Bénard-Taylor ferroconvection with time-dependent sinusoidal boundary temperatures

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Abstract. The combined effect of centrifugal acceleration and time-varying boundary temperatures on the onset of convective instability of a rotating magnetic fluid layer is investigated by means of the regular perturbation method. A perturbation expansion in terms of the amplitude of applied temperature field is implemented to effectively deal with the effects of temperature modulation. The criterion for the threshold is established based on the condition of stationary instability manifesting prior to oscillatory convection. The modulated critical Rayleigh number is computed in terms of Prandtl number, magnetic parameters, Taylor number and the frequency of thermal modulation. It is shown that subcritical motion exists only for symmetric excitation and the destabilizing effect of magnetic mechanism is perceived only for asymmetric and bottom wall excitations. It is also delineated that, for bottom wall modulation, rotation tends to stabilize the system at low frequencies and the opposite is true for moderate and large frequencies. Furthermore, it is established that, notwithstanding the type of thermal excitation, the modulation mechanism attenuates the influences of both magnetic stresses and rotation for moderate and large frequencies.

1. Introduction
Ferromagnetic convective instability has been one of the subjects of inordinate interest because of their engrossing physical properties and noteworthy commercial applications (Popplewell [1], Berkovskii et al [2] and Horng et al [3]). Ferrofluids are also used in a variety of thermoelectric cooling modules which prove instrumental for the refrigeration of semiconductor process equipment, laser diodes, medical treatment and optical communication equipment. A comprehensive research work on ferroconvection was first carried out by Finlayson [4] who carefully explicated how a non-uniform magnetic body force resulting from magnetization leads to ferroconvection. Gotoh and Yamada [5], Stiles and Kagan [6] and Russell et al [7] extended the pioneering work of Finlayson [4] to deal with the influence of a strong magnetic field and large wave number convection. Gupta and Gupta [8] and Saravanaman [9] investigated the ferroconvection problem with centrifugal acceleration. It is proved that oscillatory ferroconvection cannot occur if the Prandtl number exceeds unity. Maruthamanikandan [10] employed the Rayleigh-Ritz technique to examine the problem of onset of Bénard convection in a horizontal layer of a radiating ferromagnetic fluid. The effect of
viscosity variation on non-Darcy ferroconvection was paid attention by Soya Mathew and Maruthamanikandan [11] and Maruthamanikandan et al [12].

In many systems, such as charges in electrostatic field and ferromagnetic resonance, modulation of a suitable parameter can have marked effects on the motion and can result in increased stability of the system. Venezian [13] investigated the stability of a horizontal layer of fluid heated from below when, besides a steady temperature difference between the walls of the layer, a time-dependent sinusoidal perturbation is applied to the wall temperatures. He showed that at low frequencies the equilibrium state becomes unstable because at that frequency the disturbances grow to a sufficient size so that the inertia effect becomes important. Malashetty and Wadi [14] investigated the stability of a Boussinesq fluid saturated horizontal porous layer with time-dependent wall temperatures. It is shown that the system is most stable when the boundary temperature is modulated out of phase.

Floquet theory was made use of by Aniss et al [15] and Bajaj [16] to analyse respectively the effect of magnetic and gravity modulations on the threshold of ferroconvection. The regions of harmonic and sub-harmonic modes have been obtained. Singh and Bajaj [17] investigated numerically the effect of frequency of temperature modulation on the onset of a periodic flow in the ferrofluid layer. Depending on the parameters, the flow patterns at the onset of instability are found to consist of time-periodically oscillating vertical magnetoconvective rolls. Nisha Mary and Maruthamanikandan [18] used regular perturbation technique to address the non-Darcy ferroconvection problem with gravity modulation. It is made clear that gravity modulation and magnetic mechanisms have opposing influence.

The critical Rayleigh number in thermal modulation problems relies on the frequency of modulation and it proves to be possible to hasten or delay the onset of instability by tuning the frequency of modulation. In the present study we aim at investigating the problem of convective instability in a rotating layer of ferromagnetic fluid subject to time-periodic boundary temperatures with the intention of exploring the possibility of subcritical or supercritical motions.

2. Mathematical Formulation

We consider a ferromagnetic fluid layer confined between two infinite horizontal surfaces with \( h \) as the height. A vertical downward gravity force acts on the fluid together with a uniform, vertical magnetic field \( \vec{H}_0 \). A Cartesian frame of reference is chosen the \( z \)-axis vertically upwards. The fluid layer is subjected to rotation about the \( z \)-axis with \( \vec{\Omega} \) being the uniform angular velocity. The Boussinesq approximation is incorporated to account for the effect of density variation. The governing equations of the present study therefore take the form

\[
\nabla \cdot \vec{q} = 0 \tag{2.1}
\]

\[
\rho_R \left[ \frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla) \vec{q} + 2 \vec{\Omega} \times \vec{q} \right] = -\nabla p + \rho \vec{g} + \nabla \cdot \left( \vec{H} \cdot \vec{B} \right) + \mu \nabla^2 \vec{q} \tag{2.2}
\]

\[
\rho_R C_{\nu H} \frac{DT}{Dt} - \mu_0 \frac{D\vec{H}}{Dt} + \mu_0 T \left( \frac{\partial \vec{M}}{\partial T} \right)_{\nu, H} \cdot \frac{D\vec{H}}{Dt} = K \nabla^2 T \tag{2.3}
\]

\[
\rho = \rho_R [1 - \alpha (T - T_R)] \tag{2.4}
\]

where \( \vec{q} = (u, v, w) \) is the fluid velocity, \( t \) is the time, \( \rho_R \) is a reference density, \( p \) is the pressure, \( \vec{g} \) is the acceleration due to gravity, \( \vec{H} \) is the magnetic induction and \( \mu \) is the coefficient of viscosity, \( C_{\nu H} \) is the specific heat at constant volume and constant magnetic field, \( \mu_0 \) is the magnetic permeability, \( T \) is the temperature, \( \vec{M} \) is the magnetization, \( K_1 \) is the thermal conductivity, \( \alpha \) is the coefficient of thermal expansion and \( T_R \) is a reference temperature.
Maxwell’s equations simplified for a non-conducting fluid with no displacement current are considered and the magnetic equation of state is linearized about the magnetic field and the reference temperature to become

\[ M = M_0 + \chi (H - H_0) - K_m (T - T_R) \tag{2.5} \]

where \( \chi \) is the magnetic susceptibility and \( K_m \) is the pyromagnetic coefficient. The surface temperatures are taken to be

\[ T_R + \frac{\Delta T}{2} \left[ 1 + \epsilon \cos (\delta t) \right] \quad \text{at} \quad z = 0 \tag{2.6} \]

\[ T_R - \frac{\Delta T}{2} \left[ 1 - \epsilon \cos (\delta t + \phi) \right] \quad \text{at} \quad z = h \tag{2.7} \]

where \( \Delta T \) is the temperature difference between the two surfaces in the unmodulated case, \( \epsilon \) is the amplitude of the thermal modulation, \( \delta \) is the frequency and \( \phi \) is the phase angle.

3. Stability Analysis

On applying the method of small perturbation and introducing the magnetic potential \( \Psi \), we obtain the following stability equations (Finlayson [4], Venezian [13] and Nisha Mary and Maruthamanikandan [18])

\[
\left( \frac{1}{Pr} \frac{\partial}{\partial t} - \nabla^2 \right) \nabla^2 w + \sqrt{Ta} \frac{\partial \zeta}{\partial z} = R \left[ 1 - M_1 (\epsilon f - 1) \right] \nabla^2 T + RM_1 (\epsilon f - 1) \frac{\partial}{\partial z}\left( \nabla^2 \Psi \right) \tag{3.1}
\]

\[
\frac{\partial T}{\partial t} + (\epsilon f - 1) w = \nabla^2 T \tag{3.2}
\]

\[
\left( \frac{1}{Pr} \frac{\partial}{\partial t} - \nabla^2 \right) \zeta = \sqrt{Ta} \frac{\partial w}{\partial z} \tag{3.3}
\]

\[
\left( \frac{\partial^2}{\partial z^2} + M_3 \nabla_1^2 \right) \Psi = \frac{\partial T}{\partial z} \tag{3.4}
\]

where \( f = \text{Re} \left\{ A(\lambda) e^{\lambda z} + A(-\lambda) e^{-\lambda z} \right\} e^{-i\omega t} \), \( A(\lambda) = \frac{\lambda}{2} \left( \frac{e^{-i\phi} - e^{-\lambda}}{e^\lambda - e^{-\lambda}} \right) \), \( \lambda = (1-i) \left( \frac{\omega}{2} \right)^{1/2} \), \( \omega \) is the dimensionless frequency of modulation, \( \nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) and \( \nabla^2 = \nabla_1^2 + \frac{\partial^2}{\partial z^2} \). The dimensionless parameters are \( Pr \) the Prandtl number, \( R \) the Rayleigh number, \( M_1 \) the buoyancy-magnetization parameter, \( M_3 \) the non-buoyancy magnetization parameter and \( Ta \) the Taylor number.

The boundary conditions are taken to be

\[ w = \frac{\partial^2 w}{\partial z^2} = T = \frac{\partial \Psi}{\partial z} = \frac{\partial \zeta}{\partial z} = 0 \quad \text{at} \quad z = 0, 1. \tag{3.5} \]
To this end, we would like to mention that it is advantageous to express the entire system of equations in terms of the vertical component \( w \) of the fluid velocity. On combining Eqs. (3.1) through (3.4), we obtain the following equation for the vertical component of the velocity \( w \) in the form

\[
\left( \frac{1}{\text{Pr}} \frac{\partial}{\partial t} - \nabla^2 \right) \left( \frac{1}{\text{Pr}} \frac{\partial}{\partial t} - \nabla^2 \right) \left( \frac{\partial^2}{\partial z^2} + M_3 \nabla_1^2 \right) \nabla^2 w + Ta \left( \frac{\partial}{\partial t} - \nabla^2 \right) \left( \frac{\partial^2}{\partial z^2} + M_3 \nabla_1^2 \right) \frac{\partial^2 w}{\partial z^2} \\
= - R \left( \frac{1}{\text{Pr}} \frac{\partial}{\partial t} - \nabla^2 \right) \left( \frac{\partial^2}{\partial z^2} + M_3 \nabla_1^2 \right) \left( \epsilon f - 1 \right) \nabla_1^2 w + R M_1 M_3 \left( \frac{1}{\text{Pr}} \frac{\partial}{\partial t} - \nabla^2 \right) (1 - 2 \epsilon f) \nabla_1^4 w
\]

(3.6)

4. Method of Solution

The eigenfunctions and eigenvalues of the present study differ from those of ferroconvection with rotation by quantities of order \( \epsilon \). We therefore assume the solution of Eq. (3.6) in the form (Venezian [13] and Malashetty and Wadi [14])

\[
w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + \cdots \quad \text{(4.1)}
\]

\[
R = R_0 + \epsilon R_1 + \epsilon^2 R_2 + \cdots
\]

(4.2)

where \( R_0 \) is the critical Rayleigh number for the corresponding unmodulated problem. The stability analysis of the rotating ferroconvection problem was investigated by Gupta and Gupta [8], who showed that overstability cannot occur when the Prandtl number \( \text{Pr} > 1 \). Hence the principle of exchange of stabilities is valid for the unmodulated problem as long as \( \text{Pr} > 1 \). The expression for the Rayleigh number \( R_0 \) is given by

\[
R_0 = \frac{\left( \pi^2 + M_3 \alpha^2 \right) \left( \pi^2 + \alpha^2 \right)^3 + Ta \pi^2}{\alpha^2 \left[ \pi^2 + M_3 (1 + M_1) \alpha^2 \right]}. \quad \text{(4.3)}
\]

Following the analysis of Venezian [13], we obtain the following expression for \( R_2 \)

\[
R_2 = K_2 \sum_{n=1}^{\infty} \left( n^2 \pi^2 + \alpha^2 \right) \left( n^2 \pi^2 + M_3 (1 + 2 M_1) \alpha^2 \right) \frac{|p_n|^2 D_n}{E_n} \quad \text{(4.4)}
\]

where

\[
K_2 = \frac{R_0^2 \alpha^2 \left[ \pi^2 + M_3 (1 + 2 M_1) \alpha^2 \right]}{2 \left[ \pi^2 + M_3 (1 + M_1) \alpha^2 \right]^2}, \quad |p_n|^2 = \frac{16 \pi^4 n^2 \omega^2}{\left[ \omega^2 + (n+1)^4 \pi^4 \right] \left[ \omega^2 + (n-1)^4 \pi^4 \right]},
\]

4
\[
D_n = \frac{\alpha^2}{Pr} \left( n^2 \pi^2 + \alpha^2 \right)^2 \left( n^2 \pi^2 + M_3 \alpha^2 \right) \left( 2 + \frac{1}{Pr} \right) - \left( n^2 \pi^2 + \alpha^2 \right)^4 \left( n^2 \pi^2 + M_3 \alpha^2 \right) \\
- n^2 \pi^2 Ta \left( n^2 \pi^2 + \alpha^2 \right) \left( n^2 \pi^2 + M_3 \alpha^2 \right) + R_0 \alpha^2 \left( n^2 \pi^2 + \alpha^2 \right) \left[ n^2 \pi^2 + M_3 (1 + M_1) \alpha^2 \right] \\
E_n = P_1^2 + P_2^2
\]

\[
P_1 = \frac{\alpha^2}{Pr} \left( n^2 \pi^2 + \alpha^2 \right)^2 \left( n^2 \pi^2 + M_3 \alpha^2 \right) \left( 2 + \frac{1}{Pr} \right) - \left( n^2 \pi^2 + \alpha^2 \right)^4 \left( n^2 \pi^2 + M_3 \alpha^2 \right) \\
- n^2 \pi^2 Ta \left( n^2 \pi^2 + \alpha^2 \right) \left( n^2 \pi^2 + M_3 \alpha^2 \right) + R_0 \alpha^2 \left( n^2 \pi^2 + \alpha^2 \right) \left[ n^2 \pi^2 + M_3 (1 + M_1) \alpha^2 \right] \\
P_2 = \frac{\alpha^3}{Pr^2} \left( n^2 \pi^2 + \alpha^2 \right) \left( n^2 \pi^2 + M_3 \alpha^2 \right) - \omega \left( n^2 \pi^2 + \alpha^2 \right)^3 \left( n^2 \pi^2 + M_3 \alpha^2 \right) \left( 1 + \frac{2}{Pr} \right) \\
- \omega \left[ n^2 \pi^2 Ta \left( n^2 \pi^2 + M_3 \alpha^2 \right) - \frac{R_0}{Pr} \alpha^2 \left[ (n^2 \pi^2 + M_3 (1 + M_1) \alpha^2) \right] \right].
\]

It should be remarked that supercritical instability exists if \( R_{2c} \) is positive and subcritical instability occurs when \( R_{2c} \) turns out to be negative. We calculate \( R_{2c} \) for the following cases:

Case (i): the oscillating temperature field is symmetric when the wall temperatures are modulated in phase (\( \phi = 0 \)).

Case (ii): the oscillating temperature field is asymmetric when there is an out-of-phase modulation (\( \phi = \pi \)).

Case (iii): when only the temperature of the bottom wall is modulated (\( \phi = -i \infty \)).

5. Results and Discussion

The problem investigated is that of figuring out the criteria for the onset of thermal convection in a ferromagnetic fluid layer with centrifugal acceleration and time-periodic boundary temperatures. The underlying stability analysis relies on the condition of the smallness of the amplitude of thermal modulation. It is well known that the results of temperature modulation problem are tenable only when the modulation frequency \( \omega \) takes on moderate values (Venezian [13] and Malashetty and Wadi [14]). The regular perturbation technique is embraced to effectively deal with the underlying eigenvalue problem. Three different thermal excitations, namely in-phase, out-of-phase and bottom wall modulation are examined. The results of the present study are expounded with the help of figures 1 through 12. The results relating to symmetric modulation are exhibited in figures 1 through 4, asymmetric modulation in figures 5 through 8 and bottom wall modulation in figures 9 through 12. The parameter \( M_1 \) is the ratio of magnetic force to gravitational force. The parameter \( M_3 \) measures the departure of linearity in the magnetic equation of state. The Taylor number \( Ta \) characterizes the dominance of centrifugal force relative to viscous force. The Prandtl number \( Pr \) is the ratio of the speed of momentum propagation to that of heat transport.
Figure 1. Plot of $\omega$ versus $R_{2c}$ with variations in $M_1$ concerning symmetric modulation.

Since $R_{2c}$ is negative only for in-phase modulation, as seen in figures 1 through 12, it follows that subcritical motion is non-existent for asymmetric and bottom wall modulation. This could be a consequence of the symmetric modulation resulting in a nonlinear imposed temperature gradient.

Figure 2. Plot of $\omega$ versus $R_{2c}$ with variations in $M_3$ concerning symmetric modulation.
Figure 3. Plot of $\omega$ versus $R_{2c}$ with variations in $Ta$ concerning symmetric modulation.

It is demonstrated in figures 1 through 4 that the parameters $M_1$, $M_3$ and $Ta$ tend to stabilize the system and the opposite behaviour applies to the Prandtl number $Pr$. It is intriguing to perceive that symmetric modulation can also lead to supercritical instability for low Prandtl number ferrofluids provided the frequency $\omega$ of modulation is moderate in value.

Figure 4. Plot of $\omega$ versus $R_{2c}$ with variations in $Pr$ concerning symmetric modulation.
As for the out-of-phase modulation, as seen in figures 5 through 8, the parameters $M_1$, $M_3$ and $Pr$ have the tendency to destabilize the system and the opposite behaviour is applicable to the Taylor number $Ta$. As pointed out earlier, only supercritical instability is existent for asymmetric modulation.

**Figure 5.** Plot of $\omega$ versus $R_{2c}$ with variations in $M_1$ concerning asymmetric modulation.

**Figure 6.** Plot of $\omega$ versus $R_{2c}$ with variations in $M_3$ concerning asymmetric modulation.
Figure 7. Plot of $\omega$ versus $R_{2c}$ with variations in $T_\alpha$ concerning asymmetric modulation.

Figures 9 through 12 link to the bottom wall thermal modulation. It is understood that the influences of the parameters $M_1$, $M_3$ and $Pr$ are akin to that of asymmetric modulation. Nonetheless, the rotation mechanism tends to stabilize the system for low values of $\omega$ and the opposite behaviour occurs when $\omega$ is moderate in value.

Figure 8. Plot of $\omega$ versus $R_{2c}$ with variations in $Pr$ concerning asymmetric modulation.
As for the effect of $\omega$ on the stability of the system, it has the tendency of destabilizing the system for asymmetric and bottom wall modulation. Nevertheless, for symmetric modulation, it happens to destabilize the system for small values and the opposite occurs for moderate and large values.
Figure 11. Plot of $\omega$ versus $R_{2c}$ with variations in $Ta$ concerning bottom wall modulation.

Figure 12. Plot of $\omega$ versus $R_{2c}$ with variations in $Pr$ concerning bottom wall modulation.

Furthermore, moderate and large values of $\omega$ happen to scale down the influences of both magnetic and rotation mechanisms. Moreover, as with other research work on thermal modulation, temperature modulation effect is downright insignificant when the frequency $\omega$ of modulation is sufficiently large.
6. Concluding Remarks
The combined effect of thermal modulation and rotation on the onset of convection in a rotating ferromagnetic fluid layer is investigated and the following conclusions are drawn:

1. Magnetic mechanism tends to stabilize the system when the wall temperatures are modulated in phase and to destabilize the system in the case of asymmetric and bottom wall modulation.
2. When the temperature modulation is symmetric, supercritical motion is possible for low Prandtl number ferromagnetic fluids.
3. Rotation tends to stabilize the system when the excitation is both symmetric and asymmetric. However, for bottom wall temperature modulation, the effect of rotation is to stabilize the system for low frequencies and to destabilize the system for moderate and large values.
4. Symmetric modulation leads to manifestation of subcritical motion.
5. Moderate and large values of the frequency of modulation tends to scale down the influences of both magnetic and rotation mechanisms.

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