EARLY-TIME VELOCITY AUTOCORRELATION FOR CHARGED PARTICLES DIFFUSION AND DRIFT IN STATIC MAGNETIC TURBULENCE

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ABSTRACT

Using test-particle simulations, we investigate the temporal dependence of the two-point velocity correlation function for charged particles scattering in a time-independent spatially fluctuating magnetic field derived from a three-dimensional isotropic turbulence power spectrum. Such a correlation function allowed us to compute the spatial coefficients of diffusion both parallel and perpendicular to the average magnetic field. Our simulations confirm the dependence of the perpendicular diffusion coefficient on turbulence energy density and particle energy predicted previously by a model for early-time charged particle transport. Using the computed diffusion coefficients, we exploit the particle velocity autocorrelation to investigate the timescale over which the particles “decorrelate” from the solution to the unperturbed equation of motion. Decorrelation timescales are evaluated for parallel and perpendicular motions, including the drift of the particles from the local magnetic field line. The regimes of strong and weak magnetic turbulence are compared for various values of the ratio of the particle gyroradius to the correlation length of the magnetic turbulence. Our simulation parameters can be applied to energetic particles in the interplanetary space, cosmic rays at the supernova shocks, and cosmic-rays transport in the intergalactic medium.

Key words: cosmic rays – ISM: magnetic fields – turbulence

Online-only material: color figures

1. INTRODUCTION

In several physical systems containing a fluctuating magnetic field in a turbulent plasma, the diffusion of charged particles in the direction perpendicular to the average magnetic field has been recognized to be important. First, in the heliospheric environment, solar energetic particles associated with impulsive solar flares, or compact point-like sources, have been observed by multiple spacecraft orbiting on the heliospheric ecliptic plane, e.g., STEREO A/B and Solar and Heliospheric Observatory, widely separated in longitude; this has been interpreted as evidence for strong cross-field diffusion (Wibberenz & Cane 2006). Significant perpendicular diffusion has been also invoked to explain recurrent cosmic-ray variations at very high heliographic latitude made by Ulysses (Kota & Jokipii 1998). Observations of solar energetic particles at high heliographic latitude made by Ulysses have also been interpreted as evidence for cross-field diffusion (Zhang et al. 2003; Dalla et al. 2003).

Second, in non-relativistic collisionless shocks, e.g., interplanetary shocks or supernova remnant (SNR) shocks, the acceleration rate depends on the shock obliquity (Jokipii 1987), i.e., the angle between the magnetic field at the shock and the direction normal to the shock surface. At a perpendicular shock, since the magnetic field lines are frozen with the plasma flow, the transport perpendicular to the magnetic field lines allows the particles to remain near the shock enhancing their acceleration. Recent multiwavelength campaigns of SNRs (from radio up to γ-rays) has not yet constrained the magnetic field obliquity at the SNR shock despite a burgeoning number of observational evidences of the magnetic field amplification.

A single model unifying the transport in the directions parallel and perpendicular to the average magnetic field is needed to understand the propagation of energetic particles in a broad class of environments, i.e., interplanetary space, SNR shocks, and interstellar space; however, such a model is still missing. Perpendicular diffusion has been also studied recently disentangling two different contributions: field line random walk and gradient/curvature drift of the particle guiding center from the local field line (Fraschetti & Jokipii 2011). In the limit of weak turbulence and of low ratio of particle gyroradius to the magnetic turbulence coherence length, the usual assumption of particle magnetization, i.e., particle following field lines, is found to fail even in a simple and idealized turbulence geometry, such as the three-dimensional (3D) isotropic model. For this particular turbulence, this result must a fortiori hold for high-energy particles, because, due to larger gyroradius, high-energy particles decorrelate from the initial field line earlier than low-energy particles.

In the present paper we investigate via first principles, using Monte Carlo numerical simulations, the transport of charged test particles in the direction perpendicular to the average magnetic field at early times, prior to the spatial diffusion phase. The magnetic turbulence induces a decorrelation of the particle velocity from its unperturbed evolution. The two-point particle velocity correlation function, used here indistinguishably from particle velocity autocorrelation, would be constant in time in the absence of field fluctuations. The decorrelation time, defined as the time beyond which the Lorentz force experienced by the charged particle in the ambient magnetic field is uncorrelated with the force initially acting on it, depends physically on the magnetic energy density and on the ratio of the gyroradius to the turbulence correlation length. For a broad range of the parameters studied in this paper, the perpendicular particle velocity decorrelates within a few gyroperiods, i.e., much earlier than the typical diffusion timescale.

In the numerical experiments performed in the present paper, we relate the time dependence of the two-point parallel and perpendicular particle velocity correlation function to
the diffusion coefficient as given by the TGK formalism (Taylor 1921; Green 1951; Kubo 1957). We explore the regime of strong turbulence and use a broad range for the ratio of particle gyroradius to turbulence correlation length; our results apply both to non-relativistic and relativistic particles. Our simulations confirm the exponential, or Markovian, decay of the pitch-angle correlation. However, we find an unexpectedly fast decay of the perpendicular correlation, occurring within a few gyroperiods, which is not accounted for in any phenomenological transport model.

This work is organized as follows: in Section 2, we describe previous models for the velocity autocorrelation; in Section 3, we specify the particular turbulence model and the parameters for the test particles used in our simulations; in Section 4, we summarize the results for the behavior of diffusion coefficient at strong turbulence and high ratio of gyroradius to the turbulence correlation length; we also show that our previous weak-turbulence model based on the separation between field line random walk and gradient/curvature drift is in agreement with our simulations; in Section 5, we discuss our results for the pitch-angle correlation function and the perpendicular velocity autocorrelation; in Section 6, we describe some astrophysical applications; in Section 7, we summarize and conclude with the observational implications of our results; in Appendices A and B some details of the numerical code are outlined.

2. SPATIAL DIFFUSION MODELS BASED ON VELOCITY AUTOCORRELATION

The investigation of the particle motion in times prior to the diffusion phase can be conveniently based on the particle velocity autocorrelation, named Lagrangian as it makes use of the instantaneous particle velocity, well defined at times as short as the early ballistic phase. Moreover, the velocity autocorrelation offers the most direct approach to the long-range correlation in the so-called anomalous transport regime (Bakunin 2004). The Lagrangian velocity autocorrelation has not been frequently applied to particle transport for two reasons: (1) experimentally it is simpler to measure the Eulerian velocity autocorrelation, relating two points in space separated by a fixed coordinate distance, than the Lagrangian velocity autocorrelation, relating the velocity of the same physical particle at two different times; (2) numerical simulations require using a large number of particles to obtain good statistics.

The random walk of charged particles in a static, i.e., time-independent, turbulent magnetic field is commonly represented as a sequence of many stochastic events, independent of one another, i.e., a compound Poissonian process (Feller 1971). In a magnetic turbulence, every scattering encounter has a probability of not occurring exponentially decreasing in time, since a scattering event will certainly occur as the time proceeds. An exponential decay implies that every scattering event is independent from any other and the transport is endowed with complete lack of memory (Markovian process).

An exponential form for the autocorrelation of particle velocity \( v(t) \) moving in a turbulent fluid \( \langle v(t)v(0) \rangle = v_0^2 e^{-t/T} \) was first heuristically used by Taylor (1921) in a seminal discussion of the diffusion coefficient in the absence of magnetic field; here \( T \) has the meaning of characteristic timescale beyond which the fluid density variations or turbulent fluid motions cause a jump of the particle velocity to a value uncorrelated with the previous value. If magnetic turbulence is included, more recent theoretical arguments led to the exponential form of the particle velocity correlation function in the direction along the average magnetic field, or, equivalently, pitch-angle correlation function \( \langle \mu(0)\mu(t) \rangle = 3\langle v_\parallel(0)v_\parallel(t) \rangle / v_0^2 \) in a static magnetic field (Earl 1974). The exponential form of \( \langle \mu(t)\mu(0) \rangle \) has been shown (Forman 1977) to correspond to the closed form for the pitch-angle coefficient diffusion in Jokipii (1966) for a quasi-isotropic scattering.

We briefly recall some previous approaches to the particle velocity autocorrelation to estimate the diffusion coefficients \( \kappa \). In weak-turbulence regime and for particles isotropically scattering in all three space directions, the pitch-angle correlation is found to decrease exponentially in time (Forman 1977), i.e., the scattering is Markovian, with scattering timescale \( \tau_s \) related to the parallel mean free path \( \lambda_\parallel \) by \( \lambda_\parallel = v\tau_s \) and to the parallel diffusion coefficient by \( \kappa_\parallel = (v^2/3)\tau_s \). In the presence of strong turbulence, numerical simulations (Casse et al. 2002) found an empirical scaling of the scattering frequency \( (1/\tau_s) \) with the turbulence magnetic energy density. This scaling is based on two assumptions: (1) the motion perpendicular to the local field line, and the eventual decorrelation from the field line at high rigidity, is dominated by the turbulent scales smaller than the gyroradius; (2) the decorrelation time \( \tau_\perp \) is smaller than the scattering time \( \tau_s \). In Fraschetti & Jokipii (2011), the scale separation in the former assumption (1) of Cassé et al. (2002) is found to hold only for slab turbulence: the decorrelation from the field line is governed by the scales smaller than the correlation length, but much larger than the gyroscale (it should be noted also that Fraschetti & Jokipii 2011 assume a negligible power in the magnetic turbulence at the gyroscale). In contrast, for a 3D isotropic magnetic fluctuation, Fraschetti & Jokipii (2011) found that the decorrelation from the local field line has equal contributions from turbulent scales both larger and smaller than the correlation length. Second, in a strong turbulence the fluctuating magnetic field transversal to the average field becomes comparable or larger than \( B_0 \) and the assumption (2) in Cassé et al. (2002) might be violated.

Bieber & Matthaeus (1997) proposed a model (hereafter BAM model) for the perpendicular transport of high energy (a few \( GV \) rigidity for typical interplanetary conditions at 1 AU) charged particles based on a specific ansatz on the form of the perpendicular particle velocity autocorrelation, by supposing that particles simply move along the field lines. Numerical simulations (Giacalone & Jokipii 1999) have shown that the standard quasi-linear theory (Jokipii 1966) provides a diffusion coefficient in the perpendicular direction larger than its numerical estimate at protons energy between 1 MeV and 100 MeV, whereas the diffusion coefficients in classical scattering theory (Forman & Gleeson 1975) or the BAM model are too small. A more recent theory for perpendicular transport closer to numerical findings than the two aforementioned models is the NLGC model (Matthaeus et al. 2003). For magnetized plasma flows, NLGC relies on an approximate relation (Corrsin 1959) between the Eulerian and the Lagrangian velocity autocorrelations. The Corrsin relation assumes a diffusive nature of the particle displacement between the two points in space where the correlation is computed. However, such a diffusive assumption has been numerically found to hold only for purely hydrodynamic flows as in Kraichnan (1977) but not for MHD flows, as far as we are aware. Moreover, the NLGC, being intrinsically diffusive, cannot apply to non-diffusive magnetic turbulence geometries: different turbulence geometries are found to have different effects on the particle diffusion (see, for example, Fraschetti & Jokipii 2011). In addition, the field line random walk relies on a free parameter
(named “a” in Matthaeus et al. 2003) which is not provided by any other auxiliary model and must be empirically determined from numerical simulations. Within the idealized compounded diffusion model, where particles are strictly tied to magnetic field lines, but scatter along them and trace back along the same field line, the temporal evolution of the perpendicular velocity autocorrelation and the consequent diffusion coefficient have been determined analytically, confirming the expected subdiffusion (Kótá & Jokipii 2000).

The velocity autocorrelation \( \langle v(t)v(0) \rangle \) is known to be related to the symmetric part of the instantaneous diffusion tensor \( \kappa_{ij}(t) \) by the TGK formalism: \( \kappa_{ij}(t) = \int_0^t d\xi \langle v(\xi)v(t) \rangle \). The heuristic and physically motivated \( \text{BAM} \) forms of the perpendicular velocity autocorrelations at gyroperiod scale are:

- \( R_{\bot}(t) = \langle v_\bot(t)v_\bot(0) \rangle \propto (v^2/3)e^{-t/\tau_\bot} \cos(\Omega t) \)
- \( R_{\parallel}(t) = \langle v_\parallel(t)v_\parallel(0) \rangle \propto (v^2/3)e^{-t/\tau_\parallel} \sin(\Omega t) \)

where \( \Omega = eB_0/mc \) is the relativistic particle gyrofrequency corresponding to the unperturbed field \( B_0 \). From that assumption it follows that the perpendicular diffusion coefficients (symmetric and antisymmetric parts) are \( \kappa_\bot = \kappa_B \Omega_\perp \tau_\bot/[1 + (\Omega_\perp \tau_\bot)^2] \) and \( \kappa_A = \kappa_B (\Omega_\perp \tau_\bot)^2/[1 + (\Omega_\perp \tau_\bot)^2] \), where \( \tau_\bot \) and \( \tau_\parallel \) are the decorrelation timescales (\( \tau_\bot = \tau_\parallel \) as assumed in the \( \text{BAM} \) model). Such a form of \( \kappa_\bot \) generalizes the result of Forman et al. (1974) down to low rigidities (below 4 GV in intergalactic medium). A comparison of the timescale \( \tau_\parallel \) in the two aforementioned works allows to relate the turbulent power at zero wavenumber to the intrinsic spatial coefficient diffusion for field line random walk.

In the \( \text{BAM} \) model, it turns out also that \( \kappa_\perp/\kappa_\parallel = (\tau_\perp/\tau_\parallel)[1 + (\Omega_\perp \tau_\parallel)^2] \), which collapses to the billiard ball scattering picture (Forman & Gleeson 1975) if \( \tau = \tau_\parallel = \tau_\perp \). In this paper, we explore various regimes of the velocity autocorrelation. The result is contrasted with the \( \text{BAM} \) model within broad ranges of \( r_g/L_c \) and \( \sigma^2 \), where \( r_g \) is the particle gyroradius, \( L_c \) is the turbulence correlation length, and \( \sigma^2 = \langle \delta B^2 \rangle \) is the magnetic turbulence normalized energy density.

Note that the most general form of the total perpendicular coefficient of diffusion (symmetric part) comprises both the field line meandering part, typically dominant contribution at low rigidity (\( \kappa_{\text{MFL}} \)), and the departure from the local field line (\( \kappa_p \)), due to gradient/curvature drift: \( \kappa_\perp = \kappa_{\text{D}} + \kappa_{\text{MFL}} + \kappa_p v_\parallel \) (Fraschetti & Jokipii 2011). We estimate in this paper the correlation time for the motion perpendicular to the direction of the average field as \( \tau_\perp \approx (2/3)r_g^2/\kappa_{\text{D}} \), by using the numerically determined value of \( \kappa_{\text{D}} \) (for the factor (2/3) see Section 5.2).

3. NUMERICAL METHOD

In a series of numerical experiments, we consider a population of charged test particles gyrating in a magnetic field described as follows: we assume a 3D magnetic field of the form \( \mathbf{B}(\mathbf{x}) = \mathbf{B}_0 + \delta \mathbf{B}(\mathbf{x}) \), with an average component \( \mathbf{B}_0 = B_0 \mathbf{e}_z \), and a random component \( \delta \mathbf{B} = \delta \mathbf{B}(x, y, z) \) having a zero mean \( \langle \delta \mathbf{B}(\mathbf{x}) \rangle = 0 \), and a turbulence correlation length \( L_c \). We assume in the inertial range a scale invariant, or Kolmogorov, power spectrum in the three space dimensions: \( G(k) \propto k^{\beta - 5/3} \), where \( k \) is the wavenumber magnitude, \( \beta = 5/3 \) is the one-dimensional power-law Kolmogorov index and the additional two accounts for the dimensionality of the turbulence (for more details see Appendix A). The assumption of a static magnetic field is reasonable if the particle speed largely exceeds the Alfvén wave speed.

We perform numerical integration of equations of motion of charged particles combining the code used in Fraschetti & Melia (2008) with the prescription for the turbulence introduced in Giacalone & Jokipii (1994), and widely exploited in the last two decades in various astrophysics contexts (Giacalone & Jokipii 1999; Qin et al. 2002; Bykov et al. 2008; Fraschetti & Melia 2008). We determine the particle trajectory by numerically integrating the equation of motion using the Lorentz force determined at the instantaneous particle position (see Appendix A for details).

We follow particle trajectories in a 3D spatially unbounded region, since particles escaped from a bounded computational domain would be removed from the simulation and could artificially modify the estimate of the instantaneous diffusion coefficient. Charged particles are evolved in various realizations of the magnetic turbulence (see Appendix A for details). The final time of our simulation runs is empirically determined as the computational time where the asymptotic value for the diffusion coefficient is attained.

The relevant parameters are the ratio of the particle gyroradius to the turbulence correlations length, i.e., \( r_g/L_c \), and the normalized turbulence energy density \( \sigma^2 \). Therefore our treatment applies to energetic particles in various astrophysical environments: from the interplanetary space, to the SNR shocks, and to the intergalactic medium.

4. DIFFUSION COEFFICIENTS

We report in this section the coefficients of diffusion parallel and perpendicular, both symmetric and anti-symmetric, resulting from our numerical simulations (see Appendix B for details). In Section 5, we will discuss the autocorrelation of the particle velocity making use of the diffusion coefficients computed in the present section to estimate the characteristic timescale for the pitch-angle and the perpendicular scattering.

Figure 1 shows the parallel (\( \kappa_{\parallel} \)), perpendicular (\( \kappa_{\perp} \)), and anti-symmetric term (\( \kappa_A = \kappa_{xy} = -\kappa_{yx} \), see also Appendix B and Figure 11 (right panel), computed as \( \kappa_{ij} = \langle vi\Delta x_j \rangle \) of the diffusion tensor as a function of \( \sigma^2 \) at fixed particle energy conveniently scaled in units of Bohm diffusion coefficient \( \kappa_B = v_{\text{c}}r_g/3 \). In the weak-turbulence limit, the behavior of the
diffusion tensor as a function of $\sigma^2$, for $r_g/L_c = 0.2$. Simulated values are compared with the solid line and the dashed line, representing, respectively, the field line random walk and the departure from the local field line, or gradient/curvature drift diffusion coefficient, analytically found in Fraschetti & Jokipii (2011), here FJ11, Equations (49) and (45), and the dotted line, representing the quasi-linear result, explicit in Equation (B5) of Giacalone & Jokipii (1999), here GJ99.

Figure 3 focuses on the dependence of $\kappa_B$, as a function of $r_g/L_c$, for $\sigma^2 = 1.0$. Simulated values are compared with the solid line and the dashed line, representing, respectively, the field line random walk and the departure from the local field line, or gradient/curvature drift diffusion coefficient, analytically found in Fraschetti & Jokipii (2011), here FJ11, Equations (49) and (45), and the dotted line, representing the quasi-linear result, explicit in Equation (B5) of Giacalone & Jokipii (1999), here GJ99.

5. VELOCITY CORRELATION

In this section, we study the time dependence of the Lagrangian particle velocity autocorrelation $(v_i(t)v_i(0))$. The resulting simulations are compared with models for pitch-angle scattering and perpendicular decorrelation. The decay timescale of $(v_i(t)v_i(0))$ predicted in these models depends on the particle diffusion coefficients parallel and perpendicular to the average magnetic field, that we have estimated in the previous section.

5.1. Pitch-angle Scattering

In Figures 4–6, our simulations show that the pitch-angle correlation drops to zero as a function of time, or $t\Omega$, where $\Omega = eB_0/m\gamma c$ is the relativistic particle gyrofrequency corresponding to the unperturbed field $B_0$, for different values of $\sigma^2$ and for various particle energies: the stronger the turbulence and larger the particle energy, the faster the correlation decay even within a gyropериod scale. The subpanels in Figures 4–6 compare the early-time decay from our simulations with the isotropic scattering form $e^{-t/\tau}$ (Earl 1974). The scattering time $\tau_i$ is estimated by using $\tau_i = (3/v^2)\kappa_i$, where $\kappa_i$ is computed in our simulation runs and compared with the values from Giacalone & Jokipii (1999). The subpanel inside Figure 4 confirms the exponential decay of the pitch-angle correlation for $\sigma^2 \ll 1$. The bumps in the pitch-angle correlation at multiple integers of gyroperiod $t_g = 2\pi/\Omega$ suggest that, in its helicoidal trajectory, the particle velocity keeps higher order harmonics of its perturbed periodic motion (see also Casse et al. 2002).

Figure 5 shows that for $r_g/L_c = 0.75$ the bump at $t\Omega = 2\pi$ is smeared out because particles at such an energy (1 GeV in the interplanetary medium) can travel a distance as large as $L_c$ experiencing therefore several scatterings within one gyroperiod (compare also with the uppermost panel in Figure 8).
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Figure 4. Simulated pitch-angle correlation function for \( \sigma^2 = 0.1 \) and various particle energies as a function of \( t \Omega \). Subpanel compares our simulations with a purely exponential decay having \( \tau_\parallel \) as characteristic timescale. At early time, simulations (in black) agree with a purely exponential decay by making use of both values of \( \kappa_\parallel = (v^2/3)\tau_\parallel \), reported in this paper (in red) and in Giacalone & Jokipii (1999; in blue).

Figure 5. Pitch-angle correlation function for \( \sigma^2 = 1.0 \) and various particle energies as a function of \( t \Omega \). As in Figure 4, the subpanel focuses onto the early-time dependence. Intermediate-energy density of turbulence can be quite satisfactorily reproduced by an exponential decay with \( \tau_\parallel \) as characteristic timescale (see also caption of Figure 4).

The subpanel in Figure 6 clearly shows for the first time directly in terms of \( \langle \mu(0)\mu(t) \rangle \) that, for strong turbulence, the simulated correlation drops to zero significantly slower than \( e^{-t/\tau_\parallel} \). Such a non-Markovian behavior is found to hold independently on the particle rigidity. We have shown in Figure 1 that \( \kappa_\parallel \) (and therefore \( \tau_\parallel \)) saturates to a constant value for \( \sigma^2 \gg 1 \) at fixed energy; therefore, the deviation from the Markovian scattering shown in the subpanel in Figure 6 cannot be simply accounted for with an ad hoc modification of the \( \sigma^2 \) dependence of \( \tau_\parallel \). Within the quasi-linear limit, i.e., for small magnetic field fluctuations, Forman (1977) showed that the weighted sum over all the characteristic times that expresses \( \kappa_\parallel \) in Equation (18) of Earl (1974) is equivalent to the time integral up to infinity of the parallel velocity autocorrelation. The lowest order term of the Earl’s series for \( \kappa_\parallel \), corresponding to the case of purely isotropic scattering, can be obtained as time integral of a simply exponential decay correlation function, with characteristic time given by \( \tau_\parallel \). Higher order terms, that correspond to anisotropies in the scattering, would produce a long-duration tail in the correlation similar to what we find in the subpanel in Figure 6 (see also Equation (32) et seq. in Forman 1977). We infer from Figure 6 that enhanced turbulence produces a “memory effect” in the pitch-angle scattering comparable to the effect of the anisotropy terms in the weak turbulence. However, we do not perform a fit of our simulations due to the theoretical uncertainties underlying the higher-order characteristic timescales in the Earl series. We note that a pure exponential \( e^{-t/\tau_\parallel} \) underestimates the pitch-angle correlation at early time and therefore the asymptotic value of the diffusion coefficient \( \kappa_\parallel \). Since this paper focuses on test-particle simulations we will not develop an analytic discussion here.

5.2. Field Line Decorrelation

In this section, we present our results for the perpendicular velocity autocorrelation on the gyroperiod scale. Figure 7 shows that the simulated autocorrelation deviates from the BAM model ansatz even for relatively small values of magnetic fluctuations and of \( r_g/L_c \), within the expected regime of validity of the BAM model. The two curves corresponding to \( \kappa_\perp \) computed in this paper and to the value in Giacalone & Jokipii (1999) agree in both the middle and lower panels. For weak turbulence (lower panel), BAM model predicts a much less effective attenuation: the simulated correlation is completely smeared out beyond \( t\Omega \sim 6r_g \), or \( t \sim 3r_g \). At intermediate fluctuation level (mid panel), the exponential suppression occurs faster; this is due to the fact that large fluctuations increase the statistical value of the particle gyrofrequency (\( \Omega = eB/m\gamma c \)), computed as an ensemble-average, thus reducing the gyroradius \( r_g = v/\Omega \), i.e., the instantaneous radius of curvature of the particle trajectory; thus, the minimum in the velocity component perpendicular to the average field is reached at an earlier time (see Figure 7 mid panel). We note also that the increase of the gyrofrequency \( \Omega \) at larger magnetic fluctuations shortens the

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2 An additional numerical factor is present in Bieber & Matthaeus (1997) to agree with the QLT limit: \( \Omega r_g = 12/3\gamma_\parallel/\sigma_\parallel \) (analogous to Equation (14) in their paper). However, the rate of decrease of the correlation does not depend significantly on the particular numerical factor used. We use \( \tau_\perp = (2/3)v_g^2/\kappa_\perp \) to compare our simulations with the BAM model.
period of the oscillations, in absolute time. These effects are enhanced at larger turbulence (upper panel in Figure 7), where the oscillations are smeared out and the shape approaches a purely exponential decay with a surprisingly fast drop.

The energy dependence of the $R_⊥(t)$ is depicted in Figure 8. We find a good agreement between our simulations and the BAM model only at large particle energies ($E \simeq 10$ GeV or $r_g/L_c = 4.9$ for interplanetary medium, uppermost panel), where the circular motion of the particle is weakly affected by the field line meandering, occurring at smaller scale. However, as the rigidity is reduced, the particle motion resonates with a larger range of turbulence scales; the velocity autocorrelation cannot be retained.

For completeness, Figure 9 compares our simulations for $R_A(t)$ with the BAM model. Here, $t_A$ is estimated as $t_A = τ_⊥ = (2/3)r_g^2/κ_⊥$. Our simulations confirm the BAM ansatz for the velocity autocorrelation with the assumption $τ_⊥ = t_A$ only for weak turbulence within the gyropertime scale ($t \sim 2r_g$ for $σ^2 = 0.1$, compare Figure 7, lower and upper panels).

Note that the instantaneous diffusion coefficient in the direction perpendicular to the average field is the time integral of $(v_⊥(t)v_⊥(0))$, from zero to infinity. If the decay is sufficiently fast, as shown in Figures 7 and 8 even for low values of $r_g/L_c$ and small fluctuations ($σ^2 \ll 1$), a substantial correction to the BAM modulated exponential is required. Currently no analytic models can describe such a rapid drop in the correlation.

Figure 10 compares the instantaneous $κ_⊥(t)$ from the BAM model ($κ_⊥(t) = 1/T ∫_0^T dτ R_⊥(τ)$) with our simulations. Clearly, the BAM model is only applicable for long times due to sinusoidal factors, which make $R(t)$ assuming positive and negative values (see Figure 10). We point out in this paper that the early-time oscillations in $R(t)$ cause the $κ_⊥$ underestimate previously found (Giacalone & Jokipii 1999). Therefore, we conclude that a model for the perpendicular diffusion needs to ascertain an adequate description of the early-time propagation. On the other hand, the drawbacks of a diffusion model can be identified through a study of the early-time propagation similarly to the study presented here.

6. SOME ASTROPHYSICAL APPLICATIONS

In the preceding sections, the charged particle motion has been described prior to the diffusion regime in terms of the particle velocity autocorrelation. This treatment applies to various astrophysical environments: energetic particles ($E = 0.1–10^4$ MeV) in interplanetary space, at distance of 1 AU from the Sun, gyrating in an ordered magnetic field $B_0 \sim 5$ nT, with a turbulence $σ^2 \ll 1$ and a turbulence correlation length $L_c = 10^{-2}$ AU; particles at energy $E = 5.1 \times 10^{13}$ eV to $E = 4.1 \times 10^{16}$ eV diffusing upstream or downstream of the non-relativistic shock of SNRs, likely in an ordered field $B_0 = 3 \mu G$ with an amplified turbulence $σ^2 \ll 1$, and a turbulence correlation length $L_c \sim 3$ pc; particles at energy $E = 1.7 \times 10^{14}$ eV to $E = 1.4 \times 10^{17}$ eV transported in the turbulent galactic medium with $B_0 = 3 \mu G$, $σ^2 \ll 1$, and $L_c \sim 10$ pc.

As an example of application of our simulations, we consider the problem of propagation of energetic particles in the interplanetary medium detected by a spacecraft measuring the...
in situ magnetic field. Consider a solar energetic particle, i.e., proton, with a kinetic energy \( T = 15 \text{ MeV} \) in an approximately static solar wind Alfvénic perturbation, released by the coronal mass ejection shock propagating from a gradual event. At the location of particle release from the acceleration region, if the turbulence can be described by a 3D isotropic power spectrum, with \( L_c = 10^{-3} \text{ AU} \) and \( B_0 = 5 \text{ nT} \) \( (r_g/L_c = 0.75) \), Figure 8 shows that, due to perpendicular diffusion, the correlation drops on the gyroradius scale \( (t_g = 2\pi/\Omega \sim 14 \text{ s}) \). Likewise, if a ground level enhancement proton with \( T = 10 \text{ GeV} \) is released in a turbulent interplanetary space with \( L_c = 10^{-2} \text{ AU} \) and \( B_0 = 5 \text{ nT} \) \( (r_g/L_c = 4.9) \), the decorrelation will drop on the scale of minutes \( (t_g \sim 152 \text{ s}) \). Therefore, particle decorrelation from turbulent field lines needs to be considered in tracing the energetic particle trajectories in the interplanetary medium. The anisotropic phase of the observed flux from solar particle events can be strongly affected.

The above result has also application to the galactic cosmic rays below the “knee” of the cosmic-ray spectrum, thought to be accelerated at the SNR shock, and escaping into the turbulent interstellar medium. At those shocks an efficient magnetic field amplification has been inferred up to values largely exceeding the Rankine–Hugoniot jump across the shock through various independent methods, e.g., from the shape of radio synchrotron spectra of energetic electrons (Reynolds & Ellison 1992) to X-ray rims in the remnant interior (Berezhko et al. 2003). Various explanations have been proposed in the literature for such a large amplified magnetic field: instability by non-resonant cosines-rays streaming upstream of the shock (Bell 2004) or vortical turbulent motion seeding downstream magnetic field amplification (Giacalone & Jokipii 2007). In our simulations, assuming a large magnetic fluctuations, i.e., \( \langle \delta B/B_0 \rangle^2 \sim 10^4 \), \( L_c = 3 \text{ pc} \), and \( B_0 = 1 \mu \text{G} \), an energetic proton with energy \( E = 2.1 \times 10^{13} \text{ eV} \) \( (r_g/L_c = 0.75) \), will decorrelate on timescale of \( 2\tau_g \sim 92 \text{ yr} \) (see, for example, Figure 8, corresponding to \( \sigma^2 = 1.0 \), panel with \( r_g/L_c = 0.75) \). Our simulations do not include synchrotron energy losses, negligible for protons, but they are likely to be relevant within a decorrelation time in the case of energetic electrons.

### 7. SUMMARY AND CONCLUSION

In this paper, we have performed Monte Carlo simulations of test-particles gyrating in a static spatially turbulent magnetic field with a 3D isotropic power spectrum. First, we have computed the dependence of the diffusion coefficients in the directions parallel and perpendicular to the average magnetic field on the turbulence energy density. For weak turbulence, the prediction of our previous model for perpendicular transport based on the separation between the field line meandering and the gradient/curvature drift from the local field line is found to be in better agreement with our numerical results than the standard quasi-linear theory. For the particular power spectrum that we have considered and at fixed particle energy, the drift contribution is about one order of magnitude smaller than the contribution from field line meandering. For strong turbulence, the diffusion tensor becomes isotropic. We have also computed the dependence of the diffusion tensor on the particle energy, specifically the ratio \( r_g/L_c \). Also in this case the field-line meandering is found to be in better agreement with our numerical results than the standard quasi-linear theory. The gradient/curvature drift term turns out to be important with the increase of particle energy; a drift-dominated diffusion cannot be ruled out in different turbulent power spectra.

Second, we computed the dependence of the particle velocity autocorrelation in the directions parallel and perpendicular to the average magnetic field on the particle energy and on the turbulence energy density. The pitch-angle correlation drops exponentially in time for weak fluctuations, as predicted for the isotropic scattering case. We also found that for the case of strong turbulence the decay is not exponential. The deviation from the exponential decay in strong turbulence cannot be accounted for by current models. The perpendicular velocity autocorrelation decays faster than an exponentially modulated oscillation predicted by previous models. Even in the weak-turbulence case and for gyroradius smaller than the turbulence correlation scale, no significant correlation is found beyond...
three gyroperiods, whereas the modulated exponential decays much slower. Although in a strong turbulence the particle gyroradius cannot be uniquely defined, our simulations show that the statistical effect of strong turbulence is reducing the instantaneous radius of curvature of the particle orbit; the correlation is lost within a fraction of gyroperiod. However, for gyroradii larger than the correlation length the exponential modulation agrees with the simulations: on this scale the effect of the turbulence is not relevant and the particle pursues a quasi-helical motion in a uniform field. However, for smaller gyroradii, the estimate of the decorrelation time and the magnitude shows the need of a new model.

We do not provide here a phenomenological fit or analytical model for the velocity autocorrelation. Our work shows that the underestimate of the diffusion coefficient previously found in the classical scattering theory can be explained in terms of lack of consistent model for the scattering and decorrelation times, and sheds light for the future investigations. A model for the diffusion coefficient based on the velocity autocorrelation needs to provide accurate description of the early-time transport, because the contributions to the diffusion beyond the gyroperiod scale average out to zero.

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APPENDIX A
NUMERICAL SETUP

The global magnetic field is written as a sum of a background field $B_0$, constant and statistically uniform, and a turbulent field varying in space and independent on time. The equation of motion of a test particle with charge $e$ and mass $m$ moving with velocity $v(t)$ in a magnetic field $B(x)$ is the Lorentz equation

$$\frac{du(t)}{dt} = u(t) \times \Omega(x), \quad (A1)$$

where $\Omega(x) = eB(x)/(mc\gamma)$ with $\gamma = 1/\sqrt{1-(v/c)^2}$ the Lorentz factor, $c$ is the speed of light in vacuum and where $u(t) = \gamma v(t)/c$. We calculate the trajectory of the particle in a magnetic field as a solution of Equation (A1), where $t$ is the time in the rest frame of the propagation region. The quantity $\Omega(x)$ in Equation (A1) is given by $\Omega(x) = \Omega_0 + \delta \Omega(x)$ where $\Omega_0 = (e/ mc\gamma) B_0$, in terms of the background magnetic field $B_0$, and $\delta \Omega(x)$ is the turbulent magnetic field. We verified that the particle energy is conserved with a relative accuracy of $10^{-5}$. We ignore any large-scale background electric fields.

The fluctuating field comprises equal intensity circularly polarized left-handed and right-handed Alfvén waves. The procedure of building the turbulence calls for the random generation of a given number $N$ of transverse waves $k_i, i = 1, \ldots, N$ at every point of physical space where the particle is found, each with a random amplitude, phase, and orientation. We use the form of the power spectrum provided in Giacalone & Jokipii (1999), Equations (3)–(7). The wavenumber $k_i$ of the $N$ wave modes ranges from $k_{min} = 2\pi/L_{max}$ to $k_{max} = 2\pi/L_{min}$, spanning over four decades $L_{max}/L_{min} = 10^4$. The coherence scale $L_c$, or bend-over scale, for the turbulence power spectrum, is chosen such that $L_c = 10^2L_{min}$. The values of $k_i$ magnitude are logarithmically equspaced, $\Delta k/k = constant$. We sampled the power spectrum over a discrete number of modes $N_m$. We choose $N_m = 188$ modes of magnitude $k_i$; the resulting coefficients of diffusion do not depend significantly on the sampling resolution in $k$ space for $N_m > N_m^*$. We inject a large number of particles at the same point of space, i.e., the origin of the coordinate system, with randomly oriented velocity, but fixed energy. We integrate the Equation (A1) by using a time-step adjustable Burlisch–Stoer method (Press et al. 1986), which in our numerical solution of second-order ordinary differential equation resulted more accurate than a Runge–Kutta fifth order. This approach is well suited to investigate the particle transport in turbulence. A different approach of computing different realizations of the turbulent magnetic field in every cell of a structured Cartesian grid is much more time consuming. The latter approach would also require adapting the lattice spacing in order to maintain the same space resolution in physical space.

As in Giacalone & Jokipii (1999) the randomization is performed over the initial particle velocity orientation and over the turbulence realization, shuffled every 50 particles. Including the fluctuating field statistics ensures a meaningful comparison with a theoretically computed turbulence power spectrum, which is by definition an average over an ensemble of field realizations.

APPENDIX B
COMPUTATION OF THE DIFFUSION COEFFICIENTS

We computed the parallel ($\kappa_\parallel$) and symmetric perpendicular ($\kappa_\perp$) coefficients of diffusion in the previously specified turbulence using two different methods. The first method uses the formula for the instantaneous coefficient diffusion based on the ensemble-averaged mean square displacement:

$$d_{ij}^I(t) = \frac{1}{2N_p} \sum_{n=1}^{N_p} \frac{(x_i(t) - x_i(t_0))(x_j(t) - x_j(t_0))}{t - t_0}, \quad (B1)$$

where $N_p$ is the total number of particles (the turbulence realization being shuffled every 50 particles) and $x_i(t)$ is the particle trajectory along the $i$th space coordinate and $t_0$ injection time.

The second method involves the particle velocity autocorrelation given by the TGK formula:

$$d_{ij}^{II}(t) = \frac{1}{2} \frac{d}{dt} \langle \Delta x_i \Delta x_j \rangle = \int_0^t d\xi \langle v_i(\xi)v_j(0) \rangle, \quad (B2)$$

where $v_i(t)$ is particle velocity along the space coordinate $x_i$. The last equivalence is valid for fluctuating velocity statistically homogeneous for sufficiently long time.

In both cases, the coefficient of diffusion is estimated as the apparent asymptotic value in the diagram representing our numerical simulations (see Figure 11 (left panel)), that is attained after a large number of scatterings off the magnetic disturbance: $d_{ij}(t) \to \kappa_{ij}$ for $t \to \infty$. For $r_g \ll L_c$ it takes many gyroperiods before the force acting upon the particle decorrelate from the initial force. For $r_g \gtrsim L_c$, the force decorrelate much earlier, allowing eventually the particle to gyrate around a field line different from the field line it was released on. Thus, the diffusion timescale ranges from $10^7\tau\Omega$ (for $r_g \ll L_c$) to $10^2\tau\Omega$ (for $r_g \gtrsim L_c$).
By using the first method (Equation (B1)), we found results numerically convergent and consistent with Giacalone & Jokipii (1999) by injecting only \(N_p = 2500\) particles. However, the second method (Equation (B2)), due to the large numerical fluctuations around zero of the velocity correlation at \(t \gg \Omega^{-1}\), requires a much larger number of particles, i.e., \(N_p \approx 50,000\), for the results of the two methods to be comparable (see Figure 11 (left panel)). The chosen number of particles compromises between reducing the computation time and smearing out the numerical fluctuations in the tail at \(t \Omega > 10^{-2}\) in Figures 4–6.

Equations (B1) and (B2) are symmetric in the indexes \(i - j\), therefore not suitable to define the anti-symmetric part of the diffusion tensor. We follow here the argument in Giacalone et al. (1999), valid for a nearly isotropic population of non-relativistic particles streaming without convection, to write the anti-symmetric part of the diffusion tensor: \(\kappa_{ij} = \langle v_i \Delta x_j \rangle\), where \(v_i\) is the velocity along the \(i\)-direction and \(\Delta x_j\) is the space displacement along the \(j\)-direction. In Figure 11 (right panel), the averages of \(\kappa_{xy}\) and \(\kappa_{yx} = -\kappa_{xy}\) are plotted as a function of time, in order to smear out the large fluctuation due to velocity oscillations.

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Figure 11. Left: as a function of time, the diagram compares the coefficients of diffusion \(\kappa_{\perp}\) and \(\kappa_{\parallel}\) computed by using \((\Delta v_i)^2/2\tau\) (in dashed) and \(\int_0^\infty (v_i(\xi)v_i(0))d\xi\), or TGK (in solid) for \(r_g/L_c = 0.2\) and \(\sigma^2 = 1.0\). If the TGK formula is used, the number of particles necessary to reduce numerical fluctuations is \(N_p \approx 50,000\). Right: as a function of time, the diagram compares the anti-diagonal terms of the diffusion tensor, i.e., \(\kappa_{xy}\) (in units of \(\kappa_B\)), anti-symmetric as expected (\(\kappa_{xy} = -\kappa_{yx}\)). Here \(r_g/L_c = 0.1, \sigma^2 = 0.316,\) and the number of particles used is \(N_p = 50,000\). We draw the time averages to clearly show the asymptotic value.