack thereof, is at best logically orthogonal to the questions addressed in this article.
7 Renormalization group flow of the Pauli sum rules

Now consider the effect of interactions, and their impact on the Pauli sum rules via the renormalization group flow of the particle masses. In a completely standard manner, in terms of the renormalization scale \( \mu \), let us define the dimensionless \( \gamma \) functions as

\[
\gamma_n = \frac{\partial \ln m_n}{\partial \ln \mu} = \frac{\mu}{m_n} \frac{\partial m_n}{\partial \mu}.
\]  

(7.1)

Then the renormalization group flow for the second and third Pauli sum rules go as

\[
\mu \frac{d}{d\mu} \left( \sum_n (-1)^{2S_n} g_n m_n^2 \right) = 2 \sum_n \left\{ (-1)^{2S_n} g_n m_n^2 \gamma_n \right\};
\]  

(7.2)

\[
\mu \frac{d}{d\mu} \left( \sum_n (-1)^{2S_n} g_n m_n^4 \right) = 4 \sum_n \left\{ (-1)^{2S_n} g_n m_n^4 \gamma_n \right\}.
\]  

(7.3)

(Note that the first Pauli sum rule, being proportional to \( m^0 \), is automatically preserved under renormalization group flow.) There are then a number of situations under which the second and third Pauli sum rules are also preserved under the action of the renormalization group flow:

- The finite QFTs developed in the mid 1980’s [18–23] have all of the \( \gamma_m = 0 \), so they automatically provide a quite natural framework for a class of QFTs in which Pauli’s sum rules are perturbatively stable against radiative corrections. The explicit but soft supersymmetry breaking provides a custodial not-quite-symmetry protecting the Pauli sum rules. Indeed Pauli’s three sum rules are a necessary condition for the known finite QFTs, but they are typically not sufficient.

- A weaker condition is this: Even if the full QFT is not finite, as long as mass-renormalization is trivial, then the \( \gamma_n \) all vanish (while the \( \beta \)-functions need not vanish), then this is still enough to guarantee preservation of the Pauli sum rules under renormalization group flow.

- An even weaker condition is this: Suppose merely that all the \( \gamma_m \) are equal, \( \gamma_m = \gamma \). Physically this corresponds to mass ratios not being renormalized, while an overall mass scale does evolve under the renormalization group flow. Under this condition:

\[
\mu \frac{d}{d\mu} \left( \sum_n (-1)^{2S_n} g_n m_n^2 \right) = 2\gamma \sum_n \left\{ (-1)^{2S_n} g_n m_n^2 \right\};
\]  

(7.4)
\[
\frac{d}{d\mu} \left( \sum_n (-1)^{2S_n} g_n m_n^4 \right) = 4\gamma \sum_n \left\{ (-1)^{2S_n} g_n m_n^4 \right\}.
\] (7.5)

So even this milder condition is still enough to guarantee preservation of the Pauli sum rules under renormalization group flow.

- Finally, let us define what we might call “Pauli-sum-rule-compatible QFTs”, or more simply “Pauli-compatible QFTs”, by those QFTs that satisfy the conditions:

\[
\sum_n \left\{ (-1)^{2S_n} g_n m_n^2 \gamma_n \right\} = 0; \quad (7.6)
\]

\[
\sum_n \left\{ (-1)^{2S_n} g_n m_n^4 \gamma_n \right\} = 0. \quad (7.7)
\]

For this entire class of QFTs the Pauli sum rules are guaranteed to be preserved under renormalization group flow. These constraints are certainly stronger than the Pauli sum rules themselves, and effectively amount to the requirement that the Pauli sum rules should hold not just on-shell, but also for the running particle masses. I emphasize that since these conditions are a certainly relaxation of the conditions for the existence of finite QFTs, and because we know that finite QFTs exist [18–23], then we know that these “Pauli-compatible QFTs” certainly exist.

8 Some implications

The analysis above impacts on a number of wider issues:

- Beyond standard model (BSM) physics.
- Naive estimates of the cosmological constant.
- Renormalization group running of the cosmological constant.
- Sakharov-style induced gravity.

8.1 Beyond standard model physics

Let us now take the Pauli sum rules seriously as real physics applied to the real universe. Then to describe reality we should take the standard model and embed it into one of the “Pauli-compatible QFTs” as discussed above.
The three polynomial-in-mass constraints, because they involve the entire particle physics spectrum, certainly impact on BSM physics. In fact by dividing the spectrum into SM and BSM sectors we can write the Pauli constraints as

\[
\sum_{BSM} (-1)^{2S_n} g_n m_n^4 \ln(m_n^2/\mu_{BSM}^2) = 0; \quad (8.4)
\]

\[
\sum_{SM} (-1)^{2S_n} g_n m_n^4 \ln(m_n^2/\mu_{SM}^2) = 0. \quad (8.5)
\]

That is, enforcing Lorentz invariance of the zero-point on-shell stress-energy tensor, and adopting an overall framework (the “Pauli-sum-rule-compatible QFTs”) in which these conditions are invariant under renormalization group flow, makes some definite predictions for the spectrum of BSM particles, not least being the fact that there must be BSM particles. That is, merely using the very fundamental symmetry principle of Lorentz invariance gives us extremely useful information regarding BSM physics. (The current level of analysis is insufficient to deduce anything more specific regarding the interactions of BSM particles, either with each other or with the SM sector.)

8.2 Cosmological constant

It is commonly asserted that the cosmological constant should be identified with the zero-point energy density, and very naively asserted that it should be estimated by setting \( \rho_{cc} = \rho_{zpe} \sim M_{Planck}^4 \), with \( M_{Planck} \) playing the role of the high-energy cutoff \( K \) used at intermediate stages of our argument above. This very naive estimate disagrees with empirical observation by a factor of approximately \( 10^{123} \) and is famously referred to as the worst prediction in particle physics.

But this is also a dangerously misleading estimate — we know that fermions exist, so there will at the very least be some cancellations in the zero-point energy density. Furthermore imposing Lorentz invariance has given us a rather definite finite and cutoff-independent estimate for the cosmological constant.

Let us take this analysis a little further. Let us define two energy scales \( \mu_{SM} \) and \( \mu_{BSM} \), characteristic of the SM and BSM particle spectra, by setting

\[
\sum_{SM} (-1)^{2S_n} g_n m_n^4 \ln(m_n^2/\mu_{SM}^2) = 0; \quad (8.6)
\]

\[
\sum_{BSM} (-1)^{2S_n} g_n m_n^4 \ln(m_n^2/\mu_{BSM}^2) = 0. \quad (8.7)
\]
We shall now see that, (at least as far as the cosmological constant is concerned), all of the unknown BSM physics can be summarized by the single parameter $\mu_{BSM}$.

We have:

\[
\rho_{cc} = \rho_{zpe} = -p_{zpe} = \frac{\hbar}{64\pi^2} \sum_n (-1)^{2S_n} g_n m_n^4 \ln(m_n^2/\mu_{BSM}^2) = \frac{\hbar}{64\pi^2} \sum_{SM} (-1)^{2S_n} g_n m_n^4 \ln(m_n^2/\mu_{BSM}^2)
\]

That is:

\[
\rho_{cc} = \rho_{zpe} = -p_{zpe} = -\frac{\hbar}{64\pi^2} \left\{ \sum_{SM} (-1)^{2S_n} g_n m_n^4 \right\} \ln(\mu_{BSM}^2/\mu_{SM}^2). \tag{8.6}
\]

This is a very clean and elegant result. As promised, $\mu_{BSM}$ is the only place that unknown BSM physics now enters into the cosmological constant. Without fine tuning of the BSM physics, this might still be astrophysically large, but it will certainly not be $10^{123}$ times too large.

One can of course always, (but perhaps somewhat artificially), tune the cosmological constant to zero (or any empirically supported small quantity), by now introducing a (unitary) variant of the “Pauli–Villars” regularization proposal dating back to the 1950s — merely introduce a sufficient number of non-interacting BSM particles of appropriate mass and statistics. (That is, non-interacting except through gravity, and because you are free to choose the statistics appropriately there is no need to violate unitarity.) This observation reduces the particle-physics “cosmological constant problem” to a minor irritation, rather than a major embarrassment.

That Lorentz invariance might help ameliorate the quantitative size of the zero-point energy-density contribution to the cosmological constant has previously been mooted. (See for instance references [3–11].) It is the combination of Lorentz invariance with Pauli’s sum over all particle species that is central to the current analysis.
8.3 Renormalization group flow of the cosmological constant

Under the renormalization group flow, the cosmological constant naively runs as

\[ \frac{d\rho_{zpe}}{d\mu} = \frac{\hbar}{16\pi^2} \sum_n (-1)^{2s_n} g_n m_n^4 \ln(m_n^2/\mu^2) \gamma_n + \frac{\hbar}{32\pi^2} \sum_n (-1)^{2s_n} g_n m_n^2 \gamma_n. \]  

(8.8)

But the second term is zero for all Pauli-compatible QFTs, so more simply

\[ \frac{d\rho_{zpe}}{d\mu} = \frac{\hbar}{16\pi^2} \sum_n (-1)^{2s_n} g_n m_n^4 \ln(m_n^2/\mu^2) \gamma_n. \]  

(8.9)

Furthermore, for Pauli-compatible QFTs, this is independent of the arbitrary-but-fixed parameter \( \mu_* \).

Therefore, for the finite QFTs, or even for those QFTs with no mass renormalization, the cosmological constant is likewise unrenormalized. If mass ratios are unrenormalized, (so all \( \gamma_n \) are equal, \( \gamma_n = \gamma \)), then

\[ \frac{d\rho_{zpe}}{d\mu} = 4\gamma \rho_{zpe}, \]  

(8.10)

which has the simple solution

\[ \rho_{zpe}(\mu) = \left[ \frac{m(\mu)}{m(\mu_0)} \right]^4 \rho_{zpe}(\mu_0), \]  

(8.11)

implying a simple scaling in line with the overall mass scale. But in the general case, even for Pauli-compatible QFTs one must deal with equation (8.9).

8.4 Sakharov-style induced gravity

The preceding analysis is strictly speaking a flat-space Minkowski result, but due to the locally Euclidean nature of spacetime it will still govern the dominant short-distance physics in curved spacetime. There will certainly be sub-leading curvature-dependent terms — which are more easily dealt with by a short-distance asymptotic expansion of the heat kernel in terms of Seeley–DeWitt coefficients. This naturally leads to the concept of Sakharov-like induced gravity [26], see particularly the discussion in reference [25]. The present analysis could easily be modified and extended to further elucidate the induced gravity scenario.

However, note that some care must be taken to add and subtract only finite regulated physically meaningful quantities, before sending the regulator to infinity. See, for example, reference [27]. Over-enthusiastic application of curved space (or even flat space) renormalization techniques can easily eliminate the interesting parts of the physics.
See for instance reference [28] for a discussion of some of the potential pitfalls. Note particularly equations (19) and (23), and how they relate to Pauli’s analysis in the flat space limit, and equations (26) and how they relate to Sakharov-like induced gravity in curved spacetime. A somewhat related analysis in terms of a curved spacetime version of the Källén–Lehmann spectral decomposition, and related spectral sum rules, is given in reference [29]. See also related discussion in [30].

9 Conclusions

The key observation of the current article is the central importance of Lorentz invariance in controlling the finiteness of the zero-point stress-energy tensor:

- Lorentz invariance ⇒ the three polynomial-in-mass Pauli constraints ⇒ finiteness.
- Finiteness ⇒ the three polynomial-in-mass Pauli constraints ⇒ Lorentz invariance.

This deep and intimate connection between the fundamental physical issues of symmetry and finiteness seems rather oddly to not have previously been developed to the extent that it could.

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A The trace of the zero-point stress-energy tensor

It is sometimes useful to consider the trace of the zero-point stress-energy tensor [2]

\[ T_{zpe} = -\rho_{zpe} + 3p_{zpe} = \sum_n \left\{ (-1)^{2S_n} g_n \hbar \int \frac{d^3k}{2\sqrt{m_n^2 + k^2}} \frac{1}{(2\pi)^3} \right\} \left( -\omega_n(k)^2 + k^2 \right) \]  

(A.1)

Doing this, and (indirectly) imposing Lorentz invariance gives

\[ (T_{zpe})^{ab} = \frac{T_{zpe}}{4} \eta^{ab}. \]  

(A.2)
That is, we could evaluate the zero-point contribution to cosmological constant directly in terms of the trace:

\[ T_{\text{zpe}} = -\rho_{\text{zpe}} + 3p_{\text{zpe}} = -\sum_n \left\{ (-1)^{2S_n} g_n \hbar \int \frac{d^3k}{2\sqrt{m_n^2 + k^2}} \frac{m_n^2}{(2\pi)^3} \right\} \].  \hspace{1cm} (A.3)

Now observe that

\[
\int_0^K \frac{d^3k}{\sqrt{m^2 + k^2}} m^2 = 4\pi \int_0^K \frac{dk}{\sqrt{m^2 + k^2}} m^2 k^2 \\
= \pi \left\{ 2m^2 K \sqrt{m^2 + K^2} - 2m^4 \ln \left( \frac{K + \sqrt{m^2 + K^2}}{m} \right) \right\} \\
= \pi \left\{ 2m^2 K^2 + m^4 - 2m^4 \ln \left( \frac{2K}{m} \right) \right\} + O \left( \frac{1}{K^2} \right). \hspace{1cm} (A.4)
\]

The only minor oddity here is that there is no \( K^4 \) term. (It is not all that odd an oddity: The explicit presence of the factor of \( m^2 \) in the integral above guarantees that no \( K^4 \) term can possibly arise. Taking the trace is a useful trick for evading the biggest potentially divergent contribution, the quartic term. See related discussion in reference [2].)

Thus:

- Finiteness of the trace \( T_{\text{zpe}} = -\rho_{\text{zpe}} + 3p_{\text{zpe}} \) requires only the second and the third of the polynomial-in-mass constraint conditions of equation (1.4). If we impose these two constraints then

\[ T_{\text{zpe}} = -\rho_{\text{zpe}} + 3p_{\text{zpe}} = -\frac{1}{16\pi^2} \sum_n \left\{ (-1)^{2S_n} g_n \hbar m_n^4 \ln(m_n^2/\mu_n^2) \right\}. \hspace{1cm} (A.5)\]

- Vanishing of the trace \( T_{\text{zpe}} = -\rho_{\text{zpe}} + 3p_{\text{zpe}} \) requires only the second and the third of the polynomial-in-mass constraint conditions of equation (1.4), plus the logarithmic-in-mass constraint of equation (1.5).

- Unfortunately, controlling the trace in this manner is not enough to render the zero-point stress-energy tensor Lorentz invariant; that requires all three of the polynomial-in-mass constraints as discussed above. (See somewhat related discussion in [2].)
References

[1] Wolfgang Pauli, *Pauli Lectures on Physics: Vol 6, Selected Topics in Field Quantization*, MIT Press, 1971 (editor C.P. Enz).
(Translation of “Feldquantisierung” 1950–1951; see especially page 33 of the English translation.)

[2] M. Visser, “Lorentzian wormholes: From Einstein to Hawking”, (AIP Press, now Springer–Verlag, 1995). See especially pages 82–84.

[3] E. K. Akhmedov, “Vacuum energy and relativistic invariance”, hep-th/0204048.

[4] G. Ossola and A. Sirlin, “Considerations concerning the contributions of fundamental particles to the vacuum energy density”, Eur. Phys. J. C 31 (2003) 165 doi:10.1140/epjc/s2003-01337-7 [hep-ph/0305050].

[5] H. Culetu, “The zero point energy and gravitation”, hep-th/0410133.
See especially equations (3.1)–(3.2), (3.5)–(3.6), and (3.12).

[6] A. Y. Kamenshchik, A. A. Starobinsky, A. Tronconi, G. P. Vacca, and G. Venturi, “Vacuum energy, Standard Model physics and the 750 GeV Diphoton Excess at the LHC”, arXiv:1604.02371 [hep-ph].

[7] P. D. Mannheim, “Intrinsically quantum-mechanical gravity and the cosmological constant problem”, Mod. Phys. Lett. A 26 (2011) 2375 doi:10.1142/S0217732311036875 [arXiv:1005.5108 [hep-th]]. See especially equation (6).

[8] P. D. Mannheim, “Mass generation, the cosmological constant problem, conformal symmetry, and the Higgs boson”, arXiv:1610.08907 [hep-ph]. See especially equation (150).

[9] G. L. Alberghi, A. Y. Kamenshchik, A. Tronconi, G. P. Vacca and G. Venturi, “Vacuum energy, cosmological constant and standard model physics”, JETP Lett. 88 (2008) 705. doi:10.1134/S002136400823001X

[10] J. F. Koksma and T. Prokopec, “The cosmological constant and Lorentz invariance of the vacuum state”, arXiv:1105.6296 [gr-qc].

[11] M. Asorey, P. M. Lavrov, B. J. Ribeiro and I. L. Shapiro, “Vacuum stress-tensor in SSB theories”, Phys. Rev. D 85 (2012) 104001 doi:10.1103/PhysRevD.85.104001 [arXiv:1202.4235 [hep-th]].
[12] Y. A. Golfand and E. P. Likhtman,
“Extension of the algebra of Poincare group generators and violation of $P$ invariance”,
JETP Lett. 13 (1971) 323 [Pisma Zh. Eksp. Teor. Fiz. 13 (1971) 452].

[13] D. V. Volkov and V. P. Akulov, “Possible universal neutrino interaction”,
JETP Lett. 16 (1972) 438 [Pisma Zh. Eksp. Teor. Fiz. 16 (1972) 621].

[14] D. V. Volkov and V. P. Akulov, “Is the Neutrino a Goldstone particle?”,
Phys. Lett. 46B (1973) 109. doi:10.1016/0370-2693(73)90490-5

[15] D. V. Volkov and V. A. Soroka, “Higgs effect for Goldstone particles with spin 1/2”,
JETP Lett. 18 (1973) 312 [Pisma Zh. Eksp. Teor. Fiz. 18 (1973) 529].

[16] J. Wess and B. Zumino,
“A Lagrangian model invariant under supergauge transformations”,
Phys. Lett. 49B (1974) 52. doi:10.1016/0370-2693(74)90578-4

[17] J. Wess and B. Zumino,
“Supergauge invariant extension of quantum electrodynamics”,
Nucl. Phys. B 78 (1974) 1. doi:10.1016/0550-3213(74)90112-6

[18] P. S. Howe, K. S. Stelle and P. C. West,
“A Class of Finite Four-Dimensional Supersymmetric Field Theories”,
Phys. Lett. 124B (1983) 55. doi:10.1016/0370-2693(83)91402-8

[19] A. Parkes and P. C. West,
“Explicit supersymmetry breaking can preserve finiteness in rigid $N = 2$
supersymmetric theories”,
Phys. Lett. 127B (1983) 353. doi:10.1016/0370-2693(83)91016-X

[20] A. Parkes and P. C. West,
“Finiteness in Rigid Supersymmetric Theories”,
Phys. Lett. 138B (1984) 99. doi:10.1016/0370-2693(84)91881-1

[21] D. I. Kazakov, M. Y. Kalmykov, I. N. Kondrashuk and A. V. Gladyshev,
“Softly broken finite supersymmetric grand unified theory”,
Nucl. Phys. B 471 (1996) 389 doi:10.1016/0550-3213(96)00180-0 [hep-ph/9511419].

[22] O. Piguet, “Supersymmetry, ultraviolet finiteness and grand unification”,
hep-th/9606045.

[23] T. Kobayashi, J. Kubo, M. Mondragon and G. Zoupanos,
“Constraints on finite soft supersymmetry breaking terms”,
Nucl. Phys. B 511 (1998) 45 doi:10.1016/S0550-3213(97)00765-7 [hep-ph/9707425].

[24] P. C. West,
“The Supersymmetric Effective Potential”,
Nucl. Phys. B 106 (1976) 219. doi:10.1016/0550-3213(76)90378-3
[25] M. Visser, “Sakharov’s induced gravity: A modern perspective”,
Mod. Phys. Lett. A 17 (2002) 977 doi:10.1142/S0217732302006886 [gr-qc/0204062].

[26] A. D. Sakharov,
“Vacuum quantum fluctuations in curved space and the theory of gravitation”,
Sov. Phys. Dokl. 12 (1968) 1040 [Dokl. Akad. Nauk Ser. Fiz. 177 (1967) 70]
[Sov. Phys. Usp. 34 (1991) 394] [Gen. Rel. Grav. 32 (2000) 365].

[27] M. Visser,
“Why are Casimir energy differences so often finite?”,
arXiv:1601.01374 [quant-ph].

[28] M. Elías and F. D. Mazzitelli,
“Ultraviolet cutoffs for quantum fields in cosmological spacetimes”,
Phys. Rev. D 91 (2015) 124051, doi:10.1103/PhysRevD.91.124051
[arXiv:1504.02993 [gr-qc]].

[29] A. Y. Kamenshchik, A. Tronconi, G. P. Vacca and G. Venturi,
“Vacuum energy and spectral function sum rules”,
Phys. Rev. D 75 (2007) 083514 doi:10.1103/PhysRevD.75.083514 [hep-th/0612206].

[30] C. Gruber and H. Kleinert,
“Observed cosmological re-expansion in minimal QFT with Bose and Fermi fields”,
Astropart. Phys. 61 (2014) 72 doi:10.1016/j.astropartphys.2014.06.012
[arXiv:1407.3667 [gr-qc]].