GEOMETRY OF ENTANGLED STATES, BLOCH SPHERES AND HOPF FIBRATIONS

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ABSTRACT. We discuss a generalization to 2 qubits of the standard Bloch sphere representation for a single qubit, in the framework of Hopf fibrations of high dimensional spheres by lower dimensional spheres. The single qubit Hilbert space is the 3-dimensional sphere $S^3$. The $S^2$ base space of a suitably oriented $S^3$ Hopf fibration is nothing but the Bloch sphere, while the circular fibres represent the qubit overall phase degree of freedom. For the two qubits case, the Hilbert space is a 7-dimensional sphere $S^7$, which also allows for a Hopf fibration, with $S^3$ fibres and a $S^4$ base. A main striking result is that suitably oriented $S^7$ Hopf fibrations are entanglement sensitive. The relation with the standard Schmidt decomposition is also discussed.

1. INTRODUCTION

The interest in two-level systems, or coupled two-level systems, is as old as quantum mechanics itself, with the analysis of the electron spin sector in the helium atom. The ubiquitous two-level systems have gained a renewed interest in the past ten years, owing to the fascinating perspectives in quantum manipulation of information and quantum computation. These two-level quantum systems are now called qubits, grouped and coupled into q-registers, and manipulated by sophisticated means. It is of interest to describe their quantum evolution in a suitable representation space, in order to get some insight into the subtleties of this complicated problem. A well known tool in quantum optics is the Bloch sphere representation, where the simple qubit state is faithfully represented, up to its overall phase, by a point on a standard sphere $S^2$, whose coordinates are expectation values of physically interesting operators for the given quantum state. Our aim here is to build an adequate representation space for bi-partite systems. We are guided by the relation of the standard Bloch sphere to a geometric object called the Hopf fibration of the $S^3$ hypersphere (identified to the spin $\frac{1}{2}$ Hilbert space). Since the two qubits Hilbert space is the 7-dimensional sphere $S^7$, which also allows for a Hopf fibration, it is tempting to mimic the Bloch sphere representation in that case. An a priori unexpected result is that the $S^7$ Hopf fibration is entanglement sensitive and therefore provides a kind of "stratification" for the 2 qubits states with respect to their entanglement content.

This paper is organised as follows. We first briefly recall known facts about the Bloch sphere representation and its relation to the $S^3$ Hopf fibration. We then proceed to the 2-qubits states and recall what entanglement and Schmidt decomposition consist in. The $S^7$ Hopf fibration is then introduced and related to the 2-qubits Hilbert space. Although several papers have appeared (especially in the recent period) which aimed at a geometrical analysis of qubits Hilbert space, we are not aware of an alternative use of the $S^7$ Hopf fibration in that context. As far
as computation is concerned, going from the $S^3$ to the $S^7$ fibration merely amounts to replacing complex numbers by quaternions. This is why we give in appendix a brief introduction to quaternion numbers. Note that using quaternions is not strictly necessary here, but they provide an elegant way to put the calculations into a compact form, and have (by nature) an easy geometrical interpretation.

2. Single qubit, Bloch sphere and the $S^3$ Hopf Fibration

2.1. The Bloch sphere representation. A (single) qubit state reads

\[ |\Psi\rangle = \alpha |0\rangle + \beta |1\rangle, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1 \quad (1) \]

In the spin $\frac{1}{2}$ context, the orthonormal basis \{0, 1\} is composed of the two eigenvectors of the (say) $\sigma_z$ (Pauli spin) operator. A convenient way to represent $|\Psi\rangle$ (up to a global phase) is then provided by the Bloch sphere. The set of states $\exp i\varphi |\Psi\rangle (\varphi \in [0, 2\pi])$ is mapped onto a point on $S^2$ (the usual sphere in $\mathbb{R}^3$) with coordinates $(X, Y, Z)$

\[
X = \langle \sigma_x \rangle_{\Psi} = 2 \Re(\overline{\alpha} \beta) \\
Y = \langle \sigma_y \rangle_{\Psi} = 2 \Im(\overline{\alpha} \beta) \\
Z = \langle \sigma_z \rangle_{\Psi} = |\alpha|^2 - |\beta|^2
\]

where $\overline{\alpha}$ is the complex conjugate of $\alpha$. Recall also the relation between Bloch sphere coordinates and the pure state density matrix $\rho_{|\Psi\rangle}$:

\[
\rho_{|\Psi\rangle} = \rho_{\exp i\varphi |\Psi\rangle} = |\Psi\rangle \langle \Psi | = \frac{1}{2} \begin{pmatrix}
1 + Z & X - iY \\
X + iY & 1 - Z
\end{pmatrix} \quad (3)
\]

For mixed states, the density matrices are in one-to-one correspondence with points in the Bloch "ball", the interior of the pure state Bloch sphere.

The single qubit Hilbert space is the unit sphere $S^3$ embedded in $\mathbb{R}^4$. Indeed, writing $\alpha = x_1 + ix_2$, and $\beta = x_3 + ix_4$, the state normalisation condition translates into $\sum x_i^2 = 1$ defining the $S^3$ sphere. In this space, the set of states $\exp i\varphi |\Psi\rangle$ is a circle parametrized by $\varphi$. The projective Hilbert space is such that all states differing by a global phase are identified, and therefore corresponds to the above Bloch sphere. It can also be identified to the Hopf fibration basis as follows now.

2.2. The $S^3$ Hopf fibration. Briefly said, a space is fibred if it has a sub-space (the fibre) which can be shifted by a displacement, so that any point of the space belongs to one, and only one, fibre. For example, the euclidean space $\mathbb{R}^3$ can be seen as a fibre bundle of parallel straight lines, but also of parallel planes (the fibre need not to be one-dimensional). More precisely, a fibred space $E$ is defined by a (many-to-one) map from $E$ to the so-called "base space" $B$, all points of a given fibre $F$ being mapped onto a single base point. In the preceeding $\mathbb{R}^3$ example, the base is either a plane cutting the whole set of parallel lines, or, in the second case, a line cutting the set of parallel planes. We are facing here "trivial fibrations", in the sense that the base $B$ is embedded in the fibred space $E$, the latter being faithfully described as the direct product of the base and the fibre: $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ or $\mathbb{R}^3 = R \times R^2$.

The simplest, and most famous, example of a non trivial fibration is the Hopf fibration of $S^3$ by great circles $S^1$ and base space $S^2$. For the qubit Hilbert space purpose, the fibre represents the global phase degree of freedom, and the base $S^2$
is identified as the Bloch sphere. The standard notation for a fibred space is that of a map $E \rightarrow B$, which here reads $S^3 \rightarrow S^2$. Its non trivial character implies that $S^3 \neq S^2 \times S^1$. This non trivial character translates here into the known failure in ascribing consistently a definite phase to each representing point on the Bloch sphere.

To describe this fibration in an analytical form, we go back to the definition of $S^3$ as pairs of complex numbers $(\alpha, \beta)$ which satisfy $|\alpha|^2 + |\beta|^2 = 1$. The Hopf map is defined as the composition of a map $h_1$ from $S^3$ to $R^2 (+\infty)$, followed by an inverse stereographic map from $R^2$ to $S^2$:

$$h_1 : S^3 \rightarrow R^2 + \{\infty\} \quad (\alpha, \beta) \rightarrow \alpha \beta^{-1} \quad \alpha, \beta \in \mathbb{C}$$

$$h_2 : R^2 + \{\infty\} \rightarrow S^2 \rightarrow M(X,Y,Z) \quad X^2 + Y^2 + Z^2 = 1$$

The first map $h_1$ clearly shows that the full $S^3$ great circle, parametrized by $(\alpha \exp i\varphi, \beta \exp i\varphi)$ is mapped on the same single point with complex coordinate $C$. Note that the complex conjugation, in the above definition for the Hopf map $h_1$, is not necessary to represent a great circle fibration. It is used here on purpose to get an exact one-to-one relation with the above Bloch sphere coordinates, and to prepare the generalization to higher dimension discussed in the next paragraph. It is indeed a simple exercise to show that, with $R^2$ cutting the unit radius $S^2$ along the equator, and the north pole as the stereographic projection pole, the $S^2$ Hopf fibration base coordinates coincide with the above $S^2$ Bloch sphere coordinates.

Although there is nothing really new in this correspondence [3], it is probably poorly known in both communities (quantum optics and geometry). It is striking that the simplest non trivial object of quantum physics, the two-level system, bears such an intimate relation with the simplest non trivial fibred space.

It is tempting to try to visualize the full ($S^3$) Hilbert space with its fibre structure. This can be achieved by doing a (direct) stereographic map from $S^3$ to $R^3$ (nice pictures can be found in [3,4]). Each $S^3$ circular fibre is mapped onto a circle in $R^3$, with an exceptional straight line, image of the unique $S^3$ great circle passing through the projection pole. The great circle arrangement is intricate, since each circular fibre threads all others in $S^3$.

### 3. Two qubits, entanglement and the $S^7$ Hopf fibration

#### 3.1. The two-qubit Hilbert space

We now proceed one step further, and investigate pure states for two qubits. The Hilbert space $E$ for the compound system is the tensor product of the individual Hilbert spaces $E_1 \otimes E_2$, with a direct product basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. A two qubit pure state reads

$$|\Psi\rangle = \alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle + \delta |11\rangle$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, and $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$.

$|\Psi\rangle$ is said "separable" if, at the price of possible basis changes in $E_1$ and $E_2$ separately, it can be written as a single product. As is well known, the separability condition reads: $\alpha \delta = \beta \gamma$. A generic state is not separable, and is said to be "entangled". The Schmidt decomposition [3] implies that a suitable basis exists...
such that $|\Psi\rangle$ can be written as a sum of only two (bi-orthonormal) terms with real positive coefficients. Note that the apparently obvious decomposition

$$|\Psi\rangle = \cos \Omega (|0\rangle_1 \otimes |u_2\rangle_2) + \sin \Omega (|1\rangle_1 \otimes |v_2\rangle_2)$$

(6) with $|u_2\rangle_2 = \frac{1}{\cos \Omega} (\alpha |0\rangle_2 + \beta |1\rangle_2)$, $|v_2\rangle_2 = \frac{1}{\sin \Omega} (\gamma |0\rangle_2 + \delta |1\rangle_2)$,

$$\cos \Omega = \left( |\alpha|^2 + |\beta|^2 \right)^{\frac{1}{2}} \quad \text{and} \quad \sin \Omega = \left( |\gamma|^2 + |\delta|^2 \right)^{\frac{1}{2}}, \quad \Omega \in [0, \pi/2]$$

is generically not a Schmidt decomposition since $\langle u_2 | v_2 \rangle$ need not vanish, a point on which we return below.

The $|\Psi\rangle$ normalization condition $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$ identifies $E$ to the 7-dimensional sphere $S^7$, embedded in $R^8$. It is therefore tempting to see whether the known $S^7$ Hopf fibration (with fibres $S^3$ and base $S^4$) can play any role in the Hilbert space description.

3.2. The $S^7$ Hopf fibration. The simplest way to introduce this fibration is to proceed along the same line as for the $S^3$ case, but using quaternions instead of complex numbers (see appendix). We write

$$|\psi\rangle = q_1 + \beta \mathbf{j} q_2 = \gamma + \delta \mathbf{j}, \quad q_1, q_2 \in \mathbb{Q},$$

and a point (representing the state $|\psi\rangle$) on the unit radius $S^7$ as a pair of quaternions $(q_1, q_2)$ satisfying $|q_1|^2 + |q_2|^2 = 1$. The Hopf map from $S^7$ to the base $S^4$ is the composition of a map $h_1$ from $S^7$ to $R^4 + \{\infty\}$, followed by an inverse stereographic map from $R^4$ to $S^4$:

$$h_1 : (q_1, q_2) \rightarrow Q = \bar{q}_1 q_2, \quad q_1, q_2 \in \mathbb{Q}$$

(8)

$$h_2 : R^4 + \{\infty\} \rightarrow S^4 \quad Q \rightarrow M(x_i) \quad \sum_{i=4}^{l=0} x_i^2 = 1$$

This Hopf fibration therefore reads $S^7 \rightarrow S^3$. The base space $S^4$ is not embedded in $S^7$: the fibration is again not trivial. The fibre is a unit $S^3$ sphere as can easily be seen by noting that the $S^7$ points $(q_1, q_2)$ and $(q_1 q, q_2 q)$, with $q$ a unit quaternion (geometrically a $S^3$ sphere) are mapped onto the same $Q$ value. A main difference with the previous $S^3$ Hopf fibration comes from the non commutative nature of the quaternion product. The Hopf map $h_1'$, defined by $Q = \bar{q}_2^{-1} q_1$ is distinct from $h_1$, while still corresponding to a fibration $S^7 \rightarrow S^4$. One must stress that, due to the tensor product nature of the two-qubits Hilbert space, the latter inherits a form of "anisotropy" (in $S^7$) which translates into specific choices for the definition of the pair $(q_1, q_2)$, as will be discussed below.

Let us now look closer to the base space $S^4$, by working out explicitly the Hopf map. The $h_1$ map leads to

$$Q = \bar{q}_2^{-1} = \frac{1}{\sin^2 \Omega} \left[ (\alpha + \beta \mathbf{j})(\gamma - \delta \mathbf{j}) \right], \quad \frac{1}{\sin^2 \Omega} (C_1 + C_2 \mathbf{j})$$

with $C_1 = (\alpha \gamma + \beta \delta), \quad C_2 = (\alpha \delta - \beta \gamma)$ and $C_1, C_2 \in \mathbb{C}$.

Now comes a first striking result: the Hopf map is entanglement sensitive! Indeed, separable states satisfy $\alpha \delta = \beta \gamma$ and therefore map onto the subset of pure complex numbers in the quaternion field (both being completed by $\infty$ when
Geometrically, this means that non-entangled states map from $S^7$ onto a 2-dimensional planar subspace of the target space $R^4$. Note however that the latter property heavily depends on the chosen analytical form of the $h_1$ map (in other words, on the "orientation" of the fibres with respect to $S^7$). A second interesting point concerns states whose Hopf map is confined in the 2-dimensional subspace orthogonal to the previous one (such that $C_1 = 0$). This plane gathers the images of states with the above trivial Schmidt decomposition: indeed, $C_1 = 0$ implies $\langle u_2 | v_2 \rangle = 0$.

The second map $h_2$ sends states onto points on $S^4$, with coordinates $x_l$, with $l$ running from to 0 to 4. With the inverse stereographic pole located on the $S^4$ "north" pole ($x_0 = +1$), and the target space $R^4$ cutting $S^4$ along the equator, we get the following coordinate expressions

\begin{align}
    x_0 &= \cos 2\Omega = |q_1|^2 - |q_2|^2 \\
    x_1 &= \sin 2\Omega \ S(Q') = 2 \text{Re} (\overline{\alpha \gamma} + \overline{\beta \delta}) \\
    x_2 &= \sin 2\Omega \ V_1(Q') = 2 \text{Im} (\overline{\alpha \gamma} + \overline{\beta \delta}) \\
    x_3 &= \sin 2\Omega \ V_2(Q') = 2 \text{Re} (\overline{\alpha \delta} - \overline{\beta \gamma}) \\
    x_4 &= \sin 2\Omega \ V_3(Q') = 2 \text{Im} (\overline{\alpha \delta} - \overline{\beta \gamma})
\end{align}

with $\sin 2\Omega = 2 |q_1| |q_2|$. $Q'$ is the normalized image of the $h_1$ map ($Q' = Q/|Q|$), $S(Q')$ and $V_{1,2,3}(Q')$ being respectively the scalar and vectorial parts of $Q'$ (see appendix). We recalled above that the $X, Y$ and $Z$ Bloch sphere coordinates are expectation values of the spin operators $\sigma$ in the one-qubit state $|\Psi\rangle$ (eq.1). It happens that the $x_l$ coordinates are also expectation values of simple operators in the two-qubits state (eq.5)

An obvious one is $x_0$ which corresponds to $\langle \sigma_2 \otimes \text{Id} \rangle_{\Psi}$. Indeed,

\begin{align}
    \langle \sigma_2 \otimes \text{Id} \rangle_{\Psi} &= (\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \\
    &= |q_1|^2 - |q_2|^2 = \cos 2\Omega = x_0
\end{align}

The two next coordinates are also easily recovered as

\begin{align}
    x_1 &= \langle \sigma_x \otimes \text{Id} \rangle_{\Psi} = (\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}) \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = 2 \text{Re} (\overline{\alpha \gamma} + \overline{\beta \delta}) \\
    x_2 &= \langle \sigma_y \otimes \text{Id} \rangle_{\Psi} = (\overline{\alpha}, \overline{\beta}, \overline{\gamma}, \overline{\delta}) \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} = 2 \text{Im} (\overline{\alpha \gamma} + \overline{\beta \delta})
\end{align}

The remaining two coordinates, $x_3$ and $x_4$, are also recovered as expectation values of an operator acting on $E$, but in a more subtle way. Define $J$ as the (anti-linear) "conjugator", an operator which takes the complex conjugate of all complex numbers involved in an expression (here acting on the left in the scalar product
below). Form then the antilinear operator \( \mathbf{E} \) (for "entanglor"): \( \mathbf{E} = -\mathbf{J}(\sigma_y \otimes \sigma_y). \) One finds
\[
\begin{align*}
x_3 &= \Re(\mathbf{E})_\Psi \\
x_4 &= \Im(\mathbf{E})_\Psi.
\end{align*}
\]

Note that \( (\mathbf{E})_\Psi \) vanishes for non entangled states, and takes its maximal norm (equals to 1) for maximally entangled states, hence the name "entanglor". Such an operator, which is nothing but the time reversal operator for two spins \( \mathbf{J}/2 \), has been already largely used in quantifying entanglement[3], with a quantity \( c \), called the "concurrence", which here equals \( c = 2 |\mathbf{E}| \).

3.3. Discussion. Let us now see what has been gained in fibrating the \( \mathcal{E} \) two-qubit Hilbert space. It is clear that, at some point, we will need to focus on the "projective" Hilbert space, taking into account the global phase freedom. Is it nevertheless interesting to first stay at the Hilbert space level, and foliate the base "projective" Hilbert space, taking into account the global phase freedom. Is it is expected to be the product of two 2-dimensional spheres \( S^2_A \times S^2_B \), each sphere being the Bloch sphere associated with the given qubit. This property is clearly displayed here. The unit \( S^4 \) base space reduces in a unit \( S^2 \) sphere (since \( x_3 = x_4 = 0 \)) which is nothing but the Bloch sphere for the first qubit, as seen on the coordinate expressions,
\[ \langle \sigma_{x,y} \otimes I \rangle_Q, \text{ for } x_0, x_1 \text{ and } x_2 \text{ respectively. This } S^2 \text{ sphere is the aggregate of parallel circles, of radius } \sin 2\Omega \text{ in the plane } (x_1, x_2), \text{ each one on a parallel } S^3_\Omega \text{ sphere, and with the extremal poles corresponding to } \Omega = 0 \text{ and } \Omega = \pi/2. \]

For the second qubit Bloch sphere, one needs to define a coordinate system on the fibres. We choose the two orthogonal states \( |0\rangle_Q \) and \( |1\rangle_Q \) respectively corresponding to \( q = 1 \) and \( q = j \) in the above expression for \( \Psi_Q \). They read, in the initial \( \mathcal{E} \) product state basis

\[ \begin{align*}
|0\rangle_Q &= (\cos \Omega \exp(-i\theta/2) |0\rangle_1 + \sin \Omega \exp(i\theta/2) |1\rangle_1) \otimes |0\rangle_2 \\
|1\rangle_Q &= (\cos \Omega \exp(-i\theta/2) |0\rangle_1 + \sin \Omega \exp(i\theta/2) |1\rangle_1) \otimes |1\rangle_2
\end{align*} \]

A generic state \( \Psi_Q \) in the \( S^3 \) fibre reads \( |\Psi_Q\rangle = a |0\rangle_Q + b |1\rangle_Q \), with \( a, b \in \mathbb{C} \), \( |a|^2 + |b|^2 = 1 \). We can now iterate the fibration process on the fibre itself and get the (Hopf fibration base)-(Bloch sphere) coordinates for this two-level system. It is now a simple exercise to verify that the obtained 3 real coordinates are precisely \( \langle i \sigma_{x,y,z} \rangle_Q \), that is to say the second qubit Bloch sphere.

In summary, for non entangled qubits, the \( S^7 \) Hopf fibration, with base \( S^4 \) and fibre \( S^3 \), simplifies to the simple product of a \( S^2 \) sub-sphere of the base (the first qubit Bloch sphere) by a second \( S^2 \) (the second qubit Bloch sphere) obtained as the base of a \( S^3 \) Hopf fibration applied to the fibre itself. Let us stress that this last iterated fibration is necessary to take into account the global phase of the two qubit system.

The fact that these two \( S^2 \) spheres play a symmetrical role (although one is related to the base and the other to the fibre) can be understood in the following way. We grouped together \( \alpha \) and \( \beta \) on one hand, and \( \gamma \) and \( \delta \) on the other hand, to form the quaternions \( q_1 \) and \( q_2 \), and then define the Hopf map \( h_1 \) as the ratio of these two quaternions (plus a complex conjugation). Had we grouped \( \alpha \) and \( \gamma \), and \( \beta \) and \( \delta \), to form two new quaternions, and use the same definition for the Hopf map, we would also get a \( S^7 \) Hopf fibration, but differently oriented. We let as an exercise to compute the base and fibre coordinates in that case. The net effect is to interchange the role of the two qubits: the second qubit Bloch sphere is now part of the \( S^3 \) base, while the first qubit Bloch sphere is obtained from the \( S^3 \) fibre.

3.3.2. \textit{Entangled states.} As expected, the entangled case is more complicated. From an analytical point of view, this is related to the fact that the pure imaginary quaternion \( t \) does not reduce to the standard \( i \) imaginary unit. Let us focus first on the maximally entangled states. They correspond to the complex number \( C_2 \) having maximal norm 1/2. This in turn implies that the Hopf map base reduces to a unit circle in the plane \( (x_3, x_4) \), parametrized by the unit complex number \( 2C_2 \).

The projective Hilbert space for these M.E.S. is known to be \( S^3/Z_2 \), a \( S^3 \) sphere with opposite points identified \( \Theta \) (this is linked to the fact that all M.E.S. can be related by a local operation on one sub-system, since \( S^3/Z_2 = SO(3) \)). In order to recover this result in the present framework, one can follow the trajectory of a representative point on the base and on the fibre while the state is multiplied by an overall phase \( \exp(i\varphi) \). The expression for \( C_2 = \alpha \delta - \beta \gamma \) shows that the point on the base turns by twice the angle \( \varphi \). Only when \( \varphi = \pi \) does the corresponding state belongs to the same fiber (e.g. maps onto the same value on the base). The fact that the fibre is a \( S^3 \) sphere, and this two-to-one correspondence between the fibre and the base under a global phase change, explains the \( S^3/Z_2 \) topology for the M.E.S. projective Hilbert space.
In that case, \( \Psi_Q \) reads

\[
\frac{1}{\sqrt{2}} \left( \exp \left( -\pi C_2 / 2 \right) q, \exp \left( \pi C_2 / 2 \right) q \right).
\]

The standard four Bell states are recovered with \( q = (1 \pm j) / \sqrt{2} \) and \( C_2 = \pm \frac{1}{2} \).

Now, it is clear that the previous discussion can be repeated for any \( \Omega \). M.E.S., which have therefore the same \( S^3 / Z_2 \) topology, and reads

\[
\left( \cos \Omega \exp \left( \frac{\pi C_2}{4 |C_2|^j} \right) q, \sin \Omega \exp \left( \frac{\pi C_2}{4 |C_2|^j} \right) q \right).
\]

Finally, we consider generic two-qubit states, which are neither separable nor maximally entangled. For each value of \( \Omega \) (e.g., of \( \langle \sigma_z \otimes \text{Id} \rangle \psi \)), these states are mapped (through the \( h_1 \) map) onto a torus of the \( S^3 \) torus foliation. Such a torus reduces to a circle of radius \( 2 |C_1| \) for the projective version. To each point of these circles is again associated a \( S^3 / Z_2 \) manifold in the fibre part.

### 3.3.3. A tentative generalization of the Bloch sphere representation.

We are now led to consider a tentative (and partly well known, see below) generalization of the Bloch sphere for the two-qubit projective Hilbert space. Clearly, the present Hopf fibration description suggests a splitting of the representation space in a product of base and fibres sub-spaces. Of the base space \( S^4 \), we propose to only keep the first three coordinates

\[
(x_0, x_1, x_2) = (\langle \sigma_z \otimes \text{Id} \rangle \psi, \langle \sigma_x \otimes \text{Id} \rangle \psi, \langle \sigma_y \otimes \text{Id} \rangle \psi)
\]

All states map into a standard ball \( B^3 \) of radius 1, where the set of separable states forms the \( S^2 \) boundary (the usual first qubit Bloch sphere), and the centre corresponds to maximally entangled states. Concentric spherical shells around the centre correspond to states of equal concurrence \( c \) (maximal at the centre, zero on the surface), the radius of the spherical shell being equal to \( \sqrt{1 - c^2} \). The idea of slicing the 2-qubit Hilbert space into manifolds of equal concurrence is not new \[2\]. What is nice here is that, under the Hopf map (and a projection onto the 3d subspace of the base spanned by the first three coordinates), these manifolds transforms into concentric \( S^2 \) shells which fill the unit ball. We must now describe which subset of the \( S^3 \) fibres has to attached to each point in this base-related unit ball. Looking to the previous discussions, it is clear that it is a \( S^3 / Z_2 \) space for each interior point, and a \( S^3 \) space (the second qubit Bloch sphere) for the boundary of the unit ball.

The Bloch ball single qubit mixed state representation was recalled above. The centre of the Bloch ball corresponds to maximally mixed states. The reader might be surprised to find here (in the two-qubit case) a second unit radius ball, with maximally entangled states at the centre. There is no mystery here, and it corresponds to a known relation between partially traced two-qubit pure states and one-qubit mixed state. Indeed the partially traced density matrix \( \rho_1 \) is in fact deeply related to the \( S^7 \) Hopf map:

\[
\rho_1 = \begin{pmatrix}
|q_1|^2 & C_1 \\
C_1^* & |q_2|^2
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 + x_0 & x_1 - i x_2 \\
x_1 + i x_2 & 1 - x_0
\end{pmatrix}
\]

with unit trace and \( \det \rho_1 = |C_2|^2 \). The partial \( \rho_1 \) represents a pure state density matrix whenever \( C_2 \) vanishes (the separable case), and allows for a unit Bloch
sphere (that associated to the first qubit). It is assimilated to a mixed state density matrix as soon as $|C_2| > 0$ (and an entangled state for the two qubit state). The other partially traced density matrix $\rho_2$ is related to the other $S^7$ Hopf fibration which was discussed above. With all this in mind, one also expect a simple generic relation between Hopf map and the Schmidt decomposition. Indeed the two common eigenvalues of $\rho_1$ and $\rho_2$ read

$$\lambda_{\pm} = \frac{1 + \sqrt{1 - 4|C_2|^2}}{2} = \frac{1 + \sqrt{1 - \alpha^2}}{2}$$

(24)

from which we get the Schmidt weights $\cos \frac{\varphi}{2} = \sqrt{\lambda_+}$ and $\sin \frac{\varphi}{2} = \sqrt{\lambda_-}$. The Schmidt states are easily obtained from the diagonalization of $\rho_1$ and $\rho_2$. Their decomposition in the original product state basis can therefore also be given in term of the Hopf map parameters. As an example, the two (unnormalized) states for the first qubit simply read

$$|\varphi_{\pm}\rangle = \overline{q_1} |0\rangle_1 + \left(\lambda_{\pm} - |q_1|^2\right) |1\rangle_1$$

(25)

4. Conclusion

Our main goal in this paper was to provide a geometrical representation of the two-qubit Hilbert space pure states. We hope to have convinced the reader that the $S^7$ Hopf fibration plays a natural role in that case: it provides a adapted “skeleton” for preparing a meaningfull slicing of the full Hilbert space. It is rather fascinating that the two main Hopf fibrations (of $S^3$ and $S^7$) are so nicely related to the two important ingredients of the quantum objects, which are the global phase freedom and entanglement.

In going further, one may ask two natural questions. The first concerns the geometry of two-qubit mixed states, and the second the generalization to more than two qubits. Note that these two questions are not necessarily completly independant, as we just saw in the previous paragraph. Up to now, we have mainly focused on the second question, and addressed the 3-qubit case. Indeed, the Hilbert space is now $S^{15}$, which also allow for a Hopf fibration, with a $S^8$ base and $S^7$ fibres. From an analytical point of view, the Hopf map takes the same form, but quaternions need to be replaced by “octonions”. This is not a harmless change, since octonion algebra is not only non-commutative, but also non-associative (forbidding a matrix representation for instance). But this analysis can be done, and provides interesting insights in the Hilbert space geometry. Note that in the three Hopf fibrations sequence ($S^{15}, S^7, S^3$), the fibre in the larger dimensional space is the full space in next case. This offers the possibility of further nesting the fibrations, a possibility that we already used in the present analysis of two-qubit separable case.

Appendix: Quaternions

Quaternions are usually presented with the imaginary units $i, j$ et $k$ in the form :

$q = x_0 + x_1 i + x_2 j + x_3 k, \quad x_0, x_1, x_2, x_3 \in \mathbb{R} \quad with \quad i^2 = j^2 = k^2 = ijk = -1,$

the latter ”Hamilton” relations defining the quaternion multiplication rules which are non-commutative. They can also be defined equivalently, using the complex numbers $c_1 = x_0 + x_1 i$ and $c_2 = x_2 + x_3 i$, in the form $q = c_1 + c_2 j$.

The conjugate of a quaternion $q$ is $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k = \overline{c_1} - c_2 j$ and its squared norm reads $N_q = qq$. 
Another way in which $q$ can be written is as a scalar part $S(q)$ and a vectorial part $V(q)$:

$$q = S(q) + V(q), \quad S(q) = x_0, \quad V(q) = x_1i + x_2j + x_3k,$$

with the relations

$$S(q) = \frac{1}{2}(q + \bar{q}), \quad V(q) = \frac{1}{2}(q - \bar{q}).$$

A quaternion is said to be real if $V(q) = 0$, and pure imaginary if $S(q) = 0$. We shall also write $V_{i,j,k}(q)$ for the component of $V(q)$ along $i, j, k$. Finally, and as for complex numbers, a quaternion can be noted in an exponential form as

$$q = |q| \exp \varphi t = |q| (\cos \varphi + \sin \varphi t),$$

where $t$ is a unit pure imaginary quaternion. When $t = i$, usual complex numbers are recovered. Note that quaternion multiplication is non-commutative so that

$$\exp \varphi t \exp \lambda u = \exp(\varphi t + \lambda u)$$

only if $t = u$.

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While completing the present written version, we met a very interesting preprint by I. Bengtsson, J. Brännlund and K. Zyczkowski (quant-phys/0108064) on the geometry of qubits projective Hilbert space, with $S^3$ Hopf fibrations occasionally used to describe submanifolds. Beside a different perspective in slicing the Hilbert space, a main difference with our work is the present use of the $S^7$ Hopf fibration applied to the whole space.

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