MECHANICS OF COSMIC RINGS

Brandon Carter

Département d’Astrophysique Relativiste et de Cosmologie, C.N.R.S, Observatoire de Paris, 92 195 Meudon, France.

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Abstract.
In a flat background, simple non-conducting string loops have no strictly stationary equilibrium states, but for cosmic string loops of superconducting kind such “vorton” states will exist, with rotating circular configurations appropriately describable as “cosmic rings” (which may conceivably be a significant contributor to the material content of the universe) whose equilibrium is obtained for zero angular velocity of retrograde characteristics. Such a “cosmic ring” is characterised by its mass $M$ and angular momentum $J$ say, corresponding at a microscopic level to two independent quantum numbers, namely charge number $C$ and a topological phase number $N$ whose product determines the angular momentum, $J = CN$, while their ratio determines the local intrinsic state of the string, which may be qualitatively classified as being of “electric” or “magnetic” type depending on whether $C/N$ is greater or less than a critical value dependent on the underlying theoretical model.

1. Introduction
The purpose of this work is to set up the elements of the theory of what we shall refer to as “cosmic rings” in a framework [1,2] that is general in the sense of being independent of the precise details of the underlying quantum field theory that may be postulated to give rise to the cosmic string model under consideration. We use the term “cosmic ring” to describe a circular equilibrium state of a local cosmic string loop in a flat background. Our present discussion is based on the supposition that gravitational and electromagnetic self-interaction are sufficiently weak to be neglected. It is of course well known that no such strictly stationary equilibrium states can exist for a simple Goto-Nambu type cosmic string loop. However, for a cosmic string loop of the superconducting kind [3] the possibility of centrifugally supported circular equilibrium states has been clearly recognised by Davis and Shellard [4], who have drawn attention to the interest of such rings as examples of “vorton” type semi-topological solitons, which may conceivably have been produced in considerable numbers in the early universe (perhaps even so copiously as to pose a hidden matter problem analogous to that of monopoles).

Davis and Shellard have plotted numerical results [4] for a number of relevant
quantities for such “cosmic ring” configurations on the basis of a particular field theoretical model of the kind commonly postulated as an underlying framework for the theory of superconducting strings as introduced by Witten [3]. Such models [5,6] can be considered at a macroscopic level as particular examples within the broad category of electromagnetic string models describable by the recently introduced covariant formalism [1,2] which provides a convenient foundation for a general derivation of the essential elements of the theory of cosmic ring states, making it possible to extend the work of Davis and Shellard, complementing their numerical results by general analytic results applicable to the entire class of conceivably relevant models.

In the discussion of Davis and Shellard [4], terms such as “static” and “stable” were used rather loosely for what in stricter terminology would be characterised as “stationary” and “equilibrium”. The question of strict global stability of the equilibrium states under consideration has not yet been dealt with rigorously, but the necessity at least of the standard local stability conditions [2] is evident, while a further condition that may be conjectured to be sufficient is derived below. The stationary rotating states we are considering here are not to be confused with the strictly static states whose conceivable existence has recently been the subject of discussion by Hindmarsh, Turok and coworkers [7,8], but whose existence is only possible for the restricted subclass of string models that allow vanishing tension, \( T = 0 \), which is necessary for equilibrium in the strictly static case. It is to be mentioned that the term “cosmic spring” has sometimes been employed in this latter context, entailing the implicit suggestion that such states might be locally stable with respect to small oscillations between states of small positive tension and compressed states of small negative tension. However, the introduction of the term “spring” in the present context is misleading except for loops that are so short compared with the microscopic thickness of the underlying quantum vortex that description as a “string” is no longer appropriate. For any loop sufficiently long compared with its microscopic thickness for description as a “string” to be reasonable, states of negative tension will inevitably be locally unstable [2] so that the intermediate “relaxed” states of zero tension, if allowed at all by the underlying field theory, would be at most marginally stable, and as such may appropriately be described not as “springs” but just as “loose strings”. In so far as it has been suggested that such static “loose string” states may conceivably be cosmologically important, the same applies a fortiori to the “ring” states under consideration in the present work, since the latter can exist for any superconducting string model, not just for a restricted subclass. It is to be noticed that their properties are not unsimilar to those recently postulated for the charges ultra-massive particles (“chumps”) that have recently been proposed as dark matter candidates [9].

In order to be effectively thin, so as to be appropriately qualified at a macroscopic level as a “string” (and hence for its axisymmetric “vorton” equilibrium states to be describable as “rings” in the technical sense used here), a vortex defect of the vacuum for a field theory with spontaneously broken symmetry should be of “local” rather than “global type”. For the superconducting string models in question [1], the macroscopic action (as obtained after integration over the microscopic cross section of the local vacuum defect region) has the form of an integral over a 2-dimensional world sheet (specifying the mean, macroscopic, localisation of the string) of an effective lagrangian

\[ L = \int_\Sigma \mathcal{L}_\text{eff} \]
function \( L \) that depends only on the pseudo-metric norm \( w \) of the gauge covariant derivative within the world sheet of a scalar phase field \( \phi \). It will be convenient to introduce an important auxiliary function \( K \) constructed from the lagrangian by differentiation according to the prescription

\[
K = -2dL/dw, \quad w = |D\phi|^2
\]  

(1.1)

and it will also be useful to construct from \( L \) the corresponding “dual” lagrangian function \( \tilde{L} \) in which the roles of space and time are interchanges [1], according to the prescription

\[
\tilde{L} = L + wK.
\]  

(1.2)

With our chosen sign convention for the spacetime pseudo-metric, a positive value of \( w \) characterises a “magnetic” string regime [1] for which the gauge covariant derivative is space-like, so that the string has a preferred rest frame density \( U \) and a tension \( T \) given by \( U = -L \) and \( T = -\tilde{L} \), with an associated stream number density \( \nu \) and an effective mass \( \mu \) given respectively by \( \nu^2 = w \) and \( \nu = K\mu \). On the other hand a negative value of \( w \) characterises an “electric” regime for which the covariant phase derivative is timelike, so that the corresponding expressions are \( T = -L \) and \( U = -\tilde{L} \) for the tension and energy density, with \( \mu^2 = -w \) and \( \nu = K\mu \) for the effective mass density and number density.

In both the “magnetic” and “electric” regimes the string 2-surface stress-energy tensor is given in terms of time-like and space-like unit vectors \( u^\mu \) and \( v^\mu \) of the intrinsically preferred orthonormal basis in the string 2-surface by an expression of the form

\[
T^{\mu\nu} = (U - T)u^\mu u^\nu - T\eta^{\mu\nu}, \quad \eta^{\mu\nu} = -u^\mu u^\nu + v^\mu v^\nu,
\]  

(1.3)

where \( \eta^{\mu\nu} \) is the “fundamental tensor” of the world sheet [1]. The expression (3) is thus valid everywhere except perhaps at a “transluminal boundary” locus on which the string passes through a degenerate intermediate regime characterised by \( T = U = -L = -\tilde{L} \), where the stress-energy tensor will take the form \( T^{\mu\nu} = l^\mu l^\nu - T\eta^{\mu\nu} \), for some appropriately normalised null eigenvector \( l^\mu \), the unit vectors \( u^\mu \) and \( v^\mu \) being indeterminate.

It is to be noticed that the physical relevance of the foregoing formalism (and hence of the results to be described below) is not restricted to strings of “cosmic” origin but applies also to the elastic string models that are appropriate for familiar terrestrial applications to ordinary ropes and wires (always in the limit when they can be considered to be sufficiently thin compared with the lengths involved), though in such cases the relevant scalar variable \( \phi \) turns up merely as an abstract auxiliary (Clebsch type) potential, lacking the interpretation as a topologically periodic quantum phase that applies in the cosmic case.

### 2. Circular string loops

Our purpose here is to consider the simplest case of stationary circular configurations characterised by a radius, \( r \) say, and an angular velocity \( \Omega \) say, of the preferred rest frame vector \( u^\mu \), subject to the subluminal rotation conditions \( 0 < \Omega^2 < r^{-2} \).
(the limits $\Omega^2 = 0$ and $\Omega^2 = r^{-2}$ correspond to the static limit and the transluminal limit respectively). The dually associated (superluminal rotation) angular velocity that is associated with the preferred space-like vector $v^\mu$ will be given by

$$\tilde{\Omega} = \frac{1}{r^2} \Omega.$$

Under the foregoing conditions, it can be seen that, with respect to an appropriately oriented local stationary background frame, the components of the intrinsically preferred string 2-surface frame vectors will be given by expressions of the form $u^0 = v^1 = 1/\sqrt{1 - r^2 \Omega^2}$ and $u^1 = v^0 = r \Omega / \sqrt{1 - r^2 \Omega^2}$. The ensuing expressions for the stationary background components of the stress-energy tensor, namely

$$T^{00} = \frac{U - r^2 \Omega^2 T}{1 - r^2 \Omega^2}, \quad T^{01} = \frac{r \Omega (U - T)}{1 - r^2 \Omega^2}, \quad T^{11} = \frac{r^2 \Omega^2 U - T}{1 - r^2 \Omega^2},$$

enable us to evaluate the relevant mass function as given by

$$M = 2\pi r T^{00}$$

and the corresponding angular momentum

$$J = 2\pi r^2 T^{01}.$$

It is apparent from these expressions that the angular momentum and mass functions are related by the dually alternative expressions

$$M = \Omega J + 2\pi r U = \tilde{\Omega} J + 2\pi r T.$$

Since we are supposing that the coupling is sufficiently weak for external forces on the string to be negligible we may choose to work in a gauge such that the Maxwellian connection form $A_\mu$ vanishes so that the components of the covariant derivative are simply $D^0 \phi = \omega$ and $D^1 \phi = k$ giving the pseudo-norm $w$ that plays the role of the independent intrinsic state variable in the form

$$w = k^2 - w^2,$$

where $\omega$ is the frequency and $k$ the wavenumber of the phase field. Under these conditions the electric current vector $I^\mu$ will have components given with respect to the stationary background frame by

$$I^0 = e K \omega, \quad I^1 = e K k,$$

where $e$ is the charge coupling constant. On the understanding that in the present section we are using units such as to give unit value not only for the speed of light $c$ (as has been assumed throughout) but also for the Dirac-Planck constant $\hbar$, the
corresponding magnetic dipole moment, \( D \) say, and the electric monopole moment, \( Q \) say, for the ring will be given by
\[
D = \pi r^2 I^1 = r Q k / 2 \omega \tag{2.8}
\]
and
\[
Q = 2 \pi r I^0 = e C \tag{2.9}
\]
where the conserved charge number \( C \) is an integer valued quantity given by
\[
C = 2 \pi r K \omega \tag{2.10}
\]
the qualitative physical interpretation of this expression being dependent on whether we are dealing with the “magnetic” case for which \( K = \mu / \nu \) or the “electric” case for which \( K = \nu / \mu \). The independent topologically conserved quantum number representing the winding number of the phae round the ring can be seen to be expressible even more simply by
\[
N = r k \tag{2.11}
\]
Using the fact that the phase velocity \( \omega / k \) directly determines the angular velocity by the proportionality relation \( \omega / k = r \Omega \) in the “magnetic” case, i.e. for \( w > 0 \), and that it analogously determines the dual angular velocity \( \omega / k = r \bar{\Omega} \) in the “electric” case, i.e. for \( w < 0 \), we see from the expressions above that the angular momentum quantum number will be given in either case simply as the product of the charge number and the phase winding number: we shall always have
\[
J = C N \tag{2.12}
\]

3. The standing wave condition for equilibrium

The configurations that we are considering have up to this stage not only been dependent on the two independent parameters, \( \omega \) and \( k \) say (which together determine the intrinsic state parameter \( w \)) but are also dependent on a third independent overall scale parameter which may conveniently be taken to be the radius \( r \). However, the number of independent parameters reduces from three to two when one takes account of the condition for centrifugal equilibrium, which is equivalent to the condition that the mass function be stationary with respect to variations subject to the constraints that the separate quantum numbers \( C \) and \( N \) (and hence also \( J \)) are held constant. It is to be remarked that these constraints are such as to ensure automatically the preservation of the qualitative “magnetic” or “electric” character of the ring which cannot be affected by any continuous variation that preserves the ratio \( C / N = 2 \pi K \omega / k \), “magnetic” and “electric” regimes being separated by the critical values \( C / N = \pm 2 \pi K_0 \), where \( K_0 \) is the value of the function \( K \) where its argument vanishes, i.e. for \( w = 0 \), which would be attained if the phase velocity passed through the speed of light, \( \omega / k = \pm 1 \). (The qualitatively intermediate case of what might be termed a “transluminal” ring would be envisageable as a physically attainable intermediate state, within the framework of the simple string model under consideration,
if the underlying theory were such as to provide a lagrangian function for which $2\pi K_0$ happened to be a rational ratio, so that the positive and negative critical values of $C/N$ could be realised for integer values of $C$ and $N$.) On the presumption that $K$ is a positive but decreasing function of $w$ in the neighbourhood of the critical value $w = 0$, it can be seen that the “magnetic” side is characterised by $(C/N)^2 < (2\pi K_0)^2$ and the electric side by $(C/N)^2 > (2\pi K_0)^2$.

In order to deal with both the “magnetic” and the “electric” cases conjointly it is convenient to replace the mutually dual expressions given above for the mass function by yet another equivalent expression, namely

$$M = C\omega - 2\pi r L, \quad (3.1)$$

whose continuous variation leads directly to

$$dM = \omega dC + 2\pi K k dN - 2\pi T^{11} dr \quad (3.2)$$

with

$$T^{11} = L + k^2 K = \tilde{L} + \omega^2 K, \quad (3.3)$$

which can be seen to be consistent with the expression in (2.2) for the relevant (space) component, with respect to the local stationary background frame, of the stress-energy tensor. The condition for mechanical equilibrium of the ring (namely $dM = 0$ for $dC = dN = 0$) is thus seen from (3.3) to be simply that the spatial stress-energy component should vanish, i.e.

$$T^{11} = 0. \quad (3.4)$$

This may be seen from (2.2) to be equivalent (in both the “magnetic” and “electric” cases) to the requirement that the string tension $T$ be related to its energy $U$ in the intrinsically preferred (corotating) frame by

$$T = r^2 \Omega^2 U, \quad (3.5)$$

which can be seen to be interpretable as the condition that the rotation velocity should coincide with the extrinsic characteristic velocity $c_T$, the “kink speed”, i.e.

$$r^2 \Omega^2 = c_T^2, \quad (3.6)$$

where this speed $c_T$ of propagation (relative to the locally preferred frame) of transverse perturbations is given [2] by

$$c_T^2 = T/U. \quad (3.7)$$

We thus arrive at a theorem to the effect that the condition for equilibrium is that relatively backward moving perturbations should appear as standing waves, i.e. their angular velocity $\Omega_-$ should vanish, the forward moving perturbations therefore having angular velocity, $\Omega_+$ say, that in the non-relativistic limit would evidently have to be twice $\Omega$, the exact result for the general case being

$$\Omega_- = 0, \quad \Omega_+ = 2\Omega/(1 + r^2 \Omega^2). \quad (3.8)$$
4. Gyro-inertial and gyro-magnetic ratios

Subject to the foregoing equilibrium condition we are left with a family of ring states determined by only two independent parameters, which may be taken to be just the pair of conserved numbers $C$ and $N$. Alternatively, in order to obtain all the relevant quantities in a more explicit form, it may be more convenient (at least at a classical level when one is dealing with value sufficiently large for the parametrisation to be considered as continuous) to take the independent parameters to be the angular velocity $\omega$ and the state variable $w$, since the latter immediately determines $\omega$ and $k$ (in magnitude if not in sign) and hence also the phase velocity $\omega/k$, via the mutually dual relations

$$\omega^2 = -\tilde{L}/K, \quad k^2 = -L/K,$$

the value of the angular velocity variable $\Omega$ then determining both the sign of the phase velocity and the magnitude of the necessarily positive radius variable $r$ by the relation $\Omega r = \omega/k$ in the “magnetic” case for which the phase velocity is subluminal, and $\Omega r = k/\omega$ in the “electric” case for which the phase velocity is superluminal.

It is to be remarked that when the equilibrium condition (3.5) is taken into account, the formula for the angular momentum of the ring can be converted into either of the dually related alternative forms

$$J = 2\pi r^3 U \Omega = 2\pi r^3 T \tilde{\Omega},$$

while the mass of the ring can be expressed by the manifestly self dual formula

$$M = (\Omega + \tilde{\Omega})J.$$  \hspace{1cm} (4.3)

The angular momentum can also be expressed in manifestly self dual form as

$$J = \pm 2\pi r^2 \sqrt{UT},$$

the analogous expression for the mass being

$$M = 2\pi r(U + T).$$

Just as in the theory of pure vacuum black hole equilibrium states [10], the solution is qualitatively determined (modulo an overall scale factor) by the dimensionless ratio of mass $M$ to specific angular momentum $a = J/M$, so analogously in the present context this same dimensionless ratio $M^2/J$ (like $C/N$) fully determines the intrinsic state of the string as given by $w$ (leaving only the scale of the ring to be determined by an independent parameter such as $\Omega$ or $r$ itself): we have

$$M^2/J = \pm 2\pi(U + T)^2/\sqrt{UT}.$$  \hspace{1cm} (4.6)

(It can be seen that independently of $r$ this mass to specific angular momentum ratio must always tend to the fixed value $8\pi T_0$ in the low current limit as $w$ tends to zero, and for which both $T$ and $U$ tend to the common null state value $T_0$, which
is consistent with finite charge for arbitrarily large values of \( r \). Using the order of magnitude estimate \( T_0 \sim m^2 \), where \( m \) is the relevant symmetry breaking mass scale, one thus obtains a crude estimate for the numerical value of the ring mass as \( M \sim |J|^{1/2}m \), the corresponding estimate for the radius being \( r \sim |J|^{1/2}m^{-1} \). In the superconducting string applications one has in mind, it is not the grand unification mass scale that is contemplated, but less extreme “charged ultra-massive particle” scales that have been suggested in the literature \([9]\). The Compton wavelength \( m^{-1} \) that is expected to characterise the string thickness will satisfy the requirement of being small compared with the estimated radius \( r \) in the classical regime for which \( |J| \) is large compared with unity, but it can be seen that the “string” description will break down in the quantum regime of small integer values of \( |J| \), for which one will obtain a “vorton” more appropriately describable as a thick torus than as a thin ring.

Another scale independent function only of the intrinsic state of the string is the gyromagnetic factor, \( g = 2D/Qa \), which works out simply as

\[
g = 2MD/QJ = 1 + L/\bar{L},
\]

whose translation into terms of \( U \) and \( T \) depends on whether we are dealing with a “magnetic” ring, in which case we shall have \( g = 1 + c^{-2}_T \), or an “electric” ring, in which case we shall have \( g = 1 + c^2_T \). It is of interest to notice that the intermediate case of a “transluminal” ring, i.e. in the limit as \( U \) and \( T \) both tend to \( T_0 \) so that \( c_T \) tends to unity, the gyromagnetic ratio tends to the familiar value \( g = 2 \) that characterises an electron in the simple Dirac theory, which is the same as has been observed \([10]\) to apply to charged black hole equilibrium states. This value, \( g = 2 \), has been rather inappropriately described as “anomalous” merely because it differs from the familiar value \( g = 1 \) that holds for an ordinary massive charged rotating ring in the classical (nonrelativistic) limit. What is made apparent by the foregoing analysis is that in relativistic theory a stationary subluminally rotating string loop with a corotating electric charge, which is the case to which \([1]\) the “electric” ring model applies (whether its underlying structure be that of a vortex defect of the vacuum or something more mundane such as the twisted bunch of molecular chains that constitutes an ordinary rope), then (in view of the stability and causality requirements \( 0 \leq T \leq U \) \([2]\)) any value in the range \( 1 \leq g \leq 2 \) is possible, the lower “electrostatic” limit being the classical value, and the upper “transluminal” limit being the Dirac value. On the other hand for a “magnetic” ring (corresponding to the case most commonly studied in recent work on cosmic string theory) the allowed range of values is \( 2 \leq g \leq \infty \), the “transluminal” Dirac value being now the lower limit.

5. A stability criterion

We conclude by remarking that the physical interest of the foregoing result is dependent on the existence of parameter ranges for which the stationary ring configurations are actually stable. We have been considering states for which the derivatives of the mass function (3.1) with respect to the radius at fixed values of the charge number \( C \) and winding number \( N \) are given by

\[
\frac{dM}{dr} = 0, \quad \frac{d^2M}{dr^2} = \frac{2\pi U (3 - c^2_T)c^2_T - (3c^2_T - 1)c^2_L}{1 - c^2_Tc^2_L}
\]
where $c_L^2$ is the longitudinal characteristic speed as given [2] by

$$c_L^2 = \frac{-dT}{dU}.$$  \hspace{1cm} (5.2)

It may be conjectured that in addition to the local stability conditions, the positivity of the second derivative in (5.1) is not only necessary but also sufficient for the corresponding equilibrium state to be truly stable. This condition, i.e. the requirement that the mass function has not just a critical value but actually a non-degenerate minimum can be seen to be expressible as the inequality

$$\left(\frac{c_T}{c_L}\right)^2 > \frac{3c_T^2 - 1}{3 - c_T^2},$$  \hspace{1cm} (5.3)

whose validity is guaranteed by the positivity of the right hand side whenever $c_T^2 < \frac{1}{3}$ (which will always be the case for classical string loops with $T \ll U$) but which may conceivably fail in certain cases (though not the most obvious ones) in the relativistic context that is relevant to “cosmic” strings.

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