STANDARD YOUNG TABLEAUX AND LATTICE PATHS

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Abstract. Using lattice path counting arguments, we reproduce a well known formula for the number of standard Young tableaux. We also produce an interesting new formula for tableaux of height \( \leq 3 \) using the Fourier methods of Ault and Kicey.

1. Definitions and Statement of Theorem

In this short paper, we use lattice path counting arguments to develop a generating function for the number of standard Young tableaux of shape \( \lambda \), which we denote \( f^{\lambda} \). From there, a well-known formula for \( f^{\lambda} \) can be derived. The formula is not new, and neither do we claim to have discovered a new generating function. What is new (as far as can be determined) is the connection between lattice paths and \( f^{\lambda} \). Moreover, the Fourier methods of Kicey and the author \cite{1} provide an interesting (albeit not necessarily useful) formula for \( f^{\lambda} \) where the height of \( \lambda \) is no greater than 3.

We assume that the reader is familiar enough with the basic ideas of Young tableaux. No deep representation theory is required. The following definitions are fairly standard in the literature (e.g. \cite{7, 8, 10}). Fix \( r \in \mathbb{N} \). A partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) will have the property that \( \lambda_i \geq \lambda_j \) as long as \( i \leq j \). For convenience, we allow \( \lambda_i = 0 \). Necessarily, all zero parts will occur at the end of the sequence. If \( \lambda_r > 0 \), then we say that \( \lambda \) has height \( r \). Denote the size of \( \lambda \) by \( |\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_r \). The number of standard tableaux of shape \( \lambda \) will be denoted by \( f^{\lambda} \). Denote by \( \Sigma_r \) the symmetric group on the \( r \) letters, which acts on \( r \)-tuples \( x = (x_1, x_2, \ldots, x_r) \) by permuting entries:

\[ \sigma x = (x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(r)}). \]

For any \( r \in \mathbb{N} \), let \( r = (0, 1, 2, \ldots, r - 1) \), and denote by \( r^* \) the reversed \( r \)-tuple, \( (r - 1, r - 2, \ldots, 1, 0) \). Let \( x_1, x_2, \ldots, x_r \) be formal commuting variables, and let \( x = (x_1, x_2, \ldots, x_r) \). Furthermore, if \( m = (m_1, m_2, \ldots, m_r) \), then let \( x^m = x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} \). Let \( V_r \) be the Vandermonde polynomial,

\[ V_r = \sum_{\sigma \in \Sigma_r} \text{sgn}(\sigma) x^{\sigma(r^*)} = \prod_{i<j} (x_i - x_j) \]

Finally, let \( t_r = \sum_{i=1}^r x_i \). We will be working with Laurent polynomials in the variables \( \{x_1, x_2, \ldots, x_r\} \). For such a function \( f \), let \( [f]_m \) be the coefficient of the term \( x^m \). The following theorem is already present (at least implicitly) in the literature. For example, Fulton and Harris develop the tools in §4 of \cite{4}.

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Theorem 1. The number of standard tableaux of shape $\lambda$ whose height is no greater than $r$ is equal to the $x^\mu$ coefficient of $i^\mu_r V_r$, where $n = |\lambda|$ and $\mu = \lambda + r^*$. Equivalently,

\[
f^\lambda = \left[ \frac{i^\mu_r V_r}{x^{\mu^*}} \right]_\lambda
\]

Theorem 1 implies that $F_{n,r} = \frac{i^\mu_r V_r}{x^{\mu^*}}$ is a generating function for the numbers $f^\lambda$ where $\lambda$ is a partition of $n$ having up to $r$ components. Theorem 1 directly implies the following formula, which is well-known in the literature [7].

\[
f^\lambda = \begin{pmatrix} \mu_1 & \mu_2 & n & \mu_3 & \cdots & \mu_r \end{pmatrix} \prod_{i<j}(\mu_i - \mu_j),
\]

where $\mu_k = \lambda_k + r - k$.

2. Proof of Theorem 1 by Way of Lattice Paths

Fix $r \in \mathbb{N}$ and consider the set,

\[
\Lambda^r = \{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{Z}^r \mid \lambda_i \geq \lambda_j \geq 0 \text{ for all } i \leq j \}.
\]

We may consider elements $\lambda \in \Lambda^r \subseteq \mathbb{N}_0^r$ to be partitions, points, or vectors as necessary. Each $\lambda \in \Lambda^r$ fits into a directed graph determined by the order in which the number labels are inserted into the Ferrers diagram of the Young tableau. For example, for $\lambda = (3,1)$ there are $f^\lambda = 3$ ways to fill in the Young tableau, as suggested by Figure 1. (Note, the shape $(3,1)$ is equivalent to $(3,1,0)$). Each step in the directed graph adds 1 to one of the components of the vector, as long as the vector components remain non-increasing after the addition. Thus, if $\lambda = (1,1)$, then $\lambda + (1,0) = (2,1)$ is legal, but $\lambda + (0,1) = (1,2)$ is not.

Let

\[
\delta_i = (0, \ldots, 1, \ldots, 0).
\]

There is a recursive formula,

\[
f^\lambda = \sum_{i=1}^{r} f^{\lambda - \delta_i},
\]

where $f^\mu = 0$ whenever $\mu$ is not a legal partition – that is, whenever $\mu \notin \Lambda^r$. We take the convention that there is exactly one trivial Young tableau, corresponding to $\lambda = 0 = (0,0,\ldots,0)$, so we have $f^0 = 1$.

The key to our argument is to interpret partitions $\lambda$ as points in the lattice $\mathbb{N}_0^r$ with move set $\{\delta_i \mid 1 \leq i \leq r\}$, where a move in the direction $\delta_i$ is viewed as appending and labeling a new square to the tableau at row $i$. Thus each path that remains entirely within $\Lambda^r$ (the legal partitions) beginning at the origin and ending at the point $\lambda$ represents a distinct way to fill in the Ferrers digram of shape $\lambda$, and conversely every standard Young tableau has an associated lattice path within $\Lambda^r$ determined by its numeric labels. Therefore, the count of all such lattice paths from $0$ to $\lambda$ is equal to $f^\lambda$. See Figure 2 for examples of the lattice $\Lambda^r$ for $r = 2$ and $r = 3$.

Counting paths in $\Lambda^r$ is complicated by the restrictions on legal partitions $\lambda$. A simple shift of the lattice by $r^*$ suffices to make things easier. Indeed, this shift
Figure 1. A portion of the underlying graph $\Lambda^3$ showing individual standard tableaux. Different arrowheads signify the addition of different unit vectors.

Figure 2. $\Lambda^2$ with $\lambda_i \leq 4$, left; $\Lambda^3$ with $\lambda_i \leq 3$, right. The vertex numbers count number of paths from the origin to each point. These numbers coincide with $f^\lambda$.

will also allow Eqn. (4) to be defined more explicitly in terms of identifying exactly when $\lambda - \delta_i$ is a legal partition. Let

(5) $\hat{\Lambda} = \Lambda^r + r^* = \{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{Z}^r \mid \lambda_i > \lambda_j \text{ for all } i \leq j, \text{ and } \lambda_r \geq 0 \}$.

Observe, if $\lambda \in \Lambda^r$ and $\lambda - \delta_i \notin \Lambda^r$, then it must necessarily be the case that $\lambda - \delta_i + r^*$ has either an entry equal to $-1$ or a repeated entry.
Inspired by methods found in Ault and Kicey [2, 1], consider functions \( v : \tilde{\Lambda}^r \rightarrow \mathbb{Z} \), which may be extended to \( v : N_0^r \rightarrow \mathbb{Z} \) in a manner to be explained presently.

**Definition 2.** A function \( v : N_0^r \rightarrow \mathbb{Z} \) will be called \( \Sigma \)-admissible if it satisfies \( v(\sigma x) = \text{sgn}(\sigma) v(x) \) for all \( \sigma \in \Sigma_r \).

**Lemma 3.** If \( v \) is \( \Sigma \)-admissible, and if \( x \) has a repeated entry, then \( v(x) = 0 \).

*Proof.* Suppose \( x_i = x_j \). Let \( \sigma \in \Sigma_r \) be the transposition \((i, j)\). Then \( v(x) = v(\sigma x) = -v(x) \), which implies that \( v(x) = 0 \).

For convenience, we assume all \( \Sigma \)-admissible functions \( v \) take the value \( v(x) = 0 \) if any component of \( x \) is negative. Thus, if \( v : \tilde{\Lambda}^r \rightarrow \mathbb{Z} \), then there is a unique, well-defined extension of \( v \) to a function (also called \( v \)) such that \( v : \mathbb{Z}^r \rightarrow \mathbb{Z} \) and \( v \) is \( \Sigma \)-admissible; namely, set \( v(x) = 0 \) whenever \( x \) has negative or repeated entries and \( v(\sigma x) = \text{sgn}(\sigma) v(x) \) whenever \( x \notin \tilde{\Lambda}^r \) has distinct non-negative entries.

**Example 4.** If we defined \( v_0(2, 1, 0) = 1 \) (where \( r = 3 \)), and \( v_0(x) = 0 \) for all \( x \in \tilde{\Lambda}^3 \) for which \( x \neq (2, 1, 0) \), then \( v_0 \) may be extended to be \( \Sigma \)-admissible by setting \( v_0(0, 2, 1) = v_0(1, 0, 2) = v_0(2, 1, 0) = 1 \), \( v_0(0, 1, 2) = v_0(2, 0, 1) = v_0(1, 2, 0) = -1 \), and \( v_0(x) = 0 \) on all other points of \( \tilde{\Lambda}^3 \).

For each \( 1 \leq i \leq r \), let \( R_i \) “right-shift” operator defined on functions \( v : \mathbb{Z}^r \rightarrow \mathbb{Z} \) by

\[
R_i[v](x_1, \ldots, x_r) = v(x_1, \ldots, x_i - 1, \ldots, x_r).
\]

Let \( T_r = \sum_{i=1}^r R_i \).

**Lemma 5.** If \( v \) is \( \Sigma \)-admissible, then so is \( T_r[v] \).

*Proof.* It suffices to show that \( T_r[v](\sigma x) = -T_r[v](x) \) for a transposition \( \sigma \). Let \( \sigma = (j, k) \) for some \( 1 \leq j < k \leq r \), and let \( x_{jk} = \sigma x \). Clearly, \( x_{jk} \) has the same entries as \( x \) but with entries \( j \) and \( k \) swapped.

\[
T_r[v](\sigma x) = \sum_{i=1}^r R_i[v](x_{jk}) = \sum_{i=1}^r v(x_{jk} - \delta_i)
\]

Note that if \( i \neq j, k \), then we have by \( \Sigma \)-admissibility of \( v \),

\[
v(x_{jk} - \delta_i) = -v(x - \delta_i)
\]

Moreover, for \( i = j \) and \( i = k \), we have:

\[
v(x_{jk} - \delta_j) = -v(x - \delta_k)
\]

\[
v(x_{jk} - \delta_k) = -v(x - \delta_j)
\]

Therefore, \( T_r[v](\sigma x) = \sum_{i=1}^r (-v(x - \delta_i)) = -T_r[v](x) \), as required.

Define \( v_0 : \tilde{\Lambda}^r \rightarrow \mathbb{Z} \) such that \( v_0(r^*) = 1 \) and \( v_0(x) = 0 \) for \( x \neq r^* \), and extend \( v_0 \) to a \( \Sigma \)-admissible function. For each \( n \in \mathbb{N} \), let \( v_n = T_r^n[v_0] \), which by Lemma 5 is \( \Sigma \)-admissible.

**Lemma 6.** \( v_n(x) \) counts the number of paths of length \( n \) from \( r^* \) to \( x \) that remain entirely in the lattice \( \tilde{\Lambda}^r \), using the move set \{ \( \delta_i \mid 1 \leq i \leq r \) \}. 
Proof. There is one 0-length path beginning and ending at \( r^* \); therefore \( v_0 \) counts all 0-length paths. now fix \( n \in \mathbb{N} \) and suppose that \( v_n(x) \) counts all \( n \)-length paths from \( r^* \) to \( x \) within \( \tilde{\Lambda}^r \). Because \( v_n \) is \( \Sigma \)-admissible, we have \( v_n(x) = 0 \) on all boundary points \( x \) adjacent to the lattice. Thus, for any \( x \in \tilde{\Lambda}^r \), we have \[
v_{n+1}(x) = T[v_n](x) = \sum_{i=1}^{r} v_n(x - \delta_i),
\]
where \( v_n(x - \delta_i) = 0 \) for any case in which \( x - \delta_i \notin \tilde{\Lambda}^r \). Thus \( v_{n+1} \) counts all paths from \( r^* \) to \( x \) of length \( n + 1 \) that remain entirely within \( \tilde{\Lambda}^r \). \( \square \)

Lemma 7. If \( |\lambda| = n \), then \( v_n(\lambda + r^*) = f^\lambda \).

Proof. Let \( 0 = (0, 0, \ldots, 0) \) be the empty partition. By \( \text{[3]} \) and Lemma \( \text{[6]} \), we have \( v_0(0 + r^*) = 1 = f^0 \). Next, assume that Lemma \( \text{[7]} \) is true for all \( \lambda \) of length \( |\lambda| = n \) (for some fixed \( n \geq 0 \)). Making use of Eqn. \( \text{[4]} \), we may derive the formula for \( \lambda \) with \( |\lambda| = n + 1 \) as follows. Lemma \( \text{[3]} \) shows that \( v_k \) is zero on the boundaries where \( \lambda - \delta_i + r^* \notin \tilde{\Lambda}^r \), which is equivalent to \( \lambda - \delta_i \notin \Lambda^r \); in other words, whenever \( \lambda - \delta_i \) is not a legal partition due to having negative or repeated entries. Then it follows that:

\[
f^\lambda = \sum_{i=1}^{r} f^{\lambda - \delta_i} = \sum_{i=1}^{r} v_n(\lambda - \delta_i + r^*) = T_r[v_n](\lambda + r^*) = v_{n+1}(\lambda + r^*).\]

\( \square \)

Finally, we make the connection to Eqn. \( \text{[2]} \). Recall \( V_r \) from Eqn. \( \text{[1]} \). There are exactly \( r! \) nonzero terms, including \( x^r \), whose coefficient is 1. It should be clear from its form as a sum of terms of the form \( \text{sgn}(\sigma)x^{|\sigma|}r^{|\sigma|} \) that \( V_r \) is the (ordinary) generating function corresponding to \( v_0 \). Moreover, the transition operator \( T_r \) corresponds to multiplication by \( t_r = x_1 + \cdots + x_r \). Thus, the generating function that counts the number of \( n \)-length paths in \( \tilde{\Lambda}^r \) from \( r^* \) to \( x \) is precisely \( t_n^r V_r \).

That is, \( t_n^r V_r \) is the generating function for \( v_n \). Lemma \( \text{[7]} \) then implies that a shift of exponents is needed to get the generating function for \( f^\lambda \); namely, reduction of all multi-exponents by \( r^* \), which corresponds to division by \( x^r \) in the generating function. This completes the argument and proves Theorem \( \text{[4]} \).

3. Examples

Example 8. Counting \( f^{(k,\ell)} \), or standard tableaux with at most two rows and \( k \geq \ell \). Here, \( r = (0,1), V_r = V_2 = x_1 - x_2, \) and \( t_r = t_2 = x_1 + x_2 \).

\[
F_{n,2} = \frac{t_n^r V_2}{x_1^1 x_2^n} = \frac{(x_1 + x_2)^n (x_1 - x_2)}{x_1} = (x_1 + x_2)^n \left( 1 - \frac{x_2}{x_1} \right)
\]
Thus, with $n$ walks in Weyl alcoves. For $n$ plane can be viewed as an affine lattice $A$ where $\lambda$ simple roots $\alpha$ do not easily scale beyond $\Lambda$. Observe that $\Lambda$ interior points of an alcove in $\Lambda$. This completes the proof.

Note that the terms whose exponents are multi-indeces that correspond to legal partitions $\lambda$ have the expected coefficients. For example, we can look at $F_{7,2}$. We shall focus on the case $r = 3$, that is, we shall develop a formula to count $f^\lambda$ where $\lambda = (a, b, c)$. As this is precisely the case for which methods of Ault-Kicey [1] will be useful and interesting. (Unfortunately, the Fourier methods of Ault-Kicey do not easily scale beyond $A_2$ lattices. For more about walks in arbitrary Weyl lattices. For more about walks in arbitrary Weyl

The first few of these are:

$F_{1,2} = x_1 - \frac{x_2}{x_1}$

$F_{2,2} = x_1^2 + x_1x_2 - \frac{x_3}{x_1}$

$F_{3,2} = x_1^3 + 2x_1^2x_2 - 2x_3^2 - \frac{x_4}{x_1}$

$F_{4,2} = x_1^4 + 3x_1^3x_2 + 2x_1^2x_3 - 2x_1x_2^2 - 2x_3^3 - 3x_4^2 - \frac{x_5}{x_1}$

$F_{5,2} = x_1^5 + 4x_1^4x_2 + 5x_1^3x_3^2 - 5x_1x_3x_2^4 - 4x_2^5 - \frac{x_6}{x_1}$

$F_{6,2} = x_1^6 + 5x_1^5x_2 + 9x_1^4x_3^2 + 5x_1^3x_2x_4 - 5x_1x_2^3x_3^2 - 5x_2^4 - 6x_4^5 - \frac{x_7}{x_1}$

$F_{7,2} = x_1^7 + 6x_1^6x_2 + 14x_1^5x_3^2 + 14x_1^4x_2x_4 - 14x_1^3x_2^3x_3^2 + 14x_1x_2^4x_3^4 - 6x_2^6 - 6x_7^5 - \frac{x_8}{x_1}$

In this simple case, the Binomial Theorem can be used to produce an explicit formula.

$F_{n,2} = \left(1 - \frac{x_2}{x_1}\right) \sum_{k=0}^{n} \binom{n}{k} x_1^{k} x_2^{n-k}$

$= \sum_{k=0}^{n} \binom{n}{k} x_1^{k} x_2^{n-k} - \sum_{k=0}^{n} \binom{n}{k} x_1^{k-1} x_2^{n-k+1}$

$= \sum_{k=0}^{n} \binom{n}{k} x_1^{k} x_2^{n-k} - \sum_{k=1}^{n-1} \binom{n}{k+1} x_1^{k} x_2^{n-k}$

Thus, with $\ell = n - k$, if $k \geq \ell$, then the above yields:

$f^{(k,\ell)} = \binom{k+\ell}{\ell} - \binom{k+\ell}{k+1}$

4. SLICES OF THE LATTICE

The functions $v_n(x)$ defined above may be interpreted in a different way – as walks in Weyl alcoves. For $n \in \mathbb{N}_0$, let $\Lambda^r_n = \{ \lambda \in \Lambda \mid |\lambda| = n \}$, which we call the $n$th slice of $\Lambda^r$. Observe that $\Lambda^r_n$ is a subset of the plane $x_1 + x_2 + \cdots + x_r = n$. This plane can be viewed as an affine lattice $A_{n-1}$, whose move set is determined by the simple roots $\alpha_i = \delta_i - \delta_{i+1}$. Then $\Lambda_n^r$, with an appropriate shift, exists within the interior points of an alcove in $A_{n-1}$.

We shall focus on the case $r = 3$, that is, we shall develop a formula to count $f^\lambda$ where $\lambda = (a, b, c)$, as this is precisely the case for which methods of Ault-Kicey [1] will be useful and interesting. (Unfortunately, the Fourier methods of Ault-Kicey do not easily scale beyond $A_2$ lattices. For more about walks in arbitrary Weyl
alcoves and related structures, see [4, 5, 6, 9].) Let $A_2$ be defined in the usual way as a lattice in the plane generated by roots $\alpha = (1,0)$ and $\beta = (-1/2, \sqrt{3}/2)$, as shown in Figure 3. Thus, all points of $A_2$ take the form $\langle u,v \rangle = u\alpha + v\beta$ for $u,v \in \mathbb{Z}$.

Consider the affine transformation,

$$P : \Lambda_3^3 \rightarrow A_2$$

$$P(x,y,z) = \langle x - z + 2, y - z + 1 \rangle$$

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

With $u = x-z+2$ and $v = y-z+1$ as defined by Eqn. (6), we have $u > v$ and $v > 0$. This implies that $P(x,y,z)$ lies in the interior of the region bounded by the $\alpha$-axis and $\alpha + \beta$-axis. We also find that since $x \leq n$, then $u = x - z + 2 \leq n + 2$. That is, $P(\Lambda_3^3)$ lies entirely within the triangular region determined by $u > v$, $v > 0$, and $u < n + 3$ in $A_2$.

Moreover, the three moves, $\delta_1 = (1,0,0)$, $\delta_2 = (0,1,0)$, and $\delta_3 = (0,0,1)$, in $A^3$ transform to the following moves in $A_2$:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \delta_1 = (1,0) = \alpha, \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \delta_2 = (0,1) = \beta,$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \delta_3 = (-1,-1) = -\alpha - \beta$$

Thus, counting paths in $\Lambda^3$ is equivalent to counting paths in a triangular lattice of sufficiently large size, say $a + 1$ points along a side (where $\lambda = (a,b,c)$ is the...
shape for which we wish to count \( f(a,b,c) \), where \( a + b + c = n \) with move set \( \{ \alpha, \beta, -\alpha - \beta \} \). Referring to the notation and methods established in Appendix B of [1], the relevant transition operator is \( T^+ = R_\alpha + R_\beta + R_{-\alpha - \beta} \), and so we have on the Fourier side,

\[
\hat{T}^+(\omega_1, \omega_2) = e^{-2\pi i (\frac{\omega_1}{a+1} - \frac{2\omega_2}{a+1})} + e^{-2\pi i (\frac{\omega_1}{a+1} + \frac{\omega_2}{a+1})}
\]

The Fourier-transformed initial state is defined by

\[
V_0(\omega_1, \omega_2) = 2i \left[ -\sin \left( \frac{2\pi \omega_1}{a+1} \right) - \sin \left( 2\pi \left( \frac{\omega_1 - \omega_2}{a+1} \right) \right) + \sin \left( 2\pi \left( \frac{2\omega_1 - \omega_2}{a+1} \right) \right) \right]
\]

Now, according to Theorem B.1 of [1], \( v_n = \mathcal{F}^{-1} \left[ \hat{T}^+ V_0 \right] \) is the vertex function that counts the paths in the wedge that we are interested in. Here \( \mathcal{F} \) is a discrete Fourier transform, and \( \mathcal{F}^{-1} \) is its inverse. Adjusting the input by the affine transformation Eqn. (6), we have produced an interesting formula for height-3 standard Young tableaux.

\[
f(a,b,c) = \frac{1}{3(a+1)^2} \sum_{\omega_1=0}^{a} \sum_{\omega_2=0}^{3a+2} e^{2\pi i \left( \frac{(a+1\omega_1 + (b+c+1)\omega_2)}{a+1} \right)} \hat{T}^+(\omega_1, \omega_2)V_0(\omega_1, \omega_2)
\]

The author concedes that Eqn. (8) may be of very little practical use to those who actually wish to count \( f^\lambda \), except perhaps as a curiosity. However, the formula is fairly easy to code into a computer algebra system such as Sage [11], and has been verified to produce correct counts of \( f^\lambda \).

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