The Diffractive Quantum Limits of Particle Colliders

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Quantum Mechanics places limits on achievable transverse beam spot sizes of particle accelerators. We estimate this limit for a linear collider to be
\[
\Delta x \geq \frac{\hbar cf}{E \delta_0}
\]
where $f$ is the final focal length, $E$ the beam energy, and $\delta_0$ the intrinsic transverse Gaussian width of the electron wave-function. $\delta_0$ is determined in the phase space damping rings, and we find $\delta_0 \approx \sqrt{\frac{\hbar c}{eB}}$ where $B$ is the typical wiggler magnetic field strength in this system. For the NLC $\delta_0 \sim 25\text{nm}$, and $\Delta x \sim \mathcal{O}(0.06)\text{ nm}$, about two orders of magnitude smaller than the design goal. We can recover a crude estimate of the classical result when we include radiative relaxation effects. We also consider a synchrotron and obtain $\Delta x \geq \sqrt{\frac{\hbar cf}{E}} \sim \mathcal{O}(1.0)\text{ nm}$, We discuss formulation of quantum beam optics relevant to these issues.
1 Introduction

Particle accelerators are designed and built, based essentially upon the classical theory of point charges interacting with electromagnetism. Nevertheless, particles are described by wave-functions, and diffractive limits must exist as to how well they can be localized in a given optical apparatus. The first quantum mechanical effects to arise in a potentially limiting way might be expected to be diffractive in nature. In this paper we take a first look at the problem of estimating the quantum diffractive limits of accelerators. We begin with an important system, an NLC-class machine. We are inspired to consider this because the desired goals for the NLC beam spot size are ambitious. To achieve the desired luminosity requires a ∼ 5 nm beam spot in one transverse dimension, (the vertical or y direction in the NLC reports [1]). We will find that this criterion is about two orders of magnitude above the quantum limit. Indeed, we will describe how to estimate the classical design result for the beam spot size itself from quantum mechanics, obtaining rough agreement with the NLC specifications.

We obtain a conceptually simple result. The diffractive limit on the beam spot size in the $x_\perp$ direction is given by the Rayleigh formula for a (massless) wave of energy $E$ which has passed through an effective “aperature” $\delta_0$ and focused over a focal length $\sim f$. That is:

$$\Delta x \geq \frac{\hbar cf}{E\delta_0}$$

where $f$ is the final focal length, $E$ the beam energy (the result varies somewhat in a compound lens system, see Section 3). We emphasize that the “aperature” $\delta_0$ is not a mechanical aperature, e.g., it is not the beam pipe size. $\delta_0$ is actually the initial state Gaussian width of the transverse quantum wave-function as it enters the linac upstream from the damping rings, where the wave-function has been prepared (we assume an “ideal linac,” in which there is negligible further synchrotron radiation downstream; this is not necessarily a good approximation, and corrections to the effective $\delta_0$ are expected). The initial state can be considered to be an ensemble of particles, each in simple harmonic oscillator (SHO) transverse wave-functions, where the Gaussian envelope (groundstate) width is determined by the damping ring wiggler system. This is given by:

$$\delta_0 = \sqrt{\frac{\hbar c}{eB}}$$

where $B$ is the typical magnetic field in the damping system, of order 1 Tesla. Taking $f = 2m$, $E = 250$ GeV, $B = 1$ Tesla, yields $\delta_0 \approx 25.7$ nm, and thus, $\Delta x > 0.062$ nm as
a diffractive limit. Hence, the NLC would appear to be safely above the quantum limit by about two orders of magnitude. We remark, however, that this is the extremal lower limit which saturates the Heisenberg uncertainty relationship, and holds in our idealized limit.

In general the individual particle initial state is an excited SHO transverse wave-function of average principle quantum number \( \bar{n} \). This increases the expected diffractive spot size to \( \Delta x \approx \sqrt{\bar{n}f/E_0} \). In fact, since \( \bar{n} \propto 1/\hbar \) it is easily seen that this result is independent of \( \hbar \), and therefore should be equivalent to a classical derivation of the beam spot size. \( \bar{n} \) can be crudely estimated from radiation relaxation following the original arguments of Sands [3], and others. This yields a result of \( \Delta x \approx 2 \text{ nm} \), roughly consistent with the NLC design report calculations for the vertical beam spot [1].

The subject of quantum beam dynamics for particle accelerators is fairly novel [2]. Gaussian optics is a preferred formalism for tackling the problem considered here. Presently we construct a transverse Gaussian wave-packet, with a longitudinal plane wave structure, and propagate it through an optical system. Gaussians extremalize the Heisenberg uncertainty relation, and they are also the groundstate solutions in continuous linear focusing channels, and e.g., magnetic lenses, wigglers, to a reasonable approximation, etc., and can be approached by synchrotron radiation relaxation [3], [4], [5], [6]. Remarkably, Gaussian transverse wave-functions, which solve the quantum Schrödinger equation for propagation through the optical system, (neglecting synchrotron radiation), are controlled entirely by the classical lens matrices of the system. While Gaussian optics is a standard formalism in treating electron microscopy [7], [8], [9], to our knowledge, the behavior of a quantum Gaussian beam in a synchrotron has not been previously formulated, and we will indicate the self-replicating solution to a synchrotron by an application of lens matrix methods.

First consider the problem of a relativistic electron wave-function passing though a lens. Spin is an inessential complication [10], [11], so we can use the Klein-Gordon (KG) equation. Assume for simplicity that there is only one spatial transverse dimension, \( x_\perp \), and let \( z \) be the longitudinal spatial dimension. In the KG equation we include a transverse simple harmonic oscillator (SHO) potential term which is dependent upon \( z \) (For the analysis, we set \( \hbar = c = 1 \)):

\[
\partial^2 \phi + m^2 \phi + \tilde{K}(z)x_\perp^2 \phi = 0 \tag{1.3}
\]

Then, with \( \phi = \exp[-i(Et - p_z z)] \hat{\phi} \) and \( E^2 = p_z^2 + m^2 \), the KG equation becomes the
transverse Schrödinger equation:

\[ i \frac{\partial}{\partial z} \hat{\phi} + \frac{1}{2E} (\vec{\nabla}_\perp)^2 \hat{\phi} - \frac{K(z) x^2}{2} \hat{\phi} = 0 \]  

(1.4)

where:

\[ K \equiv \tilde{K}/E. \]  

(1.5)

This is a standard construction in optics [8], and \( \hat{\phi}(z) \) has the conventional interpretation with \( z \) replacing time. The parameter \( K(z) \) is \( z \)-dependent, corresponding to the finite longitudinal structure of the lens system. For a single thin lens we take \( K(0) = 0 \) for \( z < -\delta z \) and \( z > 0 \), and \( \tilde{K} = K_0 \) for \( -\delta z \leq z \leq 0 \).

Let us now postulate a Gaussian form for the wave-function centered at the transverse position \( x_\perp \), carrying a transverse momentum \( p_\perp \):

\[ \hat{\phi} = \exp \left( -\frac{1}{2} A(z)(x - x_\perp)^2 + ip_\perp x + C \right) \]  

(1.6)

In this expression, \( A(z) \) is the complex Gaussian kernel, \( x_\perp \) and \( p_\perp \) are real, and \( C \) simply parameterizes the overall normalization. Hence, the Gaussian wave-function has four real parameters. After substituting this ansatz, the Schroedinger equation, eq.(1.4), yields the following equations of motion for the width:

\[ i \frac{\partial A}{\partial z} = \frac{A^2}{E} - K(z) \]  

(1.7)

and for \( x_\perp \) and \( p_\perp \) we obtain the classical Hamilton equations:

\[ \frac{\partial p_\perp}{\partial z} = -Kx_\perp; \quad \frac{\partial x_\perp}{\partial z} = \frac{p_\perp}{E}; \]  

(1.8)

Note that the centroid \( x_\perp \) and centroid momentum \( p_\perp \) motions are decoupled from that of the Gaussian kernel \( A(z) \) and vice versa. (anharmonic effects would generally couple these quantities). Moreover, the boundary conditions on \( A(z) \) and of the centroid \( x_\perp \) and centroid momentum \( p_\perp \) are independent. Note that the last equation is just the “\( z \)-velocity” expressed in terms of the momentum for a particle of “mass” \( E \). Remarkably, eq.(1.7) can be seen to be equivalent to the classical Hamilton equations [8] by identifying \( A(z) \equiv iP(z)/X(z) \) where \( P(z) \) and \( X(z) \) are generalized (complex) momenta and positions which satisfy the eqs.(1.8).

Now, we impose an initial condition at \( z = -L \) that the particle has been prepared into a transverse Gaussian wave-packet, specified to have a pure real width \( \delta_0 \) given by:

\[ A_0 = Re[A(-L)] = 1/(\delta_0)^2; \quad Im[A(-L)] = 0 \]  

(1.9)
We assume that the centroid of the initial wave-packet is moving parallel to the \( z \)-axis, thus \( p_\perp = 0 \), and \( x_\perp = x_0 \) is initially arbitrary at \( z = -L \).

The wave-packet enters the lens at \( z = -\delta z \) and exits at \( z = 0 \). Upon entry of the lens \( A(-\delta z) \) is given by the free drift solution of eq.(1.7) from \( z = -L \) to the lens, over the drift distance \( L \):

\[
A(-\delta z) = \frac{A_0}{1 + iA_0L/E}
\]  
(1.10)

where we assume a thin lens, \( \delta z/L << 1 \).

In the thin lens, to a good approximation for small \( \delta z \), we have from eq.(1.7):

\[
A(0) = A(-\delta z) + i\delta z K_0
\]
(1.11)

Here we neglected the term \( i\delta z A^2/E \) in the differential equation for \( A(z) \) which only gives negligible free particle spreading in the lens. Moreover, the classical centroid motion of the wave-packet is found from eq.(1.8):

\[
p_\perp(0) = -K_0x_0\delta z; \quad x_\perp(0) = x_0.
\]
(1.12)

Upon exiting the lens the particle propagates again in free space a distance \( \ell \) with \( K = 0 \). Hence we find:

\[
A(\ell) = \frac{A(0)}{1 + iA(0)\ell/E}
\]
(1.13)

and:

\[
p_\perp(\ell) = -K_0x_0\delta z; \quad x_\perp(\ell) = x_0 - p_\perp/\ell/E.
\]
(1.14)

Note that the classical trajectory of the off-axis particle is deflected back toward the lens axis, \( x_\perp = 0 \).

The focal length, \( f \) is defined such that \( x_\perp(f) = 0 \), hence:

\[
f = \frac{E}{K_0\delta z}
\]
(1.15)

The kernel of the wave-packet can now be obtained by solving eqs.(1.9,1.10,1.11,1.13) recursively to obtain:

\[
A(\ell) = \left[ \frac{1 - i[\ell/(E\delta_0^2) - ((\delta_0)^2 E/f)]}{((\delta_0)^2(1 - \ell/f) + i(\ell/E + L/E - L\ellEf/EF))} \right]
\]
(1.16)

Note that the Gaussian kernel has an imaginary part which changes from positive (focusing) to negative (defocusing) upon passage of the geometrical focal length, \( L > f \).
Figure 1: The numerical evolution of a Gaussian wave-function with \( L = 3.0 \) evolving through a lens; \( E = 1.0, K_0 = 10.0, \) \( \delta z = 0.1 \) and \( A(-3.0) = 3.0, \) hence \( \delta_0 = 0.577, \) and the geometrical focal length, \( f = 1. \) (a) dashed (black) line is \( x_\perp(z) \) crossing the axis at the geometrical focal point; (b) dotted (red) Gaussian width \( 1/\sqrt{\text{Re}(A(z))} \) which focuses at \( f_q; \) Also shown are \( \text{Re}(A(z)) \) (green solid) and \( \text{Im}(A(z)) \) (blue dot-dashed); note the \( \text{Im}(A(z)) \) receiving a positive kick upon passing through the lens.

Thus, the transverse probability distribution becomes:

\[
|\hat{\phi}|^2 = \mathcal{N}(z) \exp \left\{ -\frac{(x - x_\perp)^2}{(\delta_0)^2(1 - \ell/f)^2 + (\ell/E\delta_0)^2[1 - L(\ell - f)/f\ell]^2} \right\}
\]

and, the transverse size of the wave-packet is given by:

\[
\delta^2(\ell) = (\delta_0)^2(1 - \ell/f)^2 + (\ell/E\delta_0)^2[1 - L(\ell - f)/f\ell]^2
\]

In Figure 1 we give a numerical integration of the Schroedinger equation in the preceding discussion, which confirms the validity of our solution. Note that for finite \( L \) the Gaussian width is focused to a minimum at \( z = f + f^2/L + \ldots \) In the limit \( L \to \infty \) the transverse size of the wave-packet reaches a minimum at the focal point \( \ell = f, \) where the new effective transverse size is:

\[
\Delta x = \frac{f}{E\delta_0}
\]

This is the usual Rayleigh diffractive minimum, \( f\lambda/a \) if we regard \( a \sim \delta_0 \) as an “effective aperture size” through which the beam has passed, and \( E = p_z c = \hbar c/\lambda, \) the usual quantum wavelength of the particle.

What if the initial prepared wave-function is not the groundstate of a SHO (pure Gaussian), but is rather an excited eigenstate of principle quantum number \( n? \) Hence, at \( z = -L, \) neglecting \( x_\perp \) and \( p_\perp, \) we assume:

\[
\hat{\psi}(-L) = H_n(x/\sqrt{2\delta_0}) \exp \left( -\frac{1}{2}(x/\delta_0)^2 + C \right)
\]
where $H_n(\xi)$ is the $n$th Hermite polynomial.

First, we note that the Gaussian solution eq.(1.6) contains the generating function for Hermite polynomials [12]:

$$
\hat{\phi} = \exp \left( -\frac{1}{2} A(z)(x)^2 + i p_\perp(z)x + C \right) \left( \sum_{n=0}^{\infty} \frac{H_n(\sqrt{A(z)/2}x)}{n!} \left[ \sqrt{A(z)/2} x_\perp(z) \right]^n \right)
$$

(1.21)

For a freely drifting particle, if we choose $p_\perp(-L) = 0$, and initial $x_\perp(-L) = x_0$, in $\hat{\phi}$, then we see that our solution $\hat{\psi}(z)$ is determined for any $z$:

$$
\hat{\psi}(z) = \frac{\partial^n}{\partial x_0^n} \phi(z)|_{x_0=0}
$$

(1.22)

After passing through an arbitrary lens system, the solution for $\hat{\psi}(z)$ becomes messy, and in general $x_\perp(z)$ and $p_\perp(z)$ are arbitrary, and we cannot so easily differentiate with respect to $x_0$ to pull out our solution. However, both $x_\perp(z)$ and $p_\perp(z)$ are proportional to $x_0$ by the linearity of the Hamilton equations. At a focal point we have $x_\perp(f) = 0$ (for any $x_0$, owing to linearity), and $p_\perp(f) \propto x_0$. Hence, the solution at a focal point simplifies:

$$
\hat{\psi}(f) = \frac{\partial^n}{\partial x_0^n} \phi(f)|_{x_0=0}
= \frac{\partial^n}{\partial x_0^n} \exp \left( -\frac{1}{2} A(z)(x)^2 + i p_\perp(z)x + C \right) |_{x_0=0}
\propto x^n \exp \left( -\frac{1}{2} A(z)(x)^2 \right)
$$

(1.23)

and thus, the arbitrary solution is focused to a Gaussian times a power of $x$. This gives a focal spot size:

$$
\Delta x = \frac{\sqrt{n} f}{E \delta_0}
$$

(1.24)

Now this result may seem counterintuitive; we are starting with a broader initial distribution by the factor $\sqrt{n}$, and we might guess that this would produce a smaller focal point by an amount $1/\sqrt{n}$. The wave-function, however, is not smooth in $x$, i.e., the Hermite polynomial yields a distribution of transverse momentum, and the initial state has “ears”, each of typical Gaussian width $\delta_0$, but displaced off the optical axis by $\sqrt{n}$. These produce the $\sqrt{n}$ enhancement of the focal spot. Yet another way to see this is to note that one can make a classical off-axis centroid motion of the groundstate Gaussian by superimposing large $n$ states, and the Gaussian width will yield the minimal $f/\delta_0E$ result. Of course, the quantum state of interest to us will typically have a large value of $n$ determined by radiative relaxation. We consider this in the next section.
The actual linear acceleration phase is inconsequential to this result. The above discussion assumed a uniform drift in the longitudinal z-direction, i.e., constant energy $E$. If the particle is accelerating linearly, then $E$ becomes z-dependent, $E(z) = (E_f - E_0)z/L + E_0$. It is easily seen that the only effect on our solution is to replace $L$ by $L \ln((E_f - E_0)/E_0)$, where $E_0$ is the initial energy, $E_f$ the final energy, and $E$ in the above expressions is everywhere replaced by $E_f$. For the first NLC, we have $E_f \sim 250 \text{ GeV} >> E_0 \sim 2 \text{ GeV}$.

The linear acceleration phase is thus equivalent to free drift through an effective distance of $L \ln(E_f/E_0) \sim 45 \text{ km}$ where $L = 10 \text{ km}$. The amount by which a wave-packet of initial size of 25 nm spreads throughout the NLC acceleration phase is about a factor of 6. However, this spreading is irrelevant to computing the final diffractive limit as seen in eq. (1.19) where the free drift length $L$ has completely cancelled from the expression at the classical focal point, and only the initial quantity $\delta_0$ (together with the local quantities $f$ and $E$) controls the diffractive limit.

Thus, the ultimate diffractive limit is controlled by the initial boundary conditions on the wave-function size, i.e. by $\delta_0$, and not by the intervening unitary lens system. What in general determines $\delta_0$? For the NLC the initial wave-function, as well as the initial classical distribution, is prepared in the “damping rings.” Damping rings are essentially a system of magnets arranged as wigglers which induce synchrotron radiation and cool the classical beam bunches of electrons. They are designed to produce roughly a four order of magnitude reduction in one of the transverse dimension phase space volumes, i.e., $\sim \Delta x \Delta p_x$ (the transverse emittance). As the system cools classically, it is also relaxing quantum mechanically. This occurs because the particles in the wiggler chain experience a transverse SHO potential, and synchrotron radiation pushes highly excited wave-functions toward the Gaussian groundstate in this potential. However, there are also re-excitation transitions which eventually come into equilibrium, and a typical average SHO principle quantum number is established. While this is certainly an oversimplified view of the actual system, we will use it as a starting point to estimate $\delta_0$.

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1We remark that the proper way to view the quantum spreading in the transverse phase-space is to use Wigner functions, which depend upon both $x$ and a quantum momentum $p$. The Wigner function isocontours deform in a manner that is conformal to the classical emittance envelope, so while the wave-functions spread in $x$ the Wigner envelopes actually shear in $x$ and $p$ and remain contained in the transverse phase-space.
2 Magnetic Focusing and Damping

We now summarize the details of the motion of a transverse wave-packet in a magnetic field. This is discussed in detail in the classic work of Sokolov and Ternov [4]. We will use the more transparent WKB approximation, expanding about the classical radius of motion. Hence, one should use caution in comparing solutions, e.g., principle quantum numbers refer to different things. For example, large \( n \) in the usual framework [4] corresponds to small \( n \), but large classical radius \( j_z \) presently.

Consider a particle moving in a planar orbit in a uniform magnetic field, aligned in the \( \hat{z} \) direction, \( \vec{B} = \hat{z}B_0 \) in a cylindrical coordinate system \((r, \phi, z)\). The vector potential can be chosen as \( A_\phi = rB_0/2 \) with \( A_r = 0 \) and \( A_z = 0 \). We examine the transverse motion of a relativistic electron in the plane \( z = 0 \). For an anzatz of the form \( e^{-iEt/\hbar}\psi(r, \phi) \), the KG equation becomes:

\[
\begin{bmatrix}
-E^2 + m^2 - \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \left(\frac{i}{r} \frac{\partial}{\partial \phi} - eA_\phi\right)^2
\end{bmatrix} \psi = 0
\tag{2.25}
\]

where a possible momentum component \( p_z \) has been set to zero. Consider a state of with a large “pseudo-angular momentum,” \( \ell \), and scale out a factor of \( 1/\sqrt{r} \):

\[
\psi = \frac{1}{\sqrt{r}} e^{i\ell \phi} \hat{\chi}(r, t).
\tag{2.26}
\]

\( \ell \) is not the physical angular momentum because it is gauge dependent due to presence of the vector potential; the physical angular momentum in the present case is \( 2\ell \), as we will see below). Hence:

\[
\begin{bmatrix}
-E^2 + m^2 - \frac{\partial^2}{\partial r^2} + \left(\frac{-\ell}{r} - \frac{1}{2} erB_0\right)^2 + \frac{1}{4r^2}
\end{bmatrix} \hat{\chi} = 0
\tag{2.27}
\]

This now has the apparent form of a one-dimensional Schrödinger equation with an effective potential:

\[
V(r) = \left[\left(\frac{\ell}{r} + \frac{1}{2} erB_0\right)^2 + \frac{1}{4r^2}\right]
\tag{2.28}
\]

The potential has a minimum at:

\[
r = R_0 \equiv \left(\frac{\sqrt{4\ell^2 + 1}}{eB_0}\right)^{1/2} \to \sqrt{\frac{2\ell}{eB_0}}
\tag{2.29}
\]
where the latter expression corresponds to $\ell >> 1$. Consider henceforth $\ell >> 1$. We consider small radial fluctuations around the large orbital radius $R_0$ as $r = R_0 + x$ and expand:

$$\left[-E^2 + m^2 - \frac{\partial^2}{\partial x^2} + V(R_0) + \frac{1}{2}x^2V''(R_0)\right] \hat{\chi} = 0 \quad (2.30)$$

and:

$$V(R_0) = 2eB_0\ell \quad V''(R_0) = 2e^2B_0^2 \quad (2.31)$$

Thus the high orbital angular momentum Landau levels are approximate eigenstates of the SHO potential defined by $\hat{V}(x) = \frac{1}{2}x^2V''(R_0)$, or $K = e^2B_0^2/2E$. The states are labelled by $(n, \ell)$ where $n$ is a principle SHO quantum number; the energy eigenvalues of these levels are given by:

$$E^2 = m^2 + eB_0(2\ell + n + \frac{1}{2}) \quad (2.32)$$

In the presence of the gauge interaction the physical angular momentum is $L_z = -i\partial/\partial\phi + eR_0A_\phi(R_0)$. Hence, the physical angular momentum is:

$$j_z = \ell + \frac{1}{2}eB_0R_0^2 = 2\ell \quad (2.33)$$

where we use the explicit solution for $R_0$ from eq.(2.29) in the latter expression. Therefore, to make the classical correspondence, we identify the angular momentum with that of an entering beam particle of momentum $p_\phi$, to obtain $R_0p_\phi = j = 2\ell$. This yields consistency with the familiar expression for the classical orbital radius and the total energy:

$$R_0 = \frac{p_\phi}{eB_0} \quad E^2 = m^2 + eB_0(j_z + n + \frac{1}{2}) \quad (2.34)$$

Transitions that increase $n$, but decrease $j_z$ are allowed; hence synchrotron radiation can be excitatory as well as relaxational. The fact that the energy is degenerate, depending upon the combination $j_z + n$ is a consequence of the symmetry in the choice of the classical orbital center. (Note that the solution formed with the anzatz $e^{-i\ell\phi}$ for large $\ell$ is actually a solution of vanishing physical momentum; it is a zero-mode associated with the translational invariance of the center of the particle’s orbit).

The groundstate in the transverse dimension is a Gaussian with $A = |eB_0|$, given by:

$$\delta_0 = \frac{1}{\sqrt{Re(A)}} = (eB_0/\hbar c)^{-1/2} \quad (2.35)$$

For a typical field strength of 1 Tesla we obtain $\delta_0 \sim 25$ nm. The “spring constant” is $O(e^2)$, hence we say that the dipole magnet is weakly focusing (for quadrupoles $V''(R_0) \sim$
where \( a \) defines a gradient, hence “strong focusing”). This description applies to wigglers, even though the dipole magnet field is alternating in \( z \), if the magnitude of the \( B \) field is roughly constant.

It has been known for a long time that an equilibrium between deexcitationary and excitatory transitions for a particle in a damping system (or synchrotron) will be established \([3]\), and there will be an equilibrium value of \( n \). This value is roughly estimated as follows. The typical energy of synchrotron radiated photons is \([3], [4]\):

\[
E \gamma \sim \frac{1}{R_0} \left( \frac{E}{m_e} \right)^3
\]

A unit step in a quantum number \( n \) or \( j_z \) produces only a small energy change, \( \sim eB_0/E \sim 1/R_0 \). The dipole approximation selection rules imply large allowed changes in \( j_z \), but only unit steps in \( n \):

\[
\Delta j_z \sim \frac{E}{R_0 eB_0} \left( \frac{E}{m_e} \right)^3 \sim \left( \frac{E}{m_e} \right)^3 \Delta n \sim \pm 1
\]

Bear in mind that we are treating \( n \) as the principle quantum number in the WKB focusing channel defined by expanding about \( R_0 \), and dipole transitions involving the operator \( \vec{A} \cdot \vec{\nabla} \) will change \( n \) by a unit (these can be excitatory). Then \( \Delta j_z/R_0 \) is essentially the change in longitudinal electron momentum, imparted to the photon. In transitions, though \( R_0 \) changes, there is no sudden translation in the transverse position of the electron wavefunction, only a transition in motion, i.e., the virtual center of the orbit changes \([4, 5]\).

Over a radiative energy loss time interval the number of emitted photons is:

\[
n_{\gamma} \sim \frac{E}{E \gamma} \sim \frac{m_e^3}{eB_0 E} \sim R_0 \left( \frac{m_e^3}{E^2} \right)
\]

The principal quantum number \( n \) undergoes a random walk by roughly \( \sqrt{n_{\gamma}} \), hence the equilibrium \( \bar{n} \) is of order \( \sim \sqrt{n_{\gamma}} \). Using \( E \sim 2 \text{ GeV} \), and \( B_0 \sim 1 \text{ Tesla} \), whence \( R_0 \sim 6.6 \text{ m} \), we find \( n_{\gamma} = 1.12 \times 10^6 \) and \( \bar{n} \sim 1.06 \times 10^3 \).

Hence, our diffractive limit is now increased by \( \sqrt{\bar{n}} \sim 0.33 \times 10^2 \), and we thus have a beam spot size \( \sqrt{\bar{n}} \times 0.06 \sim 2.0 \text{ nm} \). Why is this result so close to the design goals of the NLC that are obtained by classical physics? Indeed, we believe that this result is a quantum derivation of the classical result! The quantum number \( n \) scales as \( 1/h \), while our diffractive limit scales as \( \Delta x \propto \sqrt{n} \), hence the product \( \sqrt{n} \Delta x \) is independent of \( h \). This, moreover, assures us that the ultimate quantum limit is of order \( 1/\sqrt{\bar{n}} \) smaller than the minimal classical analysis. (We note that the Oide effect \([3]\) may be understood as a
blowing up of \( \vec{n} \) in intense final focus magnets, where large transverse energy photons are radiated.

A more detailed discussion of synchrotron radiation relaxation is beyond the scope of the present paper. Excellent treatments can be found in [4], [6], and the pioneering work of [3].

3 Quantum Particle in a Synchrotron

The solution to the Schrödinger equation for passage of a free particle through a lens, eq. (1.4), can be completely described by the simple classical optical matrix methods. If one passes a classical ray moving in the \( \hat{z} \) direction through a lens system, the outgoing state of the transverse \( \hat{x} \) canonical variables may be written as [13]:

\[
\begin{pmatrix}
  x(\ell) \\
  p(\ell)/E
\end{pmatrix}
_{\text{out}} = \mathcal{M}
\begin{pmatrix}
  x(-\delta z) \\
  p(-\delta z)/E
\end{pmatrix}
_{\text{in}}
\]

where: \( \det \mathcal{M} = 1 \).

The unimodular matrix \( \mathcal{M} \) for a compound sequence of lens elements is the corresponding sequential product of the individual matrices of the elements. For example, a sequence of free propagation (distance \( L \)), followed by defocusing lens (focal length \( -f \)), followed by a space \( a \), followed by a focusing lens (focal length \( f \)), followed by free propagation (distance \( \ell \)) yields the result:

\[
\begin{pmatrix}
  x(\ell) \\
  p(\ell)/E
\end{pmatrix}
= \left( 1 + \frac{a}{f} - \frac{af}{f^2} \right) \left( L + a + \ell \right) + \frac{af}{f} - \frac{aL}{f^2} - \frac{aL}{f} - \frac{a}{f} - \frac{aL}{f^2} - \frac{a}{f} - \frac{aL}{f^2}
\]

\[
\begin{pmatrix}
  x(-\delta z) \\
  p(-\delta z)/E
\end{pmatrix}
\]

(3.40)

The zero of the \((11)\) matrix element in \( \ell = F \equiv f + f^2/a \) implies the system is net focusing with composite focal length \( F \) (e.g., see ref. [13]).

The effect of this particular lens system in quantum mechanics, e.g., on the Gaussian kernel \( A \) as defined in eq. (1.9), can be easily derived from the Schrödinger equation:

\[
A(\ell) = \frac{A_0(1 - a/f - aL/f^2) + iEa/f^2}{1 + a/f - af/f^2 + iA_0/E[L + a + \ell + AL/f - af/f - (aL/f^2)]}
\]

(3.41)

The focal length, \( F \), is where \( \mathcal{M}_{11} = 0 \), and, at the focal length we obtain the width:

\[
\delta(F) = \frac{f^2}{aE\delta_0} \quad \text{where: } Re(A(0)) = \frac{1}{\delta^2_0}
\]

(3.42)
This result is the minimal diffractive quantum limit for the composite lens system, and it is again determined by the initial width of the quantum state.

Of course, beyond $\delta_0$, there is actually no new information in the above formula for $A$ than is already present in the lens matrix for the classical ray optics. If the lens matrix is $\mathcal{M}_{ij}$, then we see by comparison with eq.(3.41) the general result for the Gaussian kernel ([14], [8]):

**Theorem I:**

$$A_{out} = \frac{\mathcal{M}_{22}A_{in} - i\mathcal{M}_{21}E}{\mathcal{M}_{11} + i\mathcal{M}_{12}A_{in}/E}$$  \hspace{1cm} (3.43)

That is, if we write the out-amplitude as:

$$A_{out} = -iE \left( \frac{N}{D} \right)$$  \hspace{1cm} (3.44)

then:

$$\left( \begin{array}{c} D \\ N \end{array} \right) = \left( \begin{array}{cc} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{array} \right) \left( \begin{array}{c} -iE \\ A_{in} \end{array} \right)$$  \hspace{1cm} (3.45)

From this result we can easily derive the quantum limit on beam size at the classical focal length. The classical focal length occurs where $\mathcal{M}_{11} = 0$. At this point we have $\mathcal{M}_{12} = -\mathcal{M}_{21}^{-1}$. Hence we readily find:

**Theorem II:**

$$ReA_f = \frac{E^2 ReA_0}{\mathcal{M}_{12}^2[(ReA_0)^2 + (ImA_0)^2]} = \frac{1}{(\Delta x)^2}.$$  \hspace{1cm} (3.46)

If $A_0 = \hbar/\delta_0$ is pure real, then:

$$\Delta x = \frac{\mathcal{M}_{12}}{E\delta_0} = \frac{\tilde{F}}{E\delta_0}$$  \hspace{1cm} (3.47)

noting that $\mathcal{M}_{12} \equiv \tilde{F} \sim f$ is a length scale comparable to the focal length at the classical focal length, e.g., $\tilde{F} = f^2/a$ in our previous compound lens example (see previous footnote).

Now, if the magnet system is periodic, as in a synchrotron, we expect quantum states that are approximately periodic solutions in the matrix. Periodic solutions must be eigenstates of the matrix $\mathcal{M}$. Consider first the motion within a very thick lens, i.e., a continuous transverse SHO potential. For an infinitesimal displacement in the $z$-direction, the lens matrix is:

$$\mathcal{M}_{SHO} = \begin{pmatrix} 1 & \delta z \\ -\frac{K\delta z}{E} & 1 \end{pmatrix}$$  \hspace{1cm} (3.48)

---

2Here we might imagine taking $a \to \infty$ holding $f$ fixed to cause $\delta(F) \to 0$; however, for $a \gg f$ the longitudinal dimension ($\Delta z$) of the focal point becomes small as $\sim f/a$ owing to the $aL/f$ and $a\ell L/f^2$ terms in $\mathcal{M}_{12}$, and the finite longitudinal distribution of the beams becomes problematic; we have not looked in detail at optimization of this.
which is unimodular to $O((\delta z)^2)$. The eigenvalues of $M_h$ are $\lambda_\pm = 1 \pm i\delta z\sqrt{K/E}$. The stable, quantum solutions in the lens are therefore the eigenvectors:

$$
\lambda_\pm \begin{pmatrix} -iE \\ A_0 \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} -iE \\ A_0 \end{pmatrix}
$$

hence we find:

$$
A_0 = \pm \sqrt{KE}
$$

Hence, the stable solution in the linear focusing channel is, indeed, the Gaussian ground-state solution in the SHO potential. We have simply recovered the usual Gaussian ground-state in this limit.

Consider now a “synchrotron,” i.e., periodic magnet lens lattice based upon the above lens configuration. We assume an infinite series of alternating dipoles with spacing $a$ and focal lengths $\pm f$. Replacing $L = a - \ell$ in the matrix elements $M_{ij}$ of eq.(3.40) gives the lens matrix for the synchrotron:

$$
\mathcal{M} = \begin{pmatrix}
1 + \frac{a^2}{f^2} - \frac{a\ell}{f^2} & 2a + \frac{a^2}{f^2} - 2\frac{a\ell}{f^2} - \frac{a^2\ell}{f^2} + \frac{a\ell^2}{f^2} \\
-\frac{a\ell}{f^2} & 1 - \frac{a^2}{f^2} - \frac{a\ell}{f^2} + \frac{a\ell^2}{f^2}
\end{pmatrix}
$$

The condition that we have a periodic solution is:

$$
A_\ell = \frac{M_{22}A_\ell - iM_{21}E}{M_{11} + iM_{12}A_\ell/E}
$$

Using $\det \mathcal{M} = 1$ we find:

$$
A_\ell = \frac{-iE}{2M_{12}} \left[ M_{22} - M_{11} \pm \left( (M_{22} + M_{11})^2 - 4 \right)^{1/2} \right]
$$

Stable quantum solutions (solutions that are normalizeable Gaussians for all $\ell$) therefore require:

$$
-1 \leq \frac{1}{2} Tr(\mathcal{M}) \leq 1
$$

This is, of course, the familiar stability condition for the classical motion. Note that $Tr(\mathcal{M}) = 2 - a^2/f^2$, which is $\ell$ independent, thus when the condition is met for particular choices of $f$ and $a$ it holds everywhere. The stability condition is the usual one, $f \geq a/2$.

The solution for $A(\ell)$ is:

$$
A(\ell) = \frac{E}{(2f + a)(1 - \ell/f) + \ell^2/f} \left[ \left( 1 - \frac{a^2}{4f^2} \right)^{1/2} + i \left( 1 + \frac{a}{2f} - \frac{\ell}{f} \right) \right]
$$
where \( f \geq a/2 \) and \( 0 \leq \ell \leq a \).

Consider the special case of a system in which \( f = a \). We see that the minimum Gaussian width occurs at \( \ell = a \), given by:

\[
\min(1/ReA(\ell)) = \frac{2f}{\sqrt{3E}} \equiv (\Delta x)^2
\]

This implies that the minimum achievable beam spot size in a synchrotron is \( \sim \sqrt{f/E} \). If a focusing magnet of focal length \( f' \) is inserted into the synchrotron magnet lattice and, then we obtain the minimum spot size \( \sim f'/E \Delta x \sim f'/\sqrt{EF} \). There is no initial parameter \( \delta_0 \) since we have assumed that synchrotron radiation relaxes the quantum state into the stable, periodic solution. Here we see a potential advantage of a linear collider over a synchrotron, in that the linear collider has a much larger \( \delta_0 \sim \sqrt{f/E_0} \) prepared in the low energy damping ring, which makes the quantum diffractive limit smaller, \( \sim f/E \delta_0 \), while in the synchrotron \( \delta_0 \sim \sqrt{f/E} \), where \( E \) is the larger beam energy, thus giving a larger diffractive limit \( \sim \sqrt{f/E} \).

4 Conclusion

In conclusion, we estimate the minimal quantum beam spot size achievable in a linear collider to be given by:

\[
\Delta x \geq \frac{\hbar c f}{E \delta_0}
\]

where \( f \) is the final focal length, \( E \) the beam energy, and \( \delta_0 \) is the initial transverse size of the wave-functions prior to acceleration. This may be viewed as a direct transcription of the Heisenberg uncertainty principle. \( \delta_0 \) is prepared in the synchronous damping rings, typically wigglers, and \( \delta_0 \sim 1/\sqrt{eB/\hbar c} \) where \( B \) is the magnetic field strength. We have for an NLC-class machine, \( B \sim 1 \) Tesla, \( f \sim 2 \) m, \( E \sim 250 \) GeV hence \( \delta_0 \sim 25 \) nm, and \( \Delta x \sim \mathcal{O}(0.06) \) nm.

Radiation damping implies that the initial state wave-function is not a groundstate, and has an average equilibrium principle quantum number \( \bar{n} \). Then our result is modified:

\[
\Delta x = \sqrt{\bar{n}} \frac{\hbar c f}{E \delta_0}
\]

\( \bar{n} \) is estimated to be

\[
\bar{n} \sim \left( \frac{m^3}{eB_0 E} \right)^{1/2}
\]
or, for the above parameters, \( \bar{n} \sim 0.11 \times 10^4 \). Eq.(4.59) is essentially classical, and yields \( \Delta x \sim 2 \text{ nm} \), roughly consistent with the classical vertical final focus beam spot size of the NLC, \( \sim 5 \text{ nm} \). A more precise analysis of this latter effect is, however, certainly required. This may be a useful way to approach other phenomena, such as the Oide effect, in which large fluctuation in \( \bar{n} \) in strong final focus magnets can occur, broadening the beam spot size.

We have examined the quantum solution in a simple FODO synchrotron model. In a synchrotron information about the initial \( \delta_0 \) is lost, and the minimal transverse beam spot size is:

\[
\Delta x = \sqrt{\frac{\hbar c f}{E}}
\]

which is \( \mathcal{O}(1) \text{ nm} \) for most high energy synchrotrons, e.g., LEP and Tevatron, in operation at present. Again, a factor of \( \bar{n} \) would yield the classical result. Presumably proton synchrotrons are far from the equilibrium, with \( n \gg \bar{n} \).

A tantalizing question is: can quantum diffractive effects be observed? More generally, our discussion has been motivated by the belief that quantum optics may be the preferred way to analyze futuristic machines. A more general formalism more symmetrical in \( p_\perp \) and \( x_\perp \) for the study of the quantum phase space, perhaps based upon Wigner’s formulation of quantum mechanics, is desired.

**Acknowledgements**

We wish to thank W. Bardeen, D. Burke, J. D. Jackson, R. Noble, C. Quigg, A. Tollestrup, and especially P. Chen, D. Finley, L. Michelotti, and R. Raja for useful discussions.
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