Problems in Black Hole Entropy interpretation

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ABSTRACT

In this work some proposals for black hole entropy interpretation are exposed and investigated. In particular I will firstly consider the so called “entanglement entropy” interpretation, in the framework of the brick wall model [1], and the divergence problem arising in the one loop calculations of various thermodynamical quantities, like entropy, internal energy and heat capacity. It is shown that the assumption of equality of entanglement entropy and Bekenstein-Hawking one appears to give inconsistent results. These will be a starting point for a different interpretation of black hole entropy based on peculiar topological structures of manifolds with “intrinsic” thermodynamical features. It is possible to show an exact relation between black hole gravitational entropy (tree level contribution in path integral approach) and topology of these Euclidean space-times. The expression for the Euler characteristic, through the Gauss-Bonnet integral, and the one for entropy for gravitational instantons are proposed in a form which makes the relation between these self evident. Using this relations I shall propose a generalization of Bekenstein-Hawking entropy in which the former and Euler characteristic are related in the equation: \( S = \chi A / 8 \). The results, quoted above, are more largely exposed in previous works [2,3]. Finally I’ll try to expose some conclusions and hypotheses about possible further development of this research.
1. Introduction: Black Holes Thermodynamics

At the beginning of the seventies the research on black hole physics achieved a series of theoretical results that brought to an elegant and impressive formulation of some General Relativity laws as thermodynamical ones (Bardeen, Bekenstein, Carter, Christodoulou, Hawking, Ruffini[4, 5, 6]). It was found that for each classical thermodynamical law it is possible to formulate a correspondent one for black holes. These results are resumed in the frame below

0th Law: The surface gravity $\kappa$ is constant on the event horizon of a stationary black hole

1st Law: The mass of a black hole is bounded to its area $A$, surface gravity $\kappa$ and angular momentum $J$ by the relation ($\Omega =$ angular velocity of the black hole)

$$M = \frac{\kappa}{2\pi} A + \Omega J$$

2nd Law: The area of black hole event horizon can never decrease for processes satisfying the weak energy condition

#1 Consider a space-time with curvature tensor $R_{ab}$ and matter described by a stress-energy tensor $T_{ab}$. If $\xi^a$ is a null or timelike vector then from Einstein equations one has

$$R_{ab}\xi^a\xi^b = 8\pi \left[T_{ab} - \frac{1}{2} T g_{ab}\right] \xi^a\xi^b = 8\pi \left[T_{ab}\xi^a\xi^b - \frac{1}{2} T\right]$$

It is possible to see $T_{ab}\xi^a\xi^b$ as the energy density measured by an observer with 4-velocity $\xi^a$ along the geodesic. It is commonly accepted that for classical matter this energy density has not to be negative definite that is $T_{ab}\xi^a\xi^b \geq 0$ for all the $\xi^a$ timelike. This is called the weak energy condition. It may also be requested that the stress energy tensor of matter cannot be so big and negative to render the 1st member of the preceding relation negative. One can then impose the strong energy condition

$$T_{ab}\xi^a\xi^b \geq -\frac{1}{2} T$$

for all the vectors $\xi^a$ timelike.
3rd Law: It is not possible to render the surface gravity of a black hole null through physical transformations.

Although these laws soon appeared as a strong hint towards a thermodynamical behaviour of black hole, the picture became fully consistent only when Hawking found his famous results about black hole radiation by an application of quantum field theory in curved space. It is in fact impossible to define a temperature for classical black holes, thus it is not even reasonable to talk about entropy in this case. Hawking suggested that, due to the polarization of the vacuum in proximity of the black hole horizon, there is a flux of radiation flowing out towards infinity. He demonstrated this by an ingenious derivation based on Bogoliubov coefficients technique [7]. By the first law and the Hawking temperature one finds the Bekenstein-Hawking entropy of black hole to be one quarter of its area [7].

Thermodynamical aspects of black holes appear more evident in Euclidean path-integral approach[8]. Considering the generating functional of the Euclidean theory with the action equal to the Einstein-Hilbert one plus the matter contribution, in a semiclassical approach the tree level contribution is due only to gravitational part. Instantons are non singular solutions of the classical equations in 4-dimensional Euclidean space. For a wide class of black holespace-times, metrics that extremize the Euclidean action are gravitational instantons if one removes the conical singularity at the horizon. This forces to fix a period for the imaginary time. It is well known that Euclidean quantum field theory with periodic imaginary time is equivalent to a finite temperature quantum field theory in pseudoeuclidean $(-,+,+,+)$ space-time, the temperature being the inverse of imaginary time pe-
period. Thermodynamics appears in this way as a request of consistence of quantum field theory on black hole space-times, or better, on space-times with Killing horizon.

1.1. Interpretation problems for black hole entropy.

As it often happens in the history of science, the discovery of such a complex structure opened a whole set of new questions to which complete answers still lack. Some questions are in order:

1 Which dynamical degrees of freedom could be associated with BH entropy?
2 Is there any information loss in black hole dynamics?
3 How does General Relativity know about black hole thermodynamics?

I will now expose more extensively these points which appear as the main problematics opened by the research in quantum aspects of black hole.

1 The problem of a dynamical origin for black hole entropy is the effort to achieve a statistical mechanics explanation of it. This means to give an interpretation of horizon thermodynamics in a “familiar” way, in the sense that it could be seen as related to dynamical degrees of freedom associated with black hole nature.

2 The information loss is related to the fact that black hole evaporation by mean of emission of a thermal particle spectrum, appears to produce a destruction of quantum information by converting pure state in mixed ones. That is hardly acceptable by the great part of physicists because it would imply a non-unitary evolution of quantum states in presence of strong gravitational fields.

3 The last point is, I believe, the most important one. It in fact appears as the main question one has to answer in order to understand the real nature of horizon
thermodynamics. As we have just seen, the four laws of black hole thermodynamics have been formulated some years before Hawking’s discovery of quantum radiation. Most of these appear in fact as General Relativity theorems of black hole dynamics and so are already “encoded” at a classical level. But in spite of this, the consistence of this frame is achieved only at the quantum stage by the introduction of Hawking radiation. How can geometry know about quantum matter behaviour is still an unsolved question. We shall see how the attempt to explain tree level (classical) gravitational contribution as due to matter one (one loop) appears to fail.

In the last twenty years many authors tried to give answers to these questions. What is common to most part of them is the conviction that all these problems are related and that a clear comprehension of black hole entropy origin would be a great achievement in order to solve them. So we shall start considering the first point of black hole entropy, $S = A/4$, statistical interpretation. Here I shall only quote some major proposals:

1 Bekenstein[9] - The black hole entropy can be seen as $S = \ln W$ where $W$ is the number of possible microscopical configurations of astrophysical body that generate the same BH (relation to the “no hair” theorem).

2 York[10] - The dynamical degrees of freedom at the origin of black hole entropy are identified with BH “quasi normal modes”.

3 Wald[11] - Black hole entropy is identified with the Noether charge, associated to a diffeomorphism invariant theory, in presence of bifurcate Killing horizons\(^2\).

\(^2\) A Killing horizon is null hypersurface whose null generators are orbits of a Killing vector field. In General Relativity it has been demonstrated that the event horizon of a stationary black hole is always a Killing horizon. If the generators of the horizon are geodetically complete to the past (and the surface gravity of the black hole is different from zero) then it contains a 2-dimensional (in four dimensional space-times) space-like cross section $B$ on which the Killing vector is null. $B$ is called “surface of bifurcation”, is a fixed point for
The black hole entropy is generated by dynamical degree of freedom, excited at a certain time, associated to the matter in BH interior near the horizon through non-causal correlation (EPR) with external matter. The ignorance of an observer outside the black hole about the modes inside the horizon is associated with a so-called entanglement entropy which is identified with Bekenstein-Hawking one.

Black hole entropy has a topological origin. The topological structure of space-time determines the presence of gravitational entropy. The first two lines of research appear at the moment not successful due to their appeal to a count over the entire life of black hole. This point has been stigmatized by a recent Gedanken experiment by Frolov and Novikov[16]. In this work a wormhole is used as a device to explore the backside of the horizon, the behaviour found seems to show a deep relation between some sort of dynamical degrees of freedom living behind the horizon and black hole entropy. The third proposal (that of Noether charge) shows a deep link between rescaling properties of the (gravitational plus matter) action and the presence of thermodynamical behaviour in presence of bifurcate Killing horizons. Unfortunately, although very impressive, it lacks in giving a properly statistical interpretation of black hole entropy. In this sense it is more a way of recovering Bekenstein-Hawking results that casts a new light on the nature of the problem than an interpretative frame. So in the rest of this work we are going to study the last two proposals in order to understand what the right answer to our questions might be.

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the Killing flow and on it the Killing vector vanishes. $B$ lies at the intersection of the two hypersurfaces (past and future) forming the complete horizon. A bifurcate Killing horizon is an horizon with a “surface of bifurcation” $B$. 

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2. Entanglement Entropy

Let us consider a global Hilbert space $\mathcal{H}$ composed of two uncorrelated ones $\mathcal{H}_1, \mathcal{H}_2$.

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

A general state on $\mathcal{H}$ can be described as a linear superposition of states on the two Hilbert spaces

$$|\psi\rangle = \sum_{a,b} \psi(a,b)|a\rangle|b\rangle \quad (2.1)$$

One can define a global density matrix

$$\rho(a, a', b, b') = \psi(a, b)\psi^*(a', b')$$

The reduced density matrix for the subsystem $a$ is given by

$$\rho(a, a') = \sum_{b, b'} \psi(a, b)\psi^*(a', b') \quad (2.2)$$

Note that even if the general state (2.1) defined on the global Hilbert space is a pure one, the form of the reduced density matrix (2.2) shows that the corresponding state defined only in one subspace is a mixed one. This corresponds to an information loss that can be properly described by defining an entropy associated to mixed states which is null for pure ones, that is the so called von Neumann entropy

$$S(a) = -Tr(\rho_a \ln \rho_a)$$

Let us consider now the same problem in black hole case.
Let us consider a stationary black hole and define $\hat{\rho}^{iniz}$ the density matrix describing, in Heisenberg representation, the initial state of quantum matter propagating on its background. For an external observer the system consist of two parts: black hole and radiation outside of it. By defining a spacelike hypersurface we can consider quantum modes of radiation at a given time so that they can be separated in external and internal to the black hole. The state for external radiation is obtainable from $\hat{\rho}^{iniz}$ by tracing on all the state of matter inside the event horizon and so inaccessible to the external observer

$$\hat{\rho}^{rad} = Tr^{inv} \hat{\rho}^{iniz}$$

For a black hole alone this density matrix would describe its Hawking radiation at infinity. We can also define the density matrix for the black hole state

$$\rho^{BH} = Tr^{vis} \hat{\rho}^{iniz}$$

where one now performs the trace on external degrees of freedom. From this matrix is it possible to find the related von Neumann entropy

$$S^{BH} = -Tr^{inv}(\hat{\rho}^{BH} \ln \hat{\rho}^{BH})$$

This is exactly what is called “entanglement entropy”. It is important to note that this definition is invariant in the sense that independent changes of definitions of vacuum for “external” and “internal” states do not change the value of $S^{BH}$. The calculation of entanglement entropy can be resumed in few steps

- Introduction of null-like or space-like hypersurface (achronal “slice”) on which perform matter quantization.
- 2nd quantization of matter field on BH background.

- Definition of the global ground state.

- Trace of the global density matrix on internal or external states.

- Computation of entanglement entropy entropy as von Neumann entropy of reduced density matrix.

Calculations on entanglement entropy were performed by various authors in a wide class of situations, in flat spaces as in curved ones [12], [13], [14]. In spite of this we find two common points that appear as proper characteristics of entanglement entropy.

1 Entanglement entropy is proportional to the “division” plane (for BH is the event horizon).

2 Entanglement entropy is always divergent on division plane due to the presence of modes of arbitrary high modes near the horizon. The divergence form is general and independent on the kind of field.

While the first point is a strong hint towards the identification of entanglement entropy with Bekenstein-Hawking one, the second is a deep problem casting some shadows on the real understanding of the meaning of what the computation of entanglement entropy is effectively probing about black hole physics.

The last years have seen different approaches for the divergence problem resolution. Roughly they can be summarized in two kind of proposals that of regularization (t’Hooft; Frolov, Novikov; Barvinsky, Frolov, Zelnikov) and that of renormalization (Susskind, Uglum; Fursaev, Solodukhin).

The main idea of regularization [14] approach is to apply a physical cut-off to entropy by justifying it through the quantum fluctuations (Zitterbewegung) of the
event horizon. This cut-off has been estimated by Frolov and Novikov [14] and it is of the order of the Planck length. Remarkably the introduction of such a cut-off gives a value of entanglement entropy of the same order of the Bekenstein-Hawking one.

The second approach is instead based on elimination of divergences through a renormalization of gravitational coupling constant [18] and of constants related to second order curvature terms[19]. This way appears very interesting for its relation to elementary particle physics but is penalized by its necessity of a renormalizable theory of quantum gravity in order to give exact results.

We shall now consider the regularization approach in order to probe its consistency. The first idea of regularization of entanglement entropy was implicitly proposed by ‘t Hooft in 1985 [1] as applied to his “brick wall model”. In a certain sense cut-off dependent models [14,17] are up to date versions of the former. One of the problems ‘t Hooft proposed in his seminal work was the divergence of not only entropy but also of quantum matter contribute to internal energy of the black hole, which has to be regularized by using the same cut-off one has to introduce for entropy. He found that, fixing the cut-off in order to obtain $S_{ent} = S_{Bek-Haw} = A/4$, one obtains $U = \frac{3}{8}M$. So matter contribution to internal energy appeared to be a very consistent fraction of the black hole mass $M$. As ‘t Hooft underlined, this is a signal for a strong back-reaction effect, not a good aim for a model based on semiclassical (negligible back-reaction) approximation.

We shall see that the same problem is present in Barvinsky, Frolov, Zelnikov (BFZ) model [17] and that a surprising behaviour of heat-capacity is also found.
3. Entanglement Entropy and BFZ Model

In BFZ work entropy is computed from global vacuum density matrix by tracing over the degrees of freedom of matter outside the black hole. In so doing one obtains a mixed state density matrix for matter inside the black hole.

BFZ define the global wave function of the black hole as

\[ \Psi = \exp(\Gamma/2) \langle \phi_- | \exp(-\beta \hat{H}/2) | \phi_+ \rangle \]  

(3.1)

and

\[ \hat{\rho} = |\Psi\rangle \langle \Psi| \]  

(3.2)

as the related density matrix. Here \( |\phi_+\rangle \) (\( |\phi_-\rangle \)) are the external (internal) states of matter (a massless scalar field for simplicity) on the black hole fixed background.

Tracing over \( |\phi_+\rangle \) gives the internal density matrix

\[ \rho_{\text{int}}(\phi'_-, \phi_-) = \langle \phi'_- | \hat{\rho} | \phi_- \rangle \]

\[ = \int \mathcal{D} \phi_+ \Psi^* (\phi'_-, \phi_+ \Psi (\phi_-, \phi_+) \]

(3.3)

\[ = \exp(\Gamma_\beta) \langle \phi'_- | \exp(-\beta \hat{H}) | \phi_- \rangle. \]

Entanglement entropy associated to this reduced density matrix is

\[ S_{\text{ent}} = -Tr_{\text{int}}(\rho_{\text{int}} \ln \rho_{\text{int}}) \]

Here \( \Gamma_\beta \) is a normalization factor fixed in order to obtain \( tr \rho = 1 \), but it also corresponds to the 1 loop effective action

\[ \Gamma_\beta = -\ln \left[ \int \mathcal{D} \phi_- \langle \phi_- | \exp(-\beta \hat{H}) | \phi_- \rangle \right] \]

(3.4)

\[ = -\frac{1}{2} \ln \det \left[ \frac{\delta(x - y)}{2(cosh \beta \omega - 1)} \right] \]

where \( \hat{\omega} \) is the operator associated with the frequency of field modes.
BFZ calculate the entanglement entropy (in WKB approximation) as the trace over the internal modes of $-\rho_{\text{int}} \ln \rho_{\text{int}}$, so their calculation is relative to the internal degrees of freedom. Instead, in the common definition of entanglement entropy, one usually refers to the trace over the external degrees of freedom of $-\rho_{\text{ext}} \ln \rho_{\text{ext}}$. In the following, I shall assume that, given the symmetry existing in BFZ study between internal and external variables, the two definitions of entanglement entropy coincide. Moreover it is possible to demonstrate [2] that all other thermodynamical quantities, as internal energy and heat capacity, share the same property.

From the definitions one has

$$
\rho_{\text{ext}} = Tr_- |\Psi \rangle \langle \Psi | = \int D\phi_- \Psi^* (\phi_+^\prime, \phi_-) \Psi (\phi_-, \phi_+) = \\
= \int D\phi_- |\phi_+ \rangle \exp \left(-\frac{\beta \hat{H}}{2}\right) |\phi_- \rangle \exp \left(-\frac{\beta \hat{H}}{2}\right) |\phi_+ \rangle \exp (\Gamma_{\text{ext}}) \\
\rho_{\text{int}} = Tr_+ |\Psi \rangle \langle \Psi | = \int D\phi_+ \Psi^* (\phi_-^\prime, \phi_+) \Psi (\phi_+, \phi_-) = \\
= \int D\phi_+ |\phi_- \rangle \exp \left(-\frac{\beta \hat{H}}{2}\right) |\phi_+ \rangle \exp \left(-\frac{\beta \hat{H}}{2}\right) |\phi_- \rangle \exp (\Gamma_{\text{int}}) \\
$$

(3.5)

For $\Gamma_\beta$, from (3.4), one obtains (using where we used the property $\ln \det A = Tr \ln A$)

$$
\Gamma_\beta = \int dx \left[ \ln \left(2 \sinh \frac{\beta \hat{\omega}}{2}\right) \delta(x - y) \right]_{y=x} \\
$$

(3.6)

Following BFZ, one can calculate the expression below by expanding all the func-

#3 For a demonstration of this assumption see [20].
tions $\phi(x)$ in terms of eigenfunctions $R_\lambda(x)$ of the operator $\hat{\omega}$

$$\phi(x) = \sum_\lambda \phi_\lambda R_\lambda(x)$$

$$\hat{\omega}^2 R_\lambda(x) = \omega_\lambda^2 R_\lambda(x)$$

$$\delta(x-y) = \sum_\lambda g^{00} g^{1/2} R_\lambda(x) R_\lambda(y)$$

where $\sum_\lambda$ denotes the sum over all quantum numbers, $g^{00}$ is the timelike component of the metric tensor and $g = \det g_{\mu\nu} = g^{00} \det g^{ab}$ ($a, b, \ldots = 1, 2, 3$).

Hence

$$\Gamma_\beta = \int_{2M}^{r_{box}} \frac{dr}{(r-2M)} \int_0^\infty \sum_{l=0}^\infty \int_0^\infty d\omega (2l+1) R_{\lambda\omega}^2(r) \gamma(\beta\omega)$$

where

$$\gamma(\beta\omega) = \frac{\beta}{2\omega} + \ln(1 - e^{-\beta\omega})$$

and where $R_{\lambda\omega}(r)$ are the radial eigenfunctions. We are interested in the behaviour of $\Gamma_\beta$ near the horizon; using BFZ result

$$\sum_{l=0}^\infty (2l+1) R_{\lambda\omega}^2(r) \sim \frac{4\omega^2 M}{\pi (r-2M)}$$

one gets

$$\Gamma_\beta \sim \frac{4M}{\pi} \int_{2M}^{r_{box}} \frac{dr}{(r-2M)^2} \int_0^\infty d\omega \omega^2 \gamma(\beta\omega)$$

where $r_{box}$ is the radius of the box in which we have to put the black hole to regularize infrared divergences.
To compute the second integral one has to subtract the zero–point term from (3.9).

Finally one finds the following leading term near the horizon

$$\Gamma_\beta = \beta F(\beta) \sim -\frac{32\pi^3 M^4}{45} \frac{1}{\beta^3 \hbar}$$

(3.12)

where the cut–off is defined as \(h \equiv \text{Inf}(r - 2M)\).

From the free energy (3.12), it is possible to find the other thermodynamical quantities (not only entropy but also internal energy \(U\) and heat capacity \(c\)) by using the well known relations between free energy and these ones in canonical ensemble.

So one obtains

$$S \sim \frac{128\pi^3 M^4}{45} \frac{1}{\beta^3 \hbar}$$

$$U \sim \frac{32\pi^3 M^4}{15} \frac{1}{\beta^4 \hbar}$$

$$c \sim \frac{128\pi^3 M^4}{15} \frac{1}{\beta^3 \hbar}$$

(3.13)

Rewriting the above formulas in terms of a proper distance cut–off

$$\epsilon \sim 2\sqrt{r_{bh}}h \Leftrightarrow h \sim \frac{\epsilon^2}{4r_{bh}}$$

(3.14)

we find for \(F, U, S\) and \(c\) at the Hawking temperature \(\beta_H = \frac{1}{8\pi M}\)

$$F(\beta_h) \sim -\frac{M}{720\pi} \frac{1}{\epsilon^2}$$

$$S(\beta_h) \sim \frac{2M^2}{45} \frac{1}{\epsilon^2}$$

(3.15)

$$U(\beta_h) \sim -\frac{M}{240\pi} \frac{1}{\epsilon^2}$$

$$c(\beta_h) \sim \frac{4M^2}{30\epsilon^2}$$

The entropy in (3.15) is exactly the same than in BFZ.
3.1. Interpretative Problems

At this point a brief clarification seems necessary about the interpretation one would give to the quantities just found. The above divergences for the entropy and the other thermodynamical quantities requires a renormalization scheme or a brick-wall cut-off. The standard position consists in identifying the black hole entropy with the leading divergent regularized term

\[ S_{bh} \equiv S_{\text{radiation, leading}}. \]  

(3.16)

But what about the regularized terms for the other thermodynamical quantities? The cut-off present in (3.15) is the same for all the thermodynamical quantities so we have to fix the same value of \( \epsilon \) for all of them. We shall see that the values so obtained for \( F, U, c \) are very different from the classical black hole ones: this means that a straightforward identification between e.g. \( E_{bh} \) and \( E_{\text{radiation, leading}} \) doesn’t seem possible. The same kind of problem, even worse, exists for the specific heat as we’ll see further on.

The identification of Bekenstein-Hawking entropy with the entanglement one generates a problem of interpretation of classical (tree level) entropy due to gravity in the path-integral approach. The first aim of entanglement approach is to explain all black hole entropy as dynamical matter entropy. The matter leading term is not a new one-loop contribution to be added to the tree level one. So it appears as a necessary complement of this program to give a clear explanation for ignoring the presence of the tree level contribution of gravity. As a matter of fact, in literature this problem appears to be often ignored or gone around. We can quote in this sense only a work by Jacobson [21]. Here I’d like to put in evidence that this
crucial point is the same as the third question we encountered in section 1. The explanation one can give for ignoring tree level in entanglement entropy approach require an answer to the problem of relation between classical and quantum aspects in black hole thermodynamics.

Following the most part of papers on the same problem [17,18,1,22,23,24,25,26] we can try to check if the identification of Bekenstein-Hawking entropy with the entanglement one gives self-consistent results at the level of the other thermodynamical quantities. My discussion is here limited to the brick-wall regularization of the divergences [17,1,26].

3.2. Free Energy, Internal Energy and Heat Capacity

The cut-off fixing necessary to obtain the required value

\[ S_{\text{ent}} = S_{\text{Bek-Haw}} = A/4 \]

is

\[ \epsilon^2 = \frac{1}{90\pi} \]  

(3.17)

this brings to the following values for free energy and internal energy

\[ F = -\frac{1}{8} M \]

\[ U = \frac{3}{8} M. \]  

(3.18)

The results in (3.18) are the same obtained by t’Hooft in his pioneering paper [1] and exactly the same are found [2] if one calculates \( U \) and \( F \) with heat kernel expansion truncated to the first De Witt coefficient in the optical metric [27]. So, for the internal energy, the identification of the brick-wall value with the tree level one (\( M \) the mass of the black hole) seems impossible; on the other hand, it does not seem possible to understand the radiation term as a perturbative contribution.
to the black hole tree level one. It happens in fact that the quantum contribution is of the same order of the classical one. Moreover there would exist a further problem with the special form of the internal energy radiation contribution, the horizon contribution being not of the form expected for a massless gas.

I shall now check the behaviour of black hole heat capacity in brick wall model. For heat capacity, imposing again the cut-off value (3.17), one obtains from (3.15)\footnote{Also in this case, it is possible to obtain the same value from ‘t Hooft results \cite{tHooft} with a simple computation.}

\[
c = +12\pi M^2
\]  
(3.19)

It is important to note that this is a positive value and bigger in module than the classical well-known result

\[
c_{\text{class}} = -8\pi M^2
\]  
(3.20)

So, if we accept the brick-wall model plus entanglement entropy frame as dynamical explanation of black hole entropy, we find, in the most naive interpretation of (3.19), that black holes are stabilized by one loop contribution of matter.

4. Possible explanations and proposal

The results we have just found sound like a “warning bell” for proposal of entanglement entropy. The conclusions one can draw from them might be rather radical. In fact if we accept the results (3.19) for black hole heat capacity they seem to imply a “stabilization” of the black hole by quantum matter\footnote{It is well known that a negative heat capacity (like the black hole classical one) is characteristic of thermodynamical instable system}. Another possibility is that back reaction is never negligible in study of quantum effect on black hole backgrounds.
The hypothesis that it is wrong to impose entanglement entropy equal to the Bekenstein-Hawking one appears more realistic. These considerations bring to consider more deeply the fifth proposal for black hole entropy explanation we encountered in section 1. In this interpretative frame Bekenstein-Hawking entropy is seen as related to topology structure of black hole manifolds.

Recent works by Hawking, Horowitz and Ross [28,29] have demonstrated that the usual Bekenstein-Hawking law for black hole entropy fails in the case of extreme black holes. For these kinds of object we have a null entropy in spite of a non null area of the event horizon. These authors observed that this change in extreme case in respect of non-extreme one is mainly due to the different nature of event horizon in the former. One in fact finds that, in these cases, the presence of the event horizon is not associated with a non-trivial topology of space-time. Euler characteristic is in fact zero (not two) for this kind of black holes. This radical difference in extreme black hole physics seems a strong hints towards a point of view that particular case of black hole solutions (for example extreme Reissner-Nördstrom black hole is “just” the case $Q^2 = M^2$ of the general solution) is to be considered a rather different object from the non-extreme one. Nevertheless it is possible to understand extreme case as a particular case of black hole without requiring a limitation of black hole thermodynamics laws. The guiding idea (originally proposed by Gibbons and Hawking [30]) is that thermodynamical features of space-times like the Schwarzschild one are explainable as an effect due to their non-trivial topological structure and above all to the nature of their boundaries. In particular Euler characteristic and entropy have the same dependence on the boundaries of the manifold and we will relate them in a general formula. This relation (although demonstrated only for a certain class of metrics) would be valid
for every compact manifold on which Gauss-Bonnet theorem can be extended.

5. Euler characteristic and manifold structure

The Gauss-Bonnet theorem proves that it is possible to obtain the Euler characteristic of a 4-dimensional compact Riemannian manifold $M$ without boundaries by the volume integral of the 4-dimensional metric curvature

$$S_{GB} = \frac{1}{32\pi^2} \int_M \epsilon_{abcd} R^{ab} \wedge R^{cd}$$

with $R$ bound to the spin-connections $\omega$ of the manifold by the relations

$$R^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$$

Chern [31,32] showed that the differential $n$-form $\Omega$ of Gauss-Bonnet integral

$$\Omega = \frac{(-1)^p}{2^{2p} p!} \epsilon_{a_1...a_p} R^{a_1 a_2} \wedge \ldots \wedge R^{a_{2p+1} a_{2p}}$$

defined on $M^n$ can be defined on a manifold $M^{2n-1}$ which is the image of $M^n$ through the flux of its unitary vector field. Then he was able to express $\Omega$ as the exterior derivative of a differential $n-1$-form in $M^{2n-1}$

$$\Omega = d\Pi$$

He also demonstrated that the original $\Omega$ integral on $M^n$ can be performed on a submanifold $V^n$ of $M^{2n-1}$ whose boundaries are the set of singular points of the
unitary vector field previously cited. By Stoke’s theorem we then obtain

\[
S_{GB}^{vol} = \int_{M^n} \Omega = \int_{V^n} \Omega = \int_{\partial V^n} \Pi
\]

For manifold with boundaries this formula can be generalized [33]

\[
S_{GB} = S_{GB}^{vol} + S_{GB}^{bou} = \int_{M^n} \Omega - \int_{\partial M^n} \Pi = \int_{\partial V^n} \Pi - \int_{\partial M^n} \Pi \quad (5.1)
\]

This expression implies that the Euler characteristic of a manifold \( M^n \) with boundaries becomes null in the case that its contours would be the same as that of the submanifold \( V^n \) of \( M^{2n-1} \).

We shall now use (5.1) for black hole manifold. We always work in Euclidean manifolds after imaginary time compactification necessary in order to remove conical singularities on the horizons.

For non-extreme black holes the boundaries of the manifold \( V \) are set by the extreme values of the range of radius coordinate that are \( r = r_h \) and \( r = r_0 = \infty \). The physical manifold \( M \) instead has just one boundary at infinity because, after removal of conical singularity, the black hole horizon \( r = r_h \) is not a border of space-time. So

\[
(5.1) = \int_{r_0}^{r_h} \Pi - \int_{r_h}^{r_0} \Pi - \int_{r_0}^{r_h} \Pi = -\int_{r_h}^{r_0} \Pi
\]

It is possible to use this formula to calculate Euler number and for example in Schwarzschild case one correctly obtains \( \chi_{euler} = 2 \) which is the expected value for \( S^2 \times R^2 \) topology.

For extreme black holes the boundaries of the manifold \( V \) are the same as the one for the ordinary case \( r = r_h \) and \( r = r_0 = \infty \). On the other hand the
physical manifold $M$ has now two boundaries at infinity represented by the usual spatial infinity $r = r_0 = \infty$ and by the horizon $r = r_h$. In fact, in this case the time-affine Killing vector has a set of fixed points only at infinity (it becomes null only asymptotically at infinity, in this sense one says that the black hole horizon for extreme black hole is at infinity). So

$$\begin{align*}
(5.1) = & \int_{r_0}^{r_h} \Pi - \int_{r_0}^{r_h} \Pi - \int_{r_0}^{r_h} \Pi + \int_{r_0}^{r_h} \Pi = 0
\end{align*}$$

This shows that for extreme black hole the Euler characteristic is always null.

6. Entropy for manifolds with boundaries

We will follow the definition of black hole entropy adopted by Kallosh, Ortin, Peet [34].

Let us consider a thermodynamical system with conserved charges $C_i$ and relative potentials $\mu_i$ so that we work in grand ensemble.

$$Z = \text{Tr} \ e^{-\beta H - \mu_i C_i}$$

$$Z = e^{-W}$$

$$W = E - TS - \mu_i C_i$$

we obtain

$$S = \beta(E - \mu_i C_i) + \ln Z$$

Gibbons-Hawking demonstrated that at the tree level

$$Z \sim e^{-I_E}$$

$$I_E = \frac{1}{16\pi} \int_M (-R + L_{\text{matter}}) + \frac{1}{8\pi} \int_{\partial M} [K]$$

Here $I_E$ is the “on-shell” Euclidean action.
In calculating $Z$, and hence $I_E$, it is important to correctly evaluate the boundaries of our manifold $M$.

For no-extreme black hole we have just one boundary at infinity $r_0 \to \infty$ (after the removal of conical singularity, the metric is regular on the horizon $r = r_h$).

For extreme black hole we have a drastic change in boundaries structure. Metrics do not present conical singularity so we cannot fix imaginary time value. The horizon is at an infinite distance from the external observer and so it is like an “internal” boundary of our space-time (we can say that the coordinate of this internal boundary is $r_h$).

In order to determine $S$ we also have to compute $\beta(E - \mu_i C_i)$.

From Gibbons-Hawking [8] we know that for two fixed hypersurfaces at $\tau = cost$ ($\tau =$imaginary time), $\tau_1 \text{ e } \tau_2$, one has

$$\langle \tau_1 | \tau_2 \rangle = e^{-(\tau_2 - \tau_1)(E - \mu_i C_i)} \approx e^{-I_E}$$

In this case it is necessary to understand that the time-affine Killing vector $\partial/\partial\tau$ has two sets of fixed points, one at infinity and the other on the horizon. So an hypersurface at $\tau = cost$ has two boundaries in corresponding to these sets, independently of the position of horizon (which can be at infinity for extreme black hole ).

So one obtains

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#6 Note that, for metrics under our consideration, $V_{bulk} = M_{bulk}$ so the bulk part of the entropy always cancels also for metrics which are not Ricci-flat (as de Sitter case). All the entropy depends on boundary values of extrinsic curvature.
\[ S = \beta(E - \mu_i C_i) + \ln Z = \]
\[ = I_{E^\infty_{r_h}} - I_{E^\infty_{boun}} \]
\[ = \frac{1}{8\pi} \left( \int_{\partial V} [K] - \int_{\partial M} [K] \right) = \]  
\[ = \frac{1}{8\pi} \left( \int_{r_0}^{r_h} [K] - \int_{r_0}^{r_h} [K] - \int_{r_0}^{r_h} [K] + \int_{r_0}^{r_h} [K] \right) \]
\[
(6.1)
\]

The deep similarity between (6.1) and (5.1) is self-evident.

In the case of extreme black hole, we don’t have an internal boundary for \( M \) and so we don’t have \( r_{boun} \) in (6.1). Hence

\[ S = \frac{1}{8\pi} \left[ \int_{\infty}^{r_h} [K] - \int_{r_h}^{\infty} [K] \right] = -\frac{1}{8\pi} \int_{r_h}^{\infty} [K] = A/4 \]

On the contrary, in the case of extreme black hole the horizon is at infinity and \( M \) has two boundaries in \( r = \infty \) and \( r_b = r_h \)

\[ S = \frac{1}{8\pi} \left[ \int_{\infty}^{r_h} [K] - \int_{r_h}^{\infty} [K] - \int_{r_h}^{\infty} [K] + \int_{r_h}^{\infty} [K] \right] = 0 \]

Some comments on derivation (6.1) are in order. It was in fact derived in a grand ensemble but for extreme black hole there is no conical singularity so there is no \( \beta \) fixing and consequently no intrinsic thermodynamics of the manifold. We conjecture that the correct procedure we have to follow is exactly the inverse. The last line of (6.1) is the general expression of entropy for manifold with boundaries. The lack of intrinsic thermodynamics is deducible from (6.1) by consideration of boundary structure. It is not possible to fix \( \beta \) because boundary changes in extreme
case, not the contrary. Thus (6.1) is generalizable to a large class of Riemannian manifolds with boundaries and the similarity in boundary dependence with Gauss-Bonnet integral is a strong hint towards the evidence of a link between entropy and topology for gravitational instantons.

7. General case: spherically symmetric metrics

In order to find a general relation linking Euler characteristic of the manifold to the gravitational entropy, we consider, for simplicity, metrics of the form

\[ ds^2 = -e^{2U(r)} dt^2 + e^{2U(r)} dr^2 + R^2(r) d^2 \Omega \]  \hspace{1cm} (7.1)

On having

\[ A = 4\pi R^2(r_h) \]
\[ \beta = 4\pi ((e^{2U})'_{r=r_h})^{-1} \]
\[ S = \frac{\beta R}{2} \left[ (U' R + 2R') e^U - \frac{2R}{r} e^U \right]_{r=r_h} \]
\[ \chi = \frac{\beta}{2\pi} (2U' e^{2U}) (1 - e^{2U} R^2) \left|_{r=r_h} \right. \]

one finds

\[ S = 2\pi \chi (2U' e^{2U})_{r=r_h}^{-1} (1 - e^{2U} R^2)_{r=r_h}^{-1} \left[ (U' R + 2R') e^U - \frac{2R}{r} e^U \right]_{r=r_h} = \]
\[ = \pi \chi [(e^{2U})' - R^2 e^{2U} (e^{2U})']_{r=r_h}^{-1} \left[ \frac{\beta R}{2} (e^{2U})' + 2R' e^{2U} - \frac{\beta 2R}{r_h} e^U \right]_{r=r_h} \]

being \( e^{2U} \left|_{r=r_h} = 0. \)
Finally one has

\[ S = \pi \chi R(r_h) \left\{ \left( \frac{e^{2U}}{r} \right)' \right\}_{r=r_h} = \]

\[ = \frac{\pi \chi R^2(r_h)}{2} - \frac{\chi (4\pi R^2(r_h))}{8} = \frac{\chi A}{8} \]  

(7.2)

Such a relation points out the deep link between the gravitational entropy and the topological structure of the manifold.

Note that this formula gives the correct result for the extreme black hole cases (for which usual Bekenstein-Hawking formula fails) and that it also has a general validity for all the metrics of the form (7.1)

\[#7\]

8. Conclusions and Perspectives

From this analysis, we deduce that the interpretation of black hole entropy as entanglement entropy brings to problematic results for internal energy and heat capacity. These are not identifiable with their background counterparts so these have to be seen (differently from the entropy situation) as quantum correction to the corresponding tree level gravitational terms.

The internal energy, as ‘t Hooft remarked [1], is of the same order of magnitude of black hole mass: at this point one must question the applicability of the assumption of the negligible back-reaction. Even if we pass over this problem, we still find that the one loop contribute of matter to black hole heat capacity is positive and so it would stabilize the black hole. But we believe it is an inconsistent result because quantum correction is, for the heat capacity, bigger than its background

\[#7\] The extension of (7.2) to metrics of more general form is at the moment under investigation [35].
counterpart; it could be more plausible if we would have quadratic terms in curvature tensor in the gravitational action, but this is not our case. Our results can be interpreted either as a proof of the inconsistency of the identification of entanglement entropy and Bekenstein-Hawking one, or as a structural “bug” embedded in the brick-wall problem approach to black hole thermodynamics if one ignores the back-reaction of matter field on the gravitational background.

On the other side relation (7.2) appears to hold in a wide class of manifolds. It seems that this formulation of black hole entropy sheds new light on the behaviour of the extreme black holes by interpreting the gravitational entropy as a topological effect (in this sense it confirms Hawking’s position).

Unfortunately it appears rather difficult to find a dynamical explanation of this (topological) entropy. We conjecture that this relation of the gravitational entropy with boundary structure of the space-time is in a certain sense a hint towards an interpretation based on dynamical degrees of freedom associated to the vacuum states in non-trivial topologies.

In particular one may propose a deep relation between deformation due to topological changes in zero-point modes of quantum fields and thermal effect on black holes space-times [36].

Moreover this approach seems to require, as a consistency condition, a point of view towards gravitational action which gives a possible answer to the problem we exposed previously about explanation of presence of black hole laws already in General Relativity. In fact it would force, in some sense, an interpretation of gravitational action as an effective one someway induced by a more fundamental quantum matter level #8. In this sense it would be meaningless to speak about

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#8 Perhaps in a way similar to Sakharov “induced gravity” program [37]
“pure gravitational” action and paradoxes encountered in identification of tree level contribution with 1 loop one might be solved. Moreover this appears consistent with the recent Jacobson hypothesis [38] about the interpretation of General Relativity as the thermodynamical limit of a more fundamental theory.

As a matter of fact investigation about thermal nature of horizon characterized space-times appears as a crucial step toward a deeper comprehension of the essence of gravitation.

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REFERENCES

1. ‘t Hooft G., Nucl. Phys. B256 (1985), 727.

2. Belgiorno F., Liberati S.- *Divergences problem in black hole brick-wall model* - Preprint gr-qc/9503022.

3. Liberati S., Pollifrone G. - *Entropy and topology for manifolds with boundaries* - Preprint hep-th/9509093.

4. Bardeen J.M., Carter B., Hawking S.W., Comm. Math. Phys. 31 (1973), 161.

5. Christodoulou D., Phys. Rev. Lett. 25 (1970), 1596.

6. Christodoulou D., Ruffini R., Phys. Rev. D4 (1971), 3552.

7. Hawking S.W., Comm. Math. Phys. 43 (1975), 199.

8. Gibbons G.W., Hawking S.W., Phys. Rev. D15 (1977), 2752.

9. Bekenstein J.D., Phys. Rev. D9 (1974), 3292.

10. York J.W., Phys. Rev. D28 (1983), 2929.

11. Wald R.M., Phys. Rev. D48 (1993), 3427.

12. Srednicki M., Phys. Rev. Lett. 71 (1993), 666.

13. Bombelli L., Koul R.K., Lee J. and Sorkin R.D., Phys. Rev. D34 (1986), 373.

14. Frolov V.P., Novikov I., Phys. Rev. D48 (1993), 4545.

15. Hawking S.W., Horowitz G.T., Ross S.F. - *Entropy, area and black hole pairs* - Preprint gr-qc/9409013.

16. Frolov V., Novikov I., Phys. Rev. D48 (1993), 1607.

17. Barvinsky A.O., Frolov V.P., Zelnikov A.I., Phys. Rev. D51 (1995), 1741.
18. Susskind L. and Uglum J., *Phys. Rev.* **D50** (1994), 2700.

19. Fursaev D.V., Solodukhin S.N. - *On one-loop renormalization of black hole entropy* - Preprint hep-th/9412020.

20. Bekenstein J.D. - *Do we understand black hole entropy?* - Preprint gr-qc/9409013.

21. Jacobson T. - *Black hole entropy and induced gravity* - Preprint gr-qc/9404039.

22. Fursaev D.V., *Mod. Phys. Lett.* **A10** (1995), 649.

23. Solodukhin S.N., *Phys. Rev.* **D51** (1995), 609.

24. Solodukhin S.N., *Phys. Rev.* **D51** (1995), 618.

25. Fursaev D.V. and Solodukhin S.N.- *On One-Loop Renormalization of Black Hole Entropy* - Preprint hep-th/9412020.

26. Frolov V.P.- *Why the Entropy of a Black Hole is A/4?* - Preprint Alberta-Thy-22-94, gr-qc/9406037.

27. Dowker J.S., Kennedy G., *Journ. Phys.* **A11** (1978), 895.

28. Hawking S.W., Horowitz G.T., Ross S.F.- *Entropy, Area and black hole pairs* - Preprint gr-qc/9409013.

29. Hawking S.W., Horowitz G.T.- *The gravitational hamiltonian, action, entropy and surface terms* - Preprint gr-qc/9501014.

30. Gibbons G.W., Hawking S.W., *Commun. Math. Phys.* **66** (1979), 291.

31. Chern S., *Annals of Math.* **45** (1944), 747.

32. Chern S., *Annals of Math.* **46** (1945), 674.

33. Eguchi T., Gilkey P.B., Hanson A.J., *Phys. Rep.* **66, 6** (1980). .

34. Kallosh R., Ortin T., Peet A., *Phys. Rev.* **D47** (1993), 5400.
35. Liberati S., Pollifrone G. - *Work in progress*.

36. Belgiorno F., Liberati S. - *Work in progress*.

37. Sakharov A.D., *Soviet Physics - Doklady* **12**, (1968), 1040.

38. Jacobson T., *Phys. Rev. Lett.* **75** (1995), 1260.