Shannon Entropy and Fisher Information of the one-dimensional Klein-Gordon oscillator with energy-dependent potential

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In this paper, we studied, at first, the influence of the energy-dependent potentials on the one-dimensional Klein-Gordon oscillator. Then, the Shannon entropy and Fisher information of this system are investigated. The position and momentum information entropies for the low-lying states \( n = 0, 1, 2 \) are calculated. Some interesting features of both Fisher and Shannon densities as well as the probability densities are demonstrated. Finally, the Stam, Cramer–Rao and Bialynicki-Birula–Mycielski (BBM) have been checked, and their comparison with the regarding results have been reported. We showed that the BBM inequality is still valid in the form \( S_x + S_p \geq 1 + \ln \pi \) as well as in ordinary quantum mechanics.

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I. INTRODUCTION

Wave equations with energy dependent potentials have been come to view for long time. They can be seen in Klein-Gordon equation considering particle in an external electromagnetic field[1]. Arising from momentum dependent interactions, they also can be appeared in non-relativistic quantum mechanics, as shown by Green[2] for instance Pauli-Schrödinger equation possess another example [3, 4]. Sazdjian [5] and Formanek et al. [6] have noted that the density probability, or the scalar product, has to be modified with respect to the usual definition, in order to have a conserved norm. Garcia-Martinez et al [7] and Lombard [8] made an investigation on Schrödinger equation with energy-dependent potentials by solving them exactly in one and three dimensions. Hassanabadi et al. [9] studied the D-dimensional Schrödinger equation for an energy-dependent Hamiltonian that linearly depends on energy and quadratic on the relative distance. They also studied the Dirac equation for an energy-dependent potential in the presence of spin and pseudospin symmetries with arbitrary spin-orbit quantum number. They calculate the corresponding eigenfunctions and eigenvalues of a nonrelativistic energy-dependent system was done in [10]. A many-body energy-dependent system was studied by Lombard and Mareš [11]. They considered systems of N bosons bounded by two-body harmonic interactions, whose frequency depends on the total energy of the system. Other interesting related works can be found in [12–15] and references therein. So, the Presence of the energy dependent potential in a wave equation has several non-trivial implications. The most obvious one is the modification of the scalar product, necessary to ensure the conservation of the norm. This modification can modified some behavior or physical properties of a physical system: this question, in best of our knowledge, has not been considered in the literature.

The relativistic harmonic oscillator is one of the most important quantum system, as it is one of the very few that can be solved exactly. The Dirac relativistic oscillator (DO) interaction is an important potential both for theory and application. It was for the first time studied by Ito et al [16]. They considered a Dirac equation in which the momentum $\vec{p}$ is replaced by $\vec{p} - \text{im} \beta \omega \vec{r}$, with $\vec{r}$ being the position vector, $m$ the mass of particle, and $\omega$ the frequency of the oscillator. The interest in the problem was revived by Moshinsky and Szczepaniak [17], who gave it the name of DO because, in the non-relativistic limit, it becomes a harmonic oscillator with a very strong spin-orbit coupling term. Physically, it can be shown that the DO interaction is a physical system, which can be interpreted as the interaction of the anomalous magnetic moment with a linear electric field [18, 19]. The electromagnetic potential associated with the DO has been found by Benitez et al [20]. The DO has attracted a lot of interests both because it provides one of the examples of the Dirac’s equation exact solvability and because of its numerous physical applications [21–26]. Finally, Franco-Villafane et al [27] have exposed the proposal of the first experimental microwave realization of the one-dimensional DO.

The main goal of this paper is studying the effects of the modified scalar product arising in the energy-dependent Klein-Gordon oscillator problem. For this, we are focused on the study of: (i) the form of the spectrum of energy of the one-dimensional Klein-Gordon oscillator and, (ii) the Fisher and Shannon parameters of quantum information and the corresponding solutions, and (iii) the validity of Stam [28], Cramer–Rao [29, 30] and BBM [31] uncertainly relations for this type of potential.

II. THE ONE-DIMENSIONAL KLEIN-GORDON OSCILLATOR WITH AN ENERGY-DEPENDENT POTENTIAL

A. The solutions

In coordinate space, the free Klein-Gordon equation is ($\hbar = m = \omega = c = 1$):

$$\left\{ p^2_x - (E^2 - 1) \right\} \psi = 0 \tag{1}$$

In the presence of the interaction of the type of Dirac oscillator $p_x \to p_x + ix$, it becomes [32]:

$$\left\{ (p_x + ix)(p_x - ix) - (E^2 - 1) \right\} \psi(x) = 0,$n

The presence of a potential with energy-dependent potential is shown by the substituting $p_x$ with $p_x \to p_x + i(1 + \gamma E)x$ with $\gamma$ is a parameter is not that small and not that big. In this case, we have

$$\left( \frac{p^2_x}{2} + \frac{x^2}{2} \right) \psi(x, E) = \left( \frac{E^2 - 1}{2} + \frac{1}{2} (1 + \gamma E) \right) \psi(x, E) \tag{2}$$

With the following substitutions

$$\lambda = \sqrt{1 + \gamma E}. \tag{3}$$
The equation (2) represent a equation of a harmonic oscillator in one-dimensional, and the corresponding eigensolutions are

$$\psi (x, E) = C_n H_n \left( \sqrt{\lambda} x \right) \exp \left( -\frac{\lambda}{2} x^2 \right), \quad (4)$$

$$E^4 - 2E^2 - 4n^2 - 4n^2\gamma E + 1 = 0. \quad (5)$$

where $C_n$

$$(C_n^2) = \frac{1}{2^n n!} \left( 1 + \gamma E \right)^n \frac{\lambda}{\sqrt{\pi}} \left( E - \frac{\gamma}{2\sqrt{1 + \gamma E}} \left(n + \frac{1}{2}\right) \right)^{-1}.$$  

is the normalization constant, and $H_n$ it is the Hermite polynomials. Now, in momentum space, where we have $p_x \rightarrow p_x$, and $x \rightarrow i\frac{\partial}{\partial p_x}$, the equation of Klein-Gordon oscillator has the same form as

$$\left( -\frac{1}{2} \frac{\partial^2}{\partial p_x^2} + \frac{p_x^2}{2\lambda^2} \right) \psi (p_x, E) = \frac{E^2 + \lambda}{2\lambda} - 1 \psi (p_x, E) \quad (6)$$

with the corresponding eigensolutions are given by

$$\psi (p_x, E) = C'_n H_n \left( \frac{p_x}{\sqrt{\lambda}} \right) \exp \left( \frac{p_x^2}{2\lambda} \right), \quad (7)$$

$$E^4 - 2E^2 - 4n^2 - 4n^2\gamma E + 1 = 0. \quad (8)$$

with

$$(C'_n)^2 = \frac{1}{2^n n! \lambda \sqrt{\pi}} \left\{ E + \frac{\gamma}{2\lambda^3} \left(n + \frac{1}{2}\right) \right\}^{-1}.$$  

The density $\rho_{KG}$ can be expressed by [5]

$$\rho_{KG}(x, E) = \psi^* (x, E) \left( E - \frac{\partial V (x, E)}{\partial E} \right) \psi (x, E),$$

$$= |\psi (x, E)|^2 \left( E - \frac{1}{2} \gamma x^2 \right), \quad (9)$$

in the coordinate space. In the momentum space, this form is transformed into following equation

$$\rho_{KG}(p_x, E) = \psi^* (p_x, E) \left( E - \frac{\partial V (p_x, E)}{\partial E} \right) \psi (p_x, E)$$

$$= |\psi (p_x, E)|^2 \left( E + \frac{\gamma}{2(1 + \gamma E)^2 p_x^2} \right). \quad (10)$$

The equations [5] and [8] are an algebraic equation of the degree 4 having of the real and complex solutions. The complex solutions which are not physical, and by the two other real solution we have plotted Figure. [1]

In order to represent a physical system, two possibilities, for both coordinate and momentum spaces, can be made following the sign of $\rho_{KG}$:

- if $\rho_{KG} < 0$, then we have $\gamma > 0$ for the particles ($E > 0$)
- now, in the other case where $\rho_{KG} > 0$, we obtain that have $\gamma > 0$ for the anti-particles ($E < 0$).

This imposes constraints on the energy dependence for the theory to be coherent: by this, we mean a theory that have the following properties: (i) the necessary modification of the definition of probability density, (ii) The vectors corresponding to stationary states with different energies must be orthogonal, (ii) The formulation of the closure rule in terms of wave functions of stationary states justifies their standardization, (iv) finally, the operators of observable are all self-adjoint (Hermitian). In Figure. [1] we have plotted the energy $E$ versus quantum number $n$ for some different $\gamma$ values in both \{x\} and \{p\} configuration

Now, we are ready to discuss some interesting results that are not well comments in the literature. From this Figure, the asymptotic limits for both form of energies are $\frac{1}{n!}$ as in the non-relativistic case. These limits have been reproduced for both cases in Figure. [1]. Following this figure, some remarks can be made:
In what follow, we (i) studied the influence of the dependence of the potential with energies on the Fisher and Shannon parameters $F$, and (ii) checked the validity of Stam, Cramer–Rao and BBM uncertainty relations for some values of $\gamma$.

III. THE INFLUENCE OF THE $\gamma$ PARAMETER ON THE FISHER AND SHANNON INFORMATION MEASURES

A. Fisher information

The Fisher information is a quality of an efficient measurement procedure used for estimating ultimate quantum limits. It was introduced by Fisher as a measure of intrinsic accuracy in statistical estimation theory but its basic properties are not completely well known yet, despite its early origin in 1925. Also, it is the main theoretic tool of the extreme physical information principle, a general variational principle which allows one to derive numerous fundamental equations of physics: Maxwell equations, the Einstein field equations, the Dirac and Klein-Gordon equations, various laws of statistical physics and some laws governing nearly incompressible turbulent fluid flows [34–38]. Fisher information has been very useful and has been applied in different areas in quantum physics [39–44, 46–52].

In our case: the Fisher information of one-dimensional Klein-Gordon oscillator with energy-dependent potential is

$$F_x = \int \rho_n(x, E) \left[ \frac{d\ln(\rho_n(x, E))}{dx} \right]^2 dx.$$ (11)
By using the properties of the Hermite functions, we found that

\[
F_x = C_n^2 \int_{-\infty}^{+\infty} (16n^2E) H_n^2(\sqrt{\lambda}x) \exp(-\lambda x^2) dx + C_n^2 \int_{-\infty}^{+\infty} (16n\lambda E + 8n\gamma x) H_n^2(\sqrt{\lambda}x) \exp(-\lambda x^2) dx + \]

\[
C_n^2 \int_{-\infty}^{+\infty} (4\lambda^2 - 8n^2\gamma + 4\gamma \lambda) x^2 (H_n^2(\sqrt{\lambda}x) \exp(-\lambda x^2) dx + C_n^2 \int_{-\infty}^{+\infty} 8n\lambda x^3 (H_n^2(\sqrt{\lambda}x) \exp(-\lambda x^2) dx + \]

\[
C_n^2 \int_{-\infty}^{+\infty} -2\omega_2^2 \gamma x^4 (H_n^2(\sqrt{\lambda}x) \exp(-\lambda x^2) dx + C_n^2 \int_{-\infty}^{+\infty} \left(\frac{\gamma^2 x^2}{(E - \frac{1}{2} \gamma x^2)}\right) (H_n^2(\sqrt{\lambda}x) \exp(-\lambda x^2) dx. \]

The evaluation of different terms give

\[
I \to C_n^2 16n^2E 2^n n! \frac{\sqrt{n}}{\sqrt{\lambda}},
\]

\[
II \to 0,
\]

\[
III \to C_n^2 (4\lambda^2 E - 8n^2\gamma + 4\gamma \lambda) \frac{1}{\lambda} 2^n n! \frac{\sqrt{n}}{\sqrt{\lambda}} \left( n + \frac{1}{2} \right),
\]

\[
IV \to 0,
\]

\[
V \to -C_n^2 2^{-n} n! \frac{\sqrt{n}}{\sqrt{\lambda}} \left( \frac{(2n + 1)^2 + 2}{4} \right).
\]

The last term

\[
VI \to C_n^2 \int_{-\infty}^{+\infty} \left( \frac{\gamma^2 x^2}{(E - \frac{1}{2} \gamma x^2)} \right) (H_n^2(\sqrt{\lambda}x) \exp(-\lambda x^2) dx,
\]

is calculated numerically.

Hence, the final form of the Fisher parameter is written by:

\[
F_x = \left( E - \frac{\gamma}{2\sqrt{1 + \gamma}} \left( n + \frac{1}{2} \right) \right)^{-1} \left\{ 16n^2E + (4(1 + \gamma E) E - 8n^2\gamma + 4\gamma \sqrt{1 + \gamma} E) \frac{1}{\sqrt{1 + \gamma}} \left( n + \frac{1}{2} \right) - 2\gamma \left[ \frac{(2n + 1)^2 + 2}{4} \right] \right. \]

\[
+ \left. \frac{1}{\sqrt{2\lambda}} \left( \frac{1 + \gamma \lambda}{\sqrt{1 + \gamma \lambda}} \right) \right\} \left( E - \frac{\gamma}{2\sqrt{1 + \gamma}} \left( n + \frac{1}{2} \right) \right)^{-1} \int_{-\infty}^{+\infty} \left( \frac{\gamma^2 x^2}{(E - \frac{1}{2} \gamma x^2)} \right) (H_n^2(\sqrt{\lambda}x) \exp(-\lambda x^2) dx.
\]

Let’s now go to the momentum space: in this case we have

\[
F_p = \int \rho_n(\rho_n(p_x, E)) \left[ \frac{\ln(\rho_n(p_x, E))}{dp_x} \right]^2 dp_x.
\]

After some calculations, we obtain

\[
F_p = C_n^2 \int_{-\infty}^{+\infty} (16n^2E) H_n^2\left( \frac{p_x}{\sqrt{\lambda}} \right) \exp\left( \frac{p_x^2}{\lambda} \right) dp_x + C_n^2 \int_{-\infty}^{+\infty} \left( \frac{2\gamma}{\lambda^4} - \frac{4nE}{\lambda^4} \right) p_x H_n^2\left( \frac{p_x}{\sqrt{\lambda}} \right) \exp\left( \frac{-p_x^2}{\lambda} \right) dp_x + \]

\[
C_n^2 \int_{-\infty}^{+\infty} \left( \frac{4E}{\lambda^2} + \frac{8n^2\gamma}{\lambda^2} \right) p_x^2 H_n^2\left( \frac{p_x}{\sqrt{\lambda}} \right) \exp\left( \frac{-p_x^2}{\lambda} \right) dp_x - C_n^2 \int_{-\infty}^{+\infty} \frac{4n\gamma}{2\lambda^2} p_x^3 H_n^2\left( \frac{p_x}{\sqrt{\lambda}} \right) \exp\left( \frac{-p_x^2}{\lambda} \right) dp_x + \]

\[
C_n^2 \int_{-\infty}^{+\infty} \frac{4\gamma}{2\lambda^2} p_x^2 H_n^2\left( \frac{p_x}{\sqrt{\lambda}} \right) \exp\left( \frac{-p_x^2}{\lambda} \right) dp_x + C_n^2 \int_{-\infty}^{+\infty} \left( \frac{2\gamma^2 p_x^2}{\lambda^4 (2E\lambda^4 + \gamma p_x^2)} \right) H_n^2\left( \frac{p_x}{\sqrt{\lambda}} \right) \exp\left( \frac{-p_x^2}{\lambda} \right) dp_x.
\]
When we evaluate the terms appear in Eq. (15),

\[ I \rightarrow C_n^2 16n^2 E^2 n! \sqrt{\pi} \sqrt{\lambda}, \]

\[ II \rightarrow 0, \]

\[ III \rightarrow C_n^2 \left( \frac{4E}{\lambda} + \frac{8n^2 \gamma}{\lambda^3} \right) \frac{1}{\lambda} 2^n n! \sqrt{\pi} \sqrt{\lambda} \left( n + \frac{1}{2} \right), \]

\[ IV \rightarrow 0, \]

\[ V \rightarrow C_n^2 2n^2 \gamma \lambda \sqrt{\pi} \sqrt{\lambda} \left( \frac{(2n+1)^2 + 2}{4} \right), \]

we arrive at the final form of Fisher parameter where

\[ F_p = \left( E + \frac{\gamma}{\lambda^2} (n + \frac{1}{2}) \right)^{-1} \left\{ 16n^2 E + \left( \frac{4E}{\lambda} + \frac{8n^2 \gamma}{\lambda^3} \right) \left( n + \frac{1}{2} \right) + \frac{2n^2 \lambda^2}{\lambda^4} \left( \frac{(2n+1)^2 + 2}{4} \right) \right\} \]

\[ + \frac{1}{2^n n! \lambda \sqrt{\pi}} \left( E + \frac{\gamma}{\lambda^2} (n + \frac{1}{2}) \right)^{-1} \int_{-\infty}^{+\infty} \left( \frac{2n^2 \rho_x^2}{\sigma_x^2} \right) H_n^2 \left( \frac{\rho_x}{\sigma_x} \right) \exp \left( -\frac{\rho_x^2}{2\sigma_x^2} \right) d\rho_x. \]

(16)

The last term in equation is calculate numerically.

### B. Shannon entropy

Entropic measures provide analytic tools to help us to understand correlations in quantum systems. Shannon has introduced entropy to measure the uncertainty. Now, it has become a universal concept in statistical physics. The Shannon entropy has applications in several branches of physics because of its possible applications in a wide range of area (see Ref. 53 and references therein).

The position space information entropies for the one-dimensional can be calculated by using

\[ S_x = - \int |\psi(x)|^2 \ln |\psi(x)|^2 \ dx, \]

(17)

In our case, the above equation becomes

\[ S_x = - \int \rho_n(x, \gamma) \ln \rho_n(x, \gamma) \ dx, \]

(18)

with \( \rho_n(x, \gamma) \) is defined by the equations (7) and (8). In general, explicit derivations of the information entropy are quite difficult. In particular, the derivation of analytical expression for the \( S_x \) is almost impossible. The overcome this difficulties, we (i) use a numerical calculation of this integral, and (ii) represent the Shannon and Fisher information entropy densities, respectively.

The form of this parameter, in the momentum space, is written by

\[ S_p = - \int \rho_n(p_x, E) \ln \rho_n(p_x, E) \ dp_x. \]

(19)

### C. Results and discussions

In Figure 2, we show the Fisher parameter versus a \( \gamma \) for both coordinate and momentum spaces: the case of coordinate space, \( F_x \) decreases contrarily, in the momentum space where it increases. Moreover, this situation is inverted in the Figure 3: the Shannon parameter increases in the \( \{x\} \) configuration, whereas it decreases in the \( \{p\} \) configuration. We note here, that these behavior is the same for the particles and anti-particles.

The information entropy and fisher densities are defined as,

\[ \langle (\rho_F)_{\{a\}} \rangle = \rho_n(a, E) \left[ \frac{d\ln (\rho_n(a, E))}{da} \right]^2, \]

(20)

for Fisher information, and

\[ \langle (\rho_S)_{\{a\}} \rangle = \rho_n(a, E) \ln \rho_n(a, E), \]

(21)
Figure 2: Fisher information of one-dimensional Klein-Gordon oscillator versus $\gamma$ for both coordinate and momentum spaces.

Figure 3: Shannon entropy of one-dimensional Klein-Gordon oscillator versus $\gamma$ for both coordinate and momentum spaces.

for the Shannon entropies: $\{a\}$ denotes the appropriate configuration. The behavior of $(\rho_F)_{\{a\}}$ and $(\rho_S)_{\{a\}}$ is illustrated in Figures 4 and 5 for $n = 0, 1, 3$ and several values of the parameter $\gamma$.

Now, we are ready to discuss the Heisenberg uncertainty relation (HUR) and its analogue in the framework of quantum information: the HUR in quantum mechanics is an inequality between position and momentum. In recent years, a new uncertainty principles are introduced, and they are originate from the information theory: let us mention that this information-theoretic quantity and its quantum extension, not yet sufficiently well known for physicists, has been used to set up a number of relevant inequalities such as Stam and Cramer–Rao and uncertainty relations. The Cramer-Rao inequality belongs to a natural family of information-theoretic inequalities which play a relevant role.
in a great variety of scientific and technological fields ranging from probability theory, communication theory, signal processing and approximation theory to quantum physics of D-dimensional systems with a finite number of particles [56–61].

Also, the Fisher information of single-particle systems has been only recently determined in closed form in terms of the quantum numbers characterizing the involved physical state for both position and momentum spaces. These relevant inequalities which involve the Fisher information in a given space (Cramer–Rao) or the conjugate (Stam) space. They are the Stam uncertainty relations [53]

\[ F_x \leq 4 \langle p^2 \rangle, \quad F_p \leq 4 \langle x^2 \rangle, \quad (22) \]

and the Cramer–Rao inequalities

\[ F_x \geq \frac{1}{\langle x^2 \rangle}, \quad F_p \geq \frac{1}{\langle p^2 \rangle}. \quad (23) \]

In addition, for a general monodimensional systems we have that

\[ F_x F_p \geq 4. \quad (24) \]

Table. II shows a numerical results for the uncertainty relation and Fisher information measure of 1D Klein-Gordon oscillator for three levels \((n = 0, 1, 2)\) for some choice of parameter \(\gamma\). Following this Table, we observe that

- the Stam inequalities, and Cramer–Rao ones are fulfilled,
- the following relation

\[ F_x F_p \geq 4, \quad (25) \]

with D is the space dimension is well-established.
- and, finally, as the results indicate that the sum of the entropies is in consistency with BBM inequality, possesses the stipulated that

\[ S_x + S_p \geq D (1 + \ln \pi). \quad (26) \]

In our case, we have \(S_x + S_p \geq 2.14422\).
### Table I: Numerical results for the uncertainty relation and Fisher information and Shannon entropy of 1D Klein-Gordon oscillator with energy-dependent potential.

| $n$ | $\gamma$ | $\langle x^2 \rangle$ | $\Delta x$ | $\langle p_x^2 \rangle$ | $\Delta p_x$ | $\Delta x \Delta p_x$ | $F_{x,S}$ | $F_{p,S}$ | $S_x$ | $S_p$ | $S_x + S_p$ |
|-----|---------|----------------------|----------|------------------------|----------|---------------------|---------|---------|-------|-------|-------------|
| 0   | 0       | 0.5000               | 0.7071   | 0.5000                 | 0.7071   | 0.5000              | 2.0000  | 2.0000  | 4.0000 | 1.0724| 1.0724      | 2.1448    |
|     | -0.16   | 0.5918               | 0.7693   | 0.4080                 | 0.638    | 0.4908              | 1.6916  | 2.4516  | 4.1472 | 1.1561| 0.9707      | 2.1268    |
|     | -0.32   | 0.7136               | 0.8447   | 0.2737                 | 0.5232   | 0.4419              | 1.4026  | 3.7700  | 5.2878 | 1.2501| 0.8054      | 2.0555    |
|     | -0.48   | 0.8912               | 0.9440   | 0.0188                 | 0.1371   | 0.1295              | 1.1276  | 6.9355  | 8.0505 | 1.3608| 0.7162      | 2.0770    |
|     | -0.64   | 1.1842               | 1.0882   | 0.4410                 | 0.6641   | 0.7227              | 0.8607  | 8.2659  | 9.1264 | 1.5014| 0.9599      | 2.4613    |
|     | -0.80   | 1.8090               | 1.3450   | 0.6658                 | 0.8160   | 1.0975              | 0.5927  | 12.2000 | 13.7925| 2.1803| 1.0148      | 2.7266    |
| 1   | 0       | 1.5000               | 1.2247   | 0.15                   | 1.2247   | 1.5000              | 6.0000  | 6.0000  | 12.0000| 3.1427| 3.1427      | 6.2854    |
|     | -0.16   | 1.8392               | 1.3562   | 1.1756                 | 1.0842   | 1.4704              | 4.8935  | 7.6593  | 12.5528| 4.4334| 2.1167      | 6.5651    |
|     | -0.32   | 2.3685               | 1.5390   | 0.46099                | 0.6933   | 1.0670              | 3.8012  | 13.4774 | 16.2786| 5.1230| 1.5644      | 6.7892    |
|     | -0.64   | 5.1536               | 2.2702   | 1.0225                 | 1.0112   | 2.2956              | 1.7716  | 13.5291 | 15.3007| 3.9824| 0.8693      | 2.7815    |
|     | -0.80   | 10.4112              | 3.2266   | 0.533192               | 0.7302   | 2.3561              | 0.9166  | 22.3000 | 24.2166| 6.4200| 0.5957      | 2.5805    |
| 2   | 0       | 2.5                  | 1.5811   | 2.5000                 | 1.5811   | 2.5000              | 10.0000 | 10.0000 | 20.0000| 4.9864| 4.9864      | 9.9728    |
|     | -0.16   | 3.23687              | 1.7991   | 1.3414                 | 2.4133   | 7.7103              | 13.8006 | 106.4068| 110.2174| 2.9631| 1.3503      | 3.3134    |
|     | -0.32   | 4.51644              | 2.1252   | 0.9625                 | 0.9811   | 2.0580              | 5.4827  | 17.7237 | 23.2064| 5.7137| 1.3125      | 3.0358    |
|     | -0.48   | 7.00704              | 2.6471   | 1.7410                 | 1.3195   | 3.4928              | 3.4664  | 9.8800  | 13.3464| 3.8915| 1.0208      | 2.9403    |
|     | -0.64   | 12.5885              | 3.5480   | 1.0710                 | 1.0349   | 3.6718              | 1.8689  | 19.9000 | 21.7689| 5.0111| 0.6893      | 2.8194    |
|     | -0.80   | 29.3421              | 5.4168   | 0.4937                 | 0.7026   | 3.8085              | 0.7776  | 48.9000 | 59.6776| 12.6883| 0.3027      | 2.5479    |
