Prescribing Symmetries and Automorphisms for Polytopes

Egon Schulte∗†
Northeastern University,
Department of Mathematics,
Boston, MA 02115, USA

Pablo Soberón‡§
Baruch College, City University of New York,
Department of Mathematics,
New York, New York 10010, USA

Gordon Ian Williams ¶
University of Alaska Fairbanks
Department of Mathematics and Statistics,
Fairbanks, AK 99709, USA

February 15, 2019

Abstract

We study the groups for which it is possible to find a convex polytope with that group as automorphism group with additional geometric conditions on the action of the group or its subgroups. In particular, we prove that for every abelian group $G$ of even order and an involution $s$ of $G$, there is a centrally symmetric convex polytope whose automorphism group is $G$ and such that $s$ corresponds to the central symmetry.

Keywords: convex polytope, abstract polytope, automorphism group, symmetry group.

Subject classification: Primary 52B15; Secondary 52B11, 51M20

∗Partially supported by Simons Foundation award no. 420718.
†Email: schulte@neu.edu
‡Partially supported by NSF Grant DMS 1851420
§Email: pablo.soberon-bravo@baruch.cuny.edu
¶Email: giwilliams@alaska.edu
1 Introduction

The study of polytopes is largely motivated by their symmetries. Indeed, polytopes provide an aesthetic way to see their automorphism groups. Every polytope has two automorphism groups: its geometric automorphism group, which is the set of isometries that preserve it, and its combinatorial automorphism group, which is the groups of symmetries of its face lattice. It is natural to ask if we can go in the opposite direction: is every group the automorphism group of a convex polytope?

This question was answered positively by Schulte and Williams [SW15], and later a simpler proof was found by Doignon [Doi18]. In this paper we are interested in studying variations of this question with additional restrictions on the polytopes in question. The main motivation for this paper was to characterize which groups are the automorphism group of a centrally symmetric polytope. Of course, a centrally symmetric polytope in \( \mathbb{R}^d \) has the reflection by the origin \( f(x) = -x \) as one of its symmetries, so the automorphism group must have an involution. In Theorem 4.1 we give show that abelian groups satisfy this property: given an abelian group \( \Gamma \) with an involution \( \sigma \), there is a centrally symmetric polytope whose automorphism group is \( \Gamma \) where \( \sigma \) corresponds to the central symmetry.

The main tools on the proof is an extension of the methods in [SW15]. Given a group \( \Gamma \) with an involution \( \sigma \), we first find a polytope whose automorphism group contains \( \Gamma \) and where \( \sigma \) corresponds to a central symmetry. Then, we add vertices to the polytope to get rid of undesired symmetries, while preserving \( \Gamma \) and any fixed isometries that preserve the polytope. The full extent to which this symmetry-breaking argument can be done is contained in Section 3, and extends to abstract polytopes (see [MS02]). In Section 4 we adapt those arguments to centrally symmetric convex polytopes. Finally, in Section 5 we discuss some open problems.

The question of finding polytopes with prescribed automorphism groups had also been asked as motivated by representation theory, see [Lad16, BL18, FL18]. The family of polytopes studied there are called orbit polytopes, which are the orbit of a finite group acting affinely on a real vector space. Since not every group is the affine automorphism group of an orbit polytope, it seems that symmetry-breaking processes as we describe cannot be completely avoided. The question whether or not a given group is the automorphism group of a geometric, combinatorial, algebraic, or topological structure of a specified kind has been studied quite extensively. For a recent article describing the common characteristics of the approaches see the recent article [Jon18] by Jones.

2 Basic Notions

We begin by recalling some basic definitions from the theory of convex and abstract polytopes (see [Grü03, MS02, Zie95]).

An abstract polytope of rank \( n \) is a ranked poset \( P \) with the following properties. The elements of \( P \) are called faces, and the possible face ranks are \(-1, 0, \ldots, n\). A face is a \( j \)-face if its rank is \( j \). Faces of ranks \( 0, 1 \) or \( n - 1 \) are also called vertices, edges or facets of \( P \), respectively. The poset \( P \) has a smallest face (of rank \(-1\)) and largest
face (of rank \(n\)). Each flag (maximal totally ordered subset) \(\Phi\) of \(\mathcal{P}\) contains exactly \(n + 2\) faces, one for each rank \(j\). Two flags are said to be adjacent if they differ in just one face; they are \(j\)-adjacent if this face has rank \(j\). The poset \(\mathcal{P}\) is strongly flag-connected, meaning that any two flags \(\Phi\) and \(\Psi\) can be joined by a sequence of flags \(\Phi = \Phi_0, \Phi_1, \ldots, \Phi_k = \Psi\), all containing \(\Phi \cap \Psi\), such that any two successive flags \(\Phi_{i−1}\) and \(\Phi_i\) are adjacent. Finally, \(\mathcal{P}\) satisfies the diamond condition: whenever \(F \leq G\), with \(\text{rank}(F) = j − 1\) and \(\text{rank}(G) = j + 1\), there are exactly two faces \(H\) of rank \(j\) such that \(F \leq H \leq G\). Thus, for \(j = 0, \ldots, n − 1\), a flag of \(\mathcal{P}\) has exactly one \(j\)-adjacent flag.

If \(F\) and \(G\) are faces with \(F \leq G\), then \(G/F := \{H \mid F \leq H \leq G\}\) is called a section of \(\mathcal{P}\) and is a polytope in its own right. For a face \(F\), we also call \(F_n/F\) the co-face of \(\mathcal{P}\) at \(F\), or the vertex-figure of \(\mathcal{P}\) at \(F\) if \(F\) is a vertex.

The face lattice of a convex polytope is an example of an abstract polytope. Recall that a convex polytope \(P\) is the convex hull of finitely many points in Euclidean \(n\)-space \(\mathbb{E}^n\). A (proper) face of \(P\) is the intersection of \(P\) with a supporting hyperplane of \(P\); the latter is any hyperplane \(H\) such that \(P\) lies entirely in one of the two closed half-spaces bounded by \(H\) and has points in common with \(H\). The empty set \(\emptyset\), and \(P\) itself, are also called (improper) faces of \(P\). The set of all (proper and improper) faces of a convex polytope \(P\), ordered by inclusion, forms a lattice called the face lattice of \(P\). This is an abstract polytope, of rank \(n\) if \(P\) has dimension \(n\). The boundary complex of a convex \(n\)-polytope \(P\), denoted \(\text{bd}(P)\), is the set of all faces of \(P\) of rank less than \(n\), partially ordered by inclusion (see [Gri03, p. 40]); this complex tessellates the boundary \(\partial P\) of \(P\) and is topologically a sphere.

Let \(P\) be a convex \(n\)-polytope. The (standard) barycentric subdivision of \(P\) is the geometric simplicial complex of dimension \(n\), whose \(n\)-simplices are precisely the convex hulls of the centroids of the faces in a flag of \(P\) (see [Bay88, GOR04] or [MS02, Sect. 2C]). We use the term “barycentric subdivision” more broadly and allow the centroid of a face to be replaced by a relative interior point of that face. Thus a barycentric subdivision of \(P\) is an \(n\)-dimensional geometric simplicial complex with one vertex in the relative interior of each non-empty face of \(P\), and with one \(n\)-dimensional simplex per flag of \(P\), such that the vertices of an \(n\)-simplex are precisely the relative interior points chosen in the faces of the corresponding flag. Each barycentric subdivision of \(P\) is isomorphic (as an abstract simplicial complex) to the order complex of the face lattice of \(P\) (with the empty face removed); in particular, any two barycentric subdivisions are isomorphic.

There is a similar notion of barycentric subdivision for the boundary complex of a convex polytope. By \(\mathcal{C}(P)\) we denote the barycentric subdivision of the boundary complex \(\text{bd}(P)\) of \(P\). This is an \((n − 1)\)-dimensional complex.

The order complex of an abstract polytope \(\mathcal{P}\) similarly can be viewed as a “combinatorial barycentric subdivision” of \(\mathcal{P}\) (see [MS02, Sect. 2C]).

The \(k\)-skeleton \(\text{skel}_k(\mathcal{P})\) of an abstract polytope \(\mathcal{P}\) is the poset consisting of all proper faces of \(\mathcal{P}\) of rank at most \(k\) (together with the induced partial order). If \(\mathcal{P}\) is the face lattice of a convex polytope \(P\), we write \(\text{skel}_k(P)\) instead.
3 Preassigning symmetry groups

In this section we establish the following theorem.

**Theorem 3.1** Let $d \geq 3$, let $Q$ be a convex $d$-polytope with combinatorial automorphism group $\Gamma(Q)$, and let $\Gamma$ be a subgroup of $\Gamma(Q)$. Then there exists a finite abstract $d$-polytope $P$ with the following properties:

(a) $\Gamma(P) = \Gamma$.

(b) $P$ is isomorphic to a face-to-face tessellation $\mathcal{T}$ of the $(d-1)$-sphere $S^{d-1}$ by spherical convex $(d-1)$-polytopes.

(c) $\text{skel}_{d-2}(C(Q))$ is a subcomplex of $\text{skel}_{d-2}(P)$.

(d) If $\Gamma$ is a subgroup of the geometric symmetry group $G(Q)$ of $Q$, then the tessellation $\mathcal{T}$ on $S^{d-1}$ in part (b) can be chosen in such a way that $G(\mathcal{T}) = \Gamma = \Gamma(\mathcal{T})$.

First notice that the statement is wrong for $d = 2$, since the combinatorial automorphism group of a finite abstract 2-polytope (polygon) is necessarily dihedral and in particular cannot be cyclic. Hence we must require $d \geq 3$.

**Proof.** We begin with the first three parts of the theorem and later refine our arguments to settle the fourth part. Our strategy is to refine the structure of the given convex polytope $Q$ in such a way that all automorphisms in $\Gamma(Q)$ outside $\Gamma$ are destroyed. The result will be a spherical (abstract) polytope whose automorphism group is given by $\Gamma$. The method is interesting in its own right.

Consider the (standard or any other) barycentric subdivision $C(Q)$ of the boundary complex $\text{bd}(Q)$ of $Q$ in $d$-space $E^d$. This is a simplicial $(d-1)$-complex that refines $\text{bd}(Q)$ and is a realization of the order complex of $\text{bd}(Q)$ (see [MS02, Sect. 2C]). Its simplices correspond to chains (totally ordered subsets) in the poset $\text{bd}(Q)$, with the chambers (maximal simplices) corresponding to the flags of $\text{bd}(Q)$. In particular, $C(Q)$ has the structure of a labelled simplicial complex, in which every simplex is labelled by the set of ranks of the faces in the chain of $\text{bd}(Q)$ represented by the simplex. Thus the vertices of $C(Q)$ can be labelled by $0, \ldots, d-1$. The vertices of $Q$ are the vertices of $C(Q)$ with label 0. The vertices of each chamber are labelled $0, \ldots, d-1$ such that no two vertices are labelled the same. Note that $\Gamma(Q)$ (and hence $\Gamma$) acts on $C(Q)$ as a group of automorphisms of a labelled simplicial complex (labels of simplices are preserved), and that the action on the chambers is free.

As in the proof of [SW15, Theorem 1], a key step in the construction consists of chamber replacement, by complexes made up of Schlegel diagrams of convex polytopes. These complexes are inserted into the chambers of $C(Q)$. In fact, our proof basically consists of adapting the proof of [SW15, Theorem 1] to the more general situation at hand. (In that theorem, the corresponding subgroup $\Gamma$ acted simply vertex-transitively on a special convex $d$-polytope $Q$ constructed from a suitable permutation representation of $\Gamma$. In the present context, $Q$ can be an arbitrary polytope and $\Gamma$ need not act vertex-transitively.)

The complexes inserted into the chambers are constructed in exactly the same manner as in [SW15]. Each complex is built from a Schlegel diagram $D$ of a $d$-crosspolytope supported on an $(d-1)$-dimensional (outer) simplex $D$ with vertices $u_1, \ldots, u_d$, by inserting
Schlegel diagrams of certain convex $d$-polytopes (affine images of the polytopes $R_i$ and $L$ described in [SW15]) into the $(d - 1)$-simplices of $D$ corresponding to certain facets of the $d$-crosspolytope. The resulting $(d - 1)$-dimensional complex, which in [SW15] was denoted $R^L$, then also is supported on $D$ and has the $(d - 2)$-skeleton of $D$ as a subcomplex. The particular choice of the polytopes whose Schlegel diagrams are inserted into $(d - 1)$-simplices of $D$ is quite delicate and is designed in such a way that the vertices $u_1, \ldots, u_d$ of the outer simplex $D$ acquire very high valencies in $R^L$ compared with those in the interior, and that the valencies of the $d$ vertices of $D$ in $R^L$ are integers “very far apart” from each other. (In [SW15] this is accomplished through a clever choice of the parameters $m_0, \ldots, m_{d-1}$ which determine the valencies at the vertices of $D$.) These conditions on the insertion process later prevent the existence of unwanted automorphisms. In particular, $R^L$ itself only admits the trivial automorphism.

Now recall that $Q$ is a convex $d$-polytope in $\mathbb{E}^d$ on which the subgroup $\Gamma$ of $\Gamma(Q)$ acts as a group of combinatorial automorphisms. Then both $\Gamma(Q)$ and $\Gamma$ also act on the barycentric subdivision $\mathcal{C}(Q)$ as groups of combinatorial automorphisms; recall here that the simplices in $\mathcal{C}(Q)$ are just the chains of the boundary complex $\text{bd}(Q)$ of $Q$. For a chamber $C$ of $\mathcal{C}(Q)$, we let $o(C)$ denote the orbit of $C$ under $\Gamma$ in its action on $\mathcal{C}(Q)$.

The chamber replacement now proceeds as follows. First, for each chamber orbit $o(C)$ choose a simplicial convex $d$-polytope $L_{o(C)}$, in such a way that no two such polytopes have the same number of vertices and thus no two of the corresponding $(d-1)$-dimensional complexes $R_{L_{o(C)}}$ are isomorphic. Recall that a convex polytope is called simplicial if all its proper faces are simplifies.

In the final step of the construction we first replace, for each chamber orbit $o(C)$, one of its chambers, $C$ (say), by an affine copy of the corresponding complex $R_{L_{o(C)}}$ such that, for each $i = 0, \ldots, d - 1$, the vertex $u_i$ of $D$ is mapped onto the vertex of $C$ labelled $i$. We then exploit $\Gamma$ to carry this new structure to all the other chambers in an orbit, and therefore to all chambers of $\mathcal{C}(Q)$. Recall that $\Gamma(Q)$, and hence $\Gamma$, acts freely and in a label preserving manner on the chambers of $\mathcal{C}(Q)$ (flags of $Q$). More explicitly, if $C'$ is a chamber in the same orbit as $C$, that is, $o(C') = o(C)$, then we replace $C'$ by an affine copy of the complex $R_{L_{o(C)}}$ that we used for $C$, such that, for each $i = 0, \ldots, d - 1$, the vertex $u_i$ of $D$ is mapped onto the vertex of $C'$ labelled $i$. In short, with respect to insertion of diagrams we treat $C$ and $C'$ in the same manner. The resulting $(d - 1)$-dimensional complex $C'$ is a refinement of the $\mathcal{C}(Q)$ and has the full $(d - 2)$-skeleton of $\mathcal{C}(Q)$ as a subcomplex, unrefined. In particular, $C'$ tiles the boundary $\partial Q$ of $Q$ and hence is topologically a $(d - 1)$-sphere. By construction, $\Gamma$ acts on $C'$ as a group of automorphisms.

Clearly we may project the complex $C'$ radially onto the circumsphere of $Q$, and rescale the sphere (if need be), to obtain an isomorphic complex $\mathcal{T}$ which tiles the unit sphere $S^{d-1}$ in a face-to-face manner by spherical convex polytopes.

Finally, by adjoining suitable improper faces (of ranks $-1$ and $d$) to $\mathcal{T}$ we arrive at a spherical abstract $d$-polytope, denoted $\mathcal{P}$.

It remains to verify the properties of $\mathcal{P}$ described in parts (a), (b), (c) and (d). Parts (b) and (c) are clear by construction.

For the proof of part (a) we can mostly proceed as in [SW15], specifically Lemma 2. Suppose that $\gamma$ is an automorphism of $\mathcal{P}$. We want to show that $\gamma$ lies in $\Gamma$. The initial
steps of the proof are the same (almost word for word) as those in [SW15, pp. 451-452]. In particular, one can show that \( \gamma \) arises from an automorphism of \( Q \). In other words, \( \Gamma(\mathcal{P}) \) can be viewed as a subgroup of \( \Gamma(Q) \) containing \( \Gamma \). The final step then consists in showing that \( \Gamma(\mathcal{P}) = \Gamma \). Here the arguments of [SW15, pp. 452-453] need to be modified as follows.

Suppose that \( \Gamma \) is a proper subgroup of \( \Gamma(\mathcal{P}) \). Then since \( \Gamma(Q) \) acts freely on the chambers of \( \mathcal{C}(Q) \), and \( \Gamma(\mathcal{P}) \) is a subgroup of \( \Gamma(Q) \), the orbits of chambers \( C \) of \( \mathcal{C}(Q) \) under \( \Gamma(\mathcal{P}) \) are strictly larger than those under \( \Gamma \). In particular, there are two different orbits \( o(C_1) \) and \( o(C_2) \) of chambers \( C_1 \) and \( C_2 \) under \( \Gamma \), which lie in the same orbit under \( \Gamma(\mathcal{P}) \). Any automorphism \( \gamma \) of \( \Gamma(\mathcal{P}) \) which maps a chamber \( C'_1 \) in \( o(C_1) \) to a chamber \( C'_2 \) in \( o(C_2) \) induces an isomorphism between the corresponding complexes \( \mathcal{R}^{L_0(C_1)} \) and \( \mathcal{R}^{L_0(C_2)} \) inserted into \( C'_1 \) and \( C'_2 \), respectively. However, any two complexes \( \mathcal{R}^{L_0(C)} \) are non-isomorphic, by our choice of the polytopes \( L_0(C) \). Thus \( \Gamma(\mathcal{P}) = \Gamma \).

For the proof of (d) we must refine our arguments. So let \( \Gamma \) be a subgroup of the geometric symmetry group \( G(Q) \) of \( Q \). In this case we choose the standard barycentric subdivision for \( \mathcal{C}(Q) \) (with the vertices of \( \mathcal{C}(Q) \) given by the centroids of faces of \( Q \)). Then \( \mathcal{C}(Q) \) is invariant under \( \Gamma \). Next we proceed as before and replace, for each chamber orbit \( o(C) \), one of its chambers, \( C \) (say), by an affine copy of the corresponding complex \( \mathcal{R}^{L_0(C)} \) such that, for each \( i = 0, \ldots, d - 1 \), the vertex \( u_i \) of \( D \) is mapped onto the vertex of \( C \) labelled \( i \). In order to define the chamber replacement for the remaining chambers of \( \mathcal{C}(Q) \) we use as transfer maps the elements of \( \Gamma \), which now are geometric symmetries of \( \mathcal{C}(Q) \). More explicitly, if \( C' \) is a chamber of \( \mathcal{C}(Q) \) with \( o(C'') = o(C) \), and \( \gamma \) is the (unique, labeling preserving) symmetry that maps \( C \) to \( C'' \), then we replace \( C'' \) by the image of \( \mathcal{R}^{L_0(C)} \) under \( \gamma \). Then the overall structure is also invariant under \( \Gamma \), and the same holds for its (scaled) projected image \( \mathcal{T} \) on \( S^{d-1} \). Note that \( \mathcal{T} \) cannot acquire geometric symmetries which do not belong to \( \Gamma \), since these would also give combinatorial symmetries, which is impossible by part (a). This completes the proof of (d).

**Theorem 3.2** Let \( d \geq 3 \), let \( Q \) be a convex \( d \)-polytope, and let \( \Gamma \) be a subgroup of \( \Gamma(Q) \). The abstract polytope \( \mathcal{P} \) of Theorem 3.1 may be realized by a convex \( d \)-polytope \( P \). Moreover, if \( \Gamma \) is a subgroup of \( G(Q) \), then \( P \) can be chosen in such a way that \( G(P) = \Gamma = \Gamma(\mathcal{P}) \).

**Proof.** The proof of the first statement is the same as the proof of [SW15, Theorem 4.2]: first the complex \( \mathcal{C}(Q) \) is realized by a convex \( d \)-polytope \( R \), and then all subsequent modifications to the boundary required for the construction of \( \mathcal{P} \) are achieved by gluing projective copies of convex polytopes to the facets of \( R \) that are sufficiently thin in the direction of the outward facing normal to the facet. The result is a convex \( d \)-polytope \( P \).

The proof of the second statement is similar. First observe that \( R \) can be chosen in such a way that \( \Gamma \) lies in \( G(R) \). In fact, the construction of \( R \) described in the proof of [SW15, Lemma 3] respects symmetries and leads to a convex \( d \)-polytope \( R \) whose symmetry group contains \( \Gamma \) as a subgroup. The chamber replacement can again be realized by gluing thin projective copies of convex polytopes to facets of \( R \). Now this is done in two steps. First, we only glue copies to the facets of \( R \) which correspond to
chambers in a system of representatives for the chamber orbits $o(C)$ on $C(Q)$. Second, we use the symmetries in $\Gamma$ to attach copies to the remaining facets of $R$, such that facets of $R$ equivalent under $\Gamma$ receive projective copies which are also equivalent under $\Gamma$. Bear in mind that the boundary complex of $R$ has the structure of a labeled simplicial complex on which $\Gamma$ acts freely in a label-preserving manner. If the projective copies used in the first step are sufficiently thin, then the resulting structure is a convex $d$-polytope. By construction this polytope is invariant under $\Gamma$.  

Parts of Theorem 3.1 hold more generally for finite abstract polytopes. With a very similar proof we can establish

**Theorem 3.3**  
Let $d \geq 3$, let $Q$ be a finite abstract $d$-polytope, and let $\Gamma$ be a subgroup of $\Gamma(Q)$. Then there exists a finite abstract $d$-polytope $P$ with the following properties:  
(a) $\Gamma(P) = \Gamma$.  
(b) $P$ is isomorphic to a face-to-face tessellation on the topological space $|C(Q)|$ of the order complex $C(Q)$ of $Q$ by topological copies of convex polytopes.  
(b) $\text{skel}_{d-2}(C(Q))$ is a subcomplex of $\text{skel}_{d-2}(P)$.

4  Prescribing involutions as central symmetries

As an application of Theorem 3.2 we consider the following problem. Given a finite group $\Gamma$ and a subgroup $\Lambda$ of $\Gamma$, can we find a convex polytope $P$ such that

- $\Gamma(P) = \Gamma$ and
- $\Lambda$ acts on $P$ in a predetermined way?

We are particularly interested in the case where $\Lambda = C_2$ and $\Lambda$ is generated by a central involution $\sigma$ of $\Gamma$. We wish to find a polytope $P$ such that $\sigma$ acts on $P$ as a central symmetry; that is, abusing notation, $\sigma(x) = -x$ for all $x \in P$. Thus $P$ would be centrally symmetric under the central symmetry $\sigma$. A positive answer would give a centrally symmetric version of the results of [SW15]. Here we show that the answer is always positive for finite abelian groups containing an involution, that is, for abelian groups of even order.

**Theorem 4.1**  
Let $\Gamma$ be a finite abelian group of even order, and let $\sigma$ be an involution of $\Gamma$. Then there is a positive integer $n$ and a centrally symmetric convex $n$-polytope $P$ in $\mathbb{E}^n$, such that $G(P) = \Gamma(P) = \Gamma$ and $\sigma$ is realized as the central symmetry of $P$, that is, $\sigma(x) = -x$ for all $x \in \mathbb{E}^n$.

**Proof.** Let us begin with the case where $\Gamma$ is a cyclic group of even order with generator $\gamma$. Thus $\Gamma = C_{2m}$ for some $m \geq 1$, and $\sigma = \gamma^m$. We show that there exists a polytope of the desired kind in dimension $n = 4$ (see also Theorem ?? below). Consider the action of $\Gamma$ as a group of isometries on $\mathbb{E}^4$, here viewed as complex 2-space $\mathbb{C}^2$ (with $x \in \mathbb{E}^4$ corresponding to $(u, v) \in \mathbb{C}^2$), defined by letting $\gamma$ act as the mapping

$$(u, v) \mapsto (e^{\pi i/m} u, e^{\pi i/m} v).$$
Notice that for all \( x \in \mathbb{E}^4 \), \( ||\gamma(x)|| = ||x|| \). If we take a large enough finite set of points \( S \) in \( \mathbb{S}^3 \) (a five-element subset \( S \) in general position suffices if \( m \geq 3 \)), then the convex hull of the orbit set

\[
\Gamma \cdot S := \{ \varphi(x) \mid \varphi \in \Gamma, \ x \in S \}
\]

is a convex 4-polytope \( Q \) such that \( \Gamma \leq G(Q) \) and \( \sigma(x) = -x \) for all \( x \in Q \). We then apply the construction process underlying Theorem 3.2 to construct the desired convex 4-polytope \( P \). In other words, we get rid of all excess combinatorial symmetries outside of \( \Gamma \) while preserving each element of \( \Gamma \) as a geometric symmetry for \( P \), including in particular the involution \( \sigma \) as the central symmetry for \( P \). Thus \( G(P) = \Gamma = G(P) \). This settles the case when \( \Gamma \) is cyclic. (Note that we cannot work with \( \mathbb{E}^2 \) in place of \( \mathbb{E}^4 \) since the corresponding statement of Theorem 3.2 is false for \( n = 2 \).

If \( \Gamma \) is abelian but not cyclic, then, by the fundamental theorem of abelian groups, we can write \( \Gamma \) as a direct product of \( k + 1 \) abelian groups \( \Gamma = \Gamma_1 \times \ldots \times \Gamma_k \times \Gamma_{k+1} \) for some \( k \geq 1 \), so that

- \( \Gamma_1, \ldots, \Gamma_k \) are cyclic and of even order, and
- \( \sigma = (\sigma_1, \ldots, \sigma_k, 1) \), where \( \sigma_i \) is an involution in \( \Gamma_i \) for all \( 1 \leq i \leq k \).

The idea is to manufacture a suitable polytope for each direct factor of \( \Gamma \) and then combine these polytopes into a single polytope for \( \Gamma \) itself.

We know from the above that for each direct factor \( \Gamma_i \), with \( 1 \leq i \leq k \), there is a centrally symmetric 4-polytope \( P_i \) in \( \mathbb{E}^4 \) such that \( G(P_i) = \Gamma_i = \Gamma(P_i) \) and \( \sigma_i \) is the central symmetry for \( P_i \). Consider the cartesian product polytope \( P' := P_1 \times \ldots \times P_k \) in \( \mathbb{E}^{4k} \), whose vertex set is the cartesian product of the vertex sets of the component polytopes. Clearly, \( P' \) is a centrally symmetric 4\( k \)-polytope, and the direct product \( \Gamma' := \Gamma_1 \times \ldots \times \Gamma_k \) acts on \( P' \) as a group of symmetries such that each factor \( \Gamma_i \) acts on the ambient 4-dimensional subspace of the component polytope \( P_i \). Under this action, \( (\sigma_1, \ldots, \sigma_k) \) is the central symmetry for \( P' \).

For the last direct factor, \( \Gamma_{k+1} \), we embed \( \Gamma' := \Gamma_{k+1} \) into a symmetric group, \( S_{l+1} \) for some \( l \), and then take a regular \( l \)-simplex \( P'' \) in \( \mathbb{E}^d \) centered at the origin. Then \( \Gamma'' \) is a (generally proper) subgroup of \( G(P'') = \Gamma(P'') = S_{l+1} \). Any excess symmetries that \( P'' \) might have, will be trimmed at a later stage.

We now combine these polytopes. Set \( n := 4kl \). Let \( V \) denote the set of points in \( \mathbb{E}^n = \mathbb{E}^{4k} \otimes \mathbb{E}^d \) of the form \( u \otimes v \), where \( u \) and \( v \) are vertices of \( P' \) and \( P'' \), respectively, and \( \otimes \) denotes the standard tensor product (given by \( u \cdot v^T \) if \( u \) and \( v \) are viewed as column vectors). Then \( V \) is a centrally symmetric point set, since the vertex set of \( P' \) is centrally symmetric and \( (\sigma, \sigma) \) is an involution in \( \Gamma \) for all \( 1 \leq i \leq k \).

By construction, the actions of \( \Gamma' \) on \( P' \) and \( \Gamma'' \) on \( P'' \) induce an action of \( \Gamma = \Gamma' \times \Gamma'' \) on \( P \) as a group of geometric symmetries. Thus \( \Gamma \) is a subgroup of \( G(P) \). If we write the given involution \( \sigma \) of \( \Gamma \) in the form \( \sigma = (\sigma', \sigma'') \) with \( \sigma' = (\sigma_1, \ldots, \sigma_k) \in \Gamma' \) and \( \sigma'' := 1 \in \Gamma'' \), then under this action, \( \sigma \) maps each vertex \( u \otimes v \) of \( P \) to \( (\sigma', \sigma) \) and thus acts on \( P \) as central symmetry \( -\text{id} \), as desired.
In the final step, if $P$ has any extra symmetries outside of $\Gamma$ (as will usually be the case), we can trim them down using Theorem 3.2(d). This then produces the desired centrally symmetric polytope.

For cyclic groups (of even order), the construction underlying Theorem 4.1 produced convex polytopes in dimension 4. The reader might wonder if a suitable geometric representation of these groups in 3-space $E^3$ can not also give a convex 3-polytope. As the following theorem shows, the answer is negative for many abelian groups. Dimension 4 is optimal in many cases.

**Theorem 4.2** If $\Gamma = C_{4m} = \langle \gamma \rangle$ for some $m \geq 1$, and $\sigma := \gamma^{2m}$, then there is no centrally symmetric 3-polytope $P$ in $E^3$ such that $\Gamma(P) = \Gamma$ and $\sigma$ is realized as the central symmetry of $P$.

**Proof.** Suppose to the contrary that such a 3-polytope $P$ exists. Then, since $\partial P$ is homeomorphic to $S^2$, we can view $\Gamma$ as a group of homeomorphisms of $S^2$. In particular, $\lambda := \gamma^m$ is a homeomorphism of $S^2$ with $\lambda^2 = \sigma = -\text{id}$ and thus its topological degree must be positive. On the other hand, the topological degree of the homeomorphism $-\text{id}$ of the $k$-sphere $S^k$ is $(-1)^{k+1}$, which is $-1$ when $k = 2$. This leaves no possibility for the topological degree of $\lambda$. Thus $P$ cannot exist (and dimension 4 is optimal if $\Gamma = C_{4m}$ and $\sigma := \gamma^{2m}$).

Note that if in Theorem 4.2 we had insisted on achieving $G(P) = \Gamma$ (rather than $\Gamma(P) = \Gamma$), we could have argued similarly by using the determinant of linear mappings (rather than the topological degree of homeomorphisms) to rule out the existence of $P$. In fact, the determinant of $\lambda^2$ would have to be positive, but the central inversion $-\text{id}$ has determinant $-1$ in dimension 3.

On the other hand, for cyclic groups of the form $\Gamma = C_{2m} = \langle \gamma \rangle$ with $n$ odd, and $\sigma := \gamma^m$, we can indeed find a convex 3-polytope in $E^3$ such that $\Gamma(P) = \Gamma$ and $\sigma$ is realized as the central symmetry of $P$. This can be obtained as follows. Consider a bipyramid $P'$ in $E^3$ over a regular $2m$-gon in the $xy$-plane centered at the origin $o$, where the two apices lie symmetrically on the $z$-axis on different sides of the $xy$-plane. Clearly, $P'$ is invariant under the rotatory reflection $\gamma$ of order $2m$ which is the product of the rotation by $\pi/m$ about the $z$-axis and the reflection in the $xy$-plane. Thus $C_{2m} = \langle \gamma \rangle \leq G(P')$ and $\gamma^m = -\text{id}$. Note that $G(P')$ is strictly larger than $C_{2m}$, since it also contains the reflection in the $xy$-plane but $C_{2m}$ does not. (In fact, $G(P') \cong D_{2m} \times C_2$.) Thus $P'$ itself does not have the required properties. However, a simple application of Theorem 3.2 allows us to find a polytope $P$ by getting rid of the additional symmetries while preserving the action of $C_{2m}$. Alternatively, we can construct a polytope $P$ directly from $P'$ by attaching sufficiently thin pyramids to the facets of $P'$ in one facet orbit of $P'$ under $C_{2m}$.
5 Some open problems

Our previous discussion invites a number of open problems concerning the dimension of polytopes with preassigned symmetry groups or automorphism groups. Usually, given the group $\Gamma$ the interest is in finding polytopes of small dimension realizing $\Gamma$.

For a finite group $\Gamma$, we define the (combinatorial) convex polytope dimension of $\Gamma$, denoted $\text{cpd}(\Gamma)$, as the smallest dimension $d$ for which there exists a convex $d$-polytope $P$ whose combinatorial automorphism group is $\Gamma$, that is, $\Gamma(P) = \Gamma$. Note that the results of [SW15, Doi18] are saying that for every finite group $\Gamma$, we have $\text{cpd}(\Gamma) < \infty$.

Similarly, the geometric convex polytope dimension of $\Gamma$, denoted $\text{gcpd}(\Gamma)$, is defined to be the smallest dimension $d$ for which there is a convex $d$-polytope $P$ whose geometric symmetry group is $\Gamma$, that is, $G(P) = \Gamma$. The results of [Doi18] also imply $\text{gcpd}(\Gamma) < \infty$.

Open Question 1 For each $n$, is there a finite group $\Gamma$ such that $\text{cpd}(\Gamma_n) \geq n$?

Open Question 2 For each $n$, is there a finite group $\Gamma$ such that $\text{gcpd}(\Gamma_n) \geq n$?

Open Question 3 Does Theorem 4.1 hold for non-abelian groups $\Gamma$ and central involutions $\sigma$ of $\Gamma$? In other words, given a finite group $\Gamma$ and a central involution $\sigma$ of $\Gamma$, is there a centrally symmetric convex polytope $P$ with $G(P) = \Gamma(P) = \Gamma$ such that $\sigma$ is realized as the central symmetry $-\text{id}$ of $P$?

Note that the proof of Theorem 4.1 carries over to finite groups of the form $\Gamma = \Gamma_1 \times \Gamma_2$ where $\Gamma_1$ is abelian, and central involutions of $\Gamma$ of the form $\sigma = (\sigma_1, 1)$ where $\sigma_1$ is a central involution of $\Gamma_1$.

References

[BL18] B. Baumeister and F. Ladish A property of the Birkhoff Polytope *Algebraic Combinatorics* 1(2):275–281, 2018. 1

[Bay88] M.M. Bayer. Barycentric subdivisions. *Pacific J. Math.* 135(1):1–16, 1988. 2

[Cox73] H.S.M. Coxeter. *Regular Polytopes*. Dover Publications, 1973.

[Doi16] J.-P. Doignon. A convex polytope and an antimatroid for any given, finite group. *Electronic Notes in Discrete Mathematics* 54:21–25, 2016.

[Doi18] J.-P. Doignon. Any finite group is the group of some binary, convex polytope. *Discrete Comput Geom.* 59:451–460, 2018. 1, 5

[FL18] E. Friese and F. Ladish Classification of affine symmetry groups of orbit polytopes *J. Algebr. Comb.* 48:481–509, 2018. 1

[Fr38] R. Frucht. Herstellung von Graphen mit vorgegebener abstrakter Gruppe. *Compositio Math.* 6:239–250, 1938.
[GOR04] J.E. Goodman and J. O’Rourke, Editors. Handbook of Discrete and Computational Geometry. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, second edition, 2004.

[Grü03] B. Grünbaum. Convex Polytopes, Volume 221 of Graduate Texts in Mathematics. Springer-Verlag, New York, second edition, 2003. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

[Jon18] G. A. Jones. Realisation of groups as automorphism groups in categories arXiv Preprint arXiv:1807.00547 [math.GR]

[Lad16] F. Ladisch. Realizations of abstract regular polytopes from a representation theoretic view Aequat. Math. 90:1169–1193, 2016.

[MS02] P. McMullen and E. Schulte. Abstract Regular Polytopes. Cambridge University Press, 2002.

[SW15] E. Schulte and G.I. Williams. Polytopes with Preassigned Automorphism Groups. Discrete Computational Geometry 54:444–458, 2015.

[Zie95] G.M. Ziegler. Lectures on polytopes. Graduate Texts in Mathematics, Volume 152, Springer-Verlag, New York, 1995.