A Similarity Solution of Rear Stagnation-point Flow over a Flat Plate in Two Dimensions

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Abstract
This paper investigates the nature of the development of vortex shedding for two-dimensional unsteady flow of an incompressible fluid at the rear stagnation point.

Keywords: Rear Stagnation-point Flow, Third-order Partial Differential Equation, Analytical Solution, Numerical Solution

1. Introduction

The classical two-dimensional steady stagnation-point flow on the plane boundary \( y = 0 \) can be analysed exactly by Hiemenz [1]. At a rear stagnation point, on a circular cylinder say, the external flow is extracted away from the rear stagnation point. Common observation shows that when the flow is everywhere irrotational, a vortex sheet is formed near the plane and reversed flow develops in the region of vortical flow.

Forward stagnation point at which a balance is achieved between diffusion of vorticity and the inertia results in a steady solution. Rear stagnation-point flows against an infinite flat wall do not have analytic solution in two dimensions, but certain reverse flows have solution in three dimensions [2].

Proudman and Johnson [3] first suggested that the convection terms dominate in considering the inviscid equation in the body of the fluid. By introducing a very simple function of a particular similarity variable and neglecting the viscous forces in their analytic result for region sufficient far from the wall, they obtained an asymptotic solution in reversed stagnation-point flow, describing the development of the region of separated flow for large time \( t \).

The general feature of the predicted streamline pattern is sketched in Figure [1]

2. Flow Analysis Model

We shall demonstrate the flow analysis for the two-dimensional case. We begin with writing the governing equations in conservative velocity form in the Cartesian coordinates:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1a)
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (1b)
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (1c)
\]
The equation of continuity (1a) is integrated by introducing the stream function \( \psi \):

\[
\begin{align*}
\frac{\partial \psi}{\partial y} & \quad \text{and} \quad \frac{\partial \psi}{\partial x} \\
\end{align*}
\]

In rear stagnation flow without friction (ideal fluid flow), the stream function may be written as

\[
\psi = \psi_\infty = -A_\infty xy
\]

where \( A_\infty \) is a constant and from which

\[
\begin{align*}
U_\infty & = -A_\infty x \\
V_\infty & = A_\infty y.
\end{align*}
\]

We have \( U_\infty = 0 \) at \( x = 0 \) and \( V_\infty = 0 \) at \( y = 0 \), but the no-slip boundary at wall \( (y = 0) \) cannot be satisfied.

In a (real) viscous fluid the flow motion, Proudman and Johnson [3] model the flows by considering a very simple function of a particular similarity variable

\[
\begin{align*}
\psi & = -\sqrt{A \nu x} f(\eta, \tau) \\
\eta & = \sqrt{\frac{A}{\nu} y} \\
\tau & = A t
\end{align*}
\]

Equation (1) then gives, for \( f(\eta, \tau) \),

\[
f_{\eta\tau} - (f_\eta)^2 + \eta f_{\eta\eta} - \eta f_{\eta\eta\eta} = -1,
\]

with the boundary conditions

\[
\begin{align*}
f(0, \tau) & = f_\eta(0, \tau) = 0 \\
f_\eta(\infty, \tau) & = 1.
\end{align*}
\]
Considering unsteady similarity variables \( \eta \) in the form of \( \eta \) only, we have

\[
\psi = -\sqrt{A(t) \nu x f(\eta, \tau)} \tag{8a}
\]

\[
\eta = \sqrt{\frac{A(t)}{\nu y}} \tag{8b}
\]

Recall the governing equation \((1b)\)

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

and unsteady similarity variables \((5)\) yields

\[
-\frac{1}{2} \dot{A} \eta f'' - \dot{A} x f' + A^2 x (f')^2 - A^2 x f f'' = -\frac{1}{\rho} \frac{\partial p}{\partial x} - A^2 x f''
\]

or

\[
- f''' + ff'' - (f')^2 = -\frac{\dot{A}}{A^2} \left( f' + \frac{1}{2} \eta f'' \right) + \frac{1}{A^2 x \rho} \frac{\partial p}{\partial x} \tag{9}
\]

Substitution in boundary conditions \((7)\) gives the equation

\[
f''' - ff'' - 1 + (f')^2 = \kappa \left( f' + \frac{1}{2} \eta f'' - 1 \right) \tag{10a}
\]

\[
f(0) = f'(0) = 0 \tag{10b}
\]

\[
f'(\infty) = 1 \tag{10c}
\]

where

\[
\kappa = \frac{\dot{A}}{A^2} \tag{11}
\]

is required to be constant in time, and then integrates to give

\[
A(t) = \frac{1}{\kappa (t_0 - t)} \tag{12}
\]

approaching a singularity at finite time \( t = t_0 \).

Noting that \( \kappa = \dot{A}/A^2 = f L/U_\infty \) is equivalent to Strouhal number, describing the ratio between inertial forces due to the local acceleration and the inertial forces due to the convective acceleration in unsteady flow. \( f \) is the frequency of vortex shedding, \( L \) is the characteristic length and \( U_\infty \) is the external flow velocity.

It should be noted that the dimensionless velocity distribution \( f_\eta \) is, from \((5)\), independent of the length \( x \), and thus equaiton \((10)\) is a similarity equation of the full Navier-Stokes equation at two-dimension rear stagnation point.

3. Insolubility when \( \kappa = 0 \)

It is proven that all of the solutions, however, do not satisfy the boundary conditions when \( \kappa = 0 \).
Lemma 1. No solution \( f'(\eta) \) exists which has stationary value of 1 for finite \( \eta \) when \( \kappa = 0 \).

Proof. Set \( \kappa = 0 \) and rearrange Eq. (10) yields

\[
f''' = 1 - (f')^2 + f f''
\]  

(13)

Suppose for \( \eta = \eta_0 \), we have \( f'(\eta_0) = 1 \) and \( f''(\eta_0) = 0 \). Afterwards, it follows from the derivatives of Eq. (13) that \( f''' \) and all higher derivatives are zero when \( \eta = \eta_0 \). Considering a variable transformation

\[
\lambda(\eta) = f'(\eta) \\
\lambda(\eta_0) = 1
\]

(14)

Expand the function into Taylor’s series near \( \eta_0 \), we have

\[
f'(\eta) = \lambda(\eta) = \sum_{n=0}^{\infty} \frac{\lambda^{(n)}(\eta_0)}{n!} (\eta - \eta_0)^n
\]

\[
= \lambda(\eta_0) + \sum_{n=1}^{\infty} \frac{\lambda^{(n)}(\eta_0)}{n!} (\eta - \eta_0)^n
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{\lambda^{(n)}(\eta_0)}{n!} (\eta - \eta_0)^n
\]

\[
\equiv 1
\]

Hence, the boundary condition \( f'(0) = 0 \) is thus not satisfied and the Lemma is proved.

Lemma 2. When \( f' \) has a stationary value, if \( |f'| < 1 \) it is a minimum and if \( |f'| > 1 \) it is a maximum.

Proof: From Eq. (13), when \( f' \) has a stationary value, it means \( f'' = 0 \) and Eq. (13) becomes

\[
f''' = 1 - (f')^2
\]

(15)

If \( |f'| < 1 \), \( f'' > 0 \) and it is minima. Else if \( |f'| > 1 \), \( f''' < 0 \) and it is maxima. Eventually, the lemma is proved.

Theorem 1. Given any \( f'(\eta) \to 1 \) as \( \eta \to \infty \), no solution of Eq. (10) exists when \( \kappa = 0 \).

Proof: When \( |f'| < 1 \), since \( f' \to 1 \) as \( \eta \to \infty \), then \( f'' \) must be greater than zero. Hence, recall from Eq. (13),

\[
f''' = 1 - (f')^2 + f f'' > 0.
\]

for all \( \eta > \eta_0 \). After integrating \( f'''(\eta) > 0 \) from \( \eta_0 \) to \( \eta > \eta_0 \), we have

\[
f''(\eta) > f''(\eta_0) = K > 0.
\]

Another integration from \( \eta_0 \) to \( \eta > \eta_0 \) yields

\[
f'(\eta) > f'(\eta_0) + K(\eta - \eta_0).
\]

By Lemma 2, \( f'(\eta) \) has at most one stationary value because one cannot have two consecutive stationary values which are both minima. Since \( f''(\eta) > 0 \), when \( \eta \to \infty \), \( f'(\eta) \to \infty \). It violates that \( f'(\eta) \to 1 \). A similar argument shows that a solution cannot approach to 1 when \( |f'| > 1 \).
4. Similarity Analysis

Equation (10) is a third-order nonlinear ordinary differential equation. A crucial step in obtaining an analytical solution involves rearranging the equation as an autonomous differential equation. In mathematics, an autonomous differential equation is a system of ordinary differential equations which does not explicitly depend on the independent variable.

In order to omit the variable $\eta$ in the differential equation (10), it is generally accepted as a change of variable

$$F = f + \frac{\kappa}{2} \eta$$

and the equation becomes

$$F''' - FF'' + (F')^2 - 2\kappa F' - \frac{3}{4}\kappa^2 - 1 + \kappa = 0$$

$$F(0) = 0, \quad F'(0) = \frac{\kappa}{2}, \quad F'(\infty) = 0.$$

Noting that the equation (17) becomes to an autonomous differential equation when $-\frac{3}{4}\kappa^2 - 1 + \kappa = 0$. In another word, if $\kappa = -2$ or $\kappa = \frac{2}{3}$, the equation (17) could be expressed into a differential equation without dependent variable.

4.1. Case of $\kappa = \frac{2}{3}$

The differential equation (17) changes into the form as

$$F''' - FF'' + (F')^2 - \frac{4}{3}F' = 0$$

The solution of differential equation (18) can be obtained similarly by rewriting as another polynomial

$$F = a + be^{-\delta \eta}$$

$$a = \frac{1}{\delta}$$

$$b = -\frac{1}{\delta}$$

Substituting (19) into equation (18), and comparing the coefficients in the powers of $e^{-\delta \eta}$ leads to the determination of $\delta = \pm 1$. This results into another velocity function as

$$f(\eta) = -\frac{1}{3} \eta \pm \frac{1}{3} (1 - e^{\mp \eta})$$

$$f'(\eta) = -\frac{1}{3} (1 - e^{\mp \eta})$$

where $f'$ tends exponentially to a negative constant as $\eta \to \infty$. We get an immediate contradiction that the flow field is not able to remain unchanged at sufficient distances far away from the wall at any finite time. As a result, no solution to equation (18) exists.
4.2. Case of $\kappa = -2$

The differential equation (17) changes into the form as

$$F''' - FF'' + (F')^2 + 4F' = 0 \quad (21)$$

Equation (21) is analytically solvable that the solution might be expressed as a low order polynomial. It is suggested that

$$F = a + be^{-\delta \eta} \quad (22a)$$

$$a = -\frac{1}{\delta} \quad (22b)$$

$$b = \frac{1}{\delta} \quad (22c)$$

Substituting (22) into equation (21), and comparing the coefficients in the powers of $e^{-\delta \eta}$ leads to the determination of $\delta = \pm \sqrt{3i}$. Collecting results, the velocity function becomes

$$f(\eta) = \eta \mp \frac{1}{\sqrt{3i}} (1 - e^{\mp \sqrt{3i} \eta}) \quad (23a)$$

$$f'(\eta) = 1 - e^{\mp \sqrt{3i} \eta} \quad (23b)$$

The particular solution (23) is noteworthy in that it is completely analytical and can be expressed into the complex form as

$$f(\eta) = \eta - \frac{1}{\sqrt{3}} \sin \sqrt{3} \eta \mp \frac{i}{\sqrt{3}} (1 - \cos \sqrt{3} \eta) \quad (24a)$$

$$f'(\eta) = 1 - \cos \sqrt{3} \eta \pm i \sin \sqrt{3} \eta \quad (24b)$$

In view of solution (23) and (24), the flow far away from the boundary becomes

$$\lim_{\eta \to \infty} f(\eta) = \eta$$

which implies that the flow field remains unchanged at sufficient distances and the potential flow can be satisfied as the boundary condition as $\eta \to \infty$. No trouble arose from the idealization of Proudman and Johnson to neglecting the viscous term in their analytic result for region sufficient far from the wall.

The solution is obtained in the similarity transformation for unsteady viscous flows. The first term of (24) shows that the external flow is directed toward the $y-$axis and away from the wall. The appearance of a negative value in the second term in (24) describes a periodic velocity directed toward the wall.

5. Numerical Solution

Equation (17) is a third-order nonlinear ordinary differential equation. In numerical analysis, the Runge-Kutta methods are an important family of implicit and explicit iterative methods for the approximation of solutions of ordinary differential equations. This method
applies a trial step at the midpoint of an interval to cancel out lower-order error terms, besides; Runge–Kutta formulas are the methods of solving initial value problems for ordinary differential equations.

For example solving an \( n \)-th order problem numerically is common practice to reduce the equation to a system of \( n \) first-order equations. Then, by defining \( y_1 = F, \ y_2 = F', \ y_3 = F'' \), the ODE reduces to the form

\[
\frac{dy}{d\varsigma} = \begin{bmatrix} y_2 \\ y_3 \\ 3/4\kappa^2 + 1 - \kappa + y_1y_3 - y_2^2 + 2\kappa y_2 \end{bmatrix}
\]  

(25)

The first task is to reduce the equation above to a system of first-order equations and define in MATLAB a function to return these. The relevant MATLAB expression for Eq. (25) would be:

**Listing 1: System of first-order equations**

```matlab
function dy = stagnation(t,y)
k=-2;
dy = zeros(3,1);
dy(1) = y(2);
dy(2) = y(3);
dy(3) = 1-y(2)*y(2)+y(1)*y(3)+k*(y(2)+0.5*t*y(3)-1);
```

Later, it is required to apply `ode23`, an ode solver in MATLAB, to solve the initial value problem. The commands written in MATLAB would be

**Listing 2: ODE solver**

```matlab
function main
x=20;
[T,Y] = ode23(@stagnation,[0 x],[0 0 0]);
X = Y;
plot(T,Y(:,1),'-r',T,Y(:,2),'-g',T,Y(:,3),'-b')
```

The complete solutions of two-dimensional rear stagnation-point flow with different values of \( \kappa \) are shown from Figures (2) to (8). In these figures the similarity stream function \( f \), the velocity profile \( f' \) and the shear stress \( f'' \) are represented.

The result looks interesting from both theoretical and engineering points of view. A single dividing streamline plane separates streamlines approaching the plate from external flow streamlines. It is observed that if \( \kappa \leq -2 \), the similarity solution remains unchanged at distance far away from wall. When \( -2 < \kappa \leq -1.5 \), the solution becomes periodic on the entire flow region. And when \( \kappa > -1.5 \), the solution behaves frustrated and does not remain unchanged far away from the wall.

It is reasonable to state that, in general, separation will occur near the wall as \( \eta \to 0 \). The phenomenon of reversed flow with boundary-layer separation occurred and the region of reversed flow will move outward away from the wall periodically as \( \kappa < -2 \).

6. Conclusion

Analytical and numerical analysis of rear stagnation-point flow is studied. This study provides that the similarity solution of rear stagnation-point flow does not exist in particular
Figure 2: Numerical solution as $\kappa = -5$

Figure 3: Numerical solution as $\kappa = -3$
Figure 4: Numerical solution as $\kappa = -2$

Figure 5: Numerical solution as $\kappa = -1.8$
cases, because the governing equation does not satisfy the boundary conditions. Similarity equations are solved and the flow solution is provided as an exact solution of Navier-Stokes equations.

References

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Figure 7: Numerical solution as $\kappa = -1.385$

Figure 8: Numerical solution as $\kappa = -1.38$