Quantum Temporal Logic and Decoherence Functionals in the Histories Approach to Generalised Quantum Theory

C.J. Isham
Isaac Newton Institute for Mathematical Sciences
University of Cambridge
20 Clarkson Road
Cambridge CB3 0EH
United Kingdom

and

N. Linden
D.A.M.T.P.
University of Cambridge
Cambridge CB3 9EW
United Kingdom

April 1994

Abstract

We analyse and develop the recent suggestion that a temporal form of quantum logic provides the natural mathematical framework within which to discuss the proposal by Gell-Mann and Hartle for a generalised form of quantum theory based on the ideas of histories and

1 Permanent address: Blackett Laboratory, Imperial College, South Kensington, London SW7 2BZ; email: c.isham@ic.ac.uk
2email: n.linden@newton.cam.ac.uk
decoherence functionals. Particular stress is placed on properties of the space of decoherence functionals, including one way in which certain global and topological properties of a classical system are reflected in a quantum history theory.
1 Introduction

The main aim of this paper is to explore the algebraic structure underlying the generalised quantum scheme proposed by Gell-Mann and Hartle [1, 2, 3, 4, 5, 6, 7]. This scheme is a natural development of the decoherent histories approach to standard quantum theory put forward by Griffiths [8] and by Omnès [9, 10, 11, 12, 13, 14]. We shall describe how the algebraic structure underlying this scheme parallels that of quantum logic [15], with particular emphasis on the space of decoherence functionals viewed as the analogue of the space of states in standard quantum theory.

A major motivation for studying schemes of this type in the context of quantum gravity is the well-known ‘problem of time’. In particular, the unsolved question of whether the normal concept of time is fundamental or one that emerges only above the Planck scale in some coarse-grained way. If the latter is true it seems reasonable to suggest that the usual notion of a continuum space manifold may also be applicable only in some semi-classical sense: a view that is difficult to reconcile with the standard canonical approach to quantum gravity which employs a fixed background three-manifold.

This difficulty reinforces the idea of developing a more spacetime-oriented approach to quantum gravity (what is sometimes called the ‘covariant’, as opposed to ‘canonical’, perspective [16, 17]) albeit one in which the classical ideas of space and time play no fundamental role. A non-standard quantum scheme of this sort could permit the introduction of very generalised spacetime concepts like quantum topology, causal sets [18], and various other discrete models. A key supposition is that the familiar spacetime concepts ‘emerge’ well above the Planck scale, as must also the Hilbert space mathematical formalism of normal quantum theory which is tied so closely to the standard picture of space and time. This suggests that the infamous Copenhagen interpretation might also apply only in some approximate sense. Indeed, a major driving force behind the original studies of decohering histories was a desire to find an interpretation of quantum theory with a philosophical flavour that is more realist than is the usual one. Such a move from observables to ‘beables’ [19] is particularly attractive in any theory that aspires to address issues in quantum cosmology.

In this paper we focus on the approach to generalised quantum theory advocated by Gell-Mann and Hartle in which the notion of a ‘history’ may not just be a time-ordered string of events or propositions: rather, it can
appear as a fundamental theoretical entity in its own right. For example, a Lorentzian manifold \((\mathcal{M}, g)\) that is not globally-hyperbolic cannot be regarded as a time-ordered sequence of three-geometries as there is no global time function with which to construct an appropriate one-parameter family of foliations of \(\mathcal{M}\). In this sense, the Lorentzian metric \(g\) is not a ‘history’ but, nevertheless, \(g\) may be perfectly acceptable as a geometry of space-time, and hence as a ‘generalised history’ of the universe.

This example illustrates the main point well: in classical or quantum gravity a ‘history’ is simply a potential configuration for the universe as a whole (or, perhaps more precisely, a proposition about such configurations), including any quasi space-time structure it may possess. Thus a ‘history’ might involve a causal set or other discrete structure, or it could just be a simple continuum manifold with associated geometric structure.

Our approach to the Gell-Mann and Hartle (hereafter abbreviated to GH) scheme is based on \([15]\) whose central idea is to use ‘quasi-temporal’ quantum logic to provide a firm mathematical footing for the scheme. Indeed, as we shall see, by rewriting the GH rules in a certain way, the potential connection with quantum logic becomes very clear.

The programme of standard quantum logic began with a seminal paper by von Neumann and Birkhoff \([20]\), and has been developed by many authors since (an excellent review is \([21]\)). The starting point was the claim by von Neumann and Birkhoff that the essential differences between a classical system and a quantum system are captured by the structures of their associated spaces of propositions about the system at a fixed time: in classical physics this is the Boolean algebra of measurable subsets of the classical phase space; in the quantum case it is the non-distributive lattice of projection operators on the Hilbert space \(\mathcal{H}\) or, equivalently, the lattice of closed subsets of \(\mathcal{H}\).

In the latter context we recall that the following lattice operations are defined (where \(\mathcal{H}_P \subseteq \mathcal{H}\) and \(\mathcal{H}_R \subseteq \mathcal{H}\) denote the ranges of projection operators \(P\) and \(R\) respectively): (i) the partial ordering operation \(P \leq R\) if and only if \(\mathcal{H}_P \subseteq \mathcal{H}_R\); (ii) the complementation (\(i.e., \) ‘not’) operation \(\neg P := 1 - P\) (so that \(\mathcal{H}_{\neg P} = \mathcal{H}_P^\perp\)); (iii) the meet \(P \land R\) is defined as the projector onto \(\mathcal{H}_P \cap \mathcal{H}_R\); and (iv) the join \(P \lor R\) is defined as the projector onto the closure of the linear span of \(\mathcal{H}_P \cup \mathcal{H}_R\). This structure is interpreted physically by saying that if the state of the system is the density matrix \(\rho\) and if a measurement is made of the proposition represented by a projection operator \(P\),
then the probability that the proposition will be found to be true is \( \text{tr}(P \rho) \).

One aim of the quantum logic programme was to explore the possibility of associating the propositions of a quantum system with a lattice \( \mathcal{L} \) that is not just the projection lattice \( \mathcal{P}(\mathcal{H}) \) of a Hilbert space \( \mathcal{H} \). Thus, dual to such a lattice \( \mathcal{L} \), there is a space \( \mathcal{S} \) of states where \( \sigma \in \mathcal{S} \) is defined to be a real-valued function on \( \mathcal{L} \) with the properties

1. **Positivity**: \( \sigma(P) \geq 0 \) for all \( P \in \mathcal{L} \) and \( \sigma \in \mathcal{S} \).

2. **Additivity**: if \( P \) and \( R \) are disjoint (defined to mean that \( P \leq \neg R \)) then, for all \( \sigma \in \mathcal{S} \), \( \sigma(P \lor R) = \sigma(P) + \sigma(R) \). This requirement is usually extended to include countable collections of propositions.

3. **Normalisation**: \( \sigma(1) = 1 \)

where \( 1 \) is the unit proposition for which \( \mathcal{H}_1 = \mathcal{H} \) (there is also a null proposition \( 0 \) with \( \mathcal{H}_0 = \{0\} \)). Physically, \( \sigma(P) \) is interpreted as the probability that proposition \( P \) is true (or, more operationally, will be found to be true if an appropriate measurement is made) in the state \( \sigma \). The trivial propositions \( 0 \) and \( 1 \) are respectively false and true with probability one in any state.

For the purposes of our later discussion we note that there has been considerable debate about the physical meaning of the \( \lor \) and \( \land \) operations when applied to propositions that are not compatible. This has lead a number of authors to suggest weakening the idea of a lattice so that, for example, the join operation is defined only on pairs of propositions \( P \) and \( R \) that are **disjoint**, in which case it is customary to write the join as \( P \oplus R \) rather than \( P \lor R \). Thus, in schemes of this type, one has only (i) a partial ordering operation; (ii) an orthocomplementation operation \( \neg \); and (iii) a notion that certain pairs \( P, R \in \mathcal{L} \) are **disjoint**, written \( P \perp R \), and that such pairs can be combined to give a new proposition \( P \oplus R \) interpreted as ‘\( P \) or \( R \)’. The minimal useful structure of this type appears to be an **orthoalgebra** in which the partial-ordering and disjoint-join operations are related by the condition \( P \leq R \) if and only if there exists \( V \in \mathcal{L} \) such that \( R = P \oplus V \).

A slightly stronger concept is of an **orthomodular poset**, which is essentially an orthoalgebra with the extra requirement that \( P \oplus R \) is the unique least upper bound for \( P \) and \( R \).

In standard quantum logic, time evolution appears as a one-parameter family of automorphisms of the lattice of propositions; in a Hilbert space
this is implemented by the familiar family \( t \mapsto U(t) := e^{-iHt/\hbar} \) of unitary operators. However, our scheme is quite different since we propose to use an orthoalgebra to model the space of propositions about histories of the system; a proposal that suggests the use of a quantum version of temporal logic, rather than the single-time logic employed in standard quantum theory.

Temporal logic has not featured strongly in existing studies of quantum logic with the notable exception of the work of Mittelstaedt and Stachow [23, 24] who developed a theory of the logic of sequential propositions using sequential conjunctions of the type “\( A \) is true and then \( B \) is true and then \( \ldots \)”, plus other temporal connectives such as ‘or then’ and ‘sequential implication’. The approach adopted here and in [15] is partly an extension of this work to include ‘quasi-temporal’ situations in which the notion of time-evolution is replaced by something that is much broader and which, for example, could include space-time causal relations as well as ideas of quantum topology. Ultimately we are interested in situations where there is no prior temporal structure at all. However, in all cases, a key ingredient in our approach is the observation that the statement that a certain universe (i.e., history) is ‘realised’ is itself a proposition, and therefore the set \( \mathcal{UP} \) of all such history-propositions might possess the structure of an orthoalgebra or lattice that is analogous to the lattice of single-time propositions in standard quantum logic. In particular, a history proposition might be representable by a projection operator in some Hilbert space. But, whether this is true or not, the heart of our approach is a claim that, if \( \mathcal{D} \) denotes the set of decoherence functionals (a decoherence functional is a complex-valued function of pairs of histories that measures their mutual quantum interference), the pair \((\mathcal{UP}, \mathcal{D})\) plays a role in history theory that is analogous to that of the pair \((\mathcal{L}, \mathcal{S})\) in normal, fixed-time quantum theory.

The plan of the paper is as follows. We begin with a version of the Gell-Mann and Hartle rules for generalised quantum theory that is based on the ideas in [15] but presented in a way that is intended to make the use of quantum-logical techniques particularly natural. Then, using a simple two-time model, we show how the history-propositions in standard quantum theory can indeed be represented by projection operators on a new Hilbert space, and that this throws useful light on some of the manipulations performed by Gell-Mann and Hartle in their development of the scheme. We also review the way in which ‘quasi-temporal’ structure can be incorporated as a generalisation of the partial semigroup that is associated with the standard
This is followed by a discussion of several properties of decoherence functionals in standard quantum theory with a view to seeing what can, or should, be incorporated into the general scheme. We consider several ways in which decoherence functionals can be constructed, and we show that several natural-sounding properties that might be posited in the general theory are in fact violated even in standard quantum theory. This is followed by a discussion of how certain global or topological properties of the classical system are reflected in a quantum history theory. A key issue which we address is how the familiar $\pi_1(Q)$ effects in path-integrals over a configuration space $Q$ are manifested in the more abstract histories approach to the quantum theory. This framework throws a new light on the origin of such effects and how they relate to general global properties of the classical system.

Finally, a word about notation. It is sometimes useful to distinguish between a proposition $P$ and the projection operator that represents it, in which case the latter will be written as $\hat{P}$. But in general ‘hats’ will be avoided for the sake of typographical clarity.

\section{Generalised Quantum Theory}

\subsection{Histories in standard quantum theory}

In developing their abstract notion of a ‘history’, Gell-Mann and Hartle were guided partly by ideas that arise naturally in path-integral quantisation using paths in a configuration space $Q$, and partly by the idea of a history as a time-ordered sequence of projectors in a Hilbert-space based quantum theory. Both approaches suggest certain logical-type operations in the space of all histories.

Let us begin by reviewing briefly how histories are handled in a standard quantum theory defined on a Hilbert space $\mathcal{H}$. As usual, observable quantities are represented by self-adjoint operators on $\mathcal{H}$ and, to avoid ambiguities associated with overall unitary transformations, we will suppose that (i) a fixed labelling of operators with specific physical quantities has been established; and (ii) the Schrödinger representation is used throughout unless the contrary is indicated explicitly. A proposition concerning the values of an observable is represented mathematically by a projection operator $P$ on $\mathcal{H}$.  

6
Note that, by virtue of our assumptions, the representation of a particular proposition by a specific projector does not depend on time: the time dependence is carried solely by the evolving state.

Suppose now we are concerned with a sequence of propositions \( \alpha := (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n}) \) that refer to the results of measurements made at times \( t_1, t_2, \ldots, t_n \) where \( t_1 < t_2 < \cdots < t_n \), and which is such that each proposition is non-trivial, i.e., not equal to 0 or 1. We shall refer to this set as a potential homogeneous history of the system with the understanding that the history is ‘realised’ if each proposition \( \alpha_{t_i} \) is found to be true at the corresponding time \( t_i \). Thus, in the language of temporal logic, \( (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n}) \) is the sequential conjunction “\( \alpha_{t_1} \) is true at time \( t_1 \) and then \( \alpha_{t_2} \) is true at time \( t_2 \) and then \( \ldots \) and then \( \alpha_{t_n} \) is true at time \( t_n \)”.

It is convenient to think of a homogeneous history \( \alpha \) as a proposition-valued function \( t \mapsto \alpha_t \) on the time-line \( \mathbb{R} \) which is equal to the unit proposition 1 for all but a finite number of time values. We shall call this set \( \{t_1, t_2, \ldots, t_n\} \) at which the history is ‘active’ the temporal support of the homogeneous history. The situation can arise in which the result of some logical operation on homogeneous histories is a history in which at least one of the propositions in the sequence is the null proposition 0. All such histories are defined to be equivalent to each other and to the null history. Note that the subscript \( t_i \) serves merely to label the different propositions and to remind us of the times at which they are asserted: the associated projection operators \( \hat{\alpha}_{t_i} \) are in the Schrödinger representation.

A standard result in conventional quantum theory using state-vector reduction is that if the state at some initial time \( t_0 \) is the density matrix \( \rho \), then the joint probability of finding all the associated properties in an appropriate sequence of measurements is

\[
\text{Prob}(\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n}; \rho_{t_0}) = \text{tr}(\tilde{C}_\alpha^\dagger \rho \tilde{C}_\alpha) \quad (2.1)
\]

where the ‘class’ operator \( \tilde{C}_\alpha \) is defined by

\[
\tilde{C}_\alpha := U(t_0, t_1)\alpha_{t_1}U(t_1, t_2)\alpha_{t_2} \ldots U(t_{n-1}, t_n)\alpha_{t_n}U(t_n, t_0) \quad (2.2)
\]

and where \( U(t, t') = e^{-i(t-t')H/\hbar} \) is the unitary time-evolution operator from time \( t \) to \( t' \); in particular, \( U \) satisfies the evolution law \( U(t_1, t_2)U(t_2, t_3) = U(t_1, t_3) \) for all \( t_1, t_2 \) and \( t_3 \). Note that, to ease the later discussion of the relation with temporal logic, our operator \( \tilde{C}_\alpha \) is the adjoint of the operator.
$C_\alpha$ used by Gell-Mann and Hartle. We note also in passing that $\tilde{C}_\alpha$ is often written as the product of projection operators

$$\tilde{C}_\alpha = \alpha_{t_1}(t_1)\alpha_{t_2}(t_2)\ldots\alpha_{t_n}(t_n) \quad (2.3)$$

where $\alpha_{t_i}(t_i) := U(t_0, t_i)\alpha_i U(t_0, t_i)\dagger$ is the Heisenberg picture operator defined with respect to the fiducial time $t_0$.

The main assumption of the consistent-histories interpretation of quantum theory is that, under appropriate conditions, the probability assignment (2.1) is still meaningful for a closed system, with no external observers or associated measurement-induced state-vector reductions (thus signalling a move from ‘observables’ to ‘beables’). The satisfaction or otherwise of these conditions (the ‘consistency’ of a complete set of histories: see below) is determined by the behaviour of the decoherence functional $d_{(H,\rho)}$. This is the complex-valued function of pairs of homogeneous histories $\alpha = (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ and $\beta = (\beta_{t_1}, \beta_{t_2}, \ldots, \beta_{t_m})$ defined as

$$d_{(H,\rho)}(\alpha, \beta) = \text{tr}(\tilde{C}_\alpha^\dagger \rho \tilde{C}_\beta) \quad (2.4)$$

where the temporal supports of $\alpha$ and $\beta$ need not be the same. Note that, as suggested by the notation $d_{(H,\rho)}$, both the initial state and the dynamical structure (i.e., the Hamiltonian $H$) are coded in the decoherence functional. In our approach, a history $(\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ itself is just a ‘passive’, time-ordered sequence of propositions.

Two important concepts in the history formalism are coarse-graining and disjointness. It will be helpful at this point to give the relevant definitions in a way that emphasises their relation to logical operations.

- A homogeneous history $\beta := (\beta_{t_1}, \beta_{t_2}, \ldots, \beta_{t_n})$ is coarser than another such $\alpha := (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ if the temporal support of $\beta$ is equal to, or a proper subset of, the temporal support of $\alpha$, and if for every $t_i$ in the support of $\beta$, $\alpha_{t_i} \leq \beta_{t_i}$ where $\leq$ denotes the usual ordering operation on projection operators. The ensuing partial ordering on the set of homogeneous histories is written as $\alpha \leq \beta$.

- Two homogeneous histories $\alpha := (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ and $\beta := (\beta_{t_1}, \beta_{t_2}, \ldots, \beta_{t_n})$ are disjoint if their temporal supports have at least one point in common, and if for at least one such point $t_i$ the proposition $\beta_{t_i}$ is disjoint from $\alpha_{t_i}$, i.e., the ranges of the two associated projection operators are orthogonal subspaces of $\mathcal{H}$ so that $\alpha_{t_i}\beta_{t_i} = 0 = \beta_{t_i}\alpha_{t_i}$.
• The unit history assigns the unit proposition to every time $t$. The null history is represented by any history in which at least one of the propositions is the zero proposition (all such null histories are regarded as equivalent). It follows that $0 \leq \alpha \leq 1$ for all homogeneous histories $\alpha$.

A central role in the GH-scheme is played by the possibility of taking the join $\alpha \oplus \beta$ of a pair $\alpha, \beta$ of disjoint histories with the heuristic idea that $\alpha \oplus \beta$ is ‘realised’ if and only if either $\alpha$ is realised or $\beta$ is realised. This operation is straightforward if $\alpha$ and $\beta$ have the same temporal support and differ in their values at a single time point $t_i$ only. In this case, if $\alpha_{t_i}$ and $\beta_{t_i}$ are disjoint projectors then $\alpha \oplus \beta$ is equal to $\alpha$ (and therefore $\beta$) at all time points except $t_i$, where it is $\alpha_{t_i} + \beta_{t_i}$. Thus $\alpha \oplus \beta$ is a homogeneous history and satisfies the relation $\tilde{C}_{\alpha \oplus \beta} = \tilde{C}_\alpha + \tilde{C}_\beta$.

However, it is important to be able to define $\alpha \oplus \beta$ for homogeneous histories $\alpha, \beta$ that are disjoint in a less restrictive way. Similarly, one would like to have available a negation operation $\neg$ so that, heuristically, $\neg \alpha$ is realised if and only if $\alpha$ is definitely not realised. But both operations take us outside the class of homogeneous histories.

Gell-Mann and Hartle approached this issue by considering the case of a path-integral on a configuration space $Q$. The analogue of the decoherence functional (2.4) is

$$d(\alpha, \beta) := \int_{q \in \alpha, q' \in \beta} Dq Dq' e^{-i(S[q]-S[q'])/\hbar} \delta(q(t_1), q'(t_1)) \rho((q(t_0), q'(t_0))$$

(2.5)

where the integral is over paths that start at time $t_0$ and end at time $t_1$, and where $\alpha$ and $\beta$ are subsets of paths in $Q$. In this case, to say that a pair of histories $\alpha$ and $\beta$ is disjoint means simply that they are disjoint subsets of the path space of $Q$, in which case $d$ clearly possesses the additivity property

$$d(\alpha \oplus \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma)$$

(2.6)

for all subsets $\gamma$ of the path space. Similarly, $\neg \alpha$ is represented by the complement of the subset $\alpha$ of path space, in which case the decoherence functional satisfies

$$d(\neg \alpha, \gamma) = d(1, \gamma) - d(\alpha, \gamma)$$

(2.7)

where 1 denotes the entire path space (the ‘unit’ history).
Gell-Mann and Hartle noted that these properties can be replicated for the decoherence functionals on strings of projectors by postulating that, when computing a decoherence functional, one should use the class operators $\tilde{C}_{\alpha \oplus \beta}$ and $\tilde{C}_{\neg \alpha}$ that are essentially defined by

\begin{equation}
\tilde{C}_{\alpha \oplus \beta} := \tilde{C}_\alpha + \tilde{C}_\beta
\end{equation}

and

\begin{equation}
\tilde{C}_{\neg \alpha} := 1 - \tilde{C}_\alpha
\end{equation}

This procedure sounds reasonable but in some respects it is rather curious. For example, the right hand side of (2.8) is a concrete operator that ‘represents’ $\alpha \oplus \beta$ as far as calculating decoherence functionals is concerned, but it is not clear what it is that is actually being represented! A homogeneous history $\alpha$ is a time-ordered sequence $(\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ of propositions $\alpha_{t_i}$ but there is no immediate analogue for the ‘inhomogeneous’ history $\alpha \oplus \beta$. A similar remark applies to the negation $\neg \alpha$ of a homogeneous history $\alpha$.

On the other hand, (2.8–2.9) look familiar if one considers the standard operations in normal single-time quantum logic. Thus if the projection operators $\hat{P}$ and $\hat{R}$ represent single-time propositions $P$ and $R$, and if $\hat{P}$ and $\hat{R}$ are disjoint, then the proposition $P \oplus R$ is represented by $\hat{P} + \hat{R}$. Similarly, the proposition $\neg P$ is represented by the projection operator $1 - \hat{P}$. Thus (2.8) and (2.9) do suggest that some sort of logical operation is involved, but they cannot simply be justified by invoking the standard operations on projectors since, generally speaking, a product of projection operators like $\tilde{C}_\alpha$ is not itself a projector.

On reflection, these observations suggest that what is needed is a quantum form of temporal logic and, as we shall show later, this is indeed the case. However, less us first present the GH-rules for generalised quantum theory in a way that demonstrate their natural connection with the types of algebraic structure used in conventional quantum logic [15].

### 2.2 A minimal version of the Gell-Mann and Hartle scheme

The Gell-Mann and Hartle ‘axioms’ [7] constitute a new approach to quantum theory in which the notion of history is ascribed a fundamental role; i.e., a ‘history’ in this generalised sense (which is intended to include analogues of
both homogeneous and inhomogeneous histories) is an irreducible structural entity in its own right that is not necessarily to be derived from time-ordered strings of single-time propositions. We shall present our version of these axioms in a way that brings out the relation to quantum logic.

1. The fundamental ingredients in the theory are (i) a space $\mathcal{UP}$ of propositions about possible ‘histories’ (or ‘universes’); and (ii) a space $\mathcal{D}$ of decoherence functionals. A decoherence-functional is a complex-valued function of pairs $\alpha, \beta \in \mathcal{UP}$ whose value $d(\alpha, \beta)$ is a measure of the extent to which the history-propositions $\alpha$ and $\beta$ are ‘mutually incompatible’. The pair $(\mathcal{UP}, \mathcal{D})$ is to be regarded as the generalised-history analogue of the pair $(\mathcal{L}, \mathcal{S})$ in standard quantum theory.

2. The set $\mathcal{UP}$ of history-propositions is equipped with the following logical-type, algebraic operations:

   (a) A partial order $\leq$. If $\alpha \leq \beta$ then $\beta$ is said to be coarser than $\alpha$, or a coarse-graining of $\alpha$; equivalently, $\alpha$ is finer than $\beta$, or a fine-graining of $\beta$. The heuristic meaning of this relation is that $\alpha$ provides a more precise affirmation of ‘the way the universe is’ (in a transtemporal sense) than does $\beta$.

   The set $\mathcal{UP}$ possesses a unit history-proposition $1$ (heuristically, the proposition about possible histories/universes that is always true) and a null history-proposition $0$ (heuristically, the proposition that is always false). For all $\alpha \in \mathcal{UP}$ we have $0 \leq \alpha \leq 1$.

   (b) There is a notion of two history-propositions $\alpha, \beta$ being disjoint, written $\alpha \perp \beta$. Heuristically, if $\alpha \perp \beta$ then if either $\alpha$ or $\beta$ is ‘realised’ the other certainly cannot be.

   Two disjoint history-propositions $\alpha, \beta$ can be combined to form a new proposition $\alpha \oplus \beta$ which, heuristically, is the proposition ‘$\alpha$ or $\beta$’. This partial binary operation is assumed to be commutative and associative, i.e., $\alpha \oplus \beta = \beta \oplus \alpha$, and $\alpha \oplus (\beta \oplus \gamma) = (\alpha \oplus \beta) \oplus \gamma$ whenever these expressions are meaningful.

   (c) There is a negation operation $\neg \alpha$ such that, for all $\alpha \in \mathcal{UP}$, $\neg(\neg \alpha) = \alpha$. 


A crucial question is how the operations ≤, ⊕ and ¬ are to be related. This issue is not addressed explicitly in the original GH papers but we shall postulate the following, minimal, requirements:

1) ¬α is the unique element in UP such that α ⊥ ¬α with α ⊕ ¬α = 1;

2) α ≤ β if and only if there exists γ ∈ UP such that β = α ⊕ γ.

(2.10)

These conditions are certainly true of, for example, subsets of paths in a configuration space Q and, together with the other requirements above, essentially say that UP is an orthoalgebra; for a full definition see [22]. One consequence is that

α ⊥ β if and only if α ≤ ¬β.  

(2.11)

An orthoalgebra is probably the minimal useful mathematical structure that can be placed on UP, but of course that does not prohibit the occurrence of a stronger one; in particular UP could be a lattice. However, a property that could tell in favour of adopting the weaker structure is that it does not seem possible to define a satisfactory tensor product for lattices whereas this is possible for orthoalgebras [22].

3. Any decoherence functional d : UP × UP → C satisfies the following conditions:

(a) Null triviality: d(0, α) = 0 for all α.

(b) Hermiticity: d(α, β) = d(β, α)* for all α, β.

(c) Positivity: d(α, α) ≥ 0 for all α.

(d) Additivity: if α ⊥ β then, for all γ, d(α ⊕ β, γ) = d(α, γ) + d(β, γ).

If appropriate, this can be extended to countable sums.

(e) Normalisation: d(1, 1) = 1.

In addition to the above we adopt the following GH-definition: A set of history-propositions α¹, α², ..., αᴺ is said to be exclusive if αᵢ ⊥ αⱼ for all i, j = 1, 2, ..., N. The set is exhaustive (or complete) if it is exclusive and if α¹ ⊕ α² ⊕ ... ⊕ αᴺ = 1.

It must be emphasised that, within this scheme, only consistent sets of histories are graced with an immediate physical interpretation. A complete
set \( \mathcal{C} \) of history-propositions is said to be (strongly) consistent with respect to a particular decoherence functional \( d \) if \( d(\alpha, \beta) = 0 \) for all \( \alpha, \beta \in \mathcal{C} \) such that \( \alpha \neq \beta \). Under these circumstances \( d(\alpha, \alpha) \) is regarded as the probability that the history proposition \( \alpha \) is true. The GH-axioms then guarantee that the usual Kolmogoroff probability sum rules will be satisfied. However, there is currently much debate about the precise meaning in the history formalism of words like ‘realised’, ‘true’, ‘probability’ etc, and the precise, contextual ontological status of a history proposition remains elusive. This issue is of fundamental importance, but we shall not enter the lists in the present paper.

Our rules differ from the original GH set in several respects. Firstly, Gell-Mann and Hartle make considerable use of the idea of ‘fine-grained’ histories. In our language, a fine-grained history is an atom in the orthoalgebra \( \mathcal{UP} \), but we have not invoked this concept in a fundamental way since it is not clear \textit{a priori} that the algebras we might wish to consider are all necessarily atomic. Even in the history version of standard quantum theory this is problematic if one concentrates on the results of observables whose spectra are all continuous.

A more subtle difference between the two schemes is reflected by our emphasis on \textit{propositions} about histories, rather than histories themselves. This distinction may seem pedantic but it is appropriate when discussing inhomogeneous histories. For example, a two-time history proposition \((\alpha_{t_1}, \alpha_{t_2})\) is the sequential conjunction “\( \alpha_{t_1} \) is true at time \( t_1 \), and then \( \alpha_{t_2} \) is true at time \( t_2 \)” which corresponds directly to what one might want to call a history itself. On the other hand, if \( \alpha \) and \( \beta \) are two disjoint homogeneous histories, the inhomogeneous history proposition \( \alpha \oplus \beta \) affirms that “either history \( \alpha \) is realised (\textit{i.e.}, the sequential conjunction \( \alpha \) is true), or history \( \beta \) is realised” and, generally speaking, this is \textit{not} itself a sequential conjunction but rather a proposition about the pair \( \alpha, \beta \) of such. In other words, we feel that an ‘actual universe/history’ corresponds to a sequential conjunction, or analogue thereof; all other ‘histories’ are best viewed as propositions about the former.

We hope that the presentation above shows how naturally ideas of quantum logic are suggested by the GH-scheme. Specifically, the former uses the pair \((\mathcal{L}, \mathcal{S})\) of single-time propositions and states; the latter uses the pair \((\mathcal{UP}, \mathcal{D})\) of history-propositions and decoherence functionals. The sets \( \mathcal{L} \) and \( \mathcal{UP} \) are both orthoalgebras, and the axioms for states in \( \mathcal{S} \) and decoherence functionals in \( \mathcal{D} \) possess striking similarities.
Of course, the notions of time and dynamics arise in a quite different way within the two frameworks. In standard quantum logic, dynamical evolution appears as a one-parameter family of automorphisms of $\mathcal{L}$ whereas, in the history scheme—and in so far as the concept is applicable at all—dynamical evolution (and the associated notion of an ‘initial’ state) is coded in the properties of a decoherence functional.

Finally, we note several key questions that need to be addressed:

1. In some instances of the general scheme there may exist a subset $\mathcal{U}$ of $\mathcal{UP}$ that plays a role analogous to that of the homogeneous histories (i.e., sequential conjunctions) in standard quantum theory. It is tempting to identify elements of $\mathcal{U}$ as ‘possible universes’ and to expect that all elements of $\mathcal{UP}$ can be generated from $\mathcal{U}$ by logical operations. The case of standard Hamiltonian quantum theory certainly suggests that the $\oplus$ and $\neg$ operations will generally map elements of $\mathcal{U}$ out of $\mathcal{U}$ and into the full space $\mathcal{UP}$. Whether or not such a preferred subset $\mathcal{U}$ exists is closely connected with the question of whether there is any quasi-temporal structure with respect to which an analogue of the notion of sequential conjunction can be defined. We shall return to this issue in section 3.2.

2. In normal quantum logic, properties like orthomodularity or atomicity are closely related to specific physical assumptions about relationships between states and observables/propositions. For example, the seemingly innocuous relation $P \leq R$ implies $\sigma(P) \leq \sigma(R)$ for all states $\sigma \in \mathcal{S}$ has significant ramifications. An important question for the GH-scheme is whether the rules above can, or should, be strengthened by appending additional relations of this type. For example, should we add the requirement that $\alpha \leq \beta$ implies $d(\alpha, \alpha) \leq d(\beta, \beta)$ for all $d \in \mathcal{D}$? This issue will be addressed in section 4.1.

3. An important challenge is to classify the set of decoherence functionals for a given history-algebra $\mathcal{UP}$. For example, if the algebra $\mathcal{UP}$ for some system is known to be the projection lattice $\mathcal{P}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$, what is the associated space $\mathcal{D}$ of all decoherence functionals? In standard quantum logic, Gleason’s famous theorem asserts that (provided $\dim \mathcal{H} > 2$) the set of states on $\mathcal{P}(\mathcal{H})$ is exhausted by the set of density matrices. Is there an analogous result for decoherence
functionals, or is the situation more $H$-dependent than it is in standard quantum logic? Clearly, the problem of classifying decoherence functionals will depend closely on what extra rules might be appended to the minimal set given above.

4. In a standard path-integral quantum theory on a configuration space $Q$, certain topological properties of $Q$ are reflected in the decomposition of the path integral into sectors corresponding to different homotopy classes for the paths $[28, 29]$. This leads to an associated decomposition of a decoherence functional like (2.5). Can some general features be extracted from this special case that could play the analogue of topological properties for the general scheme? A discussion of this problem is given in section 4.4.

Finally, we must ask of the extent to which it really is possible to use a temporal form of quantum logic to put the suggestions above on a sound footing. In particular, does the scheme work in standard quantum theory? A key step there is to clarify the meaning of an inhomogeneous history, and to show that the collection of homogeneous and inhomogeneous histories can indeed be fitted together to form an orthoalgebra, or even perhaps a lattice. The next section is devoted to this task.

3 A Lattice Structure for History Propositions

3.1 Temporal logic and tensor products in standard quantum theory

An important step in justifying our version of the GH scheme is to show that the set of all history propositions in standard quantum theory can indeed be given the structure of a lattice. As this is discussed in detail in [15], we will here summarise only the essential ideas with the aid of two-time histories.

As we have seen, one aspect of the problem is to justify the definitions $\tilde{C}_{\alpha \oplus \beta} := \tilde{C}_\alpha + \tilde{C}_\beta$ and $\tilde{C}_{\neg \alpha} := 1 - \tilde{C}_\alpha$ which, although reminiscent of quantum logic operations on single-time projectors ($P \oplus Q = \hat{P} + \hat{Q}$ and $\neg P = 1 - \hat{P}$) cannot be justified as such since a product of projection operators like $\tilde{C}_\alpha$ is generally not itself a projector.
To illustrate how temporal logic arises naturally consider a two-time history $\alpha := (\alpha_1, \alpha_2)$ and define $D_\alpha := \alpha_1 \alpha_2$. This differs from $\tilde{C}_\alpha$ in that it does not involve the evolution operator $U(t_1, t_2)$ (since the operators $\alpha_1$ and $\alpha_2$ are in the Schrödinger picture) and hence depends only on the propositions themselves.

Then $D^\dagger_\alpha = \alpha_2 \alpha_1$ and hence $D_\alpha$ is not a projector unless $\alpha_1$ and $\alpha_2$ commute. If, nevertheless, we define $D_{\sim \alpha} := 1 - D_\alpha$ we get

$$D_{\sim \alpha} = 1 - \alpha_1 \alpha_2 \equiv (1 - \alpha_1) \alpha_2 + \alpha_1 (1 - \alpha_2) + (1 - \alpha_1)(1 - \alpha_2)$$

$$= -\alpha_1 \alpha_2 + \alpha_1 - \alpha_2 + -\alpha_1 - \alpha_2.$$  \hfill (3.1)

On the other hand, consider a temporal-logic sequential conjunction $A \sqcap B$ to be read as “$A$ is true and then $B$ is true”. Then $A \sqcap B$ is false if (i) $A$ is false and then $B$ is true, or (ii) $A$ is true and then $B$ is false, or (iii) $A$ is false and then $B$ is false; symbolically:

$$\neg (A \sqcap B) = \neg A \sqcap B \text{ or } A \sqcap \neg B \text{ or } \neg A \sqcap \neg B.$$  \hfill (3.2)

This equation is remarkably similar to (3.1) and suggests that there is indeed some close connection between temporal logic and the general assignment $D_{\sim \alpha} := 1 - D_\alpha$ even though the latter was only justified a priori by reference to integration over paths in a configuration space $Q$.

Next we observe that something resembling (3.2) also appears in the quantum theory of a pair of spin-half particles. More precisely, let $| \uparrow \rangle$ and $| \downarrow \rangle$ denote the spin-up and spin-down states for a single particle, and let $| \uparrow \rangle | \uparrow \rangle$ be the state of a pair of particles, both of which are spin-up. Thus $| \uparrow \rangle | \uparrow \rangle$ belongs to the tensor product $\mathcal{H} \otimes \mathcal{H}$ of two copies of the Hilbert space $\mathcal{H}$ of a single spin-half particle. Then a pair of particles which is not in this state is represented by

$$\neg(| \uparrow \rangle | \uparrow \rangle) = | \downarrow \rangle | \uparrow \rangle \text{ or } | \uparrow \rangle | \downarrow \rangle \text{ or } | \downarrow \rangle | \downarrow \rangle.$$  \hfill (3.3)

whose appearance is suggestively similar to that of (3.2). Of course, (3.3) is not really a meaningful quantum theory statement: at the very least, one would probably say that a state that is not $| \uparrow \rangle | \uparrow \rangle$ is some linear combination of the states on the right hand side of (3.3). Nevertheless, the similarity between (3.1), (3.2), and (3.3) is sufficient to suggest the key idea that, as far as its logical structure is concerned, a two-time homogeneous history...
$(\alpha_1, \alpha_2)$ can be realised by the operator $\alpha_1 \otimes \alpha_2$ on the tensor product $\mathcal{H} \otimes \mathcal{H}$ of two copies of the Hilbert space $\mathcal{H}$ of the original quantum theory.

In fact, this assignment does everything that is needed. For example:

- Unlike the simple product $\alpha_1 \alpha_2$, the tensor product $\alpha_1 \otimes \alpha_2$ is always a projection operator since the product of homogeneous operators on $\mathcal{H} \otimes \mathcal{H}$ is defined as $(A \otimes B)(C \otimes D) := AC \otimes BD$, while the adjoint operation is $(A \otimes B)^\dagger := A^\dagger \otimes B^\dagger$.

- Since $\alpha_1 \otimes \alpha_2$ is a genuine projection operator it is now perfectly correct to write $\neg (\alpha_1 \otimes \alpha_2) = 1 - \alpha_1 \otimes \alpha_2$ on $\mathcal{H} \otimes \mathcal{H}$, and so

$$\neg (\alpha_1 \otimes \alpha_2) = 1 - \alpha_1 \otimes \alpha_2 = (1 - \alpha_1) \otimes \alpha_2 + \alpha_1 \otimes (1 - \alpha_2) + (1 - \alpha_1) \otimes (1 - \alpha_2) = \neg \alpha_1 \otimes \alpha_2 + \alpha_1 \otimes \neg \alpha_2 + \neg \alpha_1 \otimes \neg \alpha_2 \quad (3.4)$$

which looks very much indeed like the equation (3.1) we are trying to justify.

- Let $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$ be a pair of homogeneous histories that are disjoint, i.e., $\alpha_1 \beta_1 = 0$, or $\alpha_2 \beta_2 = 0$, or both. Then, on $\mathcal{H} \otimes \mathcal{H}$, we have $(\alpha_1 \otimes \alpha_2)(\beta_1 \otimes \beta_2) = 0$ and so the projection operators $\alpha_1 \otimes \alpha_2$ and $\beta_1 \otimes \beta_2$ are disjoint. This gives rise to the bona fide equation

$$\left(\alpha_1 \otimes \alpha_2\right) \oplus \left(\beta_1 \otimes \beta_2\right) = \alpha_1 \otimes \alpha_2 + \beta_1 \otimes \beta_2 \quad (3.5)$$

which looks strikingly like the other equation $\tilde{C}_{\alpha \oplus \beta} = \tilde{C}_\alpha + \tilde{C}_\beta$ we wish to justify.

In summary, the correct logical structure of two-time propositions is obtained if they are represented on the tensor-product space $\mathcal{H} \otimes \mathcal{H}$. In particular, a homogeneous history $(\alpha_1, \alpha_2)$ is represented by a homogeneous projector $\alpha_1 \otimes \alpha_2$ and corresponds to a sequential conjunction of single-time propositions. However, other projectors exist on a tensor product space that are not of this simple form: specifically, inhomogeneous projectors like $\alpha_1 \otimes \alpha_2 + \beta_1 \otimes \beta_2$. These new projectors serve to represent what we earlier called ‘inhomogeneous’ history propositions.
Note that the class operator $\tilde{C}$ can now be viewed as a genuine map from the projectors on $\mathcal{H} \otimes \mathcal{H}$ (regarded as a subset of all linear operators on $\mathcal{H} \otimes \mathcal{H}$) to the operators on $\mathcal{H}$ that is defined on homogeneous projectors by

$$\tilde{C}(\alpha_1 \otimes \alpha_2) := U(t_0, t_1)\alpha_1 U(t_1, t_2)\alpha_2 U(t_2, t_0) \quad (3.6)$$

and then extended by linearity to the set of all projectors. This map satisfies the relations $\tilde{C}(\alpha \oplus \beta) = \tilde{C}(\alpha) + \tilde{C}(\beta)$ and $\tilde{C}(\neg \alpha) = 1 - \tilde{C}(\alpha)$ as genuine equations: it is no longer necessary to postulate them via an invocation of the analogous situation in a path-integral quantum theory.

It is straightforward to extend the discussion above to arbitrary $n$-time histories so that a homogeneous history proposition $(\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ is represented by the projector $\alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_{t_n}$ on the tensor product $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$ of $n$ copies of the original Hilbert space $\mathcal{H}$. This needs to be done for all possible choices of temporal support, and then the whole is glued together (see [13] for details) to give an infinite tensor product $\otimes_{\infty}\mathcal{H}$ whose projectors carry an accurate realisation of the space $\mathcal{UP}$ of all history propositions for this Hamiltonian quantum system. In particular, the space $\mathcal{UP}$ is thereby equipped with the structure of a non-distributive lattice.

To avoid confusion it should be emphasised that, according to the discussion above, a standard quantum-mechanical system is equipped with two quite different orthoalgebras. The first is the normal quantum-logic lattice $\mathcal{L}$ of single-time propositions associated with the projection lattice $\mathcal{P}(\mathcal{H})$ of the Hilbert space $\mathcal{H}$ of the quantum theory, and where probabilities are given by states $\sigma : \mathcal{L} \rightarrow [0, 1] \subset \mathbb{R}$. The second orthoalgebra is the lattice $\mathcal{UP}$ of ‘multi-time’ propositions that is associated with the projection lattice $\mathcal{P}(\otimes_{\infty}\mathcal{H})$ of the (much ‘larger’) Hilbert space $\otimes_{\infty}\mathcal{H}$ and where probabilities are derivable from the values of decoherence functionals $d : \mathcal{UP} \otimes \mathcal{UP} \rightarrow \mathbb{C}$ for consistent histories. It is important not to confuse these two algebraic structures.

Several constructions commonly used in the consistent histories formalism become rather natural in the language above. For example, a complete set of exhaustive and exclusive histories corresponds simply to a resolution of the identity operator in the space $\otimes_{\infty}\mathcal{H}$. In particular, this accommodates rather nicely the idea of ‘branch dependence’.

To see this, consider a two-time history system represented on $\mathcal{H}_1 \otimes \mathcal{H}_2$ in which a pair $\{P_1, P_2\}$ of disjoint propositions are the only questions that
can be asked at time \(t_1\) (so that \(\hat{P}_1 + \hat{P}_2 = 1_{\mathcal{H}_1}\)). Suppose we decide that if proposition \(P_1\) is found to be true at time \(t_1\) then at time \(t_2\) we will test an exhaustive set of disjoint propositions \(\{Q_1, Q_2, Q_3\}\) (so that \(\hat{Q}_1 + \hat{Q}_2 + \hat{Q}_3 = 1_{\mathcal{H}_2}\)), whereas if \(P_2\) is realised we will test a set \(\{R_1, R_2\}\) with \(\hat{R}_1 + \hat{R}_2 = 1_{\mathcal{H}_2}\). Then, in this simple example of branch dependence, the five possible homogeneous histories are the sequential conjunctions \(P_1 \cap Q_1, P_1 \cap Q_2, P_1 \cap Q_3, P_2 \cap R_1\) and \(P_2 \cap R_2\), which are represented respectively by the projection operators \(\hat{P}_1 \otimes \hat{Q}_1, \hat{P}_1 \otimes \hat{Q}_2, \hat{P}_1 \otimes \hat{Q}_3, \hat{P}_2 \otimes \hat{R}_1\) and \(\hat{P}_2 \otimes \hat{R}_2\) on the Hilbert space \(\mathcal{H}_1 \otimes \mathcal{H}_2\). This set of projectors is pairwise disjoint and is indeed a resolution of the identity since

\[
\hat{P}_1 \otimes \hat{Q}_1 + \hat{P}_1 \otimes \hat{Q}_2 + \hat{P}_1 \otimes \hat{Q}_3 + \hat{P}_2 \otimes \hat{R}_1 + \hat{P}_2 \otimes \hat{R}_2 = 1_{\mathcal{H}_1 \otimes \mathcal{H}_2}. \tag{3.7}
\]

It should be noted that the idea of ‘spreading out’ time points by using a tensor product was suggested earlier by Finkelstein in the context of his plexor theory [30, 31]. Using his notation, a homogeneous history \(\alpha\) would be represented by the linear graph in Figure 1 in which the vertex \(t_i\) is to be associated with the projection operator \(\alpha_{t_i}\). In computing the decoherence functional one associates the unitary operator \(U(t_i, t_{i+1})\) with the link that joins vertex \(t_i\) to vertex \(t_{i+1}\). In the context of his own study of quantum processes Finkelstein suggested generalising graphs of this type to include ones with branches as in Figure 2, and the same might be done here. For example, Figure 2 corresponds to a situation in which two systems, each with its own internal measure of time, interact at the vertex and thereafter become a single evolving system: a model perhaps for the interaction of two relativistic particles.

To form a decoherence functional using graphs like Figure 2 involves joining the free ends with an appropriate trace-type operation, and also using a vertex-operator to link together the three linear chains that meet. Another example of this type would be a consistent histories version of the spin-network ideas proposed by Penrose some years ago [32, 33, 34]. In this case the vertex-operator is constructed using an appropriate Clebsch-Gordon coefficient.

Various generalisations of the ideas above are possible. For example, a role might be found for self-adjoint operators on \(\otimes^\infty \mathcal{H}\) other than projectors: presumably they represent ‘multi-time’ observables of some sort. Also, it is not clear that every projector is generated by \(\oplus\) and \(\neg\) operations on
the homogeneous projectors; projectors that are not of this type represent
generalised history-propositions whose structure and significance remain to
be elucidated. It should also be possible to construct an analogous scheme
using a continuous tensor product. This could carry the logical structure of
propositions involving the values of ‘time-smeared’ observables like
\( \hat{q}(f) := \int dt \hat{q}(t)f(t) \).

3.2 Quasi-temporal structure

Our discussion has shown how the space \( \mathcal{UP} \) of all history propositions in
standard quantum theory can be given the structure of a non-distributive
lattice. This justifies the postulate that, in any generalised history theory,
the space \( \mathcal{UP} \) is at least an orthoalgebra, if not a full lattice.

The proof made heavy use of the existence of an ordering time variable
and, of course, that is precisely what we do not expect to be present in a
full theory of quantum gravity. However, the notions of ‘time’ and ‘homoge-
nenous history as a temporal sequence’ can be generalised considerably whilst
maintaining the ability to construct lattice models for \( \mathcal{UP} \), and structures of
this type could appear in quantum gravity in a variety of ways. This ge-
genralisation was discussed in detail in [15] and here we shall sketch only the
main ideas.

The most characteristic property of a standard sequential conjunction
\( (i.e., \) a homogeneous history) is the possibility of dividing it into two such,
one of which ‘follows’ the other. Conversely, if one sequential conjunction \( \beta \)
follows another \( \alpha \), they can be combined to give a new sequential conjunction
\( \alpha \circ \beta \) that can be read as “\( \alpha \) and then \( \beta \)”. More precisely, the time-ordered
sequence of non-trivial propositions \( \beta := (\beta_{t_1}, \beta_{t_2}, \ldots, \beta_{t_m}) \) is said to follow
\( \alpha := (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n}) \) if \( t_n < t'_1 \), in which case we define the combined se-
quence \( \alpha \circ \beta \) as

\[
\alpha \circ \beta := (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n}, \beta_{t'_1}, \beta_{t'_2}, \ldots, \beta_{t'_m}).
\]  

(3.8)

This operation satisfies the associative law \( \alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma \) whenever
both sides are defined. Thus the space \( \mathcal{U} \) of homogeneous history propositions
is a partial semi-group with respect to the combination law \( \circ \).

It is clear that the temporal properties of a homogeneous history proposi-
tion \( \alpha \in \mathcal{U} \) are encoded in its temporal support. The set \( \text{Sup} \) of all temporal
supports is itself a partial semi-group with respect to the composition

\[ s_1 \circ s_2 := \{ t_1, t_2, \ldots, t_n, t'_1, t'_2, \ldots, t'_m \}. \tag{3.9} \]

which is defined whenever the support \( s_2 := \{ t'_1, t'_2, \ldots, t'_m \} \) follows the support \( s_1 := \{ t_1, t_2, \ldots, t_n \} \), i.e., when \( t_n < t'_1 \). The relation between the structures on \( U \) and \( Sup \) is captured by the support map \( \nu : U \to Sup \) that associates with each homogeneous history its temporal support. This map is a homomorphism between these two partial semi-groups, i.e., \( \nu(\alpha \circ \beta) = \nu(\alpha) \circ \nu(\beta) \) whenever both sides of the equation are defined (in this operation, the null and unit histories are assigned to a ‘base point’ \( * \) in \( Sup \) which satisfies \( s \circ * = * \circ s = s \) for all \( s \in Sup \)).

Note that, ab initio, a temporal support is assigned only to homogeneous histories. Generally speaking the concept, is not useful outside this class; for example if \( \alpha \) and \( \beta \) are disjoint homogeneous histories with different temporal supports then the inhomogeneous history proposition \( \alpha \oplus \beta \) means that either \( \alpha \) is realised, or \( \beta \) is realised, which refers to two different sets of time points.

This way of looking at the role of time in standard quantum theory can be generalised considerably to give a ‘quasi-temporal’ theory in which the basic ingredients are (i) a partial semigroup \( U \) of possible universes (a generalisation of the idea of a sequential conjunction); (ii) a partial semigroup \( Sup \) of temporal supports (a generalisation of the space of finite ordered subsets of the timeline \( \mathbb{R} \)); (iii) a homomorphism \( \nu : U \to Sup \); and (iv) an orthoalgebra (possibly a full lattice) \( UP \) of ‘propositions concerning possible universes’ that is generated from the preferred subset \( U \) by the logical operations \( \oplus \) and \( \neg \).

If \( \alpha, \beta \in U \) are such that they can be combined to form \( \alpha \circ \beta \) we say that \( \alpha \) precedes \( \beta \) (written as \( \alpha \prec \beta \)), or \( \beta \) follows \( \alpha \). Broadly speaking, the idea is that if \( \beta \) follows \( \alpha \) then there is a possibility of some causal influence (perhaps in a rather unusual sense) of events ‘localised’ in the history \( \alpha \) on those localised in \( \beta \).

It should be emphasised that the relation of ‘following’ may not be transitive; i.e., \( \alpha \prec \beta \) and \( \beta \prec \gamma \) need not imply \( \alpha \prec \gamma \); in particular, we do not suppose that \( \prec \) is a partial order on \( U \). This is because the set of events ‘localised’ in \( \beta \) that are causally affected by events in \( \alpha \) need not include any events that causally affect events in \( \gamma \); an explicit example of this phenomenon is the causal relations between compact regions in a Lorentzian spacetime \[35\]. We
might also allow the possibility of chains $\alpha^1 \triangleleft \alpha^2 \triangleleft \cdots \triangleleft \alpha^n \triangleleft \alpha^1$, representing a generalisation of the idea of a closed time-like loop.

In standard quantum theory, any homogeneous history $\alpha := (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$ with support $\{t_1, t_2, \ldots, t_n\}$ can be written as the composition

$$\alpha = \alpha_{t_1} \circ \alpha_{t_2} \circ \ldots \circ \alpha_{t_n} \quad (3.10)$$

in which each single-time proposition $\alpha_{t_i}$, $i = 1, 2, \ldots, n$ is regarded as a homogeneous history whose temporal support is the singleton set $\{t_i\}$.

In a general history theory it is important to know when a homogeneous history proposition can be decomposed into the product of others. In this context we say that $\alpha \in U$ is a \textit{nuclear} proposition if it cannot be written in the form $\alpha = \alpha_1 \circ \alpha_2$ with both constituents $\alpha_1, \alpha_2 \in U$ being different from the unit history 1. Similarly, a support $s \in Sup$ is nuclear if it cannot be written in the form $s = s_1 \circ s_2$ with $s_1, s_2 \in Sup$ being different from the unit support $\ast$. Finally, we say that a decomposition of $\alpha$ of the form $\alpha = \alpha_1 \circ \alpha_2 \circ \ldots \alpha_N$ is \textit{irreducible} if the constituent history propositions $\alpha_i$, $i = 1 \ldots N$, are all nuclear.

Note that in the decomposition (3.10) of standard quantum theory, the constituents $\alpha_{t_i}$ are nuclear histories in the sense above, and $\{t_i\}$ is a nuclear support. Thus, in a general history theory, a nuclear support is an analogue of a ‘point of time’; in particular, it admits no further temporal-type subdivisions. Similarly, a nuclear history proposition is a general analogue of a single-time proposition. This observation raises the possibility of finding a direct analogue of the tensor-product construction discussed above for standard Hamiltonian physics. By this means one is enabled to construct explicit lattice realisations for the space of history propositions of a wide range of hypothetical systems. However note that even if nuclear supports exist there is no \textit{a priori} reason why the propositions at different ‘points of time’ should belong to isomorphic structures, in which case one needs an obvious generalisation of the tensor-product structure used above where the Hilbert spaces $\mathcal{H}_t$ are all naturally isomorphic.

4 The Space $\mathcal{D}$ of Decoherence Functionals

22
4.1 Some counter-examples in standard quantum theory

The postulated properties of decoherence functionals (see section 2.2) are

1. **Null triviality**: \( d(0, \alpha) = 0 \) for all \( \alpha \in \mathcal{UP} \).

2. **Hermiticity**: \( d(\alpha, \beta) = d(\beta, \alpha)^* \) for all \( \alpha, \beta \in \mathcal{UP} \).

3. **Positivity**: \( d(\alpha, \alpha) \geq 0 \) for all \( \alpha \in \mathcal{UP} \).

4. **Additivity**: if \( \alpha \perp \beta \) then, for all \( \gamma \), \( d(\alpha \oplus \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma) \). If appropriate, this can be extended to countable sums.

5. **Normalisation**: \( d(1, 1) = 1 \).

This is a minimal set of requirements and it is important to know if any further conditions should be added to the list, particularly vis-a-vis the problem of constructing and classifying decoherence functionals.

Of particular importance are potential relations between the partial-order operation on \( \mathcal{UP} \) and the ordering of the real numbers. For example, in standard quantum logic suppose \( P, R \in \mathcal{L} \) satisfy \( P \leq R \). Then, since \( \mathcal{L} \) is an orthoalgebra, there exists \( V \in \mathcal{L} \) such that \( R = P \oplus V \) and hence, if \( \sigma \in \mathcal{S} \) is any state, the additivity property \( \sigma(P \oplus V) = \sigma(P) + \sigma(V) \) implies at once that \( \sigma(P) \leq \sigma(R) \). This is correct physically since \( P \leq R \) means that \( P \) implies \( R \), and it is reasonable to interpret this as saying that, in all states, the probability that \( P \) is true is less than the probability that \( R \) is true.

It is a salutary experience to explore analogous relations in the history formalism, and we shall now consider several plausible-sounding inequalities of this type, all of which turn out to be false in standard quantum theory!

**Posited inequality 1.**

For all \( d \in \mathcal{D} \) and for all \( \alpha, \beta \) with \( \alpha \leq \beta \) we have \( d(\alpha, \alpha) \leq d(\beta, \beta) \). (4.1)

This inequality seems reasonable, not least because it clearly is true when applied to sequences of projectors onto subsets of configuration space in a path-integral quantum theory. Since, by assumption, \( \alpha \leq 1 \) for all \( \alpha \in \mathcal{UP} \), the inequality—if true—would also imply \( d(\alpha, \alpha) \leq 1 \), which again looks
plausible if one bears in mind the potential probabilistic interpretation of $d(\alpha, \alpha)$.

On the other hand, for any given decoherence functional $d$ and history proposition $\alpha$, the real number $d(\alpha, \alpha)$ is interpretable as a probability only if $\alpha$ is a member of a set that is consistent with respect to $d$. Thus, although $\alpha \leq \beta$ is supposed to mean that $\alpha$ implies $\beta$ (in the sense that the latter is a ‘coarse-graining’ of the former, and hence constitutes a less restrictive specification of the ‘history of the universe’) it is not clear that this should necessarily imply $d(\alpha, \alpha) \leq d(\beta, \beta)$; this certainly cannot be derived as a theorem from the GH rules.

In fact, the suggested inequality (4.1) is violated in standard quantum theory. To see this, consider a model system with an initial state $\rho := |\phi\rangle\langle\phi|$ (all vectors are assumed to be of unit length) and a two-time history $(\alpha_1, \alpha_2)$ where $\alpha_1(t_1) := |\chi\rangle\langle\chi|$, $\alpha_2(t_2) := |\psi\rangle\langle\psi|$, and where $|\psi\rangle$ is chosen so that $\langle\phi|\psi\rangle = 0$ and $|\chi\rangle$ is defined by $|\chi\rangle := \frac{1}{\sqrt{2}}(|\phi\rangle + |\psi\rangle)$ as illustrated in Figure 3. Then

$$\alpha_2(t_2)\alpha_1(t_1)\rho\alpha_1(t_1)\alpha_2(t_2) = |\psi\rangle\langle\psi|\langle\chi|\phi\rangle\langle\phi|\chi\rangle\langle\chi|\psi\rangle\langle\psi| = \frac{1}{4}|\psi\rangle\langle\psi|$$

and so the corresponding decoherence functional has the value

$$d(\alpha, \alpha) = \text{tr}(\alpha_2(t_2)\alpha_1(t_1)\rho\alpha_1(t_1)\alpha_2(t_2)) = \frac{1}{4}\text{tr}(|\psi\rangle\langle\psi|) = \frac{1}{4}. \quad (4.3)$$

Now coarse-grain $\alpha$ by replacing the first projector $\alpha_1$ with the unit operator so that the new history is $\beta := (1, \alpha_2)$ (this is really a one-time history with temporal support $\{t_2\}$). Then, by definition, $\alpha \leq \beta$. On the other hand, the condition $\langle\phi|\psi\rangle = 0$ shows at once that $d(\beta, \beta) = 0$, hence violating the supposed inequality $d(\alpha, \alpha) \leq d(\beta, \beta)$. Physically, this is the classic quantum-mechanical effect whereby a plane-polarised wave can be rotated through an angle of 90° by first inserting a polarisor at 45° to the plane of the wave, and then inserting another at 45° to the first!

Posited inequality 2. \[ \alpha \perp \beta \implies d(\alpha, \alpha) + d(\beta, \beta) \leq 1 \text{ for all } d \in \mathcal{D}. \quad (4.4) \]
The analogue of this relation in standard quantum logic is $P \perp R$ implies $\sigma(P) + \sigma(R) \leq 1$. This implies that if $P$ is true with probability 1 in a state $\sigma$ (so that $\sigma(P) = 1$) then $R$ is necessarily false (i.e., $\sigma(R) = 0$), and vice-versa. This gives a very acceptable physical meaning of what it means to say that two propositions are disjoint. It also plays a central role in Mackey’s justification of the assumption that $L$ means to say that two propositions are disjoint. This gives a very acceptable physical meaning of what it means to say that two propositions are disjoint. It also plays a central role in Mackey’s justification of the assumption that $L$ is orthomodular [36], and it would be nice if our assumption that $\mathcal{UP}$ is orthomodular could be justified in a similar way. However, as we shall see, the inequality (4.4) is violated by normal quantum theory.

To find a suitable counter-example consider a pair of two-time histories $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$ so that

$$d(\alpha, \alpha) + d(\beta, \beta) = \text{tr}(\alpha_2(t_2)\alpha_1(t_1)\rho\alpha_1(t_1)\alpha_2(t_2)) + \text{tr}(\beta_2(t_2)\beta_1(t_1)\rho\beta_1(t_1)\beta_2(t_2)), \quad (4.5)$$

and make them disjoint by setting $\beta_2(t_2) = 1 - \alpha_2(t_2)$. Now make the following additional choices: $\rho = |\phi\rangle\langle\phi|$, $\alpha_2(t_2) := |\psi\rangle\langle\psi|$ and $\beta_1(t_1) := |\phi\rangle\langle\phi|$, where $\langle\psi|\phi\rangle = 0$ and all vectors are assumed to be of unit length. Then

$$\beta_2(t_2)\beta_1(t_1) = (1 - |\psi\rangle\langle\psi|)|\phi\rangle\langle\phi| = |\phi\rangle\langle\phi| = \beta_1(t_1)$$
$$\beta_1(t_1)\beta_2(t_2) = |\phi\rangle\langle\phi|(1 - |\psi\rangle\langle\psi|) = |\phi\rangle\langle\phi| = \beta_1(t_1) \quad (4.6)$$

so that $\beta_2(t_2)\beta_1(t_1)\rho\beta_1(t_1)\beta_2(t_2) = |\phi\rangle\langle\phi|$, whose trace is one. Thus, to violate the posited inequality (4.4) it suffices to find an $\alpha_1$ so that the trace satisfies $\text{tr}(\alpha_2(t_2)\alpha_1(t_1)\rho\alpha_1(t_1)\alpha_2(t_2)) > 0$. But this is easy: use the trick employed above in finding a counter-example to (4.4) and select $\alpha_1(t_1)$ to be the projector onto the normalised vector $\frac{1}{\sqrt{2}}(|\phi\rangle + |\psi\rangle)$. Then

$$\alpha_1(t_1)\rho\alpha_1(t_1) = \frac{1}{4}(|\phi\rangle + |\psi\rangle)(\langle\psi| + \langle\phi|)|\phi\rangle\langle\phi|(|\phi\rangle + |\psi\rangle)((\psi| + \langle\phi|) = \frac{1}{4}(|\phi\rangle + |\psi\rangle)(\langle\psi| + \langle\phi|) \quad (4.7)$$

and so $\alpha_2(t_2)\alpha_1(t_1)\rho\alpha_1(t_1)\alpha_2(t_2) = \frac{1}{4}|\psi\rangle\langle\psi|$ whose trace is $\frac{1}{4}$. Thus $d(\alpha, \alpha) + d(\beta, \beta) = \frac{2}{4}$ which violates the suggested inequality (4.4).

Posited inequality 3.

For all $d \in \mathcal{D}$ and all $\gamma \in \mathcal{UP}$ we have $d(\gamma, \gamma) \leq 1. \quad (4.8)$
This cannot be derived from (4.1) as we have already shown that the latter is violated by normal quantum theory. However (4.8) is much weaker than (4.1) and therefore might be correct even if the latter is false. The inequality (4.8) is certainly true for any homogeneous history in standard quantum theory for any unitary-evolution type of decoherence functional \( d_{(H,\rho)} \) (see (2.4)) because the norm of any projection operator is bounded above by 1.

However the inequality (4.8) is not true for all inhomogeneous histories. Indeed, the proof that (4.4) is violated employed a pair of histories \( \alpha, \beta \) that are disjoint by virtue of the fact that \( \alpha_2 \beta_2 = 0 \). This means that \( d(\alpha, \beta) = 0 \), and hence

\[
\begin{align*}
\text{d}(\alpha \oplus \beta, \alpha \oplus \beta) &= \text{d}(\alpha, \alpha) + \text{d}(\beta, \beta).
\end{align*}
\] (4.9)

But the right hand side of (4.9) was shown to have a value of \( \frac{5}{4} \), and hence the posited inequality (4.8) is violated. Of course, this does not mean that \( d(\gamma, \gamma) > 1 \) for all inhomogeneous histories: using the type of history constructed in the counter-example to inequality (4.1) it is easy to find inhomogeneous histories of the form \( \gamma = \alpha \oplus \beta \) for which \( d(\gamma, \gamma) < 1 \).

We note in passing that there exist ‘non-standard’ decoherence functionals based on non-unitary time evolution for which (4.4) may be violated even for homogeneous histories; see the later discussion concerning (4.19).

The fact that the, otherwise attractive, postulated inequalities are all violated by standard quantum theory does not necessarily prohibit one or more of them being appended to the GH rules for the generalised theory. Indeed, one reason for invoking this scheme as a possible framework for quantum gravity was to suggest that normal quantum theory emerges only in some coarse-grained sense. If so, there is no a priori reason for requiring standard Hilbert-space based quantum theory to satisfy exactly all the rules of the generalised theory: it suffices that they are obeyed in the appropriate limit. However, this raises so many new possibilities that, at least in this paper, we shall adopt the cautious position of postulating no properties for the decoherence functionals that are not possessed exactly by normal quantum theory.

### 4.2 Construction of decoherence functionals

With this in mind we turn now to the problem of what can be said in general about the construction of decoherence functionals that satisfy just the
minimum requirements in the GH rules. One property that follows immediately from these is that (like the set $S$ of states in normal quantum logic) the space $D$ of decoherence functionals is a real convex set, i.e., if $d_1$ and $d_2$ are decoherence functionals then so is $rd_1 + (1-r)d_2$ for any real number $r$ such that $0 \leq r \leq 1$. However, as emphasised already, we do not even know the full classification of decoherence functionals when $\mathcal{UP}$ is the lattice of subspaces of a Hilbert space, and so our answer to the general question is necessarily very incomplete. Nevertheless, there are a few general remarks worth making.

Of the five listed requirements for a decoherence functional, the most important (and potentially therefore the hardest to achieve) is the additivity condition

$$d(\alpha \oplus \beta, \gamma) = d(\alpha, \gamma) + d(\beta, \gamma).$$

Focussing on this, one class of decoherence functionals can be obtained in the following way.

Let $\chi : \mathcal{UP} \to \mathbb{C}$ be normalised by $|\chi(1)|^2 = 1$ and $\chi(0) = 0$, and satisfy the additive relation

$$\chi(\alpha \oplus \beta) = \chi(\alpha) + \chi(\beta) \quad (4.10)$$

so that $\chi$ is a character for the partial semigroup associated with the $\oplus$ operation. Then a function satisfying all the conditions for a decoherence functional can be defined by

$$d_\chi(\alpha, \beta) := \chi(\alpha)^* \chi(\beta). \quad (4.11)$$

For example, suppose the ‘histories’ $\alpha$ are all subsets of some space $X$ equipped with a probability measure $\mu$ i.e., $\int_X d\mu(x) = 1$. Then

$$\chi_\mu(\alpha) := \int_{x \in \alpha} d\mu(x) \quad (4.12)$$

satisfies (4.10) by virtue of the additivity of an integral over disjoint subsets of the space on which it is defined. The decoherence functionals of path-integral quantum theory are essentially of this form. This example also suggests that if quasi-temporal structure exists in the general theory it is appropriate to add to (4.10) the requirement that if $\beta$ follows $\alpha$ then

$$\chi(\alpha \circ \beta) = \chi(\alpha) \chi(\beta). \quad (4.13)$$

In the case of path-integral quantum theory this reflects the basic propagator property of integrals over paths in a configuration space.
This method of generating decoherence functionals can be generalised substantially. Specifically, let $\mathcal{A}$ be any von Neumann algebra of operators equipped with a complex-valued function $\phi: \mathcal{A} \to \mathbb{C}$ which satisfies $\phi(AB) = \phi(BA)$ and $\phi(A^\dagger) = \phi(A)^*$ (i.e., $\phi$ is a trace on $\mathcal{A}$). Now let $\chi: \mathcal{U} \to \mathcal{A}$ be such that

\begin{align}
\text{Additivity} & : \chi(\alpha + \beta) = \chi(\alpha) + \chi(b) \\
\text{Evolution} & : \alpha \triangleleft \beta \implies \chi(\alpha \circ \beta) = \chi(\alpha)\chi(\beta)
\end{align}

and with $\chi(0) = 0$. Then, for each such pair $(\chi, \phi)$, a decoherence functional can be defined by

$$d_{(\chi,\phi)}(\alpha, \beta) := \phi(\chi(\alpha)^\dagger \chi(\beta))$$

provided $\chi$ and $\phi$ are normalised so that $\phi(\chi(1)^\dagger \chi(1)) = 1$.

This construction includes standard quantum theory if we make the following special choices: (i) $\mathcal{A}$ is the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on the Hilbert space $\mathcal{H}$ of the quantum mechanical system; (ii) if $\rho$ is the initial state then $\phi(A) := \text{tr}(A\rho)$; (iii) the map $\chi$ is the $\tilde{C}$-map of (2.2). From this perspective one sees that the use of a non-trivial von Neumann algebra $\mathcal{A}$ corresponds to a type of non-commutative version of the integral (4.12). This general mathematical idea has been developed in depth by Connes [37].

This example of standard quantum theory can itself be generalised in a significant way. Consider the following map on a $n$-time homogeneous history $\alpha = (\alpha_{t_1}, \alpha_{t_2}, \ldots, \alpha_{t_n})$

$$\chi(\alpha) := K(t_0, t_1)\alpha_{t_1}K(t_1, t_2)\alpha_{t_2}K(t_2, t_3) \ldots K(t_{n-1}, t_n)\alpha_{t_n}K(t_n, t_{n+1})$$

where, for the moment, $K(t_{i-1}, t_i)$, $i = 1, 2, \ldots, n+1$, is an arbitrary set of operators on $\mathcal{H}$, and $t_{n+1} > t_n$ is some fiducial ‘final’ time (a ‘final’ analogue of the initial time $t_0$). Equation (4.17) defines a $\mathcal{B}(\mathcal{H})$-valued multilinear function of the operators on $\mathcal{H}_{t_1} \otimes \mathcal{H}_{t_2} \otimes \cdots \otimes \mathcal{H}_{t_n}$ and hence can be extended uniquely to a function on the inhomogeneous histories as well.

As it stands, if used in a decoherence functional, this function violates a condition that seems very desirable. Namely, let $\alpha$ be a homogeneous history proposition $(\alpha_1, \alpha_2, \ldots, \alpha_{t-1}, \alpha_t, \alpha_{t+1}, \ldots, \alpha_n)$ and consider the value $d(\alpha, \beta)$ as $\alpha_t$ tends towards the unit proposition 1. The insertion of the unit proposition at any given time should make no difference, and hence, for any $\beta$, in the limit as $\alpha_t = 1$, $d(\alpha, \beta)$ should equal $d(\alpha', \beta)$ where $\alpha' := \alpha_1 \cdots \alpha_{t-1} \alpha_{t+1} \cdots \alpha_n$.
$(\alpha_1, \alpha_2, \ldots, \alpha_{t_i-1}, \alpha_{t_i+1}, \ldots, \alpha_{t_n})$. However, the map defined in (1.17) satisfies this requirement only if the $K$-operators obey the evolution property

$$K(t_{i-1}, t_i)K(t_i, t_{i+1}) = K(t_{i-1}, t_{i+1})$$

(4.18)

which we shall now assume to be the case. Then, provided the positive operator $K(t_0, t_f) \rho K(t_0, t_f) \geq 0$ has a non-zero trace, a decoherence functional can be defined by

$$d_{(K, \rho)}(\alpha, \beta) := \frac{1}{\text{tr}(K(t_0, t_f) \rho K(t_0, t_f))} \left\{ \text{tr}(K(t_n, t_f) \rho K(t_0, t_f)) \alpha_{t_n}K(t_{n-1}, t_n) \ldots \right.$$

$$
\ldots \alpha_{t_2} K(t_1, t_2) \rho K(t_0, t_1) \beta_{t_1} K(t'_1, t'_2) \beta_{t'_2} \ldots
$$

$$\ldots K(t'_{m-1}, t'_m) \beta_{t'_m} K(t'_m, t_f)) \right\}$$

(4.19)

where $t_f$ is a choice for final time that is greater than both $t_n$ and $t'_m$. It would also be possible to add a ‘final’ density matrix $\rho$ at the time $t_f$ but, in effect, this is already covered by the arbitrary nature of the final $K$-operators $K(t_n, t_f)$ and $K(t'_m, t_f)$.

Note that the ‘evolution’ operators $K(t, s)$ need not be unitary for the construction above to work. Thus (4.19) could describe a system with non-unitary time evolution. Indeed, Hartle has used a decoherence functional of precisely this form in his analysis of the application of the generalised quantum theory to spacetimes with closed timelike curves [38]. We note also in passing that a construction of this type would work equally well for any quasi-temporal theory that admits nuclear supports.

We mentioned earlier that for a standard decoherence functional of the type $d_{(H, \rho)}$ in (2.4) (i.e., corresponding to a unitary evolution) it is trivial to see that $d_{(H, \rho)}(\alpha, \alpha) \leq 1$ for all homogeneous histories $\alpha$. However, this is no longer the case for a general $K$. Using techniques similar to those employed in discussing (4.1) it is not difficult to construct examples in which the numerator of (4.19) is bounded below whilst the denominator becomes arbitrarily small. Thus, even for homogeneous histories $\alpha$, $d_{(K, \rho)}(\alpha, \alpha)$ can be unbounded for non-unitary time evolution.

It is tempting to speculate that (4.19) is the most general non-trivial form of decoherence functional in standard quantum theory. If not, it would be interesting to know what additional properties of $d \in \mathcal{D}$ must added to the GH rules to ensure that it is so.
4.3 The adequacy of configuration space

One of the striking features of standard path-integration is that when the state space of the classical system is just a cotangent bundle $T^*Q$, all the quantum theory results can be obtained from an integral over paths that lie just in the configuration space $Q$. This suggests that in the analogous history formalism using time-ordered strings of projectors there must be situations in which the values of a decoherence functional $d$ on strings of projectors onto subsets of $Q$ determines the values of $d$ on arbitrary homogeneous histories. The collection of homogeneous histories constructed from projections only onto subsets of $Q$ forms a subalgebra of $\mathcal{U}$ that is Boolean, which suggests that in the general history scheme there may be situations in which values of a decoherence functional can be recovered from its values on some preferred Boolean subalgebra of $\mathcal{U}$.

We will illustrate this effect in standard quantum theory and show the precise sense in which a Boolean subalgebra of $\mathcal{U}$ is sufficient. To this end consider any maximal abelian subalgebra of the observables in the quantum theory. The maximality condition implies the existence of a selfadjoint operator $A$ whose spectrum is totally non-degenerate and with the property that any member of the subalgebra can be written as a function of $A$ (a nice discussion from a physicist’s perspective is contained in [39]). For ease of exposition we will suppose that the spectrum of $A$ is discrete with spectral projectors $P_m = |m\rangle\langle m|$ so that $A = \sum m a_m P_m$ and $\sum_m P_m = 1$.

The analogue of a string of projectors onto subsets of $Q$ is a homogeneous history that is a time-ordered string of the projectors $P_n$ only. Then the results of evaluating decoherence functionals on pairs of this sort includes all the numbers

$$G_{nm} := \text{tr}(XP_nU(t,t')P_m)$$

where $t < t'$ are neighbouring time points and $X$ denotes the product of all the other operators that make up the two homogeneous histories and the initial state $\rho$.

Now suppose that $R$ is any projector on the Hilbert space. Then

$$\text{tr}(XR) = \sum_{n,m} X_{nm} R_{mn}$$

where $X_{nm} := \langle n | X | m \rangle$ and $R_{mn} := \langle m | R | n \rangle$. On the other hand

$$G_{nm} = \text{tr}(X|n\rangle\langle n|U(t,t')|m\rangle\langle m|) = X_{mn} U_{nm}$$
which, if \( U_{nm} \neq 0 \), can be solved for \( X_{mn} \) as \( X_{mn} = G_{nm}/U_{nm} \) (no sum over \( n \) or \( m \)), and then substituted into (4.21) to give

\[
\text{tr}(XR) = \sum_{n,m} \frac{G_{mn}}{U_{mn}} R_{mn}.
\] (4.23)

Bearing in mind the original meaning of \( X \), we see that (4.23) is itself a decoherence functional of two homogeneous histories but with the projector \( R \) replacing the pair \( P_nU(t,t')P_m \) in the appropriate original history. Conversely, a decoherence functional of any pair of homogeneous histories can be constructed by following the procedure above in reverse and inserting extra time slots and \( P_nU(t,t')P_m \) factors as required. In this way any decoherence functional can be calculated from its values on homogeneous histories of strings of \( P_n \) projectors.

This proves our claim that it is sufficient to know the values of a decoherence functional on strings of projectors belonging to a maximal abelian subalgebra. The set of all such strings form a Boolean sublattice of \( \mathcal{UP} \). The only proviso is that the dynamical evolution and maximal abelian subalgebra must be such that the appropriate matrix elements \( \langle n|U(t,s)|m \rangle \) are non-zero. In practice this is easy to ensure. For example, the case of paths in a configuration space \( Q \) can be incorporated by generalising the situation above to include operators with continuous spectra, in which case the object of interest is the propagator kernel

\[
U(q,t; q', t') := \langle q'|e^{-i(t'-t)H/\hbar}|q \rangle
\] (4.24)

and it is easy to see that the non-vanishing condition is satisfied for a typical non-relativistic Hamiltonian \( H = p^2/2m + V(q) \).

It is worth noting that the potential role of a preferred Boolean subalgebra of \( \mathcal{UP} \) can be seen from another perspective. For any given Hamiltonian there is a natural automorphism \( \tau \) of \( \mathcal{UP} \) generated by the map of homogeneous histories defined by

\[
\alpha_{t_1} \otimes \alpha_{t_2} \otimes \cdots \otimes \alpha_{t_n} \mapsto \alpha_{t_1}(t_1) \otimes \alpha_{t_2}(t_2) \otimes \cdots \otimes \alpha_{t_n}(t_n)
\] (4.25)

where \( \alpha_{t_i}(t_i) \) are the Heisenberg picture operators associated with the Hamiltonian. It is a curious fact that this single automorphism of \( \mathcal{UP} \) encodes all the information about the dynamical evolution at all times! In particular, if
one starts with a homogeneous history $\alpha$ made up of strings of projectors onto a particular abelian subalgebra then, for reasons similar to those discussed above, the transformed history $\tau(\alpha)$ will generally lie outside this preferred subset. Thus the lattice operations between $\alpha$ and $\tau(\alpha)$ code into the algebraic structure of UP important information about the dynamical structure of the system. This could be an important idea for future developments of the general scheme.

4.4 Topological effects

Potential roles for topological structure in quantum theory arise in a variety of ways. For example, the $\theta$-angle for a system with a non simply-connected configuration space $Q$ can be justified by considering the definition of a path-integral on $Q$. In particular, this suggests\cite{28, 29} the existence of a parametrised family of propagator functions $K_\chi$ of the form

$$K_\chi(q_1, t_1; q_2, t_2) = \sum_{\gamma} \chi(\gamma) K_\gamma(q_1, t_1; q_2, t_2)$$

(4.26)

where $\chi$ is a character of the fundamental group $\pi_1(Q)$ of $Q$, and the ‘partial’ propagator $K_\gamma(q_1, t_1; q_2, t_2)$ is constructed by integrating over paths in $Q$ that belong only to a specific homotopy class $\gamma$ (identified as an element of $\pi_1(Q)$ via, for example, a choice of reference path from $q_1$ to $q_2$).

The same group arises in various approaches to canonical quantisation. For example, the quantum version of a classical theory with a configuration space $Q$ might be specified by defining the Hilbert space of states to be the vector space of cross-sections of some flat vector bundle over $Q$—the flatness condition guarantees that no potential-energy term occurs in the quantum Hamiltonian (at least, locally in $Q$) that was not present already in the classical theory. The group $\pi_1(Q)$ appears because flat $\mathbb{C}^n$ bundles over $Q$ are classified by the group of homomorphisms of $\pi_1(Q)$ into $U(n)$\cite{40}. A similar structure arises in the pre-quantisation phase of geometric quantisation in which certain types of vector bundle are constructed over the phase space of the classical system.

It is important to note that global properties of a classical system can have important quantum effects even though they are not literally ‘topological’. A good example is if $Q$ is the (contractible) space of positive-definite, $n \times n$ symmetric matrices: a natural model for the quantisation of a metric function
in quantum gravity. This space is trivial topologically, but important effects arise from identifying the canonical group of the system as the semidirect product of \( \mathbb{R}^{n(n+1)} \) with \( GL^+(n, \mathbb{R}) \) rather than the Heisenberg group that is appropriate for a vector space of the same dimension [41, 42, 43].

This example belongs to the general class of systems whose configuration space is a homogeneous space \( G/H \). Systems of this type have been quantised canonically in a variety of ways, all of which agree on the existence of a number of inequivalent quantum versions of the given classical theory (defined by its Poisson algebra of observables) that are labelled by the irreducible representations of \( H \), and which therefore generally depend only distantly on topological properties of \( G/H \).

Note that the effects mentioned above all arise from quantising a given classical system. It is a quite different matter to ask for analogues of global or topological effects in a quantum theory that is constructed \textit{ab initio} in its own right. This question is particularly interesting in the context of the GH scheme. However, if one starts with a general \((\mathcal{U}\mathcal{P}, \mathcal{D})\) pair it is difficult to know how to proceed. The space \( \mathcal{D} \) of decoherence functionals is contractible, and hence rather uninteresting topologically. The situation with \( \mathcal{U}\mathcal{P} \) is different since a space of this type could well be equipped with a non-trivial topology. For example, the projectors on a finite-dimensional vector space are naturally elements of various Grassmann manifolds \( U(n + m)/U(n) \times U(m) \) which are topologically complex. However, there is no obvious role for topological structure of this type other than, perhaps, requiring decoherence functionals to be \textit{continuous} functions on \( \mathcal{U}\mathcal{P} \) (as we assumed implicitly in the discussion justifying the assumption that the \( K \) operators satisfy the product law \([15, 16])\). In fact, the group-theoretical examples cited above suggest that important global effects may arise that are not related directly to any topological properties.

In this paper we shall concentrate only on the easier question of identifying an analogue in the history formalism of the topological and global properties of conventional path-integrals. The perspective afforded by the histories approach has the additional benefit of allowing us to exhibit a generalisation of the \( \pi_1(Q) \) effects of conventional quantum theory to a situation in which one has an arbitrary principle bundle \( H \to P \to Q \) over \( Q \); the \( \pi_1(Q) \) effects come from the special case in which this is the familiar bundle \( \pi_1(Q) \to \tilde{Q} \to Q \) where \( \tilde{Q} \) is the universal covering manifold of \( Q \).

We shall restrict our attention to systems with a quasi-temporal struc-
ture, in which case the dynamical information is coded in the decoherence functionals (or, at least, those constructed using a form like (4.19)) via the evolution operators $K$ that act on the Hilbert space $\mathcal{H}$ of the single-time quantum system (i.e., the Hilbert space associated with single-time propositions in the sense of nuclear supports) and which, for all $s_1$ and $s_2$ such that $s_1 \prec s_2$, must satisfy the relation

$$K(s_1)K(s_2) = K(s_1 \circ s_2)$$  \hspace{1cm} (4.27)

in order to be consistent with the insertion of the unit single-time proposition. Thus $K: \text{Sup} \to \mathcal{B}(\mathcal{H})$ is a representation on $\mathcal{H}$ of the partial semigroup $\text{Sup}$ of quasi-temporal supports (in general we might have a more sheaf-like structure in which each ‘time-point’ is associated with its own Hilbert space and logic of single-time propositions; in this case $K(s)$ is to be regarded as a linear map between the Hilbert spaces associated with the two ‘ends’ of the temporal support $s$).

The analogous object in a path integral can be decomposed into a sum (4.26) of partial propagators $K_\gamma$ obtained by integrating only over paths in the homotopy class $\gamma$. In particular, for $Q \simeq S^1$ we have $\pi_1(Q) \simeq \mathbb{Z}$, and the characters $\chi: \mathbb{Z} \to U(1)$ are of the form $\chi_\theta(n) := e^{i\theta n}$ where the label $\theta$ runs from 0 to $2\pi$. In this case, the partial-propagator kernels satisfy the relation

$$K_m(q,t;q'',t'') = \sum_{n \in \mathbb{Z}} \int_{Q} dq' K_{m-n}(t,q;t',q')K_{n}(t',q';t'',q'')$$  \hspace{1cm} (4.28)

which reflects the fact that a path from $q$ to $q''$ belonging to a given homotopy class $m$ can be formed by joining together paths from $q$ to $q'$ and $q'$ to $q''$ subject only to the requirement that the sum of the homotopy classes of the two constituent paths is $m$.

The situation in which we are interested is far more general but, nevertheless, (4.28) suggests one way in which ‘topological’ or ‘global’ effects might be recognised in any quasi-temporal history scheme centered on propagators satisfying (4.27). Namely, the situation may arise in which operators exist that do not satisfy (4.27) but rather a law that is an analogue of (4.28) in the form

$$K_{h'}(s_1 \circ s_2) = \int_{H} d\mu(h) K_{h'h^{-1}}(s_1)K_h(s_2)$$  \hspace{1cm} (4.29)

where $H$ is a group with a left-invariant measure $d\mu$. In effect, (4.29) is the
group convolution law of $H$; indeed, (4.29) can be written succinctly as
\[ K(s_1 \circ s_2) = K(s_1) \ast K(s_2) \] (4.30)
where the $\ast$ operation is the usual convolution product defined by $(f_1 \ast f_2)(h') := \int_H d\mu(h) f_1(h'h^{-1}) f_2(h)$ but applied to functions whose values lie in $\mathcal{B}(\mathcal{H})$ rather than in the complex numbers.

At this stage two important questions must be asked. The first is how to get from a ‘$\ast$’ representation (4.30) of the temporal semigroup $Sup$ to a genuine representation satisfying (4.27); the second is why such objects should arise in the first place.

The answer to the first question uses spectral theory on the group $H$. This involves an analogue of a Fourier transform, $f \mapsto \hat{f}$, with the property that $(f_1 \ast f_2) \hat{f} = \hat{f}_1 \hat{f}_2$. For example, suppose that $H$ is a compact Lie group, and therefore with a countable family $\mathcal{R}$ of irreducible unitary representations. Then the ‘Fourier transform’ [14] of $f \in L^1(G, \mathbb{C})$ is a function $\hat{f}$ of $R \in \mathcal{R}$ whose value $\hat{f}(R)$ is a bounded operator on the Hilbert space $\mathcal{H}_R$ which carries the representation $R$ of $H$. More precisely,
\[ \hat{f}(R) := \int_H d\nu(h) f(h) R(h) \] (4.31)
where $R(h)$ is the operator on $\mathcal{H}_R$ that represents $h \in H$. An inverse transform also exists of the form
\[ f(h) = \sum_{R \in \mathcal{R}} \text{tr}(\hat{f}(R) R(h)). \] (4.32)

We use this formalism in the following way. Suppose we are given a family of operators $K_h(s)$, $h \in H$, on a Hilbert space $\mathcal{H}$ that satisfies the convolution relation (4.29–4.30). On the Hilbert space $\mathcal{H} \otimes \mathcal{H}_R$ we can construct the ‘Fourier-transformed’ operator
\[ K_R(s) := \int_H d\nu(h) K_h(s) \otimes R(h) \] (4.33)
which, as required, satisfies the product law
\[ K_R(s_1) K_R(s_2) = K_R(s_1 \circ s_2) \] (4.34)
by virtue of the general property that “the Fourier transform of a convolution is the product of the Fourier transforms”.

Thus, we arrive at a family
$K_R$, $R \in \mathcal{R}$, of genuine propagator operators that can be employed in the
construction of decoherence functionals.

The spectral theory above can be adapted easily enough to include well-behaved
discrete groups $H$; indeed, the expansion (4.26) is a particular case of (4.33) with $H = \pi_1(Q)$
and the integral over $H$ is a sum over the irreducible one-dimensional representations of this group.

This result is gratifying, but it begs the question of whether situations ever arise in which a compact Lie group is an appropriate choice for $H$. In fact, this is commonly the case in the canonical quantisation of a classical system whose configuration space is a homogeneous manifold $G/H$. In the ‘group-theoretical’ approach to canonical quantisation, the appropriate canonical group for this system is identified as a semidirect product of $W$ with $G$ where $W$ is a vector space that carries an representation of $G$ with the property that one of the $G$-orbits in $W$ is diffeomorphic to $Q \cong G/H$ [43]. In [43] an extensive investigation was made of the analogue of this situation in the context of path integral quantisation, and it was shown there that each representation $R$ of $H$ generates a genuine propagator function $K_R$ defined formally by

$$K_R(q_1, t_1; q_2, t_2) := \int_Q Dq e^{iS[q]/\hbar} (Pe^{i\int A[q]}) (4.35)$$

which takes its values in the operators on the Hilbert space $\mathcal{H}_R$ (see also [44, 47]). The key quantity in (4.35) is $(Pe^{i\int A[q]})$ which is the holonomy of the natural $H$-invariant connection $A$ on $G/H$ along the path $q$.

The object defined by (4.35) has some obvious properties in common with those of the $K_R$ operator constructed above, and that the former is indeed an example of the latter can be seen by taking the inverse Fourier transform of (4.35) in the form

$$K_h(q_1, t_1; q_2, t_2)(h) := \sum_{R \in \mathcal{R}} \text{tr}_R(K_R(q_1, t_1; q_2, t_2)R(h)) (4.36)$$

which, as shown in [48], is essentially the path integral $\int Dq e^{iS[q]}$ where the paths are restricted to those whose holonomy is the given element $h \in H$. This is therefore a considerable generalisation of the $K_n$ function in (4.26) whose paths are required to lie in the homotopy class $n$, i.e., they have a specific holonomy with respect to the canonical flat connection on the universal covering bundle of $Q$ with fibre $\pi_1(Q)$. 

36
5 Conclusion

We have seen how the rules proposed by Gell-Mann and Hartle for a generalised history theory suggest strongly the use of a form of quasi-temporal logic in which the pair \((U\mathcal{P}, D)\) of history-propositions and decoherence functionals plays a role analogous to that of the pair \((\mathcal{L}, \mathcal{S})\) of single-time propositions and states in standard quantum theory. In particular, we used a simple two-time example to illustrate the way in which the set \(U\mathcal{P}\) of all propositions about histories in standard quantum theory can be realised as projectors on a tensor product Hilbert space.

We discussed the problem of finding extensions of the GH rules for decoherence functionals and showed how certain plausible-sounding inequalities are in fact false in standard quantum theory. This problem is important in the context of finding a complete classification scheme of decoherence functionals for any given orthoalgebra \(U\mathcal{P}\). The non-triviality of this task is illustrated by the existence of ‘non-standard’ decoherence functionals \((1.19)\) associated with a non-unitary time evolution. It is important to know if decoherence functionals that correspond to unitary evolution possess special properties such as, for example, \(d(\alpha, \alpha) \leq 1\) for all homogeneous \(\alpha\). This is particularly significant in the context of any quantum gravity theory from which standard Hamiltonian quantum theory is supposed to emerge in some coarse-grained sense.

Another problem discussed in the paper is the way that ‘topological’ properties might appear in a formalism in which decoherence functionals play a basic role. We have argued that, in a quasi-temporal situation, the natural place to see the analogues of such effects is in the properties of the generalised propagators \(K(s)\) that give an operator realisation \((1.27)\) of the semi-group \(\mathcal{S}\) of temporal supports.

Many problems remain to be confronted, not the least of which is to try
to apply the formalism to genuine problems in quantum gravity that involve the use of non-standard models for space-time such as causal sets, discrete models for quantum topology, spin networks, and the like. In particular, we need to understand the structure of the space $\mathcal{UP}$ of all ‘history-propositions’ in situations of this type. A key issue here is whether or not there is some preferred Boolean sub-algebra of $\mathcal{UP}$ which essentially generates the whole theory, as we saw was the case in standard Hamiltonian quantum theory.

Acknowledgements
We would like to thank Jim Hartle for many helpful discussions. We gratefully acknowledge support from the SERC and Trinity College, Cambridge (CJI) and the Leverhulme and Newton Trusts (NL). We are both grateful to the staff of the Isaac Newton Institute for providing such a stimulating working environment.

References

[1] M. Gell-Mann and J. Hartle. Quantum mechanics in the light of quantum cosmology. In S. Kobayashi, H. Ezawa, Y. Murayama, and S. Nomura, editors, Proceedings of the Third International Symposium on the Foundations of Quantum Mechanics in the Light of New Technology, pages 321–343. Physical Society of Japan, Tokyo, 1990.

[2] M. Gell-Mann and J. Hartle. Quantum mechanics in the light of quantum cosmology. In W. Zurek, editor, Complexity, Entropy and the Physics of Information, SFI Studies in the Science of Complexity, Vol. VIII, pages 425–458. Addison-Wesley, Reading, 1990.

[3] M. Gell-Mann and J. Hartle. Alternative decohering histories in quantum mechanics. In K.K. Phua and Y. Yamaguchi, editors, Proceedings of the 25th International Conference on High Energy Physics, Singapore, August, 2–8, 1990, Singapore, 1990. World Scientific.

[4] J. Hartle. The quantum mechanics of cosmology. In S. Coleman, J. Hartle, T. Piran, and S. Weinberg, editors, Quantum Cosmology and Baby Universes. World Scientific, Singapore, 1991.
[5] J. Hartle. Spacetime grainings in nonrelativistic quantum mechanics. *Phys. Rev.*, D44:3173–3195, 1991.

[6] M. Gell-Mann and J. Hartle. Classical equations for quantum systems. 1992. UCSB preprint UCSBTH-91-15.

[7] J. Hartle. Spacetime quantum mechanics and the quantum mechanics of spacetime. In *Proceedings on the 1992 Les Houches School, Gravitation and Quantisation*. 1993.

[8] R.B. Griffiths. Consistent histories and the interpretation of quantum mechanics. *J. Stat. Phys.*, 36:219–272, 1984.

[9] R. Omnès. Logical reformulation of quantum mechanics. I. Foundations. *J. Stat. Phys.*, 53:893–932, 1988.

[10] R. Omnès. Logical reformulation of quantum mechanics. II. Interferences and the Einstein-Podolsky-Rosen experiment. *J. Stat. Phys.*, 53:933–955, 1988.

[11] R. Omnès. Logical reformulation of quantum mechanics. III. Classical limit and irreversibility. *J. Stat. Phys.*, 53:957–975, 1988.

[12] R. Omnès. Logical reformulation of quantum mechanics. III. Projectors in semiclassical physics. *J. Stat. Phys.*, 57:357–382, 1989.

[13] R. Omnès. From Hilbert space to common sense: A synthesis of recent progress in the interpretation of quantum mechanics. *Ann. Phys. (NY)*, 201:354–447, 1990.

[14] R. Omnès. Consistent interpretations of quantum mechanics. *Rev. Mod. Phys.*, 64:339–382, 1992.

[15] C. Isham. Quantum logic and the histories approach to quantum theory. *J. Math. Phys.*, 1994.

[16] B.S. DeWitt. Quantum theory of gravity. I. The canonical theory. *Phys. Rev.*, 160:1113–1148, 1967.

[17] B.S. DeWitt. Quantum theory of gravity. II. The manifestly covariant theory. *Phys. Rev.*, 160:1195–1238, 1967.
[18] R.D. Sorkin. Spacetime and causal sets. In J.C. D’Olivo, E. Nahmad-Achar, M. Rosenbaum, M.P. Ryan, L.F. Urrutia, and F. Zertuche, editors, Relativity and Gravitation: Classical and Quantum, pages 150–173, Singapore, 1991. World Scientific.

[19] J.S. Bell. Speakable and unspeakable in quantum mechanics. Cambridge University Press, Cambridge, 1987.

[20] J. von Neumann and G. Birkhoff. The logic of quantum mechanics. Annals of Mathematics, 37:823–843, 1936.

[21] E.G. Beltrametti and G. Cassinelli. The Logic of Quantum Mechanics. Addison-Wesley, London, 1981.

[22] D.J. Foulis, R.J. Greechie, and G.T. Rüttimann. Filters and supports in orthoalgebras. Int. J. Theor. Phys., 31:789–807, 1992.

[23] P. Mittelstaedt. Time dependent propositions and quantum logic. Jour. Phil. Logic, 6:463–472, 1977.

[24] P. Mittelstaedt. Quantum Logic. D. Reidel, Holland, 1978.

[25] E.-W. Stachow. Logical foundations of quantum mechanics. Int. J. Theor. Phys., 19:251–304, 1980.

[26] E.-W. Stachow. Sequential quantum logic. In E.G. Beltrametti and B.C. van Fraassen, editors, Current Issues in Quantum Logic, pages 173–191. Plenum Press, New York, 1981.

[27] A.M. Gleason. Measures on the closed subspaces of a Hilbert space. Journal of Mathematics and Mechanics, pages 885–893, 1957.

[28] C. DeWitt-Morette. Ann. Inst. Henri Poincaré, 11:11–, 1969.

[29] L.S. Schulman. Techniques and applications of path integration. Wiley, New York, 1981.

[30] D. Finkelstein. Space-time code. III. Phys. Rev., D5:2922–2931, 1972.

[31] D. Finkelstein. Process philosophy and quantum dynamics. In C. Hooker, editor, Physical Theory as Logico-Operational Structure, pages 1–18. D. Reidel, Dordrecht, Holland, 1978.
[32] R. Penrose. Angular momentum: an approach to combinatorial spacetime. In E. Bastin, editor, Quantum Theory and Beyond. Cambridge University Press, Cambridge, 1971.

[33] R. Penrose. On the nature of quantum gravity. In J.R. Klauder, editor, Magic without Magic. Freeman, San Francisco, 1972.

[34] L. Hughston and R. Ward. Advances in Twistor theory. Pitman, New York, 1979.

[35] R.D. Sorkin. Impossible measurements on quantum fields. In B.S. Hu and T.A. Jacobson, editors, Directions in General Relativity, Vol. II: a Collection of Essays in honour of Dieter Brill’s Sixtieth Birthday. Cambridge University Press, Cambridge, 1993.

[36] G.W. Mackey. The Mathematical Foundations of Quantum Mechanics. W.A. Benjamin, New York, 1963.

[37] A. Connes. Non Commutative Geometry. Academic Press, New York, 1994.

[38] J.B. Hartle. Unitarity and causality in generalized quantum mechanics for nonchronal spacetimes. Phys. Rev., D49, 1994.

[39] M. Redhead. Incompleteness, Nonlocality, and Realism. Clarendon Press, Oxford, 1989.

[40] J. Milnor. Comm. Math. Helv., 32:215–, 1957.

[41] J. Klauder. Soluble models of quantum gravitation. In M.S. Carmeli, S.I. Flicker, and Witten, editors, Relativity. Plenum, New York, 1970.

[42] M. Pilati. Strong coupling quantum gravity I: solution in a particular gauge. Phys. Rev., D26:2645–2663, 1982.

[43] C.J. Isham. Topological and global aspects of quantum theory. In B.S. DeWitt and R. Stora, editors, Relativity, Groups and Topology II, pages 1062–1290. North-Holland, Amsterdarm, 1984.

[44] R.E. Edwards. Integration and Harmonic Analysis on Compact Groups. Cambridge University Press, Cambridge, 1994.
[45] N.P. Landsman and N. Linden. The geometry of inequivalent quantizations. *Nucl. Phys.*, B365:121–160, 1991.

[46] A.M. Polyakov. *Gauge Fields and Strings*. Harwood Academic Publishers, London, 1987.

[47] D.M. McMullan and T. Tsutsui. On the emergence of gauge structures and generalized spin when quantizing on a coset space. 1993. Dublin Institute of Advanced Studies preprint.

[48] C.J. Isham and N. Linden. 1994. In preparation.