Application of a coupled-channel complex scaling method with Feshbach projection to the $K^-pp$ system

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Kaonic nuclei (nuclear systems with anti-kaons) are an interesting subject in hadron and strange nuclear physics, because the strong attraction between anti-kaons and nucleons might bring exotic properties to those systems. In this article, we investigate $K^-pp$ as a prototype of kaonic nuclei. Here, $K^-pp$ is a three-body resonant state in the $KNN-\piYN$ coupled channels, where $Y$ represents hyperons $\Lambda$ and $\Sigma$. In order to treat resonant states in a coupled-channel system properly, we propose a coupled-channel complex scaling method combined with the Feshbach projection, namely, the ccCSM + Feshbach method. In this method, the Feshbach projection is realized with the help of the so-called extended closure relation held in the complex scaling method, and a complicated coupled-channel problem is reduced to a simple single-channel problem, which one can treat easily. First, we confirm that the ccCSM + Feshbach method completely reproduces the results of a full coupled-channel calculation in the case of the two-body $KNN-\piYN$ system. We then proceed to study the three-body $KNN-\piYN$ system, and success- fully find solutions of the $K^-pp$ resonance by imposing self-consistency for the complex $KNN$ energy. The obtained binding energy of $K^-pp$ is well converged around 27 MeV, with an energy-dependent $KNN-\piYN$ potential based on the chiral SU(3) theory, independently of ansätze for self-consistency. This binding energy is as small as those reported in earlier studies based on chiral models. On the other hand, the decay width of $K^-pp$ strongly depends on the ansatz. We also calculate the correlation density of $NN$ and $KNN$ pairs by using the obtained complex-scaled wave function of the $K^-pp$ resonance. The effect of the repulsive core of the $NN$ potential is seen in the $NN$ correlation density. In the $KNN$ correlation density, we can confirm the survival of the $\Lambda^*$ resonance ($I^G = 0 \ KNN$ resonance) in the three-body resonance.

1. Introduction

In hadron and strange nuclear physics, nuclear systems with anti-kaons ($\bar{K}$ mesons = ($K^-, \bar{K}^0$)) are a hot issue, since the anti-kaon is expected to cause several interesting phenomena in finite nuclear systems due to the strong attraction between the anti-kaon and the nucleon. In particular, the anti-kaon could be a key to access dense nuclear matter, for which partial restoration of chiral symmetry [1–3] and kaon condensation [4–6] have been discussed for a long time.
Both theoretical and experimental studies indicate that the $\bar{K}N$ interaction in the isospin $I = 0$ channel is strongly attractive. On the other hand, it is known that the mass of the excited hyperon $\Lambda(1405)$ cannot be reproduced in a naive quark model with $P$-wave excitation [7]; namely, the mass of $\Lambda(1405)$ is exceptionally predicted to be about 100 MeV larger than the Particle Data Group [8] value. Consequently, $\Lambda(1405)$ is considered to be a quasi-bound state of an anti-kaon and a nucleon since it exists at only $\sim$30 MeV below the $\bar{K}N$ threshold. For example, the chiral unitary model [9], based on meson-baryon dynamics, has successfully explained various properties of the $\Lambda(1405)$. Thus, the $\Lambda(1405)$ is getting recognized as a $\bar{K}N$ quasi-bound state, rather than a genuine three-quark state. The attractive nature of the $\bar{K}N$ interaction with a quasi-bound state is consistent with the repulsive nature of the low-energy $\bar{K}N$ scattering data [10] and the $1s$-level energy shift of the kaonic hydrogen atom [11,12], which has been updated precisely [13,14].

For the $\bar{K}N$ interaction, there are two kinds of potential: phenomenological energy-independent potentials (see, e.g., Refs. [15,16]) and chiral SU(3)-based energy-dependent potentials [17,18]. Both are fitted to observables of the $\bar{K}N$ system available at this moment, and are applied and discussed intensively. The former type of potential is more strongly attractive than the latter type in the $\bar{K}N$ subthreshold region. One study, based on a phenomenological $\bar{K}N$ potential [15,16], argues the possibility of so-called deeply bound kaonic nuclei where an anti-kaon is deeply bound in finite nuclei with a binding energy of more than 100 MeV, and such a deeply bound state could exist as a quasi-stable state since the main decay channel $\pi \Sigma$ is closed. In such a state, nucleons are drawn to the anti-kaon by the strong $\bar{K}N$ attraction, and hence a dense system is generated. In case of light $p$-shell nuclei, studies with the antisymmetrized molecular dynamics (AMD) method have shown that the average density amounts to $\sim$4$\rho_0$ when an anti-kaon is added [19,20] ($\rho_0$: normal nuclear density, around 0.17 fm$^{-3}$). In the case of medium and heavy nuclei, studies with the relativistic mean field (RMF) approach have been carried out, and it is shown that $K^-$ mesons can create a dense state inside the nucleus [21–24]. Therefore, we can expect kaonic nuclei to be a doorway to dense nuclear matter.

Thus, kaonic nuclei are considered to be an exotic system involving several interesting aspects from the viewpoint of hadron and nuclear physics. To reveal the nature of kaonic nuclei, a prototype system of kaonic nuclei, “$K^-pp$”, has been studied extensively.\(^1\) Since $K^-pp$ is a three-body system composed of a single $K^-$ meson and two protons, various approaches are adopted. As summarized in Ref. [25], the resulting binding energy and decay width of $K^-pp$ change depending on the combination of methods [variational approach or Faddeev calculation using the Alt-Grassberger-Sandhas form (Faddeev-AGS)] and potentials (phenomenological or chiral-theory based) [26–30]. Since that time, further studies on $K^-pp$ have been carried out. A variational calculation with the hyperspherical harmonics basis function is reported in Ref. [31], where the result agrees with that of an earlier study of another variational calculation [29,30] when they use the same $\bar{K}N$ potential. A Faddeev–AGS calculation with an energy-dependent type of chiral SU(3)-based $\bar{K}N$ potential is reported in Ref. [32], where a $K^-pp$ state with small binding energy and large decay width is obtained similarly to the variational calculation [29,30]. On the other hand, there are several experiments to search for $K^-pp$ states. Actually, the experimental result reported by the FINUDA Collaboration [33] triggered studies on $K^-pp$, although several questions were raised about their interpretation of the result [34].

\(^1\) Actually, in past theoretical studies [26–30], the $\bar{K}NN-\pi \Sigma N-\pi \Delta N$ coupled-channel system with quantum numbers $J^\pi = 0^-$ and $I = 1/2$ was considered. Such a three-body system is denoted symbolically and representatively as “$K^-pp$” in this article.
The DISTO Collaboration [35] reported a bump structure found in a $\Lambda p$ invariant-mass distribution from analysis of the past data on a $p + p$ reaction. These two collaborations claim that, if the observed state is a $K^- pp$ bound state, $K^- pp$ is strongly bound with a binding energy of more than 100 MeV, although its decay width is rather different between the two. Thus, although much effort has been devoted to the study of $K^- pp$, a definite conclusion has not yet been achieved in either the theoretical or experimental studies. However, on the theoretical side, we have one consensus: $K^- pp$ exists as a resonance between the $\bar{K}NN$ and $\pi \Sigma N$ thresholds, as commonly reported in all those calculations.

From those theoretical studies, we believe that the following two ingredients are important in theoretical studies of the $K^- pp$ system: 1. Coupled-channel problem and 2. Resonant state. We employ a coupled-channel complex scaling method (ccCSM) to study $K^- pp$, since both the ingredients can be dealt with in this method simultaneously. Here, the complex scaling method (CSM) is an established powerful tool to investigate resonant states; it has already succeeded greatly in the studies of resonant states of stable/unstable nuclei [36,37]. The CSM has several advantages in the investigation of finite nuclear systems as follows: First, we can handle resonant states in the same way as bound states, since the resonant wave function in the CSM can be represented by using only $L^2$ basis functions such as Gaussian basis functions, which have often been used in bound-state studies. Second, it is straightforward to increase the number of particles in the CSM, which means that we can apply the CSM to various many-body systems. In addition, detailed properties of resonant states can be investigated by analyzing the obtained CSM wave function, as is usually done for bound states.

As the first attempt, we applied the ccCSM to the two-body $\bar{K}N-\pi Y$ system in our previous paper [38] ($Y$ means $\Lambda$ and $\Sigma$ hyperons). Through the study of scattering states as well as the resonant state $\Lambda(1405)$, we have confirmed that the ccCSM is quite useful to look into such a hadronic system. In that work, we constructed a $\bar{K}N-\pi Y$ potential based on the chiral SU(3) theory, which has a Gaussian form factor in the coordinate space and energy dependence. It is shown in studies based on chiral models [9] that $\Lambda(1405)$ should possess a so-called double-pole structure. We reconfirmed such a structure with our Gaussian-type potential. We have successfully identified the lower pole as a broad resonant state, in addition to the higher pole, by using an improved Gaussian basis function in the ccCSM [39].

Since we have confirmed that the ccCSM is quite effective for the two-body system of $\bar{K}N-\pi Y$, we tackle the three-body kaonic nucleus $K^- pp$ in this article. To study $K^- pp$, normally we have to solve an equation in the coupled-channel $\bar{K}NN$, $\pi \Sigma N$, and $\pi \Lambda N$. However, in this paper, we propose a convenient method to reduce such a coupled-channel problem to a single-channel problem; namely, we combine the ccCSM and the Feshbach projection method [40,41]. With this method, we can handle the complicated $\bar{K}NN-\pi \Sigma N-\pi \Lambda N$ system effectively as a simple $\bar{K}NN$ system without losing the effect of the decay to two other open channels. Actually, we study the $K^- pp$ as a Gamow state and obtain the eigenstate as a definite pole on the complex-energy plane. Thus, $K^- pp$ is correctly treated as a resonant state in this study. In contrast, in the earlier studies of $K^- pp$ with variational approaches [29–31], $K^- pp$ was investigated within a bound-state approximation and the decay width was perturbatively estimated with the obtained wave function.

This article is organized as follows. In the next section, we explain our new method of ccCSM + Feshbach projection in detail and give all the tools for the present calculation of $K^- pp$. In Sect. 3, we examine our method by solving the two-body $\bar{K}N-\pi Y$ system. The main results of this paper, i.e., application of our method to the three-body $K^- pp$, are shown in Sect. 4. Section 5 is devoted to a summary of the present study and discussion of our future plans.
2. Methodology

2.1. Essence of the complex scaling method

Here, we give a brief explanation of the usual complex scaling method (CSM) on which the present study is based [36,37]. In the CSM, the Hamiltonian \( \hat{H} \) and wave function \( |\Phi_1\rangle \) are transformed with the complex scaling (complex rotation) operator \( U(\theta) = U(\theta) \hat{H} U^{-1}(\theta) \) and \( |\Phi_\theta\rangle = U(\theta)|\Phi_1\rangle \), respectively. With the complex scaling, the coordinate \( r \) and the conjugate momentum \( p \) in the Hamiltonian and wave function are transformed as

\[
 r \rightarrow r e^{i\theta}, \quad p \rightarrow p e^{-i\theta}, \tag{1}
\]

where the variable \( \theta \) is called the scaling angle.

In eigenvalues of the complex-scaled Hamiltonian, those of scattering continuum states appear along the so-called \( 2\theta \) line on the complex-energy plane, which satisfies the relation \( \tan^{-1}(\text{Im} \ E/\text{Re} \ E) = -2\theta \) (the variable \( E \) means a complex eigenenergy with \( H_\theta |\Phi_\theta\rangle = E |\Phi_\theta\rangle \)). Namely, they are dependent on the scaling angle. On the other hand, eigenvalues of bound and resonant states are proven to be independent of the scaling angle. In addition, as is easily checked, if we choose appropriate values of the scaling angle \( \theta \), wave functions of resonant states are transformed to become square-integrable, although they are originally not so. Therefore, the resonant-state wave function, which is complex-scaled, can be expanded with a square-integrable \( L^2 \) basis function such as a Gaussian basis function, similarly to the bound-state wave functions.

Due to the nature of the CSM, we can obtain the eigenenergies and eigen wave functions of resonant states by diagonalizing the complex-scaled Hamiltonian with Gaussian basis functions. A detailed explanation of the complex scaling method is given in Refs. [36,37].

2.2. Feshbach projection on the coupled-channel complex scaling method

In the present study, we reduce a coupled-channel problem to a single-channel problem for an economical calculation, based on the Feshbach projection method [40,41]. In the Feshbach method, a model space (\( P \) space) and outer space of the model space (\( Q \) space) are assigned with \( P + Q = 1 \) and \( PQ = 0 \). The Schrödinger equation is given as a coupled equation of the wave functions for the \( P \) and \( Q \) spaces as

\[
 \begin{pmatrix} H_{PP} & V_{PQ} \\ V_{QP} & H_{QQ} \end{pmatrix} \begin{pmatrix} \Phi_P \\ \Phi_Q \end{pmatrix} = E \begin{pmatrix} \Phi_P \\ \Phi_Q \end{pmatrix}, \tag{2}
\]

in which \( \Phi_P \) and \( \Phi_Q \) denote \( P \)- and \( Q \)-space wave functions, respectively. By elimination of the \( Q \)-space wave function, an equation for the \( P \)-space wave function is derived from Eq. (2):

\[
 \begin{cases} H_{PP} + V_{PQ} G_Q(E) V_{QP} \end{cases} \Phi_P = E \Phi_P \quad \text{with} \quad G_Q(E) = \frac{1}{E - H_{QQ}}. \tag{3}
\]

Since the Hamiltonian for the \( P \) space \( H_{PP} \) is composed of the kinetic energy term \( T_P \) and the potential term \( V_P \), the above equation can be written as

\[
 \begin{cases} T_P + U_{P}^{\text{eff}}(E) \end{cases} \Phi_P = E \Phi_P \quad \text{with} \quad U_{P}^{\text{eff}}(E) = V_P + V_{PQ} G_Q(E) V_{QP}. \tag{4}
\]

Here, the term \( U_{P}^{\text{eff}}(E) \) is regarded as an effective potential for the \( P \) space. Thus, we obtain a single-channel Schrödinger equation for the \( P \)-space wave function in a formal way.

In application of the Feshbach method to actual studies, the problem is how to represent the \( Q \)-space Green function \( G_Q(E) \) in Eq. (3). We realize the Feshbach method with help from the nature of
the complex scaling method (CSM) as follows. It is proven that the closure relation holds in the CSM, which includes explicitly resonant states as well as continuum scattering states and bound states \cite{42,43} (extended closure relation, ECR, proposed by Berggren \cite{44}). The ECR is shown to be useful to represent the Green function of a system \cite{45}. In addition, the ECR is well described approximately with a set of finite numbers of the complex-scaled eigenstates \{\phi^n_\theta\}, which are obtained by the diagonalization of a complex-scaled Hamiltonian \(\hat{H}_\theta\) with Gaussian basis functions \{\gamma_a\} \cite{46}:

\[
\hat{H}_\theta \phi^n_\theta = e^n_\theta \phi^n_\theta \\
\text{with} \quad \phi^n_\theta = \sum_{a=1}^M C^n_a \phi_a \implies \sum_{n=1}^N \phi^n_\theta \langle \phi^n_\theta \rangle \approx 1,
\]

where \(n\) is the state index and complex parameters \{\(C^n_a\)\} are determined by a diagonalization of \(\hat{H}_\theta\).

We incorporate the ECR on the \(Q\) space into the Feshbach method. First, we consider the complex-scaled Green function for the \(Q\)-space, \(G^\theta_Q(E) = U(\theta) G_Q(E) U^{-1}(\theta)\). With the application of the ECR shown in Eq. (5), it is given approximately as

\[
G^\theta_Q(E) = \frac{1}{E - H^\theta_{QQ}} \approx \sum_{n=1}^N \phi^n_{Q,n} \left| \phi^n_{Q,n} \right\rangle \frac{1}{E - e^n_{Q,n}} \langle \phi^n_{Q,n} \rangle,
\]

where eigenenergies \{\(e^n_{Q,n}\)\} and eigenstates \{\(\phi^n_{Q,n}\)\} of the complex-scaled Hamiltonian \(H^\theta_{QQ}\) are calculated with Gaussian basis functions \{\(\gamma_a\)\}. By the inverse transformation \(U^{-1}(\theta)\), we obtain the non-scaled Green function \(G_Q(E)\) from the complex-scaled one \(G^\theta_Q(E)\):

\(G_Q(E) = U^{-1}(\theta) G^\theta_Q(E) U(\theta)\). Substituting the obtained \(G_Q(E)\) into Eq. (4), we can represent the effective \(P\)-space potential as

\[
U^\text{eff}_P(E) = V_P + \sum_{n=1}^N U^{-1}(\theta) V^\theta_{PQ} \left| \phi^n_{Q,n} \right\rangle \frac{1}{E - e^n_{Q,n}} \langle \phi^n_{Q,n} \rangle V^\theta_{QP} U(\theta),
\]

where \(V^\theta_{PQ}(QP) = U(\theta) V_{PQ}(QP) U^{-1}(\theta)\). Since the eigenstates \{\(\phi^n_{Q,n}\)\} are expanded with the Gaussian basis function, the effective potential is expressed with Gaussian functions when the original coupled-channel potential is given in a Gaussian form. Therefore, the effective potential \(U^\text{eff}_P(E)\) derived in this way is easily handled in conventional many-body calculations with Gaussian basis functions.

Thus, we reduce a coupled-channel problem to a single-channel problem with the Feshbach projection method, which is assisted by the unique nature of the complex scaling method. We call this method the coupled-channel complex scaling method with Feshbach projection, which is hereafter called the “ccCSM + Feshbach method” for short.

### 2.3. Hamiltonian and trial wave function for the single \(\bar{K}NN\) channel

In theoretical studies, the \(K^-pp\) system is treated as a coupled-channel system of \(\bar{K}NN\), \(\pi \Sigma N\), and \(\pi \Lambda N\), involving quantum numbers \(J^\pi = 0^-\) and \(I = 1/2\). We apply the ccCSM + Feshbach method to the \(\bar{K}NN-\pi Y N\) coupled-channel problem to reduce it to a \(\bar{K}NN\) single-channel problem \((\bar{Y} = \Lambda, \Sigma)\).

First, we consider the two-body system of \(\bar{K}N-\pi Y\). When we set the \(\bar{K}N\) channel to \(P\) space and the \(\pi Y\) channels to \(Q\) space, we can derive an effective \(\bar{K}N\) potential for each isospin state \((I = 0, 1)\)
with the ccCSM + Feshbach method:

\[
U_{K_N (I)}^{\text{eff}}(E_{K_N}) = V_{K_N, K_N (I)} + \sum_{Y(I)}^{N} \sum_{n=1}^{N} U^{-1}(\theta_{\bar{Q}}) V_{K_N, K_N (I)}^{\theta_{\bar{Q}}} \phi_{Y(I), n}^{\theta_{\bar{Q}}}
\]

\[
\times \frac{1}{E_{K_N} - e_{\phi_{Y(I), n}^{\theta_{\bar{Q}}}}} \phi_{Y(I), n}^{\theta_{\bar{Q}}} V_{K_N, K_N (I)}^{\theta_{\bar{Q}}} (U(\theta_{\bar{Q}}), \bar{Q})
\]

(8)

where the index \( Y(I) \) indicates \( \Sigma \) for \( I = 0 \) and \( (\Lambda, \Sigma) \) for \( I = 1 \). As explained in the previous section, eigenstates and eigenenergies \( \{|\phi_{Y(I), n}^{\theta_{\bar{Q}}}, e_{\phi_{Y(I), n}^{\theta_{\bar{Q}}}}\} \) are calculated with diagonalization of the complex-scaled Hamiltonian \( H_{\pi Y, I}^{\theta_{\bar{Q}}} \) for each isospin \( I \) channel, which is

\[
H_{\pi Y, I}^{\theta_{\bar{Q}}} = \sum_{\alpha=\pi Y(I)} \left( \Delta M_\alpha + \hat{T}_\alpha^{\theta_{\bar{Q}}} \right) |\alpha\rangle \langle \alpha| + \sum_{\alpha, \beta=\pi Y(I)} V_{\alpha\beta}^{(I), \theta_{\bar{Q}}} |\alpha\rangle \langle \beta|,
\]

(9)

where the channel indices \( \alpha \) and \( \beta \) are \( \pi \) for \( I = 0 \) and \( (\pi \Lambda, \pi \Sigma) \) for \( I = 1 \). The terms of \( \Delta M_\alpha \) and \( \hat{T}_\alpha^{\theta_{\bar{Q}}} \) are the mass of the \( \alpha \) channel measured from the \( \bar{K}_N \) threshold and the relative kinetic energy term of the \( \alpha \) channel, which is complex-scaled, respectively. The last term \( V_{\alpha\beta}^{(I), \theta_{\bar{Q}}} \) is a complex-scaled potential coupling between channels \( \alpha \) and \( \beta \) with isospin \( I \). Note that hereafter in this article the variable \( \theta_{\bar{Q}} \) means the scaling angle that is used to construct an effective potential by elimination of the \( Q \)-space components, as explained in the previous section.

With the effective \( \bar{K}_N \) potential \( U_{K_N (I)}^{\text{eff}}(E_{K_N}) \) plugged in, a three-body Hamiltonian for the single \( \bar{K}NN \) channel is constructed as

\[
\hat{H}_{\bar{K}NN} = \hat{p}_1^2 / 2\mu_{\bar{K}N} + \hat{p}_2^2 / 2\mu_{\bar{K}NN} + \hat{V}_{NN} + \sum_{i=1,2} \sum_{I=0,1} \hat{U}_{(K_N)I}^{\text{eff}}(E_{K_N}).
\]

(10)

The first two terms are kinetic-energy operators with respect to a Jacobi coordinate, \( x_1 = r_{N2} - r_{N1} \) and \( x_2 = r_K - (r_{N1} + r_{N2}) / 2 \). The term \( \hat{V}_{NN} \) is a nucleon–nucleon potential. The last term is the effective \( \bar{K}_N \) potential for a \( \bar{K}_N \) pair with isospin \( I \). A detailed explanation of the \( NN \) and \( \bar{K}_NN \) potentials will be given at the beginning of Sects. 3 and 4.

A trial wave function of the \( \bar{K}NN \) system with quantum numbers \((J^P, T) = (0^-, 1/2)\) is constructed in a similar way to an earlier study with a variational approach [29,30]. Since the total spin of the two nucleons is assumed to be zero, the trial wave function consists of two components that satisfy the antisymmetrization for the two nucleons. In one component, the \( NN \) state has even parity and isospin 1; in the other component, it has odd parity and isospin 0:

\[
| \Phi_{\bar{K}NN} \rangle = | \Phi_{\bar{K}NN}^{(+)}(x_1, x_2), S_{NN} = 0 \rangle \langle [\bar{K}[NN]_1]_{T, T_z} = (1/2, 1/2) \rangle
\]

\[
+ | \Phi_{\bar{K}NN}^{(-)}(x_1, x_2), S_{NN} = 0 \rangle \langle [\bar{K}[NN]_0]_{T, T_z} = (1/2, 1/2) \rangle.
\]

(11)

In the present study, the spatial part of the wave function \( \Phi_{\bar{K}NN}^{(\pm)}(x_1, x_2) \) is expanded with correlated Gaussian basis functions [47], so that the \( NN \) parity is realized correctly in each part:

\[
\Phi_{\bar{K}NN}^{(\pm)}(x_1, x_2) = \sum_i C_i^{(\pm)} G_i^{(\pm)}(x_1, x_2)
\]

with \( G_i^{(\pm)}(x_1, x_2) = G_i(x_1, x_2) \pm G_i(-x_1, x_2) \).

(12)

The variables \( \{C_i^{(\pm)}\} \) are complex-valued parameters that are determined by the diagonalization of the complex-scaled Hamiltonian. Here, the correlated Gaussian function is

\[
G_i(x_1, x_2) = N_i \exp \left[ -\hat{x}^T A_i \hat{x} \right].
\]

(13)
where \( \mathbf{x}^T \) indicates a Jacobi coordinate \((x_1, x_2)\), \( A_i \) is a real-symmetric \(2 \times 2\) matrix, and \( \mathcal{N}_i \) is a normalization factor. We comment that the basis functions, \( G^{(+)}_i(x_1, x_2) \) and \( G^{(-)}_i(x_1, x_2) \), are even- and odd-parity functions for the exchange of two nucleons, respectively.

Resonant states of the \( \bar{K}NN \) system are obtained in the usual way with the complex scaling method. The Hamiltonian for the \( \bar{K}NN \) system, \( \hat{H}_{\bar{K}NN} \), given in Eq. (10), is complex-scaled with a scaling angle \( \theta_P \). The complex-scaled Hamiltonian, \( \hat{H}^{\theta_P}_{\bar{K}NN} = U(\theta_P) \hat{H}_{\bar{K}NN} U^{-1}(\theta_P) \), is diagonalized with the basis functions \( \{ G^{(\pm)}_i(x_1, x_2) \} \) involving spin–isospin wave functions. It is remarked that the scaling angle used to find resonant states of the \( \bar{K}NN \) system is denoted as “\( \theta_P \)” hereafter, to distinguish it from the scaling angle \( \theta_Q \), which is used for the construction of the effective potential.

2.4. Treatment of an energy dependence of the effective potential

The effective \( \bar{K}N \) potential \( U_{\bar{K}NN}^{\text{eff}}(E_{\bar{K}N}) \), which is constructed with ccCSM + Feshbach, has an energy dependence. As shown in Eq. (8), the potential depends on a \( \bar{K}N \) energy \( (E_{\bar{K}N}) \), which means the energy of a \( \bar{K}N \) system included in the total system that we are considering. In other words, to determine the potential strength, we need to know the energy of a \( \bar{K}N \) two-body system in the \( \bar{K}NN \) three-body system. However, the energy of such a subsystem in a total system cannot be determined uniquely in principle. We deal with the energy dependence of the effective potential, following a procedure proposed in a former study \([29,30]\), in which the same issue was considered.

We calculate the so-called kaon’s binding energy \( B_K \) as an auxiliary quantity, which is obtained by subtracting the \( NN \) energy from the \( \bar{K}NN \) energy:

\[
B_K = -\left\{ \Phi_{\bar{K}NN}^{\theta_P} \right| \hat{H}^{\theta_P}_{\bar{K}NN} - \hat{H}^{\theta_P}_{NN} \left| \Phi_{\bar{K}NN}^{\theta_P} \right\},
\]

where the term \( \hat{H}^{\theta_P}_{NN} \) is a complex-scaled Hamiltonian for the \( NN \) system. The \( NN \) Hamiltonian is given as \( \hat{H}_{NN} = p_i^2/2\mu_{NN} + \hat{V}_{NN} \). Using the kaon’s binding energy, we estimate the \( \bar{K}N \) energy with two ansatzes based on two extreme concepts:

\[
\sqrt{s}_{\bar{K}N} = \begin{bmatrix}
M_N + m_K - B_K & \cdots & \text{field picture} \\
M_N + m_K - B_K/2 & \cdots & \text{particle picture}
\end{bmatrix},
\]

where \( M_N \) and \( m_K \) are the nucleon and anti-kaon masses, respectively. In one ansatz, we consider the anti-kaon as a field that carries the kaon’s binding energy (see the left panel of Fig. 1). Since the anti-kaon with the energy \( \omega_K = m_K - B_K \) interacts with each nucleon here, the \( \bar{K}N \) energy \( \sqrt{s}_{\bar{K}N} \) is equal to \( M_N + \omega_K \), namely, \( M_N + m_K - B_K \), with the static approximation applied to nucleons. In the other ansatz, we treat the anti-kaon as a particle. Since the anti-kaon is bound by two nucleons and the kaon’s binding energy is provided by them, the binding energy per \( \bar{K}N \) bond should be half of \( B_K \) (see the right panel of Fig. 1). Therefore, the energy of a \( \bar{K}N \) pair is equal to \( M_N + m_K - B_K/2 \). Hereafter, we denote the first ansatz as the field picture and the latter ansatz as the particle picture. For convenience, we refer to the \( \bar{K}N \) energy measured from the \( \bar{K}N \) threshold: \( E(\bar{K}N) \equiv \sqrt{s}_{\bar{K}N} - M_N - m_K \).

When bound and resonant states of the \( \bar{K}NN \) system are considered with such energy-dependent potentials, the self-consistency for the \( \bar{K}N \) energy has to be taken into account, as explained in the previous study \([29,30]\). The \( \bar{K}N \) energy set in the effective \( \bar{K}N \) potential should finally coincide with that estimated with the obtained wave function by following the above-mentioned ansatz. It is noted that the self-consistency is realized for the complex \( \bar{K}N \) energy in the current study. Here, we treat a resonant \( \bar{K}NN \) state as a Gamow state with the correct boundary condition. Since the pole
energy on the complex-energy plane is explicitly considered, the $\bar{K}N$ energy is treated as a complex value. On the other hand, such a self-consistency is considered only for the real energy in the previous study with a variational approach [29,30], since $\bar{K}NN$ is treated within a bound-state approximation having a real binding energy.

3. Nature of the ccCSM + Feshbach method in the $\bar{K}N–\pi Y$ system

Before the investigation of the $K^−pp$ system, we study the nature of our method in the two-body $\bar{K}N–\pi Y$ system. As the $\bar{K}N(−\pi Y)$ potential $V_{\bar{K}N,\pi Y(I)}$ shown in Eq. (8), which is the origin of the effective $\bar{K}N$ potential, we use a chiral SU(3)-based potential that was proposed in our previous study [38]. Our $\bar{K}N(−\pi Y)$ potential is energy-dependent and is given with a single-range Gaussian form in the coordinate space. A non-relativistic version of this potential, called $NRv2c$, is employed in this section.

3.1. Test calculation of ccCSM + Feshbach projection on a two-body $\bar{K}N–\pi Y$ system

First, we examine the ccCSM + Feshbach projection method on a two-body $\bar{K}N–\pi Y$ system. Table 1 shows the results of scattering and resonance properties of the $I = 0$ channel obtained with both the ccCSM and ccCSM + Feshbach methods. Here, the test calculation is performed with an energy-dependent potential ($NRv2c$). In the ccCSM, both the $\bar{K}N$ and $\pi \Sigma$ components are explicitly treated as the model space ($P$ space), while, in the ccCSM + Feshbach, one component is set as $P$ space and the other component is considered as $Q$ space to be eliminated. In the two-body case, the calculation of the ccCSM + Feshbach via an effective two-body potential is completely equivalent to that of the ccCSM treating all channels explicitly. In principle, the results of both calculations should agree with each other. However, the $Q$-space Green function used in ccCSM + Feshbach is approximately represented with a finite number of Gaussian bases, as explained in Sect. 2.2. Under this approximation, the ccCSM + Feshbach is confirmed to reproduce quite well the ccCSM results of both scattering lengths and resonance properties.

The scattering lengths are calculated with the complex-scaling wave function (CS-WF) method, which is a method to solve scattering problems with the help of ccCSM, as explained in Sect. 2.3 of our previous paper [38]. We plug the effective potential derived with ccCSM + Feshbach into the CS-WF method with a single $\bar{K}N/\pi \Sigma$ channel. It is noted that the scattering amplitudes of these components are confirmed to be identical between the two methods over a wide energy region of $−200$ MeV to $50$ MeV measured from the $\bar{K}N$ threshold. Furthermore, in the $I = 1$ case, in which the $\pi \Lambda$ channel is additionally coupled with the $\bar{K}N$ and $\pi \Sigma$ channels, the ccCSM + Feshbach reproduces all the $\bar{K}N$, $\pi \Sigma$, and $\pi \Lambda$ scattering amplitudes obtained with the ccCSM.
Table 1. Quantities of an $I = 0$ $\bar{K}N$–$\pi\Sigma$ system calculated with ccCSM and ccCSM + Feshbach. The quantities of $a_{\bar{K}N}(i = 0)$ and $a_{\pi\Sigma}(i = 0)$ are $\bar{K}N$ and $\pi\Sigma$ scattering lengths, respectively. The values $(-B(\bar{K}N), -\Gamma/2)$ indicate the resonance position of the system on the complex energy plane. $\sqrt{(r^2)_{\bar{K}N}}$ and $\sqrt{(r^2)_{\pi\Sigma}}$ are the meson–baryon mean distance of each component of the resonant state. Here, energies and lengths are given in units of MeV and fm, respectively.

| $P$ space | ccCSM | ccCSM + Feshbach |
|-----------|--------|-----------------|
| $\bar{K}N$, $\pi\Sigma$ | $a_{\bar{K}N}(i = 0)$ | $a_{\pi\Sigma}(i = 0)$ | $\bar{K}N$ | $\pi\Sigma$ |
| $1.700 + 0.68i$ | $0.724$ | $-1.700 + 0.68i$ | $-1.700 + 0.68i$ | $-$ |
| $B(\bar{K}N)$ | $17.150$ | $17.156$ | $17.415$ |
| $\Gamma/2$ | $16.608$ | $16.611$ | $16.346$ |
| $\sqrt{(r^2)_{\bar{K}N}}$ | $1.280 - 0.403i$ | $1.280 - 0.403i$ | $-$ |
| $\sqrt{(r^2)_{\pi\Sigma}}$ | $0.233 + 0.930i$ | $-$ | $0.234 + 0.932i$ |

In the calculation of a resonance pole, the self-consistency for the $\bar{K}N$ energy needs to be taken into account in both methods. As shown in Table 1, the pole position obtained self-consistently with the ccCSM + Feshbach is found to agree with that obtained with the ccCSM, whichever channel, $\bar{K}N$ or $\pi\Sigma$, is chosen as the $P$ space. The meson–baryon mean distance in the $\bar{K}N$ and $\pi\Sigma$ components also coincides in both methods, when the normalization of each component in the ccCSM is appropriately considered:

$$\langle r^2 \rangle_{MB} = \left\langle \tilde{\phi}_{MB}^\theta | r^2 | \tilde{\phi}_{MB}^\theta \right\rangle / \langle \tilde{\phi}_{MB}^\theta | \tilde{\phi}_{MB}^\theta \rangle,$$

(16)

where $\tilde{\phi}_{MB}^\theta$ is a complex-scaled wave function of the $MB$ component of the resonant state and $r^2$ indicates the complex-scaled operator of the meson–baryon distance.

The resonance pole given in Table 1 is the higher pole of the double pole obtained with our energy-dependent potential. It should be noted that the ccCSM + Feshbach also reproduces well the result of the ccCSM for the other pole, namely, the lower pole [39]. The energies ($B(\bar{K}N), \Gamma/2$) are (37.954, 135.943) MeV and (38.134, 136.169) MeV when the $\bar{K}N$ and $\pi\Sigma$ channels are selected to be the $P$ space in the ccCSM + Feshbach calculation, respectively. These values agree well with ($B(\bar{K}N), \Gamma/2$) = (38.128, 136.166) MeV obtained by the ccCSM.

3.2. Dependence of two kinds of scaling angles, $\theta_P$ and $\theta_Q$

We have made further investigation into the properties of the ccCSM + Feshbach method. As explained in Sect. 2, there are two kinds of scaling angle, $\theta_P$ and $\theta_Q$, in the method. The scaling angle $\theta_Q$ is introduced when we construct an effective potential by eliminating the $Q$-space components with the Feshbach method in Eqs. (6) and (7). The other scaling angle $\theta_P$ is used to find resonance states by means of the complex scaling method for the $P$-space Hamiltonian, which involves the effective potential. Table 2 shows the dependence of the energy of the $\Lambda^*$ resonant state on these angles, where $\Lambda^*$ means the $I = 0$ resonance of the $\bar{K}N$–$\pi\Sigma$ system. In the upper table, we investigate the $\theta_Q$ dependence by fixing the angle $\theta_P$. In principle, the resonance energy should be independent of the angle $\theta_Q$, since the complex-scaled Green function for the $Q$ space $G_Q^{\theta_Q}(E)$ is inversely transformed to be a non-scaled Green function, $G_Q(E) = U^{-1}(\theta_Q) G_Q^{\theta_Q}(E) U(\theta_Q)$, in the construction of the effective potential (see Eq. (7)). Certainly, the resonance energy is confirmed to
Table 2. Dependence of the $\Lambda^*$ eigenenergy on the scaling angles $\theta_P$ and $\theta_Q$. In the upper (lower) table, the scaling angle $\theta_Q (\theta_P)$ is varied with $\theta_P (\theta_Q)$ fixed to 30°. “NF” means that no solutions are found below the $\bar{K}N$ threshold. The NRv2 potential ($f_\pi = 110$) is employed. Energies ($B(\bar{K}N)$ and $\Gamma/2$) are given in units of MeV. The unit of the scaling angles is degrees.

| $\theta_P$ = 30 | $\theta_Q$ | <5 | 10 | 15 | 20 | 25 | 30 |
|----------------|-----------|----|----|----|----|----|----|
| $B(\bar{K}N)$ | NF        |    |    |    |    |    |    |
| $\Gamma/2$    | NF        | 17.0982 | 17.1535 | 17.1562 | 17.1563 | 17.1563 |

| $\theta_P$ = 0 | $\theta_Q$ = 30 |
|----------------|------------------|
| $B(\bar{K}N)$ | 17.1558          |
| $\Gamma/2$    | 16.6128          |

be stable for $\theta_Q > 15^\circ$. However, around $\theta_Q = 15^\circ$ the resonance energy becomes unstable, and then cannot be obtained for small angles $\theta_Q < 5^\circ$. We consider that this is due to insufficient description of the $Q$-space Green function. At such small scaling angles, since the extended closure relation is not well approximated with finite numbers of Gaussian basis functions, the $Q$-space Green function is not correctly represented [37]. Also, in the former study of the complex scaling method where the level density was analyzed [47], it is shown that the Green function is stably described with a basis function expansion when the scaling angle is chosen to be sufficiently large. On the other hand, we check the $\theta_P$ dependence in the lower table where $\theta_Q$ is fixed. It is confirmed that the resonant energy is completely stable for the scaling angle $\theta_P$. Even at $\theta_P = 0^\circ$, namely, no scaling, the same resonance energy is obtained. Thus, once the Green function for the $Q$ space is well represented with a sufficiently large scaling angle $\theta_P$, we can obtain resonant states within $P$ space correctly, using any scaling angle $\theta_P$.

3.3. Treatment of a complex effective potential

Here, we examine a treatment of an effective potential. When the channels energetically lower than the specified state are eliminated as $Q$ space in the Feshbach method, the effective potential for $P$ space is in general a complex potential. In our case, we consider the state located between the $\bar{K}N$ and $\pi Y$ threshold energies. Therefore, the effective potential for the $\bar{K}N$ channel should be a complex potential, when the lower channels, $\pi Y$, are treated as $Q$ space. In this article, such a complex potential is treated directly as it is, and the self-consistency for the energy is also considered with a complex energy. Table 3 shows the binding energy and decay width of the $\Lambda^*$ resonant state obtained with such a full treatment of the complex potential (denoted as “(Full)”), where two versions of our energy-dependent potential and several $f_\pi$ values are examined. On the other hand, so far, the imaginary part of the complex potential has often been treated perturbatively [19,20,29–31,48].

We estimate the effect of the perturbative treatment of the imaginary potential within our method as follows. First, we construct the effective potential $U_{\bar{K}N(I=0)}^{\text{eff}}(E_{\bar{K}N})$ by the Feshbach method, as explained in Sect. 2.2. With only the real part of the effective potential, we construct a Hamiltonian for the $\bar{K}N$ system: $\hat{H}'_{\bar{K}N} = \hat{T}_{\bar{K}N} + \text{Re} U_{\bar{K}N(I=0)}^{\text{eff}}(E_{\bar{K}N})$, where $\hat{T}_{\bar{K}N}$ means the operator of $\bar{K}N$ relative kinetic energy. We diagonalize the Hamiltonian $\hat{H}'_{\bar{K}N}$ with Gaussian basis functions as usual, namely, without the complex scaling. Since the eigenenergies of the Hamiltonian
Table 3. Perturbative treatment of the imaginary part of the effective potential. “(Full)” indicates the full treatment of the complex potential, while “(Perturb)” indicates the perturbative treatment of the imaginary part of the complex potential. The two potentials, NRv1 and NRv2, are examined with a parameter $f_{\pi}$ varied from 90 to 120 MeV. Energies are given in units of MeV.

| Potential | NRv2 | NRv2 |
|-----------|------|------|
| $f_{\pi}$ |      |      |
| 90        | 15.1 | 15.2 |
| 100       | 17.0 | 17.7 |
| 110       | 17.1 | 18.4 |
| 120       | 16.6 | 18.1 |
| (Full)    |      |      |
| $B(\bar{K}N)$ | 23.1 | 26.0 |
| $\Gamma/2$ | 19.8 | 23.1 |
| (Perturb) |      |      |
| $B(\bar{K}N)$ | 19.1 | 21.2 |
| $\Gamma/2$ | 18.9 | 20.1 |

$\hat{H}_\bar{K}N$ are real values, we consider the self-consistency for the $\bar{K}N$ energy within real values. In other words, we set a real-valued energy in the effective potential. After we obtain a self-consistent solution with a binding energy $B(\bar{K}N)$, we estimate the half decay width $\Gamma/2$ by calculating the expectation value of the imaginary potential $\text{Im}\hat{U}^{\text{eff}}_{\bar{K}N(I=0)}$ with the eigen wave function.

The result of the perturbative treatment of the imaginary potential is shown in Table 3 (denoted as “(Perturb)”). Compared with the result of full treatment, as mentioned before, it is found that, when the decay width is calculated to be small with the full treatment, the perturbative treatment gives a binding energy similar to that of the full treatment. However, in the case in which a large decay width is obtained in the full treatment, there is a large difference between the two treatments. In particular, in such a case the perturbative treatment tends to give a large binding energy compared with the full treatment. Therefore, the imaginary potential is found to give a repulsive contribution to the binding energy when it is included explicitly in the calculation.

4. Results of the $K^- pp$ system with the ccCSM + Feshbach method

In this section, the results of the three-body system $K^- pp$ with the ccCSM + Feshbach method are shown. In the three-body calculation, the central part of the Av18 potential [49] is employed as an $NN$ potential $\hat{V}_{NN}$, which appears in Eq. (10). As the $\bar{K}N(-\pi Y)$ potential $V_{\bar{K}N,\pi Y(I)}$ in Eq. (8), an energy-independent potential [15,16] is used for a test calculation, and two kinds of non-relativistic versions of our energy-dependent potential, called $NRv1c$ and $NRv2c$ [38], are examined.

We should comment on the applicability of the ccCSM + Feshbach method. As explained in Sect. 2.2, the effective potential generated with the present method in Eq. (8) generally has an energy dependence due to the channel elimination. In principle, such an energy dependence of the potential violates the three-body unitarity, when it is applied to the three-body $\bar{K}NN-\pi YN$ coupled-channel system. However, we expect that the $\bar{K}NN$ calculation with the ccCSM + Feshbach method is a good approximation with small unitarity violation, when the resonance pole is located close to the $\bar{K}NN$ threshold.

We remark on the scaling angles $\theta_P$ and $\theta_Q$. As shown in the previous section, the result is found to be independent of the angle $\theta_P$ in the two-body case, if the angle $\theta_Q$ is sufficiently large to represent the $Q$-space Green function. Therefore, in the three-body calculation mentioned hereafter, we take...
Fig. 2. Eigenvalue distribution of the $\bar{K}NN$ system on the complex energy plane, which is calculated with the ccCSM + Feshbach method using a phenomenological potential [15,16]. The $\bar{K}N$ energy is fixed at $\Lambda(1405)$. “$\Lambda^*$” means the $\Lambda(1405)$ resonance. The horizontal and vertical axes correspond to the real and imaginary parts of the complex $\bar{K}NN$ energy “$E(\bar{K}NN)$”, which is measured from the $\bar{K} + N + N$ threshold energy, respectively. The energy is given in units of MeV. The scaling angle $\theta$ is taken to be 25°.

a common angle $\theta$ for $\theta_P$ and $\theta_Q$: $\theta_P = \theta_Q = \theta$. Similarly to the two-body calculation, the results of the three-body calculation are confirmed to be independent of the scaling angle $\theta$.

4.1. Comparison of the “field picture” and “particle picture” in an energy-independent potential case

To consider the three-body system of $K^-pp$, we investigate how the self-consistency for the $\bar{K}N$ energy is accomplished with field and particle pictures, which are explained in Sect. 2.4. For simplicity, an energy-independent $\bar{K}N(\pi Y)$ potential, which is phenomenologically constructed [15,16], is employed here.

First, Fig. 2 shows the distribution of complex eigenvalues obtained with the ccCSM + Feshbach method, when the $\bar{K}N$ energy is fixed at that of $\Lambda(1405)$. In this condition, the $\bar{K}N$ energy set in the $\bar{K}NN$ effective potential is not consistent for the three-body $\bar{K}NN$ system, but it is consistent for the $I = 0$ $\bar{K}N$ two-body system of $\Lambda(1405)$. In the figure, the origin corresponds to the $\bar{K}N-N$ three-body threshold. Since eigenvalues of scattering continuum states are known to appear along the so-called $2\theta$ line in the complex scaling method, the eigenvalues along a line running from the origin indicate the $\bar{K}N-N$ scattering continuum states. There is another line, which starts from $E(\bar{K}NN) = (-27.7, -20.4)$ MeV (marked with a blue-dashed circle in the figure). This energy is almost equal to the $\Lambda(1405)$ energy of $E(\bar{K}N) = (-28.2, -20.1)$ MeV, which is set in the effective

2 As shown in Eq. (8), the complex-scaling operator $U(\theta_Q)$ remains in the effective $\bar{K}N$ potential $U^{\text{eff}}_{\bar{K}NN(I)}$. When the Hamiltonian for $\bar{K}NN$, $\hat{H}_{\bar{K}NN}$, given in Eq. (10), is complex-scaled with the operator $U(\theta_P)$, the effective potential is also transformed as $U(\theta_P)U_{\bar{K}NN(I)}^{\text{eff}}U^{-1}(\theta_P)$. If both the scaling angles $\theta_P$ and $\theta_Q$ are chosen to be equal, as mentioned above, the operator $U(\theta_Q)$ included in the effective potential is canceled out by the operator $U(\theta_P)$. With such a choice of the scaling angles, we can calculate matrix elements of the complex-scaled potential with Gaussian basis functions as usual. Otherwise, since an operator like $U(\theta_Q)U^{-1}(\theta_P) = U(\theta_Q - \theta_P)$ remains in the effective potential, we need to consider the transformation of the basis functions with that operator. In other words, setting $\theta_Q = \theta_P$, we are free from such transformation of the basis functions.
The $\bar{K}N$ potential. Therefore, eigenvalues along this line indicate $\Lambda(1405)$–$N$ scattering continuum states. There is an eigenvalue isolated from the two energy lines mentioned above (marked with a red-dashed circle in the figure). This state corresponds to a $\bar{K}NN$ resonance. In the case that the $\bar{K}N$ energy is fixed to $\Lambda(1405)$, the $\bar{K}NN$ resonance is obtained to be $E(\bar{K}NN) = (-44.5, -28.7)$ MeV. Thus, the $\bar{K}NN$ resonance can be identified with the ccCSM + Feshbach method.

Next, we calculate the pole energy of the $\bar{K}NN$ resonance, taking into account the self-consistency for the $\bar{K}N$ energy in the $\bar{K}NN$ three-body system. As explained in Sect. 2.4, we consider that a self-consistent solution is obtained when the $\bar{K}N$ energy set in the effective potential ($E(\bar{K}N)_{\text{In}}$) coincides with that calculated with the obtained wave function ($E(\bar{K}N)_{\text{Cal}}$). Figure 3 is a contour plot of the difference between the inputted and obtained $\bar{K}N$ energies, $|E(\bar{K}N)_{\text{Cal}} - E(\bar{K}N)_{\text{In}}|$, at each inputted complex $\bar{K}N$ energy. In the field-picture case, as shown in the left panel, three self-consistent solutions are found at $E(\bar{K}N)_{\text{In}} = (-79, -22), (-65, -39)$, and $(-43, -65)$ MeV, which are marked with star, diamond, and cross symbols in the figure, respectively. However, when the stability of the solution for the scaling angle $\theta$ is examined, it is found that only the solution with $E(\bar{K}N)_{\text{In}} = (-79, -22)$ MeV is stable; the others are unstable. On the other hand, in the particle-picture case, as shown in the right panel of Fig. 3, a single self-consistent solution is prominently found at $E(\bar{K}N)_{\text{In}} = (-38, -18)$ MeV (marked with a star symbol in the figure) and it is confirmed to be stable for the $\theta$ variation.

Details of the self-consistent solutions with the ccCSM + Feshbach method using the two pictures are given in Table 4. The binding energy of $\bar{K}NN$ ($B(\bar{K}NN)$), which is a real part of the resonance-pole energy, is not so different between the two pictures. However, the half decay width ($\Gamma/2$), which is an imaginary part of the resonance-pole energy, is very different between them. The field picture gives a decay width half as large as that of the particle picture. The spatial configuration is almost the same in these pictures, as indicated by the mean distance between two nucleons ($R(NN)$) and that between the anti-kaon and the center of mass of two nucleons ($R(\bar{K}–[NN])$).

In the last column of the table, the result of an earlier study with the ATMS$^3$ method using the same $\bar{K}N$ potential [28] is listed, for comparison with the present result. Between the two ansatzes,

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Footnote 3: This is a variational method, called amalgamation of two-body correlations into multiple scattering process. See the reference [26] therein.
Table 4. Self-consistent solutions with two ansatzes, “field picture” and “particle picture”. The $\bar{K}N$ energy of the obtained resonant state is given as $E(\bar{K}N)$, and “Re $E(\bar{K}N)$” and “Im $E(\bar{K}N)$” are its real and imaginary parts, respectively. Values with parentheses mean the $\bar{K}N$ energy set in the effective $\bar{K}N$ potential. The binding energy and half decay width are given as “$B(\bar{K}NN)$” and “$\Gamma/2$”, respectively. All energies are given in units of MeV. “$R(\bar{K}–[NN])$” means that between the anti-kaon and the center of mass of two nucleons. These lengths are given in fm. The last column lists a result from a different method, “ATMS”, which is quoted from Ref. [28].

| Method | ccCSM + Feshbach | ATMS [28] |
|--------|------------------|-----------|
| $NN$ potential | Av18 (Central) | G3RS $^\dagger$ |
| Ansatz | Field pict. | Particle pict. | |
| Re $E(\bar{K}N)$ | $-78.8$ ($-79$) | $-37.7$ ($-38$) | $-$ |
| Im $E(\bar{K}N)$ | $-22.1$ ($-22$) | $-18.0$ ($-18$) | $-$ |
| $B(\bar{K}NN)$ | 48.9 | 45.8 | 48 |
| $\Gamma/2$ | 16.6 | 27.2 | 30.5 |
| $R(\bar{K}–[NN])$ | $1.89 - 0.19i$ | $1.88 - 0.29i$ | 1.90 |
| $R(\bar{K}–[NN])$ | $1.27 - 0.14i$ | $1.25 - 0.21i$ | 1.35 |

$^\dagger$The $^1E$ potential is commonly applied to the $^1O$ state. See the text.

deepener功力the particle picture apparently provides a decay width close to that of the ATMS result. Note that they use a different $NN$ potential from the Av18 central potential that we use in the current study. In addition, they commonly apply the $^1E$ channel of the $NN$ potential to the $^1O$ state. Since the $^1E$ potential is more attractive than the $^1O$ potential, their calculation is expected to give slightly deeper binding than our calculation. Taking into account such a difference on $NN$ potentials, the particle picture is considered to give a binding energy consistent with the ATMS result rather than the field picture. Thus, in the case of an energy-independent potential [15,16], our calculation of the ccCSM + Feshbach method with the particle picture is found to give similar results to the former study with the ATMS method.

4.2. The three-body $K^- pp$ system calculated with chiral SU(3)-based potentials

In this section, we investigate the $K^- pp$ system with the present method using a chiral SU(3)-based $\bar{K}N(–\pi Y)$ potential. Here, two versions of non-relativistic $\bar{K}N$ potentials, NRv1c and NRv2c, are employed. It should be noted that these potentials themselves have an energy dependence due to the chiral dynamics. In other words, the original coupled-channel potentials are already an energy-dependent potential, before they are converted to effective single-channel potentials that involve an energy dependence due to the channel elimination by the Feshbach projection. We comment on the difference of the energy dependence between these potentials. In the NRv2c potential, the energy dependence is completely attributed to the chiral dynamics. On the other hand, in the NRv1c potential, another energy dependence is additionally involved, which comes from the so-called flux factor that gives weak energy dependence (see Eqs. (7) and (8) in Ref. [38]).

Table 5 is the summary of the present calculation, which gives the binding energy and half decay width of the $\bar{K}NN$ resonance. In Fig. 4 the pole position of the $\bar{K}NN$ resonance, ($-B(\bar{K}NN)$, $-\Gamma/2$), is depicted on the complex energy plane, when a parameter $f_\pi$ in the potentials is varied from 90 to 120 MeV. The left panel is the result obtained with the NRv2c potential. From the figure, it is found that $\bar{K}NN$ is bound more deeply and the decay width becomes wider as the parameter $f_\pi$.
Table 5. Summary of self-consistent solutions of the $\bar{K}NN$ system with chiral SU(3)-based potentials, NRv1c and NRv2c. “NF” means that no $\theta$-stable solutions are found. All energies are given in units of MeV.

| $\bar{K}N$ pot. | NRv2c | NRv1c |
|------------------|-------|-------|
| $f_\pi$          | 90    | 100   | 110  | 120  | 90    | 100   | 110  | 120  |
| Field pict.      |       |       |      |      |       |       |      |      |
| $B(\bar{K}NN)$  | NF    | 32.2  | 25.6 | 21.2 | NF    | 42.1  | 33.2 | 27.5 |
| $\Gamma/2$      | NF    | 16.1  | 11.6 | 9.0  | NF    | 16.1  | 13.3 | 10.7 |
| Particle pict.   |       |       |      |      |       |       |      |      |
| $B(\bar{K}NN)$  | 30.4  | 29.9  | 27.3 | 24.7 | 32.9  | 33.6  | 31.0 | 28.3 |
| $\Gamma/2$      | 31.7  | 24.2  | 18.9 | 15.2 | 36.8  | 28.5  | 22.1 | 17.7 |

Fig. 4. Pole energy of the self-consistent solution of the $\bar{K}NN$ resonance. Two versions of the non-relativistic chiral SU(3)-based potential are employed: NRv2c potential (left) and NRv1c potential (right). A parameter $f_\pi$ of the potentials is varied from 90 to 120 MeV. The dashed black line with filled circles and dashed red line with filled diamonds indicate the field and particle pictures, respectively. The result obtained with the “$\Lambda^*$ fixed” ansatz is also depicted by a blue-dotted line with asterisks.

decreases. However, the binding energy is not so large; it is 32 MeV at most. The binding energy is not so dependent on the pictures for the $\bar{K}N$-energy self-consistency. On the other hand, the decay width depends strongly on the pictures. Similarly to the case of an energy-independent potential, as mentioned in the previous section, the field picture gives nearly half of the decay width, compared with the particle picture. The result of the NRv1c potential has a similar tendency to that of the NRv2c potential. In this potential, the binding energy is slightly larger compared with the NRv2c case, and it amounts to 42 MeV at $f_\pi = 100$ MeV with the field picture. We consider that this is because the NRv1c potential is more attractive than the NRv2c potential, since the scattering amplitudes of the NRv1c potential show a more attractive nature in the $\bar{K}N$ subthreshold region than those of the NRv2c potential, as shown in Ref. [38]. It should be noted that we cannot find any self-consistent solution in either potential that is stable for the $\theta$ variation at $f_\pi = 90$ MeV with the field picture. As a reference, we also show the result when the $\bar{K}N$ energy set in the potential is fixed to the energy of the $I = 0 \bar{K}N$ resonance (“$\Lambda^*$ fixed” in the figure). The result with this ansatz is quite similar to that with the particle picture. The binding energy $B(\bar{K}NN)$ and half decay width $\Gamma/2$ are summarized as

$$
(B(\bar{K}NN), \Gamma/2) = \begin{cases} 
(21.2–32.2, & 9.0–16.1) \quad \cdots \text{field picture} \\
(24.7–30.4, & 15.2–31.7) \quad \cdots \text{particle picture}
\end{cases}
$$

(17)
in the case of NRv2c, and

\[
(B(\bar{K}NN), \Gamma/2) = \begin{cases} 
(27.5-42.1, \quad 10.7-16.1) & \text{field picture} \\
(28.3-33.6, \quad 17.7-36.8) & \text{particle picture}
\end{cases}
\]

(18)

in the case of NRv1c.

We have investigated the structure of the obtained $\bar{K}NN$ resonance. Table 6 shows details of the self-consistent solution of $\bar{K}NN$ calculated with the NRv2c potential with $f_\pi = 110$ MeV, as a typical result. For the spatial configuration, several kinds of mean distances are given in the table. The mean distance between two nucleons ($R(NN)$) and that between the anti-kaon and the center of mass of two nucleons ($R(\bar{K}–[NN])$) are calculated with the obtained complex-scaled wave function of the $\bar{K}NN$ resonance $|\Phi^\theta_{\bar{K}NN}\rangle$ as

\[
R(NN)^2 \equiv \langle \Phi^\theta_{\bar{K}NN} | \hat{x}_1^2 | \Phi^\theta_{\bar{K}NN} \rangle, \quad R(\bar{K}–[NN])^2 \equiv \langle \Phi^\theta_{\bar{K}NN} | \hat{x}_2^2 | \Phi^\theta_{\bar{K}NN} \rangle,
\]

(19)

where $\hat{x}_1, \theta$ and $\hat{x}_2, \theta$ indicate Jacobi-coordinate operators that are complex-scaled. The mean distance of a $\bar{K}N$ pair with isospin $I$ ($R(\bar{K}N, I)$) is calculated as

\[
R(\bar{K}N, I)^2 \equiv \langle \Phi^\theta_{\bar{K}NN} | \hat{P}_{\bar{K}N(I)}^2 | \Phi^\theta_{\bar{K}NN} \rangle / \langle \Phi^\theta_{\bar{K}NN} | \hat{P}_{\bar{K}N(I)} | \Phi^\theta_{\bar{K}NN} \rangle,
\]

(20)

where $\hat{P}_{\bar{K}N(I)}$ is a $\bar{K}N$ isospin projector. It is noted that the two mean distances are obtained independently of $\theta$ [36–39]. In the complex scaling method, expectation values of distances are necessarily complex-valued since resonance states are treated as Gamow states. However, we refer to such complex-valued distances because we expect that they are useful guides for the spatial configuration of the resonant states, especially when the imaginary part of them is small compared with the real part.

As a result of the calculation of these mean distances, it is found that there is not so large a difference between the results of the two potentials. In both cases, the imaginary part of all mean distances is small compared to the real part. When we see the real part, the mean distance is about 2.2 fm. For the comparison, the result of an earlier study with a variational calculation using a chiral SU(3)-based $\bar{K}N$ potential [30] is shown in the last column in the table. The $NN$ mean distance of the present study is found to be equal to that of the variational calculation. As mentioned in Ref. [30], this $NN$ distance is almost identical to the mean distance between two nucleons in nuclear matter with normal density. As for the $\bar{K}N$ distance, the mean distance for the $I = 0$ component is smaller than that for the $I = 1$ component. This is due to the strong $\bar{K}N$ attraction in the $I = 0$ channel. The variational calculation gives a similar result. However, the present values of the $\bar{K}N$ distance are smaller than those of the variational calculation. The same tendency has also been found in our previous study of the $\Lambda(1405)$ resonance treated as a two-body system of $I = 0 \bar{K}N–\pi \Sigma$ [38]. We consider that such a difference is caused by the treatment of the $\bar{K}NN$ resonance: In the present study it is treated as a Gamow state, whereas it is approximately treated as a bound state in the variational study. Incidentally, compared with the in-vacuum case, where the $I = 0 \bar{K}N$ pair forms a $\Lambda^*$ resonance, the mean distance of the $I = 0 \bar{K}N$ pair is slightly larger in the case of the $\bar{K}NN$ resonance. We consider that such a small elongation is due to the attraction from the other nucleon.

As mentioned in the introduction, the SIDDHARTA Collaboration reported a precise value of the $1s$-level energy shift of a kaonic hydrogen atom [13,14]. Their result gives a strong constraint on the $K^–p$ scattering length. Here, we consider the $K^–pp$ system, taking into account the SIDDHARTA data. We have constructed $\bar{K}N(–\pi Y)$ potentials similarly to the NRv2c potential that is mainly used
in the present study. Combining the $K^- p$ scattering length deduced from the SIDDHARTA data with the $I = 0/I = 1 \tilde{K}N$ scattering length by Martin’s analysis, we have constructed two potentials: One is constrained with the SIDDHARTA $K^- p$ and Martin’s $I = 1 \tilde{K}N$ scattering lengths (denoted as NRv2c-SM1), and the other is constrained with the SIDDHARTA $K^- p$ and Martin’s $I = 0 \tilde{K}N$ scattering lengths (denoted as NRv2c-SM0). A detailed explanation of these potentials is given in Appendix A. We have carried out a ccCSM + Feshbach calculation of the $K^- pp$ system with these potentials. The obtained result is summarized in Table 7. In the case of the NRv2c-SM1, the binding energy of $K^- pp$ becomes slightly smaller and the decay width increases, compared with the NRv2c. On the other hand, the NRv2c-SM0 gives almost the same result as the NRv2c. These results can be understood with the properties of these potentials. As explained in Appendix A, the NRv2c-SM1, in particular its $I = 0$ part that gives a dominant contribution in $K^- pp$, is found to be slightly less attractive and more absorptive than the NRv2c. The NRv2c-SM0 has quite a similar nature to the NRv2c. Thus, both the NRv2c-SM1 and NRv2c-SM0 potentials result in similar binding energies and decay widths to those of the NRv2c potential. Therefore, we consider that the nature of the $K^- pp$ system does not change so much in our treatment, even if the SIDDHARTA data are taken into account.

4.3. $NN/\tilde{K}N$ correlation density in $K^- pp$ with the complex scaling method

We investigate the spatial configuration of the $\tilde{K}NN$ resonance in more detail. Similarly to a former study with a variational approach [30], we calculate the correlation density for $NN$ and $\tilde{K}N$ pairs with the CSM wave function of the $\tilde{K}NN$ resonance to visualize its structure.

We give a brief explanation of the calculation of such densities, since they have to be calculated carefully in the complex scaling method. Here, we consider the case of the $NN$ correlation density as
Table 7. Result of the $K^-pp$ calculation considering the SIDDHARTA data. NRv2c-SM1 and NRv2c-SM0 are the potentials where the SIDDHARTA result is taken into account, while NRv2c is the standard potential in the present study. $f_\pi = 110 \text{ MeV}$. The binding energy $B(KNN)$ and half decay width $\Gamma/2$ are shown as $(B(KNN), \Gamma/2)$ in the table. The energy is given in units of MeV.

|                  | NRv2c-SM1  | NRv2c-SM0  | NRv2c     |
|------------------|------------|------------|-----------|
| Field pict.      | (23.2, 15.5) | (26.2, 11.6) | (25.6, 11.6) |
| Particle pict.   | (20.6, 23.6) | (28.1, 19.2) | (27.3, 18.9) |

Fig. 5. $NN$ correlation density $\rho_N(d)$ in the $KNN$ resonance obtained with the NRv2c potential ($f_\pi = 110 \text{ MeV}$) and the particle picture. The variable $d$ means the $NN$ relative distance. The real and imaginary parts of $\rho_N(d)$ are drawn with solid black and dashed red lines, respectively. The density is normalized to unity. Length and density are given in units of fm and fm$^{-3}$, respectively. The scaling angle is set at 15$^\circ$.

As explained above, we calculate the $NN$ and $\bar{K}N$ correlation densities with the obtained CSM wave function of the $\bar{K}NN$ resonance. Here, a typical result of the NRv2c potential ($f_\pi = 110 \text{ MeV}$) with the particle picture is displayed in Figs. 5–7. The scaling angle is set at a smaller angle of 15$^\circ$ for the calculation of those densities. The pole position of the resonance is confirmed not to differ so much when the scaling angle is changed to 15$^\circ$ from 30$^\circ$ at which we have calculated so far: $(-B(\bar{K}NN), -\Gamma/2)$ are $(-27.4, -19.4)\text{ MeV}$ at $\theta = 15^\circ$, while they are $(-27.3, -18.9)\text{ MeV}$ at $\theta = 30^\circ$. 

\begin{equation}
\rho_N(d) = \int dx_1 dx_2 \Phi^\theta_{\bar{K}NN}(x_1, x_2) \delta^3(x_1 e^{i\theta} - d) \Phi^\theta_{\bar{K}NN}(x_1, x_2) = e^{-3i\theta} \int dx_2 \Phi^\theta_{\bar{K}NN}(d e^{-i\theta}, x_2)^2. \tag{21}\end{equation}

In the same way, the isospin-separated $\bar{K}N$ correlation density $\rho_{\bar{K}N(1)}(d)$ can be calculated, beginning with the operator $\hat{\rho}_{\bar{K}N(1)}(d) \equiv \delta^3(\hat{r}_{\bar{K}N} - d) \hat{P}_{\bar{K}N(1)}$, where $\hat{r}_{\bar{K}N}$ is a $\bar{K}N$ relative coordinate operator and $\hat{P}_{\bar{K}N(1)}$ is a $\bar{K}N$ isospin projector.
Fig. 6. Isospin-separated $\bar{K}N$ correlation densities $\rho_{\bar{K}N(I)}(d)$ in the $\bar{K}NN$ resonance obtained with the same condition as Fig. 5. The variable $d$ means the $\bar{K}N$ relative distance. The densities of each isospin component are normalized to unity. The correlation densities of the $I = 0$ and $I = 1$ components are depicted in blue and red colors, respectively. The real and imaginary parts of the density with each isospin component are drawn with solid and dashed lines, respectively. The right panel depicts the correlation densities multiplied by $d^2$ and the unit of the vertical axis is fm$^{-1}$.

Fig. 7. Comparison of $I = 0 \bar{K}N$ correlation density in the $\bar{K}NN$ resonance and that of the $\Lambda^*$ resonance. The $I = 0 \bar{K}N$ correlation density of the $\Lambda^*$ resonance is shown with green diamonds, together with that in the $\bar{K}NN$ resonance shown in Fig. 6. The densities of both resonances are normalized to unity for display.

Figure 5 shows the $NN$ correlation density. In a short distance, both the real and imaginary parts of the density are confirmed as being suppressed due to the strong repulsive core of the Av18 $NN$ potential. In Fig. 6, the $\bar{K}N$ correlation densities for the isospin 0 and 1 components are displayed. Both densities are normalized to unity for comparison. In the figure, we can directly confirm the consequence of the strong $\bar{K}N$ attraction in the $I = 0$ channel, as mentioned in the previous section. In other words, the $I = 0 \bar{K}N$ component is found to distribute more compactly compared with the $I = 1$ component. Furthermore, the $I = 0 \bar{K}N$ correlation density is compared with the density of the $\Lambda^*$ resonance. The $\Lambda^*$ density is the $\bar{K}N$ density of the two-body $\bar{K}N$ system with $I = 0$ that is calculated with the ccCSM + Feshbach method using the same potential and the same scaling angle. As displayed in Fig. 7, both densities look rather similar to each other. Therefore, the present study with the ccCSM + Feshbach method also indicates that the $\Lambda^*$ resonance still survives in the $K^-pp$ resonance, as pointed out in the former study with a variational approach [30].
5. Summary and future plans

We have proposed a new method where the coupled-channel complex scaling method and the Feshbach projection are combined, and applied it (ccCSM + Feshbach method) to a three-body kaonic nucleus $K^- pp$. Originally, $K^- pp$ was a $\bar{K}NN - \pi \Sigma N - \pi \Lambda N$ coupled-channel system, but it has been effectively reduced to a single-channel problem of $\bar{K}NN$ by channel elimination in the method. Recall that, since the ccCSM + Feshbach method is based on the complex scaling method, the $\bar{K}NN$ resonance has been regarded as a Gamow state with the correct boundary condition as a resonance.

In the ccCSM + Feshbach method, the extended closure relation (ECR), which is held in the complex scaling method (CSM), is essentially important. In fact, Green’s function for $Q$-space (outer space of the model space $P$), needed to eliminate the $Q$ space, has been obtained easily from ECR, since the ECR is well realized with the $L^2$ Gaussian basis function only. First, we have tested the ccCSM + Feshbach method in a two-body $\bar{K}N - \pi Y$ coupled-channel system, and confirmed that it completely reproduces the results of a calculation treating all channels explicitly for both scattering and resonant problems.

At the first application of the ccCSM + Feshbach method to the three-body system of $K^- pp$, we have examined an energy-independent $\bar{K}N(-\pi Y)$ potential derived phenomenologically. Even when the original potential is energy-independent, the effective $\bar{K}N$ potential appearing in the ccCSM + Feshbach method has an energy dependence as a result of channel elimination by the Feshbach projection, and the self-consistency for the complex $\bar{K}N$ energy has to be considered when we search for resonance states. We have tested two ansatzes for the self-consistency, the field picture and the particle picture, and successfully obtained a self-consistent solution for each ansatz. We have found that the binding energy of $\bar{K}NN$ does not depend on the ansatz, while the decay width strongly depends on the ansatz: the field picture gives a decay width as small as half that of particle picture. Compared with the result of an earlier study with the ATMS method using the same potential [28], the result of the particle picture is found to be close to the past result: $(B(\bar{K}NN), \Gamma/2) = (45.8, 27.2) \text{ MeV}$.

For more theoretical investigation of the $K^- pp$ system, we have used an energy-dependent $\bar{K}N(-\pi Y)$ potential, which was proposed in our previous work based on the chiral SU(3) theory. Two versions of non-relativistic potentials, NRv1c and NRv2c [38], have been examined. Also, for these energy-dependent potentials, self-consistent solutions of the $\bar{K}NN$ resonance with the ccCSM + Feshbach method have been found. Similarly to the case of the energy-independent potential, the decay width is rather small in the field picture compared with that in the particle picture. In the case of the NRv2c potential, the binding energy and half decay width of the $\bar{K}NN$ resonance are obtained as $(21.2 - 32.2, 9.0 - 16.1) \text{ MeV}$ with the field picture and $(24.7 - 30.4, 15.2 - 31.7) \text{ MeV}$ with the particle picture. The NRv1c potential gives a slightly larger binding energy. In addition, it is confirmed that the $\bar{K}NN$ resonance energy is not so varied, even if the precise value of $K^- p$ scattering length deduced from the SIDDHARTA experiment [13,14] is taken into account.

As for the spatial configuration of the $\bar{K}NN$ resonance, we have calculated $NN$ and $\bar{K}N$ correlation densities from the wave function, carefully following the procedure of the complex scaling method. These densities are useful tools for intuitive understanding of the structure of $\bar{K}NN$ although they are given as complex values in the complex scaling method. The $NN$ correlation density is strongly suppressed at short distances as a result of the $NN$ repulsive core. The $NN$ mean distance is found to be about 2.2 fm with a small imaginary part. This distance is almost equal to the $NN$
distance of nuclear matter with normal density. Concerning the $\bar{K}N$ correlation density, we have reconfirmed the survival of the $\Lambda^*$ resonance ($I = 0 \bar{K}N$ resonance) in the $\bar{K}NN$ resonance, which was pointed out in a previous study with a variational approach [30].

Thus, through the present study with the ccCSM + Feshbach method, we have confirmed that the $K^- pp$ system is shallowly bound with a chiral SU(3)-based energy-dependent potential, as reported in earlier studies employing the same type of $\bar{K}N$ potential [29–32]. However, in the case of the NRv2c potential with the particle picture, we have always obtained another quasi self-consistent solution around the $\bar{K}N$ energy $E(\bar{K}N) \sim \left(-60, -60\right)$ MeV (“quasi self-consistent” means a local minimum for the quantity $|E(\bar{K}N)_{\text{In}} - E(\bar{K}N)_{\text{Cal}}|$, which is an indicator for the self-consistency, as explained in Sect. 4.1). In such quasi self-consistent solutions, a $\bar{K}NN$ resonance appears near the $\pi / \Sigma N$ threshold with large decay width. Since $\Lambda(1405)$ has a double pole structure with the NRv2c potential [39], as many studies with chiral SU(3) models show [8], we think that those quasi-consistent solutions are probably related to the lower pole of $\Lambda(1405)$, while the solutions reported in the previous section must be related to the higher pole. In other words, $K^- pp$ is supposed to have a double pole structure, as does $\Lambda(1405)$, as suggested in an earlier work with the Faddeev–AGS approach [50]. In order to reach a definite conclusion on this issue, we need a more delicate calculation for the deeper pole, since it has a large imaginary part.

In this article, we have successfully obtained the solution of the $K^- pp$ resonance with the ccCSM + Feshbach method. However, since we have eliminated the $\pi YN$ channel by the Feshbach projection, some of the $\pi YN$ dynamics might be lost in the present study. Toward a more decisive conclusion on the $K^- pp$ problem, we will carry out a coupled-channel three-body calculation with explicit $\pi YN$ channels. The coupled-channel calculation will clarify the detailed properties of the $K^- pp$, such as the composition of the $K^- pp$ resonant state. On the experimental side, results of the $K^- pp$ search are going to be reported from two experimental groups at J-PARC (E15 [51,52] and E27 [53]). We hope that these experimental results will provide us with useful information on $K^- pp$.

Since extending the application of the ccCSM + Feshbach method to four-body systems is straightforward, we can investigate rather easily four-body systems such as a kaonic nucleus $K^- pnn$ and a double kaonic nucleus $K^- K^- pp$, which have been investigated with a variational method [31] and the Faddeev–Yakubovsky approach [54]. Generally, the method can be applied to various kinds of mesonic nuclei that involve some decay modes, e.g., mesonic nuclei with $\eta$ [55,56], $\eta'$ [57], $\omega$ mesons, and $D$ mesons in the charm sector [58]. These are interesting systems and fall within the scope of our study with the ccCSM + Feshbach method.

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Appendix A. Chiral SU(3)-based $\bar{K}N(–\pi Y)$ potentials constrained with SIDDHARTA data

In this appendix, we explain the details of our chiral SU(3)-based potential in which the SIDDHARTA data on the kaonic hydrogen atom [13,14] are taken into account.
Table A1. Properties of the NRv2c-SM1 and NRv2c-SM0 potentials \( (f_\pi = 110 \text{ MeV}) \). The upper and lower tables are for the isospin \( I = 0 \) and \( I = 1 \) channels, respectively. The range parameters of the Gaussian form factor of the potential, \( d_{KN,KN}^{(I=0)} \), \( d_{\pi \Sigma,\pi \Sigma}^{(I=0)} \), \( d_{KN,\pi \Lambda}^{(I=0)} \), and \( d_{KN,\pi \Lambda}^{(I=1)} \), are given in units of fm. They are specified in the same way as in Tables 1 and 3 in our previous work [38]. The scattering length \( a_{\bar{K}N(i)} \) is calculated with these potentials in each isospin channel. All lengths are in units of fm. Complex values of \( Z_H \) and \( Z_L \) in the upper table are the complex energies of higher and lower poles of the \( I = 0 \) \( \bar{K}N-\pi \Sigma \) system, respectively. These energies are measured from the \( \bar{K}N \) threshold and are given in units of MeV.

| \( I = 0 \) | \( d_{KN,KN}^{(I=0)} \) | \( d_{\pi \Sigma,\pi \Sigma}^{(I=0)} \) | \( a_{\bar{K}N(i=0)} \) | \( Z_H \) | \( Z_L \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| NRv2c-SM1       | 0.458           | 0.587           | -1.670 + 1.028i | -8.5 - 20.7i | -31.9 - 108.5i |
| NRv2c-SM0       | 0.438           | 0.636           | -1.700 + 0.681i | -17.2 - 16.6i | -39.8 - 137.9i |

| \( I = 1 \) | \( d_{KN,KN}^{(I=1)} \) | \( d_{\pi \Sigma,\pi \Sigma}^{(I=1)} \) | \( a_{\bar{K}N(i=1)} \) |
|-----------------|-----------------|-----------------|-----------------|
| NRv2c-SM1       | 0.458           | 0.587           | 0.444           | 0.635 + 0.60i |
| NRv2c-SM0       | 0.438           | 0.636           | 0.363           | 0.690 + 0.93i |
| NRv2c           | 0.438           | 0.636           | 0.445           | 0.657 + 0.599i |

We can deduce the \( K^- p \) scattering length from the 1s-level energy shift of the kaonic hydrogen atom measured in the SIDDHARTA experiment. With help of the improved Deser–Trueman formula [59], the \( K^- p \) scattering length is obtained to be \( a_{K^-p} = -0.65 + 0.81i \) fm [60, 61]. To determine the parameters of our potential in each isospin channel, we need the \( I = 0 \) and \( I = 1 \) \( \bar{K}N \) scattering lengths individually, although the \( K^- p \) scattering length is given by the average of the two isospin cases of \( I = 0 \) and \( I = 1 \): \( a_{K^-p} = \frac{a_{\bar{K}N(i=0)} + a_{\bar{K}N(i=1)}}{2} \). Here, combining this with the \( \bar{K}N \) scattering length by Martin’s analysis of old data [10], we determine the \( \bar{K}N \) scattering length of each isospin channel from the SIDDHARTA \( K^- p \) scattering length in the following two cases:

| Case      | SIDDHARTA | \( a_{\bar{K}N(i=0)} \) [fm] | \( a_{\bar{K}N(i=1)} \) [fm] |
|-----------|-----------|-----------------|-----------------|
| Case 1    | SIDDHARTA | -0.65 + 0.81i   | 0.37 + 0.60i   |

| Case 2    | SIDDHARTA | Martin         | \( a_{\bar{K}N(i=1)} \) [fm] |
|-----------|-----------|-----------------|-----------------|
| Case 2    | SIDDHARTA | Martin         | 0.40 + 0.94i   |

Note that Martin’s scattering length is given for each isospin channel and has been used to determine our potentials used in this article [38].

Here, we begin with our standard potential, NRv2c with \( f_\pi = 110 \text{ MeV} \). To reproduce the above-mentioned \( \bar{K}N \) scattering lengths, we adjust the range parameters of the NRv2c potential, which are defined in Eq. (8) of Ref. [38]. We denote the potentials that are constrained with the \( \bar{K}N \) scattering lengths of Case 1 and Case 2 in the above table as \( \text{NRv2c-SM1} \) and \( \text{NRv2c-SM0} \), respectively. The range parameters of those potentials and the \( \bar{K}N \) scattering lengths calculated with them are summarized in Table A1. We make two remarks: 1) These potentials reproduce only the imaginary part of the scattering length in the \( I = 1 \) channel, following the guidelines for construction of the NRv2c potential [38]. 2) As for the \( I = 0 \) channel, the \( \text{NRv2c-SM0} \) potential is the same as the \( \text{NRv2c} \) potential, since the range parameters of the \( I = 0 \) channel in both potentials are fixed to reproduce Martin’s \( I = 0 \) value (see Case 2 in the above table).
We have investigated the properties of these potentials. The pole positions of the $I = 0$ $\bar{K}N$ resonances are given in the upper part of Table A1. These potentials have two poles, a higher pole $Z_H$ and a lower pole $Z_L$, in the $I = 0$ channel, similarly to the NRv2c potential [39]. The resonant poles of the NRv2c-SM1 potential are less strongly bound than those of the NRv2c potential. In the $I = 0$ $\bar{K}N$ scattering amplitude, the resonance structure is certainly shifted to the weaker binding side (namely, closer to the $\bar{K}N$ threshold), compared with the NRv2c potential (see the left panel of Fig. A1). In addition, the $I = 0$ $\bar{K}N$ scattering length has a large imaginary part in the case of the NRv2c-SM1 potential. Reflecting this fact, the higher pole of the $I = 0$ resonances, which is strongly coupled to the $\bar{K}N$ channel, also has a large imaginary energy. Incidentally, the NRv2c-SM0 potential shows the same poles and scattering amplitudes as the NRv2c potential, since the $I = 0$ parts of both potentials are constrained by Martin’s value, as explained above. Thus, it is found that our chiral SU(3)-based potential becomes less attractive and more absorptive in the $I = 0$ channel, when the SIDDHARTA result is taken into consideration.

As for the $I = 1$ channel, the NRv2c-SM0 potential has a large imaginary part for the $\bar{K}N$ scattering length, compared with the NRv2c-SM1 and NRv2c potentials, although all three potentials give a similar real part for the $\bar{K}N$ scattering length (lower part of Table A1). Figure A2 depicts the...
\( \bar{K}N, \pi \Sigma, \) and \( \pi \Lambda \) scattering amplitudes with \( I = 1 \). The NRv2c-SM1 potential gives almost the same amplitudes as the NRv2c potential (the amplitudes of both potentials overlap in the figure). The \( \bar{K}N \) amplitude of the NRv2c-SM0 potential is slightly more attractive than that of the other potentials. However, the difference between the scattering amplitudes of the three potentials is tiny. Therefore, the property of our potential in the \( I = 1 \) channel is found not to change so much with the SIDDHARTA result.

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