A removal singularity theorem of the
Donaldson–Thomas instanton on compact Kähler
threefolds

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Abstract

We consider a perturbed Hermitian–Einstein equation, which we
call the Donaldson–Thomas equation, on compact Kähler threefolds.
In [Ta2], we analysed some analytic properties of solutions to the
equation, in particular, we proved that a sequence of solutions to
the Donaldson–Thomas equation has a subsequence which smoothly
converges to a solution to the Donaldson–Thomas equation outside a
closed subset of the Hausdorff dimension two. In this article, we prove
that some of these singularities can be removed.

1 Introduction

In [Ta1], we introduced a perturbed Hermitian–Einstein equation on sym-
plectic 6-manifolds in order to analytically approach the Donaldson–Thomas
invariants developed in [Th], [JS], [KS1], and [KS2]. In [Ta1], we described
the local structures of the moduli space of the Donaldson–Thomas instan-
tons, and the moment map description of the moduli space. Sub sequently,
in [Ta2], we proved a weak convergence theorem of the Donaldson–Thomas
instantons on compact Kähler threefolds. This article is a sequel of [Ta2].
We prove that some of singularities which appeared in [Ta2] are removable.

Firstly, let us introduce the equations. Let $Z$ be a compact Kähler
threefold with Kähler form $\omega$, and let $E$ be a unitary vector bundle over
$Z$ of rank $r$. A complex structure on $Z$ gives the splitting of the space of
the complexified two forms as $\Lambda^2 \otimes \mathbb{C} = \Lambda^{1,1} \oplus \Lambda^{2,0} \oplus \Lambda^{0,2}$, and $\Lambda^{1,1}$ further decomposes into $\mathbb{C}\langle\omega\rangle \oplus \Lambda^{1,1}_0$. We consider the following equations for a
connection $A$ of $E$, and an $u(E)$-valued \((0,3)\)-form $u$ on $Z$.

\begin{align}
F_A^{0,2} &= 0, \quad \bar{\partial}_A^* u = 0, \\
F_A^{1,1} \wedge \omega^2 + [u, \bar{u}] + \frac{i}{3} \lambda(E) Id_E \omega^3 &= 0,
\end{align}

where $\lambda(E)$ is a constant defined by $\lambda(E) := 6\pi (c_1(E) \cdot [\omega^2])/r[\omega^3]$. We call these equations the Donaldson–Thomas equations, and a solution $(A, u)$ to these equations a Donaldson–Thomas instanton (or a D–T instanton for short).

In [Ta2], we proved the following weak convergence theorem for the Donaldson–Thomas instantons on compact Kähler threefolds.

**Theorem 1.1** ([Ta2]). Let $Z$ be a compact Kähler threefold, and let $E$ be a unitary vector bundle over $Z$. Let $\{(A_n, u_n)\}$ be a sequence of D–T instantons of $E$. We assume that $\int_Z |u_n|^2 dV_g$ are uniformly bounded. Then there exist a subsequence $\{(A_{n_j}, u_{n_j})\}$ of $\{(A_n, u_n)\}$, a closed subset $S$ of $Z$ whose real two-dimensional Hausdorff measure is finite, and a sequence of gauge transformations $\{\sigma_j\}$ over $Z \setminus S$ such that $\{\sigma_j^*(A_{n_j}, u_{n_j})\}$ smoothly converges to a D–T instanton over $Z \setminus S$.

This leads to introducing the following notion of an admissible D–T instanton, as in [Ti].

**Definition 1.2.** A smooth D–T instanton $(A, u)$ defined outside a closed subset $S$ in $Z$ is called an admissible D–T instanton if $H^2(S \cap K) < \infty$ for any compact subset $K \subset Z$, and $\int_{Z \setminus S} |F_A|^2 dV_g < \infty$.

In this article, we prove the following.

**Theorem 1.3.** Let $B_r(0) \subset \mathbb{C}^3$ be a ball of radius $r$ centred at the origin with Kähler metric $g$. We assume that the metric is compatible with the standard metric by a constant $\Lambda$. Let $E$ be a vector bundle over $B_r(0)$, and let $(A, u)$ be an admissible D–T instanton of $E$ with $\int_{B_r(0)} |u|^2 dV < \infty$. Then, there exists a constant $\varepsilon > 0$ such that if $\frac{1}{r^2} \int_{B_r(0)} |F_A|^2 dV_g \leq \varepsilon$, there exists a smooth gauge transformation $\sigma$ on $B_{\frac{r}{2}}(0) \setminus S$ such that $\sigma(A, u)$ smoothly extends over $B_{\frac{r}{2}}(0)$.

Theorem 1.3 implies that the “top stratum” of the singular set $S$, which we denote by $S^{(2)}$ (see Section 2 for its definition), can be removed. Our argument goes through in a similar way to that of Tian–Yang [TY], and we follow Nakajima [N] and Uhlenbeck [U1] for removing isolated singularities,
except that we deal with an additional nonlinear term coming from the extra field \( u \). In Section 2, we bring some results from [Ta2], which will be used in this article, and describe some structures of the singular sets. We then prove Theorem 1.3 in Section 3.

**Notation.** Throughout this article, \( C, C' \), and \( C'' \) are positive constants, but they can be different each time they occur.

## 2 Weak convergence and the singularities

In this section, we bring results from [Ta2], and describe some properties of the singular set in Theorem 1.1.

Firstly, we recall the following monotonicity formula for the Donaldson–Thomas instantons on compact Kähler threefolds.

**Proposition 2.1 ([Ta2]).** Let \((A, u)\) be a D–T instanton of a unitary vector bundle \( E \) over a compact Kähler threefold \( Z \). Then, for any \( z \in Z \), there exists a positive constant \( r_z \) such that for any \( 0 < \sigma < \rho < r_z \), the following holds.

\[
\frac{1}{\rho^2} e^{a\rho^2} \int_{B_{r}(z)} m(A, u) dV_g - \frac{1}{\sigma^2} e^{a\sigma^2} \int_{B_{\sigma}(z)} m(A, u) dV_g \\
\geq \int_{\sigma}^{\rho} 8\tau^{-3} e^{a\tau^2} \int_{B_{\tau}(z)} |[u, \bar{u}]|^2 dV_g d\tau \\
+ \int_{B_{\rho}(z) \setminus B_{\sigma}(z)} r^{-2} e^{a r^2} \left\{ 4 \left| \frac{\partial}{\partial r} F_A^\perp \right|^2 - 12 \left| \frac{\partial}{\partial r} [u, \bar{u}] \right|^2 \right\} dV_g ,
\]

(2.1)

where \( m(A, u) := |F_A^\perp|^2 - |[u, \bar{u}]|^2 \), and \( a \) is a constant which depends only on \( Z \).

Another advantage of working on compact Kähler threefolds is the following estimate on the extra field \( u \):

**Proposition 2.2 ([Ta2]).** Let \((A, u)\) be a D–T instanton of a unitary vector bundle \( E \) over a compact Kähler threefold \( Z \). Then we have

\[
||u||_{L^\infty} \leq C ||u||_{L^2},
\]

(2.2)

where \( C > 0 \) is a positive constant which depends only on \( Z \).
In this article, we use the following curvature estimate essentially proved in [Ta2] by using the monotonicity formula and the estimate on $u$ above.

Proposition 2.3. Let $(A, u)$ be a D–T instanton of a unitary vector bundle $E$ over a compact Kähler threefold $Z$. Then there exist constants $\varepsilon > 0$ and $C_1 > 0$ which depend only on $Z$ such that for any $z \in Z$ and $0 < r < r_z$, where $r_z$ is the constant in Proposition 2.1, if

\[
\frac{1}{r^2} \int_{B_r(z)} |F_A|^2 \, dV_g \leq \varepsilon \text{ and } \int_{B_r(z)} |u|^2 \, dV_g < \varepsilon,
\]

then

\[
|F_A|(z) \leq \frac{C_1}{r^2} \left( \frac{1}{r^2} \int_{B_r(z)} |F_A|^2 \, dV_g \right)^{\frac{1}{2}} + C_1 \varepsilon r.
\]

With these above in mind, we next recall that the singular set $S$ in Theorem 1.1 is given by

\[
S := S(\{(A_n, u_n)\}) := \bigcap_{r > 0} \left\{ z \in Z : \liminf_{n \to \infty} \frac{1}{r^2} \int_{B_r(z)} m(A_n, u_n) \, dV_g \geq \varepsilon \right\}.
\]

We note that this set $S$ is 2-rectifiable, which can be proved by using a result by Preiss [P] (see also [Mos]) as follows. Let $\mu_n := m(A_n, u_n) \, dV_g$ be a sequence of Radon measures. Then, we may assume that $\mu_n$ converges to a Radon measure $\mu$ on $Z$ after taking a subsequence if necessary, that is, for any continuous function $\phi$ of compact support over $Z$,

\[
\lim_{n \to \infty} \int_Z \phi \, d\mu_n = \int_Z \phi \, d\mu.
\]

We can write $\mu = m(A, u) \, dV_g + \nu$, where $\nu$ is a non-negative Radon measure on $Z$. From Proposition 2.1, for any $z \in Z$, $e^{ar^2 - 2^2 \mu(B_r(z))}$ is a non-decreasing function of $r$, hence the density $\Theta(\mu, z) := \lim_{r \to 0^+} r^2 \mu(B_r(z))$ exists for all $z \in Z$, and from the definition of $S$, $z \in S$ if and only if $\Theta(\mu, z) \geq \varepsilon$. Hence, for $\mathcal{H}^2$-a.e. $z \in S(\{(A_n, u_n)\})$, we can write $\nu(z) = \Theta(\mu, z) \mathcal{H}^2 |S(\{(A_n, u_n)\})|$. Note that we have $\Theta(\mu, z) \leq 4^2 r_z^{-2} e^{ar^2} C$ from the monotonicity formula. We also have the following:

Proposition 2.4. $\mathcal{H}^2$-a.e. $z \in S$.

\[
\lim_{r \to 0^+} \frac{1}{r^2} \int_{B_r(z)} |F_A|^2 \, dV_g = 0.
\]

Proof. We follow an argument by Tian [Ti, pp. 222] (see also [S, §1.7 Cor. 3]). We consider $S_j := \{ z : \lim_{r \to 0^+} \int_{B_r(z)} |F_A|^2 \, dV_g > j^{-1} \}$, and prove that $\mathcal{H}^2(S_j) = 0$ for each $j \geq 1$. 

For $\delta > 0$, we take a covering $\{B_\delta(z_\alpha)\}_{\alpha=1,\ldots,N}$ of $S_j$ such that $z_\alpha \in S_j$ ($\alpha = 1, \ldots, N$), and $B_{\delta/2}(z_\alpha) \cap B_{\delta/2}(z_\beta) = \emptyset$ if $\alpha \neq \beta$. By definition, for any $z \in S_j$, we have $\frac{1}{r^2} \int_{B_r(z)} |F_A|^2 dV_g > j^{-1}$ for $r < \delta$. Thus, we have

$$N \left( \frac{\delta}{2} \right)^2 \leq j \int \bigcup_{B_{\delta/2} \left( z_\alpha \right)} |F_A|^2 dV_g \leq j \int_{S_j} |F_A|^2 dV_g, \tag{2.3}$$

where $S_j^\delta = \{ z \in Z : \text{dist}(z, S_j) < \delta \}$. Hence, we get

$$N \delta^6 \leq 2^2 \delta^4 j \int_{S_j^\delta} |F_A|^2 dV_g. \tag{2.4}$$

Since $B_\delta(z_\alpha)$’s cover $S_j$, (2.4) implies that $\mathcal{H}^6(S_j) \leq C \delta^4 \int_{S_j^\delta} |F_A|^2 dV_g$. Thus, as we can take $\delta \downarrow 0$, $S_j$ has the Lebesgue measure zero. Then by the dominated convergence theorem, we get $\lim_{\delta \to 0} \int_{S_j^\delta} |F_A|^2 dV_g \to 0$. Therefore, by (2.3), we conclude $\mathcal{H}^2(S_j) = 0$. \hfill \square

In order to see that the singular set $S$ is 2-rectifiable, we invoke the following theorem by Preiss.

**Theorem 2.5** (Preiss [P], see also [Ma], [D]). If $0 \leq m \leq p$ are integers, $\Omega$ is a Borel measure on $\mathbb{R}^p$ such that $0 < \lim_{r \to 0} \frac{\Omega(B_r(x))}{r^m} < \infty$ for almost all $x \in \Omega$, then $\Omega$ is $m$-rectifiable.

Theorem 2.5 tells us that the singular set $S$ of a weak convergence sequence $\{(A_n, u_n)\}$ in Theorem 1.1 is 2-rectifiable. In particular, for $\mathcal{H}^2$-a.e. $s \in S$ there exists a unique tangent plane $T_sS$.

We remark few more on the structure of the singular sets in Theorem 1.1. We put

$$S^{(2)} := \{ z \in S(\{(A_n, u_n)\}) | \Theta(\mu, z) > 0, \lim_{r \to 0+} \frac{1}{r^2} \int_{B_r(z)} |F_A|^2 dV_g = 0 \}. \tag{2.5}$$

As we describe in Section 3, the limit solution $(A, u)$ extends across this set $S^{(2)}$. We define $S^{(0)} = S \setminus S^{(2)}$, namely,

$$S^{(0)} = \{ z \in S(\{(A_n, u_n)\}) | \Theta(\mu, z) > 0, \lim_{r \to 0+} \frac{1}{r^2} \int_{B_r(z)} |F_A|^2 dV_g \geq \varepsilon \}.$$

This $S^{(0)}$ may be seen as the set of “unremovable” singularities, however, we have the following for the size of this $S^{(0)}$ from Proposition 2.4 at the moment.
Corollary 2.6.

\[ H^2(S^{(0)}) = 0. \]

We further expect that \( H^0(S^{(0)}) < \infty \), and it is discrete. This will be discussed somewhere else.

## 3 Removable singularity

In this section, we prove the following removal singularity theorem for D–T instantons.

**Theorem 3.1** (Theorem 1.1). Let \( B_r(0) \subset \mathbb{C}^3 \) be a ball of radius \( r \) centered at the origin with Kähler metric \( g \). We assume that the metric is compatible with the standard metric by a constant \( \Lambda \). Let \( E \) be a vector bundle over \( B_r(0) \), and let \((A,u)\) be an admissible D–T instanton of \( E \) with \( \int_{B_r(0)} |u|^2dV < \infty \). Then, there exists a constant \( \varepsilon > 0 \) such that if \( \frac{1}{r^2} \int_{B_r(0)} |F_A|^2dV_g \leq \varepsilon \), there exists a smooth gauge transformation \( \sigma \) on \( B_{r^2}(0) \setminus S \) such that \( \sigma(A,u) \) smoothly extends over \( B_{r^2}(0) \).

An immediate corollary of this is the following.

**Corollary 3.2.** Let \((A,u)\) be the limit solution in Theorem 1.1, and let \( S^{(2)} \) be the top stratum of the singular set \( S \) defined by (2.5). Then \((A,u)\) extends smoothly across \( S^{(2)} \).

**Proof of Theorem 3.1.** First, we recall the following result by Bando and Siu.

**Theorem 3.3** ([BS]). Let \( E \) be a holomorphic vector bundle with Hermitian metric \( h \) over a Kähler manifold \( Z \) (not necessarily compact nor complete) outside a closed subset \( S \) of \( Z \) with locally finite Hausdorff measure of real codimension four. We assume that its curvature tensor \( F \) is locally square integrable on \( Z \). Then

(a) \( E \) extends to the whole \( Z \) as a reflexive sheaf \( \mathcal{E} \), and for any local section \( s \in \Gamma(U, \mathcal{E}) \), \( \log^+ h(s,s) \) belongs to \( H^1_{\text{loc}} \).

(b) If \( \Lambda F \) is locally bounded, then \( h(s,s) \) is locally bounded, and \( h \) belongs to \( L^p_{2,\text{loc}} \) for any finite \( p \) where \( \mathcal{E} \) is locally-free.
Combining Theorem 3.3 with standard elliptic theory, we deduce that the limiting D–T instanton in Theorem 1.1 extends smoothly on the locally-free part of a reflexive sheaf over \(Z\). Thus we consider removing isolated singularities.

As the condition \(r^{-2} \int_{B_r(0)} |F_A|^2 dV \leq \varepsilon\) is scale-invariant, we assume that \(r = 1\) below. We prove the following in the rest of this section.

**Proposition 3.4.** Let \(B \subset \mathbb{C}^3\) be the unit ball centred at the origin with Kähler metric \(g\). We assume that the metric is compatible with the standard metric by a constant \(\Lambda\). Let \(E\) be a vector bundle over \(B \setminus \{0\}\), and let \((A,u)\) be a D–T instanton of \(E\) with \(\int_{B \setminus \{0\}} |u|^2 dV < \infty\). Then, there exists a constant \(\varepsilon > 0\) such that if \(\int_{B} |F_A|^2 dV_g \leq \varepsilon\), there exists a smooth gauge transformation \(\sigma\) on \(B \setminus \{0\}\) such that \(\sigma(A,u)\) smoothly extends to a smooth D–T instanton over \(B\).

**Proof.** We follow a proof by Nakajima [N] (see also Uhlenbeck [U1]) for Yang–Mills connections, except that we deal with an additional nonlinear term coming from the extra field \(u\). First, we prove the following.

**Lemma 3.5.** Let \((A,u)\) be a D–T instanton of a vector bundle \(E\) over \(B \setminus \{0\}\) with \(\int_{B \setminus \{0\}} |u|^2 dV < \infty\). Then, there exist constants \(\varepsilon > 0\) and \(C > 0\) such that if \(\int_{B} |F_A|^2 dV_g \leq \varepsilon\), then \(|z|^4 |F_A|^2(z) \leq C \varepsilon\) for \(z \in B_{1/2} \setminus \{0\}\).

**Proof.** By the monotonicity formula (2.1) and Proposition 2.2, we have

\[
\begin{aligned}
\frac{1}{|z|^2} \int_{B_{|z|}(z)} |F_A|^2 dV &= \frac{1}{|z|^2} \int_{B_{|z|}(z)} m(A,u) dV + \frac{1}{|z|^2} \int_{B_{|z|}(z)} |[u, \bar{u}]|^2 dV \\
&\leq \frac{1}{|z|^2} \int_{B_{2|z|}(0)} m(A,u) dV + \frac{1}{|z|^2} \int_{B_{2|z|}(0)} |[u, \bar{u}]|^2 dV \\
&\leq \frac{C}{|z|^2} \int_{B} m(A,u) dV + \frac{1}{|z|^2} \int_{B_{2|z|}(0)} |[u, \bar{u}]|^2 dV \\
&\leq C \varepsilon
\end{aligned}
\]

for \(z \in B_{1/2} \setminus \{0\}\). Hence, using Proposition 2.3, we obtain

\[
|F_A|^2(z) \leq \frac{C^2_1}{|z|^6} \int_{B_{|z|}(z)} |F_A|^2 dV + C^2 \varepsilon |z|^2 \leq \frac{C \varepsilon}{|z|^4}
\]

for sufficiently small \(\varepsilon > 0\). □
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Then Lemma 3.5 enables us to use a particular choice of gauge, which is vital for removing the singularities. Let $U_\ell := \{ z \in B : 2^{-\ell-1} \leq |z| \leq 2^{-\ell} \}$, and $S_\ell := \{ z \in B : |z| = 2^{-\ell} \}$ for each $\ell = 1, 2, \ldots$ so that $B_2 \setminus \{0\} = \cup_\ell U_\ell$. We recall the following from [U2].

**Definition 3.6** ([U1] pp. 25). A broken Hodge gauge for a connection $D$ of a bundle $E$ over $B_2 \setminus \{0\}$ is a gauge related continuously to the original gauge in which $D = d + A$ and $A_\ell := A|_{U_\ell}$ satisfies the following for all $\ell \geq 1$. (a) $d^* A_\ell = 0$ in $U_\ell$, (b) $A_\ell^\psi|_{S_\ell} = A_{\ell-1}^\psi|_{S_\ell}$, (c) $d^* A_\ell^\psi = 0$ on $S_\ell$ and $S_{\ell+1}$, and (d) $\int_{S_\ell} A_\ell^r = \int_{S_{\ell+1}} A_\ell^r = 0$, where the superscripts $r$ and $\psi$ indicate the radial and spherical components respectively.

Uhlenbeck proved the following existence result of the broken Hodge gauges on $\cup_\ell U_\ell$.

**Theorem 3.7** ([U1] Theorem 4.6). There exists a constant $\gamma > 0$ such that if $D$ is a smooth connection on $B_2 \setminus \{0\}$, and the growth of the curvature satisfies $|x|^2 |F(x)| \leq \gamma \leq \gamma'$, then there exists a broken Hodge gauge in $B_2 \setminus \{0\}$ satisfying (e) $|A_\ell| (x) \leq C 2^{-\ell} ||F_A||_{L^\infty} \leq C 2^{\ell+1} \gamma$, and (f) $(\lambda - C\gamma) \int_{U_\ell} |A_\ell|^2 dV \leq 2^{-2\ell} \int_{U_\ell} |F_A|^2 dV$, where $\lambda > 0$ is a constant.

Using this broken Hodge gauge, we deduce the following.

**Lemma 3.8.** There exists a constant $\varepsilon > 0$ such that if $(A,u)$ is a $D$–$T$ instanton of a bundle $E$ over $B \setminus \{0\}$ with $\int_B |F_A|^2 dV \leq \varepsilon$, then $|F_A|^2 (z) = o(|z|^{-4+\alpha})$ for some $\alpha > 0$ for all $z \in B_2 (0) \setminus \{0\}$.

**Proof.** By integration by parts, we get

$$
\int_{U_\ell} |F_A|^2 dV_g = \int_{U_\ell} \langle F_A, DA_\ell \rangle - \frac{1}{2} \int_{U_\ell} \langle F_A, [A_\ell, A_\ell] \rangle \\
= \int_{U_\ell} \langle D^* F_A, A_\ell \rangle - \frac{1}{2} \int_{U_\ell} \langle F_A, [A_\ell, A_\ell] \rangle + \left( \int_{S_{\ell-1}} - \int_{S_\ell} \right) \langle A_\ell^\psi \wedge F_A^r \rangle.
$$

Firstly, we estimate the first and second terms in the last line of (3.1). By using the equations (1.1), (1.2), the Hölder inequality, and (e) in Theorem
3.7, the first term in the last line of (3.1) becomes
\[
\int_{U_\ell} \langle D^* F_{A_\ell}, A_\ell \rangle = \int_{U_\ell} \langle \ast D(\Lambda[u, \bar{u}]), A_\ell \rangle \\
\leq \left( \int_{U_\ell} |D(\Lambda[u, \bar{u}])|^2 dV_g \right)^{\frac{1}{2}} \left( \int_{U_\ell} |A_\ell|^2 dV_g \right)^{\frac{1}{2}} \\
\leq \varepsilon C \left( \int_{U_\ell} ||u, \bar{u}||^2 dV_g \right)^{\frac{1}{2}},
\]
where \( \Lambda = (\omega \wedge)^* \). For the second term in the last line of (3.1), we again use the Hölder inequality and (e) in Theorem 3.7. Then it becomes
\[
\frac{1}{2} \int_{U_\ell} \langle F_{A_\ell}, [A_\ell, A_\ell] \rangle \leq \frac{1}{2} \left( \int_{U_\ell} |F_{A_\ell}|^2 \right)^{\frac{1}{2}} \left( \int_{U_\ell} |A_\ell|^4 \right)^{\frac{1}{2}} \\
\leq \frac{1}{2} \left( \int_{U_\ell} |F_{A_\ell}|^2 \right)^{\frac{1}{2}} \left( C \varepsilon^2 \int_{U_\ell} |F_{A_\ell}|^2 \right)^{\frac{1}{2}} \\
\leq C' \varepsilon \int_{U_\ell} |F_{A_\ell}|^2.
\]
Thus, from (3.1), (3.2), and (3.3), we get
\[
(1 - \varepsilon C') \int_{U_\ell} |F_{A_\ell}|^2 - \varepsilon C \int_{U_\ell} ||u, \bar{u}||^2 \leq \left( \int_{S_{\ell-1}} - \int_{S_{\ell}} \right) \langle A_1^\psi, F_{A_1}^{r\psi} \rangle d\sigma.
\]
Hence, taking \( \varepsilon \small{\text{small}} \), and summing up (3.4) in \( \ell \), we obtain
\[
\int_{B_{\frac{1}{2}}} |F_{A_\ell}|^2 - \int_{B_{\frac{1}{2}}} ||u, \bar{u}||^2 \leq 2 \int_{S_1} \langle A_1^\psi, F_{A_1}^{r\psi} \rangle d\sigma.
\]
We next estimate the right-hand side of (3.5). Firstly, we have
\[
\int_{S_1} \langle A_1^\psi, F_{A_1}^{r\psi} \rangle d\sigma \leq K \int_{S_1} |A_1^\psi|^2 d\sigma + \frac{1}{K} \int_{S_1} |F_{A_1}^{r\psi}|^2 d\sigma,
\]
where \( K > 0 \) is a constant determined later. By using an estimate in [U1, Th. 2.5], the first term of the right-hand side of the above inequality is bounded as follows.
\[
\int_{S_1} |A_1^\psi|^2 d\sigma \leq C \int_{S_1} |F_{A_1}^{r\psi}|^2 d\sigma \leq C \int_{S_1} |F_{A_1}|^2 d\sigma.
\]
Hence, by the rescaling $y = z/r$, we get
\[ \frac{1}{r^2} \int_{B_r} m(A, u) \leq \frac{CK}{r} \int_{\partial B_r} |F_A|^2 d\sigma + \frac{C}{Kr} \int_{\partial B_r} |F_A^{r\psi}|^2 d\sigma. \tag{3.6} \]

We then put $M(r) := e^{ar^2} r^{-2} \int_{B_r} m(A, u) dV_g$. Multiplying $e^{ar^2}$ to (3.6), and integrating it from $\rho/2$ to $\rho$, we obtain
\[ \int_{\frac{\rho}{2}}^{\rho} M(r) \, dr \leq CK e^{ar^2} \rho^{-1} \int_{B_\rho} |F_A|^2 dV_g 
+ CK^{-1} \int_{\frac{\rho}{2}}^{\rho} e^{ar^2} r^{-1} \int_{\partial B_r} |F_A^{r\psi}|^2 d\sigma dr. \tag{3.7} \]

From the monotonicity formula (2.1), we deduce that the left-hand side of (3.7) is bounded by $\frac{\rho}{2} M \left( \frac{\rho}{2} \right)$ from below. On the other hand, the second term of the right-hand side of (3.7) is estimated as follows.
\[ CK^{-1} \int_{\frac{\rho}{2}}^{\rho} e^{ar^2} r^{-1} \int_{\partial B_r} |F_A^{r\psi}|^2 d\sigma dr \]
\[ \leq CK^{-1} \rho \left( e^{ar^2} \rho^{-2} \int_{B_\rho} |F_A|^2 dV_g - 2 e^{ar^2/4} \rho^{-2} \int_{B_{\frac{\rho}{2}}} |F_A|^2 dV_g \right) \]
\[ \leq CK^{-1} \rho \left( M(\rho) - M(\rho/2) \right) + CK^{-1} e^{ar^2} \rho \left( \rho^{-2} \int_{B_\rho} ||u - \bar{u}||^2 \right). \]

Hence we get
\[ \left( \frac{1}{2} + CK^{-1} \right) M(\rho/2) \leq (CK + CK^{-1}) M(\rho) + C' \varepsilon^2 e^{ar^2} \rho^4. \]

Thus, by taking $K$ small, we obtain $\zeta M(\rho/2) \leq M(\rho) + C' \varepsilon^2 e^{ar^2} \rho^4$ for some $\zeta > 1$. Hence, by iteration, we obtain
\[ M(2^{-\ell}) \leq \left( 2^{-\ell} \right)^{\log_2 \zeta} M(1/2) + C'' \varepsilon^2 \left( 2^{-\ell} \right)^4. \]

Therefore,
\[ M(\rho) \leq C \rho^\alpha \int_B m(A, u) \, dV_g + C'' \varepsilon^2 \rho^4, \tag{3.8} \]
where $\alpha = \log_2 \zeta$. 
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Hence, form Proposition 2.3 and (3.8), we get

\[ |F_A|^2(z) \leq \frac{C}{|z|^2} \left( \frac{1}{|z|^2} \int_{B_2|z|} |F_A|^2 \, dV_g \right) + C \varepsilon^2 |z|^2 \]

\[ \leq C' |z|^{-4+\alpha} \int_B |F_A|^2 \, dV_g + C'' \varepsilon^2 + C \varepsilon^2 |z|^2. \]

Thus, Lemma 3.8 is proved. \( \square \)

From Lemma 3.8, we deduce \( F_A \in L^p \) for some \( p > 3 \). Hence, a theorem by Uhlenbeck [U2, Th. 2.1] (see also [W, Chap. 6]) tells us that there is a gauge transformation \( \sigma \in L^p_2 \) such that \( \sigma(A,u) \) smoothly extends over \( B \). As \( L^p_2 \subset C^0 \) for \( p > 3 \), and \( A \) and \( \sigma(A) \) are smooth on \( B \setminus \{0\} \), we then realise that \( \sigma \) is also smooth on \( B \setminus \{0\} \). \( \square \)

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