Spreading in Integrable and Non–integrable Many–body Systems

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We consider a finite, closed and selfbound many–body system in which a collective degree of freedom is excited. The redistribution of energy and momentum into the non–collective degrees of freedom is referred to as spreading. Thus, spreading closely relates to thermalization. We carry out numerical simulations to show that spreading can be present and absent in integrable systems. Consequently, non–integrability is not a necessary condition for spreading. We identify subtle features which determine the onset of spreading. We also compare with a non–integrable case.

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I. INTRODUCTION

Almost sixty years ago, Fermi, Pasta and Ullam [1] presented a puzzling study of a many–body system which contrary to the intuition does not thermalize: the phase space is not ergodically filled. They studied an one–dimensional chain of point particles coupled by springs with a small nonlinear force. The chain is clamped, i.e., the particles at the ends do not move. Fermi, Pasta and Ullam found numerically that the energy which was put into the lowest Fourier mode almost completely stayed there, even after very long times, rather than being distributed over many or all modes. An understanding was achieved in a continuum limit which leads to a Kortweg–de Vries equation allowing for soliton solutions. A recent review can be found in Ref. [2]. Some years ago, Kinoshita, Wenger and Weiss [3] studied a different but related problem in a real experiment. They made a quantum Newton’s cradle by letting two one–dimensional Bose gases collide and oscillate against each other. Once more, thermalization did not take place, the two Bose gases kept their shapes even after many oscillations. Subsequently this lack of thermalization was attributed to the integrability of the problem. This conclusion however was challenged later on [4] because the system is only weakly interacting and can thus be viewed as two systems (the centers of mass of the two Bose gases) which collide without effect on their inner structure.

On the other hand, generic many body systems are expected to thermalise. The exact mechanism of the thermalisation depends on the details of system hamiltonian and has been extensively studied by many authors, in particular in the condensed matter literature, see e.g., [20] for a recent work and references.

Thermalization in the above sense is intimately connected to spreading and damping. The former is very often seen in closed many–body systems that exhibit collective and incoherent single–particle motion. Energy and momentum from one distinct — in the present context collective — state is redistributed into many other states. This has to be distinguished from damping as occurring in open systems. In the latter case, the energy leaves the system because of coupling to a large or even infinite number of external degrees of, i.e., to an external bath. In the sequel, we have in mind closed, large, but finite systems. All degrees of freedom are internal, and we always talk about spreading.

Among the numerous examples for spreading of collective motion in atomic nuclei [5], the Giant Dipole Resonance is a very well–known one. The cross section of electric dipole radiation and the spectral density of the excitations show at certain energy a huge peak whose spreading width is orders of magnitudes larger than the mean level spacing. A simple, somehow semiclassical picture helps to catch the salient features of this effect: The neutrons may be viewed as confined to one sphere, the protons to another one. There is no relative motion of the nucleons inside these spheres. The two spheres move against each other, thereby carrying out a fully collective motion. This results in an enormous response function. A bit further away from the resonance energy, relative motion inside the spheres sets in which lowers the cross section. Much further away, all motion is incoherent of single–particle type, there is no collective motion anymore, and the resonance has disappeared completely. The Giant Dipole Resonance inspired our choice of model in the sequel, but many other forms of collective motion and the associated spreading exist in nuclei.

Recently, examples for collective motion have also been found in Bose–Einstein condensates [7–9] and, as already mentioned, in Ref. [3]. Thus, one may expect that the issue of spreading will find renewed interest in this field. Nevertheless, it is important to keep in mind that systems such as nuclei are selfbound, while systems such as Bose–Einstein condensates need an external confining potential. Here, we restrict ourselves to selfbound systems, because we want to make sure that all effects we see are due to the internal dynamics and not possibly to an external confining potential.

We numerically study the interacting many–body system whose more general aspects were analytically investigated in Refs. [10, 11]. Our aim is to put forward the observation that, independent of integrability or non–integrability, spreading can be present or absent. At least for closed selfbound systems this clarifies that integrability is not the decisive system feature, rather it simply
is crucial how the many particles are mutually coupled. The paper is organized as follows. In Sec. III the model is introduced and the procedure of its numerical solution are discussed. We present the results for the integrable and non–integrable cases in Secs. III and IV respectively. We conclude in Sec. V.

II. SETUP OF THE MODEL

We consider two clouds of particles in one dimension with coupling within each cloud and between the two clouds. The clouds are initially separated and then released, which does or does not lead to a spreading of the initial energy and momentum over all degrees of freedom. In the integrable case, all interactions are harmonic. We add a forth–order term to also explore a weakly non–integrable regime. We largely use the model we introduced in the previous analytical investigations [10, 11]. As it turned out convenient to slightly change some conventions for the numerical study, we compile all necessary formulae defining the Hamiltonian in Sec. II A. We discuss the collective coordinate in Sec. II B. The numerical deviations for the numerical study, we compile all necessary formulae defining the Hamiltonian in Sec. II A. We discuss the collective coordinate in Sec. II B. The numerical method and the initial conditions are explained in the Secs. II C and II D respectively.

A. Hamiltonian

The two clouds, labeled \( a = 1, 2 \), of \( N \) point particles each with equal masses \( m \), move in one dimension. Their positions are denoted \( x_i^{(a)} \), \( i = 1, \ldots, N \), their momenta \( p_i^{(a)} \), \( i = 1, \ldots, N \). The total Hamiltonian is

\[
H = H_0 + \lambda H_{\text{int}} .
\]  

The first part is the integrable Hamiltonian as discussed in Ref. [10],

\[
H_0 = H_0^{(1)} + H_0^{(2)} + \kappa H_0^{(12)} .
\]

The term \( \lambda H_{\text{int}} \) makes the total Hamiltonian non–integrable. The first two terms of the Hamiltonian [2],

\[
H_0^{(a)} = \frac{1}{2m} \sum_{i=1}^{N} (p_i^{(a)})^2 + \sum_{i \neq j}^{N} V_{ij} \left( x_i^{(a)} - x_j^{(a)} \right)^2 ,
\]

model the two harmonic clouds \( a = 1, 2 \) before they are coupled. We notice that this definition is equivalent to, but differs from the one in Ref. [11]. The coefficients \( V_{ij} \) are taken from a symmetric and positive \( N \times N \) matrix \( V \) and are assumed equal for both clouds. As already pointed out, we focus on selfbound systems. Here, we find it convenient to ensure this directly by using the translation invariant differences of the positions and the positive coefficients \( V_{ij} \) in Eq. (3). The two clouds are coupled with an interaction \( \kappa H_0^{(12)} \) where

\[
H_0^{(12)} = \sum_{i,j=1}^{N} K_{ij} \left( x_i^{(1)} - x_j^{(2)} \right)^2 .
\]

This preserves the selfboundness since we choose the coefficients \( K_{ij} \) from a symmetric and positive \( N \times N \) matrix \( K \). In contrast to Refs. [10, 11] we introduce the control parameter \( \kappa \) for tuning the overall strength ratio of the interactions within the clouds and between the clouds.

We order the positions and momenta in two \( 2N \) component vectors \( x = (x^{(1)}, x^{(2)}) \) and \( p = (p^{(1)}, p^{(2)}) \) with the \( N \) component vectors \( x^{(a)} = (x_1^{(a)}, \ldots, x_N^{(a)}) \) and \( p^{(a)} = (p_1^{(a)}, \ldots, p_N^{(a)}) \) for \( a = 1, 2 \). Furthermore, it is helpful to cast the integrable case for \( \lambda = 0 \) into a more compact form. Defining the \( 2N \times 2N \) positive, symmetric interaction matrix

\[
C = \begin{bmatrix} W & -\kappa K/2 \\ -\kappa K/2 & W \end{bmatrix} ,
\]

with

\[
W_{i,j} = \delta_{i,j} (2 \sum_{n=1}^{N} V_{i,n} + K_{i,n}) - V_{i,j} ,
\]

the potential becomes a standard bilinear form and we arrive at the expression

\[
H_0 = \frac{p^2}{2m} + x^T C x .
\]

To fix the notation, we write down the elementary transformation to normal modes explicitly. An orthogonal matrix \( U \) diagonalizes the interaction matrix. For convenience, we write

\[
C = \frac{m}{2} T^T \omega^2 U
\]

\[
\omega = \text{diag} (\omega_1, \ldots, \omega_{2N}) ,
\]

where the eigenvalues \( m \omega_i^2 / 2 \), \( i = 1, \ldots, 2N \) of \( C \) are non–negative, because the matrices \( V \) and \( K \) have positive entries. In the rotated coordinates

\[
\xi = U x \quad \text{and} \quad \pi = U p
\]

the system Hamiltonian decouples into \( 2N \) non–interacting ones,

\[
H_0 = \sum_{i=1}^{2N} \left( \frac{\pi_i^2}{2m} + \frac{1}{2} m \omega_i^2 \xi_i^2 \right) .
\]

The positive quantities \( \omega_i \) are of course the system eigenfrequencies. The coordinates \( \xi_i \) and \( \pi_i \) are not positions and momenta of the particles, rather they are weighted linear combinations and can be viewed as positions and momenta of the non–interacting composite
particles which define the normal modes. The transformation also facilitates an elementary solution of the equations of motions, allowing for a crucial check of our numerics later on.

In Ref. [11], we extended the integrable model [2] and [7] by adding a rather general translation invariant term $\lambda H_1$ which breaks the integrability but preserves the self-boundness. For our numerical study we make the special choice $\lambda H_{ni}$ with a strength parameter $\lambda$ and the fourth-order potential
\[
H_{ni} = \sum_{i,j=1}^{N} P_{ij} \left( x_i^{(1)} - x_j^{(2)} \right)^4 .
\] (11)
The coefficients $P_{ij}$ are as well taken from a positive symmetric $N \times N$ matrix $P$. This non-integrable interaction preserves translation invariance and self-boundness.

**B. Collective Coordinate**

We aim at studying the interplay between collective and incoherent single-particle motion. Many-body systems show a rich variety of collective excitations, particularly nuclei provide a zoo of examples [8]. Of course the way how the system is probed determines which collective modes are excited. As we have in mind excitation in which the two clouds are simply pulled apart and then released to oscillate against each other, the natural choice for the collective coordinate in our case is the difference of the centers of mass in each cloud
\[
\Xi = \frac{1}{N} \sum_{i=1}^{N} x_i^{(1)} - \frac{1}{N} \sum_{i=1}^{N} x_i^{(2)} .
\] (12)
Although this definition is fully equivalent to the one we used previously, we notice that the collective coordinate $X$ in Ref. [11] differs by a factor, i.e., we have $\Xi = \sqrt{N/2} X$. The time evolution $\Xi(t)$ of the collective coordinate is our most important observable. The larger the typical amplitudes $|\Xi(t)|$ of the collective motion after some time $t$, the more energy and momentum are contained in the oscillation between the two clouds. The smaller the amplitudes, the more energy and momentum are transferred to the incoherent single-particle degrees of freedom within the clouds. Importantly, the collective coordinate itself describes the most basic eigenmode of the whole system, apart from the simple translation without internal excitation.

**C. Numerical Solution**

For the numerical integration of the equations of motion we found it efficient to use the Velocity Verlet Method, see e.g. Ref. [9], a standard method in molecular dynamics. We employed various well-established techniques and tests to implement it in an optimal way for our system. In particular, we carefully checked that the energy is conserved even for very long times beyond those we were interested in. As already mentioned in Sec. IIIA we compared the exact solution of the integrable model with a direct numerical integration which turned out highly useful to eliminate even subtle errors. To further test the results of our simulation we validated that our numerical simulation remains stable under time reversal transformation.

**D. Initial Conditions**

The initial conditions are chosen such that the two clouds are separated and at rest at time $t = 0$. The initial particle positions $x_i^{(1)} = x_i^{(1)}(0)$, $x_i^{(2)} = x_i^{(2)}(0)$, $i = 1, \ldots, N$ are taken from two uniform random distributions around the arbitrary points, $r$ and $-r$, within the intervals $[r - \Delta r, r + \Delta r]$ and $[-(r) - \Delta r, -(r) + \Delta r]$, respectively. This is illustrated in Fig. 1. If any two particles are assigned the same coordinate, a new random position is chosen for one of them. In all our investigations, we used mirrored initial conditions, i.e., the symmetry $x_i^{(2)} = -x_i^{(1)}$, $i = 1, \ldots, N$. The initial particle momenta $p_i^{(1)}$, $p_i^{(2)}$, $i = 1, \ldots, N$ are always set to zero when we investigate the integrable case in Sec. III. Only in the non-integrable case to be discussed in Sec. IV we work with non-zero initial momenta, preserving the mirror symmetry. Each particle has a mass of $1 kg$.

**III. INTEGRABLE CASE**

We begin by demonstrating in Sec. IIIA the presence and absence of spreading in the integrable case. In Sec. IIIB we give a first explanation by looking at eigenfrequencies and normal modes. We study the influence of the standard deviations of the distributions for the interaction matrix elements and of the particle number in Secs. IIIC and IIID respectively.

**A. Presence and Absence of Spreading**

We chose the entries of the interaction matrices $W$ and $K$ from independent Gaussian distributions with means $\langle W \rangle$, $\langle K \rangle$ and standard deviations $\sigma_W, \sigma_K$, respectively.
The parameters for our numerical simulations are given in Tab. II.

| $N$ | $T$ | $\Delta t$ | $\langle W \rangle$ | $\sigma_W$ | $\langle K \rangle$ | $\sigma_K$ | $|r|$ | $\Delta r$ |
|-----|-----|------------|-------------------|-----------|-------------------|-----------|-----|--------|
| 500 | 30 s | 0.001 s | 4 J/m$^2$ | 0.63 J/m$^2$ | 2 J/m$^2$ | 0.48 J/m$^2$ | 2 m | 0.2 m |

Table I: Parameter set for the integrable case.

The parameter $\kappa$ defining the strength ratio of the interaction within and between the clouds is still to be fixed. In Fig. 2, the collective coordinate is shown for two different values of $\kappa$. The results differ drastically, for $\kappa = 0.75$, the energy stays in the collective oscillation, while it is spread over the other degrees of freedom for $\kappa = 1.0$. It is sufficient to plot the envelopes of the collective coordinate, this is done in Fig. 3 for four values of $\kappa$. Surprisingly, the transition from weak to almost complete spreading happens within a relatively small interval of 0.03 or so in the parameter $\kappa$. If $\kappa$ is increased beyond $\kappa = 1.0$, the spreading becomes weaker again.

B. Eigenfrequencies and Normal Modes

The above results can be explained by the structure of the interaction matrix. As the eigenfrequencies completely determine the interaction, we display their spectra for two values of $\kappa$ in Fig. 4. There is always the eigenfrequency zero, corresponding to the center–of–mass motion of the system as a whole. It is irrelevant due to the translation invariance and not shown in Fig. 4. Although both spectra show broad bulks of non–degenerate eigenfrequencies, they are distinctly different. The one for $\kappa = 1$ shows an isolated eigenfrequency left of the bulk. It is found to belong to the eigenvector that corresponds to the collective coordinate, i.e., the modulus of
all entries is equal, but the signs differ for the two clouds. Such an eigenvalue and eigenvector corresponding to the collective coordinate always exist. The decisive questions are, first, if it is isolated and, second, if and how it is excited by the initial conditions. The answer to the latter question is easily found by transforming the initial conditions \( x_0 \) and \( p_0 \) as chosen in Sec. II D into the initial conditions of the normal modes. We use mirrored initial conditions \( x_0 \) and \( p_0 \). According to Eq. (9), we have \( \xi_0 = U x_0 \) and \( \sigma_0 = U p_0 = 0 \). In Fig. 6, we display the excitation of the normal modes for four different values of \( \kappa \). As there is a one–to–one correspondence between the amplitude \( \xi_i \) and the eigenfrequency \( \omega_i \), we show the initial amplitudes \( \xi_{i0} \) versus the eigenfrequencies \( \omega_i \) to directly visualize which eigenmode is excited. Comparing with Fig. 3, we see that spreading is obviously absent if only the collective excitation is excited and that a broader excitation of other eigenmodes leads to spreading. The truly amazing observation is the subtlety of this process that happens within a 3\% change of the parameter \( \kappa \) and thus due to minor changes in the structure of the interaction. We also conclude that the isolation of the eigenfrequency corresponding to the collective coordinate is essential to prevent spreading. This is seen in Fig. 6 for \( \kappa = 1.02 \). The crucial eigenfrequency now shows up on the right hand side of the spectral bulk, as function of \( \kappa \) it wandered through the bulk, it is isolated again, leading to a supression of spreading.

We notice that half of the eigenfrequencies are not excited, which is due to the mirrored initial conditions. This symmetry makes the projection to the antisymmetric eigenvectors vanish. If using non-mirrored initial conditions, any energy stored in the corresponding degrees of freedom is completely decoupled from the collective coordinate, thus having no effect on collective coordinate dynamics. This information cannot be extracted from the figures, we counted the excited eigenmodes numerically.

We further investigate the remarkable sensitivity of spreading to slight variations of the interaction matrices \( W \) and \( K \). In view of its high dimension, we refrain from trying to explore the space of the interaction parameters systematically. We rather focus on some examples in which we modify the standard deviations \( \sigma_W \) and \( \sigma_K \) of the probability distributions for the elements interaction matrices \( W \) and \( K \). The parameters for these numerical simulations are given in Tab. II.

| \( N \) | \( T \) | \( \Delta t \) | \( \langle W \rangle \) | \( \langle K \rangle \) | \( r \) | \( \Delta r \) |
|---|---|---|---|---|---|---|
| 500 | 30 s | 0.001 s | 2 J/m² | 1 J/m² | 2 m | 0.2 m |

Table II: Parameter set for the test of the standard deviation dependence.

We notice that the relative strength parameter is now fixed to \( \kappa = 2 \). Furthermore, we found it convenient to keep the standard deviation of the interaction between the clouds constant, \( \sigma_K = 0.1 \), and to only vary the standard deviation \( \sigma_W \) of the interaction within the clouds. In Fig. 7 the envelopes of the collective coordinates are shown for four values of \( \sigma_W \). The smaller the standard deviation \( \sigma_W \), the narrower the distribution of the eigenfrequencies and the system is less likely to show spreading. Again, it is surprising that even relatively small changes in \( \sigma_W \) have a strong impact. This behavior remains the same, if \( \sigma_K \) is changed and \( \sigma_W \) is held fixed.

So far, we demonstrated that spreading is favoured by distributing the initial excitations and thus the energy evenly over all degrees of freedom. We now investigate spreading for large standard deviations, involving very broad eigenfrequency spectra. Figure 8 displays the collective coordinate for \( \sigma_W = 20 \) and \( \sigma_K = 14 \). Thus the standard deviations are much larger than the mean values \( \langle W \rangle \) and \( \langle K \rangle \). This forces us to constantly adjust \( \kappa \), for an increased standard deviation shifts the ratio of

#### C. Modifying the Standard Deviations of the Interactions

![Figure 5](Image)

**Figure 5:** Excitation of the normal modes for \( \kappa = 0.75, 0.97 \) (top) and \( \kappa = 1.0, 1.02 \) (bottom) versus the eigenfrequencies.

![Figure 6](Image)

**Figure 6:** Envelopes of the collective coordinates versus time for four different values of \( \sigma_W \). The collective coordinates are normalized to their initial values.
\[ \langle W \rangle \] and \( \langle K \rangle \). Therefore, both \( \sigma_W \) and \( \sigma_K \), are scaled. Large standard deviations result in large interaction coefficients, leading to rapid oscillations of the collective coordinate. Hence, the above simulation is carried out using a timestep size of only 1/10 of the one previously used.

The collective coordinate decays very quickly, the first bottleneck is reached after a relatively small number of oscillations. Qualitatively, this can be explained by directly employing the transformation (9). The expression \( \zeta(t) = Ux(t) \) implies that the collective coordinate \( \Xi(t) \) is a linear combination of the cosines \( \cos(\omega_i t) \) with the excitation strengths \( a_i \). It can thus be estimated by an integral average of \( \cos(\omega' t) \) with a smooth approximation to the spectral density \( s(\omega') \) as a function of the continuous variable \( \omega' \),

\[
\Xi(t) = \sum_{i=1}^{N} a_i \cos(\omega_i t)
\]

\[
= \int d\omega' \cos(\omega' t) \sum_{i=1}^{N} a_i \delta(\omega' - \omega_i)
\]

\[
\approx \int d\omega' \cos(\omega' t)s(\omega') . \tag{13}
\]

This amounts to a Fourier transform of the spectral density. As we chose the interaction matrix elements from Gaussian distributions, it is not too surprising that a Gaussian with mean \( \mu \) and width \( \gamma \) approximates the spectral density well,

\[
\Xi(t) \sim \int_{-\infty}^{+\infty} d\omega' \cos(\omega' t) \exp \left( -\frac{(\omega' - \mu)^2}{2\gamma^2} \right)
\]

\[
\sim \exp \left( -\frac{\gamma^2 t^2}{2} \right) \cos(\mu t) . \tag{14}
\]

leading to a Gaussian decay of the collective coordinate with a decay time given by \( 1/\gamma \). The standard deviations \( \sigma_W \) and \( \sigma_K \) in turn determine the width \( \gamma \) of the spectral density.

### D. Dependence on the Particle Number

One is tempted to expect, based on observations in statistical mechanics, that large particle numbers affect the collective coordinate decay. The larger the number of degrees of freedom, the larger the recurrence times and the more effective is the process of thermalization. Accordingly, large particle number should make spreading more efficient, and the position of the bottleneck should decrease with the number of particles. In Fig. 8 however, we see a different behavior. The parameters for these numerical simulations are listed in Tab. III.

| \( T \) | \( \Delta t \) | \( \langle W \rangle \) | \( \langle K \rangle \) | \( r \) | \( \Delta r \) |
|------|---------|---------|---------|-----|-------|
| 3 s  | 0.001 s | 4 J/m²  | 2 J/m²  | 2 m | 0.2 m |

Table III: Parameter set for the particle number dependence

The number of particles \( N \) strongly affects the local timescales, resulting in ever faster oscillations when \( N \) grows. This is so because the total mass of the system increases linearly with \( N \), while the number of interactions for a given particle with other particles goes with \( N^2 \). Hence, the oscillations periods decrease. The global timescale for the spreading depends, as argued above, on the standard deviations of the interactions. For the present choice of parameters, these mean that the decay time and thus position of the bottleneck is roughly the same for all particle numbers. However, another expectation from statistical mechanics manifests itself in these simulations. As Fig. 9 illustrates, the bottleneck becomes
IV. NON–INTEGRABLE CASE

In Sec. IV A, we investigate and compare non–integrable perturbations of different strengths, before we take a closer look at the trajectories in phase space by slightly varying the initial conditions in Sec. IV B.

A. Perturbations of Different Strengths

We use the full Hamiltonian \( H \) with the non–integrable part \( H_{\text{non-int}} \). The interaction matrices \( W \) and \( K \) for the integrable harmonic interaction, the parameter \( \kappa \) as well as the interaction matrix \( P \) for the non–integrable perturbation are kept fixed in all simulations to be presented here, the parameters are given in Table IV.

| \( N \) | \( T \) | \( \Delta t \) | \( \langle w \rangle \) | \( v_w \) | \( \langle k \rangle \) | \( v_k \) | \( \langle P \rangle \) |
|---|---|---|---|---|---|---|---|
| 500 | 20 s | 0.0005 s | 2 J/m² | 0.2 J/m² | 1.5 J/m² | 0.1 J/m² | 1.5 J/m² |
| \( v_P \) | \( r \) | \( \Delta r \) | \( \Delta v \) |
| 0.15 J/m² | 2 m | 0.2 m | 0.0001 m/s |

Table IV: Parameter set for the investigation of the perturbation influence.

Only the parameter \( \lambda \) is varied to investigate the impact of different perturbation strengths. The results of the simulations are displayed in Fig. 10. For reference, the integrable case corresponding to \( \lambda = 0 \) is also shown. Obviously, non–integrability helps the spreading considerably, for the strongest perturbation, the bottleneck is reached much quicker than in all previous simulations. However, as we demonstrated in Sec. III, non–integrability is not a necessary condition for spreading.

B. Slight Variations of Initial Conditions

High sensitivity of the trajectories to slight changes in the initial conditions is the prime signature of classical chaos [21]. The proper measure is the Lyapunov exponent. In this spirit, we now measure the distance in phase space of system trajectories which differ only slightly in the initial conditions. We look at the collec-
tive coordinate as well as on some single-particle trajectories. When varying the initial conditions, we carefully ensure that the total energy of the system remains unchanged to carry out a comparison on equal footing. To this end, we found it convenient to realize the changes in the initial conditions by modifying the particle momenta instead of the particle positions. We recall that in the above integrable case, the initial momenta were always zero. Here, in the non–integrable case, one randomly chosen particle is given a fixed momentum pointing towards the origin. To preserve the mirror symmetry of the system, the same momentum in opposite direction is given to the corresponding particle of the second cloud. Furthermore, we go to two spatial dimensions to better visualize the trajectories. We look at trajectories of single particles, not of normal modes. In Fig. 11 we present two single-particle trajectories which only differ in the initial conditions for the integrable case with $\lambda = 0$ and for the non–integrable case in which the perturbation is moderate, $\lambda = 10^{-3}$. As expected, the two trajectories stay very close to each other in the former and depart more strongly in the latter scenario.

This has to related to the considerable impact of the perturbation on spreading. We expect the motion of the collective coordinate to remain largely regular [11] on local time scales. In spite of this, energy and momentum must be redistributed to the incoherent, i.e., non–collective, degrees of freedom, because the amplitude of the collective coordinate decreases the faster the stronger the perturbation. To further investigate this, we look at the strongest perturbation with $\lambda = 10^{-2}$. We compare simulations with slightly different initial conditions. In Fig. 12 we plot the collective coordinates for longer times, beyond the time scale displayed in Fig. 10. The two curves are then again on top of each other, before they depart once more in the second shaded region. This confirms that the motion of collective coordinate remains largely regular. The single–particle trajectories, however, show a much stronger onset of chaotic motion, as demonstrated in Fig. 13. At short times, the two trajectories differ only little, but at larger times of about 10 seconds, they are completely apart. The number of circulations is the same, because the energies are the same for the two trajectories.

V. CONCLUSIONS

Thermalization in a narrower sense refers to systems of infinitely many particles. Among others convenient features, infinite systems have the advantage of infinite recurrence times, at least in non–integrable cases. Nevertheless, the concepts of spreading in closed and of damping in open systems of finitely many particles are intimately related to thermalization. They are of high practical relevance, since there are very many such systems, particularly nuclei, atoms, molecules and Bose–Einstein condensates. In parts of the literature, spreading, damping and thermalization are not cleanly distinguished, and the term thermalization is often used as some kind of hypernym. We studied spreading in a closed, selfbound
many–body system for particle numbers large enough to ensure that the recurrence times did not spoil our investigations.

We presented three main result. The first one is a clarification: the phenomenon of spreading and thus of thermalization in the above discussed generalized sense is not tied to chaotic motion. We clearly showed that integrable systems show or do not show spreading. Thus, chaos is not a necessary condition. We explained this by discussing the rôle of eigenfrequencies and normal modes. The details of the interactions are crucial, a normal mode corresponding to the collective coordinate in question has to exist and has to be isolated from the oher eigenfrequencies and, obviously, it must be excited by the initial conditions. Our second main result is the subtlety of the effects. Minor modifications of the relative strength parameters or in the distributions of interaction matrix elements can have large impacts. The particle number does not necessarily change the time scale on which spreading occurs.

We then turned to a non–integrable case. Not surprisingly, non–integrability, i.e., the onset of chaotic motion can considerably accelerate spreading. Our third main result is the different behavior of the collective coordinate and of the single particles. In accordance with our earlier analytical findings, the former continues to move in a largely regular fashion, at least locally, while the single particles show the onset of chaoticity much stronger.

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