EXISTENCE AND UNIQUEENESS OF THE P-GENERALIZED MODIFIED ERROR FUNCTION

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Abstract. In this article, we define a p-generalized modified error function as the solution to a non-linear ordinary differential equation of second order, with a Robin type boundary condition at \( x = 0 \). We prove existence and uniqueness of a non-negative \( C^\infty \) solution by using a fixed point argument. We show that the p-generalized modified error function converges to the p-modified error function defined as the solution to a similar problem with a Dirichlet boundary condition. In both problems, for \( p = 1 \), the generalized modified error function and the modified error function are recovered. In addition, we analyze the existence and uniqueness of solution to a problem with a Neumann boundary condition.

1. Introduction

Ceratani et al. [5] studied a fusion Stefan problem with variable thermal conductivity and a Robin boundary condition at the fixed face \( x = 0 \). They studied

\[
\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x} \left( k(T) \frac{\partial T}{\partial x} \right), \quad 0 < x < s(t), \quad t > 0,
\]

\[
k(T(0,t)) \frac{\partial T}{\partial x}(0,t) = \frac{h}{\sqrt{t}}[T(0,t) - T_0], \quad t > 0,
\]

\[
T(s(t),t) = T_f, \quad t > 0,
\]

\[
k(T(s(t),t)) \frac{\partial T}{\partial x}(s(t),t) = -\rho l \dot{s}(t), \quad t > 0,
\]

\[
s(0) = 0,
\]

where the unknown functions are the temperature \( T \) and the free boundary \( s \) separating both phases. The parameters \( \rho > 0 \) (density), \( l > 0 \) (latent heat per unit mass), \( T_f \) (phase-change temperature), \( T_0 > T_f \) (bulk temperature), \( h > 0 \) (coefficient that characterizes the heat transfer at \( x = 0 \)), and \( c \) (specific heat) are all known constants.

Problem (1.1)–(1.5) is a phase-change problem known in the literature as a Stefan problem. It corresponds to the melting of a semi-infinite material which is initially solid at the phase-change temperature \( T_f \). As \( T_0 > T_f \), a phase-change interface
$x = s(t)$, $t > 0$ is beginning at $t = 0$ with the initial position $s(0) = 0$. Then, the temperature of the liquid phase is $T = T(x, t)$ defined in the domain $0 < x < s(t)$, $t > 0$, and the temperature of the solid phase is $T = 0$ defined in the domain $x > s(t), t > 0$.

In [6], the thermal conductivity $k$ is defined as

$$k(T) = k_0 \left(1 + \delta \left(\frac{T - T_f}{T_0 - T_f}\right)\right),$$

where $\delta$ is a given positive constant and $k_0$ is the reference thermal conductivity. The existence of a solution to (1.1)–(1.5) when the thermal conductivity $k(T)$ is defined by (1.6) has been proved through the existence of what the authors in [5] called a *generalized modified error function* (GME), which is defined as the solution to the ordinary differential

$$\left[(1 + \delta y(x))y'(x)\right]' + 2xy'(x) = 0, \quad 0 < x < +\infty,$$

$$\left(1 + \delta y(0)\right)y'(0) - \gamma y(0) = 0,$$

$$y(+\infty) = 1,$$

where

$$\gamma = 2\text{Bi}, \quad \text{Bi} = \frac{h\sqrt{\alpha_0}}{k_0} \quad \text{(generalized Biot number)},$$

$$\alpha_0 = \frac{k_0}{\rho c} \quad \text{(thermal diffusivity)}.$$

The solution to (1.1)–(1.5) is given as a function of the solution of (1.7) through the similarity variable $x/(2\sqrt{\alpha_0}t)$, see [5, 6, 12]. More explanations are given in [1, 9, 14].

Motivated by [10] we define a generalized thermal conductivity as

$$k(T) = k_0 \left(1 + \delta \left(\frac{T - T_f}{T_0 - T_f}\right)^p\right), \quad p \geq 1.$$  

(1.10)

Then the existence of a solution to (1.1)–(1.5) with $k$ given by (1.10) will be studied through the *p-generalized modified error function* (p-GME) which we define as the solution to the nonlinear differential problem

$$\left[(1 + \delta y^p(x))y'(x)\right]' + 2xy'(x) = 0, \quad 0 < x < +\infty,$$

$$\left(1 + \delta y^p(0)\right)y'(0) - \gamma y(0) = 0,$$

$$y(+\infty) = 1.$$

(1.11a)

Note that when $p = 1$, we recover the problem studied in [4, 5] and originally defined in [6, 12]. Others studies for $p = 1$ can be found in [2, 13]. In that sense, the p-GME function constitutes a mathematical generalization of the GME function.

With the purpose of proving existence and uniqueness of the p-GME function, i.e. a solution to (1.11), we define a convenient contracting mapping, in Section 2. In Section 3, we study the asymptotic behavior of the p-GME function when $\gamma \to \infty$. We will show that this function converges to the solution of an ordinary differential equation that arises by changing the Robin condition at $x = 0$ [3] by a Dirichlet condition. Finally, in Section 4 we change the Robin condition by a Neumann condition in a solidification process and analyze the existence and uniqueness of a new ordinary differential problem. In conclusion, the aim of this paper is to prove existence and uniqueness of a solution to three ordinary differential problems that
have been motivated by Stefan problems. This is done imposing different boundary
conditions at the fixed face $x = 0$: Robin, Dirichlet and Neumann conditions.

2. Existence and uniqueness of the p-GME function

Let us define

$$X = \{ h : \mathbb{R}^+ \to \mathbb{R} : h \text{ is a bounded and continuous real-valued function} \}, \quad (2.1)$$

$$K = \{ h \in X : \|h\|_\infty \leq 1, \ 0 \leq h, \ h(+\infty) = 1 \}. \quad (2.2)$$

We remark that $K$ is a non-empty closed convex and bounded subset of the
Banach space $X$ with the norm

$$\|h\|_\infty = \sup_{x \in \mathbb{R}^+} |h(x)| < \infty;$$

see [7, page 2487], [8, page 152], [11, page 132].

In this section we prove existence and uniqueness of the p-GME function (prob-
lem (1.11)) by using the Banach fixed point theorem. First, we show that the
ordinary differential problem (1.11) becomes equivalent to an integral equation.

We consider that $\gamma$ is a parameter for problem (1.11), and in Section 3 we will
study the asymptotic behavior when $\gamma \to \infty$.

**Theorem 2.1.** Let $\delta \geq 0$, $\gamma > 0$, $p \geq 1$. For each $\gamma > 0$, the function $y_\gamma \in K$ is a solution to problem (1.11) if and only if $y_\gamma$ is a fixed point to the operator $T_\gamma : K \to K$ given by

$$T_\gamma(h)(x) = \frac{1 + \gamma \int_0^x f_h(\eta)d\eta}{1 + \gamma \int_0^\infty f_h(\eta)d\eta}, \quad x \geq 0, \quad (2.3)$$

with

$$f_h(x) = \frac{1}{\Psi_h(x)} \exp \left( -2 \int_0^x \frac{\xi}{\Psi_h(\xi)}d\xi \right), \quad \Psi_h(x) = 1 + \delta h^p(x). \quad (2.4)$$

**Proof.** Notice first that for each $y = y_\gamma \in K$ we can easily obtain

$$\exp(-\eta^2) \leq f_y(\eta) \leq \exp \left( - \frac{\eta^2}{1 + \delta} \right), \quad (2.5)$$

from where it follows that

$$0 < \frac{\gamma \sqrt{\pi}}{2(1 + \delta)} < 1 + \gamma \int_0^\infty f_y(\eta)d\eta \leq 1 + \frac{\gamma \sqrt{1 + \delta \sqrt{\pi}}}{2}. \quad (2.6)$$

Taking into account (2.6), $T_\gamma(y)$ is a continuous function, since $y \in X$. Also, according to (2.1)–(2.3) and (2.6), $T_\gamma(y) \in K$.

Through the substitution $v = y'$, the ordinary differential equation (1.7a) is
equivalent to

$$-\frac{\Psi_y(x) + 2x}{\Psi_y(x)} = \frac{v'(x)}{v(x)},$$

from where we obtain

$$y(x) = y(0) + c_0 \int_0^x f_y(\eta)d\eta.$$ 

Then, condition (1.7b) is satisfied if and only if $c_0 = \gamma y(0)$. In addition, from (1.7c)
we obtain

$$y(0) = \left( 1 + \gamma \int_0^\infty f_y(\eta)d\eta \right)^{-1}. \quad (2.7)$$
Therefore, \( y \) is a solution to problem (1.11) if and only if \( y \) is a fixed point of the operator \( T_\gamma \), i.e. \( y(x) = T_\gamma(y)(x) \) for all \( x \geq 0 \). Conversely, if \( y \) is a fixed point of the operator \( T_\gamma \) we obtain immediately that (1.7c) is verified, and \( y(0) \) is given by (2.7). Then, by differentiation (1.7a) and (1.7b) hold, and then \( y \) is a solution of (1.11).

**Remark 2.2.** The notation \( y_\gamma \), \( T_\gamma \) is adopted to emphasize the dependence of the solution to (1.11) on \( \gamma \), although it also depends on \( p \) and \( \delta \). This fact is going to facilitate the subsequent analysis of the asymptotic behavior of \( y_\gamma \) when \( \gamma \to \infty \), to be presented in Section 3.

By Theorem 2.1 we will focus on proving that \( T_\gamma \) is a contracting mapping on \( K \). For that purpose, we need the following lemmas.

**Lemma 2.3.** Let \( y_1, y_2 \in K \), \( \delta \geq 0 \), \( \gamma > 0 \), \( p \geq 1 \) and \( x \geq 0 \). Then, the following estimates hold:

\[
\left| \frac{\sqrt{\pi}}{2(1 + \delta)} \right| \leq \left| \int_0^\infty f(y_1(y)) \, dy \right| \leq \sqrt{1 + \delta} \frac{\sqrt{\pi}}{2}, \tag{2.8}
\]

\[
\left| \frac{1}{\Psi(y_1(y))} - \frac{1}{\Psi(y_2(y))} \right| \leq \delta \|y_1 - y_2\|_\infty, \tag{2.9}
\]

\[
\left| \exp \left( \int_0^n \frac{-2\xi}{\Psi(y_1(\xi))} \, d\xi \right) - \exp \left( \int_0^n \frac{-2\xi}{\Psi(y_2(\xi))} \, d\xi \right) \right| \leq \frac{2\delta \eta^2}{\exp(\frac{\eta^2}{1 + \delta})} \|y_1 - y_2\|_\infty, \tag{2.10}
\]

\[
\int_0^x |f(y_1(y)) - f(y_2(y))| \, dy \leq \frac{\sqrt{\pi}}{2(1 + \delta)^{5/2}} \left( \frac{1}{\gamma \sqrt{\pi}} \right) \|y_1 - y_2\|_\infty, \tag{2.11}
\]

\[
\left| \frac{1}{1 + \gamma \int_0^\infty f(y_1(y)) \, dy} - \frac{1}{1 + \gamma \int_0^\infty f(y_2(y)) \, dy} \right| \leq \frac{2(1 + \delta)^{5/2}}{\gamma \sqrt{\pi}} \|y_1 - y_2\|_\infty. \tag{2.12}
\]

**Proof.** We follow the method was developed in [1].

Inequality (2.8) follows from integrating (2.5) in \((0, +\infty)\). For inequality (2.9) we note that from the Mean Value Theorem applied to the function \( r(x) = x^p \) and the fact that \( 1 \leq \Psi(y(x)) \leq 1 + \delta \) for all \( y \in K \), we obtain

\[
\left| \frac{1}{\Psi(y_1(y))} - \frac{1}{\Psi(y_2(y))} \right| \leq \delta |y_1^p(y) - y_2^p(y)| \leq \delta \|y_1 - y_2\|_\infty.
\]

For inequality (2.10), applying the Mean Value Theorem to \( r(x) = \exp(-2x) \) we have

\[
\left| \exp \left( \int_0^n \frac{-2\xi}{\Psi(y_1(\xi))} \, d\xi \right) - \exp \left( \int_0^n \frac{-2\xi}{\Psi(y_2(\xi))} \, d\xi \right) \right| \leq 2 \exp \left( -\frac{\eta^2}{1 + \delta} \right) \int_0^n |\frac{\xi}{\Psi(y_1(\xi))} - \frac{\xi}{\Psi(y_2(\xi))}| \, d\xi \leq 2 \exp \left( -\frac{\eta^2}{1 + \delta} \right) \eta \int_0^n |\frac{1}{\Psi(y_1(\xi))} - \frac{1}{\Psi(y_2(\xi))}| \, d\xi.
\]

Taking into account (2.9) we obtain the corresponding estimate. For inequality (2.11), from items (2.9) and (2.10) we obtain

\[
\int_0^x |f(y_1(y)) - f(y_2(y))| \, dy
\]
\[
\leq \int_0^x \left\{ |f_{y_1}(\eta) - \frac{\exp(-2 \int_0^\eta \frac{\xi}{\Psi_2(\xi)} d\xi)}{\Psi_1(\eta)}| + \frac{\exp(-2 \int_0^\eta \frac{\xi}{\Psi_2(\xi)} d\xi)}{\Psi_1(\eta)} - f_{y_2}(\eta) \right\} d\eta
\]

\[
\leq \int_0^x \left\{ \frac{1}{\Psi_2(\eta)} \exp\left( \int_0^\eta -2\frac{\xi}{\Psi_1(\xi)} d\xi \right) - \exp\left( \int_0^\eta -2\frac{\xi}{\Psi_2(\xi)} d\xi \right) + \exp\left( \int_0^\eta \frac{1}{\Psi_2(\xi)} d\xi \right) \left| \frac{1}{\Psi_1(\eta)} - \frac{1}{\Psi_2(\eta)} \right| \right\} d\eta
\]

\[
\leq \|y_1 - y_2\|_\infty \delta \langle d \int_0^x \exp (-\frac{\eta^2}{1+\delta}) (2+\delta) d\eta \rangle
\]

\[
= \|y_1 - y_2\|_\infty \delta p \sqrt{1+\delta} \left[ (2+\delta) \frac{\sqrt{\pi}}{2} \left( 2 + \delta \right) \exp \left( \frac{x}{\sqrt{1+\delta}} \right) - x \sqrt{1+\delta} \exp \left( \frac{x^2}{1+\delta} \right) \right]
\]

\[
\leq \frac{\sqrt{\pi}}{2} \delta p \sqrt{1+\delta} (2+\delta) \|y_1 - y_2\|_\infty.
\]

Inequality (2.12) follows immediately by using (2.6) and (2.11). □

**Lemma 2.4.** Let \( \gamma > 0, p \geq 1 \) and

\[
g_\gamma(x) = \exp(1+x)^{3/2}\left[ (2+x)(1+(1+x)^{3/2}) + \frac{\gamma}{\gamma\sqrt{\pi}} (1+x) \right], \quad x \geq 0.
\]

Then there exist a unique \( \delta_\gamma > 0 \) such that \( g_\gamma (\delta_\gamma) = 1 \).

The above lemma follows immediately from the fact that \( g_\gamma \) is an increasing function, \( g_\gamma(0) = 0 \) and \( \lim_{x \to \infty} g_\gamma(x) = +\infty \). Now, we are able to formulate the following result.

**Theorem 2.5.** Let \( \gamma > 0 \) and \( p \geq 1 \). The problem (1.11) has a unique solution \( y_\gamma \in K \) if and only if \( 0 \leq \delta < \delta_\gamma \), where \( \delta_\gamma \) is given by Lemma 2.4. Moreover, \( y_\gamma \) is a \( C^\infty \) function in \( \mathbb{R}^+ \) with the following properties:

\[
y'_\gamma(x) > 0, \quad y''_\gamma(x) < 0, \quad \forall x \geq 0.
\]

(2.13)

**Proof.** Let \( y_1, y_2 \in K \) and \( x \geq 0 \). Taking into account Lemma 2.3 we have

\[
|T_\gamma(y_1)(x) - T_\gamma(y_2)(x)|
\]

\[
\leq \left| \frac{1 + \gamma \int_0^x f_{y_1}(\eta) d\eta}{1 + \gamma \int_0^x f_{y_2}(\eta) d\eta} - \frac{1 + \gamma \int_0^x f_{y_2}(\eta) d\eta}{1 + \gamma \int_0^x f_{y_1}(\eta) d\eta} \right| + \frac{1 + \gamma \int_0^x f_{y_1}(\eta) d\eta}{1 + \gamma \int_0^x f_{y_2}(\eta) d\eta} \left\| \frac{1}{1 + \gamma \int_0^x f_{y_1}(\eta) d\eta} \right\| - \frac{1}{1 + \gamma \int_0^x f_{y_2}(\eta) d\eta} \right|
\]

\[
\leq g_\gamma(\delta) \|y_1 - y_2\|_\infty.
\]

Then from Lemma 2.4, if \( 0 \leq \delta < \delta_\gamma \) it follows that \( T_\gamma \) is a contracting mapping what allows to apply the Banach fixed point theorem. Therefore, the problem (1.11) has a unique non-negative continuous solution. Moreover, by differentiation and easy computation the solution is a \( C^\infty \) function in \( \mathbb{R}^+ \) with the useful properties (2.13). □
3. Asymptotic behavior of p-GME function when $\gamma \to \infty$

In this section if we consider the Stefan problem (1.1)–(1.5) and we change the Robin condition (1.2) by a Dirichlet condition.

$T(0, t) = T_0 > 0,$

we obtain the ordinary differential problem

$[(1 + \delta y^p(x)) y'(x)]' + 2x y'(x) = 0, \quad 0 < x < +\infty,$

$y(0) = 0,$

$y(+\infty) = 1.$

For the special case $p = 1$, the solution to this problem is called modified error function (ME) and was considered in [2, 4, 5, 6, 12]. In [4] the existence and uniqueness of the ME function was proved. Moreover, if it is considered $\delta = 0$, the classical error function defined by

$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-z^2)dz, \quad x > 0,$

arises as a solution.

In a similar way to the above section we can analyze the existence and uniqueness of the $p$-modified error function (p-ME), which is defined as the solution to problem (3.2) and constitutes a generalization of the ME function.

Now, let us define

$K^* = \{h \in X : \|h\|_\infty \leq 1, \ 0 \leq h, \ h(0) = 0, \ h(+\infty) = 1\},$

where $X$ is given by (2.1). We remark that $K^*$ is a non-empty closed convex and bounded subset of the Banach space $X$. We will show that the ordinary differential problem (3.2) becomes equivalent to an integral equation.

**Theorem 3.1.** Let $\delta \geq 0, \ p \geq 1$. Then the function $y^* \in K^*$ is a solution to (3.2) if and only if $y^*$ is a fixed point of the operator $T^* : K^* \to K^*$ given by:

$$T^*(h)(x) = \frac{1}{\sqrt{\pi}} \int_0^x f_h(\eta)d\eta, \quad x \geq 0,$$

with $f_h$ defined by (2.4).

**Proof.** In a similar way as in the proof of Theorem 2.1, the operator $T^*$ is well defined and it is easy to see that

$$y^*(x) = y^*(0) + c_0^* \int_0^x f_y^*(\eta)d\eta,$$

with $y^*(0) = 0$ and $c_0^* = (\int_0^\infty f_h(\eta)d\eta)^{-1}$. Then, using (3.2b) and (3.2c), we obtain (3.4). Therefore, $y^*$ is a solution to (3.2) if and only if $y^*$ is a fixed point of the operator $T^*$.

To prove that the operator $T^*$ is a contracting mapping on $K^*$, we enunciate the following lemmas which proofs are analogous to Lemma 2.3 and Lemma 2.4.

**Lemma 3.2.** Let $y_1^*, y_2^* \in K^*$, $\delta \geq 0, \ p \geq 1$ and $x \geq 0$. Then

$$\left| \frac{1}{\int_0^\infty f_{y_1^*}(\eta)d\eta} - \frac{1}{\int_0^\infty f_{y_2^*}(\eta)d\eta} \right| \leq \frac{2(1 + \delta)^{5/2}}{\sqrt{\pi}} \delta p(2 + \delta) \|y_1^* - y_2^*\|_\infty.$$
Lemma 3.3. Let $p \geq 1$ and 
\[
g^*(x) = xp(1 + x)^{3/2} (2 + x) (1 + x)^{3/2}, \quad x \geq 0.
\]
Then there exists a unique $\delta^* > 0$ such that $g^*(\delta^*) = 1$.

Theorem 3.4. Problem (3.2) has a unique solution $y^* \in K$ if and only if $0 \leq \delta < \delta^*$, where $\delta^*$ is given by Lemma 3.3. Moreover, $y^*$ is a $C^\infty$ function in $\mathbb{R}^+$. 

Proof. Let $y_1^*, y_2^* \in K^+$ and $x \geq 0$. Taking into account Lemmas 2.3 and 3.2 we obtain
\[
|T^*(y_1^*)(x) - T^*(y_2^*)(x)| \leq \left| \int_0^x f_{y_1^*}(\eta) d\eta - \int_0^x f_{y_2^*}(\eta) d\eta \right| + \left| \int_0^x f_{y_2^*}(\eta) d\eta - \int_0^\infty f_{y_2^*}(\eta) d\eta \right|
\]
\[
\leq \frac{\int_0^x |f_{y_1^*}(\eta) - f_{y_2^*}(\eta)| d\eta}{\int_0^\infty f_{y_2^*}(\eta) d\eta} + \frac{\int_0^\infty f_{y_2^*}(\eta) d\eta}{\int_0^\infty f_{y_2^*}(\eta) d\eta} - \frac{1}{\int_0^\infty f_{y_2^*}(\eta) d\eta}
\]
\[
\leq g^*(\delta^*) \|y_1^* - y_2^*\|_{\infty}.
\]

Then from Lemma 3.3, if $0 \leq \delta < \delta^*$ it follows that $T^*$ is a contracting mapping what allows to apply the Banach fixed point theorem. Therefore, the problem (3.2) has a unique non-negative continuous solution which is also a $C^\infty$ function by simple differentiations in $\mathbb{R}^+$. \hfill \Box

To problem (1.11), we impose a Robin boundary condition characterized by the coefficient $\gamma > 0$ at $x = 0$. This condition constitutes a generalization of the Dirichlet condition, in the sense that taking the limit when $\gamma \to \infty$ in condition (1.7b), we obtain condition (3.2). Now, we show that the solution to problem (1.11) converges to the solution to problem (3.2) when $\gamma \to \infty$. For this purpose, first, we need the following lemmas which proofs are immediate.

Lemma 3.5. For every $p \geq 1$, when $\gamma \to \infty$, the following convergence results hold
\begin{enumerate}[(a)]
\item $T^*_\gamma(h)(x) \to T^*(h)(x)$ for every $h \in K$ and $x \geq 0$.
\item $y_\gamma(x) \to g^*(x)$ for every $x \geq 0$.
\item $\delta_\gamma \to \delta^*$.
\end{enumerate}

In addition $g^*(x) \geq g_\gamma(x)$ and $\delta_\gamma < \delta^*$ for all $x \geq 0, \gamma > 0$.

Lemma 3.6. Let $p \geq 1$ and
\[
C(x) = 2xp(1 + x)^3 (2 + x), \quad x \geq 0.
\]
Then there exists a unique $\delta > 0$ such that $C(\delta) = 1$.

Theorem 3.7. Let $p \geq 1$ and $0 \leq \delta < \min\{\delta^*, \delta_\gamma\}$. Then $\|y_\gamma - y^*\|_{\infty} \to 0$ when $\gamma \to \infty$. Furthermore, the order of convergence is $1/\gamma$ when $\gamma \to \infty$. 

Proof. First let us note that if $0 \leq \delta < \min\{\delta^*, \delta_\gamma\}$, then as $\delta_\gamma \to \delta^*$, we obtain that $y_\gamma$ and $y^*$ are well defined because of Theorems 2.5 and 3.4. Then for $x \geq 0$ we have
\[
|y_\gamma(x) - y^*(x)| = \left| \frac{(1 + \gamma \int_0^\infty f_{y_\gamma}(\eta) d\eta) \left( \int_0^\infty f_{y^*}(\eta) d\eta \right) - (1 + \gamma \int_0^\infty f_{y^*}(\eta) d\eta) \left( \int_0^\infty f_{y_\gamma}(\eta) d\eta \right)}{(1 + \gamma \int_0^\infty f_{y_\gamma}(\eta) d\eta) \left( \int_0^\infty f_{y^*}(\eta) d\eta \right)} \right|.
\]
The above inequalities are obtained by applying Lemma 2.3, and they lead to

\begin{align*}
\leq \left| \int_0^\infty f_{y^*}(\eta)\,d\eta + \gamma \left( \int_0^x f_{y^*}(\eta)\,d\eta \right) \left( \int_0^\infty f_{y^*}(\eta)\,d\eta \right) - \int_0^x f_{y^*}(\eta)\,d\eta \right|
\end{align*}

Finally, the desired convergence and order of convergence in Theorem 3.7 are obtained by noting that if $0 \leq C(\delta) < 1$ because of Lemma 3.6.

4. Existence and uniqueness considering a Neumann condition

In this section we consider a solidification Stefan problem with a Neumann condition at the fixed face $x = 0$, given by

\begin{align*}
\rho c \frac{\partial T}{\partial t} &= \frac{\partial}{\partial x} \left( k(T) \frac{\partial T}{\partial x} \right), \quad 0 < x < s(t), \ t > 0, \\
k(T(0,t)) \frac{\partial T}{\partial x} (0, t) &= \frac{q_0}{\sqrt{t}}, \quad t > 0, \\
T(s(t), t) &= T_f, \quad t > 0, \\
k(T(s(t), t)) \frac{\partial T}{\partial x} (s(t), t) &= \rho l \dot{s}(t), \quad t > 0, \\
s(0) &= 0,
\end{align*}
where the unknown functions are the temperature $T$ and the free boundary $s$ separating both phases. The parameters $\rho > 0$ (density), $l > 0$ (latent heat per unit mass), $T_f$ (phase-change temperature), $q_0 > 0$ (characterizes the heat flux on the fixed face $x = 0$ of the face-change material which can be determined experimentally) and $c > 0$ (specific heat) are all known constants. In this case, the thermal conductivity $k$ is defined as

$$k(T) = k_0 \left(1 + \delta \left(\frac{T}{T_f}\right)^p\right), \quad p \geq 1,$$

where $\delta$ is a given positive constant and $k_0$ is the reference thermal conductivity.

In a similar way as in previous sections, this Stefan problem leads us to the study the ordinary differential problem

$$[(1 + \delta y^p(x))y'(x)]' + 2xy'(x) = 0, \quad 0 < x < +\infty,$$

$$0 < y < +\infty = 1,$$

where

$$y^* = 2Bi^* \quad \text{with} \quad Bi^* = \frac{q_0 \sqrt{\alpha_0}}{k_0 T_f}.$$

In a similar way to the above sections we can state the following results:

**Theorem 4.1.** Let $\delta \geq 0$, $p \geq 1$ and $0 < \gamma^* \leq \frac{2}{\sqrt{\pi(1+\delta)}}$. Then $y_{\gamma^*} \in K$ is a solution to (4.7) if and only if $y_{\gamma^*}$ is a fixed point of the operator $T_{\gamma^*} : K \rightarrow K$ given by

$$T_{\gamma^*}(h)(x) = 1 - \gamma^* \int_{x}^{+\infty} f_h(\eta)d\eta, \quad x \geq 0.$$  

**Proof.** Given $y_{\gamma^*} \in K$ and taking into account (2.4), we obtain

$$0 < \gamma^* \frac{\text{erfc}(x)}{1 + \delta} \leq \gamma^* \int_{x}^{+\infty} f_h(\eta)d\eta < \gamma^* \sqrt{\frac{1 + \delta}{\pi}} \leq 1.$$  

Note that from (4.9) we have that $T_{\gamma^*}(y_{\gamma^*})$ is an analytic function, since $y_{\gamma^*} \in X$. Also, according to (4.9) and (4.10), $T_{\gamma^*}(y_{\gamma^*}) \in K$.

In a similar way as in the proof of Theorem 2.1, $y_{\gamma^*}$ is a solution to (4.7) if and only if $y_{\gamma^*}$ is a fixed point of the operator $T_{\gamma^*}$.

**Theorem 4.2.** Let $p \geq 1$, $\delta > 0$ and $0 < \gamma^* \leq \frac{2}{\sqrt{\pi(1+\delta)}}$. Then (4.7) has a unique $C^\infty$ solution $y_{\gamma^*} \in K$ if and only if $\delta < \delta_{\gamma^*}$, where $\delta_{\gamma^*}$ is the unique solution to the equation $g(x) = 1$, with

$$g(x) = x \frac{p}{\sqrt{\pi}} \left[(1 + x)(\sqrt{1 + x} \exp(-\frac{1}{4}) + \sqrt{\pi})\right].$$

**Proof.** Let $y_{1,\gamma^*}, y_{2,\gamma^*} \in K$ and $x \geq 0$. Taking into account (2.9) and (2.10) we obtain

$$|T_{\gamma^*}(y_{1,\gamma^*})(x) - T_{\gamma^*}(y_{2,\gamma^*})(x)|$$

$$\leq \|y_1 - y_2\|_{\infty} \delta \gamma^* \left[(1 + \delta)^{3/2} \left(\frac{x}{\sqrt{1 + \delta}} \exp\left(-\frac{x^2}{1 + \delta}\right) + \sqrt{\frac{\pi}{2}}\right) + \sqrt{1 + \delta} \frac{\sqrt{\pi}}{2}\right]$$
\[
\leq \delta \frac{\rho}{\sqrt{\pi}} \left[ (1 + \delta) \left( \sqrt{1 + \delta} \exp\left(\frac{-1}{4}\right) + \sqrt{\pi}\right) + \sqrt{\pi} \right] y_{1,\gamma^*} - y_{2,\gamma^*} \|_{\infty} \\
\leq g(\delta) \| y_{1,\gamma^*} - y_{2,\gamma^*} \|_{\infty}.
\]
Since \( g \) is an increasing function such that \( g(0) = 0 \) and \( g(+\infty) = +\infty \), there exists a unique \( \delta_{\gamma^*} > 0 \) with \( g(\delta_{\gamma^*}) = 1 \).

Then, if \( 0 \leq \delta < \delta_{\gamma^*} \) it follows that \( T_{\gamma^*} \) is a contracting mapping what allows to apply the Banach fixed point theorem. Therefore, the problem (4.7) has a unique non-negative continuous solution which is also a \( C^\infty \) function. \( \square \)

**Conclusion.** In this article, the ordinary differential problems studied in \([4, 5]\) have been generalized by defining what we call the p-GME function and the p-ME function corresponding to the case when a Robin or Dirichlet boundary condition are imposed at \( x = 0 \), respectively. In both problems, existence and uniqueness of \( C^\infty \) solution has been proved by defining convenient contracting mappings. In addition it has been studied the behavior of the p-GME function when the coefficient \( \gamma \) that characterizes the Robin condition goes to infinity, obtaining its convergence to the p-ME function with an order of convergence of the type \( 1/\gamma \) when \( \gamma \to \infty \). Finally, existence and uniqueness of a solution to a solidification problem with a Neumann condition has been studied.

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