Polyhedral realization of the crystal bases for extremal weight modules over quantized hyperbolic Kac-Moody algebras of rank 2

Ryuta Hiasa
Graduate School of Pure and Applied Sciences, University of Tsukuba,
1-1-1 Tennodai, Tsukuba, Ibaraki 305-8571, Japan
(e-mail: hiasa@math.tsukuba.ac.jp)

Abstract

Let $g$ be a hyperbolic Kac-Moody algebra of rank 2. We give a polyhedral realization of the crystal basis for the extremal weight module of extremal weight $\lambda$, where $\lambda$ is an integral weight whose Weyl group orbit has neither a dominant integral weight nor an antidominant integral weight.

1 Introduction.

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix, where $I$ is the index set. Let $g = g(A)$ be the Kac-Moody algebra associated to $A$ over $\mathbb{C}$, and $U_q(g)$ the quantized universal enveloping algebra over $\mathbb{C}(q)$ associated to $g$. We denote by $W$ the Weyl group of $g$. Let $P$ be an integral weight lattice of $g$, and $P^+$ (resp., $-P^+$) the set of dominant (resp., antidominant) integral weights in $P$. Let $\mu \in P$ be an arbitrary integral weight. The extremal weight module $V(\mu)$ of extremal weight $\mu$ is the integrable $U_q(g)$-module generated by a single element $v_\mu$ with the defining relation that $v_\mu$ is an extremal weight vector of weight $\mu$ in the sense of [7, Definition 8.1.1]. This module was introduced by Kashiwara [7] as a natural generalization of integrable highest (or lowest) weight modules; in fact, if $\mu \in P^+$ (resp., $\mu \in -P^+$), then the extremal weight module of extremal weight $\mu$ is isomorphic, as a $U_q(g)$-module, to the integrable highest (resp., lowest) weight module of highest (resp., lowest) weight $\mu$. Also, he proved that $V(\mu)$ has a crystal basis $B(\mu)$ for all $\mu \in P$. We know from [7, Proposition 8.2.2 (iv) and (v)] that $V(\mu) \cong V(w\mu)$ as $U_q(g)$-modules, and $B(\mu) \cong B(w\mu)$ as crystals for all $\mu \in P$ and $w \in W$. Hence we are interested in the case that $\mu$ is an integral weight such that

$$W\mu \cap (P^+ \cup -P^+) = \emptyset.$$ (1.1)
Assume that \( g \) is the hyperbolic Kac-Moody algebra associated to the generalized Cartan matrix

\[
A = \begin{pmatrix}
2 & -a_1 \\
-a_2 & 2
\end{pmatrix}, \quad \text{where} \ a_1, a_2 \in \mathbb{Z}_{\geq 1} \text{ with } a_1a_2 > 4.
\]

(1.2)

Sagaki and Yu [14] proved that if \( \mu = \Lambda_1 - \Lambda_2 \), where \( \Lambda_1, \Lambda_2 \) are the fundamental weights, then the crystal basis \( B(\mu) \) is isomorphic, as a crystal, to the crystal of Lakshmibai-Seshadri paths of shape \( \mu \) in the case that \( a_1, a_2 \geq 2 \); note that \( \mu = \Lambda_1 - \Lambda_2 \) does not satisfy condition (1.1) if \( a_1 = 1 \) or \( a_2 = 1 \) (see [14] Remark 3.1.2). After that, the author [12] classified the integral weights \( \mu \) satisfying condition (1.1), and then generalized the result due to Sagaki and Yu mentioned above to the case that \( \mu = k_1 \Lambda_1 - \Lambda_2 \) with \( 1 \leq k_1 < a_1 - 1 \) or \( \mu = \Lambda_1 - k_2 \Lambda_2 \) with \( 1 < k_2 \leq a_2 - 1 \). The aim of this paper is to provide an explicit polyhedral realization of \( B(\mu) \) for arbitrary integral weight \( \mu \) satisfying condition (1.1).

In this paper, we use the following realization of the crystal basis \( B(\mu) \) for the extremal weight module \( V(\mu) \); here, we explain it in general Kac-Moody setting. Let \( B(\infty) \) (resp., \( B(-\infty) \)) be the crystal basis of the negative (resp., positive) part of the semi-infinite \( \mathbb{Z} \)-lattice together with a crystal structure associated to \( \mu \). Nakashima and Zelevinsky [13] introduced an embedding \( \Psi^+_{\iota^+}: B(\infty) \hookrightarrow \mathbb{Z}^{\infty}_{\geq 0,\iota^+} \) of crystals, where \( \iota^+ \) is an infinite sequence of elements in the index set \( I \) satisfying certain condition, and \( \mathbb{Z}^{\infty}_{\geq 0,\iota^+} := \{(\ldots, x_k, \ldots, x_2, x_1) \mid x_k \in \mathbb{Z}_{\geq 0} \text{ and } x_k = 0 \text{ for } k \gg 0 \} \) is the semi-infinite \( \mathbb{Z} \)-lattice together with a crystal structure associated to \( \iota^+ \) (see 2.4 below). Assuming a certain positivity condition on \( \iota^+ \), they gave a combinatorial description of \( B(\infty) \) (which is called a polyhedral realization of \( B(\infty) \)) as a polyhedral convex cone in \( \mathbb{Z}^{\infty}_{\geq 0,\iota^+} \). Specifically, they found the set \( \Xi_{\iota^+} \) of linear functions on \( \mathbb{R}^{\infty}_{+} \) such that the image \( \text{Im}(\Psi^+_{\iota^+}) \cong B(\infty) \) is identical to the set

\[
\{ \hat{x} \in \mathbb{Z}^{\infty}_{\geq 0,\iota^+} \mid \phi(\hat{x}) \geq 0 \text{ for all } \phi \in \Xi_{\iota^+} \}.
\]

(1.3)

Similarly, there exists an embedding \( \Psi^-_{\iota^-}: B(-\infty) \hookrightarrow \mathbb{Z}^{\infty}_{\leq 0,\iota^-} \) of crystals, where \( \iota^- \) is an infinite sequence of elements in the index set \( I \) satisfying certain condition, and \( \mathbb{Z}^{\infty}_{\leq 0,\iota^-} := \{(x_0, x_{-1}, \ldots, x_k, \ldots) \mid x_k \in \mathbb{Z}_{\leq 0} \text{ and } x_k = 0 \text{ for } k \ll 0 \} \) is the semi-infinite \( \mathbb{Z} \)-lattice together with a crystal structure associated to \( \iota^- \). Hence there exists an embedding

(\[\Psi^\mu_{\iota^+}: B(\infty) \otimes T_\mu \otimes B(-\infty) \hookrightarrow \mathbb{Z}^{\infty}_{\geq 0,\iota^+} \otimes T_\mu \otimes \mathbb{Z}^{\infty}_{\leq 0,\iota^-} =: \Xi_\mu(\mu)\]

of crystals, where \( T_\mu \) is the crystal consisting of a single element of weight \( \mu \) (see 2.2 below), and \( \iota := (\iota^+, \iota^-) \). Now, in [7], Kashiwara showed that \( B(\mu) \) is isomorphic, as a crystal, to the subcrystal \( \{ b \in B(\infty) \otimes T_\mu \otimes B(-\infty) \mid b^* \text{ is extremal} \} \) of \( B(\infty) \otimes T_\mu \otimes B(-\infty) \). Therefore the crystal basis \( B(\mu) \) is isomorphic, as a crystal, to the subcrystal \( \{ \vec{x} \in \text{Im}(\Psi^\mu_{\iota^+}) \mid \vec{x}^* \text{ is extremal} \} \) of \( \text{Im}(\Psi^\mu_{\iota^+}) \cong B(\infty) \otimes T_\mu \otimes B(-\infty) \). In this paper, we give a polyhedral realization (such as (1.3)) of \( B(\mu) \hookrightarrow \text{Im}(\Psi^\mu_{\iota^+}) \) in the case that \( g \) is of rank 2, and \( \mu \) satisfies condition (1.1).

Here we turn to be our rank 2 case, where \( A \) is as (1.2) with \( I = \{1, 2\} \). Let \( \iota = (\iota^+, \iota^-) \) with \( \iota^+ = (\ldots, i_2, i_1) := (\ldots, 2, 1, 2, 1) \) and \( \iota^- = (i_0, i_{-1}, \ldots) := (2, 1, 2, 1, \ldots) \). We know
that $\mu$ satisfies condition (1.1) if and only if $W\mu$ contains $\lambda$ of the form in Theorem 3.1.

Theorem 1.1

For $k \in \mathbb{Z}$, we define the linear function $\zeta_k \in (\mathbb{R}^\infty)^*$ by $\zeta_k(\vec{x}) := x_k$ for $\vec{x} = (\ldots, x_2, x_1) \otimes t_\lambda \otimes (x_0, x_{-1}, \ldots) \in \mathbb{R}^\infty$, and set

$$\Xi_i[\lambda] = \{\gamma_0 p_0 + \gamma_0 \zeta_0 - \zeta_i, \gamma_1 p_1 + \zeta_0 - \gamma_1 \zeta_i\}$$

where the numbers $\gamma_k \in \mathbb{R} \setminus \mathbb{Q}, k \in \mathbb{Z}$, are defined by (3.1), and the sequence $\{p_m\}_{m \in \mathbb{Z}}$ are defined by (3.3) and (3.4). We set

$$\Sigma_i(\lambda) := \{\vec{x} \in \Xi_i(\lambda) \mid \varphi(\vec{x}) \geq 0 \text{ for all } \varphi \in \Xi_i[\lambda]\}.$$  

We prove the following theorems.

Theorem 1.1 (= Theorem 3.2). The set $\Sigma_i(\lambda)$ is a subcrystal of $\text{Im}(\Psi(\lambda))$.

Theorem 1.2 (= Corollary 3.5). The equality $\Sigma_i(\lambda) = \{\vec{x} \in \text{Im}(\Psi(\lambda)) \mid \vec{x}^* \text{ is extremal}\}$ holds. Therefore, $\Sigma_i(\lambda)$ is isomorphic, as a crystal, to the crystal basis $\mathcal{B}(\lambda)$ of the extremal weight module $V(\lambda)$ of extremal weight $\lambda$.

This paper is organized as follows. In §2, we fix our notation, and recall some basic facts about extremal weight modules and their crystal bases. In §3, we prove our main results, and in §4, we prove them. In Appendix A, we give some formulas of the operators $F_k$ (which is defined in §1.2) on $\Xi_i[\lambda]$. In Appendix B, we give a proof of Theorem 3.1 in the case that $a_1 = 1$ or $a_2 = 1$.

2 Review.

2.1 Kac-Moody algebras.

Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix, and $\mathfrak{g} = \mathfrak{g}(A)$ the Kac-Moody algebra associated to $A$ over $\mathbb{C}$. We denote by $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$, $\{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ the set of simple roots, and $\{\alpha_i^\vee\}_{i \in I} \subset \mathfrak{h}$ the set of simple coroots. Let $s_i$ be the simple reflection with respect to $\alpha_i$ for $i \in I$, and let $W = \langle s_i \mid i \in I \rangle$ be the Weyl group of $\mathfrak{g}$. For a positive real root $\beta$, we denote by $\beta^\vee$ the dual root of $\beta$, and by $s_\beta \in W$ the reflection with respect to $\beta$. Let $\{\Lambda_i\}_{i \in I} \subset \mathfrak{h}^*$ be the fundamental weights for $\mathfrak{g}$, i.e., $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{i,j}$ for $i, j \in I$, where $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ is the canonical pairing of $\mathfrak{h}^*$ and $\mathfrak{h}$. We take an integral weight lattice $P$ containing $\alpha_i$ and $\Lambda_i$ for all $i \in I$. We denote by $P^+$ (resp., $-P^+$) the set of dominant (resp., antidominant) integral weights in $P$.

Let $U_q(\mathfrak{g})$ be the quantized universal enveloping algebra over $\mathbb{C}(q)$ associated to $\mathfrak{g}$, and let $U_q^+(\mathfrak{g})$ (resp., $U_q^-(\mathfrak{g})$) be the positive (resp., negative) part of $U_q(\mathfrak{g})$, that is, $\mathbb{C}(q)$-subalgebra generated by the Chevalley generators $E_i$ (resp., $F_i$) of $U_q(\mathfrak{g})$ corresponding to the positive (resp., negative) simple roots $\alpha_i$ (resp., $-\alpha_i$) for $i \in I$. 

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2.2 Crystal bases and crystals.

For details on crystal bases and crystals, we refer the reader to [9] and [3]. Let \( B(\infty) \) (resp., \( B(-\infty) \)) be the crystal basis of \( U_q^{-}(\mathfrak{g}) \) (resp., \( U_q^{+}(\mathfrak{g}) \)), and let \( u_\infty \in B(\infty) \) (resp., \( u_{-\infty} \in B(-\infty) \)) be the element corresponding to \( 1 \in U_q^{-}(\mathfrak{g}) \) (resp., \( 1 \in U_q^{+}(\mathfrak{g}) \)). Denote by \( \ast : B(\pm \infty) \rightarrow B(\pm \infty) \) the \( \ast \)-operation on \( B(\pm \infty) \); see [8] Theorem 2.1.1] and [9] §8.3. For \( \mu \in P \), let \( T_\mu = \{ t_\mu \} \) be the crystal consisting of a single element \( t_\mu \) such that

\[
\text{wt}(t_\mu) = \mu, \quad \tilde{e}_i t_\mu = \tilde{f}_i t_\mu = 0, \quad \varepsilon_i(t_\mu) = \varphi_i(t_\mu) = -\infty \quad \text{for } i \in I,
\]

where 0 is an extra element not contained in any crystal.

Let \( B \) be a normal crystal in the sense of [7] §1.5. We know from [7] §7] (see also [9] Theorem 11.1] that \( B \) has the following action of the Weyl group \( W \). For \( i \in I \) and \( b \in B \), we set

\[
S_i b := \begin{cases} 
\tilde{f}_i^{(\text{wt}(b), \alpha_i^\vee)} b & \text{if } \langle \text{wt}(b), \alpha_i^\vee \rangle \geq 0, \\
\tilde{e}_i^{-(\text{wt}(b), \alpha_i^\vee)} b & \text{if } \langle \text{wt}(b), \alpha_i^\vee \rangle \leq 0.
\end{cases}
\]

Then, for \( w \in W \), we set \( S_w := S_{i_1} \cdots S_{i_k} \) if \( w = s_{i_1} \cdots s_{i_k} \). Notice that \( \text{wt}(S_w b) = w \text{wt}(b) \) for \( w \in W \) and \( b \in B \).

**Definition 2.1.** An element of a normal crystal \( B \) is said to be extremal if for each \( w \in W \) and \( i \in I \),

\[
\tilde{e}_i(S_w b) = 0 \text{ if } \langle \text{wt}(S_w b), \alpha_i^\vee \rangle \geq 0, \\
\tilde{f}_i(S_w b) = 0 \text{ if } \langle \text{wt}(S_w b), \alpha_i^\vee \rangle \leq 0.
\]

Let \( B \) be a normal crystal. For \( b \in B \) and \( i \in I \), we set

\[
\tilde{e}_i^\text{max} b := \tilde{e}_i^\varepsilon(b) b \quad \text{and} \quad \tilde{f}_i^\text{max} b := \tilde{f}_i^\varphi(b) b.
\]

2.3 Crystal bases of extremal weight modules.

Let \( \mu \in P \) be an arbitrary integral weight. The extremal weight module \( V(\mu) \) of extremal weight \( \mu \) is, by definition, the integrable \( U_q(\mathfrak{g}) \)-module generated by a single element \( v_\mu \) with the defining relation that \( v_\mu \) is an extremal weight vector of weight \( \mu \) in the sense of [7] Definition 8.1.1]. We know from [7] Proposition 8.2.2] that \( V(\mu) \) has a crystal basis \( B(\mu) \). Let \( u_\mu \) denote the element of \( B(\mu) \) corresponding to \( v_\mu \).

**Remark 2.2.** We see from [7] Proposition 8.2.2 (iv) and (v)] that \( V(\mu) \cong V(w \mu) \) as \( U_q(\mathfrak{g}) \)-modules, and \( B(\mu) \cong B(w \mu) \) as crystals for all \( \mu \in P \) and \( w \in W \). Also, we know from the comment at the end of [7] §8.2] that if \( \mu \in P^+ \) (resp., \( \mu \in -P^+ \)), then \( V(\mu) \) is isomorphic, as a \( U_q(\mathfrak{g}) \)-module, to the integrable highest (resp., lowest) weight module of highest (resp., lowest) weight \( \mu \), and \( B(\mu) \) is isomorphic, as a crystal, to its crystal basis. So, we focus on those \( \mu \in P \) satisfying the condition that

\[
W \mu \cap (P^+ \cup -P^+) = \emptyset. \tag{2.1}
\]
The crystal basis $\mathcal{B}(\mu)$ of $V(\mu)$ can be realized (as a crystal) as follows. We set

$$\mathcal{B} := \bigsqcup_{\mu \in P} \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty);$$

in fact, $\mathcal{B}$ is isomorphic, as a crystal, to the crystal basis $\mathcal{B}(\tilde{U}_q(\mathfrak{g}))$ of the modified quantized universal enveloping algebra $\tilde{U}_q(\mathfrak{g})$ associated to $\mathfrak{g}$ (see [7, Theorem 3.1.1]). Denote by $*: \mathcal{B} \to \mathcal{B}$ the $*$-operation on $\mathcal{B}$ (see [7, Theorem 4.3.2]); we know from [7, Corollary 4.3.3] that for $b_1 \in \mathcal{B}(\infty), b_2 \in \mathcal{B}(-\infty),$ and $\mu \in P,$

$$(b_1 \otimes t_{\mu} \otimes b_2)^* = b_1' \otimes t_{-\mu-wt(b_1)-wt(b_2)} \otimes b_2'^*.$$  \hfill (2.2)

**Remark 2.3.** The weight of $(b_1 \otimes t_{\mu} \otimes b_2)^*$ is equal to $-\mu$ for all $b_1 \in \mathcal{B}(\infty)$ and $b_2 \in \mathcal{B}(-\infty)$ since $wt(b_1') = wt(b_1)$ and $wt(b_2') = wt(b_2)$.

Because $\mathcal{B}$ is a normal crystal by [7; §2.1 and Theorem 3.1.1], $\mathcal{B}$ has the action of the Weyl group $W$ (see §2.2). We know the following proposition from [7; Proposition 8.2.2 (and Theorem 3.1.1)].

**Theorem 2.4.** For $\mu \in P$, the set \{ $b \in \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \mid b^*$ is extremal \} is a subcrystal of $\mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty),$ and is isomorphic, as a crystal, to the crystal basis $\mathcal{B}(\mu)$ of the extremal weight module $V(\mu)$ of extremal weight $\mu$. In particular, $u_\infty \otimes t_{\mu} \otimes u_{-\infty} \in \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty)$ is contained in the set above, and corresponds to $u_\mu \in \mathcal{B}(\mu)$ under this isomorphism.

## 2.4 Realizations of $\mathcal{B}(\pm \infty)$ and $\mathcal{B}(\mu)$.

Let us recall realizations of $\mathcal{B}(\infty)$ and $\mathcal{B}(-\infty)$ from [13]. We fix an infinite sequence $i^+ = (\ldots, i_k, \ldots, i_2, i_1)$ of elements of $I$ such that $i_k \neq i_{k+1}$ for $k \in \mathbb{Z}_{\geq 1},$ and $\#\{k \in \mathbb{Z}_{\geq 1} \mid i_k = i\} = \infty$ for each $i \in I.$ Similarly, we fix an infinite sequence $i^- = (i_0, i_{-1}, \ldots, i_k, \ldots)$ of elements of $I$ such that $i_k \neq i_{k+1}$ for $k \in \mathbb{Z}_{\leq 0},$ and $\#\{k \in \mathbb{Z}_{\leq 0} \mid i_k = i\} = \infty$ for each $i \in I.$ We set

$$\mathbb{Z}_{\geq 0}^+ := \{(\ldots, x_k, \ldots, x_2, x_1) \mid x_k \in \mathbb{Z}_{\geq 0} \text{ and } x_k = 0 \text{ for } k \gg 0\},$$

$$\mathbb{Z}_{\leq 0}^- := \{(x_0, x_{-1}, \ldots, x_k, \ldots) \mid x_k \in \mathbb{Z}_{\leq 0} \text{ and } x_k = 0 \text{ for } k \ll 0\}.$$ We endow $\mathbb{Z}_{\geq 0}^+$ and $\mathbb{Z}_{\leq 0}^-$ with crystal structures as follows. Let $\hat{x}^+ = (\ldots, x_k, \ldots, x_2, x_1) \in \mathbb{Z}_{\geq 0}^+$ and $\hat{x}^- = (x_0, x_{-1}, \ldots, x_k, \ldots) \in \mathbb{Z}_{\leq 0}^-.$ For $k \geq 1,$ we set

$$\sigma_k^+(\hat{x}^+) = x_k + \sum_{j > k} \langle \alpha_i, \alpha_i^\vee \rangle x_j,$$

and for $k \leq 0,$ we set

$$\sigma_k^-(\hat{x}^-) = -x_k - \sum_{j < k} \langle \alpha_i, \alpha_i^\vee \rangle x_j;$$

and for $k = 0$ we set

$$\sigma_0^+(\hat{x}^+) = x_k,$$

$$\sigma_0^-(\hat{x}^-) = -x_k.$$
since $x_j = 0$ for $|j| \gg 0$, we see that $\sigma_k^+ (\hat{x}^\pm) = \sigma_k^- (\hat{x}^\pm) = 0$ for $|k| \gg 0$. For $i \in I$, we set $\sigma^+_{(i)} (\hat{x}^\pm) := \max \{ \sigma_k^+ (\hat{x}^\pm) \mid k \geq 1, i_k = i \}$ and $\sigma^-_{(i)} (\hat{x}^-) := \max \{ \sigma_k^- (\hat{x}^-) \mid k \leq 0, i_k = i \}$, and define

$$
M^+_{(i)} (\hat{x}^\pm) := \{ k \mid k \geq 1, i_k = i, \sigma_k^+ (\hat{x}^\pm) = \sigma_{(i)}^+ (\hat{x}^\pm) \},
$$

$$
M^-_{(i)} (\hat{x}^-) := \{ k \mid k \leq 0, i_k = i, \sigma_k^- (\hat{x}^-) = \sigma_{(i)}^- (\hat{x}^-) \}.
$$

Note that $\sigma_{(i)}^+ (\hat{x}^\pm) \geq 0$, and that $M^+_{(i)} (\hat{x}^\pm) = \sigma_{(i)}^+ (\hat{x}^\pm)$ is a finite set if and only if $\sigma_{(i)}^+ (\hat{x}^\pm) > 0$. We define the maps $\tilde{e}_i, \tilde{f}_i : \mathbb{Z}^{\infty}_{\geq 0} \to \mathbb{Z}^{\infty}_{\geq 0} \sqcup \{ 0 \}$ and $\tilde{e}_i, \tilde{f}_i : \mathbb{Z}^{\infty}_{\leq 0} \to \mathbb{Z}^{\infty}_{\leq 0} \sqcup \{ 0 \}$ by

$$
\tilde{e}_i \hat{x}^+ := \begin{cases} 
(\ldots, x'_{k}, \ldots, x'_2, x'_1) \text{ with } x'_k := x_k - \delta_{k, \text{max} M^+_{(i)}} & \text{if } \sigma_{(i)}^+ (\hat{x}^+) > 0, \\
0 & \text{if } \sigma_{(i)}^+ (\hat{x}^+) = 0,
\end{cases}
$$

$$
\tilde{f}_i \hat{x}^+ := \begin{cases} 
(\ldots, x'_{k}, \ldots, x'_2, x'_1) \text{ with } x'_k := x_k + \delta_{k, \text{min} M^+_{(i)}} & \text{if } \sigma_{(i)}^+ (\hat{x}^+) > 0, \\
0 & \text{if } \sigma_{(i)}^+ (\hat{x}^+) = 0,
\end{cases}
$$

$$
\tilde{e}_i \hat{x}^- := (x'_0, x'_{-1}, \ldots, x'_k, \ldots) \text{ with } x'_k := x_k - \delta_{k, \text{max} M^+_{(i)}}; 
$$

$$
\tilde{f}_i \hat{x}^- := (x'_0, x'_{-1}, \ldots, x'_k, \ldots) \text{ with } x'_k := x_k + \delta_{k, \text{min} M^+_{(i)}} 
$$

respectively. Moreover, we define

$$
\text{wt}(\hat{x}^+) := - \sum_{j \geq 1} x_j \alpha_i, \quad \varepsilon_i (\hat{x}^+) := \sigma_{(i)}^+ (\hat{x}^+) = \varepsilon_i (\hat{x}^+) + \langle \text{wt}(\hat{x}^+), \alpha_i^\vee \rangle,
$$

$$
\text{wt}(\hat{x}^-) := - \sum_{j \leq 0} x_j \alpha_i, \quad \varphi_i (\hat{x}^-) := \sigma_{(i)}^- (\hat{x}^-) = \varepsilon_i (\hat{x}^-) - \langle \text{wt}(\hat{x}^-), \alpha_i^\vee \rangle.
$$

These maps make $\mathbb{Z}^{\infty}_{\geq 0}$ (resp., $\mathbb{Z}^{\infty}_{\leq 0}$) into a crystal for $g$; we denote this crystal by $\mathbb{Z}^{\infty}_{\geq 0, \pm}$ (resp., $\mathbb{Z}^{\infty}_{\leq 0, \pm}$).

**Theorem 2.5** ([13, Theorem 2.5]). There exists an embedding $\Psi_{i^+}^+: \mathcal{B}(\infty) \hookrightarrow \mathbb{Z}^{\infty}_{\geq 0, \pm}$ of crystals which sends $u_{\infty} \in \mathcal{B}(\infty)$ to $z_{\infty} := (\ldots, 0, \ldots, 0, 0) \in \mathbb{Z}^{\infty}_{\geq 0, \pm}$. Similarly, there exists an embedding $\Psi_{i^-}^-: \mathcal{B}(\infty) \hookrightarrow \mathbb{Z}^{\infty}_{\leq 0, \pm}$ of crystals which sends $u_{-\infty} \in \mathcal{B}(\infty)$ to $z_{-\infty} := (0, 0, \ldots, 0, \ldots) \in \mathbb{Z}^{\infty}_{\leq 0, \pm}$.

We define the $*$-operations on $\text{Im}(\Psi_{i^\pm}^\pm)$ by the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{B}(\pm \infty) & \longrightarrow^* & \mathcal{B}(\pm \infty) \\
\Psi_{i^\pm}^\pm \downarrow & & \downarrow \Psi_{i^\pm}^\pm \\
\text{Im}(\Psi_{i^\pm}^\pm) & \longrightarrow^* & \text{Im}(\Psi_{i^\pm}^\pm).
\end{array}
$$

We know the following proposition from [13, Remark in §2.4].

**Proposition 2.6.** Keep the notation and setting above; recall that $i^+ = (\ldots, i_2, i_1)$ and $i^- = (i_0, i_{-1}, \ldots)$. 

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(1) Let \( \hat{x} = (\ldots, x_2, x_1) \in \mathbb{Z}_{\geq 0, t}^{+} \). Then, \( \hat{x} \in \text{Im}(\Psi_{i_+}^+) \) if and only if
\[
0 = \varepsilon_{ik}(f_{ik+1}^{xk+1}f_{ik+2}^{xk+2} \cdots z_{\infty})
\]
for all \( k \geq 1 \). Furthermore, if \( \hat{x} \in \text{Im}(\Psi_{i_+}^+) \), then \( \hat{x}^* = f_{i_1}^{x_1}f_{i_2}^{x_2} \cdots z_{\infty} \), and \( x_k = \varepsilon_{ik}(\tilde{e}_{i_{k-1}}^{x_{k-1}} \cdots \tilde{e}_{i_k}^{x_k} \hat{x}^*) \) for \( k \geq 1 \).

(2) Let \( \hat{x} = (x_0, x_1, \ldots) \in \mathbb{Z}_{\leq 0, t}^{-} \). Then, \( \hat{x} \in \text{Im}(\Psi_{i_-}^-) \) if and only if
\[
0 = \varphi_{ik}(\varepsilon_{ik-1}^{-x_0} \varepsilon_{ik-2}^{x_1} \cdots z_{-\infty})
\]
for all \( k \leq 0 \). Furthermore, if \( \hat{x} \in \text{Im}(\Psi_{i_-}^-) \), then \( \hat{x}^* = \varepsilon_{i_0}^{-x_0} \varepsilon_{i_{-1}}^{-x_{-1}} \cdots z_{-\infty} \), and \( -x_k = \varphi_{ik}(\tilde{f}_{ik+1}^{-x_{k+1}} \cdots \tilde{f}_{i_0}^{-x_0} \hat{x}^*) \) for \( k \leq 0 \).

We set \( i := (i^+, i^-) \), and \( \mathbb{Z}_i(\mu) := \mathbb{Z}_{\geq 0, t}^{+} \otimes \mathcal{T}_\mu \otimes \mathbb{Z}_{\leq 0, t}^{-} \) for \( \mu \in P \). By the tensor product rule of crystals, we can describe the crystal structure of \( \mathbb{Z}_i(\mu) \) as follows. Let \( x = \hat{x}^+ \otimes t_\mu \otimes \hat{x}^- \in \mathbb{Z}_i(\mu) \) with \( \hat{x}^+ = (\ldots, x_2, x_1) \in \mathbb{Z}_{\geq 0, t}^{+} \), and \( \hat{x}^- = (x_0, x_1, \ldots) \in \mathbb{Z}_{\leq 0, t}^{-} \).

For \( k \in \mathbb{Z} \), we set
\[
\sigma_k(x) := \begin{cases} 
\sigma_k^+(\hat{x}^+) & \text{if } k \geq 1, \\
\sigma_k^-(\hat{x}^-) - \langle \text{wt}(\hat{x}), \alpha_{i_k}^\vee \rangle & \text{if } k \leq 0.
\end{cases}
\]
For \( i \in I \), we set \( \sigma_{i}(x) := \max\{\sigma_k(x) \mid k \in \mathbb{Z}, i_k = i \} \), and
\[
M_{(i)} = M_{(i)}(x) := \{k \mid i_k = i, \sigma_k(x) = \sigma_{i}(x) \}.
\]
(2.3)

Then we see that
\[
\text{wt}(x) = \mu - \sum_{j \in \mathbb{Z}} x_j \alpha_{i_j}; \quad \varepsilon_i(x) = \sigma_i(x); \quad \varphi_i(x) = \varepsilon_i(x) + \langle \text{wt}(x), \alpha_i^\vee \rangle;
\]
if \( \varepsilon_i(x) > 0 \), then
\[
\tilde{e}_i x = (\ldots, x_2', x_1') \otimes t_\mu \otimes (x_0', x_{-1}', \ldots) \quad \text{with } x'_k := x_k - \delta_{k, \text{max}M_{(i)}};
\]
if \( \varphi_i(x) > 0 \), then
\[
\tilde{f}_i x = (\ldots, x_2', x_1') \otimes t_\mu \otimes (x_0', x_{-1}', \ldots) \quad \text{with } x'_k := x_k + \delta_{k, \text{min}M_{(i)}};
\]
(2.4)
if \( \varepsilon_i(x) = 0 \), then \( \tilde{e}_i x = 0 \); if \( \varphi_i(x) = 0 \), then \( \tilde{f}_i x = 0 \). The next corollary follows immediately from Theorem 2.5.

**Corollary 2.7.** For each \( \mu \in P \), there exists an embedding \( \Psi_i^\mu := \Psi_{i_+}^+ \otimes \text{id} \otimes \Psi_{i_-}^- : \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \rightarrow \mathbb{Z}_i(\mu) \) of crystals which sends \( u_\infty \otimes t_\mu \otimes u_{-\infty} \in \mathcal{B}(\infty) \otimes \mathcal{T}_\mu \otimes \mathcal{B}(-\infty) \) to \( z_\mu := z_\infty \otimes t_\mu \otimes z_{-\infty} \in \mathbb{Z}_i(\mu) \).
We also define the $\ast$-operation on $\text{Im}(\Psi^\mu_\iota) = \text{Im}(\Psi^+_\iota) \otimes T^\mu \otimes \text{Im}(\Psi^-_\iota)$ by the following commutative diagram:

$$
\begin{array}{ccc}
B(\infty) \otimes T^\mu \otimes B(-\infty) & \xrightarrow{\ast} & B(\infty) \otimes T^\mu \otimes B(-\infty) \\
\Phi^\mu_\iota \downarrow & & \downarrow \Phi^\mu_\iota \\
\text{Im}(\Psi^+_\iota) \otimes T^\mu \otimes \text{Im}(\Psi^-_\iota) & \xrightarrow{\ast} & \text{Im}(\Psi^+_\iota) \otimes T^\mu \otimes \text{Im}(\Psi^-_\iota).
\end{array}
$$

We see by (2.2) that if $z_1 \in \text{Im}(\Psi^+_\iota)$ and $z_2 \in \text{Im}(\Psi^-_\iota)$, then

$$(z_1 \otimes t^\mu \otimes z_2)^\ast = z_1^\ast \otimes t^-_{-\mu-\text{wt}(z_1)-\text{wt}(z_2)} \otimes z_2^\ast.$$  (2.5)

The next corollary is a consequence of Theorem 2.4 and Corollary 2.7.

**Corollary 2.8.** For $\mu \in P$, the set $\{\vec{x} \in \text{Im}(\Psi^\mu_\iota) \mid \vec{x}^\ast \text{ is extremal}\}$ is a subcrystal of $\text{Im}(\Psi^\mu_\iota)$, and is isomorphic, as a crystal, to the crystal basis $\mathcal{B}(\mu)$ of the extremal weight module $V(\mu)$ of extremal weight $\mu$.

### 3 Main results.

In the following, we assume that the generalized Cartan matrix $A$ is

$$A = \begin{pmatrix} 2 & -a_1 \\ -a_2 & 2 \end{pmatrix},$$

where $a_1, a_2 \in \mathbb{Z}_{\geq 1}$ with $a_1 a_2 > 4$,

and $\iota = (\iota^+, \iota^-)$ with $\iota^+ = (\ldots, i_2, i_1) := (\ldots, 2, 1, 2, 1)$ and $\iota^- = (i_0, i_{-1}, \ldots) := (2, 1, 2, 1, \ldots)$. We set

$$\alpha := \frac{a_1 a_2 + \sqrt{a_1^2 a_2^2 - 4a_1 a_2}}{2a_2}, \quad \beta := \frac{a_1 a_2 + \sqrt{a_1^2 a_2^2 - 4a_1 a_2}}{2a_1},$$

and

$$\gamma_k := \begin{cases} 
\alpha & \text{if } k \text{ is even}, \\
\beta & \text{if } k \text{ is odd}
\end{cases}$$  (3.1)

for $k \in \mathbb{Z}$; note that $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ and $\alpha, \beta > 0$. By the definition, we have

$$\frac{1}{\gamma_k} + \gamma_{k+1} = a_{ik}.$$  (3.2)

Let $\Lambda_1, \Lambda_2$ denote the fundamental weights for $\mathfrak{g} = \mathfrak{g}(A)$; note that $P = \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\Lambda_2$.

**Theorem 3.1.** Let $\mathcal{O} := \{W_\mu \mid \mu \in P\}$ be the set of $W$-orbits in $P$.

(1) Assume that $a_1, a_2 \geq 2$. Then, $O \in \mathcal{O}$ satisfies condition (2.1), that is, $O \cap (P^+ \cup -P^+) = \emptyset$ if and only if $O$ contains an integral weight $\lambda$ of the form either [i] or [ii]:

[Insert diagram or additional text here if necessary]
above. We define the sequence \( m \) for \( a_2 \) in exactly the same way as \([1, \text{Theorem 3.1}]\); see Appendix B.

**Proof.** The assertion of part (1) is nothing but \([1, \text{Theorem 3.1}]\). We can show parts (2) and (3) in exactly the same way as \([1, \text{Theorem 3.1}]\); see Appendix B. \( \square \)

Let \( \lambda = k_1 \Lambda_1 - k_2 \Lambda_2 \in P \) be an integral weight of the form mentioned in Theorem 3.1 above. We define the sequence \( \{p_m\}_{m \in \mathbb{Z}} \) of integers by the following recursive formulas: for \( m \geq 0 \),

\[
p_0 := k_2, \quad p_1 := k_1, \quad p_{m+2} := \begin{cases} a_2 p_{m+1} - p_m & \text{if } m \text{ is even}, \\ a_1 p_{m+1} - p_m & \text{if } m \text{ is odd}; \end{cases}
\]

(3.3)

for \( m < 0 \),

\[
p_m = \begin{cases} a_2 p_{m+1} - p_{m+2} & \text{if } m \text{ is even}, \\ a_1 p_{m+1} - p_{m+2} & \text{if } m \text{ is odd}; \end{cases}
\]

(3.4)

it follows from \([1, \text{Remark 3.7}]\) (and the comment in \([15, \text{§3.1}]\)) that \( p_m > 0 \) for all \( m \in \mathbb{Z} \). We regard \( \mathbb{R}^\infty := \{ \bar{x} = (\ldots, x_2, x_1) \otimes t_\lambda \otimes (x_0, x_{-1}, \ldots) \mid x_k \in \mathbb{R} \text{ and } x_k = 0 \text{ for } |k| \gg 0 \} \) as an infinite dimensional vector space over \( \mathbb{R} \); note that \( \mathbb{Z}_i(\lambda) \subset \mathbb{R}^\infty \). Let \( (\mathbb{R}^\infty)^* := \text{Hom}_\mathbb{R}(\mathbb{R}^\infty, \mathbb{R}) \) be its dual space. For \( k \in \mathbb{Z} \), we define the linear function \( \zeta_k \in (\mathbb{R}^\infty)^* \) by \( \zeta_k(\bar{x}) := x_k \) for \( \bar{x} = (\ldots, x_2, x_1) \otimes t_\lambda \otimes (x_0, x_{-1}, \ldots) \in \mathbb{R}^\infty \). Set

\[
\Sigma_i(\lambda) := \{ \bar{x} \in \mathbb{Z}_i(\lambda) \mid \varphi(\bar{x}) \geq 0 \text{ for all } \varphi \in \Xi_i[\lambda] \},
\]

where

\[
\Xi_i[\lambda] = \{ \gamma_0 p_0 + \gamma_0 \zeta_0 - \zeta_1, \gamma_1 p_1 + \zeta_0 - \gamma_1 \zeta_1 \} \\
\cup \{ p_k - \zeta_k, \gamma_k \zeta_k - \zeta_{k+1}, \gamma_k p_{k+1} - p_k + \zeta_k - \gamma_{k+1} \zeta_{k+1} \mid k \geq 1 \} \\
\cup \{ p_k + \zeta_k, \zeta_{k-1} - \gamma_k \zeta_k, \gamma_{k-1} p_{k-1} - p_k + \gamma_{k-1} \zeta_{k-1} - \zeta_k \mid k \leq 0 \}.
\]

**Theorem 3.2** (will be proved in §4.2). The set \( \Sigma_i(\lambda) \) is a subcrystal of \( \text{Im}(\Psi_i^+) \).

Let \( \Sigma_i(\lambda)' \) be the subset of \( \Sigma_i(\lambda) \) consisting of the elements of the form \( \hat{x} \otimes t_\lambda \otimes z_{-\infty} \) with \( \hat{x} \in \text{Im}(\Psi_i^+) \).

**Theorem 3.3** (will be proved in §4.3). For \( \bar{x} \in \Sigma_i(\lambda)' \), the element \( \bar{x}^* \) is extremal.
Theorem 3.4 (will be proved in §4.4). Let \( \bar{y} \in \text{Im}(\Psi^\lambda_\iota) \). If \( \bar{y}^* \) is extremal, then there exist \( i_1, \ldots, i_t \in I \) and \( \bar{x} \in \Sigma(\lambda)' \) such that \( \bar{x} = \hat{f}_{i_t} \cdots \hat{f}_{i_1} \bar{y} \).

Corollary 3.5. The equality \( \Sigma(\lambda) = \{ \bar{x} \in \text{Im}(\Psi^\lambda_\iota) \mid \bar{y}^* \text{ is extremal} \} \) holds. Therefore, \( \Sigma(\lambda) \) is isomorphic, as a crystal, to the crystal basis \( \mathcal{B}(\lambda) \) of the extremal weight module \( V(\lambda) \) of extremal weight \( \lambda \).

Proof. Set \( B := \{ \bar{x} \in \text{Im}(\Psi^\lambda_\iota) \mid \bar{y}^* \text{ is extremal} \} \). First, we show that \( \Sigma(\lambda) \subset B \). Let \( \bar{x} = \hat{x}_1 \otimes t_\lambda \otimes \hat{x}_2 \in \Sigma(\lambda) \) with \( \hat{x}_1 \in \mathbb{Z}^{+}_{0, t_+} \) and \( \hat{x}_2 \in \mathbb{Z}^{-}_{0, t_-} \). By Theorem 3.2, we have \( \hat{x}_2 \in \text{Im}(\Psi^-_\iota) \). Since \( \text{Im}(\Psi^-_\iota) \cong \mathcal{B}(-\infty) \) as crystals, there exist \( i_1, \ldots, i_t \) such that \( \hat{f}_{i_t} \cdots \hat{f}_{i_1} \hat{x}_2 = z_{-\infty} \). Then we see by the tensor product rule of crystals that \( \tilde{y} := \hat{f}_{i_t} \cdots \hat{f}_{i_1} \bar{x} \in \Sigma(\lambda)' \). Since \( \bar{x} \in \Sigma(\lambda) \subset \text{Im}(\Psi^\lambda_\iota) \), we see that \( \tilde{y} \in \text{Im}(\Psi^\lambda_\iota) \). Also, it follows from Theorem 3.3 that \( \tilde{y}^* \) is extremal. Thus we obtain \( \tilde{y} \in B \). Since \( B \) is a subcrystal by Corollary 2.8, we obtain \( \tilde{y} \in B \).

Next, we show that \( \Sigma(\lambda) \supset B \). Let \( \tilde{y} \in \text{Im}(\Psi^\lambda_\iota) \) be such that \( \tilde{y}^* \) is extremal. We see from Theorem 3.4 that there exist \( i_1, \ldots, i_t \in I \) such that \( \hat{f}_{i_t} \cdots \hat{f}_{i_1} \bar{y} \in \Sigma(\lambda)' \subset \Sigma(\lambda) \). Therefore, by Theorem 3.2, we obtain \( \tilde{y} \in \Sigma(\lambda) \).

Thus we have proved the corollary. \( \square \)

4 Proofs.

Throughout this section, we take and fix \( \iota = (\iota^+, \iota^-) \) and \( \lambda = k_1 \Lambda_1 - k_2 \Lambda_2 \in P \) as in §3.

4.1 Polyhedral realization of \( \mathcal{B}(\pm \infty) \) in the rank 2 case.

We define the sequences \( \{c_j\}_{j \geq 0} \) and \( \{c'_j\}_{j \geq 0} \) of integers by the following recursive formulas: for \( j \geq 0 \),

\[
\begin{align*}
c_0 &:= 0, \quad c_1 := 1, \quad c_{j+2} := \begin{cases} a_1 c_{j+1} - c_j & \text{if } j \text{ is even}, \\ a_2 c_{j+1} - c_j & \text{if } j \text{ is odd}; \end{cases} \\
c'_0 &:= 0, \quad c'_1 := 1, \quad c'_{j+2} := \begin{cases} a_2 c'_{j+1} - c'_j & \text{if } j \text{ is even}, \\ a_1 c'_{j+1} - c'_j & \text{if } j \text{ is odd}; \end{cases}
\end{align*}
\]

it is easy to show that \( c_j > 0 \) and \( c'_j > 0 \) for all \( j \geq 1 \) (see also the comment before [13, Theorem 4.1]). By [4, Corollary 4.7] and the fact that \( 1/\beta = (a_1 a_2 - \sqrt{a_1^2 a_2^2 - 4a_1 a_2})/2a_2 \), we obtain the following lemma.

Lemma 4.1. The following sequences are strictly decreasing, and converge to \( \alpha \) and \( \beta \), respectively:

\[
\frac{c_2}{c_1} > \frac{c'_2}{c'_1} > \frac{c_4}{c_3} > \frac{c'_4}{c'_3} > \cdots \to \alpha, \quad \frac{c'_2}{c'_1} > \frac{c_3}{c_2} > \frac{c'_4}{c'_3} > \frac{c_5}{c_4} > \cdots \to \beta.
\]

Applying [13, Theorem 4.1] to our rank 2 case, we obtain the following explicit descriptions of the images of the maps \( \Psi^+_\iota : \mathcal{B}(\infty) \to \mathbb{Z}^{+\infty}_{\geq 0, t^+} \) and \( \Psi^-_\iota : \mathcal{B}(-\infty) \to \mathbb{Z}^{-\infty}_{\leq 0, t^-} \).
Proposition 4.2. It hold that

\[ \text{Im}(\Psi_+^l) = \{(\ldots, x_2, x_1) \in \mathbb{Z}_{\geq 0}^+ \mid c_jx_j - c_{j-1}x_{j+1} \geq 0 \text{ for } j \geq 1\}, \]

\[ \text{Im}(\Psi_-^l) = \{(x_0, x_{-1}, \ldots) \in \mathbb{Z}_{\leq 0}^- \mid c'_{j-1}x_j - c'_{j-2}x_{j-1} \leq 0 \text{ for } j \leq 0\}. \]

Recall that the sequence \( \{p_m\}_{m \in \mathbb{Z}} \) is defined by recursive formulas (3.3) and (3.4). Let \( \hat{x} = (\ldots, x_2, x_1) \in \text{Im}(\Psi_+^l) \) be such that \( x_m \leq p_m \) for all \( m \in \mathbb{Z}_{\geq 1} \). For \( l \geq 1 \), we set

\[ z_1(\hat{x}, l) := (\ldots, x_2, x_1, p_0, p_{-1}, \ldots, p_{-2l+2}, p_{-2l+1}) \in \mathbb{Z}_{\geq 0, l}^{+\infty}, \]

\[ z_2(\hat{x}, l) := (x_{2l} - p_{2l}, \ldots, x_2 - p_2, x_1 - p_1, 0, 0, \ldots) \in \mathbb{Z}_{\leq 0, l}^{-\infty}. \]

Proposition 4.3. Let \( \hat{x} = (\ldots, x_2, x_1) \in \text{Im}(\Psi_+^l) \) be such that \( x_m \leq p_m \) for all \( m \in \mathbb{Z}_{\geq 1} \), and let \( l \geq 1 \). The following are equivalent:

1. \( z_1(\hat{x}, l) \in \text{Im}(\Psi_+^l) \);
2. \( c_jx_{2l} - c_{j-1}x_{2l+1} \geq 0 \text{ for } j \geq 2l+1 \);
3. \( 0 = \varepsilon_{ij}(f_{j+1}^p \cdot f_{j+1}^p f_{j+1}^p \cdot f_{j+1}^p \cdot f_{j+1}^p \cdot z_{\infty}) \) for \( -2l + 1 \leq j \leq 0 \).

Proof. (1) \( \Leftrightarrow \) (2): By Proposition 4.2, we see that \( z_1(\hat{x}, l) \in \text{Im}(\Psi_+^l) \) if and only if

\[ c_jp_{2l} - c_{j-1}p_{2l+1} \geq 0 \text{ for } 1 \leq j \leq 2l-1, \]

\[ c_jp_{2l+1} - c_{j-1}x_{2l+1} \geq 0 \text{ for } j = 2l, \]

\[ c_jx_{2l} - c_{j-1}x_{2l+1} \geq 0 \text{ for } 2l+1 \leq j. \]

Therefore it is obvious that (1) implies (2). Assume that (2) holds; we need to show that (1) and (2). We can easily see by induction on \( j \) that

\[ c_jp_{2l} - c_{j-1}p_{2l+1} \geq 0 \text{ for } j \geq 1. \]

In particular, we get (4.1). Since \( x_1 \leq p_1 \), we see that \( c_2p_0 - c_{2l-1}x_1 \geq c_2p_0 - c_{2l-1}p_1 \). Combining this inequality and (4.3), we obtain (4.2).

(1) \( \Leftrightarrow \) (3): By Proposition 2.6 together with the fact that \( i_s = i_t \) if \( s \equiv t \mod 2 \), we see that \( z_1(\hat{x}, l) \in \text{Im}(\Psi_+^l) \) if and only if

\[ 0 = \varepsilon_{ij}(f_{i+1}^p \cdot f_{i+1}^p \cdot f_{i+1}^p \cdot f_{i+1}^p \cdot f_{i+1}^p \cdot z_{\infty}) \text{ for } -2l + 1 \leq j \leq 0, \]

\[ 0 = \varepsilon_{ij}(f_{i+1}^p \cdot f_{i+1}^p \cdot z_{\infty}) \text{ for } j \geq 1. \]

Since \( \hat{x} \in \text{Im}(\Psi_+^l) \), we have \( \hat{x}^* = \hat{f}_{i+1}^p \hat{f}_{i+1}^p \cdot z_{\infty} \) and \( 0 = \varepsilon_{ij}(f_{i+1}^p \cdot f_{i+1}^p \cdot f_{i+1}^p \cdot z_{\infty}) \) for \( j \geq 1 \). Therefore (1) is equivalent to (3).

Thus we have proved the proposition. \( \square \)

By using Propositions 2.6 and 4.2, together with the fact that \( i_s = i_t \) if \( s \equiv t \mod 2 \), we can prove the following proposition in exactly the same way as Proposition 4.3.
Proposition 4.4. Let $\hat{x} = (\ldots, x_2, x_1) \in \text{Im}(\Psi^+_{\iota})$ be such that $x_m \leq p_m$ for all $m \in \mathbb{Z}_{\geq 1}$, and let $l \geq 1$. The following are equivalent:

(1) $z_2(\hat{x}, l) \in \text{Im}(\Psi^-_{\iota})$;

(2) $c_{\iota,j+1}(x_{j+2l} - p_{j+2l}) - c_{\iota,j}(x_{j+2l-1} - p_{j+2l-1}) \leq 0$ for $-2l + 2 \leq j \leq -1$;

(3) $0 = \varphi_{ij}(c_{\iota,j+1}^{p_{j+1}-x_{j+1}} \cdots c_{\iota,1}^{p_1-x_1} z_{-\infty})$ for $1 \leq j \leq 2l$.

Proposition 4.5. Let $\hat{x} = (\ldots, x_2, x_1) \in \text{Im}(\Psi^+_{\iota})$ be such that $x_m \leq p_m$ for all $m \in \mathbb{Z}_{\geq 1}$, and let $k \geq 1$. The following are equivalent:

(1) $c_{k+2l}x_{k} - c_{k+2l-1}x_{k+1} \geq 0$ for $l \geq 1$;

(2) $\gamma_k x_k - x_{k+1} \geq 0$.

Proof. Assume that (2) holds. By Lemma 4.1 together with (3.1), we have $c_{k+2l} > \gamma_k c_{k+2l-1}$ for $l \geq 1$. Hence we obtain $c_{k+2l}x_k - c_{k+2l-1}x_{k+1} \geq c_{k+2l-1}(\gamma_k x_k - x_{k+1})$. By the assumption, we obtain (1).

Assume that (1) holds; note that $x_k \geq 0$. If $x_k = 0$, then we have $-c_{k+2l-1}x_{k+1} \geq 0$. Since $c_{k+2l-1} > 0$, we see that $x_{k+1} = 0$, which gives $\gamma_k x_k - x_{k+1} = 0$. Assume that $x_k > 0$. By the assumption, we obtain $c_{k+2l}/c_{k+2l-1} \geq x_{k+1}/x_k$ for $l \geq 1$. Since the sequence $\{c_{k+1}/c_{k+2l-1}\}_{l \geq 1}$ is strictly decreasing, and converges to $\gamma_k$ by Lemma 4.1, we see that $x_{k+1}/x_k \leq \gamma_k$, which is equivalent to (2).

Proposition 4.6. Let $\hat{x} = (\ldots, x_2, x_1) \in \text{Im}(\Psi^+_{\iota})$ be such that $x_m \leq p_m$ for all $m \in \mathbb{Z}_{\geq 1}$, and let $k \geq 1$. The following are equivalent:

(1) $c'_{\iota,l+k}(x_{k+1} - p_{k+1}) - c'_{\iota,l+k-1}(x_k - p_k) \leq 0$ for $l \geq 1$;

(2) $\gamma_{k+1}p_{k+1} - p_k + x_k - \gamma_{k+1}x_{k+1} \geq 0$.

Proof. Assume that (2) holds. By Lemma 4.1, we have $c'_{\iota,l+k} > \gamma_{k+1}c'_{\iota,l+k-1}$ for $l \geq 1$. Since $x_{k+1} - p_{k+1} \leq 0$, we see that

$$c'_{\iota,l+k}(x_{k+1} - p_{k+1}) - c'_{\iota,l+k-1}(x_k - p_k) \leq c'_{\iota,l+k-1}(\gamma_{k+1}x_{k+1} - \gamma_{k+1}p_{k+1} - x_k + p_k) \leq 0,$$

by assumption.

Assume that (1) holds; note that $(0 \leq) x_{k+1} \leq p_{k+1}$. If $x_{k+1} = p_{k+1}$, then we have $-c'_{\iota,l+k-1}(x_k - p_k) \leq 0$. Since $c'_{\iota,l+k-1} > 0$, we obtain $x_k = p_k$, which gives $\gamma_{k+1}p_{k+1} - p_k + x_k - \gamma_{k+1}x_{k+1} = 0$. Assume that $x_{k+1} < p_{k+1}$. By the assumption, we obtain $c'_{\iota,l+k}/c'_{\iota,l+k-1} \geq (x_k - p_k)/(x_{k+1} - p_{k+1})$. Since the sequence $\{c'_{\iota,l+k}/c'_{\iota,l+k-1}\}_{l \geq 1}$ is decreasing and converges to $\gamma_{k+1}$ by Lemma 4.1, we obtain $(x_k - p_k)/(x_{k+1} - p_{k+1}) \leq \gamma_{k+1}$, which is equivalent to (2).

By Propositions 4.3 - 4.6 we obtain the following corollary.
Corollary 4.7. Let \( \hat{x} = (\ldots, x_2, x_1) \in \text{Im}(\Psi_+^\ell) \) be such that \( x_m \leq p_m \) for all \( m \in \mathbb{Z}_{\geq 1} \).

1. \( \gamma_k x_k - x_{k+1} \geq 0 \) for all \( k \geq 1 \) if and only if
   \[
   \varepsilon_{ij} (\bar{f}_{i}^{p_{j}+1} \cdots \bar{f}_{i}^{p_{j}+1} f_{i}^{p_{0}}) = 0
   \]
   for all \( j \leq 0 \).

2. \( \gamma_{k+1}p_{k+1} - p_k + x_k - \gamma_{k+1}x_{k+1} \geq 0 \) for all \( k \geq 1 \) if and only if
   \[
   \varphi_{ij}(\bar{e}_{i}^{p_{j-1}-x_{j-1}} \cdots \bar{e}_{i}^{p_{j-2}-x_{j-2}} \bar{e}_{i}^{p_{j-1}}) = 0
   \]
   for all \( j \geq 1 \).

4.2 Proof of Theorem [3.2].

Lemma 4.8. It holds that \( \Sigma_i(\lambda) \subset \text{Im}(\Psi_+^\ell) \).

Proof. Let \( \bar{x} = (\ldots, x_2, x_1) \otimes t_\lambda \otimes (x_0, x_{-1}, \ldots) \in \Sigma_i(\lambda) \). By Proposition 4.2, it suffices to show that

\[
\begin{align*}
c_j x_j - c_{j-1} x_{j+1} &\geq 0 \quad \text{for } j \geq 1, \\
c'_{j+1} x_j - c'_{j-1} x_{j-1} &\leq 0 \quad \text{for } j \leq 0.
\end{align*}
\]  

(4.4)  

(4.5)

First, we verify (4.4). If \( j = 1 \), then the assertion is obvious because \( c_j = 1 \) and \( c_{j-1} = 0 \). Assume that \( j > 1 \); note that \( c_{j-1} > 0 \). It follows from Lemma 4.1 that \( \gamma_j < c_j/c_{j-1} \). Also, we have \( (\gamma_j \zeta_j - \zeta_j + 1)(\bar{x}) = \gamma_j x_j - x_{j+1} \geq 0 \) by the definition of \( \Sigma_i(\lambda) \). Hence

\[
c_j x_j - c_{j-1} x_{j+1} = c_{j-1} \left( \frac{c_j}{c_{j-1}} x_j - x_{j+1} \right) \geq c_{j-1}(\gamma_j x_j - x_{j+1}) \geq 0.
\]

Next, we verify (4.5). If \( j = 0 \), then the assertion is obvious because \( c'_{j+1} = 1 \) and \( c'_{j-1} = 0 \). Assume that \( j < 0 \); note that \( c'_{j-1} > 0 \). It follows from Lemma 4.1 that \( \gamma_j < c'_{j+1}/c'_{j-1} \). Also, we have \( (\zeta_{j-1} - \gamma_j \zeta_j)(\bar{x}) = x_{j-1} - \gamma_j x_j \geq 0 \) by the definition of \( \Sigma_i(\lambda) \). Hence

\[
c'_{j+1} x_j - c'_{j-1} x_{j-1} = c'_{j-1} \left( \frac{c'_{j+1}}{c'_{j-1}} x_j - x_{j-1} \right) \leq c'_{j-1}(\gamma_j x_j - x_{j-1}) \leq 0.
\]

Thus we have proved the lemma.

For \( k \in \mathbb{Z} \), we set \( k^{(+)} := k + 2 \) and \( k^{(-)} := k - 2 \). Also, we define the function \( \tilde{\beta}_k : \mathbb{R}^\infty \rightarrow \mathbb{R} \) by

\[
\tilde{\beta}_k = \begin{cases} 
-\langle \lambda, \alpha_i^\vee \rangle + \zeta_k - a_{ik} \zeta_{k+1} + \zeta_{k+2} & \text{if } k = -1, 0, \\
\zeta_k - a_{ik} \zeta_{k+1} + \zeta_{k+2} & \text{otherwise};
\end{cases}
\]
note that $\bar{\beta}_k(\vec{x}) = \sigma_k(\vec{x}) - \sigma_{k(+)}(\vec{x})$. Moreover, for $k \in \mathbb{Z}$, we define the operator $F_k$ on 
\{c + \sum_{i \in \mathbb{Z}} \phi_i \zeta_i \mid c, \phi_i \in \mathbb{R}\} as follows: for $\phi = c + \sum_{i \in \mathbb{Z}} \phi_i \zeta_i$ with $c, \phi_i \in \mathbb{R}$, we set

$$F_k(\phi) := \begin{cases} 
\phi - \phi_k \bar{\beta}_{k(+)} & \text{if } \phi_k \geq 0, \\
\phi - \phi_k \bar{\beta}_{k(-)} & \text{if } \phi_k < 0;
\end{cases}$$

note that $F_k(\phi) = \phi$ if $\phi_k = 0$.

**Lemma 4.9.** Let $\Xi$ be a subset of \{c + \sum_{i \in \mathbb{Z}} \phi_i \zeta_i \mid c, \phi_i \in \mathbb{R}\}. Assume that

$$F_k(\phi) \in \sum_{j \geq 1} \mathbb{R}_{\geq 0} \zeta_j + \sum_{j \leq 0} \mathbb{R}_{\geq 0} (-\zeta_j) + \sum_{\psi \in \Xi} \mathbb{R}_{\geq 0} \psi$$

(4.6)

for all $\phi \in \Xi$ and $k \in \mathbb{Z}$. Then, $\Sigma = \{\vec{x} \in \mathbb{Z}_n(\lambda) \mid \phi(\vec{x}) \geq 0 \text{ for all } \phi \in \Xi\}$ is a subcrystal

of $\mathbb{Z}_n(\lambda)$.

**Proof.** This lemma can be shown similarly to [3, Lemma 4.3]. Let $\vec{x} \in \Sigma$. We show that if $\vec{f_i} \vec{x} \neq \vec{0}$, then $\vec{f_i} \vec{x} \in \Sigma$, that is, $\phi(\vec{f_i} \vec{x}) \geq 0$ for all $\phi \in \Xi$. Let us write $\phi = c + \sum_{i \in \mathbb{Z}} \phi_i \zeta_i$ with $c, \phi_i \in \mathbb{R}$. Define $M_{(i)} = M_{(i)}(\vec{x})$ as \[2.3\], and set $k := \min M_{(i)}$. We see by \[2.4\] that $\phi(\vec{f_i} \vec{x}) = \phi(\vec{x}) + \phi_k$. If $\phi_k \geq 0$, then the assertion is obvious because $\phi(\vec{f_i} \vec{x}) = \phi(\vec{x}) + \phi_k \geq \phi(\vec{x}) \geq 0$. Assume that $\phi_k < 0$. By the definition of $M_{(i)}$ and the fact that $i_k = i_m$ if $k \equiv m \mod 2$, we have $\sigma_k(\vec{x}) > \sigma_{k-2n}(\vec{x})$ for all $n \in \mathbb{Z}_{\geq 1}$. In particular, $\sigma_k(\vec{x}) > \sigma_{k(-)}(\vec{x})$. Since $\bar{\beta}_{k(-)}(\vec{x}) = \sigma_{k(-)}(\vec{x}) - \sigma_k(\vec{x}) \in \mathbb{Z}$, we deduce that $\bar{\beta}_{k(-)}(\vec{x}) \leq -1$. It follows that

$$\phi(\vec{f_i} \vec{x}) = \phi(\vec{x}) + \phi_k \geq \phi(\vec{x}) - \phi_k \bar{\beta}_{k(-)}(\vec{x}) = (F_k(\phi))(\vec{x}).$$

By assumption \[4.6\], we see that $F_k(\phi)$ is of the form $F_k(\phi) = \sum_{j \geq 1} t_j \zeta_j + \sum_{j \leq 0} t_j (-\zeta_j) + \sum_{\psi \in \Xi} t_\psi \psi$, where $t_j, t_\psi \in \mathbb{R}_{\geq 0}$. Since $\vec{x} \in \Sigma$, we have $\psi(\vec{x}) \geq 0$ for any $\psi \in \Xi$. Therefore we see that

$$\phi(\vec{f_i} \vec{x}) \geq (F_k(\phi))(\vec{x}) = \sum_{j \geq 1} t_j x_j + \sum_{j \leq 0} t_j (-x_j) + \sum_{\psi \in \Xi} t_\psi \psi(\vec{x}) \geq 0.$$

Thus we get $\vec{f_i} \vec{x} \in \Sigma$. Similarly, we can show that $\vec{e_i} \vec{x} \in \Sigma$ if $\vec{e_i} \vec{x} \neq \vec{0}$.

Thus we have proved the lemma. \[\square\]

**Proof of Theorem 3.2.** By Lemmas 4.8 and 4.9, it suffices to show that

$$F_k(\phi) \in \sum_{j \geq 1} \mathbb{R}_{\geq 0} \zeta_j + \sum_{j \leq 0} \mathbb{R}_{\geq 0} (-\zeta_j) + \sum_{\psi \in \Xi, \lambda} \mathbb{R}_{\geq 0} \psi$$

(4.7)

for all $k \in \mathbb{Z}$ and $\phi \in \Xi, \lambda$. Here we verify \[4.7\] for the case that $\phi = \gamma_0 p_0 + \gamma_0 \zeta_0 - \zeta_1$; for the other cases, see Appendix A. If $k \neq 0, 1$, then the assertion is trivial since $F_k(\phi) = \phi$. 

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Assume that $k = 0$. We compute

$$F_0(\phi) = (\gamma_0p_0 + \gamma_0\zeta_0 - \zeta_1) - \gamma_0\tilde{\beta}_0$$

$$= (\gamma_0p_0 + \gamma_0\zeta_0 - \zeta_1) - \gamma_0(p_0 + \zeta_0 - a_2\zeta_1 + \zeta_2)$$

$$= \gamma_0\left(a_{i0} - \frac{1}{\gamma_0}\right)\zeta_1 - \zeta_2$$

$$= \gamma_0(\gamma_1\zeta_1 - \zeta_2) \quad \text{by } (3.2).$$

Assume that $k = 1$. We compute

$$F_1(\phi) = (\gamma_0p_0 + \gamma_0\zeta_0 - \zeta_1) - (-1)^{\bar{\beta}_{-1}}$$

$$= (\gamma_0p_0 + \gamma_0\zeta_0 - \zeta_1) + (-p_1 + \zeta_1 - a_{-1}\zeta_0 + \zeta_1)$$

$$= \gamma_0p_0 - p_1 + \zeta_1 + (\gamma_0 - a_{-1})\zeta_0$$

$$= \gamma_0p_0 - p_1 + \zeta_1 - \frac{1}{\gamma_1}\zeta_0 \quad \text{by } (3.2)$$

$$= -p_1 + \left(\gamma_0 + \frac{1}{\gamma_1}\right)p_0 - p_1 + \frac{1}{\gamma_1}(\gamma_{-1}p_{-1} - p_0 + \gamma_{-1}\zeta_{-1} - \zeta_0)$$

(note that $\gamma_{-1} = \gamma_1$)

$$= -p_1 + p_0 + \frac{1}{\gamma_1}(\gamma_{-1}p_{-1} - p_0 + \gamma_{-1}\zeta_{-1} - \zeta_0) \quad \text{by } (3.2)$$

$$= 0 + \frac{1}{\gamma_1}(\gamma_{-1}p_{-1} - p_0 + \gamma_{-1}\zeta_{-1} - \zeta_0) \quad \text{by } (3.4).$$

Thus we have proved Theorem 3.2.

\[\Box\]

### 4.3 Proof of Theorem 3.3

Let $\bar{z} = z_1 \otimes t_\lambda \otimes z_2 \in \text{Im}(\Psi_\lambda^\dagger)$. First, by the tensor product rule of crystals (see also [10, Appendix B]), we see that

$$\varepsilon_i(\bar{z}) = \max\{\varepsilon_i(z_1), \varphi_i(z_2) - \langle\text{wt}(\bar{z}), \alpha_i^\vee\rangle\}, \quad (4.8)$$

$$\varphi_i(\bar{z}) = \max\{\varepsilon_i(z_1) + \langle\text{wt}(\bar{z}), \alpha_i^\vee\rangle, \varphi_i(z_2)\}. \quad (4.9)$$

Moreover,

$$e_i^{\varepsilon_i(\bar{z})} \bar{z} = e_i^{\varepsilon_i(z_1)}z_1 \otimes t_\lambda \otimes e_i^{\varepsilon_i(z_2)}, \quad (4.10)$$

where $c = \max\{-\varepsilon_i(z_1) + \varphi_i(z_2) - \langle\text{wt}(\bar{z}), \alpha_i^\vee\rangle, 0\}$. Next, for $k \in \mathbb{Z}$, we set

$$w_k := \begin{cases} (s_2s_1)^n & \text{if } k = 2n \text{ with } n \in \mathbb{Z}_{>0}, \\ s_1(s_2s_1)^n & \text{if } k = 2n + 1 \text{ with } n \in \mathbb{Z}_{>0}, \\ (s_1s_2)^{-n} & \text{if } k = 2n \text{ with } n \in \mathbb{Z}_{<0}, \\ s_2(s_1s_2)^{-n} & \text{if } k = 2n - 1 \text{ with } n \in \mathbb{Z}_{<0}; \end{cases}$$

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note that $W = \{w_k \mid k \in \mathbb{Z}\}$. By \cite{1} Lemma 3.3, we have

$$w_k \lambda = \begin{cases} p_{k+1} \Lambda_1 - p_k \Lambda_2 & \text{if } k \text{ is even}, \\ -p_k \Lambda_1 + p_{k+1} \Lambda_2 & \text{if } k \text{ is odd} \end{cases}$$

for $k \in \mathbb{Z}$. Since $\text{wt}(S_{w_k} \zeta^*) = w_k \text{wt}(\zeta^*) = -w_k \lambda$, we see that

$$\langle \text{wt}(S_{w_k} \zeta^*), \alpha_i^\vee \rangle = \begin{cases} p_k & \text{if } i = i_k, \\ -p_{k+1} & \text{if } i = i_{k+1}, \end{cases}$$

and hence

$$S_{w_k} \zeta^* = e_{i_k}^p S_{w_{k-1}} \zeta^* = \tilde{f}_{i_{k+1}}^{p_{k+1}} S_{w_{k+1}} \zeta^*.$$

\textbf{Proposition 4.10.} Let $\bar{x} = \hat{x} \otimes t_\lambda \otimes z_{-\infty} \in \Sigma(\lambda)'$ with $\hat{x} = (\ldots, x_2, x_1)$. Then,

$$S_{w_k} \bar{x} = \begin{cases} e_{i_k}^p \cdot e_{i_2}^{p_2} \cdot e_{i_1}^{p_1} \bar{x}^* \otimes t_\mu \otimes e_{i_k}^{p_k} \cdots e_{i_2}^{p_2-x_k} e_{i_1}^{p_1-x_1} z_{-\infty} & \text{if } k \geq 0, \\ \tilde{f}_{i_{k+1}}^{p_{k+1}} \cdots \tilde{f}_{i_1}^{p_1} \tilde{f}_{i_0}^{\tilde{f}_0} \bar{x}^* \otimes t_\mu \otimes z_{-\infty} & \text{if } k \leq 0, \end{cases}$$

where $\mu = -\lambda - \text{wt}(\hat{x})$.

\textbf{Proof.} Since $\hat{x} \in \text{Im}(\Psi^+_{\tilde{t}_{x}})$ by Lemma \cite{4.8} it follows from Proposition \cite{2.6} that

$$x_j = \varepsilon_{i_j} (e_{i_j}^{x_j-1} \cdots e_{i_2}^{x_2} e_{i_1}^{x_1} \bar{x}^*) \quad \text{for } j \geq 1.$$  

By the definition of $\Sigma(\lambda)'$, we have $p_k - x_k \geq 0$, $\gamma_k x_k - x_{k+1} \geq 0$, and $\gamma_{k+1} p_{k+1} - p_k + x_k - \gamma_{k+1} x_{k+1} \geq 0$ for all $k \geq 1$. By Corollary \cite{4.7} we see that

$$\varepsilon_{i_j} (\tilde{f}_{i_{j+1}}^{p_{j+1}} \cdots \tilde{f}_{i_1}^{p_1} \tilde{f}_{i_0}^{\tilde{f}_0} x^*) = 0 \quad \text{for } j \leq 0,$$

and

$$\varphi_{i_j} (e_{i_{j-1}}^{\tilde{f}_{i_j}^{p_{j-1}}-x_{j-1}} \cdots e_{i_2}^{p_2-x_2} e_{i_1}^{p_1-x_1} z_{-\infty}) = 0 \quad \text{for } j \geq 1.$$  

Now, we show the assertion by induction on $|k|$. If $k = 0$, then the assertion is obvious by (4.13). Assume that $k \geq 1$. By the induction hypothesis, we obtain

$$S_{w_{k-1}} \bar{x} = e_{i_{k-1}}^p \cdots e_{i_2}^{p_2} e_{i_1}^{p_1} x^* \otimes t_\mu \otimes e_{i_{k-1}}^{p_k-1-x_{k-1}} \cdots e_{i_2}^{p_2-x_2} e_{i_1}^{p_1-x_1} z_{-\infty},$$

We have $\langle \text{wt}(S_{w_{k-1}} \bar{x}^*), \alpha_i^\vee \rangle = -p_k \leq 0$ by (4.11), $\varepsilon_{i_k} (e_{i_k}^{x_k-1} \cdots e_{i_2}^{x_2} e_{i_1}^{x_1} \bar{x}^*) = x_k$ by (4.13), and $\varphi_{i_k} (e_{i_{k+1}}^{\tilde{f}_{i_k}^{p_{k-1}}-x_{k-1}} \cdots e_{i_2}^{p_2-x_2} e_{i_1}^{p_1-x_1} z_{-\infty}) = 0$ by (4.15). Since $x_k \leq p_k$ as seen above, we see by (4.11) and (4.12) that

$$S_{w_k} \bar{x}^* = e_{i_k}^p (e_{i_{k-1}}^{x_{k-1}} \cdots e_{i_2}^{x_2} e_{i_1}^{x_1} x^* \otimes t_\mu \otimes e_{i_{k-1}}^{p_k-1-x_{k-1}} \cdots e_{i_2}^{p_2-x_2} e_{i_1}^{p_1-x_1} z_{-\infty})$$

$$= e_{i_k}^p e_{i_{k-1}}^{x_{k-1}} \cdots e_{i_2}^{x_2} e_{i_1}^{x_1} x^* \otimes t_\mu \otimes e_{i_{k-1}}^{p_k-1-x_{k-1}} \cdots e_{i_2}^{p_2-x_2} e_{i_1}^{p_1-x_1} z_{-\infty}.$$

Assume that $k \leq -1$. By the induction hypothesis, we obtain

$$S_{w_k} \bar{x}^* = \tilde{f}_{i_{k+2}}^{p_{k+2}} \cdots \tilde{f}_{i_1}^{p_1} \tilde{f}_{i_0}^{\tilde{f}_0} x^* \otimes t_\mu \otimes z_{-\infty}.$$

Since $S_{w_k} \bar{x}^* \neq 0$, we see by (4.12) that

$$S_{w_k} \bar{x}^* = \tilde{f}_{i_{k+1}}^{p_{k+1}} (\tilde{f}_{i_{k+2}}^{p_{k+2}} \cdots \tilde{f}_{i_1}^{p_1} \tilde{f}_{i_0}^{\tilde{f}_0} x^* \otimes t_\mu \otimes z_{-\infty}) = \tilde{f}_{i_{k+1}}^{p_{k+1}} \tilde{f}_{i_{k+2}}^{p_{k+2}} \cdots \tilde{f}_{i_1}^{p_1} \tilde{f}_{i_0}^{\tilde{f}_0} x^* \otimes t_\mu \otimes z_{-\infty}.$$

Thus we have proved the proposition.
Proof of Theorem 3.3. Keep the notation and setting in Proposition 4.10. We show that $\vec{x}^*$ is extremal; by (4.11), it suffices to show that $\varepsilon_{ik}(S_{w_k}\vec{x}^*) = 0$ and $\varphi_{ik+1}(S_{w_k}\vec{x}^*) = 0$ for all $k \in \mathbb{Z}$.

**Step 1.** Assume that $k \geq 0$. We show that $\varphi_{ik+1}(S_{w_k}\vec{x}^*) = 0$. We know from Proposition 4.10 that

$$S_{w_k}\vec{x}^* = e_{ik}^{\tilde{p}_{k+1}} \cdots e_{i_1}^{\tilde{p}_1} x_i^* \otimes t_{\mu} \otimes e_{i_1}^{\tilde{p}_{k-2}} \cdots e_{i_2}^{\tilde{p}_{2}} e_{i_1}^{\tilde{p}_{1}} z_{-\infty}.$$  

By the same argument as in the proof of Proposition 4.10, we see that $\langle \text{wt}(S_{w_k}\vec{x}^*), \alpha_{ik+1}^\vee \rangle = -p_{k+1} \leq 0$, $\varepsilon_{ik+1}(e_{i_k}^{\tilde{p}_{k}} \cdots e_{i_2}^{\tilde{p}_{1}} x_i^*) = x_{k+1}$, $\varphi_{ik+1}(e_{i_k}^{\tilde{p}_{k}} \cdots e_{i_2}^{\tilde{p}_{2}} e_{i_1}^{\tilde{p}_{1}} z_{-\infty}) = 0$, and $x_{k+1} \leq p_{k+1}$. Thus, by (4.9), $\varphi_{ik+1}(S_{w_k}\vec{x}^*) = \max\{x_{k+1} + (-p_{k+1}), 0\} = 0$.

**Step 2.** Assume that $k > 0$. We show that $\varepsilon_{ik}(S_{w_k}\vec{x}^*) = 0$. We have

$$\varepsilon_{ik}(S_{w_k}\vec{x}^*) = \varepsilon_{ik}(e_{i_k}^{\tilde{p}_k} S_{w_{k-1}}\vec{x}^*) = \varepsilon_{ik}(S_{w_{k-1}}\vec{x}^*) - p_k = \varphi_{ik}(S_{w_{k-1}}\vec{x}^*) - \langle \text{wt}(S_{w_{k-1}}\vec{x}^*), \alpha_{ik}^\vee \rangle - p_k = \varphi_{ik}(S_{w_{k-1}}\vec{x}^*) = 0 \quad \text{by (4.11)}.$$  

Since $\varphi_{ik}(S_{w_{k-1}}\vec{x}^*) = 0$ by Step 1, we obtain $\varepsilon_{ik}(S_{w_k}\vec{x}^*) = 0$.

**Step 3.** Assume that $k \leq 0$. We show that $\varepsilon_{ik}(S_{w_k}\vec{x}^*) = 0$. We know from Proposition 4.10 that

$$S_{w_k}\vec{x}^* = e_{i_{k+1}}^{\tilde{p}_{k+1}} \cdots e_{i_1}^{\tilde{p}_1} \tilde{f}^0 \vec{x}^* \otimes t_{\mu} \otimes z_{-\infty}.$$  

We have $\langle \text{wt}(S_{w_k}\vec{x}^*), \alpha_{ik}^\vee \rangle = p_k$ by (4.11) and $\varepsilon_{ik}(e_{i_{k+1}}^{\tilde{p}_{k+1}} \cdots e_{i_1}^{\tilde{p}_1} \tilde{f}^0 \vec{x}^*) = 0$ by (4.14). Since $\varphi_{ik}(z_{-\infty}) = 0$, we see by (4.8) that $\varepsilon_{ik}(S_{w_k}\vec{x}^*) = \max\{0, 0 - p_k\} = 0$.

**Step 4.** Assume that $k < 0$. We show that $\varphi_{ik+1}(S_{w_k}\vec{x}^*) = 0$. We have

$$\varphi_{ik+1}(S_{w_k}\vec{x}^*) = \varphi_{ik+1}(e_{i_k}^{\tilde{p}_k} S_{w_{k+1}}\vec{x}^*) = \varphi_{ik+1}(S_{w_{k+1}}\vec{x}^*) - p_{k+1} = \varepsilon_{ik+1}(S_{w_{k+1}}\vec{x}^*) + \langle \text{wt}(S_{w_{k+1}}\vec{x}^*), \alpha_{ik+1}^\vee \rangle - p_{k+1} = \varepsilon_{ik+1}(S_{w_{k+1}}\vec{x}^*) = 0 \quad \text{by (4.11)}.$$  

Since $\varepsilon_{ik+1}(S_{w_{k+1}}\vec{x}^*) = 0$ by Step 3, we obtain $\varphi_{ik+1}(S_{w_k}\vec{x}^*) = 0$.

This completes the proof of Theorem 3.3.

4.4 Proof of Theorem 3.4

Let $\vec{y} = \vec{y}_1 \otimes t_{\lambda} \otimes \vec{y}_2 \in \text{Im}(\Psi^\lambda)$ with $\vec{y}_1 \in \text{Im}(\Psi^\lambda_+) \text{ and } \vec{y}_2 \in \text{Im}(\Psi^-_\lambda)$, and assume that $\vec{y}^*$ is extremal. Since $\text{Im}(\Psi^-_\lambda) \cong \mathcal{B}(-\infty)$ as crystals, there exist $i_1, \ldots, i_t$ such that $\tilde{f}_{i_t}^{\text{max}} \cdots \tilde{f}_{i_1}^{\text{max}} \vec{y}_2 = z_{-\infty}$. By the tensor product rule of crystals, if we set $\vec{x} := \tilde{f}_{i_t}^{\text{max}} \cdots \tilde{f}_{i_1}^{\text{max}} \vec{y}$, then $\vec{x}$ is of the form $\vec{x} = \hat{x} \otimes t_{\lambda} \otimes z_{-\infty}$ with $\hat{x} \in \text{Im}(\Psi^\lambda_+)$; in order to prove Theorem 3.4 it suffices to show that $\vec{x} \in \Sigma_{\lambda}(\lambda)$. Let us write $\hat{x} = (\ldots, x_2, x_1)$. By the definition of
Assume that (4.16) holds. Then it is obvious that (4.20) holds. Moreover, we obtain
\[ \gamma_0 p_0 + \gamma_0 \cdot 0 - x_1 \geq \gamma_0 p_0 - p_1. \] Recall that \( a_1 a_2 > 4 \). Thus we obtain \( \sqrt{a_1^2 a_2^2 - 4 a_1 a_2} > a_1 a_2 - 3 \), and hence
\[ \gamma_0 = \alpha = \frac{a_1 a_2 + \sqrt{a_1^2 a_2^2 - 4 a_1 a_2}}{2 a_2} > \frac{2 a_1 a_2 - 3}{2 a_2} = a_1 - 3 a_2. \]
Assume that \( a_1, a_2 \geq 2 \). Then \( a_1 - 3/2a_2 > a_1 - 1 > 0 \). By the definition of \( \lambda \), either \( p_0 < p_1 < (a_1 - 1)p_0 \) or \( p_1 < p_0 < (a_2 - 1)p_1 \) holds. In both cases, we deduce that \( \gamma_0 p_0 - p_1 \geq 0 \).

Assume that \( a_1 = 1 \) (resp., \( a_2 = 1 \)). Then \( a_1 - 3/2a_2 > 1/2 \) (resp., \( a_1 - 3/2a_2 > a_1 - 2 \)). By the definition of \( \lambda \), we have \( 2p_1 \leq p_0 \leq (a_2 - 2)p_1 \) (resp., \( 2p_0 \leq p_1 \leq (a_1 - 2)p_0 \)). Hence we deduce that \( \gamma_0 p_0 - p_1 \geq 0 \). Thus we get (4.19). Therefore, it remains to show that (4.16), (4.17), and (4.18).

Now, since \( \{ \tilde{z} \in \text{Im}(\Psi^\lambda_i) \mid \tilde{z}^* \text{ is extremal} \} \) is a subcrystal of \( \text{Im}(\Psi^\lambda_i) \), it follows that \( \tilde{x} \in \{ \tilde{z} \in \text{Im}(\Psi^\lambda_i) \mid \tilde{z}^* \text{ is extremal} \} \). Also, by Proposition 2.6, we have
\[ x_j = \varepsilon_{ij} (\hat{e}_{ij-1} x_{j-1} \cdots e_{ij} x_{j1} \hat{x}^*) \text{ for } j \geq 1. \]

**Proposition 4.11** (proof of (4.16)). Let \( \tilde{x} = \hat{x} \otimes t \lambda \otimes z_{-\infty} \in \text{Im}(\Psi^\lambda_i) \), and write \( \hat{x} = (\ldots, x_2, x_1) \). If \( \tilde{x}^* \) is extremal, then \( p_k - x_k \geq 0 \), and
\[ S_{\mu_k} \tilde{x}^* = \hat{e}_{ik}^k \cdots \hat{e}_{i1}^x \hat{x}^* \otimes t_{\mu} \hat{e}_{ik}^{p_k-x_k} \cdots \hat{e}_{i2}^{p_2-x_2} \hat{e}_{i1}^{p_1-x_1} z_{-\infty} \]
for \( k \geq 1 \), where \( \mu := -\lambda - \text{wt}(\hat{x}) \).

**Proof.** We proceed by induction on \( k \). Assume that \( k = 1 \). Since \( \tilde{x}^* = \hat{x}^* \otimes t_{\mu} \otimes z_{-\infty} \), and \( \langle \text{wt}(\hat{x}^*), \alpha_{i1}^\vee \rangle = (\lambda, \alpha_{i1}^\vee) = -p_1 \), we see by (1.8) and (4.21) that
\[ \varepsilon_1(\tilde{x}^*) = \max\{x_1, 0 - (-p_1)\} = \max\{x_1, p_1\}. \]

Because \( \tilde{x}^* \) is extremal, the inequality \( \langle \text{wt}(\hat{x}^*), \alpha_{i1}^\vee \rangle = -p_1 \leq 0 \) implies that \( \varepsilon_1(\tilde{x}^*) = p_1 \).

By (4.22), we obtain \( p_1 = \max\{x_1, p_1\} \), and hence \( x_1 \leq p_1 \). Also we see by (4.10) that
\[ S_{\mu_1} \tilde{x}^* = \hat{e}_{i1}^{p_1} \hat{x}^* \otimes t_{\mu_1} \hat{e}_{i1}^{p_1-x_1} z_{-\infty}. \]

Let \( k \geq 2 \). By the induction hypothesis, we have
\[ S_{\mu_{k-1}} \tilde{x}^* = \hat{e}_{ik_{k-1}}^k \cdots \hat{e}_{i2}^{x_{k-2}} \hat{e}_{i1}^{x_{k-1}} \hat{x}^* \otimes t_{\mu_{k-1}} \hat{e}_{ik_{k-1}}^{p_k-x_{k-1}} \cdots \hat{e}_{i2}^{p_2-x_2} \hat{e}_{i1}^{p_1-x_1} z_{-\infty}. \]
Hence we see by \((4.11)\) that,
\[
\varepsilon_{ik}(S_{w_{k-1}} \vec{x}^*) = \max\{\varepsilon_{ik}(\vec{e}_{i_{k-1}}^{x_{k-1}} \cdots \vec{e}_{i_1}^{x_1} \vec{x}^*), \varphi_{ik}(\vec{e}_{i_{k-1}}^{x_{k-1}} \cdots \vec{e}_{i_2}^{x_2} \vec{e}_{i_1}^{p_{1-x_1}} z_{-\infty}) - \langle \text{wt}(S_{w_{k-1}} \vec{x}^*), \alpha_{ik}^\vee \rangle\}
= \max\{x_k, m_k\}.
\]

By \((4.11)\), we have \(\langle \text{wt}(S_{w_{k-1}} \vec{x}^*), \alpha_{ik}^\vee \rangle = -p_k\). Because \(\vec{x}^*\) is extremal, the inequality \(\langle \text{wt}(S_{w_{k-1}} \vec{x}^*), \alpha_{ik}^\vee \rangle = -p_k \leq 0\) implies that \(\varepsilon_{ik}(S_{w_{k-1}} \vec{x}^*) = p_k\). Hence we obtain \(p_k = \max\{x_k, m_k\}\), which implies \(x_k \leq p_k\).

Therefore we see by \((4.10)\) and \((4.12)\) that
\[
S_{w_{k-1}} \vec{x}^* = \varepsilon_{ik}^{\text{p}k} S_{w_{k-1}} \vec{x}^* = \varepsilon_{ik}^{\text{p}k} \varepsilon_{i_{k-1}}^{x_{k-1}} \cdots \varepsilon_{i_2}^{x_2} \varepsilon_{i_1}^{x_1} \vec{x}^* \otimes \varepsilon_{i_{k-1}}^{p_{1-x_{k-1}}} \cdots \varepsilon_{i_2}^{p_{2-x_2}} \varepsilon_{i_1}^{p_{1-x_1}} z_{-\infty}
= \varepsilon_{ik} \varepsilon_{i_{k-1}}^{x_{k-1}} \cdots \varepsilon_{i_2}^{x_2} \varepsilon_{i_1}^{x_1} \vec{x}^* \otimes \varepsilon_{i_{k-1}}^{p_{1-x_{k-1}}} \cdots \varepsilon_{i_2}^{p_{2-x_2}} \varepsilon_{i_1}^{p_{1-x_1}} z_{-\infty},
\]

Thus we have proved the proposition. \(\square\)

**Proposition 4.12** (proof of \((4.18)\)). Let \(\vec{x} = \hat{x} \otimes t_\lambda \otimes z_{-\infty} \in \text{Im}(\Psi^\lambda_t)\), and write \(\hat{x} = (\ldots, x_2, x_1)\). If \(\vec{x}^*\) is extremal, then \(\gamma_{k+1}p_{k+1} - p_k + x_k - \gamma_{k+1}x_{k+1} \geq 0\) for \(k \geq 1\).

**Proof.** By Corollary \((4.7)\), it suffices to show that \(\varphi_{ij}(\vec{e}_{i_{j-1}}^{p_{j-1-x_{j-1}}} \cdots \varepsilon_{i_2}^{p_{2-x_2}} \varepsilon_{i_1}^{p_{1-x_1}} z_{-\infty}) = 0\) for all \(j \geq 1\). Let \(j \geq 1\). Since \(\vec{x}^*\) is extremal, and since \(\langle \text{wt}(S_{w_{j-1}} \vec{x}^*), \alpha_{ij}^\vee \rangle = -p_j \leq 0\) by \((4.11)\), we see that \(\varphi_{ij}(S_{w_{j-1}} \vec{x}^*) = 0\). We know from Proposition 4.11 that
\[
S_{w_{j-1}} \vec{x}^* = \varepsilon_{i_{j-1}}^{x_{j-1}} \cdots \varepsilon_{i_2}^{x_2} \varepsilon_{i_1}^{x_1} \vec{x}^* \otimes \varepsilon_{i_{j-1}}^{p_{j-1-x_{j-1}}} \cdots \varepsilon_{i_2}^{p_{2-x_2}} \varepsilon_{i_1}^{p_{1-x_1}} z_{-\infty}.
\]

We see by \((4.9)\) that
\[
0 = \varphi_{ij}(S_{w_{j-1}} \vec{x}^*) = \max\{\varepsilon_{ij}(\varepsilon_{i_{j-1}}^{x_{j-1}} \cdots \varepsilon_{i_1}^{x_1} \vec{x}^*), \varphi_{ij-1}(\varepsilon_{i_{j-1}}^{p_{j-1-x_{j-1}}} \cdots \varepsilon_{i_1}^{p_{1-x_1}} z_{-\infty})\}, \varphi_{ij}(\varepsilon_{i_{j-1}}^{p_{j-1-x_{j-1}}} \cdots \varepsilon_{i_1}^{p_{1-x_1}} z_{-\infty}).
\]

Hence we obtain \(0 \geq \varphi_{ij}(\varepsilon_{i_{j-1}}^{p_{j-1-x_{j-1}}} \cdots \varepsilon_{i_2}^{p_{2-x_2}} \varepsilon_{i_1}^{p_{1-x_1}} z_{-\infty})\). Because \(\varphi_{ij}(\hat{z}) \geq 0\) for all \(i \in I\) and \(\hat{z} \in \mathbb{Z}^{-\infty}_{\leq 0, t}\), we conclude that \(0 = \varphi_{ij}(\varepsilon_{i_{j-1}}^{p_{j-1-x_{j-1}}} \cdots \varepsilon_{i_2}^{p_{2-x_2}} \varepsilon_{i_1}^{p_{1-x_1}} z_{-\infty})\). Thus we have proved the proposition. \(\square\)

**Proposition 4.13** (proof of \((4.17)\)). Let \(\vec{x} = \hat{x} \otimes t_\lambda \otimes z_{-\infty} \in \text{Im}(\Psi^\lambda_t)\), and write \(\hat{x} = (\ldots, x_2, x_1)\). If \(\vec{x}^*\) is extremal, then \(\gamma_kx_k - x_{k+1} \geq 0\) for \(k \geq 1\).

**Proof.** By Corollary \((4.7)\), it suffices to show that \(\varepsilon_{ij}(\vec{f}_{i_{j+1}}^{p_{j+1}} \cdots \vec{f}_{i_0}^{p_0} \hat{x}^*) = 0\) for all \(j \leq 0\). Let \(j \leq 0\). Since \(\vec{x}^*\) is extremal, and since \(\langle \text{wt}(S_{w_j} \vec{x}^*), \alpha_{ij}^\vee \rangle = p_j \geq 0\) by \((4.11)\), we see that \(\varepsilon_{ij}(S_{w_j} \vec{x}^*) = 0\). We see by \((4.12)\) that
\[
S_{w_j} \vec{x}^* = \vec{f}_{i_{j+1}}^{p_{j+1}} \cdots \vec{f}_{i_0}^{p_0} S_{w_0} \vec{x}^* = \vec{f}_{i_{j+1}}^{p_{j+1}} \cdots \vec{f}_{i_1}^{p_1} \vec{f}_{i_0}^{p_0} (\hat{x}^* \otimes t_\mu \otimes z_{-\infty}).
\]
Since $S_{w_j} \bar{x}^* \neq 0$, and since $\tilde{f}_i z_{-\infty} = 0$ for all $i \in I$, we see that

$$S_{w_j} \bar{x}^* = \tilde{f}_{i_{j+1}}^{p_{j+1}} \cdot \tilde{f}_{i_{j+1}}^{p_{j-1}} \tilde{f}_{i_0}^{p_0} \bar{x}^* \otimes t_\mu \otimes z_{-\infty}.$$  

It follows from (4.8) that $0 = \varepsilon_{i_j}(S_{w_j} \bar{x}^*) = \max \{ \varepsilon_{i_j}(\tilde{f}_{i_{j+1}}^{p_{j+1}} \cdot \tilde{f}_{i_{j+1}}^{p_{j-1}} \tilde{f}_{i_0}^{p_0} \bar{x}^*), 0 - p_j \}$. Since $-p_j < 0$, and since $\varepsilon_{i}(\tilde{z}) \geq 0$ for all $i \in I$ and $\tilde{z} \in \mathbb{Z}_{\geq 0, \lambda}^+$, we obtain

$$\varepsilon_{i_k}(\tilde{f}_{i_{j+1}}^{p_{j+2}} \cdot \tilde{f}_{i_{j+1}}^{p_{j+1}} \cdot \tilde{f}_{i_{j-1}}^{p_{j-1}} \tilde{f}_{i_0}^{p_0} \bar{x}^*) = 0.$$  

Thus we have proved the proposition. \hfill \Box

**Appendices.**

**A  Action of $F_k$ on $\Xi_{i}[\lambda]$.**

In this appendix, we compute $F_k(\phi)$, $k \in \mathbb{Z}$, for $\phi = c + \sum_{i \in \mathbb{Z}} \phi_i \zeta_i \in \Xi_{i}[\lambda]$; recall that $F_k(\phi) = \phi$ for $k \in \mathbb{Z}$ such that $\phi_k = 0$.

$$F_0(\gamma_0p_0 + \gamma_0 \zeta_0 - \zeta_1) = \alpha(\gamma_1 \zeta_1 - \zeta_2).$$

$$F_1(\gamma_0p_0 + \gamma_0 \zeta_0 - \zeta_1) = \frac{1}{\beta}(\gamma_{-1}p_{-1} - p_0 + \gamma_{-1} \zeta_{-1} - \zeta_0).$$

$$F_0(\gamma_1p_1 + \zeta_0 - \gamma_1 \zeta_1) = \frac{1}{\alpha}(\gamma_2 p_2 - p_1 + \zeta_1 - \gamma_2 \zeta_2).$$

$$F_1(\gamma_1p_1 + \zeta_0 - \gamma_1 \zeta_1) = \beta(\zeta_1 - \gamma_0 \zeta_0).$$

For $k \geq 1$,

$$F_k(p_k - \zeta_k) = \begin{cases}
(\zeta_{-1} - \gamma_0 \zeta_0) + \frac{1}{\beta}(-\zeta_0) & \text{if } k = 1, \\
\frac{1}{\alpha}(p_1 - \zeta_1) + (\gamma_1 p_1 + \zeta_0 - \gamma_1 \zeta_1) & \text{if } k = 2, \\
\frac{1}{\gamma_{k-1}}(p_{k-1} - \zeta_{k-1}) + (\gamma_{k-1}p_{k-1} - p_k - \zeta_{k-2} + \gamma_{k-1} \zeta_{k-1}) & \text{if } k \geq 3.
\end{cases}$$

$$F_k(\gamma_k \zeta_k - \zeta_{k+1}) = \gamma_k(\gamma_{k+1} \zeta_{k+1} - \zeta_{k+2}).$$

$$F_{k+1}(\gamma_k \zeta_k - \zeta_{k+1}) = \begin{cases}
\frac{1}{\alpha}(\gamma_0 p_0 + \gamma_0 \zeta_0 - \zeta_1) & \text{if } k = 1, \\
\frac{1}{\gamma_{k-1}}(\gamma_{k-1} p_{k-1} - \zeta_k) & \text{if } k \geq 2.
\end{cases}$$

$$F_k(\gamma_{k+1}p_{k+1} - p_k + \zeta_k - \gamma_{k+1} \zeta_{k+1}) = \frac{1}{\gamma_{k+2}}(\gamma_{k+2}p_{k+2} - p_{k+1} + \zeta_{k+1} - \gamma_{k+2} \zeta_{k+2}).$$

$$F_{k+1}(\gamma_{k+1}p_{k+1} - p_k + \zeta_k - \gamma_{k+1} \zeta_{k+1}) = \begin{cases}
\alpha(\gamma_1 p_1 + \zeta_0 - \gamma_1 \zeta_1) & \text{if } k = 1, \\
\gamma_{k+1}(\gamma_k p_k - p_{k-1} + \zeta_{k+1} - \gamma_k \zeta_k) & \text{if } k \geq 2.
\end{cases}$$
For $k \leq 0$,

\[
F_k(p_k + \zeta_k) = \begin{cases} 
\frac{1}{\alpha}(-\zeta_1 + (\gamma_1\zeta_1 - \zeta_2) & \text{if } k = 0, \\
\frac{1}{\beta}(p_0 + \zeta_0 + (\gamma_0p_0 + \gamma_0\zeta_0 - \zeta_1) & \text{if } k = -1, \\
\frac{1}{\gamma_k}(p_{k+1} + \zeta_{k+1}) + (\gamma_{k+1}p_{k+1} - p_{k+2} + \gamma_{k+1}\zeta_{k+1} - \zeta_{k+2}) & \text{if } k \leq -2.
\end{cases}
\]

\[
F_{k-1}(\zeta_{k-1} - \gamma_k\zeta_k) = \begin{cases} 
\frac{1}{\beta}(\gamma_1p_1 + \zeta_0 - \gamma_1\zeta_1) & \text{if } k = 0, \\
\frac{1}{\gamma_k+1}(\zeta_k - \gamma_{k+1}\zeta_{k+1}) & \text{if } k \leq -1.
\end{cases}
\]

\[
F_k(\zeta_{k-1} - \gamma_k\zeta_k) = \gamma_k(\zeta_{k-2} - \gamma_{k-1}\zeta_{k-1}).
\]

\[
F_{k-1}(\gamma_{k-1}p_{k-1} + \gamma_{k-1}\zeta_{k-1} - p_k + \zeta_k) = \begin{cases} 
\beta(\gamma_0p_0 + \gamma_0\zeta_0 - \zeta_1) & \text{if } k = 0, \\
\gamma_{k-1}(\gamma_kp_k - p_{k+1} + \gamma_k\zeta_k - \zeta_{k+1}) & \text{if } k \leq -1.
\end{cases}
\]

\[
F_k(\gamma_{k-1}p_{k-1} + \gamma_{k-1}\zeta_{k-1} - p_k + \zeta_k) = \frac{1}{\gamma_k-2}(\gamma_{k-2}p_{k-2} - p_{k-1} + \gamma_{k-2}\zeta_{k-2} - \zeta_{k-1}).
\]

**B Proof of Theorem 3.1** in the case that $a_1 = 1$ or $a_2 = 1$.

We give a proof only for the case that $a_2 = 1$ (i.e., part (3)); the proof for the case that $a_1 = 1$ (i.e., part (2)) is similar. For $\mu = k\Lambda_1 - l\Lambda_2 \in P$, we define the sequence $\{p^\mu_m\}_{m \in \mathbb{Z}}$ of integers by the following recursive formulas: for $m \geq 0$,

\[
p^\mu_0 := l, \quad p^\mu_1 := k, \quad p^\mu_{m+2} := \begin{cases} 
a_2p^\mu_{m+1} - p^\mu_m & \text{if } m \text{ is even}, \\
a_1p^\mu_{m+1} - p^\mu_m & \text{if } m \text{ is odd};
\end{cases}
\]

for $m < 0$,

\[
p^\mu_m = \begin{cases} 
a_2p^\mu_{m+1} - p^\mu_{m+2} & \text{if } m \text{ is even}, \\
a_1p^\mu_{m+1} - p^\mu_{m+2} & \text{if } m \text{ is odd};
\end{cases}
\]

Note that for $m \in \mathbb{Z}$,

\[
w_{m\mu} = \begin{cases} 
p^\mu_{m+1}\Lambda_1 - p^\mu_m\Lambda_2 & \text{if } m \text{ is even}, \\
-p^\mu_m\Lambda_1 + p^\mu_{m+1}\Lambda_2 & \text{if } m \text{ is odd}.
\end{cases}
\]

**Lemma B.1.** Assume that $a_1 \geq 5$ and $a_2 = 1$. Let $\mu \in P$.

1. If there exists $n \in \mathbb{Z}$ such that $0 < p^\mu_{2n} \leq p^\mu_{2n+2}$, then $0 < p^\mu_{2m} \leq p^\mu_{2m+2}$ for all $m \geq n$.
2. If there exists $n \in \mathbb{Z}$ such that $0 < p^\mu_{2n} \leq p^\mu_{2n-2}$, then $0 < p^\mu_{2m} \leq p^\mu_{2m-2}$ for all $m \leq n$. 

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Proof. We give a proof only for part (1); the proof for part (2) is similar. We proceed by induction on \( m \). If \( m = n \), then the assertion is trivial. Assume that \( m > n \). By (B.1) and (B.2), we have
\[
p_{2m+2}^\mu - p_{2m}^\mu = (a_1 - 3)(p_{2m}^\mu - p_{2m-2}^\mu) + (a_1 - 4)p_{2m-2}^\mu.
\]
Since \( p_{2m}^\mu - p_{2m-2}^\mu \geq 0 \) and \( p_{2m-2}^\mu > 0 \) by the induction hypothesis, we obtain \( p_{2m+2}^\mu - p_{2m}^\mu > 0 \).

Proof of Theorem 3.1 (3). Assume that \( O \in \mathcal{O} \) satisfies condition (2.1). We can take \( \mu = k\Lambda_1 - l\Lambda_2 \in O \) such that \( k, l > 0 \). Then we see by the assumption and (B.3) that \( p_{m}^\mu > 0 \) for all \( m \in \mathbb{Z} \). Hence it follows from Lemma B.1 that there exists \( n \in \mathbb{Z} \) such that
\[
\cdots \geq p_{2n-4}^\mu \geq p_{2n-2}^\mu \geq p_{2n}^\mu \leq p_{2n+2}^\mu \leq p_{2n+4}^\mu \leq \cdots \quad \text{(B.4)}
\]
By (B.1) and (B.2), we have
\[
p_{2n-2}^\mu - p_{2n}^\mu = (a_1 - 2)p_{2n}^\mu - p_{2n+1}^\mu \quad \text{and} \quad p_{2n+2}^\mu - p_{2n}^\mu = p_{2n+1}^\mu - 2p_{2n}^\mu.
\]
Hence we see by (B.4) that \( 2p_{2n}^\mu \leq p_{2n+1}^\mu \leq (a_1 - 2)p_{2n}^\mu \). Then, \( \lambda := w_{2n}\mu = p_{2n+1}^\mu \Lambda_1 - p_{2n}^\mu \Lambda_2 \in W\mu = O \) satisfies the desired condition.

Let \( \lambda = k_1\Lambda_1 - k_2\Lambda_2 \) for some \( k_1, k_2 \in \mathbb{Z} > 0 \) such that \( 2k_2 \leq k_1 \leq (a_1 - 2)k_2 \); we show that \( O := W\lambda \) satisfies condition (2.1). By (B.3), it suffices to show that \( p_{m}^\lambda > 0 \) for all \( m \in \mathbb{Z} \). By (B.1), (B.2), and the assumption that \( 2k_2 \leq k_1 \leq (a_1 - 2)k_2 \), we obtain \( p_{2}^\lambda - p_{0}^\lambda = p_{1}^\lambda - 2p_{0}^\lambda = k_1 - 2k_2 \geq 0 \) and \( p_{3}^\lambda - p_{1}^\lambda = (a_1 - 2)p_{0}^\lambda - p_{1}^\lambda = (a_1 - 2)k_2 - k_1 \geq 0 \). Hence we see by Lemma (B.1) that \( p_{2}^\lambda > 0 \) for all \( m \in \mathbb{Z} \). Note that \( p_{2m}^\lambda = p_{2m+2}^\lambda + p_{2m}^\lambda \) by (B.1) and (B.2). Since \( p_{2m}^\lambda, p_{2m+2}^\lambda > 0 \) as seen above, we get \( p_{2m}^\lambda > 0 \) for all \( m \in \mathbb{Z} \).

Thus we have proved part (3) of Theorem 3.1.

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