Analytical solutions of the coupled Gross-Pitaevskii equations for the three-species Bose-Einstein condensates

Y.M. Liu$^{1,3}$ and C.G. Bao

$^1$Department of Physics, Shaoquan University, Shaoquan, 512005, P. R. China
$^2$State Key Laboratory of Optoelectronic Materials and Technologies, School of Physics and Engineering, Sun Yat-Sen University, Guangzhou, P. R. China and
$^3$State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing, 100190, China

The coupled Gross-Pitaevskii equations for the g.s. of the three-species condensates (3-BEC) have been solved analytically under the Thomas-Fermi approximation. Six types of spatial configurations in miscible phase are found. The whole parameter-space has been divided into zones each supports a specific configuration (miscible or immiscible). The borders of the zones are described by analytical formulae. Due to the division, the variation of the spatial configuration against the parameters can be visualized, and the effects of the parameters can be thereby understood. There are regions in the parameter-space where the configuration is highly sensitive to the parameters. These regions are tunable and valuable for the determination of the parameters.

PACS numbers: 03.75.Mn, 03.75.Kk

I. INTRODUCTION

In recent years there are a number of literatures dedicated to the theoretical [1–11], and experimental [12–17] study of the two-species Bose-Einstein condensates (2-BEC) (also refer to the references in [17]). The 2-BEC provides an important tool to clarify the inter-species and intra-species interactions. The miscible and immiscible phases of the ground state (g.s.) have been predicted and have been experimentally confirmed [14]. On the other hand, the study on the condensates with more than two species is very scarce [18, 19]. Since the multi-species BEC in principle experimentally achievable [20], a primary theoretical attempt might be worthy to see whether interesting physics is involved and whether this new field deserves a further study. This paper is dedicated to this purpose, namely, a primary theoretical study on the three-species Bose-Einstein condensates (3-BEC).

The spatial configurations of the 3-BEC, as in 2-BEC, are expected to have also three phases: miscible, immiscible, and asymmetric phases (as shown below). For the first phase the atoms of each kind of species are compactly distributed surrounding the center of the trap, the distribution of some species is broader and some narrower. For the second either at least one species leaves completely from the center or at least one species is distributed in more than one disconnected spatial domains. For the first and second, the distribution keeps the symmetry as the trap. For the third, the distribution does not keep the symmetry of the trap. This paper is dedicated to the g.s. in miscible phase. The emphasis is placed on the qualitative aspect. The trap is assumed to be isotropic. The spin-degrees of freedom are frozen. By introducing the Thomas-Fermi approximation (TFA, in which the kinetic energy is neglected), the coupled Gross-Pitaevskii equations (CGP) for the g.s. are solved analytically. This enable us to carry on the analysis in an analytical way. According to the relative distributions of the three species, the miscible phase has been further classified into six types. The analytical formalism derived in the follows enable us to divide the whole parameter-space into zones, each supports a specific type. Based on the division, the variety of the spatial configurations and their variation against the parameters (the intra- and inter-species interactions, the particle numbers, masses, and those for the trap) can be visualized, and thereby the effect of these parameters can be clarified.

II. HAMILTONIAN AND THE COUPLED GROSS-PITAEVSKII EQUATIONS

We consider three kinds of atoms $N_A$ A-atoms with mass $m_A$ and interacting via $V_A = c_A \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j)$, $N_B$ B-atoms with $m_B$, $V_B$, and $c_B$, and $N_C$ C-atoms with $m_C$, $V_C$, and $c_C$. The particle numbers are assumed to be huge (say, larger than 10000). The interspecies interactions are $V_{AB} = c_{AB} \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j)$ with the strength $c_{AB}$, $V_{BC}$ with $c_{BC}$, and $V_{CA}$ with $c_{CA}$. These atoms are confined by the harmonic traps $\frac{1}{2} m_s \omega_s^2 r^2$ ($s = A, B$ or $C$). We introduce a mass $m_o$ and a frequency $\omega$. Then, $\hbar \omega$ and $\lambda \equiv \sqrt{\hbar/(m_o \omega)}$ are used as units for energy and length in this paper. The total Hamiltonian is

\[
H = H_A + H_B + H_C + V_{AB} + V_{BC} + V_{CA}
\]

\[
H_A = \sum_{i=1}^{N_A} \left( -\frac{m_o}{2m_A} \nabla_i^2 + \frac{1}{2} \gamma_A r_i^2 + V_A \right)
\]

where $\gamma_A = (m_A/m_o)(\omega_A/\omega)^2$. $H_B$ and $H_C$ are similarly defined.

* Corresponding author: C.G. Bao, stsbcg@mail.sysu.edu.cn
We consider the g.s. in which no spatial excitations are involved. Thus, each kind of atoms are fully condensed into a state which is most advantageous for binding (otherwise, the energy would be higher). Accordingly, the total wave function of the g.s. can be written as

$$\Psi = \prod_{i=1}^{N_A} \frac{u_1(r_i)}{\sqrt{4\pi r_i}} \prod_{j=1}^{N_B} \frac{u_2(r_j)}{\sqrt{4\pi r_j}} \prod_{k=1}^{N_C} \frac{u_3(r_k)}{\sqrt{4\pi r_k}}$$  \hspace{1cm} (2)

where $u_1$, $u_2$, and $u_3$ are for the A-, B-, and C-atoms, respectively.

From minimizing the total energy, we obtain the set of CGP. One of them is

$$\frac{m_o}{2m_A} \nabla^2 + \frac{1}{2} \gamma A r^2 + N_A \epsilon_A \frac{u_1^2}{4\pi r^2} + N_B \epsilon_{AB} \frac{u_2^2}{4\pi r^2} + N_C \epsilon_{CA} \frac{u_3^2}{4\pi r^2} - \epsilon_A u_1 = 0$$ \hspace{1cm} (3)

where $\epsilon_A$ is the chemical potential. Via cyclic permutations of the three indexes ($A, B, C$) and the three ($u_1, u_2, u_3$), from eq. \ref{eq:3} we obtain the other two equations. It is emphasized that the three equations of normalization $\int u_i^2 dr = 1 \ (i=1, 2, \text{and } 3)$ should hold.

### III. FORMAL SOLUTIONS UNDER THE THOMAS-FERMI APPROXIMATION

Since $N_A$, $N_B$, and $N_C$ are considered to be large, the approximation TFA has been adopted. A recent numerical evaluation of this approximation is referred to the papers [11, 21]. Under the TFA, the CGP become

$$\frac{x^2}{2} + \alpha_{11} \frac{u_1^2}{r^2} + \alpha_{12} \frac{u_2^2}{r^2} + \alpha_{13} \frac{u_3^2}{r^2} - \epsilon_1 u_1 = 0$$

$$\frac{x^2}{2} + \alpha_{21} \frac{u_1^2}{r^2} + \alpha_{22} \frac{u_2^2}{r^2} + \alpha_{23} \frac{u_3^2}{r^2} - \epsilon_2 u_2 = 0$$

$$\frac{x^2}{2} + \alpha_{31} \frac{u_1^2}{r^2} + \alpha_{32} \frac{u_2^2}{r^2} + \alpha_{33} \frac{u_3^2}{r^2} - \epsilon_3 u_3 = 0$$ \hspace{1cm} (4)

where $\alpha_{11} = N_A \epsilon_A/(4\pi \gamma_A)$, $\alpha_{22} = N_B \epsilon_{AB}/(4\pi \gamma_B)$, $\alpha_{33} = N_C \epsilon_{CA}/(4\pi \gamma_C)$, $\alpha_{12} = N_B \epsilon_{CA}/(4\pi \gamma_A)$, $\alpha_{21} = N_A \epsilon_{AB}/(4\pi \gamma_B)$, $\alpha_{13} = N_C \epsilon_{AB}/(4\pi \gamma_A)$, $\alpha_{31} = N_A \epsilon_{CA}/(4\pi \gamma_C)$, $\alpha_{23} = N_C \epsilon_{BC}/(4\pi \gamma_B)$, and $\alpha_{32} = N_B \epsilon_{BC}/(4\pi \gamma_C)$, they are called the weighted strengths (W-strengths). $\epsilon_1 = \epsilon_A/\gamma_A$, $\epsilon_2 = \epsilon_B/\gamma_B$, $\epsilon_3 = \epsilon_C/\gamma_C$, they are the weighted energies for a single particle. In this paper all the interactions are considered as repulsive. Accordingly, all the W-strengths are positive. Furthermore, it is safe to assume that all the $u_i/r$ are always non-negative. Recall that there are originally 15 parameters ($N_A, m_o, \omega, c$, $\alpha_i$). From eq. \ref{eq:4} we know that their combined effects are fully represented by the nine $\alpha_i$. Among them, only eight are independent because they are related as $\alpha_{12} \alpha_{23} \alpha_{31} = \alpha_{21} \alpha_{32} \alpha_{13}$. Thus, based on the W-strengths, related analysis could be simpler.

We define a matrix $\mathfrak{M}$ with its element $(\mathfrak{M})_{ll'} = \alpha_{ll'}$. The determinant of $\mathfrak{M}$ is denoted by $\mathfrak{D}$. The algebraic cominor of $\alpha_{ll'}$ is denoted as $d_{ll'}$. Obviously, the element of the inverse matrix $(\mathfrak{M}^{-1})_{ll'} = d_{ll'}/\mathfrak{D}$.

The set of equations \ref{eq:4} has four forms of formal solution, each holds in a specific domain of $r$:

(i) Form III: When all the three wave functions are nonzero in a domain, they must have the form as

$$u_i^2/r^2 = X_i - Y_i r^2$$ \hspace{1cm} (5)

where

$$X_i = \mathfrak{D} Y_i / \mathfrak{D}$$ \hspace{1cm} (6)

$$\mathfrak{D} Y_i$$ is a determinant obtained by changing the $l$ column of $\mathfrak{D}$ from $(\alpha_{11}, \alpha_{21}, \alpha_{31})$ to $(\epsilon_1, \epsilon_2, \epsilon_3)$.

(ii) Form II: Let $(l, m, n)$ be a cyclic permutation of $(1, 2, 3)$, the same in the follows. When one and only one of the wave functions is zero inside the domain (say, $u_n/r = 0$), the other two must have the form as

$$u_i^2/r^2 = X_i^{(n)} - Y_i^{(n)} r^2$$ \hspace{1cm} (8)

where

$$X_i^{(n)} = (\alpha_{mm} \epsilon_l - \alpha_{ml} \epsilon_m)/d_{nn}$$

$$Y_i^{(n)} = \frac{1}{2} (\alpha_{mm} - \alpha_{ml})/d_{nn}$$

Once the parameters are given, the six $Y_i^{(n)} \ (n' \neq n)$ are known because they depend only on $\alpha_{ll'}$. When $Y_i^{(n)}$ is positive (negative), $u_{n'}/r$ goes down (up) with $r$. Obviously, once $Y_i^{(n)}$ is positive, $X_i^{(n)}$ must be large enough to prevent $u_i/r$ to be negative.

(iii) Form I: When one and only one of the wave functions is nonzero in a domain (say, $u_i/r \neq 0$), it must have the form as

$$u_i^2/r^2 = \frac{1}{\alpha_{ll'}} (\epsilon_l - r^2/2)$$ \hspace{1cm} (10)
Obviously, \( u_i/r \) in this form must descend with \( r \). This form could emerge only if \( \varepsilon_i \) is positive and sufficiently large.

(iv) Form 0: In this form all the three wave functions are zero.

If \( u_i/r \) is nonzero in a domain but becomes zero when \( r = r_o \), then a downward form-transition (say, from Form III to II) will occur at \( r_o \). Whereas if \( u_i/r \) is zero in a domain but becomes nonzero when \( r = r_o \), then an upward form-transition (say, from Form II to III) will occur at \( r_o \). In this way the formal solutions will link up continuously to form an entire solution. They are continuous at the transition points because the wave functions satisfy exactly the same set of nonlinear equations at \( r_o \). However, their derivatives are in general not continuous at \( r_o \).

When all the W-strengths are given, however, there are three unknowns \( \varepsilon_i \) contained in the entire solution. Once they are known all the \( X_i \) and \( X_i^{(l)} \) can also be known. Due to the requirement of normalization, we have three additional equations. They are sufficient to determine the three \( \varepsilon_i \) as shown below.

### IV. THREE LEMMAS

There are three lemmas related to the linking of formal solutions.

**Lemma I:** The three \( Y_1 \) can not all be negative.

Let us define a vector \( \Omega_t = \alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3 \), where \( \{n_1, n_2, n_3\} \) is a set of orthogonal unit vectors, and all the \( \alpha_{ij} \) are assumed to be positive as mentioned. Therefore \( \Omega_t \) is situated inside the first octant. It can be rewritten as \( \Omega_t = |\Omega_t| \vec{q}_t \), where \( \vec{q}_t \) is also a unit vector in the first octant. We define further \( \vec{n} = \frac{1}{2}(n_1 + n_2 + n_3) \). Then,

\[
\vec{Y}_1 = \frac{\vec{n}}{|\Omega_t|} (\vec{q}_t \times \vec{q}_m) \times \vec{q}_n.
\]

The three \( \vec{q}_t, \vec{q}_m, \) and \( \vec{q}_n \) should be linearly independent (otherwise, the determinant \( \mathcal{D} \) is zero and the Form III does not exist). Then, \( \vec{n} \) can be expanded as

\[
\vec{n} = n_t \vec{q}_t + n_m \vec{q}_m + n_n \vec{q}_n
\]

and accordingly

\[
Y_1 = \frac{n_t}{|\Omega_t|}
\]

Thus, the sign of \( Y_1 \) is determined by \( n_t \).

Since all the three \( \vec{q}_t \) to \( \vec{q}_n \) are inside the first octant, if all the three \( n_t, n_m, \) and \( n_n \) were negative, -\( \vec{n} \) would be in the first octant. This is in contradiction with the definition of \( \vec{n} \). Thus the three \( Y_1 \) can not all be negative, and the lemma is proved.

This lemma implies that Form III must transform to Form II somewhere because at least one of the \( Y_i \) is positive, and therefore at least one of the wave functions is descending and eventually arrives at zero.

**Lemma II:** \( Y^{(l)}_1 \) and \( Y^{(l)}_3 \) can not both be negative.

When \( l = 3 \), we define three 2-dimensional vectors \( \vec{\omega}_t = \alpha_{1t} n_1 + \alpha_{2t} n_2 \) (\( t = 1, 2 \)) and \( \vec{n}_1 = \frac{1}{2}(n_1 + n_2) \). All of them are situated in the first quadrant. Then, \( Y^{(3)}_1 = \frac{n_1 \cdot (\vec{\omega}_t \times \vec{\omega}_m)}{n_3 \cdot (\vec{\omega}_t \times \vec{\omega}_n)} \) and \( Y^{(3)}_2 = \frac{n_1 \cdot (\vec{\omega}_t \times \vec{n}_3)}{n_3 \cdot (\vec{\omega}_t \times \vec{\omega}_n)} \). \( Y^{(3)}_1 < 0 \) implies that, on the \( n_1 - n_2 \) plane, the polar angle of \( \vec{\omega}_2 \) should lie between those of \( n_1 \) and \( \vec{\omega}_1 \). Whereas \( Y^{(3)}_2 < 0 \) implies that the polar angle of \( \vec{\omega}_1 \) should lie between those of \( n_1 \) and \( \vec{\omega}_2 \). These two requirements are in contradiction. The cases with \( l \neq 3 \) are similar. Thus the lemma is proved. In fact, this lemma can also be directly proved via the definition of \( Y^{(l)}_m \) and \( Y^{(l)}_n \).

This lemma implies that Form II will transform to Form I somewhere because at least one of the wave functions is descending. Otherwise, it will transform to Form III if the missing wave function emerges. This lemma implies that Form II will either transform to Form I somewhere because at least one of the wave functions (say, \( u_n/r \)) is descending, or transform to Form III if the missing wave function emerges earlier than the vanish of \( u_n/r \).

**Lemma III:** In a domain (or at a point) where all the three \( u_i/r \) are zero, no wave function can emerge and becomes nonzero in this domain (at the point).

If \( u_i/r \) emerges singly, then it must have the form eq. (10), therefore \( u_i/r \) must descend with \( r \) and the emergence fails. If \( u_i/r \) and \( u_m/r \) emerge in pair at the same place, then both \( Y^{(n)}_m \) and \( Y^{(m)}_n \) should be negative to assure the uprising. This fails due to Lemma II. If all the three \( u_i/r \) emerge together at the same place, then all the three \( Y_i \) should be negative to assure the uprising. This fails due to Lemma I. Thus, the Lemma III is proved.

Due to Lemma III, once the unique nonzero wave function in Form I arrives at zero, say, \( u_i/r = 0 \) when \( r = r_{out} \), then \( r_{out} \) will be the outmost border for all kinds of atoms.

### V. LINKING THE FORMAL SOLUTIONS TO FORM AN ENTIRE SOLUTION IN MISCEBL PHASE

With the three lemmas, we are going to link up the formal solutions to form an entire solution. To this aim, we will first make some presumptions so that the formal solutions can be linked up in a specific way. Then, we find out a subspace in the whole parameter-space. When the parameters are chosen inside this subspace, all the presumptions can be recovered so that the entire solution stands. In this way the whole space is divided into zones each supports a specific spatial configuration of the g.s. Based on the division, we are able to obtain various types of phase-diagrams to demonstrate the variation of the g.s. against the parameters.

For the miscible phase, the first domain (starting from \( r = 0 \)) must have Form III. Therefore, the three \( X_i > 0 \)
should be presumed. Due to Lemma I, there is at least a positive $Y_l$. Without loss of generality, it is assigned that $X_i/Y_l$ is the smallest positive ratio among the three ratios. Accordingly, among the three wave functions, $u_i/r$ will arrive at zero first (refer to eq. (1)). Thus, the first domain ends at $r_a = \sqrt{X_i/Y_l}$, where a downward form-transition occurs. For miscible phase $u_i/r$ is not allowed to emerge again because it is not allowed to distribute in disconnected region. Therefore $u_i/r$ is distributed only in $(0, r_a)$. From the normalization $\int_0^{r_a} u_i^2 dr = \frac{15}{2Y_l}/a X_m^{2/5} r_m^{2/5}$ and

$$r_a^2 = \left(\frac{15}{2Y_l}\right)^{2/5}$$ (11)

This equation implies that the W-strengths should be so chosen that $Y_l \geq Y_m$ and $Y_l \geq Y_n$ hold. This choice assures that $u_i/r$ will arrive at zero first and the presumption $X_i > 0$ can be recovered.

The second domain will have the Form II and starts from $r_a$. Since at least one of the two wave functions must descend with $r$ (Lemma II), we can assign the one that arrives at zero first with the index $m$, and we define $r_b = \sqrt{X_m/Y_m}$ (refer to eq. (8)). Then, the equation of normalization for $u_m$ is

$$1 = \int_0^{r_a} (X_m r^2 - Y_m r^4) dr + \int_{r_a}^{r_b} (X_m^{(l)} r^2 - Y_m^{(l)} r^4) dr$$

Making use of the continuity at $r_a$, namely, $X_m - Y_m r_a^2 = X_m^{(l)} - Y_m^{(l)} r_a^2$, we obtain

$$r_b = r_a \left(\frac{Y_l - Y_m + Y_m^{(l)}}{Y_m^{(l)}}\right)^{1/5}$$ (12)

and $X_m^{(l)} = Y_m^{(l)} r_b^2$. It is clear that, in order to have $u_i/r$ descending in the second domain, $Y_m^{(l)} > 0$ is necessary to be presumed. Together with the previously mentioned condition $Y_l \geq Y_m$, $r_b$ is well defined from eq. (12) and $r_b > r_a$ holds. Furthermore, once $X_m^{(l)}$ is known, $X_m$ can be known from the continuity at $r_a$ as

$$X_m = Y_m^{(l)} r_b^2 + (Y_m - Y_m^{(l)}) r_a^2$$

Recall that $X_m > 0$ has been presumed. In order to recover this presumption, the W-strengths should be so chosen to ensure

$$u_n/r = \frac{1}{\alpha_{nn}}(\epsilon_n - r^2/2).$$

When $r = r_c$, $u_n/r$ arrives also at zero. Due to Lemma III, $r_c$ is the outmost border for all the atoms. Making use of the continuity at $r_a$, $r_b$, $X_n$, $X_n^{(l)}$ and $\epsilon_n$ are related as $X_n = X_n^{(l)} + (Y_n - Y_n^{(l)}) r_a^2$ and $X_n^{(l)} = \frac{1}{\alpha_{nn}}[\epsilon_n - (1/2 - \alpha_{nn} Y_n^{(l)}) r_a^2]$. Inserting these two relations into the normalization $\int_0^{r_a} u_n^2 dr = 1$, we have

$$\epsilon_n = \frac{X_l}{2Y_l} 2\alpha_{nn} (Y_l - Y_m - (Y_l - Y_m) Y_n^{(l)}/Y_m^{(l)}) + 1 + (Y_l - Y_m)/Y_m^{(l)}/2/5$$ (14)

Thus, $\epsilon_n$ can be known when all the W-strengths are given. After $\epsilon_n$ is known, $X_n^{(l)}$ and $X_n$ can be known from the continuity as shown above. Thus $u_n/r$ is obtained. Furthermore, making use of eq. (4) we have

$$\epsilon_l = \sum u_l X_l.$$ Thus, when all $X_l$ are known, all $\epsilon_l$ can also be known.

In the above form of $u_n/r$, $\epsilon_n > r_a^2/2$ is required to assure $\frac{\partial u_n}{\partial r}|_{r=r_b} > 0$. Thus, from eqs. (14, 12) the W-strengths should be so chosen that

$$Y_l - Y_m > (Y_l - Y_m) Y_n^{(l)}/Y_m^{(l)}$$ (15)

is satisfied. In order to have $\frac{\partial u_n}{\partial r}|_{r=r_a} > 0$ (i.e., $X_n^{(l)} > Y_n^{(l)} r_a^2$)

$$\epsilon_n > \alpha_{nn} Y_n^{(l)} r_a^2 + (1/2 - \alpha_{nn} Y_n^{(l)}) r_b^2$$ (16)

should be satisfied. In order to have $\frac{\partial u_n}{\partial r}|_{r=0} > 0$ (i.e.,

$$\epsilon_n > \alpha_{nn} (Y_n^{(l)} - Y_m) r_a^2 + (1/2 - \alpha_{nn} Y_n^{(l)}) r_b^2$$ (17)

should be satisfied.

Thus, the miscible phase with all the three $u_i/r$ compactly surrounding the center and with the ranges $r_a \leq r_b \leq r_c$ will appear when the W-strengths are so given that the conditions (i) $Y_l \geq Y_m$ and $Y_l \geq Y_n$. (ii) $Y_m^{(l)} > 0$, and (iii) eqs. (14, 15, 10, 17) are satisfied. This specific miscible phase is denoted as $\{l,m,n\}$ to demonstrate that $u_i/r$ has a narrowest distribution and $u_n/r$ has a broadest distribution.

VI. DIVISION OF THE PARAMETER-SPACE

Obviously, the above inequalities together define a specific zone in the multi-dimensional space of parameters. The borders of the zone are given by the surfaces defined by the equalities arising from changing each of the above inequality to equality. Note that the labels $(l,m,n)$ can be any permutation of $(1,2,3)$. Therefore, there are six types of miscible states and, correspondingly, six types of zones. Let the zones associated with $\{l,m,n\} = \{1,2,3\}$, $\{1,3,2\}$, $\{2,1,3\}$, $\{2,3,1\}$, $\{3,1,2\}$, and $\{3,2,1\}$ be denoted
as Zone I to Zone VI, respectively. The zone not belonging to the above six is for the immiscible phase and is denoted as Zone 0. Once the whole space has been divided into zones, one can select any subset of parameters as variables while the others are fixed. This leads to various types of phase-diagrams that demonstrate the variation of the spatial configuration against the selected parameters. An example is given in Fig. 1. Note that it was found in 2-BEC that the g.s. might be asymmetric to various types of phase-diagrams that demonstrate the symmetry inherent in the CGP assures the g.s. can be denoted by either one of them at the axis \( x = 1/2 \). Accordingly, once Zone II appears in one side of the axis, Zone V will also appear in the other side as its partner. Similarly, III and IV are partners. In general, the axis \( x = 1/2 \) is replaced by a surface \( Y_1 = Y_3 \) in the parameter-space. On the surface \( u_1 \) and \( u_3 \) overlap.

(iii) Similarly, when the axis \( y = 1/2 \) be the common border of two neighboring zones, the labels for these two zones are related to each other by an interchange of 1 and 2 (say, \( \{1, 2, 3\} \) and \( \{2, 1, 3\} \)). Thus, I and III are partners. V and VI also. As before, one can prove that, \( u_1 \) and \( u_2 \) overlap at the horizontal line \( y = 1/2 \) (or, in general, on the surface \( Y_1 = Y_2 \)).

(iv) The point \( O \) is the intersection of the \( x = 1/2 \) and the \( y = 1/2 \) axes (in general, the intersection of the two surfaces \( Y_1 = Y_3 \) and \( Y_2 = Y_3 \), where all the \( c_{AB} \) are equal, and the three wave functions \( u_1 \) overlap. Accordingly, in the neighborhood of \( O \) all the six types \( \{l, m, n\} \) have an equal probability to appear as shown in the figure.

Since the solutions have been obtained in an analytical way, it is straightforward to plot the wave functions. Examples are shown in Fig. 2. In 2a the wave functions are associated with the point \( P_2 \) marked by a cross in Fig.1, where the g.s. is in the \( \{1, 3, 2\} \) phase. The pattern associated with \( P_1 \) is identical with that of 2a but \( u_2 \) and \( u_3 \) interchange. The pattern associated with \( P_3 \) is close to 2a (not exactly the same) but \( u_1 \) and \( u_3 \) interchange. All the three points are not far away from the point \( O \). Therefore the three wave functions are not remarkably different from each other. Otherwise, they might be very different. 2b is associated with \( P_4 \), where the g.s. is also in the \( \{1, 3, 2\} \) phase. However, due to \( P_4 \) is very close to the zone of immiscible phase, the B-atoms tend to leave completely from the center and tend to form a shell as shown by the dash curve.

It was found that in the neighborhood of the border separating the miscible and immiscible phases, the con-
configuration is very sensitive to the variation of parameters. For an example, $P_4$ (marked in Fig.1) has $x = 0.71$, and accordingly $(u_2/r)_{r=0} = 0.048$ (it implies that the B-atoms are very few at the center). When $P_4$ shifts a little away from the border so that $x$ becomes 0.70, $(u_2/r)_{r=0}$ becomes 0.207. Thus, the neighborhood of the above border is a region of sensitivity. In this region a tiny change in the parameters might cause an explicit change in the configuration. The existence of regions of sensitivity in the parameter-space is a notable phenomenon.

VII. FINAL REMARKS

We have succeeded to derive the analytical solutions of the CGP for the 3-BEC under the TFA. Thereby the parameter-space has been divided into zones each supports a specific configuration. Based on the division, various types of phase diagrams can be plotted, and the variation of the spatial configurations against the parameters can be visualized. From the experience of 2-BEC, when the particle numbers are large and when both kinds of atoms are distributed surrounding the center (i.e., $u_1$ are nonzero when $r = 0$), the wave functions obtained under TFA and beyond TFA overlap nearly completely (refer to Fig.1a and 1b of [11], where a detailed discussion on the accuracy of the TFA is made). Since this paper concerns only this case, the TFA is believed to be applicable. Nonetheless, this remains to be further clarified.

Obviously, this paper is far from a complete description of the 3-BEC. Note that, when the inter- and intra-species interactions are close in strengths or the former is stronger than the latter, Symmetric immiscible states and asymmetric states may emerge. The details and the classification of these states remain to be studied.

The variety of the spatial configurations of 3-BEC implies that rich physics is involved. In particular, just as in 2-BEC, regions of sensitivity have been found. When a realistic parameter falls in a region of sensitivity, it can be more accurately determined. Obviously, these regions of 2-BEC and 3-BEC are different. Thus, in addition to the 2-BEC, the 3-BEC will be helpful in the determination of parameters. Recall that the BEC are a valuable tool because they are tunable. One can consider that the addition of the third kind of atoms into a 2-BEC is an one more way to tune the system.

Incidentally, the above analytical approach is quite general and can be generalized to deal with the condensates with more than three species.

ACKNOWLEDGMENTS

Supported by the National Natural Science Foundation of China under Grants No.11372122, 11274393, 11574404, and 11275279; the Open Project Program of State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, China(No.Y4KF201CJ1); and the National Basic Research Program of China (2013CB933601).

[1] T. L. Ho and V. B. Shenoy, Phys. Rev. Lett. 77, 3276 (1996).
[2] B. D. Esry, C.H.Greene, J.P.Burke, J.L.Bohn, Phys. Rev. Lett. 78, 3594 (1997).
[3] H. Pu and N. P. Bigelow, Phys. Rev. Lett. 80, 1130 (1998).
[4] M. Luo, Z.B. Li, and C.G. Bao, Phys. Rev. A 75, 043609 (2007).
[5] P.N.Galteland, E.Babaev and A.Sudbø, New J. Phys. 17 103040(2015).
[6] B.VanSchaeybroeck and J.O.Indekeu, Phys. Rev. A 91, 013626 (2015).
[7] Joseph O. Indekeu, Chang-You Lin, Nguyen Van Thu, Bert Van Schaeybroeck, and Tran Huu Phat, Phys. Rev. A 91, 033615 (2015).
[8] Pekko Kuopanportti, Natalia V. Orlova, and Milorad V. Milošević, Phys. Rev. A 91, 043605 (2015).
[9] Arko Roy and D. Angom, Phys.Rev. A 92, 011601(R) (2015).
[10] Ma Luo, Chengguang Bao and Zhibing Li, Phys. B: At. Mol. Opt. Phys. 41, 245301(2008).
[11] J.Polo and V.Ahufinger, P. Mason, S. Sridhar, T. P. Bilkam, and S. A. Gardiner, Phys. Rev. A 91, 053626 (2015).
[12] C. J. Myatt, E.A.Burt, R.W.Ghrist, E.A.Cornell, and C.E.Wieman, Phys. Rev. Lett. 78, 586 (1997).
[13] M. Anderlini, E. Courtade, M. Cristiani, D. Cossart, D. Ciampini, C. Sias, O. Morsch, and E. Arimondo, Phys. Rev. A 71, 061401(R) (2005).
[14] K.-K. Ni, S. Ospelkaus, M. H. G. de Miranda, A. Pe’er, B. Neyenhuis, J. J. Zirbel, S. Kotochigova, P. S. Julienne, D. S. Jin, and J. Ye, Science 322, 231 (2008).
[15] K. Pilch, A. D. Lange, A. Prantner, G. Kern, F. Ferlaino, H.-C. Nägerl, and R. Grimm, Phys. Rev. A 79, 042718 (2009).
[16] N. Nemitz, F. Baumer, F. Münchow, S. Tassy, A. Görlitz, Phys. Rev. A 79, 061403(2009).
[17] L. Wacker, N. B. Jørgensen, D. Birkmose, R. Horchani, W. Ertmer, C. Klempt, N. Winter, J. Sherson, and J. J. Arlt, Phys. Rev. A 92, 053602 (2015).
[18] M.Caliari and M.Squassina, Electronic Journal of Differential Equations, No.79 (2008).
[19] K.Manikandan, P.Muruganandam, M.Senthilvelan, and M.Lakshmanan, Phys. Rev. E 93, 032212 (2016).
[20] M.Tagliber, A.-C. Voigt, T.Aoki, T.W.Hänsch, and K.Dieckmann, Phys. Rev. Lett. 100, 010401 (2008).
[21] Y.Z.He, Y.M.Liu, and C.G.Bao, Phys. Rev. A 91, 033620(2015).