ON THE LENGTH OF COHOMOLOGY SPHERES

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Abstract. In [2], T. Bartsch provided detailed and broad exposition of a numerical cohomological index theory for $G$-spaces, known as the length, where $G$ is a compact Lie group. We present the length of $G$-spaces which are cohomology spheres and $G$ is a $p$-torus or a torus group, where $p$ is a prime. As a consequence, we obtain Borsuk-Ulam and Bourgin-Yang type theorems in this context. A sharper version of the Bourgin-Yang theorem for topological manifolds is also proved. Also, we give some general results regarding the upper and lower bound for the length.

1. Introduction

Let $G$ be a compact Lie group. In [2], Thomas Bartsch discuss in details the properties and results of a numerical cohomological index theory known as $(A, h^*, I)$-length, or simply, the length $\ell$ (Definition 2.1). Like all index theories, the length is a powerful tool in the study of equivariant maps and, in recent years, it has been used to prove different versions of Borsuk-Ulam and Bourgin-Yang type theorems [4, 18, 17]

In the study of critical points of functions with symmetry, Bartsch and Clapp [1] computed the value of the length of representation spheres $S(V)$ [1 Proposition 2.4] for $p$-torus or torus groups, i.e., $G = (\mathbb{Z}_p)^k$ or $(S^1)^k$, where $k \geq 1$ and $p$ prime. Considering such groups, we present a more general result by providing the length of the pair $(X, X^G)$, where $X$ is a compact $G$-space that is a cohomology spheres and $X^G$ is the fixed point set (Theorem 3.1). This is achieved by considering the splitting principle in the Euler class $e(X, X^G)$ of the oriented pair $(X, X^G)$ associated with the fibration $X^G \to BG$ given by the Borel construction [12, 9].

As an immediate consequence of monotonicity of the length, a Borsuk-Ulam type result is obtained (Corollary 3.4). This provides necessary conditions on the existence of equivariant maps between cohomology spheres under the actions of $p$-torus or torus. Motivated by results in [3, 18], regarding the sufficient conditions to the Borsuk-Ulam theorem, we provide a result for equivariant maps between $G$-ANR spaces which are cohomology spheres and representation spheres in the case $G = (\mathbb{Z}_p)^k$ (Theorem 3.9). We remark that an alternative proof could be done by using the results provided by [2, Chapter II] and nicely presented in [3, Theorem 3.2] for this context. Our proof rely on the calculation of the equivariant Lusternik-Schnirelmann $A$-cat and $A$-genus [2 Definitions 2.6 and 2.8] of such cohomology spheres. As a corollary we conclude that the Euler class of a $(\mathbb{Z}_p)^k$-ANR cohomology sphere will be polynomial, where $p$ is an odd prime (Corollary 3.11).

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Yang [23, 24] and, independently, Bourgin [6] proved that if \( f: S^{n-1} \to \mathbb{R}^m \) is a \( \mathbb{Z}_2 \)-equivariant map then \( \dim Z_f \geq n - m - 1 \), where \( Z_f = f^{-1}(0) \) and “dim” stands for covering dimension (this is the so-called Bourgin-Yang theorem). Consequently, if \( n > m \), then \( \dim Z_f \neq \emptyset \). Hence, there is no \( \mathbb{Z}_2 \)-equivariant map \( S^n \to S^m \) with respect to the antipodal action which implies the classical Borsuk-Ulam theorem [5]. Blaszczzyk et. al. [4] presented abstract and general versions of the Bourgin-Yang theorem in different settings by making use of the length. These results depend on estimations of lower- and upper-bounds for \( \ell \). In this sense, we present some general estimations. Namely, we give an upper-bound for the length (Theorem 4.3) of compact \( G \)-spaces with no fixed points in terms of covering dimension where \( G \) is any compact Lie group. This allows us to give a certain type of Bourgin-Yang theorem considering the \( p \)-torus and torus groups and recover some classical results in particular cases. In spite the fact this is a rough estimation, we remark is the best one could get with our choices for the definition of length (Remark 4.5). In the case the domain \( X \) is a topological closed orientable manifold and \( n \)-acyclic over the field corresponding to the \( p \)-torus or torus group, and using a totally different technique, following [18] Theorem 2.1, a version of this theorem with optimal estimate is obtained.

At last, we remark that the length can be used to obtain results related with the classical Borel formula (Theorem 2.3). We present an lower-bound for the length of any compact \( G \)-spaces in terms of the length of fixed points set \( X^H \), where \( H \) are subtorus of rank \( k - 1 \) of \( p \)-torus or torus group \( G \) of rank \( k \) (Theorem 2.3).

2. Preliminaries

Let \( G \) be a \( p \)-torus or a torus group of rank \( k \geq 1 \). We will distinguish the cases by the following: \( G = (\mathbb{Z}_p)^k \) for \( p \geq 2 \) or \( G = (S^1)^k \) for \( p = 0 \). A \( p \)-subtorus of rank \( t \leq k \) of \( G \) will be a subgroup \((S^1)^t \) (for \( p = 0 \)) or \((\mathbb{Z}_p)^t \) (for \( p \geq 2 \)). We consider the category of paracompact \( G \)-pairs \((X, A)\), where \( X \) is a paracompact Hausdorff space and \( A \) is a closed subspace. The isotropy subgroup of \( x \in X \) is \( G_x = \{ x \in X | gx = x, \forall g \in G \} \) and the orbit of \( x \) is the \( G \)-subspace \( G(x) = \{ gx | g \in G \} \cong G/G_x \). The orbit space of the \( G \)-space \( X \) will be denoted by \( X/G \). For any closed subgroup \( H \) of \( G \), \( X^H = \{ x \in X | hx = x, \forall h \in H \} \) is the set of fixed points in \( X \) by the induced \( H \)-action.

Let \( X_G = (EG \times X)/G \) be the Borel space where is \( EG \) is the total space of the universal principal \( G \)-bundle \( EG \to BG \) and \( BG = EG/G \) is the universal classifying space. For a \( G \)-pair \((X, A)\), we denote \( H^*_G(X, A; F) = H^*(X_G, A_G; F) \), the Borel equivariant cohomology, where \( H^* \) will always be the Čech cohomology and \( F = \mathbb{Z}_p \) or \( \mathbb{Q} \) whether \( p \geq 2 \) or \( p = 0 \). The map \( p_X^*: H^*(BG; F) \to H^*_G(X; F) \), induced by \( X \to \{pt\} \), gives a \( H^*(BG; F) \)-module structure on \( H^*_G(X; F) \) (also on \( H^*_G(X, A; F) \)) by \( xy := p_X^*(x) \cup y \in H^*_{G}+n(X; F) \), for \( x \in H^m(BG; F) \) and \( y \in H^*_{G}(X; F) \). We mainly deal with the following cohomology rings \( H^*(BG; F) \):

- If \( G = (\mathbb{Z}_2)^k \) then \( H^*(BG; \mathbb{Z}_2) \cong \mathbb{Z}_2[t_1, \ldots, t_k] \), where \( t_i \in H^1(BG; \mathbb{Z}_2) \).
- If \( G = (\mathbb{Z}_p)^k \) then \( H^*(BG; \mathbb{Z}_p) \cong \mathbb{Z}_p[t_1, \ldots, t_k] \otimes_{\mathbb{Z}_p} \Lambda(s_1, \ldots, s_k) \), where \( p > 2 \) and \( s_i \in H^1(BG; \mathbb{Z}_p) \).
- If \( G = (S^1)^k \) then \( H^*(BG; \mathbb{Q}) \cong \mathbb{Q}[t_1, \ldots, t_k] \), where \( t_i \in H^2(BG; \mathbb{Q}) \).

In the case \( p > 2 \), we set \( P^*(G) = \mathbb{Z}[t_1, \ldots, t_k] \) the polynomial part of \( H^*(BG; \mathbb{Z}_p) \). Since we mainly deal with \( p \)-tori groups, we shall suppress the coefficient field \( F \) and keep the choices as above.
For any subtorus $H$ of $G$, we have $H^*_G(G/H) \cong H^*(BH)$. In the particular case $p = 0$ (or 2) and $H$ is a subtorus of rank $k - 1$, the kernel of the map $H^*(BG) \to H^*(BH)$, induced by $G/H \to G/G$, is a principal ideal $(s_H)$, where $s_H \in H^2(BG)$ (or $s_H \in H^1(BG)$). For $p > 2$, $P^*(G) \cap \ker[H^*(BG) \to H^*(BH)] = (s_H)$ and $s_H \in H^2(BG)$.

2.1. The length. Fix a set $A$ of $G$-spaces and $I$ an ideal of the cohomology ring $H^*(BG)$. Let $(X, A)$ be a $G$-pair. The $(A, H^*_G, I)$-length of $(X, A)$ is the smallest integer $\lambda \geq 0$ such that there exist $A_1, \ldots, A_\lambda \in A$ that for any $\omega_i \in I \cap \ker[H^*(BG) \to H^*_G(A_i)]$, 1 $\leq i \leq \lambda$, we have $\omega_1 \cdot \cdots \cdot \omega_\lambda \cdot \gamma = 0 \in H^*_G(X, A)$, for all $\gamma \in H^*_G(X, A)$. If such $\lambda$ does not exist, we write that $(A, H^*_G, I)$-length of $(X, A)$ is $\infty$.

We shall make standard choices for $A$ and $I$:
- $A = \{G/H; H \subset G$ closed subgroup$\}$. This is equivalent, in terms of the value of the length, to $A' = \{G/H; H$ has rank $k - 1\}$ [2 Observation 5.5];
- For $p = 0, 2$, $I = H^*(BG)$ and for $p > 2$, $I = P^*(G)$ the polynomial part.
- For a subtorus $H$ of rank $k - 1$, we set $A_H = \{G/H\}$ and $I_H = (s_H)$.

We simply write $\ell$ (or $\ell_H$) instead of $(A, H^*_G, I)$-length (or $(A_H, H^*_G, I_H)$-length). Also, in the case $A = \emptyset$, we write $\ell(X)$ (or $\ell_H(X)$) instead of $\ell(X, \emptyset)$ (or $\ell_H(X, \emptyset)$).

Proposition 2.1 ([2 Proposition 4.7 and Corollary 4.9]). Let $X, Y$ be two $G$-spaces:

i) If $f : X \to Y$ is a $G$-equivariant map, then $\ell(X) \leq \ell(Y)$.

ii) $\ell(X) \leq A$-genus$(X)$, where the $A$-genus of $X$ is the least integer $t \geq 0$ such that there exists a $G$-equivariant map $X \to A_1 \star \cdots \star A_t$, where $A_i \in A$ for $i = 1, \ldots, t$ and "\star" means the join operation.

iii) If $X$ is a compact $G$-space such that $X^G = \emptyset$, then $\ell(X) < \infty$.

Remark 2.2. Here we only mentioned the results and choices on the definition of the length we are going to use. The length is defined in a very broad scenario and has many properties [2 Chapter 4]. The given definition of $A$-genus is one of its characterizations. It can be seen as a particular case of the definition of the equivariant Lusternik-Schnirelmann category $A$-cat [2 Definition 2.6]. We have the inequality $A$-genus$(X) \leq A$-cat$(X)$ [2 Proposition 2.10].

2.2. Cohomology spheres and Euler classes. A compact Hausdorff space $X$ is a (mod $p$)-cohomology $n$-sphere when $H^*(X; F) \cong H^*(S^n; F)$, where $F = \mathbb{Z}_p$ or $\mathbb{Q}$ depending on whether $p > 2$ or $p = 0$. For $G = (\mathbb{Z}_p)^k$ or $(S^1)^k$, the classical Smith theorem states that $X^G$ is a (mod $p$)-cohomology $r$-sphere, where $-1 \leq r \leq n$, and $r = -1$ when $X^G = \emptyset$. Let $e(X, X^G)$ be the Euler class of the oriented pair $(X, X^G)$ as defined in [9 Chapter III, 4.25]. We shall make use of the results:

Theorem 2.3. Consider $G$ and $X$ as above:

i) ([9 Chapter III, page 205]). There exists a $H^*(BG)$-isomorphism $H^*_G(X, X^G) \cong H^*(BG)/(e(X, X^G))$.

ii) ([9 Chapter III, Theorem 4.40]) Borel Formula. Let $\mathcal{H} = \{H \subset G; H$ has rank $k - 1\}$ then $n - r = \sum_{H \in \mathcal{H}} n(H) - r$, where $n(H)$ is the dimension of the cohomology sphere $X^H$. 
Remark 2.4. Since $H^*_G(X, X^G) \cong H^*(BG)/\langle e \rangle$, where $e = e(X, X^G)$, we have that the annihilator set of $H^*_G(X, X^G)$ is generated by $(e)$, i.e., $A = \{ a \in H^*(BG) | a \gamma = 0 \text{ for all } \gamma \in H^*_G(X, X^G) \} = \langle e \rangle$. In the particular case, $G = (\mathbb{Z}_p)^k$ and $e$ is a polynomial element in $P^*(G) \subset H^*(BG)$, we have $A \cap P^*(G) = \langle e \rangle$. If $e$ is not polynomial, we can write $e = z + n$, where $n$ stands for the nilpotent part, and we have $z^2 = e(z - n) \in A \cap P^*(G)$.

3. The length of cohomology spheres

Theorem 3.1. Let $G = (\mathbb{Z}_p)^k$ or $(S^1)^k$, $X$ be a $(\mod p)$-cohomology $n$-sphere and $r < n$ be the dimension of the $(\mod p)$-cohomology sphere $X^G$, then:

$$\ell(X, X^G) = \begin{cases} n-r, & \text{if } p = 2, \\ n-r, & \text{if } p = 0 \text{ or } p > 2 \text{ and } e(X, X^G) \text{ is polynomial}. \end{cases}$$

For $p > 2$, in the case $e(X, X^G)$ is not polynomial, we have $\frac{n+1}{2} \leq \ell(X, X^G) \leq n+1$.

Proof. All the cases are proved in the same way considering that the Euler class is polynomial when $p > 2$.

Essentially, the length of $(X, X^G)$ resumes in finding an element $\omega \in H^*(BG)$ that annihilates $H^*_G(X, X^G)$ and can be written as a product of generators of certain ideals $I \cap \ker[H^*(BG) \to H^*(BH)]$, for homogeneous spaces $G/H \in A$, where the number of factors in this product is the least as possible. By Remark 2.4, this is the same as to find such $\omega$ that is in the ideal ring $(e)$.

This is already the case for the Euler class $e$. It is a well-known fact [9] Chapter III, Section 4] that $e = (s_{H_1})^{k_1} \cdots (s_{H_t})^{k_t}$, where $(s_{H_i}) = I \cap \ker[H^*(BG; R) \to H^*(BH_i; R)]$ and $H_i$ are subtori of rank $k - 1$ such that $X^{H_i} \neq \emptyset$. Here we have that $k_i = n(H_i) - r$, for $p = 2$, and $k_i = \frac{n(H_i) - r}{2}$, for $p = 0$ or $p > 2$, where $n(H)$ is the dimension of the cohomology sphere $X^{H_i}$. So, by choosing $k_i$ times $G/H_i \in A, i = 1, \ldots, s$, for all $\omega_i^1, \ldots, \omega_i^{k_i} \in (s_{H_i}) \in I \cap \ker[H^*(BG) \to H^*(G/H_i)]$, with $i = 1, \ldots, t$, we have that $\omega = \prod_{i=1}^t \omega_i^1 \cdot \cdots \cdot \omega_i^{k_i} \in (e)$, which implies that $\ell(X, X^G) = k_1 + \cdots + k_t = \deg(e)$ and this completes the proof.

Now, if $p > 2$ and $e(X, X^G)$ is polynomial, we should have that $e = z + n$, where $z = (s_{H_1})^{k_1} \cdots (s_{H_t})^{k_t}$ and $n$ is nilpotent. As mentioned in Remark 2.4, $z^2 \in (e)$, then $\ell(X, X^G) \leq n - r$. Clearly, $\ell(X, X^G) \geq \frac{n}{p-1}$ because $\deg(e) = n - r$.

The next remark points out some situations where the Euler class $e$ is polynomial for $p > 2$.

Remark 3.2. If we take a subtorus $H$ of rank $k - 1$ such that $X^H \neq \emptyset$, we should have that $\ell(X^H, X^{G}) = \ell(H(X^H, X^G)) = \frac{n(H) - r}{2}$, where $\ell_H$ is as in 2.1. Indeed, in the case, $e(X^H, X^{G})$ is equal to $(s_H)^{n(H)}$ which is polynomial [9] Theorem 4.40]. So in the particular case that $G = \mathbb{Z}_p$ and $X^G = \emptyset$, we should have $e$ polynomial and $\ell(X) = \frac{p+1}{2}$. We shall see in Corollary 3.11 that when $X$ is a $G$-ANR space, we also have that $e$ is polynomial and $\ell(X) = \frac{p+1}{2}$.

Remark 3.3. The Theorem 3.1 generalizes the result [1] Proposition 2.4 by Bartsch and Clapp in the context of representation spheres. In [4] Proposition 3.6], the authors give a lower bound for a $G$-space $X$ compact (or paracompact with finite covering dimension) such that $H^i(X) = 0$, when $0 < i < n$. Namely, $\ell(X) \geq n + 1$ for $p = 2$ and $\ell(X) \geq \frac{n+1}{p-1}$, otherwise.
We derive the following version of the Borsuk-Ulam theorem.

**Corollary 3.4 (Borsuk-Ulam).** Let $G = (\mathbb{Z}_p)^k$ or $(S^1)^k$ and $X, Y$ two (mod $p$)-cohomology spheres of dimension $n$ and $m$, respectively. Suppose that $X^G = Y^G = \emptyset$. If there is a $G$-equivariant map $f : X \to Y$, then $\dim X^H \leq \dim Y^H$, for all $H < G$ of rank $k - 1$. In particular, $n \leq m$. Thus, if $n > m$, there is no $G$-equivariant map from $X$ to $Y$.

**Proof.** Suppose that exists a $G$-equivariant map $f : X \to Y$. Note that $f^H := f|_{X^H} : X^H \to Y^H$ is $G/H$-equivariant map, for any $H < G$. When $H$ is a subtorus of rank $k - 1$, from monotonicity of the length (Proposition 2.1(i)) and Theorem 3.1 (including Remark 3.2), we have that $\ell_H(X^H) = \dim(X^H) = \frac{\dim(Y^H) + 1}{2}$. This proves the first part.

Now, suppose $n > m$. By the Borel formula, $n + 1 = \sum_H (\dim X^H + 1) > \sum_H (\dim Y^H + 1) = m + 1$, then we should have $\dim X^H \geq \dim Y^H$ for at least one subtorus $H$ of rank $k - 1$. Thus, there is no $G$-equivariant map from $X$ to $Y$. □

**Remark 3.5.** For the conclusion, one could extends the result to spaces $X$ such that $H^i(X) = 0$, for $0 < i < n$, by considering the estimations of Proposition 3.6.

### 3.1. A converse for the Borsuk-Ulam

Here $G = (\mathbb{Z}_p)^k$, for $p \geq 2$. We will prove a certain converse for the Borsuk-Ulam theorem, finding sufficient conditions for the existence of $G$-equivariant maps between a (mod $p$)-cohomology sphere $X$, that is also a $G$-ANR space, and a representation sphere $S(V)$ of same dimension. For that we use the $A$-genus of $X$. As a corollary, we show that the Euler class $e(X) = c(X, \emptyset)$ is always polynomial, when $p > 2$.

**Lemma 3.6.** Let $G = (\mathbb{Z}_p)^k$ and suppose that $X$ is a (mod $p$)-cohomology $n$-sphere and $G$-ANR space. Then $A\text{-}\text{cat}(X) = A\text{-}\text{genus}(X) = n + 1$.

**Proof.** From [2] Chapter 2 and [8] Proposition 3.7 we have that $A\text{-}\text{cat}(X) \leq (\dim(X) + 1) \cdot \max_{H \subseteq G} c(H)$, where $c(H)$ is the number of connected components of $X^H/NH$ and $H \subseteq G$ are closed subgroups. Here “dim” stands for covering dimension. Since $c(H) = 1$ [16] Lemma 2.2, then $A\text{-}\text{cat}(X) \leq \dim(X) + 1 = n + 1$.

As pointed out in Remark 2.2, $\ell(X) \leq A\text{-}\text{cat}(X) \leq A\text{-}\text{genus}(X)$. We shall verify then that $n + 1 \leq A\text{-}\text{genus}(X)$. For the case $p = 2$, from Proposition 2.1 ii), $A\text{-}\text{genus}(X) \geq n + 1$ since we have $\ell(X) = n + 1$.

Let us analyze now the case $p > 2$. Suppose that $A\text{-}\text{genus}(X) = t$, then there is a $G$-equivariant map $X \to G/H_1 \star \cdots \star G/H_t$, where $G/H_i \cong \mathbb{Z}_p$. Thus we have a $G$-equivariant map $\varphi : X \star X \to (G/H_1 \star G/H_1) \star \cdots \star (G/H_t \star G/H_t)$. By a well-known trick [2] Remark 5.6, we can map $G/H_1 \star G/H_t$ to $S(V_t)$ equivariantly, where $V_t$ is an irreducible representation given by the character $G/H_t \to S^1$. Let $W = V_1 \oplus \cdots \oplus V_t$. Note that $W$ is a vector space of real dimension $2t$, then $SW$ has $2t - 1$ dimension. The map $\varphi$ induces a $G$-equivariant map between $X \star X$ and $S(W)$. Since $X \star X$ is a cohomology sphere of dimension $2n + 1$ and $\ell(SW) = t$ [4] Proposition 2.4, we see that $n + 1 = \ell(X \star X) \leq \ell(S(W)) = t$, where the first inequality is given by Theorem 3.1. Thus, $n + 1 \leq A\text{-}\text{genus}(X) \leq A\text{-}\text{genus}(X)$. □

**Remark 3.7.** For any subtorus $H$ of rank $k - 1$ of $G = (\mathbb{Z}_p)^k$ such that $X^G = \emptyset$, if we consider $A_H = \{G/H\}$, then $A_H\text{-}\text{genus}(X^H) = n(H) + 1$, where $n(H)$ is the dimension of the cohomology sphere $X^H$. 

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Lemma 3.8 (Existence). Let $G = (\mathbb{Z}_p)^k$ and $X$ be a (mod $p$)-cohomology $n$-sphere such that $X$ is a $G$-ANR space and $X^G = \emptyset$. Then there exists a $G$-equivariant map between $X$ and a representation sphere of same dimension.

Proof. It follows from Corollary 5.6 that $A$-genus($X$) = $n + 1$ and, by definition (Proposition 2.1(ii)), there exists a $G$-equivariant map $f : X \to G/K_1 \ast \cdots \ast G/K_{n+1}$, where $G/K_i \in A$. For $p = 2$, the result follows already because $G/K_i \cong \mathbb{Z}_2 \cong S^0$ and then the $G$-equivariant map is $X \to S^0 \ast \cdots \ast S^0 \cong SV$, where dim $SV = n$.

Suppose $p > 2$. For every subtorus $H$ of rank $k - 1$ such that $X^H \neq \emptyset$, we have the $G/H$-equivariant map $f|_{X^H} : X^H \to (G/K_1 \ast \cdots \ast G/K_{n+1})^H \cong G/H \ast \cdots \ast G/H$, where (by Remark 6.7 and Borel Formula) the number of copies of $G/H$ in the join must be $n(H) + 1$.

Reordering and regrouping the factors of the join $G/K_1 \ast \cdots \ast G/K_{n+1}$, we may write it as $M_1 \ast \cdots \ast M_t$, where $M_i \cong G/H_1 \ast \cdots \ast G/H_i$ ($n(H_i) + 1$ times). By the same trick [2, Remark 5.6] used in the previously lemma, every pair of join in $M_1$, $G/H_i \ast G/H_i$, can be mapped equivariantly to $S(V_i)$, where $V_i$ a irreducible representation given by the character $G/H_i \cong \mathbb{Z}_p \hookrightarrow S^1$. Since $n(H_i) + 1$ is an even number, we should have $M_i \cong S(V_i) \oplus \cdots \oplus S(V_i) \cong S(V_{H_i})$, where dim$_C S(V_{H_i}) = n(H_i)$.

Thus $f : X \to G/K_1 \ast \cdots \ast G/K_{n+1} \cong S(V_{H_1}) \ast \cdots \ast S(V_{H_t}) \cong S(V)$, where dim$_C S(V) = \sum n(H_i) = n$. □

Theorem 3.9. Let $G = (\mathbb{Z}_p)^k$ and $X$ be a (mod $p$)-cohomology $n$-sphere such that $X$ is a $G$-ANR space and $X^G = \emptyset$. There exist a $G$-equivariant map between $X$ and a representation sphere $S(V)$ of $G$ with $V^G = \{0\}$ if, and only if, dim $X^H \leq$ dim $SV^H$, for all $H$ subtori of rank $k - 1$ such that $X^H \neq \emptyset$.

Proof. From Corollary 3.3 we have a necessary condition for the existence of the map. Now suppose that dim $X^H \leq$ dim$_C SV^H$, for all $H$. From Lemma 3.8 there exists a $G$-equivariant map $X \to SW$, where dim$_C SW = \dim X$ such that dim $X^H \leq$ dim$_C SW^H$, for all subtori $H$ of rank $k - 1$ with $X^H \neq \emptyset$. By [15] Theorem 2.5, there exists a $G$-equivariant map between $SW$ and $SV$, thus there exists a $G$-equivariant map between $X$ and $SV$. □

Remark 3.10. For an alternative proof, considering that a $G$-ANR space is $G$-homotopy equivalent to a $G$-CW complex [15] Theorem 13.3] one could use the result given in [3, Theorem 3.2].

As a consequence we obtain that the Euler class of a (mod $p$)-cohomology sphere $X$ such that $X^G = \emptyset$ will be polynomial when $X$ is $(\mathbb{Z}_p)^k$-ANR and $p > 2$.

Corollary 3.11. Let $G = (\mathbb{Z}_p)^k$, for $p > 2$, and $X$ a (mod $p$)-cohomology $n$-sphere such that $X^G = \emptyset$ and $X$ is a $G$-ANR space. Then

a) $\ell(X) = \frac{n + 1}{2}$.

b) the Euler class $e$ of $X$ is polynomial.

Proof. For the item a), from Theorem 3.11 or [4] Proposition 3.6] we already have that $\frac{n+1}{2} \leq \ell(X)$. Now, since exists $f : X \to SW$ such that dim$_C SW = \dim X$ and $\ell(SW) = \frac{n+1}{2}$, it follows that $\ell(X) \leq \ell(SW) \leq \frac{n+1}{2}$. This implies that there exists a polynomial element $\alpha \in (e) \cap P^*(G)$ with same degree as $e$. Then $(\alpha) = (e)$ and part b) follows. □
4. An upper-bound for the length and Bourgin-Yang theorems

As mentioned in the introduction, we will study an upper-bound for the length, first considering any compact Lie group, and then we specialize to the $p$-torus and torus case to obtain a Bourgin-Yang type result. Let $G$ be a compact Lie group and $X$ a compact $G$-space. In [21, 22] one can find, in the context of equivariant $K$-theory $K^*_G$, a filtration for $K^*_G(X)$. The construction can be easily adapted for the Borel cohomology theory and is given as follows. Let $R$ be a commutative ring with unity.

For any finite $G$-closed cover $\mathcal{U}$ of $X$, let $N_\mathcal{U}$ be the nerve of such cover. We can associate a $G$-compact space $X_\mathcal{U} = \bigcup_{x \in N_\mathcal{U}} (U_x \times \{x\}) \subset X \times |N_\mathcal{U}|$, where $U_x = \bigcap_{\sigma \in \mathcal{U}} U_\sigma \neq \emptyset$ and $|N_\mathcal{U}|$ is the geometric realization of $N_\mathcal{U}$. Let $X^p_n = \bigcup_{\dim(\sigma) \leq p} (U_\sigma \times \{x\})$. This gives a filtration of $G$-subspaces $X^0_\mathcal{U} \subset X^1_\mathcal{U} \subset \cdots \subset X_\mathcal{U}$. We say that an element of $H^*_G(X; R)$ is in $H^*_G,\mathcal{s}(X; R)$ if, for some finite $G$-closed cover $\mathcal{W}$ of $X$, the element is in $\ker[H^*_G(X; R) \to H^*_G(X^{x-1}; R)]$.

Lemma 4.1. Let $G$ be a compact Lie group and $X$ a compact $G$-space. For any finite $G$-closed cover $\mathcal{U}$ of $X$:

a) The projection on the first coordinate $\pi_1 : X_\mathcal{U} \to X$ induces an isomorphism $\pi_1^* : H^*_G(X_\mathcal{U}; R) \to H^*_G(X; R)$.

b) If $V$ is a refinement of $\mathcal{U}$, there exists a $G$-equivariant map $X_V \to X_\mathcal{U}$, defined up to $G$-homotopy, that respects the filtrations and the projections onto $X$. Thus, $H^*_G(X; R) = H^*_G,0(X; R) \supseteq H^*_G,1(X; R) \supseteq \cdots \supseteq H^*_G,\mathcal{s}(X; R) \cdots$

c) $H^*_G,\mathcal{s}(X; R) \cdot H^*_G,\mathcal{s}(X; R) \subset H^*_G,\mathcal{s},\mathcal{s}(X; R)$, where “$\cdot$” represents the multiplication between rings.

d) $H^*_G,1(X; R) = \bigcap_{x \in X} \ker[H^*_G(X; R) \to H^*_G(G/G_s; R)]$.

Proof. For items a), b), c), d) we refer to [21], and for d) to [22]. □

Let us suppress the ring of coefficients from our notation and keep the standards choices when we specialize to the $p$-torus cases.

Remark 4.2. If $\dim(X) = n < \infty$ (covering dimension), then $H^*_G,\dim X+1(X) = 0$. Indeed, there is a cover $\mathcal{U}$ of $X$ such that $X_\mathcal{U} = X^m$, for all $m \leq \dim X$. Thus, by Lemma 4.1 a), $H^*_G,\dim X+1(X) = \ker[H^*_G(X) \to H^*_G(X_\mathcal{U})] = \{0\}$.

Now we present some relations of this construction with the length.

Corollary 4.3. Let $G$ be a compact Lie group and $X$ a compact $G$-space. Consider $\mathcal{H} = \{H_\gamma\}_{\gamma \in \Gamma}$ the collection of all maximal isotropy subgroups in $G$, i.e., if $x \in X^H$, we have $G_x = H_\gamma$, for all $\gamma \in \Gamma$. Then $H^*_G,1(X) = \bigcap_{\gamma \in \Gamma} \ker[H^*_G(X) \to H^*(BH_\gamma)]$.

Proof. Given $x \in X$, if $G_x$ is not a maximal isotropy subgroup, then there exists a $\gamma \in \Gamma$ such that $G_x < H_\gamma$. Then $G/G_x \to G/H_\gamma$ and, we conclude that, $\ker[H^*_G(X) \to H^*(BH_\gamma)] \subset \ker[H^*_G(X) \to H^*_G(G/G_x)]$. Thus $\bigcap_{\gamma \in \Gamma} \ker[H^*_G(X) \to H^*_G(BH_\gamma)] \subseteq \bigcap_{\gamma \in \Gamma} \ker[H^*_G(X) \to H^*_G(G/G_x)]$. On the other hand, we have that $\bigcap_{\gamma \in \Gamma} \ker[H^*_G(X) \to H^*_G(G/G_x)] \subseteq \bigcap_{\gamma \in \Gamma} \ker[H^*_G(X) \to H^*(BH_\gamma)]$. □

Theorem 4.4. Let $G$ be a compact Lie group and $X$ a compact $G$-space. Suppose that $X^G = \emptyset$ and the number $\alpha(X)$ of maximal isotropy subgroups of $X$ is finite. Then $\ell(X) \leq \alpha(X) \cdot (\dim X + 1)$. 
Proof. Let us say that $\mathcal{H} = \{H_i\}_{i \in \Gamma}$ is the collection of all maximal isotropy subgroups of $X$ and $|\Gamma| = \alpha(X) = s$. For any $x \in X$, the composition $G/G_x \rightharpoonup X \dashv \{\text{pt}\}$ induces the diagram

$$\begin{array}{ccc}
H^*(BG) & \xrightarrow{p_X^*} & H^*(G) \\
\downarrow & & \downarrow \ \\
H^*_G(X) & \xrightarrow{f_x} & H^*(BG_x).
\end{array}$$

Given $w \in \bigcap_{i=1}^s \ker[H^*(BG) \to H^*(BH_i)]$, $f_x \circ p_X^* (w) = 0$, for all $x \in X$ and, thus $p_X^*(w) \in \bigcap_{x \in X} \ker[H^*_G(X) \to H^*(BG_x)] = \bigcap_{i=1}^s \ker[H^*_G(X) \to H^*_G(BH_i)] = H^*_G(X)$. By Remark 4.2, we have $p_X^*(w)_{\dim X+1} \in H^1_{G, \dim X+1} (X) = \{0\}$ and then $\omega_{\dim X+1} \cdot 1_X = 0 \in H^*_G(X)$. Considering $\omega_i \in \ker[H^*_G(BG) \to H^*_G(BH_i)]$, for $i = 1, \ldots, s$, then $\omega = \omega_1 \cdots \omega_s \in \bigcap_{i=1}^s \ker[H^*(BG) \to H^*(BH_i)]$. So $p_X^*(\omega)_{\dim X+1} = 0$ which implies $\ell(X) \leq \dim X + 1$. \hfill \Box

Remark 4.5. As commented in [4] Example 6.1, one cannot expect better upper bound for this choice of length. For a family of distinct maximal subtori $H_i \subset G = (\mathbb{Z}_p)^k$, we have that for $X = \bigcup_{i=1}^s G/H$, $\alpha(X) = t$ and $\dim(\bigcup_{i=1}^s G/H_i) = \dim(X) = 0$, so $\ell(X) = t = \alpha(X)(\dim(X) + 1)$. In the case $G = \mathbb{Z}_p$, we should have $\ell(X) \leq \dim X + 1$. Also, by choosing a different collection for $\mathcal{A}$, $\mathcal{A} = \{A_1 \sqcup \cdots \sqcup A_t \in \mathcal{A}, t \geq 1\}$, the family of all finite disjoint union of orbit spaces in the definition of length, we have $\ell_{\mathcal{A}}(X) \leq \dim X + 1$. Theorem 4.6 (Bourgin-Yang). Let $G = (\mathbb{Z}_p)^k$ or $(S^1)^k$, $X$ a $(\text{mod } p)$-cohomology $n$-sphere and $Y$ a $G$-space, where $X^G = \emptyset$ and $Y - Y^G$ is a $(\text{mod } p)$-cohomology $m$-sphere. Given a $G$-equivariant map $f : X \to Y$ and considering $Z_f = f^{-1}(Y^G)$, then the number $\alpha = \alpha(Z_f)$ of subtori $H \subset G$ of rank $k - 1$ such that $X^H \subset Z_f \neq \emptyset$ is nonzero and

$$\dim Z_f \geq \begin{cases} 
\frac{n-m}{\alpha} - 1, & \text{if } p = 2, \\
\frac{n-m}{\alpha} - 1, & \text{if } p = 0 \text{ or } p > 2 \text{ and } e(Y - Y^G) \text{ is polynomial.}
\end{cases}$$

In particular, if $n > m$, there is no $G$-equivariant map $X \to Y - Y^G$.

Proof. Specializing [4] Theorem 3.1 to our case yields $\ell(Z_f) \geq \ell(X) - \ell(Y - Y^G)$. Now, combining Theorem 4.4 and 4.5 gives us the desired inequalities.

In the case $n > m$, we shall have $\dim(Z_f) \geq 0$ and, therefore $Z_f \neq \emptyset$ which implies the non-existence of a $G$-equivariant map between $X$ and $Y - Y^G$. \hfill \Box

Remark 4.7. When $k = 1$, we obtain a better estimate for $p \neq 2$. Indeed, [4] Theorem 3.1 and Theorem 4.4 gives us $\ell(Z_f) \geq \frac{n-m}{2}$ (or $\ell(Z_f) \geq n - m$ for $p = 2$) which implies that $\bigoplus_{i=0}^{m-n-1} H^i(BG) \to H^*_G(Z_f)$ is a monomorphism. Now, if the $\mathbb{Z}_p$-action is free, we conclude that $\text{cohdim}(Z_f) \geq n - m - 1$, where “cohdim” stands for cohomological dimension. This conclusion is a particular case found in the literature [10] [8] [14] [20]. The hypothesis that $e(Y - Y^G)$ is polynomial can be removed (see Remark 4.5).

4.1 Bourgin-Yang theorem for topological manifolds. A version that offers an optimal estimate for the Bourgin-Yang theorem in the context of cohomology spheres could be obtained when we add to $X$, the domain of the $G$-equivariant
map, the hypothesis that it is also a closed and orientable topological manifold. In fact, the result will be stated in a more general setting, by supposing that $X$ is a closed orientable topological manifold and $(n-1)$-acyclic, i.e., $H^i(X, \mathbb{F}) = 0$, when $1 < i < n - 1$, over the field $\mathbb{F}$ corresponding to $p$.

**Theorem 4.8.** Let $G = (\mathbb{Z}_p)^k$ or $(S^1)^k$ and $X, Y$ two $G$-spaces where:

i) $X$ is a closed orientable topological manifold such that $H^i(X) = 0$ for $1 < i < n - 1$.

ii) $Y$ is $G$-CW complex and there exists $A \subseteq Y$ such that $Y - A$ is compact (or paracompact with finite converging dimensional) and $H^i(Y - A) = 0$ for $i \geq m$. Additionally $(Y - A)^G = \emptyset$.

In the case $p = 0$, suppose that $Y$ has finitely many orbit type.

If $f : X \to Y$ is a $G$-equivariant map, we have $\dim(f^{-1}(A)) \geq n - m - 1$.

**Proof.** Let $Z := f^{-1}(A)$. By hypothesis, $f|_{X-Z} : X-Z \to Y - A$ is a $G$-equivariant map. Suppose by contradiction that $\dim(Z) < n - m - 1$.

Thus $H^i(Z) = 0$, for $i \geq n - m - 1$ and, from Alexander-Poincaré-Lefschetz duality, $0 = H^i(Z) = H_{n-m-i}(X, X-Z)$. From the hypothesis on the homology of $X$ and the long exact sequence of the pair $(X, X-Z)$ we obtain that $0 = H^i(Z) = H_{n-1-i}(X, X-Z) = \tilde{H}_{n-2-i}(X-Z)$, for $n - 2 - i \leq m - 1$.

Then we have that $0 = \tilde{H}_q(X-Z) = H_q(X-Z)$, for $q < m$, and, that $0 = H_q(Y-A) = H_q(Y-A)$, for $q \geq m$. It follows from [7, Theorem 6.4] there is no $G$-equivariant from $X - Z$ to $Y - A$, which is a contradiction.

**Remark 4.9.** This result is an immediate generalization of [13, Theorem 2.5].

5. **General Remarks about the Length for $p$-torus Group**

Let us consider the following result proved in [11, Proposition 3.1]. Here, we only state it for $p$-tori groups.

**Proposition 5.1.** Let $G = (\mathbb{Z}_p)^k$ or $(S^1)^k$ and $G_1, G_2$ two subtorus such that $G = G_1 \times G_2$. Consider a $G_1$-equivariant map $p_X : X_1 \to B_1$, for $i = 1, 2$, and a $G$-equivariant map $p_X : X \to B$, where $X = X_1 \times X_2, B = B_1 \times B_2$ are $G$-spaces through diagonal actions and $p_X = (p_{X_1}, p_{X_2})$. Assume that all spaces are paracompact. Then we have $\ker p_{X}^* = \ker(p_{X_1}^*) \otimes_R H^*_{G_2}(B_2) + H^*_{G_1}(B_1) \otimes_R \ker(p_{X_2}^*)$.

**Corollary 5.2.** Let $G = (\mathbb{Z}_p)^k$ or $(S^1)^k$ and $X$ a paracompact $G$-space. For a subtorus $H$ of rank $k - 1$ of $G$ such that $X^H \neq \emptyset$, we have that $\ker(p_{X^H}^*) = H^*(BH) \otimes \ker (q_{X^H}^*)$, where $p_{X^H} : X^H \to \{pt\}$ is a $G$-equivariant map and $q_{X^H} : X^H \to \{pt\}$ is a $L \cong G/H$-equivariant map.

**Proof.** The result follows from Proposition 5.1 considering the spaces $G_1 = H, G_2 = L \cong G/H, X_2 = X^H, X_1 = B_1 = B_2 = \{pt\}$ and the equivariant maps $p_{X_1} = q_{X^H}$ and $p_{X_2} = p_{X^H}$.

Now we state a lower bound for the length $\ell$ in terms of the length $\ell_H$ of subtori of rank $k - 1$.

**Theorem 5.3.** Let $G = (\mathbb{Z}_p)^k$ or $(S^1)^k$ and $X$ a compact $G$-space such that $X^G = \emptyset$. We have $\sum_{H \in \mathcal{H}} \ell_H(X^H) \leq \ell(X)$, where $\mathcal{H}$ is the collection of all subtori of rank $k - 1$ of $G$ such that $X^H \neq \emptyset$. 


Proof. Let us consider the case \( p > 2 \). Fix \( H \) a subtorus of rank \( k - 1 \). Then, \( G/H \to X^H \to X \to \{ pt \} \) induces the following commutative diagram.

\[
\begin{xy}
 0 & 0 & 0 & H^*(BG) & H^*(BH) & H^*(X) & H^*(X^H) \\
0 & 0 & 0 & p_X & p_X^H & * & * \\
0 & 0 & 0 & H_G(X) & H_G(X^H) & & \\
\end{xy}
\]

Thus, \( \ker p_X^* \subseteq \ker p_{X^H}^* \subseteq \ker i_H^* \) and, therefore, \( I \cap \ker p_X^* \subseteq I \cap \ker p_{X^H}^* \subseteq I \cap \ker i_H^* = (s_H^i) \), where \( I = P^*(G) \cong \mathbb{Z}_p[t_1, \ldots, t_k] \). This implies that \( I \cap \ker p_{X^H}^* = (s_H^i) \), where \( b = \ell_H(X^H) \). Indeed:

From Corollary 5.2 we have that \( \ker p_{X^H}^* \cong H^*(BH; \mathbb{Z}_p) \otimes \mathbb{Z}_p \ker q_{X^H}^* \), where the \( G/H \)-equivariant map \( q_{X^H}^*: X^H \to \{ pt \} \) induces \( q_{X^H}: H^*(B(G/H)) \to H_{G/H}^*(X^H) \).

Note that, since \( G/H \cong \mathbb{Z}_p \), we should have \( P^*(G/H) \cap \ker q_{X^H}^* = (t^b) \), \( t \in \mathbb{Z}_p[t] \otimes \Lambda(s) \cong H^*(B(G/H)) \). Then, \( I \cap \ker p_{X^H}^* \cong I \cap (H^*(BH; \mathbb{Z}_p) \otimes \mathbb{Z}_p \ker q_{X^H}^*) \cong P^*(H) \cap (t^b) \cong (s_H^b) \). Considering the definition of the length, we should have \( \ell_H(X^H) = \min \{ \lambda : p_{X^H}(t^\lambda) = 0 \} = b \).

Assuming that \( H = \{ H_1, \ldots, H_s \} \), we should have that \( I \cap \ker p_X^* \subseteq (s_{H_1}^{b_1}) \cap \cdots \cap (s_{H_s}^{b_s}) \), where \( \ell_{H_i}(X^{H_i}) = b_i \). Now, given an element \( z \in I \cap \ker p_X^* \), this corresponds to an polynomial element multiple of the polynomials \( s_{H_1}^{b_1} \cdots s_{H_s}^{b_s} \) in \( H^*(BG) \), which implies that \( \ell(X) \geq \sum b_i \ell_{H_i}(X^{H_i}) \).

For \( p = 0 \) or 2, we carry out the proof in the same way and do we not need to care about taking intersections with the polynomial ring part.

\[ \square \]

Remark 5.4. We shall have the equality à la Borel Formula, \( \ell(X) = \sum_{H \in \mathcal{H}} \ell_H(X^H) \) if, and only if, the kernel \( \ker p_X^* \) contains the polynomial element \( s_{H_1}^{b_1} \cdots s_{H_s}^{b_s} \).

The hypothesis that \( e((Y - Y^G)^H) \) is polynomial in Theorem 5.3 can be removed. Indeed, considering the statement and notations there, we can work with \( \ell(Z_I) \geq \sum \ell_H(Z_I^H) \geq \sum \ell(H^H) - \ell_H((Y - Y^G)^H) \) and here \( e((Y - Y^G)^H) \) will be polynomial.

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References

[1] Bartsch, T., Clapp, M.: Bifurcation theory for symmetric potential operators and the equivariant cup-length. Math. Z. 204, 341-356 (1990).
[2] Bartsch, T.: Topological Methods for Variational Problems with Symmetries. Lecture Notes in Mathematics 1560, Springer-Verlag Berlin Heidelberg (1993).
[3] Błaszczyk, Z., Marzantowicz, W., Singh, M.: Equivariant maps between representation spheres. Bull. Belg. Math. Soc. Simon Stevin 24, N. 4, 621-630 (2017).
[4] Błaszczyk, Z., Marzantowicz, W., Singh, M.: General Bourgin-Yang theorems. Topology and its Applications, 249, 112-126 (2018).
[5] Borsuk, K.: Drei Stze ber die n-dimensionale euklidische Sphäre, Fund. Math. 20, 177-190, (1933).
[6] Bourgin, D. G.: On some separation and mapping theorems. Comment. Math. Helv. 29, 199-214 (1955).
[7] Clapp, M., Puppe, D.: Critical point theory with symmetries. J. Reine Angew. Math. 418, 1-29 (1991).
[8] Deo, S., Tripathi, S.: Compact Lie Group Action on Finitistic Spaces. Topology 21, No. 4, pp. 393-399 (1982).
[9] Dieck, T. T.: Transformation Groups, de Gruyter, Berlin (1987).
[10] Dold, A.: Parametrized Borsuk-Ulam theorems. Comment. Math. Helv. 63, n. 2, 275-285 (1988).
[11] Fadell, E., Husseini, S. Y.: An ideal-valued cohomological index theory with applications to Borsuk-Ulam and Bourgin-Yang theorems. Ergodic Theory Dynam. Systems 8, Charles Conley Memorial Issue, 73-85 (1988).
[12] Hsiang, W. Y.: Cohomology Theory of Topological Transformation Groups. Springer-Verlag, Berlin, Heidelberg, New York (1975).
[13] Izydorek, M., Rybicki, S.: On parametrized Borsuk-Ulam theorem for free $\mathbb{Z}_p$-action. Algebraic topology (San Feliu de Guixols 1990) 227-234, Lecture Notes in Mathematics 1509, Springer, Berlin, (1992).
[14] Jaworowski, J.: A continuous version of the Borsuk-Ulam theorem. Proc. Amer. Math. Soc., 82(1):112-114, (1981).
[15] Marzantowicz, W.: Borsuk-Ulam theorem for any compact Lie groups. J. Lond. Math. Soc. 49, 195208 (1994).
[16] Marzantowicz, W., Izydorek, M.: The Borsuk-Ulam properties for cyclic groups. Topol. Methods in Nonlinear Anal. 516, 65-72 (2000).
[17] Marzantowicz, M., de Mattos, D., dos Santos, E. L.: Bourgin-Yang version of the Borsuk-Ulam theorem for $\mathbb{Z}_{p^k}$-equivariant maps. Algebraic and Geometric Topology 12, 2146-2158 (2012).
[18] Marzantowicz, M., de Mattos, D., dos Santos, E. L.: Bourgin-Yang versions of the Borsuk-Ulam theorem for $p$-toral groups. Journal of Fixed Point Theory and Applications, 19, p.1427-1437, (2017).
[19] Murayama, M.: On $G$-ANRs and their $G$-homotopy type. textitOsaka J. Math. 20, 479-512 (1983).
[20] Nakaoka, M.: Parametrized Borsuk-Ulam theorems and characteristic polynomials. Topological fixed point theory and applications (Tianjin, 1988), 155-170. Lecture Notes in Mathematics 1411, Springer, (1989).
[21] Segal, G.: Classifying Spaces and Spectral Sequences. Publications Mathématiques de l’Institut des Hautes Études Scientifiques January, Volume 34, Issue 1, pp 105-112 (1968).
[22] Segal, G.: Equivariant $K$-theory. Publ. Math. IHES 34, 129-151 (1968).
[23] Yang, C. T.: On the theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujob and Dyson, I. Ann. of Math. 60, 262-282 (1954).
[24] Yang, C. T.: On Theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujob and Dyson, II. Annals of Mathematics, Second Series, 62, No. 2 (Sep., 1955), pp. 271-283.

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