2-LC triangulated manifolds are exponentially many

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• **Facets**: inclusion-maximal faces of a complex.

• **Pure complex**: all facets of the same dimension.

• **Star** of a face $\sigma$: the smallest subcomplex containing all facets that contain $\sigma$.

• $\text{link}(\sigma, K) := \{ \tau \in \text{star}(\sigma, K) : \tau \cap \sigma = \emptyset \}$

• **Triangulation of a smooth $d$-manifold $M$**: a $d$-dim simplicial complex whose underlying space is homeomorphic to $M$.

• **$d$-sphere**: a triangulation of the $d$-dimensional sphere.

• **$d$-pseudomanifold**: a $d$-dim pure simplicial regular CW-complex where each $(d - 1)$-cell is in $\leq 2$ facets.
Gromov’s question (2000)

How many triangulations of the 3-sphere with $N$ tetrahedra are there?

- Two triangulations are equivalent $\iff$ same face poset.
- Exponentially many?
- Crucial for discrete version of quantum gravity
  - If yes, all good
  - If no, divergence issues
Theorem (Folklore)

There are more than exponentially many surfaces with $N$ triangles.

Corollary (via coning)

There are more than exponentially many 3-pseudomanifolds with $N$ tetrahedra.
• LC manifolds are those obtainable from a tree of \(d\)-simplices by recursively gluing two adjacent boundary facets.
• Mogami manifolds: ... gluing two incident ...
• All shellable spheres are LC.
Locally constructible picture

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• LC manifolds are those obtainable from a tree of $d$-simplices by recursively gluing two *adjacent* boundary facets.
• Mogami manifolds: ... gluing two *incident* ...
• All shellable spheres are LC.
Theorem (Durhuus–Jonsson 1995; Benedetti–Ziegler 2011)

LC triangulations of $d$-manifolds with $N$ facets are at most $2^{d^2N}$.

- Works also for LC pseudommanifolds.

Theorem (Mogami 1995)

Mogami triangulations of 3-manifolds with $N$ facets are exponentially many.
Later pictures

Classes of exponential size

- (d=2) Surfaces with fixed genus (Tutte 1962)
- (d=3) Causal triangulations (Durhuus–Jonsson 2014)
- (any d) Bounded geometry (Adiprasito–Benedetti 2020)

- Triangulations with bounded discrete Morse vector (Benedetti 2012)
  - contains all classes above
  - does not contain Mogami triangulations
Definition (Benedetti–P. 2022)

$t$-LC $d$-manifolds are those obtainable from a tree of $d$-simplices by recursively gluing two boundary facets whose intersection has dimension at least $d - 1 - t$.

- 1-LC the same as LC
- 1-LC $\subset$ 2-LC $\subset \cdots \subset d$-LC
- All connected $d$-manifolds are $d$-LC
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**Main Theorem (Benedetti–P. 2022)**

$2$-LC triangulations of $d$-manifolds with $N$ facets are at most $2^{\frac{d^3}{2}} N$. 
Theorem (Benedetti–P. 2022)
Cones over $t$-LC $d$-pseudomanifolds are $t$-LC.

⇒ 2-LC $d$-pseudomanifolds more than exponentially many!
  • Unlike the Benedetti-Ziegler result, our result really uses the manifold assumption: without it, it’s false.

Crucial facts for our proof
• Links of $(d – 3)$-faces in a manifold are homeomorphic to $S^2$ or a disk.
• Planar gluings lead to count by Catalan numbers.
• Our proof makes precise and extends to all dimensions the intuition for $d = 3$ by Mogami.
A $d$-dimensional complex $C$ is called [homotopy-]Cohen–Macaulay if for any face $F$, for all $i < \dim \text{link}(F, C)$, $[\pi_i(\text{link}(F, C)) = 0] \ H_i(\text{link}(F, C)) = 0$.

Constructible simplicial complex is defined inductively:
- every simplex, and every 0-complex, is constructible;
- a $d$-dim pure simplicial complex $C$ that is not a simplex is constructible if and only if it can be written as $C = C_1 \cup C_2$, where $C_1$ and $C_2$ are constructible $d$-complexes, and $C_1 \cap C_2$ is a constructible $(d-1)$-complex.
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\textbf{Results to generalize} 

• Constructible manifolds are LC. (Benedetti–Ziegler 2011)

• Constructible complexes are homotopy-Cohen–Macaulay. (Hochster 1972)
Definition (Benedetti–P. 2022)

Let $0 < t \leq d$ be integers. $t$-constructible $d$-dimensional simplicial complexes defined inductively:

- every simplex is $t$-constructible;
- a 1-dimensional complex is $t$-constructible if connected;
- a $d$-dimensional pure simplicial complex $C$ that is not a simplex is $t$-constructible if $C = C_1 \cup C_2$, where $C_1$ and $C_2$ are $t$-constructible $d$-complexes, and $C_1 \cap C_2$ is a $(d - 1)$-complex whose $(d - t)$-skeleton is constructible.
Theorem (Benedetti–P. 2022)

$t$-constructible pseudomanifolds are $t$-LC.

| Theorem (Benedetti–P. 2022) | The $(d - t + 1)$-skeleton of a $t$-constructible $d$-complex is homotopy-Cohen–Macaulay. |
|-----------------------------|---------------------------------------------------------------------------------|
|                             | (In other words, $t$-constructible $d$-complexes have (homotopic) depth $> d - t$.) |