A "saddle-node" bifurcation scenario for birth or destruction of a Smale–Williams solenoid

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Formation or destruction of hyperbolic chaotic attractor under parameter variation is considered with an example represented by Smale–Williams solenoid in stroboscopic Poincaré map of two alternately excited non-autonomous van der Pol oscillators. The transition occupies a narrow but finite parameter interval and progresses in such way that periodic orbits constituting a "skeleton" of the attractor undergo saddle-node bifurcation events involving partner orbits from the attractor and from a non-attracting invariant set, which forms together with its stable manifold a basin boundary of the attractor.

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I. INTRODUCTION

Uniformly hyperbolic strange attractors have strong chaotic properties and are structurally stable, i.e. insensitive to variations of functions and parameters in the dynamical equations [1, 2]. Formal examples are Smale–Williams solenoid, Plykin attractor, and some other mathematical constructions suggested mainly in 1960th–1970th [3–6]. Recently a number of physically realizable systems with hyperbolic attractors were proposed [10–13].

One promising direction of search for real-world non-linear dissipative systems with the hyperbolic chaotic attractors may be based on consideration of scenarios of their appearance under variation of control parameters. (Note that this is an issue certainly different from the commonly known scenarios of transition to chaos, like that of Feigenbaum, which lead to non-hyperbolic attractors.) Possible appearance of hyperbolic chaotic attractors was discussed by Ruelle and Takens in the context of the onset of turbulence [14, 15], but they did not consider concrete examples of dynamical equations demonstrating the phenomenon. More recently, Shilnikov and Turaev [16, 17] have suggested a scenario of emergence of attractor of Smale–Williams type in a kind of the so-called blue sky catastrophe. An example of dynamical equations manifesting this scenario was presented and studied numerically in Ref. [18].

In this paper we discuss an alternative scenario for the onset or destruction of attractor of Smale–Williams type associated with collision of two chaotic invariant sets, an attracting and a non-attracting one. Concretely, we consider the phenomenon in a physically realizable system composed of two alternately activated non-autonomous self-oscillators, which pass the excitation each other in such way that the phases of oscillations at successive stages of activity evolve chaotically in accordance with an expanding circle map (Bernoulli map) [10, 19]. In this system attractor of Smale–Williams type occurs in the stroboscopic Poincaré map. At certain parameters the uniformly hyperbolic nature of this attractor was verified in computations [10, 12, 19, 20]. Here we consider birth or disappearance of the chaotic attractor under variation of a parameter controlling a relative duration of the stages of activity and silence. The scenario may be thought of as a multitude of pairwise collisions of orbits relating to the attractor with those from some non-attracting invariant set happening in a narrow but finite parameter interval. In Appendix we shortly discuss a transparent example of a simple model map demonstrating qualitatively the same scenario of birth and destruction of chaotic attractor.

II. THE BASIC MODEL

The system we will study is governed by a set of differential equations [10, 19]

\[
\begin{align*}
\dot{x} &= \omega_0 u, \\
\dot{u} &= (h + A\cos 2\pi t - x^2)u - \omega_0 x + \frac{\varepsilon}{\omega_0} y \cos \omega_0 t, \\
\dot{y} &= 2\omega_0 v, \\
\dot{v} &= (h - A\cos 2\pi t - y^2)v - 2\omega_0 y + \frac{\varepsilon}{\omega_0} x^2,
\end{align*}
\]

slightly modified by introducing an additional parameter \(h\). The model is composed of two subsystems, the van der Pol oscillators with characteristic frequencies \(\omega_0\) and \(2\omega_0\). The dynamical variables \(x\) and \(y\) are the generalized coordinates of the oscillators while \(u\) and \(v\) are the normalized velocities. In each oscillator the parameter responsible for the birth of the limit cycle is forced to vary slowly with period \(T\) and amplitude \(A\), in opposite phases for the two subsystems, which become active turn by turn. The newly introduced parameter \(h\) controls the relative duration of the stages of activity and damping. The coupling between the subsystems characterized by coefficient \(\varepsilon\) is established in such special way that the excitation transfer between the subsystems is accompanied by doubling of the phase shift attributed to the os-
cillations on each next cycle of the transfer. Due to this, the stroboscopic map of the system (1) in a wide parameter range gives rise to a hyperbolic chaotic attractor of Smale–Williams type as it was verified in computations based on the cone criterion [12, 19] and on the statistics of angles between the stable and unstable manifolds [11, 12]. At a particular parameter set $h = 0$, $\omega_0 = 2\pi$, $T = 6$, $A = 5$, $\epsilon = 0.5$ the uniformly hyperbolic nature of the attractor was confirmed with a mathematically rigorous computer-assisted proof [20].

III. NUMERICAL RESULTS AND DISCUSSION

Let us start from the case $h = 0$, where the operation mode corresponding to the Smale–Williams attractor certainly takes place, and move to the domain $h < 0$ increasing the absolute value of $h$. Then, duration of the activity stages will decrease, and at some place the transferred excitation level becomes insufficient to recover the amplitude of the oscillations. Then, the system drops after some transient to the trivial stable state of zero amplitude. Figure 1 shows the Lyapunov exponents of the Poincaré map versus parameter $h$. The right-hand part of the plot corresponds to situation of existence of chaotic uniformly hyperbolic attractor of Smale–Williams type. A characteristic feature is that the largest Lyapunov exponent is positive, close to $\ln 2 \approx 0.693$ while others are negative, smoothly depending on the parameter. In fact, at $h < 0$ this attractor coexists with another one, the trivial stable zero state. (It is so because the coupling term in the last equation (1) has quadratic dependence on the dynamical variable; thus, with initial amplitude small enough, the system inevitably relaxes to zero state.) As seen from Fig. 1, with decrease of $h$, the breakdown of the chaotic attractor occurs at certain parameter value. As the remaining attractor is the fixed point in the origin, after the breakup of the chaotic attractor all trajectories tend there.

The sudden destruction of the chaotic attractor indicates the crises-like ("dangerous") character of the transition and makes one to suspect that the event is associated with a collision of two invariant sets (cf. [21]). To bring a qualitative basis for this observation let us develop some approximate approach to stroboscopic description of the dynamics. It is convenient to introduce polar coordinates on the plane of the dynamical variables of the first oscillator. We set \( \{x_0, x_1\} = \{x, u/0.9 - x/2\} \) and introduce the amplitude (or radial component) and the phase (or angular coordinate) in such way that $x_0 + i x_1 = r e^{i \varphi}$. (In definition of the variable change the coefficients are chosen to get the form of the attractor close to a circular one in the plane projection.) Now, we construct a one-dimensional amplitude map for the first oscillator as follows. Starting with zero amplitude of the second oscillator and assigning some amplitude $r$ and phase $\varphi$ to the first oscillator, we evaluate the amplitude $r_{\text{new}}$ after one period of the parameter modulation and plot it as a function of the initial amplitude. The constructed map is not so bad tool for sketchy description of the stroboscopic amplitude dynamics: since the subsystems are excited alternately, each epoch of activity for one of them corresponds to silence period for another one, whose amplitude in the rough approximation can be neglected there.

Figure 2 shows the plots for the amplitude map at four different values of $h$ as obtained from numerical integration of equations (1). Actually, instead of curves we get the widened strips on the panel (a); the reason is that the positions of the plotted points depend not only on amplitudes but also to some extent on the initial phases, varied from 0 to $2\pi$. Roughly, this widening measures a degree of incorrectness of the description in terms of the amplitude map. On the panel (b) this widening is excluded by means of averaging over the initial phases.

As seen from Fig. 2b, with variation of $h$ a tangent bifurcation occurs. At larger $h$ the map possesses three fixed points (on the plot they correspond to crossings of the curve with the bisector). One is located in the origin that is the stable stationary zero state $O$; two others correspond to some finite amplitudes. The fixed point $A$
at larger amplitude is stable while that with less amplitude $S$ is unstable. As value of $h$ decreases, the fixed points $A$ and $S$ approach each other, merge (approximately at $h \approx -1.23$) and then disappear. Of course, this is only a preliminary rough picture, and we must discuss now what happens actually nearby this parameter value in the original system.

The fixed point $A$ for the original system \cite{1} is associated with the Smale–Williams attractor: while the amplitude evolves almost periodically (with the period of the parameter modulation) the phases at successive periods behave in accordance with the chaotic Bernoulli map. Of the same kind is dynamics on a non-attracting invariant set $S$ belonging to the set. The only difference is the additional instability of orbits in this set in respect to the amplitude variations, as evident from Fig. 2b. Henceforth, we use the designations $A$ for the attractor and $S$ for the non-attracting invariant set of the original system rather than for the fixed points of the amplitude map. Figure 3 illustrates the phase dynamics on these sets with the computed iteration diagrams. The branches shown in black relate to the attractor, and the gray ones to the non-attractive invariant set. For the attractor the plot is obtained in straightforward way in the course of numerical integration of the equations \cite{10, 12, 19}: we evaluate the phases $\varphi_n = \text{arg}[x_0(t_n) + ix_1(t_n)]$ from the instant states for the first oscillator at $t_n = nT$ and plot $\varphi_{n+1}$ versus $\varphi_n$. To do the same thing for the non-attractive invariant set $S$ special measures are needed to keep the trajectory on it; actually, the picture was obtained from a collection of data for periodic orbits belonging to the set.

Accounting that the dynamics for the phases on both invariant sets $A$ and $S$ correspond to topologically equivalent Bernoulli maps, the orbits belonging to each of them are in one-to-one correspondence with a set of infinite two-symbol (binary) sequences.

It is well known that a chaotic attractor contains a dense set of unstable periodic orbits which constitutes a kind of "skeleton" for the attractor \cite{22}. In terms of symbolic dynamics, it corresponds to a class of the orbits associated with periodic binary sequences. This "skeleton" is a convenient object to comprehend the dynamics on the whole attractor $A$. The same is true for the non-attracting invariant set $S$, which possesses its own "skeleton" of unstable periodic orbits. The both sets are in one-to-one correspondence (as follows from identical symbolic dynamics): each periodic orbit on the attractor has a partner ("dual orbit") belonging to the invariant set $S$. Dealing with the "skeletons" it is possible to look over all periodic orbits (at least up to some reasonably large period) and analyze their bifurcations in dependence on the parameter $h$.

A periodic orbit of the stroboscopic map belonging to the attractor $A$ has a three-dimensional stable manifold and a one-dimensional unstable manifold (actually, the last one coincides with the whole attractor). The Floquet multiplier responsible for the instability may be evaluated as $\mu_0^A \approx 2^p$, where $p$ is the period of the orbit. The next multiplier, associated with the amplitude perturbation $\mu_2^A$ is less than 1. (It may be estimated as $|\tilde{f}'(x_A)|^p$, where $\tilde{f}$ is the function plotted in Fig. 2b, and $r_A$ designates the position of the stable fixed point.)

If the description in terms of the amplitude map of Fig. 2 be exact, the bifurcations would happen simultaneously for all pairs of the dual orbits, but actually it is not so: the bifurcation values $h_{\text{bif}}$ occupy a finite, although narrow interval. (Roughly, this interval $-1.22 < h < 1.24$ may be thought as associated with a finite width of the "curves" visible in the panel (a) of Fig. 2)

The drawn picture is supported by computations with Eqs. 1: we have traced the bifurcations for a large number of pairs of dual orbits associated with periodic symbolic codes. Table 1 collects the bifurcation values $h_{\text{bif}}$ for the orbits of period from 1 to 5 along with their binary codes.

According to the bifurcation theory \cite{23}, the fold bifurcation may be regarded as corresponding to a turning point of a curve, which represents a pair of periodic orbits in the extended space, which is the state space complemented with the coordinate axis of the parameter $h$. This interpretation reflects the fact that for $h > h_{\text{bif}}$ each saddle cycle embedded in the attractor has, as a counterpart, a dual saddle cycle from the non-attracting invariant set.

FIG. 3: Diagrams illustrating dynamics of phase on successive stages of excitation of the first oscillator on the attractor $A$ (black) and on the non-attracting invariant set $S$ (gray).
TABLE I: Bifurcation values of parameter $h$ for the unstable periodical orbits of the system (1) at $\omega_0 = 2\pi$, $A = 6.5$, $T = 6$, $\varepsilon = 0.5$.

| p | Symbolic codes | $h_{bif}$ |
|---|----------------|-----------|
| 1 | R              | -1.22583188 |
| 2 | LR             | -1.23358210 |
| 3 | LRR            | -1.22879290 |
|    | LLL            | -1.23489088 |
| 4 | LRRR           | -1.22693359 |
|    | LLRL           | -1.23226821 |
|    | RLLL           | -1.23371422 |
| 5 | LRRLR          | -1.23070842 |
|    | RRRRL          | -1.22628724 |
|    | LRRL           | -1.23012599 |
|    | RRLLL          | -1.23249362 |
|    | LLRRL          | -1.23263428 |
|    | RLLRL          | -1.23467205 |

At the bifurcation point they both merge and annihilate, or, to put it another way, the respective branches of the curve continue each other.

Figure 3a plots coordinates of points of the periodic orbits of periods $p \leq 10$ for the stroboscopic map of the system (1) versus parameter $h$ in the three-dimensional extended phase parameter space $\{x_0, x_1, h\}$. On this picture attractor corresponds to the outer part of the tube-like formation, while the non-attracting invariant set corresponds to the inner part. It looks as going inside out in order to be eventually transformed into the Smale–Williams attractor after passage of the turning points.

Figure 3b plots radial (amplitude) component for the same set of periodic orbits versus parameter $h$; it may be regarded as a longitudinal slice of the object of panel (a).

The structure observed at fixed $h = -1.15$ is illustrated in Fig. 5. Panel (a) represents, in fact, a transversal section of the object shown in Fig. 3. Observe two formations looking in rough approximation as closed circular curves but actually having fine transversal fractal structure. The outer one represents the chaotic attractor $A$, while the inner one is the non-attracting invariant set $S$ consisting of saddle orbits dual to those embedded in the attractor. It should be remembered, that the picture actually is a projection of the corresponding sets from the four-dimensional state space of the stroboscopic Poincaré map onto the plane. In Fig. 5 the three-dimensional version of the diagram is shown, and one may see more details of the fractal-like transversal structure of both sets $A$ and $S$. Evidently, the invariant set $S$ and its stable invariant manifold separate the basin of attraction of the Smale–Williams solenoid and that of the attractive fixed point in the origin. (In Fig. 5a the last one corresponds roughly to gray area inside the inner closed "curve".)

Objects similar to the invariant set $S$ are encountered in the theory of complex maps being known as Julia sets [24, 25]. This analogy is rather deep: far from the bifurcation transition, when the size of the set $S$ becomes small enough one can neglect all nonlinearities in the equations, beside that in the coupling term. Then, in a frame of description of the oscillators in terms of complex amplitudes, this set really turns to a Julia set for some complex map [26].

Qualitatively, the content of the considered scenario is illustrated by a simple artificially constructed model map in Appendix (in a similar way like one-dimensional maps illustrate the Feigenbaum or intermittent scenarios).

IV. CONCLUSION

Understanding scenarios for appearance of uniformly hyperbolic attractors under variation of control parameters seems relevant for a search for real-world systems with such attractors, which will be of special practical and theoretical interest because of structural stability of the generated chaos. In this article, basing on numerical computations, we reveal a nature of the bifurcation scenario of birth or destruction of a uniformly hyperbolic at-
FIG. 5: Attractor $A$ and non-attracting invariant set $S$ in the stroboscopic Poincaré section shown on the plane of variables of the first oscillator (a) and in three-dimensional plot (b) accounting a variable of the second oscillator at $\omega_0 = 2\pi$, $A = 6.5$, $T = 6$, $\varepsilon = 0.5$, and $h = -1.15$. The gray color on the panel (a) indicates roughly the basin of attraction of the stable fixed point in the origin.

Attractor of Smale–Williams type in stroboscopic Poincaré map of two alternately excited non-autonomous van der Pol oscillators under variation of a parameter controlling relative duration of stages of activity and silence of the oscillators. This is the second discussed in the literature scenario relating to formation of a uniformly hyperbolic attractor, after that of Shil’nikov and Turaev [16–18] corresponding to a kind of the so-called blue-sky catastrophe. In our scenario, the birth or destruction of the Smale–Williams solenoid appears not as a single bifurcation event, but occupies a finite parameter interval; it may be thought as a multitude of saddle-node bifurcations each involving a pair of orbits, one from the attractor, and another from the non-attracting invariant set.

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Appendix

It is a good way to illustrate the considered scenario with a simple model. Let us start with the equation:

$$z_{n+1} = \frac{Rz_n^2}{\sqrt{1 + |z_n|^4}},$$

where $z$ is a complex variable, and $R$ is a real parameter. Setting $z = re^{i\phi}$ we rewrite the map as

$$r_{n+1} = R\frac{r_n^2}{\sqrt{1 + r_n^4}}, \quad \phi_{n+1} = 2\phi_n \pmod{2\pi}. \quad (3)$$

At small $R$ the map for the radial variable $r$ has only a single fixed point at zero. With increase of $R$ at some instant ($R = \sqrt{2}$), a tangent bifurcation occurs accompanied with appearance of a pair of fixed points $r_{1,2} = \sqrt{\frac{1}{2}R^2 \pm \sqrt{\frac{1}{4}R^4 - 1}}$, one stable and another unstable (see Fig. 6). For the complex map the stable fixed point corresponds to attractor $A$ placed on a circle of radius $r_2$ while the angular coordinate is governed by the Bernoulli map $\phi_{n+1} = 2\phi_n \pmod{2\pi}$. The unstable point corresponds to an unstable invariant curve that is a circle of radius $r_1$ on which the angular variable evolves in accordance with the same Bernoulli map, and this curve is a boundary separating basins of two coexisting attractors, the stable point $O$ and the attractor $A$.

Now, consider a slightly modified version of the map

$$z_{n+1} = \frac{Rz_n(z_n + \varepsilon)}{\sqrt{1 + |z_n(z_n + \varepsilon)|^2}}, \quad (4)$$

which reduces to (2) at $\varepsilon = 0$, and look what happens under increase of parameter $R$ with $\varepsilon \neq 0$. At small $R$ the unique attractor is a zero fixed point $O$. For $R$ great enough there is another attractor $A$, situated in the region of relatively large $|z|$. Concerning dynamics on this attractor, the additional terms in the map (4) do not violate the nature of the dynamics of the angular variable, which follows a map of the same topological type as the Bernoulli map. However, now the invariant set $A$ does not coincide precisely with the circle, but acquires a transversal split and looks similar to a projection of the Smale–Williams solenoid. The border between the basins of attractors $O$ and $A$ is represented now by some
non-trivial invariant set $S$, which may be computed using backward iterations of the map (1):

$$z_n = -\frac{\varepsilon}{2} \pm \sqrt{\frac{\varepsilon^2}{4} + \frac{z_{n+1}}{\sqrt{R^2 - |z_{n+1}|^2}}}$$

(5)

where a sign at the square root is selected on each next iteration randomly. Visually, the set $S$ looks similar to a Julia set of complex quadratic map. (It is not surprising: neglecting the nonlinear term in the denominator of (1) we get just the complex analytic map $z_{n+1} = R(z_n^2 + \varepsilon z_n)$ providing a reasonable approximation in the domain of location of the set $S$ while $|z|$ is small.)

Dynamics on the sets $A$ and $S$ may be described with the two-symbol Bernoulli scheme. (For the set $S$ it is obvious from the above algorithm of backward iterations: a symbolic sequence of an orbit corresponds to the sequence of signs selected in the course of the procedure.)

With decrease of parameter $R$ a size of the attractor $A$ reduces while a size of the set $S$ grows (Fig. 7). As the sets $A$ and $S$ approach each other, the transverse structure of $S$ develops in the radial direction, and becomes less akin to the original Julia set. For some $R = R_1$ a touch of $A$ and $S$ happens, and it corresponds to the moment of crisis of the attractor $A$ (the touch of the basin boundary, cf. [21]).

Each of the sets $A$ and $S$ contains a dense set of unstable periodic orbits. A periodic orbit on the attractor $A$ has a multiplier $\mu_1^A \approx 2p > 1$, where $p$ is a period of the orbit, and a multiplier $\mu_2^A < 1$. For an orbit on the invariant set $S$ the multipliers are $\mu_1^S \approx 2p > 1$ and $\mu_2^S > 1$.

As symbolic representations of the orbits in $A$ and $S$ are of the same type, each orbit in $A$ can be related to a partner dual orbit in $S$ and vice versa. With decreasing $R$ the dual orbits approach each other, then merge and disappear via the fold (tangent) bifurcation, where $\mu_2^S = \mu_2^A = 1$. For different periodic orbits this bifurcation occurs at different values of $R$, so that the process of disappearance of orbits remaining on the set $A$ occupies a finite parameter interval $[R_2, R_1]$. Only in the degenerate case $\varepsilon = 0$, all orbits belonging to the sets $A$ and $S$ undergo the bifurcations simultaneously, just at $R = \sqrt{2}$.

The above consideration is in clear qualitative correspondence with that in the main part of the article. Some inconsistence is that the model map (1) is non-invertible (in contrast to the Poincaré map of the system (1)). This issue could be repaired with a Hénon-like modification of the model: $z_{n+1} = \frac{Rz_n(z_n + \varepsilon)}{\sqrt{1 + |z_n(z_n + \varepsilon)|^2}} - bz_{n-1}$, but in this case the analysis and explanations become much less transparent. (We can keep up appearances saying that we deal with this invertible map, but assign exclusively small value to the parameter $b$.)
FIG. 7: Mutual disposition of attractors $A$ and $O$ and of a non-attractive invariant set ($S$) of the map (4) on the plane of complex variable $z$ for different values of parameter $R$ at $\varepsilon = 0.2$.

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