Methods of Federated Filtering in the Problems of Navigation Data Processing

V.A. Tupyshev and Yu.A. Litvinenko

1 Dr. Sci. JSC "Elektropribor", State University of Aerospace Instrumentation, St.-Petersburg, Russia
2 Cand. Sci. JSC "Elektropribor", ITMO University, St.-Petersburg, Russia

E-mail: ya_litvinenko@mail.ru

Abstract. The paper gives an overview of federated filtering methods used in navigation data processing. The principles of designing federated filters are considered and their main properties are discussed.

1. Introduction

Currently, modern navigation systems use both centralized and decentralized methods to process measurements. In the first case, navigation parameters are generated in a single filter, which allows desired parameters to be calculated with a minimum variance [1–6]. In the second case, methods of federated filtering (FF) are widely practiced in navigation systems with modular architecture [7–10]. The essence of FF methods consists in calculation of local estimates using Kalman-type filters and subsequent generation of navigation parameters by inertialess averaging of local parameters [9–14]. Different types of filters can be conventionally divided into two groups: filters with and without reset of local filters. Generally, federated filters are not optimal; however, when adjusted properly, they can provide guaranteed estimation in which the calculated covariance matrix of the estimate error obtained by averaging the respective covariance matrices of local filters is an upper bound for the real covariance matrix of the estimate error [14]. This property of the calculated covariance matrix allows it to be used as an accuracy measure of the global estimate.

In practice, using FF methods in navigation systems, we should bear in mind that Kalman-type filters implemented in measuring modules are obtained due to linearization in the general case of nonlinear equations of dynamics and measurements [15]. It is assumed that the linearization error can be neglected. At the same time, this assumption is not always valid for a number of navigation problems. In particular, for the solution of a map-aided navigation problem and processing of substantially nonlinear measurements, linearization errors are so significant that they cannot be neglected [2,15]. In this regard, it makes sense to consider the implementation of nonlinear filtering algorithms in federated filters and study the possibility of using the so-called polynomial filters in local measuring modules. Polynomial filters allow nonlinear measurements to be described by polynomials of the second and higher orders and, as a consequence, the accuracy of local estimates increases [16–18].

This paper gives an overview of federated filtering techniques. It consists of six sections. The first two sections consider the general formulation of the nonlinear filtering problem and discuss the principles and features of centralized and federated filtering methods. Section 3 describes the principle of reproduction of the processes being estimated and suboptimality of decentralized methods of data
processing. Section 4 considers the features of the federated filtering methods in a linear statement, and Section 5, the implementation of these methods in the case of a nonlinear statement of the estimation problem. Section 6 provides an illustrative example of data processing in a navigation system containing several sensors.

2. General statement of the filtering problem
It is known that in the general case, calculation of navigation and dynamic parameters can be reduced to the solution of the estimation problem in the following statement. Assume that the behavior of a dynamic system, for example, error equations of the inertial navigation system (INS) or dead reckoning used to generate navigation parameters [15] are described as follows:

$$X_0(k) = \psi(k, X_0(k-1)) + w_0(k),$$

where $\psi(X)$ is a known nonlinear function, $w_0(k) \in N\{0, Q_0(k)\}$, $X_0(0) \in N\{X_0, P_0(0)\}$.

In this case, measurements that are taken in the modules at discrete instants of time

$$Y_i(k) = \varphi_i(k, X_0(k), C_i(k)) + v_i(k), \text{ where } v_i(k) \in N\{0, R_i(k)\}, i \in 1, m,$$

contain nonwhite noise errors which are described in a rather general form by the equations:

$$C_i(k) = \Phi_{C_i}(k)C_i(k-1) + w_{C_i}(k), \text{ where } w_{C_i}(k) \in N\{0, Q_{C_i}(k)\}, C_i(0) \in N\{C_i, P_i(0)\}.$$

In models (1)–(3), parameters $X_0(0), C_i(0), w_0(k), v(k) = [v_1^T, ..., v_m^T]^T, w_{C}(k) = [w_{C1}^T, ..., w_{Cm}^T]^T$ are assumed to be Gaussian and independent of each other, and matrices $Q_0, Q_{C_i}, R_i$ are assumed to be known.

Thus, the filtering problem is reduced to the problem of estimating the errors $X_0(k)$ of the reference system (INS or dead reckoning) from measurements (2). The resulting estimate of error vector $X_0(k)$ is then used to correct the errors of the INS (or dead reckoning system).

3. Design features of federated filters
Note that in this statement, optimal estimation can in principle be provided with the use of a centralized filter with state vector $X = [X_0^T, C_1^T, C_2^T, ..., C_m^T]^T$ and processing of all measurements in this filter.

At the same time, federated filtering techniques have also become widespread and are used along with the centralized filter in navigation systems. They are widely accepted for navigation systems with modular architecture. In this case, a bank of filters is used to process measurements obtained in the modules with the subsequent weighing of local estimates in the master filter. Processing of measurements in a federated filter can be illustrated by the block diagram in Fig. 1.

Note that filters are classed as those with and without reset, depending on the information generated in the master filter. Filters with reset, shown by a dotted line in Fig. 1, are actually filters with feedback, which makes it possible to redistribute information between local filters.
Local filters are filters with a state vector $X_i(k) = [X_{0i}(k), C_i^T]^T$, which includes the vector of the reference system errors and nonwhite-noise errors of measurements obtained in the measurement modules. The model for the adjustment of local filters has the form:

$$
X_0(k) = \psi(k, X_0(k-1)) + w_0(k), \quad X_0(0) \in N[\bar{X}, P_0(0)].
$$

The parameters of the local filters $Q_{0i}(k)$ and $P_{0i}(0)$ usually satisfy the so-called conditions of consistent adjustment of local filters:

$$
\sum Q_{0i}^{-1}(k) = Q_0^{-1}(k), \quad \sum P_{0i}^{-1}(0) = P_0^{-1}(0).
$$

The global estimate of the state vector $\hat{X}_0^F(k)$ and the calculated covariance matrix of the estimate error $P_0^F(k)$ in the federated filter are calculated in the master filter using the following equations:

$$
\hat{X}_0^F(k) = P_0^F(k) \sum_{i=1}^{m} P_{0i}^{-1}(k) \hat{X}_0^i(k), \quad P_0^F(k) = \left[ \sum_{i=1}^{m} P_{0i}^{-1}(k) \right]^{-1},
$$

where $\hat{X}_0^i(k), P_0^i(k)$ are the estimates and the calculated covariance matrices of the local filters corresponding to the state subvector $X_0(k)$.

4. Principle of reproduction of the processes being estimated

The basis for obtaining FF algorithms and their study is the principle of reproduction of the processes being estimated, according to which the problems of optimal estimation are considered with state vector $X = [X_0^T, C_1^T, C_2^T, \ldots, C_m^T]^T$ and, in an extended state space, with state vector $X_p = [X_{01}^T, C_1^T, X_{02}^T, C_2^T, \ldots, X_{0m}^T, C_m^T]^T$, which includes the state vectors of the local filters. In [13,14] this principle is formulated for linear dynamics equations.

Figure 1. A block diagram of a federated filter.
With regard to nonlinear equations of dynamics and measurements, the principle of processes reproduction can be formulated as follows. The model described by Equations (1)–(3) and the model in the extended state space with the state vector including the state vectors of local filters (4), under the conditions of the matched adjustment (5) and taking into account the relations
\[ 0 = X_{0(i+1)}(k) - X_0(k), \quad i \in [1, m-1], \]
considered as error-free measurements, ensure the coincidence of optimal estimates, i.e.,
\[ \hat{X}_0(k) = \hat{X}_0(k). \]
The proof of this statement is given in Appendix A.

It can also be shown that inertial averaging (6) used in the FF methods for generation of global parameters is the result of processing of constraint equations (7) only at the last \( k \)-th step, and, therefore, FF is a suboptimal estimation algorithm. Generation of an optimal estimate of the state vector \( X_0(k) \) is ensured by processing of the constraint equations at all estimation steps, which allows us to conclude that the loss in the accuracy of vector \( X_0(k) \) estimated by FF methods is due to the rejection of some of the information in the constraint equations at the previous steps of measurement processing.

A detailed proof of this statement is given in Appendix B.

5. Features of FF methods for the case of a linear statement

Note that solution of navigation problems often implies that it is possible to linearize the equations of dynamics and measurements assuming that the linearization error can be neglected [1,2]. In this case, the nonlinear functions in (1)–(2) can be written as:
\[ \psi(k, X_0(k - 1)) = \Phi_0 X_0(k - 1); \]
\[ \varphi_i(k, X_0(k), C_i(k)) = H_0(k) X_0(k) + B_{C_i}(k) \]
where \( \Phi_0, H_0, B_{C_i} \) are the known matrices.

Then (1)–(3) can be represented by equivalent equations for the state vector
\[ X = [X_0^T, C_1^T, C_2^T, \ldots, C_m^T]^T; \]
\[ X(k) = \Phi(k) X(k - 1) + \xi(k), \]
\[ \xi(k) = [w_0^T(k), w_{C_1}^T(k), \ldots, w_{C_m}^T(k)]^T, \]
\[ Y(k) = H(k) X(k) + v(k), \quad v(k) \in N \{0, R(k)\}, \]
where \( \Phi(k), Q(k), P(0), R(k) \) are block-diagonal matrices:
\[ \Phi(k) = diag \{ \Phi_0, \Phi_{C_1}, \Phi_{C_2}, \ldots, \Phi_{C_m} \}, \quad Q(k) = diag \{ Q_0, Q_{C_1}, Q_{C_2}, \ldots, Q_{C_m} \}, \]
\[ P(0) = diag \{ P_0(0), P_{C_1}(0), P_{C_2}(0), \ldots, P_{C_m}(0) \}, \quad R(k) = diag \{ R_1, R_2, \ldots, R_m \}, \]
\[ H(k) = \begin{bmatrix}
    H_{01} & B_{C_1} & 0 & 0 \\
    H_{02} & 0 & B_{C_2} & 0 \\
    H_{03} & 0 & 0 & 0 \\
    H_{0m} & 0 & 0 & B_{C_m}
\end{bmatrix}. \]

In this case, optimal estimation of the state vector \( X(k) \) can be ensured by using a centralized Kalman filter with processing of all measurements in this filter. Similarly, the state vectors of the local filters will have the form \( X_i(k) = \begin{bmatrix} X_{0i}(k) \\ C_i^T \end{bmatrix} \) with the following models for their adjustment.
\[ X_i(k) = \Phi_i X_i(k-1) + \xi_i(k), \quad \xi_i(k) \in N \left\{ 0, Q_i(k) \right\}, \quad X_i(0) \in N \left\{ \tilde{X}_{i0}(0), P_i(0) \right\}, \]
\[ Y_i(k) = H_i(k) X_i(k) + v_i(k), \quad v_i(k) \in N \left\{ 0, R_i(k) \right\}, \quad \xi_i(k) = \left[ w_{0i}^T(k), w_{ci}^T(k) \right]^T, \]
where \( \Phi_i(k) = \begin{bmatrix} \Phi_0 & 0 \\ 0 & \Phi_{ci} \end{bmatrix}, \quad Q_i(k) = \begin{bmatrix} Q_{0i} & 0 \\ 0 & Q_{ci} \end{bmatrix}, \quad P_i(0) = \begin{bmatrix} P_{0i} & 0 \\ 0 & P_{ci} \end{bmatrix}, \quad H_i(k) = [H_{0i}, B_{ci}]. \]

The conditions for the adjustment of local filters remain the same (5), while the global parameters are generated with the use of formulas (6).

The following result, important from the practical standpoint, was obtained within the framework of research on FF [12, 19, 20]. It is shown that in the linear statement of estimation with matched adjustment of local filters with and without reset of local filters, under certain reset conditions, guaranteed estimation of the state vector is provided, understood in the sense that the real estimation error covariance matrix is less than the calculated covariance matrix obtained in the master filter:
\[ D_0(k) \leq P_0^F(k), \]
where \( D_0(k) \) is a real error covariance matrix of the global estimate \( \hat{X}_0(k) \) derived by the weighed averaging of estimates of local filters with the use of formulas (5). Applied to the filters without reset, the conditions for the matched adjustment can be generalized as follows:
\[ S^T \left[ Q_p(k) \right]^{-1} S = Q^{-1}(k), \quad S^T \left[ P_p(0) \right]^{-1} S = P^{-1}(0). \]

The proof of this statement can be found in [10, 17].

It should be noted that the accuracy of global parameters depends on the choice of conditions for the adjustment of local filters that provide guaranteed estimation. Such studies were carried out in [21].

6. Features of FF methods for the case of nonlinearity in the equations of measurements and dynamics
The traditional approach to the solution of estimation problems in navigation systems with nonlinearity in the equations of dynamics and measurements is their linearization with an assumption that the linearization error can be neglected. If this assumption is valid, we can consider the methods of centralized and federated filtering in the linear statement that were described above. On the other hand, as already noted in the introduction, methods of polynomial filtering are increasingly used when processing of navigation information is aimed to improve accuracy [16–18]. Consider the features of these methods in the extended state space for special cases of nonlinearity only in the equations of measurements, only in the equations of dynamics and nonlinearity, both in the equations of dynamics and equations of measurements. The essence of polynomial filtering methods consists in the expansion of nonlinear functions of dynamics and measurements in Taylor series with preservation of nonlinear expansion terms and determination of optimal estimates in the class of linear estimates the determination of which involves calculation of the mathematical expectations and blocks of covariance matrices [18, 22, 23]. In order to make the procedures for their calculation simpler, we use the Gaussian approximation of the posterior density and the prediction density. Note that the simplest case of polynomial filters is a second-order filter that involves expansion of nonlinear functions accurate to the terms of the second infinitesimal order. First, consider the case of nonlinearity of measurements:
\[ Y(k) = \varphi(k, X_p(k)) + \nu(k) \]
assuming that the equations of dynamics are linear and the posterior density was subjected to Gaussian approximation at the previous step of estimation with parameters \( \hat{X}_p(k-1) \) and \( P_p(k-1) \). In this case
case, by the time of measurement processing in local filters, the prediction density will be Gaussian with an estimate and a prediction covariance matrix:

\[ \hat{X}_p(k) = \Phi_p \hat{X}_p(k-1) \]  

\[ L_p(k) = \Phi_p P_p(k-1) \Phi_p^T + Q_p(k) \]  

Now we proceed to the problem of generating an optimal estimate in the class of linear estimates. For nonlinear measurements (16), the solution of such a problem is known [1,22,23]; it is based on the formulas for the estimate and the covariance matrix:

\[ \hat{X}_p(k) = \hat{X}_p(k) + P_p^{XY} (P_p^{YY})^{-1} (Y(k) - \bar{Y}(k)) \]

\[ P_p(k) = L_p(k) - P_p^{XY} (P_p^{YY})^{-1} P_p^{XY} \]

in which \( \bar{Y}_p(k) = \int \psi(k, X_p(k)) f(X_p(k)) dX_p(k) \),

\[ P_p^{XY}(k) = \int (X(k) - \bar{X})(\psi(k, X(k)) - \bar{Y}(k))^T f(X(k)) dX(k) \]

\[ P_p^{YY}(k) = \int (\psi(k, X(k)) - \bar{Y}(k))(\psi(k, X(k)) - \bar{Y}(k))^T f(X(k)) dX(k) + R(k) \]

where under the sign of integrals is the Gaussian prediction density with parameters \( \hat{X}_p(k) \).

Note that the Gaussian approximation makes it much easier to calculate integrals if we use a polynomial approximation of the nonlinear measurement function.

Consider the case when the dynamics equations are nonlinear (1) and the measurement equations are linear [15]. Note that with the Gaussian approximation of the posterior density, the prediction density will not be Gaussian; however, it is possible to calculate the mathematical expectation of vector \( X_p(k) \) and its covariance matrix \( L_p(k) \) using the following formulas:

\[ \hat{X}_p(k) = \int \psi(X_p(k-1)) f(X_p(k-1)) dX_p(k-1) \]  

\[ L_p(k) = \int (\psi(X_p(k-1)) - \hat{X}_p)(\psi(X_p(k-1)) - \hat{X}_p)^T f(X_p(k-1)) dX_p(k-1) + Q_p(k) \]  

We take advantage of the fact that measurements are linear. It is known [1] that in this case, we need to know the prediction estimate and the prediction covariance matrix to be able to calculate the optimal estimate. Then the optimal estimate in the class of linear estimates and the covariance matrix after processing of the measurements are determined by the formulas:

\[ \hat{X}_p = \bar{X} + L_p H_p^T (H_p L_p H_p^T + R)^{-1} (Y(k) - H_p \hat{X}_p(k)) \]

\[ P_p(k) = L_p - L_p H_p^T (H_p L_p H_p^T + R)^{-1} H_p L_p \]

where \( f(X_p(k-1)) \) is the Gaussian approximation of the posterior density with parameters \( \hat{X}_p(k-1) \) and \( P_p(k-1) \).

Obviously, it is much simpler to calculate the integrals with the Gaussian approximation of the posterior density if we use the polynomial approximation of the nonlinear dynamic function.

Note that in the case of nonlinearity in the equations of dynamics and measurements, local filters can be represented by filters of different complexities; however, due to the principle of reproduction of the processes being estimated, the processes in local filters are considered as independent, and in this
case, the generalization of FF methods with nonlinearity in the equations of dynamics and measurements is reduced to the generation of estimates and calculated covariance matrices in local filters, followed by the generation of global parameters by inertialess averaging of local parameters.

7. An example of navigation data processing with the use of FF methods
A practical example of navigation data processing considered here is the problem of generating navigation parameters based on the data from an INS, terrain referenced navigation (TRN), GPS, and a synthetic aperture radar (SAR) discussed in [9]. Note that the state vector of the centralized optimal filter in the case of processing of the whole set of measurements Y(k) for the above-listed set of measurement systems and facilities will have the form:

\[ X(k) = [x_{INS}^T(k), x_{GPS}^T(k), C_{GPS}^T, C_{SAR}^T, C_{TRN}^T]^T, \]

where \( x_{INS} \) is the subvector of the INS navigation data error, \( x_{INS} \) is the subvector of the INS navigation sensor error, \( C_{GPS} \) is the subvector of the error in the generation of velocity based on the GPS data, \( C_{TRN} \) is the subvector of the error in the generation of coordinates based on the TRN data, \( C_{SAR} \) is the subvector of the error in the generation of velocity based on the SAR data.

In the case of a federated filter, suboptimal Kalman-type local filters are implemented as part of each i-th \((i = 1 \ldots 3)\) separate module, each of which calculates a local estimate \( \hat{X}_i(k) \). The state vectors of the local filters take the form:

\[ X_1(k) = [X_{INS}^T(k), X_{GPS}^T(k), C_{GPS}^T, C_{SAR}^T, C_{TRN}^T]^T, \]
\[ X_2(k) = [X_{INS}^T(k), X_{GPS}^T(k), C_{GPS}^T, C_{SAR}^T, C_{TRN}^T]^T, \]
\[ X_3(k) = [X_{INS}^T(k), X_{GPS}^T(k), C_{GPS}^T, C_{SAR}^T, C_{TRN}^T]^T. \]

Global parameters—INS navigation data errors, are generated in the federated filter using the following formulas:

\[ P_{INS}^{-1}(k) = \left( X_{INS}^{-1}(k) \right)^{-1}, \]
\[ \hat{X}_{INS}(k) = P_{INS}^{-1}(k) \sum_{i=1}^{3} P_{INS}^{-1}(k) \hat{X}_{INS}(k). \]

It should be noted that if the conditions for the consistent adjustment of local filters (5) are met, the global error \( \hat{X}_{INS}(k) \) will ensure guaranteed estimation. This is confirmed by the simulation results of the example considered above (see Fig. 2). We can see the curves showing the heading errors generated in the optimal centralized filter \((P = \text{Opt})\), the calculated and real values of the FF error in the conditions of consistent adjustment of the local filters \((P = \text{calculated}, D = \text{real})\). It can be seen that the calculated value of the heading error \( P_{INS}^{-1} \) is an upper estimate for the real error \( D_{INS} \).

8. Conclusions
An overview of federated filtering methods that have found wide application in navigation systems with modular architecture is presented. The statement of the nonlinear filtering problem is considered. The general principles and features of the implementation of federated filtering methods are discussed, the main of which are processing of measurements in a bank of local filters with the generation of global estimates by inertialess averaging of local estimates. For the general case of the filtering problem statement, the principle of reproduction of the processes being estimated has been proved, which establishes the relationship between the optimal estimates in the ordinary and extended state spaces. This principle is the basis for obtaining federated filtering algorithms and their study. The implementation of federated filters in a linear formulation of the filtering problem has been considered.
and conditions for guaranteed estimation for the cases with and without reset of local filters have been described. The implementations of federated filters with nonlinearity in the equations of dynamics and measurements with the Gaussian approximation of the posterior density and the prediction density with the generation of estimates in the class of linear estimates have been discussed.

**Figure 2.** Errors of heading generated in the INS.

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**Appendix A**

The principle of reproduction of the processes being estimated

With regard to nonlinear equations of dynamics and measurements, the principle of reproduction of the processes being estimated can be formulated as follows.

Under the conditions of the consistent adjustment (5) and taking into account the relations

$$0 = X_{0(i+1)}(k) - X_{0i}(k), \quad i \in 1, m-1,$$

considered as error-free measurements, the model described by Equations (1)–(3) and the model in the extended state space with the state vector, including the state vectors of the local filters (4), ensure coincidence of optimal estimates, i.e., $\hat{X}_0(k) = \hat{X}_{01}(k)$.

The proof is given for the case of $m = 2$ in such a way that, based on the logic of the proof, the generalization for $m \neq 2$ is obvious. For the proof, consider the joint distribution density $f((X_{01})_0, (X_{02})_0, (C_1)_0, (C_2)_0, (Y_{11})_0, (Y_{12})_0, (Y_{21})_0, (Y_{22})_0)$ in the extended state space, taking into account independence of the parameters in the following form, using the density in the form of the $\delta$-function for the constraint equations (8):

$$f((X_{01})_0, (X_{02})_0, (C_1)_0, (C_2)_0, (Y_{11})_0, (Y_{12})_0, (Y_{21})_0, (Y_{22})_0) = f_1(X_{01}(0)) f_2(X_{02}(0)) f_{C1}(C_1)_0 f_{C2}(C_2)_0 \prod_{u} f_{1u}(X_{01}(u) - \psi(u, X_{01}(u-1))) \prod_{u} f_{2u}(X_{02}(u) - \psi(u, X_{02}(u-1))) \prod_{u} f_{1u}(Y_{11}(u) - \varphi(u, X_{01}(u), C_1(u))) \prod_{u} f_{2u}(Y_{21}(u) - \varphi(u, X_{02}(u), C_2(u))) \prod_{u} \delta(X_{01}(u) - X_{02}(u)).$$

(A.2)

In these formulas we use the notation adopted in [15]:

$$X_0^k = \Bigl[ X^T(0), X^T(1), X^T(2), \ldots, X^T(k) \Bigr]^T, \quad Y_t^k = \Bigl[ Y^T(1), Y^T(2), \ldots, Y^T(k) \Bigr]^T,$$

$f(X_0^k, Y_t^k)$ is the joint distribution density of vectors $X_0^k$ and $Y_t^k$.

In what follows, integrals are understood as multidimensional, and differentials from vectors, as products of the differentials of their components; the joint density $f(X_0^k, Y_t^k)$ is understood as the
joint density of the state vector of a dynamical system and measurements at successive points in time
0, 1, 2, ..., k: \( f(X_k^0, Y_k^0) = f(X(0), X(1), ..., X(k), Y(1), ..., Y(k)) \).

Let us define the optimal estimate \( \hat{X}_{0|k}(k) \) of the state vector \( X_{0|k}(k) \) in the extended state space as a conditional mathematical expectation

\[
\hat{X}_{0|k}(k) = \frac{\int f((X_{0|k})_0^k, (X_{0|k})_1^k, (Y_{0|k})_0^k, (Y_{0|k})_1^k) d(X_{0|k})_0^k d(X_{0|k})_1^k d(C_{1|0}^k) d(C_{2|0}^k) \int f((X_{0|k})_0^k, (X_{0|k})_1^k, (Y_{0|k})_0^k, (Y_{0|k})_1^k) d(X_{0|k})_0^k d(X_{0|k})_1^k d(C_{1|0}^k) d(C_{2|0}^k)}{\int f((X_{0|k})_0^k, (X_{0|k})_1^k, (Y_{0|k})_0^k, (Y_{0|k})_1^k) d(X_{0|k})_0^k d(X_{0|k})_1^k d(C_{1|0}^k) d(C_{2|0}^k)} .
\]

Due to the independence of the disturbances and measurement errors, the density fragments can be represented as

\[
f((X_i)_0^k, (Y_i)_1^k) = f((X_i)_0(0)) \prod_a f_{iw}(X_{0i}(u) - \psi(u, X_{0i}(u - 1))) \prod_a f_{ir}(Y_{0i}(u) - \phi(u, X_{0i}(u), C_i(u))).
\]

After integration with respect to the parameter \( (X_2)_0^k \), using the property of the integral of the \( \delta \)-function, we obtain the following formula for the joint density of parameters

\[
f((X_1)_0^k, (C_1)_0^k, (Y_1)_0^k, (Y_2)_1^k) = f_1((X_{01}(0)) f_2((X_{01}(0)) f_{c_1}((C_1)_0^k) f_{c_2}((C_2)_0^k)
\]

\[
\prod_a f_{iw}(X_{0i}(u) - \psi(u, X_{0i}(u - 1))) \prod_a f_{i2}(X_{0i}(u) - \psi(u, X_{0i}(u - 1))) (\prod_a f_{ir}(Y_{0i}(u) - \phi(u, X_{0i}(u), C_i(u))) \prod_a f_{i2}(Y_{0i}(u) - \phi(u, X_{0i}(u), C_i(u))).
\]

Now, consider the fragment of density (26)

\[
J = f_{iw}(X_i(u) - \psi(u, X_i(u - 1)) f_{i2}(X_i(u) - \psi(u, X_i(u - 1))) .
\]

Taking into consideration the fact that \( f_{iw}(\cdot) \) and \( f_{i2}(\cdot) \) are Gaussian distribution densities, we derive:

\[
J = c_1 \exp(-0.5((X_0)_{0}, (X_1)_{0}, (X_1)_{0} - (X_1)_{0}) (Q_{0|0}^{-1} (X_0)_{0}(X_1)_{0} - (X_1)_{0})(X_0)_{0} (X_1)_{0} - (X_1)_{0}))
\]

\[
c_2 \exp(-0.5((X_0)_{0}, (X_1)_{0}, (X_1)_{0} - (X_1)_{0}) (Q_{0|0}^{-1} (X_0)_{0}(X_1)_{0} - (X_1)_{0})(X_0)_{0} (X_1)_{0} - (X_1)_{0})); \]

\[
c_1 c_2 \exp(-0.5((X_0)_{0}, (X_1)_{0}, (X_1)_{0} - (X_1)_{0}) (\sum Q_{i|0}^{-1} (X_0)_{0}(X_1)_{0} - (X_1)_{0})(X_0)_{0} (X_1)_{0} - (X_1)_{0})).
\]

where \( c_{1,2} \) are normalizing constants.

Based on the assumption, the condition of consistent adjustment \( \sum Q_{i|0}^{-1} (X_0)_{0} = Q_{0}^{-1} (X_0)_{0} \) is satisfied, therefore, the formula for fragment of joint density \( J \) can also be written in the form:

\[
J = c_1 c_2 \exp(-0.5((X_0)_{0}, (X_1)_{0}, (X_1)_{0} - (X_1)_{0}) (Q_{0|0}^{-1} (X_0)_{0}(X_1)_{0} - (X_1)_{0})(X_0)_{0} (X_1)_{0} - (X_1)_{0})).
\]

By similar reasoning about the fragment of the joint density \( J_0 = f_1((X_{01}(0)) f_2((X_{01}(0)), J_0 \) can be represented as:

\[
J_0 = c_0 c_{22} \exp(-0.5((X_0)_{0}, (X_1)_{0}, (X_1)_{0} - (X_1)_{0}) (P_{0|0}^{-1} (X_0)_{0}(X_1)_{0} - (X_1)_{0})(X_0)_{0} (X_1)_{0} - (X_1)_{0})).
\]

Let us now consider the joint density \( f((X_0)_0^k, (C_0)_0^k, (Y_0)_0^k, (Y_1)_0^k, (Y_2)_1^k) \) in the nonextended state space. Obviously, taking into account the measurement model (2), the joint density \( f((X_0)_0^k, (C_0)_0^k, (Y_0)_0^k, (Y_1)_0^k, (Y_2)_1^k) \) can be represented in the form:
Due to independent initial conditions and \\
\[ f \left( X_0^k, (C_1)_0^k, (C_2)_0^k, (Y_1)_0^k, (Y_2)_0^k \right) = f \left( X(0) \right) \prod_u f_u(X(u) - \psi(u, X(u - 1))) f \left( (C_1)_0^k, (C_2)_0^k \right) \]
\[ \prod_u f_u( Y_1(u) - \varphi(X(u), u)) \prod_u f_u( Y_2(u) - \varphi(X(u), u)), \]
and equality for the covariance matrices of \\
\[ A.7 \]
\[ \text{A.8} \]
\[ A.9 \]
\[ A.10 \]
\[ A.11 \]
where \( c_{0,u} \) are normalizing constants.

Comparing (A.7) and (A.11), (A.8) and (A.10), it is easy to verify that joint densities \\
\[ f \left( (X_0)_0^k, (C_1)_0^k, (C_2)_0^k, (Y_1)_0^k, (Y_2)_0^k \right) \]
\[ f \left( (X_0)_0^k, (X_0)_0^k, (C_1)_0^k, (C_2)_0^k, (Y_1)_0^k, (Y_2)_0^k \right) \]
coincide with an accuracy of the normalizing constants and notation, and hence, the posteriori densities also coincide.

As a consequence, optimal estimates \( \hat{X}_0^k(k) \) and \( \hat{X}_0^k(k) \) in the ordinary and extended state spaces will be equal, which proves the equivalence of the models with state vectors

\[ X = \begin{bmatrix} X_0^T, C_1^T, C_2^T, \ldots, C_m^T \end{bmatrix}^T \quad \text{and} \quad X_p = \begin{bmatrix} X_{p_1}^T, C_{p_1}^T, X_{p_2}^T, C_{p_2}^T, \ldots, X_{p_m}^T, C_{p_m}^T \end{bmatrix}^T. \]

In a similar way, we establish equality for \( \hat{X}_0^k(k) = \hat{X}_{02}^k(k) \) and equality for the covariance matrices of the errors of these estimates \( P_0(k) = P_{01}(k) = P_{02}(k) \).

Generalizing the results obtained, it is easy to establish the following connection between estimates and covariance matrices for the general case in the ordinary and extended state spaces

\[ X_p(k) = SX(k), \quad P_p(k) = SP(k)S^T, \]
where \( S \) is a matrix that reflects vector \( X(k) \) to vector \( X_p(k) \).

**Appendix B**

**Suboptimality of FF methods**

It is easy to verify that by step \( k \), due to independent initial conditions and disturbances, as well as rejection of the constraint equations up to the \( k \)-th step, the calculated covariance matrix in the extended state space will be a block-diagonal one, with blocks representing the calculated covariance matrices of local filters \( P_p(k) = \text{diag} \{ P_i(k) \} \), and the vector of estimates will take the form: \( \hat{X}_p(k) = \begin{bmatrix} \hat{X}_1^T(k), \ldots, \hat{X}_m^T(k) \end{bmatrix}^T \). It will consist of the vectors of estimates of the local filters obtained after processing measurements in the local filters.

Let us now process the equation of constraints at the \( k \)-th step as an error-free measurement, whose model has the form:

\[ 0 = \Gamma X_p(k), \quad \text{where} \quad \Gamma = \begin{bmatrix} E & 0 & -E & 0 & 0 & 0 \\ 0 & 0 & E & 0 & -E & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & E & 0 & -E \end{bmatrix}; \]

defining the optimal estimate in the class of linear estimates. Since pseudo-measurements are linear, after processing the pseudo-measurements (8), the estimate will be defined by the formula

\[ \hat{X}_p(k) = \hat{X}_p(k) + K_p(0 - \Gamma_p(k) \hat{X}_p(k)). \]
Determining the optimal value $K_P$ that provides a minimum of the estimation error in the class of linear estimates, we can obtain an expression for the optimal gain:

$$K_P(k) = P_P(k) R^T (R P_P(k) R^T)^{-1}, \quad (B.3)$$

where $P_P(k)$ is a block covariance matrix in the extended state space after processing of measurements in local filters. It is the a priori matrix used in the processing of the constraint equation.

As a consequence, the formula for estimating the extended state vector and its posterior covariance matrix after processing of pseudo-measurements can be written as

$$\hat{X}_P(k) = \hat{X}_P(k) + P_p(k) R^T (R P_P(k) R^T)^{-1} (0 - R \hat{X}_P(k)), \quad (B.4)$$

$$P_p'(k) = P_p(k) R^T (R P_P(k) R^T)^{-1} R P_p(k). \quad (B.5)$$

Let us now show that the estimates of subvectors $\hat{X}_0'(k)$ and blocks of the covariance matrices $P_0'(k)$ can be represented as inertialess averaging of parameters (6).

It is known [11, 12] that, taking into account the properties of rectangular matrices, formulas (37) and (38) are reduced to the form:

$$\hat{X}_P(k) = S (S^T [P_p(k)]^{-1} S)^{-1} S^T [P_p(k)]^{-1} \hat{X}_P(k) = S \hat{X}_P(k), \quad (B.7)$$

where

$$S = \begin{bmatrix}
E & 0 & 0 & 0 \\
0 & E_{C_1} & 0 & 0 \\
E & 0 & 0 & 0 \\
0 & 0 & E_{C_2} & 0 \\
E & 0 & 0 & 0 \\
0 & 0 & 0 & E_{C_m}
\end{bmatrix}$$

$E$ and $E_{C_i}$ are unit matrices corresponding to vectors $X_0(k)$ and $C_i(k)$, $[P_p(k)]^{-1}$ is now a block-diagonal matrix consisting of blocks $[P_i(k)]^{-1}$, and $P_{FF}(k) = (S^T [P_p(k)]^{-1} S)^{-1} \hat{X}_P(k) = P_{FF}(k) S^T [P_P(k)]^{-1} \hat{X}_P(k)$.

Let us introduce the notation for the calculated covariance matrices of the local filters:

$$P_i(k) = \begin{bmatrix}
P_0 & P_{21} \\
P_{21} & P_{11}
\end{bmatrix}, \quad [P_i(k)]^{-1} = \begin{bmatrix}
P_0^{-1} & P_{21}^{-1} \\
P_{21}^{-1} & P_{11}^{-1}
\end{bmatrix}, \quad (B.8)$$

where blocks $P_0'(k)$ correspond to vectors $X_0'(k)$.

Note that in accordance with the formulas for the inversion of block matrices, the parameters of the direct and inverse matrices are related by the known equation [24]:

$$P_{2i}(^{-1}) \hat{X}_0' + P_{1i}(^{-1}) \hat{C}_i' = \tilde{P}_{2i}(^{-1}) \hat{X}_0' + \tilde{P}_{1i}(^{-1}) \hat{C}_i'. \quad (B.9)$$

Using now the block representation of the matrix and multiplying the block matrices, it is easy to verify that matrix $S^T [P_p(k)]^{-1} S$ has the following form:

$$S^T [P_p(k)]^{-1} S = \begin{bmatrix}
\sum P_{0i}^{-(-1)} & P_{21}^{-(-1)} & \ldots & P_{2m}^{-(-1)} \\
\vdots & \vdots & \ddots & \vdots \\
P_{2m}^{-(-T)} & 0 & 0 & P_{1m}^{-(-1)}
\end{bmatrix}, \quad (B.10)$$

Let us introduce the notation for the calculated covariance matrices of the local filters:
where the lower block corresponding to the vector of systematic measurement errors is a block-diagonal matrix.

By inverting matrix \( P_f(k) = (S^T \left[ P(k) \right]^{-1} S)^{-1} \) in block form, we derive the following formula for block \( P_0(k) \) of matrix \( (S^T \left[ P(k) \right]^{-1} S)^{-1} \) corresponding to vector\( k \):

\[
\sum_{i=1}^{m} \tilde{P}_0(k) = P_0^{-1}(k), \quad (B.11)
\]

coinciding with (6).

Carrying out similar transformations for estimates \( \hat{X}_0(k) \) of vector \( X_0(k) \) corresponding to vectors \( X_0(k) \), we can also derive formula (6) for the global estimate \( \hat{X}_0(k) \) of the state vector \( X_0(k) \).

Based on the facts that the formulas for estimate \( \hat{X}_0(k) \) and the calculated error matrix of estimate \( P_0(k) \) used in the FF were obtained as a result of processing of the constraint equations (8) as zero “measurements” only at the last step, and the generation of an optimal estimate of the state vector \( X_0(k) \) is provided by processing of the equations at all estimation steps, we can conclude that the loss in the accuracy of estimation of vector \( X_0(k) \) by FF methods is caused by the rejection of part of the information contained in the equations of constraints at the previous steps of measurement processing.

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