Two Groups in a Curie-Weiss Model

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Abstract
We analyse a Curie-Weiss model with two disjoint groups of spins with homogeneous coupling. We show that similarly to the single-group Curie-Weiss model a bivariate law of large numbers holds for the normed sums of both groups’ spin variables. We also show central limit theorem in the high temperature regime.

Keywords Curie-Weiss model · Central limit theorem · Statistical physics

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1 Introduction
The Curie-Weiss model is probably the easiest model of magnetism which shows a phase transition between a diamagnetic and a ferromagnetic phase. In this model the spins can take values in \{-1, 1\} (or up/down), each spin interacts with all the others in the same way. More precisely, for finitely many spins \((X_1, X_2, \ldots, X_N) \in \{-1, 1\}^N\) the energy of the spins is given by

\[
H = H(X_1, \ldots, X_N) := -\frac{1}{2N} \left( \sum_{j=1}^N X_j \right)^2. \tag{1}
\]

Consequently, in the ‘canonical ensemble’ with inverse temperature \(\beta \geq 0\) the probability of a spin configuration is given by

\[
P(X_1 = x_1, \ldots, X_N = x_N) := Z^{-1} e^{-\beta H(x_1, \ldots, x_N)} \tag{2}
\]
where \(x_i \in \{-1, 1\}\) and \(Z\) is a normalization constant which depends on \(N\) and \(\beta\).
The quantity
\[ S_N = \sum_{j=1}^{N} X_j \] (3)
is called the (total) magnetization. It is well known (see e.g. Ellis [4] or [10]) that the Curie-Weiss model has a phase transition at \( \beta = 1 \) in the following sense
\[ \frac{1}{N} S_N \implies \frac{1}{2} (\delta_{-m(\beta)} + \delta_{m(\beta)}) \] (4)
where \( \implies \) denotes convergence in distribution, \( \delta_x \) the Dirac measure in \( x \).

For \( \beta \leq 1 \) we have \( m(\beta) = 0 \) which is the unique solution of
\[ \tanh(\beta x) = x \] (5)
for this case.

If \( \beta > 1 \) (5) has exactly three solutions and \( m(\beta) \) is the unique positive one. Equation (4) is a substitute for the law of large numbers for i.i.d. random variables. Moreover, for \( \beta < 1 \) there is a central limit theorem, i.e.
\[ \frac{1}{\sqrt{N}} S_N \implies N\left(0, \frac{1}{1-\beta}\right) \] (6)
For \( \beta = 1 \) there is no such central limit theorem. In fact, the random variables
\[ \frac{1}{N^{3/4}} S_N \] (7)
converge in distribution to a limit which is not a normal distribution.

The Curie-Weiss model is also called the Husimi-Temperley model. It was first introduced by Husimi [7] and Temperley [15]. Subsequently it was discussed by Kac [9], Thompson [16], and Ellis [4]. More recently, the Curie-Weiss model has been used in the context of social and political interactions. See e.g. [3] and [11].

We mention that the Curie-Weiss model is also used to describe the behaviour of voters who have the choice to vote ‘Yea’ (spin=1, say) or ‘Nay’ (spin=-1) (see [11]).

In this paper, we partition the set of all \( N \) Curie-Weiss spins into two disjoint groups \( X_1, \ldots, X_{N_1} \) and \( Y_1, \ldots, Y_{N_2} \) with \( N_1 + N_2 = N \). We let \( N_1 \) and \( N_2 \) depend on \( N \) in such a way that both \( N_1 \) and \( N_2 \) go to infinity as \( N \) does. We consider the asymptotic behaviour of the two-dimensional random variables
\[ \left( \sum_{i=1}^{N_1} X_i , \sum_{j=1}^{N_2} Y_j \right) \] (8)
as \( N \) goes to infinity and prove results similar to the one-group case considered above. We will use the method of moments which was used in [12] for the one-group case. We rely on the method developed there.

Note that all spins are coupled by the same constant \( \beta \geq 0 \). We could also define the model using a homogeneous coupling matrix
\[ J = \beta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} . \]
For an analysis of heterogeneous coupling matrices see [13].

We prove
Theorem 1 (Law of Large Numbers) If $N_1, N_2 \to \infty$ as $N \to \infty$, then we have for all $\beta$

$$
\left( \frac{1}{N_1} \sum_{i=1}^{N_1} X_i, \frac{1}{N_2} \sum_{j=1}^{N_2} Y_j \right) \overset{N \to \infty}{\Longrightarrow} \frac{1}{2} \left( \delta(-m(\beta),-m(\beta)) + \delta(m(\beta),m(\beta)) \right).
$$

Above ‘$\Longrightarrow$’ denotes convergence in distribution of the 2-dimensional random variable on the left hand side and $m(\beta)$ is again the largest solution of (5), in particular $m(\beta) = 0$ for $\beta \leq 1$ and $m(\beta) > 0$ for $\beta > 1$.

Remark 2 If we consider a model without interaction between the groups $X_i$ and $Y_j$ then the limit in (9) is

$$
\frac{1}{4} \left( \delta(-m(\beta),-m(\beta)) + \delta(-m(\beta),m(\beta)) + \delta(m(\beta),-m(\beta)) + \delta(m(\beta),m(\beta)) \right)
$$

For $\beta < 1$ we also have a central limit theorem. The covariance of the limiting normal distribution depends on the growth rate of $N_1$ and $N_2$. We set

$$
\alpha_1 = \lim_{N \to \infty} \frac{N_1}{N} \quad \alpha_2 = \lim_{N \to \infty} \frac{N_2}{N}
$$

and assume that these limits exist.

Theorem 3 (Central Limit Theorem) If $\beta < 1$, then

$$
\left( \frac{1}{\sqrt{N_1}} \sum_{i=1}^{N_1} X_i, \frac{1}{\sqrt{N_2}} \sum_{j=1}^{N_2} Y_j \right) \overset{N \to \infty}{\Longrightarrow} \mathcal{N}'((0, 0), C),
$$

where the covariance matrix $C$ is given by

$$
C = \left[ \begin{array}{c} 1 + \alpha_1 \beta \\ \sqrt{\alpha_1 \alpha_2 \beta} \end{array} \right]
$$

In particular, for sublinear growth of either $N_1$ or $N_2$, i.e. if $\alpha_1 = 0$ or $\alpha_2 = 0$, the standardized sums in (11) are asymptotically independent.

In the proof of both results we employ the moment method (see e.g. [1] or [10]). In [12] the method of moments is used to prove limit theorems for the (one-group) Curie-Weiss Model.

Thus, to show the convergence in distribution of a sequence $(X_n, Y_n)$ of two-dimensional random variables to a measure $\mu$ on $\mathbb{R}^2$ we prove that

$$
\mathbb{E} \left( X^K_n \cdot Y^L_n \right) \to \int x^K y^L \mu(dx, dy)
$$

for all $K, L \in \mathbb{N}$. 

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Equation (13) implies convergence in distribution if the moments of $\mu$ grow only moderately, namely if for some constants $A$ and $C$ and all $K, L$

$$\int |x|^K |y|^L \mu(dx, dy) \leq AC^{K+L} (K + L)!$$ (14)

holds. For the multidimensional case of the moment method we refer to [14].

After publishing the first version of this paper on arXiv, we became aware of the articles [5] and [6] which contain the above results as special cases. The methods used by those authors are different from ours. In [6], the authors show a central limit theorem based on the assumption that a certain function has one or more global minima. The article [5] shows how for two groups, under asymmetric scaling of the sums of spins, a central limit theorem can be proved. We are grateful to Francesca Collet for drawing our attention to the papers [2], [5] and [6].

2 Preparation

We have to evaluate terms of the form

$$\mathbb{E} \left( \left( \sum_{i=1}^{N_1} X_i \right)^K \left( \sum_{j=1}^{N_2} Y_j \right)^L \right)$$

$$= \sum_{i_1, i_2, \ldots, i_K=1}^{N_1} \sum_{j_1, j_2, \ldots, j_L=1}^{N_2} \mathbb{E} \left( X_{i_1} \cdot X_{i_2} \cdots X_{i_K} \cdot Y_{j_1} \cdot \ldots \cdot Y_{j_L} \right)$$ (15)

To shorten notation we set

$$\bar{i} = (i_1, \ldots, i_K) \in \{1, 2, \ldots, N_1\}^K$$

$$\bar{j} = (j_1, \ldots, j_L) \in \{1, 2, \ldots, N_1\}^L$$

and $W_{Q, M} := \{1, 2, \ldots, M\}^Q$. We also denote for $\bar{i} \in W_{K, N_1}$

$$X(\bar{i}) := X_{i_1} \cdot X_{i_2} \cdot \ldots \cdot X_{i_K}$$

and similar for $Y(\bar{j})$. So, (15) reads

$$\sum_{\bar{i} \in W_{K, N_1}} \sum_{\bar{j} \in W_{L, N_2}} \mathbb{E} \left( X(\bar{i})Y(\bar{j}) \right)$$ (16)

The energy function $H$ and hence the probability measure $\mathbb{P}$ are invariant with respect to permutation of indices. Thus we observe that

$$\mathbb{E} \left( X_{i_1} \cdot \ldots \cdot X_{i_K} \cdot Y_{j_1} \cdot \ldots \cdot Y_{j_L} \right) = \mathbb{E} \left( X_1 \cdot \ldots \cdot X_K \cdot Y_1 \cdot \ldots \cdot Y_L \right)$$

whenever both $i_1, \ldots, i_K$ and $j_1, \ldots, j_L$ are pairwise distinct. Since

$$X_i^{2m} = 1, \quad X_i^{2m+1} = X_i$$
and similar for $Y_j$, we conclude that

$$\mathbb{E}\left(X(i)Y(j)\right) = \mathbb{E}\left(X_1 \cdot \ldots \cdot X_{\rho(i)} \cdot Y_1 \cdot \ldots \cdot Y_{\rho(j)}\right)$$ (17)

where $\rho(i)$ denotes the number of indices $i_K$ which occur an odd number of times in $i$. So, to compute sums of the form (15) we need good estimates of expectations (‘correlations’) as in (17). Such estimates are provided in [12] (see also [11]).

**Proposition 4** For $K + L$ even we have

1) for $\beta < 1$:

$$\mathbb{E}\left(X_1 \cdot X_2 \cdot \ldots \cdot X_K \cdot Y_1 \cdot Y_2 \cdot \ldots \cdot Y_L\right) \approx (K + L - 1)!! \left(\frac{\beta}{1-\beta}\right)^{\frac{K+L}{2}} N^{-\frac{K+L}{2}}$$

2) for $\beta = 1$:

$$\mathbb{E}\left(X_1 \cdot X_2 \cdot \ldots \cdot X_K \cdot Y_1 \cdot \ldots \cdot Y_L\right) \approx \int t^{K+L} e^{-\frac{1}{12}t^4} dt \int e^{-\frac{1}{12}t^4} dt \cdot N^{-\frac{K+L}{4}}$$

3) for $\beta > 1$:

$$\mathbb{E}\left(X_1 \cdot X_2 \cdot \ldots \cdot X_K \cdot Y_1 \cdot \ldots \cdot Y_L\right) \approx m(\beta)^{K+L},$$

where $m(\beta)$ is the unique (strictly) positive solution of $t = \tanh(\beta t)$. For $K + L$ odd:

$$\mathbb{E}\left(X_1 \cdot X_2 \cdot \ldots \cdot X_K \cdot Y_1 \cdot \ldots \cdot Y_L\right) = 0$$

Above $a_n \approx b_n$ means $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$ and $(2n - 1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n - 3) \cdot (2n - 1)$. For a proof of Proposition 4 see [12].

3 **Laws of Large Numbers**

To prove Theorem 1 we consider

$$\frac{1}{N_1^K} \frac{1}{N_2^L} \sum_{i \in W_{K,N_1}} \sum_{j \in W_{L,N_2}} \mathbb{E}\left(X(i)Y(j)\right)$$ (18)

We assume $\beta \leq 1$ first.

By $U_{K,N_1}$ we denote the set of those $\underline{i} \in W_{K,N_1}$ for which the $i_v$ are pairwise distinct. Note that for $\underline{i} \in U_{K,N_1}$ and $\underline{j} \in W_{L,N_2}$ we have

$$\mathbb{E}\left(X(\underline{i})Y(\underline{j})\right) \leq C N^{-\frac{K}{4}}$$
by Proposition 4. We single out the following Lemma (where $|M|$ denotes the cardinality of the set $M$).

**Lemma 5**

1) $|U_{K,N_1}| = \frac{N_1!}{(N_1 - K)!} \approx N_1^K$  
   \hspace{2cm} \text{(19)}

2) $|W_{K,N_1} \setminus U_{K,N_1}| \leq K! N_1^{K-1}$  
   \hspace{2cm} \text{(20)}

**Proof** 1) is obvious.

2) If $\bar{i} \in W_{K,N_1} \setminus U_{K,N_1}$, then there are at most $K - 1$ distinct indices $i_v$ belonging to $\bar{i}$. We have consequently at most $N_1^{K-1}$ choices for the $i_v$. There are at most $K!$ ways to position them in the multiindex $\bar{i}$.

We estimate

$$\sum_{i \in U_{K,N_1}} \sum_{j \in W_{L,N_2}} \mathbb{E}\left(X(i)Y(j)\right) + \sum_{i \in W_{K,N_1} \setminus U_{K,N_1}} \sum_{j \in W_{L,N_2}} \mathbb{E}\left(X(i)Y(j)\right)$$

$$\leq N_1^K \frac{1}{N_1} \frac{1}{N_2} K! N_1^{K-1} L! N_2^L + \frac{1}{N_1^K} \frac{1}{N_2^L} K! N_1^{K-1} L! N_2^L$$

For $\beta > 1$ we prove in a similar way that

$$\frac{1}{N_1^K} \frac{1}{N_2^L} \left( \sum_{i \in U_{K,N_1}} \sum_{j \in W_{L,N_2}} \mathbb{E}\left(X(i)Y(j)\right) + \sum_{i \in W_{K,N_1} \setminus U_{K,N_1}} \sum_{j \in W_{L,N_2}} \mathbb{E}\left(X(i)Y(j)\right) \right)$$

goes to zero.

Consequently

$$\frac{1}{N_1^K} \frac{1}{N_2^L} \sum_{i \in W_{K,N_1} \setminus U_{K,N_1}} \sum_{j \in W_{L,N_2}} \mathbb{E}\left(X(i)Y(j)\right)$$

$$\approx \frac{1}{N_1^K} \frac{1}{N_2^L} \sum_{i \in U_{K,N_1}} \sum_{j \in U_{L,N_2}} \mathbb{E}\left(X(i)Y(j)\right)$$

$$\rightarrow \begin{cases} m(\beta)K + L, & \text{if } K + L \text{ even} \\ 0, & \text{otherwise} \end{cases}$$

which are the moments of the measure $\frac{1}{2} \left( \delta(-m(\beta),-m(\beta)) + \delta(+m(\beta),+m(\beta)) \right)$. 

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4 The Central Limit Theorem

We start this section by computing the moments of a centred two-dimensional normal distribution. Let \((Z_1, Z_2)\) be normally distributed with mean zero and covariance matrix

\[
\Sigma = \begin{pmatrix} \sigma_1^2 & \bar{\sigma} \\ \bar{\sigma} & \sigma_2^2 \end{pmatrix}
\] (21)

**Lemma 6** We have

\[
\mathbb{E}\left( Z_1^{2K} Z_2^{2L} \right) = \sum_{r=0}^{K \land L} \binom{2K}{2K-2r} \binom{2L}{2L-2r} (2r)! (2K - 2r - 1)! (2L - 2r - 1)! \cdot (\sigma_1^2)^{K-r} \bar{\sigma}^{2r} (\sigma_2^2)^{L-r} \quad (22)
\]

and

\[
\mathbb{E}\left( Z_1^{2K+1} Z_2^{2L+1} \right) = \sum_{r=0}^{K \land L} \binom{2K+1}{2K-(2r+1)} \binom{2L+1}{2L-(2r+1)} (2r+1)! (2K - 2r - 1)! (2L - 2r - 1)! \cdot (\sigma_1^2)^{K-r} \bar{\sigma}^{2r+1} (\sigma_2^2)^{L-r} \quad (23)
\]

A proof is given in the Appendix.

For this section we assume \(\beta < 1\). We have to evaluate

\[
\frac{1}{N_{1}^{K/2}} \frac{1}{N_{2}^{L/2}} \sum_{i \in W_{K, N_{1}}} \sum_{j \in W_{L, N_{2}}} \mathbb{E}\left( X(i) Y(j) \right) . \quad (24)
\]

As for the law of large numbers, many multiindices \(i, j\) are negligible. However, this time the selection of the leading terms is more subtle.

**Definition 7** Let us set

\[
W_{Q, M} := \{1, 2, \ldots, M\}^Q,
\]

\[
W_{Q, M}(r) := \{i \in W_{Q, M} \mid \text{exactly } r \text{ indices } i_v \text{ occur exactly once in } i\},
\]

\[
W_{Q, M}^+(r) := \{i \in W_{Q, M}(r) \mid \text{at least one index } i_v \text{ occurs more than twice in } i\},
\]

\[
W_{Q, M}^0(r) := W_{Q, M}(r) \setminus W_{Q, M}^+(r).
\]

Moreover by \(w_{Q, M}(r) \left( w_{Q, M}^+(r), w_{Q, M}^0(r) \right) \) we denote

\[
|W_{Q, M}(r)| \left( |W_{Q, M}^+(r)|, |W_{Q, M}^0(r)| \right).
\]

\(W_{Q, M}^0(r)\) consists of those multiindices for which (exactly) \(r\) indices occur once and the remaining indices occur exactly twice. It follows that \(W_{Q, M}^0(r) = \emptyset\) if \(Q - r\) is odd.

We note the following combinatorial Proposition:
Proposition 8

\[ W_{Q,M}(Q) = \frac{M!}{(M-Q)!} \approx M^Q, \]

\[ W_{Q,M}(r) \leq Q! M^{\frac{Q+r}{2}}, \]

\[ W^+_Q, M(r) \leq Q! M^{\frac{Q+r}{2}} - \frac{1}{2}, \]

\[ W_{Q,M}^0(r) = \begin{cases} 
\frac{M!}{(M-Q+r)!} & \text{if } Q-r \text{ is even} \\
0 & \text{otherwise}
\end{cases} \]

For a proof see the Appendix or [12].

Now, we split the sum (24) in four parts.

(24)

\[ = \frac{1}{N_1^{K/2}} \frac{1}{N_2^{L/2}} \sum_{r=0}^{K} \sum_{i \in W^0_{K,N_1}(r)} \sum_{s=0}^{L} \sum_{j \in W^0_{L,N_2}(s)} \mathbb{E} \left( X(i) Y(j) \right) \]

\[ + \frac{1}{N_1^{K/2}} \frac{1}{N_2^{L/2}} \sum_{r=0}^{K} \sum_{i \in W^+_{K,N_1}(r)} \sum_{s=0}^{L} \sum_{j \in W^0_{L,N_2}(s)} \mathbb{E} \left( X(i) Y(j) \right) \]

\[ + \frac{1}{N_1^{K/2}} \frac{1}{N_2^{L/2}} \sum_{r=0}^{K} \sum_{i \in W^+_{K,N_1}(r)} \sum_{s=0}^{L} \sum_{j \in W^0_{L,N_2}(s)} \mathbb{E} \left( X(i) Y(j) \right) \]

\[ + \frac{1}{N_1^{K/2}} \frac{1}{N_2^{L/2}} \sum_{r=0}^{K} \sum_{i \in W^+_{K,N_1}(r)} \sum_{s=0}^{L} \sum_{j \in W^+_{L,N_2}(s)} \mathbb{E} \left( X(i) Y(j) \right) \]

\[ = A_1 + A_2 + A_3 + A_4 \]

Lemma 9 \( A_2, A_3, A_4 \rightarrow 0 \) as \( N \rightarrow \infty \)

Proof If \( i \in W^0_{K,N_1}(r), j \in W^0_{L,N_2}(s) \) then \( \mathbb{E} \left( X(i) Y(j) \right) \leq C N^{-\frac{r+s}{2}} \) by Proposition 4. Moreover

\[ w^0_{K,N_1}(r) \leq C_1 N_1^{\frac{K+r}{2}} \]

\[ w^+_{L,N_2}(s) \leq C_2 N_2^{\frac{L+s-1}{2}} \]

by Proposition 4. Thus

\[ \frac{1}{N_1^{K/2}} \frac{1}{N_2^{L/2}} \sum_{i \in W^0_{K,N_1}(r)} \sum_{j \in W^0_{L,N_2}(s)} \mathbb{E} \left( X(i) Y(j) \right) \leq C' N^{-\frac{1}{2}} \]

It follows that \( A_2 \rightarrow 0 \). The proofs for \( A_3 \) and \( A_4 \) are similar. \( \square \)
We turn to the asymptotic computation of $A_1$.

$$A_1 = \sum_{r=0}^{K} \sum_{i \in W_{K,N_1}^0(r)} \sum_{s=0}^{L} \sum_{j \in W_{L,N_2}^0(s)} \mathbb{E} \left( X(i) Y(j) \right)$$

First, we observe that for $K + L$ odd $\mathbb{E} \left( X(i) Y(j) \right) = 0$ by Proposition 4. So, we may assume that $K$ and $L$ are both even or both odd. If $K$ and $L$ are even (odd) then both $r$ and $s$ have to be even (odd) otherwise $W_{K,N_1}^0(r) = \emptyset$ or $W_{L,N_2}^0(s) = \emptyset$. In the following we will treat the case $K$ and $L$ even, the other one being similar.

So we assume $K = 2k$ and $L = 2\ell$ and rewrite $A_1$:

$$\sum_{r=0}^{k} \sum_{s=0}^{\ell} \sum_{i \in W_{2k,N_1}^0(2r)} \sum_{j \in W_{2\ell,N_2}^0(2s)} \mathbb{E} \left( X(i) Y(j) \right)$$

If $i \in W_{2k,N_1}^0(2r)$ and $j \in W_{2\ell,N_2}^0(2s)$ then by Proposition 4

$$\mathbb{E} \left( X(i) Y(j) \right) \approx (2r + 2s - 1)!! \left( \frac{\beta}{1-\beta} \right)^{r+s} N_1^{-r} N_2^{-s}$$

using Proposition 8 we get:

$$A_1 \approx \frac{1}{N_1^2} \frac{1}{N_2^2} \sum_{r=0}^{k} \sum_{s=0}^{\ell} \frac{N_1!}{(N_1!-k-r)!} \frac{N_2!}{(N_2!-\ell-s)!} \frac{2k!}{2r!(k-r)!2\ell!} \frac{2\ell!}{2s!(\ell-s)!2r!} \cdot (2r + 2s - 1)!! \left( \frac{\alpha_1 \beta}{1-\beta} \right)^r \left( \frac{\alpha_2 \beta}{1-\beta} \right)^s (25)$$

where we set $\tilde{\beta} := \frac{\beta}{1-\beta}$.

This proves that all moments converge, hence $\left( \frac{1}{\sqrt{N_1}} \sum_{i=1}^{N_1} X_i, \frac{1}{\sqrt{N_2}} \sum_{j=1}^{N_2} Y_j \right)$ converge in distribution. In the following we have to identify the limit measure.

First we consider the case that $\alpha_1 = 0$ or $\alpha_2 = 0$. Let’s assume $\alpha_2 = 0$. Then (25) simplifies to

$$\sum_{r=0}^{k} \frac{(2k)!}{(2r)!(k-r)!2^{k-r}} \frac{(2\ell)!}{\ell!2^\ell} \frac{(2r)!}{r!2^r} \left( \alpha_1 \tilde{\beta} \right)^r$$

$$= \frac{(2k)!}{k!2^k} \frac{(2\ell)!}{\ell!2^\ell} \sum_{r=0}^{k} \left( \frac{k}{r} \right) \left( \alpha_1 \tilde{\beta} \right)^r$$

$$= (K-1)!! (L-1)!! \left( 1 + \alpha_1 \tilde{\beta} \right)^{K/2}$$
The last expression is the \((K,L)\)-moment of a centred normal distribution with covariance matrix
\[
\begin{pmatrix}
1 + \frac{\alpha_1 \beta}{1 - \beta} & 0 \\
0 & 1
\end{pmatrix}.
\] (26)

Now, we turn to the general case \((\alpha_2 \geq 0)\).
We use the following combinatorial identity which is Corollary 13 in the Appendix. A proof is given there.

\[
(2r + 2s - 1)!! = \sum_{t=0}^{r \wedge s} \frac{2r!}{(2r - 2t)!2t!} \frac{2s!}{(2s - 2t)!} \frac{(2r - 2t)!}{(r - t)!2^{r - t}} \frac{(2s - 2t)!}{(s - t)!2^{s - t}}. \tag{27}
\]

Substituting this expression into (25) we get:
\[
\begin{align*}
A_1 & \approx \sum_{r=0}^{k} \sum_{s=0}^{\ell} \alpha_1^r \alpha_2^s \frac{2k!}{2r!(k - r)!2^{k - r}} \frac{2\ell!}{2s!(\ell - s)!2^{\ell - s}} \\
& \quad \cdot \sum_{t=0}^{r \wedge s} \frac{2r!}{(2r - 2t)!2t!} \frac{2s!}{(2s - 2t)!} \frac{(2r - 2t)!}{(r - t)!2^{r - t}} \frac{(2s - 2t)!}{(s - t)!2^{s - t}} \bar{\beta}^{r+s} \\
& = \sum_{r=0}^{k} \sum_{s=0}^{\ell} \sum_{t=0}^{r \wedge s} \alpha_1^r \alpha_2^s \frac{2k!}{(k - r)!2^{k - r}} \frac{2\ell!}{(\ell - s)!2^{\ell - s}} \cdot \\
& \quad \cdot \frac{1}{2t!(r - t)!2^{r - t}} \frac{1}{(s - t)!2^{s - t}} \bar{\beta}^{r+s} \\
& = \sum_{t=0}^{k \wedge \ell} \sum_{r=t}^{k} \sum_{s=t}^{\ell} \sum_{m=0}^{k-t} \sum_{n=0}^{\ell-t} \frac{2k!}{(k - m - t)!m!2^{k - m - t}} \frac{2\ell!}{(\ell - n - t)!n!2^{\ell - n - t}} \frac{1}{2t!} \frac{1}{2^{r - t}} \frac{1}{2^{s - t}} \bar{\beta}^{m+n+2t} \\
& = \sum_{t=0}^{k \wedge \ell} \frac{2k!}{2^{k-t}} \frac{2\ell!}{2^{\ell-t}} \frac{1}{2t!} \left(\sqrt{\alpha_1 \alpha_2 \beta}\right)^{2t} \\
& \quad \cdot \sum_{m=0}^{k-t} \frac{1}{(k - m - t)!m!} (\alpha_1 \bar{\beta})^{m} \sum_{n=0}^{\ell-t} \frac{1}{(\ell - n - t)!n!} (\alpha_2 \bar{\beta})^{n} \\
& = \sum_{t=0}^{k \wedge \ell} \frac{2k!}{(2k - 2t)!2t!} \frac{2\ell!}{(2\ell - 2t)!} \frac{(2k - 2t)!}{(k - t)!2^{k-t}} \frac{(2\ell - 2t)!}{(\ell - t)!2^{\ell-t}} \\
& \quad \cdot \left(1 + \alpha_1 \bar{\beta}\right)^{k-t} \left(\sqrt{\alpha_1 \alpha_2 \beta}\right)^{2t} (1 + \alpha_2 \bar{\beta})^{\ell-t}.
\]
The last expression is, according to Lemma 6, the moment $E(Z_1^{2k}Z_2^{2\ell})$ of a bivariate normal distribution with zero mean and covariance matrix
\[
\begin{pmatrix}
1 + \alpha_1\beta \
\sqrt{\alpha_1\alpha_2}\beta
\end{pmatrix}.
\]

This concludes the proof of the Central Limit Theorem 3.

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Appendix

In this appendix we discuss some combinatorics in connection with the moment method and the Theorem of Isserlis.

Let us denote by $P_{L}(r)$ the set of all partitions $\Pi = \{\pi_1, \pi_2, ..., \pi_{\ell}\}$ of $\{1, 2, ..., L\}$ with $r$ sets $\pi_i$ with $|\pi_i| = 1$ and $\ell - r$ sets $\pi_j$ with $|\pi_j| = 2$. In particular, $P_{L}(0)$ is the set of pair partitions of $\{1, 2, ..., L\}$.

We show

**Lemma 10**

\[
|P_{2L}(0)| = (2L - 1)!! = \frac{2L!}{L!2^L}
\]

and

\[
|P_{2L+1}(0)| = 0.
\]

**Proof** The claim is true for $L = 1$. Suppose $P_{2L}(0) = (2L - 1)!!$. Then to build $\{\pi_1, ..., \pi_{2L+1}\}$ we can match the number $2L + 2$ with any of the other $2L - 1$ numbers. Thus, by induction hypothesis, we have $(2L - 1)!!$ choices to build pair partitions from the remaining $2L$ unmatched elements.

**Proposition 11** Let $L, r \in \mathbb{N}$.

If $L - r$ is even, then

\[
p_L(r) := |P_L(r)| = \binom{L}{r} (L - r - 1)!!
\]

and if $L - r$ is odd, then $p_L(r) = 0$.

**Proof** If $\{\pi_1, ..., \pi_{\ell}\} \in P_{L}(r)$ then $2\ell - r = 2(\ell - r) + r = L$, so $L - r$ is even. This proves the second assertion of the Proposition. Let $L - r$ be even. Then we have
choices for the sets $\pi_i$ with $|\pi_i| = 1$. There remain $L - r$ elements to build pair partitions from. By Lemma 10 this can be done in $(L - r - 1)!$ ways.

**Proof (Proposition 8):** We prove the final assertion of the proposition, the other assertions are easier to prove. There are $2^r$ different indices to choose from a total of $M$ possible choices. This gives $\frac{M!}{(M - 2^r)!}$ possibilities. To distribute them on the $Q$ indices with $r$ single and $\frac{Q - r}{2}$ double occurrences gives

$$\binom{Q}{r} (Q - r - 1)! = \frac{Q!}{(Q - r)!r!} \frac{(Q - r)!}{\left(\frac{Q - r}{2}\right)!2^r}.$$ Multiplying gives the assertion.

Now, we consider a disjoint partition $\{1, 2, \ldots, 2L\}$ into two sets $K$ and $M$ with $k = |K|$ and $m = |M|$, and look for all pair partitions $\Pi = (\pi_1, \ldots, \pi_L)$ such that exactly $r$ of the $\pi_i$ are mixed, i.e. $\pi_i \cap K \neq \emptyset$ and $\pi_i \cap M \neq \emptyset$.

**Proposition 12** The number $p_{K,M}(r)$ of pair partitions $\Pi = (\pi_1, \ldots, \pi_L)$ of $\{1, 2, \ldots, 2L\}$ with exactly $r$ mixed $\pi_i$ is given by

$$p_{K,M}(r) = \frac{k!}{(k - r)!} \frac{m!}{(m - r)!} \frac{1}{r!} (k - r - 1)! (m - r - 1)!$$

if both $k - r$ and $m - r$ are even and non-negative, and

$$p_{K,M}(r) = 0 \quad \text{otherwise.}$$

**Proof** There are $\frac{k!}{(k - r)!r!}$ choices for elements of $K$ in the mixed $\pi_i$ and $\frac{m!}{(m - r)!}$ ways to fill them with elements from $M$. The choices for the $(k - r)$ pure $\pi_i$ from $K$ and $(m - r)$ pure $\pi_i$ from $M$ are given by Lemma 10.

**Corollary 13** If both $k$ and $m$ are even, then

$$(k + m - 1)! = \sum_{\rho=0}^{\frac{k}{2}} \sum_{\rho=0}^{\frac{m}{2}} \frac{k! m!}{(k - 2\rho)! (m - 2\rho)! 2\rho!} (k - 2\rho)!! (m - 2\rho)!!$$

If both $k$ and $m$ are odd, then

$$(k + m - 1)! = \sum_{\rho=0}^{\frac{k-1}{2}} \sum_{\rho=0}^{\frac{m-1}{2}} \frac{k! m!}{(k - 2\rho + 1)! (m - 2\rho + 1)! (2\rho + 1)!} (k - 2\rho + 1)!! (m - 2\rho + 1)!!$$
This Corollary follows by summing $p_{k,m}(\rho)$ over all possible $\rho$. We end this appendix with the proof of Lemma 6.

**Proof (Lemma 6):** By the Theorem of Isserlis [8], we have

$$E \left( Z_1^{2K} Z_2^{2L} \right) = \sum_{\prod = (\pi_1, \ldots, \pi_{K+L}) \in \mathcal{P}_{2K+2L}(0)} \prod_{j=1}^{K+L} E_{\pi j},$$

where

$$E_{(\kappa, \lambda)} = \begin{cases} E \left( Z_1^2 \right) = \sigma_1^2, & \text{if } \kappa, \lambda \leq 2K \\ E \left( Z_2^2 \right) = \sigma_2^2, & \text{if } \kappa, \lambda > 2K \\ E \left( Z_1 Z_2 \right) = \bar{\sigma} & \text{otherwise.} \end{cases}$$

Therefore,

$$= \sum_{r=0}^{K \wedge L} \sum_{\prod \in \mathcal{P}_{2K+2L}(2r)} (\sigma_1^2)^{K-r} \bar{\sigma}^{2r} (\sigma_2^2)^{L-r}$$

$$= \sum_{r=0}^{K \wedge L} \frac{2K}{2K-2r} \frac{2L}{2L-2r} (2r)! (2K - 2r - 1)!! (2L - 2r - 1)!!$$

$$\cdot (\sigma_1^2)^{K-r} \bar{\sigma}^{2r} (\sigma_2^2)^{L-r}$$

by Proposition 12. Equation (23) can be proved analogously. 

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