SPECTRAL THEORY
OF PSEUDO-ERGODIC OPERATORS

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Abstract

We define a class of pseudo-ergodic non-self-adjoint Schrödinger operators acting in spaces $l^2(X)$ and prove some general theorems about their spectral properties. We then apply these to study the spectrum of a non-self-adjoint Anderson model acting on $l^2(\mathbb{Z})$, and find the precise condition for 0 to lie in the spectrum of the operator. We also introduce the notion of localized spectrum for such operators.

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1 Introduction

Recent papers have obtained some striking results concerning the spectral properties of the non-self-adjoint (nsa) Anderson model, which models the growth of bacteria in an inhomogeneous environment.\cite{10,11,12,13,5}. To be more precise the authors have determined the asymptotic limit of the spectrum of a nsa random finite periodic chain almost surely as the length of the chain increases to infinity. In a later paper the author considered the same random operator $H$ acting on $l^2(\mathbb{Z})$, and found that the spectrum is very different from that obtained by the cited authors,\cite{8}. The reason for this is that the spectral properties of nsa operators are highly unstable, and infinite volume limits should be examined using pseudospectral ideas,\cite{1,2,3,8,7,14,15,16,17}. More specifically if $\lambda$ lies in the spectrum of the infinite volume nsa Anderson model, it need not be close to the spectrum of the finite volume periodic Anderson model; one expects rather that the norm of the resolvent operator $(H - \lambda)^{-1}$ of the finite volume model will
diverge as the volume increases. These pseudospectral ideas have been worked out in detail for a random \textit{bidiagonal} model, which is in a certain sense exactly soluble, \cite{4, 9, 18}. Our results may therefore be interpreted as finding the region in the complex plane for which the finite volume nsa periodic Anderson model has very large resolvent norm.

In the present paper we reconsider such problems in a more general context, in which the probabilistic aspects have been eliminated in favour of what we call pseudo-ergodic ideas. As well as making the subject more accessible to those without a probabilistic training, this emphasizes the fact that the spectral matters which we consider depend only on the support of the relevant probability measure. On the other hand the asymptotics of the spectrum of the finite volume periodic nsa Anderson model does depend on the probability measure. We finally carry out a more detailed spectral analysis of the infinite volume nsa Anderson operator, and find precise conditions under which zero almost surely lies in the spectrum. We also obtain further results on the location of the spectrum, which come close to a complete determination in many cases. In the final section we consider the possibility that there may be constraints on the pair of values of the potential at two neighbouring points which are absolute rather than just probabilistic.

2 \hspace{1cm} The general context

The operators which we consider act on the Hilbert space \(l^2(X, \mathcal{K}) \sim l^2(X) \otimes \mathcal{K}\), where \(X\) is a countable set on which a group \(\Gamma\) acts by permutations. The simplest choice of the auxiliary Hilbert space \(\mathcal{K}\) is \(\mathbb{C}\), but other choices are needed in some applications; see the end of Section 3. Many of the results presented here apply to \(l^p(X, \mathcal{K})\) with \(p \neq 2\) without modification (the case \(p = 1\) is of probabilistic importance), but this does not apply to those involving numerical ranges. We define the unitary operators \(U_\gamma\) for \(\gamma \in \Gamma\) by \(U_\gamma f(x) = f(\gamma^{-1}x)\). The bounded operators which we study are of the form \(H = H_0 \otimes I + V\). Here \(H_0\) acts on \(l^2(X)\) and commutes with the action of \(\Gamma\) in the sense that \(H_0 U_\gamma = U_\gamma H_0\) for all \(\gamma \in \Gamma\), or equivalently

\[
H_0(\gamma x, \gamma y) = H_0(x, y)
\]

for all \(\gamma \in \Gamma\) and all \(x, y \in X\), where \(H_0(x, y)\) is the infinite matrix associated with \(H_0\). We assume that the spectrum \(E\) of \(H_0\) is known. From this point onwards we write \(H_0\) for \(H_0 \otimes I\).

Given a norm closed, bounded set \(\mathcal{M} \subseteq \mathcal{L}(\mathcal{K})\), we assume that the operator \(V\) is of the form

\[
(Vf)(x) = V(x)f(x)
\]

where \(V(x) \in \mathcal{M}\) for all \(x \in X\). We say that \(V\) is \((\Gamma, \mathcal{M})\) pseudo-ergodic if its set
of spatial translates is dense in the following sense. For every \( \varepsilon > 0 \), every finite subset \( F \subset X \) and every \( W : F \to \mathcal{M} \), there exists \( \gamma \in \Gamma \) such that

\[
\| W(x) - V(\gamma x) \| < \varepsilon
\]

for all \( x \in F \). It is well known that a large class of suitably defined random potentials have this property almost surely, but we consider a single potential, and do not need to introduce any probabilistic ideas. The same class of pseudo-ergodic potentials is applicable to a variety of different random models, as we explain in more detail in the final section.

The above definition suffices for our purposes, but it does not capture the full sense of random behaviour and may be refined as follows. We define a direction \( U \) to be an infinite subset of \( X \) such that for every finite \( F \subset X \) there exists \( \gamma \in \Gamma \) such that \( \gamma F \subset U \). We then say that \( V \) is \((\Gamma, \mathcal{M})\) pseudo-ergodic in the direction \( U \) if for every \( \varepsilon > 0 \), every finite subset \( F \subset X \) and every \( W : F \to \mathcal{M} \), there exists \( \gamma \in \Gamma \) such that \( \gamma F \subset U \) and

\[
\| W(x) - V(\gamma x) \| < \varepsilon
\]

for all \( x \in F \). Suitably defined random potentials have this property for every choice of direction almost surely, and therefore have the property simultaneously for any countable set of directions almost surely. The property itself, however, is defined for a single potential and makes no mention of probability.

The following theorem is an adaptation of a well-known result of Pastur for random potentials. We will use it to approximate \( \text{Spec}(H) \) from inside by making suitable choices of \( W \).

**Theorem 1** If \( H = H_0 + V \) where \( V \) is \((\Gamma, \mathcal{M})\) pseudo-ergodic and \( K = H_0 + W \) where \( W : X \to \mathcal{M} \) is arbitrary, then

\[
\text{Spec}(K) \subseteq \text{Spec}(H).
\]

In particular if \( V, W \) are both \((\Gamma, \mathcal{M})\) pseudo-ergodic then they have the same spectrum.

**Proof** If \( \lambda \in \text{Spec}(K) \) then there exists a sequence \( f_n \in l^2(X, K) \) with \( \|f_n\| = 1 \) and either \( \|Kf_n - \lambda f_n\| \to 0 \) or \( \|K^*f_n - \lambda f_n\| \to 0 \); we consider only the former case, the latter being similar. Given \( \varepsilon > 0 \) a truncation procedure shows that there exists \( f \) with finite support \( F \) in \( X \) such that \( \|f\| = 1 \) and \( \|Kf - \lambda f\| < \varepsilon/2 \). Since \( V \) is pseudo-ergodic there exists \( \gamma \in \Gamma \) such that \( \|H_\gamma f - Kf\| < \varepsilon/2 \), where

\[
H_\gamma = U_\gamma^{-1}HU_\gamma = H_0 + V(\cdot \gamma).
\]
Putting \( f_\varepsilon = U_\gamma f \) we deduce that
\[
\| H f_\varepsilon - \lambda f_\varepsilon \| = \| U_\gamma^{-1} H U_\gamma f - \lambda f \| < \varepsilon
\]
and the arbitrariness of \( \varepsilon > 0 \) implies that \( \lambda \in \text{Spec}(H) \).

**Corollary 2** If \( H = H_0 + V \) where \( V \) is \((\Gamma, \mathcal{M})\) pseudo-ergodic then
\[
\text{Spec}(H) = \bigcup \{ \text{Spec}(H_0 + W) : W \in \mathcal{M}^X \}.
\]
If also \( \tilde{H} = H_0 + \tilde{V} \) where \( \tilde{V} \) is \((\Gamma, \tilde{\mathcal{M}})\) pseudo-ergodic with \( \mathcal{M} \subseteq \tilde{\mathcal{M}} \) then
\[
\text{Spec}(H) \subseteq \text{Spec}(\tilde{H}).
\]

From this point we assume that \( H = H_0 + V \) where \( V \) is \((\Gamma, \mathcal{M})\) pseudo-ergodic. We put
\[
\text{Spec}(\mathcal{M}) = \bigcup_{A \in \mathcal{M}} \text{Spec}(A)
\]
and
\[
\text{Num}(\mathcal{M}) = \bigcup_{A \in \mathcal{M}} \text{Num}(A)
\]
where \( \text{Num} \) denotes the closure of the numerical range.

**Theorem 3** The spectrum of \( H \) satisfies
\[
E + \text{Spec}(\mathcal{M}) \subseteq \text{Spec}(H) \subseteq \text{Num}(H_0) + \text{Conv}(\text{Num}(\mathcal{M}))
\]
where \( \text{Conv} \) denotes the closed convex hull. If \( H_0 \) is normal and \( A \) is normal for every \( A \in \mathcal{M} \) then
\[
\text{Spec}(H) \subseteq \text{Conv}(E) + \text{Conv}(\text{Spec}(\mathcal{M})) \tag{1}
\]

**Proof** Theorem 1 implies that for each \( A \in \mathcal{M} \)
\[
E + \text{Spec}(A) = \text{Spec}(H_0 \otimes I + I \otimes A) \subseteq \text{Spec}(H)
\]
and this yields the first inclusion. The second depends on use of the numerical range to give
\[
\text{Spec}(H) \subseteq \text{Num}(H) \subseteq \text{Num}(H_0) + \text{Num}(V).
\]
Now \( z \) lies in the numerical range of \( V \) if and only if there exists \( f \in \ell^2(X, \mathcal{K}) \) of norm 1 such that \( z = \langle Vf, f \rangle \). Putting \( g_x = f(x)/\|f(x)\| \), provided this is non-zero, and \( \mu_x = \|f(x)\|^2 \), we see that \( \mu \) is a probability measure on \( X \) and that
\[
z = \sum_{x \in X} \mu_x \langle V_x g_x, g_x \rangle \in \text{Conv}(\text{Num}(M)).
\]
Hence $\text{Num}(V) \subseteq \text{Conv}(\text{Num}(M))$, and the first statement of the theorem follows. The second statement is a consequence of the fact that $\text{Num}(B)$ equals $\text{Conv}(\text{Spec}(B))$ for any normal operator $B$.

Let $B(x, r)$ denote the closed ball $\{y : |x - y| \leq r\}$. The next theorem complements Theorem 3.

**Theorem 4** If $A$ is normal for every $A \in \mathcal{M}$ then the spectrum of $H$ satisfies

$$\text{Spec}(H) \subseteq \text{Spec}(\mathcal{M}) + B(0, e)$$

where $e = \|H_0\|$. If $H_0$ is normal then

$$\text{Spec}(H) \subseteq E + B(0, \mu)$$

where $\mu = \max\{\|A\| : A \in \mathcal{M}\}$.

**Proof** If $V$ is normal then using $\text{Spec}(V) \subseteq \text{Spec}(\mathcal{M})$ we see that

$$\|(V - zI)^{-1}\| = \text{dist}(z, \text{Spec}(V))^{-1} \leq \text{dist}(z, \text{Spec}(\mathcal{M}))^{-1}$$

for all $z \notin \mathcal{M}$. Since $z \notin \mathcal{M} + B(0, e)$ is equivalent to $\text{dist}\{z, \text{Spec}(\mathcal{M})\} > \|H_0\|$, it implies

$$\|H_0\|(V - zI)^{-1} < 1$$

and the resolvent expansion for $(H_0 + V - zI)^{-1}$ is norm convergent. The proof of the second part of the theorem is similar.

We also wish to classify the spectrum of nsa operators acting on $l^2(X, \mathcal{K})$, and for this purpose we assume that $X$ is provided with a metric $d$ such that every ball $B(x, r) = \{y \in X : d(x, y) \leq r\}$ is finite and such that $\Gamma$ acts as a group of isometries of $X$. Given a function $f : X \to \mathcal{K}$ with $\|f\|_2 = 1$ we define its variance by

$$\text{var}(f) = \min_{y \in X} \sum_{x \in X} d(x, y)^2 |f(x)|^2$$

and its expectation to be any of the points in $X$ at which the minimum is achieved. The following theorems have analogues in which the variance is replaced by higher order moments, or suitable subexponential weights.

**Lemma 5** If $\|f\|_2 = 1$ and

$$v(x) = \sum_{y \in X} d(x, y)^2 |f(y)|^2$$

is finite for some $x \in X$ then it is finite for every $x \in X$ and $v(x)$ increases indefinitely as $x \to \infty$. Thus the minimum of $v(\cdot)$ is achieved at a finite number of points only. If $x_i, i = 1, 2$, are points at which $v$ has the same minimum value $s$ then $d(x_1, x_2) \leq 2s^{1/2}$. 
Proof If $v(x) < \infty$ then for any $u \in X$ we have
\[
v(u) \leq 2 \sum_{y \in X} \{d(x, y)^2 + d(x, u)^2\}|f(y)|^2 = 2\{v(x) + d(x, u)^2\} < \infty.
\]
by the triangle inequality. If the finite set $F$ satisfies
\[
\sum_{y \in F} |f(y)|^2 \geq \frac{1}{2}
\]
then
\[
v(x) \geq \sum_{y \in F} d(x, y)^2 |f(y)|^2 \geq \frac{1}{2} d(x, F)^2
\]
which increases indefinitely as $x \to \infty$ because of our assumption that all balls of finite radius contain only a finite number of points.

Now suppose that $s = \min\{v(x) : x \in X\}$ and that $v(x_1) = v(x_2) = s$. Then by the triangle inequality
\[
2s = \sum_{y \in X} \{d(x_1, y)^2 + d(x_2, y)^2\}|f(y)|^2
\geq \frac{1}{2} \sum_{y \in X} d(x_1, x_2)^2 |f(y)|^2
= \frac{1}{2} d(x_1, x_2)^2
\]
which implies the second statement of the lemma.

Following [8] we define the localized spectrum $\sigma_{\text{loc}}(A)$ of any bounded operator $A$ on $l^2(X, \mathcal{K})$ to be the set of all $\lambda \in \mathbb{C}$ such that there exists a sequence $f_n \in l^2(X, \mathcal{K})$ of unit vectors such that $||Af_n - \lambda f_n|| \to 0$ while $\text{var}(f_n)$ remains uniformly bounded. If $\lambda$ is an eigenvalue then one would expect its corresponding eigenfunction to decrease rapidly at infinity and hence to have finite variance, in which case $\lambda$ would lie in $\sigma_{\text{loc}}(A)$. What is more surprising is that $\sigma_{\text{loc}}(A)$ can be much larger than the set of eigenvalues of $A$.

**Theorem 6** If $H = H_0 + V$ where $V$ is $(\Gamma, \mathcal{M})$ pseudo-ergodic and $K = H_0 + W$ where $W : X \to \mathcal{M}$ is arbitrary, then
\[
\sigma_{\text{loc}}(K) \subseteq \sigma_{\text{loc}}(H).
\]
Thus every eigenvalue of $K$ lies in the localized spectrum of $H$. Moreover if $V, W$ are both $(\Gamma, \mathcal{M})$ pseudo-ergodic then they have the same localized spectrum.

**Proof** First note that if $f \in l^2(X, \mathcal{K})$ has unit norm and $\gamma \in \Gamma$ then $g = U_{\gamma}f$ has the same variance as $f$ because $\Gamma$ acts as a group of isometries of $X$. It is
a consequence of the definition of pseudo-ergodicity that there exists a sequence γ(n) ∈ Γ such that Hn = Uγ(n)−1HUγ(n) converges strongly to K. Now let \( \|f_m\| = 1 \), \( \text{var}(f_m) \leq s \) and \( \|Kf_m - \lambda f_m\| < \frac{1}{m} \) for all \( m \in \mathbb{Z}^+ \). Given \( m \)

\[
\begin{align*}
\|H(U_{\gamma(n)}f_m) - \lambda(U_{\gamma(n)}f_m)\| &= \|U_{\gamma(n)}^{-1}HU_{\gamma(n)}f_m - \lambda f_m\| \\
&= \|Hnf_m - \lambda f_m\| \\
&\to \|Kf_m - \lambda f_m\| < \frac{1}{m}
\end{align*}
\]

as \( n \to \infty \). Therefore there exists \( n(m) \) such that \( g_m = U_{\gamma(n(m))}f_m \) satisfies

\[
\|Hg_m - \lambda g_m\| < \frac{1}{m}
\]

for all \( m \in \mathbb{Z}^+ \). Since \( \text{var}(g_m) \leq s \) for all \( m \) it follows that \( \lambda \in \sigma_{\text{loc}}(H) \).

We next turn to the essential spectrum. We say that \( z \) lies in the essential spectrum of a bounded operator \( A \) if \( A - zI \) is not a Fredholm operator. We will need the following known result.

**Proposition 7** Suppose that \( z \in \mathbb{C} \) and for all \( \varepsilon > 0 \) and all finite \( N \) there exists an orthonormal set \( f_1, \ldots, f_N \) such that \( \|Af_n - zf_n\| < \varepsilon \) for all \( 1 \leq n \leq N \). Then \( z \) lies in the essential spectrum of \( A \).

**Proof** Suppose that \( z \in \mathbb{C} \) satisfies the conditions of the proposition. If \( \ker(A - zI) \) is infinite dimensional then \( A - zI \) is obviously not Fredholm, so let \( \dim(\ker(A - zI)) < N \) where \( N \) is finite. The assumption implies that for all \( \varepsilon > 0 \) there exists an \( N \)-dimensional subspace \( L \) such that \( f \in L \) implies

\[
\|Af - zf\| < \varepsilon\|f\|.
\]

Because \( \dim(L) > \dim(\ker(A - zI)) \) there exists \( f \perp \ker(A - zI) \) such that (4) holds. Since \( \varepsilon > 0 \) is arbitrary, \( A - zI \) cannot be Fredholm.

**Lemma 8** Suppose that there exists a \((\Gamma, M)\) pseudo-ergodic potential \( V \) on \( X \) where \( M \subseteq L(K) \) contains more than one point. Then for any finite subset \( F \) of \( X \) and any finite \( N \) there exist \( \gamma_1, \ldots, \gamma_N \in \Gamma \) such that \( \{\gamma_n F\}_{n=1}^N \) are pairwise disjoint.

**Proof** Let us first put \( N = 2 \). Let \( m_1, m_2 \in M \) and \( m_1 - m_2 \| = 2\delta > 0 \). Also let \( W : F \to L(K) \) satisfy \( W(x) = m_i \) for all \( x \in F \). Since \( V \) is \((\Gamma, M)\) pseudo-ergodic there exist \( \gamma_i \in \Gamma \) such that \( \|V(\gamma_1) - m_i\| < \delta \) for all \( x \in F \), or equivalently \( \|V(y) - m_i\| < \delta \) for all \( y \in \gamma_1 F \). This implies that \( \gamma_1 F \cap \gamma_2 F = \emptyset \).

We next prove that if the lemma holds for \( N \) then it holds for \( 2N \); we can then complete the proof by the use of induction. We put \( \tilde{F} = \bigcup_{j=1}^N \gamma_j F \) and let \( \beta_1, \beta_2 \in \Gamma \).
be such that $\beta_1 \tilde{F} \cap \beta_2 \tilde{F} = \emptyset$. This yields the statement of the lemma for the sets $\beta_i \gamma_j F$ where $i = 1, 2$ and $1 \leq j \leq N$.

**Theorem 9** If $H = H_0 + V$ where $V$ is $(\Gamma, M)$ pseudo-ergodic and $M$ contains more than one point, then $H$ has no inessential spectrum.

**Proof** If $\lambda \in \text{Spec}(H)$ then either (i) for every $\varepsilon > 0$ there exists $f \in l^2(X, K)$ such that $\|f\| = 1$ and $\|Hf - \lambda f\| < \varepsilon$, or (ii) for every $\varepsilon > 0$ there exists $f \in l^2(X, K)$ such that $\|f\| = 1$ and $\|H^* f - \overline{\lambda} f\| < \varepsilon$. We assume (i), the proof for (ii) being similar. By approximation we may assume that each $f$ has finite support $F$. Now for any $\varepsilon > 0$ and any finite $N$ let $\gamma_1, ..., \gamma_N \in \Gamma$ be such that $\gamma_i F$ are pairwise disjoint. Put $\tilde{F} = \bigcup_{i=1}^{N} \gamma_i F$ and define $W : \tilde{F} \to M$ by $W(\gamma_i x) = V(x)$ for all $x \in F$. Since $V$ is $(\Gamma, M)$ pseudo-ergodic there exists $\gamma \in \Gamma$ such that

$$\|V(\gamma y) - W(y)\| < \varepsilon$$

for all $y \in \tilde{F}$. Thus

$$\|V(\gamma_i x) - V(x)\| < \varepsilon \quad (5)$$

for all $x \in F$ and $1 \leq i \leq N$. We now put $f_i(x) = f(\gamma_i^{-1} \gamma^{-1} x)$ for all $x \in X$ and observe that $f_i$ have supports within $\gamma \gamma_i F$, which are disjoint, so $\{f_i\}_{i=1}^{N}$ form an orthonormal set. It follows from condition (i) and (5) that

$$\|H f_i - \lambda f_i\| < 2\varepsilon$$

for all $1 \leq i \leq N$. This implies that $\lambda$ lies in the essential spectrum of $H$ by Proposition 7.

**3 The nsa Anderson model**

In this section we apply the above ideas to an example of physical and biological importance. We first consider the one-dimensional nsa Anderson operator

$$H f_n = e^{-g} f_{n-1} + e^{g} f_{n+1} + V_n f_n \quad (6)$$

acting on $l^2(\mathbb{Z})$ (so that $K = C$), where $g > 0$ and $V$ is a $(\Gamma, M)$ pseudo-ergodic potential, $\Gamma$ being the group of all translations of $\mathbb{Z}$ and $M$ being a compact subset of $C$. The potential $V$ may be generated by assuming that its values at different points are independent and identically distributed according to a probability law which has compact support $M$.

Fourier analysis quickly establishes that $H_0$ is normal with spectrum the ellipse

$$E = \{e^{g+i\theta} + e^{-g-i\theta} : \theta \in [0, 2\pi]\} \quad (7)$$
following which Theorem 3 implies that
\[ E + M \subseteq \text{Spec}(H) \subseteq \text{Conv}(E) + \text{Conv}(M). \]  

A more precise determination of Spec\( (H) \) depends upon the size of \( g \), the choice of \( M \) and the use of Theorem 6, extending what we already proved in [8]. Given any finite sequence \( \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n-1}) \in M^n \) let \( W_\alpha \) be the periodic potential such that \( W_{\alpha,m} = \alpha_r \) if \( m = r \mod n \). The eigenvalue equation
\[
e^{-g}f_{m-1} + W_{\alpha,m}f_m + e^{g}f_{m+1} = \lambda f_m
\] may be rewritten in terms of \( w_m = (f_{m-1}, f_m) \in \mathbb{C}^2 \) as \( w_{m+1} = w_mA_m \) where
\[
A_m = \begin{bmatrix} 0 & -e^{-2g} \\ 1 & e^{-g}(\lambda - W_{\alpha,m}) \end{bmatrix}.
\]
Thus
\[
w_{n(r+1)} = w_{nr}B
\]
for all \( r \in \mathbb{Z} \) where \( B \) is the transfer matrix
\[
B = A_0A_1 \ldots A_{n-1}.
\]
Since
\[
det(B) = \prod_{r=0}^{n-1} \det(A_r) = e^{-2ng}
\]
it follows that at least one of the two eigenvalues \( \mu_1, \mu_2 \) of \( B \) satisfies \( |\mu_i| < 1 \). If we write
\[
B = \begin{bmatrix} b_{11}(\lambda) & b_{12}(\lambda) \\ b_{21}(\lambda) & b_{22}(\lambda) \end{bmatrix}
\]
then one may prove by induction that \( b_{22}(\lambda) \) is a polynomial of degree \( n \) in \( \lambda \) while the other coefficients are of lower degree.

The solution \( f \) of (9) corresponding to an eigenvalue \( \mu \) of \( B \) is exponentially increasing or decreasing on \( \mathbb{Z} \) according to whether \( |\mu| > 1 \) or \( |\mu| < 1 \) respectively.

**Theorem 10** Let \( E^n \) denote the ellipse
\[
E^n = \{ e^{i\theta} + e^{-2ng-i\theta} : \theta \in [-\pi, \pi] \}\]
and let
\[
E_\alpha = \{ \lambda : b_{11}(\lambda) + b_{22}(\lambda) \in E^n \}. \tag{10}
\]
Then \( B \) has an eigenvalue of modulus 1 if and only if \( \lambda \in E_\alpha \). Moreover \( E_\alpha \) is closed and bounded with
\[
E_\alpha \subseteq \text{Spec}(H).
\]
Proof If $\mu_1 = e^{i\theta}$ for some $\theta \in [-\pi, \pi]$ then $\mu_2 = e^{-2\pi i - i\theta}$, and
\[ b_{11}(\lambda) + b_{22}(\lambda) = e^{i\theta} + e^{-2\pi i - i\theta}. \]
or equivalently $\lambda \in E_\alpha$. The converse also holds. Our comments above on the degrees of $b_{ij}(\lambda)$ imply that $|\mu_1 + \mu_2|$ increases indefinitely as $|\lambda|$ grows. Therefore one of the $\mu_i$ must have modulus greater than 1 for large enough $|\lambda|$ and such $\lambda$ cannot lie in $E_\alpha$; therefore $E_\alpha$ must be bounded. The fact that $E_\alpha$ is closed follows directly from its definition.

Corresponding to any $\lambda \in E_\alpha$ there exists a solution $f$ of (9) such that $f_{m+n} = e^{i\theta} f_m$ for some $\theta \in \mathbb{R}$ and all $m \in \mathbb{Z}$. This $f$ is bounded but its $l^2$ norm is infinite. If we put
\[ f_{\varepsilon,m} = e^{-\varepsilon|m|} f_m \]
then a direct and well-known calculation shows that $\|f_{\varepsilon}\|_2 \to \infty$ and
\[ \frac{\| (H_0 + W) f_{\varepsilon} - \lambda f_{\varepsilon} \|_2}{\| f_{\varepsilon} \|_2} \to 0 \]
as $\varepsilon \to 0$. Applying Theorem 1 we deduce that
\[ \lambda \in \text{Spec}(H_0 + W) \subseteq \text{Spec}(H). \]

The set $C \setminus E_\alpha$ is the union of disjoint components and the number of eigenvalues $\mu_j$ of $B$ which have modulus less than 1 cannot change within each component, because the eigenvalues depend continuously on $\lambda$. This number must be either 1 or 2, and within the unbounded component it is 1. The following theorem joins the components into two sets.

**Theorem 11** If $\lambda$ lies in
\[ I_\alpha = \{ \lambda : b_{11}(\lambda) + b_{22}(\lambda) \in \text{int}(E^n) \} \] (11)
then all solutions of (9) are exponentially decreasing. If, however, $\lambda$ lies in
\[ O_\alpha = \{ \lambda : b_{11}(\lambda) + b_{22}(\lambda) \in \text{ext}(E^n) \} \] (12)
then there is an exponentially increasing solution of (9). The three sets $I_\alpha$, $O_\alpha$ and $E_\alpha$ are disjoint and cover $C$.

**Proof** The condition (11) holds if and only if both $\mu_i$ have modulus less than 1, and this implies that every solution of (9) is exponentially decreasing on $\mathbb{Z}$. Similarly The condition (12) holds if and only if one $\mu_i$ has modulus greater than 1, and this implies that one non-zero solution of (9) is exponentially increasing on $\mathbb{Z}$. 

10
The explicit description of the above sets depends upon the value of \( n \). For \( n = 1 \) we have \( \alpha \in M \) and

\[ E_\alpha = E + \alpha. \]

If \( n = 2 \) and \( \alpha = (\alpha_0, \alpha_1) \in M^2 \) then

\[ B = \begin{bmatrix} -e^{-2g} & -e^{-3g}(\lambda - \alpha_1) \\ e^{-g}(\lambda - \alpha_0) & e^{-2g}((\lambda - \alpha_0)(\lambda - \alpha_1) - 1) \end{bmatrix} \]

and \( E_\alpha \) is the set of \( \lambda \) such that

\[ e^{-2g}\{(\lambda - \alpha_0)(\lambda - \alpha_1) - 2\} \in E^2. \] (13)

This equation may be solved to present \( \lambda \) explicitly as a function of \( \theta \). For larger values of \( n \) it is probably only practicable to find \( E_\alpha \) numerically.

The special case \( n = p = 1 \) of the following theorem was proved in \([8]\). The idea owes much to the theory of block Toeplitz matrices \([1, 2, 3, 15]\).

**Theorem 12** Let \( H \) be defined by (6) where \( g > 0 \) and \( V \) is a \((\mathbb{Z}, M)\) pseudo-ergodic potential. If \( \alpha \in M^n \) and \( \beta \in M^p \) then

\[ I_\alpha \cap O_\beta \subseteq \sigma_{loc}(H). \]

**Proof** We consider the operator \( K = H_0 + W \) acting on \( l^2(\mathbb{Z}) \) where

\[ W_m = \begin{cases} \alpha_r & \text{if } m \geq 0 \text{ and } m = r \mod n \\ \beta_r & \text{if } m < 0 \text{ and } m = r \mod p. \end{cases} \]

We then consider the solutions of

\[ e^{-g}f_{m-1} + W_mf_m + e^{g}f_{m+1} = \lambda f_m \]

where \( \lambda \in I_\alpha \cap O_\beta \). Since \( \lambda \in O_\beta \) there exists a solution \( f \) which is exponentially growing for \( m < 0 \), i.e. which decreases exponentially as \( m \to -\infty \). Continuing this solution to positive \( m \) it follows from \( \lambda \in I_\alpha \) that \( f \) also decreases exponentially as \( m \to \infty \). Hence \( f \) is an eigenvector of finite variance and \( \lambda \in \sigma_{loc}(K) \subseteq \sigma_{loc}(H) \).

**Theorem 13** If in addition to the hypotheses of the last theorem we put \( M = [-\mu, \mu] \) then \( \text{Spec}(H) = E + [-\mu, \mu] \) for all \( \mu \geq e^g + e^{-g} \). Moreover \( 0 \in \text{Spec}(H) \) if and only if \( \mu \geq e^g - e^{-g} \).

**Proof** The first statement only needs the observation that the two sides of (8) coincide under the given condition. If \( \mu < e^g - e^{-g} \) then \( 0 \notin \text{Spec}(H) \) by Theorem 4. Now \( 0 \in E_{(-\mu, \mu)} \) if and only if \( e^{-2g}(-\mu^2 - 2) \in E^2 \) by (13), and this is equivalent to \( \mu = e^g - e^{-g} \); for such \( \mu \) one has \( 0 \in \text{Spec}(H) \) by Theorem 10. For smaller \( \mu \) we
have $0 \in I(-\mu, \mu)$ and for larger $\mu$ we have $0 \in O(-\mu, \mu)$. Therefore $0 \in I(0) \cap O(-\mu, \mu)$ for $\mu > e^g - e^{-g}$, and $0 \in \sigma_{loc}(H)$ by Theorem 12.

If $M = [-\mu, \mu]$ the above theorems admit the possibility that there are two holes in the spectrum on either side of the origin for

$$e^g - e^{-g} < \mu < e^g + e^{-g}.$$ 

We nevertheless conjecture that one has $\text{Spec}(H) = \text{Conv}(E + M)$ for all $\mu \geq e^g - e^{-g}$.

We contrast the above with the case in which $M = \{\pm \mu\}$. The following theorem completely determines the real part of $\text{Spec}(H)$ under the stated conditions.

**Theorem 14** If $M = \{\pm \mu\}$ and $\mu > e^g + e^{-g}$ then

$$(\text{Conv}(E) + \mu) \cup (\text{Conv}(E) - \mu) \subseteq \text{Spec}(H) \subseteq B(\mu, e^g + e^{-g}) \cup B(-\mu, e^g + e^{-g})$$

and

$$\text{Spec}(H) \subseteq \text{Conv}(E) + [-\mu, \mu].$$

**Proof** The first inclusion of the statement follows from the case $n = p = 1$ of Theorem 12 as in [8]. The second follows from the first half of Theorem 4, and the final one follows from Theorem 3.

We conjecture that the first inclusion is actually an equality.

**Corollary 15** If $M = \{\pm \mu\}$ then $0 \in \text{Spec}(H)$ if and only if

$$e^g - e^{-g} \leq \mu \leq e^g + e^{-g}.$$ 

**Proof** If $\mu < e^g - e^{-g}$ then $0 \notin \text{Spec}(H)$ by combining Corollary 2 and Theorem 13. If $\mu > e^g + e^{-g}$ then $0 \notin \text{Spec}(H)$ by Theorem 14. If $\mu = e^g - e^{-g}$ then $0 \in E_{(-\mu, \mu)} \subseteq \text{Spec}(H)$ by Theorem 10. If $\mu = e^g + e^{-g}$ then $0 \in E_{\mu} \subseteq \text{Spec}(H)$ by Theorem 10. Finally if $e^g - e^{-g} < \mu < e^g - e^{-g}$ then $0 \in O_{(-\mu, \mu)} \cap I_{\mu} \subseteq \text{Spec}(H)$ by Theorem 12.

We next turn to the nsa Anderson model in $\mathbb{Z}^n$. The operator $H$ on $l^2(\mathbb{Z}^n)$ is defined by

$$(Hf)(m, n) = (H_0f)(m, n) + V(m, n)f(m, n)$$

where

$$(H_0f)(m, n) = e^g f(m + 1, n) + e^{-g} f(m - 1, n) + f(m, n + 1) + f(m, n - 1)$$

for some $g > 0$. We assume that $V$ is real-valued and pseudo-ergodic with values in $M = [-\mu, \mu]$. It follows by Fourier transform methods that $H_0$ is normal with spectrum equal to

$$\tilde{E} = E + [-2(n - 1), 2(n - 1)]$$
where $E$ is the ellipse defined by (7). This set is connected with a hole around the origin if $n = 2$ but it may or may not have such a hole for $n \geq 3$. This phenomenon is a result of the particular choice of lattice used to discretize the Laplacian. If $\mu$ is sufficiently small the same applies to $\text{Spec}(H)$.

**Theorem 16** If $\mu \geq e^g + e^{-g} - 2(n-1)$ then $\text{Spec}(H)$ is the convex set

$$E + [-\mu - 2(n-1), \mu + 2(n-1)].$$

**Proof** As in the one-dimensional case we need only observe that the two sides of (8) are equal under the hypotheses.

We next mention the same operator acting in $l^2(X)$ where

$$X = \{(m, n) : m \in \mathbb{Z}, 1 \leq n \leq N\}$$

subject to Dirichlet boundary conditions; the Neumann case is similar. We may carry out an analysis similar to that above if we are only concerned to determine the spectrum, but more detailed spectral information is obtained by putting $l^2(X) = l^2(\mathbb{Z}, \mathcal{K})$ where $\mathcal{K} = \mathbb{C}^N$. We then put

$$(H_0f)(m) = e^g f(m + 1) + e^{-g} f(m - 1)$$

and

$$\tilde{V}(m)(r, s) = \begin{cases} 1 & \text{if } |r - s| = 1 \\ V(m, r) & \text{if } r = s \\ 0 & \text{otherwise} \end{cases}$$

where $1 \leq r, s \leq N$ in all cases. Note that $H_0$ is normal and $\tilde{V}(m)$ is a self-adjoint matrix for all $m \in \mathbb{Z}$, so all of the theorems of Section 2 apply. Using such ideas it is possible to analyze the localized spectrum of $H$ as in the one-dimensional case.

We finally comment that certain random bidiagonal operators can also be treated by the methods of this paper by making the appropriate choice of $H_0$, as can a variety of other operators whose matrix coefficients depend only on $m - n$ whenever $m \neq n$. See [4, 9, 18], which use probabilistic rather than pseudo-ergodic methods.

### 4 Resolvent Norms

The spectral behaviour of a bounded operator $A$ acting on a Hilbert space $\mathcal{H}$ can be measured in several ways. In pseudospectral theory one examines the contours of the function

$$s(A, z) = \begin{cases} \|(A - z)^{-1}\|^{-1} & \text{if } z \notin \text{Spec}(A) \\ 0 & \text{if } z \in \text{Spec}(A). \end{cases}$$
This function converges to zero as $z$ approaches the spectrum of $A$ because of the upper bound

$$s(A, z) \leq \text{dist}(z, \text{Spec}(A))$$

and the case of most interest is when $s(A, z)$ is very small for $z$ far from the spectrum. The determination of the pseudospectra, defined as the family of sets $\{z : s(z) < \varepsilon\}$ for all positive $\varepsilon$, is computationally heavy, but the family carries much more information than the spectrum alone [1, 3, 14, 15, 16, 17].

**Lemma 17** *The function $s(A, \cdot)$ satisfies the Lipschitz inequality*

$$|s(A, z) - s(A, w)| \leq |z - w|$$

*for all $z, w \in \mathbb{C}$.*

The proof uses the formula

$$s(A, z) = \inf \{\| (A - z) f \| / \| f \| : 0 \neq f \in \mathcal{H}\}$$

(valid for all $z \notin \text{Spec}(A)$). Note that this may be false for $z \in \text{Spec}(A)$, as one may see by considering the operator $\hat{A}$ on $l^2(\mathbb{Z}^+)$ defined by

$$\hat{A}f(n) = \begin{cases} 0 & \text{if } n = 1 \\ f(n-1) & \text{if } n \geq 2. \end{cases}$$

The next theorem provides an upper bound on $s(A, \cdot)$ which may be used to compute it numerically. Let $L$ be a finite-dimensional subspace of $\mathcal{H}$ and let $P$ be the orthogonal projection onto $L$. We define $B(A, L, z)$ to be the restriction of

$$P(A - zI)^*P(A - zI)P + PA^*(I - P)AP$$

to the subspace $L$, and $\sigma(A, L, z)$ to be the square root of the smallest eigenvalue of $B(A, L, z)$.

**Theorem 18** *If we put*

$$s(A, L, z) = \min \{\sigma(A, L, z), \sigma(A^*, L, \overline{z})\}$$

*then*

$$s(A, L, z) \geq s(A, z).$$

*The functions $s(A, L, \cdot)$ decrease monotonically and locally uniformly to $s(A, \cdot)$ as the subspaces increase.*
Proof It follows from its definition that
\[ \sigma(A, L, z) = \min \{ \|(A - z)f\|/\|f\| : 0 \neq f \in L \} . \]

It is clear from this that \( \sigma(A, L, z) \) decreases monotonically and pointwise to
\[ \sigma(A, z) = \inf \{ \|(A - z)f\|/\|f\| : 0 \neq f \in \mathcal{H} \} . \]

If \( z \notin \text{Spec}(A) \) this equals \( s(A, z) \). Similar comments apply with \( A \) replaced by \( A^* \), and we also have \( \sigma(A, z) = \sigma(A^*, \overline{z}) \) for all \( z \).

On the other hand if \( z \in \text{Spec}(A) \) we have either \( \sigma(A, z) = 0 \) or \( \sigma(A^*, \overline{z}) = 0 \), or both. This implies that \( s(A, L, z) \) converges monotonically and pointwise to 0. Since all the functions involved are Lipschitz continuous with Lipschitz constant 1, the convergence must be locally uniform.

Now suppose that \( \mathcal{H} \) equals \( l^2(X, \mathcal{K}) \) and \( L \) is defined as the space of all functions with support in a particular finite region \( \Omega \). The above theorem is better than the mere computation of the spectrum of \( PAP \) restricted to \( L \) (possibly subject to certain boundary conditions on \( \partial \Omega \)) because it gives rigorous upper bounds to \( s(A, z) \) rather than uncontrolled approximations. Another advantage is that it provides an upper bound for \( s(A, z) \) for every extension of the operator \( A \) beyond the subspace \( L \). Because of its approximate nature one cannot determine the spectrum of \( A \) exactly using the above theorem, but it may be possible to get good approximations to the pseudospectra, which are often of greater importance for such operators.

We now turn to pseudo-ergodic operators, working in the technical context of Section 2. The following theorem indicates how one may get rigorous upper bounds and approximations to the pseudospectra by selecting appropriate potentials \( W \).

**Theorem 19** If \( H = H_0 + V \) where \( V \) is \((\Gamma, M)\) pseudo-ergodic and \( K = H_0 + W \) where \( W : X \to M \) is arbitrary, then
\[ \|(H - zI)^{-1}\| \geq \|(K - zI)^{-1}\| \]
for all \( z \in \mathbb{C} \). Therefore
\[ s(H, z) = \min \{ s(H_0 + W, z) : W \in M^X \} . \]

If \( V, W \) are both \((\Gamma, M)\) pseudo-ergodic then the resolvent norms and hence pseudospectra of \( H \) and \( K \) are equal.

**Proof** By Theorem 1 we need only consider the case in which \( z \) does not lie in the spectrum of either operator. If \( s(K, z) < c \) then there exists \( f \in l^2(X, \mathcal{K}) \) such that \( \|(K - z)f\| < c\|f\| \) and by approximation we may assume that \( f \) has finite support.
Using the pseudo-ergodic property of $H$ there exists $g$ of finite support such that $\| (H - z) g \| < c \| g \|$ and this implies that $s(H, z) < c$. Hence $s(H, z) \leq s(K, z)$. The remainder of the proof follows Theorem 1 or Corollary 2.

For the nsa periodic Anderson model with $M = [-\mu, \mu]$ the asymptotic limit of the finite volume spectrum has been determined [10], and it is seen that for certain ranges of the parameter $\mu$ zero does not lie in the asymptotic spectrum, which is the union of a set of complex curves. On the other hand the spectrum of the same operator on any finite interval subject to Dirichlet boundary conditions is entirely real. It has been suggested in [5] that for periodic boundary conditions there is no pseudospectral pathology of the type which occurs for Dirichlet boundary conditions. However, our results demonstrate that spatially rare special sections of a random potential have a dominant effect on the spectrum of the infinite volume nsa Anderson operator. This should not be taken as an indication that our results are unphysical: it is well known that the behaviour of bulk materials is often radically affected by the presence of low concentrations of impurities and/or defects, and one should expect the mathematics to reflect this.

We have implemented the above ideas numerically using Matlab for the operator $H$ defined by (6) where $e^g = 2$ and $V_n$ are independent random variables uniformly distributed on $[-3, 3]$. We took $L$ to be the subspace of all sequences with support in $[1, 100]$ and computed the minimum value of $\sigma(H, L, x)$ over 1000 different choices of the potential $V$. We chose to study real $x \in [0, 6]$, but complex values of $x$ in any region can be accommodated by the same method. This yielded the upper bounds $\sigma(H, x)$ as follows (the omitted values of $\sigma(H, x)$ all vanish to the given accuracy).

| $x$ | $\sigma(H, x)$ |
|-----|----------------|
| 0.0 | 0.0283         |
| 0.5 | 0.0203         |
| 1.0 | 0.0084         |
| 1.5 | 0.0015         |
| ... | ...            |
| 4.5 | 0.0044         |
| 5.0 | 0.2233         |
| 5.5 | 0.6259         |
| 6.0 | 1.0817         |

Our general theory shows that the real part of the spectrum of this operator is $[-5.5, 5.5]$, which is consistent with the numerical conclusion that

$$\| (H - xI)^{-1} \| \geq 10^2$$

for all $1.0 \leq x \leq 4.5$ and

$$\| (H - xI)^{-1} \| \geq 10^4$$
for all $2.0 \leq x \leq 4.0$. (Of course the numerical calculation can also be carried out in cases in which one does not have a prior theoretical solution!) The eigenvectors of $B(A, L, x)$ corresponding to the smallest eigenvalues were also computed for several values of $x$. As expected from the theory of localized spectrum, they were all highly concentrated around some point in the interior of $[1, 100]$, and negligible at the ends of the interval.

We finally examine the behaviour of the resolvent norm at the point $z = 0$. To be precise we consider the nsa Anderson model with $M = [-\mu, \mu]$ acting on $l^2(\mathbb{Z})$ for various values of $\mu$. Recall that Theorem 13 states that $0 \in \text{Spec}(H)$ if and only if $\mu \geq e^g - e^{-g}$.

**Theorem 20** If $0 \leq \mu < e^g - e^{-g}$ then

$$\|H^{-1}\|^{-1} = e^g - e^{-g} - \lambda.$$ 

**Proof** If we exhibit the $\mu$ dependence of $H$ explicitly and put $t(\mu) = s(H_\mu, 0)$ then it follows from (14) that

$$|t(\mu) - t(\nu)| \leq |\mu - \nu|$$

for any $0 \leq \mu, \nu < e^g - e^{-g}$. Since $t(0) = e^g - e^{-g}$ and $t(e^g - e^{-g}) = 0$ we conclude that

$$t(\mu) = e^g - e^{-g} - \mu$$

for all $0 \leq \mu < e^g - e^{-g}$.

## 5 Constrained Potentials

We have avoided the use of any probabilistic methods by the introduction of the concept of pseudo-ergodicity. We now explore the variety of situations in which our ideas are applicable. The obvious possibility is to assume that $\mu$ is a probability measure with support equal to the set $\mathcal{M} \subseteq \mathcal{L}(\mathcal{K})$ and to assume that $V_x$ are independent random variables as $x \in X$ varies and that each is distributed according to $\mu$. However, even if we assume that $V_x$ are independent, we may permit each $V_x$ to be distributed according to a different probability measure $\mu_x$ with support equal to $\mathcal{M}$. These measures need not even be $\Gamma$-stationary, but they must satisfy the following condition. For every open set $U \subseteq \mathcal{L}(\mathcal{K})$ such that $U \cap \mathcal{M} \neq \emptyset$ there must exist a constant $c_U > 0$ such that $\mu_x(U) \geq c_U$ for all $x \in X$. This is sufficient to imply that $V$ is $(\Gamma, \mathcal{M})$ pseudo-ergodic almost surely by the usual probabilistic argument. For all such probabilistic models the spectrum (or localized spectrum) of the operator $\hat{H}$ is the same.

Similar remarks apply to a variety of other probabilistic models in which the values $V_x$ are not independent. There is one situation, however, in which changes in
the spectrum may arise. We say that a potential $V$ satisfies the local constraints $Q = (\mathcal{M}, \gamma_1, ..., \gamma_k, \mathcal{N}_1, ..., \mathcal{N}_k)$ where $\gamma_i \in \Gamma$ and $\mathcal{M}, \mathcal{N}_i$ are closed, bounded subsets of $\mathcal{L}(\mathcal{K})$ under the following conditions. For all $x \in X$ we require that $V_x \in \mathcal{M}$ and also that

$$V_x - V_{\gamma_i x} \in \mathcal{N}_i$$

for all $i = 1, ..., k$. Even more general constraints can be formulated. We then say that $V$ is $(\Gamma, Q)$ pseudo-ergodic if it satisfies the constraints $Q$ and for any other potential $W$ which satisfies the same constraints and any finite subset $F$ of $X$ and any $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that

$$\|V_{\gamma x} - W_x\| < \varepsilon$$

for all $x \in F$. These constraints force a relationship between the values of $V_x$ at neighbouring points which is stronger than a mere probabilistic correlation.

**Lemma 21** If $H_j = H_0 + V_j$ where $V_1$ is $(\Gamma, \mathcal{M})$ pseudo-ergodic and $H_2$ is $(\Gamma, Q)$ pseudo-ergodic, then

$$\text{Spec}(H_2) \subseteq \text{Spec}(H_1).$$

Any two $(\Gamma, Q)$ pseudo-ergodic operators have the same spectrum.

**Proof** The first statement is a consequence of Theorem 1. The second involves adapting the proof of the same theorem.

We now apply the above ideas in a simple context. We assume that $X = \mathbb{Z}$, that $\Gamma$ is the usual translation group acting on $\mathbb{Z}$, and that $\mathcal{K} = \mathbb{C}$. We assume that $a, b$ are two positive constants and impose attractive constraints $Q_1$ of the form

$$-a \leq V_n \leq a, \quad |V_n - V_{n+1}| \leq b$$

for all $n \in \mathbb{Z}$. Although we are not able to prove Theorem 12 in full generality under such conditions the important special case $n = p = 1$ is still valid.

**Theorem 22** Let $H$ be defined by (6) where $g > 0$ and $V$ is a $(\mathbb{Z}, Q_1)$ pseudo-ergodic potential. We have

$$E + [-a, a] \subseteq \text{Spec}(H) \subseteq \text{Conv}(E) + [-a, a]$$

where $E$ is given by (7). If $\alpha, \beta \in [-a, a]$ then

$$I_{\alpha} \cap O_{\beta} \subseteq \sigma_{\text{loc}}(H).$$

**Proof** The first statement of the theorem is proved as in Theorem 3. For the second part we follow the method of Theorem 12 but for the operator $K = H_0 + W$ acting on $l^2(\mathbb{Z})$ where

$$W_n = \begin{cases} 
\alpha & \text{if } n > N \\
\beta & \text{if } n < 0 \\
\beta + n(\alpha - \beta)/N & \text{if } 0 \leq n \leq N.
\end{cases}$$
Here we take $N$ large enough to ensure that $W$ satisfies the constraints $Q_1$. A more interesting variation upon our earlier theory occurs if we impose the repulsive constraint $Q_2$ defined by

$$-a \leq V_n \leq a, \quad |V_n - V_{n+1}| \geq b$$

for all $n \in \mathbb{Z}$, where $0 < b \leq 2a$. This excludes constant potentials, thus rendering the first inclusion of Theorem 3 invalid. The range of a $(\Gamma, Q_2)$ pseudo-ergodic potential $V$ is equal to $M = [-a, a-b] \cup [b-a, a]$.

The spectrum of the Anderson model (11) is easy to determine in the self-adjoint case, and we start with this.

**Theorem 23** If $g = 0$ and $V$ is $(\Gamma, Q_2)$ pseudo-ergodic then the spectrum of the operator $H = H_0 + V$ defined by (11) is given by

$$\text{Spec}(H) = T \cup (-T)$$

where

$$T = \left[ b - a, a - b \frac{b^2}{2} + \sqrt{\frac{b^2}{4} + 4} \right] .$$

Thus $\text{Spec}(H)$ has a spectral gap if and only if $a < b \leq 2a$.

**Proof** Let $W$ be the potential $W_n = (-1)^n b/2$, so that $W + sI$ satisfies the constraints $Q_2$ for all real $s$ such that $|s| \leq a - b/2$. It follows from Theorem 1 that if $S = \text{Spec}(H_0 + W)$ then

$$S + [b/2 - a, a - b/2] \subseteq \text{Spec}(H). \quad (15)$$

Conversely $\|V - W\| \leq a - b/2$, so the perturbation theoretic argument used in Theorem 4 implies that

$$\text{Spec}(H) \subseteq S + [b/2 - a, a - b/2] .$$

We deduce that

$$\text{Spec}(H) = S + [b/2 - a, a - b/2]$$

and complete the proof by using a Bloch wave analysis to compute the set $S$.

Now let us denote the same operator by $L_g$ for $g \geq 0$. We may regard $L_g$ as a perturbation of $L_0$ and use the argument of Theorem 4 to show that

$$\text{Spec}(L_g) \subseteq \text{Spec}(L_0) + B(0, e^g - 1) .$$

We may also use Theorem 4 as it stands to obtain an outer estimate of $\text{Spec}(L_g)$. We may obtain inner estimates by the method of Section 3 provided we are careful to avoid the use of constant potentials.
**Theorem 24** We have

\[ S + \left[ \frac{b}{2} - a, a - \frac{b}{2} \right] \subseteq \text{Spec}(L_g) \]

where

\[ S = \left\{ \pm \sqrt{\frac{b^2}{4} + 2 + e^{2g+i\theta} + e^{-2g-i\theta}} : \theta \in [-\pi, \pi] \right\}. \]

**Proof** if we put \( \alpha_0 = -b/2 \) and \( \alpha_1 = b/2 \) and solve (13) for \( \lambda \) we obtain

\[ E_{(-b/2,b/2)} = S. \]

The remainder of the proof follows Theorem 23, using the last part of Theorem 10. Note that for small positive \( g \), \( S \) consists of two closed curves on opposite sides of the \( y \)-axis, but for large \( g \) it is a single curve enclosing the origin.

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