Experimental Results of the Search for Unitals in the Projective Planes of Order 25

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Abstract
In this paper we present the results from a program developed by the author that finds the unitals of the known 193 projective planes of order 25. There are several planes for which we have not found any unital. One or more than one unitals have been found for most of the planes. The found unitals for a given plane are nonisomorphic each other. There are a few unitals isomorphic to a unital of another plane.

A $t$-$(v; k; \lambda)$ design $D$ is a set $X$ of points together with a family $B$ of $k$-subsets of $X$ called blocks with the property that every $t$ points are contained in exactly $\lambda$ blocks. The design with $t = 2$ is called a block-design. The block-design is symmetric if the role of the points and blocks can be changed and the resulting configuration is still a block-design. A projective plane of order $n$ is a symmetric 2-design with $v = n^2 + n + 1$, $k = n + 1$, $\lambda = 1$. The blocks of such a design are called lines. A unital in a projective plane of order $n = q^2$ is a set $U$ of $q^3 + 1$ points that meet every line in one or $q + 1$ points.

In the case of projective planes of order $n = 25$ we have: $q = 5$, the projective plane is $2 - (651; 26; 1)$ design, the unital is a subset of $5^3 + 1 = 126$ points and every line meets 1 or 6 points from the subset.

Key words: projective plane, design, graph, isomorphism, automorphism, group, stabilizer, exact algorithm, heuristic algorithm, partition, generators, orbits and order of the graph automorphism group.

Article Outline

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1. Introduction

We assume familiarity with the basics of the combinatorial designs (cf., e.g. [1]). A $t$-$(v; k; \lambda)$ design $D$ [1] is a set $X$ of points together with a family $B$ of $k$-subsets of $X$ called blocks with the property that every $t$ points are contained in exactly $\lambda$ blocks. The design with $t = 2$ is called a block-design. The block-design is symmetric if the role of the points and blocks can be changed and the resulting configuration is still a block-design. A projective plane of order $n$ is a symmetric 2-design with $v = n^2 + n + 1$, $k = n + 1$, $\lambda = 1$. The blocks of such a design are called lines. A unital in a projective plane of order $n = q^2$ is a set $U$ of $q^3 + 1$ points that meet every line in one or $q + 1$ points.

In the case of projective planes of order $n = 25$ we have: $q = 5$, the projective plane is $2 - (651; 26; 1)$ design, the unital is a subset of $5^3 + 1 = 126$ points and every line meets 1 or 6 points from the subset.
The experimental results of the algorithm in [5] for the known planes of order 16 [4] are given in [6].

2. Experimental results

In [8] we present the text of the current paper and the experimental results (in Appendix) from a program that finds the unitals in all 193 known projective planes of order 25 from the website [3]. The vertex labels in this websites start from 0, but in our program and results the starting label is 1. The program we use in present paper is based on the algorithm described in [5]. In [5] we describe the algorithm for finding unitals and maximal arcs in projective planes of order 16. In this paper we use the same algorithm but with parameters for a plane of order 25. The algorithm is heuristic – it does not guarantee finding all possible unitals of a given plane. All fond unitals of an unital are nonisomorphic each other. There are planes with no found unitals.

The results for each plane contain values for the following variables:

1. Name of the plane
2. Order of the plane automorphism group
3. Number of the orbits of the plane automorphism group
4. Order of the unital automorphism group
5. Number of the orbits of the unital automorphism group
6. Sizes of the orbits of the unital automorphism group
7. Labels of the vertices of the unital

The information for 1 to 3 is present for each plane and the information for 4 to 7 is present for each found unital.

Example (for the plane A1.HTM, see the text below - from the line with ‘A1.HTM’ : The order of the plane automorphism group is 360000 and the number of its orbits is 8. Then, the results for all 9 found unitals follow. The orders of their automorphism groups are 144, 24, 20, 15, 10, 10, 10.

A1.HTM
ORDER OF THE PLANE AUTOMORPHISM GROUP              360000
NUMBER OF THE ORBITS OF THE PLANE AUTOMORPHISM GTOUP=    8
ORDER OF THE UNITAL AUTOMORPHISM GROUP=                 144
NUMBER OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP =                   3
SIZES OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP=   1-  72   2-  48
3-   6
UNITAL= 6    7    8    9   10   11   30   31   32   34   35   39  181  182  183
184
209
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609
611
615
616
617
618
619
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621
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643
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647
648
651
ORDER OF THE UNITAL AUTOMORPHISM GROUP=                 24
NUMBER OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP =                   8
SIZES OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP=   1-  24   2-  24
3-  24   4-  24   5-  24   6-   4   7-   1   8-   1
UNITAL=


ORDER OF THE UNITAL AUTOMORPHISM GROUP = 20
NUMBER OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP = 8
SIZES OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP = 1- 20 2- 20 3- 20 4- 20 5- 20 6- 20 7- 5 8- 1
UNITAL =

ORDER OF THE UNITAL AUTOMORPHISM GROUP = 15
NUMBER OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP = 10
SIZES OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP = 1- 15 2- 15 3- 15 4- 15 5- 15 6- 15 7- 15 8- 15 9- 5 10- 1
UNITAL =

ORDER OF THE UNITAL AUTOMORPHISM GROUP =
NUMBER OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP =
SIZES OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP =
ORDER OF THE UNITAL AUTOMORPHISM GROUP = 10
NUMBER OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP = 14
SIZES OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP=
1-  10   2-  10   3-  10   4-  10   5-  10   6-  10   7-  10   8-  10   9-  10   10-  10   11-  10   12-  10   13-  5   14-  1
UNITAL=
10   27   31   37   39   43   46   54   56   61   64   67   72   78   85
188
95  100  114  116  118  131  133  147  148  153  155  163  174  182  190
201
209  212  214  216  217  226  234  244  246  254  256  258  260  263
264
283  285  287  294  302  303  306  310  315  320  326  327  343  346  349
350
356  359  363  364  370  373  376  394  398  401  411  413  418  424  425
433  443  448  452  458  463  465  467  470  475  479  495  504  511  516
518
524  530  536  539  541  544  546  549  553  555  564  565  568  570  572
593
595  597  601  603  606  608  610  622  629  630  634  639  645  648
ORDER OF THE UNITAL AUTOMORPHISM GROUP = 10
NUMBER OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP = 14
SIZES OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP=
1-  10   2-  10   3-  10   4-  10   5-  10   6-  10   7-  10   8-  10   9-  10   10-  10   11-  10   12-  10   13-  5   14-  1
UNITAL=
10   27   30   35   36   41   48   57   58   59   69   75   78   81   83
85
87  91  99  103  106  114  117  131  138  145  148  153  158  161  162
171
176  184  188  191  192  197  200  211  216  220  222  223  230  233  251
254
259  268  269  276  278  283  284  291  304  307  309  311  319  322  325
341
359  361  366  379  380  384  388  396  398  403  409  421  422  423  429
441
442  447  449  450  451  457  469  473  474  476  485  488  491  493  499
509
510  512  513  520  528  530  535  539  544  550  551  564  568  574  576
580
581  584  594  595  597  603  609  613  614  615  620  627  642  643
ORDER OF THE UNITAL AUTOMORPHISM GROUP = 10
NUMBER OF THE ORBITS OF THE UNITAL AUTOMORPHISM GROUP = 14
3. Concluding remarks

By our algorithm we have found new unitals in projective plane of order 25, but not all of them.
The following approaches can be used to find more or all unitals: (a) Development of improved algorithms by finding new conditions for pruning the search tree; (b) Transformation of the solution for one plane to solution for another plane (R. Mathon's approach - in private communication); (c) Development of parallel algorithms.

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