ESTIMATING THE TRACE-FREE RICCI TENSOR IN RICCI FLOW

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Abstract. An important and natural question in the analysis of Ricci flow singularity formation in all dimensions \( n \geq 4 \) is the following: What are the weakest conditions that provide control of the norm of the Riemann curvature tensor? In this short note, we show that the trace-free Ricci tensor is controlled in a precise fashion by the other components of the irreducible decomposition of the curvature tensor, for all compact solutions in all dimensions \( n \geq 3 \), without any hypotheses on the initial data.

1. Introduction

Standard short-time existence results imply that a solution \((M^n, g(t))\) of Ricci flow on a compact manifold becomes singular at \( T < \infty \) if and only if
\[
\lim_{t \to T} \max_{x \in M^n} |Rm(x, t)| = \infty.
\]
This suggests the following natural question: What are the weakest conditions that provide control of the norm of the full curvature tensor on a manifold evolving by Ricci flow?

In any dimension, it is true that a finite-time singularity happens if and only if
\[
\limsup_{t \to T} \max_{x \in M^n} |Rc(x, t)| = \infty.
\]
Nataša Šešum has given a direct proof [4]. The claim also follows from independent results of Miles Simon [5] by a short argument.

In dimension three, an eminently satisfactory answer is given by the well known pinching theorem obtained independently by Ivey [3] and Hamilton [2]. Their estimate implies in particular that the scalar curvature dominates the full curvature tensor of any Ricci flow solution on a compact 3-manifold with normalized initial data.

Xiuxiong Chen has expressed hope that an appropriate bound on the scalar curvature might be sufficient to rule out singularity formation in all dimensions.

1. Let \((M^n, g(t))\) be a solution of Ricci flow on a compact manifold such that \( g(t) \) is smooth for \( t \in [0, T) \), where \( T < \infty \). If \( \limsup_{t \to T} (\max_{x \in M^n} |Rc(x, t)|) < \infty \), then \[ Lemma 14.2 \] guarantees existence of a complete \( C^0 \) limit metric \( g(T) \). One may then apply \[ Theorem 1.1 \], choosing a background metric \( \tilde{g} := g(T - \delta) \) such that \((1 - \epsilon) \tilde{g} \leq g \leq (1 + \epsilon) \tilde{g} \), where \( \epsilon = \epsilon(n) \) and \( \delta = \delta(\epsilon) \). Let \( \tilde{K} = \max_{x \in M^n} |Rm(\tilde{g})|_\tilde{g} \). Simon’s theorem implies that there exists \( \eta = \eta(n, \tilde{K}) \) such that for any \( \theta \in [0, \delta] \), a solution \( \hat{g}(s) \) of harmonic-map-coupled Ricci flow exists for \( 0 \leq s < \eta \) and satisfies \( \hat{g}(0) = g(T - \theta) \); moreover, \( \hat{g}(s) \) is smooth for \( 0 < s < \eta \). Since harmonic-map-coupled Ricci flow is equivalent to Ricci flow modulo diffeomorphisms, the claim follows by taking \( \theta = \eta/2 \).

The Hamilton–Ivery pinching estimate implies the much stronger result that any rescaled limit of a finite time singularity in dimension three must have nonnegative sectional curvature.
Partial progress toward this conjecture was made recently by Bing Wang [6]. He proved that if the Ricci tensor is uniformly bounded from below on $[0, T)$ and if an integral bound
\[ \int_0^T \int_{\mathcal{M}^n} |R|^\alpha \, d\mu \, dt < \infty \]
holds for some $\alpha \geq (n + 2)/2$, then no singularity occurs at time $T < \infty$.

Recall that in any dimension $n \geq 3$, the Riemann curvature tensor admits an orthogonal decomposition
\[ Rm = U + V + W \]
into irreducible components
\[ U = \frac{1}{2n(n-1)} R(g \cdot \bar{g}) , \quad V = \frac{1}{n-2} (F \cdot \bar{g}) , \quad W = \text{Weyl tensor}, \]
where $\bar{g}$ denotes the Kulkarni–Nomizu product of symmetric tensors and $F$ denotes the trace-free Ricci tensor. The purpose of this short note is to observe that $V$ is always dominated by the other components in the following sense:

**Main Theorem.** If $(\mathcal{M}^n, g(\cdot))$ is a solution of Ricci flow on a compact manifold of dimension $n \geq 3$, then there exist constants $c(g_0) \geq 0$, $C_1(n, g_0) > 0$, and $C_2(n) > 0$ such that for all $t \geq 0$ that a solution exists, one has $R + c > 0$ and
\[ \frac{|V|}{R + c} \leq C_1 + C_2 \max_{s \in [0, t]} \frac{|W|_{\max}(s)}{R_{\min}(s)} + c. \]

2. **Proof of the main theorem**

Define
\[ a = |F| = \frac{\sqrt{n-2}}{2} |V| , \]
noting that $a$ is smooth wherever it is strictly positive. Choose $c \geq 0$ large enough so that $R_{\min}(0) + c > 0$ and define
\[ b = R + c , \]
noting that $b > 0$ for as long as a solution exists.

In any dimension $n \geq 3$, one has
\[ \frac{\partial}{\partial t} |F|^2 = \Delta |F|^2 - 2 |\nabla F|^2 + \frac{4(n-2)}{n(n-1)} R |F|^2 - \frac{8}{n-2} \text{tr} F^3 + 4 W(F, F), \]
where $\text{tr} F^3 = F_i^j F_k^l F_l^i$ and $W(F, F) = W_{ijkl} F^{i\ell} F^{jk}$. It follows from Cauchy–Schwarz that $a$ obeys the differential inequality
\[ a_t \leq \Delta a + \frac{2(n-2)}{n(n-1)} a(b - c) - \frac{4}{n-2} a^{-1} \text{tr} F^3 + 2a^{-1} W(F, F). \]

The positive quantity $b$ evolves by
\[ b_t = \Delta b + 2a^2 + \frac{2}{n} (b - c)^2. \]

To prove the theorem, it will suffice to bound the scale-invariant non-negative quantity
\[ \varphi = \frac{a}{b}. \]
Because \( \Delta \varphi = b^{-1}(\Delta a - \varphi \Delta b) - 2 \langle \nabla \varphi, \nabla \log b \rangle \), one has
\[
\varphi_t \leq \Delta \varphi + 2 \langle \nabla \varphi, \nabla \log b \rangle + 2 \rho \varphi,
\]
where the reaction term is
\[
\rho = \frac{n - 2}{n(n - 1)}(b - c) - \frac{2}{n - 2} \frac{\text{tr} F^3}{a^2} + \frac{W(F, F)}{a^2} - \frac{(b - c)^2}{nb} - a \varphi.
\]
There exist positive constants \( c_1, c_2 \) depending only on \( n \geq 3 \) such that
\[
\left| \frac{2}{n - 2} \text{tr} F^3 \right| \leq c_1 a^3 \quad \text{and} \quad |W(F, F)| \leq c_2 |W| a^2.
\]
Hence
\[
\rho \leq \frac{n - 2}{n(n - 1)}(b - c) + c_1 a + c_2 |W| - \frac{(b - c)^2}{nb} - a \varphi.
\]
Define constants \( \alpha, \beta, \gamma \) by
\[
\alpha^2 = c_1, \quad \beta^2 = \frac{n - 2}{n(n - 1)}, \quad \text{and} \quad \gamma^2 = c_2. \]
Fix \( \varepsilon > 0 \) and choose \( C_1 = \max\{\alpha^2 + \beta, \varphi_{\max}(0) + \varepsilon\} \) and \( C_2 = \gamma \). Consider the barrier function
\[
\Phi(t) = C_1 + C_2 \max_{s \in [0, t]} \sqrt{\frac{|W|_{\max}(s)}{b_{\min}(s)}},
\]
noting that \( \Phi \) is monotone nondecreasing. If \( \varphi_{\max}(t) \geq \Phi(t) \) at some \( t > 0 \), then at any \((x, t)\) where \( \varphi \) attains its spatial maximum, one has
\[
a \geq (\alpha^2 + \beta)b + \gamma \sqrt{b} |W|,
\]
which implies that
\[
a^2 \geq \alpha^2 ab + \beta^2 b^2 + \gamma^2 b |W| \geq \frac{n - 2}{n(n - 1)}(b^2 - bc) + (c_1 a + c_2 |W|) b - \frac{1}{n}(b - c)^2,
\]
hence that \( \rho \leq 0 \), hence that \( \varphi_t \leq 0 \), hence that \( \frac{d}{dt} \varphi_{\max}(t) \leq 0 \), understood in the usual sense as the lim sup of difference quotients. It follows that \( \varphi_{\max}(t) \leq \Phi(t) \) for as long as a solution exists.

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