Abstract. The purpose of the present paper is to develop the enumerative geometry of dormant $G$-opers for a semisimple algebraic group $G$. In the present paper, we construct a compact moduli stack admitting a perfect obstruction theory by introducing the notion of a dormant faithful twisted $G$-oper (or a “$G$-do’per” for short). Moreover, by means of the resulting virtual fundamental class, we obtain a semisimple CohFT (= cohomological field theory) valued in the $\ell$-adic étale cohomology of the moduli stack classifying pointed stable curves in positive characteristic. This CohFT gives an analogue of the Witten-Kontsevich theorem describing the intersection numbers of psi classes on the moduli stack of $G$-do’pers.

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Introduction

The purpose of the present paper is to develop the enumerative geometry of the moduli of dormant $G$-opers (i.e., $G$-opers with vanishing $p$-curvature) for a semisimple algebraic group $G$ in positive characteristic. The formulations and background knowledge of dormant $G$-opers used in the present paper under the assumption that $G$ is of adjoint type were discussed in the author’s paper [55].

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In the present paper, we generalize the previous work and construct a compact moduli stack admitting a perfect obstruction theory by introducing the notion of a dormant faithful twisted $G$-oper (or a “$G$-do’per” for short) defined on a stacky log-curve. By means of the resulting virtual fundamental class, we obtain, for the cases of some classical types $G$, a semisimple CohFT (= cohomological field theory) valued in the $l$-adic étale cohomology of the moduli stack classifying pointed stable curves in positive characteristic. It, moreover, specifies a 2d TQFT (= 2-dimensional topological quantum field theory), and an explicit description of (the characters of) the corresponding Frobenius algebra allows us to perform a computation for counting problem of $G$-do’pers (i.e., the Verlinde formula for $G$-do’pers). Also, the CohFT gives an analogue of the Witten-Kontsevich theorem describing the intersection numbers of psi classes on the moduli stack of $G$-do’pers. In the rest of this Introduction, we shall provide more detailed discussions, including the content of the present paper.

0.1. Recall that a dormant $G$-oper is, roughly speaking, a $G$-torsor over an algebraic curve in characteristic $p > 0$ equipped with a connection satisfying certain conditions, including the condition that its $p$-curvature vanishes identically. Various properties of dormant PGL$_2$-opers and their moduli were discussed by S. Mochizuki (cf. [40], [41]) in the context of $p$-adic Teichmüller theory. If $G = \text{PGL}_n$ or $\text{SL}_n$ for a general $n$ (but the underlying curve is assumed to be unpointed and smooth over an algebraically closed field), then the study of these objects has been carried out by K. Joshi, S. Ramanan, E. Z. Xia, J. K. Yu, C. Pauly, T. H. Chen, X. Zhu et al. (cf. [20], [21], [22], [12]). As carried out in these references, dormant $G$-opers and their moduli, which are our principal objects, contain diverse aspects and occur naturally in mathematics. A detailed understanding of them in the point of view of enumerative geometry will be directly applied to, say, counting problems of objects or structures in various areas linked to the theory of (dormant) opers in positive characteristic (cf., e.g., [37], [53], and [48], §4).

0.2. As an example, let us explain (in §§0.2-0.3) one aspect of the enumerative geometry of dormant $G$-opers, concerning the algebraic-solution problem of a linear differential equations in positive characteristic. Let $X$ be a connected proper smooth curve over $\overline{\mathbb{F}}_p$ with function field $K$, where $\overline{\mathbb{F}}_p$ denotes the algebraic closure of the field $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. Consider a monic linear homogeneous ordinary differential operator $D$ of order $n > 1$ with regular singularities defined on $X$, which may be expressed locally as follows:

$$D := \frac{d^n}{dx^n} + q_1 \frac{d^{n-1}}{dx^{n-1}} + \cdots + q_{n-1} \frac{d}{dx} + q_n,$$

(1)
where \( q_1, \cdots, q_n \in K \) and \( x \) denotes a local coordinate in \( X \). Since the \( p \)-powers \( K^p \) of elements of \( K \) coincides with the constant field in \( K \) (i.e., the kernel of the universal derivation \( d : K \to \Omega_{K/k} \)), the set of solutions of \( Dy = 0 \) in \( K \) forms a \( K^p \)-vector space. We shall say that the differential equation \( Dy = 0 \) has a full set of algebraic solutions if it has \( n \) solutions in \( K \) which are linearly independent over \( K \) (equivalently, the set of solutions forms a \( K^p \)-vector space of dimension \( n \)). The study of differential equations all of whose solutions are algebraic was originally considered in the complex case, tackled and developed in the 1870s by many mathematicians: H. A. Schwarz (for the hypergeometric equations), L. I. Fuchs, P. Gordan, and C. F. Klein (for the second order equations), C. Jordan (for the \( n \)-th order). We are interested in the analogue in positive characteristic of this classical case, and, in particular, want to know how many differential equations \( Dy = 0 \) (associated with \( D \) as in (1)) admit a full set of algebraic solutions.

Let us consider a special case, i.e., the case of (monic, linear, and homogeneous) second order differential operators on the projective line \( \mathbb{P}_{F_p} := \text{Proj}(F_p[s, t]) \) having at most three regular singular points. The classical theory of Riemann schemes shows that by pulling-back via an automorphism of \( \mathbb{P}_{F_p} \), any such equation is isomorphic to a hypergeometric differential operator

\[
D_{a,b,c} := \frac{d^2}{dx^2} + \left( \frac{c}{x} + \frac{1 - c + a + b}{x - 1} \right) \frac{d}{dx} + \frac{ab}{x(x - 1)}
\]

determined by some triple \( (a, b, c) \in \mathbb{F}_p^3 \) (where \( x := s/t \)). According to [17], §1.6 (or [25], §6.4), the equation \( D_{a,b,c} y = 0 \) has a full set of algebraic solutions if and only if \( (a, b, c) \) lies in \( \mathbb{F}_p^3 \) and either \( \tilde{b} \geq \tilde{c} > \tilde{a} \) or \( \tilde{a} \geq \tilde{c} > \tilde{b} \) is satisfied, where \( (\quad) \) denotes the inverse of the bijective restriction \( \{1, \cdots, p\} \to \mathbb{F}_p \) of the natural quotient \( \mathbb{Z} \to \mathbb{F}_p \). If \( D_{a,b,c} y = 0 \) has a full set of algebraic solutions, then the set \( \{y_{a,b,c}(x), x^{-c}y_{a-c+1,b-c+1,2-c}(x)\} \subseteq \mathbb{F}_p(x) \) forms a basis of the solutions. Here, \( y_{a,b,c}(x) \) denotes a polynomial of \( x \) defined by the following truncated hypergeometric series

\[
y_{a,b,c}(x) := 1 + \frac{a \cdot b}{1 \cdot c} x + \frac{a \cdot (a + 1) \cdot b \cdot (b + 1)}{1 \cdot 2 \cdot c \cdot (c + 1)} x^2 + \cdots,
\]

where we stop the series as soon as the numerator vanishes. In particular, after a straightforward calculation, we see that there exists precisely \( \frac{p^3 - p}{3} \) hypergeometric equations admitting a full set of algebraic solutions.
Next, let us explain the relationship between dormant opers and such differential operators by describing them in terms of connections on vector bundles. To each differential operator \( D \) as in (1), one associate, in a well-known manner, a (n) (integrable) connection on a vector bundle expressed locally as follows:

\[
\nabla = \frac{d}{dx} - \begin{pmatrix}
-q_1 & -q_2 & -q_3 & \cdots & -q_{n-1} & -q_n \\
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0
\end{pmatrix}.
\]

The assignment \( y \mapsto \left( e^{x-1}y, \ldots, \frac{d^n}{dx^n} y \right) \) gives a bijective correspondence between the solutions of the equation \( D y = 0 \) and the horizontal (relative to \( \nabla \)) sections of this vector bundle. A vector bundle equipped with such a connection is nothing but an example of an oper. Indeed, \( \text{PGL}_n \)-opers may be identified with equivalence classes (relative to a certain equivalence relation) of connections as displayed above (cf. [55], Theorem D). The point is that the differential equation \( D y = 0 \) (associated with \( D \) as in (1)) has a full set of algebraic solutions if and only if the corresponding connection has vanishing \( p \)-curvature, i.e., the corresponding \( \text{PGL}_n \)-oper is dormant (cf. [25], (6.0.5), Proposition). Thus, the algebraic-solution problem under consideration reduces to the study of the moduli of dormant \( \text{PGL}_n \)-opers.

Let us go back to the hypergeometric case. Denote by \( \mathcal{E}_{a,b,c}^\bullet \) the dormant \( \text{PGL}_2 \)-oper on \( \mathbb{P}^*_p := (\mathbb{P}_p, \{0, 1, \infty\}) \) (i.e., the projective line with three marked points determined by 0, 1, and \( \infty \)) corresponding to \( D_{a,b,c} \). One verifies from observing the Riemann schemes of hypergeometric equations that \( \mathcal{E}_{a,b,c}^\bullet \cong \mathcal{E}_{a',b',c'}^\bullet \) if and only if

\[
1 - c = \pm (1 - c'), \quad c - a - b = \pm (c' - a' - b'), \quad b - a = \pm (b' - a').
\]

Hence, the assignment \( D_{a,b,c} \mapsto \mathcal{E}_{a,b,c}^\bullet \) determines a \( 2^3 \)-to-1 correspondence between the set of hypergeometric differential operators \( D_{a,b,c} \) such that the equation \( D_{a,b,c} y = 0 \) has a full set of algebraic solutions and the set of (isomorphism classes of) dormant \( \text{PGL}_2 \)-opers on \( \mathbb{P}^*_p \). In particular, the italicized assertion at the end of the previous subsection implies that the number of (isomorphism classes of) dormant \( \text{PGL}_2 \)-opers on \( \mathbb{P}^*_p \) consists precisely with \( \frac{p^3-1}{24} \). This computation was verified (by different methods) by S. Mochizuki (cf. [41], Chap. V, §3.2, Corollary 3.7), H. Lange-C. Pauly (cf. [34], Theorem 2), and B. Osserman (cf. [23], Theorem 1.2). If \( n \) and \( X \) are general, then it is difficult to apply or generalize the result of [17] in order to compute the number of differential operators of interest. But, by applying results obtained in [55] or in the present...
paper related to the factorization property of the moduli stacks of dormant opers, one may obtain a formula for computing the number of them.

0.4. Let us describe briefly the results obtained in the present paper. To develop the enumerative geometry of dormant opers, we introduce, in §2, a twisted version of the notion of an oper. Moreover, if $G$ is a connected semisimple algebraic group over $\mathbb{F}_p$, then a dormant faithful twisted $G$-oper (or a $G$-do’per for short) is defined to be a $G$-oper on a stacky log-curve with vanishing $p$-curvature satisfying a certain representability condition (cf. Definitions 2.2.2, 2.3.1, 3.1.1). Under the assumption that $G$ is of adjoint type, the notion of a $G$-do’per is equivalent to the classical notion of a $g$-oper (where $g$ denotes the Lie algebra of $G$) defined in [55], Definition 2.2.1 (i) (cf. Remark 2.3.3). If $\overline{M}_{g,r,\mathbb{F}_p}$ denotes the usual moduli stack classifying pointed stable curves in characteristic $p$ of type $(g, r)$, then one obtains the category fibered in groupoids

\[ \mathcal{D}_p^{\text{Zas...}} G,g,r,\mathbb{F}_p \]

(cf. (113)) over $\overline{M}_{g,r,\mathbb{F}_p}$ classifying $G$-do’pers on pointed curves. Here, let us consider two additional conditions $(\ast)_G, (\ast\ast)_G$ on $G$ and $p$ described as follows:

$(\ast)_G : p > 2h$, where $h$ denotes the Coxeter number of $G$.

$(\ast\ast)_G : G$ is of classical type $A_n$ (with $2n < p - 2$), $B_l$ (with $4l < p$), or $C_m$ (with $4m < p$).

(Notice that $(\ast\ast)_G$ implies $(\ast)_G$.) Then, the first main result of the present paper concerns the structure of the moduli space $\mathcal{D}_p^{\text{Zas...}} G,g,r,\mathbb{F}_p$, as described in the following theorem.

**Theorem A** ( = Theorem 3.2.2, Theorem 4.2.1).

(i) Assume that the condition $(\ast)_G$ is satisfied. Then, $\mathcal{D}_p^{\text{Zas...}} G,g,r,\mathbb{F}_p$ may be represented by a nonempty proper Deligne-Mumford stack over $\mathbb{F}_p$ which is finite over $\overline{M}_{g,r}$, and has an irreducible component that dominates $\overline{M}_{g,r,\mathbb{F}_p}$. Moreover, $\mathcal{D}_p^{\text{Zas...}} G,g,r,\mathbb{F}_p$ admits a perfect obstruction theory, and hence, has a virtual fundamental class $[\mathcal{D}_p^{\text{Zas...}} G,g,r,\mathbb{F}_p]_{\text{vir}}$.

(ii) Assume further that $(\ast\ast)_G$ is satisfied. Then, $\mathcal{D}_p^{\text{Zas...}} G,g,r,\mathbb{F}_p$ is generically étale over $\overline{M}_{g,r,\mathbb{F}_p}$ (i.e., any irreducible component of $\mathcal{D}_p^{\text{Zas...}} G,g,r,\mathbb{F}_p$ that dominates $\overline{M}_{g,r,\mathbb{F}_p}$ has a dense open substack which is étale over $\overline{M}_{g,r,\mathbb{F}_p}$), and moreover, has generic stabilizer isomorphic to the center $Z$ of $G$. 
By the above assertion, we obtain a compact moduli space of dormant $G$-opers (for an arbitrary semisimple $G$) admitting a virtual fundamental class. (Recall that virtual fundamental classes of moduli stacks play a central role in enumerative geometry as they represent a major ingredient in the construction of deformation invariants, e.g., Gromov-Witten and Donaldson-Thomas invariants.)

0.5. Next, by means of the resulting virtual fundamental class, we construct a cohomological field theory (or just a CohFT) for $G$-do’pers. Cohomological field theories were introduced by M. Kontsevich and Y. Manin in [32] to axiomatize the properties of Gromov-Witten classes of a given target variety over the field of complex numbers $\mathbb{C}$. As it turns out, this notion is actually more general, in the sense that not all CohFTs come from Gromov-Witten theory. Indeed, one may find some examples of CohFTs, including the trivial CohFT, Hodge CohFTs, and CohFTs arising from Witten’s $r$-spin classes or Fan-Jarvis-Ruan-Witten (FJRW) theory.

In §5, we introduce (cf. Definition 5.2.1) the definition of a CohFT valued in the $l$-adic étale cohomology of $\overline{\mathcal{M}}_{g,r,F_p} := \mathcal{M}_{g,r,F_p} \otimes \overline{\mathbb{F}}_p$. A key ingredient in the construction of the desired CohFT is the quotient scheme $t^F_{\text{reg}}/W$ (by the natural action of the Weyl group $W$) of the subscheme $t^F_{\text{reg}}$ (cf. (115)) of the Lie algebra $t$ of the maximal torus in $G$ consisting of the Frobenius-invariant regular elements. The trivial $\mathbb{Z}$-action (where $\mathbb{Z}$ denotes the center of $G$) gives rise to the quotient stack $[(t^F_{\text{reg}}/W)/\mathbb{Z}]$ and then the stack of cyclotomic gerbes $\mathcal{I}_\mu([(t^F_{\text{reg}}/W)/\mathbb{Z}])$ in $[(t^F_{\text{reg}}/W)/\mathbb{Z}]$ (cf. the beginning of §1.4). Thus, we obtain a diagram of stacks

$$\begin{align*}
\mathcal{O}_{p,G,g,r,F_p}^{\text{zar}} &\xrightarrow{\pi_{g,r}} \overline{\mathcal{M}}_{g,r,F_p} \\
\mathcal{I}_\mu([(t^F_{\text{reg}}/W)/\mathbb{Z}]) &\otimes \overline{\mathbb{F}}_p,
\end{align*}$$

where $\pi_{g,r}$ denotes the forgetting morphism and $ev_i$ ($i = 1, \cdots, r$) denotes the $i$-th evaluation morphism (cf. (173)). Given a prime $l$ different from $p$, let us consider the $l$-adic étale cohomology

$$\mathcal{V} := \tilde{H}^*_\text{et}(\mathcal{I}_\mu([(t^F_{\text{reg}}/W)/\mathbb{Z}]) \otimes \overline{\mathbb{F}}_p, \overline{\mathbb{Q}}_l)$$

of the stack $\mathcal{I}_\mu([(t^F_{\text{reg}}/W)/\mathbb{Z}]) \otimes \overline{\mathbb{F}}_p$, where $\overline{\mathbb{Q}}_l$ denotes the algebraic closure of $\mathbb{Q}_l$ and $\tilde{H}^*_\text{et}(-, \overline{\mathbb{Q}}_l) := \bigoplus_{i=0}^{\infty} H^i_{\text{et}}(-, \overline{\mathbb{Q}}_l([\frac{1}{2}]))$. $\mathcal{V}$ is canonically isomorphic to the $\overline{\mathbb{Q}}_l$-vector space $\bigoplus_{\rho \in \Delta} \overline{\mathbb{Q}}_l e_{\rho}$ with basis $\{e_{\rho}\}_{\rho \in \Delta}$ indexed by a certain finite set $\Delta$ (cf. (117)). Let us consider a collection $\{\Lambda_{G,g,r}^{\overline{\mathbb{Q}}_l} \}_{g,r \geq 2, g-2+r > 0}$ consisting of
the $\mathbb{Q}_l$-linear morphisms
\begin{equation}
\Lambda_{G,g,r}^{z_\ast} : \mathcal{V}^\otimes r \rightarrow \mathcal{H}^0_{\text{et}}(\mathcal{O}_{G,g,r,p,\mathbb{Q}_l})
\end{equation}
\begin{equation}
\bigotimes_{i=1}^r v_i \mapsto \left( \pi_{g,r,*}^{\text{hom}} \left( \prod_{i=1}^r ev_i^* \left( v_i \right) \right) \right)^* \left( \bigcap \mathcal{D}^{3g-3+r \left( \left( \mathcal{D}_{G,g,r,p,\mathbb{Q}_l}^{z_\ast} \right)^{\text{vir}} \right)} \right)
\end{equation}
(c.f. (175) for its precise definition). Then, the second main result is as follows.

**Theorem B** (= Theorem 5.3.1 (ii)).
Assume that the condition (***)$G,p$ is satisfied. Then, the collection of data
\begin{equation}
\Lambda_{G} := \left( \mathcal{V}, \eta, \{ \Lambda_{G,g,r}^{z_\ast} \}_{g,r \geq 0, 2g-2+r > 0} \right)
\end{equation}
obtained in (176) forms a CohFT valued in $\mathcal{H}^0_{\text{et}}(\mathcal{O}_{G,g,r,p,\mathbb{Q}_l})$, namely, forms a 2d TQFT over $\mathbb{Q}_l$. Moreover, the corresponding Frobenius algebra $(\mathcal{V}, \eta)$ is semisimple.

For each $\tilde{\rho} := (\rho_i)_{i=1}^r \in \Delta^{\times r}$, let $\mathcal{D}_{G,g,r,p,\mathbb{Q}_l}^{z_\ast}$ (c.f. (122)) denote the closed substack of $\mathcal{D}_{G,g,r,p,\mathbb{Q}_l}^{z_\ast} := \mathcal{D}_{G,g,r,F_p}^{z_\ast} \otimes F_p$ classifying $G$-do’pers of radii $\tilde{\rho}$ (c.f. Definition 2.4.1 for the definition of the radius of a twisted $G$-oper). According to Theorem 5.3.1 (i), the value $\Lambda_{G,g,r}^{z_\ast} \left( \bigotimes_{i=1}^r \mathcal{D}_{G,g,r,p,\mathbb{Q}_l}^{z_\ast} \right)$ coincides with the generic degree of $\mathcal{D}_{G,g,r,p,\mathbb{Q}_l}^{z_\ast}$ over $\mathcal{M}_{g,r,F_p}$. In particular, this value consists precisely with $\frac{1}{|Z|}$ times the number of (isomorphism classes of) $G$-do’pers of radii $\tilde{\rho}$ on a sufficiently general pointed smooth curve of type $(g, r)$. On the other hand, the factorization properties in the definition of a CohFT allows us to compute it by means of an explicit description of (the characters of) the Frobenius algebra $(\mathcal{V}, \eta)$. For example, by comparing with the fusion ring of the SL$_2$ Wess-Zumino-Witten model, one can understand the structure of $(\mathcal{V}, \eta)$ in the case of $G = \text{PGL}_2$ (c.f. §6.3). Under the identification $\Delta = \{ 0, 1, \cdots, \frac{p-2}{2} \}$, the following equality holds (c.f. Corollary 6.3.1):
\begin{equation}
\Lambda_{\text{PGL}_2,g,r}^{z_\ast} \left( \bigotimes_{i=1}^r \mathcal{D}_{\text{PGL}_2,g,r,p,\mathbb{Q}_l}^{z_\ast} \right) = \deg \left( \mathcal{D}_{\text{PGL}_2,g,r,p,\mathbb{Q}_l}^{z_\ast} \right)
= \frac{p^{g-1}}{2^{2g-1}} \prod_{j=1}^{p-1} \sin \left( \frac{(2n_j+1)j\pi}{p} \right) \sin \left( \frac{2g-2+r \cdot \pi}{p} \right)
\end{equation}
(where $(n_i)_{i=1}^r \in \Delta^{\times r}$). It may be thought of as a generalization of the main result of [54]. Notice also that there is an explicit formula for the case of $G = \text{PGL}_n$ for a general $n$ (c.f. [55], Theorem H, or Remark 5.3.2 (ii) in the present paper).
0.6. In the last section (i.e., § 6) of the present paper, we consider an analogue for the CohFT $\mathcal{Z}_G$ of the Witten-Kontsevich theorem (cf. [50], [31], [20], [28], [39], and [45]), which is one of the landmark results concerning the intersection theory on moduli spaces of curves. The well-known Witten-Kontsevich theorem states that the intersection theory of psi classes on the moduli spaces of Riemann surfaces is equivalent to the Hermitian matrix model of 2-dimensional gravity. This implies that the partition function of the trivial CohFT is a tau function of the KdV hierarchy, in other words, it satisfies a certain series of partial differential equations. The partition function that we deal with is defined as follows (cf. Definition 6.2.1):

\[
Z_G^{z_*} := \exp \left( \sum_{g,r \geq 0} \frac{\hbar^{2g-2}}{r!} \sum_{d_1, \ldots, d_r \geq 0} \left( \int_{\mathbb{P} G_{g,r}^{z_*}} \prod_{i=1}^r e v_i^* (e_{\rho_i}) \hat{\psi}_i^d \right) \prod_{i=1}^n t_{d_i, \rho_i} \right)
\]

\[
= \exp \left( \sum_{g,r \geq 0} \frac{\hbar^{2g-2}}{r!} \sum_{d_1, \ldots, d_r \geq 0} \left( \int_{\mathbb{P} G_{g,r}^{z_*}} \mathcal{A}_{G_{g,r}}^{z_*} \prod_{i=1}^r \psi_i^d \prod_{i=1}^n t_{d_i, \rho_i} \right) \right)
\]

\[
\in \mathcal{Q}((\hbar))[[\{ t_{d, \rho} \}_{d \geq 0, \rho \in \Delta}]];
\]

where $\hat{\psi}_i$ and $\psi_i$ ($i = 1, \ldots, r$) denote the $i$-th psi classes on $\mathbb{P} G_{g,r}^{z_*}$ and $\mathbb{P}^{z_*} G_{g,r}$, respectively, and the equality in the parenthesis follows from Proposition 6.1.1. Then, the final main result of the present paper is as follows.

**Theorem C** (cf. Theorem 6.2.4).
Assume that the condition $(\star\star)_{G,p}$ is satisfied and $l$ is sufficiently large relative to $g$, $r$, and $p$. For each $n \geq -1$, let us consider a differential operator

\[
L_n := -\frac{(2n+3)!!}{2^{n+1}} \frac{\partial}{\partial t_{n+1, \rho}} + \sum_{i=0}^{\infty} \frac{(2i + 2n + 1)!!}{(2i - 1)!!2^{n+1}} \left( \sum_{\rho \in \Delta} t_{i, \rho} \frac{\partial}{\partial t_{i+n, \rho}} \right) \]

\[
+ \frac{|Z| \hbar}{2} \sum_{i=0}^{n-1} \frac{(2i + 1)!!(2n - 2i - 1)!!}{2^{n+1}} \left( \sum_{\rho \in \Delta} \frac{\partial^2}{\partial t_{i, \rho} \partial t_{n-i, \rho}} \right) \]

\[
+ \delta_{n,-1} \frac{\hbar^2}{2|Z|} \left( \sum_{\rho \in \Delta} t_{0, \rho} \rho_1 \right) + \delta_{n,0} \frac{|\Delta|}{16}.
\]

Then, for any $n \geq -1$, the following equality holds:

\[
L_n Z_G^{z_*} = 0.
\]
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Notation and conventions
Let us introduce some notation and conventions used in the present paper. We fix a field \( k \), and denote by \( \text{Sch}_k \) the category of schemes over \( k \).

We use the word stack to mean algebraic stack in the sense of the appendix in [52]. Let \( X \) be a Deligne-Mumford stack. By a sheaf on \( X \), we mean, unless otherwise stated, a sheaf on the small \( \acute{e}tale \) site of \( X \). In particular, one obtains the structure sheaf \( \mathcal{O}_X \) of \( X \). Under the assumption that \( X \) is of finite type over \( k \), we denote by \( |X| \) the coarse moduli space of \( X \), which has a natural projection \( \text{coa}_X : X \to |X| \). Also, for each morphism \( f : X \to Y \) of Deligne-Mumford stacks of finite type over \( k \), we shall write \( |f| : |X| \to |Y| \) for the morphism between the coarse moduli spaces associated with \( f \). For each \( n \in \mathbb{Z}_{\geq 0} \), denote by \( A_n(X)_\mathbb{Q} \) (cf. [52], (3.4), Definition) the rational Chow group of cycles of dimension \( n \) on \( X \) modulo rational equivalence tensored with \( \mathbb{Q} \).

Basic references for the notion and properties of a log scheme (or, more generally, a log stack) are [24], [18], and [23]. For a log stack indicated, say, by \( Y^\log \), we shall write \( Y \) for the underlying stack of \( Y^\log \). For a morphism \( Y^\log \to \mathcal{T}^\log \) of fs log Deligne-Mumford stacks, let us write \( T_{Y^\log /\mathcal{T}^\log} \) for the sheaf of logarithmic derivations of \( Y^\log \) over \( \mathcal{T}^\log \), and write \( \Omega_{Y^\log /\mathcal{T}^\log} \) for its dual \( T_{Y^\log /\mathcal{T}^\log}^\vee \), i.e., the sheaf of logarithmic 1-forms of \( Y^\log \) over \( \mathcal{T}^\log \).

For each positive integer \( l \), we denote by \( \mu_l \) the group of \( l \)-th roots of unity in the algebraic closure of \( k \). If \( G \) is an algebraic group over \( k \), then let \( BG \) denote the classifying stack of \( G \), which is defined as the quotient stack \( BG := [pt/G] \) for the trivial action of \( G \) on \( pt := \text{Spec}(k) \). If \( E \) is a (right) \( G \)-torsor over a \( k \)-stack \( Y \) in the \( \acute{e}tale \) topology and \( \mathfrak{h} \) is a \( k \)-vector space equipped with a (left) \( G \)-action, then we shall write \( \mathfrak{h}_E \) for the vector bundle on \( Y \) associated with the relative affine space \( E \times^G \mathfrak{h} \) := \( (E \times_k \mathfrak{h})/G \). Denote by \( \mathbb{G}_m \) the multiplicative group over \( k \).

Let \( S \) be a scheme. By a nodal curve over \( S \), we mean a flat morphism \( f : X \to S \) of schemes whose geometric fibers are reduced 1-dimensional schemes with at most nodal points as singularities. (For simplicity, we often write \( X \) instead of \( f : X \to S \).) For a nonnegative integer \( r \), an \( r \)-pointed nodal curve
over $S$ is defined to be a collection $X^\star := (f : X \to S, \{\sigma_{X,i} : S \to X\}_{i=1}^r)$ consisting of a nodal curve $f : X \to S$ over $S$ and sections $\sigma_{X,i} : S \to X$ $(i = 1, \cdots, r)$ such that $\text{Im}(\sigma_{X,i})$ (for any $i$) lies in the smooth locus of $X$ (relative to $S$) and $S \times_{\sigma_{X,i}X, \sigma_{X,j}S} S = \emptyset$ (for any pair $(i, j)$ with $i \neq j$).

1. Extended spin structures on twisted curves

First, we introduce some sort of natural generalization of spin structures defined on a twisted curve, called extended spin structures (cf. Definition 1.3.1). The notion of an extended spin structure will be used to describe the gap between the structure of faithful twisted $G$-oper (defined in Definitions 2.2.2 and 2.3.1 later) and the structure of $G_{\text{ad}}$-oper (where $G_{\text{ad}}$ denotes the adjoint group of $G$). In this section, we prove some properties of the moduli stack of extended spin structures (cf., e.g., Theorem 1.3.5, Propositions 1.5.2 and 1.6.2).

1.1. The moduli stack of pointed stable curves.

Given nonnegative integers $g, r$ with $2g - 2 + r > 0$ and a commutative ring $R$, we denote by $\mathcal{M}_{g,r,R}$ the moduli stack of $r$-pointed stable curves over $R$ of genus $g$ (i.e., of type $(g, r)$). Namely, $\mathcal{M}_{g,r,R}$ classifies the pointed stable curves, which are collections of data

$$X^\star := (f : X \to S, \{\sigma_{X,i} : S \to X\}_{i=1}^r)$$

consisting of a (proper) nodal curve $f : X \to S$ over an $R$-scheme $S$ of genus $g$ and $r$ marked points $\sigma_{X,i} : S \to X$ $(i = 1, \cdots, r)$ satisfying certain conditions (cf. [33], Definition 1.1). Recall (cf. [33], Corollary 2.6 and Theorem 2.7; [13], §5) that $\mathcal{M}_{g,r,R}$ may be represented by a geometrically connected, proper, and smooth Deligne-Mumford stack over $R$ of dimension $3g - 3 + r$. Denote by $\mathcal{M}_{g,r,R}$ the dense open substack of $\mathcal{M}_{g,r,R}$ classifying smooth curves.

Given nonnegative integers $g_1, g_2, r_1, r_2$ with $2g_i - 1 + r_i > 0$ $(i = 1, 2)$, we shall write

$$\Phi_{\text{tree}} : \mathcal{M}_{g_1, r_1 + 1, R} \times_R \mathcal{M}_{g_2, r_2 + 1, R} \to \mathcal{M}_{g_1 + g_2, r_1 + r_2, R}$$

for the gluing map obtained by attaching the respective last marked points of curves classified by $\mathcal{M}_{g_1, r_1 + 1, R}$ and $\mathcal{M}_{g_2, r_2 + 1, R}$ to form a node.

Similarly, given nonnegative integers $g, r$ with $2g + r > 0$, we shall write

$$\Phi_{\text{loop}} : \mathcal{M}_{g, r + 2, R} \to \mathcal{M}_{g + 1, r, R}$$

for the gluing map obtained by attaching the last two marked points of each curve classified by $\mathcal{M}_{g, r + 2, R}$ to form a node.
Finally, if $2g - 2 + r > 0$, we define
\[ \Phi_{\text{tail}} : \overline{\mathcal{M}}_{g,r+1,R} \to \overline{\mathcal{M}}_{g,r,R} \]
(18)

as the morphism obtained by forgetting the last marked point and successively contracting any resulting unstable components of each curve classified by $\overline{\mathcal{M}}_{g,r+1,R}$.

1.2. Twisted curves.

Let us recall (cf. Definitions 1.2.2 and 1.2.4) the notion of a (pointed) twisted curve and the log structure equipped with it. Here, recall that a Deligne-Mumford stack $\mathcal{X}$ over $k$ is called tame if, for any algebraically closed field $\overline{k}$ over $k$ and any morphism $\overline{x} : \text{Spec}(\overline{k}) \to \mathcal{X}$, the stabilizer group $\text{Stab}_{\mathcal{X}}(\overline{x})$ of $\overline{x}$ has order invertible in $k$.

**Definition 1.2.1.**

Let $T^{\log}$ be an fs log scheme over $k$. A **stacky log-curve** over $T^{\log}$ is an fs log Deligne-Mumford stack $\mathcal{Y}^{\log}$ together with a log smooth integrable morphism $f^{\log} : \mathcal{Y}^{\log} \to T^{\log}$ such that the geometric fibers of the underlying morphism $f : \mathcal{Y} \to T$ of stacks are reduced, connected, and 1-dimensional. (In particular, both $\mathcal{T}_{\mathcal{Y}^{\log}}/T^{\log}$ and $\Omega_{\mathcal{Y}^{\log}/T^{\log}}$ are line bundles on $\mathcal{Y}$.)

**Definition 1.2.2** (cf. [46], Definition 1.2).

Let $S$ be a $k$-scheme.

(i) A (balanced) **twisted curve** over $S$ is a proper flat morphism $f : \mathcal{X} \to S$ of tame Deligne-Mumford stacks over $k$ satisfying the following conditions:

- The geometric fibers of $f : \mathcal{X} \to S$ are purely 1-dimensional and connected with at most nodal singularities.
- If $[\mathcal{X}]^{\text{sm}}$ denotes the open subscheme of $[\mathcal{X}]$ where $|f| : [\mathcal{X}] \to S$ is smooth, then the inverse image $\mathcal{X} \times [\mathcal{X}] [\mathcal{X}]^{\text{sm}}$ of $\mathcal{X}$ coincides with the open substack of $\mathcal{X}$ where $f$ is smooth.
- For any geometric point $\overline{s} \to S$, the map $\text{coa}_{\mathcal{X}} \times \text{id}_{\overline{s}} : \mathcal{X} \times_S \overline{s} \to [\mathcal{X}] \times_S \overline{s}$ is an isomorphism over some dense open subscheme of $[\mathcal{X}] \times_S \overline{s}$.
- Consider a geometric point $\overline{s} \to [\mathcal{X}]$ mapping to a node, where there exist an affine open neighborhood $T (:= \text{Spec}(R)) \subseteq S$ of $|f|([\overline{s}])$, an affine étale neighborhood $\text{Spec}(A) \to |f|^{-1}(T) (\subseteq [\mathcal{X}])$ of $\overline{s}$, and an étale morphism

\[ \text{Spec}(A) \to \text{Spec}(R[s_0,t_0]/(s_0t_0 - u_0)) \]
(19)
over $R$ for some $u_0 \in R$. Then, the pull-back $\mathfrak{X} \times_{|\mathfrak{X}|} \text{Spec}(A)$ is isomorphic to the quotient stack

$$[\text{Spec}(A[s_1, t_1]/(s_1 t_1 - u_1, s_1 - s_0, t_1 - t_0))/\mu_l]$$

for some $u_1 \in R$ and some positive integer $l$ invertible in $k$ such that $\zeta \in \mu_l$ acts by $(s_1, t_1) \mapsto (\zeta s_1, \zeta^{-1} t_1)$.

(ii) Let $g$ be a nonnegative integer. We shall say that a twisted curve $f : \mathfrak{X} \to S$ is of genus $g$ if the genus of every fiber of the proper nodal curve $|f| : |\mathfrak{X}| \to S$ coincides with $g$.

**Definition 1.2.3** (cf. [5], Definition 4.1.2).

Let $g$ and $r$ be nonnegative integers and $S$ a $k$-scheme.

(i) An $r$-pointed twisted curve (of genus $g$) over $S$ is a collection of data

$$\mathfrak{X}^\bullet := (f : \mathfrak{X} \to S, \{\sigma_{X,i} : \mathfrak{S}_i \to \mathfrak{X}\}_{i=1}^r)$$

consisting of a twisted curve $f : \mathfrak{X} \to S$ (of genus $g$) and disjoint closed substacks $\sigma_{X,i} : \mathfrak{S}_i \to \mathfrak{X}$ ($i = 1, \cdots, r$) satisfying the following conditions:

- Each $\text{Im}(\sigma_{X,i})$ is contained in the smooth locus in $\mathfrak{X}$ (relative to $S$).
- The stacks $\mathfrak{S}_i$ are étale gerbes over $S$.
- If $\mathfrak{X}^{\text{gen}}$ denotes the complement of the union of $\text{Im}(\sigma_{X,i})$ ($i = 1, \cdots, r$) in the smooth locus in $\mathfrak{X}$ (relative to $S$), then $\mathfrak{X}^{\text{gen}}$ may be represented by a scheme.

(ii) Let $\mathfrak{X}^\bullet_j := (f_j : \mathfrak{X}_j \to S_j, \{\sigma_{X,j,i} : \mathfrak{S}_{j,i} \to \mathfrak{X}_j\}_{i=1}^r)$ ($j = 1, 2$) be $r$-pointed twisted curves. A morphism of $r$-pointed twisted curves from $\mathfrak{X}^\bullet_1$ to $\mathfrak{X}^\bullet_2$ is a pair of morphisms

$$\alpha_S : S_1 \to S_2, \alpha_X : \mathfrak{X}_1 \to \mathfrak{X}_2$$

such that $\alpha_X^{-1}(\mathfrak{S}_{2,i}) = \mathfrak{S}_{1,i}$ for any $i = 1, \cdots, r$, and moreover, the square diagram

$$
\begin{array}{ccc}
\mathfrak{X}_1 & \xrightarrow{f_1} & S_1 \\
\alpha_X \downarrow & & \downarrow \alpha_S \\
\mathfrak{X}_2 & \xrightarrow{f_2} & S_2
\end{array}
$$

is commutative and cartesian.

(iii) Suppose that $2g - 2 + r > 0$. A twisted stable curve of type $(g, r)$ over $S$ is an $r$-pointed twisted curve $\mathfrak{X}^\bullet := (f : \mathfrak{X} \to S, \{\sigma_{X,i} : \mathfrak{S}_i \to \mathfrak{X}\}_{i=1}^r)$ over $S$ whose coarse moduli space

$$|\mathfrak{X}^\bullet| := (|f| : |\mathfrak{X}| \to S, \{|\sigma_{X,i}| : |\mathfrak{S}_i| \to |\mathfrak{X}|\}_{i=1}^r)$$
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of $X^\star$ forms a pointed stable curve of type $(g, r)$.

**Definition 1.2.4.**

Let $r$ be a nonnegative integer and $S$ a $k$-scheme.

(i) Let $X^\star := (f : X \to S, \{\sigma_{X,i}\}_{i=1}^r)$ be an $r$-pointed nodal curve over $S$.

A **twistification** of $X^\star$ is a pair

$$(X^\star, \gamma)$$

consisting of an $r$-pointed twisted curve $X^\star := (f : X \to S, \{\sigma_{X,i}\}_{i=1}^r)$ and an $S$-morphism $\gamma : X \to X$ such that the induced morphism $|\gamma| : |X| \to X$ is an isomorphism, and moreover, $\gamma \circ \sigma_{X,i} = \sigma_{X,i}$ for any $i = 1, \ldots, r$.

(ii) Let $X^\star$ be an $r$-pointed nodal curve over $S$ and, for each $j \in \{1, 2\}$, let $(X_j^\star, \gamma_j)$ be a twistification of $X^\star$. An **isomorphism of twistifications** from $(X_1^\star, \gamma_1)$ to $(X_2^\star, \gamma_2)$ is an isomorphism of $r$-pointed twisted curves from $X_1^\star$ to $X_2^\star$ compatible with $\gamma_1$ and $\gamma_2$.

Let $r$ be a nonnegative integers and $X^\star := (f : X \to S, \{\sigma_{X,i}\}_{i=1}^r)$ an $r$-pointed twisted curve. According to [10], §3.10, there exists canonical log structures on $X$ and $S$ (obtained by the log structures described in Theorem 3.6 in loc. cit. and the closed substacks $\sigma_{X,i}$). Thus, we obtain the log stacks $X^\log$ and $S^\log$. The morphism $f : X \to S$ extends to a log smooth morphism $f^\log : X^\log \to S^\log$ of log stacks, by which $X^\log$ specifies a stacky log-curve over $S^\log$. If $X$ is smooth, then $S^\log = S$.

**1.3. Extended spin structures.**

In the rest of this section, we assume that $\text{char}(k) > 2$. Also, assume that we are given a pair $(Z, \delta)$ consisting of a finite abelian group $Z$ (identified with a finite group scheme over $k$) which has order invertible in $k$ and a morphism $\delta : \{\pm 1\} \to Z$ of groups. Write

$$(26) \quad \overline{\delta} : \mu_2 := \mu_2/\text{Ker}(\delta) \hookrightarrow Z$$

(where $\overline{\delta} = 1$ or 2) for the injection induced naturally by $\delta$. Denote by

$$(27) \quad \widehat{Z}_\delta$$

the cofiber product of $\delta$ and the natural inclusion $\mu_2 \hookrightarrow \mathbb{G}_m$, which fits into the following morphism of short exact sequences of algebraic groups:

$$(28) \quad 1 \longrightarrow \mu_2 \overset{\text{incl.}}{\longrightarrow} \mathbb{G}_m \overset{x \mapsto x^2}{\longrightarrow} \mathbb{G}_m \longrightarrow 1$$

$$\downarrow \delta \quad \downarrow \delta_{\mathbb{G}_m} \quad \downarrow \text{id}_{\mathbb{G}_m}$$

$$1 \longrightarrow Z \longrightarrow \widehat{Z}_\delta \overset{\nu}{\longrightarrow} \mathbb{G}_m \longrightarrow 1.$$
**Definition 1.3.1.**

(i) Let $S^{\log}$ be an fs log scheme over $k$ and $U^{\log}$ a stacky log-curve over $S^{\log}$. An extended $(Z, \delta)$-spin structure (or just a $(Z, \delta)$-structure) on $U^{\log}/S^{\log}$ is a $\hat{Z}_{\delta}$-torsor

\[ \pi_G : G_{Z, \delta} \to U^{\log}/S^{\log} \]

over $U$ in the étale topology whose classifying morphism $U \to B\hat{Z}_{\delta}$ is representable, and moreover, which fits into the following 1-commutative diagram:

\[ \begin{array}{ccc}
U & \xrightarrow{\Omega_{U^{\log}/S^{\log}}} & B\mathbb{G}_m \\
\downarrow & & \downarrow \\
B\hat{Z}_{\delta} & & B\nu
\end{array} \]

where the upper horizontal morphism $[\Omega_{U^{\log}/S^{\log}}]$ denotes the classifying morphism of the line bundle $\Omega_{U^{\log}/S^{\log}}$ and $B\nu$ denotes the morphism induced naturally by $\nu$ (cf. (28)).

(ii) Let $r$ be a nonnegative integer, $S$ a $k$-scheme, and $X^{\bullet} := (X, \{\sigma_{X,i}\})_{i=1}^r$ an $r$-pointed nodal curve over $S$. A $(Z, \delta)$-structure on $X^{\bullet}$ is a collection of data

\[ (X^{\bullet}, \gamma, \pi_G : G_{Z, \delta} \to X) \]

consisting of a twistification $(X^{\bullet}, \gamma)$ of $X^{\bullet}$ and a $(Z, \delta)$-structure $\pi_G : G_{Z, \delta} \to X$ on $X^{\log}/S^{\log}$.

**Definition 1.3.2.**

In (i)-(iii) below, let $r$ be a nonnegative integer.

(i) Let $S$ be a $k$-scheme. An $r$-pointed $(Z, \delta)$-spin curve over $S$ is a collection of data

\[ X^{\bullet} := (X^{\bullet}, X^{\bullet}, \gamma, \pi_G : G_{Z, \delta} \to X) \]

where $X^{\bullet}$ denotes an $r$-pointed nodal curve over $S$ and $(X^{\bullet}, \gamma, \pi_G : G_{Z, \delta} \to X)$ denotes a $(Z, \delta)$-structure on $X^{\bullet}$.

(ii) For each $j \in \{1, 2\}$, let $S_j$ be a $k$-scheme and $X_j^{\bullet} := (X_j^{\bullet}, X_j^{\bullet}, \gamma_j, \pi_{G,j} : G_{Z,\delta,j} \to X_j)$ (where $X_j^{\bullet} := (f_j : X_j \to S_j, \{\sigma_{X,j,i}\}_{i=1}^r)$) an $r$-pointed $(Z, \delta)$-spin curve over $S_j$. A 1-morphism (or just a morphism) of $r$-pointed $(Z, \delta)$-spin curves from $X_1^{\bullet}$ to $X_2^{\bullet}$ is a triple

\[ \alpha^{\bullet} := (\alpha_S, \alpha_X, \alpha_E) \]
of morphisms which make the following diagram 1-commutative:

\[
\begin{array}{ccc}
G_{Z,\delta,1} & \xrightarrow{\pi_{g,1}} & X_1 \\
\downarrow{\alpha_E} & & \downarrow{\alpha_X} \\
G_{Z,\delta,2} & \xrightarrow{\pi_{g,2}} & X_2
\end{array}
\]

(34)

where

- the right-hand square forms a morphism of \(r\)-pointed twisted curves,
- the left-hand square is cartesian, and \(\alpha_E\) is compatible with the respective actions of \(\hat{Z}_\delta\) on \(G_{Z,\delta,1}\) and \(G_{Z,\delta,2}\).

In particular, by taking coarse moduli spaces, one may associate, to each morphism \(\alpha_{\star} : X_1 \rightarrow X_2\) of \(r\)-pointed \((Z,\delta)\)-spin curves, a morphism \(\alpha_X : X_1^\star \rightarrow X_2^\star\) between the underlying \(r\)-pointed nodal curves.

(iii) Let \(S_j, X_j^\star (j = 1, 2)\) be as in (ii) and \(\alpha_{\star} := (\alpha_{S,1}, \alpha_{X,1}, \alpha_{E,1}) (l = 1, 2)\) be morphisms \(X_1^\star \rightarrow X_2^\star\) of \(r\)-pointed \((Z,\delta)\)-spin curves. A 2-morphism from \(\alpha_1^\star\) to \(\alpha_2^\star\) is a triple

\[
\alpha_{\star} := (\alpha_{S,1} \xrightarrow{a_{S}} \alpha_{S,2}, \alpha_{X,1} \xrightarrow{a_{X}} \alpha_{X,2}, \alpha_{E,1} \xrightarrow{a_{E}} \alpha_{E,2})
\]

(35)

of natural transformations compatible with each other (hence, \(\alpha_S\) coincides with the identity natural transformation).

(iv) Let \(g, r\) be nonnegative integers with \(2g - 2 + r > 0\) and \(S\) a \(k\)-scheme. A **pointed stable \((Z,\delta)\)-spin curve of type \((g,r)\)** over \(S\) is an \(r\)-pointed \((Z,\delta)\)-spin curve \(X^\star\) over \(S\) whose underlying pointed nodal curve is a pointed stable curve of type \((g,r)\).

In the way of Definition 1.3.2 above, \(r\)-pointed \((Z,\delta)\)-spin curves form a 2-category. It is verified that this 2-category is equivalent to the (1-)category fibered in groupoids over \(\mathcal{S}ch_{/k}\) whose fiber over \(S \in \text{Ob}(\mathcal{S}ch_{/k})\) forms the groupoid classifying \(r\)-pointed \((Z,\delta)\)-spin curves over \(S\) and 2-isomorphism classes of (1-)morphisms between them. (Indeed, since all 2-morphisms are invertible, the result of [5], Lemma 4.2.3 implies that any 1-morphism in that 2-category cannot have nontrivial automorphisms.) Thus, for each pair of nonnegative integers \((g, r)\) with \(2g - 2 + r > 0\), we obtain the (1-)category

\[
\mathcal{S}p_{Z,\delta,g,r,k}.
\]

(36)

consisting of pointed stable \((Z,\delta)\)-spin curves of type \((g, r)\) over \(k\)-schemes. The assignment \(X^\star := (X^\star, \gamma, \pi_{\mathcal{G}} : \mathcal{G}_{Z,\delta} \rightarrow \mathcal{X}) \mapsto X^\star\) determines a functor

\[
\pi_{\mathcal{G},r} : \mathcal{S}p_{Z,\delta,g,r,k} \rightarrow \overline{\mathcal{M}}_{g,r,k}.
\]

(37)
Remark 1.3.3.
Let $r$ be a positive integer and $X^\bullet$ (resp., $X^\star$) an $r$-pointed nodal curve (resp., an $r$-pointed twisted curve) over a $k$-scheme $S$. In the following discussion, we shall refer to each 2-isomorphism class of a 1-isomorphism of pointed $(Z, \delta)$-spin curves inducing the identity morphism of $X^\bullet$ (resp., of $X^\star$) as an isomorphism of $(Z, \delta)$-structures on $X^\bullet$ (resp., on $X^\star$). In particular, we obtain the groupoid of $(Z, \delta)$-structures on $X^\bullet$ (resp., on $X^\star$).

Remark 1.3.4.
Certain special cases of “$(Z, \delta)$” may be immediately understood or already appear in the previous works, as explained as follows.

(i) First, let us consider the case where $(Z, \delta) = (\mu_2, \text{id}_{\mu_2})$. Then, the notion of a pointed stable $(\mu_2, \text{id}_{\mu_2})$-spin curve is evidently equivalent to the notion of a twisted 2-spin curve in the sense of [4], §1.4. In particular, $\mathcal{Gp}_{\mu_2, \text{id}_{\mu_2}, g, r, k}$ is equivalent to the category $\mathcal{B}_{g,r}(\mathbb{G}_m, \omega_{\log}^{1/2})$ described in loc. cit.. If $(Z, \delta)$ is arbitrary, then changing the structure group of the underlying torsors via $\delta_{\mathbb{G}_m}$ gives an assignment from each twisted 2-spin curve (i.e., a pointed stable $(\mu_2, \text{id}_{\mu_2})$-spin curve) to a pointed stable $(Z, \delta)$-spin curve. This assignment determines a functor

$$\mathcal{Gp}_{\mu_2, \text{id}_{\mu_2}, g, r, k} \rightarrow \mathcal{Gp}_{Z, \delta, g, r, k}$$

over $\mathcal{M}_{g,r}$. Since $\mathcal{Gp}_{\mu_2, \text{id}_{\mu_2}, g, r, k}$ is nonempty (cf. [11], Corollary 4.11 and Proposition 4.19), $\mathcal{Gp}_{Z, \delta, g, r, k}$ is verified to be nonempty.

(ii) Next, let us assume that $Z = \{1\}$ (hence, $\delta$ must be the zero map $0$ and $\hat{Z}_\delta \cong \mathbb{G}_m$). Then, there exists only one $(\{1\}, 0)$-structure on each pointed stable curve $X^\star/S$ given by the $\mathbb{G}_m$-torsor corresponding to $\Omega_{X^\star/S}^{\log}$. In particular, $\pi_{g, r}^{\mathcal{Gp}}$ is an equivalence of categories.

(iii) More generally, let us consider the case where $Z$ is arbitrary but $\delta$ coincides with the zero map. Then, there exists a natural isomorphism $\hat{Z}_\delta \xrightarrow{\sim} \mathbb{G} \times_k Z$ and the surjection $\nu$ (cf. (28)) may be identified (relative to this isomorphism) with the first projection $\mathbb{G}_m \times_k Z \rightarrow \mathbb{G}_m$. It follows immediately that $\mathcal{Gp}_{Z, \delta, g, r, k}$ is naturally isomorphic to the moduli stack $\overline{\mathcal{M}}_{g,r}(\mathcal{B}G)$ of twisted stable maps into $\mathcal{B}G$, studied by D. Abramovich, T. Graber, and A. Vistoli (cf. [2], [3], [1], and [19]).

According to [1], Theorems 2.1.7 and 3.0.2, $\overline{\mathcal{M}}_{g,r}(\mathcal{B}G)$ may be represented by a proper smooth Deligne-Mumford stack with projective coarse moduli space. By applying an argument similar to the argument in the proof of this result, one may obtain Theorem 1.3.5 below, which may be thought of as its generalization to the case of an arbitrary $(Z, \delta)$. 

\[38\]
Theorem 1.3.5.
Let $g, r$ be nonnegative integers with $2g - 2 + r > 0$. Then, $\mathfrak{Sp}_{Z, \delta, g, r, k}$ may be represented by a nonempty proper smooth Deligne-Mumford stack over $k$ admitting a projective coarse moduli space. The forgetting morphism $\pi_{g, r}^{\mathfrak{Sp}} : \mathfrak{Sp}_{Z, \delta, g, r, k} \to \mathfrak{M}_{g, r, k}$ is finite and flat. Moreover, its restriction $\mathfrak{Sp}_{Z, \delta, g, r, k} \times \mathfrak{M}_{g, r, k} \to \mathfrak{M}_{g, r, k}$ is étale.

Proof. Since the relative cotangent complex of $B \nu : B \hat{\mathbb{Z}}_\delta \to B \mathbb{G}_m$ is verified to be trivial, the assertion follows from the arguments in [4], §1.5, §2.1, and §2.2 (or the argument in the proof of [4], Theorem 3.0.2), where $B_{g, n}(\mathbb{G}_m, \omega_1^{1/r})$ and $\kappa_r : B \mathbb{G}_m \to B \mathbb{G}_m$ in loc. cit. are replaced with $\mathfrak{Sp}_{Z, \delta, g, r, k}$ and $B \nu : B \hat{\mathbb{Z}}_\delta \to B \mathbb{G}_m$ respectively. The nonemptiness follows from the discussion in Remark 1.3.4 (i) above. Moreover, the last assertion follows from [46], Theorem 1.8, and the italicized assertion described above, which implies that deformations and obstructions of a pointed stable $(Z, \delta)$-spin curve are identical to those of the underlying twistification. □

The following proposition will be used in the proof of Proposition 1.7.1.

Proposition 1.3.6.
Let $g, r$ be nonnegative integers with $2g - 2 + r > 0$, and $X^\star := (X^\star, \chi^\star, \gamma, G_{Z, \delta})$ a pointed stable $(Z, \delta)$-spin curve of type $(g, r)$ over an algebraically closed field. Denote by $\text{Aut}_{X^\star}(X^\star)$ the automorphism group of $X^\star$ inducing the identity of $X^\star$ and by $e_1, \cdots, e_m$ the nodes in the underlying curve of $X^\star$. Then, we have a natural isomorphism of groups

$\text{Aut}_{X^\star}(X^\star) \cong \mathbb{Z} \times \prod_{i=1}^m \mu_{l_i}$,  

where $l_1, \cdots, l_m$ are the orders of the stabilizers in $X$ at $e_1, \cdots, e_m$ respectively.

Proof. The assertion follows from [1], Proposition 7.1.1, and the fact that any automorphism of the $(Z, \delta)$-structure $G_{Z, \delta}$ may be expressed as the automorphism given by translation by a unique element of $Z$. □

1.4. Radii of extended spin structures.
We shall define the notion of radius of a $(Z, \delta)$-structure defined at each marked point (cf. Definition 14.1). Given a Deligne-Mumford stack $\mathfrak{X}$ of finite type over $k$, we have the stack of cyclotomic gerbes $\mathfrak{T}_\mu(\mathfrak{X})$ in $\mathfrak{X}$, as described in [3], Definition 3.3.6. By definition, $\mathfrak{T}_\mu(\mathfrak{X})$ is the disjoint union $\coprod_{l \geq 1} \mathfrak{T}_{\mu_l}(\mathfrak{X})$, where $\mathfrak{T}_{\mu_l}(\mathfrak{X})$ (for each positive integer $l$) denotes the category fibered in groupoids over $\mathfrak{Sch}_{/k}$ whose fiber over $S \in \text{Ob}(\mathfrak{Sch}_{/k})$ is the groupoid
classifying pairs \((\mathcal{G}, \phi)\) consisting of a gerbe \(\mathcal{G}\) over \(S\) banded by \(\mu_l\) (hence \(|\mathcal{G}| \cong S\)) and a representable morphism \(\phi: \mathcal{G} \to \mathcal{X}\) over \(k\).

Let us consider the stack of cyclotomic gerbes \(\mathcal{I}_\mu(BZ)\) in the classifying stack \(BZ\). Denote by
\[
\text{Inj}(\mu, Z)
\]
the set of injective morphisms of groups \(\mu_l \to Z\), where \(l\) is some positive integer. It is a finite set because \(Z\) is finite. Given each element \(\kappa: \mu_l \to Z\) of \(\text{Inj}(\mu, Z)\), we obtain an open and closed substack
\[
\mathcal{I}_\mu(BZ)_\kappa
\]
of \(\mathcal{I}_\mu(BZ)\) classifying representable morphisms \(\phi: S \to BZ\) which are, étale locally on \(|S|\), identified with the composite \(|S| \times_k B\mu_l \to BZ\) of the second projection \(|S| \times_k B\mu_l \to B\mu_l\) and the morphism \(B\kappa: B\mu_l \to BZ\) arising from \(\kappa\). Note that \(\mathcal{I}_\mu(BZ)_\kappa\) is canonically isomorphic to \(BZ\). The stack \(\mathcal{I}_\mu(BZ)\) decomposes into the disjoint union
\[
\mathcal{I}_\mu(BZ) = \bigcoprod_{\kappa \in \text{Inj}(\mu, Z)} \mathcal{I}_\mu(BZ)_\kappa.
\]
It gives rise to a decomposition
\[
\mathcal{I}_\mu(BZ)^{\times r} = \prod_{(\kappa_i)_{i=1}^r \in \text{Inj}(\mu, Z)^{\times r}} \prod_{i=1}^r \mathcal{I}_\mu(BZ)_{\kappa_i},
\]
(where \((-)^{\times r}\) denotes the product of \(r\) copies of \((-)\)).

Now, let \(g, r\) be nonnegative integers with \(2g - 2 + r > 0, r > 0\), and \(X^* := (X, \{\sigma_{X,i}: \mathcal{G}_i \to X\}_{i=1}^r)\) an \(r\)-pointed twisted curve of genus \(g\) over a \(k\)-scheme \(S\). Also, let \(G_{Z,\delta}\) be a \((Z, \delta)\)-structure on \(X^{\log}/S^{\log}\). For each \(i = 1, \ldots, r\), the restriction \(\sigma_{X,i}^*: \mathcal{G}_{Z,\delta}\) to \(\mathcal{G}_i\) corresponds to a representable morphism \(\mathcal{G}_i \to B\tilde{Z}_\delta\). Notice that the composite \(\mathcal{G}_i \to B\mathcal{G}_{Z,m}\) of this morphism and \(B\nu: B\tilde{Z}_\delta \to B\mathcal{G}_{Z,m}\) classifies the line bundle \(\sigma_{X,i}^*: (\Omega_{X^{\log}/S^{\log}})\), which is canonically identified with the trivial line bundle \(\mathcal{O}_{\mathcal{G}_i}\) via the residue map. Hence, the morphism \(\mathcal{G}_i \to B\tilde{Z}_\delta\) factors through \(BZ \to B\tilde{Z}_\delta\). If \(\phi_{G_{Z,\delta},i}: \mathcal{G}_i \to BZ\) denotes the resulting morphism, then we obtain an object
\[
\tilde{\kappa}_{G_{Z,\delta},i} := (\mathcal{G}_i, \phi_{G_{Z,\delta},i}) \in \text{Ob}(\mathcal{I}_\mu(BZ)(S)).
\]
There exists a unique element
\[
\kappa_{G_{Z,\delta},i} \in \text{Inj}(\mu, Z)
\]
such that \(\tilde{\kappa}_{G_{Z,\delta},i}\) lies in \(\text{Ob}(\mathcal{I}_\mu(BZ)_{\kappa_{G_{Z,\delta},i}}(S))\).

**Definition 1.4.1.**

We shall refer to \(\kappa_{G_{Z,\delta},i}\) as the **radius** of the \((Z, \delta)\)-structure \(G_{Z,\delta}\) at the marked point \(\sigma_{X,i}\).
Also, each morphism $X_1^* \to X_2^*$, where $X_j^* := (X_j^*, \{\gamma_j\}, G_{Z,j})$ ($j \in \{1, 2\}$), of $r$-pointed $(Z, \delta)$-spin curves induces, in a natural way, a morphism $\tilde{\kappa}_{Z,s,i} \to \tilde{\kappa}_{Z,s,2,i}$ in $\mathcal{T}_\mu(BZ)$. Hence, the assignment $X^* \mapsto \tilde{\kappa}_{Z,s,i}$ gives rise to a morphism

$$
(46) \quad ev_i^\langle \mathbb{S} \rangle : \mathbb{S}_{Z,\delta,g,r,k} \to \mathcal{T}_\mu(BZ)
$$

of $k$-stacks. The morphisms $ev_i^\langle \mathbb{S} \rangle$ ($i = 1, \ldots, r$) determines a morphism

$$
(47) \quad ev^\langle \mathbb{S} \rangle := (ev_1^\langle \mathbb{S} \rangle, \ldots, ev_r^\langle \mathbb{S} \rangle) : \mathbb{S}_{Z,\delta,g,r,k} \to \mathcal{T}_\mu(BZ)^{\times r}.
$$

Thus, for each $\tilde{\kappa} := (\kappa_i)_{i=1}^r \in \text{Inj}(\mu, Z)^{\times r}$, we obtain an open and closed sub-stack

$$
(48) \quad \mathbb{S}_{Z,\delta,g,r,\tilde{\kappa},k} := (ev^\langle \mathbb{S} \rangle)^{-1}(\prod_{i=1}^r \mathcal{T}_\mu(BZ)_{\kappa_i})
$$

of $\mathbb{S}_{Z,\delta,g,r,k}$. That is to say, $\mathbb{S}_{Z,\delta,g,r,\tilde{\kappa},k}$ classifies pointed stable $(Z, \delta)$-spin curves of type $(g, r)$ over $k$ whose $(Z, \delta)$-structure is of radii $\tilde{\kappa}$.

1.5. Gluing $(Z, \delta)$-structures.

In this subsection, we observe (cf. Proposition 1.5.2) certain factorization properties of the moduli stack of pointed stable $(Z, \delta)$-spin curves according to the gluing maps $\Phi_{\text{tree}}$ (cf. (16)) and $\Phi_{\text{loop}}$ (cf. (17)). Let $G$ be a gerbe over a $k$-scheme $S$ bandled by $\mu_l$ ($l > 0$). One may change the banding of the gerbe through the inversion automorphism $\zeta \mapsto \zeta^{-1}$ of $\mu_l$. Denote by $G^{\vee}$ the resulting gerbe and by $\text{inv}_G : G \xrightarrow{\sim} G^{\vee}$ the isomorphism over $S$ arising from the inversion. The assignment $(G, \phi) \mapsto (G^{\vee}, \phi^{\vee})$ (where $\phi^{\vee} := \phi \circ \text{inv}_G^{-1}$) induces an involution of $\mathcal{T}_{\mu_l}(BZ)$. By applying this argument to each piece $\mathcal{T}_{\mu_l}(BZ)$ ($l \geq 1$) of $\mathcal{T}_\mu(BZ)$ separately, we obtain an involution

$$
(49) \quad (-)^{\vee} : \mathcal{T}_{\mu}(BZ) \xrightarrow{\sim} \mathcal{T}_{\mu}(BZ)
$$

of $\mathcal{T}_{\mu}(BZ)$. For each injective morphism $\kappa : \mu_l \to Z$ in $\text{Inj}(\mu, Z)$, we shall write $\kappa^\vee$ for the injective morphism $\mu_l \to Z$ in $\text{Inj}(\mu, Z)$ given by $\zeta \mapsto \kappa(\zeta^{-1})$. Then, the involution $(-)^{\vee}$ obtained above restricts to an isomorphism $\mathcal{T}_{\mu_l}(BZ)_\kappa \xrightarrow{\sim} \mathcal{T}_{\mu_l}(BZ)_{\kappa^\vee}$.

Let $S$ be as above. For each $j = 1, 2$, let $(g_j, r_j)$ be a pair of nonnegative integers with $2g_j - 1 + r_j > 0$, $X_j^*$ a pointed stable curve over $S$ of type $(g_j, r_j + 1)$, and $(X_j^*, \gamma_j)$ (where $X_j^* := (X_j, \{\sigma_{X_j,i} : G_{Z,j} \to X_j\}_{i=1}^{r_j+1})$) a twistorification of $X_j^*$. Denote by $X_{\text{tree}}^*$ the pointed stable curve of type $(g_1 + g_2, r_1 + r_2)$ obtained by attaching the respective last marked points of $X_1^*$ and $X_2^*$ to form a node. Suppose that we are given an isomorphism $\epsilon : G_{1,r_1+1} \xrightarrow{\sim} G_{2,r_2+1}$ over $S$ compatible with the bands. Then, $(X_1^*, \gamma_1)$ and $(X_2^*, \gamma_2)$ may be glued
together, by means of \( \epsilon \) (cf. the \textit{balanced-ness} described in the fourth condition in Definition 1.2.2 (i)), to a twistification

\[(X^\bullet_{\text{tree}}, \gamma_{\text{tree}})\]

(50)

(where \( X^\bullet_{\text{tree}} := (X_{\text{tree}}, \{ \sigma_{\text{tree},i} \}_{i=1}^{r_1+r_2}) \) of \( X^\bullet \). In particular, we have closed immersions \( cl_j : X_j \to X_{\text{tree}} \) \((j = 1, 2)\), which fit into the following 1-commutative diagram:

\[\begin{array}{ccc}
G_{1,r_1+1} & \xrightarrow{\epsilon} & G_{2,r_2+1} \\
\downarrow{cl_1 \circ \sigma_{X_1,r_1+1}} & & \downarrow{cl_2 \circ \sigma_{X_2,r_2+1} \circ inv^{-1}_{r_2+1}} \\
X_{\text{tree}} & & \\
\end{array}\]

(51)

It is immediately verified that the morphism

\[cl_j^\ast(\Omega_{X^\log_{\text{tree}}/S^\log}) \to \Omega_{X_j^\log/S^\log}\]

induced by \( cl_j \) is an isomorphism. That is to say, we have a canonical 2-isomorphism

\[\left[ \Omega_{X^\log_{\text{tree}}/S^\log} \right] \circ cl_j \cong \left[ \Omega_{X_j^\log/S^\log} \right]\]

(52)

(cf. (30)). The following proposition may be immediately verified.

\textbf{Proposition 1.5.1.}

\textit{Let us keep the above notation.}

(i) Let \( \pi_G : G \to X^\bullet_{\text{tree}} \) be a \((Z, \delta)\)-structure on \( X^\log_{\text{tree}}/S^\log \). Then, for each \( j = 1, 2 \), the pull-back \( \pi_{G_j} : G_j := cl_j^\ast(G) \to X^\bullet_j \) of \( \pi_G \) via \( cl_j \) forms \((via (52))\) a \((Z, \delta)\)-structure on \( X^\log_j/S^\log \). Moreover, by restricting \( G_j \) to \( \sigma_{X_j,r_j+1} \) and applying the commutativity of diagram (51), we obtain an isomorphism

\[\epsilon_G := (\epsilon, \epsilon_\phi) : \tilde{\alpha}_{G_{1,r_1+1}} \overset{\sim}{\Rightarrow} \tilde{\alpha}_{G_{2,r_2+1}} \overset{\sim}{\Rightarrow} (G_{2,r_2+1}, \delta_G_{2,r_2+1}) \]

in \( \mathcal{T}_a(BZ) \), where \( \epsilon \) is as in (51) and \( \epsilon_\phi \) denotes a 2-isomorphism \( \phi_{G_{1,r_1+1}} \Rightarrow \phi_{G_{2,r_2+1}} \circ \epsilon \).

(ii) Conversely, for each \( j \in \{1, 2\} \), let \( \pi_{G,j} : G_j \to X^\bullet_j \) be a \((Z, \delta)\)-structure on \( X^\log_j/S^\log \). Also, assume that we are given an isomorphism of the form \( (\epsilon, \epsilon_\phi) : \tilde{\alpha}_{G_{1,r_1+1}} \overset{\sim}{\Rightarrow} \tilde{\alpha}_{G_{2,r_2+1}} \). Then, there exists a unique \((up to isomorphism)\) \((Z, \delta)\)-structure \( \pi_{G,\text{tree}} : G_{\text{tree}} \to X^\bullet_{\text{tree}} \) on \( X^\log_{\text{tree}}/S^\log \) together with isomorphisms \( cl_j^\ast(G_{\text{tree}}) \overset{\sim}{\Rightarrow} G_j \) \((j = 1, 2)\) of \((Z, \delta)\)-structures via which the equality \( (\epsilon, \epsilon_\phi) = \epsilon_{G_{\text{tree}}} \) (cf. (54) above) holds.
Similarly, let \( g, r \) be nonnegative integers with \( 2g + r > 0 \), \( \mathcal{X}^* \) a pointed stable curve over \( S \) of type \((g, r + 2)\), and \((\mathcal{X}^*, \gamma)\) (where \( \mathcal{X}^* := (\mathcal{X}, \{\sigma_{\mathcal{X}, i} : \mathcal{S}_i \rightarrow \mathcal{X}\}_{i=1}^j) \)) a twistification of \( \mathcal{X}^* \). Denote by \( \mathcal{X}_{\text{loop}}^* \) the pointed stable curve of type \((g + 1, r)\) obtained by attaching the last two marked points of \( \mathcal{X}^* \) to form a node. Suppose that we are given an isomorphism \( \epsilon : \mathcal{S}_{r+1} \isom \mathcal{S}_{r+2} \) over \( S \) compatible with the bands. By attaching the last two marked points of \( \mathcal{X}^* \) along \( \epsilon \), we obtain a twistification \((\mathcal{X}_{\text{loop}}^*, \gamma_{\text{loop}})\) of \( \mathcal{X}_{\text{loop}}^* \). By the discussion similar to the above discussion, there exists a natural bijective correspondence \( \mathcal{G} \mapsto \mathcal{G}_{\text{loop}} \) between the set of isomorphism classes of \((Z, \delta)\)-structures \( \mathcal{G} \) on \( \mathcal{X}_{\text{log}}^*/S^\text{log} \) admitting an isomorphism \( \tilde{\alpha}_{g,r+1} \isom \tilde{\alpha}_{g,r+2}^\vee \) compatible with \( \epsilon \) and the set of isomorphism classes of \((Z, \delta)\)-structures \( \mathcal{G}_{\text{loop}} \) on \( \mathcal{X}_{\text{loop}}^*/S^\text{log} \). Thus, by these discussions, we conclude the following proposition.

**Proposition 1.5.2.**

(i) Let \( g_1, g_2, r_1, \) and \( r_2 \) be nonnegative integers with \( 2g_j - 1 + r_j > 0 \) \((j = 1, 2)\), where we write \( g := g_1 + g_2, r := r_1 + r_2 \). Also, let \( \kappa_1 \in \text{Inj}(\mu, Z)^{r_1} \) and \( \kappa_2 \in \text{Inj}(\mu, Z)^{r_2} \). Then, for each \( \kappa \in \text{Inj}(\mu, Z) \), the assignment

\[
((\mathcal{X}_1^*, \mathcal{X}_2^*, \gamma_1, \mathcal{G}_1), (\mathcal{X}_2^*, \mathcal{X}_2^*, \gamma_2, \mathcal{G}_2)) \mapsto (\mathcal{X}_{\text{tree}}^*, \mathcal{X}_{\text{tree}}^*, \gamma_{\text{tree}}, \mathcal{G}_{\text{tree}})
\]

resulting from Proposition 1.5.1 determines a morphism

\[
\Phi_{\text{tree}, \kappa}^\text{Sp} : \text{Sp}_{Z, \delta, g_1, r_1+1, (\kappa_1, \kappa), k} \times \text{ev}_{r_1+1}^\text{Sp} \times_{\mu(BZ)_k} \text{Sp}_{Z, \delta, g_2, r_2+1, (\kappa_2, \kappa^\vee), k} \rightarrow \text{Sp}_{Z, \delta, g, r, (\kappa_1, \kappa_2), k}
\]

Moreover, the following square diagram is commutative and cartesian:

\[
\begin{array}{ccc}
\prod_{\kappa \in \text{Inj}(\mu, Z)} \text{Sp}_{Z, \delta, g_1, r_1+1, (\kappa_1, \kappa), k} \times_{\mu(BZ)_k} \text{Sp}_{Z, \delta, g_2, r_2+1, (\kappa_2, \kappa^\vee), k} & \xrightarrow{\prod_{\kappa} \Phi_{\text{tree}, \kappa}^\text{Sp}} & \text{Sp}_{Z, \delta, g, r, (\kappa_1, \kappa_2), k} \\
\downarrow \text{ev}_{r_1+1}^\text{Sp} \times \text{ev}_{r_2+1}^\text{Sp} & & \downarrow \text{ev}_{g, r}^\text{Sp} \\
\mathcal{M}_{g_1, r_1+1, k} \times_k \mathcal{M}_{g_2, r_2+1, k} & \xrightarrow{\Phi_{\text{tree}}} & \mathcal{M}_{g, r, k}.
\end{array}
\]

(ii) Let \( g, r \) be nonnegative integers with \( 2g + r > 0 \), and let \( \kappa \in \text{Inj}(\mu, Z)^{r} \). Then, for each \( \kappa \in \text{Inj}(\mu, Z) \), the assignment

\[
(\mathcal{X}^*, \mathcal{X}^*, \gamma, \mathcal{G}) \mapsto (\mathcal{X}_{\text{loop}}^*, \mathcal{X}_{\text{loop}}^*, \gamma_{\text{loop}}, \mathcal{G}_{\text{loop}})
\]

resulting from the above discussion determines a morphism

\[
\Phi_{\text{loop}, \kappa}^\text{Sp} : \text{Ker} \left( \text{Sp}_{Z, \delta, g, r+2, (\kappa, \kappa, \kappa^\vee), k} \xrightarrow{\text{ev}_{r+1}^\text{Sp} \times_{\mu(BZ)_k}} \mathcal{M}_{g, r, k} \right) \rightarrow \text{Sp}_{Z, \delta, g, r, (\kappa, \kappa, \kappa^\vee), k}.
\]
Moreover, the following square diagram is commutative and cartesian:

\[
\begin{array}{ccc}
\coprod_{\kappa \in \text{Inj}(\mu, Z)} \text{Ker} \left( \mathcal{Gp}_{Z, \delta, g, r + 2, (\bar{z}, \kappa, \kappa, \pm)} \right) & \xrightarrow{\Phi_{\text{loop}, \kappa}} & \mathcal{Gp}_{Z, \delta, g + 1, r, \bar{z}, k} \\
\downarrow \Phi_{g, r + 2} & & \downarrow \Phi_{g + 1, r} \\
\overline{\mathcal{M}}_{g, r + 2, k} & \xrightarrow{\xi_{g + 1, r}} & \overline{\mathcal{M}}_{g + 1, r, k}.
\end{array}
\]

1.6. Forgetting tails.

Next, we observe the factorization of $\mathcal{Gp}_{Z, \delta, g, r, \bar{z}, k}$ according to the forgetting-tails map $\Phi_{\text{tail}}$ (cf. [18]). Let $r$ be a nonnegative integer, $S$ a $k$-scheme, and $X^\bullet := (X, \{\sigma_{X,i}\}_{i=1}^{r+1})$ an $(r+1)$-pointed smooth curve over $S$. Write $X^\bullet_{\text{tail}} := (X, \{\sigma_{X,i}\}_{i=1}^{r})$, i.e., the $r$-pointed curve obtained from $X^\bullet$ by forgetting the last marked point.

Let $(X^\bullet_{\text{tail}}, \gamma_{\text{tail}})$ (where $X^\bullet_{\text{tail}} := (X_{\text{tail}}, \{\sigma_{X_{\text{tail}}, i}\}_{i=1}^{r})$) be a twistification of $X^\bullet_{\text{tail}}$.

**Lemma 1.6.1.**

There exists a unique (up to isomorphism) twistification $(X^\bullet_{+, \delta}, \gamma_{+, \delta})$ of $X^\bullet$ which is isomorphic to $(X^\bullet_{\text{tail}}, \gamma_{\text{tail}})$ when restricted to $X \setminus \text{Im}(\sigma_{X,r+1})$ and such that the stabilizer at any geometric point in $X^\bullet_{+, \delta}$ ($:= \text{the underlying curve of } X^\bullet_{+, \delta}$) lying over $\text{Im}(\sigma_{X,r+1})$ ($\subseteq X$) has order 2.

**Proof.** Let us construct of the desired twistification. Consider the category $X[\text{Im}(\sigma_{X,r+1})/2]$ (cf. [13], Definition 2.2) consisting of collections $(S, M, j, s)$, where

- $S$ is an $X$-scheme $S \rightarrow X$,
- $M$ is a line bundle on $S$,
- $j$ is an isomorphism between $M^{\otimes 2}$ and the pull-back of $O_X(\text{Im}(\sigma_{X,r+1}))$ on $S$,
- $s$ is a global section of $M$ such that $j(s^{\otimes 2})$ equals the tautological section of $O_X(\text{Im}(\sigma_{X,r+1}))$ vanishing along $\sigma_{X,r+1}$.

The morphisms are defined in the obvious way. $X[\text{Im}(\sigma_{X,r+1})/2]$ may be represented by a Deligne-Mumford stack over $k$ with coarse moduli space $X$. The projection $\text{coa} : X[\text{Im}(\sigma_{X,r+1})/2] \rightarrow X$ (i.e., $\text{coa} := \text{coa}_X[\text{Im}(\sigma_{X,r+1})/2]$) is an isomorphism over $X \setminus \text{Im}(\sigma_{X,r+1})$. Also, $X[\text{Im}(\sigma_{X,r+1})/2]$ is equipped with a tautological line bundle $\mathcal{M}$ and an isomorphism $\mathcal{M}^{\otimes 2} \xrightarrow{\sim} \text{coa}^*(O_X(\text{Im}(\sigma_{X,r+1})))$.

Let us write $X^\bullet_{+, \delta} := X^\bullet_{\text{tail}} \times_X X[\text{Im}(\sigma_{X,r+1})/2]$ and write $\xi : X^\bullet_{+, \delta} \rightarrow X^\bullet_{\text{tail}}$ for the projection to the first factor. There exists a unique closed immersion $\sigma_{X_{+, \delta}, r+1} : \mathcal{G}_{r+1} \rightarrow X^\bullet_{+, \delta}$ (where $\mathcal{G}_{r+1}$ is an étale gerbe over $S$ banded by $\mu_2$) inducing the closed immersion $\sigma_{X,r+1}$ between the coarse moduli spaces. Since
Next, let \( G_{\text{tail}} \) be a \((Z, \delta)\)-structure on \( X_{\log}^{\text{log}}/S \). By means of it, we shall construct a \((Z, \delta)\)-structure on \( X^{\log}_{+,+}/S \), as follows. Observe that

\[
\Omega_{X^{\log}_{+,+}/S} \cong \xi^*(\Omega_{X^{\log}_{+,+}/S_{\text{log}}}) \otimes (\text{coa}_{X_{\text{tail}}} \circ \xi)^*(\mathcal{O}_X(\text{Im}(\sigma_{X,r+1})))
\]

\[
\cong \xi^*(\Omega_{X_{\text{log}}}/S) \otimes (\text{coa} \circ \text{coa}')^*(\mathcal{O}_X(\text{Im}(\sigma_{X,r+1})))
\]

\[
\cong \xi^*(\Omega_{X_{\text{log}}}/S) \otimes \text{coa}^*(\mathcal{M})^{\otimes 2},
\]

where \( \text{coa}' \) denotes the projection \( X_{+\delta} \to X[\text{Im}(\sigma_{X,r+1})/2] \). Hence, by twisting \( G_{\text{tail}} \) by the \( G_m \)-torsor corresponding to \( \text{coa}^*(\mathcal{M}) \), we obtain a \( \hat{\mathbb{Z}}_\delta \)-torsor \( G_{+,+} \), which forms a \((Z, \delta)\)-structure on \( X^{\log}_{+,+}/S \). Since the nontrivial automorphism of \( \mathcal{G}_{r+1} \) over \( S \) arises from the automorphism of \( \mathcal{M} \) given by multiplication by \((-1) \in \mu_2 \), the commutativity of the left-hand square in (28) implies that the radius of \( G_{+,+} \) at \( \sigma_{X,+r+1} \) coincides with \( \delta : \mu_2 \to \mathbb{Z} (\subseteq \hat{\mathbb{Z}}_\delta) \). Thus, we have obtained a \((Z, \delta)\)-structure

\[
(\mathcal{X}_{-,+}^{\star}, \gamma_{-,+}, G_{-,+})
\]

on \( X^\star \) with \( \kappa_{G_{-,+},r+1} = \overline{\delta} \).

Conversely, suppose that we are given a \((Z, \delta)\)-structure \((\mathcal{X}^\star, \gamma, G)\) on \( X^\star \) with \( \kappa_{G,r+1} = \overline{\delta} \). One may find a unique (up to isomorphism) twistification \((\mathcal{X}_{-,+}^{\star}, \gamma_{-,+})\) (where \( \mathcal{X}_{-,+}^{\star} := (\mathcal{X}_{-,+}, \{\sigma_{X,-,i}\}_{i=1}^r) \)) of \( X_{\text{tail}} \) such that there exists an isomorphism \( \mathcal{X} \cong \mathcal{X}_{-,+} \times_X X[\text{Im}(\sigma_{X,r+1})/2] \) whose composite with \( \gamma_{-,+} \times \cos : \mathcal{X}_{-,+} \times_X X[\text{Im}(\sigma_{X,r+1})/2] \to X \) coincides with \( \gamma \). If \( G \cdot \mathcal{M} \) denotes the twist of \( G \) by the pull-back of \( \mathcal{M} \) via the (composite) projection \( \mathcal{X} \to X[\text{Im}(\sigma_{X,r+1})/2] \), then \( 2\overline{\delta} = 0 \in \text{Inj}(\mu, Z) \) implies that the automorphism group of \( \mathcal{G}_{r+1} \) acts on \( \sigma_{X,r+1}^*(G \cdot \mathcal{M}) \) trivially. Hence, \( G \cdot \mathcal{M} \) comes from a (unique up to isomorphism) \((Z, \delta)\)-structure \( G_{-,+} \) on \( X^{\log}_{-,+}/S \) via pull-back by the projection \( \mathcal{X} \to \mathcal{X}_{-,+} \). Consequently, we have obtained the following proposition.
Proposition 1.6.2.

(i) The assignments $(X_{\text{tail}}, \gamma_{\text{tail}}, G) \mapsto (X^\star, \gamma^\star, G^\delta)$ and $(X^\star, \gamma, G) \mapsto (X^\star, \gamma^\star, G_{-\delta})$ discussed above determine an equivalence of categories between the groupoid of $(Z, \delta)$-structures on $X_{\text{tail}}$ and the groupoid of $(Z, \delta)$-structures on $X^\star$ whose radius at $\sigma_{X, r+1}$ coincides with $\delta$.

(ii) Let $g$, $r$ be nonnegative integers with $2g - 1 + r > 0$, and let $\tilde{\kappa} \in \text{Inj}(\mu, Z)^{2r}$. Then, the assignment

$$
(X^\star, (X_{\text{tail}}, X^\star, \gamma_{\text{tail}}, G)) \mapsto (X^\star, (X^\star, \gamma^\star, G_{-\delta}))
$$

determines an isomorphism

$$
\mathcal{M}_{g, r+1, k} \times \mathfrak{M}_{g, r, k} \times \mathcal{G}_{Z, \delta, g, r+1, k} \simeq \mathcal{M}_{g, r+1, k} \times \mathfrak{M}_{g, r+1, k} \times \mathcal{G}_{Z, \delta, g, r+1, (\tilde{\kappa}, \delta), k}
$$

over $\mathcal{M}_{g, r+1, k}$.

1.7. $(Z, \iota)$-structures on the 2-pointed projective line.

In this last subsection, we shall study the $(Z, \delta)$-structures on the 2-pointed projective line. For a $k$-scheme $S$, let $\mathbb{P}_S := \mathcal{P}\text{roj} (\mathcal{O}_S[x, y])$ denote the projective line over $S$, i.e., the moduli space classifying ratios $[x : y]$ (for $x, y \in \mathcal{O}_S$ with $(x, y) \neq (0, 0)$). Denote by $\sigma_{\mathbb{P}, 1}$ and $\sigma_{\mathbb{P}, 2}$ the marked points of $\mathbb{P}_S$ determined by the values $0 (= [0 : 1])$ and $\infty (= [1 : 0])$ respectively. In particular, we have a 2-pointed curve $P^\star_S := (\mathbb{P}_S, \{\sigma_{\mathbb{P}, 1}, \sigma_{\mathbb{P}, 2}\})$ over $S$. $\mathbb{P}_S$ has two open subschemes $U_1 := \mathbb{P}_S \setminus \text{Im}(\sigma_{\mathbb{P}, 2}) = \text{Spec} (\mathcal{O}_S[s_1])$ (where $s_1 := x/y$) and $U_2 := \mathbb{P}_S \setminus \text{Im}(\sigma_{\mathbb{P}, 1}) = \text{Spec} (\mathcal{O}_S[s_2])$ (where $s_2 := y/x$).

Proposition 1.7.1.

(i) Let $(\mathbb{P}^\star, \gamma, G)$ be a $(Z, \delta)$-structure on $P^\star_S$. Then, the equality $\kappa_{G, 1} = \kappa_{G, 2}$ ($\in \text{Inj}(\mu, Z)$) holds.

(ii) Let $\kappa \in \mathcal{G}_{\mu, Z}$. Then, the assignment $(\mathbb{P}^\star, \gamma, G) \mapsto \widetilde{\kappa}_{G, 1}$ determines an equivalence of categories between the groupoid of $(Z, \delta)$-structures $(\mathbb{P}^\star, \gamma, G)$ on $P^\star_S$ with $\kappa = \kappa_{G, 1} = \kappa_{G, 2}$ and the groupoid $\mathcal{T}_\mu (\mathcal{B}Z)_\kappa (S)$ ($\simeq \mathcal{B}Z(S)$).

Proof. By taking account of descent with respect to étale coverings of $S$, we see that assertion (i) and (ii) are of local nature. Hence, one may assume, without loss of generality, that $\mu_l \subseteq \Gamma(S, \mathcal{O}_S)$ for any $l$ dividing $|Z|$.

Before proceeding, let us introduce some notation. Given a $k$-scheme $V$, we denote by $Z^\text{triv}_V$ the trivial $Z$-torsor over $V$. For each $j = 1, 2$ and a positive integer $l$, we shall define $U_{j, l}$ to be the $U_j$-scheme $\text{Spec} (\mathcal{O}_S[s_{j/l}])$, equipped with the $\mu_l$-action given by $s_{j/l} \mapsto \zeta s_{j/l}$ ($\zeta \in \mu_l$); given each $\zeta \in \mu_l$, we denote the corresponding automorphism of $U_{j, l}$ by $\alpha_{j, l, \zeta}$. Denote by $\pi_{j, l} : U_{j, l} \rightarrow [U_{j, l}/\mu_l]$
the natural projection. Finally, given $\xi \in Z$, we denote by $\beta_\xi : Z \xrightarrow{\sim} Z$ the translation of $Z$ by $\xi$.

Now, we shall consider assertion (i). To this end, one may assume, without loss of generality, that $S = \text{Spec}(k)$ and $k$ is algebraically closed. Let $(\mathfrak{P}^\star, \gamma, \mathcal{G})$ (where $\mathfrak{P}^\star := (\mathfrak{P}, \{\sigma_{\mathfrak{P},1}, \sigma_{\mathfrak{P},2}\})$) be a $(\mathcal{Z}, \delta)$-structure on $\mathfrak{P}_k^\star$. Since

$$\Omega_{\mathfrak{P}^\star,k} \cong \gamma^\star(\Omega_{\mathfrak{P}^\star,k}) \cong \gamma^\star(\mathcal{O}_{\mathfrak{P},k}) \cong \mathcal{O}_{\mathfrak{P}},$$

the classifying morphism $\mathfrak{P} \to \mathcal{B}\hat{Z}_\delta$ of $\mathcal{G}$ factors through $\mathcal{B}Z \to \mathcal{B}\hat{Z}_\delta$. That is to say, $\mathcal{G}$ may be thought of as a $Z$-torsor over $\mathfrak{P}$. Denote by $l_j$ (for each $j = 1, 2$) the order of the stabilizer in $\mathfrak{P}$ at $\sigma_{\mathfrak{P},j}$. Then, there exists an isomorphism $t_{j,l_j} : [U_{j,l_j}/\mu_{l_j}] \xrightarrow{\sim} \mathfrak{P} \setminus \text{Im}(\sigma_{\mathfrak{P},j})$ of stacks over $U_j$. Since we have assumed that $Z$ has order invertible in $k$ and $k$ is algebraically closed, $(t_{j,l_j} \circ \pi_{j,l_j})^\star(\mathcal{G})$ turns out to be trivial. Let us identify $(t_{j,l_j} \circ \pi_{j,l_j})^\star(\mathcal{G})$ with $Z_{U_{j,l_j}}^{\text{triv}}$ by a fixed isomorphism. For each $\zeta \in \mu_L$, the equality $(t_{j,l_j} \circ \pi_{j,l_j}) \circ \alpha_{j,l_j,\zeta} = (t_{j,l_j} \circ \pi_{j,l_j})$ implies that $\alpha_{j,l_j,\zeta}$ induces an automorphism of $(t_{j,l_j} \circ \pi_{j,l_j})^\star(\mathcal{G})$ over $U_{j,l_j}$. Under the identification $(t_{j,l_j} \circ \pi_{j,l_j})^\star(\mathcal{G}) = Z_{U_{j,l_j}}^{\text{triv}} = U_{j,l_j} \times_k Z$, this automorphism may be expressed (by the definition of $\kappa_{j,l_j}$) as $\alpha_{j,l_j,\zeta} \times \beta_{\kappa_{j,l_j}}(\zeta)$.

Next, let us consider the commutative diagram

$$\begin{array}{ccc}
U & \xrightarrow{v_1} & U_1, l_1 \\
\downarrow & & \downarrow \quad \downarrow \quad \downarrow \\
U_2, l_2 & \xrightarrow{v_2} & \mathfrak{P},
\end{array}$$

where $U := \text{Spec}(k[s_1^{1/L}, \ldots, s_L^{1/L}]) = \text{Spec}(k[s_1^{\pm 1/L}, \ldots, s_L^{\pm 1/L}])$. If $\tau_j : \mu_L \to \mu_{l_j}$ denotes the surjection given by $\zeta \mapsto \zeta^{l_j/L}$, then, for each $\zeta \in \mu_L$, the following equality of automorphisms of $Z_{U_{j,l_j}}^{\text{triv}} = v_j^\star(Z_{U_{j,l_j}}^{\text{triv}})$ holds:

$$v_j^\star(\alpha_{j,l_j,\tau_j(\zeta)} \times \beta_{\kappa_{j,l_j}}(\zeta)) = (\alpha_{j,l_j,\zeta}|_{U}) \times \beta(\kappa_{j,l_j,\zeta}(\tau_j(\zeta))).$$

By the commutativity of (66), there exists an isomorphism

$$Z_{U}^{\text{triv}} = v_1^\star(Z_{U_{1,l_1}}^{\text{triv}}) \xrightarrow{\sim} v_2^\star(Z_{U_{2,l_2}}^{\text{triv}}) = Z_{U}^{\text{triv}}$$

over $U$ (given by $\beta_\xi$ for some $\xi \in Z$). On the other hand, the equality $s_1^{1/L} = s_2^{-1/L}$ implies $\alpha_{1,l_j,\zeta}|_U = \alpha_{2,l_j,\zeta^{-1}}|_U$. By passing to (68) and taking account of (67), we obtain the equalities

$$\beta(\kappa_{j,l_j,\zeta}(\tau_j(\zeta)) = \left(\beta^{-1}_\xi \circ \beta(\kappa_{j,l_j,\zeta}(\tau_j(\zeta)) \circ \beta_\xi = \beta(\kappa_{j,l_j,\zeta}(\tau_j(\zeta)))^{-1} = \beta(\kappa_{j,l_j,\zeta}(\tau_j(\zeta))).$$
(for all $\zeta \in \mu_L$). Thus, the equality $\kappa_{G,l_1} \circ \tau_1 = (\kappa_{G,2} \circ \tau_2)^{\psi}$ of morphisms $\mu_L \to Z$ holds. Since both $\kappa_{G,l_1}$ and $\kappa_{G,l_2}$ are surjective, we have $l_1 = l_2$ and hence, $\kappa_{G,l_1} = \kappa_{G,2}^{\psi}$. This completes the proof of assertion (i).

Next, let us consider assertion (ii). Let $\kappa : \mu_l \to Z$ be an element of $\text{Inj}(\mu, Z)$. Denote by $\widetilde{\kappa} : S \times k \mathcal{B} \mu_l \to \mathcal{B} \mu$ the object of $\mathcal{F}(\mathcal{B} \mu)_n$ defined as the composite of the second projection $S \times k \mathcal{B} \mu_l \to \mathcal{B} \mu_l$ and $\mathcal{B} \kappa : \mathcal{B} \mu_l \to \mathcal{B} \mu$. Since $[U_{1,l}/\mu] \times_{\mathcal{F}_2} (U_1 \cap U_2) \cong U_1 \cap U_2 \cong [U_{2,l}/\mu] \times_{\mathcal{F}_2} (U_1 \cap U_2)$, the two stacks $[U_{1,l}/\mu]$ and $[U_{2,l}/\mu]$ may be glued together to a twisted curve $\mathcal{P}_Z$ over $S$ equipped with two marked points $\sigma_{p,1}$, $\sigma_{p,2}$ over $\mathcal{P}_{p,1}$, $\mathcal{P}_{p,2}$ respectively. Denote by $\mathcal{P}^\star_Z := (\mathcal{P}_Z, \{\sigma_{p,1}, \sigma_{p,2}\})$ the resulting pointed twisted curve and by $\gamma_\mathcal{P} : \mathcal{P}_Z \to \mathcal{P}_S$ the natural projection (i.e., $\gamma_\mathcal{P} := \text{cod}_{\mathcal{P}_Z}$). Observe that there exists a unique (up to isomorphism) $Z(= \mathcal{Z}_\delta)$-torsor over $\mathcal{P}_Z$ whose restriction to $\mathcal{P}_{p,1}$ and $\mathcal{P}_{p,2}$ are classified by $\widetilde{\kappa}$ and $\widetilde{\kappa}_{\mathcal{P}}$ respectively. Indeed, for each $j = 1, 2$, consider the trivial $Z$-torsor $Z^\text{triv}_{U_{j,l}}$ on $U_{j,l}$. Let us equip $Z^\text{triv}_{U_{j,l}}$ (resp., $Z^\text{triv}_{U_{2,l}}$) with the $\mu_l$-action given by $\alpha_{j,l,\zeta} \times \beta_{\kappa_{\mu_l}(\zeta)}$ (resp., $\alpha_{2,l,\zeta} \times \beta_{\kappa_{\mu_l}(\zeta)}$) for each $\zeta \in \mu_l$. These $\mu_l$-actions on $Z^\text{triv}_{U_{1,l}}$ and $Z^\text{triv}_{U_{2,l}}$ are compatible when restricted to $U := U_{1,l} \times_{\mathcal{F}_2} U_{2,l}$. By means of these actions and the identity morphism $(Z^\text{triv}_U =) Z^\text{triv}_{U_{1,l}|U} \cong Z^\text{triv}_{U_{2,l}|U} (= Z^\text{triv}_U)$ of $Z^\text{triv}_U$, the torsors $Z^\text{triv}_{U_{1,l}}$, $Z^\text{triv}_{U_{2,l}}$ may be glued together to the desired $Z$-torsor $\mathcal{G}_Z$. The injectivity of $\kappa$ implies that the classifying morphism $\mathcal{P}_Z \to \mathcal{B} \mu$ of $\mathcal{G}_Z$ is representable. Thus, we obtain a $(Z, \delta)$-structure $(\mathcal{P}^\star_Z, \gamma_\mathcal{P}, \mathcal{G}_Z)$ on $P^\star_S$. It is verified immediately that the assignment $\widetilde{\kappa} \mapsto (\mathcal{P}^\star_Z, \gamma_\mathcal{P}, \mathcal{G}_Z)$ determines the inverse of $(\mathcal{P}^\star_Z, \gamma, \mathcal{G}) \mapsto \widetilde{\kappa}_{\mathcal{G},1}$. (Indeed, by Proposition 1.3.6, the functor resulting from the assignment $\widetilde{\kappa} \mapsto (\mathcal{P}^\star_Z, \gamma_\mathcal{P}, \mathcal{G}_Z)$ is fully faithful.) This completes the proof of assertion (ii). □

Corollary 1.7.2.

\begin{align}
\mathcal{S}^\kappa_{Z,\delta,0,3,(\kappa_1,\kappa_2,\tau),k} & \cong \begin{cases} BZ & \text{if } \kappa_1 = \kappa_2^{\psi} \\ \emptyset & \text{if otherwise.} \end{cases} \\
\end{align}

Proof. The assertion follows from Propositions 1.6.2(i) and 1.7.1 □

2. Twisted opers on pointed stable curves

In this section, we define (cf. Definition(s) 2.2.2 and 2.3.1) the notion of a (faithful) twisted $G$-oper on a pointed stable curve and construct the moduli space, denoted by $\mathcal{D}P_{G,G,T,\{\rho\},k}$, classifying pointed stable curves paired with a faithful twisted $G$-oper (of prescribed radii $\rho$) on it. The main result of this section asserts (cf. Theorems 2.3.5 and 2.4.2) that this moduli space may be represented by a (nonempty) smooth Deligne-Mumford stack which is flat (of
constant relative dimension) over $\overline{\mathcal{M}}_{g,r,k}$. Also, we study the faithful twisted $G$-opers on the projective line with two or three marked points (cf. Proposition 2.6.1 and Corollary 2.6.2).

2.1. Algebraic groups and Lie algebras.

Let $G$ be a connected semisimple algebraic group over $k$. In this section, we shall assume that $\text{char}(k) = 0$ or $\text{char}(k) = p$ for some prime $p$ satisfying the condition $(\ast)_{G,p}$ described in Introduction. We fix a maximal torus $T$ of $G$ and a Borel subgroup $B$ containing $T$. Denote by $g$, $t$, and $b$ the Lie algebras of $G$, $T$, and $B$ respectively (hence, $t \subseteq b \subseteq g$). For each character $\beta$ of $T$, we write

$$(71) \quad g^\beta := \{ x \in g \mid \text{ad}(t)(x) = \beta(t) \cdot x \text{ for all } t \in T \}.$$  

Let $\Gamma$ denote the set of simple roots in $\mathfrak{g}$, and $\tilde{\omega}$ denotes the fundamental coweight of $\alpha$. One may find a unique collection $(y_{\alpha})_{\alpha \in \Gamma}$, where $y_{\alpha}$ is a generator of $g^{-\alpha}$, such that if we write $p_{-1} := \sum_{\alpha \in \Gamma} y_{\alpha}$, then the set $\{ p_{-1}, 2 \rho, p_1 \}$ forms an $\mathfrak{sl}_2$-triple. Finally, recall a canonical decreasing filtration $\{ g^j \}_{j \in \mathbb{Z}}$ on $g$ such that $g^0 = b$, $g^0/g^1 = \bigoplus_{\alpha \in \Gamma} g^\alpha$, and $[g^{j_1}, g^{j_2}] \subseteq g^{j_1+j_2}$ for $j_1, j_2 \in \mathbb{Z}$.

2.2. Twisted $G$-opers on stacky log-curves.

In the following, we shall define the notion of a twisted $G$-oper on a family of stacky log-curves (cf. Definition 2.2.2). Let $S^{\log}$ be an fs log scheme (or, more generally, an fs log stack) over $k$, $\mathcal{U}^{\log}$ a stacky log-curve over $S^{\log}$, and $\pi : \mathcal{E} \to \mathcal{U}$ a (right) $G$-torsor over $\mathcal{U}$. By pulling-back the log structure of $\mathcal{U}^{\log}$ via $\pi$, one may obtain a log structure on $\mathcal{E}$; we denote the resulting log stack by $\mathcal{E}^{\log}$. The $G$-action on $\mathcal{E}$ carries a $G$-action on the direct image $\pi_* (\mathcal{T}_{\mathcal{E}^{\log}/S^{\log}})$ of $\mathcal{T}_{\mathcal{E}^{\log}/S^{\log}}$. Denote by $\mathcal{T}_{\mathcal{E}^{\log}/S^{\log}}$ the subsheaf of $G$-invariant sections of $\pi_* (\mathcal{T}_{\mathcal{E}^{\log}/S^{\log}})$. The differential of $\pi$ gives rises to a short exact sequence

$$(72) \quad 0 \longrightarrow g_{\mathcal{E}} \longrightarrow \mathcal{T}_{\mathcal{E}^{\log}/S^{\log}} \xrightarrow{\nu_{\mathcal{E}}} \mathcal{T}_{\mathcal{U}^{\log}/S^{\log}} \longrightarrow 0$$

of $\mathcal{O}_\mathcal{U}$-modules.

**Definition 2.2.1.**

An $S^{\log}$-connection on $\mathcal{E}$ is an $\mathcal{O}_\mathcal{U}$-linear morphism $\nabla_\mathcal{E} : \mathcal{T}_{\mathcal{U}^{\log}/S^{\log}} \to \mathcal{T}_{\mathcal{E}^{\log}/S^{\log}}$ with $\mathfrak{d}_{\mathcal{E}} \circ \nabla_\mathcal{E} = \text{id}_{\mathcal{T}_{\mathcal{U}^{\log}/S^{\log}}}$.

Now, suppose that we are given a (right) $B$-torsor $\pi_B : \mathcal{E}_B \to \mathcal{U}$ over $\mathcal{U}$. Denote by $\pi_G : (\mathcal{E}_B \times^B G =:) \mathcal{E}_G \to \mathcal{U}$ the $G$-torsor over $\mathcal{U}$ obtained by change of structure group via the inclusion $B \hookrightarrow G$. The natural morphism $\mathcal{E}_B \to \mathcal{E}_G$
yields a canonical isomorphism $\mathfrak{g}_B \sim \mathfrak{g}_G$ and moreover a morphism between short exact sequences:

\[
\begin{array}{cclcl}
0 & \longrightarrow & b_E & \longrightarrow & T_{E_B}^{\log}/S_\log & \xrightarrow{a_{E_B}^{\log}} & T_{\mathfrak{U}^{\log}}^{\log} & \longrightarrow & 0 \\
\downarrow t_{g/b} & & \downarrow \tilde{t}_{g/b} & & \downarrow \text{id} \\
0 & \longrightarrow & \mathfrak{g}_G & \longrightarrow & \tilde{T}_{E_G}^{\log}/S_\log & \xrightarrow{a_{E_G}^{\log}} & T_{\mathfrak{U}^{\log}}^{\log} & \longrightarrow & 0,
\end{array}
\]

(73)

where the upper and lower horizontal sequences are (72) applied to $E_B$ and $E_G$ respectively. Since $\mathfrak{g}^i (\subseteq \mathfrak{g})$ is closed under the adjoint action of $B$, one obtains vector bundles $\mathfrak{g}_B^i (j \in \mathbb{Z})$ associated with $E_B \times_B \mathfrak{g}^j$, which form a decreasing filtration on $\mathfrak{g}_B (\cong \mathfrak{g}_G)$. On the other hand, diagram (73) induces a composite isomorphism

\[
\mathfrak{g}_B^0 \xrightarrow{\sim} \mathfrak{g}_G / t_{g/b}(b_E) \xrightarrow{\sim} \tilde{T}_{E_G}^{\log}/S_\log / t_{g/b}(\tilde{T}_{E_B}^{\log}/S_\log).
\]

(74)

The filtration $\{\mathfrak{g}_B^j\}_{j \leq 0}$ carries, via this composite isomorphism, a decreasing filtration $\{\tilde{T}_j^{\log}/S_j^{\log}\}_{j \leq 0}$ on $\tilde{T}_{E_G}^{\log}/S_\log$ in such a way that $\tilde{T}_0^{\log}/S_\log = t_{g/b}(\tilde{T}_{E_B}^{\log}/S_\log)$ and the resulting morphism $\mathfrak{g}_B^{-1} / \mathfrak{g}_E^0 \rightarrow \tilde{T}_j^{\log}/S_j^{\log} / \tilde{T}_0^{\log}/S_\log$ is an isomorphism. Since each $\mathfrak{g}^{-\alpha}$ ($\alpha \in \Gamma$) is closed under the action of $B$ (defined to be the composite $B \rightarrow T \xrightarrow{\text{adj. rep.}} \text{Aut}(\mathfrak{g}^{-\alpha}))$, the canonical decomposition $\mathfrak{g}^{-1} / \mathfrak{g}^0 = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^{-\alpha}$ gives rise to a decomposition

\[
\tilde{T}_0^{\log}/S_\log / \tilde{T}_0^{\log}/S_\log = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_B^{-\alpha}.
\]

(75)

**Definition 2.2.2.**

(i) A twisted $G$-oper on $\mathfrak{U}^{\log}/S_\log$ is a pair

\[
\mathcal{E} := (\pi_B : E_B \rightarrow \mathfrak{U}, \nabla_E : T_{\mathfrak{U}^{\log}}^{\log} \rightarrow \tilde{T}_{E_G}^{\log}/S_\log)
\]

(76)

consisting of a $B$-torsor $E_B$ over $\mathfrak{U}$ and an $S_\log$-connection $\nabla_E$ on the $G$-torsor $E_B : E_G \rightarrow \mathfrak{U}$ induced by $E_B$ satisfying the following two conditions:

- $\nabla_E(T_{\mathfrak{U}^{\log}}^{\log}) \subseteq \tilde{T}_{E_G}^{-1}/S_\log$;
- For any $\alpha \in \Gamma$, the composite

\[
T_{\mathfrak{U}^{\log}}^{\log} \xrightarrow{\nabla_E} \tilde{T}_{E_G}^{-1}/S_\log \rightarrow \tilde{T}^{-1}/S_\log \rightarrow \tilde{T}_{E_G}^{-1}/S_\log \rightarrow \mathfrak{g}_{E_B}
\]

is an isomorphism, where the third arrow denotes the natural projection relative to decomposition (75).
If $\log / S^{\log} = X^{\log} / S^{\log}$ for a pointed twisted curve $X^\bullet := (X/S, \{\sigma_{X,i}\}_i)$, then we shall refer to any twisted $G$-oper on $X^{\log} / S^{\log}$ as a twisted $G$-oper on $X^\bullet$.

If $X^\bullet$ is a pointed nodal curve, then we define a twisted $G$-oper on $X^\bullet$ to be a collection of data

$$\mathcal{E}^\bullet := (X^\bullet, \gamma, \mathcal{E}^\bullet),$$

consisting of a twistification $(X^\bullet, \gamma)$ of $X^\bullet$ and a twisted $G$-oper on $X^\bullet$.

\[(ii)\] For each $j = 1, 2$, let $S_j$ be a $k$-scheme and $X_j^\bullet := (X_j^\bullet, f_j, \sigma_{X_j,i})$ (where $X_j^\bullet := (f_j : X_j \to S_j, \{\sigma_{X_j,i}\}_i)$) a collection of data consisting of a pointed nodal curve $X_j^\bullet$ and a twisted $G$-oper $(X_j^\bullet, \gamma_j, \mathcal{E}_j^\bullet)$

on it. A 1-morphism (or just a morphism) from $X_1^\bullet$ to $X_2^\bullet$ is a triple

$$\alpha^\bullet := (\alpha^\bullet, \alpha^\bullet, \alpha^\bullet)$$

of morphisms which make the following diagram 1-commutative:

$$\begin{array}{c}
\mathcal{E}_{B,1} \xrightarrow{\pi_{B,1}} X_1 \xrightarrow{f_1} S_1 \\
\downarrow \alpha^\bullet \quad \downarrow \alpha^\bullet \quad \downarrow \alpha^\bullet \\
\mathcal{E}_{B,2} \xrightarrow{\pi_{B,2}} X_2 \xrightarrow{f_2} S_2,
\end{array}$$

where

- the right-hand square diagram forms a morphism of pointed twisted curves (cf. Definition 1.2.4 (ii)),
- the left-hand square is cartesian, and $\alpha^\bullet$ is compatible with the respective $B$-actions of $\mathcal{E}_{B,1}$ and $\mathcal{E}_{B,2},$
- the morphism $\mathcal{E}_{G,1} := \mathcal{E}_{B,1} \times B G \to \mathcal{E}_{G,2} := \mathcal{E}_{B,2} \times B G$ induced by $\alpha^\bullet$ is compatible with the respective connections $\nabla_{\mathcal{E},1}$ and $\nabla_{\mathcal{E},2}.$

In particular, one may associate, to such a morphism $\alpha^\bullet$, a morphism $\alpha^\bullet : X_1^\bullet \to X_2^\bullet$ between the underlying pointed nodal curves.

\[(iii)\] Let $X_j^\bullet$ ($j = 1, 2$) be as in (ii) and $\alpha^\bullet := (\alpha^\bullet, \alpha^\bullet, \alpha^\bullet)$ ($l = 1, 2$) (1-)morphisms from $X_1^\bullet$ to $X_2^\bullet$. A 2-morphism from $\alpha^\bullet$ to $\alpha^\bullet$ is a triple

$$a^\bullet := (\alpha^\bullet, a^\bullet, a^\bullet, a^\bullet, a^\bullet, a^\bullet)$$

of natural transformations compatible with each other (hence, $a^\bullet$ coincides with the identity natural transformation).
2.3. Faithful twisted \( G \)-opers and their moduli.

Next, we shall introduce the notion of a faithful twisted \( G \)-oper, as follows. Let \( \mathcal{U}^{\text{log}}/S^{\text{log}} \) as before.

**Definition 2.3.1.**
We shall say that a twisted \( G \)-oper \( \mathcal{E}^\bullet := (\mathcal{E}_B, \nabla_\mathcal{E}) \) on \( \mathcal{U}^{\text{log}}/S^{\text{log}} \) is faithful if the classifying morphism \( \mathcal{U} \to BT \) of the \( T \)-torsor \( \mathcal{E}_B \times^B T \) is representable. Also, let \( \mathcal{E}^\bullet := (X^\bullet, \gamma, \mathcal{E}^\bullet) \) be a twisted \( G \)-oper on a pointed nodal curve. Then, we shall say that \( \mathcal{E}^\bullet \) is faithful if \( \mathcal{E}^\bullet \) is faithful.

We shall observe the relationship with the notion of an extended spin structure discussed in \( \S \) 1. Denote by \( Z \) the center of \( G \), which is finite and has order invertible in \( k \) (because of the assumption imposed at the beginning of \( \S \) 2.1). Write \( G_{\text{ad}} := G/Z \) (i.e., the adjoint group of \( G \)) and \( T_{\text{ad}} := T/Z \).

Let \( \lambda : G_m \to T_{\text{ad}} \) (cf. \( \S \) 2.3, Definition 2.3.1). Write \( \mathcal{E}_B \) as multiplication by \( t \), i.e., the morphism determined by the condition that for any \( \alpha \in G \), \( \lambda(t) \) acts on \( g^\alpha \) (via the adjoint representation \( (T_{\text{ad}} \subseteq G_{\text{ad}} \to \text{GL}(g)) \)) as multiplication by \( t \). Then, one may find a unique morphism \( \lambda^\bullet : G_m \to T \) (cf. \( \S \) 10, (54)) such that \( \lambda(t)^2 = \lambda^\bullet(t) \) mod \( Z \) for any \( t \in G_m \). \( \lambda^\bullet \) restricts to a morphism \( \delta^\bullet : \mu_2 \to Z \) (\( \subseteq T \)). Observe that the following square diagram is commutative:

\[
\begin{array}{ccc}
\mu_2 & \xrightarrow{\text{incl.}} & G_m \\
\downarrow & & \downarrow \\
Z & \xrightarrow{z \mapsto (z,e)} & T \times_{T_{\text{ad}},\lambda} G_m \\
\end{array}
\]

(82)

where \( e \) denotes the unit of \( G_m \). Hence, this diagram determines a morphism

\[
\tilde{Z}_{\delta^\bullet} \ (\cong Z \times^\bullet \mu_2 G_m) \to T \times_{T_{\text{ad}},\lambda} G_m,
\]

which is an isomorphism since it fits into the following morphism of short exact sequences:

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & Z & \longrightarrow & \tilde{Z}_{\delta^\bullet} & \longrightarrow & G_m & \longrightarrow & 0 \\
& & \downarrow \text{id}_Z & & \downarrow \text{Z}^{(83)} & & \downarrow \text{id}_{G_m} & & \\
0 & \longrightarrow & Z & \xrightarrow{z \mapsto (z,e)} & T \times_{T_{\text{ad}},\lambda} G_m & \xrightarrow{(h,\psi) \mapsto \psi} & G_m & \longrightarrow & 0,
\end{array}
\]

(84)

where the upper horizontal sequence is the lower horizontal sequence in (28).

Now, let \( \mathcal{E}^\bullet := (\mathcal{E}_B, \nabla_\mathcal{E}) \) be a twisted \( G \)-oper on \( \mathcal{U}^{\text{log}}/S^{\text{log}} \). If \( (\mathcal{U}^{\text{log}}/S^{\text{log}})^\times \) denotes the \( G_m \)-torsor over \( \mathcal{U} \) corresponding to the line bundle \( \mathcal{U}^{\text{log}}/S^{\text{log}} \), then the \( T_{\text{ad}} \)-torsor \( \mathcal{E}_B \times^B T_{\text{ad}} \) (obtained from \( \mathcal{E}_B \) via change of structure group by the composite \( B \to B/[B,B] \) \( \cong T \) \( \to T_{\text{ad}} \)) is isomorphic to \( (\mathcal{U}^{\text{log}}/S^{\text{log}})^\times \times^{G_m,\lambda} T_{\text{ad}} \) (cf. the discussion in \( \S \) 10, §3.4.1). Hence, by passing to (83), one may use the
$T$-torsor $E_B \times^B T$ and the $\mathbb{G}_m$-torsor $(\Omega_{\mathcal{U}^{\log}/S^{\log}})^{\times}$ in order to obtain a $\hat{Z}_{\delta g}$-torsor

$$\pi_{G, E} := G_{Z, \delta g, E} \to \mathcal{U}.$$  

By definition, $G_{Z, \delta g, E} \times \hat{Z}_{\delta g} \mathbb{G}_m$ is isomorphic to $(\Omega_{\mathcal{U}^{\log}/S^{\log}})^{\times}$.

**Proposition 2.3.2.**

$E^\bullet$ is faithful if and only if $G_{Z, \delta g, E}$ forms a $(Z, \delta^*)$-structure on $\mathcal{U}^{\log}/S^{\log}$.

**Proof.** Let us consider the composite

$$\hat{Z}_{\delta g} \overset{(83)}{\longrightarrow} T \times_{T_{ad}} \mathbb{G}_m \longrightarrow T,$$

where the second arrow denotes the first projection. The composite of the classifying morphism $[G_{Z, \delta g, E}]: \mathcal{U} \to \hat{Z}_{\delta g}$ of $G_{Z, \delta g, E}$ and the morphism $B\hat{Z}_{\delta g} \to BT$ induced by $(86)$ coincides with the classifying morphism $[E_B \times^B T]: \mathcal{U} \to BT$ of $E_B \times^B T$. Since $(86)$ is a closed immersion, it follows from [5], Lemma 4.4.3, that $[G_{Z, \delta g, E}]$ is representable if and only if $[E_B \times^B T]$ is representable. This completes the proof of Proposition 2.3.2. □

**Remark 2.3.3.**

Let us consider the case where $G$ is of adjoint type, i.e., $G = G_{ad}$. Then, the definition of a faithful twisted $G$-oper introduced above may be identified with the classical definition of a $\frak{g}$-oper in the sense of [55], Definition 2.2.1 (i). Indeed, let $X^\bullet$ be a pointed nodal curve and $E^\bullet := (X^\bullet, \gamma, E^\bullet)$ a twisted $G$-oper on $X^\bullet$. The representability of the classifying morphism $X^\bullet \to \hat{Z}_{\delta g}$ of $G_{\mathcal{E}^\bullet}$ and the assumption $Z = \{1\}$ imply (cf. [5], Lemma 4.4.3) that the stabilizers of the nodes and the marked points of $X^\bullet$ are trivial. Hence, the morphism $\gamma: X^\bullet \to X$ must be an isomorphism and $E^\bullet$ may be thought of as a $\frak{g}$-oper (in the classical sense) on $X^\bullet$. Conversely, given a $\frak{g}$-oper $E^\bullet$ on $X^\bullet$, we obtain a faithful twisted $G$-oper $(X^\bullet, \text{id}_X, E^\bullet)$ on $X^\bullet$. In this way, we shall not distinguish between faithful twisted $G$-opers (in the case of $G = G_{ad}$) and $\frak{g}$-opers.

In the way of Definition 2.2.2 (ii) and (iii), the collections $(X^\bullet, E^\bullet)$ (where $X^\bullet$ denotes a pointed nodal curve and $E^\bullet$ denotes a twisted $G$-oper on it) form a 2-category. Just as in the case of $\mathcal{Sp}_{Z, \delta g, r, k}$ (cf. the discussion following Definition 1.3.2), this 2-category is equivalent to the 1-category consisting of $(X^\bullet, E^\bullet)$'s and 2-isomorphism classes of 1-morphisms between them. If $g, r$ are nonnegative integers with $2g - 2 + r > 0$, then we shall denote by

$$(87) \quad \mathcal{O}p_{G, g, r, k}$$
the (1-)category of pairs \((X^\star, \mathfrak{c})\) consisting of a pointed stable curve \(X^\star\) of type \((g, r)\) over a \(k\)-scheme and a faithful twisted \(G\)-oper \(\mathfrak{c}\) on it. The assignments \((X^\star, \mathfrak{c}) \mapsto X^\star\) and \((X^\star, \mathfrak{c}) \mapsto (X^\star, G_{Z, \delta^\star, \mathfrak{c}})\) determine functors

\[
\mathcal{O}p_{G, g, r, k} \rightarrow \overline{M}_{g, r, k} \quad \text{and} \quad \mathcal{O}p_{G, g, r, k} \rightarrow \mathcal{S}p_{Z, \delta^\star, g, r, k}
\]

respectively. Also, by change of structure group via \(G \rightarrow \hat{G}\), we obtain a functor

\[
op_{\hat{G}, g, r, k} : \mathcal{O}p_{G, g, r, k} \rightarrow \mathcal{O}p_{\hat{G}, g, r, k}
\]

over \(\overline{M}_{g, r, k}\).

**Remark 2.3.4.**

Let \(X^\star\) be a pointed stable curve. Then, we shall refer to each 2-isomorphism class of a 1-isomorphism defined in Definition 2.2.2 (ii) inducing the identity morphism of \(X^\star\) as an isomorphism of faithful twisted \(G\)-opers on \(X^\star\). In this way, we obtain the groupoid of faithful twisted \(G\)-opers on \(X^\star\).

**Theorem 2.3.5.**

(i) The functor

\[
\mathcal{O}p_{G, g, r, k} \tilde{\rightarrow} \mathcal{O}p_{\hat{G}, g, r, k} \times M_{g, r, k} \mathcal{S}p_{Z, \delta^\star, g, r, k}
\]

induced by (89) and (the second morphism in) (88) is an equivalence of categories over \(\overline{M}_{g, r, k}\).

(ii) \(\mathcal{O}p_{G, g, r, k}\) may be represented by a nonempty smooth Deligne-Mumford stack over \(k\) which is flat over \(\overline{M}_{g, r, k}\) of relative dimension

\[
(g - 1) \cdot \dim(G) + \frac{r}{2} \cdot (\dim(G) + \text{rk}(G)),
\]

where \(\dim(G)\) and \(\text{rk}(G)\) denote the dimension and the rank of \(G\) respectively.

**Proof.** We shall consider assertion (i). Let \(X^\star\) be a pointed stable curve of type \((g, r)\) over a \(k\)-scheme \(S\). Suppose that we are given a \(G_{ad}\)-oper \(\mathcal{E}_{ad} := (E_{B_{ad}}, \nabla_{\mathcal{E}})\) on \(X^\star\) and a \((Z, \delta^\star)\)-structure \((X^\star, \gamma, G_{Z, \delta^\star})\) on \(X^\star\) (hence, \(X^\star := (X^\star, X^\star, \gamma, G_{Z, \delta^\star})\) forms a pointed stable \((Z, \delta^\star)\)-spin curve). Both the \(B_{ad}\)-torsor \(\gamma^*(E_{B_{ad}})\) (over \(X\)) and the \(T\)-torsor \(G_{Z, \delta^\star} \times \hat{T}_T\) \((\text{obtained from} \ G_{Z, \delta^\star} \text{via change of structure group by the composite of (83) and the first projection} \ T \times T_{ad}(\text{of} \ G_m \rightarrow T))\) induce the same \(T_{ad}\)-torsor \((\Omega_{X^\log/S^\log} \times G_m)^{\times} \times^{\text{Gm,} \lambda} T_{ad}\). Since \(B \cong B_{ad} \times T_{ad}\), these torsors give rise to a \(B\)-torsor \(\mathcal{E}_B\) over \(X\). The natural morphism \(\mathcal{E}_G := \mathcal{E}_B \times B G \rightarrow \mathcal{E}_{G_{ad}}\) induces an isomorphism \(\tilde{T}_{\mathcal{E}_G^\log/S^\log} \tilde{\rightarrow} \tilde{T}_{\mathcal{E}_{G_{ad}}^\log/S^\log}\), by which \(\nabla_{\mathcal{E}}\) may be thought of as an \(S^\log\)-connection on \(\mathcal{E}_G\). By construction, the collection \(\mathfrak{c} := (X^\star, \gamma, \mathcal{E}_G)\)
(where $\mathcal{E}^\star := (\mathcal{E}_B, \nabla_\mathcal{E})$) is verified to form a faithful twisted $G$-oper on $X^\star$. The assignment $((X^\star, \mathcal{E}^\star_{\text{ad}}), \mathcal{X}^\star) \mapsto (X^\star, \mathcal{E}^\star)$ determines a functor

$$\mathcal{D}p_{G_{\text{ad}}, g, r, k} \times \mathcal{D}p_{\mathcal{M}_{g, r, k}} \mathcal{D}p_{Z, \delta^\tau, g, r} \to \mathcal{D}p_{G, g, r, k}. \tag{92}$$

This functor specifies the inverse of (90), and hence, we complete the proof of assertion (i). Assertion (ii) follows from Theorem 1.3.5 and [55], Theorem A. □

2.4. Radii of twisted $G$-opers.

Let us generalize the notion of radius discussed in [55], §2, Definition 2.9.2, to our case (i.e., $G$ is not necessarily of adjoint type). Let us write $c := \mathfrak{g}/G$, i.e., the GIT quotient of $\mathfrak{g}$ by the adjoint action of $G$. Also, write $\chi : \mathfrak{g} \to c$ for the natural projection. $c$ has the involution $\varrho \mapsto \varrho^\tau$ arising from the automorphism of $\mathfrak{g}$ given by multiplication by $(-1)$. (It follows from [55], Lemma 7.3.4, that this involution coincides with the involution defined in §5.8 of loc. cit., which was considered for some cases of $G$.) This involution induces an involution $(-)^\tau$ on $\mathfrak{c}(S) \times \text{Inj}(\mu, Z)$ defined by $(\varrho, \kappa)^\tau := (\varrho^\tau, \kappa^\tau)$. Let us write

$$\varepsilon := \chi(-\varrho) \in c(k), \quad \varrho := (\varepsilon, \delta) \in c(k) \times \text{Inj}(\mu, Z). \tag{93}$$

It is immediately verified that $\varepsilon^\tau = \varepsilon$ and $\varrho^\tau = \varrho$.

Next, by equipping $c$ with the trivial $Z$-action, we obtain the quotient stack $[c/Z]$ of $c$, which is canonically isomorphic to $c \times_k BZ$. By means of the composite of natural isomorphisms

$$\widetilde{T}_\mu([c/Z]) \sim \mathcal{T}_\mu(c \times_k BZ) \sim c \times_k \mathcal{T}_\mu(BZ), \tag{94}$$

we identify the stack of cyclotomic gerbes $\mathcal{T}_\mu([c/Z])$ in $[c/Z]$ with $c \times_k \mathcal{T}_\mu(BZ)$: Moreover, according to decomposition (122), $\mathcal{T}_\mu([c/Z])$ may be identified with $\coprod_{c \in \text{Inj}(\mu, Z)} c \times_k \mathcal{T}_\mu(BZ)_\kappa$. Given $(\varrho, \kappa) \in c(k) \times \text{Inj}(\mu, Z)$, we obtain a closed substack

$$\mathcal{T}_\mu([c/Z])_\varrho \tag{95}$$

of $\mathcal{T}_\mu([c/Z])$ corresponding, via (122), to the close immersion $\mathcal{T}_\mu(BZ)_\kappa = \text{Spec}(k) \times_k \mathcal{T}_\mu(BZ)_\kappa \hookrightarrow c \times_k \mathcal{T}_\mu(BZ)$ defined as the product of $\varrho$ and the inclusion $\mathcal{T}_\mu(BZ)_\kappa \hookrightarrow \mathcal{T}_\mu(BZ)$.

Let $r$ be a positive integer, $X^\star := (X, \{\sigma_{X,i}\}_{i=1}^r)$ an $r$-pointed nodal curve over a $k$-scheme $S$, and $\mathcal{E}^\star := (X^\star, \gamma, \mathcal{E}^\star)$ a faithful twisted $G$-oper on $X^\star$. Let $(\mathcal{E}_{\text{ad}}, G_{Z, \delta^\tau})$ be the pair of a faithful twisted $G_{\text{ad}}$-oper on $X^\star$ and a $(Z, \delta^\tau)$-structure on $X^\star$ corresponding to $\mathcal{E}^\star$ via equivalence of categories (100).

Recall from [55], §2, Definition 2.9.1, that the radius of $\mathcal{E}^\star_{\text{ad}} := (\mathcal{E}_{B, \text{ad}}, \nabla_\mathcal{E})$ at each $\sigma_{X,i}$ $(i = 1, \ldots, r)$ is defined as a certain element of $c(S)$, which we shall denote by $\rho_{\mathcal{E}^\star_{\text{ad}}, i}$. That is to say, $\rho_{\mathcal{E}^\star_{\text{ad}}, i}$ is defined as the image, via
the natural projection \([g/G] \to \mathfrak{c}\), of the pair \((\sigma_{X,i}^{\chi_i}(\mathcal{E}_{G,\text{ad}}), \mu_i^{(\mathcal{E}_{G,\text{ad}}, \nabla \varepsilon)})\) (where \(\mathcal{E}_{G,\text{ad}} := \mathcal{E}_{B,\text{ad}} \times_{\text{Rad} G} \mathcal{E}_{\text{ad}}\) and \(\mu_i^{(\mathcal{E}_{G,\text{ad}}, \nabla \varepsilon)}\) denotes the monodromy of \((\mathcal{E}_{G,\text{ad}}, \nabla \varepsilon)\) at \(\sigma_{X,i}\) in the sense of [55], Definition 1.6.1). Write
\[
\rho_{\mathcal{E}^\bullet, i} := (\rho_{\mathcal{E}^\bullet, i}, \kappa_{\mathcal{E}^\bullet, i}) \in c(S) \times \text{Inj}(\mu, Z),
\]
\[
\tilde{\rho}_{\mathcal{E}^\bullet, i} := (\rho_{\mathcal{E}^\bullet, i}, \tilde{\kappa}_{\mathcal{E}^\bullet, i}) \in \left( c(S) \times \text{Ob}(\mathcal{T}_\mu([\mathcal{E}/Z])(S)) \right)^{-1} \text{Ob}(\mathcal{T}_\mu([\mathcal{E}/Z])(S)).
\]
We shall refer to \(\rho_{\mathcal{E}^\bullet, i}\) as the radius of \(\mathcal{E}^\bullet\) at \(\sigma_{X,i}\). (In particular, if \(G = G_{\text{ad}}\), then the notion of radius coincides with the classical definition discussed in loc. cit.)

**Definition 2.4.1.**
Let \(X^\bullet\) be as above and \(\bar{\rho} := (\rho_i)_{i=1}^r\) an element of \((c(S) \times \text{Inj}(\mu, Z))^\times r\). Then, we shall say that a faithful twisted \(G\)-oper \(\mathcal{E}^\bullet\) on \(X^\bullet\) is of radii \(\bar{\rho}\) if \(\rho_{\mathcal{E}^\bullet, i} = \rho_i\) for any \(i = 1, \ldots, r\).

For each \(i = 1, \ldots, r\), the assignment \(\mathcal{E}^\bullet \mapsto \tilde{\rho}_{\mathcal{E}^\bullet, i}\) determines a morphism
\[
ev_{i}^{O}: \mathcal{O}p_{G, g, r, k} \to \mathcal{T}_\mu([\mathcal{E}/Z]),
\]
which factors through the closed immersion \(\mathcal{T}_\mu([\mathcal{E}/Z])_{\rho_{\mathcal{E}^\bullet, i}} \hookrightarrow \mathcal{T}_\mu([\mathcal{E}/Z]).\) Moreover, the morphisms \(ev_{i}^{O}\) determine a morphism
\[
ev^{O} := (ev_1^{O}, \ldots, ev_r^{O}) : \mathcal{O}p_{G, g, r, k} \to \mathcal{T}_\mu([\mathcal{E}/Z])^\times r.
\]
Given an element \(\bar{\rho} := (\rho_i)_{i=1}^r \in (c(k) \times \text{Inj}(\mu, Z))^\times r\), we obtain the closed substack
\[
\mathcal{O}p_{G, g, r, \bar{\rho}, k} := (ev^{O})^{-1}(\prod_{i=1}^r \mathcal{T}_\mu([\mathcal{E}/Z])_{\rho_i}),
\]
the stack classifying faithful twisted \(G\)-opers of radii \(\bar{\rho}\).

**Theorem 2.4.2.**
Let \(\bar{\rho} := ((\rho_i, \kappa_i))_{i=1}^r \in (c(k) \times \text{Inj}(\mu, Z))^\times r\). Then, isomorphism ([74]) restricts to an isomorphism
\[
\mathcal{O}p_{G, g, r, \bar{\rho}, k} \cong \mathcal{O}p_{G_{\text{ad}}, g, r, (\rho_i)}_{i=1}^r \times_{\mathbb{M}_{g, r, k}} \mathcal{O}p_{Z^{\bullet}, g, r, (\kappa_i)}_{i=1}^r.
\]
In particular, \(\mathcal{O}p_{G, g, r, \bar{\rho}, k}\) may be represented by a smooth Deligne-Mumford stack over \(k\) which is flat over \(\mathbb{M}_{g, r, k}\) of relative dimension
\[
(g - 1) \cdot \dim(G) + \frac{r}{2} \cdot (\dim(G) - \text{rk}(G)).
\]
Proof. The assertion follows from the various definitions involved together with Theorem 2.3.5 and [55], Theorem A.

2.5. Forgetting tails.

Let $r$ be a nonnegative integer, $\tilde{\rho}$ an element of $(c(S) \times \text{Inj}(\mu, Z))^r$, and $X^\bullet := (X, \{\sigma_{X,i}\}^r_{i=1})$ an $(r+1)$-pointed smooth curve over a $k$-scheme $S$ (hence $S^{\log} = S$). Write $X^\bullet_{\text{tail}} := (X_{\text{tail}}, \{\sigma_{X_{\text{tail}},i}\}_{i=1}^r)$, i.e., the $r$-pointed curve obtained from $X^\bullet$ by forgetting the last marked point (hence, $X_{\text{tail}} = X$ and $\sigma_{X_{\text{tail}},i} = \sigma_{X,i}$ for any $i \in \{1, \cdots, r\}$). In the following, we construct an equivalence of categories between the groupoid of faithful twisted $G$-opers on $X^\bullet_{\text{tail}}$ of radii $\tilde{\rho}$, denoted by $\mathcal{D}_pX^\bullet_{\text{tail}},\tilde{\rho}$, and the groupoid of faithful twisted $G$-opers on $X^\bullet$ of radii $(\tilde{\rho}, \epsilon)$ (cf. (123)), denoted by $\mathcal{D}_pX^\bullet, (\tilde{\rho}, \epsilon)$.

First, let us consider the case where $G$ is of adjoint type (hence, $\text{Inj}([\mu, Z]) = \{\}$ and $\epsilon = \epsilon$). Denote by $\mathcal{E}^1_{B,X^\bullet}$ and $\mathcal{E}^\dagger_{B,X^\bullet}$ the $B$-torsors of $G$-opers on $X^\bullet$ of radii $\tilde{\rho}$, denoted by $\mathcal{D}_pX^\bullet,\tilde{\rho}$ and the groupoid of faithful twisted $G$-opers on $X^\bullet_{\text{tail}}$ of radii $(\tilde{\rho}, \epsilon)$, respectively. (Although such $B$-torsors were constructed over a scheme which is étale over a pointed stable curves, one may evidently generalize, in the same manner, this construction to our case, i.e., the case where the base space is a pointed nodal curve.) Write $\mathcal{E}^1_{G,X^\bullet} := \mathcal{E}^1_{B,X^\bullet} \times B G$ and $\mathcal{E}^\dagger_{G,X^\bullet} := \mathcal{E}^\dagger_{B,X^\bullet} \times B G$.

Now, let $\mathcal{E}^\bullet_{\text{tail}} := (\mathcal{E}_{\text{tail}}, \nabla_{\mathcal{E}_{\text{tail}}})$ be a faithful twisted $G$-oper (i.e., a $\mathfrak{g}$-oper in the sense of [55], Definition 2.2.1) on $X^\bullet_{\text{tail}}$ of radii $\tilde{\rho} \in c(S)^r$. According to [55], Proposition 2.7.3, there exists a unique pair $(\mathcal{E}^\bullet_{\text{tail}}, \text{can}_{\mathcal{E}^\bullet})$ consisting of a faithful twisted $G$-oper $\mathcal{E}^\bullet_{\text{tail}} := (\mathcal{E}^1_{B,X^\bullet_{\text{tail}}}, \nabla_{\tilde{\rho}}_{\mathcal{E}_{\text{tail}}})$ on $X^\bullet_{\text{tail}}$ of canonical type and an isomorphism $\text{can}_{\mathcal{E}^\bullet} : \mathcal{E}^\bullet_{\text{tail}} \sim \mathcal{E}^\bullet_{\text{tail}}$ of $G$-opers (cf. Definition 2.7.1 in loc. cit. for the definition of a faithful twisted $G$-oper of canonical type under our assumption). By means of $\text{can}_{\mathcal{E}^\bullet}$, we shall identify $\mathcal{E}^\bullet_{\text{tail}}$ with $\mathcal{E}^\bullet_{\text{tail}}$, i.e., assume that $\mathcal{E}^\bullet_{\text{tail}}$ is of canonical type. In the following, we shall construct, from $\nabla_{\mathcal{E}_{\text{tail}}}$, an $S$-connection on $\mathcal{E}^\dagger_{G,X^\bullet}$.

Let us take a pair $U = (U, t)$ consisting of an open subscheme $U$ of $X$ with $U \cap \text{Im}(\sigma_{X,r+1}) \neq \emptyset$, $U \cap \text{Im}(\sigma_{X,i}) = \emptyset$ ($i = 1, \cdots, r$) and an element $t \in \Gamma(U, \mathcal{O}_X)$ defining the closed subscheme $\sigma_{X,r+1}$ such that $d\log(t)$ generates $\Omega_{X_{\text{log}}/S}|_U$. Hence, $t \cdot d\log(t) (= dt)$ generates $\Omega_{X_{\text{log}}/S}|_U$. According to [55], (127), the pair $tU := (U, dt)$ determines a trivialization $\text{triv}_{tU} : \mathcal{E}^t_{B,X^\bullet_{\text{tail}}}|_U \sim U \times_k B$ of the $B$-torsor $\mathcal{E}^t_{B,X^\bullet_{\text{tail}}}|_U$. The differential of this trivialization specifies an $\mathcal{O}_U$-isomorphism

$$
\mathcal{T}_{G,X^\bullet_{\text{tail}}/S}|_U \sim \mathcal{T}|_U \oplus (\mathcal{O}_U \otimes_k \mathfrak{g}).
$$
Consider the isomorphism

\[(103) \quad T \cong \prod_{\alpha \in \Gamma} \text{GL}(g^\alpha),\]

which, to each point \(h \in T\), assigns the collection \((h_\alpha)_{\alpha \in \Gamma}\), where each \(h_\alpha\) denotes the automorphism of \(g^\alpha\) given by multiplication by \(\alpha(h) \in \mathbb{G}_m\). Write \(U^0 := U \setminus \text{Im}(\sigma_{X,r+1})\). Also, write \(\mu_U : U^0 \rightarrow U\) for the natural open immersion and \(m_{a,t} \ (\alpha \in \Gamma)\) for the automorphism of \(\mathcal{O}_{U^0} \otimes_k g^\alpha\) given by multiplication by \(\mu_U^*(t) \in \Gamma(U^0, \mathcal{O}_{X}^*)\).

Let \(m_U\) be the automorphism of the trivial (right) \(B\)-torsor \(U^0 \times_k B\) over \(U^0\) defined as the (left-)translation by the element of \(T(U^0)\) corresponding to \((m_{a,t})_{a \in \Gamma} \in \prod_{\alpha \in \Gamma} \text{GL}(g^\alpha)\) via isomorphism \([103]\). One verifies that the \(S\)-connection

\[(104) \quad \nabla_{U/S} \rightarrow \nabla_{G,X} \quad \nabla_{U/S} \rightarrow \nabla_{U/S} + (\mathcal{O}_{U^0} \otimes_k g) \rightarrow \nabla_{U/S} + (\mathcal{O}_{U^0} \otimes_k g)\]
on \(U^0 \times_k G\) (extends (uniquely) to an \(S\)-connection on \(U \times_k G\) (equipped with the log structure pulled-back from \(X^\log\)). Moreover, the monodromy \(\mu_{r+1}^{(U \times_k G, \nabla_U)} \in \mathfrak{g}(U \times_k G, \nabla_U)\) at \(\sigma_{X,r+1}\) (cf. [55], Definition 1.6.1 for the definition of monodromy) coincides with \(-\rho \in \mathfrak{g}\) (cf. [14], Proposition 9.2.1). The pair \((U \times_k B, \nabla_U)\) forms a faithful twisted \(G\)-oper on \(X^\log|U/S\) whose radius at \(\sigma_{X,r+1}\) coincides with \(\epsilon = \chi(-\rho)\). Hence, (by [55], Proposition 2.7.3) there exists a unique faithful twisted \(G\)-oper \(\mathcal{E}_{G,X} = (\mathcal{E}_{G,X}^\dagger | U, \nabla_U)\) on \(X^\log|U/S\) of canonical type which is isomorphic to \((U \times_k B, \nabla_U)\).

On the other hand, since \(\mathcal{E}_{G,X}^\dagger | X^\log|U^0\) may be naturally identified with \(\mathcal{E}_{G,X}^\dagger | X^\log|\sigma_{X,r+1}\), the restriction \(\nabla_{\mathcal{E},\text{tail}} | X^\log|\sigma_{X,r+1}\) of \(\nabla_{\mathcal{E},\text{tail}}\) to \(X \setminus \text{Im}(\sigma_{X,r+1})\) may be thought of as an \(S\)-connection on \(\mathcal{E}_{G,X}^\dagger | X^\log|\sigma_{X,r+1}\). Since \(\nabla_{\mathcal{E},\text{tail}} | U^0 = \nabla_{\mathcal{E},\text{tail}} | U^0\), the \(S\)-connection \(\nabla_{\mathcal{E},\text{tail}} | X^\log|\sigma_{X,r+1}\) and the various \(S\)-connections \(\nabla_{\mathcal{E},\text{tail}} \) (where \(U\) ranges over the pairs \(U = (U, t)\) as above) may be glued together to an \(S\)-connection \(\nabla_{\mathcal{E},\epsilon}\) on \(\mathcal{E}_{G,X}^\dagger\). One verifies that the pair

\[(106) \quad \mathcal{E}_{\epsilon} := (\mathcal{E}_{G,X}^\dagger, \nabla_{\mathcal{E},\epsilon})\]

forms a faithful twisted \(G\)-oper on \(X^\star\) of radii \((\tilde{\rho}, \epsilon) \in \mathfrak{c}(S)^{\times(r+1)}\). By construction, the assignment \(\mathcal{E}_{\text{tail}}^\dagger \rightarrow \mathcal{E}_{\epsilon,\text{tail}}^\dagger\) is verified to determine an equivalence of categories \(\mathcal{D}p_{X^\star,\tilde{\rho}} \cong \mathcal{D}p_{X^\star,\epsilon}\).

Next, let us remove the assumption that \(G\) is of adjoint type. Let \(\mathcal{E}_{\text{tail}}^\dagger\) be a faithful twisted \(G\)-oper on \(X_{\text{tail}}^\star\); it corresponds, via [50], to a pair \((\mathcal{E}_{\text{tail}}^\star, (X_{\text{tail}}^\star, \gamma_{\text{tail}}, G_{Z,\gamma_{\text{tail}}})\) consisting of a faithful twisted \(G_{\text{ad}}\)-oper \(\mathcal{E}_{\text{tail}}^\dagger\) on \(X_{\text{tail}}^\star\) and a
(Z, δ)-structure (X_{\text{tail}}^*, \gamma_{\text{tail}}, G_{Z,\delta, \text{tail}}) on X_{\text{tail}}^*. According to the above discussion and the discussion in §1.6, they induce a faithful twisted \mathcal{E}_{\epsilon} on X^* and a (Z, δ)-structure (X^*_{+, \delta}, \gamma_{+, \delta}, G_{+, \delta}) on X^* respectively. Then, we obtain a faithful twisted G-oper

$$\mathcal{E}_{\epsilon}$$

on X^* corresponding, via the inverse of (\text{III}), to the pair (\mathcal{E}_{+\epsilon}, (X^*_{+, \delta}, \gamma_{+, \delta}, G_{+, \delta})).

The above discussion and Proposition 1.6.2 imply the following proposition.

**Proposition 2.5.1.**

(i) Let X^* and X_{\text{tail}}^* be as above. Then, the assignment \mathcal{E}_{\text{tail}} \mapsto \mathcal{E}_{+\epsilon} determines an equivalence of categories \mathcal{O}_{p, X_{\text{tail}}} \sim \mathcal{O}_{p, X^*, (\bar{\rho}, \epsilon)}.

(ii) Let g, r be nonnegative integers with 2g - 1 + r > 0, and let \bar{\rho} \in (c(k) \times \text{Inj}(\mu, Z))^\times r. Then, there exists an isomorphism of stacks

$$M_{g,r+1,k} \times \mathcal{O}_{p, G, g, r, \bar{\rho}, k} \sim M_{g,r+1,k} \times \mathcal{O}_{p, G, g, r+1, (\bar{\rho}, \epsilon), k}$$

over M_{g,r+1,k}.

2.6. Faithful twisted G-opers on the 2-pointed projective line.

In this subsection, we shall study the faithful twisted G-opers on the 2-pointed projective line. Let us keep the notation in §1.7.

**Proposition 2.6.1.**

(i) If \mathcal{E}^* is a faithful twisted G-oper on P_S^*, then we have the equality

$$\rho_{\mathcal{E}^*, 1} = \rho_{\mathcal{E}^*, 2}.$$

(ii) Let \rho \in c(S) \times \text{Inj}(\mu, Z). Then, the assignment \mathcal{E}^* \mapsto \bar{\rho}_{\mathcal{E}^*, 1} determines an equivalence of categories between the groupoid \mathcal{O}_{p, P_S^*, (\rho, \rho^z)} of faithful twisted G-opers on P_S^* with \rho = \rho_{\mathcal{E}^*, 1} = \rho_{\mathcal{E}^*, 2} and the groupoid \mathcal{I}_\mu([c/Z])_\rho(S).

**Proof.** By Proposition 1.7.1 and Theorem 2.4.2 one may assume that G = G_{ad} (hence \mathcal{I}_\mu([c/Z]) = c). First, we shall consider assertion (i). Let \mathcal{E}^\diamond := (\mathcal{E}_{B, P_S^*}^\top, \nabla^\circ_{\epsilon, \mathcal{E}}) be a faithful twisted G-oper on P_S^* of canonical type. The global section \frac{ds_1}{s_1} \left( = \frac{-ds_2}{s_2} \right) \in \Gamma(P_S, \Omega_{P_S^{\text{log}}/S}) generates globally \Omega_{P_S^{\text{log}}/S}. It follows that the pair (P_S, \frac{ds_1}{s_1}) gives a trivialization \mathcal{E}^\top_{G, P_S^*} \sim P_S \times_k G (cf. [53], §2.5,
(125)). By means of the natural identifications
\[
\text{Hom}(\mathcal{T}_{P_S^{\text{log}}/S}, \tilde{\mathcal{T}}_{(P_S^{\text{log}} \times G)/S}) = \Gamma(P_S, \Omega_{P_S^{\text{log}}/S} \otimes (\mathcal{T}_{P_S^{\text{log}}/S} \otimes (\mathcal{O}_{P_S} \otimes_k g)))
\]
the $S$-connection $\nabla'$ on $P_S \times_k G$ corresponding to $\nabla^\otimes_S$ via this trivialization may be expressed as
\[
(110) \quad \left(1, \frac{ds_1}{s_1} \otimes v\right) \quad \left(1, -\frac{ds_2}{s_2} \otimes v\right) \quad \in \Gamma(P_S, \mathcal{O}_{P_S} \oplus (\Omega_{P_S^{\text{log}}/S} \otimes g))
\]
for some $v \in g(S)$. Hence, we obtain the sequence of equalities
\[
(111) \quad \rho(P_S \times G, \nabla'), 1 = \chi \left(\sigma^*_{P,1} \left(\frac{ds_1}{s_1} \otimes v\right)\right)
\]
\[
= \chi \left(\sigma^*_{P,2} \left(-\frac{ds_2}{s_2} \otimes v\right)\right)
\]
\[
= \chi \left(\sigma^*_{P,2} \left(\frac{ds_2}{s_2} \otimes v\right)\right)^\circ
\]
\[
= \rho^\circ(P_S \times G, \nabla'), 2.
\]
This completes the proof of assertion (i).

Next, let us consider assertion (ii). Let $\rho \in \mathfrak{c}(S)$. Recall (cf. [42], Lemma 1.2.1) that (since $\text{char}(k) = 0$ or $\text{char}(k) = p > 0$ satisfying $(*)_{G,p}$) the morphism $\text{kos} : g^{\text{ad}(p^1)} \to \mathfrak{c}$ given by $s \mapsto \chi(p^1 + s)$ (for each $s \in g^{\text{ad}(p^1)}$) is an isomorphism. Hence, there exists a unique $v \in g^{\text{ad}(p^1)}(S)$ with $\text{kos}(v) = \rho$. The pair $\mathcal{E}_\rho := (P_S \times_k B, \nabla_\rho)$ forms a faithful twisted $G$-oper on $P_S^\bullet$, where $\nabla_\rho$ denotes the $S$-connection on $P_S \times_k G$ expressed, by means of the element $v$ just obtained, as (110). One verifies (by taking account of the above discussion) that the assignment $\rho \mapsto \mathcal{E}_\rho^\bullet$ specifies the inverse of $\mathcal{E}_\rho^\bullet \mapsto \rho_{\mathcal{E}_\rho^\bullet, 1}$. This completes the proof of assertion (ii).

\[\text{Corollary 2.6.2.}\]
\[
(112) \quad \mathcal{O}_P G, 0, 3, \rho_1, \rho_2, \epsilon, k \cong \begin{cases} 
\mathcal{O}Z & \text{if } \rho_1 = \rho_2^\circ \\
\emptyset & \text{if otherwise.}
\end{cases}
\]

\[\text{Proof.}\] The assertion follows from Propositions 2.5.1 and 2.6.1.
3. Do’pers and their moduli

In this section, we define (cf. Definition 3.1.1) the notion of a $G$-do’per ($= \text{a dormant faithful twisted } G\text{-oper}$). We prove that the moduli stack of $G$-do’pers may be represented by a proper Deligne-Mumford stack (cf. Theorem 3.2.2) and satisfies certain factorization properties according to clutching morphisms of the stacks $\mathcal{M}_{g,r,k}$ for various pairs $(g,r)$ (cf. Proposition 3.2.3).

In this section, we assume that $G$ satisfies the condition $(\ast)_{G,p}$ and moreover $k = F_p$.

3.1. $G$-do’pers.

We recall the definition of the $p$-curvature of a logarithmic connection (cf., e.g., [54], § 3). Let $S_{\log}$ be an fs log scheme over $F_p$, $U_{\log}$ a stacky log-curve over $S_{\log}$, and $(\pi : E \to U_{\log}, \nabla_E)$ a $G$-torsor $\pi : E \to U_{\log}$ over $U_{\log}$ paired with an $S_{\log}$-connection $\nabla_E$ on $E$. If $\partial$ is a logarithmic derivation corresponding to a local section of $T_{U_{\log}/S_{\log}}$ (resp., $\tilde{T}_{E_{\log}/S_{\log}} := (\pi_*(T_{E_{\log}/S_{\log}}))^G$), then we shall denote by $\partial[p]$ the $p$-th symbolic power of $\partial$ (i.e., “$\partial \mapsto \partial[p]$” asserted in [44], Proposition 1.2.1), which is also a logarithmic derivation corresponding to a local section of $T_{U_{\log}/S_{\log}}$ (resp., $\tilde{T}_{E_{\log}/S_{\log}}$). (Notice that although the $p$-th symbolic power of $\partial$ was defined, in [44], for the case where $U_{\log}$ may be represented by a log scheme, one may generalize, in an evident fashion, the definition to our case.) Then, one may find (cf. [55], § 3.2) a unique $O_U$-linear morphism $\psi(\xi,\nabla) : T_{U_{\log}/S_{\log}}^{\otimes p} \to g_E (\subseteq \tilde{T}_{E_{\log}/S_{\log}})$ determined by $\partial^{\otimes p} \mapsto (\nabla_E(\partial))^{[p]} - \nabla_E(\partial^{[p]})$. We shall refer $\psi(\xi,\nabla)$ to as the $p$-curvature map of $(E, \nabla_E)$.

Definition 3.1.1.

We shall say that a twisted $G$-oper $E^\bullet := (E_B, \nabla_E)$ on $U_{\log}/S_{\log}$ is dormant if $\psi(\xi_E,\nabla_E) \equiv 0$ on $U_{\log}$. Let $X^\bullet$ be a pointed nodal curve. A $G$-do’per on $X^\bullet$ is a faithful twisted $G$-oper $E^\bullet := (X^\bullet, \gamma, E^\bullet)$ on $X^\bullet$ such that $E^\bullet$ is dormant.

Let $g$, $r$ be nonnegative integers with $2g - 2 + r > 0$. Write

$$\Omega_{G,g,r,F_p}^{zar\ldots}$$

for the closed substack of $\Omega_{G,g,r,F_p}$ classifying $G$-do’pers. Also, write

$$\pi_{g,r} : \Omega_{G,g,r,F_p}^{zar\ldots} \to \mathcal{M}_{g,r,k}, \quad l_{g,r} : \Omega_{G,g,r,F_p}^{zar\ldots} \to \Omega_{G,g,r,F_p}$$

for the forgetting morphism and the natural closed immersion respectively.
3.2. Radii of $G$-do'pers.

Next, let us consider $G$-do'pers of prescribed radii. Let $t_{\text{reg}}$ denote the open subscheme of $t$ of the regular elements. Hence, $t_{\text{reg}} = \{ t \in t \mid d\alpha(t) \neq 0 \text{ for any root } \alpha \in G \}$ (cf. [27], Theorem 7.2). We shall write

\[(115) \quad t^F_{\text{reg}} := \text{Ker} \left( t_{\text{reg}} \xrightarrow{F_{\text{reg}} \cdot \text{id}} t_{\text{reg}} \right), \]

where $F_{\text{reg}}$ denotes the $p$-th power Frobenius endomorphism of $t_{\text{reg}}$. If $W$ denotes the Weyl group of $(G, T)$, then the natural $W$-action on $t$ restricts to a $W$-action on $t^F_{\text{reg}}$. The composite $t \mapsto g \mapsto c$ induces a closed immersion from the resulting quotient $t^F_{\text{reg}}/W$ to $c$. In particular, the finite set

\[(116) \quad (t^F_{\text{reg}}/W)(\mathbb{F}_p) \quad (= t^F_{\text{reg}}(\mathbb{F}_p)/W)\]

may be considered as a subset of $c(S)$ for any $\mathbb{F}_p$-scheme $S$. The $\mathbb{F}_p$-scheme $t^F_{\text{reg}}/W$ decomposes into the disjoint union of copies of the schemes $\text{Spec}(\mathbb{F}_p) =: \text{Spec}(\mathbb{F}_p)_\rho$ indexed by the elements $\rho \in (t^F_{\text{reg}}/W)(\mathbb{F}_p)$. By equipping $t^F_{\text{reg}}/W$ with the trivial $\mathbb{Z}$-action, we obtain a closed substack $\mathcal{I}_\mu([((t^F_{\text{reg}}/W)/\mathbb{Z})]$ of $\mathcal{I}_\mu([c/\mathbb{Z}]])$.

Let us define a finite set $\Delta$ to be

\[(117) \quad \Delta := (t^F_{\text{reg}}/W)(\mathbb{F}_p) \times \text{Inj}(\mu, \mathbb{Z}). \]

Decomposition (112) determines a decomposition

\[(118) \quad \mathcal{I}_\mu([((t^F_{\text{reg}}/W)/\mathbb{Z})] = \mathcal{I}_\mu([((t^F_{\text{reg}}/W) \times \mathbb{Z})] = \prod_{\rho \in (t^F_{\text{reg}}/W)(\mathbb{F}_p)} \text{Spec}(\mathbb{F}_p) \times \prod_{\kappa \in \text{Inj}(\mu, \mathbb{Z})} \mathcal{I}_\mu(\mathbb{Z})_\kappa = \prod_{\rho \in \Delta} \mathcal{I}_\mu(\mathbb{Z})_\rho, \]

where $\mathcal{I}_\mu(\mathbb{Z})_{(\rho, \kappa)} := \text{Spec}(\mathbb{F}_p)_\rho \times \mathcal{I}_\mu(\mathbb{Z})_\kappa$.

**Proposition 3.2.1.**

Let $r$ be a positive integer, $S$ a $k$-scheme, $X^\bullet := (X, \{ \sigma_{X,i} \}_{i=1}^r)$ an $r$-pointed nodal curve over $S$, and $\mathcal{E}^\bullet$ a $G$-do'per on $X^\bullet$. Then, for each $i \in \{ 1, \cdots, r \}$, the radius $\rho_{\mathcal{E}^\bullet, i} \in c(S) \times \text{Inj}(\mu, \mathbb{Z})$ of $\mathcal{E}^\bullet$ at $\sigma_{X,i}$ lies in $\Delta$.

**Proof.** One may assume, without loss of generality, that $G = G_{\text{ad}}$ and $\mathcal{E}^\bullet = \mathcal{E}^\bullet$, where $\mathcal{E}^\bullet := (\mathcal{E}_B, \nabla \mathcal{E})$ is a faithful twisted $G(= G_{\text{ad}})$-oper of canonical type. By [55], Proposition 3.5.2 (i), $\rho_{\mathcal{E}^\bullet, i} \in c(\mathbb{F}_p)$. Hence, it suffices to consider the case where $S = \text{Spec}(\mathbb{K})$ for some algebraic closed field $\mathbb{K}$ (over $\mathbb{F}_p$). Let us consider a Jordan decomposition of the monodromy $\mu|_{(\mathcal{E}_B, \nabla \mathcal{E}), i} = \mu_s + \mu_n \in g(\mathbb{K})$.
with $\mu_s$ semisimple and $\mu_n$ nilpotent. Denote by $\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g})$ the adjoint representation of $\mathfrak{g}$, which is injective and compatible with the respective restricted structures, i.e., $p$-power maps. One may find an isomorphism $\alpha : \text{End}(\mathfrak{g}) \cong \mathfrak{g}|_{\dim(\mathfrak{g})}$ of restricted Lie algebras which sends $\alpha(\text{ad}(\mu_{(E,\nabla)})_{i,j}) = (\alpha(\text{ad}(\mu_s)) + \alpha(\text{ad}(\mu_n)))$ to a Jordan normal form. Namely, $\alpha(\text{ad}(\mu_s))$ is diagonal and every entry of $\alpha(\text{ad}(\mu_s))$ except the superdiagonal is 0. Let us observe the following sequence of equalities:

$$
(119) \quad \alpha(\text{ad}(\mu_s)) + \alpha(\text{ad}(\mu_n)) = \alpha(\text{ad}((\mu_s + \mu_n)[p])) = \alpha(\text{ad}(\mu_s + \mu_n))^p = (\alpha(\text{ad}(\mu_s)) + \alpha(\text{ad}(\mu_n)))^p,
$$

where the second equality follows from the assumption that $\mathcal{E}^\ast$ has vanishing $p$-curvature (and $[53], (229)$). By an explicit computation of $\alpha(\text{ad}(\mu_s)) + \alpha(\text{ad}(\mu_n))^p$, $(119)$ implies that $\alpha(\text{ad}(\mu_n)) = 0$ (hence $\mu_n = 0$), namely, $\mu_{(E,\nabla),i}$ is conjugate to some $v \in \mathfrak{t}(k)$. $(119)$ also imply the equality $v = (v[p]) = F^*(v)$, where $F_i$ denotes the $p$-power Frobenius endomorphism of $\mathfrak{t}$. Hence, $v$ lies in $\mathfrak{t}(\mathbb{F}_p)$. Moreover, note that (since $\mathcal{E}^\ast$ is of canonical type) $\mu_{(E,\nabla),i}$ equals $\alpha + u$ for some $u \in \Gamma(S, \mathcal{O}_S \otimes \mathfrak{g}^{\text{ad}(\mu_n)})$, and hence, is regular (cf. $[42]$, the comment preceding Lemma 1.2.3). It follows that $v \in \mathfrak{t}_{\text{reg}}(\mathbb{F}_p)$. Consequently, $\rho_{(E,\nabla),i} (\chi(\mu_{(E,\nabla),i}) = \chi(v)) \in \mathfrak{t}_{\text{reg}}(\mathbb{F}_p)/\mathfrak{W}$, as desired. $
$
Because of Proposition 3.2.1 above, the morphism $\psi_{i}^{\mathcal{D}_{\mathfrak{p}}}$ ($i = 1, \cdots, r$) and $\psi^{\mathcal{D}_{\mathfrak{p}}}$ restrict to morphisms

$$
(120) \quad \psi_{i}^{Zz} : \mathcal{D}_{\mathfrak{p}} \to \prod_{\rho \in \Delta} \mathcal{I}_\rho(BZ)_{\rho}
$$

and

$$
(121) \quad \psi^{Zz} : \mathcal{D}_{\mathfrak{p}} \to \prod_{(\rho)} \mathcal{I}_\rho(BZ)_{\rho}
$$

respectively. Given $\tilde{\rho} := (\rho_i)_{i=1}^{r} \in \Delta^{\times r}$, we obtain an open and closed substack

$$
(122) \quad \mathcal{D}_{\mathfrak{p}}^{Zz} := (\psi_{i}^{Zz})_{i=1}^{r}^{-1}(\prod_{i=1}^{r} \mathcal{I}_\rho(BZ)_{\rho}) = \mathcal{D}_{\mathfrak{p}}^{Zz} \cap \mathcal{D}_{\mathfrak{p}}^{Zz}
$$

of $\mathcal{D}_{\mathfrak{p}}^{Zz}$. The stack $\mathcal{D}_{\mathfrak{p}}^{Zz}$ decomposes into the disjoint union

$$
(123) \quad \mathcal{D}_{\mathfrak{p}}^{Zz} = \prod_{\rho \in \Delta^{\times r}} \mathcal{D}_{\mathfrak{p}}^{Zz}
$$

Of course, $\mathcal{D}_{\mathfrak{p}}^{Zz}$ may be empty for some choices of $\tilde{\rho}$.

Theorem 3.2.2.
(i) Isomorphism (100) restricts to an isomorphism

\[
\mathcal{O}_p^{\text{Zax...}}_{G,g,r,F_p} \sim \mathcal{O}_p^{\text{Zax...}}_{G_{ad},g,r,F_p} \times \mathcal{M}_{g,r,F_p} \mathcal{O}_p^{Z,g,r,F_p}.
\]

In particular, the morphism \(\text{op}_{ad}\) (cf. (89)) restricts to a morphism

\[
\text{op}_{ad} : \mathcal{O}_p^{\text{Zax...}}_{G,g,r,F_p} \to \mathcal{O}_p^{\text{Zax...}}_{G_{ad},g,r,F_p},
\]

which makes the following commutative diagram cartesian:

\[
\begin{array}{ccc}
\mathcal{O}_p^{\text{Zax...}}_{G,g,r,F_p} & \xrightarrow{\text{op}_{ad}} & \mathcal{O}_p^{\text{Zax...}}_{G_{ad},g,r,F_p} \\
\downarrow & & \downarrow \\
\mathcal{O}_p^{\text{Zax...}}_{G,g,r,F_p} & \xrightarrow{\text{op}_{ad}} & \mathcal{O}_p^{\text{Zax...}}_{G_{ad},g,r,F_p}
\end{array}
\]

Moreover, for each \(\tilde{\rho} := ((\rho_i, \kappa_i))_{i=1}^{r} \in \Delta^{\times r}\), isomorphism (124) restricts to an isomorphism

\[
\mathcal{O}_p^{\text{Zax...}}_{G,g,r,\tilde{\rho},F_p} \sim \mathcal{O}_p^{\text{Zax...}}_{G_{ad},g,r,\tilde{\rho},F_p} \times \mathcal{M}_{g,r,F_p} \mathcal{O}_p^{Z,g,\tilde{\rho},g,r,\kappa,F_p}
\]

(ii) \(\mathcal{O}_p^{\text{Zax...}}_{G,g,r,\tilde{\rho},F_p}\) and \(\mathcal{O}_p^{\text{Zax...}}_{G,g,r,F_p}\) (for each \(\tilde{\rho} \in \Delta^{\times r}\)) may be represented by (possibly empty) proper Deligne-Mumford stacks over \(k\) which are finite over \(\mathcal{M}_{g,r,F_p}\). Moreover, \(\mathcal{O}_p^{\text{Zax...}}_{G,g,r,F_p}\) is nonempty and has an irreducible component that dominates \(\mathcal{M}_{g,r,F_p}\).

(iii) Let us assume further that \(G\) satisfies the condition (**\(G,p\)) described in Introduction, § 0.4. Then, \(\mathcal{O}_p^{\text{Zax...}}_{G,g,r,\tilde{\rho},F_p}\) is étale over the points of \(\mathcal{M}_{g,r,F_p}\) classifying pointed totally degenerate curves (cf. [55], Definition 7.5.1, for the definition of a pointed totally degenerate curve). In particular, \(\mathcal{O}_p^{\text{Zax...}}_{G,g,r,\tilde{\rho},F_p}\) is generically étale over \(\mathcal{M}_{g,r,F_p}\) (i.e., any irreducible component that dominates \(\mathcal{M}_{g,r,F_p}\) admits a dense open subscheme which is étale over \(\mathcal{M}_{g,r,F_p}\)), and moreover, has generic stabilizer isomorphic to the center \(Z\) of \(G\).

(iv) The following assertion holds (without the assumption imposed in (iii)):

\[
\mathcal{O}_p^{\text{Zax...}}_{G,g,r,\rho,3,F_p} \cong \begin{cases} 
BZ & \text{if } \rho_1 \in \Delta \text{ and } \rho_1 = \rho_2, \\
\emptyset & \text{if otherwise.}
\end{cases}
\]

\[
\begin{aligned}
\text{Proof.} & \quad \text{Assertion (i) follows from the various definitions involved. Assertion (ii) follows from assertion (i) and [55], Theorem C, (i). Assertion (iii) follows from assertion (i), Theorem 1.3.5, and Theorem G in loc. cit. Finally, assertion (iv) follows from Corollary 2.6.2 and Proposition 3.2.1.} \\
\end{aligned}
\]

Proposition 3.2.3.

Assume that the condition (**\(G,p\)) is satisfied.
(i) Let \( g_1, g_2, r_1, \) and \( r_2 \) be nonnegative integers with \( 2g_j - 1 + r_j > 0 \) \((j = 1, 2)\), and let \( \bar{\rho}_j \in \Delta^{\times r_j} \). Write \( g := g_1 + g_2, r = r_1 + r_2 \). Then, given each \( \rho \in \Delta \), there exists a morphism

\[
\Phi_{\text{tree}, \rho} : \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_1}} \times \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_2}} \rightarrow \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_1}} \times \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_2}}
\]

obtained by gluing together two faithful twisted \( G \)-opers along the fibers over the respective last marked points of the underlying curves. Moreover, the following square diagram is commutative and cartesian:

\[
\begin{array}{ccc}
\mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_1}} \times \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_2}} & \xrightarrow{\Phi_{\text{tree}, \rho}} & \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_1}} \times \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_2}} \\
\Pi_{\rho \in \Delta} \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_1}} \times \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_2}} & \xrightarrow{\Pi_{\rho \in \Delta} \Phi_{\text{tree}, \rho}} & \Pi_{\rho \in \Delta} \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_1}} \times \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_2}} \\
\end{array}
\]

(ii) Let \( g, r \) be nonnegative integers with \( 2g + r > 0 \), and let \( \bar{\rho} \in \Delta^{\times r} \). Then, given each \( \rho \in \Delta \), there exists a morphism

\[
\Phi_{\text{loop}, \rho} : \text{Ker} \left( \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_1}} \times \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_2}} \rightarrow \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_1}} \times \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_2}} \right)
\]

obtained by gluing each \( G \)-oper along the fibers over the last two marked points of the underlying curve. Moreover, the following square diagram is commutative and cartesian:

\[
\begin{array}{ccc}
\mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_1}} \times \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_2}} & \xrightarrow{\Phi_{\text{loop}, \rho}} & \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_1}} \times \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_2}} \\
\Pi_{\rho \in \Delta} \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_1}} \times \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_2}} & \xrightarrow{\Pi_{\rho \in \Delta} \Phi_{\text{loop}, \rho}} & \Pi_{\rho \in \Delta} \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_1}} \times \mathcal{O}_{\mathfrak{p}}^{\Delta^{\times r_2}} \\
\end{array}
\]

(iii) Let \( g, r \) be nonnegative integers with \( 2g - 1 + r > 0 \), and let \( \bar{\rho} \in \Delta^{\times r} \). Then, there exists an isomorphism

\[
\mathcal{M}_{g,r+1,F_p} \times \mathcal{M}_{g,r,F_p} \rightarrow \mathcal{M}_{g,r+1,F_p} \times \mathcal{M}_{g,r+1,F_p}
\]

over \( \mathcal{M}_{g,r+1,F_p} \).
Proof. The assertions follow from Proposition 3.2.2 (i) and 7.4.1.

Remark 3.2.4.
At the time of writing the present paper, the author does not know to what extent one can weaken the condition \((**)_{G,p}\) imposed in Theorem 3.2.2 (iii) and Proposition 3.2.3.

4. The virtual fundamental class on the moduli of do’pers

In this section, we construct a perfect obstruction theory for the moduli stack \(\mathcal{D}_p^{Z_{\text{reg}}}\). As a result, we obtain a virtual fundamental class on that moduli stack (cf. (149)). In this section, we assume that the condition \((*)_{G,p}\) is satisfied.

4.1. General definition of a perfect obstruction theory.

First, we shall recall from \([9]\) the notions of a perfect obstruction theory and the virtual fundamental class associated with it. Let \(\mathcal{X}\) be a separated Deligne-Mumford stack, locally of finite type over a field \(k\). Denote by \(D(\mathcal{O}_X)\) the derived category of the category \(\text{Mod}(\mathcal{O}_X)\) of \(\mathcal{O}_X\)-modules and by \(L^*_X \in \text{Ob}(D(\mathcal{O}_X))\) the cotangent complex of \(\mathcal{X}\) relative to \(k\). For a morphism \(E^0_{\text{fl}} \to E^1_{\text{fl}}\) of abelian sheaves on the big fppf site \(\mathcal{X}_{\text{fl}}\) of \(\mathcal{X}\), one obtains the quotient stack \([E^1_{\text{fl}}/E^0_{\text{fl}}]\). That is to say, for an object \(T \in \text{Ob}(\mathcal{X}_{\text{fl}})\) the groupoid \([E^1_{\text{fl}}/E^0_{\text{fl}}](T)\) of sections over \(T\) is the category of pairs \((P,f)\), where \(P\) is an \(E^0_{\text{fl}}\)-torsor over \(T\) and \(f\) is an \(E^0_{\text{fl}}\)-equivariant morphism \(P \to E^1_{\text{fl}}|_T\) of sheaves on \(\mathcal{X}_{\text{fl}}\). If \(E^*_{\text{fl}}\) is a complex of arbitrary length of abelian sheaves on \(\mathcal{X}_{\text{fl}}\), we shall write

\[
h^1/h^0(E^*_{\text{fl}}) := [Z^1/C^0],
\]

where \(Z^1 := \ker(E^1_{\text{fl}} \to E^2_{\text{fl}})\) and \(C^0 := \operatorname{coker}(E^{-1}_{\text{fl}} \to E^0_{\text{fl}})\). Denote by \(\mathfrak{N}_{\mathcal{X}}\) the stack defined to be \(\mathfrak{N}_{\mathcal{X}} := h^1/h^0(((L^*_X)_{\text{fl}})^\vee)\), where for each complex \(E^*\) on (the small étale site of) \(\mathcal{X}\), we shall write \(E^*_{\text{fl}}\) for the complex on \(\mathcal{X}_{\text{fl}}\) associated with \(E^*\). Let us take a diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\iota} & M \\
\pi \downarrow & & \downarrow \\
\mathcal{X} & & 
\end{array}
\]
where \( U \) denotes an affine \( k \)-scheme of finite type, \( M \) denotes a smooth affine \( k \)-scheme of finite type, \( \iota \) is a closed immersion, and \( \pi \) is an étale morphism. Then, there exists a natural quasi-isomorphism

\[
(136) \quad \phi : L^\bullet_X |_U \xrightarrow{\sim} [I/I^2 \to \iota^*(\Omega_{M/k})],
\]

where \( I \) denotes the ideal sheaf on \( M \) defining the closed subscheme \( U \) and we consider the natural morphism \( I/I^2 \to \iota^*(\Omega_{M/k}) \) as a complex \([I/I^2 \to \iota^*(\Omega_{M/k})]\) concentrated in degrees \(-1 \) and \(0\). Write \( T_M := \text{Spec}(\text{Sym}_{\Omega_{M/k}}(\Omega_{M/k})) \) (i.e., the total space of the tangent bundle \( T_{M/k} \)) and \( N_{U/M} := \text{Spec}(\text{Sym}_{\Omega_C}(I/I^2)) \), where \( N_{U/M} \) admits an \( \iota^*(T_M) \)-action induced from the morphism \( I/I^2 \to \iota^*(\Omega_{M/k}) \). Quasi-isomorphism (136) gives rise to an isomorphism

\[
(137) \quad [N_{U/M}/\iota^*(T_M)] \xrightarrow{\sim} \mathfrak{R}_X|_U.
\]

The intrinsic normal cone of \( X \) (cf. [9], Definition 3.10) is defined as a unique closed substack \( \mathfrak{C}_X \) of \( \mathfrak{R}_X \) determined by the condition that if we are given a diagram as in (135), then \( \mathfrak{C}_X|_U \) may be identified, via (137), with the closed substack \( [C_{U/M}/\iota^*(T_M)] \), where \( C_{U/M} \) denotes the normal cone \( \text{Spec}(\bigoplus_{n \geq 0} I^n/I^{n+1}) \) of \( U \) in \( M \). A perfect obstruction theory for \( X \) (cf. [9], Definitions 4.4 and 5.1) is a morphism \( \phi : E^* \to L^\bullet_X \) in \( D(\mathcal{O}_X) \) satisfying the following conditions:

- \( h^0(\phi) \) is an isomorphism and \( h^{-1}(\phi) \) is surjective.
- \( E^* \) is of perfect amplitude contained in \([-1, 0]\), i.e., is locally isomorphic (in \( D(\mathcal{O}_X) \)) to a complex \([E^{-1} \to E^0]\) of locally free sheaves of finite rank.

Let \( \phi : E^* \to L^\bullet_X \) be a perfect obstruction theory for \( X \). The virtual dimension of \( X \) with respect to \( \phi : E^* \to L^\bullet_X \) is a well-defined locally constant function on \( X \), denoted by \( \text{rk}(E^*) \), defined in such a way that if \( E^* \) is locally written as a complex of vector bundles \([E^{-1} \to E^0]\), then \( \text{rk}(E^*) := \dim(E^0) - \dim(E^{-1}) \).

Let us suppose further that \( \text{rk}(E^*) \) is constant and that \( E^* \) has a global resolution, i.e., has a morphism of vector bundles \( F^* := \{F^{-1} \to F^0\} \) considered as a complex concentrated in degrees \(-1 \) and \(0\) together with an isomorphism \( F^* \xrightarrow{\sim} E^* \) in \( D(\mathcal{O}_X) \). Since \( h^1/h^0((E^\bullet)^\vee) \) is isomorphic to \([F^{-1}_\text{pec}]/(F^0_\text{pec}) \), the relative affine space \( \mathfrak{V} \) associated with \((F^{-1}_\text{pec})^\vee \) specifies a global presentation \( \mathfrak{V} \to h^1/h^0((E^\bullet)^\vee) \). Let \( \mathcal{W} \) be the fiber product

\[
\begin{array}{ccc}
\mathfrak{V} & \longrightarrow & \mathfrak{W} \\
\downarrow & & \downarrow \\
\mathfrak{C}_X & \longrightarrow & h^1/h^0((E^\bullet)^\vee),
\end{array}
\]

where the lower horizontal arrow denotes the composite of the closed immersion \( \mathfrak{C}_X \to \mathfrak{R}_X \) and the morphism \( \mathfrak{R}_X \to h^1/h^0((E^\bullet)^\vee) \) induced by \( \phi \), being a closed immersion (cf. [9], Theorem 4.5). We define the virtual fundamental class
\[ [\mathcal{X}, E^\bullet]^{\text{virt}} \] to be the intersection of \( \mathfrak{M} \) with the zero section \( 0_\mathfrak{M} : \mathcal{X} \to \mathfrak{M} \), i.e.,

\[
(139) \quad [\mathcal{X}, E^\bullet]^{\text{virt}} := 0_\mathfrak{M}[\mathfrak{M}],
\]

which is an element of \( A_{\text{rk}(E^\bullet)}(\mathcal{X})_Q \). This class is independent of the global resolution \( F^\bullet \) used to construct it. If \( \mathcal{X} \) is smooth, then the virtual fundamental class \( [\mathcal{X}, L^\bullet]^{\text{virt}} \) is equal to the usual fundamental class \( [\mathcal{X}] \).

### 4.2. The perfect obstruction theory for the moduli of G-do'pers.

Now, we construct a perfect obstruction theory for \( \mathcal{D}_{G, g, r, \mathbb{F}_p}^{\text{zar}} \). Denote by \( \text{Conn}_{G, g, r, \mathbb{F}_p} \) the category classifying pairs \( (X^\bullet, [(\mathcal{E}, \nabla_\mathcal{E})]) \) consisting of a pointed stable curve \( X^\bullet := (X, \{\sigma_{X,i}\})_i \) of type \( (g, r) \) over an \( \mathbb{F}_p \)-scheme \( S \) and the isomorphism class of a \( G \)-torsor \( \mathcal{E} \) over \( X \) paired with an \( \mathcal{S} \)-connection \( \nabla_\mathcal{E} \) on \( \mathcal{E} \); it may be represented by a Deligne-Mumford stack over \( \mathbb{F}_p \) and the forgetting morphism \( \text{Conn}_{G, g, r, \mathbb{F}_p} \to \mathfrak{M}_{g, r, \mathbb{F}_p} \) is representable. Also, denote by \( \text{Conn}_{G, g, r, \mathbb{F}_p}^{\text{zar}} \) the closed substack of \( \text{Conn}_{G, g, r, \mathbb{F}_p} \) classifying pairs \( (X^\bullet, [(\mathcal{E}, \nabla_\mathcal{E})]) \) with \( \psi(\mathcal{E}, \nabla_\mathcal{E}) = 0 \). The assignment from each pair \( (X^\bullet, \mathcal{E}^\bullet) \) (where \( X^\bullet \) denotes a pointed stable curve and \( \mathcal{E}^\bullet \) denotes a faithful twisted \( G \)-oper on \( X^\bullet \)) to \( (X^\bullet, [(\mathcal{E}^{\mathfrak{f}}_{X^\bullet}, \nabla^{\mathfrak{f}}_{X^\bullet})]) \) (where let \( \mathcal{E}^{\mathfrak{f}}_{X^\bullet} := (\mathcal{E}^{\mathfrak{f}}_{B, X^\bullet}, \nabla^{\mathfrak{f}}_{X^\bullet}) \) be a unique faithful twisted \( G \)-oper of canonical type isomorphic to \( \mathcal{E}^\bullet \)) determines morphisms

\[
(140) \quad \xi : \mathcal{D}_{G, g, r, \mathbb{F}_p} \to \text{Conn}_{G, g, r, \mathbb{F}_p}, \quad \xi^{\text{zar}} : \mathcal{D}_{G, g, r, \mathbb{F}_p}^{\text{zar}} \to \text{Conn}_{G, g, r, \mathbb{F}_p}^{\text{zar}}
\]

over \( \mathfrak{M}_{g, r, \mathbb{F}_p} \). These morphisms make the following square diagram commutative and cartesian:

\[
(141) \quad \begin{array}{ccc}
\text{Conn}_{G, g, r, \mathbb{F}_p}^{\text{zar}} & \xrightarrow{\xi^{\text{zar}, g, r}} & \text{Conn}_{G, g, r, \mathbb{F}_p} \\
\uparrow{\xi} & & \uparrow{\xi} \\
\mathcal{D}_{G, g, r, \mathbb{F}_p}^{\text{zar}} & \xrightarrow{i_{g, r}^{\mathfrak{f}}} & \mathcal{D}_{G, g, r, \mathbb{F}_p}
\end{array}
\]

where the upper horizontal arrow \( i_{g, r}^{\mathfrak{f}} \) denotes the natural closed immersion. Denote by \( \mathcal{I}_G \) (resp., \( \hat{\mathcal{I}}_G \)) the ideal sheaf on \( \mathcal{D}_{G, g, r, \mathbb{F}_p} \) (resp., \( \text{Conn}_{G, g, r, \mathbb{F}_p} \)) defining the closed substack \( \mathcal{D}_{G, g, r, \mathbb{F}_p}^{\text{zar}} \) (resp., \( \text{Conn}_{G, g, r, \mathbb{F}_p}^{\text{zar}} \)). Diagram \((141)\) induces a commutative diagram of coherent sheaves on \( \mathcal{D}_{G, g, r, \mathbb{F}_p}^{\text{zar}} \):

\[
(142) \quad \begin{array}{ccc}
\xi^{\text{zar}, g, r}(\hat{\mathcal{I}}_G/\mathcal{I}_G^2) & \xrightarrow{i_{g, r}^{\mathfrak{f}}} & (i_{g, r}^{\mathfrak{f}} \circ \xi^{\text{zar}, g, r} \ast (\Omega_{\text{Conn}_{G, g, r, \mathbb{F}_p}^{\text{zar}}/\mathbb{F}_p}) \\
\xi^{\text{zar}, g, r} \downarrow & & \downarrow{\xi} \\
\mathcal{I}_G/\mathcal{I}_G^2 & \xrightarrow{i_{g, r}^{\mathfrak{f}}} & \mathcal{I}_G/\mathcal{I}_G^2
\end{array}
\]
Since (141) is cartesian, $ξ^Z$ is verified to be surjective. Let us consider the composite $ι^Zζ \circ ξ^Z = (ξ \circ ι^Zζ)$ as a complex

$$E^* := [ξ^Zζ(\widehat{I}_G/\widehat{I}_G^2) \to i^Zg,rζ(Ω_{\mathcal{OP} G,g,r,F_p/F_p})]$$

concentrated in degrees $-1$ and $0$. Here, note that since $\mathcal{OP} G,g,r,F_p$ is smooth (cf. Theorem 2.3.5 (ii)), the cotangent complex $L^•_{\mathcal{OP} G,g,r,F_p}$ is naturally isomorphic (in $D(\mathcal{OP} G,g,r,F_p)$) to the complex $[I_G/I_G^2 \to ι^Zζg,rζ(Ω_{\mathcal{OP} G,g,r,F_p/F_p})]$. Thus, the pair of $ξ^Zζ$ and the identity morphism of $ι^Zζg,rζ(Ω_{\mathcal{OP} G,g,r,F_p/F_p})$ specifies a morphism

$$φ : E^* \to L^•_{\mathcal{OP} G,g,r,F_p}$$

in $D(\mathcal{OP} G,g,r,F_p)$.

**Theorem 4.2.1.**
The morphism $φ$ just obtained forms a perfect obstruction theory for $\mathcal{OP} G,g,r,F_p$ of constant virtual dimension $3g - 3 + r$ with $E^*$ perfect.

**Proof.** It follows from Theorem 2.3.5 (ii) that $Ω_{\mathcal{OP} G,g,r,F_p}$ is locally free of rank $3g - 3 + r + N$, where $N := (g - 1) \cdot \text{dim}(G) + \frac{3}{2} \cdot (\text{dim}(G) + \text{rk}(G))$. Hence, in order to complete the proof, it suffices to prove that $ξ^Zζ(\widehat{I}_G/\widehat{I}_G^2)$ is locally free of rank $N$.

For simplicity, let us denote by $X^• := (f : X \to S, \{σ_X,i\}_{i=1}^\infty)$ the tautological family of pointed stable curves over $S := \mathcal{OP} G,g,r,F_p$. Also, denote by $E^• := (E_Γ, Γ)$ the tautological $G$-do’per on $X^•$. Let $∇_Γ^E : g_E \to Ω_X^{\log}/Γ_Ω^{\log} ⊗ g_E$ be the $S^{\log}$-connection on the adjoint vector bundle $g_E$ induced by $∇_E$; we consider it as a complex $K^•(∇_Γ^E)$ concentrated in degrees $0$ and $1$. According to [55], §6.11, (681), there exists a short exact sequence

$$0 \to \mathbb{R}^1f_*(\text{Ker}(∇_Γ^E)) \to \mathbb{R}^1f_*(K^•(∇_Γ^E)) \to f_*(\text{Coker}(∇_Γ^E)) \to 0$$

of $\mathcal{O}_S$-modules. It follows from well-known generalities of deformation theory that one may construct a canonical isomorphism

$$(i^Zg,rζ)^*(\mathcal{T}_{\mathcal{OP} G,g,r,F_p/F_p}) \sim \mathbb{R}^1f_*(K^•(∇_Γ^E)).$$

By [55], Proposition 6.8.1, it restricts to an isomorphism

$$ξ^Zζ(\mathcal{T}_{\mathcal{OP} G,g,r,F_p/F_p}) \sim \mathbb{R}^1f_*(\text{Ker}(∇_Γ^E)).$$
On the other hand, it follows from [55, Proposition 6.8.1], that $f_*(\text{Coker}(\nabla^{\text{ad}}_E))$ is locally free of rank $N$. Hence, the dual of (146) and (147) induces a commutative diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & f_*(\text{Coker}(\nabla^{\text{ad}}_E))^v \\
\xi^z_*((\hat{\mathcal{L}}_G/\hat{\mathcal{I}}_G)) & \downarrow & \downarrow \\
\xi^z_*((\mathcal{O}_{\text{Conn}}^{g,r,F_p}/\mathcal{M}_{g,r,F_p}))^{(146)^{-1}} & \rightarrow & \mathbb{R}f_*(\mathcal{O}^\bullet(\nabla^{\text{ad}}_E))^v \\
\xi^z_*((\mathcal{O}_{\text{Conn}}^{g,r,F_p}/\mathcal{M}_{g,r,F_p}))^{(147)^{-1}} & \downarrow & \downarrow \\
0 & \downarrow & 0,
\end{array}
$$

where the both sides of vertical sequences are exact. Here, observe the equalities $\mathbb{R}^2f_*(\mathcal{O}^\bullet(\nabla^{\text{ad}}_E)) = 0$ (cf. [55, Proposition 6.2.2, (iii)]) and $\mathbb{R}^2f_*(\text{Ker}(\nabla^{\text{ad}}_E)) = 0$ (which follows from $\dim(X/S) = 1$). By taking account of these equalities and the constructions of (146) and (147), we see that both $\mathcal{O}_{\text{Conn}}^{g,r,F_p}$ and $\mathcal{O}_{\text{Conn}}^{g,r,F_p}$ are smooth over $\mathcal{M}_{g,r,F_p}$ at the points lying over $S$. It follows that the left-hand top vertical arrow in (148) is injective. That is to say, the top horizontal arrow $\xi^z_*((\hat{\mathcal{L}}_G/\hat{\mathcal{I}}_G)) \rightarrow f_*(\text{Coker}(\nabla^{\text{ad}}_E))^v$ is an isomorphism. Consequently, $\xi^z_*((\hat{\mathcal{L}}_G/\hat{\mathcal{I}}_G))$ is verified to be locally free of rank $N$. This completes the proof of the theorem.

As a result of the above theorem, we obtain the virtual fundamental class

$$[\mathcal{O}_p^{g,r,F_p}]^{\text{vir}} := [\mathcal{O}_p^{g,r,F_p}, E^\bullet]^{\text{vir}} \in A_{3g-3+r}(\mathcal{O}_p^{g,r,F_p}) \otimes \mathbb{Q}
$$

associated with $\phi : E^\bullet \rightarrow L^\bullet_{\mathcal{O}_p^{g,r,F_p}}$.

**Remark 4.2.2.**
Assume that $G = \text{PGL}_2$. Then, it follows from [11, Chap. II, Theorem 2.8], that $\mathcal{O}_p^{g,r,F_p}$ is smooth over $\mathbb{F}_p$. In particular, the virtual fundamental class $[\mathcal{O}_p^{g,r,F_p}]^{\text{vir}}$ coincides with the usual fundamental class $[\mathcal{O}_p^{g,r,F_p}]$. 
5. CohFTs associated with do’pers

In this section, we define the notion of a CohFT (= a cohomological field theory) mapped into the $l$-adic étale cohomology of the moduli stack classifying pointed stable curves in positive characteristic (cf. Definition 5.2.1). Also, by means of the virtual fundamental class obtained in the previous section and the factorization properties resulting from Proposition 3.2.3, we construct a CohFT for $G$-do’pers (cf. Theorem 5.3.1). It also forms a 2d TQFT whose corresponding Frobenius algebra is semisimple. In this section, we assume that $G$ satisfies the condition (**)$_{G,p}$ and moreover $k = F_p$. Denote by $F_p$ the algebraic closure of $F_p$. (In the following discussions, we are alway free to replace $F_p$ by any algebraic closed field over $F_p$.)

5.1. $l$-adic cohomology and Borel-Moore homology.

Let $\mathcal{M}$ be a finite type separated Deligne-Mumford stack over $F_p$, and $l$ a prime invertible in $F_p$. Denote by $D^b_c(\mathcal{M}, \mathbb{Q}_l)$ the derived category of constructible $\mathbb{Q}_l$-modules on $\mathcal{M}$. Hence, for each complex $\mathcal{L}$ in $D^b_c(\mathcal{M}, \mathbb{Q}_l)$, we obtain its étale cohomology $H^i_{\text{ét}}(\mathcal{M}, \mathcal{L})$ ($i \in \mathbb{Z}$). Also, we write

$$\tilde{H}^i_{\text{ét}}(\mathcal{M}, \mathcal{L}) := H^i_{\text{ét}}(\mathcal{M}, \mathcal{L}(\lfloor \frac{i}{2} \rfloor)),$$

$$\tilde{H}^*_\text{ét}(\mathcal{M}, \mathcal{L}) := \bigoplus_{i \in \mathbb{Z}} \tilde{H}^i_{\text{ét}}(\mathcal{M}, \mathcal{L}).$$

(150)

For each $i \in \mathbb{Z}$, write $H^i_{BM}(\mathcal{M}, \mathcal{O}_l)$ for the $i$-th Borel-Moore homology of $\mathcal{M}$ defined in [47], Definition 2.2. That is to say, we set

$$H^i_{BM}(\mathcal{M}, \mathcal{O}_l) := H^{-i}_{\text{ét}}(\mathcal{M}, \omega_{\mathcal{M}}),$$

(151)

where $\omega_{\mathcal{M}} \in D^b_c(\mathcal{M}, \mathcal{O}_l)$ denotes the $l$-adic dualizing complex of $\mathcal{M}$. Also, set

$$\tilde{H}^i_{BM}(\mathcal{M}, \mathcal{O}_l) := H^i_{BM}(\mathcal{M}, \mathcal{O}_l)(-\lceil \frac{i}{2} \rceil), \quad \tilde{H}^*_BM(\mathcal{M}, \mathcal{O}_l) := \bigoplus_{i \in \mathbb{Z}} \tilde{H}^i_{BM}(\mathcal{M}, \mathcal{O}_l).$$

(152)

The $i$-th cycle map is the $\mathbb{Q}$-linear map

$$cl^i : A_*(\mathcal{M})_{\mathbb{Q}} \to \tilde{H}^i_{BM}(\mathcal{M}, \mathcal{O}_l)$$

mentioned in [47], §2.10. If $\mathcal{M}$ is smooth of dimension $d$, then $\omega_{\mathcal{M}} \cong \mathcal{O}_l(d)[2d]$ and hence, we obtain a composite of natural isomorphisms

$$(-)^* : \tilde{H}^*_BM(\mathcal{M}, \mathcal{O}_l) \cong \bigoplus_{i \geq 0} H^{2d-i}_{\text{ét}}(\mathcal{M}, \mathcal{O}_l(d)[2d])(-\lceil \frac{i}{2} \rceil)
\cong \bigoplus_{i \geq 0} H^{2d-i}_{\text{ét}}(\mathcal{M}, \mathcal{O}_l(\lfloor \frac{2d-i}{2} \rfloor))$$

$$\cong \tilde{H}^{2d-*}_{\text{ét}}(\mathcal{M}, \mathcal{O}_l).$$

(154)
Then, the cycle map $cl_i$ coincides, via (154), with the classical definition of the cycle map $A_i(M) \rightarrow \widetilde{H}_{2i}(M, \mathbb{Q}_l)$.

By the definition of $\widetilde{H}_{BM}(\mathcal{M}, \mathbb{Q}_l)$, the cup product in $l$-adic cohomology induces a natural pairing

$$(-) \cap (-) : \widetilde{H}_{2j}(M, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \widetilde{H}_{2i}(M, \mathbb{Q}_l) \rightarrow \widetilde{H}_{2i-j}(M, \mathbb{Q}_l).$$

If $\mathcal{N}$ be another separated Deligne-Mumford stack of finite type over $\mathbb{F}_p$ and $f : \mathcal{M} \rightarrow \mathcal{N}$ is a proper morphism over $\mathbb{F}_p$, then there exists the pushforward map

$$f^\text{hom}_* : \widetilde{H}_{BM}(\mathcal{M}, \mathbb{Q}_l) \rightarrow \widetilde{H}_{BM}(\mathcal{N}, \mathbb{Q}_l)$$

along $f$ described in [47], §2.10. The following projection formula may be immediately verified:

$$(\alpha \cap (\cdot)) \cap (\cdot) : \widetilde{H}_{BM}(\mathcal{M}, \mathbb{Q}_l) \otimes_{\mathbb{Q}_l} \widetilde{H}_{BM}(\mathcal{N}, \mathbb{Q}_l) \rightarrow \widetilde{H}_{BM}(\mathcal{N}, \mathbb{Q}_l).$$

By assigning $v \mapsto 0$ for any $v \in \widetilde{H}_j(M, \mathbb{Q}_l)$ with $j \neq 2i$, we shall regard $\int_\alpha$ as a morphism $\widetilde{H}_{2i}(M, \mathbb{Q}_l) \rightarrow \mathbb{Q}_l$.}

### 5.2. Cohomological field theories.

We shall describe the definition of a cohomological field theory by means of the $l$-adic étale cohomologies $\widetilde{H}_{et}(\mathcal{M}, \mathbb{Q}_l)$ of $\mathcal{M}$.

**Definition 5.2.1.**

An $(l$-adic) cohomological field theory (with flat identity), abbreviated CohFT, is a collection of data

$$\Delta := (\mathcal{H}, \eta, 1, \{\Lambda_{g,r}\}_{g,r \geq 0, 2g-2+r>0})$$

consisting of

- a finite dimensional $\mathbb{Q}_l$-vector space $\mathcal{H}$ (called the state space) with basis $\mathbf{e} := \{e_1, \ldots, e_{\dim(\mathcal{H})}\}$,
- a symmetric nondegenerate pairing $\eta : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{Q}_l$ (called the metric),
- an element $1$ of $\mathcal{H}$,
\begin{itemize}
\item \( \Omega_r \)-linear morphisms \( \Lambda_{g,r} : \mathcal{H}^{\otimes r} \to \widetilde{H}^*_\mathrm{et}(\mathcal{M}_{g,r}, \mathcal{Q}_l) \) (called the \textbf{correlators}), where \( \mathcal{H}^{\otimes 0} := \mathcal{Q}_l \),
\end{itemize}

satisfying the following conditions:

\begin{itemize}
\item Each \( \Lambda_{g,r} \) is compatible with the respective actions of the symmetric group \( S_r \) on \( \mathcal{H}^{\otimes r} \) and \( \widetilde{H}^*_\mathrm{et}(\mathcal{M}_{g,r}, \mathcal{Q}_l) \) arising from permutations of the \( r \) factors in \( \mathcal{H}^{\otimes r} \) and the \( r \) punctures in the tautological family of curves over \( \mathcal{M}_{g,r} \).
\item For any \( v_1, v_2 \in \mathcal{H} \), the following equality holds:
\begin{equation}
\eta(v_1, v_2) = \int_{[\mathcal{M}_{0,3}, \mathcal{Q}_l]} \Lambda_{0,3}(v_1 \otimes v_2 \otimes 1).
\end{equation}
\item For any \( v_1, \ldots, v_{s+r} \in \mathcal{H} \), the following equality holds:
\begin{equation}
\Phi^*_\mathrm{tree}(\Lambda_{g_1+g_2, r_1+r_2}(v_1 \otimes \cdots \otimes v_{r_1+r_2})) = \sum_{e_1, e_2 \in \mathcal{E}} \Lambda_{g_1, r_1+1}(v_1 \otimes \cdots \otimes v_r \otimes e_1) \eta^{e_1 e_2} \otimes \Lambda_{g_2, r_2+1}(v_{r+1} \otimes \cdots \otimes v_{r_1+r_2}),
\end{equation}
where \( \Phi^*_\mathrm{tree} \) denotes the morphism
\begin{equation}
\widetilde{H}^*_\mathrm{et}(\mathcal{M}_{g_1+g_2}, \mathcal{Q}_l) \to \widetilde{H}^*_\mathrm{et}(\mathcal{M}_{g_1}, \mathcal{Q}_l) \otimes \mathcal{Q}_l \to \widetilde{H}^*_\mathrm{et}(\mathcal{M}_{g_2}, \mathcal{Q}_l)
\end{equation}
induced by \( \Phi^*_\mathrm{tree} \) (cf. \ref{16}).
\item For any \( v_1, \ldots, v_r \in \mathcal{H} \), the following equality holds:
\begin{equation}
\Phi^*_\mathrm{loop}(\Lambda_{g+1, r}(v_1 \otimes \cdots \otimes v_r)) = \sum_{e_1, e_2 \in \mathcal{E}} \Lambda_{g, r+1}(v_1 \otimes \cdots \otimes v_r \otimes e_1 \otimes e_2) \eta^{e_1 e_2},
\end{equation}
where \( \Phi^*_\mathrm{loop} \) denotes the morphism
\begin{equation}
\widetilde{H}^*_e(\mathcal{M}_{g+1}, \mathcal{Q}_l) \to \widetilde{H}^*_e(\mathcal{M}_{g}, \mathcal{Q}_l)
\end{equation}
induced by \( \Phi^*_\mathrm{loop} \) (cf. \ref{17}).
\item For any \( v_1, \ldots, v_r \in \mathcal{H} \), the following equality holds:
\begin{equation}
\Phi^*_\mathrm{tail}(\Lambda_{g, r}(v_1 \otimes \cdots \otimes v_r)) = \Lambda_{g, r+1}(v_1 \otimes \cdots \otimes v_r \otimes 1),
\end{equation}
where \( \Phi^*_\mathrm{tail} \) denotes the morphism
\begin{equation}
\widetilde{H}^*_\mathrm{et}(\mathcal{M}_{g, r}, \mathcal{Q}_l) \to \widetilde{H}^*_\mathrm{et}(\mathcal{M}_{g, r+1}, \mathcal{Q}_l)
\end{equation}
induced by \( \Phi^*_\mathrm{tail} \) (cf. \ref{18}).
\end{itemize}

A CohFT whose correlators \( \Lambda_{g,r} \) are all valued in \( \widetilde{H}^*_\mathrm{et}(\mathcal{M}_{g,r}, \mathcal{Q}_l) = \mathcal{Q}_l \) may be thought of as a \textbf{2d TQFT} (= a 2-dimensional topological quantum field theory) over \( \mathcal{Q}_l \) (cf. \cite{29}, §1.3.32 for the definition of an \( n \)-dimensional topological quantum field theory). Moreover, it is well-known that 2d TQFTs correspond to Frobenius algebras. Recall (cf. \cite{29}, §2.2.5) that a \textbf{Frobenius}
algebra over $\mathbb{Q}_l$ is a pair $(\mathcal{H}, \eta)$ consisting of a unital, associative, and commutative $\mathbb{Q}_l$-algebra $\mathcal{H}$ of finite dimension and a nondegenerate $\mathbb{Q}_l$-bilinear pairing $\eta : \mathcal{H} \times \mathcal{H} \to \mathbb{Q}_l$ such that
\begin{equation}
\eta(v_1, (v_2 \times v_3)) = \eta((v_1 \times v_2), v_3)
\end{equation}
for any $v_1, v_2, v_3 \in \mathcal{H}$, where $\times$ denotes the multiplication in $\mathcal{H}$. We shall say that a Frobenius algebra $(\mathcal{H}, \eta)$ is 
\emph{semisimple} if there exists a basis $e^\dagger := \{e^\dagger_a\}_{a \in I}$ of $\mathcal{H}$ such that
\begin{equation}
e^\dagger_a \times e^\dagger_b = \delta_{ab} e^\dagger_a \quad \text{and} \quad \eta(e^\dagger_a, e^\dagger_b) = \delta_{ab} \nu_a
\end{equation}
for any $a, b \in I$, where $\nu_a$ is some nonzero element of $\mathbb{Q}_l$. We shall refer to $e^\dagger$ as a \emph{canonical basis} of $(\mathcal{H}, \eta)$.

Now, let $\Lambda := (\mathcal{H}, \eta, 1, \{\Lambda_{g,r}\}_{g,r}^\dagger)$ be a CohFT (where let $e := \{e_a\}_{a \in I}$ denote the distinguished basis of $\mathcal{H}$). There exists a multiplication $\times : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ on $\mathcal{H}$ given by
\begin{equation}
u := \sum_{a,b \in I} \Lambda_{0,3}(u \otimes v \otimes e_a) \eta_{ab} e_b.
\end{equation}
The $\mathbb{Q}_l$-vector space $\mathcal{H}$ together with this multiplication forms a unital, associative, and commutative $\mathbb{Q}_l$-algebra, in which $1$ is the unit. Moreover, one verifies that the pair $(\mathcal{H}, \eta)$ forms a Frobenius algebra over $\mathbb{Q}_l$, which we shall refer to as the \emph{Frobenius algebra associated with $\Lambda$}.

5.3. CohFTs associated with the moduli of do’pers.

In the following, we shall construct a CohFT associated with the moduli of do’pers. As described in Introduction, we write
\begin{equation}
\mathcal{V} := \tilde{H}_{\text{et}}^*(\mathcal{I}_\mu([((t^F_{\text{reg}}/W)) \otimes \mathbb{F}_p, \mathbb{Q}_l]).
\end{equation}
Decomposition (118) gives rise to a sequence of natural isomorphisms
\begin{equation}
\mathcal{V} \simeq \tilde{H}_{\text{et}}^*(\coprod_{\rho \in \Delta} \mathcal{I}_\mu(BZ)_\rho \otimes \mathbb{F}_p, \mathbb{Q}_l) \simeq \bigoplus_{\rho \in \Delta} \tilde{H}_{\text{et}}^*(\mathcal{I}_\mu(BZ)_\rho \otimes \mathbb{F}_p, \mathbb{Q}_l) \simeq \bigoplus_{\rho \in \Delta} \mathbb{Q}_l e_\rho,
\end{equation}
where $e_\rho$ (for each $\rho \in \Delta$) denotes the element of $\tilde{H}_{\text{et}}^0(\mathcal{I}_\mu(BZ)_\rho \otimes \mathbb{F}_p, \mathbb{Q}_l) (\subseteq \mathcal{V})$ corresponding to $1 \in \mathbb{Q}_l$ via the following composite isomorphism:
\begin{equation}
\tilde{H}_{\text{et}}^*(\mathcal{I}_\mu(BZ)_\rho \otimes \mathbb{F}_p, \mathbb{Q}_l) \simeq \tilde{H}_{\text{et}}^*(BZ \otimes \mathbb{F}_p, \mathbb{Q}_l) \simeq \tilde{H}_{\text{et}}^0(BZ \otimes \mathbb{F}_p, \mathbb{Q}_l) = \mathbb{Q}_l.
\end{equation}
Let us choose $e_\Delta := \{e_\rho\}_{\rho \in \Delta}$ as a distinguished basis of $\mathcal{V}$. 
Next, write

$$\eta : \mathcal{V} \times \mathcal{V} \to \overline{\mathbb{Q}}_l$$

for the $\overline{\mathbb{Q}}_l$-bilinear pairing determined by

$$\eta_{\rho_1, \rho_2} := \eta(\rho_1, \rho_2) = \begin{cases} \frac{1}{|\mathcal{Z}|} & \text{if } \rho_1 = \rho_2^\vee, \\ 0 & \text{if otherwise}. \end{cases}$$

It is clear that $\eta$ is symmetric and nondegenerate.

Finally, let $g, r$ be nonnegative integers with $2g - 2 + r > 0$. Denote by $[\mathcal{O}_{G,g,r,F}^{\text{vir}}]_{\mathbb{Q}_l}$ the pull-back of $[\mathcal{O}_{G,g,r,F}]_{\mathbb{Q}_l}$. Write

$$\pi_{g,r} : \mathcal{O}_{G,g,r,F}^{\text{vir}} \to \mathcal{M}_{g,r,F}$$

for each $i = 1, \ldots, r$ for the pull-backs of $\pi_{g,r}^Z$ and $e_{v_i}$. We define a $\overline{\mathbb{Q}}_l$-linear morphism

$$\Lambda_{G,g,r}^{Z} : \mathcal{V}^r \to \widetilde{H}_0^\text{et}(\mathcal{M}_{g,r,F}, \overline{\mathbb{Q}}_l)$$

to be the morphism determined uniquely by

$$\Lambda_{G,g,r}^{Z}(\bigotimes_{i=1}^r v_i) = \left( \left( \prod_{i=1}^r e_{v_i}(v_i) \right) \cap c_1^{3g-3+r}(\frac{[\mathcal{O}_{G,g,r,F}^{\text{vir}}]}{[\mathcal{O}_{G,g,r,F}]_{\mathbb{Q}_l}}) \right),$$

where $v_1, \ldots, v_r \in \mathcal{V}$. Thus, we obtain a collection of data

$$\mathcal{M}_G := (\mathcal{V}, \eta, e, \{\Lambda_{G,g,r}^{Z}\}_{g,r \geq 0, 2g - 2 + r > 0}).$$

**Theorem 5.3.1.**

(Recall that we have assumed the condition (**)$_{G,p}$).

(i) For each $\bar{\rho} := (\rho_i)_{i=1}^r \in \Delta^r$, $\Lambda_{G,g,r}^{Z}(\bigotimes_{i=1}^r e_{\rho_i})$ lies in $\widetilde{H}_0^\text{et}(\mathcal{M}_{g,r,F}, \overline{\mathbb{Q}}_l) (= \overline{\mathbb{Q}}_l)$ and the following equality holds:

$$\Lambda_{G,g,r}^{Z}(\bigotimes_{i=1}^r e_{\rho_i}) = \deg(\mathcal{O}_{G,g,r,F}^{\text{vir}})_{\mathbb{Q}_l} \mathcal{M}_{g,r,F}, \overline{\mathbb{Q}}_l),$$

where the right-hand side denotes the generic degree of $\mathcal{O}_{G,g,r,F}^{\text{vir}}$ over $\mathcal{M}_{g,r,F}$ (cf. Theorem 3.2.2 (iii)).

(ii) The collection of data $\mathcal{M}_G$ forms a CohFT valued in $\widetilde{H}_0^\text{et}(\mathcal{M}_{g,r,F}, \overline{\mathbb{Q}}_l) (= \overline{\mathbb{Q}}_l)$, namely, forms a 2d TQFT over $\overline{\mathbb{Q}}_l$. Moreover, the corresponding Frobenius algebra $(\mathcal{V}, \eta)$ is semisimple.
Proof. Let us consider assertion (i). First, observe that, by the definition of $N_{g,r}$, the following equality holds:

\begin{equation}
N_{g,r} (\bigotimes_{i=1}^{r} e_{\rho_i}) = \left( n_{g,r}^{\text{hom}} \left( c^{g-3+r} \left( \left[ D_p Z_{g,r} \right]_{\text{vir}} \mid \left[ D_p Z_{g,r} \right]_{\text{vir}} \right) \right) \right)^{\dagger},
\end{equation}

where $(-) |_{D_p Z_{g,r}}$ denotes the restriction of the class $(-)$ to the component $D_p Z_{g,r}$. Also, since the square diagram

\begin{equation}
\begin{array}{ccc}
A_{3g-3+r}(D_p Z_{g,r})_Q & \xrightarrow{c^{g-3+r}} & H_{6g-6+2r}(D_p Z_{g,r})_F,
\end{array}
\end{equation}

is commutative (cf. [17], § 2.10), we have

\begin{equation}
\begin{aligned}
\pi_{g,r} & \left( c^{g-3+r} \left( \left[ D_p Z_{g,r} \right]_{\text{vir}} \mid \left[ D_p Z_{g,r} \right]_{\text{vir}} \right) \right) \\
& = c^{g-3+r} \left( \pi_{g,r} \left( \left[ D_p Z_{g,r} \right]_{\text{vir}} \mid \left[ D_p Z_{g,r} \right]_{\text{vir}} \right) \right).
\end{aligned}
\end{equation}

Next, since $D_p Z_{g,r}$ is finite over $\overline{M}_{g,r}$ (cf. Theorem 3.2.2 (ii)) and $\overline{M}_{g,r}$ is an irreducible stack of dimension $3g-3+r$, any prime cycle in $A_{3g-3+r}(D_p Z_{g,r})_Q$ dominates $\overline{M}_{g,r}$. On the other hand, the generic étaleness of $D_p Z_{g,r}$ (where we express $\rho$ as $(\rho_1, \kappa_1)$) over $\overline{M}_{g,r}$ (cf. [55], Theorem G) implies the generic étaleness of $D_p Z_{g,r} (\cong D_p Z_{g,r}, (\rho_1, \kappa_1))_Q \times \overline{M}_{g,r}$ by Theorem 3.2.2 (i) over $D_p Z_{g,r}(\kappa_1)_Q$. Hence, by the smoothness of $D_p Z_{g,r}(\kappa_1)_Q$ (cf. Theorem 1.3.5), $D_p Z_{g,r}$ turns out to be generically smooth over $\overline{F}_p$. More precisely, any irreducible component $\mathfrak{N}$ of $D_p Z_{g,r}$ dominating $\overline{M}_{g,r}$ has a dense open substack $\mathfrak{N}^0$ (which does not intersect any other irreducible components and) which is smooth over $\overline{F}_p$. The restriction of $\left[ D_p Z_{g,r} \right]_{\text{vir}}$ to $\mathfrak{N}^0$ coincides with the (usual) fundamental class $[\mathfrak{N}^0]$. By the observations made so far and the definition of the pushforward map $\pi_{g,r}$ between rational Chow groups, the following equality holds:

\begin{equation}
\pi_{g,r} \left( D_p Z_{g,r} \right)_{\text{vir}} = \deg(D_p Z_{g,r} \cap \overline{M}_{g,r}) \cdot \left[ \overline{M}_{g,r} \right].
\end{equation}
Thus, (178), (180), and (181) give the following sequence of equalities:

\[(182)\qquad \Lambda_{G,g,r}^{Z_{\mathfrak{g},r}}(\bigotimes_{i=1}^{r} e_{\rho}) \overset{178}{=} \left(\pi_{g,r}^{\text{hom}} \left(\text{cl}^{3g-3+r} \left(\frac{\partial p_{\mathfrak{g},g,r,\rho}}{\partial p_{\mathfrak{g},g,r,\rho}}\right)\right)\right) \overset{178}{=} \left(\pi_{g,r}^{\text{hom}} \left(\text{cl}^{3g-3+r} \left(\frac{\partial p_{\mathfrak{g},g,r,\rho}}{\partial p_{\mathfrak{g},g,r,\rho}}\right)\right)\right) \overset{181}{=} \left(\text{cl}^{3g-3+r} \left(\deg(\partial p_{\mathfrak{g},g,r,\rho}^{Z_{\mathfrak{g},r}}/\mathcal{M}_{g,r})\right)\right)\]

This completes the proof of assertion (i).

The former assertion of (ii) follows from assertion (i) and Proposition 3.2.3 (i), (ii), and (iii).

Finally, we shall consider the latter assertion of (ii), i.e., the semisimplicity of the Frobenius algebra \((\mathcal{V}, \eta)\). To this end, it suffices to prove that the \(\mathbb{Q}_{\ell}\)-algebra \(\mathcal{V}\) is reduced. Let us fix an isomorphism \(\mathcal{O}_{\ell} \overset{\sim}{\rightarrow} \mathbb{C}\). Denote by \(\gamma\) the involution on \(\mathcal{V}\) (viewed as an \(\mathbb{R}\)-algebra) given by \(\sum_{\rho \in \Delta} v_{\rho} e_{\rho} \mapsto \sum_{\rho \in \Delta} \overline{v_{\rho}} e_{\rho}\), where \(v_{\rho} \in \mathbb{C}\) for each \(\rho \in \Delta\) and \((-)\) denotes the complex conjugation. Note that for each \(x := \sum_{\rho \in \Delta} v_{\rho} e_{\rho} \in \mathcal{V}\), we have \(\eta(x, \gamma(x)) = \frac{1}{|\Delta|} \sum_{\rho \in \Delta} |v_{\rho}|^2\). Hence, \(\eta(x, \gamma(x)) = 0\) implies \(x = 0\). Now, let us take an element \(x := \sum_{\rho \in \Delta} v_{\rho} e_{\rho} \in \mathcal{V}\) with \(x \times x = 0\). Then,

\[(183)\qquad \eta(x \times \gamma(x), \gamma(x \times \gamma(x))) = \eta(x \times \gamma(x), \gamma(x) \times x) = \eta(x \times x, \gamma(x) \times \gamma(x)) = 0,\]

which implies \(x \times \gamma(x) = 0\). It follows that \(\eta(x, \gamma(x)) = 0\), and hence, that \(x = 0\). Consequently, \(\mathcal{V}\) is verified to be reduced, and this completes the proof of the latter assertion of (ii).

\[\square\]

**Remark 5.3.2.**

(i) Since the Frobenius algebra \((\mathcal{V}, \eta)\) associated with \(\mathbb{A}_{\mathfrak{g},r}^{Z_{\mathfrak{g},r}}\) is semisimple, there exists a canonical basis \(\{e_{\rho}^i\}_{\rho \in \Delta}\) of \(\mathcal{V}\). Let us write \(e_{\rho}^i := (\nu_{\rho}^{1/2})^{-1} e_{\rho}^i\) (where each \(\nu_{\rho}^{1/2}\) is an element of \(\mathbb{Q}_{\ell}\) with \((\nu_{\rho}^{1/2})^2 = v_{\rho}\)). The \(S\)-matrix \(S := (S_{\rho \lambda})_{\rho \lambda} \in \text{GL}(\mathcal{V})\) is defined in such a way that \(e_{\rho} = \sum_{\lambda \in \Delta} S_{\rho \lambda} e_{\lambda}^i\) (i.e., \(S_{\rho \lambda} := \frac{S_{\rho \lambda}}{v_{\rho}}\) in the sense of [9], §3.3.5, (20)) for any \(\rho \in \Delta\). Hence, \(\nu_{\rho} = S_{\rho \lambda}^2 (\rho \in \Delta)\) (cf. [9], §3.3.8). By applying the discussion in [9], §4.3, we obtain the Verlinde formula for G-do’pers. That is to say, for
each \((\rho_i)_{i=1}^r \in \Delta \times \Delta\), the following equality holds:

\[
(184) \quad \Lambda^{Z_{\Delta}}_{G,g,r} \left( \bigotimes_{i=1}^r e_{\rho_i} \right) = \deg(\mathcal{O}_{\mathcal{P}_{G,g,r}(\rho_i)_{i=1}^r, \mathbb{Z} / p} / \mathcal{O}_{g,r, \mathbb{Z}, \mathbb{P}}) = \sum_{\lambda \in \Delta} \prod_{i=1}^r S_{\rho_i}^{\lambda_{\rho_i}} .
\]

This formula contains the case of \(g = 0\), i.e.,

\[
(185) \quad \Lambda^{Z_{\Delta}}_{G,g,r} (1) = \deg(\mathcal{O}_{\mathcal{P}_{G,g,0,r}, \mathbb{Z} / p} / \mathcal{O}_{g,0,r, \mathbb{Z}, \mathbb{P}}) = \sum_{\rho \in \Delta} \chi^{-g - 1} \prod_{i=1}^r \chi(\rho_i) .
\]

(ii) Let us assume further that \(G\) is of adjoint type (i.e., \(|Z| = 1\)). Then, \(\Lambda^{Z_{\Delta}}_{G,g,r}\) forms (under the fixed isomorphism \(\mathbb{Q}_l \sim \mathbb{C}\)) a fusion ring in the sense of \([50]\), Definition 11.10. Formula (184) may be expressed as

\[
(186) \quad \Lambda^{Z_{\Delta}}_{G,g,r} \left( \bigotimes_{i=1}^r e_{\rho_i} \right) = \sum_{\chi \in \mathfrak{S}} \chi(\mathcal{C}_{as} G)^g - 1 \prod_{i=1}^r \chi(\rho_i) .
\]

(cf. \([55]\), Theorem F), where \(\mathcal{C}_{as} G := \sum_{\rho \in \Delta} \rho \times \rho \) and \(\mathfrak{S}\) denotes the set of characters (i.e., morphisms of \(\mathbb{Q}_l\)-algebras) \(\mathcal{V} \to \mathbb{Q}_l\). Here, recall that the forgetting morphism \(\mathcal{O}_{\mathcal{P}_{G,g,r}(\rho_i)_{i=1}^r, \mathbb{Z} / p} \to \mathcal{O}_{g,r, \mathbb{Z}, \mathbb{P}}\) is representable, finite, and generically étale. Hence, for a sufficiently general curve \(X^*\) in \(\mathcal{M}_{g,r, \mathbb{P}}\), the number of \(G\)-do’pers of radii \((\rho_i)_{i=1}^r\) on \(X^*\) consists precisely with the value \(\Lambda^{Z_{\Delta}}_{G,g,r} \left( \bigotimes_{i=1}^r e_{\rho_i} \right)\).

If, moreover, \(G = PGL_n\) for a small \(n\) (relative to \(g\) and \(p\)), then the result of \([55]\), Theorem H, allows us to compute the value \(\Lambda^{Z_{\Delta}}_{PGL_n,g,0,r} (1)\) without explicit knowledge of the characters \(\mathcal{V} \to \mathbb{Q}_l\). Indeed, under the assumption that \(p > n \cdot \max\{g - 1, 2\}\), the following formula holds:

\[
(187) \quad \Lambda^{Z_{\Delta}}_{PGL_n,g,0,r} (1) = \deg(\mathcal{O}_{\mathcal{P}_{PGL_n,g,0,r}, \mathbb{Z} / p} / \mathcal{O}_{g,0,r, \mathbb{Z}, \mathbb{P}}) = \frac{p^{(n-1)(g-1)-1}}{n!} \cdot \sum_{(\zeta_{1, \ldots, \zeta_n} \in \mathbb{C}^n \setminus \zeta_i = 1, \zeta_i \neq \zeta_j)} \prod_{i \neq j} (\zeta_i - \zeta_j)^{g-1} .
\]

6. THE WITTEN-KONSTSEVICH THEOREM FOR DO’PERS

In this last section, we introduce the partition function for do’pers and apply the discussion in \([19]\), §4, in order to obtain a result analogous to the Witten-Kontsevich theorem, which gives nontrivial relationships among the intersection numbers of the psi classes on \(\mathcal{O}_{\mathcal{P}_{G,g,r}, \mathbb{Z}, \mathbb{P}}\) (cf. Theorem 6.2.4). Finally, we shall conclude the paper with an explicit computation of the CohFT
for $\mathrm{PGL}_2$-do’pers (cf. Corollary 6.3.1). Let us keep the assumption imposed at the beginning of §5.

6.1. Correlator functions.

Denote by $\psi_i \in \tilde{H}^2_{\mathrm{et}}(\mathcal{M}_{g,r},\mathbb{Q}_l)$ ($i = 1, \cdots, r$) the $i$-th psi class on $\mathcal{M}_{g,r}$. Given a pair of nonnegative integers $(g,r)$ and an $r$-tuple of nonnegative integers $d_1, \cdots, d_r$, we shall recall (cf., e.g., [39], Introduction) the invariants

$$\langle \tau_{d_1} \cdots \tau_{d_r} \rangle_g = \langle \prod_{i=1}^r \tau_{d_i} \rangle_g := \int_{[\mathcal{M}_{g,r}]} \prod_{i=1}^r \psi_i^{d_i} \in \mathbb{Q}_l \quad (188)$$

(where if $r = 0$ or $2g - 2 + r \leq 0$, then $\langle - \rangle_g = 0$).

Next, let us define the classe $\widehat{\psi}_i \in \tilde{H}^2_{\mathrm{et}}(\mathcal{O}_p, \mathbb{Q}_l)$ to be the pull-back of $\psi_i$. Given a pair of nonnegative integers $(g,r)$, an $r$-tuple of nonnegative integers $d_1, \cdots, d_r$, and $v_1, \cdots, v_r \in \mathcal{V}$, we write

$$\langle \tau_{d_1}(v_1) \cdots \tau_{d_r}(v_r) \rangle_{G,g} := \int_{[\mathcal{O}_p]} \prod_{i=1}^r ev_i^*(v_i)\widehat{\psi}_i^{d_i} \in \mathbb{Q}_l \quad (189)$$

(where if $r = 0$ or $2g - 2 + r \leq 0$, then we set $\langle - \rangle_{G,g} := 0$). The invariants $\langle \tau_{d_1}(v_1) \cdots \tau_{d_r}(v_r) \rangle_{G,g}$ are called the $r$-point correlators.

Proposition 6.1.1.

Let $g, r, d_1, \cdots, d_r$, and $v_1, \cdots, v_r$ be as above. Then, the following equality holds:

$$\langle \tau_{d_1}(v_1), \cdots, \tau_{d_r}(v_r) \rangle_{G,g} = \Lambda_{G,g,r}^{Z_{2,\mathbb{Z}}(\boxtimes_{i=1}^r v_i)\langle \tau_{d_1} \cdots \tau_{d_r} \rangle_g. \quad (190)$$
Proof. The assertion follows from the following sequence of equalities:

\[(\tau_{d_1}(v_1)\cdots\tau_{d_r}(v_r))_{G,g}\]

\[= (f \circ \pi_{g,r})_\ast \left(\prod_{i=1}^r \psi_i^{d_i}(v_i)\right) \cap \mathcal{O}^{Zg-3+r}([\mathcal{M}_{G,g,r,\mathbb{F}_p}^{\text{vir}}])\]

\[= f^\ast \circ \pi_{g,r}^\ast \left(\prod_{i=1}^r \psi_i^{d_i}(v_i)\right) \cap \mathcal{O}^{Zg-3+r}([\mathcal{M}_{G,g,r,\mathbb{F}_p}^{\text{vir}}])\]

\[= f^\ast \left(\prod_{i=1}^r \psi_i^{d_i}(v_i)\right) \cap \mathcal{O}^{Zg-3+r}([\mathcal{M}_{G,g,r,\mathbb{F}_p}^{\text{vir}}])\]

\[= \Lambda_{G,g,r}(\bigotimes_{i=1}^r \psi_i^{d_i}(v_i))\]

where \(f\) denotes the structure morphism \(\mathcal{M}_{g,r,\mathbb{F}_p} \to \text{Spec}(\mathbb{F}_p)\) of \(\mathcal{M}_{g,r,\mathbb{F}_p}\) and the third equality follows from the projection formula (cf. (157)). □

6.2. The partition function of \(G\)-do’pers.

Let \(h\) and \(t_{d,r} (d \in \mathbb{Z}_{\geq 0}, \rho \in \Delta)\) be formal parameters. Given each basis \(e' := \{e'_{d,\rho}\}_{\rho \in \Delta}\) of \(\mathcal{V}\) and each \(g \geq 0\), we set

\[\Phi_{G,g,e'} := \langle \exp(\sum_{d \in \mathbb{Z}_{\geq 0}, \rho \in \Delta} \tau_d(e'_{d,\rho}t_{d,\rho})) \rangle_{G,g}\]

\[= \sum_{r \geq 0} \frac{1}{r!} \sum_{d_1,\ldots,d_r \geq 0} \langle \prod_{i=1}^r \tau_{d_i}(e'_{d,\rho}) \rangle_{G,g} \prod_{i=1}^n t_{d_i,\rho}.\]

\[= \sum_{(s_{d,\rho})_{d,\rho} \geq 0} \langle \prod_{d,\rho} \tau_d(e'_{d,\rho})^{s_{d,\rho}} \rangle_{G,g} \prod_{d,\rho \in \Delta} t_{d,\rho}^{s_{d,\rho}}\]

\[\in \mathbb{T}(\{\{t_{d,\rho}\}_{d \in \mathbb{Z}_{\geq 0}, \rho \in \Delta}\})\]

where the sum is run over the set of sequences of nonnegative integers \((s_{d,\rho})\) indexed by the elements of \(\mathbb{Z}_{\geq 0} \times \Delta\) with finitely many nonzero integers. Also,
write
\begin{equation}
\Phi_{G,e'} := \sum_{g \geq 0} \Phi_{G,g,e} h^{2g-2} \left( \in \mathbb{C}_l[[h]][\{t_{d,\rho}\}_{d \in \mathbb{Z}_{\geq 0}, \rho \in \Delta}]\right),
\end{equation}
\begin{equation}
Z_{G,e'}^{z_{\Delta}} := \exp(\Phi_{G,e'}) \left( \in \mathbb{C}_l((h))[[\{t_{d,\rho}\}_{d \in \mathbb{Z}_{\geq 0}, \rho \in \Delta}]\right).
\end{equation}
If \(e''\) is another basis of \(V\), the change of basis from \(e'\) to \(e''\) induces naturally an automorphism of the \(l\)-algebra \(\mathbb{C}_l((h))[[\{t_{d,\rho}\}_{d \in \mathbb{Z}_{\geq 0}, \rho \in \Delta}]\), by which \(Z_{G,e'}^{z_{\Delta}}\) is mapped to \(Z_{G,e''}^{z_{\Delta}}\). Notice that \(Z_{G,e'}^{z_{\Delta}}\) coincides with \(Z_{G}^{z_{\Delta}}\) described in Introduction.

**Definition 6.2.1.**
We shall refer to \(Z_{G}^{z_{\Delta}}\) (\(= Z_{G,e'}^{z_{\Delta}}\)) as the partition function of \(G\)-do'pers.

Next, let us fix a canonical base \(e^\dagger := \{e^\dagger_\rho\}_{\rho \in \Delta}\) (with \(\eta(e^\dagger_\rho,e^\dagger_\rho) = \nu_\rho\) of the Frobenius algebra corresponding to the CohFT \(\Delta_G^{z_{\Delta}} := (V,\eta,e\varepsilon,\{\Lambda_G^{z_{\Delta}}\}_{g,r})\).

Also, fix elements \(\nu^{1/3}_\rho\) of \(\mathbb{Q}\) with \((\nu^{1/3}_\rho)^3 = \nu_\rho\). For each \(\rho \in \Delta\) and \(n \geq -1\), we shall write
\begin{equation}
L_{n}(\rho) := -\frac{(2n+3)!!}{2^{n+1}}(\nu^{1/3}_\rho)^n \frac{\partial}{\partial \rho,n+1} + \sum_{i=0}^{\infty} \frac{(2i+2n+1)!!}{(2i-1)!!2^{n+1}}(\nu^{1/3}_\rho)^n \frac{t_{i}}{i} \frac{\partial}{\partial \rho,i+n}
+ \frac{\hbar^2}{2} \sum_{i=0}^{n-1} \frac{(2i+1)!!(2n-2i-1)!!}{2^{n+1}}(\nu^{1/3}_\rho)^{n-3} \frac{\partial^2}{\partial \rho,i \partial \rho,n-1-i}
+ \frac{\hbar^2}{2} (\nu^{1/3}_\rho)^2 t_{0} \delta_{n,0} + \frac{1}{16} \delta_{n,0}.
\end{equation}

There operators satisfy \([L_{n+1},L_{m}] = (n-m)\delta_{\rho_1,\rho_2} L_{n+m}\) for any \(n, m \geq -1\) and \(\rho_1, \rho_2 \in \Delta\) (cf. [19], §4.3). Similarly, if \(L_{n} \ (n \geq -1)\) are the differential operators defined in Introduction (cf. [13]), then \([L_n,L_m] = (n-m)L_{n+m}\) for any \(n, m \geq -1\).

**Theorem 6.2.2.**
Suppose that \(l\) is sufficiently large relative to \(g, r,\) and \(p\) (cf. Remark 6.2.3 below).

(i) Given \(\rho \in \Delta\), \(n \geq -1\), we obtain the following equality:
\begin{equation}
L_{n}(\rho) Z_{G,e'}^{z_{\Delta}} = 0.
\end{equation}
Moreover, these equations completely determine \(Z_{G}^{z_{\Delta}}\) (cf. the discussion preceding Definition [6.2.1]).

(ii) For any \(n \geq -1\), the following equality holds:
\begin{equation}
L_{n} Z_{G}^{z_{\Delta}} = 0.
\end{equation}
Proof. The assertions follow from Proposition 6.1.1 and [19], Proposition 4.4 applied to our CohFT $\Lambda\Lambda_{Zz}$. Indeed, by passing to a fixed isomorphism $\mathbb{Q}_l \cong \mathbb{C}$, one may apply Proposition 4.4 in loc. cit. because of the comparison between the $l$-adic cohomology $H^\ast(\overline{M}_{g,r,p}, \mathbb{Q}_l)$ and the complex cohomology $H^\ast(M_{g,r}^{\text{top}}, \mathbb{C})$ arising from the cospecialization map (cf. Remark 6.2.3 for a more detailed discussion). □

Remark 6.2.3.
We claim that the moduli stack $\overline{M}_{g,r,Z_p}$ (where $Z_p$ denotes the ring of $p$-adic integers) has $l$-prime inertia (in the sense of [57], Definition 1.4) for any sufficiently large (relative to $g$, $r$, and $p$) prime $l$. Indeed, let $X$ be a proper smooth curve of genus $g$ $(\geq 0)$ over an algebraically closed field $k$ and $\text{Aut}_k(X)$ the automorphism group of $X$ over $k$. Write $N_X := |\text{Aut}_k(X)|$. If $g > 1$ and $k$ has zero characteristic, then the classical Hurwitz bound for $X$ asserts that $N_X \leq 84(g - 1) (\leq 16g^4)$. On the other hand, according to [51], if $g > 1$ and $k$ is of characteristic $p$, then $N_X < 16g^4$ unless $X$ is isomorphic to the Hermitian curve: $V(y^{p^m} + y - x^{p^m+1}) \subseteq \mathbb{P}^2$ with $g = \frac{p^m(p^m-1)}{2}$. The order of the automorphism group of the Hermitian curve $X$ is equal to the value

$$U(m) := p^{3m}(p^{3m+1}+1)(p^{2m}-1).$$

(We refer to [16] for a more sharper bound of the order $N_X$.) Also, it is well-known (cf. [15], Chap. IV, §4, Corollary 4.7) that the stack $\overline{M}_{g,r,Z_p}$ with $g = 0$ (resp., $g = 1$) has $l$-prime inertia for any $l$ (resp., for any $l$ with $l > 4$). Hence, for any $X$ classified by a geometric point in $\overline{M}_{g,r,Z_p}$, the inequality $N_X \leq N_{g,p}$ holds, where

$$N_{g,p} := \begin{cases} 0 & \text{if } g = 0, \\ 4 & \text{if } g = 1, \\ \max \{16g^4 - 1, p^{3m}(p^{3m}+1)(p^{2m}-1)\} & \text{if } g = \frac{p^m(p^m-1)}{2} \text{ for some } m > 1, \\ 16g^4 - 1 & \text{otherwise.} \end{cases}$$

(197)

Next, consider the case where $X$ is classified by a geometric point in $\overline{M}_{g,r,Z_p} \setminus \overline{M}_{g,r,Z_p}$ (i.e., $X$ has nodal singularities). If $m$ denotes the number of components in the normalization $\tilde{X}$ of $X$, then it holds that $m \leq 2g - 2 + r$. Since any automorphism of $X$ induces a unique bijection from the set of components in $\tilde{X}$ onto itself, we have

$$N_X \leq m! \cdot \max \left\{N_{X'| X' \text{ is a component in } \tilde{X}} \right\} \leq m! \cdot \max \left\{N_{g',p} \mid 0 \leq g' \leq g\right\} \leq (2g - 2 + r)! \cdot \max \left\{4, 16g^4 - 1, U\left(\log_p \left(\frac{1+\sqrt{1+8g}}{2}\right)\right)\right\}.\]
Consequently, if we write

\[ N^\dagger_{g,r,p} := (2g - 2 + r)! \cdot \max \left\{ 4, 16g^4 - 1, U(\left\lceil \frac{1+\sqrt{1+8g}}{2} \right\rceil) \right\}, \]

then (since \( N_{g,p} \leq N^\dagger_{g,r,p} \)) the inequality \( N_X \leq N^\dagger_{g,r,p} \) holds for any \( X \) classified by a geometric point in \( \mathfrak{M}_{g,r,Z_p} \). That is to say, \( \mathfrak{M}_{g,r,Z_p} \) has \( l \)-prime inertia for any prime \( l \) with \( l > N^\dagger_{g,r,p} \).

Now, let us fix an isomorphism \( \mathfrak{Q}_l \simeq \mathbb{C} \) and suppose that \( \mathfrak{M}_{g,r,Z_p} \) is \( l \)-prime inertia for some prime \( l \) different from \( p \). Then, we obtain the following two isomorphisms

\[ H^*_\text{ét}(\mathfrak{M}_{g,r,F_p}, \mathfrak{Q}_l) \simeq H^*_\text{ét}(\mathfrak{M}_{g,r,C}, \mathfrak{Q}_l) \simeq H^*(\mathfrak{M}_{g,r,C}, \mathbb{C}), \]

where

- the first isomorphism follows from the specialization theorem (obtained from [37], Propositions 2.12, 2.15 (i) and an argument similar to the argument in the proof of [38], Corollary 4.2) for the cohomology of the fibers of the proper smooth morphism \( \mathfrak{M}_{g,r,Z_p} \to \mathbb{Z}_p \);
- \( \mathfrak{M}_{g,r,C} \) denotes the orbifold (or, topological stack in the sense of [43]) associated with \( \mathfrak{M}_{g,r,C} \), and \( H^*(\mathfrak{M}_{g,r,C}, \mathbb{C}) \) denotes the usual complex cohomology of \( \mathfrak{M}_{g,r,C} \);
- the second isomorphism denotes the comparison isomorphism obtained from the fixed isomorphism \( \mathfrak{Q}_l \simeq \mathbb{C} \) and the argument in [38], Theorem 3.12, together with Riemann’s existence theorem for stacks (cf. [43], Theorem 20.4).

The composite of these two isomorphisms is, regardless of the choice of \( \mathfrak{Q}_l \simeq \mathbb{C} \), compatible with the Poincare dualities, and moreover, with the cycle maps. In particular, the intersection numbers \( \langle \tau_{d_1} \cdots \tau_{d_r} \rangle \) \((d_1, \cdots, d_r \geq 0)\) defined in \( H^*_\text{ét}(\mathfrak{M}_{g,r,F_p}, \mathfrak{Q}_l) \) coincide with the corresponding intersection numbers defined in \( H^*(\mathfrak{M}_{g,r,C}, \mathbb{C}) \). In particular, the Witten-Kontsevich theorem may be thought of as the result for the \( l \)-adic cohomology.

Let \((g, r)\) be a pair of nonnegative integers, \((d_1, \cdots, d_r)\) an \( r \)-tuple of nonnegative integers, and \((v_1, \cdots, v_r)\) an \( r \)-tuple of elements of \( \mathcal{V} \). Then, we shall write

\[ \langle \langle \tau_{d_1}(v_1) \cdots \tau_{d_r}(v_r) \rangle \rangle_{G,g} := \langle \langle \tau_{d_1}(v_1) \cdots \tau_{d_r}(v_r) \rangle \exp \left( \sum_{d \in \mathbb{Z}_{\geq 0}, \mathbf{p} \in \Delta} \tau_{d}(e_{\mathbf{p}}) e_{d,\mathbf{p}} \right) \rangle_{G,g} \]

(where if \( r = 0 \) or \( 2g - 2 + r \leq 0 \), then we set \( \langle \langle - \rangle \rangle_{G,g} := 0 \)). Also, write

\[ \langle \langle \tau_{d_1}(v_1) \cdots \tau_{d_r}(v_r) \rangle \rangle_G := \sum_{g \geq 0} h^{2g-2} \langle \langle \tau_{d_1}(v_1) \cdots \tau_{d_r}(v_r) \rangle \rangle_{G,g}. \]
Then, by [19], Proposition 4.6, the following proposition holds.

**Theorem 6.2.4.**

For any \( d \in \mathbb{Z}_{\geq 0} \), \( v \in \mathcal{V} \), and \( \rho_1, \rho_2, \rho_3, \rho_4 \in \Delta \), the following equation holds:

\[
\begin{align*}
2d + 1 & \left( \frac{1}{\hbar} \langle \tau_0(e_{\rho_1}) \tau_0(e_{\rho_2}) \rangle \right) \mathcal{G}_{\rho_1,\rho_2}^{\rho_3,\rho_4} \\
& = \langle \tau_0(e_{\rho_1}) \tau_0(e_{\rho_2}) \rangle \mathcal{G}_{\rho_1,\rho_2}^{\rho_3,\rho_4} + 2 \langle \tau_0(e_{\rho_1}) \tau_0(e_{\rho_2}) \rangle \mathcal{G}_{\rho_1,\rho_2}^{\rho_3,\rho_4} \mathcal{G}_{\rho_1,\rho_2}^{\rho_3,\rho_4} + \frac{1}{4} \langle \tau_0(e_{\rho_1}) \tau_0(e_{\rho_2}) \rangle \mathcal{G}_{\rho_1,\rho_2}^{\rho_3,\rho_4} \mathcal{G}_{\rho_1,\rho_2}^{\rho_3,\rho_4}.
\end{align*}
\]

Moreover, equation (204) and the fact that \( L_{-1}^{|\rho|} Z_{\rho,\epsilon}^{\text{SL}_2} = 0 \) \( (\rho \in \Delta) \) completely determine \( \Phi_{\mathcal{G},\epsilon} \).

### 6.3. The CohFT for PGL\(_{2}\)-do’pers.

In this last subsection, let us study the relationship between the Frobenius algebra corresponding to the CohFT \( \mathbb{A}_{\text{PGL}_{2},\text{SL}_2} \) and the fusion ring of the SL\(_2\) WZW (= Wess-Zumino-Witten) model (of level \( p - 2 \)). By means of this relationship, we describe (cf. Corollary 6.3.1) explicitly the values of \( \mathbb{A}_{\text{SL}_2,\text{SL}_2} \) and the partition function associated with \( \mathbb{A}_{\text{SL}_2} \). Write \( (\mathcal{U}, \eta^{(\mathcal{U})}) \) for the fusion ring associated with the SL\(_2\) WZW model of level \( p - 2 \), which corresponds to a CohFT valued in \( H^1(\text{PGL}_{2},\mathbb{C}, \mathbb{C}) \). Also, write \( Z_{\text{SL}_2}^{\text{WZW},p-2} \) for the partition function associated with this CohFT.

First, let us recall the structure of the \( \mathbb{C} \)-algebra \( \mathcal{U} \). Denote by \( \theta \) the highest root of \( \mathfrak{sl}_2(\mathbb{C}) \) and by \( (-,-) \) the Killing form on \( \mathfrak{sl}_2(\mathbb{C}) \) normalized as \( (\theta, \theta) = 2 \). Also, denote by \( P_{p-2} \) the set of dominant weights \( \lambda \) of \( \mathfrak{g} \) with \( 0 \leq (\theta, \lambda) \leq p-2 \). Since \( P_{p-2} = \{0, \frac{1}{2} \alpha, \alpha, \cdots, \frac{p-2}{2} \alpha\} \) for a unique dominant weight \( \alpha \), we shall identify, via the correspondence \( j \alpha \leftrightarrow j, \ P_{p-2} \) with the set \( \{0, \frac{1}{2}, \cdots, \frac{p-2}{2}\} \) \((\subseteq \frac{1}{2} \mathbb{Z})\). Then, \( \mathcal{U} \) may be expressed as the \( \mathbb{C} \)-vector space \( \bigoplus_{j \in P_{p-2}} \mathbb{C} \mathcal{U}^j \) with basis \( \{\mathcal{U}^j\}_{j \in P_{p-2}} \) indexed by \( P_{p-2} \). (In particular, \( Z_{\text{SL}_2}^{\text{WZW},p-2} \) may be considered as an element of \( \mathbb{C}((\hbar))[[\{t_{d,m}\}_{d \geq 0, m \in P_{p-2}}]]\).) Also, the multiplication “\( \times \)” in \( \mathcal{U} \) is determined as follows: if \( \{N_{a,b,c} \in \mathbb{C} \mid a, b, c \in P_{p-2}\} \) denotes a collection defined in such a way that \( e_a \times e_b = \sum_{c \in P_{p-2}} N_{a,b,c} e_c \), then \( N_{a,b,c} \in \{0, 1\} \) for any triple \((a, b, c)\), and \( N_{a,b,c} = 1 \) if and only if \((a, b, c)\) satisfies the condition that

\[
a + b + c \in \mathbb{Z}, \quad a + b + c \leq \frac{p-2}{2}, \quad \text{and} \quad |b - c| \leq a \leq b + c.
\]
On the other hand, let \((V, \eta^V)\) be the Frobenius algebra associated with \(\Lambda_{PGL_2}^Z\) and write

\[ P^Z_{p-2} := \left\{ m \in \mathbb{Z} \mid 0 \leq j \leq \frac{p-2}{2} \right\} \quad (= P_{p-2} \cap \mathbb{Z}). \]

According to the discussion in [55, §7.11 (or [41], Introduction, §1.2, Theorem 1.3)], \(V\) is isomorphic to the \(\mathbb{Q}_l\)-vector space \(\bigoplus_{m \in P^Z_{p-2}} \mathbb{Q}_l e^V_m\) (where \(\{e^V_m\}_{m \in P^Z_{p-2}}\) is a basis indexed by \(P^Z_{p-2}\)) with multiplication characterized uniquely by the following condition: if we identify \(\mathbb{Q}_l\) with \(\mathbb{C}\) by means of an arbitrary isomorphism \(\mathbb{Q}_l \sim \hookrightarrow \mathbb{C}\), then the \(\mathbb{C}\)-linear morphism \(incl: V \rightarrow \mathcal{U}\) given by \(e^V_m \mapsto e^U_m\) forms an injective morphism of \(\mathbb{C}\)-algebras.

Next, let us describe the characters \(V \rightarrow \mathbb{C}\) (\(\cong \mathbb{Q}_l\)) of \(V\). By [7], Lemma 9.3, the set of characters \(\mathcal{U} \rightarrow \mathbb{C}\) coincides with the set \(\{\text{Tr}^*_j\}_{j=1}^{p-1}\), where each \(\text{Tr}^*_j\) is given by \(\text{Tr}^*_j : e^U_m \mapsto \sin\left(\frac{(2m+1)j\pi}{p}\right)\sin\left(\frac{j\pi}{p}\right)\) \((m \in P_{p-2})\).

The equality \(\text{Tr}^*_j \circ incl = \text{Tr}^*_j \circ incl\) holds if and only if \(j_1 + j_2 = p\). It follows that the set \(\mathcal{S}\) of characters \(V \rightarrow \mathbb{C}\) (cf. Remark 5.3.2 (ii)) coincides with \(\{\text{Tr}^*_j \circ incl\}_{j=1,2,\ldots,p-1}\).

By applying equality (138), one may compute the values of \(\Lambda_{PGL_2,g,r}^Z\). Indeed, for each \(j \in \{1, 2, \ldots, \frac{p-1}{2}\}\), we have \(|\text{Tr}^*_j(e^U_m)| = |\text{Tr}^*_j(e^U_{m-2})|\) \((m \in P_{p-2})\), and hence,

\[ (\text{Tr}^*_j \circ incl)(\text{Cas}_{PGL_2}) = \sum_{m \in P^Z_{p-2}} |(\text{Tr}^*_j \circ incl)(e^V_m)|^2 \]

\[ = \sum_{m \in P^Z_{p-2}} \frac{1}{2} \left( |\text{Tr}^*_j(e^U_m)|^2 + |\text{Tr}^*_j(e^U_{m-2})|^2 \right) \]

\[ = \frac{1}{2} \sum_{m \in P_{p-2}} |\text{Tr}^*_j(e^U_m)|^2 \]

\[ = \frac{1}{2} \cdot \frac{2p}{\left(2 \sin\left(\frac{j\pi}{p}\right)\right)^2} \]

\[ = \frac{p}{\left(2 \sin\left(\frac{j\pi}{p}\right)\right)^2}. \]
where the fourth equality follows from [7], Lemma 9.7. It follows that, for each pair of nonnegative integers \((g, r)\) with \(2g - 2 + r > 0\) and each \((n_i)_{i=1}^r \in (P_{p-2})^r\), the following sequence of equality holds:

\[
\Lambda_{\mathbb{PGL}_{2}, g, r}^{\mathbb{Z}_2} \left( \bigotimes_{i=1}^r e_{n_i}^V \right) = \sum_{\chi \in \mathcal{S}} \chi(\text{Cas}_{\mathbb{PGL}_2})^{g-1} \prod_{i=1}^r \chi(e_{n_i}^V)
\]

\[
= \sum_{j \in \{1, \ldots, \frac{p-1}{2}\}} \prod_{i=1}^r \sin \left( \frac{(2n_i+1)j\pi}{p} \right) \sin^{2g-2+r} \left( \frac{j\pi}{p} \right)
\]

(209)

where the third equality follows from (207) and (208). Consequently, we have obtained the following corollary.

**Corollary 6.3.1.**

(i) For each pair of nonnegative integers \((g, r)\) with \(2g - 2 + r > 0\) and each \((n_i)_{i=1}^r \in (P_{p-2})^r\), the following equality holds:

\[
\Lambda_{\mathbb{PGL}_{2}, g, r}^{\mathbb{Z}_2} \left( \bigotimes_{i=1}^r e_{n_i}^V \right) = \deg(\Delta_{\mathbb{PGL}_{2}, g, r, (n_i)_{i=1}^r \in \mathbb{P}_{p-2}}^{\mathbb{Z}_2})
\]

\[
= \frac{p^{r-1}}{2^{2g-2}} \sum_{j=1}^{p-1} \prod_{i=1}^r \sin \left( \frac{(2n_i+1)j\pi}{p} \right) \sin^{2g-2+r} \left( \frac{j\pi}{p} \right)
\]

(210)

(ii) Let us consider the surjective morphism of \(\mathbb{C}\)-algebras

\[
\alpha : \mathbb{C}((h)) \rightarrow \mathbb{C}((h)) \rightarrow \mathbb{C}((h))[[\{t_{d,m} \}_{d \geq 0, m \in P_{p-2}}]]
\]

given by \(h \mapsto h^{\frac{1}{2}}, t_{d,m} \mapsto t_{d,m}\) for any \(m \in P_{p-2}\), and \(t_{d,m} \mapsto 0\) for any \(m \in P_{p-2} \setminus P_{p-2}\). Then, the following equality holds:

\[
Z_{\mathbb{PGL}_2}^{\mathbb{Z}_2} = \alpha(Z_{\mathbb{SL}_2}^{\mathbb{Z}_{WZW}, p-2}).
\]

(211)

**Proof.** Assertion (ii) follows from assertion (i), which implies that if we write \(\Lambda_{\mathbb{PGL}_2, p-2}^{\mathbb{Z}_{WZW}} (g, r \geq 0, 2g - 2 + r > 0)\) for the correlators of the CohFT
corresponding to the Frobenius algebra $(\mathcal{U}, \eta^U)$, then
\[
\Lambda_{WZW, p-2}^{WZ}\left(\bigotimes_{i=1}^r e_{n_i}^V\right) = 2^{g-1} \cdot \Lambda_{\text{PGL}_{2,p,r}}^{\mathbb{Z}} \left(\bigotimes_{i=1}^r e_{n_i}^V\right) \quad (\text{for any } n_1, \cdots, n_r \in \mathbb{Z}_{p-2}).
\]
□

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