The No-Hair Theorem for the Abelian Higgs Model

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Abstract
We consider the general procedure for proving no-hair theorems for static, spherically symmetric black holes. We apply this method to the abelian Higgs model and find a proof of the no-hair conjecture that circumvents the objections raised against the original proof due to Adler and Pearson.

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1. Introduction

Classical “no-hair” theorems [1] state that a stationary black hole is characterized by a small number of parameters – its mass, angular momentum, and charges corresponding to long-range gauge fields. It has long been known [2,3] that static black holes carry no hair, or external field, corresponding to massless or massive scalar fields or Proca-massive gauge fields. It was also shown sometime later [4] that if an abelian gauge field acquires a mass via the Higgs mechanism, the corresponding gauge field must vanish outside the horizon. However, the arguments used there have recently been criticized [5] as being too restrictive. Therefore it is necessary to find a more rigorous proof of the no-hair conjecture before one can believe in it.

On the other hand, a failure to find such a proof, or more precisely, evidence that a rigorous proof cannot exist, will have very interesting consequences. It has been recognized [6-7] that the classical no-hair theorems do not rule out the possibility that black holes may carry charges that are detectable only through experiments of a quantum nature. The $Z_N$ quantum hair as it is called [7] arises from spontaneous symmetry breaking in an abelian Higgs model, where the Higgs condensate has charge $N\hbar$, $\hbar$ being the charge quantum of the theory. The dynamical effect of this quantum hair is expected to be non-perturbative in $\hbar$, with a serious effect on the thermodynamics of a static black hole [8,9]. The partition function of a black hole at temperature $\beta^{-1}$ is given by [12] the path-integral of the Euclidean action over asymptotically flat, topologically $R^2 \times S^2$ configurations that are periodic in the imaginary time $\tau$ with period $\beta\hbar$. The saddle points of this path-integral include classical Euclidean black hole solutions coupled to non-trivial gauge and Higgs field configurations. It was conjectured [8,9] that such solutions indeed exist, and they behave like vortices on the $\tau - r$ plane on an ‘almost Schwarzschild’ Euclidean background. More precise calculations followed [10,11], and even though no exact solution was found, stronger arguments were given for their existence. If the no-hair conjecture for the abelian Higgs model is found to be incorrect, one will have found more evidence for the existence of these solutions. On the other hand, if it is correct, one has to demonstrate the failure of the proof for Euclidean backgrounds in order for these ‘Euclidean’ vortices to exist.

Another reason for looking for a rigorous proof of the no-hair conjecture is the following. It was found in [3] that a static black hole can carry a topological charge corresponding to the surface integral of an antisymmetric tensor potential, and later it was shown [13] that this special ‘hair’ persists even when this potential becomes massive via a coupling
to a massless abelian gauge field. The action used for this purpose can also have an interpretation by which the gauge field becomes massive after absorbing the degree of freedom in the tensor potential \cite{14,15}. Thus, contrary to prevalent belief, a black hole can carry some information apart from its mass and angular momentum in the presence of a gauge field that acquires a gauge-invariant mass. Even though this does not imply the failure of the no-hair conjecture for the abelian Higgs model, it does raise some skepticism.

In light of the various results mentioned above, we propose to take a renewed look at the classical no-hair theorems. We consider the general procedure used to prove such theorems, specifying all the assumptions that are used. We apply this procedure to the abelian Higgs model coupled to gravity and find a rigorous proof for the no-hair conjecture. This proof fails when the space-time metric has a Euclidean signature, corroborating the arguments of \cite{8-11}.

2. General Setup

We restrict ourselves to a 3 + 1-dimensional static, spherically symmetric, asymptotically flat space-time with a horizon. This implies making the following assumptions:

\text{(i)} The space-time is endowed with a timelike Killing vector $\xi^\mu$ with $\xi^\mu \xi_\mu = -\lambda^2(r)$ which obeys $\xi_{[\mu} \nabla_{\nu} \xi_{\lambda]} = 0$; (it follows \cite{1} that there is a spacelike hypersurface $\Sigma$ — the space ‘outside’ the horizon — which is everywhere orthogonal to $\xi^\mu$.)

\text{(ii)} The hypersurface $\Sigma$ allows a coordinatization isomorphic to the flat-space spherical coordinates (the space-time metric may be written as $ds^2 = -\lambda^2(r) dt^2 + h^2(r) dr^2 + r^2 d\Omega$);

\text{(iii)} $\lambda$ vanishes at a finite value $r_H$ of the radial coordinate $r$, thus defining the horizon;

\text{(iv)} $\lambda \to 1 + O(1/r)$ as $r \to \infty$ (asymptotic flatness).

These are all the assumptions we will need to make about the space-time, now we turn our attention to the fields that live on this space-time. The crucial assumption that goes into proving the standard no-hair theorems is that the squared norm of the stress-energy tensor is bounded at the horizon and vanishes suitably rapidly at infinity \cite{2-4}. This may be seen as being dictated by Einstein’s equations. If the stress-energy tensor $T_{\mu\nu}$ has unbounded norm at any point, the Einstein tensor and therefore the curvature must also become unbounded there, giving rise to a singularity. The horizon is not, however, a curvature singularity, but only a coordinate singularity. Therefore the stress-energy tensor must remain bounded at the horizon. Similarly, asymptotic flatness dictates that the metric approaches the Schwarzschild metric as $r \to \infty$. It follows that $T_{\mu\nu}$ must vanish in
this region. Similar arguments show that $T_{\mu\nu}$ must also be static, i.e., have vanishing Lie derivative with respect to $\xi^\mu$.

We will need one more result for proving no-hair theorems, which we write down here. Let us denote the projection operator that projects down to $\Sigma$ by $\Pi^\mu_\mu := \delta^\mu_\mu + \lambda^{-2} \xi^\mu \xi^\nu$. Let us also denote the space-time connection by $\nabla_\mu$ and the induced connection on $\Sigma$ by $\tilde{\nabla}_\mu$. Then for a rank $p$ antisymmetric tensor $\Omega$ whose $\Sigma$-projection is $\omega$ and $\mathcal{L}_\xi \Omega = 0$, it can be shown that

$$\tilde{\nabla}_\alpha (\lambda \omega^{\alpha \mu \cdots \nu}) = \lambda \nabla_\alpha \Omega^{\alpha \mu \cdots \nu} \Pi^\mu_\mu \cdots \Pi^\nu_\nu. \quad (2.1)$$

Physically this may be understood as the statement that the 4-divergence of $\Omega$ is equal to its 3-divergence when the metric and $\omega$ are time-independent.

The algorithm for proving no-hair theorems may be seen in our first example of a real scalar field $\rho$ moving in a potential $U(\rho)$. The Lagrangian is

$$\mathcal{L} = -\left(\frac{1}{2} \nabla_\mu \rho \nabla^\mu \rho + U(\rho)\right), \quad (2.2)$$

and the equations of motion are

$$\nabla_\mu \nabla^\mu \rho = \frac{\partial U}{\partial \rho} \equiv U'(\rho). \quad (2.3)$$

Using the divergence relation (2.1) we can write

$$\tilde{\nabla}_\mu (\lambda \tilde{\nabla}^\mu \rho) = \lambda U'(\rho). \quad (2.4)$$

Multiplying both sides by $\rho$ and integrating over the space-like region $\Sigma$ between the horizon and infinity, we get

$$\int_{\partial \Sigma} \lambda \rho \tilde{\nabla}_\mu \rho n^\mu - \int_{\Sigma} \lambda (\tilde{\nabla}_\mu \rho \tilde{\nabla}^\mu \rho + \rho U'(\rho)) = 0, \quad (2.5)$$

where $\partial \Sigma$ is composed of the spheres at the horizon and at infinity, and $n^\mu$ is the outward pointing space-like unit normal on these two spheres. $\tilde{\nabla}_\mu \rho \tilde{\nabla}^\mu \rho$ appears in $T_{\mu\nu}$, so must be bounded at the horizon and vanish at infinity. Since the metric on $\Sigma$ is positive definite, we may apply Schwarz inequality, which says $|\tilde{\nabla}_\mu \rho n^\mu|^2 \leq (\tilde{\nabla}_\mu \rho \tilde{\nabla}^\mu \rho)(n_\mu n^\mu) = (\tilde{\nabla}_\mu \rho \tilde{\nabla}^\mu \rho)$, since $n^\mu$ is a unit vector. It follows that $\tilde{\nabla}_\mu \rho n^\mu$ has to obey the said boundedness conditions. If $\rho$ is massive ($U(\rho) = \frac{1}{2} m^2 \rho^2$), the behavior of $T_{\mu\nu}$ also dictates that $\rho$ has to

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1 See Appendix A.
be bounded everywhere on or outside the horizon. It follows that the integral over the boundary $\partial \Sigma$ vanishes, the volume integral is an integral of a sum of squares (the metric on $\Sigma$ is positive definite) and therefore $\rho$ must be trivial outside the horizon. The same result also holds when $U = \alpha \rho^4$ with $\alpha > 0$ (or any other convex potential). The situation is different when $U$ is a double-well potential $U = \frac{\alpha}{4}(\rho^2 - v^2)^2$. There is no known rigorous proof of the no-hair conjecture for the scalar field in such a potential \cite{3}.

(Note: If $\rho$ is a conformal scalar field, $U = \frac{1}{12}R\rho^2$, one cannot impose boundedness conditions on $\rho$ because of the local conformal symmetry of the theory. It follows that the boundary integrals need not vanish. Thus even if we are looking for solutions with $R = 0$, we can find a non-trivial conformal scalar field outside the horizon \cite{10}.)

3. The Abelian Higgs Model

Now we are ready to look for a proof of the no-hair conjecture in the case of the Abelian Higgs model. We will work with the Lagrangian

$$\mathcal{L} = -(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(D_\mu \Phi)^* D^\mu \Phi + U(\Phi)).$$

(3.1)

Here $\Phi$ is a complex scalar field, $F_{\mu\nu} = \nabla_{[\mu} A_{\nu]}$ is the field strength of the Abelian gauge field $A_\mu$, $D_\mu \Phi = (\nabla_\mu + iq A_\mu)\Phi$ is the gauge covariant derivative, and $U(\Phi) = \frac{\alpha}{4}(|\Phi|^2 - v^2)^2$, $\alpha > 0$, is the Higgs potential. If we parametrize $\Phi$ as $\Phi = \rho e^{i\eta/v}$, we can see that the Lagrangian is left invariant by gauge transformations $A_\mu \to A_\mu + \nabla_\mu \chi, \eta \to \eta - ivq\chi$. In terms of $\rho$ and $\eta$, the Lagrangian reads

$$\mathcal{L} = -(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} \nabla_\mu \rho \nabla^\mu \rho + \frac{1}{2} \rho^2 q^2 (A_\mu + \frac{1}{qv} \nabla_\mu \eta) (A^\mu + \frac{1}{qv} \nabla^\mu \eta) + \frac{\alpha}{4}(\rho^2 - v^2)^2).$$

(3.2)

One of the objections raised in \cite{3} about the proof given in \cite{4} was that there may not be any non-singular gauge choice in which both $A_\mu$ and $\Phi$ are static and/or have bounded norm at the horizon. However, the squared norms of the temporal and spatial components of the combination $(A_\mu + \frac{1}{qv} \nabla_\mu \eta)$ appear in $T_{\mu\nu}$. Therefore, we can take $(A_\mu + \frac{1}{qv} \nabla_\mu \eta)$ to be static as well as of bounded norm at the horizon and vanishing norm at infinity.

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\footnote{For the massless scalar, this is enforced by demanding that $\rho$ remains measurable and thus bounded at the horizon \cite{2,3}.}
Let us denote the $\Sigma$-projections of $A_\mu$ and $F_{\mu\nu}$ by $a_\mu$ and $f_{\mu\nu}$, respectively. The electric potential $\phi := \lambda^{-1} \xi^\alpha A_\alpha$ and the electric field $e^\mu := \lambda^{-1} \xi^\alpha F_{\mu\alpha}$ satisfy the relations

$$\tilde{\nabla}_\mu e^\mu = \lambda^{-1} \xi^\alpha \nabla_\mu F_{\mu\alpha}, \quad \tilde{\nabla}_\mu (\lambda \phi) = \lambda e_\mu + \mathcal{L}_\xi A_\mu.$$  \hspace{1cm} (3.3)

We follow the procedure used for a real scalar field, first writing down the equations of motion for the $\Sigma$-projections of the fields plus the one coming from (3.3), multiply by the appropriate fields and integrate over the region between the horizon and infinity. We concentrate on one of the field equations,

$$\tilde{\nabla}_\mu e^\mu - q^2 \rho^2 (\phi + \frac{1}{qv} \lambda^{-1} \dot{\eta}) = 0,$$  \hspace{1cm} (3.4)

where we have written $\xi^\mu \nabla_\mu \eta$ as $\dot{\eta}$. We multiply this equation by $\lambda (\phi + \frac{1}{qv} \lambda^{-1} \dot{\eta})$ and integrate over the region between the horizon and infinity. Including all terms, the resulting equation is

$$\int_{\partial \Sigma} \lambda (\phi + \frac{1}{qv} \lambda^{-1} \dot{\eta}) e^\mu n_\mu \, d\Sigma - \int_{\Sigma} (\lambda e^\mu e_\mu + e^\mu \mathcal{L}_\xi (A_\mu + \frac{1}{qv} \nabla_\mu \eta) + \lambda q^2 \rho^2 (\phi + \frac{1}{qv} \lambda^{-1} \dot{\eta})^2) \, d\Sigma = 0.$$  \hspace{1cm} (3.5)

Since the squared norms of the spatial and temporal components of $(A_\mu + \frac{1}{qv} \nabla_\mu \eta)$ appear in $T_{\mu\nu}$, this quantity must be static. Therefore the second term in the volume integral vanishes. The remaining terms of the integrand are positive indefinite, and must vanish if the surface integral is zero. The stress tensor $T_{\mu\nu}$ contains the product $\rho^2 (\phi + \frac{1}{qv} \lambda^{-1} \dot{\eta})^2$, and the boundary conditions on $T_{\mu\nu}$ imply the following. Since $\rho \to v$ as $r \to \infty$, we must have $(\phi + \frac{1}{qv} \lambda^{-1} \dot{\eta}) \to 0$ (as well as $e^\mu n_\mu \to 0$) as $r \to \infty$, and therefore the contribution to the surface integral from the sphere at infinity must vanish. The contribution from the horizon may be non-zero, but then it follows from the boundedness of $T_{\mu\nu}$ that $\rho \to 0$ as $r \to r_H$. The remainder of the proof consists of showing that $\rho \neq 0$ at $r = r_H$.

The $\Sigma$-projection of the equation of motion for $\rho$ is

$$\tilde{\nabla}_\mu \lambda \tilde{\nabla}^\mu \rho = \lambda \alpha \rho (\rho^2 - v^2) - \lambda q^2 \rho (\phi + \frac{1}{qv} \lambda^{-1} \dot{\eta})^2,$$  \hspace{1cm} (3.6)

assuming for the moment that $(a_\mu + \frac{1}{qv} \tilde{\nabla}_\mu \eta) = 0$. If $\rho = 0$ at $r = r_H$ and $\rho \to v$ as $r \to \infty$, there are only the following possibilities:

(i) $v \geq \rho \geq 0$ for all $r \geq r_H$, $\rho \to v$ monotonically as $r \to \infty$;

(ii) $v \geq \rho \geq 0$ for $r_v \geq r \geq r_H$, $\rho = v$ at $r = r_v$;
(iii) \( v \geq \rho \geq 0 \) for \( r_{\text{max}} \geq r \geq r_H \), \( \rho \) has a local maximum at \( r = r_{\text{max}} \) \((n^\mu \tilde{\nabla}_\mu \rho |_{r_{\text{max}}} = 0)\);
(iv) \(-v \leq \rho \leq 0 \) for \( r_{-v} \geq r \geq r_H \), \( \rho = -v \) at \( r = r_{-v} \);
(v) \(-v \leq \rho \leq 0 \) for \( r_{\text{min}} \geq r \geq r_H \), \( \rho \) has a local minimum at \( r = r_{\text{min}} \) \((n^\mu \tilde{\nabla}_\mu \rho |_{r_{\text{min}}} = 0)\).

For cases (i)–(iii), we multiply \((3.6)\) by \( (\rho - v) \) and integrate over a region \( V \) between the horizon and the spheres respectively at (i) infinity, (ii) \( r_v \), and (iii) \( r_{\text{max}} \). The resulting equation is
\[
\int_{\partial V} \lambda (\rho - v) n^\mu \tilde{\nabla}_\mu \rho - \int_V \lambda (\tilde{\nabla}_\mu \rho \tilde{\nabla}^\mu \rho + (\rho - v)^2 \rho (\rho + v) - \rho (\rho - v) (\phi + \frac{1}{qv} \lambda^{-1} \eta)^2) = 0. \tag{3.7}
\]
By our choice of the region \( V \), the surface integral over \( \partial V \) vanishes in each case \((\lambda = 0 \) and \( \rho, n^\mu \tilde{\nabla}_\mu \rho < \infty \) at the horizon\), and since \( v \geq \rho \geq 0 \) everywhere on \( V \), the integrand is necessarily positive definite, \( i.e., \) we have a contradiction. For the cases (iv) and (v), we multiply \((3.6)\) by \( (\rho + v) \) and integrate over the region between the horizon and (iv) \( r_{-v} \), (v) \( r_{\text{min}} \), respectively. Again we find that the integral of a positive definite quantity must be zero. Since the cases (i)–(v) exhaust the possible behaviors of \( \rho \) if it has to vanish at the horizon and reach \( v \) at infinity, we conclude that there is no such solution. It follows that the surface integral in \((3.5)\) must vanish as well \((\text{finiteness of } T_{\mu \nu} \text{ and the non-vanishing of } \rho \text{ demands the finiteness of } (\phi + \frac{1}{qv} \lambda^{-1} \eta) \text{ at the horizon})\), and the black hole carries no gauge hair.

We also need to justify the assumption that \((a_\mu + \frac{1}{qv} \tilde{\nabla}_\mu \eta) = 0\), and the justification is the following. The equation of motion for \( a_\mu \) leads to the integral
\[
\int_{\partial \Sigma} \lambda (a_\mu + \frac{1}{qv} \tilde{\nabla}_\mu \eta) f^{\mu \nu} n_\nu - \int_{\Sigma} \lambda (\frac{1}{2} f^{\mu \nu} f_{\mu \nu} + q^2 \rho^2 (a_\mu + \frac{1}{qv} \tilde{\nabla}_\mu \eta)^2) = 0. \tag{3.8}
\]
Obviously, the only way to have \((a_\mu + \frac{1}{qv} \tilde{\nabla}_\mu \eta) \) non-vanishing outside the horizon is to allow it to diverge at least as fast as \( \lambda^{-1} \) at the horizon. But the \( f_{\mu \nu} = \tilde{\nabla}_{[\mu} a_{\nu]} \) will diverge as fast as \( \lambda^{-2} \) and \( f^{\mu \nu} f_{\mu \nu} \), a quantity appearing in \( T_{\mu \nu} \), will also diverge. (The appearance, in a coordinate basis, of \( g^{rr} \) in \( f^{\mu \nu} f_{\mu \nu} \) cannot nullify this divergence because \( \sqrt{g^{rr}} \) appears in the surface integral as well. Then, assuming \( g^{rr} \) vanishes faster than \( \lambda^2 \), we find that \((a_\mu + \frac{1}{qv} \tilde{\nabla}_\mu \eta) \) has to diverge as \( 1/(\lambda \sqrt{g^{rr}}) \), \( i.e., \) \( f^{\mu \nu} f_{\mu \nu} \) diverges as \( 1/(\lambda^2 g^{rr}) \).) Since this is contradictory to our assumption that \( T_{\mu \nu} \) is bounded at the horizon, we conclude that \((a_\mu + \frac{1}{qv} \tilde{\nabla}_\mu \eta) \) cannot diverge at the horizon and therefore must vanish everywhere on \( \Sigma \) according to \((3.8)\).

Thus we have proven the no-hair conjecture for the abelian Higgs model coupled to a static, spherically symmetric black hole without making the restrictive assumptions made
in the original proof. This proof is gauge-invariant, so the objections raised in [5] have been taken care of.

Finally, the above proof of the no-hair conjecture for the abelian Higgs model does not apply to a metric with Euclidean signature. The equations (3.5) and (3.8) hold, as well as the arguments following them, but the equation (3.6) is now different (the last term changes sign), and the arguments following it do not hold any more.

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Appendix A. Derivation of (2.1)

Frobenius’ condition for hypersurface-orthogonality for a Killing vector $\xi^\mu$ states that $[\xi_{[\mu}, \nabla_{\nu} \xi_{\lambda]} = 0$. Contracting with $\xi^\lambda$ it follows that

$$\Pi^\mu_{\mu'} \nabla_{\mu} \xi_{\nu} = 0. \quad (A.1)$$

Also, from the definition of $\lambda$,

$$\tilde{\nabla}_{\mu} \lambda = \Pi^\mu_{\mu'} \nabla_{\mu'} \lambda \equiv \nabla_{\mu} \lambda = -\lambda^{-1} \xi_\alpha \nabla_{\mu} \xi^\alpha. \quad (A.2)$$

Now we are ready to look at the left hand side of (2.1),

$$\tilde{\nabla}_{\alpha}(\lambda \omega^{\alpha\mu_1\cdots\nu}) = \lambda \tilde{\nabla}_{\alpha} \omega^{\alpha\mu_1\cdots\nu} + \omega^{\alpha\mu_1\cdots\nu} \tilde{\nabla}_{\alpha} \lambda$$

$$= \lambda \Pi^\alpha_{\mu_1} \cdots \Pi^\nu_{\nu'} \nabla_{\alpha} \Omega^{\alpha_2 \mu_2 \cdots \nu'} + \omega^{\alpha\mu_1\cdots\nu} \tilde{\nabla}_{\alpha} \lambda$$

$$= \lambda \nabla_{\alpha} \Omega^{\alpha_2 \mu_2 \cdots \nu'} \Pi^\mu_{\mu'} \cdots \Pi^\nu_{\nu'} + \lambda^{-1} \xi_\alpha \xi_\alpha' \nabla_{\alpha} \Omega^{\alpha_2 \mu_2 \cdots \nu'} \Pi^\mu_{\mu'} \cdots \Pi^\nu_{\nu'}$$

$$+ \Pi^\alpha_{\alpha'} \Pi^\mu_{\mu'} \cdots \Pi^\nu_{\nu'} (- \lambda^{-1} \xi_\rho \nabla_{\alpha} \xi_\rho) \Omega^{\alpha_2 \mu_2 \cdots \nu'}.$$

(A.3)

By our assumptions, $\mathcal{L}_\xi \Omega = 0$ (this may be thought of as the statement that $\Omega$ is time-independent if one sets up a system of coordinates where time is parametrized along $\xi$). Thus we can make a substitution for the second term of the last line of the above equation,

$$\tilde{\nabla}_{\alpha}(\lambda \omega^{\alpha\mu_1\cdots\nu}) = \lambda \nabla_{\alpha} \Omega^{\alpha_2 \mu_2 \cdots \nu'} \Pi^\mu_{\mu'} \cdots \Pi^\nu_{\nu'} + \lambda^{-1} \xi_\alpha \Omega^{\alpha_2 \mu_2 \cdots \nu'} \Pi^\mu_{\mu'} \cdots \Pi^\nu_{\nu'} \nabla_{\mu'} \xi_{\mu'}$$

$$+ \cdots + \lambda^{-1} \xi_\alpha' \Omega^{\alpha_2 \mu_2 \cdots \nu'} \Pi^\mu_{\mu'} \cdots \Pi^\nu_{\nu'} \nabla_{\nu'} \xi_{\nu'}.$$

(A.4)
Since $\Omega$ is a form, one can insert the projection operators $\Pi$ to replace the $\mu' \cdots \nu'$ contractions in the last terms (since there is already one $\xi$ contracted with $\Omega$), and we get

$$
\tilde{\nabla}_\alpha (\lambda \omega^{\alpha \mu' \cdots \nu'}) = \lambda \nabla_\alpha \Omega^{\alpha \mu' \cdots \nu'} \Pi_{\mu'} \cdots \Pi_{\nu'} + \lambda^{-1} \xi \alpha' \Omega^{\alpha' \mu''' \cdots \nu'} \Pi_{\mu} \cdots \Pi_{\nu'} \Pi_{\mu''} \cdots \Pi_{\nu'''} \nabla_{\mu'''} \xi_{\mu'} + \cdots \tag{A.5}
$$

The second term and similar terms (represented by the dots) vanish by (A.1) and we are left with

$$
\tilde{\nabla}_\alpha (\lambda \omega^{\alpha \mu' \cdots \nu'}) = \lambda \Pi_{\mu'} \cdots \Pi_{\nu'} \nabla_\alpha \Omega^{\alpha' \mu' \cdots \nu'}. \tag{2.1}
$$
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