Equivalent Properties of Two Kinds of Hardy-Type Integral Inequalities

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Abstract: In this paper, using weight functions as well as employing various techniques from real analysis, we establish a few equivalent conditions of two kinds of Hardy-type integral inequalities with nonhomogeneous kernel. To prove our results, we also deduce a few equivalent conditions of two kinds of Hardy-type integral inequalities with a homogeneous kernel in the form of applications. We additionally consider operator expressions. Analytic inequalities of this nature and especially the techniques involved have far reaching applications in various areas in which symmetry plays a prominent role, including aspects of physics and engineering.

Keywords: Hardy-type integral inequality; weight function; equivalent form; operator; norm

MSC: 26D15; 47A05

1. Introduction

In 1925, by introducing one pair of conjugate exponents \((p, q)\), Hardy [1] established a well-known extension of Hilbert’s integral inequality as follows. If \(p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0,\)

\[
0 < \int_0^\infty f^p(x)dx < \infty \text{ and } 0 < \int_0^\infty g^q(y)dy < \infty,
\]

then:

\[
\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} \, dxdy < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left( \int_0^\infty g^q(y)dy \right)^{\frac{1}{q}},
\]

(1)

where the constant factor \(\frac{\pi}{\sin(\pi/p)}\) is the best possible.

Inequalities (1) as well as Hilbert’s integral inequality (for \(p = q = 2\) in (1), cf. [2]) are important in analysis and its applications (cf. [3,4]).

Almost ten years later, in 1934, Hardy et al. proved an extension of (1) with the general homogeneous kernel of degree \(-1\) as \(k_1(x, y)\) (cf. [3], Theorem 319). The following Hilbert-type integral inequality with the general nonhomogeneous kernel was established. If \(h(u) > 0, \phi(\sigma) = \int_0^\infty h(u)u^{\sigma-1}du \in \mathbb{R}^+\), then:
\[
\int_0^\infty \int_0^\infty h(xy)f(x)g(y)dx
dy < \phi\left(\frac{1}{p}\right)\left(\int_0^\infty x^{p-2}f^p(x)dx\right)^\frac{1}{p}\left(\int_0^\infty g^q(y)dy\right)^\frac{1}{q},
\]
where the constant factor \(\phi\left(\frac{1}{p}\right)\) is the best possible (cf. [3], Theorem 350).

In 1998, by introducing an independent parameter \(\lambda > 0\), Yang proved an extension of Hilbert’s integral inequality with the kernel \(\frac{1}{(x+y)^\lambda}\) (cf. [5,6]). In 2004, by introducing another pair of conjugate exponents \((r,s)\), Yang [7] was able to establish an extension of (1) with the kernel \(\frac{1}{x+y}^\lambda\) (\(\lambda > 0\)). In the paper [8], a further extension of (1) was proved along with the result of the paper [5] with the kernel \(\frac{1}{x+y}\). Several papers (cf. [9–14]) provided some extensions of (1) with parameters. In 2009, Yang presented the following extension of (1) (cf. [15,16]).

If \(\lambda_1 + \lambda_2 = \lambda \in \mathbb{R} = (-\infty, \infty)\), \(k_\lambda(x,y)\) is a non-negative homogeneous function of degree \(-\lambda\), satisfying:

\[k_\lambda(ux, uy) = u^{-\lambda}k_\lambda(x, y) \quad (u, x, y > 0),\]

and:

\[k(\lambda_1) = \int_0^\infty k_\lambda(u, 1)u^{\lambda_1-1}du \in \mathbb{R}_+ = (0, \infty),\]

then we have:

\[
\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)dx\,dy < \lambda_1\left(\int_0^\infty x^{\sigma(1-\lambda_1)-1}f^p(x)dx\right)^\frac{1}{p}\left(\int_0^\infty y^{\sigma(1-\lambda_2)-1}g^q(y)dy\right)^\frac{1}{q},
\]
where the constant factor \(k(\lambda_1)\) is the best possible.

For \(\lambda = 1\), \(k_\lambda(x, y) = \frac{1}{x+y}\), \(\lambda_1 = \frac{1}{q}\), \(\lambda_2 = \frac{1}{p}\), (3) reduces to (1). The following extension of (2) was proven:

\[
\int_0^\infty \int_0^\infty h(xy)f(x)g(y)dx\,dy < \sigma\left(\int_0^\infty x^{\sigma(1-\sigma)-1}f^p(x)dx\right)^\frac{1}{p}\left(\int_0^\infty y^{\sigma(1-\sigma)-1}g^q(y)dy\right)^\frac{1}{q},
\]
where the constant factor \(\phi(\sigma)\) is the best possible (cf. [17]).

For \(\sigma = \frac{1}{2}\), (4) reduces to (2).

Some equivalent inequalities of (3) and (4) are considered in [16]. In 2013, Yang [17] also studied the equivalency between (3) and (4) by adding a condition. In 2017, Hong [18] proved an equivalent condition between (3) and a few parameters. Some similar results were obtained in [19–28].

**Remark 1** (cf. [17]). If \(h(xy) = 0\), for \(xy > 1\), then:

\[\phi(\sigma) = \int_0^1 h(u)u^{\sigma-1}du = \phi_1(\sigma) \in \mathbb{R}_+,
\]
and (4) reduces to the following Hardy-type integral inequality with nonhomogeneous kernel:
\[
\int_0^\infty g(y) \left( \int_0^y h(xy)f(x) \, dx \right) \, dy \\
< \phi_1(\sigma) \left( \int_0^\infty x^{p(1-\sigma)-1} f^p(x) \, dx \right)^{\frac{1}{p}} \left( \int_0^\infty y^{q(1-\sigma)-1} g^q(y) \, dy \right)^{\frac{1}{q}},
\]

where the constant factor \( \phi_1(\sigma) \) is the best possible.

If \( h(xy) = 0 \), for \( xy < 1 \), then:

\[
\phi(\sigma) = \int_1^\infty h(u) u^{\sigma-1} \, du = \phi_2(\sigma) \in \mathbb{R}_+,
\]

and (4) reduces to the following kind of Hardy-type integral inequality with nonhomogeneous kernel:

\[
\int_0^\infty g(y) \left( \int_0^\infty h(xy)f(x) \, dx \right) \, dy \\
< \phi_2(\sigma) \left( \int_0^\infty x^{p(1-\sigma)-1} f^p(x) \, dx \right)^{\frac{1}{p}} \left( \int_0^\infty y^{q(1-\sigma)-1} g^q(y) \, dy \right)^{\frac{1}{q}},
\]

where the constant factor \( \phi_2(\sigma) \) is the best possible.

In this paper, using weight functions as well as employing various techniques from real analysis, we establish a few equivalent conditions of two kinds of Hardy-type integral inequalities with the nonhomogeneous kernel:

\[
\frac{|\ln xy|^{\beta}}{(xy)^{\lambda+1}} (\beta > -1, \lambda > 0).
\]

To prove our results, we also deduce a few equivalent conditions of two kinds of Hardy-type integral inequalities with a homogeneous kernel in the form of applications. We additionally consider operator expressions. Analytic inequalities of this nature and especially the techniques involved have far reaching applications in various areas in which symmetry plays a prominent role, including aspects of physics and engineering.

2. Two Lemmas

For \( \beta > -1, \lambda > 0 \), we set

\[
h(u) := \frac{|\ln u|^{\beta}}{u^{\lambda+1}} (u > 0).
\]

For \( \sigma > 0 \), by the Lebesgue term-by-term integration theorem, we derive that:

\[
k_1(\sigma) := \int_0^1 h(u) u^{\sigma-1} \, du = \int_0^1 \left( -\ln u \right)^{\beta} u^{\sigma-1} \, du \\
= \int_0^1 (\ln u)^{\beta} \sum_{k=0}^{\infty} (-1)^k u^{k\lambda+\sigma-1} \, du \\
= \int_0^1 (\ln u)^{\beta} \sum_{k=0}^{\infty} (u^{2i\lambda} - u^{(2i+1)\lambda}) u^{\sigma-1} \, du \\
= \sum_{i=0}^{\infty} \int_0^1 (\ln u)^{\beta} (u^{2i\lambda} - u^{(2i+1)\lambda}) u^{\sigma-1} \, du \\
= \sum_{k=0}^{\infty} (-1)^k \int_0^1 (\ln u)^{\beta} u^{k\lambda+\sigma-1} \, du.
\]
Setting $v = (k\lambda + \sigma)(-\ln u)$ in the above integral, we obtain:
\[
k_1(\sigma) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k\lambda + \sigma)\beta + 1} \int_0^\infty v^\beta e^{-v} dv
= \frac{\Gamma(\beta + 1)}{\lambda^{\beta + 1}} \xi\left(\beta + 1, \frac{\sigma}{\lambda}\right) \in \mathbb{R}_+,
\]
(7)

where:
\[
\Gamma(\eta) := \int_0^\infty v^{\eta-1} e^{-v} dv \ (\eta > 0)
\]
stands for the gamma function and:
\[
\xi(s,a) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(k + a)^s} \ (Re(s), \ a > 0),
\]
which is a function very well known for its applications in analytic number theory.

For $0 < \sigma < \lambda, \mu = \lambda - \sigma > 0$, setting $v = \frac{1}{\eta}$, by (7), we obtain that:
\[
k_2(\sigma) := \int_1^\infty h(u) u^{\sigma-1} du
= \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda + 1} u^{\sigma-1} du = \int_0^1 \frac{(-\ln u)^\beta}{u^\lambda + 1} u^{\beta-1} du
= \frac{\Gamma(\beta + 1)}{\lambda^{\beta + 1}} \xi(\beta + 1, \frac{\mu}{\lambda}) = k_1(\mu) \in \mathbb{R}_+.
\]
(8)

In the sequel, we assume that $p > 1, \frac{1}{\sigma} + \frac{1}{q} = 1, \sigma_1, \mu_1 \in \mathbb{R}$.

**Lemma 1.** If $\beta > -1, \sigma, \lambda > 0$, there exists a constant $M_1$, such that for any non-negative measurable functions $f(x)$ and $g(y)$ in $(0, \infty)$, the following inequality:
\[
\int_0^\infty g(y) \left[ \int_0^{x^\sigma} \frac{|\ln xy|^{\beta}}{(xy)^{\lambda} + 1} f(x) dx \right] dy
\leq M_1 \left[ \int_0^\infty x^{p(1-\sigma)-1} f_p(x) dx \right]^\frac{1}{\sigma} \left[ \int_0^\infty y^{q(1-\sigma)-1} g_q(y) dy \right]^\frac{1}{q}
\]
(9)

holds true. Then, we have $\sigma_1 = \sigma$, and $M_1 \geq k_1(\sigma)$.

**Proof.** If $\sigma_1 > \sigma$, then for $n \geq \frac{1}{\sigma_1 - \sigma} \ (n \in \mathbb{N})$, we set the following two functions:
\[
f_n(x) := \begin{cases} x^{\sigma_1 - \frac{1}{\sigma_1 - \sigma}} & 0 < x \leq 1 \\
0 & x > 1 \end{cases},
\]
\[
g_n(y) := \begin{cases} 0, 0 < y < 1 \\
y^{\sigma_1 - \frac{1}{\sigma_1 - \sigma}} & y \geq 1 \end{cases},
\]
and deduce that:
\[
I_1 := \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^\frac{1}{\sigma} \left[ \int_0^\infty y^{q(1-\sigma)-1} g^q(y) dy \right]^\frac{1}{q} = n.
\]

Setting $u = xy$, we obtain:
\[ I_1 := \int_0^\infty g_n(y) \left( \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{(xy)^\lambda + 1} f_n(x) \, dx \right) \, dy \]

\[ = \int_1^\infty \left( \int_0^{\frac{1}{y}} \frac{(-\ln xy)^\beta}{(xy)^\lambda + 1} \, dx \right)^{\gamma} \, dy \]

\[ = \int_1^\infty y^{(\gamma - 1)} \int_0^{1/y} \frac{(-\ln u)^\beta}{u^{\lambda + 1}} \, du, \]

and then by (9), we have:

\[ \int_1^\infty y^{(\gamma - 1)} \int_0^{1/y} \frac{(-\ln u)^\beta}{u^{\lambda + 1}} \, du \leq I \leq M_1 I_1 = M_1 n < \infty. \]  

(10)

Since \( (\gamma - 1) - \frac{1}{n} \geq 0 \), it follows that:

\[ \int_1^\infty y^{(\gamma - 1)} \, dy = \infty. \]

By (10), in view of:

\[ \int_0^1 \frac{(-\ln u)^\beta}{u^{\lambda + 1}} u^{\gamma - \frac{1}{n}} \, du > 0, \]

we deduce that \( \infty < \infty \), which is a contradiction.

If \( \gamma < \lambda \), then for \( n \geq \frac{1}{\gamma - \epsilon} \) \( (n \in \mathbb{N}) \), we set the following two functions:

\[ \tilde{f}_n(x) := \begin{cases} 0, & 0 < x < 1 \\ x^{\gamma - \frac{1}{n} - 1}, & x \geq 1 \end{cases}, \quad \tilde{g}_n(y) := \begin{cases} y^{\gamma + \frac{1}{n} - 1}, & 0 < y \leq 1 \\ 0, & y > 1 \end{cases}, \]

and obtain:

\[ \tilde{I}_1 := \left[ \int_0^\infty x^{(\gamma - 1)} \tilde{f}_n(x) \, dx \right]^{\frac{1}{\gamma}} \left[ \int_0^\infty y^{(\gamma - 1)} \tilde{g}_n(y) \, dy \right]^{\frac{1}{\gamma}} = n. \]

Setting \( u = xy \), we obtain:

\[ \tilde{I}_1 := \int_0^\infty \tilde{f}_n(x) \left[ \int_0^{\frac{1}{x}} \frac{|\ln xy|^\beta}{(xy)^\lambda + 1} \tilde{g}_n(y) \, dy \right] \, dx \]

\[ = \int_1^\infty \left[ \int_0^{\frac{1}{x}} \frac{(-\ln xy)^\beta}{(xy)^\lambda + 1} \, dy \right] x^{(\gamma - 1) - \frac{1}{n}} \, dx \]

\[ = \int_1^\infty x^{(\gamma - 1) - \frac{1}{n}} \, dx \int_0^{1/x} \frac{(-\ln u)^\beta}{u^{\lambda + 1}} \, du, \]

and then by Fubini’s theorem and (9), we have:

\[ \int_1^\infty x^{(\gamma - 1) - \frac{1}{n}} \, dx \int_0^{1/x} \frac{(-\ln u)^\beta}{u^{\lambda + 1}} \, du \leq \tilde{I}_1 = \int_0^\infty \tilde{g}_n(y) \left[ \int_0^{\frac{1}{y}} \frac{|\ln xy|^\beta}{(xy)^\lambda + 1} \tilde{f}_n(x) \, dx \right] \, dy \leq M_1 \tilde{I}_1 = M_1 n. \]  

(11)

Since \( (\gamma - 1) - \frac{1}{n} \geq 0 \), it follows that:

\[ \int_1^\infty x^{(\gamma - 1) - \frac{1}{n}} \, dx = \infty. \]

By (11), in view of the fact that
we obtain that $\infty < \infty$, which is a contradiction.

Hence, we conclude that $\sigma_1 = \sigma$.

For $\sigma_1 = \sigma$, we reduce (11) as follows:

$$
M_1 \geq \int_0^1 \frac{(-\ln u)^\beta}{u^\lambda + 1} u^{\sigma_1 + \frac{1}{\beta} - 1} du. 
$$

(12)

Since:

$$
\begin{cases}
(-\ln u)^\beta & u^{\sigma_1 + \frac{1}{\beta} - 1} \\
u^\lambda + 1
\end{cases}
$$

is non-negative and increasing in $(0,1]$, by Levi’s theorem, we derive that:

$$
M_1 \geq \lim_{n \to \infty} \int_0^1 \frac{(-\ln u)^\beta}{u^\lambda + 1} u^{\sigma_1 + \frac{1}{\beta} - 1} du
$$

$$
= \int_0^1 \lim_{n \to \infty} \frac{(-\ln u)^\beta}{u^\lambda + 1} u^{\sigma_1 + \frac{1}{\beta} - 1} du = k_1(\sigma).
$$

This completes the proof of the lemma.  \( \square \)

**Lemma 2.** If $\beta > -1, 0 < \sigma < \lambda$, there exists a constant $M_2$, such that for any non-negative measurable functions $f(x)$ and $g(y)$ in $(0,\infty)$, the following inequality:

$$
\int_0^\infty g(y) \left[ \int_0^\infty \frac{|\ln xy|^\beta}{(xy)^\lambda + 1} f(x)dx \right] dy 
$$

$$
\leq M_2 \left[ \int_0^\infty x^{\beta(1-\sigma) - 1} f_p(x)dx \right] \frac{1}{p} \left[ \int_0^\infty y^{\beta(1-\sigma) - 1} g^\alpha(y)dy \right] \frac{1}{\alpha} 
$$

(13)

holds true. Then, we have $\sigma_1 = \sigma$, and $M_2 \geq k_2(\sigma)$.

**Proof.** If $\sigma_1 < \sigma$, then for $n \geq \frac{1}{\sigma_1 - \sigma}$ ($n \in \mathbb{N}$), we set two functions $\tilde{f}_n(x)$ and $\tilde{g}_n(y)$ as in Lemma 1, and derive that:

$$
\tilde{f}_1 = \left[ \int_0^\infty x^{\beta(1-\sigma) - 1} \tilde{f}_p(x)dx \right] \frac{1}{p} \left[ \int_0^\infty y^{\beta(1-\sigma) - 1} \tilde{g}_n(y)dy \right] \frac{1}{\alpha} = n.
$$

Setting $u = xy$, we obtain:

$$
\tilde{f}_2 = \int_0^\infty \tilde{g}_n(y) \left[ \int_0^\infty \frac{|\ln xy|^\beta}{(xy)^\lambda + 1} \tilde{f}_n(x)dx \right] dy
$$

$$
= \int_0^1 \left[ \int_0^\infty \frac{|\ln xy|^\beta}{(xy)^\lambda + 1} x^{\sigma - \frac{1}{\beta} - 1} dx \right] y^{\sigma + \frac{1}{\beta} - 1} dy
$$

$$
= \int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{\beta} - 1} dy \int_1^\infty \frac{|\ln u|^\beta}{u^\lambda + 1} u^{\sigma - \frac{1}{\beta} - 1} du,
$$

and then by (13), we deduce that:

$$
\int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{\beta} - 1} dy \int_1^\infty \frac{|\ln u|^\beta}{u^\lambda + 1} u^{\sigma - \frac{1}{\beta} - 1} du
$$

$$
= \tilde{f}_2 \leq M_2 \tilde{f}_1 = M_2 n < \infty. 
$$

(14)
Since \((\sigma_1 - \sigma) + \frac{1}{n} \leq 0\), it follows that:
\[
\int_0^1 y^{(\sigma_1 - \sigma) + \frac{1}{n} - 1} dy = \infty.
\]

By (14), in view of
\[
\int_1^\infty \frac{(\ln u)^\beta}{u^\lambda + 1} u^{\sigma_1 - \frac{1}{n} - 1} du > 0,
\]
we have \(\infty < \infty\), which is a contradiction.

If \(\sigma_1 > \sigma\), then for \(n \geq \frac{1}{\sigma_1 - \sigma} (n \in \mathbb{N})\), we set two sequences of \(f_n(x)\) and \(g_n(y)\) as in Lemma 1, and obtain:
\[
J_1 = \left(\int_0^\infty x^{p(1-\sigma) - 1} f_n^p(x) dx\right)^\frac{1}{p} \left[\int_0^\infty y^{q(1-\sigma_1) - 1} g_n^q(y) dy\right]^\frac{1}{q} = n.
\]

Setting \(u = xy\), we obtain:
\[
I_2 = \int_0^\infty x^{(\sigma - \sigma_1) + \frac{1}{n} - 1} d x \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda + 1} u^{\sigma_1 - \frac{1}{n} - 1} du,
\]
and then, by Fubini’s theorem and (13), we have:
\[
\int_0^1 x^{(\sigma - \sigma_1) + \frac{1}{n} - 1} dx \int_1^\infty \frac{(\ln u)^\beta}{u^\lambda + 1} u^{\sigma_1 - \frac{1}{n} - 1} du \leq M_2 J_1 = M_2 n. \tag{15}
\]

Since \((\sigma - \sigma_1) + \frac{1}{n} \leq 0\), it follows that
\[
\int_0^1 x^{(\sigma - \sigma_1) + \frac{1}{n} - 1} dx = \infty.
\]

By (15), in view of the fact that:
\[
\int_1^\infty \frac{(\ln u)^\beta}{u^\lambda + 1} u^{\sigma_1 - \frac{1}{n} - 1} du > 0,
\]
we have \(\infty < \infty\), which is a contradiction.

Hence, we conclude the fact that \(\sigma_1 = \sigma\).

For \(\sigma_1 = \sigma\), we reduce (15) as follows:
\[
M_2 \geq \int_1^\infty \left(\frac{(\ln u)^\beta}{u^\lambda + 1} u^{\sigma_1 - \frac{1}{n} - 1}\right) du. \tag{16}
\]

Since:
\[
\left\{\left(\frac{(\ln u)^\beta}{u^\lambda + 1} u^{\sigma_1 - \frac{1}{n} - 1}\right)\right\}_{n=1}^\infty
\]
is non-negative and increasing in \([1, \infty)\), still by Levi’s theorem, we have:

\[
M_2 \geq \lim_{n \to \infty} \int_1^\infty \frac{(\ln u)^\beta}{u^{\lambda} + 1} u^{\sigma - \frac{1}{\beta} - 1} du
\]

\[
= \int_1^\infty \lim_{n \to \infty} \frac{(\ln u)^\beta}{u^{\lambda} + 1} u^{\sigma - \frac{1}{\beta} - 1} du = k_2(\sigma).
\]

This completes the proof of the Lemma. \(\Box\)

### 3. Main Results and Corollaries

**Theorem 1.** If \(\beta > -1, \sigma, \lambda > 0\), then the following conditions are equivalent.

(i) There exists a constant \(M_1\), such that for any \(f(x) \geq 0\), satisfying:

\[
0 < \int_0^\infty x^{p(1-\sigma)-1} f_p(x) dx < \infty,
\]

we have the following Hardy-type integral inequality of the first kind with nonhomogeneous kernel:

\[
J := \left\{ \int_0^\infty y^{p_1 - 1} \left[ \int_0^\frac{1}{y} \left| \frac{\ln xy}{(xy)^{\lambda} + 1} f(x) dx \right|^p dy \right]^{\frac{1}{p}} \right\}^{\frac{1}{p}} < M_1 \left[ \int_0^\infty x^{p(1-\sigma)-1} f_p(x) dx \right]^{\frac{1}{p}}.
\]

(ii) There exists a constant \(M_1\), such that for any \(f(x), g(y) \geq 0\), satisfying:

\[
0 < \int_0^\infty x^{p(1-\sigma)-1} f_p(x) dx < \infty \text{ and } 0 < \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy < \infty,
\]

we have the following inequality:

\[
I := \int_0^\infty g(y) \left[ \int_0^\frac{1}{y} \left| \frac{\ln xy}{(xy)^{\lambda} + 1} f(x) dx \right|^p dy \right] \leq M_1 \left[ \int_0^\infty x^{p(1-\sigma)-1} f_p(x) dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}.
\]

(iii) \(\sigma_1 = \sigma\).

If Condition (iii) holds, then \(M_1 \geq k_1(\sigma)\) and the constant factor:

\[
M_1 = k_1(\sigma) = \frac{\Gamma(\beta + 1)}{\lambda^{\beta + 1}} \xi(\beta + 1, \sigma, \lambda)
\]

in (17) and (18) is the best possible.

**Proof.** (i) \(\Rightarrow\) (ii). By Hölder’s inequality (cf. [29,30]), we obtain:

\[
I = \int_0^\infty \left[ y^{\sigma_1 - \frac{1}{p}} \int_0^\frac{1}{y} \left| \frac{\ln xy}{(xy)^{\lambda} + 1} f(x) dx \right|^p \left( y^{\frac{1}{p} - \sigma_1} g(y) \right) dy \right] \leq \left[ \int_0^\infty y^{q(1-\sigma_1)-1} g^q(y) dy \right]^{\frac{1}{q}}.
\]

Then by (17), we have (18).

(ii) \(\Rightarrow\) (iii). By Lemma 1, we have \(\sigma_1 = \sigma\).
(iii) $\Rightarrow$ (i). Setting $u = xy$, we obtain the following weight function:

$$\omega_1(x, y) = y^\frac{1}{p} \int_0^\frac{1}{y} \left[ \frac{\ln xy}{(xy)^\lambda + 1} ight]^{\frac{1}{p}} dx$$

By Hölder’s inequality with weight and (20), for $y \in (0, \infty)$, we have:

$$\omega_1(x, y) = \left[ \int_0^\frac{1}{y} \left[ \frac{\ln xy}{(xy)^\lambda + 1} \right]^{\frac{1}{p}} f(x) dx \right]^p$$

$$\leq \left[ \frac{y^{\sigma(1 - \epsilon) - 1}}{(xy)^\lambda + 1} \frac{y^{\sigma(1 - \epsilon) - 1} f_p(x)}{dx} \int_0^\frac{1}{y} \left[ \frac{\ln xy}{(xy)^\lambda + 1} \right]^{\frac{1}{p}} f(x) dx \right]^{p - 1}$$

$$= \left( k_1(x) \right)^{p - 1} y^{(1 - \epsilon) - 1} \int_0^\frac{1}{y} \left[ \frac{\ln xy}{(xy)^\lambda + 1} \right]^{\frac{1}{p}} f_p(x) dx.$$

If (21) takes the form of equality for some $y \in (0, \infty)$, then (cf. [30]) there exist constants $A$ and $B$, such that they are not all zero and:

$$A \frac{y^{\sigma(1 - \epsilon) - 1} f_p(x)}{dx} = B \frac{x^{\sigma(1 - \epsilon) - 1}}{y^{\sigma(1 - \epsilon) - 1}} \text{ a.e. in } R_+.$$

We suppose that $A \neq 0$ (otherwise $B = A = 0$). It follows that:

$$x^{p(1 - \epsilon) - 1} f_p(x) = y^{\sigma(1 - \epsilon) - 1} B A x \text{ a.e. in } R_+,$$

which contradicts the fact that:

$$0 < \int_0^\infty x^{p(1 - \epsilon) - 1} f_p(x) dx < \infty.$$

Hence, (21) takes the form of strict inequality.

For $\sigma_1 = \sigma$, by (21) and Fubini’s theorem, we obtain:

$$J = \left( k_1(x) \right)^{\frac{1}{2}} \left[ \int_0^\infty \left[ \int_0^\frac{1}{y} \left[ \frac{\ln xy}{(xy)^\lambda + 1} \right]^{\frac{1}{p}} f_p(x) dx \right] dy \right]^{\frac{1}{2}}$$

$$= \left( k_1(x) \right)^{\frac{1}{2}} \left[ \int_0^\infty \left[ \int_0^\frac{1}{y} \left[ \frac{\ln xy}{(xy)^\lambda + 1} \right]^{\frac{1}{p}} f_p(x) dx \right]^{\frac{1}{p}} dy \right]^{\frac{1}{p}}$$

$$= \left( k_1(x) \right)^{\frac{1}{2}} \left[ \int_0^\infty \omega_1(x, y) x^{p(1 - \epsilon) - 1} f_p(x) dx \right]^{\frac{1}{2}}$$

$$= k_1(x) \left[ \int_0^\infty x^{p(1 - \epsilon) - 1} f_p(x) dx \right]^{\frac{1}{2}}.$$

Setting $M_1 = k_1(x)$, (17) follows.

Therefore, Condition (i), Condition (ii) and Condition (iii) are equivalent.
When Condition (iii) is satisfied, if there exists a constant \(M_1 \leq k_1(\sigma)\), such that (18) is valid, then by Lemma 1, we have \(M_1 \geq k_1(\sigma)\). Then, the constant factor \(M_1 = k_1(\sigma)\) in (18) is the best possible. The constant factor \(M_1 = \lambda - \mu\) in (17) is still the best possible. Otherwise, by (19) (for \(\sigma_1 = \sigma\)), we can conclude that the constant factor \(M_1 = k_1(\sigma)\) in (18) is not the best possible. 

Setting \(y = \frac{1}{\sigma}\), \(G(Y) = Y^{\lambda-2}g(Y)\), \(\mu_1 = \lambda - \sigma_1\) in Theorem 1, then replacing \(Y\) (resp. \(G(Y)\)) by \(y\) (resp. \(g(y)\)), we derive the following Corollary.

**Corollary 1.** If \(\beta > -1, \sigma, \lambda > 0\), then the following conditions are equivalent.

(i) There exists a constant \(M_1\), such that for any \(f(x) \geq 0\), satisfying:

\[
0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty
\]

we have the following Hardy-type inequality of the first kind with homogeneous kernel:

\[
\left\{ \int_0^1 y^{p-1} \left[ \int_0^1 \frac{|\ln(x/y)|^\beta}{x^\lambda + y^\lambda} f(x) dx \right]^p dy \right\}^{\frac{1}{p}} < M_1 \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{2}},
\]

(22)

(ii) There exists a constant \(M_1\), such that for any \(f(x), g(y) \geq 0\), satisfying:

\[
0 < \int_0^\infty x^{\sigma(1-\sigma)-1} f(x) dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{\mu(1-\mu)-1} g(y) dy < \infty,
\]

we have the following inequality:

\[
\int_0^\infty g(y) \left[ \int_0^y \frac{|\ln(x/y)|^\beta}{x^\lambda + y^\lambda} f(x) dx \right] dy < M_1 \left[ \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx \right]^{\frac{1}{2}} \left[ \int_0^\infty y^{\mu(1-\mu)-1} g^\mu(y) dy \right]^{\frac{1}{2}};
\]

(23)

(iii) \(\mu_1 = \mu\).

If Condition (iii) holds, then we have \(M_1 \geq k_1(\sigma)\), and the constant \(M_1 = k_1(\sigma)\) in (22) and (23) is the best possible.

Similarly, we obtain the following weight function:

\[
\omega_2(\sigma, y) = y^\sigma \int_0^\infty \frac{|\ln(xy)|^\beta x^{\sigma-1}}{(xy)^\lambda + 1} dx
\]

\[
= \int_1^\infty \frac{\ln^\beta u}{u^\lambda + 1} u^{\sigma-1} du = k_2(\sigma)(y > 0),
\]

and then in view of Lemma 2 and in a similar manner, we obtain the following theorem:

**Theorem 2.** If \(\beta > -1, 0 < \sigma = \lambda - \mu < \lambda\), then the following conditions are equivalent.

(i) There exists a constant \(M_2\), such that for any \(f(x) \geq 0\), satisfying:

\[
0 < \int_0^\infty x^{p(1-\sigma)-1} f^p(x) dx < \infty,
\]
we have the following Hardy-type inequality of the second kind with the nonhomogeneous kernel:

\[
\left\{ \int_0^\infty y^{p \nu - 1} \left[ \int_y^\infty \frac{\ln xy|\beta}{(xy)^\lambda + 1} f(x) \, dx \right]^p \, dy \right\}^{\frac{1}{p}} < M_2 \left[ \int_0^\infty x^{p(1-\sigma) - 1} f^p(x) \, dx \right]^{\frac{1}{p}}.
\]  

(24)

(ii) There exists a constant \(M_2\), such that for any \(f(x), g(y) \geq 0\), satisfying:

\[
0 < \int_0^\infty x^{p(1-\sigma) - 1} f^p(x) \, dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{p(1-\sigma) - 1} g^p(y) \, dy < \infty,
\]

we have the following inequality:

\[
\int_0^\infty g(y) \left[ \int_y^\infty \frac{\ln xy|\beta}{(xy)^\lambda + 1} f(x) \, dx \right] \, dy < M_2 \left[ \int_0^\infty x^{p(1-\sigma) - 1} f^p(x) \, dx \right]^{\frac{1}{p}} \left[ \int_0^\infty y^{p(1-\sigma) - 1} g^p(y) \, dy \right]^{\frac{1}{p}}.
\]  

(25)

(iii) \(\sigma_1 = \sigma\).

If Condition (iii) holds, then we have \(M_2 \geq k_2(\sigma)\), and the constant factor:

\[
M_2 = k_2(\sigma) = \frac{\Gamma(\beta + 1)}{\lambda^{p+1}} \left( \beta + 1, \frac{\mu}{\lambda} \right) = k_1(\mu)
\]

in (24) and (25) is the best possible.

Setting:

\[
y = \frac{1}{Y}, \quad G(Y) = \gamma^{\lambda - 2} \left( \frac{1}{Y} \right), \quad \mu_1 = \lambda - \sigma_1
\]

in Theorem 2, then replacing \(Y\) (resp. \(G(Y)\)) by \(y\) (resp. \(g(y)\)), we derive the following Corollary.

**Corollary 2.** If \(\beta > -1, 0 < \sigma = \lambda - \mu < \lambda\), then the following conditions are equivalent.

(i) There exists a constant \(M_2\), such that for any \(f(x) \geq 0\), satisfying:

\[
0 < \int_0^\infty x^{p(1-\sigma) - 1} f^p(x) \, dx < \infty,
\]

we have the following Hardy-type inequality of the second kind with homogeneous kernel:

\[
\left\{ \int_0^\infty y^{p \nu - 1} \left[ \int_y^\infty \frac{\ln(x/y)|\beta}{x^\lambda + y^\lambda} f(x) \, dx \right]^p \, dy \right\}^{\frac{1}{p}} < M_2 \left[ \int_0^\infty x^{p(1-\sigma) - 1} f^p(x) \, dx \right]^{\frac{1}{p}}.
\]  

(26)

(ii) There exists a constant \(M_2\), such that for any \(f(x), g(y) \geq 0\), satisfying

\[
0 < \int_0^\infty x^{p(1-\sigma) - 1} f^p(x) \, dx < \infty \quad \text{and} \quad 0 < \int_0^\infty y^{p(1-\mu_1) - 1} g^p(y) \, dy < \infty,
\]
we have the following inequality:

\[
\int_0^\infty g(y) \left[ \int_y^\infty \frac{|\ln(x/y)|^\beta}{x^\lambda + y^\lambda} f(x) \, dx \right] \, dy < M_2 \left[ \int_0^\infty x^{(1-\sigma)-1} f(x) \, dx \right]^\frac{\sigma}{\lambda} \left[ \int_0^\infty y^{(1-\mu)-1} g(y) \, dy \right]^\frac{\mu}{\lambda}. \tag{27}
\]

(iii) \( \mu_2 = \mu \).

If Condition (iii) holds, then we have \( M_2 \geq k_2(\sigma) \), and the constant \( M_2 = k_2(\sigma) = k_1(\mu) \) in (26) and (27) is the best possible.

4. Operator Expressions

For \( \sigma, \lambda > 0, \mu = \lambda - \sigma \), we set the following functions:

\[ \phi(x) := x^{p(1-\sigma)-1}, \quad \psi(y) := y^{q(1-\sigma)-1}, \quad \phi(y) := y^{q(1-\mu)-1}, \]

and:

\[ \psi^{1-p}(y) = y^{p\sigma-1}, \quad \phi^{1-p}(y) = y^{p\mu-1} \quad (x, y \in R_+). \]

Define the following real normed linear spaces:

\[ L_{p,\psi}(R_+) := \left\{ f : ||f||_{p,\psi} := \left( \int_0^\infty \phi(x)|f(x)|^p \, dx \right)^\frac{1}{p} < \infty \right\}, \]

\[ L_{q,\phi}(R_+) = \left\{ g : ||g||_{q,\phi} := \left( \int_0^\infty \psi(y)|g(y)|^q \, dy \right)^\frac{1}{q} < \infty \right\}, \]

\[ L_{q,\phi}(R_+) = \left\{ g : ||g||_{q,\phi} := \left( \int_0^\infty \phi(y)|g(y)|^q \, dy \right)^\frac{1}{q} < \infty \right\}, \]

\[ L_{p,q^{1-p}}(R_+) = \left\{ h : ||h||_{p,q^{1-p}} = \left( \int_0^\infty \psi^{1-p}(y)|h(y)|^p \, dy \right)^\frac{1}{p} < \infty \right\}, \]

\[ L_{q^{1-p},p}(R_+) = \left\{ h : ||h||_{q^{1-p},p} = \left( \int_0^\infty \phi^{1-p}(y)|h(y)|^q \, dy \right)^\frac{1}{q} < \infty \right\}. \]

(a) In view of Theorem 1 (setting \( c_1 = c \)), for \( f \in L_{p,\psi}(R_+), \) setting:

\[ h_1(y) := \int_0^y \frac{|\ln xy|}{(xy)^\lambda + 1} f(x) \, dx \quad (y \in R_+), \]

by (17), we have:

\[ ||h_1||_{p,q^{1-p}} = \left[ \int_0^\infty \psi^{1-p}(y)h_1^p(y) \, dy \right]^\frac{1}{p} < M_1 ||f||_{p,\psi} < \infty. \tag{28} \]

**Definition 1.** Define a Hardy-type integral operator of the first kind with the nonhomogeneous kernel:

\[ T_1^{(1)} : L_{p,\psi}(R_+) \to L_{p,q^{1-p}}(R_+) \]

as follows.

For any \( f \in L_{p,\psi}(R_+) \), there exists a unique representation:

\[ T_1^{(1)} f = h_1 \in L_{p,q^{1-p}}(R_+), \]
Theorem 3. For any \( f \in T \) and then the operator \( T_1^{(1)} \) is bounded satisfying
\[
\|T_1^{(1)} f\|_{p,\varphi^{1-p}} = \|h_1\|_{p,\varphi^{1-p}} \leq M_1 \|f\|_{p,\varphi},
\]
and then we can rewrite Theorem 1 as follows.

If we define the formal inner product of \( T_1^{(1)} f \) and \( g \) as follows:
\[
\langle T_1^{(1)} f, g \rangle := \int_0^\infty \left[ \int_0^1 \frac{\ln x y \varphi^\beta}{(x y)^{1+\lambda}} f(x) dx \right] g(y) dy,
\]
then we can rewrite Theorem 1 as follows.

**Theorem 3.** For \( \beta > -1, \sigma, \lambda > 0 \), the following conditions are equivalent.

(i) There exists a constant \( M_1 \), such that for any \( f(x) \geq 0, f \in L_{p,\varphi}(R_+) \), \( \|f\|_{p,\varphi} > 0 \), we have the following inequality:
\[
\|T_1^{(1)} f\|_{p,\varphi^{1-p}} < M_1 \|f\|_{p,\varphi}. \tag{29}
\]

(ii) There exists a constant \( M_2 \), such that for any \( f(x) \geq 0, f \in L_{p,\varphi}(R_+), g \in L_{q,\varphi}(R_+) \), \( \|f\|_{p,\varphi}, \|g\|_{q,\varphi} > 0 \), we have the following inequality:
\[
\langle T_1^{(1)} f, g \rangle < M_1 \|f\|_{p,\varphi} \|g\|_{q,\varphi}. \tag{30}
\]

We also have that \( \|T_1^{(1)} \| = k_1(\sigma) \leq M_1 \).

(b) In view of Corollary 1 (setting \( \mu_1 = \mu \)), for \( f \in L_{p,\varphi}(R_+) \), considering the function:
\[
h_2(y) := \int_0^y \frac{\ln(x/y)^\beta}{x^\lambda + y^\lambda} f(x) dx \ (y \in R_+),
\]
by (22), we have:
\[
\|h_2\|_{p,\varphi^{1-p}} = \left[ \int_0^\infty \Phi^{1-p}(y) h_2^\varphi(y) dy \right]^{\frac{p}{1-p}} < M_1 \|f\|_{p,\varphi} < \infty. \tag{31}
\]

**Definition 2.** Define a Hardy-type integral operator of the first kind with the homogeneous kernel:
\[
T_1^{(2)} : L_{p,\varphi}(R_+) \to L_{p,\varphi^{1-p}}(R_+)
\]
as follows.

For any \( f \in L_{p,\varphi}(R) \), there exists a unique representation:
\[
T_1^{(2)} f = h_2 \in L_{p,\varphi^{1-p}}(R_+),
\]
satisfying \( T_1^{(2)} f(y) = h_2(y) \), for any \( y \in R_+ \).

In view of (31), it follows that:
\[ ||T_1^{(2)}f||_{p,\phi^{1-p}} = ||h_2||_{p,\phi^{1-p}} \leq M_1 ||f||_{p,\phi}, \]

and then the operator \( T_1^{(2)} \) is bounded satisfying:

\[ ||T_1^{(2)}|| = \sup_{f(\neq 0) \in L_{p,\phi}(\mathbb{R}_+)} \frac{||T_1^{(2)}f||_{p,\phi^{1-p}}}{||f||_{p,\phi}} \leq M_1. \]

If we define the formal inner product of \( T_1^{(2)}f \) and \( g \) as follows:

\[(T_1^{(2)}f, g) := \int_{0}^{\infty} \int_{0}^{\infty} \frac{|\ln(x/y)|^\beta}{x^\lambda + y^\lambda} f(x)dx \ g(y)dy,\]

then we can rewrite Corollary 1 as follows.

**Corollary 3.** For \( \beta > -1, \sigma, \lambda > 0 \), the following conditions are equivalent.

(i) There exists a constant \( M_1 \), such that for any \( f(x) \geq 0, f \in L_{p,\phi}(\mathbb{R}_+) \), \( ||f||_{p,\phi} > 0 \), we have the following inequality:

\[ ||T_1^{(2)}f||_{p,\phi^{1-p}} < M_1 ||f||_{p,\phi}. \] (32)

(ii) There exists a constant \( M_1 \), such that for any \( f(x), g(y) \geq 0, f \in L_{p,\phi}(\mathbb{R}_+), g \in L_{q,\phi}(\mathbb{R}_+) \), \( ||f||_{p,\phi}, ||g||_{q,\phi} > 0 \), we have the following inequality:

\[ (T_1^{(2)}f, g) < M_1 ||f||_{p,\phi} ||g||_{q,\phi}. \] (33)

We still have \( ||T_1^{(2)}|| = k_1(\sigma) \leq M_1 \).

(c) In view of Theorem 2 (setting \( \sigma_1 = \sigma \)), for \( f \in L_{p,\phi}(\mathbb{R}_+) \), considering the function:

\[ H_1(y) := \int_{\frac{y}{2}}^{\infty} \frac{|\ln(xy)|^\beta}{(xy)^\lambda + 1} f(x)dx \ (y \in \mathbb{R}_+), \]

by (24), we have:

\[ ||H_1||_{p,\phi^{1-p}} = \left[ \int_{0}^{\infty} \phi^{1-p}(y)H_1^p(y)dy \right]^{\frac{1}{p}} < M_2 ||f||_{p,\phi} < \infty. \] (34)

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**Definition 3.** Define a Hardy-type integral operator of the second kind with the nonhomogeneous kernel:

\[ T_2^{(1)} : L_{p,\phi}(\mathbb{R}_+) \to L_{p,\phi^{1-p}}(\mathbb{R}_+) \]

as follows.

For any \( f \in L_{p,\phi}(\mathbb{R}_+) \), there exists a unique representation:

\[ T_2^{(1)}f = H_1 \in L_{p,\phi^{1-p}}(\mathbb{R}_+), \]

satisfying \( T_2^{(1)}f(y) = H_1(y) \), for any \( y \in \mathbb{R}_+ \).

In view of (34), it follows that:

\[ ||T_2^{(1)}f||_{p,\phi^{1-p}} = ||H_1||_{p,\phi^{1-p}} \leq M_2 ||f||_{p,\phi}, \]
Theorem 4. Define a Hardy-type integral operator of the second kind with the homogeneous kernel:

\[
||T_2^{(1)} f|| = \sup_{f \neq 0} \frac{||T_2^{(1)} f||_{p, \varphi^{1-p}}}{||f||_{p, \varphi}} \leq M_2.
\]

If we define the formal inner product of \(T_2^{(1)} f\) and \(g\) as follows.

\[
(T_2^{(1)} f, g) := \int_0^\infty \left[ \int_0^\infty \frac{(\ln xy)^\beta}{(xy)^\lambda + 1} f(x) dx \right] g(y) dy,
\]

then we can rewrite Theorem 2 as follows.

**Theorem 4.** For \(\beta > -1, 0 < \sigma = \lambda - \mu < \lambda\), the following conditions are equivalent.

(i) There exists a constant \(M_2\), such that for any \(f(x) \geq 0, f \in L_{p, \varphi}(R_+)\), \(||f||_{p, \varphi} > 0\), we have the following inequality:

\[
||T_2^{(1)} f||_{p, \varphi^{1-p}} < M_2 ||f||_{p, \varphi}.
\] (35)

(ii) There exists a constant \(M_2\), such that for any \(f(x), g(y) \geq 0, f \in L_{p, \varphi}(R_+), g \in L_{q, \psi}(R_+), \)

\(||f||_{p, \varphi}, ||g||_{q, \psi} > 0\), we have the following inequality:

\[
(T_2^{(1)} f, g) < M_2 ||f||_{p, \varphi} ||g||_{q, \psi}.
\] (36)

We still have \(||T_2^{(1)} f|| = k_2(\sigma) \leq M_2\).

(d) In view of Corollary 2 (setting \(\mu_1 = \mu\), for \(f \in L_{p, \varphi}(R_+)\), considering the function:

\[
H_2(y) := \int_y^\infty \frac{\ln(x/y)^\beta}{x^\lambda + y^\lambda} f(x) dx \ (y \in R_+),
\]

by \(36\), we have:

\[
||H_2||_{p, \varphi^{1-p}} = \left[ \int_0^\infty \phi^{1-p}(y) H_2^p(y) dy \right]^{\frac{1}{p}} < M_2 ||f||_{p, \varphi} < \infty.
\] (37)

**Definition 4.** Define a Hardy-type integral operator of the second kind with the homogeneous kernel:

\[
T_2^{(2)} : L_{p, \varphi}(R_+) \to L_{p, \varphi^{1-p}}(R_+)
\]

as follows.

For any \(f \in L_{p, \varphi}(R)\), there exists a unique representation:

\[
T_2^{(2)} f = H_2 \in L_{p, \varphi^{1-p}}(R_+),
\]

satisfying \(T_2^{(2)} f(y) = H_2(y)\), for any \(y \in R_+\).

In view of \(37\), it follows that:

\[
||T_2^{(2)} f||_{p, \varphi^{1-p}} = ||H_2||_{p, \varphi^{1-p}} \leq M_2 ||f||_{p, \varphi}.
\]
and then the operator $T_2^{(2)}$ is bounded satisfying

$$
\|T_2^{(2)}\| = \sup_{f(\neq 0) \in L_p, q} \frac{\|T_2^{(2)} f\|_{p, q}}{\|f\|_{p, q}} \leq M_2.
$$

If we define the formal inner product of $T_2^{(2)} f$ and $g$ as follows:

$$(T_2^{(2)} f, g) := \int_0^\infty \left[ \int_y^\infty \frac{\ln(x/y)}{x^{\lambda + y^{\lambda}} f(x)dx} \right] g(y)dy,$$

then we can rewrite Corollary 2 as follows.

**Corollary 4.** For $\beta > -1, 0 < \sigma = \lambda - \mu < \lambda$, the following conditions are equivalent.

(i) There exists a constant $M_2$, such that for any $f(x) \geq 0, f \in L_{p, q}(\mathbb{R}_+), \|f\|_{p, q} > 0$, we have the following inequality:

$$
\|T_2^{(2)} f\|_{p, q}^{1-\sigma} \leq M_2 \|f\|_{p, q}. \quad (38)
$$

(ii) There exists a constant $M_2$, such that for any $f(x), g(y) \geq 0, f \in L_{p, q}(\mathbb{R}_+), g \in L_{q, \phi}(\mathbb{R}_+), \|f\|_{p, q}, \|g\|_{q, \phi} > 0$, we have the following inequality:

$$
(T_2^{(2)} f, g) < M_2 \|f\|_{p, q} \|g\|_{q, \phi}. \quad (39)
$$

We still have $\|T_2^{(2)}\| = k_2(\sigma) = k_1(\mu) \leq M_2$.

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