INSTABILITY AND SPECTRUM OF THE LINEARIZED TWO-PHASE FLUIDS INTERFACE PROBLEM AT SHEAR FLOWS

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Abstract. This paper is concerned with the 2-dim two-phase interface Euler equation linearized at a pair of monotone shear flows in both fluids. We extend the Howard’s Semicircle Theorem and study the eigenvalue distribution of the linearized Euler system. Under certain conditions, there are exactly two eigenvalues for each fixed wave number $k \in \mathbb{R}$ in the whole complex plane. We provide sufficient conditions for spectral instability arising from some boundary values of the shear flow velocity. A typical mode is the ocean-air system in which the density ratio of the fluids is sufficiently small. We give a complete picture of eigenvalue distribution for a certain class of shear flows in the ocean-air system.

1. Introduction

In this paper, We study two-phase interface system which is modeled by the two dimensional Euler problem. Both fluids are considered immiscible and incompressible (see, for example, [9]). We use superscript ”+” to denote notations in the upper fluid and subscript ”−” for the lower one. At time $t \geq 0$, the above fluid occupies $\Omega^+_t$, and the fluid below inhabits the region $\Omega^-_t$. The fixed constants $h_\pm > 0$ are the locations of flat lid of the above fluid (see, for example, [9]) and flat bed of the fluid below. We denote $S_t := \partial \Omega^+_t \cap \partial \Omega^-_t = \{(x, t)|x_2 = \eta(t, x_1)\}$ as the interface, which is considered as a graph of a smooth function $\eta$. In addition, we assume that

$$\Omega^+_t \cup \Omega^-_t \cup S_t = \mathbb{T} \times (-h_-, h_+),$$

where $\mathbb{T} := \mathbb{R} \setminus 2\pi \mathbb{Z}$. Suppose the densities of both fluids are constants $\rho^\pm > 0$ for $x \in \Omega^\pm$. Let $v^\pm = v^\pm(t, x) \in \mathbb{R}^2$ be the velocity field and $p^\pm = p^\pm(t, x) \in \mathbb{R}$ be the pressure. Then $v^\pm$ and $p^\pm$ satisfy the two dimensional incompressible Euler system as following.

$$\partial_t v^\pm + v^\pm \cdot \nabla v^\pm + \frac{1}{\rho^\pm} \nabla p^\pm + g e_2 = 0 \quad \text{in} \quad \Omega^\pm,$$

$$\nabla \cdot v^\pm = 0 \quad \text{in} \quad \Omega^\pm,$$

where $g$ is the gravitational acceleration and $e_2$ is the unit vector in $x_2$-axis. Since $v^\pm$ restricted to $S_t$ is a boundary velocity, the motion of the interface satisfies the kinematic boundary condition.

$$\partial_t \eta = v^\pm \cdot (-\partial x) \eta, 1)^T \quad x \in S_t.$$

Considering the surface tension, we also impose the dynamic boundary condition

$$p^+(t, x) - p^-(t, x) = \sigma \kappa \quad x \in S_t,$$

where $\sigma > 0$ and $\kappa = \frac{\eta_x}{(1 + \eta_x^2)^{3/2}}$ is the mean curvature of $S_t$ at $x$ which corresponds to the surface tension. We assume that the upper fluid has finite altitude. It is justified that the behavior of the flow does not strongly affect the dynamics near the lower fluid if it evanesces at high altitude (e.g. [9]). The rigid flat lid of the upper fluid and bed of the lower fluid imply that

$$v^\pm \cdot e_2 = 0 \quad x_2 = \pm h_\pm.$$

The local well-posedness theory for the interface problem (1.1) has been well studied. When surface tension is considered, the full nonlinear problem is locally well-posed (cf., [2], [18]). For both rotational and irrotational problem, the linearized system is ill-posed if surface tension is ignored [8].

It is well known that shear flows are a fundamental class of stationary solutions in the form of

$$v^\pm(t, x) = (U^\pm(x_2), 0), \quad S_t = \mathbb{T} \times \{0\}.$$
Our goal is to analyze this two-phase fluids interface system linearized at a pair of uniformly monotone shear flows.

\[(1.3) \quad U^{\pm} \in C^{0, l_0}, \, l_0 \geq 6, \, (U^{+})' \neq 0, \, x_2 \in [0, h_+], \, \text{and} \, (U^{-})' \neq 0, \, x_2 \in [-h_-, 0].\]

1.1. Linearization. In this subsection, we linearize the Euler system (1.1) near shear flows (1.2). The linearization was previously obtained in [1], but we adopt a slightly more direct approach as in [11]. We consider an one-parameter family of solutions \((S_t(\theta), v^{\pm}(\theta, t, x), p^{\pm}(\theta, t, x))\) of system (1.1) with

\[
(S_t(0), v^{\pm}(0, t, x), \nabla p^{\pm}(0, t, x)) = (S_*, v^{\pm}_*, -g\rho^{\pm} c_2).
\]

We abuse the notations and let \(v^{\pm}, p^{\pm}, \eta\) be the linearized solution in the following way.

\[
v^{\pm}(t, x) = (v_1^{\pm}(t, x), v_2^{\pm}(t, x)) := \lim_{\theta \to 0} \partial_\theta v^{\pm}(t, x),
\]

\[
p^{\pm}(t, x) := \lim_{\theta \to 0} \partial_\theta p^{\pm}(t, x), \, \eta(t, x_1) := \lim_{\theta \to 0} \partial_\theta \eta(t, x_1).
\]

We differentiate (1.1a) with respect to \(\theta\) and send \(\theta\) to 0,

\[(1.4) \quad \partial_t v^{\pm} + v^{\pm}_* \cdot \nabla v^{\pm} + \nabla v^{\pm}_* \cdot v^{\pm} + \frac{1}{\rho^{\pm}} \nabla p^{\pm} = 0.\]

Considering the second component of this equation, we have

\[(1.5a) \quad \partial_t v_2^{\pm} + U^{\pm}(x_2) \partial_{x_2} v_2^{\pm} + \frac{1}{\rho^{\pm}} \partial_{x_2} p^{\pm} = 0, \quad (x_1, \pm x_2) \in \mathbb{T} \times (0, h_\pm).
\]

We apply the divergence operator to (1.4) to obtain that

\[
\nabla \cdot (v^{\pm}_* \cdot \nabla v^{\pm} + v^{\pm} \cdot \nabla v^{\pm}_*) + \frac{1}{\rho^{\pm}} \Delta p^{\pm} = 0.
\]

Sending \(\theta\) to 0 in the above equation, we obtain that

\[(1.5b) \quad -\frac{1}{\rho^{\pm}} \Delta p^{\pm} = 2(U^{\pm})'(x_2) \partial_{x_2} v_2^{\pm}, \quad (x_1, \pm x_2) \in \mathbb{T} \times (0, h_\pm).
\]

We differentiate (1.1c) in \(\theta\) to obtain that

\[(1.5c) \quad \partial_\theta \eta = v_2^{\pm} - U^{\pm}(x_2) \partial_{x_2} \eta, \quad \{x_2 = 0\}.
\]

Now we consider the variation of \(S_t\). We notice that (1.1d) means that

\[
p^{\pm}(\theta, t, x_1, \eta(\theta, t, x_1)) = p^{-}(\theta, t, x_1, \eta(\theta, t, x_1)) + \sigma \kappa(\theta, t, x_1, \eta(\theta, t, x_1)).
\]

Taking derivative with respect to \(\theta\) and sending \(\theta\) to 0, we obtain that

\[(1.5d) \quad p^{+}(0) - p^{-}(0) = g\eta(\rho^{+} - \rho^{-}) + \sigma \partial_{x_2}^2 \eta.
\]

Linearizing (1.1e), we have

\[(1.5e) \quad v_2^{\pm}(x_1, \pm h_\pm, t) = 0.
\]

We restrict (1.5c) to \(\pm x_2 = h_\pm\) and obtain that

\[(1.5f) \quad \partial_{x_2} p^{\pm} = 0, \quad \pm x_2 = h_\pm.
\]

The system (1.5) form a two-phase fluids interface problem linearized at a pair of shear flows (1.2). In the capillary gravity water wave problem \((\rho^{\pm} = 0)\), the pressure \(p\) can be covered by a boundary value problem of elliptic system (1.5b), (1.5d), and (1.5f). In this two-phase fluids interface problem \((\rho^{\pm} > 0)\), the pressure can be expressed in terms of velocity. We refer the readers to Section 2 in [17]. Hence, the linearized system (1.5) can be viewed as an evolutionary problem of two unknowns \((v_2, \eta)\). The goal of this paper is to study the instability of monotone shear flows.
1.2. Background. We notice that the variable coefficients in the linearized system (1.5) depend only on $x_2$. Hence, each Fourier mode in $x_1$ is decoupled from other modes. To study the linearized system, it is natural to seek eigenvalues and eigenfunctions in the form of

$$(v_2^+(t,x_1,x_2), \eta(t,x_1), p^+(t,x_1,x_2)) = (v_2^+(x_2), \eta, p^+(x_2)) e^{ik(x_1-ct)}, \quad k \in \mathbb{Z} \setminus \{0\}$$

where the eigenvalues take the form $\lambda = -ikc$ with the wave speed $c = c_R + ic_I \in \mathbb{C}$ and wave number $k$. The linear solution is spectrally unstable if a wave speed $c$, which appears in conjugate pairs, has a positive imaginary part and wave number $k > 0$. Due to the symmetry of the spectrum, to find instability it suffices to consider the case of $k > 0$ and seek solution with $c_I > 0$.

Due to the significance in mathematics and physics, the two-phase fluids interface problem linearized at shear flows has been studied by some mathematicians. The simplest situation is the classical Kelvin-Helmholtz model where the upper layer fluid is air and the ocean stays below. In this model, the ocean at shear flows has been studied by some mathematicians. The simplest situation is the classical Kelvin-channel (cf., [1]).

There are also some results regarding the linearized free boundary problem of gravity waves at shear flows. For a class of shear flows, the rigorous bifurcation of unstable eigenvalues was proved, e.g., in [4, 10]. Lin showed that for a certain class of shear flows, the neutral limiting wave speed must be an inflection value of the velocity profile and proved the global bifurcation of unstable modes from neutral modes rigorously [10]. There are also some results regarding the linearized free boundary problem of gravity waves at shear flows. Yih [20] showed that if the shear velocity profile $U$ is monotone without inflection value, there are no singular neutral modes with $c$ in the interior of the range of $U$. He also proved the existence of non-singular neutral modes with real wave speed outside the range of $U$ and extended the Semicircle Theorem to the linearized gravity water wave near shear flows. For a certain class of shear flows, Hur and Lin [6, 7] proved that the linear instability may occur only when the wave speed $c$ is near the value of $U$ at bottom, critical values, or inflection values of $U$. They also gave an open set of wave numbers where linear instability exists. M. Renardy and Y. Renardy [16] found examples showing that instability can arise from these three locations through numerical methods. In the author’s recent work with Zeng [11], we considered capillary gravity water waves linearized at monotone shear flows and proved that in contrast to the fixed channel flow, there are exactly two non-singular modes (i.e. the corresponding eigenvalue problem has nontrivial solution when $c$ is outside the range of $U$) for each wave number $k$ under certain conditions which are real for all high wave numbers. We also obtained the bifurcation of unstable eigenvalues from inflection values of $U$ and value of $U$ at bottom. Moreover, we gave a complete picture of eigenvalue distribution for a certain class of monotone shear flows.

1.3. Main Results. The goal of this paper is to study the eigenvalue distribution of the linear system (1.5) and seek linear instability. Particularly, we consider the upper layer fluid is lighter than the bottom fluid and the perturbed free surface $\eta \neq 0$. If $\eta = 0$, system (2.2) is equivalent to the system which consists of
two Rayleigh equations with fixed boundary without interface motion. In that case of the Euler equation
linearized at monotone shear flows, the only possible locations of $c$ where instability arises are inflection
values of $U^+$ or $U^-$ (e.g. [10]). We first extend the Howard’s Semicircle theorem to the two-phase fluid
interface problem under a mild condition which holds if the fluid above is no heavier than the one below, i.e.
$\rho^+ \leq \rho^-$. Throughout this paper, we let
\[
(1.7) \quad a := \min\{U^-([-h_-,0]) \cup U^+([0,h_+])\}, \quad b := \max\{U^-([-h_-,0]) \cup U^+([0,h_+])\}.
\]

**Theorem 1.1.** Assume that $\eta \neq 0$ and the following holds.
\[
(1.8) \quad -g(\rho^+ - \rho^-) + \sigma k^2 \geq 0.
\]
Then any unstable wave speed $c = c_R + ic_I \in \mathbb{C}$ with $c_I > 0$ must stay in the following upper semicircle
\[
(1.9) \quad (c_R - \frac{a+b}{2})^2 + c_I^2 \leq \left(\frac{b-a}{2}\right)^2, \quad c_I > 0.
\]

In seeking solutions in the form of $(1.6)$, we treat the wave number $k \in \mathbb{R}$ as a parameter. The next
theorem shows that when $|k|$ is large, there are exactly two eigenvalues in the whole complex plane with two
branches of wave speed $c^\pm(k) \in \mathbb{R}$. For a pair of monotone shear flows without inflection points, we discuss
two cases, depending on whether the intersection of the range of $U^-$ and the one of $U^+$ is empty. For each
case, we prove that under a certain condition, there is no singular mode and both $c^\pm(k)$ can be extended to
be even and analytic functions for all $k \in \mathbb{R}$. Basically, singular modes correspond to embedding eigenvalues
with wave speed $c \in U^-([-h_-,0]) \cup U^+([0,h_+])$. One can see the precise definition in Definition 2.1.

**Theorem 1.2.** Suppose $U^\pm \in C^3$, $(U^\pm)' \neq 0$, $\eta \neq 0$, and $0 < \rho^+ < \rho^-$. Let $a, b$ be defined in (1.7). Then
the following hold.

1. There exists $k_0 > 0$ such that for any $|k| > k_0$, there are no singular modes and exist exactly two
non-singular modes with $c^\pm(k) \in \mathbb{C}$. Moreover, $c^+(k) > b$ and $c^-(k) < a.$

2. $\lim_{|k| \to \infty} c^\pm(k) / \sqrt{\frac{\sigma |k|}{\rho^+ + \rho^-}} = \pm 1.

3. Assume that $(U^\pm)'' \neq 0$ and
\[
\min_{c \in (U^-(0), U^-(-h_-))} \int_{-h_-}^{0} \frac{1}{(U^-(x_2) - c)}^2 dx_2, \quad \min_{c \in (U^+(0), U^+(h_+))} \int_{0}^{h_+} \frac{1}{(U^+(x_2) - c)}^2 dx_2 > \frac{1}{g}.
\]
If one the following holds,

(a) $U^-([-h_-,0]) \cup U^+((0,h_+)) = \emptyset$ and
\[
\max_{c \in (U^\pm(0), U^\pm(\pm h_\pm))} \left\{ \rho^+ \int_{0}^{h_+} (U^+(x_2) - c)^2 dx_2 + \rho^- \int_{-h_-}^{0} (U^-(x_2) - c)^2 dx_2 \right\} < \sigma,
\]

(b) For all $c \in U^-([-h_-,0]) \cup U^+((0,h_+)) \neq \emptyset$, $(U^\pm)''(U^\pm)' > 0$, and
\[
\max_{c \in (a,b)} \left\{ \rho^+ \int_{0}^{h_+} (U^+(x_2) - c)^2 dx_2 + \rho^- \int_{-h_-}^{0} (U^-(x_2) - c)^2 dx_2 \right\} < \sigma,
\]
then the two branches $c^\pm(k)$ can be extended to be even and analytic functions for all $k \in \mathbb{R}$. $c^+(k) > b$
and $c^-(k) < a$ for all $k \in \mathbb{R}$. Moreover, $c^\pm(k)$ correspond to the only singular and non-singular modes
of the linearized Euler system $(1.5)$.

**Remark 1.1.** When $\rho^+ = 0$ and $(U^-)' \neq 0$, then results in Theorem 1.2(1) and (2) coincide the ones of
capillary gravity water wave linearized at monotone shear flows which is obtained in [11].

We notice that $c = a$ is on the boundary of the domain where the Rayleigh equation keeps regularity.
Based on Theorem 1.1, it might correspond to an isolated singular mode or a neutral limiting mode which
is defined as the limit of a sequence of unstable modes (see Definition 2.1). In the following theorem, we
consider the case where the lighter fluid stays on top of the heavier fluid. For the case of monotone shear
flows which have no inflection value, we study the behavior of branch $c^-(k)$ which is obtained in Theorem
1.2(1) as $|k|$ tends to 0 from infinity. We show that there exists $g_* \geq 0$ such that the value of $g$ relative to
$g_*$ tells us when the instability happens near $c = a$, i.e. the minimal value of $U^+(0,h_+) \cup U^-(0,h_-)$. More
precisely, if $g \geq g_*$, there is no instability when $c$ is near $a$. If $g < g_*$, under an additional assumption
that \((U^\pm)'' > 0\), instability arises from \(c = a\) under some conditions. In this case, if \((U^\pm)'' < 0\), there is no instability near \(c = a\). In the capillary gravity water wave problem linearized at monotone shear flows, according to Theorem 1.1 in [11], there exists \(g_\#\) which provides a sharp condition for linear instability arising from the value of velocity profile at the bottom. More precisely, when \(g \in (0, g_\#)\), an increasing convex shear flow is spectrally unstable and an increasing concave shear flow is spectrally stable. When \(g \geq g_\#\), monotone shear flow is always spectrally stable if the basic velocity profile has no inflection value. Our result for \(\rho^+ = 0\) coincides this result and in this case \(g_* = g_\#\).

**Theorem 1.3.** Assume that \(U^\pm \in C^6, (U^\pm)' \neq 0, (U^\pm)'' \neq 0, \eta \neq 0, \) and \(0 < \rho^+ < \rho^-\). Let \(c^-(k)\) be obtained in Theorem 1.2(1) and \(a = \min U^-([-h_-,0]) \cup U^+([0,h_+])\). Then, there exists \(g_*>0\) such that the following hold.

1. If \(g > g_*\), then \(c^-(k)\) can be extended as an even and analytic function such that for all \(k \in \mathbb{R}\), \(c^-(k) < a\). There is no singular mode at \(c = a\) for all \(k \in \mathbb{R}\).
2. If \(g = g_*\), there exists a unique \(k_* > 0\) such that \(c^-(k)\) can be extended to be an even and \(C^{1,\alpha}\) function for all \(k \in \mathbb{R}\) (for any \(\alpha \in [0,1)\)) such that it is analytic everywhere except \(k = \pm k_*\) where \(c^-(\pm k_*) = a\) for all \(k \in \mathbb{R}\).
3. If one of the following assumptions is further satisfied,
   
   (a) \(a = U^+(0) \notin U^-([-h_-,0])\) and \(\frac{\rho^-}{g(\rho^- - \rho^+)} < \int_{-h_-}^0 \frac{1}{(U^-(x_2) - U^+(0))^2} dx_2;\)
   
   (b) \(a = U^-(-h_-)\) or \(U^-(0)\);
   
   (c) \(a = U^+(h_+) \notin U^-([-h_-,0])\) and \(\frac{\rho^-}{g(\rho^- - \rho^+)} < \int_{-h_-}^0 \frac{1}{(U^-(x_2) - U^+(h_+))^2} dx_2;\)

   then for \(g \in (0, g_*]\), there exist \(k^+_\ast > k^-_\ast > 0\) and \(\delta > 0\) such that we have the following.

   (i) Assume that \((U^\pm)' = 0\). Then \(c^-(k)\) can be extended to be an even and \(C^{1,\alpha}\) function for all \(|k| \geq k^+\ast - \delta\) and analytic except at \(k = \pm k^+_\ast\) and it satisfies that

   \[
   (1.10) \quad c^-(\pm k^+_\ast) = a, \quad c^-(k) < a, \quad \forall |k| > k^+_\ast, \\
   (1.11) \quad c^\# - (k) > 0, \quad c^\#_R(k) > a, \quad \forall |k| \in [k^-_\ast - \delta, k^+_\ast). \\
   
   And there exists a \(C^{1,\alpha}\) function \(C(k)\) on \([0, k^-_\ast + \delta]\) and analytic except at \(k = \pm k^-_\ast\) such that

   \[
   (1.12) \quad C(\pm k^-_\ast) = a, \quad C(k) < a, \quad \forall |k| \in [0, k^-_\ast), \\
   (1.13) \quad C_\#(k) > 0, \quad C_\#R(k) > a, \quad \forall |k| \in (k^-_\ast, k^+_\ast + \delta].
   
   Moreover, there exists \(\gamma > 0\) such that for \(C_\# < a + \gamma\), all singular and non-singular modes with \(|k| \geq k^+_\ast - \delta\) are \(c^-(k)\) and the ones with \(|k| \in [0, k^-_\ast + \delta]\) are \(C(k)\).

   (ii) Assume that \((U^\pm)' = 0\). Then \(c^-(k)\) can be extended to be an even and \(C^{1,\alpha}\) function for all \(|k| \geq k^+_\ast\), analytic except at \(k = \pm k^+_\ast\), and (1.10) holds. And there exists a \(C^{1,\alpha}\) function \(C(k)\) on \([0, k^+_\ast]\) and analytic except at \(k = \pm k^-_\ast\) such that (1.12) holds. In addition, for \(c \leq a, \) all singular and non-singular modes with \(|k| \geq k^+_\ast\) are \(c^-(k)\) and the ones with \(|k| \in [0, k^-_\ast]\) are \(C(k)\).

**Remark 1.2.** 1. In Theorem 1.3(3), \(c^-(k)\) may not be extended to be a \(C^{1,\alpha}\) function for all \(k \in \mathbb{R}\) since there might exist an isolated neutral limiting mode (see Definition 2.1) with \(c \neq a\) (see Lemma 3.4), which may break the continuation of the bifurcation curve. 2. Let \(\rho^+ = 0\). The result in Theorem 1.3(1), (2), and (3) for \(a = U^-(-h_-)\) coincides the one of capillary gravity water wave linearized at monotonically increasing and convex shear flows obtained in [11].

When \(\epsilon := \frac{\rho^+}{\rho^-} \ll 1\), we consider the two-phase fluid interface problem as a perturbation of capillary gravity water wave linearized at monotone shear flows. In the following, we discuss a special case where the speed of the monotonically increasing shear flow in lower fluid is strictly slower than the one in upper fluid. We prove that if the one fluid free boundary problem has a neutral mode with wave speed \(c > U^-(0)\) in
the range of $U^+$, which means that there exists a critical layer location $x_2 \in [0, h_\pm]$, then small $\epsilon$ leads to unstable mode near such wave speed in the ocean-air system. This is due to the resonance between shear flows in the air and ocean. Such situation can be viewed as an extension of the critical layer phenomenon studied by Miles [12, 13, 14] and Bühler-Shatah-Walsh-Zeng [1]. We know that in the one fluid problem, the right branch of wave speed $c_0^+(k)$ is real and larger than $U^-(0)$ for all $k \in \mathbb{R}$. The eigenvalue distribution of the ocean-air system depend on the location of the neutral mode with zero wave number (i.e. $c_0$ defined in the below) in the one fluid free boundary problem and the sign of $(U^+)^\nu$. If $(U^+)^\nu < 0$, then the existence of critical layer in the above sense in the range of $U^+$ leads to instability. We prove that if $(U^+)^\nu > 0$ and $c^+(0) \in (U^+(0), U^+(h_\pm))$, then then for small $\epsilon > 0$, as $|k|$ becomes small and tends to 0, the right branch $c^+(k)$ of the two-phase interface problem disappears when it reaches $U^+(h_\pm)$. If $(U^+)^\nu > 0$ and $c_0^+(0) \in (U^-(0), U^+(0))$, then $c^+(k)$ of disappears when it reaches $U^+(h_\pm)$ and reappears at $U^+(0)$. Meanwhile, we have more precise result of the other branch $c^-(k)$ compared with the one in Theorem 1.3, which depends on the relative value of $g$ and $g_\#$(mentioned before Theorem 1.3). Under a certain condition at $c = U^-(h_\pm)$, if $g > g_\#$, for small $\epsilon$, $c^-(k)$ can still be extended for all $k \in \mathbb{R}$ and $c^-(k) < U^-(h_\pm)$. When $g \in [0, g_\#]$ with a small assumption when $g = g_\#$, if $(U^-)^\nu > 0$, instability occurs when $c_R^-(k)$ enters and then leaves the range of $U^-$ through $U^-(-h_-)$ as $|k|$ tends to 0 from infinity for $0 < \epsilon \ll 1$. If $(U^-)^\nu < 0$, $c^-(k)$ disappears when it touches $U^-(-h_-)$ and then reappears for small $|k|$.

**Theorem 1.4.** Assume that $U^\pm$ satisfies (1.3), $(U^\pm)' > 0, U^-(0) < U^+(0), (U^\pm)^\nu \neq 0$, and $\eta \neq 0$.

1. Assume that

\begin{equation}
\int_{-h_-}^{0} (U^-(x_2) - U^+(h_\pm))^2 dx_2 < \frac{\sigma}{\rho^-}.
\end{equation}

Let $c_0 > U^-(0)$ be the unique solution to

\begin{equation}
\int_{-h_-}^{0} \frac{1}{(U^-(x_2) - c^2)^2} dx_2 = \frac{1}{g}.
\end{equation}

Let $c^+(k)$ be obtained in Theorem 1.2. Then the following hold.

1. If $c_0 > U^+(h_\pm)$, there exists $c_0 > 0$ such that for any $\frac{U^-}{\rho^-} \in (0, c_0]$, $c^+(k)$ can be extended to be an even and analytic function for all $k \in \mathbb{R}$ and $c^+(k) > U^+(h_\pm)$. It is the only singular and non-singular mode for $c^+ > U^+(0)$.
2. If $c_0 \in (U^+(0), U^+(h_\pm))$, there exists $c_1 > 0$ such that for any $\frac{U^-}{\rho^-} \in (0, c_1]$, there exists a unique $k_* > 0$ such that $c^+(\pm k_*) = U^+(h_\pm)$ and the following holds.
   1. If $(U^+)^\nu < 0$, then $c^+(k)$ can be extended to be an even and $C^{1,\alpha}$ (for all $\alpha \in (0, 1)$) function for all $k \in \mathbb{R}$ and analytic in $k$ except at $\pm k_*$. $c^+(k)$ satisfies
      \begin{align*}
      c^+(k) > U^+(h_\pm), \forall |k| \in (k_*, \infty), \quad c^+_f(k) > 0, c^+_R(k) \in (U^+(0), U^+(h_\pm)), \forall |k| \in (0, k_*].
      \end{align*}
   2. If $(U^+)^\nu > 0$, $c^+(k)$ can be extended to be a real valued $C^{1,\alpha}$ function (for all $\alpha \in (0, 1)$) for $|k| > k_*$ and analytic in $k$ if $|k| > k_*$. Moreover, $c^+(k)$ is the only singular and non-singular mode for $c^+ > U^+(0)$.
3. Suppose $c_0 \in (U^-(-h_-), U^+(0))$. Then there exists $c_2 > 0$ such that for any $\epsilon \in (0, c_2]$, there exist $k^* > k^* > 0$ such that $c^+(k^* \pm) = U^+(h_\pm)$, $c^+(k^*) = U^-(0)$.
   1. If $(U^+)^\nu < 0$, then $c^+(k)$ can be extended as an even $C^{1,\alpha}$ function (for $\alpha \in [0, 1)$) for all $k \in \mathbb{R}$ and analytic in $k$ except at $k = \pm k^*_\pm$ such that
      \begin{align*}
      c^+_f(k) > 0, \quad c^+_R(k) \in (U^+(0), U^+(h_\pm)), \forall |k| \in (k^*_\pm, k^*_\pm],
      \end{align*}
   2. If $(U^+)^\nu > 0$, then $c^+(k)$ can be extended as an even $C^{1,\alpha}$ real valued function (for $\alpha \in [0, 1)$) for $|k| \geq k^*_\pm$ and analytic in $k$ if $k \neq \pm k^*_\pm$. And $c^+(k) > U^+(h_\pm)$ for all $|k| > k^*_\pm$. In addition, there exists an even real function $C(k)$ which is $C^{1,\alpha}$ in $k$ for all $|k| \leq k^*_\pm$ and analytic in $k$ except at $k = \pm k^*_\pm$ such that $C(\pm k^*_\pm) = U^+(0)$. And $C(k) \in (U^-(-h_-), U^+(0))$ for all $|k| < k^*_\pm$.

Moreover, for $c^+_R > U^-(-h_-)$, all the singular and non-singular modes are $c^+(k)$ and $C(k)$.\]
Remark 1.3. 1. If there exists $\gamma > 0$ such that $\gamma < \theta \min\{U^+(h_+) - U^+(0), U^+(0) - U^-(0)\}$ for any $\theta \in (0, 1)$ and $c_0 \in [U^-(0) + \gamma, U^+(0)] \cup [U^+(0) + \gamma, U^+(h_+)] \cup [U^+(h_+) + \gamma, \infty)$, then $c_0$ in (1), $\epsilon_1$ in (2), and $\epsilon_2$ in (3) can be the same and independent of location of $c_0$.

1.4. Outline of the proof. The proof of Theorem 1.1 is presented in Section 2. To prove Theorem 1.2, we first generalize some existing results we obtained in [11] on the fundamental solutions $y^\pm(c, x, k, x_2)$ to classical Rayleigh equation and crucial quantities $Y^\pm(c, k) := \frac{y^\pm(c, x, k, x_2)}{y^\pm(0, x, k, x_2)}(c, k)$. The free boundary condition on the interface (2.14) which gives the relation between wave speed $c$ and wave number $k$ can be written in terms of $Y^\pm(c, k)$ which are well defined if both $(U^+)'' \neq 0$ and $(U^-)' \neq 0$ or $|k|$ is large. This allows us to analyze the function $F(c, k, \epsilon)$ which is defined in (3.1) based on the properties of $Y^\pm(c, k)$. The zeros of $F(c, k, \epsilon)$ correspond to the eigenvalues of the linearized system (1.5). To discuss the eigenvalue distribution, we treat the wave number $k$ as a parameter and start with large $|k|$. We apply the Contraction Mapping Theorem on the region outside the range of $U^\pm$ to prove statement (1) and (2)(See Lemma 3.2). Finally, after ruling out all the possible singular modes under certain conditions, we prove that without singular modes, the two branches wave speed $c^\pm(k)$ can be extended for all $k \in \mathbb{R}$ in statement (3). This follows directly from Lemma 3.5.

To prove Theorem 1.3, we study the properties of $Y^\pm(c, k)$ and explicit representation of $y^+(c, 0)$ (resp. $y^-(c, 0)$) when $c \notin U^+(0, h_+)$ (resp. $c \notin U^-(0, h_-)$). Thanks to the monotonicity of $\partial_k Y^\pm$ and concavity of $F(c, k, \epsilon)$ for $k \geq 0$, we reduce the analysis to the case of $k = 0$. And then we apply a bifurcation analysis near $c = a$ to study the behavior of $c^-(k)$ locally. Finally, by applying a complex continuation argument of a holomorphic function, we extend $c^-(k)$ as $|k|$ becomes small. The proof of Theorem 1.3 follows from a series of Lemmas in Section 3.

To prove Theorem 1.4, we consider the interface problem as a perturbation of the one fluid free boundary problem and study a special case where both speed of wind and the one of ocean are monotonically increasing and have no inflection value, and at the interface, the speed of ocean is strictly slower than the one of air. After overcoming the degeneracy at $c = U^+(0)$ and $c = U^+(h_+)$, we prove the instability arising from these locations under a certain condition by applying the Implicit Function Theorem and Mean Value Theorem. Based on the eigenvalue distribution of the capillary gravity wave wave linearized at monotone shear flows (see [11]), we apply complex continuation argument of a holomorphic function and give a complete picture of eigenvalue distribution. The proof of Theorem 1.4 is completed in Section 4.

2. Preliminary

To study the linear system (1.5), we seek eigenvalues and eigenfunctions in the form of (1.6) and consider the corresponding eigenvalue problem. Using (1.6) in (1.5b), (1.5a) and (1.5d), we have

\begin{equation}
\frac{1}{\rho^\pm}(k^2 - \partial^2_{x_2})p^\pm(x_2) = 2ik(U^\pm)'(x_2)c^\pm_2(x_2),
\end{equation}
\[(2.1b) \quad ik(U^\pm(x_2) - c)v_2^\pm(x_2) + \frac{1}{\rho}(p^\pm)'(x_2) = 0,\]

\[(2.1c) \quad p^+(0) - p^-(0) = g\eta(p^+ - p^-) - \sigma\eta k^2.\]

We compute \(\partial_x(2.1a) - (k^2 - \partial^2_x)(2.1b)\) to obtain the following classical Rayleigh equations.

\[(2.2a) \quad -(v_2^\pm)''(x_2) + (k^2 + \frac{(U^\pm)''(x_2)}{U^\pm(x_2) - c})v_2^\pm(x_2) = 0, \quad \pm x_2 \in [0, h_\pm].\]

From (1.5e) and (1.5c), we obtain that

\[(2.2b) \quad v_2^\pm(\pm h_\pm) = 0\]

\[(2.2c) \quad i\eta(U^\pm(0) - c) = v_2^\pm(0).\]

We evaluate \(\partial_x(2.1b) + (2.1a)\) at \(x_2 = 0\). Combining with the last equation, we obtain the boundary condition at the interface.

\[(2.2d) \quad i\eta(g(\rho^+ - \rho^-) - \sigma k^2) = \rho^- U^-(0)v_2^- + \rho^+ U^+(0)v_2^+ + p^+(v_2^+)'(0)(U^+(0) - c) - \rho^- v_2^-'(0)(U^-(0) - c).\]

**Remark 2.1.** To seek unstable solutions, we consider \(k \neq 0\) in (1.6) because zeroth mode corresponds to variation towards nearby shear flows. But we may still study the solution of system (2.2) with \(k = 0\) since it is helpful to obtain a picture of the eigenvalue distribution for all integer \(k\).

We are interested in looking for linearized unstable solutions with \(\eta \neq 0\) and \(k \neq 0\). Without loss of generality, we normalize it by taking

\[i\eta = 1.\]

Meanwhile, we let \(\epsilon := \frac{\rho^+}{\rho^-}\) be the density ratio. Then we can rewrite (2.2c) and (2.2d) in the following.

\[(2.3a) \quad v_2^\pm(0) = U^\pm(0) - c,\]

\[(2.3b) \quad g(\epsilon - 1) - \frac{\sigma}{\rho}k^2 = (U^-)'(0)v_2^-(0) - \epsilon(U^+)'(0)v_2^+(0) + \epsilon(v_2^+)'(0)(U^+(0) - c) - (v_2^-)'(0)(U^-(0) - c).\]

We summarize the above calculation in the following.

**Lemma 2.1.** For \(k \in \mathbb{Z} \setminus \{0\}\), \(-i\eta c\) with \(c \notin U^-([-h_-,0]) \cup U^+([0, h_+])\) is an eigenvalue of the linearized Euler system (1.5) at the shear flow \(v_2 = (U^\pm(x_2),0)\) if there exists a non-trivial solution of (2.2a), (2.2b), and (2.3).

**Remark 2.2.** The solutions to system (2.2a), (2.2b), and (2.3) are even in \(k\). In addition, the complex conjugate of solutions are still solutions and \(c\) is replaced by \(\bar{c}\).

Using the symmetry in Remark 2.2, we notice that the existence of a solution to system (2.2a), (2.2b), and (2.3) with \(c_1 > 0\) and \(k > 0\) implies exponential linear instability of Euler system. In the rest of paper, \(k\) is assumed non-negative mostly. To study instability, we introduce some concepts as following.

**Definition 2.1.** A pair \((c,k)\) with real \(c\), positive \(k\) is said to be a neutral mode if there exists a pair of non-trivial solutions to (2.2a), (2.2b), and (2.3). It is called an unstable mode if \(c_1 > 0\). We call it a neutral limiting mode if it is the limit of a sequence of unstable modes \(\{(c_n, k_n)\}\). Furthermore, if \(c \in U^+([0, h_+]) \cup U^-([-h_-,0])\), it is called a singular mode. Otherwise, it is a non-singular mode.

In the following, we prove Theorem 1.1 which shows that any unstable mode must have wave speed in an upper semicircle under a mild condition.

**Proof of Theorem 1.1.** Suppose \(c_1 \neq 0\). Let \(\psi^\pm(x_2)\) be defined by \(v_2^\pm(x_2) = (c - U^\pm(x_2))\psi^\pm(x_2)\). Then \(\psi^\pm(x_2)\) satisfy

\[(2.4) \quad (\psi^\pm)'' + k^2(\psi^\pm)' + c_1^2(\psi^\pm)' = 0,\]
along with
\begin{equation}
\psi^\pm(\pm h_{\pm}) = 0, \quad \psi^\pm(0) = -1,
\end{equation}
\begin{equation}
-g(\rho^+ - \rho^-) + \sigma k^2 = \rho^+(U^+(0) - c)^2(\psi^+(0) - \rho^-) - \rho^-(U^-(0) - c)^2(\psi^-)'(0).
\end{equation}
We multiply (2.4) by $\psi^\pm$ and integrate from 0 to $\pm h_{\pm}$ respectively and obtain that
\begin{equation}
-(U^-(0) - c)^2(\psi^-)'(0) = \int_{-h_-}^0 (U^--c)^2(k^2|\psi^-|^2 + |(\psi^-)'|^2)dx_2,
\end{equation}
\begin{equation}
(U^+(0) - c)^2(\psi^+)'(0) = \int_{h_+}^0 (U^+-c)^2(k^2|\psi^+|^2 + |(\psi^+)'|^2)dx_2.
\end{equation}
Let $Q^\pm := k^2|\psi^\pm|^2 + |(\psi^\pm)'|^2$. Then $Q^\pm > 0$. We consider imaginary part of (2.7a) and (2.7b).
\begin{equation}
Im\big((U^-(0) - c)^2(\psi^-)'(0)\big) = -2c_I\int_{-h_-}^0 (U^- - c_R)Q^-dx_2,
\end{equation}
\begin{equation}
Im\big((U^+(0) - c)^2(\psi^+)'(0)\big) = -2c_I\int_{h_+}^0 (U^+ - c_R)Q^+dx_2.
\end{equation}
We compute $\rho^-(2.8a) + \rho^+(2.8b)$, use equation (2.6), and obtain that
\begin{equation}
\int_{0}^{h_+} \rho^+U^+Q^+dx_2 + \int_{-h_-}^0 \rho^-U^-Q^-dx_2 = c_R\left(\int_{0}^{h_+} \rho^+Q^+dx_2 + \int_{-h_-}^0 \rho^-Q^-dx_2\right),
\end{equation}
Similarly, we consider the real part of (2.7a) and (2.7b).
\begin{equation}
Re\big((U^-(0) - c)^2(\psi^-)'(0)\big) = \int_{-h_-}^0 ((U^- - c_R)^2 - c_I^2)Q^-dx_2,
\end{equation}
\begin{equation}
Re\big((U^+(0) - c)^2(\psi^+)'(0)\big) = \int_{h_+}^0 ((U^+ - c_R)^2 - c_I^2)Q^+dx_2.
\end{equation}
We compute $\rho^+(2.10b) + \rho^-(2.10a)$ and use (2.6) to obtain that
\begin{equation}
\int_{0}^{h_+} U^+\rho^+Q^+dx_2 + \int_{-h_-}^0 U^-\rho^-Q^-dx_2 = (c_R^2 + c_I^2)(\int_{0}^{h_+} \rho^+Q^+dx_2 + \int_{-h_-}^0 \rho^-Q^-dx_2)
\end{equation}
\begin{equation}
-g(\rho^+ - \rho^-) + \sigma k^2,
\end{equation}
where we used equation (2.9) and real part of (2.6). Now we consider the inequality
\begin{equation}
\int_{0}^{h_+} \rho^+Q^+(U^+ - a)(U^+ - b)dx_2 + \int_{-h_-}^0 \rho^-Q^-(U^- - a)(U^- - b)dx_2 \leq 0,
\end{equation}
where $a$ and $b$ are defined in (1.7). Substituting (2.9) and (2.11), we have
\begin{equation}
(c_R^2 + c_I^2 - (a+b)c_R + ab)(\int_{0}^{h_+} \rho^+Q^+dx_2 + \int_{-h_-}^0 \rho^-Q^-dx_2) \leq -(a+b)(c_R^2 + c_I^2) \leq 0.
\end{equation}
Since $Q^\pm \geq 0$ and can not be both identically zero, $(c_R - \frac{a+b}{2})^2 + c_I^2 \leq (\frac{a-b}{2})^2$.

To study the instability, we will address the possible locations of the limit of a sequence of unstable solutions to (2.2a), (2.2b), and (2.3), i.e. neutral limiting mode. In fact, not every neutral mode serves as a neutral limiting mode. First, we consider and study the fundamental solutions of classical Rayleigh equations. Some results about the fundamental solutions were obtained in [11]. Let $y^+(c, k, x_2)$ be the solution to (2.2a) on $[0, h_+]$ under the following boundary condition.
\begin{equation}
y^+(h_+) = 0, \quad (y^+)'(h_+) = 1.
\end{equation}
Suppose that $y^-(c, k, x_2)$ be the solution to (2.2a) on $[-h_-, 0]$ satisfying
\begin{equation}
y^-(h_-) = 0, \quad (y^-)'(h_-) = 1.
\end{equation}
When \( c \in U^+([0,h^+]) \) (or \( U^-((-h_,0]) \)), \( y^+(c,k,x_2) \) (resp. \( y^-(c,k,x_2) \)) is defined as \( \lim_{\theta \to 0^+} y^+(c+i\theta,k,x_2) \) (resp. \( \lim_{\theta \to 0^+} y^-(c+i\theta,k,x_2) \)). It satisfies the following condition for \( U^+(x_2) \neq c \) (resp. \( U^-(x_2) \neq c \)).

\[
\lim_{\theta \to 0^+} \left( (y^+_R)'(c,k,x_2+\theta) - (y^+_R)'(c,k,x_2-\theta) \right) = 0,
\]
\[
\lim_{\theta \to 0^+} (y^-_R)'(c,k,x_2-\theta) = \frac{\pi}{(U^+_R)'(x_2)} y^+_R(x_2),
\]

where \( x_c = (U^-)^{-1}(c) \) is the preimage of \( c \in U^-([-h_-,0]) \) (resp. \( \lim_{\theta \to 0^+} ((y^-_R)'(c,k,x_2+\theta) - (y^-_R)'(c,k,x_2-\theta)) = 0 \)), \( \lim_{\theta \to 0^+} (y^-_R)'(c,k,x_2-\theta) = \pi(U^-)'(x_2) y^-(x_2), \) \( x_c = (U^-)^{-1}(c) \). Moreover, because of the singularity of (2.2a), \( (y^+_R)'(x_2) \) has a logarithmic singularity \( \log|x_2 - x_c| \) near \( x_c \).

We also define the following crucial quantities which are related to Reynold stress. We let

\[
Y^-(c,k) = Y^-_R(c,k) + iY^-_I(c,k) := \frac{(y^-)'(0)}{y^-(0)}(c,k), \quad c \in \mathbb{C} \setminus U^+([-h_-,0])
\]
\[
Y^-(c,k) = \lim_{\beta \to 0^+} Y^- (c + i\beta, k) \quad c \in U^-([-h_-,0]),
\]
\[
Y^+(c,k) = Y^+_R(c,k) + iY^+_I(c,k) := \frac{(y^+)'(0)}{y^+(0)}(c,k), \quad c \in \mathbb{C} \setminus U^+([0,h_+]),
\]
\[
Y^+(c,k) = \lim_{\beta \to 0^+} Y^+ (c + i\beta, k) \quad c \in U^+([0,h_+]).
\]

We should notice that \( Y^- \) (or \( Y^+ \)) is not well defined at \( c = U^-(0) \) (resp. \( c = U^+(0) \)) because of the singularity of (2.2a) at \( x_2 = 0 \). The domain of \( Y^\pm \) is given by

\[
D(Y^\pm) = \{ (c,k) \in \mathbb{C} \times \mathbb{R} \mid c \neq U^\pm(0), y^\pm(c,k,0) \neq 0 \}.
\]

In \( D(Y^\pm) \), we can simplify the free boundary condition of our eigenvalue problem (2.3b).

\[
g(1-\epsilon) + \frac{\sigma}{\rho} k^2 = \epsilon \left( (U^+)'(0)(U^+(0)-c) - Y^+(U^+(0)-c)^2 \right) - (U^-)'(0)(U^-(0)-c) + Y^-(U^-(0)-c)^2.
\]

This allows us to focus on the system consisting of (2.2a), (2.11), and (2.14). We summarize the analysis above in the following lemma.

**Lemma 2.2.** The linearization of the interface problem (1.1) at the shear flow (1.2) has an eigenvalue \( -ikc \) if the solution \( y^\pm(c,k,x_2) \) of (2.2a) satisfy (2.11) and the pair \( (c,k) \) satisfies (2.14).

For convenience, we introduce notations \( x^\pm_c \) as following.

**Definition 2.2.** If \( c \in U^-([-h_-,0]) \), let \( x^-_c \) be \( c = U^-(x^-_c) \). If \( c \in U^+([0,h_+]) \), define \( x^+_c \) as \( c = U^+(x^+_c) \).

**Remark 2.3.** It is possible that \( c \in U^-([-h_-,0]) \cap U^+([0,h_+]) \neq \emptyset. \) In this case, \( c = U^+(x^+_c) = U^-(x^-_c) \), but \( x^+_c \) is not equal to \( x^-_c \).

We shall use the following results about \( y^- \) and \( Y^- \) which were obtained in [11] and the analogous properties of \( y^+ \) and \( Y^+ \).

**Lemma 2.3.** ([11], Lemma 3.9, Lemma 3.19(1),(3), Lemma 3.20(3), Lemma 3.22, Lemma 3.24) Assume that \( U^- \in C^{\infty}, l_0 \geq 6 \) and \( (U^-)' \neq 0 \). Let \( x_c \) be defined in Definition 2.2. Then \( Y^- \) is analytic in both \( (c,k) \in D(Y^-) \setminus \{(c,k) \in \mathbb{R} \times \mathbb{C} \setminus U^-([-h_-,0]) \} \), \( C^{\infty} \) in \( k \), and locally \( C^\alpha \) in \( (c,k) \in D(Y^-) \setminus \{(c,k) \mid c_1 \geq 0 \} \) for any \( \alpha \in [0,1) \). And the following hold.

1. For any \( \beta \in (0,\frac{1}{2}) \), there exists \( C > 0 \) depending only on \( \beta, |(U^-)'|_{C^2}, \) and \( 1/(U^-)'|_{C^0} \) such that, for any \( c \in \mathbb{C} \),

\[
|y^-(x_2)|/(\sqrt{k^2+1} - \sinh(x_2+h_+\sqrt{k^2+1})) \leq C(k^2+1)^\frac{\beta}{2} \sinh(x_2+h_-\sqrt{k^2+1}).
\]

2. For any \( k \in \mathbb{R} \),

\[
y^-(c,k,x_2) > 0, \quad \forall x_2 \in (-h_0,0], \quad c \in \mathbb{R} \setminus U^-((-h_-,0]), \quad y^-(x^-_c) > 0 \text{ if } c \in U^-((-h_-,0])
\]
There exists $C > 0$ depending only on $U^−$ such that for any $k \in \mathbb{R}$,
\begin{equation}
|y^−(c, k, 0)| \geq \frac{C}{k^2 + 1} |y^−(x^−_c)| \sinh \frac{(x^−_c + h_c)}{\sqrt{k^2 + 1}} \sinh \frac{|x^−_c|}{\sqrt{k^2 + 1}}, \quad c \in U^−([-h_−, 0]),
\end{equation}
\begin{equation}
\text{for any } \beta \in (0, \frac{1}{2}), \text{ there exist } k_0 > 0 \text{ and } C > 0 \text{ depending only on } \beta, \ |(U^−)'|_{C^2}, \text{ and } \frac{1}{|U^−|^2 |c|^6} \text{ such that,}
\end{equation}
\begin{equation}
|Y^−(c, k)| \leq C \left( \sqrt{1 + k^2} + |\log \min \{1, |U^−(0) − c|\}\right), \quad \forall k \in \mathbb{R}, \ |c − U^−(0)| \leq \rho.
\end{equation}
\begin{equation}
Y^−(c, k) = 0 \quad \text{for } c \in \mathbb{R} \setminus U^−((-h_−, 0]). \quad \text{If } y^−(c, k, 0) \neq 0,
\end{equation}
\begin{equation}
Y^−(c, k) = \frac{\pi(U^−)''(x^−_c) y^−(c, k, x^−_c)^2}{|(U^−)'(x^−_c)||y^−(c, k, 0)|^2}, \quad c \in U^−((-h_−, 0)).
\end{equation}
\begin{equation}
Y^−(c, k) = \begin{cases} \frac{1}{\pi} \int_{U^−([-h_−, 0])} \frac{Y^−(c, k')}{c - c'} dc' + \coth(kh_−), & c \notin U^−([-h_−, 0]), \\ -\mathcal{H}(Y^−_I(·, k))(c) + iY^−_I(c, k) + \coth(kh_−), & c \in U^−([-h_−, 0]). \end{cases}
\end{equation}
\begin{equation}
\text{Here, } \mathcal{H} \text{ means Hilbert transform in } c \in \mathbb{R}.
\end{equation}

We also have $y^+$ and $Y^+$ satisfy the analogous properties as following.

**Lemma 2.4.** Assume that $U^+$ satisfies (1.3). Let $x^+_c$ be defined in Definition 2.2. Then $Y^+$ is $C^{0−3}$ in $k$ and locally $C^\alpha$ in $(c, k) \in D(Y^+) \cap \{(c, k) | c_I \geq 0\}$ for any $\alpha \in [0, 1)$ and analytic in $(c, k) \in D(Y^+) \setminus \{(c, k) | c_I \geq 0\}$ and the following hold.

1. For any $\beta \in (0, \frac{1}{2})$, there exists $C > 0$ depending only on $\beta$, $|(U^+)'|_{C^2}$, and $1/(U^+)'|_{C^6}$ such that, for any $c \in \mathbb{C} \setminus U^+([0, h_+])$,
\begin{equation}
|y^+(x_2)| / \sqrt{k^2 + 1} - \sinh \frac{x_2 - h_+}{\sqrt{k^2 + 1}} \leq C(k^2 + 1)^{-\beta} \sinh \frac{h_+ - x_2}{\sqrt{k^2 + 1}}.
\end{equation}

2. For any $k \in \mathbb{R}$,
\begin{equation}
y^+(c, k, x_2) < 0, \forall x_2 \in [0, h_+), \ c \in \mathbb{R} \setminus U^+([0, h_+)), \ y^+(x^+_c) < 0 \text{ if } c \in U^+([0, h_+)).
\end{equation}

3. There exists $C > 0$ depending only on $U^+$ such that for any $k \in \mathbb{R}$,
\begin{equation}
|y^+(x^+_c, 0)| \geq \frac{C}{k^2 + 1} |(U^+)''(x^+_c)| \sinh \frac{h_+ - x_+}{\sqrt{k^2 + 1}} \sinh \frac{|x^+_c|}{\sqrt{k^2 + 1}}, \quad c \in U^+([0, h_+)).
\end{equation}

4. There exists $C, \rho > 0$ depending only on $U^+$ such that
\begin{equation}
|Y^+(c, k)| \leq C \left( \sqrt{1 + k^2} + |\log \min \{1, |U^+(0) − c|\}\right), \quad \forall k \in \mathbb{R}, \ |c − U^+(0)| \leq \rho.
\end{equation}
\begin{equation}
\text{for any } \beta \in (0, \frac{1}{2}), \text{ there exist } k_0 > 0 \text{ and } C > 0 \text{ depending only on } \beta, \ |(U^+)'|_{C^2}, \text{ and } \frac{1}{|U^+(0)|^2 |c|^6} \text{ such that,}
\end{equation}
\begin{equation}
|Y^+(c, k) + \coth(kh_+)| \leq C \left( (k^2 + 1)^{1/2} + |\log \min \{1, |U^+(0) − c|\}\right), \quad \forall |k| \geq k_0, c \neq U^+(0).
\end{equation}
\begin{equation}
\text{For } c \in \mathbb{R} \setminus U^+([0, h_+)), \ Y^+_I(c, k) = 0. \text{ If } y^+(c, k, 0) \neq 0,
\end{equation}
\begin{equation}
Y^+_I(c, k) = \frac{\pi(U^+)''(x^+_c) y^+(c, k, x^+_c)^2}{|(U^+)''(x^+_c)||y^+(c, k, 0)|^2}, \quad c \in U^+([0, h_+)).
\end{equation}
\begin{equation}
Y^+(c, k) = \begin{cases} \frac{1}{\pi} \int_{U^+([0, h_+))} \frac{Y^+_I(c, k')}{c' - c} dc' − \coth(kh_+), & c \notin U^+([0, h_+)), \\ -\mathcal{H}(Y^+_I(·, k))(c) + iY^+_I(c, k) − \coth(kh_+), & c \in U^+([0, h_+)). \end{cases}
\end{equation}
Lemma 3.2. Where $c$ is defined as $\lim_{\beta \to 0^+} Y^-(c-i\beta, k)$, then $\text{sgn}(Y^-)$ is different, i.e., $-Y^-$ satisfies (2.20). $Y^+$ has an analogous property. In this paper, we shall only use the properties of $Y^+_i$ for $c_i \geq 0$. Based on (2.17), $Y^-$ is well defined for $c \in U^\cdot((-h_-,0))$ under the assumption that $(U^-)'' \neq 0$. According to (2.24), $Y^+$ is well defined for $c \in U^\cdot([0,h_+))$ if $(U^+)'' \neq 0$.

3. Distribution of Eigenvalues

In this section, we shall consider the situation where the upper layer fluid is lighter than the lower layer fluid. In other words, we shall discuss the linear instability of shear flows (1.2) which satisfy (1.3) for $0 \leq \rho^+ \leq \rho^-$. Some of the results are also true for large $\epsilon = \frac{\rho^+}{\rho^-}$. Since the linear system (1.5) preserves Fourier mode $e^{ikx}$ for any $k \in \mathbb{R}$, we will treat the wave number $k \in \mathbb{R}$ as a parameter. According to Lemma 2.1, $-ikc$ with $c \in \mathbb{C}\setminus\left(U^\cdot((-h_-,0))\cup U^\cdot([0,h_+))\right)$ is an eigenvalue of (1.5) with parameter $k$ if

\begin{equation}
F(c,k,\epsilon) := \epsilon \left( (U^+)'(0)(U^+(0) - c) - Y^+(U^+(0) - c)^2 - (U^-)'(0)(U^-(0) - c) + Y^-(U^-(0) - c)^2 - (g(1-\epsilon) + \frac{\sigma}{\rho}k^2) \right) = 0,
\end{equation}

where $Y^\pm$ are defined in (2.12). According to (2.18), (2.20), (2.25), and (2.27), we also define that

\begin{equation}
F(c,k,\epsilon) := \lim_{\beta \to 0^+} F(c+i\beta,k,\epsilon) = \lim_{\beta \to 0^+} F(c-i\beta,k,\epsilon), \quad \forall c \in U^\cdot((-h_-,0)) \cup U^\cdot([0,h_+)).
\end{equation}

Then, the zeros of $F(c,k,\epsilon)$ correspond to singular or non-singular mode $(c,k)$ which is defined in Definition 2.1. Based on Lemma 4.1 in [11], Lemma 2.3, and Lemma 2.4, we give some basic properties of $F(c,k,\epsilon)$ for each $\epsilon \geq 0$.

Lemma 3.1. Assume $U^\pm$ satisfy (1.3). Then the following hold.

1. $F(c,k,\epsilon)$ is well defined for all $k \in \mathbb{R}$, $c \in \mathbb{C}$, and $\epsilon \geq 0$.
2. When restricted to $c_i \geq 0$, $F(c,k,\epsilon)$ is $C^6$ in $k$, $C^{1,\alpha}$ in $c$, for any $\alpha \in [0,1)$, and $C^\infty$ in $\epsilon \geq 0$.

To study the eigenvalue distribution, we consider wave number $k$ as a parameter. We address the location of wave speed $c \in \mathbb{C}$ by starting from large $|k|$. And then we study what happens to the eigenvalues as $|k|$ decreases from infinity.

Lemma 3.2. Assume that $U^\pm$ satisfy (1.3). Then there exist some $k_0 > 0$ and $C > 0$ depending only on $|U^\pm|_{C^2}$ and $1/|U^\pm|_{C^3}$, such that for each $\epsilon \in [0,1]$ and all $|k| > k_0$, $F(c,k,\epsilon)$ defined by (3.1) has exactly two solutions $c^\pm(k)$ depending on $k$ analytically. Moreover,

\begin{equation}
c^\pm(k) \in \mathbb{R}, \quad c^\pm(k) = c^\pm(-k),
\end{equation}

\begin{equation}
|c^\pm(k) - \frac{U^+(0)\epsilon + U^-(0)}{1+\epsilon}| \leq \sqrt{\frac{\epsilon(U^+(0) - U^-(0))^2}{(1+\epsilon)^2} + \frac{\sigma|k|}{\rho^{-2}(1+\epsilon)}} \leq C,
\end{equation}

\begin{equation}
|\partial_cF(c^\pm(k),k,\epsilon) \mp 2|k|^2 \sqrt{\sigma(1+\epsilon)/\rho^-} | \leq C|k|.
\end{equation}

Proof. Since $\epsilon \in [0,1]$, (1.8) holds for all $k \in \mathbb{R}$. According to (2.19), and (2.26), $Y^\pm(c,k)$ are comparable to $\mp|k|$ as $|k|$ tends to $\infty$. Then, for large $|k|$, $F(c,k,\epsilon)$ behaves more or less like a quadratic function in $k$ with a negative leading coefficient. Hence, there exist $N_1, \gamma > 0$ such that for all $|k| > N_1$, if $y^\pm(c,k,x_2)$ with $(c_R - \frac{a+b}{2})^2 + c_J^2 \leq b + 2\sigma - a$, solves the system (2.11a) and (2.11b), then $F < -\frac{\sigma}{2\rho}k^2 < 0$. Theorem 1.1 implies that $-ikc$ is an eigenvalue with $c$ lying outside the semicircle (1.9), then $c \in \mathbb{R}\setminus[a,b]$. Therefore, if $(c,k,y^\pm(c,k,x_2))$ with $|k| > N_1$ solves the system (2.11a), (2.11b), and (3.1a), then $c_R \in [a-\gamma,b+\gamma]$, where $a$ and $b$ are defined in (1.7). Let

\begin{equation}
S_+ := \{c > b + \gamma\}, \quad S_- := \{c < a - \gamma\}.
\end{equation}
By the definition of $F(c, k, \epsilon)$, $F$ is $C^\infty$ in $(c, k) \in S_\pm \times \mathbb{R}$. For each $\epsilon \in [0, 1]$, we consider $F(c, k, \epsilon) = 0$ as a quadratic equation of $c$. Its roots satisfy

$$
c = f^\pm(c, k) = \frac{-B \pm \sqrt{B^2 - 4(Y^-(c, k) - \epsilon Y^+(c, k))A}}{2(Y^-(c, k) - \epsilon Y^+(c, k))},$$

where

$$A := \epsilon((U^+)^'(0)U^+(0) - Y^+U^+(0))^2 - (U^-)^'(0)U^-(0) + Y^-U^-(0)^2 - g(1 - \epsilon) - \frac{\sigma}{\rho}k^2,$$

$$B := (U^+)^'(0) - 2U^-(0)Y^- - (U^-)^'(0)\epsilon + 2Y^+U^+(0).$$

Using formulas (2.21) and (2.20) with estimate (2.15), we obtain that there exist $C > 0$ and large $N_1 > 0$ such that for all $c \in S_+ \cup S_-$ and $|k| > N_1$,

$$|Y^-(c, k) - k\coth kh| = \frac{1}{\pi} \left| \int_{U^-(\{h_-, 0\})} \frac{Y^-(c', k)}{c - c'} dc' \right| \leq C \int_{U^-[\{h_-, 0\}]} \frac{y^-(c, k, x_0)^2}{|y^-(c, k, 0)|^2} dc \leq C \int_{-h_-}^0 e^{k^2|T_+|} dc \leq C(|k| + 1)^{-1}.$$

A similar computation based on (2.28), (2.22), and (2.27) leads to

$$|Y^+(c, k) + k\coth kh| \leq C(|k| + 1)^{-1}.$$

According to these two estimates and the fact that $\coth(x) = 1 + 2(e^{2x} - 1)^{-1}$, there exists $k_0 > N_1$ and $C > 0$ depending only on $|(U^\pm)|_{C^2}$, and $1/(|U^\pm|)_{C^3}$, such that for all $|k| > k_0$,

$$\left| f^\pm(c, k) - \frac{U^+(0)\epsilon + U^-(0)}{1 + \epsilon} \pm \sqrt{\frac{\epsilon(U^+(0) - U^-(0))^2}{(1 + \epsilon)^2} + \frac{\sigma|k|}{\rho(1 + \epsilon)}} \right| \leq C.$$

Then, $f^\pm(c, k) : S_+ \to S_\pm$ for $|k| > k_0$. It remains to evaluate $\partial_c f^\pm(c, k)$ and show $f^\pm(c, k)$ is a contraction acting on $S_\pm$. We first compute $\partial_c Y^\pm$. While $\partial_c Y^-$ was obtained and evaluated in [11], we compute $\partial_c Y^+$ here for self-completeness.

We differentiate (2.2a) in $c \in \mathbb{R} \setminus U^-(\{h_-, 0\})$ and $c \in \mathbb{R} \setminus U^+(\{0, h_+\})$ respectively,

(3.6) \begin{align*}
\partial_c (y^\pm)'(x_2) + (k^2 + \frac{(U^\pm)^''}{U^\pm - c}) \partial_c y^\pm(x_2) &= -\frac{(U^\pm)^''}{(U^\pm - c)^2} y^\pm(x_2), \\
\partial_c y^\pm(\pm h_\pm) &= \partial_c (y^\pm)'(\pm h_\pm) = 0.
\end{align*}

We compute (2.2a) $\partial_c y^\pm(x_2) - (3.6) y^\pm(x_2)$ and obtain that,

$$-(y^\pm)'' \partial_c y^\pm + \partial_c (y^\pm)' y^\pm = \frac{(U^\pm)''}{(U^\pm - c)^2} (y^\pm)^2, \quad \pm x_2 \in (0, h_\pm).$$

Using the boundary conditions in (3.6), we integrate the above equation on $[-h_-, 0]$ and $[0, h_+]$ respectively, and obtain that

$$(y^+)'(0) \partial_c y^+(0) + \partial_c (y^+)'(0)y^+(0) = -\int_0^{h_+} \frac{(U^+)^''}{(U^+ - c)^2} (y^+)^2 dx_2,$$

$$-(y^-)'(0) \partial_c y^-(0) + \partial_c (y^-)'(0)y^-(0) = \int_{-h_-}^0 \frac{(U^-)^''}{(U^- - c)^2} (y^-)^2 dx_2.$$ 

This implies that

(3.7a) \begin{align*}
\partial_c Y^+(c, k) &= -\frac{1}{y^+(0)^2} \int_0^{h_+} \frac{(U^+)^''}{(U^+ - c)^2} (y^+)^2 dx_2, \\
(3.7b) \quad \partial_c Y^-(c, k) &= \frac{1}{y^-(0)^2} \int_{-h_-}^0 \frac{(U^-)^''}{(U^- - c)^2} (y^-)^2 dx_2.
\end{align*}

Using (2.15) and (2.22), we obtain that there exist $C_1 > 0$ such that for all $|k| > k_0$ and $c \in S_+ \cup S_-$,

$$|\partial_c Y^\pm| \leq \frac{C_1}{|k|}.$$
We compute $\frac{df^\pm(k)}{dk}$. Based on (2.19), (2.26), and the estimate of $|\partial_t Y^\pm|$, we can choose $k_0$ large such that the term involving $\frac{dA}{dk}$ dominates in $\frac{df^\pm(k)}{dk}$. Then one may check that

$$\left| \frac{df^\pm(k)}{dk} \right| \leq \frac{C}{\sqrt{|k|}} < 1, \quad \forall |k| \geq k_0,$$

where $C$ is independent of $k$, $c$, and $\epsilon$. Therefore, $f^\pm(c, k)$ are contractions acting on $S^\pm$ respectively. Then their fixed points $c^\pm(k)$ are the only solutions to system (2.2a), (2.2b), and (2.3) on $S^\pm$. Moreover, they are analytic in $k$. Finally, one may compute

$$\partial_c F = \epsilon \left( - (U^+)'(0) + 2(U^+(0) - c)Y^+ - \partial_c Y^+(U^+(0) - c)^2 \right) + \partial_c Y^-(U^-(0) - c)^2 - 2(U^-(0) - c)Y^- + (U^-)'(0).$$

Then by using the above estimates of $|Y^\pm \pm k \coth kh|$, $\partial_c Y^\pm$, and $c^\pm(k)$, the last desired estimate of $\partial_c F(c^\pm(k), k, \epsilon)$ can be obtained. This completes the proof of the lemma. \hfill $\square$

**Remark 3.1.** 1) When $c = 0$, $c^\pm(k)$ obtained in Lemma 3.2 coincide with those obtained in [11]. Later in Section 4, this fact will be used.

2) Recall that $a$ and $b$ are defined in (1.7). Suppose that $k_+ \geq 0$ is the biggest $|k|$ such that $F(b, k_+, \epsilon) = 0$, and $k_- \geq 0$ is the biggest $|k|$ such that $F(a, k_-, \epsilon) = 0$. By Lemma 4.3 in [11] and Theorem 1.1, for each $\epsilon \in [0, 1]$, $c^\pm(k)$ (resp. $c^-(k)$) obtained in Lemma 3.2 can be extended to be an even and analytic function of $k$ for all $|k| \in [k_+, \infty)$ (resp. $|k| \in [k_-, \infty)$) such that

$$F(c^+(k), k, \epsilon) = 0, \quad \forall |k| \in [k_+, \infty), \quad c^+(k) > b, \quad \forall |k| \in (k_+, \infty),$$

$$F(c^-(k), k, \epsilon) = 0, \quad \forall |k| \in [k_-, \infty), \quad c^-(k) < a, \quad \forall |k| \in (k_-, \infty).$$

Moreover, $c^+(k)$ (resp. $c^-(k)$) can be extended to be an analytic function for all $|k| \in [0, \infty)$ if for any $k \in \mathbb{R}$, $(b, k)$ (resp. $(a, k)$) is not a neutral mode. In addition, for $c_R > b$ and $c_I \in \mathbb{R}$, $c^+(k)$ is the unique root of $F(\cdot, k, \epsilon) = 0$, (resp. $c_R < a$ and $c_I \in \mathbb{R}$, $c^-(k)$ is the unique root of $F(\cdot, k, \epsilon) = 0$.)

Since $F(c, k, \epsilon) \in \mathbb{R}$ for all $c \in \mathbb{R} \setminus [a, b]$, $\partial_c F(c^+(k), k, \epsilon)$ does not change sign along these simple roots. Hence, the signs of $\partial_c F$ in (3.4), Lemma 2.3, Lemma 2.4, and Lemma 3.1 imply that

$$\partial_c F(c^+(k), k, \epsilon) > 0, \quad \forall |k| \in [k_+, \infty), \quad \partial_c F(c^-(k), k, \epsilon) < 0, \quad \forall |k| \in [k_-, \infty).$$

Particularly, since $Y^-(-c, k)$ has logarithmic singularity $\log|U^-(0) - c|$ near $c = U^-(0)$,

$$F(U^-(0), k, 0) = -g - \frac{\sigma}{\rho}k^2 < 0, \quad \partial_c F(c^+(k), k, 0) > 0, \quad \forall k \in \mathbb{R}.$$

According to Remark 3.1(2), $c^\pm(k)$ can keep continuity in $k$ as simple roots of the analytic function $F(\cdot, k, \epsilon)$ as $|k|$ gets smaller. We shall track $c^\pm(k)$ as $|k|$ decreases to study possible bifurcation and seek instability. Before studying the eigenvalue distribution as $|k|$ decreases, we prove some properties of $F(c, k, \epsilon)$ in the following lemma. Even though some properties of $Y^-(-c, k)$ were obtained in [11], we present the proof for self-completeness.

**Lemma 3.3.** Assume that $U^\pm$ satisfy (1.3). Suppose that $K = k^2$ and $\epsilon \geq 0$. Let $a$ and $b$ be defined in (1.7). Then for $c \in \mathbb{R} \setminus \left( U^-((-h_-, 0)) \cup U^+([0, h_+]) \right)$, the following hold.

(1)

$$\partial_{KK} F(c, k, \epsilon) < 0.$$

(2) If both $(U^+)' > 0$ and $(U^-)' < 0$, then $\partial_{KK} F(c, k, \epsilon) < 0$ for all $c \leq a$ and $k \in \mathbb{R}$.

(3) If both $(U^+)' < 0$ and $(U^-)' > 0$, then $\partial_{KK} F(c, k, \epsilon) > 0$ for all $c \geq b$ and $k \in \mathbb{R}$.

**Proof.** Let $K := k^2$. We notice that for $c \in \mathbb{C} \setminus U^-([-h_-, 0])$, $\partial_c y^-$ satisfies the following equations.

$$\begin{cases}
-\partial_{K} (y^-)'(x_2) + (k^2 + \frac{(U^-)}{(U^-)^2}) \partial_{K} y^-(x_2) = -y^-(x_2), \\
\partial_{K} y^-(-h_-) = \partial_{K} (y^-)'(-h_-) = 0.
\end{cases}$$

$$\begin{aligned}
\partial_{KK} F(c, k, \epsilon) = & \frac{\partial}{\partial K} \frac{\partial}{\partial K} F(c, k, \epsilon) \\
= & \frac{\partial}{\partial K} \left( \frac{\partial}{\partial K} F(c, k, \epsilon) \right) \\
= & -\partial_{KK} F(c, k, \epsilon) < 0.
\end{aligned}$$
For $c \in \mathbb{C} \setminus U^+([0, h_+])$, $\partial_K y^+$ satisfies the following system.

\begin{equation}
\begin{aligned}
&-\partial_K (y^+)''(x_2) + (k^2 + \frac{(U^+)'}{(U^+)n}) \partial_K y^+(x_2) = -y^+(x_2), \quad x_2 \in (0, h_+), \\
&\partial_K y^+(h_+) = \partial_K (y^+)''(h_+) = 0.
\end{aligned}
\end{equation}

We consider $c \in \mathbb{R} \setminus (U^-([-h_-, 0]) \cup U^+([0, h_+]))$ and do (2.2a) $\partial_K y^\pm(x_2) - (3.13) y^\pm(x_2)$ respectively and obtain that

\begin{equation}
\begin{aligned}
&\partial_K y^-(x_2) = -\frac{1}{y_0} \int_{-h_-}^0 y^-(x_2)^2 \, dx_2 > 0, \\
&\partial_K y^+(x_2) = -\frac{1}{y_0} \int_0^{h_+} y^+(x_2)^2 \, dx_2 < 0.
\end{aligned}
\end{equation}

Similarly, we integrate (3.14a) on $[-h_-, 0]$ and integrate (3.14b) on $[0, h_+]$, and then divide it by $y^\pm(x_2)$ respectively. We obtain that

\begin{equation}
\begin{aligned}
&\frac{d}{dx_2} \left( \frac{\partial_K y^-(x_2)}{y^-(x_2)} \right) = \frac{1}{y^-(x_2)^2} \int_{-h_-}^{x_2} y^-(x_2)^2 \, dx_2' > 0, \quad x_2 \in [-h_-, 0], \\
&\frac{d}{dx_2} \left( \frac{\partial_K y^+(x_2)}{y^+(x_2)} \right) = -\frac{1}{y^+(x_2)^2} \int_{x_2}^{h_+} y^+(x_2)^2 \, dx_2' < 0, \quad x_2 \in [0, h_+].
\end{aligned}
\end{equation}

These imply that

\begin{equation}
\begin{aligned}
&\partial_K y^-(x_2) = -2 \int_{-h_-}^0 \frac{y^-(x_2)^2}{y_0^2} \left( \frac{\partial_K y^-(0)}{y^-(0)} - \frac{\partial_K y^-(x_2)}{y^-(x_2)} \right) \, dx_2 < 0, \\
&\partial_K y^+(x_2) = 2 \int_0^{h_+} \frac{y^+(x_2)^2}{y^+(0)^2} \left( \frac{\partial_K y^+(0)}{y^+(0)} - \frac{\partial_K y^+(x_2)}{y^+(x_2)} \right) \, dx_2 > 0.
\end{aligned}
\end{equation}

Therefore, for $c \in \mathbb{R} \setminus \left( U^-((-h_-, 0)) \cup U^+([0, h_+]) \right)$,

\[ \partial_K K^F(c, k, c) = -\epsilon \partial_K K^Y^+(c, k)(U^+(0) - c)^2 + \partial_K K^Y^-(c, k)(U^-(0) - c)^2 < 0. \]

Suppose that $c \in \mathbb{R} \setminus \left( U^-((-h_-, 0)) \cup U^+([0, h_+]) \right)$. We differentiate (3.7a) in $K$ to obtain that

\[ \partial_K K^F(c, k, c) = 2 \frac{y^+(0)}{y^+(0)^2} \int_0^{h_+} \left( \frac{y^+(x_2)}{y^+(0)} \right) \, dx_2 > 0. \]

Using (1.65), we conclude that

\begin{equation}
\begin{aligned}
&\text{sgn}(\partial_K K^F) = \text{sgn}(U^+)^n. \\
\end{aligned}
\end{equation}

A similar computation leads to

\begin{equation}
\begin{aligned}
&\text{sgn}(\partial_K K^Y^+) = \text{sgn}(U^+)'. \\
\end{aligned}
\end{equation}

We consider the following function.

\begin{equation}
\begin{aligned}
&\partial_K F = \partial_K Y^-(U^-(0) - c)^2 - 2(U^-(0) - c)\partial_K Y^- + \epsilon \left( -\partial_K Y^+ (U^+(0) - c)^2 + 2(U^+(0) - c)\partial_K Y^+ \right).
\end{aligned}
\end{equation}

Suppose that $c \leq a$, $(U^+)' > 0$, and $(U^-)' > 0$. Then, based on (3.19), we use (3.15) and (3.18) to obtain that $\partial_K F(c, k, e) < 0$. Lemma 2.3, Lemma 2.4, and Lemma 3.1 imply that $\partial_K F$ is well defined at $c = a$ and then $\partial_K F(a, k, c) < 0$. A similar proof of statement (3) for the case of $c \geq b$ and both $(U^+)' < 0$ and $(U^-)' < 0$ is similar.
As $|k|$ gets smaller from infinity, there might exist singular modes. Lemma 3.3 implies that we can use the monotonicity of $\partial_k F(c, k, \epsilon)$ and $\partial \bar{c} F(c, k, \epsilon)$ in $K$ and reduce some computations to the case of $k = 0$. Hence, it is worth paying closer attention to the special case of $k = 0$. When $k = 0$, $y^\pm(x_2)$ which are solutions to (2.2a) and (2.11) have explicit representations. A direction computation shows that if $c = U^+(h_+)$,

\[(3.20)\]
\[y^+(x_2) = \frac{U^+(x_2) - U^+(h_+)}{(U^+)'(h_+)}.\]

If $c \in \mathbb{R} \setminus U^+([0, h_+])$,

\[(3.21)\]
\[y^+(x_2) = (U^+(h_+) - c)(U^+(x_2) - c) \int_{x_2}^{h_+} \frac{-1}{(U^+(x_2') - c)^2} dx_2'.\]

If $c = U^-(h_-)$,

\[(3.22)\]
\[y^-(x_2) = \frac{U^-(x_2) - U^-(h_-)}{(U^-)'(h_-)}.\]

If $c \in \mathbb{R} \setminus U^-([-h_-, 0])$,

\[(3.23)\]
\[y^-(x_2) = (U^-(x_2) - c)(U^-(h_-) - c) \int_{-h_-}^{x_2} \frac{1}{(U^- - c)^2} dx_2.\]

In the next lemma, we address the possible locations of neutral limiting modes for given $\epsilon \in (0, 1)$ and small $\epsilon$.

**Lemma 3.4.** Assume that $U^\pm$ satisfy (1.3), and $(U^\pm)'' \neq 0$. Let

\[(3.24)\]
\[\mathcal{E} := \{U^+(h_+), U^+(0), U^-(h_-), U^-(0)\}, \quad \mathcal{I} := U^+((0, h_+)) \cap U^-((-h_-, 0))\]

Then the following holds.

1. Suppose that $\epsilon \in (0, 1)$ and there is a sequence of unstable modes $(c_n, k_n)$ converges to $(c_\infty, k_\infty) \in \mathbb{R}^2$ as $n \to \infty$, and

\[(3.25)\]
\[g(1 - \epsilon) + \frac{\sigma}{\rho} k_\infty^2 > 0.\]

Then the following holds.

   a. If $\mathcal{I} = \emptyset$, then $c_\infty \in [a, b] \backslash \left\{ U^-(((-h_-, 0)) \cup U^+((0, h_+)) \right\}$. Under additional assumption that $U^+(0) = U^- (0)$, $c_\infty \in \mathcal{E} \setminus \{U^\pm(0)\}$.
   
   b. Assume $\mathcal{I} \neq \emptyset$. Then $c_\infty \in \mathcal{E} \cup \mathcal{I}$ if $U^+(0) \neq U^-(0)$. And if $U^+(0) = U^-(0)$, $c_\infty \in \mathcal{E} \cup \mathcal{I} \setminus \{U^\pm(0)\}$. In addition, if $(U^+)'(U^-)' > 0$, then $c_\infty \in \{a, b\}$, where $a, b$ are defined in (1.7).

   2. Suppose that $\mathcal{I} = \emptyset$. There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, if an unstable sequence $(c_n, k_n) \to (c_\infty, k_\infty) \in \mathbb{R}^2$ as $n \to \infty$, (3.25) holds, and we further assume one of the following holds:
   
   a. $(U^-)'(U^-)' > 0$, and $\max_{[0, h_+]} U^+(x_2) < U^-(h_-)$; or
   
   b. $(U^-)'(U^-)' > 0$, and $\min_{[0, h_+]} U^+(x_2) > U^- (0)$, then $c_\infty \in \mathcal{E}$.

**Proof.** Suppose that $\epsilon \in (0, 1)$ is given. Lemma 3.1 implies that

\[F(c_\infty, k_\infty, \epsilon) = \lim_{n \to \infty} F(c_n, k_n, \epsilon) = 0.\]

To address the possible locations of the neutral limiting mode for each $\epsilon > 0$, we need to consider the imaginary part of $F(c_\infty, k_\infty, \epsilon)$.

\[(3.26)\]
\[F_1(c_\infty, k_\infty, \epsilon) = -\epsilon Y_1^+(c_\infty, k_\infty)(U^+(0) - c_\infty)^2 + Y_1^-(c_\infty, k_\infty)(U^- (0) - c_\infty)^2.\]

Since (3.25) holds, Theorem 1.1 implies that $c_\infty \in [a, b]$, where $a$ and $b$ are defined in (1.7). We shall discuss the cases as following.

* Case (1a). $\mathcal{I} = \emptyset$. Since $(U^\pm)'' \neq 0$, according to Lemma 2.3 and Lemma 2.4, $Y_1^- \neq 0$ if $c \in U^-((-h_-, 0))$. And $Y_1^- \equiv 0$ if $c \in \mathbb{R} \setminus U^-((-h_-, 0))$, $Y_1^+ \neq 0$ if $c \in U^+((0, h_+))$, and $Y_1^+ \equiv 0$ if $c \in \mathbb{R} \setminus U^+((0, h_+))$. Hence, $F_1(c_\infty, k_\infty, \epsilon) \neq 0$ for $c_\infty \in U^-((-h_-, 0)) \cup U^+((0, h_+))$. Particularly, if we further assume that $U^+(0) = U^-(0)$, then (2.19) and (2.26) imply that $F(c_\infty, k, \epsilon) = -g(1 - \epsilon) - \frac{\sigma}{\rho} k^2 < 0$ for all $k \in \mathbb{R}$ and $\epsilon \in (0, 1)$. Hence, in this case, $c \notin U^\pm(0)$. 
Hence, a direction computation leads to
\[
F_l(c_\infty, k_\infty, \epsilon) = \frac{\pi (U^+)''(x_{c_\infty}^+) y'(x_{c_\infty})^2 (U^+(0) - c_\infty)^2}{(U^+(0) - c_\infty)^2} + \frac{\pi (U^-)''(x_{c_\infty}^-) y'(x_{c_\infty})^2}{\left|(U^-)'(x_{c_\infty})\right|^2} (U^+(0) - c_\infty)^2.
\]
Hence, if \( \text{sgn}((U^-)'') = \text{sgn}((U^+)''') \), \( F_l(c_\infty, k_\infty, \epsilon) \neq 0 \). Statement (1) is proved.

2. Now we consider \( \mathcal{I} = \emptyset \) and will prove \( c_\infty \in \mathcal{E} \) under the assumption of (2a) or (2b). To prove (2), we will show that in the following cases, there exists \( \epsilon_0 > 0 \) such that for any \( \epsilon \in (0, \epsilon_0) \), if \( F(c, k, \epsilon) = 0 \) then \( \partial_c F(c, k, \epsilon) \neq 0 \) under some conditions. Let \( K := k^2 \) and \( K_\infty = k^2_{\infty} \).

Case (a). \( (U^-)' > 0, (U^-)'' > 0 \), and \( \max_{[0, h_+]} U^+(x_2) < U^-(h_-) \). Assume that
\[
(3.27) \quad c \in \left( \max_{[0, h_+]} U^+(x_2), U^-(h_-) \right).
\]
Since \( (U^-)'' > 0, (3.18) \) implies that \( \partial_K Y^- < 0 \). By (3.15a), \( \partial_K Y^- > 0 \). Hence,
\[
\partial_K F(c, k, 0) = ((U^-)'(0) - c)^2 \partial_K Y^- + 2 (U^-)'(0) - c \partial_K Y^- (c, k) < 0.
\]
This implies that \( \partial_c F(c, k, 0) < \partial_c F(c, 0, 0) \) for all \( k > 0 \). If \( c = U^-(h_-) \), \( y^- (x_2) \) is in the form of (3.22). Using (3.7b), we compute that
\[
(3.28) \quad (U^-)'(0) - (U^-)(-h_-))^2 \partial_c Y^- (U^-)(-h_-) = \int_{-h_-}^0 (U^-)' dx_2 = (U^-)'(0) - (U^-)'(h_-),
\]
\[
(3.29) \quad Y^- (U^-)(-h_-) = \frac{(U^-)'(0)}{U^-}'(0) - U^-(h_-).
\]
Hence, we obtain that
\[
(3.30) \quad \partial_c F(U^-(h_-), 0, 0) = -(U^-)'(h_-) < 0.
\]
Similarly, if \( c \in \mathbb{R} \setminus U^-([-h_-, 0]) \), \( y^- (x_2) \) is in the form of (3.23). Then we compute
\[
(3.31) \quad (U^-)'(0) - c)^2 \partial_c Y^- (c, 0) = \frac{1}{(U^-)(-c)} \int_{-h_-}^0 (U^-)'' \left( \int_{-h_-}^0 \frac{1}{(U^-)(-c)^2} dx_2 \right)^2 dx_2
\]
\[
\leq \int_{-h_-}^0 (U^-)' dx_2 = (U^-)'(0) - (U^-)'(h_-),
\]
\[
(3.32) \quad Y^- (c, 0) = \frac{(U^-)'(0)}{U^-)'(0) - c} + \frac{1}{(U^-)(-c)^2} \int_{-h_-}^0 \frac{1}{(U^-)(-c)^2} dx_2.
\]
Then, we obtain that
\[
(3.33) \quad \partial_c F(c, 0, 0) \leq -(U^-)'(h_-) - \frac{2}{(U^-)(-c)^2} \int_{-h_-}^0 \frac{1}{(U^-)(-c)^2} dx_2 < 0.
\]
Hence, \( \partial_c F(c, 0, 0) < 0 \) if \( c \leq U^-(h_-) \). By (3.12), there exists \( \epsilon_0 > 0 \) such that \( \partial_c F(c, k, \epsilon) = \partial_c F(c, k, 0) + O(\epsilon) < \partial_c F(c, 0, 0) + O(\epsilon) < 0 \) for any \( \epsilon \in (0, \epsilon_0) \) and \( c \) satisfies (3.27). In fact, this result does not require that \( F(c, k, \epsilon) = 0 \).

Case (b). \( (U^-)' > 0, (U^-)'' > 0 \), and \( U^-)'(0) < \min_{[0, h_+]} U^+(x_2) \). Assume that
\[
(3.34) \quad c \in \left( U^-)'(0), \min_{[0, h_+]} U^+(x_2) \right).
\]
Since \( F(c, k, \epsilon) = 0 \), by the smoothness of \( F \) in \( \epsilon \), we obtain that
\[
F(c, k, \epsilon) = F(c, k, 0) + O(\epsilon).
\]
Hence, a direction computation leads to
\[
Y^- (c, k) = g + \frac{\pi k^2}{(U^-)'(0)^2} + (U^-)'(0) / (U^-)'(0) - c.
\]
Then we compute that
\[
\partial_t F(c, k, \epsilon) = (U^-(0) - c)^2 \partial_t Y^-(c, k) + (U^-)'(0) - 2 \frac{g + \frac{\sigma}{\rho}k^2 + O(\epsilon)}{U^-(0) - c} + O(\epsilon).
\]
Using the assumption that \((U^-)' > 0\), the fact that \(\partial_t Y^- > 0\) if \((U^-)'' > 0\) (see (3.7b)), and the smallness of \(\epsilon\), we have \(\partial_t F(c, k, \epsilon) > 0\) in this case.

We let
\[
\mathcal{N} := [a, b] \setminus \left( U^-([-h_-, 0]) \cup U^+([0, h_+]) \right).
\]
By the proof in (1), to prove (2), it remains to consider \(c \in \mathcal{N}\). We suppose that for some \(\epsilon \in (0, \epsilon_0)\), there is a non-existence of singular modes. Lemma 3.2 implies that \(k_\infty\) cannot be sufficiently large. Hence, \(\epsilon_0\) can be chosen independent of \(k_\infty\) by the smoothness of \(F\) in \((c, k, \epsilon) \in \mathcal{N} \times \mathbb{R} \times (0, 1)\). Since \(F(c, k, \epsilon)\) is analytic near \(c \in \mathcal{N}\), we use the Cauchy-Riemann equation to compute the \(2 \times 2\) Jacobian matrix of \(D_t F\).
\[
D_t F(c_\infty, k_\infty, \epsilon) = \begin{pmatrix}
\partial_{cR} F_{cR} & \partial_{cI} F_{cR} \\
\partial_{cR} F_{cI} & \partial_{cI} F_{cI}
\end{pmatrix}_{(c_\infty, k_\infty, \epsilon)} = \partial_t F(c_\infty, k_\infty, \epsilon) I_{2 \times 2}.
\]
For both case (a) and (b), \(\partial_t F(c_\infty, k_\infty, \epsilon) \neq 0\). Hence, by the Implicit Function Theorem, there exists a smooth complex-valued function \(C(k)\) such that all the roots of \(F(\cdot, \epsilon)\) near \((c_\infty, k_\infty)\) are in the form of \((C(k), \epsilon)\) in \(\mathcal{N}\). We will show that \(C(k) \in \mathbb{R}\) to complete the proof. Since \(F_R\) is smooth for \(c \in \mathcal{N}\) and \(\partial_{cR} F_R(c_\infty, k_\infty, \epsilon) = \partial_t F(c_\infty, k_\infty, \epsilon) \neq 0\), we apply the Implicit Function Theorem on \(F_R\). Then there exists a smooth real-valued function \(C_1(k)\) for \(k \to k_\infty\) such that \(C_1(k_\infty) = c_\infty\) and \(F_R(C_1(k), k, \epsilon) = 0\). Since \(c_\infty \in \mathcal{N}\), \(F_1(c_\infty, k, \epsilon) = 0\) near \(c_\infty\). Hence, by the uniqueness of solutions obtained by the Implicit Function Theorem, \(C(k) = C_1(k) \in \mathbb{R}\) near \(c_\infty\). Hence, there is no unstable mode near \((c_\infty, k_\infty)\). The lemma is proved.

According to Lemma 3.4, for each \(\epsilon \in (0, 1)\), neutral limiting modes might happen at the endpoints of the range of \(U^\pm\) or the intersection of the range of \(U^\pm\). In either case, \((c, k)\) is a singular mode. By Remark 3.1, we notice that the number of eigenvalues \(\lambda = -\imath k c\) may be changed if \(c^\pm(k)\) obtained in Lemma 3.2 touches the range of \(U^\pm\) as \(|k|\) decreases. On the other hand, if we can rule out the existence of any singular modes, then \(c^\pm(k)\) can be extended to all \(k \in \mathbb{R}\). We would like to study how the eigenvalue distributes when \(|k|\) gets smaller. The rest of this section includes two main directions. One is to prove the stability of shear flows by ruling out all singular modes under some certain conditions. The other one is to seek instability occurring near singular modes for general \(\epsilon \geq 0\). In the following lemma, we provide some sufficient conditions for the non-existence of singular modes.

**Lemma 3.5.** Assume that \(U^\pm\) satisfies (1.3), \((U^\pm)'' \neq 0\), and \(\epsilon \in (0, 1)\). Let \(a, b\) be defined in (1.7), \(E, \mathcal{I}\) be defined in (3.24), and
\[
m(c) := \epsilon \int_0^{h_+} (U^+(x_2) - c)^2 dx_2 + \int_{-h_--}^0 (U^-(x_2) - c)^2 dx_2.
\]
If
\[
\min_{c \in (U^-(-h_--), U^-(-h_+))} \int_{-h_--}^0 \frac{1}{(U^- - c)^2} dx_2 > \frac{1}{g}, \quad \min_{c \in (U^+(0), U^+(h_+))} \int_0^{h_+} \frac{1}{(U^+ - c)^2} dx_2 > \frac{\epsilon}{g},
\]
and one of the following holds,
\[
(1) \mathcal{I} = \emptyset \quad \text{and} \quad \max_{E \in \mathcal{I}} m(E) < \frac{\sigma}{\rho};
\]
\[
(2) \mathcal{I} \neq \emptyset, \quad \max_{E \in \mathcal{I}} m(E) < \frac{\sigma}{\rho}, \quad \text{and} \quad (U^+''(U^-)'' > 0),
\]
then for each \(k > 0\), \(c^\pm(k)\) obtained in Lemma 3.2 can be extended to be even and analytic functions for all \(k \in \mathbb{R}\) and the following hold
\[
F(c^+(k), k, \epsilon) = 0, \quad c^+(k) > b, \quad c^-(k) < a, \quad k \partial_k F(k, c^\pm(k)) > 0.
\]

**Proof.** According to (2.19)-(2.21), \((U^- - c)^2 \partial_{k}^j Y^-(c, k), j \in \mathbb{N}\) is well defined and \(C^1\) near \(c = U^-(-h_-)\). Similarly, (2.26)-(2.28) imply that \((U^+ - c)^2 \partial_{k}^j Y^+(c, k), j \in \mathbb{N}\) is well defined and \(C^1\) near \(c = U^+(0)\) and \(U^+(h_+)\). In the following, we prove the non-existence of singular neutral mode under the
Hence, 
\[
-\epsilon \partial K Y^+(c, 0, 0) (U^+(0) - c) = \epsilon \int_0^{h_+} (U^+(x) - U^+(0))^2 dx_2.
\]

Similarly, using the above estimates and logarithmic singularity of \(\partial c K\), we can write
\[
\lim_{c \to U^-(0)} \partial K Y^-(c, 0) (U^-(0) - c) = 0.
\]

Now, we compute \(\partial K F(c, 0, \epsilon)\) for each \(c \in E\). If \(c = U^-(h_-) \notin U^+([0, h_+])\), we use the formula (3.38) and estimates (3.41) and (3.40) to compute
\[
\partial K F(c, 0, \epsilon) \leq \int_{-h_-}^{0} (U^-(x) - U^-(h_-))^2 dx_2 + \epsilon \int_0^{h_+} (U^+(x) - U^-(h_-))^2 dx_2 - \frac{\sigma}{\rho},
\]

which is less than 0.

According to Lemma 3.3, we have \(\partial K F(c, \epsilon) < \partial K F(c, 0, \epsilon)\) for \(c \in \mathbb{R} \setminus \{U^-((-h_-), 0) \cup U^+((0, h_+))\}\). Hence, \(\partial K F(U^-(h_-), k, \epsilon) < \partial K F(U^-((-h_-), 0), 0, \epsilon) < 0\).

If \(c = U^-(0) \notin U^+([0, h_+])\), we use (3.17b), (3.43), and (3.40) to compute that
\[
\partial K F(U^-(0), k, \epsilon) = -\epsilon \partial K Y^+(U^-(0), k) (U^+(0) - c)^2 \leq -\epsilon \partial K Y^+(U^-(0), 0) (U^+(0) - c)^2
\]

which is less than 0.
Step 2. We will show that $F(c, 0, \epsilon) < 0$ if $c \in \mathcal{E}$. If $c \notin U^+([0, h_+])$, using (3.21), we compute

$$Y^+(c, 0) = \frac{(U^+)'(0)}{U^+(0)} - c - \frac{1}{(U^+(0) - c)^2} \int_0^{h_+} \frac{1}{(U^+(0) - c)^2} dx.$$ 

If $c = U^+(-h_-) \notin U^+([0, h_+])$, we use (3.22), (3.32), and (3.36) to obtain that

$$F(U^+(-h_-), 0, 0) = -g + \epsilon \frac{1}{\int_0^{h_+} (U^+(0) - c)^2 dx} < 0.$$ 

If $c = U^+(-h_-) = U^+(0)$ or $c = U^+(-h_-) = U^+(h_+)$, then $F(c, 0, \epsilon) = -g < 0$. Following a similar argument for $c = U^+(0)$ and $U^+(h_+)$, we have $F(c, 0, \epsilon) < 0$ for $c \in \mathcal{E}$ and $I = \emptyset$.

Combining the results from step 1 and step 2, we obtain that $F(c, k, \epsilon) < 0$ for $c \in \mathcal{E}$ and all $k \in \mathbb{R}$ if $I = \emptyset$ by Lemma 3.3 and the logarithmic singularity of $Y^+(c, k)$ near $c = U^+(0)$ (resp. $Y^-(c, k)$ near $c = U^-(0)$). In other words, there is no singular mode at $c \in \mathcal{E}$.

Now, we first consider the case of $\max_{[-h_-, 0]} U^- < \min_{[0, h_+]} U^+$. Let $k_0$ be defined in Lemma 3.2. There are exactly two wave speed $c^\pm(k_0) \in \mathbb{C}$ such that $F(c^\pm(k_0), k_0, \epsilon) = 0$. According to Lemma 2.3 and Lemma 2.4, $F_1(c, k, \epsilon) \neq 0$ for $c \in U^-((-h_-, 0)) \cup U^+(0, h_+)$. Because of the non-existence of singular modes at $c \in \mathcal{E}$, $F(c, k, \epsilon) \neq 0$ for all $c \in U^-((-h_-, 0)) \cup U^+(0, h_+)$. By the compactness and continuity of $F$, there exist open sets $B_{1,2} \subset \mathbb{C}$ satisfying $U^-((-h_-, 0)) \subset B_1$ and $U^+(0, h_+) \subset B_2$ such that $F(c, k, \epsilon) \neq 0$ whenever $c \in B_1 \cup B_2$ and $k \in \mathbb{R}$. By (2.20), (2.21), and (2.15), we notice that if $c$ is real and $|c|$ is large, the leading term of $(U^+(0) - c)^2 Y^+(c, k)$ is $(U^+(0) - c)^2 k \coth(kh_-)$. Similarly, we use (2.27), (2.28), and (2.22) to obtain the leading term of $(U^+(0) - c)^2 Y^+(c, k)$ is $-\left[(U^+(0) - c)^2 k \coth(kh_+)\right]$. Hence, for $|k| \in [0, k_0]$, $F(c, k, \epsilon)$ behaves like a quadratic function in $c$ with a positive uniformly bounded leading coefficient if $c$ is real and $|c|$ is large. Therefore, there exists $B_3 > 0$ such that if $F(c, k, \epsilon) = 0$ and $|k| \in [0, k_0]$, then $|c| < B_3$.

We choose $B_3 \subset \mathbb{C}$ be a disk centered at the origin with radius larger than $B_3$ such that $c^\pm(k_0) \in B_3$ and $B_{1,2} \subset B_3$. Then $F(c, k, \epsilon) \neq 0$ for all $c \in \partial B_3$ and $|k| \in [0, k_0]$. Let $\Omega \subset B_3 \setminus \{B_1 \cup B_2\}$ containing $(\mathbb{R} \cap B_3) \setminus (B_1 \cup B_2)$ such that $\partial \Omega$ is sufficiently close to $\partial B_3$, $\partial B_2$, and $\partial B_1$. Then $\Omega$ can be chosen such that $F(c, k, \epsilon) \neq 0$ for all $c \in \partial \Omega$ and $|k| \in [0, k_0]$. We consider

$$n(k) := \int_{\partial \Omega} \frac{\partial_c F}{F}(c, k, \epsilon) dc.$$ 

Since $c^\pm(k_0)$ are the unique roots of $F(c, k_0, \epsilon) = 0$, $n(k_0) = 2$. Since $n(k)$ is continuous in $k$, $n(k) = 2$ for all $|k| \in [0, k_0]$. Hence, for all $k \in \mathbb{R}$, $n(k) \equiv 2$. Moreover, by Remark 3.1, we obtain that $c^\pm(k)$ obtained in Lemma 3.2 can be extended to be even and analytic functions for all $k \in \mathbb{R}$ and they are the only roots of $F(c, k, \epsilon) = 0$. Since $F(c, k, \epsilon) \in \mathbb{R}$ for all $c \in \mathbb{R} \setminus \{U^-((-h_-, 0)) \cup U^+(0, h_+))\}$, $\partial_c F(c^\pm(k), k, \epsilon)$ does not change sign and theirs signs are the same as the ones given in Lemma 3.2.

Let us consider the remaining case of $\max_{[-h_-, 0]} U^- = \min_{[0, h_+]} U^+$. Since $F(c, k, \epsilon) \neq 0$ for all $k \in \mathbb{R}$ and $c \in [\min U^-, \max U^+]$, there exists $B_4 \subset \mathbb{C}$ containing $[\min U^-, \max U^+]$ such that $F(c, k, \epsilon) \neq 0$ for all $k \in \mathbb{R}$ and $c \in B_4$. Then, there exists a big enough $\Omega_+ \subset \{c \geq \max U^+\} \setminus B_4$ containing $c^\pm(k_0)$ which is a bounded region such that $F(c, k, \epsilon) \neq 0$ for all $c \in \partial \Omega_+$ and $|k| \in [0, k_0]$. We can apply (3.46) and a similar argument as above on such $\Omega_+$ to obtain that $c^\pm(k)$ which is obtained in Lemma 3.2 can be extended to be an even and analytic function for all $k \in \mathbb{R}$. Similarly, we can choose $\Omega_- \subset \mathbb{C}$ big enough containing $c^\pm(k_0)$ such that $F(c, k, \epsilon) \neq 0$ on $\partial \Omega_-$. And obtain that $c^\pm(k)$ can be extended to all $k \in \mathbb{R}$. Moreover, let $\Omega$ be a large bounded region containing $c^\pm(k_0)$ and apply (3.46) on $(\Omega \setminus B_4) \cap \{c_1 > -\alpha\}$ where $0 < \alpha < 1$ and $(\Omega \setminus B_4) \cap \{c_1 < \alpha\}$ respectively. We can obtain that $c^\pm(k)$ are the only roots of $F(c, k, \epsilon) = 0$ for all $k \in \mathbb{R}$.

* Case 2. $I \neq \emptyset$. Without loss of generality, we can choose $\Omega_- \subset \mathbb{C}$ such that $\Omega_- \cap B_4 = \emptyset$ and $(U^+)'(U^-)' > 0$. Following the same argument of Case 1, it remains to consider $c \in I$. We notice that $F_1(c, k, \epsilon) \neq 0$ for any $c \in I$ if $sgn(U^-) = -sgn(U^+_1)$. Lemma 2.3 and Lemma 2.4 imply that if $(U^-)'(U^-)' > 0$, then $F(c, k, \epsilon) \neq 0$ for $c \in I$. The rest of the proof can be completed by using the same continuation arguments of $n(k)$ as the one in Case 1.

We notice that $c = a$ is a point on the boundary of the domain of analyticity of $F(c, k, \epsilon)$. Even more, it is possible that it corresponds to a neutral limiting mode. In the following lemma, we shall study the behavior of $F$ near $c = a$ for any $\epsilon \geq 0$.

**Lemma 3.6.** Assume that $U^\pm \in C^6$, $(U^\pm)'$, and $(U^\pm)'' > 0$. For each $\epsilon \geq 0$, if one of the following holds,
implies that both $(U^+(0) - c)Y^+$ and $\partial_c Y^+(U^+(0) - c)^2$ vanish at $c = U^+(0)$. Since $(U^-)' > 0$, we compute (3.8) to obtain that

$$\partial_c F(a, 0, \epsilon) = -(U^-)'(-h_-) - \epsilon (U^-)'(0) < 0, \quad \forall \epsilon \geq 0.$$ 

If $U^-(h_-) \notin U^+([0, h_+])$, we use (2.31) and (3.7a) to compute that

$$-(U^+(0) - a)^2 \partial_c Y^+(a, 0) = \frac{1}{(U^+(0) - a)^2} \int_0^{h_+} (U^+(x))^2 \left( \int_0^{h_+} \frac{1}{(U^+ - a)\xi} dx_2 \right)^2 dx_2 \leq \int_0^{h_+} (U^+(x))^2 dx_2 = (U^+(h_+))' - (U^+(0))'.$$

Using (3.47), this implies that

$$\partial_c F(a, 0, \epsilon) \leq -(U^-)'(-h_-) + \epsilon \left( \frac{2}{(U^+(0) - U^-(-h_-)) \int_0^{h_+} \frac{1}{(U^+ - U^-(-h_-))} dx_2} \right) < 0.$$

* Case 2. $a = U^+(0)$. It remains to consider $U^+(0) \notin U^+([-h_-, 0])$. Using (3.31) and (3.32), we compute

$$\partial_c F(a, 0, \epsilon) \leq -(U^-)'(-h_-) + \epsilon \left( \frac{2}{(U^+(0) - U^+(0)) \int_{-h_-}^{0} \frac{1}{(U^+ - U^+(0))} dx_2} \right) - \epsilon (U^+(0))' < 0.$$ 

□

Remark 3.2. When $(U^+)' > 0$, $(U^+)' \neq 0$, and $a = U^-(-h_-) < U^+(0)$, there exists $\epsilon_0 > 0$ such that (3.47) holds for all $\epsilon \in [0, \epsilon_0]$. Particularly, when $\epsilon = 0$, the result in Lemma 3.6 coincides the result for the capillary gravity water wave linearized at monotone convex shear flows.

According to Lemma 3.6, we can apply bifurcation analysis near $c = a$ and seek instability if $F(a, k, \epsilon) = 0$. We introduce $g_*$ which tells us when a bifurcation may happen at $c = a$. In the following lemma, we first study the roots of $F(a, k, \epsilon) = 0$ based on the value of $g$ relative to $g_*$. 

Lemma 3.7. Assume that $U^\pm$ satisfies (1.3) and $(U^\pm)'' \neq 0$. Let a be defined in (1.7). Let

$$g_* = \max \{ F(a, k, \epsilon) + g \mid k \in \mathbb{R} \},$$

then the following holds.

1. $g_* > 0$ for any $\epsilon > 0$.
2. If $g > g_*$, then $F(a, k, \epsilon) < 0$ for all $k \in \mathbb{R}$ and $\epsilon \geq 0$.
3. If one of the following cases holds,
   a. $a = U^+(0) \notin U^-([-h_-, 0])$ and
   b. $a = U^+(h_+) \notin U^-([-h_-, 0])$ and

   $$\begin{align*}
   0 &< \frac{1}{g(1 - \epsilon)} < \int_{-h_-}^{0} \frac{1}{(U^-(x_2) - U^+(0))^2} dx_2, \\
   0 &< \frac{1}{g(1 - \epsilon)} < \int_{-h_-}^{0} \frac{1}{(U^-(x_2) - U^+(h_+))^2} dx_2
   
   \end{align*}$$

   then the following holds.

   (1) $a = U^+(0) \leq U^-(h_-)$, or
   (2) $a = U^-(h_-) < U^+(0)$ and
Proof. Since $Y^\pm(c, k)$ are comparable to $\pm k$ as $|k|$ becomes large, $F(a, k, \epsilon)$ behaves like a quadratic function of $k$ with a negative leading coefficient. Hence, $g_\ast$ exists. If $a = U^-(0)$, $Y^-(a, 0)$ is in the form of (3.32). If $a = U^-(0)$, (2.18) implies that $(U^-(0) - a)^2 Y^-(a, 0) = 0$. If $a \notin U^-([-h_-, 0])$, then one can use (3.32) to compute $F(a, 0, \epsilon)$. If $a = U^+(0)$, (2.25) implies that $(U^+(0) - a)^2 Y^+(a, 0) = 0$. If $a = U^+(h_+)$, we apply (3.20) to compute

$$Y^+(a, 0) = \frac{(U^+(a, 0))'}{(U^+(a, 0) - a)}.$$  

If $a \notin U^+([0, h_+])$, we apply (3.21) to obtain that

$$Y^+(a, 0) = \frac{(U^+(0))'}{U^+(0) - a} + \frac{1}{(U^+(0) - a)^2} \int_0^{h_+} \frac{1}{(U^+(0) - a)^2} dx_2.$$  

Using these formulas, one can obtain that $g_\ast \geq F(a, 0, \epsilon) + g \geq \epsilon g > 0$ since $\epsilon > 0$. If $a = U^+(0) \notin U^-([-h_-, 0])$,

$$F(a, 0, \epsilon) = g(\epsilon - 1) + \frac{1}{\int_{-h_-}^0 \frac{1}{(U^+(0) - a)^2} dx_2}.$$  

Hence, $F(a, 0, \epsilon) < 0$ if (3.49) holds. If $a = U^+(h_+) \notin U^-([-h_-, 0])$, $F(a, 0, \epsilon)$ is in the form of (3.53) and $F(a, 0, \epsilon) < 0$ if (3.50) holds. If $a = U^-(0) \notin U^+([0, h_+])$,

$$F(a, 0, \epsilon) = g(\epsilon - 1) - \frac{1}{(U^+(0) - a)^2} \int_0^{h_+} \frac{1}{(U^+(0) - a)^2} dx_2 < 0.$$  

If $a = U^-(0) = U^+(0)$ or $U^-(0) = U^+(h_+)$, then $F(a, 0, \epsilon) = g(\epsilon - 1) < 0$. If $a = U^-(0) \notin U^+([0, h_+])$, then $F(a, 0, \epsilon)$ is in the form of (3.54) and $F(a, 0, \epsilon) < 0$. If $a = U^-(0) = U^+(0)$ or $U^-(0) = U^+(h_+)$, then $F(a, 0, \epsilon) = g(\epsilon - 1) < 0$. The rest of the proof for (3) follows directly from the concavity of $F(a, k, \epsilon)$ in $K = k^2$ by (3.12) and the definition of $g_\ast$. \hfill \Box

In the next lemma, we analyze the roots of $F(\cdot, k, \epsilon) = 0$ near $c = a$ if $F(a, k_\ast, \epsilon) = 0$ for some $k_\ast > 0$ based on the different properties of $F$ at $(a, k_\ast)$. This is a local result near $(a, k_\ast)$ for all $\epsilon \geq 0$.

**Lemma 3.8.** Assume that $U^\pm$ satisfies (1.3) and $(U^\pm)'' > 0$, and $F(a, k_\ast, \epsilon) = 0$ for some $k_\ast > 0, \epsilon \geq 0$. If $\partial c F(a, k_\ast, \epsilon) < 0$, then there exist $\delta > 0$ and a unique $C^\ast \in C^{1, \alpha}([k_\ast - \delta, k_\ast + \delta], \mathbb{C})$ for any $\alpha \in [0, 1)$ such that

$$C^\ast(k) = a, \quad F(C^\ast(k), k) = 0 \quad \forall k \in [k_\ast - \delta, k_\ast + \delta].$$  

And $C^\ast(k)$ satisfies the following.

1. If $\partial k F(a, k_\ast, \epsilon) = 0$, then $(C^\ast)'(k_\ast) = 0, C^\ast_{\ast} \equiv 0$ and $C^\ast(k) < a$ for all $0 < |k - k_\ast| < \delta$. Moreover, $k_\ast$ are the only roots of $F(a, , \epsilon) = 0$.
2. If $\pm \partial k F(a, k_\ast, \epsilon) > 0$, then $\pm (C^\ast_{\ast})'(k_\ast) > 0$. Moreover,

$$C^\ast_R(k) < a, \quad C^\ast_L(k) = 0, \quad 0 < C^\ast_R(k) \leq \delta,$$

$$C^\ast_R(k) > a, \quad C^\ast_L(k) > 0, \quad 0 < C^\ast_R(k) \leq \delta.$$
Proof. Recall that $\mathcal{I}$ is defined in (3.24). According to Lemma 2.3 and Lemma 2.4, $F_I$ is not continuous at $c \in U^-(I_{-} h_{-}) \cup U^+(I_{+} h_{+}) \subseteq \mathbb{C}$ in general. Hence, we extend $F(c, k, \epsilon)$ by letting
\[
F(c, k, \epsilon) = \tilde{F}_R(c, k, \epsilon) + i\tilde{F}_I(c, k, \epsilon) \in \mathbb{C}, \quad \tilde{F} = F \text{ for } c_I \geq 0,
\]
be a $C^{1,\alpha}$ extension of $F$ into a neighborhood of $(a, k) \in \mathbb{C} \times \mathbb{R}$ for any $\alpha \in [0, 1)$ and each $\epsilon \geq 0$. By Lemma 3.1, since Cauchy-Riemann equations hold and $F(c, k, \epsilon) \in \mathbb{R}$ for all $c < a$, the $2 \times 2$ Jacobian matrix of $D_c \tilde{F}$ satisfies
\[
D_c \tilde{F}(a, k_\epsilon) = \left( \frac{\partial_{c_R} \tilde{F}_R}{\partial_{c_I} \tilde{F}_I}, \frac{\partial_{c_I} \tilde{F}_R}{\partial_{c_I} \tilde{F}_I} \right) = \frac{\partial_k \tilde{F}(a, k_\epsilon)}{\partial_{c_I} \tilde{F}_I} I_{2 \times 2}.
\]
Since $\partial_k F(a, k_\epsilon) < 0$, we apply the Implicit Function Theorem and obtain a $C^{1,\alpha}$ complex-valued function $C^-(k)$ such that $C^-(k_\epsilon) = a$ and $\tilde{F}(C^-(k), k_\epsilon) = 0$ on $[k_\epsilon - \delta, k_\epsilon + \delta]$. We know that $\tilde{F}_R \in C^1$ and $\partial_{c_R} \tilde{F}_R(a, k_\epsilon) = \partial_k F(a, k_\epsilon) < 0$. According to the Implicit Function Theorem, there exist $\delta > 0$ and a $C^{1,\alpha}$ real-valued function $\tilde{C}(k)$ such that $\tilde{C}(k_\epsilon) = a$ and $\tilde{F}_R(\tilde{C}(k), k_\epsilon) = 0$. Since $F$ is real-valued if $c \leq a$, the uniqueness of solutions by the Implicit Function Theorem ensures that $C^-(k) = \tilde{C}(k) \in \mathbb{R}$ if $\tilde{C}(k_\epsilon) \leq 0$. Moreover, $F(C^-(k), k_\epsilon) = \tilde{F}(C^-(k), k_\epsilon) = 0$.

We first consider the case of $\partial_k F(a, k_\epsilon) = 0$ with $k_\epsilon > 0$. The uniqueness of $k_\epsilon$ is ensured by (3.12). Since $\partial_{c_R} \tilde{F}_R(a, k_\epsilon) = \partial_k F(a, k_\epsilon) < 0$ and $F(a, k_\epsilon)$ is concave in $K = k^2$, we obtain that
\[
F_R(a, k_\epsilon) > 0, \quad \forall 0 < a < c \ll 1, \quad F_R(a, k_\epsilon) < 0, \quad \forall k \in \mathbb{R} \setminus \{k_\epsilon\}.
\]
Then there exist $k$ near $k_\epsilon$ and $c$ near $a$ with $c < a$ such that $F_R(c, k_\epsilon) = 0$. Moreover, the uniqueness of solutions by the Implicit Function Theorem ensures that $c = \tilde{C}(k)$. Therefore, $C^-(k) = \tilde{C}(k) < a$ for $k \neq k_\epsilon$ close to $k_\epsilon$.

Now, we consider the case of $\partial_k F(a, k_\epsilon) < 0$. The proof for the case of $\partial_k F(a, k_\epsilon) < 0$ is similar. Since $\partial_k F(a, k_\epsilon) < 0$,
\[
(C^-)'(k_\epsilon) = -\frac{\partial_k F(a, k_\epsilon)}{\partial_{c_R} F(a, k_\epsilon)} > 0.
\]
Hence, $C_R^-(k) > a$ for $k \in (k_\epsilon, k_\epsilon + \delta]$. By the Mean Value Theorem, there exists $\gamma$ between 0 and $C_R^-(k)$ such that
\[
0 = F_I(C^-(k), k_\epsilon) = F_I(C_R^-(k), k_\epsilon) + C_I^-(k)\partial_{c_I} F_I(C_R^-(k) + i\gamma, k_\epsilon).
\]
By the $C^{1,\alpha}$ regularity of $F$ and $C^-(k)$, we obtain that
\[
C_I^-(k) = -\frac{F_I(C_R^-(k), k_\epsilon)}{\partial_{c_I} F_I(C_R^-(k) + i\gamma, k_\epsilon)} = -\frac{Y_I^{-}(C_R^-(k), k_\epsilon)(U^{-}(0) - C_R^-(k))^2 - \epsilon Y_I^{+}(C_R^-(k), k_\epsilon)(U^{+}(0) - C_R^-(k))^2}{\partial_{c_I} F_I(C_R^-(k), k_\epsilon) + O(|C_I^{-}(k)|^\alpha)} \quad \frac{C^-(k)}{k, \epsilon} = 0.
\]
Since $U^+'' > 0$, using (2.20) and (2.27), we have that the numerator is always positive for small $\delta$. This implies that $C_R^-(k) > 0$ when $0 < k - k_\epsilon \ll 1$. 

\[ \square \]

Remark 3.3. 1. If $\partial_k F(a, k_\epsilon) > 0$, to obtain the same results above, the assumption in Lemma 3.8(2) should be changed to $\pm \partial_k F(a, k_\epsilon) < 0$. 2. If $U^{+''} < 0$, the signs in the result of Lemma 3.8(2) should be reversed.

We are in the position to prove Theorem 1.3.

Proof of Theorem 1.3. It remains to prove the last part of Theorem 1.3(2)(b)(ii). The rest follows directly from Lemma 3.7, Lemma 3.8, and Remark 3.1. Suppose that $a = U^-(I_{-} h_{-}) \notin U^+(I_{+} h_{+})$. Then (3.45) shows that $g_s \geq F(U^-(I_{-} h_{-}), 0, \epsilon) + g > 0$ if $\epsilon > 0$. Similarly, one may check that $F(U^{+}(0), 0, \epsilon) + g > 0$ by using (3.23) if $a = U^{+}(0) \notin U^-(I_{-} h_{-}, 0)$ and $\epsilon > 0$. In the case of $a = U^{-}(-h_{-}) = U^{+}(0)$, since $F(a, 0, \epsilon) + g > 0$, $g_s = 0$ and only if $\partial_K F(a, 0, \epsilon) \leq 0$, where $K := k^2$. The proof of the theorem is completed. 

\[ \square \]
4. Ocean-Air Model

In this section, we study the spectral distribution for the case of $0 \leq \epsilon \ll 1$. We shall treat the interface problem as a perturbation to the capillary-gravity water wave problem. According to Theorem 1.1, all the unstable modes must have wave speed lie in a big upper semicircle (1.9) if (1.8) is satisfied. We first locate all the possible locations of neutral limiting modes inside $[a, b]$ where $a, b$ are defined in (1.7) as $\epsilon$ tends to zero, and then seek unstable modes nearby when $\epsilon$ is perturbed. We give a complete picture of the eigenvalue distribution for a special case of $(U^+) > 0, (U^+)^\prime \neq 0$, and $U^-(0) < U^+(0)$. In the following lemma, we first address the possible locations of neutral limiting mode as $\epsilon$ tends to $0$.

**Lemma 4.1.** Assume that $U^\pm$ satisfy (1.3). Let $I$ be defined in (3.24) and $\mathcal{I} = \emptyset$. If a sequence of unstable modes $(c_n, k_n, \epsilon_n)$ with $\epsilon_n > 0$ converges to $(c_*, k_*, 0)$ as $n \to \infty$, then $c_* = \lim_{n \to \infty} c_n \in \{U^-(h_-)\} \cup U^+(0, h_+) \setminus U^-(0)$ or

$$
\Omega := \{c_R + ic_I, c_R \in \mathbb{R}, c_I \in \mathbb{R}^+ \mid (c_R - \frac{U^-(h_-) + U^-(0)}{2})^2 + c_I^2 < \frac{(U^-(h_-) - U^-(0))^2}{2}\}.
$$

**Proof.** Recall that $F(c, k, \epsilon)$ is defined in (3.1). Then, Lemma 3.1 implies that $\lim_{n \to \infty} F(c_n, k_n, \epsilon_n) = F(c_*, k_*, 0) = 0$. According to Theorem 1.1 in [11], $c_* \in \mathbb{R} \setminus U^-(h_-, 0) \cup \Omega$. Since $F(c_*, k_*, \epsilon_n) = 0$, Theorem 1.1 implies that $c_*$ stays in the upper semicircle (1.9). By the definition of $(c_n, k_n, \epsilon_n)$, for any $0 < \alpha \ll 1$, there exists $N_0 > 0$ such that whenever $n > N_0, \epsilon_n, |k_n - k_*| < \alpha$. Hence, $c_* \in (a, b] \setminus U^-(h_-, 0) \cup \Omega$, where $a$ and $b$ are defined in (1.7). We shall discuss the following possible cases to shrink this region.

**Case 1.** $U^-(0) < U^+(0)$. Suppose that $c_* \in (U^-(0), U^+(0))$. By Remark 3.1, when $\epsilon = 0$, $c^+(k)$ obtained in Lemma 3.2 can be extended to all $k \in \mathbb{R}$. This shows that $c_*$ is a simple root of $F(c_*, k_*, 0)$ and according to (3.10), $\partial_c F(c_*, k_*, 0) > 0$. Since $F(c, k, \epsilon)$ is analytic near $c = c_*$, there exist $\delta, \epsilon_0 > 0$ and a smooth function $C(k, \epsilon) : [k_* - \delta, k_* + \delta] \to \mathbb{C}$ such that

$$
F(C(k, \epsilon), k, \epsilon) = 0, \quad C(k_*, 0) = c_*.
$$

By the uniqueness of solutions ensured by the Implicit Function Theorem and the definition of the sequence $\{(c_n, k_n, \epsilon_n)\}$, there exists $N > 0$ such that for all $n \geq N$, $C(k_n, \epsilon_n) = c_n$. This implies that $C^I(k_n, \epsilon_n) > 0$.

Since $c_* \notin U^-(h_-, 0) \cup U^+(0, h_+)$, $F_R \in C^\infty$ and $\partial_{c_R} F_R(k_*, 0) = 0$. Then the Implicit Function Theorem implies that there exists a smooth real-valued function $\tilde{C}(k, \epsilon)$ for $k$ near $k_*$ and $\epsilon$ near $0$ such that

$$
\tilde{C}(k_*, 0) = c_*, \quad F_R(\tilde{C}(k, \epsilon), k, \epsilon) = 0.
$$

Since $F_I(k, \epsilon) = 0$ if $c \in (U^-(0), U^+(0))$, the uniqueness of solutions by the Implicit Function Theorem implies that $C(k, \epsilon) = \tilde{C}(k, \epsilon) \in \mathbb{R}$ if $\tilde{C} \in (U^-(0), U^+(0))$. Hence, $C(k_n, \epsilon_n) \in \mathbb{R}$ for all $n > N$, which contradicts to the previous argument that $C^I(k_n, \epsilon_n) > 0$. Therefore, $c_* \notin (U^-(0), U^+(0))$.

**Case 2.** $U^+(h_+) < U^-(h_-)$. It suffices to consider $U^+(h_+) < U^-(h_-)$. According to Remark 3.1, $\partial_{c_R} F(k_*, 0) < 0$. By a similar argument as above, the lemma is proved. \hfill \Box

This lemma implies that instability might occur near $U^-(h_-)$ and $U^+(0, h_+]$ for $0 < \epsilon \ll 1$. We shall analyze the bifurcation near $U^+(h_+)$ and $U^+(0)$ and give a complete picture of the eigenvalue distribution when $0 < \epsilon \ll 1$. The strategy can be adapted to analyze some other cases including all the cases of $(U^+) \prime, (U^+)^\prime \neq 0$ and $\mathcal{I} \neq \emptyset$, where $\mathcal{I}$ is defined in (3.24).

**Lemma 4.2.** Suppose that $U^\pm$ satisfy (1.3), $(U^\pm)^\prime > 0, U^-(0) < U^+(0), (U^+)^\prime \neq 0$, and $F(U^+(h_+), k_0, 0) = 0$ for some $k_0 > 0$. Then the following hold.

(1) Suppose that $\partial_k F(U^+(h_+), k_0, 0) \neq 0$. Then there exist $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$ there exist $\delta_0 > 0$ such that $F(U^+(h_+), k_*, 0) = 0$, $k_0 = k_0$, $F(C^+(k_*) = 0, \forall k \in [k_*(\epsilon) - \delta_0, k_*(\epsilon) + \delta_0]$, $C^+(\pm k_0) = U^+(h_+)$. And for $c_I \in [0, U^+(h_+) - U^+(0)]$ and $|c_R - U^+(h_+) - U^+(0)| \leq U^+(h_+) - U^+(0)$, $C^+(k)$ is the unique root of $F(c, k, \epsilon) = 0$ with $|k| \leq (k_*(\epsilon) - \delta_0, k_*(\epsilon) + \delta_0)$. Moreover, the following hold.
(a) If \( \partial_k F(U^+(h_+), k_0, 0) < 0 \), then
(i) \( k'(0) > 0 \) and \( k(e) > k_0 \);
(ii) \( C^+(k) > U^+(h_+) \) for any \( k \in (k(e), k(e) + \delta_0) \) and
\[
(U^+)'' C^+_R(k) < 0, \quad C^+_R(k) < U^+(h_+), \ \forall |k| \in [k(e) - \delta_0, k(e)].
\]
(b) If \( \partial_k F(U^+(h_+), k_0, 0) > 0 \), then
(i) \( k'(0) < 0 \) and \( k(e) < k_0 \);
(ii) \( C^+(k) > U^+(h_+) \) for any \( k \in [k(e) - \delta_0, k(e)] \) and
\[
(U^+)'' C^+_R(k) < 0, \quad C^+_R(k) < U^+(h_+), \ |k| \in (k(e), k(e) + \delta_0].
\]
(2) Suppose that \( \partial_k F(U^+(h_+), k_0, 0) = 0 \). Then there exist \( \epsilon_1 > 0 \) such that for each \( \epsilon \in (0, \epsilon_1) \) there exist \( k_2(\epsilon) > k_0 > k_1(\epsilon) \geq 0, \ \delta > 0 \), and \( C^+_{1,2} \in C^{1,\alpha}([k_1,2(\epsilon) - \delta, k_1,2(\epsilon) + \delta], \mathbb{C}) \) for any \( \alpha \in [0,1) \) such that
\[
F(U^+(h_+), \pm k_1,2(\epsilon), \epsilon) = 0, \quad F(C^+_{1,2}(k), \pm k_1,2(\epsilon), \epsilon) = U^+(h_+).
\]
Moreover, \( C^+_{1,2}(k) \) satisfy that
\[
\begin{align*}
C^+_{1}(k) &= U^+(h_+), \ \forall |k| \in (k_1(\epsilon) - \delta, k_1(\epsilon) + \delta), \\
C^+_{2}(k) &= U^+(h_+), \ \forall |k| \in (k_2(\epsilon), k_2(\epsilon) + \delta), \\
C^+_{R}(k) &= U^+(h_+), \ (U^+)'' C^+_{R}(k) < 0, \ \forall |k| \in (k_1(\epsilon), k_1(\epsilon) + \delta), \\
C_{2R}(k) &= U^+(h_+), \ (U^+)'' C_{2R}(k) < 0, \ \forall |k| \in (k_2(\epsilon) - \delta, k_2(\epsilon) + \delta).
\end{align*}
\]
In addition, for \( |c_R - U^+(h_+)| < U^+(h_+) - U^+(0) \) and \( c_1 \in [0, U^+(h_+) - U^+(0)] \), \( C^+_{1} \) is the unique root of \( F(\cdot, k, \epsilon) = 0 \) with \( |k| \in (k_1(\epsilon) - \delta, k_1(\epsilon) + \delta) \), \( C^+_{2} \) is the unique root of \( F(\cdot, k, \epsilon) = 0 \) with \( |k| \in (k_2(\epsilon) - \delta, k_2(\epsilon) + \delta) \).

Proof. Let \( K := k^2 \). We shall prove the case of \( \partial_K F(U^+(h_+), k_0, 0) < 0 \). The proof for the case of \( \partial_K F(U^+(h_+), k_0, 0) > 0 \) is similar. Since \( \partial_k F(U^+(h_+), k_0, 0) < 0 \) and \( k_0 > 0, \ \partial_K F(U^+(h_+), k_0, 0) < 0 \). And there exist \( 0 < \epsilon_0 \ll 1 \) and a real-valued smooth function \( K(\epsilon) \) such that \( F(U^+(h_+), K(\epsilon), \epsilon) = 0 \) for all \( \epsilon \in [-\epsilon_0, \epsilon_0] \) and \( K(0) = k_0^2 \). We compute that
\[
\partial_r F(\cdot, k, \epsilon) = -(U^+(0) - c)^2 Y^+(c, k) + (U^+)'(0)(U^+(0) - c) + g.
\]
We apply (3.51) to obtain that
\[
\partial_r F(U^+(h_+), 0, 0) = g > 0
\]
According to (3.15b),
\[
\partial_r K F(U^+(h_+), k, \epsilon) = -(U^+(0) - U^+(h_+))^2 \partial_K Y^+(U^+(h_+), k) > 0,
\]
for all \( k \in \mathbb{R} \) and \( \epsilon \geq 0 \). Hence, \( \partial_r F(U^+(h_+), k_0, 0) > \partial_r F(U^+(h_+), 0, 0) > 0 \). Using the assumption that \( \partial_K F(U^+(h_+), k_0, 0) < 0 \), we obtain that
\[
K'(0) = -\frac{\partial_r F}{\partial_K F}(U^+(h_+), k_0, 0) > 0.
\]
Then \( \epsilon_0 \) can be chosen small enough such that \( K'(\epsilon) > 0 \) for all \( \epsilon \in [0, \epsilon_0] \).

For each \( \epsilon \in (0, \epsilon_0) \), we consider \( F(U^+(h_+), k_0, \epsilon) \), where \( k(\epsilon) = \sqrt{K(\epsilon)} \). We extend \( F(c, k, \epsilon) \) by letting \( \tilde{F} \) be defined in (3.55). Then \( \tilde{F} \) is a \( C^{1,\alpha} \) extension of \( F \) into a neighborhood of \( (U^+(h_+), k_0, \epsilon) \) \( \in \mathbb{C} \times \mathbb{R} \) for any \( \alpha \in [0,1) \). Since \( F(c, k, \epsilon) \in \mathbb{R} \) for all \( c \geq U^+(h_+) \), we use Cauchy-Riemann equations to compute
\[
D_c \tilde{F}(U^+(h_+), k_0, \epsilon) = \begin{pmatrix}
\partial_{c_\text{Re}} \tilde{F}_R & \partial_{c_\text{Im}} \tilde{F}_R \\
\partial_{c_\text{Re}} \tilde{F}_I & \partial_{c_\text{Im}} \tilde{F}_I
\end{pmatrix}
(U^+(h_+), k_0, \epsilon) = \partial_r F(U^+(h_+), k_0, 0) I_{2 \times 2}.
\]
We apply the Mean Value Theorem and the smoothness of \( K(\epsilon) \) to obtain that
\[
\partial_c F(U^+(h_+), k_0, 0) = \partial_c F(U^+(h_+), k_0, 0) + \partial_c K F(U^+(h_+), k_0, 0) (K(\epsilon) - k_0^2) + O(\epsilon)
\]
\[
= \partial_c F(U^+(h_+), k_0, 0) + O(\epsilon),
\]
where $\theta_1 \in (0, \epsilon)$ and $\theta_2 \in (0, k(\epsilon) - k_0).$ By (3.11), $\partial_c F(U^+(h_+), k_0, 0) > 0.$ Since $\epsilon_0 \ll 1,$

(4.4) 
\[ \partial_c F(U^+(h_+), k(\epsilon), \epsilon) > 0, \quad \forall \epsilon \in [0, \epsilon_0]. \]

Therefore, there exist $0 < \beta \ll 1$ and a $C^{1, \alpha}$ function $C^+(k) : [k(\epsilon) - \beta, k(\epsilon) + \beta] \to \mathbb{C}$ for any $\alpha \in (0, 1)$ such that 

\[ \hat{F}(C^+(k), k, \epsilon) = 0, \quad C^+ (k(\epsilon)) = U^+(h_+), \]

(4.5) 
\[ C^+_R(k) > U^-(0). \]

Since $F_R \in C^1$ and $\partial_{cR} F_R(U^+(h_+), k(\epsilon), \epsilon) = \partial_c F(U^+(h_+), k(\epsilon), \epsilon) > 0,$ there is a $C^1$ real-valued function $\hat{C}(k)$ for $k$ near $k(\epsilon)$ such that 

\[ \hat{C}(k(\epsilon)) = U^+(h_+), \quad F_R(\hat{C}(k), k, \epsilon) = 0. \]

Since $F_I(c, k, \epsilon) = 0$ for all $c \geq U^+(h_+),$ the uniqueness of solutions by the Implicit Function Theorem ensure that $\hat{C}(k) = C^+(k) \in \mathbb{R}$ if $\hat{C} \geq U^+(h_+).$

By the Mean Value Theorem and smoothness of $K(\epsilon),$ we use the smallness of $\epsilon_0$ to obtain that 

(4.6) 
\[ \partial_c F(U^+(h_+), k(\epsilon), \epsilon) = \partial_c F(U^+(h_+), k_0, 0) + O(\epsilon) < 0. \]

Hence, we compute 

\[ \hat{C}'(k(\epsilon)) = (C^+)'(k(\epsilon)) = -\frac{2k(\epsilon)\partial_c F}{\partial_c F}(U^+(h_+), k(\epsilon), \epsilon) > 0. \]

Hence, $\hat{C}(k) = C^+(k) > U^+(h_+)$ for $k$ slightly larger than $k(\epsilon).$ When $k \in [k(\epsilon) - \beta, k(\epsilon)],$ $C^+_R(k) < U^+(h_+).$

In this case, we consider 

\[ 0 = \tilde{F}_I(C^+(k), k, \epsilon) = F_I(C^+_R(k), k, \epsilon) + C^+_I(k)\partial_c \tilde{F}_I(C^+_R(k), i\theta, k, \epsilon), \]

where $|\theta| \in (0, C^+_I(k)).$ then 

\[ C^+_I(k) = -\frac{F_I(C^+_R(k), k, \epsilon)}{\partial_c \tilde{F}_I(C^+_R(k), i\theta, k, \epsilon)} \]

\[ = -\frac{(U^+(0) - C^+_R(k))^2 Y_{-}^I(C^+_R(k), k) - \epsilon(U^+(0) - C^+_R(k))^2 Y_{+}^I(C^+_R(k), k)}{\partial_c \tilde{F}_I(C^+_R(k), k, \epsilon) + O(|C^+_I(k)|^\alpha) + O(\epsilon)} \]

By (2.20), (4.5), and (2.27) $Y_{-}^I(C^+_R(k), k, \epsilon) = 0.$ Hence, 

\[ C^+_I(k) = \frac{\epsilon(U^+(0) - C^+_R(k))^2 Y_{+}^I(C^+_R(k), k)}{\partial_c \tilde{F}_I(U^+(h_+), k(\epsilon), \epsilon) + O(|k - k(\epsilon)|^\alpha)} \]

\[ = \frac{\epsilon(U^+(0) - U^+(h_+)) + O(|k - k(\epsilon)|^\alpha)}{\partial_c F(U^+(h_+), k(\epsilon), \epsilon) + O(|k - k(\epsilon)|^\alpha)} \]

By (2.27), the statement (1) is proved.

Now we consider the case of $\partial c F(U^+(h_+), k_0, 0) = 0.$ Since $\partial c F(U^+(h_+), k_0, 0) > 0,$ there exist $0 < \delta \ll 1$ and $\Gamma(k) \in C^\infty([k_0 - \delta, k_0 + \delta], \mathbb{R})$ such that 

\[ F(U^+(h_+), k, \Gamma(k)) = 0, \quad \Gamma(k_0) = 0. \]

Since $\partial RF(U^+(h_+), k_0, 0) = 0,$ we use (3.12) to obtain that, 

\[ \partial_{KK} \Gamma(k_0) = -\frac{\partial_{KK} F}{\partial RF}(U^+(h_+), k_0, 0) > 0. \]

Hence, $\Gamma(k) \geq 0$ on $[k_0 - \delta, k_0 + \delta].$ Let $\epsilon_1 := \sup_{|k| \in [k_0 - \delta, k_0 + \delta]} \Gamma(k).$ Then for each $\epsilon \in (0, \epsilon_1),$ there exist $k_2(\epsilon) > k_0 > k_1(\epsilon) > 0$ such that 

\[ F(U^+(h_+), k_2(\epsilon), \epsilon) = F(U^+(h_+), k_1(\epsilon), \epsilon) = 0. \]

Thanks to (3.12), we have 

\[ \partial RF(U^+(h_+), k_1(\epsilon), \epsilon) < 0, \quad \partial RF(U^+(h_+), k_2(\epsilon), \epsilon) > 0. \]
For such $\epsilon$, we apply the bifurcation analysis near $(U^+(h_+), k_1(\epsilon), \epsilon)$ and $(U^+(h_+), k_2(\epsilon), \epsilon)$ respectively, which is similar as the proof for the case of $\partial_R F(U^+(h_+), k_0, \epsilon) < 0$. The proof is completed.

When $U^+(h_+) \notin U^-([-h_-, 0])$ is the maximum speed of both fluids, we also discuss about the possible bifurcation near $c = U^+(h_+)$ if $(U^+(h_+), 0)$ is a neutral mode in the capillary gravity water wave problem.

**Lemma 4.3.** Suppose that $U^\pm$ satisfy (1.3), $U^-(0) < U^+(0)$, $(U^\pm)' > 0$, $(U^\pm)'' \neq 0$, and $F(U^+(h_+), 0, 0) = 0$. Then the following hold.

1. There exist $\epsilon_0 > 0$ and $\bar{C}(\epsilon) \in \mathcal{C}^{1, \alpha}([0, \epsilon_0], \mathbb{C})$ for any $\alpha \in (0, 1)$ such that
   \[ F(\bar{C}(\epsilon), 0, \epsilon) = 0, \quad \forall \epsilon \in [0, \epsilon_0], \quad \bar{C}(0) = U^+(h_+). \]
   Moreover, $\bar{C}$ satisfies that
   \[ \bar{C}_R(\epsilon) < U^+(h_+), \quad (U^\pm)''(\epsilon) < 0, \quad \forall \epsilon \in (0, \epsilon_0]. \]

2. Let $K := k^2$. If $\partial_R F(U^+(h_+), 0, 0) \leq 0$, then there exist $\epsilon_1 > 0$ and $k(\epsilon) \in \mathcal{C}^\infty([0, \epsilon_1], \mathbb{R}^+ \cup \{0\})$ such that for each $\epsilon \in (0, \epsilon_1),$
   \[ F(U^+(h_+), \pm k(\epsilon), \epsilon) = 0, \quad k(\epsilon) > 0, \quad \text{and} \quad k(0) = 0. \]
   Moreover, there exist $\delta_0 > 0$ and $C^+ \in \mathcal{C}^{1, \beta}([k(\epsilon) - \delta_0, k(\epsilon) + \delta_0], \mathbb{C})$ for any $\beta \in (0, 1)$ such that
   \[ F(\bar{C}^+(\epsilon), \pm k(\epsilon), \epsilon) = 0, \quad C^+(k(\epsilon)) = U^+(h_+), \quad C^+(k) > U^+(h_+), \quad \forall k \in [k(\epsilon), k(\epsilon) + \delta_0], \]
   \[ (U^\pm)''(\epsilon)C^+_R(\epsilon) < 0, \quad (U^\pm)''(\epsilon) < 0, \quad \forall k \in [k(\epsilon) - \delta_0, k(\epsilon)]. \]

In addition, for $c_1 \in [0, U^+(h_+) - U^+(0)], |c_R - U^+(h_+)| < U^+(h_+) - U^+(0)$, $C^+(k)$ is the only root of $F(\cdot, k, \epsilon) = 0$ with $|k| \in [k(\epsilon) - \delta_0, k(\epsilon) + \delta_0]$.

**Proof.** We extend $F(c, k, \epsilon)$ into a neighborhood of $(U^+(h_+), 0) \in \mathbb{C} \times \mathbb{R}$ by letting $\bar{F}$ be defined in (3.35). Using Cauchy-Riemann equations, $D_\bar{c} \bar{F}(U^+(h_+), 0, 0)$ is in the form of (4.3). By (3.11), $\partial_c F(U^+(h_+), 0, 0) > 0$. Hence, there exist $0 < \epsilon_0 < 1$ and a $C^{1, \alpha}$ function $\bar{C}(\epsilon) : [-\epsilon_0, \epsilon_0) \rightarrow \mathbb{C}$ for any $\alpha \in (0, 1)$ such that
   \[ \bar{F}(\bar{C}(\epsilon), 0, \epsilon) = 0, \quad \bar{C}(0) = U^+(h_+). \]

By a similar argument as the proof in Lemma 4.2, we obtain that, $\bar{F}(\bar{C}(\epsilon), 0, \epsilon) = \bar{F}(\bar{C}(\epsilon), 0, \epsilon)$. Using (4.2), we obtain that $(\bar{C}'(0) = -\partial_R F(U^+(h_+), 0, 0) < 0$. Then, for each $\epsilon \in (0, \epsilon_0)$, $\bar{C}_R(\epsilon) < U^+(h_+)$.

Moreover, from the Mean Value Theorem, for such $\epsilon$, there exists $\theta \in (0, \epsilon)$ such that
   \[ 0 = \bar{F}_I(\bar{C}(\epsilon), 0, \epsilon) = F_I(\bar{C}_R(\epsilon), 0, \epsilon) + \bar{C}_I(\epsilon)\partial_{\epsilon_1}\bar{F}_I(\bar{C}_R(\epsilon) + i\theta, 0, \epsilon). \]

Similar as the argument in the proof of Lemma 4.2, using the $C^{1, \alpha}$ regularity of $F$ and $\bar{C}(\epsilon)$, we obtain that
   \[ \bar{C}_I(\epsilon) = \frac{F_I(\bar{C}_R(\epsilon), 0, \epsilon) + \epsilon Y_I(\bar{C}_R(\epsilon), 0)(U^+(0) - U^+(h_+)) + O(\epsilon)^2}{\partial_c F(U^+(h_+), 0, 0) + O(\epsilon)}. \]

The proof of statement (1) is completed.

Let $K := k^2$. To prove the statement (2), we notice that the proof for the case of $\partial_R F(U^+(h_+), 0, 0) < 0$ is similar as the proof in Lemma 4.2. It remains to consider the case of $\partial_R F(U^+(h_+), 0, 0) = 0$. By (4.2), there exist $0 < K_0 < 1$ and $\Gamma \in \mathcal{C}^\infty([-K_0, K_0], \mathbb{R})$ such that
   \[ F(U^+(0), k, \Gamma(K)) = 0, \quad \Gamma(0) = 0. \]

Moreover, by a direct computation, $\partial_R k \Gamma(0) = -\partial_R F(U^+(h_+), 0, 0) > 0$. Let $k_0 := \sqrt{K_0}$. For any $k \in (0, k_0)$, we have $\Gamma(K) > 0$ and $\Gamma'(K) > 0$. Let $\epsilon_0 := \sup_{(0, k_0)} \Gamma(k)$, $K(\epsilon) := \Gamma^{-1}(\epsilon) > 0$, and $k(\epsilon) := \sqrt{K(\epsilon)}$.

Then for each $\epsilon \in (0, \epsilon_0),$
   \[ F(U^+(h_+), k(\epsilon), \epsilon) = 0, \quad \partial_\epsilon F(U^+(h_+), k(\epsilon), \epsilon) + \epsilon Y_I(\bar{C}_R(\epsilon), 0)(U^+(0) - U^+(h_+)) + O(\epsilon) = 0. \]

Moreover, we use (3.15b) to compute that
   \[ \partial_R F(U^+(h_+), k(\epsilon), \epsilon) = \partial_\epsilon F(U^+(h_+), k(\epsilon), \epsilon) - \epsilon \partial_R Y_I(U^+(0) - U^+(h_+)) + O(\epsilon) < 0. \]
The rest of the proof is similar as the proof for Lemma 4.2(a). \hfill \Box

Using a similar strategy, we can also prove the following result for small $\epsilon$ if the one fluid free boundary problem has a neutral mode at $(U^+(0), k_0)$ with $k_0 > 0$ or $(U^+(0), 0)$.

**Lemma 4.4.** Suppose that $U^\pm$ satisfy (1.3), $(U^\pm)' > 0$, $U^-(0) < U^+(0)$, $(U^\pm)'' \neq 0$, and $F(U^+(0), k_0, 0) = 0$ for some $k_0 > 0$. Then the following hold.

1. Suppose that $\partial_k F(U^+(0), k_0, 0) \neq 0$. Then there exist $\epsilon_0 > 0$ such that for each $\epsilon \in (0, \epsilon_0)$, there exist $\delta_0 > 0$, $k(\epsilon) \in C^1,\alpha([0, \epsilon_0] \times \mathbb{R}^+)$, and $C^+(k') \in C^1,\alpha([k(\epsilon) - \delta_0, k(\epsilon) + \delta_0])$ for any $\alpha \in [0, 1)$ such that
   \[ F(U^+(0), \pm k(\epsilon), \epsilon) = 0, \quad k(0) = k_0. \]
   Let $\rho := \min\{U^+(h^+_{1,2}) - U^+(0), U^+(0) - U^-(0)\}$. For $c_1 \in (0, \rho]$ and $|c_R - U^+(0)| \leq \rho$, $C^+(k)$ is the only root of $F(\cdot, k, \epsilon) = 0$ with $|k| \in [k(\epsilon) - \delta_0, k(\epsilon) + \delta_0]$. Moreover, the following hold.
   (a) If $\partial_k F(U^+(0), k_0, 0) < 0$, then
      (i) $k'(\epsilon) > 0$ and $k(\epsilon) < k_0$;
      (ii) $C^+(k) < U^+(0)$ for any $|k| \in [k(\epsilon) - \delta_0, k(\epsilon)]$ and
           \[ (U^+)''C^+_1(k) > 0, \quad C^+_R(k) > U^+(0), \quad \forall |k| \in (k(\epsilon), k(\epsilon) + \delta_0). \]
   (b) If $\partial_k F(U^+(0), k_0, 0) > 0$, then
      (i) $k'(\epsilon) < 0$ and $k(\epsilon) > k_0$;
      (ii) $C^+(k) < U^+(0)$ for any $|k| \in (k(\epsilon), k(\epsilon) + \delta_0]$ and
           \[ (U^+)''C^+_1(k) < 0, \quad C^+_R(k) > U^+(0), \quad \forall |k| \in [k(\epsilon) - \delta_0, k(\epsilon)). \]

2. Suppose that $\partial_k F(U^+(0), k_0, 0) = 0$. Then there exist $\epsilon_1 > 0$ such that for each $\epsilon \in (0, \epsilon_1)$, there exist $k_{2,1}(\epsilon) > k_0 > k_{1,2}(\epsilon) \geq 0$, $\delta > 0$, and $C^1,\alpha$ functions $C_{1,2} : [k_{1,2}(\epsilon) - \delta, k_{1,2}(\epsilon) + \delta] \to \mathbb{C}$ for any $\alpha \in [0, 1)$ such that
   \[ F(U^+(0), \pm k_{1,2}(\epsilon), \epsilon) = 0, \quad C^+_1(\pm k_{1,2}(\epsilon)) = U^+(0), \]
   \[ F(C^+_1(\epsilon), \pm k(\epsilon), \epsilon) = 0, \quad \forall |k| \in (k_{1,2}(\epsilon) - \delta, k_{1,2}(\epsilon) + \delta]. \]
   In addition, $C^+_1(k)$ satisfy that
   \[ C^+_1(k) < U^+(0), \quad \forall |k| \in (k_1(e), k_1(e) + \delta), \quad C^+_1(k) < U^+(0), \quad \forall |k| \in [k_2(e) - \delta, k_2(e)], \]
   \[ C^+_R(k) > U^+(0), \quad (U^+)''C^+_1(k) < 0, \quad \forall |k| \in (k_1(e) - \delta, k_1(e)), \]
   \[ C^+_R(k) > U^+(0), \quad (U^+)''C^+_1(k) < 0, \quad \forall |k| \in (k_2(e), k_2(e) + \delta). \]
   Let $\rho := \min\{U^+(h^+_{1,2}) - U^+(0), U^+(0) - U^-(0)\}$. Then for $|c_R - U^+(0)| < \rho$ and $c_1 \in (0, \rho]$, $C^+_1(k)$ is the unique root of $F(\cdot, k, \epsilon) = 0$ with $|k| \in (k_1(e) - \delta, k_1(e) + \delta]$ and $C^+_1(k)$ is the unique root of $F(\cdot, k, \epsilon) = 0$ with $|k| \in [k_2(e) - \delta, k_2(e) + \delta]$.

**Lemma 4.5.** Suppose that $U^\pm$ satisfy (1.3), $U^-(0) < U^+(0)$, $(U^\pm)' > 0$, $(U^\pm)'' < 0$, and $F(U^+(0), 0, 0) = 0$. Then the following hold.

1. There exist $\epsilon_0 > 0$ and $C^1,\alpha$ function $\tilde{C}(\epsilon) : [0, \epsilon_0] \to \mathbb{C}$ for any $\alpha \in [0, 1)$ such that
   \[ F(\tilde{C}(\epsilon), 0, \epsilon) = 0, \quad \tilde{C}(0) = U^+(0), \]
   \[ \tilde{C}(\epsilon) \in (U^-(0), U^+(0)), \quad \forall \epsilon \in (0, \epsilon_0). \]

2. Let $K := k^2$. If $\partial_k F(U^+(0), 0, 0) \leq 0$, then there exist $\epsilon_1 > 0$ and smooth function $k(\epsilon) : [0, \epsilon_1] \to \mathbb{R}$ such that for each $\epsilon \in (0, \epsilon_1)$,
   \[ F(U^+(0), \pm k(\epsilon), \epsilon) = 0, \quad k(\epsilon) > 0, \quad \text{and} \quad k(0) = 0. \]
   In addition, there exist $\delta_0 > 0$ and $C^{1,\beta}$ function $C^+(k) : [k(\epsilon) - \delta_0, k(\epsilon) + \delta_0] \to \mathbb{C}$ for any $\beta \in [0, 1)$ such that
   \[ F(C^+(k), \pm k, \epsilon) = 0, \quad C^+(k) = U^+(0), \quad C^+(k) < U^+(0), \quad \forall |k| \in [k(\epsilon) - \delta_0, k(\epsilon)]. \]
can be extended to be an analytic function for all 
\[ k \in (k(\epsilon), k(\epsilon) + \delta_0), \] 

Let \( \rho := \min \{U^+(0) - U^-(0), U^+(h_0) - U^+(0)\} \). For \( c_1 \in [0, \rho), |c_R - U^+(0)| < \rho, C^+(k) \) is the

In the following, we prove Theorem 1.4.

Proof of Theorem 1.4. When \( k = 0, \epsilon = 0 \) and \( c > U^-(0) \), \( y^- \) is in the form of (3.32). Then \( F(c, 0, 0) = 0 \) corresponds to

\[ (\int_{-h_-}^0 \frac{1}{(U - c)^2} dx_2)^{-1} = g. \]

Hence, \( c_0 \) is the only root of \( F(\epsilon, 0, 0) = 0 \) in \( \{c_R > U^-(0)\} \).

We first consider the case of \( c_0 > U^+(h_+). \) By (3.11),

\[ F(c, 0, 0) < F(U^+(h_+), 0, 0) < F(c_0, 0, 0) = 0, \forall c \in (U^-(0), U^+(h_+)). \]

There exists \( 0 < c_0 < 1 \) such that for each \( \epsilon \in (0, c_0), |F(c, 0, \epsilon)| > 0 \) where \( \epsilon \in \{U^+(0), U^+(h_+)\}. \) Let \( m(c) \) be defined in (3.35). Then (3.36) holds for small \( \epsilon. \) Let \( c^+(k) \) be the

Next, we shall prove statement (1b). Let \( K := k^2. \) Suppose that \( c_0 \in \{U^+(0), U^+(h_+)\}. \) Let \( c_0^+(k) \) be the

Using (3.42) and the assumption (1.14), we obtain that

\[ \partial_K F(U^+(h_+), k_0, 0) < \partial_K F(U^+(h_+), 0, 0) = (U^-(0) - U^+(h_+)) \partial_K Y^-(U^+(h_+), 0) - \frac{\sigma}{\rho}. \]

This implies that

\[ \partial_K F(U^+(h_+), k_0, 0) < 0. \]

For \( \epsilon = 0, \) we use (3.11) to obtain that

\[ (c_0^+)'(k_0) = -\frac{2k\partial_K F(U^+(h_+), k_0, 0)}{\partial_{\epsilon} F(U^+(h_+), k_0, 0)} > 0. \]

Since \( c_0^+(k) \) is analytic in \( k \in \mathbb{R}, \) this implies that \( k_0 \) is the unique root of \( F(U^+(h_+), \cdot, 0) = 0. \) By Lemma 4.2, there exist \( \delta_0 > 0, k(\epsilon) \in C^{1, \beta}([0, \epsilon_1], \mathbb{R}^+), \) and \( C^+(k) \in C^{1, \alpha}([k(\epsilon) - \delta_0, k(\epsilon) + \delta_0], \mathbb{C}) \) for any \( \alpha, \beta \in [0, 1) \) such that \( k(\epsilon) > k_0, F(U^+(h_+), k(\epsilon), \epsilon) = 0, F(C^+(k), k, \epsilon) = 0, \) and \( C^+(k(\epsilon)) = U^+(h_+). \) Since \( (U^+)^{''} < 0, C_0^+(k) > 0, C_0^+(k) < U^+(h_+) \) for any \( k \in [k(\epsilon) - \delta_0, k(\epsilon)] \) and \( C^+(k) > U^+(h_+) \) for any \( k \in (k(\epsilon), k(\epsilon) + \delta_0). \) By Remark 3.1, \( c^+(k) \) obtained in Lemma 3.2 can be extended to be an analytic function for all \( |k| > k(\epsilon) \) and \( C^+(k) \) for any \( \alpha \in [0, 1) \) in \( k \geq k(\epsilon) \) such that \( F(c^+(k), k, \epsilon) = 0. \) The uniqueness of solutions by the Implicit Function Theorem ensures that \( C^+(k) = c^+(k) \) for any \( k \in (k(\epsilon), k(\epsilon) + \delta_0). \) Moreover, there exists an open

By the smallness of \( \epsilon_1, (4.6) \) holds. Then (3.12) implies that \( k(\epsilon) \) is the unique root of \( F(U^+(h_+), \cdot, \epsilon) = 0 \) for all \( k \in \mathbb{R}. \) We use (3.42), the assumption (1.14), \( (U^-)^{''} > 0, \) and \( U^-(0) < U^+(0) \) to compute that

\[ \partial_K F(c, 0, 0) = (U^-(0) - c)^2 \partial_K Y^-(c, 0) - \frac{\sigma}{\rho}. \]

\[ \leq \int_{-h_-}^{0} (U^-(x_2) - c)^2 dx_2 - \frac{\sigma}{\rho} \]

\[ \leq \int_{-h_-}^{0} (U^-(x_2) - U^+(h_+))^2 dx_2 - \frac{\sigma}{\rho} < 0, \forall c \in (U^-(0), U^+(h_+)). \]
Hence, by (3.12), for any \( k \in \mathbb{R} \) and \( c \in (U^-(0), U^+(h_+)) \), \( \partial_R F(c, k, 0) < \partial_R F(c, 0, 0) < 0 \). Since \( 0 < \epsilon_1 < 1 \), for any \( c \in [0, \epsilon_1) \), \( \partial_R F(c, k, \epsilon_1) < 0 \) for all \( k \in \mathbb{R} \) and \( c \in (U^-(0), U^+(h_+)) \). Since \( F(c, k, \epsilon) = F(c, k, 0) + \epsilon g \) and \( c_0 > U^+(0) \), using the smallness of \( \epsilon_1 \) and (4.7), we obtain that
\[
F(c, k, \epsilon) < 0, \quad \forall \epsilon \in [0, \epsilon_1), \quad \forall k \in \mathbb{R}, \quad \forall c \in (U^-(0), U^+(0)].
\]
Since \( F_I(c, k, \epsilon) \neq 0 \) for any \( c \in U^+(0, h_+) \), \( F(c, k, \epsilon) \neq 0 \) for any \( c \in U^+(0, h_+) \), \( |k| \leq k(\epsilon) - \frac{\delta}{\epsilon_1} \). Hence, there exists an open set \( B_1 \) satisfying that \( U^+(0, h_+) \subset B_1 \subset \mathbb{C} \) and \( B_1 \cap B_0 \neq \emptyset \) such that \( F(c, k, \epsilon) \neq 0 \) for any \( c \in B_1 \) and \( |k| \leq k(\epsilon) - \frac{\delta}{\epsilon_1} \). Because of (4.9) and Lemma 4.1, there exists \( \gamma > 0 \) such that \( |F(c, k, \epsilon)| > 0 \) for any \( c \in B_2 := \{ U^+(0) - \gamma \leq c_R \leq U^+(0) + \gamma \} \) and \( k \in \mathbb{R} \). The smallness of \( \epsilon_1 \) implies that \( |F(c, k, \epsilon)| > 0 \) for \( c \in B_2 \) and \( k \in \mathbb{R} \). Let \( B_3 := [U^+(0), U^+(h_+)] \times [0, U^+(h_+) - U^-(0)] \subset \mathbb{C} \). Consider \( \Omega := B_3 \setminus (B_0 \cup B_1 \cup B_2) \). Thanks to Theorem 1.1, we obtain that \( F(c, k, \epsilon) \neq 0 \) for any \( c \in \partial \Omega \) and \( |k| \leq k(\epsilon) - \frac{\delta}{\epsilon_1} \). We consider \( n(k(\epsilon)) \) which is defined in (3.46). Lemma 4.2 implies that \( n(k(\epsilon)) = 1 \). Since \( n(k) \) is continuous for \( |k| \leq k(\epsilon) - \frac{\delta}{\epsilon_1} \), \( n(k) \equiv 1 \) for \( |k| \leq k(\epsilon) - \frac{\delta}{\epsilon_1} \). Lemma 4.2 shows that \( \mathcal{C}^+(k) \) is the unique zero of \( F(\cdot, k, \epsilon) \) for \( |k| \in [k(\epsilon) - \delta_0, k(\epsilon) + \delta_0] \). Hence, \( n(k) \equiv 1 \) for all \( |k| \leq k(\epsilon) \). This implies that \( \mathcal{C}^+(k) \) can be extended to be a \( C^{1,\alpha} \) function (for any \( \alpha \in [0, 1) \)) for all \( |k| \leq k(\epsilon) \) and analytic except at \( k = k(\epsilon) \). Let \( \mathcal{C}^+(k) = \mathcal{C}^+(k) \) for \( |k| < k(\epsilon) \). Statement (1b) is proved. The proof for statement (1c) is similar.

To prove statement (2), let
\[
g_{\#} := \max\{ F(c, k, 0) + g \}.
\]
Then statement (2a) directly follows from Theorem 1.1 in [11] and a similar proof as the one for Statement (1a). The proof for Statement (2b) is similar as the one for Statement (1b).

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