On category $\mathcal{O}$ for affine Grassmannian slices and categorified tensor products

Joel Kamnitzer, Peter Tingley, Ben Webster, Alex Weekes, and Oded Yacobi

Abstract. Truncated shifted Yangians are a family of algebras which naturally quantize slices in the affine Grassmannian. These algebras depend on a choice of two weights $\lambda$ and $\mu$ for a Lie algebra $\mathfrak{g}$, which we will assume is simply-laced. In this paper, we relate the category $\mathcal{O}$ over truncated shifted Yangians to categorified tensor products: for a generic integral choice of parameters, category $\mathcal{O}$ is equivalent to a weight space in the categorification of a tensor product of fundamental representations defined by the third author using KLRW algebras. We also give a precise description of category $\mathcal{O}$ for arbitrary parameters using a new algebra which we call the parity KLRW algebra. In particular, we confirm the conjecture of the authors that the highest weights of category $\mathcal{O}$ are in canonical bijection with a product monomial crystal depending on the choice of parameters.

This work also has interesting applications to classical representation theory. In particular, it allows us to give a classification of simple Gelfand-Tsetlin modules of $U(\mathfrak{gl}_n)$ and its associated W-algebras.

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1. Introduction

1.1. Two geometric models and symplectic duality. Let $G$ be a simply-laced semisimple group with Lie algebra $\mathfrak{g}$. There are two geometric constructions of the finite-dimensional irreducible representations of $\mathfrak{g}$. First we have the geometric Satake correspondence (due to Lusztig [Lus83], Ginzburg [Gin90], and Mirkovic-Vilonen [MV07]) which constructs representations using the affine Grassmannian $Gr = G^{\vee}((z))/G^{\vee}[[z]]$ of the Langlands dual group of $G$. The second construction involves the cohomology of quiver varieties constructed using the Dynkin diagram of $\mathfrak{g}$ (due to Nakajima [Nak98]). One goal of this paper is to answer the following question.

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Question 1.1. What is the relationship between these two geometric constructions?

In [BLPW16], Braden, Licata, Proudfoot, and the third author proposed that the framework of symplectic duality could be used to explain the connection between these two geometric models. In particular, they conjectured that there should exist a Koszul duality between certain categories of modules over the quantizations of these varieties.

More precisely, let $\lambda, \mu$ be a pair of dominant weights for $\mathfrak{g}$, such that $\lambda - \mu = \sum_i m_i \alpha_i$, with $m_i \in \mathbb{N}$. Then we can consider the affine Grassmannian slice

$$\text{Gr}_\lambda^\mu := G^{\vee}[[z]]z^\lambda \cap G^{\vee}_1 [z^{-1}]z^\mu \subset \text{Gr}.$$

By the geometric Satake correspondence, the intersection cohomology of $\text{Gr}_\lambda^\mu$ is (non-canonically) isomorphic to $V(\lambda)_\mu$, the $\mu$-weight space of the irreducible representation of $\mathfrak{g}$ of highest weight $\lambda$.

In [KWWY14], 80% of the authors proved that $\text{Gr}_\lambda^\mu$ is an affine Poisson variety. We also introduced the truncated shifted Yangian $Y_\lambda^\mu$, an algebra which quantizes $\text{Gr}_\lambda^\mu$. In fact, there are a family of such algebras, $Y_\lambda^\mu(R)$, depending on a parameter $R \in \prod_i \mathbb{C}^{\lambda_i}$, where $\lambda = \sum_i \lambda_i \omega_i$ (Definition 4.8). In [BFNa], the definitions of $\text{Gr}_\lambda^\mu$ and $Y_\lambda^\mu$ were generalized to the case of non-dominant $\mu$.

The algebra $Y_\lambda^\mu$ contains a polynomial subalgebra and thus we can speak about weight modules for $Y_\lambda^\mu$ (see Definition 5.1). Category $\mathcal{O}$ for this algebra consists of those weight modules whose weights are bounded above. This category will be the main object of study in this paper.

On the other hand, associated to $\lambda, \mu$, we have the Nakajima quiver variety $M(\lambda, \mu)$. Its top cohomology is also isomorphic to $V(\lambda)_\mu$. This quiver variety is defined as the Hamiltonian reduction of the cotangent bundle of a vector space of framed quiver representations by the group $H = \prod_i GL_{m_i}$.

As a Hamiltonian reduction, the quiver variety admits a natural quantization. The category $\mathcal{O}$ of highest weight modules for this quantization has been studied extensively. In particular, in [Web17b] Th. A', the third author proved (building on work of Rouquier and Varagnolo-Vasserot) that it is Koszul dual to the category of modules of a certain combinatorial/diagramatic algebra, called a KLRW algebra.

1.2. Parity KLRW algebras. KLRW algebras (also called tensor product categorifications or red-black algebras) were introduced by the third author in [Web17a], following the foundational work of Khovanov-Lauda and Rouquier [KL09, Rou]. These are algebras of string diagrams, containing red and black strands, modulo some local relations. These algebras are used to categorify tensor products of irreducible representations of $\mathfrak{g}$. More precisely, we have the KLRW algebra $\text{-}T$, whose category of modules carries a categorical $\mathfrak{g}$-action. The Grothendieck group of $\text{-}T$-mod is isomorphic to a tensor product of fundamental representations $\otimes_i V(\varpi_i)^{\otimes \lambda_i}$.

Let $R$ be an integral set of parameters (cf. Section 2.1). In this paper, we introduce the parity KLRW algebra which is a subalgebra $\text{-}P^R \subset \text{-}T$ (Definition 3.7). The categorical $\mathfrak{g}$-action on $\text{-}T$-mod preserves $\text{-}P^R$-mod (Lemma 3.11), and thus the category of $\text{-}P^R$-mod categorifies a subrepresentation of $\otimes_i V(\varpi_i)^{\otimes \lambda_i}$. For generic values of these parameters, this representation is the full tensor product, whereas for very special
values, it is just the irreducible representation $V(\lambda)$. As with all such categorifications, $-P_{\mu}^R$ is a direct sum of algebras $-P_{\mu}^R$, such that $-P_{\mu}^R$-mod corresponds to the $\mu$-weight space of the Grothendieck group.

The main result of this paper is the following:

**Theorem 1.2.** There is an equivalence of categories from category $\mathcal{O}$ over $Y_{\lambda}^\lambda(R)$ to $-P_{\mu}^R$-mod.

This theorem has an immediate corollary which establishes the categorical symplectic duality between affine Grassmannian slices and quiver varieties.

**Corollary 1.3.** For a generic integral choice of $R$, the quadratic dual of category $\mathcal{O}$ over $Y_{\lambda}^\lambda(R)$ is the category $\mathcal{O}$ for the quiver variety $\mathcal{M}(\lambda, \mu)$ (associated to a $\mathbb{C}^*$-action depending on $R$). If $\lambda$ is a sum of minuscule weights, then both category $\mathcal{O}$’s are Koszul, and they are Koszul dual to each other.

*Proof.* We will just sketch the argument, since this duality is not the main focus of the current paper and since very similar arguments are used in [Webb] to establish Koszul duality in more general settings.

Let $R$ be a generic set of parameters (cf. [KWWY14, Section 2.5]). In this case, we have that $-P_{\mu}^R = -T_{\mu}^R$, and by Theorem 1.2, the quadratic dual of the category $\mathcal{O}$ of $Y_{\mu}^\lambda(R)$ is the quadratic dual of $-T_{\mu}^R$-mod. We refer the reader to [MOS09] for the definition of quadratic dual; in down to earth terms, $-T_{\mu}^R$ is Morita equivalent to a (very difficult to describe) positively graded algebra, and we mean the representations of the quadratic dual of that algebra.

By [MOS09] Theorem 12 the quadratic dual of $-T_{\mu}^R$-mod is equivalent to the category $LCP(-T_{\mu}^R)$ of linear complexes of projective objects in $-T_{\mu}^R$-mod. By [Webb17b, Theorem A'] this latter category is equivalent to (a graded lift of) the category $\mathcal{O}$ of $\mathcal{M}(\lambda, \mu)$. In the special case of minuscule weights, [Webb17b, Theorem B] implies that $-T_{\mu}^R$-mod is Koszul, and thus quadratic dual and Koszul dual coincide. □

### 1.3. Highest weights and monomial crystals

The polynomial subalgebra of $Y_{\mu}^\lambda$ which we use to define category $\mathcal{O}$ is isomorphic to $P(\Sigma)$, where $P$ is a polynomial ring and $\Sigma$ is a product of symmetric groups acting on the variables. Since $P(\Sigma)$ is a partially symmetrized polynomial ring, we can think of the weights of $Y_{\mu}^\lambda$ as collections of multisets $S$.

The algebra $Y_{\mu}^\lambda(R)$ is a quotient of the shifted Yangian $Y_{\mu}^\lambda$ (Definition 4.1). The representation theory of the shifted Yangian is much simpler, since $Y_{\mu}^\lambda$ admits a PBW basis. In particular, $Y_{\mu}^\lambda$ has Verma modules for any choice of multisets $S$. On the other hand, the Verma modules for $Y_{\mu}^\lambda(R)$ are much more difficult to understand; there are only finitely many of them for each $R$.

**Question 1.4.** For which $S$ does $Y_{\mu}^\lambda(R)$ have a Verma module with highest weight $S$?

In [KTW+], we formulated a conjectural answer using the product monomial crystal $\mathcal{B}(R)$. This is a $g$-crystal whose elements are collections of rational monomials in variables $a_{i,k}$, where $i \in I$ and $k \in \mathbb{Z}$. It depends in a subtle way on the parameters $R$; for generic values it is isomorphic to the tensor product crystal $\bigotimes \mathcal{B}(\varpi_i)^{\otimes \lambda_i}$, but for
special values it is isomorphic to the irreducible crystal $\mathcal{B}(\lambda)$ (cf. Theorem 2.3). In this paper, we prove our conjecture from [KTW].

**Theorem 1.5** (Corollary 5.22). Let $R$ be an integral set of parameters. There is a map $S \mapsto a_R b^{-1}$ which gives a bijection between the possible highest weights for $Y_\mu^\lambda(R)$ and the product monomial crystal $\mathcal{B}(R)$.

As explained in [KTW], this theorem is motivated from Nakajima’s equivariant version of the Hikita conjecture. Thus in proving this theorem, we have proved a weak form of the equivariant Hikita conjecture for the symplectic dual pair $\text{Gr}_\lambda \times \text{M}(\lambda, \mu)$.

There is a crystal structure on $\text{Irr}(\mathcal{O}_R)$, the set of equivalence classes of simple $\mathcal{O}_R$-modules. We prove that taking “highest weight” induces a crystal isomorphism $\text{Irr}(\mathcal{O}_R) \cong \mathcal{B}(R)$ (Theorem 3.16). Given this identification of crystals, Theorem 1.5 follows from Theorem 1.2.

### 1.4. The categorical action.

By transport de structure, Theorem 1.2 defines a categorical $g$-action on the direct sum over all $\mu$ of category $O$ for $Y_\mu^\lambda$. We note that this is an abelian action; the Chevalley generators $E_i, F_i$ act by exact functors. In this way, this category $g$-action is similar to the famous Bernstein-Frenkel-Khovanov [BFK99] action of $\mathfrak{sl}_2$ on blocks of category $O$ for $\mathfrak{sl}_n$. In fact, our work can be seen as a direct generalization of their construction (though the link is not immediate due to differences in the definition of category $O$.)

Of course, it would be preferable to describe this categorical action without using the equivalence from Theorem 1.2. In a forthcoming paper [KTW], we will construct quantum Hamiltonian reductions relating truncated shifted Yangians. We will prove that these quantum Hamiltonian reductions give rise to the categorical $g$ action via induction and restriction functors. These functors generalize the Bezrukavnikov-Etingof [BE09] induction and restriction functors for modules over Cherednik algebras.

### 1.5. Coulomb branch algebras.

Given a group $H$ and a representation $V$, Braverman-Finkelberg-Nakajima [BFN] defined the Coulomb branch of the 3d $\mathcal{N} = 4$ supersymmetric gauge theory associated to the pair $(H, V)$. They also defined an algebra $A(H, V)$ quantizing this Coulomb branch (here we specialize $h = 1$). In [BFNa], it is proved that when $H = \prod GL_{m_i}$ and $V$ is the vector space of framed quiver representations corresponding to $\lambda$ and $\mu$, then the quantized Coulomb branch $A(H, V)$ admits a homomorphism from a truncated shifted Yangian, and that this is an isomorphism in finite ADE type when $\mu$ is dominant. Both of these restrictions are removed in a recent paper of the fourth author: by [Wee19, Theorem A], the quantum Coulomb branch for any simply-laced quiver gauge theory is a truncated shifted Yangian, as in Definition 4.8.

In this paper, we assume that $g$ is a simply-laced Kac-Moody Lie algebra whose Dynkin diagram is bipartite (note that this generalizes the finite-dimensional simply-laced simple Lie algebras). This is the setting in which we prove Theorems 1.2 and 1.5. These assumptions are mostly for the purposes of simplifying the combinatorics; Theorem 1.2 can be generalized to the symmetrizable case by using weighted KLRW algebras [Web]; we will prove this in future work [KTW].

In [Weba], the third author considered the case of arbitrary $(H, V)$. He introduced a combinatorially defined algebra depending on $(H, V)$ and proved an equivalence similar
to Corollary 1.3 in this context. The present paper can be thought of a specialization of \cite{Weba} to the quiver case. Indeed our methods are similar; however, we emphasize that the present paper can be read independently of \cite{Weba} and \cite{BFNa,BFNb} with no need to explicitly use the Coulomb machinery.

We remark that Braverman-Finkelberg-Nakajima have formulated a geometric Satake conjecture in the context of generalized affine Grassmannian slices for arbitrary symmetric Kac-Moody Lie algebras (\cite[Conjecture 3.25]{BFNa}). In future work \cite{KTW}, we plan to prove this conjecture using techniques similar to those of this paper.

1.6. Gelfand-Tsetlin modules. In the case where $\lambda = N\omega_1$ is a multiple of the first fundamental weight, the algebras $Y^\lambda_\mu$ for $\mu$ dominant are exactly the W-algebras of $\mathfrak{gl}_N$. The weight modules of $Y^\lambda_\mu$ in this case are the Gelfand-Tsetlin modules of the corresponding W-algebra.

The results of this paper give a classification of the simple Gelfand-Tsetlin modules in terms of crystal combinatorics (cf. Cor. 6.5), and a combinatorial description of representation theoretic quantities such as the weight multiplicities of simples. We describe this application of our work in Section 6. We will study in more detail the relationship of this approach to other works on Gelfand-Tsetlin modules in the future.

1.7. Outline of the proof of Theorem 1.2. The bulk of this paper is devoted to proving Theorem 1.2. The proof proceeds by introducing several related algebras. The following diagram gives an overview of the relationships between their module categories:

\[
\begin{array}{cccccc}
\tilde{T}^R_\mu -\text{mod} & 5.19 & \rightarrow & \mathcal{Y}(R) -\text{wtmod} & \\
\downarrow & & & & \\
\mathcal{F}^R_\mu -\text{mod} & 5.2 & \rightarrow & FY^\lambda_\mu(R) -\mathcal{O}^- & 4.3 & \rightarrow & Y^\lambda_\mu(R) -\mathcal{O}^-
\end{array}
\]

All the horizontal functors are equivalences, and the two vertical ones are induced by idempotent truncation. At the rightmost node of the diagram is the category $\mathcal{O}$ for $Y^\lambda_\mu(R)$, and recall our aim is to prove the equivalence $\mathcal{F}^R_\mu -\text{mod} \cong Y^\lambda_\mu -\mathcal{O}^-$. Our argument is structured as follows. First we observe that the parity KLRW algebra is Morita equivalent to the metric KLRW algebra $\tilde{\mathcal{F}}^R_\mu$ (Definition 3.23). Roughly speaking, $\tilde{\mathcal{F}}^R_\mu$ consists of KLRW diagrams where the red strands carry additional data coming from $R$, which we call longitudes, and the black strands also admit longitudes in such a way that the longitudes are weakly increasing as we move from left to right. We can think of the longitude as the “x”-coordinate of a strand.

We also also enlarge $Y^\lambda_\mu(R)$ to a Morita equivalent algebra $FY^\lambda_\mu(R)$, called the flag Yangian (Definition 4.16). The Morita equivalence restricts to an equivalence between their category $\mathcal{O}$'s.

Thus it suffices to prove the equivalence $\mathcal{F}^R_\mu -\text{mod} \cong FY^\lambda_\mu -\mathcal{O}^-$, which we do by relating the metric KLRW algebra and the flag Yangian to yet larger algebras. The linchpin is the KLR-Yangian algebra $\mathcal{Y}(R)$ (Definition 4.19), which is a version of the extended BFN category from \cite{Weba}. It consists of KLR-like diagrams drawn on a
cylinder, with additional relations along the “seam” of the cylinder. We then deduce Theorem 1.2 using an equivalence between \( \mathcal{H}(R) \)-weight modules and modules over \( \tilde{T}_R \), the coarse metric KLRW algebra (Definition 3.28). This latter algebra is a further generalization of the metric KLRW algebra, in which we weaken the weakly increasing condition on longitudes.

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2. Background

2.1. Notation. Fix a bipartite graph with vertex set \( I = I_0 \cup I_1 \). We call vertices in \( I_0 \) even and those in \( I_1 \) odd. We write \( i \sim j \) if \( i \) and \( j \) are connected. We orient the graph so that arrows always point from even vertices to odd ones and write \( i \rightarrow j \) for the oriented edges. We also fix a total order on \( I \), such that all even vertices come before all odd vertices.

Let \( g \) be the derived simply-laced Kac-Moody Lie algebra whose Dynkin diagram is \( I \). We fix a triangular decomposition \( g = n_− \oplus h \oplus n_+ \). Since \( g = [g, g] \), we have that \( h \) has a basis given by the simple coroots \( \alpha^∨_i \). As usual, we call the \( \mathbb{Z} \)-span \( \Lambda^∨ \) of these the coroot lattice, and its dual \( \Lambda \subset h^∗ \) the weight lattice of \( g \). The weight lattice has a basis of fundamental weights \( \varpi_i \) dual to the simple coroot basis of \( \Lambda^∨ \). For a dominant weight \( \lambda \), we have that \( \lambda = \sum_i \lambda_i \varpi_i \) for unique non-negative integers \( \lambda_i \). We let \( V(\lambda) \) be the irreducible representation of highest weight \( \lambda \).

We let \( C^\lambda = \prod_i \mathbb{C}^{\lambda_i}/\Sigma_{\lambda_i} \), the set of all collections of multisets of sizes \( (\lambda_i)_{i \in I} \). A point in \( C^\lambda \) will be written as \( R = (R_i)_{i \in I} \) where \( R_i \) is a multiset of size \( \lambda_i \) and it will be called a set of parameters of weight \( \lambda \). Thus, we say that \( i \in I \) and \( k \in \mathbb{Z} \) have the same parity if \( i \in I_k \).

We say that \( R \) is integral, if for all \( i \), all elements of \( R_i \) are integers and have the same parity as \( i \). So an integral set of parameters consists of a multiset of \( \lambda_i \) even integers for every \( i \in I_0 \) and \( \lambda_i \) odd integers for every \( i \in I_1 \).

2.2. Definition of the monomial crystal. Recall that a crystal for \( g \) is a set \( B \), along with a partial inverse permutations \( \tilde{e}_i, \tilde{f}_i : B \rightarrow B \), for all \( i \in I \), and a weight map \( wt : B \rightarrow \Lambda \). For each dominant weight \( \lambda \), there is a crystal \( B(\lambda) \) corresponding to the irreducible representation \( V(\lambda) \). We say that a crystal \( B \) is normal if it is the disjoint union of these crystals \( B(\lambda) \) (for varying \( \lambda \)).
The most basic operation on crystals is the tensor product of crystals. We must be careful about this definition since different papers in the literature use different conventions.

**Definition 2.1.** Given crystals $B_1, B_2$, their tensor product is the Cartesian product of the underlying sets, with the weight function given by the sum, and

$$
\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} 
\tilde{e}_1(b_1) \otimes b_2 & \varphi_i(b_1) \geq \epsilon_i(b_2) \\
b_1 \otimes \tilde{e}_2(b_2) & \varphi_i(b_1) < \epsilon_i(b_2) 
\end{cases}
$$

$$
\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} 
\tilde{f}_1(b_1) \otimes b_2 & \varphi_i(b_1) > \epsilon_i(b_2) \\
b_1 \otimes \tilde{f}_2(b_2) & \varphi_i(b_1) \leq \epsilon_i(b_2) 
\end{cases}
$$

Here $\varphi_i(b) = \max\{n \geq 0 \mid \tilde{f}_i^n b \neq 0\}$, and $\epsilon_i(b) = \max\{n \geq 0 \mid \tilde{e}_i^n b \neq 0\}$.

Note this is the opposite of the definition in [LW15, §7]; thus, in these conventions, [LW15 Th. 7.2] proves that the natural crystal structure on the simples of a tensor product categorification for $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)})$ is naturally isomorphic to $B(\lambda^{(1)}) \otimes \cdots \otimes B(\lambda^{(\ell)})$.

Let $\mathcal{M}$ denote the set of all monomials in the variables $a_{i,k}$, for $k \in \mathbb{Z}$, such that $i, k$ have the same parity. Let

$$b_{i,k} = \frac{a_{i,k} a_{i,k+2}}{\prod_{j \sim i} a_{j,k+1}}$$

For a variable $a_{i,k}$ or $b_{i,k}$ the second index $k$ is called the *longitude* of the variable.

Given a monomial $p = \prod_{i,k} a_{i,k}^{d_{i,k}}$, let

$$wt(p) = \sum_{i,k} d_{i,k} \omega_i \quad \varepsilon_i^m(p) = -\sum_{l \leq m} d_{i,l} \quad \varphi_i^m(p) = \sum_{l \geq m} d_{i,l}$$

and

$$\varepsilon_i(p) = \max_m \varepsilon_i^m(p) \quad \varphi_i(p) = \max_m \varphi_i^m(p)$$

We can define the Kashiwara operators on this set of monomials by the rules:

$$\tilde{e}_i(p) = \begin{cases} 
0 & \text{if } \varepsilon_i(p) = 0 \\
b_{i,m} p & \text{for } m \text{ minimal such that } \varepsilon_i^m(p) = \varepsilon_i(p) > 0
\end{cases}$$

$$\tilde{f}_i(p) = \begin{cases} 
0 & \text{if } \varphi_i(p) = 0 \\
b_{i,m-2p}^{-1} & \text{for } m \text{ maximal such that } \varphi_i^m(p) = \varphi_i(p) > 0
\end{cases}$$

The following result is due to Kashiwara [Kas03 Proposition 3.1].

**Theorem 2.2.** $\mathcal{M}$ is a normal crystal.
2.3. **Product monomial crystals.** For any \( c \in \mathbb{Z} \) and \( i \in I \) of the same parity, the monomial \( a_{i,c} \) is clearly highest weight and we can consider the monomial subcrystal \( \mathcal{B}(\varpi_i, c) \) generated by \( a_{i,c} \). Since \( a_{i,c} \) has weight \( \varpi_i \), we see that \( \mathcal{B}(\varpi_i, c) \cong \mathcal{B}(\varpi_i) \). The fundamental monomial crystals for different \( c \) all look the same, they differ simply by translating the variables.

Given a dominant weight \( \lambda \) and an integral set of parameters \( R \) of weight \( \lambda \) as above, following \([KTW+]^b\), we define the product monomial crystal \( \mathcal{B}(R) \) by

\[
\mathcal{B}(R) = \prod_{i \in I, c \in R_i} \mathcal{B}(\varpi_i, c)
\]

In other words, for each parameter \( c \in R_i \), we form its monomial crystal \( \mathcal{B}(\varpi_i, c) \) and then take the product of all monomials appearing in all these crystals. In \([KTW+^b]\), we proved the following result as a consequence of the link between \( \mathcal{B}(R) \) and graded quiver varieties. It also follows as an immediate corollary of the link between the monomial crystal and the parity KLRW algebra, see Theorem 3.16 of the current paper.

**Theorem 2.3.** \( \mathcal{B}(R) \) is a subcrystal of \( \mathcal{M} \). In particular it is a normal crystal. Moreover, there exists embeddings \( \mathcal{B}(\lambda) \subseteq \mathcal{B}(R) \subseteq \otimes_i \mathcal{B}(\varpi_i)^{\otimes \lambda_i} \).

Thus \( \mathcal{B}(R) \) is a crystal which depends on the set of parameters \( R \) and lies between the crystal of the irreducible representation and the crystal of the corresponding tensor product of fundamental representations (for an example see Section 2.6 in \([KTW+^b]\)).

2.4. **Collections of multisets and monomials.** Given a collection of multisets \( S = (S_i)_{i \in I} \), we can define

\[
a_S = \prod_{i, k \in S_i} a_{i,k}, \quad b_S = \prod_{i, k \in S_i} b_{i,k}.
\]

Here we still require that the elements of \( S_i \) are integers of the same parity as \( i \), but we don’t put any restriction on the cardinalities of the \( S_i \) so this is not necessarily a set of parameters of weight \( \lambda \).

From the definition of the monomial crystal, it is easy to see that every monomial \( p \) in \( \mathcal{B}(R) \) is of the form

\[
p = a_R b_S^{-1} = \prod_{i, k \in R_i} a_{i,k} \prod_{i, k \in S_i} \frac{\prod_{j \sim i} a_{j,k+1}}{a_{i,k} a_{i,k+2}}
\]

for some collection of multisets \( S \). Thus an alternative combinatorics for labeling elements of the monomial crystal are these collections of multisets \( S \).

**Remark 2.4.** For \( p \in \mathcal{B}(R) \), \( S \) is uniquely determined. In fact for any tuples of multisets \( S \) and \( S' \), \( b_S = b_{S'} \) implies \( S = S' \).

3. **Variations on KLRW algebras**

3.1. **Recollection on KLRW algebras.** Consider a sequence \( \lambda = (\lambda^{(1)}, \ldots, \lambda^{(\ell)}) \) of dominant integral weights. We recall now the third author’s construction of the KLRW algebra \( \tilde{T}^{\lambda} \) (alias tensor product algebra, alias red-black algebra, alias Webster algebra).
The algebra $\hat{T}^\lambda$ is defined as a span of Stendhal diagrams, modulo some local relations. More precisely, a Stendhal diagram [Web17a Definition 4.1] defining an element of $\hat{T}^\lambda$ is a collection of finitely many curves in $\mathbb{R} \times [0,1]$, where each curve is either red and labeled by one of the $\lambda^{(j)}$, or is black and labeled by $i \in I$. Each curve has one endpoint on $\mathbb{R} \times \{0\}$ and one on $\mathbb{R} \times \{1\}$. The black curves can also be decorated with finitely many dots. The diagrams are considered up to isotopy, and must be locally of the form

Note that no red strands can ever cross. In [Web17a] the diagrams are oriented. We only consider downward oriented strands, so we omit the orientation here. The diagrams in $\hat{T}^\lambda$ each have $\ell$ red strands, labeled by $\lambda^{(1)}, ..., \lambda^{(\ell)}$ from left to right. For convenience, we let $y_k$ denote the dot on the $k$th black strand read from left to right. The diagrams satisfy the following relations:

- the KLR relations (3.1a–3.1g) for $\Gamma$ with

$$X_{ij}(u,v) = \begin{cases} 1 & i \not\leftrightarrow j \\ u - v & i \leftrightarrow j \end{cases}$$

$$Q_{ij}(u,v) = X_{ij}(u,v)X_{ji}(v,u) = \begin{cases} 1 & i \not\leftrightarrow j \\ u - v & i \leftrightarrow j \\ v - u & i \rightarrow j \end{cases}$$

(3.1a) \quad \begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram_3.1a.pdf}
\end{array}
\quad \begin{array}{c}
\text{unless } i = j
\end{array}

(3.1b) \quad \begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram_3.1b.pdf}
\end{array}
\quad \begin{array}{c}
\text{unless } i = j
\end{array}

(3.1c) \quad \begin{array}{c}
\includegraphics[width=0.5\textwidth]{diagram_3.1c.pdf}
\end{array}
\quad \begin{array}{c}
\end{array}
(3.1d) \[ \begin{array}{cccc}
  & \bullet & & \\
  & i & & i \\
  & i & + & i
\end{array} \]

(3.1e) \[ \begin{array}{ccc}
  & \bullet & \\
  & i & i \\
  & i & j & = 0 \text{ and } & i & j & = Q_{ij}(y_1, y_2)
\end{array} \]

(3.1f) \[ \begin{array}{ccc}
  & \bullet & \\
  & i & j & k \\
  & i & j & k
\end{array} \]

(3.1g) \[ \begin{array}{ccc}
  & \bullet & \\
  & i & j & i \\
  & i & j & i
\end{array} \]

- All black crossings and dots can pass through red lines. For the latter two relations (3.2b, 3.2c), we also include their mirror images:

(3.2a) \[ \begin{array}{ccc}
  & j & \lambda & i \\
  & j & \lambda & i
\end{array} \]

(3.2b) \[ \begin{array}{ccc}
  & & \ \\
  & & \\
\end{array} \]

(3.2c) \[ \begin{array}{ccc}
  & & \\
  & & \\
\end{array} \]
• The “cost” of separating a red $\lambda$-strand and a black $i$-strand is adding $\lambda_i$ dots to the black strand.

\[
\begin{array}{c}
\lambda \\
\downarrow \\
\lambda_i
\end{array}
= 
\begin{array}{c}
i \\
\downarrow \\
i
\end{array}
\]

\(3.3\)

\[
\begin{array}{c}
i \\
\downarrow \\
i
\end{array}
= 
\begin{array}{c}
\lambda_i \\
\downarrow \\
\lambda
\end{array}
\]

The algebra $\tilde{T}^\lambda$ is graded with degree given by [Web17a, Def. 4.4]:

\[
\deg \begin{array}{c}
i \\
\downarrow \\
\lambda
\end{array} = \langle \alpha_i, \lambda \rangle \\
\deg i = -2 \\
\deg \begin{array}{c}
i \\
\downarrow \\
j
\end{array} = -\langle \alpha_i, \alpha_j \rangle
\]

The identity of $\tilde{T}^\lambda$ is a sum of idempotents corresponding to sequences of red and black strands. More carefully, we associate an idempotent $e(i, \kappa)$ to a sequence $i = (i_1, \ldots, i_n)$, which labels the black strands in order, and a weakly increasing function $\kappa: [1, \ell] \rightarrow [0, n]$ such that the $k$th red strand is between the $\kappa(k)$th and $(\kappa(k) + 1)$th black strands. We interpret $\kappa(k) = 0$ to mean this red strand is left of all black strands, and $\kappa(k) = n$ to mean it is right of all black strands. Recall that in $\tilde{T}^\lambda$ the red strands are always labeled $\lambda^{(1)}, \ldots, \lambda^{(\ell)}$ from left to right, so the pair $(i, \kappa)$ uniquely determines the idempotent.

As in [Web17a, Lemma 4.12], $\tilde{T}^\lambda$ acts on

\[\text{Pol} = \bigoplus_{(i, \kappa)} \text{Pol}(i, \kappa), \quad \text{where} \quad \text{Pol}(i, \kappa) = \mathbb{C}[Y_1(i, \kappa), \ldots, Y_n(i, \kappa)]\]

The idempotent $e(i, \kappa)$ acts by projection onto $\text{Pol}(i, \kappa)$, and

\[
\begin{array}{c}
i \\
\downarrow \\
\lambda
\end{array}
\cdot f = Y_k^{\lambda} \cdot f \\
\begin{array}{c}
i \\
\downarrow \\
\lambda
\end{array} 
\cdot f = f \\
\begin{array}{c}
i \\
\downarrow \\
\lambda
\end{array}
\cdot f = Y_k \cdot f
\]

\(3.4a\)

\[
\begin{array}{c}
\begin{array}{c}
j \\
\downarrow \\
i
\end{array}
\cdot f = \begin{cases} 
X_{ij}(Y_{k+1}, Y_k)f^{s_k} & i \neq j \\
f^{s_k} - f & i = j
\end{cases}
\end{array}
\]

\(3.4b\)

In the equations above, the black $i$-strand is always the $k$th black strand from left to right.
3.2. Some statements about categorification. Set \( \lambda = \sum_j \lambda^{(j)} \). For \( \mu \in \Lambda \), let \( \tilde{T}_\mu^\Lambda \) be the subalgebra of \( \tilde{T}^\Lambda \) where the sum of the roots associated to the black strands is \( \lambda - \mu \). We have a non-unital algebra homomorphism \( \iota_\mu^\pm : \tilde{T}_\mu^\Lambda \to \tilde{T}_{\mu-\alpha_i}^\Lambda \), which adds a black \( i \)-strand to the right of the diagram, and its mirror image \( \iota_{\mu}^- \) adding a strand at the left.

Let \( \tilde{T}_\mu^\Lambda \)-mod be the category of finitely generated \( \tilde{T}_\mu^\Lambda \)-modules. Consider the functors

\[
\begin{align*}
F_i^+: \tilde{T}_\mu^\Lambda \text{-mod} & \to \tilde{T}_{\mu-\alpha_i}^\Lambda \text{-mod} \\
F_i^-: \tilde{T}_\mu^\Lambda \text{-mod} & \to \tilde{T}_{\mu-\alpha_i}^\Lambda \text{-mod}
\end{align*}
\]

given by induction along the maps \( \iota_\mu^\pm \). More specifically, \( F_i^+(M) = M \otimes_{\tilde{T}_\mu^\Lambda} \tilde{T}_{\mu-\alpha_i}^\Lambda \). Let \( \mathcal{E}_i^\pm \) be the adjoint functors of \( F_i^\pm \) given by restriction along the maps \( \iota_\mu^\pm \). Note that setting \( e_i^\pm_i \) (respectively \( e_i^-_i \)) to be the sum of idempotents whose rightmost (respectively leftmost) strand is a black \( i \)-strand, then \( \mathcal{E}_i^+(M) = e_i^+M \) and \( \mathcal{E}_i^-(M) = e_i^-M \).

Let \( K^C(\tilde{T}^\Lambda) \) denote the Grothendieck group of finitely generated \( \tilde{T}^\Lambda \)-modules, and set \( K^C(-) := K^C(-) \otimes_{\mathbb{Z}} \mathbb{C} \). Let \( V(\lambda) = V(\lambda^{(1)}) \otimes \cdots \otimes V(\lambda^{(\ell)}) \). By [MW18, Prop. 2.18], we have:

**Proposition 3.1.** We have an isomorphism of \( (U(n_-), U(n_-)) \)-bimodules

\[
\mathcal{G}: U(\lambda) := U(n_-) \otimes V(\lambda) \to K^C(\tilde{T}^\Lambda)
\]

which maps \( K^C(\tilde{T}^\Lambda) \) to the \( \mu \)-weight space of \( U(\lambda) \). Under this isomorphism, the functors \( F_i^+, F_i^- \) categorify the actions:

\[
\begin{align*}
f_i^+ \cdot (u \otimes v_1 \otimes \cdots \otimes v_\ell) &= f_i u \otimes v_1 \otimes \cdots \otimes v_\ell \\
&\quad + u \otimes f_i v_1 \otimes \cdots \otimes v_\ell + \cdots + u \otimes v_1 \otimes \cdots \otimes f_i v_\ell \\
(u \otimes v_1 \otimes \cdots \otimes v_\ell) \cdot f_i^- &= u f_i v_1 \otimes \cdots \otimes v_\ell
\end{align*}
\]

This isomorphism is uniquely fixed by the action property above, and the fact that the functor \( I_\lambda \) induced by adding a red strand at the far right (i.e. like \( F_i^+ \), but with a different color) is induced by

\[
I_\lambda(u \otimes v_1 \otimes \cdots \otimes v_\ell) \mapsto u \otimes v_1 \otimes \cdots \otimes v_\ell \otimes v_{\lambda,\text{high}}
\]

where \( v_{\lambda,\text{high}} \) is the highest weight vector of \( V(\lambda) \). We let \( \tilde{p}_i^* = [\tilde{T}_e(i, \kappa)] \), following the notation of [Web17a, §4.7]; this is easily calculated from the rules above.

**Remark 3.2.** The association of \( F_i^+ \) with the left action and \( F_i^- \) with the right here might seem a little peculiar, since they involve adding strands on the right and left sides of the diagram. This is due to the dyslexia of the conventions in [Web17a]: unfortunately, this is a sin of one of authors for which all of us, readers included, must do penance.

The algebra \( \tilde{T}^\Lambda \) has a natural triple of quotients; we let \( \iota^\Lambda \tilde{T}^\Lambda \) be the quotient by the two-sided ideal generated by all idempotents where \( \kappa(1) > 0 \) (i.e. a black strand is to the left of all red strands), \( \iota^\Lambda \tilde{T}^\Lambda \) be the quotient by the two-sided ideal generated by all idempotents where \( \kappa(\ell) < n \) (i.e. a black strand is to the right of all red strands), and \( \iota^\Lambda \tilde{T}^\Lambda \) the quotient by the sum of these ideals.
Note that \( \mathcal{F}_i^+ \) descends to a well-defined functor on \( + T^\lambda \)-mod and \( \mathcal{F}_i^- \) to one on \( - T^\lambda \)-mod, still both defined by extension of scalars along an inclusion of algebras. Since the quotients \( + T^\lambda \) and \( - T^\lambda \) are finite dimensional, the adjoint functors \( \mathcal{E}_i^+ \) and \( \mathcal{E}_i^- \) preserve finitely generated modules, and less obviously, they preserve the category of projective modules (this follows from the Morita equivalence of \([\text{Web}17a, \text{Th. 4.30}]\) by allowing us to rewrite it as induction to a Morita equivalent algebra).

These quotient algebras correspond to killing the action of the augmentation ideal of \( U(n_-) \) for the left action, right action, or both.

**Proposition 3.3.** We have isomorphisms of Grothendieck groups

\[
(3.5a) \quad K^C( + T^\lambda) \cong U(\lambda)/n_+ U(\lambda) \cong V(\lambda)
\]

\[
(3.5b) \quad K^C( - T^\lambda) \cong U(\lambda)/U(\lambda)n_- \cong V(\lambda')
\]

\[
(3.5c) \quad K^C(0 T^\lambda) \cong U(\lambda)/(U(\lambda)n_- + n_- U(\lambda)) \cong V(\lambda)/n_- V(\lambda) \cong (V(\lambda))^n
\]

In \((3.5a)\), \( X \) denotes the list of dominant weight in reverse order \((\lambda^{(1)}, \ldots, \lambda^{(1)})\). The action of the functors \( \mathcal{E}_i^\pm, \mathcal{F}_i^\pm \) on \( + T \)-mod categorify the action of the Chevalley generators \( e_i, f_i \) on \( V(\lambda) \).

**Proof.** The isomorphism \((3.5a)\) is a consequence of \([\text{Web}17a, \text{Th. B}]\). Note that the relations \((3.1a), (3.3)\) are preserved by reflecting in a vertical axis and negating crossings \( \psi_i \) of strands with the same labels. This gives an isomorphism of \( - T^\lambda \) to \( + T^\lambda \)-mod with the order of labels on red strands reversed. Thus \((3.5b)\) follows.

The isomorphism \((3.5c)\) is not explicitly discussed in \([\text{Web}17a]\), but is easily derived from the definitions there: the modules over \( 0 T^\lambda \) are the same as those over \( + T^\lambda \) killed by all \( \mathcal{E}_i^+ \) for all \( i \), and the number of simple modules killed by these functors is the same as the number of highest weight elements of the corresponding tensor product crystal by \([\text{LW}15, \text{Th. 7.2}]\), so the classes of these simples span the space of highest weight vectors. The description of the highest weight vectors as the quotient \( V/n_- V \) is valid for any finite dimensional \( g \)-module \( V \). \(\square\)

Let \( \mathcal{U} \) be the 2-Kac-Moody algebra categorifying \( U(g) \), as defined by Khovanov-Lauda \([\text{KL}10]\) and Rouquier \([\text{Rou}]\). For details see Definition 2.4 in \([\text{Web}17a]\). Note that in the definition we use the same matrix of polynomials \( Q_{i,j}(u, v) \) which appears above.

**Proposition 3.4 (\([\text{Web}17a, \text{Th. B}]\)).** The functors \( \mathcal{E}_i^\pm, \mathcal{F}_i^\pm \) induce an action of \( \mathcal{U} \) on the categories \( + T^\lambda \)-mod, which each categorify \( V(\lambda) \) and \( V(\lambda') \).

If we instead consider the graded Grothendieck group of \( \tilde{T}, + T \), we will arrive at the obvious quantum analogues of the objects under consideration. This is significant because these quantum analogues allow us to define a canonical basis of the spaces \( U(\lambda) \) and \( V(\lambda) \), and when the conventions are chosen correctly, the isomorphisms of Propositions \((3.1), (3.3)\) send the classes of indecomposable projectives bijectively to canonical basis by the argument of \([\text{Web}15, \text{Th. 8.7}]\); note that the statement of loc. cit. only covers \( V(\lambda) \), but that is derived as a consequence of the fact that it holds for \( T^\lambda \) and \( U(\lambda) \).

We can convert this into a more concrete statement about multiplicities with a little algebra:
Lemma 3.5. The dimension $m_{i, \kappa, b} = \dim e(i, \kappa)L_b$ of the image of the idempotent $e(i, \kappa)$ in the unique simple quotient of the indecomposable projective $P_b$ such that $[P_b] = b$ for $b$ a canonical basis vector is the coefficient of $b$ when $[\hat{T}e(i, \kappa)]$ is expanded in the canonical basis. That is:

$$p^b_i = [\hat{T}e(i, \kappa)] = \sum_b m_{i, \kappa, b} \cdot b.$$

Proof. Of course, $m_{i, \kappa, b} = \dim \text{Hom}(\hat{T}e(i, \kappa), L_b)$. Since $\dim \text{Hom}(P_{b'}, L_b) = \delta_{b, b'}$, the result follows. \hfill \Box

Unfortunately, some real care is needed here about conventions. It might seem strange that in (3.5a) and (3.5b) we used opposite orders of the tensor product, since representations of $g$ are a symmetric tensor category and thus $V(\lambda)$ and $V(\lambda')$ are canonically isomorphic. However, this is no longer true after quantization.

This manifests on the level of categorification: reflecting in a vertical axis (and inserting a few signs) instead gives an isomorphism $+T^\lambda \cong -T^{\lambda'}$. In fact, by [LW15, Th. 3.12], this implies that $+T^\lambda \text{-mod}$ is a tensor product categorification (in the sense of loc. cit.) for $V(\lambda)$ while $-T^\lambda \text{-mod}$ is for $V(\lambda')$.

3.3. A KLRW algebra associated to $R$. Fix a dominant integral weight $\lambda$ and write $\lambda = \sum_i \lambda_i \varpi_i$. We fix also an integral set of parameters $R = \{R_i\}_{i \in I}$ of weight $\lambda$. It will be convenient at times to encode the parameters using multiplicity functions $\rho_i : \mathbb{Z} \to \mathbb{N}$ defined by

$$\rho_i(q) = \begin{cases} \text{multiplicity of } 2q \text{ in } R_i, & \text{if } i \text{ is even,} \\ \text{multiplicity of } 2q + 1 \text{ in } R_i, & \text{if } i \text{ is odd.} \end{cases}$$

Recalling that we’ve fixed a total order on $I$ (cf. Section 2.1), let $I = \{i_1 < \cdots < i_r\}$. Define the following sequences of fundamental weights:

$$\Xi_R(q) := (\Xi_{i_1}^{\rho_{i_1}(q)}, \ldots, \Xi_{i_r}^{\rho_{i_r}(q)})$$

where $\Xi_i^k$ denotes the sequence $\varpi_i, \ldots, \varpi_i$ of length $k$. Concatenate these together to form

$$\Xi_R := (\ldots, \Xi_R(q-1), \Xi_R(q), \ldots)$$

Since $R$ is finite, clearly $\Xi_R$ is a finite sequence of fundamental weights:

$$\Xi_R = (\Xi_{j_1}, \ldots, \Xi_{j_{\ell}})$$

for some $j_1, \ldots, j_{\ell} \in I$. We let

$$\hat{T} = \hat{T}^R := \hat{T}^{\Xi_R}$$

Of course $\hat{T}$ depends on $R$, but we fix this choice and usually suppress this from our notation. We warn the reader that $\hat{T}$ here is different from the algebra with same name in [Web17a].

Note that in $\hat{T}$ all diagrams have red strands labeled by $j_1, \ldots, j_{\ell}$ from left to right. Let $r_1, \ldots, r_{\ell}$ be the weakly increasing ordering of the elements of $R$ (with multiplicity). We say $r_k$ is the longitude of $j_k$. 
Example 3.6. Let \( g = \mathfrak{s}_4 \), and \( \lambda = \varpi_1 + 2\varpi_2 + 2\varpi_3 \). We order the nodes of the Dynkin diagram by \( 2 < 1 < 3 \). Consider an integral set of parameters \( R_1 = \{1\}, R_2 = \{0,4\} \), and \( R_3 = \{-1,1\} \). Then
\[
\varpi_R = (\varpi_3, \varpi_2, \varpi_1, \varpi_3, \varpi_2).
\]
Now take \( i = (i_1, i_2, i_3) \) and \( \kappa : 1 \mapsto 0, 2 \mapsto 1, 3 \mapsto 1, 4 \mapsto 3, 5 \mapsto 3 \). Then
\[
e(i, \kappa) = \begin{bmatrix}
2 & 1 & 3 \\
1 & 2 & 1 \\
i_1 & i_2 & i_3 \\
2 & 3 & 2
\end{bmatrix}
\]

The longitudes of the red strands are \(-1,0,1,1,4\) from left to right.

3.4. The parity KLRW algebra. We now introduce a subalgebra of \( \tilde{T} \) whose representation theory describes the weight modules over the truncated shifted Yangian. This will allow us to deduce some theorems about the highest weight theory of the latter algebra (cf. Theorem 3.21).

Consider an idempotent \( e \in \tilde{T} \). Since a strand of \( e \) (of either color) is labeled by an element of \( I \), we can talk about its parity. We will define the parity distance \( \delta \) between the strands of \( e \) as follows. First for consecutive strands \( p, p' \) we set:
\[
\delta(p, p') = \begin{cases}
2 & \text{if } p \text{ and } p' \text{ have the same parity, } p \text{ is black and } p' \text{ is red} \\
1 & \text{if } p \text{ and } p' \text{ have different parity} \\
0 & \text{otherwise}
\end{cases}
\]

We then extend to all pairs of strands by sharp triangle inequality \( \delta(p, p') = \delta(p, p'') + \delta(p'', p') \), where \( p \) is left of \( p' \) and \( p'' \) is any strand between \( p \) and \( p' \).

Definition 3.7. We say an idempotent \( e \in \tilde{T} \) is parity, if for any pair of red strands \( p, p' \) of \( e \), we have \( \delta(p, p') \leq |r - r'| \), where \( r \) is the longitude of \( p \) and \( r' \) the longitude of \( p' \). Let \( e_R \) be the sum of all parity idempotents. We call an indecomposable projective \( \tilde{T} \)-module \( Q \) and its corresponding canonical basis vector parity if it is a summand of \( \tilde{T}e_R \).

The parity KLRW algebra is the subalgebra \( \tilde{P} = \tilde{P}^R = e_R \tilde{T}e_R \), and \( \tilde{P}_\mu = \tilde{P} \cap \tilde{T}_\mu \).

Of course, the Grothendieck groups \( K^0(\tilde{P} \text{-mod}), K^0(\pm P \text{-mod}) \) are naturally isomorphic to the span of parity canonical basis vectors in \( U(\lambda), V(\lambda), V(\lambda') \) as appropriate; we denote these subspaces by \( U(R), V(R), V(R') \).

Note that these algebras will be Morita equivalent if and only if \( \tilde{T} = \tilde{T}e_R \tilde{T} \), which will hold if every simple module of \( \tilde{T} \) is not killed by \( e_R \) (and thus, does not factor through \( \tilde{T}/\tilde{T}e_R \tilde{T} \)). As for \( \tilde{T} \), the algebra \( \tilde{P} \) has a natural triple of quotients: we let \( \pm P_R \) be the quotient by the two-sided ideal generated by all idempotents where \( \kappa(1) > 0 \), \( _{-} P_R \) be the quotient by the two-sided ideal generated by all idempotents where \( \kappa(\ell) < n \), and \( qP_R \) the quotient by the sum of these ideals. If \( R \) is clear from context we usually omit it. Similarly, we also have algebras \( \pm P_\mu \).

Example 3.8. If \( |r - r'| \gg n + \ell \) for all \( r, r' \in R \), then all idempotents with at most \( n \) black strands are parity. In particular, if \( \text{ht}(\lambda - \mu) \leq n \) then \( \tilde{P}_\mu = \tilde{T}_\mu \).

On the other hand, if all red strands have the same parity, and we have \( r_1 = \cdots = r_\ell \), then for an idempotent \( e(i, \kappa) \) to be parity we must have \( \kappa(1) = \cdots = \kappa(\ell) \in \{0, n\} \).
Example 3.9. Suppose $\lambda = N\varpi_i$ for some $i \in I$. The set of parameters is a single multiset $R = \{c_1^1, \ldots, c_q^q\}$, where $c_1 < \cdots < c_q$, $c_i$ occurs $t_i$ times in $R$, and $\sum t_i = N$.

The parity algebra $\tilde{P}$ is isomorphic to the KLRW $\tilde{T}^\lambda$, where $\lambda = (t_1\varpi_i, \ldots, t_q\varpi_i)$. Indeed, the parity idempotents in this case are all of the form:

$$
\begin{array}{c}
\cdots \\
\overline{t_1} \\
\cdots \\
\overline{t_q} \\
\cdots
\end{array}
$$

Here the red strands are grouped into $q$ “packets”, and the black strands can appear only in between the packets.

Under the isomorphism between $\tilde{P}$ and $\tilde{T}^\lambda$, this idempotent maps to the one in $\tilde{T}^\lambda$ where the $k$-th packet of red strands is collapsed to a single red strand labeled by $t_k\varpi_i$. Note that in particular when $g = \mathfrak{sl}_2$, the parity algebra $\tilde{P}$ is isomorphic to the KLRW algebra.

Example 3.10. Consider the case where $g = \mathfrak{sl}_3$ and $R_1 = \{1\}$ and $R_2 = \{2\}$. In this case, the idempotent $1 1 2 \ 2$ is not parity, since the parity distance between 1 and 2 is 3, but $|r_1 - r_2| = 1$. On the other hand $2 1 1 2$ is parity, since the parity distance between the two red strands is 1. (Here, and in Example 3.15 below, we are using the obvious shorthand for idempotent diagrams: each number corresponds to a strand in the same order, the color of a number corresponds to the color of the strand, and the number itself is the label on the strand.)

We remark that the algebra $\tilde{T}$ has a one dimensional representation where $e(1, 1, 2, 2)$ acts by the identity, and every other homogeneous element of the algebra acts by 1. This simple is killed by every parity idempotent, which shows that $\tilde{P}$ and $\tilde{T}$ are not Morita equivalent in this case.

Lemma 3.11. The categorical actions on $\pm T$–mod of Proposition 3.4 are inherited by the subcategories $\pm P$–mod.

Proof. The maps $\iota^\pm$ add black strands that are not between any two red strands. Thus, they do not change the parity distance between any two reds, and they send parity idempotents to parity idempotents.

Thus, if we have a $\pm T$-module $M$ which is killed by all parity idempotents, then the action of a parity idempotent $e$ on $E_iM$ is given by that of the parity idempotent $\iota^+(e)$, which is still parity and thus acts by zero.

Every element of $F_iM$ lies in the span of $x \otimes m$ where $m \in M$, and $x$ is a straight-line diagram pulling a black strand with label $i$ from the far right at $y = 0$ to some position at $y = 1$, leaving all other strands in place. If the top of $x$ is a parity idempotent, then the bottom is as well (since we can only decrease parity distance by moving a strand to the far right), and thus $x \otimes m = 0$. Thus, $F_iM$ is killed by parity idempotents.

Since $\pm P$–mod is the quotient of $\pm T$–mod by elements killed by parity idempotents, the result follows. \qed

This immediately implies:
**Corollary 3.12.** The subspace $U(R)$ is a $(U(n_-, U(n_-))$-subbimodule, and $V(R), V(R')$ are $U(g)$-submodules.

Note, these spaces can be naturally quantized to give based modules in the sense of Lusztig; we will not discuss this in any detail, since we will not use anything other than the existence of the canonical basis.

Let $\text{Irr}(\_ P)$ denote the set of isomorphism classes of simple $\_ P$-modules. This has a crystal structure with the Kashiwara operators defined by

$$\hat{e}_i L = \text{hd}(E_i^- L) \quad \hat{f}_i L = \text{hd}(F_i^- L).$$

This is a subcrystal of $\text{Irr}(\_ T^\lambda)$ which is isomorphic to the corresponding tensor product crystal by [LW15, Thm. 7.2].

Similarly, the set $\text{Irr}(\_ P')$ of nilpotent simple modules over $\_ P'$ carries a bicrystal structure defined by

$$\hat{f}_i L = \text{hd}(F_i^- L) \quad \hat{f}_i^* L = \text{hd}(F_i^+ L).$$

We need to impose the nilpotent condition since $\_ P'$ is finite over its center, which is a polynomial ring; thus, for any maximal ideal of the center, there are simples killed by that maximal ideal, with the nilpotent simples corresponding to unique graded maximal ideal. These are also the simple modules which will be relevant to the representation theory of the Yangian via Theorem 5.2.

Note that many other sources on KLR algebras, such as [KL09], have typically studied graded modules instead. However, nilpotent simple modules are in bijection with graded simple modules considered up to grading shift:

**Lemma 3.13.** A simple module over $\_ P$ is nilpotent if and only if it is gradeable. Every indecomposable projective $\_ P$-module has a unique nilpotent simple quotient.

**Proof.** Since $\_ P$ is finitely generated over a commutative subalgebra, its simple modules must be finite dimensional, so any element of non-zero degree must be nilpotent on any simple.

On the other hand, the subalgebra $A$ of polynomials in the dots which is symmetric under permutations of strands and their labels is a central and $\_ P$ is finitely generated over $A$. As usual, any central subalgebra acts semi-simply on any simple module. Thus, a simple is nilpotent if it factors through the quotient by the unique graded maximal ideal $m$ of this central subalgebra. The result then follows from the fact that simple modules over finite dimensional graded algebras are always gradeable uniquely up to grading shift.

Similarly, if we have an indecomposable projective $Q$ over $\_ P$, then $Q/mQ$ is a projective module over $\_ P/m\_ P$. Let $L$ be a simple module in the cosocle of $Q/mQ$. As discussed above, $L$ is gradeable, and since $\_ P$ has finite dimensional degree 0 part, $L$ has a unique graded projective cover $Q'$. The projective property induces a surjective map $Q \to Q'$, which must be an isomorphism by the indecomposability of $Q$. Thus, $Q/mQ = Q'/mQ'$ is the projective cover of $L$ as a $\_ P/m\_ P$-module, and so $Q$ has no other nilpotent simple modules as quotients. 

$\square$
3.5. **A crystal isomorphism.** In this section we’ll show that the crystals \( \text{Irr}(\_P) \) and \( \mathcal{B}(\mathbf{R}) \) are isomorphic.

Presently, we describe how to construct an idempotent \( e(S) \in \_P \) from a monomial \( a_{\mathbf{R}} b_{\mathbf{S}}^{-1} \in \mathcal{B}(\mathbf{R}) \). First, we order the elements of \( S \) so that \( s_1 \leq \cdots \leq s_n \). Then, we define a sequence \( i = (i_1, \ldots, i_n) \) by \( s_m \in S_{i_m} \) for all \( m \). (Note if elements occur in \( S \) with multiplicity, there is not a unique such \( i \), but this choice won’t affect the isomorphism class of \( e(S) \).) We call \( s_m \) the **longitude** of \( i_m \). (Recall that we’ve already defined the longitudes of red strands in Section 3.3.)

The idempotent \( e(S) \) is given by interlacing according to longitude the \( \ell \) red strands with the \( n \) black strands. Of course the red strands are labeled \( j_1, \ldots, j_{\ell} \) from left to right, while the black strands are labeled by \( i_1, \ldots, i_n \) from left to right. If the longitude of a red strand agrees with the longitude of a black one, then the red strand goes to the left of the black one. As mentioned above, we also have to make a choice of the order of black strands with the same longitude, but reordering these gives idempotents which are isomorphic by the obvious straight-line diagram.

Going in the other direction, given an idempotent \( e \in \_P \) we construct \( S \) as follows: for every black \( i \)-strand we add an element to \( S \) equal to:

\[
(\text{longitude of closest red strand to its right}) - (\text{parity distance between these strands})
\]

Note that such a red strand exists since we are working in \( \_P \). This construction gives the unique \( S \) such that \( e(S) = e \) with \( x(S) \) is maximal, where:

**Definition 3.14.** For \( a_{\mathbf{R}} b_{\mathbf{S}}^{-1} \in \mathcal{B}(\mathbf{R}) \) set \( x(S) = \sum_{s \in S} s \).

**Example 3.15.** Consider \( \mathfrak{g} = \mathfrak{sl}_3 \) and \( \lambda = \varpi_1 + \varpi_2 \). If we take \( R_1 = \{-1\} \) and \( R_2 = \{2\} \), the elements of the product monomial crystal are
The corresponding idempotents are

\[
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
1 \\
2
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
1 \\
2
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
1 \\
2
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
1 \\
2
\end{array}
\end{array}
\quad \begin{array}{c}
\begin{array}{c}
1 \\
2 \\
1 \\
2
\end{array}
\end{array}
\end{array}
\]

Note that all the idempotents appearing here are parity. This is a special case of Theorem 3.16 below.

Given a simple \(_P\)-module \(L\), there is an \(S\) with \(x(S)\) maximal such that \(e(S)L \neq 0\). We call this the **highest weight** of \(L\). It’s not obvious the highest weight is unique, but we will show this below.

**Theorem 3.16.** Every simple \(_P\)-module has a unique highest weight. This induces a crystal isomorphism \(\varphi : \text{Irr}(_P) \to \mathcal{B}(\mathcal{R})\). This bijection sends the simple modules that factor through \(_P\) to the highest weight elements of \(\mathcal{B}(\mathcal{R})\).

We will prove this theorem after developing the representation theory of \(_P\) a bit further.

**Definition 3.17.** Given an idempotent \(e(S)\), let \(\Delta(S)\) be the quotient of \(_Pe(S)\) by

\[
\sum _{-}Pe(S')_Pe(S)
\]

where the sum is over \(S'\) such that \(x(S') < x(S)\).

**Lemma 3.18.**

1. Suppose that \(\Delta(S)\) is nonzero. Then it has a unique simple quotient \(L(S)\) which has \(S\) as its unique highest weight.

2. There is an injective map \(\varphi : \text{Irr}(_P) \to \mathcal{B}(\mathcal{R})\), given by \(L(S) \mapsto a_{RbS}^{-1}\).

**Proof.** First we prove (1). Assume that \(x(S') = x(S)\). We will first show that \(e(S')\Delta(S)\) is zero unless \(S' = S\). Indeed, suppose \(S' \neq S\) and we have a nonzero diagram \(D \in e(S')\Delta(S)\). Note that attached to a black strand \(s\) in \(D\) there are two pieces of additional data, namely a top longitude coming from \(S'\), and a bottom longitude coming from \(S\). We’ll denote these \(\text{top}(s)\) and \(\text{bot}(s)\) respectively.

Since \(S' \neq S\) there is some black strand \(s\) in \(D\) such that \(\text{bot}(s) > \text{top}(s)\). Suppose this strand is labeled \(i\). Define \(S''\) by

\[
S''_j = \begin{cases} 
S_i \setminus \{\text{bot}(s)\} \cup \{\text{top}(s)\} & \text{if } j = i \\
S_j & \text{otherwise}
\end{cases}
\]
Let $D'' \in e(S'')_P e(S)$ be the diagram equal to $e(S)$ except for one black $i$-strand whose bottom longitude is $\text{bot}(s)$ and top longitude is $\text{top}(s)$. Then there is a diagram $D' \in e(S')_P e(S'')$ such that in $e(S')_P e(S)$ we have

$$D = D'D'' + \text{(diagrams with fewer crossings)}$$

Since $x(S'') < x(S)$, we have that $D'D'' = 0$ in $\Delta(S)$. Therefore, by induction, we may now assume that no strand in $D$ has bigger bottom longitude than top longitude. This is only possible if $S = S'$, contradicting our assumption. This shows that $e(S')\Delta(S) = 0$.

Note that $e(S)\Delta(S)$ is a quotient of the subalgebra $A$ in $e(S')\tilde{P} e(S)$ where all strands preserve their longitude. The algebra $A$ is isomorphic to a tensor product of affine nilHecke algebras. Thus it is Morita equivalent to a tensor product of symmetric polynomials on alphabets corresponding to like-colored strands with a given longitude. The module $e(S)\Delta(S)$ is cyclic over $A$, and thus has a unique simple quotient (since the same is true of a finite dimensional graded quotient of a polynomial ring). A standard argument as in [Web17a, Lem. 5.9] shows that the same is true of $\Delta(S)$.

It remains to show that $L(S)$ has $S$ as its unique highest weight. But this is clear, since $e(S)L(S) \neq 0$, and $e(S')\Delta(S) = 0$ for any $S' \neq S$ such that $x(S') \leq x(S)$. The proof of (2) is now immediate since every simple module is a quotient of some $\Delta(S)$.

**Proposition 3.19.** The map $\varphi : \text{Irr}(\_P) \rightarrow B(R)$ is a morphism of $\mathfrak{g}$-crystals.

**Proof.** Fix $i \in I$ and an idempotent $e(U) \in \_P$ such that $U_i = \emptyset$. Let $e = \sum e(U')$, where the sum is over all idempotents such that $U_j = U_j$ for $j \neq i$.

We wish to relate the simple $\_P$-modules on which this idempotent acts non-trivially to modules over the tensor product algebra for the root $\mathfrak{sl}_2$ subalgebra of $\mathfrak{g}$ corresponding to $i$. We consider a set of parameters for $\mathfrak{sl}_2$ given by $R' = \{R_i \cup \bigcup_{j \neq i} (U_j + 1)\}$; note that since we are working with $\mathfrak{sl}_2$ a set of parameters consists of a single multiset. Let $\_P'$ be the $\mathfrak{sl}_2$ parity algebra associated with $R'$. To remind ourselves which root subalgebra we’re working with, we label the black and red strands with $i$.

Let $A_U$ be the subalgebra of $\tilde{P}$ spanned by diagrams where all red strands and all black strands with labels other than $i$ are straight vertical in the position of $e(U)$. We have a natural map $A_U \rightarrow e_{\_P} e$, which makes $e_{\_P}$ into a right $A_U$-module. We have a “redification” map $g_i : A_U \rightarrow \_P'$ defined on diagrams as follows:

- Black and red $i$-strands remain in their positions.
- For $j \sim i$, a dotless black $j$-strand maps to a red $i$-strand.
- A dot on a black $j$-strand is sent to 0.
- Remove any black $j$-strand if $j \not\sim i$ and $j \neq i$.
- For $j \not\sim i$, remove any red $j$-strand.

Note that for a diagram in $A_U$, any crossing must involve a black $i$-strand. Therefore the nilHecke relations [3.1c][3.1e] only involving one color are only relevant for the label $i$, and they are sent to the same relation in $\_P'$. When strands of more than one color are involved, $g_i$ sends the relations [3.1a][3.1b] to [3.2c], the relation [3.1e] to [3.3] and the relations [3.1f][3.1g] to [3.2a][3.2b]. Therefore $g_i$ is indeed a map of algebras.

Thus, given a $\_P'$-module $M$, we can consider the associated module $e_{\_P} \otimes_{A_U} M$. Note that it is quite possible that this tensor product is 0; we are free to ignore these cases.
We define a functor $\mathbb{X} : \text{irr} P' \to \text{irr} P$ as follows: $\mathbb{X}(M)$ is the quotient of $Pe \otimes A_U M$ by the submodule generated the image of all idempotents where the average longitude of the black strands with labels other than $i$ is higher than $x(U)$. The image $e\mathbb{X}(M)$ is naturally an $A_U$-module, and as in the proof of Lemma 3.18 we see that $e\mathbb{X}(M)$ is a quotient of $M$. In particular, if $M$ is simple, then applying [Web17, Lem. 5.9] shows that $\mathbb{X}(M)$ has a unique simple quotient $L(M)$ (if it is non-zero). One can also easily confirm that $L$ commutes with the functors $E_i, F_i$ of the categorical $\mathfrak{sl}_2$-action on both categories. Thus, the induced map on simples commutes with the categorical crystal operators for $e_i$ and $f_i$. In particular, if a simple in the crystal is sent to 0, then its whole component is.

Let $\mathcal{B}(R')$ be the product monomial crystal for $\mathfrak{sl}_2$. Note that we have a crystal map $L : \mathcal{B}(R') \to \mathcal{B}(R)$ given by $a_{R'}b_{T}^{-1} \mapsto a_{R}b_{S,T}^{-1}$. Since the numbers $a_i$ are unchanged by this map, $L$ is a map of $\mathfrak{sl}_2$ crystals. Thus we have a diagram of maps, where the left hand map is dashed since it is only partially defined:

$$\begin{array}{ccc}
\text{irr}(P') & \xrightarrow{\varphi'} & \mathcal{B}(R') \\
\downarrow L & & \downarrow L \\
\text{irr}(P) & \xrightarrow{\varphi} & \mathcal{B}(R)
\end{array}$$

This diagram commutes on objects where $L$ is defined. Both vertical maps are morphisms of $\mathfrak{sl}_2$-crystals, and the maps $\varphi$ and $\varphi'$ are both injective. Furthermore, every element of $\text{irr}(P)$ is in the image of $L$ for some choice of $U$; in fact, this implies that its whole root string under $e_i, f_i$ is.

Thus to show that $\varphi$ is a crystal map, it suffices to prove this for $\varphi'$ for an arbitrary choice of $i$ and $U$. That is, it suffices to prove the result in the case where $g = \mathfrak{sl}_2$. In this case, the monomial crystal structure is simply a tensor product crystal, and $P$ is an $\mathfrak{sl}_2$ tensor product algebra (cf. Example 3.9). The crystals for these match by [LW15, Th. 7.2].

\textbf{Proof of Theorem 3.16.} To complete the proof of the bijection, we need to show that $\Delta(S)$ is non-trivial if and only if $a_{R}b_{S}^{-1}$ lies in $\mathcal{B}(R)$. We’ll prove this by induction on $n = \sum_i |S_i|$.

For $n = 0$ we have that $S = \emptyset$. Since $\Delta(\emptyset) = C e(\emptyset) \neq 0$ and $a_R \in \mathcal{B}(R)$, the claim follows.

Now let $n > 0$. Suppose first that there is an element of $S$ which is less than or equal to every element of $R$. If $\Delta(S)$ is non-trivial then $L(S)$ is not killed by $E_i$, for some $i$. Thus, $L(S) = f_i L(S')$ for some highest weight $S'$. By induction we can assume that $a_{R}b_{S'}^{-1} \in \mathcal{B}(R)$, and hence by Proposition 3.19 $a_{R}b_{S'}^{-1} = f_i(a_{R}b_{S}^{-1})$ is in $\mathcal{B}(R)$ as well. Conversely, if $a_{R}b_{S}^{-1} \in \mathcal{B}(R)$ then $e_i(a_{R}b_{S}^{-1}) \neq 0$, and reversing the argument above we get that $L(S) \neq 0$.

It remains to consider the case where there is $r \in R_i$ strictly less than all elements of $S$. We can assume $r$ corresponds to the leftmost red strand in $e(S)$. Let $R'$ be the
set of parameters of weight \( \lambda - \varpi_i \) obtained from \( R \) by removing \( r \) from \( R_i \). Let \( \_P' \) be the associated algebra. We have a map \( \_P' \to \_P \) given by adding a red \( i \)-strand on the left. Let \( e \in \_P \) be the sum of idempotents whose leftmost strand is red. If \( \Delta(S) \) is non-trivial then \( eL(S) \) is a nontrivial \( \_P' \)-module, which has highest weight \( S \). Therefore some simple composition factor of this module has highest weight \( S \). By induction we then have that \( aRbS^{-1} \in B(R') \). Now note that multiplication by \( a_{i,r} \) defines an injective set map \( \iota: B(R') \to B(R) \). Hence \( aRbS^{-1} \in B(R) \).

Conversely, if \( aRbS^{-1} \in B(R) \) then it’s in the image of \( \iota \). Hence \( aRbS^{-1} \in B(R') \). By induction we then have that the irreducible \( \_P' \)-module \( L'(S) \neq 0 \). By Frobenius reciprocity there exists a nonzero map \( \_P \otimes \_P', L'(S) \to L(S) \), and so \( \Delta(S) \neq 0 \).

This completes the proof that the map \( L(S) \mapsto aRbS^{-1} \) is a crystal isomorphism from the set of isomorphism classes of simple \( \_P \)-modules to \( B(R) \).

To complete the proof of the theorem, assume that a simple \( L(S) \) is a highest weight element of the crystal. This holds if and only if \( E_i L(S) = 0 \) for all \( i \), which holds if and only if \( L(S) \) is killed by all idempotents whose leftmost strand is black. This is, of course, equivalent to factoring through the quotient \( aP \).

Note, this proves an interesting and non-trivial statement: there is a natural embedding of the monomial product crystal in the tensor product crystal for any ordering compatible with the longitudes. This embedding is uniquely characterized by the fact that adding a new element of \( R \) larger than all the longitudes of elements in \( S \) matches tensoring with the highest weight element of a new tensor factor.

We can extend this result to understand the sets of nilpotent simple modules \( \text{Irr}(\tilde{T}^R) \) and \( \text{Irr}(\tilde{P}^R) \) as crystals. In the case where \( R = \emptyset \), we have that \( \tilde{T}^\emptyset \) is the original KLR algebra, and work of Lauda and Vazirani [LV11, Th. 7.4] identifies \( \text{Irr}(\tilde{T}^\emptyset) \) with the crystal \( B(\infty) \).

Recall from [Web17a, Def. 5.4] that we have a partial standardization functor
\[
\mathfrak{s}^{R,\emptyset}: \_T^R \otimes \tilde{T}^\emptyset \text{-mod} \to \tilde{T}^R \text{-mod}.
\]
Note that since we are using \( \_T \), we swap left and right everywhere compared to [Web17a], in particular in the definition of standard modules. This correctly accounts for the fact that we use the opposite convention for crystal tensor product from [LW15, Web17a]. As argued in [Web17a, Th. 5.8], the map
\[
h: \text{Irr}(\_T^R) \times \text{Irr}(\tilde{T}^\emptyset) \to \text{Irr}(\tilde{T}^R) \quad h(L, L') \mapsto \text{hd}(\mathfrak{s}^{R,\emptyset}(L \otimes L'))
\]
is a bijection. As discussed before, [Web17a] deals with graded simple modules, but the same results hold for nilpotent modules by Lemma 3.13.

**Lemma 3.20.** The map \( h \) induces a crystal isomorphism \( \text{Irr}(\tilde{P}^R) \cong B(R) \otimes B(\infty) \).

**Proof.** We can make sense of parity algebras where we allow \( \pm \infty \) as longitudes (for these purposes, \( \infty \) is both even and odd). Note that the parity distance between any two strands with longitude \( \infty \) is 0.

Fix a large integer \( n \), and let \( R_+ \) be obtained from \( R \) by adding \( n \) copies of \( \infty \) to each \( R_i \). Note that we have a surjective map \( \tilde{P}^R \to \_P^{R_+} \) by adding \( n \) strands labeled by each fundamental weight at the far right of the diagram; no black strands are allowed.
between these new red strands, since the parity distance between them is 0. Let \( I \) be the kernel of this map. We have a surjective map
\[
\text{Irr}(\tilde{\mathcal{P}}^R) \to \text{Irr}(\mathcal{P}^{R+}) \sqcup \{0\}
\]
defined by \( L \mapsto L/IL \).

Furthermore, this map is a bijection in any given weight \( \mu \) for sufficiently large \( n \); thus these maps for all \( n \geq 0 \) uniquely fixed the crystal structure on \( \text{Irr}(\tilde{\mathcal{P}}^R) \).

Straightforward modification of [Web17a, Th. 5.8] shows that we have a natural bijection \( B(\mathbb{R}^+) \cong B(\mathbb{R}) \otimes B(n\rho) \) compatible with \( h \), and [LW15, Th. 7.2] shows this is a crystal isomorphism. Taking inverse limit as \( n \to \infty \), we obtain that \( h \) is a crystal map as well. \( \square \)

3.6. Metric KLRW algebras. We’ll now consider a different generalization of the KLRW algebras of [Web17a, §4].

**Definition 3.21.** Consider a sequence \( i \in I^n \) and a function \( \kappa : [1, \ell] \to [0, n] \) fixing the position of the red strands. A (metric) longitude compatible with this data is a sequence \( a = (a_1, \ldots, a_n) \in \mathbb{Z}^n \) such that for all \( k \),

1. \( i_k \) and \( a_k \) have the same parity,
2. \( a_k \leq a_{k+1} \), and
3. \( a_k \geq r_p \) if and only if \( k > \kappa(p) \).

The longitude conditions say roughly that we can isotope the idempotent \( e(i, \kappa) \in \tilde{T} \) to have a black strand of label \( i_k \) at \( x = a_k \), and red strands at \( x = r_p \); condition (2) implies that the black strands are in the correct order, and condition (3) implies that red and black strands are in the correct relative position. Note that there are some border cases that keep this from being strictly true: consecutive black strands can have the same longitude, but cannot have the same \( x \)-value, and similarly, a red strand and the black to its right (but not its left) can have the same longitude, but not the same \( x \)-value.

We let \( \text{Long} \) be the set of possible combinations of data \( (i, \kappa) \) as above and the longitudes \( (a_1, \ldots, a_n) \). Note that the information of \( a \) is enough to uniquely fix the choice of \( \kappa \). We can also speak of the integer this attaches to a single strand as its longitude; the longitude of a red strand is by convention the corresponding element \( r_p \).

**Definition 3.22.** A metric Stendhal diagram is a Stendhal diagram together with a choice of a longitudes \( a_{\text{top}} \) and \( a_{\text{bottom}} \) on the data \( (i_{\text{top}}, \kappa_{\text{top}}) \) and \( (i_{\text{bottom}}, \kappa_{\text{bottom}}) \) at the top and bottom of the diagram.

As in [Web17a, Def. 4.2], we can define a product on the formal span of all metric Stendhal diagrams considered up to isotopy. Note that in taking this product, we must require that longitudes in addition to labels in \( I \) match, or else we take the product to be 0. There are idempotent diagrams in this algebra given by straight line diagrams with the same longitudes at the top and bottom. The isomorphism type of this idempotent only depends on \( i \) and \( \kappa \), since we can label the top and bottom of this diagram with any compatible longitude.
Definition 3.23. The metric KLRW algebra $\tilde{T} = \tilde{T}^R$ consists of finite spans of metric Stendhal diagrams, modulo the local relations (3.1a–3.3). We let $\pm T$ denote its quotients by diagrams that violate on left and right respectively, and as usual $\pm T_\mu \subset \pm T$ denote the subalgebras where the sum of the roots labeling black strands is equal to $\lambda - \mu$.

Example 3.24. Let $I = \{x, y\}$, connected by a single edge $x \rightarrow y$. In this case, $\mathfrak{g} = \mathfrak{sl}_3$, and we take $\lambda = \varpi_x + \varpi_y, R_x = \{-1\}$, and $R_y = \{4\}$. Here is a typical element of the metric KLRW algebra $\tilde{T}^R$.

\[
\begin{array}{ccccc}
x & x & x & y & y \\
-1 & -1 & 3 & 4 & 6 \\
\end{array}
\]

On the top and bottom we depict the string labels and longitudes in two rows. So for instance, the bottom longitudes of the black strands in the above diagram are $-3, 2, 5$. Notice that the longitude of a black strand can change from bottom to top, but the longitudes on the red strands are fixed by the choice of $R$.

Note that a metric structure on the idempotent $e(i, \kappa)$ is precisely the same as a choice of $S$ such that $e(i, \kappa) = e(S)$. We let $d(S) \in \tilde{T}$ be the idempotent whose diagram is the same as $e(S)$, with the (metric) longitudes given by the longitudes of $S$. The key connection between the metric and parity KLR algebras is:

Lemma 3.25. An idempotent in $\tilde{T}$ has a compatible metric longitude if and only if it is a parity idempotent.

Proof. Suppose an idempotent in $\tilde{T}$ has a compatible longitude. Let’s call the difference between the metric longitude of two strands their “metric distance”. We claim that the metric distance is always greater than or equal to parity distance. This will show that the idempotent is parity.

Both distances satisfy the strict triangle identity, so we need only check this for consecutive strands. If two consecutive black strands have the same parity, then the parity distance between them is 0. If they have opposite parity, then their metric longitudes have opposite parity as well, so the metric distance between them is at least 1, and thus greater than or equal to the parity distance. If we have a black strand and then a red of the same parity, then they cannot have the same metric longitude by Definition 3.21(3), so they must differ by at least 2, which is the parity distance. Thus, indeed, the parity distance is a lower bound on the metric distance.

Conversely, given a parity idempotent, we have a corresponding monomial as described in Section 3.5, which gives a choice of compatible metric longitudes. □

We have a natural bimodule between $\tilde{T}$ and $\tilde{T}$ where we only label the top of strands with a compatible longitude, and do not choose one on the bottom. Multiplying on the right by $e_R$, we obtain a bimodule between $\tilde{T}$ and $\tilde{P}$. One can easily confirm that:
Lemma 3.26. The bimodule above induces a Morita equivalence between \( \tilde{T} \) and \( \tilde{P} \), and between \( \pm \tilde{T} \) and \( \pm \tilde{P} \).

Proof. For each parity idempotent, we choose a single compatible longitude, and let \( e' \) be the sum of the corresponding idempotents in \( \tilde{T} \). We obtain an isomorphism \( \tilde{P} \to e' \tilde{T} e' \) by labeling the top and bottom of each diagram with the corresponding longitude.

On the other hand, we have that \( \tilde{T} e' \tilde{T} = \tilde{T} \), since any idempotent \( e \in \tilde{T} \) can be written as product of two idempotents, with labels at the extremes being arbitrary and in the center being the fixed longitudes for \( e \). □

As with the parity algebra, a simple \( _{-T} \)-module \( L \) has highest weight \( S \) if \( x(S) \) is maximal such that \( d(S)L \neq 0 \). By Theorem 3.16 we obtain:

Corollary 3.27. There is a bijection, which (by slight abuse of notation) we denote \( \varphi \), between the simple modules of \( _{-T} \) and the product monomial crystal \( B(\mathbf{R}) \), sending a simple \( L \) to \( a_{\mathbf{R}} b_{S}^{-1} \), where \( S \) is the highest weight of \( L \).

3.7. Coarse metric KLRW algebras. We’ll also want to consider a more general notion of coarse longitudes. These are useful for the combinatorics relating the representation theory of Stendhal algebras to the representations of the KLR Yangian algebra \( \mathcal{A} \) we will introduce in Section 4.4.

We’ll define a preorder on the integers by \( a \succeq b \) if and only if \( \lfloor a^2 \rfloor \geq \lfloor b^2 \rfloor \). This is a coarsening of the standard order, where we also have \( 2q \succeq 2q+1 \) for any integer \( q \). We write \( a \approx b \) if \( a \preceq b \) and \( b \preceq a \) (that is, \( a,b \in \{2p,2p+1\} \) for some integer \( p \)).

Definition 3.28. Consider a sequence \( \mathbf{i} \in I^n \) and a function \( \kappa : [1, \ell] \to [0, n] \) fixing the position of the red strands. A coarse longitude compatible with this data is a sequence \( \mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n \) which satisfies the conditions of a longitude for the order \( \succeq \), that is:

1. \( i_k \) and \( a_k \) have the same parity,
2. \( a_k \preceq a_{k+1} \), and
3. if \( k \leq \kappa(p) \) then \( a_k < r_p \), and if \( k > \kappa(p) \) then \( a_k \succeq r_p \).

We call \( (\mathbf{i}, \kappa, \mathbf{a}) \) a coarse longitude triple.

Note that every longitude is a coarse longitude, but not vice versa. A coarse metric Stendhal diagram is a Stendhal diagram together with a choice of coarse longitude triples on the top and bottom of the diagram. As with the metric KLRW algebra, we define a product on the formal span of coarse metric Stendhal diagrams, requiring that coarse longitudes in addition to labels in \( I \) must match, or else we take the product to be zero. The coarse metric KLRW algebra \( \tilde{T}_L = \tilde{T}_L^R \) consists of finite spans of coarse metric Stendhal diagrams, modulo the local relations (3.1a, 3.3).
For an example of an element of $\tilde{T}_L$ which is not in $\check{T}$, consider the diagram of Example 3.24 but with the bottom longitudes of the black strands changed to $-3, -2, 5$:

\[
\begin{array}{cccc}
\times & \times & x & y \\
-1 & -1 & 3 & 4 \\
\times & \times & y & x \\
-3 & -1 & 2 & 4 \\
\end{array}
\]

3.7.1. Generators. One reason introducing the coarse metric KLRW algebra is that its generators are easier to describe, as compared to the metric KLRW algebra.

We let $e(i, \kappa, a) \in \tilde{T}_L$ be the straight line diagram with $(i, \kappa, a)$ the coarse longitude triple assigned to both the top and bottom of the diagram. We’ll draw these diagrams with the label $i_k$ and longitude $a_k$ written at the start and end of each strand. The element $y_k(i, \kappa, a) \in \tilde{T}_L$ is obtained from $e(i, \kappa, a)$ by placing a dot on the $k$th black strand.

We also have generators which cross strands; due to the presence of coarse longitudes we have to be more careful than usual with these.

The crossing $\psi_k$ of two neighbouring black strands is only defined when $a_k \approx a_{k+1}$. In that case we assign $(i, \kappa, a)$ as the bottom longitude, and $(s_ki, \kappa, s_ka)$ as the top longitude.

The red-black crossings are defined in two cases. In the first case, the rightmost black strand with longitude $a_k \approx 2q$ for fixed $q \in \mathbb{Z}$ increases longitude by 2, and moves rightward by crossing all red strands with longitude $\preceq 2q + 2$. (Of course, there may be no such red strands, in which case this generator doesn’t contain any crossings and just changes the longitude.) All other longitudes remain the same. We denote this generator by $\phi^+_k = \phi^+_k(i, \kappa, a)$, where $(i, \kappa, a)$ is the coarse longitude triple on the bottom of the
The second case is the mirror image - it consists of the diagram where the leftmost strand with longitude \( a_k \approx 2q \) decreases longitude by 2, crossing all red strands with longitude \( \approx 2q \), and all other longitudes remain the same. We denote this generator by \( \phi_k^- = \phi_k^-(i, \kappa, a) \), where \((i, \kappa, a)\) is the coarse longitude triple on the bottom of the diagram.

3.7.2. **Polynomial representation.** Later in the paper, we will need to use the polynomial representation of this algebra, which is a little complicated since we need many sets of variables.

The polynomial representation of \( \tilde{T}_L \) is given by

\[
\text{Pol}_L = \bigoplus_{(i, a, \kappa)} \text{Pol}(i, a, \kappa), \quad \text{where} \quad \text{Pol}(i, a, \kappa) = \mathbb{C}[Y_1(i, a, \kappa), \ldots, Y_n(i, a, \kappa)]
\]

where the sum is over all coarse longitude triples. The action is given by the formulae (3.4a, 3.4b). More precisely, \( x \in e(i', a', \kappa') \tilde{T}_L e(i, a, \kappa) \) defines an operator

\[
x : \text{Pol}(i, a, \kappa) \to \text{Pol}(i', a', \kappa')
\]

which acts by ignoring longitudes, acting by (3.4a, 3.4b), and then replacing the correct longitudes in the variables. Since \( \text{Pol} \) is a faithful \( \tilde{T}_L \)-module for any \( \lambda \), it follows that \( \text{Pol}_L \) is a faithful \( \tilde{T}_L \)-module. In fact, the modules \( \text{Pol}_L \) and \( \text{Pol} \) are identified under the Morita equivalence below.

3.7.3. **An equivalence.** It turns out that (unlike the metric KLRW algebra or parity KLRW algebra) the coarse metric KLRW algebra is always Morita equivalent to a usual KLRW algebra, for a sequence \( \lambda \) which depends on \( R \).
Recall that to $\mathbb{R}$ we associate multiplicity functions $\rho_i : \mathbb{Z} \to \mathbb{N}$ (cf. Section 3.3). For $q \in \mathbb{Z}$ define $\lambda(q) = \sum_i \rho_i(q) \omega_i$, and set $\lambda = (\ldots, \lambda(q-1), \lambda(q), \ldots)$. Note that generically, $\lambda = \omega_{\mathbb{R}}$, but in general it is different.

There is a natural inclusion $\tilde{T}_\lambda \to \tilde{T}$, which splits the red strand labeled $\lambda(q)$ into $\sum_i \rho_i(q)$ red strands with labels from left to right given by the sequence \((3.6)\). Observe that the idempotents in the image of this map will never have a black strand between two red strands

- where both red strands have longitude $2q$, or
- where both red strands have longitude $2q + 1$, or
- where the left red strand has longitude $2q$ and right one has longitude $2q + 1$.

**Proposition 3.29.** The algebras $\tilde{T}_L$ and $\tilde{T}_\lambda$ are Morita equivalent.

**Proof.** As in the proof of Lemma 3.26, our desired result reduces to showing that the idempotents that carry coarse longitudes are precisely those isomorphic to ones in the image of $\tilde{T}_\lambda$.

Given an idempotent with a coarse longitude, the only obstruction to it lying in $\tilde{T}_\lambda$ is if one of the three cases above occurs. First notice that the first one cannot occur since that would not support a coarse longitude. In the latter two cases, there must be a black strand of longitude $2q$ in between the red strands. Crossing this black strand over the right red strand is irrelevant since the strands have opposite parities. Thus, this idempotent is isomorphic to one in $\tilde{T}_\lambda$.

On the other hand, given an idempotent in $\tilde{T}_\lambda$, we’ll show that it carries a coarse longitude. Consider a black strand between two red strands with longitudes $r_p$ and $r_{p+1}$. Note that $|r_p - r_{p+1}| \neq 0$. If $|r_p - r_{p+1}| = 1$ then we must have $r_p = 2q - 1$ and $r_{p+1} = 2q$ for some $q$. In this case we let the longitude of the black strand be whichever element of $\{2q - 2, 2q - 1\}$ has the correct parity. If $|r_p - r_{p+1}| > 1$ then we can just choose a longitude $a$ of the correct parity such that $r_p \leq a \leq r_{p+1}$. Clearly it’s possible to do this for all black strands so that the coarse longitude conditions are met. □

## 4. Truncated shifted Yangians and KLR Yangian algebras

Throughout this section we let $\lambda$ be a dominant weight, $\mu$ a weight with $\lambda \geq \mu$, and let $\mathbb{R}$ be a set of parameters $\mathbb{R}$ of weight $\lambda$.

### 4.1. The shifted Yangian

Following [BFNa, Appendix B], we now define a variation on the shifted Yangian $Y_\mu$, but without the assumption that $\mathfrak{g}$ is finite type.
Definition 4.1. The shifted Yangian \( Y_\mu = Y_\mu(\mathfrak{g}, \mathbb{R}) \) is defined to be the \( \mathbb{C} \)-algebra with generators \( E_i^{(q)}(\mathfrak{g}), F_i^{(q)}(\mathfrak{g}), A_i^{(q)}(\mathfrak{g}) \) for \( i \in I, \ q > 0 \), with relations
\[
\begin{align*}
[A_i^{(p)}, A_j^{(q)}] &= 0, \\
[E_i^{(p)}, F_j^{(q)}] &= 2\delta_{ij} H_i^{(p+q-1)}, \\
[A_i^{(p+1)}, E_j^{(q)}] - [A_i^{(p)}, E_j^{(q+1)}] &= -\delta_{ij} A_i^{(p)} E_j^{(q)}, \\
[A_i^{(p+1)}, F_j^{(q)}] - [A_i^{(p)}, F_j^{(q+1)}] &= \delta_{ij} F_j^{(q)} A_i^{(p)}, \\
[E_i^{(p+1)}, E_j^{(q)}] - [E_i^{(p)}, E_j^{(q+1)}] &= \alpha_i \cdot \alpha_j (E_i^{(p)} E_j^{(q)} + E_j^{(q)} E_i^{(p)}), \\
[F_i^{(p+1)}, F_j^{(q)}] - [F_i^{(p)}, F_j^{(q+1)}] &= -\alpha_i \cdot \alpha_j (F_i^{(p)} F_j^{(q)} + F_j^{(q)} F_i^{(p)}), \\
i \neq j, N = 1 - \alpha_i \cdot \alpha_j &\Rightarrow \text{sym} [E_i^{(p1)}, [E_i^{(p2)}, \ldots [E_i^{(pN)}, E_j^{(q)}] \ldots] = 0, \\
i \neq j, N = 1 - \alpha_i \cdot \alpha_j &\Rightarrow \text{sym} [F_i^{(p1)}, [F_i^{(p2)}, \ldots [F_i^{(pN)}, F_j^{(q)}] \ldots] = 0.
\end{align*}
\]

Here we define elements \( H_i^{(q)}(\mathfrak{g}) \in Y_\mu(\mathbb{R}) \) by the rule
\[
H_i(u) = p_i(u) \prod_{j \sim i} (u - 1)^{m_j} \prod_{j \sim i} A_j(u - 1) A_i(u) A_i(u - 2),
\]
where \( p_i(u) = \prod_{c \in R_i} (u - c) \) and
\[
H_i(u) = u^{\langle \mu, \omega_i \rangle} + \sum_{q > -\langle \mu, \omega_i \rangle} H_i^{(q)} u^{-q}, \quad A_i(u) = 1 + \sum_{p > 0} A_i^{(p)} u^{-p}.
\]

For simplicity we will write \( Y_\mu \), but we note again that this definition depends in general on the choice of \( \mu \) and \( \mathbb{R} \) (and so implicitly on \( \lambda \)), as opposed to the more usual dependence solely on \( \mu \) in [BFNa, Definition B.2]. For this reason, as well as our choice of generators, our definition may seem unfamiliar to the reader. First, let us note:

**Lemma 4.2.** In \( Y_\mu \), there are relations
\[
\begin{align*}
[H_i^{(p+1)}, E_j^{(q)}] - [H_i^{(p)}, E_j^{(q+1)}] &= \alpha_i \cdot \alpha_j (H_i^{(p)} E_j^{(q)} + E_j^{(q)} H_i^{(p)}), \\
[H_i^{(p+1)}, F_j^{(q)}] - [H_i^{(p)}, F_j^{(q+1)}] &= -\alpha_i \cdot \alpha_j (H_i^{(p)} F_j^{(q)} + F_j^{(q)} H_i^{(p)}).
\end{align*}
\]

**Proof.** Similar to part of the proof of [FT17, Theorem 6.6].

The definition of shifted Yangians given in [BFNa, Definition B.2] also makes sense for arbitrary simply-laced \( \mathfrak{g} \). That is, we could take generators \( H_i^{(q)} \) instead of \( A_i^{(p)} \), with the relations from the lemma. Denoting the resulting algebra by \( Y_\mu^{\text{BFN}} \), it follows from the lemma that there is a homomorphism
\[
Y_\mu^{\text{BFN}} \longrightarrow Y_\mu
\]
defined by sending the generators \( E_i^{(p)}, F_i^{(p)}, H_i^{(q)} \) to the same-named elements. In general this map is neither injective nor surjective, because of a discrepancy between respective commutative subalgebras: the \( H_i^{(r)} \) are like simple coroots for \( \mathfrak{g} \), while the \( A_i^{(q)} \) are like (negatives of) fundamental coweights. Since the Cartan matrix need not be
invertible, we cannot necessarily express the fundamental coweights in terms of coroots, and similarly we may not be able to express the $A_i^{(p)}$ in terms of $H_i^{(q)}$. The coroots written na"{i}vely in terms of fundamental coweights may not be linearly independent, and likewise our $H_i^{(q)} \in Y_\mu$ may not be algebraically independent. That said, we believe our definition of $Y_\mu$ is a reasonable one because it makes contact with quantized Coulomb branches in general type, as we will see below.

In finite type our definition is equivalent to the usual one:

**Lemma 4.3.** If $\mathfrak{g}$ is of finite ADE type, then $Y_\mu^{\text{BFN}} \sim Y_\mu$.

**Proof.** This follows from results of [GKLO05]: the equation (4.1) can be inverted to express the generators $A_i^{(p)}$ in terms of the $H_j^{(q)}$, and the relations in Lemma 4.2 imply those from Definition 4.1. □

**Remark 4.4.** Recall that we take $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ to be the derived subalgebra of a Kac-Moody Lie algebra, and that the full Kac-Moody algebra has a larger Cartan. It seems likely that by appropriately incorporating the full Cartan into the definition of $Y_\mu$ (c.f. [GNW17, Definition 2.1]), we could avoid having to choose between generators $A_i^{(p)}$ and $H_i^{(q)}$.

**Remark 4.5.** Our conventions regarding Yangians differ slightly from the literature in another way: simply put, we have taken $\hbar = 2$ instead of the more standard choice of $\hbar = 1$. This is not an essential difference, as is well-known, since these choices are isomorphic via a simple rescaling of the generators: roughly speaking, we map $X_i^{(r)} \mapsto (\frac{1}{2})^r X_i^{(r)}$ for $X = E, H, F$.

4.1.1. **Action on symmetric polynomials.** Since $\lambda \geq \mu$, we may write $\lambda - \mu = \sum_{i} m_i \alpha_i$ with all $m_i \geq 0$. Consider the polynomial ring

$$P = \mathbb{C}[z_{i,j} : i \in I, 1 \leq j \leq m_i]$$

Let $\Sigma = \prod_{i} \Sigma_{m_i}$ be the product of symmetric groups, acting on $P$ by permuting each set of variables. We will be interested in the $\Sigma$-invariant polynomials $P^\Sigma$. Denote

$$W_i(u) = \prod_{r=1}^{m_i} (u - z_{i,r}) \quad \text{and} \quad W_{i,r}(u) = \prod_{\substack{s=1 \atop s \neq r}}^{m_i} (u - z_{i,s}).$$

Let $\beta_{i,r}$ denote the algebra morphism $P \to P$ that shifts the variable $z_{i,r}$ by 2, i.e. $\beta_{i,r}(z_{j,s}) = z_{j,s} + 2\delta_{ij}\delta_{rs}$.

**Theorem 4.6.** There is an action of $Y_\mu$ on $P^\Sigma$, defined by

\begin{align*}
(A_i^{(p)}(u)) & \mapsto z^{-m_i} W_i(u), \\
(E_i^{(p)}(u)) & \mapsto -\sum_{r=1}^{m_i} \frac{\prod_{j=1}^{m_i} W_j(z_{i,r} - 1)}{(u - z_{i,r}) W_{i,r}(z_{i,r})} \beta_{i,r}^{-1}, \\
(F_i^{(p)}(u)) & \mapsto \sum_{r=1}^{m_i} \frac{p_i(z_{i,r} + 2) \prod_{j=1}^{m_i} W_j(z_{i,r} + 1)}{(u - z_{i,r} - 2) W_{i,r}(z_{i,r})} \beta_{i,r}.
\end{align*}
This theorem is inspired by the results of \cite{GKLO05}. As in \cite[Theorem B.15]{BFNa}, we can see that the above formulas define a homomorphism

\begin{equation}
\Phi^\lambda : Y_\mu \to \mathcal{A}
\end{equation}

to a suitable localization $\mathcal{A}$ of the algebra of difference operators on $P$. Thus the above map certainly defines an action of $Y_\mu$ on the field of fractions $\text{Frac}(P)$. To prove the theorem, it remains to show that the action actually preserves $P^\Sigma$. This can be proven geometrically using results comparing $Y_\mu$ to the quantized Coulomb branch algebra, and identifying $P^\Sigma$ with the equivariant cohomology of a point. It can also be proven algebraically, as we do now.

Let us introduce the following notation. Let $R$ be a commutative domain over \(\mathbb{C}\), and consider the polynomial ring $R[z_1, \ldots, z_m]$ as well as its field of fractions $R(z_1, \ldots, z_m)$.

For $1 \leq r \leq m$, we define an algebra endomorphism $\beta_r \in \text{End}_R(R(z_1, \ldots, z_m))$ by $\beta_r(z_s) = z_s + 2r\delta_{r,s}$. For $1 \leq r < m$, we define the divided difference operator $\partial_r \in \text{End}_R(R(z_1, \ldots, z_m))$ by $\partial_r = \frac{1}{z_m+1-z_r}(s_r - 1)$ where $s_r$ denotes the transposition permuting the variables $z_r, z_{r+1}$ and fixing all others.

**Lemma 4.7.** With notation as above, fix a polynomial $c(t) \in R[t]$. Consider the following two endomorphisms of $R(z_1, \ldots, z_m)$:

\begin{align}
(4.4) \quad \sum_{r=1}^m \prod_{1 \leq s \leq m, s \neq r} \frac{c(z_r)}{(z_r - z_s)^\beta_r} \\
(4.5) \quad \partial_{m-1} \cdots \partial_1 c(z_1) \beta_1
\end{align}

Applied to any element of $R(z_1, \ldots, z_m)^\Sigma_m$, these two operators agree. In particular, both preserve $R[z_1, \ldots, z_m]^\Sigma_m$.

**Proof.** We prove the first claim by induction on $m$, for any domain of coefficients $R$. The case $m = 1$ is trivial, so let us assume the claim is true for $m$. In particular, letting $R' = R(z_{m+1})$ we can think of any $f \in R(z_1, \ldots, z_m, z_{m+1})^{\Sigma_{m+1}} \subset R'(z_1, \ldots, z_m)^{\Sigma_m}$.

Then by our inductive assumption,

$$
\partial_m \partial_{m-1} \cdots \partial_1 c(z_1) \beta_1 \cdot f = \partial_m \cdot (\partial_{m-1} \cdots \partial_1 c(z_1) \beta_1 \cdot f)
$$

$$
= \partial_m \cdot \sum_{r=1}^m \prod_{1 \leq s \leq m, s \neq r} \frac{c(z_r)}{(z_r - z_s)^{\beta_r}} f(z_1, \ldots, z_r + 2, \ldots, z_m, z_{m+1})
$$

Using the fact that $f$ is symmetric under $\Sigma_{m+1}$, applying $\partial_m$ to a summand where $1 \leq r < m$ we get

$$
\frac{1}{z_m+1-z_m} \left( \frac{1}{z_r - z_{m+1}} - \frac{1}{z_r - z_m} \right) \prod_{1 \leq s < m, s \neq r} \frac{c(z_r)}{(z_r - z_s)^{\beta_r}} f(z_1, \ldots, z_r + 2, \ldots, z_{m+1})
$$

$$
= \frac{c(z_r)}{\prod_{1 \leq s \leq m+1, s \neq r} (z_r - z_s)} f(z_1, \ldots, z_r + 2, \ldots, z_{m+1})
$$
Next, applying $\partial_m$ to the summand where $r = m$, we get
\[
\frac{c(z_{m+1})}{(z_{m+1} - z_m)} \prod_{1 \leq s < m} (z_{m+1} - z_s) f(z_1, \ldots, z_{m+1} + 2, z_m)
\]
\[
+ \frac{c(z_m)}{(z_m - z_{m+1})} \prod_{1 \leq s < m} (z_m - z_s) f(z_1, \ldots, z_m + 2, z_{m+1})
\]

Putting all summands together, this proves the claim.

For the second claim, note on the one hand that the operator (4.4) preserves $R(z_1, \ldots, z_m)\Sigma_m$, as it is equivariant for the action of $\Sigma_m$. On the other hand, the operator (4.5) preserves $R[z_1, \ldots, z_m]$ since this is true of divided difference operators. Together with the first claim, it follows that these operators map $R[z_1, \ldots, z_m] \Sigma_m$ into $R(z_1, \ldots, z_m)\Sigma_m$.

Proof of Theorem 4.6. Clearly the image of (4.2a) preserves $P^{\Sigma}$, so let us prove that (4.2c) does as well (the case (4.2b) being similar). This follows from Lemma 4.7, applied to
\[
R = \mathbb{C}[z_{j,s} : j \neq i, 1 \leq j \leq m_j]^{\Sigma_m}_{i \neq m_i}
\]
\[
c(t) = (t + 2)^{r-1} p_i(t + 2) \prod_{j < i} W_j(t + 1) \in R[t]
\]

In this case the operator (4.4) from the lemma agrees with the action of $F_i^{(r)}$ from (4.2c) (i.e., the coefficient of $u^{-r}$, under expansion at $u = \infty$).

4.2. The truncated shifted Yangian. We now define the truncated shifted Yangian $Y^\lambda_\mu = Y^\lambda_\mu(R)$. By Theorem 4.6, we have a homomorphism
\[Y_\mu \to \text{End}_C(P^{\Sigma})\]

Definition 4.8. The truncated shifted Yangian $Y^\lambda_\mu = Y^\lambda_\mu(R)$ is the image of $Y_\mu$ in $\text{End}_C(P^{\Sigma})$.

From the definition, $Y^\lambda_\mu$ has a faithful action on $P^{\Sigma}$. Note that by (4.2a), we see that $A_i^{(s)} = 0$ in $Y^\lambda_\mu$ for $s > m_i$. We can identify the image $\mathbb{C}[A_i^{(s)}] \subset Y^\lambda_\mu$ with $P^{\Sigma}$.

When $\mathfrak{g}$ is finite type, the definition of $Y^\lambda_\mu$ first appeared in [KWWY14, Section 4C] in the case when $\mu$ is dominant, and in [BFNa, Appendix B] for all $\mu$. Our present definition of $Y^\lambda_\mu$ is a straightforward generalization to arbitrary simply-laced Kac-Moody type.

Remark 4.9. More precisely, in both [KWWY14, Section 4C], [BFNa, B(viii)], $Y^\lambda_\mu$ is defined as the image $\text{Im}(\Phi^\lambda_\mu) \subset \widetilde{A}$ of the map $\Phi^\lambda_\mu : Y_\mu \to \widetilde{A}$ into an algebra of difference operators, as in (4.2a). But this definition is isomorphic: the action of $Y_\mu$ on $P^{\Sigma}$ factors
through $\text{Im}(\Phi^\lambda_\mu)$, and in fact there is a diagram

\[
\begin{array}{c}
Y_\mu \longrightarrow Y_\mu^\lambda \subset \text{End}_\mathbb{C}(P^\Sigma) \\
\downarrow \quad \quad \quad \quad \quad \quad \quad \downarrow \\
\text{Im}(\Phi^\lambda_\mu)
\end{array}
\]

Indeed, the action of $\text{Im}(\Phi^\lambda_\mu)$ on $P^\Sigma$ is faithful, since difference operators act faithfully on polynomials.

**Remark 4.10.** Yet another variation on the definition of truncated shifted Yangian appears in [KTW+b], where it is defined instead as the quotient

\[
Y_\mu / \langle A_i^{(r)} : i \in I, r > m_i \rangle
\]

This algebra surjects onto $Y_\mu^\lambda$ (as defined in the present paper), and we conjecture that this surjection is an isomorphism, c.f. [KWWY14, Theorem 4.10]. This conjecture is proven in finite type $A$ in [KMWY18].

Our interest in $Y_\mu^\lambda$ comes from deformation quantization. When $\mathfrak{g}$ is finite type there is a filtration on $Y_\mu$, which is defined explicitly in terms of a PBW basis, see [BFNa, Appendix B(i)] or [FKP+18, Section 5.4]. Consider the corresponding quotient filtration on $Y_\mu^\lambda$. When $\mu$ is dominant, in [KWWY14, Theorem 4.8] it was shown that $Y_\mu^\lambda$ quantizes a scheme supported on the Poisson variety $\text{Gr}_\mu^\lambda$, a transverse slice to a spherical Schubert variety in the affine Grassmannian. It was conjectured that $Y_\mu^\lambda$ quantizes precisely $\text{Gr}_\mu^\lambda$, i.e.

\[
\text{gr} Y_\mu^\lambda \cong \mathbb{C}[\text{Gr}_\mu^\lambda]
\]

For $\mathfrak{g}$ finite type and $\mu$ dominant, this conjecture was established by [BFNa, Corollary B.28] using the theory of Coulomb branches.

**Remark 4.11.** When $\mathfrak{g}$ is not finite type, we do not know whether an analogous filtration on $Y_\mu$ exists, although this seems very reasonable. The existence of this filtration in finite type comes from Drinfeld-Gavarini duality (alias quantum duality principle), which ultimately uses the Hopf algebra structure on the Yangian, see [KWWY14, Section 3C], [FT18, Appendix A].

Coulomb branches are developed mathematically in [BFNa], where an affine variety $\mathcal{M}_C = \mathcal{M}_C(H, V)$ called the Coulomb branch, is associated to each pair $(H, V)$ consisting of a reductive group $H$ and a representation $V$ of $H$. Its coordinate ring admits a natural deformation quantization $A_h = A_h(H, V)$ over $\mathbb{C}[\hbar]$; in particular $\mathcal{M}_C(H, V)$ has a Poisson structure. The case of greatest present interest to us is when the pair $(H, V)$ is

\[
H = \prod_{i \in I} \text{GL}(m_i), \quad V = \bigoplus_{i \rightarrow j} \text{Hom}(\mathbb{C}^{m_i}, \mathbb{C}^{m_j}) \oplus \bigoplus_i \text{Hom}(\mathbb{C}^{\lambda_i}, \mathbb{C}^{m_i})
\]

By [BFNa, Theorem 3.10], if $\mathfrak{g}$ is finite type, then for $(H, V)$ as above $\mathcal{M}_C$ is a generalized slice $\overline{W}_\mu^\lambda$. Note that when $\mu$ is dominant there is an isomorphism $\overline{W}_\mu^\lambda \cong \text{Gr}_\mu^\lambda$. 
It is not a priori clear what Coulomb branches have to do with Yangians. However, a connection is established by \cite[Appendix B]{BFNa}: there is a homomorphism of algebras $Y_\mu \rightarrow A_{h=2}$, for any simply-laced Kac-Moody $g$. This map comes from the homomorphism \eqref{homomorphism}, and in particular there is an embedding of algebras $Y_\mu \hookrightarrow A_{h=2}$

When $g$ is finite type, \cite[Appendix B]{BFNa} shows that this map respects filtrations, and that it is an isomorphism of filtered algebras when $\mu$ is dominant. These results have been extended by the fourth author:

Theorem 4.12 (\cite[Theorem A]{Wee19}). Let $g$ be any simply-laced Kac-Moody type. For any $\lambda, \mu$ and $R$:

(a) There is an isomorphism of algebras $Y_\mu^{\lambda} \sim \rightarrow A_{h=2}$

(b) If $g$ is finite ADE type, then the above is an isomorphism of filtered algebras. In particular, $\text{gr} Y_\mu^{\lambda} \cong \mathbb{C}[\mathcal{W}^{\lambda}]$.

Remark 4.13. There is a subtlety which we have neglected, namely the role of the parameters $R$. In terms of Coulomb branches, by working equivariantly for the flavour symmetry group $G_F = \prod_{i \in I} GL_{\lambda_i}$, we can deform $A_h$ to an algebra $A_{h,F}$ over $H^*_{G_F}(pt)$. A choice of parameters $R$ defines a homomorphism $H^*_{G_F \times \mathbb{C} \times (pt)} = \mathbb{C} \otimes \bigotimes_{i \in I} H^*_{GL_{\lambda_i}}(pt) \rightarrow \mathbb{C}$, which sends $h \rightarrow 2$ and specializes $H^*_{GL_{\lambda_i}}(pt) \rightarrow \mathbb{C}$ according to the multiset $R_i$. Then more precisely, Theorem 4.12 states that there is an isomorphism $Y_\mu^{\lambda}(R) \sim \rightarrow A_{h,F} \otimes H^*_{G_F \times \mathbb{C} \times (pt)} \mathbb{C}$, and that this is an isomorphism of filtered algebras in finite ADE type.

4.3. The flag Yangian. We will now study a matrix algebra over $Y_\mu^{\lambda}$. We begin with the following general setup.

Assume we have a commutative complex algebra $A$ and an algebra $B \subset \text{End}_\mathbb{C}(A)$ containing $A$. We consider $\text{End}_\mathbb{C}(A^n)$. We can embed $B \subset \text{End}_\mathbb{C}(A^n)$ by acting on just the first copy of $A$. On the other hand, the matrix algebras $M_n(A) = \text{End}_A(A^n)$ and $M_n(B)$ also embed in $\text{End}_\mathbb{C}(A^n)$. We have the following observation.

Lemma 4.14. With the above setup, the subalgebra of $\text{End}_\mathbb{C}(A^n)$ generated by $B$ and by $M_n(A)$ equals $M_n(B)$.

Now, we recall the following standard results. For each $i \in I, 1 \leq r \leq m_i$, we define the divided difference operator $\partial_{i,r} : P \rightarrow P$ by $\partial_{i,r} = \frac{1}{z_{i,r+1} - z_{i,r}} (s_{i,r} - 1)$ where $s_{i,k}$ denotes the operator permuting the variables $z_{i,r}, z_{i,r+1}$.

Proposition 4.15.

(1) $P$ is a rank $\prod m_i^1$ free module over $P^\Sigma$. 

(2) The algebra $\text{End}_{P^\Sigma} P$ is generated by the operators of multiplication by elements of $P$ and by the divided difference operators $\{\partial_{i,r}\}$.

In particular, $\text{End}_{P^\Sigma} P$ contains the projection operator onto the subspace $P^\Sigma$. This projection is given by

$$e'_m := \prod_i \frac{1}{m_i!} \partial_{i,w_0} \prod_{1 \leq k < l \leq m_i} (z_{i,k} - z_{i,l})$$

where $\partial_{i,w_0}$ denotes the product of the divided difference operators $\partial_{i,r}$ corresponding to the longest element in $\Sigma_{m_i}$. Note that $m = (m_i)_{i \in I}$.

**Definition 4.16.** The flag truncated shifted Yangian $FY^\lambda_\mu = FY^\lambda_\mu(R)$ is defined to be the subalgebra of $\text{End}(P)$ generated by the divided difference operators, multiplication by elements of $P$ and by $Y^\lambda_\mu(R)$.

Thus, applying Lemma 4.14 to $A = P^\Sigma, B = Y^\lambda_\mu$ and $n = m_i!$, we immediately see that $FY^\lambda_\mu$ is a rank $\prod m_i$ matrix algebra over $Y^\lambda_\mu$. Moreover, the module $e'_m FY^\lambda_\mu \cong Y^\lambda_\mu \otimes_{P^\Sigma} P$ induces a Morita equivalence between $FY^\lambda_\mu$ and $Y^\lambda_\mu$.

We can embed $\Sigma$ into $FY^\lambda_\mu$ using the elements $s_{i,k} = (z_{i,k+1} - z_{i,k})\partial_{i,k} + 1$. When we do this, we see that the idempotent $e'_m$ equal the usual idempotent of $\Sigma$. Thus we see that the Morita equivalence between $FY^\lambda_\mu$-modules and $Y^\lambda_\mu$-modules is just given by $M \mapsto M^\Sigma$.

### 4.4. The KLR Yangian algebra

Fix $\lambda$ and an integral set of parameters $R$ of weight $\lambda$. We will now introduce a yet bigger algebra (depending on $R$), which will contain $FY^\lambda_\mu$ as a subalgebra, for all $\mu$.

**Definition 4.17.** A cylindrical KLR diagram is a collection of finitely many black curves in a cylinder $\mathbb{R}/\mathbb{Z} \times [0,1]$. Each curve is labeled with $i \in I$ and decorated with finitely many dots. The diagram must be locally of the form

```
  x
```

with each curve oriented in the negative direction (we don’t depict the orientation). The curves must meet the circles at $y = 0$ and $y = 1$ at distinct points with $x \neq 0$. We consider these up to isotopy preserving the conditions above.

We draw cylindrical KLR diagrams on the plane, using a dashed line at $x = 0$ to depict the gluing of the cylinder. We refer to the dashed line as the “seam” of the cylinder. Here is an example of a cylindrical KLR diagram:
As usual, the product of two such diagrams is defined by stacking them on top of each other if their labels match, and is zero otherwise.

Given a list \( i \in I^m \), we let \( e(i) \) be the cylindrical KLR diagram with vertical strands with these labels in order. We write \( z_k(i) \) for the same diagram where the \( k \)th strand carries a dot, and \( \psi_k(i) \) for the diagram where the \( k \)th and \( k+1 \)st strands cross. (When the labels are clear from the context we will simply write \( z_k \) and \( \psi_k \).) Diagrammatically if \( i = (i_1, \ldots, i_m) \) then \( e(i) \), \( z_k(i) \), and \( \psi_k(i) \) are given by:

\[
\begin{align*}
e(i) = & \quad \begin{array}{c|c|c|c|c|c}
i_1 & \ldots & \ldots & \ldots & i_m \\
\end{array} \\
z_k(i) = & \quad \begin{array}{c|c|c|c|c|c}
i_1 & \ldots & \ldots & \ldots & i_k \\
\end{array} \\
\psi_k(i) = & \quad \begin{array}{c|c|c|c|c|c}
i_1 & \ldots & i_k & i_{k+1} & i_m \\
\end{array}
\end{align*}
\]

We also have seam crossing diagrams \( \sigma_{\pm}(i) \):

\[
\begin{align*}
\sigma_+(i) = & \quad \begin{array}{c}
\ldots \\
i_1 & i_{m-1} & i_m \\
\end{array} \\
\sigma_-(i) = & \quad \begin{array}{c}
\ldots \\
i_1 & i_2 & i_m \\
\end{array}
\end{align*}
\]

Let \( p_{i,+}(u) = p_i(u + 2) \) and let

\[
\begin{align*}
\bar{X}_{ij}(u,v) &= \begin{cases} u - v - 1 & i \leftarrow j \\ 1 & \text{otherwise} \end{cases} \\
\bar{Q}_{ij}(u,v) &= \bar{X}_{ij}(u,v) \bar{X}_{ji}(v,u) = \begin{cases} u - v - 1 & i \leftarrow j \\ v - u - 1 & i \rightarrow j \\ 1 & i \not\leftrightarrow j \end{cases}
\end{align*}
\]

**Remark 4.18.** The -1 in the formula for \( \bar{X}_{ij} \) might look strange to readers used to KLR algebras; in fact, we can get rid of it by a carefully chosen automorphism of the algebra \( \mathcal{R} \) defined below. However, in order to relate \( \mathcal{R} \) to the usual presentation of the Yangian (cf. Theorem 4.22), it is more convenient to use this shifted version.

**Definition 4.19.** The KLR Yangian algebra \( \mathcal{R} = \mathcal{R}(\mathbb{R}^\mathbb{L}) \) is the quotient of the span of cylindrical KLR diagrams by

- the usual KLR algebra relations \( \text{(3.1a–3.1g)} \) for \( I \) using the polynomials \( \bar{Q}_{ij} \); since these relations are local, we can apply them in a disk that avoids \( x = 0 \).
- the following relations around \( x = 0 \):

(4.7a)

\[
\begin{align*}
\begin{array}{c|c|c}
i & -2 & i \\
\end{array} &= \begin{array}{c|c|c}
i & +2 & i \\
\end{array}
\end{align*}
\]

\(^1\)Following the Cyrillic spelling “Янгиан.” This is, of course, pronounced “Ya.”
Because diagrams can only be multiplied if their labels match, we have an obvious decomposition of the algebra according to multiplicities of the simple roots in the label. The diagrams \( e(\mathbf{i}) \) give the idempotents corresponding to this decomposition.

Let

\[
\text{Pol}_R = \bigoplus_{m \geq 1, \mathbf{i} \in I^m} \text{Pol}_i, \quad \text{Pol}_i := \mathbb{C}[Z_1(\mathbf{i}), \ldots, Z_m(\mathbf{i})]
\]

be a direct sum of polynomial rings, one for each list \( \mathbf{i} \).

Let \( \sigma : I^m \to I^m \) be the rotation, so that \( \sigma(i_1, \ldots, i_m) = (i_m, i_1, \ldots, i_{m-1}) \) and extend this to an algebra automorphism of \( \text{Pol}_R \) by

\[
\sigma(Z_k(\mathbf{i})) = \begin{cases} 
Z_{k+1}(\sigma(\mathbf{i})) & k < m \\
Z_1(\sigma(\mathbf{i})) - 2 & k = m.
\end{cases}
\]

Also for any \( 1 \leq k < m \), we write \( s_k : I^m \to I^m \) for the usual transposition, extended to an algebra automorphism of \( \text{Pol}_R \) by \( s_k(Z_r(\mathbf{i})) = Z_{s_k(r)}(s_k(\mathbf{i})) \).

If we have a list \( \mathbf{i} \) with \( i_k = i_{k+1} \), then we define the divided difference operator \( \partial_k : \text{Pol}_i \to \text{Pol}_i \) by \( \partial_k = \frac{1}{Z_{k+1}(\mathbf{i}) - Z_k(\mathbf{i})} (s_k - 1) \). (Note that in this formula \( s_k \) acts on \( \text{Pol}_i \) since \( i_k = i_{k+1} \).)

**Theorem 4.20.** We have a faithful action of \( \mathfrak{g} \) on \( \text{Pol}_R \) sending:

- the idempotent \( e(\mathbf{i}) \) to the projection onto \( \text{Pol}_i \)
- the dot \( z_k(\mathbf{i}) \) to the multiplication operator \( Z_k(\mathbf{i}) \)
- the crossing \( \psi_k(\mathbf{i}) \) to the Demazure operator or multiplication operator, depending on whether \( i_k = i_{k+1} \) or not:

\[
\psi_k(\mathbf{i}) \mapsto \begin{cases} 
\partial_k & i_k = i_{k+1} \\
s_k \circ X_{ij}(Z_k(\mathbf{i}), Z_{k+1}(\mathbf{i})) & i_k \neq i_{k+1}
\end{cases}
\]
• the seam crossings $\sigma_{\pm}(i)$ to the automorphism $\sigma$ or its inverse times a polynomial, depending on the sign:

$$\sigma_{\pm}(i) \mapsto \sigma|_{\text{Pol}_i}$$

$$\sigma_{-}(i) \mapsto \left( p_{i,+}(Z_m(i)) \circ \sigma^{-1} \right) |_{\text{Pol}_i} = \left( \sigma^{-1} \circ p_{i}(Z_1(\sigma(i))) \right) |_{\text{Pol}_i}$$

Proof. The KLR relations hold by comparison with the usual polynomial relation of the KLR algebra (for example, as in [Rou]). The other relations are straightforward to check.

Thus, it remains to check that the representation is injective. Assume that we have an element of the kernel. This is given by a sum of a finite number of cylindrical diagrams, and the strands of each diagram trace out a word in the length 1 and length 0 generators in an extended affine symmetric group. First note that the relations (3.1f–3.1g, 4.7c–4.7d) allow to replace the element for one word with the diagram for the word with a braid relation apply to it, plus diagrams with shorter words.

If one of these words is not reduced, then we can use braid relations until we have two consecutive appearances of a generator. At this point, we can apply the relations (3.1e, 4.7b) to reduce the length. Thus, using the relations of the algebra, we can assume this element of the kernel is a sum of diagrams corresponding to reduced words of different elements of the affine symmetric group, times polynomials in dots. Note that the action of each of such diagram in the representation is the action of the corresponding symmetric group element, times a rational function, plus similar terms of shorter length.

Now consider an element $w$ of maximal length with non-zero coefficient on its diagram. In this case, the image of this element in the representation must be a sum of the usual action of $w$ with non-zero coefficient and shorter elements of the affine symmetric group. Thus, this image is non-zero, contradicting the assumption.

□

Remark 4.21. Let $\text{Pol}_m = \bigoplus_{i \in I} \text{Pol}_i$ so that $\text{Pol}_\mathcal{R} = \bigoplus_{m \geq 0} \text{Pol}_m$. It’s clear from the formulas that $\text{Pol}_m$ is $\mathcal{R}$-invariant for any $m$.

4.5. Relating the KLR Yangian and the flag Yangian algebras. As in the previous section, we fix $\lambda$ and an integral set of parameters $\mathcal{R}$. In addition we fix a weight $\mu \leq \lambda$ and let $m = (m_i)_{i \in I}$. Recall, that we previously fixed a total order on the set $I$ such that all vertices in $I_0$ come before those in $I_1$. Consider the sequence $i_m$ where the nodes of $I$ appear in the above order, with $i$ appearing with multiplicity $m_i$. Let $e(i_m)$ be the corresponding idempotent in $\mathcal{R}$. By the previous theorem, we can view $e(i_m)\mathcal{R}e(i_m)$ as a subalgebra of endomorphisms of $\text{Pol}_{i_m}$. Using the above total order on $I$, we will identify the variables $z_{i,k}$ with the variables $z_1(i_m), \ldots, z_m(i_m)$. In this way, we identify $\text{Pol}_{i_m} = P$.

The algebra $e(i_m)\mathcal{R}e(i_m)$ also naturally contains a tensor product $\bigotimes \text{NH}_{m_i}$ of nil-Hecke algebras. This embeds as the subalgebra where we only permit crossings between pairs of strands which have equal labels (and no crossing of the seam $x = 0$). We may thus identify the element $e'_m \in e(i_m)\mathcal{R}e(i_m)$ from $(4.6)$. 
Theorem 4.22. As subalgebras of $\text{End}_C P$, we have $FY^\lambda_\mu = e(i_m)R(e(i_m))$. Moreover, the elements $E_i^{(r)}, F_i^{(r)} \in Y^\lambda_\mu \subset FY^\lambda_\mu$ correspond to elements of $R$ as follows:

\begin{align}
E_i^{(r)} &= -w_{i,+} (z_{m_1+\ldots+m_i} + 2)^{r-1} e_m' \\
F_i^{(r)} &= (-1)^{\sum_{i\to j} m_j} w_{i,-} (z_{m_1+\ldots+m_i} + 2)^{r-1} e_m' 
\end{align}

Here $w_{i, \pm}$ is the diagram wrapping the rightmost (resp. leftmost) strand with label $i$ a full positive (resp. negative) revolution around the cylinder.

For example, diagrammatically:

\[ \text{Diagram} \]

where the braces at the bottom denote labels from $I$, appearing in the order $i_m$. Meanwhile, the element $z_{m_1+\ldots+m_i}$ corresponds to a dot on the right-most strand labeled $i$.

Proof. First, let us confirm that the left and right sides of the equations (4.8a, 4.8b) act in the same way on $P^\Sigma$, i.e. that the right-hand sides match (4.2b, 4.2c).

We will prove (4.8a) carefully, and leave it to the reader to derive (4.8b) similarly. First, we note that $(z_{m_1+\ldots+m_i} + 2)^{r-1}$ acts on $P$ by $(z_{i,m_i} + 2)^{r-1}$. Then, as we move the strand to the right, we pass strands with label $j > i$. By (4.7a), this has no effect besides permuting the labels of variables, since either there is no arrow between the nodes, or $i \to j$ (by our choice of orientation, cf. Section 2.1). Then, we cross rightward over $x = 0$, which by (4.7f) contributes an action of $\sigma$. Next, we cross rightward over the strands with labels $j < i$. Those with $j \to i$ contribute factors of $P_{ij}$ in appropriate variables. The crossing of all strands labelled $j < i$ contributes a factor of

\[ \prod_{j \to i} W_j(z_{i,1} - 1) \rho_i(z_{i,m_i} + 2)^{r-1} \]

Note that, although $i_m$ is permuted during the process of crossing strands, after crossing all those discussed above we are back at $i_m$.

Thus far, we have wrapped the strand almost all the way around the cylinder, but without crossing any of the other strands with label $i$. We can summarize the action of the element defined thus far by

\[ \prod_{j \to i} W_j(z_{i,1} - 1) \rho_i(z_{i,m_i} + 2)^{r-1} = \prod_{j \to i} W_j(z_{i,1} - 1) z_{i,1}^{r-1} \rho_i \]
where \( \rho_i \) is the algebra automorphism of \( P \) defined by

\[
\rho_i(z_{j,k}) = \begin{cases} 
  z_{j,k} & i \neq j \\
  z_{i,k+1} & i = j, k \neq m_i \\
  z_{i,1} - 2 & i = j, k = m_i
\end{cases}
\]

Note that \( \rho_i \) has the same effect on \( P^\Sigma \) as \( \beta_{i,1}^{-1} \).

Finally, in crossing over the remaining strands with label \( i \), by applying (4.7e) again, we get a contribution of \( \partial_{i,m_i-1} \cdots \partial_{i,2} \partial_{i,1} \). Altogether, we see that \( w_{i,+}(z_{m_1+...+m_i + 2}) \) acts on \( P^\Sigma \) by

\[
\partial_{i,m_i-1} \cdots \partial_{i,1} z_{m_i}^{-1} \prod_{j \to i} W_j(z_{i,1} - 1) \beta_{i,1}^{-1}
\]

By Lemma 4.7 (see also the proof of Theorem 4.6), this agrees with the action of \(-E_i^{(r)}\) on \( P^\Sigma \) under (4.2b). This shows that the both sides of (4.8a) agree as operators \( P^\Sigma \to P \).

Note that equations (4.8a, 4.8b) imply that, as subalgebras of \( \text{End}(P) \),

\[
FY^\lambda_\mu \subset e(i_m)Y_{\mathfrak{e}}(i_m)
\]

Indeed these equations prove that the generators of \( Y^\lambda_\mu \) appear in \( e(i_m)Y_{\mathfrak{e}}(i_m) \). The inclusion follows since \( e(i_m)Y_{\mathfrak{e}}(i_m) \) also contains the tensor product of nil-Hecke algebras corresponding to packets of like-colored strands. It remains to check that this inclusion is an equality. Our proof of this requires a detour into the theory of Coulomb branches, and appears as [Wee19, Corollary 4.12].

**Remark 4.23.** One can reasonably wonder if there is an interpretation of the entirety of \( \mathfrak{R} \) in terms of the other aspects of this theory; one such interpretation is given in [Wee19, Thm. 4.11] in terms of the extended BFN category of [Weba, Def. 3.5]. This generalizes the “affine KLR algebra” which is sketched out in [Fmi18].

**Remark 4.24.** It is possible to define a variant of \( \mathfrak{R} \) in order to give a diagrammatic description of \( Y^\lambda_\mu \). To accomplish this, we use thick calculus in the sense of [KLMS12]. We consider diagrams in \( \mathfrak{R} \) which start and end with the idempotent \( e(i_m) \) and then we use a splitter to collect all strands with the same label \( i \), in the sense of [KLMS12, §2]. We can consider these diagrams modulo the relations of Definition 4.19 away from the splitters, as well as the relation that crossing two strands entering the splitter gives 0:

\[
= 0
\]

We compose these diagrams by stacking them, and resolving the product of splitters into a half twist of the strands passing through it, as in [KLMS12, Cor. 2.6]. This algebra has the unusual feature that it is not easy to write the identity in it, though it is possible using [KLMS12, (2.70)]; we can extend our diagrammatic calculus further by allowing thick strands that represent several strands with a single label “zipped together” (in [KLMS12], these are drawn as green). We will not go into details about the relations between these
diagrams, as they can get quite complicated. In this framework, we can write \( E_i^{(s)} \) and \( F_i^{(s)} \) as simpler diagrams:

\[
E_i^{(s)} = - i_{i_1} \ldots i_{i_r} s - 1 \\
F_i^{(s)} = \pm i_{i_1} \ldots i_{i_r} s - 1
\]

5. Weight modules and parity KLR algebras

5.1. Weight modules for (flag) truncated shifted Yangians. The algebras \( Y_\mu^\lambda, FY_\mu^\lambda \) have natural polynomial subalgebras, so we can consider weight modules.

The algebra \( Y_\mu^\lambda \) contains the polynomial subalgebra \( P^\Sigma = \mathbb{C}[A_i^{(s)}] \). We identify \( \text{Spec} P^\Sigma \) with the set of collections \( S = (S_i)_{i \in I} \) of multisets, where \( |S_i| = m_i \). Here \( S \) gives rise to the map \( \operatorname{ev}_S : P^\Sigma \to \mathbb{C} \) taking \( A_i^{(s)} \) to the \( s \)th elementary symmetric function of \( S_i \).

The algebra \( FY_\mu^\lambda \) contains the polynomial subalgebra \( P = \mathbb{C}[z_{i,k}] \). We identify \( \text{Spec} P = \prod_i \mathbb{C}^{m_i} \), where \( \nu = (\nu_i)_{i \in I} \in \text{Spec} P \) gives rise to the map \( \operatorname{ev}_\nu : P \to \mathbb{C} \) taking \( z_{i,k} \) to \( \nu_{i,k} \).

**Definition 5.1.** A weight module over \( Y_\mu^\lambda \) (resp. \( FY_\mu^\lambda \)) is a module \( M \) for which the subalgebra \( P^\Sigma \) (resp. \( P \)) acts locally finitely with finite-dimensional generalized eigenspaces.

Let \( M \) be a weight module for \( Y_\mu^\lambda \). So we can write

\[
M = \bigoplus_{S \in \text{Spec} P^\Sigma} W_S(M)
\]

\[
W_S(M) = \{ v \in M : \exists N \text{ s.t. } (f - \operatorname{ev}_S(f))^N v = 0 \text{ for all } f \in P^\Sigma \}
\]

We refer to \( W_S(M) \) as the \( S \)-weight space of \( M \) and we say that \( M \mapsto W_S(M) \) is a weight functor. Similarly, for a module \( M \) over \( FY_\mu^\lambda \), we define \( W_\nu(M) \) and we have weight modules where \( M = \bigoplus_\nu W_\nu(M) \).

Note that under the Morita equivalence, weight modules over \( FY_\mu^\lambda \) correspond to weight modules over \( Y_\mu^\lambda \). Now, there is an obvious map from \( \text{Spec} P \to \text{Spec} P^\Sigma \) which takes the tuples \( \nu_i \) and turns them into multisets \( S(\nu) = (S_i(\nu)) \). Let \( M \) be a weight module over \( FY_\mu^\lambda \) and recall that its image under the Morita equivalence is \( M^\Sigma \), where \( \Sigma \) is the product of symmetric groups. The group \( \Sigma \) also acts on the space \( \text{Spec} P = \prod_i \mathbb{C}^{m_i} \) of weights for \( M \) and we can consider the stabilizer of a weight \( \nu \). It is easy to see that

\[
W_{S(\nu)}(M^\Sigma) = W_\nu(M)^{\text{Stab}_\Sigma(\nu)}.
\]

We say that \( \nu \in \prod_i \mathbb{C}^{m_i} \) is an integral weight, if for each \( i \in I \) and \( k \leq m_i \), \( \nu_{i,k} \) is an integer of the same parity as \( i \). We say that a \( FY_\mu^\lambda \) weight module \( M \) has integral
weights. If \( W_\nu(M) \neq 0 \) implies that \( \nu \) is an integral weight. Let \( FY_\mu^\lambda \)-wtmod denote the category of weight modules over \( FY_\mu^\lambda \) with integral weights.

Considering the adjoint action of the dots on the flag Yangian, we see that if a module \( M \) is generated by finitely many weight vectors, then every other weight that appears will lie in the affine Weyl group orbit of the original weights. Since integral weights are closed under the action of the affine Weyl group, we could equivalently define \( FY_\mu^\lambda \)-wtmod to be the category of weight modules generated by vectors of integral weight.

5.2. An equivalence relating the flag Yangian and the metric KLRW algebras. Recall that idempotents in the metric KLRW algebra \( \tilde{T} = \tilde{T}_R \) are indexed by longitude triples \((i, a, \kappa)\) satisfying the conditions of Definition 3.21. For each integral weight \( \nu \), let \( d(\nu) = d(S(\nu)) \) be the idempotent in \( \tilde{T} \). Let \( \Sigma_\nu \) be the subgroup of \( \Sigma \) that stabilizes \( \nu \); this is a product of symmetric groups which permute the groups of strands that have the same label and longitude. Using the formula in (4.6), we can find an idempotent in \( d(\nu) \tilde{T} d(\nu) \) which acts in the polynomial representation \( \text{Pol}(i, a, \kappa) \) by projecting to invariants for this group. We denote this idempotent by \( d'(\nu) \). Since the nilHecke algebra of \( \Sigma_\nu \) is a matrix algebra, any other primitive idempotent in it will be isomorphic to \( d'(\nu) \).

Let \( \tilde{T}_\mu^R \)-modnil be the subcategory of finitely generated modules over \( \tilde{T}_\mu^R \) where the dots act nilpotently; since \( \tilde{T}_\mu^R \) is finitely generated as a module over the dots, such a module is necessarily finite dimensional.

**Theorem 5.2.** There is an equivalence \( \Theta: \tilde{T}_\mu^R \)-modnil \( \to \) \( FY_\mu^\lambda \)-wtmod such that

\[
WS(\Theta(M)) = d'(S)M \quad \text{and} \quad W_\nu(\Theta(M)) = d(\nu)M.
\]

This is actually a consequence of a more general equivalence, which we’ll discuss below. While this equivalence factors through objects which are more auxiliary from our perspective, it is easier to prove since it involves categories with simpler generators and relations.

Let us note one interesting corollary. Let \( \tilde{p}(S) = [\tilde{T}d'(S)] \); this is equal to the corresponding vector \( \tilde{p}_\kappa \) up to division by the size of the stabilizer of \( \nu \) in \( \Sigma \). Transferring Lemma 3.5 through this equivalence, we find that:

**Corollary 5.3.** We have

\[
\tilde{p}(S) = \sum_b \left( \dim WS(\Theta(L_b)) \right) \cdot b
\]

where the sum is over parity canonical basis vectors \( b \), and \( L_b \) is the corresponding simple \( \tilde{T} \)-module.

5.3. Weight modules for \( \mathcal{Y} \). Let \( M \) be a module for \( \mathcal{Y} \). Such a module comes with a decomposition \( M = \bigoplus_i e(i)M \), according to the idempotents \( e(i) \) of \( \mathcal{Y} \). We say that \( M \) is a weight module, if for all sequences \( i \in I^m \), we have

\[
e(i)M = \bigoplus_{a \in C^n} W_{i,a}(M)
\]
where this is a decomposition into generalized eigenspaces
\[ W_{i,a}(M) = \{ v \in e(i)M : \exists N \text{ s.t. } (z_k(i) - a_k)^N v = 0, \text{ for } k = 1, \ldots, m \} \]

We call the pair \((i, a) \in I^m \times \mathbb{C}^m\) integral, if for each \(k, a_k\) is an integer of the same parity as \(i_k\). We say that a \(R\)-module \(M\) has integral weights if \(W_{i,a}(M) \neq 0\) implies that \((i, a)\) is integral. Let \(R\text{-wtmod}\) denote the category of \(R\) weight modules with integral weights.

We have a quotient functor from \(R\text{-wtmod}\) to \(FY_\lambda^\pi\text{-wtmod}\) given by multiplication by the idempotent \(e(i)\) defined earlier. This quotient functor has a fully faithful left adjoint sending \(M\) to \(R \otimes_{FY} M\).

The category \(R\text{-wtmod}\) has weight functors \(W_{i,a}\) indexed by integral pairs \((i, a)\) as above. We will be interested in morphisms between these weight functors, the most important (for our purposes) we call the neutral crossing which goes between pairs connected by admissible permutations.

### 5.4. Admissible permutations and neutral crossings

Recall that the usual action of the symmetric group \(S_m\) on \(m\)-tuples is given by the rule \(\pi(a) = (a_{\pi^{-1}(1)}, \ldots, a_{\pi^{-1}(m)})\).

**Definition 5.4.** Let \((i, a) \in I^m \times \mathbb{C}^m\). We say that a pair of distinct indices \(k, l \in \{1, \ldots, m\}\) are not \((i, a)\)-switchable if \(i_k \leq i_l\) and \(a_k = a_l + 1\); otherwise, we say that they are switchable. We say that a permutation \(\pi \in \Sigma_m\) is \((i, a)\)-admissible if whenever \((k, l)\) are not \((i, a)\)-switchable, then \(\pi(k), \pi(l)\) are in the same order as \(k, l\).

Note that \(k, k+1\) are not \((i, a)\) switchable if and only if \(X_{i_k, i_{k+1}}(a_k, a_{k+1}) = 0\). Observe also that the simple transposition \(s_k\) is admissible if and only if \(k, k+1\) are switchable, which means

- if \(i_k \to i_{k+1}\) then \(a_k \neq a_{k+1} - 1\) and if \(i_k \leftarrow i_{k+1}\), then \(a_k \neq a_{k+1} + 1\).

**Lemma 5.5.** Let \(\pi_1, \pi_2\) be two permutations. Assume that \(\pi_1\) is \((i, a)\)-admissible and \(\pi_2 = (\pi_1(i), \pi_1(a))\)-admissible. Then \(\pi_2 \circ \pi_1\) is \((i, a)\)-admissible.

**Proof.** Suppose \(k < l\) are such that \(i_k \leq i_l\) and \(a_k = a_l + 1\). Then \(\pi_1(k) < \pi_1(l)\) since \(\pi_1\) is \((i, a)\)-admissible. Letting \(j = \pi_1(i)\) and \(b = \pi_1(a)\), we then have that \(\bar{j}_{\pi_1(k)} = i_k \leftarrow i_l = j_{\pi_1(l)}\) and \(b_{\pi_1(k)} = a_k = a_l + 1 = b_{\pi_1(l)} + 1\). Since \(\pi_2\) is \((j, b)\)-admissible it follows that \(\pi_2\pi_1(k) < \pi_2\pi_1(l)\). The analogous argument holds if \(k > l\). \(\square\)

**Lemma 5.6.** Let \(\pi\) be a permutation which is \((i, a)\)-admissible. Assume that \(\ell(\pi s_k) < \ell(\pi)\). Then \(s_k\) is \((s_k(i), s_k(a))\)-admissible.

**Proof.** Since \(\ell(\pi s_k) < \ell(\pi)\), we see that \(\pi(k + 1) < \pi(k)\). Hence \(k, k + 1\) are \((i, a)\)-switchable. This implies the desired result. \(\square\)

The following corollary follows immediately from the previous two lemmas.

**Corollary 5.7.** If \(\pi\) is an \((i, a)\)-admissible permutation, and if \(\pi = s_{k_r} \cdots s_{k_1}\) is a reduced word, then each \(s_{k_p}\) is \((s_{k_p} \cdots s_{k_1} i, s_{k_p} \cdots s_{k_1} a)\)-admissible.

The following examples of admissible permutations will be important later.
**Lemma 5.8.** Let \((i, a)\) be an integral pair and assume that \(a\) is weakly increasing (with respect to the usual partial order), and that up to reordering \(i = i_m\). Let \(\pi = \pi_i\) be the shortest permutation such that \(\pi(i) = i_m\). Then \(\pi\) is \((i, a)\)-admissible.

**Proof.** Suppose that \(i_k \leftarrow i_l\). Then \(i_l\) is even and \(i_k\) is odd. By our choice of the ordering of \(I\), \(i_l\) must come before \(i_k\) in \(i_m\), and hence \(\pi(l) < \pi(k)\). Now if \(a_k = a_l + 1\) then \(a_l \leq a_k\) and so \(l < k\). Therefore \(\pi(k), \pi(l)\) are in the same order as \(k, l\) and so \(\pi\) is admissible.

**Lemma 5.9.** Let \((i, a)\) be an integral pair, and let \(\pi = \pi_a\) be the shortest permutation of \(\{1, \ldots, m\}\) such that \(\pi(a)\) is weakly increasing in the order \(\preceq\) (cf. Section 3.7). Then \(\pi\) is \((i, a)\)-admissible.

**Proof.** Suppose \(i_k \leftarrow i_l\) and \(a_k = a_l + 1\). Then \(i_l\) is even node and \(i_k\) is odd. Hence \(a_l = 2q\) and \(a_k = 2q + 1\) for some \(q\), and so \(a_k \approx a_l\). Therefore, by construction, \(\pi\) does not change the relative order of \(k\) and \(l\).

**Definition 5.10.** Fix \((i, a)\). Suppose that \(s_k\) is \((i, a)\)-admissible. We define the neutral crossing \(\chi_k : W_{i,a} \rightarrow W_{s_k i, s_k a}\) as follows. For a \(\mathcal{R}\) weight module \(M\) and \(v \in W_{i,a}(M)\), we define

\[
\chi_k(v) = \begin{cases}
\psi_k \frac{1}{X_{i_k, i_{k+1}}(z_k, z_{k+1})} v & \text{if } i_k \neq i_{k+1} \\
(z_k - z_{k+1}) \psi_k v + v & \text{if } i_k = i_{k+1}.
\end{cases}
\]

Note that by the above admissible assumption \(X_{i_k, i_{k+1}}(z_k, z_{k+1})\) is invertible on \(W_{i,a}(M)\).

Though \(\text{Pol}\) is not a weight module, we have the following result which is proved by an elementary computation.

**Lemma 5.11.** The neutral crossing \(\chi_k\) is well-defined on \(\text{Pol}\) and acts by algebra automorphism \(s_k\).

**Lemma 5.12.** Suppose that \(s_k\) is admissible for \((i, a)\) and let \(1 \leq l \leq m\). Then

\[
e(i) \chi_k = \chi_k e(s_k i) \quad z_l \chi_k = \chi_k z_{s_k(l)} \quad \psi_l \chi_k = \chi_k \psi_{s_k(l)}
\]

In fact, this lemma implies that \(\chi_k\) maps \(W_{i,a}\) to \(W_{s_k i, s_k a}\).

**Proof.** Because \(\text{Pol}\) is a faithful representation, we can check these relations in \(\text{End}(\text{Pol})\) where they are obvious.

**Proposition 5.13.** The neutral crossings obey the relations in the symmetric group in the following sense.

1. For any admissible elementary transposition, we have \(\chi_k^2 = I\).
2. If \(s_k\) and \(s_l\) are both admissible and \(|k - l| > 1\), then \(\chi_k \chi_l = \chi_l \chi_k\).
3. If \(s_k\) and \(s_{k+1}\) are both admissible, then we have \(\chi_k \chi_{k+1} \chi_k = \chi_{k+1} \chi_k \chi_{k+1}\).

**Proof.** As in the proof of the previous lemma, we can check these relations in \(\text{Pol}\).

As a consequence of this proposition and of Corollary 5.7, we obtain the following.
Corollary 5.14. Suppose that $\pi$ is $(i, a)$-admissible and let $i' = \pi(i), a' = \pi(a)$. Using a composition of neutral crossings, we get an isomorphism $\chi_{\pi} : W_{i,a} \sim W_{i',a'}$ which depends only on $\pi$.

Lemma 5.15. Suppose that $\pi$ is $(i, a)$-admissible. Let $1 \leq l \leq m$. Then

$$e(i)\chi_{\pi} = \chi_{\pi}e(i), \quad z\chi_{\pi} = \chi_{\pi}z_{\pi(l)}, \quad \psi\chi_{\pi} = \chi_{\pi}\psi_{\pi(l)}$$

There are other morphisms between weight functors which will be useful to us, which we call the seam crossings.

Definition 5.16. Given $(i, a) \in I^m \times \mathbb{C}^m$, we have $\sigma(i) = (i_m, i_1, \ldots, i_{m-1})$ and $\sigma(a) = (a_m - 2, a_1, \ldots, a_{m-1})$. We define the rightward seam crossing $\hat{\sigma}_+ : W_{i,a} \to W_{\sigma(i),\sigma(a)}$ by applying the rightward crossing of the dashed line $x = 0$. We similarly define the leftward seam crossing $\hat{\sigma}_- : W_{i,a} \to W_{\sigma^{-1}(i),\sigma^{-1}(a)}$.

We remark that the appearance of the minus 2 in the definition of $\sigma(a)$ above is due to relation (4.7a), and ensures that $\hat{\sigma}_+$ defines a natural transformation $W_{i,a} \to W_{\sigma(i),\sigma(a)}$.

5.5. An equivalence relating the coarse metric KLRW algebra and $\mathfrak{H}$. Given an integral pair $(i, a)$, we will define an idempotent $d(i, a) \in \hat{\mathcal{T}}_{\mathcal{L}}$ as follows. We let $\kappa(p)$ be the number of indices $i$ such that $a_i < r_p$; note that $\kappa(2q) = \kappa(2q + 1)$ for all $q \in \mathbb{Z}$. Set $\pi = \pi_a$ (cf. Lemma [5.9]). Then $(\pi(i), \kappa, \pi(a))$ is a coarse longitude triple, and we define $d(i, a) := e(\pi(i), \kappa, \pi(a)) \in \hat{\mathcal{T}}_{\mathcal{L}}$. This defines a map

$$\{ \text{integral pairs } (i, a) \} \to \{ \text{idempotents for } \hat{\mathcal{T}}_{\mathcal{L}} \}$$

This map admits a 1-sided inverse by simply mapping the idempotent $e(i, \kappa, a)$ to $(i, a)$.

Let $\hat{\mathcal{T}}_{\mathcal{L}}$-$\text{mod}_{\text{nil}}$ denote the category of finitely-generated $\hat{\mathcal{T}}_{\mathcal{L}}$-modules where the dots act nilpotently. Given $M \in \hat{\mathcal{T}}_{\mathcal{L}}$-$\text{mod}_{\text{nil}}$, we consider the action of various generators of $\hat{\mathcal{T}}_{\mathcal{L}}$ on $M$. Firstly, for any integral pair $(i, a)$ and any $1 \leq k \leq m$ we have an operator

$$y_k : d(i, a)M \to d(i, a)M$$

given by the action of $y_k(\pi(i), \kappa, \pi(a)) \in \hat{\mathcal{T}}_{\mathcal{L}}$ (cf. the discussion following Definition 3.28).

Secondly, we consider the action of the crossing $\psi_{\pi(k)}$ on $d(i, a)M$, where as above $\pi = \pi_a$. Since we are working in $\hat{\mathcal{T}}_{\mathcal{L}}$ this crossing must carry coarse longitudes on the top and bottom. If we hope to define a nonzero operator, on the bottom $\psi_{\pi(k)}$ must carry $(\pi(i), \kappa, \pi(a))$. On the top the natural choice is $(s_{\pi(k)}(\pi(i)), \kappa, s_{\pi(k)}(\pi(a)))$. But note that if $a_k \neq s_k$ then this is not necessarily a coarse longitude. Therefore we consider the operator $\psi_{\pi(k)}$ on $d(i, a)M$ only when $a_k \approx a_{k+1}$, and the definition of this operator breaks up into two cases:

$$\psi_{\pi(k)} : d(i, a)M \to d(i, a)M \quad \text{if } i_k = i_{k+1},$$

$$\psi_{\pi(k)} : d(i, a)M \to d(s_ki, s_ka)M \quad \text{if } i_k \neq i_{k+1}$$

In the first case the crossing is given top longitude $(\pi(i), \kappa, \pi(a))$, and in the second case $(\pi(s_ki), \kappa, \pi(s_ka))$. Notice that since $a_k \approx a_{k+1}$, by the definition of $\pi = \pi_a$, we have that $\pi(k + 1) = \pi(k) + 1$. 


Lemma 5.17. There is a functor $\Theta: \tilde{\mathcal{T}}_\mathcal{L}\text{-mod}_{\text{nil}} \to \mathcal{W}_\mathcal{W}\text{-wtmod}$, such that for any integral pair $(i, a)$,
\begin{equation}
W_{i,a}(\Theta(M)) = d(i,a)M.
\end{equation}

Proof. Let $M \in \tilde{\mathcal{T}}_\mathcal{L}\text{-mod}_{\text{nil}}$. We begin by defining $\Theta(M)$ as a vector space by
\[
\Theta(M) = \bigoplus_{(i,a)\text{ integral}} d(i,a)M\delta(i,a)
\]
where $\delta(i,a)$ is a formal symbol. This formal symbol is introduced to distinguish between different pairs $(i,a)$ which give rise to the same idempotent $d(i,a)$.

Recall that the algebra $\mathcal{W}$ is generated by the dots $z_k$ and the crossings $\psi_i$, and the seam crossings of $\sigma_\pm$. On $d(i,a)M\delta(i,a)$ we consider operators as in (5.4-5.6), and define the action of $\mathcal{W}$ on $\Theta(M)$ as follows:
- We let $z_k \in \mathcal{W}$ act by $z_kv\delta(i,a) = (y_{\pi a(k)} + a_k)v\delta(i,a)$
  Note that this has the correct eigenvalue, since $y_{\pi a(k)}$ is nilpotent.
- We let the crossing $\psi_i$ of two strands act by
  \begin{equation}
  \psi_i v\delta(i,a) = \begin{cases}
  \psi_{\pi(i)}v\delta(i,a) & i_k = i_k+1, a_k \approx a_k+1 \\
  \frac{1}{y_{\pi(k+1)}-y_{\pi(k)}+a_k-a_k+1}v\delta(i,a) & i_k = i_k+1, a_k \not\approx a_k+1 \\
  \frac{1}{y_{\pi(k)}-y_{\pi(k+1)}+a_k-a_k+1}v\delta(i,a) & i_k \neq i_k+1, a_k \approx a_k+1 \\
  \psi_{\pi(k)}v\delta(s_k i, s_k a) & i_k \neq i_k+1, a_k \not\approx a_k+1 \\
  X_{i, i+1} (y_{\pi a(k+1)} + a_k) & i_k \neq i_k+1, a_k \approx a_k+1 \\
  y_{\pi a(k+1)} + a_k + 1) v\delta(s_k i, s_k a) & i_k \neq i_k+1, a_k \not\approx a_k+1
  \end{cases}
  \end{equation}
  As a sanity check on these formulae, note that when we apply $\psi_k$ in $\mathcal{W}$ to a weight vector, we get a sum of weight vectors for the weights $(s_k i, s_k a)$ and $(s_k i, s_k a)$ by (3.1a-3.1d). If $i_k \neq i_k+1$, then we only have a non-zero term in the second case; if $i_k = i_k+1$ and $a_k = a_k+1$, then these two weight spaces are the same. This will only change the idempotent $d(i,a)$ if $a_k \approx a_k+1$. Note also that in the last case (where $i_k \neq i_k+1$ and $a_k \not\approx a_k+1$) that $d(s_k i, s_k a) = d(i,a)$, so given $v\delta(i,a)$ then $v\delta(s_k i, s_k a)$ also makes sense.
- We send $\sigma_+$ to $\phi^+_{\pi a(m)}$ as defined in Section 3.7 which is the unique straight-line diagram attaching $d(i,a)$ to $d((i_1, i_1 + 1, ..., i_{m-1}), (a_{m+2}, a_1, a_{m-1}))$ with the least number of crossings. The only crossings in this diagram involve the single black strand with bottom longitude $a_m$, and it only crosses the red strands for elements of $R_j$ in the interval $a_m < r \leq a_m + 2$.
- We send $\sigma_-$ to the precomposition of the polynomial
  \[
  \prod_{r \in R, r \neq a_1} (y_{\pi(1)} - r + a_1)
  \]
with the diagram $\phi^-_{\pi a(1)}$, which is the straight-line diagram that joins $d(i,a)$ to $d((i_2, ..., i_m, i_1), (a_2, ..., a_m, a_1 - 2))$. The only crossings in this diagram involve
the single black strand with bottom longitude \(a_1\), and it only crosses the red strands for elements of \(R_j\) in the interval \(a_1 - 2 < r \leq a_1\).

We will now check that these formulae define an action of \(\mathcal{H}\) on \(\Theta(M)\). Consider the subcategory of \(\tilde{T}_L\)-modules for which this does define a \(\mathcal{H}\)-action. Obviously, this set is closed under taking submodules, quotients, and direct sums.

Recall that we have a faithful \(\tilde{T}_L\)-module \(\text{Pol}_L\). Let \(\text{Pol}_L^{(N)}\) denote the quotient of \(\text{Pol}_L\) by the ideal \(I^{(N)}\) generated by all symmetric polynomials in the \(Y_i\)’s of degree \(\geq N\). It’s clear from (3.4a–3.4b) that \(I^{(N)}\) is invariant under \(\tilde{T}_L\), hence \(\text{Pol}_L^{(N)}\) has a well-defined \(\tilde{T}_L\) action. Note that any \(Y_i\) must be nilpotent on this module, since the coinvariant algebra is finite dimensional.

Since \(\tilde{T}_L\) is faithful on the inverse limit of this system of modules, every module on which the dots are nilpotent is a subquotient of the direct sum of these modules. Thus, it suffices to check that we have a \(\mathcal{H}\) action when applying the construction \(\Theta\) to \(\text{Pol}_L^{(N)}\) for all \(N\).

We have that \(\Theta(\text{Pol}_L^{(N)}) = \bigoplus d(i, a)\text{Pol}_L^{(N)} \delta(i, a)\). Applying the formulae (3.4a–3.4b), we find that on \(f\delta(i, a) \in d(i, a)\text{Pol}_L^{(N)} \delta(i, a)\):

- The generator \(z_k\) acts by multiplication by \(Y_{\pi_a(k)} + a_k\).
- The crossing \(\psi_k\) maps \(f\delta(i, a)\) to:

\[
\begin{aligned}
\frac{f - f(\pi_a(k), \pi_a(k+1))}{Y_{\pi_a(k)} - Y_{\pi_a(k+1)}} \delta(i, a) & \quad i_k = i_{k+1}, a_k \approx a_{k+1} \\
\frac{Y_{\pi_a(k)} - Y_{\pi_a(k+1)}}{f\delta(i, a)} & \quad i_k = i_{k+1}, a_k \not\approx a_{k+1} \\
Y_{\pi_a(k+1)} + a_k - a_{k+1} + \frac{f\delta(i, s_k a)}{f(\pi_a(k), \pi_a(k+1))} & \quad i_k \neq i_{k+1}, a_k \approx a_{k+1} \\
X_i k_{i+1}(Y_{\pi_a(k)}, Y_{\pi_a(k+1)}) f(\pi_a(k), \pi_a(k+1)) \delta(s_k i, s_k a) & \quad i_k \neq i_{k+1}, a_k \not\approx a_{k+1} \\
X_i k_{i+1}(Y_{\pi_a(k)} + a_k, Y_{\pi_a(k+1)} + a_{k+1}) f(\pi_a(k), \pi_a(k+1)) \delta(s_k i, s_k a) & \quad i_k \neq i_{k+1}, a_k \not\approx a_{k+1}
\end{aligned}
\]

- The seam crossings \(\sigma_{\pm}\) act by

\[
\begin{aligned}
\sigma_+ \cdot f\delta(i, a) &= f\delta((i_m, i_1, \ldots, i_{m-1}), (a_m + 2, a_1, \ldots, a_{m-1})) \\
\sigma_- \cdot f\delta(i, a) &= \prod_{r \in R_i} (Y_{\pi(1)} - r + a_1)f\delta((i_2, \ldots, i_{m}, i_1), (a_2, \ldots, a_m, a_1 - 2))
\end{aligned}
\]

since the leftward crossing crosses one red strand with the same label for each time \(a_1\) appears in \(r\).
If we change basis by the formula \((Y_{\pi a(k)} + a_k)\delta(\mathbf{i}, \mathbf{a}) = Z_k\delta(\mathbf{i}, \mathbf{a})\), then these formulas become

\[
z_k \cdot f(\mathbf{Z})\delta(\mathbf{i}, \mathbf{a}) = Z_k f(\mathbf{Z})\delta(\mathbf{i}, \mathbf{a})
\]

\[
\psi_k \cdot f(\mathbf{Z})\delta(\mathbf{i}, \mathbf{a}) = \begin{cases}
\frac{f - f_{s_k}}{Z_k - Z_{k+1}}\delta(\mathbf{i}, \mathbf{a}) & i_k = i_{k+1}, a_k \approx a_{k+1} \\
\frac{Z_k - Z_{k+1}}{z_k - Z_{k+1}}\delta(\mathbf{i}, \mathbf{a}) + \frac{f_{s_k}}{z_k - Z_{k+1}}\delta(\mathbf{i}, s_k \mathbf{a}) & i_k = i_{k+1}, a_k \not\approx a_{k+1} \\
\chi_{i_{i_k+1}}(Z_k, Z_{k+1}) f_{s_k}\delta(s_k \mathbf{i}, s_k \mathbf{a}) & i_k \neq i_{k+1}
\end{cases}
\]

\[
\sigma_+ \cdot f(\mathbf{Z})\delta(\mathbf{i}, \mathbf{a}) = \sigma(f)\delta(i_m, i_1, \ldots, i_{m-1}, (a_m + 2, a_1, \ldots, a_{m-1}))
\]

\[
\sigma_- \cdot f(\mathbf{Z})\delta(\mathbf{i}, \mathbf{a}) = \prod_{r \in R_i} (Z_1 - r)\sigma^{-1}(f)\delta(i_2, \ldots, i_m, i_1, (a_2, \ldots, a_m, a_1 - 2))
\]

Note that in the case of the \(\psi_k\) action, the formula breaks up into three cases (instead of four), since if \(i_k \leftarrow i_{k+1}\) and \(a_k \approx a_{k+1}\) then \(a_k = 2q + 1\) and \(a_{k+1} = 2q\) for some \(q\). Hence \(X_{i_k, i_{k+1}}(Y_{\pi_a(k)}, Y_{\pi_a(k+1)}) = X_{i_k, i_{k+1}}(Z_k, Z_{k+1})\) by the equality

\[
X_{i_k, i_{k+1}}(Y_{\pi_a(k)}, Y_{\pi_a(k+1)}) = X_{i_k, i_{k+1}}(Z_k - a_k, Z_{k+1} - a_k + 1) = \tilde{X}_{ij}(Z_k, Z_{k+1} + a_k - a_{k+1} - 1)
\]

Finally note that the quotient of polynomials in \(\mathbb{C}[Y_1, \ldots, Y_m]\) by symmetric polynomials of degree \(\geq N\) is equal to the quotient of polynomials \(\mathbb{C}[Z_1, \ldots, Z_m]\) by the symmetric polynomials of degree \(\geq N\) in the shifted variables \(\{Z_k - a_k\}\). That is, we find \(\Theta(\text{Pol}_l^N)\) is the sum over integral weights \(\mathbf{a} = (a_1, \ldots, a_m)\) of the quotient of \(\text{Pol}_m^l\) by \(I^N\) shifted to this point (cf. Remark 3.21). Comparing with the formulas for the polynomial action defined in (4.7c, 4.7g) shows that the action defined via \(\Theta\) agrees with the induced action of \(\mathcal{H}\) by those formulas. Thus, we indeed get a \(\mathcal{H}\) action in this case, so we do in the general case as well. \(\square\)

**Lemma 5.18.** There is a functor \(\Gamma : \mathcal{H}-\text{wtmod} \to \tilde{\mathcal{F}}_L-\text{mod}_{\text{nil}}\) such that for all coarse longitude triples, we have

\[
e(\mathbf{i}, \kappa, \mathbf{a})\Gamma(N) = W_{\mathbf{i}, \mathbf{a}}(N)
\]

**Proof.** Let \(N\) be a \(\mathcal{H}\) weight module with integral weights. We’ll define an action of \(\tilde{\mathcal{F}}_L\) on the vector space

\[
\Gamma(N) = \bigoplus_{\mathbf{i}, \mathbf{a} \text{ weakly } \preceq \text{-increasing}} W_{\mathbf{i}, \mathbf{a}}(N)
\]

Note that \(\Gamma(N)\) will in general be smaller than \(N\), since typically \(N\) will have weights which are not weakly \(\preceq\)-increasing.

Recall the generators of the algebra \(\tilde{\mathcal{F}}_L\) defined in Section 3.7. We let:

- the dots \(y_k \in \tilde{\mathcal{F}}_L\) act by the nilpotent part \(z_k - a_k \in \mathcal{H}\).
- the crossing \(\psi_k \in \tilde{\mathcal{F}}_L\) of two strands with longitudes satisfying \(a_k \approx a_{k+1}\) is simply given by \(\psi_k \in \mathcal{H}\)
- The generator \(\phi_k^+\) is sent to the composition of morphisms between weight functors

\[
(5.9) \quad \chi_{k-1} \cdots \chi_1 \sigma + \chi_{m-1} \cdots \chi_k
\]
In other words, in $\mathcal{R}$, we wrap the $k$th strand one full rightward twist around the cylinder crossing each other strand once, using a neutral crossing each time.

- The generator $\phi_k^-$ is sent to the composition

$$\prod_{r \in R_i \atop r \neq a_k} (y_k - r)^{-1} \chi_k \cdots \chi_{m-1} \sigma_1 \cdots \chi_{k-1}$$

In other words, in $\mathcal{R}$, we wrap the $k$th strand one full leftward twist around the cylinder, crossing each other strand once, using a neutral crossing each time, multiplied by the above rational function.

As in the proof of Lemma 5.17, we can check this by considering the subcategory of weight modules on which this does define a $\tilde{T}_L$ action. We would like to claim that the polynomial representation $\text{Pol}_R$ lies in this subcategory. This is slightly tricky since of course, the polynomial representation is not a weight module, but aiming to invert the proof of Lemma 5.17 shows how we can get around this difficulty. Let $I_a(N)$ be the ideal in $\mathbb{C}[Z]$, generated by symmetric functions of degree $\geq N$ in $Z_i - a_i$. The action of $\mathcal{R}$ on $\text{Pol}_R$ induces one on the sum of quotients $\sum a \text{Pol}_R / I_a(N)$ which is a weight module. The result of applying the functor $\Gamma$ just gives the polynomial representation of $\tilde{T}_L$ modulo the ideal $I(N)$, since the formulas above just invert those of Lemma 5.17. Thus, these modules lie in the category where $\Gamma$ defines a $\tilde{T}_L$-module structure; the inverse limit of this system of modules is faithful, so every weight module is a subquotient of a direct sum of them.

Thus, $\Gamma$ is well-defined as a functor. By direct construction, this functor sends weight modules to modules with finite dimensional images of $e(i, a)$ where the dots are nilpotent, and has the desired action on weight spaces. $\square$

**Theorem 5.19.** The functors $\Theta : \tilde{T}_L\text{-mod}_{\text{nil}} \to \mathcal{R}\text{-wtmod}$ and $\Gamma : \mathcal{R}\text{-wtmod} \to \tilde{T}_L\text{-mod}_{\text{nil}}$ give mutually inverse equivalences

$$\mathcal{R}\text{-wtmod} \cong \tilde{T}_L\text{-mod}_{\text{nil}}$$

**Proof.** Let $N \in \mathcal{R}\text{-wtmod}$. Unpacking the definitions from the previous two lemmas,

$$\Theta(\Gamma(N)) = \bigoplus_{(i,a) \text{ integral}} W_{\pi_a(i), \pi_a(a)}(N) \delta(i, a)$$

That is, the $(i, a)$-weight space of $\Theta(\Gamma(N))$ is given by the $(\pi_a(i), \pi_a(a))$-weight space of $N$. We must construct an isomorphism $\Theta(\Gamma(N)) \cong N$, which will be constructed one weight space at a time as an isomorphism $W_{i,a}(N) \cong W_{\pi_a(i), \pi_a(a)}(N)$.

This will be supplied by the neutral crossings. For each $i, a$, $\pi_a$ is an admissible permutation and so we have a neutral crossing map $\chi_{\pi_a} : W_{i,a}(N) \to W_{\pi_a(i), \pi_a(a)}(N)$.

We write $\chi : N \to \Theta(\Gamma(N))$ for the direct sum of these maps. We must prove that $\chi$ is a map of $\mathcal{R}$-modules. To do this we will show that for each of the generators $r$ of $\mathcal{R}$, we have $\chi(rv) = r\chi(v)$ for all $v \in N$.

Because the modules $N, \Gamma(N)$ and $\Theta(\Gamma(N))$ have closely related underlying vector spaces, we will introduce the following temporary notation. Given an element $u \in W_{\pi_a(i), \pi_a(a)}(N)$, we will write $\overline{u}$ when it is regarded as an element of $\Gamma(N)$ and $\overline{\pi\delta(i, a)}$
will check these generators in turn. Since and on the other hand, we have space, whereas the second term lies in the not be a weight vector and in order to compute is its expansion as a sum of weight vectors; the first term lies in the (5.11) holds for the dots. Applying Lemma 5.15 we have and on the other hand, we have and so 5.11 holds for the dots.

Now we consider \( r = \psi_k \). There will be three cases to consider.

**Case 1:** \( a_k \approx a_{k+1} \). In this case, we find that \( \pi_a = \pi_{s_k a} \) and we write \( \pi = \pi_a \). Thus applying Lemma 5.15 we have and on the other hand applying (5.8), we have so 5.11 holds in this case.

**Case 2:** \( i_k = i_{k+1} \) and \( a_k \not\approx a_{k+1} \). This case is slightly more complicated since \( \psi_k v \) will not be a weight vector and in order to compute \( \chi(\psi_k v) \) we need to write it as a sum of weight vectors. Applying the “dot/crossing” relation and the definition of the neutral crossing, we see that is its expansion as a sum of weight vectors; the first term lies in the \( (s_k i, s_k a) \) weight space, whereas the second term lies in the \( (s_k i, a) \) weight space.

There are actually two subcases to consider here, depending on which of \( a_k, a_{k+1} \) is larger. These two cases are very similar, so let us assume that \( a_{k+1} \) is larger, so that \( \pi_{s_k a} = \pi_a s_k \). In particular, this means that

\[
\chi(\psi_k v) = \chi(\psi_k v) = \chi(\psi_k v) = \chi(\psi_k v)
\]

On the other hand applying the definitions from the previous two lemmas, especially 5.8, we see that

\[
\psi_k(\chi(v)) = \frac{1}{z_k - z_{k+1}} \chi(v) \delta(s_k i, s_k a) + \frac{1}{z_{k+1} - z_k} \chi(v) \delta(s_k i, s_k a)
\]

Since \( \chi_k^2 = 1 \) and applying Lemma 5.15, we see that 5.11 holds in this case.
Case 3: \( i_k \neq i_{k+1} \) and \( a_k \neq a_{k+1} \). In this case \( \chi_k v \) is a weight vector of weight \((s_k i, s_k a)\). Again, we have two subcases and we assume that \( a_{k+1} \) is larger which leads us to \( \pi_{s_k a} = \pi_a s_k \). Then we have
$$
\chi(\psi_k(v)) = \chi^\pi \chi_k(\psi_k(v)) \delta(s_k i, s_k a)
$$
and applying the definitions, we have
$$
\psi_k(\chi(v)) = [s_k i, s_k a](z_{\pi(k)}, z_{\pi(k)+1}) v \delta(s_k i, s_k a))
$$
The result follows upon noting that \( \chi_k \psi_k = X_{s_k i, s_k a}(z_{\pi(k)}, z_{\pi(k)+1}) \) by the definition of the neutral crossing and the \( \chi_k^2 \) relation.

Finally, we consider the positive seam crossing \( \sigma_+ \) (the negative one is similar). As above let \( v = W_{i,a}(N) \) and let \( \pi = \pi_a \). Then \( \sigma_+(v) \in W_{\sigma(i), \sigma(a)} \) where as before
$$
\sigma(i) = (i_m, \ldots, i_1), \quad \sigma(a) = (a_m + 2, a_1, \ldots, a_{m-1})
$$
Let us write \( (b_1, \ldots, b_m) = \pi(a) \) and let \( k \) be the maximal index such that \( b_k = m \) (so that \( \pi(m) = k \)). By the definition of partial order \( \leq \), we have that
$$
\pi_{\sigma(a)}(\sigma(a)) = (b_1, \ldots, b_k, b_k + b_{k+1}, \ldots, b_m)
$$
(the strange definition of \( \leq \) ensures that this is weakly-\( \leq \) increasing). We conclude
$$
\pi_{\sigma(a)} = \pi s_{m-1} \cdots s_1.
$$
Now, let us examine the LHS of (5.11) in this case. We see that it equals
$$
\chi^\pi \chi_{m-1} \cdots \chi_1 \sigma_+ v \delta(\sigma(i), \sigma(a))
$$
On the other hand, following the definitions of the previous two lemmas, the RHS of (5.11) is
$$
\sigma_+ \chi^\pi \chi_{m-1} \cdots \chi_1 v \delta(\sigma(i), \sigma(a))
$$
and so the result follows.

On the other hand, let \( M \in \mathcal{L}_-^{\text{modnil}} \). Then from the definitions, we naturally have \( \Gamma(\Theta(M)) = M \). Checking that the module action is correct just runs the logic of the previous paragraph backwards.

**Proof of Theorem 5.2.** The algebra \( \mathcal{L}_- \) is a subalgebra of \( \mathcal{T}_- \) and we have an idempotent \( e_\ell \in \mathcal{T}_- \) defined as
$$
e_\ell = \sum_{(i, \kappa, a) \text{ longitude triple}} e(i, \kappa, a)
$$
This satisfies \( \mathcal{L}_- = e_\ell \mathcal{T}_- e_\ell \).

We have a pair of quotient functors
$$
e_\ell \mathcal{T}_- \otimes - : \mathcal{T}_-^{\text{modnil}} \rightarrow \mathcal{L}_-^{\text{modnil}} \quad e(i_m) \mathcal{Y}_- \otimes - : \mathcal{Y}_{\text{wtmod}} \rightarrow FY_\mu^{\lambda_{\gamma}} \mathcal{Y}_{\text{wtmod}}.
$$

We have mutually inverse equivalences \( \Theta : \mathcal{L}_-^{\text{modnil}} \rightarrow \mathcal{Y}_{\text{wtmod}} \) and \( \Gamma : \mathcal{Y}_{\text{wtmod}} \rightarrow \mathcal{T}_-^{\text{modnil}} \). Thus it suffices to show that
$$
\text{for } M \in \mathcal{T}_-^{\text{modnil}}, \text{ if } e_\ell M = 0, \text{ then } e(i_m) \Theta(M) = 0
$$
and
$$
\text{for } N \in \mathcal{Y}_{\text{wtmod}}, \text{ if } e(i_m) N = 0, \text{ then } e_\ell \Gamma(N) = 0
$$
Suppose that \( M \in \mathcal{F}_L\text{-mod}_{\text{nil}} \) and \( e_\ell M = 0 \). Then
\[
e(i_m)\Theta(M) = \bigoplus_{a \text{ s.t. } (i_m,a) \text{ integral}} d(i_m,a)M \delta(i_m,a)
\]
so it suffices to show that for all integral \((i_m,a)\) we have that \(d(i_m,a)M = 0\). Now, \(i_m\) has the property that if \(j\) is even and \(i\) is odd, then all appearances of \(j\) come before any appearance of \(i\) in \(i_m\). Assume that \(p < q\) and \((i_m)_p = j\) and \((i_m)_q = i\). Since we cannot have \(a_q\) even and \(a_p\) odd, if \(a_q \geq a_p\) then we must have \(a_q \geq a_p\). Thus, in the idempotent \(d(i_m,a)\), the longitudes are weakly increasing. Thus \(d(i_m,a)\) is a summand of \(e_\ell\) and so \(d(i_m,a)M = 0\) as desired.

On the other hand, suppose that \( N \in \mathfrak{H}\text{-wtmod} \) and \( e(i_m)N = 0 \). We need to show that for any \(a\) weakly \(\leq\)-increasing, we have \(e(i,\kappa,a)\Gamma(N) = 0\). Fix \(a\) weakly increasing. Since \(e(i,\kappa,a)\Gamma(N) = W_{i,a}(N)\), we need to show that these weight spaces vanish. By Lemma 5.8 and Corollary 5.14, there exists \(a'\) such that \(W_{i,a}(N) \cong W_{i,a'}(N)\) and thus we conclude that \(W_{i,a}(N) = 0\) as desired.

Finally, note that the quotient functor \(\mathcal{F}_L\text{-mod}_{\text{nil}} \to \mathcal{F}\text{-mod}_{\text{nil}}\) is split by the inclusion functor \(\mathfrak{F}\text{-mod}_{\text{nil}} \to \mathcal{F}_L\text{-mod}_{\text{nil}}\) which is given by taking a \(\mathfrak{F}\)-module \(M\) to \(\mathcal{F}_L e_\ell \otimes \mathfrak{J} M\). Thus, the equivalence \(\Theta\) takes a module \(M \in \mathcal{F}\text{-mod}_{\text{nil}}\) to
\[
e(i_m)\Theta(\mathcal{F}_L e_\ell \otimes \mathfrak{J} M) = \bigoplus_{a \text{ s.t. } (i_m,a) \text{ integral}} d(i_m,a)M \delta(i_m,a)
\]
and in particular, we see that \(W_a(\Theta(M)) = d(a)M\).

These functors have a natural analogue for category \(\mathcal{O}\) and for the finite-dimensional modules. Let
\[
x(\nu) = \sum_{i \in I} \sum_{k=1}^{m_i} \nu_{i,k}.
\]

**Definition 5.20.** Let \(FY_\mu^\lambda\text{-mod}_{\text{id}}\) be the category of finite dimensional \(FY_\mu^\lambda\) modules with integral weights. Let \(FY_\mu^\lambda\text{-O}^-\) be the full subcategory of \(FY_\mu^\lambda\text{-wtmod}\) consisting of those objects \(M\), for which there exists \(N \subseteq \mathbb{R}\), such that \(W_\mu(M) = 0\) whenever \(x(\nu) < N\). Similarly we define \(FY_\mu^\lambda\text{-O}^+\) using the condition \(x(\nu) > N\).

Similarly, let \(Y_\mu^\lambda\text{-O}^-\) be the full subcategory of \(Y_\mu^\lambda\text{-wtmod}\) consisting of those objects \(M\), for which there exists \(N \subseteq \mathbb{R}\), such that \(W_\mu(M) = 0\) whenever \(x(S) < N\). Similarly we define \(Y_\mu^\lambda\text{-O}^+\) using the condition \(x(S) > N\).

**Theorem 5.21.** The equivalence \(\Theta\): \(\mathcal{F}_\mu^R\text{-mod}_{\text{nil}} \to FY_\mu^\lambda\text{-wtmod}\) restricts to equivalences
\[
\pm\mathcal{F}_\mu\text{-mod} \cong FY_\mu^\lambda\text{-O}^\pm, \quad 0\mathcal{F}_\mu\text{-mod} \cong FY_\mu^\lambda\text{-mod}_{\text{id}}
\]

**Proof.** We will prove the statement about \(FY_\mu^\lambda\text{-O}^-\). The statement about \(FY_\mu^\lambda\text{-O}^+\) is proved similarly and then the one for \(FY_\mu^\lambda\text{-mod}_{\text{id}}\) is obtained by intersecting.

Suppose that \(M \in \pm\mathcal{F}_\mu\text{-mod}\). Then for any \(\nu\), we have \(W_\mu(\Theta(M)) = e(\nu)(M)\). Choose some \(\nu\) such that \(d(\nu)(M) \neq 0\). Now, \(d(\nu) = (i,\kappa,a)\) where \(a\) is obtained by putting the elements of \(\nu\) in order. Since \(M \in \pm\mathcal{F}_\mu\text{-mod}\), for all \(1 \leq k \leq m\), we have...
$k < \kappa(\ell)$ (as the rightmost strand must be red). Thus for all $k$, we have $a_k < r_\ell$ and thus we see that $x(\nu) = \sum a_k < mr_\ell$. So we can choose $N = \max(mr : r \in \mathbb{R})$ and we are done.

Suppose that $\Theta(M) \in FY_\mu^\lambda \cdot \mathcal{O}^{-}$. Suppose that $M$ does not lie in $\mathcal{T}_{\mu}^{-}\cdot \text{mod}$. Then for some triple $(i, \kappa, a)$ not having a rightmost red strand, we have $e(i, \kappa, a)M \neq 0$. Since the rightmost strand in $e(i, \kappa, a)$ is black, we can increase the label on this black strand and still preserve the longitude condition. The straight line diagram which increases a longitude is invertible in $\mathcal{T}$. Thus we conclude that for all $p \in \mathbb{N}$, we have $e(i, \kappa, a + 2p\varepsilon_m)M \neq 0$ (where $\varepsilon_m = (0, \ldots, 0, 1)$). Now as in the proof of Theorem 5.2 there exists $\nu$ such that $W_{\nu}(\Theta(M)) = e(i, \kappa, a + 2p\varepsilon_m)M \neq 0$. Then $x(\nu) = \sum a_k + 2p$. Since $\Theta(M) \in FY_\mu^\lambda \cdot \mathcal{O}^{-}$, this is a contradiction. Thus, we see that $M \in \mathcal{T}_{\mu}^{-}\cdot \text{mod}$.

As noted before, this equivalence goes between a category with a well-known categorical action of the Lie algebra $\mathfrak{g}$ and one where no such action is apparent. We’ll correct this issue by constructing the counterpart action on category $\mathcal{O}$ in a future paper [KTW$^+$a].

Finally, we obtain that this equivalence matches highest weights of simples in $\mathcal{O}$ to the product monomial crystal, as conjectured in [KTW$^+$b].

**Corollary 5.22.** The map $\varphi$ induces a bijection between the highest weights of simple modules in $Y_\mu^\lambda \cdot \mathcal{O}^{-}$ and $\mathcal{B}(\mathbb{R})_\mu$, the $\mu$ weight set of the product monomial crystal $\mathcal{B}(\mathbb{R})$.

This bijection maps the simple $Y_\mu^\lambda(\mathbb{R})$-module of highest weight $S$ to $a_\mathbb{R}b_S^{-1} \in \mathcal{B}(\mathbb{R})$.

**Proof.** By Corollary 3.27 we have a bijection $\varphi : \text{Irr}(\mathcal{T}_{\mu}^{-}\cdot \text{mod}) \to \mathcal{B}(\mathbb{R})_\mu$. By Theorem 5.21 $\text{Irr}(\mathcal{T}_{\mu}^{-}\cdot \text{mod}) = \text{Irr}(FY_\mu^\lambda \cdot \mathcal{O}^{-})$. Now the equivalence $M \to M^\Sigma$ between $FY_\mu^\lambda \cdot \text{mod}$ and $Y_\mu^\lambda \cdot \text{mod}$, induces an equivalence between the respective category $\mathcal{O}$’s. Hence we have that $\text{Irr}(FY_\mu^\lambda \cdot \mathcal{O}^{-}) = \text{Irr}(Y_\mu^\lambda \cdot \mathcal{O}^{-})$, and $\varphi$ induces the desired bijection.

Now let $L(S)$ be the simple $Y_\mu^\lambda$-module of highest weight $S$. Recall this means that $W_\mathbb{S}(L(S)) \neq 0$, and if $W_T(L(S)) \neq 0$ then $x(T) < x(S)$. Let $L = FY_\mu^\lambda e_m \otimes Y_\mu^\lambda L(S)$ be the simple $FY_\mu^\lambda$-module corresponding to $L(S)$ under the Morita equivalence. By Theorem 5.21 there is a simple $\mathcal{T}_{\mu}$-module $M$ such that $\Theta(M) = L$, and hence $\Theta(M)^\Sigma = L(S)$. By Theorem 5.2 we have that $d(S)M \neq 0$. This implies that $d(S)M \neq 0$, and $S$ is the highest monomial that doesn’t kill $M$. Hence, by Corollary 3.27 $\varphi([M]) = a_\mathbb{R}b_S^{-1}$, which implies under the identification $\text{Irr}(\mathcal{T}_{\mu}^{-}\cdot \text{mod}) = \text{Irr}(Y_\mu^\lambda \cdot \mathcal{O}^{-})$ that $\varphi([L(S)]) = a_\mathbb{R}b_S^{-1}$.

\[ \square \]

### 5.6. The non-integral case

In the preceding sections, we have only considered integral choices of $\mathbb{R}$ and integral weights of shifted Yangians. This was mostly for simplicity of presentation rather than due to any mathematical difficulties. In this section we briefly describe how to generalize our results to the non-integral case.

First, we relax our assumptions to let $\mathbb{R}$ contain arbitrary elements of $\mathbb{C}$; we could generalize even further by considering any field of characteristic 0 without changing any of the discussion below. Elements of $r \in R_i$ and $r' \in R_j$ “interact” in an interesting
way if $r - r'$ is an integer whose parity is even (odd) if $i$ and $j$ have the same (different) parity. Let $\tilde{i}$ be the parity of $i \in I$, i.e., the element of $\mathbb{Z}/2\mathbb{Z}$ such that $i \in I_\tilde{i}$. This motivates the following definition:

**Definition 5.23.** We write $(r, i) \sim (r', j)$ if $r - r' \equiv \tilde{i} - \tilde{j}$ (mod $2\mathbb{Z}$). This defines an equivalence relation on $\mathbb{C} \times I$.

The equivalence classes for this relation map injectively to $\mathbb{C}/2\mathbb{Z}$ by sending $(r, i) \mapsto \tilde{r} + \tilde{i}$. Note that $R$ is integral if and only if its image in $\mathbb{C}/2\mathbb{Z}$ is $\{0\}$.

We can naturally decompose $Y_\mu^\lambda$ weight modules into “integrality classes.” For each $i \in I$, fix a size $m_i$, multi-subset $\s_i$ of $\mathbb{C}/2\mathbb{Z}$. This collection $\s$ defines a point in $\prod_i (\mathbb{C}/2\mathbb{Z})^{m_i}/\Sigma$. Equivalently, it defines an orbit of the semi-direct product $\Sigma \ltimes \prod_i (2\mathbb{Z})^{m_i}$, acting on $\prod_i \mathbb{C}^{m_i}$, which we can think of as the extended affine Weyl group of $\mathfrak{g}$.

**Definition 5.24.** A weight module $M$ for $Y_\mu^\lambda$ has **integrality class** $\s$ if whenever $W_S(M) \neq 0$, then $\overline{\s} = \s$ (reduction modulo $2\mathbb{Z}$). Similarly a weight module $M$ for $FY_\mu^\lambda$ has **integrality class** $\nu$ if whenever $W_V(M) \neq 0$, then $\nu \in \s$ (regarded as a orbit).

We write $Y_\mu^\lambda$-$\text{wtmod}_\s$ (respectively $Y_\mu^\lambda$-$\mathcal{O}_\s^\pm$, $Y_\mu^\lambda$-$\text{mod}_\s$) for the categories of weight (respectively $\mathcal{O}_\s^\pm$, finite-dimensional) modules over $Y_\mu^\lambda$ having integrality class $\s$.

The Morita equivalence between $Y_\mu^\lambda$ and $FY_\mu^\lambda$ restricts to an equivalence between $Y_\mu^\lambda$-$\text{wtmod}_\s$ and $FY_\mu^\lambda$-$\text{wtmod}_\s$.

By studying the adjoint action on the algebra $Y_\mu^\lambda$ itself, it is easy to deduce the following result.

**Lemma 5.25.** Every indecomposable weight module for $Y_\mu^\lambda$ has an integrality class.

We can extend the notion of integrality class $\s$ to $\mathfrak{g}$ by considering weight modules where the weight space for a pair $(i, a)$ is only non-zero if the multiiset of residues mod $2\mathbb{Z}$ of the complex numbers $a_k$, such that $i_k = i$, equals $\s_i$. Let $X$ be the union in $\mathbb{C}/2\mathbb{Z}$ of images of $R_i + i$ and of $\s_i + i$ for $i \in I$. If $R$ is integral, then the integral weight modules we have studied previously are weight modules with integrality class given by $\s_i = \{i, \ldots, i\}$. In this case $X = \{0\}$.

Consider the Dynkin diagram $I \times X$ and the associated Lie algebra $\mathfrak{g}_X$, which is just $|X|$ copies of $\mathfrak{g}$. We define weights $\lambda_R = \sum_i \sum_{r \in R_i} \omega_{i, \tilde{r} + i}$ and $\mu_\s = \lambda_R - \sum_i \sum_{s \in \s} \alpha_{i, i + s}$.

The integrality class $\s$ is called **$R$-integral** if $X$ coincides with the set of equivalence classes for the equivalence relation on $R$.

**Example 5.26.** Let $I = \{x, y\}$, connected by a single edge $x \to y$ with $\tilde{x} = 0$ and $\tilde{y} = 1$; as before $\mathfrak{g} = \mathfrak{sl}_3$. Consider $R_x = \{0, 1/2, 2 + 3i\}$ and $R_y = \{0, -1/2\}$. Note that the only pair of the elements of $R$ which are equivalent is $(x, 1/2) \sim (y, -1/2)$, so there are 4 equivalence classes.

Thus, an integrality class is **$R$-integral** if the elements of $\s_x$ only contain elements of $\{0, 1/2, 1, i\} \subset \mathbb{C}/2\mathbb{Z}$, and $\s_y$ only contains elements of $\{0, -1/2, 1, 1 + i\}$.
For example, if $\mathcal{S}_x = \{1, \overline{1/2}\}$ and $\mathcal{S}_y = \{0, -\overline{1/2}\}$, then this is an $\mathbf{R}$-integral class and $X = \{0, 1/2, 1, i\}$. We have that

$$
\lambda_\mathcal{R} = \omega_{x,0} + \omega_{x,\overline{1/2}} + \omega_{x,i} + \omega_{y,1} + \omega_{y,\overline{1/2}}
$$

$$
\mu_\mathcal{S} = \lambda_\mathcal{R} - \alpha_{x,1} - \alpha_{y,1} - \alpha_{x,i/2} - \alpha_{y,\overline{1/2}}
$$

Attached to $\mathbf{R}$ and $\mathcal{S}$, we can define metric KLRW algebras $\tilde{\mathfrak{S}}^R_{\mu_\mathcal{S}}$, $\pm \tilde{\mathfrak{S}}^R_{\mu_\mathcal{S}}$, $0 \tilde{\mathfrak{S}}^R_{\mu_\mathcal{S}}$ associated to the Dynkin diagram $I \times X$, where the longitudes on strands with label $(i, s)$ for $s \in \mathbb{C}/\mathbb{Z}$ are required to live in the coset corresponding to $s$ in $\mathbb{C}$; in Definition 3.21, we change the inequalities in (2) and (3) to equalities between real parts of the corresponding complex numbers. Note that these algebras $\tilde{\mathfrak{S}}^R_{\mu_\mathcal{S}}$ depend on the integrality class $\mathcal{S}$, since $X$ depends on $\mathcal{S}$.

We define parity distance (as in Section 3.3) between strands labeled with nodes $(i, x)$ and $(j, x)$ corresponding to the same element of $X$; we only count changes in parity between strands in the same equivalence class as those we are comparing. Given this, we can easily confirm the generalization of Lemma 3.25 and define the parity algebra $\tilde{\mathfrak{P}}^R_{\mu_\mathcal{S}}$ (respectively, $\pm \tilde{\mathfrak{P}}^R_{\mu_\mathcal{S}}$, $0 \tilde{\mathfrak{P}}^R_{\mu_\mathcal{S}}$) Morita equivalent to $\tilde{\mathfrak{S}}^R_{\mu_\mathcal{S}}$ (respectively, $\pm \tilde{\mathfrak{S}}^R_{\mu_\mathcal{S}}$, $0 \tilde{\mathfrak{S}}^R_{\mu_\mathcal{S}}$).

If $\mathcal{S}$ is not $\mathbf{R}$-integral, then that means that some black strand has a label that is not in the same component of $I \times X$ with any red strands. By (3.1c 3.3), every idempotent in $\tilde{\mathfrak{P}}^R_{\mu_\mathcal{S}}$ is isomorphic to one where we pull all strands labeled with this element of $X$ to the far left or far right. Thus, we have that $\pm \tilde{\mathfrak{P}}^R_{\mu_\mathcal{S}} = 0$.

We can now state the generalization of Theorem 1.2 to the non-integral case. The proof of this result follows in the same way as Theorem 1.2, using the above metric (and coarse metric) KLRW algebras and using $\mathfrak{Y}$-modules having integrality class $\mathcal{S}$.

**Theorem 5.27.** Let $\mathbf{R}, \mathcal{S}$ be as above. The categories $Y^\lambda_\mu\text{-wmod}_\mathcal{S}, Y^\lambda_\mu\text{-O}^\pm_\mathcal{S}$ are equivalent to the categories $\tilde{\mathfrak{P}}^R_{\mu_\mathcal{S}}\text{-mod}_{\text{nil}}, \pm \tilde{\mathfrak{P}}^R_{\mu_\mathcal{S}}\text{-mod}_{\text{nil}}, 0 \tilde{\mathfrak{P}}^R_{\mu_\mathcal{S}}\text{-mod}$ of modules over parity KLRW algebras associated to the Dynkin diagram $I \times X$. In particular, if $\mathcal{S}$ is not $\mathbf{R}$-integral, then $Y^\lambda_\mu\text{-O}^\pm_\mathcal{S}$ is trivial.

We can prove this last assertion in a more lower tech way by noting which pairs of weights are related by admissible permutations. Note that if $a_k$ and $a_k \pm 1$ are not equivalent under the relation discussed above, then the permutation $s_k$ is always admissible. Similarly, if $a_m$ is not equivalent to any element of $\mathbf{R}$, then the rightward seam crossing $\sigma_+: W_{i,a} \to W_{\sigma(i)\sigma(a)}$ is an isomorphism (since $\sigma_-\sigma_+$ is multiplication by an invertible polynomial in the dots). Combining these, let $Q$ be a subset of $I \times \mathbb{C}/\mathbb{Z}$ which is closed under $\sim$, and contains no elements of $\mathbf{R}$. If we let $a^{(N)}$ be defined by

$$
a^{(N)}_k = \begin{cases} a_k & (i_k, a_k) \notin Q \\ a_k + 2N & (i_k, a_k) \in Q \end{cases}
$$

then the isomorphisms above show $W_{i,a} \cong W_{i, a^{(N)}}$ for all $N \in \mathbb{Z}$, but lying in $\mathcal{O}^\pm$ requires the latter weight space must be $0$ for $N \gg 0$ (resp. $N \ll 0$).
6. GELFAND-TSETLIN MODULES

Theorem 5.2 allows us to address a number of more classical questions in the representation theory of $U(\mathfrak{gl}_N)$ and its $W$-algebras.

$W$-algebras arise in our setting as truncated shifted Yangians for $\mathfrak{g} = \mathfrak{sl}_n$, and the weight $\lambda = N\omega_1$. Here, we follow the conventions of [WWY], with the exception of the fact that we have specialized $\hbar = 2$ in the Yangian rather than $\hbar = 1$. We will note when this convention causes issues.

Thus, we identify the Dynkin diagram of $\mathfrak{g}$ with $I = \{1, \ldots, n-1\}$. We fix $\lambda = N\omega_1$, and consider a weight $\mu$ with

$$\mu = \sum_{i=1}^{n-1} m_i \alpha_{n-i}, \quad \lambda - \mu = \sum_{i=1}^{n-1} m_i \alpha_{n-i}.$$  
(Notice that in order to match the conventions of [WWY] we have to index these coefficients in a slightly strange way.)

Consider the partition $\pi \vdash N$, defined by

$$p_1 = m_1, p_2 = m_2 - m_1, \ldots, p_{n-1} = m_{n-1} - m_{n-2}, p_n = N - m_{n-1}.$$

Let $W(\pi)$ be the finite $W$-algebra quantizing the Slodowy slice to the nilpotent orbit in $\mathfrak{gl}_N$ with Jordan type given by $\pi$. In particular, $U(\mathfrak{gl}_N)$ itself corresponds to $\mu = 0$ and $\pi = (1, \ldots, 1)$. In general, if $\mu$ is arbitrary, then the result is the OGZ algebras introduced by Mazorchuk in [Maz99].

In [WWY] we give a modification of Brundan and Kleshchev [BK08] which matches the truncated shifted Yangians in type A in our conventions to parabolic $W$-algebras. Applied to this case we get an isomorphism of algebras:

$$Y^\lambda_{\mu} \approx W(\pi).$$

This isomorphism is defined explicitly in [WWY] Section 4.2. It should be emphasized that on the LHS of this equation, the coefficients of the polynomial $p_{n-1}(u)$ which appears in the definition of the $A^{(r)}_i$ (Equation 4.1) are here formal parameters $R^{(1)}_{n-1}, \ldots, R^{(N)}_{n-1}$. Under (6.3), these are sent to generators of the center of $W(\pi)$ (which is isomorphic to that of $U(\mathfrak{gl}_N)$). To avoid confusion, in this section we will not suppress our choice of a set of parameters, so if we specialise $p_{n-1}(u)$ to have complex coefficients with roots given by a multiset $R$ of weight $N\omega_{n-1}$, then we will write $Y^\lambda_{\mu}(R)$ for the corresponding truncated shifted Yangian.

We also note that our isomorphism $\mathbb{C}[R^{(1)}_{n-1}, \ldots, R^{(N)}_{n-1}] \approx Z(\mathfrak{gl}_N)$ is unchanged from [WWY]: our changes of conventions have had cancelling effects. This isomorphism is fixed by the fact that if we specialize $p_{n-1}(u)$ to have roots $r_1 < \cdots < r_N$, then the corresponding ideal in $Z(\mathfrak{gl}_N)$ kills the modules with highest weight

$$1 \frac{1}{2} (r_1 - N + 1, r_2 - N + 3, \ldots, r_N + N - 1)$$

\textsuperscript{2}We have fixed the partition $\tau$ defined in [WWY] §1.2 to be $(1^N)$.\)
We have a chain of algebras $W_1 \subset W_2 \subset \cdots \subset W_n = W(\pi)$ where $W_k = W((p_1, \ldots, p_k))$

**Definition 6.1.** The Gelfand-Tsetlin subalgebra $\Gamma \subset W(\pi)$ is the subalgebra generated by the centers of all $W_k$ inside $W(\pi)$ [EMO10].

Via the isomorphism (6.3), the inclusion of $W_k \subset W_n$ induces a map $Y_{\mu}^{m_{k}\omega_1} \rightarrow Y_{\mu}^{N\omega_1}$ sending $p_{k-1}(u+1) \mapsto A_k(u)$. Here $Y_{\mu}^{m_{k}\omega_1}$ is a truncated shifted Yangian for $\mathfrak{sl}_k$, and $\mu_k = m_k\omega_1 - \sum_{i=1}^{k-1} m_i \alpha_{k-i}$.

Thus, the center of $W_k$ is sent to the subalgebra of $Y_{\mu}^{N\omega_1}$ generated by the $A_k^{(r)}$ for all $r$, and $\Gamma$ is sent to the subalgebra generated by $A_k^{(r)}$, $R_k^{(r)}$ for all $k, r$, which we call the Gelfand-Tsetlin subalgebra of $Y_{\mu}^{N\omega_1}$. This matches the algebra of the same name defined in [BK08].

**Definition 6.2.** We call a finitely generated $W(\pi)$ module a Gelfand-Tsetlin module (or GT module for short) if $\Gamma$ acts locally finitely.

As we’ll now explain, under the isomorphism (6.3), these correspond to weight modules of truncated shifted Yangians, as in Definition 5.1. Theorem 5.2 will therefore allow us to give a description of the category of GT modules for the $W$-algebra $W(\pi)$.

First note that fixing a maximal ideal of the center of $W(\pi)$ corresponds to a set of parameters $R$ of weight $N\omega_{n-1}$. Any simple GT module factors through such a specialization.

Having fixed $R$ we can then further decompose the category of GT modules according to a choice of integrality class $\mathcal{I}$ corresponding to $\pi$. Here $\mathcal{I} = (\mathcal{I}_i)_{i \in I}$, and $\mathcal{I}_i$ is a multi-subset of $\mathbb{C}/2\mathbb{Z}$ of cardinality $m_i$. Every maximal ideal of $\Gamma$ has a corresponding integrality class as in Definition 5.2. Following this definition, we say that a GT module $M$ has **integrality class** $\mathcal{I}$ if all the maximal ideals of $\Gamma$ with non-zero weight spaces are congruent modulo $2\mathbb{Z}$ to $\mathcal{I}$ (that is, $\mathcal{I}$ is their integrality class).

In the terminology of Futorny, Molev and Osienko [EMO10], the irreducible Gelfand-Tsetlin modules in this subcategory are precisely those with central character $R$ that are extended from maximal ideals of $m \subset \Gamma$ with this integrality class. Also, in the study of GT modules there has been some recent focus on non-singular and singular modules (see e.g. [FGR17]). In our setting $m$ is non-singular if the elements of $\mathcal{I}_i$ are all distinct, and singular of index $\ell$ if $\ell$ is the maximal multiplicity of an element of $\mathcal{I}_i$ for some $i$.

**Definition 6.3.** For $R$ a set of parameters of weight $N\omega_{n-1}$ and $\mathcal{I}$ an integrality class corresponding to $\pi$, we let $\text{GT}(\pi, R, \mathcal{I})$ be the category of Gelfand-Tsetlin modules of integrality class $\mathcal{I}$ and central character corresponding to $R$.

Note that (6.3) induces an equivalence $\text{GT}(\pi, R, \mathcal{I}) \cong Y_\mu^{N\omega_1}(R) \text{-wtmod}_\mathcal{I}$, where $\mu$ is determined from $\pi$ as above. Now to relate the category $\text{GT}(\pi, R, \mathcal{I})$ to KLRW algebras, recall that attached to the choices of $R$ and $\mathcal{I}$ we have the set $X \subset \mathbb{C}/2\mathbb{Z}$ as in Section 5.5. Consider the Lie algebra $\mathfrak{sl}_n^X$ given by maps from $X$ to $\mathfrak{sl}_n$. Let $r_1, r_2, \ldots, r_h$ be the distinct complex numbers that appear in $R_{n-1}$, and let $g_1, \ldots, g_h$ be the multiplicity with which they appear. Let $\lambda_k = g_k\omega_1, r_k+1$ be the highest weight of $\text{Sym}^{g_k}(C^k)$, considered as a module over $\mathfrak{sl}_n^X$ with the copy of $\mathfrak{sl}_n$ corresponding to $r_k + 1 \in \mathbb{C}/2\mathbb{Z}$ acting by the natural representation, and all others acting trivially.
Let \( \lambda = (\lambda_1, \ldots, \lambda_h) \), and consider the tensor product
\[
U(\lambda') = U(n) \otimes V(\lambda_h) \otimes \cdots \otimes V(\lambda_1).
\]
Recall that by Proposition 3.1, \( U(\lambda') \) is categorified by the category of finitely generated \( \tilde{T}^{\lambda} \)-modules. Define \( \lambda_R = \sum_{k=1}^{h} \lambda_k \), and \( \mu_\mathcal{S} = \lambda_R - \alpha_\mathcal{S} \), where
\[
\alpha_\mathcal{S} = \sum_{i=1}^{h} \sum_{s \in \mathcal{S}_i} \alpha_{i,s} + \bar{i}.
\]

**Theorem 6.4.** There is an equivalence of categories from \( GT(\pi, R, \mathcal{S}) \) to the category of nilpotent representations of the KLRW algebra \( \tilde{T}_R^\lambda \).

**Proof.** We have:
\[
GT(\pi, R, \mathcal{S}) \cong \text{Y}_{\mu}^{N \omega_1}(R)\text{-wtmod}_\mathcal{S}
\]
\[
\cong \tilde{p}_R^{\mu \mathcal{S}}\text{-mod}_{\text{nil}}
\]
\[
\cong \tilde{T}_R^\lambda\text{-mod}_{\text{nil}} \tag{6.3}
\]
\[
\cong \tilde{T}_R^{\lambda_\mathcal{S}}\text{-mod}_{\text{nil}}. \tag{6.9}
\]

As promised in the introduction, we will leave most detailed discussion of the consequences of this observation to later work, but will note that the results of our earlier sections resolve several important questions about Gelfand-Tsetlin modules.

The first clear consequence is that \( GT(\pi, R, \mathcal{S}) \) is a Deligne tensor product of the categories \( GT(\pi_x, R_x, \mathcal{S}_x) \) where we decompose our data according to the cosets of \( \mathbb{C}/2\mathbb{Z} \) (i.e. it is the representation category of the tensor product of the endomorphism algebras of projective generators in these categories). Thus, all questions about the structure of simples, extensions, weight multiplicities, etc. can be derived from the integral case.

Perhaps of more direct interest, Lemma 3.20 gives a classification of simple Gelfand-Tsetlin modules:

**Corollary 6.5.** There are canonical bijections between:

1. the set of simple objects of \( GT(\pi, R, \mathcal{S}) \);
2. the elements of weight \( \mu_\mathcal{S} \) in the tensor product crystal \( B(\lambda) \otimes B(\infty) \) for \( \mathfrak{sl}_N^X \).

**Example 6.6.** Consider the example \( \pi = (1, \ldots, 1) \); in this case, we have that \( W(\pi) = U(\mathfrak{gl}_N) \). Let us explain how our result applies in this case to modules extended from integral maximal ideals of \( \Gamma \), since this is the most difficult case to attack with conventional methods. First we fix an integral central character \( U(\mathfrak{gl}_N) \). Following our conventions, this corresponds to a choice of distinct integers \( r_1, \ldots, r_h \) such that \( r_i \neq (N \mod 2) \), and positive multiplicities \( g_1, \ldots, g_h \) such that \( \sum g_i = N \). Let \( \mathfrak{m} \) be the corresponding ideal of \( Z(\mathfrak{gl}_N) \); note that there is a finite dimensional module with this central character if and only if \( g_i = 1 \) for all \( i \).

Then we have a canonical bijection between integral irreducible GT modules with central character \( \mathfrak{m} \) and the zero weight space of the \( \mathfrak{sl}_N \)-crystal
\[
B(g_1 \omega_1) \otimes \cdots \otimes B(g_h \omega_1) \otimes B(\infty)
\]
The reader might naturally wonder if this is related to the appearance of tensor product crystals in the structure of category $\mathcal{O}$, for example as in [BK08, Th. 4.4 & 4.5]. While not unrelated, this is not the same as the crystal structure induced by translation functors; rather, it is Howe dual in an appropriate sense (much as in [MW18]).

Furthermore, the weight multiplicities of these simple modules have a natural interpretation. Associated to each metric longitude, we have a projective $\widehat{T}e(S)$, and thus a class in $\widehat{p}(S) \in K^0(\widehat{T}^\lambda) \cong U(\lambda)$.

**Corollary 6.7.** The weight multiplicity of a Gelfand-Tsetlin weight corresponding to $S$ in a simple is the coefficient of the corresponding canonical basis vector when $\widehat{p}(S)$ is expanded in the canonical basis of $U(\lambda)$.

One can use this together with analysis of canonical bases to understand a number of special cases of $\text{GT}(\pi, R, S)$, such as nonsingular or 1-singular modules or non-critical modules, but this result also shows why concrete classification of Gelfand-Tsetlin modules is so hard to do by hand: it essentially requires construction of these canonical bases, and thus an understanding of Kazhdan-Lusztig theory in type A which one cannot expect to achieve so explicitly.

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J. Kamnitzer: Department of Mathematics, University of Toronto, Canada
E-mail address: jkamnitz@math.toronto.edu

P. Tingley: Department of Mathematics and Statistics, Loyola University, Chicago, United States
E-mail address: ptingley@luc.edu

B. Webster: Department of Pure Mathematics, University of Waterloo & Perimeter Institute for Theoretical Physics, Canada
E-mail address: ben.webster@uwaterloo.ca

A. Weekes: Perimeter Institute for Theoretical Physics, Canada
E-mail address: aweekes@perimeterinstitute.ca

O. Yacobi: School of Mathematics and Statistics, University of Sydney, Australia
E-mail address: oded.yacobi@sydney.edu.au