Structure Based Extended Resolution for Constraint Programming

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Abstract. Nogood learning is a powerful approach to reducing search in Constraint Programming (CP) solvers. The current state of the art, called Lazy Clause Generation (LCG), uses resolution to derive nogoods expressing the reasons for each search failure. Such nogoods can prune other parts of the search tree, producing exponential speedups on a wide variety of problems. Nogood learning solvers can be seen as resolution proof systems. The stronger the proof system, the faster it can solve a CP problem. It has recently been shown that the proof system used in LCG is at least as strong as general resolution. However, stronger proof systems such as extended resolution exist. Extended resolution allows for literals expressing arbitrary logical concepts over existing variables to be introduced and can allow exponentially smaller proofs than general resolution. The primary problem in using extended resolution is to figure out exactly which literals are useful to introduce. In this paper, we show that we can use the structural information contained in a CP model in order to introduce useful literals, and that this can translate into significant speedups on a range of problems.

1 Introduction

Nogood learning is a powerful approach to reducing search in Constraint Programming (CP) solvers. The current state of the art is Lazy Clause Generation (LCG). LCG adapts the clause learning techniques from Boolean Satisfiability (SAT) to the more generic domain of CP problems where we have finite domain variables and global constraints. Each propagator in an LCG solver is instrumented so that it is able to explain each of its propagations using a clause. These clauses form an implication graph. When a search failure occurs, the implication graph is analyzed and resolution is performed on the clauses in the implication graph in order to derive a nogood which explains the reasons for the failure. These nogoods can then be propagated in order to prune other parts of the search tree. Nogood learning is very effective on structured problems and can often provide orders of magnitude speedup over a non-learning solver.

A complete search CP solver can be seen as a proof system which is trying to prove that no solution exists in a satisfiability problem, or that no solution better than a certain objective value exists in an optimization problem. It either finds a counter example (i.e., a solution) during the proof process, or it succeeds
in proving that no solution exists. The size of the search tree is bounded from below by the size of the smallest proof possible in the proof system. Thus in general, the stronger the proof system used, the faster a CP solver can solve the problem. The proof system used in an LCG solver is much stronger than the one used in a non-learning solver. This is because when a non-learning CP solver fails a subtree, it has only proved that that particular subtree fails. On the other hand, when a LCG solver fails a subtree, it uses resolution to derive a nogood that proves that this subtree fails, but also that other similar subtrees (i.e., those which satisfy the conditions in the nogood) also fail.

It has recently been proved [22] that a SAT solver performing conflict directed clause learning and restarts has a proof system that is as powerful as general resolution [23]. The proof system of LCG solvers, which inherits the resolution based learning of SAT solvers and the possibly non-resolution based inferences of CP’s global propagators, is even more powerful. However, the power of the resolution part of the proof system is constrained by the set of literals that it is allowed to use in the proof. We call this set of literals the language of the resolution proof system. Extended resolution [26] is a proof system even stronger than general resolution and is one of the most powerful proof systems for propositional logic [27]. It allows the language of the resolution proof system to be dynamically extended during runtime by introducing new variables representing arbitrary logical expressions over the existing ones. It is well known that there can be an exponential separation between the size of the proof generated by extended resolution and general resolution on certain problems [7]. Clearly, utilizing extended resolution in nogood learning could be a very effective way to improve the speed of a CP solver.

While extended resolution offers the potential for significant speedups, such speedups are often difficult to realize in practice. This is because in order for extended resolution to produce shorter proofs, it is necessary for the system to make the right extensions, i.e., introduce variables expressing the right logical concepts. It is typically very difficult to know which particular extensions are needed to speed up a proof, so it is difficult to use extended resolution effectively. There have been several attempts at augmenting SAT solvers with extended resolution capabilities with varying success (e.g., [2, 13]). Constraint Programming provides a unique opportunity for the effective usage of extended resolution. In Constraint Programming, problems are modeled in terms of high level constraints which preserve much of the structure of the problem. This structural information provides important information regarding which extensions will be useful, and allows us to exploit the potential speedups made possible by extended resolution.

Most of the major advances in resolution based nogood learning in CP has come about due to an extension of the language of the resolution proof system in order to exploit the structure of finite domain integer variables. The earlier works on nogood learning (see e.g. [8], chapter 6) only considers equality literals of the form $x = v$ where $x$ is a variable and $v$ is a value. Later on, disequality literals of the form $x \neq v$ were introduced [14, 15]. More recently, inequality literals of the form $x \geq v, x \leq v$ were introduced in LCG [21]. Each extension significantly increased the expressiveness of the nogoods and the power of the proof system, resulting in significant speedups compare to previous versions. In this paper, we
look at other types of structure that can be found in CP problems and consider how they can be exploited via the introduction of additional literals into the language. Our main contributions are as follows:

– We provide a framework for assessing the generality of an explanation generated by an LCG propagator, given a fixed language of resolution $L$.

– We examine the internal structure of commonly used global constraints to see which kinds of language extensions can be useful for improving the power of the resolution proof system.

– We show that the global structure of the problem can be used in order to decide which language extension to make.

2 Definitions and Background

Let $\equiv$ denote syntactic identity, $\Rightarrow$ denote logical implication and $\Leftrightarrow$ denote logical equivalence. A constraint satisfaction problem (CSP) is a tuple $P \equiv (V, D, C)$, where $V$ is a set of variables, $D$ is a set of (unary) domain constraints, and $C$ is a set of (n-ary) constraints. An assignment $\theta$ is a solution of $P$ if it satisfies every constraint in $D$ and $C$. In an abuse of notation, if a symbol $C$ refers to a set of constraints $\{c_1, \ldots, c_n\}$, we will often also use the symbol $C$ to refer to the conjunction $c_1 \land \ldots \land c_n$.

CP solvers solve CSP’s by interleaving search with inference. We begin with the original problem at the root of the search tree. At each node in the search tree, we propagate the constraints to try to infer variable/value pairs which can no longer be taken in any solution in this subtree. Such pairs are removed from the current domain. If some variable’s domain becomes empty, then the subtree has no solution and the solver backtracks. If all the variables are assigned and no constraint is violated, then a solution has been found and the solver can terminate. If inference is unable to detect either of the above two cases, the solver further divides the problem into a number of more constrained subproblems and searches each of those in turn.

A CP solver implementing LCG has a number of additional features which allow it to perform nogood learning. Firstly, for each integer variable $x$ with initial domain $\{l, \ldots, u\}$, the solver adds Boolean variables to represent the truth value of the logical expressions $x = v$ for $v = l, \ldots, u$ and $x \geq v$ for $v = l + 1, \ldots, u$. We use $[e]$ to denote the Boolean variable which represents the truth value of logical expression $e$. The solver enforces the channeling constraint $[e] \leftrightarrow e$ for each such variable. So for example, the Boolean variable $[x = 5]$ is true iff $x = 5$ is implied by the current domain $D$. For convenience, we also use $[x \neq v]$ to refer to $\neg [x = v]$ and $[x \leq v]$ to refer to $\neg [x \geq v + 1]$. We call literals of form $[x = v]$ equality literals, literals of form $[x \neq v]$ disequality literals, and literals of form $[x \geq v]$ or $[x \leq v]$ inequality literals.

In an LCG solver (and indeed most CP solvers), the only allowed kinds of domain changes are: fixing a variable to a value, removing a value, increasing the lower bound, or decreasing the upper bound. Each of these can be expressed as setting one of the literals $[x = v]$, $[x \neq v]$, $[x \geq v]$ or $[x \leq v]$ true. Each propagator in an LCG solver is instrumented in order to explain each of its domain changes with a clause called the explanation.
Definition 1. Given current domain \( D \), suppose the propagator for constraint \( c \) makes an inference \( p \), i.e., \( c \land D \Rightarrow p \). An explanation for this inference is a clause: \( \text{expl}(p) \equiv l_1 \land \ldots \land l_k \rightarrow p \) where \( l_i \) and \( p \) are literals, s.t. \( c \Rightarrow \text{expl}(p) \) and \( D \Rightarrow l_1 \land \ldots \land l_k \).

For example, given constraint \( x \leq y \) and current domain \( x \in \{3, 4, 5\} \), the propagator may infer that \( y \geq 3 \), with the explanation \([x \geq 3] \rightarrow [y \geq 3] \). The explanation \( \text{expl}(p) \) explains why \( p \) has to hold given \( c \) and the current domain \( D \). We can consider \( \text{expl}(p) \) as the fragment of the constraint \( c \) from which we inferred that \( p \) has to hold. We call the set of literals available for forming explanations the language of resolution for the system.

As propagation proceeds, these explanations form an acyclic implication graph. Whenever a conflict is found by an LCG solver, the implication graph can be analyzed in order to derive a set of sufficient conditions for the conflict to reoccur. Just as most current state of the art SAT solvers, LCG solvers derive the first unique implication point (1UIP) nogood. This is done by repeatedly resolving the conflicting clause (the clause explaining the conflict) with the explanation clause for the latest inferred literal until the clause contains only one literal from the current decision level. The resulting clause, or nogood as it is more commonly called in CP, is an implied constraint of the problem which proves that this particular subtree failed. However, this nogood often also proves that other subtrees fail for a similar reason to the current one. Thus we can add the nogood as a propagator to prune other parts of the search tree.

Example 1. Consider a simple constraint problem with variables \( x_1, x_2, x_3, x_4, x_5, x_6 \) with all initial domain \( \{0, 1, 2, 3, 4, 5, 6, 7\} \), and three constraints: \( x_1 + 2x_2 + 3x_3 + 4x_4 + 4x_5 \leq 30 \), \( x_4 \leq 4 \rightarrow x_6 = 1 \), and \( x_5 \leq 4 \rightarrow x_6 = 0 \). Suppose we make the decisions: \( x_1 \geq 1 \) (nothing propagates), \( x_2 \geq 2 \) (propagates \( x_4 \leq 6 \) and \( x_5 \leq 6 \)), and \( x_3 \geq 3 \). This propagates \( x_4 \leq 4 \) and \( x_5 \leq 4 \), which in turn propagates \( x_6 = 1 \) and causes the constraint \( x_5 \leq 4 \rightarrow x_6 = 0 \) to fail. Figure 1 shows the implication graph when the conflict occurs. The double boxes indicate decision literals while the dashed lines partition literals into decision levels. Dotted lines are literals that are irrelevant to the failure. To obtain the 1UIP nogood we start with the conflict nogood \([x_5 \leq 2] \land [x_6 = 1] \rightarrow \text{false} \) which contains every literal directly connected to the \text{false} conclusion. We have two literals from the last decision level \(([x_5 \leq 4]) \) and \([x_6 = 1]) \). Since \([x_6 = 1]\) was the last literal to be inferred of those two, we resolve the current nogood with \( \text{expl}([x_6 = 1]) = [x_4 \leq 4] \) obtaining \([x_4 \leq 4] \land [x_5 \leq 4] \rightarrow \text{false} \). We still have two literals of the last decision level so we replace \([x_5 \leq 4] \) by \( \text{expl}([x_5 \leq 4]) = [x_1 \geq 1] \land [x_2 \geq 2] \land [x_3 \geq 3] \) obtaining \([x_1 \geq 1] \land [x_2 \geq 2] \land [x_3 \geq 3] \land [x_4 \leq 4] \rightarrow \text{false} \). We then replace \([x_4 \leq 4] \) by \( \text{expl}([x_4 \leq 4]) = [x_1 \geq 1] \land [x_2 \geq 2] \land [x_3 \geq 4] \) obtaining \([x_1 \geq 1] \land [x_2 \geq 2] \land [x_3 \geq 3] \rightarrow \text{false} \). This is the 1UIP nogood since it contains only one literal from level 3.

3 Generality of Explanations

Since the nogoods derived by the resolution proof system are formed by resolving the explanations generated by the propagators, the more general the explanations are, the more general the nogood derived will be. Using better explanations...
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means that for the same amount of search, we can derive stronger nogoods that prove that a greater part of the search space is failed. The following definitions allow us to compare and assess how good an explanation is:

**Definition 2.** Given two possible explanations $E \equiv l_1 \land \ldots \land l_n \rightarrow p$ and $E' \equiv k_1 \land \ldots \land k_m \rightarrow p$ for the inference $p$, $E'$ is strictly more general than $E$ iff:

$$\land_{i=1}^n l_i \Rightarrow \land_{i=1}^m k_i \text{ and } \land_{i=1}^m k_i \not\Rightarrow \land_{i=1}^n l_i.$$  

**Definition 3.** An explanation $E \equiv l_1 \land \ldots \land l_n \rightarrow p$ for the inference $p$ is maximally general w.r.t. language of resolution $L$, if there does not exist another explanation $E'$ in $L$ which is strictly more general than $E$.  

Note that maximally general explanations are not necessarily unique.

**Example 2.** Consider a linear constraint $x_1 + 2x_2 + 3x_3 + 4x_4 \leq 30$ and a current domain of $x_1 = 1, x_2 = 2, x_3 = 3$. The propagator can infer that $x_4 \leq 4$. There are many possible explanations. For example, $\lbrack x_1 = 1 \rbrack \land \lbrack x_2 = 2 \rbrack \land \lbrack x_3 = 3 \rbrack \rightarrow \lbrack x_4 \leq 4 \rbrack$ is a perfectly valid explanation. However, it is not very general. A strictly more general explanation is $\lbrack x_1 \geq 1 \rbrack \land \lbrack x_2 \geq 2 \rbrack \land \lbrack x_3 \geq 3 \rbrack \rightarrow \lbrack x_4 \leq 4 \rbrack$. However, this is still not maximally general in the standard LCG language. For example, $\lbrack x_2 \geq 1 \rbrack \land \lbrack x_3 \geq 3 \rbrack \rightarrow \lbrack x_4 \leq 4 \rbrack$ is a maximally general explanation which is more general than the one before. Similarly, $\lbrack x_1 \geq 1 \rbrack \land \lbrack x_2 \geq 2 \rbrack \land \lbrack x_3 \geq 2 \rbrack \rightarrow \lbrack x_4 \leq 4 \rbrack$ is another maximally general explanation.

Clearly, a good starting point for making the resolution proof system stronger is to ensure that the LCG solver is using maximally general explanations, so that we are making the most out of the existing language.

**Definition 4.** An explanation $E \equiv l_1 \land \ldots \land l_n \rightarrow p$ for the inference $p$ is universally maximally general if it is maximally general w.r.t. to the universal language $L$ consisting of all possible logical expressions.

If the universally maximally general explanation for an inference cannot be expressed as a conjunction of literals in the existing language, then it is a good indication that a language extension may be useful for increasing the generality of the explanations for this constraint.
Example 3. Consider the inference from Example 2. The universally maximal general explanation is: \( \llbracket x_1 + 2x_2 + 3x_3 \geq 11 \rrbracket \rightarrow \llbracket x_4 \leq 4 \rrbracket \), since \( x_1 + 2x_2 + 3x_3 \geq 11 \) is a necessary and sufficient condition on the domain for us to infer \( x_4 \leq 4 \) from \( x_1 + 2x_2 + 3x_3 + 4x_4 \leq 30 \). Clearly, there is no way that \( \llbracket x_1 + 2x_2 + 3x_3 \geq 11 \rrbracket \) can be expressed equivalently as a conjunction of equality, disequality or inequality literals on \( x_1, x_2, x_3, x_4 \). So a language extension may be useful here.

4 Extending the Language

We now consider how we can extend the language of resolution to give more general explanations. We first give a simple motivating example.

Example 4. Consider the 0-1 knapsack problem, given by \( x_1, \ldots, x_n \in \{0, 1\}, \sum_{i=1}^{n} w_i x_i \leq W, \sum_{i=1}^{n} p_i x_i \geq f \), where \( f \) is to be maximized, \( w_i \) represents the weights of item \( i \), \( W \) is the total capacity of the knapsack, and \( p_i \) is the profit of item \( i \). A normal CP solver will require \( O(2^n) \) to solve this problem. Using an LCG solver does no better, because the size of the smallest proof of optimality using only equality, disequality and inequality literals on the \( x_i \) is still exponential in \( n \). On the other hand, suppose we introduced literals to represent partial sums of form: \( \sum_{i=1}^{k} w_i x_i \geq W' \) and \( \sum_{i=1}^{k} p_i x_i \leq f' - p' \) where \( k, W', f', p' \) are arbitrary constants. Then, it becomes possible to prove optimality in \( O(nWP) \), where \( P = \sum_{i=1}^{n} p_i \). This is because it is now possible to express nogoods such as: \( \left[ \sum_{i=1}^{k} w_i x_i \geq W - W' \right] \land \left[ \sum_{i=1}^{k} p_i x_i \leq f - p' \right] \rightarrow \text{false} \) which represent that we have proved that given only a weight limit of \( w_i \) for the items \( k+1 \) to \( n \), there is no way we can pick a subset of them such that their profit sum to at least \( p' \). If we modify our LCG solver to use these new partial sum literals the amount of search required can be \( O(nWP) \) which is pseudo-polynomial rather than exponential complexity.

When deciding on language extensions, there are two main factors we have to consider:

- Are they going to improve the size of the resolution proof?
- What is the overhead of introducing the literal into the system?

When we extend the language by introducing a literal \( \llbracket e \rrbracket \) where \( e \) is some logical expression over existing variables, we have to keep track of the truth value of \( \llbracket e \rrbracket \) so that we can propagate any nogoods with this literal in it. This is accomplished by enforcing a channeling constraint \( \llbracket e \rrbracket \leftrightarrow e \). For example, if we introduced a literal \( \llbracket x_1 + 2x_2 + 3x_3 \geq 10 \rrbracket \), we would have to enforce the channeling constraint: \( \llbracket x_1 + 2x_2 + 3x_3 \geq 10 \rrbracket \leftrightarrow x_1 + 2x_2 + 3x_3 \geq 10 \). Depending on what the expression is, this could be cheap or expensive.

As mentioned in the previous section, literals which allow global propagators to explain their inferences in a more general way are prime candidates for language extensions. Another benefit of such literals is that they often represent intermediate logical concepts in the propagation algorithm which the propagator is already keeping track of, and thus the channeling propagation required...
to enforce $[e] \leftrightarrow e$ can be “piggy-backed” onto the original propagator at little extra cost.

Ideally we can add a set of literals which allow the universally maximally general explanation for each inference to be described. Unfortunately, this is not always possible as the universally maximally general explanation may be some complicated logical expression that we cannot easily check the truth value of during search. Instead, we may have to settle for less general but more practical language extensions. We now analyze a number of global constraints to see what the maximally general explanations are given the standard LCG language consisting of equality, disequality and inequality literals on existing variables, and show the language extensions which can provide stronger explanations.

### 4.1 Linear

Linear constraints are by far the most common constraint appearing in models.

**Example 5.** Consider the linear constraint $x_1 + 2x_2 + 3x_3 + 4x_4 + 4x_5 \leq 30$ of Example 1. Given $x_1 \geq 1, x_2 \geq 2, x_3 \geq 3$, we can infer $x_4 \leq 4$. There are multiple possible maximally general explanation in the standard LCG language, e.g., $[x_2 \geq 1] \land [x_3 \geq 3] \rightarrow [x_4 \leq 4]$ or $[x_1 \geq 1] \land [x_2 \geq 2] \land [x_3 \geq 2] \rightarrow [x_4 \leq 4]$. However, none of them are the most general explanation possible. If we extended the language with literals representing partial sums, we can now use the universally maximally general explanation: $[x_1 + 2x_2 + 3x_3 \geq 11] \rightarrow [x_4 \leq 4]$. Using this intermediate literal the implication graph for Example 1 changes to that shown in Figure 2. Both $[x_4 \leq 4]$ and $[x_5 \leq 4]$ are explained by $[x_1 + 2x_2 + 3x_3 \geq 11]$. The new 1UIP is simply $[x_1 + 2x_2 + 3x_3 \geq 11] \rightarrow false$. This is a much stronger nogood that will prune more of the search space. □

The channelling propagation which enforces the consistency of a partial sum literal and the variables in the partial sum must itself be explained, and we can similarly use the partial sum literals to give more general explanations.

**Example 6.** Consider Example 1 again. If we only generate explanations on demand during conflict analysis, we can introduce new partial sum literals to explain other partial sum literals in a maximally general fashion. When $x_4 \leq 4$ is inferred by the linear constraint $x_1 + 2x_2 + 3x_3 + 4x_4 + 4x_5 \leq 30$, the chain of
explanations going backwards from \[ x_4 \leq 4 \] would be: \[ x_1 + 2x_2 + 3x_3 \geq 11 \] \[ x_4 \leq 4 \], \[ x_1 + 2x_2 \geq 2 \] \[ x_3 \geq 3 \] \[ x_1 + 2x_2 + 3x_3 \geq 11 \] \[ x_2 \geq 1 \] \[ x_4 \geq 4 \]. The implication graph is shown in Figure 2. □

There are also a significant number of global constraints which are composed of linear constraints along with other primitive constraints like channeling constraints (e.g., among, at_most, at_least, sliding_sum, gcc, etc). These can similarly benefit from partial sum literal language extensions on the lines they are composed from.

### 4.2 Lex

Consider the global lexicographical constraint: \textit{lex_less}([x_1, \ldots, x_n], [y_1, \ldots, y_n]) which constrains the sequence x_1, \ldots, x_n to be lexicographically less than y_1, \ldots, y_n, i.e., \( x_1 < y_1 \lor (x_1 = y_1 \land x_2 < y_2) \lor \ldots \lor (x_1 = y_1 \land \ldots \land x_{n-1} = y_{n-1} \land x_n < y_n) \).

Consider a partial assignment \( x_1 = 1, y_1 = 1, x_2 = 2, y_2 = 2, x_3 = 3 \). From the constraint, we can infer that \( y_3 \geq 3 \). The maximally general explanation in the standard LCG language is: \[ x_1 = 1 \] \[ y_1 = 1 \] \[ x_2 = 2 \] \[ y_2 = 2 \] \[ x_3 = 3 \] \[ y_3 = 3 \]. However, if we extend the language with literals to represent things such as \( x_i \geq y_i \), we can explain it using: \[ x_1 \geq y_1 \] \[ x_2 \geq y_2 \] \[ x_3 \geq y_3 \] \[ y_3 = 3 \]. This second explanation is strictly more general and can produce a more general nogood. For example, if in another branch, we had \( x_1 = 2, y_1 = 2, x_2 = 1, y_2 = 1, x_3 = 3 \), the first nogood cannot propagate since \( x_1 = 1 \) is not true, but the second one can since \( x_1 = y_1, y_2 \leq x_2 \) are true. The other mode of propagation for a \textit{lex_less} constraint can make use of new literals of the form: \( x_i > y_i \).

### 4.3 Disjunctive

Consider a disjunctive constraint \textit{disjunctive}([s_1, s_2], [5, 5]) over two tasks with start times having current domains of \( s_1 \in \{2, \ldots, 8\}, s_2 \in \{0, \ldots, 4\} \), and durations \( d_1 = d_2 = 5 \). A global propagator would reason that task 1 must be scheduled after task 2, and therefore that \( s_1 \geq 5 \). There are multiple maximally general explanation in the standard LCG language, e.g., \[ s_1 \geq 0 \] \[ s_2 \leq 4 \] \[ s_2 \geq 0 \] \[ s_1 \geq 5 \]. We can extend the language with literals to represent that task \( i \) runs before task \( j \), written as \( i \ll j \) and channeled via: \( i \ll j \) \[ s_i + d_i \leq s_j, \sim [i \ll j] \rightarrow s_j + d_j \leq s_i \]. Then we can explain the inference via: \[ 2 \ll 1 \] \[ s_2 \geq 0 \] \[ s_1 \geq 5 \]. A nogood created from this explanation have \[ 2 \ll 1 \] rather than \[ s_1 \geq 0 \] \[ s_2 \leq 4 \] in it and will be more general. For example, if we have another domain with \( s_1 \geq 3, s_2 \leq 7, s_2 \geq 0 \), a nogood created from the first two explanations would not be able to propagate, but the second one might since \( s_1 \geq 3, s_2 \leq 7 \) will cause \( 2 \ll 1 \) to become true.

### 4.4 Table

Consider a table constraint \textit{table}([x_1, x_2, x_3, x_4], [[1, 2, 3, 4], [4, 3, 2, 1], [1, 2, 2, 3], [3, 1, 2, 1], [1, 1, 1, 1]]). Suppose we have \( x_1 = 1, x_2 = 2 \). Among other things,
propagation will infer \(x_4 \neq 1\). There are a number of different maximally general explanations in the standard LCG language, e.g., \([x_1 \neq 4] \land [x_1 \neq 3] \land [x_2 \neq 1] \rightarrow [x_4 \neq 4]\), or \([x_2 \neq 3] \land [x_2 \neq 1] \land [x_2 \neq 1] \rightarrow [x_4 \neq 4]\). However, none of these give the most general reason for \([x_4 \neq 4]\). Suppose we extend the language with literals \(r_i\) which represent whether the \(i\)th tuple is taken or not, i.e., \(r_1 \equiv [x_1 = 1 \wedge x_2 = 2 \wedge x_3 = 3 \wedge x_4 = 4]\), \(r_2 \equiv [x_4 = 1 \wedge x_2 = 3 \wedge x_3 = 2 \wedge x_4 = 1]\), etc. Then we can explain \([x_4 \neq 4]\) using \(-r_2 \wedge -r_4 \wedge -r_5 \rightarrow [x_4 \neq 4]\). This is a universally maximally general explanation, i.e., if any domain knocks out tuples 2, 4 and 5 (which are the only ones that support \(x_4 = 1\)), then \(x_4 \neq 1\).

Similarly, the explanations for the regular constraint can be improved by introducing literals representing the intermediate states of the automata, and the explanations for binary/multi-decision diagram constraints (BDD/MDD) can be improved by introducing literals representing whether we take a particular node in the BDD/MDD or not. The global constraints alldiff, circuit, and many others also have language extensions that can give stronger explanations. However, the extensions we can think of are most likely impractical due to the expense of the channelling constraints.

5 Exploiting Global Structure for Linears

For lex, table, disjunctive, regular and bdd/mdd, the number of useful literals identified in Section 4 is only linear or quadratic in the size of the constraint. Furthermore, all of those logical expressions are already things maintained internally by the global propagator and it is easy to alter the propagators to channel these literals and use them in explanations. Thus the overhead of adding these literals is fairly low and it is fine to simply add them all to the language. Linear on the other hand is more difficult. Linear constraints are extremely common and we know for certain that language extensions can be useful for this constraint. On the other hand, there are \(O(adn2^n)\) possible partial sum literals for a length \(n\) linear with largest coefficient \(a\) and maximum domain size \(d\). If we add too many of them, the cost of channeling them will swamp out any benefit we may get from search space reduction. In the worse case, we may have to calculate an exponential number of partial sums at each node just to channel them. We propose to only add the partial sum literals along a certain ordering of the terms in the linear, and to use the global structure of the problem in order to pick the ordering we use.

Suppose that we had a particular ordering of the variables and suppose we had a linear constraint \(\sum_{i=1}^n a_i x_i \leq a_0\). Without loss of generality assume that for each \(i\), \(x_i\) is before \(x_{i+1}\) in our chosen ordering (if not, just move the terms in the linear around and relabel the indices). We propose to add only partial sum literals of the form \(\sum_{i=1}^k a_i x_i \geq v\) for \(1 \leq k < n\), e.g., \([a_1 x_1 + a_2 x_2 \geq 3]\), but not \([a_3 x_1 + a_4 x_3 \geq 3]\). The benefit here is that a single forward and backward pass through the terms is sufficient to channel the values of all these literals. On the forward pass, we aggregate the lower bound on \(\sum_{i=1}^k a_i x_i\), where we start with \(\sum_{i=1}^0 a_i x_i \geq 0\) and each subsequent term \(\sum_{i=1}^k a_i x_i\) is greater than or equal to either the lower bound of the previous term \(\sum_{i=1}^{k-1} a_i x_i\) plus the lower
bound of $a_kx_k$, or to $v$ where $v$ is the largest value such that $\left[ \sum_{i=1}^{k} a_ix_i \geq v \right]$ is currently true. Similarly, on the backward pass, we aggregate the maximum value of $\sum_{i=1}^{k} a_ix_i$, where we start with $\sum_{i=1}^{n} a_ix_i \leq a_0$, and each subsequent term $\sum_{i=1}^{k} a_ix_i$ is less than or equal to either the upper bound of the previous term $\sum_{i=1}^{k+1} a_ix_i$ minus the lower bound of $a_kx_k$, or to $v-1$ where $v$ is the smallest value such that $\left[ \sum_{i=1}^{k} a_ix_i \geq v \right]$ is currently false. These allow us to fix any of the values of the partial sum literals which should be fixed, and we can propagate upper bounds on $a_kx_k$ using the difference between the lower bound of $\sum_{i=1}^{k} a_ix_i$ and the upper bound of $\sum_{i=1}^{k+1} a_ix_i$.

These partial sum literals can be lazily introduced only as needed, i.e., when we need to use one of them in a nogood. Whenever the nogood database is cleaned to remove inactive nogoods, we also remove any partial sum literal that is no longer in any nogood. To reduce overhead even further, we can also only allow partial sum literals to be introduced at regular intervals. For example, if the interval was 5, we would only allow literals of the form $\left[ \sum_{i=1}^{5} a_ix_i \geq v \right]$,

$$\sum_{i=1}^{10} a_ix_i \geq v,$$

etc, to be introduced. We claim that this is often sufficient to get most of the benefit of the language extension. Thus we can trade off less overhead for a smaller reduction in proof size.

Now we need to pick the ordering that gives us the most useful partial sum literals. Many constraint problems have structures such that each variable is only strongly related to a small subset of other variables. For example, in a disjunctive scheduling problem, tasks which have overlapping time intervals may be strongly related, while tasks whose time intervals are far apart may be weakly related. Or in a graph colouring problem, adjacent nodes are strongly related, but nodes far apart in the graph are weakly related. Such structure can be exploited in order to give smaller resolution proofs. A good search strategy will label variables which are strongly related to those already fixed, rather than to pick some completely random variable to label. This improves propagation and also allows stronger nogoods to be derived. It is well known that techniques such as caching [25], variable elimination [16], dynamic programming [4], and nogood learning allows a problem to be solved with a complexity that is only exponential in the width of the search order (assuming sufficient memory). We claim that the ordering which minimizes this width is also the ideal ordering to use in order to introduce the partial sum literals as it provides the most generality to the nogoods.

**Example 7.** Consider a simple problem with variables $x_1, \ldots, x_{10} \in \{1, \ldots, 5\}$, constraints $\text{all\_diff}\left(x_i, x_{i+1}, x_{i+2}\right)$ for $i = 1, \ldots, 8$ and objective function $\sum x_i$ to be minimized. Given that the $\text{all\_diff}$ constraints constrain sets of consecutive variables, an ordering which minimizes the width is to label $x_1, \ldots, x_{10}$ in order. Suppose we are trying to find a solution with objective $\leq 21$. Suppose we made the decisions $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4, x_5 = 5$ in that order. Propagation forces $x_6, \ldots, x_{10} \leq 2$, causing the $\text{all\_diff}$’s to fail. If we used the ordering $x_1, \ldots, x_{10}$ to introduce partial sum literals, the 1UIP nogood would be: $\left[ \sum_{i=1}^{5} x_i \geq 15 \right] \rightarrow \text{false}$. If we used the ordering $x_1, x_3, x_5, x_7, x_9, x_2, x_4, x_6, x_8, x_{10}$
to introduce partial sum literals instead, the 1UIP would be: $\left[ x_2 \geq 2 \right] \land \left[ x_4 \geq 4 \right] \land \left[ x_1 + x_3 + x_5 \geq 9 \right] \rightarrow false$, which is far less general.

In our experiments, we manually find a low width ordering of the variables. However, it is easy to automate this by using an approximate algorithm for calculating the pathwidth of the constraint graph (e.g., [5]) to give a good variable ordering.

6 Experiments

We perform 4 sets of experiments. The experiments were performed on Xeon Pro 2.4GHz processors using the state of the art LCG solver Chuffed. We use 8 problems. For brevity we only describe the global constraints and the structural order we used to create partial sum literals in each problem with linear constraints. MiniZinc models of these problems can be found at www.cs.mu.oz.au/~pjs/ext-res/. The knapsack problem has linear constraints. We pick an ordering which sorts the items such that the profit to weight ratio is descending. The concert hall problem [17] and talent scheduling problem [11] are scheduling problems with linear constraints. We pick an ordering based on time from earliest to latest. The maximum density still life problem (CSPLib prob032) is a board type problem with linear constraints. We pick an ordering which goes row by row from top to bottom and left to right. The PC-board problem [18] and the balanced incomplete block design problem (BIBD) [19] are matrix problems with linear constraints. BIBD also has lex lesseq symmetry breaking constraints. We use an ordering which goes row by row from top to bottom. The nonograms problem [28] is a board type problem with regular constraints. The jobshop scheduling problem [12] has disjunctive constraints. We use a timeout of 600 seconds. In each table, fails is the geometric mean of the the number of fails in the search, time is the geometric mean of the time of search in seconds (with timeouts counting as 600), and svd is the number of instances solved to optimality. Note that we use propagators of the exact same propagation strength in all the methods for all experiments, so any difference is purely due to nogood learning.

The first experiment compares a CP solver without nogood learning (no-ngl) with nogood learning using the basic language of equality, disequality and inequality literals on existing variables (basic-ngl), and nogood learning with the language extensions described in Section 4 (er-ngl). For the 6 problems with linear constraints, we use a fixed order search based on the structural ordering we described above. We will test the same 6 problems with a dynamic search in the fourth experiment. For nonograms and jobshop, we use the weighted degree search heuristic [6], which works well for these two problems. The results are shown in Table 1. Clearly, we can get very significant reductions in node counts on a wide variety of problems. However, depending on how large the node reduction is and the overhead of extending the language, we may not always get a speedup (e.g., BIBD). The node reduction tends to grow exponentially with problem size.

In the second set of experiments, we test what happens if we only introduce partial sum literals every 5, 10, 20 or 50 variables as described in Section 5.
Table 1. Comparison of the solver without nogood learning (no-<span class="xvariables" style="background-color: #DDEEEE;">ngl</span>) with nogood learning using the basic language of equality, disequality and inequality literals on existing variables (basic-<span class="xvariables" style="background-color: #DDEEEE;">ngl</span>), and nogood learning with the language extensions described in Section 4 (er-<span class="xvariables" style="background-color: #DDEEEE;">ngl</span>).

| Problem          | no-<span class="xvariables" style="background-color: #DDEEEE;">ngl</span> | basic-<span class="xvariables" style="background-color: #DDEEEE;">ngl</span> | er-<span class="xvariables" style="background-color: #DDEEEE;">ngl</span> |
|------------------|--------------|----------------|----------------|
|                  | fails | time | svd | fails | time | svd | fails | time | svd |
| Knapsack-30      | 24712 | 0.10 | 20 | 24526 | 0.55 | 20 | 207  | 0.04 | 20 |
| Knapsack-40      | 2549810 | 7.91 | 20 | 2548993 | 68.14 | 20 | 685  | 0.17 | 20 |
| Knapsack-100     | 119925896 | 600  | 0  | 10855544 | 600  | 0  | 12100 | 32.44 | 20 |
| Concert-Hall-35  | 93231 | 2.95 | 20 | 22450 | 1.62 | 20 | 1529  | 0.31 | 20 |
| Concert-Hall-40  | 1814248 | 63.12 | 18 | 389799 | 35.14 | 20 | 8751  | 2.45 | 20 |
| Concert-Hall-45  | 10186173 | 375.1 | 7  | 3187357 | 329.7 | 9  | 34433 | 14.95 | 20 |
| Talent-14        | 81220 | 2.33 | 20 | 17543 | 1.17 | 20 | 6362  | 0.66 | 20 |
| Talent-16        | 572341 | 17.67 | 20 | 111403 | 11.12 | 20 | 20629 | 2.98 | 20 |
| Talent-18        | 8369293 | 256.3 | 16 | 1535814 | 215.1 | 16 | 89813 | 20.81 | 20 |
| Still-Life-9     | 726722 | 39.42 | 1  | 123544 | 13.43 | 1  | 13687 | 3.07  | 1  |
| Still-Life-10    | 3390292 | 189.25 | 1  | 478182 | 57.14 | 1  | 10165 | 2.39  | 1  |
| Still-Life-11    | 9727533 | 600  | 0  | 4329170 | 600  | 0  | 76225 | 37.44 | 1  |
| PC-Board         | 16944535 | 405.8 | 21 | 68946 | 7.35 | 97 | 34064 | 6.66 | 97 |
| BIBD             | 4660894 | 126.3 | 7  | 125689 | 25.66 | 13 | 32588 | 28.57 | 15 |
| Nonogram-small   | 13566 | 8.02 | 11 | 1827 | 2.17 | 11 | 854  | 1.13 | 11 |
| Nonogram-medium  | 299092 | 254.2 | 3  | 2822 | 5.47 | 4  | 1491  | 2.97  | 4  |
| Nonogram-large   | 829090 | 600  | 0  | 51449 | 73.18 | 4  | 15166 | 27.61 | 4  |
| Jobshop-8        | 16459 | 0.65 | 20 | 489  | 0.08 | 20 | 357  | 0.06  | 20 |
| Jobshop-10       | 2260393 | 167.6 | 13 | 6266 | 1.74 | 20 | 3404  | 1.02  | 20 |
| Jobshop-12       | 6848535 | 596.1 | 1  | 87619 | 41.93 | 20 | 40825 | 18.43 | 20 |

For ease of comparison, we also repeat the er-<span class="xvariables" style="background-color: #DDEEEE;">ngl</span> column from above, where we introduced partial sum literals after every variable. The results are shown in Table 2. The trend is very clear here. The fewer partial sum literals we add, the less reduction in node count we have. However, it also requires less overhead. For many of the instances, introducing partial sum literals every 5 to 10 variables is optimal.

We now compare different ways of picking the order for creating partial sum literals. We compare the structure based ordering (struct) which we used in the previous experiments, a random ordering (random), and an ordering based on sorting on the size of the coefficients in descending order (coeff). We also repeat the no language extension column basic and the er-<span class="xvariables" style="background-color: #DDEEEE;">ngl</span> (renamed to struct) from the first experiment for ease of comparison. The results are shown in Table 3. It can be seen that even when we use an ordering which is inconsistent with the structure of the problem, we can still get some reduction in node count. However, the much smaller reduction in node count means that the overhead may often swamp out any benefit from the reduced search. A random ordering generally gives the least reduction in node count out of the three and the structure based ordering generally gives the most. An ordering based on the size of the coefficients is somewhere in the middle, depending on whether the coefficients happen to follow the structure of the problem or not.

Finally, we test whether the node reduction we gain from partial sum literals is dependent on a specific search order, or whether we will benefit even if we use a dynamic search strategy. We use the variable state independent decaying sum
Table 2. Comparison of introducing partial sum literals after every 1, 5, 10, 20 or 50 terms in the linear constraints.

| Problem       | 1 fails time | 5 fails time | 10 fails time | 20 fails time | 50 fails time |
|---------------|--------------|--------------|---------------|---------------|---------------|
| Knapsack-30   | 207 0.04     | 461 0.02     | 1157 0.05     | 12301 0.37    | 24626 0.56    |
| Knapsack-40   | 685 0.17     | 1957 0.14    | 8961 0.41     | 86123 3.24    | 2548994 67.07 |
| Knapsack-100  | 12100 32.44  | 4677 42.87   | 387626 166.02 | 4689069 600   | 12267788 600  |
| Concert-Hall-35| 1529 0.31    | 1973 0.20    | 3042 0.26     | 4502 0.34     | 22450 1.62    |
| Concert-Hall-40| 8751 2.45    | 11125 1.37   | 16703 1.75    | 29582 2.97    | 389800 34.65  |
| Concert-Hall-45| 34433 14.95  | 44658 7.78   | 66910 10.69   | 178391 30.19  | 319693 329.77 |

Table 3. Comparison of different orderings for generating partial sum literals.

| Problem       | basic fails time | struct fails time | random fails time | coeff fails time |
|---------------|------------------|-------------------|-------------------|------------------|
| Knapsack-30   | 24526 0.55       | 207 0.04          | 15873 4.02        | 9882 2.47        |
| Knapsack-40   | 2548994 68.14    | 685 0.17          | 849769 350.1      | 429410 196.4     |
| Knapsack-100  | 135955 20.26     | 12100 32.44       | 1535554 216.1     | 1532865 216.5    |
| Concert-Hall-35| 13687 3.07       | 13952 2.07        | 14979 2.04        | 20556 2.62       |
| Concert-Hall-40| 849769 350.1     | 9252 1.37         | 11090 1.46        | 130490 3.89      |
| Concert-Hall-45| 701622 20.08     | 34433 14.95       | 217273 77.46      | 251276 77.91     |
| Talent-14     | 6362 0.66        | 9196 0.67         | 17543 1.19        | 17543 1.20       |
| Talent-16     | 20629 2.98       | 31906 3.27        | 111403 11.02      | 111403 10.96     |
| Talent-18     | 89813 20.81      | 135955 20.26      | 1535554 216.1     | 1532865 216.5    |
| Still-Life-9   | 13687 3.07       | 13952 2.07        | 14979 2.04        | 20556 2.62       |
| Still-Life-10  | 10165 2.39       | 9252 1.37         | 11090 1.46        | 130490 3.89      |
| Still-Life-11  | 701622 20.08     | 34433 14.95       | 217273 77.46      | 251276 77.91     |
| Still-Life-14  | 34064 6.66       | 46387 6.32        | 111403 11.08      | 111403 10.96     |
| Still-Life-16  | 111403 11.12     | 20629 2.98        | 167449 59.63      | 111403 10.96     |
| Still-Life-18  | 1535814 215.1    | 89813 20.81       | 1535554 216.1     | 1532865 216.5    |
| PC-Board      | 34064 6.66       | 46387 6.32        | 111403 11.08      | 111403 10.96     |
| BIBD          | 125689 25.66     | 325888 28.57      | 55281 14.63       | 94860 22.75      |

(VSIDS) heuristic [20] adapted from SAT. This is the standard search heuristic used in most current state of the art SAT solvers and is also very effective for some CP problems. For easy comparison, we also repeat the column er-ngl from Table 1, which we rename to er-ngl-fixed. We call VSIDS on the basic language basic-ngl-vsids and on the extended language er-ngl-vsids. The results are shown in Table 4. It can be seen that extending the language reduces the node count on all the problems tested. However, VSIDS is not as capable of exploiting the new literals as a fixed order search using the structure based ordering in knapsack, concert hall, talent scheduling or maximum density still life. On PC board and BIBD, VSIDS is far superior to the fixed order search even without any language extension. The language extension does result in a reduced node count, but the overhead swamps out any benefits.
Table 4. Comparison between VSIDS search heuristic on the basic language and the extended language.

| Problem          | er-ngl-fixed | basic-ngl-vsids | er-ngl-vsids |
|------------------|--------------|-----------------|--------------|
|                  | fails | time | svid | fails | time | svid | fails | time | svid |
| Knapsack-30      | 207   | 0.04 | 20   | 6562  | 0.11 | 20   | 7051  | 0.54 | 20   |
| Knapsack-40      | 685   | 0.17 | 20   | 179932| 5.71 | 20   | 18821 | 5.89 | 20   |
| Knapsack-100     | 12100 | 32.44| 20   | 701750| 600  | 0    | 140533| 600  | 0    |
| Concert-Hall-35  | 1529  | 0.31 | 20   | 84535 | 6.76 | 20   | 17165 | 4.74 | 20   |
| Concert-Hall-40  | 8751  | 2.45 | 20   | 170386| 156.94| 18  | 113479| 45.60 | 20 |
| Concert-Hall-45  | 34433 | 14.95| 20   | 5087797| 510.95| 4   | 502381| 280.84| 15 |
| Talent-14        | 6362  | 0.66 | 20   | 21981 | 1.41 | 20   | 10904 | 2.57 | 20   |
| Talent-16        | 20629 | 2.98 | 20   | 139837| 8.86 | 20   | 29529 | 6.61 | 20   |
| Talent-18        | 89813 | 20.81| 20   | 632778| 45.74| 19   | 202930| 46.83| 18   |
| Still-Life-9      | 13687 | 3.07 | 1    | 2412735| 318.1| 1   | 46829 | 12.96 | 1   |
| Still-Life-10     | 10165 | 2.39 | 1    | 3813974| 600  | 0    | 86837 | 27.17 | 1   |
| Still-Life-11     | 76225 | 37.44| 1    | 4341884| 600  | 0    | 195013| 81.6  | 1   |
| PC-Board          | 34064 | 6.66 | 97   | 11803 | 0.99 | 100  | 11576 | 1.73 | 100  |
| BIBD              | 32588 | 28.57| 15   | 3489  | 0.38 | 28   | 2207  | 1.21 | 28   |

7 Related Work

The work presented here is closely related to the body of work on Boolean encodings for global constraints in SAT. Better Boolean encodings can be smaller in size and can also improve the power of the proof system, leading to faster solves (e.g., [24, 3, 9]). However, our approach has several advantages. When developing Boolean encodings for global constraints for use in a SAT solver, the primary concerns are: 1) the size of the encoding, and 2) its propagation strength. All of the literals and clauses in the encoding must be statically created in the SAT solver. As a result, the encoding has to be reasonably small or the SAT solver will run out of memory or slow to a crawl. These concerns are far less important in the context of LCG. This is because an LCG solver does not ever need to produce a static Boolean encoding of the global constraint. Instead, the high level CP global propagator is fully responsible for all propagation (so we always get the full propagation strength), and it lazily creates literals and clauses as needed. As a result, even if there are potentially an exponential number of possible literals and clauses, it is typically still fine, because during any one solve, only a very small proportion of those literals and clauses will need to be created, and they can be thrown away as soon as they are no longer useful [10]. This flexibility means that we are able to consider other important factors such as which literals will improve the power of the proof system. For example, when a SAT solver encodes a pseudo Boolean constraint into clauses via a BDD translation, the variable order must be chosen to make the BDD small, which may not be ideal for the power of the proof system. On the other hand, we can pick an order which is better for the power of the proof system even if the potential number of literals introduced is very large.

The closest related work to that presented here is conflict directed lazy decomposition [1]. Lazy decomposition treats a global propagator for \( c \) as a black box that hides a SAT encoding that implements \( c \). As computation progresses the propagator for \( c \) lazily exposes more and more of this SAT encoding if an
activity heuristic indicates that this may be beneficial. These exposed Boolean variables give us a structure-based extension to the language. However, lazy decomposition is complex to apply, as one must be able to effectively split a propagator into two parts. The only constraints for which lazy decomposition is defined in [1] are cardinality and pseudo-Boolean constraints. Lazy decomposition has the advantage that it uses the activity of literals in the search as a heuristic to determine whether adding the partial sum literals will be useful. It has the disadvantage that it does not use the global structure to determine which intermediate literals to add. Instead, for pseudo-Boolean constraints it uses the size of the coefficients to order the partial sums, as this tends to reduce the size of the Boolean encoding. However, as Table 3 shows, using an ordering different from the structural one can nullify most of the search space reduction.

There have been several works on using extended resolution in clause learning SAT solvers. In [2], the extension considered are of the form: if \( l_1 \lor \alpha \) and \( l_2 \lor \alpha \) are two successively derived nogoods, then add the new literal \( z \) defined via \( z \leftrightarrow l_1 \lor l_2 \). In [13], another extension rule is proposed where they add a new literal \( z \leftrightarrow d_1 \lor \ldots \lor d_k \) where the \( d_i \) are a subset of the assignments which led to a conflict. While these extensions appear useful for some SAT instances, it seems unlikely that these methods will be able to generate the right extensions for CP problems. For example, a partial sum literal such as \( [x_1 + x_2 \geq 5] \) is defined by \( [x_1 + x_2 \geq 5] \leftrightarrow \ldots \lor ([x_1 \geq 0] \land [x_2 \geq 5]) \lor ([x_1 \geq 1] \land [x_2 \geq 4]) \lor \ldots \). Considering the size of the definition of a general partial sum literal, it will take an incredible amount of luck for one of the above methods to introduce a literal that means exactly a partial sum literal. It is far more effective to keep the structural information contained in a high-level model and to use that to introduce useful literals. Once a problem has been converted into conjunctive normal form, so much of the structural information has been lost that it is very difficult for any automated methods to be able to “rediscover” the literals that matter.

8 Conclusion

A significant amount of research in CP has focused on improving the power of the proof systems used in CP solvers by developing more powerful global propagators. However, such research may well be nearing their limits as the optimal propagators for most commonly used constraints are already known. Nogood learning provides an orthogonal way in which to improve the power of the proof systems used in CP solvers. Extending the language of resolution changes the power of the resolution proof system used in nogood learning and can exponentially reduce the size of a proof of unsatisfiability or optimality, leading to much faster CP solving. The primary difficulty of using extended resolution is in finding the right language extensions to make. We have given a framework for analyzing the generality of explanations made by global propagators in LCG solvers, and shown that language extensions which improve the generality of these explanations are excellent candidates for language extension. Experiments show that such structure-based extended resolution can be highly beneficial in solving a wide range of combinatorial optimization problems.
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