Zero-range potentials with Inner structure:
fitting parameters for resonance scattering

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Abstract

The solution of the classical Fermi problem of low-energy neutron scattering by nuclei, when the excitations of the nuclei in scattering processes are taken into account, is found by the method of zero-range potentials with inner structure. This model is a generalization of the Fermi zero-range potential obtained by adding a non-trivial inner Hamiltonian and inner space with indefinite metric. We propose a general principle of analyticity of the Caley-transform of the S-scattering matrix, written as a function of wave number $k$. This permits us to evaluate all parameters of the model, including the indefinite metric tensor of the inner space, once the spectrum of the inner Hamiltonian, the scattering length and the effective radius are chosen.

PACS numbers: 03.65.Nk, 82.20.Fd, 28.20.Cz

Keywords: zero-range potential
I. INTRODUCTION

Modelling the scattering problem of nucleons by nuclei using a complex-valued potential leads to a basic physical model applicable in the region of low energies \[1, 2\]. This approach is called the optical model and it is often used for the analysis of experimental data. The parameters of the complex-valued potential are fitted by solving the associated Schrödinger equation and then attempting to obtain the best agreement with the experimental scattering cross-section. This method requires laborious numerical calculations \[1\].

In this paper we develop another model, called the zero-range potential with inner structure, which serves as a generalization of the Fermi zero-range potential model describing low-energy neutron scattering by nuclei \[3\]. The Fermi zero-range potential is generalization by adding a non-trivial inner Hamiltonian and appropriate inner space \[4\] supplied with an indefinite metric. This zero-range model with inner structure effectively takes into account multiparticle interactions of the scattering particles which is important for such problems as scattering of electrons by atoms and scattering of nucleons by nuclei. The zero-range model with inner structure is a solvable model which is an important feature of our approach. This is the main advantage of our model over the optical model. Moreover we do not introduce a non-Hermitian Hamiltonian as in the optical model, hence the S-scattering matrix of our model is unitary and has the correct placement of poles and zeros in the complex plane of the wave number \(k\).

It is known also \[5\] that the descriptions of the resonance processes and collisions with redistribution are the most difficult problems in the optical model. In the present work we consider the resonance scattering in the s-channel where all results are given in explicit analytical form. In this section we examine the main works starting from the fundamental paper \[3\] which laid the foundation for the development of the zero-range model. It is useful to study the Fermi model, assuming that the nucleus potential is 3-D finite square well:

\[
V(r) = \begin{cases} 
-V_0, & \text{if } r \leq r_0 \\
0, & \text{if } r > r_0 
\end{cases}
\]  

(1)

where \(V_0 > 0\). Let us consider the bound state of the particle in this well with the energy \(E = -\frac{\hbar^2 \kappa^2}{2\mu} < 0\), where \(\mu = \frac{m_1 m_2}{m_1 + m_2}\) is the reduced mass of scattering particles. Then the solution of the Schrödinger equation decreasing as \(r \to \infty\) is:

\[
\psi(r) = \frac{C}{r} \exp(-\kappa r).
\]  

(2)

Defining the function \(\chi(r) = r\psi(r)\) and using Eq. (2) we find the condition on the boundary of the 3-D square well:

\[
\left. \frac{1}{\chi(r)} \frac{d}{dr} \chi(r) \right|_{r = r_0} = -\kappa.
\]  

(3)

Let us consider the special case, i.e. the weakly bound state: \(|E| << V_0\) or equivalently \(\kappa r_0 << 1\). This holds, for example, in the case of deuteron and negatively charged ions. If in this case \(\kappa\), or the energy of the bound state \(E < 0\), is fixed and the inequalities \(\kappa r_0 << 1\)
and \( kr_0 << 1 \) hold, where \( k \) is the wave number, then the boundary condition (3) may be reduced to the condition:

\[
\left( \frac{1}{\chi(r)} \frac{d}{dr} \chi(r) \right)_{r=0} = -\kappa. \tag{4}
\]

This boundary condition was introduced by Fermi [3] in the scattering problem of neutrons by nuclei and means that the wave function \( \psi(r) \) is singular at \( r = 0 \) and in the vicinity of the zero point \( r = 0 \) has the form:

\[
\psi(r) \approx C \left( \frac{1}{r} - \kappa \right). \tag{5}
\]

Introducing the Fermi s-wave scattering length \( a = \kappa - \frac{r_0}{2} \kappa^2 \), one can find that if \( k \to 0 \) then the total cross-section will be \( \sigma = 4\pi a^2 \). The boundary condition (4) determines the so called the Fermi zero-range potential model [3]. It is important that the boundary condition (4) holds in the low-energy limit \( k \to 0 \) for a deep well with arbitrary form when the bound state with the energy \( E = -\frac{k^2}{2\mu} \kappa^2 < 0 \) is a weakly bound state.

We note that, in the general case, the Laplace operator is symmetric on the scattering waves of the Fermi zero-range potential. Really, if the asymptotic of the scattering waves at the origin \( r = 0 \) is:

\[
\psi_i(r) = A_i \frac{1}{4\pi r} + B_i + o(1), \tag{6}
\]

then the corresponding boundary form \( J_{ik} \) for general smooth elements \( \Psi_i(r) \) possessing the above asymptotic at the origin \( r = 0 \) can be written

\[
J_{ik} = \langle -\triangle \Psi_i, \Psi_k \rangle - \langle \Psi_i, -\triangle \Psi_k \rangle = \bar{A}_i B_k - A_k \bar{B}_i, \tag{7}
\]

where

\[
\langle \Psi_i, \Psi_k \rangle = \int \bar{\psi}_i(r) \psi_k(r) d^3r.
\]

Due to Eq. (5) the values \( A_i \) and \( B_i \) satisfy to the boundary condition

\[
\kappa A_i + 4\pi B_i = 0, \tag{8}
\]

hence the boundary form (7) is zero: \( J_{ik} \equiv 0 \).

In the case of square well when the conditions \( \kappa r_0 << 1 \) and \( k r_0 << 1 \) hold one can make a small correction to the Fermi result:

\[
\frac{1}{a} = \kappa - \frac{r_0}{2} \kappa^2, \tag{9}
\]

where \( a^{-1} = -(k \cot \delta(k))_{k=0} \) and the function \( k \cot \delta(k) \) has the form:

\[
k \cot \delta(k) = -\frac{1}{a} + \frac{r_0}{2} \kappa^2. \tag{10}
\]

Here \( \delta(k) \) is the scattering phase in the s-channel \( (l = 0) \). Eqs. (9) and (10) are also correct in the low-energy limit when the well has an arbitrary shape but the top bound state with the energy \( E = -\frac{k^2}{2\mu} \kappa^2 < 0 \) is the weakly bound state. In this case the parameter \( r_0 \) is the effective radius of the theory and is only in an approximate sense the well radius [3].
A single standard Fermi zero-range potential defines an operator with only trivial spectral structure (one negative eigenvalue). Still the question remained until recently: is it possible to extend the construction of Fermi to include operators with rich spectrum? A version of this question appeared in the paper [6] by Wiegner, where it was noticed that for compactly supported potentials $V(r)$ ($V(r) = 0$ for $r > r_0$) the scattering waves may be calculated for $r > r_0$ if we use an appropriate energy-dependent boundary condition on the sphere $r = r_0$:

$$
\left( \frac{\partial}{\partial r} \chi(r, k) + \alpha(k) \chi(r, k) \right)_{r=r_0} = 0,
$$

where $\chi(r, k) = r \psi(r, k)$. Unfortunately this condition contains the full scattering data, which correspond to compactly supported potentials. Really, the solution of the Schrödinger equation leads to the $s$-scattering waves at $r > r_0$ for compactly supported potentials in the form:

$$
\psi(r, k) = \frac{C}{r} \sin(kr + \delta(k)),
$$

where $\delta(k)$ is the scattering phase in the $s$-channel ($l = 0$). Combining Eqs. (11) and (12) we find:

$$
\alpha(k) = -k \cot(kr_0 + \delta(k)).
$$

Hence to calculate $\alpha(k)$ we should know $\delta(k)$ which is equivalent to the solution of the Schrödinger equation. We note that the suggestion of Wigner inspired numerous attempts to construct the energy-dependent potentials and appropriate boundary conditions, see also [7, 8].

The authors of the paper [9] suggested a new version of the zero-range potential in terms of von-Neumann Operator-Extension techniques. All books and papers on zero-range solvable models in quantum mechanics, see for instance [10, 11], are based on this important paper [9].

Thereby Gelfand attracted [12] attention of mathematicians to the necessity of reconsidering the main results of von-Neumann’s Operator-Extension theory for general Hermitian operators in terms of symplectic Hermitian forms similar to the boundary form of a differential operator. The above form (14), for instance in $R_1$, is:

$$
\int_0^\infty (-\bar{\psi}_i''(x)\psi_k(x) + \bar{\psi}_i(x)\psi_k''(x))dx = (\bar{\psi}_i'(x)\psi_k(x) - \bar{\psi}_i(x)\psi_k'(x))_{x=0}.
$$

Realization of this plan would imply some sort of “integration by parts” for abstract Hermitian operators, see for instance section 2 below. The first step in the development of symplectic techniques in Operator-Extension theory was done in [4] with the intention of creating a quantum mechanical solvable model with a reasonably rich and algebraically computable discrete spectrum and resonances. This program was developed in a series of publications, see for instance [11, 13, 14, 15].

In [4] the “zero-range potentials with inner structure” were suggested as an application of the classical Krein formula [16, 17] to “spectral modelling” of quantum systems. This prospect was anticipated by M. Krein himself long before [18]. In particular the operator extension approach permits one to construct simple solvable models of quantum systems.
with resonance properties without solving sophisticated boundary problems, but operating with finite matrices (see also [19, 21, 22] and more references in [11]).

Unfortunately zero-range solvable models, in particular models with inner structure, contain a large number of parameters which have no physical interpretation and hence are not fitting parameters. We propose a solution of this problem in this paper which has application for low-energy resonance scattering of neutron by nuclei. The generalization of the Fermi zero-range potential is obtained by a symplectic version of the operator-extension technique and assuming that the zero-range potential has inner structure which arises by adding some “inner” Hamiltonian and “inner” space $E$ with a special indefinite metric. Though our treatment is based on Operator-Extention techniques we are able to define generalized zero-range potentials by special boundary conditions at the origin $r = 0$, which is a generalization of the Fermi condition (11):

$$\left( \frac{1}{\chi(r, k)} \frac{\partial}{\partial r} \chi(r, k) \right)_{r=0} = k \cot \delta(k).$$

(15)

This condition can be found formally if we suppose that the scattering wave has the form (12) at the origin $r = 0$—which is not trivial because, for conventional potentials, it is not generally true. It is important to understand that this boundary condition does not define the particular zero-range potential but it defines a whole class of zero-range potentials. In section 5 we show that the zero-range potentials derived in our paper do in fact satisfy the boundary condition (15).

We note that the boundary condition (15) follows from the expansion of the wave function $\psi(r, k)$ at the origin $r = 0$ as:

$$\psi(r, k) = C \sin \delta(k) \left( \frac{1}{r} + k \cot \delta(k) + \ldots \right).$$

(16)

Another, most important, point of our treatment is the principle of analyticity of the function:

$$k \cot \delta(k) = -\frac{1}{a} + \frac{r_0}{2} k^2 + \ldots = \sum_{n=0}^{\infty} g_n k^{2n}. \tag{17}$$

Thus we require that this function is an entire function of the complex variable $k$ which permits us to evaluate all the extra parameters of the zero-range model with inner structure.

In the next section we develop symplectic techniques for the general extension procedure of symmetric operators in a Pontryagin space (this may be thought of as an abstract version of integration by parts). In the third section we calculate the S-scattering matrix based on a variant of the Krein formula. Then in section 4 we study a special class of zero-range models of scattering systems for which the scattering matrices fulfil the analyticity condition formulated for the function $k \cot \delta(k)$. It appears that for the models of this class, if the scattering length, effective radius and the eigenvalues of the inner Hamiltonian are fixed, then all other parameters of the model, in particular the boundary parameters and the deficiency vector in the finite-dimensional case $\dim E = N$ may be calculated directly. Moreover, the type of the indefinite metric in the inner space $E$ is also pre-defined by the above condition of analyticity.

Then in section 5 we apply the results to a typical problem of resonance s-scattering. In
Appendix A we give a straightforward derivation of the basic Krein formula. One usually assumes that the simplest zero-range potential is the singular potential proportional to the Dirac $\delta$-function, which is often used in various physics problems. Consequently, we provide additional on this case in Appendix B.

II. SYMPLECTIC VERSION OF THE OPERATOR-EXTENSION TECHNIQUES

At first we define the dimensionless coordinates $x_i$ in $\mathbb{R}^3$ space by the equation $v = k_0 \mathbf{r}$ assuming that $k_0$ is some characteristic wave number. We also assume that the wave functions $u(x)$ are dimensionless then the Schrödinger equation for scattering problem can be written:

$$\left(\Delta + \lambda - \tilde{V}(x)\right)u(x) = 0,$$

where $\lambda = E/E_0$ is a dimensionless energy with $E = \frac{k^2 k^2}{2\mu}$ and $\tilde{V}(x)$ is the dimensionless potential

$$\tilde{V}(x) = E_0^{-1} V(r), \quad E_0 = \frac{k^2 k_0^2}{2\mu}.$$

Here $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of the scattering particles. Below we will use conventional notations

$$x := \mathbf{x} = k_0 \mathbf{r}, \quad \lambda := \frac{k^2}{k_0^2},$$

then the equation for the Greens function of free evolution is:

$$-(\Delta + \lambda) G_{\lambda}(x, x') = \delta^{(3)}(x - x'),$$

and the appropriate Greens function for outgoing waves has the form:

$$G_{\lambda}(x, x') = -(\Delta + \lambda)^{-1} \delta^{(3)}(x - x') = \frac{e^{i\sqrt{\lambda} \vert x - x' \vert}}{4\pi \vert x - x' \vert}.$$

We develop here the symplectic technique for the zero-range potential of a pair of quantum particles in three-dimensional space, assuming that in the course of interaction the pair may enter an “inner” space $E$ supplied with an indefinite metric. This interaction is determined by an inner Hamiltonian $B$, which is a Hermitian operator with respect to the properly defined dot-product $[\cdot, \cdot]$ in $E$. The two-body Hamiltonian in the outer space, with respect to coordinates defined in terms of the center of mass of the pair, may be given by a three-dimensional self-adjoint “non-perturbed” Schrödinger operator in the “outer” space $L^2(R_3)$ with the conventional dot-product $\langle u, \alpha v \rangle = \alpha \langle u, v \rangle = \langle \bar{\alpha}u, v \rangle$:

$$lu = -\Delta u,$$

defined on the Sobolev class $W^2_2(R_3)$ of all square-integrable functions $u$ possessing square integrable second-order derivatives. In the course of our analysis the spectral parameter may take negative and complex values. The operator $l = -\Delta$ is defined with respect to
the dimensionless space variable $x$. Other variables $\tilde{x}, \tilde{y}$ will be used for vectors in the inner space $E$ of the model, see below.

Following [9] we restrict the operator $l$ to the domain consisting of all smooth functions vanishing near $x = 0$. The closure $l_0$ of the restricted operator is defined on the domain $D_0$ which consists of all $W^2_2$-functions vanishing just at the origin $x = 0$. This operator is symmetric and has deficiency indices $(1, 1)$:

$$\overline{(l_0 - iI)D_0} = L_2(R_3) \ominus N_i, \quad (l_0 + iI)D_0 = L_2(R_3) \ominus N_{-i},$$

$\dim N_i = \dim N_{-i} = 1$. The adjoint operator $l^+_0$ is defined, due to the von-Neumann theorem, see [22], by the same differential expression (23) on the domain $D^+_0 = D_0 + N_i + N_{-i}$, where the one-dimensional subspaces $N_i, N_{-i}$ are spanned by the corresponding non-perturbed Green functions $G_\lambda(x, 0)$ at $\text{Im} \sqrt{\lambda} > 0$ and $\lambda = \pm i$ :

$$N_i = \{G_{-i}(\ast, 0)\}, \quad N_{-i} = \{G_i(\ast, 0)\},$$

$$G_{-i}(x, 0) = \frac{e^{i\sqrt{-\lambda} x}}{4\pi |x|}, \quad G_i(x, 0) = \frac{e^{i\sqrt{\lambda} x}}{4\pi |x|} = \frac{l + iI}{l - iI}G_{-i}(x, 0),$$

(24)

where we define the branch of the square root by the condition $\text{Im} \sqrt{\lambda} > 0$.

The above description of the domain of the adjoint operator based on the von-Neumann formulae was used in [9, 22]. The other description of the domain of the adjoint operator as the set of singular elements described above (6) was probably invented by Fermi [3] and is now commonly used in the physical literature, see for instance [10, 23]. Each element $u$ from the domain of the adjoint operator is characterized by the asymptotic boundary values $A, B$ for $|x| \to 0$:

$$u(x) = \frac{A^u}{4\pi |x|} + B^u + o(1).$$

(25)

It is well known that these representations are equivalent. Nevertheless we have a good reason, see Lemma 2.2, to derive this fact from the decomposition of elements of the defect $N_i + N_{-i}$ with respect to the symplectic basis

$$W_+(x) = \frac{1}{2} [G_{-i}(x, 0) + G_i(x, 0)] = \frac{l}{l - iI}G_{-i}(x, 0),$$

$$W_-(x) = \frac{1}{2i} [G_{-i}(x, 0) - G_i(x, 0)] = - \frac{l}{l - iI}G_{-i}(x, 0).$$

We will present the von-Neumann formula for elements of the domain of the adjoint operator in terms of a decomposition with respect to the basis $W_\pm$ :

$$u = u_0 + \eta^u_+ W_+ + \eta^u_- W_-,$$

(26)

where $u_0$ is an element from the domain of the closure $l_0$ of the restricted operator and $\eta^u_\pm$ are complex coefficients. The deficiency elements $G_{\pm\ast}(\ast, 0)$ are the eigenvectors of the adjoint operator with eigenvalues $\mp i$ respectively. Then the elements $W_\pm$ are transformed by the adjoint operator as:

$$l^+_0 W_+ = W_-, \quad l^+_0 W_- = -W_+.$$  

(27)
Lemma II.1 The boundary form

\[ \langle l_0^+ u, v \rangle - \langle u, l_0^+ v \rangle := J_l(u, v), \]

of the adjoint operator \( l_0^+ \) depends only on the terms \( u_d, v_d \) of the elements \( u, v \) in the defect. It is an Hermitian symplectic form in the variables \( \eta_{\pm} \) and may be presented as

\[ J_l(u, v) = J_l(u_d, v_d) = \left< \eta_{\pm}^u \eta_{\pm}^v - \eta_u \eta_{\pm}^v \right> . \tag{28} \]

The symplectic variables \( \eta_{\pm} \) are connected with the asymptotic boundary values \( A, B \) via the transformation:

\[ T \begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & -4\pi \sqrt{2} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} . \tag{29} \]

Then the Hermitian symplectic form is given by \( J_l(u, v) = (B_v A - A_v B) \).

Remark The above representation of the “outer” boundary form in terms of the symplectic variables \( A, B \) is usually obtained, see for instance \([3, 9, 10, 23]\) via straightforward integration by parts. We use the construction suggested in the above lemma to introduce new symplectic variables which are useful in the symplectic version of operator-extension theory for general “abstract” operators, see below.

Proof of the formula (28) may be obtained via direct application of the adjoint operator to the above version (26) of the von-Neumann decomposition of elements from \( D_0^+ \):

\[ l_0^+ u = l_0 u_0 + \eta_+^u W_+ - \eta_-^u W_+, \]

in the boundary form of the adjoint operator:

\[ \langle l_0^+ u, v \rangle - \langle u, l_0^+ v \rangle := J_l(u, v). \tag{30} \]

One can easily see that the above “outer” boundary form depends only on the parts of the elements \( u, v \) in the sum of the deficiency subspaces (on the “defect”), \( u_d = \eta_+^u W_+ + \eta_-^u W_- \), \( v_d = \eta_+^v W_+ + \eta_-^v W_- \):

\[ J_l(u, v) = J_l(u_d, v_d). \]

Now the boundary form may be calculated (using (27)):

\[ \langle l_0^+ u_d, v_d \rangle - \langle u_d, l_0^+ v_d \rangle = (\eta_+^u \eta_-^v - \eta_-^u \eta_+^v) \int_{R_3} |G_i(\xi, 0)|^2 d^3 \xi. \]

The integral is evaluated as \( 1/(4\pi \sqrt{2}) \) giving us the desired formula (28). The second formula is based on the calculation of the asymptotic behavior of the non-perturbed Greens function as \( x \to 0 \):

\[ W_+(x) = \frac{1}{4\pi |x|} - \frac{1}{4\pi \sqrt{2}} + o(1), \quad W_-(x) = -\frac{1}{4\pi \sqrt{2}} + o(1). \]
Hence for \( x \to 0 \) we have the asymptotic behavior:

\[
\eta_+ W_+(x) + \eta_- W_-(x) = \frac{\eta_+}{4\pi|x|} - \frac{\eta_- + \eta_+}{4\pi\sqrt{2}} + o(1).
\]

The second announced formula follows immediately from the above asymptotics.

Self-adjoint extensions of \( l_0 \) were obtained in [3, 4] by requiring that the asymptotic boundary values \( A, B \) satisfy Fermi boundary conditions \( B = \gamma A \) parametrized by a real \( \gamma = \bar{\gamma} \). In [4] an orthogonal sum of two operators \( l_0 \oplus B_0 \) with an abstract operator \( B \) was restricted and then extended based on an abstract version of the above symplectic “integration by parts” (28) in a Hilbert space \( E \) with an indefinite metric \( g \).

Here \( \overline{\bullet} \) product. It is a Pontryagin space with indefinite metric \( g \). Now develop the symplectic technique for a \( g \)-Hermitian operator \( B \) in a Hilbert Space space with an indefinite metric.

Assume that \( P_\pm \) are two complementary orthogonal projections in a finite-dimensional Hilbert space \( E \), \( \dim E = N \) with conventional dot-product \( \langle *, * \rangle \) and consider the operator 

\[
g = P_+ - P_-
\]

defining an indefinite dot product \( (g\text{-dot product}) \):

\[
\langle g \tilde{x}, \tilde{y} \rangle = \langle \tilde{x}, \tilde{y} \rangle = \langle P_+ \tilde{x}, P_+ \tilde{y} \rangle - \langle P_- \tilde{x}, P_- \tilde{y} \rangle.
\]

Here \( [\alpha \tilde{x}, \tilde{y}] = \alpha [\tilde{x}, \tilde{y}] = [\tilde{x}, \alpha \tilde{y}] \). We denote by \( E_g \) the space \( E \) supplied with the \( g \)-dot product. It is a Pontryagin space with indefinite metric \( g \) if \( \dim P_- \) is finite, see for instance [24, 25]. We develop below a symplectic version of the operator-extension technique in the Pontryagin space which differs from the above techniques in \( L_2(R_3) \) by minor details only.

A bounded or densely defined closed operator \( B \) is called \( g \)-symmetric (symmetric with respect to a \( g \)-dot product) if \( [B \tilde{x}, \tilde{y}] = \tilde{B} \tilde{x}, \tilde{y} \] at least on the domain of \( B \) in \( E \). This condition is equivalent to the condition \( gB \subset B^+ g \), where \( B^+ \) is the adjoint operator with respect to the conventional dot-product \( \langle *, * \rangle \) in \( E \). Without loss of generality we may assume that the operator \( B - iI \) is invertible and the inverse is bounded.

Eventually we will apply the symplectic scheme to bounded operators which are symmetric with respect to the conventional dot-product in \( E \) and commute with \( g \) in the finite-dimensional space \( E \) (\( \dim E = N \)). We assume that the spectrum of the operator \( B \) consists of a finite number of simple real eigenvalues \( \lambda_s \):

\[
(B - \lambda_s) e_s = 0.
\]

One may choose a normalized generating vector \( e \in E \), \( [e, e] = 1 \) such that the vectors \( e \) and \( e' = \frac{B + iI}{B - iI} e \) are connected by the above \( g \)-isometry \( \frac{B + iI}{B - iI} / e \) and form a linearly-independent pair (\( Be \neq 0 \)). We define the domain \( D_{B_0} \) of the restricted operator as \( D_{B_0} = \{ B - iI \}^{-1} \{ E \cap g \{ e \} \} \) and set \( B_0 = B|_{D_{B_0}} \). Here the orthogonal difference with respect to the \( g \)-dot product is denoted by \( \cap g \). The vector \( e \) plays, in this construction, the role of the deficiency vector of the restricted operator at the spectral point \( i \):

\[
([B - iI] D_{B_0}, e) = 0,
\]

and the vector \( e' \left( [e', e'] = [e, e] \right) \) plays a role of the deficiency vector at the spectral point \(-i\):

\[
([B + iI] D_{B_0}, e') = 0.
\]
The sum $M$ of two one-dimensional subspaces $M_i, M_{-i}$ spanned by the vectors $e, e'$ respectively is called the defect of the operator $B_0$. At any other two complex conjugated regular points $\lambda, \bar{\lambda}$ of the operator $B$ the deficiency subspaces $N_{\lambda}, N_{\bar{\lambda}}$ are one-dimensional. The corresponding deficiency vectors are calculated as

$$e_{\lambda} = \frac{B + iI}{B - \lambda I} e, \quad e_{\bar{\lambda}} = \frac{B + iI}{B - \bar{\lambda} I} e,$$

and the defect $M_{\lambda} + M_{\bar{\lambda}}$ is two-dimensional. Let us define the $g$-adjoint operator $B_0^+$ by the formula

$$[B_0 \tilde{x}, \tilde{y}] = [\tilde{x}, B_0^+ \tilde{y}],$$

on the elements $\tilde{y}$ for which the $g$-dot product in the left hand side may be continued onto the whole space $E$ as an “anti-linear” functional [37] of $\tilde{x}$. For the densely-defined operator $B$ and $e$ chosen from the complement of the domain $D_B$ this condition implies

$$(B_0^+ + iI) e = 0, \quad (B_0^+ - iI) e' = 0. \quad (31)$$

For a bounded operator $B$ we just define the adjoint operator on the defect by the above formula (31). Then for any complex value of the spectral parameter the deficiency vector is the eigenvector of the adjoint operator at the adjoint spectral point:

$$\left( (B_0^+ - \lambda I) e \right)_{\lambda} = 0, \quad \left( (B_0^+ - \bar{\lambda} I) e \right)_{\lambda} = 0.$$

This is similar to the above fact for Laplacian:

$$\tilde{l}_0^+ G_{\lambda} = \lambda G_{\lambda},$$

on the space of square integrable functions $L_2(R_3)$.

An analog of the von-Neumann representation of the domain of the adjoint operator remains true in a Ponryagin space, see [24]. So, for elements from the domain of the adjoint operator we have the representation $\tilde{x} = \tilde{x}_0 + \alpha e + \beta e'$, $\tilde{x}_0 \in D_0$ with

$$B_0^+ (\tilde{x}_0 + \alpha e + \beta e') = B \tilde{x}_0 - i \alpha e + i \beta e'.$$

Note that in the case when the original operator $B$ commutes with $g$ and is self-adjoint with respect to the conventional dot-product, the operator $\frac{B+iI}{B-iI}$ is isometric both in the conventional and $g$-metric.

Consider a new basis for the defect $M = M_i + M_{-i}$ of the operator $B_0$, which is similar to the above basis consisting of linear combinations of Greens-functions:

$$w_+ = \frac{e + e'}{2} = \frac{B}{B - iI} e, \quad w_- = \frac{e - e'}{2i} = \frac{-I}{B - iI} e.$$

We see from (31) that

$$B_0^+ w_+ = w_-, \quad B_0^+ w_- = -w_+.$$

In the symplectic version of operator-extension theory we use the new basis to represent elements from the domain of the adjoint operator via new dimensionless symplectic variables $\xi_\pm$ which play a role analogous to the above pair $\eta_\pm$:

$$\tilde{x} = \tilde{x}_0 + \xi_+ w_+ + \xi_- w_-.$$
Lemma II.2 The boundary form of the adjoint operator may be given by an Hermitian symplectic form in terms of the variables $\xi_{\pm}$:

$$K(\bar{x}, \bar{y}) = [B^+, \bar{x}, \bar{y}] - [\bar{x}, B^+] = \xi_+^y \xi_-^y - \xi_-^y \xi_+^y.$$  \hspace{1cm} (32)

Proof exactly follows the pattern of the previous lemma concerning the representation of the boundary form in the “outer” space $L_2(R_3)$. Really, the value of the boundary form depends only on the parts of the vectors $\bar{x}, \bar{y}$ in the defect,

$$K(\bar{x}, \bar{y}) =$$

$$[B^+_0(\xi_+^x w_+ + \xi_-^x w_-), (\xi_+^y w_+ + \xi_-^y w_-)] - [(\xi_+^x w_+ + \xi_-^x w_-), B^+_0(\xi_+^y w_+ + \xi_-^y w_-)] =$$

$$[(\xi_+^x w_- - \xi_-^x w_+), (\xi_+^y w_+ + \xi_-^y w_-)] - [(\xi_+^x w_+ + \xi_-^x w_-), (\xi_+^y w_- - \xi_-^y w_+)] =$$

$$\xi_+^y \xi_-^y ([w_-, w_-] + [w_+, w_+]) - \xi_-^y \xi_+^y ([w_-, w_-] + [w_+, w_+]) = \xi_+^y \xi_-^y - \xi_-^y \xi_+^y,$$

since the coefficients $[[w_-, w_-] - [w_+, w_+]]$ in front of $\xi_+^y \xi_-^y$ are equal to zero. Thus we have:

$$w_+ = \frac{B}{B - iI}e, \quad w_- = \frac{-I}{B - iI}e,$$

which implies $([w_-, w_-] + [w_+, w_+]) = 1$ and $([w_-, w_+] - [w_+, w_-]) = 0$ and the announced abstract integration by parts formula (32).

Consider now the orthogonal sum of operators $l \oplus B$ in the Pontryagin space $L_2(R_3) \oplus E_g$ with elements $U = (u, \bar{x}), u \in L_2(R), \bar{x} \in E_g$. Restricting both operators as above we obtain the symmetric operator $l_0 \oplus B_0$ with defect $N \oplus M$ and deficiency index $(2, 2)$. We define the adjoint operator as an orthogonal sum of adjoints $l_0^* \oplus B_0^*$ and calculate the boundary form of it as a sum of boundary forms of the outer and inner components:

$$J(U, V) = J(u, v) + K(\bar{x}, \bar{y}) = (B^u \bar{A}^v - A^u \bar{B}^v) + (\xi_+^x \xi_-^y - \xi_-^x \xi_+^y),$$

where we choose the symplectic variables $A, B$ for the “outer” boundary form. The boundary form $J(U, V)$ of the orthogonal sum of operators $l_0^* \oplus B_0^*$ in the Pontryagin space $L_2 \oplus E_g$ is a symplectic Hermitian form. Hermitian extensions of the operator $l_0 \oplus B_0$ (or Hermitian restrictions of the operator $l_0^\dagger \oplus B_0^\dagger$) may be constructed via imposing boundary conditions on symplectic variables $A, B, \xi_\pm$ such that the total boundary form vanishes on the corresponding planes. For instance the condition defined by a Hermitian matrix is

$$\Gamma = \Gamma^+ = \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{pmatrix},$$

$$\begin{pmatrix} B \\ -\xi_- \end{pmatrix} = \begin{pmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{pmatrix} \begin{pmatrix} A \\ \xi_+ \end{pmatrix}. \hspace{1cm} (33)$$

The planes in the four-dimensional space of complex symplectic variables $A, B, \xi_\pm$ where these boundary conditions are fulfilled and hence the boundary form vanishes, are called
Lagrangian planes of the symplectic form $\mathcal{J}(U,V)$. We assume here that the elements $\gamma_{ik}$ have numerical values.

All Lagrangian planes of the above form may be constructed either with a help of a Hermitian matrix $\Gamma$, as above, or obtained from the constructed plane by a proper $\mathcal{J}$-unitary transformation, for instance:

$$A \rightarrow A', \quad B \rightarrow B', \quad \xi_- \rightarrow \xi'_-, \quad \xi_+ \rightarrow \xi'_+$$

which leaves the form $\mathcal{J}(U,V)$ unchanged. One may show that all Lagrangian planes of the form $\mathcal{J}(U,V)$ may be obtained via the composition of Hermitian transformations (33) and the fundamental $\mathcal{J}$-unitary transformation $A' = A, \quad B' = B, \quad \xi'_- = \xi_-, \quad \xi'_+ = -\xi_-$.

The matrix elements $\gamma_{ik}$ are called “the boundary parameters”. They do not have any straightforward physical interpretation. One of the most important problems of the theory of solvable models based on operator-extension methods is the problem of fitting these parameters in a physically meaningful way. We solve this problem for the general case in section 5.

III. KREIN FORMULA AND THE S-SCATTERING MATRIX

The expression for the non-stationary S-scattering matrix obtained as a product of wave-operators may be derived [21] from the corresponding Krein formula [16, 17]. This formula gives a description of all resolvents of an extended operator in terms of the boundary conditions (33) and some functional parameter—the so called Krein’s $Q$-function, see [22]. In this paper we focus on the derivation of an expression for the stationary S-scattering matrix, just fitting the properly constructed ansatz for scattered waves to the above boundary condition (33). In the course of the calculation of the coefficient in front of the divergent wave in the ansatz (the scattering amplitude) we will use the $Q$-function of the inner operator $B_0^+$. This parameter appears in the course of the solution of the homogeneous equation for the adjoint operator as a map of symplectic variables. The Krein’s $Q$-function is an abstract analog of the Weyl-Titchmarsh $m$-function [26].

**Lemma III.1** The $Q$-function for the operator $B$ has the form:

$$[e, \frac{I + \lambda B}{B - \lambda I} e],$$

where the symplectic coordinates $\xi_\pm$ of the solution $e_\lambda$ of the adjoint homogeneous equation are related by

$$\xi_- = -[e, \frac{I + \lambda B}{B - \lambda I} e] \xi_+.$$

**Proof.** Consider the $g$-orthogonal projections $P_g = e][e$ and $I - P_g$ in the $g$-metric onto the one-dimensional deficiency subspace $N_i$ and onto the complementary subspace $E \ominus g N_i$, respectively. Then the solution of the adjoint homogeneous equation $e_\lambda$ may be written as:

$$e_\lambda = \frac{B + iI}{B - \lambda I} e = \frac{B}{B - iI} e + \frac{I}{B - iI} e [e, \frac{I + \lambda B}{B - \lambda I} e ] + \ldots$$
\[ \frac{I}{B - iI}(I - P_g) \frac{I + \lambda B}{B - \lambda I} e. \]  

(34)

The last term in the right-hand side is an element \( u_0 \) from the domain of the restricted operator \( B_0 \), but the first two terms belong to the defect \( M \) so the whole linear combination may be written as

\[ \frac{B + iI}{B - \lambda I} e = w_+ - Q(\lambda)w_- + u_0, \]

where \( Q(\lambda) = [e, \frac{I + \lambda B}{B - \lambda I} e]. \)

\[ \square \]

**Remark** The Krein’s \( Q \)-function for a Hermitian operator in a space with a *positive* metric \( g > 0 \) is a dimensionless Nevanlinna-class function with positive imaginary part in the upper half-plane \( \text{Im} \lambda > 0 \). If \( B \) is a finite diagonal matrix \( \text{diag}\{\lambda_s\}_{s=1}^N \), \( N = \dim E \), and the metric tensor \( g \) is trivial \( g = I \), then

\[ Q(\lambda) = \sum_{s=1}^{N} \frac{1 + \lambda \lambda_s}{\lambda_s - \lambda} |e^s|^2, \]

where the \( \lambda_s \) are non-negative eigenvalues of \( B \) and \( |e^s|^2 \) are the squares of the moduli of the components \( e^s \) of the deficiency vector \( e \) with respect to the (standard) basis of eigenvectors of \( B \) in \( E \).

Now consider an indefinite metric tensor \( g \) defined by a diagonal matrix, for instance \( \{g_{ss}\} = \pm 1, s = 1, 2, \ldots N \), a real positive diagonal matrix \( B = \{\lambda_s\} , s = 1, 2, \ldots \dim E \) and a \( g \)-normalized deficiency vector \( e \ ([e, e] = 1) \) with non-zero components \( e^s \) with respect to the same standard basis. Then the corresponding \( Q \)-function will have a form

\[ Q(\lambda) = \sum_{s=1}^{N} \frac{1 + \lambda \lambda_s}{\lambda_s - \lambda} g_{ss} |e^s|^2 = \sum_{s=1}^{N} \frac{1 + \lambda \lambda_s}{\lambda_s - \lambda} P_s, \]  

(35)

with *real* coefficients \( P_s = g_{ss} |e^s|^2 \) and non-negative eigenvalues \( \lambda_s \). The \( Q \)-function (35) has poles of first order at the eigenvalues \( \lambda_s \) of \( B \).

The previous lemma permits us to solve the adjoint non-homogeneous equation and derive the Krein formula for the resolvent of the self-adjoint (or at least self-adjoint in the \( g \)-metric) extension \( \mathcal{A} \) of the symmetric operator \( l_0 \oplus B_0 \) which is defined by the boundary condition (33) (see the simple derivation of the Krein formula for the operator \( B \) in Appendix A). The \( S \)-scattering matrix may be derived from it in a rather standard way, see for instance [21].

We concentrate now on the direct derivation of an expression for the scattering amplitudes and the construction of special solutions—scattered waves—of the homogeneous equation

\[ \mathcal{A} \Psi = (l_0^+ \oplus B_0^+) \Psi = \lambda \Psi, \]

which satisfy the boundary condition (33). The scattered waves serve as eigenfunctions of the self-adjoint extension \( \mathcal{A} \) of the restricted operator \( l_0 \oplus B_0 \) and may be found in the form

\[ \Psi = \begin{pmatrix} \psi \\ \psi_E \end{pmatrix}, \]
where the ansatz for the “outer” component $\psi$ of the scattered wave in $R^3$ is a sum of incoming and outgoing waves:

$$\psi_\lambda(x, \nu) = e^{i\sqrt{\lambda}(\nu,x)} + T(\nu, \sqrt{\lambda}) \frac{e^{i\sqrt{\lambda}|x|}}{4\pi|x|},$$

(36)

with the amplitude $T(\nu, \sqrt{\lambda})$ in front of the outgoing wave. We note that this dimensionless amplitude $T$ is proportional to the standard “physical” amplitude $f : T = 4\pi k_0 f$. The component $\psi_E$ of the scattered wave in the inner space $E$ is just proportional to the limit value of the solution $e_\lambda$ of the adjoint homogeneous equation on the real axis of the spectral parameter:

$$\psi_E = T_E e_\lambda = T_E [w_+ - Q(\lambda)w_- + u_0].$$

One may show, see for instance [10, 11], that the boundary conditions formulated for elements of the domain of the extension of the operator $l_0 \oplus B_0$ are valid for the corresponding scattered waves. The symplectic variables in the above ansatz for components of the scattered wave in the outer and the inner spaces are:

$$B = \left(1 + i\frac{\sqrt{\lambda}}{4\pi}T\right), \; A = T, \; \xi_- = -QT_E, \; \xi_+ = T_E.$$

(37)

This gives the following equation for the amplitudes $T$ and $T_E$:

$$\begin{bmatrix}
1 + i\frac{\sqrt{\lambda}}{4\pi}T \\
QT_E
\end{bmatrix} = \begin{bmatrix}
\gamma_{00} & \gamma_{01} \\
\gamma_{10} & \gamma_{11}
\end{bmatrix} \begin{bmatrix}
T \\
T_E
\end{bmatrix}.$$ 

(38)

Solving this equation we obtain the following expressions for these amplitudes.

**Theorem III.1** The amplitudes $T$, and $T_E$ for real positive values of the spectral variable $\lambda$ are equal to

$$T(\sqrt{\lambda}) = \frac{1}{\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q} - i\sqrt{\lambda}}, \; T_E = \frac{\gamma_{10}}{Q - \gamma_{11}}T,$$

(39)

where

$$Q(\lambda) = \sum_{s=1}^{N} \frac{1 + \lambda\lambda_s}{\lambda_s - \lambda}|e^s|^2g_{ss}.$$

**Proof** may be obtained from the previous discussion.

**Remark** Note that the amplitude

$$T(\sqrt{\lambda}) = \frac{1}{\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q} - i\sqrt{\lambda}},$$

contains essential information on the spectral properties of the extension $\mathcal{A}$. For example, the solution of the corresponding homogeneous equation

$$\mathcal{A}\Psi = \lambda\Psi,$$
\[ \Psi = \begin{pmatrix} 
\frac{e^{i\sqrt{|x|}}}{4\pi|x|} \\
T_e e^{\lambda} 
\end{pmatrix} \]

yields the equation

\[ \begin{pmatrix} 
i\sqrt{\lambda} \\
QT_e, 
\end{pmatrix} = \begin{pmatrix} 
\gamma_{00} & \gamma_{01} \\
\gamma_{10} & \gamma_{11} 
\end{pmatrix} \begin{pmatrix} 
1 \\
T_e 
\end{pmatrix} \]

which has a solution \( \Psi \) with \( T_e = \frac{-\gamma_{10}}{\gamma_{11} - Q(\lambda)} \) if and only if the \textit{dispersion equation} is fulfilled:

\[ \frac{i\sqrt{\lambda}}{4\pi} = \gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q(\lambda)}. \]

The negative roots \( \lambda_s < 0 \) of the above equation on the spectral sheet \( \text{Im}\sqrt{\lambda} > 0 \) of the variable \( \sqrt{\lambda} \) correspond to the square-integrable first component of the solution \( \Psi \), hence they are the eigenvalues of the extended operator \( A \).

\[ \text{IV. A SPECIAL CLASS OF ZERO-RANGE POTENTIALS WITH INNER STRUCTURE} \]

In this section we study a special class of zero-range models for which the function \( F(k) = k\cot\delta(k) \) satisfies an analyticity condition with respect to the wave number \( k \) \((|k| = \text{cm}^{-1})\).

We will show that for models from this class the boundary parameters and the deficiency vector in the finite-dimensional case \( \text{dim } E = N \) may be calculated directly. Moreover, the type of indefinite metric in the inner space \( E \) is also pre-determined by the above condition of analyticity.

We will also use the dimensional \textit{physical scattering amplitude} \( f(k) \) with orbital angular momentum \( l = 0 \), see [27], connected to the above dimensionless amplitude \( T(\sqrt{\lambda}) \) in the “outer” space. Using Eqs.(36) and (39) we find this \( s \)-scattering amplitude \( f(k) \) as:

\[ f(k) = \frac{1}{4\pi k_0} T(\sqrt{\lambda}) = \left( 4\pi k_0 \gamma_{00} - \frac{4\pi k_0 |\gamma_{01}|^2}{\gamma_{11} - Q(\lambda)} - ik \right)^{-1}. \] (40)

The scattering matrix \( S(k) = \exp(2i\delta(k)) \) where \( \delta(k) \) is the scattering phase in the \( s \)-channel has the form:

\[ S(k) = \frac{\cot \delta(k) + i}{\cot \delta(k) - i} = 1 + 2ikf(k), \] (41)

hence the scattering amplitude \( f(k) \) is:

\[ f(k) = \frac{1}{k\cot \delta(k) - ik}. \] (42)
Combining Eqs. (40), (41) and (42) one may find the S-scattering matrix in the form:

\[
S(k) = 1 + \frac{2ik}{4\pi k_0 [\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q(\lambda)}]} - ik \frac{F(k) + ik}{F(k) - ik},
\]  

(43)

with

\[
F(k) = 4\pi k_0 [\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q(\lambda)}].
\]  

(44)

We may rewrite the Q-function calculated in lemma 3.1 in terms of the wave number \( k \), assuming that all eigenvalues \( \lambda_s = (k_s/k_0)^2 \) of the inner Hamiltonian \( \mathbf{B} \) are positive. Then it takes the following form

\[
Q(\lambda) = \frac{1}{k_0^2} \sum_{s=1}^{N} \frac{k_0^4 + k_s^2 k_0^2}{k_s^2 - k_0^2} P_s,
\]  

(45)

where the weights \( P_s = g_{ss} |e_s|^2 \) are not necessary positive! Note that we have introduced the resonance values \( k_s > 0 \) of the wave number \( k \) through the equation \( \lambda_s = (k_s/k_0)^2 \), hence \( [k_s] = cm^{-1} \). Using Eqs. (44) and (45) we obtain a similar representation for the function \( F(k) \)

\[
F(k) \equiv k\cot \delta(k) = 4\pi \gamma_{00} k_0 - \frac{4\pi |\gamma_{01}|^2 k_0}{\gamma_{11} - \frac{1}{k_0^2} \sum_{s=1}^{N} \frac{k_0^4 + k_s^2 k_0^2}{k_s^2 - k_0^2} P_s}.
\]  

We consider a special type of potential with inner structure for which the function \( F(k) \) defined by the above formula is an entire function of the variable \( k \).

The behavior of the function \( F(k) \) defines the scattering process and can be represented by a Taylor expansion at the origin \( k = 0 \):

\[
F(k) = - \frac{1}{a} + \frac{r_0}{2} k^2 + \ldots = \sum_{n=0}^{\infty} g_n k^{2n},
\]  

(46)

where \( a \) is the scattering length and \( r_0 \) is the effective radius.

We may also impose on the parameters of our model the requirement that the scattering matrix satisfy the following “physically reasonable” behaviour: \( S(k) \to 1 \) when \( k^2 \to \infty \). This condition is fulfilled if and only if the denominator \( \gamma_{11} - Q(\lambda) \) tends to zero as \( 1/\lambda \) when \( \lambda = (k/k_0)^2 \to \infty \) which yields:

\[
\gamma_{11} + \sum_{s=1}^{N} \frac{k_s^2}{k_0^2} P_s = 0.
\]  

(47)

We will show that for the introduced class of zero-range potentials the S-scattering matrix satisfies this “physically reasonable” condition for large energy automatically.

**Lemma IV.1** *The function*

\[
F(k) = k\cot \delta(k) = 4\pi k_0 \left[ \frac{\gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - \frac{1}{k_0^2} \sum_{s=1}^{N} \frac{k_0^4 + k_s^2 k_0^2}{k_s^2 - k_0^2} P_s}}{\gamma_{11} - \frac{1}{k_0^2} \sum_{s=1}^{N} \frac{k_0^4 + k_s^2 k_0^2}{k_s^2 - k_0^2} P_s} \right]
\]  

(48)

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is an entire function of \( k \) (a polynomial of degree \( N \) in \( k^2 \))

\[
F(k) \approx -4\pi|\gamma_0|^2(-1)^N \frac{k^{2N}}{\Lambda k_0^{2N-1}}, \quad k \to \infty,
\]

with non-zero normalization constant \( \Lambda \), if and only if the values of all weights \( P_s \) are defined by the formula:

\[
P_s = (-1)^s \frac{k_0^{2N+2} \Lambda}{(k_0^4 + k_s^4) \prod_{t(t \neq s)} |k_t^2 - k_s^2|}.
\] (49)

Here the product \( \prod_{t(t \neq s)} \) is spread over all indices \( t, 1 \leq t \leq N \), except one, \( t \neq s \). This analyticity requirement automatically yields the boundary parameter \( \gamma_{11} \) as

\[
\gamma_{11} = -\sum_{s=1}^{N} \frac{k_s^2}{k_0^2} P_s.
\]

Proof. We may consider the denominator \( D \) of the \( F \)-function as a function of the variable \( \lambda = k^2/k_0^2 \) with parameters \( \lambda_s = k_s^2/k_0^2 \):

\[
D(\lambda) = \gamma_{11} - Q(\lambda) = \gamma_{11} + \sum_{s=1}^{N} \lambda_s P_s - \sum_{s=1}^{N} \frac{1 + \lambda_s^2}{\lambda_s - \lambda} P_s,
\] (50)

where \( N = \dim E \). The analyticity of the fraction \( \frac{1}{D(\lambda)} \) on the whole plane \( \lambda \) means that there are no zeroes of the denominator in any compact domain of the complex \( \lambda \) plane. Hence \( D(\lambda) \), being a rational function with \( N \) simple poles must have only one zero of multiplicity \( N \) at infinity and hence both \( 1/D(\lambda) \) and \( F(\lambda) \) are polynomials of degree \( N \):

\[
D(\lambda) \approx \frac{\Lambda}{\lambda^N} (-1)^N, \quad \lambda \to \infty,
\]

where we introduce the sign-factor \( (-1)^N \) to give the final formula a simple form. Then the asymptotic of the polynomial at infinity is defined by the highest order coefficient :

\[
\Lambda \neq 0, \quad F(k) \approx -4\pi k_0|\gamma_0|^2(-1)^N \frac{\Lambda^N}{\lambda}.\]

The constant \( \Lambda \) plays an essential role below. We will call it the normalization constant.

On the other hand, expanding the fractions in the above representation (50) of \( D \) in Laurent series we obtain \( D(\lambda) \) in another form:

\[
D(\lambda) = \gamma_{11} + \sum_{s=1}^{N} \lambda_s P_s + \frac{1}{\lambda} \sum_{s=1}^{N} (1 + \lambda_s^2) P_s + \frac{1}{\lambda^2} \sum_{s=1}^{N} (1 + \lambda_s^2) \lambda_s P_s
\]

\[+ \ldots + \frac{1}{\lambda^N} \sum_{s=1}^{N} (1 + \lambda_s^2) \lambda_s^{N-1} P_s + o(\lambda^{-N}) \approx (-1)^N \frac{\Lambda^N}{\lambda^N}, \quad \lambda \to \infty.\] (51)
This means that all coefficients in front of the powers \(\lambda^{-l}\) \((0 \leq l < N)\) are equal to zero and the coefficient in front of \(\lambda^{-N}\) is equal to the normalization constant. Thus the variables \(q_s = (1 + \lambda^2_s)P_s\) \((s = 1, 2, \ldots, N)\) fulfill the linear system obtained by comparison of the coefficients in front of the powers \(\frac{1}{\lambda^l}\) \((l = 1, 2, \ldots, N)\) on the left and right hand sides of the last equation (51). This system has a positive Vandermonde determinant \(W = W(\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_N)\) if the parameters \(\lambda_s\) are arranged in order of increasing \(\lambda_1 < \lambda_2 < \lambda_3 < \ldots \lambda_N\). The solution of it may be found as a ratio of the determinant and minors \(W_s = W(\lambda_1, \lambda_2, \ldots, \lambda_{s-1}, \lambda_{s+1} \ldots \lambda_N) > 0:\)

\[
(1 + \lambda^2_s)P_s = (-1)^s \frac{W_s}{W} \Lambda = \frac{(-1)^s}{\prod_{t(t \neq s)} |\lambda_s - \lambda_t|} \Lambda, \quad s = 1, 2, \ldots N.
\] (52)

The announced result is obtained from the last formula by inserting wave numbers \(k_s, k\) instead of \(\lambda_s, \lambda\).

\[\square\]

**Corollary 1** We may assume (see theorem 4.1), that the metric tensor is defined as \(g_{ss} = (-1)^s \text{sgn} \Lambda\). Then the squares of the components \(|e^s|^2\) of the deficiency vector \(e\) are given by

\[
|e^s|^2 = \frac{1}{1 + \lambda^2_s} \frac{|\Lambda|}{\prod_{t(t \neq s)} |\lambda_t - \lambda_s|} = \frac{|\Lambda|k^2_0(2^{(N+1)})}{(k^2_0 + k^2_4) \prod_{t(t \neq s)} |k^2_s - k^2_t|}.
\] (53)

**Remark.** The basic formula for the \(Q\)-function was derived under the assumption that the deficiency vector \(e\) is non-degenerate with respect to the indefinite metric form and \([e, e] = 1\). In the "generic case"

\[
\sum_{s=1}^{N} \frac{1}{1 + \lambda^2_s} \frac{(-1)^s}{\prod_{t(t \neq s)} |\lambda_t - \lambda_s|} \neq 0,
\] (54)

so we may choose the normalization constant as

\[
\Lambda = \frac{1}{\sum_{s=1}^{N} \frac{1}{1 + \lambda^2_s} \frac{(-1)^s}{\prod_{t(t \neq s)} |\lambda_t - \lambda_s|}} = \frac{1}{\sum_{s=1}^{N} \frac{k^2_0(2^{(N+1)})(-1)^s}{(k^2_0 + k^2_4) \prod_{t(t \neq s)} |k^2_s - k^2_t|}},
\]

and use the normalized deficiency vector in the formula for the \(Q\)-function. We assume that the non-degeneracy condition (54) is fulfilled.

**Theorem IV.1** The necessary condition for the analyticity of the function \(F(k) = k \cot \delta(k)\) on a whole complex plane of \(k\) for zero-range potentials with inner structure is the alternation of the diagonal components of the metric tensor \(g\):

\[
g_{ss} = (-1)^s \text{sgn} \Lambda,
\] (55)

where the order of the components \((s = 1, 2, \ldots)\) is defined by the ordering of the parameters \(k_s: k^2_1 < k^2_2 < \ldots\). The only parameters which remain free in the model are the scattering length, the effective radius, and the eigenvalues of the inner Hamiltonian which commutes.
with the metric tensor. The boundary parameters $\gamma_{00}, |\gamma_{01}|$ are defined as functions of the scattering length $a$, effective radius $r_0$, and $k_s^2$ ($s = 1, 2, \ldots, N$):

$$a = \left( -4\pi \gamma_{00} k_0 + \frac{4\pi |\gamma_{01}|^2}{k_0^{2N-1} \Lambda} \prod_{s=1}^N k_s^2 \right)^{-1}, r_0 = \frac{8\pi |\gamma_{01}|^2}{\Lambda k_0^{2N-1}} \left( \prod_{t=1}^N k_t^2 \right) \sum_{s=1}^N k_s^{-2}.$$

In particular, if the effective radius is positive, $r_0 > 0$, then $\Lambda > 0$ and $g_{ss} = (-1)^s$. All other essential parameters of the model: the moduli of the components of the deficiency vector and the the inner boundary parameter $\gamma_{11}$ are uniquely defined from the condition of analyticity of the $F$-function on the whole $k$ plane.

**Proof.** Using Eq.(49) and our condition $k_1^2 < k_2^2 < \ldots$ one see that

$$\text{sign} P_s = (-1)^s \text{sign} \Lambda. \quad (56)$$

Combining this equation with our definition $P_s = g_{ss} \epsilon^s |e^s|^2$ we find the necessary condition for the metric tensor $g$ given by Eq.(55). The explicit form of the function $F(k)$ follows from Eqs.(48) and (49):

$$F(k) = k \cot \delta(k) = \varepsilon - \gamma \prod_{s=1}^N (k_s^2 - k^2), \quad (57)$$

where we have introduced the notations

$$\varepsilon = 4\pi \gamma_{00} k_0, \quad \gamma = \frac{4\pi |\gamma_{01}|^2}{k_0^{2N-1} \Lambda}. \quad (58)$$

In particular, using (57), (58) and definition (46) we find the scattering length $a$ and effective radius $r_0$:

$$a = \left( -\varepsilon + \gamma \prod_{s=1}^N k_s^2 \right)^{-1}, r_0 = 2\gamma \sum_{s=1}^N \prod_{t(\neq s)} k_t^2. \quad (59)$$

This completes the theorem.

\[ \square \]

**Corollary 2** It is remarkable that, using (43) and (57), we may write the scattering matrix $S$ in terms of the parameters $k_s$, $\varepsilon$ and $\gamma$ as:

$$S(k) = 1 + \frac{2ik}{\varepsilon - ik - \gamma \prod_{s=1}^N (k_s^2 - k^2)}. \quad (60)$$

The total scattering cross-section is $\sigma(k) = 4\pi |f(k)|^2$ or can be written in explicit form (see next section) using equations (42) and (57). This expression for the $S$-scattering matrix describes the resonance scattering of particles with resonances defined by the spectral properties of the inner Hamiltonian (again see the next section).
V. RESONANCE S-SCATTERING ON ZERO-RANGE POTENTIALS WITH INNER STRUCTURE

In this last section we consider the main physical properties of the resonance scattering problem on zero-range potentials with inner structure. This problem has application mainly to low-energy neutron scattering by nuclei.

First we will show that zero-range potentials with inner structure, as defined above, fulfil our condition for generalized zero-range potentials (15), i.e. the special boundary condition at the origin \( r = 0 \). Due to the fact that the condition (15) follows from the expansion (16) we will consider Eq. (16) which can be rewritten in dimensionless form (25) as:

\[
 u(x, k) = \frac{A(k)}{4\pi |x|} + B(k) + o(1),
\]

where

\[
 A(k) = 4\pi k_0 C \sin \delta(k), \quad B(k) = k C \cos \delta(k).
\]

Hence the special boundary condition (15) is equivalent to the equation:

\[
 \frac{4\pi B(k)}{A(k)} = \frac{k}{k_0} \cot \delta(k). \tag{63}
\]

On the other hand, as follows from Eqs. (37) and (39), the left side of Eq. (63) for zero-range potentials with inner structure is:

\[
 \frac{4\pi B(k)}{A(k)} = 4\pi \frac{A(k)}{A(k)} + i\sqrt{\lambda} = 4\pi \left( \gamma_{00} - \frac{|\gamma_{01}|^2}{\gamma_{11} - Q(\lambda)} \right). \tag{64}
\]

Using Eq. (44) we find that the right side of Eq. (64) is \( k_0^{-1} F(k) = (k/k_0) \cot \delta(k) \), hence equations (63) and (15) are satisfied for zero-range potentials with inner structure. Thus our definition (15) is correct for the zero-range potentials derived in this paper.

Let us show that our resonance scattering model depends only the scattering length \( a \), effective radius \( r_0 \), and the spectrum \( k_s (s = 1, 2, \ldots, N) \) of the inner Hamiltonian. It is convenient to define the typical wave-number \( k_0 \) and a dimensionless parameter \( \alpha \) by:

\[
 k_0 := -\frac{1}{4\pi \gamma_{00} a_0}, \quad \alpha := -\frac{4\pi |\gamma_{01}|^2}{\Lambda} \left( 4\pi \gamma_{00} \right)^{2N-1},
\]

where \( a_0 \) is a new parameter with dimension \( ([a_0] = cm) \). Then the function \( F(k) = k \cot \delta(k) \) for our zero-range potential is:

\[
 F(k) = -\frac{1}{a_0} - \frac{\alpha}{a_0} \prod_{s=1}^{N} \left( a_0^2 k_s^2 - a_0^2 k^2 \right). \tag{66}
\]

Defining the scattering length \( a \) from the equation \( F(0) = -a^{-1} \), we obtain:

\[
 a = \frac{a_0}{1 + \alpha \prod_{s=1}^{N} a_0^2 k_s^2}. \tag{67}
\]
Hence the parameter $a_0$ is the scattering length for the zero-range potential without inner structure, i.e. when $\alpha = 0$. Evidently the above equation (67) is the renormalization equation since it expresses the renormalized scattering length $a$, via the non-renormalized scattering length $a_0$, taking into account resonance scattering.

Thus the function $F(k)$ given by Eq. (66) depends on $N + 2$ parameters: $a_0, \alpha$ and $k_s, s = 1, 2, \ldots, N$. The effective radius may be found from (66) as

$$r_0 = \frac{\alpha a_0}{2} \sum_{n=1}^{N} \prod_{s(n \neq n)} a_0^2 k_s^2. \quad (68)$$

Consequently, the non-renormalized scattering length $a_0$, the dimensionless parameter $\alpha$ and the spectrum $k_s (s = 1, 2, \ldots, N)$ of the inner Hamiltonian define the scattering length $a$ and effective radius $r_0$ of the model. Vice versa, we may choose the renormalized scattering length $a$, the effective radius $r_0$, and the spectrum $k_s (s = 1, 2, \ldots, N)$ of the inner Hamiltonian as independent parameters, and define $a_0$ and $\alpha$ by Eqs.(67) and (68) but the former option is more useful and will be used below.

The total cross-section for spherically-symmetric scattering is calculated as

$$\sigma(k) = 4\pi |f(k)|^2 = \frac{4\pi}{|F(k) - ik|^2}; \quad (69)$$

which implies, due to (66), the explicit formula:

$$\sigma(k) = \frac{4\pi a_0^2}{1 + a_0^2 k^2 + 2\alpha \prod_{s=1}^{N} (a_0^2 k_s^2 - a_0^2 k^2) + \alpha^2 \prod_{s=1}^{N} (a_0^2 k_s^2 - a_0^2 k^2)^2}. \quad (70)$$

The latter equation exhibits clear resonance properties: for $k = k_s$ the products in the denominator vanish and the whole cross-section is reduced to

$$\sigma(k_s) = \frac{4\pi a_0^2}{1 + a_0^2 k_s^2}, \quad s = 1, 2, \ldots, N,$$

which coincides with the Wiegner formula $\sigma(k) = \frac{4\pi a_0^2}{1 + a_0^2 k^2}$ for $k = k_s$.

Consider now the analytic structure of the scattering matrix (43)

$$S(k) = \frac{F(k) + ik}{F(k) - ik}$$

in the complex $k$ plane. Then the equation for the poles of the scattering matrix is:

$$F(k) - ik = 0.$$

One may see that all solutions of this equation may be written as $k = i\kappa$, where $\kappa$ are the zeroes of another polynomial with real coefficients:

$$\alpha \prod_{s=1}^{N} \left( a_0^2 k_s^2 + a_0^2 \kappa^2 \right) - a_0 \kappa + 1 = 0. \quad (71)$$
In the simplest case $N = 0$ one may see from (66), (67), and (71):

$$a_0 = a, \quad \kappa = \frac{1}{a}, \quad F(k) = -\frac{1}{a} = -\kappa,$$

which is in full agreement with the Fermi model, see [3] and the Introduction above. From Eq. (70) it follows that the integral cross-section in this case is given by the classical Wiegner formula: $\sigma(k) = \frac{4\pi a^2}{1+a^2 k^2}$, where $a = a_0$.

For $N = 1$ one may find from (71):

$$\frac{r_0}{2} \kappa^2 - \kappa + \frac{1}{a} = 0 \quad (72)$$

with

$$\frac{1}{a} = \frac{1}{a_0} + \alpha a_0 k_1^2, \quad \frac{r_0}{2} = \alpha a_0.$$

Hence Eq. (66) in this case yields:

$$F(k) = -\frac{1}{a} + \frac{r_0}{2} k^2.$$

These results and Eq. (70) for $N = 1$ coincide with well known results for low-energy resonance scattering, see [27] and Eqs. (9) and (10).

In the case $N = 2$ one may show from equations (71), (67), and (68) that:

$$\frac{r_0}{2} \kappa^4 + \frac{r_0}{2} \kappa^2 - \kappa + \frac{1}{a} = 0, \quad (73)$$

with

$$\frac{1}{a} = \frac{1}{a_0} + \alpha a_0^3 k_1^2 k_2^2, \quad \frac{r_0}{2} = \alpha a_0^3 (k_1^2 + k_2^2).$$

In this case the total cross-section is given by Eq. (70) for $N = 2$. In the general case $N > 2$ it follows from Eq. (71) that some of the poles of the S-scattering matrix are situated on the real axis $\kappa$ (imaginary axis $k = i\kappa = p + iq$) and the other sit at complex-conjugated points $\kappa_s = q_s - ip_s, \bar{\kappa}_s = q_s + ip_s$, i.e. symmetric with respect to the real axis in $k$-plane, because all coefficients of the algebraic equation (71) are real. The zeroes of the S-scattering matrix are defined by the adjoint equation $F(k) + ik = 0$, hence if a pole of the S-scattering matrix is situated at the point $k_s = p_s + iq_s$ then the appropriate zero is situated at the complex-conjugated point $\bar{k}_s = p_s - iq_s$. Note that these are universal properties of the S-scattering matrix and are a consequence of unitarity and causality, but we have derived them using only Eq. (71). The imaginary poles $k_s = iq_s$ may lie on both the positive or negative imaginary semi-axes ($q_s > 0$ or $q_s < 0$), but all other poles (when $p_s \neq 0$) should lie in the lower half-plane. This property is connected with general spectral properties of selfadjoint operators in spaces with indefinite metric, see for instance [24, 25, 28]. Thus the properties of the function $F(k)$ given by Eq. (66) guarantee the correct analytic structure of the S-scattering matrix in the complex plane of the wave-number $k$.

For our model the poles $k_n = p_n + iq_n$ of the S-scattering matrix may be classified according to their positions using the following scheme:
1. If \( p_n = 0 \) then for \( q_n > 0 \) the pole corresponds to a bound state with the energy \( E_n = -\frac{\hbar^2}{2\mu}q_n^2 \) and for \( q_n < 0 \) the pole corresponds to a virtual state with energy \( E_n = -\frac{\hbar^2}{2\mu}q_n^2 \).

2. Poles with \( p_n > 0, \ q_n < 0 \) correspond to metastable states with complex energy \( E_n = E'_n - i\frac{\Gamma_n}{2} \), where:

\[
E'_n = \frac{\hbar^2}{2\mu} \left[ (p_n)^2 - (q_n)^2 \right], \quad \Gamma_n = -\frac{\hbar^2}{2\mu}p_nq_n > 0
\]

and the rate of decay of the metastable state is \( w_n = \frac{\Gamma_n}{\hbar} \). Here \( E'_n > 0 \) if \( p_n > |q_n| \).

3. Poles with \( p_n < 0, \ q_n < 0 \) correspond to resonance trapping. These poles and the corresponding poles of the metastable states are symmetrically situated with respect to the imaginary axis.

No other poles are present in our zero-range model since all poles may be found as zeroes of a polynomial with real coefficients.

In conclusion we point out the main problems which should be examined in the future in connection with the model of zero-range potentials with inner structure. Besides pure resonance scattering which we have considered in this paper, scattering theory of nucleons by nuclei must include:

- Potential scattering connected with consideration of the interaction around the nucleus surface and with the Coulomb interaction in the case of proton scattering by nuclei.
- The spin-orbit interaction describing polarization effects.
- The generalization of the model for scattering with an arbitrary quantum number \( l \).

VI. ACNOWLEDGEMENTS

The authors are grateful to Professor H. Langer for discussions of the structure and properties of the positive invariant subspace of self-adjoint operators in Pontryagin space and to doctors Y. Shondin and P. Kurasov for attracting our attention to the paper [29] where the question of fitting parameters was considered for a class of zero-range potentials without inner structure. Following advice from them the authors included the Appendix which contains a simple derivation of the classical Krein Formula for resolvents of self-adjoint extensions in Pontryagin spaces. B.P. is grateful to the Japanese Foundation JSPS which supported his stay at the Solid State Computer Laboratory of Aizu University, Japan, where a part of this paper was written, and to Professor V. Ryzhii for useful advice.

VII. APPENDIX A: DERIVATION OF THE KREIN FORMULA WITH SYMPLECTIC COORDINATES

In this Appendix, using the notations \( B_0 = B_0 \) and \( B = B \), we give a simple derivation of the classical Krein formula for the \( g \)-symmetric operator \( B_0 \) in the Pontryagin space \( E \). This derivation follows the pattern of [14] where similar formulae for operators in spaces with positive metric is presented and may be easily modified for extensions of the operator \( l_0 \oplus B_0 \) in Pontryagin space.
Consider a \( g \)-hermitian operator \( B \) in the Pontryagin space \( E \) with an indefinite dot-product \([.,.] = \langle g.,. \rangle \). Assume that the operators \( B \pm iI \) are invertible, and \( E \subset E \) is a finite-dimensional subspace which will play the role of the deficiency subspace \( M_i \) for the reduced operator \( B_0 \) defined on \( D_0 = \{ x : [B - iI) D_0, x] = 0 \} \). The deficiency subspace at the point \(-i\) may be obtained from \( E \) via a \( g \)-unitary transformation \( M_{-i} = \frac{B + iI}{B - iI} E \). If the operator \( B_0 \) is densely defined then the subspaces form a positive angle. If the operator \( B_0 \) is not densely defined, (for instance, if \( B \) is bounded) then we assume that the angle between \( M_{\pm i} \) is positive. In both cases one may define an adjoint of the formally self-adjoint operator, \( B_0^+ \), at least on the defect \( M = M_i + M_{-i} \). In fact the extension of the reduced operator \( B_0 \) must be constructed on the defect and hence we may derive the Krein formula below only on the defect.

The subspace \( E \) is mapped into the defect by the transformations

\[
\begin{align*}
e & \rightarrow \frac{e + \frac{B + iI}{B - iI} e}{\sqrt{2}} = \frac{B}{B - iI} e := w_+, \\
\frac{e - \frac{B + iI}{B - iI} e}{\sqrt{2} i} & = \frac{-1}{B - iI} e := w_-,
\end{align*}
\]

where

\[
B_0^+ w_+ = w_-, \quad B_0^- w_- = -w_+.
\]

Then the solution \( u = u_0 + \frac{B}{B - iI} \xi_+ - \frac{I}{B - iI} \xi_- \) of the non-homogeneous adjoint equation

\[
B^+ u = Bu_0 - \frac{I}{B - iI} \xi_+ - \frac{B}{B - iI} \xi_- = \lambda u + f
\]

may be found from the condition:

\[
(B - \lambda I) u_0 - \frac{I + \lambda B}{B - iI} \xi_+ + \frac{B - \lambda I}{B - iI} \xi_- = f.
\]

We require that the symplectic coordinates \( \xi_\pm \) satisfy the following boundary condition in terms of the \( g \)-Hermitian operator \( \Gamma \):

\[
\xi_- + \Gamma \xi_- = 0,
\]

and find an explicit formula for the resolvent of the of the corresponding extension \( B_\Gamma \). Assuming that the operator \( B - \lambda I \) is invertible we may multiply the above equation by \( \frac{B - iI}{B - \lambda I} \), thus obtaining

\[
(B - iI) u_0 = \frac{B - iI}{B - \lambda I} f + \frac{I + \lambda B}{B - \lambda I} \xi_+ + \xi_-.
\]

Substituting here \( \xi_- \) from the above boundary condition and projecting orthogonally (in the Pontryagin space) onto the subspace \( E \) we obtain an important connection between the symplectic coordinate \( \xi_+ \) of the solution \( u \) and \( f \):

\[
\xi_+ = \frac{I}{\Gamma - P_\xi \frac{I + \lambda B}{B - \lambda I} P_\xi} \frac{B - iI}{B - \lambda I} f.
\]

(74)
This connection permits us to calculate the resolvent of the extension:

\[ u = [B_\Gamma - \lambda I]^{-1} f = \]

\[
\begin{align*}
&\frac{I + \lambda B}{B - \lambda I} \frac{I}{B - i\lambda} \xi_+ - \frac{I}{B - i\lambda} \Gamma \xi_+ + \frac{I}{B - \lambda I} f + \frac{B}{B - i\lambda} \xi_+ - \frac{I}{B - i\lambda} \\
&\quad = \frac{I}{B - \lambda I} f + \frac{I + B^2}{(B - \lambda I)(B - i\lambda)} \xi_+ = \frac{I}{B - \lambda I} f + \frac{B + i\lambda}{B - \lambda I} \Gamma - \frac{i\lambda B}{B - \lambda I} P_{\epsilon} B - i\lambda f. 
\end{align*}
\]

(75)

The last expression gives the Krein formula for the resolvent of the extension \( B_\epsilon \) of the reduced operator.

VIII. APPENDIX B: SINGULAR \( \delta \)-POTENTIALS

We consider here potentials which are often used as an approximation of, for example, a hard sphere potential or a 3-dimensional deep square well potential \[30\] when the atom-atom interactions are effectively weak and dominated by elastic s-wave scattering:

\[ V(r - r') = U_0 \delta(r - r'), \]  

(76)

where

\[ U_0 = \frac{2\pi a \hbar^2}{\mu}, \quad \frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}. \]  

(77)

Here \( \mu \) is the reduced mass (\( \mu = m/2 \) when the scattering particles are identical) and \( a \) is the s-wave scattering length. The sign of the parameter \( a \) (\( a > 0 \) or \( a < 0 \) for repulsive and attractive interactions respectively) depends sensitively on the precise details of the interatomic potential. This potential is considered a good approximation when three-body processes can be neglected and hence we need take only binary collisions into account. Note that expression \((77)\) for \( U_0 \) is found by the Born approximation in the case of low-energy \( (k \to 0) \) scattering (though this approximation is not valid here, see below). Moreover one can suppose that the potential \((76)\) yields the total cross-section as \( \sigma = 4\pi a^2 \) in the limit \( k \to 0 \) when the scattering particles are different and \( \sigma = 8\pi a^2 \) for identical particles (Bose statistics). Evidently this potential is pointlike and hence, in some sense, one may consider it as a simple example of a zero-range potential. For \( U_0 < 0 \) this potential represents a deep well, see the Introduction of this paper. Nevertheless we show below that the potential \((76)\) is not the correct approximation for atom-atom collisions in both cases \( a > 0 \) and \( a < 0 \) and hence cannot represent some kind of zero-range potential. Actually this fact is well known. However, we examine it in detail as it demonstrates some non-trivial aspects of the theory of zero-range potentials and because this \( \delta \)-singular potential is still very popular \[30\].

We may consider the Dirac \( \delta \)-function in \((76)\) as a \( \delta \)-sequence \( \delta_\epsilon(r - r') \) given by:

\[
\delta_\epsilon(r - r') = \begin{cases} 
\frac{3}{4\pi \epsilon r_0^3}, & \text{if } |r - r'| \leq \epsilon r_0 \\
0, & \text{if } |r - r'| > \epsilon r_0.
\end{cases}
\]  

(78)
Then the potential (76) is \( V(r) = V_0 \) if \( r \leq \varepsilon r_0 \), and \( V(0) = 0 \) if \( r > \varepsilon r_0 \), where

\[
V_0 = \frac{3a\hbar^2}{2\mu\varepsilon^3 r_0^3}.
\]

(79)

In case 1 when \( U_0 > 0 \ (a > 0) \) and \( k \to 0 \) the total cross-section is \( \sigma \):

\[
\sigma = 4\pi\varepsilon^2 r_0^2 \left( \frac{\tanh(\varepsilon q_0 r_0)}{\varepsilon q_0 r_0} - 1 \right)^2,
\]

(80)

where \( q_0 = \sqrt{2\mu V_0 / \hbar^2} \sim \varepsilon^{-3/2} \).

For \( \varepsilon \to 0 \) we have \( \sigma \sim \varepsilon^2 \) and hence the total cross-section \( \sigma \) for the singular potential (76) is zero which means that no scattering occurs in this case!

We note that the conventional derivation of the Gross-Pitaevskii equation is based on the potential (76). Fortunately this equation may be found [31] without using the \( \delta \)-approximation, see also [32].

In case 2 when \( U_0 < 0 \ (a < 0) \) the potential (76) represents an infinitely deep well: \( V(r) = -|V_0| \) if \( r \leq \varepsilon r_0 \), and \( V(0) = 0 \) if \( r > \varepsilon r_0 \). Then the total cross-section for \( k \to 0 \) is:

\[
\sigma = 4\pi\varepsilon^2 r_0^2 \left( \frac{\tanh(\varepsilon q_0 r_0)}{\varepsilon q_0 r_0} - 1 \right)^2,
\]

(81)

with \( q_0 = \sqrt{2\mu |V_0| / \hbar^2} \sim \varepsilon^{-3/2} \). The limit \( \varepsilon \to 0 \) does not exist in this equation which means that the potential (76) is not correct in the case \( U_0 < 0 \ (a < 0) \) at all.

Thus the singular potential (76) is not a proper approximation for scattering processes in both cases 1 and 2. Actually the correct approximation of the hard sphere potential (case 1) is given in the papers [33] and [34], see also [35], and the correct approximation for the deep well (case 2) was developed by Fermi [3], see also the Introduction.

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[36] “Generating” means: non-orthogonal to all eigenvectors $e_s$ of the operator $B$, $[e, e_s] \neq 0$, $s = 1, 2, \ldots N$.

[37] Recall that we are using the “physical” notations for the dot-product $[\tilde{\alpha} \tilde{x}, \tilde{y}] = \alpha [\tilde{x}, \tilde{y}] = [\tilde{x}, \alpha \tilde{y}]$