Ellipsoids and elliptic hyperboloids in the Euclidean space $\mathbb{E}^{n+1}$

Dong-Soo Kim *
Department of Mathematics, Chonnam National University,
Kwangju 500-757, Korea

Abstract

We establish some characterizations of elliptic hyperboloids (resp., ellipsoids) in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$, using the $n$-dimensional area of the sections cut off by hyperplanes and the $(n+1)$-dimensional volume of regions between parallel hyperplanes. We also give a few characterizations of elliptic paraboloids in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$.

1. Introduction

In what follows we will say that a convex hypersurface of $\mathbb{R}^{n+1}$ is strictly convex if the hypersurface is of positive normal curvatures with respect to the unit normal $N$ pointing to the convex side. In particular, the Gauss-Kronecker curvature $K$ is positive with respect to the unit normal $N$. We will also say that a convex function $f : \mathbb{R}^n \to \mathbb{R}$ is strictly convex if the graph of $f$ is strictly convex with respect to the upward unit normal $N$.

Consider a smooth function $g : \mathbb{R}^{n+1} \to \mathbb{R}$. We denote by $R_g$ the set of all regular values of the function $g$. We assume that there exists an interval $S_g \subset R_g$ such that for every $k \in S_g$, the level hypersurface $M_k = g^{-1}(k)$ is a smooth strictly convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$. We also denote by $S_g$ the maximal interval in $R_g$ which satisfies the above property.

2000 Mathematics Subject Classification. 53A07.
Key words and phrases. Ellipsoid, elliptic hyperboloid, $(n+1)$-dimensional volume, $n$-dimensional surface area, level hypersurface, Gauss-Kronecker curvature.

* was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (2010-0022926). E-mail: dosokim@chonnam.ac.kr
If \( k \in S_g \), then we may choose a maximal interval \( I_k \subset S_g \) so that each \( M_{k+h} \) with \( k+h \in I_k \)
lies in the convex side of \( M_k \). Note that \( I_k \) is of the form \( (k, a) \) with \( a > k \) or \( (b, k) \) with \( b < k \)
according as the gradient \( \nabla g \) of the function \( g \) points to the convex side of \( M_k \) or not.

For examples, consider two functions \( g_\pm : \mathbb{R}^{n+1} \to \mathbb{R} \) defined by \( g(x, z) = z^2 \pm (a_1^2 x_1^2 + \cdots + a_n^2 x_n^2) \) with positive constants \( a_1, \cdots, a_n \). Then, for the function \( g_- \) we have \( R_{g_-} = R - \{0\} \), 
\( S_{g_-} = (0, \infty) \) and \( I_k = (k, \infty) \), \( k \in S_{g_-} \). For \( g_+ \), we get \( R_{g_+} = S_{g_+} = (0, \infty) \) and \( I_k = (0, k) \)
with \( k \in S_{g_+} \).

For a fixed point \( p \in M_k \) with \( k \in S_g \) and a sufficiently small \( h \) with \( k+h \in I_k \), we consider the tangent hyperplane \( \Phi \) of \( M_{k+h} \) at some point \( v \in M_{k+h} \), which is parallel to the tangent hyperplane \( \Psi \) of \( M_k \) at \( p \in M_k \). We denote by \( A_p^*(k, h), V_p^*(k, h) \) and \( S_p^*(k, h) \) the \( n \)-dimensional area of the section in \( \Phi \) enclosed by \( \Phi \cap M_k \), the \( (n+1) \)-dimensional volume of the region bounded by \( M_k \) and the hyperplane \( \Phi \), and the \( n \)-dimensional surface area of the region of \( M_k \)
between the two hyperplanes \( \Phi \) and \( \Psi \), respectively.

In [3], the author and Y. H. Kim studied hypersurfaces in the \( (n+1) \)-dimensional Euclidean space \( \mathbb{E}^{n+1} \) defined by the graph of some function \( f : \mathbb{R}^n \to \mathbb{R} \). In our notations, they proved the following characterization theorem for elliptic paraboloids in the \( (n+1) \)-dimensional Euclidean space \( \mathbb{E}^{n+1} \), which extends a result in [2].

**Proposition 1.** Suppose that \( f : \mathbb{R}^n \to \mathbb{R} \) is a strictly convex function. We consider the function \( g : \mathbb{R}^{n+1} \to \mathbb{R} \) defined by \( g(x, z) = z - f(x), x = (x_1, \cdots, x_n) \). Then, the following are equivalent.

1) For a fixed \( k \in R, V_p^*(k, h) \) is a nonnegative function \( \phi(h) \), which depends only on \( h \).
2) For a fixed \( k \in R, A_p^*(k, h)/|\nabla g(p)| \) is a nonnegative function \( \psi(h) \), which depends only on \( h \). Here \( \nabla g \) denotes the gradient of \( g \).
3) The function \( f(x) \) is a quadratic polynomial given by \( f(x) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2 \) with \( a_i > 0, i = 1, 2, \cdots, n \), and hence every level hypersurface \( M_k \) of \( g \) is an elliptic paraboloid.

Note that in the above proposition, \( R_g = S_g = R \) and \( I_k = (k, \infty) \).

In particular, when \( n = 2 \), in a long series of propositions, Archimedes proved that every level surface \( M_k \) (paraboloid of rotation) of the function \( g(x, y, z) = z - a^2(x^2 + y^2) \) in the 3-dimensional Euclidean space \( \mathbb{E}^3 \) satisfies \( V_p^*(k, h) = c h^2 \) for some constant \( c \) ([5], p.66 and Appendix A and B).
In this paper, we study the family of strictly convex level hypersurfaces $M_k, k \in S_g$ of a function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ which satisfies the following conditions.

\((V^*)\): For $k \in S_g$ with $k + h \in I_k$, $V^*_p(k, h)$ with $p \in M_k$ is a nonnegative function $\phi_k(h)$, which depends only on $k$ and $h$.

\((A^*)\): For $k \in S_g$ with $k + h \in I_k$, $A^*_p(k, h)/|\nabla g(p)|$ with $p \in M_k$ is a nonnegative function $\psi_k(h)$, which depends only on $k$ and $h$.

\((S^*)\): For $k \in S_g$ with $k + h \in I_k$, $S^*_p(k, h)/|\nabla g(p)|$ with $p \in M_k$ is a nonnegative function $\eta_k(h)$, which depends only on $k$ and $h$.

As a result, first of all, we establish the following characterizations of elliptic hyperboloids.

**Theorem 2.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative strictly convex function. For a nonzero real number $\alpha \in R$ with $\alpha \neq 1$, let’s denote by $g$ the function defined by $g(x, z) = z^\alpha - f(x)$. Suppose that the level hypersurfaces $M_k(k \in S_g)$ of $g$ in the $(n + 1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ are strictly convex. Then the following are equivalent.

1) The function $g$ satisfies Condition $(V^*)$.
2) The function $g$ satisfies Condition $(A^*)$.
3) For $k \in S_g$, $K(p)|\nabla g(p)|^{n+2} = c(k)$ is constant on $M_k$, where $K(p)$ denotes the Gauss-Kronecker curvature of $M_k$ at $p \in M_k$ with respect to the unit normal pointing to the convex side.
4) The function $g$ is given by

$$g(x, z) = z^2 - (a_1^2x_1^2 + \cdots + a_n^2x_n^2),$$

where $a_i > 0, i = 1, 2, \cdots, n$. In this case, $R_g = R - \{0\}$, $S_g = (0, \infty)$ and $I_k = (k, \infty), k \in S_g$.

Next, in the similar way to the proof of Theorem 2, we prove the following characterizations of ellipsoids.

**Theorem 3.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a nonnegative strictly convex function. For a nonzero real number $\alpha \in R$ with $\alpha \neq 1$, let’s denote by $g$ the function defined by $g(x, z) = z^\alpha + f(x)$. Suppose that the level hypersurfaces $M_k(k \in S_g)$ of $g$ in the $(n + 1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ are strictly convex. Then the following are equivalent.

1) The function $g$ satisfies Condition $(V^*)$.
2) The function $g$ satisfies Condition $(A^*)$.
3) For \( k \in S_g \), \( K(p)|\nabla g(p)|^{n+2} = c(k) \) is constant on \( M_k \), where \( K(p) \) denotes the Gauss-Kronecker curvature of \( M_k \) at \( p \in M_k \) with respect to the unit normal pointing to the convex side.

4) The function \( g \) is given by
\[
g(x, z) = z^2 + a_1^2 x_1^2 + \cdots + a_n^2 x_n^2,
\]
where \( a_i > 0, i = 1, 2, \cdots, n \). In this case, \( R_g = S_g = (0, \infty) \) and \( I_k = (0, k), k \in S_g \).

In view of the above theorems and Lemma 9 in Section 2, it is natural to ask the following question.

**Question 4.** Which functions \( g : \mathbb{R}^{n+1} \to \mathbb{R} \) satisfy Condition \((S^*)\)?

Partially, we answer Question 4 as follows.

**Theorem 5.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a nonnegative strictly convex function. For a nonzero real number \( \alpha \in \mathbb{R} \) with \( \alpha \neq 1 \), let’s denote by \( g \) the function defined by \( g(x, z) = z^\alpha - f(x) \). Suppose that the level hypersurfaces \( M_k(k \in S_g) \) of \( g \) in the \((n+1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \) are strictly convex. Then, the function \( g \) does not satisfy Condition \((S^*)\).

In [3], using harmonic function theory, the author and Y. H. Kim proved Theorem 5 when \( \alpha = 1 \).

Finally, we generalize the characterization theorem of [3] for elliptic paraboloids in the \((n+1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \) as follows.

**Theorem 6.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a nonnegative strictly convex function. For a nonzero real number \( \alpha \in \mathbb{R} \) with \( \alpha \neq 2 \), let’s denote by \( g \) the function defined by \( g(x, z) = z^\alpha - f(x) \). Suppose that the level hypersurfaces \( M_k(k \in S_g) \) of \( g \) in the \((n+1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \) are strictly convex. Then the following are equivalent.

1) The function \( g \) satisfies Condition \((V^*)\).
2) The function \( g \) satisfies Condition \((A^*)\).
3) For \( k \in S_g \), \( K(p)|\nabla g(p)|^{n+2} = c(k) \) is constant on \( M_k \).
4) The function \( g \) is given by
\[
g(x, z) = z - a_1^2 x_1^2 - \cdots - a_n^2 x_n^2.
\]
where $a_1, \cdots, a_n$ are positive constants. In this case, $R_g = S_g = R$ and $I_k = (k, \infty)$.

Throughout this article, all objects are smooth and connected, unless otherwise mentioned.

2. Preliminaries

Suppose that $M$ is a smooth strictly convex hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ with the unit normal $N$ pointing to the convex side. For a fixed point $p \in M$ and for a sufficiently small $t > 0$, consider the hyperplane $\Phi$ passing through the point $p + tN(p)$ which is parallel to the tangent hyperplane $\Psi$ of $M$ at $p$.

We denote by $A_p(t), V_p(t)$ and $S_p(t)$ the $n$-dimensional area of the section in $\Phi$ enclosed by $\Phi \cap M$, the $(n+1)$-dimensional volume of the region bounded by the hypersurface and the hyperplane $\Phi$ and the $n$-dimensional surface area of the region of $M$ between the two hyperplanes $\Phi$ and $\Psi$, respectively.

Now, we may introduce a coordinate system $(x, z) = (x_1, x_2, \cdots, x_n, z)$ of $\mathbb{E}^{n+1}$ with the origin $p$, the tangent hyperplane of $M$ at $p$ is the hyperplane $z = 0$. Furthermore, we may assume that $M$ is locally the graph of a non-negative strictly convex function $f : \mathbb{R}^n \to \mathbb{R}$. Hence $N$ is the unit normal pointing upward.

Then, for a sufficiently small $t > 0$ we have

\[ A_p(t) = \int_{f(x) < t} 1 \, dx, \quad (2.1) \]

\[ V_p(t) = \int_{f(x) < t} \{t - f(x)\} \, dx \quad (2.2) \]

and

\[ S_p(t) = \int_{f(x) < t} \sqrt{1 + |\nabla f|^2} \, dx, \quad (2.3) \]

where $x = (x_1, x_2, \cdots, x_n)$, $dx = dx_1 dx_2 \cdots dx_n$ and $\nabla f$ denote the gradient vector of the function $f$.

Note that we also have

\[ V_p(t) = \int_{f(x) < t} \{t - f(x)\} \, dx = \int_{z = 0}^t \int_{f(x) < z} 1 \, dx \, dz. \quad (2.4) \]

Hence, together with the fundamental theorem of calculus, (2.4) shows that

\[ V'_p(t) = \int_{f(x) < t} 1 \, dx = A_p(t). \quad (2.5) \]
In order to prove our theorems, first of all, we need the following.

**Lemma 7.** Suppose that the Gauss-Kronecker curvature $K(p)$ of $M$ at $p$ is positive with respect to the unit normal $N$ pointing to the convex side of $M$. Then we have the following.

1) \[ \lim_{t \to 0} \frac{1}{(\sqrt{t})^n} A_p(t) = \frac{(\sqrt{2})^n \omega_n}{\sqrt{K(p)}}, \] (2.6)

2) \[ \lim_{t \to 0} \frac{1}{(\sqrt{t})^{n+2}} V_p(t) = \frac{(\sqrt{2})^{n+2} \omega_n}{(n+2)\sqrt{K(p)}}, \] (2.7)

3) \[ \lim_{t \to 0} \frac{1}{(\sqrt{t})^n} S_p(t) = \frac{(\sqrt{2})^n \omega_n}{\sqrt{K(p)}}, \] (2.8)

where $\omega_n$ denotes the volume of the $n$-dimensional unit ball.

**Proof.** For proofs of 1), 2) and 3) with $n=2$, see Lemma 7 of [2]. For a proof of 3) with arbitrary $n$, see Lemma 8 of [3].

Now, we prove the following.

**Lemma 8.** Consider the family of strictly convex level hypersurfaces $M_k = g^{-1}(k)$ of a function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ which satisfies Condition $(V^*)$. Then, for each $k \in S_g$, on $M_k$ we have

\[ K(p)|\nabla g(p)||^{n+2} = c(k), \] (2.9)

which is independent of $p \in M_k$, where $K(p)$ is the Gauss-Kronecker curvature of $M_k$ at $p$ with respect to the unit normal $N$ pointing to the convex side and $\nabla g(p)$ denotes the gradient of $g$ at $p$.

**Proof.** By considering $-g$ if necessary, we may assume that $I_k$ is of the form $[k, a]$ with $a > k$, that is, $N = \nabla g/|\nabla g|$ on $M_k$. For a fixed point $p \in M_k$ and a small $t > 0$, we have

\[ V_p(t) = V_p^*(k, h(t)) = \phi_k(h(t)), \]

where $h = h(t)$ is a positive function with $h(0) = 0$. By differentiating with respect to $t$, we get

\[ A_p(t) = V_p'(t) = \phi_k'(h)h'(t), \] (2.10)
where $\phi'_k(h)$ denotes the derivative of $\phi_k$ with respect to $h$. Hence we obtain

$$\frac{1}{(\sqrt{t})^\alpha} A_p(t) = \frac{\phi'_k(h)}{(\sqrt{h})^n} \left( \frac{h(t)}{t} \right)^n h'(t). \quad (2.11)$$

Now we claim that

$$\lim_{t \to 0} h'(t) = |\nabla g(p)|. \quad (2.12)$$

Assuming (2.12), we also get

$$\lim_{t \to 0} \sqrt{\frac{h(t)}{t}} = \sqrt{|\nabla g(p)|}. \quad (2.13)$$

Let us put $\lim_{h \to 0} \phi'_k(h)/(\sqrt{h})^n = \gamma(k)$, which is independent of $p$. Then it follows from (2.11), (2.12), (2.13) and Lemma 7 that

$$K(p)|\nabla g(p)|^{n+2} = \frac{2^n \omega_2^2}{\gamma(k)^2}, \quad (2.14)$$

which is constant on the level hypersurface $M_k$. Thus it suffices to show that (2.12) holds.

In order to prove (2.12), we consider an orthonormal basis $E_1, \cdots, E_n, N(p)$ of $\mathbb{R}^{n+1}$ at $p \in M_k$, where $E_1, \cdots, E_n \in T_p(M_k)$ and $N(p) = \nabla g(p)/|\nabla g(p)|$ is the unit normal pointing to the convex side. We consider a 1-parameter family $\Phi_t$ of hyperplanes $\Phi_t(s_1, \cdots, s_n) = p + tN(p) + \sum_{i=1}^n s_i E_i$, which are parallel to the tangent hyperplane $\Phi_0$ of $M_k$ at $p$. For small $t > 0$, there exist $s_i = s_i(t), i = 1, 2, \cdots, n$ with $s_i(0) = 0$ such that $\Phi_t$ is tangent to $M_k + h(t)$ at $s_i = s_i(t), i = 1, 2, \cdots, n$. Hence we have

$$k + h(t) = g(\Phi_t(s_1, \cdots, s_n)) = g(p + tN(p) + \sum_{i=1}^n s_i(t) E_i). \quad (2.15)$$

Thus, by differentiating with respect to $t$, we get

$$\frac{dh}{dt} = \left( \nabla g(\Phi_t(s_1, \cdots, s_n)), N(p) + \sum_{i=1}^n \frac{ds_i}{dt} E_i \right). \quad (2.16)$$

Note that $\Phi_t(s_1, \cdots, s_n) \to p$ as $t$ tends to 0. Therefore, we obtain

$$\lim_{t \to 0} \frac{dh}{dt} = \lim_{t \to 0} \langle \nabla g(p), N(p) \rangle = |\nabla g(p)|, \quad (2.17)$$

which completes the proof of (2.12). This completes the proof. □

Similarly to the proof of Lemma 8, we may obtain
Lemma 9. Consider the family of strictly convex level hypersurfaces $M_k = g^{-1}(k)$ of a function $g: \mathbb{R}^{n+1} \to \mathbb{R}$ which satisfies either Condition $(A^*)$ or Condition $(S^*)$. Then, on $M_k$ with $k \in S_g$ we have

$$K(p)|\nabla g(p)|^{n+2} = d(k),$$

which is independent of $p \in M_k$, where $K(p)$ is the Gauss-Kronecker curvature of $M_k$ at $p$ with respect to the unit normal $N$ pointing to the convex side and $\nabla g(p)$ denotes the gradient of $g$ at $p$.

**Proof.** As in the proof of Lemma 8, we may assume that $I_k$ is of the form $[k, a]$ with $a > k$, that is, $N = \nabla g/|\nabla g|$ on $M_k$. For a fixed point $p \in M_k$ and a small $t > 0$, we have $A_p(t) = A_p^*(k, h(t))$ for some positive function $h = h(t)$ with $h(0) = 0$. Suppose that $M_k$ satisfies Condition $(A^*)$. Then, we have

$$A_p(t) = A_p^*(k, h(t)) = \psi_k(h(t))|\nabla g(p)|.$$

Hence we obtain

$$\frac{1}{(\sqrt{t})^n}A_p(t) = \frac{\psi_k(h)}{(\sqrt{h})^n}(\sqrt{\frac{h(t)}{t}})^n|\nabla g(p)|.$$

We put $\lim_{h \to 0}\psi_k(h)/(\sqrt{h})^n = \beta(k)$, which is independent of $p \in M_k$. Then it follows from (2.13), (2.20) and Lemma 7 that

$$K(p)|\nabla g(p)|^{n+2} = \frac{2^n \omega_n^2}{\beta(k)^2},$$

which is constant on the level hypersurface $M_k$.

The remaining case can be treated similarly. This completes the proof. □

3. Ellipsoids and elliptic hyperboloids

In this section, first of all, we prove Theorem 2.

For a nonzero real number $\alpha$ with $\alpha \neq 1$ and a nonnegative convex function $f(x)$ defined on $\mathbb{R}^n$, we consider the function $g(x, z) = z^\alpha - f(x)$. We assume that the level hypersurfaces $M_k, k \in S_g$ defined by $g(x, z) = k$ are all strictly convex, and hence each $M_k, k \in S_g$ has positive Gauss-Kronecker curvature $K$ with respect to the unit normal $N$ pointing to the convex side.

On each $M_k$, by differentiating, we have for a fixed point $p = (x, z) \in M_k$,

$$\nabla f = \alpha z^{\alpha-1} \nabla z,$$

$$|\nabla g(p)|^2 = \alpha^2 z^{2\alpha-2} + |\nabla f(x)|^2,$$

$$z_{ij} = \frac{1}{\alpha^2 z^{2\alpha-1}}(\alpha z^\alpha f_{ij} - (\alpha - 1)f_if_j), \quad i, j = 1, 2, \cdots, n,$$  

(3.1)
where \( z_i \) denotes the partial derivative of \( z \) with respect to \( x_i, i = 1, 2, \cdots, n \), and so on. The Gauss-Kronecker curvature \( K(p) \) of \( M_k \) at \( p \) is given by ([6])

\[
K(p) = \frac{\det(z_{ij})}{(\sqrt{1 + |\nabla z|^2})^{n+2}} = \frac{\alpha^{n+2} z^{(\alpha-1)(n+2)} \det(z_{ij})}{(\sqrt{\alpha^2 z^{2\alpha-2} + |\nabla f(x)|^2})^{n+2}},
\]

where the second equality follows from (3.1). Thus, it follows from (3.1) and (3.2) that

\[
K(p)|\nabla g|^n = \frac{\alpha^{n+2} z^{(\alpha-1)(n+2)} \det(z_{ij})}{\alpha^{n-2} z^{\alpha n-2\alpha+2}} \det(\alpha z^{\alpha} f_{ij} - (\alpha - 1)f_i f_j).
\]

First, suppose that the function \( g(x, z) = z^\alpha - f(x) \) satisfies Condition \((V^*)\) or Condition \((A^*)\). Then, it follows from Lemma 8 or Lemma 9 that the function \( g \) satisfies the condition 3) of Theorem 2. That is, there exists a constant \( c(k) \) depending on \( k \) such that

\[
\det(\alpha z^{\alpha} f_{ij} - (\alpha - 1)f_i f_j) = \alpha^{n-2} c(k) z^{\alpha n-2\alpha+2}.
\]

By substituting \( z^\alpha = f(x) + k \) into (3.4), we see that \( f(x) \) satisfies

\[
\det(\alpha f(x) + k) f_{ij} - (\alpha - 1)f_i f_j) = \alpha^{n-2} c(k) (f(x) + k)^{n-2+2/\alpha}.
\]

We denote by \( A_i, i = 1, 2, \cdots, n \) the \( i \)-th column vector of the matrix in the left hand side of (3.5). Then we have

\[
A_i = \alpha(f(x) + k) B_i - C_i,
\]

where

\[
B_i = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = \nabla f_i, \quad C_i = (\alpha - 1)f_i \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix} = (\alpha - 1)f_i \nabla f.
\]

Hence, it follows from the multilinear alternating property of determinant function that

\[
\det(A_1, \cdots, A_n) = \alpha^n (f(x) + k)^n \det(B_1, \cdots, B_n)
- \alpha^{n-1} (f(x) + k)^{n-1} \{ \det(C_1, B_2, \cdots, B_n)
+ \cdots + \det(B_1, B_2, \cdots, B_{n-1}, C_n) \}.
\]
Since \( \det(f_{ij}) = \det(B_1, \cdots, B_n) \), it follows from (3.5) and (3.8) that
\[
\begin{align*}
c(k)(f(x) + k)^{2-\alpha} &= \alpha^2 (f(x) + k) \det(f_{ij}) - \alpha \{ \det(C_1, B_2, \cdots, B_n) \\
& \quad + \cdots + \det(B_1, B_2, \cdots, B_{n-1}, C_n) \} \\
& = A(x)k + B(x),
\end{align*}
\]
where we use the following notations.
\[
A(x) = \alpha^2 \det(f_{ij}), \\
B(x) = \alpha^2 f(x) \det(f_{ij}) - \alpha \{ \det(C_1, B_2, \cdots, B_n) \\
& \quad + \cdots + \det(B_1, B_2, \cdots, B_{n-1}, C_n) \}.
\]
Note that the right hand side of (3.9) is a linear polynomial in \( k \) with functions in \( x = (x_1, \cdots, x_n) \) as coefficients. Furthermore, note that for each \( k \), \( c(k) \) is positive and \( f(x) + k \) is a nonconstant function in \( x \). It follows from (3.9) that
\[
c(k)^\alpha = (A(x)k + B(x))^{\alpha} (k + f(x))^{\alpha-2}.
\]
Suppose that \( \alpha \) is a nonzero real number with \( \alpha \neq 1, 2 \). Then, by using logarithmic differentiation of (3.11) with respect to \( x_i, i = 1, 2, \cdots, n \), we get
\[
\alpha(\nabla A(x)k + \nabla B(x))(k + f(x)) + (\alpha - 2)(A(x)k + B(x))\nabla f(x) = 0,
\]
which is a quadratic polynomial in \( k \). It follows from (3.12) and the assumption \( \alpha \neq 0, 1, 2 \) that \( \nabla f(x) = 0 \), which is a contradiction.

Thus, by assumption, we see that \( \alpha = 2 \) is the only possible case. In this case, (3.9) implies that for some constants \( a \) and \( b \), \( c(k) = ak + b \) with \( A(x) = a \) and \( B(x) = b \). It follows from (3.10) that
\[
\det(f_{ij}) = \frac{a}{4},
\]
and
\[
\det(C_1, B_2, \cdots, B_n) + \cdots + \det(B_1, B_2, \cdots, B_{n-1}, C_n) = \frac{1}{2} (af(x) - b),
\]
where
\[
B_i = \nabla f_i, \quad C_i = f_i \nabla f, \quad i = 1, 2, \cdots, n.
\]

Since \( f(x) \) is a nonnegative strictly convex function, (3.13) shows that \( \det(f_{ij}) \) is a positive constant on \( \mathbb{R}^n \). Hence \( f(x) \) is a quadratic polynomial given by ([1], [4])
\[
f(x_1, \cdots, x_n) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2, \quad a_1, \cdots, a_n > 0.
\]
Thus, the level hypersurfaces must be the elliptic hyperboloids $M_k = g^{-1}(k)$, where $g(x, z) = z^2 - (a_1^2 x_1^2 + \cdots + a_n^2 x_n^2), z > 0$ with $k > 0$ and $a_1, \cdots, a_n > 0$.

Conversely, consider the function $g$ given by $g(x, z) = z^2 - f(x), z > 0$ with $k > 0$, where $f(x) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2, a_1, \cdots, a_n > 0$. For the function $g$, we have $R_g = R - \{0\}, S_g = (0, \infty)$ and $I_k = (k, \infty), k \in S_g$.

For a fixed $k > 0$ and a small $h > 0$, consider the tangent hyperplane $\Psi$ of $M_k$ at a point $p \in M_k$. There exists a point $v \in M_{k+h}$ such that the tangent hyperplane $\Phi$ of $M_{k+h}$ at $v$ is parallel to the hyperplane $\Psi$. The two points $p$ and $v$ of tangency are related by

$$v = \frac{\sqrt{k+h}}{\sqrt{k}} p, \quad p = (p_1, \cdots, p_n, \sqrt{r^2 + k}), \quad r^2 = a_1^2 p_1^2 + \cdots + a_n^2 p_n^2. \quad (3.17)$$

Note that $V^*_p(k, h)$ denote the $(n + 1)$-dimensional volume of the region of $M_k$ cut off by the hyperplane $\Phi$.

Then the linear mapping

$$T_1(x_1, x_2, \cdots, x_n, z) = (a_1 x_1, a_2 x_2, \cdots, a_n x_n, z) \quad (3.18)$$

transforms $M_k$ (resp., $M_{k+h}$) onto a hyperboloid of revolution $M'_k: z^2 = x_1^2 + x_2^2 + \cdots + x_n^2 + k$, (resp., $M'_{k+h}: z^2 = x_1^2 + x_2^2 + \cdots + x_n^2 + h$), $\Phi$ to a hyperplane $\Phi'$, $p \in M_k$ and $v \in M_{k+h}$ to points of tangency $p' = (p'_1, \cdots, p'_n, \sqrt{(r')^2 + k}) \in M'_k$ and $v' = (\sqrt{k+h}/\sqrt{k})p' \in M'_{k+h}$, respectively, where $(r')^2 = \Sigma(p'_i)^2$. If we let $V^*_{p'}(k, h)$ denote the volume of the region of $M'_k$ cut off by the hyperplane $\Phi'$, then we get

$$V^*_p(k, h) = a_1 \cdots a_n V^*_{p'}(k, h). \quad (3.19)$$

Let’s consider the rotation $A$ around the $z$-axis which maps the point $p'$ of tangency to $p'' = (0, \cdots, 0, \sqrt{(r')^2 + k})$. Then the rotation $A$ takes $v'$ to $v'' = (\sqrt{k+h}/\sqrt{k})p''$. Note that the 1-parameter group $B(t)$ on the $x_n$-$z$-plane defined by

$$B(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \quad (3.20)$$

takes the upper hyperbola $z^2 = x_n^2 + k, z > 0$ (resp., $z^2 = x_n^2 + k + h, z > 0$) onto itself. Hence, there exists a parameter $t_0$ such that $B(t_0)$ maps $p''$ to $p''' = (0, \cdots, 0, \sqrt{k})$ (resp., $v''$ to $v''' = (0, \cdots, 0, \sqrt{k+h})$).

We consider the linear mapping $T_2 = \tilde{B}(t_0) \circ A$ of $\mathbb{R}^{n+1}$, where $\tilde{B}(t_0)$ denotes the extended linear mapping of $B(t_0)$ on $\mathbb{R}^{n+1}$ fixing $x_1 \cdots x_{n-1}$-plane. Then the linear mapping $T_2$ takes
the hyperboloid of rotation $M'_k$ (resp., $M'_{k+h}$) onto itself, $p'$ and $v'$ to the points of tangency $p''' = (0, \ldots, 0, \sqrt{k})$ and $v''' = (0, \ldots, 0, \sqrt{k + h})$, $\Phi'$ to the hyperplane $\Phi'' : z = \sqrt{k + h}$.

Due to the volume-preserving property of $T_2$, we obtain

$$V''_p(k, h) = V^*(k, h), \quad (3.21)$$

where $V^*(k, h)$ denotes the volume of the region of $M'_k$ cut off by the hyperplane $\Phi''$. Together with (3.19), it follows from (3.21) that

$$V^*_p(k, h) = \frac{\omega_n}{a_1 \cdots a_n} \{ \sqrt{k + h} h^{n/2} - n \int_0^{\sqrt{n}} \sqrt{r^2 + k} r^{n-1} dr \}, \quad (3.22)$$

where $\omega_n$ denotes the volume of the $n$-dimensional unit ball. Hence, we see that $V^*_p(k, h)$ is independent of the point $p \in M_k$, which is denoted by $\phi_k(h)$. Thus the function $g$ given by $g(x, z) = z^2 - f(x), z > 0$ satisfies Condition ($V^*$).

Finally, we show that the function $g$ given by $g(x, z) = z^2 - f(x), z > 0$ with $k > 0$, where $f(x) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2, a_1, \cdots, a_n > 0$ satisfies Condition ($A^*$). For a fixed point $p = (p_1, \cdots, p_n, \sqrt{r^2 + k}) \in M_k$, where $r^2 = a_1^2 p_1^2 + \cdots + a_n^2 p_n^2$, and a small $t \in R$, we have $V_p(t) = \phi_k(h(t))$ for some $h = h(t)$ with $h(0) = 0$. By differentiating with respect to $t$, from (2.5) we get

$$A_p(t) = V'_p(t) = \phi'_k(h) h'(t), \quad (3.23)$$

where $\phi'_k(h)$ denotes the derivative of $\phi_k$ with respect to $h$.

With the aid of (3.17), it is straightforward to show that

$$t = \frac{2\sqrt{k} (\sqrt{k + h} - \sqrt{k})}{|\nabla g(p)|}. \quad (3.24)$$

Hence we get

$$h(t) = \frac{1}{4k} |\nabla g(p)|^2 t^2 + |\nabla g(p)| t. \quad (3.25)$$

It follows from (3.24), (3.25) and (3.23) that

$$A^*_p(k, h) = A_p(t) = \frac{\sqrt{k + h}}{\sqrt{k}} \phi'_k(h) |\nabla g(p)|, \quad (3.26)$$

which shows that the function $g$ given by $g(x, z) = z^2 - f(x), z > 0$ satisfies Condition ($A^*$).

It follows from (3.9) and (3.10) with $\alpha = 2$ that the function $g$ given by $g(x, z) = z^2 - f(x), z > 0$ satisfies

$$K(p)|\nabla g(p)|^{n+2} = c(k) = 2^{n+2} a_1^2 a_2^2 \cdots a_n^2 k. \quad (3.27)$$
This completes the proof of Theorem 2.

Next, we prove Theorem 3 as follows.

For a nonzero real number $\alpha$ with $\alpha \neq 1$ and a nonnegative convex function $f(x)$ defined on $\mathbb{R}^n$, we consider the function $g(x,z) = z^\alpha + f(x)$. We assume that the level hypersurfaces $M_k, k \in S_g$ defined by $g(x,z) = k$ are all strictly convex, and hence each $M_k, k \in S_g$ has positive Gauss-Kronecker curvature $K$ with respect to the unit normal $N$ pointing to the convex side.

Suppose that the function $g$ satisfies Condition $(V^*)$ or Condition $(A^*)$. Then, it follows from Lemma 8 or Lemma 9 that $M_k$ satisfies the condition 3) of Theorem 3. Then, changing $f(x)$ by $-f(x)$ in the proof of Theorem 2, (3.11) shows that $f(x)$ satisfies

$$(-1)^\alpha c(k)^\alpha = (A(x)k + B(x))^{\alpha}(k - f(x))^{\alpha - 2},$$

where

$$A(x) = \alpha^2 \det(f_{ij}),$$
$$B(x) = -\alpha^2 f(x) \det(f_{ij}) + \alpha \{ \det(C_1, B_2, \cdots, B_n)$$
$$+ \cdots + \det(B_1, B_2, \cdots, B_{n-1}, C_n) \},$$

and $B_k = \nabla f_i, C_i = (\alpha - 1)f_i \nabla f, i = 1, 2, \cdots, n$. Since $c(k) > 0$, the logarithmic differentiation of (3.28) shows that $\alpha = 2$, and hence for some constants $a$ and $b$, we get $(-1)^\alpha c(k) = ak + b$ with $\det(f_{ij}) = a/4$. Since $f(x)$ is a nonnegative strictly convex function, this implies that $\det(f_{ij})$ is a positive constant on $\mathbb{R}^n$. By the same argument as in the proof of Theorem 2, we see that $f(x)$ is a quadratic polynomial given by

$$f(x_1, \cdots, x_n) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2, \quad a_1, \cdots, a_n > 0. \quad (3.30)$$

Thus, the level hypersurfaces must be the ellipsoids given by $g(x,z) = z^2 + a_1^2 x_1^2 + \cdots + a_n^2 x_n^2 = k$ with $k > 0$ and $a_1, \cdots, a_n > 0$.

Conversely, we consider the function $g$ given by $g(x,z) = z^2 + a_1^2 x_1^2 + \cdots + a_n^2 x_n^2$ with $a_1, \cdots, a_n > 0$. For the function $g$, we have $R_g = S_g = (0, \infty)$ and $I_k = (0, k), k \in S_g$.

For a fixed $k > 0$ and a point $p \in M_k$, consider the tangent hyperplane $\Psi$ of $M_k$ at $p$. For a sufficiently small $h < 0$ with $k + h > 0$, there exists a point $v \in M_{k+h}$ such that the tangent hyperplane $\Phi$ of $M_{k+h}$ at $v$ is parallel to the hyperplane $\Psi$. The two points $p$ and $v$ of tangency are related by

$$v = \frac{\sqrt{k+h}}{\sqrt{k}} p, \quad p = (p_1, \cdots, p_n, \sqrt{k-r^2}), \quad r^2 = a_1^2 p_1^2 + \cdots + a_n^2 p_n^2. \quad (3.31)$$
Then the linear mapping

\[ T_1(x_1, x_2, \cdots, x_n, z) = (a_1 x_1, a_2 x_2, \cdots, a_n x_n, z) \]  \hspace{1cm} (3.32)

transforms \( M_k \) (resp., \( M_{k+h} \)) onto a hypersphere \( M'_k : x_1^2 + x_2^2 + \cdots + x_n^2 + z^2 = k \), (resp., \( M'_{k+h} : x_1^2 + x_2^2 + \cdots + x_n^2 + z^2 = k + h \)), \( \Phi \) to a hyperplane \( \Phi' \), \( p \in M_k \) and \( v \in M_{k+h} \) to points of tangency \( p' = (p'_1, \cdots, p'_n, \sqrt{k - (r')^2}) \in M'_k \) and \( v' = (\sqrt{k + h}/\sqrt{k})p' \in M'_{k+h} \), respectively, where \((r')^2 = \Sigma(p'_i)^2\). The corresponding volume \( V_p^\infty(k, h) \) is given by

\[ V_p^\infty(k, h) = a_1 \cdots a_n V_p^\infty(k, h). \]  \hspace{1cm} (3.33)

By the symmetry of hyperspheres \( M'_k \) and \( M'_{k+h} \) centered at the origin, we see that \( V_p^\infty(k, h) \) is independent of the point \( p' \). This, together with (3.33), shows that the function \( g \) satisfies Condition \( (V^*) \).

Finally, we show that the function \( g \) given by \( g(x, z) = z^2 + a_1^2 x_1^2 + \cdots + a_n^2 x_n^2 \) with \( a_1, \cdots, a_n > 0 \) satisfies Condition \((A^*)\).

For a fixed point \( p = (p_1, \cdots, p_n, \sqrt{k - r^2}) \in M_k \), where \( r^2 = a_1^2 p_1^2 + \cdots + a_n^2 p_n^2 \), and a small \( t > 0 \), we have \( V_p(t) = \phi_k(h(t)) \) for some negative function \( h = h(t) \) with \( h(0) = 0 \). By differentiating with respect to \( t \), we get from (2.5)

\[ A_p(t) = V_p'(t) = \phi'_k(h) h'(t), \]  \hspace{1cm} (3.34)

where \( \phi'_k(h) \) denotes the derivative of \( \phi_k \) with respect to \( h \).

With the help of (3.31), it is straightforward to show that the distance \( t \) from \( p \in M_k \) to the tangent hyperplane \( \Phi \) to \( M_{k+h} \) at \( v \in M_{k+h} \) is given by

\[ t = \frac{2 \sqrt{k}(\sqrt{k} - \sqrt{k + h})}{|\nabla g(p)|}. \]  \hspace{1cm} (3.35)

Hence we get

\[ h(t) = \frac{1}{4k} |\nabla g(p)|^2 t^2 - |\nabla g(p)| t. \]  \hspace{1cm} (3.36)

It follows from (3.34), (3.35) and (3.36) that

\[ A_p^*(k, h) = A_p(t) = -\frac{\sqrt{k + h}}{\sqrt{k}} \phi'_k(h) |\nabla g(p)|. \]  \hspace{1cm} (3.37)

This shows that the function \( g \) given by \( g(x, z) = z^2 + a_1^2 x_1^2 + \cdots + a_n^2 x_n^2 \) with \( a_1, \cdots, a_n > 0 \) satisfies Condition \((A^*)\).
It follows from (3.28) and (3.29) with \( \alpha = 2 \) that the family \( M_k \) of ellipsoids satisfies

\[
K(p)\|\nabla g(p)\|^2 = c(k) = 2^{n+2}a_1^2a_2^2\cdots a_n^2k. \tag{3.38}
\]

This completes the proof of Theorem 3.

4. Condition \((S^*)\)

In this section, we prove Theorem 5.

Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a nonnegative strictly convex function. For a real number \( \alpha \in R \) with \( \alpha \neq 0,1 \), we consider the function \( g : \mathbb{R}^{n+1} \to \mathbb{R} \) defined by \( g(x,z) = z^2 - f(x) \).

Suppose that the level hypersurfaces \( M_k, k \in S_\alpha \) of \( g \) in the \((n+1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \) are strictly convex and that the function \( g \) satisfies Condition \((S^*)\). Then, as in the proof of Theorem 2, we can show that the function \( g \) is given by \( g(x,z) = z^2 - f(x), z > 0 \) with \( k > 0 \), where \( f(x) = a_1^2x_1^2 + \cdots + a_n^2x_n^2, a_1, \ldots, a_n > 0 \).

For a fixed \( k > 0 \) and a small \( h > 0 \), consider the tangent hyperplane \( \Phi \) of \( M_{k+h} \) at a point \( v \in M_{k+h} \) which is parallel to the tangent hyperplane \( \Psi \) of \( M_k \) at \( p \in M_k \). The two points \( p \) and \( v \) of tangency are related by

\[
v = \frac{\sqrt{k+h}}{\sqrt{k}}p = \frac{\sqrt{k+h}}{\sqrt{k}}(p_1, \cdots, p_n, \sqrt{r^2 + k}), r^2 = a_1^2p_1^2 + \cdots + a_n^2p_n^2. \tag{4.1}
\]

The tangent hyperplane \( \Phi \) of \( M_{k+h} \) at \( v \in M_{k+h} \) is given by

\[
\Phi : z = \frac{1}{\sqrt{r^2 + k}}\{a_1^2p_1x_1 + \cdots + a_n^2p_nx_n + \sqrt{k(k+h)}\}. \tag{4.2}
\]

The linear transformation \( y_1 = a_1x_1, \ldots, y_n = a_nx_n, z = z \) transforms \( M_k \) to \( M'_k : z^2 = |y|^2 + k, p = (p_1, \cdots, p_n, \sqrt{r^2 + k}) \) to \( q = (a_1p_1, \cdots, a_np_n, \sqrt{r^2 + k}) \), and \( \Phi \) to the hyperplane \( \Phi' \) defined by

\[
\Phi' : z = \frac{1}{\sqrt{|y|^2 + k}}\{\langle q, y \rangle + \sqrt{k(k+h)}\}. \tag{4.3}
\]

Hence the \( n \)-dimensional surface area \( S^*_p(k,h) \) of the region of \( M_k \) between the two hyperplanes \( \Phi \) and \( \Psi \) is given by

\[
S^*_p(k,h) = \frac{1}{a} \int_{D_q(k,h)} \frac{|(a_1^2 + 1)y_1^2 + \cdots + (a_n^2 + 1)y_n^2 + k|^{1/2}}{|y|^2 + k} dy, \tag{4.4}
\]

where \( a = a_1 \cdots a_n \) and

\[
D_q(k,h) : (|y|^2 + k)(|y|^2 + k) \leq (\langle q, y \rangle + \sqrt{k(k+h)})^2. \tag{4.5}
\]
By assumption, \( S_p^*(k, h) / |\nabla g(p)| = \eta_k(h) \) is independent of \( p \). Since we have

\[
|\nabla g(p)|^2 = 4((a_1^2 + 1)q_1^2 + \cdots + (a_n^2 + 1)q_n^2 + k),
\]

we see that

\[
\int_{D_q(k, h)} H(y)dy = 2a\sqrt{|q|^2 + k}H(q)\eta_k(h), \quad (4.6)
\]

where we denote

\[
H(y) = \frac{((a_1^2 + 1)y_1^2 + \cdots + (a_n^2 + 1)y_n^2 + k)^{1/2}}{\sqrt{|y|^2 + k}}. \quad (4.7)
\]

It is straightforward to show that \( D_q(k, h) \) is an ellipsoid centered at \( \sqrt{(k + h)/kq} \) and its canonical form is given by

\[
\frac{ky_1^2}{h(|q|^2 + k)} + \frac{y_2^2 + \cdots + y_n^2}{h} \leq 1. \quad (4.8)
\]

This shows that the volume of \( D_q(k, h) \) is given by

\[
V(D_q(k, h)) = \frac{1}{\sqrt{k}(\sqrt{h})^n} \sqrt{|q|^2 + k\omega_n}. \quad (4.9)
\]

Let’s denote by \( \theta_k(h) \) the function defined by

\[
\theta_k(h) = \frac{2a\sqrt{k}}{\omega_n(\sqrt{h})^n} \eta_k(h). \quad (4.10)
\]

Then, it follows from (4.6) and (4.9) that \( H(y) \) satisfies

\[
\frac{1}{V(D_q(k, h))} \int_{D_q(k, h)} H(y)dy = H(q)\theta_k(h). \quad (4.11)
\]

When \( q = 0 \), \( H(0) = 1 \) and \( D_0(k, h) \) is the ball \( B_0(\sqrt{h}) \) of radius \( \sqrt{h} \) centered at \( y = 0 \). Hence, (4.11) implies that for any positive numbers \( k \) and \( h \)

\[
\theta_k(h) = \frac{1}{V(B_0(\sqrt{h}))} \int_{B_0(\sqrt{h})} H(y)dy > 1. \quad (4.12)
\]

If we let \( a_1 = \max\{a_i\mid i = 1, 2, \cdots, n\} \), then we have from (4.7)

\[
1 = H(0) \leq H(y) < \sqrt{a_1^2 + 1}, \quad \lim_{y_1 \to \infty} H(y_1, 0, \cdots, 0) = \sqrt{a_1^2 + 1}. \quad (4.13)
\]
Thus, the left hand side of (4.11) is less than $\sqrt{a_1^2 + 1}$ for any positive numbers $k, h$ and $q \in \mathbb{R}^n$. But, since $\theta_k(h) > 1$, for $q = (q_1, 0, \cdots, 0)$ with sufficiently large $q_1$, the right hand side of (4.11) is greater than $\sqrt{a_1^2 + 1}$. This contradiction completes the proof of Theorem 5.

5. Elliptic paraboloids

In this section, we prove Theorem 6.

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a nonnegative strictly convex function. For a real number $\alpha \in \mathbb{R}$ with $\alpha \neq 0, 2$, let's consider the function $g$ defined by $g(x, z) = z^\alpha - f(x)$. We suppose that the level hypersurfaces $M_k, k \in S_g$ of $g$ in the $(n + 1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ are strictly convex.

Suppose that the function $g$ satisfies the condition 1) or 2) in Theorem 6. Then, it follows from Lemma 8 or Lemma 9 that $g$ satisfies the condition 3) in Theorem 6.

First, we show that the condition 3) in Theorem 6 implies 4) as follows. As in the proof of Theorem 2 in Section 3, we can show that if $\alpha \neq 0, 1, 2$, then (3.12) leads to a contradiction. Since $\alpha \neq 0, 2$, the remaining case is for $\alpha = 1$. In this case, it follows from (3.4) that

$$\det(f_{ij}(x)) = c(k).$$

(5.1)

Hence we see that $\det(f_{ij})$ is a positive constant $c$ with $c(k) = c$. Thus, $f(x)$ is a quadratic polynomial given by ([1], [4])

$$f(x_1, \cdots, x_n) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2, \quad a_1, \cdots, a_n > 0.$$  

(5.2)

Conversely, suppose that the level hypersurfaces are given by $M_k : z = f(x) + k, z > 0$ with $k > 0$, where $f(x) = a_1^2 x_1^2 + \cdots + a_n^2 x_n^2, a_1, \cdots, a_n > 0$. In this case, we have $R_g = S_g = R$ and $I_k = (k, \infty)$. From the proof of Theorem 5 in [3], we get

$$V_p^*(k, h) = \gamma_n h^{(n+2)/2} \quad \text{and} \quad A_p^*(k, h) = \frac{n+2}{2} \gamma_n |\nabla g(p)| h^{n/2},$$

(5.3)

where

$$\gamma_n = \frac{2\sigma_{n-1}}{n(n + 2)a_1a_2\cdots a_n}$$

(5.4)

and $\sigma_{n-1}$ denotes the surface area of the $(n - 1)$-dimensional unit sphere. It also follows from (3.3) with $\alpha = 1$ that

$$K(p)|\nabla g(p)|^{n+2} = 2^n a_1^2 a_2^2 \cdots a_n^2.$$  

(5.5)

This completes the proof of Theorem 6.
Corollary 10. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a nonnegative strictly convex function. For a nonzero real number \( \alpha \in \mathbb{R} \), let’s denote by \( g \) the function defined by \( g(x,z) = z^\alpha - f(x) \). Suppose that \( R_g = R \) and the level hypersurfaces \( M_k \) of \( g \) in the \((n+1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \) are strictly convex for all \( k \in R \). Then the following are equivalent.

1) The function \( g \) satisfies Condition \((V^\ast)\).
2) The function \( g \) satisfies Condition \((A^\ast)\).
3) \( K(p)|\nabla g(p)|^{n+2} = c(k) \) is constant on each \( M_k \).
4) For some positive constants \( a_1, \ldots, a_n \),
   \[
g(x,z) = z - (a_1^2x_1^2 + \cdots + a_n^2x_n^2).
   \]

**Proof.** Suppose that the function \( g \) satisfies one of the conditions 1), 2) and 3). Then as above, we have \( \alpha = 1 \) or \( \alpha = 2 \). In case \( \alpha = 2 \), \( c(k) \) is a nonconstant linear function in \( k \). This contradicts to the positivity of \( c(k) \). Hence we have \( \alpha = 1 \). Thus, Theorem 6 completes the proof. \( \square \)

**References**

[1] Jörgens, K., *Über die Lösungen der Differentialgleichung rt−s^2 = 1*, Math. Ann. 127(1954), 130-134.

[2] Kim, D.-S. and Kim, Y. H., *Some characterizations of spheres and elliptic paraboloids*, Linear Algebra Appl. 437 (2012), no. 1, 113-120.

[3] Kim, D.-S. and Kim, Y. H., *Some characterizations of spheres and elliptic paraboloids II*, Linear Algebra Appl. 438 (2013), no. 3, 1356-1364.

[4] Pogorelov, A. V., *On the improper convex affine hyperspheres*, Geom. Dedicata 1(1972), no. 1, 33-46.

[5] Stein, S., *Archimedes. What did he do besides cry Eureka?*, Mathematical Association of America, Washington, DC, 1999.

[6] Thorpe, J. A., *Elementary topics in differential geometry*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1979.