On deformations of curves supported on rigid divisors

Víctor González-Alonso

Institut für Algebraische Geometrie, Leibniz Universität Hannover
Welfengarten 1, 30167 Hannover, Germany
gonzalez@math.uni-hannover.de

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Abstract
Motivated by a conjecture of Xiao, we study supporting divisors of fibred surfaces. On the one hand, after developing a formalism to treat one-dimensional families of varieties of any dimension, we give a structure theorem for fibred surfaces supported on relatively rigid divisors. On the other hand, we study how to produce supporting divisors by constructing a global adjoint map for a fibration over a curve (generalizing the infinitesimal constructions of Collino, Pirola, Rizzi and Zucconi).

1 Introduction

The study of fibrations of algebraic varieties, or more generally, of flat families and deformations, is quite a general problem in algebraic geometry. From the infinitesimal point of view, there is a well understood deformation theory, with the Kodaira-Spencer map playing a central role. Another useful tool to study first-order deformations are supporting divisors, which roughly speaking encode where in the variety the deformation is taking place, and can be somehow used to measure how far a given deformation is from being trivial. In the case of irregular varieties, supporting divisors are closely related to adjoint images (or maps), introduced by Collino and Pirola [5] for curves, extended later to higher dimensions by Pirola and Zucconi [10], and to higher-dimensional base spaces (but still with curves as fibres) by Pirola and Rizzi [9].

In this article we focus on families of varieties over a smooth curve, generalizing both the notion of supporting divisor and the construction of adjoint maps to the non-infinitesimal case. The initial motivation of this work was the following conjecture about the relative irregularity of non-isotrivial fibrations.

Conjecture 1.1 (modified Xiao’s conjecture). For any non-trivial fibration $f : S \to B$ with fibres of genus $g$ and relative irregularity $q_f = q(S) - g(B)$, one has

$$q_f \leq \frac{g}{2} + 1.$$
be useful to study deformations of higher-dimensional varieties, we have developed as many results as possible in this general setting. There are, however, some better results in the specific case of curves that have no obvious higher-dimensional analogue.

The paper is divided into two main sections. In the first one we introduce supporting divisors and their *spans* in the non-infinitesimal case, giving a language to study one-dimensional families of varieties of any dimension. The main result in this section is the following theorem about fibred surfaces.

**Theorem 2.27.** Let \( S \) be a compact surface, and \( f : S \to B \) a fibration by curves of genus \( g \) and relative irregularity \( q_f \geq 2 \). Suppose \( f \) is supported on an effective divisor \( D \) such that \( D \cdot C < 2g - 2 \) and \( h^0(C, \mathcal{O}_C (D|_C)) = 1 \) for some smooth fibre \( C \). Then, possibly after finitely many blow-ups and a change of base, there is a different fibration \( h : S \to B' \) over a curve of genus \( g(B') = q_f \). In particular \( S \) is a covering of the product \( B \times B' \), and both surfaces have the same irregularity.

For us, a supporting divisor for a family \( f : \mathcal{X} \to B \) is a divisor \( D \) on the total space \( \mathcal{X} \) whose restriction to a general fibre supports the corresponding infinitesimal deformation (Definition 3.6). This means that the pull-back sequence (where \( \mathcal{L}_D = \ker(\Omega^1_{\mathcal{X}/B} \to \Omega^1_{\mathcal{X}/B(D)}) \)).

\[
\begin{align*}
\xi_D : & \quad 0 \longrightarrow f^*\omega_B \longrightarrow \mathcal{F}_D \longrightarrow \mathcal{L}_D \longrightarrow 0 \\
\xi : & \quad 0 \longrightarrow f^*\omega_B \longrightarrow \Omega^1_{\mathcal{X}} \longrightarrow \Omega^1_{\mathcal{X}/B} \longrightarrow 0
\end{align*}
\]

is split around a general fibre, but it does not necessarily split globally. Fortunately, if the fibres are curves and the divisor has low degree with respect to the fibres, then the local splitting implies the global splitting (Corollary 2.19). Using this global splitting, it is just a matter of technicalities to prove Proposition 2.25 which shows that (up to change of base) we can replace the splitting subsheaf \( \mathcal{L}_D \) by a line bundle with much better properties. After these results, the proof of Theorem 2.27 follows easily with the help of the classical Castelnuovo-de Franchis theorem.

In the second section we consider the problem of finding supporting divisors for a given family. We first consider the infinitesimal case, introducing the *adjoint bundle* of a variety (Definition 3.6) and computing its top Chern class. This gives a numerical condition on a variety which is sufficient for the existence of a subspace with vanishing adjoint image (Theorem 3.7). Then we construct, for any base \( B \) but considering only curves as fibres, a *global adjoint map*, which glues the adjoint maps on the smooth fibres in a coherent way (covering also the singular fibres). Putting together this construction and the numerical condition mentioned above, we obtain Theorem 3.13 and its following

**Corollary 3.15.** If \( f : S \to B \) is a fibration of genus \( g \) such that \( q_f > \frac{g+1}{2} \), then, possibly after a base change \( f' : S' \to B' \), there is a divisor \( D \subset S' \) supporting \( f' \) and such that \( D \cdot C_b < 2g - 2 \) for any fibre \( C_b = (f')^{-1}(b) \).

That is, if the relative irregularity of the fibration is too big (in particular, it does not verify Xiao’s conjecture), then there is a divisor satisfying the first hypothesis in Theorem 2.27 (up to change of base). In this way, both sections of the paper get clearly connected, and its relation with Xiao’s conjecture is evident.

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## 2 One-dimensional families of varieties

Let \( X \) be a smooth compact complex variety of dimension \( d \), and \( \xi \in H^1(X, T_X) \) the Kodaira-Spencer class of a first-order infinitesimal deformation \( X \), given by the extension of vector bundles

\[
\begin{align*}
\xi : & \quad 0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{F} \longrightarrow \Omega^1_X \longrightarrow 0.
\end{align*}
\]
Definition 2.1. The deformation $\xi$ is said to be supported on an effective divisor $D$ of $X$ if it lies on $K_D = \ker \left( H^1 \left( X, T_X \right) \to H^1 \left( X, T_X \left( D \right) \right) \right)$, or equivalently, if the pull-back sequence

$$
\begin{array}{c}
0 \to \mathcal{O}_X \to \mathcal{F}_D \to \Omega^1_X (-D) \to 0 \\
\xi_D : \\
0 \to \mathcal{O}_X \to \mathcal{F} \to \Omega^1_X \to 0
\end{array}
$$

splits.

Equivalently, assuming it is non-trivial, $\xi$ is supported on $D$ if the point $[\xi] \in \mathbb{P} = \mathbb{P} \left( H^1 \left( X, T_X \right) \right)$ lies in the linear subvariety $\mathbb{P}(K_D)$.

Definition 2.2 (Span of a divisor). The linear subvariety $\mathbb{P}(K_D) \subseteq \mathbb{P}$ is called the span of $D$.

The following Lemma follows tautologically from the definition.

Lemma 2.3. The ideal sheaf of $\mathbb{P}(K_D)$ in $\mathbb{P}$ is generated by the first order of the map $H^1 \left( X, T_X \left( D \right) \right)^{\vee} \to H^1 \left( X, T_X \right)^{\vee} = H^0 \left( \mathbb{P}, \mathcal{O}_\mathbb{P} \left( 1 \right) \right)$.

Remark 2.4. In the case $X = C$ is a curve, $\mathbb{P}$ is its bicanonical space (assuming $g(C) \geq 2$), and the span of $D$ as defined above is precisely the linear span of its image by the bicanonical embedding.

The aim of this section is to glue these constructions to the case of a smooth variety of dimension $d + 1$ fibred over a curve (that is, a one-dimensional family of $d$-dimensional varieties), extending them also to the singular fibres. The previous Lemma is fundamental to generalize the span of a divisor. Some of the ideas used here also appear in [11].

2.1 General setting

From now on, let $f : \mathcal{X} \to B$ be a proper morphism of smooth complex varieties, where the base $B$ is a smooth curve (not necessarily compact), and $\mathcal{X}$ has dimension $d + 1$. This can be considered as a family of $d$-dimensional complex spaces $X_b = \mathcal{X} \times_B \text{Spec} \mathbb{C} (b)$, $b \in B$. Denote by $B^0 \subseteq B$ the open set of regular values, so that $X_b$ is smooth if and only if $b \in B^0$, and denote also by $\mathcal{X}^0 = f^{-1} (B^0)$ the union of the smooth fibres. We will assume that $f$ is non-isotrivial, that is, the smooth fibres are not mutually isomorphic.

For every smooth fibre $X = X_b$, the fibration $f$ induces an infinitesimal deformation, whose Kodaira-Spencer class $\xi \in H^1 \left( X, T_X \right) \otimes T_{B,b}^{\vee} \cong \text{Ext}^1_{\mathcal{O}_X} \left( \Omega^1_X, \mathcal{O}_X \otimes T_{B,b}^{\vee} \right)$ is the extension class of

$$
0 \to N^\vee_{\mathcal{X}/X} = \mathcal{O}_X \otimes T_{B,b}^{\vee} \to \Omega^1_{\mathcal{X}/X} \to \Omega^1_X \to 0,
$$

obtained by restricting the sequence

$$
\xi : \\
0 \to f^* \omega_B \to \Omega^1_X \to \Omega^1_{\mathcal{X}/B} \to 0
$$

(1)

defining the sheaf of relative differentials $\Omega^1_{\mathcal{X}/B}$. Indeed, $\Omega^1_{\mathcal{X}/B}$ is locally free of rank $d$ on $\mathcal{X}^0$, and restricts to the cotangent bundle of the smooth fibres. Its dual is

$$
T_{\mathcal{X}/B} = \mathcal{H}om_{\mathcal{O}_X} \left( \Omega^1_{\mathcal{X}/B}, \mathcal{O}_X \right) = \ker \left( T_X \to f^* T_B \right),
$$

the relative tangent sheaf. It is also locally free of rank $d$ on $\mathcal{X}^0$, and restricts to the tangent bundle of the smooth fibres. We will also consider the relative dualizing sheaf $\omega_{\mathcal{X}/B} = \omega_X \otimes f^* \omega_B^{\vee}$, which is a line bundle on $\mathcal{X}$ and restricts to the canonical bundle of the smooth fibres.
Remark 2.5. Usually, one chooses a generator of $T_{B,b}$ and considers $\xi_b \in H^1(X,T_X)$. However, this might not be done globally on $B$, so we will keep the natural twist by $T_{B,b}^\vee$.

Since the fibration $f$ is not isotrivial, $\xi_b \neq 0$ for general $b \in B^o$ and hence we can consider the point $[\xi_b] \in P_b := \mathbb{P} (H^1(X_b,T_{X_b}))$. Furthermore, if $D$ is any effective divisor on $X$, we can also ask whether $\xi_b$ is supported on $D_b = D|_{C_b}$, and if the answer is positive, what consequences for the fibration $f$ does it have.

We will now globalize the ambient space of the deformations: the vector space $H^1(X,T_X)$ and its projectivization.

Definition 2.6. Let $E$ be the sheaf on $B$ defined as (see the Appendix for the definition and main properties of the relative Ext sheaves)

$$E = \mathcal{E}xt^1_f \left( \Omega^1_{X/B}, f^* \omega_B \right) \cong \mathcal{E}xt^1_f \left( \Omega^1_{X/B}, \mathcal{O}_X \right) \otimes \omega_B,$$

and let

$$\mathbb{P} = \text{Proj}_{\mathcal{O}_B} (\text{Sym}^* E^\vee)$$

be the associated projective bundle, with projection $\pi : \mathbb{P} \to B$.

Note that $E^\vee$ is torsion free over a smooth curve, hence it is locally free and $\mathbb{P}$ is actually a projective bundle.

Lemma 2.7. There is an injection

$$R^1 f_* T_{X/B} \otimes \omega_B \hookrightarrow E$$

which is an isomorphism over an open dense subset of $B^o$. In particular, for a general regular value $b \in B^o$ there is a natural isomorphism

$$E \otimes \mathbb{C} (b) \cong H^1(X_b,T_{X_b}) \otimes T^\vee_{B,b}.$$ 

Proof. The injection is obtained directly from the local-global spectral sequence

$$R^p f_* \mathcal{E}xt^q_{\mathcal{O}_X} \left( \Omega^1_{X/B}, f^* \omega_B \right) \Rightarrow \mathcal{E}xt^{p+q}_f \left( \Omega^1_{X/B}, f^* \omega_B \right),$$

since the beginning of the corresponding five-term exact sequence is

$$R^1 f_* \left( \text{Hom}_{\mathcal{O}_X} \left( \Omega^1_{X/B}, f^* \omega_B \right) \right) \hookrightarrow \mathcal{E}xt^1_f \left( \Omega^1_{X/B}, f^* \omega_B \right) = E,$$

and the projection formula gives

$$R^1 f_* \left( \text{Hom}_{\mathcal{O}_X} \left( \Omega^1_{X/B}, f^* \omega_B \right) \right) \cong R^1 f_* \left( T_{X/B} \otimes f^* \omega_B \right) \cong R^1 f_* T_{X/B} \otimes \omega_B.$$

As for the statement about the general regular values, note that $\Omega^1_{X/B|X^o}$ and $T_{X/B|X^o} = \left( \Omega^1_{X/B|X^o} \right)^\vee$ are both locally free. Therefore, using Theorem 4.3 in the Appendix, we get

$$\mathcal{E}xt^1_f \left( \Omega^1_{X/B|X^o}, \mathcal{O}_{X^o} \right) \cong \mathcal{E}xt^1_f \left( \mathcal{O}_{X^o}, T_{X/B|X^o} \right) \cong R^1 f_* \left( T_{X/B|X^o} \right) = (R^1 f_* T_{X/B})|_{B^o}.$$

Finally, $T_{X/B|X^o} = T_{X_b}$ for any smooth fibre, and the base-change map $E \otimes \mathbb{C} (b) \to H^1(X_b,T_{X_b}) \otimes T^\vee_{B,b}$ is an isomorphism on the open set of $B$ where the function $b \mapsto h^1(X_b,T_{X_b})$ is constant.

By the previous Lemma, the fibre of $\mathbb{P}$ over a general regular value $b$ is isomorphic to

$$\mathbb{P} \left( H^1(X_b,T_{X_b}) \otimes T^\vee_{B,b} \right) = \mathbb{P} \left( H^1(X_b,T_{X_b}) \right),$$

as wanted.
We define now a morphism $\gamma : B \to \mathbb{P}$ (in fact, a section of $\pi : \mathbb{P} \to B$), which maps a general $b \in B^0$ to $[\xi_b]$. Recall that the fibration $f : \mathcal{X} \to B$ defines an element $\xi \in \text{Ext}^1_{\mathcal{O}_X}(\Omega^1_{\mathcal{X}/B}, f^*\omega_B)$ (the extension class of $[\mathcal{X}_b]$). Now, the spectral sequence

$$E^p_q = H^p(B, \text{Ext}^q_f(\Omega^1_{\mathcal{X}/B}, f^*\omega_B)) = \text{Ext}^{p+q}_{\mathcal{O}_X}(\Omega^1_{\mathcal{X}/B}, f^*\omega_B)$$

(2)

gives the map

$$\rho : \text{Ext}^1_{\mathcal{O}_X}(\Omega^1_{\mathcal{X}/B}, f^*\omega_B) \to H^0(B, \text{Ext}^1_f(\Omega^1_{\mathcal{X}/B}, f^*\omega_B)) = H^0(B, \mathcal{E}).$$

(3)

By construction, the section $\rho(\xi)$ of $\mathcal{E}$ maps general value $b \in B^0$ to the Kodaira-Spencer class $\xi_b$ of the deformation of $X_b$. Since we assumed the fibration $f$ to be non-isotrivial, $\rho(\xi)$ does not vanish identically and induces the wanted section $\gamma : B \to \mathbb{P}$.

**Remark 2.8.** We can construct $\gamma : B \to \mathbb{P}$ more formally as follows. Consider the evaluation of $\rho(\xi)$

$$\mathcal{C}(\rho(\xi)) \otimes \mathcal{O}_B \cong \mathcal{O}_B \to \mathcal{E},$$

and let $\mathcal{M} \subseteq \mathcal{O}_B$ be the image of its dual $\mathcal{E}' \to \mathcal{O}_B$. According to [9], Proposition II.7.12, the surjection $\mathcal{E}' \to \mathcal{M}$ corresponds to a map $\gamma : B \to \mathbb{P}$ such that $\gamma^*\mathcal{O}_{\mathbb{P}}(1) = \mathcal{M}$, and it is easy to see that it is the section we want.

Finally, we present a way to globalize the span of a divisor on a fibre.

**Definition 2.9.** For any divisor $\mathcal{D} \subset \mathcal{X}$ (considered as a closed subscheme), define

$$\mathcal{L}_\mathcal{D} = \ker\left(\Omega^1_{\mathcal{X}/B} \to \Omega^1_{\mathcal{X}/B}[\mathcal{D}]\right),$$

and

$$\mathcal{E}_\mathcal{D} = \text{Ext}^1_f(\mathcal{L}_\mathcal{D}, f^*\omega_B).$$

The inclusion $\mathcal{L}_\mathcal{D} \subseteq \Omega^1_{\mathcal{X}/B}$ induces maps of sheaves

$$\mathcal{E} \to \mathcal{E}_\mathcal{D} \quad \text{and its dual} \quad \mathcal{E}_\mathcal{D}' \to \mathcal{E}'. \quad (4)$$

Using the sheaf $\mathcal{E}_\mathcal{D}$ we construct now a subvariety $\mathbb{P}_\mathcal{D} \subseteq \mathbb{P}$ with the property that $\rho(\xi)$ belongs to the kernel of $H^0(B, \mathcal{E}) \to H^0(B, \mathcal{E}_\mathcal{D})$ if and only if the image of $\gamma$ is contained in $\mathbb{P}_\mathcal{D}$ (see Definition 2.13 and Proposition 2.14). In this way, we generalize the notion of a deformation being supported on a divisor on a smooth fibre.

Composing the pull-back of $\mathcal{E}_\mathcal{D}' \to \mathcal{E}'$ by $\pi$ with the natural surjection $\pi^*\mathcal{E}' \to \mathcal{O}_{\mathbb{P}}(1)$ we obtain a map

$$\mu_{\mathcal{D}}(1) : \pi^*\mathcal{E}'_\mathcal{D} \to \mathcal{O}_{\mathbb{P}}(1).$$

**Definition 2.10.** We define $\overline{\mathbb{P}_\mathcal{D}} \subseteq \mathbb{P}$ as the closed subscheme whose sheaf of ideals $\mathcal{J}_\mathcal{D}$ is the image of

$$\mu_{\mathcal{D}} : \pi^*\mathcal{E}'_\mathcal{D} \otimes \mathcal{O}_{\mathbb{P}}(-1) \to \mathcal{O}_{\mathbb{P}}.$$ (5)

The subscheme $\overline{\mathbb{P}_\mathcal{D}}$ is a first generalization of the span of a divisor on a fibre. However, it is not fine enough for us, since it may contain several irreducible components which do not dominate $B$ and hence cannot contain the curve of deformations $\gamma(B)$.

**Lemma 2.11.** The subscheme $\overline{\mathbb{P}_\mathcal{D}}$ contains a unique irreducible component $\overline{\mathbb{P}_\mathcal{D}}$ dominating $B$. The fibre of $\overline{\mathbb{P}_\mathcal{D}}$ over a general value $b \in B^0$ is precisely the span of $\mathcal{D}_\mathcal{X}_b$ (in the sense of Definition 2.3). Moreover, if $\mathcal{D}' \subseteq \mathcal{X}$ is another subscheme with the same components as $\mathcal{D}$ dominating $B$, then $\overline{\mathbb{P}_{\mathcal{D}'}} = \overline{\mathbb{P}_\mathcal{D}}$. 

5
Proof. Let $D_0 \subset S$ be the union of the components of $\mathcal{D}$ that dominate $B$, and let $U \subseteq B^\circ$ be the open set such that $D_{f^{-1}(U)} = D_{0|f^{-1}(U)}$ (the complement in $B^\circ$ of the image of the components of $\mathcal{D}$ not dominating $B$). For any $b \in U$, let $D_b$ denote the restriction of either $D$ or $D_0$ to $X_b$. Then,
\[
\mathcal{L}_{D_{f^{-1}(U)}} \cong \left( \Omega_{X/B}^1(-D_0) \right)_{(f^{-1}(U))} \quad \text{and} \quad \mathcal{E}_{D|U} \cong R^{d-1}f_* \left( \omega_{X/B} \otimes \Omega_{X/B}^1(-D_0) \right)_{|U} \otimes T_U.
\]

Let $V \subseteq U$ be the open set where the function $b \mapsto h^{d-1} \left( X_b, \omega_{X_b} \otimes \Omega_{X_b}^1 (-D_b) \right)$ is constant. For any $b \in V$, the base-change map gives an isomorphism
\[
\mathcal{E}_{D|V} \otimes \mathbb{C} (b) \cong H^{d-1} \left( X_b, \omega_{X_b} \otimes \Omega_{X_b}^1 (-D_b) \right) \otimes T_{B,b}.
\]

Therefore, the map $\mu_D$ restricts to
\[
\mu_{D|b} : H^{d-1} \left( X_b, \omega_{X_b} \otimes \Omega_{X_b}^1 (-D_b) \right) \otimes T_{B,b} \otimes \mathcal{O}_{b, -1} \rightarrow \mathcal{O}_{b, -1},
\]

which, after choosing a generator of $T_{B,b}$, coincides up to scalar with the map in Lemma 2.3. This shows that the fibres of $\mathcal{P}_D$ over any $b \in V$ are the spans of $D_b$ and, shrinking $V$ if necessary, all of them have the same dimension. Hence $\mathcal{P}_D \cap \pi^{-1}(V)$ is irreducible, and we define $\mathcal{P}_D$ to be its closure in $\mathbb{P}$.

The last assertion follows because $\mathcal{P}_D$ is determined by the components of $\mathcal{D}$ dominating $B$. \qed

Definition 2.12 (Span of an effective divisor). Given an effective divisor $\mathcal{D} \subset \mathcal{X}$, we define its span as the subvariety $\mathcal{P}_\mathcal{D}$ of Lemma 2.11.

Note that by Lemma 2.11 the span of a subscheme is determined only by its components not contained in fibres.

Definition 2.13. Analogously to the case of an infinitesimal deformation, we say that the deformation $\xi$ (or also the fibration $f$) is supported on $\mathcal{D}$ if, for general $b \in B$, the infinitesimal deformation $\xi_b$ is supported in $D_{X_b}$. Equivalently, assuming $f$ is not isotrivial, it is supported on $\mathcal{D}$ if the image of $\gamma : B \rightarrow \mathbb{P}$ lies in $\mathcal{P}_\mathcal{D}$.

Remark 2.14. It follows from Lemma 2.11 that, if $\mathcal{D}, \mathcal{D}' \subset \mathcal{X}$ are two effective divisors with exactly the same components dominating $B$, then $\xi$ is supported on $\mathcal{D}$ if and only if it is supported on $\mathcal{D}'$.

Since the definition is local around general smooth fibres, it is easy to prove the following

Lemma 2.15. Let $p : B' \rightarrow B$ be a finite morphism, let $\mathcal{X}'$ be a desingularization of $\mathcal{X} \times_B B'$, and consider the induced commutative diagram

\[
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{p'} & \mathcal{X} \\
\downarrow f' & & \downarrow f \\
B' & \xrightarrow{p} & B
\end{array}
\]

Suppose that $f$ is supported on a divisor $\mathcal{D}$. Then $f'$ is supported on $\mathcal{D}' = (p')^* \mathcal{D}$.

Proposition 2.16. The fibration $f$ is supported on $\mathcal{D} \subset \mathcal{X}$ if and only if $\rho (\xi)$ it is mapped to zero by the map

\[
H^0 \left( B, \mathcal{E} \right) \rightarrow H^0 \left( B, \mathcal{E}_\mathcal{D} \right)
\]

associated to $[\mathcal{D}]$.

Proof. Since $\mathcal{P}_\mathcal{D}$ is the only component of $\mathcal{P}_\mathcal{D}$ dominating $B$, and $\gamma (B)$ dominates $B$, the statement is equivalent to prove that $\rho (\xi)$ maps to zero in $H^0 \left( B, \mathcal{E}_\mathcal{D} \right)$ if and only if $\gamma (B) \subseteq \mathcal{P}_\mathcal{D}$. To this aim, consider
the commutative diagram
\[
\begin{array}{ccc}
\operatorname{Ext}^1_{\mathcal{O}_X}(\Omega^1_{X/B}, f^* \omega_B) & \xrightarrow{\iota^*} & \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{L}_D, f^* \omega_B) \\
\rho & & \rho_D \\
H^0(B, \mathcal{E}) & \xrightarrow{\iota^*} & H^0(B, \mathcal{E}_D)
\end{array}
\]
where the vertical maps are given by the corresponding local-global spectral sequences, and the horizontal ones are induced by the inclusion of sheaves \(\iota : \mathcal{L}_D \hookrightarrow \Omega^1_{X/B}\).

We want to show that \(\gamma(B) \subseteq \tilde{P}_D\) if and only if the section \(\tilde{\xi}_D := \tilde{\iota}^*(\rho(\xi)) = \rho_D(\iota^*(\xi)) \in H^0(B, \mathcal{E}_D)\) is zero. Recall that the morphism \(\gamma : B \to \mathbb{P}\) was defined by the evaluation of \(\rho(\xi)\), so that (see Remark 2.8)
\[\gamma^* \mathcal{O}_B(1) \cong \mathcal{M} = \operatorname{im} (\mathcal{E}_D^\vee \to \mathcal{O}_B) \subseteq \mathcal{O}_B.\]

Recall also that the ideal sheaf \(\mathcal{J}_D\) of \(\tilde{P}_D\) is the image of the composition
\[\pi^* \mathcal{E}_D^\vee (-1) \to \pi^* \mathcal{E}_D^\vee (-1) \to \mathcal{O}_\pi,\]
so \(\gamma^* \mathcal{J}_D(1)\) is generated by the image of the composition
\[\mathcal{E}_D^\vee \to \mathcal{E}_D^\vee \to \mathcal{M} \hookrightarrow \mathcal{O}_B.\]

But this composition is dual to the composition of the evaluation of \(\rho(\xi)\) and the map \(\mathcal{E} \to \mathcal{E}_D\), which is precisely the evaluation of \(\xi_D\).

Therefore, \(\tilde{\xi}_D = 0\) if and only if the map \(\mathcal{E}_D^\vee \to \mathcal{M}\) vanishes. By the previous discussion, this is equivalent to the vanishing of \(\gamma^* \mathcal{J}_D \to \mathcal{O}_B\), which means precisely that the image of \(\gamma\) is (schematically) contained in \(\tilde{P}_D\), finishing the proof.

Remark 2.17. By definition, \(f\) is supported on \(D\) if the pull-back sequence
\[
\begin{array}{cccccc}
\xi_D : & 0 & \to & f^* \omega_B & \to & \mathcal{F}_D & \to & \mathcal{L}_D & \to & 0 \\
\xi : & 0 & \to & f^* \omega_B & \to & \Omega^1_X & \to & \Omega^1_{X/B} & \to & 0
\end{array}
\]
(5)
splits around a general fibre. Of course, this does not imply in general that the pull-back sequence is itself split.

In the language of the previous proof, this means that \(\rho_D\) is not necessarily injective. In fact,
\[\ker \rho_D = H^1(B, f_* \mathcal{H}om(\mathcal{L}_D, f^* \omega_B)),\]
which might not vanish if \(B\) is a compact curve. However, in the next section we will see (Corollary 2.19) that if \(X\) is a surface and \(D\) is not “too big”, then \(\rho_D\) is injective and the local splitting of \(\xi_D\) implies that it splits globally.

2.2 The special case of fibred surfaces

From now until the end of the section, we will assume that \(X = S\) is a surface. We will denote the fibres by \(C_b\) (instead of \(X_b\) or \(S_b\)), and \(g\) will stand for the genus of the general (smooth) fibres.

In this particular case we find a close relation between the sheaves \(\Omega^1_{S/B}\) and \(\omega_{S/B}\), which one can expect since they coincide on the smooth fibres (they both restrict to the canonical bundle). More
precisely, there is a natural map \( \alpha : \Omega_{S/B}^1 \longrightarrow \omega_{S/B} \) which is not available in the general setting above. Also, as we announced in Remark \[2.21\] in this case it is possible to control whether the local splitting of a pullback \([2\) is in fact a splitting over the whole \( S \).

We will first focus in this last question, whose key result is the following Lemma. Recall the definitions of \( L_D \) and \( E_D \) from the previous section.

**Lemma 2.18.** Let \( D \subseteq S \) be an effective divisor such that \( D \cdot C_b < 2g - 2 \). Then the map \( \rho_D : \text{Ext}^1_{D,S}(L_D, f^*\omega_B) \longrightarrow H^0(B, E_D) \)

is an isomorphism.

**Proof.** By the five-term exact sequence associated to the spectral sequence \[2.22\]

we have

\[
\ker \rho_D = H^1(B, f_* \text{Hom}_{O_S}(L_D, f^*\omega_B)) \quad \text{and} \quad \text{coker} \rho_D = H^2(B, f_* \text{Hom}_S(L_D, f^*\omega_B)).
\]

Since \( \dim B = 1 \), it is clear that \( \text{coker} \rho_D = 0 \). It remains to show that \( \ker \rho_D = 0 \), and we will directly show that \( f_* \text{Hom}_{O_S}(L_D, f^*\omega_B) = 0 \). In fact, \( f_* \text{Hom}(L_D, f^*\omega_B) = (f_*L_D) \otimes \omega_B \), so it will be enough to prove that \( f_*L_D = 0 \). The dual \( L_D^\vee \) is torsion-free, so its direct image \( f_*L_D^\vee \) is also torsion-free, hence it is a vector bundle. Therefore, we will be done if we see that \( (f_*L_D^\vee) \otimes C(b) = 0 \) for general \( b \in B \).

As in the proof of Lemma \[2.11\] for a general smooth fibre \( C_b \) we have \( L_{D/C_b} = \omega_{C,(-D_b)} \) and

\[
(f_*L_D^\vee) \otimes C(b) = H^0(C_b, T_{C_b}(D_b)).
\]

To finish, the second term vanishes because \( D \cdot C_b < 2g - 2 \) is equivalent to \( \deg(T_{C_b}(D_b)) < 0 \).

**Corollary 2.19.** If \( D \cdot C_b < 2g - 2 \) for some fibre \( C_b \), then \( f \) is supported on \( D \) if and only if the pull-back sequence \([2\) splits.

We study now the relation between \( \Omega_{S/B}^1 \) and \( \omega_{S/B} \), which is crucial to prove the technical Proposition \[2.23\] and, therefore, Theorem \[2.24\]. First we need an elementary

**Definition 2.20.** The Jacobian ideal sheaf of \( f \) is

\[
J := \text{im}(T_f : T_S \longrightarrow f^*T_B) \otimes f^*\omega_B \subseteq O_S.
\]

It is the ideal of a subscheme \( Z \) supported on the critical points of \( f \). Denote by \( Z_d \) the union of the divisorial components of \( Z \), and by \( Z_p \) the residual subscheme supported on points.

**Remark 2.21.** In \[12\], Serrano defined a sheaf (also denoted by \( J \)) which is essentially our Jacobian ideal sheaf, but without the twisting by \( f^*\omega_B \).

In order to state the main properties relating \( \Omega_{S/B}^1, T_{S/B}, J \) and \( \omega_{S/B} \), we need a little bit more of notation: let \( \{E_i\} \) be the set of irreducible components of the singular fibers, and let \( \nu_i \) be the multiplicity of \( E_i \) as a component of the corresponding singular fibre.

**Lemma 2.22** (\[12\] Lemma 1.1). 1. The relative tangent sheaf \( T_{S/B} \) is an invertible sheaf, whose inverse is

\[
\left( \Omega_{S/B}^1 \right)^\vee = T_{S/B}^\vee \cong \omega_{S/B} \left(-\sum_i (\nu_i - 1) E_i \right).
\]

2. \( J^\vee \cong O_S \left(-\sum_i (\nu_i - 1) E_i \right) \). Therefore \( Z_d = \sum_i (\nu_i - 1) E_i \) and

\[
J = J^\vee \otimes I_{Z_p} = I_{Z_p} \left(-\sum_i (\nu_i - 1) E_i \right).
\]
Lemma 2.23. The sheaves $\Omega^1_{S/B}$ and $\omega_{S/B}$ fit into the exact sequence
\[ 0 \longrightarrow (f^*\omega_B(Z_d))|Z_d \longrightarrow \Omega^1_{S/B} \longrightarrow \omega_{S/B} \longrightarrow \omega_{S/B}|Z \longrightarrow 0. \]
In particular, if all the fibres of $f$ are reduced, then $Z = Z_p$, $\alpha$ is injective and $\Omega^1_{S/B} \cong \omega_{S/B} \otimes J$ is torsion-free. In general, $\omega_{S/B} \otimes J$ is the quotient of $\Omega^1_{S/B}$ by its torsion subsheaf.

Proof. Let us first recall the construction of the map $\alpha : \Omega^1_{S/B} \to \omega_{S/B}$. Twisting the exact sequence defining $\Omega^1_{S/B}$ by $f^*\omega_B$, one gets
\[ 0 \longrightarrow (f^*\omega_B)^{\otimes 2} \longrightarrow f^*\omega_B \otimes \Omega^1_{S/B} \longrightarrow f^*\omega_B \otimes \Omega^1_{S/B} \longrightarrow 0. \]
Wedge product induces a map $\tilde{\beta} : f^*\omega_B \otimes \Omega^1_{S/B} \to \omega_{S/B}$ that maps $(f^*\omega_B)^{\otimes 2}$ to zero. Therefore, $\tilde{\beta}$ induces a map $\beta : f^*\omega_B \otimes \Omega^1_{S/B} \to \omega_{S/B}$. The map $\alpha$ in the statement is precisely $\beta$ twisted by $(f^*\omega_B)^{\vee}$. Denoting by $\tilde{\alpha}$ the corresponding twist of $\beta$, we get the following diagram with exact rows:
\[ \begin{array}{cccccccc}
0 & \longrightarrow & f^*\omega_B & \longrightarrow & \Omega^1_{S/B} & \longrightarrow & \Omega^1_{S/B} & \longrightarrow & 0 \\
& & \downarrow{\tilde{\alpha}} & & \downarrow{\alpha} & & \downarrow{\alpha} & & \\
0 & \longrightarrow & \omega_{S/B} & \longrightarrow & \omega_{S/B} & \longrightarrow & 0 & & \\
\end{array} \]
The snake lemma gives $\text{coker} \alpha = \text{coker} \tilde{\alpha}$ and $\ker \alpha = (\ker \tilde{\alpha})/(f^*\omega_B)$, so it is enough to study the map $\tilde{\alpha}$. But $\tilde{\alpha}$ is exactly the tangent map $T_f$ twisted by $\omega_S$, so by the definition of $J$, $Z$ and $Z_d$ we get
\[ \ker \tilde{\alpha} = \ker (T_f) \otimes \omega_S = T_{S/B} \otimes \omega_{S/B} \otimes f^*\omega_B = f^*\omega_B (Z_d) \]
and
\[ \text{coker} \tilde{\alpha} = \text{coker} (T_f) \otimes \omega_S = (\mathcal{O}_S/J) \otimes \omega_{S/B} = \omega_{S/B}|Z. \]
To conclude, just note that the inclusion $f^*\omega_B \hookrightarrow \ker \tilde{\alpha}$ is induced by the natural map $\mathcal{O}_S \hookrightarrow \mathcal{O}_S(Z_d)$, so
\[ \ker \alpha = f^*\omega_B \otimes (\mathcal{O}_S(Z_d) / \mathcal{O}_S) = (f^*\omega_B(Z_d))|Z_d. \]
\[ \square \]

Before going through the proof of Theorem 2.27, we need a technical result (Proposition 2.25) about inclusions $\mathcal{L} \hookrightarrow \Omega^1_{S/B}$ or $\omega_S \hookrightarrow \omega_{S/B}$ lifting to $\Omega^1_{S}$ (see Definition 2.22 below). It will allow us to improve the properties of any divisor supporting a fibration $f$.

Since this notion will appear very often through the rest of the section, we make first the next

Definition 2.24. We say that a rank-one subsheaf $\mathcal{L}$ of $\Omega^1_{S/B}$ (resp. $\omega_{S/B}$) lifts to $\Omega^1_{S}$ if the inclusion can be factored as an injection $\mathcal{L} \hookrightarrow \Omega^1_{S}$ followed by the natural projection $\Omega^1_{S} \to \Omega^1_{S/B}$ (resp. the same projection composed with $\alpha : \Omega^1_{S/B} \to \omega_{S/B}$). Equivalently, $\mathcal{L} \hookrightarrow \Omega^1_{S/B}$ lifts to $\Omega^1_{S}$ if the pull-back
\[ \xi_{\mathcal{L}} : \begin{array}{cccccccc}
0 & \longrightarrow & f^*\omega_B & \longrightarrow & F_{\mathcal{L}} & \longrightarrow & \mathcal{L} & \longrightarrow & 0 \\
& & \downarrow{\xi} & & \downarrow{\xi} & & \downarrow{\xi} & & \\
0 & \longrightarrow & f^*\omega_B & \longrightarrow & \Omega^1_{S} & \longrightarrow & \Omega^1_{S/B} & \longrightarrow & 0 \\
\end{array} \]
is split.

Recall the notation
\[ \mathcal{L}_D = \ker\left(\Omega^1_{S/B} \to \Omega^1_{S/B}|D\right) \]
introduced in Definition 2.23 for any divisor $D \subseteq S$.

We are now ready to state the announced
Proposition 2.25. Let $f : S \to B$ be a fibration with reduced fibres. If a rank-one subsheaf $L \hookrightarrow \Omega^1_{S/B}$ lifts to $\Omega^1_S$ and satisfies $\deg(L|_{C_b}) > 0$ for some smooth fibre $C_b$, then there exists an effective divisor $D$ on $S$ such that

1. the inclusions $L \hookrightarrow \Omega^1_{S/B}$ and $\omega_{S/B}(-D) \hookrightarrow \omega_{S/B}$ fit into the following chain
   $L \hookrightarrow \omega_{S/B}(-D) \hookrightarrow \Omega^1_{S/B} \xrightarrow{\alpha} \omega_{S/B}$,
2. the injection $\omega_{S/B}(-D) \hookrightarrow \Omega^1_{S/B}$ lifts to $\Omega^1_S$,
3. $D \cdot C_b < 2g-2$ for any fibre $C_b$,
4. $D$ has no component contracted by $f$, and
5. the quotient $\Omega^1_S/\omega_{S/B}(-D)$ is isomorphic to
   $f^*\omega_B \otimes O_S(D) \otimes I_Z$
for some finite subscheme $Z \subset S$, hence torsion-free.

Proof. We proceed in several steps.

Step 1: Obtaining a first divisor $E$ satisfying 1, 2 and 3.

We first show that the double dual $L^{\vee\vee}$ still injects into $\Omega^1_{S/B}$ and lifts to $\Omega^1_S$. Indeed, on the one hand, the lifting $L \hookrightarrow \Omega^1_S$ induces an injective map $L^{\vee\vee} \hookrightarrow \Omega^1_S$. On the other hand, $L$ also injects in $\omega_{S/B}$ because $\alpha$ is injective, hence there is a second injection $L^{\vee\vee} \hookrightarrow \omega_{S/B}$. Both injections fit into the commutative diagram

```
\begin{tikzcd}
L^{\vee\vee} \ar[dr] \ar[dd] & & \Omega^1_S \ar[dl] & \Omega^1_{S/B} \ar[dl, mapsto, near start] \ar[dd] \ar[dl] & \omega_{S/B} \ar[dl, mapsto, near end]
\end{tikzcd}
```

so that the composition $L^{\vee\vee} \hookrightarrow \Omega^1_S \to \Omega^1_{S/B}$ must still be injective, as claimed, and it clearly lifts to $\Omega^1_S$ by construction.

Therefore, we have the sequence of nested sheaves

$L \hookrightarrow L^{\vee\vee} \hookrightarrow \Omega^1_{S/B} \xrightarrow{\alpha} \omega_{S/B}$.

But $L^{\vee\vee}$ is a locally free (reflexive of rank one) subsheaf of $\omega_{S/B}$, hence of the form $\omega_{S/B}(-E)$ for a unique effective divisor $E$. As for the inequality $E \cdot C_b < 2g-2$ for some (any) fibre $C_b$, it follows directly from the hypothesis $\deg(L|_{C_b}) > 0$.

Step 2: Removing the vertical components.

As a previous step, we see that $\xi$ is supported on $E$. Indeed, since $\omega_{S/B}(-E) \hookrightarrow \Omega^1_{S/B}$ lifts to $\Omega^1_S$, the pull-back sequence

```
\begin{tikzcd}
\zeta_E : 0 \arrow[r] & f^*\omega_B \arrow[rr, mapsto] & & \tilde{F}_E \arrow[r] & \omega_{S/B}(-E) \arrow[r] & 0
\end{tikzcd}
```

```
\begin{tikzcd}
\xi : 0 \arrow[r] & f^*\omega_B \arrow[r] & \Omega^1_S \arrow[r] & \Omega^1_{S/B} \arrow[r] & 0
\end{tikzcd}
```

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splits. Now, completing the diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{L}_E & \longrightarrow & \Omega^1_{S/B} & \longrightarrow & \Omega^1_{S/B|E} & \longrightarrow & 0 \\
| & & | & & \alpha & & \downarrow & & \\
0 & \longrightarrow & \omega_{S/B}(-E) & \longrightarrow & \omega_{S/B} & \longrightarrow & \omega_{S/B|E} & \longrightarrow & 0
\end{array}
\]

we obtain an injective map \( \iota_E : \mathcal{L}_E \hookrightarrow \omega_{S/B}(-E) \), and

\[
\xi_E : 0 \longrightarrow f^*\omega_B \longrightarrow \mathcal{F}_E \longrightarrow \mathcal{L}_E \longrightarrow 0
\]

is the pull-back of \( \xi_E \) by \( \iota_E \). Therefore, \( \xi_E \) is split, and thus \( f \) is supported on \( E \) (Corollary 2.19), as claimed.

Denote now by \( E' \leq E \) the divisor obtained by removing from \( E \) the components contracted by \( f \). Clearly, \( f \) is also supported on \( E' \) (Remark 2.14). Furthermore, since \( E' \cdot C_b = E \cdot C_b < 2g - 2 \) for any fibre \( C_b \), Corollary 2.19 again implies that \( \xi_{E'} : 0 \longrightarrow f^*\omega_B \longrightarrow \mathcal{F}_{E'} \longrightarrow \mathcal{L}_{E'} \longrightarrow 0 \) is also split. Hence \( \mathcal{L}_{E'} \hookrightarrow \Omega^1_{S/B} \) lifts to \( \Omega^1_S \), and analogously as we showed in Step 1, \( \mathcal{L}_{E'}^{\vee\vee} \hookrightarrow \Omega^1_{S/B} \) also lifts to \( \Omega^1_S \). To finish, we prove that \( \mathcal{L}_{E'}^{\vee\vee} \cong \omega_{S/B}(-E') \), so that \( E' \) will satisfy conditions 1 through 4. Indeed, the injection \( \iota_{E'} : \mathcal{L}_{E'} \hookrightarrow \omega_{S/B}(-E') \) and its double dual \( \iota_{E'}^{\vee\vee} : \mathcal{L}_{E'}^{\vee\vee} \hookrightarrow \omega_{S/B}(-E') \) are isomorphisms away from the critical points of \( f \). But the critical points form a set of codimension 2 because \( f \) has reduced fibres, hence \( \iota_{E'}^{\vee\vee} \) is an isomorphism, as wanted.

**Step 3: Removing the torsion of the cokernel.**

Up to now, we have an effective divisor \( E' \) satisfying conditions 1 through 4. In particular, \( \omega_{S/B}(-E') \) lifts to \( \Omega^1_S \). Denote by \( M_0 \subseteq \Omega^1_S \) its image, and by \( \tilde{K} \) the quotient \( \Omega^1_S/M_0 \). Let \( T \) be the torsion subsheaf of \( \tilde{K} \), and \( \mathcal{K} = \tilde{K}/T \) its torsion-free quotient. Finally, let \( \mathcal{M} \) be the kernel of the composition of surjections \( \Omega^1_S \twoheadrightarrow \mathcal{K} \rightrightarrows \mathcal{K} \). We want to see that \( \mathcal{M} \) is isomorphic to \( \omega_{S/B}(-D) \) for some divisor \( 0 \leq D \leq E' \).

We first show that \( \mathcal{M} \) is locally free. Clearly, it is torsion-free, and the inclusion \( \mathcal{M} \hookrightarrow \Omega^1_S \) factors as \( \mathcal{M} \hookrightarrow \mathcal{M}^{\vee\vee} \hookrightarrow \Omega^1_S \). Consider now the exact diagram

\[
\begin{array}{cccccccccc}
0 & & 0 & & 0 & & \mathcal{G} & & 0 \\
0 & \longrightarrow & \mathcal{M} & \longrightarrow & \mathcal{M}^{\vee\vee} & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
0 & \longrightarrow & \Omega^1_S & \longrightarrow & \Omega^1_S & \longrightarrow & \mathcal{G} & \longrightarrow & 0 \\
0 & \longrightarrow & \mathcal{G} & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
0 & & 0 & & 0
\end{array}
\]

where we have used the snake lemma to identify the cokernel of the first row and the kernel of the last row. On the one hand, both \( \mathcal{M} \) and \( \mathcal{M}^{\vee\vee} \) have rank one, so \( \mathcal{G} \) is a torsion sheaf and, on the other hand, \( \mathcal{G} \) is torsion free since \( \mathcal{K} \) is. Therefore \( \mathcal{G} = 0 \) and \( \mathcal{M} \cong \mathcal{M}^{\vee\vee} \) is locally free.
To finish, the composition\[
M \hookrightarrow \Omega^1_S \rightarrow \omega_{S/B}
\]
is injective. Indeed, the image $\tilde{M}$ is of rank 1 because $M_0 \subseteq M$ and the image of $M_0$ is $\omega_{S/B} (−E')$, so the kernel of $M \rightarrow \omega_{S/B}$ is a rank-zero subsheaf of a torsion-free sheaf, hence zero. Therefore,
\[
M \cong \tilde{M} = \omega_{S/B} (−D)
\]
with $D \leq E'$ because by construction $\omega_{S/B} (−E') \subseteq \tilde{M}$.

For the other assertion about $K = \Omega^1_S/\omega_{S/B} (−D)$, we first compute the Chern class
\[
c_1 (K) = c_1 (\Omega^1_S) - c_1 (\omega_{S/B} (−D)) = c_1 (f^*\omega_B \otimes \mathcal{O}_S (D)) .
\]
Since $K$ is torsion-free, this means that $K \cong f^*\omega_B \otimes \mathcal{O}_S (D) \otimes L \otimes I_Z$ for some finite subscheme
\[
Z \subset S
\]
and some $L \in \text{Pic}^0 (S)$.

Consider now the diagram of exact rows
\[
\begin{array}{c}
0 \rightarrow \omega_{S/B} (−D) \rightarrow \Omega^1_S \rightarrow K \rightarrow 0 \\
0 \rightarrow \omega_{S/B} (−D) \rightarrow \omega_{S/B} \rightarrow \omega_{S/B[D]} \rightarrow 0
\end{array}
\]
(6)

Since $f$ has reduced fibres, the map $\alpha : \Omega^1_{S/B} \rightarrow \omega_{S/B}$ is injective and its cokernel is supported on the finite subscheme $Z'$ of critical points of $f$ (Lemma 2.23). Hence the central map in (6) has kernel $f^*\omega_B$ and cokernel $\omega_{S/B[Z']}$, and the snake lemma leads to the exact sequence
\[
0 \rightarrow f^*\omega_B \rightarrow K \rightarrow \omega_{S/B[D]} \rightarrow \omega_{S/B[Z']} \rightarrow 0 .
\]
The first map corresponds to a section
\[
\sigma \in H^0 (S, \omega_S (D) \otimes L \otimes I_Z) \subset H^0 (S, \mathcal{O}_S (D) \otimes L)
\]
whose zero scheme is $D$. Indeed, the zero scheme $Z (\sigma)$ is contained in $D$, and coincides with it outside the finite subscheme $Z'$. This implies that $L \cong \mathcal{O}_S$ and we are done.

\[
\square
\]

**Remark 2.26** (About Step 3 in the proof of Proposition 2.25). If a subsheaf of the form $M_0 = \omega_{S/B} (−E') \subseteq \omega_{S/B}$ lifts to $\Omega^1_S$, there is an easy geometric interpretation of the support of the divisor $E'$: it is the locus where $M_0 \subseteq \Omega^1_S$ is not transverse to $f^*\omega_B$, that is
\[
\text{Supp } E' = \{ p \mid \text{ im } (f^*\omega_B \otimes M_0)_p \rightarrow \Omega^1_{S,p} \neq \Omega^1_{S,p} \} \quad = \{ p \mid \text{ im } ((f^*\omega_B \otimes M_0) \otimes \mathcal{C} (p) \rightarrow \Omega^1_{S,p} \otimes \mathcal{C} (p)) \neq \Omega^1_{S,p} \otimes \mathcal{C} (p) \} .
\]

The failure of the transversality at some regular point $p \in E'$ may occur either because

1. the images of $M_0 \otimes \mathcal{C} (p)$ and $(f^*\omega_B) \otimes \mathcal{C} (p)$ in $\Omega^1_{S,p} \otimes \mathcal{C} (p)$ coincide, or

2. because $M_0 \otimes \mathcal{C} (p)$ maps to zero in $\Omega^1_{S,p} \otimes \mathcal{C} (p)$.

The first case means that not all local sections of $M_0$ vanish at $p$, but their values are proportional to pull-backs of 1-forms on $B$, while the second case means that all local sections of $M_0$ vanish at $p$. An immediate computation in local coordinates shows that if the second case happens along some components $E'_k$ of $E'$, the quotient sheaf $\Omega^1_S/M_0$ would have torsion supported on $E'_k$. The last step in the proof of Proposition 2.25 replaces $E'$ by $D = E' - E'_k$.  

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We close now this section with Theorem 2.27 about the structure of fibrations supported on relatively rigid divisors (that is, divisors whose restriction to a general fibre is rigid).

**Theorem 2.27.** Let \( S \) be a compact surface, and \( f : S \to B \) a fibration by curves of genus \( g \) and relative irregularity \( q_f = q(S) - q(B) \geq 2 \). Suppose \( f \) is supported on an effective divisor \( D \) such that \( D \cdot C < 2g - 2 \) and \( h^0(C, \mathcal{O}_C(D_C)) = 1 \) for some smooth fibre \( C \). Then, possibly after finitely many blow-ups and a change of base, there is a different fibration \( h : S \to B' \) over a curve of genus \( g(B') \geq q_f \). In particular \( S \) is a covering of the product \( B \times B' \), and both surfaces have the same irregularity.

**Proof.** By general results on fibred surfaces (e.g. [2] Thm III.10.3), after blowing-up some points and a change of base, we can assume that \( f \) has reduced fibres and still satisfies the rest of the hypotheses. Also, the new fibration is isotrivial if and only if the original one was.

Now, \( \deg(\mathcal{L}_{D|C}) = 2g - 2 - (D \cdot C) > 0 \) for a general fibre \( C \), because \( D \cdot C < 2g - 2 \). The same inequality gives, by Corollary 2.19, that the inclusion \( \mathcal{L}_D \to \Omega^1_{S/B} \) lifts to \( \Omega^1_S \). Applying Proposition 2.26, we can replace \( D \) by a divisor (still called \( D \) for simplicity) and assume that \( \omega_{S/B}(-D) \) lifts to \( \Omega^1_S \). \( D \) has no component contracted by \( f \) and that the cokernel \( \mathcal{K} = K_D \) of the lifting is torsion-free, isomorphic to \( f^*\omega_B \otimes \mathcal{O}_S(D) \otimes I_Z \) for some finite subscheme \( Z \subset S \). Since we have replaced \( D \) by a divisor, it still holds that \( h^0(C, \mathcal{O}_C(D_C)) = 1 \) for some (hence a general) smooth fibre \( C \).

**Claim:** \( h^0(S, \omega_{S/B}(-D)) \geq q_f \). Indeed, it follows from the exact sequence

\[
0 \to \omega_{S/B}(-D) \to \Omega^1_S \to \mathcal{K} \to 0
\]

that \( h^0(\omega_{S/B}(-D)) \geq h^0(\Omega^1_S) - h^0(\mathcal{K}) = q(S) - h^0(\mathcal{K}) \), so it is enough to prove that \( h^0(\mathcal{K}) = g(B) \).

Since \( f \) has reduced fibres, \( \Omega^1_{S/B} \) is a subsheaf of \( \omega_{S/B} \) (Lemma 2.28) and the sequence

\[
0 \to f^*\omega_B \to \Omega^1_S \to \omega_{S/B}
\]

is exact. Applying the snake lemma to the diagram of exact rows

\[
\begin{array}{ccccccc}
0 & \to & \omega_{S/B}(-D) & \to & \Omega^1_S & \to & \mathcal{K} & \to & 0 \\
| & & | & & | & & | & & |
0 & \to & \omega_{S/B}(-D) & \to & \omega_{S/B} & \to & \omega_{S/B|D} & \to & 0
\end{array}
\]

we get that the kernel of \( \mathcal{K} \to \omega_{S/B|D} \) is also \( f^*\omega_B \). Therefore, taking direct images, we obtain the following exact sequence of sheaves on \( B \)

\[
0 \to \omega_B \to f_*\mathcal{K} \to f_*\omega_{S/B|D}.
\]

Since \( \mathcal{K} \cong f^*\omega_B \otimes \mathcal{O}_S(D) \otimes I_Z \) is torsion-free and \( D|C \) is rigid for a general fibre \( C \),

\[
f_*\mathcal{K} = \omega_B \otimes f_*(\mathcal{O}_S(D) \otimes I_Z)
\]

is a vector bundle (torsion-free over a curve) of rank one. Therefore, the cokernel of the injection \( \omega_B \hookrightarrow f_*\mathcal{K} \) must be a torsion sheaf of \( f_*\omega_{S/B|D} \). But the latter is torsion-free because \( D \) has no component contracted by \( f \) (see Lemma 2.28 below), so the injection \( \omega_B \hookrightarrow f_*\mathcal{K} \) is in fact an isomorphism, and

\[
h^0(S, \mathcal{K}) = h^0(B, f_*\mathcal{K}) = h^0(B, \omega_B) = g(B),
\]

finishing the proof of the claim.

Since the lifting of \( \omega_{S/B}(-D) \) to \( \Omega^1_S \) is a line bundle \( \mathcal{L} \), the wedge product of any two of its sections is zero. Therefore, since we have just seen that \( h^0(\mathcal{L}) \geq q_f \geq 2 \), the Castelnuovo-de Franchis Theorem ([1], Theorem 1.9) implies the existence of the fibration \( h : S \to B' \) over a curve \( B' \) of genus \( g(B') \geq q_f \).
It remains to show that $g(B') = q_f$, which follows from the last structural statement. In fact, the two fibrations give a covering $\pi$ completing the diagram

$$
\begin{array}{ccc}
S & \xrightarrow{f} & B \\
\downarrow{\pi} & & \downarrow{h} \\
B \times B' & \xrightarrow{\varphi} & B'
\end{array}
$$

Since $\pi$ is surjective, $q(S) \geq q(B \times B') = g(B) + g(B')$, hence $g(B') \leq q_f$, and the proof is finished. \[\square\]

Lemma 2.28. If $f : S \to B$ is any fibration, $D$ is an effective divisor on $S$ without components contracted by $f$, and $L$ is any line bundle on $S$, then $f_* (L|_D)$ is a torsion-free sheaf on $B$.

Proof. We have to show that, given any open subset $U \subseteq B$ and any non-zero section

$$
\alpha \in H^0 (U, f_* (L|_D)) = H^0 (f^{-1} (U), L|_D),
$$

the condition

$$
\phi_\alpha = 0 \in H^0 (U, f_* (L|_D)) = H^0 (f^{-1} (U), L|_D)
$$

for some $\phi \in H^0 (U, \mathcal{O}_B)$ implies that $\phi = 0$.

Let $p \in D \cap f^{-1} (U)$ be any point, $R = \mathcal{O}_{S,p}$ and $T = \mathcal{O}_{B, f(p)}$ the local rings at $p$ and $f (p)$, and $\mathfrak{m}_R$ and $\mathfrak{m}_T$ the corresponding maximal ideals. Recall that both $R$ and $T$ are factorial rings because $S$ and $B$ are smooth varieties, and also that $f$ induces an injection $f^* : T \to R$ (because it is surjective).

Let $d \in \mathfrak{m}_R$ be a local equation for $D$ near $p$, which has no factors in $f^* \mathfrak{m}_T$ because $D$ has no component contracted by $f$. Let

$$
\tilde{\alpha} \in L_p \cong R
$$

be a germ of section of $L$ at $p$ that restricts to the germ of $\alpha$ in

$$
(L|_D)_{p} \cong R/ \langle d \rangle.
$$

The condition $\phi_\alpha = 0$ means that $(f^* \phi_p) \tilde{\alpha} \in \langle d \rangle$. But the factoriality of $R$ and the fact that $d$ has no factors in $f^* \mathfrak{m}_T$ imply that either $\tilde{\alpha} \in \langle d \rangle$ or $f^* \phi_p = 0$. But the first condition cannot happen for every $p \in D \cap f^{-1} (U)$, since it would imply that $\alpha = 0$. Hence we obtain that for some $p$ we have $\phi_p = 0$, and therefore $\phi = 0$, as wanted. \[\square\]

3 The global adjoint map and supporting divisors

The main topic of this Section are adjoint images, which have proved to be a useful tool to study both infinitesimal and local deformations of irregular varieties. They were introduced in the study of curves by Collino and Pirola in [5], and then extended to higher-dimensional varieties by Pirola and Zucconi in [10]. A more intrinsical construction, in terms of adjoint maps, was given later by Pirola and Rizzi in [9] for smooth families of curves. The aim of this section is to generalize the construction of the adjoint map both to first-order deformations of higher-dimensional (irregular) varieties, and to all the fibres of a family of curves (in particular, to fibred surfaces). This second generalization will allow us to produce supporting divisors under suitable assumptions on the fibration.

3.1 Adjoint map of an infinitesimal deformation

We first introduce adjoint maps. Although as in the previous Section, their main applications deal with curves, most of the constructions and basic results also work for higher dimensions. Hence, we present adjoint maps in their most general form, for varieties of arbitrary dimension.
Let $X$ be a smooth projective variety of dimension $d$. For any integer $k = 1, \ldots, d$ we consider the map

$$
\psi_k : \bigwedge^k H^0 (X, \Omega_X^1) \longrightarrow H^0 (X, \Omega_X^k)
$$
given by wedge product (for $k = 1$ it is simply the identity). Given a linear subspace $W \subseteq H^0 (X, \Omega_X^1)$, we define

$$
W^k = \psi_k \left( \bigwedge^k W \right) \subseteq H^0 (X, \Omega_X^k).
$$

In particular, for $k = d$, $W^d \subseteq H^0 (X, \omega_X)$.

**Definition 3.1.** If $W^d \neq 0$, denote by $D_W$ the base divisor of the linear series $|W^d| \subseteq |\omega_X|$.

Consider now a first-order infinitesimal deformation $\mathcal{X} \rightarrow \Delta = \text{Spec} \mathbb{C}[[\epsilon]] / (\epsilon^2)$ of $X$, corresponding to an extension class $\xi \in \text{Ext}^1_{\mathcal{O}_X} (\Omega_X^1, \mathcal{O}_X \otimes T^\vee_{\Delta,0}) \cong H^1 (X, T_X) \otimes T^\vee_{\Delta,0} \cong H^1 (X, T_X)$, the Kodaira-Spencer class of the sequence of vector bundles

$$
0 \longrightarrow N^\vee_{X/\mathcal{X}} = \mathcal{O}_X \otimes T^\vee_{\Delta,0} \cong \mathcal{O}_X \longrightarrow \Omega^1_{X/\mathcal{X}} \longrightarrow \Omega^1_X \longrightarrow 0.
$$

The corresponding connecting homomorphism

$$
\partial_\xi = \cup \xi : H^0 (X, \Omega_X^1) \longrightarrow H^1 (X, \mathcal{O}_X) \otimes T^\vee_{\Delta,0}
$$
is given by cup-product with $\xi$. Denote by

$$
K = K_\xi = \ker \partial_\xi = \text{im} \left( H^0 (X, \Omega^1_{X/\mathcal{X}}) \longrightarrow H^0 (X, \Omega_X^1) \right)
$$
the subspace of 1-forms on $X$ that can be extended to $\mathcal{X}$, and assume $\dim K_\xi \geq d + 1$ (in particular, $q (X) \geq d + 1$).

Given any subspace $W \subseteq K_\xi$, denote by $W^\vee \subseteq H^0 (X, \Omega^1_X)$ its preimage, so that we have the following exact sequence

$$
0 \longrightarrow T^\vee_{\Delta,0} \longrightarrow W^\vee \longrightarrow W \longrightarrow 0,
$$
from which we obtain the presentation

$$
T^\vee_{\Delta,0} \otimes \bigwedge^d W^\vee \longrightarrow \bigwedge^{d+1} W^\vee \longrightarrow \bigwedge^{d+1} W \longrightarrow 0.
$$

Wedge product induces also a map

$$
\bigwedge^{d+1} W \longrightarrow H^0 (X, \Omega_X^{d+1}) \cong T^\vee_{\Delta,0} \otimes H^0 (X, \omega_X),
$$
and it is clear that the image of $T^\vee_{\Delta,0} \otimes \bigwedge^d W^\vee$ maps precisely to $T^\vee_{\Delta,0} \otimes W^d$. Hence, there is a well-defined map

$$
\nu_W : \bigwedge^{d+1} W \longrightarrow T^\vee_{\Delta,0} \otimes (H^0 (X, \omega_X) / W^d)
$$
completing the diagram below.

$$
\begin{array}{cccccc}
T^\vee_{\Delta,0} \otimes \bigwedge^d W^\vee & \longrightarrow & \bigwedge^{d+1} W^\vee & \longrightarrow & \bigwedge^{d+1} W & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
T^\vee_{\Delta,0} \otimes W^d & \longrightarrow & T^\vee_{\Delta,0} \otimes H^0 (X, \omega_X) & \longrightarrow & T^\vee_{\Delta,0} \otimes (H^0 (X, \omega_X) / W^d) & \longrightarrow & 0
\end{array}
$$
Definition 3.2. The map \( \nu_W \) in (7) is the adjoint map associated to \( W \).

In their works [5], [10], Collino, Pirola and Zucconi restrict to the case \( \dim W = d + 1 \), while the above construction works for any subspace \( W \) of dimension at least \( d + 1 \) (in fact, if \( \dim W \leq d \), then \( \bigwedge^{d+1} W = 0 \) and \( \nu_W \) is the zero map). They also start from a basis \( \eta_1, \ldots, \eta_{d+1} \) of \( W \), choosing arbitrary preimages \( s_1, \ldots, s_{d+1} \in H^0 X, \Omega^1_X \), and showing that the class \([w]\) of \( s_1 \wedge \cdots \wedge s_{d+1} \) in \( H^0 (X, \omega_S)/W^d \) is well-defined up to scalar (depending on a chosen isomorphism \( T_{\Delta,0} \cong \mathbb{C} \) and the basis \( \{\eta_i\} \)). They call the class \([w]\) an adjoint class of \( W \), which in our setting is precisely the image of \( \eta_1 \wedge \cdots \wedge \eta_{d+1} \) by the adjoint map \( \nu_W \).

We can now state one of the most powerful results about adjoint images (or maps): the Adjoint Theorem. It was first proven by Collino and Pirola for curves ([5] Th. 1.1.8), and then it was generalized to arbitrary dimensions by Pirola and Zucconi ([10] Th. 1.5.1). Recall that the deformation \( \xi \) is said to be supported on an effective divisor \( D \subset X \) if

\[
\mathbb{C} = \ker (H^1(X, T_X) \rightarrow H^1(X, T_X(D)))
\]

Theorem 3.3 (Adjoint Theorem, [10] Th. 1.5.1, [5] Th. 1.1.8 for curves). Let \( W \subseteq K_\xi \subseteq H^0(X, \omega_X) \) be a \((d+1)\)-dimensional subspace such that \( W^d \neq 0 \), and let \( D = D_W \) be the base locus of the corresponding linear series \([W^d] \subseteq [\omega_X]\). If the adjoint map of \( W \) is the zero map, then \( \xi \) is supported on \( D \).

Our next objective is to use the Adjoint Theorem to produce supporting divisors of a given deformation \( \xi \). In particular, we want to give a condition on \( \dim K_\xi \) that guarantees the existence of a \((d+1)\)-dimensional subspace \( W \) with vanishing adjoint map. Some of the following constructions and results are inspired by the study of special deformations carried out by Collino and Pirola in [5], Section 1.3.

We will need the following

Extra assumption 3.4. For any \((d+1)\)-dimensional subspace, \( \bigwedge^d W \) injects into \( H^0(X, \omega_X) \). According to the Generalized Castelnuovo-de Franchis Theorem ([4], Theorem 1.9), this can be more geometrically rephrased as “\( X \) does not admit higher-irrational pencils” (i.e. fibrations over varieties whose Albanese map is birational and not surjective).

Since this condition holds automatically if \( X = C \) is a curve, and our main applications are for curves, this is not really a restrictive condition for our purposes. Furthermore, the condition can be relaxed to “the images \( W^d \subseteq H^0(X, \omega_X) \) have all the same dimension”.

Let \( \mathbb{G} \) be the Grassmannian of \((d+1)\)-dimensional subspaces of \( K = K_\xi \). For any vector space \( E \), denote by \( E_\mathbb{G} = E \otimes \mathbb{G} \) the trivial vector bundle on \( \mathbb{G} \) with fibre \( E \). As customary, denote by \( S \subseteq K_\mathbb{G} \) and \( Q = K_\mathbb{G}/S \) the tautological subbundle and quotient bundle. Note that the extra assumption above implies that \( \bigwedge^d S \) injects in \( H^0(X, \omega_X)_\mathbb{G} \) as a vector bundle of rank \( d + 1 \), and the quotient is also a vector bundle (of rank \( p_{q_1}(X) - (d + 1) \)).

Lemma 3.5. The adjoint maps \( \nu_W \) depend holomorphically on \( W \in \mathbb{G} \). More precisely, there exists a map of vector bundles

\[
\nu : \bigwedge^{d+1} S \rightarrow T_{\Delta,0}^{\vee} \otimes \left( H^0(X, \omega_X)_\mathbb{G} / \bigwedge^d S \right).
\]

such that \( \nu \otimes \mathbb{C}(W) = \nu_W \).

Proof. The proof is quite immediate. One only has to mimick the construction of the \( \nu_W \) replacing \( W \) by the tautological subbundle \( S \).

Denote by \( \tilde{S} \subseteq H^0 X, \Omega^1_X \) the preimage of \( S \subseteq K_\mathbb{G} \) by the natural projection \( \pi : H^0 X, \Omega^1_X \rightarrow K \), which is a vector bundle of rank \( d + 2 \) and fits into the exact sequence

\[
0 \rightarrow T_{\mathbb{G}}^{\vee} \rightarrow \tilde{S} \rightarrow S \rightarrow 0,
\]
(where $T = T_{\Delta,0}$). The analogue to the diagram \([\insetbox] is

\[
\begin{array}{ccc}
T^\vee \otimes \Lambda^d \tilde{S} & \rightarrow & \Lambda^{d+1} \tilde{S} \\
\downarrow & & \downarrow \nu \\
T^\vee \otimes \Lambda^d S & \rightarrow & T^\vee \otimes H^0(X, \omega_X)_G \\
\downarrow & & \downarrow \\
T^\vee & \rightarrow & T^\vee \otimes \left( H^0(X, \omega_X)_G / \Lambda^d S \right) \\
\end{array}
\]

where the central vertical arrow is also given by wedge product and the isomorphism $\Omega^{d+1}_{X|X} \cong T^\vee \otimes \omega_X$.

It is immediate to check that the map $\nu$ gives the adjoint map $\nu_W$ at any point $W$.

**Definition 3.6.** We call the map $\nu$ constructed in the previous Lemma simply the adjoint map of the deformation $\xi$. It can be seen as a section of the vector bundle

\[
\mathcal{A} = T_{\Delta,0}^\vee \otimes \Lambda^d S \otimes \left( H^0(X, \omega_X)_G / \Lambda^d S \right),
\]

which we call the adjoint bundle.

Computing the Chern class of $\mathcal{A}$ leads to the proof of the following

**Theorem 3.7.** If $V \subseteq K_{\xi}$ has dimension $\dim V \geq \frac{r_g(X)}{d+1} + d$, then there exists some $(d+1)$-dimensional subspace $W \subseteq V$ such that $\nu_W = 0$.

**Proof.** Let $G_V = Gr(d+1, V) \subseteq G$ be the subvariety of $G$ consisting of the $(d+1)$-dimensional subspaces of $K$ contained in $V$, which is in turn a Grassmannian variety. Furthermore, the tautological subbundle $S_V$ of $G_V$ is the restriction of $S$, and the adjoint map $\nu$ restricts to

\[
\nu_V : \Lambda^{d+1} S_V \rightarrow T^\vee \otimes \left( H^0(X, \omega_X)_G / \Lambda^d S_V \right)
\]

(as above, we have simplified $T = T_{\Delta,0}$) which is a section of

\[
\mathcal{A}_V = T^\vee \otimes \Lambda^d S_V \otimes \left( H^0(X, \omega_X)_G / \Lambda^d S_V \right) = \mathcal{A}|_V.
\]

Denoting by $Z = Z(\nu) \subseteq G$ the zero locus of $\nu$, and by $Z_V$ the zero locus of $\nu_V$, it is clear that $Z_V = Z \cap G_V$.

With these notations, the theorem says that $Z_V \neq \emptyset$. In order to prove that, we will compute the top Chern class of $\mathcal{A}_V$ and show that it does not vanish. This is enough, since if a vector bundle admits a nowhere vanishing section, then its top Chern class is zero.

First of all, our only hypothesis is equivalent to

\[
r = \text{rk} \mathcal{A}_V = p_g(X) - (d+1) \leq (d+1) (\dim V - (d+1)) = \dim G_V,
\]

so it is indeed possible that $c_r(\mathcal{A}_V) \neq 0$.

Secondly, up to the trivial twisting by $T^\vee$, $\mathcal{A}_V$ is the globally generated bundle

\[
\mathcal{G} = H^0(X, \omega_X)_{G_V} / \Lambda^d S_V
\]

twisted by the line bundle $\Lambda^{d+1} S_V \cong \mathcal{O}_{G_V}(1)$, the very ample line bundle inducing the Plücker embedding.
Therefore, we can compute (see [7] Remark 3.2.3.(b)) Summing up, since all the Chern classes of \( G \) are represented by zero or effective cycles (because it is globally generated), we obtain
\[
c_r (A_V) = \sum_{i=0}^{r} c_{r-i} (G) c_1 (O_{G_V} (1)) = c_1 (O_{G_V} (1)) + (\text{effective classes}) \neq 0
\]
because \( O_{G_V} (1) \) is very ample.

\[\text{Corollary 3.8.} \quad \text{If } X = C \text{ is a curve of genus } g, \text{ and } V \subseteq K_X \text{ has dimension greater than } \frac{g+1}{2}, \text{ then there exists a two-dimensional subspace } W \subseteq V \text{ whose adjoint class vanishes. In particular, the deformation is supported on a divisor } D \text{ of degree } \deg D < 2g - 2.\]

\[\text{Proof.} \quad \text{The first assertion follows directly from Theorem}\ [\text{[4]}\] taking } d = 1. \text{ The second assertion, take } D \text{ the base divisor of } |W| \subseteq |\omega_C|, \text{ which obviously has degree smaller than } 2g - 2.\]

\[\text{Remark 3.9.} \quad \text{For higher dimensions, the inequality } \dim V \geq \frac{2q(X)}{q(X) + 1} + d, \text{ combined with the non-existence of higher irrational pencils (assumption } [\text{[4]}], \text{ becomes a quite restrictive condition. For example, the only surfaces to which this method could be applied are those satisfying}
\]
\[2q(X) - 3 \leq p_g (X) \leq 3 (q(X) - 2),\]
\[\text{where the first inequality is the Castelnuovo-de Franchis inequality.}\]

\[\text{3.2 Global adjoint map}\]

In this last section we extend the previous constructions to the case of a fibration over a compact curve. Since the extra assumption [5] is quite restrictive, we will only consider the case when the fibres are curves (though some constructions carry over to some cases with higher-dimensional fibres).

Therefore, let \( f : S \to B \) be a fibration of a surface \( S \) over a compact curve \( B \), and denote by
\[
V = V_f = H^0 (S, \Omega^1_S) / f^* H^0 (B, \omega_B),
\]
which has dimension \( q_f \), the relative irregularity of \( f \). It is easy to see that \( V \) naturally injects into \( H^0 (C, \omega_C) \) for any smooth fibre \( C \) of \( f \). Furthermore, if \( \xi \in H^1 (C, T_C) \) is the infinitesimal deformation of \( C \) induced by \( f \), then \( V \) is contained in the kernel \( K_\xi \) of the cup-product map
\[
\cup \xi : H^0 (C, \omega_C) \to H^1 (C, \Omega^1_C).
\]

We have previously constructed the adjoint map associated to any subspace of \( K_\xi \). In order to obtain a "global" adjoint map valid for all the fibres, we restrict now to a slightly less general version, considering only subspaces \( W \subseteq V \) of \( V \).

All the injections \( V \subseteq H^0 (C, \omega_C) \) for smooth fibres glue together into an inclusion of vector bundles
\[
V_B = V \otimes \mathcal{O}_B \hookrightarrow f_* \omega_S|_B \tag{9}
\]
whose cokernel \( G \) is also a vector bundle (see [8] Theorem 3.1 and its proof). In fact, the results of Fujita say moreover that the inclusion splits (so \( f_* \omega_S|_B \equiv V_B \oplus G \)) and \( G \) has some good cohomological properties, but we will not use them in the sequel.

The inclusion \( \tag{9} \) can be more explicitly constructed as follows. First of all, wedge product gives a natural map \( H^0 (S, \Omega^1_S) \otimes \omega_B \to f_* \omega_S \). Clearly \( (f^* H^0 (B, \omega_B)) \otimes \omega_B \) maps to zero, so there is an induced map
\[
V \otimes \omega_B \to f_* \omega_S = (f_* \omega_S|_B) \otimes \omega_B.
\]
Since it is injective over a generic \( b \in B \), it is everywhere injective (as a map of sheaves), and cancelling the twist by \( \omega_B \) we obtain the inclusion \( \tag{9} \).

Denote now by \( G = Gr (2, V) \) the Grassmannian of 2-planes of \( V \), and by \( S_V \subseteq V \otimes \mathcal{O}_G \) the tautological subbundle. Consider the product \( Y = B \times G \), and denote by \( p_1 : Y \to B \) and \( p_2 : Y \to G \) the natural
projections. The variety $Y$ is the Grassmann bundle of $2$-dimensional subspaces of $V_B$, and $S = p_2^* S_Y$ is the corresponding tautological subbundle. Clearly, $S$ is a vector subbundle of $V_Y = V \otimes O_Y = p_1^* V_B$, hence also of $p_1^* f_* \omega_{S/B}$.

We will now reproduce the construction of the adjoint map for an infinitesimal deformation. Denote by $\tilde{S} \subseteq H^0 (S, \Omega^1_Y) \otimes O_Y$ the natural preimage of $S$, so that

$$0 \rightarrow H^0 (B, \omega_B) \otimes O_Y \xrightarrow{f^*} \tilde{S} \rightarrow S \rightarrow 0$$

is an exact sequence of vector bundles on $Y$. Therefore, we obtain the following presentation of $\wedge^2 S$,

$$\tilde{S} \otimes H^0 (B, \omega_B) \rightarrow \wedge^2 \tilde{S} \rightarrow \wedge^2 S \rightarrow 0 \quad (10)$$

The wedge product $\wedge^2 H^0 (S, \Omega^1_Y) \rightarrow H^0 (S, \omega_S)$ coincides with the adjoint map constructed in Definition 3.6, restricted to the Grassmannian subvariety $Gr (2, V)$. Hence, according to equation (10), $\tilde{v}$ induces a well-defined map of vector bundles on $Y$:

$$\tilde{v} : \wedge^2 \tilde{S} \rightarrow p_1^* f_* \omega_S.$$ 

Clearly, this map sends the image of $\tilde{S} \otimes H^0 (B, \omega_B)$ into the subsheaf $S \otimes p_1^* \omega_B$. Hence, according to equation (10), $\tilde{v}$ induces a well-defined map of vector bundles on $Y$:

$$\nu : \wedge^2 S \rightarrow (p_1^* f_* \omega_S) / (S \otimes p_1^* \omega_B). \quad (11)$$

Definition 3.10 (Global Adjoint Map). The map $\nu$ in (11) is the global adjoint map of the fibration $f$.

Remark 3.11. It is clear from the construction that if $C = f^{-1} (b)$ is a smooth fibre of $f$, the restriction $\nu_{|\{b\} \times G}$ coincides with the adjoint map constructed in Definition 3.6 restricted to the Grassmannian subvariety $Gr (2, V)$.

To close both this section and the article, we will combine the global adjoint map with Corollary 3.8, considering only vector subbundles of rank two $\mathcal{W} \subseteq V \otimes O_B$. Such a vector subbundle defines a section

$$\eta_{\mathcal{W}} : B \rightarrow Y$$

of $p_1$, such that $\eta_{\mathcal{W}} (b)$ is the subspace $\mathcal{W} \otimes \mathbb{C} (b) \subseteq V$. Conversely, given any section $\eta : B \rightarrow Y$ of $p_1$, it defines the vector subbundle

$$\mathcal{W}_{\eta} = \eta^* S \hookrightarrow \eta^* (V \otimes O_Y) = V \otimes O_B.$$ 

Clearly, the assignations $\mathcal{W} \mapsto \eta_{\mathcal{W}}$ and $\eta \mapsto \mathcal{W}_\eta$ are mutually inverse, giving a one-to-one correspondence between the sets of vector subbundles of $V \otimes O_B$ of rank $2$ and the sections of $p_1 : Y \rightarrow B$.

Now, given a vector subbundle $\mathcal{W}$ as above, we can consider the restriction $\nu_{\mathcal{W}}$ of the adjoint map $\nu$ to the curve $\eta_{\mathcal{W}} (B) \cong B$, which can be seen as a map of vector bundles on $B$:

$$\nu_{\mathcal{W}} : \wedge^2 \mathcal{W} \rightarrow (f_* \omega_S) / (\mathcal{W} \otimes \omega_B). \quad (12)$$

Definition 3.12 (Global Adjoint Map associated to a subbundle). We call the map $\nu_{\mathcal{W}}$ in equation (12) the global adjoint map associated to the subbundle $\mathcal{W}$.

We are now ready to state the wanted global result.

---

1By a vector subbundle of a vector bundle $V$ we mean a locally free subsheaf whose quotient is also locally free.
Theorem 4.3. If
\[ q_f > \frac{g+1}{2}, \]
then there exist a finite base change \( \pi : B' \to B \) and a rank-two vector subbundle \( W \subseteq V \otimes O_B \) whose associated global adjoint map vanishes identically.

Proof. Let \( Z \subseteq Y \) be the zero set of the global adjoint map \( \nu \), which is an analytic subvariety. By Remark 3.13 for any regular value \( b \), the set \( Z_b = Z \cap (\{b\} \times \mathcal{G}) \) is the vanishing set of the adjoint map of \( C_b \), which is non-empty by Corollary 3.8. Therefore, there is a component of \( Z \) dominating \( B \), hence it is possible to choose an irreducible curve \( B' \subseteq Z \) dominating \( B \). Let \( \mu : B' \to B \) be the normalization of \( B' \), and define \( \pi \) as the composition \( p_1 \circ \mu : B' \to B \). As for the vector subbundle, let \( \eta : B' \to B' \times_B Y \cong B' \times \mathbb{G} \) be the section induced from the map \( B' \to B \to Y \), and let \( W = W_{\eta} \). Since the image of \( \eta \) is contained in the zero locus of the adjoint map associated to the fibration \( S' = S \times_B B' \to B' \), (see next Remark), it is tautological that the global adjoint map associated to \( W \) vanishes identically. \( \square \)

Remark 3.14 (Global Adjoint Maps and base change). Consider any finite base change \( \pi : B' \to B \). Denote by \( f' : S' = S \times_B B' \to B' \) the fibration obtained after change of base and desingularization, and by \( V' = V_{f'} = H^0(S', \Omega_{S'}^1) / (f')^* H^0(B', \omega_B^g) \) the corresponding space of relative 1-forms. Define also \( G' = Gr(2, V') \) and \( Y' = B' \times G' \), and let \( \nu' \) be the global adjoint map of \( f' \).

Clearly \( V \) injects into \( V' \), and therefore \( B' \times \mathbb{G} \) is naturally a subvariety of \( Y' \). Furthermore, the pull-back (by \( \pi \times \text{id}_{\mathbb{G}} \)) of \( \nu \) is the restriction of \( \nu' \) to \( B' \times \mathbb{G} \). Hence, the zero locus of \( \nu' \) contains the preimage of the zero locus of \( \nu \).

Corollary 3.15. If \( f : S \to B \) is a fibration of genus \( g \) such that \( q_f > \frac{2g+1}{3} \), then after a base change \( f' : S' \to B' \) as above, there is a divisor \( D \subseteq S' \) supporting \( f \) and such that \( D \cdot C_b < 2g - 2 \).

Proof. Let \( W \subseteq V \otimes O_B \subseteq f_* \omega_{S/B} \) be the rank-two vector subbundle provided by Theorem 3.13 and consider the relative evaluation map
\[ f^* W \to f^* f_* \omega_{S/B} \to \omega_{S/B}. \]
Then it is enough to take \( D \) as the union of the divisorial components of its base locus. \( \square \)

4 Appendix: Relative \( \mathcal{E}xt \) sheaves

Since they play a central role in the technical part of Section 2 we include here a summary of the definition and some of the main properties of the relative \( \mathcal{E}xt \) sheaves, which can be found in the first chapter of [3].

Definition 4.1 (Relative ext sheaves, [3] Def. 1.1.1). Given a morphism of schemes (or more generally, of ringed spaces) \( f : X \to Y \), and an \( \mathcal{O}_X \)-module \( \mathcal{F} \), we define \( \mathcal{E}xt^p_f (\mathcal{F}, -) \) as the \( p \)-th right derived functor of the left-exact functor \( f_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, -) \).

Example 4.2 ([3], Def. Rem. 1.1.2). Some particular cases:

1. If \( Y = \text{Spec} \mathbb{C} \) is a point, then \( \mathcal{E}xt^p_f (\mathcal{F}, -) = \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, -) \), the global \( \mathcal{E}xt \) functor. If furthermore \( \mathcal{F} = \mathcal{O}_X \), \( \mathcal{E}xt^p_f (\mathcal{O}_X, -) = H^p(X, -) \) is the usual sheaf cohomology.

2. If \( f \) is the identity (hence \( Y = X \)), then \( \mathcal{E}xt^p_f (\mathcal{F}, -) = \mathcal{E}xt^p_{\mathcal{O}_X}(\mathcal{F}, -) \) is the usual local \( \mathcal{E}xt \) functor.

3. If \( \mathcal{F} = \mathcal{O}_X \), then \( f_* \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, -) = f_* \) is the usual push-forward functor, so that \( \mathcal{E}xt^p_f (\mathcal{O}_X, -) = R^p f_* \), are the higher-direct image functors.

Theorem 4.3. Some properties:
1. ([3] Th. 1.1.3) For any \( O_X \)-modules \( F, G \), \( \text{Ext}_f (F, G) \) is the sheaf associated to the presheaf
\[
U \mapsto \text{Ext}^p_{O_{f^{-1}(U)}} (F|_{f^{-1}(U)}, G|_{f^{-1}(U)}).
\]
In particular, for any open subset \( W \subseteq Y \),
\[
\text{Ext}^p_f (F, G)|_{f^{-1}(W)} \cong \text{Ext}^p_f (F|_{f^{-1}(W)}, G|_{f^{-1}(W)}).
\]

2. ([3] Th. 1.1.4) If \( L \) and \( N \) are locally free sheaves of finite rank on \( X \) and \( Y \), respectively, then
\[
\text{Ext}^p_f (F \otimes L, - \otimes f^* N) \cong \text{Ext}^p_f (F, - \otimes L^\vee \otimes f^* N) \cong \text{Ext}^p_f (F, - \otimes L^\vee) \otimes N.
\]

3. ([3] Th. 1.1.5) If \( 0 \to F' \to F \to F'' \to 0 \) is an exact sequence of \( O_X \)-modules, and \( G \) is another \( O_X \)-module, then there is a long exact sequence
\[
\cdots \to \text{Ext}^{p-1}_f (F', G) \to \text{Ext}^p_f (F'', G) \to \text{Ext}^p_f (F, G) \to \text{Ext}^p_f (F', G) \to \cdots
\]

4. (Local to global spectral sequence, [3] Th. 1.2.1) Suppose \( g : Y \to Z \) is another morphism, and denote \( h = g \circ f \). For any \( O_X \)-modules \( F, G \) there is a spectral sequence
\[
E_2^{p,q} = R^p g_* \text{Ext}^q_f (F, G) \Rightarrow \text{Ext}^{p+q}_h (F, G).
\]

5. (Coherence, [3] Th. 1.3.1) If \( f \) is projective and \( F, G \) are coherent \( O_X \)-modules, then \( \text{Ext}^p_f (F, G) \) is a coherent \( O_Y \)-module.

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