The factorization method for the inverse acoustic scattering problems with limited aperture data

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Abstract

We study the factorization method for the inverse acoustic scattering problems in the case of limited aperture data. In this case, the factorization of the far field operator is not symmetric. So, we cannot apply the original factorization method which requires the symmetricity of the factorization. In this paper, we provide a new functional analytic theorem corresponding to the case of limited aperture data, and by applying it, we give the characterizations of the unknown domain.

1 Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set, and assume that its exterior $\mathbb{R}^3 \setminus \overline{\Omega}$ is connected. Let $k > 0$ be the wave number, and let $\theta \in \mathbb{S}^2$ be incident direction where $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ denotes the unit sphere in $\mathbb{R}^3$. We consider the following two problems.

(O) $\Omega$ is an impenetrable obstacle with Dirichlet boundary condition. Find $u^s \in H^1_{\text{loc}}(\mathbb{R}^3 \setminus \overline{\Omega})$ such that

$$\Delta u + k^2 u = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega},$$

$$u(x) = e^{ik\theta \cdot x} + u^s(x),$$

$$u = 0 \text{ on } \partial \Omega$$

$$\lim_{r \to \infty} r \left( \frac{\partial u^s}{\partial r} -iku^s \right) = 0,$$

where $r = |x|$, and (1.4) is the Sommerfeld radiation condition.

(M) $\Omega$ is a penetrable medium modeled by a contrast function $q \in L^\infty(\Omega)$ with $\text{Im } q \geq 0$. Find $u^s \in H^1_{\text{loc}}(\mathbb{R}^3)$ such that

$$\Delta u + k^2 (1 + q) u = 0 \text{ in } \mathbb{R}^3,$$

$$u(x) = e^{ik\theta \cdot x} + u^s(x),$$
\[ \lim_{r \to \infty} r \left( \frac{\partial u^s}{\partial r} - iku^s \right) = 0. \]  

(1.7)

Note that supp\( q = \Omega \), and we extend \( q \) by zero outside \( \Omega \).

In both problems, it is well known that there exists a unique solution \( u^s \) and it has the following asymptotic behavior \([1]\):

\[ u^s(x) = \frac{e^{ik|x|}}{4\pi|x|} u^\infty(\hat{x}, \theta, k) + O \left( \frac{1}{|x|^2} \right), \quad |x| \to \infty, \quad \hat{x} := \frac{x}{|x|}. \]  

(1.8)

The function \( u^\infty \) is called the far field pattern of \( u^s \). The inverse acoustic scattering problem we usually consider is to characterize the unknown domain \( \Omega \) from the data \( u^\infty(\hat{x}, \theta, k) \) with all \( \hat{x} \in \mathbb{S}^2 \), all \( \hat{x} \in \mathbb{S}^2 \), and one \( k > 0 \). However, in this paper, we consider the case of limited aperture data, that is, \( u^\infty(\hat{x}, \theta, k) \) with all \( \hat{x} \in \Gamma_m \), all \( \hat{x} \in \Gamma_s \), and one \( k > 0 \), where \( \Gamma_m \) and \( \Gamma_s \) denote some open sets in \( \mathbb{S}^2 \).

In order to solve the inverse acoustic scattering problem, the factorization method introduced by Kirsch \([2]\) is commonly used. The feature of this method is to set the functional analytic theorem which generalizes the properties of the far field operator for each problem, and to characterize the unknown target by applying it in each case. The original functional analytic theorem (e.g., Theorem 2.15 in \([4]\), Theorem 2.1 in \([5]\)) requires that the factorization of the far field operator is symmetric. However, in the case of limited aperture data that is not always symmetric. So, we cannot apply the original one as it is to this case.

In this paper, we provide a new functional analytic theorem corresponding to the case of limited aperture data, and by applying it, we give the characterizations of \( \Omega \) for an arbitrary choice of \( \Gamma_m \) and \( \Gamma_s \subset \mathbb{S}^2 \) (See Theorem 4.1 for this functional analytic theorem, and this characterizations of \( \Omega \) are given in Theorems 5.2 and 5.4).

Furthermore, the contribution of this paper is not only to give the characterizations of \( \Omega \) with limited aperture data, but also to reduce a priori assumptions of \( q \) in the medium problem \((1.5)-(1.7)\). (Compare the assumption in Theorem 3.3 with that in Theorem 5.4.)

This paper is organized as follows. In Section 2, we recall the far field operator of the obstacle problem \((1.1)-(1.4)\), and the medium problem \((1.5)-(1.7)\), respectively, with limited aperture data. In Section 3, we consider the case \( \Gamma_m = \Gamma_s \). By the symmetricity of the factorization of far field operator in this case, the characterization of \( \Omega \) is given by applying the original functional analytic theorem. In Section 4, we provide a new functional analytic theorem corresponding to the case of an arbitrary choice \( \Gamma_m \) and \( \Gamma_s \). In Section 5, we give the characterizations of \( \Omega \) as an application of it.
2 Preliminary

In Section 2, we define the far field operator, and briefly summarize its properties. The operator
\[ F_{ms} : L^2(\Gamma_s) \rightarrow L^2(\Gamma_m) \]
is defined by
\[ F_{ms} g(\hat{x}) := \int_{\Gamma_s} u^\infty(\hat{x}, \theta) g(\theta) ds(\theta), \; \hat{x} \in \Gamma_m. \tag{2.1} \]
The operator \( F_{ms} \) is called the far field operator. We write the far field operator of the obstacle problem (1.1)–(1.4) as \( F_{ms} = F^{O}_{ms} \), and the medium problem (1.5)–(1.7) as \( F_{ms} = F^{M}_{ms} \), respectively.

2.1 The factorization of \( F^{O}_{ms} \)

We consider the factorization of the far field operator \( F^{O}_{ms} \) for the obstacle. The data-to-pattern operator \( G^{O}_{m(s)} : H^{1/2}(\partial \Omega) \rightarrow L^2(\Gamma_{m(s)}) \) is defined by
\[ G^{O}_{m(s)} f := v^\infty|_{\Gamma_{m(s)}}, \tag{2.2} \]
where \( v^\infty \) is the far field pattern of a radiating solution \( v \) (that is, \( v \) satisfies the Sommerfeld radiation condition) such that
\[ \Delta v + k^2 v = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega}, \tag{2.3} \]
\[ v = f \text{ on } \partial \Omega. \tag{2.4} \]
We define the single layer boundary operator \( S : H^{-1/2}(\partial \Omega) \rightarrow H^{1/2}(\Omega) \) by
\[ S \varphi(x) := \int_{\partial \Omega} \varphi(y) \Phi(x,y) ds(y), \; x \in \partial \Omega, \tag{2.5} \]
where \( \Phi(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|} \).

By these definitions, we can show that \( F^{O}_{ms} \) has a factorization of the form
\[ F^{O}_{ms} = -G^{O}_{m(s)} S^* G^{O*}_{s}. \tag{2.6} \]
The following properties of the operators \( G^{O}_{m} \) and \( S \) are given by previous works in [4]:

**Lemma 2.1** (Theorem 2.9 and Lemma 2.10 in [4]).  (a) The operator \( G^{O}_{m(s)} : H^{1/2}(\partial \Omega) \rightarrow L^2(\Gamma_{m(s)}) \) is compact, injective with dense range in \( L^2(\Gamma_{m(s)}) \).
(b) For $z \in \mathbb{R}^3$

\[ z \in \Omega \iff \phi_z \in \text{Ran}(G^{O}_{m(s)}), \quad (2.7) \]

where $\phi_z(\hat{x}) := e^{-ikz \cdot \hat{x}}$, $\hat{x} \in \Gamma_{m(s)}$.

**Lemma 2.2** (Lemma 1.14 in [4]). Assume that $k^2$ is not a Dirichlet eigenvalue of $-\Delta$ in $\Omega$. Then, the followings hold:

(a) $S$ is bijective.

(b) $\text{Im}(\varphi, S\varphi) < 0$ for all $\varphi \in H^{-1/2}(\partial\Omega)$ with $\varphi \neq 0$.

(c) Let $S_i$ be the boundary integral operator (2.5) corresponding to the wave number $k = i$. The operator $S_i$ is self-adjoint and coercive, i.e, there exists $c > 0$ such that

\[ \langle \varphi, S_i\varphi \rangle \geq c \| \varphi \|^2 \text{ for all } \varphi \in H^{-1/2}(\partial\Omega). \quad (2.8) \]

(d) $S - S_i$ is compact.

**2.2 The factorization of $F^{M}_{ms}$**

We consider the factorization of the far field operator $F^{M}_{ms}$ for the medium in the same way as Section 2.1. The data-to-pattern operator $G^{M}_{m(s)} : L^2(\Omega) \to L^2(\Gamma_{m(s)})$ is defined by

\[ G^{M}_{m(s)}f := v^\infty|_{\Gamma_{m(s)}}, \quad (2.9) \]

where $v^\infty$ is the far field pattern of a radiating solution $v$ such that

\[ \Delta v + k^2(1 + q)v = -k^2 \frac{q}{\sqrt{|q|}}f \text{ in } \mathbb{R}^3, \quad (2.10) \]

We define the operator $M : L^2(\Omega) \to L^2(\Omega)$ by

\[ M\varphi := \frac{|q|}{k^2q} \varphi - \sqrt{|q|}w|_{\Omega}, \quad (2.11) \]

where $w$ is a radiating solution such that

\[ \Delta w + k^2(1 + q)w = -\sqrt{|q|}\varphi \text{ in } \mathbb{R}^3, \quad (2.12) \]

By these definitions, we can show that $F^{M}_{ms}$ has a factorization of the form

\[ F^{M}_{ms} = G^{M}_{m}M^*G^{M}_{s}*. \quad (2.13) \]

The following properties of the operators $G^{M}_{m}$ and $M$ are given by the same argument in [4] and [5];
Lemma 2.3 (Theorem 2.9 and Lemma 2.10 in [4]). (a) The operator $G^M_{m(s)} : L^2(\Omega) \to L^2(\Gamma_m(s))$ is compact with dense range in $L^2(\Gamma_m(s))$.

(b) Assume that $|q|$ is locally bounded below in $\Omega$, i.e., for every compact subset $B \subset \Omega$, there exists $c > 0$ (depend on $B$) such that $|q| \geq c$ in $B$. Then, for $z \in \mathbb{R}^3$

$$z \in \Omega \iff \phi_z \in \text{Ran}(G^M_{m(s)}), \quad (2.14)$$

where $\phi_z(\hat{x}) := e^{-ikz \cdot \hat{x}}, \ \hat{x} \in \Gamma_m(s)$.

Lemma 2.4 (Lemma 3.4 in [5]). Assume that $q \in L^\infty(\Omega)$ with $\text{Im} q \geq 0$. Then, the followings hold:

(a) The operator $M$ is bijective.

(b) $\text{Im}(M\varphi, \varphi) \geq 0$ for all $\varphi \in L^2(\Omega)$.

(c) The operator $M$ can be in the form $M = M_0 + K$ where $M_0f := \frac{|q|}{k^2}q \cdot f$, and $K$ is some compact. Furthermore, if there exist $t \in [0, 2\pi)$ and $c > 0$ such that

$$\text{Re}(e^{-it}q) \geq c|q| \ a.e. \ in \ \Omega_1, \quad (2.15)$$

then the operator $\text{Re}(e^{-it}M_0)$ is coercive.

3 The case $\Gamma_m = \Gamma_s$

In Section 3, we consider the case $\Gamma_m = \Gamma_s$. To emphasize the symmetricity, we write $F_{mm}$ instead of $F_{ms}$. Then, they have a factorization of the form $F_{mm} = G_mT_G^*$. By the symmetricity of the factorization of $F_{mm}$, we can apply the following functional analytic theorem ([4], [5]) to this case.

**Theorem 3.1** (Theorem 2.15 in [4], Theorem 2.1 in [5]). Let $X \subset U \subset X^*$ be a Gelfand triple with a Hilbert space $U$ and a reflexive Banach space $X$ such that the imbedding is dense. Furthermore, let $Y$ be a second Hilbert space and let $F : Y \to Y$, $G : X \to Y$, $T : X^* \to X$ be linear bounded operators such that

$$F = GTG^*.$$  \quad (3.1)

We make the following assumptions:

(1) $G$ is compact with dense range in $Y$.  


(2) There exists \( t \in [0, 2\pi] \) such that \( \text{Re}(e^{it}T) \) has the form \( \text{Re}(e^{it}T) = C + K \) with some compact operator \( K \) and some self-adjoint and positive coercive operator \( C \), i.e., there exists \( c > 0 \) such that
\[
\langle \varphi, C\varphi \rangle \geq c \| \varphi \|^2 \text{ for all } \varphi \in X^*.
\] (3.2)

(3) \( \text{Im}(\varphi, T\varphi) \geq 0 \) for all \( \varphi \in X^* \).

Furthermore, we assume that one of the following assumptions:

(4) \( T \) is injective.

(5) \( \text{Im}(\varphi, T\varphi) > 0 \) for all \( \varphi \in \overline{\text{Ran}(G^*)} \) with \( \varphi \neq 0 \).

Then, the operator \( F_\# := |\text{Re}(e^{it}F)| + \text{Im}F \) is non-negative, and the ranges of \( G : X \to Y \) and \( F_\#^{1/2} : Y \to Y \) coincide with each other.

Remark that the real part and the imaginary part of an operator \( A \) are self-adjoint operators given by
\[
\text{Re}(A) = \frac{A + A^*}{2} \quad \text{and} \quad \text{Im}(A) = \frac{A - A^*}{2i}.
\] (3.3)

The outside operators \( G^O_m \) and \( G^M_m \) of \( F^O_{mm} \) and \( F^M_{mm} \), respectively, satisfy the assumptions with respect to \( G \) in Theorem 3.1 (See Lemmas 2.1 (a), 2.3 (a)). The middle operators \(-S^*\) and \( M^* \) of \( F^O_{mm} \) and \( F^M_{mm} \), respectively, also satisfy the assumptions with respect to \( T \) (See Lemmas 2.2 and 2.4). Therefore by applying Theorem 3.1 to far field operators \( F^O_{mm} = G^O_m(-S^*)G^O_{m*} \) and \( F^M_{mm} = G^O_mM^*G^O_{m*} \), and (b) of Lemmas 2.1 and 2.3, respectively, we have the following characterizations;

**Theorem 3.2** (The case of the obstacle problem (1.1)–(1.4)). Assume that \( k^2 \) is not a Dirichlet eigenvalue of \(-\Delta \) in \( \Omega \). Then, for \( z \in \mathbb{R}^3 \)
\[
z \in \Omega \iff \sum_{n=1}^{\infty} \frac{|\langle \phi_z, \varphi_n \rangle_{L^2(\Gamma_m)}|^2}{\lambda_n} < \infty,
\] (3.4)

where \( (\lambda_n, \varphi_n) \) is a complete eigensystem of \( F_\# \) given by
\[
F_\# := |\text{Re}F^O_{mm}| + \text{Im}F^O_{mm},
\] (3.5)

and the function \( \phi_z \) is defined by
\[
\phi_z(\hat{x}) := e^{-ikz \cdot \hat{x}}, \quad \hat{x} \in \Gamma_m.
\] (3.6)
**Theorem 3.3** (The case of the medium problem (1.5)–(1.7)). Assume the followings:

(i) $q \in L^\infty(\Omega)$ with $\text{Im}q \geq 0$ in $\Omega$.

(ii) $|q|$ is locally bounded below in $\Omega$.

(iii) There exists $t \in [0, 2\pi]$ and $c > 0$ such that

$$\text{Re}(e^{-it} q) \geq c|q| \text{ in } \Omega.$$  \hfill (3.7)

Then, for $z \in \mathbb{R}^3$

$$z \in \Omega \iff \sum_{n=1}^\infty \frac{|(\phi_z, \varphi_n)_{L^2(\Gamma_m)}|^2}{\lambda_n} < \infty,$$  \hfill (3.8)

where $(\lambda_n, \varphi_n)$ is a complete eigensystem of $F_\#$ given by

$$F_\# := |\text{Re}(e^{it} F_{mm})| + \text{Im} F_{mm},$$  \hfill (3.9)

and the function $\phi_z$ is defined by

$$\phi_z(\hat{x}) := e^{-ikz \cdot \hat{x}}, \quad \hat{x} \in \Gamma_m.$$  \hfill (3.10)

### 4 A new functional analytic theorem for limited aperture data

The original factorization method requires the symmetric factorization of the far field operator (see (5.1)). However, the factorization of $F_{ms}$ in the case $\Gamma_m \neq \Gamma_s$ is not symmetric (see (2.6) and (2.13)), so we fails to apply the original one.

Here, let us explain how we provide a new functional analytic theorem in the case of an arbitrary choice $\Gamma_m$ and $\Gamma_s$. First, we extend both the domain $L^2(\Gamma_m)$ and range $L^2(\Gamma_s)$ of the operator $F_{ms}$ to $L^2(\Gamma_m \cup \Gamma_s)$ by multiplying the operator $E_m$ from the left hand side, and $E_m^*$ from the right hand side. (i.e., $E_m F_{ms} E_m^* : L^2(\Gamma_m \cup \Gamma_s) \to L^2(\Gamma_m \cup \Gamma_s)$.) Here, $E_m(\hat{x}) : L^2(\Gamma_m) \to L^2(\Gamma_m \cup \Gamma_s)$ is defined by

$$E_m(\hat{x}) \varphi(\hat{x}) = \begin{cases} 
\varphi(\hat{x}) & \text{for } \hat{x} \in \Gamma_m \\
0 & \text{for } \hat{x} \in (\Gamma_m \cup \Gamma_s) \setminus \Gamma_m.
\end{cases}$$  \hfill (4.1)
(Then, $E^*_{m(s)}$ is the restrict operator from $L^2(\Gamma_m \cup \Gamma_s)$ to $L^2(\Gamma_{m(s)})$, i.e.,
$E^*_{m(s)}g := g|_{\Gamma_{m(s)}}$.) Then, the real part $\text{Re}(E_m F_{ms} E^*_s)$ of the operator $E_m F_{ms} E^*_s$ has a factorization of the form

$$\text{Re}(E_m F_{ms} E^*_s) = \frac{1}{2} (E_m G_m) T (E_s G_s)^* + \frac{1}{2} (E_s G_s) T^* (E_m G_m)^*$$

$$= (E_m G_m, E_s G_s) \begin{pmatrix} 0 & \frac{1}{2} T^* \\ \frac{1}{2} T & 0 \end{pmatrix} \begin{pmatrix} (E_m G_m)^* \\ (E_s G_s)^* \end{pmatrix}$$

$$=: GTG^*.$$  \hfill (4.2)

Although we could make a factorization of the operator $\text{Re}(E_m F_{ms} E^*_s)$, its middle operator $T := \begin{pmatrix} 0 & \frac{1}{2} T^* \\ \frac{1}{2} T & 0 \end{pmatrix}$ does not satisfy the assumptions (2), (3), and (5) in Theorem 3.1. However, the bijectivity of the middle operator $T$ still holds (See Lemmas 2.2 (a) and 2.4 (a)). So, we provide a new functional analytic theorem where the only bijectivity of the middle operator is used.

**Theorem 4.1.** Let $X$ be a reflexive Banach space with $X \subset X^*$, and $Y$ be a Hilbert space. Furthermore, let $F : Y \to Y$ be compact self-adjoint, let $G : X \to Y$ be bounded with dense range in $Y$, and let $T : X^* \to X$ be bounded bijective such that

$$F = GTG^*.$$  \hfill (4.3)

Then, the ranges of $G : X \to Y$ and $|F|^{1/2} : Y \to Y$ coincide with each other.

**Proof of Theorem 4.1.** Let $(\lambda_j, \varphi_j)_{j \in \mathbb{N}}$ be a complete eigensystem of the self-adjoint and compact operator $F$. We split $Y$ into $Y = Y^+ \oplus Y^-$ where

$$Y^+ := \text{span}\{\varphi_j; \lambda_j > 0\} \quad \text{and} \quad Y^- := \text{span}\{\varphi_j; \lambda_j \leq 0\}.$$  \hfill (4.4)

Let $P^\pm : Y \to Y^\pm$ be the orthogonal projectors onto $Y^\pm$. Define the bounded operators $Q^\pm : X^* \to X^*$ by

$$Q^\pm u = \begin{cases} u & \text{for } u \in G^* Y^\pm \\ 0 & \text{for } u \not\in G^* Y^\pm. \end{cases}$$  \hfill (4.5)

We will show

$$Q^\pm G^* (P^+ + P^-) = G^* P^\pm.$$  \hfill (4.6)

Indeed, if $G^* P^- y \in G^* Y^+$, then $Q^+ G^* P^- y = G^* P^- y \in G^* Y^+ \cap G^* Y^-$. By the injectivity of $G^*$, $G^* Y^+ \cap G^* Y^- = \{0\}$, which implies that $Q^+ G^* P^- y = 0$. Similarly, if $G^* P^- y \in G^* Y^-$, then $Q^- G^* P^- y = 0$. Therefore, $Q^\pm G^* (P^+ + P^-) = G^* P^\pm$. 

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If $G^*P^-y \not\in G^*Y^+$, then $Q^+G^*P^-y = 0$ by the definition of $Q^+$. Therefore, $Q^+G^*P^- = 0$. By the same argument, we have $Q^-G^*P^+ = 0$, which proves (4.6).

From (4.6) we obtain

$$|F| = F(P^+ - P^-) = GTG^*(P^+ - P^-)$$
$$= GT(Q^+ - Q^-)G^*(P^+ + P^-)$$
$$= GT(Q^+ - Q^-)G^* := GT_{#}G^*, \quad (4.7)$$

where $T_{#} := T(Q^+ - Q^-) : X^* \to X$. By using the relations that $Q^\pm Q^{\mp} = 0$ and $Q^\pm Q^{\mp} = Q^{\pm}$, and the bijectivity of the operator $T$, we have for $y \in Y$

$$\|G^*y\|^2 = \|(Q^+ + Q^-)G^*y\|^2 = \|(Q^+ - Q^-)(Q^+ - Q^-)G^*y\|^2$$
$$\leq C_1 \|(Q^+ - Q^-)G^*y\|^2$$
$$\leq C_2 \|T(Q^+ - Q^-)G^*y\|^2 = C_2 \|T_{#}G^*y\|^2. \quad (4.8)$$

We will show that there exists $C > 0$ such that

$$C \|T_{#}\varphi\|^2 \leq \langle \varphi, T_{#}\varphi \rangle, \quad \varphi \in G^*Y. \quad (4.9)$$

Indeed, if $\varphi = G^*y \in G^*Y$, then we have

$$\langle \varphi, T_{#}\varphi \rangle = \langle y, |F|y \rangle \geq 0, \quad (4.10)$$

and if $\varphi \notin G^*Y$, then by the definition of $Q_1$ and $Q_2$, we have $T_{#}\varphi = T(Q_1 - Q_2)\varphi = 0$. From this and (4.10), we obtain

$$\langle \varphi, T_{#}\varphi \rangle \geq 0, \quad \varphi \in X^*. \quad (4.11)$$

Let $\varphi = G^*y \in G^*Y$ and $t > 0$. Since $X \subset X^*$ we have $\varphi - tT_{#}\varphi \in X^*$, and by the non-negativity (4.11) of $T_{#}$, we have

$$0 \leq \langle (\varphi - tT_{#}\varphi), T_{#}(\varphi - tT_{#}\varphi) \rangle$$
$$= \langle \varphi, T_{#}\varphi \rangle - t\langle \varphi, T_{#}T_{#}\varphi \rangle - t(T_{#}\varphi, T_{#}T_{#}\varphi) + t^2\langle T_{#}\varphi, T_{#}T_{#}\varphi \rangle$$
$$\leq \langle \varphi, T_{#}\varphi \rangle - t\langle \varphi, T_{#}T_{#}\varphi \rangle - t \|T_{#}\varphi\|^2 + t^2 \|T_{#}\| \|T_{#}\varphi\|^2. \quad (4.12)$$

If $T_{#}\varphi = G^*y' \in G^*Y$, then by the self-adjointness of $|F|$ we have

$$\langle \varphi, T_{#}T_{#}\varphi \rangle = \langle G^*y, T_{#}G^*y' \rangle = \langle y, |F|y' \rangle$$
$$= \langle |F|y, y' \rangle = \langle T_{#}\varphi, T_{#}\varphi \rangle = \|T_{#}\varphi\|^2 \quad (4.13)$$
which implies that from (4.12) we obtain

\[ 0 \leq \langle \varphi, T\# \varphi \rangle + (-2t + t^2 \|T\#\|) \|T\# \varphi\|^2. \tag{4.14} \]

By substituting \( t = \frac{1}{\|T\#\|} \) to (4.14), we have

\[ \frac{1}{\|T\#\|} \|T\# \varphi\|^2 \leq \langle \varphi, T\# \varphi \rangle. \tag{4.15} \]

If \( T\# \varphi \notin G^*Y \), then \( T\# T\# \varphi = T(Q^+ - Q^-)T\# \varphi = 0 \). So, from (4.12) we have

\[ 0 \leq \langle \varphi, T\# \varphi \rangle + (-t + t^2 \|T\#\|) \|T\# \varphi\|^2. \tag{4.16} \]

By substituting \( t = \frac{1}{2 \|T\#\|} \) to (4.16), we have

\[ \frac{1}{4 \|T\#\|} \|T\# \varphi\|^2 \leq \langle \varphi, T\# \varphi \rangle. \tag{4.17} \]

By setting \( C := \min\{\frac{1}{\|T\#\|}, \frac{1}{4 \|T\#\|}\} \), and (4.15), (4.17), we have (4.9).

Therefore, from (4.18) and (4.9) we obtain that there exists \( C > 0 \) such that

\[ C \|\varphi\|^2 \leq \langle \varphi, T\# \varphi \rangle, \quad \varphi \in G^*Y. \tag{4.18} \]

The following lemma is given by the same argument in Lemma 2.4 in [3]. We will prove it in the Appendix.

**Lemma 4.2.** Let \( H \) be a Hilbert space and, let \( X_1 \) and \( X_2 \) be reflexive Banach spaces. Furthermore, let \( A_j : X_1^* \to X_j \) be bounded, and \( B_j : X_j \to H \) be bounded such that

\[ B_1 A_1^* B_1^* = B_2 A_2 B_2^*. \tag{4.19} \]

Assume that for \( j = 1, 2 \),

\[ |\langle \varphi, A_j \varphi \rangle| \geq C \|\varphi\|^2, \quad \varphi \in B_j^*H. \tag{4.20} \]

Then, the ranges of \( B_1 : X_1 \to H \) and \( B_2 : X_2 \to H \) coincide with each other.

By the coercivity condition (4.18), we can apply Lemma 4.2 to the following factorization;

\[ |F| = |F|^{1/2}(|F|^{1/2})^* = GT\#G^*. \tag{4.21} \]

Therefore, the ranges of \( G : X \to Y \) and \( |F|^{1/2} : Y \to Y \) coincide with each other. \( \square \)
5 Applications of Theorem 4.1

In Section 5, we consider the case of an arbitrary choice $\Gamma_m$ and $\Gamma_s$. To characterize $\Omega$, we discuss how to apply Theorem 4.1 to this case.

5.1 The case of the obstacle problem (1.1)–(1.4)

We consider how we provide the characterization of $\Omega$ by the far field operator $F_{ms}^{O}$ for the obstacle problem (1.1)–(1.4). The factorization of the operator $\text{Re}(E_mF_{ms}^{O}E_s^*)$ is given by

$$\text{Re}(E_mF_{ms}^{O}E_s^*) = G^O S G^{O*},$$

(5.1)

where $G^O : H^{1/2}(\partial \Omega) \times H^{1/2}(\partial \Omega) \to L^2(\Gamma_m \cup \Gamma_s)$ is defined by

$$G^O \left( \begin{array}{c} f \\ g \end{array} \right) := \left( E_m G^O_m, E_s G^O_s \right) \left( \begin{array}{c} f \\ g \end{array} \right) = E_m G^O_m f + E_s G^O_s g,$$

(5.2)

and $S : H^{-1/2}(\partial \Omega) \times H^{-1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega) \times H^{1/2}(\partial \Omega)$ is defined by

$$S \left( \begin{array}{c} f \\ g \end{array} \right) := \left( \begin{array}{cc} 0 & -\frac{1}{2} S^* \\ -\frac{1}{2} S & 0 \end{array} \right) \left( \begin{array}{c} f \\ g \end{array} \right) = \left( \begin{array}{c} -\frac{1}{2} S^* g \\ -\frac{1}{2} S f \end{array} \right).$$

(5.3)

Note that $S$ is bijective if $k^2$ is not a Dirichlet eigenvalue of $-\Delta$ in $\Omega$ (See Lemma 2.1). The bijectivity of $S$ is required when we apply Theorem 4.1 to the case $\text{Re}(E_mF_{ms}^{O}E_s^*) = G^O S G^{O*}$. Furthermore, $G^O$ has the following properties corresponding to the assumption $G$ in Theorem 4.1.

Lemma 5.1. (a) The operator $G^O : H^{1/2}(\partial \Omega) \times H^{1/2}(\partial \Omega) \to L^2(\Gamma_m \cup \Gamma_s)$ is compact with dense range in $L^2(\Gamma_m \cup \Gamma_s)$.

(b) For $z \in \mathbb{R}^3$

$$z \in \Omega \iff \phi_z \in \text{Ran}(G^O),$$

(5.4)

where

$$\phi_z(\hat{x}) := \begin{cases} e^{-ikz \cdot \hat{x}} & \text{for } \hat{x} \in \Gamma_m \cup \Gamma_s \setminus (\Gamma_m \cap \Gamma_s) \\ 2e^{-ikz \cdot \hat{x}} & \text{for } \hat{x} \in \Gamma_m \cap \Gamma_s. \end{cases}$$

(5.5)

Proof of Lemma 5.1. (a) The compactness of $G^O$ follows from the compactness of $G^O_m$ and $G^O_s$. We will show the denseness of $G^O$. Indeed, assume that

$$G^O \psi = \left( \begin{array}{c} G^O_m E_m^* \\ G^O_s E_s^* \end{array} \right) \psi = 0.$$

(5.6)
By the injectivity of $G^O_m$ and $G^O_s$ (see Lemma 2.1(a)), and the fact that $E^*_m(s)$ is restrict operator from $L^2(\Gamma_m \cup \Gamma_s) \to L^2(\Gamma_m(s))$ we have $\psi|_{\Gamma_m} = 0$ and $\psi|_{\Gamma_s} = 0$, which implies that $\psi = 0$. Therefore, $G^O$ is dense in $L^2(\Gamma_m \cup \Gamma_s)$.

(b) First, we consider the case $\Gamma_m \neq \Gamma_s$. Let first $z \in \Omega$. Then, 
\[ \left( \begin{array}{c} \Phi(\cdot, z)|_{\partial \Omega} \\ \Phi(\cdot, z)|_{\partial \Omega} \end{array} \right) \in H^{1/2}(\partial \Omega) \times H^{1/2}(\partial \Omega), \]
and
\[ G^O \left( \begin{array}{c} \Phi(\cdot, z)|_{\partial \Omega} \\ \Phi(\cdot, z)|_{\partial \Omega} \end{array} \right) = E_m e^{-ikz \cdot \hat{x}} + E_s e^{-ikz \cdot \hat{x}} = \phi_z, \] (5.7)
which proves $\phi_z \in \text{Ran}(G^O)$. Next, let $\phi_z \in \text{Ran}(G^O)$. Then, there exists $f, g \in H^{1/2}(\partial \Omega)$ such that
\[ \phi_z = G^O \left( \begin{array}{c} f \\ g \end{array} \right) = E_m G^O f + E_s G^O g. \] (5.8)

Since $\Gamma_m \neq \Gamma_s$, we may assume without loss of generality that there exists an open set $\Gamma$ in $\Gamma_m \setminus (\Gamma_m \cap \Gamma_s)$. Let $v^\infty$ be the far field pattern of the problem (2.3)–(2.4). By restricting (5.9) to $\Gamma$, $\phi_z|_{\Gamma} = G^O_m f|_{\Gamma}$, i.e., $e^{-ikz \cdot \hat{x}} = v^\infty(\hat{x})$, $\hat{x} \in \Gamma$. Since the far field pattern is analytic on sphere $S^2$, $e^{-ikz \cdot \hat{x}} = v^\infty(\hat{x})$ for all $\hat{x} \in S^2$. From here, we can show that $z \in \Omega$ by the same argument in Theorem 1.12 in [4].

If $\Gamma_m = \Gamma_s$, then we have $\text{Ran}(G^O_m) = \text{Ran}(G^O)$. Therefore by using Lemma 2.1 (b), we have
\[ z \in \Omega \iff \phi_z \in \text{Ran}(G^O_m) = \text{Ran}(G^O), \] (5.9)

\[ \square \]

By applying Theorem 4.1 and using Lemma 5.1 (b) we have the following characterization of $\Omega$ for arbitrary choice $\Gamma_m$ and $\Gamma_s$.

**Theorem 5.2.** Assume that $k^2$ is not a Dirichlet eigenvalue of $-\Delta$ in $\Omega$. Then, for $z \in \mathbb{R}^3$
\[ z \in \Omega \iff \sum_{n=1}^{\infty} \frac{|(\phi_z, \varphi_n)_{L^2(\Gamma_m \cup \Gamma_s)}|^2}{\lambda_n} < \infty, \] (5.10)
where $(\lambda_n, \varphi_n)$ is a complete eigensystem of $|\text{Re}(E_m F^O_m E^*_s)|$, and
\[ \phi_z(\hat{x}) := \begin{cases} 
  e^{-ikz \cdot \hat{x}} & \text{for } \hat{x} \in \Gamma_m \cup \Gamma_s \setminus (\Gamma_m \cap \Gamma_s) \\
  2e^{-ikz \cdot \hat{x}} & \text{for } \hat{x} \in \Gamma_m \cap \Gamma_s.
\end{cases} \] (5.11)
5.2 The case of the medium problem (1.5)–(1.7)

We consider the medium problem (1.5)–(1.7) in the same way as Section 5.1. The factorization of the operator \( \text{Re}(E_m F_m^* E_s) \) is given by

\[
\text{Re}(E_m F_m^* E_s) = G^M M G^{M*},
\]

where \( G^M : L^2(\Omega) \times L^2(\Omega) \to L^2(\Gamma_m \cup \Gamma_s) \) is defined by

\[
G^M \left( \begin{array}{c} f \\ g \end{array} \right) = (E_m G^M_m, E_s G^M_s) \left( \begin{array}{c} f \\ g \end{array} \right) = E_m G^M_m f + E_s G^M_s g,
\]

and \( M : L^2(\Omega) \times L^2(\Omega) \to L^2(\Omega) \times L^2(\Omega) \) is defined by

\[
M \left( \begin{array}{c} f \\ g \end{array} \right) := \left( \begin{array}{cc} 0 & \frac{1}{2} M^* \\ \frac{1}{2} M & 0 \end{array} \right) \left( \begin{array}{c} f \\ g \end{array} \right) = \left( \begin{array}{c} \frac{1}{2} M^* g \\ \frac{1}{2} M f \end{array} \right).
\]

Note that \( M \) is bijective (See Lemma 2.3). Furthermore, \( G^M \) has the following properties by the same argument in Lemma 5.1.

Lemma 5.3. (a) The operator \( G^M : L^2(\Omega) \times L^2(\Omega) \to L^2(\Gamma_m \cup \Gamma_s) \) is compact with dense range in \( L^2(\Gamma_m \cup \Gamma_s) \).

(b) For \( z \in \mathbb{R}^3 \)

\[
z \in \Omega \iff \phi_z \in \text{Ran}(G^M),
\]

where

\[
\phi_z(\hat{x}) := \begin{cases} e^{-ikz \cdot \hat{x}} & \text{for } \hat{x} \in \Gamma_m \cup \Gamma_s \setminus (\Gamma_s \cap \Gamma_m) \\ 2e^{-ikz \cdot \hat{x}} & \text{for } \hat{x} \in \Gamma_m \cap \Gamma_s. \end{cases}
\]

By applying Theorem 4.1, and using Lemma 5.3 (b), we have the following characterization of \( \Omega \) for arbitrary choice \( \Gamma_m \) and \( \Gamma_s \).

Theorem 5.4. Assume the followings:

(i) \( q \in L^\infty(\Omega) \) with \( \text{Im} q \geq 0 \) in \( \Omega \).

(ii) \( |q| \) is locally bounded below in \( \Omega \).

Then, for \( z \in \mathbb{R}^3 \)

\[
z \in \Omega \iff \sum_{n=1}^{\infty} \frac{|(\phi_z, \varphi_n)_{L^2(\Gamma_m \cup \Gamma_s)}|^2}{\lambda_n} < \infty,
\]

where \( (\lambda_n, \varphi_n) \) is a complete eigensystem of \( |\text{Re}(E_m F_m^* E_s)| \), and

\[
\phi_z(\hat{x}) := \begin{cases} e^{-ikz \cdot \hat{x}} & \text{for } \hat{x} \in \Gamma_m \cup \Gamma_s \setminus (\Gamma_m \cap \Gamma_s) \\ 2e^{-ikz \cdot \hat{x}} & \text{for } \hat{x} \in \Gamma_m \cap \Gamma_s. \end{cases}
\]
The contribution of this paper is not only to give the characterizations of \( \Omega \) with limited aperture data, but also to reduce a priori assumptions of \( q \) in the case of the medium problem. Let us compare the assumption in Theorem 3.3 with that in Theorem 5.4. Theorem 3.3 require the condition (3.7), while, Theorem 5.4 does not require it since a new functional analytic theorem (Theorem 4.1) does not need the condition such like (2) in the original one (Theorem 3.1).

6 Appendix

In this section, we prove Lemma 4.2 in the proof of Theorem 4.1.

Proof of Lemma 4.2. To prove the range identity, we assume on the contrary (without loss of generality) that there exists \( \phi = B_2 \varphi \in \text{Ran}(B_2) \) with \( \phi \notin \text{Ran}(B_1) \) and \( \|\phi\| = 1 \), and derive a contradiction. Define \( V := \{\psi \in H; \langle \psi, \phi \rangle = 0\} \). We will show that \( B_1^*(V) \) is dense in \( B_1^*H \). Indeed, assume that \( (B_1^*|_V)^*x = 0 \) for some \( x \in B_1^*H^* \). Then, we have \( \langle B_1x, \psi \rangle = 0 \) for all \( \psi \in V \), i.e., \( B_1x \in V^\perp \). From this, we can show that \( B_1x = \lambda \phi \) for some \( \lambda \in \mathbb{C} \). Since \( \phi \notin \text{Ran}(B_1) \), we have \( \lambda = 0 \), and thus \( x = 0 \) by the injectivity of the operator \( B_1^*|_V \). Therefore, \( B_1^*(V) \) is dense in \( B_1^*H \). Take a sequence \( \{\psi_n\} \subset V \) with \( B_1^*\psi_n \to -B_1^*\phi \). Then,

\[
\rho_n := |\langle \phi + \psi_n, B_1^*A_1B_2^*(\phi + \psi_n) \rangle| \leq \|A_1\| \|B_1^*(\phi + \psi_n)\| \to 0. \tag{6.1}
\]

On the other hand, by the coercivity condition (4.20) of the operator \( A_2 \) we have

\[
\rho_n = |\langle \phi + \psi_n, B_2A_2B_2^*(\phi + \psi_n) \rangle| = |\langle B_2^*(\phi + \psi_n), A_2B_2^*(\phi + \psi_n) \rangle| \geq C \|B_2^*(\phi + \psi_n)\|^2 \geq \frac{C}{\|\varphi\|^2} \|B_2^*(\phi + \psi_n), \varphi \|^2 = \frac{C}{\|\varphi\|^2} \|\langle \phi + \psi_n, \varphi \rangle\|^2. \tag{6.2}
\]

This is a contradiction. \( \square \)

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