Linear $r$–Matrix Algebra for Systems Separable in Parabolic Coordinates

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Abstract

We consider a hierarchy of many particle systems on the line with polynomial potentials separable in parabolic coordinates. Using the Lax representation, written in terms of $2 \times 2$ matrices for the whole hierarchy, we construct the associated linear $r$–matrix algebra with the $r$-matrix dependent on the dynamical variables.

A dynamical Yang-Baxter equation is discussed.

1 Introduction

It is known that it is possible to provide a $2 \times 2$ Lax operator satisfying the standard linear $r$-matrix algebra for a Hamiltonian system of natural form where the potential $U$ is of second degree in the coordinates (see e.g. [3], [6], [8], [18], [21]). In recent years the study of completely integrable systems admitting a classical $r$–matrix Poisson structure with the $r$-matrix dependent on dynamical variables has attracted some attention [3], [8], [13]. It is remarkable that the celebrated Calogero–Moser system, whose complete integrability was shown a number of years ago (c.f. [15]), was found only recently to possess a classical $r$–matrix of dynamical type [2]. In this paper we study another example of a dynamical $r$–matrix structure. The potentials described are known as a parabolic family of potentials since they are permutationally symmetric potentials separable in generalized parabolic coordinates. They were introduced in [22], [23] and their connection with restricted coupled KdV flows was studied in [17] (see also [14] for the isomorphism with KdV). However the $r$–matrix Poisson structure associated with higher degree potentials has not been discussed. The systems represent a generalization of a known hierarchy of two-particle systems with polynomial potentials separable in generalized ($n$-dimensional) parabolic coordinates (c.f. [14], [16]).
The polynomial second order spectral problem associated with the Lax representation given below has been studied in a number of papers (c.f. [7, 24]). Here we reproduce all these results in the framework of a \((2 \times 2)\) Lax representation, which is a direct generalization of that given by [14]; this being an effective technique to describe explicitly our class of integrable systems and to investigate their classical Poisson structure. We also note, that the Lax representation can be extracted from the results [1] by some limiting procedure. We consider the system within the method of separation of variables [11, 12, 21] which allows us to develop the classical theta-functional integration theory and to consider the associated quantum problem. The last problem is reduced to a set of multiparameter spectral problems which are a confluent form of ordinary differential equations of Fuchsian type.

The central result of this letter is the description of the Poisson structure of the system by a dynamical linear \(r\)-matrix algebra.

2 Lax Representation

Consider the hierarchy of the Hamiltonian systems of \(n + 1\) particles defined by the Hamiltonians

\[
H_N (p_1, \ldots, p_{n+1}; q_1, \ldots, q_{n+1}) = \frac{1}{2} \sum_{i=1}^{n+1} p_i^2 + U_N (q_1, \ldots, q_{n+1}),
\]

where the potentials \(U_N\) fix the member of hierarchy by the recurrence relation

\[
U_N = (q_{n+1} - B) U_{N-1} + \frac{1}{4} \sum_{i=1}^{n} \sum_{j=2}^{N} (-1)^j q_i^2 U_{N-j} A_i^{j-2}
\]

with the first trivial potentials given by \(U_0 = -2, U_1 = -2q_{n+1} - 2B, U_2 = -2q_{n+1}^2 - \frac{1}{2} \sum_{i=1}^{n} q_i^2\). The general expression for the potentials \(U_N\) at \(N > 2\) can be written in the form of a \((N - 2) \times (N - 2)\) determinant

\[
U_N = (-1)^{N-1} \det \begin{pmatrix}
   f_N & g_{-1} & g_0 & g_1 & \ldots & g_{N-5} \\
   f_{N-1} & -1 & g_{-1} & g_0 & \ldots & g_{N-6} \\
   f_{N-2} & 0 & -1 & g_{-1} & \ldots & g_{N-7} \\
   \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
   f_4 & 0 & \ldots & 0 & -1 & g_{-1} \\
   f_3 & 0 & \ldots & 0 & 0 & -1
\end{pmatrix},
\]

with

\[
g_{-1} = q_{n+1} - B, \quad g_m = \frac{(-1)^m}{4} \sum_{i=1}^{n} A_i^m q_i^2, \quad m = 0, \ldots
\]

\[
f_k = \sum_{l=0}^{2} U_l g_{k-l-2}, k = 3, \ldots, N,
\]

and \(U_0, U_1, U_2\) as given above. The first nontrivial potentials are

\[
U_3 = -2q_{n+1}^3 - q_{n+1} \sum_{i=1}^{n} q_i^2 + \frac{1}{2} \sum_{i=1}^{n} A_i q_i^2 + 2Bq_{n+1}^2.
\]
\[ U_4 = -\frac{1}{8} \left( \sum_{i=1}^{n} q_i^2 \right)^2 - \frac{3}{2} q_{n+1}^2 \sum_{i=1}^{n} q_i^2 - 2q_{n+1}^4 + \sum_{i=1}^{n} A_i q_i^2 \left( q_{n+1} - \frac{1}{2} A_i \right) + B \left( 4q_{n+1}^3 - 2Bq_{n+1}^2 + q_{n+1} \sum_{j=1}^{n} q_j^2 \right). \] (2.6)

The potential (2.5) is a many particle generalization of one of the known integrable cases of the Hénon-Heiles system for which \( n = 1 \) (c.f. [6, 14]). Analogously the potential (2.6) is a many particle generalization of the system “(1 : 12 : 16)”, known to be separable in parabolic coordinates (c.f. [10, 16]). The system with the potential (2.5) possesses two remarkable reductions: a) at \( q_{n+1} = \text{const} \), it reduces to the Neuman system which describes the motion of a particle over a sphere in the field of a second order potential and b) at \( q_{n+1} = \sum_{i=1}^{n} q_i^2 \) it reduces to an anisotropic oscillator with a fourth order potential (see, for instance [16]). The same reduction can be carried out for other members of the hierarchy.

We look for a Lax representation in the form
\[
\dot{L}_N(z) = [M_N(z), L_N(z)],
\]
where
\[
L_N(z) = \begin{pmatrix} V(z) & U(z) \\ W_N(z) & -V(z) \end{pmatrix}, \quad M_N(z) = \begin{pmatrix} 0 & 1 \\ Q_N(z) & 0 \end{pmatrix},
\]
with
\[
U(z) = 4z - 4q_{n+1} + 4B - \sum_{i=1}^{n} \frac{q_i^2}{z + A_i},
\]
\[
V(z) = -\frac{1}{2}\dot{U}(z) = 2p_{n+1} + \sum_{i=1}^{n} \frac{p_i q_i}{z + A_i},
\]
\[
W_N(z) = -\frac{1}{2}\ddot{U}(z) + U(z)Q_N(z),
\]
where \( Q_N(z) \) is a polynomial of degree \( N - 2 \). The ansatz for the functions \( U(z), V(z), W(z) \) is a generalization of the corresponding ansatz constructed by Newell et al. [14] to give the Lax representation for the integrable Hénon-Heiles system. Here we introduce additional degrees of freedom \( n > 1 \), and higher degrees of the polynomial \( Q_N \) are considered. See also [12] for the link to the \( su(1,1) \)-Gaudin magnet which corresponds to a free \( n \)-dimensional particle separable in parabolic coordinates.

It is possible to show that the Lax representation (2.7) is valid for all the hierarchy of Hamiltonian systems (2.1), (2.3) with the polynomial \( Q_N(z) \) and the function \( W_N(z) \) given by the recurrence relations
\[
Q_N(z) = zQ_{N-1}(z) - \frac{1}{2} \frac{\partial U_{N-1}(q_1, \ldots, q_{n+1})}{\partial q_{n+1}},
\]
\[
W_N(z) = W_N^+(z) + W^-(z),
\]
\[
W_N^+(z) = zW_{N-1}^+ - 2U_{N-1}, \quad N = 2, \ldots,
\]
\[
W^-(z) = \sum_{i=1}^{n} \frac{p_i^2}{z + A_i},
\]
where \( U_{N-1} \) is the potential fixing the \((N - 1)\)-th member of the hierarchy and \( Q_2 = 1. \)
We can easily solve (2.11)–(2.14) to obtain formula for the functions \(W_N(z)\) and \(Q_N(z)\)

\[
Q_N(z) = z^{N-2} - \frac{1}{2} \sum_{k=0}^{N-3} \frac{\partial U_{N-k-1}}{\partial q_{n+1}} z^k,
\]

\[
W_N(z) = 4z^{N-1} - \frac{1}{2} \sum_{k=0}^{N-2} U_{N-k-1} z^k + \sum_{i=1}^{n} \frac{p_i^2}{z + A_i}.
\]  

(2.15)  

(2.16)  

For example, for the first nontrivial cases we have

\[
Q_3(z) = z + 2q_{n+1},
\]

\[
W_3(z) = 4z^2 + 4zq_{n+1} + 4Bz + 4q_{n+1}^2 + \sum_{k=1}^{n} q_k^2 + \sum_{i=1}^{n} \frac{p_i^2}{z + A_i}
\]  

(2.17)  

(2.18)  

for the many particle Hénon-Heiles system and

\[
Q_4(z) = z^2 + 2zq_{n+1} + 3q_{n+1}^2 + \frac{1}{2} \sum_{i=1}^{n} q_i^2 - 2Bq_{n+1},
\]

\[
W_4(z) = 4z^3 + 4Bz^2 + 4z^2q_{n+1} + 4zq_{n+1}^2 + z \sum_{i=1}^{n} q_i^2 + 4q_{n+1}^3 +
\]

\[
+ 2q_{n+1} \sum_{i=1}^{n} q_i^2 - \sum_{i=1}^{n} A_i q_i^2 - 4Bq_{n+1}^2 + \sum_{i=1}^{n} \frac{p_i^2}{z + A_i}
\]  

(2.19)  

(2.20)  

for the system with \(N = 4\).

The Lax representation yields the hyperelliptic curve \(C^{(N)} = (w, z)\),

\[
\text{Det} (L^{(N)}(z) - wI) = 0
\]

(2.21)  

generating the integrals of motion \(H_N, F_N^{(i)}, i = 1, \ldots, n\). We have from (2.21) and (2.8)-(2.10)

\[
w^2 = 16z^{N-2}(z + B)^2 + 8H_N + \sum_{i=1}^{n} \frac{F_N^{(i)}}{z + A_i}, \quad N = 3, \ldots,
\]

(2.22)  

where

\[
F_N^{(i)} = 2q_i^2 \sum_{j=1}^{N-1} (-1)^{j-1} A_i^j U_{N-j} + 4p_{n+1} p_i q_i - p_i^2 (A_i + 4q_{n+1} - 4B) +
\]

\[
+ \sum_{k \neq m} \frac{q_i^2}{A_m - A_k}, \quad i = 1, \ldots, n
\]  

(2.23)  

with \(l_{ij} = q_i p_j - q_j p_i, \quad i, j = 1, \ldots, n\). The integrals of motion \(H_N, F_N^{(i)}, i = 1, \ldots, n\) are independent and have vanishing Poisson brackets.
3 Separation of Variables

To define the separation variables, i.e. the canonically conjugated variables \( \pi_i, \mu_i, \ i = 1, \ldots, n+1 \) and \( n+1 \) functions \( \Phi_j \) such that

\[
\Phi_j(\pi_i, \mu_i, H_N, F^{(1)}_N, \ldots, F^{(n)}_N) = 0,
\]

where \( H_N, F^{(i)}_N \) are the integrals of motion in the involution, we use the scheme (c.f. [12, 19, 20]) according to which these variables are defined in terms of the Lax matrix as \( \pi_i = V(\mu_i), U(\mu_i) = 0 \). The set of zeros \( \mu_j, j = 1, \ldots n+1 \) of the function \( U(z) \) in the Lax representation (2.7) defines the parabolic coordinates given by the formulae

\[
q_{n+1} = \sum_{i=1}^{n+1} A_i + B + \sum_{i=1}^{n+1} \mu_i,
\]

\[
q_m^2 = -4 \frac{\prod_{j=1}^{n+1} (\mu_j + A_m)}{\prod_{k \neq m} (A_m - A_k)}, \ m = 1, \ldots, n.
\]  

The separation equations are of the form

\[
\pi_i^2 = w^2(\mu_i), \quad i = 1, \ldots, n+1,
\]

where the function \( w^2(z) \) is given by (2.22).

The separation equations have two uses – to integrate the equations of motion in terms of theta functions and to quantize the systems. We mention here that canonical quantization in the space of separation variables leads to the following multiparameter spectral problem for the wave function of the system \( \Psi = \prod_{j=1}^{n+1} \Psi_j \)

\[
[\frac{d^2}{dx^2} + 16x^{N-2}(x + B)^2 + 8\lambda_{n+1} + \sum_{i=1}^{n} \frac{\lambda_i}{x + A_i}] \Psi_j(x; \lambda_1 \ldots \lambda_{n+1}) = 0
\]

with \( j = 1, \ldots, n+1 \) and the spectral parameters \( \lambda_1, \ldots, \lambda_{n+1} \). The problem (3.4) has to be solved on \( n+1 \) different intervals – “permitted zones”.

4 \( r \)-Matrix Representation

It follows from the results of §2 and [3] that the system admits an \( r \)-matrix algebra. Let \{ \cdot, \cdot \} be the standard Poisson bracket and \{ \cdot \otimes \cdot \} be the standard Poisson bracket in the product of two linear spaces \( V^2 \otimes V^2 \). Then the classical Poisson structure for the hierarchy of dynamical systems described by the Lax operator \( L(z) \) with the entries (2.8)-(2.10) can be written in the form

\[
\{L_1^{(N)}(x) \otimes L_2^{(N)}(y)\} = [r(x - y), L_1^{(N)}(x) + L_2^{(N)}(y)] + [s^{(N)}(x, y), L_1^{(N)}(x) - L_2^{(N)}(y)],
\]  

with \( r, s \) satisfying certain \((N+1)\times(N+1)\)-dimensional matrix relations. The classical Poisson structure for the hierarchy of dynamical systems described by the Lax operator \( L(z) \) with the entries (2.8)-(2.10) can be written in the form

\[
\{L_1^{(N)}(x) \otimes L_2^{(N)}(y)\} = [r(x - y), L_1^{(N)}(x) + L_2^{(N)}(y)] + [s^{(N)}(x, y), L_1^{(N)}(x) - L_2^{(N)}(y)],
\]  

with \( r, s \) satisfying certain \((N+1)\times(N+1)\)-dimensional matrix relations.
where $L_1^{(N)}(x) = I \otimes L^{(N)}(x)$, $L_2^{(N)} = L^{(N)}(x) \otimes I$, $I$ is the $2 \times 2$ unit matrix and the matrices $r(x - y)$ and $s^{(N)}(x, y)$ are given by the formulae

$$r(x - y) = \frac{2}{x - y} P, \quad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$s^{(N)}(x, y) = 2\alpha_N(x, y)S, \quad S = \sigma_- \otimes \sigma_- \quad \sigma_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with

$$\alpha_N(x, y) = \frac{Q_N(x) - Q_N(y)}{x - y} = \frac{x^{N-1} - y^{N-1}}{x - y} - \frac{1}{2} \sum_{k=0}^{N-3} \frac{x^{N-k-1} - y^{N-k-1}}{x - y} \frac{\partial U_{N-k-1}}{\partial q_{n+1}}.$$  \hfill (4.4)

The equality (4.1) contains all the information concerning the hierarchy of dynamical systems, in particular one can obtain a simple proof of the involutivity of the integrals of motion. The representation (2.7) can also be derived from (4.1) (c.f. [3, 18]).

We write the relation (4.1) in the form

$$\{L_1^{(N)}(x) \otimes L_2^{(N)}(x)\} = [d_{12}^{(N)}(x, y), L_1^{(N)}(x)] - [d_{21}^{(N)}(x, y), L_2^{(N)}(y)]$$  \hfill (4.5)

with $d_{ij}^{(N)} = r_{ij} + s_{ij}^{(N)}$, $d_{ji}^{(N)} = s_{ij}^{(N)} - r_{ij}$ at $i < j$.

The compatibility conditions for (4.3) (Yang-Baxter equations) have the form

$$[d_{12}^{(N)}(x, y), d_{13}^{(N)}(x, z)] + [d_{12}^{(N)}(x, y), d_{23}^{(N)}(y, z)] + [d_{23}^{(N)}(y, z), d_{13}^{(N)}(x, z)]$$

$$+ \{L_2^{(N)}(y) \otimes d_{13}^{(N)}(x, z)\} - \{L_3^{(N)}(z) \otimes d_{12}^{(N)}(x, y)\}$$

$$+ [c(x, y, z), L_2^{(N)}(y) - L_3^{(N)}(z)] = 0$$  \hfill (4.6)

and cyclic permutations. In this context $L_1^{(N)}(x) = L^{(N)}(x) \otimes I \otimes I$, $L_2^{(N)}(y) = I \otimes L^{(N)}(y) \otimes I$, $L_3^{(N)}(z) = I \otimes I \otimes L^{(N)}(z)$, and $c(x, y, z)$ is some matrix dependent on the dynamical variables.

If we denote $S_{12} = S \otimes I$, $S_{23} = I \otimes S$, $S_{13} = \sigma_- \otimes I \otimes \sigma_-$, where the matrix $S$ is defined in (4.3). Then the following equality is valid for each member of the hierarchy of dynamical systems

$$\{L_2^{(N)}(y) \otimes s_{13}^{(N)}(x, z)\} = \{L_3^{(N)}(z) \otimes s_{12}^{(N)}(x, y)\}$$

$$= 2\beta_N(x, y, z)[P_{23}, S_{13} + S_{12}]$$

$$- \frac{\partial \beta_N(x, y, z)}{\partial q_{n+1}}[s, L_2^{(N)}(y) - L_3^{(N)}(z)]$$  \hfill (4.7)

with cyclic permutations. In (4.7) the matrix $s = \sigma_- \otimes \sigma_- \otimes \sigma_-$ and

$$\beta_N(x, y, z) = \frac{Q_N(x)(y - z) + Q_N(y)(z - x) + Q_N(z)(x - y)}{(x - y)(y - z)(z - x)}.$$  \hfill (4.8)
Therefore the content of this letter can be interpreted as finding a solution for the dynamical Yang-Baxter equation (4.6) which describes the evolution of the hierarchy of many particle one-dimensional Hamiltonian system separable in parabolic coordinates.

Details of all these results will be published elsewhere [1].

In conclusion we remark that the dependence of the $r$-matrix on dynamical variables can in principle be avoided by embedding the system into a completely integrable system with more degrees of freedom. This system can be described by the standard linear $r$-matrix algebra (4.1) with $s = 0$. For example, for the case $N = 3$, set the functions $\tilde{U}(z), \tilde{V}(z), \tilde{W}(z)$, to be the entries of the Lax operator $\tilde{L}(z)$

\[
\begin{align*}
\tilde{U}(z) &= U(z), \\
\tilde{V}(z) &= V(z) - 8Pz - 8BP, \\
\tilde{W}(z) &= W(z) - 16P^2x + Q - 8Pp_{n+1} + 16P^2q_{n+1},
\end{align*}
\]  

(4.9)

where the functions $U(z), V(z), W(z)$ are given by the formulae (2.8)-(2.10) at $N = 3$ and $Q, P$ are new canonically conjugated coordinates with respect to the standard Poisson bracket. It is easy to see that the corresponding system satisfies the algebra (4.1) with the matrix $s = 0$.

The corresponding enlarged dynamical system has integrals of motion $I_3, H_3, F_3^{(i)}, i = 1, \ldots, n$ given by the formulae

\[
\begin{align*}
I_3 &= 16Q - 4 \sum_{m=1}^{n} q_m^2 + 4BP^2, \\
H_3 &= H_3 + \frac{1}{8} \left[ I_3(B - q_{n+1}) + P(4 \sum_{i=1}^{n} p_i q_i + P \sum_{i=1}^{n} q_i^2) \right], \\
F_3^{(i)} &= F_3^{(i)} - \frac{1}{4} I_3 \sum_{i=1}^{n} q_i^2 + P^2(B \sum_{i=1}^{n} q_i^2 - \frac{1}{8} \sum_{i=1}^{n} A_i q_i^2 - \frac{1}{8} q_{n+1} \sum_{i=1}^{n} q_i^2) \\
&\quad + \frac{1}{2} P \left( (B - q_{n+1}) \sum_{i=1}^{n} p_i q_i - \sum_{i=1}^{n} A_i p_i q_i + \frac{1}{4} p_{n+1} \sum_{i=1}^{n} q_i^2 \right),
\end{align*}
\]  

(4.10), (4.11), (4.12)

where in (4.11), (4.12) $H_3$ and $F_3^{(i)}$ are the integrals of motions of the many-particle Hénon-Heiles system calculated by the formula (2.23). We can see that at $P = 0$, $I_3 = 0$ the system reduces to the many-particle Henon–Heiles system.

We expect that the analogous apparatus can also be developed for systems separable in elliptic coordinates, in which case we have to change the ansatz for $U(z)$ to $U(z) = 1 + \sum_{i=1}^{n} q_i^2/(z + A_i)$.

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