Abstract

We provide a direct proof of Agafonov’s theorem which states that finite state selection preserves normality. We also extend this result to the more general setting of shifts of finite type by defining selections which are compatible with the shift. A slightly more general statement is obtained as we show that any Markov measure is preserved by finite state compatible selection.

1 Introduction

Normality was introduced by Borel in [5] more than one hundred years ago to formalize the most basic form of randomness for real numbers. A number is normal to a given integer base if its expansion in that base is such that all blocks of digits of the same length occur in it with the same limiting frequency.

Although normality is a purely combinatorial property, it has close links with finite state machines. A fundamental theorem relates normality and finite automata: an infinite sequence is normal to a given alphabet if and only if it cannot be compressed by lossless finite transducers. These are deterministic finite automata with injective input-output behavior. This result was first obtained by joining a theorem by Schnorr and Stimm [16] with a theorem by Dai, Lathrop, Lutz and Mayordomo [9]. Becher and Heiber gave a direct proof in [1]. Another astonishing result is Agafonov’s theorem stating that selecting symbols in a normal sequence using a finite state machine preserves normality [1]. Agafonov’s publication [1] does not really include the proof but O’Connor [13] provided it using predictors defined from finite automata, and Broglio and Liardet [7] generalized it to arbitrary alphabets. Later Becher and Heiber gave another proof based of the characterization of normality by non-compressibility by lossless finite transducers [3]. In this paper, we provide a direct proof of Agafonov’s theorem. The proof is almost elementary but it still relies on Markov chains arguments.
The notion of normality has been extended to broader contexts like the one of dynamical systems and especially shifts of finite type [12]. When sofic shifts are irreducible and aperiodic, they have a measure of maximal entropy and a sequence is then said to be normal if the frequency of each block equals its measure. This extension to shifts meets the original aim of normality to study expansions of numbers in bases when the shift arises from a numerical systems like the \( \beta \)-shifts coming from the numeration in a non-integer base \( \beta \). Normality can be again interpreted as the good distribution of blocks of digits in the expansion of a number in a base \( \beta \). In this paper, we extend Agafonov’s theorem to the setting of shift of finite type. More precisely, we show that genericity for Markovian measure is preserved by selection with finite state state machines if the machines satisfy some compatibility condition with the measure. This result includes the case of shifts of finite type as their Parry measure is Markovian.

The paper is organized as follows. Section 2 is devoted to notation and main definitions. The link between selection and special finite-state machines called selectors is given in Section 3. Agafanov’s theorem is stated and proved in Section 4. The extension of the theorem to Markovian measures is given in Section 5. Note that the proof given that section subsumes the one given in the previous one. We keep both proofs since we think that the one in Section 4 is a nice preparation for the reader to the one in Section 5.

2 Preliminaries

2.1 Sequences, shifts and selection

We write \( \mathbb{N} \) for the set of all non-negative integers. Let \( A \) be a finite set. We let \( A^* \) and \( A^{\mathbb{N}} \) respectively denote the sets of all finite and infinite sequences over the alphabet \( A \). Similarly \( A^k \) stands for the set of sequences of length \( k \). Finite sequence are also called words. The empty word is denoted by \( \lambda \) and the length of a word \( w \) is denoted \( |w| \). The positions in finite and infinite words are numbered starting from 1. For a word \( w \) and positions \( 1 \leq i \leq j \leq |w| \), we let \( w[i] \) and \( w[i:j] \) denote respectively the symbol \( a_i \) at position \( i \) and the word \( a_i a_{i+1} \cdots a_j \) from position \( i \) to position \( j \). A word of the form \( w[i:j] \) is called a block of \( w \). A word \( u \) is a prefix (respectively suffix) of a word \( w \), denoted \( u \sqsubseteq w \), if \( w = uv \) (respectively \( w = vu \)) for some word \( v \).

For any finite set \( S \) we denote its cardinality with \( \#S \). We write \( \log \) for the base 2 logarithm.

In this article we are going to work on shift spaces, in particular shifts of finite type (SFT). Let \( A \) be a given alphabet. The full shift is the set \( A^{\mathbb{N}} \) of all (one-sided) infinite sequences \( (x_n)_{n \geq 1} \) of symbols in \( A \). The shift \( \sigma \) is the function from \( A^{\mathbb{N}} \) to \( A^{\mathbb{N}} \) which maps each sequence \( (x_n)_{n \geq 1} \) to the sequence...
$(x_{n+1})_{n \geq 1}$ obtained by removing the first symbol.

A shift space of $A^\mathbb{N}$ or simply a shift is a subset $X$ of $A^\mathbb{N}$ which is closed for the product topology and invariant under the shift operator, that is $\sigma(X) = X$. Let $F \subset A^*$ be a set of finite words called forbidden blocks. The shift $X_F$ is the subset of $A^\mathbb{N}$ made of sequences without any occurrences of blocks in $F$. More formally, it is the set

$$X_F = \{ x : x[m:n] \notin F \text{ for each } 1 \leq m \leq n \}.$$

It is well known that a shift $X$ is characterized by its forbidden blocks, that is $X = X_F$ for some set $F \subset A^*$. The shift $X$ is said to be of finite type if $X = X_F$ for some finite set $F$ of forbidden blocks [11, Def. 2.1.1]. Up to a change of alphabet, any shift space of finite type is the same as a shift space $X_F$ where any forbidden block has length 2, that is $F \subset A^2$.

For simplicity, we always assume that each forbidden block has length 2. In that case, the set $F$ is given by a $A \times A$-matrix $P = (p_{ab})_{a,b \in A}$ where $p_{ab} = 0$ if $ab \in F$ and $p_{ab} > 0$ otherwise and we write $X = X_P$. The shift $X$ is called irreducible if the graph induced by the matrix $P$ is strongly connected, that is, for each symbols $a, b \in A$, there exists an integer $n$ (depending on $a$ and $b$) such that $P^n_{ab} > 0$. The shift $X$ is called irreducible and aperiodic if there exists an integer $n$ such that $P^n_{ab} > 0$ for each symbols $a, b \in A$.

Example 1 (Golden mean shift). The golden mean shift is the shift space $X_F \subset \{0, 1\}^\mathbb{N}$ where the set of forbidden blocks is $F = \{11\}$. It is made of all sequences over $\{0, 1\}$ with no two consecutive 1. This subshift is also equal to $X_M$ where the matrix $M$ is given by $M = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Let $x = a_1a_2a_3 \cdots$ be a sequence over the alphabet $A$. Let $L \subseteq A^*$ be a set of finite words over $A$. The word obtained by oblivious prefix selection of $x$ by $L$ is $x \upharpoonright L = a_{i_1}a_{i_2}a_{i_3} \cdots$ where $i_1, i_2, i_3, \ldots$ is the enumeration in increasing order of all the integers $i$ such that the prefix $a_1a_2 \cdots a_{i-1}$ belongs to $L$. This selection rule is called oblivious because the symbol $a_i$ is not included in the considered prefix. If $L = A^*1$ is the set of words ending with a 1, the sequence $x \upharpoonright L$ is made of all symbols of $x$ occurring after a 1 in the same order as they occur in $x$.

2.2 Measures and genericity

A probability measure on $A^*$ is a function $\mu : A^* \to [0, 1]$ such that $\mu(\lambda) = 1$ and

$$\sum_{a \in A} \mu(aw) = \mu(w)$$

holds for each word $w \in A^*$. The simplest example of a probability measure is a Bernoulli measure. It is a monoid morphism from $A^*$ to $[0, 1]$ (endowed
with multiplication) such that \( \sum_{a \in A} \mu(a) = 1 \). Among the Bernoulli measures is the uniform measure which maps each word \( w \in A^* \) to \( (\#A)^{-|w|} \).

In particular, each symbol \( a \) is mapped to \( \mu(a) = 1/\#A \).

By the Carathéodory extension theorem, a measure \( \mu \) on \( A^* \) can be uniquely extended to a probability measure \( \hat{\mu} \) on \( A^\infty \) such that \( \hat{\mu}(wA^\infty) = \mu(w) \) holds for each word \( w \in A^* \). In the rest of the paper, we use the same symbol for \( \mu \) and \( \hat{\mu} \). A probability measure \( \mu \) is said to be (shift) invariant if the equality

\[
\sum_{a \in A} \mu(aw) = \mu(w)
\]

holds for each word \( w \in A^* \).

We now recall the definition of Markov measures. For a stochastic matrix \( P \) and a stationary distribution \( \pi \), that is a raw vector such that \( \pi P = \pi \), the Markov measure \( \mu_{\pi,P} \) is the invariant measure defined by the following formula \cite{10} Lemma 6.2.1.

\[
\mu_{\pi,P}(a_1a_2\cdots a_k) = \pi a_1 P_{a_1a_2} \cdots P_{a_{k-1}a_k}
\]

A measure \( \mu \) is compatible with a shift \( X_F \) if it only puts weight on blocks of \( X \), that is, \( \mu(w) > 0 \) implies \( w \notin F \) for each word \( w \). For a shift of finite type, there is a unique compatible measure with maximal entropy \cite{10} Thm. 6.2.20]. This measure is called the Parry measure and it is a Markov measure. This measure can be explicitly given as follows. Consider again the Parry measure of the golden mean shift. Consider again the golden mean shift \( X \). Its Parry measure is the Markov measure \( \mu_{\pi,P} \) where \( \pi \) is the distribution \( \pi = (\lambda^2/(1 + \lambda^2), 1/(1 + \lambda^2)) \) and \( P \) is the stochastic matrix \( P = \begin{pmatrix} 1/\lambda & 1/\lambda^2 \\ 1 & 0 \end{pmatrix} \) where \( \lambda \) is the golden mean.

Conversely, the support of an invariant measure \( \mu \) is the shift \( X_\mu = X_F \) where \( F \) is the set of words of measure zero, that is \( F = \{ w : \mu(w) = 0 \} \). If \( \mu \) is the Markovian measure \( \mu_{\pi,P} \), then its support \( X_\mu \) is a shift of finite type because it is equal to the shift \( X_P \) given by matrix \( P \).

We recall here the notion of normality and the notion of genericity. We start with the notation for the number of occurrences of a given word \( u \) within another word \( w \). For two words \( u \) and \( w \), the number \( |w|_u \) of occurrences of \( u \) in \( w \) is given by \( |w|_u = \# \{ i : w[i:i + |u| - 1] = u \} \). Borel’s definition \cite{5} of normality for a sequence \( x \in A^\infty \) is that \( x \) is normal if for each finite word \( w \in A^* \)

\[
\lim_{n \to \infty} \frac{|x[1:n]|_w}{n} = (\#A)^{-|w|}
\]
A sequence $x$ is called generic for a measure $\mu$ (or merely $\mu$-generic) if for each word $w \in A^*$
\[
\lim_{n \to \infty} \frac{|x[1:n]|_w}{n} = \mu(w)
\]
Normality is then the special case of genericity when the measure $\mu$ is the uniform measure. There are another definitions of normality and genericity taking into account only some occurrences, called aligned occurrences, of each word $w$. More precisely, the sequence $x$ is factorized $x = w_1 w_2 w_3 \cdots$ where $|w_i| = |w|$ for each $i \geq 1$ and it is required that the quotient $\#\{i \leq n : w_i = w\}/n$ converges to $\mu(w)$ when $n$ goes to infinity for each word $w$.

It is shown in [2] that the two notions coincide as long as the measure $\mu$ is Markovian.

3 Finite-state selection

In this section, we introduce the automata with output also known as transducers which are used to select symbols from a sequence. We consider deterministic transducers computing functions from sequences in a shift $X$ to sequences in a shift $Y$, that is, for a given input sequence $x \in X$, there is at most one output sequence $y \in Y$. We focus on transducer that operate in real-time, that is, they process exactly one input alphabet symbol per transition. We start with the definition of a transducer.

**Definition 3.** An input deterministic transducer $T$ is a tuple $(Q, A, B, \delta, I, F)$, where
- $Q$ is a finite set of states,
- $A$ and $B$ are the input and output alphabets, respectively,
- $\delta : Q \times A \to B^* \times Q$ is the transition function,
- $I \subseteq Q$ and $F \subseteq Q$ are the sets of initial and final states, respectively.

Input deterministic transducers are also called sequential in the litterature [16]. The relation $\delta(p, a) = (w, q)$ is written $p \overset{a}{\to} w q$ and the tuple $(p, a, w, q)$ is then called a transition of the transducer. A finite (respectively infinite) run is a finite (respectively infinite) sequence of consecutive transitions,
\[
q_0 \overset{a_1|v_1}{\to} q_1 \overset{a_2|v_2}{\to} q_2 \cdots q_n \overset{a_n|v_n}{\to} q_n.
\]
Its input and output labels are respectively $a_1 \cdots a_n$ and $v_1 \cdots v_n$. A finite run is written $q_0 \overset{a_1 \cdots a_n|v_1 \cdots v_n}{\to} q_n$. An infinite run is written $q_0 \overset{a_1a_2a_3\cdots|v_1v_2v_3\cdots}{\to} \infty$. An infinite run is accepting if its first state $q_0$ is initial. Note that there is no accepting condition. This is due to the fact that we always assume that the domain is a closed subset of $A^\mathbb{N}$.
x in $A^\mathbb{N}$. If the output label is the infinite sequence $y$, we write $y = \mathcal{T}(x)$. By a slight abuse of notation, we write $\mathcal{T}(x[m:n])$ for the output of $\mathcal{T}$ along that run while reading the block $x[m:n]$ of $x$. We always assume that all transducers are trim: each state occurs in at least one accepting run. Since transducers are input deterministic, the starting state and the input label determine the run and the ending state. For a state $p$ and a word $u$, we let $p \ast u$ and $p \cdot u$ denote respectively the run $p \xrightarrow{ui} q$ and its ending state $q$.

A selector is a deterministic transducer such that each of its transitions has one of the types $p \xrightarrow{a} q$ (type I), $p \xrightarrow{\lambda} q$ (type II) for a symbol $a \in A$. In a selector, the output of a transition is either the symbol read by the transition (type I) or the empty word (type II). Therefore, it can be always assumed that the output alphabet $B$ is the same as the input alphabet $A$.

It follows that for each run $p \xrightarrow{ui} q$, the output label $v$ is a subword, that is a subsequence, of the input label $u$.

![Figure 1: A selector](image)

A selector is oblivious if all transitions starting from a given state have the same type. The selector pictured in Figure 1 is not oblivious but the one pictured in Figure 2 is oblivious. The terminology is justified by the following relation between oblivious prefix selection and selectors. If $L \subseteq A^*$ is a rational set, the oblivious prefix selection by $L$ can be performed by an oblivious selector. There is indeed an oblivious selector $S$ such that for each input word $x$, the output $S(x)$ is the result $x \upharpoonright L$ of the selection by $L$. This selector $S$ can be obtained from any deterministic automaton $A$ accepting $L$. Replacing each transition $p \xrightarrow{a} q$ of $A$ by either $p \xrightarrow{a} q$ if the state $p$ is accepting or by $p \xrightarrow{\lambda} q$ otherwise yields the selector $S$. It can be easily verified that the obtained transducer is an oblivious selector performing the oblivious prefix selection by $L$. Conversely, each oblivious selector performs the oblivious prefix selection by $K$ where $K$ is the set of words being the input label of a run from the initial state to a state $q$ such that transitions starting from $q$ have type I.

The transducer pictured in Figure 2 is an oblivious selector that selects symbols occurring after a 1. It performs the oblivious prefix selection by $L$ where $L$ is the set $A^*1$ of words ending with a 1.

Some reasoning about transducers only involve the input labels of tran-
sitions and ignore the output labels. We call automaton a transducer where output labels of transitions are removed. This means that the transition function $\delta$ is then a function from $Q \times A$ to $Q$ where $Q$ is the state set and $A$ the alphabet.

4 Preservation of normality

In this section, we consider normality in the full shift. We give an alternative proof of Agafonov’s result [1] that finite state selection preserves normality. This means that if the sequence $x$ is normal and $L$ is a regular set of finite words, then the sequence $x \upharpoonright L$ is still normal. Since it has been remarked that selecting by a regular set is the same as using an oblivious selector, the result means that if $S$ is an oblivious selector and $x$ is normal, then $S(x)$ is also normal.

The following theorem of Agafonov states that oblivious prefix selection by a regular set preserves normality.

Theorem 4 (Agafonov [1]). If $x$ is normal and $L$ is regular, then $x \upharpoonright L$ is still normal.

The strategy of the proof is the following. We consider an oblivious selector $S$ performing selection by $L$. This means that if $x$ and $y$ are the input and output label of successful run, then $y = x \upharpoonright L$. We show then that if the input label $x$ a normal sequence, then the output of the run of $S$ is also normal. We fix a state $p$ of $S$ and an integer $\ell$. We show that for $k$ great enough, the number of runs starting from $p$ and outputting less than $\ell$ symbols is negligible. Then we show that for each words $w$ and $w'$ of length $\ell$, the number of runs outputting $w$ and $w'$ are almost the same. Finally, we show that all these runs of lengths $k$ starting from $p$ have the same frequency in a run whose input is a normal word.

The following lemma shows that the number of runs starting from a state $p$ and outputting a fixed word $w$ is not too large.

Lemma 5. Let $S$ be an oblivious selector. For each state $p$ of $S$, each integer $n \geq 0$, and word $w \in A^*$ such that $|w| \leq n$, there are at most $|A|^{n-|w|}$ runs $p \xrightarrow{uv} q$ of length $n$ such that $w$ is a prefix of the output label $v$. 

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Proof. The proof is carried out by induction on the integer \( n \). If \( n = 0 \), the only possible word is the empty word \( \lambda \). Since there is only one run of length 0, the inequality is satisfied. We now suppose that \( n \geq 1 \). Since the selector is oblivious, all transitions starting from state \( p \) have the same type, either type I or type II.

We first suppose that all transitions starting from state \( p \) have type I. Let us write \( w = aw' \) where \( a \) is a symbol and \( w' \) a word. Consider the transition \( p \xrightarrow{a} q \). All runs starting from \( p \) such that \( w \) is a prefix of the output label must use this transition as a first transition. Applying the induction hypothesis to \( q \), \( n-1 \) and \( w' \) gives the result.

We now suppose that all transitions starting from state \( p \) have type II, that is, have the form \( p \xrightarrow{a} \lambda q \) for each symbol \( a \). This implies that all runs of length \( n \) starting from \( p \) have an output label of length at most \( n-1 \). Therefore, if \( |w| = n \), there is no run such that \( w \) is prefix of its output label and the inequality is trivially satisfied. If \( |w| \leq n-1 \), applying the induction hypothesis to each \( q a \), \( n-1 \), and \( w \) gives that the number of runs starting from \( q a \) such that \( w \) is a prefix of their output label is at most \((\#A)^{n-1-|w|}\) Summing up all these inequalities for all \( q a \) gives the required inequality for \( p \).

Some of the bounds are obtained using the ergodic theorem for Markov chains [6, Thm 4.1]. For that purpose, we associate a Markov chain \( \mathcal{M} \) to each strongly connected automaton \( \mathcal{A} \). For simplicity, we assume that the state set \( Q \) of \( \mathcal{A} \) is the set \( \{1, \ldots, \#Q\} \). The state set of the Markov chain is the same set \( \{1, \ldots, \#Q\} \). The transition matrix of the Markov chain is the matrix \( P = (p_{i,j})_{1 \leq i,j \leq \#Q} \) where each entry \( p_{i,j} \) is equal to \( \#\{a : i \xrightarrow{a} j\}/\#A \). Note that \( \#\{a : i \xrightarrow{a} j\} \) is the number of transitions from \( i \) to \( j \). Since the automaton is assumed to be deterministic and complete, the matrix \( P \) is stochastic. If the automaton \( \mathcal{A} \) is strongly connected, the Markov chain is irreducible and it has therefore a unique stationary distribution \( \pi \) such that \( \pi P = \pi \). The vector \( \pi \) is called the distribution of \( \mathcal{A} \).

By a slight abuse of notation, we let \( |p * w|_q \) denote the number of occurrences of the state \( q \) in the finite run \( p * w \). The idea of the following lemma is borrowed from [16].

**Lemma 6.** Let \( \mathcal{A} \) be a strongly connected deterministic and complete automaton and let \( \pi \) be its distribution. For each real numbers \( \epsilon, \delta > 0 \), there exists an integer \( N \) such that for each integer \( n > N \)

\[
\#\{w \in A^n : \exists p, q \in Q \mid |p * w|_q/n - \pi_q| > \delta\} < \epsilon(\#A)^n
\]

**Proof.** The proof is a mere application of the ergodic theorem for Markov chains [6, Thm 4.1].

The following corollary is also borrowed from [16].
Corollary 7. Let \(A\) be a deterministic and strongly connected automaton and let \(\pi\) its distribution. Let \(\rho\) the run of \(A\) on a normal sequence \(x\). Then for each state \(q\)

\[
\lim_{n \to \infty} \frac{|\rho[1:n]|_q}{n} = \pi_q,
\]

where \(\rho[1:n]\) is the finite run made of the first \(n\) transitions of \(\rho\).

Proof. Since \(\sum_{q \in Q} \pi_q = 1\), it suffices to prove that \(\limsup_{n \to \infty} \frac{|\rho[1:n]|_q}{n} \geq \pi_q\) holds for each state \(q\).

Let \(\varepsilon > 0\) be a positive real number. Applying Lemma 6 with \(\delta = \varepsilon\) provides an integer \(k\) such that

\[
B = \{ w \in A^k : \exists p | p * w|_q/k - \pi_q| > \varepsilon \}
\]

has cardinality at most \(\varepsilon(\#A)^n\). The run \(\rho\) is then factorized

\[
\rho = p_0 \xrightarrow{w_0} p_1 \xrightarrow{w_1} p_2 \xrightarrow{w_2} p_3 \cdots = (q_0 * w_0)(q_1 * w_1)(q_2 * w_2) \cdots
\]

where each word \(w_i\) is of length \(k\) and \(x = w_0w_1w_2 \cdots\). Since \(x\) is normal, there is, by Theorem 4 in \([2]\), an integer \(N\) such that for each \(n > N\) the cardinality of the set \(\{ i < n : w_i = w \}\) is greater than \((1 - \varepsilon)n/(\#A)^k\) for each word \(w \in A^k\).

\[
\limsup_{n \to \infty} \frac{|\rho[1:n]|_q}{n} = \lim_{n \to \infty} \frac{|\rho[1:nk]|_q}{nk}
\]

\[
= \frac{1}{nk} \sum_{i=0}^{n-1} |q_i * w_i|_q
\]

\[
\geq \frac{1}{nk} \sum_{w \in A^k} \# \{ i < n : w_i = w \} \times \min_{p \in Q} |p * w|_q
\]

\[
\geq \frac{1}{nk} \sum_{w \in A^k \setminus B} ((1 - \varepsilon)n/(\#A)^k)(k(\pi_q - \varepsilon))
\]

\[
= (1 - \varepsilon)^2 (\pi_q - \varepsilon)
\]

Since this inequality holds for each real number \(\varepsilon > 0\), we have proved that \(\limsup_{n \to \infty} |\rho[1:n]|_q/n \geq \pi_q\).

Using the terminology of Markov chains, a strongly connected component (SCC) of an automaton is called recurrent if it cannot be left. This means that there is no transition \(p \xrightarrow{a} q\) where \(p\) is in that component and \(q\) is not. The following lemma is Satz 2.5 in \([10]\).

Lemma 8. Let \(A\) be an automaton and let \(\rho\) be a run of \(A\) on a normal input sequence. The run \(\rho\) reaches a recurrent SCC of \(A\).
The hypothesis that the input sequence is normal is stronger than what is required. It suffices that each block has infinitely many occurrences in the sequence.

**Lemma 9.** Let $S$ be a strongly connected selector. For each integer $k$ and each real number $\varepsilon > 0$, there exists an integer $N$ such that for each integer $n > N$, each state $p$ and each word $w$ of length $k$, the number of runs $p \xrightarrow{w|v} q$ of length $n$ such that $w$ is a prefix of the output label $v$ is between $(1 - \varepsilon)(\#A)^{n-|w|}$ and $(\#A)^{n-|w|}$.

**Proof.** Let $p$ be any state. The upper bound $(\#A)^{n-|w|}$ has been already proved in Lemma 5. It remains to prove the lower bound.

Let fix a state $q$ such that the transitions starting from $q$ are of type I. If no such state exists, all transitions of the selector outputs the empty word and the output label of any run is empty. Applying Lemma 6 with $\varepsilon/(\#A)^k$ and $\delta = \pi_q/2$ provides an integer $N_0$ such that for each $n > N_0$, the set $B = \{ u \in A^n : |p \rightarrow u_q/n - \pi_q| > \pi_q/2 \}$ has cardinality at most $\varepsilon(\#A)^{n-k}$. Fix now $N = \max(N_0, 2k/\pi_p)$ and let $n$ be such that $n > N$. If a word $u$ of length $n$ does not belong to $B$, the run $p \rightarrow u$ satisfies $|p \rightarrow u_q| > n\pi_q/2 \geq k$. This implies that the length of its output label is greater than $k$. Indeed, the state $q$ has at most $k + 1$ occurrences in the run and each transition starting from $q$ outputs one symbol.

Consider the $(\#A)^n$ runs of the form $p \rightarrow u$ for $u$ of length $n$. Among these runs, at most $\varepsilon(\#A)^{n-k}$ many of them do not have an output greater than $k$. For each $w' \neq w$, $w'$ is the prefix of the output label of at most $(\#A)^{n-k}$ many of them. It follows that $w$ is the prefix of the output label of at least $(1 - \varepsilon)(\#A)^{n-k}$ many of them. \(\square\)

Let $A$ be an automaton with state set $Q$. We now define and automaton whose states are the run of length $n$ in $A$. We let $A^n$ denote the automaton whose state set is $\{ p \rightarrow w : p \in Q, w \in A^n \}$ and whose set of transitions is defined by

$$\{(p \rightarrow bw) \rightarrow (q \rightarrow wa) : p \rightarrow q \text{ in } A, \ a, b \in A \text{ and } w \in A^{n-1}\}$$

The Markov chains associated with the automaton $A^n$ is called the *snake* Markov chains. See Problems 2.2.4, 2.4.6 and 2.5.2 (page 90) in [6] for more details. It is pure routine to check that the distribution $\xi$ of $A^n$ is given by $\xi_{p\rightarrow w} = \pi_p/(\#A)^n$ for each state $p$ and each word $w$ of length $n$.

**Proof of theorem.** Let $y$ be the output of the run of $S$ on $x$. By Lemma 8, the run of $S$ on $x$ reaches a recurrent SCC. Therefore it can be assumed without loss of generality that the selector $S$ is strongly connected.

Let $k$ be a fixed integer. We claim that for each word $w$ of length $k$

$$\lim_{n \rightarrow \infty} |y[1:n]|_w/n = 1/(\#A)^k.$$ 
With each occurrence of a word $w$ of
length $k$ in $y$, we associate the occurrence of the state $q$ in the run at which starts the transition that outputs the first symbol of $w$. Note that transitions starting from $q$ must be of type I. Conversely, with each occurrence in the run of such a state, we associate the block of length $k$ of $y$ starting from that position.

We fix a state $p$ such that transitions starting from $p$ have type I. We first claim that for each integer $n$ all runs of length $n$ starting from $p$ have the same frequency in the run. To prove this claim, we apply Corollary 7 to the automaton $A^n$ where $A$ is the automaton obtained by removing the outputs from $S$.

Let $\varepsilon > 0$ be a positive real number. By Lemma 9, there is an integer $n$ such that for each $w$ on length $k$, the number of run starting from $p$ outputting $w$ as their first $k$ symbols is between $(1 - \varepsilon)(A)^{n-k}$ and $(A)^{n-k}$. Combining this result with the fact that all these runs of length $n$ have the same frequency, we get that the frequency of each $w$ is between $(1 - \varepsilon)(A)^{n-k}$ and $(A)^{n-k}$. Since this is true for each $\varepsilon > 0$, all words of length $k$ have the same frequency after an occurrence of $p$. Since this is true for each state $p$, we get that all words of length $k$ have the same frequency in $y$. 

5 Genericity for Markov measures

In this section we extend Agafonov’s result to the more general setting of shifts of finite type. In this context, normality is defined through the Parry measure which is the unique invariant and compatible measure with maximal entropy. A sequence is said to be normal if it is generic for that measure. We actually prove a slightly stronger result by showing that genericity for any Markov measure is preserved by finite state selection as long as the selection is compatible with the measure. This includes the case of shifts of finite type because their Parry measure is Markovian.

To obtain such a result, the selection must be performed in a compatible way with the measure and its support. This boils down to putting some constraints on the selector to guarantee that if the input sequence is in the support of the measure, then the output sequence is also in that support. Insuring that the output is still in the support is not enough as it is shown by the following example. Consider the golden mean shift $X$ and the selector pictured in Figure 2. This selector selects symbols following a 1. If the input sequence $x$ is in $X$, the sequence $y$ of selected symbols is $0^N = 000 \cdots$ since $x$ has no consecutive 1s. Therefore, $y$ is always in $X$ but genericity is lost. To prevent this problematic behaviour, the selector is only allowed to select the next symbol if the last read symbol and the last selected symbol coincide. This restriction rules out the previous selector because it does satisfies this property.
We suppose that a Markov measure $\mu = \mu_{\pi, P}$ is fixed and we let $X_{\mu}$ be its support. We introduce automata and selectors which are compatible with the shift $X_{\mu}$. An automaton $A$ is compatible with $X_{\mu}$ if there exists a function $\iota$ from its state set $Q$ to $A$ such that the following condition is fulfilled.

i) If $p \xrightarrow{a} q$ is a transition of $A$, then $P_{\iota(p)a} > 0$ and $\iota(q) = a$.

The condition implies that all transitions arriving to a given state $q$ have the same label $\iota(q)$ and that the label of any path is in the shift $X_{\mu}$. Such an automaton is called $X_{\mu}$-complete if for each pair $(p, a)$ such that $P_{\iota(p)a} > 0$, there exists a transition $p \xrightarrow{a} q$ for some state $q$.

We continue by defining selectors which are compatible with $X_{\mu}$. A selector $S$ is compatible with $X_{\mu}$ if there exist two functions $\iota$ and $\eta$ from its state set $Q$ to the alphabet $A$ such that the following two conditions are fulfilled.

i) If $p \xrightarrow{a|\lambda} q$ is a transition of type I, then $P_{\iota(p)a} > 0$, $\iota(q) = \eta(q) = a$, and $\eta(p) = \iota(p)$.

ii) If $p \xrightarrow{a|\lambda} q$ is a transition of type II, then $P_{\iota(p)a} > 0$, $\iota(q) = a$ and $\eta(q) = \eta(p)$.

The condition $P_{\iota(p)a} > 0$ states that the selector can only read consecutive symbols with non-zero transition probability. The condition $\eta(p) = \iota(p)$ for the transition $p \xrightarrow{a|\lambda} q$ states the last read and last selected symbols must coincide for the selector to be able to select. The other conditions state that $\iota(q)$ is always the last read symbol, and that $\eta(q)$ is the last selected symbol if there is one and that it is equal to $\eta(p)$ otherwise.

![Figure 3: A selector compatible with the golden mean shift](image)

The selector pictured in Figure 3 is compatible with the golden mean shift. It selects symbols at even positions (starting from 1) if it is possible, that is, if the last read symbol and the last selected symbol coincide. The dashed edges are useless if the input sequence is in the golden mean shift. In that case, the output sequence is also in the golden mean shift. Each state is labelled by $prs$ where $p \in \{0, 1\}$ is the parity of the number of read
symbols so far, \( r \in \{0,1\} \) is the last read symbol and \( s \in \{0,1\} \) the last selected symbol. The two functions \( \iota \) and \( \eta \) can be defined by \( \iota(prs) = r \) and \( \eta(prs) = s \).

The following theorem states that selection with compatible selectors preserves genericity for Markov measures. The input sequence \( x \) must be assumed to be in the shift \( X_\mu \) because compatible selectors only read sequences from \( X_\mu \).

**Theorem 10.** Let \( \mu \) be a Markov measure and let \( x \) be a sequence in \( X_\mu \) which is \( \mu \)-generic. For each oblivious selector \( S \) compatible with \( X_\mu \), the output \( S(x) \) of \( S \) on \( x \) belongs to \( X_\mu \) and is \( \mu \)-generic.

The previous theorem can be applied to the Parry measure \( \mu \) of a shift \( X \) of finite type because the support of \( \mu \) is actually \( X_\mu = X \).

We start with the definition of the conditional measures induced by \( \mu \). For each symbol \( a \in A \), we let \( \mu_a \) denote the conditional measure defined by

\[
\mu_a(a_1a_2\cdots a_n) = P_{aa_1}P_{a_1a_2}\cdots P_{a_{n-1}a_n}.
\]

Note that the measures \( \mu_a \) might not be invariant. Since \( \pi \) is the stationnary distribution, the measure \( \mu \) can be recovered from the measures \( \mu_a \) by the formula \( \mu = \sum_{a \in A} \pi_a \mu_a \).

The following lemma shows that the set of runs starting from a state \( p \) and outputting a fixed word \( w \) is not too large. This is the analog of Lemma 5 in the context of Markov measures.

**Lemma 11.** Let \( S \) be an oblivious selector compatible with \( \mu \). For each state \( p \) of \( S \), each integer \( n \geq 0 \), and word \( w \in A^* \) such that \( |w| \leq n \), then the inequality \( \mu_{\iota(p)}(\{u \in A^n : p \xrightarrow{w} q \text{ and } w \subseteq v\}) < \mu_{\eta(p)}(w) \) holds.

**Proof.** Let \( U \) be the set \( \{u \in A^n : p \xrightarrow{w} q \text{ and } w \subseteq v\} \). The proof is carried out by induction on the integer \( n \). If \( n = 0 \), the set \( U = \{\lambda\} \) and \( w \) must be the empty word \( \lambda \). The inequality is then satisfied because both measures are equal to 1. We now suppose that \( n \geq 1 \). Since the selector is oblivious, all transitions starting from state \( p \) have the same type, either type I or type II. We distinguish two cases depending on the type of these transitions.

We first suppose that all transitions starting from state \( p \) have type I. Let us write \( w = aw' \) where \( a \) is a symbol and \( w' \) a word. Consider the transition \( p \xrightarrow{aw} p' \). The compatibility of \( S \) with \( \mu \) implies that \( \iota(p) = \eta(p) \) and \( \iota(p') = \eta(q) = a \). All runs starting from \( p \) such that \( w \) is a prefix of the output label must use this transition as a first transition. Applying the induction hypothesis to \( p' \), \( n - 1 \) and \( w' \) gives that \( \mu_a(U') < \mu_a(w') \) where \( U' = \{u \in A^{n-1} : p' \xrightarrow{w'} q \text{ and } w \subseteq v\} \). Since \( U = aU' \), the result follows from the equalities \( \mu_{\iota(p)}(U) = P_{\iota(p)a} \mu_a(U') \) and \( \mu_{\eta(p)}(w) = P_{\eta(p)a} \mu_a(w') \).
We now suppose that all transitions starting from state $p$ have type II, that is, have the form $p \xrightarrow{a|\lambda} p_a$ for each symbol $a$. The compatibility of $S$ with $\mu$ implies that $\iota(p_a) = a$ and $\eta(p_a) = \eta(p)$ for each $a \in A$. All runs of length $n$ starting from $p$ have an output label of length at most $n - 1$. Therefore, if $|w| = n$, there is no run such that $w$ is prefix of its output label and the inequality is trivially satisfied. If $|w| \leq n - 1$, applying the induction hypothesis to each $p_a$, $n - 1$ and $w$, gives that $\mu_a(U_a) < \mu_{\eta(p)}(w)$ where $U_a = \{u \in A^{n-1} : p_a \xrightarrow{u|\lambda} q$ and $w \sqsubseteq v\}$. Since $U = \bigcup_{a \in A} aU_a$, the result follows from the equalities $\mu_{\iota(p)}(U) = \sum_{a \in A} P_{\iota(p)a} \mu_a(U_a)$ and $\mu_{\eta(p)}(w) = \sum_{a \in A} P_{\iota(p)a} \mu_{\eta(p_a)}(w)$.

Some of the bounds are again obtained using the ergodic theorem for Markov chains [6, Thm 4.1]. For that purpose, we associate a Markov chain $M$ to each strongly connected automaton $A$ which is compatible with $X_\mu$ and $X_\mu'$-complete. This means that there is a function $\iota$ from $Q$ to $A$ such that if $p \xrightarrow{a} q$ is a transition, then $\iota(q) = a$. For simplicity, we assume that the state set $Q$ of $A$ is the set \{1, \ldots, \#Q\}.

The state set of the Markov chain is the same set \{1, \ldots, \#Q\}. The transition matrix of the Markov chain is the matrix $\hat{P} = (\hat{P}_{pq})_{1 \leq p, q \leq \#Q}$ where each entry $\hat{P}_{pq}$ is equal to $P_{\iota(p)a} = P_{\iota(p)\iota(q)}$ if $p \xrightarrow{a} q$ is a transition of $A$ and 0 otherwise. Since the automaton is assumed to be deterministic and $X_\mu'$-complete, the matrix $\hat{P}$ is stochastic. If the automaton $A$ is strongly connected, the Markov chain is irreducible and it has therefore a unique stationary distribution $\hat{\pi}$ such that $\hat{\pi} \hat{P} = \hat{\pi}$. The vector $\hat{\pi}$ is called the distribution of $A$. The matrix $\hat{P}$ and its stationary distribution $\hat{\pi}$ define a Markov measure $\hat{\mu} = \mu_{\hat{\pi}, \hat{P}}$ on finite runs of $A$. The link between the measures $\mu$ and $\hat{\mu}$ is that $\hat{\mu}(p \ast u) = \hat{\pi}_{\mu_{\iota(p)}}(u)$ for each state $p$ and each word $u$.

**Lemma 12.** Let $A$ be a strongly connected deterministic and complete automaton and let $\pi$ be its distribution. For each real numbers $\varepsilon, \delta > 0$, there exists an integer $N$ such that for each integer $n > N$

$$\mu \left( \{ u \in A^n : \exists p, q \in Q \ | \ |p \ast u|/n - \hat{\pi}_q | > \delta \} \right) < \varepsilon$$

The lemma is stated for the measure $\mu$ but the ergodic theorem is valid for any initial distribution. The result is therefore also valid for the conditional measures $\mu_a$.

**Proof.** The proof is a mere application of the ergodic theorem for Markov chains [6, Thm 4.1].

**Corollary 13.** Let $A$ be a deterministic and strongly connected automaton and let $\pi$ its distribution. Let $\rho$ be the run of $A$ on a $\mu$-generic sequence $x$. Then for each state $q$

$$\lim_{n \to \infty} \frac{|\rho[1:n]|_q}{n} = \hat{\pi}_q.$$
where $\rho[1:n]$ is the finite run made of the first $n$ transitions of $\rho$.

**Proof.** Since $\lim_{n \to \infty} |\rho[1:n]|_q/n = 1$, it suffices to prove that $\liminf_{n \to \infty} |\rho[1:n]|_q/n \geq \hat{\pi}_q$ holds for each state $q$.

Let $\varepsilon > 0$ be a positive real number. Applying Lemma 12 with $\delta = \varepsilon$ provides an integer $k$ such that

$$
\mu \left( \left\{ u \in A^k : \exists p \ |p \ast u|_q/k - \hat{\pi}_q| > \varepsilon \right\} \right) < \varepsilon.
$$

The run $\rho$ is then factorized

$$
\rho = p_0 \ u_0 \ p_1 \ u_1 \ p_2 \ u_2 \ \cdots = (p_0 \ast u_0)(p_1 \ast u_1)(p_2 \ast u_2) \ \cdots
$$

where each word $u_i$ is of length $k$ and $x = u_0 u_1 u_2 \ \cdots$. Since $x$ is $\mu$-generic, there is an integer $N$ such that for each $n > N$ the cardinality of the set $\{i < n : u_i = u\}$ is greater than $(1-\varepsilon)n\mu(u)$ for each word $u \in A^k$.

$$
\liminf_{n \to \infty} \frac{|\rho[1:n]|_q}{n} = \liminf_{n \to \infty} \frac{|\rho[1:nk]|_q}{nk} = \frac{1}{nk} \sum_{i=0}^{n-1} |p_i \ast u_i|_q \\
\geq \frac{1}{nk} \sum_{u \in A^k} \#\{i < n : u_i = u\} \times \min_{p \in Q} |p \ast u|_q \\
\geq \frac{1}{nk} \sum_{u \in A^k \setminus B} ((1-\varepsilon)n\mu(u))(k(\hat{\pi}_q - \varepsilon)) \\
= (1-\varepsilon)^2(\hat{\pi}_q - \varepsilon)
$$

Since this inequality holds for each real number $\varepsilon > 0$, we have proved that $\liminf_{n \to \infty} |\rho[1:n]|_q/n \geq \hat{\pi}_q$.

**Lemma 14.** Let $A$ be an automaton compatible with $\mu$ and let $\rho$ be a run in $A$ on a $\mu$-generic sequence in $X_\mu$. The run $\rho$ reaches a recurrent strongly connected component of $A$.

**Proof.** We claim that for each SCC $C$ which is not recurrent, there exists a word $w$ with $\mu(w) > 0$ and starting with a symbol $a$ such that from any state $q$ in $C$ such that $P_{(q)a} > 0$ the run $q \ast w$ leaves $C$.

We fix a symbol $a$. Let $\{q_1, \ldots, q_n\}$ be the set of states $q$ in $C$ such $P_{(q)a} > 0$. We construct a sequence $w_0, w_1, \ldots, w_n$ of words such that if $i \leq j$, then the run $q_i \ast w_j$ leaves $C$. We set $w_0 = \lambda$ and the statement is true. Suppose that $w_0, \ldots, w_k$ have been already chosen and consider the state $p_k = q_k \ast w_k$. If this state $p_k$ is already out of $C$, we set $w_{k+1} = w_k$. Otherwise, since $C$ is not recurrent, there is a word $v_k$ such that $p_k \ast v_k$ is out of $C$: we set $w_{k+1} = w_k v_k$ so that $q_{k+1} \ast w_{k+1} = p_k \ast v_k$ is out of $C$. 

\[ q_{k+1} \ast w_{k+1} = p_k \ast v_k \]
Fix now $N \in \mathbb{N}$ proved in Lemma 11. It remains to prove the lower bound. Let $\varepsilon \in \mathbb{R}$ word and the output label of any run is empty. Applying Lemma 12 with $\varepsilon > \eta \in \mathbb{R}$ and each real number $\varepsilon > \kappa$ is greater than $\kappa$ is less than $\kappa$. This is a contradiction because $\kappa + \varepsilon$ leaves $C$ while $C$ is supposed to be the last SCC reached by $\rho$. 

Lemma 15. Let $S$ be a strongly connected selector. For each integer $k$ and each real number $\varepsilon > 0$, there exists an integer $N$ such that for each integer $n \geq N$, each state $p$ and each word $w$ of length $k$, the inequalities $(1 - \varepsilon)\mu_{\eta(p)}(w) < \mu_{\eta(p)}(\{u \in A^n : p \xrightarrow{u} q \text{ and } w \subseteq v\}) < \mu_{\eta(p)}(w)$ hold.

Proof. Let $p$ be any state. The upper bound $\mu_{\eta(p)}(w)$ has been already proved in Lemma 12. It remains to prove the lower bound.

Let fix a state $q$ such that the transitions starting from $q$ are of type I. If no such state exists, all transitions of the selector outputs the empty word and the output label of any run is empty. Applying Lemma 12 with $\eta = \pi_q/2$ provides an integer $N_0$ such that for each $n > N_0$,

$$\mu_{\eta(p)}\left(\left\{u \in A^n : \left|\left|p \xrightarrow{u} q\right|\right| - \pi_q \right| > \pi_q/2\right\}\right) < \varepsilon \mu_{\eta(p)}(w).$$

Fix now $N = \max(N_0, 2k/\pi_q)$ and let $n$ be such that $n > N$. If a word $u$ of length $n$ does not belong to the small set above, the run $p \xrightarrow{u}$ satisfies $|p \xrightarrow{u} q| > n\pi_q/2 \geq k$ for each state $p$. This implies that the length of its output label is greater than $k$. Indeed, the state $q$ has at most $k + 1$ occurrences in the run and each transition starting from $q$ outputs one symbol.

Consider the $(\#A)^n$ runs of the form $p \xrightarrow{u}$ for $u$ of length $n$. The measure of those having an output smaller than $k$ is less than $\varepsilon \mu_{\eta(p)}(w)$. For each $w' \neq w$, the measure of those having $w'$ as prefix of length $k$ of their output label is at most $\mu_{\eta(p)}(w)$. It follows that the measure of those having $w$ as prefix of length $k$ of their output label is at most $(1 - \varepsilon)\mu_{\eta(p)}(w)$.

Let $\mathcal{A}$ be an automaton with state set $Q$. We now define an automaton whose states are the runs of length $n$ in $\mathcal{A}$. We let $A^n$ denote the automaton whose state set is $\{p \xrightarrow{u} q : p \in Q, u \in A^n\}$ and whose set of transitions is defined by

$$\{(p \xrightarrow{bu}) \xrightarrow{a} (q \xrightarrow{ua}) : p \xrightarrow{b} q \text{ in } \mathcal{A}, a, b \in A \text{ and } u \in A^{n-1}\}$$

The Markov chains associated with the automaton $A^n$ is called the snake Markov chains. See Exercises 2.2.4, 2.4.6 and 2.5.2 in [6] for more details. It is pure routine to check that the distribution $\xi$ of $\mathcal{A}^n$ is given by $\xi_{p\xrightarrow{w}} = \hat{\pi}_p \mu_{\eta(p)}(w)$ for each state $p$ and each word $w$ of length $n$. 

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Proof of theorem \[\Box\] Let \( y \) be the output of the run of \( S \) on \( x \). By Lemma \[14\] the run of \( S \) on \( x \) reaches a recurrent strongly connected component. Therefore it can be assumed without loss of generality that the selector \( S \) is strongly connected.

Let \( k \) be a fixed integer. We claim that for each word \( w \) of length \( k \),
\[
\lim_{n \to \infty} \frac{|y[1:n]|_w}{n} = \mu(w).
\]
With each occurrence of a word \( w \) of length \( k \) in \( y \), we associate the occurrence of the state \( q \) in the run from which starts the transition that outputs the first symbol of \( w \). Note that transitions starting from \( q \) must be of type I. Conversely, with each occurrence in the run of such a state, we associate the block of length \( k \) of \( y \) starting from that position.

We fix a state \( p \) such that transitions starting from \( p \) have type I. We first claim that for each integer \( n \), each run \( p \ast u \) of length \( n \) starting from \( p \) has a frequency of \( \mu_{\eta(p)}(w) \). To prove this claim, we apply Corollary \[13\] to the automaton \( A^n \) where \( A \) is the automaton obtained by removing the outputs from \( S \).

Let \( \varepsilon > 0 \) be a positive real number. By Lemma \[15\], there is an integer \( n \) such that for each \( w \) on length \( k \), the measure \( \mu_{\eta(p)} \) of all runs starting from \( p \) outputting \( w \) as their first \( k \) symbols is between \((1 - \varepsilon)\mu_{\eta(p)}(w)\) and \( \mu_{\eta(p)}(w) \).

Combining this result with the fact that each run \( p \ast u \) of length \( n \) occurs after state \( p \) with a frequency equal to \( \mu_{\eta(p)}(u) \), we get that the frequency of each word \( w \) is between \((1 - \varepsilon)\mu_{\eta(p)}(w)\) and \( \mu_{\eta(p)}(w) \). Since this is true for each \( \varepsilon > 0 \), all words of length \( k \) have a frequency after state \( p \) equal to its measure \( \mu_{\eta(p)}(w) \). Since this is true for each state \( p \), we get that each word of length \( k \) have a frequency equal to \( \mu(w) \) in \( y \). \[\Box\]

Conclusion

As a conclusion, we would like to mention a few extensions of our results. Agafanov’s theorem deals with prefix selection: a given digit is selected if the prefix of the word up to that digit belongs to a fixed set of finite words. Suffix selection is defined similarly: a given digit is selected if the suffix of the word from that digit belongs to a fixed set of sequences. It has been shown in \[3\] that suffix selection also preserves normality as long as the fixed set of sequences is regular. Let us recall that a set of sequences is regular if it can be accepted by non-deterministic Büchi or by a deterministic Muller automaton \[14\]. The proof given in \[3\] is based on the characterization of normality by non-compressibility. The proof techniques developed here to prove Agafanov’s theorem can be adapted to also prove directly the result about suffix selection.

The prefix and suffix selections considered so far are usually called oblivious because the digit to be selected is not included to either the prefix or the suffix taken into account. Non-oblivious does not preserve in gen-

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eral normality but it does for a restricted class of sets of finite words called
group languages. Group languages are sets of words which are accepted
by deterministic automata such that each symbol induces a permutation of
the states. This later property means that for each symbol \( a \), the function
which maps each state \( p \) to the state \( q \) such that \( p \xrightarrow{a} q \) is a permutation of
the state set. The techniques presented in this paper can also be adapted
to prove such a result.

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