REGULARIZING PROPERTIES OF COMPLEX MONGE-AMPÈRE FLOWS

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Abstract. We study the regularizing properties of complex Monge-Ampère flows on a Kähler manifold \((X, \omega)\) when the initial data are \(\omega\)-psh functions with zero Lelong number at all points. We prove that the general Monge-Ampère flow has a solution which is immediately smooth. We also prove the uniqueness and stability of solution.

Introduction

Let \((X, \omega)\) be a compact Kähler manifold of complex dimension \(n\) and \(\alpha \in H^{1,1}(X, \mathbb{R})\) a Kähler class with \(\omega \in \alpha\). Let \(\Omega\) be a smooth volume form on \(X\). Denote by \((\theta_t)_{t \in [0, T]}\) a family of Kähler forms on \(X\), and assume that \(\theta_0 = \omega\). The goal of this note is to prove the regularizing and stability properties of solutions to the following complex Monge-Ampère flow

\[
(CMAF) \quad \frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi_t)
\]

where \(F\) is a smooth function and \(\varphi(0, z) = \varphi_0(z)\) is a \(\omega\)-plurisubharmonic (\(\omega\)-psh) function with zero Lelong numbers at all points.

One motivation for studying this Monge-Ampère flow is that the Kähler-Ricci flow can be reduced to a particular case of \((CMAF)\). When \(F = F(z)\) and \(\theta_t = \omega + t\chi\), where \(\chi = \eta - Ric(\omega)\), then \((CMAF)\) is the local potential equation of the twisted Kähler-Ricci flow

\[
\frac{\partial \omega_t}{\partial t} = -Ric(\omega_t) + \eta,
\]

which was studied recently by Collins-Székelydi [CS12] and Guedj-Zeriahi [GZ13].

Running the Kähler-Ricci flow from a rough initial data has been the purpose of several recent works [CD07], [ST09], [SzT11], [GZ13], [BG13], [DNL14]. In [ST09], [SzT11] the authors succeeded to run \((CMAF)\) from continuous initial data, while [DNL14] and [GZ13] are running a simplified flow starting from an initial current with zero Lelong numbers. In this note we extend these latter works to deal with general \((CMAF)\) and arbitrary initial data.
A strong motivation for studying (CMAF) with degenerate initial data comes from the Analytic Minimal Model Program introduced by J. Song and G. Tian [ST09], [ST12]. It requires to study the behavior of the Kähler-Ricci flow on mildly singular varieties, and one is naturally lead to study weak solutions of degenerate complex Monge-Ampère flows (when the function $F$ in (CMAF) is not smooth but continuous). Eyssidieux-Guedj-Zeriahi have developed in [EGZ16] a viscosity theory for degenerate complex Monge-Ampère flows which allows in particular to define and study the Kähler-Ricci flow on varieties with canonical singularities.

Our main result is the following:

**Theorem A.** Let $\varphi_0$ be a $\omega$-psh function with zero Lelong numbers at all points. Let $(t, z, s) \mapsto F(t, z, s)$ be a smooth function on $[0, T] \times X \times \mathbb{R}$ such that $\frac{\partial F}{\partial s} \geq -C$ for some $C \geq 0$. Then there exists a family of smooth strictly $\omega$-psh functions $(\varphi_t)$ satisfying (CMAF) in $(0, T] \times X$, with $\varphi_t \to \varphi_0$ in $L^1(X)$, as $t \searrow 0^+$. This family is moreover unique if $C = 0$ and $|\frac{\partial F}{\partial t}| < C'$ for some $C' > 0$.

We further show that
- $\varphi_t$ converges to $\varphi_0$ in $C^0(X)$ if $\varphi_0$ is continuous.
- $\varphi_t$ converges to $\varphi_0$ in capacity if $\varphi_0$ is merely bounded.
- $\varphi_t$ converges to $\varphi_0$ in energy if $\varphi \in E^1(X, \omega)$ has finite energy.

Moreover, we also prove the following stability result:

**Theorem B.** Let $\varphi_0, \varphi_{0,j}$ be $\omega$-psh functions with zero Lelong number at all points, such that $\varphi_{0,j} \to \varphi_0$ in $L^1(X)$. Denote by $\varphi_{t,j}$ and $\varphi_j$ the corresponding solutions of (CMAF) with initial condition $\varphi_{0,j}$ and $\varphi_0$ respectively. Then for each $\varepsilon \in (0, T)$

$$\varphi_{t,j} \to \varphi_t \text{ in } C^\infty([\varepsilon, T] \times X) \text{ as } j \to +\infty.$$  

Moreover, if $\varphi_0$ and $\psi_0$ are continuous, then for any $k \geq 0$, for any $0 < \varepsilon < T$, there exists a positive constant $C(k, \varepsilon)$ depending only on $k$ and $\varepsilon$ such that

$$||\varphi - \psi||_{C^k([\varepsilon, T] \times X)} \leq C(k, \varepsilon)||\varphi_0 - \psi_0||_{L^\infty(X, \omega)}.$$  

We also prove in Section 5 that one can run the Monge-Ampère flow from a positive current representing a nef class, generalizing results from [GZ13], [DNL14].

The paper is organized as follows. In Section 1 we recall some analytic tools, and give the strategy of proof of Theorem A. In Section 2 we prove various a priori estimates for the regular case. In Section 3 we prove Theorem A using the a priori estimates from Section 2. In Section 4 we prove the uniqueness in Theorem A and Theorem B. In Section 5 we show that the Monge-Ampère flow can run from a nef class.

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1. Preliminaries and Strategy

In this section we recall some analytic tools which will be used in the sequel.

1.1. Plurisubharmonic functions and Lelong number. Let $(X, \omega)$ be a compact Kähler manifold. We define the following operators:

$$d := \partial + \bar{\partial}, \quad d^c := \frac{1}{2i\pi} (\partial - \bar{\partial}).$$

**Definition 1.1.** We let $PSH(X, \omega)$ denote the set of all $\omega$-plurisubharmonic functions ($\omega$-psh for short), i.e. the set of functions $\varphi \in L^1(X, \mathbb{R} \cup \{-\infty\})$ which can be locally written as the sum of a smooth and a plurisubharmonic function, and such that

$$\omega + dd^c \varphi \geq 0$$

in the weak sense of positive currents.

**Definition 1.2.** Let $\varphi$ be a $\omega$-psh function and $x \in X$. The Lelong number of $\varphi$ at $x$ is

$$\nu(\varphi, x) := \liminf_{z \to x} \frac{\varphi(z)}{\log |z - x|}.$$

We say $\varphi$ has a logarithmic pole of coefficient $\gamma$ at $x$ if $\nu(\varphi, x) = \gamma$.

1.2. A Laplacian inequality. Let $\alpha$ and $\omega$ be $(1, 1)$-forms on a complex manifold $X$ with $\omega > 0$. Then the trace of $\alpha$ with respect $\omega$ is defined as

$$\text{tr}_\omega(\alpha) = n \frac{\alpha \wedge \omega^{n-1}}{\omega^n}.$$

We can diagonalize $\alpha$ with respect to $\omega$ at each point of $X$, with real eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\text{tr}_\omega(\alpha) = \sum_j \lambda_j$. The Laplace of a function $\varphi$ with respect to $\omega$ is given by

$$\Delta_\omega \varphi = \text{tr}_\omega(dd^c \varphi).$$

We have the following eigenvalue estimate:

**Lemma 1.3.** If $\omega$ and $\omega'$ are two positive $(1, 1)$-forms on a complex manifold $X$ of dimension $n$, then

$$\left( \frac{\omega'^n}{\omega^n} \right)^{\frac{1}{n}} \leq \frac{1}{n} \text{tr}_\omega(\omega') \leq \left( \frac{\omega'^n}{\omega^n} \right) (\text{tr}_{\omega'}(\omega))^{n-1}.$$
Proposition 1.4 ([Siu87]). Let $\omega, \omega'$ be two Kähler forms on a compact complex manifold. If the holomorphic bisectional curvature of $\omega$ is bounded below by a constant $B \in \mathbb{R}$ on $X$, then we have
\[
\Delta_{\omega'} \log \text{tr}_{\omega} (\omega') \geq - \frac{\text{tr}_{\omega} \text{Ric}(\omega')}{\text{tr}_{\omega} (\omega')} + B \text{tr}_{\omega'} (\omega).
\]

1.3. Maximum principle and comparison theorem. We establish here a slight generalization of the comparison theorem that we will need.

Proposition 1.5. Let $\varphi, \psi \in C^\infty([0, T] \times X)$ be $\theta_t$-psh functions such that
\[
\frac{\partial \varphi}{\partial t} \leq \log \left( \frac{(\theta_t + dd^c \varphi)^n}{\Omega} \right) - F(t, z, \varphi),
\]
\[
\frac{\partial \psi}{\partial t} \geq \log \left( \frac{(\theta_t + dd^c \psi)^n}{\Omega} \right) - F(t, z, \psi),
\]
where $F(t, z, s)$ is a smooth function with $\frac{\partial F}{\partial s} \geq -\lambda$. Then
\[
\sup_{[0, T] \times X} (\varphi_t - \psi_t) \leq e^{\lambda T} \max \left\{ \sup_X (\varphi_0 - \psi_0); 0 \right\}. \tag{1.1}
\]
In particular, if $\varphi_0 \leq \psi_0$, then $\varphi_t \leq \psi_t$.

Proof. We define $u(x, t) = e^{-\lambda t} (\varphi_t - \psi_t)(x) - \varepsilon t \in C^\infty([0, T] \times X)$ where $\varepsilon > 0$ is fixed. Suppose $u$ is maximal at $(t_0, x_0) \in [0, T] \times X$. If $t_0 = 0$ then we have directly the estimate (1.1). Assume now $t_0 > 0$, using the maximum principle, we get $\dot{u} \geq 0$ and $dd^c \psi \leq 0$ at $(t_0, x_0)$, hence
\[
-\lambda e^{-\lambda t_0} (\varphi_t - \psi_t) + e^{-\lambda t_0} (\dot{\varphi}_t - \dot{\psi}_t) \geq \varepsilon > 0 \text{ and } dd^c \varphi_t \leq dd^c \psi_t.
\]
Observing that at $(t_0, x_0)$
\[
\dot{\varphi} - \dot{\psi} \leq F(t, x, \psi) - F(t, x, \varphi),
\]
we infer that
\[
0 \leq F(t, x, \psi) + \lambda \psi - [F(t, x, \varphi) + \lambda \varphi],
\]
at $(t_0, x_0)$. Since $\frac{\partial F}{\partial s} \geq -\lambda$, $F(t, x, s) + \lambda s$ is an increasing function in $s$, hence $\varphi_{t_0}(x_0) \leq \psi_{t_0}(x_0)$. Thus $u(x, t) \leq u(x_0, t_0) \leq 0$. Letting $\varepsilon \to 0$, this yields
\[
\sup_{[0, T] \times X} (\varphi_t - \psi_t) \leq e^{\lambda T} \max \left\{ \sup_X (\varphi_0 - \psi_0); 0 \right\}. \]
\[
\square
\]

The following proposition has been given for the twisted Kähler-Ricci flow by Di Nezza and Lu [DNL14]:

Proposition 1.6. Assume $\psi_t$ a smooth solution of (CMAF) with a smooth initial data $\psi_0$ and $\varphi_t$ is a subsolution of (CMAF) with initial data $\varphi_0$ which is a $\omega$-psh function with zero Lelong number at all point: i.e $\varphi_t \in C^\infty([0, T] \times X)$ satisfies
\[
\frac{\partial \varphi_t}{\partial t} \leq \log \left( \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} \right) - F(t, z, \varphi_t),
\]
and \( \varphi_t \to \varphi_0 \) in \( L^1(X) \). Suppose that \( \varphi_0 \leq \psi_0 \), then \( \varphi_t \leq \psi_t \).

**Proof.** Fix \( \epsilon > 0 \) and note that \( \varphi - \psi \) is a smooth function on \( [\epsilon, T] \times X \). It follows from Proposition 1.5 that

\[
\varphi - \psi \leq e^{\lambda T} \max \left\{ \sup_X (\varphi_\epsilon - \psi_\epsilon); 0 \right\}.
\]

We have \( (\epsilon, x) \mapsto \psi_\epsilon(x) \) is smooth by assumption. Using Hartogs’ Lemma and the fact that \( \varphi_t \) converges to \( \varphi_0 \) in \( L^1(X) \) as \( t \to 0 \), we get

\[
\sup_X (\varphi_\epsilon - \psi_\epsilon) \to \sup_X (\varphi_0 - \psi_0) \leq 0 \quad \text{as} \quad \epsilon \to 0.
\]

This implies that \( \varphi_t \leq \psi_t \) for all \( 0 \leq t \leq T \). \( \square \)

1.4. **Evans-Krylov and Schauder estimates for Monge-Ampère flow.**

The Evans-Krylov and Schauder theorems for nonlinear elliptic equations

\[
F(D^2 u) = f
\]

with \( F \) concave, have been used to show that bounds on \( u, D^2 u \) imply \( C^{2,\alpha} \) on \( u \) for some \( \alpha > 0 \) and higher order bounds on \( u \). There are also Evans-Krylov estimates for parabolic equations (see [Lie96]), but the precise version which we need is as follows

**Theorem 1.7.** Let \( U \subseteq \mathbb{C}^n \) be an open subset and \( T \in (0, +\infty) \). Suppose that \( u \in C^\infty([0, T] \times \bar{U}) \) and \( (t, x, s) \mapsto f(t, x, s) \) is a function in \( C^\infty([0, T] \times \bar{U} \times \mathbb{R}) \), satisfy

\[
\frac{\partial u}{\partial t} = \log \det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) + f(t, x, u).
\]

In addition, assume that there is a constant \( C > 0 \) such that

\[
\sup_{(0, T) \times \bar{U}} \left( |u| + \left| \frac{\partial u}{\partial t} \right| + |\nabla u| + |\Delta u| \right) \leq C.
\]

Then for any compact \( K \subseteq U \), for each \( \epsilon > 0 \) and \( p \in \mathbb{N} \),

\[
\|u\|_{C^p([\epsilon, T] \times K)} \leq C_0.
\]

where \( C_0 \) only depends on \( C \) and \( ||f||_{C^q([0, T] \times \bar{U} \times [-C, C])} \) for some \( q \geq p - 2 \).

The proof of this theorem follows the arguments of Boucksom-Guedj [BG13, Theorem 4.1.4] where the function \( f \) is independent of \( u \).

First of all, we recall the parabolic \( \alpha \)-Hölder norm of a function \( f \) on the cylinder \( Q = U \times (0, T) \):

\[
||f||_{C^\alpha(Q)} := ||f||_{C^0(Q)} + [f]_{\alpha, Q},
\]

where

\[
[f]_{\alpha, Q} := \sup_{X, Y \subseteq Q, X \neq Y} \frac{|f(X) - f(Y)|}{\rho^\alpha(X, Y)}
\]

is the \( \alpha \)-Hölder seminorm with respect to the parabolic distance

\[
\rho((x, t), (x', t')) = |x - x'| + |t - t'|^{1/2}.
\]
For each \( k \in \mathbb{N} \), the \( C^{k,\alpha} \)-norm is defined as
\[
\|f\|_{C^{k,\alpha}(Q)} := \sum_{|\alpha|+2|\beta| \leq k} \|D^\beta_u D^\gamma_t f\|_{C^\alpha(Q)}.
\]

If \((\omega_t)_{t \in (0,T)}\) is a path of differential forms on \( U \), we can similarly define \([\omega_t]_{\alpha,\beta}\) and \(\|\omega_t\|_{C^{k,\alpha}(Q)}\), with respect to the flat metric \(\omega_U\) on \( U \).

The first ingredient in the proof of Theorem 1.7 is the Schauder estimates for linear parabolic equations.

**Lemma 1.8.** ([Kry96, Theorem 8.11.1], [Lie96, Theorem 4.9]) Let \((\omega_t)_{t \in (0,T)}\) be a smooth path of Kähler metrics on \( U \) and \(\omega_U\) be the flat metric on \( U \). Define \( Q = U \times (0,T) \), and assume that \( u, g \in C^\infty(Q) \) satisfy
\[
\left( \frac{\partial}{\partial t} - \Delta_t - c(t,x) \right) u(t,x) = g(t,x),
\]
where \( \Delta_t \) is the Laplacian with respect to \(\omega_t\). Suppose also that there exist \( C > 0 \) and \( 0 < \alpha < 1 \) such that on \( Q \) we have
\[
C^{-1} \omega_U \leq \omega_t \leq C \omega_U, \quad \|c\|_{C^\alpha(Q)} \leq C \quad \text{and} \quad [\omega_t]_{\alpha,\beta} \leq C.
\]
Then for each \( Q' = U' \times (\varepsilon,T) \) with \( U' \subset U \), we can find a constant \( A \) only depending on \( U' \), \( \varepsilon \) and \( C \) such that
\[
\|u\|_{C^{2,\alpha}(Q')} \leq A(\|u\|_{C^\alpha(Q)} + \|g\|_{C^\alpha(Q)}).
\]

The second ingredient in the proof Theorem 1.7 is the following Evans-Krylov estimates type for complex Monge-Ampère flows.

**Lemma 1.9.** ([Gil11, Theorem 4.1]) Suppose \( u, g \in C^\infty(Q) \) satisfy
\[
\frac{\partial u}{\partial t} = \log \text{det} \frac{\partial^2 u}{\partial z_j \partial z_k} + g(t,x),
\]
and assume also that there exists a constant \( C > 0 \) such that
\[
C^{-1} \leq \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) \leq C \quad \text{and} \quad \left| \frac{\partial g}{\partial t} \right| + |dd^c g| \leq C.
\]
Then for each \( Q' = U' \times (\varepsilon,T) \) with \( U' \subset U \) an open subset and \( \varepsilon \in (0,T) \), we can find \( A > 0 \) and \( 0 < \alpha < 1 \) only depending on \( U' \), \( \varepsilon \) and \( C \) such that
\[
[dd^c u]_{\alpha,\beta} \leq A.
\]

**Proof of Theorem 1.7.** In the sequel of the proof, we say that a constant is under control if it is bounded by the terms of \( C, \varepsilon \) and \( \|f\|_{C^\infty([0,T] \times \bar{U} \times [-C,C])} \).

Consider the path \( \omega_t := dd^c u_t \) of Kähler forms on \( U \). Denote by \(\omega_U\) the flat metric on \( U \). It follows from 1.2 that
\[
\omega^n_t = \exp \left( \frac{\partial u}{\partial t} - f \right) \omega^n_U.
\]
Since \( \frac{\partial u}{\partial t} - f \) is bounded by a constant under control by the assumption, there exists a constant \( C_1 \) under control such that \( C^{-1}_1 \omega^n_U \leq \omega^n_t \leq C_1 \omega^n_U \). It follows
from the assumption that $\text{tr}_{\omega_U} \omega_t$ is bounded. Two latter inequalities imply that $C_2^{-1} \omega_t \leq \omega_t \leq C_2 \omega_U$ for some $C_2 > 0$ under control by considering inequalities of eigenvalues. Set $g(t, x) := f(t, x, u)$. Since $C_2^{-1} \omega_U \leq \omega_t \leq C_2 \omega_U$ and

$$\sup_{(0, T) \times U} \left( |u| + \left| \frac{\partial u}{\partial t} \right| + |\nabla u| + |\Delta u| \right) \leq C,$$

we get $|\partial u/\partial t| + |dd^c g| \leq C_3$ with $C_3$ under control. Apply Lemma 1.9 to (1.2), we obtain $[dd^c u]_{\alpha, Q}$ is under control for some $0 < \alpha < 1$.

Let $D$ be any first order differential operator with constant coefficients. Differentiating (1.2), we get

$$\left( \frac{\partial}{\partial t} - \Delta_t - \frac{\partial f}{\partial s} \right) Du = Df,$$

with $|u| + |\nabla u| + \left| \frac{\partial u}{\partial t} \right| + |\Delta u|$ and $[dd^c u]_{\alpha, Q}$ are under control, so $C^0$ norm of $Du$ is under control. Applying the parabolic Schauder estimates (Lemma 1.8) to (1.3) with $c(t, x) = \frac{df}{\partial s}(t, x, u)$, the $C^{2, \alpha}$ norm of $Du$ is thus under control. Apply $D$ to (1.3) we get

$$\left( \frac{\partial}{\partial t} - \Delta_t - \frac{\partial f}{\partial s} \right) D^2 u = D^2 f + \frac{\partial(Df)}{\partial s} Du + \sum_{j, k} (D\omega_t^{j k}) \frac{\partial^2 Du}{\partial z_j \partial \bar{z}_k}$$

$$+ \frac{\partial^2 f}{\partial s^2} |Du|^2 + D \left( \frac{\partial f}{\partial s} \right) Du,$$

where the parabolic $C^\alpha$ norm of the right-hand side is under control. Thanks to the parabolic Schauder estimates 1.8, the $C^{2, \alpha}$ norm of $D^2 u$ is under control. Iterating this procedure we complete the proof of Theorem 1.7.

1.5. Monge–Ampère capacity.

Definition 1.10. Let $K$ be a Borel subset of $X$. We set

$$\text{Cap}_{\omega}(K) = \sup \left\{ \int_K MA(\varphi); \varphi \in PSH(X, \omega), 0 \leq \varphi \leq 1 \right\}.$$

Then we call $\text{Cap}_{\omega}$ is the Monge–Ampère capacity with respect to $\omega$.

Definition 1.11. Let $(\varphi_j) \in PSH(X, \omega)$. We say that $(\varphi_j)$ converges to $\varphi$ as $j \to +\infty$ in capacity if for each $\varepsilon > 0$

$$\lim_{j \to +\infty} \text{Cap}_{\omega}(|\varphi_j - \varphi| < \varepsilon) = 0.$$

The following Proposition [GZ05, Proposition 3.7] states that decreasing sequences of $\omega$-psh functions converge in capacity.

Theorem 1.12. Let $\varphi, \varphi_j \in PSH(X, \omega) \cap L^\infty(X)$ such that $(\varphi_j)$ decreases to $\varphi$, then for each $\varepsilon > 0$

$$\text{Cap}_{\omega}(\{\varphi_j > \varphi + \varepsilon\}) \to 0 \quad \text{as} \quad j \to +\infty.$$
1.6. Monge-Ampère energy. The energy of a \( \omega \)-psh function has been introduced in [GZ07] and further studied in [BBGZ13]. For \( \phi \in PSH(X, \omega) \cap L^\infty(X) \), the Aubin-Yau energy functional is

\[
E(\phi) := \frac{1}{(n + 1) V} \sum_{j=0}^{n} \int_X \phi(\omega + dd^c \phi)^j \wedge \omega^{n-j},
\]

where

\[
V := \int_X \omega^n.
\]

For any \( \phi \in PSH(X, \omega) \), we set

\[
E(\phi) := \inf \{ E(\psi); \psi \in PSH(X, \omega) \cap L^\infty(X), \phi \leq \psi \}.
\]

**Definition 1.13.** We say that \( \phi \in PSH(X, \omega) \) has a finite energy if \( E(\phi) > -\infty \) and denote by \( \mathcal{E}^1(X, \omega) \) the set of all finite energy \( \omega \)-psh functions.

Let \( (\theta_t)_{t \in [0, T]} \) be a family of Kähler metrics on \( X \) and \( \Omega \) be a smooth volume form. We consider the following complex Monge-Ampère flow

\[
(CMAF) \begin{cases} 
\frac{\partial \varphi}{\partial t} = \log \frac{(\theta_t + dd^c \varphi)^n}{\Omega} - F(t, z, \varphi), \\
\varphi(0, .) = \varphi_0.
\end{cases}
\]

We set \( \omega_t = \theta_t + dd^c \varphi_t \).

**Definition 1.14.** Suppose \( \varphi_t \) is a solution of \( (CMAF) \). The energy for \( \varphi_t \) is

\[
E(\varphi_t) := E_{\theta_t}(\varphi_t) := \frac{1}{(n + 1) V} \sum_{j=0}^{n} \int_X \varphi_t(\theta_t + dd^c \varphi)^j \wedge \theta_t^{n-j}.
\]

In particular, when \( \theta_t = \omega \) for all \( t \in [0, T] \) we get the Aubin-Yau energy functional.

1.7. Reduction to \( \frac{\partial F}{\partial s} \geq 0. \) We now consider the complex Monge-Ampère flow

\[
(CMAF) \quad \frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi),
\]

where \( F(t, z, s) \in C^\infty([0, T] \times X \times \mathbb{R}, \mathbb{R}) \) with

\[
\frac{\partial F}{\partial s} \geq -C,
\]

for some \( C \geq 0. \)

First of all, we observe that it is sufficient to prove Theorem A with \( F \) satisfying \( F(t, z, s) \in C^\infty([0, T] \times X \times \mathbb{R}, \mathbb{R}) \) and \( s \mapsto F(t, z, s) \) is non-decreasing. Indeed, assume that \( \varphi_t \) is a solution of \( (CMAF) \) with \( \partial F/\partial s \geq -C. \) By changing variables

\[
\phi(t, z) = e^{Bt} \varphi(B^{-1}(1 - e^{-Bt}), z),
\]
we get
\[ \frac{\partial \phi_t}{\partial t} = \log \left( \tilde{\theta}_t + dd^c \phi_t \right)^n - \tilde{F}(t, z, \phi_t), \]
where \( \tilde{\theta}_t = e^{Bt} \theta_1^{-e^{-Bt}} \) and
\[ \tilde{F}(t, z, s) = -Bs + Bnt + F(B^{-1}(1 - e^{-Bt}), z, e^{-Bt}s). \]
We thus have
\[ \frac{\partial \tilde{F}}{\partial s} = -B + \frac{\partial F}{\partial s} e^{-Bt} \geq -B - Ce^{-Bt}. \]
Choosing \( B < 0 \) such that \( -B - Ce^{-Bt} \geq 0 \) or \( -Be^{Bt} \geq C \) for all \( t \in [0, T] \), we get the desired equation. Note that we can not always choose \( B \) for any \( T > 0 \) because the maximal value of \( -Be^{Bt} \) is \( 1/e^T \) at \( B = -1/T \), but in our case we can assume \( T \) is small enough such that \( C < 1/e^T \). Finally we obtain the equation
\[ \frac{\partial \phi_t}{\partial t} = \log \left( \tilde{\theta}_t + dd^c \phi_t \right)^n - \tilde{F}(t, z, \phi_t), \]
where \( \phi(0, z) = \varphi_0 \) and \( \partial \tilde{F}/\partial s \geq 0. \)

1.8. **Strategy of the proof.** We fix \( \omega \) a reference Kähler form. Since we are interested in the behavior near 0 of the flow, we can assume that for \( 0 \leq t \leq T \)
\[ \frac{\omega}{2} \leq \theta_t \leq 2\omega, \quad (1.4) \]
and there exists \( \delta > 0 \) such that
\[ \delta^{-1} \Omega \leq \theta_t^n \leq \delta \Omega, \forall t \in [0, T]. \]

We consider the complex Monge-Ampère flow
\[ (CMAF) \quad \frac{\partial \varphi_t}{\partial t} = \log \left( \theta_t + dd^c \varphi_t \right)^n - F(t, z, \varphi), \]
where \( F(t, z, s) \in C^\infty([0, T] \times X \times \mathbb{R}, \mathbb{R}) \) is such that \( \frac{\partial F}{\partial s} \geq -C \), for some \( C \geq 0 \). Our first goal is to show the following generalization of [GZ13, DNL14]:

**Theorem 1.15.** Let \( \varphi_0 \) be a \( \omega \)-psh function with zero Lelong numbers. There exists a family of smooth strictly \( \theta_t \) - psh function \( (\varphi_t) \) such that
\[ \frac{\partial \varphi_t}{\partial t} = \log \left( \theta_t + dd^c \varphi_t \right)^n - F(t, z, \varphi), \]
in \( (0, T] \times X \), with \( \varphi_t \rightarrow \varphi_0 \) in \( L^1(X) \), as \( t \searrow 0^+ \). This family is unique if \( C = 0 \) and \( |\frac{\partial F}{\partial s}| < C' \) for some \( C' > 0 \). Moreover, \( \varphi_t \rightarrow \varphi \) in energy if \( \varphi \in \mathcal{E}^1(X, \omega) \) and \( \varphi_t \) is uniformly bounded and converges to \( \varphi_0 \) in capacity if \( \varphi_0 \in L^\infty(X) \).

The strategy of the proof is as follows:
- We first reduce to the case when \( \frac{\partial F}{\partial s} \geq 0 \) following Section 1.7.
• Approximate $\varphi_0$ by a decreasing sequence $(\varphi_{0,j})$ of smooth and strictly $\omega$-psh functions by using the regularization result of Demailly [Dem92, BK07]. There exists unique solutions $\varphi_{t,j} \in PSH(X, \omega) \cap C^\infty(X)$ to the flow above with initial data $\varphi_{0,j}$.

• We then establish various priori estimates which will allow us to pass to the limit as $j \to \infty$. We prove for each $0 < \varepsilon < T$:
  1. $(t,z,j) \mapsto \varphi_{t,j}(z)$ is uniformly bounded on $[\varepsilon, T] \times X \times \mathbb{N}$,
  2. $(t,z,j) \mapsto \dot{\varphi}_{t,j}(z)$ is uniformly bounded on $[\varepsilon, T] \times X \times \mathbb{N}$,
  3. $(t,z,j) \mapsto \Delta_\omega \varphi_{t,j}(z)$ is uniformly bounded on $[\varepsilon, T] \times X \times \mathbb{N}$.

• Finally, we apply the Evans-Krylov theory and Schauder estimates to show that $\varphi_{t,j} \to \varphi_t$ in $C^\infty((0, T] \times X)$, as $j \to +\infty$ such that $\varphi_t$ satisfies (CMAF). We then check that $\varphi_t \to \varphi_0$ as $t \to 0^+$, and also study finer convergence properties:
  1. For $\varphi_0 \in L^1(X)$, we show that $\varphi_t \to \varphi_0$ in $L^1(X)$ as $t \to 0$.
  2. When $\varphi_0$ is bounded, we show that $\varphi_t \to \varphi_0$ in capacity.
  3. When $\varphi_0 \in E^1(X, \omega)$, we show that $\varphi_t$ converges to $\varphi_0$ in energy as $t \to 0$.

2. A priori estimates

In this section we prove various a priori estimates for $\varphi_t$ which satisfies

$$\frac{\partial \varphi_t}{\partial t} = \log \left( \frac{\theta_t + dd^c \varphi_t}{\Omega} \right) - F(t, z, \varphi)$$

with a smooth strictly $\omega$-psh initial data $\varphi_0$, where $(t,z,s) \mapsto F(t,z,s) \in C^\infty([0, T] \times X \times \mathbb{R}, \mathbb{R})$ with $\frac{\partial F}{\partial s} \geq 0$. Since we are interested in the behavior near $0$ of (CMAF), we can further assume that

$$\theta_t - t\dot{\theta}_t \geq 0 \text{ for } 0 \leq t \leq T. \tag{2.1}$$

This assumption will be used to bound the $\dot{\varphi}_t$ from above.

2.1. Bounding $\varphi_t$.

Lemma 2.1. We have

$$\varphi_t \leq Ct + \max \{ \sup \varphi_0, 0 \},$$

where $C = -\inf_{x \in X, t \in [0, T]} F(t, x, 0) + n \log \delta$.

Proof. Consider $\psi_t = Ct$, where $C = -\inf_{x \in X, t \in [0, T]} F(t, x, 0) + n \log \delta$. Thus we have

$$\log \left( \frac{\theta_t + dd^c \psi_t}{\Omega} \right) = \log \left( \frac{\theta^n_t}{\Omega} \right) \leq n \log \delta.$$

Now $F(t, z, \psi_t) \geq F(t, z, 0) \geq \inf_{x \in X, t \in [0, T]} F(t, x, 0)$, since we assume $s \mapsto F(\cdot, \cdot, s)$ is increasing. Therefore

$$\frac{\partial \psi_t}{\partial t} \geq \left( \frac{\theta_t + dd^c \psi_t}{\Omega} \right) - F(t, z, \psi_t).$$
Apply Proposition 1.5 for $\varphi_t$ and $\psi_t$, we get $\varphi_t \leq Ct + \max\{\sup \varphi_0, 0\}$. □

We now find a lower bound of $\varphi_t$ which does not depend on $\inf_X \varphi_0$. First, we assume that $\theta_t \geq \omega + t\chi, \forall t \in [0, T]$, for some smooth $(1, 1)$-form $\chi$. Fix $0 < \beta < +\infty$ and $0 < \alpha$ such that

$$\chi + (2\beta - \alpha)\omega \geq 0.$$  

It follows from Skoda’s integrability theorem [Sko72] that $e^{-2\beta\varphi_0}\omega^n$ is absolutely continuous with density in $L^p$ for some $p > 1$. This is where we use the crucial assumption that $\varphi_0$ has zero Lelong number at all points. Kołodziej’s uniform estimate [Koł98] insures the existence of a continuous $\omega$-psh solution $u$ of the equation

$$\alpha^n(\omega + dd^c u)^n = e^{\alpha u - 2\beta\varphi_0}\omega^n.$$  

Assume that $\phi_t$ is solution of the following equation

$$\begin{cases}
\frac{\partial \phi_t}{\partial t} = \log \left(\frac{(\omega + t\chi + dd^c \phi)^n}{\omega^n}\right), \\
\phi(0, \cdot) = \varphi_0.
\end{cases}$$

By Lemma 2.9 in [GZ13] we have

$$\phi_t(z) \geq (1 - 2\beta t)\varphi_0(z) + \alpha tu(z) + n(t \log t - t). \quad (2.2)$$

Using this we have the following lemma:

**Lemma 2.2.** For all $z \in X$ and $t \in (0, T]$, we have

$$\varphi_t(z) \geq \phi_t + At \geq (1 - 2\beta t)\varphi_0(z) + \alpha tu(z) + n(t \log t - t) + At, \quad (2.3)$$

where $A$ depend on $\sup_X \varphi_0$. In particular, there exists $c(t) \geq 0$ such that

$$\varphi_t(z) \geq \varphi_0(z) - c(t),$$

with $c(t) \searrow 0$ as $t \searrow 0$.

**Proof.** There is $\sigma > 0$ such that $\sigma^{-1}\omega^n \leq \Omega \leq \sigma\omega^n$, so we may assume that

$$\frac{\partial \phi_t}{\partial t} \leq \log \left(\frac{(\theta_t + dd^c \phi_t)^n}{\Omega}\right).$$

Thanks to Lemma 2.1, $\varphi_t \leq C_0$ with $C_0 > 0$ depends on $\sup_X \varphi_0$ and $T$. As we assume $s \mapsto F(\cdot, s)$ is increasing, $F(t, z, \varphi_t) \leq F(t, z, C_0)$. Replacing $\varphi_t$ by $\varphi_t - At$ and $F$ by $F - A$, where

$$A := \sup_{[0, T] \times X} F(t, z, C_0),$$

we can assume that

$$\sup_{[0, T] \times X} F(t, z, \sup_{[0, T] \times X} \varphi_t) \leq 0.$$
Hence we have
\[
\frac{\partial \varphi_t}{\partial t} = \log \left( \theta_t + \frac{\partial F}{\partial t}(t, z, \varphi_t) \right) - F(t, z, \varphi_t)
\geq \log \left( \omega + t \chi + \frac{\partial F}{\partial t}(t, z, \varphi_t) \right).
\]

Apply the comparison theorem (Proposition 1.5) for \( \varphi_t \) and \( \phi_t \) we have \( \varphi_t \geq \phi_t \). In general, we get
\[
\varphi_t(z) \geq \phi_t + At \geq (1 - 2\beta t)\varphi_0(z) + \alpha t u(z) + n(t \log t - t) + At.
\]

\[
\square
\]

2.2. Upper bound for \( \dot{\varphi}_t \). We now prove a crucial estimate which allows us to use the uniform version of Kolodziej’s estimates in order to get the bound of \( \text{Osc}_X \varphi_t \).

**Proposition 2.3.** Fix \( \varepsilon \in (0, T) \). There exists \( 0 < C = C(\sup_X \varphi_0, \varepsilon, T) \) such that for all \( \varepsilon \leq t \leq T \) and \( z \in X \),
\[
\dot{\varphi}_t(z) \leq -\varphi_{\varepsilon}(z) + C \leq -\frac{\phi_{\varepsilon}(z) + C}{t} - A,
\]
where \( \phi_t \) and \( A \) are as in Lemma 2.2.

**Proof.** We consider \( G(t, z) = t \dot{\varphi}_t - \varphi_t - nt + Bt^2/2 \), with \( B < \min F' \) on \([\varepsilon, T] \times X\). We obtain
\[
\frac{\partial G}{\partial t} = t \ddot{\varphi}_t - n = t \Delta_{\omega_t} \dot{\varphi} + t \text{tr}_{\omega_t} \theta_t - t \frac{\partial F}{\partial s} \dot{\varphi} - tF' - n + Bt,
\]
and
\[
\Delta_{\omega_t} G = t \Delta_{\omega_t} \dot{\varphi} = t \Delta_{\omega_t} \varphi_t - (n - \text{tr}_{\omega_t} \theta_t),
\]

hence
\[
\left( \frac{\partial}{\partial t} - \Delta_{\omega_t} \right) G = -t \dot{\varphi} \frac{\partial F}{\partial s} + t(B - F') - \text{tr}_{\omega_t}(\theta_t - t \theta_t).
\]

Since we assume that \( \theta_t - t \theta_t \geq 0 \) and \( B < \min F' \), we get
\[
\left( \frac{\partial}{\partial t} - \Delta_{\omega_t} \right) G < -t \dot{\varphi} \frac{\partial F}{\partial s}.
\]

If \( G \) attains its maximum at \( t = \varepsilon \), we have the result. Otherwise, assume that \( G \) attains its maximum at \((t_0, z_0)\) with \( t_0 > \varepsilon \), then at \((t_0, z_0)\) we have
\[
0 \leq \left( \frac{\partial}{\partial t} - \Delta_{\omega_t} \right) G < -t_0 \frac{\partial F}{\partial s} \dot{\varphi}.
\]

Since \( \frac{\partial F}{\partial s} \geq 0 \) by the hypothesis, we obtain \( \dot{\varphi}(t_0, z_0) < 0 \) and
\[
t \dot{\varphi}_t - \varphi_t - nt + Bt^2/2 \leq -\varphi_{t_0}(z_0) - nt_0 + Bt_0^2/2.
\]

Using Lemma 2.2 we get \( \varphi_{t_0} \geq \varphi_{\varepsilon} - C(\varepsilon) \), hence
\[
t \dot{\varphi}_t \leq \varphi_t - \varphi_{\varepsilon} + C_1.
\]
It follows from Lemma 2.1 that $\varphi_t \leq C_2(\sup \varphi_0, T)$, so

$$\dot{\varphi}_t(x) \leq -\frac{\varphi_\varepsilon + C}{t},$$

where $C$ depends on $\sup \varphi_0, \varepsilon, T$. Since $\varphi_\varepsilon \geq \phi_\varepsilon + At$ (Lemma 2.2), we obtain the desired inequality. \qed

2.3. Bounding the oscillation of $\varphi_t$. Once we get an upper bound for $\dot{\varphi}_t$ as in Proposition 2.3, we can bound the oscillation of $\varphi_t$ by using the uniform version of Kolodziej’s estimates. Indeed, observe that $\varphi_t$ satisfies

$$(\theta_t + dd^c \varphi_t)^n = H_t \Omega,$$

then by Proposition 2.3, for any $\varepsilon \in (0, T)$,

$$H_t = \exp(\dot{\varphi}_t + F) \leq \exp\left(-\frac{\varphi_\varepsilon + C}{t} + C'\right)$$

are uniformly in $L^2(\Omega)$ for all $t \in [\varepsilon, T]$ since $\dot{\phi}_\varepsilon$ is smooth. Thanks to the uniform version of Kolodziej’s estimates [Kol98, EGZ08], we infer that the oscillation of $\varphi_t$ is uniformly bounded:

**Theorem 2.4.** Fix $0 < t < T$. There exist $C(t) > 0$ independent of $\inf_X \varphi_0$ such that

$$\text{Osc}_X(\varphi_t) \leq C(t).$$

2.4. Lower bound for $\dot{\varphi}_t$. The next result is similar to [ST09, Lemma 3.2] and [GZ13, Proposition 3.3].

**Proposition 2.5.** Assume $\varphi_0$ is bounded. There exist constants $A > 0$ and $C = C(A, \text{Osc}_X \varphi_0) > 0$ such that for all $(x, t) \in X \times (0, T]$,

$$\dot{\varphi} \geq n \log t - A \text{Osc}_X \varphi_0 - C,$$

**Proof.** We consider $H(t, x) = \dot{\varphi}_t + A \varphi_t - \alpha(t)$, where $\alpha \in C^\infty(\mathbb{R}^+, \mathbb{R})$ will be chosen hereafter. We have

$$\frac{\partial H}{\partial t} = \ddot{\varphi}_t + A \dot{\varphi}_t - \dot{\alpha}$$

$$= \Delta_{\omega_t} \dot{\varphi}_t + \text{tr}_{\omega_t} \dot{\theta}_t - F' - \frac{\partial F}{\partial s} \dot{\varphi}_t + A \dot{\varphi}_t - \dot{\alpha},$$

and

$$\Delta_{\omega_t} H = \Delta_{\omega_t} \dot{\varphi}_t + A \Delta_{\omega_t} \varphi_t.$$ 

Therefore, we have

$$\left(\frac{\partial}{\partial t} - \Delta_{\omega_t}\right) H = A \dot{\varphi}_t + \text{tr}_{\omega_t} \dot{\theta}_t - A \text{tr}_{\omega_t}(\omega_t - \theta_t) - F' - \dot{\alpha} - \frac{\partial F}{\partial s} \dot{\varphi}_t$$

$$= A \dot{\varphi}_t + \text{tr}_{\omega_t}(A \theta_t + \dot{\theta}_t) - An - F' - \dot{\alpha} - \frac{\partial F}{\partial s} \dot{\varphi}_t$$

$$= (A - \frac{\partial F}{\partial s}) \dot{\varphi}_t + \text{tr}_{\omega_t}(A \theta_t + \dot{\theta}_t) - F' - \dot{\alpha} - An.$$
Now $A\dot{\theta} + \dot{\theta} \geq \omega$ with $A$ sufficiently large, hence
\[
\text{tr}_{\omega_t}(A\dot{\theta} + \dot{\theta}) \geq \text{tr}_{\omega_t} \omega.
\]
Using the inequality
\[
\text{tr}_{\omega_t}(\omega) \geq n \left( \frac{\omega_t^{n}}{\omega^n} \right)^{-1/n} = n \exp \left( \frac{-1}{n} (\dot{\phi} + F) \right) \left( \frac{\Omega}{\omega^n} \right)^{-1/n} \geq \sigma^{-1/n} h_t^{-1/n} \exp(- \sup_{[0,T] \times X} F(t, z, C_0)/n),
\]
where $h_t = e^{\dot{\phi}}$ and $C_0$ depends on $\sup X \varphi_0$, we have
\[
\text{tr}_{\omega_t}(A\dot{\theta} + \dot{\theta}) \geq \frac{h_t^{-1/n}}{C}.
\]
In addition, we apply the inequality $\sigma x > \log x - C_\sigma$ for all $x > 0$ with $x = h_t^{-1/n}$ and $\sigma \ll 1$ to obtain $\sigma h_t^{-1/n} = \sigma e^{-\dot{\phi}/n} > -\dot{\varphi}/n - C_\sigma$. Finally, we can choose $A$ sufficiently large and $\sigma > 0$ such that
\[
(A - \frac{\partial F}{\partial s})\dot{\phi} + \text{tr}_{\omega_t}(A\dot{\theta} + \dot{\theta}) \geq \frac{h_t^{-1/n}}{C_1} - C'_1.
\]
Since $|F'|$ is bounded by some constant $C(Osc_X \varphi_0) > 0$, we obtain
\[
\left( \frac{\partial}{\partial t} - \Delta_{\omega_t} \right) H > \frac{h_t^{-1/n}}{C_1} - \alpha'(t) - C_2,
\]
where $C_2$ depends on $Osc_X \varphi_0$.

We chose $\alpha$ such that $\alpha(0) = -\infty$. This insures that $H$ attains its minimum at $(t_0, z_0)$ with $t_0 > 0$. At $(t_0, z_0)$ we have
\[
C_1[C_2 + \alpha'(t_0)] \geq h_{t_0}^{-1/n}(z_0),
\]
hence
\[
H(t_0, z_0) \geq A\varphi_{t_0}(z_0) - \{ n \log[C_2 + \alpha'(t_0)] + \alpha(t_0) \}.
\]
From Lemma 2.1 we have $\varphi_{t_0} \leq \sup X \varphi_0 + C'$ we have
\[
\dot{\varphi} \geq \alpha(t) - A Osc_X \varphi_0 - C_3 - \{ n \log[C_2 + \alpha'(t_0)] + \alpha(t_0) \}.
\]
Choosing $\alpha(t) = n \log t$ we have
\[
n \log[C_2 + \alpha'] + \alpha \leq C_4,
\]
so obtain the inequality. □
2.5. Bounding the gradient of $\varphi$. In this section we bound the gradient of $\varphi$ using the same technique as in [SzT11, Lemma 4] (which is a parabolic version of Błocki’s estimate [Bł09]). In these articles $\theta_t = \omega$ is independent of $t$. We note that if one is interested in the special case of (twisted) Kähler-Ricci flows, then the gradient estimate is not needed.

**Proposition 2.6.** Fix $\varepsilon \in [0, T]$. There exists $C > 0$ depending on $\sup_X \varphi_0$ and $\varepsilon$ such that for all $\varepsilon \leq t \leq T$

$$|\nabla \varphi(z)|_\omega^2 < e^{C/t}. $$

**Proof.** Define

$$ K = t \log |\nabla \varphi|_\omega^2 - \gamma \circ \varphi = t \log \beta - \gamma \circ \varphi, $$

where $\beta = |\nabla \varphi|_\omega^2$ and $\gamma \in C^\infty(\mathbb{R}, \mathbb{R})$ will be chosen hereafter.

If $K$ attains its maximum for $t = \varepsilon$, $\beta$ is bounded in terms of $\sup_X \varphi_0$ and $\varepsilon$, since $|\varphi_t|$ is bounded by a constant depending on $\sup_X \varphi_0$ and $\varepsilon$ for all $t \in [\varepsilon, T]$ (Lemma 2.1 and Lemma 2.2).

We now assume that $K$ attains its maximum at $(t_0, z_0)$ in $[\varepsilon, T] \times X$ with $t_0 > \varepsilon$. Near $z_0$ we have $\omega = dd^c g$ for some smooth strongly plurisubharmonic $g$ and $\theta_t = dd^c h_t$ for some smooth function $h_t$, hence $u := h_t + \varphi$ is plurisubharmonic near $(t_0, z_0)$. We take normal coordinates for $\omega$ at $z_0$ such that

$$ g_{i\bar{k}}(z_0) = \delta_{jk}, \quad g_{i\bar{k}}(z_0) = 0, \quad u_{pq}(t_0, z_0) \text{ is diagonal}, $$

(2.4) (2.5) (2.6)

here we denote $\alpha_{j\bar{k}} := \frac{\partial^2 \alpha}{\partial z_j \partial \bar{z}_k}$.

We now compute $K_p, K_{pp}$ at $(t_0, z_0)$ in order to use the maximum principle. At $(t_0, z_0)$ we have $K_p = 0$ hence

$$ t\beta_p = \beta \gamma' \varphi_p $$

(2.7)

or

$$ \left(\frac{\beta_p}{\beta}\right)^2 = \frac{1}{t^2} (\gamma')^2 |\varphi_p|^2. $$

Therefore,

$$ K_{p\bar{p}} = t \frac{\beta_{p\bar{p}} - |\beta_p|^2}{\beta^2} - \gamma'' |\varphi_p|^2 - \gamma' \varphi_{p\bar{p}} $$

$$ = t \frac{\beta_{p\bar{p}}}{\beta} - [t^{-1} (\gamma')^2 + \gamma''] |\varphi_p|^2 - \gamma' \varphi_{p\bar{p}}. $$

Now we compute $\beta_p, \beta_{p\bar{p}}$ at $(t_0, z_0)$ with $\beta = g^{jk} \varphi_j \varphi_k$ where $(g^{jk}) = ([g_{jk}]^{-1})$. We have

$$ \beta_p = g^{jk} \varphi_j \varphi_k + \tilde{g}^{jk} \varphi_{jp} \varphi_k + \tilde{g}^{jk} \varphi_{j\bar{p}} \varphi_{k\bar{p}}. $$
At \((t_0, z_0)\), use (2.4), (2.5)

\[ g_p^{jk} = -g_p^{ji} g_{dp} g^{dk} = 0, \]

hence

\[ \beta_p = \sum \varphi_{jp} \varphi_j + \sum \varphi_{pj} \varphi_j, \quad (2.8) \]

and

\[ \beta_{pp} = g_{pp}^{jk} \varphi_j \varphi_k + 2 \Re \sum \varphi_{pp} \varphi_j + \sum |\varphi_{jp}|^2 + \sum |\varphi_{jp}|^2. \]

Note that

\[ R_{ijk} = -g_{ijk} + g^{si} g_{sk} g_{i\bar{i}}, \]

hence, at \((t_0, z_0)\) \( g_{jk}^{jk} = -g_{jkpp} = R_{jkpp} \), and

\[ \beta_{pp} = R_{jkpp} \varphi_j \varphi_k + 2 \Re \sum \varphi_{pp} \varphi_j + \sum |\varphi_{jp}|^2 + \sum |\varphi_{jp}|^2. \]

Now

\[ \Delta_{\omega_{t_0}} K = \sum_{p=1}^{n} \frac{K_{pp}}{u_{pp}}, \]

hence

\[ \Delta_{\omega_{t_0}} K = t \sum \frac{R_{jkpp} \varphi_j \varphi_k}{\beta u_{pp}} + 2 t \Re \sum \frac{\varphi_{pp} \varphi_j}{\beta u_{pp}} + t \sum \frac{|\varphi_{jp}|^2 + |\varphi_{jp}|^2}{\beta u_{pp}} \]

\[ - \left[ t^{-1} (\gamma')^2 + \gamma'' \right] \frac{|\varphi_p|^2}{u_{pp}} - \gamma' \frac{\varphi_{pp}}{u_{pp}}. \]

Since \( u_{pp} = \varphi_{pp} + h_{pp} \) near \((t_0, z_0)\), then at \((t_0, z_0)\)

\[ \sum \frac{\gamma' \varphi_{pp}}{u_{pp}} = n\gamma' - \sum \frac{\gamma' h_{pp}}{u_{pp}}. \]

Moreover, assume that the holomorphic bisectional curvature of \( \omega \) is bounded by a constant \( B \in \mathbb{R} \) on \( X \), then at \((t_0, z_0)\)

\[ t \sum \frac{R_{jkpp} \varphi_j \varphi_k}{\beta u_{pp}} \geq -B t \sum \frac{1}{u_{pp}}, \]

therefore

\[ \Delta_{\omega_{t_0}} K \geq (\gamma' - tB) \sum \frac{1}{u_{pp}} + 2 t \Re \sum \frac{\varphi_{pp} \varphi_j}{\beta u_{pp}} \]

\[ + \frac{t}{\beta} \sum \frac{|\varphi_{jp}|^2 + |\varphi_{jp}|^2}{\beta u_{pp}} - [t^{-1} (\gamma')^2 + \gamma''] \sum \frac{|\varphi_p|^2}{u_{pp}} - n\gamma' + \gamma' \sum \frac{t h_{pp}}{u_{pp}}. \]

By the maximum principle, at \((t_0, z_0)\)

\[ 0 \leq \left( \frac{\partial}{\partial t} - \Delta_{\omega} \right) K. \]
hence,

\[
0 \leq \log \beta - \gamma' \dot{\varphi} - (\gamma' - t B) \sum_p \frac{1}{u_{p\bar{p}}} + t \frac{\beta'}{\beta} - 2t \text{Re} \sum_{j,p} \frac{\varphi_{p\bar{j}} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}}
- \frac{t}{\beta} \sum_{j,p} \frac{|\varphi_{jp}|^2 + |\varphi_{j\bar{p}}|^2}{\beta u_{p\bar{p}}} + |t^{-1}(\gamma')^2 + \gamma''| \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}} + n \gamma'.
\] (2.9)

We will simplify 2.9 to get a bound for \( \beta \) at \((t_0, z_0)\). We now estimate

\[
t \frac{\beta'}{\beta} - 2t \text{Re} \sum_{j,p} \frac{\varphi_{p\bar{j}} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} \quad \text{and} \quad - \frac{t}{\beta} \sum_{j,p} \frac{|\varphi_{jp}|^2}{\beta u_{p\bar{p}}} + t^{-1}(\gamma')^2 \sum_p \frac{|\varphi_p|^2}{u_{p\bar{p}}}.
\]

For the first one, we note that near \((t_0, z_0)\)

\[
\log \det(u_{p\bar{q}}) = \dot{\varphi} + F(t, z, \varphi) + \log \Omega,
\]

hence using

\[
\frac{d}{ds} \det A = A^{\bar{i}j} \left( \frac{d}{ds} A_{i\bar{j}} \right) \det A
\]

we have at \((t_0, z_0)\)

\[
u^{p\bar{p}} u_{p\bar{j}} = \frac{u_{p\bar{j}}}{u_{p\bar{p}}} = (\dot{\varphi} + F(t, z, \varphi) + \log \Omega)_j.
\]

Therefore

\[
2t \text{Re} \sum_{j,p} \frac{\varphi_{p\bar{j}} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} = 2t \text{Re} \sum_{j,p} \frac{(u_{p\bar{j}} - h_{p\bar{j}}) \varphi_{\bar{j}}}{\beta u_{p\bar{p}}}
= \frac{2t}{\beta} \text{Re} \sum_{j,p} (\dot{\varphi} + F(t, z, \varphi) + \log \Omega)_j \varphi_{\bar{j}} - 2t \text{Re} \sum_{j,p} \frac{h_{p\bar{j}} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}}
= \frac{2t}{\beta} \text{Re} \sum_{j,p} (\dot{\varphi} j \varphi_{\bar{j}}) \varphi_{\bar{j}}
+ \frac{2t}{\beta} \text{Re} \left( (F(t, z, \varphi) + \log \Omega)_j + \frac{\partial F}{\partial r} \varphi_{\bar{j}} \right) \varphi_{\bar{j}}
- \frac{2t}{\beta} \text{Re} \sum_{j,p} \frac{h_{p\bar{j}} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}}.
\]

In addition, at \((t_0, z_0)\)

\[
t \frac{\beta'}{\beta} = \frac{t}{\beta} \sum g^{\bar{k}} \dot{\varphi}_j \varphi_{\bar{k}} + \varphi_j \dot{\varphi}_{\bar{k}}
= \frac{2t}{\beta} \text{Re}(\dot{\varphi}_j \varphi_{\bar{j}}),
\]

we infer that

\[
t \frac{\beta'}{\beta} - 2t \text{Re} \sum_{j,p} \frac{u_{p\bar{j}} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}} = -\frac{2t}{\beta} \text{Re} \sum_{j,p} (F(t, z, \varphi) + \log \Omega)_j \varphi_{\bar{j}} - \frac{2t}{\beta} \sum \frac{\partial F}{\partial s} |\varphi_j|^2
+ 2t \text{Re} \sum_{j,p} \frac{h_{p\bar{j}} \varphi_{\bar{j}}}{\beta u_{p\bar{p}}}.
\]
We may assume that \( \log \beta > 1 \) so that
\[
\frac{|\phi_j|}{\beta} < C
\]
By the hypothesis that \( \frac{\partial F}{\partial s} \geq 0 \) there exists \( C_1 \) depends on \( \sup |\phi_0| \) and \( C_2 \) depends on \( h \) and \( \varepsilon \) such that
\[
t^\beta \beta - 2t Re \sum_{j,p} \frac{u_{j\bar{p}} \phi_j}{\beta u_{p\bar{p}}} < C_1 t + C_2 t \sum \frac{1}{u_{p\bar{p}}}
\]
we now estimate
\[
-\frac{t}{\beta} \sum_{j,p} \frac{\phi_{jp}}{\beta u_{p\bar{p}}} + t^{-1}(\gamma')^2 \sum_p \frac{\phi_p}{u_{p\bar{p}}}.
\]
It follows from (2.7) and (2.8) that
\[
\beta_p = \sum \phi_{jp} \bar{\phi}_j + \sum \phi_{pj} \bar{\phi}_j,
\]
\[
t\beta_p = \beta' \phi_p
\]
then,
\[
\sum_j \phi_{jp} \bar{\phi}_j = (t^{-1} \gamma' - \phi_{pp}) \phi_p.
\]
Hence
\[
\frac{t}{\beta} \sum_{j,p} \frac{|\phi_{jp}|^2}{u_{p\bar{p}}} \geq \frac{t}{\beta^2} \sum_{j,p} \frac{|\sum \phi_{jp} \phi_j|^2}{u_{p\bar{p}}} = \frac{t}{\beta^2} \sum \frac{|t^{-1} \gamma' \beta + 1 - u_{pp}|^2 |\phi_p|^2}{u_{p\bar{p}}}
\]
\[
\geq t^{-1}(\gamma')^2 \sum \frac{|\phi_p|^2}{u_{p\bar{p}}} - C_3 \gamma',
\]
here \( C_3 \) depends on \( h \) and we assume \( \gamma' > 0 \).

We now choose
\[
\gamma(s) = As - \frac{1}{A} s^2
\]
with \( A \) so large that \( \gamma' > 0 \) and \( \gamma'' = -2/A < 0 \) for all \( s \leq \sup_{[0,T] \times X} \phi_t \).
From Lemma 2.5 we have \( \phi \geq C_0 + n \log t \), where \( C_0 \) depends on \( Ocs_X \phi_0 \). Combining this with (2.9), (2.10), (2.11) we obtain
\[
0 \leq -\frac{2}{A} \sum \frac{|\phi_p|^2}{u_{p\bar{p}}} - (\gamma' - Bt - C_2 t) \sum \frac{1}{u_{p\bar{p}}} + \log \beta + C_4 \gamma' + C_1 t,
\]
where \( C_1, C_2, C_4 \) depend on \( \sup_X |\phi_0| \), \( h_t \) and \( \varepsilon \). If \( A \) is chosen sufficiently large, we have a constant \( C_5 > 0 \) such that
\[
\sum \frac{1}{u_{p\bar{p}}} + \sum \frac{|\phi_p|^2}{u_{p\bar{p}}} \leq C_5 \log \beta,
\]
so we get \((u_{p\bar{p}})^{-1} \leq C_5 \log \beta\) for \(1 \leq p \leq n\). From (2.1) and (2.3) we have at \((t_0, z_0)\)
\[
\prod_p u_{p\bar{p}} = e^{-\varphi_t + F(t, x, \varphi_t)} \leq C_6,
\]
where \(C_6\) depends on \(\sup_X |\varphi_0|, \varepsilon\). Then we get
\[
u_{p\bar{p}} \leq C_6 (C_5 \log \beta)^{n-1},
\]
so from (2.12) we have
\[
\beta = \sum |\varphi_p|^2 \leq C_6 (C_5 \log \beta)^n,
\]
hence \(\log \beta < C_7\) at \((t_0, z_0)\). This shows that \(\beta = |\nabla \varphi(z)|^2 < e^{C/t}\) for some \(C\) depending on \(\sup |\varphi_0|\) and \(\varepsilon\). \(\square\)

**2.6. Bounding \(\Delta \varphi_t\).** We now use previous a priori estimates above to get a estimate of \(\Delta \varphi_t\). The estimate on \(|\nabla \varphi|^2\) is needed here, because \(F(t, z, \varphi)\) depends on \(\varphi\), in contrast with \([GZ13, DNL14]\).

**Lemma 2.7.** For all \(z \in X\) and \(s, t > 0\) such that \(s + t \leq T\),
\[
0 \leq t \log \text{tr}_\omega(\omega_{t+s}) \leq A\text{Osc}_X(\varphi_s) + C + [C - n \log s + A\text{Osc}_X(\varphi_s)] t
\]
for some uniform constants \(C, A > 0\).

**Proof.** We define
\[
P = t \log \text{tr}_\omega(\omega_{t+s}) - A\varphi_{t+s},
\]
and
\[
u = \text{tr}_\omega(\omega_{t+s})
\]
with \(A > 0\) to be chosen latter. We set \(\Delta_t := \Delta_{\omega_{t+s}}\). Now,
\[
\frac{\partial}{\partial t} P = \log u + t \frac{\dot{u}}{u} - A\dot{\varphi}_{t+s},
\]
\[
\Delta_t P = t \Delta_t \log u - A\Delta_t \varphi_{t+s}
\]
hence
\[
\left(\frac{\partial}{\partial t} - \Delta_t\right) P = \log u + t \frac{\dot{u}}{u} - A\varphi_{t+s} - t \Delta_t \log u + A\Delta_t \varphi_{t+s}. \tag{2.13}
\]
First, we have
\[
A\Delta_t \varphi_{t+s} = An - A\text{tr}_{\omega_{t+s}}(\theta_{t+s}) \leq An - \frac{A}{2} \text{tr}_{\omega_{t+s}}(\omega),
\]
and by Proposition 1.4
\[
-t \Delta_t \log u \leq B \text{tr}_{\omega_{t+s}}(\omega) + t \frac{\text{tr}_\omega(\text{Ric}(\omega_{t+s}))}{\text{tr}_\omega(\omega_{t+s})}.
\]
Moreover,
\[
\frac{t\dot{u}}{u} = t \frac{1}{u} \left[\Delta_\omega \left(\log \omega_{t+s}^n / \omega^n - \log \Omega / \omega^n - F(t, z, \varphi_{t+s})\right) + \text{tr}_\omega \dot{\theta}_t\right],
\]
\[
= t \frac{1}{u} \left[- \text{tr}_\omega(\text{Ric} \omega_{t+s}) + \text{tr}_\omega(\dot{\theta}_t + \text{Ric} \omega) - \Delta_\omega \left(F(t, z, \varphi) + \log \Omega / \omega^n\right)\right],
\]
with \( u = \text{tr}_\omega(\omega_{t+s}) \), and
\[
\text{tr}_{\omega_{t+s}}(\omega) \text{tr}_\omega(\omega_{t+s}) \geq n,
\]
we get
\[
-t \Delta_t \log u + \frac{t \dot{u}}{u} \leq (B + C_1)t \text{tr}_{\omega_{t+s}}(\omega) - \frac{t \Delta_\omega [F(t, z, \varphi) + \log \Omega/\omega^n]}{\text{tr}_\omega(\omega_{t+s})}. \tag{2.14}
\]
Now
\[
\Delta_\omega F(t, z, \varphi_{t+s}) = \Delta_\omega F(z, \varphi) + 2\text{Re} \left[ g^{jk} \left( \frac{\partial F}{\partial s} \right)_j \varphi_k \right] + \frac{\partial F}{\partial s} \Delta_\omega \varphi + \frac{\partial^2 F}{\partial s^2} \left| \nabla \varphi \right|^2.
\]
So there are constants \( C_2, C_3, C_4 \) such that
\[
\left| \Delta_\omega (F(t, z, \varphi_{t+s}) + \log \Omega/\omega^n) \right| \leq C_2 + C_3 \left| \nabla \varphi \right|^2 + C_4 \text{tr}_\omega \omega_{t+s}.
\]
Then we infer
\[
-t \Delta_t \log u \leq \frac{1}{n} \text{tr}_{\omega_{t+s}}(\omega)(C_2 + C_3 \left| \nabla \varphi \right|^2 + C_4),
\]
so from Lemma 2.6 and (2.14) we have
\[
-t \Delta_t \log u + \frac{t \dot{u}}{u} \leq (B + C_5)t \text{tr}_{\omega_{t+s}}(\omega) + C_6. \tag{2.15}
\]
From Lemma 1.3 and the inequality \((n - 1) \log x \leq x + C_n\),
\[
\log u = \log \text{tr}_\omega(\omega_{t+s}) \leq \log \left( n \left( \frac{\omega_{t+s}^n}{\omega^n} \right) \text{tr}_{\omega_{t+s}}(\omega)^{n-1} \right)
\]
\[
= \log n + \dot{\varphi}_{t+s} + F(t, z, \varphi) + (n - 1) \log \text{tr}_{\omega_{t+s}}(\omega)
\]
\[
\leq \dot{\varphi}_{t+s} + \text{tr}_{\omega_{t+s}}(\omega) + C_7.
\]
It follows from (2.13), (2.14) and (2.15) that
\[
\left( \frac{\partial}{\partial t} - \Delta_t \right) P \leq C_8 - (A - 1) \dot{\varphi}_{t+s} + [(B + C_5)t + 1 - A/2] \text{tr}_{\omega_{t+s}} \omega.
\]
We choose \( A \) sufficiently large such that \((B + C_5)t + 1 - A/2 < 0\). Applying Proposition 2.5,
\[
\left( \frac{\partial}{\partial t} - \Delta_t \right) P \leq C_8 - (A - 1)(n \log s - A\text{osc}_{X} \varphi_s - C).
\]
Now suppose \( P \) attains its maximum at \((t_0, z_0)\). If \( t_0 = 0 \), we get the desired inequality. Otherwise, at \((t_0, z_0)\)
\[
0 \leq \left( \frac{\partial}{\partial t} - \Delta_t \right) P \leq C_8 - (A - 1)(n \log s - A\text{osc}_{X} \varphi_s - C).
\]
Hence we get
\[
t \log \text{tr}_\omega(\omega_{t+s}) \leq A\text{osc}_{X}(\varphi_s) + C + [C - n \log s + A\text{osc}_{X}(\varphi_s)]t.
\]
\[\square\]
Corollary 2.8. For all \((t, x) \in (0, T] \times X\)

\[
0 \leq t \log \text{tr}_\omega(\omega_{t+s}) \leq 2AOsc_X(\varphi_{t/2}) + C'.
\]

2.7. Higher order estimates. For the higher order estimates, we can follow Székelyhidi-Tosatti [SzT11] by bounding

\[
S = g^{ij} \varphi_i \varphi_j g^{k\ell} \varphi_{ij k} \varphi_{\ell pq} \text{ and } |\text{Ric}(\omega_t)|\omega_t,
\]

then using the parabolic Schauder estimates in order to obtain bounds on all higher order derivatives for \(\varphi\). Besides we can also combine previous estimates with Evans-Krylov and Schauder estimates (Theorem 1.7) to get the \(C^k\) estimates for all \(k \geq 0\).

Theorem 2.9. For each \(\varepsilon > 0\) and \(k \in \mathbb{N}\), there exists \(C_k(\varepsilon)\) such that

\[
|||\varphi|||_{C^k([\varepsilon, T] \times X)} \leq C_k(\varepsilon).
\]

3. Proof of Theorem A

3.1. Convergence in \(L^1\). We approximate \(\varphi_0\) by a decreasing sequence \(\varphi_{0,j}\) of smooth \(\omega\)-psh functions (using [Dem92] or [BK07]). Denote by \(\varphi_{t,j}\) the smooth family of \(\theta_t\)-psh functions satisfying on \([0, T] \times X\)

\[
\frac{\partial \varphi_t}{\partial t} = \log \left( \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} \right) - F(t, z, \varphi)
\]

with initial data \(\varphi_{0,j}\).

It follows from the comparison principle (Proposition 1.5) that \(j \mapsto \varphi_{j,t}\) is non-increasing. Therefore we can set

\[
\varphi_t(z) := \lim_{j \to +\infty} \varphi_{t,j}(z).
\]

Thanks to Lemma 2.2 the function \(t \mapsto \sup_X \varphi_{t,j}\) is uniformly bounded, hence \(\varphi_t\) is a well-defined \(\theta_t\)-psh function. Moreover, it follows from Theorem 2.9 that \(\varphi_t\) is also smooth in \((0, T] \times X\) and satisfies

\[
\frac{\partial \varphi_t}{\partial t} = \log \left( \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} \right) - F(t, z, \varphi).
\]

Observe that \((\varphi_t)\) is relatively compact in \(L^1(X)\) as \(t \to 0^+\), we now show that \(\varphi_t \to \varphi_0\) in \(L^1(X)\) as \(t \searrow 0^+\).

First, let \(\varphi_{t_k}\) is a subsequence of \((\varphi_t)\) such that \(\varphi_{t_k}\) converges to some function \(\psi\) in \(L^1(X)\) as \(t_k \to 0^+\). By the properties of plurisubharmonic functions, for all \(z \in X\)

\[
\limsup_{t_k \to 0} \varphi_{t_k}(z) \leq \psi(z),
\]

with equality almost everywhere. We infer that for almost every \(z \in X\)

\[
\psi(z) = \limsup_{t_k \to 0} \varphi_{t_k}(z) \leq \limsup_{t_k \to 0} \varphi_{t_k,j}(z) = \varphi_{0,j}(z),
\]

by continuity of \(\varphi_{t,j}\) at \(t = 0\). Thus \(\psi \leq \varphi_0\) almost everywhere.
Moreover, it follows from Lemma 2.2 that
\[ \varphi_t(z) \geq (1 - 2\beta t)\varphi_0(z) + \alpha t u(z) + n(t \log t - t) + A t, \]
with \( u \) continuous, so
\[ \varphi_0 \leq \liminf_{t \to 0} \varphi_t. \]
Since \( \psi \leq \varphi_0 \) almost everywhere, we get \( \psi = \varphi_0 \) almost everywhere, so \( \varphi_t \to \varphi_0 \) in \( L^1 \).

We next consider some cases in which the initial condition is slightly more regular.

3.2. Uniform convergence. If the initial condition \( \varphi_0 \) is continuous then by Proposition 1.5 we get \( \varphi_t \in C^0([0, T] \times X) \), hence \( \varphi_t \) uniformly converges to \( \varphi_0 \) as \( t \to 0^+ \).

3.3. Convergence in capacity. When \( \varphi_0 \) is only bounded, we prove this convergence moreover holds in capacity (Definition 1.11). It is the strongest convergence we can expect in the bounded case (cf. [GZ05]). First, observe that it is sufficient to prove that \( u_t := \varphi_t + c(t) \) converges to \( \varphi_0 \) as \( t \to 0 \) in capacity, where \( c(t) \) satisfies \( \varphi_t + c(t) \geq \varphi_0 \) as in Proposition 2.2. Since \( \varphi_t \) converges to \( \varphi_0 \), so does \( u_t \), and we get
\[ \limsup_{t \to 0} u_t \leq \varphi_{0,j}, \]
for all \( j > 0 \), where \( (\varphi_{0,j}) \) is a family of smooth \( \omega \)-psh functions decreasing to \( \varphi_0 \) as in Section 3.1. It follows from Hartogs’ Lemma that for each \( j > 0 \) and \( \varepsilon > 0 \), there exists \( t_j > 0 \) such that
\[ u_t \leq \varphi_{0,j} + \varepsilon, \quad \forall 0 \leq t \leq t_j. \]
Therefore
\[ \text{Cap}_\omega (\{u_t > \varphi_0 + 2\varepsilon\}) \leq \text{Cap}_\omega (\{\varphi_{0,j} > \varphi_0 + \varepsilon\}), \]
for all \( t \leq t_j \). Since \( \varphi_{0,j} \) converges to \( \varphi_0 \) in capacity (Proposition 1.12), the conclusion follows.

3.4. Convergence in energy. Using the same notations as in Section 1.6 we get the following monotonicity property of the energy.

**Proposition 3.1.** Suppose \( \varphi_t \) is a solution of \( \text{CMAF} \) with initial data \( \varphi_0 \in \mathcal{E}^1(X, \omega) \). Then there exists a constant \( C \geq 0 \) such that \( t \mapsto E(\varphi_t) + C t \) is increasing on \( [0, T] \).

**Proof.** By computation we get
\[
\frac{dE(\varphi_t)}{dt} = \frac{1}{V} \int_X \dot{\varphi} t \omega_t + \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X \varphi_t \theta_t \wedge [j\theta_t + (n-j)\omega_t] \wedge \omega_t^j \wedge \theta_t^{n-j-1}.
\]
For the first term, we use the concavity of the logarithm to get
\[
\int_X \dot{\varphi} t \omega_t^n = \int_X \log \left( \frac{\omega_t^n}{e^F \Omega} \right) \frac{\omega_t^n}{V_t} \geq - \log \left( \frac{\int_X e^{F(t,z,\varphi_t)} \Omega}{V_t} \right) \geq - \log (C_0 \delta)
\]
where \( F(t, z, \varphi_t) \leq \log C_0 \) and

\[
V_t := \int_X \omega^n_t = \int_X \theta^n_t \geq \delta^{-1}V.
\]

For the second one, there is a constant \( A > 0 \) such that
\[
\dot{\theta}_t \leq A \theta_t
\]
for all \( 0 \leq t \leq T \). We note that
\[
\int_X \varphi_t(\theta_t + dd^c \varphi_t)^j \wedge \theta_t^{n-j} \leq \int_X \varphi_t(\theta_t + dd^c \varphi_t)^{j-1} \wedge \theta_t^{n-j+1},
\]
hence
\[
\frac{dE(\varphi_t)}{dt} \geq -C_1 + C_2 E(\varphi_t),
\]
for some \( C_1, C_2 > 0 \). By Lemma 2.2 we have
\[
E(\varphi_t) \geq C_3 E(\varphi_0) + C_3 \geq C_4
\]
Thus \( t \mapsto E(\varphi_t) + Ct \) is increasing on \([0, T]\) for some \( C > 0 \). □

**Proposition 3.2.** If \( \varphi_0 \in \mathcal{E}^1(X, \omega) \), then \( \varphi_t \) converges to \( \varphi_0 \) in energy as \( t \to 0 \).

**Proof.** It follows from Proposition 3.1 that \( \varphi_t \) stays in a compact subset of the class \( \mathcal{E}^1(X, \omega) \). Let \( \psi = \lim_{t_k \to 0} \varphi_{t_k} \) be a cluster point of \( \varphi_t \) as \( t \to 0 \). Reasoning as earlier, we have \( \psi \leq \varphi_0 \). Since the energy \( E(\cdot) \) is upper semi-continuous for the weak \( L^1 \)-topology (cf. [GZ07]), Proposition 3.1 and the monotonicity of Aubin-Yau energy functional yield
\[
E(\varphi_0) \leq \lim_{t_k \to 0} E(\varphi_{t_k}) \leq E(\psi) \leq E(\varphi_0),
\]
Therefore \( E(\psi) = E(\varphi_0) \), so \( \psi = \varphi_0 \) and we have \( \varphi_t \to \varphi_0 \) in energy. □

### 4. Uniqueness and Stability of Solution

We now prove the uniqueness and stability for the complex Monge-Ampère flow

\[
(CMAF) \quad \frac{\partial \varphi_t}{\partial t} = \log \frac{(\theta_t + dd^c \varphi_t)^n}{\Omega} - F(t, z, \varphi),
\]
where \( F(t, z, s) \in C^\infty([0, T] \times X \times \mathbb{R}, \mathbb{R}) \) with

\[
\frac{\partial F}{\partial s} \geq 0 \quad \text{and} \quad \left| \frac{\partial F}{\partial t} \right| \leq C',
\]
for some constant \( C' > 0 \).
4.1. **Uniqueness.** For the uniqueness and stability of solution we follow the approach of Di Nezza-Lu [DNL14]. The author thanks Eleonora Di Nezza and Hoang Chinh Lu for valuable discussion on the argument in [DNL14, Theorem 5.4].

Suppose $\varphi_t$ is a solution of

\[
\begin{aligned}
\frac{\partial \varphi}{\partial t} &= \log \left( \frac{\theta_t + \frac{dd^c \varphi}{\Omega}}{n} \right) - F(t, z, \varphi), \\
\varphi(0, \cdot) &= \varphi_0.
\end{aligned}
\tag{4.1}
\]

Consider

\[
\phi(t, z) = e^{At} \varphi \left( \frac{1 - e^{-At}}{A}, z \right),
\]

so $\phi_0 = \varphi_0$. Then

\[
\frac{\partial \phi_t}{\partial t} = \log \left( \frac{\tilde{\theta}_t + \frac{dd^c \phi}{\Omega}}{\tilde{A}} \right) + A\phi_t - H(t, z, \phi_t), \tag{4.2}
\]

where

\[
\tilde{\theta}_t = e^{At} \frac{1 - e^{-At}}{A},
\]

and

\[
H(t, z, \phi) = Ant + F(A^{-1}(1 - e^{-At}), z, e^{-At} \phi).
\]

Since

\[
\frac{\partial \tilde{\theta}_t}{\partial t} = Ae^{At} \frac{1 - e^{-At}}{A} + \frac{\tilde{\theta}_t}{\tilde{A}},
\]

we can choose $A$ so large that $\tilde{\theta}_t$ is increasing in $t$. Observe that the equation (4.1) has a unique solution if and only if the equation (4.2) has a unique solution. It follows from Lemma 2.2 that

\[
\varphi \geq \varphi_0 - c(t),
\]

where $c(t) \downarrow 0$ as $t \downarrow 0$, so for $\phi(t):

\[
\phi \geq \phi_0 - \alpha(t),
\]

with $\alpha(t) \downarrow 0$ as $t \downarrow 0$.

**Theorem 4.1.** Suppose $\psi$ and $\varphi$ are two solutions of (4.1) with $\varphi_0 \leq \psi_0$, then $\varphi_t \leq \psi_t$. In particular, the equation (4.1) has a unique solution.

**Proof.** Thanks to the previous remark, it is sufficient to prove $u \leq v$, where $u(t, z) = e^{At} \varphi \left( \frac{1 - e^{-At}}{A}, z \right)$, and $v(t, z) = e^{At} \psi \left( \frac{1 - e^{-At}}{A}, z \right)$.

Fix $\varepsilon \in (0, T)$, define

\[
\tilde{v}(t, z) = v_{t+\varepsilon} + \alpha(\varepsilon)e^{At} + n\varepsilon(e^{At} - 1).
\]
then \( \tilde{v}_0 \geq v_0 = \psi_0 \) and \( \tilde{v} \geq v_{t+\varepsilon} \). Since we choose \( A \) so large that \( \tilde{\theta}_t \) is increasing,
\[
\frac{\partial \tilde{v}}{\partial t} = \log \frac{(\tilde{\theta}_{t+\varepsilon} + dd^c v_{t+\varepsilon})^n}{\Omega} + A\tilde{v}_t - H(t, z, v_{t+s}) \\
\geq \log \frac{(\tilde{\theta}_t + dd^c \tilde{v}_t)^n}{\Omega} + A\tilde{v}_t - H(t, z, v_{t+s})
\]
Where
\[
H(t, z, v_{t+\varepsilon}) = 2A t - A n(t + \varepsilon) + F(A^{-1}(1 - e^{-A(t+\varepsilon)})), z, e^{-A(t+\varepsilon)} v_{t+\varepsilon})
\]
It follows from the monotonicity of \( F \) in the third variable that
\[
F(A^{-1}(1 - e^{-A(t+\varepsilon)}), z, e^{-A(t+\varepsilon)} v_{t+\varepsilon}) \leq F(A^{-1}(1 - e^{-A(t+\varepsilon)}), z, e^{-At} \tilde{v}_t)
\]
By the assumption \( \left| \frac{\partial F}{\partial s} \right| < C' \), we choose \( A > C' \) and get
\[
s \mapsto -A(t + s) + F(A^{-1}(1 - e^{-A(t+s)}), z, e^{-At} \tilde{v}_t)
\]
is decreasing. Thus
\[
H(t, z, v_{t+\varepsilon}) \leq A t + F(A^{-1}(1 - e^{-At}), z, e^{-At} \tilde{v}_t)
\]
and
\[
\frac{\partial \tilde{v}}{\partial t} \geq \log \frac{(\tilde{\theta}_t + dd^c \tilde{v}_t)^n}{\Omega} + A\tilde{v}_t - H(t, z, \tilde{v}_t).
\]
Therefore \( \tilde{v} \) is the supersolution of (4.2). It follows from Proposition 1.6 that \( u_t \leq \tilde{v}_t, \forall t \in [0, T] \). Letting \( \varepsilon \to 0 \), we get \( u_t \leq v_t \), so \( \varphi_t \leq \psi_t \).

\textbf{Remark 4.2.} For \( \theta_t(x) = \omega(x), \Omega = \omega^n, F(t, z, s) = -2|s|^{1/2} \) and \( \varphi_0 = 0 \), we obtain two distinct solutions to (CMAF), \( \varphi_t(z) \equiv 0 \) and \( \varphi_t(z) = t^2 \). Here \( \frac{\partial F}{\partial s} \) is negative and \( F \) is not smooth along \( (s = 0) \).

We now prove the following qualitative stability result:

\textbf{Theorem 4.3.} Fix \( \varepsilon > 0 \). Let \( \varphi_{0,j} \) be a sequence of \( \omega \)-psh functions with zero Lelong number at all points, such that \( \varphi_{0,j} \to \varphi_0 \) in \( L^1(X) \). Denote by \( \varphi_{t,j} \) and \( \varphi_j \) the solutions of (4.1) with the initial condition \( \varphi_{0,j} \) and \( \varphi_0 \) respectively. Then
\[
\varphi_{t,j} \to \varphi_t \text{ in } C^\infty([\varepsilon, T] \times X) \text{ as } j \to +\infty.
\]

\textbf{Proof.} Observe that we can use previous techniques in Section 2 to obtain estimates of \( \varphi_{t,j} \) in \( C^k([\varepsilon, T] \times X) \) for all \( k \geq 0 \). In particular, for the \( C^0 \) estimate, we need to have the uniform bound for \( H_{t,j} = \exp(\varphi_{t,j} + F) \) in order to use the uniform version of Kolodziej’s estimates [Kol98, EGZ08]. By Lemma 2.3 we have
\[
H_{t,j} = \exp(\varphi_{t,j} + F) \leq \exp \left( \frac{-\phi_0 + C}{t} + C' \right),
\]
where $C, C'$ depend on $\varepsilon, \sup_X \varphi_{0,j}$. Since $\varphi_{0,j}$ decreases to $\varphi_0$, we have the sup$_X \varphi_{0,j}$ is uniformly bounded in term of sup$_X \varphi_0$ for all $j$, so we can choose $C, C'$ to be independent of $j$. Hence there is a constant $A(t, \varepsilon)$ depending on $t$ and $\varepsilon$ such that $|H_{t,j}|_{L^2(L)}$ is uniformly bounded by $A(t, \varepsilon)$ for all $t \in [\varepsilon, T]$. By the Arzela-Ascoli theorem we can extract a subsequence $\varphi_{j_k}$ that converges to $\phi_t$ in $C^\infty([\varepsilon, T] \times X)$. Note that
\[
\frac{\partial \phi_t}{\partial t} = \log \frac{(\theta_t + dd^c \phi_t)^n}{\Omega} - F(t, z, \phi_t).
\]
We now prove $\phi_t = \varphi_t$. From Lemma 2.2 we get
\[
\varphi_{t,j_k} \geq (1 - \beta_t) \varphi_{0,j_k} - C(t),
\]
where $C(t) \searrow 0$ as $t \to 0$. Let $j_k \to +\infty$ we get $\phi_t \geq (1 - \beta_t) \varphi_0 - C(t)$, hence
\[
\lim_{t \to 0} \inf \phi_t \geq \varphi_0.
\]
It follows from Theorem 4.1 that $\phi_t \geq \varphi_t$. For proving $\phi_t \leq \varphi_t$, we consider $\psi_{0,k} = \left(\sup_{j \geq k} \varphi_{0,j}\right)^*$, hence $\psi_{0,k} \searrow \varphi_0$ by Hartogs theorem. Denote by $\psi_{t,k}$ the solution of (4.1) with initial condition $\psi_{0,j}$. It follows from Theorem 4.1 that
\[
\psi_{t,j} \geq \varphi_{t,j}.
\]
Moreover, thanks to the same arguments for proving the existence of a solution in Sections 2 and 3 by using a decreasing approximation of $\varphi_0$, we have that $\psi_{t,j}$ decreases to $\varphi_t$. Thus we infer that $\phi_t \leq \varphi_t$ and the proof is complete.

4.2. Quantitative stability estimate. In this section, we prove the following stability result when the initial condition is continuous.

**Theorem 4.4.** If $\varphi, \psi \in C^\infty((0, T] \times X)$ are solutions of (CMAF) with continuous initial data $\varphi_0$ and $\psi_0$, then
\[
||\varphi - \psi||_{C^k([\varepsilon, T] \times X)} \leq C(k, \varepsilon)||\varphi_0 - \psi_0||_{L^\infty(X)}.
\]

*Proof.* **Step 1.** It follows from Demailly’s approximation result (cf. [Dem92]) that there exist two sequences $\{\varphi_{0,j}\}, \{\psi_{0,j}\} \subset PSH(X, \omega) \cap C^\infty(X)$ such that
\[
\lim_{j \to \infty} ||\varphi_{0,j} - \varphi_0||_{L^\infty(X)} = 0 \quad \text{and} \quad \lim_{j \to \infty} ||\psi_{0,j} - \psi_0||_{L^\infty(X)} = 0.
\]
Denote by $\varphi_{t,j}, \psi_{t,j}$ solution of (CMAF) corresponding to initial data $\varphi_{0,j}, \psi_{0,j}$. Moreover, thanks to Theorem 4.3 we obtain
\[
\lim_{j \to \infty} ||\varphi_{t,j} - \varphi_t||_{C^k([\varepsilon, T] \times X)} = 0 \quad \text{and} \quad \lim_{j \to \infty} ||\psi_{t,j} - \psi_t||_{C^k([\varepsilon, T] \times X)} = 0.
\]
Thus it is sufficient to prove (4.3) with smooth functions $\varphi_0, \psi_0$.

**Step 2.** We now assume that $\varphi_0$ and $\psi_0$ are smooth. For each $\lambda \in [0, 1]$,
there is a unique solution \( \varphi^\lambda_t \in C^\infty((0,T] \times X) \) for the complex Monge-Ampère flow

\[
\begin{aligned}
\frac{\partial \varphi^\lambda_t}{\partial t} &= \log \left( \frac{\theta_t + dd^c \varphi^\lambda_t}{\Omega} \right) - F(t, z, \varphi^\lambda), \\
\varphi^\lambda(0,.) &= (1 - \lambda)\varphi_0 + \lambda \psi_0.
\end{aligned}
\]

(4.4)

By the local existence theorem, \( \varphi^\lambda \) depends smoothly on the parameter \( \lambda \).

We denote by \( \Delta^\lambda_t \) the Laplacian with respect to the Kähler form

\[
\omega^\lambda := \theta_t + dd^c \varphi^\lambda.
\]

Observe that

\[
\left( \frac{\partial}{\partial t} - \Delta^\lambda_t \right) \frac{\partial \varphi^\lambda}{\partial \lambda} = -\frac{\partial F}{\partial s} \frac{\partial \varphi^\lambda}{\partial \lambda},
\]

so

\[
\left( \frac{\partial}{\partial t} - \Delta^\lambda_t \right) u^\lambda_t + g_\lambda(t, z) u^\lambda_t = 0,
\]

(4.5)

where \( u^\lambda_t = \frac{\partial \varphi^\lambda}{\partial \lambda} \) and \( g_\lambda(t, z) = \frac{\partial F}{\partial s}(t, z, \varphi^\lambda) \geq 0 \). Moreover

\[
\psi_t - \varphi_t = \int_0^1 u^\lambda d\lambda,
\]

thus it is sufficient to show that

\[
\|u^\lambda_t\|_{C^k([\varepsilon,T] \times X)} \leq C(k, \varepsilon)\|u^\lambda_0\|_{L^\infty(X)} = C(k, \varepsilon)\|\psi_0 - \varphi_0\|_{L^\infty}.
\]

**Step 3.** It follows from Theorem 2.9 that for each \( k \geq 0 \),

\[
\|g_\lambda\|_{C^k([\varepsilon,T] \times X)} \leq C_1(k, \varepsilon) \quad \text{and} \quad \|\omega^\lambda_t\|_{C^k([\varepsilon,T] \times X)} \leq C_2(k, \varepsilon),
\]

for all \( \lambda \in [0,1] \). Using the parabolic Schauder estimates [Kry96, Theorem 8.12.1] for the equation (4.5) we get

\[
\|u^\lambda_t\|_{C^k([\varepsilon,T] \times X)} \leq C(k, \varepsilon)\|u^\lambda_0\|_{L^\infty(X)}.
\]

**Step 4.** Proving

\[
\|u^\lambda_t\|_{L^\infty(X)} \leq \|u^\lambda_0\|_{L^\infty(X)}.
\]

Indeed, suppose that \( u^\lambda \) attains its maximum at \( (t_0, z_0) \). If \( t_0 = 0 \), we obtain the desired inequality. Otherwise, by the maximum principle, at \( (t_0, z_0) \)

\[
0 \leq \left( \frac{\partial}{\partial t} - \Delta^\lambda_t \right) u^\lambda_t = -g_\lambda(t_0, z_0) u^\lambda_{t_0}.
\]

Since \( g_\lambda \geq 0 \), we get

\[
u^\lambda_t \leq \max \left\{ 0, \max_X u^\lambda_0 \right\}.
\]

Similarly, we obtain

\[
u^\lambda_t \geq \min \left\{ 0, \min_X u^\lambda_0 \right\},
\]

hence

\[
\|u^\lambda_t\|_{L^\infty(X)} \leq \|u^\lambda_0\|_{L^\infty(X)}.
\]
Finally,

\[ \|\varphi - \psi\|_{C^k([\varepsilon,T] \times X)} \leq \int_0^1 \|u^\lambda\|_{C^k([\varepsilon,T] \times X)} d\lambda \leq C(k,\varepsilon) \|\varphi_0 - \psi_0\|_{L^\infty(X)}. \]

The proof of Theorem B is therefore complete. \qed

5. Starting from a nef class

Let \((X,\omega)\) be a compact Kähler manifold. In [GZ13], the authors proved that the twisted Kähler-Ricci flow can smooth out a positive current \(T_0\) with zero Lelong numbers belonging to a nef class \(\alpha_0\). At the level of potentials it satisfies the Monge-Ampère flow

\[ \frac{\partial \varphi_t}{\partial t} = \log \left( \frac{(\theta_0 + t \omega + dd^c \varphi_t)^n}{\omega^n} \right) - F(t, z, \varphi_t), \]

where \(\theta_0\) is a smooth differential closed \((1,1)\)-form representing a nef class \(\alpha_0\) and \(\varphi_0 \in PSH(X,\theta_0)\) is a \(\theta_0\)-psh potential for \(T_0\), i.e. \(T_0 = \theta_0 + dd^c \varphi_0\). We prove here this is still true for more general flows we have considered:

**Theorem 5.1.** Let \(\theta_0\) be a smooth closed \((1,1)\)-form representing a nef class \(\alpha_0\) and \(\varphi_0\) be a \(\theta_0\)-psh function with zero Lelong number at all points. Set \(\varphi_t := \theta_0 + t \omega\). Then there exists a unique family \((\varphi_t)_{t \in [0,T]}\) of smooth \((\theta_t)\)-psh functions satisfying

\[ \frac{\partial \varphi_t}{\partial t} = \log \left( \frac{(\theta_t + dd^c \varphi_t)^n}{\omega^n} \right) - F(t, z, \varphi_t), \]

such that \(\varphi_t\) converges to \(\varphi_0\) in \(L^1\).

**Proof.** First, observe that for \(\varepsilon > 0\), \(\theta_0 + \varepsilon \omega\) is a Kähler form. Thanks to Theorem A, there exists a family \(\varphi_{t,\varepsilon}\) of \((\theta_0 + \varepsilon \omega)\)-psh functions satisfying

\[ \frac{\partial \varphi_{t,\varepsilon}}{\partial t} = \log \left( \frac{(\theta_0 + \varepsilon \omega + dd^c \varphi_{t,\varepsilon})^n}{\omega^n} \right) - F(t, z, \varphi_{t,\varepsilon}) \]

with initial data \(\varphi_0\) which is a \((\theta_0 + \varepsilon \omega)\)-psh function with zero Lelong numbers.

First, we prove that \(\varphi_{t,\varepsilon}\) is decreasing in \(\varepsilon\). Indeed, for any \(\varepsilon' > \varepsilon\)

\[ \frac{\partial \varphi_{t,\varepsilon'}}{\partial t} = \log \left( \frac{(\theta_t + \varepsilon' \omega + dd^c \varphi_{t,\varepsilon'})^n}{\omega^n} \right) - F(t, z, \varphi_{t,\varepsilon'}) \]

\[ \geq \log \left( \frac{(\theta_t + \varepsilon \omega + dd^c \varphi_{t,\varepsilon})^n}{\omega^n} \right) - F(t, z, \varphi_{t,\varepsilon}) \]

hence \(\varphi_{t,\varepsilon'} \geq \varphi_{t,\varepsilon}\) by the comparison principle (Proposition 1.5). Then we consider

\[ \varphi_t := \lim_{\varepsilon \to 0^+} \varphi_{t,\varepsilon}. \]

We now show that \(\varphi_t\) is bounded below (so it is not \(\to -\infty\)). Thanks to [GZ13, Theorem 7.1], there exist a family \((\phi_t)\) of \((\theta_0 + t \omega)\)-psh functions
such that
\[
\frac{\partial \phi_t}{\partial t} = \log \frac{(\theta_0 + t \omega + dd^c \phi_t)^n}{\omega^n}
\]
There is \(\sigma > 0\) such that \(\sigma^{-1} \omega^n \leq \Omega \leq \sigma \omega^n\), so we may assume that
\[
\frac{\partial \phi_t}{\partial t} \leq \log \frac{(\theta_0 + t \omega + dd^c \phi_t)^n}{\Omega}.
\]
Moreover, \(\varphi_{t, \varepsilon} \leq C\), where \(C\) only depends on \(\sup_X \varphi_0\), hence assume that \(F(t, z, \varphi_{t, \varepsilon}) \leq A\) for all \(\varepsilon\) small. Changing variables, we can assume that
\[
\frac{\partial \varphi_{t, \varepsilon}}{\partial t} \geq \log \frac{(\theta_0 + t \omega + dd^c \varphi_{t, \varepsilon})^n}{\Omega}.
\]
Using the comparison principle (Theorem 1.5) again, we get \(\varphi_{t, \varepsilon} \geq \phi_t\) for all \(\varepsilon > 0\) small, so \(\varphi_t \geq \phi_t\).

For the essential uniform bound of \(\varphi_t\), we use the method of Guedj-Zeriahi. For \(\delta > 0\), we fix \(\omega_\delta\) a Kähler form such that \(\theta_0 + \delta \omega = \omega_\delta + dd^c h_\delta\) for some smooth function \(h_\delta\). Our equation can be rewritten, for \(t \geq \delta\)
\[
(\omega_\delta + (t - \delta) \omega + dd^c (\varphi_t + h_\delta))^n = H_t \Omega \tag{5.3}
\]
where
\[
H_t = e^{\dot{\varphi_t} + F(t, x, \varphi_t)}
\]
are uniformly in \(L^2\), since
\[
\varphi_t \leq \frac{-\phi_\delta + C}{t} + C,
\]
for \(t \geq \delta\) as in Lemma 2.2. Kolodziej’s estimates now yields that \(\varphi_t + h_\delta\) is uniformly bounded for \(t \geq \delta\), so is \(\varphi_t\).

Now apply the arguments in Section 2 to the equation (5.3) we obtain the bounds for the time derivative, gradient, Laplacian and higher order derivatives of \(\varphi_t + h_\delta\) in \([\delta, T] \times X\). We thus obtain a priori estimates for \(\varphi_t\) which allow us get the existence of solution of (5.2) and the convergence to the initial convergence in \(L^1(X)\).

\[
\square
\]

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Institut Mathématiques de Toulouse,, Université Paul Sabatier, 31062 Toulouse cedex 09, France

E-mail address: tat-dat.to@math.univ-toulouse.fr