ON THE MICROSCOPIC SPACETIME CONVEXITY PRINCIPLE FOR FULLY NONLINEAR PARABOLIC EQUATIONS II:
SPACETIME QUASICONCAVE SOLUTIONS

CHUANQIANG CHEN*

Department of Applied Mathematics
Zhejiang University of Technology
Hangzhou, 310023, Zhejiang Province, China

(Communicated by Alessio Figalli)

Abstract. In [13], Chen-Ma-Salani established the strict convexity of spacetime level sets of solutions to heat equation in convex rings, using the constant rank theorem and a deformation method. In this paper, we generalize the constant rank theorem in [13] to fully nonlinear parabolic equations, that is, establish the corresponding microscopic spacetime convexity principles for spacetime level sets. In fact, the results hold for fully nonlinear parabolic equations under a general structural condition, including the $p$-Laplacian parabolic equations ($p > 1$) and some mean curvature type parabolic equations.

1. Introduction. This paper is devoted to the microscopic spacetime convexity principle for the second fundamental forms of the spatial and spacetime level sets of the solutions to fully nonlinear parabolic equations. In this paper, we consider the spatial convexity and the spacetime convexity of the level sets of the spacetime quasiconcave solutions to the heat equation. A continuous function $u(x,t)$ in $\Omega \times (0,T]$ is called spacetime quasiconcave if the spacetime superlevel sets $\{(x,t) \in \Omega \times (0,T) | u(x,t) \geq c\}$ are convex for each constant $c$.

Spacetime convexity is a basic geometric property of the solutions of parabolic equations. In [5, 6, 7], Borell used the Brownian motion to study certain spacetime convexities of the solutions of diffusion equations and the level sets of the solution to a heat equations with Schrödinger potential. Ishige-Salani introduced some notions of parabolic quasiconcavity in [18, 19] and parabolic power concavity in [20], which are some kinds of spacetime convexity. In [18, 19, 20], they studied the corresponding parabolic boundary value problems using the convex envelope method, which is a macroscopic method. At the same time, Hu-Ma [17] established a constant rank theorem for the space-time Hessian of space-time convex solutions to the heat equation, which is the microscopic method. Chen-Hu [12] generalized the microscopic spacetime convexity principle to fully nonlinear parabolic equations. Recently, Chen-Ma-Salani [13] established the strict convexity of spacetime

2010 Mathematics Subject Classification. Primary: 35K10; Secondary: 35B99.
Key words and phrases. Spacetime level set, constant rank theorem, spacetime quasiconcave solution.

The author is supported by the National Natural Science Foundation of China (NO. 11301497 and NO. 11471188).

* Corresponding author: Chuanqiang Chen.
level sets of solutions to the heat equation in convex rings, using the constant rank theorem and a deformation process. In this paper, we generalize the constant rank theorem in [13] to fully nonlinear parabolic equations. And the results hold for fully nonlinear parabolic equations under a general structural condition, including the $p$-Laplacian parabolic equations ($p > 1$) and some mean curvature type parabolic equations. As in [13], this approach can be used to establish some spacetime convexity of the solutions of some parabolic equations in convex rings, by combining a deformation process.

The convexity of the level sets of the solutions to elliptic partial differential equations has been studied extensively. For instance, Ahlfors [1] contains the well-known result that level curves of Green function on simply connected convex domain in the plane are the convex Jordan curves. In 1956, Shiffman [29] studied the minimal annulus in $\mathbb{R}^3$ whose boundary consists of two closed convex curves in parallel planes $P_1, P_2$. He proved that the intersection of the surface with any parallel plane $P$, between $P_1$ and $P_2$, is a convex Jordan curve. In 1957, Gabriel [15] proved that the level sets of the Green function on a 3-dimensional bounded convex domain are strictly convex. In 1977, Lewis [23] extended Gabriel’s result to $p$-harmonic functions in higher dimensions. Caffarelli-Spruck [9] generalized the Lewis [23] results to a class of semilinear elliptic partial differential equations. Motivated by the result of Caffarelli-Friedman [8], Korevaar [22] gave a new proof on the results of Gabriel and Lewis by applying the deformation process and the constant rank theorem of the second fundamental form of the convex level sets of $p$-harmonic function. A survey of this subject is given by Kawohl [21]. For more recent related extensions, please see the papers by Bianchini-Longinetti-Salani [4], Bian-Guan [2], Xu [31] and Bian-Guan-Ma-Xu [3].

For the convexity of spacetime level sets, Ishige-Salani [18, 19] introduced some notions of parabolic quasiconcavity, and studied the corresponding parabolic boundary value problems using the convex envelope method. Recently, Chen-Ma-Salani made a breakthrough in the strict convexity of the heat equation in [13] using the constant rank theorem method.

There is also an extensive literature on the curvature estimates of the level sets of the solutions to elliptic partial differential equations, see [28], [25], [26], [27], [10], [16] and references therein.

Let us introduce some notations.

**Definition 1.1.** For each $\theta \in S^{n-1}$, denote $\theta^\perp$ the linear subspace in $\mathbb{R}^n$ which is orthogonal to $\theta$. Define $S^-(\theta)$ to be the class of $n \times n$ symmetric real matrices which are negative definite on $\theta^\perp$. Denote $S^{0-} (\theta)$ the subclass of $S^-(\theta)$ of matrices that have $\theta$ as eigenvector with corresponding null eigenvalue. For any $b \in \mathbb{R}^n$ with $s = \langle b, \theta \rangle > 0$, define

$$B^b_{\theta} (\Upsilon) = \left\{ B \in S^{n+1} | B = \begin{pmatrix} \tilde{B} & b^T \\ b & \chi \end{pmatrix} \text{ with } \tilde{B} \in S^{0-} (\theta) \cap \Upsilon, \chi \in \mathbb{R} \right\},$$

where $S^{n+1}$ denote the space of real symmetric $(n + 1) \times (n + 1)$ matrices, and $\Upsilon \subset S^n$ be an open set.

Denote $J = (I_n | 0)$ the $n \times (n + 1)$ matrix, where $I_n$ is the $n \times n$ identity matrix and $0$ is the null vector in $\mathbb{R}^n$.

In this paper, we consider the spacetime quasiconcave solution to fully nonlinear parabolic equation as follows,
Theorem 1.2. Suppose $u$ is a spacetime quasiconcave solution to fully nonlinear parabolic equation (1), and $F$ satisfies conditions (2)-(4). Then the second fundamental form of spacetime level sets $\Sigma^c = \{(x,t) \in \Omega \times (0,T) \mid u(x,t) = c\}$ has the same constant rank in $\Omega$ for each fixed $t \in (0,T)$. Moreover, let $l(s)$ be the minimal rank of the second fundamental form in $\Omega$, then $l(s) \leq l(t)$ for all $0 < s \leq t \leq T$.

For the study of the spacetime level sets of fully nonlinear equation, we should consider the spatial level sets first. Suppose $u$ is the spacetime quasiconcave solution to fully nonlinear parabolic equation (1), then $u$ is also spatial quasiconcave, that is the spatial level sets $\Sigma^c = \{x \in \Omega \mid u(x,t) = c\}$ are all convex. And we get the following constant rank theorem for the second fundamental form of the spatial level sets.

Theorem 1.3. Suppose $u \in C^{3,1}((\Omega \times (0,T)))$ is a spacetime quasiconcave solution to fully nonlinear parabolic equation (1), and $F$ satisfies conditions (2)-(4). Then the second fundamental form of spatial level sets $\Sigma^c = \{x \in \Omega \mid u(x,t) = c\}$ has the same constant rank in $\Omega$ for each fixed $t \in (0,T)$. Moreover, let $l(t)$ be the minimal rank of the second fundamental form in $\Omega$, then $l(s) \leq l(t)$ for all $0 < s \leq t \leq T$.

As it is well known, one needs to choose a suitable coordinate system to simplify the calculations in the proof of the constant rank theorem. In [13], the proof for the heat equation is based on a coordinate system such that the spatial second fundamental form $a(x,t)$ (see (11)) is diagonalized at each point. In this paper, we generalize the constant rank theorem in [13] to fully nonlinear parabolic equations, and we give a more technical proof under the coordinate system such that the spacetime second fundamental form $\hat{a}(x,t)$ (see (15)) is diagonalized at each point. As in [17, 12] and [11], the key difficulties of two calculations are the same, and the processes of the two proofs are also the same. So the corresponding proof holds for fully nonlinear equations based on the coordinate system such that the spatial second fundamental form $a(x,t)$ is diagonalized at each point, and the calculations must be more complicate than [13].

Remark 1. In fact, in the proof of Theorem 1.3, we just need a weaker structural condition as follows instead of (3),

$$F(s^{-1}JB^{-1}J^T, s^{-1}\theta, u, x, t) \text{ is locally concave in } (B, x)$$

(5)
for fixed \((\theta, u) \in S^{n-1} \times \mathbb{R}\). But the condition that \(u\) is spacetime quasiconcave is necessary, and it is the main difference between Theorem 1.3 and the result in [14]. That is, if \(u\) is spacetime quasiconcave, the constant rank theorem for spatial level sets holds for the parabolic equations with (5). Otherwise, if \(u\) is spatial quasiconcave, Chen-Shi [14] established the constant rank theorem for spatial level sets holds for the parabolic equations with a totally different structural condition as follows

\[ F(s^2 A, s \theta, u, t) \text{ is locally concave in } (A, s), \quad \text{for fixed } (\theta, u, t) \in S^{n-1} \times \mathbb{R} \times \mathbb{R}_+, \]

where \(A \in S^n\) and \(s > 0\) do not depend on \(A\).

In fact, the \(p\)-Laplacian operator and the mean curvature operators, that is, the parabolic equations

\begin{align*}
    u_t &= \text{div} \left( |\nabla u|^{p-2} \nabla u \right), \quad p > 1, \quad (6) \\
    u_t &= \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right), \quad (7) \\
    u_t &= (1 + |\nabla u|^2)^{\frac{3}{2}} \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right), \quad (8)
\end{align*}

do not satisfy the structural condition (3) or (5). But we have the following theorem.

**Theorem 1.4.** Theorem 1.2 and Theorem 1.3 holds for the spacetime quasiconcave solutions to the parabolic equations (6), (7) and (8).

**Remark 2.** Theorem 1.3 can be looked as a parabolic version for Theorem 1.1 in [3].

**Remark 3.** The microscopic spacetime convexity principle can be used to establish the spacetime convexity of the solutions of some parabolic equations in convex rings, by combining the deformation process. For example, Chen-Ma-Salani [13] consider the heat equation in convex rings, and get the strict convexity of the spacetime level sets, with some compatible conditions on the initial data and the convex rings.

The rest of the paper is organized as follows. Section 2 contains some preliminaries. In Section 3, we prove the constant rank theorem of the spatial second fundamental form, that is Theorem 1.3 and the corresponding part of Theorem 1.4. The constant rank theorem of the spacetime second fundamental form is proved in Section 4, including Theorem 1.2 and the corresponding part of Theorem 1.4.

2. Preliminaries. In this section, we will give some preliminaries.

First, we introduce the definitions of spatial quasiconcave and spacetime quasiconcave.

**Definition 2.1.** A continuous function \(u(x, t)\) on \(\Omega \times (0, T]\) is called **spatial quasiconcave** if its superlevel sets \(\{x \in \Omega | u(x, t) \geq c\}\) are convex for each constant \(c\) and any fixed \(t \in (0, T]\). And \(u(x, t)\) is called **spacetime quasiconcave** if its superlevel sets \(\{(x, t) \in \Omega \times (0, T] | u(x, t) \geq c\}\) are convex for each constant \(c\).

In the following, we always assume \(\nabla u = (u_1, \cdots, u_n)\) is the spatial gradient of \(u\) and \(Du = (u_1, \cdots, u_n, u_t)\) the spacetime gradient.
2.1. Spatial level sets and the spatial second fundamental form. Suppose \( u(x, t) \in C^2(\Omega \times (0, T)) \), and \( u_n \neq 0 \) for any fixed \((x, t) \in \Omega \times (0, T)\). It follows that the upward inner normal direction of the spatial level sets \( \Sigma^c = \{ x \in \Omega | u(x, t) = c \} \) is

\[
\vec{\nu} = \frac{|u_n|}{|\nabla u|u_n} (u_1, u_2, \cdots, u_{n-1}, u_n),
\]

(9)

where \( \nabla u = (u_1, u_2, \cdots, u_{n-1}, u_n) \) is the spatial gradient of \( u \).

The second fundamental form \( H \) of the spatial level sets of function \( u \) with respect to the upward normal direction (9) is

\[
b_{ij} = -\frac{|u_n|(u^2_{ij}u_n + u_{jj}u_{i}u_n - u_{nn}u_{ij}u_n - u_{n}u_{ij}u_{jn})}{|\nabla u|u_n^2}, \quad 1 \leq i, j \leq n - 1.
\]

(10)

Set

\[
h_{ij} = u^2_{ij}u_n + u_{nn}u_{ij} - u_{nj}u_{in} - u_{n}u_{ij}u_{jn}, \quad 1 \leq i, j \leq n - 1,
\]

then we may write

\[
b_{ij} = -\frac{|u_n|h_{ij}}{|\nabla u|u_n^2}.
\]

Note that if \( \Sigma^c = \{ x \in \Omega | u(x, t) = c \} \) is locally convex, then the second fundamental form of \( \Sigma^c \) is semipositive definite with respect to the upward normal direction (9). Let \( a(x, t) = (a_{ij}(x, t)) \) be the symmetric Weingarten tensor of \( \Sigma^c = \{ x \in \Omega | u(x, t) = c \} \), then \( a \) is semipositive definite. As computed in [3], if \( u_n \neq 0 \), and the Weingarten tensor is

\[
a_{ij} = -\frac{|u_n|}{|\nabla u|u_n^3} A_{ij}, \quad 1 \leq i, j \leq n - 1,
\]

(11)

where

\[
A_{ij} = h_{ij} - \frac{u_iu_jh_{ij}}{W(1 + W)u_n^2} - \frac{u_ju_ih_{ij}}{W(1 + W)u_n^2} + \frac{u_{ij}u_ku_lh_{kl}}{W^2(1 + W)^2u_n^4}, \quad W = \frac{|\nabla u|}{u_n},
\]

(12)

and Einstein summation convention is used. With the above notations, at the point \((x, t)\) where \( u_n(x, t) = |\nabla u(x, t)| > 0, u_t(x, t) = 0, i = 1, \cdots, n - 1, a_{ij, k} \) is commutative, that is, they satisfy the Codazzi property \( a_{ij, k} = a_{ik, j}, \forall i, j, k \leq n - 1 \).

2.2. Spacetime level sets and the spacetime second fundamental form. Suppose \( u(x, t) \in C^2(\Omega \times (0, T)) \), and \( u_t \neq 0 \) for any fixed \((x, t) \in \Omega \times (0, T)\). It follows that the upward inner normal direction of the spatial level sets \( \hat{\Sigma}^c = \{(x, t) \in \Omega \times (0, T) | u(x, t) = c \} \) is

\[
\vec{\nu} = \frac{|u_t|}{|Du|u_t} (u_1, u_2, \cdots, u_{n-1}, u_n, u_t),
\]

(13)

where \( Du = (u_1, u_2, \cdots, u_{n-1}, u_n, u_t) \) is the spacetime gradient of \( u \).

The second fundamental form \( \hat{H} \) of the spacetime level sets of function \( u \) with respect to the upward normal direction (13) is

\[
\hat{b}_{\alpha\beta} = -\frac{|u_t|(u^2_{\alpha\beta}u_{\alpha\beta} + u_{tt}u_{\alpha\alpha}u_{\beta\beta} - u_{tt}u_{\alpha\alpha}u_{\beta\beta} - u_{tt}u_{\alpha\beta}u_{\alpha\beta})}{|Du|u_t^2}, \quad 1 \leq \alpha, \beta \leq n.
\]

(14)

Set

\[
\hat{h}_{\alpha\beta} = u^2_{\alpha\beta}u_{\alpha\beta} + u_{tt}u_{\alpha\alpha}u_{\beta\beta} - u_{tt}u_{\alpha\alpha}u_{\beta\beta} - u_{tt}u_{\alpha\beta}u_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n,
\]

then we may write

\[
\hat{b}_{\alpha\beta} = -\frac{|u_t|\hat{h}_{\alpha\beta}}{|Du|u_t^2}.
\]
Note that if $\Sigma^c = \{(x, t) \in \Omega \times (0, T)|u(x, t) = c\}$ is locally convex, then the second fundamental form of $\Sigma^c$ is semipositive definite with respect to the upward normal direction (13). Let $\hat{a}(x, t) = (\hat{a}_{ij}(x, t))$ be the symmetric Weingarten tensor of $\Sigma^c = \{(x, t) \in \Omega \times (0, T)|u(x, t) = c\}$, then $\hat{a}$ is semipositive definite. If $u_t \neq 0$, and the Weingarten tensor is

$$\hat{a}_{\alpha\beta} = -\frac{|u_t|}{D|u_t|^2} \hat{A}_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n,$$

where

$$\hat{A}_{\alpha\beta} = \hat{h}_{\alpha\beta} - \frac{u_{\alpha} u_{\gamma} \hat{h}_{\beta\gamma}}{W(1 + W) u_t^2} - \frac{u_{\beta} u_{\gamma} \hat{h}_{\alpha\gamma}}{W(1 + W) u_t^2} + \frac{u_{\alpha} u_{\beta} u_{\gamma} u_{\eta} \hat{h}_{\gamma\eta}}{W^2 (1 + W)^2 u_t^4}, \quad \hat{W} = \frac{|Du|}{|u_t|}. \quad (16)$$

With the above notations, at the point $(x, t)$ where $u_t(x, t) > 0$, $u_n(x, t) = |\nabla u(x, t)| > 0$, $u_i(x, t) = 0, i = 1, \ldots, n - 1$, we get

$$1 - \frac{u_n^2}{W(1 + W) u_t^2} = \frac{W u_t^2 + W^2 u_t^2 - u_n^2}{W(1 + W) u_t^2} = \frac{1}{W}. \quad (17)$$

So

$$\hat{A}_{\alpha\beta} = \hat{h}_{\alpha\beta} = u_t^2 u_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n - 1; \quad (18)$$

$$\hat{A}_{\alpha n} = \hat{h}_{\alpha n} = \frac{u_n^2 h_{\alpha n}}{W(1 + W) u_t^2} = \frac{1}{W} h_{\alpha n} = \frac{1}{W} [u_t^2 u_{\alpha n} - u_t u_n u_{\alpha t}], \quad 1 \leq \alpha \leq n - 1; \quad (19)$$

$$\hat{A}_{nn} = \hat{h}_{nn} - 2 \frac{u_n^2 h_{nn}}{W(1 + W) u_t^2} + \frac{u_n^4 h_{nn}}{W^2 (1 + W)^2 u_t^4} = \frac{1}{W^2} \hat{h}_{nn}$$

$$= \frac{1}{W^2} [u_n^2 u_{nn} + u_n^2 u_{nt} - 2 u_t u_n u_{nt}]. \quad (20)$$

Also, at any point $(x, t)$, we can translate the spacetime coordinate systems. When we choose the coordinates $y = (y_1, \ldots, y_n, y_{n+1})$ as a new spacetime coordinates, such that $u_{y_{n+1}} > 0$, the Weingarten tensor is

$$\bar{a}_{\alpha\beta} = -\frac{|u_{y_{n+1}}|}{D|u_{y_{n+1}}|^2} \bar{A}_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n,$$  

where

$$\bar{A}_{\alpha\beta} = \bar{h}_{\alpha\beta} - \frac{u_{\alpha} u_{y_{\gamma}} \bar{h}_{\beta\gamma}}{W(1 + W) u_{y_{n+1}}^2} - \frac{u_{\beta} u_{y_{\gamma}} \bar{h}_{\alpha\gamma}}{W(1 + W) u_{y_{n+1}}^2} + \frac{u_{\alpha} u_{\beta} u_{y_{\gamma}} u_{y_{\eta}} \bar{h}_{\gamma\eta}}{W^2 (1 + W)^2 u_{y_{n+1}}^4}, \quad (22)$$

$$\bar{W} = \frac{|D|u_{y_{n+1}}|}{|u_{y_{n+1}}|}$$

and

$$\bar{h}_{\alpha\beta} = u_{y_{n+1}}^2 u_{\alpha y_{\beta}} + u_{y_{n+1}} u_{y_{\alpha}y_{\beta}} u_{y_{\gamma}} - u_{y_{n+1}} u_{y_{\alpha}y_{\beta}} u_{y_{\gamma}} u_{y_{\eta}} - u_{y_{n+1}} u_{y_{\alpha}y_{\eta}} u_{y_{\beta}y_{\gamma}}, \quad (23)$$

for $1 \leq \alpha, \beta \leq n$. With the above notations, at the point $(x, t)$ with the new coordinates $y$ such that $u_y = 0$ for any $1 \leq i \leq n$ and $u_{y_{n+1}} = |D|u| > 0$, we get

$$\bar{A}_{\alpha\beta} = \bar{h}_{\alpha\beta} = u_{y_{n+1}}^2 u_{\alpha y_{\beta}}, \quad 1 \leq \alpha, \beta \leq n, \quad (24)$$
2.3. Elementary symmetric functions. In this subsection, we recall the definition and some basic properties of elementary symmetric functions, which could be found in [24].

**Definition 2.2.** For any \( k = 1, 2, \cdots, n \), we set
\[
\sigma_k(\lambda) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}, \quad \text{for any } \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n. \tag{25}
\]

We also set \( \sigma_0 = 1 \) and \( \sigma_k = 0 \) for \( k > n \).

We denote by \( \sigma_k(\lambda|i) \) the symmetric function with \( \lambda_i = 0 \) and \( \sigma_k(\lambda|ij) \) the symmetric function with \( \lambda_i = \lambda_j = 0 \).

We need the following standard formulas of elementary symmetric functions.

**Proposition 1.** Let \( \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n \) and \( k = 0, 1, \cdots, n \), then
\[
\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i\sigma_{k-1}(\lambda|i), \quad \forall 1 \leq i \leq n,
\]
\[
\sum_{i=1}^{n} \lambda_i\sigma_{k-1}(\lambda|i) = k\sigma_k(\lambda),
\]
\[
\sum_{i=1}^{n} \sigma_k(\lambda|i) = (n-k)\sigma_k(\lambda).
\]

The definition can be extended to symmetric matrices by letting \( \sigma_k(W) = \sigma_k(\lambda(W)) \), where \( \lambda(W) = (\lambda_1(W), \lambda_2(W), \cdots, \lambda_n(W)) \) are the eigenvalues of the symmetric matrix \( W \). We also denote by \( \sigma_k(W|i) \) the symmetric function with \( W \) deleting the \( i \)-row and \( i \)-column and \( \sigma_k(W|ij) \) the symmetric function with \( W \) deleting the \( i,j \)-rows and \( i,j \)-columns. Then we have the following identities.

**Proposition 2.** Suppose \( W = (W_{ij}) \) is diagonal, and \( m \) is a positive integer, then
\[
\frac{\partial \sigma_m(W)}{\partial W_{ij}} = \begin{cases} 
\sigma_{m-1}(W|i), & \text{if } i = j, \\
0, & \text{if } i \neq j.
\end{cases} \tag{26}
\]

and
\[
\frac{\partial^2 \sigma_m(W)}{\partial W_{ij}\partial W_{kl}} = \begin{cases} 
\sigma_{m-2}(W|ik), & \text{if } i = j, k = l, i \neq k, \\
-\sigma_{m-2}(W|ik), & \text{if } i = l, j = k, i \neq j, \\
0, & \text{otherwise}.
\end{cases} \tag{27}
\]

To study the rank of the spacetime second fundamental form \( \hat{a} \), we need the following simple lemma.

**Lemma 2.3.** Suppose \( \hat{a} \geq 0 \), and \( l = \text{Rank}\{\hat{a}(x_0,t_0)\} \leq n - 1 \), and \( \hat{a}_{\alpha\beta} \) is diagonal at \( (x_0,t_0) \) with \( \hat{a}_{11} \geq \hat{a}_{22} \geq \cdots \geq \hat{a}_{n-1n-1} \), then at \( (x_0,t_0) \), there is a positive constant \( C_0 \) such that

**Case 1.**
\[
\hat{a}_{11} \geq \cdots \geq \hat{a}_{l-1l-1} \geq C_0, \quad \hat{a}_{ll} = \cdots = \hat{a}_{n-1n-1} = 0,
\]
\[
\hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{ii}^2}{\hat{a}_{ii}} \geq C_0, \quad \hat{a}_{in} = 0, \quad l \leq i \leq n - 1.
\]
Case 2.

\[
\hat{a}_{11} \geq \cdots \geq \hat{a}_{ll} \geq C_0, \quad \hat{a}_{l+1,1+1} = \cdots = \hat{a}_{n-1,n-1} = 0,
\]

\[
\hat{a}_{nn} = \sum_{i=1}^{l-1} \frac{\hat{a}_{ii}^2}{\hat{a}_{ii}}, \quad \hat{a}_{ii} = 0, \quad l + 1 \leq i \leq n - 1.
\]

**Proof.** Set \( M = \left( \hat{a}_{\alpha \beta}(x_0, t_0) \right)_{(n-1) \times (n-1)} \) \( = \text{diag}(\hat{a}_{11}, \hat{a}_{22}, \cdots, \hat{a}_{n-1,n-1}) \geq 0 \) and we can assume \( \text{Rank}\{M\} = k \) at \( (x_0, t_0) \), then we can obtain \( k = l - 1 \) or \( k = l \). Otherwise, if \( k < l - 1 \), we know

\[
\hat{a}_{l-1,l-1} = \cdots = \hat{a}_{n-1,n-1} = 0 \quad \text{at} \quad (x_0, t_0),
\]

and from \( \hat{a}(x_0, t_0) \geq 0 \), we get

\[
\hat{a}_{l-1,l-1} = \cdots = \hat{a}_{n-1,n-1} = 0 \quad \text{at} \quad (x_0, t_0).
\]

So \( \text{Rank}\{\hat{a}\} \leq l - 1 \), contradiction. If \( k > l \), we have \( l = \text{Rank}\{\hat{a}\} \geq \text{Rank}\{M\} = k \geq l + 1 \) \( \text{at} \quad (x_0, t_0) \).

This is impossible.

For \( k = l - 1 \), we know at \( (x_0, t_0) \)

\[
\hat{a}_{11} \geq \cdots \geq \hat{a}_{l-1,l-1} > 0, \quad \hat{a}_{ll} = \cdots = \hat{a}_{n-1,n-1} = 0,
\]

and due to \( \hat{a}(x_0, t_0) \geq 0 \), we get

\[
\hat{a}_{ll} = \cdots = \hat{a}_{n-1,n-1} = 0.
\]

Since \( \text{Rank}\{\hat{a}\} = l \), then \( \sigma_l(\hat{a}) > 0 \). Direct computation yields

\[
\sigma_l(\hat{a}) = \hat{a}_{nn}\sigma_{l-1}(M) - \sum_{i=1}^{l-1} \hat{a}_{ni}\hat{a}_{in}\sigma_{l-2}(M|i) = \sigma_{l-1}(M)[\hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{ii}^2}{\hat{a}_{ii}}] > 0,
\]

so we have

\[
\hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{ii}^2}{\hat{a}_{ii}} > 0.
\]

This is Case 1.

For \( k = l \), we know at \( (x_0, t_0) \)

\[
\hat{a}_{11} \geq \cdots \geq \hat{a}_{ll} > 0, \quad \hat{a}_{l+1,l+1} = \cdots = \hat{a}_{n-1,n-1} = 0,
\]

and due to \( \hat{a}(x_0, t_0) \geq 0 \), we get

\[
\hat{a}_{l+1,l+1} = \cdots = \hat{a}_{n-1,n-1} = 0.
\]

Since \( \text{Rank}\{\hat{a}\} = l \), then \( \sigma_{l+1}(\hat{a}) = 0 \). Direct computation yields

\[
\sigma_{l+1}(\hat{a}) = \hat{a}_{nn}\sigma_{l}(M) - \sum_{i=1}^{l} \hat{a}_{ni}\hat{a}_{in}\sigma_{l-1}(M|i) = \sigma_{l}(M)[\hat{a}_{nn} - \sum_{i=1}^{l} \frac{\hat{a}_{ii}^2}{\hat{a}_{ii}}] = 0,
\]

so we have

\[
\hat{a}_{nn} - \sum_{i=1}^{l} \frac{\hat{a}_{ii}^2}{\hat{a}_{ii}} = 0.
\]

This is Case 2.\( \square \)
2.4. Structural conditions (3) and (5). Now we discuss the structural conditions (3) and (5).

First, we introduce the following set to study the matrix $B$.

**Definition 2.4.** For each $\theta \in S^{n-1}$, define $\mathcal{A}_\theta^-$ as follows

$$
\mathcal{A}_\theta^-(\Upsilon) = \left\{ A \in S^{n+1} | A = \begin{pmatrix}
\bar{A} \\
\mu \theta^T \\
0
\end{pmatrix}
\ text{with}
\bar{A} \in S^-_n(\theta) \cap \Upsilon, \mu > 0 \right\}. \hspace{1cm} (28)
$$

Properties of $\mathcal{A}_\theta^-$, $\mathcal{B}_\theta^-$ and their relationship have been studied in [4]. In particular, if $\theta = (0, \cdots, 0, 1)$,

$$
B = \begin{pmatrix}
0 & \times \\
\times & \cdots & \times \\
\times & \cdots & \times & s & \times & \mu & 0
\end{pmatrix}
\in \mathcal{B}_\theta^-(\Upsilon), \hspace{1cm} (29)
$$

then

$$
A = B^{-1} = \begin{pmatrix}
\times & 0 \\
\times & \cdots & \times \\
\times & \cdots & \times & \mu & 0
\end{pmatrix}
\in \mathcal{A}_\theta^-(\Upsilon). \hspace{1cm} (30)
$$

where the $(n-1) \times (n-1)$ matrix $(a_{ij})$ is negative definite and can be assumed diagonal, $(a_{ij})$ is the inverse matrix of $(a_{ij})$, $s = B_{n+1,n} = \frac{1}{\theta} > 0$. The values at the positions denoted by “$\times$” which are not important in the calculations.

For any given $V = (X_{ij}), Y, (Z_i), D) \in S^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$, we define a quadratic form

$$
Q^*(V, V) = F^{ij,kl} X_{ij} X_{kl} + 2 F^{ij,ui} \theta_i X_{ij} Y + 2 F^{ij,xk} X_{ij} Z_k + 2 F^{ij,t} X_{ij} D + F^{u,ui} \theta_i Y^2 + 2 F^{u,xk} \theta_k Y Z_l + 2 F^{u,t} Y D + F^{x,x} Z_k Z_l + 2 F^{z,x} Z_k + F^{t,t} D^2 + 6 s F_{ij} X_{ij} Y - 6 s F^{ij} A_{ij} Y^2 + 2 s \sum_{i \in T} \frac{F^{kl}}{A_{ij}} [X_{ik} - 2 A_{ik} Y][X_{il} - 2 A_{il} Y], \hspace{1cm} (31)
$$

where Einstein summation convention is used, the derivative functions of $F$ are evaluated at $(s^{-1} \bar{A}, s^{-1} \theta, u, x, t)$, $T := \{1, 2, \cdots, n-1\}$, and

$$
F^{ij} = \frac{\partial F(\nabla^2 u, \nabla u, u, x, t)}{\partial u_{ij}},
F^{ui} = \frac{\partial F(\nabla^2 u, \nabla u, u, x, t)}{\partial u_k},
F^{u} = \frac{\partial F(\nabla^2 u, \nabla u, u, x, t)}{\partial u},
F^{x} = \frac{\partial F(\nabla^2 u, \nabla u, u, x, t)}{\partial x_k},
F^{t} = \frac{\partial F(\nabla^2 u, \nabla u, u, x, t)}{\partial t}, \text{ etc.}
$$

Through direct calculations, we can get

**Lemma 2.5.** $F$ satisfies the condition (3) if and only if for each $p \in \mathbb{R}^n$

$$
Q^*(V, V) \leq 0, \hspace{0.5cm} \forall \hspace{0.2cm} V = (X_{ij}), (Z_i), D) \in S^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \hspace{1cm} (32)
$$

where the derivative functions of $F$ are evaluated at $(s^{-1} \bar{A}, s^{-1} \theta, u, x, t)$, and $Q^*$ is defined in (31).
The proof of Lemma 2.5 is similar to the discussion in [2], and we omit it.

**Remark 4.** $F$ satisfies the condition (5) if and only if for each fixed $p \in \mathbb{R}^n$, and for any \( \bar{V} = ((X_{ij}), Y,(Z_{ij}), 0) \in \mathcal{S}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \)

\[
Q^*(\bar{V}, \bar{V}) = F^{ij,kl} X_{ij} X_{kl} + 2 F^{ij,ui} \theta_i X_{ij} Y + 2 F^{ij,xi} X_{ij} Z_k + F^{ui,ui} \theta_i Y^2 + 2 F^{ui,xl} \theta_l Z_i + 2 s F^{ui,ki} \theta_k Y^2 + 6 s F^{ij} X_{ij} Y
\]

\[\leq 0,
\]

where the derivative functions of $F$ are evaluated at \((s^{-1}A, s^{-1}T, u, x, t)\), and $Q^*$ is defined in (31). Obviously, the condition (5) is weaker than the condition (3).

**2.5. An auxiliary lemma.** Similarly to the Lemma 2.5 in Bian-Guan[2], we have

**Lemma 2.6.** Suppose \( W(x) = (W_{ij}(x))_{N \times N} \geq 0 \) for every $x \in \Omega \subset \mathbb{R}^n$, and $W_{ij}(x) \in C^{1,1}(\Omega)$, then for every $\mathcal{O} \subset \subset \Omega$, there exists a positive constant $C$ depending only on the dist\{\mathcal{O}, \partial \Omega\} and $\|W\|_{C^{1,1}(\Omega)}$ such that

\[
|\nabla W_{ij}| \leq C(W_{ii} W_{jj})^{\frac{1}{4}},
\]

for every $x \in \mathcal{O}$ and $1 \leq i, j \leq N$.

**Proof.** The same arguments as in the proof of Lemma 2.5 in [2] carry through with a small modification since $W$ is a general matrix instead of a Hessian of a convex function.

It’s known that for any nonnegative $C^{1,1}$ function $h$, $|\nabla h(x)| \leq C h^{\frac{1}{2}}(x)$ for all $x \in \mathcal{O}$, where $C$ depends only on $\|h\|_{C^{1,1}(\Omega)}$ and dist\{\mathcal{O}, \partial \Omega\} (see [30]).

Since $W(x) \geq 0$, so we choose $h(x) = W_{ii}(x) \geq 0$. Then we can get from the above argument

\[
|\nabla W_{ii}| \leq C_1(W_{ii})^{\frac{1}{2}} = C_1(W_{ii} W_{jj})^{\frac{1}{4}}
\]

so (34) holds for $i = j$.

Similarly, for $i \neq j$, we choose $h = \sqrt{W_{ii} W_{jj}} \geq 0$, then we get

\[
|\nabla \sqrt{W_{ii} W_{jj}}| \leq C_2(\sqrt{W_{ii} W_{jj}})^{\frac{1}{2}} = C_2(W_{ii} W_{jj})^{\frac{1}{4}}.
\]

And for $h = \sqrt{W_{ii} W_{jj}} - W_{ij}$, we have

\[
|\nabla (\sqrt{W_{ii} W_{jj}} - W_{ij})| \leq C_3(\sqrt{W_{ii} W_{jj}} - W_{ij})^{\frac{1}{2}} \leq C_3(W_{ii} W_{jj})^{\frac{1}{4}}.
\]

So from (35) and (36), we get

\[
|\nabla W_{ij}| = |\nabla \sqrt{W_{ii} W_{jj}} - \nabla (\sqrt{W_{ii} W_{jj}} - W_{ij})|
\leq \left|\nabla \sqrt{W_{ii} W_{jj}}\right| + \left|\nabla (\sqrt{W_{ii} W_{jj}} - W_{ij})\right|
\leq (C_2 + C_3)(W_{ii} W_{jj})^{\frac{1}{4}}.
\]

So (34) holds for $i \neq j$. \(\square\)

**Remark 5.** If $W(x, t) = (W_{ij}(x, t))_{N \times N} \geq 0$ for every $(x, t) \in \Omega \times (0, T)$, and $W_{ij}(x, t) \in C^{1,1}(\Omega \times (0, T))$, then for every $\mathcal{O} \times (t_0 - \delta, t_0) \subset \mathcal{O} \times (0, T)$ with...
Suppose structural condition (5) as follows. The constant rank theorem for the spatial second fundamental form diagonal, and (1) nonlinear parabolic equation \( t \in (t_0 - \delta, t_0) \) and 1 \( \leq i, j \leq N \). Notice that \( DW_{ij} = (\nabla_{\xi}W_{ij}, \partial_t W_{ij}) \). In fact, if \( t_0 = T \), it only holds

\[
|\nabla_{\xi}W_{ij}| \leq C(W_{ii}W_{jj})^{\frac{1}{2}},
\]

for every \((x, t) \in \Omega \times (t_0 - \delta, t_0)\) and 1 \( \leq i, j \leq N \).

3. Constant rank theorem of the spatial second fundamental form. In this section, we consider the spatial level sets \( \Sigma^c = \{ x \in \Omega | u(x, t) = c \} \). Since \( u \) is the spacetime quasiconcave solution to fully nonlinear parabolic equation (1), \( u \) is also spatial quasiconcave, that is the spatial level sets \( \Sigma^c \) are all convex for \( t \in (0, T] \), that is the spatial second fundamental form \( a \geq 0 \). We will establish the constant rank theorem for the spatial second fundamental form \( a \) under the structural condition (5) as follows.

**Theorem 3.1.** Suppose \( u \in C^{3,1}(\Omega \times (0, T]) \) is a spacetime quasiconcave to fully nonlinear parabolic equation (1), and \( F \) satisfies conditions (2), (4) and (5). Then the second fundamental form of spatial level sets \( \Sigma^c = \{ x \in \Omega | u(x, t) = c \} \) has the same constant rank in \( \Omega \) for any fixed \( t \in (0, T] \). Moreover, let \( l(t) \) be the minimal rank of the second fundamental form in \( \Omega \), then \( l(s) \leq l(t) \) for all \( 0 < s \leq t \leq T \).

From the discussion in Section 2, the structural condition (5) is weaker than the structural condition (3), then Theorem 1.3 holds directly from Theorem 3.1.

In the following of this section, we will prove Theorem 3.1, and discuss some constant rank properties of the spatial second fundamental form \( a \). And we will prove the constant rank theorem of the spatial fundamental form of the spacetime quasiconcave solutions to the parabolic equations (6)-(8).

3.1. Proof of Theorem 3.1. Suppose \( a(x, t) \) attains minimal rank \( l \) at some point \((x_0, t_0) \in \Omega \times (0, T]\). We may assume \( l \leq n - 2 \), otherwise there is nothing to prove. And we assume \( u \in C^4(\Omega \times (0, T]) \) and \( u_n(x_0, t_0) > 0 \). So there is a neighborhood \( \Omega \times (t_0 - \delta, t_0] \) of \((x_0, t_0) \), such that there are \( l \) “good” eigenvalues of \((a_{ij})\) which are bounded below by a positive constant, and the other \( n - 1 - l \) “bad” eigenvalues of \((a_{ij})\) are very small. Denote \( G \) be the index set of these “good” eigenvalues and \( B \) be the index set of “bad” eigenvalues. And for any fixed point \((x, t) \in \Omega \times (t_0 - \delta, t_0] \), we may express \((a_{ij})\) in a form of (11), by choosing \( e_1, \cdots, e_{n-1}, e_n \) such that

\[
|\nabla u(x, t)| = u_n(x, t) > 0 \quad \text{and} \quad \left(u_{ij}\right)_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x, t).
\]

Without loss of generality we assume \( u_{11} \leq u_{22} \leq \cdots \leq u_{n-1,n-1} \). So, at \((x, t) \in \Omega \times (t_0 - \delta, t_0] \), from (10)-(12), we have the matrix \( \left(a_{ij}\right)_{1 \leq i, j \leq n-1} \) is also diagonal, and \( a_{11} \geq a_{22} \geq \cdots \geq a_{n-1,n-1} \). There is a positive constant \( C > 0 \) depending only on \( ||u||_{C^2} \) and \( \Omega \times (t_0 - \delta, t_0] \), such that \( a_{11} \geq a_{22} \geq \cdots \geq a_{ll} > C \) for all \((x, t) \in \Omega \times (t_0 - \delta, t_0] \). For convenience we denote \( G = \{1, \cdots, l\} \) and \( B = \{l+1, \cdots, n-1\} \) be the “good” and “bad” sets of indices respectively. If there is no confusion, we also denote

\[
G = \{a_{11}, \cdots, a_{ll}\}, \quad B = \{a_{l+1,l+1}, \cdots, a_{n-1,n-1}\}.
\]
Note that for any $\delta > 0$, we may choose $\mathcal{O} \times (t_0 - \delta, t_0]$ small enough such that $a_{jj} < \delta$ for all $j \in \mathcal{B}$ and $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$.

For each $c$, let $a = (a_{ij})$ be the symmetric Weingarten tensor of $\Sigma^c$. Set

$$p(a) = \sigma_{t+1}(a_{ij}), \quad q(a) = \begin{cases} \frac{\sigma_{t+2}(a_{ij})}{\sigma_{t+1}(a_{ij})}, & \text{if } \sigma_{t+1}(a_{ij}) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

(41)

Since we are dealing with general fully nonlinear equation (1), as in the case for the convexity of solutions in [2], there are technical difficulties to deal with $p(a)$ alone. A key idea in [2] is the introduction of function $q$ as in (41) and explore some crucial concavity properties of $q$. We consider function

$$\phi(x, t) = p(a) + q(a),$$

(42)

where $p$ and $q$ as in (41). We will prove the differential inequality

$$\sum_{\alpha, \beta=1}^n F^{\alpha\beta} \phi_{\alpha\beta}(x, t) - \phi_t \leq C(\phi + |\nabla \phi|), \quad \forall (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0],$$

(43)

where $C$ is a positive constant independent of $\phi$. Combining with the conditions

$$\phi \geq 0, \quad \phi(x_0, t_0) = 0,$$

we can get by the strong maximum principle

$$\phi \equiv 0, \quad \forall (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0].$$

Hence

$$\sigma_{t+1}(a) \equiv 0, \quad \forall (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0].$$

By the method of continuity, Theorem 3.1 holds. In the following, we prove the differential inequality (43).

To get around $\sigma_{t+1}(a) = 0$ in $q(a)$, for $\varepsilon > 0$ sufficiently small, we instead consider

$$\phi_\varepsilon(a) = \phi(a_\varepsilon),$$

(44)

where $a_\varepsilon = a + \varepsilon I$. We will also denote $G_\varepsilon = \{a_{ii} + \varepsilon, i \in \mathcal{B}\}$, $B_\varepsilon = \{a_{ii} + \varepsilon, i \in \mathcal{B}\}$.

To simplify the notations, we will drop subindex $\varepsilon$ with the understanding that all the estimates will be independent of $\varepsilon$. In this setting, if we pick $\mathcal{O} \times (t_0 - \delta, t_0]$ small enough, there is $C > 0$ independent of $\varepsilon$ such that

$$\phi(a(x, t)) \geq C\varepsilon, \quad \sigma_1(B) \geq C\varepsilon, \quad \forall (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0].$$

(45)

In the following, we denote

$$H_\phi = \sum_{i,j \in B} |\nabla a_{ij}| + \phi.$$

We will use notion $h = O(H_\phi)$ if $|h(x, t)| \leq C H_\phi$ for $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ with positive constant $C$ under control.

For any fixed point $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$, we choose a coordinate system as in (39) so that $|\nabla u| = u_n > 0$ and the matrix $(a_{ij}(x, t))$ is diagonal for $1 \leq i, j \leq n - 1$ and semipositive definite. From the definition of $\phi$, we get

$$\phi \geq \sigma_1(G) \sum_{i \in B} a_{ii} \geq 0,$$

so

$$a_{ii} = O(\phi) = O(H_\phi), \quad \forall i \in B.$$  

(46)
And from (10) - (12), we get
\[ a_{ii} = -\frac{h_{ii}}{u_n^3} = -\frac{u_{ii}}{u_n}, \]
so
\[ h_{ii} = O(\mathcal{H}_\phi), u_{ii} = O(\mathcal{H}_\phi), \forall i \in B. \tag{47} \]
From the definition of \(a_{ij}\), and \(u_k = 0\) for \(k = 1, \ldots, n - 1\), we can get
\begin{align*}
a_{ij,\alpha} &= \left( -\frac{|u_n|}{\sqrt[3]{u_n^3}} \right) a_{ij,\alpha}^t + \left( -\frac{|u_n|}{\sqrt[3]{u_n^3}} \right) a_{ij,\alpha}^t \\
&= 3u_n^{-4}u_{ij}^2 - u_n^{-3}[u_{ij}^2u_{ij}^{-2} + 2u_nu_{n,\alpha}u_{ij} - u_{,\alpha}u_nu_{ij} - u_{j,\alpha}u_{n,ij}] \\
&= -u_n^{-2}[u_{ij}u_{ij}^{-2} - u_{n,\alpha}u_{ij} - u_{ij}u_{n,\alpha} - u_{ij}u_{n,ij}], \tag{48} \end{align*}
so for \(i, j \in B\), we get
\begin{align*}
u_{ij,\alpha} &= O(\mathcal{H}_\phi), \forall \alpha < n, \tag{49} \\
u_{ij,\alpha} &= 2u_{in}u_{jn} + O(\mathcal{H}_\phi). \tag{50} \end{align*}
In fact, from (14)-(16),
\[ \hat{a}_{jj} = -\frac{|u_t|}{|Du||u|^3} \hat{h}_{jj} = -\frac{u_{jj}}{|Du|} = O(\mathcal{H}_\phi), \space \forall j \in B, \]
and from the spacetime convexity, we can get
\[ \hat{a}_{jn}^2 = \left[ -\frac{|u_t|}{|Du||u|^3} \frac{1}{W} \hat{h}_{jn} \right]^2 \leq \hat{a}_{jj} \hat{a}_{nn} = O(\mathcal{H}_\phi), \space \forall j \in B, \]
so it yields
\[ \hat{h}_{jn}^2 = O(\mathcal{H}_\phi), \forall j \in B. \tag{51} \]
Following the proof of Lemma 2.1 in [14], we can get
\begin{align*}
\phi_t &= \sum_{ij=1}^{n-1} \frac{\partial \phi}{\partial u_{ij}} a_{ij,t} \\
&= -\frac{1}{u_n^3} \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma_l^2(B) - \sigma_2(B)}{\sigma_l^2(B)} \right] \left[ u_n^2u_{jj,l} - 2u_nu_{jn}u_{jn,l} \right] + O(\mathcal{H}_\phi), \tag{52} \end{align*}
and
\begin{align*}
\sum_{\alpha, \beta=1}^{n} F_{\alpha\beta} \phi_{\alpha\beta} &= \sum_{\alpha, \beta=1}^{n} F_{\alpha\beta} \left[ \sum_{ij=1}^{n-1} \frac{\partial \phi}{\partial u_{ij}} a_{ij,\alpha\beta} + \sum_{ijkl=1}^{n-1} \frac{\partial^2 \phi}{\partial a_{ij}\partial a_{kl}} a_{ij,\alpha\beta}a_{kl,\alpha\beta} \right] \\
&= \frac{1}{u_n^3} \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma_l^2(B) - \sigma_2(B)}{\sigma_l^2(B)} \right] \left( -u_n^2 \sum_{\alpha, \beta=1}^{n} F_{\alpha\beta} u_{\alpha\beta,jj} \right) \\
&\quad + 6u_nu_{nj} \sum_{\alpha, \beta=1}^{n} F_{\alpha\beta} u_{\alpha\beta,j} - 6u_n^2u_{nj} \sum_{\alpha, \beta=1}^{n} F_{\alpha\beta} u_{\alpha\beta,jn} \\
&\quad + \frac{2}{u_n^3} \sum_{j \in B, i \in G} \left[ \sigma_l(G) + \frac{\sigma_l^2(B) - \sigma_2(B)}{\sigma_l^2(B)} \right] \sum_{\alpha, \beta=1}^{n} F_{\alpha\beta} \left( \frac{1}{u_{ii}} [u_{n,ij,\alpha} - 2u_{n,ij,\alpha,j}] ight) \\
&\quad \cdot [u_{n,ij\beta} - 2u_{ij\beta}u_{jn}] \right) \end{align*}
\begin{align*}
- \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta=1}^{n} \sum_{i \neq j \in B} F^{\alpha\beta} \left[ \sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j \in B} a_{jj,\beta} \right] \left[ \sigma_1(B) a_{ii,\beta} - a_{ii} \sum_{j \in B} a_{jj,\beta} \right] (53)

- \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta=1}^{n} \sum_{i \neq j \in B} F^{\alpha\beta} a_{ij,\alpha} a_{ij,\beta} + O(\mathcal{H}_\phi).

\text{Hence}
\sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} \phi_{\alpha\beta} - \phi_t
\begin{aligned}
= - \frac{1}{u_t^n} \sum_{j \in B} \left[ \sigma_1(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] \left[ u_t^n \left( \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{jj,\alpha\beta} - u_{jjt} \right) \right. \\
+ 2u_n u_{jn} u_{jt} - 6u_n u_{jn} \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{j\alpha\beta} + 6u_{jn}^2 \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} u_{\alpha\beta} \\
+ \left. \frac{2}{u_t^3} \sum_{j \in B, i \in G} \left[ \sigma_1(G) + \frac{\sigma_1^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] \sum_{\alpha,\beta=1}^{n} F^{\alpha\beta} \left( \frac{1}{u_t} [u_n u_{ij\alpha} - 2u_{i\alpha} u_{jn}] \right) \right]
\end{aligned}
\begin{aligned}
\cdot [u_n u_{ij\beta} - 2u_{i\beta} u_{jn}] \\
- \frac{1}{\sigma_1^2(B)} \sum_{\alpha,\beta=1}^{n} \sum_{i \neq j \in B} F^{\alpha\beta} \left[ \sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j \in B} a_{jj,\beta} \right] \left[ \sigma_1(B) a_{ii,\beta} - a_{ii} \sum_{j \in B} a_{jj,\beta} \right] \\
- \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta=1}^{n} \sum_{i \neq j \in B} F^{\alpha\beta} a_{ij,\alpha} a_{ij,\beta} + O(\mathcal{H}_\phi).
\end{aligned}
(54)

From \( \hat{h}_{jn} = u_t^2 u_{jn} - u_n u_{tjn} \), we have
\begin{align*}
2u_n u_{jn} u_{jt} &= - \frac{1}{u_t} \left[ \frac{\hat{h}_{jn}}{u_t} \right]^2 - \left( u_t u_{jn} \right)^2 - \left( u_n u_{jt} \right)^2 \\
&= O(\mathcal{H}_\phi) + \frac{1}{u_t} \left[ (u_t u_{jn})^2 + (u_n u_{jt})^2 \right],
\end{align*}
(55)
where the second \( \approx \) holds from (51).

For each \( j \in B \), differentiating equation (1) in \( e_j \) direction at \( x \),
\begin{align*}
u_{jjt} &= \sum_{k,l=1}^{n} F_{kl} u_{kjl} + \sum_{i=1}^{n} F_{ui} u_{ij} + F^u u_{jj} \\
&+ \sum_{k,l,p,q=1}^{n} F_{kl,pq} u_{kjl} u_{pqj} + 2 \sum_{i=1}^{n} F_{kl,u} u_{kij} u_{ij} + 2 \sum_{k,l=1}^{n} F_{kl,u} u_{kjl} u_{jq} \\
&+ 2 \sum_{k,l=1}^{n} F_{kl,x_j} u_{kjl} + \sum_{i=1}^{n} F_{ui,x_k} u_{ij} u_{kj} + 2 \sum_{i=1}^{n} F_{ui,x_k} u_{ij} u_{jq} \\
&+ 2 \sum_{i=1}^{n} F_{ui,x_j} u_{ij} + F^u u_{jj}^2 + 2 F^u x_j u_{jj} + F^x x_j.
\end{align*}
\[
\begin{align*}
= \sum_{kl=1}^{n} F_{kl} u_{klj} + 2 \frac{F_{un}}{u_n} u_{jn}^2 + \sum_{klpq=1}^{n} F_{klpq} u_{klj} u_{pqj} + 2 \sum_{kl=1}^{n} F_{kl} u_{klj} u_{jn} \\
+ 2 \sum_{kl=1}^{n} F_{kl} u_{klj} + F_{un} u_{jn}^2 + 2 F_{un} x_j u_{jn} + F_{xj} x_j + O(\mathcal{H}_\phi). 
\end{align*}
\]

Set

\[
Q_j = \sum_{klpq=1}^{n} F_{klpq} u_{klj} u_{pqj} + 2 \sum_{kl=1}^{n} F_{kl} u_{klj} u_{jn}^2 + 2 \sum_{kl=1}^{n} F_{kl} u_{klj} u_{jn} \\
+ F_{un} u_{jn}^2 + 2 F_{un} x_j u_{jn} + F_{xj} x_j u_{jn}^2 + 2 F_{un} u_{jn}^2 \\
+ 6 u_n u_{jn} \sum_{\alpha,\beta=1}^{n} F_{\alpha\beta} u_{\alpha\beta} - 6 u_{jn}^2 \sum_{\alpha,\beta=1}^{n} F_{\alpha\beta} u_{\alpha\beta} \\
+ 2 \sum_{i=1}^{n} \sum_{\alpha,\beta=1}^{n} F_{\alpha\beta} \frac{1}{u_{ii}} [u_{ii} - 2 u_{i\alpha} u_{jn}] [u_{i\alpha} - 2 u_{i\beta} u_{jn}],
\]

and denote

\[
s = \frac{1}{u_n} = \frac{1}{|\nabla u|}, A_{ij} = u_{ij} = \frac{u_{ij}}{u_n}, \theta = (0, 0, \ldots, 0, 1); \]

\[
X_{\alpha\beta} = 2 u_{\alpha\beta} u_{jn}, \quad \alpha \in B \text{ or } \beta \in B; \]

\[
X_{\alpha\beta} = u_{\alpha\beta} u_{ni}, \quad \text{otherwise}; \]

\[
Y = u_{jn} u_n; \]

\[
Z_i = \delta_{ij}, \quad i = 1, 2, \ldots, n; \]

\[
\tilde{V} = ((X_{\alpha\beta}), Y, (Z_i), 0) \in \mathcal{S} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R};
\]

then we can get

\[
X_{i\alpha} - 2 A_{i\alpha} Y = 0, \quad i \in B.
\]

So it yields

\[
Q_j = Q^*(\tilde{V}, \tilde{V}),
\]

where \(Q^*(\tilde{V}, \tilde{V})\) is defined in (33).

From (54) - (56), it yields

\[
F_{\alpha\beta} \phi_{\alpha\beta} - \phi_t
\]

\[
= u_n^{-3} \sum_{j \in B} \left[ \sigma_t(G) + \frac{\sigma^2_1(B) - \sigma_2(B)}{\sigma^2_1(B)} \right] \left( Q_j - 2 u_n u_{jn} u_{jt} \right)
\]

\[
- \frac{1}{\sigma^2_1(B)} \sum_{\alpha,\beta=1}^{n} \sum_{i=1}^{n} F_{\alpha\beta} \sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j \in B} a_{jj,\alpha} \sigma_1(B) a_{ii,\beta} - a_{ii} \sum_{j \in B} a_{jj,\beta} \sigma_1(B)
\]

\[
- \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta=1}^{n} \sum_{i \neq j}^{n} F_{\alpha\beta} a_{ij,\alpha} a_{ij,\beta} + O(\mathcal{H}_\phi)
\]

\[
\leq u_n^{-3} \sum_{j \in B} \left[ \sigma_t(G) + \frac{\sigma^2_1(B) - \sigma_2(B)}{\sigma^2_1(B)} \right] Q_j
\]
\[ -\frac{1}{\sigma_1(B)} \sum_{\alpha, \beta = 1}^{n} \sum_{i \in B} F_{\alpha \beta}[\sigma(B)a_{ii, \alpha} - a_{ii} \sum_{j \in B} a_{jj, \alpha}][\sigma(B)a_{ii, \beta} - a_{ii} \sum_{j \in B} a_{jj, \beta}] \]

\[ -\frac{1}{\sigma_1(B)} \sum_{\alpha, \beta = 1}^{n} \sum_{i \neq j, i, j \in B} F_{\alpha \beta}a_{ij, \alpha}a_{ij, \beta} + O(\mathcal{H}_\phi). \]  

(59)

From the structural condition (5) (i.e. Remark 4), it implies

\[ Q^*(\tilde{V}, \tilde{\tilde{V}}) \leq 0. \]

so for \( j \in B \), we get

\[ Q_j = Q^*(\tilde{V}, \tilde{\tilde{V}}) \leq 0. \]  

(60)

Condition (2) implies

\[ (F_{\alpha \beta}) \geq \delta_0 I_n, \quad \text{for some } \delta_0 > 0, \text{ and } \forall x \in \mathcal{O}. \]  

(61)

Set

\[ V_{i\alpha} = \sigma_1(B)a_{ii, \alpha} - a_{ii} \sum_{j \in B} a_{jj, \alpha}. \]

Combining (59), (60) and (61),

\[ F_{\alpha \beta}\phi_{\alpha \beta} \leq C(\phi + \sum_{i, j \in B} |\nabla a_{ij}|) - \delta_0\left[\frac{\sum_{i \neq j \in B, d \alpha} a_{ij, \alpha}^2}{\sigma_1(B)} + \frac{\sum_{d \in B, d, a \alpha} V_{d\alpha}^2}{\sigma_1^3(B)}\right]. \]  

(62)

By Lemma 3.3 in [2], for each \( M \geq 1 \), for any \( M \geq |\gamma_i| \geq \frac{1}{17} \), there is a constant \( C \) depending only on \( n \) and \( M \) such that, \( \forall \alpha, \)

\[ \sum_{i, j \in B} |a_{ij, \alpha}| \leq C(1 + \frac{1}{\delta_0})(\sigma_1(B) + |\sum_{d \in B} \gamma_{d} a_{d, \alpha}|) \]

\[ + \delta_0 \left[\frac{\sum_{i \neq j \in B} |a_{ij, \alpha}|^2}{\sigma_1(B)} + \frac{\sum_{d \in B} V_{d\alpha}^2}{\sigma_1^3(B)}\right]. \]  

(63)

Taking \( \gamma_i = \sigma_{l}(G) + \frac{\sigma_2^{(B[ii])} - \sigma_2(B[ii])}{\sigma_1^2(B)} \) for each \( i \in B \), the Newton-MacLaurine inequality implies

\[ \sigma_{l}(G) + 1 \geq \sigma_{l}(G) + \frac{\sigma_2^{(B[j])} - \sigma_2(B[j])}{\sigma_1^2(B)} \geq \sigma_{l}(G), \quad \forall j \in B. \]

and

\[ \phi_{\alpha} = \sum_{i, j = 1}^{n-1} \frac{\partial \phi}{\partial a_{ij}} a_{ij, \alpha} = \sum_{d \in B} \gamma_{d} a_{d, \alpha} + O(\phi). \]  

(64)

Therefore we conclude from (63) and (64) that \( \sum_{i, j \in B} |\nabla a_{ij}| \) can be controlled by the rest terms on the right hand side in (62) and \( \phi + |\nabla \phi| \). So (43) holds, and the proof of Theorem 3.1 is complete. \( \square \)
3.2. Constant rank properties of $a$. In the proof of Theorem 3.1, we can get for any $(x,t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ with the suitable coordinate (39),

\[
\sum_{\alpha,\beta = 1}^{n} F^{\alpha\beta} \phi_{\alpha\beta} - \phi_t
= u_n^{-3} \sum_{j \in B, i \in G} \left[ \sigma_1(G) + \frac{\sigma_1^2(B) - \sigma_2(B)}{\sigma_1^2(B)} \right] \left( Q_j - \frac{1}{u_t}((u_t u_{jn})^2 + (u_n u_{jt})^2) \right)
- \frac{1}{\sigma_1^2(B)} \sum_{\alpha,\beta = 1}^{n} \sum_{i \in B} F^{\alpha\beta} [\sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j \in B} a_{jj,\alpha}] [\sigma_1(B) a_{ii,\beta} - a_{jj} \sum_{j \in B} a_{jj,\beta}]
- \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta = 1}^{n} \sum_{i \neq j, i, j \in B} F^{\alpha\beta} a_{ij,\alpha} a_{ij,\beta} + O(H) \tag{65}
\leq C(\phi + |\nabla \phi|),
\]

and by the strong maximum principle,

\[\phi = 0 \quad \text{for} \quad (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]. \tag{66}\]

So it must have for any $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ with the suitable coordinate (39)

\[a_{jj} = 0, \quad \text{for} \quad j \in B. \tag{67}\]

In fact, we can get more information from the differential inequality, and the constant rank properties is as follows

**Corollary 1.** For any $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ with the suitable coordinate (39)

\[u_{jn} = u_{jt} = 0, |Du_j| = 0, \quad \text{for} \quad j \in B; \tag{68}\]

\[\sum_{k,l = 1}^{n} F^{kl} u_{kljj} - u_{jjt} = 2 \sum_{i \in G} \sum_{\alpha,\beta = 1}^{n} F^{\alpha\beta} u_{ij}^2 u_{ij} u_{ij}^2, \quad \text{for} \quad j \in B; \tag{69}\]

\[|Du_{ij}| = 0, \quad \text{for} \quad i \in B, j = 1, 2, \ldots, n - 1. \tag{70}\]

**Proof.** For $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$ with the suitable coordinate (39), we have from (11) and (67)

\[u_{jj} = 0, \quad \text{for} \quad j \in B. \]

and from (66) and (65), we get for $j \in B$,

\[Q_j = 0, \quad u_t u_{jn} = u_n u_{jt} = 0. \]

So

\[u_{jn} = u_{jt} = 0, \quad \text{for} \quad j \in B, \tag{71}\]

then

\[|Du_j| = 0, \quad \text{for} \quad j \in B. \tag{72}\]
From the definition of $Q_j$ and (66), (70), (71), we get

$$0 = Q_j = \sum_{k,l,m,n=1} F^{kl, mn} u_{klj} u_{mnj} u_n^2 + 2 \sum_{k,l=1} F^{k,l, w} u_{klij} u_{jn} u_n^2 + 2 \sum_{k,l=1} F^{k,l, x} u_{klij} u_n^2 + 2 F^{w, x} u_{jn} u_n^2 + F_x(x_j x_j u_n^2 + 2 \sum_{i \in G} \sum_{\alpha, \beta = 1} F^{\alpha, \beta} u_n^2 u_{ij\alpha} u_{ij\beta}}/u_{ii}, \quad \text{for } j \in B.$$

Also, we can get from (66), and Lemma 2.6 (i.e. Remark 18)

$$|Du_{ij}| = 0, \quad \text{for } i \in B, j = 1, 2, \ldots, n - 1,$$

then from (48) and (70)

$$|Du_{ij}| = 0, \quad \text{for } i \in B, j = 1, 2, \ldots, n - 1.$$

So the proof is complete. \hfill \Box

3.3. Constant rank theorem of the spatial second fundamental form for the equation (6). In this subsection, we consider the $p$-Laplacian parabolic equation, that is

$$u_t = \text{div}(|\nabla u|^{p-2} \nabla u) = L_{\alpha\beta}(\nabla u) u_{\alpha\beta}, \quad \text{in } \Omega \times (0, T],$$

where

$$L_{\alpha\beta}(\nabla u) = |\nabla u|^{p-2} \delta_{\alpha\beta} + (p-2)|\nabla u|^{p-4} u_{\alpha} u_{\beta}, \quad 1 \leq \alpha, \beta \leq n.$$  \hfill (74)

It is easy to know the equation (73) is parabolic when $p > 1$ and $|\nabla u| > 0$ in $\Omega \times [0, T]$. We will establish the constant rank theorem for the spatial second fundamental form $a$ as follows.

**Theorem 3.2.** Suppose $u \in C^{3,1}(\Omega \times (0, T])$ is a spacetime quasiconcave to the parabolic equation (73) and satisfies (4). Then the second fundamental form of spatial level sets $\Sigma^t = \{ x \in \Omega | u(x, t) = c \}$ has the same constant rank in $\Omega$ for any fixed $t \in (0, T)$. Moreover, let $l(t)$ be the minimal rank of the second fundamental form in $\Omega$, then $l(s) \leq l(t)$ for all $0 < s < t \leq T$.

**Proof.** The proof is similar to the the proof of Theorem 3.1, with some modifications.

Suppose the spatial second fundamental form $a(x, t)$ attains minimal rank $l$ at some point $(x_0, t_0) \in \Omega \times (0, T]$. We may assume $l \leq n - 2$, otherwise there is nothing to prove. So there is a small neighborhood $\mathcal{O} \times (t_0 - \delta, t_0]$ of $(x_0, t_0)$, such that there are $l$ “good” eigenvalues of $(a_{ij})$ which are bounded below by a positive constant, and the other $n - 1 - l$ “bad” eigenvalues of $(a_{ij})$ are very small. Denote $G$ be the index set of these “good” eigenvalues and $B$ be the index set of “bad” eigenvalues. We will prove the differential inequality

$$\sum_{\alpha, \beta = 1} F_{\alpha\beta}(x, t) - \phi_t \leq C(\phi + |\nabla \phi|), \quad \forall (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0],$$

where $\phi$ is defined in (42) and $C$ is a positive constant independent of $\phi$. Then by the strong maximum principle and the method of continuity, Theorem 3.2 holds.
Following the proof of Theorem 3.1, we get from (59), by choosing $e_1, \cdots, e_{n-1}, e_n$ such that

$$|\nabla u(x,t)| = u_n(x,t) > 0 \text{ and } (u_{ij})_{1 \leq i,j \leq n-1} \text{ is diagonal at } (x,t). \quad (76)$$

For any fixed point $(x,t) \in \mathcal{O} \times (t_0 - \delta, t_0]$, we may express $(a_{ij})$ in a form of (11), we get

$$L_{\alpha \beta} \phi_{\alpha \beta} - \phi_t = u_n^{-3} \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma_l^2(B)}{\sigma_l^2(B)} \right] \left( Q_j - 2u_n u_n u_{ij} \right)$$

$$- \frac{1}{\sigma_1(B)} \sum_{\alpha, \beta = 1}^{n} \sum_{i \in B} L_{\alpha \beta} \left[ \sigma_1(B) a_{ii, \alpha} - a_{ii} \sum_{j \in B} a_{jj, \beta} \right]$$

$$- \frac{1}{\sigma_1(B)} \sum_{\alpha, \beta = 1}^{n} \sum_{i \neq j, j \in B} L_{\alpha \beta} a_{ij, \alpha} a_{ij, \beta} + O(H_\phi), \quad (77)$$

where

$$Q_j = 2 \sum_{k,l = 1}^{n} \frac{\partial L_{kl}}{\partial u_n} u_{kl} u_{ij} u_n^2 + \sum_{k,l = 1}^{n} \frac{\partial^2 L_{kl}}{\partial u_n^2} u_{kl}^2 u_{ij}^2 u_n^2$$

$$+ 2 \sum_{k,l = 1}^{n} \frac{\partial L_{kl}}{\partial u_n} u_{kl} u_n u_{ij} + 6 u_n u_{ij} \sum_{k,l = 1}^{n} L_{kl} u_{klj} - 6 u_n^2 \sum_{k,l = 1}^{n} \frac{L_{kl}}{u_n}$$

$$+ 2 \sum_{i \in G} \sum_{\alpha, \beta = 1}^{n} \frac{1}{u_{ij}} L_{\alpha \beta} \left[ u_n u_{ij, \alpha} - 2 u_{ij, \alpha} u_n \right] \left[ u_n u_{ij, \beta} - 2 u_{ij, \beta} u_n \right].$$

Under the coordinate (76), we get

$$L_{kl} = 0, k \neq l; \quad L_{kk} = u_n^{-2}, k < n; \quad L_{nn} = (p - 1) u_n^{-2}; \quad (78)$$

$$\frac{\partial L_{kl}}{\partial u_n} = 0, \quad k \neq l; \quad (79)$$

$$\frac{\partial L_{kk}}{\partial u_n} = (p - 2) u_n^{-3} = (p - 2) \frac{L_{kk}}{u_n}, \quad k < n; \quad (80)$$

$$\frac{\partial L_{nn}}{\partial u_n} = (p - 1)(p - 2) u_n^{-3} = (p - 2) \frac{L_{nn}}{u_n}. \quad (81)$$

and

$$\frac{\partial^2 L_{kl}}{\partial u_n^2} = 0, k \neq l; \quad (82)$$

$$\frac{\partial^2 L_{kk}}{\partial u_n^2} = (p - 2)(p - 3) u_n^{-4} = (p - 2)(p - 3) \frac{L_{kk}}{u_n^2}, \quad k < n; \quad (83)$$

$$\frac{\partial^2 L_{nn}}{\partial u_n^2} = (p - 1)(p - 2)(p - 3) u_n^{-4} = (p - 2)(p - 3) \frac{L_{nn}}{u_n^2}. \quad (84)$$

From the equation (73), we know

$$u_t = L_{kk} u_{kk},$$
Hence, and for $j \in B$

$$u_{tj} = L_{kk}u_{kkj} + \frac{\partial L_{kl}}{\partial u_p}u_{pj}u_{kl} = L_{kk}u_{kkj} + \frac{\partial L_{kl}}{\partial u_j}u_{jj}u_{kl} + \frac{\partial L_{kl}}{\partial u_n}u_{nj}u_{kl}$$

$$= L_{kk}u_{kkj} + O(\mathcal{H}_\phi) + \frac{\partial L_{kk}}{\partial u_n}u_{nj}u_{kk} = L_{kk}u_{kkj} + (p - 2)\frac{L_{kk}}{u_n}u_{nj}u_{kk} + O(\mathcal{H}_\phi)$$

$$= L_{kk}u_{kkj} + (p - 2)\frac{u_t}{u_n}u_{nj} + O(\mathcal{H}_\phi). \quad (85)$$

And from (51), we get

$$\hat{h}_{jn}^2 = O(\mathcal{H}_\phi), \forall j \in B. \quad (86)$$

So

$$Q_j = 2\sum_{k=1}^{n}(p - 2)\frac{L_{kk}}{u_n}u_{kkj}u_{jn}u_n^2 + \sum_{k=1}^{n}(p - 2)(p - 3)\frac{L_{kk}}{u_n^2}u_{kkj}u_{jn}u_n^2$$

$$+ 2\sum_{k=1}^{n}(p - 2)\frac{L_{kk}}{u_n}u_{kkj}u_{jn}u_n^2 + 6u_nu_j\sum_{k=1}^{n}L_{kk}u_{kkj} - 6u_{jn}^2u_t$$

$$+ 2\sum_{i \in G}\sum_{\alpha = 1}^{n}\frac{1}{u_{ii}}L_{i\alpha\alpha}[u_nu_{i\alpha} - 2u_{i\alpha}u_{jn}]^2$$

$$= 2(p - 2)[u_{tj} - (p - 2)\frac{u_t}{u_n}u_{nj}]u_{jn}u_n + (p - 2)(p - 3)u_{tj}u_{jn}^2$$

$$+ 2(p - 2)u_tu_{jn}^2 + 6u_nu_j[u_{tj} - (p - 2)\frac{u_t}{u_n}u_{nj}] - 6u_{jn}^2u_t$$

$$+ 2\sum_{i \in G}\sum_{\alpha = 1}^{n}\frac{1}{u_{ii}}L_{i\alpha\alpha}[u_nu_{i\alpha} - 2u_{i\alpha}u_{jn}]^2 + O(\mathcal{H}_\phi)$$

$$= (2p + 2)u_nu_{jn}u_{tj} - (p^2 + p)u_tu_{jn}^2$$

$$+ 2\sum_{i \in G}\sum_{\alpha = 1}^{n}\frac{1}{u_{ii}}L_{i\alpha\alpha}[u_nu_{i\alpha} - 2u_{i\alpha}u_{jn}]^2 + O(\mathcal{H}_\phi).$$

Hence

$$Q_j - 2u_nu_{jn}u_{tj} = 2pu_nu_{jn}u_{tj} - (p^2 + p)u_tu_{jn}^2$$

$$+ 2\sum_{i \in G}\sum_{\alpha = 1}^{n}\frac{1}{u_{ii}}L_{i\alpha\alpha}[u_nu_{i\alpha} - 2u_{i\alpha}u_{jn}]^2 + O(\mathcal{H}_\phi)$$

$$= 2pu_{jn}[u_tu_{jn} - \frac{\hat{h}_{jn}}{u_t}] - (p^2 + p)u_tu_{jn}^2$$

$$+ 2\sum_{i \in G}\sum_{\alpha = 1}^{n}\frac{1}{u_{ii}}L_{i\alpha\alpha}[u_nu_{i\alpha} - 2u_{i\alpha}u_{jn}]^2 + O(\mathcal{H}_\phi)$$

$$= \frac{p}{u_t}[\frac{1}{p - 1} - \hat{h}_{jn}^2] - (p - 1)(u_tu_{jn} + \frac{1}{p - 1}u_t)^2$$

$$+ 2\sum_{i \in G}\sum_{\alpha = 1}^{n}\frac{1}{u_{ii}}L_{i\alpha\alpha}[u_nu_{i\alpha} - 2u_{i\alpha}u_{jn}]^2 + O(\mathcal{H}_\phi).$$
So we can get
\[
\frac{\partial}{\partial t} \phi_{\alpha\beta} - \phi_t \leq C(\phi + \sum_{i,j \in B} |\nabla a_{ij}|)
\]
\[
- \frac{1}{\sigma_1(B)} \sum_{\alpha, \beta = 1}^n L_{\alpha\beta} \left[ \sigma_1(B) a_{ii,\alpha} - a_{ii} \sum_{j \in B} a_{jj,\alpha} \right] - \frac{1}{\sigma_1(B)} \sum_{\alpha, \beta = 1}^n \sum_{i \notin j, i, j \in B} L_{\alpha\beta} a_{ij,\alpha} a_{ij,\beta}.
\]
(87)

Following the proof of Theorem 3.1, we get (75).

Remark 6. The constant rank properties (that is, Corollary 1) still holds for the equation (6).

3.4. Constant rank theorem of the spatial second fundamental form for the equation (7). In this subsection, we consider the mean curvature parabolic equation, that is
\[
u_t = \text{div} \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = \sum_{\alpha, \beta = 1}^n m_{\alpha\beta}(\nabla u) u_{\alpha\beta}, \text{ in } \Omega \times (0, T],
\]
(88)
where
\[
m_{\alpha\beta}(\nabla u) = (1 + |\nabla u|^2)^{-\frac{1}{2}} \delta_{\alpha\beta} - (1 + |\nabla u|^2)^{-\frac{3}{2}} u_{\alpha} u_{\beta}.
\]
(89)

We will establish the constant rank theorem for the spatial second fundamental form \(a\) as follows.

Theorem 3.3. Suppose \(u \in C^3(\Omega \times (0, T))\) is a spacetime quasiconcave to the parabolic equation (88) and satisfies (4). Then the second fundamental form of spatial level sets \(\Sigma^c = \{ x \in \Omega | u(x, t) = c \}\) has the same constant rank in \(\Omega\) for any fixed \(t \in (0, T]\). Moreover, let \(l(t)\) be the minimal rank of the second fundamental form in \(\Omega\), then \(l(s) \leq l(t)\) for all \(0 < s \leq t \leq T\).

Proof. The proof is similar to the the proof of Theorem 3.1 and Theorem 3.2, with some modifications.

Suppose the spatial second fundamental form \(a(x, t)\) attains minimal rank \(l\) at some point \((x_0, t_0) \in \Omega \times (0, T]\). We may assume \(l \leq n - 2\), otherwise there is nothing to prove. So there is a small neighborhood \(\mathcal{O} \times (t_0 - \delta, t_0)\) of \((x_0, t_0)\), such that there are \(l\) “good” eigenvalues of \((a_{ij})\) which are bounded below by a positive constant, and the other \(n - 1 - l\) “bad” eigenvalues of \((a_{ij})\) are very small. Denote \(G\) be the index set of these “good” eigenvalues and \(B\) be the index set of “bad” eigenvalues. We will prove the differential inequality
\[
\sum_{\alpha, \beta = 1}^n m_{\alpha\beta}(x, t) - \phi_t \leq C(\phi + |\nabla \phi|), \quad \forall (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0],
\]
(90)
where \(\phi\) is defined in (42) and \(C\) is a positive constant independent of \(\phi\). Then by the strong maximum principle and the method of continuity, Theorem 3.3 holds.
For any fixed point \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]\), we may express \((a_{ij})\) in a form of (11), by choosing \(e_1, \ldots, e_{n-1}, e_n\) such that
\[
|\nabla u(x, t)| = u_n(x, t) > 0 \quad \text{and} \quad \left(u_{ij}\right)_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x, t).
\] (91)

Following the proof of Theorem 3.1, we get from (59)
\[
m_{\alpha\beta}\phi_{\alpha\beta} = \phi_t
\]
\[
= u_n^{-3} \sum_{j \in B} \left[ \sigma_t(G) + \frac{\sigma_2^2(B|j) - \sigma_2(B|j)}{\sigma_1^2(B)} \right] (Q_j - 2u_n u_{jn} u_{jj})
\]
\[
- \frac{1}{\sigma_1^2(B)} \sum_{\alpha, \beta=1}^n \sum_{i \in B} m_{\alpha\beta} \sigma_1(B) a_{ii, \alpha} - a_{ii} \sum_{j \in B} a_{jj, \alpha} \sigma_1(B) a_{ii, \beta} - a_{ii} \sum_{j \in B} a_{jj, \beta}
\]
\[
- \frac{1}{\sigma_1(B)} \sum_{\alpha, \beta=1, i \neq j, i, j \in B}^n m_{\alpha\beta} a_{ij, \alpha} a_{ij, \beta} + O(\mathcal{H}_\phi),
\] (92)

where
\[
Q_j = 2 \sum_{k=1}^n \frac{\partial m_{kl}}{\partial u_n} u_{kl} u_{jn} u_n^2 + \sum_{k=1}^n \frac{\partial^2 m_{kl}}{\partial u_n^2} u_{kl} u_{jn}^2 u_n^2
\]
\[
+ 2 \sum_{k=1}^n \frac{\partial m_{kl}}{\partial u_n} u_{kl} u_{jn} + 6u_n u_{jn} \sum_{k=1}^n m_{kl} u_{kl} - 6u_{jn}^2 \sum_{k=1}^n m_{kl} u_{kl}
\]
\[
+ 2 \sum_{i \in B} \sum_{\alpha, \beta=1}^n \frac{1}{u_{ii}} m_{\alpha\beta} [u_n u_{ji, \alpha} - 2u_{ii} u_{jj}] [u_n u_{ij, \beta} - 2u_{ii} u_{jn}].
\]

Under the coordinate (91), we get
\[
m_{kl} = 0, \quad k \neq l; \quad m_{kk} = (1 + u_n^2)^{-\frac{3}{2}}, \quad k < n; \quad m_{nn} = (1 + u_n^2)^{-\frac{5}{2}};
\]
\[
\frac{\partial m_{kl}}{\partial u_n} = 0, \quad k \neq l;
\]
\[
\frac{\partial m_{kk}}{\partial u_n} = -(1 + u_n^2)^{-\frac{3}{2}} u_n, \quad k < n;
\]
\[
\frac{\partial m_{nn}}{\partial u_n} = -3(1 + u_n^2)^{-\frac{5}{2}} u_n;
\]

and
\[
\frac{\partial^2 m_{kl}}{\partial u_n^2} = 0, \quad k \neq l;
\]
\[
\frac{\partial^2 m_{kk}}{\partial u_n^2} = -(1 + u_n^2)^{-\frac{3}{2}} + 3(1 + u_n^2)^{-\frac{5}{2}} u_n^2, \quad k < n;
\]
\[
\frac{\partial^2 m_{nn}}{\partial u_n^2} = -3(1 + u_n^2)^{-\frac{5}{2}} + 15(1 + u_n^2)^{-\frac{7}{2}} u_n^2.
\]

From (47), and (49)
\[
u_{kk} = O(\mathcal{H}_\phi), \quad u_{kkj} = O(\mathcal{H}_\phi), \quad \forall k \in B, j \in B,
\] (93)

From (51), we get
\[
\hat{h}_{jn}^2 = O(\mathcal{H}_\phi), \forall j \in B.
\] (94)
From the equation (88), we know

\[ u_t = \sum_{k=1}^{n} m_{kk} u_{kk} = (1 + u_n^2)^{-\frac{1}{2}} \sum_{k=1}^{n-1} u_{kk} + (1 + u_n^2)^{-\frac{3}{2}} u_{nn}, \]

so we get

\[ u_t - (1 + u_n^2)^{-\frac{3}{2}} u_{nn} = (1 + u_n^2)^{-\frac{1}{2}} \sum_{k=1}^{n-1} u_{kk} \]

\[ = (1 + u_n^2)^{-\frac{1}{2}} \sum_{k \in G} u_{kk} + O(H_0), \]

and since \( u_{kk} \leq 0 \) for \( k < n \), it yields

\[ (1 + u_n^2)^{-\frac{3}{2}} u_{nn} \geq u_t. \]

Hence we can get

\[ \sum_{kl=1}^{n} \partial m_{kl} \frac{u_{kl}}{u_n} = \sum_{k=1}^{n} \partial m_{kk} \frac{u_{kk}}{u_n} = \sum_{k=1}^{n-1} \partial m_{kk} \frac{u_{kk}}{u_n} + \partial m_{nn} \frac{u_{nn}}{u_n} \]

\[ = - (1 + u_n^2)^{-\frac{3}{2}} u_n \sum_{k=1}^{n-1} u_{kk} - 3(1 + u_n^2)^{-\frac{3}{2}} u_n u_{nn} \]

\[ = - (1 + u_n^2)^{-\frac{3}{2}} u_n \sum_{k=1}^{n-1} u_{kk} - 3(1 + u_n^2)^{-\frac{3}{2}} u_n u_{nn} \]

\[ = - (1 + u_n^2)^{-\frac{3}{2}} u_n [u_t - (1 + u_n^2)^{-\frac{3}{2}} u_{nn}] - 3(1 + u_n^2)^{-\frac{3}{2}} u_n u_{nn} \]

\[ = - (1 + u_n^2)^{-\frac{3}{2}} u_n [u_t - 2(1 + u_n^2)^{-\frac{5}{2}} u_n u_{nn}]. \]

and

\[ \sum_{kl=1}^{n} \partial^2 m_{kl} \frac{u_{kl}}{u_n^2} = \sum_{k=1}^{n} \partial^2 m_{kk} \frac{u_{kk}}{u_n^2} = \sum_{k=1}^{n-1} \partial^2 m_{kk} \frac{u_{kk}}{u_n^2} + \partial^2 m_{nn} \frac{u_{nn}}{u_n^2} \]

\[ = \left[ - (1 + u_n^2)^{-\frac{3}{2}} + 3(1 + u_n^2)^{-\frac{3}{2}} u_n \right] \sum_{k=1}^{n-1} u_{kk} \]

\[ + \left[ - 3(1 + u_n^2)^{-\frac{3}{2}} + 15(1 + u_n^2)^{-\frac{3}{2}} u_n \right] u_{nn} \]

\[ = \left[ - (1 + u_n^2)^{-1} + 3(1 + u_n^2)^{-2} u_n^2 \right] [u_t - (1 + u_n^2)^{-\frac{3}{2}} u_{nn}] \]

\[ + \left[ - 3(1 + u_n^2)^{-1} + 15(1 + u_n^2)^{-\frac{3}{2}} u_n \right] u_{nn} \]

\[ = - (1 + u_n^2)^{-1} u_t + 3(1 + u_n^2)^{-2} u_n u_t \]

\[ + \left[ - 2(1 + u_n^2)^{-\frac{5}{2}} + 12(1 + u_n^2)^{-\frac{7}{2}} u_n \right] u_{nn}. \]

For \( j \in B \), differentiating the equation once in \( x_j \), we get

\[ u_{tj} = \sum_{k=1}^{n} m_{kk} u_{kjk} + \sum_{k=1}^{n} \frac{\partial m_{kk}}{\partial u_n} u_{nj} u_{kk} + O(H_0), \]

so

\[ \sum_{kl=1}^{n} m_{kl} u_{klj} = \sum_{k=1}^{n} m_{kk} u_{kjk} = u_{tj} - \sum_{k=1}^{n} \frac{\partial m_{kl}}{\partial u_n} u_{nj} u_{kl} - \sum_{k=1}^{n} \frac{\partial m_{kl}}{\partial u_j} u_{jj} u_{kl} \]
\[ u_{tj} - \sum_{k=1}^{n} \frac{\partial m_{kk}}{\partial u_n} u_{nj} u_{kk} + O(\mathcal{H}_\phi) \]
\[ = u_{tj} + (1 + u_n^2)^{-\frac{3}{2}} u_n u_{nj} \sum_{k=1}^{n-1} u_{kk} + 3(1 + u_n^2)^{-\frac{5}{2}} u_n u_{nj} u_{nn} + O(\mathcal{H}_\phi) \]
\[ = u_{tj} + (1 + u_n^2)^{-1} u_n u_{nj} [u_t - (1 + u_n^2)^{-\frac{3}{2}} u_{nn}] \]
\[ + 3(1 + u_n^2)^{-\frac{5}{2}} u_n u_{nj} u_{nn} + O(\mathcal{H}_\phi) \]
\[ = u_{tj} + (1 + u_n^2)^{-1} u_t u_n u_{nj} + 2(1 + u_n^2)^{-\frac{3}{2}} u_n u_{nj} u_{nn} + O(\mathcal{H}_\phi), \]

and from (94),
\[ (1 + u_n^2)^{-\frac{3}{2}} u_{nnj} = u_{tj} - (1 + u_n^2)^{-\frac{3}{2}} \sum_{k=1}^{n-1} u_{kkj} + (1 + u_n^2)^{-1} u_t u_n u_{nj} \]
\[ + 2(1 + u_n^2)^{-\frac{3}{2}} u_n u_{nj} u_{nn} + O(\mathcal{H}_\phi) \]
\[ = u_{tj} - (1 + u_n^2)^{-\frac{3}{2}} \sum_{k=1}^{n-1} u_{kkj} + (1 + u_n^2)^{-1} u_t u_n u_{nj} \]
\[ + 2(1 + u_n^2)^{-\frac{3}{2}} u_n u_{nj} u_{nn} + O(\mathcal{H}_\phi). \]

Hence
\[ \sum_{kl=1}^{n} \frac{\partial m_{kl}}{\partial u_n} u_{klj} = \sum_{k=1}^{n} \frac{\partial m_{kk}}{\partial u_n} u_{kkj} = \sum_{k=1}^{n-1} \frac{\partial m_{kk}}{\partial u_n} u_{kkj} + \frac{\partial m_{nn}}{\partial u_n} u_{nnj} \]
\[ = -(1 + u_n^2)^{-\frac{3}{2}} u_n \sum_{k=1}^{n-1} u_{kkj} - 3(1 + u_n^2)^{-\frac{5}{2}} u_n u_{nnj} \]
\[ = -(1 + u_n^2)^{-\frac{3}{2}} u_n \sum_{k=1}^{n-1} u_{kkj} + O(\mathcal{H}_\phi) \]
\[ - 3(1 + u_n^2)^{-1} u_n \left[ u_{tj} - (1 + u_n^2)^{-\frac{3}{2}} \sum_{k=1}^{n-1} u_{kkj} + (1 + u_n^2)^{-1} u_t u_n u_{nj} \right. \]
\[ \left. + 2(1 + u_n^2)^{-\frac{3}{2}} u_n u_{nj} u_{nn} \right] \]
\[ = -3(1 + u_n^2)^{-1} u_n u_{tj} + 2(1 + u_n^2)^{-\frac{3}{2}} u_n \sum_{k=1}^{n-1} u_{kkj} \]
\[ - 3(1 + u_n^2)^{-2} u_t u_n^2 u_{nj} - 6(1 + u_n^2)^{-\frac{5}{2}} u_n^2 u_{nj} u_{nn} + O(\mathcal{H}_\phi). \]

So
\[ Q_j = 4(1 + u_n^2)^{-\frac{3}{2}} u_n \sum_{k \in G} u_{kkj} \cdot u_{jn} u_n^2 + 2(1 + u_n^2)^{-\frac{3}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2 u_{kkj} u_{jn}]^2 \]
\[ + 2 \left[-3 \right(1 + u_n^2)^{-1} u_n u_{tj} - 3(1 + u_n^2)^{-2} u_t u_n^2 u_{nj} - 6(1 + u_n^2)^{-\frac{5}{2}} u_n^2 u_{nj} u_{nn} \right] u_n^2 u_{nn} \]
\[ + \left[\frac{-1}{1 + u_n^2} u_t + \frac{3}{(1 + u_n^2)^2} u_n^2 u_{t} + \left(\frac{-2}{1 + u_n^2} + \frac{12}{(1 + u_n^2)^2} \right) u_n^2 \right] u_n^2 u_{nn} \]
\[ + 2 \left[-(1 + u_n^2)^{-1} u_t u_n - 2(1 + u_n^2)^{-\frac{3}{2}} u_n u_{nn} \right] u_n^2 u_{nn} \]
\[ + 6u_n u_{jn} \left[u_{tj} + (1 + u_n^2)^{-1} u_t u_n u_{nj} + 2(1 + u_n^2)^{-\frac{3}{2}} u_n u_{nj} u_{nn} \right] - 6u_n^2 u_{nn}. \]
Hence So we can get
\[\begin{align*}
&=4(1 + u_n^2)^{-\frac{3}{2}}u_n \sum_{k \in G} u_{kk} \cdot u_{jn} u_n^2 + 2(1 + u_n^2)^{-\frac{3}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn}]^2 \\
&\quad - (1 + u_n^2)^{-1} u_n u_{nj} - 3(1 + u_n^2)^{-2} u_n^2 \cdot u_n u_{nj}^2 + 3(1 + u_n^2)^{-1} u_n^2 \cdot u_{n} u_{nj}^2 \\
&\quad + 6(1 + u_n^2)^{-\frac{3}{2}} u_n^2 \cdot u_{nn} u_{jn}^2 + 6u_n u_{jn} u_{tt} - 6u_{jn}^2 u_t \\
&+ 2 \sum_{i \in G, a = 1, a \neq i} \frac{n}{u_{ii}} m_{\alpha \alpha} [u_n u_{ij} - 2u_{i\alpha} u_{jn}]^2 + O(H_\phi),
\end{align*}\]
where
\[\begin{align*}
&=2 \left(1 + u_n^2\right)^{-\frac{3}{2}} u_n \sum_{k \in G} u_{kk} \cdot u_{jn} u_n^2 + 2(1 + u_n^2)^{-\frac{3}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn}]^2 \\
&=2 \left(1 + u_n^2\right)^{-\frac{3}{2}} \left(\sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn}]^2 \right) + 2u_n^2 u_{jn} + 2 \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + 2u_{kk} u_{jn}] \\
&\quad + 8(1 + u_n^2)^{-\frac{3}{2}} u_n^2 u_{jn} \sum_{k \in G} u_{kk}
\end{align*}\]
\[\begin{align*}
&=2(1 + u_n^2)^{-\frac{1}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + (1 + u_n^2)^{-1} u_n^2 u_n u_{kk}]^2 \\
&\quad - 2(1 + u_n^2)^{-\frac{1}{2}} u_n^2 u_{jn} \sum_{k \in G} u_{kk} + 8(1 + u_n^2)^{-\frac{1}{2}} u_n^2 u_{jn} \sum_{k \in G} u_{kk}
\end{align*}\]
\[\begin{align*}
&=2(1 + u_n^2)^{-\frac{1}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + (1 + u_n^2)^{-1} u_n^2 u_n u_{kk}]^2 \\
&\quad - 2(1 + u_n^2)^{-2} u_n^4 u_{jn} [u_t - (1 + u_n^2)^{-\frac{1}{2}} u_{nn}] \\
&\quad + 8(1 + u_n^2)^{-1} u_n^2 u_n^2 [u_t - (1 + u_n^2)^{-\frac{1}{2}} u_{nn}] + O(H_\phi)
\end{align*}\]
\[\begin{align*}
&=2(1 + u_n^2)^{-\frac{1}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + (1 + u_n^2)^{-1} u_n^2 u_{nn} u_{kn}]^2 + O(H_\phi) \\
&\quad - 2(1 + u_n^2)^{-2} u_n^4 \cdot u_n u_{jn}^2 + 8(1 + u_n^2)^{-1} u_n^2 \cdot u_t u_{jn} \\
&\quad + [2(1 + u_n^2)^{-\frac{1}{2}} u_n^4 - 8(1 + u_n^2)^{-\frac{1}{2}} u_n^2] u_{nn} u_{jn}^2.
\end{align*}\]
So we can get
\[Q_j - 2u_n u_{jn} u_{ttj} = 2(1 + u_n^2)^{-\frac{1}{2}} \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kk} u_{jn} + (1 + u_n^2)^{-1} u_n^2 u_{nn} u_{kn}]^2 \\
\quad - 6(1 + u_n^2)^{-1} u_n^2 \cdot u_n u_{tj} u_{jn} - 5(1 + u_n^2)^{-2} u_n^4 \cdot u_t u_{nj} \\
\quad + 11(1 + u_n^2)^{-1} u_n^2 \cdot u_t u_{nj} + 4u_n u_{jn} u_{tt} - 6u_{jn}^2 u_t \\
\quad + [2(1 + u_n^2)^{-\frac{1}{2}} u_n^4 - 2(1 + u_n^2)^{-\frac{1}{2}} u_n^2] u_{nn} u_{jn}^2 \\
\quad + 2 \sum_{i \in G, a = 1, a \neq i} \frac{n}{u_{ii}} m_{\alpha \alpha} [u_n u_{ij} - 2u_{i\alpha} u_{jn}]^2 + O(H_\phi).\]

Hence
\[Q_j - 2u_n u_{jn} u_{ttj} \leq - 6(1 + u_n^2)^{-1} u_n^2 \cdot u_n u_{tj} u_{jn} - 5(1 + u_n^2)^{-2} u_n^4 \cdot u_t u_{nj} \]
\[+ 11(1 + u_n^2)^{-1} u_n^2 \cdot u_t u_{nj} + 4u_n u_{jn} u_{tt} - 6u_{jn}^2 u_t +\]
+ 11(1 + u_n^2)^{-1}u_n^2 \cdot u_nu_n^2 - 6u_n^2u_t \\
+ [2(1 + u_n^2)^{-2}u_n^{-4} - 2(1 + u_n^2)^{-1}u_n^2 \cdot u_nu_n^2 + O(H_\phi) \\
= - 6(1 + u_n^2)^{-1}u_n^2 \cdot u_n[u_tu_n - \frac{\hat{h}_{jn}}{u_t}] - 3(1 + u_n^2)^{-2}u_n^4 \cdot u_nu_n^2 \\
+ 9(1 + u_n^2)^{-1}u_n^2 \cdot u_nu_n^2 + 4u_n[u_tu_n - \frac{\hat{h}_{jn}}{u_t}] - 6u_n^2u_t + O(H_\phi) \\
= \frac{\hat{h}_{jn}}{u_t}[6(1 + u_n^2)^{-1}u_n^2 - 4] \cdot u_n \\
+ [-2 + 3(1 + u_n^2)^{-1}u_n^2 - 3(1 + u_n^2)^{-2}u_n^4 \cdot u_nu_n^2 + O(H_\phi) \\
= \frac{\hat{h}_{jn}}{u_t}[6(1 + u_n^2)^{-1}u_n^2 - 4] \cdot u_n - \frac{5}{4}u_nu_n^2 \\
- 3[(1 + u_n^2)^{-1}u_n^2 - \frac{1}{2}]^2 \cdot u_nu_n^2 + O(H_\phi) \\
= - u_t(\frac{\hat{h}_{jn}}{u_t}[3(1 + u_n^2)^{-1}u_n^2 - 2] - u_jn)^2 \\
+ u_t[3(1 + u_n^2)^{-1}u_n^2 - 2]^2 \frac{\hat{h}_{jn}}{u_t} - \frac{1}{4}u_nu_n^2 \\
- 3[(1 + u_n^2)^{-1}u_n^2 - \frac{1}{2}]^2 \cdot u_nu_n^2 + O(H_\phi) \\
\leq u_t[3(1 + u_n^2)^{-1}u_n^2 - 2]^2 \frac{\hat{h}_{jn}}{u_t} + O(H_\phi) = O(H_\phi). 

So we can get

$$m_{\alpha\beta}\phi_{\alpha\beta} - \phi_t$$

\begin{equation}
\leq C(\phi + \sum_{i,j \in B} |\nabla a_{ij}|) - \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta=1}^{\sigma_1(B)} \sum_{i \neq j, i, j \in B} m_{\alpha\beta}a_{ij,\alpha}a_{ij,\beta} + \frac{1}{\sigma_1(B)} \sum_{\alpha,\beta=1}^{\sigma_1(B)} m_{\alpha\beta}[\sigma_1(B)a_{ii,\alpha} - a_{ii} \sum_{j \in B} a_{jj,\alpha}] [\sigma_1(B)a_{ii,\beta} - a_{ii} \sum_{j \in B} a_{jj,\beta}].
\end{equation}

Following the proof of Theorem 3.1, we get (90). \hfill \Box

**Remark 7.** The constant rank properties (that is, Corollary 1) still holds for the equation (7).

3.5. **Constant rank theorem of the spatial second fundamental form for the equation (8).** In this subsection, we consider the mean curvature parabolic equation, that is

$$u_t = (1 + |\nabla u|)^2 \text{div}(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}) = \sum_{\alpha,\beta=1}^{n} M_{\alpha\beta}(\nabla u)u_{\alpha\beta}, \text{ in } \Omega \times (0, T),$$

where

$$M_{\alpha\beta}(\nabla u) = (1 + |\nabla u|^2) \delta_{\alpha\beta} - u_{\alpha}u_{\beta}.$$ 

We will establish the constant rank theorem for the spatial second fundamental form $a$ as follows.
\textbf{Theorem 3.4.} Suppose \( u \in C^{3,1}(\Omega \times (0,T]) \) is a spacetime quasiconcave to the parabolic equation (96) and satisfies (4). Then the second fundamental form of spatial level sets \( \Sigma = \{ x \in \Omega | u(x,t) = c \} \) has the same constant rank in \( \Omega \) for any fixed \( t \in (0,T) \). Moreover, let \( l(t) \) be the minimal rank of the second fundamental form in \( \Omega \), then \( l(s) \leq l(t) \) for all \( 0 < s \leq t \leq T \).

\textit{Proof.} The proof is similar to the the proof of Theorem 3.1, with some modifications.

Suppose the spatial second fundamental form \( a(x,t) \) attains minimal rank \( l \) at some point \( (x_0, t_0) \in \Omega \times (0,T] \). We may assume \( l \leq n - 2 \), otherwise there is nothing to prove. So there is a small neighborhood \( \mathcal{O} \times (t_0 - \delta, t_0) \) of \( (x_0, t_0) \), such that there are \( l \) “good” eigenvalues of \( (a_{ij}) \) which are bounded below by a positive constant, and the other \( n - 1 - l \) “bad” eigenvalues of \( (a_{ij}) \) are very small. Denote \( G \) be the index set of these “good” eigenvalues and \( B \) be the index set of “bad” eigenvalues. We will prove the differential inequality

\begin{equation}
\sum_{\alpha, \beta = 1}^{n} M_{\alpha \beta} \phi_{\alpha \beta}(x,t) - \phi_t \leq C(\phi + |\nabla \phi|), \quad \forall (x,t) \in \mathcal{O} \times (t_0 - \delta, t_0], \tag{98}
\end{equation}

where \( \phi \) is defined in (42) and \( C \) is a positive constant independent of \( \phi \). Then by the strong maximum principle and the method of continuity, Theorem 3.4 holds.

Following the proof of Theorem 3.1, we get from (59)

\[ M_{\alpha \beta} \phi_{\alpha \beta} - \phi_t \]

\[ = -u^n^3 \sum_{j \in B} \left[ \sigma_l(G) + \frac{\sigma^2_l(B|j) - \sigma_2(B|j)}{\sigma^2_l(B)} \right] \left( Q_j - 2u_n u_j u_j \right) \]

\[ - \frac{1}{\sigma^2_1(B)} \sum_{\alpha, \beta = 1, \alpha \neq j, j < B}^{n} M_{\alpha \beta} [\sigma_1(B)a_{i\alpha} - \sigma_2(B)a_{i\beta}] + O(\mathcal{H}_\phi), \tag{100} \]

where

\[ Q_j = 2 \sum_{kl=1}^{n} \frac{\partial M_{kl}}{\partial u_n} u_{kl} u_{jn} u^n + \sum_{kl=1}^{n} \frac{\partial^2 M_{kl}}{\partial u_n^2} u_{kl} u^n u_j^2 + 2 \sum_{kl=1}^{n} \frac{\partial M_{kl}}{\partial u_n} u_{kl} u_{jn}^2 + 6u_n u_{jn} \sum_{kl=1}^{n} M_{kl} u_{kl} - 6u_n u_{jn} \sum_{kl=1}^{n} M_{kl} u_{kl} \]

\[ + 2 \sum_{i \in G} \sum_{\alpha, \beta = 1}^{n} \frac{1}{u_i} m_{ij} [u_{n} u_{ij\alpha} - 2u_{n} u_{ij\beta}] [u_{n} u_{ij\alpha} - 2u_{n} u_{ij\beta} u_{jn}], \]

Under the coordinate (99), we get

\[ M_{kl} = 0, \quad k \neq l; \quad M_{kk} = 1 + u^2_n, \quad k < n; \quad M_{nn} = 1; \]

\[ \frac{\partial M_{kl}}{\partial u_n} = 0, \quad k \neq l; \quad \frac{\partial M_{kk}}{\partial u_n} = 2u_n, \quad k < n; \quad \frac{\partial M_{nn}}{\partial u_n} = 0; \]
and
\[ \frac{\partial^2 M_{kl}}{\partial u_n^2} = 0, \quad k \neq l; \quad \frac{\partial^2 M_{kk}}{\partial u_n^2} = 2, \quad k < n; \quad \frac{\partial^2 M_{nn}}{\partial u_n^2} = 0. \]

From (47), (49)
\[ u_{kk} = O(\mathcal{H}_\phi), \quad u_{kkj} = O(\mathcal{H}_\phi), \quad \forall k \in B, j \in B. \quad (101) \]

From (51), we get
\[ \hat{h}_{jn}^2 = O(\mathcal{H}_\phi), \quad \forall j \in B. \quad (102) \]

From the equation (96), we know
\[ u_t = \sum_{k=1}^{n} M_{kk} u_{kk} = (1 + u_n^2) \sum_{k=1}^{n-1} u_{kk} + u_{nn}, \]
so we get
\[ (1 + u_n^2) \sum_{k=1}^{n-1} u_{kk} = u_t - u_{nn}, \]
and by \( u_{kk} \leq 0 \) for \( k < n \), it yields
\[ u_{nn} \geq u_t. \]

Hence we can get
\[ \sum_{kl=1}^{n} \frac{\partial M_{kl}}{\partial u_n} u_{kl} = \sum_{k=1}^{n} \frac{\partial M_{kk}}{\partial u_n} u_{kk} = \sum_{k=1}^{n-1} \frac{\partial M_{kk}}{\partial u_n} u_{kk} + \frac{\partial M_{nn}}{\partial u_n} u_{nn} \]
\[ = 2u_n \sum_{k=1}^{n-1} u_{kk} \]
\[ = 2u_n(1 + u_n^2)^{-1}[u_t - u_{nn}] \]
\[ = 2(1 + u_n^2)^{-1} u_t u_n - 2(1 + u_n^2)^{-1} u_n u_{nn}. \]

and
\[ \sum_{kl=1}^{n} \frac{\partial^2 M_{kl}}{\partial u_n^2} u_{kl} = \sum_{k=1}^{n} \frac{\partial^2 M_{kk}}{\partial u_n^2} u_{kk} = \sum_{k=1}^{n-1} \frac{\partial^2 M_{kk}}{\partial u_n^2} u_{kk} + \frac{\partial^2 M_{nn}}{\partial u_n^2} u_{nn} \]
\[ = 2 \sum_{k=1}^{n-1} u_{kk} = 2(1 + u_n^2)^{-1}[u_t - u_{nn}] \]
\[ = 2(1 + u_n^2)^{-1} u_t - 2(1 + u_n^2)^{-1} u_{nn}. \]

For \( j \in B \), differentiating the equation once in \( x_j \), we get
\[ u_{tj} = \sum_{k=1}^{n} M_{kk} u_{kkj} + \sum_{kl=1}^{n} \frac{\partial M_{kl}}{\partial u_p} u_{pj} u_{kl} \]
\[ = \sum_{k=1}^{n} M_{kk} u_{kkj} + \sum_{kl=1}^{n} \frac{\partial M_{kl}}{\partial u_j} u_{jj} u_{kl} + \sum_{kl=1}^{n} \frac{\partial M_{kl}}{\partial u_n} u_{nj} u_{kl} \]
\[ = \sum_{k=1}^{n} M_{kk} u_{kkj} + \sum_{kl=1}^{n} \frac{\partial M_{kk}}{\partial u_n} u_{nj} u_{kk} + O(\mathcal{H}_\phi), \]
\[
\sum_{kl=1}^{n} M_{kl} u_{klj} = \sum_{k=1}^{n} M_{kk} u_{kkj} \\
= u_{ij} - \sum_{k=1}^{n} \frac{\partial M_{kk}}{\partial u_n} u_{nj} u_{kk} + O(\mathcal{H}_\phi) \\
= u_{ij} - 2u_n u_{nj} \sum_{k=1}^{n-1} u_{kk} + O(\mathcal{H}_\phi) \\
= u_{ij} - 2u_n u_{nj} (1 + u_n^2)^{-1} [u_t - u_{nn}] + O(\mathcal{H}_\phi) \\
= u_{ij} - 2(1 + u_n^2)^{-1} u_t u_n u_{nj} + 2(1 + u_n^2)^{-1} u_n u_{nj} u_{nn} + O(\mathcal{H}_\phi).
\]

Hence from (101),
\[
\sum_{kl=1}^{n} \frac{\partial M_{kl}}{\partial u_a} u_{klj} = \sum_{k=1}^{n} \frac{\partial M_{kk}}{\partial u_n} u_{kkj} = \sum_{k=1}^{n-1} \frac{\partial M_{kk}}{\partial u_n} u_{kkj} + \frac{\partial M_{nn}}{\partial u_n} u_{nnj} \\
= 2u_n \sum_{k=1}^{n-1} u_{kkj} \\
= 2u_n \sum_{k \in G} u_{kkj} + O(\mathcal{H}_\phi).
\]

So
\[
Q_j = 4u_n \sum_{k \in G} u_{kkj} \cdot u_{jn} u_n^2 + 2(1 + u_n^2)^{-1} [u_t - u_{nn}] u_n^2 u_{jn} \\
+ 4u_n (1 + u_n^2)^{-1} [u_t - u_{nn}] u_n u_{jn}^2 \\
+ 6u_n u_{jn} [u_{ij} - 2(1 + u_n^2)^{-1} u_t u_n u_{nj} + 2(1 + u_n^2)^{-1} u_n u_{nj} u_{nn} ] \\
- 6u_{jn} u_t + 2(1 + u_n^2) \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kkj} u_{jn}]^2 \\
+ 2 \sum_{i \in G} \sum_{\alpha=1, \alpha \neq i}^{n} \frac{1}{u_{ii}} M_{\alpha\alpha} [u_n u_{\alpha j} - 2u_{\alpha j} u_{jn}]^2 + O(\mathcal{H}_\phi) \\
= 4u_n \sum_{k \in G} u_{kkj} \cdot u_{jn} u_n^2 + 2(1 + u_n^2) \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kkj} u_{jn}]^2 \\
- 6(1 + u_n^2)^{-1} u_n^2 \cdot u_{nj}^2 [u_t - u_{nn}] + 6u_n u_{jn} u_{jt} - 6u_{jn}^2 u_t \\
+ 2 \sum_{i \in G} \sum_{\alpha=1, \alpha \neq i}^{n} \frac{1}{u_{ii}} M_{\alpha\alpha} [u_n u_{\alpha j} - 2u_{\alpha j} u_{jn}]^2 + O(\mathcal{H}_\phi),
\]

where
\[
4u_n \sum_{k \in G} u_{kkj} \cdot u_{jn} u_n^2 + 2(1 + u_n^2) \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kkj} u_{jn}]^2 \\
= 2(1 + u_n^2) \left( \sum_{k \in G} \frac{1}{u_{kk}} [u_n u_{kkj} - 2u_{kkj} u_{jn}]^2 \right)
\]
Following the proof of Theorem 3.1, we get (98).

**Remark 8.** The constant rank properties (that is, Corollary 1) still holds for the equation (8).
4. Constant rank theorem of the spacetime second fundamental form.

In this section, we start to consider the spacetime level sets \( \Sigma_c = \{(x, t) \in \Omega \times (0, T)|u(x, t) = c\} \), and as in Section 2, the Weingarten tensor is

\[
\hat{A}_{\alpha\beta} = -\frac{\|u_t\|}{|Du|u_t^\alpha} \hat{A}_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq n, \quad (104)
\]

where

\[
\hat{A}_{\alpha\beta} = \hat{h}_{\alpha\beta} - \frac{u_{\alpha\gamma}u_{\beta\eta}\hat{h}_{\gamma\eta}}{W(1 + W)u_t^2} - \frac{u_{\alpha\gamma}u_{\beta\gamma}\hat{h}_{\gamma\gamma}}{W(1 + W)u_t^2} + \frac{u_{\alpha\beta}u_{\gamma\eta}\hat{h}_{\gamma\eta}}{W^2(1 + W)^2u_t^4}, \quad \hat{W} = \frac{|Du|}{|u_t|}. \quad (105)
\]

Suppose \( \hat{a}(x, t) = (\hat{a}_{\alpha\beta})_{n \times n} \) attains the minimal rank \( l \) at some point \((x_0, t_0) \in \Omega \times (0, T)\). We may assume \( l \leq n - 1 \), otherwise there is nothing to prove. At \((x_0, t_0)\), we may choose \( e_1, \ldots, e_{n-l}, e_n \) such that

\[
|\nabla u(x_0, t_0)| = u_n(x_0, t_0) > 0 \quad \text{and} \quad \left( u_{ij} \right)_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x_0, t_0). \quad (106)
\]

Without loss of generality we assume \( u_{11} \leq u_{22} \leq \cdots \leq u_{n-1,n-1} \). So, at \((x_0, t_0)\), from (104), we have the matrix \( (\hat{a}_{ij})_{1 \leq i, j \leq n-1} \) is also diagonal, and \( \hat{a}_{11} \geq \hat{a}_{22} \geq \cdots \geq \hat{a}_{n-1,n-1} \). From lemma 2.3, there is a positive constant \( C_0 \) such that at \((x_0, t_0)\)

**Case 1.**

\[
\hat{a}_{11} \geq \cdots \geq \hat{a}_{l-l-1} \geq C_0, \quad \hat{a}_{ll} = \cdots = \hat{a}_{n-1,n-1} = 0, \\
\hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{ii}^2}{\hat{a}_{ii}} \geq C_0, \quad \hat{a}_{in} = 0, \quad l \leq i \leq n-1.
\]

**Case 2.**

\[
\hat{a}_{11} \geq \cdots \geq \hat{a}_{ll} \geq C_0, \quad \hat{a}_{l+1,l+1} = \cdots = \hat{a}_{n-1,n-1} = 0, \\
\hat{a}_{nn} - \sum_{i=1}^{l} \frac{\hat{a}_{ii}^2}{\hat{a}_{ii}} \geq C_0, \quad \hat{a}_{in} = 0, \quad l + 1 \leq i \leq n-1.
\]

4.1. Case 1. In this subsection, we consider Case 1, that is, at \((x_0, t_0)\), we have

\[
\hat{a}_{11} \geq \cdots \geq \hat{a}_{l-l-1} \geq C_0, \quad \hat{a}_{ll} = \cdots = \hat{a}_{n-1,n-1} = 0, \\
\hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{ii}^2}{\hat{a}_{ii}} \geq C_0, \quad \hat{a}_{in} = 0, \quad l \leq i \leq n-1.
\]

Then there is a neighborhood \( \mathcal{O} \times (t_0 - \delta, t_0] \) of \((x_0, t_0)\), such that for any fixed point \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]\), we may choose \( e_1, \ldots, e_{n-l}, e_n \) such that

\[
|\nabla u(x, t)| = u_n(x, t) > 0 \quad \text{and} \quad \left( u_{ij} \right)_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x, t). \quad (107)
\]

Similarly we assume \( u_{11} \leq u_{22} \leq \cdots \leq u_{n-1,n-1} \). So, at \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]\), from (104), we have the matrix \( (\hat{a}_{ij})_{1 \leq i, j \leq n-1} \) is also diagonal, and \( \hat{a}_{11} \geq \hat{a}_{22} \geq \cdots \geq \hat{a}_{n-1,n-1} \). There is a positive constant \( C_0 > 0 \) depending only on \( \|u\|_{C^2} \) and \( \mathcal{O} \times (t_0 - \delta, t_0] \), such that

\[
\hat{a}_{11} \geq \cdots \geq \hat{a}_{l-l-1} \geq C_0, \\
\hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{ii}^2}{\hat{a}_{ii}} \geq C_0.
\]
for \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]\). For convenience we denote \(G = \{1, \ldots, l - 1\}\) and 
\(B = \{l, \ldots, n - 1\}\) be the “good” and “bad” sets of indices respectively.

Since
\[
\hat{a}_{ij} = \left| \frac{\nabla u}{Du} \right| a_{ij}, 1 \leq i, j \leq n - 1,
\] (108)
there is a positive constant \(C > 0\) depending only on \(\|u\|_{C^2}\) and \(\mathcal{O} \times (t_0 - \delta, t_0]\), such that
\[
a_{11} \geq \cdots \geq a_{l-1, l-1} \geq C, \quad (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0],
\] (109)
and
\[
a_{ll}(x_0, t_0) = \cdots = a_{n-1n-1}(x_0, t_0) = 0.
\] (110)
So the spatial second fundamental form \(a = (a_{ij})_{(n-1)\times(n-1)}\) attains the minimal rank \(l - 1\) at \((x_0, t_0)\). From Theorem 3.1, the constant rank theorem holds for the spatial second fundamental form \(a\). So we can get \(a_{ii} = 0, \forall i \in B\) for any \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]\). Furthermore,
\[
a_{ii} = 0, \forall i \in B.
\] (111)

We denote \(M = (\hat{a}_{ij})_{1 \leq i, j \leq n-1}\), so
\[
\sigma_{l+1}(M) = \sigma_l(M) \equiv 0, \quad \text{for every} \quad (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0].
\] (112)
Then we have
\[
0 \leq \sigma_{l+1}(\hat{a}) \leq \sigma_{l+1}(M) + \hat{a}_{nn} \sigma_l(M) = 0.
\] (113)
So
\[
\sigma_{l+1}(\hat{a}) \equiv 0, \quad \text{for every} \quad (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0].
\] (114)

By the continuity method, Theorem 1.2 holds under the Case 1.

4.2. Case 2. In this subsection, we consider Case 2. From Lemma 2.3, \(\hat{a}(x, t) = (\hat{a}_{ij})_{n \times n}\) attains the minimal rank \(l\) at some point \((x_0, t_0) \in \Omega \times (0, T]\) and at \((x_0, t_0)\), we may choose \(e_1, \ldots, e_{n-1}, e_n\) such that
\[
|\nabla u| = u_n > 0 \quad \text{and} \quad (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x_0, t_0)\.
\]
Then we have
\[
\hat{a}_{11} \geq \cdots \geq \hat{a}_{ll} \geq C_0, \quad \hat{a}_{l+1, l+1} = \cdots = \hat{a}_{n-1n-1} = 0,
\]
\[
\hat{a}_{nn} = \sum_{i=1}^{l} \hat{a}_{ii}^2, \quad \hat{a}_{in} = 0, \quad l + 1 \leq i \leq n - 1.
\]
Then there is a small enough neighborhood \(\mathcal{O} \times (t_0 - \delta, t_0]\) of \((x_0, t_0)\), such that for any fixed point \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]\), we may choose \(e_1, \ldots, e_{n-1}, e_n\) such that
\[
|\nabla u(x, t)| = u_n(x, t) > 0 \quad \text{and} \quad (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x, t).
\] (115)
Similarly we assume \(u_{11} \leq u_{22} \leq \cdots \leq u_{n-1n-1}\). So, at \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]\), from (104), we have the matrix \((\hat{a}_{ij})_{1 \leq i, j \leq n-1}\) is also diagonal, and \(\hat{a}_{11} \geq \hat{a}_{22} \geq \cdots \geq \hat{a}_{n-1n-1}\). There is a positive constant \(C > 0\) depending only on \(\|u\|_{C^2}\) and \(\mathcal{O} \times (t_0 - \delta, t_0]\), such that
\[
\hat{a}_{11} \geq \cdots \geq \hat{a}_{ll} \geq C,
\]
for all \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]\). For convenience we denote \(G = \{1, \ldots, l\}\) and 
\(B = \{l + 1, \ldots, n - 1\}\) be the “good” and “bad” sets of indices respectively. Since
\[
\hat{a}_{ij} = \left| \frac{\nabla u}{Du} \right| a_{ij}, 1 \leq i, j \leq n - 1,
\]
there is a positive constant $C > 0$ depending only on $\|u\|_{C^2}$ and $\mathcal{O} \times (t_0 - \delta, t_0)$, such that

$$a_{11} \geq \cdots \geq a_{l-1,l-1} \geq C, \quad (x, t) \in \mathcal{O} \times (t_0 - \delta, t_0),$$

and

$$a_{ll}(x_0, t_0) = \cdots = a_{n-1n-1}(x_0, t_0) = 0.$$ 

So the spatial second fundamental form $a = (a_{ij})_{n-1 \times (n-1)}$ attains the minimal rank $l$ at $(x_0, t_0)$. From Theorem 3.1, the constant rank theorem holds for the spatial second fundamental form $a$. So we can get $a_{ii} = 0, \forall i \in B$ for any $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0)$. Furthermore, $u_{x_1x_i} = 0, \forall i \in B$.

In order to simplify the calculations, we need a new spacetime coordinate system. For any fixed point $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0)$, let $\{e_1, \cdots, e_n, e_n+1\}$ be the coordinate satisfying (115) and (116), and $e_{n+1}$ is the time coordinate. First, by translating $\{e_n, e_{n+1}\}$, we get the coordinate $\{\hat{e}_1, \cdots, \hat{e}_{n-1}, \hat{e}_n, \hat{e}_{n+1}\}$ with

$$z = (z_1, \cdots, z_n, z_{n+1}) = (x, t)O,$$

where

$$O = (O_{ab})_{(n+1) \times (n+1)} = \begin{pmatrix}
1 & & & \\
& \ddots & & \\
& & 1 & \\
& & & \cos \theta \sin \theta \\
& & & -\sin \theta \cos \theta
\end{pmatrix},$$

such that

$$u_{z_{n+1}} = |Du| > 0, \quad u_{z_1} = \cdots = u_{z_n} = 0, \quad \text{at} \ (x, t).$$

So from (4), we have $\theta \in (0, \frac{\pi}{2})$. Now we fix the coordinate $\{\epsilon_{l+1}, \cdots, e_{n-1}, \hat{e}_{n+1}\}$ and translate $\{e_1, \cdots, e_l, \hat{e}_n\}$, and we get the coordinates $\{\epsilon_1, \cdots, \epsilon_l, e_{l+1}, \cdots, e_{n-1}, \hat{e}_n, \hat{e}_{n+1}\}$ with

$$y = (y_1, \cdots, y_n, y_{n+1}) = (z_1, \cdots, z_n, z_{n+1})T,$$

where

$$T = (T_{\alpha\beta})_{(n+1) \times (n+1)} = \begin{pmatrix}
T_{11} & \cdots & T_{1l} & T_{1n} \\
\vdots & \ddots & \vdots & \vdots \\
T_{l1} & \cdots & T_{ll} & T_{ln} \\
1 & & & 0 \\
& \ddots & & \\
& & 1 & 0 \\
T_{n1} & \cdots & T_{nl} & 0 & \cdots & 0 & 1
\end{pmatrix},$$

such that

$$\begin{pmatrix}
u_{y_{\alpha\beta}}\end{pmatrix}_{1 \leq \alpha, \beta \leq n} \text{ is diagonal at } (x, t).$$

Finally, we get a new spacetime coordinate $\{\bar{e}_1, \cdots, \bar{e}_l, e_{l+1}, \cdots, e_{n-1}, \bar{e}_n, \bar{e}_{n+1}\}$ with

$$y = (y_1, \cdots, y_n, y_{n+1}) = (x, t)P, \quad P = (P_{\alpha\beta})_{(n+1) \times (n+1)} = OT,$$
such that
\[ u_{y_{n+1}} = |Du| > 0, \quad u_{y_1} = \cdots = u_{y_n} = 0, \quad \text{at } (x,t), \quad (124) \]
\[ \begin{pmatrix} u_{y_\alpha y_\beta} \end{pmatrix}_{1 \leq \alpha, \beta \leq n} \text{ is diagonal at } (x,t). \quad (125) \]

From (21)-(23), we get
\[ \bar{a}_{\alpha \beta} = -\frac{1}{u_{y_{n+1}}} u_{y_\alpha y_\beta}, \quad 1 \leq \alpha, \beta \leq n, \quad (126) \]

Without loss of generality, we can assume
\[ \frac{\partial^2 u}{\partial y_1 \partial y_l} \leq \cdots \leq \frac{\partial^2 u}{\partial y_l \partial y_l} \leq -C < 0, \]
where the positive constant \( C > 0 \) depending only on \( \|u\|_{C^2} \). Then we have
\[ \bar{a}_{11} \geq \cdots \geq \bar{a}_{ll} \geq C. \quad (127) \]

In the following, we will prove a differential inequality
\[ \sum_{ij=1}^{n} F^{ij} \phi_{ij} - \phi_t \leq C(\phi + |\nabla_x \phi|) \quad \text{in } \mathcal{O} \times (t_0 - \delta, t_0). \quad (129) \]

In fact, if \( t_0 = T \) and \( (x,t) \in \mathcal{O} \times \{t_0\} \), we only have (38) instead of (37) from Lemma 2.6 (see Remark 5). So in order to utilizing (37), we just prove (129) holds for any \( (x,t) \in \mathcal{O} \times (t_0 - \delta, t_0) \), with a constant \( C \) independent of \( \text{dist}(\mathcal{O} \times (t_0 - \delta, t_0), \partial (\mathcal{O} \times (0, T))) \) and then by a approximation, (129) holds for \( t = t_0 \). Then by the strong maximum principle and the method of continuity, we can prove Theorem 1.2 under Case 2.

For convenience, we will use \( i, j, k, l = 1, \cdots, n \) to represent the \( x \) coordinates, \( t \) still the time coordinate, and \( \alpha, \beta, \gamma, \eta = 1, \cdots, n + 1 \) the \( y \) coordinates. And we have
\[ \frac{\partial y_\alpha}{\partial x_i} = P_{i\alpha}, \quad (130) \]
\[ \frac{\partial y_\alpha}{\partial t} = P_{n+1\alpha}. \quad (131) \]

In the following, we always denote
\[ u_t = \frac{\partial u}{\partial t}, \quad u_{\alpha t} = \frac{\partial u}{\partial y_\alpha}, \quad u_{y_{n+1}} = \frac{\partial u}{\partial y_{n+1}}, \]
\[ u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}, \quad u_{i\alpha} = \frac{\partial^2 u}{\partial x_i \partial y_\alpha}, \quad u_{\alpha \beta} = \frac{\partial^2 u}{\partial y_\alpha \partial y_\beta}, \quad \text{etc.} \]

Also, we will use notion \( h = O(\phi) \) if \( |h(x,t)| \leq C(\phi) \) for \( (x,t) \in \mathcal{O} \times (t_0 - \delta, t_0) \) with positive constant \( C \) under control, and \( h = O(\phi + |\nabla \phi|) \) has a similar meaning.
From the above discussions, for any \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0)\) with the coordinate (124) and (125), we get
\[
u_{\alpha\alpha} = \frac{\partial^2 u}{\partial y_{\alpha} \partial y_{\alpha}} = \frac{\partial^2 u}{\partial x_{\alpha} \partial x_{\alpha}} = 0, \quad \forall \alpha \in B. \quad (132)
\]

Under above assumptions, we can get

**Proposition 3.** For any \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0)\) with the coordinate (124) and (125), we can get
\[
\bar{a}_{\alpha\alpha}(x, t) = 0, \quad \alpha \in B. \quad (133)
\]

Furthermore, we have from the semipositive definite of \(\bar{a}\),
\[
\bar{a}_{\alpha\beta}(x, t) = 0, \quad \alpha \in B, \beta \in B \cup G, \quad (134)
\]
\[
\bar{a}_{\alpha\alpha}(x, t) = \bar{a}_{\alpha\alpha}(x, t) = 0, \quad \alpha \in B, \quad (135)
\]
\[
D\bar{a}_{\alpha\beta}(x, t) = 0, \quad \alpha \in B, \beta \in B \cup G, \quad (136)
\]
\[
D\bar{a}_{\alpha\alpha}(x, t) = 0, \quad \alpha \in B. \quad (137)
\]

**Proof.** The proof is directly from the constant rank theorem of \(a\) and Lemma 2.6.

For any \((x, t) \in \mathcal{O} \times (t_0 - \delta, t_0)\) with the coordinate (124) and (125), we can get from (132)
\[
\bar{a}_{\alpha\alpha}(x, t) = -\frac{1}{|Du|} u_{\alpha\alpha} = 0, \forall \alpha \in B.
\]

From the positive definite of \(\bar{a}\) at \((x, t)\), we get
\[
\bar{a}_{\alpha\beta}(x, t) = 0, \quad \alpha \in B, \beta \in B \cup G,
\]
\[
\bar{a}_{\alpha\alpha}(x, t) = 0, \forall \alpha \in B.
\]

And from Lemma 2.6 (i.e. Remark 5), we get
\[
|D\bar{a}_{\alpha\beta}|(x, t) = 0, \quad \alpha \in B, \beta \in B \cup G,
\]
\[
|D\bar{a}_{\alpha\alpha}|(x, t) = 0, \forall \alpha \in B.
\]

So Proposition 3 holds. \(\square\)

**Lemma 4.1.** Under the above conditions, we have
\[
Du_{\alpha} = 0, \quad \alpha \in B, \quad (138)
\]
\[
Du_{\alpha\beta} = 0, \quad \alpha \in B, \beta \in B, \quad (139)
\]
\[
Du_{\alpha\beta} = 0, \quad \alpha \in B, \beta \in G \cup \{n\}. \quad (140)
\]

**Proof.** By the constant rank properties Corollary 1, (138) holds since the \(y_{\alpha}\) coordinate is the \(x_{\alpha}\) coordinate for \(\alpha \in B\).

By (21), (22), we can get for \(\alpha \in B\) and \(\beta = 1, \cdots, n\),
\[
\bar{a}_{\alpha\beta} = 0, \quad \bar{A}_{\alpha\beta} = 0, \quad \bar{h}_{\alpha\beta} = 0, \quad u_{\alpha\beta} = 0,
\]
then from (136) and (137), we get
\[
0 = D\bar{a}_{\alpha\beta} = D(-\frac{|u_{n+1}|}{|Du||u_{n+1}|} \bar{A}_{\alpha\beta} - \frac{|u_{n+1}|}{|Du||u_{n+1}|^3} D\bar{A}_{\alpha\beta}
\]
\[
= 0 - \frac{|u_{n+1}|}{|Du||u_{n+1}|^3} D\bar{h}_{\alpha\beta} = -\frac{|u_{n+1}|}{|Du||u_{n+1}|^3} D\bar{h}_{\alpha\beta}.
\]
and from (23) and (124), we get
\[
0 = D\bar{h}_{\alpha\beta} = D(u^2_{n+1}u_{\alpha\beta} + u_{n+1}u_{\alpha}u_{\beta} - u_{n+1}u_{\beta}u_{\alpha} + u_{n+1}u_{\alpha}u_{\beta})
\]
\[
= u^2_{n+1}Du_{\alpha\beta} + 2u_{n+1}Du_{\alpha}u_{\beta} - u_{n+1}u_{n+1}Du_{\beta} - u_{n+1}Du_{n+1}Du_{\alpha}
\]
\[
= u^2_{n+1}Du_{\alpha\beta} + 0 - 0 - 0
\]
\[
= u^2_{n+1}Du_{\alpha\beta}, \quad \alpha \in B, \quad \beta = 1, \ldots, n.
\]
Hence the lemma holds.

Lemma 4.2. Under the above conditions, we have
\[
u_{yn,yn} = O(\phi),
\]
(141)
\[
u_{x,yn} = O(\phi), i < n;
\]
(142)
\[
u_{yn,yn,x_i} = O(\phi + |\nabla\phi|), \quad i = 1, \ldots, n-1,
\]
(143)
\[
u_{yn,yn,x_n} = 2\frac{1}{u_{y_n+1}}u_{yn,x_n}u_{yn,yn+1} + O(\phi + |\nabla\phi|)
\]
(144)
Proof. In the $y$ coordinates, we have from (133)
\[
\phi = \sigma_{l+1}(\bar{\pi}) = \sigma_l(G)\bar{\pi}_{nn} \geq 0,
\]
so we have
\[
\bar{\pi}_{nn} = O(\phi).
\]
By (21)-(23), we have
\[
\bar{a}_{nn} = -\frac{|u_{yn+1}|}{|Du|u_{yn+1}}^3\bar{A}_{nn} = -\frac{|u_{yn+1}|}{|Du|u_{yn+1}}^3\bar{h}_{nn} = -\frac{|u_{yn+1}|}{|Du|u_{yn+1}}^3u_{n+1}^2u_{yn,yn},
\]
so
\[
\bar{A}_{nn} = O(\phi), \quad \bar{h}_{nn} = O(\phi), \quad u_{yn,yn} = O(\phi).
\]
(145)
Taking the first derivatives of $\phi$ in $x$, we have
\[
\phi_x = \frac{\partial \phi}{\partial x_i} = \sum_{\alpha=1}^n \sigma_l(\bar{\pi})\bar{a}_{\alpha,i}
\]
\[
= \sum_{\alpha \in G} \sigma_l(\bar{\pi})\bar{a}_{\alpha,i} + \sum_{\alpha \in B} \sigma_l(\bar{\pi})\bar{a}_{\alpha,i} + \sum_{\alpha = n} \sigma_l(\bar{\pi})\bar{a}_{\alpha,i}
\]
\[
= \sigma_l(G)\bar{a}_{nn,i},
\]
(146)
since $\sigma_l(\bar{\pi}) = 0$ if $\alpha \in G$ and $\bar{a}_{\alpha,i} = 0$ if $\alpha \in B$ from (134) and (136). Moreover, from (21)-(23) and (146),
\[
\bar{a}_{nn,i} = (\frac{|u_{yn+1}|}{|Du|u_{yn+1}^3})i\bar{A}_{nn} = -\frac{|u_{yn+1}|}{|Du|u_{yn+1}^3}\bar{A}_{nn,i} = O(\phi) - \frac{|u_{yn+1}|}{|Du|u_{yn+1}^3}\bar{A}_{nn,i}
\]
\[
= -\frac{|u_{yn+1}|}{|Du|u_{yn+1}^3}\bar{h}_{nn,i} + O(\phi)
\]
\[
= -\frac{|u_{yn+1}|}{|Du|u_{yn+1}^3}(u^2_{yn+1}u_{yn,yn+1}x_i - 2u_{yn+1}u_{yn,x_i}u_{yn,yn+1}) + O(\phi),
\]
(147)
so
\[
\bar{a}_{nn,i} = O(\phi + |\nabla\phi|), \quad \bar{A}_{nn,i} = O(\phi + |\nabla\phi|), \quad \bar{h}_{nn,i} = O(\phi + |\nabla\phi|),
\]
(148)
and
\[ u_{y_n y_n x_i} = \frac{1}{u_{y_{n+1}}} u_{y_n x_i} + O(\phi + |\nabla \phi|), \]  
(150)
It is easy to know for \( i = 1, \ldots, n - 1 \),
\[ u_{y_n x_i} = u_{y_n y_n} \frac{\partial y_n}{\partial x_i} = u_{y_n y_{n+1}} \frac{\partial y_{n+1}}{\partial x_i} = O(\phi) + u_{y_n y_{n+1}} \frac{\partial y_{n+1}}{\partial x_i} \]
(151)
Hence the lemma holds from (150) and (151).

Lemma 4.3.
\[ u_{y_{n+1} y_n x_n} = u_{x_n} u_{y_{n+1}} + O(\phi), \]
(152)
\[ u_{y_{n+1} y_n y_n} = u_{y_{n+1}} u_{y_{n+1}} + O(\phi). \]
(153)
Proof. By the chain rule, we get
\[ u_{x_n} u_{y_{n+1}} = u_{y_n \frac{\partial y_n}{\partial x_n}} u_{y_{n+1}} = u_{y_{n+1}} \frac{\partial y_{n+1}}{\partial x_n} u_{y_{n+1}}, \]
and
\[ u_{y_{n+1} y_n} = u_{y_{n+1}} u_{y_{n+1}} \frac{\partial y_n}{\partial x_n} = u_{y_{n+1}} \frac{\partial y_n}{\partial x_n} + u_{y_{n+1}} \frac{\partial y_{n+1}}{\partial x_n} = O(\phi) + u_{y_{n+1}} \frac{\partial y_{n+1}}{\partial x_n}. \]
so (152) holds.

Similarly, we have
\[ u_{y_{n+1} y_n} = u_{y_{n+1}} u_{y_{n+1}} \frac{\partial y_n}{\partial t} = u_{y_{n+1}} \frac{\partial y_n}{\partial t} + u_{y_{n+1}} \frac{\partial y_{n+1}}{\partial t} = 0 + O(\phi) \]
(153)
so (153) holds.

Lemma 4.4.
\[ \sum_{k,l=1}^n F^{kl} u_{kly_n y_\gamma} = 0, \quad \text{for} \quad \gamma \in B. \]  
(154)
Proof. From (69) (i.e. the constant rank properties Corollary 1) and (139), we have for \( \gamma \in B \)
\[ \sum_{k,l=1}^n F^{kl} u_{kly_n y_\gamma} = \sum_{k,l=1}^n F^{kl} u_{kly_n y_\gamma} - u_{y_n y_\gamma t} = \sum_{k,l=1}^n F^{kl} u_{kly_n x_\gamma} - u_{x_n t} = 2 \sum_{i \in G} \sum_{k,l=1}^n F^{kl} u_{x_n} u_{ikx_n} u_{ilx_n}. \]  
(155)
In fact, for $i \in G, \gamma \in B$, we have from (139) and (140),

$$u_{ikx} = u_{iky} = u_{y_kx+iy} \frac{\partial y_\alpha}{\partial x_i} = \sum_{\alpha \leq n} \frac{\partial u_{y_ky}}{\partial x_k} \frac{\partial y_\alpha}{\partial x_i} + u_{y_n+1kx} \frac{\partial y_n+1}{\partial x_i} = 0 + u_{y_n+1ky} \frac{\partial z_n+1}{\partial x_i} = 0.$$  \quad (156)

So (154) holds from (155) and (156).

**Lemma 4.5.**

$$\sum_{ij=1}^{n} F^{ij} \phi_{ij} = \sigma_l(G) \sum_{ij=1}^{n} F^{ij} \bar{a}_{nn,ij} - 2 \sigma_l(G) \sum_{ij=1}^{n} F^{ij} \sum_{\eta \in G} \frac{\bar{a}_{nn,\eta} \bar{a}_{\eta j}}{\bar{a}_{\eta \eta}} + O(\phi + |\nabla \phi|).$$  \quad (157)

**Proof.** Taking the second derivatives of $\phi$ in $y$ coordinates, we have

$$\phi_{\alpha\beta} = \frac{\partial^2 \phi}{\partial y_\alpha \partial y_\beta} = \sum_{\gamma=1}^{n} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta} + \sum_{\gamma \neq \eta} \frac{\partial^2 \sigma_l+1}{\partial \bar{a}_{\gamma \gamma} \partial \bar{a}_{\eta \eta}} \bar{a}_{\gamma \gamma,\alpha\beta} + \sum_{\gamma \neq \eta} \frac{\partial^2 \sigma_l+1}{\partial \bar{a}_{\gamma \gamma} \partial \bar{a}_{\eta \eta}} \bar{a}_{\gamma \gamma,\alpha\beta}$$

$$= \sum_{\gamma=1}^{n} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta} + \sum_{\gamma \neq \eta} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta} - \sum_{\gamma \neq \eta} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta},$$  \quad (158)

where

$$\sum_{\gamma=1}^{n} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta} = \sum_{\gamma \in G} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta} + \sum_{\gamma \in B} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta} + \sum_{\gamma=1}^{n} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta}$$

$$= \sigma_l(G) \sum_{\gamma \in B} \bar{a}_{\gamma \gamma,\alpha\beta} + \sigma_l(G) \bar{a}_{nn,\alpha\beta} + O(\phi),$$  \quad (159)

$$\sum_{\gamma \neq \eta} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta} = \sum_{\gamma \neq \eta} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta} + \sum_{\gamma=1}^{n} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta} + \sum_{\gamma=1}^{n} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta}$$

$$+ \sum_{\eta \in G} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta}$$

$$= O(\phi) + \sum_{\eta \in G} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta} \bar{a}_{nn,\alpha}$$

$$+ \sum_{\gamma \in G} \sigma_l(\bar{a}_{\gamma \gamma}) \bar{a}_{\gamma \gamma,\alpha\beta} \bar{a}_{nn,\beta}$$

$$= \sigma_l(G) \left[ \sum_{\eta \in G} \frac{\bar{a}_{\eta \eta}}{\bar{a}_{\gamma \gamma}} \bar{a}_{nn,\alpha} + \sum_{\gamma \in G} \frac{\bar{a}_{\gamma \gamma}}{\bar{a}_{\gamma \gamma}} \bar{a}_{nn,\beta} \right] + O(\phi),$$  \quad (160)
and
\[
\sum_{\gamma \neq \eta} \sigma_{t-1}(\gamma) \pi_{\gamma,\alpha} \pi_{\gamma,\beta} = \sum_{\gamma,\eta \in G} \sigma_{t-1}(\gamma) \pi_{\gamma,\alpha} \pi_{\gamma,\beta} + \sum_{\gamma,\eta \in G} \sigma_{t-1}(\gamma) \pi_{\gamma,\alpha} \pi_{\gamma,\beta} + \sum_{\gamma,\eta \in G} \sigma_{t-1}(\gamma) \pi_{\gamma,\alpha} \pi_{\gamma,\beta}
\]
\[
+ \sum_{\gamma \in G} \sigma_{t-1}(\gamma) \pi_{\gamma,\alpha} \pi_{\gamma,\beta}
\]
\[
= O(\phi) + \sum_{\eta \in G} \sigma_{t-1}(G) \pi_{\eta,\alpha} \pi_{\eta,\beta}
\]
\[
+ \sum_{\gamma \in G} \sigma_{t-1}(G) \pi_{\gamma,\alpha} \pi_{\gamma,\beta}
\]
\[
= 2\sigma_t(G) \sum_{\eta \in G} \frac{\pi_{\eta,\alpha} \pi_{\eta,\beta}}{\pi_{\eta\eta}} + O(\phi).
\]

So we have
\[
\phi_{\alpha\beta} = \sigma_t(G) \sum_{\gamma \in B} \pi_{\gamma,\alpha} \pi_{\gamma,\beta} + \sigma_t(G) \pi_{\eta,\alpha} \pi_{\eta,\beta} - 2\sigma_t(G) \sum_{\eta \in G} \frac{\pi_{\eta,\alpha} \pi_{\eta,\beta}}{\pi_{\eta\eta}}
\]
\[
+ \sigma_t(G) \sum_{\gamma \in G} \frac{\pi_{\gamma,\alpha} \pi_{\gamma,\beta}}{\pi_{\gamma\gamma}} + O(\phi)
\]

So we have
\[
\sum_{ij=1}^{n} F_{ij} \phi_{ij} = \sum_{ij=1}^{n} F_{ij} \sum_{\alpha,\beta=1}^{n+1} P_{\alpha\beta} P_{j\beta} \phi_{\alpha\beta}
\]
\[
= \sigma_t(G) \sum_{\gamma \in B} \sum_{ij=1}^{n} F_{ij}^{\gamma} \sum_{\alpha,\beta=1}^{n+1} P_{\alpha\beta} P_{j\beta} \pi_{\gamma,\alpha} \pi_{\gamma,\beta} + \sigma_t(G) \sum_{ij=1}^{n} F_{ij} \sum_{\alpha,\beta=1}^{n+1} P_{\alpha\beta} P_{j\beta} \pi_{\eta,\alpha} \pi_{\eta,\beta}
\]
\[
- 2\sigma_t(G) \sum_{ij=1}^{n} F_{ij} \sum_{\eta \in G} \frac{[\sum_{\alpha=1}^{n+1} P_{\alpha\beta} \pi_{\eta,\alpha}] [\sum_{\beta=1}^{n+1} P_{j\beta} \pi_{\eta,\beta}]}{\pi_{\eta\eta}}
\]
\[
+ \sigma_t(G) \sum_{ij=1}^{n} F_{ij} \sum_{\eta \in B} \frac{P_{j\beta} \pi_{\eta,\beta}}{\pi_{\eta\eta}} \sum_{\alpha=1}^{n+1} P_{\alpha\beta} \pi_{\eta,\alpha} + \sum_{\gamma \in G} \frac{[\sum_{\alpha=1}^{n+1} P_{\alpha\gamma} \pi_{\eta,\alpha}] [\sum_{\beta=1}^{n+1} P_{j\beta} \pi_{\eta,\beta}]}{\pi_{\eta\eta}}
\]
\[
+ O(\phi)
\]
\[
= \sigma_t(G) \sum_{\gamma \in B} \sum_{ij=1}^{n} F_{ij}^{\gamma} \pi_{\gamma,ij} + \sigma_t(G) \sum_{ij=1}^{n} F_{ij} \pi_{\eta,ij} - 2\sigma_t(G) \sum_{ij=1}^{n} F_{ij} \sum_{\eta \in G} \frac{\pi_{\eta,ij} \pi_{\eta,n,j}}{\pi_{\eta\eta}}
\]
\[
+ \sigma_t(G) \sum_{ij=1}^{n} F_{ij} \sum_{\eta \in G} \pi_{\eta,ij} \pi_{\eta,n,i} + \sum_{\gamma \in G} \pi_{\gamma,ij} \pi_{\gamma,n,j} + O(\phi)
\]
\[
= \sigma_t(G) \sum_{\gamma \in B} \sum_{ij=1}^{n} F_{ij}^{\gamma} \pi_{\gamma,ij} + \sigma_t(G) \sum_{ij=1}^{n} F_{ij} \pi_{\eta,ij}
\]
\[
- 2\sigma_t(G) \sum_{ij=1}^{n} F_{ij} \sum_{\eta \in G} \frac{\pi_{\eta,ij} \pi_{\eta,n,j}}{\pi_{\eta\eta}} + O(\phi + |\nabla_\phi|).
\]
For $\gamma \in B$, we have
\[ \bar{a}_{\gamma,ij} = -\frac{|u_{n+1}|}{|Du|u_{n+1}} A_{\gamma,ij} - \frac{|u_{n+1}|}{|Du|u_{n+1}^3} \bar{h}_{\gamma,ij} = -\frac{|u_{n+1}|}{|Du|u_{n+1}^3} \bar{u}_{y_{n+1}} u_{y_{n+1}} x_i x_j, \]
so
\[ \sigma_l(G) \sum_{\gamma \in B} F^{ij}_l \pi_{\gamma,ij} = -\frac{|u_{n+1}|}{|Du|u_{n+1}^3} \sigma_l(G) \sum_{\gamma \in B} u_{y_{n+1}}^2 \sum_{ij=1}^n F^{ij} u_{y_{n+1}} x_i x_j \]
\[ = 0. \quad (164) \]
From (163) and (164), (157) holds. \hfill \Box

**Lemma 4.6.**

\[ \phi_t = -u_{y_{n+1}}^{-3} \sigma_l(G)[u_{y_{n+1}} u_{y_{n+1}} x_i x_j - 2u_{y_{n+1}} u_{y_{n+1}} u_{y_{n+1}} x_i x_j] + O(\phi), \quad (165) \]
and
\[ \sum_{i,j=1}^n F^{ij} \phi_{ij} = u_{y_{n+1}}^{-3} \sigma_l(G)[u_{y_{n+1}}^2 \sum_{i,j=1}^n F^{ij} u_{y_{n+1}} x_i x_j \]
\[ + 6u_{y_{n+1}} u_{y_{n+1}} \sum_{i,j=1}^n F^{ij} u_{y_{n+1}} - 6u_{y_{n+1}}^2 \sum_{i,j=1}^n F^{ij} u_{y_{n+1}} \]
\[ + 2u_{y_{n+1}}^{-3} \sigma_l(G) \sum_{\alpha \in G} \sum_{i,j=1}^n F^{ij} \left( \frac{1}{u_{y_{n+1}}} u_{y_{n+1}} - 2u_{y_{n+1}} u_{y_{n+1}} \right) \]
\[ + O(\phi + |\nabla_x \phi|). \quad (166) \]

**Proof.** Similarly with (147), taking the first derivatives of $\phi$ in $t$, we have
\[ \phi_t = \frac{\partial \phi}{\partial t} = \sum_{\alpha \in 1}^n \sigma_l(\pi(\alpha) \pi_{\alpha,ij} = \sigma_l(G) \pi_{\alpha,ij} + O(\phi) \]
\[ = -\frac{|u_{n+1}|}{|Du|u_{n+1}^3} \sigma_l(G) \bar{h}_{\alpha,ij} + O(\phi) \]
\[ = -\frac{1}{u_{n+1}^3} \sigma_l(G)[u_{y_{n+1}} u_{y_{n+1}} x_i x_j - 2u_{y_{n+1}} u_{y_{n+1}} u_{y_{n+1}} x_i x_j] + O(\phi). \quad (167) \]
In the following, we prove (166). In fact, the calculation is similar as in [3] and [16].

It is easy to know
\[ \sum_{i,j=1}^n F^{ij} \pi_{n,n,ij} - 2 \sum_{i,j=1}^n F^{ij} \sum_{\eta \in G} \frac{\pi_{n,i} \pi_{m,j}}{\pi_{\eta \eta}} \]
\[ = \sum_{\alpha \epsilon 1}^{n+1} G_{\alpha \beta} \pi_{n,n,\alpha \beta} - 2 \sum_{\alpha \beta = 1}^{n+1} G_{\alpha \beta} \sum_{\eta \in G} \frac{\pi_{n,n,\alpha} \pi_{\eta \eta}}{\pi_{\eta \eta}}, \quad (168) \]
where
\[ G_{\alpha \beta} = \sum_{i,j=1}^n F^{ij} P_{\alpha \beta}. \quad (169) \]
By (21)-(23), we have
\[
\alpha_{nn,\alpha} = \left( -\frac{|u_{n+1}|}{Du|u_{n+1}|^3} \right) \alpha \tilde{A}_{nn} - \left( -\frac{|u_{n+1}|}{Du|u_{n+1}|^3} \right) \beta \tilde{A}_{nn,\alpha} = O(\phi) - \left( -\frac{|u_{n+1}|}{Du|u_{n+1}|^3} \right) \tilde{A}_{nn,\alpha}
\]
\[
= -\frac{|u_{n+1}|}{Du|u_{n+1}|^3} \tilde{h}_{nn,\alpha} + O(\phi)
\]
\[
= -\frac{1}{u_{n+1}^3} \left[ u_{y_{n+1}} u_{y_{y_n} y_n} - 2u_{y_{n+1}} u_{y_{y_n} y_n} u_{y_{y_{n+1}}} \right] + O(\phi), \quad (170)
\]
and
\[
\tilde{a}_{nn,\alpha\beta} = \left( -\frac{|u_{n+1}|}{Du|u_{n+1}|^3} \right) \alpha \tilde{A}_{nn} + \left( -\frac{|u_{n+1}|}{Du|u_{n+1}|^3} \right) \beta \tilde{A}_{nn,\alpha} + \left( -\frac{|u_{n+1}|}{Du|u_{n+1}|^3} \right) \tilde{A}_{nn,\alpha\beta}
\]
\[
= O(\phi) + \left( -\frac{|u_{n+1}|}{Du|u_{n+1}|^3} \right) \alpha \tilde{A}_{nn,\alpha} + \left( -\frac{|u_{n+1}|}{Du|u_{n+1}|^3} \right) \beta \tilde{A}_{nn,\alpha} + \left( -\frac{|u_{n+1}|}{Du|u_{n+1}|^3} \right) \tilde{A}_{nn,\alpha\beta}
\]
\[
= (-\frac{|u_{n+1}|}{Du|u_{n+1}|^3} \alpha \tilde{A}_{nn} + (-\frac{|u_{n+1}|}{Du|u_{n+1}|^3} )\beta \tilde{A}_{nn,\alpha} - \frac{1}{u_{n+1}^3} \tilde{h}_{nn,\alpha\beta} + O(\phi). \quad (171)
\]
and
\[
\tilde{h}_{nn,\alpha\beta} = u_{n+1}^2 u_{y_{y_n} y_n,\alpha\beta} + 2u_{n+1} u_{y_{y_{n+1}} y_{y_n}} u_{y_{y_n} y_n\beta} + 2u_{n+1} u_{y_{y_{n+1}} y_{y_n}} u_{y_{y_n} y_n\beta}
\]
\[
+ 2u_{y_{y_{n+1} y_{y_n}} y_{y_{n+1}}} u_{y_{y_{n+1}} y_{y_n}} u_{y_{y_n} y_n\beta} - 2u_{y_{y_{n+1} y_{y_n}} y_{y_{n+1}}} u_{y_{y_{n+1}} y_{y_n}} u_{y_{y_n} y_n\beta}
\]
\[
- 2u_{y_{n+1} y_{y_n} y_{y_n}} u_{y_{y_{n+1}} y_{y_n}} u_{y_{y_n} y_n\beta} + 2u_{y_{n+1} y_{y_n} y_{y_n}} u_{y_{y_{n+1}} y_{y_n}} u_{y_{y_n} y_n\beta}
\]
\[
+ 2u_{y_{n+1} y_{y_n} y_{y_n}} u_{y_{y_{n+1}} y_{y_n}} u_{y_{y_n} y_n\beta} - 2u_{y_{n+1} y_{y_n} y_{y_n}} u_{y_{y_{n+1}} y_{y_n}} u_{y_{y_n} y_n\beta}
\]
\[
- 2u_{y_{n+1} y_{y_n} y_{y_n}} u_{y_{y_{n+1}} y_{y_n}} u_{y_{y_n} y_n\beta} + 2u_{y_{n+1} y_{y_n}} \left[ -u_{y_{n+1} \tilde{a}_{nn,\beta}} \right]
\]
\[
+ 2u_{y_{n+1} y_{y_n}} \left[ -u_{y_{n+1} \tilde{a}_{nn,\alpha}} \right] + O(\phi). \quad (172)
\]
Hence,
\[
\sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} \tilde{a}_{nn,\alpha\beta} = \sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} \left[ -\frac{1}{u_{n+1}^3} \tilde{h}_{nn,\alpha\beta} \right] + O(\phi + |\nabla_x \phi|)
\]
\[
= -\frac{1}{u_{n+1}^3} \sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} \left[ u_{n+1}^2 u_{y_{y_n} y_n,\alpha\beta} + 4u_{y_{n+1} y_{y_n}} u_{y_{y_n} y_n\beta} u_{y_{y_n} y_{n+1}}
\]
\[
+ 2u_{y_{n+1} y_{n+1}} u_{y_{y_n} y_{y_n}} u_{y_{y_n} y_{n+1}} - 2u_{y_{n+1} y_{y_n}} u_{y_{y_n} y_{n+1}} u_{y_{y_n} y_{n+1}}
\]
\[
- 4u_{y_{n+1} y_{y_n}} u_{y_{y_n} y_{y_n}} u_{y_{y_n} y_{n+1}} \right] + O(\phi + |\nabla_x \phi|). \quad (173)
\]
where
\[
\sum_{\alpha,\beta=1}^{n+1} G^{\alpha\beta} u_{y_{n+1} y_{y_n}} u_{y_{y_n} y_{n+1}} = u_{n+1}^2 \sum_{\alpha=1}^{n+1} G^{\alpha n+1} u_{y_{n+1} y_{y_n}} + O(\phi)
\]
So

\[ u_{n+1} \sum_{\alpha, \beta=1}^{n+1} G^{\alpha \beta} a_{n,\alpha \beta} = - u_{n+1}^2 \sum_{\alpha, \beta=1}^{n+1} G^{\alpha \beta} u_{\alpha \beta} + 6u_{n+1} u_{n, \alpha \beta} \sum_{\alpha, \beta=1}^{n+1} G^{\alpha \beta} u_{\alpha \beta} - 6u_{y_{n+1} y_{n+1}}^2 \sum_{\alpha, \beta=1}^{n+1} G^{\alpha \beta} u_{\alpha \beta} - 4u_{n+1} u_{n, \alpha \beta} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n+1} G^{\alpha \beta} u_{\alpha \beta} + 8u_{n+1}^2 \sum_{\alpha=1}^{n} G^{\alpha \beta} u_{\alpha \beta} + 6u_{y_{n+1} y_{n+1}}^2 \sum_{\alpha, \beta=1}^{n} G^{\alpha \beta} u_{\alpha \beta} + O(\phi). \]  

(177)

and

\[ u_{y_{n+1}} u_{y_{n+1}} \sum_{\alpha=1}^{n+1} \sum_{\beta=1}^{n+1} G^{\alpha \beta} u_{\alpha \beta} = u_{y_{n+1}} u_{y_{n+1}} \left( \sum_{\alpha \in B} G^{\alpha \beta} u_{\alpha \beta} + \sum_{\alpha \in G} G^{\alpha \beta} u_{\alpha \beta} \right) 
= u_{y_{n+1}} u_{y_{n+1}} \sum_{\alpha \in G} G^{\alpha \beta} u_{\alpha \beta} 
= u_{y_{n+1}} u_{y_{n+1}} \sum_{\alpha \in G} G^{\alpha \beta} [-u_{y_{n+1}}^2 a_{n,\alpha \beta} + u_{y_{n+1} y_{n+1}} u_{\alpha \beta} + u_{y_{n+1} y_{n+1}} u_{y_{n+1} y_{n+1}}] + O(\phi) \]  

(178)
\[
- u_{y_{n+1}}^2 u_{y_{n+1}} y_{n+1} \sum_{\alpha \in G}^{n+1} G^{\alpha \beta} \bar{a}_{n \alpha \beta} + u_{y_{n+1}}^2 \sum_{\alpha \in G}^{n+1} G^{\alpha \alpha} u_{y_{\alpha \alpha}} \\
+ 2 u_{y_{n+1}}^2 \sum_{\alpha \in G} \frac{G^{\alpha \alpha + 1} u_{y_{\alpha \alpha + 1}}}{1} + O(\phi). \quad (178)
\]

(177) and (178) yield

\[
\sum_{\alpha, \beta = 1}^{n+1} G^{\alpha \beta} \bar{a}_{n \alpha \beta}^2 \]

\[
= - u_{y_{n+1}}^2 \sum_{\alpha, \beta = 1}^{n+1} G^{\alpha \beta} u_{y_{\alpha \alpha}} y_{\beta} + 6 u_{y_{n+1}} u_{y_{n+1}} \sum_{\alpha, \beta = 1}^{n+1} G^{\alpha \beta} u_{y_{\alpha \beta}} \\
- 6 u_{y_{n+1}}^2 \sum_{\alpha, \beta = 1}^{n+1} G^{\alpha \beta} u_{\alpha \beta} + 4 u_{y_{n+1}}^2 u_{y_{n+1}} \sum_{\alpha, \beta = 1}^{n+1} G^{\alpha \beta} \bar{a}_{n \alpha \beta} \\
+ 2 u_{y_{n+1}}^2 \sum_{\alpha \in G} \frac{G^{\alpha \alpha + 1} u_{y_{\alpha \alpha + 1}}}{1} + O(\phi). \quad (179)
\]

So,

\[
\sum_{\alpha, \beta = 1}^{n+1} G^{\alpha \beta} \bar{a}_{n \alpha \beta}^2 - 2 \sum_{\alpha, \beta = 1}^{n+1} G^{\alpha \beta} \frac{\bar{\eta}_{n \alpha \beta} \bar{\eta}_{n \alpha \beta}}{1} - 6 u_{y_{n+1}}^2 \sum_{\alpha, \beta = 1}^{n+1} G^{\alpha \beta} u_{\alpha \beta} \\
- 2 \sum_{\eta \in G} \left( \frac{\sum_{\alpha, \beta = 1}^{n+1} G^{\alpha \beta} \bar{\eta}_{n \alpha \beta} \bar{\eta}_{n \alpha \beta}}{1} - \frac{u_{y_{n+1}}^2}{1} \sum_{\beta = 1}^{n+1} G^{\alpha \beta} \bar{a}_{n \alpha \beta} - \frac{u_{y_{n+1}}^2}{1} \sum_{\eta \in G} \frac{G^{\eta \eta} u_{y_{\eta \eta}}}{1} \right) \\
+ O(\phi). \quad (180)
\]

In fact, for any \( \eta \in G, \)

\[
\sum_{\alpha, \beta = 1}^{n+1} \frac{G^{\alpha \beta} \bar{\eta}_{n \alpha \beta} \bar{\eta}_{n \alpha \beta}}{1} - \frac{u_{y_{n+1}}^2}{1} \sum_{\beta = 1}^{n+1} G^{\alpha \beta} \bar{a}_{n \alpha \beta} - \frac{u_{y_{n+1}}^2}{1} \sum_{\eta \in G} \frac{G^{\eta \eta} u_{y_{\eta \eta}}}{1} \\
= - \frac{1}{1} \frac{u_{y_{n+1}}^2}{1} \left( \sum_{\alpha, \beta = 1}^{n+1} \frac{G^{\alpha \beta} \bar{\eta}_{n \alpha \beta} \bar{\eta}_{n \alpha \beta}}{1} - \frac{u_{y_{n+1}}^2}{1} \sum_{\beta = 1}^{n+1} G^{\alpha \beta} \bar{a}_{n \alpha \beta} - \frac{u_{y_{n+1}}^2}{1} \sum_{\eta \in G} \frac{G^{\eta \eta} u_{y_{\eta \eta}}}{1} \right) \\
= - \frac{1}{1} \frac{u_{y_{n+1}}^2}{1} \left[ u_{y_{n+1}} u_{\eta \eta}^2 - 2 u_{\eta \eta} u_{y_{n+1}}^2 y_{n+1} \right] u_{y_{n+1}} u_{\eta \eta}^2 - 2 u_{\eta \eta} u_{y_{n+1}}^2 y_{n+1} \right] + O(\phi). \]

Then we get

\[
\sum_{\alpha, \beta = 1}^{n+1} G^{\alpha \beta} \bar{a}_{n \alpha \beta}^2 - 2 \sum_{\alpha, \beta = 1}^{n+1} G^{\alpha \beta} \frac{\bar{\eta}_{n \alpha \beta} \bar{\eta}_{n \alpha \beta}}{1} \\
= - u_{y_{n+1}}^2 u_{y_{n+1}} y_{n+1} \sum_{\alpha \in G}^{n+1} G^{\alpha \beta} \bar{a}_{n \alpha \beta} + u_{y_{n+1}}^2 \sum_{\alpha \in G}^{n+1} G^{\alpha \alpha} u_{y_{\alpha \alpha}} \\
+ 2 u_{y_{n+1}}^2 \sum_{\alpha \in G} \frac{G^{\alpha \alpha + 1} u_{y_{\alpha \alpha + 1}}}{1} + O(\phi). \quad (180)
\]
Proof. First, we consider a special case: $F^{ij} = \delta_{ij}$. That is, we need to prove

$$\sum_{\alpha \in G} \sum_{i,j=1}^{n} F^{ij} \frac{1}{u_{\alpha,y_{\alpha}}} [u_{y_{\alpha+1}, i} y_{\alpha+1}]^2 + O(\phi + |\nabla x \phi|).$$

Form (138)-(140), (143) and (144), we have

$$u_{y_{\alpha+1}, i} y_{\alpha+1} = 0, \quad \alpha \in G, i \in B;$$

$$u_{y_{\alpha+1}, i} y_{\alpha+1} = 0, \quad \alpha \in B, i \in G \cup \{n\};$$

$$u_{y_{\alpha+1}, i} y_{\alpha+1} = O(\phi + |\nabla x \phi|), \quad i \in G \cup \{n\}.$$

Since $\nabla_y^2 u = \left(\begin{array}{ccc} u_{y_{\alpha+1}, i} y_{\alpha+1} & \cdots & u_{y_{\alpha+1}, i} y_{\alpha+1} \\ \vdots & \ddots & \vdots \\ u_{y_{\alpha+1}, i} y_{\alpha+1} & \cdots & u_{y_{\alpha+1}, i} y_{\alpha+1} \end{array}\right) \leq 0$ is diagonal, by the approximation, we have for any $i \in G \cup \{n\}$,

$$\sum_{\alpha \in G} \sum_{i,j=1}^{n} \frac{1}{u_{\alpha,y_{\alpha}}} [u_{y_{\alpha+1}, i} y_{\alpha+1}]^2 = \lim_{\varepsilon \to 0^+} (\nabla_y^2 u - \varepsilon I_n)\left(\begin{array}{ccc} u_{y_{\alpha+1}, i} y_{\alpha+1} & \cdots & u_{y_{\alpha+1}, i} y_{\alpha+1} \\ \vdots & \ddots & \vdots \\ u_{y_{\alpha+1}, i} y_{\alpha+1} & \cdots & u_{y_{\alpha+1}, i} y_{\alpha+1} \end{array}\right) + O(\phi + |\nabla x \phi|),$$

where $\varepsilon > 0$ small, and

$$(\nabla_y^2 u - \varepsilon I_n)\left(\begin{array}{ccc} u_{y_{\alpha+1}, i} y_{\alpha+1} & \cdots & u_{y_{\alpha+1}, i} y_{\alpha+1} \\ \vdots & \ddots & \vdots \\ u_{y_{\alpha+1}, i} y_{\alpha+1} & \cdots & u_{y_{\alpha+1}, i} y_{\alpha+1} \end{array}\right) + O(\phi + |\nabla x \phi|).$$

Denote

$$C := u_{z_i z_n} - \varepsilon - \sum_{i=1}^{l} \frac{u_{z_i z_n}^2}{u_{z_i z_n} - \varepsilon} < 0,$$
Hence, (183) holds from (184) and (191).

\[
\left(\nabla^2 u - \epsilon I_n\right)^{-1} = \frac{1}{C} \left( - \frac{u_{z_1 z_n}}{u_{z_1 z_1} - \epsilon}, \ldots, - \frac{u_{z_1 z_n}}{u_{z_1 z_1} - \epsilon}, 0, \ldots, 1 \right)^T \left( - \frac{u_{z_1 z_n}}{u_{z_1 z_1} - \epsilon}, \ldots, - \frac{u_{z_1 z_n}}{u_{z_1 z_1} - \epsilon}, 0, \ldots, 1 \right)
\]

\[+ \text{diag}(\frac{1}{u_{z_1 z_1} - \epsilon}, \ldots, 1, 0, \ldots, 0).\]

So

\[
\left(\nabla^2 u - \epsilon I_n\right)^{-1} \left(u_{y_{n+1}} \nabla^2 u - \epsilon I_n\right)^{-1} (u_{y_{n+1}} - 2 \nabla u_{y_{n+1}})^T \leq \sum_{k \in G} \frac{1}{u_{z_1 z_1} - \epsilon} \left[u_{y_{n+1}} - 2 u_{x_k z_k} u_{y_{n+1}}\right]^2
\]

\[= \sum_{k \in G} \frac{1}{u_{z_1 z_1} - \epsilon} \left[u_{y_{n+1}} - 2 u_{x_k z_k} u_{y_{n+1}}\right]^2. \quad (190)
\]

Then we have for \( i \in G \cup \{n\} \)

\[
\lim_{\epsilon \to 0^+} \sum_{k \in G} \frac{1}{u_{z_1 z_1} - \epsilon} \left[u_{y_{n+1}} - 2 u_{x_k z_k} u_{y_{n+1}}\right]^2 + O(\phi + |\nabla_x \phi|) = \sum_{k \in G} \frac{1}{u_{z_1 z_1} - \epsilon} \left[u_{y_{n+1}} - 2 u_{x_k z_k} u_{y_{n+1}}\right]^2 + O(\phi + |\nabla_x \phi|). \quad (191)
\]

Hence, (183) holds from (184) and (191).

For the general case, the CLAIM also holds following the above proof.

\[\text{Theorem 4.8. Under the assumptions of Theorem 1.2 and the above notations, we have for any fixed point } (x, t) \in O \times (t_0 - \delta, t_0), \]

\[
\sum_{i,j=1}^{n} F^{ij} \phi_{ij} - \phi_t \leq c_0(\phi + |\nabla_x \phi|). \quad (192)
\]

Proof. From (165), (166) and (182),

\[
\sum_{i,j=1}^{n} F^{ij} \phi_{ij} - \phi_t \leq u_{y_{n+1}}^{-3} \sigma_1(G) \left[ - u_{y_{n+1}}^2 \left( \sum_{i,j=1}^{n} F^{ij} u_{ijy_{n+1} - u_{y_{n+1}} u_{y_{n+1}}} - 2 u_{ijy_{n+1}} u_{y_{n+1}} \right) + 6 u_{y_{n+1}} u_{y_{n+1}} \right] + 6 u_{y_{n+1}} u_{y_{n+1}} \sum_{i,j=1}^{n} F^{ij} u_{ijy_{n+1}} \right]
\]

\[
+ 2 \sigma_1(G) \sum_{k \in G} \sum_{i,j=1}^{n} F^{ij} u_{kz_k z_k} u_{y_{n+1}} u_{y_{n+1}} u_{y_{n+1}} u_{y_{n+1}} u_{y_{n+1}} + O(\phi + |\nabla_x \phi|). \quad (193)
\]
From the equation (1), we get

\[ u_{yn,nt} = \sum_{ij=1}^{n} F^{ij}_{yn,yn,ij} + \sum_{k=1}^{n} F^{u,k}_{yk,yn,yn} + \sum_{ijkl=1}^{n} F^{ij,kl}_{yn,yn,yn} + \frac{\partial x_k}{\partial y_n} \]

\[ \sum_{i,j=1}^{n} F^{ij}_{yn,yn,ij} + \sum_{i,j=1}^{n} F^{ij,uk}_{yn,yn,yn} + \sum_{k=1}^{n} F^{u,k,uk}_{yn,yn,yn} = \sum_{i,j=1}^{n} F^{ij,uk}_{yn,yn,yn} + \sum_{k=1}^{n} F^{u,k,uk}_{yn,yn,yn} + \sum_{i,j=1}^{n} F^{ij,uk}_{yn,yn,yn} + \sum_{k=1}^{n} F^{u,k,uk}_{yn,yn,yn} \]

And from (152),

\[ u_{xn,yn} = u_{yn+1,yn} + O(\phi) \]

so

\[ u_{xn,yn} = u_{yn+1,yn} + O(\phi) \]

Hence

\[ \sum_{ij=1}^{n} F^{ij}_{uijy,ukly,yn} u^{2}_{yn,yn+1} + \sum_{ij=1}^{n} F^{ij,uk}_{yn,yn,yn} u^{2}_{yn,yn+1} + 2 \sum_{ij=1}^{n} F^{ij,uk}_{yn,yn,yn} u^{2}_{yn,yn+1} \]

\[ Q = \sum_{i,j=1}^{n} F^{ij,uk}_{yn,yn,yn} u^{2}_{yn,yn+1} + 2 \sum_{ij=1}^{n} F^{ij,uk}_{yn,yn,yn} u^{2}_{yn,yn+1} \]
SPACETIME QUASICONECAVE SOLUTIONS 4807

\[ + 2 \sum_{i,j,k=1}^{n} F^{ij,jk} u_{ij,y} \frac{\partial x_{k}}{\partial y_{n}} u_{n+1}^{2} + 2 \sum_{i,j=1}^{n} F^{ij,t} u_{ij,y} \frac{\partial t}{\partial y_{n}} u_{n+1}^{2} + F^{u,u} \frac{u_{n+1}^{2}}{u_{x_{n},y_{n}}^{2}} \]

\[ + 2 \sum_{i=1}^{l} F^{u,x} \frac{\partial x_{i}}{\partial y_{n}} u_{n+1}^{2} + 2 F^{u,t} u_{x_{n},y_{n}} \frac{\partial t}{\partial y_{n}} u_{n+1}^{2} + 2 F^{u} u_{x_{n},y_{n}} \frac{u_{n+1}^{2}}{u_{x_{n},y_{n}}^{2}} \]

\[ + 6 \sum_{i,j=1}^{n} F^{ij} \frac{u_{n+1}}{u_{x_{n}}} \sum_{i,j=1}^{n} F^{ij} u_{ij} - 6 u_{n+1}^{2} u_{x_{n}}^{2} \sum_{i,j=1}^{n} F^{ij} u_{ij} \]

(198)

From (153), we have

\[ u_{y_{n+1}} u_{y_{n+1}} u_{y_{n+1}} = u_{i} u_{y_{n+1}}^{2} + O(\phi) \geq O(\phi), \quad (199) \]

Set

\[ s = \frac{1}{u_{x_{n}}}, A_{ij} = su_{ij}, \theta = (0, \cdots, 0, 1), \]

\[ X_{\alpha \beta} = u_{x_{\alpha} x_{\beta}} y_{n}, \]

\[ Y = u_{x_{n}, y_{n}} u_{x_{n}}, \]

\[ Z_{k} = \frac{\partial x_{k}}{\partial y_{n}} u_{x_{n}}, \]

\[ D = \frac{\partial t}{\partial y_{n}} u_{x_{n}}, \]

\[ V = (X_{\alpha \beta}, Y, (Z_{i}), D) \in S^{n} \times \mathbb{R} \times \mathbb{R}^{n} \times \mathbb{R}; \]

then we get

\[ X_{i \alpha} = 0, \quad A_{i \alpha} = 0, \quad X_{i \alpha} - 2 A_{i \alpha} Y = 0, \quad i \in B. \]

So it yields

\[ Q = \frac{u_{n+1}^{2}}{u_{x_{n}}^{2}} Q^{*}(V, V), \quad (200) \]

where \( Q^{*}(V, V) \) is defined in (31).

From the structural condition (3) (i.e. Lemma 2.5),

\[ Q^{*}(V, V) \leq 0. \]

so we get

\[ Q = \frac{u_{n+1}^{2}}{u_{x_{n}}^{2}} Q^{*}(V, V) \leq 0. \quad (201) \]

Hence (192) holds from (197), (199) and (201).
4.3. Constant rank theorem of the spacetime second fundamental form for the equations (6)-(8). Following the proof of Theorem 1.2, we establish the constant rank theorem for the spacetime fundamental form for the equations (6)-(8) in this subsection as follows.

**Theorem 4.9.** Suppose $u \in C^{3,1}(\Omega \times [0, T])$ is a spacetime quasiconcave to the parabolic equation (6)-(8), and satisfies the condition (4). Then the second fundamental form of spatial level sets $\Sigma^c = \{(x, t) \in \Omega \times (0, T)|u(x, t) = c\}$ has the same constant rank in $\Omega$ for any fixed $t \in (0, T)$. Moreover, let $(t)$ be the minimal rank of the second fundamental form in $\Omega$, then $l(s) \leq l(t)$ for all $0 < s \leq t \leq T$.

**Proof.** The proof is following the proof of Theorem 1.2.

Suppose $\hat{a}(x, t) = (\hat{a}_{\alpha\beta})_{n \times n}$ attains the minimal rank $l$ at some point $(x_0, t_0) \in \Omega \times (0, T)$. We may assume $l \leq n - 1$, otherwise there is nothing to prove. At $(x_0, t_0)$, we may choose $e_1, \cdots, e_{n-1}, e_n$ such that

$$|\nabla u(x_0, t_0)| = u_n(x_0, t_0) > 0 \text{ and } (u_{ij})_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x_0, t_0). \tag{202}$$

Without loss of generality we assume $u_{11} \leq u_{22} \leq \cdots \leq u_{n-1,n-1}$. So, at $(x_0, t_0)$, from (104), we have the matrix $(\hat{a}_{ij})_{1 \leq i, j \leq n-1}$ is also diagonal, and $\hat{a}_{11} \geq \hat{a}_{22} \geq \cdots \geq \hat{a}_{n-1,n-1}$. From lemma 2.3, there is a positive constant $C_0$ such that at $(x_0, t_0)$

**Case 1.**

$$\hat{a}_{11} \geq \cdots \geq \hat{a}_{l-1,l-1} \geq C_0, \quad \hat{a}_{ll} = \cdots = \hat{a}_{n-1,n-1} = 0,$$

$$\hat{a}_{nn} - \sum_{i=1}^{l-1} \frac{\hat{a}_{1i}^2}{\hat{a}_{ii}} \geq C_0, \quad \hat{a}_{in} = 0, \quad l \leq i \leq n - 1.$$

**Case 2.**

$$\hat{a}_{11} \geq \cdots \geq \hat{a}_{l,l} \geq C_0, \quad \hat{a}_{l+1,l+1} = \cdots = \hat{a}_{n-1,n-1} = 0,$$

$$\hat{a}_{ll} = \sum_{i=1}^{l} \frac{\hat{a}_{1i}^2}{\hat{a}_{ii}}, \quad \hat{a}_{in} = 0, \quad l + 1 \leq i \leq n - 1.$$

For the Case 1, the theorem holds from Subsection 4.1 and Theorem 3.3, Theorem 3.5, Theorem 3.7.

For the Case 2, we need to prove the differential inequality (192), which is similar to the proof of Theorem 3.3, Theorem 3.5, and Theorem 3.7, with some modifications. In the following, we just prove (192) for the equation (6). And for the equation (7) and (8), the proofs follow from the proofs of Theorem 3.5 and Theorem 3.7 with the same modifications.

For the equation (6), following the assumptions and notations, we need to prove

$$\sum_{\alpha,\beta=1}^{n} L_{\alpha\beta} \phi_{\alpha\beta}(x, t) - \phi_t \leq C(|\phi| + |\nabla_x \phi|), \quad \forall (x, t) \in \mathcal{O} \times (t_0 - \delta_0, t_0), \tag{203}$$

where $\phi$ is defined in (128) and $C$ is a positive constant independent of $\phi$. Then by a approximation, (203) holds for $t = t_0$. Then by the strong maximum principle and the method of continuity, we can prove Theorem 4.9 under Case 2.

For any fixed point $(x, t) \in \mathcal{O} \times (t_0 - \delta, t_0]$, following the Subsection 4.2, we first choose the space coordinates $e_1, \cdots, e_{n-1}, e_n, e_{n+1}$ still the time coordinate with $y = (y_1, \cdots, y_n, y_{n+1}) = (x, t)P, \quad P = (P_{\alpha\beta})_{(n+1) \times (n+1)} = OT$, then
such that
\[
|\nabla u(x, t)| = u_n(x, t) > 0 \text{ and } \left( \frac{\partial}{\partial y_i} u_j \right)_{1 \leq i, j \leq n-1} \text{ is diagonal at } (x, t). \tag{204}
\]

Finally, we get a new spacetime coordinate \( \{ \tilde{e}_1, \cdots, \tilde{e}_l, \tilde{e}_{l+1}, \cdots, \tilde{e}_{n-1}, \tilde{e}_n, \tilde{e}_{n+1} \} \) such that
\[
|Du| > 0, \quad u_{y_1} = \cdots = u_{y_n} = 0, \quad \text{at } (x, t), \tag{205}
\]
\[
\left( \frac{\partial}{\partial y_i} u_j \right)_{1 \leq i, j \leq n} \text{ is diagonal at } (x, t). \tag{206}
\]

Also we will use \( i, j, k, l = 1, \cdots, n \) to represent the \( x \) coordinates, \( t \) still the time coordinate, and \( \alpha, \beta, \gamma, \eta = 1, \cdots, n+1 \) the \( y \) coordinates.

Following the proof of Theorem 4.8, we get from (197)
\[
\sum_{i,j=1}^{n} L_{ij} \phi_{ij} - \phi_t \leq u_{y_{n+1}}^{-3} \sigma_i(G) \left( Q - 2u_{y_{n+1}} u_{y_n y_{n+1}} u_{y_{n+1} t} \right) + O(\phi + |\nabla x \phi|)
\]
\[
= u_{y_{n+1}}^{-3} \sigma_i(G) \left( Q^*(V, V) - 2u_{x_n} u_{x_n y_n} u_{y_n t} \right) + O(\phi + |\nabla x \phi|), \tag{207}
\]

where
\[
Q^*(V, V) = 2 \sum_{k,l=1}^{n} \frac{\partial L_{kl}}{\partial u_n} u_{kly_n} u_{ny_n} u_n^2 + \sum_{k,l=1}^{n} \frac{\partial^2 L_{kl}}{\partial u_n^2} u_{kly_n} u_{ny_n} u_n^2
\]
\[
+ 2 \sum_{k,l=1}^{n} \frac{\partial L_{kl}}{\partial u_n} u_{kly_n} u_{ny_n}^2 + 6u_n u_{ny_n} \sum_{k,l=1}^{n} L_{kl} u_{kly_n} - 6u_{ny_n}^2 \sum_{k,l=1}^{n} L_{kl} u_{kly_n}
\]
\[
+ 2 \sum_{i \in G} \sum_{\alpha, \beta=1}^{n} \frac{1}{u_{ij}} L_{i\alpha \beta} [u_n u_{i\alpha y_n} - 2u_{i\alpha n y_n}] [u_n u_{i\beta y_n} - 2u_{i\beta n y_n}].
\]

Under the coordinate (204) (i.e. (76)), we still have (78) - (84). So from the equation (6), we know
\[
u_t = L_{kk} u_{kk},
\]
and differentiating the equation in \( y_n \),
\[
u_{t y_n} = L_{kk} u_{kk y_n} + \frac{\partial L_{kl}}{\partial u_n} u_{kly_n} u_{kl} = L_{kk} u_{kk y_n} + \frac{\partial L_{kl}}{\partial u_n} u_{ny_n} u_{kl} + O(\phi)
\]
\[
= L_{kk} u_{kk y_n} + (p - 2) \frac{L_{kk}}{u_n} u_{ny_n} u_{kk} + O(\phi)
\]
\[
= L_{kk} u_{kk y_n} + (p - 2) \frac{u_t}{u_n} u_{ny_n} + O(\phi). \tag{208}
\]

So
\[
Q^*(V, V) = 2 \sum_{k=1}^{n} (p - 2) \frac{L_{kk}}{u_n} u_{kk y_n} u_{ny_n} u_n^2 + \sum_{k=1}^{n} (p - 2)(p - 3) \frac{L_{kk}}{u_n^2} u_{kk y_n} u_{ny_n}^2 u_n^2
\]
\[
+ 2 \sum_{k=1}^{n} (p - 2) \frac{L_{kk}}{u_n} u_{kk y_n} u_{ny_n}^2 + 6u_n u_{ny_n} \sum_{k=1}^{n} L_{kk} u_{kk y_n} - 6u_{ny_n}^2 u_t
\]
Hence from (152) and (153),

\[ Q^*(V, V) - 2u_{ny}u_{ty} - (p^2 + p)u_tu_{ny}^2 + 2 \left( \sum_{G \alpha \in G} u_{ii} \right) \frac{1}{u_{ii}} L_{\alpha \alpha} \left[ u_{iay} - 2u_{iay} - 2u_{iay}^2 \right] + O(\phi) \]

\[ = 2(p - 2) \left[ u_{ty} - (p - 2) \frac{u_t}{u_n} u_{ny} \right] u_{ny} u_n + (p - 2)(p - 3)u_t u_{ny}^2 \]

\[ + 2(p - 2)u_t u_{ny}^2 + 6u_{ny} \left[ u_{ty} - (p - 2) \frac{u_t}{u_n} u_{ny} \right] - 6u_{ny}^2 u_t \]

\[ + 2 \sum_{G \alpha \in G} \frac{1}{u_{ii}} L_{\alpha \alpha} \left[ u_{iay} - 2u_{iay} \right]^2 + O(\phi) \]

\[ = (2p + 2)u_{ny} u_{ny} u_{ty} - (p^2 + p)u_t u_{ny}^2 \]

\[ + 2 \sum_{G \alpha \in G} \frac{1}{u_{ii}} L_{\alpha \alpha} \left[ u_{iay} - 2u_{iay} \right]^2 + O(\phi). \]

So we get (203). \qed

**Acknowledgments.** The author would like to thank Prof. Xi-Nan Ma and Prof. Paolo Salani for the joint work [13] and the enlightenment in this paper. Also, the author would like to express sincere gratitude to Prof. Pengfei Guan for the advice on the choice of coordinates in Dec. 2012. At last, the author is very grateful to the referee for the careful reading and the valuable suggestions.

**REFERENCES**

[1] L. V. Ahlfors, *Conformal Invariants: Topics in Geometric Function Theory*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1973.

[2] B. Bian and P. Guan, A microscopic convexity principle for nonlinear partial differential equations, *Inventiones Math.*, **177** (2009), 307–335.

[3] B. Bian, P. Guan, X. N. Ma and L. Xu, A constant rank theorem for quasiconcave solutions of fully nonlinear partial differential equations, *Indiana Univ. Math. J.*, **60** (2011), 101–119.

[4] C. Bianchini, M. Longinetti and P. Salani, Quasiconcave solutions to elliptic problems in convex rings, *Indiana Univ. Math. J.*, **58** (2009), 1565–1589.

[5] C. Borell, Brownian motion in a convex ring and quasiconcavity, *Comm. Math. Phys.*, **86** (1982), 143–147.

[6] C. Borell, A note on parabolic convexity and heat conduction, *Ann. Inst. H. Poincaré Probab. Statist.*, **32** (1996), 387–393.

[7] C. Borell, Diffusion equations and geometric inequalities, *Potential Anal.*, **12** (2000), 49–71.
[8] L. Caffarelli and A. Friedman, Convexity of solutions of some semilinear elliptic equations, *Duke Math. J.*, 52 (1985), 431–456.
[9] L. Caffarelli and J. Spruck, Convexity properties of solutions to some classical variational problems, *Comm. Part. Diff. Eq.*, 7 (1982), 1337–1379.
[10] S.-Y. A. Chang, X. N. Ma and P. Yang, Principal curvature estimates for the convex level sets of semilinear elliptic equations, *Discrete Contin. Dyn. Syst.*, 28 (2010), 1151–1164.
[11] C. Q. Chen, On the microscopic spacetime convexity principle of fully nonlinear parabolic equations I: Spacetime convex solutions, *Discrete Contin. Dyn. Syst. A*, 34 (2014), 3383–3402.
[12] C. Q. Chen and B. W. Hu, A microscopic convexity principle for spacetime convex solutions of fully nonlinear parabolic equations, *Acta Mathematica Sinica, English Series*, 29 (2013), 651–674.
[13] C. Q. Chen, X. N. Ma and P. Salani, On the spacetime quasiconcave solutions of the heat equation, preprint, arXiv:1405.6373.
[14] C. Q. Chen and S. J. Shi, Curvature estimates for the level sets of spatial quasiconcave solutions to a class of parabolic equations, *Science China Mathematics*, 54 (2011), 2063–2080.
[15] R. Gabriel, A result concerning convex level surfaces of 3-dimensional harmonic functions, *J. London Math. Soc.*, 32 (1957), 286–294.
[16] P. Guan and L. Xu, Convexity estimates for level surfaces of quasiconcave solutions to fully nonlinear elliptic equations, *J. Reine Angew. Math.*, 680 (2013), 41–67.
[17] B. W. Hu and X. N. Ma, Constant rank theorem of the spacetime convex solution of heat equation, *manuscripta math.*, 138 (2012), 89–118.
[18] K. Ishige and P. Salani, Parabolic quasi-concavity for solutions to parabolic problems in convex rings, *Math. Nachr.*, 283 (2010), 1526–1548.
[19] K. Ishige and P. Salani, On a new kind of convexity for solutions of parabolic problems, *Discret. Contin. Dyn. Syst. Ser. S*, 4 (2011), 851–864.
[20] K. Ishige and P. Salani, Parabolic power concavity and parabolic boundary value problems, *Math. Ann.*, 358 (2014), 1091–1117.
[21] B. Kawhol, *Rearrangements and Convexity of Level Sets in PDE*, Springer Lecture Notes in Math. 1150, 1985.
[22] N. Korevaar, Convexity of level sets for solutions to elliptic ring problems, *Comm. Part. Diff. Eq.*, 15 (1990), 541–556.
[23] J. Lewis, Capacitary functions in convex rings, *Arch. Rat. Mech. Anal.*, 66 (1977), 201–224.
[24] G. Lieberman, *Second Order Parabolic Differential Equations*, World Scientific, 1996.
[25] M. Longinetti, Convexity of the level lines of harmonic functions, *(Italian) Boll. Un. Mat. Ital. A*, 2 (1983), 71–75.
[26] M. Longinetti, On minimal surfaces bounded by two convex curves in parallel planes, *J. Diff. Equations*, 67 (1987), 344–358.
[27] X. N. Ma, Q. Z. Ou and W. Zhang, Gaussian curvature estimates for the convex level sets of p-harmonic functions, *Comm. Pure Appl. Math.*, 63 (2010), 935–971.
[28] M. Ortel and W. Schneider, Curvature of level curves of harmonic functions, *Canad. Math. Bull.*, 26 (1983), 399–405.
[29] M. Shiffman, On surfaces of stationary area bounded by two circles or convex curves in parallel planes, *Annals of Math.*, 63 (1956), 77–90.
[30] F. Treves, A new method of proof of the subelliptic estimates, *Commun. Pure Appl. Math.*, 24 (1971), 71–115.
[31] L. Xu, A microscopic convexity theorem of level sets for solutions to elliptic equations, *Cal. Var. PDE*, 40 (2011), 51–63.

Received November 2014; revised March 2016.
E-mail address: cqchen@mail.ustc.edu.cn