ANOTHER PRODUCT FOR A BORCHERDS FORM

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In a celebrated pair of papers \[1\] and \[2\], Borcherds constructed meromorphic modular forms on the locally symmetric varieties associated to rational quadratic spaces $V$ of signature $(n, 2)$. More precisely, for an even lattice $M$ with respect to the symmetric bilinear form $(\ , \ )$, there is a finite Weil representation $\rho_M$ of an extension $\Gamma'$ of $\text{SL}_2(\mathbb{Z})$ on the group algebra $S_M = \mathbb{C}[M^\vee/M]$, where $M^\vee$ is the dual lattice of $M$. A weakly holomorphic modular form $F$ of weight $1 - \frac{n}{2}$ and type $\rho_M$ is an $S_M$-valued holomorphic function of $\tau \in \mathcal{H}$, the upper half-plane, with transformation law

$$F(\gamma'(\tau)) = (c\tau + d)^{1 - \frac{n}{2}} \rho_M(\gamma') F(\tau),$$

for $\gamma' \in \Gamma'$, and with Fourier expansion of the form

$$F(\tau) = \sum_m c(m) q^m, \quad c(m) \in S_M,$$

where $m \in \mathbb{Q}$ and where there are only a finite number of nonvanishing terms with $m < 0$, i.e., $F$ is meromorphic at the cusp. Let $D$ be one component of the space of oriented negative 2-planes in $V(\mathbb{R}) = M \otimes_{\mathbb{Z}} \mathbb{R}$. Assuming that the $c(m)$ for $m \leq 0$ lie in $\mathbb{Z}[M^\vee/M]$, Borcherds constructs a meromorphic modular form $\Psi(F)$ on $D$ of weight $\frac{1}{2} c(0)(0)$ with respect to an arithmetic group $\Gamma_M$ in $\text{Aut}(M)$. The divisor of $\Psi(F)$ is given explicitly in terms of the $c(m)$'s for $m < 0$ and, most remarkably, in a suitable neighborhood of any point boundary component, $\Psi(F)$ is given by an explicit infinite product.

In the present paper, assuming that the rational quadratic space $V = M \otimes_{\mathbb{Z}} \mathbb{Q}$ contains isotropic 2-planes, we give another family of product formulas for $\Psi(F)$, each valid in a neighborhood of the 1-dimensional boundary component associated to such a 2-plane $U$.

In the simplest case, suppose that $M = L$ is an even unimodular lattice of signature $(n, 2)$ and that there is a Witt decomposition

\[(0.1) \quad V = U + V_0 + U', \]

of $V$ such that\[3\]

$$L = L_U + L_0 + L_U',$$

where $L_U = L \cap U$, $L_U' = L \cap U'$, and $L_0 = L \cap V_0$. Note that $L_0$ is even unimodular and positive definite. In this case, $S_L = \mathbb{C} \varphi_0$ is one dimensional, with basis vector $\varphi_0$ associated to the zero element of $L^\vee/L$, and we can write the input form $F = F_0 \varphi_0$ where $F_0$ is scalar valued. Write $c(m) = c_0(m) \varphi_0$. Also associated to the decomposition \[(0.1)\] and a choice

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1 Equivalently, Borcherds usually works with signature $(2, n)$.

2 In a common terminology, ‘L splits two unimodular hyperbolic planes.’
of basis $e_1$ and $e_2$ for $M_U$ and dual basis $e'_1$ and $e'_2$ for $U'$, is a realization of $D$ as a Siegel domain of the third kind:

$$D \simeq \{(\tau_1, \tau_2', w_0) \in \mathcal{H} \times \mathbb{C} \times V_0(\mathbb{C}) \mid 4v_1v'_2 + Q(w_0 - \bar{w}_0) > 0\},$$

where $v_1 = \text{Im}(\tau_1)$, $v'_2 = \text{Im}(\tau_2')$ and $Q(x) = \frac{1}{2}(x, x)$. We write $q_1 = e(\tau_1)$ and $q_2 = e(\tau_2')$, where $e(t) = e^{2\pi it}$. In these coordinates, our product formula has the following form.

**Theorem A.** In a suitable neighborhood of the 1-dimensional boundary component associated to $U$, the associated Borcherds form $\Psi(F)$ is the product of the factors

$$(0.2) \prod \prod \prod_{a \in \mathbb{Z}, b \in \mathbb{Z}, x_0 \in L_0} (1 - q_2^a q_1^b e(-(x_0, w_0)))^{c_0(ab - Q(x_0))}$$

and

$$(0.3) \kappa q_2^{l_0} \eta(\tau_1)^{c_0(0)} \prod_{x_0 \in L_0, x_0 \neq 0} \left( \frac{\vartheta_1(-(x_0, w_0), \tau_1)}{\eta(\tau_1)} \right)^{c_0(-Q(x_0))/2}$$

where $\kappa$ is a scalar of absolute value 1 and

$$I_0 = -\sum_m \sum_{x_0 \in L_0} c_0(-m) \sigma_1(m - Q(x_0)).$$

Here $\eta(\tau)$ is the Dedekind eta-function, $\vartheta_1(\tau, z)$ is the Jacobi theta function $[11,17]$, and $\sigma_1$ is the usual divisor function extended by the conventions $\sigma_1(r) = 0$ if $r \notin \mathbb{Z}_{\geq 0}$ and $\sigma_1(0) = -\frac{1}{24}$. Note that in the product $[11,3]$, $x_0$ runs over a finite set. The result in the general case, i.e., for any $M$ and $U$, has a similar shape, cf. Theorem $[2.1]$ and Corollary $[2.3]$ in section 2. Note that the scalar $\kappa$ arises due to the fact that $\Psi(F)$ is only defined up to such a factor. Of course, if there are several inequivalent isotropic planes, it remains to determine how these factors vary.

Our proof of the product formula is a variant of that of Borcherds $[2]$. There he computes the regularized theta lift of $F$ in the tube domain coordinates associated to the maximal parabolic subgroup stabilizing an isotropic line. He observes that, in a suitable neighborhood of the cusp and up to terms ultimately arising from a Petersson norm, the regularized theta integral is the log $| \cdot |^2$ of a holomorphic function on that neighborhood. Since, up to an explicit singularity along some special divisors, the regularized integral is globally defined and automorphic, Borcherds is able to conclude the existence of the meromorphic modular form $\Psi(F)$ with the given product expansions.

Analogously, we compute the regularized theta lift in the (Siegel domain of the third kind) coordinates associated to the maximal parabolic subgroup stabilizing an isotropic 2-plane $U$. Again in a suitable neighborhood of the 1-dimensional boundary component associated to $U$, we find that the regularized lift is the log $| \cdot |^2$ of a meromorphic function with a product formula, as described in a special case in Theorem A. One main difference between our product and that of Borcherds is that our expression includes the finite product $(0.3)$, defined on all of $D$, of functions having zeros and poles in our neighborhood. In effect, this factor accounts for some of the singularities which limit the convergence of the classical Borcherds product.
and require the introduction of Weyl chambers in its description. With these singularities absorbed in (0.3), our product is valid in a much simpler region depending only on the Witt decomposition (0.1) and the choice of a basis $e_1, e_2$ for $M_U$.

The difference between the two products may be viewed as a reflection of the geometry. Suppose that $\Gamma \subset \text{Aut}(M)$ is a neat subgroup of finite index. Then in a smooth toroidal compactification $\tilde{X}$ of $X = \Gamma \backslash D$, the inverse image of a 1-dimensional boundary component in the Bailey-Borel compactification $X^{BB}$ is a Kuga-Sato variety over a modular curve. This component of the compactifying divisor arises from the fact that $\Gamma_U \backslash D$, where $\Gamma_U$ is the stabilizer of $U$ in $\Gamma$, can be viewed as a line bundle, minus its zero section, on such a Kuga-Sato variety. A compactifying chart is obtained by filling in the zero section. In our coordinates, the boundary component in $X^{BB}$ is the modular curve $\bar{\Gamma} \backslash U \backslash H$, where $\bar{\Gamma}$ is the subgroup of $\text{SL}(U)$ obtained by restricting elements of $\Gamma_U$ to $U$ and $\tau_1 \in \tilde{\mathcal{H}}$. The coordinate $w_0$ is the fiber coordinate of the Kuga-Sato variety and $q_2 = e(\tau_2^2)$ is the fiber coordinate for the line bundle over it. In particular, the product formula of Theorem A shows that $\Psi(F)$ extends to this compactifying divisor provided $q_1^{-l_0}$ does (this will depend on the intersection of $\Gamma$ with the center of $P_U$), and the order of vanishing of the extension along the compactifying divisor can be read off. Since the factor (0.2) goes to 1 as $q_2$ goes to zero, the (regularized) value of $\Psi(F)$ on the compactifying divisor is given by

$$\Psi_0(\tau_1, w_0) = \lim_{q_2 \to 0} q_2^{-l_0} \Psi(F) = \kappa \eta(\tau_1)^c_{o(0)} \prod_{x_0 \in L_0} \left( \frac{\vartheta_1(-((x_0, w_0), \tau_1))}{\eta(\tau_1)} \right)^{c_{o(-Q(x_0)/2)}}.$$

In contrast, the description of the inverse image in $\tilde{X}$ of a point boundary component in $X^{BB}$ involves the machinery of torus embeddings, in particular the choice of a system of rational polyhedral cones in the negative light cone associated to an isotropic line, [16]. The classical Borcherds products, which depend on the choice of a Weyl chamber, should give a description of $\Psi(F)$ in the various associated coordinate charts. The combinatorics in this situation are considerably more complicated than those required for the 1-dimensional boundary components. It is also worth noting that Bruinier and Freitag [4] investigated the behavior of Borcherds products locally in a neighborhood of a generic point of a rational 1-dimensional boundary component and that the factor (0.3) in Theorem A is closely related to what they call a local Borcherds product, cf. section 2.5 below.

Product formulas like that of Theorem A already occur in Borcherds [1] and in work of Gritsenko [7]. Indeed, in Borcherds original approach and in the construction of [7], the input data is a suitable Jacobi form and the associated modular form for an arithmetic subgroup $\Gamma$ in $O(n, 2)$ is constructed by applying an infinite sum of Hecke operators to it, cf. the discussion on pp.191–2 of [1], especially the third displayed equation on p.192. This method requires information about the generators for $\Gamma$ and the theory of Jacobi forms. The method of regularized theta integrals developed by Borcherds in his subsequent paper [2], stimulated by ideas of Harvey and Moore [13], takes a vector valued form $F$ as discussed above as input and works greater generality. In particular, the modularity of the output ultimately follows from the transformation properties of the theta kernel involved.
Our product formula can be viewed as providing an analogue of the expressions arising in [1] and [7] in the general case. In the case of a unimodular lattice $L$ as in Theorem A, we have

$$
\Psi(F)(w) = q_2^{l_0} \Psi_0(\tau_1, w_0) \exp \left( - \sum_{n=1}^{\infty} \frac{1}{n} \sum_{a=1}^{\infty} q_2^{an} \Theta_{a,n}(F)(\tau_1, w_0) \right).
$$

where

$$
\Theta_{a,n}(F)(\tau_1, w_0) = \sum_m c_\alpha(m) q_1^{a^{-1}mn} \sum_{x_0 \in L_0} q_1^{a^{-1}nQ(x_0)} e^{-a(x_0, w_0)}.
$$

Note that one obtains the Fourier-Jacobi expansion of $\Psi(F)$ by expanding the exponential function; for example, the next such coefficient is $\Psi_0 \cdot \Theta_{1,1}(F)$. The analogue of (0.5) for any Borcherds lift $\Psi(F)$ is given in Corollary 3.2 which thus shows that every Borcherds lift has such a product.

As already explained, our construction is based on the method of regularized theta integrals and makes no use of the theory of Jacobi forms or of generators for $\Gamma$. It is amusing to note that the eta-function and Jacobi theta function come into our formula due to the first and second Kroecker limit formulas which turn up in our calculation precisely in the form discussed in [18]. The infinite sum of Hecke operators occurring in [1] and [7] is implicit in our computation as well, for example in the non-singular orbits in [4,11], but we have not tried to include this in our formulation.

We now discuss the contents of various section. Section 1 sets up the notation, in particular the realization of $D$ as a Siegel domain of the third kind determined by a Witt decomposition (0.1) for an isotropic 2-plane $U$. We also explain a convenient choice of a sublattice $L \subset M$ compatible with (0.1). The main calculations are then done for $S_L$-valued forms $F$. In section 2, we review the regularized theta integral construction of the Borcherds form $\Psi(F)$ and state the first form of our product formula (Theorem 2.1). Then we give a more intrinsic description of the index sets which yields a formula for general lattices $M$. The final formula depends only on $M$, the choice of Witt decomposition (0.1), and the choice of a basis $e_1$ and $e_2$ for $M \cap U$. In section 3, which is the technical core of the paper, we compute the regularized theta integral. The key point is to express the theta kernel in terms of a mixed model for the Weil representation determined by the Witt decomposition (0.1). From a classical point of view, this amounts to taking a certain partial Fourier transform of theta kernel. Precisely the same trick is an essential part of Borcherds’ calculation in section 7 of [2], where the relevant Witt decomposition involves an isotropic line. In the mixed model, the theta integral has an orbit decomposition (4.4) which allows a further unfolding argument. There are non-singular terms, terms of rank 1, and the zero orbit, and these eventually give rise to the various factors in Theorem 2.1. The calculation for the rank 1 orbits is very pleasant, as it leads almost immediately to precisely the expressions evaluated by means of the first and second Kronecker limit formulas in Siegel [18]. The contribution of the zero orbit is already essentially determined by Borcherds. It is worth noting that in most of our calculation, we use the coordinates on $D$ that come from the action of the real points of the unipotent radical of the maximal parabolic $P_U$, whereas the natural complex coordinates involve a shift (1.6). To get our final product formula expressed in these holomorphic coordinates, we
need to combine the contribution of the zero orbit with some of the factors occurring in the
Kronecker limit formula terms, cf. \((4.30)\) and \((4.31)\). That this is possible depends essentially
on the identity of Proposition \((4.6)\) (Borcherds’ quadratic identity), which seems to lie at the
heart of the theory of Borcherds forms, cf. the comments on p.536 of [2] and Lemma 2.2 of
[12], for example. In section 4, we check that our formula yields several examples from the
literature. For more recent work using the Jacobi form method cf. Cléry-Gritsenko [5] and
the references given there. In section 5, we explain how to pass from our product formula to
one of those given by Borcherds for a particular choice of Weyl chamber. In this case, the
Weyl vector in the Borcherds product arises in a natural way from the factors in our formula.
The Borcherds products for other Weyl chambers do not seem to be accessible in this way.

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1. Complex coordinates and lattices

1.1. The Siegel domain of the third kind. Let \(V\) be a rational quadratic space of signa-
ture \((n,2)\) and fix a Witt decomposition \((0.1)\), with \(\dim U = 2\). Choose a basis \(e_1, e_2\) for \(U\)
and dual basis \(e'_1, e'_2\) for \(U'\), and write \(x = x_0 + x_{11}e'_1 + x_{12}e'_2 + x_{21}e_1 + x_{22}e_2\) as
\[
(1.1) \quad x = \begin{pmatrix} x_2 \\ x_0 \\ x_1 \end{pmatrix} \in V,
\]
with \(x_1, x_2 \in \mathbb{Q}^2\) (column vectors) and \(x_0 \in V_0\). Then
\[
(x,x) = (x_0,x_0) + 2x_1 \cdot x_2,
\]
where the second term is the dot product. The unipotent radical of the parabolic subgroup
\(P_U\) of \(G = O(V)\) stabilizing \(U\) is
\[
n(b,c) = \begin{pmatrix} 1_2 & -b^* & cJ - Q(b) \\ 1_{V_0} & b \\ 1_2 \end{pmatrix},
\]
where \(b = [b_1,b_2] \in V_0^2, c \in \mathbb{R},\) and
\[
J = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]
Here \(b^*\) is the element of \(\text{Hom}(V_0, \mathbb{Q}^2)\) defined by
\[
b^*(v_0) = \begin{pmatrix} (b_1,v_0) \\ (b_2,v_0) \end{pmatrix},
\]
and \(Q(b) = \frac{1}{2}(b_i,b_j)\). In particular,
\[
(1.2) \quad n(b,c)x = \begin{pmatrix} x_2 - (b,x_0) + (cJ - Q(b))x_1 \\ x_0 + bx_1 \\ x_1 \end{pmatrix}.
\]
The Levi factor of $P_U$ determined by (0.1) is

$$M_U \simeq GL(U) \times O(V_0).$$

Here, for example, if $\alpha \in GL_2$ and $h \in O(V_0)$, we have

$$m_U(\alpha, h)x = \begin{pmatrix} \alpha x_2 \\ h x_0 \\ t^{-1} \alpha^{-1} x_1 \end{pmatrix}. $$

We review the realization of the space of oriented negative 2-planes in $V(\mathbb{R})$, as a Siegel domain of the third kind associated to the Witt decomposition (0.1). For a more elegant treatment, cf. [16]. First recall that for an oriented negative 2-plane $z$, we can view the orientation as a complex structure $j_z$ on $z$ preserving the inner product. The isomorphism of the space of oriented negative 2-planes with

$$(1.3) \quad \{ w \in V(\mathbb{C}) \mid (w, w) = 0, (w, \bar{w}) < 0 \} / \mathbb{C} \times \subset \mathbb{P}(V(\mathbb{C}))$$

is realized by sending $z$ to $w(z)$, the $+i$-eigenspace of $j_z$ on the complexification $z_\mathbb{C}$.

Note that $U^\perp = V_0 + U$ is positive semidefinite, so the projection of $V$ to $U'$ with kernel $V_0 + U$ induces an isomorphism of any negative 2-plane $z$ with $U'$. In particular, an orientation of $z$ induces an orientation on $U'$ and on $U$. The projections $pr_{U'}(w(z))$ and $pr_{U'}(\bar{w}(z)) \in U'_\mathbb{C}$ form a basis, so that, up to scaling, we can write

$$w = \begin{pmatrix} u \\ w_0 \\ \tau_1 \\ 1 \end{pmatrix}, \quad u \in \mathbb{C}^2, \ w_0 \in V_0(\mathbb{C}), \ \tau_1 \in \mathbb{C} - \mathbb{R}. $$

We assume that the orientations are chosen so that $D$ is the component for which $\tau_1 \in \mathfrak{h}$ and we write $Q$ for the corresponding component of (1.3).

For a pair $\tau_1$ and $\tau_2 \in \mathfrak{h}$, let

$$w(\tau_1, \tau_2) = \begin{pmatrix} -\tau_2 \\ \tau_1 \tau_2 \\ 0 \\ \tau_1 \end{pmatrix} = \begin{pmatrix} -\tau_2 J \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \tau_1 \\ 1 \end{pmatrix}. $$

Note that $(w, \bar{w}) = -4v_1v_2$. In particular, $|y|^2 = 2v_1v_2$ in the notation of [14], (1.10), p.299.

Then there is an isomorphism

$$(1.4) \quad i : \mathfrak{h} \times \mathfrak{h} \times V_0^2(\mathbb{R}) \xrightarrow{\sim} Q, $$

defined by

$$i(\tau_1, \tau_2, v_0) = n(v_0, 0) \cdot w(\tau_1, \tau_2) = \begin{pmatrix} -\tau_2 \mathbb{I} - Q(v_0) \\ v_0 \end{pmatrix} \begin{pmatrix} \tau_1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\tau_2 - \frac{1}{2}(v_0, \bar{v}_0) \\ \tau_1 \tau_2 - \frac{1}{2}(v_0, \bar{v}_0) \\ v_0 \end{pmatrix} \begin{pmatrix} \tau_1 \\ 1 \end{pmatrix}, $$
where \( w_0 = v_0 \begin{pmatrix} \tau_1 \\ 1 \end{pmatrix} \). Note that the top entries do not depend holomorphically on \( w_0 \). The problem is that \( \tau_2 \) is not a natural holomorphic coordinate when \( \tau_2 \neq 0 \). To fix this, we write our vector as

\[
(1.5) \quad w = \begin{pmatrix} -\tau'_2 \\ \tau_1 \tau'_2 - Q(w_0) \\ w_0 \\ \tau_1 \\ 1 \end{pmatrix},
\]

where

\[
(1.6) \quad \tau'_2 = \tau_2 + \frac{1}{2}(v_{01}, w_0) \in \mathbb{C}.
\]

This then satisfies \( (w, w) = 0 \) and varies holomorphically with \( \tau_1 \in \mathfrak{H}, \tau'_2 \in \mathbb{C} \) and \( w_0 \in V_0(\mathbb{C}) \), subject to

\[
(1.7) \quad 0 > (w, \bar{w}) = -Q(w_0 - \bar{w}_0) - 4v_1 v'_2.
\]

Since \( Q(w_0 - \bar{w}_0) = -4v_1^2 Q(v_{01}) \), this just amounts to the condition

\[
(1.8) \quad v'_2 > v_1 Q(v_{01}).
\]

**Remark 1.1.** In the case of signature \((3, 2)\) and quadratic form

\[
(1.9) \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix},
\]

we have \( Q(v_{01}) = v_{01}^2 \) and condition \((1.8)\) just says that

\[
\begin{pmatrix} \tau_1 \\ w_0 \\ \tau'_2 \end{pmatrix} \in \mathfrak{H}_2,
\]

the Siegel space of genus 2.

In the calculations that follow, we have chosen to work with the ‘group action’ coordinates \((1.4)\) and to recover the ‘holomorphic coordinates’ \((1.5)\) by a substitution at the end. One could, alternatively, work with the holomorphic coordinates throughout.

### 1.2. Boundary components

Let \( \overline{Q} \) be the closure of \( Q \) in \( \mathbb{P}(V(\mathbb{C})) \) (in the complex topology), and note that the set \( \partial Q = \overline{Q} - Q \) consists of certain isotropic lines in \( V(\mathbb{C}) \). The rational point boundary components in \( \partial Q \) are the isotropic lines in \( V(\mathbb{Q}) \). If \( U \subset V \) is an isotropic plane, then the associated rational 1-dimensional boundary component is the set

\[
C(U) = \{ w \in U(\mathbb{C}) \mid U(\mathbb{C}) = \text{span}\{w, \bar{w}\} \}/\mathbb{C}^\times \subset \partial Q.
\]

If a basis \( e_1, e_2 \) for \( U \) is chosen, then there is an isomorphism

\[
\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}) \overset{\sim}{\twoheadrightarrow} C(U), \quad \tau_1 \mapsto \mathbb{C}(\tau_1 e_2 - e_1),
\]

and the rational isotropic lines in \( U \) correspond to points of \( \mathbb{P}^1(\mathbb{Q}) \) and to the rational point boundary components in the closure of \( C(U) \). For a choice of \( U' \) with dual basis \( e'_1 \) and \( e'_2 \), we have Siegel domain coordinates \((\tau_1, \tau'_2, w_0)\) as above, and, as \( v'_2 \) goes to infinity, the line
in $Q$ spanned by the vector $w$ given by (1.5) goes to the isotropic line $C(\tau_1 e_2 - e_1)$ in $C(U)$. Finally, for a point $C(\tau_2 e_2 - e_1)$ in $C(U)$, a basis for the open neighborhoods in the Satake topology is given by

$$\{(\tau_1, \tau_2, w_0) \in Q \mid |\tau_1 - \tau| < \epsilon, w_0 \in V_0(\mathbb{C}), Q(w_0 - \bar{w}_0) + 4v_1v_2 > \frac{1}{\epsilon}\},$$

for $\epsilon > 0$ and $\epsilon > 0$, cf., for example, [4], p.10, or [16], p.542.

1.3. Lattices. Suppose that $M$, $(\ , \ )$, is an even integral lattice in $V$ with dual lattice $M^\vee \supset M$. Let $S_M \subset S(V(A_f))$ be the subspace of functions supported in $M^\vee \otimes \hat{\mathbb{Z}}$ which are translation invariant under $\hat{M} = M \otimes \hat{\mathbb{Z}}$. This space is spanned by the characteristic functions $\varphi_\lambda$ of the cosets $\lambda + \hat{M}$, for $\lambda \in M^\vee/M$. Note that, if $L \subset M$ is a sublattice, then $S_M \subset S_L$.

For a given Witt decomposition (0.1), we construct a compatible sublattice $L$ of $M$ as follows. Note that

$$M \supset M_{U'} + M_0 + M_U,$$

where $M_U = M \cap U$, $M_{U'} = M \cap U'$ and $M_0 = M \cap V_0$. Let

$$M_U' = \{u \in U \mid (u, M_{U'}) \in \mathbb{Z}\},$$

so that $M_U' \supset M_U$ and define $M_{U''}^\vee \supset M_{U''}$ analogously. Let $N \in \mathbb{Z}_{>0}$ be such that $N \cdot M_{U''}^\vee \subset M_{U''}$, and let

$$L = N \cdot M_{U'}^\vee + M_0 + M_U = L_{U'} + L_0 + L_U.$$

Then,

$$L^\vee = N^{-1}L_{U'} + L_0^\vee + N^{-1}L_U,$$

and, taking $e_1$ and $e_2$ a basis for $L_U$, with dual basis $e_1'$ and $e_2'$ for $U'$, as before, in our coordinates (1.1), $x$ will be in $L$ for $x_2 \in \mathbb{Z}^2$, $x_0 \in L_0$ and $x_1 \in N\mathbb{Z}^2$.

Let $\Gamma_M$ be the subgroup of $\text{Aut}(M)$ that acts trivially on $M^\vee/M$ and define $\Gamma_L$ analogously. Since

$$L \subset M \subset M^\vee \subset L^\vee,$$

we have $\Gamma_L \subset \Gamma_M$ of finite index. Thus, automorphic forms on $D$ with respect to $\Gamma_M$ can be viewed as automorphic forms with respect to $\Gamma_L$ with some additional conditions. We will sometimes work with a neat subgroup $\Gamma \subset \Gamma_M$ of finite index. This allows us to avoid orbifold issues when discussing the geometry.

3Strictly speaking, we are describing the intersection of such an open set with $Q$.

4We could require that $N$ be the smallest such integer.
2. Theta series and the Borcherds lift

2.1. The Borcherds lift. In working with the Borcherds lift, we use the adelic setup and notation of [14] to which we refer the reader for unexplained notation. In particular, \( G'_{\mathbb{A}} \) (resp. \( G'_{\mathbb{R}} \)) is the metaplectic cover of \( SL_2(\mathbb{A}) \) (resp. \( SL_2(\mathbb{R}) \)) and \( \Gamma' \) is the inverse image of \( SL_2(\mathbb{Z}) \) in \( G'_{\mathbb{R}} \).

The input to our Borcherd lift will be a weakly holomorphic modular form \( F \) on \( G'_{\mathbb{A}} \) valued in \( S_M \) of weight \( \ell = 1 - \frac{n}{2} \) whose Fourier expansion is

\[
F(g'_{\tau}) = v^{-\ell/2} \sum_m c(m) q^m,
\]

where \( c(m) \in S_M \). For any sublattice \( L \subset M \), we can write

\[
c(m) = \sum_{\lambda \in L^\perp/L} c_\lambda(m) \varphi_\lambda
\]

with respect to the coset basis \( \varphi_\lambda \) for \( S_L \). For an oriented negative 2-plane \( z \in D \), let

\[
(x, x)_z = (x, x) + 2R(x, z), \quad R(x, z) = |(pr_z(x), pr_z(x))|,
\]

be the corresponding majorant, where \( pr_z(x) \) is the \( z \)-component of \( x \) with respect to the decomposition \( V(\mathbb{R}) = z^\perp + z \). Let

\[
\varphi_\infty(x, z) = \exp(-\pi(x, x)_z),
\]

be the corresponding Gaussian. For a Schwartz function \( \varphi \in S(V(\mathbb{A}_f)) \) and \( \tau \in \mathcal{H} \), there is a theta function

\[
\theta(g'_{\tau}, \varphi_\infty(z)\varphi) = \sum_{x \in V(\mathbb{Q})} \omega(g'_{\tau}) \varphi_\infty(x, z) \varphi(x).
\]

We can view this as defining a family of distributions \( \theta(g'_{\tau}, \varphi_\infty(z)) \) on \( S(V(\mathbb{A}_f)) \), depending on \( \tau \) and \( z \), and it will be convenient to write \( \langle \varphi, \theta(g'_{\tau}, \varphi_\infty(z)) \rangle \) for the pairing of such a distribution with \( \varphi \). Pairing with the \( S(V(\mathbb{A}_f)) \)-valued function \( F \), we get an \( SL_2(\mathbb{Z}) \)-invariant function \( \langle F(g'_{\tau}), \theta(g'_{\tau}, \varphi_\infty) \rangle \) on \( \mathcal{H} \). We want to compute the regularized theta lift

\[
\Phi(z; F) = \int_{\Gamma' \setminus \mathcal{H}} \langle F(g'_{\tau}), \theta(g'_{\tau}, \varphi_\infty(z)) \rangle v^{-2} \, du \, dv
\]

in the coordinates of section [14.1] associated to a 1-dimensional boundary component.

Recall that the regularization used by Borcherds is defined as follows. Let \( \xi \) be a \( \Gamma' \) invariant (smooth) function on \( \mathcal{H} \), satisfying the following two conditions:

1. There exists a constant \( \sigma \) such that the limit
   \[
   \phi(s, \xi) = \lim_{T \to \infty} \int_{F_T} \xi(\tau) \, v^{-s-2} \, du \, dv
   \]
   exists for \( \text{Re}(s) > \sigma \) and defines a holomorphic function of \( s \) in that half plane.
2. The function \( \phi(s, \xi) \) has a meromorphic continuation to a half plane \( \text{Re}(s) > -\epsilon \) for some \( \epsilon > 0 \).
Then the regularized integral
\[ \int_{\Gamma \backslash \mathfrak{g}}^{\text{reg}} \xi(\tau) v^{-2} \, du \, dv \]
is defined to be the constant term of the Laurent expansion of \( \phi(s, \xi) \) at \( s = 0 \).

2.2. Another product formula. One of Borcherds’ main results in \cite{2} is that the regularized theta integral \( \Phi(z; F) \) can be written as
\[ \Phi(z; F) = -2 \log |\Psi(z; F)|^2 - c_0(0)( \log |y|^2 + \log(2\pi) - \gamma), \]
where \( \Psi(F) \) is a meromorphic modular form of weight \( c_0(0)/2 \) on \( D \) and \( y \) is the imaginary part of \( z \) in a tube domain model associated to an isotropic line. In a suitable neighborhood of the corresponding point rational boundary component, he shows that \( \Psi(z; F) \) has a product expansion. Our main result is another product expansion for \( \Psi(z; F) \), valid in a neighborhood of a \( 1 \)-dimensional rational boundary component. We will explain the relation between the two products in section 6. Our computation is, in fact, quite analogous to that given in \cite{2} and, as a byproduct, we also derive (2.5) and another proof of the existence of \( \Psi(z; F) \).

Here is our main result.

**Theorem 2.1.** Suppose that the lattice \( L \) is chosen as in section \( 1.3 \) and that
\[ F(\tau) = \sum_m c(m) q^m \]
is a weakly holomorphic \( SL \)-valued modular form of weight \( -\ell = 1 - \frac{n}{2} \), type \( \rho_L \) and integral coefficients for \( m \leq 0 \). There are positive constants \( A \) and \( B \), depending on \( F \) and on the Witt decomposition (1.10), cf. Lemma 4.1, such that in a region of the form
\[ v_2' > (A + Q(v_0))v_1 + Bv_1^{-1}, \]
the Borcherds form \( \Psi(F) \) is the product of the three factors:

(i)
\[ \prod_{\lambda, \lambda_{12}=0} \prod_{m} \left( \prod_{a \in \Lambda_1 + N\mathbb{Z}} \prod_{a_0 \in \Lambda_0 + L_0} (1 - q_1^a q_2^a e(- (x_0, w_0) - \Lambda_2)) \right)^{c_{\lambda}(m)}, \]

where \( q_1 = e(\tau_1), q_2 = e(\tau_2'), b = a^{-1}(m + Q(x_0) + a \lambda_{21}) \) and \( \Lambda_2 = \lambda_{21} \tau_1 + \lambda_{22} \),

(ii)
\[ \prod_{m} \prod_{\lambda, \lambda_{1}=0} \left( \prod_{x_0 \in \Lambda_0 + L_0} \frac{\eta_{\lambda_{1}}(- (x_0, w_0) - \Lambda_2, \tau_1) \eta(\tau_1)}{\eta(\tau_1)} e( (x_0, w_0) + \frac{1}{2} \Lambda_2 \lambda_{21}) \right)^{c_{\lambda}(-m)/2}, \]

where the factor for \( m = 0 \) and \( \lambda = 0 \) is omitted, and

(iii)
\[ \kappa \eta(\tau_1)^{c_0(0)} q_2^{I_0}, \]

where \( \kappa \) is a constant of absolute value 1 and
\[ I_0 = - \sum_{m} \sum_{\lambda, \lambda_{1}=0} \sum_{x_0 \in \Lambda_0 + L_0} c_{\lambda}(-m) \sigma_1(m - Q(x_0)), \]
with $\sigma_1(0) = -\frac{1}{24}$ and $\sigma_1(r) = 0$ for $r \notin \mathbb{Z}_{\geq 0}$.

**Remark 2.2.** (1) The factor (i) converges and, in particular, has no zeroes or poles in our region near the boundary. Moreover, its limit as $q_2 \to 0$ is 1.

(2) The finite product in factor (ii) is independent of $\tau_2'$, and, in expanded form (4.19), has evident zeroes or poles on the set of $(w_0, \tau_1)'s$ where $(x_0, w_0) + \Lambda_2 = 0$ for some $x_0 \in L_0$ with $c_\lambda(-Q(x_0)) \neq 0$. The regularized integral itself is actually finite on these ‘walls’. This is because, just as in Borcherds’ case, the integral is ‘over-regularized’. Its values on the walls can be computed by using the expression in (4.23) to calculate the contribution of each term [4.22] for which $(x_0, w_0) + \Lambda_2 = 0$. We omit this calculation.

(3) Finally, the factor (iii) gives the order of the pole or zero of the Borcherds form along the compactifying divisor, whose (semi-)local equation is $q_2 = 0$. The regularized value along this divisor, obtained by removing the factor $q_2^{\tau_0}$, is given by the product of theta functions in factor (ii) and the factor $\kappa \eta(\tau_1)c^{\alpha(0)}$.

### 2.3. A more intrinsic variant.

In the statement of Theorem [2.1], we have written our product formula more or less in the expanded form that arises from the computations of section 3. We next describe an alternative, more intrinsic version.

First note that if $x \in \lambda + L$ with $(x, e_2) = 0$, then $\lambda_{12} = 0$, and we have

\[(2.6) \quad x = x_0 + ae'_1 + (\lambda_{21} - b) e_1 + (\lambda_{22} - c) e_2,\]

where $x_0 \in \lambda_0 + L_0$, $a \in \lambda_{11} + N\mathbb{Z}$, $b, c \in \mathbb{Z}$. Then, for $w$ as in (1.5),

\[-(x, w) = a\tau'_2 + b\tau_1 + c - (x_0, w_0) - \lambda_{21}\tau_1 - \lambda_{22},\]

and $e(-(x, w))$ is independent of $c$. Note that $Ze_2 = L \cap \mathbb{Q}e_2$.

Therefore the factor in (i) of Theorem 2.1 can be written as

\[(2.7) \quad \prod_{\substack{x \in L^* \\ (x, e_2) = 0 \\ (x, e_1) > 0 \\ \text{mod} \, L \cap \mathbb{Q}e_2}} (1 - e(-(x, w)))^{c(-Q(x))(x)}.\]

Here, recall that $c(m) \in S_L \subset S(V(\Lambda_f))$ so that $c(-Q(x))(x)$ is simply the value of the Schwartz function $c(-Q(x))$ at $x$, i.e., is $c_\lambda(-Q(x))$ if $x \in \lambda + L$ and 0 otherwise. The expression (2.7) depends only on the choice of $U$ and of the basis $e_1, e_2$ for $L \cap U$. This choice of basis might be viewed as the analogue in our situation of the choice of Weyl chamber which occurs in the standard Borcherds product.

The factor in (ii) of Theorem 2.1 also has a more intrinsic expression. First we examine the range of the product. Recall that the isotropic 2-plane $U$ determines a filtration $0 \subset U \subset U^\perp \subset V$. A vector $x \in L^*$ lies in $L^* \cap U^\perp$ precisely when it is given as in (2.6) with $a = 0$. The vector $x$ then lies in $L^* \cap U$ precisely when $Q(x) = Q(x_0) = 0$, since this condition implies that $x_0 = 0$.

For a given $x \in L^* \cap U^\perp$, we have $\lambda_{21} = (x, e'_1) + (x, e'_2)$, and

\[\lambda_2 = (w, e_2)^{-1}( (x, e'_1)(w, e_1) + (x, e'_2)(w, e_2)) = (w, e_2)^{-1}(xU, w),\]
where \((x_U, w)\) is the pairing of the \(U\)-component \(x_U\) of \(x\) with the \(w\), a quantity which, for a given \(x\) and \(w\), depends only on the Witt decomposition and not on the choice of basis \(e_1, e_2\). Here we have written an expression for \(\Lambda_2\) that does not depend on the normalization \((w, e_2) = 1\) of \(w\).

Retaining the normalization \((w, e_2) = 1\), the factor in (ii) can be written as the product of two factors,

\[
(2.8) \prod_{x \in L \cap U^+ \mod L \cap U \atop (x, w) \neq 0} \left( \frac{\vartheta_1(-(x, w), \tau_1)}{\eta(\tau_1)} \right) e(\frac{1}{2} (x, w)^2) e(Q(x)(x)/2),
\]

and a factor arising from \(x\) with \(Q(x) = 0\), i.e., \(x_0 = 0\), so that \(x = x_U\),

\[
(2.9) \prod_{x \in L \cap U^+ \mod L \cap U \atop (x, w) \neq 0} \left( \frac{\vartheta_1(-(x, w), \tau_1)}{\eta(\tau_1)} \right) e(\frac{1}{2} (x, w)^2) e(0(x)/2).
\]

We have separated out the factor (2.9) since it depends only on \(\tau_1\).

In both (2.8) and (2.9) a square root has been taken, since it is only assumed that the Fourier coefficients \(c_\lambda(-m)\) of \(F\) for \(m \in \mathbb{Z}_{>0}\) are integers. On the other hand, we know that \(c_\lambda(-m) = c_{-\lambda}(-m)\) for all \(m\). Thus we can choose a particular square root in (2.8) as follows. Let

\[
(2.10) \quad R_0(F) = \{ \alpha_0 \in L_0^\vee \mid Q(\alpha_0) > 0, \ c(-Q(\alpha_0))(\alpha_0 + \alpha_2) \neq 0 \}
\]

These are precisely the \(x_0\) components of vectors \(x\) that appear in the product (2.8). Let \(W_0\) be a connected component of the complement of the hyperplanes, \(\alpha_0 \perp, \alpha_0 \in R_0(F)\), in \(V_0(\mathbb{R})\). We refer to \(W_0\) as a Weyl chamber in \(V_0(\mathbb{R})\).

Then we can write (2.8) as

\[
(2.11) \quad \pm i^\ast \prod_{x \in L \cap U^+ \mod U \cap \mathbb{L} \atop (x, W_0) > 0} \left( \frac{\vartheta_1(-(x, w), \tau_1)}{\eta(\tau_1)} \right) e(\frac{1}{2} (x, w)^2) e(-Q(x)(x))/2,
\]

where the sign depends on the choice of square roots in (2.8) and

\[
* = \sum_{x \in L \cap U^+ \mod U \cap \mathbb{L} \atop (x, W_0) > 0} c(-Q(x)(x)).
\]

A change in the choice of \(W_0\) simply changes (2.11) by a sign.

Next recall that, for any even integral lattice \(M \subset M^\vee\) and Witt decomposition (1.1), we have associated, in section 1.3, a lattice \(L \subset M\) that is compatible with the Witt decomposition, so that (1.10) and (1.11) hold. Note that, by construction, \(L \cap U = M \cap U\) and hence \(L \cap \mathbb{Q}e_2 = M \cap \mathbb{Q}e_2\). Since \(S_M \subset S_L\), a weakly holomorphic form \(F\) valued in \(S_M\) can
be viewed as a weakly holomorphic form valued in $S_L$ and, in a neighborhood of the 1-dimensional boundary component associated to $U$, the Borcherds form $\Psi(F)$ is given as the product of the factors just described. Note that, since $c(m) \in S_M$, it follows that if $c(m)(x) \neq 0$ for some $x \in V(\mathbb{Q})$ then $x \in M^\vee$. Thus, all of the expressions just given for the factors of $\Psi(F)$ can be rewritten in terms of $M$, and, we obtain a more intrinsic version of our product formula.

**Corollary 2.3.** Let $M$ be an even integral lattice in $V$ and let $F$ be an $S_M$-valued weakly holomorphic form with associated Borcherds form $\Psi(F)$. Let $U \subset V$ be an isotropic 2-plane and choose a Witt decomposition (0.1) and a $\mathbb{Z}$-basis $e_1$ and $e_2$ for $M \cap U$ with dual basis $e'_1$, $e'_2$ for $U'$. Suppose that $w$ is normalized so that $(w,e_2) = 1$ and let $(w,e_1) = \tau_1$. Then $\Psi(F)(w)$ is the product of four terms:

(a) \[ \prod_{x \in M^\vee \atop (x,e_2) = 0 \atop (x,e_1) > 0 \atop \text{mod } M \cap \mathbb{Q} e_2} \left( 1 - e(-(x, w)) \right) e^{(-Q(x))(x)}. \]

(b) \[ \prod_{x \in M^\vee \cap U^\perp \atop \text{mod } M \cap U \atop (x,W_0) > 0} \left( \frac{\vartheta_1(-(x, w), \tau_1)}{\eta(\tau_1)} e((x, w) - \frac{1}{2}(x_U, w))^{(x,e'_1)} \right) e^{(-Q(x))(x)}, \]

where $x_U = (x, e'_1)e_1 + (x, e'_2)e_2$ is the $U$-component of $x$ and $W_0$ is a ‘Weyl chamber’ in $V_0(\mathbb{R})$.

(c) \[ \prod_{x \in M^\vee \cap U/M \cap U \atop x \neq 0} \left( \frac{\vartheta_1(-(x, w), \tau_1)}{\eta(\tau_1)} e^{1/2((x, w))^{(x,e'_1)}} \right) e^{0(x)/2}, \]

(d) and \[ \kappa \eta(\tau_1)^{c(0)(0)} q^{I_0}, \]

where $\kappa$ is a scalar of absolute value 1, and

\[ I_0 = - \sum_m \sum_{x \in M^\vee \cap U^\perp \atop \text{mod } M \cap U} c(-m)(x) \sigma_1(m - Q(x)). \]

Here the constant $\kappa$ may differ from that in Theorem 2.1 due to the slight shift in the factor (b). Notice that a nice feature of this version is that we do not need to worry about coordinates on $D$. The value $\Psi(F)(z)$ is simply given by evaluating on the (unique) $w$ in (1.3) associated to $z$ scaled so that $(w,e_2) = 1$.

### 2.4. Theta translates.

Next we would like to clarify the meaning of the, at first sight peculiar, factor which occurs together with the function $\eta(\tau_1)^{-1} \vartheta(-(x, w), \tau_1)$ in factors (b) and (c). We first recall some basic facts about theta functions, following the conventions of Mumford, [17], Chapter 1. The Jacobi theta function $\vartheta_1(z, \tau)$ coincides with $\vartheta_{11}(z)$ in the
Theorem 2.1 as a normalized translate by
In view of these remarks, we may write the expression occurring in the product in (ii) of the Proposition on p.84 of [17]. It is easy to check that

This just amounts to the isomorphism

is then a basis vector for the space \( \text{Th} \). For example, (5) p.72 of [19] is precisely (2.12).

For \( \eta = \eta_1 + \eta_2 \) with \( \eta_1 \) and \( \eta_2 \in \mathbb{R} \) and for \( \theta \in \text{Th}(L_\tau, H_\tau, \alpha) \), let

Then \( \theta_\eta \in \text{Th}(L_\tau, H_\tau, \alpha \gamma_\eta) \), where \( \gamma_\eta : L \to \mathbb{C} \) is the character defined by

This just amounts to the isomorphism

of the Proposition on p.84 of [17]. It is easy to check that

so that \( (\theta_\eta)^* \) is a renormalized translate of \( \theta^* \). However, since the quantity \( B(\eta, \eta) \) does not depend holomorphically on \( \tau \), it is better to include an extra factor (independent of \( z \)) and set

(2.13)

Then \( (\theta^*)^\sharp \) is again a basis for the space \( \text{Th}^*(L_\tau, H_\tau, \alpha \gamma_\eta) \).

In view of these remarks, we may write the expression occurring in the product in (ii) of Theorem 2.1 as a normalized translate by \( \eta = -\lambda_2 = -\lambda_2 \tau_1 + \lambda_2 \). More precisely, setting

\[
\Theta_1[\eta](z, \tau) = (\theta_{11})^\sharp \eta(z),
\]
for convenience, and inspecting expression $^{(2.13)}$ for $(\theta^*)^2_\eta$, we have
\[ \vartheta_1(-(x_0, w_0) - \Lambda_2, \tau_1) e((x_0, w_0)\lambda_{21} + \Lambda_2 \lambda_{21}) = \Theta_1[-\Lambda_2](-(x_0, w_0), \tau_1). \]

In the general case of Corollary $^{2.3}$ we have a identification, $U(\mathbb{R}) \sim \mathbb{C}$, and $U(\mathbb{R})/M \cap U \sim \mathbb{C}/L_{\tau_1} = E_{\tau_1}$, $u \mapsto (u, w)$.

If $x \in M^\vee \cap U^\perp$, then the point $(x_U, w)$ attached to the $U$-component of $x$ determines a torsion point of $E_{\tau_1}$. Writing
\[ (x, w) = (x_0, w_0) + (x_U, w), \]
we have the expression
\[ \vartheta_1(-(x, w), \tau_1) e((x, w) - \frac{1}{2}(x_U, w))^{(x, \epsilon')_1} = \Theta_1[-(x_U, w)](-(x_0, w_0), \tau_1). \]

in the factors in (b) and (c). In particular, the factors
\[ \vartheta_1(-(x, w), \tau_1) e((x, w) - \frac{1}{2}(x_U, w))^{(x, \epsilon')_1} = \Theta_1[-(x_U, w)](0, \tau_1) \]
occurring in (c) are thetanullwerte. An easy computation using $^{(2.13)}$ shows that, for $\ell \in L_{\tau_1}$,
\[ \Theta_1[\eta + \ell](z, \tau_1) = \alpha(\ell) e(\frac{1}{2}E(\eta, \ell)) \Theta_1[\eta](z, \tau_1). \]

Thus, if the coset representatives in (b) and (c) are changed by elements of $M \cap U$, the theta translates are changed by certain roots of unity.

2.5. Local Borcherds products. In $^{[4]}$, Bruinier and Freitag considered the local Picard group in a neighborhood of a generic point of a rational 1-dimensional boundary component associated to an isotropic 2-plane $U$. In particular, they introduced local Borcherds products attached to vectors $x \in M^\vee \cap U^\perp$, Definition 4.2, p.16. In our notation, such a product is given by
\[ \Psi_x(w) = (1 - e((x, w))) \prod_{a>0} (1 - q^a_1 e((x, w))) (1 - q^a_0 e(-(x, w))). \]

On the other hand, by the classical product formula $^{(4.18)}$, we have
\[ \frac{\theta_1(-(x, w), \tau_1)}{\eta(\tau_1)} = -iq^{\frac{1}{2}} \Psi_x(w), \]
so that factor (b) in Corollary $^{2.3}$ is essentially a product of such local Borcherds products. Of course, this factor accounts for the divisor of $\Psi(F)$ in a neighborhood of the boundary component.

3. Fourier-Jacobi expansions

In this section, we make explicit the information about the Fourier-Jacobi expansion of $\Psi(F)$ that is contained in our product formula as given in Corollary $^{2.3}$. For simplicity we normalize $\Psi(F)$ so that $\kappa = 1$, for our fixed $U$, and write the Fourier-Jacobi expansion as
\[ \Psi(F)(w) = q^{I_0}_2 \sum_{k \geq 0} \Psi_k(\tau_1, w_0) q^k_2. \]
Then the leading coefficient is the product of the factors in (b), (c) and (d), with the power of \( q_2 \) in (d) omitted:

\[
\Psi_0(\tau_1, w_0) = \eta(\tau_1)^{c(0)(0)} \prod_{x \in M^\vee \cap U/M \cap U \atop x \neq 0 \mod M\cap U} \left( \frac{\vartheta_1(-(x, w), \tau_1)}{\eta(\tau_1)} e\left(\frac{1}{2}(x, w)\right)^{(x,e'_1)} \right)^{c(0)(x)/2}
\times \prod_{x \in M^\vee \cap U^\perp \atop (x, W_0) > 0 \mod M\cap U} \left( \frac{\vartheta_1(-(x, w), \tau_1)}{\eta(\tau_1)} e\left((x, w) - \frac{1}{2}(x_U, w)\right)^{(x,e'_1)} \right)^{c(-Q(x))(x)}
\]

Note that, in the product on the first line of this formula, \((x, w)\) does not depend on \( \tau'_2 \) or \( w_0 \); it has the form \( \alpha \tau_1 + \beta \) for \( \alpha, \beta \in \mathbb{Q} \) and hence \( \vartheta_1(-(x, w), \tau_1) \) is a division point value of the Jacobi theta function. The second line of the product gives the dependence on \( w_0 \). We will make this more explicit in a moment.

To compute more Fourier-Jacobi coefficients, we consider the product in (a) of Corollary 2.3, which we write in the form

\[
\exp\left( -\sum_x c(-Q(x))(x) \sum_{n=1}^{\infty} \frac{1}{n} e(-n(x, w)) \right).
\]

Here \( x \) runs over the same index set as in (a). Note that, since \( x \in M^\vee \) and \( e_1 \) is a primitive vector in \( M \), we have \((x, e_1) = a \in \mathbb{Z}_{>0} \) and we can write

\[
x = ae'_1 + \dot{x}
\]

where \( \dot{x} \in U^\perp \). If \( e'_1 \in M^\vee \), then \( \dot{x} \in M^\vee \cap U^\perp \), but this need not always be the case. In any case, the set of components \( \dot{x} \) arising for \( x \in M^\vee \) with \((x, e_2) = 0 \) is a union of \( M \cap U \) cosets. We can write the product (a) as

\[
\exp\left( -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{a=1}^{\infty} q_2^{an} \sum_{\dot{x} \in U^\perp \atop (x, W_0) > 0 \mod M\cap U} c(-Q(x))(x) e(-n(\dot{x}, w)) \right),
\]

where \( x = \dot{x} + ae'_1 \). Note that in this expression, we are evaluating \( c(m) \in S_M \subset S(V(\mathbb{A}_f)) \), for \( m = -Q(\dot{x} + ae'_1) \), on the vector \( \dot{x} + ae'_1 \), and hence are imposing, in particular, the condition that \( \dot{x} + ae'_1 \in M^\vee \). Thus, we obtain the following striking formula.

**Corollary 3.1.**

\begin{equation}
\Psi(F)(w) = q_2^{l_0} \Psi_0(\tau_1, w_0) \exp\left( -\sum_{n=1}^{\infty} \frac{1}{n} \sum_{a=1}^{\infty} q_2^{an} \Theta_{a,n}(F)(\tau_1, w_0) \right).
\end{equation}

where

\begin{equation}
\Theta_{a,n}(F)(\tau_1, w_0) = \sum_{\dot{x} \in U^\perp \atop (x, W_0) > 0 \mod M\cap U} c(-Q(x))(x) e(-n(\dot{x}, w)).
\end{equation}

Here \( \dot{x} + ae'_1 \).
Formulas of this sort occur frequently in the work of Gritsenko, [8], Gritsenko-Nikulin, [10] [11] [12], Cléry-Gritsenko, [5], and others. Indeed, in these papers, (3.2) is essentially taken as the definition of a lift from suitable space of Jacobi forms to modular forms for $O(n, 2)$, and the modularity is proved by using information about generators for the group $\Gamma_L$, as was the case in the original paper of Borcherds, [1]. Here we obtain these expansion form the regularized theta lift defined in Borcherds second paper, [2] and hence we see that every Borcherds lift $\Psi(F)$ from that paper has such an expression.

Expanding the exponential series, we obtain expressions for the Fourier-Jacobi coefficients of $\Psi(F)$.

**Corollary 3.2.** Writing $\Psi_k = \Psi_k(\tau_1, w_0)$ and $\Theta_{a,n} = \Theta_{a,n}(F)(\tau_1, w_0)$,

$$\frac{\Psi_1}{\Psi_0} = \Theta_{1,1},$$

\begin{equation}
\frac{\Psi_2}{\Psi_0} = -\Theta_{2,1} - \frac{1}{2} \Theta_{1,2} + \frac{1}{2} \Theta_{1,1}^2,
\end{equation}

$$\frac{\Psi_3}{\Psi_0} = -\Theta_{3,1} - \frac{1}{3} \Theta_{1,3} + \Theta_{1,1} \Theta_{2,1} + \frac{1}{2} \Theta_{1,1} \Theta_{1,2} - \frac{1}{3!} \Theta_{1,1}^3,$$

$$\frac{\Psi_4}{\Psi_0} = -\Theta_{4,1} - \frac{1}{2} \Theta_{2,2} - \frac{1}{4} \Theta_{1,4} + \cdots + \frac{1}{4!} \Theta_{1,1}^4$$

\ldots .

To make these series more explicit, we choose $L \subset M$ as in (1.10) and write

$$c(m) = \sum_{\lambda \in L' \cap L} c_\lambda (m) \varphi_\lambda,$$

as in (2.2). Recall that $M \cap U = L \cap U$. For $x = \dot{x} + ae_1'$, we have $\varphi_\lambda(x) \neq 0$ implies that $\lambda_{12} = 0$ and $\lambda_{11} \equiv a \mod N$. Then we write

$$x = \dot{x} + ae_1' = (\lambda_{21} - b)e_1 + (\lambda_{22} - c)e_2 + x_0 + ae_1'$$

so that

$$m = -Q(x) = -Q(\dot{x}) - a(e_1', \dot{x}) = -Q(x_0) + ab - a\lambda_{21}$$

where $b$ and $c \in \mathbb{Z}$. Hence $a \mid (m + Q(x_0) + a\lambda_{21})$. Also

$$(\dot{x}, w) = (x_0, w_0) + (\lambda_{21} - b)\tau_1 + \lambda_{22} - c.$$

With this notation, we can write (3.3) as

$$\Theta_{a,n}(F)(\tau_1, w_0) = \sum_{\lambda} \sum_{\lambda_{12} = 0} c_\lambda (m) q_1^{a-nm} \sum_{\lambda_{11} \equiv a \mod (N)} \sum_{x_0 \equiv 0 \mod L_0} q_1^{a-nQ(x_0)} e(-n(x_0, w_0) - n\Lambda_2).$$

where $\Lambda_2 = \lambda_{21}\tau_1 + \lambda_{22}$. For $a = 1$ and $n = 1$, this is simply

$$\Theta_{1,1}(F)(\tau_1, w_0) = \sum_{\lambda} \sum_{\lambda_{12} = 0} c_\lambda (m) q_1^m \sum_{x_0 \equiv 0 \mod L_0} q_1^{Q(x_0)} e(-(x_0, w_0) - \Lambda_2).$$
Here the divisibility condition in the inner sum has been dropped. Indeed, if \( x \in \lambda + L \), we have \( Q(x) \equiv Q(\lambda) \mod \mathbb{Z} \), and if \( c_\lambda(m) \neq 0 \) we have \( m + Q(\lambda) \in \mathbb{Z} \). Thus, for \( x \) as above with \( a = 1, m + Q(x_0) + \lambda_{21} \in \mathbb{Z} \).

The following transformation law is not difficult to check.

**Lemma 3.3.** Assume that \( L_0 \) is even integral, and for \( b_1 \) and \( b_2 \in L_0 \), let \( \Lambda_b = b_1 \tau_1 + b_2 \).

Then

\[
\Theta_{a,n}(F)(\tau_1, w_0 + \Lambda_b) = e(-anQ(b_1)\tau_1 - an(w_0, b_1)) \Theta_{a,n}(F)(\tau_1, w_0).
\]

**Proof.** Noting that \((\dot{x}, b_i) = (x_0, b_i) \in \mathbb{Z}, \text{ since } x_0 \in L_0^\vee \) and \( b_i \in L_0 \), we can write

\[
q_1^{-nQ(x_0)} e(-n(x_0, w_0 + \Lambda_b) - n\Lambda_2) = q_1^{-nQ(x_0 - ab_1) - anQ(b_1)} e(-n(x_0 - ab_1, w_0) - n\Lambda_2) e(-an(b_1, w_0)).
\]

so that all summands scale in the same way. \( \square \)

We will omit the transformation law under \( \text{SL}_2(\mathbb{Z}) \) and simply note that the weight of \( F \) is \( 1 - \frac{n}{2} \) and that of the theta function associated to \( L_0 \) is \( \frac{n}{2} - 1 \), so that \( \Theta_{a,n}(F) \) is a generalized (weak) Jacobi form of weight 0 and index \( an \), cf. for example, [8], [5].

Finally, with the same notation, we can write

\[
\Psi_0(\tau_1, w_0) = \eta(\tau_1)^{c_0(0)} \prod_{\lambda_{21}, \lambda_{22} \in N^{-1}Z/\mathbb{Z}} \left( \frac{\vartheta_1(-\Lambda_2, \tau_1)}{\eta(\tau_1)} e(\frac{1}{2} \Lambda_2 \lambda_{21}) \right)^{c_\lambda(0)/2} \\
\times \prod_{\lambda \in M^\vee \cap U^\perp \mod M \cap U \atop \langle x, W_0 \rangle > 0} \left( \frac{\vartheta_1(-\langle x, w \rangle, \tau_1)}{\eta(\tau_1)} e(\langle x, w \rangle - \frac{1}{2} \langle x_U, w \rangle) \right)^{\langle -Q(x) \rangle(\tau)} e^{(\langle x, \varepsilon_1 \rangle)}.
\]

Here, in the first line, \( \Lambda_2 = \lambda_{21} \tau_1 + \lambda_{22} \) and the prime indicates that \( \lambda_{21} \) and \( \lambda_{22} \) are not both zero.

4. A computation of the regularized integral

4.1. **Passage to a mixed model.** To obtain his product formulas, Borcherds computes that Fourier expansion of the regularized theta lift along the maximal parabolic which is the stabilizer of an isotropic line in \( V \). We compute, instead, the expansion with respect to the maximal parabolic \( G_U \) stabilizing the isotropic 2-plane \( U \). To do this, we switch to a model of the Weil representation associated to a polarization arising from \( U \).

Let \( W = X + Y, \langle , \rangle \), be the standard 2 dimensional symplectic vector space with polarization. Choosing basis vectors \( e_X \) for \( X \) and \( e_Y \) for \( Y \) with \( \langle e_X, e_Y \rangle = 1 \), we have \( W(\mathbb{Q}) = \mathbb{Q}^2 \) (row
vectors) with the right action of $\text{Sp}(W) = \text{SL}_2(\mathbb{Q})$. The symplectic vector space $V \otimes W$, $\langle \cdot, \cdot \rangle$ has two polarizations

$$V \otimes W = V \otimes X + V \otimes Y = (V_0 \otimes X + \text{U} \otimes W) + (V_0 \otimes Y + U \otimes W).$$

For the first of these, we have the standard Schrödinger model of the Weil representation on $S(V \otimes X(\mathbb{A})) = S(V(\mathbb{A}))$, the Schwartz space of $V(\mathbb{A})$. For the second, we have a mixed model of the Weil representation on the space $S((V_0 \otimes X + \text{U} \otimes W)(\mathbb{A}))$. We change model of the Weil representation using a partial Fourier transform. We write $\varphi \in S(V(\mathbb{A}))$ as a function of $(x_0, x_1, x_2)$ where $x_0 \in V_0 \otimes X(\mathbb{A}) = V_0(\mathbb{A})$, $x_1 \in U' \otimes X(\mathbb{A}) = U'(\mathbb{A}) = \mathbb{A}^2$ and $x_2 \in U \otimes X(\mathbb{A}) = U(\mathbb{A}) = \mathbb{A}^2$, via our choice of bases. Then define

$$S(V(\mathbb{A})) \overset{\sim}{\longrightarrow} S(V_0(\mathbb{A})) \otimes S(U' \otimes W(\mathbb{A})), \quad \varphi \mapsto \hat{\varphi},$$

where we take $\psi$ to be the standard additive character of $\mathbb{A}/\mathbb{Q}$ that is trivial on $\hat{\mathbb{Z}}$ and restricts to $x \mapsto e(x)$ on $\mathbb{R}$. Here $\eta_2 \in U' \otimes Y(\mathbb{A}) = U'(\mathbb{A}) = \mathbb{A}^2$, and the natural pairing of $U \otimes X$ and $U' \otimes Y$, defined by the restriction of $(\cdot, \cdot) \otimes (\cdot, \cdot)$, becomes the dot product on $\mathbb{A}^2$. For an element $g' \in G'_\mathbb{A}$, we have

$$\frac{\omega(g') \varphi(x_0, \eta_1, \eta_2) = \omega_0(g') \hat{\varphi}(x_0, [\eta_1, \eta_2]g')}{\text{where } \omega_0 \text{ is the Weil representation for } V_0. \text{ Similarly, for an element of the Levi factor of } P_U, \text{ we have }}$$

$$m(h \alpha) \varphi(x_0, \eta) = \hat{\varphi}(h^{-1} x_0, ^t \alpha \eta).$$

We will view the argument $\eta = [\eta_1, \eta_2]$ as an element of $U' \otimes W(\mathbb{A}) = \text{Hom}(U, W)(\mathbb{A}) = M_2(\mathbb{A})$.

Note that, under this transformation there is an identity of theta distributions

$$\sum_{x \in V(\mathbb{Q})} \varphi(x) = \Theta(\varphi) = \hat{\Theta}(\hat{\varphi}) = \sum_{x_0 \in V_0(\mathbb{Q}), \eta \in M_2(\mathbb{Q})} \hat{\varphi}(x_0, \eta).$$

Since the regularized theta lift involves an integral over $\Gamma' \backslash \mathbb{G}$, we decompose according to $\Gamma'$-orbits:

$$\theta(g', \varphi) = \sum_{x_0 \in V_0(\mathbb{Q}), \eta \in M_2(\mathbb{Q})} \omega_0(g') \hat{\varphi}(x_0, \eta g')$$

$$= \sum_{\eta} \sum_{\gamma \in \Gamma' \backslash \Gamma'} \theta_\gamma(\gamma g', \varphi),$$

where

$$\theta_\gamma(g', \varphi) = \sum_{x_0 \in V_0(\mathbb{Q})} \omega_0(g') \hat{\varphi}(x_0, \eta g').$$
A set of orbit representatives for $\text{SL}_2(\mathbb{Z})$ acting on $M_2(\mathbb{Q})$ by right multiplication is given by:

\[(4.1) \quad 0, \quad \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}, \quad a > 0, \text{ or } a = 0, b > 0, \text{ in } \mathbb{Q}, \quad \begin{pmatrix} a & b \\ 0 & \alpha \end{pmatrix}, \quad a, \alpha \in \mathbb{Q}^\times, \quad a > 0, \quad b \in \mathbb{Q} \mod a\mathbb{Z}.\]

As stabilizers, we have $\text{SL}_2(\mathbb{Z})$, $\left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mid n \in \mathbb{Z} \right\}$, and $1$ respectively, and we write $\Gamma_\eta'$ for their inverse images in $\Gamma'$.

Thus, we get a decomposition

\[(4.2) \quad \langle F(g'_r), \theta(g'_r, \varphi_\infty) \rangle = \sum_{\eta/\sim} \sum_{\gamma \in \Gamma_\eta' \setminus \Gamma'} \langle F(g'_{\gamma r}), \theta_\eta(g'_{\gamma r}, z) \rangle.\]

Note that, for the terms with $\eta \neq 0$, the contributions of $\gamma$ and $-\gamma$ are identical since $-1_2$ acts trivially on $H$. This will result in a factor of $2$ for such terms when we unfold.

We will apply this identity to functions of the form $\varphi_{r,z} \otimes \varphi$ for $\varphi \in S(V(\mathbb{A}_F))$ and

\[(4.3) \quad \varphi_{r,z}(x) = \omega(g'_r) \varphi_\infty(x, z) = v^{n+2} e(\tau Q(x)) \exp(-2\pi v R(x, z)).\]

Now we return to the decomposition

\[(4.4) \quad \langle F(g'_r), \theta(g'_r, \varphi_\infty(z)) \rangle = \sum_{\eta/\sim} \sum_{\gamma \in \Gamma_\eta' \setminus \Gamma'} \langle F(g'_{\gamma r}), \theta_\eta(g'_{\gamma r}, z) \rangle.\]

and we break this into three blocks according to the rank of $\eta$:

\[(4.5) \quad \langle F(g'_r), \theta(g'_r, \varphi_\infty(z)) \rangle = \sum_{i=0}^2 \sum_{\eta/\sim} \sum_{\gamma \in \Gamma_\eta' \setminus \Gamma'} \langle F(g'_{\gamma r}), \theta_\eta(g'_{\gamma r}, z) \rangle v^{-2i} du dv.\]

Note that each block defines a $\Gamma'$-invariant function on $\mathfrak{h}_r$. Moreover, for our choice of representatives, all $\eta$ of a given rank have the same stabilizer $\Gamma_\eta'$ in $\Gamma'$. We obtain a corresponding decomposition of the regularized theta integral (2.4)

\[\Phi(z; F) = \sum_{i=0}^2 \Phi_i(z; F),\]

where

\[\Phi_i(z; F) = \int_{\Gamma' \setminus \mathfrak{h}_r}^\text{reg} \sum_{\eta/\sim} \sum_{\gamma \in \Gamma_\eta' \setminus \Gamma'} \langle F(g'_{\gamma r}), \theta_\eta(g'_{\gamma r}, z) \rangle v^{-2i} du dv.\]

The case $i = 0$, where $\eta = 0$, was essentially already treated by Borcherds, [2], and we will review the result in section 4.4 below.
For $i = 1$ and 2, we need to show that

$$
\phi_i(s, z) = \lim_{T \to \infty} \int_{F_T} \sum_{\eta/ \sim \atop \text{rank}((\eta)) = i} \sum_{\gamma \in \Gamma_0} \langle F(g_\gamma(\tau)), \theta_{\eta}(g_\gamma(\tau), z) \rangle v^{-s-2} \, du \, dv
$$

defines a holomorphic function of $s$ in a right half plane, to prove analytic continuation to a neighborhood of $s = 0$, and to compute the constant term there.

4.2. Non-singular terms. We will need to restrict $z$ to a certain open subset $D^o$ of $D$. To describe it, we need to introduce some basic constants. It is a standard fact that the Fourier coefficients of a weakly holomorphic modular form have sub-exponential growth, i.e., there is a positive constant $c_F$, depending on $F$, such that for large $m$,

$$
|c_\lambda(m)| = O(e^{2\pi c_F \sqrt{m}}),
$$

for all $\lambda \in L'/L$. The Fourier coefficients $c(m)$ of $F$ lie in $S(V(\mathbb{A}_f))$, and we write $\hat{c}(m)$ for their images under the partial Fourier transform

$$
\hat{c}(m)(x_0, \eta) = \int_{\mathbb{A}_f^2} c(m)(x_0, \eta_1, x_2) \psi(x_2 \cdot \eta_2) \, dx_2.
$$

Let $B_a$ (resp. $B_\alpha$) be a lower bound for the set of $a, a > 0$, (resp. $|\alpha|$) occurring as components of a rank 2 orbit representative $\eta$ for which $\hat{c}(m)(\cdot, \eta) \neq 0$ for some $m$. Finally, let $B_m$ be an upper bound on the set of $m > 0$ for which $c(-m) \neq 0$.

By some tedious estimates, which we omit, we obtain the following.

**Lemma 4.1.** Suppose that $z$ lies in the region $D^o$ in $D$ where

$$
v_2 > \max \left( \frac{8B_m}{B_a^2} v_1, 3 \frac{c_F^2}{2B_a^2} v_1^{-1} \right).
$$

Then $\phi_2(s, z)$ defines an entire function of $s$. Moreover, its value at $s = 0$ can be computed by unfolding and is given by

$$
\phi_2(0, z) = 2 \sum_{\eta/ \sim} \sum_m \int_{\mathbb{A}_f} \left( \hat{c}(m) \cdot \varphi_{\tau, z} \right)(x_0, \eta) q^m v^{-\ell/2-2} \, du \, dv.
$$

Here note that

$$
\hat{c}(m) \cdot \varphi_{\tau, z} \in S(V(\mathbb{A}_f)) \otimes S(M_2(\mathbb{A}_f)).
$$

The first step is to determine $\varphi_{\tau, z}$. The majorant can be expressed as follows.

**Lemma 4.2.** (i)

$$
R(x, z) = 2|(w, \bar{w})|^{-1} |(x, w)|^2.
$$

(ii)

$$
(x, w) = \left( -^t x_1(\tau_2 J + Q(v_0)) + (x_0, v_0) + ^t x_2 \right) \left( \begin{array}{c} \tau_1 \\ 1 \end{array} \right).
$$
Here the expression in the first factor on the right side is a row vector.

(iii) 
\[ |(x, w)|^2 = \left| t(x_2 - B) \begin{pmatrix} \tau_1 \\ 1 \end{pmatrix} \right|^2, \]

where 
\[ B = (Q(v_0) - \tau_2 J)x_1 - (v_0, x_0). \]

Using these expressions and a straightforward computation of the partial Fourier transform, we obtain the following.

**Lemma 4.3.** Write \( \eta = [\eta_1, \eta_2] \in M_2(\mathbb{R}) \) and let \( \eta_r = \eta \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \tau \eta_1 + \eta_2 \). Then
\[ \tilde{\varphi}_{\tau, z}(x_0, \eta) = \frac{v^{\frac{a-2}{4}}}{v} e(Q(x_0)\tau) e(B \cdot \eta_r) \exp(-\pi v_2 v_1^{-1} |(1, -\tau_1) \eta_r|^2), \]

where \( B = (Q(v_0) - \tau_2 J) \eta_1 - (v_0, x_0) \).

**Lemma 4.4.** Suppose that
\[ m + Q(x_0 - av_01) + a^2 v_1^{-1} v_2 > 0. \]

Then the value of the integral
\[ \int_{\mathbb{R}} \tilde{\varphi}_{\tau, z}(x_0, \begin{pmatrix} a \\ b \\ \alpha \end{pmatrix}) q^m v^{-s-\ell/2-2} du dv \]

at \( s = 0 \) is
\[ a^{-1}|\alpha|^{-1} e \left( \alpha \left( a \tau_2 - (x_0, w_0) + a^{-1}(m + Q(x_0))\tau_1 \right) \right) e(-a^{-1}b(m + Q(x_0))), \]

if \( aa > 0 \) and
\[ a^{-1}|\alpha|^{-1} e \left( \alpha \left( a \tau_2 - (x_0, w_0) + a^{-1}(m + Q(x_0))\tau_1 \right) \right) e(-a^{-1}b(m + Q(x_0))), \]

if \( aa < 0 \).

**Proof.** In the integrand here
\[ B \cdot \eta_r = \left( a Q(v_01) - (v_01, x_0) \right) (a \tau + b) + \left( a \frac{1}{2}(v_02, v_01) + a \tau_2 - (v_02, x_0) \right) \alpha, \]

so that
\[ \tilde{\varphi}_{\tau, z}(x_0, \eta) q^m v^{-s-\ell/2-2} = v^{-s-2} v_2 e(C\tau + B\alpha + B') \exp \left( -\pi v_2^{-1} v_1^{-1} |a \tau + b - \alpha \tau_1|^2 \right), \]

where, for simplicity, we let
\[ C = m + Q(x_0 - av_01), \]
\[ B = a \frac{1}{2}(v_02, v_01) + a \tau_2 - (v_02, x_0), \quad \text{and} \quad B' = \left( a Q(v_01) - (v_01, x_0) \right) b. \]

We first compute the integral over \( \mathbb{R} \) with respect to \( u \) to obtain
\[ v^{-s-2} v_2 e(B\alpha + B') \exp(-2\pi C v - \pi v_1^{-1} v_2 (av - \alpha v_1)^2) \]
\[ \times e(C a^{-1}(\alpha u_1 - b)) (v_1^{-1} v_2)^{-\frac{1}{2}} a^{-1} \exp(-\pi (v_1^{-1} v_2)^{-1} a^{-2} C^2). \]
Next we have to compute the integral over \((0, \infty)\) with respect to \(v\). First we pull out the factor
\[
a^{-1} v_2 (v_1^{-1} v_2)^{-\frac{3}{2}} e(B\alpha + B') \exp(2\pi a\alpha v_2) e(Ca^{-1}(\alpha u_1 - b)),
\]
which has no dependence on \(v\). The integral of the remaining factor is
\[
\int_0^\infty v^{-s-\frac{3}{2}} \exp(-\pi vv_1 v_2^{-1} (a^{-1} C + a v_1^{-1} v_2)^2 - \pi v^{-1} v_1 v_2 \alpha^2) \, dv
\]
\[
= 2 \left( \frac{|C + a^2 v_1^{-1} v_2|}{v_2 |\alpha|} \right)^{s+\frac{1}{2}} K_{-s-\frac{1}{2}}(2\pi v_1 |\alpha||a|^{-1} |C + a^2 v_1^{-1} v_2|),
\]
using the formula
\[
\int_0^\infty v^{\nu-1} \exp(-av - bv^{-1}) \, dv = 2 \left( \frac{a}{b} \right)^{-\nu} K_\nu(2\sqrt{ab}).
\]
Collecting terms, we have
\[
a^{-1} (v_1 v_2)^{\frac{1}{2}} e(B\alpha + B' - ia \alpha v_2 + Ca^{-1}(\alpha u_1 - b))
\]
\[
\times 2 \left( \frac{|C + a^2 v_1^{-1} v_2|}{v_2 |\alpha|} \right)^{s+\frac{1}{2}} K_{-s-\frac{1}{2}}(2\pi v_1 |\alpha||a|^{-1} |C + a^2 v_1^{-1} v_2|).
\]
Next setting \(s = 0\), and recalling that
\[
K_{-\frac{1}{2}}(2\pi r) = \frac{1}{2} r^{-\frac{1}{4}} e^{-2\pi r},
\]
and simplifying, we have
\[
a^{-1} |\alpha|^{-1} e(B\alpha + B' - ia \alpha v_2 + Ca^{-1}(\alpha u_1 - b) + iv_1 |\alpha||a|^{-1} |C + a^2 v_1^{-1} v_2|).
\]
Suppose that
\[
C + a^2 v_1^{-1} v_2 = m + Q(x_0 - av_0) + a^2 v_1^{-1} v_2 > 0.
\]
Then we have
\[
a^{-1} |\alpha|^{-1} e(B\alpha + B' - ia \alpha v_2 + Ca^{-1}(\alpha u_1 - b) + iv_1 |\alpha||a|^{-1} |C + i|a||\alpha| v_2)
\]
\[
= \begin{cases} a^{-1} |\alpha|^{-1} e(B\alpha + B' + Ca^{-1}(\alpha \tau_1 - b)) & \text{if } a \alpha > 0, \\ a^{-1} |\alpha|^{-1} e(B\alpha + B' + Ca^{-1}(\alpha \tau_2 - b)) & \text{if } a \alpha < 0. \end{cases}
\]
Rewriting in terms of holomorphic coordinates, we obtain the claimed expressions. \(\Box\)

We now suppose that the lattice \(L\) is chosen as in section 1.3 above. Coset representatives \(\lambda \in L^\vee/L\) then have the form \(\lambda = \lambda_0 + \lambda_1 + \lambda_2\) with \(\lambda_0 \in L_0^\vee\), \(\lambda_1 \in (\mathbb{Z}/N\mathbb{Z})^2\) and \(\lambda_2 \in (N^{-1}\mathbb{Z}/\mathbb{Z})^2\). Then
\[
\hat{c}(m) = \sum_\lambda c_\lambda(m) \varphi_\lambda
\]
and an easy computation shows that
\[
\varphi_\lambda(x_0, \eta_1, \eta_2) = e(-\lambda_2 \cdot \eta_2) \varphi_{\lambda_0}(x_0) \varphi_{\lambda_1}(\eta_1) \varphi_{\lambda_2}(\eta_2).
\]
For our orbit representative $\eta$, this will vanish unless $\alpha \in \mathbb{Z}$, $a \in \lambda_1 + N\mathbb{Z}$, $\lambda_{12} = 0$, and $b \in \mathbb{Z}$, in which case it has the value

$$e(-\lambda_{21} b - \lambda_{22} a) \varphi_{\lambda_0}(x_0).$$

For fixed $\eta$ with $\alpha > 0$, we have

$$\sum_{\lambda_1=0} \sum_{\lambda_{12}=0} \sum_{m} c_\lambda(m) \sum_{x_0 \in V_0(Q)} a^{-1} |\alpha|^{-1} e\left( \alpha \left( a\tau_2' - (x_0, w_0) + a^{-1}(m + Q(x_0))\tau_1 \right) \right)$$

$$\times e(-a^{-1}b(m + Q(x_0))) e(-\lambda_{21} b - \lambda_{22} a) \varphi_{\lambda_0}(x_0) \varphi_{\lambda_{11}}(a) \varphi_{\lambda_{12}}(\alpha).$$

Now the transformation properties of $F$ imply that $m + Q(x_0) + \lambda_{21} a \in \mathbb{Z}$, for the terms occurring in this sum\footnote{i.e., for $\begin{pmatrix} a \\ b \end{pmatrix} \in \lambda_1 + N\mathbb{Z}^2$ and $x_0 \in \lambda_0 + L_0$, we have $c_\lambda(m) \neq 0$ implies that $m + Q(\lambda_0) + \lambda_1 \cdot \lambda_2 \in \mathbb{Z}$.}. Taking the sum on $\delta$ modulo $a\mathbb{Z}$, we obtain

$$\sum_{\lambda_1=0} \sum_{\lambda_{12}=0} \sum_{m} c_\lambda(m) \sum_{x_0 \in V_0(Q)} a^{-1} |\alpha|^{-1} e\left( \alpha \left( a\tau_2' - (x_0, w_0) + a^{-1}(m + Q(x_0))\tau_1 \right) - \alpha \lambda_{22} \right)$$

$$\times \varphi_{\lambda_0}(x_0) \varphi_{\lambda_{11}}(a) \varphi_{\lambda_{12}}(\alpha).$$

The analogous contribution for $\alpha < 0$ is the same except that $\tau_2'$ and $\tau_1$ are replaced by $\bar{\tau_2'}$ and $\bar{\tau_1}$ respectively. Now the sum on $\alpha > 0$ yields

$$\sum_{\lambda_1=0} \sum_{\lambda_{12}=0} \sum_{m} c_\lambda(m) \sum_{x_0 \in V_0(Q)} \log(1 - e\left( \alpha \left( a\tau_2' - (x_0, w_0) + a^{-1}(m + Q(x_0))\tau_1 \right) - \lambda_{22} \right))$$

$$\times \varphi_{\lambda_0}(x_0) \varphi_{\lambda_{11}}(a),$$

while the analogous sum for $\alpha < 0$ yields its complex conjugate.

Thus, the whole contribution will be

$$-2 \sum_{\lambda_1=0} \sum_{\lambda_{12}=0} c_\lambda(m) \sum_{x_0 \in V_0(Q)} \sum_{a} \log |1 - e\left( \alpha \left( a\tau_2' - (x_0, w_0) + a^{-1}(m + Q(x_0))\tau_1 \right) - \lambda_{22} \right)|^2$$

$$\times \varphi_{\lambda_0}(x_0) \varphi_{\lambda_{11}}(a).$$

Note that, as remarked before, the factor of 2 arises since, in the unfolding, $\gamma$ and $-\gamma$ make the same contribution.
Now \((4.13)\) is \(-2 \log |z|\) of the product
\[
(4.14) \prod_{\lambda} \prod_{m} \left( \prod_{a \in \lambda + NZ \atop a > 0} \prod_{x_0 \in \lambda_0 + L_0 \atop a | (m + Q(x_0)) + a \lambda_21} \left( 1 - e(ar_2' - (x_0, w_0)) + a^{-1}(m + Q(x_0)) \tau_1 - \lambda_22 \right) \right)^{c_\lambda(m)}. 
\]

It is easy to check that no factor in this product can vanish in the region
\[v_2 = v_2' - Q(v_01) > B_m v_1.\]

It is also not difficult to check the absolute (uniform) convergence of this product in a region of the form
\[v_2 > B_m v_1 + c_F v_1^{-1}.
\]

It is interesting to remark that, in this calculation the conjugate pair of factors arise naturally for each \(x_0\) and there is no choice of Weyl chamber involved. This is consistent with the fact that the expansion we are computing is associated to a 1-dimensional boundary component where no choice of rational polyhedral cone is being made. In contrast, the formulas of Borcherds associated to a 0-dimensional boundary component involve a choice of Weyl chamber.

It remains to compute the terms for the other two types of orbits.

4.3. **Rank 1 terms.** Suppose that \(\eta = [0, \eta_2]\) for \(\eta_2 \in \mathbb{Q}^2\) nonzero.

Here we have to compute a regularization of
\[
\sum_{\eta = [0, \eta_2]} \int_{\Gamma_{\infty} \setminus \delta} (F(g'_r), \theta_\eta(g'_r, \varphi_\infty)) v^{-2} du dv.
\]

This comes to taking the constant term at \(s = 0\) of the sum on \(\eta_2\) of the integrals
\[
\sum_m \sum_{x_0 \in V_0(\mathbb{Q})} \int_{\Gamma_{\infty} \setminus \delta} \hat{c}(m) \cdot \varphi_{r, z}(x_0, 0, \eta_2) q^m v^{-s-t/2-2} du dv.
\]

Now in the integrand
\[B \cdot \eta_r = -(v_01, x_0) a - (v_02, x_0) b\]
is independent of \(\tau\), and we have
\[\varphi_{r, z}(x_0, \eta) = v^{\frac{a_2}{2}} v_2 e(Q(x_0) \tau) e(B \cdot \eta_r) \exp(-\pi v^{-1} v_1^{-1} v_2 |b\tau_1 - a|^2).\]

Thus,
\[
\sum_m \sum_{x_0 \in V_0(\mathbb{Q})} \int_{\Gamma_{\infty} \setminus \delta} \hat{c}(m)(x_0, 0, \eta_2) v^{\frac{a_2}{2}} v_2 e(Q(x_0) \tau) \\
\times e(B \cdot \eta_r) \exp(-\pi v^{-1} v_1^{-1} v_2 |b\tau_1 - a|^2) e(m \tau) v^{-s-t/2-2} du dv \\
= \Gamma(s + 1) (\pi v_1^{-1} v_2)^{-s-1} v_2 \sum_m \sum_{x_0 \in V_0(\mathbb{Q})} \sum_{Q(x_0) = m} \hat{c}(m)(x_0, 0, \eta_2) e(B \cdot \eta_r) |b\tau_1 - a|^{-2s-2}.
\]
The sum here is finite, since only a finite number of \( \hat{c}(-m) \) for \( m \geq 0 \) are nonzero and the corresponding set of \( x_0 \)'s is also finite.

We must still sum on \( \eta \). Again we take \( L \) to be the lattice defined in section 4.13 Then by (4.9) and (4.10),

\[
\hat{c}(m)(x_0, 0, \eta_2) = \sum_{\lambda \in \mathbb{L}^\vee / L \atop \lambda_1 = 0} c_\lambda(-m) e(-\lambda_2 \cdot \eta_2) \varphi_{\lambda_0}(x_0) \varphi_{\lambda_1}(\eta_2),
\]

so that \( \eta_2 \) will run over non-zero elements of \( \mathbb{Z}^2 \). Note that, by taking the sum in this way, we are implicitly including the factor of 2 coming from the identical contributions of \( \gamma \) and \( -\gamma \) in the unfolding. For fixed \( m, \lambda \) and \( x_0 \), we must compute the constant term at \( s = 0 \) of

\[
(4.15) \quad \Gamma(s + 1)(\pi v_1^{-1} v_2)^{-s-1} v_2 \sum_{a,b} e(C_0 a + C_1 b) |b \tau_1 - a|^{-2s-2}
\]

where

\[
C_0 = -(v_{01}, x_0) - \lambda_{21}, \quad C_1 = -(v_{02}, x_0) - \lambda_{22}.
\]

First suppose that \( C_0 \) and \( C_1 \) are not both zero. Note that, if \( m \neq 0 \) so that \( x_0 \neq 0 \), this will generically be the case. We can apply the second Kronecker limit formula, Siegel [18], (39), p.32,

\[
\frac{z - \bar{z}}{-2\pi i} \sum_{m,n} e^{2\pi i (m \mu + n \nu)} |m + nz|^2 = \log \left| \frac{\vartheta_1(v - uz, z)}{\eta(z)} e^{\pi i uz^2} \right|^2.
\]

Setting \( s = 0 \) in (4.15), we have

\[
\pi^{-1} v_1 \sum_{a,b} e(C_0 a + C_1 b) |b \tau_1 - a|^{-2} = -\log \left| \frac{\vartheta_1(C_1 + C_0 \tau_1, \tau_1)}{\eta(\tau_1)} e^{\pi i \tau_1 C_0^2} \right|^2.
\]

Here note that

\[
C_1 + C_0 \tau_1 = -(x_0, w_0) - \Lambda_2, \quad \text{ where } \Lambda_2 = \lambda_2 \cdot \left( \begin{array}{c} \tau_1 \\ 1 \end{array} \right) = \lambda_{21} \tau_1 + \lambda_{22}.
\]

The full contribution of these terms is then

\[
(4.16) \quad -\sum_{m} \sum_{\lambda \atop \lambda_1 = 0} c_\lambda(-m) \sum_{x_0 \in \mathbb{L}^\vee + L_0 \atop Q(x_0) = m} \log \left| \frac{\vartheta_1(-(x_0, w_0) - \Lambda_2, \tau_1)}{\eta(\tau_1)} e^{\pi i \tau_1 C_0^2} \right|^2.
\]

Recall that the theta series

\[
(4.17) \quad \vartheta_1(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i (n + \frac{1}{2})^2 \tau + 2\pi i (n + \frac{1}{2})(z - \frac{1}{2})},
\]

has a product expansion, [18], (36), p30,

\[
(4.18) \quad \vartheta_1(z, \tau) = -ie^{\pi i (\tau/4)} (e^{\pi i z} - e^{-\pi i z}) \prod_{n=1}^{\infty} (1 - e^{2\pi i (z+n\tau)}) (1 - e^{-2\pi i (z-n\tau)}) (1 - e^{2\pi i n \tau}).
\]
We may then write the contribution of these rank 1-orbits as \(-\log |z|^2\) of the following product

\[
\prod_m \prod_{\lambda, \lambda_1 = 0} \left( \prod_{x_0 \in \lambda_0 + L_0 \atop Q(x_0) = m} \frac{\vartheta_1(-(x_0, w_0) - \Lambda_2, \tau_1)}{\eta(\tau_1)} e^{\pi i \tau_1 C_0^2} \right) c_\lambda(-m),
\]

or in a fully expanded version which will be useful in section 6

\[(4.19) \prod_m \prod_{\lambda, \lambda_1 = 0} \left( \prod_{x_0 \in \lambda_0 + L_0 \atop Q(x_0) = m} e^{\pi i \tau_1 C_0^2} (e\left(\frac{1}{2}((x_0, w_0) + \Lambda_2)\right) - e\left(-\frac{1}{2}((x_0, w_0) + \Lambda_2)\right))
\]

\[\times q_1^{\frac{1}{2}} \sum_{n=1}^{\infty} \left( 1 - e(-(x_0, w_0) - \Lambda_2) q_1^n \right) \left( 1 - e((x_0, w_0) + \Lambda_2) q_1^n \right) c_\lambda(-m).\]

Here we will want to extract the factor

\[(4.20) \prod_m \prod_{\lambda, \lambda_1 = 0} \left( \prod_{x_0 \in \lambda_0 + L_0 \atop Q(x_0) = m} e^{\pi i \tau_1 C_0^2} \right) c_\lambda(-m)
\]

whose \(-\log |z|^2\) is

\[(4.21) 2\pi v_1 \sum_m \sum_{\lambda, \lambda_1 = 0} \sum c_\lambda(-m) ((x_0, v_{01} + \lambda_21)^2).\]

Next suppose that \(C_0 = C_1 = 0\). This will always occur when \(\lambda = 0\) and \(m = 0\), so that \(x_0 = 0\). It can also occur when \(m \neq 0\) and \((w_0, \tau_1)\) lies on certain affine hyperplanes. In this case (4.15) reduces to the Eisenstein series, and we have

\[(4.22) \Gamma(s + 1) \pi^{-s-1} v_2^{-s} v_1^{s+1} \sum_{a, b} |b\tau_1 - a|^{-2s-2}
\]

\[= \Gamma(s + 1) \pi^{-s-1} v_2^{-s} \left( \frac{\pi}{s} + 2\pi (\gamma - \log 2 - \log(v_1^\frac{1}{2} |\eta(\tau_1)|^2) + O(s) \right),\]

by the first Kronecker limit formula, [18], p.14. This has a pole with residue 1 at \(s = 0\) and the constant term there is

\[(4.23) \gamma - \log(4\pi v_2) - 2\log(v_1^\frac{1}{2} |\eta(\tau_1)|^2).\]

Thus, in the generic case, i.e., when \((w_0, \tau_1)\) is not on any singular hyperplane, we obtain an additional contribution:

\[(4.24) - c_0(0) \left( \log(4\pi v_1 v_2) - \gamma + 2\log |\eta(\tau_1)|^2 \right).\]

Note that the quantity \(-c_0(0) (\log(4\pi v_1 v_2) - \gamma)\) is part of the normalized Petersson inner product in (2.5).
4.4. The zero orbit. Finally, we have the term for \( \eta = 0 \). In this case,
\[
\hat{\varphi}_{\tau,z}(x_0, 0) = v^{n-2} v_2 e(Q(x_0) \tau).
\]
and we need to compute
\[
(4.25) \quad \int_{\Gamma \setminus \mathfrak{H}}^{\text{reg}} (F(g'_\tau), \theta_0(g'_\tau, \varphi_\infty)) \, v^{-2} \, du \, dv.
\]
This integral is essentially the Rankin product of \( F \) with a positive definite theta series attached to \( V_0 \). More precisely, write
\[
(4.26) \quad F_0^\circ(\tau) = \sum_m \hat{c}(m, 0) q^m = \sum_m \sum_{\lambda} c_\lambda(m) q^m \hat{\varphi}_\lambda(\cdot, 0),
\]
so that \( F_0^\circ : \mathfrak{H} \to S(V_0(\mathbb{H})) \) is a weakly holomorphic form of weight \(-\ell = 1 - \frac{n}{2}\).
Also note that only terms with \( \lambda_1 = 0 \) contribute to this sum, and that, for such a \( \lambda \),
\[
\hat{\varphi}_\lambda(x_0, 0) = \varphi_{\lambda_0}(x_0).
\]
Thus,
\[
F_0^\circ(\tau) = \sum_{\lambda_0} F_{\lambda_0}(\tau) \varphi_{\lambda_0}
\]
where
\[
F_{\lambda_0}(\tau) = \sum_m \sum_{\lambda_2} c_{\lambda_0 + \lambda_2}(m) q^m.
\]
For \( \lambda_0 \in L_0^\vee/L_0 \), we have a theta series of weight \( \ell \)
\[
\theta(\tau, \varphi_{\lambda_0}) = \sum_{x_0 \in \lambda_0 + L_0} e(Q(x_0) \tau).
\]
By Corollary 9.3 of [1], (4.25) is equal to
\[
\pi \frac{1}{3} v_2 \, \text{CT} \left[ E_2(\tau) \sum_{\lambda_0} F_{\lambda_0}(\tau) \theta(\tau, \varphi_{\lambda_0}) \right] = -8\pi v_2 \sum_m \sum_{\lambda} \sum_{x_0 \in \lambda_0 + L_0} c_\lambda(-m) \sigma_1(m - Q(x_0)),
\]
where CT means the constant term in the \( q \)-expansion and
\[
E_2(\tau) = 1 - 24 \sum_{m=1}^{\infty} \sigma_1(m) q^m.
\]
We set \( \sigma_1(0) = -\frac{1}{27} \) and \( \sigma_1(r) = 0 \) for \( r \notin \mathbb{Z}_{\geq 0} \). In particular, only terms with \( m \geq 0 \) occur and the sum is finite. For convenience, we write
\[
(4.27) \quad I_0 := -\sum_m \sum_{\lambda} \sum_{x_0 \in \lambda_0 + L_0} c_\lambda(-m) \sigma_1(m - Q(x_0)) = \text{CT} \left[ E_2(\tau) \sum_{\lambda_0} F_{\lambda_0}(\tau) \theta(\tau, \varphi_{\lambda_0}) / 24 \right]
\]
so that the contribution from \( \eta = 0 \) is simply \( 8\pi v_2 I_0 \). Note that \( 24 I_0 \) is an integer, since the Fourier coefficients \( c_\lambda(-m) \) for \( m \geq 0 \) of the original input form \( F \) are required to be integers.
4.5. **Borcherds’ vector system identity.** At this point, to obtain our final formula, we need to combine the contribution (4.25), in the form just given, with the quantity (4.21), using a version of Borcherds’ vector system identity, [2], p. 536, Theorem 10.5. In order to describe this identity in our present case, we consider another partial Fourier transform map. Let

$$V_{00} = Qe_1 + V_0 + Qe'_1,$$

so that $V_{00}$ has signature $(n-1,1)$ and we have a Witt decomposition

$$V = Qe_2 + V_{00} + Qe'_2.$$

Define a map

(4.28) \[ S(V(Af)) \rightarrow S(V_{00}(Af)), \quad \varphi \mapsto \hat{\varphi}^{oo}, \]

where

$$\hat{\varphi}^{oo}(x_{00}) = \int_{Af} \varphi(x_{00} + ye_2) \, dy.$$

Let

$$L_{00} = Z e_1 + L_0 + NZ e'_1,$$

so that $L_{00}$ has signature $(n-1,1)$ and

$$L = Z e_2 + L_{00} + NZ e'_2.$$

Also, by analogy with (4.10), for \( \lambda = \lambda_0 + \lambda_1 + \lambda_2 \), we have

$$\hat{\varphi}^{oo}_\lambda(x_0 + ae_1 + a' e'_1) = \varphi_{\lambda_0}(x_0) \varphi_{\lambda_1}(a') \varphi_{\lambda_2}(a) \varphi_{\lambda_2}(0).$$

Let $\lambda_{00} = \lambda_0 + \lambda_2 e_1 + \lambda_1 e'_1$ and set

$$c_{\lambda_{00}}(m) = \sum_{\lambda = \lambda_{00} + \lambda_2 e_2} c_\lambda(m).$$

Then the image of $c(m) \in S_L$ under the partial Fourier transform (4.28) is

$$\hat{c}^{oo}(m) = \sum_{\lambda_{00}} c_{\lambda_{00}}(m) \varphi_{\lambda_{00}},$$

and the image of $F$ under this partial Fourier transform is an $S_{L_{00}}$-valued weakly holomorphic modular form $F^{oo}$ with Fourier expansion

$$F^{oo}(\tau) = \sum_m \sum_{\lambda_{00} \in L_{00}/L_0} c_{\lambda_{00}}(m) q^m \varphi_{\lambda_{00}}.$$

Note that the function $F^o$ of (4.20) can be obtained from $F^{oo}$ by applying a second partial Fourier transform. As explained in [2], p.536, the fact that the Borcherds lift of $F^{oo}$ defines a piecewise linear function on the negative cone in $V_{00}(\mathbb{R})$ amounts to the following relation for all vectors $v_{01} \in V_0(\mathbb{R})$.

**Proposition 4.5.** *(Borcherds’ vector system identity)*

(4.29) \[ 4 I_0 \cdot Q(v_{01}) = \sum_{m > 0} \sum_{\lambda_{00}} c_{\lambda_{00}}(-m) \sum_{x_0 \in \lambda_0 + L_0 \atop Q(x_0) = m} (x_0, v_{01})^2. \]

We can rewrite this in terms of the original coefficients as follows.
Corollary 4.6. For any $v_{01} \in V_0(\mathbb{R})$,

$$4 I_0 \cdot Q(v_{01}) = \sum_{m>0} \sum_{\lambda} c_\lambda(-m) \sum_{x_0 \in \lambda_0 + L_0 \atop Q(x_0)=m} (x_0, v_{01})^2.$$ 

Thus the sum of (4.21) and (4.25) is

$$8 \pi (v_1 Q(v_{01}) + v_2) I_0 = 8 \pi v_2' I_0$$

plus the additional term

$$2 \pi \sum_m \sum_{\lambda} \sum_{x_0 \in \lambda_0 + L_0 \atop Q(x_0)=m} c_\lambda(-m) \lambda_{21} \left(2(x_0, v_{01}) v_1 + \lambda_{21} v_1 \right).$$

Thus, this quantity is $-2 \log | |^2$ of

$$q_2^{-\alpha} \prod_m \prod_{\lambda} \left( \prod_{x_0 \in \lambda_0 + L_0 \atop Q(x_0)=m} e( (x_0, w_0) + \frac{1}{2} A_2 ) \right)^{c_\lambda(-m)\lambda_{21}/2}.$$ 

Collecting all contributions, we obtain the result stated in Theorem 2.1.

5. Examples

1. In the simplest case where $L$ is self-dual, we consider a weakly holomorphic form $F = \sum_m c_0(m) q^m \varphi_0$, with corresponding Borcherds form $\Psi(F)$. Suppose that $L$ has a Witt decomposition as in (1.10), with $N = 1$. Then, our product formula for the Borcherds form $\Psi(F)$ reduces to that given in Theorem A of the introduction. Let

$$R_0(F) = \{ \alpha_0 \in L_0 \mid Q(\alpha_0) > 0, c_0(-Q(\alpha_0)) \neq 0 \},$$

and let $W_0$ be a connected component of the complement of the hyperplanes, $\alpha_0^\perp, \alpha_0 \in R_0(F)$, in $V_0(\mathbb{R})$. Let we can write the factor (0.3) as

$$\pm i B q_2^{-\alpha} \eta(\tau_1)^{c_0(0)} \prod_{x_0 \in L_0 \atop (x_0, w_0) > 0} \left( \frac{\theta_1(-(x_0, w_0), \tau_1)}{\eta(\tau_1)} \right)^{c_0(-Q(x_0))},$$

where

$$B = \frac{1}{2} \sum_{x_0 \in L_0 \atop x_0 \neq 0} c_0(-Q(x_0)).$$

1.0. The simplest case of all is when $L_0 = 0$ and $F_0(\tau) = j(\tau) - 744 = q^{-1} + O(q)$. In this case, $I_0 = -1$ and $c_0(0) = 0$, so that our product reduces to

$$j(\tau_2) - j(\tau_1) = q_2^{-1} \prod_{b > 0} \prod_a (1 - q_2^a q_2^b)^{c_\alpha(ab)}.$$
This is the example mentioned on p.163 of [1]. The left side of this identity is a meromorphic function on $X(1) \times X(1)$, where $X(1) = \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{H}^*$ is the compactified modular curve. Once the factor $q_2^{-1}$ has been removed, the remaining product is convergent for $v_1$ in a bounded set and $v_2$ large, i.e., in a neighborhood of the 1-dimensional boundary component $Y(1) \times \{i\infty\}$.

1.1. One of the most beautiful examples is the case $L = \prod_{26}^{2}$ and

$$F_0(\tau) = \eta(\tau)^{-24} = q^{-1} \sum_{r=0}^{\infty} p_{24}(r) q^{r} = q^{-1} + 24 + 324 q + \ldots,$$

so that $\Psi(F)$ has weight 12. Here we recover some of the results of [9]. For any positive definite even unimodular lattice $L_0$ of rank 24, i.e., any Niemeier lattice, we have an isomorphism $L \simeq L_0 + H^2$, where $H$ is a rank 2 hyperbolic lattice. It follows that, up to the action of $\text{Aut}(L)$, there are 24 such decompositions, determined by the isometry class of $L_0$, and we obtain a product formula for $\Psi(F)$ for each of them. Analogously, there is only one orbit of Witt decompositions of the form $L = L_0 + H$ and associated 0-dimensional boundary component. Let $N_2(L_0) = 24 h$ be the number of lattice vectors of norm 2 in $L_0$; the values of $h$ are listed Table 16.1, p.407 of [6]. The quantity $I_0$ is given by

$$I_0 = \frac{1}{24} N_2(L_0) = h.$$

Up to a scalar of absolute value 1, in a neighborhood of the 1-dimensional cusp associated to $L_0$, $\Psi(F)$ is the product of two factors,

$$\prod_{a=1}^{\infty} \prod_{b=1}^{\infty} \prod_{x_0 \in L_0} \left(1 - q_2^{a} q_1^{b} e(-(x_0, w_0)) \right) c_{\nu(ab - Q(x_0))},$$

and

$$q_2^h \eta(\tau_1)^{24-h} \prod_{x_0 \in L_0 \atop Q(x_0) = 1 \atop (x_0, W_0) > 0} \vartheta_1(-(x_0, w_0), \tau_1),$$

where $W_0$ is any connected component of the complement of the hyperplanes $x_0 \perp$ in $V_0(\mathbb{R})$ as $x_0$ runs over the vectors with $Q(x_0) = 1$ in $L_0$, the ‘roots’ of $L_0$. For example, $h = 0$ only when $L_0$ is the Leech lattice and, in this case, the second factor reduces to $\eta(\tau_1)^{24}$.

It would be most natural to normalize $\Psi(F)$ by taking it to be equal to $\eta(\tau_1)^{24}$ times the second factor (5.5) in the neighborhood of the boundary component corresponding to the Leech lattice. It is then an interesting question to determine the scalar factors arising in the other product expansions.

The compactifying divisor for the 1-dimensional boundary component indexed by a lattice $L_0$ is the abelian scheme

$$\mathcal{E}(L_0) := L_0 \otimes_{\mathbb{Z}} \mathcal{E} \longrightarrow Y(1) = \text{SL}_2(\mathbb{Z}) \backslash \mathfrak{H},$$

of relative dimension 24, where $\mathcal{E} \longrightarrow Y(1)$ is the universal elliptic curve and the tensor product is the Serre construction. The function (5.5), with the $q_2$ factor removed, is a section $^{6}$

$^{6}$Up to orbifold aspects.
of a certain line bundle over $E(L_0)$. For example, in the case of the Leech lattice, this bundle is just the pullback from the base of the line bundle of modular forms of weight 12. In general, the Borcherds form $\Psi(F)$ extends to the smooth toroidal (partial) compactification obtained by adding these compactifying divisors for the 1-dimensional boundary components and the multiplicity of the divisor associated to a lattice $L_0$ in $\text{div}(\Psi(F))$ is $h = N_2(L_0)/24$, the Coxeter number of $L_0$. The theta function occurring in (5.5) is the analogue of that considered by Looijenga, [15], p.31, in the case of a root lattice. Its divisor is the union of the ‘root’ hypertori and is invariant under the group $\text{Aut}(L_0)$, whose natural action on $E(L_0)$ extends to the relevant line bundle.

2. We consider the example of Gritsenko-Nikulin, [10], discussed in section 5 of [14]. In this case, we have $L = \mathbb{Z}^5$ with inner product defined by (1.9), so that the signature is $(3,2)$, $N = 1$ and $L_0 = \langle 2 \rangle$. The $S_L$-valued input form $F$ is obtained from the Jacobi form $\phi_{0,1}(\tau, z) = \phi_{12,1}(\tau, z)\eta(\tau)^{-24}$, cf. (5.27) of [14]. It has weight $-\frac{1}{2}$ and the associated Borcherds form $\Psi(F)$ is $2^{-6}\Delta_5(z)$, where $\Delta_5$ is the Siegel cusp form of weight 5. Here $L' = L'_0/L_0$ so that $\lambda_1 = 0$, $\lambda_2 = 0$, and $\lambda_0 = 0$ or $\frac{1}{2}$. We write $\varphi_0$ and $\varphi_1$ for the corresponding coset functions and let

$$
F = F_0\varphi_0 + F_1\varphi_1, \quad F_0(\tau) = 10 + 108q + \ldots, \quad F_1(\tau) = q^{-\frac{1}{2}} - 64q^{\frac{3}{4}} + \ldots.
$$

We then find that $I_0 = 1/2$ and that the product formula of Theorem 2.1 for $\Psi(F)$ reduces to

$$
(5.6) \quad i\eta(\tau_1)^{10} q_2^{\frac{1}{2}} \eta(\tau_1) \prod_{a,b \in \mathbb{Z}^3, \ a > 0} (1 - q_0^a q_1^b q_2^c)^{c(r^2 - 4ab)}.
$$

Here $q_0 = e(w_0)$, $r = 2x_0$, and we use the convention that, for an integer $d$ congruent to 0 or 1 modulo 4, $c(d) = c_0(-d/4)$ for $d \equiv 0 \mod 4$ and $c(d) = c_1(-d/4)$ for $d \equiv 1 \mod 4$. This is essentially equivalent to the product formula given in [12], (2.7), p.234, and (2.16), p.239, noting that the Fourier coefficients of their Jacobi form are given by the relation $f(n, \ell) = c(\ell^2 - 4n)$.

6. Comparison

In this section, we explain the relation between our product formula, associated to an isotropic 2-plane and that of Borcherds, associated to an isotropic line and a particular choice of Weyl chamber.

Suppose that $\ell$ is an isotropic line in $V$ which is contained in an isotropic plane $U$. If $M$ is an even integral lattice in $V$, we take basis $e_1$ and $e_2$ for $M_U = M \cap U$ such that $\ell \cap M = \mathbb{Z}e_2$. We then get compatible Witt decompositions (0.1) and

$$
(6.1) \quad V = \ell + V_{00} + \ell',
$$

where $\ell' = \mathbb{Q}e'_2$ and

$$
(6.2) \quad V_{00} = \mathbb{Q}e_1 + V_0 + \mathbb{Q}e'_1.
$$
As explained in section [1.3], we may choose a lattice $L \subset M$ with $L_U = M_U$ compatible with these Witt decompositions. With respect to (6.1) and (6.2), a vector $x$ with coordinates as in (1.1) becomes

$$x = \begin{pmatrix} x_{22} \\ x_{00} \\ x_{12} \end{pmatrix}, \quad x_{00} = \begin{pmatrix} x_{21} \\ x_0 \\ x_{11} \end{pmatrix} \in V_00(\mathbb{Q}).$$

Now our vector $w$ as in (1.5) can be written as

$$w = z + e'_2 - Q(z) e_2, \quad z \in V_00(\mathbb{C}),$$

so that

$$z = \begin{pmatrix} -\tau'_2 \\ w_0 \\ \tau_1 \end{pmatrix}, \quad Q(z) = Q(w_0) - \tau_1 \tau'_2.$$

For simplicity, we assume that $L = M$ is unimodular so that

$$L = \mathbb{Z}e_2 + L_{00} + \mathbb{Z}e'_2, \quad \text{with} \quad L_{00} = \mathbb{Z}e_1 + L_0 + \mathbb{Z}e'_1.$$  

Note that

$$\vartheta_1(z, \tau) = i q^{\frac{1}{2}} e(-\frac{1}{2} z) (1 - e(z)) \prod_{n=1}^{\infty} (1 - q^n e(z))(1 - q^n e(-z)).$$

Then we can write the product (0.3) as the product of the quantities

(6.3) \[ \prod_{b>0} \prod_{x_0 \in L_0} (1 - q^b e(-(x_0, w_0))) c_0(-Q(x_0)), \]

(6.4) \[ \prod_{x_0 \in L_0} (1 - e(-(x_0, w_0))) c_0(-Q(x_0)), \]

and

(6.5) \[ (-1)^{B/2} q^{B} \prod_{x_0 \in L_0} e((x_0, w_0)) c_0(-Q(x_0))^{1/2}, \]

where $B$ is given by (5.3). Note that, in each case, the product on $x_0$ is taken over a finite set of vectors.

To relate this product expansion to that of Borcherds, we need some information about his Weyl chambers. Let

(6.6) \[ R_{00}(F) = \{ \alpha \in L_{00} \mid Q(\alpha) > 0, \ c(-Q(\alpha)) \neq 0 \}, \]

be the set of ‘roots’ in $L_{00}$ for $F$. The walls in $V_{00}(\mathbb{R})$ are the hyperplanes $\alpha^\perp$ given by $(\alpha, y) = 0$ for $\alpha \in R_{00}(F)$. Let $C_{00}$ be the component of cone of negative vectors in $V_{00}(\mathbb{R})$ determined by $D$. The Weyl chambers in Borcherds are the connected components of the complement

$$C_{00} - \bigcup_{\alpha \in R_{00}(F)} C_{00} \cap \alpha^\perp.$$
Let $m_{\text{max}}$ be the largest positive integer such that $c_o(-m) \neq 0$, and let $W_0$ be a connected component of the set

$$V_0(\mathbb{R}) - \bigcup_{\alpha_0 \in R_0(F)} \alpha_0^\perp,$$

where $R_0(F)$ is given by (5.1). Let

$$R_0(F)^+ = \{ \alpha \in R_0(F) \mid (\alpha_0, W_0) > 0 \},$$

so that

$$R_0(F) = R_0(F)^+ \sqcup (-R_0(F)^+).$$

The crucial facts for us are the following.

**Lemma 6.1.** There is a unique Weyl chamber $W_{00}$ in $C_{00}$ containing a vector $y$ with $y_1 = 1$, $y_2 > 4m_{\text{max}} + 2$, and with

$$0 < (\alpha_0, y_0) < \frac{1}{2}, \quad \forall \alpha_0 \in R_0(F)^-.$$

**Lemma 6.2.** For the Weyl chamber $W_{00}$ characterized in the previous lemma, the set

$$\left\{ x_{00} = \begin{pmatrix} b \\ -x_0 \\ -a \end{pmatrix} \in L_{00} \mid c_o(-Q(x_{00})) \neq 0, \quad (x_{00}, W_{00}) > 0 \right\}$$

is given by

$$\left\{ x_{00} = \begin{pmatrix} b \\ -x_0 \\ -a \end{pmatrix} \in L_{00} \mid c_o(-Q(x_{00})) \neq 0, \quad (x_{00}, W_{00}) > 0 \right\}.$$

Noting that $Q(x_{00}) = Q(x_0) - ab$ and that

$$(x_{00}, \tilde{z}) = -(x_0, w_0) + ar^2 + b\tau_1,$$

we can write the product of the factors (0.2), (6.3), (6.4), and (6.5) as

$$(-1)^{B/2} \Gamma^B e((\rho_{00}, \tilde{z})) \prod_{x_{00} \in L_{00}} (1 - e((x_{00}, \tilde{z}))) c_o(-Q(x_{00})),$$

where $\rho_{00}$ is the ‘Weyl vector’

$$\rho_{00} = \frac{1}{2} \sum_{x_0 \in L_0 \atop (x_0, W_0) > 0} c_o(-Q(x_0)) x_0 - \frac{1}{2} I_0 \epsilon_1' + \frac{1}{24} (c_o(0) + 2B) \epsilon_1.$$

associated to $W_{00}$. This is precisely the product of Theorem 13.3 in Borcherds [2] with respect to the Weyl chamber $W_{00}$ or Theorem 10.1 of [1]. Note that, up to some differences in sign conventions, our Weyl vector coincides with that of Theorem 10.4 of [1]. In particular, the vector system in $V_{00}$ associated to $F^{\infty}$ of section 4.5, has index $m = I_0$, via (4.27), and ‘dimension’ $d = c_o(0) + 2B$, where these invariants are explained in section 6 of [1].
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