Comparative analysis of three MCMC methods for estimating GARCH models

D B Nugroho
Department of Mathematics, Satya Wacana Christian University,
Jln. Diponegoro 52–60 Salatiga 50711 Central Java, Indonesia

didit.budinugroho@staff.uksw.edu

Abstract. GARCH model have been considered as an important and widely employed tool to analyse and forecast variance of the financial market. This study develops three MCMC methods, namely adaptive random walk Metropolis, Hamiltonian Monte Carlo, and Independence Chain Metropolis–Hastings algorithms. It is used to estimate GARCH (1,1) under Normal and Student-t distributions for conditional return distribution. Results on real financial market data indicate that the best method is the approach based on the Independence Chain Metropolis–Hastings algorithm.

1. Introduction
Financial markets have played an important role in global economic growth. On this subject one of the important aspects of the markets is to model and estimate the fluctuative financial market. It is caused by the volatility estimation, is often relied by policy makers as a barometer for the sensitivity of financial markets [1]. Meanwhile, in stock exchange markets, the volatility of stock price is an important indicator of the stock price fluctuations [2]. Volatility can be determined by using statistical measures of standard deviation (square root of variance) between returns from that same asset. Commonly, the high volatility (the wide range of fluctuations) reflects high trading risk. In a rising market, however, high volatility can favor the return potential of an investment [3].

The popular class of volatility model for the financial time series that exhibit heteroscedasticity (refers to the volatility fluctuation) is represented by the Autoregressive Conditional Heteroskedasticity (ARCH) model of [4] and the Generalized ARCH (GARCH) of [5]. On the empirical level, for example, [6,7,8,9,10,11,12,13] provided several applications of GARCH-type models for estimating and forecasting volatility in the stock and foreign exchange markets.

In order to infer the model parameters from GARCH models, the numerical procedures based on a maximum likelihood (ML) are often employed. For example, [14,15,16] developed the ML estimation for the GARCH-type models with normal and Student-\(t\) distributions and [17] proposed the non-Gaussian Quasi-Maximum Likelihood estimation for the GARCH model with Student-\(t\) distribution.

Although the ML estimation is conceptually simple, in practice it may be possible to encounter difficulties in applying the ML-based methods for the GARCH-type models, see the detail in [18]. To overcome the computational difficulties of ML-based methods, [18] proposed Bayesian method via Markov Chain Monte Carlo (MCMC) estimation method. A variety of algorithms which might implement MCMC methods effectively for the GARCH-type models have been proposed and applied. For example, the Gibbs sampler is used in [19], the Metropolis–Hastings-based algorithms are used in
[15,18,20,21,22,23,24,25,26], and the Hamiltonian Monte Carlo method is used in [27,28]. In particular, [15] reported that the GARCH(1,1) parameters obtained by the ML and Metropolis–Hastings methods are close to each other. Furthermore, [20,29] showed that the Bayesian approach via the MCMC methods may lead to more reliable and suitable than the ML-based methods for estimating the GARCH model based on the use of real financial data. Hence this study focuses on applying MCMC algorithms to GARCH models.

Since the implementation of the MCMC algorithm is not unique and may involve high-dimensional integration, it is important to consider how rapidly the convergence of the Markov chain produced by MCMC algorithms to its stationary distribution. It is therefore necessary to investigate the convergence performance of MCMC methods in estimating the GARCH parameters. The convergence property of these methods is revealed by measure of the sampling efficiency and Markov chain mixing, which are defined by integrated autocorrelation time (IACT) and numerical standard error (NSE), respectively. The literature on comparison of the MCMC methods in estimating the GARCH models is much limited. In the GARCH context, only [23] numerically compared some of MCMC methods, including Griddy Gibbs, Metropolis–Hastings, adaptive rejection Metropolis, and acceptance-rejection Metropolis–Hastings.

In the context of seeking a method that draws samples efficiently from any univariate conditional density function, this study has two main contributions. The first is a novel performance comparison of three MCMC methods, including adaptive random walk Metropolis (ARWM), independence chain Metropolis–Hastings (ICMH), and Hamiltonian Monte Carlo (HMC), for the estimation of the GARCH parameters. Our survey indicate that none of research apply those methods separately to estimate the Student-t GARCH model in particular and even compare their performance. The ARWM method is proposed by [30] for improving the efficiency of the random walk Metropolis, which is the simplest and the most widely used MCMC method [31]. The ICMH method is proposed by [32] and among the most popular alternative Metropolis–Hastings (MH) methods. This method works best when there is another probability measure which is easy to sample from. The HMC method is proposed by [33] and has better convergence properties than MH and Gibbs sampler [34]. The second one is to contribute to extend the empirical study on GARCH estimation in emerging financial markets.

Although the extension of the standard GARCH is numerous, this study chooses the GARCH(1,1) models with normal and Student-t distributions, without limiting the methodological potential of estimation procedures, because of their application to many financial market data. An empirical application that compares the MCMC methods is illustrated by fitting the models to the daily volatility of currency exchange rate returns on the Indonesian Currency Exchange Market. This is to provide a performance comparison of the proposed algorithms in terms of autocorrelation time and standard error. This study is structured into four sections as follows. Immediately preceding Introduction in Section 1 is Section 2, which outlines the methodology and real data. Section 3 presents the empirical results, and Section 4 provides the concluding remarks.

2. Methodology and Real Data

2.1. Methodology

The Bayesian framework was chosen because this allows the application of MCMC methods which are known as very powerful techniques for calculating the numerical integration of the joint density. The key steps of our proposed framework are as follows. First, within a particular GARCH model, a Markov chain having an identical stationary distribution to the posterior density of parameter is constructed by using an MCMC method. After discarding early samples and sufficient iterations, the Markov chain provides samples from the posterior distribution of parameter. Second, for a given set of competing MCMC methods, we employ each method separately and base their comparison in terms of autocorrelation time and numerical standard error.
Let $R_t$ be a serially uncorrelated return whose mean equation is given by $R_t = \sigma_t \varepsilon_t$, where $\{\varepsilon_t\}$ is a sequence of iid random innovations (errors) with mean 0 and variance 1. The GARCH(1,1) approach estimates the current variance, $\sigma_t^2$, as a linear function of past squared innovation and past variance, i.e.,

$$\sigma_t^2 = \omega + \alpha R_{t-1}^2 + \beta \sigma_{t-1}^2,$$

where $\omega, \alpha, \beta \geq 0$ to ensure the positivity of the conditional variance and $\alpha + \beta < 1$ to ensure stationarity; see [5] for details. The above model is then called the normal GARCH (GARCHn hereafter) model. By considering $\mathbf{R} = (R_1, R_2, \ldots, R_T)$ and let $p(\omega, \alpha, \beta)$ and $f(\mathbf{R}|\omega, \alpha, \beta)$ be a joint prior density and a likelihood function, respectively, the logarithm of the joint posterior density of parameters can be obtained by combining the log joint prior density with the log likelihood function via Bayes’ Theorem, i.e.,

$$L = \log p(\omega, \alpha, \beta|\mathbf{R}) = \log p(\omega, \alpha, \beta) - \frac{1}{2} T \log(2\pi) - \frac{1}{2} \sum_{t=1}^{T} \log(\sigma_t^2) - \frac{1}{2} \sum_{t=1}^{T} R_t^2. \quad (1)$$

If $\varepsilon_t$ has the Student-$t$ distribution with degrees of freedom $\nu$, the model is called the Student-$t$ GARCH (GARCH$^t$ hereafter) model. In this case, the logarithm of the joint posterior density of parameters $(\omega, \alpha, \beta, \nu)$ is similar to that given by [35],

$$L = \log p(\omega, \alpha, \beta, \nu|\mathbf{R}) = \log p(\omega, \alpha, \beta, \nu) + T \log \Gamma\left(\frac{\nu+1}{2}\right) - T \log \Gamma\left(\frac{\nu}{2}\right) - T \log \Gamma\left(\frac{1}{2}\right)$$

$$- \frac{1}{2} \sum_{t=1}^{T} \log(\sigma_t^2 (\nu - 2)) - \frac{\nu+1}{2} \sum_{t=1}^{T} \log \left(1 + \frac{R_t^2}{\sigma_t^2 (\nu - 2)}\right). \quad (2)$$

This study develops the procedure to sample the parameter vector $\theta = (\omega, \alpha, \beta, \nu)$ in the GARCH$^t$ model from the log joint posterior density (2) at each sampling stage in the MCMC algorithm as follows.

1. Draw $\omega$ from $p(\omega|\alpha, \beta, \nu, \mathbf{R})$.
2. Draw $\alpha$ from $p(\alpha|\omega, \beta, \nu, \mathbf{R})$.
3. Draw $\beta$ from $p(\beta|\omega, \alpha, \nu, \mathbf{R})$.
4. Draw $\nu$ from $p(\nu|\omega, \alpha, \beta, \mathbf{R})$.

In the GARCH$n$ case, the sampling procedure is similar to the above but ignoring parameter $\nu$ and on the basis of the joint posterior (1). Since the above conditional posteriors are not of standard form, we draw each parameter using the same method in a MCMC simulation.

Two popular MH chains are the random walk chain and the independence chain. The random walk chain centers the proposal density over the current value of the chain. Improving the performance of Random Walk Metropolis algorithm has been proposed by [30] through ARWM algorithm which automatically handles the the proposal value. In contrast to the random walk chain, “independence” of ICMH refers to the fact that the proposal value is not necessary depend on the current value. For this method to work well, this study adapts the proposal value using the method which is based on the behavior of the density about its mode (see [36]). In the case of the sampling of degrees of freedom $\nu$, the first and second derivatives of $L(\nu)$ are respectively given as in [35].

An alternative MH chain is the HMC method which was proposed in the statistical physics literature. This method avoids the random walk behavior by combining Gibbs updates with Metropolis updates. The proposal value is computed based on a trajectory which is guided by first-order derivative of the log of the posterior density. In particular, this study applies the similar algorithm which was employed by [37, 38, 39] for estimating the degrees of freedom $\nu$.

For the unknown parameters in the GARCH models, following standard practice, we complete the models by the truncated normal priors with mean 0 and variance 1000 on $(\omega, \alpha, \beta)$ and the exponential prior with parameter $\lambda = 0.01$ on $\nu$. For initial parameter values, we use $\omega_0 = 0.01$, $\alpha_0 = 0.2$, $\beta_0 = 0.7$, and $\nu_0 = 0.01$. For the HMC updates, we apply 100 leapfrog iterations with a step size of 0.01.

Desirable properties for method in the MCMC algorithm are high efficiency and well chain mixing, which yield fast convergence. In this study, the efficiency is measured by the integrated autocorrelation time (IACB) which is estimated using the Sokal’s adaptive truncated periodogram estimator [40]. The quantity can be roughly interpreted as the number of iterations required to produce one independent draws. A lower value indicates a higher sampling efficiency in a set of draws while large values indicate
a poor sampling efficiency (little movement between successive draws). Meanwhile, the mixing performance is checked by assessing the Monte Carlo error that can be accomplished by estimating the variance which is also known as the numerical standard error (NSE); see [41] for details.

2.2 Data for analysis

For illustrative and comparative purposes, this study uses the series of daily closing prices \( \{ P_t \} \) of the EUR, JPY, and USD selling exchange rates to IDR over a period from January 2010 to December 2016. The series of return are daily percentage returns given by

\[
R_t = 100 \times (\log P_t - \log P_{t-1}).
\]

Notice that our daily returns are neither serially correlated and normally distributed on the basis of Ljung–Box correlation and Jarque–Bera normality tests.

3. Empirical Results

For each sampling method, we ran two MCMC simulations, i.e., for 6000 iterations (first 1000 iterations discarded as burn-in) and 15000 iterations (first 5000 iterations discarded as burn-in). This runs are long enough to obtain stationary distribution, and so follow realistic statistical practice. The remaining iterations are then used for calculating the posterior mean, IACT, and NSE. The MCMC methods were implemented in Matlab R2013a by writing own code and the simulations were ran on a personal machine with Intel Core i7-4790 CPU 3.60GHz x 16.0 GB of memory on a 64-bit Windows 10 Pro operating system.

| Stat. Method | N= 6000, Burn-in = 1000 | N= 15000, Burn-in = 5000 |
|--------------|--------------------------|----------------------------|
|              | \( \omega \) \( \alpha \) \( \beta \) \( v \) Time (sec.) | \( \omega \) \( \alpha \) \( \beta \) \( v \) Time (sec.) |
| ARWM         | 0.0243 0.0621 0.8833 2.6 0.0184 0.0557 0.9029 6.9 |
| Mean HMC     | 0.0373 0.0781 0.8380 8.8 4.9 0.0583 0.0960 0.7735 8.7 | 3.02 |
| Mean HMC     | 0.0221 0.0585 0.8918 256 0.2323 0.0612 0.8865 410 |
| ICMH         | 0.0452 0.0840 0.8142 8.7 507 0.0428 0.0800 0.8238 8.7 | 4.59 |
| ICMH         | 0.0130 0.0474 0.9231 66 0.0146 0.0504 0.9166 178 |
| ICMH         | 0.0280 0.0687 0.8683 8.4 152 0.0346 0.0750 0.8472 8.3 | 6.58 |
| IACT HMC     | 219.9 130.3 216.7 245.3 197.2 259.1 193.5 140.1 191.7 6.4 | 221.4 173.1 223.2 6.3 |
| IACT HMC     | 116.2 96.5 129.7 131.2 83.7 133.0 150.9 113.4 161.0 15.8 | 198.2 170.7 202.3 |
| ICMH         | 72.3 55.1 75.1 99.3 71.0 111.7 94.3 70.2 95.2 1.8 | 94.1 64.0 90.0 |
| ICMH         | 3.02 2.90 9.30 67.2 3.96 3.15 11.9 36.9 |
| NSE (x10^-3) | 1.82 5.06 | 1.37 3.47 |
| NSE (x10^-3) | 1.55 0.60 | 7.50 |
| NSE (x10^-3) | 1.39 4.58 | 1.48 4.63 |
| NSE (x10^-3) | 3.00 8.34 | 4.06 12.10 |
| NSE (x10^-3) | 0.49 1.80 | 0.41 1.56 |
| NSE (x10^-3) | 1.37 4.59 | 2.13 6.58 |
| NSE (x10^-3) | 1.58 22.3 | 1.86 20.8 |

Tables 1–3 show estimation results for the GARCHn(1, 1) and GARCHt(1,1) models adopting IDR/EUR, IDR/JPY and IDR/USD return series, respectively. Although the methods produce similar parameter estimates, the ICMH method is most efficient and gives the best mixing in terms of IACT and NSE, followed by the HMC method and the ARWM method in the case of sampling \( \omega, \alpha, \beta \). On estimating the parameter degrees of freedom \( v \), all methods experimentally show to have high efficiency, where the ARWM and HMC methods are very competitive. Regarding computational time, the HMC
method runs much slower than the others caused by the use of leapfrog method. Among three competing methods, the ARWM method is least time-demanding.

| Stat. | Method | $N=6000$, Burn-in = 1000 | $N=15000$, Burn-in = 5000 |
|-------|--------|--------------------------|---------------------------|
|       | $\omega$ | $\alpha$ | $\beta$ | $\nu$ | Time (sec.) | $\omega$ | $\alpha$ | $\beta$ | $\nu$ | Time (sec.) |
| Mean  | ARWM    | 0.0377 | 0.0984 | 0.8428 | 2.6 | 0.0333 | 0.0925 | 0.8554 | 6.8 |
|       |         | 0.0399 | 0.1117 | 0.8285 | 5.1 | 0.0426 | 0.1152 | 0.8212 | 5.1 |
|       | HMC     | 0.0300 | 0.0880 | 0.8653 | 246.6 | 0.0335 | 0.0928 | 0.8548 | 401.8 |
|       |         | 0.0379 | 0.1065 | 0.8374 | 5.1 | 0.0416 | 0.1115 | 0.8267 | 5.1 |
|       | ICMH    | 0.0255 | 0.0818 | 0.8783 | 65.7 | 0.0281 | 0.0855 | 0.8704 | 157.6 |
|       |         | 0.0280 | 0.0920 | 0.8668 | 5.0 | 0.0299 | 0.0958 | 0.8603 | 5.0 |

| NSE (x10^{-3}) | ARWM | 223.8 | 166.2 | 206.9 | 236.0 | 264.3 | 295.7 |
| IACT   | HMC   | 67.9  | 56.9  | 75.7  | 94.2  | 81.0  | 99.5  |
|        |       | 146.9 | 107.0 | 143.7 | 103.3 | 95.3  | 110.2 |
|        | ICMH  | 48.2  | 47.7  | 53.0  | 58.5  | 51.6  | 60.8  |
|        |       | 62.5  | 56.0  | 67.1  | 83.2  | 72.5  | 93.2  |

| NSE (x10^{-3}) | ARWM | 2.42  | 3.60  | 7.10  | 1.72  | 2.44  | 5.03  |
| IACT   | HMC   | 1.03  | 1.64  | 3.16  | 1.50  | 2.32  | 4.67  |
|        |       | 2.37  | 3.58  | 7.06  | 1.76  | 2.70  | 5.21  |
|        | ICMH  | 0.57  | 1.13  | 1.98  | 0.91  | 1.43  | 2.78  |
|        |       | 0.83  | 1.54  | 2.77  | 0.80  | 1.42  | 2.58  |

| Stat. | Method | $N=6000$, Burn-in = 1000 | $N=15000$, Burn-in = 5000 |
|-------|--------|--------------------------|---------------------------|
|       | $\omega$ | $\alpha$ | $\beta$ | $\nu$ | Time (sec.) | $\omega$ | $\alpha$ | $\beta$ | $\nu$ | Time (sec.) |
| Mean  | ARWM    | 0.0092 | 0.2065 | 0.7719 | 2.4 | 0.0098 | 0.2138 | 0.7631 | 6.2 |
|       |         | 0.0040 | 0.2005 | 0.7939 | 4.2 | 0.0045 | 0.2137 | 0.7802 | 4.2 |
|       | HMC     | 0.0098 | 0.2145 | 0.7628 | 243.8 | 0.0097 | 0.2113 | 0.7657 | 416.1 |
|       |         | 0.0039 | 0.1970 | 0.7972 | 4.2 | 0.0040 | 0.1961 | 0.7979 | 4.2 |
|       | ICMH    | 0.0089 | 0.2063 | 0.7743 | 65.7 | 0.0093 | 0.2135 | 0.7670 | 155.6 |
|       |         | 0.0038 | 0.2015 | 0.7946 | 4.1 | 0.0036 | 0.1936 | 0.8026 | 4.1 |
|       |         |         |         |       | 148  |       |       |       | 246  |
| IACT  | ARWM    | 109.4  | 144.0  | 152.9 | 105.2 | 142.3 | 160.6 |
|       |         | 211.2  | 267.8  | 266.5 | 213.7 | 277.6 | 279.3 |
|       |         | 46.0   | 43.8   | 52.4  | 33.0  | 36.0  | 38.5  |
|       |         | 80.0   | 125.9  | 124.4 | 90.7  | 176.6 | 176.2 |
|       | ICMH    | 24.2   | 37.2   | 37.7  | 23.5  | 31.0  | 34.7  |
|       |         | 68.4   | 95.1   | 95.8  | 51.8  | 96.9  | 97.3  |
|       |         |         |         |       | 51.8  | 96.9  | 97.3  |
|       |         |         |         |       | 51.8  | 96.9  | 97.3  |
|       |         |         |         |       | 51.8  | 96.9  | 97.3  |
| NSE (x10^{-3}) | ARWM | 0.23  | 3.92  | 4.19  | 0.17 | 2.84  | 3.00  |
| HMC   | 0.15  | 4.65  | 4.77  | 13.6 | 0.16 | 4.30  | 4.39  |
|       | 0.18  | 2.56  | 2.86  | 2.86 | 0.15 | 2.33  | 2.55  |
|       | 0.12  | 4.03  | 4.06  | 12.0 | 0.09 | 2.61  | 2.65  |
|       | 0.08  | 1.55  | 1.55  | 1.55 | 0.09 | 1.360 | 1.451 |
|       | 0.08  | 2.67  | 2.71  | 4.5  | 0.06 | 2.08  | 2.12  |

Table 2. Parameter estimates and convergence speeds in the GARCH$n$ and GARCH$t$ models adopting IDR/JPY return series.
4. Conclusions
This study adopted three popular MCMC methods for estimating the GARCH models with normal and Student-t distributions for the return errors. For each method, we describe the difference in proposing a candidate value. The comparison of the proposed methods performance was grounded on the real data. All proposed methods work well and provide similar accuracy. The results declare that the ICMH is the best method in terms of the efficiency, mixing, and computational time of the MCMC algorithm.

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