Spin-1/2 Particles in Non-Inertial Reference Frames: Low- and High-Energy Approximations

D. Singh\textsuperscript{1}, G. Papini\textsuperscript{2}

\textsuperscript{1}Department of Physics, University of Alberta, Edmonton, Alberta, T6H 2J1, Canada
\textsuperscript{2}Department of Physics, University of Regina, Regina, Saskatchewan, S4S 0A2, Canada

Abstract

Spin-1/2 particles can be used to study inertial and gravitational effects by means of interferometers, particle accelerators, and ultimately quantum systems. These studies require, in general, knowledge of the Hamiltonian and of the inertial and gravitational quantum phases. The procedure followed gives both in the low- and high-energy approximations. The latter affords a more consistent treatment of mass at high energies. The procedure is based on general relativity and on a solution of the Dirac equation that is exact to first-order in the metric deviation. Several previously known acceleration and rotation induced effects are re-derived in a comprehensive, unified way. Several new effects involve spin, electromagnetic and inertial/gravitational fields in different combinations.
1 Introduction

The behaviour of quantum systems in inertial and gravitational fields is of interest in investigations regarding the structure of spacetime at the quantum level [1]. Quantum objects are in fact finer and more appropriate probes of structures that appear classically as results of limiting procedures. Though a definitive answer to questions regarding the fundamental structure of spacetime may only come from a successful quantum theory of gravity, the extrapolation of general relativity from planetary lengths, over which it is well established, to Planck’s length requires a leap of faith in its validity of over forty orders of magnitude and the resolution of difficult quantization problems. The alternative, performing experiments at Planck’s length, appears remote indeed. A more modest, but realistic approach consists in verifying the theory at intermediate lengths. This may be accomplished, to some extent, by considering the interaction of classical inertial and gravitational fields with quantum objects. A vast array of effects can be predicted in this instance and a unified treatment is afforded by Einstein’s theory. General relativity incorporates the equivalence principle from the outset and observations, where feasible, do confirm that inertia and gravity interact with quantum systems in ways that are compatible with Einstein’s views. This is borne out of measurements on superconducting electrons [2] and on neutrons [3] which are certainly not tests of general relativity per se, but offer tangible evidence that the effect of inertia and Newtonian gravity on wave functions down to lengths of $10^{-3}$ and $10^{-13}$ cm respectively is that predicted by wave equations compliant with general relativity [4, 5].

Inertial effects must be identified with great accuracy. This is dictated by their unavoidable presence in Earth-bound and near-space experiments of ever increasing accuracy aimed at testing fundamental theories. They also provide a guide in the study of relativity because, in all instances where non-locality is not an issue, the equivalence principle, in some of its forms [6, 7], ensures the existence of a gravitational effect for each inertial effect.
Among the quantum mechanical probes, spin-1/2 particles play a prominent role and in reality some of the most precise experiments in physics involve Dirac particles. They are very versatile tools that can be used in a variety of experimental situations and energy ranges while still retaining essentially a non-classical behaviour. Within the context of general relativity comprehensive studies of the Dirac equation were conducted by De Oliveira and Tiomno \[8\] and Peres \[9\]. More recently, spin-inertia and spin-gravity interactions \[10\] have been shown to have non-trivial physical and astrophysical consequences. This is the case for neutrinos whose inertia/gravity interactions are just starting to be studied \[11\]. Superconducting and neutron interferometers of large dimensions \[12, 13\] hold great promise in many of these investigations. They can provide accurate measurements of quantum phases, whose role is important in gyroscopy, and possibly in testing general relativity \[14, 15\]. It is anticipated that similar studies will be performed with particle accelerators \[16\]. The forerunner of this second group of investigations is the work of Bell and Leinaas \[17\] in which evidence was found for the coupling of spin to rotation.

For most studies involving non-relativistic Dirac particles, the inertial/gravitational phase is of paramount importance. For other problems, such as the interaction of inertia/gravitation at the atomic level \[18\], the knowledge of the Hamiltonian is of greater importance. The derivation of the Hamiltonian is usually accomplished by following a sequence of Foldy-Wouthuysen (FW) transformations \[8\]. It has been recently presented in comprehensive form by Hehl and Ni \[19\] purely within the framework of special relativity and in the local frame of the fermion. In the present work, the non-relativistic case is tackled by means of a procedure that renders the quantum phase manifest and can, therefore, be applied to both types of problems.

For tests involving accelerators, a high-energy approximation corresponding to the FW-transformation was given long ago by Cini and Touschek (CT) \[20\] for free Dirac particles. Their work is extended here to include external electromagnetic and gravitational fields and quantum phases to first order. The derivation of the Hamiltonian can then be accomplished in a standard
way.

The method and derivation of the Hamiltonian are given in Section 2. The low-energy approximation in Section 3 uses the FW-transformation and follows the standard textbook approach \[21\]. In contrast, the high-energy approximation in Section 4 uses the CT-transformation and involves a non-standard procedure.

# 2 Derivation of the Dirac Hamiltonian

We use the formalism of general relativity, that encompasses both inertial and true gravitational fields, to derive the Hamiltonian of a fermion in a non-inertial frame.

The starting point is represented by the covariant Dirac equation \[22\]

\[
\left(i\gamma^\mu D_\mu - \frac{mc}{\hbar}\right)\psi(x) = 0, \tag{2.1}
\]

where the generalized matrices \(\gamma^\mu(x)\) satisfy the relation \(\{\gamma^\mu(x), \gamma^\nu(x)\} = 2g^{\mu\nu}(x)\) and are related to the usual Dirac matrices \(\gamma^\alpha\) by means of the vierbeins \(e^\mu_\alpha(x)\). Caratted indices refer to the observer’s local inertial frame. In (2.1), \(D_\mu = \nabla_\mu + i\Gamma_\mu\), \(\psi(x)\) is the wavefunction defined for a general co-ordinate frame, \(\nabla_\mu\) represents the usual covariant derivative, \(\Gamma_\mu\) is the spinor connection which follows from \(D_\mu \gamma_\nu = 0\) and is given by

\[
\Gamma_\mu = \frac{i}{4}\gamma^\nu(\nabla_\mu \gamma_\nu) = -\frac{1}{4}\sigma^{\alpha\beta}\epsilon^{\nu}_{\alpha\beta}(\nabla_\mu e_{\mu\beta}), \tag{2.2}
\]

where \(\sigma^{\alpha\beta} = \frac{i}{2}[\gamma^\alpha, \gamma^\beta]\). Obviously, \(\nabla_\mu \psi(x) = \partial_\mu \psi(x)\), where \(\partial_\mu\) indicates partial differentiation.

It is possible to define a local co-ordinate frame according to an orthonormal tetrad with three-acceleration \(a\) along a particle’s world-line and three-rotation \(\omega\) of the spatial triad, subject to Fermi-Walker transport. This tetrad \(e_\mu\), is related to the general co-ordinate tetrad \(e_\mu\) by

\[
e_0 = \left(1 + \frac{a \cdot x}{c^2}\right)^{-1} \left[ e_0 - \frac{1}{c}(\omega \times x)^k e_k \right], \tag{2.3}
\]
\[ e_i = e_i. \]  \hspace{1cm} (2.4)

The corresponding vierbeins relating the two frames are then

\[ e^0_0 = \left(1 + \frac{a \cdot x}{c^2}\right)^{-1}, \]  \hspace{1cm} (2.5)

\[ e^k_0 = -\frac{1}{c} \left(1 + \frac{a \cdot x}{c^2}\right)^{-1} \epsilon^{ijk} \omega_i x_j, \]  \hspace{1cm} (2.6)

\[ e^0_i = 0, \]  \hspace{1cm} (2.7)

\[ e^k_i = \delta^k_i. \]  \hspace{1cm} (2.8)

Similarly, by inverting (2.3) and (2.4), we find the inverse vierbeins

\[ e^0_0 = \left(1 + \frac{a \cdot x}{c^2}\right) \]  \hspace{1cm} (2.9)

\[ e^k_0 = \frac{1}{c} \epsilon^{ijk} \omega_i x_j, \]  \hspace{1cm} (2.10)

\[ e^0_i = 0, \]  \hspace{1cm} (2.11)

\[ e^k_i = \delta^k_i. \]  \hspace{1cm} (2.12)

The vierbeins satisfy the orthonormality conditions

\[ \delta_{\hat{\alpha}} \hat{\mu} = e^{\nu}_{\hat{\mu}} e_{\nu}, \]  \hspace{1cm} (2.13)

\[ \delta^{\alpha}_{\mu} = e_{\mu}^{\nu} e^{\alpha}_{\nu}. \]  \hspace{1cm} (2.14)
It follows that the metric tensor components are

\[ g_{00} = \left( 1 + \frac{a \cdot x}{c^2} \right)^2 + \frac{2}{c^2} \left[ (\omega \cdot \omega) (x \cdot x) - (\omega \cdot x)^2 \right], \]  

\[ g_{0j} = -\frac{1}{c} (\omega \times x)_j, \]  

\[ g_{jk} = \eta_{jk}, \]

where \( \eta_{\mu\nu} \) represents the Minkowski metric of signature \(-2\). Equation (2.1) has an exact solution to first order in the weak-field approximation defined by \( g_{\mu\nu}(x) = \eta_{\mu\nu} + \gamma_{\mu\nu}(x) \), where the metric deviation \( \gamma_{\mu\nu} \) is a small quantity of first order.

In fact, a new spinor \( \tilde{\psi}(x) \) defined by

\[ \tilde{\psi}(x) \equiv e^{i\Phi_S/\hbar} \psi(x), \]

where

\[ \Phi_S \equiv \hbar \mathcal{P} \int_x^x d\lambda \Gamma_\lambda(z), \]

satisfies the equation

\[ \left( i\tilde{\gamma}^\mu(x) \nabla_\mu - \frac{mc}{\hbar} \right) \tilde{\psi}(x) = 0, \]

where \( \mathcal{P} \) refers to path ordering and

\[ \tilde{\gamma}^\mu(x) \equiv e^{i\Phi_S/\hbar} \gamma^\mu(x) e^{-i\Phi_S/\hbar}. \]

By multiplying (2.20) on the left by \((-i\tilde{\gamma}^\nu(x) \nabla_\nu - mc/\hbar)\), we obtain the equation

\[ \left( g^{\mu\nu} \nabla_\mu \nabla_\nu + \frac{m^2 c^2}{\hbar^2} \right) \tilde{\psi}(x) = 0. \]

This last equation can be solved, in the weak-field approximation, for every component of \( \tilde{\psi}(x) \). Here, \( \tilde{\psi}(x) \) is a solution of a second-order equation and does, of course, contain redundant
solutions. These can be eliminated by writing the appropriate solution as

\[ \tilde{\psi}(x) = \left( -i \tilde{\gamma}^\mu(x) \nabla_\mu - \frac{mc}{\hbar} \right) e^{-i \Phi_G/\hbar} \Psi_0, \quad (2.23) \]

where

\[ \Phi_G \equiv \frac{1}{2} \int_x^x dz^\lambda \gamma_{\alpha\lambda}(z) P^\alpha \]

\[ -\frac{1}{4} \int_x^x dz^\lambda (\gamma_{\alpha\lambda,\beta}(z) - \gamma_{\beta\lambda,\alpha}(z)) L^{\alpha\beta}(z), \quad (2.24) \]

and \(\Psi_0\) satisfies the Klein-Gordon equation for a free particle in Minkowski space.

For (2.24),

\[ [L^{\alpha\beta}(z), \Psi_0] \equiv \{ (x^\alpha - z^\alpha) P^\beta - (x^\beta - z^\beta) P^\alpha \} \Psi_0, \quad (2.25) \]

\[ [P^\alpha, \Psi_0] \equiv \{ i\hbar \partial^\alpha \} \Psi_0, \quad (2.26) \]

where \(P^\mu\) is the momentum operator of the free particle. In (2.23), \(\Phi_G\) plays the role of Berry’s phase [25] because spacetime co-ordinates are just parameters and spacetime becomes simply Berry’s parameter space [13]. The solution of (2.1) is, therefore, \(\psi(x) = \exp (-i\Phi_S/\hbar) \tilde{\psi}(x)\) and is exact to first-order in the metric deviation.

It is interesting to notice that, by multiplying (2.1) on the left with \((-i\gamma^\nu(x)D_\nu - mc/\hbar\), we obtain the second-order equation

\[ \left( \gamma^\mu(x)\gamma^\nu(x)D_\mu D_\nu + \frac{m^2c^2}{\hbar^2} \right) \psi(x) = 0, \quad (2.27) \]

which, on using the relations \([D_\mu, D_\nu] = -\frac{4}{3} \sigma^{\alpha\beta} R_{\alpha\beta\mu\nu}\) and \(\sigma^{\mu\nu}\sigma^{\alpha\beta} R_{\mu\nu\alpha\beta} = 2R [23, 8]\), reduces to

\[ \left( g^{\mu\nu} D_\mu D_\nu - \frac{R}{4} + \frac{m^2c^2}{\hbar^2} \right) \psi(x) = 0. \quad (2.28) \]
This equation does not contain any spin-curvature coupling for pure gravitational fields \((R = 0)\) and the gyro-gravitational g-factor is, therefore, zero \([24]\). When \(R = -\frac{8\pi G T}{c^4}\), the g-factor (coefficient of \(R\)) is \(\frac{1}{4}\) \([23]\). An equation that also yields a value for the orbital g-factor can be obtained from \((2.1)\) by means of the transformation

\[
\psi'(x) = e^{-i\Phi_G/\hbar} \psi(x).
\]  
(2.29)

It is easy to show that \(\psi'(x)\) satisfies the equation

\[
\left[ i\gamma^\mu(x) \left( \nabla^\mu + \frac{i}{\hbar}(\nabla^\mu \Phi_G) \right) - mc/\hbar \right] \psi'(x) = 0.
\]  
(2.30)

Equation \((2.30)\) can be immediately extended to include the electromagnetic fields by adding to \(\Phi_G\) the phase

\[
\Phi_{EM} \equiv \frac{e}{c} \int_X dz^\lambda A^\lambda(z),
\]  
(2.31)

and can be written in the form

\[
\left[ i\gamma^\mu(x) \left( \nabla^\mu + \frac{i}{\hbar}(\nabla^\mu \Theta) \right) - mc/\hbar \right] \psi'(x) = 0,
\]  
(2.32)

where \(\Theta \equiv \Phi_G + \Phi_{EM} + \Phi_S\).

By operating from the left with \([-i\gamma^\nu(x) \left( D^\nu + \frac{i}{\hbar}(\nabla^\nu \Phi_G) \right) - mc/\hbar\] on both sides of \((2.30)\), we obtain to first-order in the metric deviation

\[
\left[ g^{\mu\nu} D^\mu D^\nu - \frac{i}{2} \sigma^{\mu\nu} ([D^\mu, D^\nu] + iG_{\mu\nu}) + \frac{m^2 c^2}{\hbar^2} \right] \psi'(x) = 0,
\]  
(2.33)

where \(\frac{1}{\hbar}(\nabla^\mu \Phi_G) \equiv \kappa^\mu\) and \(G_{\mu\nu} \equiv \kappa_{\nu;\mu} - \kappa_{\mu;\nu}\). On the other hand, \(\frac{1}{4} G_{\mu\nu} = \frac{1}{4} R_{\mu\nu\alpha\beta} L^{\alpha\beta}\) and the resulting orbital g-factor is also \(\frac{1}{4}\). This confirms that the gyro-gravitational ratio of a spin-1/2 particle is 1 as shown in \([8, 26, 27]\).

Equation \((2.32)\) can now be used to derive a Hamiltonian in general co-ordinates, taking the form

\[
i\hbar c \nabla_0 \psi'(x) = (g^{00}(x))^{-1} \left[ c \gamma^0(x) \gamma^j(x) (-i\hbar \nabla_j) + mc^2 \gamma^0(x) \right]
\]
\[ + c \gamma^0(x) \gamma^\mu(x) (\nabla_\mu \Theta) \psi'(x) = H \psi'(x), \] (2.34)

where

\[ g^{00}(x) = (e_0^0)^2 \eta^{00} = \left(1 + \frac{a \cdot x}{c^2}\right)^{-2}, \] (2.35)

\[ \gamma^0(x) = e_0^0 \gamma^0 = \left(1 + \frac{a \cdot x}{c^2}\right)^{-1} \beta, \] (2.36)

\[ \gamma^0(x) \gamma^j(x) = e_0^0 \left( \gamma^0 \gamma^j + e_0^j \right) \]

\[ = \left(1 + \frac{a \cdot x}{c^2}\right)^{-2} \left[ (1 + \frac{a \cdot x}{c^2})^2 - \frac{1}{c^2} e^{ijkl} \omega_k x_l \right]. \] (2.37)

Since \( \Phi_G \) is correct only to first-order, this is also a constraint on the validity of (2.34). In what follows, terms of higher order in the metric deviation will be dropped. Explicit evaluation of \( \nabla_\mu \Theta \) shows that

\[ (\nabla_\mu \Theta) = \nabla_\mu (\Phi_{EM} + \Phi_S + \Phi_G) = \frac{c}{e} A_\mu + \hbar \Gamma_\mu + (\nabla_\mu \Phi_G), \] (2.38)

where

\[ \Gamma_0 = - \frac{i}{2e^2} (a \cdot \alpha) - \frac{1}{2e^2} \omega \cdot \sigma, \] (2.39)

\[ \Gamma_j = 0, \] (2.40)

and

\[ (\nabla_\mu \Phi_G) = \frac{1}{2} \gamma_{\alpha\mu}(x)p^\alpha - \frac{1}{2} \int_X dz^\lambda (\gamma_{\mu\lambda,\beta}(z) - \gamma_{\beta\lambda,\mu}(z)) p^\beta, \] (2.41)

where \( p^\mu \) is the momentum eigenvalue of the free particle.
It follows that, to first-order in $a$ and $\omega$, the Dirac Hamiltonian in the general co-ordinate frame is

$$H \approx c(\alpha \cdot p) + mc^2 \beta + V(x), \quad (2.42)$$

where

$$V(x) = \frac{1}{c}(a \cdot x)(\alpha \cdot p) + m(a \cdot x)\beta - \omega \cdot (L + S) - \frac{i\hbar}{2c}(a \cdot \alpha)$$

$$- e \left(1 + \frac{a \cdot x}{c^2}\right)(\alpha \cdot A) + \frac{e}{c} \omega \cdot (x \times A) + e \varphi$$

$$+ c \alpha \cdot (\nabla \Phi_G) + c (\nabla_0 \Phi_G), \quad (2.43)$$

the $\alpha, \beta, \sigma$ matrices are those of Minkowski space, and $L = x \times p$ and $S = \hbar \sigma/2$ are the orbital and spin angular momenta, respectively.

### 3 Low-Energy Approximation

Although the Dirac Hamiltonian as described by (2.42) and (2.43) is useful as is, there is some benefit in considering approximations which emphasize both the low- and high-energy limits in a particle's range of motion. In this section, the low-energy approximation is being considered.

According to the FW transformation technique, it is possible to group the Dirac Hamiltonian into the form

$$H = mc^2 \beta + O + E, \quad (3.1)$$

where the “odd” and “even” operators $O$ and $E$, respectively satisfy $\{O, \beta\} = [E, \beta] = 0$. For this derivation, we introduce by hand the anomalous magnetic moment

$$\frac{\kappa e \hbar}{2mc} F_{\mu\nu} = \frac{\kappa e \hbar}{2mc} (i \alpha \cdot E - \sigma \cdot B), \quad (3.2)$$
with \( \kappa \equiv (g - 2)/2 \), as another term in \( V(x) \), by means of the substitution

\[
mc^2 \beta \to \beta \left[ mc^2 + \frac{\kappa e\hbar}{2mc} (i\alpha \cdot E - \sigma \cdot B) \right].
\]

Then, given (2.42) and (2.43), it is possible to identify with (3.1), where

\[
\mathcal{O} = c\alpha \cdot \left[ \left( 1 + \frac{a \cdot x}{c^2} \right) \pi + (\nabla \Phi_G) - \frac{i\kappa e\hbar}{2mc^2} \beta E - \frac{i\hbar}{2c^2} \alpha \right],
\]

\[
\mathcal{E} = \left[ m(a \cdot x) - \frac{\kappa e\hbar}{2mc} \left( 1 + \frac{a \cdot x}{c^2} \right) (\sigma \cdot B) \right] \beta
\]

\[
- \omega \cdot (x \times \pi) - \omega \cdot \left( \frac{\hbar}{2\sigma} \right) + e\varphi + c(\nabla_0 \Phi_G),
\]

where \( \pi = p - eA/c \).

Following the procedure given by Bjorken and Drell [21], the transformed Hamiltonian is represented by a series expansion of \( S \), according to a unitary transformation

\[
H' = UHU^{-1}
\]

\[
\approx H + i[S,H] - \frac{1}{2} [S,[S,H]] - \frac{i}{6} [S,[S,[S,H]]]
\]

\[
+ \frac{mc^2}{24} [S,[S,[S,[S,\beta]]]] - \hbar \dot{S} - \frac{ih}{2} [S,\dot{S}],
\]

and \( S = O(1/m) \) is the Hermitian exponent of a unitary transformation operator \( U \equiv \exp(iS) \).

By three successive applications of (3.6) for the choice

\[
S \equiv S_{FW} = -\frac{i}{2mc^2} \beta \mathcal{O},
\]

the final transformed Hamiltonian becomes

\[
H_{FW} = mc^2 \beta + \mathcal{E}'
\]
\[
= \beta \left( mc^2 + \frac{1}{2mc^2} \mathcal{O}^2 - \frac{1}{8m^3e^3} \mathcal{O}^4 \right) + \mathcal{E}
\]

\[- \frac{1}{8m^2c^4} [\mathcal{O}, [\mathcal{O}, \mathcal{E}]] - \frac{i\hbar}{8m^2c^4} [\mathcal{O}, \dot{\mathcal{O}}]. \tag{3.8}\]

To determine the gravitational corrections in (3.8), it is necessary to isolate the external electromagnetic potentials within the definition of odd and even operators. This implies that
\[
\mathcal{O} \equiv \mathcal{O}_0 + \mathcal{O}_1 \quad \text{and} \quad \mathcal{E} \equiv \mathcal{E}_0 + \mathcal{E}_1,
\]
where
\[
\mathcal{O}_0 = c \alpha \cdot \pi, \tag{3.9}
\]
\[
\mathcal{E}_0 = e \phi. \tag{3.10}
\]

Therefore,
\[
\mathcal{O}_1 = c \alpha \cdot \left[ \left( \frac{a \cdot x}{c^2} \right) \pi + (\nabla \Phi_G) - \frac{i \kappa e \hbar}{2mc^2} \beta \mathcal{E} - \frac{i\hbar}{2c^2} a \right], \tag{3.11}
\]
\[
\mathcal{E}_1 = \left[ m(a \cdot x) - \frac{\kappa e \hbar}{2mc^2} \left( 1 + \frac{a \cdot x}{c^2} \right) (\sigma \cdot B) \right] \beta
\]
\[
- \omega \cdot (x \times \pi) - \omega \cdot \left( \frac{\hbar}{2\sigma} \right) + c(\nabla_0 \Phi_G). \tag{3.12}
\]

Neglecting the \(\mathcal{O}^4\) contribution and considering only terms up to first-order in \(a\), \(\omega\), and \(1/m^2\), it follows that
\[
\mathcal{O}^2 = \mathcal{O}_0^2 + \mathcal{O}_1^2 + \{\mathcal{O}_0, \mathcal{O}_1\}, \tag{3.13}
\]
\[
[\mathcal{O}, [\mathcal{O}, \mathcal{E}]] \approx [\mathcal{O}_0, [\mathcal{O}_0, \mathcal{E}_0]] + [\mathcal{O}_0, [\mathcal{O}_1, \mathcal{E}_0]] + [\mathcal{O}_0, [\mathcal{O}_0, \mathcal{E}_1]] + [\mathcal{O}_1, [\mathcal{O}_0, \mathcal{E}_0]], \tag{3.14}
\]
\[
[\mathcal{O}, \dot{\mathcal{O}}] \approx [\mathcal{O}_0, \dot{\mathcal{O}}_0] + [\mathcal{O}_0, \dot{\mathcal{O}}_1] + [\mathcal{O}_1, \dot{\mathcal{O}}_0]. \tag{3.15}
\]

From the zeroth-order terms in (3.8), it can be shown that \[H_{FW(0)} = mc^2 \beta + \frac{1}{2mc^2} \beta O_0^0 + e \varphi - \frac{1}{8m^2c^4} [\mathcal{O}_0, [\mathcal{O}_0, \mathcal{E}_0]] - \frac{i\hbar}{8m^2c^4} [\mathcal{O}_0, \dot{\mathcal{O}}_0]\]

\[
= mc^2 \beta + \left[ \frac{1}{2m} \pi^2 - \frac{e \hbar}{2mc} \sigma \cdot B \right] \beta - \frac{e \hbar}{4m^2c^2} \sigma \cdot (E \times \pi)
- \frac{e \hbar^2}{8m^2c^2} [(\nabla \cdot E) + i \sigma \cdot (\nabla \times E)] + e \varphi, \tag{3.16}
\]

where the third term coupled to \( \beta \) is the magnetic dipole energy, and the following term is the spin-orbit energy.

Neglecting the time-dependent contributions from (3.15) and considering only those terms up to second-order in \( \pi \), it follows that

\[
O_1^2 = -\frac{i e \hbar}{m} \beta \left[ \left( \frac{a \cdot x}{c^2} \right) (E \cdot \pi) + (\nabla \Phi_G) \cdot E \right] - \frac{\kappa e \hbar^2}{2mc^2} (a \cdot E)
- \frac{\kappa e \hbar^2}{2m} \left( \frac{a \cdot c^2}{a^2} \right) \beta (\nabla \cdot E) - \frac{\kappa e \hbar^2}{2m} \left( \frac{a \cdot x}{c^2} \right) \beta \sigma \cdot (\nabla \times E), \tag{3.17}
\]

\[
\{\mathcal{O}_0, O_1\} = (a \cdot x) \pi^2 + 2c^2 ((\nabla \Phi_G) \cdot \pi) - \frac{i e \hbar}{m} \beta (E \cdot \pi) - i \hbar (a \cdot \pi)
+ c^2 \pi \left( \frac{a \cdot x}{c^2} \right) \cdot \pi - i \hbar c^2 (\nabla^2 \Phi_G) - \frac{\kappa e \hbar^2}{2m} (\nabla \cdot E) - \frac{\kappa e \hbar}{m} \beta \sigma \cdot (E \times \pi)
+ \hbar \sigma \cdot (a \times \pi) - 2e \hbar c \left( \frac{a \cdot x}{c^2} \right) \sigma \cdot (\nabla \times A) - \frac{i \kappa e \hbar^2}{2m} \beta \sigma \cdot (\nabla \times E), \tag{3.18}
\]

\[
[\mathcal{O}_0, [O_1, \mathcal{E}_0]] = i e \hbar^2 a \cdot \nabla \varphi - e \hbar^2 (a \cdot x) \nabla^2 \varphi - 2e \hbar (a \cdot x) \sigma \cdot (\nabla \varphi \times \pi), \tag{3.19}
\]

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\[
\begin{align*}
[\mathcal{O}_0, [\mathcal{O}_0, \mathcal{E}_1]] &= -4ie\hbar^2 \beta (\mathbf{a} \cdot \mathbf{\pi}) - \hbar^2 e^2 \nabla^2 (\nabla_0 \Phi_G) - i e \hbar^2 c \sigma \cdot (\mathbf{\omega} \times (\nabla \times \mathbf{A})) \\
&+ 2mc^2 \hbar \beta (\mathbf{a} \times \mathbf{\pi}) + 4mc^2 \beta (\mathbf{a} \cdot \mathbf{x}) \mathbf{\pi}^2 - 4e m \hbar \beta (\mathbf{a} \cdot \mathbf{x}) \sigma \cdot (\nabla \times \mathbf{A}) \\
&- 2\hbar^2 \sigma \cdot (\nabla (\nabla_0 \Phi_G) \times \mathbf{\pi}) + 2\hbar^2 \sigma \cdot (\mathbf{\omega} \times \pi) \times \pi,
\end{align*}
\]

After neglecting the non-Hermitian terms in (3.17) - (3.21), it becomes evident that the low-energy approximation for the Dirac Hamiltonian is

\[
H_{FW} \approx mc^2 \beta + \left[ \frac{1}{2m} \pi^2 - \frac{e \hbar}{2mc} \sigma \cdot \mathbf{B} \right] \beta - \frac{(g - 1)e \hbar}{4m^2 c^2} \sigma \cdot (\mathbf{E} \times \pi) \\
- \frac{e \hbar^2}{8m^2 c^2} [\nabla \cdot \mathbf{E} + i \sigma \cdot (\nabla \times \mathbf{E})] + e \varphi \\
+ \left[ m(\mathbf{a} \cdot \mathbf{x}) - \frac{e \hbar}{2mc} \left\{ \kappa \left( 1 + \frac{a \cdot x}{c^2} \right) + \left( \frac{a \cdot x}{c^2} \right) \right\} (\sigma \cdot \mathbf{B}) \right] \beta \\
- \mathbf{\omega} \cdot (\mathbf{x} \times \mathbf{\pi}) - \mathbf{\omega} \cdot \left( \frac{\hbar}{2\sigma} \right) + c(\nabla_0 \Phi_G) \\
+ \frac{1}{2m} \beta \mathbf{\pi} \left( \frac{a \cdot x}{c^2} \right) \cdot \mathbf{\pi} + \frac{\hbar}{4mc^2} \beta \sigma \cdot (\mathbf{a} \times \mathbf{\pi}) - \frac{\hbar}{4mc^2} \sigma \cdot (\mathbf{\omega} \times \mathbf{\pi}) \times \mathbf{\pi}
\]
\[-\frac{\kappa e h^2}{4m^2c^2} \left( 1 + \frac{a \cdot x}{c^2} \right) \left[ (\nabla \cdot E) + i \sigma \cdot (\nabla \times E) \right] \]

\[-\frac{\kappa e h^2}{4m^2c^2} (a \cdot E) + \frac{e h^2}{4m^2c^2} (a \cdot \frac{x}{c^2}) \nabla^2 \varphi \]

\[+ \frac{1}{m} \beta (\nabla \Phi_G) \cdot \pi + \frac{eh}{4m^2c^2} \sigma \cdot \left( \nabla \varphi \times \left[ (\nabla \Phi_G) + \left( \frac{a \cdot x}{c^2} \right) \pi \right] \right) \]

\[+ \frac{\hbar^2}{8m^2c} \nabla^2 (\nabla_0 \Phi_G) + \frac{\hbar}{4m^2c} \sigma \cdot (\nabla (\nabla_0 \Phi_G) \times \pi). \quad (3.22) \]

The occurrence of non-Hermitian terms, here neglected, is a well known phenomenon, likely connected with the breakdown of the single-particle interpretation of the Dirac equation in the presence of time-dependent inertial and gravitational fields [18, 28].

4 High-Energy Approximation

Though used less often than the FW-transformation, the CT-transformation follows the same mathematical principles as the former to arrive at the high-energy limit for the Dirac Hamiltonian. Although the FW-transformation can be successfully applied when non-trivial potential energy terms are present, it is far from obvious that the CT-transformation can do the same. This is because it is not clear how to classify the Dirac Hamiltonian into a high-energy analogue of odd and even operators, as found in the FW approach, in order to systematically remove the undesired terms. An unfortunate consequence of this impasse is that it becomes impossible to analyze the motion and properties of fast-moving massive particles in the presence of fields without arbitrarily setting its mass equal to zero within the Hamiltonian. Clearly, this precludes any opportunity to compare the behaviour of these particles with that of strictly massless particles.
It is shown below, however, that it is possible to derive a high-energy approximation of the
generalized Dirac Hamiltonian for a spin-1/2 particle moving in a potential defined by both elec-
tromagnetic and gravitational fields. For this derivation, the final expression for the Hamiltonian
is also in terms of a general co-ordinate frame. The steps now proceed backward from the Dirac
equation or corresponding second-order equation in locally Minkowski space.

Given that

\[
\begin{align*}
\bar{\hbar} c \hat{\nabla}_0 \Psi_0 &= H_0 \Psi_0 \\
&= \left[ -c \alpha \hat{\mathcal{P}}^2 + m e^2 \beta \right] \Psi_0,
\end{align*}
\]

(4.1)
it is possible to define a new wavefunction \( \Psi_0' \equiv \exp(iS_{\text{CT}})\Psi_0 \), where

\[
S_{\text{CT}} \approx -\frac{im}{2\mu^2} \beta(\alpha^2 \mathcal{P}_j),
\]

(4.2)
and \([\nabla_\mu, S_{\text{CT}}] = 0\). It then follows from the CT-transformation that

\[
\begin{align*}
\bar{\hbar} c \hat{\nabla}_0 \Psi_0' &= \left[ e^{iS_{\text{CT}}} H_0 e^{-iS_{\text{CT}}} \right] \Psi_0' \\
&\approx H_{\text{CT}(0)} \Psi_0',
\end{align*}
\]

(4.3)
where

\[
H_{\text{CT}(0)} \approx -\frac{E}{p} (\alpha^2 \mathcal{P}_j)
\]

(4.4)
is the CT Hamiltonian for the free particle. Since \( \Psi_0 \) can be related to \( \tilde{\psi}(x) \) by (2.23), it can be shown that

\[
\begin{align*}
\Psi_0' &= e^{iS_{\text{CT}}} \Psi_0 \\
&= e^{iS_{\text{CT}}} \left[ e^{i\Theta/\hbar} \psi'(x) \right],
\end{align*}
\]

(4.5)
where

\[ \psi''(x) = e^{-i\Phi_S/h} \left( -i\gamma^\mu(x) \nabla_\mu - \frac{mc}{\hbar} \right)^{-1} e^{i\Phi_S/h} \psi(x). \]  

(4.6)

Substituting (4.5) into (4.3), we obtain

\[ -i\bar{\hbar} c \left[ E_{pc} \left( e^{-iS_{CT}} \alpha^j e^{iS_{CT}} \right) \nabla_j + \nabla_0 \right] e^{i\Phi_S/h} \psi''(x) = 0. \]  

(4.7)

By using the vierbeins, it becomes possible to describe (4.7) in terms of the general co-ordinate frame, so that

\[ -i\bar{\hbar} c \left[ E_{pc} \left( e^{-iS_{CT}} \alpha^j e^{iS_{CT}} \right) e^{\mu_j} + e^{\mu_0} \right] \nabla_\mu \left( e^{i\Phi_S/h} \psi''(x) \right) = 0. \]  

(4.8)

It is a straightforward matter to evaluate the transformation of \( \alpha^j \). Though it is possible to perform the expansion for higher-order terms, it is only necessary to consider the zeroth- and first-order terms. It follows that

\[ e^{-iS_{CT}} \alpha^j e^{iS_{CT}} \approx \alpha^j - i[S_{CT}, \alpha^j] \]

\[ \approx \alpha^j + \left( \frac{mc}{p^2} \right) \beta P^j. \]  

(4.9)

Substituting (4.9) into (4.7), we obtain the expression

\[ i\hbar c \nabla_0 \psi''(x) = \frac{E}{p} \left( 1 + \frac{a \cdot x}{c^2} \right) \left[ \alpha'' \cdot p + \alpha'' \cdot \nabla \Theta + \left( \frac{mc}{p^2} \right) \beta'' \left( p \cdot p + p \cdot \nabla \Theta \right) \right] \psi''(x) \]

\[ - \omega \cdot [x \times (p + \nabla \Theta)] \psi''(x) + c \nabla_0 \Theta \psi''(x) \]

\[ = H_{CT} \psi''(x), \]  

(4.10)

where

\[ \alpha'' = e^{-i\Phi_S/h} \alpha e^{i\Phi_S/h}, \]  

(4.11)
\[ \beta'' = e^{-i\Phi_S/\hbar} \beta e^{i\Phi_S/\hbar}. \]  

(4.12)

To first-order in \( a \) and \( \omega \), the \( \nabla_\mu \Theta \) contribution is given by (2.38). The only remaining term to be evaluated is \( p \cdot \nabla \Theta \). It follows that

\[ p \cdot \nabla \Theta = -\frac{e}{c} (p \cdot A) + p \cdot (\nabla \Phi_G). \]  

(4.13)

Therefore, the final expression for the CT-Hamiltonian with electromagnetic and gravitational fields present is

\[ H_{CT} \approx \frac{E}{p} (\alpha'' \cdot p) + V(x)_{CT}, \]  

(4.14)

where

\[ V(x)_{CT} \approx \frac{E}{pc^2} (a \cdot x) (\alpha'' \cdot p) + \frac{E}{p} \left( 1 + \frac{a \cdot x}{c^2} \right) \left( \frac{mc}{p^2} \right) \beta'' (p \cdot \rho) - \omega \cdot (L + S) - \frac{i\hbar}{2c} (a \cdot \alpha) \]

\[ - \frac{Ee}{pc} \left( 1 + \frac{a \cdot x}{c^2} \right) \alpha'' \cdot A + \frac{e}{c} \omega \cdot (x \times A) + e \varphi + \frac{E}{p} \alpha'' \cdot (\nabla \Phi_G) + c (\nabla_0 \Phi_G) \]

\[ + \frac{E}{p} \left( 1 + \frac{a \cdot x}{c^2} \right) \left( \frac{mc}{p^2} \right) \beta'' \left[ p \cdot (\nabla \Phi_G) - \frac{e}{c} (p \cdot A) \right]. \]  

(4.15)

From (2.19), (2.39) - (2.40), (4.11) and (4.12), the first-order expansions of \( \alpha'' \) and \( \beta'' \) are

\[ \alpha'' \approx \alpha + \mathcal{P} \int_X dz^0 \left[ \frac{i}{c^2} (a \times \sigma) + \frac{1}{2c} (\omega \times \alpha) \right], \]  

(4.16)

\[ \beta'' \approx \beta + \frac{1}{c^2} \mathcal{P} \int_X dz^0 \beta (a \cdot \alpha). \]  

(4.17)

5 Conclusions

The study of inertia/gravity requires in general knowledge of the phase factors \( \Phi_G \), along with \( \Phi_S \) and the Hamiltonian. The procedure followed gives both. It also leads to a solution of the
Dirac equation that, as $\Phi_G$, is exact to first-order in the metric deviation. As well, $\Phi_S$ is exact.

The Hamiltonian for both the non-relativistic and extreme relativistic cases are represented by (2.42) – (2.43) and (4.14) – (4.15), respectively. Low- and high-energy approximations can be further developed by following well-known procedures. The low-energy approximation only has been derived in detail. A comparison of (2.43) with (4.15) indicates differences only in terms proportional to the mass. This is understandable in view of the different expansions of the energy implied by the corresponding approximations.

Some of the terms that appear in (2.43) and (4.15) are identical and can be identified with corresponding terms that appear in the Hamiltonian of Hehl and Ni. One such instance is represented by the term $-\omega \cdot S$, the Mashhoon effect [29, 30], wrongly interpreted by Bell and Leinaas as a version of the Unruh effect. The term $-\omega \cdot L$ is the Page-Werner effect [31], while $m(a \cdot x)\beta$ represents the Bonse-Wroblewski effect [32]. Both effects have been tested experimentally. Also present is the term $(a \cdot x)(\alpha \cdot p)/c$, which is an energy-momentum redshift.

While several terms in (4.15) vanish in the limit $m \to 0$, the Mashhoon effect is not affected by this limit. This leads one to conclude that the rotation-helicity effects discussed by Cai and Papini [10, 11] for massive neutrinos persist in the vanishing mass limit. The Mashhoon effect is obviously a prime candidate for experiments with accelerators and interferometers. In the latter case, $\Theta$ can be applied to a spacetime loop that is effectively closed because of the coherence of the particle wavefunctions. The result is manifestly gauge invariant and is given by

$$\Theta = \frac{1}{4} \int_\Sigma R_{\mu\nu\alpha\beta} J^{\alpha\beta} d\tau^{\mu\nu} + \frac{e}{c} \int_\Sigma F_{\mu\nu} d\tau^{\mu\nu},$$

where $J^{\alpha\beta}$ is the total angular momentum of the particle, $R_{\mu\nu\alpha\beta}$ is the linearized Riemann tensor, $F_{\mu\nu}$ is the electromagnetic field tensor and $\Sigma$ is a surface bounded by the loop. Contributions by the second order derivatives of the metric therefore appear in measurable phases also in the case of inertial fields. Eq. (5.1) clearly indicates that the phase shifts of quantum interferometry depend on the masses of the particles involved and that a strong form of the equivalence principle
cannot be present at the quantum level \[1\]. Nonetheless Eq. (2.22), on which our solution is based, still implies that gravitational fields can be simulated locally by acceleration fields \[7\].

Eq. (3.22) for the low-energy Hamiltonian can be now compared with the results obtained by other authors. Let us neglect, for simplicity, the anomalous magnetic moment contributions introduced in Section 3. The Bonse-Wroblewski, Page-Werner and Mashhoon terms can be immediately recognized by inspection. They correspond to the eighth, tenth and eleventh terms respectively. The fifteenth term contains electromagnetic and momentum corrections to the Mashhoon effect. The thirteenth term represents the redshift effect of the kinetic energy already mentioned, but here in the company of its electromagnetic corrections. These also appear in the new inertial spin-orbit term found by Hehl and Ni (the fourteenth term). The fourth and sixth terms represent spin-orbit coupling and are discussed, for instance, by Bjorken and Drell. The third term is also wellknown and represents the magnetic dipole interaction. The Darwin term is the fifth and the nineteenth represents an acceleration correction to it. All remaining terms are proportional to the derivatives of \( \Phi_G \) (see Eq. (2.41)). Among these \( c(\nabla_0 \Phi_G) + \frac{1}{m} \beta (\nabla \Phi_G) \cdot \pi \) appear to dominate. The integral-dependent part of (2.41) yields contributions that are small for small paths and low particle momenta. They produce, in general, curvature contributions for closed spacetime paths, as in interferometry, and lead to (5.1) above. The largest contributions come from the first part of (2.41) which contains the terms \( \frac{1}{2} mc^2 \gamma_{00}(x) \) and \( mc \gamma_{0i}(x)p^i \) already discussed by De Witt and Papini in connection with the behaviour of superconductors in weak inertial and gravitational fields.

In view of the above, Eq. (2.32) appears remarkably successful in dealing with all the inertial and gravitational effects discussed in the literature.

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