POLARIZATION AND DEFORMATIONS OF GENERALIZED DENDRIFORM ALGEBRAS

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Abstract. We generalize three results of M. Aguiar, which are valid for Loday’s dendriform algebras, to arbitrary dendriform algebras, i.e., dendriform algebras associated to algebras satisfying any given set of relations. We define these dendriform algebras using a bimodule property and show how the dendriform relations are easily determined. An important concept which we use is the notion of polarization of an algebra, which we generalize here to (arbitrary) dendriform algebras: it leads to a generalization of two of Aguiar’s results, dealing with deformations and filtrations of dendriform algebras. We also introduce weak Rota-Baxter operators for arbitrary algebras, which lead to the construction of generalized dendriform algebras and to a generalization of Aguiar’s third result, which provides an interpretation of the natural relation between infinitesimal bialgebras and pre-Lie algebras in terms of dendriform algebras. Throughout the text, we give many examples and show how they are related.

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1. Introduction

Dendriform algebras were introduced by J.-L. Loday in [16] as a dichotomized version of associative algebras. By definition, a Loday dendriform algebra is an algebra \((A, \prec, \succ)\) satisfying, for all \(a, b, c \in A\), the relations
\[
\begin{align*}
(a \prec b) \prec c &= a \prec (b \prec c + b \succ c), \\
(a \succ b) \prec c &= a \succ (b \prec c), \\
(a \prec b + a \succ b) \succ c &= a \succ (b \succ c).
\end{align*}
\]

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Summing up these three equations and setting $a \star b := a \prec b + a \succ b$ for all $a, b \in A$, one sees that $\star$ is associative, so that $(A, \star)$ is an associative algebra, on which the dendriform operations provide some extra structure. In [3], M. Aguiar introduced the notion of deformation for a commutative dendriform algebra $(A, \prec, \succ)$, where commutativity means that $a \prec b = b \succ a$ for all $a, b \in A$. He shows that such a deformation makes $(A, \times, \circ)$ into a pre-Poisson algebra, where $\times$ stands for $\succ$ and where the new product $\circ$ on $A$ is constructed from the first order deformation terms of $\prec$ and $\succ$. The notion of a pre-Poisson algebra was also introduced in loc. cit.: $(A, \times, \circ)$ is a pre-Poisson algebra if the following relations are satisfied, for all $a, b, c \in A$:

$$ a \times (b \times c) = (a \times b + b \times a) \times c, $$

$$ (a \times b + b \times a) \circ c = a \times (b \circ c) + b \times (a \circ c), $$

$$ (a \circ b - b \circ a) \times c = a \times (b \circ c) - b \circ (a \times c), $$

$$ (a \circ b - b \circ a) \circ c = a \circ (b \circ c) - b \circ (a \circ c). $$

This result is a dendriform version of the well-known fact that the skew-symmetrization of the first deformation term of a deformation of an associative algebra $A$ is a Poisson bracket on $A$. Aguiar also establishes a similar result for filtered dendriform algebras, also a dendriform version of a well-known result. Even if these results can easily be proven by a direct computation, these computations lack a conceptual understanding, which we will provide in this paper. We do this by generalizing these results to arbitrary dendriform algebras; a key element is the notion of polarization for (arbitrary) dendriform algebras, which we will introduce.

We define generalized dendriform algebras as follows. Let $\mathcal{C}$ denote the category of all algebras $(A, \mu)$ which satisfy a given set of relations $R_1 = 0, \ldots, R_k = 0$. An algebra $(A, \prec, \succ)$ is said to be a $\mathcal{C}$-dendriform algebra if $(A \times A, \otimes) \in \mathcal{C}$, where $\otimes$ is defined for $(a, x), (b, y) \in A \times A$, by $(a, x) \otimes (b, y) := (a \prec b + a \succ b, a \succ y + x \prec b)$. This property can also be expressed as a bimodule property. The $\mathcal{C}$-dendriform algebras form a category $\mathcal{C}_{dend}$ with algebra homomorphisms as morphisms. Taking as relation associativity, we recover the definition of a Loday dendriform algebra. We show that when all relations are multilinear, the relations which every $\mathcal{C}$-dendriform algebra must satisfy are easily obtained from the relations $R_i = 0$. Generalized dendriform algebras have already been considered from the operadic point of view in [5], but we will not use or need this formalism since the phenomena and properties which we present are most naturally expressed in terms of the basic algebraic language which we use.

In order to construct (interesting) examples of generalized dendriform algebras, we introduce the notion of a weak Rota-Baxter operator. Given any algebra $A$ (satisfying a given set of relations), a linear map $R : A \to A$ is said to be a weak Rota-Baxter operator of $A$ if, for all $a, b \in A$, the element

$$ R(aR(b) + R(a)b - R(a)R(b)) \quad (1.1) $$
commutes with all elements of \( A \). We show that \(( A, \prec, \succ)\) becomes a generalized dendriform algebra upon setting \( a \succ b := R(a)b \) and \( a \prec b := aR(b) \) for all \( a, b \in A \). More precisely, we show which relations \( \prec \) and \( \succ \) will satisfy.

When (1.1) is zero for all \( a, b \in A \) (in which case \( R \) is called a Rota-Baxter operator) these relations are precisely the dendrification of the relations satisfied by \( A \); the same is true for arbitrary weak Rota-Baxter operators in case the relations can be written in commutator form (see Section 3.2 for the definition of this notion). As an application, we generalize yet another result by M. Aguiar [4], which states that the natural functor which associates to any \( \epsilon \)-bialgebra \(( A, \mu, \Delta)\) the corresponding pre-Lie algebra \(( A, \circ)\), restricted to the category of quasi-triangular \( \epsilon \)-bialgebras, admits a natural factorization through the category of dendriform algebras, i.e., the following diagram is commutative (see Section 3.3 for details):

\[
\begin{array}{ccc}
\text{QT} \text{ \( \epsilon \)-bialg} & \xrightarrow{r_{\text{a} \rightarrow \text{a} \rightarrow r}} & \epsilon \text{-bialg, } \mu, \Delta \\
\text{Assoc} \text{dend}, \prec, \succ & \xrightarrow{a \succ b \rightarrow b \rightarrow a} & \text{pre-Lie, } \circ \\
\end{array}
\]

We show that this diagram can be generalized to coboundary \( \epsilon \)-bialgebras by replacing in it the two leftmost entries by coboundary \( \epsilon \)-bialgebras and \( A_3 \)-dendriform algebras, without changing the arrows; recall from [11] that an algebra \(( A, \mu)\) is said to be \( A_3 \)-associative if for all \( a, b, c \in A, \)

\[(ab)c + (bc)a + (ca)b = a(bc) + b(ca) + c(ab) ;\]

the corresponding dendriform algebras are called \( A_3 \)-dendriform algebras.

For algebras \(( A, \mu)\) (with one operation), the notion of polarization has been introduced in [18]: the product \( \mu \) is decomposed into its symmetric and antisymmetric parts, yielding an algebra \(( A, \cdot, [\cdot, \cdot])\) for which the relations are obtained from the relations satisfied by \( \mu \). This definition is easily adapted to generalized dendriform algebras, as indicated in the following commutative diagram of categories and functors, where the horizontal arrows are isomorphisms of categories (see Section 4 for the notation):

\[
\begin{array}{ccc}
\mathcal{C}, \mu & \xrightarrow{(ab+ba)/2, (ab-ba)/2} & C_{\text{pol}}, [\cdot, \cdot] \\
\mathcal{C}_{\text{dend}}, \prec, \succ & \xrightarrow{a \prec b \rightarrow [a, b]} & C_{\text{pol}}, [\cdot, \cdot] \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{C}_{\text{dend}}, \prec, \succ & \xrightarrow{a \prec b \rightarrow b \rightarrow a, a \succ b \rightarrow a \succ b} & C_{\text{pol}}, \ast, \circ \\
\mathcal{C}_{\text{dend}}, \prec, \succ & \xrightarrow{a \prec b \rightarrow b \rightarrow a, a \succ b \rightarrow a \succ b} & C_{\text{pol}}, \ast, \circ \\
\end{array}
\]

Thanks to the commutativity of the diagram we can obtain the relations of \( \mathcal{C}_{\text{pol}} \) from the relations of \( \mathcal{C}_{\text{pol}} \) in case the latter are multilinear. For example,
if $\mathcal{C}_{\text{pol}}$ denotes the category of Poisson algebras, $\mathcal{C}_{\text{dend}}^{\text{dend}}$ is the category of pre-Poisson algebras, mentioned above.

We give two applications of polarization. Suppose that $(A[[\nu]], \prec, \succ)$ is a deformation of a commutative algebra $(A, \prec_0, \succ_0)$ in $\mathcal{C}_{\text{dend}}^{\text{dend}}$ and consider the algebra $(A, \times, \circ)$, where $\times$ stands for $\succ_0$ and $\circ$ is defined for $a, b \in A$ by

$$a \circ b := \frac{a \succ b - b \prec a}{2\nu} \big|_{\nu=0}.$$

We show that $(A, \times, \circ) \in \mathcal{C}_{\text{pol}}^{\text{dend}}$, where $\mathcal{C}_{\text{pol}}^{\text{dend}}$ is the category of all polarized dendriform algebras $(A, \ast, \circ)$ satisfying for all relations $R = 0$ of $\mathcal{C}_{\text{pol}}^{\text{dend}}$ the relation $\mathbb{R}_R = 0$; here, $\mathbb{R}_R$ stands for the lowest weight part of $\mathbb{R}$, where the weight of a monomial in $A$ is defined as being the number of operations $\circ$ that it contains. Notice that these relations can easily be computed. A prime example is the case in which $(A, \prec, \succ)$ is a Loday dendriform algebra; we then recover Aguiar’s result since then $\mathcal{C}_{\text{pol}}^{\text{dend}}$ is the category of pre-Poisson algebras. As a second application, we show that a similar result holds for filtered commutative algebras in $\mathcal{C}_{\text{dend}}^{\text{dend}}$. Both applications admit also an anticommutative version.

The structure of the paper is as follows. We introduce in Section 2 the notion of a $C$-dendriform algebra and we show how the relations satisfied by all $C$-dendriform algebras can be obtained from the relations in $C$. We give several examples and we show how they are related, both in their original and in their dendrified form. Rota-Baxter and weak Rota-Baxter operators are shown in Section 3 to provide constructions of $C$-dendriform algebras; we give an application of it to $\epsilon$-bialgebras. The notion of polarization for dendriform algebras is introduced in Section 4 and again we show how for polarized algebras, defined by multilinear relations, the relations satisfied by the corresponding dendriform algebras are obtained. As an application, we give a conceptual proof of the generalization to $C$-dendriform algebras of Aguiar’s results, stated at the beginning of the introduction; this yields, in particular, a conceptual proof of these results. All our results extend to $C$-tridendriform algebras; throughout the paper, we will indicate these generalizations in some short remarks.

**Conventions.** All algebraic structures which we consider (algebras, modules, bialgebras, etc.) are defined over a commutative ring $R$ in which 2 is invertible. Since the base ring $R$ will never change, we denote $\otimes_R$ simply by $\otimes$. By “$R$-algebra”, which we call simply “algebra”, we mean an $(n+1)$-tuple $(A, \mu_1, \ldots, \mu_n)$, where $A$ is an $R$-module and $\mu_i : A \otimes A \to A$ is a product, i.e., a linear map, for $i = 1, \ldots, n$. By an algebra homomorphism between two algebras $(A, \mu_1, \ldots, \mu_n)$ and $(A', \mu'_1, \ldots, \mu'_n)$ we mean a linear map $f : A \to A'$ such that $f(\mu_i(a \otimes b)) = \mu'_i(f(a) \otimes f(b))$ for all $a, b \in A$ and all $i = 1, \ldots, n$. Unless otherwise specified, the products $\mu_i$ are not assumed to have any extra properties. In the case of an algebra $(A, \mu)$ with
one product, we usually write \( ab \) for \( \mu(a \otimes b) \). We use the standard notations \( S_n \) and \( A_n \) for the symmetric and alternating groups of degree \( n \).

## 2. Dendriform algebras

In this section, we recall the notion of a Loday dendriform algebra and we show that it naturally generalizes to algebras \((A, \mu)\), defined by any finite collection of relations; Loday dendriform algebras correspond to associative algebras, which are defined by a single relation, namely associativity. We also show that when the relations in the original algebra are multilinear, the relations which hold in the corresponding dendriform algebras are easily determined. Recall that we write \( ab \) for \( \mu(a \otimes b) \).

### 2.1. Loday dendriform algebras.

We first recall from [16] the notion of a Loday dendriform algebra.

**Definition 2.1.** A Loday dendriform algebra is an algebra \((A, \prec, \succ)\) satisfying for all \(a, b, c \in A\) the following relations:

\[
(a \prec b) \prec c = a \prec (b \prec c + b \succ c) ,
\]

\[
(a \succ b) \prec c = a \succ (b \prec c) ,
\]

\[
(a \prec b + a \succ b) \succ c = a \succ (b \succ c) .
\]

The terminology **dendriform** comes from the shape of the free Loday dendriform algebra, which is naturally described in terms of planar binary trees [loc. cit. Sections 5.4 and 5.7]. Dendriform algebras can be considered as a dichotomized version of an associative algebra: defining \( a \star b := a \prec b + a \succ b \) for all \( a, b \in A \), the newly formed algebra \((A, \star)\) is associative. In fact, Loday dendriform algebras can be characterized as follows (see [7]):

**Proposition 2.2.** Let \((A, \prec, \succ)\) be an algebra and let \( \star \) denote the sum of \( \prec \) and \( \succ \). Then \((A, \prec, \succ)\) is a dendriform algebra if and only if the following conditions are satisfied:

1. \((A, \star)\) is an associative algebra;
2. \((A, \succ, \prec)\) is an \((A, \star)\)-bimodule.

In this characterization, the notion of bimodule (over an associative algebra) is the standard one; see the lines following Definition 2.3 below for the more general concept of a bimodule over other types of algebras.

Conditions (1) and (2) of the proposition can be restated as the single condition that \((A \times A, \boxtimes)\) is associative, where the product \( \boxtimes \) is defined, for \((a, x), (b, y) \in A \times A\), by

\[
(a, x) \boxtimes (b, y) := (a \star b, a \succ y + x \prec b) .
\]

The proof of the equivalence is a direct consequence of the following formulas, valid for all \((a, x), (b, y), (c, z) \in A \times A\):

\[
((a, x) \boxtimes (b, y)) \boxtimes (c, z) = ((a \star b) \star c, (a \star b) \succ z + (a \succ y) \prec c + (x \prec b) \prec c) ,
\]

\[
((a, x) \boxtimes (b, y)) (c, z) = ((a \star b) \star c, (a \star b) \succ z + (a \succ y) \prec c + (x \prec b) \prec c) ,
\]

\[
(a, x) \boxtimes ((b, y) \boxtimes (c, z)) = (a \star (b \star c), a \succ (y \succ z) + (x \prec b) \prec c) .
\]

\[
(a, x) \boxtimes ((b, y) \boxtimes (c, z)) = (a \star (b \star c), a \succ (y \succ z) + (x \prec b) \prec c) .
\]
\[(a, x) \boxtimes ((b, y) \boxtimes (c, z)) = (a \ast (b \ast c), a \succ (b \succ z) + a \succ (y \prec c) + x \prec (b \ast c)) .\]

It follows that a Loday dendriform algebra can equivalently be defined as an algebra \((A, \prec, \succ)\) such that \((A \times A, \boxtimes)\) is associative, where \(\boxtimes\) is defined by (2.4). It is this more conceptual definition which we will generalize.

2.2. \(C\)-dendriform algebras. Let \(R_1 = 0, \ldots, R_k = 0\) be given relations and denote by \(C\) the category of all algebras (with one operation) which satisfy these relations called, by a slight abuse of language, \(C\)-relations. Morphisms in \(C\) are algebra homomorphisms. If \((A, \mu)\) is an object of \(C\), we write \((A, \mu) \in C\).

**Definition 2.3.** An algebra \((A, \prec, \succ)\) is said to be a \(C\)-dendriform algebra if \((A \times A, \boxtimes) \in C\), where \(\boxtimes\) is defined for \((a, x)\), \((b, y) \in A \times A\), by
\[
(a, x) \boxtimes (b, y) := (a \prec b + a \succ b, a \succ y + x \prec b) .
\]

Taking \(x = y = 0\) in (2.5), it is clear that if \((A, \prec, \succ)\) is a \(C\)-dendriform algebra, then \((A, \ast) \in C\), where \(\ast\) denotes the sum of \(\prec\) and \(\succ\). In the language of bimodules (for general algebras, not necessarily associative), the property that \((A \times A, \boxtimes)\) belongs to \(C\), where \(\boxtimes\) is defined by (2.5), is by definition precisely the condition that \((A, \ast) \in C\) and that \((A, \succ, \prec)\) is an \((A, \ast)\)-bimodule with respect to \(C\) (see [20]).

**Remark 2.4.** Definition 2.3 admits the following natural generalization: using the notations and under the assumptions of that definition, an algebra \((A, \prec, \succ, \cdot, \cdot)\) is said to be a \(C\)-tridendriform algebra if \((A \times A, \boxtimes) \in C\), where \(\boxtimes\) is now defined for \((a, x), (b, y) \in A \times A\), by
\[
(a, x) \boxtimes (b, y) := (a \prec b + a \succ b + a \cdot b, a \succ y + x \prec b + x \cdot y) .
\]

In the particular case when \(a \cdot b = 0\) for all \(a, b \in A\), one recovers the above definition of a \(C\)-dendriform algebra. Also, taking for \(C\) the category of all associative algebras, one recovers the classical notion of a tridendriform algebra, as first introduced by J.-L. Loday and M. Ronco in [17] (for a proof, see [7] in which our definition of a \(C\)-tridendriform algebra appears in the associative case as a characterization of a tridendriform algebra).

2.3. Algebras defined by multilinear relations. The relations, satisfied by the algebras which we will consider, are multilinear and we will show how for such relations we can easily obtain the corresponding relations which must be satisfied by the corresponding dendriform algebras; we do this for one relation at a time. Our method is based on the fact that, by multilinearity, the condition that \((A \times A, \boxtimes)\) belongs to \(C\) is equivalent to the conditions obtained by demanding that the relations are satisfied for all possible \(n\)-tuplets (for an \(n\)-linear relation) of elements of \(A \times A\), taken from a generating set of \(A \times A\). We take this generating set to be the union of \(A_0 := A \times \{0\}\) and \(A_1 := \{0\} \times A\). We will find it convenient to use for any \(a \in A\) the following notation: \(a_0 := (a, 0)\) and \(a_1 := (0, a)\); also, when we
consider elements \( a_0 \in A_0 \) or \( a_1 \in A_1 \) we implicitly assume that \( a \in A \). In this notation, (2.5) is equivalently described by the following table, in which \( a \) and \( b \) stand for arbitrary elements of \( A \):

| ⊠ 0 | a \ast b_1 | a \bowtie b_1 |
|-----|------------|----------------|
| a_0 | a \ast b_1 | a \bowtie b_1 |
| a_1 | a \bowtie b_1 | (0, 0) |

Table 1. The product \( \boxtimes \) for generators of \( A \times A \).

We explain the procedure in the case of a trilinear relation, the case of a bilinear relation being too simple\(^1\) to illustrate how it works; see Remark 2.5 below for the case of an \( n \)-linear relation. By a trilinear relation on an algebra \((A, \mu)\), we mean a relation in three variables which is linear in each of the variables, i.e., the relation is of the form \( R = 0 \), where

\[
R(a_1, a_2, a_3) = \sum_{\sigma \in S_3} l_{\sigma} (a_{\sigma(1)} a_{\sigma(2)}) a_{\sigma(3)} + \sum_{\sigma \in S_3} l'_{\sigma} a_{\sigma(1)} (a_{\sigma(2)} a_{\sigma(3)}) . \tag{2.7}
\]

The 12 constants \( l_{\sigma} \) and \( l'_{\sigma} \) belong to the base ring \( R \). The associativity relation, \((ab)c - a(bc) = 0\), is an example; there are many other such relations, such as the ones defining Leibniz algebras, NAP algebras, pre-Lie algebras, Lie-admissible algebras, and so on. Several of these, and some others, will be considered below, where their definition will be recalled.

Let \( R = 0 \) be a trilinear relation and let us denote by \( R_{\boxtimes} \) (resp. \( R_\ast \)) the formula \( R \) in which the product \( \mu \) is replaced by \( \boxtimes \) (resp. \( \ast \)). We show how to obtain the corresponding relations for a \( C \)-dendriform algebra.

- If we take three arbitrary elements \( a_0, 0, 0 \) in \( A_0 \), then

\[
(a_0 \boxtimes 0) \boxtimes 0 = (a \ast b) \ast c_0 , \text{ and } a_0 \boxtimes (0 \boxtimes 0) = a \ast (b \ast c_0) ,
\]

so that \( R_{\boxtimes}(a_0, 0, 0) = R_\ast(a, b, c) \), for all \( a, b, c \in A \). Therefore, the relation which we find is that \( R_\ast = 0 \), i.e., that \( (A, \ast) \in C \). As we will see in the next item, this relation needs not be stated explicitly, because it follows from the other relations.

- When we take two elements in \( A_0 \) and one in \( A_1 \), we get from \( R_{\boxtimes} = 0 \) three non-trivial relations which may be linearly dependent. Notice that

\[
(a_0 \boxtimes 0) \boxtimes 1 + (a_0 \boxtimes 1) \boxtimes 0 + (a_1 \boxtimes 0) \boxtimes 0
= (a \ast b) > c + (a > b) < c + (a < b) < c = (a \ast b) \ast c_1 ,
\]

for any \( a, b, c \in A \), and similarly with the opposite parenthesizing,

\[
a_0 \boxtimes (0 \boxtimes 1) + a_0 \boxtimes (1 \boxtimes 0) + a_1 \boxtimes (0 \boxtimes 0) = a \ast (b \ast c)_1 .
\]

---

\(^1\)When \( R \) is a field, the only non-trivial bilinear relations are commutativity and anticommutativity.
If we write $R$ as in (2.7), then it follows from these two equations that

$$
R_{\otimes}(a_1, a_2, a_3) + R_{\otimes}(a_1, a_2, a_3) + R_{\otimes}(a_1, a_2, a_3)
$$

$$
= \sum_{\sigma \in S_3} l_\sigma(a_{\sigma(1)} \star a_{\sigma(2)}) \star a_{\sigma(3)} + \sum_{\sigma \in S_3} l_\sigma(a_{\sigma(1)} \star a_{\sigma(2)}) \star a_{\sigma(3)}
$$

$$
+ \sum_{\sigma \in S_3} l'_\sigma a_{\sigma(1)} \star (a_{\sigma(2)} \star a_{\sigma(3)}) + \sum_{\sigma \in S_3} l'_\sigma a_{\sigma(1)} \star (a_{\sigma(2)} \star a_{\sigma(3)})
$$

$$
= R_+(a_1, a_2, a_3),
$$

(2.8)

and so the sum of the three relations which we just found for $<$ and $>$ is precisely the corresponding relation for their sum $\star$, as stated above.

- Taking at most one element in $A_0$ and the other ones in $A_1$ gives trivial relations, because a triple product in $(A \times A, \otimes)$ vanishes as soon as at least two of its factors belong to $A_1$, as follows at once from the definition of $\otimes$.

The upshot is that a trilinear relation $R = 0$ gives rise to at most three independent relations, which are found by considering $R_{\otimes}$ for a triplet of elements in $A \times A$, where two of them are arbitrary elements in $A_0$ and the other one in $A_1$. Notice that, when $R$ is invariant under a cyclic permutation in its three variables, the three obtained relations will be the same, so that only one such triplet has to be considered; similarly, when $R$ is invariant under a transposition of two of the three variables, only two triplets need to be considered. Since the defining relations of many types of algebras are quite symmetric, we will see below several examples of this.

Remark 2.5. The above analysis is also valid for $n$-linear relations, with $n > 3$: in order to obtain all $C$-dendriform relations, it suffices to substitute $n - 1$ elements from $A_0$ and one from $A_1$, and this in the $n$ possible ways. To see this, notice first that if one substitutes in any monomial $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ (with any parentheses) at least two elements from $A_1$ and the other ones from $A_0$, one always gets zero, because $A_0 \otimes A_1$ and $A_1 \otimes A_0$ are contained in $A_1$ and $A_1 \otimes A_1 = \{(0, 0)\}$. It remains to be shown that the relation, which is obtained by substituting $n$ elements from $A_0$, follows from the $n$ relations which are obtained by substituting $n - 1$ elements from $A_0$ and one element from $A_1$. Consider a monomial $a_1 a_2 \cdots a_n$ in $A$, with some parentheses, and denote for $i = 1, 2, \ldots, n$,

$$
X := a_{10} \otimes a_{20} \otimes \cdots \otimes a_{n0} = a_1 \star a_2 \star \cdots \star a_{n0},
$$

$$
X_i := a_{10} \otimes a_{20} \otimes \cdots \otimes a_{i-10} \otimes a_{i1} \otimes a_{i+10} \otimes \cdots \otimes a_{n0},
$$

with the same parentheses. Notice that $X \in A_0$ and that $X_i \in A_1$ for $i = 1, 2, \ldots, n$. Defining $a \in A$ by $X = a_0$ (i.e., $a = a_1 \star a_2 \star \cdots \star a_n$, with the same parentheses), we show that $\sum_{i=1}^n X_i = a$. We do this by induction on $n$, the case of $n = 3$ already being proven above. We can write $X$ (uniquely, as dictated by the parentheses) as $X = X' \otimes X''$, where

$$
X' = a_{10} \otimes a_{20} \otimes \cdots \otimes a_{m0}, \quad X'' = a_{m+10} \otimes a_{m+20} \otimes \cdots \otimes a_{n0},
$$
with $1 \leq m < n$, and both $X'$ and $X''$ come with a parenthesizing inherited from the one of $X$. We define for $i = 1, \ldots, m$ (resp. for $i = m + 1, \ldots, n$) the element $X'_i$ (resp. $X''_i$) analogously to the definition of $X_i$ above. If we apply the induction hypothesis to $X'$ and $X''$, we get $\sum_{i=1}^m X'_i = a'_1$ and $\sum_{i=m+1}^n X''_i = a''_1$, where $X' = a'_0$ and $X'' = a''_0$. It follows that

$$\sum_{i=1}^n X_i = \sum_{i=1}^m X'_i \otimes X'' + X' \otimes \sum_{i=m+1}^n X''_i = a'_1 \otimes a''_0 + a'_0 \otimes a''_1,$$

while $X = X' \otimes X'' = a'_0 \otimes a''_0 = a' \ast a''$, so that $\sum_{i=1}^n X_i = a_1$ where $X = a_0$. It proves the announced property for $n$-linear relations, for all $n$.

**Remark 2.6.** For relations which are sums of $k$-linear relations, with $k$ varying from 1 to $n$, the above procedure can be adapted, but there is no need to do this since for $k = 1, \ldots, n$ the $k$-linear part of such a relation $R = 0$ is itself a relation. To show this, one shows that the leading ($n$-linear) part is a relation, which follows by substituting successively $a_i = 0$ for $i = 1, \ldots, n$.

**Remark 2.7.** For $C$-tridendriform algebras (see Remark 2.3, where $C$ is defined by multilinear relations, the relations are obtained in the same way as in the case of $C$-dendriform algebras, but there will be many more relations. Indeed, given an $n$-linear relation $R = 0$, substituting in $R_{2\otimes} = 0$ two or more elements from $A_1$ and the other ones from $A_0$ will lead to a non-trivial relation, contrary to what we have seen in the case of a $C$-dendriform algebra. We will therefore get $2^n$ relations for a $C$-tridendriform algebra, rather than $n$. It can be shown that the relation, obtained by substituting in $R_{2\otimes}$ only elements from $A_0$, is the sum of all $2^n - 1$ relations obtained by substituting in $R_{2\otimes}$ at least one element from $A_1$ and the other elements from $A_0$. However, apart from this, these $2^n$ relations are in general independent.

2.4. **Examples.** We illustrate the above procedure in the following examples. In Section 2.6, we will show how these examples are related.

**Example 2.8.** We start with the case of a Loday dendriform algebra, which we recalled in Section 2.1, here, the only relation is associativity. We show how we obtain the relations of Definition 2.1 from the associativity of $\boxtimes$. First, take $a_0$, 0 in $A_0$ and 1 in $A_1$. Then, by the associativity of $\boxtimes$ and by Table 1,

$$(a \ast b) \triangleright c_1 = (a_0 \boxtimes 0) \boxtimes 1 = a_0 \boxtimes (0 \boxtimes 1) = a > (b \triangleright c)_1,$$

so that $(a \ast b) \triangleright c = a \triangleright (b \triangleright c)$, which is (2.3). Relations (2.2) and (2.1) are similarly obtained by taking $a_0$, 0 in $A_0$ and 1 in $A_1$ (resp. 0, 0 in $A_0$ and $a_1$ in $A_1$).

**Example 2.9.** A pre-Lie algebra $(A, \mu)$ is an algebra for which the associator, defined by $(a,b,c) := (ab)c - a(bc)$ is symmetric in its first two variables,
\((a, b, c) = (b, a, c)\) for all \(a, b, c \in A\). Thus, the trilinear relation which defines pre-Lie algebras is given by

\[(ab)c - a(bc) = (ba)c - b(ac) .\]  

Let \(\mathcal{C}_{pL}\) denote the category of all pre-Lie algebras. Using the above procedure, we obtain the relations which any \(\mathcal{C}_{pL}\)-dendriform algebra \((A, \prec, \succ)\) must satisfy, by substituting in the relation

\[((a, x) \boxtimes (b, y)) \boxtimes (c, z) - (a, x) \boxtimes ((b, y) \boxtimes (c, z))
= ((b, y) \boxtimes (a, x)) \boxtimes (c, z) - (b, y) \boxtimes ((a, x) \boxtimes (c, z)) , \]

two elements from \(A_0\) and one from \(A_1\). Substituting \(a_0, 0\) and 1 in (2.10), we get, using Table 1

\[(a * b) \succ c_1 - a \succ (b \succ c)_1 = (b * a) \succ c_1 - b \succ (a \succ c)_1 , \]

which leads to the relation

\[(a * b) \succ c - a \succ (b \succ c) = (b * a) \succ c - b \succ (a \succ c) . \]

Similarly, substituting \(a_0, 1\) and 0 in (2.10), we get

\[(a \succ b) \prec c - a \succ (b \prec c) = (b < a) \prec c - b < (a * c) . \]

Since (2.9) is invariant under the transposition which permutes \(a\) and \(b\), we have obtained all relations, and so the relations for a \(\mathcal{C}_{pL}\)-dendriform algebra are given by (2.11) and (2.12). In the literature, such dendriform algebras are known as \(L\)-dendriform algebras (see [8], where they have been introduced). One should keep in mind that, from our point of view, the \(L\) in \(L\)-dendriform stands for pre-Lie.

**Example 2.10.** The defining relation for an \(A_3\)-associative algebra \((A, \mu)\) is

\[(ab)c + (bc)a + (ca)b = a(bc) + b(ca) + c(ab) . \]

It can be written in terms of associators in the following compact form:

\[\sum_{\sigma \in A_3} (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) = 0 \]

(2.14)

where \(a_1, a_2, a_3 \in A\). The symmetric form of (2.14) is at the origin of the terminology “\(A_3\)” (see [11]); this form is often useful in computations, as we will see below. Since (2.14) is invariant under a cyclic permutation of \(a_1, a_2, a_3\), the corresponding dendriform algebras, which we will call \(A_3\)-dendriform algebras, need to satisfy only one relation. We obtain it by substituting \(a_0, 0\) and 1 for \((a_1, x_1), (a_2, x_2)\) and \((a_3, x_3)\), in the relation

\[\sum_{\sigma \in A_3} ((a_{\sigma(1)}, x_{\sigma(1)}), (a_{\sigma(2)}, x_{\sigma(2)}), (a_{\sigma(3)}, x_{\sigma(3)})) = 0 , \]

where \((\ldots)_\boxtimes\) stands for the associator of the product \(\boxtimes\). The resulting relation defining \(A_3\)-dendriform algebras is given by

\[a \succ (b \succ c) - (c < a) \prec b + c < (a * b) = (a * b) \succ c - b \succ (c < a) + (b \succ c) < a . \]

(2.15)
Notice that, upon defining \( a \circ b := a \triangleright b \triangleleft a \) for all \( a, b \in A \), the latter relation can be rewritten in the following simple form:

\[
(a \circ b) \circ c - b \circ (c \triangleleft a) - a \circ (b \triangleright c) = 0 .
\] (2.16)

We determine for this case also the relations of the corresponding tridendriform algebras. To do this, we need to substitute in \( R_{22} = 0 \) at least one element from \( A_1 \) and the other ones from \( A_0 \). Notice that, if one substitutes only one element from \( A_1 \), one obtains exactly the dendriform relations, with \( \ast \) standing now for \( a \ast b := a \triangleleft b + a \triangleright b + a.b \), so these relations do not have to be computed again. Also, as above, there is only one relation obtained by substituting two elements from \( A_1 \) and one from \( A_0 \), namely

\[
(a.b) \triangleleft c + (b \triangleleft c) . a + (c \triangleright a) . b = a. (b \triangleleft c) + b . (c \triangleright a) + c \triangleright (a.b) .
\] (2.17)

A final relation is obtained by substituting three elements from \( A_1 \). It is clear that the found relation just says that \((A, \circ)\) is \( A_3 \)-associative.

**Example 2.11.** A **Lie-admissible algebra** (or \( LA \)-algebra) is classically defined as an algebra \((A, \mu)\) for which the anticommutative product \([\cdot, \cdot]\), defined as the commutator \([a, b] := ab - ba\), is a Lie bracket, i.e., satisfies the Jacobi identity. The trilinear relation which characterizes Lie-admissible algebras is therefore given by

\[
\sum_{\sigma \in A_3} \left( (a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)}) - (a_{\sigma(2)}, a_{\sigma(1)}, a_{\sigma(3)}) \right) = 0 .
\] (2.18)

The relation (2.18) is invariant under the full symmetry group \( S_3 \), so the corresponding dendriform algebras, which we call \( LA \)-dendriform algebras, are defined by a single relation, as in the case of \( A_3 \)-dendriform algebras. It is obtained in the same way as in that case, and is given by

\[
a \triangleright (b \triangleright c - c \triangleleft b) - (b \triangleright c - c \triangleleft b) \triangleleft a - b \triangleright (a \triangleright c - c \triangleleft a) + (a \triangleright c - c \triangleleft a) \triangleleft b + c \triangleleft (a \ast b - b \ast a) - (a \ast b - b \ast a) \triangleright c = 0 ,
\] (2.19)

where \( \ast \) stands again for the sum of \( \triangleleft \) and \( \triangleright \). As above, we define \( a \circ b := a \triangleright b - b \triangleleft a \) for all \( a, b \in A \) and observe that \( a \ast b - b \ast a = a \circ b - b \circ a \), for all \( a, b \in A \). Then the relation defining \( LA \)-dendriform algebras can be rewritten in the following simple form:

\[
a \circ (b \circ c) - b \circ (a \circ c) - (a \circ b - b \circ a) \circ c = 0 .
\] (2.20)

It is equivalent to saying that \((A, \circ)\) is a pre-Lie algebra (see Example 2.9).

**Example 2.12.** An **associative-admissible algebra** (or \( AA \)-algebra) is similarly defined as an algebra \((A, \mu)\) for which the commutative product \([\cdot, \cdot]^+\), defined as the anticommutator \([a, b]^+ := ab + ba\), is associative. They are in a certain sense the commutative analogs of \( LA \)-algebras. \( AA \)-algebras are characterized by the trilinear relation

\[
(ab + ba)c + c(ab + ba) = a(bc + cb) + (bc + cb)a .
\] (2.21)
The relation (2.21) is again invariant under the full symmetry group $S_3$, so the corresponding dendriform algebras, $AA$-dendriform algebras, are defined by a single relation. It is most easily obtained from the compact form \([a,b]^+ , c]^+ = [a,[b,c]^+]^+\) of the relation (2.21). Indeed, let us denote by \([\cdot, \cdot]^+\) the anticommutator of $\mathbb{K}$, and let $a \ast b := a \triangleright b + b \triangleright a$ for all $a, b \in A$ (not to be confused with $a \ast b = a \triangleright b + a \triangleleft b$). Using the obvious identity $a \ast b + b \ast a = a \ast b + b \ast a$ it is easy to derive from Table 11 that

\[
[a_0,0]^+_\mathbb{K} = a \ast b + b \ast a_0 , \quad [a_0,1]^+_\mathbb{K} = a \ast b_1 ,
\]

for $a, b \in A$. Substituted in $[a_0,0]^+_\mathbb{K}, 1]^+_\mathbb{K} = [a_0, [0, 0]^+_\mathbb{K} = ]^+_\mathbb{K}$, we obtain the following relation for $AA$-dendriform algebras:

\[
(a \ast b + b \ast a) \ast c = a \ast (b \ast c) .
\]

(2.22)

This property is known as the Zinbiel property, see [16].

Example 2.13. Our last example is closely related to Poisson algebras (see Examples 4.3 and 4.9). Consider the following relation:

\[
3(ab)c = 3a(bc) + (ac)b + (bc)a - (ba)c - (ca)b .
\]

(2.23)

In view of the mentioned relation to Poisson algebras, we call any algebra satisfying this relation a $P$-algebra. The category of all P-algebras is denoted by $P$. It was shown in [12] that P-algebras are $A_3$-associative. Since (2.23) admits no symmetry (when the variables $a, b, c$ are permuted), we get three relations for the corresponding dendriform algebras, which we call $P$-dendriform algebras. They are given by the following formulas, where the first one is obtained by substituting $a_0, 0$ and 1 for $a, b$ and $c$ in (2.23), where the product $\mu$ has been replaced by $\mathbb{K}$, and similarly for the other two, where one substitutes $a_0, 1, 0$ and $a_1, 0, 0$ respectively:

\[
3 (a \ast b) \triangleright c = 3 a \triangleright (b \triangleright c) + (a \triangleright c) \triangleleft b + (b \triangleright c) \triangleleft a
\]

\[
- (b \ast a) \triangleright c - (c \triangleleft a) \triangleright b ,
\]

(2.24)

\[
3 (a \triangleright b) \triangleleft c = 3 a \triangleright (b \triangleleft c) + (a \ast c) \triangleright b + (b \triangleleft c) \triangleleft a
\]

\[
- (b \ast a) \triangleleft c - (c \triangleleft a) \triangleright b ,
\]

(2.25)

\[
3(a \triangleleft b) \triangleleft c = 3 a \triangleleft (b \ast c) + (a \ast c) \triangleleft b + (b \triangleleft c) \triangleright a
\]

\[
- (b \triangleright a) \triangleleft c - (c \triangleright a) \triangleleft b .
\]

(2.26)

In these formulas, $\ast$ stands again for the sum of $\prec$ and $\triangleright$.  

2.5. **Commutative and anticommutative dendriform algebras.** Many algebras of interest are commutative or anticommutative, i.e., they satisfy the relation $ab = ba$ or $ab = -ba$, besides satisfying some other relations. It follows at once from the defining relations that:

(1) Associative, pre-Lie, AA and P-algebras which are commutative, are precisely commutative associative algebras;
(2) $A_3$-associative and LA-algebras which are commutative, are just arbitrary (commutative) algebras; similarly, AA-algebras which are anticommutative are arbitrary (anticommutative) algebras:

(3) $A_3$-associative, pre-Lie, LA and P-algebras which are anticommutative, are precisely Lie algebras;

(4) Associative algebras which are anticommutative, are precisely (left and right) 2-step nilpotent algebras, i.e., algebras $A$ satisfying $(ab)c = a(bc) = 0$ for all $a, b, c \in A$.

It is clear from (2.5) that the corresponding dendriform algebras must satisfy the relation $a \ll b = b \gg a$, respectively $a \ll b = -b \gg a$. It leads to the following definition.

**Definition 2.14.** A $C$-dendriform algebra $(A, \ll, \gg)$ is said to be commutative (resp. anticommutative) if it satisfies $b \gg a = a \ll b$ (resp. $b \gg a = -a \ll b$) for all $a, b \in A$.

In these cases it is natural to view $A$ as an algebra with only one product, by setting for all $a, b \in A$, $a \times b := a \gg b$ and the relations which $\times$ has to satisfy follow easily by substituting in the already found dendriform relations everywhere $a \times b$ for $a \gg b$ and for $\pm b \ll a$, the sign depending on whether commutativity or anticommutativity is considered. We give a few examples, based on the examples from Section 2.4.

**Example 2.15.** We start with (1) above: to obtain the relations of a commutative associative dendriform algebra, we substitute $a \times b$ for $a \gg b$ and for $b \ll a$ in the relations (2.1) – (2.3), to find the relations

\[
(a \times b + b \times a) \times c = a \times (b \times c), \quad c \times (a \times b) = a \times (c \times b).
\]  

The first property is the Zinbiel property (see Example 2.12). The second property is known as the NAP (for non-associative, permutative) property, see [15]. Since the Zinbiel property implies the NAP property, commutative associative dendriform algebras are, written in terms of a single product, the same as Zinbiel algebras.

**Example 2.16.** For (2) above, arbitrary (anti-) commutative algebras, one only gets the dendriform relation $a \ll b = \pm b \gg a$, with no relation for $\times$.

**Example 2.17.** For Lie algebras (case (3) above), the quickest way to obtain the relation which $\times$ must satisfy is by substituting $2a \times b$ (or just $a \times b$) for $a \circ b$ in (2.20), so we get the pre-Lie relation (2.9). Thus, Lie dendriform algebras are, written in terms of a single product, pre-Lie algebras.

**Example 2.18.** By definition, (right and left) 2-step nilpotent algebras (case (4) above) satisfy $(ab)c = a(bc) = 0$. Their dendriform algebras satisfy the following six relations:

\[
(a \ll b) \gg c = (a \gg b) \ll c = (a \ll b) \ll c = 0, \quad c \ll (b \gg a) = c \gg (b < a) = c \gg (b > a) = 0.
\]
It follows that anticommutative associative dendriform algebras are, in terms of a single product, also (right and left) 2-step nilpotent algebras, as they satisfy the relation \((a \times b) \times c = a \times (b \times c) = 0\).

**Remark 2.19.** Similarly, a tridendriform algebra is said to be **commutative** or **anticommutative** if it satisfies the relations \(a \succ b = \pm b \prec a\) and \(a.b = \pm b.a\), with the plus sign of course corresponding to the commutative case. Such tridendriform algebras are naturally seen as algebras with two operations “\(\times\)” and “\(\pm\)”, upon setting \(a \times b := a \succ b\), while keeping “\(^\pm\)”. 

**Example 2.20.** We give an example of an anticommutative tridendriform algebra: a Lie tridendriform algebra. We obtain the relations from the relations of an \(A_3\)-tridendriform algebra, given in Example 2.11 by replacing in them \(a \succ b - b \prec a\), in particular \(a \star b\) by \(a \times b - b \times a + a.b\) and \(a \circ b\) by \(2a \times b\), and using that \(a.b = -b.a\). After some trivial simplifications, one finds that a Lie tridendriform algebra is a Lie algebra, satisfying the following two relations, obtained from (2.16) and (2.17):

\[
(a.b) \times c = a \times (b \times c) - (a \times b) \times c - b \times (a \times c) + (b \times a) \times c ,
\]

\[
c \times (a.b) = (c \times b).a - (c \times a).b .
\]

In the literature, Lie tridendriform algebras are known as Post-Lie algebras (see [6][21]).

### 2.6. Categories of generalized dendriform algebras.

Let, as before, \(R_1 = 0, \ldots, R_k = 0\) be given relations. Recall that we denote by \(C\) the category of all algebras \((A, \mu)\) over \(R\) which satisfy these relations, with algebra homomorphisms as morphisms in \(C\). Clearly, the class of all \(C\)-dendriform algebras (over \(R\)) also form a category \(C^{dend}\) with morphisms the algebra homomorphisms. For example, the category of Loday dendriform algebras (constructed from associative algebras) is denoted by \(Assoc^{dend}\) and the category of \(P\)-dendriform algebras is denoted by \(P^{dend}\).

By the above, \(C^{dend}\) is constructed out of \(C\), but that does not mean that we know how to associate to algebras in \(C\) dendriform algebras in \(C^{dend}\); we have on the contrary a (faithful) functor \(C^{dend} \to C\), which on objects \((A, \prec, \succ)\) is defined by \((A, \prec, \succ) \mapsto (A, \star)\), where \(\star\) denotes, as in the case of a Loday dendriform algebra, the sum of the products \(\prec\) and \(\succ\); on morphisms, the functor is just the identity in the sense that it sends the map underlying a morphism to itself.

Suppose that we have a second collection of relations \(R'_1 = 0, \ldots, R'_k = 0\), where every \(R_i\) is a linear combination of \(R'_1, \ldots, R'_l\). It is clear that every algebra satisfying all relations \(R'_1 = 0, \ldots, R'_{l} = 0\) satisfies all relations \(R_i = 0\), and so \(C'\), the category of all algebras satisfying the relations \(R'_1 = 0, \ldots, R'_{l} = 0\), is a subcategory of \(C\). Then \(C^{dend}\) is a subcategory of \(C^{dend}\), since the relations \(R'_i = 0\) can be seen as a subset of the relations \(R_i = 0\), and similarly for the dendriform relations obtained from the relations \(R'_i = 0\) and \(R_j = 0\). Thus, we have the following commutative diagram of categories:
In this diagram, the horizontal arrows are inclusions and the products denote typical products of the objects of the respective categories.

As a first application, we denote by \( C_{\text{com}} \) (resp. by \( C_{\text{com}}^{\text{dend}} \)) the subcategory of \( C \) (resp. of \( C^{\text{dend}} \)) consisting of all commutative algebras in the respective category. Then we have the following commutative diagram of categories:

\[
\begin{array}{ccc}
C_{\text{com}}, \mu & \longrightarrow & C, \mu \\
\downarrow a < b + a > b & & \downarrow a < b + a > b \\
C_{\text{com}}^{\text{dend}}, \preccurlyeq, \succcurlyeq & \xleftarrow{\preccurlyeq, \succcurlyeq} & C^{\text{dend}}, \preccurlyeq, \succcurlyeq
\end{array}
\]

Indeed, we can view the commutative algebras in \( C \) as being those which satisfy the extra condition of commutativity, and this relation leads to the condition of commutativity for the corresponding \( C \)-dendriform algebras, by the above observation. The same applies, of course, to anticommutative algebras.

As a second application, we show how the above examples of \( C \)-dendriform algebras are related. We have the following strict inclusion relations between the original category of algebras on the left; they lead to inclusion relations between their corresponding categories of dendriform algebras on the right.

We have not included AA-algebras and their dendriform algebras, because there are no apparent inclusion relations between the category of AA-algebras and any of the other categories that we considered.

The following table shows that the induced inclusions in the rightmost diagram are also strict and that there is no inclusion relation between \( A_3^{\text{dend}} \) or \( \mathcal{P}^{\text{dend}} \) and \( L^{\text{dend}} \). In the table, the algebra \((A, \preccurlyeq, \succcurlyeq)\) is a free module of
rank at least two and \( a \) and \( b \) are elements of a basis of \( A \). The first two columns describe the products \( \prec \) and \( \succ \) on some of the basis elements; it is understood that all other products between elements of the basis are zero.

| \( \prec \) | \( \succ \) | \( \ast \) | of type | not of type |
|---|---|---|---|---|
| \( a \prec a = -b \) | \( a \succ a = a + b \) | \( a \ast a = a \) | \( A_3\)-dendri | L-dendri |
| | \( b \succ a = b \) | \( b \ast a = b \) | | P-dendri |
| \( a \prec b = b \) | \( b \succ a = b \) | \( a \ast b = b \) | \( A_3\)-dendri | L-dendri |
| | \( b \succ b = b \) | \( b \ast b = b \) | | |
| — | \( b \succ a = a \) | \( b \ast a = a \) | L-dendri | \( A_3\)-dendri |
| \( a \prec b = -a \) | \( b \succ a = a \) | \( a \ast b = -a \) | P-dendri | L-dendri |
| | | \( b \ast a = a \) | | |
| \( a \prec a = a + b \) | — | \( a \ast a = a + b \) | dendri | P-dendri |

Table 2. Some examples of generalized dendriform algebras.

3. **(Weak) Rota-Baxter operators**

In this section, we introduce the notion of a weak Rota-Baxter operator, which generalizes the notion of a Rota-Baxter operator. We show how such operators can be used to construct generalized dendriform algebras and give an application to coboundary \( \epsilon \)-bialgebras.

3.1. **Dendriform algebras from Rota-Baxter operators.** We start with the definition of a Rota-Baxter operator (on an arbitrary algebra), see [13].

**Definition 3.1.** Let \((A, \mu)\) be any algebra, let \( \mathcal{R} : A \to A \) be a linear map and let \( l \in R \). One says that \( \mathcal{R} \) is a **Rota-Baxter operator of weight** \( l \) **of** \( A \) if \( \mathcal{R} \) satisfies the **Rota-Baxter equation**

\[
\mathcal{R}(a \mathcal{R}(b) + \mathcal{R}(a)b + lab) - \mathcal{R}(a)\mathcal{R}(b) = 0 ,
\]

for all \( a, b, \in A \). When \( l = 0 \) one simply speaks of a **Rota-Baxter operator**.

Let \( \mathcal{C} \) be the category of all algebras satisfying a given collection of multilinear relations \( \mathcal{R}_1 = 0, \ldots, \mathcal{R}_k = 0 \). We show in the following proposition how any Rota-Baxter operator (of weight zero) on any algebra \((A, \mu)\) of \( \mathcal{C} \) leads to a \( \mathcal{C} \)-dendriform algebra \((A, \prec, \succ)\).
Proposition 3.2. Let \( \mathcal{R} \) be a Rota-Baxter operator on an algebra \((A, \mu)\) which belongs to \(C\). For \(a, b \in A\), let \(a \triangleright b := \mathcal{R}(a)b\) and \(a \prec b := a\mathcal{R}(b)\). Then \((A, \prec, \triangleright)\) is a \(C\)-dendriform algebra.

Proof. We will give the proof for a trilinear relation \(\mathcal{R} = 0\); it is easily generalized to \(n\)-linear relations by induction on \(n\). Recall from Section \(2\) that \(\mathcal{R} = 0\) leads to 3 dendriform relations which are obtained by substituting two elements from \(A_0\) and one element from \(A_1\) in \(\mathcal{R} \boxtimes 0 = 0\), where \(\boxtimes\) is the product on \(A \times A\), defined by (2.4). Recall also that we write \(a_0\) for \((a, 0)\) and \(a_1\) for \((0, a)\), where \(a \in A\).

We show that such a substitution in \(\mathcal{R} \boxtimes 0\) amounts to writing \(\mathcal{R}\) for three elements of \(A\), on two of which \(\mathcal{R}\) has been applied, and rewriting the result in terms of the dendriform operations. To show this, we compare the effect of these substitutions on the two types of monomials \((ab)c\) and \(a(bc)\), where each time we consider the three possible substitutions. In view of Table 1, the definition of \(\prec\) and \(\triangleright\), and the Rota-Baxter equation (3.1), we get for the first type the following correspondence:

\[
\begin{align*}
(a_0 \boxtimes 0) \boxtimes 0 & = (a \prec b) \prec c_1 = (a\mathcal{R}(b))\mathcal{R}(c)_1, \\
(a_0 \boxtimes 1) \boxtimes 0 & = (a \triangleright b) \prec c_1 = (\mathcal{R}(a)b)\mathcal{R}(c)_1, \\
(a_0 \boxtimes 0) \boxtimes 1 & = (a \triangleright b) \triangleright c_1 = (\mathcal{R}(a)\mathcal{R}(b))c_1,
\end{align*}
\]

and similarly for the other type. In the third line we have used (3.1) with \(l = 0\), which says that \(\mathcal{R} : (A, \ast) \to (A, \mu)\) is a morphism. \(\square\)

Remark 3.3. Our proof shows that the \(C\)-dendriform relations can also formally be obtained from the relations \(\mathcal{R}_i = 0\) by formally applying \(\mathcal{R}\) to two of the variables and rewriting the resulting expression in terms of the dendriform operations (using the Rota-Baxter equation). Our proof also explains where the particular form of the Rota-Baxter equation comes from.

As a direct consequence of Proposition 3.2, we have the following result, which is well-known in the case of an associative or Lie algebra:

Corollary 3.4. Let \(\mathcal{R}\) be a Rota-Baxter operator on an algebra \((A, \mu)\) in \(C\). For \(a, b \in A\), let \(a \ast b := a\mathcal{R}(b) + \mathcal{R}(a)b\). Then \((A, \ast)\) also belongs to \(C\).

Remark 3.5. The proof of Proposition 3.2 is easily adapted to prove the following generalization of Proposition 3.2. If \(\mathcal{R}\) is a Rota-Baxter operator of weight \(l\) on an algebra \((A, \mu)\) which belongs to \(C\), then \((A, \prec, \triangleright, \ast)\) is a \(C\)-tridendriform algebra, upon defining \(a \triangleright b := \mathcal{R}(a)b\) and \(a \prec b := a\mathcal{R}(b)\) and \(a \ast b := lab\), for all \(a, b \in A\). If fact, it suffices to change in the proof the meaning of \(a \ast b\), which should now stand for \(a \triangleright b + a \prec b + a.b\).

Remark 3.6. In the case of Lie algebras, one encounters also the following equation, generalizing the Rota-Baxter equation (of weight zero):

\[
\mathcal{R}(a\mathcal{R}(b) + \mathcal{R}(a)b) = \mathcal{R}(a)\mathcal{R}(b) + \nu ab,
\]

(3.2)
where $\nu \in R$ is a constant. Equation (3.2) is known as the modified Yang-Baxter equation and has many application in the theory of integrable systems (see [1, Sect. 4.4.3]). The statement and proof of Proposition 3.2 and hence also Corollary 3.4 generalize easily to this case, so if $(A, \mu) \in C$ is equipped with a solution $\mathcal{R}$ of the modified Yang-Baxter equation (3.2) then $(A, \prec, \succ)$ is a $C$-dendriform algebra, where $a \succ b := \mathcal{R}(a)b$ and $a \prec b := a\mathcal{R}(b)$ for all $a, b \in A$. It is clear that in the display in the proof of the proposition, we only need to replace in line 3, $\mathcal{R}(a)\mathcal{R}(b)$ by $\mathcal{R}(a)\mathcal{R}(b) + \nu a b$. For the rest the proof is unchanged: these extra terms will disappear because the original product $\mu$ satisfies the relation $\mathcal{R} = 0$.

**Example 3.7.** The prime example of a solution to the modified Yang-Baxter equation is based on the notion of a Lie algebra splitting (see [1, Sect. 4.4.1]). It naturally generalizes as follows. Let $C$ be, as before, the category of all algebras satisfying a given set of relations. A $C$-algebra splitting of $(A, \mu) \in C$ is a module direct sum decomposition $A = A_+ \oplus A_-$ of $A$, where $A_+$ and $A_-$ are subalgebras of $A$. If one denotes by $P_+$ and $P_-$ projection on $A_+$ and $A_-$, then $\mathcal{R} := P_+ - P_-$ is a solution to (3.2), with $\nu = 1$. Indeed, upon setting $a_+ := P_+(a)$ and $a_- := P_-(a)$ for $a \in A$, one has, for any $a, b \in A$,

$$\mathcal{R}(a\mathcal{R}(b)) + \mathcal{R}(a)b = 2\mathcal{R}(a_+ b_+ - a_- b_-) = 2(a_+ b_+ + a_- b_-),$$

where we have used in the last step that $A_+$ and $A_-$ are subalgebras of $A$; this is clearly equal to

$$(a_+ - a_-)(b_+ - b_-) + (a_+ + a_-)(b_+ + b_-) = \mathcal{R}(a)\mathcal{R}(b) + a b.$$

It follows that a $C$-algebra splitting of $(A, \mu) \in C$ yields an algebra $(A, \star) \in C$, where $a \star b := a\mathcal{R}(b) + \mathcal{R}(a)b = a(b_+ - b_-) + (a_+ - a_-)b$, for $a, b \in A$.

### 3.2. Dendriform algebras from weak Rota-Baxter operators

We now introduce the notion of a weak Rota-Baxter operator, which generalizes the notion of a Rota-Baxter operator. For any algebra $(A, \mu)$, we denote by $C(A)$ the set of elements $c$ of $A$ which commute with all elements in $A$. It is a submodule of $A$ but is in general not a subalgebra of $A$.

**Definition 3.8.** Let $\mathcal{R} : A \to A$ be a linear map and let $l \in R$. One says that $\mathcal{R}$ is a weak Rota-Baxter operator of weight $l$ of $A$ if, for all $a, b \in A$,

$$\mathcal{R}(a\mathcal{R}(b)) + \mathcal{R}(a)b + l a b - \mathcal{R}(a)\mathcal{R}(b) \in C(A).$$

(3.3)

When $l = 0$ one simply speaks of a weak Rota-Baxter operator of $A$.

We show how Proposition 3.2 can be generalized to the case of weak Rota-Baxter operators (of weight zero). For clarity, and in view of the examples, we will restrict ourselves to the case of trilinear relations. We say that a trilinear relation $\mathcal{R} = 0$ has commutator form if it can be written as a linear

---

2For a general algebra, $C(A)$ strictly contains the center $Z(A)$, whose elements are required to have the extra property that any associator containing them vanishes.
is said to have commutator form for some constants $c$. Let $R$ be a collection of trilinear relations which are assumed to have commutator form. Let $R = 0$ be the category of all algebras satisfying these relations. Let $\mathfrak{R}$ be a weak Rota-Baxter operator on an algebra $(A, \mu)$ which belongs to $\mathcal{C}$. For $a, b \in A$, define $a \succ b := \mathfrak{R}(ab)$ and $a \prec b := a\mathfrak{R}(b)$. Then $(A, \prec, \succ)$ is a $\mathcal{C}$-dendriform algebra.

**Proposition 3.9.** Let $R_1 = 0, \ldots, R_k = 0$ be a collection of trilinear relations which are assumed to have commutator form. Let $\mathcal{C}$ be the category of all algebras satisfying these relations. Let $\mathfrak{R}$ be a weak Rota-Baxter operator on an algebra $(A, \mu)$ which belongs to $\mathcal{C}$. For $a, b \in A$, define $a \succ b := \mathfrak{R}(ab)$ and $a \prec b := a\mathfrak{R}(b)$. Then $(A, \prec, \succ)$ is a $\mathcal{C}$-dendriform algebra.

**Proof.** By the assumption, we may assume that $R_1 = 0, \ldots, R_k = 0$ have commutator form. Let $R = 0$ be one of these relations. We can repeat for $R$ the proof of Proposition 3.2 except that we need to show how to express the terms of the form $(\mathfrak{R}(a)\mathfrak{R}(b))c$ and $c(\mathfrak{R}(a)\mathfrak{R}(b))$ in terms of the dendriform operations and that by this procedure the same terms are obtained as by substituting in $R$ two terms from $A_0$ and one term from $A_1$. To do this, first observe that (3.3) can (for $l = 0$) be equivalently written as the condition that $[\mathfrak{R}(a \ast b), c] = [\mathfrak{R}(a)\mathfrak{R}(b), c]$, where $a \ast b = a \succ b + a \prec b = a\mathfrak{R}(b) + \mathfrak{R}(a)b$, leading to the following correspondence:

$$[a_0 \boxtimes 0, 1]_{\boxtimes} = (a \ast b) \succ c - c \prec (a \ast b)_{\boxtimes} = [\mathfrak{R}(a)\mathfrak{R}(b), c]_{\boxtimes},$$

where $[\cdot, \cdot]_{\boxtimes}$ stands for the commutator of the product $\boxtimes$. For the two other possible substitutions, it is not necessary to use the commutator form and one can simply rely on the formulas given in the proof of Proposition 3.2. Yet, for completeness, we also express them in commutator form:

$$[a_0 \boxtimes 1, 0]_{\boxtimes} = (a \succ b) \prec c - c \succ (a \succ b)_{\boxtimes} = [\mathfrak{R}(a)b, \mathfrak{R}(c)]_{\boxtimes},$$

$$[a_1 \boxtimes 0, 0]_{\boxtimes} = (a \prec b) \succ c - c \succ (a \prec b)_{\boxtimes} = [a\mathfrak{R}(b), \mathfrak{R}(c)]_{\boxtimes}.$$

It follows that the $\mathcal{C}$-dendriform relation $R = 0$ is satisfied by the products $\prec$ and $\succ$, defined by the weak Rota-Baxter operator $\mathfrak{R}$. 

The above theorem can be applied to $A_3$-associative algebras and Lie admissible algebras, since (2.13) and (2.18) can be respectively rewritten in the commutator forms

$$[ab, c] + [bc, a] + [ca, b] = 0,$$

$$\sum_{\sigma \in A_3} [a_{\sigma(1)}a_{\sigma(2)} - a_{\sigma(2)}a_{\sigma(1)}, a_{\sigma(3)}] = 0.$$

However, many relations cannot be written in commutator form. The associativity relation, $a(bc) = (ab)c$, is a prime example; other examples are the derivation property $a(bc) = (ab)c + b(ac)$, the Zinbiel property...
\(a(bc) = (ab + ba)c\) and the NAP property \(a(bc) = b(ac)\), just to mention a few. In such cases, when the relations of \(\mathcal{C}\) imply a relation \(\mathcal{R} = 0\) which can be written in commutator form, any dendriform algebra \((A, \prec, \succ)\) obtained by using a weak Rota-Baxter operator on an algebra \((A, \mu)\) in \(\mathcal{C}\) will satisfy (at least) the \(\mathcal{C}\)-dendriform relation, derived from \(\mathcal{R} = 0\). Moreover, any relation \(\mathcal{R} = 0\) which does not involve a product of two of the variables leads to a (single) dendriform relation. We illustrate this in the following example, on which we will elaborate in the following subsection.

**Example 3.10.** The associativity relation, \(a(bc) = (ab)c\) can clearly not be written in commutator form. Summing up three instances of this relation it implies however \((ab)c + (bc)a + (ca)b = a(bc) + b(ca) + c(ab)\), which is the relation of \(A_3\)-associativity, which we wrote in commutator form in (3.5).

Therefore, if \(\mathcal{R}\) is a weak Rota-Baxter operator on an associative algebra \((A, \mu)\) then \((A, \prec, \succ)\), with \(\prec\) and \(\succ\) defined by \(a \prec b := a\mathcal{R}(b)\) and \(a \succ b := \mathcal{R}(a)b\) is a priori not a Loday dendriform algebra, but it is an \(A_3\)-dendriform algebra. Moreover, the associativity relation \(a(bc) = (ab)c\) does not contain a product of \(a\) and \(c\), so we do not need to use the weak Rota-Baxter equation to rewrite \(\mathcal{R}(a)(b\mathcal{R}(c)) = (\mathcal{R}(a)b)\mathcal{R}(c)\) in terms of the dendriform products. The resulting relation \(a \succ (b \prec c) = (a \succ b) \prec c\) of \((A, \prec, \succ)\) is called inner-associativity.

It follows that a weak Rota-Baxter operator on an associative algebra leads to an inner-associative \(A_3\)-dendriform algebra. We show in the following example that in general the latter is not a Loday dendriform algebra.

**Example 3.11.** Let \(A\) be a commutative associative algebra. Every linear map \(\mathcal{R} : A \to A\) is a weak Rota-Baxter operator since \(C(A) = A\), hence leads to an inner-associative \(A_3\)-dendriform algebra. To see that it may not be a classical dendriform algebra, take \(\mathcal{R} = \text{Id}_A\). Then \(a \prec b = a \succ b = ab\) and (2.1) cannot be satisfied, unless \(abc = 0\) for all \(a, b, c \in A\).

**Remark 3.12.** The proof of Proposition 3.9 is easily adapted to prove the following generalization of Proposition 3.9 under the same assumptions on the relations of \(\mathcal{C}\), any weak Rota-Baxter operator \(\mathcal{R}\) of weight \(l\) on an algebra \((A, \mu) \in \mathcal{C}\) leads to a \(\mathcal{C}\)-tridendriform algebra, upon setting \(a \prec b := a\mathcal{R}(b)\) and \(a \prec b := a\mathcal{R}(b)\) and \(a \prec b := lab\), for all \(a, b \in A\). Again, it suffices to change in the proof the meaning of \(a \star b\), which should now stand for \(a \succ b + a \prec b + a.b\). The comments made about relations which cannot be written in commutator form apply here without modification.

**Remark 3.13.** If we denote by \(C'(A)\) the set of elements \(c\) of \(A\) which anticommute with all elements of \(A\), i.e., \(ac = -ca\) for all \(a \in A\), we can also consider operators \(\mathcal{R}\) satisfying (3.3), with \(C(A)\) replaced by \(C'(A)\). The results of this section are easily adapted to the case of such operators. For example, the conclusion of Proposition 3.9 still holds for such an operator \(\mathcal{R}\) when the relations have anticommutator form. An example of such a relation is the relation (2.21) defining \(\mathcal{A}\)-algebras.
3.3. **Application: coboundary \( \epsilon \)-bialgebras.** As an application of weak Rota-Baxter operators, we now generalize a result obtained by M. Aguiar in [4], which we will recall. We first recall the definition of an \( \epsilon \)-bialgebra:

**Definition 3.14.** An \( \epsilon \)-bialgebra is a triple \((A, \mu, \Delta)\), where \( A \) is an \( R \)-module and \( \mu : A \otimes A \to A \) and \( \Delta : A \to A \otimes A \) are linear maps, such that

1. \( \mu \) is associative;
2. \( \Delta \) is coassociative;
3. \( \Delta \) is a derivation: \( \Delta(ab) = a \cdot \Delta(b) + \Delta(a) \cdot b \), for all \( a, b \in A \).

In item (3), we have used a dot to denote the natural left, resp. right action of \( A \) on \( A \otimes A \); later on in this section, it will also be used for the natural left and right actions of \( A \) on \( A \otimes A \otimes A \).

Let \((A, \mu, \Delta)\) be an \( \epsilon \)-bialgebra and let us write \( \Delta(a) = \sum_i (a) a_i (1) \otimes a_i (2) \) for all \( a \in A \) (Sweedler’s notation). It is shown in [4] that if one defines \( a \circ b := \sum_i (b) b_i (1) a_i (2) b_i (2) \) for all \( a, b \in A \), then \((A, \circ)\) is a pre-Lie algebra. This yields a functor which associates to any \( \epsilon \)-bialgebra \((A, \mu, \Delta)\) the corresponding pre-Lie algebra \((A, \circ)\), and which is identity on morphisms.

The fundamental observation of Aguiar is that the restriction of this functor to quasi-triangular \( \epsilon \)-bialgebras factors in a natural way through the category of Loday dendriform algebras, as in the following diagram:

\[
\begin{array}{ccc}
\text{QT} \epsilon\text{-bialg}, \mu, r & \xrightarrow{r-a-a-r} & \epsilon\text{-bialg}, \mu, \Delta \\
\downarrow \text{Assoc}^{\text{dend}}, <, > & & \downarrow \text{pre-Lie}, \circ \\
\sum_i a_i b_i, \sum_i u_i a_i b_i & & \sum_i b_i (1) a_i (2)
\end{array}
\]

In order to explain this diagram, we first recall from [4] that a quasi-triangular \( \epsilon \)-bialgebra is a triple \((A, \mu, r)\), where \((A, \mu)\) is an associative algebra and \( r \in A \otimes A \) is a solution of the associative Yang-Baxter equation

\[
\text{AYB}(r) := r_{13} r_{12} - r_{12} r_{23} + r_{23} r_{13} = 0 .
\]

Let \((A, \mu, r)\) be a quasi-triangular \( \epsilon \)-bialgebra and write \( r = \sum_i u_i \otimes v_i \).

On the one hand, setting for all \( a \in A \)

\[
\Delta_r(a) := r \cdot a - a \cdot r ,
\]

we get an \( \epsilon \)-bialgebra \((A, \mu, \Delta_r)\). On the other hand, the map \( \mathfrak{R} : A \to A \), defined for all \( a \in A \) by \( \mathfrak{R}(a) = \sum_i u_i av_i \), is a Rota-Baxter operator for \( A \), and so, by Proposition 3.2 the products \(<, >\) defined for all \( a, b \in A \) by

\[
a < b := \sum_i a_i bv_i , \quad a > b := \sum_i u_i av_i b ,
\]

make \((A, <, >)\) into a Loday dendriform algebra. The above construction which associates to a solution of the associative Yang-Baxter equation an \( \epsilon \)-bialgebra has a natural generalization, given by the next proposition.
gives necessary and sufficient conditions on \( r \in A \otimes A \) so that the triplet \((A, \mu, \Delta_r)\) is an \( \epsilon \)-bialgebra, with \( \Delta_r \) defined by \((3.7)\).

**Proposition 3.15** \((2)\). Let \((A, \mu)\) be an associative algebra and let \( r \in A \otimes A \). Then \((A, \mu, \Delta_r)\) is an \( \epsilon \)-bialgebra if and only if \( \text{AYB}(r) \) is invariant, i.e., \( a \cdot \text{AYB}(r) = \text{AYB}(r) \cdot a \), for all \( a \in A \). One then says that \((A, \mu, r)\) is a coboundary \( \epsilon \)-bialgebra.

**Proposition 3.16.** Let \((A, \mu, \sum_i u_i \otimes v_i)\) be a coboundary \( \epsilon \)-bialgebra.

1. The linear map \( \mathcal{R} : A \to A \), defined for all \( a, b \in A \) by \( \mathcal{R}(a) := \sum_i u_i a v_i \), is a weak Rota-Baxter operator for \( A \).

2. For \( a, b \in A \), let \( a \prec b := \mathcal{R}(a)b = \sum_i u_i a v_i b \) and \( a \succ b := a \mathcal{R}(b) = \sum_i au_i b v_i \). Then \((A, \prec, \succ)\) is an inner-associative \( A_3 \)-dendriform algebra.

**Proof.** We only need to prove (1), because (2) follows from it by Example \([3.10]\). To do this, we show that the linear map \( \omega : A \otimes A \to A \), defined for \( a, b \in A \) by \( \omega(a \otimes b) := \mathcal{R}(a) \mathcal{R}(b) - \mathcal{R}(a \mathcal{R}(b) + \mathcal{R}(a)b) \) satisfies \( \omega(a \otimes b)c = c \omega(a \otimes b) \) for all \( a, b, c \in A \). We do this by relating \( \omega \) with \( \text{AYB}(r) \). Without loss of generality, we may assume that the associative algebra \( A \) has a unit, denoted \( 1_A \). Writing \( r = \sum_i u_i \otimes v_i \),

\[
\text{AYB}(r) = r_{13}r_{12} - r_{12}r_{23} + r_{23}r_{13}
\]

\[
= \sum_{i,j} (u_i \otimes 1_A \otimes v_j)(u_j \otimes v_j \otimes 1_A) - \sum_{i,j} (u_i \otimes v_i \otimes 1_A)(1_A \otimes u_j \otimes v_j)
\]

\[
+ \sum_{i,j} (1_A \otimes u_i \otimes v_j)(u_j \otimes 1_A \otimes v_j)
\]

\[
= \sum_{i,j} (u_i u_j \otimes v_j \otimes v_i - u_i \otimes u_j v_j \otimes v_i + u_j \otimes u_i \otimes v_i v_j)
\];

\[
\omega(a \otimes b) = \sum_{i,j} u_i a v_i u_j b v_j - \mathcal{R} \left( \sum_i a u_i b v_i + \sum_i u_i a v_i b \right)
\]

\[
= - \sum_{i,j} (u_j a u_i b v_i v_j + u_j u_i a v_i b v_j - u_i a v_i u_j b v_j)
\]

\[
= - \sum_{i,j} (u_i u_j a v_i b v_j - u_i a v_i u_j b v_j + u_j a u_i b v_i v_j)
\].

If we compare these two expressions and we write \( \text{AYB}(r) \) as \( \text{AYB}(r) = \sum_k X_k \otimes Y_k \otimes Z_k \), then we see that \( \omega(a \otimes b) = - \sum_k X_k a Y_k b Z_k \). The invariance of \( \text{AYB}(r) \), which can be written as \( \sum_k cX_k \otimes Y_k \otimes Z_k = \sum_k X_k \otimes Y_k \otimes Z_k c \) for all \( c \in A \) therefore yields \( \omega(a \otimes b)c = - \sum_k X_k a Y_k b Z_k c = - \sum_k cX_k a Y_k b Z_k = c \omega(a \otimes b) \), as was to be shown. \( \square \)

**Proposition 3.16** leads to the following commutative diagram, generalizing Aguiar’s commutative diagram:
3.4. Curved Rota-Baxter systems. We show in this paragraph that curved Rota-Baxter systems also provide examples of inner-associative $A_3$-dendriform algebras. We first recall the definition of such systems (see [22]).

Definition 3.17. Let $A$ be an associative algebra endowed with linear maps $\mathcal{R}, \mathcal{S} : A \to A$ and $\omega : A \otimes A \to A$. The 4-tuple $(A, \mathcal{R}, \mathcal{S}, \omega)$ is called a curved Rota-Baxter system if the following conditions are satisfied, for all $a, b \in A$:

\begin{align*}
\mathcal{R}(a)b & = \mathcal{R}(\mathcal{R}(a)b + a\mathcal{S}(b)) + \omega(a \otimes b), \quad (3.9) \\
\mathcal{S}(a)b & = \mathcal{S}(\mathcal{R}(a)b + a\mathcal{S}(b)) + \omega(a \otimes b). \quad (3.10)
\end{align*}

The definition is easily generalized to arbitrary algebras, but not the results which follow; this is why we consider only the case of associative algebras. Notice that weak Rota-Baxter operators on an associative algebra $A$ correspond to curved Rota-Baxter systems $(A, \mathcal{R}, \mathcal{S}, \omega)$ with $\mathcal{R} = \mathcal{S}$ and having the property that $\omega$ takes values in $Z(A)$, the center of $A$ (which coincides with $C(A)$ because $A$ is associative). Under this correspondence, the following proposition generalizes item (2) of Proposition 3.16.

Proposition 3.18. Let $(A, \mathcal{R}, \mathcal{S}, \omega)$ be a curved Rota-Baxter system. Define two new products on $A$ by setting $a \triangleright b := \mathcal{R}(a)b$ and $a \prec b = a\mathcal{S}(b)$, for all $a, b \in A$. Then $(A, \prec, \triangleright)$ is an $A_3$-dendriform algebra if and only if $\omega$ takes values in $Z(A)$. In any case, $(A, \prec, \triangleright)$ is inner-associative.

Proof. $(A, \prec, \triangleright)$ is inner-associative, since for all $a, b, c \in A$,

\[(a \triangleright b) \prec c = (\mathcal{R}(a)b) \prec c = \mathcal{R}(\mathcal{R}(a)b + a\mathcal{S}(c)) = a \triangleright (b\mathcal{S}(c)) = a \triangleright (b \prec c).\]

Using (3.10) we find that

\[(a \prec b) \prec c - a \prec (b \prec c + b \triangleright c) = a\mathcal{S}(b)\mathcal{S}(c) - a\mathcal{S}(b\mathcal{S}(c) + \mathcal{R}(b)c) = a\omega(b \otimes c),\]

and similarly, using (3.9),

\[b \triangleright (c \triangleright a) - (b \triangleright c + b \triangleright c) \triangleright a = \omega(b \otimes c)a.\]

So, (2.15) is satisfied (i.e., $(A, \prec, \triangleright)$ is an $A_3$-dendriform algebra) if and only if $a\omega(b \otimes c) = \omega(b \otimes c)a$, for all $a, b, c \in A$; in turn, this is equivalent to $\omega(b \otimes c) \in Z(A)$, for all $b, c \in A$. □

The proof also shows that when $\omega = 0$ the $A_3$-dendriform algebra which is obtained is a Loday dendriform algebra; this was already observed in [10].

It was proven in [9] that, if $(A, \mathcal{R}, \mathcal{S}, \omega)$ is a curved Rota-Baxter system and we define a new product on $A$ by $a \circ b = \mathcal{R}(a)b - b\mathcal{S}(a)$, then $(A, \circ)$ is a
pre-Lie algebra if and only if \( \omega(ab - ba) \in Z(A) \), for all \( a, b \in A \). In particular, \((A, \circ)\) is a pre-Lie algebra when \( \omega \) takes values in \( Z(A) \). We recover this result as a direct consequence of Example 2.10 and Proposition 3.18.

**Example 3.19.** Let \( A \) be an associative algebra and \( R, S : A \to A \) be a left, respectively right Baxter operator, i.e., \( R(a)R(b) = R(R(a)b) \) and \( S(a)S(b) = S(aS(b)) \), for all \( a, b \in A \), satisfying the extra condition that \( R(a)S(b) = R(aS(b)) = S(R(a)b) \) for all \( a, b \in A \). Then \((A, R, S, \omega)\) is a curved Rota-Baxter system, where \( \omega : A \otimes A \to A \) is defined by \( \omega(a \otimes b) = -R(a)S(b) \). If moreover \( R(a), S(a) \in Z(A) \) for all \( a \in A \), then \( \omega \) takes values in \( Z(A) \), hence Proposition 3.18 can be applied to yield an (inner-associative) \( A_3 \)-dendriform algebra. A particular case of this example already appears in [9], where it is shown that if \( r = \sum_i x_i \otimes y_i \) and \( s = \sum_j z_j \otimes w_j \) are invariant, then the linear maps \( R, S : A \to A \) and \( \omega : A \otimes A \to A \), defined for \( a \in A \) by

\[
R(a) := \sum_i x_i ay_i \quad \text{and} \quad S(a) := \sum_j z_j aw_j \quad \text{and} \quad \omega(a \otimes b) = -R(a)S(b),
\]

make \((A, R, S, \omega)\) into a curved Rota-Baxter system.

**4. Dendriform algebras in polarized form**

In this section, we introduce the notion of a dendriform algebra for algebras \((A, \cdot, [\cdot, \cdot])\), where \( \cdot \) is commutative and \([\cdot, \cdot]\) is anticommutative, satisfying again any finite collection of (extra) relations. It will be shown that this notion of a dendriform algebra corresponds to the one introduced in Section 2 via a polarization functor which we will also introduce.

**4.1. Polarized algebras.** We first define the class of algebras which we will consider in this section.

**Definition 4.1.** An algebra \((A, \cdot, [\cdot, \cdot])\) is said to be a **polarized algebra** when \( \cdot \) is commutative and \([\cdot, \cdot]\) is anticommutative, i.e., for all \( a, b \in A \),

\[
b \cdot a = a \cdot b, \quad \text{and} \quad [b, a] = -[a, b].
\]

The choice of the terminology **polarized** will become clear in Section 4.4 when we will see how we can obtain polarized algebras from algebras with one product by using a procedure called **polarization**.

**Example 4.2.** If \((A, \cdot)\) is a commutative algebra, we can make it into a polarized algebra \((A, \cdot, [\cdot, \cdot])\) simply by adding any anticommutative product \([\cdot, \cdot]\) on \( A \), for example the trivial (zero) product. Similarly, any anticommutative algebra \((A, [\cdot, \cdot])\) can be made into a polarized algebra.
Example 4.3. Recall (for example from [14]) that an algebra \((A, \cdot, \{\cdot, \cdot\})\) is a Poisson algebra if \((A, \cdot)\) is a commutative associative algebra, \((A, \{\cdot, \cdot\})\) is a Lie algebra and the two products are compatible in the sense that
\[
\{a \cdot b, c\} = a \cdot \{b, c\} + \{a, c\} \cdot b ,
\]
for all \(a, b, c \in A\). The latter condition can also be formulated by saying that the product \(\{\cdot, \cdot\}\), usually referred to as the Poisson bracket, is a derivation in each one of its arguments. Clearly, every Poisson algebra \((A, \cdot, \{\cdot, \cdot\})\) is a polarized algebra. We will come back several times to this example.

4.2. Polarized \(C\)-dendriform algebras. In analogy with Definition 2.3, we now define the notion of a dendriform algebra for a polarized algebra.

Here, \(\mathcal{R}_1 = 0, \ldots, \mathcal{R}_k = 0\) are given relations involving the products “\(\cdot\)” and \([\cdot, \cdot]\) (only). The category of all polarized algebras satisfying these relations is denoted by \(\mathcal{C}_{\text{pol}}\). The morphisms in \(\mathcal{C}_{\text{pol}}\) are the algebra homomorphisms.

Definition 4.4. An algebra \((A, \ast, \circ)\) is said to be a polarized \(C\)-dendriform algebra if \((A \times A, \circ, \{\cdot, \cdot\}) \in \mathcal{C}_{\text{pol}}\), where \(\circ\) and \([\cdot, \cdot]\) are defined, for \((a, x)\) and \((b, y)\) in \(A \times A\), by
\[
(a, x) \circ (b, y) := (a \ast b + b \ast a, a \ast y + b \ast x) , \quad (4.1)
\]
\[
[(a, x), (b, y)] := (a \circ b - b \circ a, a \circ y - b \circ x) . \quad (4.2)
\]

The category of all polarized \(C\)-dendriform algebras (over \(R\)) is denoted by \(\mathcal{C}_{\text{pol}}^{\text{dend}}\). The morphisms in this category are the algebra homomorphisms. Setting \(x = y = 0\) in (4.1) and in (4.2), we see that we have again a faithful functor \(\mathcal{C}_{\text{pol}}^{\text{dend}} \to \mathcal{C}_{\text{pol}}\), defined on objects by \((A, \ast, \circ) \mapsto (A, \cdot, \{\cdot, \cdot\})\), where the two new products on \(A\) are defined, for all \(a, b \in A\), by
\[
a \cdot b := a \ast b + b \ast a , \quad \text{and} \quad [a, b] := a \circ b - b \circ a . \quad (4.3)
\]

Remark 4.5. The above definition of a polarized \(C\)-dendriform algebra admits the following natural generalization. An algebra \((A, \ast, \circ, |, \square)\) is said to be a polarized \(C\)-tridendriform algebra if \((A, |, \square)\) is a polarized algebra and \((A \times A, \circ, \{\cdot, \cdot\}, |, \square) \in \mathcal{C}_{\text{pol}}\), where \(\circ\) and \([\cdot, \cdot]\) are defined for \((a, x)\) and \((b, y)\) in \(A \times A\), by
\[
(a, x) \circ (b, y) := (a \ast b + b \ast a + a | b, a \ast y + b \ast x + x | y), \quad (4.4)
\]
\[
[(a, x), (b, y)] := (a \circ b - b \circ a + a \square b, a \circ y - b \circ x + x \square y) . \quad (4.5)
\]

We have a functor from the category \(\mathcal{C}_{\text{pol}}^{\text{trid}}\) of all polarized \(C\)-tridendriform algebras to \(\mathcal{C}_{\text{pol}}\), defined on objects by \((A, \ast, \circ, |, \square) \mapsto (A, \cdot, \{\cdot, \cdot\})\), where
\[
a \cdot b := a \ast b + b \ast a + a | b , \quad \text{and} \quad [a, b] := a \circ b - b \circ a + a \square b ,
\]
for all \(a, b \in A\). Any polarized \(C\)-dendriform algebra \((A, \ast, \circ)\) can be seen in a natural way as a polarized \(C\)-tridendriform algebra by considering \((A, \ast, \circ, |, \square)\), where the products \(|\) and \(\square\) are trivial.
4.3. Algebras defined by multilinear relations. As in the case of a $C$-dendriform algebra (see Section 2.3), the relations which every polarized $C$-dendriform algebra must satisfy, can be easily computed when the relations $R_i = 0$ of $C_{pol}$ are multilinear. We show again how these relations can be computed for a trilinear relation $R = 0$. Thanks to commutativity and anticommutativity, $R$ is of the form

$$R(a_1, a_2, a_3) = \sum_{\sigma \in A_3} l_\sigma(a_{\sigma(1)} \cdot a_{\sigma(2)}) \cdot a_{\sigma(3)} + \sum_{\sigma \in A_3} l'_\sigma [a_{\sigma(1)}, [a_{\sigma(2)}, a_{\sigma(3)}]]$$

$$+ \sum_{\sigma \in A_3} l''_\sigma [a_{\sigma(1)}, a_{\sigma(2)}] \cdot a_{\sigma(3)} + \sum_{\sigma \in A_3} l'''_\sigma [a_{\sigma(1)} \cdot a_{\sigma(2)}, a_{\sigma(3)}],$$

where the constants $l_\sigma, \ldots, l'''_\sigma$ belong to the base ring $R$. Notice that we have the same number of constants as in the case of an algebra with one product, namely 12; we will see the reason for this in Section 4.4.

By trilinearity, the relations which must be satisfied by every algebra in $C_{dend}$ are obtained by demanding that the relations are satisfied on all possible triplets of elements of $A \times A$, taken from the union of $A_0$ and $A_1$, which is a generating set of $A \times A$. In the following two tables, we exhibit the products $\circ$ and $[\cdot, \cdot]$ in terms of these generators:

$$\begin{array}{c|cc}
\circ & 0 & 1 \\
\hline
a_0 & a \cdot b + b \cdot a_0 & a \cdot b_1 \\
a_1 & b \cdot a_1 & (0, 0)
\end{array}$$

$$\begin{array}{c|cc}
[\cdot, \cdot] & 0 & 1 \\
\hline
a_0 & a \circ b - b \circ a_0 & a \circ b_1 \\
a_1 & -b \circ a_1 & (0, 0)
\end{array}$$

Table 3. The products $\circ$ and $[\cdot, \cdot]$ for generators of $A \times A$.

The observations made in the case of algebras with one product are, mutatis mutandis, also valid here, namely the relations are trivially satisfied when one takes at least two elements in $A_1$, and the relation which is obtained by taking all elements in $A_0$ is a consequence of the relations which are obtained by taking two elements in $A_0$ and taking the other element in $A_1$. To see the latter claim, it suffices to consider, as in (2.8), the following formulas, which follow easily from Table 3:

$$(a_0 \circ 0) \circ 1 + (a_0 \circ 1) \circ 0 + (a_1 \circ 0) \circ 0 = (a \cdot b) \cdot c_0,$$

$$[[a_0, 0] \circ 1 + [a_0, 1] \circ 0 + [a_1, 0] \circ 0] = [a, b] \cdot c_1;$$

$$[[a_0 \circ 0, 1] + [a_0 \circ 1, 0] + [a_1 \circ 0, 0]] = [a \cdot b, c],$$

$$[[[a_0, 0], 1] + [[[a_0, 1], 0] + [[[a_1, 0], 0]]] = [[a, b], c],$$

together with the four formulas, corresponding to the other parenthesizing. We have used (4.3) to write the above formulas in a compact form.
Example 4.6. We return to the example of a Poisson algebra (see Example 2.15). We show how to obtain the relations which an algebra \((A, *, \circ)\) must satisfy in order to belong to the corresponding dendriform category, which we denote by \(P_{\text{pol}}\). We have three trilinear relations defining a Poisson algebra, namely the associativity of ",", the biderivation property and the Jacobi identity. We start with associativity of \(\circ\), taking first \(a_0, 0 \in A_0\) and \(1 \in A_1\), from which we find
\[
(a \ast b + b \ast a) \ast c_1 = (a_0 \circ 0) \circ 1 = a_0 \circ (0 \circ 1) = a \ast (b \ast c)_1,
\]
so that
\[
a \ast (b \ast c) = (a \ast b + b \ast a) \ast c,
\]
for all \(a, b, c \in A\), which means that \((A, \ast)\) is a Zinbiel algebra (see Example 2.15). Similarly, taking \(a_0, 0 \in A_0\) and \(1 \in A_1\), we find
\[
c \ast (a \ast b)_1 = (a_0 \circ 1) \circ 0 = a_0 \circ (1 \circ 0) = a \ast (c \ast b)_1,
\]
so that \(c \ast (a \ast b) = a \ast (c \ast b)\) for all \(a, b, c \in A\), which means that \((A, \ast)\) is a NAP algebra (see Example 2.15). Since every Zinbiel algebra is a NAP algebra, we don’t need to state the NAP condition for \(\ast\). By symmetry (recall that "\(\cdot\)" is commutative) we also don’t need to consider the case of \(0, 0 \in A_0\) and \(a_1 \in A_1\). Similarly, the derivation property \(a \cdot b, c = [a, c] \cdot b + a \cdot [b, c]\) is symmetric in \(a\) and \(b\), so we get by the above procedure only two equations, which can be written in the following symmetric form:
\[
(a \ast b + b \ast a) \circ c = a \ast (b \circ c) + b \ast (a \circ c), \quad (4.7)
\]
\[
(a \circ b - b \circ a) \ast c = a \ast (b \circ c) - b \circ (a \ast c). \quad (4.8)
\]
Finally, because the Jacobi identity is symmetric in all of its variables, we get only one equation from the Jacobi identity, namely the pre-Lie condition
\[
(a \circ b - b \circ a) \circ c = a \circ (b \circ c) - b \circ (a \circ c). \quad (4.9)
\]
It follows that equations (4.6) – (4.9) are the four relations of \(P_{\text{pol}}\).

An algebra \((A, \ast, \circ)\) which satisfies (4.6) – (4.9) (i.e., an algebra in \(P_{\text{pol}}\)) is exactly what M. Aguiar in [3] calls a pre-Poisson algebra. Thus, our general procedure to obtain \(C_{\text{pol}}\) from \(C_{\text{pol}}\) yields a canonical way to obtain the concept of a pre-Poisson algebra from the concept of a Poisson algebra.

Remark 4.7. The relations which every polarized \(C\)-tridendriform algebra must satisfy are similarly obtained when the relations \(R_i\) are multilinear, but as in the case of \(C\)-dendriform algebras, the relations obtained by substituting general elements from the union of \(A_0\) and \(A_1\) in \(R_i\) are all non-trivial, so there are many more relations for a polarized \(C\)-tridendriform algebra than for a polarized \(C\)-dendriform algebra. The only relation which we don’t need to consider is the one obtained by substituting only elements from \(A_0\) in \(R_i\), since the obtained relation is the sum of all the other relations obtained by substituting elements from the union of \(A_0\) and \(A_1\) in \(R_i\).
Example 4.8. We continue example 4.6 and give the relations which an algebra \((A, *, \circ, |, \Box)\) must satisfy in order to be a polarized \(\mathcal{P}\)-tridendriform algebra. We get the following three equations from associativity, where the first one is obtained using the same substitutions as (4.6), while the two other equations are obtained respectively by substituting in the associativity relation two or three elements from \(A_0\):

\[
\begin{align*}
  a \ast (b \ast c) &= (a \ast b) \ast c + (b \ast a) \ast c + (a \mid b) \ast c, \\
  a \ast (b \mid c) &= (a \ast b) \mid c, \\
  a \mid (b \mid c) &= (a \mid b) \mid c.
\end{align*}
\]

By symmetry, the Jacobi identity implies that we only get three relations from it, by substituting respectively one, two or three elements from \(A_0\):

\[
\begin{align*}
  a \circ (b \circ c) - b \circ (a \circ c) &= (a \circ b - b \circ a + a \Box b) \circ c, \\
  (a \circ b) \Box c &= a \circ (b \Box c) + (a \circ c) \Box b, \\
  0 &= (a \Box b) \Box c + b \Box (c \Box a) + c \Box (a \Box b).
\end{align*}
\]

Finally, the derivation property leads to the following five relations:

\[
\begin{align*}
  a \ast (b \circ c) + b \ast (a \circ c) &= (a \ast b + b \ast a + a \mid b) \circ c, \\
  a \ast (b \circ c) - b \circ (a \ast c) &= (a \circ b - b \circ a + a \Box b) \ast c, \\
  (a \ast b) \Box c &= a \ast (b \Box c) + b \mid (a \circ c), \\
  c \circ (a \mid b) &= a \mid (c \circ b) + b \mid (c \circ a), \\
  (a \mid b) \Box c &= a \mid (b \Box c) + b \mid (a \Box c).
\end{align*}
\]

These 11 equations are, together with the commutativity and anticommutativity of \(|\) and \(\Box\), precisely the 13 relations [19, Eqs. 48–60] which define the notion of a post-Poisson algebra.

4.4. Polarization. We show in this subsection how the two notions of dendriform algebras, introduced in Sections 2.2 and 4.2, are related via a process of polarization. We first recall from [18] the notion of polarization for an algebra \((A, \mu)\). Two new products “\(\cdot\)” and \([\cdot, \cdot]\) are defined on \(A\) by setting

\[
\begin{align*}
  a \cdot b &= \frac{1}{2} (ab + ba), \\
  [a, b] &= \frac{1}{2} (ab - ba),
\end{align*}
\]

for all \(a, b \in A\) (recall that 2 is assumed invertible in the base ring \(R\)). This procedure is called polarization. Notice that we can easily reconstruct \(\mu\) from the two products “\(\cdot\)” and \([\cdot, \cdot]\), because \(ab = a \cdot b + [a, b]\), for all \(a, b \in A\); this is what is called depolarization. Thus, we have a natural way to associate to each algebra \((A, \mu)\) a polarized algebra \((A, \cdot, [\cdot, \cdot])\) and vice-versa. Obviously, a commutative algebra corresponds to a polarized algebra with \([\cdot, \cdot] = 0\) and vice-versa, and similarly for an anticommutative algebra, so we will only be interested in polarized algebras for which both products are non-trivial.
Example 4.9. The P-algebras introduced in Example 2.13 correspond by polarization/depolarization to Poisson algebras, see [18].

Let $\mathcal{C}$ be the category of all algebras (with one product) which satisfy a given collection of relations $R_1 = 0, \ldots, R_k = 0$. Applying polarization to all objects of $\mathcal{C}$ leads to a category $\mathcal{C}_{pol}$ of polarized algebras; the morphisms in this new category are the algebra homomorphisms. Thus, by definition, \((A, \cdot, [\cdot, \cdot]) \in \mathcal{C}_{pol}\) if and only if \((A, \mu) \in \mathcal{C}\), with $\mu(a, b) := a \cdot b + [a, b]$ for all $a, b \in A$. Alternatively, we can polarize the relations $R_i = 0$ of $\mathcal{C}$ by substituting in $R$ for $ab = \mu(a, b)$ the sum $a \cdot b + [a, b]$. Then $\mathcal{C}_{pol}$ can also be described as the category of all polarized algebras, satisfying these relations. Notice that the relations in $\mathcal{C}$ are multilinear if and only if the polarized relations are multilinear. The above polarization and depolarization procedures define inverse functors $\mathcal{C} \rightarrow \mathcal{C}_{pol}$ and $\mathcal{C}_{pol} \rightarrow \mathcal{C}$ which make $\mathcal{C}$ and $\mathcal{C}_{pol}$ into isomorphic categories.

For given relations $R_1 = 0, \ldots, R_k = 0$ (in one operation) we have constructed four categories $\mathcal{C}$, $\mathcal{C}_{pol}$, $\mathcal{C}_{dend}$ and $\mathcal{C}_{dend pol}$ and three functors, as in the following diagram, which we completed into a square by adding a pair of inverse arrows between $\mathcal{C}_{dend}$ and $\mathcal{C}_{dend pol}$; the commutativity of the diagram is easily established.

\[
\begin{array}{ccc}
\mathcal{C}, \mu & \xlongleftarrow{(ab+ba)/2,(ab-ba)/2} & \mathcal{C}_{pol}, [\cdot, \cdot] \\
\xlongleftarrow{a-b+[a,b]} & & \xlongrightarrow{a+b+a, aoa-boa}
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{C}_{dend}, \prec, \succ & \xlongleftrightarrow{a+b+b \succ a, a+b \prec a} & \mathcal{C}_{dend pol}, \ast, \circ \\
\xlongleftrightarrow{b \ast a - boa, a \ast b + ab} & & \xlongleftarrow{b \ast a - boa, a \ast b + ab}
\end{array}
\] (4.11)

In analogy with the upper arrows, we call the lower arrows polarization and depolarization. These arrows define functors which are isomorphisms of categories, just like the upper arrows. Notice that by commutativity of the diagram, a polarized $\mathcal{C}$-dendriform algebra can also be defined as an algebra $(A, \ast, \circ)$ whose depolarized form $(A, \prec, \succ)$ is a $\mathcal{C}$-dendriform algebra (which justifies the terminology). Indeed, according to the definition and by depolarization, $(A, \ast, \circ) \in \mathcal{C}_{pol}$ if and only if $(A \times A, \bullet) \in \mathcal{C}$, with

$$(a, x) \bullet (b, y) = (a, x) \odot (b, y) + [(a, x), (b, y)]$$

$$= (b \ast a - b \circ a + a \ast b + a \circ b, a \ast y + a \circ y + b \ast x - b \circ x)$$

$$= (a \prec b + a \succ b, a \succ y + x < b).$$

We have obtained exactly the condition that the depolarized form $(A, \prec, \succ)$ of $(A, \ast, \circ)$ belongs to $\mathcal{C}_{dend}$ (see Definition 2.3), showing our claim.

Remark 4.10. Polarization and depolarization can also be defined for tridendriform and polarized tridendriform algebras, leading for any category of algebras $\mathcal{C}$ as above, to an isomorphism of the category $\mathcal{C}_{trid}$ of $\mathcal{C}$-tridendriform
algebras and the category $C_{\text{trid}}^{\text{pol}}$ of polarized $C$-tridendriform algebras. On objects, the pair of inverse isomorphisms is given by

$$ C_{\text{trid}}, \prec, \succ, \rightarrow C_{\text{trid}}^{\text{pol}}, *, \circ, \Box. \quad (4.12) $$

They extend the pair of lower arrows in (4.11) and lead to a commutative diagram, as in (4.11).

**Example 4.11.** We return once more to the case of $P$-algebras and Poisson algebras which, as we recall, correspond under polarization; this is why we also refer to Poisson algebras as polarized $P$-algebras, and similarly for their dendriform and tridendriform algebras. Specialized to this case, the above results can be summarized in the following commutative diagram, in which the horizontal arrows are given by the horizontal arrows in (4.11) and (4.12):

```
P, \mu \quad \longrightarrow \quad P_{\text{pol}}, *, [\cdot, \cdot] 
```

It was already pointed out by M. Aguiar in [3] that, if $(A, *, \circ) \in P_{\text{pol}}$, i.e., is a pre-Poisson algebra, and we define new operations on $A$ by $a \times b = a \succ b + b \prec a$ and $\{a, b\} = a \circ b - b \circ a$, for all $a, b \in A$, then $(A, \times, \{\cdot, \cdot\})$ is a Poisson algebra. It corresponds to the composition of the two right arrows in the diagram.

### 4.5. Application I: deformations of dendriform algebras.

In [3], M. Aguiar introduced the notion of deformation for a commutative Loday dendriform algebra $(A, \prec, \succ)$ and he showed that such a deformation makes $(A, \times, \circ)$ into a pre-Poisson algebra, where $\times$ stands for $\succ$ and where the product $\circ$ on $A$ is constructed from the first order deformation terms of the products $\prec$ and $\succ$. In this section we generalize this result to arbitrary $C$-dendriform algebras, giving a conceptual proof of Aguiar’s result.

As before, $C$ denotes in this section the category of all $R$-algebras satisfying a fixed set of relations $R_1 = 0, \ldots, R_k = 0$. Let $\nu$ be an indeterminate and let $R^\nu$ denote the ring of formal power series $R[[\nu]]$. More generally, for any $R$-module $A$ we denote by $A^\nu$ the $R^\nu$-module of formal power series in $\nu$ with coefficients in $A$. For a formal power series $X \in A^\nu$ its evaluation at 0, which is the constant term of $X$, is denoted by $X_0$.

**Definition 4.12.** Let $(A, \prec_0, \succ_0)$ be a commutative $C$-dendriform algebra and denote $a \times b := a \succ_0 b = b \prec_0 a$ for all $a, b \in A$. An $R^\nu$-algebra
$(A^\nu, \prec, \succ)$ is said to be a formal deformation of $(A, \prec_0, \succ_0)$ if $(A^\nu, \prec, \succ)$ is a $C$-dendriform algebra over $R^\nu$ and for any $a, b \in A$,

$$(a \succ b)_0 = a \succ_0 b \quad \text{and} \quad (a \prec b)_0 = a \prec_0 b.$$  

We can then define a new product on $A$ by setting, for all $a, b \in A$,

$$a \circ b := \frac{a \succ b - b \prec a}{2\nu} \bigg|_{\nu=0} \quad \text{(4.13)}$$

The algebra $(A, \times, \mathcal{O})$ is called the infinitesimal algebra of the deformation.

The question which we study here is to which category the infinitesimal algebra $(A, \times, \mathcal{O})$ belongs. When $C$ is the category of associative algebras the answer is provided by Aguiar [3], who showed that $(A, \times, \mathcal{O})$ is a pre-Poisson algebra.

In order to answer the above question in general, we first introduce a few more notions and notations. Let $M$ be a monomial which involves the (commutative and anticommutative) products “$\cdot$” and $[\cdot, \cdot]$ only. We define the weight of $M$ as the number of operations $[\cdot, \cdot]$ in $M$. Similarly, for a monomial $M$ in the products $\ast$ and $\circ$, its weight is the number of operations $\circ$ in $M$. In either case, a sum $\mathcal{R}$ of such monomials is said to be homogeneous of weight $m$ if each of its terms has weight $m$. The lowest weight part of $\mathcal{R}$ is denoted by $\mathcal{R}_0$. Finally, we denote by $C_{\text{pol}}$ (resp. by $C_{\text{dend}}$) the category of all $R$-algebras satisfying all relations $\mathcal{R} = 0$, where $\mathcal{R}$ runs through the linear space of relations of $C_{\text{pol}}$ (resp. of $C_{\text{dend}}$).

**Proposition 4.13.** Let $(A^\nu, \prec, \succ)$ be a formal deformation of a commutative algebra $(A, \prec_0, \succ_0) \in C_{\text{dend}}$, with deformation algebra $(A, \times, \mathcal{O})$. Then

$$(A, \times, \mathcal{O}) \in C_{\text{pol}}.$$  

In particular, when the relations of $C_{\text{dend}}$ are generated by weight homogeneous relations, then $(A, \times, \mathcal{O}) \in C_{\text{pol}}$. Also, when the relations of $C$ are multilinear, $C_{\text{dend}} = (C_{\text{pol}})^{\text{dend}}$, so that $(A, \times, \mathcal{O}) = (C_{\text{pol}})^{\text{dend}}$.

**Proof.** We will only prove here that $(A, \times, \mathcal{O}) \in C_{\text{dend}}$ leaving the more technical proof that $C_{\text{dend}} = (C_{\text{pol}})^{\text{dend}}$ to the end of the section.

Given a formal deformation $(A^\nu, \prec, \succ)$ we can construct by polarization (which, as we recall, is an isomorphism of categories) an algebra $(A^\nu, \ast, \circ)$, which is a polarized dendriform algebra over $R^\nu$. We define new products $\ast_i$ and $\circ_i$ on $A$ by setting for all $a, b \in A$,

\[
a \ast b = a \ast_0 b + a \ast_1 b \nu + a \ast_2 b \nu^2 + \cdots, \\
a \circ b = a \circ_0 b + a \circ_1 b \nu + a \circ_2 b \nu^2 + \cdots. \quad \text{(4.15)}
\]

Since, by polarization, $a \circ b = (a \succ b - b \prec a)/2$ and $a \ast b = (a \succ b + b \prec a)/2$ (see (4.11)), we have by commutativity of $(A, \prec, \succ)$ that $a \ast_0 b = a \times b$ and
that \( a \circ_0 b = 0 \); also, the definition of \( \circ \) implies that \( a \circ_1 b = a \circ b \) for all \( a, b \in A \). Hence, (4.15) can be rewritten as
\[
a \ast b = a \times b + a \ast_1 b \nu + a \ast_2 b \nu^2 + \cdots ,
\]
(4.16)
\[
a \circ b = a \circ b \nu + a \circ_2 b \nu^2 + \cdots ,
\]
(4.17)
where the dots stand for terms containing \( \nu^i \) with \( i > 2 \). Suppose now that \( R = 0 \) is a relation of \( C_{\text{dend}} \). Writing \( R = R \ast \circ \) to indicate the products which are involved, we may also consider \( R \ast \circ \). We need to show that \( R \ast \circ (a_1, \ldots, a_n) = 0 \) for all \( a_1, \ldots, a_n \in A \). To do this, consider the relation \( R \ast \circ (a_1, \ldots, a_n) = 0 \). In view of (4.16) and (4.17),
\[
R \ast \circ (a_1, a_2, \ldots, a_n) = R \ast \circ (a_1, a_2, \ldots, a_n) \nu^d + \cdots ,
\]
(4.18)
where \( d \) denotes the lowest weight of the terms of \( R \), i.e. the weight of \( R \). It follows that \( (A, \times, \circ) \) satisfies the relation \( R \ast \circ = 0 \), as was to be shown. □

**Example 4.14.** Let \( C \) be the category of all associative algebras (over \( R \)). Then, by polarization, the following are the relations in \( C_{\text{pol}} \) (see [18]):
\[
[a \cdot b, c] = a \cdot [b, c] + [a, c] \cdot b ,
\]
(4.19)
\[
[[a, b], c] = (b \cdot c) \cdot a - (c \cdot a) \cdot b .
\]
(4.20)
Recall that (4.20) implies the Jacobi identity, which is weight homogeneous (of weight 2), just like the derivation property (4.19) (of weight 1). Notice that the lowest weight part of (4.20) is \( (b \cdot c) \cdot a = (c \cdot a) \cdot b \), which is associativity (since “\( \cdot \)” commutative). It follows that \( C_{\text{pol}} \) is the category of Poisson algebras, hence that \( C_{\text{dend}} \) is the category of pre-Poisson algebras. This shows that the infinitesimal algebra of a deformation of a Loday dendriform algebra is a pre-Poisson algebra, as was first shown by Aguiar [3].

**Example 4.15.** The relations which define Poisson algebras (see Example [4.3]) are 3-linear and homogeneous: associativity is of weight 0, the derivation property is of weight 1 and the Jacobi identity is of weight two. For \( A_3 \)-associative algebras and \( LA \)-algebras in polarized form, the relations are also easily written in homogeneous form. It follows that the second part of Proposition 4.13 can be applied to these algebras: in each of these cases, the infinitesimal algebra \( (A, \times, \circ) \) of the deformation belongs to \( C_{\text{pol}} \). 

**Remark 4.16.** Proposition 4.13 is easily adapted to the classical case of formal deformations \( (A, \mu) \) of commutative algebras \( (A, \mu_0) \in C \). The infinitesimal algebra is then defined as \( (A, \mu_0, \circ) \), where
\[
a \circ b := \lim_{\nu \to 0} \frac{\mu(a, b) - \mu(b, a)}{2\nu} .
\]
One shows as in the proof of Proposition 4.13 that \( (A, \mu_0, \circ) \in C_{\text{pol}} \). In the case of associative algebras, \( C_{\text{pol}} \) is the category of Poisson algebras (see Example [4.14]), so we recover the classical result that the infinitesimal algebra of a deformation of an associative algebra is a Poisson algebra.
Remark 4.17. One may also consider more generally deformations of $\mathcal{C}$-tridendriform algebras. Recall that in a commutative $\mathcal{C}$-tridendriform algebra $(A, <, >, \cdot)$, one also requires the last product to be commutative. The weight of a relation $R = R_{*,0,|,\square}$ is now defined such that $*$ and $|$ have weight 0, while $\circ$ and $\square$ have weight 1. It is clear that all the above results generalize to this case. The infinitesimal algebra has now four operations. For example, when $\mathcal{C}$ is the category of associative algebras, the infinitesimal algebra is a post-Poisson algebra (see Example 4.8).

Remark 4.18. We have considered deformations of commutative dendriform algebras, but everything can be easily adapted to anticommutative dendriform algebras: the roles of $*$ and $\circ$ are exchanged in the sense that one will have now that $*_0 = 0$, that $*_1 = \times$ and $\circ_0 = \mathcal{O}$, where $(A, \mathcal{O})$ is the original anticommutative dendriform algebra (written as an algebra with one operation). As we have seen in Section 2.5, $A_3$-associative, LA and P-algebras which are anticommutative are Lie algebras, so there are many natural examples of this case.

To finish this section, we prove that when the relations of $\mathcal{C}$ are multilinear, $\mathcal{C}_{\text{pol}}^{\text{dend}} = (\mathcal{C}_{\text{pol}})^{\text{dend}}$, as stated in (4.14). The property says that the lowest weight parts of all relations in $\mathcal{C}_{\text{pol}}^{\text{dend}}$ are obtained by dendrifying the lowest weight parts of all relations in $\mathcal{C}_{\text{pol}}^{\text{dend}}$. Notice that since each dendrification of a monomial of weight $k$ (involving the products $\cdot$ and $[\cdot,\cdot]$ only) is homogeneous of weight $k$, one has that all algebras in $\mathcal{C}_{\text{pol}}^{\text{dend}}$ are also algebras of $\mathcal{C}_{\text{pol}}^{\text{dend}}$. We therefore only need to prove the reciprocal inclusion.

Notice also that we may restrict ourselves to $n$-linear relations, for a fixed $n$, since the dendrification of a $k$-linear relation is $k$-linear, i.e. we may suppose that all relations $R_1, \ldots, R_k$ of $\mathcal{C}_{\text{pol}}^{\text{dend}}$, and hence also of $\mathcal{C}_{\text{pol}}$, are $n$-linear.

For $0 \leq \ell \leq n$, consider the free $R$-modules $\mathcal{M}_\ell$ and $\mathcal{M}_\ell$, generated by all $\ell$-linear monomials $M$ involving only the (commutative and anticommutative) products $\cdot$ and $[\cdot,\cdot]$ only, respectively generated by all $\ell$-linear monomials $\tilde{M}$ involving only the products $*$ and $\circ$ in $n$ variables, say $x_1, \ldots, x_n$. Their direct sums are denoted $\mathcal{M}$ and $\tilde{\mathcal{M}}$ respectively. Elements of $\mathcal{M}_\ell$ and $\tilde{\mathcal{M}}_\ell$ are also said to be of length $\ell$; notice that the weight of a monomial of length $\ell$ is between 0 and $\ell - 1$ (included). The modules $\mathcal{M}_\ell$ and $\tilde{\mathcal{M}}_\ell$ admit natural decompositions

$$\mathcal{M}_\ell = \mathcal{M}_\ell^0 \oplus \cdots \oplus \mathcal{M}_\ell^{\ell-1} \quad \text{and} \quad \tilde{\mathcal{M}}_\ell = \tilde{\mathcal{M}}_\ell^0 \oplus \cdots \oplus \tilde{\mathcal{M}}_\ell^{\ell-1},$$

where $\mathcal{M}_\ell^i \subset \mathcal{M}_\ell$ and $\tilde{\mathcal{M}}_\ell^i \subset \tilde{\mathcal{M}}_\ell$, are the submodules generated by the monomials of weight $i$. Each monomial $M$ of $\mathcal{M}_\ell$ of length at least two can be decomposed as $M = M_1 \cdot M_2$ or $M = [M_1, M_2]$; this decomposition is unique up to the order of the factors.
We describe the process of dendrification of multilinear relations of $\mathcal{C}_{\text{pol}}$, introduced and studied in Section 4.3, in terms of the linear maps

$$
\varphi_0, \varphi_1, \ldots, \varphi_n : \mathcal{M} \to \tilde{\mathcal{M}},
$$

which we define on monomials $M$, using induction on the length of $M$:

$$
\varphi_0(M) := \begin{cases} 
x_i & \text{if } M = x_i; 
\varphi_0(M_1) * \varphi_0(M_2) + \varphi_0(M_2) * \varphi_0(M_1) & \text{if } M = M_1 \cdot M_2; 
\varphi_0(M_1) \circ \varphi_0(M_2) - \varphi_0(M_2) \circ \varphi_0(M_1) & \text{if } M = [M_1, M_2],
\end{cases}
$$

and for $p = 1, \ldots, n$ we define

$$
\varphi_p(M) := \begin{cases} 
0 & \text{if } M \text{ is independent of } x_p; 
x_p & \text{if } M = x_p; 
\varphi_0(M_1) * \varphi_p(M_2) & \text{if } M = M_1 \cdot M_2 \text{ and } M_2 \text{ depends on } x_p; 
\varphi_0(M_1) \circ \varphi_p(M_2) & \text{if } M = [M_1, M_2] \text{ and } M_2 \text{ depends on } x_p.
\end{cases}
$$

It is clear that these maps are well-defined and that they preserve the length and the weight of a monomial. Notice that, by construction, in all terms of $\varphi_p(M)$ the variable $x_p$ is located at the last position. Therefore, the images of the maps $\varphi_1, \ldots, \varphi_n$ are in direct sum.

To see the relation with dendrification, let $R = 0$ be an $n$-linear relation of $\mathcal{C}_{\text{pol}}$. Then $R \in \mathcal{M}$ and for $p = 1, \ldots, n$, the relation $\varphi_p(R) = 0$ is precisely the relation obtained by substituting in $R_{\cdot[i, \cdot]}$ for the $p$-th variable $(0, x_p)$ and for the $q$-th variable $(x_q, 0)$, where $q \neq p$.

**Lemma 4.19.** The maps $\varphi_0, \ldots, \varphi_n$ are injective.

**Proof.** Let $M$ be a monomial of $\tilde{\mathcal{M}}$. We show that there exists a unique monomial $M \in \mathcal{M}$ such that $M$ is a term of $\varphi_0(M)$; from it the injectivity of $\varphi_0$ is clear.

We do this by induction on the length of $M$. When $M$ is of length 1, the claim is trivially true, so let us assume that the claim is true for monomials of length strictly less than some $\ell \geq 2$. Let $M$ be a monomial of $\tilde{\mathcal{M}}$ of length $\ell$. We can write $M$ uniquely as $M = M_1 \ast M_2$ or $M = M_1 \circ M_2$, up to the order of the factors. By the induction hypothesis there exists a unique couple $(M_1, M_2)$ such that $\tilde{M}_1$ and $\tilde{M}_2$ are terms of $\varphi_0(M_1)$ and $\varphi_0(M_2)$ respectively, and hence such that $\tilde{M}$ is a term of $\varphi_0(M_1) \ast \varphi_0(M_2)$ or $\varphi_0(M_1) \circ \varphi_0(M_2)$, depending on whether $\tilde{M} = \tilde{M}_1 \ast \tilde{M}_2$ or $\tilde{M} = \tilde{M}_1 \circ \tilde{M}_2$.

It follows that, if we define $M := M_1 \cdot M_2$ or $M := [M_1, M_2]$, depending on whether $\tilde{M} = \tilde{M}_1 \ast \tilde{M}_2$ or $\tilde{M} = \tilde{M}_1 \circ \tilde{M}_2$, then $\varphi_0(M) = M$. Since the decomposition of $M$ is unique up to the order of the factors, $M$ is unique. This shows the claim, and hence the injectivity of $\varphi_0$.

In order to show the injectivity of the other maps $\varphi_1, \ldots, \varphi_n$ one proceeds in a similar way: one shows as above that given any monomial $M$ of $\tilde{\mathcal{M}}$ there exists a unique monomial $M$ of $\mathcal{M}$ and a unique integer $p \in \{1, \ldots, n\}$ such that $M$ is a term of $\varphi_p(M)$. \qed
Lemma 4.20. Let $\mathcal{R}_1, \ldots, \mathcal{R}_k \in \mathcal{M}_n$. For any constants $l^p_i \in R$ ($1 \leq i \leq k$ and $p = 1, \ldots, n$), not all equal to zero,

$$\sum_{i=1}^{k} \sum_{p=1}^{\nu} \lambda^p_i \varphi_p(\mathcal{R}_i) = \sum_{p=1}^{n} \varphi_p \left( \sum_{i=1}^{k} \lambda^p_i \mathcal{R}_i \right). \quad (4.21)$$

Proof. For $i = 1, \ldots, k$, let $\mathcal{R}_i = \mathcal{R}_i^0 + \cdots + \mathcal{R}_i^{n-1}$ be the weight decomposition of $\mathcal{R}_i$. By $R$-linearity of the maps $\varphi_p$,

$$\sum_{i=1}^{k} \sum_{p=1}^{\nu} \lambda^p_i \varphi_p(\mathcal{R}_i) = \sum_{\ell=m}^{n-1} A_\ell, \quad \text{where} \quad A_\ell = \sum_{p=1}^{n} \varphi_p \left( \sum_{i=1}^{k} \lambda^p_i \mathcal{R}_i^\ell \right),$$

and where $m$ is chosen such that $A_0, \ldots, A_{m-1} = 0$ and $A_m \neq 0$. Since the maps $\varphi_p$ are weight-preserving, $A_\ell$ is homogeneous of weight $\ell$, and so $A_m$ is equal to the left hand side of $(4.21)$. Let $0 \leq \ell < m$. Then $\sum_{p=1}^{\nu} \varphi_p \left( \sum_{i=1}^{k} \lambda^p_i \mathcal{R}_i^\ell \right) = A_\ell = 0$, so that $\varphi_p \left( \sum_{i=1}^{k} \lambda^p_i \mathcal{R}_i^\ell \right)$ for all $p$, since the images of the maps $\varphi_1, \ldots, \varphi_n$ are in direct sum. Since the maps $\varphi_p$ are injective (Lemma 4.19), this implies that $\sum_{i=1}^{k} \lambda^p_i \mathcal{R}_i^\ell = 0$ for $\ell = 0, \ldots, m-1$. Also, $\sum_{i} \lambda^p_i \mathcal{R}_i^m \neq 0$ since $A_m \neq 0$. It follows that

$$\sum_{i=1}^{k} \lambda^p_i \mathcal{R}_i = \sum_{i=1}^{k} \sum_{\ell=0}^{n-1} \lambda^p_i \mathcal{R}_i^\ell = \sum_{i=1}^{k} \lambda^p_i \mathcal{R}_i^m,$$

so that $A_m$ is also equal to the right hand side of $(4.21)$. \qed

We use Lemma 4.20 to show that all algebras in $\mathcal{C}_{\text{pol}}^\dend$ are also algebras of $(\mathcal{C}_{\text{pol}})^\dend$, so that $\mathcal{C}_{\text{pol}}^\dend = (\mathcal{C}_{\text{pol}})^\dend$. Suppose that $\mathcal{R}_1 = 0, \ldots, \mathcal{R}_k = 0$ is a basis for the module of all $n$-linear relations of $\mathcal{C}_{\text{pol}}$. Let $\mathcal{R} = 0$ be a relation of $\mathcal{C}_{\text{pol}}^\dend$. By definition, $\mathcal{R}$ is the lowest weight part of $\sum_{i=1}^{k} \sum_{p=1}^{n} \lambda^p_i \varphi_p(\mathcal{R}_i)$, for some constants $\lambda^p_i$. In view of the lemma, $\mathcal{R}$ is obtained by dendrification of some relations in $\mathcal{C}_{\text{pol}}$, namely the $p$ relations $\sum_{i=1}^{k} \lambda^p_i \mathcal{R}_i = 0$, for $p = 1, \ldots, n$. This shows that $\mathcal{R} = 0$ is a relation of $(\mathcal{C}_{\text{pol}})^\dend$.

4.6. Application II: filtered dendriform algebras. As a second application of polarized dendriform algebras, we generalize another result of Aguilar [3], which is itself an analogue for Loday dendriform algebras of the well-known result which says that the graded algebra associated to an almost commutative filtered associative algebra is a Poisson algebra.

Let $(A, <, >)$ be an algebra. An (increasing) filtration on $A$ is an increasing sequence of subspaces $A_0 \subset A_1 \subset A_2 \subset \cdots$ such that

$$A = \bigcup_{i \geq 0} A_i \quad \text{and} \quad (A_i < A_j + A_i \succ A_j) \subseteq A_{i+j},$$
for all $i, j \geq 0$. Then $A$ is called a filtered algebra. It is convenient to set $A_i := \{0\}$ for $i < 0$. The associated graded algebra is, as an $R$-module,

$$\text{gr}(A) := \bigoplus_{i \geq 0} A_i/A_{i-1}$$

and inherits two products from $\prec$ and $\succ$, which are still denoted by $\prec$ and $\succ$. They are (well-) defined by setting, for $a \in A_i$ and $b \in A_j$, with $i, j \geq 0$,

$$(a + A_{i-1}) \times (b + A_{j-1}) := (a \prec b + A_{i+j-1}) \in \frac{A_{i+j}}{A_{i+j-1}},$$

and similarly for $\succ$. As in the case of algebras with one operation, $A$ and $\text{gr}(A)$ are canonically isomorphic as $R$-modules, but not as algebras. It is however clear that any $n$-linear relation which is satisfied by the original products $\prec$ and $\succ$ will be satisfied by the induced products.

We will be interested in almost commutative filtered algebras, which have the property that the associated graded algebra is commutative, i.e., $a \prec b = b \succ a$ for all $a, b \in \text{gr}(A)$. As before, we then view $\text{gr}(A)$ as an algebra with one operation $\times$ (setting as usual $\times := \succ$), and $\text{gr}(A)$ can be equipped with another product, defined for $a \in A_i$ and $b \in A_j$, with $i, j \geq 0$ by

$$(a + A_{i-1}) \circ (b + A_{j-1}) := (a \succ b - b \prec a + A_{i+j-2}) \in \frac{A_{i+j-1}}{A_{i+j-2}}. \quad (4.22)$$

The question is now again to which category $(\text{gr}(A), \times, \circ)$ belongs. When $C$ is the category of associative algebras, Aguiar’s answer is that $(\text{gr}(A), \times, \circ)$ is a pre-Poisson algebra, as in the case of deformations (see [2]). We will give here the answer for arbitrary algebras; as we will see, the result is very similar to the result which we obtained for deformations (Section 4.5). The definitions and assumptions are the same as in the latter section, except that the relations of $C$ (and hence of $C_{\text{pol}}$) are supposed here to be multilinear.

**Proposition 4.21.** Suppose that the relations of $C$ are multilinear. Let $(A = \bigcup_i A_i, \prec, \succ)$ be a commutative filtered algebra in $C_{\text{den}}$. On $\text{gr}(A)$, consider the product $\times$, defined for $a, b \in \text{gr}(A)$ by $a \times b := a \succ b$, as well as the product $\circ$, defined by $(4.22)$. Then

$$(\text{gr}(A), \times, \circ) \in C_{\text{pol}} = (C_{\text{pol}})_{\text{den}}.$$

**Proof.** As in the proof of Proposition 4.13 we use polarization to transform the deformation into an algebra of $C_{\text{den}}$. Namely, by polarization, we have a filtered algebra $(A, *, \circ) \in C_{\text{den}}$, having the property that

$$A_i * A_j \subset A_{i+j}, \quad \text{and} \quad A_i \circ A_j \subset A_{i+j-1}. \quad (4.23)$$

In terms of $*$ and $\circ$, the above definitions of $\times$ and $\circ$ now amount to setting, for $a \in A_i$ and $b \in A_j$,

$$(a + A_{i-1}) \times (b + A_{j-1}) := a * b + A_{i+j-1}, \quad (4.24)$$

$$(a + A_{i-1}) \circ (b + A_{j-1}) := a \circ b + A_{i+j-2}. \quad (4.25)$$
Suppose now that $\mathcal{R} = \mathcal{R}_{\times,\circ}$ is an $n$-linear relation of $\mathcal{C}_{\mathrm{pol}}^{\mathrm{dend}}$ and recall that we denote the lowest weight part of $\mathcal{R}$ by $\mathcal{R}_\circ$. The weight of $\mathcal{R}_\circ$ is denoted by $d$. Let $a_1, a_2, \ldots, a_n \in A$ with $a_i \in A_j$, for $i = 1, \ldots, n$. Then
\[
\mathcal{R}_{\times,\circ}(a_1 + A_{j_1-1}, \ldots, a_n + A_{j_n-1}) \\
= \mathcal{R}_{\times,\circ}(a_1, \ldots, a_n) + A_{j_1+\cdots+j_n-d-1} \\
= \mathcal{R}_{\times,\circ}(a_1, \ldots, a_n) + A_{j_1+\cdots+j_n-d-1} = A_{j_1+\cdots+j_n-d-1}.
\]
where we used in the last step that $(A, \times, \circ)$ satisfies $\mathcal{R}$. It follows that $(\mathfrak{gr}(A), \times, \circ)$ satisfies the relation $\mathcal{R}_\circ = 0$. Therefore, $(\mathfrak{gr}(A), \times, \circ)$ satisfies all relations of $\mathcal{C}_{\mathrm{pol}}^{\mathrm{dend}}$, and so $(\mathfrak{gr}(A), \times, \circ) \in \mathcal{C}_{\mathrm{pol}}^{\mathrm{dend}}$.

**Example 4.22.** We return once more to the case where $\mathcal{C}$ is the category of associative algebras. We have already analyzed the relations defining $\mathcal{C}_{\mathrm{pol}}^{\mathrm{dend}}$ in Example 4.14 where we have shown that the lowest weight terms of the relations are the relations which define a pre-Poisson algebra. Hence, we find that if $(A, \prec, \succ)$ is an almost commutative filtered Loday dendriform algebra, then $(\mathfrak{gr}(A), \times, \circ)$ is a pre-Poisson algebra. We thereby recover Aguiar’s result, cited above.

The strong similarity between our results on filtrations and on deformations is not accidental. Indeed, let $(A^\nu, \prec, \succ)$ be a formal deformation of a commutative algebra $(A, \prec_0, \succ_0) \in \mathcal{C}_{\mathrm{dend}}$, where we assume that the relations which define $\mathcal{C}$ are multilinear. Setting $A^\nu_i := \nu^i A^\nu$ for all $i \in \mathbb{N}$ it is clear that $(A^\nu, \prec, \succ)$ is a filtered $\mathcal{C}$-dendriform algebra. Notice that the filtration is **descending**, so that $(\mathfrak{gr}(A^\nu))$ is now defined as $(\mathfrak{gr}(A^\nu)) := \bigoplus_{i \geq 0} A^\nu_i / A^\nu_{i+1}$, and that $(\mathfrak{gr}(A^\nu))$ is commutative. Though ascending and descending filtrations (indexed by $\mathbb{N}$) are from many points of view different, it is easily verified that the above results on ascending filtrations hold also for descending filtrations. In particular, $(\mathfrak{gr}(A^\nu), \times, \circ) \in \mathcal{C}_{\mathrm{pol}}^{\mathrm{dend}}$, as in Proposition 4.21. Under the canonical isomorphisms $A^\nu_i / A^\nu_{i+1} \simeq A$, valid for all $i \in \mathbb{N}$, we get that $(A, \times, \circ) \in \mathcal{C}_{\mathrm{pol}}^{\mathrm{dend}}$, where the latter products on $A$ are inherited from the products on $\mathfrak{gr}(A)$. It is easily checked that $(A, \times, \circ)$ is the deformation algebra of $(A^\nu, \prec, \succ)$. This shows that under the extra assumption that the relations defining $\mathcal{C}_{\mathrm{pol}}^{\mathrm{dend}}$ are multilinear, Proposition 4.13 is a consequence of 4.21. It should now be clear that all remarks made in Section 4.5 also apply to almost commutative (or anticommutative) filtered algebras (always under the assumption that the relations defining $\mathcal{C}_{\mathrm{pol}}^{\mathrm{dend}}$ are multilinear).

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