Some classes of renormalizable tensor models

Joseph Ben Geloun\textsuperscript{1,2} and Etera R. Livine\textsuperscript{3,1,†}

\textsuperscript{1}Perimeter Institute for Theoretical Physics, 31 Caroline St, Waterloo, ON, Canada
\textsuperscript{2}International Chair in Mathematical Physics and Applications, ICMPA-UNESCO Chair, 072BP50, Cotonou, Rep. of Benin
\textsuperscript{3}Laboratoire de Physique, ENS Lyon, CNRS-UMR 5672, 46 Allée d’Italie, Lyon 69007, France

\textit{(Dated: May 2, 2014)}

We identify new families of renormalizable of tensor models from anterior renormalizable tensor models via a mapping capable of reducing or increasing the rank of the theory without having an effect on the renormalizability property. Mainly, a version of the rank 3 tensor model as defined in [arXiv:1201.0176 [hep-th]], the Grosse-Wulkenhaar model in 4D and 2D generate three different classes of renormalizable models. The proof of the renormalizability is fully performed for the first reduced model. The same procedure can be applied for the remaining cases. Interestingly, we find that, due to the peculiar behavior of anisotropic wave function renormalizations, the rank 3 tensor model reduced to a matrix model generates a simple super-renormalizable vector model.

Pacs numbers: 11.10.Gh, 04.60.-m, 02.10.Ox
Key words: Renormalization, tensor models, matrix models.
pi-qg-285 and ICMPA/MPA/2012/12
May 2, 2014

CONTENTS

I. Introduction 1

II. A just normalizable matrix analogue of a rank three tensor model 3
   A. Rank 3 tensor model and its matrix reduction 3
   B. Multiscale analysis 6
   C. Divergence degree analysis 8
   D. Renormalization 10
   E. RG flow of wave function renormalizations and a super-renormalizable vector model 13

III. New classes of renormalizable models from previous models 14
   A. Classes of GW models 14
   B. Families of renormalizable tensor models 17

IV. Outlook 19

Acknowledgements 20

Appendix: One-loop $\beta$-function of the $\left(ap^{a+\epsilon} + bq^{a}\right)$–model 20

References 21

I. INTRODUCTION

Tensorial Group Field Theory (TGFT) \cite{1,2} is a recently built quantum field theoretical framework pertaining to the "discrete to continuum" scenario for quantum gravity. In such an instance, fields are tensors of rank $D$ labeled by abstract group representations which are viewed as $D$ simplexes and the interactions are of the form of $D + 1$ simplexes. According to quantum field theory rules, tensor fields of rank $D$ or $D$ simplexes are glued along their $D - 1$ simplexes and interact in $D + 1$ simplexes. The simplicial approach for discussing quantum gravity is in fact
well known and has led to the famous study of matrix models in lower dimensional statistical mechanics \cite{5, 6}. TGFT should be regarded as the higher rank extension of random matrix models with a bonus: the genuine feature to be renormalizable.

Within the above TGFT framework, new classes of renormalizable models involving tensor fields have been highlighted \cite{7, 12}. The renormalization procedure for these models involves an extended notion of multiscale analysis \cite{13} which steers a new kind power counting theorem and locality principle. It is noteworthy that all interactions are nonlocal (precisely, they happen in a region of the abstract group manifold or position space) and build around Guram’s 1/N expansion \cite{14, 21} for higher rank colored theories \cite{22, 23}. Remarkably, renormalizable TGFTs shed as well more light on anterior results in renormalization of quantum matrix models on noncommutative spaces \cite{24, 26}. Indeed, at the perturbative level, Feynman graphs of TGFT models are generated by vertices and propagators spanned by stranded graphs representing higher rank extension of ribbon vertices and propagators as used in the matrix formulation of the Grosse-Wulkenhaar (GW) model \cite{25, 26}. Power counting theorems and locality principle in TGFT extend several similar notions found in the matrix case.

At quantum gravity energy scale, some axioms or principles of ordinary quantum theory should be, if not drastically revised, at least profoundly rethought. Several theoretical frameworks address the fact that locality should no longer hold in that regime (string theory, noncommutative geometry, etc...). Let us focus on the particular forms of nonlocality as appear in TGFTs. In a broader sense, the models cited above belong to the class of models endowed with nonlocal interactions. Field arguments in the interaction term might be paired in many possible ways. Specific forms have to be physically motivated and tractable. For instance, in noncommutative field theory induced by noncommutative Moyal field algebra, four fields interact in a region (a parallelogram) the area of which is the Planck length square \cite{24}. The recent tensor models in \cite{7, 8} possess interactions of the form of 3 and 4-simplex which generate, through a path integral formalism, simplicial pseudo-manifolds in 3D and 4D.

In a nonlocal field theory, it could happen that the interaction is of a definite form which can be called “partially cyclic”, namely, in the interaction pairing, a tensor field only share (at least two) arguments with at most two other fields (a “totally cyclic” or simply “cyclic” interaction would be an interaction having this property valid for all fields which define it). Consider some tensors $T^I$ and an interaction defined by contractions of indices of the $T^I$’s such that

$$I = \sum_{[J],[K],...} T^1_{[J],...} T^2_{[J],[K],...} T^3_{[K],...} \cdots,$$

where $[J]$ and $[K]$ are block indices or arguments. Consider now a similar but simpler interaction where indices $i$ and $k$ replace block indices $[J]$ and $[K]$ such that

$$\tilde{I} = \sum_{j,k,...} T^1_{...,j,...} T^2_{...,j,k,...} T^3_{...,k,...} \cdots$$

It is natural to ask: how starting from $I$, one generates $\tilde{I}$ and what is the main feature of the reduced model described by $\tilde{I}$? A quantum field theory being not only defined by interactions what implications has such a reduction on the dynamics? These questions might be very intricate and, more to the point, even more complex if one would like to preserve some nice properties such as renormalizability or symmetry aspects of the initial theory. If all features of the model described by $I$ are fully represented in $\tilde{I}$, then $\tilde{I}$ could provide a much simpler model than $I$.

These questions could find also an importance in the double scaling limit analysis of tensor models \cite{27}. In this latter work, the author projects the rank $D$ tensor random variables $T_{a_1,a_2,...,a_D}$ onto a simplified rank 2 tensor $\tilde{T}_{a_1,a_2}$ so that particular cyclic tensor interactions can be merely seen as matrix trace invariants. This has led to the discovery of new multi-critical points and a novel double scaling limit for matrix models.

Several points about TGFT have yet to be addressed. TGFT might generate a wealth models with several possible interactions and kinetic terms. We definitely need a guidance towards true physical models which should incorporate the geometry seed of a theory of General Relativity. Including renormalizability in the game might be one selection criterion. Under the Renormalization Group (RG) flow, only microscopic physically robust models with long-lived logarithmic flow couplings would resist to the several layer scales and would finally give macroscopic observable effects. This is indeed what a quantum theory for gravity would demand and this is indeed why our first intention is to preserve the renormalizability feature of any TGFT models generated. In particular, if the reduction $I \rightarrow \tilde{I}$ respects that feature then it could provide an important simplification worth to be investigated.

In this paper, we show that, at least three particular classes of tensor models equipped with cyclic interactions can be projected back to reduced rank models and, reciprocally, any tensor model of this kind can be extended to a higher rank tensor model with a cyclic interaction in a sector. During the process, we are able to identify new classes of renormalizable models (perturbatively and at all orders). If the projection and extension of these models can be somehow understood either by dividing or by multiplying the number of indices, what we actually show is that
this mapping preserves the renormalizability of the initial model. The proof is fully established for a reduced rank 3 tensor model issued from [8]. The model considered is an independent not identically distributed random matrix model. In other words, it is anisotropic in the sense that its strands are not equally weighted from the point of view of the measure (the GW model with a magnetic field [28] is likewise but not totally similar). The renormalization procedure for tensor models is always a delicate issue. However, we show here that the full renormalization program (from the multi-scale analysis to the renormalization of divergences) applies for this peculiar model. The proof also applies to the GW model in 2D [26] and 4D [25], and, from these, the universal feature of our formalism can be easily inferred. Theorem 1 and Theorem 3 are our main results and they mainly allow us to identify three different families of renormalizable tensor models. Interestingly, we find that there exists a continuum of perturbatively renormalizable theories linking the three classes.

We would like to stress also the fact that the projection-extension mechanism can be used to reduce a piece of the rank 4 model of [7] but cannot reduce the entire interaction to a matrix interaction. Thus, there exist actually renormalizable tensor models which are not partially cyclic. Another point that has been also not entirely covered in this paper is whether or not the procedure generates models stable under the RG flow. In some situations, it may actually happen that the initial model flows towards a different model hence it is not stable. However, in all situations carried out below, the choice of coupling constants of the model is made in such a way that this odd feature is simply avoided.

The plan of this paper is the following: In the next section, as a complete test of the above ideas, we carry out the full renormalization program for a new matrix model built from a renormalizable rank 3 renormalizable model [9]. Section II is devoted to the identification of new families of renormalizable tensor models issued from GW models in 4D and 2D by applying the program of Section II to these models. We infer the existence of three different families of renormalizable models (rank 3 tensor, GW 4D and GW 2D models) having three different renormalizable matrix models as roots. An appendix discussing the one-loop $\beta$-function or UV behavior of some of the models treated closes the paper.

II. A JUST NORMALIZABLE MATRIX ANALOGUE OF A RANK THREE TENSOR MODEL

A. Rank 3 tensor model and its matrix reduction

Let us recall the main features of the the model defined in [8] henceforth called T3 model. The following results and transformations will find consistent analogues for other types of tensor models discussed in the remaining sections.

Consider complex fields $\varphi: U(1)^3 \to \mathbb{C}$ which can be equivalently described after Fourier mode decomposition as tensors

$$\varphi(g_1, g_2, g_3) = \sum_{p_i} \varphi_{[p]} e^{i p_1 \theta_1} e^{i p_2 \theta_2} e^{i p_3 \theta_3}, \quad g_k = e^{i p_k \theta_k}, \quad \theta_k \in [0, 2\pi),$$

where $[p] = (p_1, p_2, p_3), p_i \in \mathbb{Z}$. It is possible to restrict the discussion for positive mode fields, i.e. we assume that tensor fields satisfying the symmetry

$$\varphi_{p_1, p_2, p_3} = \varphi_{-p_1, -p_2, p_3} = \varphi_{p_1, -p_2, p_3} = \varphi_{p_1, p_2, -p_3}.$$  

Thus, we will consider only fields such that $p_i \in \mathbb{N}$, namely fields can be regarded as living in $(U(1)/\mathbb{Z}_2)^3$. Note such a restriction is made by sake of simplicity. Hence, such a prescription will have no consequence on the subsequent analysis (this point will be emphasized later on).

Using now these tensor components such that $\varphi_{p_1, p_2, p_3} \in \mathbb{C}, p_i \in \mathbb{N}$, the T3 model possesses a kinetic term given by

$$S_{\text{kin}} = \sum_{p_j} \bar{\varphi}_{p_1, p_2, p_3} \left( \sum_{s=1}^{3} a_s p_s + \mu \right) \varphi_{p_1, p_2, p_3},$$

with some mass $\mu, a_s$ are wave-function couplings associated with the theory propagator

$$\hat{C}([p_1]; [p_i]) = \prod_{i=1}^{3} \delta_{p_i, \tilde{p}_i} / \left( \sum_{s=1}^{3} a_s p_s + \mu \right).$$

Hence the Gaussian measure $d\mu_G[\varphi]$ of the model has a covariance $C = 1/(\sum_{s=1}^{3} a_s p_s + \mu)$. The interaction of the T3 model is of $\varphi^4$-type given by

$$S_{\text{int}} = \sum_{p_j} \varphi_{p_1, p_2, p_3} \bar{\varphi}_{p_1, p_2, p_3} \varphi_{p_1', p_2', p_3} \bar{\varphi}_{p_1', p_2', p_3} + \text{permutations},$$
where “permutations” refers to other terms induced by color symmetry on strand indices. Propagator and vertices of the model are pictured in Fig. 1.

![Diagram of propagator and vertices of the type $\varphi^4$ of the rank 3 tensor model.](image)

**FIG. 1.** Propagator and vertices of the type $\varphi^4$ of the rank 3 tensor model.

On of the main theorem proved in [8] consists in the following statement: Introducing a UV cut-off $\Lambda$ on the propagator $C \to C^\Lambda$, the multi-scale analysis of graph amplitudes [13] proves that the action

$$S^\Lambda = \lambda^3 S_{int}^{\Lambda} + CT^{\Lambda}_{2;1} S_{2;1} + \sum_{s=1,\ldots,3} CT^{\Lambda}_{s;2} S_{s;2},$$  

(8)

where $CT^{\Lambda}_{2;1} S_{2;1}$ is a mass counter-term and $CT^{\Lambda}_{s;2} S_{s;2}$, $s = 1, 2, 3$, are wave-function counter-terms, and the related a partition function

$$Z = \int d\mu S^\Lambda$$  

(9)

define a model which is renormalizable at all orders of perturbation theory.

We introduce now the “anisotropic” model defined by the interaction

$$S_{T3}^{int} = \sum_{p} \varphi_{p_1,p_2,p_3} \tilde{\varphi}_{p_1',p_2',p_3} \varphi_{p_1',p_2',p_3'} \tilde{\varphi}_{p_1,p_2,p_3},$$  

(10)

and keeping still the same kinetic term given by (5). Thus, we have explicitly broken the strand symmetry by choosing such an interaction. This breaking enforces a particular choice of wave function couplings (see discussion of Section 5.3 in [8]) affording a proper notion of wave function renormalization. The choice of $a_1 \neq a_2 = a_3$ is therefore of great significance because it is only under these conditions that the model defined by $S^{kin}$ and $\lambda S_{T3}^{int}$ turns out to be just renormalizable [8]. In other words, other choices might lead to instability under the RG flow.

Henceforth, we will restrict our analysis to the model with the unique interaction (10). This is nothing but the first vertex depicted in Fig. 1.

A striking feature of the interaction (10) is that it maps to a pure matrix interaction using any bijection $\tilde{\sigma} : \mathbb{N}^2 \to \mathbb{N}$. Indeed, consider the following field redefinition:

$$\varphi_{p_1,p_2,p_3} \mapsto \phi_{p,n}, \quad p_1 = p, \quad \tilde{\sigma}(p_2,p_3) = n$$  

(11)

to which, given $\tilde{\sigma}(p,q) = n$ and its inverse noted as $\tilde{\sigma}_1^{-1}(n) = p$ and $\tilde{\sigma}_2^{-1}(n) = q$, corresponds the following transformed actions (from now on, $a_1 = a$, $a_2 = a_3 = b$),

$$S_{T3}^{kin} = \sum_{p,n} \phi_{p,n} \left( a p + b (\tilde{\sigma}_1^{-1}(n) + \tilde{\sigma}_2^{-1}(n)) + \mu \right) \phi_{p,n},$$  

(12)

$$S_{T3}^{int} = \sum_{p_1,p_2,n_1,n_2} \phi_{p_1,n_1} \tilde{\phi}_{p_2,n_1} \phi_{p_2,n_2} \tilde{\phi}_{p_1,n_2},$$  

(13)

We reduce the $(p_2,p_3)$-momentum sector according to the fact that the interaction is cyclic with respect to this couple of indices.

Being unique and of the matrix kind, the interaction (13) cannot clearly generate under the RG flow any other coupling than itself. It can be represented in the ordinary form of Fig 2.

We now investigate the implications induced by the reduction procedure on the kinetic term. We introduce

$$N_n = \tilde{\sigma}_1^{-1}(n) + \tilde{\sigma}_2^{-1}(n).$$  

(14)
In particular, the standard choice $\sigma$ for $\tilde{\sigma}$ is defined by ordering the pairs of integers $(p,q)$ in $\mathbb{N}^2$ along the diagonal (at constant $p+q$) and numbering them from bottom to top, as illustrated in Fig. 3.

Explicitly, this map reads:

$$\sigma(p,q) = \frac{1}{2}(p+q)(p+q+1)+q, \quad \sigma(0,0) = 0 \quad (15)$$

and its inverse can be characterized as

$$\sigma^{-1}(n) = (p,q) \in \mathbb{N}^2, \quad N = p+q, \quad N(N+1) \leq 2n \leq N(N+3), \quad q = 2n - N(N+1), \quad p = N - q, \quad (16)$$

where the inequality uniquely determines $N = N_n \in \mathbb{N}$. We denote $\sigma_1^{-1}(n) = p$ and $\sigma_2^{-1}(n) = q$. It can be easily shown that the following (optimal) bounds holds:

$$\sqrt{\frac{9+8n-3}{2}} \leq N \leq \sqrt{\frac{1+8n-1}{2}},$$

$$\left(\frac{\sqrt{9+8n-3}}{8}\right)^2 \leq \frac{N^2}{2} \leq p^2+q^2 \leq \left[\left(p+q\right)^2 = N^2\right] \leq \frac{\left(\sqrt{1+8n-1}\right)^2}{4} \leq 2n. \quad (17)$$

Hence, for large $n$ (such a condition will find a motivation later as the UV limit where field modes proliferate),

$$\sqrt{2n} - \frac{3}{2} \leq \sqrt{2n} \left[\frac{9}{8n} + 1 - \frac{3}{\sqrt{8n}}\right] \leq N \leq \sqrt{\frac{1+8n-1}{2}} \leq \sqrt{2n}, \quad (18)$$

the approximation $N \sim \sqrt{2n}$ is correct. We could of course introduce another choice of bijection between $\mathbb{N}^2$ and $\mathbb{N}$, but the map $\sigma$ is, in some sense, the most compact choice for which we have a natural simple estimate of $N = p+q$ in terms of $n = \sigma(p,q)$. For a more general map, we would reshuffle the labeling of points $(p,q) \in \mathbb{N}^2$ by the integer $n$, which would lead to a more random behavior of $N$ that would wildly fluctuate away from $\sqrt{2n}$.

Let us point out that we can easily adapt our analysis to $\mathbb{Z}^2$ and introduce a generalized map $\sigma: \mathbb{Z}^2 \rightarrow \mathbb{N}$ as shown in Fig. 3, labeling points $(p,q) \in \mathbb{Z}^2$ along the “circles” of constant $N = |p| + |q|$. This would lead to a similar behavior of $N$ scaling proportionality to $\sqrt{n}$. This also sustains the fact that we could have let all field modes $p_i \in \mathbb{Z}$ (without assuming any symmetry [4]) and still the analysis will be valid.

What boils down in the above algebra is that we can introduce the following kinetic term

$$S_{\sigma T3}^{\text{kin}} = \sum_{p,n} \tilde{\phi}_{p,n} \left( ap + b\sqrt{n} + \mu \right) \phi_{p,n} \quad (19)$$

associated with a new propagator (which draws as in Figure 2)

$$\tilde{C}([p,n],[\tilde{p},\tilde{n}]) = \delta_{p,\tilde{p}} \delta_{n,\tilde{n}} / \left( ap + b\sqrt{n} + \mu \right). \quad (20)$$
Next we introduce a UV cut-off $\Lambda$ on the new propagator \[20\] and $CT_{2s,0}^\Lambda S_{2s,0}$ a mass counter-term and $CT_{2s,s}^\Lambda S_{2s,s}$, $s = 1, 2$, two wave function counter-terms defined as

$$S_{2s,0} = \sum \phi_{p,n} \phi_{p,n} \, , \quad S_{2s,2} = \sum \phi_{p,n} p \phi_{p,n} \, , \quad S_{2s,1} = \sum \phi_{p,n} \sqrt{n} \phi_{p,n} \, .$$ \((21)\)

Our remaining task is to prove that, after the tensor-to-matrix reduction procedure followed by the approximation of $N_n \sim \sqrt{2n}$, the following statement holds:

**Theorem 1** (Renormalizability of $\sigma T3$). The $\sigma T3$ model defined by

$$S^\Lambda = \lambda S_{T3}^{\text{int}} + \sum_{s=0,1,2} CT_{2s,s}^\Lambda S_{2s,s}$$ \(22\)

with Gaussian measure $d\mu_C^\Lambda(\phi)$ with covariance associated with the kinetic term $S_{\sigma T3}^{\text{kin}}$, is renormalizable at all orders of perturbation theory.

The proof of this statement will follow a multi-scale analysis leading to a power counting theorem and renormalization procedure for divergent terms in the way of \[13\].

**B. Multiscale analysis**

**Propagator bound.** Using Schwinger’s kernel, we rewrite the propagator \[20\] and its slice decomposition \[13\] as

$$C(p, n) = \frac{1}{ap + b\sqrt{n} + \mu} = \int_0^\infty d\alpha e^{-\alpha(ap+b\sqrt{n}+\mu)} \, ,$$

$$C_i(p, n) = \int_{M^{-i}}^{M^{-i+1}} d\alpha e^{-\alpha(ap+b\sqrt{n}+\mu)} \, , \quad C_0(p, n) = \int_1^\infty d\alpha e^{-\alpha(ap+b\sqrt{n}+\mu)} \, .$$ \((23)\)

for some large constant $M \in \mathbb{N}$. One has $C = \sum_{i=0}^\infty C_i$. We get the following bounds on the sliced propagators:

$$C_i(p, n) \leq KM^{-i}e^{-M^{-i}(ap+b\sqrt{n}+\mu)} \, , \quad C_0(p, n) \leq Ke^{-ap+b\sqrt{n}+\mu} \, .$$ \((24)\)

for $K \geq 0$. Hence, a slice $i \gg 1$ probe high momenta either of order $M^i$ or of order $M^{2i}$. A UV cut-off can be introduced such that the cut-offed propagator is $C^\Lambda = \sum_{i=0}^\Lambda C_i$ and the UV limit is obtained by taking $\Lambda \to \infty$. It is common to refer $C_0$ and $C_\Lambda$ to as the IR and UV propagator slice, respectively. The subscript $\Lambda$ will be dropped in the following.

**Optimal bound on an amplitude.** Given a connected graph $\mathcal{G}$ with set of vertices $\mathcal{V}$, $\mathcal{V} = |\mathcal{V}|$, set of lines $\mathcal{L}$, $\mathcal{L} = |\mathcal{L}|$, for evaluating the optimal bound on the bare amplitude associated with such a graph, we proceed in the usual way \[13\] but we must take into account the effect introduced by the new propagator. First write the bare amplitude of the graph as

$$A_\mathcal{G} = \sum_{\mu} f_\mu(\lambda, CT) A_{\mathcal{G}, \mu} \, , \quad A_{\mathcal{G}, \mu} = \sum_{p_{v,s}} \left[ \prod_{\ell \in \mathcal{L}} C_{\ell}(\mu) \{p_{v(\ell), s}\} \{p_{v'(\ell), s}\} \right] \prod_{v \in \mathcal{V}_s} \delta_{p_{v,s}, p_{v,s'}} \, ,$$ \((25)\)

where $\mu = \mu(i_1, i_2, \ldots, i_q)$ is called momentum assignment and gives to each propagator of each internal line $\ell$ a scale $i_\ell \in [0, \Lambda]$; the sum over $\mu$ is performed on all assignments and can only be done after renormalization in the way of \[13\]. The function $f_\mu$ contains product of coupling constants as well as symmetry factor of the graph. Given a line $\ell$, its propagator momenta $p_{v(\ell), s}$ relate a vertex $v(\ell)$ to another vertex $v'(\ell)$ and possess also a strand index $s = 1, 2$. The vertex operator is simply a collection of delta functions identifying entering and exiting momenta. If we do have external lines hooked on the graph $\mathcal{G}$, we could fix all external line indices to $i_{ext} = -1$.

The next stage is to perform in a optimal way the sum on $p_{v,s}$ thereby getting an optimal bound on $A_{\mathcal{G}, \mu}$ which is the quantity of interest. This optimal sum can be done by introducing the so-called quasi-local subgraphs, key ingredients for the multiscale analysis \[13\]. Given $\mu$ and a scale $i$, we consider the complete list of the connected components $\mathcal{G}_i^k$, $k = 1, 2, \ldots, k(i)$, of the subgraph $\mathcal{G}_i$ made of all lines in $\mathcal{G}$ with the scale attribution $j \leq i$ in $\mu$, with $\mathcal{G}_0 = \mathcal{G}$. The set of $\{\mathcal{G}_i^k\}_{k,i}$ is partially ordered by inclusion. The Gallavotti-Nicolò tree \[29\] is an abstract tree made with nodes the $\mathcal{G}_i^k$’s associated with that partial order such that there is a link between two nodes if and only if one is included in the other. Such a tree has obviously a root $\mathcal{G}_0 = \mathcal{G}$. We refer the reader to Figure 3 in \[8\] for
a complete illustration of this tree in the T3 model from which the similar notion in the present reduced framework should be clear. The key point is to choose a spanning tree of lines in the graph \( G \) and to perform the sum associated with momenta of these lines in such a way “to be compatible” with the Gallavotti-Nicolò tree. This compatibility condition entails an optimal bound which must be specified.

The vertex operator and the propagator contain both a bunch of delta functions which make that the amplitude factorizes along closed and open strands that are called faces. Thus the set \( \mathcal{F} \) of faces divides in \( \mathcal{F}_{\text{int}} \) set of internal or closed faces with cardinal \( \mathcal{F}_{\text{int}} = |\mathcal{F}_{\text{int}}| \), and \( \mathcal{F}_{\text{ext}} \) set of external or open faces with cardinal \( \mathcal{F}_{\text{ext}} = |\mathcal{F}_{\text{ext}}| \). Moreover, since the momentum associated with such faces can be \( p \) or \( \sqrt{l} \), we introduce another discrepancy between the faces: those indexed by a momentum \( p_f \) and belonging to the set \( \mathcal{F}_{\text{int}} \) and those coined by \( \sqrt{l} \) and belonging to the set \( \mathcal{F}_{\text{ext}} \).

Consequently, we can also have other types of subsets given by \( \mathcal{F}_{\text{int}}^\pm = \mathcal{F}_{\text{int}} \cap \mathcal{F} \), \( \bullet = \text{int, ext} \). We write \( |\mathcal{F}_{\text{int}}^\pm| = |\mathcal{F}_{\text{int}}| \) and \( |\mathcal{F}_{\text{ext}}^\pm| = |\mathcal{F}_{\text{ext}}| \).

Then, using the sliced propagator bound \[(24)\], after summing over all delta functions, one comes to
\[
|A_G| \leq K' L \left[ \prod_{\ell \in L} M^{-\mu} e^{-M^{-\mu} \mu} \right] \left[ \prod_{n_f, p_f} \left[ \prod_{f \in \mathcal{F}_{\text{int}}} e^{-M^{-\mu} b \sqrt{\delta'}} \right] \left[ \prod_{f \in \mathcal{F}_{\text{ext}}} e^{-M^{-\mu} |F|} \right] \right] \leq K'^{2L} K'^{2M_{\text{int}}} \left[ \prod_{\ell \in L} M^{-\mu} \right] \left[ \prod_{f \in \mathcal{F}_{\text{int}}} M^{2|f|} \right] \left[ \prod_{f \in \mathcal{F}_{\text{ext}}} M^{|f|} \right].
\]

where the set \( \{ \ell \in f \} \) denotes the set of strand lines involved in the face \( f \), and \( K' \) some constant. We must consider several cases

(i) If \( f \in \mathcal{F}_{\text{int}}^+ \), then the face amplitude is of the form \( \sum_{n_f} e^{-\sum_{\ell \in f} M^{-\mu} b \sqrt{\delta'}} \) the sum on \( n_f \) can be optimized by choosing \( i_f = \min_{\ell \in f} i_\ell \) and
\[
\sum_{n_f} e^{-\sum_{\ell \in f} M^{-\mu} b \sqrt{\delta'}} = \delta' M^{2|f|} + O(M') , \quad \delta = |\{ \ell \in f \}| , \quad \delta' = \frac{2}{\delta^2 b^2} .
\]

(ii) If \( f \in \mathcal{F}_{\text{int}}^- \), the face amplitude becomes \( \sum_{p_f} e^{-\sum_{\ell \in f} M^{-\mu} |F|} \) and it is optimal by choosing \( i_f = \min_{\ell \in f} i_\ell \) such that
\[
\sum_{p_f} e^{-\sum_{\ell \in f} M^{-\mu} |F|} = \delta' M^{2|f|} + O(M') , \quad \delta = |\{ \ell \in f \}| , \quad \delta' = \frac{1}{\delta^2 a} .
\]

(iii) Assume now that \( f \in \mathcal{F}_{\text{ext}} \), all intermediate momenta can be summed and yield \( O(1) \).

We therefore obtain
\[
|A_G| \leq K'^{2L} K'^{2M_{\text{int}}} \left[ \prod_{\ell \in L} M^{-\mu} \right] \left[ \prod_{f \in \mathcal{F}_{\text{int}}} M^{2|f|} \right] \left[ \prod_{f \in \mathcal{F}_{\text{ext}}} M^{|f|} \right] .
\]

where \( K' \) is some constant. The above amplitude rewrites using the \( G_k \)'s as
\[
|A_G| \leq K'^{2L} K'^{2M_{\text{int}}} \left[ \prod_{\ell \in L} \prod_{i=1}^{N} M^{-i} \right] \left[ \prod_{f \in \mathcal{F}_{\text{int}}} M^{2|f|} \right] \left[ \prod_{f \in \mathcal{F}_{\text{ext}}} M^{|f|} \right] ,
\]

where \( l_f \) is the strand in \( f \) such that \( t_{l_f} = i_f = \min_{\ell \in f} i_\ell \). Then, we have
\[
|A_G| \leq K'^{2L} K'^{2M_{\text{int}}} \left[ \prod_{(i,k) \in \mathcal{N}} \prod_{i} M^{-i} \right] \left[ \prod_{(i,k) \in \mathcal{N}} \prod_{f \in \mathcal{F} \cap G_{i}^k} M^{|f|} \right] \left[ \prod_{(i,k) \in \mathcal{N}} \prod_{|f| \in G_{i}^k} M^{|f|} \right] ,
\]

\[ (30) \]
where $K(Power$ $counting)$ counter-terms. These are two-point graphs formed with two-leg vertices of the missing faces after the tensor to matrix reduction and, from that, one recovers the balanced power-counting of $F$ in $F$. From (32), the sole remaining point is simply solved by $\omega(d(G)) = -L(G) + V_{2,1}(G) + V_{2,2}(G) + 2F_{\text{int}}(G)$.

**Proof.** From (32), the sole remaining point is simply solved by

$$2F_{\text{int}}^+(G) + F_{\text{int}}^-(G) \leq 2F_{\text{int}}(G).$$

The bound by $\omega(d(G)) \leq -L(G) + 2F_{\text{int}}(G)$ is, in fact, optimal since there are some configurations where $F_{\text{int}}(G) = F_{\text{int}}^+(G)$.

**C. Divergence degree analysis**

Divergence degree in topological terms. In order to re-express the divergence degree of a graph in term of its topological components, we first need to introduce the notion of “pinching” of an open graph or a graph with external legs. This notion has been initially defined for an arbitrary rank $D$ colored tensor graph in the reference [30]. Here, we apply the pinching procedure for a ribbon graph with external legs.

The pinching procedure of a ribbon graph $G$ with external legs consists in the construction of another graph $\tilde{G}$ by the gluing of 2-leg vertices at each external leg of $G$. See Figure 3 for an illustration. After pinching a open ribbon graph $G$, one obtains a closed ribbon graph $\tilde{G}$, with the same set of vertices and set of edges but a different set of faces. There are two types of faces in $\tilde{G}$, those coming from $G$ (see $f_1$ in Figure 4) and others coming from the external faces of $G$ (see $f_2$ in Figure 4). Hence

$$F_{\text{int}}(G) = F_{\text{int}}^+(G) \cup F_{\text{ext}}^+, \quad |F_{\text{int}}(G)| = F_{\text{int}}^+(G) + |F_{\text{ext}}^+|.$$
Consider now the pinched graph \( \tilde{\phi} \), any of internal lines, \( F \) vertices of mass renormalization, and \( V \) are determined by primitively divergent graphs.

**Proof.** Let the number of external legs of \( G \), the set \( \partial G \) of external faces of \( G \) (see an illustration in Figure 4). Therefore, \( V(\partial G) = N_{\text{ext}}(G), \quad L(\partial G) = F_{\text{ext}}(G). \) (37)

There is no difficulty to infer that \( F_{\text{ext}} = C_{\partial G}. \) (38) where \( C_{\partial G} \) is the number of connected components of \( \partial G \).

**Proposition 1 (Divergence degree).** Let \( G \) be a connected graph. The divergence degree of \( G \) is

\[
\omega_d(G) = -V_2 - \frac{1}{2} [N_{\text{ext}} - 4] - 4g_{\tilde{\phi}} - 2(C_{\partial G} - 1),
\]

(39)

where \( V_2 \) is the number of mass renormalization vertices, \( g_{\tilde{\phi}} \) is the genus of the closed pinched graph \( \tilde{G} \) extending the initial graph \( G \), \( C_{\partial G} \) is the number of connected components of \( \partial G \) the boundary graph associated with \( G \) and \( N_{\text{ext}} \) is the number of external legs of \( G \).

**Proof.** Let \( G \) be a ribbon graph of the theory. It has \( V_4(G) \) number of vertices of the \( \varphi^4 \) type, \( V_{2,0}(G) \), number of vertices of mass renormalization, and \( V_{2,s=1,2}(G) \), number of vertices of wave function renormalization, \( L(G) \) number of internal lines, \( F_{\text{int}}(G) \) number closed faces, \( F_{\text{ext}}(G) \) number of open faces. The following expression relates as, in any \( \varphi^4 \) theory, the number of vertices and lines:

\[
V(G) = V_4(G) + \sum_{s=0,1,2} V_{2,s}(G),
\]

\[4V_4(G) + 2 \sum_{s=0,1,2} V_{2,s}(G) = 2L(G) + N_{\text{ext}}(G), \quad L(G) - 2V_4(G) = \sum_{s=0,1,2} V_{2,s}(G) - \frac{1}{2} N_{\text{ext}}(G). \] (40)

Consider now the pinched graph \( \tilde{G} \) and its Euler characteristics:

\[
2 - 2g_{\tilde{\phi}} = V(\tilde{G}) - L(\tilde{G}) + F_{\text{int}}(\tilde{G}),
\]

(41)

where \( g_{\tilde{\phi}} \) denotes the genus of \( \tilde{G} \). Combining both (36) and (38), we infer from (41) that

\[
F_{\text{int}}(G) = 2 - 2g_{\tilde{\phi}} - [V(G) - L(G) + C_{\partial G}].
\]

(42)

The degree of divergence of \( G \) (34) therefore can be rewritten as

\[
\omega_d(G) = -L(G) + 4 - 4g_{\tilde{\phi}} - 2[V(G) - L(G) + C_{\partial G}] + \sum_{s=1,2} V_{2,s}(G)
\]

\[= 4 - 4g_{\tilde{\phi}} - V_{2,0}(G) - \frac{1}{2} N_{\text{ext}}(G) - 2C_{\partial G}, \] (43)

where, in the last step, we used (40). Hence (39) is immediate.

**Primitively divergent graphs.** From (39), one realizes that the marginal or log-divergent graphs with \( N_{\text{ext}} = 4 \) are determined by

\[
\omega_d(G) = 0: \quad g_{\tilde{\phi}} = 0, \quad C_{\partial G} = 1, \quad V_2 = 0.
\]

(44)
They should correspond to a vertex renormalization. For \( N_{\text{ext}} = 2 \), there are two categories of divergent graphs:

\[
\begin{align*}
\omega_d(\mathcal{G}) = 1 : & \quad g_0 = 0, \quad C_{\partial \mathcal{G}} = 1, \quad V_2 = 0, \\
\omega_d(\mathcal{G}) = 0 : & \quad g_0 = 0, \quad C_{\partial \mathcal{G}} = 1, \quad V_2 = 1.
\end{align*}
\]

These should contribute to a mass and wave function renormalizations.

A remarkable fact is that all these divergent contributions are planar graphs. We remark also that this power-counting is in agreement with the one of the GW model in 4D for which the renormalization procedure identifies as relevant graphs only planar graphs with one broken external face with at most four external legs [20].

There is now a clear effect induced by the projection map. In the renormalization procedure of \( T3 \) [8], the dominant contributions, at given \( N_{\text{ext}} = 4, 2 \), were specific graphs called “melonic” [18] with melonic boundary graph \( \partial \mathcal{G} \) having a unique connected component. In the present context, one realizes the fact that dominant contributions should be melonic has been washed away and gets replaced just by a planarity condition. This can be indeed expected since the new model is a matrix theory which can only be sensitive to an ordinary planarity condition. Nevertheless, for melonic graphs \( F_{\text{int}}^+ \) should be the one which contributes so that one would rather have a divergence degree much more restrictive and of the form \( \tilde{\omega}_d(\mathcal{G}) = -L(\mathcal{G}) + 2F_{\text{int}}^+(\mathcal{G}) \) so that many planar graphs which would be not melonic would be simply cast away for being convergent if one use such a power counting theorem.

### D. Renormalization

We can now study the subtraction terms and provide the proof that \( N_{\text{ext}} \)-point functions expand in divergent parts of the form of the initial Lagrangian terms plus a convergent remainder. It is at this stage that one finds important that the kinetic term is of form \( a p^\mu + bn^\beta \) in order to obtain the correct wave function counter-terms.

#### Renormalization of the four-point functions

Consider a 4-point quasi local subgraph \( G^k_i \) as determined by the conditions [14] with four external propagators. The external momenta of this graph should follow the pattern of the vertex as given by Figure [2]. We shall denote the external momenta associated to each of the four external faces counting is in agreement with the one of the GW model in 4D for which the renormalization procedure identifies as

\[
F \in \int_0^\epsilon \frac{d\epsilon}{\epsilon} \left\{ \begin{array}{l}
\chi \sum_{\alpha_i} a_i \, a_i' \, q_i \, q'_{i}\, e^{-\mu \epsilon - \mu \epsilon} \, e^{-\sum_{\alpha_i} a_i \, a_i' \, q_i \, q'_{i}} \times \left( \prod_{\alpha_i} e^{-\sum_{\alpha_i} a_i \, a_i' \, q_i \, q'_{i}} \right) \right\}.
\]
Scrutinizing the term depending on the external momenta \( q_{f,x}^\pm \), we find that this factorizes and yields

\[
A_4\{(q_{f,x}^\pm);0\} = C_{ji}(\{q_{f,j_i}^\pm\})C_{j2}(\{q_{f,2}^\pm\})C_{j3}(\{q_{f,3}^\pm\})C_{j4}(\{q_{f,4}^\pm\}) \times \\
\sum_{q_{f,i}^\pm} \int \prod_{\ell \neq i} d\alpha_\ell e^{-\alpha_\ell \mu} \prod_{\epsilon = \pm} \left( \prod_{f \in F_{\text{ext}}} e^{-\sum_{\epsilon (f/\ell) \neq 1} \alpha_\ell a_f q_{f,x}^\epsilon} \right) (50)
\]

This amplitude is log-divergent and corresponds to a graph with 4 propagator lines the external data of which coincide with a vertex of the model. In other word, \( A_4\{(q_{f,x}^\pm);0\} \) contributes to the vertex renormalization.

Focusing on the remainder, we have the following bound

\[
|R_4| = \left| \int_0^1 \left\{ \sum_{q_{f,i}^\pm} \int \prod_{\ell} d\alpha_\ell e^{-\alpha_\ell \mu} \sum_{\epsilon = \pm; f \in F_{\text{ext}}} \left[ \int_{\alpha \epsilon} [\sum_{\ell \neq f; \ell \neq i} \alpha_\ell] a_f q_{f,x}^\epsilon \right] \prod_{\epsilon = \pm} \left( \prod_{f \in F_{\text{ext}}} e^{-\sum_{\epsilon (f/\ell) \neq 1} \alpha_\ell a_f q_{f,x}^\epsilon} \right) \right| \right|
\]

(51)

where, passing from the equality to the inequality, we use \( A|X|e^{-A|X|} \leq e^{-A|X|/2} \) with \( |X| = q_{f,x} \) and \( A = a_\epsilon (\alpha_\ell + \alpha_\ell') \). We can optimize the bound on \( |R_4| \) by choosing

\[
e(G_i^k) = \sup_{i \in G_i^k} j_i, \quad \alpha_\ell \geq M^{-e(G_i^k)}, \quad i(G_i^k) = \inf_{i \in G_i^k} i_\ell, \quad \alpha_\ell \leq M^{-i(G_i^k)}, \quad \ell \neq i,
\]

(52)

such that there exists some constant \( c \) such that

\[
|R_4| \leq c M^{-(i(G_i^k) - e(G_i^k))} \left\{ \sum_{q_{f,i}^\pm} \int \prod_{\ell} d\alpha_\ell e^{-\alpha_\ell \mu} \sum_{\epsilon = \pm} \left( \prod_{f \in F_{\text{ext}}} e^{-\frac{1}{2} \alpha_\ell a_f q_{f,x}^\epsilon} \left[ \prod_{f \in F_{\text{ext}}} e^{-\sum_{\epsilon (f/\ell) \neq 1} \alpha_\ell a_f q_{f,x}^\epsilon} \right] \right) \right\}
\]

(53)

The factor \( M^{-(i(G_i^k) - e(G_i^k))} \) brings an additional decay ensuring both an improved power counting and the final summability over the scale \( \mu \).

**Renormalization of the two-point functions.** We now focus on 2-point quasi local subgraphs \( G_i^k \) as given by \( G_i^k \) with two external propagators. We will however not treat both cases but only the linearly divergent graph. The logarithmically divergent contribution can be recovered by simple inference.

The external momenta of the graph should follow the pattern of mass vertex with two external momenta associated to each of the two external faces \( f_0^- \) and \( f_0^+ \), by \( p_{f_0}^-, n_{f_0}^+ \). As above, we use a compact script \( q_{0}^- = p_{f_0}^- \) and \( q_{0}^+ = \sqrt{n_{f_0}^+} \). We use the same conventions and notations for internal and external scales as done before.

The amplitude of \( G_i^k \) writes

\[
A_2\{(q_{0}^\pm);0\} = \sum_{q_{f,i}^\pm} \int \prod_{\ell} d\alpha_\ell e^{-\alpha_\ell \mu} \left[ \prod_{\epsilon = \pm} e^{-\sum_{\epsilon (f/\ell) \neq 1} \alpha_\ell a_f q_{f,x}^\epsilon} \right] \prod_{\epsilon = \pm} \left( \prod_{f \in F_{\text{int}}} e^{-\sum_{\epsilon (f/\ell) \neq 1} \alpha_\ell a_f q_{f,x}^\epsilon} \right),
\]

(55)

where \( \alpha_\ell \in [M^{-j_\ell + 1}, M^{j_\ell}] \) and, for external propagators, \( \alpha_\ell \in [M^{-j_\ell + 1}, M^{j_\ell}] \) with \( j_\ell \ll i \).

Using again a parameterized amplitude

\[
A_2\{(q_{0}^\pm);0\};\ell = \sum_{q_{f,i}^\pm} \int \prod_{\ell} d\alpha_\ell e^{-\alpha_\ell \mu} \left[ \prod_{\epsilon = \pm} e^{-(\alpha_\ell + \alpha_\ell') a_f q_{0}^\pm} \left[ \prod_{\epsilon = \pm} \left( \prod_{f \in F_{\text{int}}} e^{-\sum_{\epsilon (f/\ell) \neq 1} \alpha_\ell a_f q_{f,x}^\epsilon} \right) \right] \right],
\]

(56)

where \( \alpha_\ell \in [M^{-j_\ell + 1}, M^{j_\ell}] \) and, for external propagators, \( \alpha_\ell \in [M^{-j_\ell + 1}, M^{j_\ell}] \) with \( j_\ell \ll i \).
providing an interpolation from $A_2[\{q^+_0, q^-_0\}; t = 1]$ to $A_2[\{q^+_0, q^-_0\}; t = 0]$, we perform a Taylor expansion such that

$$A_2[\{q^+_0, q^-_0\}; t] \bigg|_{t=1} = A_2[\{q^+_0, q^-_0\}; t] \bigg|_{t=0} + \frac{d}{dt} A_2[\{q^+_0, q^-_0\}; t] \bigg|_{t=0} + \int_0^1 (1-t) \frac{d^2}{dt^2} A_2[\{q^+_0, q^-_0\}; t] dt.$$  \hfill (57)

We focus on the first term:

$$A_2[\{q^+_0, q^-_0\}; 0] = \int \prod_q (t_a) e^{-\alpha_\mu} \left[ \prod_{\epsilon=\pm} \sum_{\ell \neq \ell'} e^{-(\alpha_{\ell+\alpha_{\ell'}}) a_\ell q_{\ell'}} \prod_{\epsilon=\pm} \sum_{f \in F_{\text{int}}} e^{-[\sum_{\ell \in f} \alpha_\ell] a_\ell q_{f}} \right]$$

$$= C[q^+_0, q^-_0] C[q^-_0, q^+_0] \sum_{q_{f, i}} \left[ \prod_{\ell \neq \ell'} (t_a) e^{-\alpha_\mu} \left[ \prod_{\epsilon=\pm} \sum_{f \in F_{\text{int}}} e^{-[\sum_{\ell \in f} \alpha_\ell] a_\ell q_{f}} \right] \right]$$

which is linearly divergent and, clearly, renormalizes the mass in the model.

The second term can be written in the form

$$A_2'[\{q^+_0, q^-_0\}; 0] = \sum_{q_{f, i}} \left[ \prod_{\ell \neq \ell'} (t_a) e^{-\alpha_\mu} \left[ \prod_{\epsilon=\pm} \sum_{f \in F_{\text{int}}} e^{-[\sum_{\ell \in f} \alpha_\ell] a_\ell q_{f}} \right] \right]$$

$$= \frac{1}{\prod_q (t_a) e^{-\alpha_\mu}} \left[ \prod_{\ell \neq \ell'} e^{-(\alpha_{\ell+\alpha_{\ell'}}) a_\ell q_{\ell'}} \right]$$

$$\sum_{\epsilon=\pm} \left[ \prod_{\ell \neq \ell'} (t_a) e^{-\alpha_\mu} \left[ \prod_{\epsilon=\pm} \sum_{f \in F_{\text{int}}} e^{-[\sum_{\ell \in f} \alpha_\ell] a_\ell q_{f}} \right] \right]$$

This term contributes to the wave function renormalization and should separately provide two log-divergent contributions to $a_+ \sqrt{n}$ and to $a_- p$ of the initial kinetic term. Indeed, the first integral in $d\alpha_\ell$ yields again two factorized external propagators and the integral in $d\alpha_\ell$ yields a logarithmic divergence by the power counting. This is clearly the case since the sum $\sum_{\ell \in f} \alpha_\ell \leq k M^{-1} = k \prod_{i(k)/f \in G^i} M^{-1}$, lowers the linear divergence of one unit.

It is definitely at this stage that the introduction of two wave function couplings, namely $a_{\pm}$, plays a drastic role by providing the necessary freedom to renormalize wave functions with different coefficients. Indeed, due to the fact that the model is no longer symmetric in strands, it is not true that the contributions to the wave function renormalization of $a_+$ might find corresponding contributions in the wave function renormalization for $a_-$. As a simple illustration, at one-loop, consider the unique tadpole graphs $T^+$ and $T^-$ given in Figure 2 in the appendix which are involved in the self-energy. At this order of perturbation, $T^+$ contribute to the wave function renormalization associated with $a p$. Meanwhile the contribution of $T^-$ becomes finite. There is therefore, at this order and at any higher order, no term restoring the symmetry between for the wave function renormalizations $a p \delta Z_1$ and $b \sqrt{n} \delta Z_2$.

The last term needed to be analyzed in (57) is the reminder

$$|R_2| = \left| \int_0^1 (1-t) dt \sum_{q_{f, i}} \prod_{\ell \neq \ell'} (t_a) e^{-\alpha_\mu} \left[ \prod_{\epsilon=\pm} \left[ - \left[ \sum_{\ell \in f} \alpha_\ell \right] a_\ell q_{\ell'} \right] \right] e^{-[\sum_{\ell \in f} \alpha_\ell] a_\ell q_{f}} \prod_{\epsilon=\pm} \left[ \sum_{f \in F_{\text{int}}} e^{-[\sum_{\ell \in f} \alpha_\ell] a_\ell q_{f}} \right] \right|$$

$$\leq \left| \int_0^1 (1-t) dt \sum_{q_{f, i}} \prod_{\ell \neq \ell'} (t_a) e^{-\alpha_\mu} \left[ \prod_{\epsilon=\pm} \left[ \sum_{\ell \in f} \alpha_\ell \right] (a_\ell + a_{\ell'})^{-1} \right] e^{-[\sum_{\ell \in f} \alpha_\ell] a_\ell q_{f}} \prod_{\epsilon=\pm} \left[ \sum_{f \in F_{\text{int}}} e^{-[\sum_{\ell \in f} \alpha_\ell] a_\ell q_{f}} \right] \right|$$

$$\leq c M^{-2((G^i_n)^-c(G^i_n))} \sum_{q_{f, i}} \prod_{\ell \neq \ell'} (t_a) e^{-\alpha_\mu} \left[ \prod_{\epsilon=\pm} \left[ e^{-(\alpha_\ell + a_{\ell'}) a_\ell q_{\ell'}} / 4 \right] e^{-[\sum_{\ell \in f} \alpha_\ell] a_\ell q_{f}} \right] \right|$$

$$\times \int_0^1 (1-t) dt \left[ \prod_{\epsilon=\pm} e^{-[\sum_{\ell \in f} \alpha_\ell] a_\ell q_{f}} \right].$$  \hfill (61)
The integrals in $t$ and in $d\alpha_l$ yield a mere $O(1)$ factor. Thus, one proves that this remainder $|R_2|$ is bounded and provides an additional decay of $M^{-2(i(G_k^1) - c(G_k^1))}$ to $M^{\omega_d(G_k^1)=1}$ which will ensure the final summability on momentum attribution $\mu$ in the standard way of \[3\].

This achieves the proof of the renormalizability of the model and hence Theorem \[1\] holds.

\section*{E. RG flow of wave function renormalizations and a super-renormalizable vector model}

The $T3$ model with symmetric interactions has been proved asymptotically free \[3\]. The projection on the matrix model proves to preserve that behavior and the model $\sigma T3$ is also asymptotically free. The proof of this statement has been provided in the appendix. There is however more that we can say about the latter situation.

One notices that two wave function renormalizations should be introduced at this level

$$Z_b = 1 - \frac{1}{b} \partial_p \Sigma(p, \sqrt{n})|_{p=0=n}, \quad Z_a = 1 - \frac{1}{a} \partial_p \Sigma(p, \sqrt{n})|_{p=0=n},$$

where $\Sigma(p, \sqrt{n})$ is the so-called self-energy or sum of all two-point one-particle irreducible (1PI) contributions. In the appendix, we discuss at first order how behave these terms. At one-loop computations, one finds that the contribution to $\frac{1}{b} \partial_p \Sigma(p, \sqrt{n})|_{p=0=n}$ is finite (see the appendix). In contrast, $Z_a$ is a log-divergent quantity and should be used to renormalize $a$. It becomes immediate that, the renormalized quantity associated with $b$, namely $b^{\text{ren}}$, is such that $b^{\text{ren}} = b(Z_b/Z_a)^{1/2} \to 0$. Thus, the model flows towards a free model with vanishing $b$ in its kinetic term.

Having this remark in mind, one could ask if the above model does not imply the existence of another renormalizable model flowing towards the same Gaussian fixed point with vanishing $b$. This question is certainly non trivial. The simplest way to think about this is to start building a model with the same kinetic term as for the $\sigma T3$ but with vanishing $b$. We furthermore need to gauge fix the sector $n$ in all field $\phi_{p,n}$, otherwise, there will be a free mode sum in this sector. Using, for instance, the $n = 0$ mode sector, one builds a the kinetic term and interaction of the form

$$S^\text{kin,0} = \sum_p \phi_{p,0}(ap + \mu)\phi_{p,0}, \quad S^\text{int,0} = \left(\sum_p \phi_{p,0}^2\phi_{p,0}\right)^2$$

which describes, surprisingly, a vector model. Propagator and vertices can be represented by Figure 5.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig5.png}
\caption{Propagator and vertex $v_4$ of the vector model.}
\end{figure}

We can perform a multi-scale analysis in a similar way that was introduced in Subsection \[11\]. It is simple to obtain this limit case since it correspond to set $b = 0$ in the analysis therein. One gets propagator bound given by \[24\] at $b = 0$ and, for any connected graph $G$ and for a momentum scale attribution $\mu$, an optimal bound on amplitude as

$$|A_{\mu}| \leq K^n \prod_{(i,k)} M^{\omega_d(G_k^i)}, \quad \omega_d(G) = -L(G_k^i) + F_{\text{int}}(G_k^i),$$

where, once again, closed strands are the sources of divergences.

Since the vertex $\[33\]$ is apparently disconnected from the point of view of its external legs (it is clearly a factor of two pieces), we introduce $v_4'$ made with two external legs and being half of this vertex $v_4$. Consider now a graph $G$ connected with respect to only vertices of the form $v_4'$. Call $V'$ the number of vertices of this graph, $L$ its number of lines and $F_{\text{int}}$ is number of closed loops, $C$ its number of connected components, $N_{\text{ext}}$ its number of external legs. We have the relations

$$2V' = 2L + N_{\text{ext}}, \quad C = F_{\text{int}} + C'',$n

where $C'$ counts the number of external strands. For $G$, it is immediate to translate the above divergence degree as

$$\omega_d(G) = -L + F_{\text{int}} = -V' + \frac{1}{2}N_{\text{ext}} + C - C'.$$
Having assumed that $G$ is connected, then $C = 1$. A connected component in the theory is either an open strand or a closed one, then either $N_{\text{ext}} = 2$, $C' = 1$, $F_{\text{int}} = 0$ or $N_{\text{ext}} = 0$, $C' = 0$ and $F_{\text{int}} = 1$. In all cases, the divergence degree recasts as

$$\omega_d(G) = -V' + C.$$  \hfill (67)

From this, we can give the list of all primitively divergent graphs which reduces to a unique type of graph:

$$C = 1, \quad \omega_d(G) = -V' + 1 = 0 \iff (V' = 1, L = 1, N_{\text{ext}} = 0).$$  \hfill (68)

This corresponds to a unique 1-loop graph which is logarithmically divergent. Coming back to the situation with graphs with vertices $v_4$, the same type of graphs is nothing but a tadpole graph (see Figure 6) which should only involve a mass renormalization. Thus this model is super-renormalizable. Clearly, this feature is similar to a ordinary scalar $\varphi^2$.

FIG. 6. Tadpole graph.

### III. NEW CLASSES OF RENORMALIZABLE MODELS FROM PREVIOUS MODELS

In this section, we use the above mechanism to reveal the existence of new renormalizable models issued from well-known renormalizable tensor and matrix models.

#### A. Classes of GW models

The GW model is the first discovered renormalizable model pertaining to both matrix models and noncommutative geometry [25]. This model proves to be renormalizable at all orders by curing a previous undesirable effect called UV/IR mixing affecting renormalization procedure on noncommutative spaces [24]. The UV/IR mixing is simply removed by adding an harmonic term of the form $\Omega^2 \tilde{x}^2$, where $\Omega$ is an harmonic frequency, $\tilde{x} = 2(\Theta^{-1})_{\mu\nu}x^\mu$, with the noncommutative structure given by $[x^\mu, x^\nu] = \Theta_{\mu\nu}$, $x^\mu \in \mathbb{R}^D$, $D = 2, 4$.

We will restrict the study to complex fields and will place ourselves at the self-dual point $\Omega = 1$ for which, in the continuum, the kinetic term of the GW model is $(-\Delta + \tilde{x}^2 + \mu)$, $\mu$ being some IR mass regulator, so that the model becomes dual in momenta and positions. In 4D, the complex GW model is given by the action (Euclidean signature) [25]

$$S_{GW;4D} = \frac{1}{2} \sum_{\vec{\rho}, \vec{\gamma}, \vec{n} \in \mathbb{N}^2} \tilde{\phi}_{\vec{\rho}, \vec{\gamma}} \left[ |p| + |q| + \mu \right] \tilde{\phi}_{\vec{\rho}, \vec{\gamma}} + \frac{\lambda}{4} \sum_{\vec{n}, \vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{q} \in \mathbb{N}^2} \tilde{\phi}_{\vec{m}, \vec{n}} \tilde{\phi}_{\vec{\alpha}, \vec{\beta}} \tilde{\phi}_{\vec{\rho}, \vec{\gamma}} \tilde{\phi}_{\vec{m}, \vec{n}} \tilde{\phi}_{\vec{n}, \vec{m}}.$$  \hfill (70)

where $\star$ denotes the Moyal star product. There exists a basis for which the above model finds another clear translation. This is the so-called matrix basis [25] where each field can be viewed as a rank 4 complex tensor $\phi_{\vec{p}, \vec{q}}$ where $\vec{p} = (p_1, p_2)$ and $\vec{q} = (q_1, q_2) \in \mathbb{N}^2$. The same GW action reads as, at the self-dual point $\Omega = 1$,

$$S_{GW;4D} = \frac{1}{2} \sum_{\vec{\rho}, \vec{\gamma} \in \mathbb{N}^2} \tilde{\phi}_{\vec{p}, \vec{q}} \left[ |p| + |q| + \mu \right] \phi_{\vec{p}, \vec{q}} + \frac{\lambda}{4} \sum_{\vec{n}, \vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{q} \in \mathbb{N}^2} \phi_{\vec{m}, \vec{n}} \phi_{\vec{\alpha}, \vec{\beta}} \phi_{\vec{p}, \vec{q}} \phi_{\vec{m}, \vec{n}}.$$

where we introduce the notation, for any $\vec{n} \in \mathbb{N}^2$, $|\vec{n}| = n_1 + n_2$. Call this the $GW_{4D}$ model. The propagator and vertex of this model can be represented as in Figure 2 but one should regard each strand as doubled. The interaction is clearly of the cyclic form so that the procedure introduced above applies naturally here.

Let us first recall few facts about the renormalizability of the $GW_{4D}$ [25]. The propagator in the slice $i$ admits the bound, for some constant $K$,

$$C_i(\vec{p}, \vec{q}) \leq K M^{-i} e^{-M^{-i}(|p|+|q|+\mu)}.$$  \hfill (71)
After a multi-scale analysis, one is led to a power counting theorem giving the divergence degree of any connected graph $\mathcal{G}$ as (up to unessential two-point vertices which can be contracted)

$$\omega_d(\mathcal{G}) = -L(\mathcal{G}) + 2F_{\text{int}}(\mathcal{G}) = -4g(\mathcal{G}) - 2(C_d(\mathcal{G}) - 1) - \frac{1}{2}(N_{\text{ext}}(\mathcal{G}) - 4),$$

(72)

using similar notations as in previous section. Thus only open ribbon graphs $\mathcal{G}$ characterized with 4 and 2 external legs, a vanishing genus of their pinching $g(\mathcal{G}) = 0$ and a unique connected component of the boundary should be the ones inducing divergent 4- and 2-point functions. 4-pt graphs should participate to the coupling constant $\lambda$ renormalization and 2-pt graphs should involve mass and wave function renormalizations.

We now start the program dealing with the rank reduction of the model. The $\sigma$ projection leads us to a new rank 3 model, called $\sigma GW_{4D}$, described by the following action:

$$S_{\sigma GW_{4D}} = \frac{1}{2} \sum_{p \in \mathbb{N}, \tilde{q} \in \mathbb{N}^2} \tilde{\phi}_{p, \tilde{q}}[a \sqrt{p} + b|q| + \mu] \phi_{\tilde{q}, p} + \frac{\lambda}{4} \sum_{m, p \in \mathbb{N}; \tilde{n}, \tilde{q} \in \mathbb{N}^2} \tilde{\phi}_{m, \tilde{n}} \phi_{\tilde{n}, p} \tilde{\phi}_{p, \tilde{q}} \phi_{\tilde{q}, m},$$

(73)

where we introduce wave coupling parameters, $a$ and $b$, in order to have a priori a proper notion of wave function renormalizations. Note that, in the new action, the mapping $\sigma$ can be applied as well on the second couple of integers $\tilde{q}$. Hence, combinatorially, we have two such $\sigma GW_{4D}$ models.

The propagator of the model (73) is given by

$$C(\{p, \tilde{q}\}; \{\tilde{p}, \tilde{q}'\}) = \delta_{p, \tilde{p}} \delta_{\tilde{q}, \tilde{q}'} / (a \sqrt{p} + b|q| + \mu).$$

(74)

Introducing a UV cut-off on momenta, we claim that this model is just renormalizable at all orders of perturbations. We shall sketch the main phases of the proof of this statement since the details of all arguments can be fully recovered from Section III.

First, we bound the propagator kernel in the slice $i$ as

$$C_i(p, \tilde{q}) \leq K M^{-i} e^{-M^{-i}(a \sqrt{p} + b|q| + \mu)}.$$

(75)

Performing the multi-scale analysis using this sliced propagator bound, it is immediate to realize that, open faces do not participate to the power-counting and that there is two types of closed faces: faces parameterized by $\sqrt{p}$ which have a double weight and faces with momentum $q_1$ or $q_2$. These latter faces go always by pairs $(q_1, q_2)$. The closed face amplitudes evaluation yields:

$$\sum_p e^{-a M^{-i}\sqrt{p}} = \frac{2}{a^2} M^{2i} (1 + O(M^{-i})), \quad \sum_{q_1, q_2} e^{-b M^{-i}(q_1 + q_2)} = \frac{1}{b^2} M^{2i} (1 + O(M^{-i})).$$

(76)

Thus (76) shows that, even though the number of faces of the reduced theory certainly decreases in $p$-sector, the amplitude of each face becomes twice greater in that sector. This will be the key feature ensuring again the renormalizability of $\sigma GW_{4D}$.

The degree of divergence of any connected graph $\mathcal{G}$ becomes, in the same anterior notations (and conventions forgetting two-point vertices):

$$\omega_d(\mathcal{G}) = -L(\mathcal{G}) + 2F_{\text{int}}^+(\mathcal{G}) + 2F_{\text{int}}^-(\mathcal{G}) = -L(\mathcal{G}) + 2F_{\text{int}}.$$

(77)

Since the arguments yielding (72) depend uniquely on the topology of graphs, and provided the topology of graphs of $GW_{4D}$ and of graphs of $\sigma GW_{4D}$ is the same, one ends up with the same degree of divergence. As a consequence, the list of divergent graphs are identical for both models. The renormalization of 4-pt and 2-pt functions can be checked in the way of Subsection III D. However, another interesting point which has to be fully inspected is the possibility of having a unique wave function renormalization for $\sigma GW_{4D}$.

Interestingly, we can prove that putting $a = b \sqrt{2}$, the new model is still renormalizable. This can be viewed as follows. Using a Taylor expansion of a general two-point function around its “local” divergence (see corresponding paragraph in Subsection III D), setting $a = b \sqrt{2}$, we must prove that the divergent contribution associated with the

$$\frac{1}{2} \sum_{p \in \mathbb{N}, \tilde{q} \in \mathbb{N}^2} \tilde{\phi}_{p, \tilde{q}}[a \sqrt{2\tilde{p}} + b(q_1 + q_2) + \mu] \phi_{\tilde{q}, p},$$

(78)

well motivated by the $\sigma$ map. In this case, one sets $a = b$ and still get the renormalizability for the subsequent model.

---

1 Note that this simply means that we could have introduced a different choice of action $S_{\sigma GW_{4D}}$ (73) with a kinetic term such that

$$\frac{1}{2} \sum_{p \in \mathbb{N}, \tilde{q} \in \mathbb{N}^2} \tilde{\phi}_{p, \tilde{q}}[a \sqrt{2\tilde{p}} + b(q_1 + q_2) + \mu] \phi_{\tilde{q}, p},$$

(78)
amplitude associate with \( G \) and \( G \). The propagator and vertex of this model match with Figure 2. The proposition of the super-renormalizability of this \( \sigma \) equation of (76), there are two types of faces coined by \( \sigma \sqrt{G} \) equation of (76), there are two types of faces coined by \( \sqrt{G} \) or \( \sqrt{q} \) with equal weight \( M^2 \). Following step by step, the above procedure, we are led to the same power counting theorem and the same type of graphs which ought to be renormalized. The renormalization of these graphs can be performed as earlier done. It is immediate that the model \( \sigma^2GW_{4D} \) is renormalizable at all orders. Note that we can put now \( a = b \) involving the possibility of having a unique wave function renormalization. This is without consequence on the renormalizability property of the model due to the restored symmetry of all strands in the propagator.

Applying another reduction in the remaining sector \( \tilde{q} \), we get finally a matrix model that we call \( \sigma^2GW_{4D} \) given by

\[
S_{\sigma^2GW_{4D}} = \frac{1}{2} \sum_{p,q \in \mathbb{N}} \tilde{\phi}_{p,q} \left[ a \sqrt{p} + b \sqrt{q} + \mu \right] \phi_{p,q} + \frac{\lambda}{4} m,q,n,p \tilde{\phi}_{m,n} \phi_{n,p} \tilde{\phi}_{p,q} \phi_{q,m}.
\] (79)

The model \( \sigma^2GW_{4D} \) is strand symmetric and can be studied along the lines of the anterior analysis. From the first equation of (76), there are two types of faces coined by \( \sqrt{p} \) or \( \sqrt{q} \) with equal weight \( M^2 \). Following step by step, the above procedure, we are led to the same power counting theorem and the same type of graphs which ought to be renormalized. The renormalization of these graphs can be performed as earlier done. It is immediate that the model \( \sigma^2GW_{4D} \) is renormalizable at all orders. Note that we can put now \( a = b \) involving the possibility of having a unique wave function renormalization. This is without consequence on the renormalizability property of the model due to the restored symmetry of all strands in the propagator.

It is noteworthy that \( S_{\sigma^2GW_{4D}} \) which is a rank 2 model does not correspond to the action of the GW model in 2D denoted \( GW_{2D} \). The \( GW_{2D} \) model is super-renormalizable and is given by the action [20]:

\[
S_{GW_{2D}} = \frac{1}{2} \sum_{p,q \in \mathbb{N}} \tilde{\phi}_{p,q} \left[ p + q + \mu \right] \phi_{p,q} + \frac{\lambda}{4} m,q,n,p \tilde{\phi}_{m,n} \phi_{n,p} \tilde{\phi}_{p,q} \phi_{q,m}.
\] (80)

The propagator and vertex of this model match with Figure 2. The proposition of the super-renormalizability of this model can be quickly reviewed. The propagator in a slice meets a bound

\[
C_i(\{p, q\}; \{\tilde{p}, \tilde{q}\}) \leq KM^{-i}e^{-M^{-i}(p+q+\mu)}.
\] (81)

From this bound and a multiscale analysis of the amplitude of a connected graph \( G \), we can write an optimal bound amplitude associate with \( G \) with degree of divergence (with similar anterior conventions)

\[
\omega_d(G) = -L(G) + F_{\text{int}}(G) = -2\tilde{q} - (V(G) - 1) - (C_{\tilde{\phi}}(G) - 1),
\] (82)

where one should use the Euler characteristics to map the different numbers. In this form, one realizes that the more the graph contains vertices the more it is convergent, a specific feature of super-renormalizability. The list of primitively divergent graphs summarizes as follows: since \( V \geq 1 \) and \( C_{\tilde{\phi}}(G) \geq 1 \) (we do not discuss vacuum graphs \( C_{\tilde{\phi}}(G) = 0 \)), the only possibility for \( \omega_d(G) \geq 0 \) reads

\[
g_{\tilde{\phi}} = 0, \quad V(G) = 1, \quad C_{\tilde{\phi}}(G) = 1, \quad \omega_d(G) = 0.
\] (83)

This is nothing but a tadpole graph yielding a log-divergent contribution which re-absorbed by a mass renormalization.

The action (80) is already in the matrix form. We can consider now the reversed of the \( \sigma \) process. Reshuffling now (80) by using the inverse \( \sigma^{-1} \) in one sector, say \( q \) without loss of generality, \( \sigma^{-1}(q) = \tilde{q} \), we get a rank 3 GW model:

\[
S_{GW_{2D}} = S'_{GW_{2D}} = \frac{1}{2} \sum_{p,q \in \mathbb{N}, \tilde{q} \in \mathbb{N}^2} \tilde{\phi}_{p,\tilde{q}} \left[ ap + b\sigma(\tilde{q}) + \mu \right] \phi_{\tilde{q},p} + \frac{\lambda}{4} m,p,n,\tilde{q} \tilde{\phi}_{m,n} \phi_{n,p} \tilde{\phi}_{\tilde{q},p} \phi_{\tilde{q},m}
\] (84)

which can be related to the new \( \sigma^{-1}GW_{2D} \) model defined as

\[
S_{\sigma^{-1}GW_{2D}} = \frac{1}{2} \sum_{p,q \in \mathbb{N}, \tilde{q} \in \mathbb{N}^2} \tilde{\phi}_{p,\tilde{q}} \left[ ap + b(q_1^2 + q_2^2) + \mu \right] \phi_{\tilde{q},p} + \frac{\lambda}{4} m,p,n,\tilde{q} \tilde{\phi}_{m,n} \phi_{n,p} \tilde{\phi}_{\tilde{q},p} \phi_{\tilde{q},m}.
\] (85)

One can check that, using (17), that the propagators (84) and (85) have same behavior in a slice. We have the propagator for \( \sigma^{-1}GW_{2D} \) given by

\[
C(\{p, \tilde{q}\}; \{\tilde{p}, \tilde{\tilde{q}}\}) = \delta_{p,\tilde{p}} \delta_{q,\tilde{\tilde{q}}} / (ap + b(q_1^2 + q_2^2) + \mu).
\] (86)
We must show that this model is again super-renormalizable using a momentum cut-off and the recipe by now used.

Bounding the propagator in a slice gives

\[ C_i((p, q); (\tilde{p}, \tilde{q})) \leq KM^{-i}e^{-M^{-1}(a p + b(q_1^2 + q_2^2) + \mu)}. \]  

(87)

The multi-scale analysis leads us to consider two types of faces. One labeled by \(ap\) and a pair of faces always labeled by \(\tilde{q}\). The closed face amplitude evaluation associated with these new faces yields:

\[ \sum_{q_1, q_2} e^{-\delta M^{-1}b(q_1^2 + q_2^2)} = \frac{\pi}{4\delta^2} M^i(1 + O(M^{-i/2})). \]  

(88)

In this specific instance, the fact that the number of faces in \(q\)-sector is increasing is merely compensated by the fact that each face will be associated with a less divergent factor of \(M^{-1/2}\). The ensuing power counting is the same as the one determined by (82), yielding the same type of divergent graphs involved only in the mass renormalization. Thus, the \(\sigma^{-1}GW_{2D}\) is super-renormalizable.

Discussing the possibility of merging wave function couplings, since the \(\sigma^{-1}GW_{2D}\) model is super-renormalizable with only mass renormalization, there is actually no point to merge or not couplings \(a\) and \(b\). A symmetric model will be however the one which has two tadpoles with exactly the same amplitude. This can be provided by the identification \(a = 4b^2/\pi\) in (85).

Applying now \(\sigma^{-1}\) in the \(p\)-sector, we can infer that the following rank 4 GW model which will be referred to as \(\sigma^{-2}GW_{2D}\):

\[ S_{\sigma^{-2}GW_{2D}} = \frac{1}{2} \sum_{\vec{p}, \vec{q} N^2} \tilde{\phi}_{\vec{p}, \vec{q}} \left[ a(p_1^2 + p_2^2) + b(q_1^2 + q_2^2) + \mu \right] \phi_{\vec{p}, \vec{q}} + \frac{\lambda}{4} \sum_{\vec{m}, \vec{n}, \vec{p}, \vec{q} N^2} \tilde{\phi}_{\vec{m}, \vec{n}} \phi_{\vec{m}, \vec{n}} \tilde{\phi}_{\vec{p}, \vec{q}} \phi_{\vec{p}, \vec{q}}. \]

(89)

The proof that this model is super-renormalizable can be easily performed according to the previous case. The interesting point is that doubling the faces in each sector \(\vec{p}\) or \(\vec{q}\) is still controlled by the fact each of the new face amplitude is less divergent and behaves like \(M^{1/2}\). This maintains the balance and allows us to recover the same super-renormalizable power counting theorem.

It can be asked the continuum models underlying (73), (79), (85) and (89) and their relation to noncommutative geometry. At this point, an answer to that question is not clear. One can investigate the particular forms of the propagators which might lead to other interesting kinetic terms extending the ordinary \((p, x)\)-duality which has led to the control of UV/IR mixing. However, as explained at the beginning, these actions might be useful in another context of nonlocal field theories called Tensorial Group Field Theory (TGFT) \([3]\) different from noncommutative field theory on Moyal spaces. Indeed, in [7], a rank 4 tensor model extending the above \(T\) tensor model has been proved to be renormalizable at all order of perturbation theory. Fields \(\varphi : U(1)^4 \rightarrow \mathbb{C}\) can be viewed as rank four tensors \(\varphi_{p_1, p_2, p_3, p_4}, p_i \in \mathbb{Z}\). The kinetic part of this model is given by closely related to the kinetic part of (89). However, the interactions of these two models are different. In [7], one type of interaction is cyclic (hence can be recast in a matrix form) and another cannot be recast in terms of matrix trace. This makes this particular higher rank TGFTs non trivial with this respect but definitely susceptible to simplified using the above analysis.

### B. Families of renormalizable tensor models

By iterating the procedure, we can generate three different families of models related either to the GW models or to the \(T3\) model. We establish that, for the three models,

\[ T3 \text{ Class: } \cdots \rightarrow \sigma^{-n}T3 \rightarrow \cdots \rightarrow \sigma^{-1}T3 \rightarrow T3 \rightarrow \sigma T3; \]

\[ GW_{4D} \text{ Class: } \cdots \rightarrow \sigma^{-n}GW_{4D} \rightarrow \cdots \rightarrow \sigma^{-2}GW_{4D} \rightarrow \sigma^{-1}GW_{4D} \rightarrow GW_{4D} \rightarrow \sigma GW_{4D} \rightarrow \sigma^2GW_{4D}; \]

\[ GW_{2D} \text{ Class: } \cdots \rightarrow \sigma^{-n}GW_{2D} \rightarrow \cdots \rightarrow \sigma^{-2}GW_{2D} \rightarrow \sigma^{-1}GW_{2D} \rightarrow GW_{2D}. \]

(90)

Note that each arrow might lead to different theories according to the choice of indices on which the reduction or extension are performed. For instance, \(T3 \rightarrow \sigma T3\) leads to a unique model whereas \(GW_{4D} \rightarrow \sigma GW_{4D} \rightarrow \sigma^2GW_{4D}\) leads to two models and \(\sigma^{-2}GW_{2D} \leftrightarrow \sigma^{-1}GW_{2D} \leftrightarrow GW_{2D}\) leads as well to two models. A way to classify all these models might be to consider as belonging to the same family or class those having a common and initial matrix model.

We claim that all models issued from \(GW_{4D}\) and \(\sigma T3\) are just renormalizable and all models from \(GW_{2D}\) are super-renormalizable. The justification of this has been in fact already established. Let us formalize that proof in full generality for the sake of clarity.
Let $G$ be a connected graph in any of the above model $\sigma^n(\cdot)$. Call $\sigma G$ and $\sigma^{-1} G$ the graphs corresponding to $G$ in $\sigma^{n+1}(\cdot)$ and $\sigma^{n-1}(\cdot)$, respectively. Both are uniquely defined by $G$. $\sigma G$ is obtained from $G$ after merging two faces corresponding to the collapse of two collated (or cyclically disposed) strand momenta in one, i.e. $a(p_1^2 + p_2^2) \to ap^{n/2}$.

Meanwhile, $\sigma^{-1} G$ is obtained after splitting of one strand momentum into two cyclic ones in $G$, say $ap^{n} \to a(p_1^2 + p_2^2)$. The proof or our claim rests on the following statement:

**Lemma 1** (Stability of degree of divergence). Let $G$ be a connected graph in any of the above model $\sigma^n(\cdot)$, $\bullet = GW_4D, T3, GW_2D$. Let $\omega_{d,n,\bullet}(G)$ the degree of divergence of $G$ in $\sigma^n(\bullet)$. Then

$$\omega_{d,n,\bullet}(G) = \omega_{d,n\pm 1,\bullet}(\sigma^{\pm 1} G).$$

**(Proof.** Given a graph $G$, the multi-scale analysis yields, uniquely from propagator bounds, the quantity $\prod_{i,k} M^{-L(G_i^k)}$. This contribution is identical for all models. For $\sigma^n(\bullet)$, let us call $F_{int,n}(G)$ the number of faces of $G$. This number divides as

$$F_{int,n}(G) = F_{int,n}(G) + F_{int,n}(G),$$

where $F_{int,n}(G)$ counts uniquely the number of faces associated with a particular strand momentum $p^n$, or two cyclic momenta such that $(p_1^n, p_2^n)$. Once the model is fixed, for a given graph $G$, $F_{int,n}^\alpha$ is known. Applying $\sigma^{\pm 1}$ on $G$, we have

$$F_{int,n}^\alpha(G) = F_{int,n\pm 1}^\alpha(\sigma^{\pm 1} G), \quad F_{int,n}^\alpha(G) = 2^{\mp 1} F_{int,n\pm 1}^\alpha(\sigma^{\pm 1} G).$$

The stability of the power counting theorem results from the following facts: For the process $\sigma^{-1}$ (resp. $\sigma$): the splitting of one closed face in two produces at the same time a reduction of the divergence of the face amplitudes by half (resp. the merging of two faces in one increases by a factor of two the divergence of that latter face). In the initial $\sigma^n(\bullet)$, consider a face with a momentum $p^n$. Then the amplitude of this closed face is, for large $i$,

$$\sum_p e^{-\delta M^{-i}ap^n} = cM^\frac{1}{2}(1 + O(M^{-\frac{1}{2}})),$$

up to some constant $c$. After applying $\sigma^{-1}$ (resp. $\sigma$), we obtain from one face two collated faces (resp. from two collated faces one face) with amplitude

$$\sum_p e^{-\delta M^{-i}ap^n} = c' M^{-i}(1 + O(M^{-\frac{1}{2}})) \quad (\text{resp. } \sum_p e^{-\delta M^{-i}ap^n} = c' M^{-i}(1 + O(M^{-\frac{1}{2}})),

for some constant $c'$. Thus, for any cases, bearing in mind $\lbrack 93 \rbrack$, the optimal bound amplitude gives a degree of divergence, for some constant $\beta$,

$$\omega_{d,n,\bullet}(G) = -L(G^k_i) + \frac{1}{\alpha} F_{int,n}(G^k_i) + \beta F_{int,n}(G^k_i)

= -L(G^k_i) + \frac{1}{\alpha} 2^{\mp 1} F_{int,n\mp 1}(G^k_i) + \beta F_{int,n\mp 1}(G^k_i) = \omega_{d,n\mp 1,\bullet}(\sigma^{\mp 1} G).$$

The above proposition shows that the degree of divergence does not depend on $n$. The proof of renormalizability in a particular class holds because, all the models in that class have an identical power counting theorem leading to an identical list of primitively divergent graphs with the same degree of divergence. The subtraction scheme and renormalization of each diverging $N$-point function can be undertaken in a standard way. Note that for the $\sigma^nGW_4D$ and $\sigma^nT3$ models, one may require to adjust properly the $a$’s and $b$’s in order to have a well defined wave function renormalization if the initial model is defined with a single wave function coupling. Otherwise, one may always insert new wave function couplings after every momentum splitting in order to define as much as wanted wave function renormalizations. This leads us to the following proposition.

**Theorem 3** (Classes of renormalizable models). The models $\sigma^nGW_4D$, $n \in (-\infty, 2]$, $\sigma^nT3$, $n \in (-\infty, 1]$, are just renormalizable at all orders. The models $\sigma^nGW_4D$, $n \in (-\infty, 0]$ are super-renormalizable at all orders.

Being interested in the change of the propagator for these different theories, we have the following table:
allows to have a similar power counting of \( \sigma_{GW} \) of \( p \in \epsilon \end{eqnarray*} R \) in \( \sqrt{\text{D}} \) renormalizations. In the UV regime, this explicit symmetry breaking therefore leads to a peculiar free model with the propagator should possess different wave function couplings \( a \) and \( b \) in order to have well-defined wave function renormalizations. In the UV regime, this explicit symmetry breaking therefore leads to a peculiar free model with kinetic term having vanishing in part. By simple inference, within the set of our basic axioms, we build a toy model corresponding to this kinetic term. The said model turns out to be a super-renormalizable vector model.

As additional insights of the above study, let us comment that, since \( GW_{2D} \) is super-renormalizable \cite{20}, by introducing a set of propagators

\[
C^\epsilon[\{p, q\}; \{\tilde{p}, \tilde{q}\}] = \delta_{p, \tilde{p}} \delta_{q, \tilde{q}}/(ap + bq^{1+\epsilon} + \mu), \quad \epsilon \in [0, 1/2],
\]

we find a continuum of theories interpolating between \( GW_{2D} \) for \( \epsilon = 1/2 \) and \( \sigma T3 \) for \( \epsilon = 0 \), all with the property from being super-renormalizable to being just-renormalizable. Indeed, following step by step our analysis, one finds a divergence degree for a connected graph \( G \) as

\[
\omega_{d, GW_{2D}}(G) \leq \omega_{d}(G) = -L(G) + F_{\text{int}; \epsilon}(G) + \left( \frac{2}{1 + 2\epsilon} \right) F_{\text{int}; \epsilon}^+(G) \leq \omega_{d, T3}(G),
\]

leading us to the fact that only tadpoles with internal face momenta \( p \) or \( q^{1+\epsilon} \) diverge. Pursuing the propagator interpolation and using the propagator

\[
C^{\epsilon'}[\{p, q\}; \{\tilde{p}, \tilde{q}\}] = \delta_{p, \tilde{p}} \delta_{q, \tilde{q}}/(ap^{1+\epsilon'} + bq^\epsilon + \mu), \quad \epsilon' \in [0, 1/2],
\]

we find another continuum of theories being all just-renormalizable leaving from \( \sigma T3 \) at \( \epsilon' = 1/2 \) to \( \sigma^{2} GW_{4D} \) at \( \epsilon' = 0 \). Indeed, in this situation, we have another divergence degree given by

\[
\omega_{d, T3}(G) \leq \omega_{d}^\epsilon(G) = -L(G) + \left( \frac{2}{1 + 2\epsilon'} \right) F_{\text{int}; \epsilon'}^-\epsilon(G) + 2F_{\text{int}; \epsilon'}^+(G) \leq \omega_{d, GW_{4D}}(G).
\]

At any \( \epsilon' \in [0, 1/2] \), \( \omega_{d, GW_{4D}}(G) \) is a upper bound which can be saturated by \( \omega_{d}^\epsilon(G) \). The reasoning becomes the same as for the proof of the renormalizability of \( \sigma T3 \). In any cases, the divergence subtraction scheme will remain the same. The \( T3 \) model is, with this respect, "critical".

One may also ask about the UV behavior of these classes of models. It can be shown that, all the models such that \( ap^{1+\epsilon} + bq^\epsilon + \mu \), for \( \epsilon \in [0, 1/2] \), are asymptotically free in the UV (see the appendix for a proof of this claim) and, at the end-point \( \epsilon = 0 \), the model becomes safe which corresponds, of course, of the well-known asymptotic safeness of the GW model in 4D \cite{22, 23}.

Finally, it would be interesting to provide a space-time representation to all these matrix/tensor theories, the same way that the GW model is defined by the action \cite{69} as a field theory living on non-commutative \( \mathbb{R}^{4} \). Since terms in \( \sqrt{p} \) in the propagator seem slightly awkward to translate as differential operators (introducing another type of nonlocality), it seems more natural to work with the theories with linear propagator, for instance \( p_{1} + p_{2} + q_{1} + q_{2} \).
for $GW_{4D}$ and $a p + b(q_1 + q_2)$ for $T_3$. Considering $T_3$, one can write it naturally as a group field theory [1] on $U(1)^3$ with the kinetic term given by the sum of the derivative with respect to each coordinate. Written as such, we lose a priori the relation with noncommutative field theory. However, one could similarly write the $GW_{4D}$ model as a group field theory on $U(1)^4$. From this perspective, it seems interesting to investigate in the future the relationship between non-commutative field theories of the Moyal-type and group field theories. A possible approach could be to push further the relation between the Moyal star-product and the non-commutativity based on group manifolds as investigated in [31].

ACKNOWLEDGEMENTS

Discussions with Razvan Gurau are gratefully acknowledged. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Research and Innovation.

APPENDIX: ONE-LOOP $\beta$-FUNCTION OF THE $(ap^{1+\epsilon} + bq^{1/2})$-MODEL

We prove in this appendix that all the models with propagators of the form $ap^{1+\epsilon} + bq^{1/2}$, $\epsilon \in [0, 1/2]$ are asymptotically free in the UV. The calculation of the $\beta$-function is made for $\epsilon = 1/2$ corresponding to the $\sigma T_3$ model. For the remaining models, the proof is totally similar.

The $\beta$-function of a $\phi^4$ theory is generally encoded in the ratio

$$\lambda_{\text{ren}} = -\frac{\Gamma_4(0, 0, 0, 0)}{Z^2},$$

(A.1)

where $\lambda_{\text{ren}}$ is the renormalized coupling (and so $\lambda$ stands for the bare coupling). $\Gamma_4(m, n, p, q)$ is the sum of amputated 1PI four-point functions truncated at one-loop and which should be computed at zero external momenta in [A.1]. $Z$ is the wave function renormalization which should involve the subleading log-divergent term obtained after the Taylor expansion of the self-energy $\Sigma$ which is the sum of the 1PI amputated two-point functions truncated at one-loop.

Coming back to our present model, $\Sigma$ involves two types of contributions called tadpoles “up” $T^+$ and “down” $T^-$ (see Fig.7).

![Diagram](image)

**FIG. 7.** Tadpoles up $T^+$ and down $T^-$ and four-point graph $F$.

We have at one-loop

$$\Sigma = A_{T^+} + A_{T^-} = \frac{(-\lambda)}{2} [2 \sum_k \frac{1}{am + b\sqrt{k} + \mu} + 2 \sum_k \frac{1}{ak + b\sqrt{n} + \mu}] + O(\lambda^3).$$

(A.2)

As noticed, $\Sigma = \Sigma(m, \sqrt{n})$, so that evaluating $\partial_m \Sigma$ or $\partial_{\sqrt{n}} \Sigma$, we only collect the log-divergent part contributing to the wave function renormalization and this is

$$Z = 1 - \frac{1}{a} \partial_m \Sigma |_{m=0} = 1 - \lambda S + O(\lambda^2), \quad S = \sum_k \frac{1}{(b\sqrt{k} + \mu)^2}. \quad (A.3)$$

To $\Gamma_4$ contribute only a unique divergent four-point function $F$ of the form given by Fig.7. We have, still at one-loop,

$$\Gamma_4(m, n, p, q) = -\lambda + \frac{1}{2} \frac{\lambda^2}{2 \cdot 2 \cdot 2} \sum_k \frac{1}{(am + b\sqrt{k} + \mu)(ap + b\sqrt{k} + \mu)} + O(\lambda^3).$$

(A.4)

Note that, at this level, the above model differs from the GW 4D model since more graphs contribute now to the $\Gamma_4$ function. This entails a combinatorial factor twice greater [24].
We are in position to compute the $\beta$-function:

\[
\chi_{\text{ren}} = -\frac{\Gamma_4(0, 0, 0, 0)}{Z^2} = -\frac{(-\lambda + \lambda^2S + O(\lambda^3))}{(1 - \lambda S + O(\lambda^2))^2} = \lambda + \lambda^2S + O(\lambda^3). \tag{A.5}
\]

Therefore $\beta = +1$ and $\sigma T3$ is asymptotically free in the UV as expected from [5]. In a similar way, all theories with propagator $\alpha p^{4+\epsilon} + b q^2$, $\epsilon \in [0, 1/2]$ will be asymptotically free (once again, by the same reasons, $T^-$ should be dropped because $\sum_k 1/(ak^{4+\epsilon} + \mu)^2$ is convergent and the same contributions of the four-point function are still convergent and should be neglected) whereas, at the end-point $\epsilon_0 = 0$, the GW$_{4D}$ model becomes safe.
[30] R. Gurau, “Topological Graph Polynomials in Colored Group Field Theory,” Annales Henri Poincare 11, 565 (2010) [arXiv:0911.1945 [hep-th]].

[31] M. Dupuis, F. Girelli and E. R. Livine, “Spinors and Voros star-product for Group Field Theory: First Contact,” arXiv:1107.5693 [gr-qc].