Abstract. Let $K$ denote a compact invariant set for a strongly monotone semiflow in an ordered Banach space $E$, satisfying standard smoothness and compactness assumptions. Suppose the semiflow restricted to $K$ is chain transitive. The main result is that either $K$ is unordered, or else $K$ is contained in totally ordered, compact arc of stationary points; and the latter cannot occur if the semiflow is real analytic and dissipative. As an application, entropy is 0 when $E = \mathbb{R}^3$. Analogous results are proved for maps. The main tools are results of Mierczyński and Tereščák.

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1 Introduction

Throughout this paper $E$ denotes an ordered Banach space whose order cone $E^+ = \{x \in E : x \geq 0\}$ has nonempty interior $\text{Int} E^+$. Our main results are Theorem 1.3 concerning smooth strongly monotone maps, and its analogue for semiflows, Theorem 1.6.

Notation and terminology. We write $x \geq y$ if $x - y \in E^+$ and $x > y$ if $x \geq y$ and $x \neq y$, or equivalently, $x - y \in E^+ \setminus \{0\}$. If $x - y \in \text{Int} E^+$ we write $x \gg 0$.

A subset $Y \subset X$ is unordered if none of its points satisfy $x < y$. We call $x$ and $y$ comparable in case $x \leq y$ or $y \leq x$.

If $A$ and $B$ are sets, notation such as $A > B$ means $a > b$ for all $a \in A, b \in B$. We abuse notation slightly by writing $A > b$ if $A$ is a set, $b$ is a point, and $A > \{b\}$.

For any subsets or points $A, B$

$$A_+ = \{x \in E : x \geq A\}, \quad A_- = \{x \in E : x \leq A\}$$

$$[A, B] = \{x \in E : A \leq x \leq B\}$$

$$[[A, B]] = \{x \in E : A \ll x \ll B\}$$

For sets $A, B$

$$A^+ = \bigcup_{x \in A} x_+, \quad A^- = \bigcup_{x \in A} x_-$$
\[ [A, B] = A^\bullet \cap B^\bullet = \bigcup_{a \in A, b \in B} [a, b] \]

If \( A \subseteq B^\bullet \), then \( B \) majorizes \( A \), while if \( A \supseteq B^\bullet \) then \( B \) minorizes \( A \).

\( S : X \to X \) always denotes a continuous map defined in an open set \( X \subset E \).

The orbit \( \gamma(x) \) of \( x \in X \) is the set \( \{ S^n x : n \in \mathbb{N} \} \) where \( \mathbb{N} = \{0, 1, \ldots \} \) is the set of natural numbers. The orbit closure of \( x \) is \( \overline{\gamma(x)} = \text{clos} \gamma(x) \). The omega limit set of \( x \) is \( \omega(x) = \bigcap_{k \geq 0} \overline{\gamma(S^k x)} \).

A point \( p \) is \( m \)-periodic if \( S^m p = p \). If also \( \omega(z) = \gamma(p) \) for some \( z > p \) then \( p \) and \( \gamma(p) \) are upper attracting. A lower attracting periodic point is defined dually (by reversing order relations).

The following hypotheses is always in force:

**Hypothesis 1.1**

(i) \( X \subset E \) is an open set that is order convex: if \( p, q \in X \) then \( X \) contains the closed order interval \([p, q]\).

(ii) \( S : X \to X \) is \( C^1 \) (continuously Frechet differentiable).

(iii) \( S \) is order compact: the image of each closed order interval \([p, q]\), \( p, q \in X \), has compact closure in \( X \).

(iv) For each \( x \in X \) the derivative of \( S \) at \( x \) is a strongly positive linear operator \( DS(x) : E \to E \), i.e., \( DS(x)u \gg 0 \) if \( u > 0 \). This implies \( S \) is strongly monotone: \( Sx \gg Sy \) if \( x > y \).

(v) \( S \) is completely continuous: the image of any bounded set has compact closure in \( X \). Moreover, each derivative \( DS(x) \) is completely continuous.

(vi) Every orbit closure is a compact subset of \( X \).

Such maps arise in ordinary, partial and functional differential equations; see e.g. Dancer & Hess [8], Hess & Poláčik [12], Hirsch [19, 20], Poláčik & Tereščák [30, 31], Smith [33], Smith & Thieme [34], Takáč [35, 36].

Our results are based on the following fundamental property of this class of maps (which does not require order compactness):

**Theorem 1.2** (Tereščák [37]) There exists a natural number \( m \geq 1 \) and an open dense set of points \( x \) such that \( \omega(x) \) is an \( m \)-periodic orbit.

Similar results under stronger hypotheses have been obtained by Takáč [35], Poláčik & Tereščák [30], Hess & Poláčik [12].

The smallest number \( m \) satisfying Theorem 1.2 is called the generic asymptotic period.
Attractor-free sets, p-arcs and the main theorem for maps. Let $T : Y \to Y$ be a continuous map. A subset $A \subset Y$ is an attractor if $A$ is invariant ($T(A) = A$) and contained in an open set $W \subset Y$ such that

$$\lim_{n \to \infty} \text{dist} (T^n w, A) = 0 \quad \text{uniformly in } w \in W.$$ 

If also $A \neq Y$ then $A$ is a proper attractor. $T$ is dissipative if there is an attractor that contains all omega limit points.

Now let $Y \subset X$ be a compact invariant set for $S$. If the map $S|Y$ does not have a proper attractor, $Y$ is attractor-free. By a theorem of Conley [6] this is equivalent to $S|Y$ being chain transitive (see Section 2). Every omega limit set is attractor-free and unordered. Attractor-free sets for semiflows occur as limit sets of several kinds of dynamic and stochastic processes (Benaïm and Hirsch [2, 3, 4], Mischaikow et al. [28], Thieme [38, 39]).

A smooth arc $J \subset E$ is the homeomorphic image of the closed unit interval under an injective $C^1$ map $h : [0, 1] \to E$ whose derivative is nonzero everywhere. We call such a $J$ a p-arc for $S$ if $S(J) = J$ and $h'(t) \gg 0$ for all $t \in [0, 1]$. This makes $J$ totally ordered.\footnote{The concept of p-arc is due to Mierczyński [27], who also allowed degenerate p-arcs, i.e., fixed points. Here we allow only nondegenerate p-arcs.} The endpoints $h(0), h(1)$ are fixed points. The set of endpoints is denoted by $\partial J$.

A p-arc is stationary if it consists of fixed points. It is easy to see that if $J$ is a stationary p-arc for $S^m$, then each point of $J$ is $m$-periodic, there is a divisor $d \geq 1$ of $m$ such that the arcs $J, S(J), \ldots, S^{d-1}(J)$ are disjoint and permuted cyclically by $S$, and if $0 \leq i < j \leq d - 1$ then no point of $S^i J$ is related to any point of $S^j J$.

We can now state our main results for maps. The following two theorems refer to a map $S : X \to X$ satisfying Hypotheses [11] with generic asymptotic period $m \geq 1$.

**Theorem 1.3** Let $K \subset X$ be an attractor-free set. Then either

(a) $K$ is unordered; or else

(b) $K = \bigcup_{0 \leq i < m} S^i(J)$ where $J$ is a stationary p-arc for $S^m$.

Moreover $K$ is unordered provided $S$ is real analytic and dissipative.

The following result complements shows that except for upper attracting $m$-periodic orbits, unordered attractor-free sets are rather unstable:

**Theorem 1.4** Let $K$ be an unordered attractor-free set. Then either $K$ is an upper attracting $m$-periodic orbit, or else there exists a lower attracting $m$-periodic point $q$ such that $\gamma(q)$ majorizes $K$ and $K$ minorizes $\gamma(q)$, and $\omega(y) = \gamma(q)$ if $x < y < u$ for some $x \in K, u \in \gamma(q)$. The dual result also holds.

Thus if $K$ is not an upper attracting $m$-periodic orbit, it lies in the upper boundary of the basin of attraction of $\gamma(q)$. 
The main theorem for semiflows. Let $S = \{S_t : X \to X\}_{0 \leq t < \infty}$ be a semiflow in $X$, i.e., $S_t x$ is continuous in $(t, x)$, $S_t \circ S_r = S_{t+r}$, and $S_0$ is the identity map of $X$. A point $p$ is an equilibrium if $S_t p = p$ for all $t \geq 0$. We always assume:

**Hypothesis 1.5**

Each map $S_t : X \to X$, $t > 0$ satisfies Hypothesis 1.1.

Let $Y \subset X$ be invariant under $S$, i.e., $S_t(Y) = Y$ for all $t$. An attractor for the restricted semiflow $S|Y = \{S_t|Y : Y \to Y\}$ is a nonempty compact invariant $A \subset Y$ having a neighborhood $V \subset Y$ such that $\lim_{t \to \infty} \text{dist}(S_t y, A) = 0$ uniformly in $v \in V$. The definitions for semiflows of proper attractor, attractor-free, and dissipative are similar to those for maps.

A smooth arc $J \subset X$ is a $p$-arc for $S$ if it is a $p$-arc for every map $S_t$. If in addition every point of $J$ is an equilibrium, then $J$ is a stationary $p$-arc for $S$.

The analogue of Theorem 1.3 for semiflows is:

**Theorem 1.6** Assume the semiflow $S$ satisfies Hypothesis 1.1 and let $K \subset X$ be an attractor-free set. Then either $K$ is unordered, or $K$ is a stationary $p$-arc for $S$. If $S$ is dissipative and real analytic, then $K$ is unordered.

This result has been applied to stochastic approximation and game theory in Benaïm & Hirsch [4].

Theorem 1.4 takes the following form for semiflows:

**Theorem 1.7** Assume the semiflow $S$ satisfies Hypothesis 1.5, and let $K \subset X$ be an unordered attractor-free set that is not an upper attracting equilibrium. Then there is a lower attracting equilibrium $q \gg K$ such that $S_t y \to q$ if $x < y < q$ for some $x \in K$.

**Application to invariant measures.** Before beginning the proof of the main theorems, we use them to investigate invariant measures.

An invariant measure $\mu$ for $S$ is a Borel probability measure on $X$ with compact support such that $\mu(U) = \mu(S^{-1}U)$ for every $\mu$-measurable set $U$. The support $\text{Supp}(\mu)$ of $\mu$ is the complement in $X$ of the union of all open sets $U$ such that $\mu(U) = 0$. Notice that $\text{Supp}(\mu)$ is an invariant set. If $\mu$ is invariant for every map in a semiflow $S$, then $\mu$ is called an invariant measure for $S$. An invariant measure is ergodic if every measurable invariant set has measure zero or one.

Chain recurrence is defined in Section 2 below.

**Proposition 1.8** Let $K$ be the support of an invariant measure $\mu$. Then every point of $K$ is chain recurrent for $S|K$. If $\mu$ is ergodic, $K$ is attractor-free.

**Proof:** We rely on the fact that $\mu(Q) = 0$ for every nonempty, relatively open set $Q \subset \text{Supp}(\mu)$. Let $R \subset K$ denotes closure of the set of recurrent points in $K$. If $K \setminus R \neq \emptyset$ then $\mu(K \setminus R) > 0$. But then $K \setminus R$ carries an invariant measure, and thus contains a recurrent point by Poincaré’s recurrence theorem (Nemytskii &
The topological entropy of a dynamical system is a much studied numerical invariant. While the definition is too complicated to give here (see e.g. Katok & Hasselblatt [26]), it can be noted that positive entropy is often used as a criterion for chaos. Conversely, zero entropy suggests nonchaotic behavior. Rigorously, zero entropy implies that the system does not contain a subsystem dynamically equivalent to a Smale horseshoe.

Under Hypotheses 1.1 or 1.5 we have the following result:

**Theorem 1.9** Assume \( E = \mathbb{R}^3 \) and \( S \) is a flow (respectively, \( E = \mathbb{R}^2 \) and \( S \) is a diffeomorphism). Then \( S \) (respectively, \( S \)) has topological entropy 0.

**Proof:** The topological entropy is the supremum of the measure theoretic entropies of ergodic invariant measures (Goodwyn [11]). Therefore it suffices to prove that every ergodic invariant measure \( \mu \) for \( S \) (or \( S \)) has measure theoretic entropy 0.

Consider a flow \( S \) in \( X \subset \mathbb{R}^3 \). Then \( K \) is attractor-free by Proposition 1.8 and Theorem 1.3 applies. It is easy to see that ergodicity precludes conclusion (b) of Theorem 1.3 so \( K \) is unordered. Therefore \( K \) lies in an invariant surface (Hirsch [18], Takáč [35]). Since every surface flow has entropy 0 (Young [41]), the proof for flows is complete.

The proof for a diffeomorphism \( S \) in \( X \subset \mathbb{R}^2 \) is similar: Theorem 1.6 implies \( K \) is unordered, \( K \) lies in an invariant 1-manifold, and every homeomorphism of a 1-manifold has entropy 0 (Adler et al. [1]).

## 2 Preliminaries

**Chain equivalence.** Let \( T : Y \to Y \) be a continuous map in a metric space. Let \( u, v \in Y \). We say \( u \) \( \epsilon \)-chains to \( v \), written \( u \sim_{\epsilon} v \), if there exist a number \( m \in \mathbb{N}_+ \) (the set of positive natural numbers) and a finite sequence in \( Y \) of the form

\[
    u = y_0, \ldots, y_m = v
\]

such that

\[
    ||Ty_{i-1} - y_i|| < \epsilon, \quad i = 1, \ldots, m
\]

The \( m + 1 \)-tuple \((y_1, \ldots, y_m)\) is an \( \epsilon \)-chain. If \( u \sim_{\epsilon} v \) for every \( \epsilon > 0 \) then \( u \) chains to \( v \), denoted by \( u \leadsto v \). If \( u \leadsto v \) and \( v \leadsto u \) then \( u \) and \( v \) are chain equivalent,
written \( u \approx v \). If \( u \sim v \) then \( u \) is chain recurrent. Define
\[
\Omega(u, T) = \{ v : u \sim v \}
\]
This set is closed and forward invariant; when \( u \) is chain recurrent, it is invariant.

The binary relation of chain equivalence is symmetric and transitive, and reflexive on the set \( CR(T) \) of chain recurrent points of \( T \). This closed invariant set contains the nonwandering set, all homoclinic and heteroclinic cycles, and all supports of invariant measures.

If every point is chain recurrent, then \( T \) is a chain recurrent map. If \( Y = CR(T) \) then we say \( Y \) is chain transitive. When \( Y \) is compact, this is equivalent to \( Y \) being attractor-free by a theorem of Conley [6] for semiflows and its analogue for maps.

Now consider a semiflow \( T = \{ T_t \}_{0 \leq t < \infty} \) in \( Y \). For \( R > 0 \) and \( \epsilon > 0 \) we say \( u \) \((R, \epsilon)\)-chains to \( v \), written \( u \sim_{R, \epsilon} v \), if there exists a natural number \( m \geq 1 \), real numbers \( t_1, \ldots, t_r \geq R \), and a finite sequence in \( Y \) of the form \( v = y_0, \ldots, y_n = u \) such that
\[
|| T_{t_i} y_{i-1} - y_i || < \epsilon \quad i = 1, \ldots, m
\]
If \( u \sim_{R, \epsilon} v \) for every \( R > 0, \epsilon > 0 \) then \( u \) chains to \( v \), denoted by \( u \sim v \). If \( u \sim u \) then \( u \) is chain recurrent for \( T \).

The definitions of chain recurrence, chain equivalence and chain transitivity for semiflows are analogous to those for maps. When \( Y \) is compact, chain transitivity is equivalent to attractor-free.

**Monotone convergence and p-arcs.** We return to the map \( S : X \to X \) satisfying Hypothesis 1.1.

A point \( x \) is convergent if \( \omega(x) \) is a singleton (necessarily a fixed point).

Suppose \( \omega(x) = p \). We say \( \gamma(x) \) eventually decreases to \( p \) and write \( \gamma(x) \searrow p \) provided there exists \( n \geq 0 \) such that \( S^n x > S^{n+1} x \), in which case
\[
S^k x \gg S^{k+1} x \gg p \text{ for all } k \geq n.
\]
If \( S^n x < S^{n+1} x \) for some \( n \geq 0 \), we say \( \gamma(x) \) eventually increases to \( p \) and write \( \gamma(x) \nearrow p \). When \( \gamma(x) \searrow p \) or \( \gamma(x) \nearrow p \), we say \( \gamma(x) \) is eventually monotone and converges eventually monotonically.

**Theorem 2.1 (Mierczyński)** Let \( p \) be a fixed point. The set of points whose orbits converge to \( p \), but not monotonically, is unordered.

**Proof:** Follows from Proposition 2.1 of [27].

For any \( x \in X \) let \( \rho(x) \) denote the spectral radius of the linear operator \( L_x = DS(x) : E \to E \). The Krein-Rutman theorem (Deimling [9]) implies \( \rho(x) \) is a simple eigenvalue of \( L_x \), whose one-dimensional eigenspace, called the principal eigendirection \( E_1(x) \), is spanned by a vector \( \gg 0 \). There is a direct sum decomposition \( E = E_1(x) \oplus E_2(x) \) invariant under \( L_x \) such that \( L_x | E_2(x) \) has spectral radius \( \rho_2(x) < \rho(x) \). Moreover \( E_2(x) \cap E^+ = \{0\} \).

It is easy to see that if \( x \) belongs to a stationary p-arc then \( \rho(x) = 1 \).
**Theorem 2.2 (Mierczyński)**

(a) Assume $p$, $q$ are fixed points with $p < q$ and the set of fixed points in $[p, q]$ is compact. Then there is a $p$-arc whose endpoints are $p$ and $q$.

(b) If $x$ is a fixed point in a $p$-arc $J$ then the tangent space $T_x J$ is the principal eigendirection $E_1(x)$.

**Proof:** Part (a) is proved in Theorem 3.16 of [27]. Part (b) is Lemma 3.6 of [27].

Let $J$ be a stationary $p$-arc. There is a continuous family of bounded, $C^1$ hypersurfaces $\{L(x)\}_{x \in J}$ with the following properties (see Proposition 3.8 of [27]):

**Lemma 2.3 (Mierczyński)**

(a) $L(x)$ is tangent to $E_2(x)$ at $x$

(b) $L(x)$ is forward invariant

(c) $\lim_{n \to \infty} S^ny = x$ uniformly for $y \in L(x)$

(d) Set $B(J) = \bigcup_{x \in J} L(x)$. Then the interior of $B(J)$ is a neighborhood of $J \setminus \partial J$, and $J$ is a global attractor for $S | B(J)$.

I call $B(J)$ a contracting collar for $J$.

Denote the two endpoints of $J$ by $e_0 \ll e_1$. Set

$$Q(J) = \lfloor L(e_0), L(e_1) \rfloor$$

**Proposition 2.4 (Mierczyński)** $B(J)$ is a neighborhood of $J$ in $Q(J)$.

**Proof:** This is Proposition 3.8(vi) of [27].

Define forward invariant sets

$$V^{-}(J) = \{ x \in X : \gamma(x) \cap (\text{Int} J_-) \neq \emptyset \},$$

$$V^{+}(J) = \{ x \in x : \gamma(x) \cap \text{Int} (J_+) \neq \emptyset \},$$

$$V(J) = X \setminus (V^{-}(J) \cup V^{+}(J))$$

Then $V^{-}(J)$ and $V^{+}(J)$ are open and $V(J)$ is closed in $X$.

**Proposition 2.5** $B(J)$ is a neighborhood of $J$ in $V(J)$.

**Proof:** By Proposition 2.4 it suffices to prove that $V(J) \cap Q(J)$ is a neighborhood of $J$ in $Q(J)$. Let $\{x_n\}$ be a sequence in $Q(J)$ converging to a point $z \in J$. We can choose $n_*$ sufficiently large that $x_k \gg e_0$ for all $k \geq n_*$, because $z \gg e_0$. For such $k$ we have $\gamma(x_k) \gg e_0$ by strong monotonicity, whence $x_k \notin V^{-}(J)$. Similarly for $V^{+}(J)$.
Proposition 2.6 \( J \) is an attractor for \( S|V(J) \).

Proof: By Lemma 2.5 it is enough to prove that \( \lim_{n \to \infty} S^n x = 0 \) uniformly for \( x \in B(J) \). This follows from 2.3(c). \( \blacksquare \)

Lemma 2.7 Let \( J \) be a stationary \( p \)-arc. Suppose \( x \in X \) and \( \omega(x) \cap J \neq \emptyset \). Then either \( \omega(x) \ll J \) or \( \omega(x) \) is a singleton in \( J \).

Proof: Suppose there exists \( p \in \omega(x) \cap J \). Then \( \omega(x) \cap B(J) \subset L(p) \), for otherwise \( \omega(x) \) would contain two points of \( J \), contradicting \( \omega(x) \) being unordered. Therefore \( p \in \partial J \), for otherwise \( \omega(x) \cap J \) would be a proper attractor in \( \omega(x) \) by Propositions 2.4 and 2.6. Thus \( \omega(x) \cap J = \{ p \} \) where \( p \) is an endpoint of \( J \); we need consider only the case \( p = \inf J \). Then \( \omega(x) \cap L(p) = \{ p \} \) because \( x \ll y \) then means \( (x, y) \in \text{Int} R \).

Consequences of chaining. In the remainder of this section \( T : X \to X \) denotes any strongly monotone continuous map. (Here \( X \) could be any metric space endowed with a closed partial order relation \( R \subset X \times X \). The notation \( x \ll y \) then means \( (x, y) \in \text{Int} R \).

Let \( v, u \in X \) denote chain recurrent points such that \( v > u \).

Proposition 2.8 Suppose there exists \( z \in X \) such that

\[ v > Tz > z \geq u \]

and

\[ \text{dist} \left( T(z_+), X \setminus \text{Int}(z_+) \right) = \epsilon > 0 \]

Then \( v \not
\ll u \). In fact, if \( 0 < \delta < \epsilon \) and \( v = x_0, x_1, \ldots, x_n \) is a \( \delta \)-chain, then \( x_n \gg u \).

Proof: \( Tv \in \text{Int} T(z_+) \), because \( Tz \gg T^2(z) \gg Tz \) by strong monotonicity. This implies \( x_1 \gg z \), i.e., \( x_1 \in \text{Int}(z_+) \), because:

\[ \text{dist} \left( x_1, X \setminus \text{Int}(z_+) \right) \geq -d(x_1, Tv) + \text{dist} \left( Tv, X \setminus \text{Int}(z_+) \right) \]
\[ \geq -\delta + \text{dist} \left( T(u)_+, X \setminus \text{Int}(z_+) \right) \]
\[ = -\delta + \epsilon > 0 \]

Thus \( x_n \gg z \geq u \), and the same calculation shows by induction on \( n \) that all \( x_n \gg u \). \( \blacksquare \)

Corollary 2.9 If \( v, u \) are chain equivalent and \( v \geq y \geq u \), then \( y \) does not converge eventually monotonically.
Proposition 2.10 Suppose $x, y$ belong to an attractor-free compact invariant set $M \subset X$ and $x < y$. Then $x$ is convergent if and only if $y$ is convergent.

Proof: Assume $y$, but not $x$, is convergent; set $\omega(y) = \{q\}$. Then $\omega(x) \leq q$, and in fact $\omega(x) < q$ because $\omega(x)$ is unordered and not a singleton. The set $N = \omega(x) \cap M$ is compact, forward invariant, nonempty because $q \in N$, and a proper nonempty subset of $M$ because $\omega(x) \not\subset N$. Strong monotonicity shows that $S(N) \subset \text{Int}_M N$, implying that $N$ contains an attractor for $S|M$; contradiction. ■

3 Proof of Theorem 1.3 for the case $m = 1$

In this section we assume Hypothesis 1.1 with $m = 1$.

Lemma 3.1 Let $e \in \text{Int} E^+$. Then

$$\text{dist} (e + E^+, E \setminus \text{Int} E^+) > 0.$$ 

Proof: Let $x \in E^+$ be arbitrary and choose $y \in E \setminus \text{Int} E^+$ so that that

$$\text{dist} (e + x, E \setminus \text{Int} E^+) = \|e + x - y\| = \text{dist} (e, y - x)$$

Now $y - x \not\in \text{Int} E^+$ because $x \in E^+$ and $x + (y - x) = y \not\in \text{Int} E^+$. Therefore, setting

$$\alpha = \text{dist} (e, E \setminus \text{Int} E^+) > 0$$

we have

$$\text{dist} (e + x, E \setminus \text{Int} E^+) \geq \alpha$$

for all $x \in E^+$.

Now suppose $a, b$ are chain equivalent and $a < b$.

Proposition 3.2 Suppose $a \leq y \leq b$ and $S^n y$ is comparable to $S^{n+1} y$ for some $n \geq 0$. Then $S^n y = S^{n+1} y$. Thus no orbit entering in $[a, b]$ is eventually monotone.

Proof: Arguing by contradiction, we suppose $S^n y < S^{n+1} y$. Setting $S^n y = z, u = S^{n+1} a, v = S^{n+1} b$, we have

$$u \leq z \ll S z \leq v.$$ 

From Lemma 3.1 with $e = S x - x$ we see that

$$\text{dist} (T(z_+), X \setminus \text{Int} z_+) > 0$$

Proposition 2.8 gives the contradiction that $u, v$ are not chain equivalent. ■

Lemma 3.3 Let $a < x < y < b$ with $\omega(x) = \{p\}, \omega(y) = \{q\}$. Then

$$\omega(a) \ll p \ll q \ll \omega(b),$$

there is a unique $p$-arc $J$ with endpoints $p$ and $q$, and any such $p$-arc is stationary.
Proof: By monotonicity,
\[ \omega(a) \leq p \leq q \leq b \]

Assume \( \omega(a) \not\ll p \); then \( \omega(a) = \{p\} \) or \( \omega(a) < p \). But the latter entails \( \omega(a) \ll p \) by strong monotonicity and invariance of \( \omega(a) \), so necessarily \( \omega(a) = \{p\} \). Now Theorem 2.1 implies that either \( \gamma(a) \) or \( \gamma(x) \) converges monotonically, contradicting Proposition 3.2. This proves \( \omega(a) \ll p \), and similar arguments prove \( p \ll q \ll \omega(b) \).

Theorem 2.2 yields a p-arc joining \( p \) to \( q \). By Lemma 2.9 no orbit in \( J \) can converge monotonically. As \( J \) is totally ordered and invariant, it follows that \( J \) is stationary. Uniqueness of \( J \) follows easily from strong monotonicity.

Lemma 3.4 Let \( a \leq u < v \leq b \). Then there is a stationary p-arc \( J \) such that \( \omega(u) \ll J \ll \omega(v) \).

Proof: Choose convergent points \( x, y \) such that \( Su \ll x \ll y \ll Sv \) (by strong monotonicity, Theorem 1.2 and the assumption \( m = 1 \)). Set \( \omega(x) = \{p\}, \omega(y) = \{q\} \) and apply Lemma 3.3.

Lemma 3.5 Every point of \([a, b]\) is convergent. If \( a \leq x < y \leq b \) then \( \omega(x) \ll \omega(y) \).

Proof: Suppose for example that \( a \leq x < b \) and \( x \) is not convergent, so that \( \omega(x) \) is a compact unordered invariant set containing more than one point. By Lemma 3.4 and the compactness assumption (Hypothesis 1.1(v)) there is a minimal fixed point \( p \) satisfying
\[ \omega(x) \ll p \leq \omega(b) \]

Pick any \( u \in \omega(x), v \in \omega(b) \). Then \( u \ll p \leq v \) and \( u \) is chain equivalent to \( v \). From Lemma 3.4 we find a fixed point \( q, u \ll q \ll p \). As this contradicts minimality of \( p \), it follows that \( x \) is convergent. The last sentence is a consequence of Lemma 3.4.

In the rest of this section we assume the attractor-free set \( K \) of Theorem 1.3 is not unordered. Therefore by Lemma 3.5 we can select \( p \ll q \in K \) with the following properties:

- \( p \) and \( q \) are respectively maximal and minimal fixed points in \( K \)
- Every point of \([p, q]\) is convergent
- If \( u < v \) in \([p, q]\) then the trajectories of \( u \) and \( v \) converge to distinct fixed points.

Lemma 3.6 \( p \) is a minimal point of \( K \) and \( q \) is a maximal point of \( K \).
Proof: Suppose there exists \( u \in K, u < p \). Then Lemma 3.4 yields a stationary p-arc \( J \) such that
\[
\omega(u) \ll J \ll \omega(p) = \{p\}
\]
But \( \omega(u) \) is a singleton, contradicting minimality of \( p \).

Lemma 3.7 The set of fixed points in \([p, q]\) is a stationary p-arc \( G \) with endpoints \( p, q \).

Proof: By Lemma 3.4 there is a stationary p-arc \( G \) joining \( p \) to \( q \). Choose \( x \in [p, q] \setminus G \); we show \( Sx \neq x \). There is a minimal \( y \in G \) such that \( y > x \); then \( y \nottp x \). Since \( y = Sy \gg Sx \), it follows that \( Sx \neq x \).

Let \( B(G) \subset X \) be a contracting collar for \( G \) (see Theorem 2.2). I claim \( B(G) \cap K \) is a neighborhood of \( G \cap K \) in \( K \). If not, there is a sequence \( \{x_n\} \) in \( K \setminus B(G) \) converging to an endpoint \( p \in G \), by Lemma 2.5. We assume \( p = \inf G \).

There exists \( y \in L(p) \) and \( k \) such that \( x_k < y \). Now \( \omega(y) = p \) by Lemma 2.3(c). Therefore \( \omega(x_k) \ll p \) by Theorem 2.1. But this contradicts Lemma 3.6.

It now follows from Lemma 2.6 that \( G \cap K \) is an attractor for \( S|K \). Since \( K \) is attractor-free, we have proved \( K = G \cap K \). Since \( K \) is an attractor-free set of stationary points, it is connected. Thus \( K \) is a stationary p-arc, showing that either (a) or (b) of Theorem 1.3 holds when \( m = 1 \).

Now assume \( S \) is real analytic and dissipative. We show there cannot be a stationary p-arc \( J \). If there is, by Zorn’s lemma there exists a set \( L \subset X \) that is a connected, totally ordered set of stationary points containing \( J \), and which is setwise maximal in these properties. Then \( L \) is compact because \( S \) is dissipative, whence \( L \) is an arc (Wilder [40]). But there can be no totally ordered compact arc of fixed points when \( S \) is real analytic (Jiang & Yu [24], Lemma 3.3 and Theorem 2). Therefore Theorem 1.3(a) holds.

4 Proof of Theorem 1.3 for the case \( m > 1 \)

Assume \( m > 1 \). Pick an arbitrary \( a \in K \) and set \( L(a) = \Omega(a, S^m|K) \). This compact subset of \( K \) is attractor-free for \( S^m \), and it can be shown that
\[
S^i L(a) = L(S^i a) = L(S^{i+m}a)
\]
Because \( S|K \) is chain transitive, \( K = \cup_{0 \leq i < m} S^i L(a) \). Moreover if \( S^i L(a) \) and \( S^j L(a) \) intersect, they coincide.

With \( a \) chosen once and for all, set \( K_i = S^i L(a), 0 \leq i \leq m - 1 \). Then \( K_i = S^i K_0 \), and \( K = \cup_{0 \leq i < m} K_i \).

Suppose there exist \( x \in K_j, y \in K_k \) with \( x < y, j \neq k \); we will reach a contradiction. Relabel the \( K_i \) so that \( j = 0 \); then \( 1 \leq k \leq m - 1 \), and \( y = S^k u, u \in K_0 \). Therefore the set
\[
M = \{x \in K_0 : \exists u \in K_0, k \in \{1, \ldots, m - 1\} \text{ with } x < S^ny\}\
\]
is nonempty. $M$ is invariant under $S^m$ and relatively open in $K_0$ by strong monotonicity. On the other hand, one can also prove $M$ compact. Therefore $M$ is an attractor for $S^m|K_0$, which implies $M = K_0$.

Compactness and strong monotonicity now imply there exists a smallest $k, 1 \leq k < m$ such that $S^k K_0 \gg K_0$. Therefore $M$ is an attractor for $S^k|K_0$, which implies $M = K_0$.

Thus $k \leq m - k$, and an induction leads to the absurdity that $1 \leq b \leq k - l m$ for all $l \in \mathbb{N}_+$. Therefore $M$ is empty, proving that no points of different $K_i$ are comparable.

Since we proved Theorem 1.3 for $S^m$, it follows that each $K_i$ is either a totally ordered arc of fixed points for $S^m$, or else it is unordered. In both cases the conclusion of Theorem 1.3 for $S$ follows.

**Remark 4.1** The proof of Theorem 1.3 contains the following result for chain transitive sets that are not necessarily internally chain transitive:

Let $S$ satisfy Hypothesis 1.1 with generic asymptotic period $m \geq 1$. Let $L \subset E$ be a compact chain transitive set. Given $a < b$ in $L$, there is a stationary $p$-arc $J$ for $S^m$ such that if $a \leq x \leq b$ then $\omega(x) = \gamma(p)$ for some $p \in J$.

I suspect, but cannot prove, that Theorem 1.3 is not true under the weaker hypothesis that $K$ is merely chain transitive.

## 5 Proof of Theorem 1.4

(Adapted from Benaim and Hirsch [4].) We assume $K$ is not an upper attracting $m$-periodic orbit. Consider the case $m = 1$. We first prove that if $x \in K^\bullet \setminus K$, then there is a fixed point $p$ such that

$$\omega(x) \geq \{p\} \gg K.$$ 

Replacing $x$ by $Sx$, we assume $x \gg y \in K$. By Theorem 1.2 we choose $z \in [[[y,x]]]$ such that $\omega(z)$ is a singleton $\{p\}$. Then

$$\omega(x) \geq p \geq \omega(y).$$

I claim $p \notin K$. For if $p \in K$ then $\omega(y) = \{p\}$ because $K$ is unordered, and therefore $\gamma(z) \searrow p$ by Theorem 2.1. Let $n \geq 0$ be such that $S^n z \gg p$. Thus $p$ is an upper attracting fixed point. Set

$$N = \{w \in K : w \leq S^n z\}$$

Strong monotonicity implies that $\omega(w) = \{p\}$ for all $w \in N$, and also that $S$ maps the compact set $N$ into its relative interior in $K$. This implies $N$ contains an attractor for $S|K$. Therefore $N = K$, yielding the contradiction that that $K = \{p\}$. Thus $p \notin K$. 

The set $C = \{ u \in K : u < p \}$ is a nonempty, forward invariant compact set. Strong monotonicity implies $S$ maps $C$ into its relative interior in $K$, which is $\text{Int}_K C = \{ u \in K : u \ll p \}$. Since $K$ is attractor-free, $C = K$. Therefore $\omega(x) \geq p \gg K$.

The compactness assumption Hypothesis 1.1(v) implies there is a minimal fixed point $q \gg K$. Suppose $x \in [K, p_1] \setminus K$. We saw above that $\gamma(x)$ converges to a fixed point $\gg K$. Since $\gamma(x) \leq q$ by monotonicity, $\omega(x) = \{q\}$. Taking $x$ sufficiently close to $K$ proves $q$ lower attracting.

Suppose $m > 1$. Then $K$ has the partition $K = K_1 \cup \cdots \cup K_d$ where $d|m$, each $K_i$ is attractor-free for $S^m$ and $S$ cyclically permutes the $K_i$. Since the generic asymptotic period for $S^m$ is 1, the case already proved yields a fixed point $q$ for $S^m$ satisfying 1.4 for $S^m$. It is easy to see this $q$ satisfies Theorem 1.4.

6 Proofs of theorems on semiflows

The proof of Theorem 1.6 is almost the same as that of Theorem 1.3. In place of Tereščák’s theorem 1.2 one uses the following result:

**Theorem 6.1** (Smith-Thieme [34]) There is a dense open set of points whose trajectories converge to equilibria.

There are also close analogues of Theorems 2.1 and 2.2. The real analytic case is based on a result in Jiang [25] that rules out setwise maximal totally ordered stationary p-arcs; see also Chow & Hale [5], p. 321. These results are put together just as in the proof of Theorem 1.3. The details are left to the reader.

**Proof of Theorem 1.7.** The proof is similar to the proof in Section 4 of the case $m = 1$ Theorem 1.4. One shows using Theorem 6.1 that if $x > y \in K$, there is an equilibrium $q_1$ such that $\omega(x) \geq q_1 \gg K$.

In this way one shows there exist a smallest equilibrium $q \gg K$ and this $q$ satisfies the theorem.

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