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Pricing in a competitive stochastic insurance market

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ABSTRACT

This paper studies a one-period stochastic game to determine the optimal premium strategies of non-life insurers in a competitive market. Specifically, the optimal premium strategy is determined by the Nash equilibrium of an n-player game, in which each player is assumed to maximise the expected utility of terminal wealth. The terminal wealth is stochastic, since the number of policies and the size of claims are considered to be random variables. The total loss of each insurer is described by the collective risk model. The expected number of policies is affected by all the premiums in the market and further investigated by two distinct demand functions. Both models have an exponential functional form, that is characterised by market and price sensitivity parameters. The demand in the first model is zero for premiums above a given threshold, whereas the second model does not include such restriction. The pure strategy Nash equilibrium premiums are given as solutions to constrained optimisation problems. For the first model we prove the existence and uniqueness of a pure strategy Nash equilibrium, whereas for the second model we provide a formula when it exists. Two numerical examples are provided to illustrate the applicability of our findings.

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1. Introduction

1.1. Motivation

In the insurance literature, there is a continuous and strong interest in the modelling of insurance premiums in a competitive market. Insurers determine their premiums in response to the premiums that are being offered by the competitor companies.

Standard actuarial approaches, particularly in non-life insurance, are almost exclusively devoted on calculating the distribution of the underlying risk, following the underlying trends in distribution instead of formulating the underwriting strategies (Emms et al., 2007). Such approaches ignore the actions made by the competitors which affect the movement of premium rates and, eventually, the exposure of the company into the corresponding market (Taylor, 1986). Due to the complex nature of price competition, several modelling and pricing challenges arise, and this is the motivation of our paper.

1.2. Literature review

The present paper formulates a stochastic model in competitive non-life insurance pricing. A standard approach is to let the premium exclusively depend on the distribution of the underlying risk (see Kaas et al., 2008). However, in this approach one does not assume that the insurance markets are competitive.

Taylor (1986) addresses the subjective nature of the insurance underwriting in a deterministic discrete-time framework. He proposes a demand function to capture how the number of policies sold by the insurer is affected by the premium choices compared to other competitors’ premiums. Under this demand function, an insurer selects the optimal response to movements of premiums in the market. In a subsequent work, Taylor (1987) shows that significant changes in the optimal premiums may happen when non-constant expense rates occur. Emms and Haberman (2005), Emms et al. (2007), Emms et al. (2007) extend the literature with a stochastic continuous-time framework. They calculate the optimal premium strategy, which maximises the expected terminal wealth of an insurer, considering a demand law and applying optimal control theory. Pantelous and Passalidou (2013, 2015) embed the insurer’s reputation and a stochastic disturbance into their stochastic discrete-time model and explicitly calculate the optimal premiums of an insurer as solutions of a polynomial. All the aforementioned models consist of single optimisation problems, in which competitors’ premiums are unaffected by the individual insurer’s premium strategy. In Table 1, we classify the papers, where the break-even and market’s premiums are considered either as deterministic or stochastic, into static/dynamic and deterministic/stochastic settings.

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Non-cooperative game theory provides tools to embark from individual optimisation procedures and aim to simultaneous optimisation solutions for all participants in the market. First cooperative game theoretical approaches in insurance can be found in Borch (1962, 1974), Bühlmann (1980, 1984), and Lemaire (1984, 1991). The non-cooperative game models that have been applied to non-life insurance markets can be distinguished into two categories: (i) the Bertrand oligopoly in which insurers set premiums and (ii) the Cournot oligopoly in which insurers choose the volume of business. Emms (2012) studies the Nash equilibria of an n-player non-cooperative differential game in which each insurer aims to maximise her expected utility of terminal wealth under a demand function. He proposes both a deterministic and a stochastic differential game regarding whether or not the break-even premium was considered uncertain. Dutang et al. (2013) use a loss model for the claims and a lapse model to describe the probabilistic behaviour of individuals as a function of insurers’ premiums. By solving a maximisation problem that consisted of a quadratic equation and a solvency constraint function, they prove the existence and uniqueness of the Nash equilibrium, as well as the existence of the Stackelberg equilibrium of a one-period non-cooperative game. Wu and Pantelous (2017) determine the optimal premium strategy by calculating and proving the existence of Nash equilibria in an n-player potential game with non-linear aggregation. Instead of applying a demand function to describe the exposure of insurers, they aggregate all the paired competitions in the market. Boonen et al. (2018) apply optimal control theory to determine the open-loop Nash equilibrium premium strategies in an n-player differential game, by including the solvency ratio in the demand function. Recently, Asmussen et al. (2019) consider the customer’s problem with market frictions and formulate stochastic differential game between two insurance companies that have a different size. Table 2 classifies the papers with non-cooperative game theoretical orientation into static/dynamic and deterministic/stochastic settings.

### 1.3. Our contribution

In actuarial practice, the resulting premium offered to insureds usually diverges from the actuarial (net) premium, producing a positive (or sometimes negative) premium loading. Many factors may contribute to this deviation, such as the customers’ affordability, insurers’ claims and expenses, past experience, and various market conditions.

As shown in Table 2, we propose a static and a stochastic model in this paper. In this sense, this paper is closest to Dutang et al. (2013). Like Dutang et al. (2013), this paper proposes a model in which the terminal wealth of each insurer is characterised by the collective risk model, which depends on the premiums in the market. We differ from Dutang et al. (2013) in the following three ways. First, in our paper, the demand function is related to how many policies are expected to be sold in a reference period. We assume that the number of policies is either Poisson-distributed, whose intensity is driven by the competition in the market, or follows a Negative Binomial distribution. Second, we propose two distinct exponential demand functions: the first one is concave, and considers that there is a premium in the market under which no policy will be sold. The second demand function, proposed by Taylor (1986), is convex and does not assume an upper bound on the premium. Third, we assume that the insurers are strictly risk-averse, and all of them maximise exponential utility functions (see, Emms, 2012). Under this setting, we calculate the best response strategies and show the existence of Nash equilibria. Moreover, we consider two numerical examples to illustrate the applicability of our treatment.

This paper is set out as follows. In Section 2, we provide the necessary notations and assumptions. Assuming a loss model for the total claim amount, Poisson or Negative Binomial distributions for the number of policies, and insurers who are exponential utility maximisers, we obtain the lower bound for the premium domain and the cost function of each insurer. In Sections 3.1 and 3.2, we introduce two families of exposure functions and find the candidate optimal premiums of each individual insurer. In Section 3.3, two results for optimality are presented. In Sections 4.1 and 4.2, we define two exponential exposure functions, with different curvatures, which we use to derive optimal premiums for each insurer. Section 5 provides a Nash equilibrium premium strategy for each model. In the first one the existence and uniqueness of a Nash equilibrium is shown, whereas in the second one there is a structure when it exists. Section 6 presents a numerical example for both models to illustrate our main results. Section 7 concludes, and further suggestions of future research are provided. The proofs are delegated to the Appendix.

### 2. Preliminaries

Let \( N = \{ 1, 2, \ldots , n \} \), \( n \in \mathbb{N} \), be the finite set of insurers in the market. Every insurer selects a certain premium \( p_i \) per policy. Given the premium of the insurer and the premiums of all other insurers in the market, the insurer underwrites a stochastic number of insurance policies, which are realised at a fixed future period in time. The key concept of our model is that premium choices affect directly the intensity of the number of policies which flow among insurers. Rather than finding the premium strategy of one insurer, our objective is to calculate the premium strategies followed by all insurers in the market in their attempt to maximise the expected utility of terminal wealth.

**Table 1**

| Papers | Discrete | Continuous | Deterministic | Stochastic |
|--------|----------|------------|---------------|-------------|
| Taylor (1986, 1987) | ✓ | | ✓ | ✓ |
| Emms and Haberman (2005) | | ✓ | | |
| Emms et al. (2007) | ✓ | | ✓ | ✓ |
| Emms (2007) | ✓ | | ✓ | ✓ |
| Pantelous and Passalidou (2013, 2015) | ✓ | | ✓ | ✓ |

**Table 2**

| Papers | Static | Dynamic | Deterministic | Stochastic |
|--------|--------|---------|---------------|-------------|
| Emms (2012) | ✓ | ✓ | ✓ | ✓ |
| Dutang et al. (2013) | ✓ | ✓ | ✓ | ✓ |
| Wu and Pantelous (2017) | ✓ | ✓ | ✓ | ✓ |
| Boonen et al. (2018) | ✓ | ✓ | ✓ | ✓ |
| Asmussen et al. (2019) | ✓ | ✓ | ✓ | ✓ |
| This paper | ✓ | ✓ | ✓ | ✓ |
2.1. Basic notations

The definitions of key parameters that we use in this paper are listed as follows:

- \( w_i \) \( \rightarrow \) Initial wealth of insurer \( i \in N \),
- \( p_i \) \( \rightarrow \) Premium value (per policy) charged by insurer \( i \) for the period under consideration,
- \( p_{-i} \) \( \rightarrow \) Strategy profile of premiums charged by all insurers except insurer \( i \),
- \( \bar{p}_{-i} \) \( \rightarrow \) Average premium charged by all insurers except insurer \( i \),
- \( q_i(p_i, p_{-i}) \) \( \rightarrow \) Exposure volume of insurer \( i \) for the period under consideration.
- \( q_j \) \( \rightarrow \) Exposure volume of insurer \( i \) from the previous period,
- \( N_i \) \( \rightarrow \) Actual number of policies underwritten by insurer \( i \) within the period under consideration,
- \( \lambda_i \) \( \rightarrow \) Risk aversion parameter of insurer \( i \),
- \( h \) \( \rightarrow \) Risk aversion parameter of the most risk-averse individual in the market,
- \( b \) \( \rightarrow \) Market scale parameter,
- \( a_i \) \( \rightarrow \) Price sensitivity parameter of insurer \( i \).

2.2. Loss model

Let us now define the collective risk model adopted in our analysis. The total claim amount faced by insurer \( i \), \( S(N_i) \), for the period under consideration, is defined as

\[
S(N_i) = \begin{cases} 
0, & N_i = 0 \\
\sum_{j=1}^{N_i} X_{ij}, & N_i \geq 1,
\end{cases}
\]

where \( X_{ij} \) is a random variable which represents the claim amount associated to claimant \( j \) for insurer \( i \), for the period under consideration. Claims \([X_{ij}]_{j=1}^{N_i}\) are non-negative random variables, and \( N_i \) reflects the number of policies sold by insurer \( i \), and this is a random variable which follows either Poisson distribution with parameter \( q_i(p_i, p_{-i}) \) or Negative Binomial distribution with parameters \( q_i(p_i, p_{-i}) \) and \( k \in (0, 1) \). Thus, \( q_i(p_i, p_{-i}) \) is related to the expected number of insurance policies that are sold. This parameter depends on all the premiums in the insurance market, and will be specified later in Section 3. Moreover, elements of the set \([X_{ij}]_{j=1}^{N_i}\) are independent and identically distributed (i.i.d.) random variables, as well independent from \( N_i \). For convenience, we denote the common distribution of \([X_{ij}]_{j=1}^{N_i}\) with \( X_i \), i.e., \( X_{ij} \sim X, \) for all \( j = 1, 2, \ldots, N_i, i = 1, 2, \ldots, n \).

Throughout this paper, we assume that all insurers are risk-averse, i.e., \( \lambda_i > 0 \) for all \( i \in N \), and the case with \( \lambda_i = 0 \) is only used to compare our results with the literature.

2.4. Lower bound on the premiums

Let each insurer \( i \) have a utility function of the form (1), with risk aversion parameter \( \lambda_i > 0 \). Then, insurer \( i \) is willing to charge a premium \( p_i \), in order to cover a potential loss \( X \), only if the expected utility of this action is greater than or equal to not undertake this risk. Hence, the premium \( p_i \) charged by insurer \( i \) should satisfy the following individual rationality constraint,

\[
E \left[ u \left( w_i^{-} + N_i p_i - S(N_i) \right) \right] \geq u \left( w_i^{-} \right),
\]

where \( w_i^{-} \) is the initial capital of insurer \( i \).

**Lemma 2.** For any distribution of \( N_i \) independent of \( X_i \) and such that \( Pr[N_i > 0] > 0 \), the premium charged by insurer \( i, p_i \), satisfies (2) if and only if

\[
p_i \geq \frac{1}{\lambda_i} \log (M_X(\lambda_i)),
\]

where \( \lambda_i \) is the risk aversion parameter of insurer \( i \).

The indifference premium for insurer \( i \) is defined by

\[
p_i^\circ = \frac{1}{\lambda_i} \log (M_X(\lambda_i)),
\]

and via the following lemma, we present some main properties of it.

**Lemma 3 (Kaas et al., 2008).** For the non-negative random variable \( X \), define the indifference premium \( p_i^\circ \) as a function of \( \lambda_i \) by \( p(\lambda_i) \). It satisfies the following properties:

1. \( p(\lambda) \) is an increasing function of \( \lambda \).
2. \( \lim_{\lambda \to 0} p(\lambda) = E[X] \).
3. Assume that \( X \) is a bounded non-negative random variable with \( \max[X] = c \), i.e., \( Pr[X \leq c] = 1 \) but for all \( \varepsilon > 0 \), \( Pr[X > c - \varepsilon] > 0 \). Then it holds \( \lim_{\lambda \to +\infty} p(\lambda) = c \).

2.5. Objective of the insurers

The objective of each insurer \( i \in N \) is to set a premium \( p_i \) per policy such that to maximise their expected utility of terminal wealth. From this point up to Section 5, we focus only on maximisation procedure regarding one insurer, conditioning on other insurers’ premiums, although we have assumed that all insurers’ premiums are unknown. In Section 5 we apply the notion of a Nash equilibrium strategy profile, and how it is calculated through simultaneous individual maximisation procedures. Recalling the necessary notation presented in Section 2.1, next we state the wealth of insurer \( i \). The wealth of insurer \( i \), for the period under investigation, is given by

\[
w_i(p_i, p_{-i}) = w_i^{-} + p_i N_i(q(p_i, p_{-i})) - S(N_i(q(p_i, p_{-i}))).
\]

Note that the number of policies that are underwritten is realised after the prices are set, and the distribution of the insurance claims does depend on this amount. It is well-known that for exponential utilities, the initial wealth does not affect the risk preferences. Since the premium profile \( (p_i, p_{-i}) \) is generally clear

...
from the context, we simplify our notation by writing \( w_i = w_i(p_i, p_{-i}) \), and \( q_i = q_i(p, p_{-i}). \)

Now, we observe that the random variable \( w_i \) for the wealth is a function of the random variables \( N_i \) and \( S_i \), whereas \( S_i \) depends on \( N_i \). Thus, we invoke the law of iterated expectations to calculate the expected utility of the wealth of insurer \( i \).

**Proposition 4.** Given that insurer \( i \)'s risk aversion parameter is \( \lambda_i > 0 \) and \( N_i \) follows a Poisson distribution with parameter \( q_i(p, p_{-i}) \), the expected utility of insurer \( i \)'s wealth is

\[
E [u_i(w_i)] = -\exp \left\{ -\lambda_i w_i^{-1} + q_i(p, p_{-i}) \left( M_k(\lambda_i) e^{-\lambda_i p_i} - 1 \right) \right\},
\]

where \( M_k(\lambda_i) \) is the moment generating function of \( \lambda_i \).

We continue considering the case where the actual number of policies underwritten by insurer \( i \), \( N_i \), follows a Poisson distribution with intensity \( q(p, p_{-i}) \geq 0 \) for all insurer \( i \) set with a premium \( p_i \) such that to maximise her expected utility of wealth. We observe from (6) that \( E [u_i(w_i)] < 0 \) for all \( p_i \). This enables us to convert the maximisation problem into a minimisation procedure, due to the following equivalent characterisation

\[
\max_{P_i} [E [u_i(w_i)]] \text{ or } \min_{P_i} \left\{ -E [u_i(w_i)] \right\},
\]

or \( \min_{P_i} \left\{ \log(-E [u_i(w_i)]) \right\}. \)

Therefore, (6) and (7) lead us to the cost (objective) function \( C_i : P \rightarrow \mathbb{R} \), which is defined by

\[
C_i(p, p_{-i}) = \log(-E [u_i(w_i)]) = -\lambda_i w_i^{-1} + q_i(p, p_{-i}) \left( M_k(\lambda_i) e^{-\lambda_i p_i} - 1 \right),
\]

for all \( i = 1, 2, \ldots, n \), where \( P = P_1 \times P_2 \times \cdots \times P_n \) is the set of all strategy profiles, \( p_i \in P_i \) is a strategy of insurer \( i \) and \( p_{-i} = (p_j : j \neq i) \in P_{-i} \) is a strategy profile of all insurers apart from \( i \).

**Remark 5.** Let \( N_i \) follow a Negative Binomial distribution\(^1\) with parameters \( q(p, p_{-i}) \) and \( 0 < k < 1 \). Instead of (8), the cost function is now equal to

\[
C_i(p, p_{-i}) = -\lambda_i w_i^{-1} + q_i(p, p_{-i}) \log \left( \frac{k M_k(\lambda_i) e^{-\lambda_i p_i}}{1 - (1 - k) M_k(\lambda_i) e^{-\lambda_i p_i}} \right).
\]

**3. Model and individual price optimisation**

The actual number of policies that insurer \( i \) underwrites, \( N_i \), follows a Poisson distribution (or a Negative Binomial distribution later in some remarks), whose expectations depend on \( q(p, p_{-i}) \), which we will specify in this section. We assume that the volume of exposure \( q(p, p_{-i}) \) is driven by the competition in the market and affected directly by changes in premiums. We capture the interaction among insurers in the market by defining for each insurer \( i \) the exposure function as in Taylor (1986), who explained that exposure in the current period should be proportional to exposure in the previous period. Define

\[
q_i(p_i, p_{-i}) = f(p_i, p_{-i}) q_i^{-1},
\]

where \( p_{-i} = \frac{1}{n} \sum_{j \neq i} p_j^2 \) (e.g., Taylor, 1986, 1987), \( f \) is a twice continuously differentiable function of \( p_i \in P_i \) and \( p_{-i} \), and \( q_i^{-1} \).

\(^1\) Detailed calculations for the Negative Binomial are omitted as they are similar to those for the Poisson case.

\(^2\) However, in our mathematical formulation, \( p_{-i} \) can be any continuous and differentiable function.
We propose the following basic properties for $\tilde{f}$:

$$\frac{\partial \tilde{f}(p_i, \tilde{p}_{-i})}{\partial p_i} < 0 \quad \text{for all } p_i \in \left[p_i^l, p_i^u\right], \tilde{p}_{-i} < p_i^u. \quad (15)$$

$$\tilde{f}(p_i, \tilde{p}_{-i}) = 0 \quad \text{at } p_i = p_i^u, \text{ for all } \tilde{p}_{-i}. \quad (16)$$

In particular, for $\tilde{p}_{-i} < p_i^u$, it holds that $\tilde{f}(p_i, \tilde{p}_{-i}) \geq 0$ for all $p_i \in P_i$ and $\tilde{f}(p_i, \tilde{p}_{-i}) > 0$ for all $p_i \in \left[p_i^l, p_i^u\right]$.

Regarding (11), the property in (15) implies positive price-elasticity of demand. Therefore, any increase in insurer $i$’s premium leads to a decrease in the volume of exposure (see, for instance, Taylor, 1986; Emms et al., 2007; Pantelous and Pasalidou, 2015). Moreover, we assume that $\tilde{q}_i(p_i, \tilde{p}_{-i})$ is a non-negative and decreasing function of $p_i$ on $P_i$, which achieves its lowest value, i.e., zero, at the highest possible value of $p_i$ on $P_i$.

We next solve the individual optimisation problem, given by

$$\min_{p_i \in P_i} C_i(p_i, p_{-i}), \quad (17)$$

where $p_{-i} \in P_{-i}$.

**Lemma 7.** Given the premium profile $p_{-i} \in P_{-i}$ such that $\tilde{p}_{-i} \neq p_{-i}^u$ and let $N_i$ follow a Poisson distribution with parameter $\tilde{q}_i(p_i, p_{-i})$, then candidate optimal premiums of (17) for insurer $i$, $\tilde{p}_i^*$, are the solutions to the equation

$$\frac{\partial \tilde{q}_i(p_i, p_{-i})}{\partial p_i} \left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right) - \lambda_i \tilde{q}_i(p_i, p_{-i})M_X(\lambda_i)e^{-\lambda_i p_i} = 0, \quad (18)$$

and belong to the interior of the premium domain $P_i$, i.e., $\tilde{p}_i^* \in (p_i^l, p_i^u)$.

### 3.2. Optimisation problem of exposure function without market restriction

We next introduce the exposure function without market restriction. Here, we denote the relative change of the expected number of policies $f$ as $\tilde{f}$ and the volume of exposure $q_i$ as $\tilde{q}_i$. We propose the following basic properties for $\tilde{f}$:

$$\tilde{f}(p_i, \tilde{p}_{-i}) > 0 \quad \text{for all } p_i \geq p_i^l, \quad (19)$$

$$\frac{\partial \tilde{f}(p_i, \tilde{p}_{-i})}{\partial p_i} < 0 \quad \text{for all } p_i \geq p_i^l. \quad (20)$$

Specifically, Taylor (1986) proposed the demand function as in (10), which satisfies properties (19)–(20). The difference with the previous demand function is that the exposure volume of insurer $i$ will be positive even for large values of her premium $p_i$. Additionally, regarding (11), the property (20) implies positive price-elasticity of demand.

In this setting, the set of all available premiums of insurer $i$ is equal to

$$P_i = \left[p_i^l, +\infty\right). \quad (21)$$

In the remaining of the paper, it will be clear of the context which set is used, and when necessary, we distinguish them. For $P_i$ as in (21), we next solve the individual optimisation problem given by

$$\min_{p_i \in P_i} C_i(p_i, p_{-i}), \quad (22)$$

where $p_{-i} \in P_{-i}$.

**Lemma 8.** Given the premium profile $p_{-i} \in P_{-i}$ and $N_i$ follows a Poisson distribution with parameter $\tilde{q}_i(p_i, p_{-i})$, candidate optimal premiums to (22) for insurer $i$, $\tilde{p}_i^*$, are the solutions to the equation

$$\frac{\partial \tilde{q}_i(p_i, p_{-i})}{\partial p_i} \left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right) - \lambda_i \tilde{q}_i(p_i, p_{-i})M_X(\lambda_i)e^{-\lambda_i p_i} = 0, \quad (23)$$

and satisfy $\tilde{p}_i^* > p_i^l$.

**Remark 9.** Let $N_i$ follow a Negative Binomial distribution with parameters $\tilde{q}_i(p_i, p_{-i})$ and $0 < k < 1$. Using similar arguments as in Lemmas 7 and 8, candidate optimal premiums are the solutions to

$$\frac{\partial \tilde{q}_i(p_i, p_{-i})}{\partial p_i} \log\left(1 - \frac{kM_X(\lambda_i)e^{-\lambda_i p_i} - \lambda_i \tilde{q}_i(p_i, p_{-i})M_X(\lambda_i)e^{-\lambda_i p_i}}{1 - kM_X(\lambda_i)e^{-\lambda_i p_i}}\right) = 0, \quad (24)$$

and belong to the interior of the corresponding premium domain, where $q_i = \tilde{q}_i$ when the market restriction is present, or $q_i = \tilde{q}_i$ otherwise. Moreover, the optimal premiums are denoted by $\tilde{p}_i^*$ and $\tilde{p}_i^*$ for the restricted and unrestricted models, respectively.

### 3.3. Convexity analysis

Let us consider the case of $N_i$ following a Poisson distribution with parameter $\tilde{q}_i(p_i, p_{-i})$. Given that $\tilde{q}_i(p_i, p_{-i})$ is twice continuously, partially differentiable with respect to $p_i$, in our attempt to minimise the cost function $C_i(p_i, p_{-i})$ with respect to $p_i$, we find all the critical points of $C_i$ by solving the first order condition, and using the second order condition we establish the extremum nature of them. In Sections 3.1 and 3.2, we showed that the optimal solutions must be in the interior of $P_i$. Generally, by Lemma 7 and 8, we get that both of them are the solutions of an equation of the form

$$\frac{\partial \tilde{q}_i(p_i, p_{-i})}{\partial p_i} \left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right) - \lambda_i \tilde{q}_i(p_i, p_{-i})M_X(\lambda_i)e^{-\lambda_i p_i} = 0, \quad (25)$$

where $q_i = \tilde{q}_i$ when the market restriction is present, or $q_i = \tilde{q}_i$ otherwise. Now, we investigate the convexity of the objective function $C_i(p_i, p_{-i})$ with respect to $p_i$. Without any loss of generality, we denote the candidate optimal solutions of (25) with $p_i^l$ and whenever it is necessary we distinguish if $p_i^l \in (p_i^l, p_i^u)$ or $p_i^l > p_i^l$. The following two results concern both exposure functions, so we keep the general notation of $q_i$.

From (8), the second partial derivative of $C_i(p_i, p_{-i})$ with respect to $p_i$ is equal to

$$\frac{\partial^2 C_i(p_i, p_{-i})}{\partial p_i^2} = \frac{\partial^2 \tilde{q}_i(p_i, p_{-i})}{\partial p_i^2} \left(M_X(\lambda_i)e^{-\lambda_i p_i} - 1\right) - 2\lambda_i \frac{\partial \tilde{q}_i(p_i, p_{-i})}{\partial p_i}M_X(\lambda_i)e^{-\lambda_i p_i} + \lambda_i^2 \tilde{q}_i(p_i, p_{-i})M_X(\lambda_i)e^{-\lambda_i p_i}. \quad (26)$$

Observing that the factors $M_X(\lambda_i)e^{-\lambda_i p_i} - 1$ and $\tilde{q}_i(p_i, p_{-i})/\partial p_i$ in (26) are non-positive, whereas the rest of them are positive, we state two results for the convexity of the cost function $C_i$, based on $\partial^2 \tilde{q}_i(p_i, p_{-i})/\partial p_i^2$. The first one is global, over all the premium domain of $p_i$, whereas the second is local, at the solution $p_i^l$ of (25).

**Theorem 10.** Let $p_{-i} \in P_{-i}, p_i^l$ be a solution which satisfies (25) and $N_i$ follow a Poisson distribution with parameter $\tilde{q}_i(p_i, p_{-i})$. Then, $p_i^l$
is a global minimum for the cost function $C_i(p_i, p_{-i})$, if for all $p_i \in P_i$ it holds
\[
\frac{\partial^2 q_i(p_i, p_{-i})}{\partial p_i^2} \leq 0.
\] (27)

**Theorem 11.** Let $p_{-i} \in P_{-i}$, $p_i^*$ be a solution which satisfies (25) and $N_i$ follow a Poisson distribution with parameter $q_i(p_i, p_{-i})$. Then, $p_i^*$ is a local minimum for the cost function $C_i(p_i, p_{-i})$, if
\[
\frac{\partial^2 q_i(p_i, p_{-i})}{\partial p_i^2} \bigg|_{p_i = p_i^*} < 0.
\]

The exposure function $q_i(p_i, p_{-i})$ describes how the expected number of policies will be sold by insurer $i$ while competing with other insurers. Section 3 provides in general the properties that the exposure functions have to satisfy in our modelling. In this section we define two specific forms of exponential exposure function $q_i(p_i, p_{-i})$, and we prove that the minimisation problems in Sections 3.1 and 3.2 have solutions for both exposure functions.

### 4. Exponential exposure function

The exposure function $q_i(p_i, p_{-i})$ describes how the expected number of policies will be sold by insurer $i$ while competing with other insurers. Section 3 provides in general the properties that the exposure functions have to satisfy in our modelling. In this section we define two specific forms of exponential exposure function $q_i(p_i, p_{-i})$, and we prove that the minimisation problems in Sections 3.1 and 3.2 have solutions for both exposure functions.

#### 4.1. Exponential exposure function with market restriction

We introduce a specific exposure function that satisfies the conditions in (15)–(16). The exponential exposure function with market restriction of insurer $i$ is defined by
\[
\tilde{q}_i(p_i, p_{-i}) = \tilde{b} \left[ 1 - \exp \left\{ -a_i \frac{p_i^{1/\lambda} - p_i}{p_i^{1/\lambda} - \bar{p}_{-i}} \right\} \right] q_i^-
\] (32)
for $p_i \in P_i$ and $\bar{p}_{-i} < p_i^\lambda$. Here, $\tilde{b} > 1$ is a market scale parameter, which represents the available potentials of the market. Larger values of $\tilde{b}$ indicate greater amount of individuals looking to be insured. Moreover, $a_i > 0$ is a price sensitivity parameter, and larger values of it are associated to insurers who possess greater market power. Our interpretation of (32) is that changes in small values of premiums have greater impact on $\tilde{q}_i(p_i, p_{-i})$ than changes in larger premium values. Therefore, the rate of decrease of $\tilde{q}_i$ with respect to $p_i$ is assumed a decreasing function of $p_i$.

The first and second partial derivatives of $\tilde{q}_i(p_i, p_{-i})$ with respect to $p_i$ are
\[
\frac{\partial \tilde{q}_i(p_i, p_{-i})}{\partial p_i} = -\tilde{b} a_i \frac{p_i^{1/\lambda} - p_i}{p_i^{1/\lambda} - \bar{p}_{-i}} \exp \left\{ -a_i \frac{p_i^{1/\lambda} - p_i}{p_i^{1/\lambda} - \bar{p}_{-i}} \right\} q_i^- < 0,
\] (33)
and
\[
\frac{\partial^2 \tilde{q}_i(p_i, p_{-i})}{\partial p_i^2} = -\tilde{b} a_i^2 \left( \frac{p_i^{1/\lambda} - p_i}{p_i^{1/\lambda} - \bar{p}_{-i}} \right)^2 \exp \left\{ -a_i \frac{p_i^{1/\lambda} - p_i}{p_i^{1/\lambda} - \bar{p}_{-i}} \right\} q_i^- < 0,
\] (34)
for all $p_i \in P_i$.

**Proposition 14.** Given the exponential exposure function in (32), the average premium $\bar{p}_{-i} < p_i^\lambda$ and $N_i$ follows a Poisson distribution with parameter $\tilde{q}_i(p_i, p_{-i})$, there is unique optimal solution, $\bar{p}_i^*$, of the minimisation problem (17), which is given by
\[
\frac{a_i}{p_i^{1/\lambda} - \bar{p}_{-i}} \exp \left\{ -a_i \frac{p_i^{1/\lambda} - p_i}{p_i^{1/\lambda} - \bar{p}_{-i}} \right\} (M_X(\lambda_i) e^{-\lambda_i p_i} - 1) + \lambda_i \left[ 1 - \exp \left\{ -a_i \frac{p_i^{1/\lambda} - p_i}{p_i^{1/\lambda} - \bar{p}_{-i}} \right\} \right] M_X(\lambda_i) e^{-\lambda_i p_i} = 0,
\] (35)
and satisfies $\bar{p}_i^* \in (p_i^\lambda, p_i^\lambda)$.

**Remark 15.** When $N_i \sim NB(\tilde{q}_i(p_i, p_{-i}), k), 0 < k < 1$, substituting (32) and (33) into (24), the unique optimal premium $\bar{p}_i^* \in (p_i^\lambda, p_i^\lambda)$ is the solution to
\[
\frac{a_i}{p_i^{1/\lambda} - \bar{p}_{-i}} \exp \left\{ -a_i \frac{p_i^{1/\lambda} - p_i}{p_i^{1/\lambda} - \bar{p}_{-i}} \right\} \log \left( \frac{kM_X(\lambda_i) e^{-\lambda_i p_i}}{1 - (1 - k)M_X(\lambda_i) e^{-\lambda_i p_i}} \right) + \lambda_i \left[ 1 - \exp \left\{ -a_i \frac{p_i^{1/\lambda} - p_i}{p_i^{1/\lambda} - \bar{p}_{-i}} \right\} \right] \left( \frac{kM_X(\lambda_i) e^{-\lambda_i p_i}}{1 - (1 - k)M_X(\lambda_i) e^{-\lambda_i p_i}} \right) = 0.
\] (36)
4.2. Exponential exposure function without market restriction

In this section, we study a well-known exponential functional form, which is introduced by Taylor (1986). The exponential exposure function without market restriction is defined as

\[
\hat{q}(p_i, p_{-i}) = \exp \left\{ -a_i \frac{p_i - p_{-i}}{\bar{p}_i - p_{-i}} \right\} q_{i-}^v,
\]

where \( p_i \in P_i \) and \( \bar{p}_{-i} \geq 0 \).

We observe that \( \hat{q}(p_i, p_{-i}) > 0 \) for all \( p_i \geq p_i^v \), and the first and second partial derivatives with respect to \( p_i \) are equal to

\[
\frac{\partial \hat{q}(p_i, p_{-i})}{\partial p_i} = -a_i \frac{p_i - p_{-i}}{\bar{p}_i - p_{-i}} \hat{q}(p_i, p_{-i}),
\]

(37)

and

\[
\frac{\partial^2 \hat{q}(p_i, p_{-i})}{\partial p_i^2} = \left( \frac{a_i}{\bar{p}_i} \right)^2 \hat{q}(p_i, p_{-i}).
\]

(38)

(39)

Proposition 16. Given the exposure function in (37), the premium profile \( p_{-i} \in P_{-i} \) and \( N_i \) follows a Poisson distribution with parameter \( \hat{q}(p_i, p_{-i}) \), there is a unique optimal solution, \( \tilde{p}_i^* \), of the minimisation problem (22), which is given by

\[
\tilde{p}_i^* = -\frac{1}{\lambda_i} \log \left( \frac{a_i}{(a_i + \lambda_i \bar{p}_{-i}) M_X(\lambda_i)} \right),
\]

(40)

and satisfies \( \tilde{p}_i^* > p_i^v \).

Remark 17. Expression (39) implies that \( \hat{q}(p_i, p_{-i}) \) is a convex function in \( p_i \), and the parameter \( a_i \) is related to the price-elasticity of demand (Malqueg, 1994). Indeed, regarding (11), the price-elasticity of demand is equal to

\[
e(p_i) = -\frac{\partial \hat{q}(p_i, p_{-i})}{\partial p_i} \frac{p_i}{\hat{q}(p_i, p_{-i})} = a_i \frac{p_i}{\bar{p}_i}.
\]

We observe that the price-elasticity of demand is an increasing function of \( a_i \), and as Taylor (1986) indicated, a decrease in price-elasticity is associated to an increase in optimal premiums. This can be proved by showing that the first derivative of (40) with respect to \( a_i \) is negative. Indeed, it derives in a straightforward manner that

\[
\frac{\partial \tilde{p}_i^*}{\partial a_i} = \frac{1}{\lambda_i} \log \left( \frac{a_i}{(a_i + \lambda_i \bar{p}_{-i}) M_X(\lambda_i)} \right) > 0,
\]

(41)

Remark 18. When \( N_i \sim NB(\hat{q}(p_i, p_{-i}), k), 0 < k < 1 \), substituting (37) and (38) into (24), optimal premiums \( \tilde{p}_i^* > p_i^v \) are the solutions to

\[
\frac{a_i}{\bar{p}_i} \log \left( \frac{k M_X(\lambda_i) e^{-\lambda_i p_i}}{1 - (1 - k) M_X(\lambda_i) e^{-\lambda_i p_i}} \right) + \lambda_i \left[ 1 - (1 - k) M_X(\lambda_i) e^{-\lambda_i p_i} \right]^{-1} = 0.
\]

Moreover, substituting (38) and (39) into (31), we verify that (31) is satisfied.

5. Nash equilibrium premiums

In the previous sections, we have presented our analysis for a single optimisation problem. However, we are interested in finding the best choice of premiums for all insurers in the market who do not cooperate and act selfishly. To do so, we next study the Nash equilibrium. In a Nash equilibrium, no insurer has an incentive to unilaterally deviate from the premium strategy. The formal definition is provided as follows.

Definition 19. A strategy profile \( p^* = (p_1^*, p_2^*, \ldots, p_n^*) \in P \) is a pure strategy Nash equilibrium if for all \( i \in N \), the premium \( p_i^* \) solves the minimisation problem

\[
\min_{p_i \in P_i} C_i(p_i, p_{-i}^*).
\]

Now, the next results state the existence of a pure strategy Nash equilibrium in the case of both exposure functions, and how it is calculated.

Theorem 20. Let \( N_i \) follow a Poisson distribution with parameter \( \hat{q}(p_i, p_{-i}) \). There exists unique pure strategy Nash equilibrium \( \hat{p}_i^* \) in the case of the exponential exposure function with market restriction (32), in which for all \( i = 1, 2, \ldots, n \), \( \hat{p}_i^* \) solves

\[
\frac{a_i}{p_i^* - \bar{p}_{-i}} \exp \left\{ -a_i \frac{p_i^* - p_i^v}{p_i^* - \bar{p}_{-i}} \right\} \left[ M_X(\lambda_i) e^{-\lambda_i p_i^v} - 1 \right] + \lambda_i \left[ 1 - \exp \left\{ -a_i \frac{p_i^* - p_i^v}{p_i^* - \bar{p}_{-i}} \right\} \right] M_X(\lambda_i) e^{-\lambda_i p_i^v} = 0,
\]

(43)

and \( \hat{p}_i^* \in (p_i^v, p_i^u) \).

Remark 21. Let \( N_i \sim NB(\hat{q}(p_i, p_{-i}), k), 0 < k < 1 \). Similar to Theorem 20, there exists unique pure strategy Nash equilibrium \( \hat{p}_i^* \) in the case of the exponential exposure function with market restriction (32), in which for all \( i = 1, 2, \ldots, n \), \( \hat{p}_i^* \in (p_i^v, p_i^u) \) solves

\[
\frac{a_i}{p_i^* - \bar{p}_{-i}} \exp \left\{ -a_i \frac{p_i^* - p_i^v}{p_i^* - \bar{p}_{-i}} \right\} \log \left( \frac{k M_X(\lambda_i) e^{-\lambda_i p_i^v}}{1 - (1 - k) M_X(\lambda_i) e^{-\lambda_i p_i^v}} \right) + \lambda_i \left[ 1 - \exp \left\{ -a_i \frac{p_i^* - p_i^v}{p_i^* - \bar{p}_{-i}} \right\} \right] \left[ 1 - (1 - k) M_X(\lambda_i) e^{-\lambda_i p_i^v} \right]^{-1} = 0.
\]

(44)

Proposition 22. Let \( N_i \) follow a Poisson distribution with parameter \( \hat{q}(p_i, p_{-i}) \). A pure strategy Nash equilibrium \( \hat{p}_i^* \) in the case of the exponential exposure function without market restriction (37) is obtained by solving for all \( i = 1, 2, \ldots, n \)

\[
p_i = -\frac{1}{\lambda_i} \log \left( \frac{a_i}{(a_i + \lambda_i \bar{p}_{-i}) M_X(\lambda_i)} \right),
\]

(45)

where \( \hat{p}_i^* > p_i^v \).

Remark 23. Let \( N_i \sim NB(\hat{q}(p_i, p_{-i}), k), 0 < k < 1 \). A pure strategy Nash equilibrium \( \hat{p}_i^* \) in the case of the exponential exposure function without market restriction (37) is obtained by solving for all \( i = 1, 2, \ldots, n \)

\[
\frac{a_i}{\bar{p}_{-i}} \log \left( \frac{k M_X(\lambda_i) e^{-\lambda_i p_i^v}}{1 - (1 - k) M_X(\lambda_i) e^{-\lambda_i p_i^v}} \right) + \lambda_i \left[ 1 - (1 - k) M_X(\lambda_i) e^{-\lambda_i p_i^v} \right]^{-1} = 0,
\]

and \( \hat{p}_i^* > p_i^v \).

Remark 24. In Theorem 20 and Proposition 22, it is found that both equilibria are affected by insurers’ risk aversion parameters, the claims distribution and the sensitivity parameter of the demand functions. On the other hand, the initial wealth does not affect insurers’ premium strategies.
6. Numerical application

In this section, we present a hypothetical insurance market to illustrate our theoretical results. We assume that there are five insurers in the market competing with each other and we call them Insurer $i$, for $i = 1, 2, \ldots, 5$. Claims $X_i$ follow an exponential distribution with mean $m = 100$, and hence the moment generating function is given by

$$M_X(t) = \frac{1}{1 - mt}, \quad t < \frac{1}{m},$$

and the number of policies, $N_i$, follows a Poisson distribution with intensity $q_i(p_i, p_{-i})$. In the following two subsections, we investigate separately the Nash equilibrium of our two models, as well as we perform a sensitivity analysis of the underlying model parameters.

6.1. Model with market restriction

Firstly, we adopt the exposure function (32), with risk aversion parameter $h$. Based on Assumptions 1 and 6, the insurers’ risk aversion parameters $\lambda_i, i = 1, 2, \ldots, 5$, and $h$ should be less than $(1/m) = 0.01$, and it holds that $\lambda_i < h, i = 1, 2, \ldots, 5$. The market power is illustrated by the shape parameter $a_i$. In our scenario, Insurer 3 possesses a leading position in the market, Insurers 2 and 4 share the same portion of the market, whereas Insurer 5 is considered the “weakest” player in the market. The set of all the necessary model parameters is summarised in Table 3. Since the insurers use exponential utilities, it is well-known that the value of $w_i$ does not affect the risk attitude of the insurer, and thus it does not affect the insurance prices in equilibrium.

The upper bound for all insurers’ premiums is given by (13) and equals $p^{U} = 172.00$, whereas the insurers’ indifference premiums is given by (4) and depicted in Table 4. Note that $p^{I}_i > m = 100, i = 1, 2, \ldots, 5$, i.e., the indifference premium is larger than the net premium, producing positive loading for all insurers.

According to Theorem 20, a unique pure strategy Nash equilibrium $\tilde{p}^* = (\tilde{p}^*_1, \ldots, \tilde{p}^*_5)$ exists, and is obtained by solving simultaneously, for all $i = 1, 2, \ldots, 5$, Eq. (43). The equilibrium premiums are illustrated in Table 4. We verify that for each insurer $i = 1, 2, \ldots, 5$ the condition $\tilde{p}^*_i (\tilde{p}^{U}_j, \tilde{p}^{U}_k)$ is satisfied. Table 5 depicts the exposure volume at Nash equilibrium.

Since Insurer 3 possesses the largest proportion of the market, she is more vulnerable to uncertain risks, and hence her risk aversion parameter is the largest one. For this reason, the resulting Nash equilibrium premium for Insurer 3 is the highest one in the market, which leads to significant reduction in her exposure volume for the period under consideration. On the other hand, the Nash equilibrium premiums of Insurers 1 and 5 are lower than the others, in order to gain more share of the market in the upcoming period. Finally, Insurer 4’s Nash equilibrium premium is larger than Insurer 2’s, since she has higher risk aversion, resulting to larger reduction in her exposure volume.

Now, maintaining all other the parameters in Table 3, we increase only the risk aversion parameter $h$. The upper premium bound is depicted in Table 6 for each value of $h$. Moreover, this table also shows the pure strategy Nash equilibrium and the corresponding exposure volumes.

As the risk aversion parameter $h$ increases, insureds are willing to pay more for their insurance coverage, which yields the upper premium bound to increase as well. Therefore, all insurers take advantage of this opportunity and increase their premiums in order to produce larger profits. Moreover, we observe that insurers follow the same reasoning in their premium strategy as in the case of $h = 0.007$, i.e weaker insurers set lower premiums to gain market power, whereas larger insurers charge relatively high premiums. For the exposure volumes when $h = 0.008$ or $0.009$, it holds that only weaker insurers’ exposure decreases, in contrast to Insurers 3 and 4, whose exposure volume increases (see Table 6). On the other hand, the losses of exposure for Insurers 1, 2 and 5 are still lower than the losses for Insurers 3 and 4. This may be due to the fact that insureds are capable of paying more money for their insurance, so they trust insurers with greater market power, which is usually accompanied by better reputation, as mentioned in Pantelous and Passalidou (2013, 2015).

Next, maintaining all the parameters as in Table 3, we vary only the mean $m$ of the exponentially distributed claims $X$. The upper premium bound and the indifference premiums are shown in Table 7. The corresponding Nash equilibrium and exposure volumes appear in Table 8. When the size of the underlying risk increases, the lower and upper premium bounds increase, too. The corresponding Nash equilibrium premiums increase as well, but following the same pattern as in the case of $m = 100$ and leading to a similar structure for the exposure volume.

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### Table 3
**Model parameters used in Section 6.**

| Parameters | Insurer 1 | Insurer 2 | Insurer 3 | Insurer 4 | Insurer 5 |
|------------|-----------|-----------|-----------|-----------|-----------|
| $q_i^m$    | 1000      | 2000      | 3000      | 2000      | 500       |
| $w_i^m$    | 10000     | 10000     | 10000     | 10000     | 10000     |
| $\lambda_i$| 0.003     | 0.004     | 0.006     | 0.005     | 0.001     |
| $a_i$      | 1.6       | 1.7       | 1.8       | 1.7       | 1.5       |

$h = 1.2, \quad m = 100, \quad h = 0.007$

### Table 4
**Nash equilibrium premiums.**

| Premium | Insurer 1 | Insurer 2 | Insurer 3 | Insurer 4 | Insurer 5 |
|---------|-----------|-----------|-----------|-----------|-----------|
| Upper bound: $p^U$ | 172.00 | 172.00 | 172.00 | 172.00 | 172.00 |
| Nash equilibrium: $\tilde{p}^*_i$ | 158.29 | 159.70 | 164.75 | 161.44 | 156.63 |
| Indifference premium: $p^I_i$ | 118.89 | 127.71 | 152.72 | 138.63 | 105.36 |

### Table 5
**Exposure volume at Nash equilibrium.**

| Exposure volume | Insurer 1 | Insurer 2 | Insurer 3 | Insurer 4 | Insurer 5 |
|----------------|-----------|-----------|-----------|-----------|-----------|
| $q_i^m$ | 1025.75 | 1996.70 | 2281.58 | 1851.96 | 526.87 |
| $q_i^m$ | 1000 | 2000 | 3000 | 2000 | 500 |

### Table 6
**Nash equilibrium premiums and exposure volumes for three values of the risk aversion parameter $h$.**

| Parameter $h$ | $p^U$ | $\tilde{p}^*_1$ | $\tilde{p}^*_2$ | $\tilde{p}^*_3$ | $\tilde{p}^*_4$ | $\tilde{p}^*_5$ |
|---------------|-------|----------------|----------------|----------------|----------------|----------------|
| $h = 0.007$   | 172.00 | 158.29 | 159.70 | 164.75 | 161.44 | 156.63 |
| $h = 0.008$   | 201.18 | 177.95 | 179.63 | 184.51 | 181.36 | 176.02 |
| $h = 0.009$   | 255.84 | 214.07 | 215.91 | 220.71 | 217.45 | 212.38 |
| $h = 0.007$   | 1025.75 | 1996.70 | 2281.58 | 1851.96 | 526.87 |
| $h = 0.008$   | 998.99 | 1973.16 | 2654.98 | 1893.67 | 506.32 |
| $h = 0.009$   | 981.53 | 1966.01 | 2833.35 | 1929.22 | 488.18 |

| Parameter $h$ | $\tilde{p}^*_1$ | $\tilde{p}^*_2$ | $\tilde{p}^*_3$ | $\tilde{p}^*_4$ | $\tilde{p}^*_5$ |
|---------------|----------------|----------------|----------------|----------------|----------------|
| $h = 0.007$   | 1025.75 | 1996.70 | 2281.58 | 1851.96 | 526.87 |
| $h = 0.008$   | 998.99 | 1973.16 | 2654.98 | 1893.67 | 506.32 |
| $h = 0.009$   | 981.53 | 1966.01 | 2833.35 | 1929.22 | 488.18 |
market power when the underlying risk increases. Individuals feel more secure with insurers who possess large market share. Insurers 2, 3, and 4 increase. This may indicate that their premiums of Insurer 3 decrease for larger values of λ, where m decreases for larger values of m, whereas the exposure volumes of the larger Insurers 2, 3, and 4 increase. This may indicate that individuals feel more secure with insurers who possess large market power when the underlying risk increases.

Next, we maintain all the parameters as in Table 3 and decrease only the risk aversion parameter of Insurer 3. The indifference premium of Insurer 3 is given in Table 9, for λ3 = 0.005, 0.006. The corresponding Nash equilibrium premiums and exposure volumes are presented in Table 8. According to Lemma 3, smaller values of the risk aversion parameter yield smaller values for the indifference premium, as verified in Table 9. Moreover, we observe a general reduction in the Nash equilibrium premiums for all insurers, with the highest decrease in Insurer 3’s premium. Since the premium of the largest Insurer 3 is now more appealing than before, her exposure volume presents an increase, whereas all the other insurers’ exposure volumes decrease.

Finally, maintaining all the parameters as in Table 3, we decrease the market share parameter of Insurer 3, a3, and present the corresponding Nash equilibrium premiums and exposure volumes in Table 10. We observe that the optimal premium for Insurer 3 increases with respect to a3. Moreover, an increase in Insurer 3’s optimal premium yields an increase in the premiums of the other insurers in the Nash equilibrium. However, the relative premium strategies are not affected, since the weaker Insurers 1 and 5 still charge lower premiums than the stronger Insurers 2, 3 and 4.

### 6.2. Model without market restriction

Now, we adopt the exposure function (37). Based on Assumption 1, insurers’ risk aversion parameters λi, i = 1, 2, . . . , 5 are less than (1/m) = 0.01. It is assumed that less market power implies a greater price sensitivity parameter a0, as indicated in Lerner (1934), Wu and Pantelous (2017), and Boonen et al. (2018). In particular, let a1 = 2.7, a2 = 2.6, a3 = 2.5, a4 = 2.6, and a5 = 2.8. All other model parameters coincide with the parameters in Table 3. For all cases that we study in this section, the pure strategy Nash equilibrium exists and is unique.

The indifference premiums for each insurer are the same as in Section 6.1 (see Table 4). The pure strategy Nash equilibrium \( \hat{p} = (\hat{p}_1, . . . , \hat{p}_5) \) is obtained by solving simultaneously for all \( i = 1, 2, . . . , 5 \) Eq. (45) and illustrated in Table 11. We observe that for each insurer \( i = 1, 2, . . . , 5 \) the condition \( \hat{p}_i > p_i^* \) is satisfied. Table 12 depicts the corresponding Nash equilibrium at which the exposure volumes in Table 15.

### Table 7

Indifference premiums and upper bound \( p^\mu \) for three values of the mean m.

| m | \( p^\mu \) | \( p^\mu_1 \) | \( p^\mu_2 \) | \( p^\mu_3 \) | \( p^\mu_4 \) | \( p^\mu_5 \) |
|---|---|---|---|---|---|
| 90 | 142.04 | 104.90 | 111.57 | 129.42 | 119.57 | 94.31 |
| 100 | 172.00 | 118.89 | 127.71 | 152.72 | 138.63 | 105.36 |
| 120 | 261.80 | 148.76 | 163.48 | 212.16 | 183.26 | 127.83 |

### Table 8

Nash equilibrium premiums and exposure volumes for three values of the mean m.

| m | \( p^*_1 \) | \( p^*_2 \) | \( p^*_3 \) | \( p^*_4 \) | \( p^*_5 \) |
|---|---|---|---|---|---|
| 90 | 132.66 | 133.69 | 137.27 | 134.95 | 131.39 |
| 100 | 158.29 | 159.70 | 164.75 | 161.44 | 156.63 |
| 120 | 230.45 | 233.02 | 243.24 | 236.29 | 227.86 |

### Table 9

Nash equilibrium premiums and exposure volumes for two values of the risk aversion parameter \( \lambda_i \). Moreover, the second column presents the indifference premium of Insurer 3.

| Parameter | \( p^*_1 \) | \( p^*_2 \) | \( p^*_3 \) | \( p^*_4 \) | \( p^*_5 \) |
|---|---|---|---|---|---|
| \( \lambda_3 = 0.006 \) | 152.72 | 159.70 | 164.75 | 161.44 | 156.63 |
| \( \lambda_3 = 0.005 \) | 138.63 | 135.37 | 141.20 | 138.99 | 135.70 |

### Table 10

Nash equilibrium premiums and exposure volumes for two values of the risk aversion parameter \( \alpha_3 \).

| Market shape parameter | \( \alpha_3 = 1.8 \) | \( \alpha_3 = 1.6 \) |
|---|---|---|
| \( p^*_1 \) | 158.29 | 159.70 |
| \( p^*_2 \) | 164.75 | 161.44 |
| \( p^*_3 \) | 161.44 | 156.63 |
| \( p^*_4 \) | 156.63 | 155.97 |
| \( p^*_5 \) | 155.97 | 155.97 |

As in Section 6.1, Insurer 3 possesses the largest proportion of the market which makes her more vulnerable to uncertain risks, and hence her risk aversion parameter is the largest one. For this reason, the resulting Nash equilibrium premium for Insurer 3 is the highest one in the market, which leads to significant reduction in her exposure volume for the period under consideration. On the other hand, the Nash equilibrium premiums of Insurers 1 and 5 are lower than the others in order to gain more share of the market in the upcoming period. Finally, Insurer 4’s Nash equilibrium premium is larger than the Nash equilibrium premium of Insurer 2, since Insurer 4 has higher risk aversion, resulting to reduction in her exposure volume.

Next, while keeping all other the parameters the same as above, we now vary the mean m of the exponentially distributed claims X. The indifference premiums coincide with the ones in Table 7. The corresponding Nash equilibrium and exposure volumes are shown in Table 13. When the size of the underlying risk increases, the indifference premiums increase too. The corresponding Nash equilibrium increase as well, but following the same reasoning as in the initial case of \( m = 100 \) and leading to an equivalent structure for the exposure volume. Under this model, we observe that insurers 3 and 4, with greater market power, face a reduction in their exposure volume, whilst the rest of them face an increase in their exposure volume. For larger values of m, this may happen because policyholders lapse from insurers 3 and 4 because they are charging high premiums, and buy policies from the other insurers that charge relatively lower premiums.

While keeping all other the parameters the same as above, we decrease only the risk aversion parameter of Insurer 3 (the largest insurer). The indifference premium of Insurer 3 coincides with the ones in Table 9, for each value of \( \lambda_3 \). The corresponding Nash equilibrium and exposure volumes are presented in Table 14. According to Lemma 3, smaller values of the risk aversion parameter yield smaller values for the indifference premium. Moreover, we observe a general reduction in the Nash equilibrium premiums for all insurers, with the highest decrease in the premium of Insurer 3. Since the premium of Insurer 3 is now more appealing than before, the exposure volume of Insurer 3 increases, whereas the exposure volumes of all other insurers decrease.

Finally, while keeping all the other parameters the same as above, we decrease the price sensitivity parameter of Insurer 3, \( \alpha_3 \), and present the corresponding Nash equilibrium and exposure volumes in Table 15. Regarding Remark 17, the parameter \( \alpha_3 \) is related to the price-elasticity of demand which is an increasing...
function of \( a_3 \). As mentioned in Taylor (1986), any decrease in \( a_3 \), i.e., in the price elasticity, causes an increase in the optimal premiums, because the loss of the number of policies due to high premiums is relatively small. This phenomenon is clearly illustrated by Table 15.

### Table 11

Nash equilibrium premiums.

| Premium | Insurer 1 | Insurer 2 | Insurer 3 | Insurer 4 | Insurer 5 |
|---------|-----------|-----------|-----------|-----------|-----------|
| Nash equilibrium: \( p_i^* \) | 184.48 | 192.89 | 214.82 | 201.34 | 173.81 |
| Indifference premium: \( p_i^\dagger \) | 118.89 | 127.71 | 152.72 | 138.63 | 105.36 |

### Table 12

Exposure volume at the Nash equilibrium.

| Exposure volume | Insurer 1 | Insurer 2 | Insurer 3 | Insurer 4 | Insurer 5 |
|-----------------|-----------|-----------|-----------|-----------|-----------|
| \( q_i^* \)     | 1167.62   | 2019.58   | 2104.31   | 1749.9    | 707.23    |
| \( q_i^\dagger \) | 1000      | 2000      | 3000      | 2000      | 500       |

### Table 13

Nash equilibrium premiums and exposure volumes for three values of the mean \( m \).

| Expected claim | \( p_i^1 \) | \( p_i^2 \) | \( p_i^3 \) | \( p_i^4 \) | \( p_i^5 \) |
|----------------|-----------|-----------|-----------|-----------|-----------|
| \( m = 90 \)  | 162.94    | 169.51    | 185.19    | 175.56    | 154.36    |
| \( m = 100 \) | 184.48    | 192.89    | 214.82    | 201.34    | 173.81    |
| \( m = 120 \) | 231.01    | 244.42    | 287.41    | 260.34    | 215.25    |

### Table 14

Nash equilibrium premiums and exposure volumes for two values of the risk aversion parameter \( \lambda_i \).

| Parameter | \( p_i^1 \) | \( p_i^2 \) | \( p_i^3 \) | \( p_i^4 \) | \( p_i^5 \) |
|-----------|-----------|-----------|-----------|-----------|-----------|
| \( \lambda_i = 0.006 \) | 1167.62   | 2019.58   | 2104.31   | 1749.9    | 707.23    |
| \( \lambda_i = 0.005 \) | 1125.32   | 1941.45   | 2446.05   | 1674.45   | 685.04    |

### Table 15

Nash equilibrium premiums and exposure volumes for two values of the price sensitivity parameter \( \alpha_3 \).

| Price sensitivity | \( p_i^1 \) | \( p_i^2 \) | \( p_i^3 \) | \( p_i^4 \) | \( p_i^5 \) |
|-------------------|-----------|-----------|-----------|-----------|-----------|
| \( \alpha_3 = 2.5 \) | 184.48    | 192.89    | 214.82    | 201.34    | 173.81    |
| \( \alpha_3 = 2 \)  | 185.75    | 194.12    | 227.70    | 202.51    | 175.19    |
| \( \alpha_3 = 2.5 \) | 1167.62   | 2019.58   | 2104.31   | 1749.9    | 707.23    |
| \( \alpha_3 = 2 \)  | 1210.4    | 2098.78   | 2001.87   | 1826.67   | 720.55    |

### Appendix. Proofs

#### Proof of Lemma 2

From (1) and (2) we obtain the following inequality

\[
E \left[ u_i (w_i + N_i p_i - S_i(N_i)) \right] = E \left[ w_i (w_i + N_i p_i - S_i(N_i)) \right] + E \left[ -e^{-\lambda_i w_i} + \sum_{j=1}^{N_i} (p_j - X_j) \right] N_i \\
= -e^{-\lambda_i w_i} E \left[ \prod_{j=1}^{N_i} e^{-\lambda_j p_j} e^{X_j} \right] N_i \\
= -e^{-\lambda_i w_i} \left[ \prod_{j=1}^{N_i} e^{-\lambda_j p_j} e^{X_j} \right] = -e^{-\lambda_i w_i} E \left[ e^{X_i} \right] \\
\]

From this and (2), we get

\[
E \left[ e^{X_i} \right] \leq 1. \tag{48}
\]

This inequality reads as \( E[c^N_i] \leq 1 \) for some \( c \geq 0 \), and since \( \Pr[N_i > 0] > 0 \) this implies \( c \leq 1 \). Hence, (48) leads to \( e^{-\lambda_i p_i} M_X(\lambda_i) \leq 1 \). From this, we readily derive (3), which concludes the proof.

#### Proof of Proposition 4

Applying (5) to (1) and taking expectations, we obtain

\[
E \left[ u_i (w_i) \right] = E \left[ -e^{-\lambda_i w_i} \right] = -e^{-\lambda_i w_i} E \left[ e^{-\lambda_i (p_i N_i - S_i)} \right]. \tag{49}
\]

Considering that \( N_i \) follows a Poisson distribution with parameter \( \lambda_i (p_i, \cdot, \cdot) \) and applying the law of iterated expectations, the second factor on the right-hand side of (49) can be written as

\[
E \left[ e^{-\lambda_i (p_i N_i - S_i)} \right] = E \left[ e^{-\lambda_i (p_i N_i - S_i)} \mid N_i \right] = \sum_{m=0}^{\infty} e^{-\lambda_i (p_i m - S_i)} P_r[N_i = m].
\]

Since \( X_1, X_2, \ldots, X_m \) are independent and identically distributed as \( X \), and the moment generating function of \( X \) is finite, we
Derive
\[
E \left[ e^{-\lambda_i (p N_i - S_i)} \right] = \sum_{m=0}^{\infty} \left[ M(x; \lambda_i) e^{-\lambda_i m} \right]^m \Pr [N_i = m] 
\]
\[= P_{N_i} (M(x; \lambda_i) e^{-\lambda_i m}) . \tag{50} \]
The probability generating function of \(N_i\) is defined by \(P_{N_i}(t) = E[t^{N_i}]\), and it always converges at least for \(|t| \leq 1\). From (48), we have that the argument of the probability generating function in (50) is positive and less than or equal to 1, and hence we obtain
\[
E \left[ e^{-\lambda_i (p N_i - S_i)} \right] = \exp \left\{ \phi_i(p, p - I) (M(x; \lambda_i) e^{-\lambda_i m}) - 1 \right\} . \tag{51} \]
Substituting (51) to (49), we obtain Eq. (6).

**Proof of Lemma 7.** Given the premium profile \(p_{-i}\) charged by all insurers except insurer \(i\), the premium \(p_i\) charged by insurer \(i\) is the solution to the constrained minimisation problem
\[
\min_{p_i \in \mathbb{P}} \left\{ -\lambda_i w_{i}^- + \hat{q}_i(p_i, p_{-i}) (M(x; \lambda_i) e^{-\lambda_i p_i} - 1) \right\} , \tag{52} \]
subject to
\[
-\hat{q}_i(p_i, p_{-i}) \leq 0, \tag{53} \]
\[
M(x; \lambda_i) e^{-\lambda_i p_i} \leq 1 . \tag{54} \]
This is a minimisation problem with inequality constraints and we solve it using the Karush–Kuhn–Tucker (KKT) conditions, which are necessary conditions for optimality. The KKT stationarity condition is
\[
\frac{\partial \hat{q}_i(p_i, p_{-i})}{\partial p_i} (M(x; \lambda_i) e^{-\lambda_i p_i} - 1) - \lambda_i \hat{q}_i(p_i, p_{-i}) M(x; \lambda_i) e^{-\lambda_i p_i} - \mu \frac{\partial \hat{q}_i(p_i, p_{-i})}{\partial p_i} - \nu \lambda_i M(x; \lambda_i) e^{-\lambda_i p_i} = 0 , \tag{55} \]
and the complementary slackness conditions are
\[
-\mu \hat{q}_i(p_i, p_{-i}) = 0, \quad v (M(x; \lambda_i) e^{-\lambda_i p_i} - 1) = 0 \tag{56} \]
\[
\mu \geq 0, \quad \nu \geq 0 \tag{57} \]
Any premium \(p_i\) which satisfies the KKT conditions (55)–(57) is only a candidate optimal solution. There exist four cases we have to investigate:

| Case | \(-\hat{q}_i(p_i, p_{-i})\) | \(M(x; \lambda_i) e^{-\lambda_i p_i} - 1\) |
|------|----------------|-------------------|
| I    | 0              | 0                 |
| II   | 0              | 0                 |
| III  | 0              | 0                 |
| IV   | 0              | 0                 |

**Case I.** Equation \(\hat{q}_i(p_i, p_{-i}) = 0\), and requirements (15) and (16) yield \(p_i = p^U_i\). From equation \(M(x; \lambda_i) e^{-\lambda_i p_i} - 1 = 0\) and Lemma 2, we obtain \(p_i = p^*_i\). Now, we should show that there exist nonnegative multipliers \(\mu\) and \(\nu\) which satisfy the KKT stationarity condition (55). From (55), we have that
\[
-\mu \frac{\partial \hat{q}_i(p_i, p_{-i})}{\partial p_i} - \nu \lambda_i = 0 . \tag{58} \]
From (15), and the fact that \(\lambda_i\) is positive, we can find nonnegative multipliers \(\mu\) and \(\nu\) to satisfy Eq. (58). So, it must hold that \(p^*_i = p^U_i\) and \(p^*_i = p^L_i\) is a candidate optimal solution to minimisation problem (52)–(54). Thus, the premium strategy set of insurer \(i\) is the point set \(p_i = (p^*_i)\). This in contradiction to \(\lambda_i < h\). Insurer \(i\) is not going to generate profits by charging \(p^*_i = p^U_i\) or \(p^*_i = p^L_i\).

**Case II.** From (56)–(57), we obtain \(v = 0\). Since \(\hat{q}_i(p_i, p_{-i}) = 0\), we have \(p_i = p^U_i\). Eq. (55) yields
\[
\left( M(x; \lambda_i) e^{-\lambda_i p^L_i} - 1 - \mu \right) \frac{\partial \hat{q}_i(p_i, p_{-i})}{\partial p_i} \bigg|_{p_i = p^L_i} = 0 \tag{59} \]
There are two sub-cases:

(i) If \(\frac{\partial \hat{q}_i(p_i, p_{-i})}{\partial p_i} < 0\), we obtain \(\mu = M(x; \lambda_i) e^{-\lambda_i p^L_i} - 1 < 0\), which is in contradiction with (57).

(ii) If \(\frac{\partial \hat{q}_i(p_i, p_{-i})}{\partial p_i} = 0\), then (59) holds for any nonnegative \(\mu\), and a candidate optimal solution is given by \(p^*_i = p^U_i\), where \(p^*_i < p^L_i\).

Now, we observe that \(\hat{q}_i(p^U_i, p_{-i}) = 0\), and from (8) we obtain \(C(p^U_i, p_{-i}) = -\lambda_i w_i^-\). However, for any \(p_i \in (p^*_i, p^U_i)\), we have \(\hat{q}_i(p_i, p_{-i}) > 0\), and \(M(x; \lambda_i) e^{-\lambda_i p_i} - 1 < 0\). Therefore, (8), and for any \(p_i \in (p^*_i, p^U_i)\), we find
\[
C(p_i, p_{-i}) = -\lambda_i w_i^- + \hat{q}_i(p_i, p_{-i}) (M(x; \lambda_i) e^{-\lambda_i p_i} - 1) < -\lambda_i w_i^- = C(p^U_i, p_{-i}) . \]
Thus, \(p^*_i = p^U_i\) is not a local solution to minimisation problem (52)–(54).

**Case III.** From (56)–(57), we obtain \(\mu = 0\). Equation \(M(x; \lambda_i) e^{-\lambda_i p_i} - 1 = 0\) yields \(p_i = p^L_i\). Eq. (55) implies \(-\lambda_i \hat{q}_i(p^L_i, p_{-i}) - \nu \lambda_i = 0\). From this, \(\lambda_i > 0\) and (15)–(16), we get \(v = -\hat{q}_i(p^L_i, p_{-i}) < 0\), which is a contradiction to (57).

**Case IV.** From (56)–(57), we obtain \(\mu = 0\), and substituting them into Eq. (55), we obtain (18). Therefore, solutions \(p^*_i\) to (18) are only candidate optimal solutions, and belong to the interior of the premium domain \(p_i\), i.e. \(p^*_i \in (p^L_i, p^U_i)\).

In conclusion of the proof, only Case IV provides candidate optimal solutions, \(p^*_i\), which are given by Eq. (18), and belong to the open interval \((p^L_i, p^U_i)\).

**Proof of Lemma 8.** Given the premium profile \(p_{-i}\) charged by all insurers except insurer \(i\), the premium \(p_i\) charged by insurer \(i\) is the solution to the constrained minimisation problem
\[
\min_{p_i \in \mathbb{P}} \left\{ -\lambda_i w_{i}^- + \hat{q}_i(p_i, p_{-i}) (M(x; \lambda_i) e^{-\lambda_i p_i} - 1) \right\} , \tag{60} \]
subject to
\[
M(x; \lambda_i) e^{-\lambda_i p_i} - 1 \leq 0 \tag{61} \]
This is a minimisation problem with inequality constraint and we solve it using the KKT conditions. The KKT stationarity condition is
\[
\frac{\partial \hat{q}_i(p_i, p_{-i})}{\partial p_i} (M(x; \lambda_i) e^{-\lambda_i p_i} - 1) - \lambda_i \hat{q}_i(p_i, p_{-i}) M(x; \lambda_i) e^{-\lambda_i p_i} - \nu \lambda_i M(x; \lambda_i) e^{-\lambda_i p_i} = 0 , \tag{62} \]
and the complementary slackness conditions are
\[
v (M(x; \lambda_i) e^{-\lambda_i p_i} - 1) = 0, \quad v \geq 0 . \tag{63} \]
Any premium \(p_i\) which satisfies the KKT conditions (62)–(63) is only a candidate optimal solution.

**Case I.** If \(M(x; \lambda_i) e^{-\lambda_i p_i} - 1 = 0\), then it follows from (4) that \(p_i = p^L_i\). From the KKT stationarity condition (62) and the fact that \(\lambda_i > 0\), we obtain
\[
-\lambda_i \hat{q}_i(p^L_i, p_{-i}) - \nu \lambda_i = 0 . \tag{64} \]
which implies
\[
v = -\hat{q}_i(p^L_i, p_{-i}) < 0 , \tag{65} \]
where the inequality is due to (19). This is in contradiction with (63).
Case II. If $M_\lambda(\lambda) e^{-\lambda p_1} - 1 < 0$, then it follows from (63) that $\nu = 0$, and substituting it into the KKT stationarity condition (62), we obtain the candidate optimal solutions, $\hat{p}_i^\ast$, by (23) and $\hat{p}_i^\ast > p_i^\ast$.

**Proof of Theorem 10.** Given inequality (27), we obtain from (26) that for all $p_i$ in the corresponding premium domain

$$\frac{\partial^2 C(p_i, p_{-i})}{\partial p_i^2} \geq 0.$$ 

Therefore, $C(p_i, p_{-i})$ is a convex function in $p_i$. Now, (27) implies that $q_i(p_i, p_{-i})$ is concave in $p_i$, and hence $-q_i(p_i, p_{-i})$ is convex in $p_i$. Moreover, $M_\lambda(\lambda) e^{-\lambda p_1} - 1$ is convex in $p_i$, too. Thus, the constraints (53) and (54) of the minimisation problem (52), and (61) of the minimisation problem (60), are convex functions of $p_i$. Since the cost function $C(p_i, p_{-i})$, and the constraints are convex in $p_i$, the KKT conditions (55)–(57) and (62)–(63) are not only necessary, but also sufficient for global optimality. Therefore, any solution $p_i^\ast$ to Eq. (25) is a global minimum. ■

**Proof of Theorem 11.** The second partial derivative of $C(p_i, p_{-i})$ with respect to $p_i$, given by (26), can be rewritten as

$$\frac{\partial^2 C(p_i, p_{-i})}{\partial p_i^2} = \frac{\partial^2 q_i(p_i, p_{-i})}{\partial p_i^2} (M_\lambda(\lambda) e^{-\lambda p_1} - 1)$$

and we also derive from (25) that

$$\left[ \frac{\partial q_i(p_i, p_{-i})}{\partial p_i} (M_\lambda(\lambda) e^{-\lambda p_1} - \lambda_i q_i(p_i, p_{-i}) M_\lambda(\lambda) e^{-\lambda p_1}) \right]_{p_i = p_i^\ast}$$

Substituting (65) and (66) into (64), we obtain

$$\frac{\partial^2 C(p_i, p_{-i})}{\partial p_i^2} \bigg|_{p_i = p_i^\ast} = \frac{\partial^2 q_i(p_i, p_{-i})}{\partial p_i^2} \left. \left( \frac{\partial q_i(p_i, p_{-i})}{\partial p_i} M_\lambda(\lambda) e^{-\lambda p_1} \right)^{-1} \right) \lambda_i q_i(p, p_{-i}) M_\lambda(\lambda) e^{-\lambda p_1} + 1 \right) \bigg|_{p_i = p_i^\ast}.$$ 

In order for $p_i^\ast$ to be a local minimum for $C_i$ with respect to $p_i$, evaluated at $p_i^\ast$, it has to be positive, i.e.

$$\frac{\partial^2 C(p_i, p_{-i})}{\partial p_i^2} \bigg|_{p_i = p_i^\ast} > 0.$$ 

For interior solutions $p_i^\ast$, we have that $\hat{q}_i(p_i^\ast, p_{-i})$ and $\hat{q}_i(p_i^\ast, p_{-i})$ are positive, and thus it is allowed to invert $q_i(p_i^\ast, p_{-i})$. Now,

$$\frac{\partial q_i(p_i, p_{-i})}{\partial p_i} \bigg|_{p_i = p_i^\ast} < 0$$

which always holds. Therefore, Theorem 11 yields that the solution $p_i^\ast$ of (69) is a local minimum of $C(p_i, p_{-i})$. Moreover, it is unique global minimum, since (69) has unique solution $p_i^\ast$, given by (40). ■

**Proof of Proposition 14.** According to Lemma 7, the candidate optimal premiums, $p_i^\ast$, in the case of the exponential exposure function with market restriction, are given by (18) and satisfy $p_i^\ast > p_i^\ast$. Substituting the exposure function (32) and its partial derivative (33) into (18), we obtain Eq. (35). Since $q_i(p_i, p_{-i}) \geq 0$, $p_i \in P_i$, and inequalities (33) and (34) are strictly on $P_i$, we obtain from (26) that

$$\frac{\partial^2 C(p_i, p_{-i})}{\partial p_i^2} > 0$$

Therefore, $C(p_i, p_{-i})$ is a strictly convex function in $p_i$. Now, (34) implies that $-q_i(p_i, p_{-i})$ is strictly convex in $p_i$. Moreover, $M_\lambda(\lambda) e^{-\lambda p_1} - 1$ is convex in $p_i$, too. Thus, the constraints (53) and (54) of the minimisation problem (52), and (61) of the minimisation problem (60), are convex functions of $p_i$. Since the constraints are convex in $p_i$, and the objective function $C(p_i, p_{-i})$ is strictly convex in $p_i$, the optimal solution $p_i^\ast$ to Eq. (35) is a unique global minimum. ■

**Proof of Proposition 16.** According to Lemma 8, the candidate optimal premiums, $p_i^\ast$, in the case of the exponential exposure function without market restriction, are given by (23) and satisfy $p_i^\ast > p_i^\ast$. Substituting (37) and (38) into (23) we obtain

$$- \frac{q_i}{p_i^\ast} (M_\lambda(\lambda) e^{-\lambda p_1} - 1) = 0,$$

which has unique solution $p_i^\ast$ given in (40). Now, from (4) and (8) we observe that the value of the cost function at $p_i = p_i^\ast$ equals $C(p_i^\ast, p_{-i}) = -\lambda_i w_i$. Since $q_i(p_i, p_{-i})$ is positive and $M_\lambda(\lambda) e^{-\lambda p_1} - 1$ is negative for all $p_i > p_i^\ast$, it holds

$$C(p_i, p_{-i}) = -\lambda_i w_i + \hat{q}_i(p_i, p_{-i}) (M_\lambda(\lambda) e^{-\lambda p_1} - 1)$$

Therefore, the lower bound $p_i^\ast$ is not optimal solution to minimisation problems (60)–(61), and $p_i^\ast > p_i^\ast$. Now it remains to prove that the solution of (23), given by (40), is indeed a global minimum point for the cost function $C(p_i, p_{-i})$. From Theorem 11, substituting (38) and (39) into (28), we verify that (28) is satisfied. Indeed, since (38) satisfies condition (29), we obtain

$$\frac{\partial^2 q_i(p_i, p_{-i})}{\partial p_i^2} \bigg|_{p_i = p_i^\ast} < \frac{M_\lambda(\lambda) e^{-\lambda p_1} + 1}{M_\lambda(\lambda) e^{-\lambda p_1}} \left( \frac{\partial q_i(p_i, p_{-i})}{\partial p_i} \bigg|_{p_i = p_i^\ast} \right)^2 \times \left( \hat{q}_i(p_i^\ast, p_{-i}) \right)^{-1},$$

which always holds. Therefore, Theorem 11 yields that the solution $p_i^\ast$ of (69) is a local minimum of $C(p_i, p_{-i})$. Moreover, it is unique global minimum, since (69) has unique solution $p_i^\ast$, given by (40). ■

**Proof of Theorem 20.** We observe that the set of all strategy profiles, $P = P_1 \times P_2 \times \cdots \times P_i$, is convex, non-empty and compact set, since $P_i = [p_i, p_i^\ast]$, $i = 1, 2, \ldots, n$. Moreover, in the proof of
Proposition 14, we show that if we substitute (32)–(34) into (26), we obtain
\[
\frac{\partial^2 C_i(p_i, p_{-i})}{\partial p_i^2} = -b \left( \frac{a_i}{p_i^U - p_i - \bar{p}_i} \right)^2 \exp \left\{ \frac{-a_i (p_i^U - p_i)}{p_i^U - p_i - \bar{p}_i} \right\} q_i^i, \\
\times \left( M_X(\lambda_i) e^{-\lambda_i \bar{p}_i} - 1 \right), \\
+ 2 \lambda_i b \frac{a_i}{p_i^U - p_i - \bar{p}_i} \exp \left\{ \frac{-a_i (p_i^U - p_i)}{p_i^U - p_i - \bar{p}_i} \right\} q_i^i, \\
\times M_X(\lambda_i) e^{-\lambda_i \bar{p}_i}, \\
+ \lambda_i^2 \left[ 1 - \exp \left\{ \frac{-a_i (p_i^U - p_i)}{p_i^U - p_i - \bar{p}_i} \right\} \right] \lambda_i^{-1} M_X(\lambda_i) e^{-\lambda_i \bar{p}_i}.
\]
In order for \( C(p_i, p_{-i}) \) to be convex in \( p_i \), it needs to hold that
\[
p_i < - \frac{1}{\lambda_i} \log \left( \frac{a_i}{a_i + \lambda_i \bar{p}_i} \right)^2 \frac{1}{M_X(\lambda_i)}.
\]
From (45) it follows that the equilibrium premium \( \hat{p}_i \) always satisfies (74).  
\]

\section*{References}

Asmussen, S., Christensen, B.J., Thøgersen, J., 2019. Nash equilibrium premium strategies for push–pull competition in a frictional non-life insurance market. Insurance Math. Econom. 87, 92–100.
Boonen, T.J., Pantelous, A.A., Wu, R., 2018. Non-cooperative dynamic games for general insurance markets. Insurance Math. Econom. 78, 123–135.
Borch, K., 1962. Application of game theory to some problems in automobile insurance. Astin Bull. 2 (2), 208–221.
Borch, K.H., 1974. The Mathematical Theory of Insurance: An Annotated Selection of Papers on Insurance Published 1960–1972. Lexington Books.
Bühlmann, H., 1980. An economic premium principle. Astin Bull. 11 (1), 52–60.
Bühlmann, H., 1984. The general economic premium principle. Astin Bull. 14 (1), 13–21.
Dutang, C., Albrecher, H., Loisel, S., 2013. Competition among non-life insurers under solvency constraints: A game-theoretic approach. European J. Oper. Res. 231 (3), 702–711.
Emms, P., 2007. Dynamic pricing of general insurance in a competitive market. Astin Bull. 37 (1), 1–34.
Emms, P., 2012. Equilibrium pricing of general insurance policies. N. Am. Actuar. J. 16 (3), 323–349.
Emms, P., Haberman, S., 2005. Pricing general insurance using optimal control theory. Astin Bull. 35 (2), 427–453.
Emms, P., Haberman, S., Savouli, I., 2007. Optimal strategies for pricing general insurance. Insurance Math. Econom. 40 (1), 15–34.
Friesz, T.L., 2010. Dynamic Optimization and Differential Games, volume 135. Springer Science & Business Media.
Kaar, R., Goovaerts, M., Dhaene, J., Denuit, M., 2008. Modern Actuarial Risk Theory: Using R, volume 128. Springer Science & Business Media.
Lemaire, J., 1984. An application of game theory: Cost allocation. Astin Bull. 14 (1), 61–81.
Lemaire, J., 1991. Co-operative game theory and its insurance applications. Astin Bull. 21 (1), 17–40.
Lerner, A.P., 1934. The concept of monopoly and the measurement of monopoly power. Rev. Econ. Stud. 1 (3), 157–175.
Malug, D.A., 1994. Monopoly output and welfare: The role of curvature of the demand function. J. Econ. Educ. 25 (3), 235–250.
Pantelous, A.A., Passalidou, E., 2013. Optimal premium pricing policy in a competitive insurance market environment. Ann. Actuar. Sci 7 (2), 175–191.
Pantelous, A.A., Passalidou, E., 2015. Optimal premium pricing strategies for competitive general insurance markets. Appl. Math. Comput. 259, 858–874.
Rosen, J., 1965. Existence and uniqueness of equilibrium points for concave n-person games. Econometrica 33 (3), 520–534.
Taylor, G.C., 1986. Underwriting strategy in a competitive insurance environment. Insurance Math. Econom. 5 (1), 59–77.
Taylor, G.C., 1987. Expenses and underwriting strategy in competition. Insurance Math. Econom. 6 (4), 275–287.
Wu, R., Pantelous, A.A., 2017. Potential games with aggregation in non-cooperative general insurance markets. Astin Bull. 47 (1), 269–302.