Growth of Schreier graphs of automaton groups

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Abstract

Every automaton group naturally acts on the space $X^\omega$ of infinite sequences over some alphabet $X$. For every $w \in X^\omega$ we consider the Schreier graph $\Gamma_w$ of the action of the group on the orbit of $w$. We prove that for a large class of automaton groups all Schreier graphs $\Gamma_w$ have subexponential growth bounded above by $n^{(\log n)^m}$ with some constant $m$. In particular, this holds for all groups generated by automata with polynomial activity growth (in terms of S. Sidki), confirming a conjecture of V. Nekrashevych. We present applications to $\omega$-periodic graphs and Hanoi graphs.

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1 Introduction

Let $G$ be a group with a generating set $S$, and acting on a set $X$. The (simplicial) Schreier graph $\Gamma(G, S, X)$ of the action $(G, X)$ is the graph with the set of vertices $X$, where two vertices $x$ and $y$ are adjacent if and only if there exists $s \in S \cup S^{-1}$ such that $s(x) = y$. The Schreier graphs are generalizations of the Cayley graph of a group, which corresponds to the action of a group on itself by the left multiplication.

The growth of Cayley graphs (growth of groups) is one of the central objects of study in geometric group theory. The celebrated theorem of M. Gromov characterizes groups of polynomial growth. Many classical groups (like linear groups, solvable groups, hyperbolic groups, etc.) have either polynomial or exponential growth. The first group of intermediate growth between polynomial and exponential was constructed by R. Grigorchuk [8]. Nowadays, the Grigorchuk group is the best known example of automaton groups (also known as self-similar groups, or groups generated by automata).

In this paper, we consider the growth of Schreier graphs of automaton groups. Every automaton group $G$ generated by an automaton $A$ over some alphabet $X$ naturally acts on the set $X^n$ for every $n \geq 1$, and on the space $X^\omega$ of right-infinite sequences over $X$. We get
the associated sequence of finite Schreier graphs $\Gamma_n(G,A)$ of the action $(G,X^n)$, and the family of orbital Schreier graphs $\Gamma_w(G,A)$ for $w \in X^\omega$ of the action of $G$ on the orbit of $w$. The Schreier graphs $\Gamma_n$ and $\Gamma_w$ provide a large source of self-similar graphs, and were studied in relation to such topics as spectrum, growth, amenability, topology of Julia sets, etc. (see [1, 4, 10, 11, 9, 7]).

It was noticed in [1] that for a few automaton groups the sequence of Schreier graphs $\Gamma_n$ converges to a nice metric space with fractal structure. Further, this observation lead to the notion of a limit space of an automaton group, and, more generally, to a limit dynamical system [14]. The expanding property of this dynamical system corresponds to the contracting property of a group, what brings us to the important class of contracting automaton groups. Namely the strong contracting properties of the Grigorchuk group was used to prove that it has intermediate growth, while in general a contracting group may have exponential growth. At the same time, all orbital Schreier graphs $\Gamma_w$ of contracting groups have polynomial growth [1, 14]. The degree of the growth is related to the asymptotic characteristics of the group such as the complexity of the word problem, the Hausdorff dimension of the limit space, the exponent of divergence of geodesics in the associated Gromov-hyperbolic self-similarity complex, etc. (see [4]).

Most of the studied automaton groups are generated by polynomial automata. These automata were introduced by S. Sidki in [16], who tried to classify automaton groups by the cyclic structure of the generating automaton. A finite automaton is called polynomial if the simple directed cycles away from the trivial state are disjoint. The term “polynomial” comes from the equivalent definition, where a finite automaton is polynomial if the number of paths of length $n$ avoiding the trivial state in the automaton grows polynomially in $n$.

It is an open problem whether contracting groups and groups generated by polynomial automata are amenable. However, it is known that these groups do not contain nonabelian free subgroups [15, 17] (but may contain free semigroups and be of exponential growth), and that groups generated by polynomial automata of degree 0 (bounded automata) are amenable [4]. In [15] V. Nekrashevych introduced a general approach to the existence of free subgroups in automaton groups, and applied it to contracting groups and to groups generated by polynomial automata. In order to eliminate one of the cases, he proved that the orbital Schreier graphs $\Gamma_w$ are amenable, and conjectured [15, page 857] that these graphs may have subexponential growth. This conjecture was based on the results from [4, 3], where it is shown that for the group generated by one of the simplest polynomial automata all Schreier graphs $\Gamma_w$ have intermediate growth. The main goal of this paper is to prove this conjecture.

**Theorem 1.** Let $G$ be a group generated by a polynomial automaton of degree $m$. There exists a constant $A$ such that all orbital Schreier graphs $\Gamma_w(G)$ for $w \in X^\omega$ have subexponential growth not greater than $A(\log n)^{m+1}$.

In order to prove Theorem 1 we establish certain weak contracting properties of groups generated by polynomial automata. In particular, we prove that the word problem in these groups is solvable in subexponential time bounded above by $B(\log n)^{m+1}$ for some constant $B$. 2
We also prove Theorem 1 for a more general class of automata (see Theorem 2 in Section 4), which generalizes both polynomial automata and contracting groups.

In Section 3 we apply Theorem 1 and its generalized version to construct automaton groups with Schreier graphs \( \Gamma_w \) of intermediate growth. In the first example we consider a class of automaton groups, whose Schreier graphs \( \Gamma_w \) are a generalized version of the \( \omega \)-periodic graph of intermediate growth studied in \( \{3, 5\} \). Basically, this example shows that for many polynomial automata the Schreier graphs \( \Gamma_w \) have intermediate growth. Another example comes from the well-known Hanoi Tower Game on \( k \) pegs. This game was modeled by automaton groups \( G_{(k)} \) in [10], and, using known estimates on the complexity of the Hanoi Tower Game, it was noticed that the orbital Schreier graph \( \Gamma_0 = (G_{(k)}) \) for \( k \geq 4 \) has intermediate growth. The automata generating groups \( G_{(k)} \) for \( k \geq 4 \) are not polynomial, but we apply similar arguments to prove that all orbital Schreier graphs \( \Gamma_w(G_{(k)}) \) have intermediate growth.

\[ \text{2 Automaton groups and their Schreier graphs} \]

In this section we recall all needed definition, see [4] for a more detailed introduction.

**Spaces of words.** Let \( X \) be a finite alphabet with at least two letters. Denote by \( X^* = \{x_1x_2...x_n|x_i \in X, n \geq 0\} \) the set of all finite words over \( X \) (including the empty word denoted \( \emptyset \)). The length of a word \( v = x_1x_2...x_n \in X^n \) is denoted by \( |v| = n \). Let \( X^\omega \) be the space of all infinite sequences (words) \( x_1x_2..., x_i \in X \), with the product topology of discrete sets \( X \). The space \( X^\omega \) is homeomorphic to the Cantor set, i.e., it is compact totally disconnected metrizable topological space without isolated points.

**Automata.** An invertible automaton \( A \) over the alphabet \( X \) is the triple \( (S, \pi, \lambda) \), where \( S \) is the set of states of the automaton, \( \lambda : S \times X \to S \) is the transition function, and \( \pi : S \times X \to X \) is the output function such that for every \( s \in S \) the map \( \pi(s, \cdot) : X \to X \) is a permutation on \( X \). All automata in the paper are invertible, and further we omit the term invertible. An automaton is finite if the set of its states is finite. An automaton \( A \) is represented by a directed labeled graph (Moore diagram), whose vertices are the states of \( A \), and for every state \( s \in S \) and every letter \( x \in X \) there is an arrow \( s \to \lambda(s, x) \) labeled by the pair \( s|\pi(s, x) \). This diagram contains complete information about the automaton, and we identify the automaton with its Moore diagram. The notation \( A \) is also used for the state set of the automaton \( A \), so that one can talk about a state \( s \in A \). A subset \( B \subset A \) (with induced edges) is a subautomaton of \( A \) if \( \lambda(s, x) \in B \) for every \( s \in B \) and \( x \in X \).

**Automaton groups.** For every state \( s \in A \) and every finite word \( v = x_1x_2...x_n \in X^* \) there exists a unique path in the automaton \( A \) starting at the state \( s \) and labeled by \( x_1|y_1, x_2|y_2, ..., x_n|y_n \) for some \( y_i \in X \). Then the word \( y_1y_2...y_n \) is called the image of \( x_1x_2...x_n \) under \( s \), and the end state of this path is denoted by \( s|v \). In other words, for every finite word \( v \in X^* \) we define the image \( s(v) \) of \( v \) under \( s \) and the state \( s|v \) recursively by the
the automaton $A$ rules

$$s|_x = \lambda(s, x), \quad s|_{xv} = s|_v, \quad \text{and} \quad s(x) = \pi(s, x), \quad s(xv) = s(x)s|_x(v)$$

for every $x \in X$ and $v \in X^*$. The action of states of $A$ on the set $X^n$ can be given by the automaton $A^{(n)}$ obtained from $A$ by passing to the power $X^n$ of the alphabet $X$. The automaton $A^{(n)}$ is defined over the alphabet $X^n$, and has the same states as $A$, but the arrows are $s \to s|_v$, labeled by $v|s(v)$ for every $s \in A$ and $v \in X^n$.

In the same way we get an action of every state $s \in A$ on the space $X^\omega$ by looking at the infinite paths in the automaton. Since the automata are invertible, all transformations $s \in A$ are also invertible. The group $G$ generated by all states $s \in A$ is called the automaton group $A$ generated by $A$. Since every state preserves the length of words in its action on $X^\omega$, the states act by isometries on the space $X^\omega$, and every automaton group is a subgroup of $\text{Iso}(X^\omega)$. The automaton groups generated by $A$ and $A^{(n)}$ coincide viewed as subgroups of $\text{Iso}(X^\omega)$ with the natural identification $X^\omega = (X^n)^\omega$. All automata in the paper are supposed to be minimized, i.e., different states of a given automaton act differently on $X^\omega$. Hence we identify the states with the respective transformations of $X^\omega$.

An alternative approach is through self-similar actions. A faithful action of a group $G$ on the set $X^* \cup X^\omega$ is called self-similar if for every $g \in G$ and $v \in X^*$ there exist $u \in X^{|v|}$ and $h \in G$ such that $g(\omega v) = uh(\omega)$ for all $w \in X^* \cup X^\omega$. The element $h$ is called the restriction (state) of $g$ at $v$ and is denoted by $g|_v$. We are using left actions, i.e., $(g_1g_2)(v) = g_1(g_2(v))$, and hence the restrictions have the property

$$(g_1g_2)|_v = g_1|_{g_2(v)}g_2|_v \quad \text{for any } g_1, g_2 \in G \text{ and } v \in X^*.$$

Every self-similar action of a group $G$ is given by the complete automaton $A(G)$ of the group, whose states are the elements of $G$, and the arrows are $g \to g|_x$ labeled by $x|g(x)$ for every $g \in G$ and $x \in X$.

An automaton group $G$ is called contracting if there exists a finite subset $\mathcal{N} \subset G$ with the property that for every $g \in G$ there exists $n \in \mathbb{N}$ such that $g|_v \in \mathcal{N}$ for all words $v \in X^*$ of length $|v| \geq n$. The smallest set $\mathcal{N}$ with this property is called the nucleus of the group. Notice that a finitely generated contracting group can be generated by a finite automaton. A group $G$ generated by a finite automaton is contracting if and only if the complete automaton $A(G)$ of the group contains only finitely many simple directed cycles.

Throughout the paper by a cycle in an automaton we mean a simple directed cycle. States that lie on cycles are called circuit. An element $g$ of an automaton group $G$ is circuit if there exists a nonempty word $v \in X^*$ such that $g|_v = g$.

**Polynomial automata.** A cycle in an automaton is called trivial if it is a loop at the state acting trivially on $X^\omega$ (trivial state, denoted $e$). A finite automaton $A$ is called polynomial if different nontrivial cycles in $A$ are disjoint. A polynomial automaton $A$ is of degree $m$ if the largest number of nontrivial cycles in $A$ connected by a directed path is equal to $m + 1$. A finite automaton $A$ is polynomial (of degree $\leq m$) if and only if the number of directed paths in $A$ of length $n$ that do not pass through the trivial state is bounded
by a polynomial in $n$ (of degree $\leq m$). The set of all states (transformations of $X^\omega$) of all polynomial automata of degree $\leq m$ forms a group $\text{Pol}(m)$ called the group of polynomial automata of degree $m$. Every finitely generated subgroup of $\text{Pol}(m)$ is a subgroup of some automaton group generated by a polynomial automaton.

We classify the states of a polynomial automaton $A$ as follows. A state $s \in A$ is finitary if there exists $n \in \mathbb{N}$ such that $s|_v = e$ for all $v \in X^n$. The finitary states are precisely the elements of $\text{Pol}(-1)$. The states from $\text{Pol}(m) \setminus \text{Pol}(m - 1)$ are called polynomial of degree $m$. For every polynomial automaton $A$ there exists $n$ such that for every $s \in A$ and $v \in X^n$ the state $s|_v$ is either circuit or has degree less than the degree of $s$. Notice, that if $s \in A$ is a nontrivial circuit state, then for every $n$ there exists precisely one word $v \in X^n$ such that $s$ and $s|_v$ have the same degree as polynomial states, and $s|_u$ for all $u \in X^n$, $u \neq v$, has degree less than the degree of $s$.

**Schreier graphs.** Let $G$ be an automaton group generated by a finite subset $S$. The Schreier graph $\Gamma_n(S) = \Gamma_n(G, S)$ is the graph with the set of vertices $X^n$, where two vertices $v$ and $u$ are adjacent if there exists $s \in S \cup S^{-1}$ such that $s(v) = u$. The orbital Schreier graph $\Gamma_w(S) = \Gamma_w(G, S)$ for $w \in X^\omega$ is the graph whose vertex set is the orbit $G(w)$ and two vertices $v$ and $u$ are adjacent if there exists $s \in S \cup S^{-1}$ such that $s(v) = u$. The orbital Schreier graphs $\Gamma_w(S)$ are precisely the connected components of the Schreier graph $\Gamma(G, S, X^\omega)$ of the action $(G, X^\omega)$. Every orbital Schreier graph $\Gamma_w$ is a limit of the finite Schreier graphs $\Gamma_n$ in the local Gromov-Hausdorff topology on pointed graphs.

**Growth of graphs.** Let $\Gamma$ be a locally finite connected graph. The growth function $\gamma_v(n)$ of $\Gamma$ with respect to its vertex $v$ is equal to the number of vertices in the closed ball $B(v, n)$ of radius $n$ centered at $v$. There is a partial order on the growth functions. Given two functions $f, g : \mathbb{N} \to \mathbb{N}$ we say that $f$ hasgrowth not greater than $g$ (denoted $f \preceq g$) if there exists a constant $C > 0$ such that $f(n) \leq g(Cn)$ for all $n \in \mathbb{N}$. If $f \preceq g$ and $g \preceq f$ then $f$ and $g$ are called equivalent $f \sim g$ and have the same growth. Formally, by the growth we can understand the equivalence class of a function. Then, for any two vertices of the graph $\Gamma$, the respective growth functions are equivalent, and one can talk about the growth of $\Gamma$.

A graph $\Gamma$ has subexponential growth if its growth function has growth not greater than the exponential growth $a^n$ with $a > 1$, and is not equivalent to it. The growth is superpolynomial if it is greater than every polynomial function, and the growth is intermediate if it is superpolynomial and subexponential.

Let $G$ be a finitely generated group with finite generating set $S$. The length $l(g)$ of $g \in G$ with respect to $S$ is equal to the distance between the trivial element $e$ and $g$ in the Cayley graph $\Gamma(G, S)$, i.e., $l(e) = 0$ and

$$l(g) = l_S(g) = \min\{n \mid g = s_1s_2\ldots s_n \text{ for } s_i \in S \cup S^{-1}\}.$$

The growth function $\gamma$ of the group $G$ is the growth function of the Cayley graph $\Gamma(G, S)$ with respect to the vertex $e$, i.e., $\gamma(n)$ is equal to the number of elements $g \in G$ of length $l(g) \leq n$. The growth functions of the Cayley graphs $\Gamma(G, S_1)$ and $\Gamma(G, S_2)$, for any two finite generating sets $S_1$ and $S_2$, are equivalent.
The growth of Schreier graphs $\Gamma_w(G, S)$ of an automaton group $G$ also does not depend on the choice of a finite generating set $S$ of the group. Working with automaton groups it is useful to assume that a generating set $S$ is self-similar (automaton), i.e., $s_i|_v \in S$ for every $s \in S$ and $v \in X^*$. For example, we will frequently use the following observation. If there is a presentation $g = s_1 s_2 \ldots s_n$ for $s_i \in S$ then for every $v \in X^*$ we get the induced presentation
\[ g|_v = s_1|_{v_1} s_2|_{v_2} \ldots s_{n-1}|_{v_{n-1}} s_n|_{v_n}, \]
where $v_n = v$, and $v_{i-1} = s_i(v_i)$ for $i = n, n-1, \ldots, 2$. In particular, if $S$ is self-similar then $s_i|_{v_i} \in S$ and $l(g|_v) \leq l(g)$. Also, it is usually assumed that $S$ is symmetric, i.e., $S = S^{-1}$.

All logarithms in the paper are with base 2, except if directly indicated. Usually, a logarithm appears as $C \log n$, and the base can be hidden in the constant $C$. Also (to avoid some multiple brackets) we use the convention that $\log 0 = 1$ and $\log 1 = 1$ so that $\log n > 0$ for all $n \in \mathbb{N} \cup \{0\}$, otherwise one can just replace $\log n$ by $\log(n+2)$.

## 3 Proof of Theorem 1

Let us recall how to prove that for contracting groups the Schreier graphs $\Gamma_w$ have polynomial growth (see [14, Section 2.13.4], [1]). Let $G$ be a finitely generated contracting group with nucleus $\mathcal{N}$, and let $S$ be a finite symmetric self-similar generating set of $G$ that contains $\mathcal{N}$. We can choose a constant $C$ such that $(s_1 s_2)|_v \in \mathcal{N}$ for all $s_1, s_2 \in S$ and every word $v \in X^*$ of length $|v| \geq C$. Consider an element $g \in G$ and let $g = s_1 s_2 \ldots s_n$ for $s_i \in S$ with $n = l(g)$. Then $(s_i s_{i+1})|_v \in \mathcal{N} \subset S$ and hence the element $g|_v$ has length $\leq (n+1)/2$. It follows that $g|_v \in \mathcal{N}$ for all words $v \in X^*$ of length $|v| \geq C \log n$. This justifies the term contracting group: the length of restrictions exponentially decreases (“contracts”) until they become elements of the nucleus.

Now consider the Schreier graph $\Gamma_w = \Gamma_w(G, S)$ for a sequence $w = x_1 x_2 \ldots \in X^w$. Let $B(w, n)$ be the ball of radius $n$ in the graph $\Gamma_w$ centered at the vertex $w$. Notice that if $g(v_1 w_1) = v_2 w_2$ for $v_1, v_2 \in X^k$ and $w_1, w_2 \in X^\omega$ then $g|_{v_1}(w_1) = w_2$. Let $k$ be the least integer greater than $C \log n$. Then each sequence in the ball $B(w, n)$ is of the form $v_2 w_2$ for some $v_2 \in X^k$ and $w_2 = h(x_{k+1} x_{k+2} \ldots)$ for some $h \in \mathcal{N}$. Hence
\[ |B(w, n)| \leq |X|^{C \log n + 1} \cdot |\mathcal{N}(x_{k+1} x_{k+2} \ldots)| \leq |X|^{C \log n + 1} \cdot |\mathcal{N}|, \]
which is a polynomial in $n$. More precise estimate can be given using the contracting coefficient of the group (see [14, Proposition 2.13.8]).

We want to apply similar arguments to groups generated by polynomial automata, and first we will establish certain weak contracting properties of these groups.

**Lemma 1.** Let $G$ be a finitely generated subgroup of $\text{Pol}(m)$. There exists a constant $C$ such that for every $g \in G$ and every word $v \in X^*$ of length $|v| \geq C (\log l(g))^{m+1}$ the state $g|_v$ is either circuit or belongs to $\text{Pol}(m-1)$. 


To prove Lemma 1 we use induction on \( m \). However there is one slight difficulty, that in order to apply the induction hypothesis we may need to consider the length of the element \( g \mid v \) for a different generating set than the length of \( g \). To overcome this problem we will make a few assumptions so that the groups involved in the induction have consistent generating sets.

Without loss of generality, we can assume that the group \( G \) is an automaton subgroup of \( \text{Pol}(m) \). Let \( S \) be a finite symmetric self-similar generating set of \( G \). Let \( G_k \) be an automaton subgroup of \( G \) generated by \( S_k = S \cap \text{Pol}(k) \) for \( k = -1, 0, \ldots, m \). The generating set \( S_k \) is again symmetric and self-similar. Notice that, in general, the group \( G_k \) may not coincide with \( G \cap \text{Pol}(k) \).

Let us prove that the length of cycles in the complete automaton \( A(G) \) of the group \( G \) is bounded. Without loss of generality, by passing to a power of the alphabet, we can assume that the group \( G \) and the generating set \( S \) satisfy the following conditions:

1) for every \( s \in S \) and \( x \in X \) the state \( s \mid x \) is either circuit or has degree less than the degree of \( s \);

2) for every nontrivial circuit state \( s \in S \) there exists (unique) \( x \in X \) such that \( s \mid x = s \) (every cycle in \( S \) is a loop).

(For example, one can pass to the alphabet \( X^k \), where \( k \) is a multiple of the length of every cycle in \( S \), and is greater than the diameter of \( S \)). Let us prove that then every cycle in the automaton \( A(G) \) is a loop, i.e., for every circuit element \( g \in G \) there exists a letter \( x \in X \) such that \( g \mid x = g \). Assume \( g \mid v = g \) for a nonempty word \( v \in X^* \) and let \( g = s_1s_2 \ldots s_n \) for \( s_i \in S \) with \( n = l(g) \). Notice that \( s_i \notin S_{i-1} \) for every \( i \), because otherwise \( g \mid v = g \) can be expressed as a product \( s_1s_2 \ldots s_i \) with \( l < n \). Over all presentations of \( g \) as a product \( s_1s_2 \ldots s_n \) we consequently for \( k = 0, 1, \ldots, m \) choose the one with the largest number of circuit generators from \( S_k \) and then with the largest number of the rest generators from \( S_k \).

Consider the induced presentation (1):

\[
g = g \mid v = s_1 \mid v_1 s_2 \mid v_2 \ldots s_n \mid v_n,
\]

where \( v_{i-1} = s_i(v_i) \) with \( v_n = v \). By our choice of the presentation \( g = s_1s_2 \ldots s_n \) we should get

\[
s_1 \mid v_1 = s_1, \quad s_2 \mid v_2 = s_2, \quad \ldots, \quad s_n \mid v_n = s_n.
\]

Then

\[
s_1 \mid x_1 = s_1, \quad s_2 \mid x_2 = s_2, \quad \ldots, \quad s_n \mid x_n = s_n,
\]

where \( x_i \) is the first letter of \( v_i \), and we get \( g \mid x_n = g \). It also follows that \( g \mid y \in G_{m-1} \) for \( y \in X \), \( y \neq x_n \), and in its induced presentation \( g \mid y = s_1 \mid y_1 s_2 \mid y_2 \ldots s_n \mid y_n \) every generator \( s_i \mid y_i \) belongs to the set \( S_{m-1} \). In particular, the length of \( g \mid y \) with respect to the generating set \( S_{m-1} \) of the group \( G_{m-1} \) is not greater than \( n \). Further we will use this property without specifying the generating set.

To the rest of this section we always assume that the group \( G \) and the generating set \( S \) satisfy the above assumptions.

We will prove a slightly stronger formulation of Lemma 1.
Lemma 1*. There exists a constant $C$ such that for every $g = s_1 s_2 \ldots s_n \in G$ for $s_i \in S$ and every word $v \in X^*$ of length $|v| \geq C (\log n)^{m+1}$ either $g|_v \in G_{m-1}$ and in the induced presentation $g|_v = s_1|_v s_2|_v \ldots s_n|_v$ every generator $s_i|_v$ belongs to $S_{m-1}$, or the element $g|_v$ is circuit, $g|_v x = g|_v$ for some $x \in X$, and the induced presentations of $g|_v$ and $g|_v x$ coincide as words in generators.

Proof. The proof goes by induction on $m$. For $m = -1$ the group $G$ is a subgroup of the group $	ext{Pol}(-1)$ of finitary elements. There exists a constant $C$ such that for all words $v$ of length $|v| \geq C$ we have $s|_v = e$ for every $s \in S$ (actually by our assumption 1) on the generating set $S$ we can take $C = 1$). Hence for every product $g = s_1 s_2 \ldots s_n \in G$ for $s_i \in S$ we get $g|_v = e$ and in the induced presentation of $g|_v$ every element is trivial.

We assume the lemma holds for the groups $G_k$ (with $m$ replaced by $k$) for $k = -1, 0, \ldots, m-1$ with some common constant $C_1$.

Let $g \in G$ be a circuit element and $g|_x = g$ for a letter $x \in X$. Let $g = s_1 s_2 \ldots s_k$, $s_i \in S$, be a presentation such that the induced presentation of $g = g|_x$ coincides with $s_1 s_2 \ldots s_k$ as a word in generators. Then for every word $v \in X^*$ either $g|_v \in G_{m-1}$ and every generator in the induced presentation of $g|_v$ belong to $S_{m-1}$, or $g|_v = g$ (in the case $v = x x \ldots x$) and the induced presentation of $g|_v$ coincides with $s_1 s_2 \ldots s_k$. Consider the product $h g$ for $h = t_1 t_2 \ldots t_n \in G_{m-1}$, $t_i \in S_{m-1}$, and its restrictions $(h g)|_v = h|_g| g|_v$ for words $v \in X^*$ of length $|v| \geq C_1 (\log n)^m$. We have the following cases. If $g|_v \in G_{m-1}$ then $(h g)|_v \in G_{m-1}$, so further assume $g|_v = g$ and let us look at $h|_g| g|_v$. By the assumption on the length of the word $v$ the element $h|_g| g|_v$ either belongs to $G_{m-2}$ or it is circuit (with the specific induced presentation).

If $h|_g| g|_v \in G_{m-2}$ then $(h g)|_v = h g'$ for $h' \in G_{m-2}$.

If $h|_g| g|_v$ is circuit and $h|_g| g|_v x = h|_g| g|_v$ then $(h g)|_v$ is circuit, here $(h g)|_v x = (h g)|_v$.

If $h|_g| g|_v$ is circuit and $h|_g| g|_v \neq h|_g| g|_v$ then $(h g)|_v x = h g'$ for $h' \in G_{m-2}$.

Hence in all cases the element $(h g)|_u$ for $u \in X^{|v|+1}$ is either circuit, or belongs to $G_{m-1}$, or it is of the form $h g'$ for $h' \in G_{m-2}$. In the last case we can apply the same arguments to the product $h g'$. After $m$ steps we get either a circuit state or an element of $G_{m-1}$. In the worst case we need to take the words of length

$$C_1 (\log n)^m + 1 + C_1 (\log n)^{m-1} + 1 + \ldots + C_1 (\log n)^0 + 1.$$ 

Choose a constant $C_2$ such that $C_2 (\log n)^m$ is greater than the value of the equation above. Then the element $(h g)|_v$ for words $v \in X^*$ of length $|v| \geq C_2 (\log n)^m$ is either circuit or belongs to $G_{m-1}$, and the conditions on the induced presentation of $(h g)|_v$ are satisfied.

Let $g_1, g_2 \in G$ be circuit elements and consider the product $g_1 h g_2$ for $h \in G_{m-1}$. Assume $g_i$ is expressed as a product in generators of length $k_i$ with the same properties as the one of $g$ above, and $h$ is expressed as a product of $n$ generators from $S_{m-1}$. Then the element $(g_1 h g_2)|_v$ for $v \in X^*$ of length $|v| \geq C_3 (\log (k_1 + n + k_2))^m$ with $C_3 = 2 C_2$ is either circuit or belongs to $G_{m-1}$ with the specific induced presentation.

Now consider an arbitrary element $g = s_1 s_2 \ldots s_n \in G$ for $s_i \in S$. Let us partition the product $s_1 s_2 \ldots s_n$ on blocks

$$g = h_0 g_1 h_1 g_2 h_2 \ldots g_i h_i,$$ (2)
such that every block $h_i$ contains only generators from $S_{m-1}$ (may be trivial), and $g_i$ is either a generator from $S \setminus S_{m-1}$, or $g_i$ is circuit and the induced presentation of $g_i|_x$ coincides with the presentation $g_i$ for some $x \in X$. Notice that the sum of lengths of all $h_i$ and $g_i$ is equal to $n$. By passing to the induced presentation of the state $g|_x$ on any letter $x \in X$ we can assume that every element $g_i$ is circuit. Consider every product $g_1h_1g_2, g_3h_3g_4$, and so on, in the presentation $\langle g \rangle$. Every such block restricted to a word of length $\geq C_3(\log n)^m$ is either circuit or belongs to $G_{m-1}$. Hence the state $g|_v$ for words $v \in X^*$ of length $|v| \geq C_3(\log n)^m + 1$ can be expressed as a product $\langle 2 \rangle$ with $\leq (l+1)/2$ positions with some circuit elements $g_i$. Applying the same procedure $\log l + 1$ times we get either a circuit state or an element of $G_{m-1}$. Choose a constant $C$ such that $C(\log n)^{m+1} \geq (C_3(\log n)^m + 1)(\log n + 1)$ (here we use $l \leq n$). Then $g|_v$ for $v \in X^*$ of length $|v| \geq C(\log n)^{m+1}$ is either circuit or belongs to $G_{m-1}$, and in both cases it has the required induced presentation. □

**Corollary 2.** The word problem in every finitely generated subgroup of $\text{Pol}(m)$ is solvable in subexponential time.

**Proof.** We use the same notations and assumptions as above.

Consider an element $g = s_1s_2 \ldots s_n \in G$ for $s_i \in S$. Let $k$ be the least integer greater than $C(\log n)^{m+1}$. The element $g$ is trivial if and only if it acts trivially on $X^k$ and every element $g|_v$ for $v \in X^k$ is trivial. Notice that the size of $X^k$ is subexponential in $n$. Consider the induced presentation $g|_v = s'_1s'_2 \ldots s'_n$ with $s'_i = s_i|_v$. If $s'_i \in S_{m-1}$ for all $i$, then the problem reduces to the word problem in the group $G_{m-1}$. Otherwise, $g|_v$ is circuit and there exists $x \in X$ such that the induced presentation $g|_{v|x} = s'_1|_{x_1}s'_2|_{x_2} \ldots s'_n|_{x_n}$ coincides with the presentation $g|_v = s'_1s'_2 \ldots s'_n$ letter by letter, i.e., $s'_i|_{x_i} = s'_i$ for all $i$. Then $g|_v$ is trivial if and only if it acts trivially on $X$ and every element $g|_v|_y \in G_{m-1}$ for $y \in X$, $y \neq x$, is trivial. Again the problem reduces to the word problem in $G_{m-1}$. By induction we conclude that the word problem is solvable in subexponential time with an upper bound $|X|^{C_1(\log n)^{m+1}}$ for some constant $C_1$. □

We are ready to prove the main result.

**Theorem 3.** Let $G$ be a finitely generated subgroup of $\text{Pol}(m)$. There exists a constant $C$ such that every orbital Schreier graph $\Gamma_w(G)$ for $w \in X^\omega$ has subexponential growth not greater than $|X|^{C(\log n)^{m+1}}$.

**Proof.** The proof goes by induction on $m$. For $m = -1$ the group $G < \text{Pol}(-1)$ is finite, and every Schreier graph $\Gamma_w$ has at most $|G|$ vertices. We can also start the induction from $m = 0$ using the fact that in this case the group $G < \text{Pol}(0)$ is contracting (see $\langle 3 \rangle$).

We suppose by induction that the statement holds for the group $G_{m-1}$ and every Schreier graph $\Gamma_{w|}(G_{m-1}, S_{m-1})$ has subexponential growth not greater than $|X|^{C_1(\log n)^m}$ with some constant $C_1$.

Fix a sequence $w = x_1x_2 \ldots \in X^\omega$ and consider the ball $B(w, n)$ in the graph $\Gamma_w = \Gamma_w(G, S)$ of radius $n$ centered at the vertex $w$. If $g(v_1w_1) = v_2w_2$ for $v_1, v_2 \in X^k$ and $w_1, w_2 \in X^\omega$ then $g|_{v_1}(w_1) = w_2$. Hence for every fixed $k$ each sequence in the ball $B(w, n)$
is of the form $v_2w_2$ for some $v_2 \in X^k$ and $w_2 = h(x_{k+1}x_{k+2} \ldots)$ for some $h \in \mathcal{N}(n, k)$, where
\[
\mathcal{N}(n, k) = \{g|_{x_1x_2\ldots x_k} : g \in G \text{ and } l(g) \leq n\}.
\]
It follows that
\[
|B(w, n)| \leq |X|^k \cdot |\mathcal{N}(n, k)(v)|,
\]
where $v = x_{k+1}x_{k+2} \ldots$.

Let $H_n = \{h \in G_{m-1} : l_{S_{m-1}}(h) \leq n\}$ be the ball of radius $n$ in the group $G_{m-1}$ with respect to its generating set $S_{m-1}$.

Let $k$ be the least integer greater than $C(\log n)^{m+1}$ given in Lemma [I]. Then for every $g \in \mathcal{N}(n, k) \setminus H_n$ there exists $x \in X$ such that $g|_x = g$. Hence
\[
\mathcal{N}(n, k) \subset \bigcup_{x \in X} \mathcal{N}^x(n, k) \cup H_n,
\]
where $\mathcal{N}^x(n, k) = \{g \in \mathcal{N}(n, k) : g|_x = g\}$ for $x \in X$. By induction hypothesis the size of the orbit $H_n(v)$ is not greater than $|X|^{C_1(\log n)^m}$. Let us estimate the size of the orbits $\mathcal{N}^x(n, k)(v)$. Let $x \in X$ be the first letter of the word $v$ and consider the following cases. If $v = xx \ldots = x^\infty$ then for $g \in \mathcal{N}^x(n, k)$ we have $g(v) = z^\infty$ for some $z \in X$; and for $g \in \mathcal{N}^y(n, k)$ with $y \in X$, $y \neq x$, we have $g(v) = zh(x^\infty)$ for some $z \in X$ and $h \in H_n$. Hence we get estimates
\[
|\mathcal{N}^x(n, k)(v)| \leq |X| \quad \text{and} \quad |\mathcal{N}^y(n, k)(v)| \leq |X| \cdot |H_n(v)|
\]
for $y \in X$, $y \neq x$. If $v = x^t x_1 v_1$ for $x_1 \neq x$ then for $g \in \mathcal{N}^x(n, k)$ we have $g(v) = z^t z_1 h(v_1)$ for some $z, z_1 \in X$ and $h \in H_n$; and for $g \in \mathcal{N}^y(n, k)$ with $y \in X$, $y \neq x$, we have $g(v) = zh(x^{t-1} x_1 v_1)$ for some $z \in X$ and $h \in H_n$. Hence we get estimates
\[
|\mathcal{N}^x(n, k)(v)| \leq |X|^2 \cdot |H_n(v_1)| \quad \text{and} \quad |\mathcal{N}^y(n, k)(v)| \leq |X| \cdot |H_n(x^{t-1} x_1 v_1)|
\]
for $y \in X$, $y \neq x$. Summarizing all estimates we get
\[
|B(w, n)| \leq |X|^{C_1(\log n)^{m+1}+1} \cdot |X|^3 \cdot |X|^{C_1(\log n)^m} \leq |X|^{C_2(\log n)^{m+1}}
\]
for some constant $C_2$.

As a corollary we recover the following result from [5, Corollary 4.6].

**Corollary 4.** The orbital Schreier graphs $\Gamma_w$ for $w \in X^\omega$ of groups generated by polynomial automata are amenable.

### 4 A generalization of Theorem [1]

Theorem [1] can be generalized to automaton groups with a certain combination of contracting and polynomial properties.
Theorem 5. Let $A$ be a finite automaton with subautomaton $B$ such that different cycles in $A \setminus B$ are disjoint, and the group generated by $B$ is contracting. Let $G$ be the automaton group generated by $A$. There exists a constant $C$ such that every orbital Schreier graph $\Gamma_w(G)$ for $w \in X^\omega$ has subexponential growth bounded above by $|X|^{C(\log n)^{m+1}}$, where $m$ is the maximal number of different cycles in $A \setminus B$ connected by a directed path.

Proof. The proof goes by induction on $m$ following the same strategy as in the proof of Theorem 3.

First, we make a few assumptions about the generating sets to make our life easier. We can assume that the generating sets $A$ and $B$ are symmetric, and $B$ contains the nucleus $\mathcal{N}$ of the contracting group $\langle B \rangle$. Let $S_k$ for $k = 0, 1, \ldots, m$ be the largest subautomaton of $A$ such that $S_k \setminus B$ contains at most $k$ cycles connected by a directed path. Then every $S_k$ is also symmetric and self-similar. We pass to a power of the alphabet so that every cycle in $A \setminus B$, and hence in $S_k \setminus S_0$, is actually a loop; and for every $s \in S_k \setminus S_0$ and $x \in X$ either $s|x \in S_{k-1}$ or $s|x$ is circuit.

Let $G_k$ be the group generated by $S_k$. In particular, $G = G_m$ and we use notation $S = S_m$. The group $G_0$ is contracting with nucleus $\mathcal{N}$. Indeed, by construction, there are no cycles in $S_0 \setminus \mathcal{N}$, and hence $s|_u \in \mathcal{N}$ for all $s \in S_0$ and all words $v$ of length greater than the diameter of $S_0 \setminus \mathcal{N}$.

Lemma 2. The length of cycles in the complete automaton $A(G)$ of the group $G$ is bounded.

Proof. Take a circuit element $g \in G$. Let $g|_v = g$ for a nonempty word $v$, and $g|_u \neq g$ for every beginning $u$ of $v$. Over all presentation of $g$ as a product $s_1s_2\ldots s_n$ for $s_i \in S$ with $n = l(g)$, we consequently choose the one with the largest number of circuit elements $s_i \in S_k$ and then with the largest number of any elements $s_i \in S_k$, for $k = 0, 1, \ldots, m$. Consider the induced presentation (1):

$$g = g|_v = s_1|_{v_1}s_2|_{v_2}\ldots s_n|_{v_n},$$

where $v_{i-1} = s_i(v_i)$ with $v_n = v$. Notice, that this presentation also satisfies the above assumptions, as is the induced presentation $g = g|_{v\ldots v}$ for every iteration $v\ldots v$ of the word $v$. In particular, every $s_i$ is actually circuit, because otherwise the presentation given by $g = g|_{v\ldots v}$ would contain more circuit generators than the chosen one.

Two consecutive elements $s_i$ and $s_{i+1}$ in the presentation cannot both lie in $\mathcal{N}$. Indeed, in this case $(s_is_{i+1})|_u \in \mathcal{N} \subset S$ for all sufficiently large $u$. Hence $g = g|_{v\ldots v}$ can be expressed as a product of less than $n$ generators for large enough iteration $v\ldots v$, contradicting $n = l(g)$.

Our choice of the presentation $g = s_1s_2\ldots s_n$ and the assumptions on the generating set $S$ force $s_i|_{v_i} = s_i$ for every $s_i \notin \mathcal{N}$. Hence $v_i = x^{m_i}$ (here $m_i = |v_i|$), where $x_i \in X$ is the unique letter such that $s_i|x_i = s_i$. If $s_i \in \mathcal{N}$ for $i < n$ then $s_{i+1} \notin \mathcal{N}$ and we still get $v_i = x^{m_i}$ with $x_i = s_{i+1}(x_{i+1})$.

Assume $s_n \notin \mathcal{N}$, and hence $v = x^{m}$ with $x = x_n$. Consider the induced presentations of $g|_{x^l}$ for every $l = 1, 2, \ldots$. If $s_i \notin \mathcal{N}$ then the $i$-th generator in the presentation of every $g|_{x^l}$ remains the same and is equal to $s_i$. Let us trace the positions with elements from $\mathcal{N}$. If $s_i \in \mathcal{N}$ then the $i$-th element in the presentation of $g|_{x^l}$ is equal to $s_i|x_i^l$. Hence these
The \( i \)-th elements change according to the vertices of the path in the nucleus \( N \) that starts at the state \( s_i \) and goes along arrows with left label \( x_i \). It follows that every position is preperiodic. We can eliminate the preperiods by passing from the presentation \( g = s_1 s_2 \ldots s_n \) to the induced presentation \( g = g|_{x^l} \) for large enough \( l \), and hence we assume that every position repeats periodically. If \( l \) is a multiple of the lengths of every cycle in \( N \), then the induced presentation of \( g|_{x^l} \) coincides with \( s_1 s_2 \ldots s_n \) letter by letter, and hence \( g = g|_{x^l} \).

It follows that the length of the word \( v \) is bounded by the least common multiple of the length of cycles in the nucleus \( N \).

If \( s_n \in N \) then \( s_{n-1} \not\in N \), \( v = s_{n-1}^{-1}(x_{n-1}^m) \), and we can apply the same arguments. \( \square \)

Let \( L \) be the upper bound on the length of cycles in \( A(G) \). Just to notice, in general, we cannot pass to a power of the alphabet to transform every cycle into a loop, i.e., to make \( L = 1 \), as was possible in the case of polynomial automata.

Further, we will use the fact proved in Lemma \( \square \) that if \( g \in G_k \) for \( k \geq 1 \) is circuit and \( g|_v = g \) then \( g|_u \in G_{k-1} \) for every \( u \in X^{|v|} \), \( u \neq v \), and the length of \( g|_u \) with respect to \( S_{k-1} \) is not greater than the length of \( g \) with respect to \( S_k \).

The formulation of Lemma 1 remains the same (with \( m \geq 1 \)).

**Lemma 3.** There exists a constant \( C \) such that for every \( g = s_1 s_2 \ldots s_n \in G \) for \( s_i \in S \) and every word \( v \in X^* \) of length \(|v| \geq C (\log n)^{n+1} \) the element \( g|_v \) is either circuit, or \( g|_v \in G_{m-1} \).

**Proof.** The proof is basically the same as the one of Lemma 1*. Let us indicate only the main argument.

We assume the lemma holds for the groups \( G_k \) for \( k = 1, \ldots, m-1 \) with some common constant \( C_1 \). Let \( g \in G \) be a circuit element. Consider the product \( hg \) for \( h \in G_{m-1} \) and its restrictions \((hg)|_v = h|_{g(v)} g|_v \) for words \( v \in X^* \) of length \(|v| \geq C_1 (\log l(h))^m \). We have the following cases. If \( g|_v \in G_{m-1} \) then \((hg)|_v \in G_{m-1} \). So further assume \( g|_v \not\in G_{m-1} \), and hence \( g|_v \) and \( g \) lie on the same cycle. There exists a nonempty word \( u \) of length \(|u| \leq L \) such that \( g|_u = g|_v \). Let us look at \( h|_{g(u)} \), which either belongs to \( G_{m-2} \) or is circuit. If \( h|_{g(u)} \in G_{m-2} \) then \((hg)|_v = h'|_g \) for circuit \( g' = g|_v \) and \( h' = h|_{g(v)} \in G_{m-2} \). Otherwise \( h|_{g(u)} \) is circuit, and let \( l \leq L \) be the length of the cycle at \( h|_{g(v)} \). Then the length of the word \( u' \) is a multiple of \(|u| \) and of \( l \). If \( h|_{g(u)} |_{g(u')} = h|_{g(v)} \) then \((h|_{g(v)} g|_v)|_{u'} = h|_{g(v)} g|_v \) and the state \((hg)|_v \) is circuit; otherwise \( h|_{g(v)} \not\in G_{m-2} \). Hence in all cases the state \((hg)|_w \) for a word \( w \) of length \(|w| \geq C_1 (\log l(h))^m + L^2 \) is either circuit, or it belongs to \( G_{m-1} \), or it is of the form \( h'g' \) for circuit \( g' \) and \( h' \in G_{m-2} \) with \( l(h') \leq (l(h)) \). Eventually, in the worst case, we will need to consider the product \( hg \) for circuit \( g \) and \( h \in N \subset S_0 \). In this case we can apply the same arguments as in the proof of Lemma \( \square \).

The rest of the proof is the same. \( \square \)

Now we can return to the growth of Schreier graphs. If \( m = 0 \) then the group \( G \) is contracting and the statement holds. We suppose by induction that the statement holds for the group \( G_{m-1} \) with some constant \( C_1 \).
Take a sequence \( w = x_1x_2\ldots \in X^\omega \) and consider the Schreier graph \( \Gamma_w = \Gamma_w(G,S) \). We will use the estimate (3).

Let \( H_n = \{ h \in G_m : l_s(m,h) \leq n \} \) be the ball of radius \( n \) in the group \( G_m \) with respect to its generating set \( S_m \).

Let \( k \) be the least integer greater than \( C(\log n)^m+1 \) from Lemma 3. Then for every \( g \in N_{(n,k)} \setminus H_n \) there exists a nonempty word \( u \) of length \( |u| \leq L \) such that \( g|_u = g \). Hence

\[
N_{(n,k)} \subset \bigcup_{|u| \leq L} N_{(n,k)}^u \cup H_n,
\]

where \( N_{(n,k)}^u = \{ g \in N_{(n,k)} : g|_u = g \} \). By induction hypothesis the size of the orbit \( H_n(v) \) is not greater than \( |X|^{|C_1(\log n)|^m} \). Let us estimate the size of the orbits \( N_{(n,k)}^u(v) \). If \( v = uu\ldots = u^\infty \) then for every \( g \in N_{(n,k)}^u \) we get \( g(v) = z^\infty \) for some \( z \in X^{|u|} \). If \( v = u'v_1u \) for \( u_1 \in X^{|u|} \), \( u_1 \neq u \), and \( l \geq 0 \), then for every \( g \in N_{(n,k)}^u \) we get \( g(v) = z^lz_1h(v_1) \) for some \( z, z_1 \in X^{|u|} \) and \( h \in H_n \). Summarizing all estimates we get

\[
|B(w,n)| \leq |X|^{C(\log n)^m+1+1} \cdot |X|^{1+2+\ldots+L} \cdot |X|^{2L} \cdot |X|^{|C_1(\log n)|^m} \leq |X|^{C_2(\log n)^m+1}
\]

for some constant \( C_2 \).

\[\square\]

5 Examples

**Omega-periodic graphs.** Let \( X = \{0,1\} \) be the binary alphabet. For every finite word \( v = x_1x_2\ldots x_k \) over \( X \) consider the automaton \( A_v \) shown in Figure 1 where we use notation \( \overline{0} = 1 \) and \( \overline{1} = 0 \). Every automaton \( A_v \) is polynomial of degree \( k = |v| \). The automaton \( A_v \) is a subautomaton of \( A_{vx} \), and we get an increasing chain of polynomial automata

\[
A_\emptyset \subset A_{x_1} \subset A_{x_1x_2} \subset \ldots,
\]

for every \( x_i \in X \). Hence every orbital Schreier graph \( \Gamma_w(A_v) \) for \( w \in X^\omega \) is a subgraph of \( \Gamma_w(A_{vx}) \). This allows us to construct the Schreier graph \( \Gamma_w(A_v) \) consequently as \( \Gamma_w(A_{x_1}) \subset \Gamma_w(A_{x_1x_2}) \subset \ldots \subset \Gamma_w(A_{x_1x_2\ldots x_k}) \), by looking at the action of the states \( a, a_1, a_2, \ldots, a_k \).
We start from the orbital Schreier graphs $\Gamma_w(A_0)$ of the automaton $A_0$. The transformation $a$ is called the adding machine, because its action on a sequence $y_1 y_2 \ldots \in X^\omega$ corresponds to the addition of 1 to the binary number $\sum_{i \geq 1} y_i 2^{i-1} \in \mathbb{Z}_2$. In particular, the infinite cyclic group generated by $A_0$ acts faithfully on every its orbit on $X^\omega$. Hence every Schreier graph $\Gamma_w(A_0)$ for $w \in X^\omega$ is a “line”, i.e., its vertices can be identified with $\mathbb{Z}$ via the map $a^m(w) \mapsto m$, and the edges are $(m, m+1)$ for all $m \in \mathbb{Z}$.

Notice that for every state $a_k$ of the automaton $A_v$ if $a_k(y_1 y_2 \ldots) = z_1 z_2 \ldots$, for $y_i, z_i \in X$, and we take the first position $n$ with $y_n \neq z_n$, then $a_k(y_1 y_2 \ldots) = y_n z_{n+1} \ldots$. It follows that each $a_k$ preserves the orbits of the action of $a$ on $X^\omega$. Hence the orbits of $(A_v, X^\omega)$ for every word $v$ coincide with the orbits of $(A_0, X^\omega)$, and we can identify the vertex set of every Schreier graph $\Gamma_w(A_v)$ for $w \in X^\omega$ with the set $\mathbb{Z}$.

Every state $a_k$ of the automaton $A_{x_1 x_2 \ldots x_k}$ acts on the infinite sequences as follows

$$a_k(x_k^{n_k} \overline{x_k} \ldots x_2^{n_2} \overline{x_2} x_1^{n_1} \overline{x_1} w) = x_k^{n_k} \overline{x_k} \ldots x_2^{n_2} \overline{x_2} x_1^{n_1} \overline{x_1} a(w)$$

for every $w \in X^\omega$ and $n_i \in \mathbb{N} \cup \{0, \infty\}$. Notice that every sequence from $X^\omega$ appears in (4) for suitable numbers $n_i$. Further, for every fixed $n_1, \ldots, n_k$, the edges defined by (4), when $w$ runs through $X^\omega$, are periodic, namely, these edges are invariant under the shift $m \mapsto m + 2^{n_1+\ldots+n_k+k}$ on $\mathbb{Z}$. It follows that all Schreier graphs $\Gamma_w(A_v)$ for $w \in X^\omega$ are unions of periodic subgraphs on $\mathbb{Z}$, and hence they are $\omega$-periodic in the terminology of [3].

The Schreier graph $\Gamma_{0^\omega}(A_0)$ (see Figure 2) was defined in [3] as an example of a graph of intermediate growth (its growth is equivalent to $n \log_4 n$) connected to the long range percolation on $\mathbb{Z}$. All orbital Schreier graphs $\Gamma_w(A_0)$ for $w \in X^\omega$ of the automaton $A_0$ were studied in [3], where it was proved that the family of these graphs contains uncountably many pairwise nonisomorphic graphs, all of them except $\Gamma_{0^\omega}$ are locally isomorphic, and all have intermediate growth. The groups generated by $A_0$ and $A_1$ are the same, and the automata $A_0, A_1$ are just different generating sets of this group. Hence the Schreier graphs...
Figure 3: A nonpolynomial automaton with Schreier graphs $\Gamma_w$ of intermediate growth

$\Gamma_w(A_1)$ also have intermediate growth. The Schreier graphs $\Gamma_w(A_v)$ for every nonempty word $v$ have growth not less than the growth of $\Gamma_w(A_0)$ or of $\Gamma_w(A_1)$. Hence their growth is superpolynomial, and then Theorem 5 implies that all orbital Schreier graphs $\Gamma_w(A_v)$ for $w \in X^\omega$ and $v \neq \emptyset$ have intermediate growth.

Actually, if we take the automaton shown in Figure 1 and change the labels of the edges passing from the states $a_1, \ldots, a_k$ in any way we want, we still get the automaton whose all Schreier graphs $\Gamma_w$ have intermediate growth. More generally, if we take any polynomial automaton $A$ over $X$ that contains $A_0$ (with any labels at $a_1$) as a subautomaton, then all orbital Schreier graphs $\Gamma_w(A)$ have intermediate growth.

Nonpolynomial example. Consider the automaton $B$ over the alphabet $X = \{0, 1\}$ shown in Figure 3. The automaton $B$ is not polynomial, but satisfies the conditions of Theorem 5. The group $H = \langle b \rangle$ is the infinite cyclic group, which coincides with the automaton group generated by the subautomaton of $B$ with states $b^{\pm 1}, b^{\pm 2}$. The group $H$ is contracting with nucleus $N = \{1, b^{\pm 1}, b^{\pm 2}, b^{\pm 3}\}$, while the group $G$ generated by $B$ is not contracting. Theorem 5 implies that all orbital Schreier graphs $\Gamma_w(B)$ for $w \in X^\omega$ have subexponential growth.

To obtain a lower bound we can use the previous example. We notice that the group $H$ acts faithfully on every its orbit on $X^\omega$. The states $b, c$ act similarly to the states $a, a_1$, namely, $b^{2n+1}(0^n1w) = 0^n1b(w)$ and $c(0^n1w) = 0^n1b(w)$ for $w \in X^\omega$ and $n \in \mathbb{N} \cup \{0, \infty\}$. It follows that the Schreier graph $\Gamma_w(A_0)$ can be seen as a subgraph of the Schreier graph $\Gamma_w(B)$ via the map $a^m(w) \mapsto b^m(w)$ for every $w \in X^\omega$. Hence all orbital Schreier graphs $\Gamma_w(B)$ have intermediate growth.

Hanoi graphs. The Hanoi Tower Game on $k$ pegs is played with $n$ disks of distinct size placed on $k$ pegs, $k \geq 3$. Initially, all disks are placed on the first peg according to their size so that the smallest disk is at the top, and the largest disk is at the bottom. A player can move only one top disk at a time from one peg to another peg, and can never place a bigger disk over a smaller disk. The goal of the game is to transfer the disks from the first peg to another peg. More generally, one can take any two configurations of disks and ask a player to transform one configuration to another one, where a configuration is any
arrangement of \( n \) disks on \( k \) pegs such that a bigger disk does not lie on a smaller disk. For more information about this game, its history, solutions, and open problems, we refer the reader to [12, 13, 18, 10] and the references therein.

The Hanoi Tower Game is modeled by the Hanoi graphs \( H_n^{(k)} \). The vertices of the graph \( H_n^{(k)} \) are the configurations of \( n \) disks on \( k \) pegs, and two configurations are adjacent if one can pass from one to another by a single disk move. Then the Hanoi Tower Game is equivalent to finding a path in the graph \( H_n^{(k)} \) between two given vertices. In particular, the diameter of the Hanoi graph gives an upper bound on the optimal number of steps in the Hanoi Tower Game.

It was noticed in [10] that the game can be also modeled by a finite automaton \( A_{(k)} \) over the alphabet \( X = \{1, 2, \ldots, k\} \), called the Hanoi Towers automaton on \( k \) pegs. In this model, the pegs are identified with the letters of the alphabet \( X \), and the disks are labeled by 1, 2, \ldots, \( n \) according to their size. Then the configurations of \( n \) disks on \( k \) pegs can be encoded by words of length \( n \) over \( X \), where the word \( x_1x_2\ldots x_n \) corresponds to the unique configuration in which the disk \( i \) is placed on the peg \( x_i \). The automaton \( A_{(k)} \) has the trivial state \( e \) and the state \( a_{(ij)} \) for every transposition \((i, j)\) on \( X \) with two arrows \( a_{(ij)} \rightarrow e \) labeled by \( i|j \) and \( j|i \), and the loop at \( a_{(ij)} \) labeled by \( x|x \) for every \( x \in X \setminus \{i, j\} \). For example, the automaton \( A_{(4)} \) is shown in Figure 4 (the loops at the trivial state \( e \) are not drawn). The action of the state \( a_{(ij)} \) on a word of length \( n \) corresponds to a single disk move between the pegs \( i \) and \( j \). Hence the Schreier graph \( \Gamma_n(A_{(k)}) \) is precisely the Hanoi graph \( H_n^{(k)} \).
The complexity of the Hanoi Tower Game highly depends on whether \( k \) is 3 or \( k \geq 4 \). In the case \( k = 3 \), the Hanoi graph \( H_n^{(3)} \) has diameter \( 2^n - 1 \), and this is the smallest number of steps to win the game. The automaton \( A_{(3)} \) is bounded and generates a contracting group. All orbital Schreier graphs \( \Gamma_w(A_{(3)}) \) for \( w \in X^\omega \) have polynomial growth of degree \( \frac{\log 3}{\log 2} \).

For \( k \geq 4 \), the Hanoi Tower Game can be solved in subexponential number of moves, asymptotically in \( \exp(n^{\frac{1}{k-2}}) \) moves (see [8, 13]). In particular, the diameter of the Hanoi graph \( H_n^{(k)} \) and the Schreier graph \( \Gamma_n(A_{(k)}) \) is asymptotically \( \exp(n^{\frac{1}{k-2}}) \). One can apply this asymptotic estimates to the orbital Schreier graphs \( \Gamma_w(A_{(k)}) \) for \( w \in X^\omega \). It is easy to see that the growth function \( \gamma \) of each graph \( \Gamma_w(A_{(k)}) \) satisfies \( \gamma(d_n) \geq k^n \), where \( d_n \) is the diameter of the graph \( \Gamma_n(A_{(k)}) \). Hence the upper bound on the diameter of \( \Gamma_n(A_{(k)}) \) implies a superpolynomial lower bound on the growth of \( \Gamma_w(A_{(k)}) \), namely \( \gamma(m) \geq k^{C(\log m)^{k-2}} \) with some constant \( C > 0 \). This was used in [10, Theorem 2.1] to conclude that the Schreier graph \( \Gamma_{0\omega}(A_{(k)}) \) for \( k \geq 4 \) has intermediate growth between \( a^{(\log m)^{k-2}} \) and \( b^{(\log m)^{k-2}} \) for some constants \( b > a > 0 \).

The automaton \( A_{(k)} \) for \( k \geq 4 \) is not polynomial, and does not satisfy the assumptions of Theorem 3. However, we can apply similar ideas to give the subexponential upper bound \( k^{(\log m)^{k-2}} \) on the growth of every Schreier graph \( \Gamma_w(A_{(k)}) \) for \( w \in X^\omega \).

**Theorem 6.** All orbital Schreier graphs \( \Gamma_w(A_{(k)}) \) for \( k \geq 4 \) and \( w \in X^\omega \) have intermediate growth.

**Proof.** To provide an upper bound we follow similar arguments as in the proofs on Theorems 3 and 4. We will show only the case \( k = 4 \), the general case is analogous.

Let \( G_{(k)} \) be the automaton group generated by \( A_{(k)} \). Notice that for every state \( s \) of the automaton \( A_{(k)} \) and every word \( v \in X^* \) if \( s|_v \neq 1 \) then \( s(v) = v \) and \( s|_v = s \). It immediately follows that every cycle in the complete automaton \( A(G_{(k)}) \) of the group \( G_{(k)} \) is a loop labeled by \( x|x \) for some letter \( x \in X \).

The group \( G_{(3)} \) is contracting with nucleus \( N = A_{(3)} \). Then for every \( g = s_1s_2\ldots s_n \in G_{(3)} \), \( s_i \in A_{(3)} \), the restriction \( g|_v \) belongs to \( N \) for all words \( v \) of length \( |v| \geq \log n \). Hence every Schreier graph \( \Gamma_w(A_{(3)}) \) has subexponential (polynomial) growth not greater than \( 4|X|^{\log n + 1} \leq |X|^{C_1 \log n} \) for some constant \( C_1 \).

Now consider the group \( G_{(4)} \) and the automaton \( A_{(4)} \). For \( x \in X \) denote by \( A_{(4)}^x \) the subautomaton of \( A_{(4)} \) consisting of states that fix the letter \( x \). The group generated by \( A_{(4)}^x \) acts on the words over \( X \setminus \{x\} \) in the same way as the group \( G_{(3)} \) acts on \( \{1, 2, 3\}^* \), and that is where we can apply the inductive arguments. In particular, if \( g = s_1s_2\ldots s_n \) for \( s_i \in A_{(4)}^x \) then for every word \( v \in X^* \) that contains at least \( \log n \) letters different from \( x \) the restriction \( g|_v \) belongs to \( A_{(4)} \), in other words, \( g|_v \) fixes one more letter except \( x \).

Take two different letters \( x, y \in X \) and consider elements \( g = s_1s_2\ldots s_k \) for \( s_i \in A_{(4)}^x \) and \( h = t_1t_2\ldots t_m \) for \( t_i \in A_{(4)}^y \). Take an arbitrary word \( v \in X^* \) of length \( |v| \geq 2\log n \) with \( n = k + m \). If the word \( v \) contains at least \( \log n \) letters not equal to \( x \) then \( g|_v \in A_{(4)} \). Otherwise, \( v \) contains at least \( \log n \) letters equal to \( x \), and then \( g|_v \) contains at least \( \log n \) letters equal to \( x \). Since \( x \neq y \), we get \( h|_{g(v)} \in A_{(4)} \). It follows that for every word of length
|v| \geq 2 \log n + 1 \text{ there exists a letter } z \in X \text{ such that every generator in the induced presentation of } (hg)|_v = h|_{g(v)}g|_v \text{ belongs to } A^z_{(4)} (\text{i.e., } (hg)|_v \text{ is circuit}).

Let us prove that there exists a constant } C_2 \text{ such that for every product } g = s_1s_2\ldots s_n \in G_{(4)}, \text{ } s_i \in A_{(4)}, \text{ and every word } v \in X^* \text{ of length } |v| \geq C_2(\log n)^2 \text{ there exists } x \in X \text{ such that every generator } s_i|_{v_i} \text{ in the induced presentation } g|_v = s_1|_{v_1}s_2|_{v_2}\ldots s_n|_{v_n} \text{ belongs to } A^z_{(4)} \text{ (hence } g|_v \text{ is circuit, here } g|_{v|x} = g|_v). \text{ We partition the presentation } g = s_1s_2\ldots s_n \text{ on blocks}
\begin{equation}
g = g_1g_2\ldots g_l
\end{equation}
such that every } g_i = s_{j_i}s_{j_i+1}\ldots s_{j_i+1-1} \text{ contains only generators from } A^z_{(4)} \text{ for some letter } x \in X \text{ (depending on } i). \text{ Consider the products } g_1g_2, g_3g_4, \ldots, \text{ and their restrictions on words } v \text{ of length } \geq 2 \log n + 1. \text{ Using the above property we get that the element } g|_v \text{ can be expressed as a product } (5) \text{ with } \leq (l + 1)/2 \text{ elements } g_i. \text{ Applying the same procedure log } l \text{ times, we get an element with needed properties. The existence of the constant } C_2 \text{ follows.}

Consider the Schreier graph } \Gamma_w(A_{(4)}) \text{ for } w = x_1x_2\ldots x_\omega. \text{ We will use estimate } (1) \text{ with } k \text{ being the least integer greater than } C_2(\log n)^2. \text{ Then}
\begin{equation}
N_{(n,k)} = \{g|_{x_1x_2\ldots x_k} : g \in G_{(4)} \text{ and } l(g) \leq n\} \subset \bigcup_{x \in X} N^x_{(n,k)},
\end{equation}
where } N^x_{(n,k)} \text{ consists of those elements } g \in N_{(n,k)} \text{ that can be expressed as a product of no more than } n \text{ elements from } A^x_{(4)}. \text{ Every element } g \in N^x_{(n,k)} \text{ fixes every occurrence of the letter } x \text{ in the sequence } x_{k+1}x_{k+2}\ldots, \text{ and changes every other letter in the same way as the group } G_{(3)} \text{ acts on } X \setminus \{x\}. \text{ Hence the orbit } N^x_{(n,k)}(x_{k+1}x_{k+2}\ldots) \text{ has subexponential growth not greater than } (|X| - 1)^C_1 \log n. \text{ We get the upper bound}
\begin{equation}
|B(w, n)| \leq |X|^{C_2(\log n)^2 + 1} \cdot |X| \cdot (|X| - 1)^{C_1 \log n} \leq |X|^{C(\log n)^2}
\end{equation}
for some constant } C. \qed

Weakly contracting groups. One of the main properties used in the proof of Theorems 1 and 3 leads us to the following definition. We say that a group } G \text{ generated by a finite automaton is } weakly contracting \text{ if the length of cycles in the complete automaton } A(G) \text{ of the group is bounded. Every finitely generated contracting group is weakly contracting. The groups generated by polynomial automata, the groups from Theorem 4, and the Hanoi Towers groups are also weakly contracting. It is natural to ask which properties can be generalized to this class of groups. Is the word problem in weakly contracting groups solvable in subexponential time? Do weakly contracting groups not contain non-abelian free subgroups? Do the orbital Schreier graphs } \Gamma_w \text{ of weakly contracting groups have subexponential growth?}
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