Convergent $\tilde{Y}$-Map for a new covariant Loop Quantum Gravity formulation and Implicit Reality Condition

Leonid Perlov
lperlov@webrealview.com
Cambridge, MA

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Abstract

One of the most important elements in a new spin-foam loop quantum gravity formulation is the map $Y : H^{SU(2)} \rightarrow H^{SL(2, C)}$. In this paper we provide an alternative improved map $\tilde{Y}$. The image of a new map $\tilde{Y}$ contains the weighted infinite sums of $SL(2, C)$ matrix coefficients. The sums are convergent and their limits are the square integrable functions of $SL(2, C)$ with the measure $L^2(g, e^{-|Y|^2/\hbar} \eta(g) du dY)$ according to the recently proved Holomorphic Peter-Weyl theorem [2]. We also discuss the consequence of a choice of a unitary principal series instead of the general principal series (sometimes called non-unitary) in EPRL model. The general principal series contains the unitary principal series as a sub-representation and becomes unitary when its parameter $\nu$ is real rather than complex. The solution of the simplicity constraint then implicitly makes the Barbero-Immirizi parameter real instead of complex. We call this - an implicit reality condition.

1 Introduction

A new covariant loop quantum gravity formulation [1] is based on the map $Y$ which is a map $H^{SU(2)} \rightarrow H^{SL(2, C)}$. $Y$ maps $H^{SU(2)}$ states $|j, q\rangle$ to $H^{SL(2, C)}$ states $|j, q', \gamma\rangle, j \in Z^+, \gamma - Barbero$-Immirzi. it maps $SU(2)$ matrix coefficients $D^j_{qq'}(u)$ to $SL(2, C)$ unitary principal series matrix coefficients $D^{(j, j\gamma)}_{qq'}(g)$. By the classical Peter-Weyl theorem any $SU(2)$ function can be decomposed into the infinite sum of Wigner matrices $D^j_{qq'}(u)$. The $Y$ maps then the functions of $SU(2)$ to the functions of $SL(2, C)$ on the following manner:

$$Y : \phi (u) = \sum_{j=0}^{\infty} \sum_{q, q'=-j}^{j} C_{jqq'} D^j_{qq'}(u) \rightarrow \psi(g) = \sum_{j=0}^{\infty} \sum_{q, q'=-j}^{j} C_{jqq'} D^{(j, j\gamma)}_{jqq'}(g) \quad (1)$$

, where on the right hand side the infinite sum of the unitary principal series matrix
coefficients is assumed to converge and to be a function of \( g \). The proof of the sum convergence does not exist. In this paper we define an alternative convergent map, which we call \( \tilde{Y} \). We prove that \( \tilde{Y} \) converges and the limit is a square integrable function of \( SL(2, \mathbb{C}) \) with the measure \( (e^{-|Y|^2/\hbar} \eta(g) du dY) \), \( g \in SL(2, \mathbb{C}), u \in SU(2), Y \in su(2) \). For proof we use the recently proved the Holomorphic Peter-Weyl theorem \([2]\) (J. Huebschmann 2008).

The paper is organized as follows. In section \([2]\) we state and discuss the Holomorphic Peter-Weyl theorem. In the subsequent section \([3]\) we introduce the convergent \( \tilde{Y} \)-Map and prove its convergence and square integrability. After that in the section \([4]\) we discuss the implicit reality conditions imposed on Barbero-Immirzi parameter by a choice of the unitary principal series representation instead of general principal series. The discussion section \([5]\) concludes the paper.

## 2 Holomorphic Peter Weyl Theorem

The holomorphic Peter Weyl theorem \([2]\) establishes the isomorphism between the Hilbert space spanned by the compact group \( K \) matrix coefficients and the Hilbert space spanned by the rational representation matrix coefficients of that group complexification \( K^C \):

\[
\phi^C(g) \rightarrow (\hbar \pi)^{\dim(K)/4} e^{\hbar |\lambda|^2/2} \phi(g)
\]

the inner products of these two Hilbert spaces are related by the following:

\[
\int_{K^C} \phi^C(g) \phi^C(g) e^{-|Y|^2/\hbar} \eta(g) du dY = (\hbar \pi)^{\dim(K)/2} e^{\hbar |\lambda+\rho|^2/2} \int_K \bar{\phi}(u) \phi(u) du
\]

, where \( g \in K^C \) and we use polar decomposition \( g = ue^{iY}, u \in K, Y \in t, \) algebra of \( K, \lambda \) is the highest weight of \( K \), while \( \rho \) is the Weyl vector of \( K \), i.e the half sum of the positive roots, the density of the measure on the left hand side is

\[
\eta(u,Y) = \left( \det \left( \frac{\sin(ad(Y))}{ad(Y)} \right) \right)^{1/2}, u \in K, Y \in t
\]

It means that we can calculate the inner product in the non-compact group rational representation Hilbert space by calculating the inner product of its isomorphic projection to the Hilbert space of its maximum compact subgroup representation. Since the map is provided by constant multiplication, all the orthonormal properties of the matrix coefficients in the compact group case propagate to the Hilbert space of its non-compact complexification.

Let \( K \) be a compact group, \( K^C \) - its complexification, \( t \) and \( t^C \) its algebras respectively. Let \( g \in K^C, u \in K, Y \in t, \eta(u,Y) \) as in \([4]\). We denote as in \([2]\) \( \hat{K}^C \) to be the set of isomorphism classes of irreducible rational representations of \( K^C \). \( \hat{K}^C \) is identified with the space of highest weights corresponding to the dominant Weyl
chamber. For the highest weight \( \lambda \), \( T_{\lambda} \) is a rational representation in a class of \( \lambda \): \( K^C \rightarrow \text{End}(V_{\lambda}) \), \( V_{\lambda} \) is a representation vector space. For \( \psi \in V_{\lambda}^* \) and \( w \in V_{\lambda} \) the function \( \Phi_{\psi w}(g) = \psi(q w) \) is a representative function on \( K^C \) and it provides a morphism: \( V_{\lambda}^* \otimes V_\lambda \rightarrow \mathbb{C}[K^C] \). We denote this morphism following \cite{2} as \( V_{\lambda}^* \otimes V_\lambda \).

**Theorem** [Holomorphic Peter-Weyl J. Huebschmann 2008]

The Hilbert space \( HL^2(K^C, e^{-|Y|^2/\hbar}\eta(g)du\,dY) \) contains the vector space \( \mathbb{C}[K^C] \) of representative functions (matrix coefficients) on \( K^C \) as a dense subspace, and as a unitary \((K \times K)\)-representation, \( HL^2(K^C, e^{-|Y|^2/\hbar}\eta(g)du\,dY) \) decomposes as the direct sum into \( K \times K \)-isotypical summands:

\[
HL^2(K^C, e^{-|Y|^2/\hbar}\eta(g)du\,dY) = \bigoplus_{\lambda \in \mathbb{Z}^C} V_{\lambda}^* \otimes V_\lambda
\]

(5)

**Theorem** [J. Huebschmann 2008 \cite{2} Theorem 5.3]

The association: \( \phi^C(g) \rightarrow (\hbar \pi)^{\dim(K)/4} e^{\hbar |\lambda + \rho|^2/2} \phi(g) \) as \( \lambda \) ranges over the highest weights induces a unitary isomorphism of unitary \((K \times K)\) representations.

\[
HL^2(K^C, e^{-|Y|^2/\hbar}\eta(g)du\,dY) \rightarrow L^2(K, dx)
\]

, where \( \phi_{\lambda} \in V_{\lambda}^* \otimes V_\lambda, \lambda \cdot \) is the highest weight of \( K \), while \( \rho \) is the Weyl vector of \( K \), i.e the half sum of the positive roots. \( dx \) is a Haar measure.

For the details and the Theorem proofs see \cite{2}. In this paper we apply these theorems to the case \( K = SU(2), K^C = SL(2, C) \) to derive a convergent \( \tilde{Y} \)-map.

### 3 \( \tilde{Y} \)-Map

The Holomorphic Peter-Weyl Theorem establishes the isomorphism between the Hilbert space spanned by the compact group \( K \) matrix coefficients and the Hilbert space spanned by the matrix coefficients of that group complexification \( K^C \). In this chapter we will use the Holomorphic Peter-Weyl theorem stated above in order to introduce \( \tilde{Y} \)-map. In our case \( K \) is \( SU(2), K^C = SL(2, C) \). Let us derive for our case \( \phi, \phi^C, \lambda, \) and \( \rho \) and substitute them into \( \phi^C(g) \).

\[
\phi^C(g) \rightarrow (\hbar \pi)^{\dim(K)/4} e^{\hbar |\lambda + \rho|^2/2} \phi(g)
\]

(7)

The corresponding matrix coefficients \( \phi \) and \( \phi^C \) are as follows:

\[
\phi(u) = D_{qq'}^{j''}(u), \quad \phi^C(g) = D_{qq'}^{(j''-j)}(g) = D_{qq'}^{(j,0)}(g) \otimes D_{qq'}^{(0,j)}(g)
\]

(8)

For \( SU(2) \) \( \dim(K) = 3 \), the highest weight \( \lambda_j \) of the finite dimensional representation is \((\dim(V) - 1) \frac{\alpha(H)}{2} \), which is \( 2j \cdot \frac{\alpha(H)}{2} \), where \( \alpha(H) \) is the only \( SU(2) \) positive root.
\[ \alpha(H) = 2h, \quad H = \text{diag}(ih, -ih). \] The Weyl vector \( \rho = \frac{\alpha(H)}{2} \). The Killing form gives the value of \( |\lambda_j + \rho|^2 = \frac{(2j+1)^2}{8} \) By substituting these values into (2) we find the matrix coefficients:

\[ D_{qq'}^{j} (u) \rightarrow \frac{1}{A_j} D_{qq'}^{(j,j)} (g) \tag{9} \]

where

\[ A_j = (\hbar \pi)^{3/2} e^{-\frac{\hbar(2j+1)^2}{8}} \tag{10} \]

By the Holomorphic Peter-Weyl theorem \( \frac{1}{A_j} D_{qq'}^{(j,j)} (g) \) are dense in \( L^2(g, e^{-|Y|^2/\hbar} \eta(g) du dY) \), where \( u \in SU(2), g \in SL(2, \mathbb{C}) \).

This provides the following improved \( \tilde{Y} \) map of the functions of \( SU(2) \) to the square integrable functions of \( SL(2, \mathbb{C}) \) with the above measure:

\[ \tilde{Y} : \phi(u) = \sum_{j=0}^{\infty} \sum_{q,q'=-j}^{j} C_{qq'} D_{qq'}^{j} (u) \rightarrow \psi(g) = \sum_{j=0}^{\infty} \sum_{q,q'=-j}^{j} \frac{1}{A_j} C_{qq'} D_{qq'}^{(j,j)} (g) \tag{11} \]

\( D_{qq'}^{(j,j)} (g) \) are the matrix coefficients of the non-unitary spinor finite dimensional \( SL(2, \mathbb{C}) \) representation. This representation in turn is contained in a general (sometimes called non-unitary) principal series representation \( H^{(n, \nu)} \), where \( n \in \mathbb{Z}, \nu \in \mathbb{C} \). We would like to emphasize the fact that the parameter \( \nu \) is complex. The non-unitary principal series becomes a unitary principal series when \( \nu \) is real. In general the matrix coefficients \( D_{q_+ q_- q'_+ q'_-}^{(j,j-\nu)} (g) \) are contained in general principal series matrix coefficients

\[ D_{q_+ q_-}^{(j,j-\nu)} (g) = D_{q_+ q_-}^{(n, \nu)} (j, j+, j-, j-) \tag{12} \]

for \( n = 2j_+ - 2j_- \), \( \nu = 2i(1 + j_+ + j_-) \). For details see for example [15]

\[ D_{q_+ q_-}^{(j,j-\nu)} (g) = D_{q_+ q_-}^{(2j_+ - 2j_-, 2(1 + j_+ + j_-))} (g) \tag{13} \]

As it was shown in [3] the diagonal simplicity constraint of the holomorphic spin-foam provides us two solutions: \( (j, 0), \gamma = -i \) and \( (0, j), \gamma = i \). Substituting them into (13) we obtain two sets of matrix coefficients:

\[ D_{qq'}^{(j,0)} (g) = D_{qq'}^{(2j, -2(1+j))} (g) \tag{14} \]

and

\[ D_{qq'}^{(0,j)} (g) = D_{qq'}^{(-2j, 2(1+j))} (g) \tag{15} \]

The off-diagonal simplicity constraint selects the second of these two solutions: \( (0, j), \gamma = i \) corresponding to (15).

After substituting it into the expression (11) for the \( \tilde{Y} \) map, we obtain:
\[ \tilde{Y} : \phi(u) = \sum_{j=0}^{j} \sum_{q,q'=-j}^{j} C_{jqq'} D_{qq'}^j(u) \rightarrow \psi(g) = \sum_{j=0}^{j} \sum_{q,q'=-j}^{j} \frac{1}{A_j} C_{jqq'} D_{jqq'}^{(2j,2j+1)}(\bar{g}) \]

(16)

After substituting the expression (10) for \( A_j \) into (16) we obtain the final expression for \( \tilde{Y} \) map:

\[ \tilde{Y} : \phi(u) = \sum_{j=0}^{j} \sum_{q,q'=-j}^{j} C_{jqq'} D_{qq'}^j(u) \rightarrow \psi(g) = \sum_{j=0}^{j} \sum_{q,q'=-j}^{j} \left( \frac{\pi}{2} \right)^{-3/2} e^{-\frac{h(2j+1)^2}{8}} C_{jqq'} D_{jqq'}^{(2j,2j+1)}(\bar{g}) \]

(17)

We would like to stress that \( A_j \) depends on \( j \). According to the Holomorphic Peter-Weyl theorem, the above sum is convergent and the function \( \psi(g) \) is a square integrable function in \( L^2(g, e^{-|Y|^2/\eta(g)} du dY) \), \( g \in SL(2, C) \), \( u \in SU(2) \), \( Y \in su(2) \).

4 Implicit Reality Conditions

Both in EPRL formalism \[7\] and in a new theory formulation \[1\] the unitary principal series representation is chosen instead of the general principal series (sometimes called non-unitary). By limiting the general principal series to its unitary sub-representation, one implicitly implies a reality constraint on the Barbero-Immirzi parameter making it real instead of complex. One can see it immediately from the simplicity constraints solution \[7\] :

\[ \nu = n\gamma \]

(18)

Where \( \nu \) and \( n \) are \( H^{(n,\nu)} \) the principal series Hilbert space parameters. For the general principal series \( n \in \mathbb{Z}, \nu \in \mathbb{C} \), while for its unitary sub-representation \( \nu \in \mathbb{R} \). It follows then from (18) that the Barbero-Immirzi parameter is complex for the general principal series representation and becomes real when one selects its unitary sub-representation since \( \gamma = \nu/n \).

Therefore a choice of the unitary sub-representation of the general principal series implicitly implies a reality constraint on the Barbero-Immirzi parameter making it real instead of complex.

5 Discussion

In the current paper we have introduced an alternative to the \( Y \)-map of a new LQG covariant formulation. We called a new map \( \tilde{Y} \). We have shown that \( \tilde{Y} \)-map is well defined and convergent in the space of square integrable functions of \( SL(2, C) \) with the measure \( e^{-|Y|^2/\hbar}(g) du dY \), as a consequence of the holomorphic Peter-Weyl theorem. We have provided the formulation of the holomorphic Peter-Weyl theorem \[2\], which we used to derive the \( \tilde{Y} \) mapping. We discussed and emphasized that the choice of the unitary principal series instead of the general (or sometimes called non-unitary
principal series) in EPRL and a new LQG covarian formulation implicitly imposes a reality condition on the Barbero-Immirzi parameter $\gamma$ making it real instead of complex.

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