Privacy-Preserving Policy Synthesis in Markov Decision Processes

Parham Gohari, Matthew Hale and Ufuk Topcu

Abstract—In decision-making problems, the actions of an agent may reveal sensitive information that drives its decisions. For instance, a corporation’s investment decisions may reveal its sensitive knowledge about market dynamics. To prevent this type of information leakage, we introduce a policy synthesis algorithm that protects the privacy of the transition probabilities in a Markov decision process. We use differential privacy as the mathematical definition of privacy. The algorithm first perturbs the transition probabilities using a mechanism that provides differential privacy. Then, based on the privatized transition probabilities, we synthesize a policy using dynamic programming. Our main contribution is to bound the “cost of privacy,” i.e., the difference between the expected total rewards with privacy and the expected total rewards without privacy. We also show that computing the cost of privacy has time complexity that is polynomial in the parameters of the problem. Moreover, we establish that the cost of privacy increases with the strength of differential privacy protections, and we quantify this increase. Finally, numerical experiments on two example environments validate the established relationship between the cost of privacy and the strength of data privacy protections.

I. INTRODUCTION

In many decision-making problems, agents desire to protect sensitive information that drives their actions from eavesdroppers and adversaries, such as applications in autonomous driving or smart power grids [1], [2]. In these applications, as well as in many other sequential decision-making problems, choosing actions can be cast as a policy-synthesis problem wherein the environment is modeled as a Markov decision process (MDP) [3], [4]. The goal in a policy-synthesis problem is to find a reward-maximizing control policy based on the transition probabilities of the underlying MDP. In this work, we study the problem of synthesizing a policy that protects the privacy of the transition probabilities.

Transition probabilities in an MDP govern the dynamics of the environment and may carry information that should be protected during policy synthesis. For example, suppose that through market research, a corporation discovers a niche in the market and decides to invest. Such an investment may alert competitors to the discovered niche and lead to other firms making similar decisions. Previous works in economics have associated higher market shares with profitability [5], [6]. Therefore, competitors’ entrance to the market may be harmful to the investing corporation. As a result, it is often crucial for a decision-maker to choose actions that do not reveal its knowledge about its environment dynamics.

We use differential privacy as the definition of privacy for an MDP’s transition probabilities. Differential privacy, first introduced in [7], is a property of an algorithm and has been used in the computer science literature as a quantitative definition of privacy for databases [8], [9]. It has also recently been used in control theory [10], [11]. Differential privacy makes it unlikely that the output of a differentially private algorithm will reveal any useful information about the individual entries of the input dataset; however, it may still pass on information about the aggregate statistics of the input dataset that are useful in down-stream analytics.

The main contribution of this paper is to develop a policy-synthesis algorithm that enforces differential privacy for transition probabilities with adjustable privacy and utility. We define the utility of a privacy-preserving policy synthesis algorithm to be the value function associated with the policy, which in an MDP is its expected total reward [12]. Utility loss due to privacy is a common phenomenon, and we follow the convention in the differential privacy literature to analyze the utility of the privacy-preserving algorithm by comparing it to its non-private counterpart [13], [14].

In order to show that the algorithm enforces differential privacy, we exploit the fact that differential privacy is immune to post-processing [13]. By immunity to post-processing, we mean that arbitrary functions of the output of a differentially private algorithm do not weaken its privacy guarantees. The algorithm first privatizes the transition probabilities via the Dirichlet mechanism [15]. We then use dynamic programming to synthesize a policy based on the privatized transition probabilities. Since the dynamic programming stage is an act of post-processing on the output of a differentially private mechanism, its output preserves the differential privacy provided to transition probabilities.

We employ the Dirichlet mechanism for privatization because it preserves the unique structure of the transition probabilities, i.e., vectors with non-negative components that sum to one. Using traditional differentially private mechanisms that add infinite-support noise to transition probabilities are ill-suited to this work as they break the structure of transition probabilities. For example, they can result in a transition probability vector with negative components. Although normalization may seem a fitting solution in order to project the perturbed vector back onto the unit simplex, we avoid normalization because it makes it difficult to quantify utility.

We introduce the “cost of privacy” as a measure of
the utility of the algorithm. We define the cost of privacy to be the difference between the expected total rewards of the policy with privacy and that of the same policy without privacy. Since we perturb the transition probabilities to enforce differential privacy, the output of the dynamic-programming stage is susceptible to suboptimality, which the cost of privacy quantifies.

We bound the cost of privacy for both finite- and infinite-horizon MDPs. For finite-horizon MDPs, we show that we can compute the cost of privacy in polynomial time via a backward-in-time recursive algorithm. For the case of infinite-horizon MDPs, we show that an algorithm similar to policy evaluation converges to the cost of privacy asymptotically. We further show that the number of iterations required to approximate the cost of privacy is polynomial in problem parameters. This work enables a decision-maker to control the level of privacy based on the utility loss that they are willing to tolerate.

In order to empirically validate the expressions that we introduce for the cost of privacy, we run the algorithm on two example MDPs. The first example is a small MDP that models a corporation’s investment. The second example is an MDP with a larger state and action space, with its transition probabilities generated randomly. We run the algorithm at a range of privacy levels and visualize our results by plotting the cost of privacy versus privacy level. Furthermore, we observe that the bounds we provide for the cost of privacy are meaningful in the sense that they empirically provide a close approximation to the cost of privacy.

Related work. The works in [16]–[18] study the problem of learning a policy in an MDP while enforcing differential privacy. The key difference between this paper and the works above is that we protect the transition probabilities which belong to the probability simplex, whereas the other works protect the sensory data that are scalars. We emphasize that although scalars can be readily privatized using traditional differentially private mechanisms, transition probabilities need to be treated specially to ensure that they remain non-negative and sum to one.

The problems of robust and distributionally robust MDPs are related to this paper. Robust policy synthesis in an MDP is the problem of synthesizing a policy that mitigates uncertainties present in transition probabilities [19], [20]. Distributionally robust MDPs assume that the planner has access to a probability measure over the uncertainty sets [21].

We base the cost of privacy bounds on a concentration bound that we derive for the output of the Dirichlet mechanism. Finding the worst-case cost of privacy coincides with lower bounding the value of a distributionally robust policy where the uncertainties in the transition probabilities adhere to the concentration bound of the Dirichlet mechanism. Despite the fact that the solutions take similar forms, this paper studies the value loss due to privacy, whereas the problem of robust policy synthesis tries to compute a policy that mitigates the effect of uncertainties.

II. PRELIMINARIES

In this section we set the notation and definitions used throughout the paper.

A. Notation

We denote the set of real numbers by \( \mathbb{R} \). Let \( (\cdot)^T \) denote the transpose of a vector. We define the unit simplex to be \( \Delta(n) := \{ x \in \mathbb{R}^n \mid 1^T x = 1, x \geq 0 \} \), where \( 1 \) is the vector of all ones in \( \mathbb{R}^n \) and the inequality is evaluated element wise. We use the notation \( \Delta^c(n) \) to denote the interior of \( \Delta(n) \). For a finite set \( A \), its cardinality is denoted by \( |A| \). \( \mathbb{E} [\cdot] \) and \( \text{Var}(\cdot) \) denote the expectation and the variance of a random variable, respectively. \( \| \cdot \|_1 \) and \( \| \cdot \|_\infty \) denote the one and infinity norm of a vector, respectively. For a vector \( p \), we use the notation \( p_i \) to denote the \( i \)th component of \( p \). We use the gamma function

\[
\Gamma(z) := \int_0^\infty x^{z-1} \exp(-x) dx.
\]

B. Markov decision processes

An MDP is a tuple \( \mathcal{M} = (\mathcal{S}, A_s, r, P, T, \gamma) \) where \( \mathcal{S} \) is the set of states, \( A_s \) is the set of available actions at state \( s \in \mathcal{S} \), and \( r : \mathcal{S} \times A_s \rightarrow \mathbb{R} \) is the reward function that indicates the one-step reward for taking action \( a \) at state \( s \). \( P := \{ P(s, a) \in \Delta(|\mathcal{S}|) \mid (s, a) \in \mathcal{S} \times A_s \} \) is the set of transition probabilities. Finally, \( T \) is the time horizon and \( \gamma \) is the discount factor.

We now define a policy, that is, a rule for making a sequence of decisions in an MDP. In particular, let \( h_t := \{ s_0, a_0, r_0, s_1, a_1, r_1, \ldots, s_t \} \) be a history until stage \( t \), and let \( \mathcal{H}_t(s_t) \) denote the set of all possible histories that end in state \( s_t \). A policy \( \pi : \mathcal{H}_t(s_t) \rightarrow \Delta(|A_s|) \) maps a history \( h_t \) to a probability distribution over the set of actions, \( A_s \).

A policy \( \pi \) is evaluated by its value function \( V^\pi_t : \mathcal{S} \rightarrow \mathbb{R} \), that is defined as

\[
V^\pi_t(s) := \mathbb{E} \left[ \sum_{t=0}^T \gamma^{t-1} r_i \mid s_t = s \right].
\]

The expectation is taken over the stochasticity of the policy \( \pi \) and transition probabilities \( P \). We study the problem of privacy-preserving policy synthesis, and in a synthesis problem, the goal is to find an optimal policy, in the sense that it achieves the highest value function beginning at initial state \( s_0 \).

In this paper, we restrict our attention to Markovian policies, i.e., the class of policies that only depend on the most recent state of the history. Markovian policies are shown to be optimal under some mild conditions [22]. We use the notation \( \pi_t(a \mid s) \) to show the probability of taking action \( a \) at state \( s \) and stage \( t \).

C. Differential privacy

For an algorithm that satisfies differential privacy, it is unlikely to tell apart nearby input datasets based on observations of the algorithm’s output. Nearby datasets are defined formally by an adjacency relationship. We first state the adjacency relationship used in this paper.
Definition 1 (From [15], Definition 1). For a constant \( b \in (0, 1] \), two vectors \( p, q \in \Delta(n) \) are said to be \( b \)-adjacent if there exist indices \( i, j \) such that
\[ p_{(i,j)} = q_{(i,j)} \text{ and } ||p - q||_1 \leq b. \]

The above definition considers two vectors in the unit simplex adjacent if they only differ in two indices, \( i, j \), by no more than \( b \) in their 1-norm. Note that the usual adjacency relationship in the differential privacy literature considers two input datasets adjacent if they only differ in one entry [13]; however, it is not possible for the elements of the unit simplex to differ in only one entry because their components must sum to one.

Definition 2 (Probabilistic differential privacy [15]). Fix a probability space \((\Omega, F, P)\) and \( b \in (0, 1] \). A mechanism \( M : \Delta(n) \times \Omega \rightarrow \Delta(n) \) is said to be probabilistically \((\epsilon, \delta)\)-differentially private if, for all \( p \in \Delta(n) \), we can partition the output space \( \Delta(n) \) into two disjoint sets, \( \Omega_1, \Omega_2 \), such that \( P[M(p) \in \Omega_2] \leq \delta \) and, for all \( q \in \Delta(n) \) \( b \)-adjacent to \( p \), we have that
\[ \log \left( \frac{P[M(p) = x]}{P[M(q) = x]} \right) \leq \epsilon, \forall x \in \Omega_1. \]

Probabilistic \((\epsilon, \delta)\)-differential privacy is known to imply ordinary \((\epsilon, \delta)\)-differential privacy [23].

D. The Dirichlet mechanism

A Dirichlet mechanism with parameter \( k > 0 \) takes as input a vector \( p \in \Delta_k \) and outputs \( x \in \Delta(n) \) according to a Dirichlet probability distribution. Fix \( k \) and let \( M_{D}^{(k)} \) denote the Dirichlet mechanism. Then
\[ P \left[ M_{D}^{(k)} (p) = x \right] = \frac{1}{B(kp)} \prod_{i=1}^{n-1} x_i^{kp_{i}-1} \left( 1 - \sum_{i=1}^{n-1} x_i \right)^{k_{p,n}-1}, \]
where
\[ B(kp) := \frac{\prod_{i=1}^{n} \Gamma(kp_{i})}{\Gamma \left( k \sum_{i=1}^{n} p_{i} \right)} \]
is the multi-variate beta function.

The Dirichlet mechanism satisfies probabilistic \((\epsilon, \delta)\)-differential privacy [15], and has the following properties. The expected value of the output is equal to the input vector, i.e., \( E \left[ M_{D}^{(k)} (p) \right] = p \). An increase in \( k \) results in weaker differential privacy protections, and in particular it increases \( \epsilon \). However, as \( k \) increases, the output becomes more concentrated around the input vector \( p \).

III. PRIVACY-PRESERVING SYNTHESIS ALGORITHM

In this section, we first present the proposed privacy-preserving synthesis algorithm. Then, we show the differential privacy of the algorithm.

Algorithm 1: Privacy-preserving synthesis algorithm

**Input:** \((S, A_s, r, P, T, \gamma)\), \( k \)

**Output:** \( \overline{\pi}, \overline{V} \)

1. Construct the set of privatized transition probabilities \( \overline{P} := \left\{ P(s,a) = M_{D}^{(k)} (P(s,a)) \mid P(s,a) \in P \right\} \).
2. Replace \( M \) with its privatized version \( \overline{M} := (\overline{S}, \overline{A}_s, r, \overline{P}, T, \gamma) \).
3. Synthesize policy \( \overline{\pi} \) for \( \overline{M} \).
4. Compute the value function of \( \overline{\pi}, \overline{V} \).

A. Algorithm

The algorithm takes as input an MDP representation \( M = (S, A_s, r, P, T, \gamma) \) and the value of \( k \) that is the parameter for the Dirichlet mechanism. It outputs a policy \( \overline{\pi} \) and its value function \( \overline{V} \). The algorithm comprises two stages. The first stage privatizes the transition probabilities by applying the Dirichlet mechanism independently on each transition probability vector in \( P \). Let \( \overline{P} := \left\{ \overline{P}(s,a) = M_{D}^{(k)} (P(s,a)) \mid P(s,a) \in P \right\} \) be the set of transition probabilities after privatization. The second stage finds an optimal policy and the optimal value of the privatized MDP \( \overline{M} := (\overline{S}, \overline{A}_s, r, \overline{P}, T, \gamma) \). An optimal policy is one that satisfies the Bellman condition of optimality, and the optimal value is the value of such policies [22]. In the case of a finite-horizon MDP, the second stage finds \( (\overline{\pi}, \overline{V}_t) \) such that for all \( t \in \{0, \ldots, T-1\} \) and all \( s \in S \),
\[ \overline{V}_t(s) = \max_{a \in A_s} \overline{\pi}(a \mid s) \left( r(s,a) + \gamma \sum_{s' \in S} \overline{P}(s,a,s') \overline{V}_{t+1}(s') \right), \]
\[ \overline{\pi}_t \in \arg \max_{a \in A_s} \overline{\pi}(a \mid s) \left( r(s,a) + \gamma \sum_{s' \in S} \overline{P}(s,a,s') \overline{V}_{t+1}(s') \right), \]
where \( \overline{P}(s,a,s') \) denotes the privatized probability that taking action \( a \) at state \( s \) takes the agent to state \( s' \). We assume that the terminal values are given by a known function \( R_T : S \rightarrow \mathbb{R} \), i.e., \( \overline{V}_T(s) = R_T(s) \), for all \( s \in S \).

For an infinite-horizon discounted MDP, it can be shown that the optimal policy is a stationary policy, i.e., a policy that adopts the same decision rule at all stages [22]. Let \( \overline{V}_\infty \) denote the optimal value of \( \overline{M} \). Then, for an infinite-horizon MDP, the second stage of Algorithm [1] computes \( (\overline{\pi}, \overline{V}_\infty) \) such that for all \( s \in S \),
\[ \overline{V}_\infty(s) = \max_{a \in A_s} \overline{\pi}(a \mid s) \left( r(s,a) + \gamma \sum_{s' \in S} \overline{P}(s,a,s') \overline{V}_\infty(s') \right), \]
\[ \overline{\pi} \in \arg \max_{a \in A_s} \overline{\pi}(a \mid s) \left( r(s,a) + \gamma \sum_{s' \in S} \overline{P}(s,a,s') \overline{V}_\infty(s') \right). \]

There are various methods suggested to efficiently compute \( \overline{\pi} \) and its value function, such as dynamic programming or linear programming [22]. The third and the fourth step of Algorithm [1] may adopt any of these methods to synthesize and evaluate an optimal policy for the privatized MDP \( \overline{M} \).
B. Proof of differential privacy

We prove that Algorithm 1 is \((\epsilon, \delta)\)-differentially private by differential privacy’s immunity to post-processing.

Lemma 1 (From [13], Proposition 2.1). Let \(\mathcal{M} : \Delta(n) \mapsto \Delta(n)\) be a mechanism that is \((\epsilon, \delta)\)-differentially private. Let \(f : \Delta(n) \mapsto \mathbb{R}\) be an arbitrary mapping. Then, \(f \circ \mathcal{M} : \Delta(n) \mapsto \mathbb{R}\) is \((\epsilon, \delta)\)-differentially private.

Recall that probabilistic \((\epsilon, \delta)\)-differential privacy implies ordinary \((\epsilon, \delta)\)-differential privacy. Let \((\epsilon, \delta)\) denote the level of the probabilistic differential privacy of the Dirichlet mechanism employed in Algorithm 1. By Lemma 1, the algorithm is \((\epsilon, \delta)\)-differentially private because the synthesis step is an instance of a post-processing mapping \(f\).

IV. UTILITY ANALYSIS

Recall that Algorithm 1 synthesizes a policy \(\bar{\pi}\) based on privatized transition probabilities in \(\mathcal{P}\). It then computes the value function of \(\bar{\pi}\), \(V^\pi\), using \(\mathcal{P}\). Let \(V^\pi : \mathcal{S} \mapsto \mathbb{R}\) be the value function that assigns to each non-private transition probabilities in \(\mathcal{P}\) assign to \(\bar{\pi}\). The utility of Algorithm 1 is equal to \(V^\pi(s_0)\). We start off the utility analysis of Algorithm 1 introducing a concentration bound on the output of the Dirichlet mechanism.

Lemma 2. Let \(\mathcal{M}_D^{(k)}\) denote a Dirichlet mechanism with parameter \(k \in \mathbb{R}_+\). Then, for all \(\beta > 0\), and all \(p \in \Delta^n\),

\[
\mathbb{P} \left( \|\mathcal{M}_D^{(k)}(p) - p\|_\infty \geq \sqrt{\frac{\log(1/\beta)}{2(k+1)}} \right) \leq \beta.
\]

Proof. See Appendix II.

The above lemma enables us to evaluate \(V^\pi(s_0)\), i.e., the conditional expectation of the value function without privacy, based on the privatized transition probabilities \(\mathcal{P}\) and \(k\). In particular, for a finite-horizon MDP we provide an upper bound on \(\mathbb{E} \left[ V^\pi_0(s_0) \mid \mathcal{P}, k \right] - \bar{V}^\pi_0(s_0)\). For an infinite-horizon MDP, we upper bound \(\mathbb{E} \left[ V^\pi_\infty(s_0) \mid \mathcal{P}, k \right] - \bar{V}^\pi_\infty(s_0)\). We refer to both expressions as the “cost of privacy.”

The bounds are based on the pessimistic and optimistic value functions that possible transition-probability vectors generate. Let \(\alpha := \sqrt{\log(1/\beta)/2(k+1)}\), then Lemma 2 implies for all \(P(s, a) \in \mathcal{P}\) and the corresponding \(\bar{P}(s, a) \in \mathcal{P}\), \(\mathbb{P} \left( \|P(s, a) - \bar{P}(s, a)\|_\infty \leq \alpha\right) \geq 1 - \beta\). We define \(\bar{\mathcal{P}}_{\alpha, \beta}\) as

\[
\{\beta P_1(s, a) + (1-\beta)P_2(s, a) \mid \|P_2(s, a) - \bar{P}(s, a)\|_\infty \leq \alpha, \ P_1(s, a), P_2(s, a) \in \Delta(|S|), (s, a) \in S \times A_s\}
\]

We use the set \(\bar{\mathcal{P}}_{\alpha, \beta}\) to compute a pessimistic and optimistic value function to bound the cost of privacy.

A. Finite-horizon MDPs

We bound the cost of privacy for a finite-horizon MDP by establishing a common upper and lower bound for both \(\mathbb{E} \left[ V^\pi_0(s_0) \mid \mathcal{P}, k \right]\) and \(\bar{V}^\pi_0(s_0)\). We first state a technical lemma that we later use to prove the main theorem of this section.

Lemma 3. Fix \(k\) and a set of transition probabilities \(\mathcal{P}\), and let \(\bar{P} := \left\{ \bar{P}(s, a) = \mathcal{M}_D^{(k)}(P(s, a)) \mid P(s, a) \in \mathcal{P}\right\}\). For any \(\beta > 0\), let \(\alpha := \sqrt{\log(1/\beta)/2(k+1)}\). Then,

\[
\mathbb{E} \left[ P(s, a) \mid \bar{P}, k \right] \in \bar{\mathcal{P}}_{\alpha, \beta}, \ \forall P(s, a) \in \mathcal{P}.
\]

Proof. See Appendix III.

Theorem 1. Let \(\mathcal{M} = (\mathcal{S}, A_s, r, P, T, \gamma)\) and \(k\) be the input, and \((\bar{\pi}, \bar{V}^\pi)\) be the output of Algorithm 1 and let \(T < \infty\). Fix \(\beta > 0\), and let \(\alpha := \sqrt{\log(1/\beta)/2(k+1)}\). Let \(R_T : \mathcal{S} \mapsto \mathbb{R}\) denote the terminal value function of \(\mathcal{M}\). Define \(\bar{r}^\pi : \mathcal{S} \mapsto \mathbb{R}\) and \(\bar{v}^\pi : \mathcal{S} \mapsto \mathbb{R}\) as follows. For all \(s \in \mathcal{S}\), let \(\bar{r}^\pi(s) := R_T(s), \bar{v}^\pi(s) := R_T(s)\), and for all \(t \in \{0, \ldots, T - 1\}\), let

\[
\bar{v}^\pi_t(s) := \sum_{a \in A_s} \bar{\pi}(a \mid s) r(s, a) + \gamma \min_{p \in \bar{\mathcal{P}}_{\alpha, \beta}} \sum_{s' \in \mathcal{S}} p(s, a, s') \bar{v}^\pi_{t+1}(s'),
\]

\[
\bar{v}^\pi_t(s) := \sum_{a \in A_s} \bar{\pi}(a \mid s) r(s, a) + \gamma \max_{p \in \bar{\mathcal{P}}_{\alpha, \beta}} \sum_{s' \in \mathcal{S}} p(s, a, s') \bar{v}^\pi_{t+1}(s').
\]

Then, we have that

\[
\mathbb{E} \left[ V^\pi_0(s_0) \mid \bar{P}, k \right] - \bar{V}^\pi_0(s_0) \leq \bar{v}^\pi_0(s_0).
\]

Proof. We first show by induction that for all stages \(t \in \{0, 1, \ldots, T\}\), and all states \(s \in \mathcal{S}\), \(\bar{v}^\pi_t(s)\) lower bounds \(\bar{V}^\pi_t(s)\). Since all terminal values are determined by \(R_T\), we have that for all states, \(\bar{V}^\pi_T(s) = \bar{v}^\pi_T(s) = R_T(s)\). Let \(\tau \in \{1, \ldots, T - 1\}\), and assume that for all states, we have that \(\bar{V}^\pi_\tau(s) \geq \bar{v}^\pi_\tau(s)\). Then, for \(t = \tau - 1\), and for all \(s \in \mathcal{S}\), we can write

\[
\bar{V}^\pi_{\tau-1}(s) = \sum_{a \in A_s} \bar{\pi}(a \mid s) r(s, a) + \gamma \sum_{s' \in \mathcal{S}} \bar{P}(s, a, s') \bar{V}^\pi_{\tau}(s') \geq \sum_{a \in A_s} \bar{\pi}(a \mid s) r(s, a) + \gamma \sum_{s' \in \mathcal{S}} \bar{P}(s, a, s') \bar{v}^\pi_{\tau}(s') \geq \sum_{a \in A_s} \bar{\pi}(a \mid s) r(s, a) + \gamma \min_{p \in \bar{\mathcal{P}}_{\alpha, \beta}} \sum_{s' \in \mathcal{S}} p(s, a, s') \bar{v}^\pi_{\tau}(s') = \bar{v}^\pi_{\tau-1}(s).
\]
The first inequality immediately results from the induction hypothesis. For the second inequality, note that by Lemma \[ \bar{V}(s,a) \in \bar{P}; \] therefore, \[ \bar{V}(s,a) \] is a feasible solution of the minimization problem in (2).

By induction, we conclude that for all \( t \in \{0, \ldots, T\} \), and all \( s \in S \), \( \bar{v}^\pi_i(s) \) lower bounds \( \bar{V}^\pi_i(s) \), which includes stage \( t = 0 \) and state \( s = s_0 \), hence, \( \bar{v}^\pi_0(s_0) \geq \bar{v}^\pi(s_0) \). An identical argument by reversing the direction of the inequalities and substituting the minimization in (2) with maximization, leads to \( \bar{v}^\pi_i(s) \) upper bounding \( \bar{V}^\pi_i(s) \), for all \( s \in S \) and all \( t \in \{0, \ldots, T\} \). So far we have established that

\[
\bar{v}^\pi(s_0) \leq \bar{V}^\pi(s_0) \leq \bar{v}^\pi(s_0).
\]

We now consider the value of \( \bar{V}^\pi \) based on the non-private transition probabilities in \( \bar{P} \), that is denoted \( \bar{V}^\pi \). Recall that for all stages \( t \in \{1, \ldots, T\} \) and all states \( s \in S \), \( \bar{V}^\pi(s) \) satisfies the following conditions:

\[
\bar{V}^\pi_{t-1}(s) = \sum_{a \in A_s} \bar{\pi}(a | s) \left( r(s, a) + \gamma \sum_{s' \in S} P(s, a, s') \bar{V}^\pi_t(s') \right).
\]

Taking the expectation of both sides, we arrive at

\[
E \left[ \bar{V}^\pi_{t-1}(s) \mid \bar{P}, k \right] = \sum_{a \in A_s} \bar{\pi}(a | s) \left( r(s, a) + \gamma \sum_{s' \in S} E \left[ P(s, a, s') \mid \bar{P}, k \right] E \left[ \bar{V}^\pi_t(s') \mid \bar{P}, k \right] \right).
\]

In order to establish an iterative relation between the values of \( E \left[ V^\pi_{t-1}(s) \mid \bar{P}, k \right] \) for different stages, we need to break the expectation on the right-hand side of the above equation. Consider the scenario wherein at each stage, the transition probabilities are independently privatized using the Dirichlet mechanism. Let \( \bar{V}^\pi \) denote the value function that is associated with \( \bar{\pi} \) according to the above scenario. Due to the independence assumption, we can write

\[
E \left[ \bar{V}^\pi_{t-1}(s) \mid \bar{P}, k \right] = \sum_{a \in A_s} \bar{\pi}(a | s) \left( r(s, a) + \gamma \sum_{s' \in S} E \left[ P(s, a, s') \mid \bar{P}, k \right] E \left[ \bar{V}^\pi_t(s') \mid \bar{P}, k \right] \right).
\]

Assume that for an arbitrary \( \tau \in \{1, \ldots, T - 1\} \) and for all \( s \in S \), it holds that \( E \left[ \bar{V}^\pi_{\tau}(s') \mid \bar{P}, k \right] \geq \bar{v}^\pi_\tau(s) \). Then, for \( t = \tau - 1 \), it holds that

\[
E \left[ V^\pi_{t-1}(s) \mid \bar{P}, k \right] \geq \sum_{a \in A_s} \bar{\pi}(a | s) \left( r(s, a) + \gamma \sum_{s' \in S} E \left[ P(s, a, s') \mid \bar{P}, k \right] \bar{v}^\pi(s') \right).
\]

By Lemma \[ \bar{P}(s,a) \in \bar{P}; \] \( E \left[ P(s, a) \mid \bar{P}, k \right] \) belongs to the set \( \bar{P}_{\alpha, \beta} \). Therefore we can write

\[
E \left[ \bar{V}^\pi_{t-1}(s) \mid \bar{P}, k \right] \geq \sum_{a \in A_s} \bar{\pi}(a | s) \left( r(s, a) + \gamma \sum_{s' \in S} \min_{p \in \bar{P}_{\alpha, \beta}} P(s, a, s') \bar{v}^\pi(s') \right) = \bar{v}^\pi_{t-1}(s).
\]

Therefore, by induction, we have shown that for all stages and all states, \( \bar{v}^\pi \) is a lower bound to \( E \left[ \bar{V}^\pi \mid \bar{P}, k \right] \).

We now revisit the independence assumption. We introduce the worst-case expected value function that possible transition probabilities can generate as the lower bound to the value function of the generated policy without privacy. The independence assumption leads to independent minimization problems at each stage. The worst-case value function without the independence assumption has the following restriction on the transition probabilities. Transition probabilities that appear as a minimization variable for the same state-action pair at different stages must take equal values. Since the feasible region of the case without independence is a subset of that of the case with independence, the lower bound for \( E \left[ \bar{V}^\pi \mid \bar{P}, k \right] \) also lower bounds \( E \left[ V^\pi \mid \bar{P}, k \right] \).

Similarly, an identical argument by reversing the direction of the inequalities, and substituting the min operators with max operators leads to \( E \left[ V^\pi \mid \bar{P}, k \right] \) also lower bounds for all stages and all states. Therefore,

\[
\bar{v}^\pi(s_0) \leq E \left[ V^\pi_0(s_0) \mid \bar{P}, k \right] \leq \bar{v}^\pi(s_0).
\]

**Proof.** See Appendix \[ III \].
Theorem 2. Let \( \mathcal{M} = (S, A_r, r, \mathcal{P}, T, \gamma) \) and \( k \) be the input, and \( (\vec{\pi}, \vec{V}_n^\pi) \) be the output of Algorithm 1 and let \( T = \infty \). Fix \( \beta > 0 \), and let \( \alpha := \sqrt{\log(1/\beta)/2(k+1)} \). For all \( s \in S \), let \( \vec{v}_{\infty}^\pi : S \rightarrow \mathbb{R} \) and \( \vec{v}_{\infty}^\pi : S \rightarrow \mathbb{R} \) satisfy
\[
\vec{v}^\pi_{\infty}(s) = \sum_{a \in A_r} \bar{\pi}(a | s) \left( r(s, a) + \gamma \min_{p \in \mathcal{P}_n, s' \in S} p(s, a, s') \vec{v}^\pi_{\infty}(s') \right),
\]
\[
\vec{v}_\pi^\infty(s) = \sum_{a \in A_r} \bar{\pi}(a | s) \left( r(s, a) + \gamma \max_{p \in \mathcal{P}_n, s' \in S} p(s, a, s') \vec{v}_\pi^\infty(s') \right).
\]
Then, we have that
\[
|E \left[ V_\infty^\pi(s_0) | \bar{\mathcal{P}}, k \right] - \bar{V}_\infty^\pi(s_0)| \leq \vec{v}^\pi_{\infty}(s_0) - \vec{v}_\pi^\infty(s_0).
\]

Proof. Consider the \( \gamma \)-contraction mappings \( L_1, L_2 \) and \( L_3 \) introduced in Lemma 3. Notice that \( \vec{v}^\pi_{\infty}, \vec{V}^\pi_{\infty} \) and \( \bar{E} \left[ V_\infty^\pi(s_0) | \bar{\mathcal{P}}, k \right] \) are the fixed points of mappings \( L_1, L_2 \) and \( L_3 \), respectively. Contraction mappings are known to admit a fixed point by Banach fixed-point theorem. Furthermore, contractive mappings converge to their fixed points by applying the mapping repeatedly to an arbitrary initial \( v \in \mathbb{R}|S| \). Let \( v_3^{(k)} \) denote the \( k \)-th iteration corresponding to \( L_1 \), and let all three mappings start from a common initial vector. Assume that at stage \( k \), it holds that \( v_1^{(k)} \leq v_2^{(k)} \), then for stage \( k+1 \), with a similar argument to the proof of Theorem 1 we can write
\[
v_2^{(k+1)}(s) = \sum_{a \in A_r} \bar{\pi}(a | s) \left( r(s, a) + \gamma \sum_{s' \in S} \bar{P}(s, a, s') v_2^{(k)}(s') \right)
\geq \sum_{a \in A_r} \bar{\pi}(a | s) \left( r(s, a) + \gamma \sum_{s' \in S} \bar{P}(s, a, s') v_1^{(k)}(s') \right)
\geq \sum_{a \in A_r} \bar{\pi}(a | s) \left( r(s, a) + \gamma \text{min}_{p \in \mathcal{P}_n, s' \in S} p(s, a, s') v_1^{(k)}(s') \right)
= v_1^{(k+1)}(s).
\]
As a result, we have that the limiting value of \( v_1^{\infty} \) and \( v_2^{\infty} \) when \( k \rightarrow \infty \) also satisfy
\[
\vec{v}_\pi^\infty = \lim_{k \rightarrow \infty} v_1^{(k)} \leq \lim_{k \rightarrow \infty} v_2^{(k)} = \bar{V}_\infty^\pi.
\]
Using a similar argument, it can be shown that \( \bar{V}_\infty^\pi \) is upper bounded by \( \vec{v}_{\infty}^\pi \). Therefore we have that
\[
\vec{v}_{\infty}^\pi \leq \bar{V}_\infty^\pi \leq \vec{v}_\pi^\infty.
\]
(5)

We now assume that at stage \( k \), it holds that \( v_1^{(k)} \leq v_2^{(k)} \). At stage \( k+1 \), we write
\[
v_3^{(k+1)}(s) = \sum_{a \in A_r} \bar{\pi}(a | s) \left( r(s, a) + \gamma \mathbb{E} \left[ P(s, a, s') v_3^{(k)}(s') | \bar{\mathcal{P}}, k \right] \right).
\]
Notice that there is no stochasticity in \( v_3^{(k)} \). Therefore, we have that
\[
v_3^{(k+1)}(s) = \sum_{a \in A_r} \bar{\pi}(a | s) \left( r(s, a) + \gamma \mathbb{E} \left[ P(s, a, s') v_3^{(k)}(s') | \bar{\mathcal{P}}, k \right] \right).
\]
The above equation combined with the induction hypothesis implies that
\[
\vec{v}_3^{(k+1)}(s) \leq \vec{v}_3^{(k+1)}(s).
\]
By Lemma 3 we have that \( v_3^{(k+1)} \geq v_3^{(k+1)} \). Therefore,
\[
\vec{v}_\pi^\infty \leq \lim_{k \rightarrow \infty} v_3^{(k)} = \lim_{k \rightarrow \infty} \mathbb{E} \left[ V_\infty^\pi(s_0) | \bar{\mathcal{P}}, k \right].
\]
With a similar argument, it can be shown that \( \vec{v}_\pi^\infty \) upper bounds \( \mathbb{E} \left[ V_\infty^\pi(s_0) | \bar{\mathcal{P}}, k \right] \), which combined by (5) and (6) concludes the proof.

V. COMPUTATIONAL COMPLEXITY

In the previous section, we introduced expressions that bound the cost of privacy for both finite- and infinite-horizon MDPs. The bounds do not take a closed form, and they are computed by iterative methods. In this section, we show that the cost of privacy for both cases can be computed efficiently. In particular we show that for both cases, the computational complexity is polynomial in problem parameters.

A. Finite-horizon MDPs

Revisiting the definition of \( \vec{v}^\pi_{\infty} \) and \( \vec{v}_\pi^\infty \) in Theorem 1 the inner minimization or maximization problem must be solved for each of \( T \) stages and \( |S| \) states. We first consider computing the lower bound \( \vec{v}^\pi_{\infty} \). Fix \( (s, a) \in S \times A_r \) and \( t \in \{0, \ldots, T - 1\} \). Then the inner minimization problem can be recast as
\[
\min_{P_1, P_2 \in \mathbb{R}|S|} p(s, a, s') v_3^{(k)}(s')
\]
subject to
\[
1^T P_1(s, a) = 1, P_1(s, a, s') \geq 0, \quad \forall s' \in S,
1^T P_2(s, a) = 1, P_2(s, a, s') \geq 0, \quad \forall s' \in S,
1^T p(s, a) = 1, p(s, a, s') \geq 0, \quad \forall s' \in S,
P_2(s, a, s') - \bar{P}(s, a, s') \leq \alpha, \quad \forall s' \in S,
P_2(s, a, s') - \bar{P}(s, a, s') \geq -\alpha, \quad \forall s' \in S,
\beta P_1(s, a) + (1 - \beta) P_2(s, a) = p(s, a).
\]
The above optimization problem is a linear program (LP) with \( 3|S| \) variables and \( 6|S| + 3 \) constraints. Similarly, the inner maximization problem in \( \vec{v}_\pi^\infty \) can be cast as an LP by negating the objective function in (P).

There exists numerous algorithms for solving an LP, each of which is associated with different computational complexities. We consider the interior-point method that is known to solve an LP in polynomial time, \( O(n^{3.5}) \), where \( n \)
In this section, we empirically validate the developments of previous sections, wherein we introduced the expressions that compute the cost of privacy and their corresponding computational complexity. We apply Algorithm 1 to two example MDPs at a range of $k$ values, which represent a range of privacy protection levels. The first is a small-sized MDP that represents a simple model for a corporation’s investment planning. The second example is an MDP with a larger state and action space, which has transition probabilities, reward function, and terminal reward function generated randomly. For both examples, the algorithm is run 50 times, and Figure 1, which depicts the results, shows the mean values alongside their standard deviations that appear as error bars.

**Example 1.** Suppose a corporation has been tracking four startups, and it has to decide which startup to acquire. Assume that the corporation’s model of each of the startup’s probability of success is given by the MDP in Figure 2.

The first empirical result of this section corresponds to applying the algorithm to the scenario described in Example 1, and is depicted in Figure 1a. The results show that as $k$ increases, the pessimistic and optimistic value functions provide better bounds for the private and non-private value functions. Therefore the cost of privacy decreases with $k$, which Figure 1 confirms.

**Example 2.** In this example, we apply the algorithm to a larger MDP in order to test its scalability. In particular, the MDP has 20 states, 5 actions available at each state, and a time horizon of 10.

Similar to the previous example, Figure 1b indicates that an increase in $k$ improves the approximations of the private and non-private value functions by $\overline{V}_0^\pi(s_0)$ and $\overline{V}_0^\pi(s_0)$. As a result, the cost of privacy must decrease with $k$, which is confirmed by Figure 1.
In Example 2, the computation of the cost of privacy for each instance of $k$ took 4.88s on a desktop computer with a 3.5GHz CPU and 24GB of RAM. Doubling the time horizon of the MDP to 20 results in 9.34s/itr, and doubling the size of the action space to $|A_s| = 10$ results in 9.68s/itr, which is consistent with the computational complexity introduced in Section V.

For both examples, the negative correlation between $k$ and the cost of privacy is consistent with the developments in Section V. By Lemma 2, an increase in $k$ results in a tighter concentration bound on the output of the Dirichlet mechanism, and it lowers $\alpha$ in Theorems 1 and 2. A smaller $\alpha$ further restricts the inner optimization problem in [2]; thus, it helps the optimistic and the pessimistic value functions to provide better approximations, which leads to a lower cost of privacy.

VII. CONCLUSION

We introduced a privacy-preserving policy synthesis algorithm that protects the privacy of the transition probabilities of its input MDP. The algorithm employs a Dirichlet mechanism to privatize the transition probabilities. We established a concentration bound on the output of the Dirichlet mechanism based on its scaling parameter $k$. We used the concentration bound to bound the cost of privacy imposed by privatizing the transition probabilities. We further showed that the cost of privacy can be computed efficiently by establishing that the computational complexity of the algorithm is polynomial in problem parameters. Finally, the simulation results validated the developments in both the soundness of the expressions we introduced for the cost of privacy and the computational complexity associated with computing them.

REFERENCES

[1] D. J. Glancy, “Privacy in autonomous vehicles,” Santa Clara L. Rev., vol. 52.
[2] Z. Guan, G. Si, X. Zhang, L. Wu, N. Guizani, X. Du, and Y. Mu, “Privacy-preserving and efficient aggregation based on blockchain for power grid communications in smart communities,” IEEE Communications Magazine, vol. 56, no. 7.
[3] S. Brechtle, T. Gindele, and R. Dillmann, “Probabilistic mdp-behavior planning for cars,” in 2011 14th International IEEE Conference on Intelligent Transportation Systems (ITSC).
[4] S. Misra, A. Mondal, B. Banik, M. Khattu, S. Bera, and M. S. Obaidat, “Residential energy management in smart grid: A markov decision process-based approach,” in 2013 IEEE International Conference on Green Computing and Communications and IEEE Internet of things and IEEE Cyber, Physical and Social Computing.
[5] D. M. Szymanski, S. G. Bharadwaj, and P. R. Varadarajan, “An analysis of the market share-profitability relationship,” Journal of Marketing.
[6] J. E. Prescott, A. K. Kohli, and N. Venkatraman, “The market share-profitability relationship: An empirical assessment of major assertions and contradictions,” Strategic Management Journal, vol. 7, no. 4.
[7] C. Dwork, F. McSherry, K. Nissim, and A. Smith, “Calibrating noise to sensitivity in private data analysis,” in Theory of cryptography conference.
[8] F. K. Dankar and K. El Emam, “The application of differential privacy to health data,” in Proceedings of the 2012 Joint EDBT/ICDT Workshops.
[9] S.-S. Ho and S. Ruan, “Differential privacy for location pattern mining,” in Proceedings of the 4th ACM SIGSPATIAL International Workshop on Security and Privacy in GIS and LBS.
[10] J. Cortés, G. E. Dullerud, S. Han, J. Le Ny, S. Mitra, and G. J. Pappas, “Differential privacy in control and network systems,” in 2016 IEEE 55th Conference on Decision and Control (CDC).
[11] S. Han and G. J. Pappas, “Privacy in control and dynamical systems,” Annual Review of Control, Robotics, and Autonomous Systems, vol. 1.
[12] R. S. Sutton and A. G. Barto, Reinforcement learning: An introduction. MIT press, 2018.
[13] C. Dwork, A. Roth et al., “The algorithmic foundations of differential privacy,” Foundations and Trends® in Theoretical Computer Science.
[14] F. McSherry and K. Talwar, “Mechanism design via differential privacy,” in 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS’07).
[15] P. Gohari, B. Wu, M. T. Hale, and U. Topcu, “The dirichlet mechanism for differential privacy on the unit simplex,” in To appear in Proc. American Control Conference (ACC), 2020.
[16] M. Zhang, J. Chen, L. Yang, and J. Zhang, “Dynamic pricing for privacy-preserving mobile crowdsensing: A reinforcement learning approach,” IEEE Network, vol. 33, no. 2.
[17] P. Venkitasubramaniam, “Privacy in stochastic control: A markov decision process perspective,” in 2013 51st Annual Allerton Conference on Communication, Control, and Computing (Allerton).
[18] B. Wang and N. Hegde, “Privacy-preserving q-learning with functional noise in continuous spaces,” in Advances in Neural Information Processing Systems.
[19] A. Nilim and L. El Ghaoui, “Robust markov decision processes with uncertain transition matrices,” Ph.D. dissertation, University of California, Berkeley, 2004.
[20] G. N. Iyengar, “Robust dynamic programming,” Mathematics of Operations Research, vol. 30, no. 2.
[21] H. Xu and S. Mannor, “Distributionally robust markov decision processes,” in Advances in Neural Information Processing Systems.
[22] M. L. Puterman, Markov decision processes: discrete stochastic dynamic programming. John Wiley & Sons, 2014.
[23] A. Machanavajjhala, D. Kifer, J. Abowd, J. Gehrke, and L. Vilhuber, “Privacy: Theory meets practice on the map,” in 2005 24th international conference on data engineering.
[24] L. Kong, M. K. Khan, F. Wu, G. Chen, and P. Zeng, “Millimeter-wave wireless communications for iot-cloud supported autonomous vehicles: Overview, design, and challenges,” IEEE Communications Magazine.
[25] Z. Li, C. Chen, and K. Wang, “Cloud computing for agent-based urban transportation systems,” IEEE Intelligent Systems, vol. 26, no. 1.
[26] J. Liu, C. Zhang, and Y. Fang, “Epic: A differential privacy framework for cloud computing,” in Proceedings of the 2012 Joint EDBT/ICDT Workshops.
[27] O. Marchal, J. Arbel et al., “On the sub-gaussianity of the beta and dirichlet distributions,” Electronic Communications in Probability, vol. 22, 2017.

APPENDIX I

PROOF OF LEMMA 2

Let $p \in \Delta^c(n)$ and $k > 0$. From the properties of the Dirichlet mechanism we have that $E \left[ M_D^{(k)}(p) \right] = p$. Let $u \in \Delta(n)$ and $\lambda \in \mathbb{R}_+$. By Theorem 3.3 in [28], we have that

$$E \left[ \exp \left( \lambda u^T \left( M_D^{(k)}(p) - E \left[ M_D^{(k)}(p) \right] \right) \right) \right] \leq \prod_{i=1}^n \left[ E \left[ \exp \left( \lambda u_i \left( M_D^{(k)}(p_j) - p_i \right) \right) \right] \right].$$

Let $\beta(a, b)$ denote the beta distribution with parameter $(a, b)$. It can be shown that each component of the Dirichlet mechanism satisfies

$$M_D^{(k)}(p_j) \sim \beta(kp_j, k(1 - p_j)).$$
Recall that a random variable $X$ is said to be $R$-subgaussian if it satisfies
\[ \mathbb{E} \left[ \exp \left( sX \right) \right] \leq \exp \left( \frac{R^2 s^2}{2} \right), \forall s \in \mathbb{R}. \]

A beta distribution with parameters $(a, b)$ is $\sqrt{1/(a + b + 1)}$-subgaussian [28]. Therefore,
\[
\prod_{i=1}^{n} \mathbb{E} \left[ \exp \left( \lambda u_i \left( \mathcal{M}_D^{(k)}(p)_i - p_i \right) \right) \right] \leq \exp \left( \frac{\lambda^2 \|u\|_2^2}{2(8(k+1))} \right).
\]

Since $u$ can be any vector in $\Delta(n)$, we consider an instance of $u$ that puts weight 1 on the component of $\mathcal{M}_D^{(k)}(p) - p$ with maximum magnitude and zero elsewhere. Considering the case where $\lambda = 1$, we can write
\[
\mathbb{E} \left[ \exp \|\mathcal{M}_D^{(k)}(p) - p\|_\infty \right] \leq \exp \left( \frac{1}{8(k+1)} \right).
\]
Finally, for all $\theta \geq 0$, we have that
\[
\mathbb{P} \left( \|\mathcal{M}_D^{(k)}(p) - p\|_\infty \geq \alpha \right) = \mathbb{P} \left( \exp \left( \theta \|\mathcal{M}_D^{(k)}(p) - p\|_\infty \right) \geq \exp (\theta \alpha) \right). 
\]
By Markov’s inequality,
\[
\mathbb{P} \left( \exp \left( \theta \|\mathcal{M}_D^{(k)}(p) - p\|_\infty \right) \geq \exp (\theta \alpha) \right) \leq \mathbb{E} \left[ \exp \left( \theta \|\mathcal{M}_D^{(k)}(p) - p\|_\infty \right) \right] \cdot \exp (-\theta \alpha).
\]
Combining the above inequality with (7), we arrive at
\[
\mathbb{P} \left( \|\mathcal{M}_D^{(k)}(p) - p\|_\infty \geq \alpha \right) \leq \exp \left( \frac{\theta^2}{8(k+1)} - \theta \alpha \right).
\]
Taking $\theta = 4\alpha(k+1)$ and a proper change of variable from $\alpha$ to $\beta$ concludes the lemma.

**APPENDIX II**

**PROOF OF LEMMA 3**

In the definition of $\mathcal{P}_{a, \beta}$ in (1), take $P_1(s, a) := \mathcal{P}(s, a)$, and $P_2(s, a) := \mathcal{P}(\bar{s}, a)$. Then, $\beta P_1(s, a) + (1-\beta)P_2(s, a)$ is within the set $\mathcal{P}_{a, \beta}$, which concludes the first result.

For a given $\beta$, by Lemma 2, there exists $\beta' \geq \beta$, such that
\[
\mathbb{P} \left( \|\mathcal{M}_D^{(k)}(p) - p\|_\infty \geq \sqrt{\frac{\log (1/\beta')}{2(k+1)}} \right) = \beta.
\]
Define $\alpha' := \sqrt{\log(1/\beta')/2(k+1)}$ and let
\[
P_1(s, a) := \mathbb{E} \left[ \left. P(s, a) \right| \mathcal{P}, k, \|P(s, a) - \mathcal{P}(s, a)\|_\infty \geq \alpha' \right],
\]
and
\[
P_2(s, a) := \mathbb{E} \left[ \left. P(s, a) \right| \mathcal{P}, k, \|P(s, a) - \mathcal{P}(s, a)\|_\infty \leq \alpha' \right].
\]
Since $\alpha' \leq \alpha$, $\|P_2(s, a) - \mathcal{P}(s, a)\|_\infty \leq \alpha$. Then, $\beta P_1(s, a) + (1-\beta)P_2(s, a) = \mathbb{E} \left[ P(s, a) \left| \mathcal{P}, k \right. \right]$ is also within the set $\mathcal{P}_{a, \beta}$, which completes the proof.

**APPENDIX III**

**PROOF OF LEMMA 3**

Let $U, V \in \mathbb{R}^{[S]}$. Then, for all $s \in S$, we can write
\[
\mathcal{L}_1 U(s) - \mathcal{L}_1 V(s) = \gamma \sum_{a \in A_s} \hat{\pi}(a | s).
\]
\[
\left( \min_{p \in \mathcal{P}_{\alpha_0, \beta_0} \in S} p(s, a, s') U(s') - \min_{p \in \mathcal{P}_{\alpha_0, \beta_0} \in S} p(s, a, s') V(s') \right).
\]
For all $(s, a) \in S \times A_s$, let
\[
p^*(s, a) = \arg\min_{p \in \mathcal{P}_{\alpha_0, \beta_0} \in S} p(s, a, s') V(s').
\]
Then,
\[
\mathcal{L}_1 U(s) - \mathcal{L}_1 V(s) \leq \gamma \sum_{a \in A_s} \hat{\pi}(a | s) \cdot \sum_{s' \in S} p^*(s, a, s') (U(s') - V(s')) \leq \gamma \max_{s \in S} |U(s) - V(s)|.
\]
(8)

For mapping $\mathcal{L}_2$, we have that
\[
\mathcal{L}_2 U(s) - \mathcal{L}_2 V(s) = \gamma \sum_{a \in A_s} \hat{\pi}(a | s) \cdot \sum_{s' \in S} \hat{P}(s, a, s') (U(s') - V(s')) \leq \gamma \max_{s \in S} |U(s) - V(s)|.
\]
(9)

Finally, for $\mathcal{L}_3$, we write
\[
\mathcal{L}_3 U(s) - \mathcal{L}_3 V(s) = \gamma \sum_{a \in A_s} \hat{\pi}(a | s) \sum_{s' \in S} \mathbb{E} \left[ P(s, a, s') \mid \mathcal{P}, k \right] \cdot \left( U(s') - V(s') \right) \leq \gamma \max_{s \in S} |U(s) - V(s)|.
\]
(10)

Notice that $\|U - V\|_\infty = \max_{s \in S} |U(s) - V(s)|$, and that (8), (9) and (10) hold for any $s \in S$. Thus, $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are $\gamma$-contraction mappings.