Singular sources in Demiański–Newman spacetimes

V S Manko\(^1\), J Martín\(^2\) and E Ruiz\(^2\)

\(^1\) Departamento de Física, Centro de Investigación y de Estudios Avanzados del IPN, AP 14-740, 07000 México DF, Mexico
\(^2\) Departamento de Física Fundamental, Universidad de Salamanca, 37008 Salamanca, Spain

Received 28 February 2006
Published 12 June 2006
Online at stacks.iop.org/CQG/23/4473

Abstract

The analysis of singular regions in the NUT solutions carried out in the recent paper (Manko and Ruiz 2005 Class. Quantum Grav. 22 3555) is now extended to Demiański–Newman (DN) vacuum and electrovacuum spacetimes. We show that the effect which produces the NUT parameter in a more general situation remains essentially the same as in the purely NUT solutions: it introduces the semi-infinite singularities of infinite angular momenta and positive or negative masses depending on the interrelations between the parameters; the presence of the electromagnetic field endows the singularities additionally with electric and magnetic charges. The exact formulae describing the mass, charges and angular momentum distributions in the Demiański–Newman solutions are obtained, and concise general expressions

\[ P_n = (m + i\nu)(i\alpha)^n, \quad Q_n = (q + ib)(i\alpha)^n \]

for the entire set of the corresponding Beig–Simon multipole moments are derived. These moments correspond to a unique choice of the integration constant in the expression of the metric function \(\omega\) which is different from the original choice made by Demiański and Newman.

PACS number: 04.20.Jb

1. Introduction

Demiański–Newman (DN) stationary axisymmetric spacetimes \[1\] are (i) the Kerr solution \[2\] endowed with the NUT parameter \[3\] (the pure vacuum case) and (ii) the Kerr–Newman solution \[4\] endowed with the NUT parameter and magnetic charge (the case of electrovacuum). The canonical Papapetrou form of the general DN class was obtained in \[5\] as the simplest \(N = 1\) specialization of the extended multi-soliton electrovacuum metric \[6\].

The presence of the non-vanishing NUT parameter makes the DN spacetimes asymptotically non-flat. This complicates the analysis of the respective mass and angular momentum distributions since, as was shown in \[7\], the singular semi-infinite regions developed by the NUT parameter have in general infinite angular momenta and non-zero masses. Moreover, the presence of the electromagnetic field requires additional consideration.
of the distributions of electric and magnetic charges in these singular regions. As an interesting aspect of the analysis of DN solutions one may consider establishing the correspondence between the multipole moments of the asymptotically NUT spacetimes calculated on a 3-manifold of trajectories of the timelike Killing vector with the aid of Simon’s procedure [8] and the physical characteristics of the entire four-dimensional metric.

The plan of our paper is as follows. In section 2 we consider the Ernst complex potentials and corresponding metric functions defining the DN spacetimes; here in particular we obtain concise formulae for the entire set of the Beig–Simon (BS) multipole moments associated with these solutions. Section 3 is devoted to the analysis of the mass and angular momentum distributions in the DN vacuum spacetimes, also known in the literature under the name Kerr–NUT metric. The more general case involving the NUT parameter and the electromagnetic field is analysed in section 4. Lastly, concluding remarks are given in section 5.

2. The Ernst potentials, metric functions and BS multipole moments of DN solutions

It is commonly known that the stationary axisymmetric electrovacuum solutions of the Einstein–Maxwell equations are most easily presentable and analysed if one uses, on the one hand, the Papapetrou canonical line element:

$$\text{d}s^2 = f^{-1}[e^{2\gamma}(\text{d}\rho^2 + \text{d}z^2) + \rho^2 \text{d}\varphi^2] - f(\text{d}t - \omega \text{d}\varphi)^2,$$

(1)

where the coefficients $f, \gamma, \omega$ are functions of the cylindrical coordinates $\rho, z$ only, and, on the other hand, the Ernst formalism developed in the papers [9, 10] according to which the knowledge of two complex potentials $E$ and $\Phi$ satisfying the Ernst equations [10]

$$\text{Re} E + \Phi \overline{\Phi} \Delta E = (\nabla E + 2 \overline{\Phi} \nabla \Phi) \nabla E,$$

$$\text{Re} E + \Phi \overline{\Phi} \Delta \Phi = (\nabla E + 2 \overline{\Phi} \nabla \Phi) \nabla \Phi$$

(2)

(a bar over a symbol denotes complex conjugation, $\nabla$ and $\Delta$ are the three-dimensional gradient and Laplace operators, respectively) is sufficient for the reconstruction of the metric coefficients entering (1), together with the electric and magnetic components of the electromagnetic 4-potential.

Demiański and Newman originally derived their solutions by employing some specific complex coordinate transformations not preserving Papapetrou’s form of the line element; the Ernst potentials defining the general class of DN metrics were obtained later in [5] using the extended multi-soliton electrovacuum solution [6] constructed with the aid of Sibgatullin’s method [11, 12]; these potentials have the form

$$E = \frac{\kappa x - m - i(ay + \nu)}{\kappa x + m - i(ay - \nu)}, \quad \Phi = \frac{q + ib}{\kappa x + m - i(ay - \nu)},$$

$$x = \frac{1}{2\kappa}(r_+ + r_-), \quad y = \frac{1}{2\kappa}(r_+ - r_-),$$

$$r_\pm = \sqrt{\rho^2 + (z \pm \kappa)^2}, \quad \kappa = \sqrt{m^2 + v^2 - a^2 - q^2 - b^2},$$

(3)

where the arbitrary real parameters $m, v, a, q, b$ can be associated, respectively, with the mass, gravomagnetic monopole (NUT parameter), angular momentum per unit mass, electric and magnetic charges. In the absence of the electromagnetic field ($q = b = 0 \Leftrightarrow \Phi = 0$), one obtains from (3) the expression for $E$ defining the vacuum DN spacetimes. When the NUT parameter $\nu$ is equal to zero, one arrives at the black-hole solutions considered by Carter [13].

The potentials (3) can be used for the calculation of the corresponding BS multipole moments [8, 14, 15]. For this purpose it is advantageous to employ the Hoenselaers–Perjés
procedure [16], recently rectified by Sotiriou and Apostolatos [17], which involves the axis expressions of the Ernst potentials. For \( \rho = 0, z > \text{Re}(\kappa) \) we have

\[
e(z) \equiv E(\rho = 0, z) = \frac{z - m - i(a + v)}{z + m - i(a - v)},
\]

\[
f(z) \equiv \Phi(\rho = 0, z) = \frac{q + i b}{z + m - i(a - v)}.
\]

Then, passing to the potentials \( \xi \) and \( \eta \) via the formulæ

\[
e(z) = \frac{1 - \xi}{1 + \xi}, \quad f(z) = \frac{\eta}{1 + \xi},
\]

and considering the expressions of the functions \( \tilde{\xi} = z \xi \) and \( \tilde{\eta} = z \eta \) in the limit \( z \to \infty \), we obtain

\[
\tilde{\xi} = \frac{(m + iv)z}{z - ia} = (m + iv) \sum_{n=0}^{\infty} (ia)^n z^{-n},
\]

\[
\tilde{\eta} = \frac{(q + ib)z}{z - ia} = (q + ib) \sum_{n=0}^{\infty} (ia)^n z^{-n}.
\]

Remarkably, it turns out that the coefficients in expansions (6) coincide exactly with the actual BS multipole moments because, as one trivially verifies, all quantities \( M_{ij}, Q_{ij}, S_{ij}, H_{ij} \) defined by formulæ (23) of [17] (they describe the deviations of the coefficients in (6) from the multipole moments) are equal to zero identically. Hence we arrive at the following elegant explicit expressions for the complex multipole moments:

\[
P_n = M_n + iJ_n = (m + iv)(ia)^n, \quad Q_n = E_n + iB_n = (q + ib)(ia)^n,
\]

where the real quantities \( M_n, J_n, E_n, B_n \) are, respectively, the mass, angular momentum, electric and magnetic multipole moments of the DN solution (3). Formulae (7) generalize in a very natural and straightforward way the Sotiriou–Apostolatos result [17] derived for the Kerr–Newman spacetime. The explicit expressions for \( M_n, J_n, E_n \) and \( B_n \) are readily obtainable from (7) by considering separately the even and odd moments:

\[
M_{2k} = (-1)^k ma^{2k}, \quad M_{2k+1} = (-1)^{k+1} va^{2k+1},
\]

\[
J_{2k} = (-1)^k va^{2k}, \quad J_{2k+1} = (-1)^k ma^{2k+1},
\]

\[
E_{2k} = (-1)^k qa^{2k}, \quad E_{2k+1} = (-1)^{k+1} ba^{2k+1},
\]

\[
B_{2k} = (-1)^k ba^{2k}, \quad B_{2k+1} = (-1)^{k+1} qa^{2k+1}, \quad k = 0, 1, \ldots.
\]

We find it appropriate to now make several comments on the multipole moments obtained. Firstly, one should remember that the BS moments, like the Geroch–Hansen multipole moments [18, 19], are only well defined for the asymptotically flat spacetimes. In the stationary axisymmetric case these are spacetimes whose metric coefficients in (1) have the following behaviour at spatial infinity:

\[
f \to 1, \quad \gamma \to 0, \quad \omega \to 0.
\]

Although the last condition on \( \omega \) does not enter explicitly into the procedure for the calculation of multipole moments because the latter are defined on a specific three-dimensional manifold, it is normally taken into account by demanding that the NUT parameter of the solution be equal to zero. As was already remarked by Simon [8], the presence of the NUT parameter and magnetic charge may cause the three- and four-dimensional descriptions of a stationary electrovac solution to be topologically incompatible.
Secondly, the multipole expansions define the corresponding Ernst potentials uniquely, which is a consequence of the mathematical theorems on analytic functions. On the other hand, the corresponding metric functions $\gamma$ and $\omega$ are constructed from the potentials $E$ and $\Phi$ up to two arbitrary real additive constants. The constant in $\gamma$ can be easily adjusted to have the desired behaviour of $\gamma$ even in the presence of the NUT parameter; besides, $\gamma$ is involved explicitly in the three-dimensional manifold which is used for the calculation of the multipole moments and hence can be always defined uniquely by imposing the asymptotic flatness condition. The unique choice of the integration constant in the expression of $\omega$ leading to $\omega \to 0$ at spatial infinity can be realized only in the absence of the NUT parameter. Therefore, an important non-trivial question arises of whether the multipole moments of the asymptotically NUT spacetimes obtainable from the Ernst potentials describe correctly the sources, and which is the precise choice of the integration constant in $\omega$ corresponding to those moments?

Thirdly, because it is clear that the analysis of sources in the asymptotically NUT solutions cannot be restricted to only the calculation of the multipole moments, one also needs a procedure of the evaluation of physical quantities which would be applicable to asymptotically non-flat spacetimes. The Komar integrals [20] then seem to be the best option as they had already proved to be very efficient for treating the black-hole properties in the external gravitational fields [21, 22]. In our recent paper [7] we have applied the Komar integrals to the analysis of the mass and angular momentum distributions in the NUT spacetimes; the main result of that paper consists in establishing the unique value of the integration constant entering the expression of $\omega$ at which the total angular momentum is a finite quantity and hence is in accordance with the multipole structure. One may expect that an analogous criterion of the choice of the integration constants can be worked out on the basis of Komar integrals for the DN spacetimes too.

Let us proceed with the description of the DN spacetimes in the Ernst picture by writing the corresponding metric functions of the electric $A_4$ and magnetic $A_3$ components of the electromagnetic 4-potential worked out in [5]:

\begin{align}
    f &= \frac{\kappa^2(x^2 - 1) - a^2(1 - y^2)}{(\kappa x + m)^2 + (ay - v)^2}, \\
    e^{2\gamma} &= \frac{\kappa^2(x^2 - 1) - a^2(1 - y^2)}{\kappa^2(x^2 - y^2)}, \\
    \omega &= 2\nu(y + C_1) - \frac{a(1 - y^2)[2(m\kappa x - vay + m^2 + v^2) - q^2 - b^2]}{\kappa^4(x^2 - 1) - a^2(1 - y^2)}, \\
    A_4 &= \frac{q(\kappa x + m) + b(v - ay)}{(\kappa x + m)^2 + (ay - v)^2}, \\
    A_3 &= b(C_2 - y) + \frac{(1 - y)(ay + a - 2v)[g(\kappa x + m) + b(v - ay)]}{(\kappa x + m)^2 + (ay - v)^2},
\end{align}

(10)

Here $C_1$ and $C_2$ are two arbitrary real (integration) constants whose concrete values corresponding to the multipole moments (7) cannot yet be pointed out. Formulae (10) describe the entire family of DN spacetimes, and it is worth mentioning that the particular choice of $C_1$ and $C_2$ made in [5] does not lead to the geometry defined by the multipoles (7).

It is convenient and instructive to consider pure vacuum DN spacetimes separately from electrovacuum ones (as was done in the original Demiański–Newman paper [1]): the simpler, vacuum case provides a solid basis for understanding the general structure of the DN sources, and we are passing to its consideration in the following section.
3. The mass and angular momentum distributions in the DN vacuum subclass

The vacuum subclass of DN spacetimes is normally known in the literature as the ‘genuine’ Demiański–Newman metric. It is also known under the name ‘(combined) Kerr–NUT metric’ given to it by Demiański and Newman themselves because it contains one additional (NUT) parameter compared to the Kerr solution [2]. Using formulae of the previous section let us write out the Ernst potential \( E \) (the other potential \( \Phi_1 \) is equal to zero identically) and corresponding metric functions determining these particular vacuum spacetimes:

\[
E = f + i\Omega = \frac{\kappa x - m - i(ay + v)}{\kappa x + m - i(ay - v)}, \quad \Omega = -\frac{2(\nu xx + may)}{(\kappa x + m)^2 + (\kappa y - v)^2},
\]

\[
f = \frac{\kappa^2(x^2 - 1) - a^2(1 - y^2)}{(\kappa x + m)^2 + (ay - v)^2}, \quad e^{2\nu} = \frac{\kappa^2(x^2 - 1) - a^2(1 - y^2)}{\kappa^2(x^2 - y^2)},
\]

\[
\omega = 2\nu(y + C) - \frac{2a(1 - y^2)(\nu xx - vay + m^2 + v^2)}{\kappa^2(x^2 - 1) - a^2(1 - y^2)},
\]

\[
x = \frac{1}{2\kappa}(r_+ + r_-), \quad y = \frac{1}{2\kappa}(r_+ - r_-), \quad r_\pm = \sqrt{\rho^2 + (z \pm \kappa)^2},
\]

\[
\kappa = \sqrt{m^2 + v^2 - a^2},
\]

where the real constant \( C_1 \) from (10) is now called \( C \), while the expression for \( \kappa \) does not contain the parameters \( q \) and \( b \) (cf formula (3)).

When \( C \neq \pm 1, v \neq 0 \), the geometries (11) are characterized by two semi-infinite singular sources located on the symmetry \( z \)-axis: \( \rho = 0, z > \text{Re}(\kappa) \) (the upper singularity) and \( \rho = 0, z < -\text{Re}(\kappa) \) (the lower singularity). In the particular cases \( C = \pm 1, v \neq 0 \), only one semi-infinite singularity is present, which is analogous to the pure NUT case [7]. If \( \kappa \) is a real non-zero quantity, i.e. \( m^2 + v^2 > a^2 \), the symmetry axis is divided into three parts: the upper region I (\( \kappa < z < \infty \)), the intermediate region II (\( |z| < \kappa \)) and the lower region III (\( -\infty < z < -\kappa \)), see figure 1(a). For \( a^2 > m^2 + v^2 \), \( \kappa \) becomes a pure imaginary quantity \( \kappa = iv(a^2 - m^2 - v^2) \), and only two different regions, I and III, will then be present on the \( z \)-axis (figure 1(b)), the cut between the points \( -\kappa \) and \( +\kappa \) representing a superextreme central object.
Since our major interest lies in the semi-infinite sources which are originated by the NUT parameter, in what follows we shall mainly concentrate on the subextreme case \( m^2 + \nu^2 > a^2 \), and we will pay less attention to the superextreme case which can be considered as a sort of degeneration of the sources structure (the region II of the symmetry axis disappears).

In the subextreme case the calculation of Komar quantities can be suitably carried out with the aid of Tomimatsu’s formulae obtained in [23]:

\[
M = -\frac{1}{2} \omega \Omega (z = z_2) - \Omega (z = z_1),
\]

\[
J = -\frac{1}{2} \omega (z_2 - z_1) - \frac{1}{8} \omega^2 \Omega (z = z_2) - \Omega (z = z_1),
\]

(12)

where \( M \) and \( J \) define, respectively, the mass and angular momentum of the part of the symmetry axis \([z_1, z_2]\) on which \( \omega \) takes the constant value.

The functions involved in formulae (12) are \( \omega \) and \( \Omega \) taken on the symmetry axis, and they have the following form there.

**Region I.** This part of the \( z \)-axis is determined by \( \rho = 0, z > \kappa \), or equivalently by \( x = z/\kappa, y = 1 \). Then, substituting these values of \( x \) and \( y \) into (11), we obtain

\[
\omega = 2\nu(1 + C), \quad \Omega = -\frac{2(\nu z + ma)}{(z + m)(z + m + (a + \nu)^2) - \nu \kappa (aC + \nu)}.
\]

(13)

**Region II.** The intermediate region which is associated with the central rotating body is defined by \( \rho = 0, |z| < \kappa \), or \( x = 1, y = z/\kappa \), thus providing us with the following values for \( \omega \) and \( \Omega \):

\[
\omega = \frac{2}{a} \left[ m(\kappa + m) + \nu(aC + \nu) \right], \quad \Omega = -\frac{2\kappa (ma \nu + \kappa^2 \nu)}{(az - \kappa \nu)^2 + \kappa^2(k + m)^2}.
\]

(14)

**Region III.** Here \( \rho = 0, z < -\kappa \), i.e. \( x = -z/\kappa, y = -1 \), and we have

\[
\omega = 2\nu(C - 1), \quad \Omega = \frac{2(\nu z + ma)}{(z - m)^2 + (a + \nu)^2}.
\]

(15)

We first apply formulae (12) to (13)–(15) for getting the masses \( M_1, M_2, M_3 \) of the corresponding regions I, II, III; the result is

\[
M_1 = \frac{\nu(1 + C)(a + \nu)}{2(k + m)}, \quad M_2 = m + \frac{\nu(aC + \nu)}{k + m}, \quad M_3 = \frac{\nu(1 - C)(a - \nu)}{2(k + m)}.
\]

(16)

It is easy to verify that the total mass \( M = M_1 + M_2 + M_3 \) is equal to \( m \).

From (12) it follows that the entire angular momenta of the semi-infinite singular regions are infinitely large quantities since in the case of region I one has to put \( z_2 = +\infty \), and in the case of region III \( z_1 = -\infty \). Therefore, as in the paper [7], we shall calculate the angular momentum \( J_1(z_0) \) of the part \( \kappa < z \leq z_0 \) of the upper singular region, the total angular momentum \( J_2 \) of the central body \((|z| \leq \kappa)\) and the part \(-z_0 \leq z < -\kappa \) of the lower
Semi-infinite singularity. Formulae (12) then give
\begin{align}
J_1(z_0) &= -\frac{v(1 + C)}{2} \left[ \frac{z_0 - \kappa}{\kappa + m} + \frac{v(1 + C)(a + v)}{\kappa + m} \right] + \frac{v^2(1 + C)^2(vz_0 + ma)}{(z_0 + m)^2 + (a - v)^2}, \\
J_2 &= (a + Cv) \left[ \frac{m}{(z_0 + m)^2 + (a + v)^2} \right], \\
J_3(z_0) &= \frac{v(1 - C)}{2} \left[ \frac{z_0 - \kappa}{\kappa + m} - \frac{v(1 - C)(a - v)}{\kappa + m} \right] - \frac{v^2(1 - C)^2(vz_0 - ma)}{(z_0 + m)^2 + (a + v)^2}. \tag{17}
\end{align}

The total angular momentum of the part $|z| \leq z_0$ of the $z$-axis thus takes the form
\begin{align}
J(z_0) &= -Cv(z_0 - 2m) + ma + \frac{v^2(1 + C)^2(vz_0 + ma)}{(z_0 + m)^2 + (a - v)^2} - \frac{v^2(1 - C)^2(vz_0 - ma)}{(z_0 + m)^2 + (a + v)^2}, \tag{18}
\end{align}
and one can see that the only choice of the constant $C$ leading to the finite value of $J$ in the limit $z_0 \to \infty$ is $C = 0$. In particular, in the original DN solution characterized by $C = -1$ the only (lower) semi-infinite singularity carries an infinite angular momentum.

In the general case, the aggregate mass of two semi-infinite singular sources is equal to
\begin{align}
M_{\text{agg}} = M_1 + M_3 &= -\frac{v(a + v)}{\kappa + m}, \tag{19}
\end{align}
and it can assume either positive or negative values. When $C = 0$, $M_{\text{agg}}$ is a positive quantity for all non-zero $m$, $v$ and $a$. However, even in the cases when $M_{\text{agg}}$ is positively defined, the negative mass is present in the singular semi-infinite sources. Let us illustrate this by taking as an example the original DN vacuum solution ($C = -1$). In this case $M_{\text{agg}} = M_3 = v(a - v)/(\kappa + m)$, so that $M_{\text{agg}}$ is positive for instance when $v > 0$, $v < a < m$.

In figure 2 we have plotted the mass $M_3(z_0)$ of the part $-z_0 < z < -\kappa$ of the semi-infinite singularity, namely
\begin{align}
M_3(z_0) &= \frac{v(a - v)}{\kappa + m} + \frac{2v(vz_0 - ma)}{(z_0 + m)^2 + (a + v)^2}. \tag{20}
\end{align}

There is a misprint in formula (15) of [7]: the first expression in parentheses on the right-hand side does not contain $a$ and should be read as $(z_0 - 2m)$. **Figure 2.** The mass distribution in the semi-infinite singularity of the subextreme DN solution. The particular choice of the parameters is $m = 5$, $v = 3$, $a = 4$, $C = -1$. 
Figure 3. The mass distribution in the semi-infinite singularity of the subextreme DN solution, the total mass of the singularity being equal to zero.

Figure 4. Dependence of the mass distribution in the original DN semi-infinite singularity on the angular momentum of the central body.

As the function of $z_0$, for the particular choice of the parameters $m = 5$, $\nu = 3$, $a = 4$. It follows that $M_3(z_0)$ is a monotonously increasing function on the interval $4.243 < z_0 < 20.272$, taking its maximal value $0.681$ at $z_0 = 20.272$, and it is a monotonously decreasing function on the interval $20.272 < z_0 < \infty$, reaching asymptotically the value $0.325$, which means that the latter interval is entirely composed of the negative mass.

Moreover, even in the special case $\nu = a$ when the total mass of the DN semi-infinite singularity is equal to zero, the qualitative picture of the mass distribution in the singularity is similar to that shown in figure 2: the interval with the positive mass is followed by the distribution of the negative mass. This is illustrated in figure 3 for the particular choice of the parameters $m = 5$, $\nu = 3$, $a = 3$, the total mass of the singularity reaching asymptotically the value $M_3 = 0$.

When the central body is a superextreme object, i.e. in the case $a^2 > m^2 + \nu^2$, the structure of singularities preserves the main characteristic features of the subextreme case: the aggregate angular momentum of the singularities assumes a finite value only when the constant $C$ is equal to zero (for the non-vanishing $m$, $\nu$ and $a$), and the semi-infinite singular regions involve negative masses. In figure 4 we have plotted the mass distribution in the singularity of the original DN vacuum solution ($C = -1$) for two particular parameter sets which have common values $m = 2$, $\nu = 1$ and differ in the value of $a$: in the first set $a = 4$ (figure 4(a)), and in the second $a = -4$ (figure 4(b)).
Singular sources in Demia´nski–Newman spacetimes

Since only the case \( C = 0 \) is consistent with the multipole moments defined by the Ernst potential (11), below we shall give the general expressions for the masses and angular momenta exclusively for this superextreme case, subindex 2 referring to the central object:

\[
M_1 = -\frac{ma}{m^2 + (a - \nu)^2}, \quad M_2 = \frac{m(m^2 + \nu^2 + a^2)^2}{(m^2 + \nu^2 + a^2)^2 - 4\alpha^2\nu^2}, \quad M_3 = \frac{ma}{m^2 + (a + \nu)^2},
\]

\[
J_1(z_0) = -\frac{vz_0}{2} - \frac{ma^2}{m^2 + (a - \nu)^2} + \frac{v^2(vz_0 + ma)}{(z_0 + m)^2 + (a - \nu)^2},
\]

\[
J_2 = ma + \frac{2mav^2(m^2 + \nu^2 + a^2)}{(m^2 + \nu^2 + a^2)^2 - 4\alpha^2\nu^2},
\]

\[
J_3(z_0) = \frac{vz_0}{2} - \frac{ma^2}{m^2 + (a + \nu)^2} - \frac{v^2(vz_0 - ma)}{(z_0 + m)^2 + (a + \nu)^2}.
\]

As before, \( M_1 \) and \( M_3 \) are the total masses of the upper and lower singular regions, \( M_2 \) and \( J_2 \) are the mass and angular momentum of the central object, while \( J_1(z_0) \) and \( J_3(z_0) \) denote angular momenta of the parts \( 0 < z \leq z_0 \) and \( -z_0 \leq z < 0 \) of the singular regions, respectively. Note that since formulae (21) have been derived under the supposition \( a^2 > m^2 + \nu^2 \), the limit \( a \to 0 \) in (21) is impossible, as \( m \) and \( \nu \) are real quantities by definition.

It is easy to see from (21) that the aggregate mass of two singularities always takes a negative value for any positive \( m \) and \( \nu \neq 0 \):

\[
M_1 + M_3 = -\frac{4ma^2\nu^2}{(m^2 + \nu^2 + a^2)^2 - 4\alpha^2\nu^2}.
\]

In figure 5 we have plotted the mass distributions in the semi-infinite singularities for a particular choice of the parameters \( m = 1, \nu = 2, a = 3 \).

4. The electrovacuum case

In the presence of the electric and magnetic charges the general picture of the singular regions due to the NUT parameter is similar to that of the vacuum case. However, the singularities will now be electrically and magnetically charged, and their masses and angular momenta will have both the gravitational and electromagnetic contributions. Since the properties of the semi-infinite singularities are practically independent of the central body, we shall restrict our consideration to only the subextreme case defined by the inequality \( m^2 + \nu^2 > a^2 + q^2 + b^2 \).

The symmetry axis is then divided into three regions, as in figure 1: \( \kappa < z < \infty \) (region I),
$|z| < \kappa$ (region II) and $-\kappa < z < -\infty$ (region III). The functions $\omega, \Omega \equiv \text{Im} \, \mathcal{E}, \Phi$ and $A_3$ which are needed for the calculation of the Komar quantities show the following behaviour in the regions I, II, III:

**Region I.** For $\rho = 0, \kappa < z < \infty$ we obtain from (3) and (10):

\[
\begin{align*}
\omega_{(1)} &= 2\nu(C_1 + 1), & \Omega_{(1)} &= -\frac{2(vz + ma)}{(z + m)^2 + (a - \nu)^2}, \\
\Phi_{(1)} &= -\frac{q + ib}{z + m - i(a - \nu)}, & A_{3(1)} &= -b(C_2 + 1),
\end{align*}
\]

where subindex 1 in parentheses (and subindices 2, 3 below) signifies that the respective function is calculated on the indicated part of the symmetry axis.

**Region II.** When $\rho = 0, |z| < \kappa$, we have

\[
\begin{align*}
\omega_{(2)} &= 2C_1\nu + \frac{(\kappa + m)^2 + a^2 + \nu^2}{a}, & \Omega_{(2)} &= -\frac{2\kappa (maz + \kappa^2 \nu)}{(az - \nu)^2 + \kappa^2 (\kappa + m)^2}, \\
\Phi_{(2)} &= -\frac{\kappa (b - iq)}{az - \kappa v + i(\kappa + m)}, & A_{3(2)} &= -C_2 b - \frac{q(\kappa + m) + nb}{a} \left[ \frac{b(az - \kappa v) - \kappa q(\kappa + m)((\kappa + m)^2 + (a - \nu)^2)}{a((az - \kappa v)^2 + \kappa^2 (\kappa + m)^2)} \right].
\end{align*}
\]

**Region III.** For $\rho = 0, \kappa < z < \infty$ one gets

\[
\begin{align*}
\omega_{(3)} &= 2\nu(C_1 - 1), & \Omega_{(3)} &= \frac{2(vz + ma)}{(z - m)^2 + (a + \nu)^2}, \\
\Phi_{(3)} &= -\frac{q + ib}{z - m - i(a + \nu)}, & A_{3(3)} &= -b(C_2 - 1) + \frac{4v[q(z - m) - b(a + \nu)]}{(z - m)^2 + (a + \nu)^2}.
\end{align*}
\]

The calculation of the masses $M_i$, electric $Q_i$ and magnetic $B_i$ charges, as well as angular momenta $J_i$ of the regions I–III can be carried out with the aid of Tomimatsu’s formulae derived in the paper [24]:

\[
M_i = M_i^G + M_i^E
\]

\[
\begin{align*}
M_i &= \frac{1}{4} \int_{d_i}^{u_i} \left[ \omega_{(i)} \Omega_{(i), z} - 2\omega_{(i)} \text{Im}(\Phi_{(i)}(\Phi_{(i), z})) \right] dz \\
&\quad - \frac{1}{2} \int_{d_i}^{u_i} \omega_{(i)} \text{Im}(\Phi_{(i)}(\Phi_{(i), z})) dz \\
&\quad - \frac{1}{4} \omega_{(i)} \left[ \Omega_{(i)}(z = u_i) - \Omega_{(i)}(z = d_i) \right]; \\
Q_i &= \frac{1}{2} \omega_{(i)} \text{Im}[\Phi_{(i)}(z = u_i) - \Phi_{(i)}(z = d_i)], \\
B_i &= -\frac{1}{2} \omega_{(i)} \text{Re}[\Phi_{(i)}(z = u_i) - \Phi_{(i)}(z = d_i)], \\
J_i &= J_i^G + J_i^E = -\frac{1}{8} \int_{d_i}^{u_i} \omega_{(i)} [2 + \omega_{(i)} \Omega_{(i), z} - 2\omega_{(i)} \text{Im}(\Phi_{(i)}(\Phi_{(i), z}))] dz \\
&\quad + \frac{1}{2} \int_{d_i}^{u_i} \omega_{(i)} A_3 \text{Im}(\Phi_{(i), z}) dz,
\end{align*}
\]

where the superscripts $G$ and $E$ denote the decomposition of masses and angular momenta into the gravitational and electromagnetic components introduced by Tomimatsu following Carter’s paper [13].
The total masses of the regions I, II and III obtainable with the aid of the above formulae (26) in which one has to set \( u_1 = \infty, d_1 = u_2 = \kappa, d_2 = u_3 = -\kappa, d_3 = -\infty \), have the form

\[
M_1 = \frac{\nu(C_1 + 1)(\kappa v + ma)}{(\kappa + m)^2 + (a - \nu)^2},
\]

\[
M_2 = \left( m + \frac{2\nu(\kappa v + ma)}{(\kappa + m)^2 + (a - \nu)^2} \right) \left( 1 + \frac{2a\nu(C_1 - 1)}{(\kappa + m)^2 + (a + \nu)^2} \right),
\]

\[
M_3 = \frac{\nu(C_1 - 1)(\kappa v - ma)}{(\kappa + m)^2 + (a + \nu)^2},
\]

and it is easy to verify that the total mass \( M_{\text{tot}} \) of the solution is simply

\[
M_{\text{tot}} = \sum_{i=1}^{3} M_i = m
\]

for all values of the constant \( C_1 \). It is worth pointing out that the cases \( C_1 = -1 \) and \( C_1 = 1 \) correspond to vanishing of the upper and lower semi-infinite singularities, respectively. At the same time, for instance, the combination of the parameters \( \kappa v + ma = 0, C_1 \neq -1 \), which also leads to \( M_1 = 0 \), does not annihilate the upper singularity, but only reflects the fact that the total gravitational and electromagnetic contributions into the mass of this singularity are equal in absolute values and have opposite signs; at the same time, these contributions calculated for any finite interval of the singularity do not give zero.

The distribution of the electric charge is described by the formulae

\[
Q_1 = \frac{\nu(C_1 + 1)[b(\kappa + m) + q(a - \nu)]}{(\kappa + m)^2 + (a - \nu)^2},
\]

\[
Q_2 = \left( q + \frac{2\nu[b(\kappa + m) + q(a - \nu)]}{(\kappa + m)^2 + (a - \nu)^2} \right) \left( 1 + \frac{2a\nu(C_1 - 1)}{(\kappa + m)^2 + (a + \nu)^2} \right),
\]

\[
Q_3 = \frac{\nu(C_1 - 1)[b(\kappa + m) - q(a + \nu)]}{(\kappa + m)^2 + (a + \nu)^2},
\]

\[
Q_{\text{tot}} = \sum_{i=1}^{3} Q_i = q,
\]

whereas for the distribution of the magnetic charge we obtain

\[
B_1 = \frac{\nu(C_1 + 1)[q(\kappa + m) - b(a - \nu)]}{(\kappa + m)^2 + (a - \nu)^2},
\]

\[
B_2 = \left( b - \frac{2\nu[q(\kappa + m) - b(a - \nu)]}{(\kappa + m)^2 + (a - \nu)^2} \right) \left( 1 + \frac{2a\nu(C_1 - 1)}{(\kappa + m)^2 + (a + \nu)^2} \right),
\]

\[
B_3 = \frac{\nu(C_1 - 1)[q(\kappa + m) + b(a + \nu)]}{(\kappa + m)^2 + (a + \nu)^2},
\]

\[
B_{\text{tot}} = \sum_{i=1}^{3} B_i = b.
\]

One can see that the total charges too do not depend on the constant \( C_1 \).

The distribution of the angular momentum in the electrovac DN solution resemble qualitatively the distribution in the pure vacuum case, but the corresponding formulae turn out to have a very cumbersome form and so will be not given here (the reader can work them out straightforwardly from (26)). We only comment that the choice \( C_1 = 0 \) is obligatory for having a finite value of the total angular momentum, and once this choice is made, the other constant \( C_2 \) can be adjusted in such a way that the total angular momentum be equal to \( ma \), in accordance with the multipole moments (7).
5. Concluding remarks

Like in the NUT solution, the semi-infinite singularities in the combined Kerr–NUT and electrovac DN spacetimes carry infinite angular momenta, and the only possibility for them to have a finite total angular momentum is assigning zero value to the integration constant in the expression for the metric function \( \omega \), in which case the two singularities will possess oppositely oriented angular momenta cancelling out the infinities in sum. The total mass of the singularities in this special case assumes a negative value. In contradistinction to the purely NUT solution where the counter-rotating singularities and a static central body form a system which is antisymmetric with respect to the equatorial plane, in the DN vacuum and electrovac spacetimes the non-zero parameter \( a \) makes impossible such an additional equatorial symmetry. The analyses carried out in the present paper and in [7] show how careful one should be when trying to establish the multipole structure of spacetimes possessing the NUT parameter. At the same time, it is surprising how elegantly the additional parameters \( \nu \) and \( b \) of the electrovacuum DN spacetime generalize the known multipole expressions derived for the Kerr–Newman solution.

Acknowledgments

VSM would like to thank the Department of Fundamental Physics of the Salamanca University where a part of this work was done for its kind hospitality and financial support. This work was partially supported by project 45946-F from CONACyT of Mexico, and by Project BFM2003-02121 from MCyT of Spain.

References

[1] Demiański M and Newman E T 1966 Bull. Acad. Pol. Sci. Ser. Math. Astron. Phys. 14 653
[2] Kerr R P 1963 Phys. Rev. Lett. 11 237
[3] Newman E, Tamburino L and Unti T 1963 J. Math. Phys. 4 915
[4] Newman E, Couch E, Chinnapared K, Exton A, Prakash A and Torrence R 1965 J. Math. Phys. 6 918
[5] Aguilar-Sánchez J A, García A A and Manko V S 2001 Grav. Cosmol. 7 149
[6] Ruiz E, Manko V S and Martín J 1995 Phys. Rev. D 51 4192
[7] Manko V S and Ruiz E 2005 Class. Quantum Grav. 22 3555
[8] Simon W 1984 J. Math. Phys. 25 1035
[9] Ernst F J 1968 Phys. Rev. 167 1175
[10] Ernst F J 1968 Phys. Rev. 168 1415
[11] Sibgatullin N R 1991 Oscillations and Waves in Strong Gravitational and Electromagnetic Fields (Berlin: Springer)
[12] Manko V S and Sibgatullin N R 1993 Class. Quantum Grav. 10 1383
[13] Carter B 1973 Black Holes ed C DeWitt and B S DeWitt (New York: Gordon and Breach Science Publishers) p 57
[14] Beig R and Simon W 1981 Proc. R. Soc. A 376 333
[15] Simon W and Beig R 1983 J. Math. Phys. 24 1163
[16] Hoenselaers C and Perjés Z 1990 Class. Quantum Grav. 7 1819
[17] Sotiriou T P and Apostolatos T A 2004 Class. Quantum Grav. 21 5727
[18] Geroch R 1970 J. Math. Phys. 11 2580
[19] Hansen R O 1974 J. Math. Phys. 15 46
[20] Komar A 1959 Phys. Rev. 113 934
[21] Tomimatsu A 1984 Phys. Lett. A 103 374
[22] Bretón N, García A, Manko V S and Denisova T E 1998 Phys. Rev. D 57 3382
[23] Tomimatsu A 1983 Prog. Theor. Phys. 70 385
[24] Tomimatsu A 1984 Prog. Theor. Phys. 72 73