Crowdsourcing: Low complexity, Minimax Optimal Algorithms

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Abstract

We consider the problem of accurately estimating the reliability of workers based on noisy labels they provide, which is a fundamental question in crowdsourcing. We propose a novel lower bound on the minimax estimation error which applies to any estimation procedure. We further propose Triangular Estimation (TE), an algorithm for estimating the reliability of workers. TE has low complexity, may be implemented in a streaming setting when labels are provided by workers in real time, and does not rely on an iterative procedure. We further prove that TE is minimax optimal and matches our lower bound. We conclude by assessing the performance of TE and other state-of-the-art algorithms on both synthetic and real-world data sets.

1 Introduction

The performance of many machine learning techniques, and in particular data classification, strongly depends on the quality of the labeled data used in the initial training phase. A common way to label new datasets is through crowdsourcing: many workers are asked to label data, typically texts or images, in exchange of some low payment. Of course, crowdsourcing is prone to errors due to the difficulty of some classification tasks, the low payment per task and the repetitive nature of the job. Some workers may even introduce errors on purpose. Thus it is essential to assign the same classification task to several workers and to learn the reliability of each worker through her past activity so as to minimize the overall error rate and to improve the quality of the labeled dataset.

Learning the reliability of each worker is a tough problem because the true label of each task, the so-called ground truth, is unknown; it is precisely the objective of crowdsourcing to

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guess the true label. Thus the reliability of each worker must be inferred from the comparison of her labels on some set of tasks with those of other workers on the same set of tasks.

In this paper, we consider binary labels and study the problem of estimating the workers reliability based on the answers they provide to tasks. We make two novel contributions to that problem:

(i) We derive a lower bound on the minimax estimation error which applies to any estimator of the workers reliability. In doing so we identify "hard" instances of the problem, and show that the minimax error depends on two factors: the reliability of the three most informative workers and the mean reliability of all workers.

(ii) We propose TE (Triangular Estimation), a novel algorithm for estimating the reliability of each worker based on the correlations between triplets of workers. We analyze the performance of TE and prove that it is minimax optimal, so that it matches the lower bound we previously derived.

TE has low complexity in terms of memory space and computation time, does not require to store the whole data set in memory and can be easily applied in a setting in which answers to tasks arrive sequentially, i.e., in a streaming setting. Finally, we compare the performance of TE to state-of-the-art algorithms through numerical experiments using both synthetic and real datasets.

2 Related Work

The first problems of data classification using independent workers appeared in the medical context, where each label refers to the state of a patient (e.g., sick or sane) and the workers are clinicians. In [5], Dawid and Skene proposed an expectation-maximization (EM) algorithm, admitting that the accuracy of the estimate was unknown. Several versions and extensions of this algorithm have since been proposed and tested in various settings [9, 19, 1, 17, 13]. Performance guarantees have been provided only recently for improved versions of the algorithm [2, 24], as discussed below.

A number of Bayesian techniques have also been proposed and applied to this problem, see [17, 22, 10, 13, 12, 11] and references therein. Of particular interest is the belief-propagation (BP) algorithm of Karger, Oh and Shah [10], which is provably order-optimal in terms of the number of workers required per task for any given target error rate, in the limit of an infinite number of tasks and an infinite population of workers.

Another family of algorithms is based on the spectral analysis of some matrix representing the correlations between tasks or workers. Gosh, Kale and McAfee [6] work on the task-task matrix whose entries correspond to the number of workers having labeled two tasks in the same manner, while Dalvi et al. [4] work on the worker-worker matrix whose entries correspond to the number of tasks labeled in the same manner by two workers. Both obtain performance guarantees by the perturbation analysis of the top eigenvector of the corresponding expected matrix. The BP algorithm of Karger, Oh and Shah is in fact closely related to these spectral algorithms: their message-passing scheme is very similar to the power-iteration method applied to the task-worker matrix, as observed in [10].
Two notable recent contributions are [2] and [24]. [2] provides performance guarantees for two versions of EM, and derives lower bounds on the attainable prediction error (the probability of estimating labels incorrectly). [24] provides lower bounds on the estimation error of the workers reliability as well as performance guarantees for an improved version of EM relying on spectral methods in the initialization phase. Our results bring some improvement to [2, 24] both in terms of lower bounds and attainable estimation error, as discussed in more details in latter sections. In fact, our estimator shares some features of the algorithm proposed in [24] to initialize EM, which suggests that the EM phase itself is not essential to attain minimax optimality.

A recent paper proposes an algorithm based on the notion of minimax conditional entropy [25], based on some probabilistic model jointly parameterized by the workers reliability and the task difficulty. The algorithm is evaluated through numerical experiments on real datasets only; no theoretical results are provided on the performance and the complexity of the algorithm.

All these algorithms require the storage of all labels in memory. To our knowledge, the only streaming algorithm that has been proposed for crowdsourced data classification is the recursive EM algorithm of Wang et al. [21], for which no performance guarantees are available.

Some authors consider slightly different versions of our problem. Ho et al. [8, 7] assume that the ground truth is known for some tasks and use it to learn the reliability of workers in the exploration phase and to assign tasks optimally in the exploitation phase. Liu and Liu [14] also look for the optimal task assignment but without the knowledge of any true label: an iterative algorithm similar to EM is used to infer the reliability of each worker, yielding a cumulative regret in $O(\ln^2 t)$ for $t$ tasks compared to the optimal decision. Finally, some authors seek to rank the workers with respect to their reliability, an information which is useful for task assignment but not easy to exploit for data classification itself [3, 16].

The remainder of the paper is organized as follows. In section 3 we state the problem and introduce our notations. In section 5 we present a lower bound on the minimax error rate of any estimator. In section 6 we present TE, discuss its complexity and prove that it is minimax optimal. In section 7 we present numerical experiments on synthetic and real-world data sets and section 8 concludes the paper. To ensure a smoother presentation the proofs of the two main results are presented in the appendix.

3 Model

Consider $n$ workers, for some integer $n \geq 3$. Each task consists in determining the answer to a binary question. The answer to task $t$, the “ground-truth”, is denoted by $G(t) \in \{+1, -1\}$. We assume that the random variables $G(1), G(2), \ldots$ are i.i.d. and centered, so that there is no bias towards one of the answers.

Each worker provides an answer with probability $\alpha \in (0, 1]$. When worker $i \in \{1, \ldots, n\}$ provides an answer, this answer is correct with probability $\frac{1}{2}(1 + \theta_i)$, independently of the other workers, for some parameter $\theta_i \in [-1, 1]$ that we refer to as the reliability of worker $i$. If
\( \theta_i > 0 \) then worker \( i \) tends to provide correct answers; if \( \theta_i < 0 \) then worker \( i \) tends to provide incorrect answers; if \( \theta_i = 0 \) then worker \( i \) is non-informative. We denote by \( \theta = (\theta_1, \ldots, \theta_n) \) the reliability vector.

Let \( X_i(t) \in \{-1, 0, 1\} \) be the output of worker \( i \) for task \( t \), where the output 0 corresponds to the absence of an answer. We have:

\[
X_i(t) = \begin{cases} 
G(t) & \text{w.p. } \alpha \frac{1+\theta_i}{2}, \\
-G(t) & \text{w.p. } \alpha \frac{1-\theta_i}{2}, \\
0 & \text{w.p. } 1 - \alpha.
\end{cases}
\]  

Since the workers are independent, the random variables \( X_1(t), \ldots, X_n(t) \) are independent given \( G(t) \), for each task \( t \). We denote by \( X(t) \) the corresponding vector. The goal is to estimate the ground-truth \( G(t) \) as accurately as possible by designing an estimator \( \hat{G}(t) \) that minimizes the error probability \( \mathbb{P}(\hat{G}(t) \neq G(t)) \). The estimator \( \hat{G}(t) \) is adaptive and may be a function of \( X(1), \ldots, X(t) \) but not on the unknown parameters \( \alpha, \theta \).

It is well-known that, given \( \theta \) and \( \alpha = 1 \), an optimal estimator of \( G(t) \) is the weighted majority vote [15, 18], namely

\[
\hat{G}(t) = 1\{W(t) > 0\} - 1\{W(t) < 0\} + Z1\{W(t) = 0\},
\]

where \( W(t) = \frac{1}{n} \sum_{i=1}^{n} w_i X_i(t) \), \( w_i = \ln(\frac{1+\theta_i}{1-\theta_i}) \) is the weight of worker \( i \) (possibly infinite), and \( Z \) is a Bernoulli random variable of parameter \( \frac{1}{2} \) over \( \{+1, -1\} \) (for random tie-breaking). We prove this result for any \( \alpha \in (0, 1] \).

**Proposition 1** Assuming \( \theta \) is known, the estimator [2] is an optimal estimator of \( G(t) \).

**Proof.** Finding an optimal estimator of \( G(t) \) amounts to finding an optimal statistical test between hypotheses \( \{G(t) = +1\} \) and \( \{G(t) = -1\} \), under a symmetry constraint so that type I and type II error probability are equal. For any \( x \in \{-1, 0, 1\}^n \), let \( L^+(x) \) and \( L^-(x) \) be the probabilities that \( X(t) = x \) under hypotheses \( \{G(t) = +1\} \) and \( \{G(t) = -1\} \), respectively. We have

\[
L^+(x) = \frac{1}{2^\ell} \alpha^\ell (1-\alpha)^{n-\ell} \prod_{i=1}^{n} (1 + \theta_i)^{1\{x_i=+1\}} (1 - \theta_i)^{1\{x_i=-1\}},
\]

\[
L^-(x) = \frac{1}{2^\ell} \alpha^\ell (1-\alpha)^{n-\ell} \prod_{i=1}^{n} (1 + \theta_i)^{1\{x_i=-1\}} (1 - \theta_i)^{1\{x_i=+1\}},
\]

where \( \ell = \sum_{i=1}^{n} |x_i| \) is the number of answers. We deduce the log-likelihood ratio,

\[
\ln \left( \frac{L^+(x)}{L^-(x)} \right) = \sum_{i=1}^{n} w_i x_i = w^T x.
\]

By the Neyman-Pearson theorem, for any level of significance, there exists \( a \) and \( b \) such that the uniformly most powerful test for that level is:

\[
1\{w^T x > a\} - 1\{w^T x < a\} + Z1\{w^T x = a\},
\]
where \(Z\) is a Bernoulli random variable of parameter \(b\) over \(\{+1, -1\}\). By symmetry, we must have \(a = 0\) and \(b = \frac{1}{2}\), which is the announced result.

This result shows that estimating the true answer \(G(t)\) reduces to estimating the unknown parameter \(\theta\), which is the focus of the paper. Note that the problem of estimating \(\theta\) is important in itself, due to the presence of "spammers" (i.e., workers with low reliability); a good estimator can be used by the crowdsourcing platform to incentivize good workers.

4 Identifiability

Estimating \(\theta\) from \(X(1), \ldots, X(t)\) is not possible unless we have identifiability, namely there cannot exist two distinct parameters \(\theta, \theta'\) under which the distribution of \(X(1), \ldots, X(t)\) is the same. Let \(X \in \{-1, 0, 1\}^n\) be any sample, for some parameters \(\alpha \in (0, 1]\) and \(\theta \in [-1, 1]^n\). The parameter \(\alpha\) is clearly identifiable since \(\alpha = \mathbb{P}(X_1 \neq 0)\). The identifiability of \(\theta\) is less obvious. Assume for instance that \(\theta_i = 0\) for all \(i \geq 3\). It follows from (1) that for any \(x \in \{-1, 0, 1\}^n\),

\[
\mathbb{P}(X = x) = \frac{1}{2^\ell} \alpha^\ell (1 - \alpha)^{n-\ell} \times \left\{ \begin{array}{ll} 
1 + \theta_1 \theta_2 & \text{if } x_1 = x_2, x_1 x_2 \neq 0 \\
1 - \theta_1 \theta_2 & \text{if } x_1 \neq x_2, x_1 x_2 \neq 0 \\
1 & \text{if } x_1 x_2 = 0,
\end{array} \right.
\]

where \(\ell = \sum_{i=1}^n |x_i|\) is the number of answers. In particular, two parameters \(\theta, \theta'\) such that \(\theta_1 \theta_2 = \theta'_1 \theta'_2\) and \(\theta_i = \theta'_i = 0\) for all \(i \geq 3\) cannot be distinguished. Similarly, by symmetry, two parameters \(\theta, \theta'\) such that \(\theta' = -\theta\) cannot be distinguished.

Let:

\[
\Theta = \left\{ \theta \in [-1, 1]^n : \sum_{i=1}^n \mathbf{1}\{\theta_i \neq 0\} \geq 3, \sum_{i=1}^n \theta_i > 0 \right\}.
\]

The first condition states that there are at least 3 informative workers, the second that the average reliability is positive.

**Proposition 2** Any parameter \(\theta \in \Theta\) is identifiable.

**Proof.** The result is a direct consequence of the fact that any parameter \(\theta \in \Theta\) can be expressed as a function of the covariance matrix of \(X\) (section 6 below): the absolute value and the sign of \(\theta\) follow from (4) and (5), respectively. \(\square\)

5 Lower bound on the minimax error

We now derive a lower bound on the minimax error of any estimator \(\hat{\theta}\) of \(\theta\). Define:

\[
||\hat{\theta} - \theta||_\infty = \max_{i=1,\ldots,n} |\hat{\theta}_i - \theta_i|.
\]
Define for all $\theta \in [-1,1]^n$:

$$A(\theta) = \min_k \max_{i,j \neq k} \sqrt{|\theta_i \theta_j|}, \quad B(\theta) = \sum_{i=1}^{n} \theta_i.$$ 

Observe that $\Theta = \{\theta \in [-1,1]^n : A(\theta) > 0, \ B(\theta) > 0\}$. This suggests that the estimation of $\theta$ becomes hard when either $A(\theta)$ or $B(\theta)$ is small. Define for any $a, b \in (0,1)$,

$$\Theta_{a,b} = \{\theta \in [-1,1]^n : A(\theta) \geq a, \ B(\theta) \geq b\}.$$ 

We have the following lower bound on the minimax error. As the proof reveals, the parameters $a$ and $b$ characterize the difficulty of estimating the absolute value and the sign of $\theta$, respectively.

**Theorem 1 (Minimax error)** Consider any estimator $\hat{\theta}$ of $\theta$. For any $\epsilon \in (0, \min(a, (1-a)/2, 1/4))$ and $\delta \in (0, 1/4)$, we have

$$\min_{\theta \in \Theta_{a,b}} \mathbb{P}(||\hat{\theta} - \theta||_\infty \geq \epsilon) \geq \delta$$

whenever $t \leq \max(T_1, T_2)$, where

$$T_1 = c_1 \frac{1-a}{\alpha^2a^4\epsilon^2} \ln \left(\frac{1}{4\delta}\right), \quad T_2 = c_2 \frac{(1-a)^4(n-4)}{\alpha^2a^2b^2} \ln \left(\frac{1}{4\delta}\right),$$

with $c_1, c_2$ two strictly positive universal constants.

**Relation with prior work.** The lower bound derived in [24] [Theorem 3] shows that the minimax error of any estimator $\hat{\theta}$ must be greater than $O((at)^{-\frac{1}{2}})$. Our lower bound is stricter, and shows that the minimax error is in fact greater than $O(a^{-2} \alpha^{-1} t^{-\frac{1}{2}})$. Another lower bound was derived in [2] [Theorems 3.4 and 3.5], but this concerns the prediction error rate, that is $\mathbb{P}(\hat{G} \neq G)$, so that it cannot be easily compared to our result.

6 Triangular estimation

We here present our estimator. The absolute value of the reliability of each worker $k$ is estimated through the correlation of her answers with those of the most informative pair $i, j \neq k$. We refer to this algorithm as triangular estimation (TE). The sign of the reliability of each worker is estimated in a second step.
Covariance matrix. Let \( X \in \{-1, 0, 1\}^n \) be any sample, for some parameters \( \alpha \in (0, 1] \) and \( \theta \in \Theta \). We shall see that the parameter \( \theta \) could be recovered exactly if the covariance matrix of \( X \) were perfectly known. For any \( i \neq j \), let \( C_{ij} \) be the covariance of \( X_i \) and \( X_j \) given that \( X_i X_j \neq 0 \) (that is, both workers \( i \) and \( j \) provide an answer). In view of (1),

\[
C_{ij} = \theta_i \theta_j.
\]

(3)

In particular, for any distinct indices \( i, j, k \),

\[
C_{ik} C_{jk} = \theta_i \theta_j \theta_k^2 = C_{ij} \theta_k^2.
\]

We deduce that, for any \( k = 1, \ldots, n \) and any pair \( i, j \neq k \) such that \( C_{ij} \neq 0 \),

\[
\theta_k^2 = \frac{C_{ik} C_{jk}}{C_{ij}}.
\]

(4)

Note that such a pair exists for each \( k \) because \( \theta \in \Theta \).

To recover the sign of \( \theta_k \), we use the fact that

\[
\theta_k \sum_{i=1}^{n} \theta_i = \theta_k^2 + \sum_{i \neq k} C_{ik}.
\]

Since \( \theta \in \Theta \), we get

\[
\text{sign}(\theta_k) = \text{sign} \left( \theta_k^2 + \sum_{i \neq k} C_{ik} \right).
\]

(5)

The TE algorithm consists in estimating the covariance matrix to recover \( \theta \) from the above expressions.

TE algorithm. At any time \( t \), define

\[
\forall i, j = 1, \ldots, n, \quad \hat{C}_{ij} = \frac{\sum_{s=1}^{t} X_i(s) X_j(s)}{\max \left( \sum_{s=1}^{t} |X_i(s) X_j(s)|, 1 \right)}.
\]

(6)

For all \( k = 1, \ldots, n \), find the most informative pair \( (i_k, j_k) \in \arg \max_{i \neq j \neq k} |\hat{C}_{ij}| \) and let

\[
|\hat{\theta}_k| = \begin{cases} \min \left( \sqrt{\frac{|\hat{C}_{i_kk} \hat{C}_{j_kk}|}{\hat{C}_{i_kj_k}}}, 1 \right) & \text{if } |\hat{C}_{i_kj_k}| > 0, \\ 0 & \text{otherwise}. \end{cases}
\]

Next, define \( k^* = \arg \max_k |\hat{\theta}_k^2 + \sum_{i \neq k} \hat{C}_{ik}| \) and let

\[
\text{sign}(\hat{\theta}_k) = \begin{cases} \text{sign}(\hat{\theta}_{k^*}^2 + \sum_{i \neq k^*} \hat{C}_{ik^*}) & \text{if } k = k^*, \\ \text{sign}(\hat{\theta}_{k^*} \hat{C}_{kk^*}) & \text{otherwise}, \end{cases}
\]

with the convention that \( \text{sign}(0) = + \).
Minimax optimality. The following result shows that the proposed estimator is minimax optimal. Namely the sample complexity of our estimator matches the lower bound up to an additive logarithmic term \( \ln(n) \) and a multiplicative constant.

**Theorem 2** Let \( \theta \in \Theta_{a,b} \) and denote by \( \hat{\theta} \) the estimator defined by \((4)-(5)\). For any \( \epsilon \in (0, \min\left(\frac{b}{3}, 1\right)) \) and \( \delta \in (0, 1) \), we have

\[
P(||\hat{\theta} - \theta||_\infty \geq \epsilon) \leq \delta.
\]

whenever \( t \geq \max(T_1', T_2') \) with

\[
T_1' = c'_1 \frac{1}{\alpha^2 a^4 a^2} \ln \left(\frac{6n^2}{\delta}\right), \quad T_2' = c'_2 \frac{n}{\alpha^2 a^2 b^2} \ln \left(\frac{4n^2}{\delta}\right),
\]

and \( c'_1, c'_2 \) two positive universal constants.

**Complexity.** First note that the TE algorithm is in fact a streaming algorithm since \( \hat{C}_{ij} \)

\[
\hat{C}_{ij} = \frac{M_{ij}}{\max(N_{ij}, 1)},
\]

with

\[
M_{ij} = \sum_{s=1}^{t} X_i(s)X_j(s), \quad N_{ij} = \sum_{s=1}^{t} |X_i(s)X_j(s)|.
\]

Thus the algorithm requires \( \mathcal{O}(n^2) \) memory space (to store the matrices \( M \) and \( N \)) and has a time complexity of \( \mathcal{O}(n^2) \) per task: \( \mathcal{O}(n^2) \) operations to update \( \hat{C} \), \( \mathcal{O}(n^2) \) operations to identify the most informative pairs for all workers, \( \mathcal{O}(n^2) \) operations to compute \( |\hat{\theta}| \), \( \mathcal{O}(n^2) \) operations to compute the sign of \( \hat{\theta} \).

**Relation with prior work.** Our upper bound brings improvement over \cite{24} as follows. Two conditions are required for the upper bound of \cite{24}[Theorem 4] to hold: (i) it is required that \( \max_i |\theta_i| < 1 \), and (ii) the number of workers \( n \) must grow with both \( \delta \) and \( t \), and in fact must depend on \( a \) and \( b \), so that \( n \) has to be large if \( b \) is smaller than \( \sqrt{n} \). Our result does not require condition (i) to hold. Further there are values of \( a \) and \( b \) such that condition (ii) is never satisfied, for instance \( n \geq 5, a = \frac{1}{2}, b = \frac{\sqrt{n-5}}{2} \) and \( \theta = (a, -a, a, -a, \frac{b}{n-4}, ..., \frac{b}{n-4}) \in \Theta_{a,b} \). For \cite{24}[Theorem 4] to hold, \( n \) should satisfy \( n \geq c_3(n/\alpha)\ln(t^2n/\delta) \) with \( c_3 \) a universal constant (see discussion in the supplement) and for \( t \) or \( 1/\delta \) large enough no such \( n \) exists. It is noted that for such values of \( a \) and \( b \), our result remains informative.

Our result does not subsume that of \cite{24} which considers a more general model (the model we consider is referred in \cite{24} as the "one-coin model"), but provide improvements for "hard" instances of the one-coin model, namely when either \( a \) or \( b \) is small. Also, our algorithm shares some features with the way the EM algorithm is initialized in \cite{24}, the critical difference being the sign estimation procedure. This suggests that the initialization phase of this algorithm is more important than the EM phase itself.
The analysis of [2] also imposes $n$ to grow with $t$ and conditions on the minimal value of $b$. Specifically the first and the last condition of [2][Theorem 3.3], require that $n \geq \ln(t)$ and that $\sum \theta_i^2 \geq 6\ln(t)$. Using the previous example (even for $t = 3$), this translates to $b \geq 2\sqrt{n} - 4$.

In fact, the value $b = O(\sqrt{n})$ seems to mark the transition between "easy" and "hard" instances of the crowdsourcing problem. Indeed, when $n$ is large and $b$ is large with respect to $\sqrt{n}$, then the majority vote outputs the truth with high probability by the Central Limit Theorem.

7 Numerical Experiments

Synthetic data. We consider three instances:

(i) $n = 50$, $t = 10^3$, $\alpha = 0.25$, $\theta_i = a$ if $i \leq n/2$ and 0 otherwise;
(ii) $n = 50$, $t = 10^4$, $\alpha = 0.25$, $\theta = (1, a, a, 0, ... 0)$;
(iii) $n = 50$, $t = 10^4$, $\alpha = 0.9$, $\theta = (a, -a, -a, \frac{b}{n-4}, ..., \frac{b}{n-4})$.

Instance (i) is an "easy" instance where half of the workers are informative, with $A(\theta) = a$ and $B(\theta) = na/2$. Instance (ii) is a "hard" instance, the difficulty being to estimate the absolute value of $\theta$ accurately by identifying the 3 informative workers. Instance (iii) is another "hard" instance, where estimating the sign of the components of $\theta$ is difficult. In particular, one must distinguish $\theta$ from $\theta' = (-a, a, -a, a, \frac{b}{n-4}, ..., \frac{b}{n-4})$, otherwise a large error occurs.

Both "hard" instances (ii) and (iii) are inspired by our derivation of the lower bound and constitute the hardest instances in $\Theta_{a,b}$. For each instance we average the performance of algorithms on $10^3$ independent runs and apply a random permutation of the components of $\theta$ before each run. We consider the following algorithms: KOS (the BP algorithm of [10]), Maj (majority voting), Oracle (weighted majority voting with optimal weights, which is the optimal estimator the ground truth), RoE (first spectral algorithm of [4]), EoR (second spectral algorithm of [4]), GKM (spectral algorithm of [6]), S-EM ($k$ (EM algorithm with spectral initialization of [24] with $k$ iterations of EM) and TE (our algorithm). We do not present the estimation error of KOS, Maj and Oracle since these algorithms only predict the ground truth but do not estimate $\theta$ directly.

| Instance | RoE | EoR | GKM | S-EM1 | S-EM10 | TE |
|----------|-----|-----|-----|-------|--------|----|
| (i) $a = 0.3$ | 0.200 | 0.131 | 0.146 | 0.100 | 0.041 | 0.134 |
| (i) $a = 0.9$ | 0.274 | 0.265 | 0.271 | **0.022** | **0.022** | 0.038 |
| (ii) $a = 0.55$ | 0.551 | 0.459 | 0.479 | 0.045 | **0.044** | 0.050 |
| (ii) $a = 0.95$ | 0.528 | 0.522 | 0.541 | 0.034 | **0.033** | 0.039 |
| (iii) $b = 1$ | 0.253 | 0.222 | 0.256 | 0.533 | 0.389 | **0.061** |
| (iii) $b = \sqrt{n}$ | 0.105 | 0.075 | 0.085 | 0.437 | **0.030** | 0.045 |

Table 1: Synthetic data: estimation error $E(||\hat{\theta} - \theta||_\infty)$. 

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The results are shown in Tables 1 and 2. We make the following observations. The spectral algorithms RoE, EoR and GKM tend to be outperformed by the other algorithms. To perform well, GKM needs $\theta_1$ to be positive and large (see [6]); whenever $\theta_1 \leq 0$ or $|\theta_1|$ is small, GKN tends to make a sign mistake causing a large error. Also the analysis of RoE and EoR assumes that the task-worker graph is a random $D$-regular graph (so that the worker-worker matrix has a large spectral gap). Here this assumption is violated and the practical performance suffers noticeably, so that this limitation is not only theoretical. KOS performs consistently well, and seems immune to sign ambiguity, see instance (iii). Further, while the analysis of KOS also assumes that the task-worker graph is random $D$-regular, its practical performance does not seem sensitive to that assumption. The performance of S-EM is good except when sign estimation is hard (instance (iii), $b = 1$). This seems due to the fact that the initialization of S-EM (see the algorithm description) is not good in this case. Hence the limitation of $b$ being of order $\sqrt{n}$ is not only theoretical but practical as well. In fact (combining our results and the ideas of [24]), this suggests a new algorithm where one uses EM with TE as the initial value of $\theta$.

Further, the number of iterations of EM brings significant gains in some cases and thus should affect the universal constants in front of the various error bounds (providing theoretical evidence for this seems non trival). TE performs consistently well except for (i) $a = 0.3$ (which we believe is due to the fact that $t$ is relatively small in that instance). In particular when sign estimation is hard TE clearly outperforms the competing algorithms. This sharp difference corroborates our idea of the existence two regimes for sign estimation: $b = O(1)$ (hard regime) and $b = O(\sqrt{n})$ (easy regime).

**Real-world data.** We next consider 6 publicly available data sets (see [23] [25] for further information), each consisting of labels provided by workers and the ground truth. A summary of each data set is found in Table 3, where the density is the average number of labels per worker, i.e., $\alpha$ in our model. The worker degree is the average number of tasks labeled by a worker.

We apply two preprocessing steps. First, for data sets with more than 2 possible label values, we split the label values into two groups and associate them with labels $-1$ and $+1$ respectively. Second, we remove any worker who provides less than 10 labels. Our
| Data Set | # Tasks | # Workers | # Labels | Density | Worker Degree |
|----------|---------|-----------|----------|---------|---------------|
| Bird     | 108     | 39        | 4,212    | 1       | 108           |
| Dog      | 807     | 109       | 8,070    | 0.09    | 74            |
| Duchenne | 159     | 64        | 1,221    | 0.12    | 19            |
| RTE      | 800     | 164       | 8,000    | 0.06    | 49            |
| Temp     | 462     | 76        | 4,620    | 0.13    | 61            |
| Web      | 2,653   | 177       | 15,539   | 0.03    | 88            |

Table 3: Summary of the real-world datasets.

Preliminary numerical experiments (not shown here for concision) show that without this, none of the studied algorithms even match the majority consistently. Workers with low degree create a dramatic amount of noise and (to the best of our knowledge) any theoretical analysis of crowdsourcing algorithms assumes that the worker degree is sufficiently large. The performance of various algorithms is reported in Table 4. No information about the workers reliability is available so we only report the prediction error $P(\hat{G} \neq G)$. Further, one cannot compare algorithms to the Oracle, so that the main goal is to outperform the majority.

| Data Set | KOS | Maj | RoE | EoR | GKM | S-EM1 | S-EM10 | TE   |
|----------|-----|-----|-----|-----|-----|-------|-------|------|
| Bird     | 0.28| 0.24| 0.29| 0.29| 0.28| 0.20  | 0.28  | 0.18 |
| Dog      | 0.19| 0.18| 0.18| 0.18| 0.20| 0.24  | 0.17  | 0.20 |
| Duchenne | 0.30| 0.28| 0.29| 0.28| 0.29| 0.28  | 0.30  | 0.26 |
| RTE      | 0.50| 0.10| 0.50| 0.89| 0.49| 0.32  | 0.16  | 0.38 |
| Temp     | 0.43| 0.06| 0.24| 0.10| 0.43| 0.06  | 0.06  | 0.08 |
| Web      | 0.02| 0.14| 0.13| 0.14| 0.02| 0.04  | 0.06  | 0.03 |

Table 4: Real-world data: prediction error $P(\hat{G} \neq G)$.

We draw the following conclusions: apart from "Bird" and "Web", none of the algorithms seem to be able to significantly outperform the majority and are sometimes noticeably worse. It is noted that "Bird" and "Web" both have a large number of labels and a high worker degree. We believe that this should serve as a cautionary note against applying sophisticated inference algorithms to small crowdsourcing datasets, where one is better off simply using a majority vote. On the other hand, for "Web" which has both the largest number of labels and a high worker degree, there is a significant gain over the majority vote, and TE, despite its low complexity, slightly outperforms S-EM and is competitive with KOS and GKM which both perform best on this dataset.
8 Conclusion

We have derived a minimax error lower bound for the crowdsourcing problem and have proposed TE, a low-complexity algorithm which matches this lower bound. We believe that our results open two questions of interest. First, while recent work has shown that one can obtain strong theoretical guarantees by combining one step of EM with a well-chosen initialization, we have shown that, at least in the case of binary labels, one can forgo the EM phase altogether and still obtain both minimax optimality and good numerical performance. It would be interesting to know if this is still possible when there are more than two possible labels. Another question of interest is whether or not, for more than two possible labels, there exists minimax optimal, low-complexity algorithms which do not involve storing the whole data set in memory.

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Appendices
We here provide the proofs of the two main results of the paper, and provide a more in-depth discussion on the relation between our upper bound and that of [24].

A Proof of Theorem 1

We use the following inequality between the Kullback-Leibler and $\chi^2$ divergences.

**Lemma 1** The Kullback-Leibler divergence of any two discrete distributions $P, Q$ satisfies

$$D(P||Q) \leq \mathbb{E} \left( \frac{(P(X) - Q(X))^2}{P(X)Q(X)} \right),$$

with $X \sim P$.

**Proof.** Using the inequality $\ln(z) \leq z - 1$, we get

$$D(P||Q) = \sum_x P(x) \ln \left( \frac{P(x)}{Q(x)} \right) \leq \sum_x P(x) \left( \frac{P(x)}{Q(x)} - 1 \right) = -1 + \sum_x \frac{P(x)^2}{Q(x)}.$$

Writting

$$P(x)^2 = Q(x)^2 + 2Q(x)(P(x) - Q(x)) + (P(x) - Q(x))^2,$$

we deduce

$$D(P||Q) \leq -1 + \sum_x Q(x) + 2 \sum_x (P(x) - Q(x)) + \sum_x \frac{(P(x) - Q(x))^2}{Q(x)},$$

$$= \sum_x \frac{(P(x) - Q(x))^2}{Q(x)},$$

$$= \sum_x P(x) \frac{(P(x) - Q(x))^2}{P(x)Q(x)}.$$

$\square$

**Proof of Theorem 1.** Let $X \in \{+1, 0, -1\}^n$ be any sample under parameters $\alpha, \theta$. We have for any distinct indices $i, j, k \in \{1, \ldots, n\}$,

$$\mathbb{P}((X_i, X_j, X_k) = (0, 1, 1)) = \mathbb{P}((X_i, X_j, X_k) = (0, -1, -1)) = (1 - \alpha) \frac{\alpha^2}{4} (1 + \theta_j \theta_k),$$

$$\mathbb{P}((X_i, X_j, X_k) = (0, 1, -1)) = \mathbb{P}((X_i, X_j, X_k) = (0, -1, 1)) = (1 - \alpha) \frac{\alpha^2}{4} (1 - \theta_j \theta_k),$$

$$\mathbb{P}((X_i, X_j, X_k) = (1, 1, 1)) = \mathbb{P}((X_i, X_j, X_k) = (-1, -1, -1)) = \alpha^3 \frac{1}{8} (1 + \theta_i \theta_j + \theta_j \theta_k + \theta_k \theta_i),$$

$$\mathbb{P}((X_i, X_j, X_k) = (1, 1, -1)) = \mathbb{P}((X_i, X_j, X_k) = (-1, -1, 1)) = \alpha^3 \frac{1}{8} (1 + \theta_i \theta_j - \theta_j \theta_k - \theta_k \theta_i).$$

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Now let \( a \in (0, 1) \) and 
\[
\theta = (1, a, a, 0, \ldots, 0), \quad \theta' = (1 - 2\epsilon, \frac{a}{1 - 2\epsilon}, \frac{a}{1 - 2\epsilon}, 0, \ldots, 0).
\]
Observe that \( \theta, \theta' \in \Theta_{a,b} \) for any \( b \in (0, 1) \). Denote by \( P, P' \) the distributions of \( X \) under parameters \( \theta, \theta' \), respectively. We use Lemma 1 to get an upper bound on the Kullback-Leibler divergence \( D(P' \| P) \) between \( P' \) and \( P \). Observe that we can restrict the analysis to the case \( n = 3 \). We calculate \( P(x), P'(x) \) for all possible values of \( x \in \{-1, 0, 1\}^3 \):

(a) If \( x_2 = 0 \) or \( x_3 = 0 \) then \( P(x) = P'(x) \).

(b) If \( x = (0, 1, 1) \) or \( x = (0, -1, -1) \),
\[
P(x) = (1 - \alpha) \frac{\alpha^2}{4} (1 + a^2), \quad P'(x) = (1 - \alpha) \frac{\alpha^2}{4} \left( 1 + \left( \frac{a}{1 - 2\epsilon} \right)^2 \right).
\]

(c) If \( x = (0, 1, -1) \) or \( x = (0, -1, 1) \),
\[
P(x) = (1 - \alpha) \frac{\alpha^2}{4} (1 - a^2), \quad P'(x) = (1 - \alpha) \frac{\alpha^2}{4} \left( 1 - \left( \frac{a}{1 - 2\epsilon} \right)^2 \right).
\]

(d) If \( x = (1, 1, 1) \) or \( x = (-1, -1, -1) \),
\[
P(x) = \frac{\alpha^3}{8} (1 + 2a + a^2), \quad P'(x) = \frac{\alpha^3}{8} \left( 1 + 2a + \left( \frac{a}{1 - 2\epsilon} \right)^2 \right).
\]

(e) If \( x = (1, -1, -1) \) or \( x = (-1, 1, 1) \),
\[
P(x) = \frac{\alpha^3}{8} (1 - 2a + a^2), \quad P'(x) = \frac{\alpha^3}{8} \left( 1 - 2a + \left( \frac{a}{1 - 2\epsilon} \right)^2 \right).
\]

(f) Otherwise,
\[
P(x) = \frac{\alpha^3}{8} (1 - a^2), \quad P'(x) = \frac{\alpha^3}{8} \left( 1 - \left( \frac{a}{1 - 2\epsilon} \right)^2 \right).
\]

Observing that
\[
\left( \frac{a}{1 - 2\epsilon} \right)^2 - a^2 = 4\epsilon(1 - \epsilon) \left( \frac{a}{1 - 2\epsilon} \right)^2 \leq 4\epsilon \left( \frac{a}{1 - 2\epsilon} \right)^2 \leq 16a^2\epsilon,
\]
we get in cases (b)-(c),
\[
\frac{(P(x) - P'(x))^2}{P(x)} \leq \frac{64\alpha^2 a^4\epsilon^2}{1 - a}.
\]

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and in cases (d)-(f),
\[
\frac{(P(x) - P'(x))^2}{P(x)} \leq \frac{32a^2d^2\epsilon^2}{1-a}.
\]
Summing, we obtain
\[
D(P'\|P) \leq \frac{512a^2d^2\epsilon^2}{1-a}.
\]
Now let \( t \leq T_1 \) with \( c_1 = \frac{1}{512} \). Denoting by \( P^t, P'^t \) the distributions of \( X(1), \ldots, X(t) \) under the respective parameters \( \theta, \theta' \), we obtain \( D(P'^t\|P^t) = tD(P'\|P) \leq \ln \left( \frac{1}{\delta} \right) \). Since \( ||\theta - \theta'||\infty = 2\epsilon \), it follows from [20, Theorem 2.2] that:
\[
\min \left( \mathbb{P}_{\theta}(||\hat{\theta} - \theta||\infty \geq \epsilon), \mathbb{P}_{\theta'}(||\hat{\theta} - \theta'||\infty \geq \epsilon) \right) \geq \frac{1}{4\exp(D(P'^t\|P^t))} \geq \delta.
\]
Since \( \theta, \theta' \in \Theta_{a,b} \), we get
\[
\min_{\theta \in \Theta_{a,b}} \mathbb{P} \left( ||\hat{\theta} - \theta||\infty \geq \epsilon \right) \geq \delta.
\]
Now assume that \( n > 4 \) so that \( T_2 > 0 \). Let \( m = n - 4 \) and \( c = b/m \). Consider the two parameters
\[
\theta = (a, a, -a, -a, c, \ldots, c), \quad \theta' = (-a, -a, a, a, c, \ldots, c).
\]
Observe that \( \theta, \theta' \in \Theta_{a,b} \). Denote by \( P \) and \( P' \) the distributions of \( X \) under parameters \( \theta, \theta' \), respectively. Again, we use Lemma 1 to get an upper bound on the Kullback-Leibler divergence between \( P \) and \( P' \).

Define \( y = (1, 1, -1, -1) \), \( k^+ = \sum_{i=1}^{4} 1\{x_i = y_i\}, k^- = \sum_{i=1}^{4} 1\{x_i = -y_i\}, k = k^+ + k^- \), \( h = k^+ - k^- = x_1 + x_2 - x_3 - x_4, \ell^+ = \sum_{i>4} 1\{x_i = +1\} \) and \( \ell^- = \sum_{i>4} 1\{x_i = -1\} \). We have:
\[
P(x) = \frac{1}{2^{k+\ell}} \alpha^{k+\ell} (1-\alpha)^{n-k-\ell} \left( (1+a)^{k^+} (1-a)^{k^-} p_{\ell^+,\ell^-} + (1+a)^{k^-} (1-a)^{k^+} p_{\ell^-,\ell^+} \right),
\]
\[
= \frac{1}{2^{k+\ell}} \alpha^{k+\ell} (1-\alpha)^{n-k-\ell} (1+a)^{k^-} (1-a)^{k^+} \left( (1+a)^h p_{\ell^+,\ell^-} + (1-a)^h p_{\ell^-,\ell^+} \right).
\]
and
\[
P'(x) = \frac{1}{2^{k+\ell}} \alpha^{k+\ell} (1-\alpha)^{n-k-\ell} \left( (1+a)^{k^+} (1-a)^{k^-} p_{\ell^-,\ell^+} + (1+a)^{k^-} (1-a)^{k^+} p_{\ell^+,\ell^-} \right),
\]
\[
= \frac{1}{2^{k+\ell}} \alpha^{k+\ell} (1-\alpha)^{n-k-\ell} (1+a)^{k^-} (1-a)^{k^+} \left( (1+a)^h p_{\ell^-,\ell^+} + (1-a)^h p_{\ell^+,\ell^-} \right),
\]
where
\[
\forall i, j \in \mathbb{N}, \quad p_{ij} = (1+c)^i (1-c)^j.
\]
Define:
\[
F(x) = \frac{(P(x) - P'(x))^2}{P(x)P'(x)}
\]
We have:

\[ F(x) = \frac{((1 + a)^h - (1 - a)^h)^2(p_{\ell^+, \ell^-} - p_{\ell^-, \ell^+})^2}{((1 + a)^h p_{\ell^+, \ell^-} + (1 - a)^h p_{\ell^-, \ell^+})(1 + a)^h p_{\ell^+, \ell^-} + (1 - a)^h p_{\ell^-, \ell^+})} \]

Notice that \( F \) is invariant (a) when \( h \) is replaced by \(-h\) and (b) when one exchanges \( \ell^+ \) and \( \ell^- \). So we can assume that \( h > 0 \) and \( \ell^+ \geq \ell^- \), so that \( p_{\ell^+, \ell^-} \geq p_{\ell^-, \ell^+} \). Now:

\[ F(x) \leq \frac{((1 + a)^h - (1 - a)^h)^2 (p_{\ell^+, \ell^-} - p_{\ell^-, \ell^+})^2}{(1 - \eta)(1 + a)^h} \]

Define \( \eta = (1 - c)/(1 + c) \). Then:

\[ \frac{(p_{\ell^+, \ell^-} - p_{\ell^-, \ell^+})^2}{p_{\ell^+, \ell^-}^2} = \left( 1 - \eta \right)^2 \leq (\ell^+ - \ell^-)^2 (1 - \eta)^2 = (\ell^+ - \ell^-)^2 \frac{4c^2}{(1 + c)^2} \leq 4c^2(\ell^+ - \ell^-)^2. \]

Moreover, using the fact that \( h \leq 4 \),

\[ \frac{(1 + a)^h - (1 - a)^h}{(1 - a)^h(1 + a)} \leq \frac{(1 + a)^4 - (1 - a)^4}{(1 - a)^4} = \frac{(4a(1 + a^2))^2}{(1 - a)^4} \leq 64 \frac{a^2}{(1 - a)^4}. \]

By Lemma 1,

\[ D(P||P') \leq 256 \frac{a^2c^2}{(1 - a)^4} \mathbb{E}(h^2(X)((\ell^+(X) - \ell^-(X))^2)), \]

where \( X \) has distribution \( P \) and we make explicit the dependency of \( h, \ell^+, \ell^- \) in the state \( X \). The random quantity \( h^2(X)((\ell^+(X) - \ell^-(X))^2 \) does not change when \( X \) is replaced by \(-X\), so:

\[ \mathbb{E}(h^2(X)((\ell^+(X) - \ell^-(X))^2)) = \mathbb{E}(h^2(X))\mathbb{E}((\ell^+(X) - \ell^-(X))^2). \]

Now

\[ \mathbb{E}(h^2(X)) = \text{var}(h(X)) = \text{var}(h(X)|G = 1) = 4\text{var}(X_1|G = 1) = 4\alpha(1 - \alpha a^2) \]

and

\[ \mathbb{E}((\ell^+(X) - \ell^-(X))^2) = \text{var}(\ell^+(X) - \ell^-(X)), \]

\[ = \text{var}(\ell^+(X) - \ell^-(X)|G = 1), \]

\[ = m\text{var}(X_2|G = 1) = m\alpha(1 - \alpha c^2) \]

so that

\[ D(P||P') \leq 1024 \frac{m\alpha^2a^2c^2}{(1 - a)^4} = 1024 \frac{\alpha^2a^2b^2}{m(1 - a)^4}. \]
Now let $t \leq T_2$ with $c_2 = \frac{1}{1024}$. Denoting by $P^t, P'^t$ the distributions of $X(1), \ldots, X(t)$ under the respective parameters $\theta, \theta'$, we obtain $D(P^t||P'^t) = tD(P||P') \leq \ln \left( \frac{1}{1024} \right)$. Since $||\theta - \theta'||_\infty = 2a$, it follows from [20, Theorem 2.2] that:

$$\min \left( \mathbb{P}_\theta( ||\hat{\theta} - \theta||_\infty \geq a), \mathbb{P}_{\theta'}( ||\hat{\theta}' - \theta'||_\infty \geq a) \right) \geq \frac{1}{4 \exp(D(P^t||P'^t))} \geq \delta.$$ 

Since $\theta, \theta' \in \Theta_{a,b}$ and $a \geq \epsilon$, we get

$$\min_{\theta \in \Theta_{a,b}} \mathbb{P}( ||\hat{\theta} - \theta||_\infty \geq \epsilon) \geq \delta. \quad \Box$$

B Proof of Theorem 2

We use the following preliminary results.

A concentration inequality. Define:

$$||\hat{C} - C||_\infty = \max_{i,j:i\neq j} |\hat{C}_{ij} - C_{ij}|.$$

Lemma 2 We have:

(i) For all $i = 1, \ldots, n$ and all $\epsilon > 0$,

$$\mathbb{P}( |\sum_{j \neq i} (\hat{C}_{ij} - C_{ij})| \geq \epsilon) \leq 2 \exp \left( -\frac{\epsilon^2 \alpha^2 t}{30 \max(B(\theta)^2, n)} \right) + 2n \exp \left( -\frac{t \alpha^2}{8(n-1)} \right).$$

(ii) For all $j \neq i$ and all $\epsilon \in (0, 1)$,

$$\mathbb{P}( |\hat{C}_{ij} - C_{ij}| \geq \epsilon) \leq 2 \exp \left( -\frac{\epsilon^2 \alpha^2 t}{120} \right) + 4 \exp \left( -\frac{t \alpha^2}{8} \right).$$

(iii) For all $\epsilon \in (0, 1)$,

$$\mathbb{P}( ||\hat{C} - C||_\infty \geq \epsilon) \leq 3n^2 \exp \left( -\frac{\epsilon^2 \alpha^2 t}{120} \right).$$

Proof. We first prove (i). Let $Z = \sum_{j \neq i} (C_{ij} - C_{ij})$, for some fixed $i$. The distribution of $Z$ is independent of $G(1), \ldots, G(t)$ so we fix $G(1) = \ldots = G(t) = 1$ until the end of the proof. Note that, given $G(t) = 1$, the random variables $X_1(t), \ldots, X_n(t)$ are independent, with respective expectations $\theta_1, \ldots, \theta_n$. Let $U = (U_j(t))_{j,t}$ be i.i.d Bernoulli random variables.
with $\mathbb{E}(U_j(t)) = \alpha$ and $V = (V_j(t))_{j,t}$ be independent random variables on $\{-1, 1\}$ with $\mathbb{E}(V_j(t)) = \theta_j$. Then $(X_j(t))_{j,t}$ has the same distribution as $(U_j(t)V_j(t))_{j,t}$. Define:

$$Z(s) = \frac{U_i(s)U_j(s)V_i(s)V_j(s)}{N_j},$$

with

$$N_j = \sum_{s=1}^t U_i(s)U_j(s).$$

Observe that $Z$ has the same distribution as $\sum_{s=1}^t Z(s)$.

**Conditioning**

Let us first condition on $U$. We denote by $\mathbb{E}_U$ and $\mathbb{P}_U$ the corresponding conditional expectation and probability. We upper bound the cumulant generating function of $Z(s)$. Consider $s$ fixed and drop $s$ for clarity. Consider $\lambda \in \mathbb{R}$ and:

$$\ln(\mathbb{E}_U(e^{\lambda Z}|V_i)) = \ln(\mathbb{E}_U(e^{\lambda U_i V_i \sum_{j \neq i} U_j V_j/N_j}|V_i))$$

$$= \sum_{j \neq i} \ln(\mathbb{E}_U(e^{\lambda U_i U_j V_j V_i/N_j}|V_i))$$

$$\leq \sum_{j \neq i} \ln(\mathbb{E}_U(e^{\lambda U_i U_j V_i \theta_j/N_j + \lambda^2 U_i U_j/(2N_j^2)}|V_i))$$

$$= \lambda^2 \sum_{j \neq i} \frac{U_i U_j}{2N_j^2} + \ln(\mathbb{E}_U(e^{\lambda U_i V_i \sum_{j \neq i} U_j \theta_j/N_j}|V_i)),$$

using the independence of $V_1, \ldots, V_n$, and the fact that, if $Y$ is a random variable with $|Y| \leq 1$, then $\ln(\mathbb{E}(e^{\lambda Y})) \leq \lambda \mathbb{E}(Y) + \lambda^2/2$ by Hoeffding’s lemma. Taking expectation over $V_i$:

$$\ln(\mathbb{E}_U(e^{\lambda U_i V_i \sum_{j \neq i} U_j \theta_j/N_j})) \leq \lambda \sum_{j \neq i} \frac{U_i U_j \theta_i \theta_j}{N_j} + \frac{\lambda^2}{2} \left( \sum_{j \neq i} \frac{U_i U_j \theta_j}{N_j} \right)^2,$$

where we have used Hoeffding’s lemma once again. Putting it together, we have proven:

$$\ln(\mathbb{E}_U(e^{\lambda Z})) \leq \lambda \sum_{j \neq i} \frac{U_i U_j \theta_i \theta_j}{N_j} + \frac{\lambda^2}{2} \left( \sum_{j \neq i} \frac{U_i U_j}{N_j^2} + \left( \sum_{j \neq i} \frac{U_i U_j \theta_j}{N_j} \right)^2 \right).$$

Define $N = \min_{j \neq i} N_j$, $S = \sum_{s=1}^t \left( \sum_{j \neq i} U_i(s)U_j(s)\theta_j \right)^2$ and $\sigma^2 = (n-1)N + S$. It is noted that $N$, $S$ and $\sigma^2$ depend on $(U_j(t))_{j,t}$ but not on $(V_j(t))_{j,t}$. 

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Using independence,
\[
\ln(\mathbb{E}_U(e^{\lambda \bar{Z}})) = \sum_{s=1}^{t} \ln(\mathbb{E}_U(e^{\lambda Z(s)})) \leq \lambda \mathbb{E}(Z) + \frac{\lambda^2}{2} \left( \sum_{j \neq i} \frac{1}{N_j} + \sum_{s=1}^{t} \left( \sum_{j \neq i} \frac{U_i(s)U_j(s)\theta_j}{N_j} \right)^2 \right),
\]
where we used the fact that \( \mathbb{E}_U(Z) = \mathbb{E}(Z) = \theta_i \sum_{j \neq i} \theta_j \).

Chernoff bound
We now derive a Chernoff bound for \( \bar{Z} \). For all \( \varepsilon > 0 \), we have
\[
P_U(Z - \mathbb{E}(Z) \geq \varepsilon) \leq \min_{\lambda \geq 0} e^{-\lambda \varepsilon \mathbb{E}_U(e^{\lambda(Z-\mathbb{E}(Z))})} \leq e^{-\varepsilon^2/2\sigma^2},
\]
the minimum being attained for \( \lambda = 2\varepsilon/\sigma^2 \).

Controlling the fluctuations of \( \sigma^2 \)
To remove the conditioning on \( U \), so that we need to control the fluctuations of \( \sigma^2 \), thus those of \( N \) and \( S \). First consider \( N \). Since \( N_j \) is the sum of \( t \) independent Bernoulli variables with expectation \( \alpha^2 \), we have by Lemma \( \ref{lemma} \)
\[
P(N_j \leq \alpha^2 t/2) \leq e^{-\alpha^2 t/8}.
\]
Using a union bound,
\[
P(N \leq \alpha^2 t/2) \leq \sum_{j \neq i} P(N_j \leq \alpha^2 t/2) \leq (n-1)e^{-\alpha^2 t/8}.
\]
We turn to \( S \). \( S \) is a sum of \( t \) positive independent variables bounded by \( (n-1)^2 \) with expectation
\[
\mu = \mathbb{E} \left( \sum_{j \neq i} U_i(t)U_j(t)\theta_j \right)^2 = \alpha^2(\alpha B_i(\theta)^2 + (1 - \alpha) \sum_{j \neq i} \theta_j^2),
\]
where \( B_i(\theta) = \sum_{j \neq i} \theta_j \). We have \( \mu \leq \bar{\mu} \equiv \alpha^2 \max(B_i(\theta)^2, n-1) \). By Lemma \( \ref{lemma} \)
\[
P(S \geq 2t\bar{\mu}) = \mathbb{P} \left( \frac{S}{(n-1)^2} \geq \frac{2t\bar{\mu}}{(n-1)^2} \right) \leq \exp \left( -tD \left( \frac{2\bar{\mu}}{(n-1)^2} \mid \frac{\mu}{(n-1)^2} \right) \right) \leq e^{-\frac{t^2}{3(n-1)^2}} \leq e^{-\frac{t^2}{3(n-1)}}.
\]
If both events $S \leq 2t\bar{\mu}$ and $N \geq \alpha^2 t/2$ occur we have:

$$
\sigma^2 \leq \frac{2(n-1)}{\alpha^2 t} + \frac{8 \max(B_i(\theta)^2, n-1)}{\alpha^2 t} \leq \frac{10 \max(B_i(\theta)^2, n-1)}{\alpha^2 t}.
$$

Estimation Error

Finally,

$$
P(\bar{Z} - E(\bar{Z}) \geq \varepsilon) \leq P\left(\bar{Z} - E(\bar{Z}) \geq \varepsilon, \sigma^2 \leq \frac{10 \max(B_i(\theta)^2, n-1)}{\alpha^2 t}\right) + P(N \leq \alpha^2 t/2) + P(S \geq 2t\bar{\mu})
$$

$$
\leq \exp\left(-\frac{\varepsilon^2 \alpha^2 t}{10 \max(B_i(\theta)^2, n-1)}\right) + (n-1) \exp\left(-\frac{t\alpha^2}{8}\right) + \exp\left(-\frac{t\alpha^2}{3(n-1)}\right)
$$

Doing the same reasoning for $P(\bar{Z} - E(\bar{Z}) \leq -\varepsilon)$ yields

$$
P(|\bar{Z} - E(\bar{Z})| \geq \varepsilon) \leq 2 \exp\left(-\frac{\varepsilon^2 \alpha^2 t}{10 \max(B_i(\theta)^2, n-1)}\right) + 2n \exp\left(-\frac{t\alpha^2}{8(n-1)}\right).
$$

Statement (i) then follows from the fact that $\max(B_i(\theta)^2, n-1) \leq 3 \max(\theta^2, n)$. Indeed, $\max(B_i(\theta)^2, n-1) \leq \max(B_i(\theta)^2, 3n)$, and if $B_i(\theta) \geq \sqrt{3n} \geq 3$,

$$
\frac{B_i(\theta)^2}{B(\theta)^2} \leq \frac{B_i(\theta)^2}{(B_i(\theta) - 1)^2} \leq \frac{9}{4} \leq 3.
$$

Statement (ii) is obtained by setting $n = 2$ in statement (i); statement (iii) follows from a union bound over all pairs $i,j$ of statement (ii), on observing that

$$
P(|\hat{C}_{ij} - C_{ij}| \geq \varepsilon) \leq 6 \exp\left(-\frac{\varepsilon^2 \alpha^2 t}{120}\right).
$$

\[\square\]

**Lemma 3 (Chernoff’s Inequality)** Let $Y_1, \ldots, Y_t$ be i.i.d. random variables on $[0, 1]$ with expectation $\mu$. Denote by $D(\mu'||\mu)$ the Kullback Leibler divergence between two Bernoulli distribution with parameters $\mu'$ and $\mu$.

(i) For all $\mu' \geq \mu$, $P(\sum_{s=1}^t Y_s \geq t\mu') \leq e^{-tD(\mu'||\mu)}$.

(ii) For all $\mu' \leq \mu$, $P(\sum_{s=1}^t Y_s \leq t\mu') \leq e^{-tD(\mu'||\mu)}$.

(iii) For all $\mu \geq 0$, $D(2\mu||\mu) \geq \mu/2$ and $D(\mu/2||\mu) \geq \mu/8$.

**Estimation of absolute value.** For all $i \neq j \neq k$, define

$$
\rho_k(i, j) = \sqrt{\frac{\hat{C}_{ik}\hat{C}_{jk}}{C_{ij}}}.
$$
Lemma 4 If \( ||\hat{C} - C||_\infty \leq \varepsilon \), then

\[
|\rho_k(i,j) - |\theta_k|| \leq 10 \frac{\varepsilon}{|C_{ij}|}.
\]

Proof. Without loss of generality, assume that \(|\theta_i| \geq |\theta_j|\) so that \(|C_{ik}| \geq |C_{jk}|\).

a) If \( \varepsilon \geq |C_{ij}|/2 \), the inequality holds since \( 10 \frac{\varepsilon}{|C_{ij}|} \geq 5 \) and \(|\rho_k(i,j) - |\theta_k|| \leq 2\).

b) Assume \( \varepsilon \geq C_{ij}/2 \) and \( \varepsilon \geq C_{jk}/2 \). Then

\[
|\theta_k| = C_{jk}/|\theta_j| \leq 2\varepsilon/|\theta_j| \leq 2\varepsilon/|C_{ij}|.
\]

Furthermore, \( |\hat{C}_{jk}| \leq |C_{jk}| + \varepsilon \), \( |\hat{C}_{ik}| \leq |C_{jk}| + \varepsilon \) and \( |\hat{C}_{ij}| \geq |C_{ij}| - \varepsilon \geq |C_{ij}|/2 \).

So

\[
\rho_k(i,j) \leq \sqrt{2(|C_{ik}| + \varepsilon)(|C_{jk}| + \varepsilon)}/C_{ij} = \sqrt{2(|\theta_k| + \varepsilon/|\theta_i|)(|\theta_k| + \varepsilon/|\theta_j|)}
\]

\[
\leq 2|\theta_k| + 2\varepsilon\left(\frac{1}{|\theta_i|} + \frac{1}{|\theta_i|}\right) \leq 2|\theta_k| + \frac{4\varepsilon}{|C_{ij}|}
\]

and

\[
|\rho_k(i,j) - |\theta_k|| \leq |\rho_k(i,j)| + |\theta_k| \leq 3|\theta_k| + \frac{4\varepsilon}{|C_{ij}|} \leq 10 \frac{\varepsilon}{|C_{ij}|}.
\]

c) Finally, let \( \varepsilon \leq \min(|C_{ij}|, |C_{ik}|, |C_{jk}|)/2 \). Define

\[
\Delta = |C_{ik}C_{jk}\hat{C}_{ij} - \hat{C}_{ik}\hat{C}_{jk}C_{ij}|.
\]

We have

\[
\Delta \leq |C_{ik}C_{jk}||\hat{C}_{ij} - C_{ij}| + |C_{ij}\hat{C}_{ik}||\hat{C}_{jk} - C_{jk}| + |C_{ij}C_{jk}||\hat{C}_{ik} - C_{ik}|,
\]

\[
\leq \varepsilon (|C_{ik}C_{jk}| + 2|C_{ij}C_{ik}| + |C_{ij}C_{jk}|).
\]

Further,

\[
|\rho_k(i,j)^2 - \theta_k^2| = \frac{\Delta}{|C_{ij}C_{ij}|} \leq \frac{2\varepsilon}{C_{ij}^2} \left(\frac{\theta_k^2}{C_{ij}} + 2|\theta_k| + |\theta_k||\theta_k|\right) \leq \frac{8\varepsilon |\theta_k|}{|C_{ij}|}.
\]

Finally, \( \rho_k(i,j)^2 - \theta_k^2 = (\rho_k(i,j) + |\theta_k|)(\rho_k(i,j) - |\theta_k|) \), so that

\[
||\rho_k(i,j)| - |\theta_k|| \leq \frac{|\rho_k(i,j)^2 - \theta_k^2|}{\rho_k(i,j) + |\theta_k|} \leq \frac{\rho_k(i,j)^2 - \theta_k^2}{|\theta_k|} \leq \frac{8\varepsilon}{|C_{ij}|}.
\]

\[
\square
\]

Lemma 5 If \( ||\hat{C} - C||_\infty \leq \varepsilon \), then

\[
||\hat{\theta}_k| - |\theta_k|| \leq \frac{20\varepsilon}{A^2(\theta)}.
\]
Proof. a) If $\varepsilon \geq A^2(\theta)/4$,
\[
||\hat{\theta}_k| - |\theta_k|| \leq 2 \leq 5 \leq \frac{20\varepsilon}{A^2(\theta)}
\]
so that the inequality holds.

b) Consider $\varepsilon \leq A^2(\theta)/4$. By definition, $\hat{\theta}_k = \rho_k(i_k, j_k)$. By assumption, there exists $i, j \neq k$ such that $C_{i,j} \geq A^2(\theta)$. Further:
\[
|C_{i_k,j_k}| + \varepsilon \geq |\hat{C}_{i_k,j_k}| \geq |\hat{C}_{i,j}| \geq |C_{i,j}| - \varepsilon \geq A^2(\theta) - \varepsilon.
\]
So $C_{i_k,j_k} \geq A^2(\theta) - 2\varepsilon \geq A^2(\theta)/2$. Using Lemma 4,
\[
||\hat{\theta}_k| - |\theta_k|| = |\rho_k(i_k, j_k)| - |\theta_k| \leq \frac{10\varepsilon}{C_{i_k,j_k}} \leq \frac{20\varepsilon}{A^2(\theta)}.
\]
□

Sign estimation. We will use the following fact:

**Fact 1** Let $u, v \in \mathbb{R}^n$ and define $\epsilon = \max_i |u_i - v_i|$, $\bar{u} = \max_i |u_i|$, $i^* \in \arg\max_i |v_i|$. If $\epsilon \leq \bar{u}/4$ then (i) $\text{sign}(v_{i^*}) = \text{sign}(u_{i^*})$ and (ii) $|u_{i^*}| \geq \bar{u}/2$.

**Proof.** (i) We proceed by contradiction. Assume that $\text{sign}(u_{i^*}) \neq \text{sign}(v_{i^*})$. Since $i^* \in \arg\max_i |v_i|$, we have $|v_{i^*}| \geq \bar{u} - \epsilon$. On the other hand, since $\text{sign}(u_{i^*}) \neq \text{sign}(v_{i^*})$, $|v_{j}| \leq \epsilon$. Hence $2\epsilon \geq \bar{u}$, a contradiction.

(ii) We have $|u_{i^*}| \geq v_{i^*} - \epsilon \geq \max_i |v_i| - \epsilon \geq \bar{u} - 2\epsilon \geq \bar{u}/2$. □

In the rest of the proof, we define $\phi = \max_i |\theta_i|$.

**Lemma 6** Assume that $||\hat{C} - C||_{\infty} \leq \varepsilon \leq A^2(\theta)/2$. Then for all $k = 1, \ldots, n$,
\[
|\hat{\theta}_k^2 - \theta_k^2| \leq \frac{8\varepsilon \phi^2}{A^2(\theta)}.
\]

**Proof.** We have
\[
|\rho_k(i, j)^2 - \theta_k^2| = \Delta \frac{\Delta}{|\hat{C}_{i,j}|^2},
\]
with
\[
\Delta = |C_{i_k}C_{j_k}\hat{C}_{i,j} - \hat{C}_{i_k}\hat{C}_{j_k}C_{i,j}|.
\]
Now,
\[
\Delta \leq |C_{i_k}C_{j_k}||\hat{C}_{i,j} - C_{i,j}| + |C_{i,j}\hat{C}_{i_k}| |\hat{C}_{j_k} - C_{j_k}| + |C_{i,j}C_{j_k}||\hat{C}_{i_k} - C_{i_k}|,
\]
\[
\leq \varepsilon (|C_{i_k}C_{j_k}| + |C_{i,j}\hat{C}_{i_k}| + |C_{i,j}C_{j_k}|),
\]
\[23\]
We have $|\hat{C}_{ik}| \leq |C_{ik} + \varepsilon| \leq \phi^2 + \varepsilon \leq 2\phi^2$, since $\phi^2 \geq A^2(\theta) \geq \varepsilon$. Further, using the fact that $|C_{ik}C_{jk}| \leq \phi^2|C_{ij}|$, we get $|C_{ij}C_{jk}| \leq \phi^2|C_{ij}|$. Replacing, we obtain

$$|C_{ij}|^2 \leq 4\phi^2\varepsilon|C_{ij}|$$

and

$$|\rho_k(i, j)^2 - \theta_k^2| = \frac{4\phi^2\varepsilon}{|C_{ij}|}.$$

We have

$$\max_{i,j\neq k} |\hat{C}_{ij}| \geq \max_{i,j\neq k} |C_{ij}| - \varepsilon \geq A^2(\theta) - A^2(\theta)/2 = A^2(\theta)/2.$$

Setting $(i, j) \in \arg\max_{i,j\neq k} |\hat{C}_{ij}|$, we get the announced result. \(\square\)

**Estimation error.** We can now control the estimation error.

**Lemma 7** Assume that $||\hat{C} - C||_\infty \leq \varepsilon \leq A^2(\theta) \min\left(\frac{1}{2}, \frac{B(\theta)}{64}\right)$ and $\max_i |\sum_{j\neq i} \hat{C}_{ij} - C_{ij}| \leq \frac{A(\theta)B(\theta)}{8}$. Then

(i) $\text{sign}(\theta_{k^*}) = \text{sign}(\hat{\theta}_{k^*})$,

(ii) for all $i$ such that $|\theta_i| \geq \frac{2\varepsilon}{A^2(\theta)}$, $\text{sign}(\hat{\theta}_i) = \text{sign}(\theta_i)$,

(iii) $||\hat{\theta} - \theta||_\infty \leq \frac{24\varepsilon}{A^2(\theta)}$.

**Proof.** (i) Let $u_i = \theta_i^2 + \sum_{j\neq i} C_{ij} = \theta_i B(\theta)$ and $v_i = \hat{\theta}_i^2 + \sum_{j\neq i} \hat{C}_{ij}$. We have

$$|u_i - v_i| \leq |\hat{\theta}_i^2 - \theta_i^2| + |\sum_{j\neq i} (\hat{C}_{ij} - C_{ij})|.$$

Using Lemma 6,

$$|\hat{\theta}_i^2 - \theta_i^2| \leq \min\left(\frac{\phi^2 B(\theta)}{8}, 4\phi^2\right) \leq \frac{\phi^2 B(\theta)}{8} \leq \frac{\phi B(\theta)}{8}.$$

Further, since $A(\theta) \leq \phi$,

$$|\sum_{j\neq i} (\hat{C}_{ij} - C_{ij})| \leq \frac{\phi B(\theta)}{8}.$$

So, for all $i$,

$$|u_i - v_i| \leq \frac{\phi B(\theta)}{4} = \max_i |u_i|.$$

Applying Fact 1 statement (i) ensures that $\text{sign}(\hat{\theta}_{k^*}) = \text{sign}(\theta_{k^*})$.

(ii) Fact 1 statement (ii) gives $|\theta_{k^*}| B(\theta) \geq \frac{\phi B(\theta)}{2}$, so that $|\theta_{k^*}| \geq \phi/2 \geq A(\theta)/2$. Consider $i \neq k^*$ and $|\theta_i| \geq 2\varepsilon/A(\theta)$. We have $|C_{ik^*}| = |\theta_i||\theta_{k^*}| \geq \varepsilon$ since $|\theta_{k^*}| \geq A(\theta)/2$. Since $|\hat{C}_{ik^*} - C_{ik^*}| \leq \varepsilon$, we have $\text{sign}(\hat{C}_{ik^*}) = \text{sign}(C_{ik^*})$. So $\text{sign}(\hat{\theta}_i) = \text{sign}(\theta_i)$ which proves the second claim.
We may readily check that
\[ |\hat{\theta}_i - \theta_i| \leq |\hat{\theta}_i| - |\theta_i| + 2|\theta_i|1\{\text{sign}(\theta_i) \neq \text{sign}(\hat{\theta}_i)\} \leq \frac{20\varepsilon}{A^2(\theta)} + \frac{4\varepsilon}{A(\theta)} \leq \frac{24\varepsilon}{A^2(\theta)}. \]
where we applied the previous statement and Lemma 5.

\[ \square \]

**Proof of Theorem 2.** Let \( \theta \in \Theta_{a,b} \), \( \epsilon \in (0, \min(b/3, 1)) \) and assume that the following two events occur:

\[ \left\{ ||\hat{C} - C||_{\infty} \leq \frac{\epsilon A^2(\theta)}{24} \right\} \quad \text{and} \quad \left\{ \max_{i=1,\ldots,n} |\sum_{j \neq i} (\hat{C}_{ij} - C_{ij})| \leq \frac{A(\theta)B(\theta)}{8} \right\}. \]

We may readily check that
\[ \frac{\epsilon A^2(\theta)}{24} \leq \frac{A^2(\theta)}{24} \min \left( \frac{b}{3}, 1 \right) \leq \frac{A^2(\theta)}{24} \min \left( \frac{B(\theta)}{3}, 1 \right) \leq A^2(\theta) \min \left( \frac{B(\theta)}{64}, \frac{1}{2} \right). \]
Thus Lemma 7 guarantees that \( ||\hat{\theta} - \theta||_{\infty} \leq \epsilon \). By a union bound,
\[ \mathbb{P}(||\hat{\theta} - \theta||_{\infty} \geq \epsilon) \leq \mathbb{P}
left( ||\hat{C} - C||_{\infty} \geq \frac{\epsilon A^2(\theta)}{24} \right) + \sum_{i=1}^{n} \mathbb{P}
left( \left| \sum_{j \neq i} (\hat{C}_{ij} - C_{ij}) \right| \geq \frac{A(\theta)B(\theta)}{8} \right). \]
Now let \( t \geq \max(T'_1, T'_2) \) with \( c'_1 = 120 \times 24^2 \) and \( c'_2 = 30 \times 8^2 \). Applying Lemma 2
\[ \mathbb{P}
left( ||\hat{C} - C||_{\infty} \geq \frac{\epsilon A^2(\theta)}{24} \right) \leq 3n^2 \exp \left( -\frac{\epsilon^2 A^4(\theta)\alpha^2 t}{120 \times 24^2} \right) \leq \frac{\delta}{2}. \]
Applying Lemma 2 once again, we get
\[ \mathbb{P}
left( \left| \sum_{j \neq i} (\hat{C}_{ij} - C_{ij}) \right| \geq \frac{A(\theta)B(\theta)}{8} \right) \leq 2 \exp \left( -\frac{A(\theta)^2 B(\theta)^2 \alpha^2 t}{30 \times 8^2 \max(B(\theta)^2, n)} \right) + 2n \exp \left( -\frac{t \alpha^2}{8(n - 1)} \right). \]
Since
\[ \frac{A(\theta)^2 B(\theta)^2 \alpha^2 t}{30 \times 8^2 \max(B(\theta)^2, n)} \geq \frac{a^2 \alpha^2 t}{30 \times 8^2} \max \left( 1, \frac{b^2}{n} \right) \geq \ln \left( \frac{4n^2}{\delta} \right) \]
and
\[ \frac{t \alpha^2}{8(n - 1)} \geq \ln \left( \frac{4n^2}{\delta} \right), \]
we get
\[ \mathbb{P}
left( \left| \sum_{j \neq i} (\hat{C}_{ij} - C_{ij}) \right| \geq \frac{A(\theta)B(\theta)}{8} \right) \leq \frac{\delta}{n^2}, \]
and summing we get \( \mathbb{P}(||\hat{\theta} - \theta||_{\infty} \geq \epsilon) \leq \delta \) as announced. \( \square \)
C Upper bounds: Relation with Prior Work

Consider $n \geq 5$, $a = 1/2$, $b = \frac{\sqrt{n-4}}{2}$ and $\theta = (a, -a, a, -a, b, \frac{b}{n-4}, ..., b)$. We have $\max_i |\theta_i| \leq \frac{1}{2}$.

For Theorem 4 of [24] to hold, one requires the following inequality to be satisfied:

$$n \geq c_4 \frac{\ln(tn/\delta)}{D}$$

where $c_4 > 0$ is a universal constant and:

$$D = \frac{\alpha}{n} \sum_{i=1}^{n} D \left( \frac{1 + \theta_i}{2} \parallel \frac{1 - \theta_i}{2} \right),$$

where $D(p||q)$ denotes the Kullback-Leibler divergence between Bernoulli distributions with parameters $p$ and $q$. From inequality $\ln(z) \leq z - 1$, we have $D(p||q) \leq \frac{(p-q)^2}{q(1-q)}$ for all $p, q$ in $(0, 1)$. Since $\max_i |\theta_i| \leq 1/2$ we get:

$$D \leq \frac{16\alpha}{3n} \sum_{i=1}^{n} \theta_i^2 = \frac{20\alpha}{3n}.$$ 

Therefore $n$ must satisfy $n \geq c_4 (3n/20\alpha) \ln(tn/\delta)$ and there can exist no such $n$ when $t$ or $1/\delta$ are sufficiently large.