The Characterizations on a Class of Weakly Weighted Einstein–Finsler Metrics

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Received: 14 February 2023 / Accepted: 29 May 2023 / Published online: 12 June 2023
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Abstract
In this paper, we mainly introduce and study the weakly weighted Einstein–Finsler metrics. First, we show that weakly weighted Einstein–Kropina metrics must be of isotropic $S$-curvature with respect to the Busemann–Hausdorff volume form under a certain condition about the weight constants. Then we characterize weakly weighted Einstein–Kropina metrics completely via their navigation expressions or via $\alpha$ and $\beta$, respectively. In particular, when $\nu \neq 0$ (or $\nu = \kappa = 0$, respectively) and $S$-curvature with respect to the Busemann–Hausdorff volume form is isotropic, we prove that a Kropina metric determined by navigation data $(h, W)$ is a weakly weighted Einstein metric if and only if the Riemann metric $h$ is a weighted Einstein–Riemann metric.

Keywords Finsler metric · Ricci curvature · Generalized weighted Ricci curvature · Weakly weighted Einstein–Finsler metric · Kropina metric · $S$-curvature

Mathematics Subject Classification 53B40 · 53C60

1 Introduction
In Riemannian geometry, the $\infty$-Bakry–Emery Ricci curvature tensor on a smooth Riemannian metric measure space $(M, g, e^{-f}d\text{vol}_g)$ is defined as

$$\text{Ric}_\infty = \text{Ric} + \text{Hess } f.$$
Here, $M$ is a complete $n$-dimensional Riemannian manifold with metric $g$, $f$ is a smooth real-valued function on $M$, and $d\text{vol}_g$ is the standard Riemannian volume form of $g$ on $M$. The equation $\text{Ric}_\infty = \lambda g$ for some constant $\lambda$ is exactly the gradient Ricci soliton equation, which plays an important role in the theory of Ricci flow. Further, for $N \in \mathbb{R}\setminus\{n\}$, the $N$-weighted Ricci curvature tensor associated with the measure $dV = e^{-f}d\text{vol}_g$ is defined as

$$\text{Ric}_N = \text{Ric} + \text{Hess}f - \frac{df \otimes df}{N-n}.$$  

The equation $\text{Ric}_N = \lambda g$ for some $\lambda \in C^\infty(M)$ is a special case of the so-called generalized quasi-Einstein equation ([2, 10]).

Recently, the study of the weighted Ricci curvature in Finsler geometry has attracted a lot of attentions. Roughly speaking, the weighted Ricci curvatures in Finsler geometry are various combinations of Ricci curvature and S-curvature, where the S-curvature was first introduced by Z. Shen when he studied Bishop–Gromov volume comparison in Finsler geometry [16]. Let $(M, F, dV)$ be an $n$-dimensional Finsler metric measure manifold with volume form $dV = \sigma(x)dx^1 \cdots dx^n$. Let $Y$ be a $C^\infty$ geodesic field on an open subset $U \subset M$ and $\hat{g} = g_Y$. Let

$$dV := e^{-\psi}d\text{vol}_{\hat{g}}, \quad d\text{vol}_{\hat{g}} = \sqrt{\det (g_{ij}(x, Y_x))}dx^1 \cdots dx^n.$$

It is easy to see that $\psi$ is given by

$$\psi(x) = \ln \frac{\sqrt{\det (g_{ij}(x, Y_x))}}{\sigma(x)} = \tau(x, Y_x),$$

which is just the distortion along $Y_x$ at $x \in M$ [8]. Let $y := Y_x \in T_xM$ (that is, $Y$ is a geodesic extension of $y \in T_xM$). Then, by the definitions of the S-curvature and the Hessian [17, 18], we have

$$S(x, y) = y[\tau(x, Y_x)] = d\psi(y), \quad \dot{S}(x, y) = y[S(x, Y)] = y[Y(\psi)] = \text{Hess}\psi(y).$$

Then, for $N \in \mathbb{R}\setminus\{n\}$, we define the N-weighted Ricci curvature in Finsler geometry by [7]

$$\text{Ric}_N(y) = \text{Ric}(y) + \text{Hess}\psi(y) - \frac{d\psi(y)^2}{N-n}. \quad (1.1)$$

Obviously, (1.1) is an analogue of the N-weighted Ricci curvature in Riemannian geometry. As the limits of $N \to \infty$, we define the $\infty$-weighted Ricci curvature in Finsler geometry as follows.

$$\text{Ric}_\infty(y) := \text{Ric}(y) + \text{Hess}\psi(y). \quad (1.2)$$
Actually, (1.1) and (1.2) are just the weighted Ricci curvatures introduced by Ohta in [15] as follows

\[ \text{Ric}_N(y) = \text{Ric}(y) + \frac{1}{N-n} S^2, \]  
(1.3)  
\[ \text{Ric}_\infty(y) = \text{Ric}(y) + \dot{S}. \]  
(1.4)

There is another weighted Ricci curvature in Finsler geometry, namely, the projective Ricci curvature

\[ \text{PRic}(y) = \text{Ric}(y) + (n-1) \left( \frac{\dot{S}}{n+1} + \frac{S^2}{(n+1)^2} \right), \]  
(1.5)

The projective Ricci curvature is a projective invariant when the volume form \( dV \) is fixed [5, 19].

More generally, we can introduce the generalized weighted Ricci curvature in Finsler geometry, which is called the \((a, b)\)-weighted Ricci curvature in [20]. Concretely, the generalized weighted Ricci curvature with weight constants \( a \) and \( c \) is defined as follow

\[ \text{Ric}_{a,c}(y) = \text{Ric}(y) + a \dot{S} - c S^2, \]  
(1.6)

where \( a, c \in \mathbb{R} \). Obviously, the weighted Ricci curvatures \( \text{Ric}_N, \text{Ric}_\infty \) and the projective Ricci curvature \( \text{PRic} \) are all special generalized weighted Ricci curvatures with special weight constants. The generalized weighted Ricci curvature with weight constants \( a \) and \( c \) can also be written in the following form

\[ \text{Ric}_{a,c} = \text{PRic} - \frac{\kappa}{n+1} \left( \dot{S} + \frac{4}{n+1} S^2 \right) + \frac{\nu}{(n+1)^2} S^2, \]  
(1.7)

where \( \kappa := (n-1) - a(n+1) \) and \( \nu := 3(n-1) - 4a(n+1) - c(n+1)^2 \).

A Finsler metric \( F \) on an \( n \)-dimensional manifold \( M \) with a volume form \( dV = e^{-\sigma} dV_{BH} \) is said to be a weakly weighted Einstein–Finsler metric with weight constants \( a \) and \( c \) if the generalized weighted Ricci curvature satisfies

\[ \text{Ric}_{a,c} = (n-1) \left( \frac{3\theta}{F} + \sigma \right) F^2, \]  
(1.8)

where \( dV_{BH} \) denotes the Busemann–Hausdorff volume form (see Sect. 2 for the definition) and \( \sigma \) is a scalar function and \( \theta = \theta_i y^i \) is a 1-form on \( M \). If \( F \) satisfies (1.8) with \( \theta = 0 \), then \( F \) is called a weighted Einstein–Finsler metric with weight constants \( a \) and \( c \). In particular, when \( a = c = 0 \), Finsler metrics satisfying (1.8) are called the weak Einstein metrics [6]. When \( a = 1, c = 0 \) and \( \theta = 0 \), (1.8) becomes \( \text{Ric}_\infty = (n-1)\sigma F^2 \) and \((M, F, dV)\) is called a Finsler gradient Ricci almost soliton. If \((M, F, dV)\) satisfies \( \text{Ric}_\infty = (n-1)\sigma F^2 \) for a constant \( \sigma \), it is called a Finsler gradient Ricci soliton. In [14], Mo-Zhu-Zhu find the sufficient and necessary conditions for a Randers measure space to be a Finsler gradient Ricci soliton. Further, Shen-Zhao...
find that weakly weighted Einstein–Randers metrics must be of isotropic S-curvature with respect to the Busemann–Hausdorff volume form when $\nu \neq 0$ and classify this class of Randers metrics completely via their navigation expressions or $\alpha$ and $\beta$, respectively, [20]. Recently, H. Zhu determines completely the structure of Berwald square metric as Finsler gradient Ricci almost soliton (which is called quasi-Einstein square metric by Zhu [24]).

In this paper, we will specialize in studying weakly weighted Einstein–Kropina metrics. We first find that every weakly weighted Einstein–Kropina metric must be of isotropic S-curvature with respect to the Busemann–Hausdorff volume form if $\nu \neq 0$ or $\nu = \kappa = 0$ (see Theorem 3.6 or Proposition 6.1, respectively). Then we characterize weakly weighted Einstein–Kropina metrics completely via their navigation expressions when $\nu \neq 0$ (see Theorem 4.1) or $\nu = \kappa = 0$ (see Theorem 6.2), respectively. In particular, when $\nu \neq 0$ (or $\nu = \kappa = 0$, respectively) and $S$-curvature with respect to the Busemann–Hausdorff volume form is isotropic, we prove that a Kropina metric $F$ with navigation data $(h, W)$ is a weakly weighted Einstein metric if and only if $h$ is a weighted Einstein–Riemann metric satisfying (4.2) [or (6.3), respectively] with respect to volume form $dV_f = e^{-(n+1)f} d\text{vol}_h$.

2 Preliminaries

Let $F$ be a Finsler metric on an $n$-dimensional manifold $M$ and $G^j$ be the geodesic coefficients of $F$, which are defined by

$$G^i := \frac{1}{4} g^{il} \left\{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \right\}.$$  \hspace{1cm} (2.1)

For any $x \in M$ and $y \in T_x M \backslash \{0\}$, the Riemann curvature $R_y := R^i_k \frac{\partial}{\partial x^i} \otimes dx^k$ is defined by

$$R^i_k := 2 \frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^m \partial y^k} y^m + 2G^m \frac{\partial^2 G^i}{\partial y^m \partial y^k} - \frac{\partial G^i}{\partial y^m} \frac{\partial G^m}{\partial y^k}.$$  \hspace{1cm} (2.2)

The Ricci curvature is defined as the trace of the Riemann curvature, that is,

$$\text{Ric}(y) := R^m_m.$$  \hspace{1cm} (2.3)

Let $(M, F, dV)$ be a Finsler measure manifold with a volume form $dV = \sigma(x) dx^1 \cdots dx^n$. For each $y \in T_x M \backslash \{0\}$, the quantity

$$\tau(x, y) := \ln \sqrt{\frac{\text{det} \left( g_{ij} \left( x, y \right) \right)}{\sigma(x)}}$$  \hspace{1cm} (2.4)
is called the distortion of $F$. Further, let $c = c(t)$ be the geodesic with $c(0) = x$ and $\dot{c}(0) = y$. The S-curvature $S$ and its change rate $\dot{S}$ along geodesic $c$ are defined by

$$
S(x, y) := \left. \frac{d}{dt} [\tau (c(t), \dot{c}(t)) ] \right|_{t=0}, \quad \dot{S}(x, y) := \left. \frac{d}{dt} [S (c(t), \dot{c}(t)) ] \right|_{t=0}
$$

(2.5)

, respectively. In short, we have

$$
S = \tau_{\mid m} (x, y) y^m, \quad \dot{S} = S_{\mid m} (x, y) y^m,
$$

(2.6)

where “$|$” denotes the horizontal covariant derivative with respect to $F$.

The Busemann–Hausdorff volume form $dV_{BH} = \sigma_{BH}(x) dx^1 \cdots dx^n$ on a Finsler manifold $(M, F)$ is defined by [8]

$$
\sigma_{BH}(x) := \frac{\text{Vol} (\mathcal{B}^n (1))}{\text{Vol} \{(y^i) \in \mathbb{R}^n \mid F(x, y) < 1\}}.
$$

In the following, we always use $S_{BH}$ to denote the S-curvature determined by Busemann–Hausdorff volume form $dV_{BH}$.

Let $f$ be a $C^2$ function on $M$. The Hessian of $f$ can be defined as a map $\text{Hess}_F f : TM \to R$ by

$$
\text{Hess}_F f(y) := \frac{d^2}{ds^2} (f \circ c) \big|_{s=0}, \quad y \in T_x M,
$$

(2.7)

where $c : (-\epsilon, \epsilon) \to M$ is the geodesic with $c(0) = x$, $\dot{c}(0) = y \in T_x M$ (see [18]). In local coordinates,

$$
\text{Hess}_F f(y) = \frac{\partial^2 f}{\partial x^i \partial x^j}(x) \dot{c}^i(0) \dot{c}^j(0) + \frac{\partial f}{\partial x^i}(x) \ddot{c}^i(0)
= \frac{\partial^2 f}{\partial x^i \partial x^j}(x) y^i y^j - 2 \frac{\partial f}{\partial x^i}(x) G^i(x, y)
= \left( \frac{\partial^2 f}{\partial x^i \partial x^j}(x) - \frac{\partial f}{\partial x^m} \Gamma^m_{ij}(x, y) \right) y^i y^j.
$$

(2.8)

Here, $\Gamma^k_{ij}(x, y)$ denote the Chern connection coefficients of $F$, which depends on the tangent vector $y \in T_x M$ usually.

Randers metrics and Kropina metrics form an important class of Finsler metrics which are both called $C$-reducible Finsler metrics when $n \geq 3$. Randers metrics and Kropina metrics can be also both expressed as the solution of the Zermelo navigation problem on some Riemannian manifold $(M, h)$ with a vector field $W$ [4, 6, 22]. Randers metrics are one of the simplest non-Riemannian Finsler metrics with the form $F = \alpha + \beta$, where $\alpha = \sqrt{a_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) y^i$ is a 1-form with $\| \beta_x \|_\alpha < 1$ on the manifold. Kropina metrics are those Finsler metrics in the form $F = \frac{\alpha^2}{\beta}$. Kropina metrics were first introduced by Berwald when he studied the two-dimensional Finsler spaces with rectilinear extremal and were investigated by
Kropina (see [11, 12]). Kropina metrics have important and interesting applications in the theory of thermodynamics. Besides, both of Randers metrics and Kropina metrics play an interesting role in the Krivan problem in ecology [1]. However, Randers metrics are regular Finsler metrics, but Kropina metrics are the Finsler metrics with singularity. In fact, Kropina metrics are not classical Finsler metrics, but conic Finsler metrics defined on the conic domain [3, 13, 21, 22]

\[ A = \{ (x, y) \in TM \mid \beta = b_i(x)y^i > 0 \} \subset TM. \]

Let

\[ r_{ij} = \frac{1}{2}(b_{i;j} + b_{j;i}), \quad s_{ij} = \frac{1}{2}(b_{i;j} - b_{j;i}), \]

where “;” denotes the covariant derivative with respect to the Levi-Civita connection of \( \alpha \). Put

\[
\begin{align*}
 r^i_j &:= a^{ik}r_{kj}, \quad r_j := b^j r_i, \quad r^i := a^{ij}r_j, \quad r := r_{ij}b^j b^i, \\
 s^i_j &:= a^{ik}s_{kj}, \quad s_j := b^j s_i, \quad s^i := a^{ij}s_j, \quad e_{ij} := r_{ij} + b_is_j + b_js_i,
\end{align*}
\]

where \((a^{ij}) := (a_{ij})^{-1}\) and \(b^j := a^{ij}b_j\). Besides, we will use the following notations: \(r_{i0} = r_{ij}y^j\), \(r_{00} = r_{i0}y^iy^j\), \(s_{i0} = s_{ij}y^j\), etc.. Obviously, \(s_{00} = 0\).

Let \(F = \frac{\alpha^2}{\beta}\) be a Kropina metric and \(G^i\) and \(G^i_\alpha\) be the geodesic coefficients of \(F\) and \(\alpha\), respectively. Then we have the following lemmas.

**Lemma 2.1** [23] For the Kropina metric \(F = \frac{\alpha^2}{\beta}\), its geodesic coefficients \(G^i\) are connected with the geodesic coefficients \(G^i_\alpha\) of \(\alpha\) by

\[ G^i = G^i_\alpha + T^i, \quad (2.9) \]

where

\[ T^i = -\frac{\alpha^2}{2\beta^2}s^0 + \frac{1}{2b^2} \left( \frac{\alpha^2}{\beta}s_0 + r_{00} \right) b^j - \frac{1}{b^2} \left( s_0 + \frac{\beta}{\alpha^2}r_{00} \right) y^i. \quad (2.10) \]

**Lemma 2.2** [21, 23] For the Kropina metric \(F = \frac{\alpha^2}{\beta}\), the Ricci curvature of \(F\) is given by

\[ \text{Ric} = \text{Ric}^\alpha + T, \quad (2.11) \]

where \(\text{Ric}^\alpha\) denotes the Ricci curvature of \(\alpha\), and

\[
T = \frac{3(n - 1)}{b^4F^2}r_{00} + \frac{n - 1}{Fb^4} \left( 2r_{00}s_0 - 4r_{00}r_0 - 4Fr_0s_0 - Fs_0^2 \right) \\
+ \frac{n - 1}{b^2F} \left( r_{00} + Fs_0 + F^2s_0s_0 \right) + \frac{1}{b^4} \left( r_0 + s_0 \right)^2 - r(r_{00} + Fs_0)
\]

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\[
\frac{1}{b^2} \left\{ (Fs_{0; k} + r_{00; k})b^k - (r_{0;0} + s_{0;0}) + (r_{00} + Fs_0)r^k_k + 2nr_{0k}s^k_0 \right.

\left. - Frk s^k_0 - Fr_{0k}s^k_0 - \frac{F^2}{2}s^k_ks^k_j - Fs_{0;k} - \frac{F^2}{4}s^j_ks^j_k. \right\}
\]

(2.12)

Kropina metrics can be expressed as the solution of the Zermelo navigation problem on some Riemannian manifold \((M, h)\) with a vector field \(W\). Concretely, assume that
\[
h = \sqrt{h_{ij}(x) y^i y^j} \quad \text{and} \quad W^i = \frac{\alpha}{\beta} \partial_{x^i}
\]
with \(\|W\|_h = 1\). Then the metric \(F\) obtained by solving the following problem
\[
h \left( x, \frac{y}{F(x, y)} - W_x \right) = 1
\]
(2.13)
is a Kropia metric given by
\[
F = \frac{h^2}{2W_0},
\]
(2.14)
where \(W_0 := W_i y^i = h(y, W_x)\), \(W_i := h_{ij} W^j\). In this case, the pair \((h, W)\) is called the navigation data of conic Kropina metric \(F = \frac{a^2}{\beta}\) and
\[
\alpha = \frac{b}{2} h, \quad \beta = \frac{b^2}{2} W_0.
\]
(2.15)
Here, \(b := \|\beta\|_\alpha\) denotes the norm of \(\beta\) with respect to \(\alpha\). In fact, \(F\) given by (2.14) is a conic Kropina metric defined on the conic domain
\[
A = \{(x, y) \in TM \mid h(y, W_x) > 0\} = \{(x, y) \in TM \mid \beta = b_i(x) y^i > 0\} \subset TM.
\]
Conversely, given a conic Kropina metric \(F = \frac{a^2}{\beta}\), put
\[
h_{ij} = \frac{4}{b^2} a_{ij}, \quad W^i = \frac{1}{2} b^i.
\]
(2.16)
Then we can get a Riemannian metric \(h\) and a vector field \(W\) with \(\|W\|_h = 1\) from (2.16) and \(F\) is just given by (2.13) for \(h\) and \(W\). Thus, there is an one-to-one correspondence between a conic Kropina metric \(F\) and a pair \((h, W)\) with \(\|W\|_h = 1\).

In the following, we just study conic Kropina metrics and we always use Kropina metric to take place of conic Kropina metric. From (2.15) or (2.16), we have
\[
a_{ij} = e^{-2\rho} h_{ij}, \quad b_i = 2e^{-2\rho} W_i, \quad b^2 = 4e^{-2\rho},
\]
(2.17)
where \(\rho := \ln \frac{2}{b}\).

For a Kropina metric \(F = \frac{a^2}{\beta}\) with navigation data \((h, W)\), let
\[
\mathcal{R}_{ij} := \frac{1}{2} \left( W_{i|j} + W_{j|i} \right), \quad S_{ij} := \frac{1}{2} \left( W_{i|j} - W_{j|i} \right),
\]
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\[ S^i_j := h^{ik} S_{kij}, \quad S_i := W^i S_{ij}, \quad R_j := W^i R_{ij}, \quad \mathcal{R} := \mathcal{R}_j W^j, \]

where “\( | \)“ denotes the covariant derivative with respect to \( h \). It is clear that \( S_j W^j = 0 \).

From (2.17), it is easy to get

\[
\begin{align*}
    r_{ij} &= 2e^{-2\rho} \left( R_{ij} - W^l \rho_l h_{ij} \right), \\
    s_{ij} &= 2e^{-2\rho} \left( S_{ij} + \rho_i W_j - \rho_j W_i \right),
\end{align*}
\]

where \( \rho_i := \frac{\partial \rho}{\partial x^i} \). Then we have

\[
\begin{align*}
    r_{00} &= 2e^{-2\rho} \left( \mathcal{R}_{00} - W^j \rho_j h^2 \right), \\
    s^i_0 &= 2 \left( S^i_0 + \rho^i W_0 - \rho_0 W^i \right), \\
    s_0 &= 4e^{-2\rho} \left( S_0 + W^j \rho_j W_0 - \rho_0 \right),
\end{align*}
\]

where \( \rho_0 := \rho_0 y^i \) [23]. By conformal relation between \( \alpha \) and \( h \) given by (2.17), the geodesic coefficients \( G^i_\alpha \) of \( \alpha \) are related to the geodesic coefficients \( G^i_h \) of \( h \) by

\[
G^i_\alpha = G^i_h + \frac{1}{2} \rho^i h^2 - \rho_0 y^i,
\]

where \( \rho^i := h^{ij} \rho_j \). By (2.9), (2.19) and (2.20), we obtain the following important result.

**Proposition 2.3** For a Kropina metric \( F \) with navigation data \((h, W)\), the geodesic coefficients \( G^i \) of \( F \) can be expressed in terms of the geodesic coefficients \( G^i_h \) of \( h \) and the covariant derivatives of \( W \) with respect to \( h \) as follow:

\[
G^i = G^i_h - F S^i_0 - \frac{1}{2F} \left( \mathcal{R}_{00} + 2FS_0 \right) \left( y^i - FW^i \right). \tag{2.21}
\]

### 3 S-curvature and Ricci Curvature of Weakly Weighted Einstein–Kropina Metrics

In this section, we mainly consider S-curvature and Ricci curvature of weakly weighted Einstein–Kropina metrics. Firstly, we need the following lemma.

**Lemma 3.1** [23] For Kropina metric \( F = \frac{\alpha^2}{b^2} \), we have

\[
S_{BH}(x, y) = \frac{n + 1}{b^2} \left( r_0 - \frac{1}{F} r_{00} \right). \tag{3.1}
\]

Now, take \( dV = e^{-(n+1)f} dV_{BH} \). Then we have

\[
S = S_{BH} + (n + 1)f_0. \tag{3.2}
\]
\[
\dot{S} = \dot{S}_{BH} + (n + 1)\text{Hess}_F f(y),
\]  
(3.3)

where \( f_0 := f_i y^i, f_i := \frac{\partial f}{\partial x^i} \) and \( \text{Hess}_F f(y) \) is defined by (2.8).

**Proposition 3.2** Let \((M, F, dV = e^{-(n+1)f}dV_{BH})\) be an \(n\)-dimensional Kropina measure space. Then

\[
\frac{1}{n + 1} \dot{S} = \frac{1}{b^2} \left( r_{0;0} - \frac{\beta}{\alpha^2} r_{00;0} + \frac{\alpha^2}{\beta} s^i 0 r_i - 2 r_{0i} s^i 0 \right) + \frac{1}{b^4} \left[ - \left( \frac{\alpha^2}{\beta} s_0 + r_0 \right) r \right.
\]
\[
+ \left. \frac{2}{\alpha^2} r_{00} (3 r_0 - s_0) + 2 r_0 (s_0 - r_0) - 4 \frac{\beta^2}{\alpha^4} r_{00}^2 \right] + \text{Hess}_F f(y).
\]  
(3.4)

**Proof** By (2.9) and a direct calculation, we have

\[
\dot{S}_{BH} = \left( \frac{\partial S_{BH}}{\partial x^k} - \frac{\partial G^j}{\partial y^k} \frac{\partial S_{BH}}{\partial y^j} \right) y^k = y^k \frac{\partial S_{BH}}{\partial x^k} - 2 G^j \frac{\partial S_{BH}}{\partial y^j}
\]
\[
= (S_{BH})_{;0} - 2 T^j (S_{BH})_{,y^j},
\]  
(3.5)

where \( T^j \) is given by (2.10) and ";" denotes the covariant derivative with respect to \( \alpha \). From (3.1), we get

\[
(S_{BH})_{;0} = (n + 1) \left[ \left( \frac{1}{b^2} \right)_{;0} \left( r_0 - \frac{1}{F} r_{00} \right) + \frac{1}{b^2} \left( r_{0;0} - \left( \frac{r_{00}}{F} \right)_{;0} \right) \right].
\]  
(3.6)

It is easy to see that

\[
\left( \frac{1}{b^2} \right)_{;0} = - \frac{2 (r_0 + s_0)}{b^4}, \quad (r_{00}/F)_{;0} = \frac{r_{00}^2 + \beta r_{00;0}}{\alpha^2},
\]

where we have used \( \beta_{;0} = r_{00} \). Substituting these into (3.6), we have

\[
(S_{BH})_{;0} = (n + 1) \left[ \frac{1}{b^2} \left( r_{0;0} - \frac{\beta}{\alpha^2} r_{00;0} - \frac{r_{00}}{\alpha^2} \right) + \frac{2}{b^4} (r_0 + s_0) \left( \frac{\beta}{\alpha^2} r_{00} - r_0 \right) \right].
\]  
(3.7)

On the other hand, it follows from (3.1) that

\[
\frac{1}{n + 1} (S_{BH})_{,y^k} = \frac{1}{b^2} \left[ r_k - \left( \frac{b_k}{\alpha^2} - \frac{2 \beta y_k}{\alpha^4} \right) r_{00} - \frac{2 \beta}{\alpha^2} r_{0k} \right],
\]  
(3.8)

where \( y_k = a_{jk} y^j \).
Plugging \((2.10), (3.7), \text{and} (3.8)\) into \((3.5)\) yields

\[
\frac{1}{n + 1} \dot{S}_{BH} = \left[ \frac{1}{b^2} \left( r_{0;0} - \frac{\beta}{\alpha^2} r_{00;0} - \frac{r_0^2}{\alpha^2} \right) + \frac{2}{b^4} (r_0 + s_0) \left( \frac{\beta}{\alpha^2} r_{00} - r_0 \right) \right] 
- \frac{2}{(n + 1)} T_j^i (\mathbf{S}_{BH})_{yi},
\]

where

\[
\frac{2}{n + 1} T_j^i (\mathbf{S}_{BH})_{yi} = - \left[ \frac{\alpha^2}{\beta} s_j^i 0 - \frac{1}{b^2} \left( \frac{\alpha^2}{\beta} s_0 + r_00 \right) b^j + \frac{2}{b^2} (s_0 + \frac{\beta}{\alpha^2} r_{00}) y^j \right] \times \frac{1}{b^2} \left[ r_j - \left( \frac{b_j}{\alpha^2} - \frac{2\beta y_j}{\alpha^4} \right) r_{00} - \frac{2\beta}{\alpha^2} r_{0j} \right] 
= - (\pi_1 + \pi_2 + \pi_3)
\]

and

\[
\pi_1 := \frac{\alpha^2}{\beta} s_j^i 0 \times \frac{1}{b^2} \left[ r_j - \left( \frac{b_j}{\alpha^2} - \frac{2\beta y_j}{\alpha^4} \right) r_{00} - \frac{2\beta}{\alpha^2} r_{0j} \right] 
= \frac{1}{b^2} \left( \frac{\alpha^2}{\beta} s_j^i 0 r_j - \frac{1}{\beta} s_0 r_{00} - 2r_0 s_j^i 0 \right),
\]

\[
\pi_2 := - \frac{1}{b^2} \left( \frac{\alpha^2}{\beta} s_0 + r_00 \right) b^j \times \frac{1}{b^2} \left[ r_j - \left( \frac{b_j}{\alpha^2} - \frac{2\beta y_j}{\alpha^4} \right) r_{00} - \frac{2\beta}{\alpha^2} r_{0j} \right] 
= \frac{1}{b^4} \left[ 2r_0 s_0 - \left( \frac{\alpha^2}{\beta} s_0 + r_00 \right) r - \frac{2\beta}{\alpha^2} (s_0 - r_0) r_{00} - \frac{2\beta^2}{\alpha^4} r_{00}^2 \right] + \frac{1}{b^2} \left( \frac{1}{\beta} r_00 s_0 + \frac{1}{\alpha^2} r_{00}^2 \right),
\]

\[
\pi_3 := \frac{2}{b^4} \left( s_0 + \frac{\beta}{\alpha^2} r_{00} \right) y^j \times \left[ r_j - \left( \frac{b_j}{\alpha^2} - \frac{2\beta y_j}{\alpha^4} \right) r_{00} - \frac{2\beta}{\alpha^2} r_{0j} \right] 
= \frac{2}{b^4} \left[ r_0 s_0 + \frac{\beta}{\alpha^2} r_{00} (r_0 - s_0) - \frac{2\beta^2}{\alpha^4} r_{00}^2 \right].
\]

Substituting \((3.10)\) into \((3.9)\) yields

\[
\frac{1}{n + 1} \dot{S}_{BH} = \frac{1}{b^2} \left( r_{0;0} - \frac{\beta}{\alpha^2} r_{00;0} + \frac{\alpha^2}{\beta} s_j^i 0 r_i - 2r_0 s_j^i 0 \right) + \frac{1}{b^4} \left[ - \left( \frac{\alpha^2}{\beta} s_0 + r_00 \right) r \right. 
\left. + \frac{2\beta}{\alpha^2} r_{00} (3r_0 - s_0) + 2r_0 (s_0 - r_0) - 4\frac{\beta^2}{\alpha^4} r_{00}^2 \right].
\]

By \((3.3)\) and \((3.11)\), we get \((3.4)\).

In order to prove our main theorems, we need the following lemmas.

Lemma 3.3 [21] For a Kropina metric \( F \) on an \( n \)-dimensional manifold \( M \), the following are equivalent.

\[ \square \] Springer
Lemma 3.4 [23] For a Kropina metric $F$ on an $n$-dimensional manifold $M$, $r_{00} = \eta \alpha^2$ is equivalent to $R_{ij} = 0$. In this case, $W^k \rho_k = -\frac{1}{2} \eta$.

Lemma 3.5 Let $F = \frac{\alpha^2}{\beta}$ be a Kropina metric with navigation data $(h, W)$. If $F$ is of isotropic $S$-curvature with respect to the Busemann–Hausdorff volume, $S_{BH} = (n + 1) c F$, then $S_j = 0$.

Now we are in the position to prove our first main result.

Theorem 3.6 Let $F$ be a weakly weighted Einstein–Kropina metric on an $n$-dimensional manifold $M$ with volume form $dV = e^{-(n+1) f} dV_{BH}$. Assume that $\nu \neq 0$. Then $F$ is of isotropic $S$-curvature with respect to the Busemann–Hausdorff volume form.

Proof By the assumption, we have

$$0 = b^4 \beta^2 \alpha^4 \left[ Ric_{a,c} - (n - 1) \left( 3 \theta F + \sigma F^2 \right) \right] = b^4 \beta^2 \alpha^4 (Ric + a S - c S^2) - (n - 1) (3 \theta b^4 \beta \alpha^6 + \sigma b^4 \alpha^8)$$

for some constants $a$ and $c$. Substituting (2.11), (3.2) and (3.4) into (3.12), we have

$$0 = \nu \beta^4 r_{00}^2 + \beta^3 \alpha^2 \left[ \kappa (b^2 r_{00}; 0 + 2 r_{00} r_0 + 2 r_{00} s_0) - 2 \nu r_{00} r_0 + 2 (n + 1) b^2 r_{00} f_0 \right]$$

$$+ \beta^2 \alpha^4 \left[ b^4 Ric^a + b^2 b^k r_{00;k} + (n - 2) b^2 s_0 : 0 + b^2 r_{00} r_k - (n - 2) s_0^2 \right]$$

$$+ (\kappa - n - 2) b^2 r_{00} + (\kappa - n) r_{00} r_0 + (-\kappa + 4 - 4) r_{00} s_0 + (n + 1) b^2 r_{00} s_0^k$$

$$+ (n - 2 - \kappa) r_0^2 + a(n + 1) b^4 Hess_F f(y) - c(n + 1)^2 (2 b^2 r_0 f_0 + b^4 f_0^2) \right]$$

$$+ \beta^6 \left[ (\kappa - n) s_0 r + b^2 b^k s_0 : k + b^2 s_0 r_k - b^4 s_0^k + (-\kappa + n - 2) b^2 r_k s_0^k \right.$$

$$- b^2 r_{00} s^k + (n - 1) b^2 s_k s^k_0 - 3 (n - 1) \theta b^4 \left] - \alpha^8 \left[ \frac{1}{2} b^2 s_k s^k_0 + \frac{1}{4} b^4 s_k s^k + (n - 1) \sigma b^4 \right] \right]$$

(3.13)

where we have used that $\kappa = (n - 1) - \alpha (n + 1)$ and $\nu = 3 (n - 1) - 4 \alpha (n + 1) - c(n + 1)^2$. Further, (3.13) can be reorganized as follows:

$$0 = \nu \beta^4 r_{00}^2 + \alpha^2 \beta^3 P_1 + \alpha^4 \beta^2 P_2 + \alpha^6 \beta P_3 + \alpha^8 P_4.$$  

(3.14)
The above equation shows that \( \nu \beta^4 r_{00}^2 \) can be divided by \( \alpha^2 \). Since \( \beta^4 \) can not be divided by \( \alpha^2 \) and \( \alpha^2 \) is an irreducible polynomial in \( y \), we conclude that there exists a scalar function \( \eta(x) \) such that

\[ r_{00} = \eta(x)\alpha^2. \]

Thus, by Lemma 3.3, the S-curvature with respect to the Busemann–Hausdorff volume form is isotropic, \( S_{BH} = 0 \). □

In the following, we will determine the Ricci curvature of weakly weighted Einstein–Kropina metrics.

**Lemma 3.7** Let \( \sqrt{h_{ij} y^i y^j} \) be a Riemannian metric on an \( n \)-dimensional manifold \( M \). Let \( W = W^i \frac{\partial}{\partial x^i} \) be a vector field on \( M \) satisfying \( R_{ij} = 0 \). Then

\[ W_m R^m_{ji} = -W_m R^m_{kj}. \]  

(3.15)

where \( R^m_{ji} \) denote the coefficients of the Riemann curvature tensor of \( h \).

**Proof** By the assumption, we have

\[ W_i | j + W_j | i = 0. \]  

(3.16)

First, differentiating (3.16) and exchanging the indices, we obtain

\[ W_i | j | k + W_j | i | k = 0, \]  

(3.17)

\[ W_j | i | j = 0, \]  

(3.18)

\[ W_k | i | j + W_i | k | j = 0. \]  

(3.19)

Adding (3.18) and (3.19) together, then the sum being subtracted by (3.17), we obtain

\[ (W_i | j + W_j | i) + (W_j | i + W_i | j) + (W_k | j | i + W_i | k | j) + 2W_k | i | j = 0. \]  

(3.20)

Using the Ricci identity, \( W_k | i | j + W_k | j | i = W_m R^m_{k ij} \), we obtain

\[ W_m R^m_{k ij} + W_m R^m_{j ki} + W_m R^m_{j ki} + 2W_k | i | j = 0. \]  

(3.21)

By applying the Bianchi identity, \( R^m_{i kj} + R^m_{k ji} + R^m_{j ik} = 0 \), we obtain (3.15). □

By (2.21), we have

\[ G^i = G^i_h + Q^i, \]

where

\[ Q^i := -FS^i_0 - \frac{1}{2F}(R_{00} + 2FS_0)(y^j - FW^j). \]
Then, by (2.2), we have

\[ R^i_k = \overline{R}^i_k + 2Q^i_{|k} - [Q^i_{|m}]_{yk} y^m + 2Q^m_{|i} y^m y^k - [Q^i_{|m}] y^m_{y^k}. \] (3.22)

Here “\(|\)” denotes the covariant differentiation with respect to \( h \).

From now on, we assume that \( F \) is of isotropic S-curvature with respect to the Busemann–Hausdorff volume form. By Lemmas (3.3, 3.4 and 3.5), \( R_{ij} = 0 \) and \( S_j = 0 \). Then the geodesic coefficients \( G^i \) are reduced to the following expression:

\[ G^i = G^i_h + Q^i, \] (3.23)

where

\[ Q^i = -FS^i_0. \] (3.24)

By (3.15), we have

\[ W_{i|j|k} = -W^p \overline{R}^i_{kpj}. \] (3.25)

Then we have

\[ S^i_{0|k} = \overline{R}^i_{pq} y^p W^q, \] (3.26)

\[ S^i_{0|0} = -\overline{R}^i_{pq} y^p y^q W^m, \] (3.27)

\[ S^i_{0|k} = S^i_{mk} S^m_{0} + W^m \overline{R}^i_{pq} y^p W^q. \] (3.28)

Observe that

\[ W_{0|k} = S^i_{0k}, \quad W^i_{|k} = S^i_{k}, \quad F_{|k} = \frac{F}{W_0} S^i_{k0}. \]

Now, for simplicity, let

\[ \xi^i := y^i - FW^i. \] (3.29)

By a direct calculation, we have

\[ Q^i_{|k} = -F \overline{R}^i_{pq} W^p W^q - F_{|k} S^i_0. \] (3.30)

Futher, we need the following formula,

\[ F_{y^k} = \frac{y_k - FW^k}{W_0} = \frac{\xi_k}{W_0}, \]

where \( y_k = h_{kj} y^j \) and \( \xi_k := h_{ik} \xi^i \).

Thus, by a series computations, we obtain that

\[ [Q^i_{|m}]_{yk} y^m = \frac{\xi_k}{W_0} \overline{R}^i_{pq} y^p W^q - F \overline{R}^i_{pq} y^q W^m + F_{|k} S^i_0. \] (3.31)

\[ Q^m_{|i} y^m y^k = \frac{F}{W_0} \left( S^i_{0k} S^m_{0} + \xi_k S^m_{0} S^i_{m} \right). \] (3.32)
\[ [Q^i]_y^m [Q^m]_y^k = F^2 S^m_k S^i_m + \frac{F}{W_0} \left( -S_{k0} S^i_0 + \xi_k S^i_m S^m_0 \right). \quad (3.33) \]

Plugging (3.30–3.33) into (3.22) yields the following result.

**Proposition 3.8** Let \( F \) be a Kropina metric expressed by (2.14). Suppose that it is of isotropic S-curvature with respect to the Busemann–Hausdorff volume form. Then the Riemann curvature of \( F \) can be expressed in terms of the Riemann curvature of \( h \) and the covariant derivatives of \( W \) with respect to \( h \) as follows:

\[
R^i_k = \overline{R}^i_k - 2 F \overline{R}^i_p k q y p W^q - \frac{\xi_k}{W_0} \overline{R}^i_p m q y p y q W^m + F \overline{R}^i_k m q y m W^q
- F^2 S^m_k S^i_m + \frac{\xi_k}{W_0} F S^m_0 S^i_m. \quad (3.34)
\]

From (3.34), we have the following

**Proposition 3.9** Let \( F \) be a Kropina metric expressed by (2.14). Suppose that it is of isotropic S-curvature with respect to the Busemann–Hausdorff volume form. Then the Ricci curvature of \( F \) can be expressed in terms of the Ricci curvature of \( h \) and the covariant derivatives of \( W \) with respect to \( h \) as follow:

\[
Ric = \overline{Ric} - 2 F \overline{R}^i_p i q y p W^q - F^2 S^m_i S^i_m, \quad (3.35)
\]

where \( \overline{Ric} \) denotes the Ricci curvature of \( h \).

**Proof** By applying the Bianchi identity, \( \overline{R}^i_i m q + \overline{R}^i_m q i + \overline{R}^i_q i m = 0 \), we have

\[
F \overline{R}^i_i m q y m W^q = F \left( -\overline{R}^i_m q i - \overline{R}^i_q i m \right) y m W^q
= F \left( \overline{R}^i_m i q - \overline{R}^i_q i m \right) y m W^q = 0. \quad (3.36)
\]

By (3.28) and \( S_j = 0 \), we have \( S^m_0 S^i_m = W^m \overline{R}^i_p m q W^p y q \). Then

\[
\frac{\xi_i}{W_0} F S^m_0 S^i_m - \frac{\xi_i}{W_0} \overline{R}^i_p m q y p y q W^m = \frac{\xi_i}{W_0} \overline{R}^i_p m q W^m y q (F W^p - y p)
= - \frac{\xi_i}{W_0} \overline{R}^i_p m q \xi p W^m y q = 0.
\]

Then (3.35) follows from (3.34).

\[\square\]

**4 The Weakly Weighted Einstein–Kropina Metrics with \( \nu \neq 0 \)**

In this section, we will firstly characterize weakly weighted Einstein–Kropina metrics via navigation data \( (h, W) \) in the case that \( \nu \neq 0 \).

\[\square\] Springer
Theorem 4.1 Let $a, c$ be two constants satisfying $v \neq 0$ and $F$ be a Kropina metric on an $n$-dimensional manifold $M$ defined by navigation data $(h, W)$. Then $F$ is a weakly weighted Einstein metric with weight constants $a$ and $c$ satisfying

$$\text{Ric}_{a,c} = (n - 1) \left( \frac{3\theta}{F} + \sigma \right) F^2$$  \hspace{1cm} (4.1)

with respect to a volume form $dV = e^{-(n+1)f} dV_{BH}$ if and only if there exists a scalar function $\mu$ on $M$, such that the Ricci curvature tensor of $h$ satisfies

$$\text{Ric}^h + a(n + 1)\text{Hess}_h f - c(n + 1)^2 (df \otimes df) = (n - 1)\mu h^2$$  \hspace{1cm} (4.2)

and $W$ satisfies $\mathcal{R}_{ij} = 0$. In this case,

$$\sigma = \mu - \frac{1}{n - 1} \left\{ \text{Ric}^h(W) + S_q^p S^q_p + a(n + 1)\text{Hess}_h f(W) - c(n + 1)^2 (f_p W^p)^2 \right\}$$  \hspace{1cm} (4.3)

and

$$\theta_i = \frac{1}{3(n - 1)} \left[ 2a(n + 1)(f_i p W^p + f_p S^p_i) - 2c(n + 1)^2 f_i f_p W^p \right].$$  \hspace{1cm} (4.4)

**Proof** Firstly, suppose that $F$ is a weakly weighted Einstein metric satisfying (4.1). By Theorem 3.6, $S_{BH} = 0$, that is, $\mathcal{R}_{ij} = 0$ and $S_j = 0$. Then by (2.21), we have

$$G^i_h - G^i = F S^i_0.$$  \hspace{1cm} (4.5)

Recall that $F(x, y) = h(x, \xi) = \tilde{h}$ and $\xi := y - FW$. We have

$$G^i_h - G^i = (\xi^i + \tilde{h} W^i) \tilde{h} S^i_j.$$  \hspace{1cm} (4.6)

In this case, $\tilde{S} = \tilde{S}_{BH} + (n + 1)\text{Hess}_f f(y) = (n + 1)\text{Hess}_f f(y)$. Further, by (2.8), we have

$$\text{Hess}_f f(y) = f_{ij}(\xi^i + \tilde{h} W^i)(\xi^j + \tilde{h} W^j) + 2f_i(G^i_h - G^i)$$

$$= f_{ij}(\xi^i + \tilde{h} W^i)(\xi^j + \tilde{h} W^j) + 2f_i(\xi^j + \tilde{h} W^j) \tilde{h} S^i_j.$$  \hspace{1cm} (4.7)

Here, $f_i := \frac{\partial f}{\partial x^i}$ and $f_{ij} := \frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^m_{ij} \frac{\partial f}{\partial x^m}$ denote the second-order covariant derivatives of $f$ with respect to $h$.

By Proposition 3.9, (4.1) is equivalent to

$$\text{Ric}^h(\xi) - \tilde{h}^2 \text{Ric}^h(W) - \tilde{h}^2 S^m_q S^i_m + a(n + 1) \left[ f_q p (\xi^p \xi^q + 2\tilde{h} \xi^p W^q + \tilde{h}^2 W^p W^q) + 2\tilde{h} f_i S^i_p \xi^p \right] - c(n + 1)^2 \left[ f_0^2 + 2\tilde{h} f_0 f_p W^p + \tilde{h}^2 (f_p W^p)^2 \right]$$

$$= (n - 1)[3\tilde{\theta} \tilde{h} + (3\theta_p W_p + \sigma) \tilde{h}^2].$$  \hspace{1cm} (4.8)
where \( \tilde{W}_0 := W_p \xi^p \), \( \tilde{S}_0 := S_p \xi^p \), \( \tilde{f}_0 := f_p \xi^p \) and \( \tilde{\theta} = \theta_i \xi^i \).

Note that \( \tilde{\theta} \) is irrational in \( \xi \). Separating rational and irrational terms in the above equation, we have

\[
\text{Ric}^h(\xi) - \tilde{\theta}^2 \text{Ric}^h(W) - \tilde{\theta}^2 S^m_i S^i_m + a(n + 1) \left[ \text{Hess}_h f(\xi) + \tilde{\theta}^2 \text{Hess}_h f(W) \right]
\]

\[
- c(n + 1)^2 \left[ f_0^2 + \tilde{\theta}^2 (f_p W_p)^2 \right] = (n - 1) (3 \theta_p W_p + \sigma) \tilde{\theta}^2
\]

(4.9)

and

\[
2a(n + 1) \xi^p (f_{pq} W_q + f_q S^q_p) - 2c(n + 1)^2 f_0 f_p W_p = 3(n - 1) \tilde{\theta}.
\]

(4.10)

From (4.9), we obtain

\[
\text{Ric}^h(\xi) + a(n + 1) \text{Hess}_h f(\xi) - c(n + 1)^2 f_0^2 = (n - 1) \mu \tilde{\theta}^2,
\]

(4.11)

where

\[
\mu = 3 \theta_p W_p + \sigma + \frac{1}{n - 1} \left[ \text{Ric}^h(W) + S^m_i S^i_m - a(n + 1) \text{Hess}_h f(W) + c(n + 1)^2 (f_p W_p)^2 \right].
\]

(4.12)

From (4.11), we get (4.2).

Conversely, suppose that (4.11) holds and \( R_{ij} = 0 \). We choose

\[
\theta_i := \frac{1}{3(n - 1)} \left[ 2a(n + 1) (f_i W_p + f_p S^p_i) - 2c(n + 1)^2 f_i f_p W_p \right].
\]

(4.13)

And then, equation (4.10) holds. Furthermore, we choose

\[
\sigma := \mu - \frac{1}{n - 1} \left[ \text{Ric}^h(W) + S^p_i S^i_p + a(n + 1) \text{Hess}_h f(W) - c(n + 1)^2 (f_p W_p)^2 \right].
\]

(4.14)

It is easy to check that equation (4.9) also holds and it follows that

\[
\text{Ric}_{a,c} = (n - 1) \left( \frac{3 \theta}{F} + \sigma \right) F^2
\]

with respect to \( dV = e^{-(n+1)F} dV_{BH} \).

(4.15)

It is notable that (4.2) is equivalent to \( h \) being a weighted Einstein-Riemann metric with weight constants \( a \) and \( c \) with respect to \( dV = e^{-(n+1)F} dV_{BH} = e^{-(n+1)F} dV_{h} \).

In the following, we will characterize weakly weighted Einstein-Kropina metrics via \( \alpha \) and \( \beta \) in the case that \( \nu \neq 0 \). Assume that \( F = \frac{\sigma^2}{p} \) is a weakly weighted Einstein-Kropina metric satisfying (1.8) and \( \nu \neq 0 \). By Theorem 3.6 and Lemma 3.3, we have

\[
r_{00} = \eta(x) \alpha^2.
\]

(4.15)
Then, it is easy to get

\[
\begin{align*}
  r_{0i} &= \eta y_i, \quad r_i = \eta b_i, \quad r = \eta b^2, \quad r_j = \eta \delta^j_i, \\
  r_{0k}s^k_0 &= 0, \quad r_{0k}s^k = \eta s_0, \quad r_0 = \eta \beta, \quad s^k_r k = \eta s_0, \\
  r_{00;k} &= \eta k \alpha^2, \quad r_{00;0} = \eta_0 \alpha^2, \quad r_{0;0} = \eta_0 \beta + \eta^2 \alpha^2. 
\end{align*}
\]  

(4.16)

Here, \( y_i = a_{ij} y^j \). Substituting (4.16) into (3.13) and dividing both sides of (3.13) by \( \alpha^4 \), we can obtain

\[
\begin{align*}
  0 &= \text{Ricc} b^4 \beta^2 + (n - 2) \beta^2 [b^2(s_{0;0} + \eta_0 \beta) - 2\eta_0 s_0 - s_0^2 - \eta^2 \beta^2] \\
  &\quad - [3\kappa - \nu - a(n + 1)] b^4 \beta^2 f_0^2 + (-\kappa + n - 1) b^4 \beta^2 \text{Hess}_F f(y) \\
  &\quad + b^2 \beta \alpha^2 [\beta b^k \eta_k + (n - 2) \eta^2 \beta + (n - 3) \eta s_0 + b^k s_{0;k} - b^2 s^k_{0;k} \\
  &\quad + (n - 1)(s_k s^k_{0;0} - 3b^2 \theta)] - b^2 \alpha^2 \left( \frac{1}{2} s^k s_k + \frac{b^2}{4} s^i_j s^j_k + (n - 1)b^2 \sigma \right). 
\end{align*}
\]  

(4.17)

From (4.17), it is easy to see that there exists some scalar function \( \lambda = \lambda(x) \) on \( M \) such that

\[
\begin{align*}
  \text{Ricc} b^4 + (n - 2)[b^2(s_{0;0} + \eta_0 \beta) - 2\eta_0 s_0 - s_0^2 - \eta^2 \beta^2] \\
  &\quad - (3\kappa - \nu - a(n + 1)) b^4 f_0^2 + (-\kappa + n - 1) b^4 \text{Hess}_F f(y) = \lambda \alpha^2.
\end{align*}
\]  

(4.18)

Then (4.17) can be simplified as

\[
\begin{align*}
  0 &= \beta \left[ \lambda \beta + b^2 \beta b^k \eta_k + (n - 2) \eta^2 \beta + (n - 3) \eta s_0 + b^k s_{0;k} - b^2 s^k_{0;k} \\
  &\quad + (n - 1)(s_k s^k_{0;0} - 3b^2 \theta) \right] - b^2 \alpha^2 \left( \frac{1}{2} s^k s_k + \frac{b^2}{4} s^j_k s^j + (n - 1)b^2 \sigma \right). 
\end{align*}
\]  

(4.19)

Since \( \alpha^2 \) is irreducible polynomial in \( y \), (4.19) implies the following equations

\[
\begin{align*}
  \lambda \beta + b^2 [(n - 3) \eta s_0 + (n - 2) \eta^2 \beta + b^k \eta_k \beta + b^k s_{0;k} \\
  &\quad - b^2 s^k_{0;k} + (n - 1)(s_k s^k_{0;0} - 3b^2 \theta)] = 0, 
\end{align*}
\]  

(4.20)

\[
\frac{1}{2} s^k s_k + \frac{b^2}{4} s^j_k s^j + (n - 1)b^2 \sigma = 0.
\]  

(4.21)

Differentiating both sides of (4.20) with respect to \( y^i \) yields

\[
\begin{align*}
  \lambda b_i + [(n - 2) \eta^2 + b^k \eta_k] b^2 b_i - 3(n - 1) b^4 \theta_i + (n - 1) b^2 s_k s^k_i + (n - 3) \eta b^2 s_i \\
  + b^2 b^k s_{i;k} - b^4 s^k_{i;k} = 0.
\end{align*}
\]  

(4.22)
Contracting (4.22) with $b^i$ gives

$$\lambda b^2 + [(n - 2)\eta^2 + b^k \eta_k]b^4 - 3(n - 1)b^4 \theta_i b^j - (n - 2)b^2 s_k s^k + b^4(s^i_{;k} + s^j_{;k}; s_i k s_j) = 0,$$

where we used

$$s_{i;:k} b^j = -\eta s_{k} - s_i s^j_{;k},$$

$$s^k_{i;:k} b^j = -s^k_{;k} - s^i_{;k}. $$

Hence, we obtain

$$\lambda = -\left[(n - 2)\eta^2 + b^k \eta_k\right]b^2 + 3(n - 1)b^2 \theta_i b^j + (n - 2)s_k s^k - b^2(s^i_{;k} + s^j_{;k}; s_i k s_j).$$

(4.24)

Plugging (4.24) into (4.20) yields

$$0 = \beta [ (n - 2)s^k s_k + 3(n - 1)b^2 \theta_i b^j - b^2(s^k_{;k} + s^i_{;k}; s_i k s_j)] + b^2 [(n - 3)\eta s_0 + b^k s_0; k - b^2 s^k_{0; k} + (n - 1)s_k s^k_0 - 3(n - 1)b^2 \theta].$$

(4.25)

Further, by (4.21), we obtain

$$\sigma = -\frac{1}{(n - 1)b^2} \left(\frac{1}{2}s^k s_k + \frac{b^2}{4}s^j_{;k} s^k_{;j}\right).$$

(4.26)

Conversely, if (4.15), (4.18), and (4.25) hold and $\lambda, \sigma$ are given by (4.24) and (4.26), respectively, then it is easy to see that (3.13) holds, that is, $F$ is a weakly weighted Einstein-Kropina metric with weight constants $a$ and $c$. Hence, we have actually proved the following

**Theorem 4.2** Let $a, c$ be two constants satisfying $\nu \neq 0$ and $F = \frac{a^2}{\beta}$ be a Kropina metrics on an $n$-dimensional manifold $M$. Then $F$ is a weakly weighted Einstein metric satisfying

$$\text{Ric}_{a,c} = (n - 1) \left(\frac{3\theta}{F} + \sigma\right) F^2$$

with respect to some volume form $dV = e^{-(n + 1)f} dV_{BH}$ if and only if equations (4.15), (4.18), and (4.25) are satisfied for some scalar functions $\lambda, \sigma$ given by (4.24) and (4.26), respectively.
5 The Weakly Weighted Einstein–Kropina Metrics with $\nu = 0$ and $\kappa \neq 0$

By the definition, when $\nu = 0$ and $\kappa \neq 0$, the generalized weighted Ricci curvature with weight constants $a$ and $c$ is given by

$$\text{Ric}_{a,c} = \text{PRic} - \kappa \left[ \frac{\hat{S}}{n+1} + \frac{4S^2}{(n+1)^2} \right].$$

In the following, we are going to derive an equivalent condition for a Kropina metric $F$ to satisfy

$$\text{Ric}_{a,c} = (n-1) \left( \frac{3\theta}{F} + \sigma \right) F^2. \tag{5.1}$$

Assume that $F$ is a weakly weighted Einstein–Kropina metric satisfying (5.1). In this case, the equation (3.14) becomes

$$\beta^3 P_1 + \beta^2 \alpha^2 P_2 + \beta \alpha^4 P_3 + \alpha^6 P_4 = 0 \tag{5.2}$$

and the polynomials $P_i$'s can be simplified a little as

$$P_1 = \kappa (b_2^2 r_{00;0} + 2 r_{00} r_0 + 2 r_{00} s_0) + 2[3 \kappa - a(n + 1)] b_2 r_{00} f_0,$$

$$P_2 = b_4 \text{Ric}^a + b_2 b_4 r_{00;0} + (n - 2) b_2 s_0;0 + b_2 r_{00} r_k - n - 2 s_0^2 + (-\kappa + n - 2) b_2 r_{00;0}$$
$$+ (\kappa - n) r_0 r + (-2 \kappa + 4 - 2n) r_0 s_0 + 2(\kappa + 1) b_2^2 r_{00} s_0^2 + (-2 \kappa + 2 - n) r_0^2$$
$$+ (-\kappa + n - 1) b_2^4 \text{Hess}_F f(y) (3 \kappa - a(n + 1)) (2 b_2 r_0 f_0 + b_4 f_0^2),$$

$$P_3 = (\kappa - n) s_0 r + b_2^2 b_4 s_0;0 + b_2^2 s_0 r_k - b_2^4 s_0^2 + (-\kappa + n - 2) b_2^2 r_k s_0^2$$
$$- b_2^2 r_{00} s_0^2 + (n - 1) b_2^2 s_0 f_0 - 3(n - 1) \theta b_4,$$

$$P_4 = -b_2^2 \left[ \frac{1}{2} s^k s_k + \frac{1}{4} b_2^2 s^j s^k_j + (n - 1) \sigma b_2 \right].$$

By (5.2), we know that there exists a 1-form $\zeta = \zeta_i(x) y^i$ such that

$$P_1 = \zeta \alpha^2,$$

which is equivalent to

$$\zeta \alpha^2 = \kappa (b_2^2 r_{00;0} + 2 r_{00} r_0 + 2 r_{00} s_0) + 2[3 \kappa - a(n + 1)] b_2 r_{00} f_0. \tag{5.3}$$

Then (5.2) can be simplified as

$$\beta^2 (\beta \zeta + P_2) + \beta \alpha^2 P_3 + \alpha^4 P_4 = 0. \tag{5.4}$$

By (5.4), we know there exists some function $u = u(x)$ such that

$$\beta \zeta + P_2 = u \alpha^2. \tag{5.5}$$
Then (5.4) can be simplified as

$$\beta(\beta u + P_3) + \alpha^2 P_4 = 0,$$

that is

$$\beta(\beta u + (\kappa - n)s_0 r + b^2 b^k s_{0:k} + b^2 s_0 r^k - b^4 s^{k}_{0:k} + (-\kappa + n - 2)b^2 r_k s^k_{0} - b^2 r_{0k}s^k + (n - 1)b^2 s_k s^k_{0} - 3(n - 1)\beta b^4) - \alpha^2 b^2 \left[\frac{1}{2}s^k s_k + \frac{1}{4}b^2 s^j k s^k_j + (n - 1)\sigma b^2 \right] = 0.$$ 

Since $\alpha^2$ can't be divided by $\beta$, we see that above equation is equivalent to the following equations

$$\beta u + (\kappa - n)s_0 r + b^2 b^k s_{0:k} + b^2 s_0 r^k - b^4 s^{k}_{0:k} + (-\kappa + n - 2)b^2 r_k s^k_{0} - b^2 r_{0k}s^k + (n - 1)b^2 s_k s^k_{0} - 3(n - 1)\beta b^4 = 0,$$

$$\frac{1}{2}s^k s_k + \frac{b^2}{4} s^j k s^k_j + (n - 1)b^2 \sigma = 0.$$  

(5.6) 

(5.7)

Differentiating both sides of (5.6) with respect to $y^i$ yields

$$ub_i + (\kappa - n)s_i r + b^2 b_i s^k r^k + (-\kappa + n - 2)b^2 r_k s^k_i + b^2 b^k s_{i:k} - b^4 r_{ik}s^k_i + (n - 1)b^2 s_k s^k_{i} - 3(n - 1)\theta_i b^4 + b^2 b^k s_{i:k} - b^4 s_i s^k_{i:k} = 0.$$  

(5.8)

Further, contracting (5.8) with $b^i$ gives

$$ub^2 + (\kappa - n)b^2 r_i s^i - (n - 2)b^2 s_i s^i - 3(n - 1)\theta_i b^i b^4 + b^4 (s_i^k + s_i^k s^i_{k} + s_i^k r^i_{k}) = 0,$$

(5.9)

where we have used

$$s_i^k b^i = -s_i (r^i_{k} + s^i_{k}),$$

$$s_i^k b^i = -(s_i^k + s^k s^i_{k} + s^k r^i_{k}).$$

Then we obtain

$$u = (n - \kappa)r_i s^i + (n - 2)s^i s_i + 3(n - 1)b^2 \theta_i b^i - b^2 (s_i^k + s_i^k s^i_{k} + s_i^k r^i_{k}).$$  

(5.10)

Plugging (5.10) into (5.6) yields

$$\beta \left[ (n - \kappa)r_i s^i + (n - 2)s^i s_i + 3(n - 1)b^2 \theta_i b^i - b^2 (s_i^k + s_i^k s^i_{k} + s_i^k r^i_{k}) \right]$$

$$+ (\kappa - n)s_0 r + b^2 b^k s_{0:k} + b^2 s_0 r^k - b^4 s^{k}_{0:k} + (n - \kappa - 2)b^2 r_k s^k - b^2 r_{0k}s^k + (n - 1)b^2 s_k s^k_{0} - 3(n - 1)\theta b^4 = 0.$$  

(5.11)
Furthermore, from (5.7), we obtain
\[
\sigma = -\frac{1}{(n - 1)b^2} \left( \frac{1}{2} s^k s_k + \frac{b^2}{4} s^j s^k s_j \right). \tag{5.12}
\]

Conversely, if (5.3) and (5.5), (5.11) hold for some 1-form \( \zeta \) with (5.10) and (5.12), then it is easy to see that \( F \) is weakly weighted Einstein-Kropina metric. Thus we have proved the following.

**Theorem 5.1** Let \( a, c \) be two constants satisfying \( \nu = 0 \) and \( \kappa \neq 0 \). Let \( F = \frac{a^2}{b^2} \) be a Kropina metric on an \( n \)-dimensional manifold \( M \). Then \( F \) is weakly weighted Einstein-Kropina metric satisfying
\[
\text{Ric}_{a,c} = (n - 1) \left( \frac{3\theta}{F} + \sigma \right) F^2
\]
with respect to some volume form \( dV = e^{-(n+1)f} dV_{BH} \) if and only if equations (5.3) and (5.5), (5.11) hold for some 1-form \( \zeta \) and \( u = u(x) \) and \( \sigma \) are determined by (5.10) and (5.12), respectively.

### 6 The Weakly Weighted Einstein–Kropina Metrics with \( \nu = 0 \) and \( \kappa = 0 \)

In this section, we shall consider the weakly weighted Einstein-Kropina metrics with \( \nu = 0 \) and \( \kappa = 0 \). In this case, the generalized weighted Ricci curvatures are just the projective Ricci curvature [5, 19]
\[ \text{Ric}_{a,c} = \text{PRic} \]
and \( a = \frac{n-1}{n+1}, \ c = -\frac{n-1}{(n+1)^2} \). Then, the polynomials \( P_i \)'s in (5.2) can be further simplified as follows.

\[
P_1 = -2(n - 1)b^2 r_{00} f_0,
\]
\[
P_2 = b^4 \text{Ric}^\alpha + b^2 b^k r_{00;k} + (n - 2)(b^2 s_{0;0} - s_0^2 + b^2 r_{0;0} - 2r_{00} - r_0^2) + b^2 r_{00} r^k_k
\]
\[\quad -nr_{00} r + 2b^2 r_{0k} s^k_0 + (n - 1)b^4 \text{Hess}_F f(y) + (n - 1) \left( 2b^2 r_{00} f_0 + b^4 f_0^2 \right),
\]
\[
P_3 = -ns_0 r + b^2 b^k s_{0;k} + b^2 s_0 r_k + r_{00} r^k_0 + (n - 2)b^2 r_k s^k_0
\]
\[\quad -b^2 r_{0k} s^k_0 + (n - 1)b^2 s_k s^k_0 - 3(n - 1)\theta b^4,
\]
\[
P_4 = -b^2 \left[ \frac{1}{2} s^k s_k + \frac{1}{4} b^2 s^j s^k s_j + (n - 1)\sigma b^2 \right].
\]

At the same time, (5.3) is reduced equivalently to
\[
\zeta \alpha^2 = -2(n - 1)b^2 r_{00} f_0. \tag{6.1}
\]
From (6.1), we find that there exists a scalar function \( \eta = \eta(x) \) on \( M \) such that

\[
  r_{00} = \eta \alpha^2. \tag{6.2}
\]

Thus, by Lemmas 3.3 and 3.4, we can conclude the following result.

**Proposition 6.1** Let \( F \) be a weakly weighted Einstein–Kropina metric with \( \nu = 0 \) and \( \kappa = 0 \). Then the S-curvature of \( F \) with respect to the Busemann-Hausdorff volume form is isotropic, that is, \( \mathcal{R}_{ij} = 0 \).

By Proposition 3.9, Proposition 6.1 and (3.2), (3.3), completely similar to the proof of Theorem 4.1, we can get the following classification theorem.

**Theorem 6.2** Let \( F \) be a Kropina metric on an n-dimensional manifold \( M \) defined by navigation data \((h, W)\). Then \( F \) is weakly weighted Einstein-Kropina metric with weight constants \( a = \frac{n-1}{n+1} \) and \( c = -\frac{n-1}{(n+1)^2} \) satisfying

\[
  \text{PRic} = (n - 1) \left( \frac{3\theta}{F} + \sigma \right) F^2
\]

with respect to some volume form \( dV = e^{-(n+1)f} dV_{BH} \) if and only if there exists a scalar function \( \mu \) on \( M \), such that the Ricci curvature tensor of \( h \) satisfies

\[
  \text{Ric}^h + (n - 1)\text{Hess}_h f + (n - 1)(df \otimes df) = (n - 1)\mu h^2 \tag{6.3}
\]

and \( W \) satisfies \( \mathcal{R}_{ij} = 0 \). In this case,

\[
  \sigma = \mu - \text{Hess}_h f(W) - (f_p W^p)^2 - \frac{1}{n - 1} \left\{ \text{Ric}^h(W) + S_q^p S^q_p \right\} \tag{6.4}
\]

and

\[
  \theta_i = \frac{2}{3} \left( f_{ip} W^p + f_p S^p_i + f_i f_p W^p \right). \tag{6.5}
\]

It is worth to note that (6.3) is equivalent to the Riemannian metric \( h \) being a weighted Einstein–Riemann metric with weight constants \( a = \frac{n-1}{n+1} \) and \( c = -\frac{n-1}{(n+1)^2} \) with respect to \( dV = e^{-(n+1)f} dV_{BH} = e^{-(n+1)f} dV_h \). Besides, the Theorem 6.2 improves the result on the Kropina metrics of isotropic projective Ricci curvature in [9].

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