CLASSIFICATION OF MAXIMAL SUBALGEBRAS OF $k[t, t^{-1}, y]$

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Abstract. Let $k$ be an algebraically closed field. We classify all maximal $k$-subalgebras of $k[t, t^{-1}, y]$. To the authors’ knowledge, this is the first such classification result for an algebra of dimension $> 1$. In the course of this study, we classify also all maximal $k$-subalgebras of $k[t, y]$ that contain a coordinate of $k[t, y]$. Furthermore, we give examples of maximal $k$-subalgebras of $k[t, y]$ that do not contain a coordinate.

1. Introduction

All rings in this article are commutative an have a unity. A minimal ring extension is a non-trivial ring extension that does not allow a proper intermediate ring. A good overview of minimal ring extensions can be found in [PPL06]. A first general treatment of minimal ring extensions was done by Ferrand and Olivier in [FO70]. They came up with the following important property of minimal ring extensions.

Theorem (see [FO70, Théorème 2.2]). Let $A \subseteq R$ be a minimal ring extension and let $\varphi : \text{Spec}(R) \to \text{Spec}(A)$ be the induced morphism on spectra. Then there exists a unique maximal ideal $m$ of $A$ such that $\varphi$ induces an isomorphism

$$\text{Spec}(R) \setminus \varphi^{-1}(m) \cong \text{Spec}(A) \setminus \{m\}.$$  

Moreover, $\varphi$ is surjective if and only if $R$ is a finite $A$-module if and only if $m = mR$.

Let $A \subseteq R$ be a minimal ring extension. Then $R$ is called a minimal overring of $A$ and $A$ is called a maximal subring of $R$. In the case where $\text{Spec}(R) \to \text{Spec}(A)$ is non-surjective, we call $A$ an extending maximal subring of $R$ and otherwise, we call it a non-extending maximal subring. Moreover, the unique maximal ideal $m$ of $A$ (from the theorem above) is called the crucial maximal ideal.

In this article we are interested in the description of all maximal subrings of a given ring $R$. Let us give an instructive example.

Example I (see Lemma 3.2). Let $k$ be an algebraically closed field and let $R = k[t, t^{-1}, y]$. The ring

$$A = k[t] + yk[t, t^{-1}, y] = k[t, y/y, y/t^2, y/t^3, \ldots]$$

is a non-extending maximal subring of $k[t, t^{-1}, y]$. The crucial maximal ideal of $A$ is given by

$$m = (t, y, y/t, y/t^2, \ldots).$$

Thus $A \subseteq k[t, t^{-1}, y]$ induces an open immersion $\mathbb{A}^1_k \times \mathbb{A}^1_k \to \text{Spec}(A)$ and the complement of the image is just $\{m\}$. Moreover, the morphism $\text{Spec}(A) \to \mathbb{A}^2_k$...
induced by $k[t, y] \subseteq A$, sends the crucial maximal ideal $m$ to the origin $(0, 0)$. So in some sense we “added” to $A_k^\ast \times A_k^1$ the point $(0, 0) \in \{0\} \times A_k^1$.

Another description of the affine scheme $\text{Spec}(A)$ is the following: It is the inverse limit of \[ \cdots \longrightarrow A_k^2 \overset{\varphi_2}{\longrightarrow} A_k^1 \overset{\varphi_1}{\longrightarrow} A_k^0 \] inside the category of affine schemes, where $\varphi_i(t, y_i) = (t, ty_i)$.

Since in the non-extending case $R$ is a finite $A$-module, one has maybe the intuition, that this is not such a difficult case. Indeed, in Section 5 we provide a classification of all non-extending maximal subrings of an arbitrary ring (up to the classification of all maximal subfields of a given field). In fact, Dobbs, Mullins, Picavet and Picavet-L’Hermitte gave in [DMPPL05] already such a classification.

In Section 4 we give in the same section some general properties of maximal subrings that we will use often in the course of this article. As the classification of all maximal subfields of a given field. Moreover, we explicitly construct all the non-extending maximal subrings of a given ring.

Thus we are left with the extending case. One can reduce the study to the case where $R$ is an integral domain, and this is explained in Section 5. Moreover, we give in the same section some general properties of maximal subrings that we will use often in the course of this article. As the classification of all extending maximal subrings of an arbitrary integral domain seems to be still a difficult task, we restrict ourselves to the case where $R$ is a finitely generated domain over an algebraically closed field $k$ (of any characteristic). Moreover, we consider then only extending maximal $k$-subalgebras.

If $R$ is one-dimensional, then we are able to describe all extending maximal $k$-subalgebras. This description is done in Section 4. Let us give a simple example.

**Example II.** If $R = k[t, t^{-1}]$, then the only extending maximal $k$-subalgebras are $k[t]$ and $k[t^{-1}]$. In fact, $\mathbb{P}_k^1$ is a smooth projective closure of $A_k^\ast$. If we identify $A_k^\ast$ with the image under the open immersion \[ A_k^\ast \longrightarrow \mathbb{P}_k^1, \quad t \longrightarrow (t : 1), \] then $k[t]$ is the subring of functions on $A_k^\ast$ that are defined at $(0 : 1) \in \mathbb{P}_k^1$ and $k[t^{-1}]$ is the subring of those functions on $A_k^\ast$ that are defined at $(1 : 0)$.

This example follows from the general description, which is very similar:

**Theorem A** (see Theorem 1.2). Let $R$ be a finitely generated one-dimensional $k$-algebra. Take a projective closure $\overline{X}$ of the affine curve $X = \text{Spec}(R)$ such that $\overline{X}$ is non-singular at every point of $\overline{X} \setminus X$. If $\overline{X} \setminus X$ contains just a single point, then $R$ has no extending maximal $k$-subalgebra. Otherwise, for any point $p \in \overline{X} \setminus X$, \[ \{ f \in R \mid f \text{ is defined at } p \} \] is an extending maximal $k$-subalgebra of $R$ and every extending maximal $k$-subalgebra of $R$ is of this form.

The main work of this article consists in the classification of all extending maximal $k$-subalgebras of $R = k[t, t^{-1}, y]$. To the authors’ knowledge, this is the first such classification result for an algebra of dimension $> 1$.

Towards this classification, we describe in Section 5 all extending maximal $k$-subalgebras of $k[t, t^{-1}, y]$ that contain $k[t, y]$. To formulate our results we introduce some notation. Let $k[[t^Q]]$ be the Hahn field over $k$ with rational exponents, i.e. the field of formal power series \[ \alpha = \sum_{s \in \mathbb{Q}} a_s t^s \quad \text{such that } \text{supp}(\alpha) = \{ s \in \mathbb{Q} \mid a_s \neq 0 \} \] is well ordered.

Moreover, we denote by $k[[t^Q]]^+$ the subring of elements $\alpha \in k[[t^Q]]$ that satisfy $\text{supp}(\alpha) \subseteq [0, \infty)$. By extending the scalars $k[t, t^{-1}]$ to the Hahn field $k[[t^Q]]$ one has then a simple classification, namely:
Theorem B (see Corollary 5.14 and Remark 5.15). We have a bijection
\[ k[[t^Q]]^+ \rightarrow \left\{ \text{extending maximal } k\text{-subalgebras of } k[[t^Q]][y] \text{ that contain } k[[t^Q]]^+[y] \right\}, \quad \alpha \mapsto k[[t^Q]]^+ + (y - \alpha) k[[t^Q]][y] \]

With the aid of this theorem, we are then able to classify all extending maximal \( k \)-subalgebras of \( k[t, t^{-1}, y] \) that contain \( k[t, y] \).

Theorem C (see Theorem 5.24). Let \( \mathcal{S} \) be the set of \( \alpha \in k[[t^Q]]^+ \) such that \( \text{supp}(\alpha) \) is contained in a strictly increasing sequence of \( Q \). Then we have a surjection
\[ \mathcal{S} \rightarrow \left\{ \text{extending maximal } k\text{-subalgebras of } k[t, t^{-1}, y] \text{ that contain } k[t, y] \right\}, \quad \alpha \mapsto A_\alpha \cap k[t, t^{-1}, y] \]
where
\[ A_\alpha = k[[t^Q]]^+ + (y - \alpha) k[[t^Q]][y]. \]
Moreover, two elements of \( \mathcal{S} \) are sent to the same \( k \)-subalgebra, if and only if they lie in the same orbit under the natural action of \( \text{Hom}(Q/\mathbb{Z}, \mathbb{K}) \) on \( \mathcal{S} \).

In Section 6 we then start with the description of all maximal \( k \)-subalgebras of \( k[t, t^{-1}, y] \). Our main result of that section is the following.

Theorem D (see Proposition 6.1). Let \( A \subseteq k[t, t^{-1}, y] \) be an extending maximal \( k \)-subalgebra. Then, exactly one of the following cases occur:
\[ \text{i) There exists an automorphism } \sigma \text{ of } k[t, t^{-1}, y] \text{ such that } \sigma(A) \text{ contains } k[t, y]; \]
\[ \text{ii) } A \text{ contains } k[t, t^{-1}]. \]

The maximal \( k \)-subalgebras of case i) are then described by Theorem C. Thus we are left with the description of the extending maximal \( k \)-subalgebras of \( k[t, t^{-1}, y] \) that contain \( k[t, t^{-1}] \). This will be done in Section 7. In order to state our result let us introduce some notation. Let \( \mathcal{M} \) be the set of extending maximal \( k \)-subalgebras of \( k[t, y] \) that contain \( k[t] \). Moreover, let \( \mathcal{N} \) be the set of extending maximal \( k \)-subalgebras \( A \) of \( k[t, y, y^{-1}] \) that contain \( k[t, y^{-1}] \) and such that
\[ A \rightarrow k[t, y, y^{-1}]/(t - \lambda) \]
is surjective, where \( \lambda \) is the unique element in \( k \) such that the crucial maximal ideal of \( A \) contains \( t - \lambda \) (this \( \lambda \) exists by Remark 7.1). The set \( \mathcal{N} \) is described by Theorem E. The maximal \( k \)-subalgebras of case ii) in Theorem D are then described by the following result.

Theorem E (see Theorem 7.2 and Proposition 8.2). With the definitions of \( \mathcal{M} \) and of \( \mathcal{N} \) from above, we have bijections \( \Theta \) and \( \Phi \)
\[ \mathcal{N} \xrightarrow{\Theta} \mathcal{M} \supseteq \left\{ B \in \mathcal{M} \text{ s.t. the crucial maximal ideal of } B \text{ does not contain } t \right\} \xrightarrow{\Phi} \left\{ \text{extending maximal } k\text{-subalgebras of } k[t, t^{-1}, y] \text{ that contain } k[t, t^{-1}] \right\} \]
given by \( \Theta(A) = A \cap k[t, y] \) and \( \Phi(A') = A' \cap k[t, y] \).

In particular, Theorem E gives us a description of the extending maximal \( k \)-subalgebras of \( k[t, y] \) that contain a coordinate of \( k[t, y] \). Thus, one may speculate that every such maximal \( k \)-subalgebra contains a coordinate. However, this is not the case. We give plenty such examples in Section 8 using techniques of birational geometry of surfaces.
2. Classification of the non-extending maximal subrings

Let $R$ be any ring and denote by $X = \text{Spec}(R)$ the corresponding affine scheme. In the sequel, we describe three procedures to construct a non-extending maximal subring of $R$.

a) Glueing two closed points transversally. Choose two different closed points $x_1, x_2 \in X$ such that their residue fields $\kappa(x_1), \kappa(x_2)$ are isomorphic and choose some isomorphism $\sigma: \kappa(x_1) \to \kappa(x_2)$. Let

$$R_{x_1,x_2} = \{ f \in R \mid \sigma(f(x_1)) = f(x_2) \}$$

(note that $R_{x_1,x_2}$ depends on $\sigma$). Then $R_{x_1,x_2}$ is a non-extending maximal subring of $R$ with crucial maximal ideal $m_{x_1,x_2} = \{ f \in R \mid \sigma(f(x_1)) = f(x_2) = 0 \}$.

The homomorphisms on residue fields $R_{x_1,x_2}/m_{x_1,x_2} \to \kappa(x_i)$ are isomorphisms and the fiber of $X \to \text{Spec}(R_{x_1,x_2})$ over $m_{x_1,x_2}$ contains only the points $x_1$ and $x_2$. Moreover, the natural linear map on tangent spaces

$$T_{x_1}X \oplus T_{x_2}X \to T_{m_{x_1,x_2}}\text{Spec}(R_{x_1,x_2})$$

is an isomorphism.

Proof. For $i = 1, 2$, let $m_i \subseteq R$ be the maximal ideal corresponding to $x_i$. The injective homomorphism

$$R_{x_1,x_2}/m_{x_1,x_2} \to R/m_i$$

is surjective. Indeed, let $f \in R$. By symmetry, we can assume that $i = 1$. Then there exists $h \in m_1$ such that $h(x_2) = \sigma(f(x_1)) - f(x_2)$, thus $f + h \in R_{x_1,x_2}$, which proves the surjectivity. Let $\kappa = R_{x_1,x_2}/m_{x_1,x_2}$. Since $m_{x_1,x_2} = m_1 \cap m_2$, the homomorphism

$$R_{x_1,x_2}/m_{x_1,x_2} \subseteq R/m_1 \cap m_2 = R/m_1 \times R/m_2$$

identifies with the diagonal homomorphism $\kappa \to \kappa \times \kappa$. Since $m_{x_1,x_2} = m_{x_1,x_2}R$, it follows that $R_{x_1,x_2}$ is a maximal subring of $R$, see Lemma 3.2. Moreover, the fiber of $X \to \text{Spec}(R_{x_1,x_2})$ over $m_{x_1,x_2}$ consists of $x_1$ and $x_2$. For the last statement we prove that the $\kappa$-linear map on cotangent spaces

$$m_{x_1,x_2}/m_{x_1,x_2}^2 \to m_1/m_1^2 \oplus m_2/m_2^2$$

is an isomorphism. Let $f_1 \in m_1$ such that $f_1(x_2) = 1$. The ideals $m_1^2$ and $m_2^2$ are coprime, since $m_1$ and $m_2$ are coprime, and thus we get $m_1^2 \cap m_2^2 = m_1^2 \cdot m_2^2 \subseteq m_{x_1,x_2}^2$. This proves the injectivity of $\{1\}$. Let $h \in m_2$. Then

$$f_1^2h \in m_{x_1,x_2}, \quad f_1^2h - h \in m_2^2 \quad \text{and} \quad f_1^2h \in m_1^2,$$

which proves that $\{0\} \oplus m_2/m_2^2$ lies in the image of $\{1\}$. By symmetry we get the surjectivity of $\{1\}$. \qed

b) Deleting a tangent direction at a closed point. Choose a closed point $x \in X$ and a derivation $\delta: \mathcal{O}_{X,x} \to \kappa(x)$ that induces a non-zero tangent vector in $T_xX$. Let

$$R_{x,\delta} = \{ f \in R \mid \delta(f) = 0 \}.$$ 

Then $R_{x,\delta}$ is a non-extending maximal subring of $R$ with crucial maximal ideal $m_{x,\delta} = \{ f \in R \mid \delta(f) = 0, f(x) = 0 \}$. The morphism $X \to \text{Spec}(R_{x,\delta})$ is bijective, maps $x$ on $m_{x,\delta}$ and induces an isomorphism on residue fields $R_{x,\delta}/m_{x,\delta} \to \kappa(x)$. Moreover there is an induced exact sequence

$$0 \to \kappa(x)v \to T_xX \to T_{m_{x,\delta}}\text{Spec}(R_{x,\delta})$$
where \( v \in T_x X \) denotes the non-zero tangent vector induced by \( \delta \).

**Proof.** Let \( m \subseteq R \) be the maximal ideal corresponding to \( x \) in \( X \). Since \( \delta \) induces a non-zero \( \kappa(x) \)-linear map \( m/m^2 \to \kappa(x) \), there exists \( e \in m \) such that \( \delta(e) = 1 \).

The injective homomorphism

\[
R_x,\delta/m_{x,\delta} \longrightarrow \kappa(x)
\]

is surjective. Indeed, if \( f \in R \), then there exists \( r \in R \) such that \( \delta(f) - r(f) = 0 \) inside \( \kappa(x) \). Hence, \( f - er \in R_x,\delta \) and \( er \in m \), which proves the surjectivity.

Let \( \kappa = R_{x,\delta}/m_{x,\delta} \). Since \( m_{x,\delta} \) is an ideal of \( R \), we get a \( \kappa \)-algebra isomorphism

\[\kappa[\varepsilon]/(\varepsilon^2) \longrightarrow R/m_{x,\delta}, \quad \varepsilon \mapsto e\]

and the map \( R_{x,\delta}/m_{x,\delta} \to R/m_{x,\delta} \) identifies with the \( \kappa \)-linear map \( \kappa \to \kappa[\varepsilon]/(\varepsilon^2) \).

By Lemma [32] it follows that \( R_{x,\delta} \) is a maximal subring of \( R \). Clearly, the induced map \( X \to \text{Spec}(R) \) is bijective and maps \( x \) to \( m_{x,\delta} \). The last statement follows from the exact sequence of \( \kappa \)-vector spaces

\[
m_{x,\delta}/m_{x,\delta}^2 \longrightarrow m/m^2 \quad \delta \longrightarrow \kappa \longrightarrow 0.
\]

\[\square\]

**Remark 2.1.**

i) There are affine schemes \( X \) such that for some closed point \( x \in X \) the tangent space \( T_x X \neq 0 \), but there exists no non-zero derivation \( \mathcal{O}_{X,x} \to \kappa(x) \). Take for example \( X = \text{Spec}(\mathbb{Z}) \) and \( x = p\mathbb{Z} \) where \( p \) is some prime number.

ii) There are affine schemes \( X \) such that for some closed point \( x \in X \) there exists a non-zero derivation \( \delta: \mathcal{O}_{X,x} \to \kappa(x) \) that induces the zero vector in \( T_x X \).

Take for example any affine scheme \( X \) and take any closed point \( x \in X \) such that there exists a non-zero derivation \( \delta_0: \kappa(x) \to \kappa(x) \). Then

\[
\mathcal{O}_{X,x} \longrightarrow \kappa(x) \xrightarrow{\delta_0} \kappa(x)
\]

is a non-zero derivation that induces the zero vector in \( T_x X \).

c) **Shrinking the residue field at a closed point.** Choose a closed point \( x \in X \) and choose a maximal subfield \( k \) of the residue field \( \kappa(x) \), i.e. a subfield \( k \subseteq \kappa(x) \) such that there exists no proper intermediate field between \( k \) and \( \kappa(x) \).

Let

\[
R_{x,k} = \{ f \in R \mid f(x) \in k \}.
\]

Then \( R_{x,k} \) is a non-extending maximal subring of \( R \) with crucial maximal ideal

\[
m_{x,k} = \{ f \in R \mid f(x) = 0 \}
\]

and the residue field \( R_{x,k}/m_{x,k} \) is \( k \). Moreover, the morphism \( X \to \text{Spec}(R_{x,k}) \) is bijective and maps \( x \) on \( m_{x,k} \).

**Proof.** Note, that a maximal subfield of a field is automatically a maximal subring of that field. Thus the maximality of \( R_{x,v} \) in \( R \) follows from the fact that

\[
k = R_{x,v}/m_{x,v} \longrightarrow R/m_{x,v} = \kappa(x)
\]

is a maximal subfield, see Lemma [32]. The other statements are clear. \[\square\]

The next result shows, that we receive every non-extending maximal subring by one of the three constructions above. In fact, a version of this result can be found in [DMPPL05 Corollary II.2]. However for the sake of completeness and the shortness of the argument, we provide here a proof. The main ingredient will be an easy, but very important Lemma of Ferrand and Olivier [FO70].
Proposition 2.2. If $A \subseteq R$ is a non-extending maximal subring, then it is one of the maximal subrings constructed in a), b) or c).

Proof. By assumption, the map $\text{Spec}(R) \to \text{Spec}(A)$ is surjective and there exists a unique maximal ideal $m \subseteq A$ such that $m = mA$. By Lemma 3.2 the field $K = A/m$ is a maximal subring of $R/m$. By [FO70, Lemme 1.2] one of the following possibilities occur:

i) The map $K \subseteq R/m$ identifies with the diagonal map $K \to K \times K$.

ii) The map $K \subseteq R/m$ identifies with the $K$-homomorphism $K \to K[\varepsilon]/(\varepsilon^2)$.

iii) $R/m$ is a field.

If we are in case i), then $A = R_{x_1,x_2}$ where $x_1, x_2 \in \text{Spec}(R)$ correspond to the two maximal ideals $\{0\} \times K$, $K \times \{0\}$ of $R/m \simeq K \times K$ and $\sigma$ is given by

$$\kappa(x_1) \xrightarrow{\varepsilon} A/m \xrightarrow{\varepsilon} \kappa(x_2).$$

If we are in case ii), then $A = R_{x,\delta}$ where $x \in \text{Spec}(R)$ corresponds to the maximal ideal $(\varepsilon)$ of $R/m \simeq K[\varepsilon]/(\varepsilon^2)$ and $\delta$ is given by

$$\delta: R \longrightarrow R/m \simeq K[\varepsilon]/(\varepsilon^2) \xrightarrow{\delta'} K$$

where $\delta'$ is the $K$-derivation that maps $\varepsilon$ to 1. If we are in case iii), then $A = R_{x,k}$ where $x$ corresponds to the maximal ideal $m$ of $R$ and $k$ is the subfield $K$ of $R/m$. This finishes the proof. \qed

To the authors’ knowledge there is no complete description of the maximal subfields of a given field. Partial results in this direction can be found in [PPL06, Proposition 2.2]. However, if we restrict ourselves to the case, where $R$ is a finitely generated algebra over an algebraically closed field and if we consider only subalgebras, then we can exclude the construction c). More precisely, we get the following result.

Corollary 2.3. Let $k$ be an algebraically closed field and let $R$ be a finitely generated $k$-algebra. Denote $X = \text{Spec}(R)$. Then

i) $\{ f \in R \mid f(x_1) = f(x_2) \text{ inside } k \}$ (where $x_1 \neq x_2 \in X$ are closed points)

ii) $\{ f \in R \mid D_x(f) = 0 \}$ (where $0 \neq \varepsilon \in T_xX$ and $x \in X$ is a closed point)

are non-extending maximal $k$-subalgebras of $R$. Moreover, every non-extending maximal $k$-subalgebra of $R$ is one of the above.

Proof. For closed points $x_1 \neq x_2 \in X$, it follows by the construction in a) that the $k$-subalgebra in i) is equal to $R_{x_1,x_2}$ (we define $\sigma: k \to k$ as the identity). For a closed point $x \in X$ and for a non-zero tangent vector $0 \neq \varepsilon \in T_xX$, it follows by the construction in b) that the $k$-subalgebra in ii) is equal to $R_{x,\delta}$ where we define $\delta$ as the $k$-derivation

$$D_\varepsilon: O_{X,x} = m_x \oplus k \cdot 1 \longrightarrow m_x \longrightarrow m_x/m_x^2 = k$$

and where $m_x \subseteq O_{X,x}$ denotes the unique maximal ideal.

Conversely, let $A \subseteq R$ be a non-extending maximal $k$-subalgebra. By Proposition 2.2, $A$ is one of the maximal subrings of $R$ constructed in a), b) or c). Since $R$ is a finitely generated $k$-algebra and since $k$ is algebraically closed, it follows that the residue field of every closed point of $X$ is $k$. Since $R$ is a finitely generated $A$-module, it follows that $X \to \text{Spec}(A)$ is surjective, and therefore the residue field of every closed point of $\text{Spec}(A)$ is $k$. Thus $A$ cannot be one of the maximal subrings constructed in c). We distinguish two cases.
• A is the maximal subring constructed in a). Then there exist two different closed points $x_1 \neq x_2$ in $X$ and an isomorphism $\sigma: \kappa(x_1) \to \kappa(x_2)$ such that

$$A = \{ f \in R \mid \sigma(f(x_1)) = f(x_2) \}.$$  

Since $k \subseteq A$, $\sigma$ commutes with the canonical isomorphisms $k \simeq \kappa(x_i)$. Thus by identifying $\kappa(x_i)$ with $k$, the isomorphism $\sigma$ is the identity.

• A is the maximal subring constructed in b). Then there exist a closed point $x \in X$ and a derivation $\delta: \mathcal{O}_{X,x} \to k$ (that induces a non-zero tangent vector at $x$) such that

$$A = \{ f \in R \mid \delta(f) = 0 \}.$$  

Since $k \subseteq A$, it follows that $\delta$ is a $k$-derivation. There exists a $k$-linear isomorphism from $T_x X$ to the vector space of $k$-derivations $\mathcal{O}_{X,x} \to k$, given by $v \mapsto D_v$. Hence, there exists $0 \neq v \in T_x X$ such that $\delta = D_v$.

This finishes the proof. □

3. Some general considerations about maximal subrings

3.1. Reduction to integral domains. The aim of this subsection is to show that one can reduce the classification of the extending maximal subrings of a ring $R$ to the case, where $R$ is an integral domain.

Let $p$ be a minimal prime ideal of $R$ and denote by $\pi: R \to R/p$ the canonical projection. If $A$ is an extending maximal subring of $R$, such that the crucial maximal ideal contains $A \cap p$, then one can easily see, that $\pi(A)$ is an extending maximal subring of $R/p$. On the other hand, if $B$ is an extending maximal subring of $R/p$, then one can easily see $\pi^{-1}(B)$ is an extending maximal subring of $R$ and its crucial maximal ideal contains $\pi^{-1}(B) \cap p$. Thus we established a bijective correspondence:

$$\begin{align*}
\{ \text{extending maximal subrings } A \subseteq R & \text{ such that the crucial maximal ideal contains } A \cap p \} & \longleftrightarrow & \{ \text{extending maximal subrings of } R/p \}.
\end{align*}$$

Note that for every extending maximal subring $A$ of $R$ there exists a minimal prime ideal $p$ of $R$ such that the crucial maximal ideal contains $A \cap p$. Thus we are reduced to the case, where $R$ is an integral domain.

3.2. Some properties of maximal subrings. In this subsection we gather some general properties of maximal subrings, that we will constantly use in the course of this article.

The first lemma says that maximal subrings behave well under localization.

**Lemma 3.1** (see [FO70 Lemme 1.3]). Let $A \subseteq R$ be a maximal subring and let $S$ be a multiplicatively closed subset of $A$. Then the localization $A_S$ is either a maximal subring of the localization $R_S$ or $A_S = R_S$.

The second lemma gives us the possibility for certain cases to reduce to quotient rings, while searching for maximal subrings. It is a direct consequence of [FO70, Lemma 1.4].

**Lemma 3.2.** Let $A \subseteq R$ be a ring extension and let $I \subseteq A$ be an ideal such that $I = IR$. Then $A$ is a maximal subring of $R$ if and only if $A/I$ is a maximal subring of $R/I$.

In particular, for every ring extension $A \subseteq R$, the conductor ideal

$$I = \{ a \in A \mid aR \subseteq A \}$$

satisfied $I = IR$. Note that every ideal $J$ of $A$ with $J = JR$ is contained in the conductor ideal $I$. 
Lemma 3.3 (see [FO70 Lemme 3.2]). Let $A \subseteq R$ be an extending maximal subring. Then the conductor ideal of $A$ in $R$ is a prime ideal of $R$.

Samuel introduced in [Sam57] the $P_2$-property for ring extensions. This property will be crucial for our classification result.

Definition 3.4. Let $A \subseteq R$ be a subring. We say that $A$ satisfies the property $P_2$ in $R$, if for all $r, q \in R$ with $rq \in A$ we have either $r \in A$ or $q \in A$.

Lemma 3.5 (see [FO70 Proposition 3.1]). Let $A \subseteq R$ be an extending maximal subring. Then $A$ satisfies the property $P_2$ in $R$.

The next lemma shows, that the extending maximal subrings of a field have a well known characterization. It is a direct consequence of [FO70 Proposition 3.3].

Lemma 3.6. Let $K$ be a field and let $R \subseteq K$ be a subring. Then, $R$ is an extending maximal subring of $K$ if and only if $R$ is a one-dimensional valuation ring of $K$.

Let us state and prove the following rather technical lemma for future use.

Lemma 3.7. Let $C$ be a Noetherian domain such that the quotient field $Q(C)$ is not a finitely generated $C$-algebra. Let $A \subseteq C[y]$ be an extending maximal subring that contains $C$ and denote by $m$ the crucial maximal ideal of $A$. Then $m \cap C \neq 0$.

Proof. Assume that $m \cap C = 0$. Then we have the following commutative diagram

\[
\begin{array}{ccc}
C & \longrightarrow & A \\
\downarrow & & \downarrow \pi \\
Q(C) & \longrightarrow & A/m.
\end{array}
\]

As $A$ is a maximal subring of $C[y]$, we have $A \neq C$ and thus there exists $f \in A$ with $\deg_y(f) > 0$. Let $f = f_ny^n + \ldots + f_1y + f_0$ where $f_i \in C$, $f_n \neq 0$. We have

\[
y(f_ny^{n-1} + \ldots + f_1) = f - f_0 \in A.
\]

Since $A$ satisfies the property $P_2$ in $C[y]$ and since $y \notin A$ we get $f_ny^{n-1} + \ldots + f_1 \in A$. Proceeding in this way it follows that there exists $0 \neq c \in C$ such that $cy \in A$. Let us define the $C$-algebra homomorphism $\sigma$ by

\[
\sigma: C[y] \longrightarrow A/m, \quad y \mapsto \frac{\pi(cy)}{\pi(c)}.
\]

We claim that $\sigma$ and $\pi$ coincide on $A$. We proceed by induction on the $y$-degree of the elements in $A$. By definition, $\sigma$ and $\pi$ coincide on $C$, i.e. they coincide on the elements of $y$-degree equal to zero. Let $g = g_ny^n + \ldots + g_1y + g_0 \in A$ and assume that $g_n \neq 0$, $n > 0$. As before, we get $y(g_ny^{n-1} + \ldots + g_1) \in A$ and $g_ny^{n-1} + \ldots + g_1 \in A$. Thus we have

\[
\pi(g) = \pi(g_ny^{n-1} + \ldots + g_1) + \pi(g_0)
= \pi(cy(g_ny^{n-1} + \ldots + g_1)) + \pi(g_0)
= \frac{\pi(cy)}{\pi(c)} \pi(g_ny^{n-1} + \ldots + g_1) + \pi(g_0)
= \sigma(y)\sigma(g_ny^{n-1} + \ldots + g_1) + \sigma(g_0)
= \sigma(g),
\]

where we used in the second last equality the induction hypothesis. This proves the claim. Since $\pi$ is surjective, $\sigma$ is surjective to. Hence there exists $a \in A/m$
which is algebraic over $Q(C)$ such that $A/m$ is generated by $a$ as a $C$-algebra. Let $h_0 + h_1x + \ldots + h_mx^m + x^{m+1}$ be the minimal polynomial of $a$ over $Q(C)$ and let

$$C_0 = C[h_0, \ldots, h_m] \subseteq Q(C).$$

As $C$ is Noetherian, $C_0$ is Noetherian. Moreover, $A/m$ is generated by $1, a, \ldots, a^m$ as a $C_0$-module. Hence, $Q(C)$ is a finitely generated $C_0$-module. Thus $Q(C)$ is a finitely generated $C$-algebra, a contradiction. \hfill $\square$

### 4. The one-dimensional case

Let $k$ be an algebraically closed field. The purpose of this section is to classify all extending maximal $k$-subalgebras of a given one-dimensional affine $k$-domain $R$.

The key ingredient is the following observation.

**Lemma 4.1.** Let $A$ be a $k$-subalgebra of the one-dimensional affine $k$-domain $R$. Then either $A = k$ or $A$ is a one-dimensional affine $k$-domain.

**Proof.** We can assume that $A \neq k$. Then there exists $a \in A \setminus k$, which is transcendental over $k$. By the Krull-Akizuki-Theorem applied to $k[a] \subseteq A$, it follows that $A$ is Noetherian, see for example [Nag75, Theorem 33.2]. By [OY82, Corollary 1.2], we have

$$\dim A = \text{tr.deg}_k A = 1.$$ 

Let $A'$ be the integral closure of $A$ in its quotient field. By [Mat86, Theorem 9.3] it follows that $\dim A' = 1$. In particular, $A'$ is equidimensional. [OY82, Theorem 3.2] implies now, that $A$ is an affine $k$-domain. \hfill $\square$

The next Theorem classifies all extending maximal $k$-subalgebras of $R$.

**Theorem 4.2.** Take a projective closure $\overline{X}$ of the affine curve $X = \text{Spec}(R)$ such that $\overline{X}$ is non-singular at every point of $\overline{X} \setminus X$ (such an $\overline{X}$ is unique up to isomorphism). Let $U \subseteq \overline{X}$ be a proper open subset that contains $X$ and the complement $U \setminus X$ is just a single point. Then the image of the map on sections $\Gamma(U, O_U) \rightarrow R$ is an extending maximal $k$-subalgebra of $R$ and every extending maximal $k$-subalgebra of $R$ is of this form.

**Proof.** First, note that $U \subseteq \overline{X}$ is an affine curve, see [Har77, Chp. IV, Ex. 1.4]. Let $A \subseteq R$ be the image of $\Gamma(U, O_U) \rightarrow R$ and consider an intermediate ring $A \subseteq B \subseteq R$. By Lemma 4.1, $B$ is a one-dimensional affine $k$-domain. Consider the induced maps

$$X \xrightarrow{f} \text{Spec}(B) \rightarrow U.$$

As this composition is an open immersion, the first map is an open immersion. As $f$ is not an isomorphism, the complement $\text{Spec}(B) \setminus f(X)$ is non-empty. As $U \setminus X$ is a single point, this implies that $\text{Spec}(B) \rightarrow U$ is surjective. In fact, since $U$ is non-singular in $U \setminus X$, this map is an isomorphism and thus we get $A = B$. This proves that $A$ is an extending maximal $k$-subalgebra of $R$.

Conversely, let $A \subseteq R$ be an extending maximal $k$-subalgebra. By Lemma 4.1, $A$ is an affine $k$-domain. Let $g: X \rightarrow \text{Spec}(A)$ be the induced map on affine varieties. It is an open immersion and $\text{Spec}(A) \setminus X$ consists only of the crucial maximal ideal $m$ of $A$. Consider the birational map

$$\text{Spec}(A) \xrightarrow{g^{-1}} X \rightarrow \overline{X},$$

which is an open immersion on $g(X)$. We have to show, that this map is an open immersion on $\text{Spec}(A)$. By [FO70, Proposition 3.3], the localization $A_m$ is a one-dimensional valuation ring. Since $A_m$ is Noetherian, it is a discrete valuation ring. Thus $\text{Spec}(A)$ is non-singular at $m$ and therefore the birational map (2) is
an injective morphism, which is an open immersion locally at \( \mathfrak{m} \) (note that \( \overline{X} \) is smooth at every point of \( \overline{X} \setminus X \)). Thus the morphism \( R \) is an open immersion.

**Example 4.3.** Consider \( R = k[x, y]/(y - x^3 + xy^2) \). The closure of \( X = \text{Spec}(R) \subseteq A^2 \) in \( \mathbb{A}^2 \) consists of the three smooth points

\[
p_1 = (0 : 1 : 0), \quad p_2 = (1 : 1 : 0), \quad p_3 = (-1 : 1 : 0).
\]

The corresponding extending maximal \( k \)-subalgebras of \( R \) are given by the images on sections of the following maps (all maps are seen as restrictions of maps \( k_x, y \rightarrow A^2_{s,t} \))

\[
p_1: \quad X \rightarrow \{ t - s^2 + t^2 \}, \quad (x, y) \mapsto (x, xy)
p_2: \quad X \rightarrow \{ s^2 - t + 2t^2 - ts^2 \}, \quad (x, y) \mapsto (x - y, (x - y)x)
p_3: \quad X \rightarrow \{ s^2 - t - 2t^2 + ts^2 \}, \quad (x, y) \mapsto (x + y, (x + y)x).
\]

5. **Classification of maximal subrings of \( k[t, t^{-1}, y] \) that contain \( k[t, y] \)**

The goal of this section is the classification of all maximal subrings of \( k[t, t^{-1}, y] \) that contain \( k[t, y] \). Let us start with a simple example.

**Example 5.1.** By using Lemma 3.2 one can see that

\[
A = k[t, y] + (y^2 - t) k[t, t^{-1}, y]
\]

is a maximal subring of \( k[t, t^{-1}, y] \), which contains \( k[t, y] \). Another description of this ring is the following

\[
A = B \cap k[t, t^{-1}, y], \text{ where } B = k[t^{1/2}, y] + (y - t^{1/2}) k[t^{1/2}, t^{-1/2}, y].
\]

By using Lemma 3.2 one can see that \( B \) is a maximal subring of \( k[t^{1/2}, t^{-1/2}, y] \), which contains \( k[t^{1/2}, y] \). However, the ring \( B \) is of a simpler form than \( A \) (we replaced \( y^2 - t \) by a linear polynomial in \( y \)).

The general strategy works in a similar way. First we “enlarge” the coefficients \( k[t] \) to some ring \( F \) in such a way, that all maximal subrings of \( F[t] \) that contain \( F[y] \) have a simple form (in the example, we replaced \( k[t] \) by \( F = k[t^{1/2}] \)). Then we prove that the intersection of such a simple maximal subring with \( k[t, t^{-1}, y] \) yields a maximal subring of \( k[t, t^{-1}, y] \) that contains \( k[t, y] \) and that we receive by this intersection-process every maximal subring that contains \( k[t, y] \).

For the “enlargement” of the coefficients we have to introduce some notation and terminology.

5.1. **Notation and terminology.** Let \( k \) be an algebraically closed field (of any characteristic). We denote by \( k[[t^Q]] \) the Hahn field over \( k \) with exponents in \( Q \), i.e. the field of all formal power series

\[
\alpha = \sum_{s \in Q} a_s t^s
\]

with coefficients \( a_s \in k \) and with the property that the support

\[
\text{supp}(\alpha) = \{ s \in Q \mid a_s \neq 0 \}
\]

is a well ordered subset of \( Q \). There exists a natural valuation on \( k[[t^Q]] \), namely

\[
\nu: k[[t^Q]] \rightarrow \mathbb{Q}, \quad \alpha \mapsto \min(\text{supp}(\alpha)).
\]

The valuation ring of \( \nu \) we denote by \( k[[t^Q]]^+ \). More generally, for any subring \( B \subseteq k[[t^Q]] \) we denote by \( B^+ \) the subring of elements with \( \nu \)-valuation \( \geq 0 \), i.e.

\[
B^+ = \{ b \in B \mid \nu(b) \geq 0 \}.
\]
Finally, for any subring $A \subseteq k[[t^Q]][y]$ we denote by $A_1$ the subset of degree one elements, i.e.

\[ A_1 = \{ a \in A \mid \deg_y(a) = 1 \} . \]

### 5.2. Organisation of the section
In Subsection 5.3, we classify all maximal subrings of $K[y]$ that contain $K^+[y]$ for any algebraically closed field $K \subseteq k[[t^Q]]$ that contains the field of rational functions $k(t)$ and satisfies the so-called cutoff-property (see Definition 5.7). For example, the Hahn field $k[[t^Q]]$, the Puiseux field $\bigcup_s k((t^{1/n}))$ or the algebraic closure of $k(t)$ enjoy the cutoff-property (see Example 5.8).

In Subsection 5.4, we prove that for any maximal subring $A \subseteq K[y]$ containing $K^+[y]$, the intersection $A \cap k[t, t^{-1}, y]$ is again a maximal subring of $k[t, t^{-1}, y]$. Moreover, we prove that any maximal subring of $k[t, t^{-1}, y]$ that contains $k[t, y]$ can be constructed as an intersection like above.

Thus we are left with the question, which of the maximal subrings of $K[y]$ that contain $K^+[y]$ give the same ring, after intersection with $k[t, t^{-1}, y]$. We give an answer to this question in Subsection 5.5.

### 5.3. Classification of maximal subrings of $K[y]$ that contain $K^+[y]$.
Throughout this subsection, we fix an algebraically closed subfield $K \subseteq k[[t^Q]]$ that contains the field of rational functions $k(t)$.

#### Proposition 5.2
Let $K^+[y] \subseteq A \subseteq K[y]$ be an intermediate ring and assume that $A \subseteq K[y]$ satisfies the property $P_2$ in $K[y]$. Then $A = K^+[A_1]$.

**Proof.** Let $a \in A$. After multiplying $a$ with a unit of $K^+$, we can assume that

\[ a = \frac{y^n}{t^s} + \text{lower degree terms in } y , \]

where $s \in \mathbb{Q}$ and $n \geq 0$ is an integer. We have to show that $a \in K^+[A_1]$. We proceed by induction on $n$. If $n = 0$, then $a \in A \cap K = K^+$. So let us assume $n > 0$. As $K$ is algebraically closed and contains $t$, there exist $\alpha_1, \ldots, \alpha_n \in K$ with

\[ a = \left( \frac{y - \alpha_1}{t^{s/n}} \right) \left( \frac{y - \alpha_2}{t^{s/n}} \right) \cdots \left( \frac{y - \alpha_n}{t^{s/n}} \right) . \]

Since $A \subseteq K[y]$ satisfies the property $P_2$, we have $(y - \alpha_i)/(t^{s/n}) \in A$ for some $i$. This implies that

\[ \frac{(y - \alpha_i)^n}{t^s} \in K^+[A_1] . \]

Thus $q = a - (y - \alpha_i)^n/t^s \in A$. By induction hypothesis we have $q \in K^+[A_1]$ and thus $a \in K^+[A_1]$. Hence $A \subseteq K^+[A_1]$, which implies the result. \qed

#### Lemma 5.3
Let $K^+[y] \subseteq E \subseteq K[y]$ be a proper subring. Then there exists a proper subring $E' \subseteq K[y]$ that satisfies the property $P_2$ and contains $E$.

**Proof.** Denote by $\tilde{E} \subseteq K[y]$ the integral closure of $E$ in $K[y]$. As $E \neq K[y]$, it follows that $\tilde{E}$ is a proper ideal of $E$. In particular, $\varphi : \text{Spec}(K[y]) \to \text{Spec}(E)$ is nonsurjective. Since $\text{Spec}(E) \to \text{Spec}(E)$ is surjective (see [Mat86] Theorem 9.3), it follows that $\tilde{E} \neq K[y]$. Hence there exists an intermediate ring $\tilde{E} \subseteq E' \subseteq K[y]$ that satisfies the property $P_2$ in $K[y]$, by [Sam57] Théorème 8. \qed

Now, we give an application of these two results to maximal subrings. Roughly speaking, the proposition says, that for rings which are generated by degree one elements, one can see the maximality already on the level of degree one elements.
Proposition 5.4. Let \( K^+[y] \subseteq A \subseteq K[y] \) be a proper subring that satisfies \( A = K^+[A_1] \). Then \( A \) is maximal in \( K[y] \) if and only if
\[
\text{for all } f \in K[y] \setminus A \text{ of degree } 1 \text{ we have } A[f] = K[y]. \tag{3}
\]

Proof. Assume that \( A \) satisfies (3). Let \( A \subseteq E \subseteq K[y] \) be an intermediate ring. We want to prove \( A = E \). By Lemma 5.3 there exists a proper subring \( E' \subseteq K[y] \) that satisfies the property \( P_2 \) and contains \( E \). Now, if there would exist \( f \in (E')_1 \setminus A_1 \), then we would have by (3)
\[
K[y] = A[f] \subseteq E' \subseteq K[y].
\]
This would imply that \( E' = K[y] \), a contradiction. Thus we have \( A_1 = (E')_1 \).
According to Proposition 5.2 we have \( A = E' \) and therefore \( A = E \).

The other implication is clear. \( \Box \)

Definition 5.5. Let \( S = \{ s_1 < s_2 < \ldots \} \) be a strictly monotone sequence in \( \mathbb{Q}_{\geq 0} \) and let \( \Lambda = \{ \alpha_1, \alpha_2, \ldots \} \) be a sequence in \( K \) such that supp\((\alpha_i)\) \( \subseteq [0, s_i) \) and supp\((\alpha_{i+1} - \alpha_i)\) \( \subseteq [s_i, s_{i+1}) \) for all \( i > 0 \). We call then \( (S, \Lambda) \) an admissible pair of \( K \).

Proposition 5.9. Assume that \( K \) satisfies the cutoff property.

Let \( K^+[y] \subseteq E \subseteq K[y] \). Then \( E \) satisfies the property \( P_2 \) in \( K[y] \) if and only if
\[
E = K^+ \left\{ \frac{y - \alpha_i}{t^{s_i}} \mid i = 1, 2, \ldots \right\}
\]
for an admissible pair \((S, \Lambda)\) of \( K \) or \( E = K^+[e] \) for some element \( e \in K[y] \) of degree 1.
Proof. Assume that $K^+[y] \subseteq E \subseteq K[y]$ is a subring that satisfies the property $P_2$ in $K[y]$. We consider the following subset of $E$:

$$N = \left\{ \frac{y-\alpha}{t^s} \in E \mid s \in \mathbb{Q}_{>0}, \alpha \in K \text{ and } \text{supp}(\alpha) \subseteq [0,s) \right\}.$$  

Using the fact, that $E \cap K = K^+$ and that $K$ satisfies the cutoff property, one can see that $N$ has the following two properties:

i) If $(y-\alpha)/t^s, (y-\alpha')/t^{s'} \in N$ and $s \leq s'$, then supp$(\alpha' - \alpha) \subseteq [s,s').$

ii) If $(y-\alpha)/t^s \in E$, $s \in \mathbb{Q}_{>0}$ and $\alpha \in K$, then $\alpha \in K^+$ and there exists $n \in N$, such that $((y-\alpha)/t^s) - n \in K^+$.

Property ii) of $N$ implies $K^+[N,y] = K^+[E_1]$.

Let $U \subseteq \mathbb{Q}_{>0}$ be the set of all $s \in \mathbb{Q}_{>0}$ such that there exists $\alpha \in K^+$ with $(y-\alpha)/t^s \in N$. Property i) of $N$ implies that for every $s \in U$ there exists a unique $\alpha_s \in K^+$ such that $(y-\alpha_s)/t^s \in N$. Now, we make the following distinction.

sup$(U) \in U$: Let $u = \text{sup}(U)$. It follows from property i) of $N$, that $(y-\alpha_s)/t^s \in K^+[(y-\alpha_u)/t^u]$ for all $s \in U$. This implies $K^+[N,y] \subseteq K^+[(y-\alpha_u)/t^u]$. Clearly, we have $K^+[(y-\alpha_u)/t^u] \subseteq K^+[N,y]$. With \[4\] and Proposition 5.2 we get the equality $E = K^+[(y-\alpha_u)/t^u]$.

sup$(U) \notin U$: Let $S = \{ s_1 < s_2 < \ldots \}$ be a sequence in $U$ such that $\lim_{i\to\infty} s_i = \text{sup}(U)$. If we set $\alpha_i = \alpha_{s_i}$ and $\Lambda = \{ \alpha_1, \alpha_2, \ldots \}$, then $(S, \Lambda)$ is an admissible pair. Let $s \in U$. As sup$(U) \notin U$, there exists $i$ with $s_i > s$. With property i) of $N$, we get now $(y-\alpha_s)/t^s \in K^+[(y-\alpha_{s_i})/t^{s_i}]$. Thus $K^+[N,y]$ is generated over $K^+$ by $(y-\alpha_s)/t^s$, $i = 1,2,\ldots$.

By \[4\] and Proposition 5.2 we get $K^+[N,y] = K^+[E_1] = E$.

Thus $E$ has the claimed form.

Now, we prove that $K^+[e]$ satisfies the property $P_2$ in $K[y]$, provided that $e \in K[y]$ has degree 1. By applying a $K$-algebra automorphism of $K[y]$, we can assume that $e = y$. Consider the following extension of the valuation $\nu|_K$ on $K$ to $K[y]$ 

$$\mu: K[y] \rightarrow \mathbb{Q}, \ f_0 + \ldots + f_n y^n \mapsto \min\{\nu(f_0), \ldots, \nu(f_n)\},$$

which extends (uniquely) to $K(y)$. Then $K^+[y]$ is exactly the set of elements in $K[y]$ with $\mu$-valuation $\geq 0$. From this it follows readily that $K^+[y]$ satisfies the property $P_2$ in $K[y]$.

Now, let $(S, \Lambda)$ be an admissible pair of $K$. Then

$$K^+ \left\{ \frac{y-\alpha_i}{t^{s_i}} \mid i = 1,2,\ldots \right\}$$

satisfies the property $P_2$ in $K[y]$ as it is the union of the increasing $P_2$-subrings

$$K^+ \left[ \frac{y-\alpha_1}{t^{s_1}} \right] \subseteq K^+ \left[ \frac{y-\alpha_2}{t^{s_2}} \right] \subseteq \cdots.$$  

With this classification result at hand, we can now achieve a classification of all maximal subrings of $K[y]$ that contain $K^+[y]$.

**Proposition 5.10.** Assume that $K$ satisfies the cutoff property. Let $(S, \Lambda)$ be an admissible pair of $K$. Assume that either $\lim s_i = \infty$ or $(S, \Lambda)$ has no limit in $K$. Then

$$K^+ \left\{ \frac{y-\alpha_i}{t^{s_i}} \mid i = 1,2,\ldots \right\}$$

is a maximal subring of $K[y]$ that contains $K^+[y]$. On the other hand, every maximal subring of $K[y]$ that contains $K^+[y]$ is of this form.
Proof. Let $B \subseteq K[y]$ be the ring of \eqref{max}. We claim that $B \neq K[y]$. Otherwise, there exists $i$ such that $1/t \in K^+[(y - \alpha_i)/t^s]$, as $(S, \Lambda)$ is an admissible pair. This would imply $1/t \in K^+$, a contradiction.

Note, that we have $B = K^+[B_1]$. Thus, according to Proposition \ref{max} it is enough to show, that $B[f] = K[y]$ for all $f \in K[y] \setminus B$ of degree $1$. Up to multiplying $f$ with a unit of $K^+$, we can assume that $f = (y - \alpha)/t^s$, with $\alpha \in K$ and $s \in \mathbb{Q}$. First, assume that $s < \lim s_i$. Hence there exists $i$ with $s < s_i$, thus we have

$$\frac{\alpha_i - \alpha}{t^s} = \frac{y - \alpha}{t^s} - \frac{y - \alpha_i}{t^s} \in K[y] \setminus B.$$ 

So the last element lies in $K \setminus K^+$. Hence we have $B[f] = K[y]$. Now, assume $s \geq \lim s_i$ (and thus $\lim s_i$ is finite). As $(S, \Lambda)$ has no limit in $K$, there exists $i$ such that $\text{supp}(\alpha - \alpha_i)$ is not contained in $[s_i, \infty)$. Thus,

$$\frac{y - \alpha}{t^s} - \frac{y - \alpha_i}{t^s} = \frac{\alpha_i - \alpha}{t^s} \in K \setminus K^+,$$

and hence we get $B[f] = K[y]$ again.

Now, let $A \subseteq K[y]$ be a maximal subring that contains $K^+[y]$, which must be an extending maximal subring. By Lemma \ref{max2} $A \subseteq K[y]$ satisfies the property $P_2$. Since $K^+[e] \subseteq K[y]$ is not a maximal subring for all $e \in K[y]$ of degree $1$, it follows from Proposition \ref{max3} that there exists an admissible pair $(S, \Lambda)$, such that

$$A = K^+ \left[ \left\{ \frac{y - \alpha_i}{t^s} \mid i = 1, 2, \ldots \right\} \right].$$

It remains to prove that $\lim s_i = \infty$ or $(S, \Lambda)$ has no limit in $K$. Assume towards a contradiction that $s = \lim s_i < \infty$ and $\alpha \in K$ is a limit of $(S, \Lambda)$. Then, it follows that $A \subseteq K^+[(y - \alpha)/t^s]$. As $K^+[(y - \alpha)/t^s]$ is certainly not a maximal subring of $K[y]$, we get a contradiction. This finishes the proof. \hfill $\Box$

Remark 5.11. Let $A$ be the maximal subring of $K[y]$ in the Proposition \ref{max1}. We describe the crucial maximal ideal of $A$. Let $n \subseteq K^+$ be the unique maximal ideal.

In fact, $n = \sum_{q \in \mathbb{Q}_{>0}} t^q K^+$. For $i \in \mathbb{N}$, let $a_i = \alpha_i + t^q a_{i+1}$ where $a_{i+1} \in k$ denotes the coefficient of $t^q$ in $\alpha_{i+1}$. Thus $A$ is generated over $K^+$ by the elements $(y - \alpha_i)/t^q$. We have the following inclusion of ideals in $A$

$$n + \sum_{i=0}^{\infty} \frac{y - \alpha_i}{t^q} A \subseteq \sum_{q \in \mathbb{Q}_{>0}} t^q A.$$ 

As every element of $A$ is an element of $k \cdot 1$ modulo the left hand ideal and the right hand ideal is proper in $A$, these ideals are the same. It follows, that the maximal ideal, has residue field $k$ and it is the crucial maximal ideal.

Proposition 5.12. For $\alpha \in K^+$, the ring $K^+ + (y - \alpha)K[y]$ is a maximal subring of $K[y]$ that contains $K^+[y]$, with non-zero conductor ideal $(y - \alpha)K[y]$. Moreover, all maximal subrings $K^+[y] \subseteq A \subseteq K[y]$ with non-zero conductor are of this form.

Proof. The first statement follows from Lemma \ref{max2}. For the second statement, let $K^+[y] \subseteq A \subseteq K[y]$ be a maximal subring and assume there exists $0 \neq f \in A$ such that $fK[y] \subseteq A$. We can assume that $f$ is monic in $y$. Let $f = f_1 \cdots f_k$ be the decomposition of $f$ into monic linear factors inside $K[y]$. As $A \subseteq K[y]$ satisfies the property $P_2$, for all $n \in \mathbb{N}$ there exists $i = i(n)$ such that $f_i / (t^n/k) \in A$. This implies that there exists $i$ such that $f_i / t^n \in A$ for all $n \in \mathbb{N}$. Let $f_i = y - \alpha_i$. Hence, $K^+[y] + (y - \alpha_i)K[y] \subseteq A$. Since $A \subseteq K[y]$ is a proper subring, we get $\alpha_i \in K^+$ and thus $A = K^+ + (y - \alpha_i)K[y]$. \hfill $\Box$
Remark 5.13. Assume that $K$ satisfies the cutoff property. Let $A \subseteq K[y]$ be a maximal subring that contains $K^+[y]$, $I \subseteq K[y]$ the conductor ideal of $A$ in $K[y]$ (which could be zero), and let $\mathfrak{m} \subseteq A$ be the crucial maximal ideal. By Proposition 3.3, the localization $(A/I)_{\mathfrak{m}}$ is a one-dimensional valuation ring. Let $(S,A)$ be an admissible pair in $K$ such that $A$ is generated over $K^+$ by $(y - \alpha_i)/t^{s_i}$ for $i = 1, 2, \ldots$, see Proposition 5.10. Let $\alpha \in k[[t^Q]]^+$ be a limit of $(S,A)$, which is not unique (however, it exists by Lemma 5.6). Using Proposition 5.10 and Proposition 5.17 one can check that the $K$-homomorphism

$$K[y]/I \to k[[t^Q]], \ f \mapsto f(\alpha)$$

is injective. Hence,

$$\omega : Q(K[y]/I) \to k; \ f \mapsto \nu(f(\alpha))$$

is a valuation on the quotient field of $K[y]/I$. With the aid of Remark 5.11, one can see, that the valuation on $(A/I)_{\mathfrak{m}}$ is given by $\omega$. In particular we have for $f \in K[y]$ $f \in A \iff \omega(\bar{f}) \geq 0$.

where $\bar{f}$ denotes the residue class modulo $I$. Moreover, we get for the crucial maximal ideal

$f \in \mathfrak{m} \iff \omega(\bar{f}) > 0$.

This characterization of $A$ and $\mathfrak{m}$ will be very important for us.

As a consequence of Proposition 5.10 and Lemma 5.6 we can now classify all the maximal subrings of $k[[t^Q]][y]$ which contain $k[[t^Q]]^+[y]$.

Corollary 5.14. If $K = k[[t^Q]]$, then for all $\alpha \in K^+$ the ring

$$K^+ \left\{ \frac{y - \alpha}{t^s} \mid s = 1, 2, \ldots \right\}$$

is maximal in $K[y]$ and contains $K^+[y]$. On the other hand, every maximal subring of $K[y]$ that contains $K^+[y]$ is of this form.

Remark 5.15. The maximal subring of $K[y]$ in Corollary 5.14 is the ring $K^+ + (y - \alpha)K[y]$. Its crucial maximal ideal is $\mathfrak{n} + (y - \alpha)K[y]$, where $\mathfrak{n} \subseteq K^+$ denotes the unique maximal ideal.

With Proposition 5.10 and Proposition 5.12 at hand, we can now give another description of the maximal subrings of $K[y]$ that contain $K^+[y]$ in the case where $K$ is the algebraic closure of $k(t)$. We just want to stress the following definition in advance.

Definition 5.16. A subset $S$ of $\mathbb{Q}$ is called a strictly increasing sequence if there exists an isomorphism of the natural numbers to $S$ that preserves the given orders.

Proposition 5.17. Let $K$ be the algebraic closure of $k(t)$ inside $k[[t^Q]]$ and let $\mathcal{S}$ be the set of $\alpha \in k[[t^Q]]^+$ such that $\text{supp}(\alpha)$ is contained in a strictly increasing sequence. Then we have bijections

$$\Xi_1 : K^+ \to \left\{ \text{maximal subrings of } K[y] \text{ with non-zero conductor that contain } K^+[y] \right\}$$

$$\Xi_2 : \mathcal{S} \setminus K^+ \to \left\{ \text{maximal subrings of } K[y] \text{ with zero conductor that contain } K^+[y] \right\}$$

given by

$$\Xi_1(\alpha) = K^+ + (y - \alpha)K[y] \quad \text{and} \quad \Xi_2(\beta) = K^+ \left\{ \frac{y - \beta_i}{t^{s_i}} \mid i \in \mathbb{N} \right\}$$

where $\{s_1 < s_2 < \ldots\} = \text{supp}(\beta)$ and $\beta_i$ is the sum of the first $i - 1$ non-zero terms of $\beta$. 
Proposition 5.19. Assume that there exist integers $a, L$. Then (S,A) is an admissible pair and $\beta$ is a limit of it. Since $\beta \not\in K^+$ and since $K$ satisfies the cutoff property, there exists no limit of (S,A) in $K^+$. Hence, by Proposition 5.10, the subring $\mathcal{E}_2(\beta)$ is maximal in $K[y]$. Thus $\mathcal{E}_2$ is well-defined.

Let $A \subseteq K[y]$ be a maximal subring with zero conductor that contains $K^+[y]$. By Proposition 5.12, there exists an admissible pair $(S', \Lambda')$ in $K$ such that

$$A = K^+ \left\{ \frac{y - \beta_i'}{t_i'} \mid i \in \mathbb{N} \right\}$$

where $S' = \{ s'_1 < s'_2 < \ldots \}$ and $\Lambda' = \{ \beta'_1, \beta'_2, \ldots \}$, and either $\lim s'_i = \infty$ or $(S', \Lambda')$ has no limit in $K$. If $(S', \Lambda')$ has a limit in $K$, then the conductor of $A \subseteq K[y]$ is non-zero. Thus $(S', \Lambda')$ has no limit in $K$. Since $K$ is the algebraic closure of $k(t)$, the support $\text{supp}(\beta_i')$ is finite for all $i$. Hence the pair $(S', \Lambda')$ has a limit $\beta'$ inside $J \setminus K^+$. Moreover, this limit satisfies $\mathcal{E}_2(\beta') = A$, which proves the surjectivity of $\mathcal{E}_2$.

Let $\gamma_1, \gamma_2 \in J \setminus K^+$ such that $\mathcal{E}_2(\gamma_1) = \mathcal{E}_2(\gamma_2)$ and denote this ring by $D$. For $k = 1, 2$, let $\{ s_{k,i} < s_{k,i+1} < \ldots \} = \text{supp}(\gamma_k)$ and let $\gamma_{k,i} \in K$ be the sum of the first $i - 1$ non-zero terms of $\gamma_k$. Let $i > 0$ be an integer. Without loss of generality we can assume that $s_{1,i} \leq s_{2,i}$. Since $(y - \gamma_{k,i})/t_{k,i} \in D$ for $k = 1, 2$, it follows that

$$\frac{\gamma_{2,i} - \gamma_{1,i}}{t_{2,i}} = \frac{y - \gamma_{1,i}}{t_{1,i}} \quad \frac{y - \gamma_{2,i}}{t_{2,i}} \in D \cap K = K^+.$$

Hence $\gamma_{2,i} = \gamma_{1,i} + t_{2,i} \eta$ where $\eta \in K^+$. However, since $\text{supp}(\gamma_{1,i})$ and $\text{supp}(\gamma_{2,i})$ have the same number of elements, it follows that $\eta = 0$. Thus $\gamma_{1,i} = \gamma_{2,i}$ for all $i$. This implies that $\gamma_1 = \gamma_2$.

5.4. Description of all maximal subrings of $k[t, t^{-1}, y]$ that contain $k[t, y]$ by “intersection”. In this subsection we still fix an algebraically closed subfield $K \subseteq k[[t^\mathbb{Q}]]$ that contains the field of rational functions $k(t)$. Moreover, we fix a subring $L \subseteq K$ that contains $k[t, t^{-1}]$. Recall that $L^+$ (respectively $K^+$) denotes the elements in $L$ (respectively $K$) of $\nu$-valuation $\geq 0$.

Lemma 5.18. The ring extension $L^+ \subseteq K^+$ is flat.

Proof. Let $n$ be the unique maximal ideal of the valuation ring $K^+$. This ideal consists of all elements in $K$ with $\nu$-valuation $> 0$. Denote by $L'$ the localization $(L^+)^{n \setminus L^+}$. We show that $K^+$ is a flat $L'$-module, which implies then the result. Clearly, $K^+$ is a torsion-free $L'$-module. By [Bou72, Chp. 1, §2, no. 4, Proposition 3], it is thus enough to prove that $L'$ is a valuation ring.

Let $g, h \in L^+$ and assume that $g \neq 0$, $\nu(h/g) \geq 0$. As the value group of $\nu$ is $\mathbb{Q}$, there exist integers $a > 0, b > 0$ such that $\nu(g/a) = a/b$. Thus we get

$$\nu \left( \frac{hg^{b-1}}{t^a} \right) \geq 0 \quad \text{and} \quad \nu \left( \frac{g^b}{t^a} \right) = 0$$

and therefore $hg^{b-1}/t^a \in L^+, g^b/t^a \in L^+ \setminus n$. This implies $h/g \in L'$. Hence, $L'$ is a valuation ring (with valuation $\nu$ on $L^+$).

Our first result says that one can construct every maximal subring of $L[y]$ that contains $L^+[y]$ by intersecting $L[y]$ with some maximal subring of $K[y]$ that contains $K^+[y]$ under a certain assumption.

Proposition 5.19. Assume that $L^+$ is a maximal subring of $L$. If $L^+[y] \subseteq B \subseteq L[y]$ is a maximal subring, then there exists a maximal subring $K^+[y] \subseteq A \subseteq K[y]$ such that $B = A \cap L[y]$. 
Remark 5.20. The assumption, that $L^+$ is a maximal subring of $L$ is satisfied for example if $L = k[t,t^{-1}]$ or $L = k(t)$.

Proof. Let $M$ be the $L^+$-module $L[y]/B$. By assumption, $M$ is non-zero. In fact, since $L^+$ is a maximal subring of $L$, we have an injection

$$L^+/tL^+ \rightarrow M, \quad \lambda \mapsto \lambda t^{-1}.$$ 

Since $K$ is a flat $L^+$-module (see Lemma 5.18), we get an injection

$$K^+/tK^+ \simeq K^+ \otimes_{L^+} (L^+/tL^+) \rightarrow K^+ \otimes_{L^+} M.$$ 

Thus $K^+ \otimes_{L^+} M$ is non-zero. Again, since $K^+$ is a flat $L^+$-module, this implies that

$$K^+ \otimes_{L^+} B \subseteq K^+ \otimes_{L^+} L[y].$$

Therefore, $K^+[B]$ is a proper subring of $K[y]$, which contains $K^+[y]$. Applying Zorn's Lemma to

$$\{ A \subseteq K[y] \mid A \supseteq K^+[B] \text{ and } t^{-1} \notin A \}$$

yields a maximal subring $A$ in $K[y]$ that lies over $K^+[B]$. Thus $B = A \cap L[y]$. \qed

In the next proposition we prove that any maximal subring of $K[y]$ that lies over $K^+[y]$ gives a maximal subring of $L[y]$ after intersection with $L[y]$.

Proposition 5.21. Assume that $K$ satisfies the cutoff property. Let $K^+[y] \subseteq A \subseteq K[y]$ be a maximal subring and let $B = A \cap L[y]$. Then

i) If $I$ denotes the conductor ideal of $A$ in $K[y]$, then $I \cap B$ is the conductor ideal of $B$ in $L[y]$.

ii) The subring $B \subseteq L[y]$ is maximal. Moreover, if $m$ denotes the crucial maximal ideal of $A$, then $m \cap L[y]$ is the crucial maximal ideal of $B$.

Proof.

i) Let $b \in I \cap B$. Then $bL[y] \subseteq A \cap L[y] = B$. Thus $b$ lies in the conductor of $B$ in $L[y]$. Now, let $f \in B$ be an element of the conductor of $B$ in $L[y]$. Then we have $fL[y] \subseteq B$ and in particular, $f/t^n \in B \subseteq A$ for all $n \in \mathbb{N}$. As $K^+[y] = A_t$, this implies that $fK[y] \subseteq A$. Thus $f \in I \cap B$.

ii) Let $I \subseteq K[y]$ be the conductor ideal of $A$ in $K[y]$. By i) the intersection $J = I \cap B$ is the conductor ideal of $B$ in $K[y]$. Let $m \subseteq A$ be the crucial maximal ideal and let $n = m \cap B$. We divide the proof in several steps

a) We claim that $(B/J)_n$ is a one-dimensional valuation ring. Since $(A/I)_m$ is a one-dimensional valuation ring (see [FO70] Proposition 3.3]), it is enough to prove that

$$(B/J)_n = (A/I)_m \cap Q(L[y]/J)$$

inside $Q(K[y]/I)$ (see also [Main3] Theorem 10.7]). Let $g,h \in L[y]/J$ be non-zero elements and assume that $h/g \in (A/I)_m$. Thus it follows for the valuation $\omega$ defined in Remark 5.19 that $\omega(h/g) \geq 0$. There exist integers $a,b$ such that $\omega(g) = a/b$ and we can assume that $b \geq 0$. Thus we have $\omega(g^b/t^a) = 0$. Since $\omega(h) \geq \omega(g)$ we get $\omega(hg^{b-1}/t^a) \geq 0$. Thus $g^b/t^a$ and $hg^{b-1}/t^a$ both lie inside $A/I$. Using the fact that

$$B/J = A/I \cap L[y]/J \subseteq K[y]/I$$

we get

$$\frac{h}{g} = \frac{h \cdot (g^{b-1}/t^a)}{g^b/t^a} \in (B/J)_n.$$

Thus we have $(A/I)_m \cap Q(L[y]/J) = (B/J)_n$. Note that the reasoning is similar to the proof of Lemma 5.18.
b) We claim that the complement of the image of \( \text{Spec } L[y] \to \text{Spec } B \) is just the point \( n \). By Remark 5.11 the residue field of the crucial maximal ideal \( m \subseteq A \) is \( k \) and thus \( n \) is a maximal ideal of \( B \). Let \( b \in n \). By Remark 5.11 we have \( m = \text{rad}(tA) \) and thus there exists an integer \( q \geq 1 \) such that \( bt^q \in tA \). Therefore \( bt^q/t \in A \). Since \( bt^q/t \in L[y] \), we get \( bt^q \in tB \). Thus we proved \( n \subseteq \text{rad}(tB) \). If \( p \subseteq B \) is a prime ideal such that \( pL[y] = L[y] \), then we get \( t \in p \) (since \( B_t = L[y] \)). Thus \( n \subseteq \text{rad}(tB) \subseteq p \) and by the maximality of \( n \) we get \( n = p \).

c) Now, we prove that \( B/J \) is a maximal subring of \( L[y]/J \). Let \( C \subseteq L[y]/J \) be a subring that lies over \( B/J \). Using b), the fact that \( J \subseteq n \) and that \( (B/J)_n = L[y]/J \), we get the following commutative diagram

\[
\begin{array}{ccc}
\text{Spec } C & \xrightarrow{\varphi} & \text{Spec } B/J \\
\text{Spec } L[y]/J & \xrightarrow{\sim} & (\text{Spec } B/J) \setminus \{n\}_\text{open} \text{ Spec } B/J.
\end{array}
\]

From this, one can easily deduce that \( \varphi \) is surjective. Let \( p \in \text{Spec } C \) with \( \varphi(p) = n \). By a) and Lemma 3.6 \( (B/J)_n \) is a maximal subring of \( Q(L[y]/J) \). Since \( t \in p \), this implies \( (B/J)_n = C_p \). Hence we have \( B/J = C \) by [Mat86, Theorem 4.7].

From c) and from Lemma 5.2 it follows that \( B \) is a maximal subring of \( L[y] \). From b) it follows that \( n = m \cap L[y] \) is the crucial maximal ideal of \( B \).

\[\Box\]

Remark 5.22. If the conductor ideal of \( A \) in \( K[y] \) is non-zero, then there exists \( \alpha \in K^+ \) such that this ideal is \( (y - \alpha)K[y] \). Now, if \( L \) is a field and \( L \subseteq K \) an algebraic field extension, then the conductor of \( B = A \cap L[y] \) is the ideal \( m_\alpha L[y] \) where \( m_\alpha \subseteq L[y] \) is the minimal polynomial of \( \alpha \) over \( L \).

In the future we will need the following consequence of the last two propositions.

Corollary 5.23. We have a bijective correspondence

\[\varphi: \left\{ \begin{array}{ll}
\text{maximal subrings of } & \text{maximal subrings of } \\
k[t, t^{-1}, y] \text{ that contain } & k(t)[y] \text{ that contain } \\
k[t, y] & k(t)^+[y]
\end{array} \right\} \xrightarrow{1:1} \left\{ \begin{array}{ll}
\text{maximal subrings of } & \text{maximal subrings of } \\
k[t, t^{-1}, y] \text{ that contain } & k(t)[y] \\
k[t, y] & k(t)^+[y]
\end{array} \right\} \]

given by \( \varphi(B) = B_S \) and \( \varphi^{-1}(A) = A \cap k[t, t^{-1}, y] \), where \( S \) denotes the multiplicative subset \( k[t] \setminus \{t\} \) of \( k[t] \).

Proof. Let \( B \subseteq k[t, t^{-1}, y] \) be a maximal subring that contains \( k[t, y] \). By Lemma 5.1, the localization \( B_S \) is a maximal subring of \( k[t, t^{-1}, y]_S \), since \( S = k[t] \setminus \{t\} \). Moreover, we have

\[B \subseteq B_S \cap k[t, t^{-1}, y] \subseteq k[t, t^{-1}, y]_S \]

and thus by the maximality of \( B \) we get the equality \( B = B_S \cap k[t, t^{-1}, y]_S \). This proves the injectivity of \( \varphi \).

Let \( A \subseteq k(t)[y] \) be a maximal subring that contains \( k(t)^+[y] \). By Proposition 5.19 there exists a maximal subring \( K^+[y] \subseteq A' \subseteq K[y] \) such that \( A' \cap k(t)[y] = A \). By Proposition 5.21 it follows that \( A \cap k[t, t^{-1}, y] \) is a maximal subring of \( k[t, t^{-1}, y] \). Clearly, \( A \cap k[t, t^{-1}, y] \) contains \( k[t, y] \). Moreover,

\[(A \cap k[t, t^{-1}, y])_S \subseteq A\]

and by the maximality of \( (A \cap k[t, t^{-1}, y])_S \) we get equality. This proves the surjectivity of \( \varphi \).
5.5. Classification of the maximal subrings of \( k[t, t^{-1}, y] \) that contain \( k[t, y] \). Throughout this subsection \( K \) denotes the algebraic closure of \( k(t) \) inside the Hahn field \( k[[t^Q]] \). In this subsection we give a classification of all maximal subrings of \( k[t, t^{-1}, y] \) that contain \( k[t, y] \).

Let \( \alpha \) be in \( k[[t^Q]]^+ \). In this subsection we denote

\[
A_\alpha = k[[t^Q]]^+ + (y - \alpha) k[[t^Q]][y].
\]

Thus \( \alpha \mapsto A_\alpha \) is a bijective correspondence between \( k[[t^Q]]^+ \) and the maximal subrings of \( k[[t^Q]][y] \) by Corollary 5.14 and Remark 5.15.

Let \( (\mathbb{Q}/\mathbb{Z})^* \) be the group of group homomorphisms \( \mathbb{Q}/\mathbb{Z} \to \mathbb{K}^* \). There exists a natural action of this group on the Hahn field, given by the homomorphism

\[
(\mathbb{Q}/\mathbb{Z})^* \to \text{Aut}(k[[t^Q]]/k((t))), \quad \sigma \mapsto \left( \sum_{s \in \mathbb{Q}} a_s t^s \mapsto \sum_{s \in \mathbb{Q}} a_s \sigma(s) t^s \right),
\]

where \( \text{Aut}(k[[t^Q]]/k((t))) \) denotes the group of field automorphisms of \( k[[t^Q]] \) that fix the subfield \( k((t)) \) pointwise (note that \( k((t)) \subseteq k[[t^Q]] \) is a Galois extension if and only if the characteristic of \( k \) is zero). The action \( (\mathbb{Q}/\mathbb{Z})^* \) commutes with the valuation \( \nu \) on \( k[[t^Q]] \). In particular we have for all \( \sigma \in (\mathbb{Q}/\mathbb{Z})^* \) and for all \( \alpha \in k[[t^Q]]^+ \)

\[
A_\alpha \cap k[t, t^{-1}, y] = A_{\sigma(\alpha)} \cap k[t, t^{-1}, y].
\]

The following result is the main theorem of this section.

**Theorem 5.24.** Let \( \mathcal{I} \) be the set of \( \alpha \in k[[t^Q]]^+ \) such that \( \text{supp}(\alpha) \) is contained in a strictly increasing sequence (see Definition 5.16). Then we have a bijection

\[
\Psi: \mathcal{I}/(\mathbb{Q}/\mathbb{Z})^* \rightarrow \left\{ \text{maximal subrings of } k[t, t^{-1}, y] \text{ that contain } k[t, y] \right\}, \quad \alpha \mapsto A_\alpha \cap k[t, t^{-1}, y].
\]

Moreover, \( \psi(\alpha) \) has non-zero conductor in \( k[t, t^{-1}, y] \) if and only if \( \alpha \in K^+ \) where \( K \) denotes the algebraic closure of \( k(t) \) inside the Hahn field \( k[[t^Q]] \).

For the proof we need some preparation. First, we reformulate the action of \( (\mathbb{Q}/\mathbb{Z})^* \) on the Hahn field. Let \( k((t^Q)) \) be the subfield of the Hahn field generated by the ground field \( k \) and the elements \( t^s, s \in \mathbb{Q} \). Then \( (\mathbb{Q}/\mathbb{Z})^* \) is isomorphic to the group \( \text{Aut}(k((t^Q))/k((t))) \) of field automorphisms of \( k((t^Q)) \) that fix \( k((t)) \) pointwise. An isomorphism is given by

\[
(\mathbb{Q}/\mathbb{Z})^* \rightarrow \text{Aut}(k((t^Q))/k((t))), \quad \sigma \mapsto (t^s \mapsto \sigma(s) t^s),
\]

and the homomorphism \( (\mathbb{Q}/\mathbb{Z})^* \) identifies then under this isomorphism with

\[
\text{Aut}(k((t^Q))/k((t))) \rightarrow \text{Aut}(k[[t^Q]]/k((t))), \quad \varphi \mapsto \left( \sum_{s \in \mathbb{Q}} a_s t^s \mapsto \sum_{s \in \mathbb{Q}} a_s \varphi(t^s) \right)
\]

(note that \( \varphi(t^s) \) is a multiple of \( t^s \) with some element of \( k^* \)). For proving the injectivity of the map \( \Psi \) in Theorem 5.24 we need two lemmas.

**Lemma 5.25.** Let \( q \in \mathbb{Q}_{\geq 0} \) and let \( \alpha, \alpha' \in k[[t^Q]]^+ \). Assume that we have decompositions

\[
\alpha = \alpha_0 + \alpha_1, \quad \alpha' = \alpha_0 + ct^q + \alpha_1' \quad \text{with} \quad \alpha_0, \alpha_1, \alpha_1' \in k[[t^Q]], \quad c \in k
\]

such that

\[
\text{supp}(\alpha_0) \subseteq [0, q], \quad \text{supp}(\alpha_1), \text{supp}(\alpha_1') \subseteq (q, \infty), \quad \text{supp}(\alpha_0) \text{ is finite}.
\]

If \( \nu(f(\alpha)) = \nu(f(\alpha')) \) for all \( f \in k((t))[y] \), then \( \alpha_0 + ct^q = \sigma(\alpha_0) \) for some \( \sigma \in \text{Aut}(k((t^Q))/k((t))) \).
Proof. Let \( m_0 \in k(t)[y] \) be the minimal polynomial of \( \alpha_0 \) over \( k(t) \). Note that \( \alpha_0 \) is algebraic over \( k(t) \) since the support of \( \alpha_0 \) is a finite set. Denote by \( \alpha_0 = \beta_0, \ldots, \beta_r \) the different elements of the set

\[
\{ \sigma(\alpha_0) \mid \sigma \in \text{Aut}(k(t^q)/k(t)) \}.
\]

As the field extension \( k(t) \subseteq k(t^q) \) is normal, there exist integers \( k_0 > 0 \) and \( k_1, \ldots, k_r \geq 0 \) such that

\[
m_0 = (y - \beta_0)^{k_0}(y - \beta_1)^{k_1} \cdots (y - \beta_r)^{k_r},
\]

see [Mor96, Theorem 3.20]. Assume towards a contradiction that \( \alpha \) satisfies

\[
\alpha \in k(t^q).
\]

Assume that \( \alpha \) is also algebraic over \( k(t) \), hence there exist \( k_0 \geq 0 \) and \( k_1, \ldots, k_r \geq 0 \) such that

\[
m_0 = (y - \beta_0)^{k_0}(y - \beta_1)^{k_1} \cdots (y - \beta_r)^{k_r},
\]

see [Mor96, Theorem 3.20]. Assume towards a contradiction that \( \alpha_0 + ct^q \neq \beta_j \) for all \( 0 \leq j \leq r \). Let \( \nu \) be an integer with \( 1 \leq \nu \leq q \). Since \( \nu(\alpha_0 - \beta_i) \leq q \), we get

\[
\nu(\alpha_0 + ct^q) = \nu(\alpha_0 - \beta_i) = \nu(ct^q - \alpha_0 - \beta_i) = \nu(-\alpha_0 - ct^q - \beta_i),
\]

where we used in the second and third equality the fact that \( ct^q + \alpha_0 \neq \beta_i \). Since \( \alpha_0 = \beta_i \neq \alpha_0 + ct^q \), the constant \( c \) is non-zero. Thus we have \( \nu(\alpha_1) > \nu(\alpha_1 + ct^q)\). In summary we get

\[
\nu(m_0(\alpha)) = k_0\nu(\alpha_1) + \sum_{i \neq 0} k_i\nu(\alpha_1 + \alpha_0 - \beta_i) > k_0\nu(\alpha_1 + ct^q) + \sum_{i \neq 0} k_i\nu(\alpha_1 + ct^q + \alpha_0 - \beta_i) = \nu(m_0(\alpha'))
\]

and thus we arrive at a contradiction.

\( \square \)

Lemma 5.26. Let \( \alpha, \alpha' \in k[[t^q]]^+ \) and assume that \( \text{supp}(\alpha), \text{supp}(\alpha') \) are contained in strictly increasing sequences (see Definition 5.16). Then \( \nu(f(\alpha)) = \nu(f(\alpha')) \) for all \( f \in k(t)[y] \) if and only if there exists \( \sigma \in \text{Aut}(k(t^q)/k(t)) \) such that \( \alpha' = \sigma(\alpha) \).

Proof. Assume that \( \nu(f(\alpha)) = \nu(f(\alpha')) \) for all \( f \in k(t)[y] \). By assumption, there exists a strictly increasing sequence \( 0 < s_1 < s_2 < \ldots \) in \( \mathbb{Q} \) such that

\[
\alpha = a_0 + \sum_{j=1}^{\infty} a_s t^{s_j} \quad \text{and} \quad \alpha' = a_0' + \sum_{j=1}^{\infty} a'_s t^{s_j}.
\]

For \( i \geq 1 \) let

\[
\alpha_i = a_0 + \sum_{j=1}^{i-1} a_s t^{s_j} \quad \text{and} \quad \alpha'_i = a_0' + \sum_{j=1}^{i-1} a'_s t^{s_j}.
\]

We define inductively \( \sigma_1, \sigma_2, \ldots \in \text{Aut}(k(t^q)/k(t)) \) such that \( \sigma_i(\alpha_i) = \alpha_i' \).

Since \( \nu(\alpha - a_0) = \nu(\alpha' - a_0') \) by assumption, it follows that \( a_0' = a_0 \). Hence \( \sigma_1 = \text{id} \) satisfies \( \sigma_1(\alpha_1) = \alpha_1' \). Assume that \( \sigma_i \in \text{Aut}(k(t^q)/k(t)) \) with \( \sigma_i(\alpha_i) = \alpha_i' \) is already constructed. For all \( f \in k(t)[y] \) we have

\[
\nu(f(\sigma_i(\alpha))) = \nu(\sigma_i(f(\alpha))) = \nu(f(\alpha)) = \nu(f(\alpha')).
\]

Since \( \alpha'_i = \sigma_i(\alpha_i) \), Lemma 5.25 implies that there exists \( \varphi_0 \in \text{Aut}(k(t^q)/k(t)) \) such that \( \alpha'_i = \varphi_0(\alpha_{i+1}) \). Thus we can define \( \alpha_{i+1} = \varphi_0 \circ \sigma_i \).

By construction, \( \sigma_{i-1} \) and \( \sigma_i \) coincide on the field

\[
K_i = k(\{t^s \mid s \in \text{supp}(\alpha_i)\}).
\]

Thus we get a well-defined automorphism of the field \( \bigcup_{i=0}^{\infty} K_i \) that restricts to \( \sigma_i \) on \( K_i \). By the normality of the extension \( k(t) \subseteq k(t^q) \) we can extend this automorphism to an automorphism \( \sigma \) of \( k(t^q) \) and we have \( \sigma(\alpha) = \alpha' \) (see [Mor96, Theorem 3.20]).
The converse of the statement is clear.

Proof of Theorem 5.21 Consider the bijections
\[
\Xi_1: \mathbb{K}^+ \longrightarrow \left\{ \text{maximal subrings of } \mathbb{K}[y] \text{ with non-zero conductor that contain } \mathbb{K}^+[y] \right\}
\]
\[
\Xi_2: \mathfrak{K} \setminus \mathbb{K}^+ \longrightarrow \left\{ \text{maximal subrings of } \mathbb{K}[y] \text{ with zero conductor that contain } \mathbb{K}^+[y] \right\}.
\]
of Proposition 5.17 For \( \alpha \in \mathbb{K}^+ \) and \( \beta \in \mathfrak{K} \setminus \mathbb{K}^+ \) we have
\[
\Xi_1(\alpha) \cap \mathbb{k}[t, t^{-1}, y] = \Psi(\alpha) \quad \text{and} \quad \Xi_2(\beta) \cap \mathbb{k}[t, t^{-1}, y] = \Psi(\beta).
\]
Using Proposition 5.21 and Remark 5.22 we see that \( \Xi_1(\alpha) \cap \mathbb{k}[t, t^{-1}, y] \) is a maximal subring of \( \mathbb{k}[t, t^{-1}, y] \) with non-zero conductor and \( \Xi_2(\beta) \cap \mathbb{k}[t, t^{-1}, y] \) is a maximal subring of \( \mathbb{k}[t, t^{-1}, y] \) with zero conductor. Thus \( \Psi \) is a well-defined map. Using Proposition 5.19, we see that \( \Psi \) is surjective.

For proving the injectivity, let \( \alpha_1, \alpha_2 \in \mathfrak{K} \) such that the rings \( A_{\alpha_1} \cap \mathbb{k}[t, t^{-1}, y], A_{\alpha_2} \cap \mathbb{k}[t, t^{-1}, y] \) are the same subsets of \( \mathbb{k}[t, t^{-1}, y] \). For \( i = 1, 2 \), \( A_{\alpha_i} \cap \mathbb{k}(t)[y] \) is a maximal subring of \( \mathbb{k}(t)[y] \), see Proposition 5.21. By Corollary 5.23, we get the equality
\[
A_{\alpha_1} \cap \mathbb{k}(t)[y] = A_{\alpha_2} \cap \mathbb{k}(t)[y].
\]
Let \( B = A_{\alpha_1} \cap \mathbb{k}(t)[y] = A_{\alpha_2} \cap \mathbb{k}(t)[y] \). Let \( \mathfrak{n} \) be the crucial maximal ideal of \( B \) and let \( \mathcal{J} \) be the conductor ideal of \( B \) in \( \mathbb{k}(t)[y] \). With Remark 5.13 we get for \( i = 1, 2 \)
\[
B = \{ f \in \mathbb{k}(t)[y] \mid \omega_i(f) \geq 0 \} \quad \text{and} \quad \mathfrak{n} = \{ f \in \mathbb{k}(t)[y] \mid \omega_i(f) > 0 \},
\]
where \( f \) denotes the residue class modulo \( \mathcal{J} \) and \( \omega_i \) denotes the valuation
\[
\omega_i: Q(\mathbb{k}(t)[y]/J) \longrightarrow \mathbb{Q}, \quad g \longrightarrow \nu(g(\alpha_i)).
\]
By \([\text{Q70}]\) Proposition 3.3, \( (B/J)_{\alpha_1} \) is a one-dimensional valuation ring of the field \( Q(\mathbb{k}(t)[y]/J) \) and therefore it is a maximal subring of \( Q(\mathbb{k}(t)[y]/J) \), see Lemma 5.10 The description above of \( B \) and \( \mathfrak{n} \) implies that \( (B/J)_{\alpha_1} \) is the valuation ring with respect to \( \omega_1 \) and with respect to \( \omega_2 \). Therefore, the valuations \( \omega_1, \omega_2 \) are the same up to an order preserving isomorphism of \( (\mathbb{Q}, +, <) \). However, since \( \omega_1(t) = 1 = \omega_2(t) \), these valuations must then be the same. Thus by Lemma 5.20 there exists \( \sigma \in \text{Aut}(\mathbb{k}(t^2)/\mathbb{k}(t)) \) such that \( \alpha_1 = \sigma(\alpha_2) \). This proves the injectivity of \( \Psi \).

6. Classification of the maximal \( \mathbb{k} \)-subalgebras of \( \mathbb{k}[t, t^{-1}, y] \)

The goal of this section is to classify all maximal \( \mathbb{k} \)-subalgebras of \( \mathbb{k}[t, t^{-1}, y] \). In fact, we reduce this problem in this section to another classification result, which will we solve then in the next section.

Proposition 6.1. Let \( A \subseteq \mathbb{k}[t, t^{-1}, y] \) be an extending maximal \( \mathbb{k} \)-subalgebra. Then, exactly one of the following cases occurs:

i) There exists an automorphism \( \sigma \) of \( \mathbb{k}[t, t^{-1}, y] \) such that \( \sigma(A) \) contains \( \mathbb{k}[t, y] \); ii) \( A \) contains \( \mathbb{k}[t, t^{-1}] \).

Proof of Proposition 6.7 Note that \( A \) satisfies the property \( P_2 \) in \( \mathbb{k}[t, t^{-1}, y] \), see Lemma 5.35 Since \( t \cdot t^{-1} = 1 \in A \), it follows that either \( t \in A \) or \( t^{-1} \in A \). Assume that we are not in case ii), i.e. assume that \( \mathbb{k}[t, t^{-1}] \) is not contained in \( A \). By applying an appropriate automorphism of \( \mathbb{k}[t, t^{-1}, y] \), we can assume that \( t \in A \) and hence \( t^{-1} \notin A \). Therefore we get \( A_t = A[t^{-1}] = \mathbb{k}[t, t^{-1}, y] \), since \( A \) is maximal. This implies that there exists an integer \( k \geq 0 \) such that \( t^ky \in A \). Thus the \( \mathbb{k}[t, t^{-1}] \)-automorphism
\[
\sigma: \mathbb{k}[t, t^{-1}, y] \longrightarrow \mathbb{k}[t, t^{-1}, y], \quad y \longmapsto t^{-k}y
\]
satisfies \( y \in \sigma(A) \). Hence we get \( \sigma(A) \supseteq \mathbb{k}[t, y] \) and therefore we are in case i).
The extending maximal $k$-subalgebras in case i) of Proposition 6.1 are then described by Theorem 5.24. Thus we are left with the description of the extending maximal $k$-subalgebras in case ii). In fact, they can be characterized in the following way:

**Proposition 6.2.** There is a bijection

$$
\Phi: \left\{ \begin{array}{l}
\text{extending maximal} \\
\text{k-subalgebras of } k[t, t^{-1}, y] \\
\text{that contain } k[t, t^{-1}] 
\end{array} \right\} \longrightarrow \left\{ \begin{array}{l}
\text{extending maximal} \\
k\text{-subalgebras of } k[t, y] \\
\text{that contain } k[t] \text{ and } t \text{ lies not in} \\
\text{the crucial maximal ideal} 
\end{array} \right\}
$$

given by $\Phi(A) = A \cap k[t, y]$.

**Proof.** Let $A$ be an extending maximal $k$-subalgebra of $k[t, t^{-1}, y]$ that contains $k[t, t^{-1}]$. By Lemma 5.7 there exists $\lambda \in k'$ such that $t - \lambda$ lies in the crucial maximal ideal $m$ of $A$. Thus $A_{t-\lambda} = k[t, t^{-1}, y]_{t-\lambda}$. Hence there exists $k \geq 1$ such that $(t - \lambda)^k y \in A$ and thus we get

$$k[t, t^{-1}, (t - \lambda)^k y] \subseteq A \subseteq k[t, t^{-1}, y].$$

This implies

$$k[t, (t - \lambda)^k y] \subseteq A \cap k[t, y] \subseteq k[t, y].$$

We claim, that $A \cap k[t, y]$ is a maximal subring of $k[t, y]$. Let therefore $A \cap k[t, y] \subseteq B \subseteq k[t, y]$ be an intermediate ring. Thus we get

$$B = B_{t-\lambda} \cap B_t \supseteq k[t, y] \cap B_t \supseteq B.$$

One can check that $A = (A \cap k[t, y])_t$. Since $A$ is maximal in $k[t, t^{-1}, y]$, we get either $B_t = A$ or $B_t = k[t, t^{-1}, y]$ and the claim follows. This proves that $\Phi$ is well-defined and injective.

Let $A'$ be an extending maximal $k$-subalgebra of $k[t, y]$ that contains $k[t]$ and the crucial maximal ideal does not contain $t$. By Lemma 5.1 it follows that $A'$ is an extending maximal $k$-subalgebra of $k[t, t^{-1}, y]$ that contains $k[t, t^{-1}]$. Moreover, we have $A' \cap k[t, y] = A$ by the maximality of $A$ in $k[t, y]$. This proves the surjectivity of $\Phi$. \hfill $\Box$

After this proposition, one is now reduced to the problem of the description of all maximal $k$-subalgebras of $k[t, y]$ that contain $k[t]$.

7. **Classification of the maximal $k$-subalgebras of $k[t, y]$ that contain $k[t]$**

Let $\mathcal{M}$ be the set of extending maximal $k$-subalgebras of $k[t, y]$ that contain $k[t]$. The goal of this section is to describe the set $\mathcal{M}$ with the aid of the classification result Theorem 5.24. For this we introduce a subset $\mathcal{N}$ of the maximal $k$-subalgebras of $k[t, y, y^{-1}]$ that contain $k[t, y^{-1}]$.

**Remark 7.1.** If $A$ is an extending maximal $k$-subalgebra of $k[t, y, y^{-1}]$ that contains $k[t, y^{-1}]$, then the residue field of the crucial maximal ideal is isomorphic to $k$, by Remark 5.1 and Theorem 5.24. Hence there exists a unique $\lambda \in k$ such that $t - \lambda$ lies in the crucial maximal ideal of $A$.

Define $\mathcal{N}$ to be the set of extending maximal $k$-subalgebras $A$ of $k[t, y, y^{-1}]$ that contain $k[t, y^{-1}]$ and such that

$$A \longrightarrow k[t, y, y^{-1}]/(t - \lambda)$$

is surjective where $\lambda$ denotes the unique element in $k$ such that the crucial maximal ideal contains $t - \lambda$ (see Remark 7.1). Now, we can formulate the main result of this section.
Theorem 7.2. The map $\Theta : \mathcal{N} \to \mathcal{M}$, $A \mapsto A \cap k[t,y]$ is bijective.

Remark 7.3. As we classified already all maximal subrings of $k[t,y,y^{-1}]$ that contain $k[t,y^{-1}]$ (see Theorem 5.24), Theorem 7.2 gives us a description of all extending maximal $k$-subalgebras of $k[t,y]$ that contain a coordinate of $k[t,y]$ (up to automorphisms of $k[t,y]$).

Remark 7.4. Lemma 3.7 implies the following: If $A$ is an extending maximal $k$-subalgebra of $k[t,y]$ which contains $k[t]$, then there exists a unique $\lambda \in k$ such that $t - \lambda$ lies in the crucial maximal ideal of $A$. Thus $\mathcal{M}$ is the disjoint union of the sets
\[
\mathcal{M}_\lambda = \{ A \in \mathcal{M} \mid t - \lambda \text{ lies in the crucial maximal ideal of } A \}, \quad \lambda \in k.
\]

By Remark 7.4, $\mathcal{N}$ is the disjoint union of the sets
\[
\mathcal{N}_\lambda = \{ A \in \mathcal{N} \mid t - \lambda \text{ lies in the crucial maximal ideal of } A \}, \quad \lambda \in k.
\]

Note that we have canonical bijections
\[
\mathcal{M}_0 \to \mathcal{M}_\lambda, \quad A \mapsto \sigma_\lambda(A) \quad \text{and} \quad \mathcal{N}_0 \to \mathcal{N}_\lambda, \quad A \mapsto \sigma_\lambda(A)
\]
where $\sigma_\lambda$ is the automorphism of $k[t,y,y^{-1}]$ given by $\sigma_\lambda(t) = t - \lambda$ and $\sigma_\lambda(y) = y$. Using the fact that for all $A \in \mathcal{M}_0$ we have
\[
\sigma_\lambda(A) \cap k[t,y] = \sigma_\lambda(A \cap k[t,y]),
\]
one is reduced for the proof of Theorem 7.2 to proving the following proposition.

Proposition 7.5. The map $\mathcal{N}_0 \to \mathcal{M}_0, A \mapsto A \cap k[t,y]$ is bijective.

For the proof of Proposition 7.5 we need several (technical) lemmas.

Lemma 7.6. Let $k[t] \subseteq Q \subseteq k[t,y]$ be an intermediate ring that satisfies the $P_2$ property in $k[t,y]$ and assume that
\[
Q \longrightarrow k[t,y]/t k[t,y]
\]
is surjective. If $p \subseteq Q$ is an ideal that contains $t$ and that does not contain $t k[t,y] \cap Q$, then there exists $h \in Q \setminus p$ such that $y^{-1} \in Q_h$.

Proof. By assumption, there exists $g \in k[t,y] \setminus y k[t,y]$ and $n \geq 0$ such that
\[ ty^n g \in Q \setminus p. \tag{8} \]
Let $g_0 \in k[t], g' \in k[t,y] \setminus y k[t,y]$ and $r \geq 1$ such that $g - g_0 = y^r g'$. If $n = 0$, we get
\[ ty^r g' = ty - t g_0 \in Q \setminus p. \]
Thus we can and will assume that $n \geq 1$. Now, choose $g \in k[t,y] \setminus y k[t,y]$ of minimal $y$-degree such that (8) is satisfied for some $n \geq 1$. We claim that $g \in Q$. Otherwise, $\deg_y(g) > 0$ and $ty^n \in Q$, since $Q$ satisfies the $P_2$ property in $k[t,y]$.

In fact, since $g$ is of minimal $y$-degree, we get $\deg_{y} ty^n \in p$. Thus we get a contradiction to the fact that
\[ ty^n g' = ty^n g - ty^n g_0 \in Q \setminus p \quad \text{and} \quad \deg_y(g') < \deg_y(g). \]

Let $h = ty^n g \in Q \setminus p$. Since $t, g \in Q$, it follows that $y^{-n} = tg/h \in Q_h$. Since $Q$ satisfies the property $P_2$ in $k[t,y]$, the localization $Q_h$ satisfies the property $P_2$ in $k[t,y]_h = k[t, t^{-1}, y, y^{-1}]$. Hence, we get $y^{-1} \in Q_h$. \hfill \Box

Lemma 7.7. Let $A \in \mathcal{N}_0$ and let $m$ be the crucial maximal ideal of $A$. Then the inclusion $A \cap k[t,y] \subseteq k[t,y]$ defines an open immersion
\[
\varphi : A^2_k \longrightarrow \text{Spec } A \cap k[t,y]
\]
on spectra and the complement of the image of $\varphi$ consists only of the maximal ideal $m \cap k[t,y]$ of $A \cap k[t,y]$. 
Remark 7.8. The proof will show the following:

a) the maximal ideal $m \cap k[t,y]$ of $A \cap k[t,y]$ contains $t$ and does not contain $t k[t,y] \cap A \cap k[t,y]$ (see iii) in the proof);
b) the homomorphism $A \cap k[t,y] \to k[t,y]/t k[t,y]$ is surjective (see i) in the proof).

Proof of Lemma 7.7. Let $A' = A \cap k[t,y]$ and let $m' = m \cap k[t,y]$. Due to Remark 7.1, the residue field $A/\mathfrak{m}$ is isomorphic to $k$. Hence, $m'$ is a maximal ideal of $A'$. We divide the proof in several steps.

i) We claim that $\varphi$ induces a closed immersion $\{0\} \times \mathbb{A}^1_k \to V_{\text{Spec}(A')}(t)$. Due to the surjection (7), there exists $f \in k[t,y,y^{-1}]$, $a \in A$ such that $y = a + tf$. Let $f = f^+ + f^-$ where $f^+ \in k[t,y]$ and $\deg_y(f^-) < 0$. We have $a + tf^- \in A$ and $tf^- \in t k[t,y]$. Thus we get

$$a + tf^- = y - tf^- \in A \cap k[t,y] = A'. $$

This implies that

$$A'/t A' \to k[t,y]/t k[t,y] = k[y]$$

is surjective, which implies the claim.

ii) We claim that $\varphi$ induces an isomorphism $\mathbb{A}_k^1 \times \mathbb{A}_k^1 \simeq \text{Spec}(A') \setminus V_{\text{Spec}(A')}(t)$.

Since $t \in m$, we have $A_t = k[t, t^{-1}, y, y^{-1}]$ and thus $t^k y \in A$ for some integer $k$. This implies $t^k y \in A'$ and thus $A'_t = k[t, t^{-1}, y]$.

iii) We claim that $\text{Spec}(A') \setminus \varphi(\mathbb{A}_k^1) = \{m'\}$. Using i) and ii) this is equivalent to show that $m'$ is the only prime ideal of $A'$ that contains $t$ and does not contain $t k[t,y] \cap A'$.

Since $m$ contains $t$ it follows that $m'$ contains $t$. Since there exists no prime ideal of $k[t,y,y^{-1}]$ that lies over $m$, the surjection (7) implies that $m$ does not contain $t k[t,y,y^{-1}] \cap A$. Hence there exists $f \in k[t,y,y^{-1}]$ such that $tf \in A \setminus m$. Since $k[t,y,y^{-1}] \subseteq m$, we can even assume that $f \in k[t,y]$. Hence $tf \in A' \setminus m'$ and therefore $m'$ does not contain $tk[t,y] \cap A'$.

As $A \subseteq k[t,y,y^{-1}]$ induces an isomorphism $\mathbb{A}_k^1 \times \mathbb{A}_k^1 \simeq \text{Spec}(A) \setminus \{m\}$ and since $t \in m$, we have

$$\text{rad}(t) = t k[t,y,y^{-1}]$$

Intersecting with $k[t,y]$ yields

$$\text{rad}(t A') = t k[t,y] \cap A' \cap m'.$$

Thus every prime ideal of $A'$ that contains $t$ and does not contain $tk[t,y] \cap A'$ must be equal to $m'$ (note that $m'$ is a maximal ideal of $A'$).

iv) We claim that $\varphi$ is an open immersion. According to Theorem 5.24 and Remark 5.14 there exists $a \in k[[l^{-1}]]^+$ such that

$$A = \{f \in k[y^{-1}, y, t] \mid \nu(f(\alpha)) > 0\}. $$

Note that $y^{-1}$ corresponds to the $t$ in Theorem 5.24 and $t$ corresponds to the $y$ in Theorem 5.24. In particular we have $\nu(y^{-1}) = 1$. If $\alpha = 0$, then $A = k[y^{-1}] + k[y^{-1}, y, t]$ and thus (7) is not surjective. Hence $\alpha \neq 0$. Let $\nu(\alpha) = a/b$ for integers $a \geq 0$, $b > 0$ and let $\lambda \in k^\*$ be the coefficient of $y^{-a/b}$ of $\alpha$. There exists $k \geq 1$ such that

$$y(\lambda^b - a^by^a)^k \in A'. $$

Indeed, $\nu(\lambda^b - a^by^a) > 0$, since $\alpha$ is equal to $\lambda(y^{-1})^{a/b}$ plus higher order terms in $y^{-1}$. Hence, there exists $k \geq 1$ such that

$$\nu(y(\lambda^b - a^by^a)^k) = -1 + k\nu(\lambda^b - a^by^a) \geq 0,$$

which yields (9).
As $A$ satisfies the property $P_2$ in $k[y^{-1}, t, y]$ (see Lemma 7.8), it follows that $A'$ satisfies the property $P_2$ in $k[y, t]$. Since $y \notin A'$ we get thus $y^h - t^hy^a \in A'$ by (10). Again by (10) we have $y \in A'_{\lambda = t^hy^a}$, which implies

$$A'_{\lambda = t^hy^a} = k[t, y]_{\lambda = t^hy^a}.$$

As the zero set of $\lambda^h - t^hy^a$ and of $t$ in $k^2 = \text{Spec} k[t, y]$ are disjoint, it follows with ii) that $\varphi: \mathbb{A}^2_k \to \text{Spec}(A')$ is locally an open immersion. However, i) and ii) imply that $\varphi$ is injective and thus $\varphi$ is an open immersion.

Lemma 7.9. If $A \in \mathcal{M}_0$, then $A \cap k[t, y]$ is a maximal subring of $k[t, y]$. Moreover, there exists $h \in A \cap k[t, y]$, depending only on $A \cap k[t, y]$, such that

$$A = (A \cap k[t, y])_h \cap k[t, y, y^{-1}].$$

Remark 7.10. If $A \in \mathcal{M}_0$ and $m \subseteq A$ is the crucial maximal ideal, then $m \cap k[t, y]$ is the crucial maximal ideal of $A \cap k[t, y]$, see Lemma 7.4.

Proof of Lemma 7.9. Let $A' = A \cap k[t, y]$ and let $m' = m \cap A$ where $m$ denotes the crucial maximal ideal of $A$. As $A$ satisfies the $P_2$ property in $k[t, y, y^{-1}]$, $A'$ satisfies the $P_2$ property in $k[t, y]$. By Remark 7.8 $m'$ contains $t$ and does not contain $t k[t, y] \cap A'$. Moreover, the homomorphism $A' \to k[t, y]/t k[t, y]$ is surjective according to Remark 7.8. Hence, by Lemma 7.9 there exists $h \in A' \setminus m'$ such that $y^{-1} \in A'_h$. We claim that $A'_h = A_h$.

Indeed, if $a = a^+ + a^- \in A$ and $a^+ \in k[t, y]$, $\deg_y(a^-) < 0$, then we get

$$a^+ = a - a^- \in A \cap k[t, y] = A'.$$

However, $a^- \in k[t, y, y^{-1}] \subseteq A'_h$ and thus $a \in A'_h$, which implies the claim. Using Lemma 3.1 and the fact that $h \in A' \setminus m'$, the claim implies that $A'_h \subseteq k[t, y, y^{-1}]_h = k[t, y]_h$ is an extending maximal subring. Now, let $A' \subseteq B \subseteq k[t, y]$ be an intermediate ring. Since $\varphi: \mathbb{A}^2_k \to \text{Spec}(A')$ is an open immersion and $\text{Spec}(A') \setminus \varphi(\mathbb{A}^2_k) = \{m'\}$ (see Lemma 7.7), it follows that $m'$ lies in the image of the morphism $\text{Spec}(B) \to \text{Spec}(A')$. Hence, there exists a prime ideal in $B$ that lies over $m'$. Since $h \in A' \setminus m'$, it follows that there exists a prime ideal of $B_h$ that lies over $m'A'_h$. In particular, $B_h \neq k[t, y]_h$. By the maximality of $A'_h$ in $k[t, y]_h$ we get $A'_h = B_h$. Thus

$$B \subseteq B_h \cap k[t, y] = A'_h \cap k[t, y] = A_h \cap k[t, y] = A'$$

where the last equality follows from the fact that

$$A_h \cap k[t, y, y^{-1}] = A \quad (10)$$

(note that $y \notin A_h$, since otherwise $y^h \in A \cap k[t, y] = A'$ for a certain integer $k$ and thus $y \in A'_h$). This proves the maximality of $A'$ in $k[t, y]$. The second statement follows from (10) and the fact that $A'_h = A_h$.

Proof of Proposition 7.5. From the first statement of Lemma 7.9 and from Remark 7.10 it follows that the map $\mathcal{M}_0 \to \mathcal{M}_0$ is well-defined. From the second statement of Lemma 7.9 it follows that $\mathcal{M}_0 \to \mathcal{M}_0$ is injective.

Now, we prove the surjectivity. Let $Q \in \mathcal{M}_0$. We have the following inclusion

$$Q/t k[t, y] \cap Q \subseteq k[t, y]/t k[t, y] = k[y].$$

(11)

On spectra, this map yields an open immersion, since $\text{Spec} k[t, y] \to \text{Spec} Q$ is an open immersion. Hence, (11) is a finite ring extension, and thus (11) must be an equality. This implies that the crucial maximal ideal $p$ of $Q$ does not contain
\( t \mathbb{k}[t, y] \cap Q \) (note that \( \text{Spec } Q \setminus \text{Spec } \mathbb{k}[t, y] = \{ p \} \)). By assumption, \( t \in p \). Moreover, \( Q \) satisfies the \( P_n \) property in \( \mathbb{k}[t, y] \) by Lemma 6.5. By Lemma 7.6 there exists \( h \in Q \setminus p \) such that \( y^{-1} \in Q_h \). Thus, Lemma 7.1 implies that
\[
Q_h \subseteq \mathbb{k}[t, y]_h = \mathbb{k}[t, y, y^{-1}]_h
\]
is an extending maximal subring. Since \( y^{-1} \in Q_h \), the ring
\[
Q' = Q_h \cap \mathbb{k}[t, y, y^{-1}] \subseteq \mathbb{k}[t, y, y^{-1}]
\]
contains \( \mathbb{k}[t, y^{-1}] \). Now, we divide the proof in several steps.

i) We claim that \( Q' \) is a maximal subring of \( \mathbb{k}[t, y, y^{-1}] \). Therefore, take an intermediate ring \( Q' \subseteq B \subseteq \mathbb{k}[t, y, y^{-1}] \). By the maximality of \( Q \) in \( \mathbb{k}[t, y] \) we get \( Q = B \cap \mathbb{k}[t, y] \) and hence
\[
Q_h = (B \cap \mathbb{k}[t, y])_h.
\]
If \( y \) would be in \( B_h \), then \( y \) would be in \( (B \cap \mathbb{k}[t, y])_h = Q_h \), a contradiction to the fact that \( Q_h \neq \mathbb{k}[t, y, y^{-1}]_h \). Hence we have \( B_h \neq \mathbb{k}[t, y, y^{-1}]_h \). The maximality of \( Q_h \) in \( \mathbb{k}[t, y, y^{-1}]_h \) implies that \( B_h = Q_h \). Hence, we have
\[
B \subseteq B_h \cap \mathbb{k}[t, y, y^{-1}] = Q' \subseteq B,
\]
which proves the maximality of \( Q' \) in \( \mathbb{k}[t, y, y^{-1}] \).

ii) We claim that \( pQ_h \cap k[t, y, y^{-1}] \) is the crucial maximal ideal of \( Q' \). Clearly, \( pQ_h \) is the crucial maximal ideal of \( Q_h \). If \( pQ_h \cap k[t, y, y^{-1}] \) would not be the crucial maximal ideal of \( Q' \), then \( \text{Spec } Q_h \to \text{Spec } Q' \) would send \( pQ_h \) to a point of the open subset \( \text{Spec } k[t, y, y^{-1}] \) of \( \text{Spec } Q' \). This would imply that \( k[t, y, y^{-1}] \subseteq Q_h \), a contradiction.

iii) We claim that \( Q' \in \mathcal{N}_0 \). By ii), \( pQ_h \cap k[t, y, y^{-1}] \) is the crucial maximal ideal of \( Q' \) and it contains \( t \). By the equality (11) we get \( y = q + tf \) for some \( q \in Q \), \( f \in k[t, y] \). Since \( Q \subseteq Q' \) and \( k[t, y^{-1}] \subseteq Q' \), the homomorphism
\[
Q' \to \mathbb{k}[t, y, y^{-1}]/t \mathbb{k}[t, y, y^{-1}]
\]
is surjective. With i) we get \( Q' \in \mathcal{N}_0 \).

iv) We claim that \( Q' \cap k[t, y] = Q \). This follows from the fact that \( Q \subseteq Q' \cap k[t, y] \subseteq k[t, y] \) and from the maximality of \( Q \) in \( k[t, y] \).

This proves the surjectivity. \( \square \)

Let us interpret the map \( \mathcal{N} \to \mathcal{M} \), \( A \mapsto A \cap k[t, y] \) in geometric terms. For this we introduce the following terminology.

**Definition 7.11.** We call a dominant morphism \( Y \to X \) of affine schemes an \textit{(extending) minimal morphism}, if \( \Gamma(X, \mathcal{O}_X) \) is an (extending) maximal subring of \( \Gamma(Y, \mathcal{O}_Y) \). Moreover, the point in \( X \) which corresponds to the crucial maximal ideal of \( \Gamma(X, \mathcal{O}_X) \) we call the \textit{crucial point} of \( X \).

Let us denote by \( \text{pr} : \mathbb{A}^2_k \to \mathbb{A}^1_k \) the projection \( (t, y) \mapsto t \). The set \( \mathcal{M} \) corresponds to the extending minimal morphisms \( \psi : \mathbb{A}^2_k \to X \) such that \( \text{pr} : \mathbb{A}^2_k \to \mathbb{A}^1_k \) factorizes as
\[
\mathbb{A}^2_k \xrightarrow{\psi} X \longrightarrow \mathbb{A}^1_k.
\]
The set \( \mathcal{N} \) corresponds to the extending minimal morphisms \( \varphi : \mathbb{A}^1_k \times \mathbb{A}^1_k \to Y \) such that the open immersion \( \mathbb{A}^1_k \times \mathbb{A}^1_k \to \mathbb{A}^2_k, (t, y) \mapsto (t, y^{-1}) \) factorizes as
\[
\mathbb{A}^1_k \times \mathbb{A}^1_k \xrightarrow{\varphi} Y \longrightarrow \mathbb{A}^2_k
\]
and such that the image of \( \{0\} \times \mathbb{A}^1_k \) under \( \varphi \) is closed in \( Y \).
Proposition 7.12. Let \( \varphi: \mathbb{A}_k^1 \times \mathbb{A}_k^* \to Y \) be an extending minimal morphism corresponding to an element \( A \in \mathcal{N} \). Then
\[
\text{Spec } A \cap k[t, y] = Y \cup_{\varphi} \mathbb{A}_k^2
\]
where \( Y \cup_{\varphi} \mathbb{A}_k^2 \) denotes the glueing via \( \mathbb{A}_k^2 \leftarrow \mathbb{A}_k^1 \times \mathbb{A}_k^* \xrightarrow{\varphi} Y \) where \( \sigma \) is the open immersion defined by \( \sigma(t, y) = (t, y) \).

Proof. By Theorem 7.2 we have the following commutative diagram
\[
\begin{array}{ccc}
\mathbb{A}_k^1 \times \mathbb{A}_k^* & \xrightarrow{\varphi} & Y \\
\sigma \downarrow \quad & & \downarrow \\
\mathbb{A}_k^1 \times \mathbb{A}_k^1 & \xrightarrow{\text{extend. minimal mor.}} & \text{Spec } A \cap k[t, y].
\end{array}
\]

By Lemma 7.9, Remark 7.10 and Remark 7.4 there exists a regular function \( h \) on \( \text{Spec } A \cap k[t, y] \) that does not vanish at the crucial point of \( \text{Spec } A \cap k[t, y] \) and we have
\[
A_h = (A \cap k[t, y])_h.
\]

Thus \( Y \to \text{Spec } A \cap k[t, y] \) restricts to an open immersion on \( Y_h \). By the commutativity of the diagram, it follows that \( Y \to \text{Spec } A \cap k[t, y] \) restricts to an open immersion on \( \varphi(\mathbb{A}_k^1 \times \mathbb{A}_k^* \cup_k \mathbb{A}_k^2) \). By Remark 7.10 and Remark 7.4, the morphism \( Y \to \text{Spec } A \cap k[t, y] \) maps the crucial point of \( Y \) to that one of \( \text{Spec } A \cap k[t, y] \). Hence, \( Y_h \) contains the crucial point of \( Y \). In summary, we get that \( Y \to \text{Spec } A \cap k[t, y] \) is an open immersion. Thus all morphisms in the diagram above are open immersions. Moreover, \( \varphi \) induces an isomorphism \( \mathbb{A}_k^1 \times \mathbb{A}_k^* \to Y \cap \mathbb{A}_k^1 \times \mathbb{A}_k^1 \) where we consider \( Y \cap \mathbb{A}_k^1 \times \mathbb{A}_k^1 \) as an open subset of \( \text{Spec } A \cap k[t, y] \). Hence \( \text{Spec } A \cap k[t, y] \) is the claimed glueing.

8. Examples of extending maximal \( k \)-subalgebras of \( k[t, y] \) that do not contain a coordinate

In the last section, we classified all extending maximal \( k \)-subalgebras of \( k[t, y] \) that contain a coordinate of \( k[t, y] \), see Theorem 7.2 and Remark 7.3. It is thus natural to ask, whether all extending maximal \( k \)-subalgebras of \( k[t, y] \) contain a coordinate. In this section we construct plenty of examples, which give a non-affirmative answer to this question. For this we use techniques of birational geometry of surfaces. As we fix the algebraically closed field \( k \), we write \( \mathbb{P}^n \) for \( \mathbb{P}^n_k \) and \( \mathbb{A}^n \) for \( \mathbb{A}^n_k \).

Definition 8.1. Let \( L \subseteq \mathbb{P}^2 \) be a line, let \( p \in L \) be a point and let \( \Gamma \subseteq \mathbb{P}^2 \) be an irreducible curve with \( \Gamma \neq L \) which passes through \( p \). We say that \( \Gamma \) is tangent to \( L \) at \( p \) of order at least \( m \), if there exists a sequence of blow-ups
\[
S_m \xrightarrow{\pi_m} S_{m-1} \xrightarrow{\pi_{m-1}} \ldots \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} \mathbb{P}^2
\]
such that \( \pi_1 \) is centered at \( p \), \( \pi_i \) is centered at a point on the exceptional divisor of \( \pi_{i-1} \) for \( i = 2, \ldots, m \) and the strict transforms of \( L \) and of \( \Gamma \) under \( \pi_1 \circ \cdots \circ \pi_m \) have an intersection point on the exceptional divisor of \( \pi_m \).

The following Lemma is crucial for our construction.

Lemma 8.2. Let \( \Gamma \subseteq \mathbb{P}^2 \) be an irreducible curve and let \( L \neq \Gamma \) be a line in \( \mathbb{P}^2 \). Fix some \( p \in \Gamma \cap L \). If \( \Gamma \) is tangent to \( L \) at \( p \) of order at least 2 and if \( \Gamma \) is smooth at \( p \), then there exists no coordinate \( f: \mathbb{A}^2 = \mathbb{P}^2 \setminus L \to \mathbb{A}^1 \) such that the rational map \( f|\Gamma: \Gamma \dashrightarrow \mathbb{A}^1 \) is defined at \( p \).
Proof. Let \( \varphi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \) be a birational map that restricts to an automorphism on \( \mathbb{P}^2 \setminus L = \mathbb{A}^2 \). Let \( pr : \mathbb{A}^2 \to \mathbb{A}^1 \) be the projection given by \( pr(x, y) = x \). We have to prove that the rational map \( pr \circ \varphi|_\Gamma : \Gamma \dashrightarrow \mathbb{A}^1 \) is not defined at \( p \). Let \( a \in \mathbb{P}^2 \) be the image of \( p \) under the rational map \( \varphi|_\Gamma : \Gamma \dashrightarrow \mathbb{P}^2 \), which is defined at \( p \) since \( \Gamma \) is smooth at \( p \). We have \( a \in L \), since either \( \varphi \) contracts the line \( L \) to some point on \( L \) or \( \varphi \) maps \( L \) isomorphically onto itself. If \( a \neq (0 : 1 : 0) \), then the map \( pr \circ \varphi|_\Gamma : \Gamma \dashrightarrow \mathbb{A}^1 \) is not defined at \( p \). Thus we can assume that \( a = (0 : 1 : 0) \).

Let \( \sigma : Bl_a(\mathbb{P}^2) \to \mathbb{P}^2 \) be the blow-up of \( \mathbb{P}^2 \) centered at \( a \). Then, \( pr \circ \varphi|_\Gamma : \Gamma \dashrightarrow \mathbb{A}^1 \) is not defined at \( p \) if and only if

\[
\Gamma \subseteq \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \xrightarrow{\sigma^{-1}} Bl_a(\mathbb{P}^2)
\]

maps \( p \) to the intersection point of the exceptional divisor of \( \sigma \) and the strict transform of \( L \) under \( \sigma \). In other words, we have to prove that \( \varphi(\Gamma) \) is tangent to \( L \) at \( a \) of order at least \( 1 \).

If \( \varphi \) is an automorphism, then the result is obvious, so we can assume that there exist base-points of \( \varphi \). By [Bluh99], Lemma 2.2] there exist birational morphisms \( \varepsilon : Y \to \mathbb{P}^2 \) and \( \eta : Y \to \mathbb{P}^2 \) such that the following is satisfied:

- We have \( \eta = \varphi \circ \varepsilon \);
- No curve of self-intersection \(-1\) of \( Y \) is contracted by both, \( \varepsilon \) and \( \eta \);
- There are decompositions
  \[
  \varepsilon = \varepsilon_1 \circ \cdots \circ \varepsilon_n : Y \longrightarrow \mathbb{P}^2 \quad \text{and} \quad \eta = \eta_1 \circ \cdots \circ \eta_n : Y \longrightarrow \mathbb{P}^2
  \]
  where \( \varepsilon_i \) (respectively \( \eta_i \)) is a blow-up centered at a point on \( L \) and \( \varepsilon_i \) (respectively \( \eta_i \)) is a blow-up centered at a point on the exceptional divisor of \( \varepsilon_{i-1} \) (respectively \( \eta_{i-1} \)) for \( i > 1 \);
- The integer \( n \) is greater than or equal to \( 3 \);
- The strict transform of \( L \) under \( \varepsilon \) (respectively \( \eta \)) has self-intersection \(-1\).

Let \( q_{i-1} \) be the center of \( \varepsilon_i \) and let \( E_i \) be the exceptional divisor of \( \varepsilon_i \) for \( i = 1, \ldots, n \). Moreover, we denote by \( L_i \) the strict transform of \( L \) under \( \varepsilon_i \circ \cdots \circ \varepsilon_1 \) for \( i = 1, \ldots, n \). Since \( (L_n)^2 = -1 \), we see that \( L_i \) passes through \( q_0 \), that \( L_1 \) passes through \( q_1 \), but \( L_i \) passes not through \( q_i \) for \( i > 1 \).

By assumption, \( \Gamma \) is tangent to \( L \) at \( p \) of order at least \( 2 \), so there exists a sequence of blow-ups

\[
S_2 \xrightarrow{\pi_2} S_1 \xrightarrow{\pi_1} \mathbb{P}^2
\]

such that \( \pi_2 \) is centered at \( p \), \( \pi_2 \) is centered at some point on the exceptional divisor of \( \pi_1 \) and the strict transforms of \( L \) and of \( \Gamma \) under \( \pi_1 \circ \pi_2 \) intersect at one point on the exceptional divisor. Denote this intersection point on \( S_2 \) by \( p_2 \). Consider the birational map

\[
\psi = \varepsilon_2^{-1} \circ \varepsilon_1^{-1} \circ \pi_1 \circ \pi_2.
\]

This map is defined at \( p_2 \) and we denote by \( \psi \) its image under \( \psi \). Since \( p_2 \in L_2 \) and since \( q_2 \not\in L_2 \) there exists exactly one point \( r \in L_n \) that is mapped onto \( p_2 \) via \( \varepsilon_n \circ \cdots \circ \varepsilon_3 \) (note that \( n \geq 3 \)). Remark that the strict transform of \( \Gamma \) under \( \psi \) passes through \( r \).

Let \( r_{i-1} \) be the center of \( \eta_i \) and let \( F_i \) be the exceptional divisor of \( \eta_i \) for \( i = 1, \ldots, n \). Since \( E_n \) and \( L_n \) are the only curves of self-intersection \(-1\) lying in \( Y \setminus \varepsilon^{-1}(\mathbb{P}^2 \setminus L) \), it follows that \( E_n \) is the strict transform of \( L \) under \( \eta \) and that \( \eta_{n} \) contracts \( L_n \) i.e. \( F_n = L_n \). Hence we have for \( i = 2, \ldots, n \)

\[
\eta_1 \circ \cdots \circ \eta_n(r) = r_{i-1} \in F_{i-1}.
\]

As \( r \in L_n \), the curve \( L_n \) is contracted by \( \eta \) onto \( \eta(r) \); this point being also the point where \( \varphi \) contracts \( L \), we get \( \eta(r) = a \in L \). Since the strict transform of \( L \) under \( \eta \) has self-intersection \(-1 \), it follows that \( r_1 \) is the intersection point of \( F_1 \)
and the strict transform of $L$ under $\eta$. As the strict transform of $\Gamma$ under $\varepsilon$ passes through $r$, its image passes through all the points $r_i$ and thus also through $r_1$. So the curve $\varphi(\Gamma)$ is tangent to $L$ at $a \in \mathbb{P}^2$ of order at least 1. $\square$

With this lemma we can construct plenty of examples of extending maximal $k$-subalgebras of $k[t,y]$ that do not contain a coordinate of $k[t,y]$.

Let $X$ be an irreducible curve of $\mathbb{A}^2$, which is defined by some polynomial $f$ in $k[t,y]$. Let $\Gamma$ be the closure of $X$ in $\mathbb{P}^2$. Assume that there exists a smooth point $p$ on $\Gamma$ that lies not in $X$ and assume that $\Gamma \setminus X$ contains more than one point. Then the ring

$$A = \{ h \in \Gamma(X, \mathcal{O}_X) \mid h \text{ is defined at } p \}$$

is an extending maximal $k$-subalgebra of $\Gamma(X, \mathcal{O}_X)$, which is finitely generated over $k$, see Theorem 4.2 and Lemma 8.2. Let $a_1, \ldots, a_k \in A$ be a set of generators and let $r_1, \ldots, r_k \in k[t,y]$ be elements such that $r_i|_X = a_i$. If $\Gamma$ is tangent to $L = \mathbb{P}^2 \setminus \mathbb{A}^2$ at $p$ of order at least 2, then

$$k[r_1, \ldots, r_k] + f k[t,y]$$

is an extending maximal $k$-subalgebra of $k[t,y]$ that does not contain a coordinate of $k[t,y]$, see Lemma 5.2 and Lemma 8.2.

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