Lazy OCO: Online Convex Optimization on a Switching Budget

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Abstract
We study a variant of online convex optimization where the player is permitted to switch decisions at most $S$ times in expectation throughout $T$ rounds. Similar problems have been addressed in prior work for the discrete decision set setting, and more recently in the continuous setting but only with an adaptive adversary. In this work, we aim to fill the gap and present computationally efficient algorithms in the more prevalent oblivious setting, establishing a regret bound of $O(T/S)$ for general convex losses and $\tilde{O}(T/S^2)$ for strongly convex losses. In addition, for stochastic i.i.d. losses, we present a simple algorithm that performs $\log T$ switches with only a multiplicative $\log T$ factor overhead in its regret in both the general and strongly convex settings. In addition, for stochastic i.i.d. losses, we present a simple algorithm that performs $\log T$ switches with only a constant factor overhead in its regret in the general convex setting, and a $\log T$ factor overhead in the strongly convex setting. Finally, we complement our algorithms with lower bounds that match our upper bounds in some of the cases we consider.

1 Introduction
We study online convex optimization with limited switching. In the classical online convex optimization (OCO) problem, a player and an adversary engage in a $T$-round game, where in each round, the player chooses a decision $w_t \in W \subseteq \mathbb{R}^d$, and the adversary responds with a loss function $f_t : W \rightarrow \mathbb{R}$. The losses $f_t$ are convex functions over $W$ which is also convex and traditionally referred to as the decision set. Each round incurs a loss of $f_t(w_t)$ against the player, whose objective is to minimize her cumulative loss. The performance of the player is then measured by her regret, defined as the difference between her cumulative loss and that of the best fixed decision in hindsight:

$$\sum_{t=1}^{T} f_t(w_t) - \min_{w \in W} \sum_{t=1}^{T} f_t(w).$$

This theoretical framework has found diverse applications in recent years, many of which benefit from player strategies that switch decisions sparingly. In adaptive network routing (Awerbuch and Kleinberg, 2008) switching decisions amounts to changing packet routes, which should be kept to a minimum as it may lead to severe networking problems (see, e.g., Feamster et al., 2014). When investing in the stock market, transactions may be associated with fixed commission costs, and thus trading strategies that change stock positions infrequently are of value. As another example, Geulen et al. (2010) approach online buffering by devising a low switching variant of the well known Multiplicative Weights algorithm. In addition, recent applications of OCO in online

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reinforcement learning and control problems involve addressing the fact that changing policies introduces short-term penalties, and thus could benefit from keeping the number of policy switches to a minimum (e.g., Cohen et al., 2018, 2019; Agarwal et al., 2019a,b; Foster and Simchowitz, 2020).

This motivates the study of regret bounds achievable when we limit the number of decision switches the player is allowed to perform. When the limit is applied to the expected number of decision switches, we arrive at a variant of the standard model we shall refer to as lazy OCO. In particular, we say that an OCO algorithm is $S$-lazy if the expected number of switches it performs over $T$ rounds is less than $S$. A closely related problem where the player is charged a fixed price $c > 0$ per switch has been the focus of several works in the past, though mainly in the context of experts or multi-armed bandit problems (Dekel et al., 2014; Geulen et al., 2010; Altschuler and Talwar, 2018). It is not hard to see this problem, which we refer to as switching-cost OCO, is effectively equivalent to lazy OCO. (We defer formal details to Appendix B.)

Perhaps the most natural approach for this problem would be to divide the $T$ rounds into $S$ equally sized time-blocks, and treat the cumulative loss of each block as a single loss function. This effectively reduces the game to the standard unconstrained OCO setting with $S$ rounds, and a Lipschitz constant larger by a factor of $T/S$. This method has been termed “blocking argument” and dates back at least to Merhav et al. (2002) who used it to obtain an $O(T/\sqrt{S})$ regret bound on the prediction with expert advice problem. It is not hard to see that this strategy also yields $O(T/\sqrt{S})$ regret in the general convex setting. Recently, Chen et al. (2019) prove that this is in fact optimal against an adaptive adversary with linear losses. However, in the oblivious adversary setting, stronger results may be achieved owed to the power of randomization. For example, several works (Kalai and Vempala, 2005; Geulen et al., 2010; Devroye et al., 2013) have obtained a stronger $O(T/S)$ bound for the experts problem by employing randomized player strategies. In the general convex setting results have been more scarce, though the same bound has been achieved by Anava et al. (2015), who adapt the method of Geulen et al. (2010) to the continuous online optimization setup.

In this work, we aim to further develop our understanding of lazy OCO by addressing a number of questions that have remained open. First, the algorithm in Anava et al. (2015) obtains a regret bound of $\tilde{O}(\sqrt{dT} + dT/S)$ where $d$ denotes the dimension of the decision set, and thus exhibits dimension dependence even in the case that $T$ switches are permitted. Second, to the best of our knowledge, no results have been established in the strongly convex lazy OCO setting, neither for adaptive adversaries nor for oblivious ones. Finally, in terms of dependence on the number of switches $S$, it is not clear a priori whether the additive $O(T/S)$ term in the regret is essential, and in particular, whether an $O(\sqrt{T})$ regret bound may be obtained with $S = o(\sqrt{T})$ switches. This work aims to fill these gaps and obtain a more coherent understanding of the lazy OCO problem.

1.1 Our contributions

We make the following contributions:

- **Regret upper bounds.** We present a computationally efficient $S$-lazy algorithm, achieving an $O(\sqrt{T} + dT/S)$ regret bound for general convex losses, and an $\tilde{O}(d^2T/S^2)$ bound for strongly convex losses. Compared to the algorithm of Anava et al. (2015), we remove the $\sqrt{d}$ factor from the $\sqrt{T}$ term, and our algorithm further extends to the strongly convex case where it obtains improved regret bounds, which does not seem to be the case for the algorithm of Anava et al. (2015).\(^1\)

\(^1\)In fact, a closer look into the regret analysis of Anava et al. (2015) reveals that it does not at all exploit the convexity of the losses: by taking a discretization of the decision set and using, e.g., the Shrinking-Dartboard algorithm
- **Regret lower bounds.** For the general convex case, we prove an $\Omega(T/S)$ lower bound for the regret of any $S$-lazy algorithm, matching our upper bound in this setting in terms of dependence on $T$ and $S$. In the strongly convex adaptive setting, we prove an $\Omega(T/S)$ lower bound, which matches up to a logarithmic factor the $O((T/S) \log S)$ upper bound obtained by a straightforward application of the blocking technique (see Appendix A.2 for details). Finally, for the strongly convex oblivious setting, we prove an $\Omega(T/S^2)$ lower bound for a certain family of algorithms, as discussed in Section 5.

- **Regret bounds for stochastic i.i.d. losses.** For the special case of a stochastic i.i.d. adversary, we present an algorithm that performs $O(\log T)$ switches while introducing only a constant multiplicative $O(\log T)$ factor overhead in the regret bound compared to unrestricted OCO in the general convex case, and only an extra logarithmic factor in the strongly convex case.

Table 1 lists our contributions compared to the relevant state-of-the-art bounds. Our upper bounds are discussed in Sections 3 and 4, and our lower bounds in Section 5.

**Addendum.** Following the initial publication of this work in COLT’21 (Sherman and Koren, 2021), an error in the arguments given there was brought to our attention by the authors of Agarwal et al. (2023) through private correspondence. Specifically, the random variable $p_t - \nabla f_t(w_t)$ in line 6 of Algorithm 2 in (Sherman and Koren, 2021) is not normally distributed (due to dependence between $p_t$ and $w_t$), while Lemma 3 (through the use of Lemmas 1 and 2) requires it to be so. The current version of the manuscript presents a corrected version (see Algorithm 2 and Lemma 3), by essentially replicating the approach proposed by Agarwal et al. (2023). The key aspect in this approach is to setup the algorithm in such a way that would allow working with the density functions of the decision variables $w_t$ directly. This is elegantly accomplished by the use of barrier functions and the change of variables formula, which ultimately allow coupling over consecutive decisions in spite of the dependence between $p_t$ and $w_t$.

For completeness, the bounds for Lazy OCO obtained in the recent work of Agarwal et al. (2023) were added to Table 1 below; since ultimately our corrected algorithm is nearly identical to theirs, our regret bounds match theirs up to small logarithmic factors. We also note that Agarwal et al. (2023) additionally give improved bounds for the case where the loss functions are in the form of generalized linear models (GLMs), which we do not reproduce here.

### 1.2 Key ideas and techniques

Our starting point for designing lazy algorithms in the (adversarial, oblivious) OCO setup is the general idea present in Follow-the-Lazy-Leader (FLL) algorithm of Kalai and Vempala (2005), where perturbations are introduced for obfuscating small changes in the player’s (unperturbed) decisions. Then, the perturbations may be correlated in such a way that preserves the marginal distribution of decisions, and at the same time have sufficient overlap in total variation, which allows for the player to avoid switching altogether across several consecutive rounds. However, unlike Kalai and Vempala (2005) who study the linear case, we are interested in the general convex and strongly convex settings, which pose a number of additional challenges.

First, the subset of perturbed objectives cannot be fixed in advance and must be determined per round in a dynamical fashion during execution of the algorithm. This is because the particular perturbation that leads to the same decision being used across rounds depends on the loss sequence of Geulen et al. (2010), one would obtain essentially the same regret guarantee (albeit not in polynomial time).

Essentially, the only difference between the two algorithms is in the implementation of the coupling mechanism used to correlate sampling of consecutive decisions.
Table 1: \( S \)-lazy OCO bounds, omitting log terms (except for the i.i.d. case) and factors other than \( S \) and \( T \). Our contributions are in boldface.

| Setting       | Adversary | Lower Bound                        | Upper Bound                        |
|---------------|-----------|------------------------------------|------------------------------------|
| Experts       | Oblivious | \( T/S \)                           | \( T/S \)                           |
|               |           | Geulen et al. (2010)               | Kalai and Vempala (2005) b         |
|               | Adaptive  | \( T/\sqrt{S} \)                   | \( T/\sqrt{S} \)                   |
|               |           | Chen et al. (2019)                 | Chen et al. (2019)                 |
| OCO           | Oblivious | \( T/S \)                           |                                    |
|               |           |                                    | \( T/S \) a                        |
|               |           |                                    | this work, Agarwal et al. (2023)   |
|               |           |                                    | Anava et al. (2015)                |
| i.i.d.        |           | \( \sqrt{T} \)                     | \( \sqrt{T} \) c                  |
| OCO           | Adaptive  | \( T/S \)                           | (\( T/S \))                        |
| Strongly Conv | Oblivious | \( T/S^2 \) d                      | \( T/S^2 \)                        |
|               |           |                                    | this work, Agarwal et al. (2023)   |
|               | i.i.d.    | \( \log T \)                       | \( \log^2 T \) c                  |

\[a\] For \( S = O(\sqrt{T}) \).
\[b\] Also Geulen et al. (2010); Devroye et al. (2013); Altschuler and Talwar (2018).
\[c\] For \( S \geq 1 + \log T \).
\[d\] For a certain restricted class of algorithms. See Section 5 for details.

In a complex way that mandates ad-hoc coupling between consecutive decisions. Even further, the unperturbed decisions need to be stabilized so that consecutive decision distributions overlap sufficiently in total variation. To that end, a regularization component is added to obtain the desired relation between regret and the number of decision switches. Thus, unlike Follow-the-Perturbed-Leader-type algorithms that introduce perturbations for promoting stability, we draw our stability properties from a regularization component while using perturbations only for inducing proximity in total variation, which in turn allows the algorithm to resample decisions less frequently.

Finally, our regret bound for strongly convex losses makes use of two additional ideas that were key in achieving the improved dependence on the switches parameter \( S \). First, perhaps surprisingly, the perturbation scale has to be increased at a rate that is in accordance with the increasing curvature of the per round minimization objective (despite the fact that the unperturbed decision actually becomes more and more stable with time). The second and more crucial observation is that the regret penalty introduced by perturbations on top of the hypothetical “be-the-leader” strategy can be bounded much more efficiently for strongly convex losses: our analysis reveals that this penalty depends on the distance between the deterministic, unperturbed minimizer and the perturbed random one; crucially, with strong convexity, this distance shrinks rapidly with the number of steps at a rate that compensates for the increased perturbation scale.
1.3 Additional related work

Prior work on low switching strategies in online learning has been mostly concerned with the switching-cost perspective. All bounds we present here pertain to algorithms with an expected number of switches bounded by $S$, so that they are easily comparable. For completeness, their equivalent original switching-cost forms can be found in Appendix B.

**Experts.** The experts problem with switching costs has been extensively studied, giving rise to several algorithms such as FLL (Kalai and Vempala, 2005), Shrinking-Dartboard (Geulen et al., 2010) and Perturbation-Random-Walk (Devroye et al., 2013), all of which achieve $O(T/S)$ regret known to be optimal due to a matching lower bound of Geulen et al. (2010). Recently, Altschuler and Talwar (2018) study experts and multi-armed bandits in the setting where the player is given a hard cap on the number of switches she is allowed (see Appendix B for a discussion of this variant of the model); they develop a framework converting Follow-the-Perturbed-Leader (FPL) type algorithms that work in expectation to ones with high probability guarantees, and leverage this result to achieve an upper bound of $\tilde{O}(T/S)$ for $S = O(\sqrt{T})$, shown to be tight.

**Multi-armed bandits.** Unlike experts, the multi-armed bandit problem has proved to exhibit a more significant dependence on the number of switches, setting apart switching-cost regret from the standard unconstrained setting. An $O(T/\sqrt{S})$ upper bound was obtained by a blocking argument (Arora et al., 2012) applied to the EXP3 algorithm (Auer et al., 2002). A matching $\Omega(T/\sqrt{S})$ lower bound was proved by Dekel et al. (2014). In the case of stochastic i.i.d. losses, Cesa-Bianchi et al. (2013) present an $\tilde{O}(\sqrt{T})$ algorithm for multi-armed bandits that performs $O(\log \log T)$ switches.

**Online convex optimization.** To the best of our knowledge, Anava et al. (2015) establish the first and only $O(T/S)$ upper bound in the general convex setting with an oblivious adversary, albeit with a computationally intensive algorithm whose running time is bounded by a high-degree polynomial in the dimension. More recently, Chen et al. (2019) study lazy OCO in the general convex setting with an adaptive adversary and prove a tight $\Theta(T/\sqrt{S})$ result. Our work is thus complementary to theirs as we study lazy OCO in the oblivious setting, where stronger upper bounds turn out to be possible. Also relevant to our work is the paper of Jaghargh et al. (2019), who propose a Poisson process based algorithm for both general and strongly convex losses, although their results are suboptimal compared to those presented here.

**Movement costs.** Also related to lazy OCO is the study of movement costs in online learning, where the player pays a switching cost proportional to the distance between consecutive decisions. This variant was studied in the context of multi-armed bandits (Koren et al., 2017a,b), and is at the core of the well known metrical-task-systems (MTS) framework in competitive analysis (Borodin et al., 1992; Borodin and El-Yaniv, 2005). In particular, the continuous variant of MTS has been the subject of several works both in the low dimensional setting (Bansal et al., 2015; Antoniadis and Schewior, 2017), and in the high dimensional setting where it has been recently termed smoothed OCO (Chen et al., 2018; Goel et al., 2019; Shi et al., 2020). MTS differs from lazy OCO in a number of important ways; we refer to Blum and Burch (2000); Buchbinder et al. (2012); Andrew et al. (2013) for an extensive discussion of the relations between competitive analysis and regret minimization.
Correlated sampling. The algorithms we present are based on a lazy sampling procedure for sampling from maximal couplings (see Section 2.3). This procedure bears similarity to the well-known correlated sampling problem (Broder, 1997; Kleinberg and Tardos, 2002), where two players are given two probability distributions and are required to produce samples with minimal disagreement probability. As the players are not allowed to communicate, this problem is crucially different than sampling from maximal couplings; see Bavarian et al. (2020) for a more elaborate discussion.

Differentially-private online learning. It has recently been observed that online learning with switching constraints is strongly related to differentially-private online learning (Asi et al., 2023b; Agarwal et al., 2023). Differential Privacy is concerned with learning mechanisms that produce outputs that reveal little about any individual data point used to train them (see Dwork et al., 2014). In the context of online learning, and more specifically OCO, this boils down to designing algorithms with the property that if a single loss function were changed, the sequence of decisions produced by the algorithm would not change by much.

There has been a long line of work studying differentially private prediction from expert advice and online learning more generally (e.g., Jain et al., 2012; Guha Thakurta and Smith, 2013; Jain and Thakurta, 2014; Agarwal and Singh, 2017; Kairouz et al., 2021; Asi et al., 2023a; Kaplan et al., 2023). Recently, Asi et al. (2023b) and subsequently Agarwal et al. (2023) used low-switching online learning as a means to design differentially private algorithms. Informally, the key idea behind this approach is that information about the loss sequence is leaked only when the online learning algorithm changes its decision; hence, an online algorithm can be transformed into a privacy-preserving one by limiting the number of switches it performs.

2 Preliminaries

We start by giving a precise definition of our model and describe techniques and basic tools we use.

2.1 Problem setup

We describe the setting of lazy OCO, within which we develop all results presented in the paper. In this setting, an oblivious adversary chooses convex Lipschitz loss functions $f_t : W \to \mathbb{R}$ over a convex domain $W \subseteq \mathbb{R}^d$. Throughout the paper (excluding Section 4), we assume losses are also twice differentiable and smooth, i.e., that $\nabla^2 f(w) \preceq \beta I$ for some parameter $\beta > 0$ and all $w \in W$. The game proceeds for $T$ rounds, where in round $t$ the player chooses $w_t \in W$, suffers loss $f_t(w_t)$, and observes $f_t$ as feedback. We denote by $\mathcal{R}_T$ the player’s regret;

$$\mathcal{R}_T := \sum_{t=1}^{T} f_t(w_t) - \min_{w \in W} \sum_{t=1}^{T} f_t(w),$$

and by $\mathcal{S}_T$ the number of decision switches she performs; $\mathcal{S}_T := \sum_{t=1}^{T-1} \mathbb{1}\{w_{t+1} \neq w_t\}$. When it is not clear from context, we may write $\mathcal{R}_T(A)$ and $\mathcal{S}_T(A)$ to make explicit which player we are referring to. We are interested in the asymptotic behavior of the player’s regret, under the restriction she is obligated to perform a limited number $S \in [T]$ of switches in expectation; $\mathbb{E}\mathcal{S}_T \leq S$. We say $A$ is an $S$-lazy algorithm if it satisfies that for any loss sequence $\mathbb{E}\mathcal{S}_T \leq S$. 

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2.2 Basic definitions and tools

We operate over \( \mathbb{R}^d \), denote the \( p \)-norm by \( \| \cdot \|_p \), and omit the subscript for the Euclidean norm; meaning \( \| \cdot \| := \| \cdot \|_2 \). The diameter of a set \( W \subseteq \mathbb{R}^d \) is defined as \( \max_{x,y \in W} \| x - y \| \). We denote by \( \Pi_W (x) := \arg \min_{w \in W} \| w - x \|^2 \) the orthogonal projection of a point \( x \in \mathbb{R}^d \) onto \( W \), but usually omit the subscript \( W \) and write \( \Pi (x) \) unless the context requires to be explicit. For two probability measures \( Q, Q' \) over a sample space \( \mathcal{X} \), we write \( \| Q - Q' \|_{TV} \) to denote their total variation distance;

\[
\| Q - Q' \|_{TV} := \max_{B \subseteq \mathcal{X}} Q(B) - Q'(B).
\]

Also, a two dimensional random variable \((X, Y)\) is a coupling of \( p \) and \( q \) if its marginals satisfy \( X \sim p \) and \( Y \sim q \). Given a scale parameter \( \sigma > 0 \), we denote by \( \text{Lap}(\sigma) \) a multivariate Laplace distribution, with density \( \nu \) given by;

\[
\nu(z) = \frac{1}{(2\sigma)^d} \exp \left(-\|z\|_1 / \sigma\right).
\]

### Truncated domain and barrier functions.

Given a convex and compact domain \( W \subset \mathbb{R}^d \) and parameter \( \gamma > 0 \), we let

\[
W_\gamma := \{ w_0 + (1 - \gamma)(w - w_0) : w \in W \}
\]

denote the \( \gamma \) truncated domain, where \( w_0 \) may be chosen arbitrarily from \( W \). Note that for any \( w \in W \), \( w - (w_0 + (1 - \gamma)(w - w_0)) = \gamma (w - w_0) \). Thus if \( D \) is the diameter of \( W \), for any \( w \in W \) there exists \( w_\gamma \in W_\gamma \) such that \( \| w - w_\gamma \| \leq \gamma D \).

We say \( B : \mathbb{R}^d \rightarrow \mathbb{R} \cup \infty \) is a barrier function (also known as Legendre; see Cesa-Bianchi and Lugosi, 2006) on \( W \) if \( B \) is non-negative, continuous, convex, and \( B(w) \in \mathbb{R} \) for \( w \in \text{int}(W) \), and \( B(w) = \infty \) otherwise. We assume we have access to a family of barrier functions \( B_{\gamma, \alpha} \) that satisfy \( B_{\gamma, \alpha}(w) \leq \alpha \) for all \( w \in W_\gamma \). This can be satisfied for any \( \alpha, \gamma \) by scaling a given barrier function with an appropriate factor.

2.3 Sampling from maximal couplings

Algorithm 1 presented below provides a mechanism to maximally couple consecutive decision distributions. A similar procedure for sampling from maximal couplings can be found in the literature in various places, see e.g., Jacob et al. (2020). The desired properties of the algorithm follow from the two lemmas stated next. For completeness, we provide their proofs in Appendix C. Throughout the paper, within an algorithmic context, we use the calligraphic font \( (\mathcal{P}, \mathcal{Q}, \text{etc.}) \) to denote computational objects that provide \( O(1) \) oracle access to evaluate the density and to sample from a probability distribution.

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**Algorithm 1 LazySample**

1: **input:** \( x, \mathcal{Q}, \mathcal{P} \)
2: Sample \( z \sim \text{Unif}[0, \mathcal{Q}(x)] \)
3: If \( \mathcal{P}(x) > z \), return \( x \)
4: Otherwise, repeat;
5: Sample \( y \sim \mathcal{P} \), and \( z' \sim \text{Unif}[0, \mathcal{P}(y)] \)
6: If \( z' > \mathcal{Q}(y) \), return \( y \)
Lemma 1. Running LazySample\((x, Q, P)\) with \(x \sim Q\), we have that \(P\) is sampled from with probability \(\|Q - P\|_{TV}\), where randomness is over choice of \(x\) and execution of the algorithm.

Lemma 2. Assume we run LazySample\((x, Q, P)\) with \(x \sim Q\), and that \(P, Q\) are density functions that can be evaluated at any point and sampled from in polynomial time. Then the algorithm generates a return value distributed according to \(P\) in expected polynomial time.

3 Lazy OCO

In this section, we present and analyze our lazy OCO algorithm for convex losses in the oblivious adversarial setup. The algorithm has a regularization component and generates decisions that are minimizers of a perturbed cumulative loss on each round. As such, it can be viewed as a natural combination of the well known Follow-the-Perturbed-Leader (FPL) algorithm (Kalai and Vempala, 2005) and Follow-the-Regularized-Leader (FTRL) meta-algorithm, with regularization being intrinsic in the strongly convex case. The resulting algorithm, given in Algorithm 2, is thus named Follow-the-Perturbed-Regularized-Lazy-Leader (FTPRLL).

The key idea is that stability introduced by regularization causes minimizers of the unperturbed objectives to move in small steps, thereby encouraging consecutive decisions—minimizers of the perturbed objectives—to overlap in total variation. This, combined with the lazy sampling subroutine Algorithm 1, produces a low switching algorithm. Importantly, we note that while in FPL the perturbations are the source of stability, here regularization accounts for stability, and the perturbations serve to obfuscate the shifts between consecutive decisions. Next, we present notation associated with our algorithm and provide its pseudocode subsequently.

Notation. Given a regularizer \(R: W \rightarrow \mathbb{R}\), a barrier function \(B: \mathbb{R}^d \rightarrow \mathbb{R} \cup \infty\), and a sequence of loss functions \(f_1, \ldots, f_{t-1}\), we define:

\[
\phi_t(w; p) := \sum_{i=1}^{t-1} f_i(w) + p^T w + R(w) + B(w),
\]

\[
w_t(p) := \arg \min_{w \in W} \phi_t(w; p).
\]

Given a random perturbation vector \(p_t \sim \nu_t\), we let \(Q_t\) denote the density function of the random variable \(w_t(p_t)\):

\[
Q_t(w) = \Pr_{p_t \sim \nu_t} (w_t(p_t) = w).
\]

With slight notation overloading, we let \(Q_t\) also denote the distribution of \(w_t(p_t)\), and write \(w_t(p_t) \sim Q_t\).

We note that given second order access to the loss functions, barrier and regularizer, Algorithm 2 is polynomial-time efficient. Indeed, the density functions \(Q_t, Q_{t+1}\) passed as arguments into Algorithm 1 may be sampled from and evaluated efficiently at any given point, as required by Lemma 2. To generate a sample from \(Q_t\), we sample \(p \sim \nu_t\) and compute \(w_t(p)\) by minimizing \(\phi_t(\cdot; p)\) over \(W\). In order to evaluate \(Q_t\) at a given \(w \in W\), we follow the same approach presented in Agarwal et al. (2023), and use a closed form expression based on the change of variables formula (see Lemma 18 in Appendix D for the details):

\[
Q_t(w) = \nu_t(-\nabla \phi_t(w; 0)) | \det (-\nabla^2 \phi_t(w; 0)) |.
\]
Algorithm 2 Follow-The-Perturbed-Regularized-Lazy-Leader (FTPRLL)

1: **input:** $\sigma_1, \ldots, \sigma_T \in \mathbb{R}$, regularizer $R$, barrier $B$
2: Sample $p_1 \sim \text{Lap}(\sigma_1)$
3: $w_1 \leftarrow w_1(p_1)$
4: for $t = 1$ to $T$ do
5: Play $w_t$, Observe $f_t$
6: Let $Q_{t+1}$ denote the density of $w_{t+1}(p_{t+1})$, where $p_{t+1} \sim \text{Lap}(\sigma_{t+1})$
7: $w_{t+1} \leftarrow \text{LazySample}(w_t, Q_t, Q_{t+1})$
8: end for

In what follows, we provide a regret analysis of Algorithm 2 in the general convex and strongly convex cases. We begin with the following lemma which establishes a bound on the total variation between consecutive decision distributions of the algorithm. The lemma follows from arguments given in Agarwal et al. (2023); for completeness, we provide a proof in Appendix D.

**Lemma 3.** Assume the loss sequence $f_1, \ldots, f_T : W \to \mathbb{R}$ is convex, $G$-Lipschitz, and $\beta$-smooth, and that $\phi_t$ is $(1/\eta_t)$-strongly convex. Then, for any $t$ such that $\sigma_{t+1} = \sigma_t$, we have

$$\|Q_{t+1} - Q_t\|_{TV} \leq \frac{\eta_t \beta d}{\sigma_t} + \sqrt{dG/\sigma_t}.$$ 

Combined with Lemma 1, this ensures the switch probability in any single round is bounded by the quantity above that can be controlled with the choice of step size and perturbation scale parameters.

### 3.1 The general convex case

We start with the general convex case where the regret analysis is simpler. Here we introduce stability into the algorithm by means of L2 regularization, with $R(w) = \frac{1}{2\eta} \|w - w_0\|^2$ for some $w_0 \in W$. Below, we state and prove our theorem giving the guarantees of Algorithm 2 when tuned for general convex losses.

**Theorem 1.** Let $W \subset \mathbb{R}^d$ be convex and of diameter $D$, and assume the loss functions $f_1, \ldots, f_T$ are $G$-Lipschitz, $\beta$-smooth and convex over $W$. Further, assume the barrier $B$ satisfies $0 \leq B(w) \leq 1$ for all $w \in W$, where $\gamma := 1/T$. Then running Algorithm 2 with $\sigma_t = \sigma$ for all $t$ and $R(w) = \frac{1}{2\eta} \|w - w_0\|^2$, $w_0 \in W$, we obtain

$$\mathbb{E} R_T \leq 2\eta G^2 T + \frac{D^2}{2\eta} + \sigma \sqrt{dD} + GD + 1 \quad \text{and} \quad \mathbb{E} S_T \leq \eta \beta d T + \sqrt{dGT/\sigma}.$$ 

In particular, setting $\sigma = 2\sqrt{dGT/S}$ and $\eta = \min\left\{D/2G\sqrt{T}, S/\beta d T\right\}$ we obtain $\mathbb{E} S_T \leq S$ and $\mathbb{E} R_T = O(\sqrt{T} + dT/S)$.

**Proof.** Define the random minimization objective at time $t$ by;

$$\tilde{\phi}_t(w) := \phi_t(w; p_t) = \sum_{i=1}^{t-1} f_i(w) + p_t^T w + R(w) + B(w),$$

and set

$$y_{t+1} := \arg\min_{w \in W} \left\{ \tilde{\phi}_t(w) + f_t(w) \right\}.$$
We have,

\[ E \mathcal{R}_T = E \left[ \sum_{t=1}^{T} f_t(w_t) - f_t(w^*) \right] = E \left[ \sum_{t=1}^{T} f_t(w_t) - f_t(y_{t+1}) \right] + E \left[ \sum_{t=1}^{T} f_t(y_{t+1}) - f_t(w^*) \right], \tag{7} \]

where the expectation is taken over randomness of the algorithm originating from random the perturbations \( p_1, \ldots, p_T \). To bound the first term in Eq. (7), consider any perturbation \( p_t \), and note that

\[ \tilde{\phi}_t(w) + f_t(w) = \sum_{i=1}^{t} f_i(w) + p_t^T w + \frac{1}{2\eta} \| w - w_0 \|^2 + B(w), \]

thus \( \tilde{\phi}_t + f_t \) is \( 1/\eta \)-strongly-convex. In addition, \( \tilde{\phi}_t + f_t \) is minimized over \( W \) by \( y_{t+1} \), and \( \tilde{\phi}_t \) is minimized by \( w_t \) over \( W \). Therefore by a standard bound on the stability of minimizers of strongly convex objectives (see Lemma 19 in Appendix E) we obtain

\[ \| y_{t+1} - w_t \| \leq \frac{2}{\eta G}, \]

and then

\[ E \left[ \sum_{t=1}^{T} f_t(y_{t+1}) - f_t(w^*) \right] \leq 2\eta G^2 T. \]

To bound the second (leaders regret) term of Eq. (7), we first obtain a bound the w.r.t. the truncated domain \( W_\gamma \).

**Lemma 4.** The hypothetical leaders regret is bounded as,

\[ E \left[ \sum_{t=1}^{T} f_t(y_{t+1}) - f_t(w^*_\gamma) \right] \leq \frac{D^2}{2\eta} \sigma \sqrt{dD} + 1, \]

for any \( w^*_\gamma \in W_\gamma \).

Now, note that for any \( w^* \in W \), there exists \( w^*_\gamma = w_0 + (1-\gamma)(w^* - w_0) \in W_\gamma \) such that \( \| w^* - w^*_\gamma \| \leq \gamma D \). Thus,

\[ E \left[ \sum_{t=1}^{T} f_t(y_{t+1}) - f_t(w^*) \right] \leq \frac{D^2}{2\eta} \sigma \sqrt{dD} + GDT\gamma + 1 = \frac{D^2}{2\eta} \sigma \sqrt{dD} + GD + 1, \]

which concludes the proof of our regret bound. For the switches bound, by Lemma 3;

\[ E \mathcal{S}_T = \sum_{t=2}^{T} \Pr(w_{t+1} \neq w_t) = \sum_{t=2}^{T} \| Q_{t+1} - Q_t \|_{TV} \leq \frac{\eta \beta dT + \sqrt{dGT}}{\sigma}, \]

as claimed. \( \square \)

The proof of Lemma 4 is a straightforward adaptation of the analysis for the linear case laid out in Kalai and Vempala (2005), and makes use of the well known Follow-the-leader Be-the-leader Lemma, stated next for completeness.
Lemma 5 (FTL-BTL, Kalai and Vempala, 2005). Let \( \mathcal{X} \subset \mathbb{R}^d \) be convex and compact, and \( h_1, \ldots, h_T: \mathcal{X} \to \mathbb{R} \) be a sequence of losses. Then, for \( w^*_t := \arg \min_{w \in \mathcal{X}} \sum_{s=1}^t h_t(w) \), we have

\[
\sum_{t=1}^T h_t(w^*_t) \leq \sum_{t=1}^T h_t(w^*_T).
\]

Proof of Lemma 4. Fix a perturbation sequence \( p_1, \ldots, p_T \), and additionally define \( p_0 = 0 \). Consider the auxiliary loss sequence \( \tilde{f}_0(w) = R(w) + B(w) \), and for \( t \geq 1 \), \( \tilde{f}_t(w) := f_t(w) + (p_t - p_{t-1})^T w \).

From Eq. (6) it follows that

\[
y_{t+1} = \arg \min_{w \in \mathcal{W}} \left\{ \sum_{i=0}^t \tilde{f}_i(w) \right\},
\]

hence the BTL Lemma (Lemma 5) we obtain (for any \( w^*_t \in \mathcal{W} \)); \( \sum_{t=0}^T \tilde{f}_t(y_{t+1}) \leq \sum_{t=0}^T \tilde{f}_t(w^*_t) \). Substituting for the definition of \( \tilde{f}_t \) and rearranging we get

\[
\sum_{t=1}^T f_t(y_{t+1}) - f_t(w^*_t) \leq R(w^*_1) - R(y_1) + B(w^*_1) - B(y_1) + \sum_{t=1}^T (p_t - p_{t-1})^T (w^*_t - y_{t+1})
\]

\[
\leq \frac{D^2}{2\eta} + 1 + \sum_{t=1}^T (p_t - p_{t-1})^T (w^*_t - y_{t+1}).
\]

Now, consider any perturbations distribution \( Q \), such that the marginals of the \( p_t \)'s under \( Q \) are the same as the marginals of the \( p_t \)'s under our actual lazy algorithm which we denote by \( A \). Recall that \( y_{t+1} \) defined by Eq. (6) depends only on randomness introduced by \( p_t \). This implies that the \( y_t \)'s are distributed the same under both \( A \) and \( Q \) as long as the marginals of the perturbations match. Hence

\[
\mathbb{E}_A \left[ \sum_{t=1}^T f_t(y_{t+1}) - f_t(w^*_t) \right] = \sum_{t=1}^T \mathbb{E}_A \left[ f_t(y_{t+1}) - f_t(w^*_t) \right]
\]

\[
= \sum_{t=1}^T \mathbb{E}_Q \left[ f_t(y_{t+1}) - f_t(w^*_t) \right] = \mathbb{E}_Q \left[ \sum_{t=1}^T f_t(y_{t+1}) - f_t(w^*_t) \right].
\]

By Lemma 2, for all \( t \) it holds that \( p_t \sim \text{Lap}(\sigma) \) when generated by our algorithm \( A \). Therefore, choosing \( Q \) by letting \( p_1 \sim \text{Lap}(\sigma) \), and setting \( p_t = p_1 \) for all \( t \geq 2 \), we achieve the same marginals as those induced by \( A \). This implies

\[
\mathbb{E}_A \left[ \sum_{t=1}^T f_t(y_{t+1}) - f_t(w^*_t) \right] \leq \frac{D^2}{2\eta} + 1 + \mathbb{E}_Q \left[ \sum_{t=1}^T (p_t - p_{t-1})^T (w^*_t - y_{t+1}) \right]
\]

\[
\leq \frac{D^2}{2\eta} + 1 + \mathbb{E}_Q \left[ \sum_{t=1}^T \|p_t - p_{t-1}\| \|w^*_t - y_{t+1}\| \right]
\]

\[
\leq \frac{D^2}{2\eta} + 1 + D \mathbb{E}_Q \left[ \sum_{t=1}^T \|p_t - p_{t-1}\| \right]
\]

\[
= \frac{D^2}{2\eta} + 1 + \sigma \sqrt{dD},
\]

as desired. \( \square \)
3.2 The strongly convex case

In this section, we state and prove Theorem 2 providing the guarantees of Algorithm 2 for the strongly convex setting. The performance here hinges on increasing the perturbations variance at a certain rate, accounting for the increasing curvature in the per round minimized objective. This, along with a careful analysis of the perturbed leaders regret, is key to achieving the quadratic gain in the guarantee.

**Theorem 2.** Let \( W \subset \mathbb{R}^d \) be convex and of diameter \( D \), and assume the loss functions \( f_1, \ldots, f_T \) are \( G \)-Lipschitz, \( \beta \)-smooth and \( \lambda \)-strongly convex over \( W \). Further, assume the barrier \( B \) satisfies \( 0 \leq B(w) \leq 1 \) for all \( w \in W_\gamma \), where \( \gamma := 1/T \), and let \( \sigma_t = \sqrt{t} \sigma \) for all \( t \in \tau := \{2, 2^1, \ldots, 2^{[\log T]}\} \), and \( \sigma_t = \sigma_{t-1} \) for \( t \notin \tau \). Then running Algorithm 2 with parameters \( \{\sigma_t\} \), \( R(w) = 0 \) and barrier \( B \), we obtain:

\[
\mathbb{E} R_T \leq \frac{4G^2 + 2d\sigma^2}{\lambda} (1 + \log T) + 1 \quad \text{and} \quad \mathbb{E} S_T \leq \frac{2\beta d}{\lambda} (1 + \log T) + \frac{G\sqrt{dT} \log T}{\sigma}.
\]

In particular, for any \( S \geq \frac{8\beta d}{\lambda} \log T \), setting \( \sigma = 6G\sqrt{dT} / S \) we obtain the switching guarantee \( \mathbb{E} S_T \leq S \) and regret \( \mathbb{E} R_T = O(d^2 T \log T / S^2) \).

**Proof.** As in the convex case, let \( y_{t+1} = \arg \min_{w \in W} \left\{ \tilde{\phi}_t(w) + f_t(w) \right\} \), where

\[
\tilde{\phi}_t(w) := \phi_t(w; \xi_t) = \sum_{i=1}^{t-1} f_i(w) + \xi_t^T w + B(w),
\]

and by following an argument similar to the proof of Theorem 1, we obtain

\[
\mathbb{E} R_T \leq \frac{2G^2}{\lambda} (1 + \log T) + \mathbb{E} \left[ \sum_{t=1}^{T} f_t(y_{t+1}) - f_t(w^*) \right],
\]

with the only difference being that \( \tilde{\phi}_t + f_t \) is now \( t\lambda \)-strongly-convex. To bound the second term in the above display, we follow the same argument given in the proof of Lemma 4 to obtain for any \( w^*_\gamma \in W_\gamma \):

\[
\mathbb{E} \left[ \sum_{t=1}^{T} f_t(y_{t+1}) - f_t(w^*_\gamma) \right] \leq 1 + \mathbb{E}_Q \left[ \sum_{t=1}^{T} (\xi_t - \xi_{t-1})^T (w^*_\gamma - y_{t+1}) \right],
\]

where this time we define \( Q \) by \( \xi \sim \text{Lap}(1) \), and set \( \xi_t = \sigma_t \xi \) for all \( t \in [T] \), so that \( \xi_t \sim \text{Lap}(\sigma_t) \) under \( Q \). Indeed, by Lemma 2 it holds that under our actual algorithm \( \xi_t \sim \text{Lap}(\sigma_t) \) as well, and therefore the marginals match those induced by \( Q \). Next, we exploit the fact that \( \xi_t \) are zero mean in order to get rid of the non-random part of \( w^*_\gamma - y_{t+1} \). To that end, set \( x_{t+1} := \arg \min_{w \in W} \left\{ \sum_{i=1}^{t} f_i(w) \right\} \), and note \( x_{t+1} \) is deterministic. Therefore,

\[
\mathbb{E} \left[ \sum_{t=1}^{T} f_t(y_{t+1}) - f_t(w^*_\gamma) \right] - 1 \leq \mathbb{E}_Q \left[ \sum_{i=1}^{T} (\xi_t - \xi_{t-1})^T (w^*_\gamma - y_{t+1}) \right]
\]

\[
= \sum_{t=1}^{T} \mathbb{E}_Q [\xi_t - \xi_{t-1}]^T (w^*_\gamma - x_{t+1}) + \mathbb{E}_Q \left[ \sum_{i=1}^{T} (\xi_t - \xi_{t-1})^T (x_{t+1} - y_{t+1}) \right]
\]

\[
= \mathbb{E}_Q \left[ \sum_{i=1}^{T} (\xi_t - \xi_{t-1})^T (x_{t+1} - y_{t+1}) \right].
\]
Now, note that $x_{t+1}$ and $y_{t+1}$ minimize the same $t\lambda$-strongly-convex objective up to the additional perturbation vector $p_t$, therefore by Lemma 19: $\|x_{t+1} - y_{t+1}\| \leq \frac{2\|p_t\|}{t\lambda}$. Thus we obtain,
\[
\mathbb{E}\left[\sum_{t=1}^{T} f_t(y_{t+1}) - f_t(w^*)\right] - 1 \leq \mathbb{E}_Q\left[\sum_{t=1}^{T} (p_t - p_{t-1})^T (x_{t+1} - y_{t+1})\right]
\leq \mathbb{E}_Q\left[\sum_{t=1}^{T} \|p_t - p_{t-1}\| \|x_{t+1} - y_{t+1}\|\right]
\leq \mathbb{E}_Q\left[\sum_{t=1}^{T} \|p_t - p_{t-1}\| \frac{2\|p_t\|}{t\lambda}\right]
\begin{equation}
= \mathbb{E}_{\xi \sim \text{Lap}(1)}\left[\sum_{t=1}^{T} \|\sigma_t \xi - \sigma_{t-1} \xi\| \frac{2\|\sigma_t \xi\|}{t\lambda}\right]
\end{equation}
\begin{equation}
= \mathbb{E}_{\xi \sim \text{Lap}(1)}[\|\xi\|^2]\left[\sum_{t=1}^{T} |\sigma_t - \sigma_{t-1}| \frac{2\|\sigma_t\|}{t\lambda}\right]
\end{equation}
\begin{equation}
= \frac{4d}{\lambda} \sum_{t=1}^{T} \frac{\sigma_t (\sigma_t - \sigma_{t-1})}{t}
\end{equation}
\begin{equation}
= \frac{4d}{\lambda} \sum_{t \in \tau} \frac{\sigma_t (\sigma_t - \sigma_{t-1})}{t}
\end{equation}
\begin{equation}
= \frac{4d\sigma^2}{\lambda} \sum_{t \in \tau} \frac{\sqrt{t} / \sqrt{t} - \sqrt{t}/2}{t}
\end{equation}
\leq \frac{2d\sigma^2}{\lambda} (1 + \log T).
\]

The above, after accounting for the truncation, gives;
\[
\mathbb{E}\left[\sum_{t=1}^{T} f_t(y_{t+1}) - f_t(w^*)\right] \leq \frac{2d\sigma^2}{\lambda} (1 + \log T) + GD + 1 \leq \frac{2d\sigma^2}{\lambda} (1 + \log T) + 2G^2 / \lambda + 1.
\]

which concludes the proof of the regret bound. For the switches guarantee, note that since losses are $\lambda$-strongly convex, we have that $\phi_t$ are $(1/\eta_t)$-strongly convex with $\eta_t = 1/(\lambda t)$. Thus, by Lemma 3;
\[
\mathbb{E}S_T = \sum_{t=2}^{T} \Pr(w_{t+1} \neq w_t) = \sum_{t=2}^{T} \|Q_{t+1} - Q_t\|_{TV} \leq 1 + \log T + \sum_{t \in \tau} t(\eta_t \beta d + \sqrt{dG} / \sigma_t)
\begin{equation}
= 1 + \log T + \sum_{t \in \tau} \frac{\beta td}{\lambda t} + \sqrt{dG} \sqrt{t} / \sigma
\end{equation}
\begin{equation}
\leq \frac{2\beta d}{\lambda} (1 + \log T) + \frac{G \sqrt{dT}}{\sigma(\sqrt{2} - 1)},
\end{equation}

where the first inequality follows since our epoch schedule contains $\lfloor \log T \rfloor$ perturbation scale changes; $\sqrt{t} / 2\sigma = \sigma_{t-1} \rightarrow \sigma_t = \sqrt{t}\sigma$ for $t \in \tau$, and the last since it is a geometric sum starting at $\sqrt{2}$. Finally, note that for any $S \geq \frac{8d^2}{\lambda} \log T$, setting $\sigma = 6G \sqrt{dT} / S$ gives
\[
\mathbb{E}S_T \leq \frac{2\beta d}{\lambda} (1 + \log T) + \frac{S}{2} \leq \frac{4\beta d}{\lambda} \log T + \frac{S}{2} \leq S,
\]
and completes the proof. \[ \square \]
4 Lazy Stochastic OCO

In this section, we present a simple algorithm for the special case where the losses are drawn i.i.d. from some distribution of convex losses $F$. The standard objective to be minimized here is the pseudo regret, defined by

$$R_T := \mathbb{E} \left[ \sum_{t=1}^{T} f_t(w_t) - f_t(w^*) \right],$$

where $w^* = \arg \min_{w \in W} \mathbb{E} [f_1(w)]$ and the expectation is over the loss distribution $F$, from which $f_1, \ldots, f_T$ are sampled i.i.d. Importantly, the minimizer of the expected loss defined above stays fixed for the duration of the game. This is in stark contrast to the situation of the general adversarial setting, and enables significantly better bounds achieved by non uniform blocking as outlined by Algorithm 3.

Next, we state and prove Theorem 3 which summarizes the guarantees of Algorithm 3.

**Algorithm 3 Lazy SGD**

1: **input:** learning rates $\eta_1, \ldots, \eta_T > 0$
2: $k \leftarrow 0$; arbitrary $x_1 \in W$
3: for $t = 1$ to $T$ do
4: if $t = 2^k$ then
5: $k \leftarrow k + 1$
6: $\tilde{w}_k \leftarrow \Pi \left( \frac{1}{t} \sum_{s=1}^{t} x_t \right)$
7: end if
8: Play $w_t = \tilde{w}_k$; Observe $f_t$
9: $x_{t+1} \leftarrow x_t - \eta_t \nabla f_t(x_t)$
10: end for

**Theorem 3.** Assume $F$ is a distribution of $G$-Lipschitz convex losses over a domain $W$ of diameter $D$. Then running Algorithm 3 with step size $\eta_t = D / G \sqrt{t}$ guarantees $S_T \leq 1 + \log T$ and

$$R_T \leq 6DG\sqrt{T}.$$

If we further assume losses sampled from $F$ are $\lambda$-strongly-convex, then running Algorithm 3 with step size $\eta_t = 1 / \lambda t$ guarantees $S_T \leq 1 + \log T$ and

$$R_T \leq \frac{2G^2}{\lambda} \log^2 T.$$

**Proof.** For the general convex case, observe that the iterates $x_t$ maintained by the algorithm are just decision variables of standard OGD with decreasing step $\eta_t = D / G \sqrt{t}$. By well known arguments (see e.g., Hazan, 2019) these obtain an any time $t \in [T]$ guarantee of

$$\sum_{s=1}^{t} f_s(x_s) - f_s(w) \leq 2DG\sqrt{t},$$

for any $w \in W$. Therefore, we have for any $t$,

$$\mathbb{E} [f_t(\tilde{w}_k) - f_t(w^*)] \leq \frac{1}{2k} \sum_{s=1}^{2^k} \mathbb{E} [f_t(x_s) - f_t(w^*)] = \frac{1}{2k} \mathbb{E} \left[ \sum_{s=1}^{2^k} f_s(x_s) - f_s(w^*) \right] \leq \frac{2DG\sqrt{2k}}{2k}.$$
Now, set $T_k := \min\{2^k, T + 1\}$ and we obtain

\[
E \left[ \sum_{t=1}^{T} f_t(w_t) - f_t(w^*) \right] = \sum_{k=0}^{[\log T]} \sum_{T_k}^{T_k+1-1} E \left[ f_t(\tilde{w}_k) - f_t(w^*) \right]
\]

\[
\leq \sum_{k=0}^{[\log T]} \sum_{T_k}^{T_k+1-1} 2DG\sqrt{2^k} \frac{2^k}{2^k}
\]

\[
\leq 2DG \sum_{k=0}^{[\log T]} \sqrt{2^k}
\]

\[
\leq 2DG^21^{[\log T]/2^k}
\]

\[
\leq 6DG\sqrt{T},
\]

which concludes the proof for the general convex case. For the strongly convex case, we similarly have;

\[
E \left[ f_t(\tilde{w}_k) - f_t(w^*) \right] \leq \frac{G^2}{\lambda2^k} (1 + \log T_k),
\]

therefore,

\[
E \left[ \sum_{t=1}^{T} f_t(w_t) - f_t(w^*) \right] \leq \sum_{k=0}^{[\log T]} \sum_{T_k}^{T_k+1-1} \frac{G^2}{\lambda2^k} (1 + \log T_k)
\]

\[
\leq \frac{G^2}{\lambda} \log T + \frac{G^2}{\lambda} \sum_{k=0}^{[\log T]} \log(2^k)
\]

\[
\leq \frac{2G^2}{\lambda} \log^2 T,
\]

which completes the proof. 

## 5 Lower Bounds

In this section, we briefly discuss our lower bounds. All proofs are deferred to Appendix A. In the general convex setting, a lower bound can be derived somewhat indirectly by previous results of Geulen et al. (2010), who prove a lower bound for online buffering problems in the experts setting. Here we provide a dedicated statement and proof for the lazy OCO setup.

**Theorem 4.** For any $S \in \mathbb{N}$, there exists a stochastic sequence of 1-Lipschitz linear losses over $W = [-1, 1]$, such that the expected regret of any $S$-lazy algorithm is $\Omega(T/S)$.

In the strongly convex setting we prove two lower bounds, for both the adaptive and oblivious settings. The result for the adaptive case stated next, establishes the blocking technique to be optimal in this setting; for details, see Appendix A.2.

**Theorem 5.** For any $S$-lazy player $A$ there exists a sequence of 1-strongly-convex, 1-Lipschitz, 1-smooth losses $\{f_t\}$ over the domain $W = [-1, 1]$ such that $R_T = \Omega(T/S)$. 

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For the strongly convex oblivious setting, we prove a lower bound for a certain restricted class of players, characterized by Assumption 1 below. Loosely speaking, these are algorithms that employ continuous decision distributions with a constant portion of mass within one standard deviation from their mean.

**Assumption 1.** There exist constants $C_1 \geq 1 \geq C_0 > 0$, such that every pair of player decision distributions $p_1, p_2 \in \Delta[-1,1]$, with means $\mu_1, \mu_2$ and variances $\sigma_1^2 \leq \sigma_2^2$, satisfy

\[
\begin{align*}
1. \quad &\|p_1 - p_2\|_{TV} \geq \frac{1}{C_1} \min \left\{ 1, \frac{|\mu_1 - \mu_2|}{\sigma_1 + \sigma_2} \right\}, \text{ or } \sigma_1 + \sigma_2 \geq \frac{1}{C_1}; \text{ and} \\
2. \quad &\|p_1 - p_2\|_{TV} \geq \frac{1}{C_1} \min \left\{ 1, C_0 - \frac{\sigma_1}{\sigma_2} \right\}, \text{ or } \sigma_1 \geq 1.
\end{align*}
\]

It is not hard to show the above assumption holds for the family of uniform distributions, and the work of Devroye et al. (2018) establishes similar properties for Gaussian distributions. In Appendix A.3.1 we prove this property is satisfied by a more general class of distributions. Subject to Assumption 1, we are able to show our upper bound (Theorem 2) is tight, as established by our next and final theorem.

**Theorem 6.** For any $S$-lazy OCO algorithm $\mathcal{A}$ with decision distributions that satisfy Assumption 1, there exists a sequence of $1$-strongly convex, $1$-Lipschitz, $1$-smooth losses on which the regret of $\mathcal{A}$ is $\Omega(T/S^2)$.

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A Lower Bounds - Proofs

In this section, we provide detailed proofs of lower bounds presented in Section 5.

A.1 General convex, oblivious adversary

In this section, we establish an $\Omega(T/S)$ lower bound on the expected regret of any $S$-lazy algorithm in the general convex setting. We denote by Ber$_p$ the Bernoulli distribution over $\{-1, 1\}$ that takes the value 1 w.p. $p \in (0, 1)$, and by Ber$_p^j$ the joint distribution of $j$ independent samples from Ber$_p$. First consider the standard unconstrained setup in the scalar case $W = [-1, 1]$. For $p, q \in (0, 1)$ sufficiently close, an adversary that plays $f_t(w) = bw$, with $b_t \sim $ Ber$_p$ is indistinguishable from one that draws $b_t \sim $ Ber$_q$, and an $\Omega(\sqrt{T})$ bound may be established. In the lazy OCO setting, the switching limit gives room for the adversary to repeat losses, effectively decreasing the amount
of samples revealed and thereby allowing for a larger deviation between the loss distributions while maintaining their indistinguishability. The proof of Theorem 4 given next, provides a formal construction of this nature.

Proof of Theorem 4. Fix $T \in \mathbb{N}$, and let $\mathcal{A}$ be an arbitrary $S$-lazy algorithm. For any $p \in (0, 1)$ we define the adversarial construction of the stochastic loss sequence $\mathcal{F}(p, S)$ as follows. Split the $T$ rounds into $J := C^2S^2$ sections with $\tau := \frac{T}{J}$ consecutive rounds in each, where $C \in \mathbb{R}$ is a universal constant that will be determined later on. At the onset of each section $j \in [J]$, draw a single sample $b_j \sim \text{Ber}_p$ and play

$$f_t(w) = b_j \quad \forall t \in [(j-1)\tau + 1, j\tau].$$

That is, the adversary commits to the loss given by the single sample $b_j$ for the entirety of section $j$. This concludes the adversarial construction $\mathcal{F}(p, S)$. Moving forward, we consider the minimizer of the expected cumulative loss

$$w^* := \arg\min_{w \in W} E_p \left[ \sum_{t=1}^{T} f_t(w) \right],$$

where we use the subscript $p$ under expectation to signify $\{f_t\}_{t=1}^{T}$ are distributed according to $\mathcal{F}(p, S)$. Clearly, $w^*$ can perform no better than the realized minimizer in hindsight, and therefore it suffices to prove our lower bound with respect to it;

$$\mathbb{E} R_T = \mathbb{E} \left[ \sum_{t=1}^{T} f_t(w_t) - \min_{w \in W} \sum_{t=1}^{T} f_t(w) \right] \geq \mathbb{E} \left[ \sum_{t=1}^{T} f_t(w_t) - \sum_{t=1}^{T} f_t(w^*) \right].$$

Now, set $\epsilon := \frac{1}{8CS^2}$ and consider the two adversaries given by $\mathcal{F}(p_+, S)$ and $\mathcal{F}(p_-, S)$, where

$$p_+ := \frac{1 + \epsilon}{2}, \quad \text{and} \quad p_- := \frac{1 - \epsilon}{2}.$$

Respectively, denote the minimizer of the expected loss of each adversary by

$$w^*_+ := \arg\min_{w \in W} \mathbb{E}_{p_+} \left[ \sum_{t=1}^{T} f_t(w) \right], \quad \text{and} \quad w^*_- := \arg\min_{w \in W} \mathbb{E}_{p_-} \left[ \sum_{t=1}^{T} f_t(w) \right].$$

The following lemma establishes that on every round where the player’s decision and the loss are independent, the regret incurred against at least one of these adversaries will be $\Omega(\epsilon)$.

Lemma 6. Denote by $w_t$ the decision of $\mathcal{A}$ on round $t \in [T]$, and assume $f_t$ and $w_t$ are independent. Then at least one of the following bounds holds;

$$\mathbb{E}_{p_+} [f_t(w_t) - f_t(w^*_+)] \geq \frac{\epsilon}{4} \quad \text{or} \quad \mathbb{E}_{p_-} [f_t(w_t) - f_t(w^*_-)] \geq \frac{\epsilon}{4}.$$

Importantly, since the player is allowed far fewer switches ($S$) than there are sections ($J = C^2S^2$), it follows that for most sections the player’s decision is indeed independent of the loss. Proceeding, we consider the decomposition of her regret into two terms;

$$R_T = R_{pos} + R_{neg}.$$
The positive regret term $R_{pos}$ includes all rounds belonging to sections where the player did not switch. These are precisely the rounds on which her decision is independent of the loss. The negative regret term $R_{neg}$ on the other hand, includes all other rounds belonging to sections on which at least one decision switch was performed. On any such section, the loss suffered is trivially lower bounded by $-1$, which implies

$$\mathbb{E} R_{neg} \geq \mathbb{E} [-S_T \cdot \tau] \geq -S \frac{T}{C^2 S^2} = - \frac{T}{C^2 S}.$$ (Recall $S_T$ denotes the random variable number of switches performed by $A$.) This leaves $J - S_T$ sections that work in favor of the adversary and contribute (positively) to $R_{pos}$. On every round of any such section, Lemma 6 applies and the player must suffer a $\frac{\epsilon}{4}$ regret penalty from at least one of the adversaries. This implies

$$\mathbb{E} R_{pos} \geq \mathbb{E} \left[ \frac{(J - S_T) \tau \epsilon}{2} \right] \geq \frac{(J - S) \tau \epsilon}{2} = \frac{T}{64CS} - \frac{T}{64C^3 S^2}.$$ Now, for a choice of $C = 128$ we obtain

$$\mathbb{E} R_T = \mathbb{E} [R_{neg} + R_{pos}] \geq \frac{T}{CS} \left( \frac{1}{64} - \frac{1}{64C^2 S} - \frac{1}{C} \right) > \frac{T}{200CS},$$

which completes the proof. □

The proof of Lemma 6 hinges on Lemma 7 stated below, whose proof follows from well known information theoretic arguments and is given in Appendix E for completeness.

**Lemma 7.** Let $\epsilon \in (0, \frac{1}{4})$, and set $p = \frac{1+\epsilon}{2}$, $q = \frac{1-\epsilon}{2}$. It holds that

$$\|\text{Ber}_p^j - \text{Ber}_q^j\|_{TV} \leq \epsilon \sqrt{2j}.$$ 

**Proof of Lemma 6.** By assumption, $w_t$ and $f_t$ are independent, therefore $b_t$ is Bernoulli distributed with the appropriate parameter for both adversaries. Hence, for $p \in \{p_+, p_-\}$ we have

$$\mathbb{E}_p [f_t(w)] = (2p - 1)w,$$

and

$$\mathbb{E}_{p_+} [f_t(w^*_+)] = \mathbb{E}_{p_-} [f_t(w^*_-)] = -\epsilon.$$ 

Therefore

$$\mathbb{E}_{p_+} [f_t(w_t) - f_t(w^*_+)] = \epsilon \mathbb{E}_{p_+} [1 + w_t]$$

$$\mathbb{E}_{p_-} [f_t(w_t) - f_t(w^*_-) = \epsilon \mathbb{E}_{p_-} [1 - w_t].$$

Now, assume by contradiction that both bounds fail to hold. Then

$$\mathbb{E}_{p_+} [1 + w_t] < \frac{1}{4}; \quad \mathbb{E}_{p_-} [1 - w_t] < \frac{1}{4}.$$
and by Markov’s inequality
\[ \Pr_{p_+}(w_t > 0) < \frac{1}{4}; \quad \Pr_{p_-}(w_t < 0) < \frac{1}{4}. \]

Hence, considering the event the player outputs a decision \( w_t > 0 \), we arrive at the conclusion the total variation between the distributions of the two adversaries is \( \geq 1/2 \). We now apply Lemma 7 to arrive at contradiction;

\[
\frac{1}{2} \leq \Pr_{p_-}(w_t > 0) - \Pr_{p_+}(w_t > 0) \\
\leq \|\text{Ber}_{p_+}^J - \text{Ber}_{p_-}^J\|_{TV} \\
\leq \epsilon \sqrt{2J} \\
= \frac{\sqrt{2C^2S^2}}{8CS} \leq \frac{1}{4},
\]

and the proof is complete. \( \square \)

### A.2 Strongly convex losses, adaptive adversary

In this section, we prove an \( \Omega(T/S) \) lower bound for an adaptive adversary with strongly convex losses. This result establishes the blocking technique (see e.g., Arora et al., 2012; Chen et al., 2019, where it is termed mini-batching) to be optimal in this setting; It is well known (see e.g., Hazan, 2019) that for \( G \)-Lipschitz \( \lambda \)-strongly-convex losses the regret guarantee of OGD with decreasing step size \( \eta_t = 1/\lambda t \) is \( O\left(\frac{G^2}{\lambda} \log T\right) \). By applying the blocking technique in this setting, we arrive at an \( S \) round unconstrained game with \( (T/S)G \)-Lipschitz \( (T/S)\lambda \)-strongly-convex losses, and with a properly adjusted step size we obtain a \( O\left(\frac{G^2}{\lambda} \log S\right) \) regret guarantee. At a high level, we make two observations. The first is that in order to exploit strong convexity and obtain low regret, following the leader is practically mandatory in the standard setup. The next lemma formalizes this idea; The leaders loss is only within an \( O(\log T) \) additive advantage over the best fixed decision in hindsight, therefore any low regret algorithm must also perform well compared to the leaders.

**Lemma 8 (Reverse BTL-lemma).** Let \( f_t \) be a sequence of 1-strongly-convex 1-Lipschitz losses. Denote by \( w^*_t := \arg \min_{w \in W} \sum_{s=1}^t f_t(w) \) the leader at time \( t \). Then

\[
\sum_{t=1}^T f_t(w^*_t) - \sum_{t=1}^T f_t(w_t) \leq 2 \sum_{t=1}^T \frac{1}{t}.
\]

The second observation is that on any round, an adaptive adversary may move the leader away from the player by \( \Omega(1/T) \). Since the player has a limited number of switches, there will be several long sections where these small steps accumulate and result in large loss. The proof of the theorem stated below and its proof formalize the above idea.

**Proof of Theorem 5.** We consider losses of the form

\[ f_t(w) := \frac{1}{2}(w - x_t)^2; \quad x_t \in [-1, 1]. \]

As a direct implication of Lemma 8 we have that

\[
\sum_{t=1}^T f_t(w_t) - f_t(w^*) \geq \sum_{t=1}^T (f_t(w_t) - f_t(w^*_t)) - 2 \log T,
\]
and therefore we will be interested in lower bounding the regret with respect to the leaders. On every round $t$, we choose $x_t$ such that the leader moves $1/2T$ away from $w_t$:

$$x_t := \begin{cases} w_{t-1}^* + \frac{t}{2T} & w_t \leq w_{t-1}^*, \\ w_{t-1}^* - \frac{t}{2T} & w_t > w_{t-1}^*. \end{cases}$$

This implies

$$w_t^* = \frac{t-1}{t} w_{t-1}^* + \frac{1}{t} x_t = \begin{cases} w_{t-1}^* + \frac{t}{2T} & w_t \leq w_{t-1}^*, \\ w_{t-1}^* - \frac{t}{2T} & w_t > w_{t-1}^*. \end{cases}$$

Since $w_t^*$ never leaves $[-\frac{1}{2}, \frac{1}{2}]$, we can be certain $x_t \in [-1, 1]$ throughout the game and therefore the construction is valid. We will now limit our attention to the second half of the game and assume w.l.o.g that $S$ switches occur there:

$$R_T + 2 \log T \geq \sum_{t=1}^{T} f_t(w_t) - f_t(w_t^*)$$

$$\geq \sum_{t>T/2} f_t(w_t) - f_t(w_t^*)$$

$$= \sum_{j=1}^{S} \sum_{t=s_j}^{s_j+\tau_j} f_t(w_{s_j}) - f_t(w_t^*),$$

where $s_j \in \{[T/2], \ldots, T\}$ denote the player’s switch rounds. To show that on each stationary section $s_j \rightarrow s_j + \tau_j$ the player’s loss is $\Omega(\tau_j^2/T)$, first observe that

$$f_t(w_t) - f_t(w_t^*) = \frac{1}{2} (w_t - x_t)^2 - \frac{1}{2} (w_t^* - x_t)^2$$

$$= \frac{1}{2} [(w_t - w_t^*)^2 + 2(w_t - w_t^*)(w_t^* - x_t)]$$

$$\geq (w_t - w_t^*)(w_t^* - x_t)$$

$$\geq \frac{|w_t - w_t^*|}{2}.$$

In addition, whenever the player stays stationary the above distances accumulate;

$$|w_k^* - w_{k+\ell}^*| \geq |w_k^* - w_{k+\ell}^*| = \frac{t}{2T}.$$

Hence, for all $j \in [S]$ we have

$$\sum_{t=s_j}^{s_j+\tau_j} f_t(w_{s_j}) - f_t(w_t^*) \geq \sum_{t=s_j}^{s_j+\tau_j} \frac{|w_t - w_t^*|}{2} \geq \frac{1}{2} \sum_{t=s_j}^{s_j+\tau_j} \frac{t}{2T} \geq \frac{\tau_j^2}{8T}.$$

Substituting the above into Eq. (10) we obtain

$$R_T + 2 \log T \geq \frac{1}{8T} \sum_{j=1}^{S} \tau_j^2.$$
To complete the proof, we recall that \( \sum_{j=1}^{S} \tau_j = T/2 \), and invoke the following well known fact. Let \( v \in \mathbb{R}^S \) with \( \sum_{i=1}^{S} v_i = C \), and \( u \in \mathbb{R}^S \) with \( u_i = \frac{C}{S} \) for all \( i \in [S] \), then
\[
\|v\|^2 = \|v - u + u\|^2 = \|v - u\|^2 + \|u\|^2 \geq \frac{S C^2}{S^2} = \frac{C^2}{S}.
\]
This implies
\[
R_T + 2 \log T \geq \frac{1}{8T} \frac{T^2}{4S} = \frac{T}{32S},
\]
and we are done.

**Proof of Lemma 8 (Reverse BTL-Lemma).** We prove by induction on \( T \). The base case is obvious, for the inductive step observe;
\[
\sum_{t=1}^{T} f_t(w^*_T) - \sum_{t=1}^{T} f_t(w^*_t) = \sum_{t=1}^{T-1} f_t(w^*_T) - \sum_{t=1}^{T-1} f_t(w^*_t)
\]
\[
= \sum_{t=1}^{T-1} f_t(w^*_T) - f_t(w^*_T) + \sum_{t=1}^{T-1} f_t(w^*_T) - f_t(w^*_t)
\]
\[
\leq \sum_{t=1}^{T-1} f_t(w^*_T) - f_t(w^*_T) + \sum_{t=1}^{T-1} \frac{2}{t},
\]
where the last inequality follows from the inductive hypothesis. In addition, we have
\[
\sum_{t=1}^{T} f_t(w^*_T) \leq \sum_{t=1}^{T} f_t(w^*_T)
\]
\[
\iff \sum_{t=1}^{T-1} f_t(w^*_T) - f_t(w^*_T) \leq f_T(w^*_T) - f_T(w^*_T) \leq \frac{2}{T},
\]
with the last inequality follows from the fact that \( \|w^*_T - w^*_T\| \leq \frac{2}{T} \). Putting this together with the previous inequality we obtain
\[
\sum_{t=1}^{T} f_t(w^*_T) - \sum_{t=1}^{T} f_t(w^*_T) \leq \frac{2}{T} + \sum_{t=1}^{T-1} \frac{2}{t} = \sum_{t=1}^{T} \frac{2}{t},
\]
as desired.

### A.3 Strongly convex losses, oblivious adversary

In this section, we prove Theorem 6, establishing a regret lower bound in the strongly convex oblivious setting. Given a randomized player, we tailor an adversary to that player exploiting knowledge of her algorithm, and in particular, of her decision distributions. At a high level, we force the player to follow the leader over a long trajectory. Since the player is given a limited switching quota, the only way she can follow the leader over a long trajectory is by using high variance decisions, which imply high regret. A link between the switching quota and the total variation between decision distributions is made formal by the next well known lemma.
Lemma 9 (max coupling). Let $X, Y$ be two random variables distributed according to $p, q$ respectively over a sample space $\Omega$. Then for any joint distribution $(X, Y)$ we have that

$$\Pr(X \neq Y) \geq \|p - q\|_{TV}.$$ 

Given the above, the next lemma now links long trajectories to high variance, subject to a switching constraint.

Lemma 10 (variance lower bound). Let $p, q$ be two probability measures with means $\mu_p, \mu_q$ and variances $\sigma_p^2, \sigma_q^2$. Then

$$\frac{(\mu_p - \mu_q)^2}{2 \|p - q\|_{TV}} \leq (\mu_p - \mu_q)^2 + \sigma_p^2 + \sigma_q^2.$$ 

In particular, if $\|p - q\|_{TV} \leq 1/4$, then

$$\frac{(\mu_p - \mu_q)^2}{2 \|p - q\|_{TV}} \leq \sigma_p^2 + \sigma_q^2.$$ 

The proofs for Lemmas 9 and 10 are provided in Appendix E. Unfortunately, Lemma 10 is insufficient to prove our lower bound, and is un-improvable in general due to the existence of distributions that do not have mass near the mean, such as Bernoulli distributions. Instead, we make use of the stronger Assumption 1 on the distributions played by the player.

Proof of Theorem 6. Throughout the proof, we denote the player’s decision distribution sequence by $\{p_t\}_{t=1}^T$, her decision expectations by $\mu_t$, and the variance of her decision by $\sigma_t^2 := \mathbb{E}_{w_t \sim p_t}[(w_t - \mu_t)^2]$.

The adversary construction is similar to the one in the adaptive case (Theorem 5). We consider losses of the form;

$$f_t(w) := \frac{1}{2} (w - x_t)^2,$$

and select $x_t$ such that the leader moves $\delta := 1/2T$ away from $\mu_t$:

$$x_t := w_{t-1}^* + \delta_t := \begin{cases} 
  w_{t-1}^* + \frac{t}{2T} & \mu_t \leq w_{t-1}^*, \\
  w_{t-1}^* - \frac{t}{2T} & \mu_t > w_{t-1}^*.
\end{cases}$$

This implies

$$w_t^* = \frac{t-1}{t} w_{t-1}^* + \frac{1}{t} x_t = \begin{cases} 
  w_{t-1}^* + \frac{1}{2T} & \mu_t \leq w_{t-1}^*, \\
  w_{t-1}^* - \frac{1}{2T} & \mu_t > w_{t-1}^*.
\end{cases}$$

Since $w_t^*$ never leaves $[-\frac{1}{2}, \frac{1}{2}]$, we can be certain $x_t \in [-1, 1]$ throughout the game and therefore the construction is valid. As a first step, we derive a per round regret lower bound in terms of the player’s decision expectation and variance.

Lemma 11. For all $t \in [T]$ it holds that

$$\mathbb{E}_{w_t \sim p_t}[f_t(w_t)] - f_t(w_t^*) \geq \sigma_t^2 + \frac{(\mu_t - w_t^*)^2}{2} + (1-t)\delta_t (\mu_t - w_{t-1}^*).$$

(We arbitrarily set $w_0^* := 0$ as a matter of convenience.)

As in the adaptive case, assume the game starts on round $T/2 + 1$, so that the linear term from Lemma 11 is $\geq 1/4; (1-t)\delta_t \geq 1/4$. (Formally, this can be accomplished for example with an adversary that plays $\delta_t = 0$ for the first $T/2$ rounds.)
We now consider the decomposition of the regret into $J \geq 1$ sections formed by the $J - 1$ rounds on which the player forces the adversary to change direction. On each of these sections, the leaders progress either constantly to the right or constantly to the left. For section $j < J$ (i.e., not the last section), on the last round the leader will be bypassed by the player, causing it to change directions. To ease notational clutter, we conveniently use shifted indexes when discussing a certain section; for example, $w_t^*$ and $p_t$ of section $j \in [J]$ that starts on round $t_j$, correspond to $w_{t_j+t}^*$ and $p_{t_j+t}$ of the global indexing scheme. For an illustration of a rightward section see Fig. 1.

**Figure 1:** A rightward path leaders section.

Our next two lemmas lower bounds the player’s regret on each such individual section. Both are formulated w.l.o.g. in terms of a rightward moving section (clearly, a leftward moving one behaves the same). The first does not make use of Assumption 1, and applies only in the case a generous switching budget was invested in the section. Intuitively, these are the long sections.

**Lemma 12.** Let $w_0^* \in W$, assume $w_t^* = w_{t-1}^* + \delta$, and $\mu_t \leq w_{t-1}^*$ for $t = 1, \ldots, \tau$. In addition, assume the player uses a switching budget of $\zeta \geq 1/2a$ over all $\tau$ rounds for some $a \geq 1$. Then it holds that:

$$\sum_{t=1}^{\tau} \mathbb{E}[f_t(w_t)] - f_t(w_t^*) \geq \tau \frac{(\tau \delta)^2}{32^3 a^2 \zeta^2}.$$

Our next lemma is used to lower bound the regret on low switching sections in which the player covers the distance traveled by the leader, and crucially uses Assumption 1.

**Lemma 13.** Let $w_0^* \in W$, assume $w_t^* = w_{t-1}^* + \delta$, $\mu_t \leq w_{t-1}^*$ for $1 \leq t \leq \tau$, and $\mu_{\tau+1} > w_\tau^*$. In addition, assume the player satisfies Assumption 1 and uses a switching budget of $\zeta$ over all $\tau$ rounds; $\sum_{t=1}^\tau \|p_t - p_{t+1}\|_{TV} = \zeta$. Then, it holds that

$$\sum_{t=1}^{\tau} \mathbb{E}[f_t(w_t)] - f_t(w_t^*) \geq \tau \frac{C_0^2}{32^3 C_1} \min \left\{ 1, \frac{(\tau \delta)^2}{\zeta^2} \right\}.$$

For the total regret minimization problem, the player faces multiple optimization problems (one per section), each of which we are able to lower bound using our above lemmas. Letting $R^j(\tau_j, \zeta_j)$ denote the player’s regret on the $j$’th section consisting of $\tau_j$ rounds and switching budget of $\zeta_j$, the total regret may be written as

$$R_T = \sum_{j=1}^J R^j(\tau_j, \zeta_j).$$

If the last section has $\geq T/4$ rounds, our result follows by Lemma 12. Assume otherwise, then the player has invested $\geq T/4$ rounds into direction swap sections where Lemma 13 applies, meaning
for all \( j \leq K := J - 1 \) it holds that

\[
\mathcal{R}^j(\tau_j, \zeta_j) \geq C \cdot \tau_j \cdot \min \left\{ 1, \frac{(\tau_j \delta_j)^2}{\zeta_j^2} \right\}, \quad \text{where } C := \frac{C_0^2}{32^{\frac{3}{2}}}.
\]

Hence, it follows that the player’s total regret is lower bounded by the optimal solution value of the following optimization problem:

\[
\min_{\tau_1, \ldots, \tau_J, \zeta_1, \ldots, \zeta_J} C \delta^2 \sum_{k=1}^{K} \frac{\tau_k^3}{\zeta_k^2}
\]

subject to:

\[
\sum_{k=1}^{K} \tau_k = T/4,
\]

\[
\sum_{k=1}^{K} \zeta_k = S.
\]

We have that

\[
C \delta^2 \sum_{k=1}^{K} \frac{\tau_k^3}{\zeta_k^2} = \frac{C T^3}{4^3 4^2 T^2} \sum_{k=1}^{K} \frac{(\tau_k/(T/4))^3}{\zeta_k^2} = \frac{C T^2}{4^4 S^2} \sum_{k=1}^{K} \frac{(\tau_k/T)^3}{(\zeta_k/S)^2}.
\]

To conclude the proof, what remains is to show the term on the right hand side is lower bounded by a constant. Indeed, the following lemma establishes it is \( \geq 1 \).

**Lemma 14.** For any \( \alpha, \beta \in \Delta_K \), it holds that \( \sum_{k=1}^{K} \frac{\alpha_k^3}{\beta_k^2} \geq 1 \).

This concludes the proof. \( \square \)

**Proof of Lemma 11.** Observe;

\[
E f_t(w_t) - f_t(w_t^*) = \frac{1}{2} \left( E \left[ (w_t - x_t)^2 \right] - (w_t^* - x_t)^2 \right)
\]

\[
= \frac{1}{2} \left( E \left[ (w_t - \mu_t + \mu_t - x_t)^2 \right] - (w_t^* - x_t)^2 \right)
\]

\[
= \frac{1}{2} \left( \sigma_t^2 + 2 E [(w_t - \mu_t)(\mu_t - x_t)] + (\mu_t - x_t)^2 - (w_t^* - x_t)^2 \right)
\]

\[
= \frac{1}{2} \sigma_t^2 + \frac{(\mu_t - w_t^*)^2}{2} + (\mu_t - w_t^*)(w_t^* - x_t).
\]

In addition,

\[
(\mu_t - w_t^*)(w_t^* - x_t) = (\mu_t - w_{t-1}^* - \delta_t)(\delta_t + w_{t-1}^* - x_t)
\]

\[
= (\mu_t - w_{t-1}^* - \delta_t)(\delta_t - t \delta_t)
\]

\[
= (\mu_t - w_{t-1}^*)(1 - t) \delta_t - (1 - t) \delta_t^2
\]

\[\geq (\mu_t - w_{t-1}^*)(1 - t) \delta_t,
\]

and the claim immediately follows. \( \square \)

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Proof of Lemma 12. First, observe that if \(|\mu_t - w^*_{t-1}| \geq (\tau \delta / 16a\zeta)^2\) on \(\tau/2\) rounds, it follows by Lemma 11 that

\[
R_t \geq (\tau/2) \frac{(\tau \delta)^2}{4(16a\zeta)^2} = \tau \frac{(\tau \delta)^2}{32^2 a^2 \zeta^2},
\]

and the result follows. Henceforth, we assume there exist \(\geq \tau/2\) rounds with \(|\mu_t - w^*_{t-1}| < (\tau \delta / 16a\zeta)^2\).

Split these rounds into \(8a\zeta\) consecutive sections of length \(M := (\tau/2)/(8a\zeta) = \tau/16a\zeta\) each. To be sure we have \(\geq 2\) rounds in each section, we assume \(\tau \geq 32a\zeta\). Otherwise, for \(T > 320a\) we have

\[
\tau \frac{(\tau \delta)^2}{32^2 a^2 \zeta^2} < \tau \delta^2 < 1/T,
\]

and therefore the statement holds trivially.

Proceeding, by Lemma 9 it must hold that the sum of total variations between consecutive distributions is \(\leq \zeta\), therefore at most \(4a\zeta\) sections may use up more than \(1/4a\) total variation (aka switching) budget. This establishes there are \(\geq 4a\zeta\) sections that must use up less than \(1/4a\) budget.

Formally, let \(\{t_i\}_{i=1}^M\) be rounds of one of these sections, then

\[
\sum_{i=1}^{M-1} \|p_{t_{i+1}} - p_{t_i}\|_{TV} \leq \frac{1}{4a}.
\]

Since the total variation defines a distance metric, by the triangle inequality this further implies that \(\|p_{t_{i+m}} - p_{t_i}\|_{TV} \leq 1/4a\) for all \(i, i + m \in [m]\). Now, by Lemma 10, for all \(i \in [M/2]\) we have that

\[
\sigma^2_{t_i} + \sigma^2_{t_{i+M/2}} \geq (M\delta/2 - (\tau \delta / 16a\zeta)^2)^2 \geq (M\delta/4)^2 = \frac{(\tau \delta)^2}{16^2 a^2 \zeta^2}.
\]

To justify the second inequality, note that \(1 \geq \tau \delta / 16a\zeta\), thus \((\tau \delta / 16a\zeta)^2 \leq \tau \delta / 16a\zeta = M\delta/4\). To conclude the proof, note we have identified a total of \(4a\zeta M/2 \geq \tau/8\) round pairs that contribute at least \((\tau \delta)^2 / 16^2 a^2 \zeta^2\) to the regret. Denote these rounds \(\{t_i\}_{i=1}^{\tau/8}\), then by Lemma 11;

\[
\frac{\tau}{\sum_{l=1}^{\tau/8} \mathbb{E}[f_t(w_t)] - f_t(w^*_t) \geq \sum_{l=1}^{\tau/8} \mathbb{E}[f_{t_l}(w_{t_l})] - f_{t_l}(w^*_{t_l}) \geq \frac{1}{2} \sum_{l=1}^{\tau/8} \sigma^2_{t_l} \geq \tau \frac{(\tau \delta)^2}{32^2 a^2 \zeta^2},
\]

and the proof is complete. \(\square\)

Proof of Lemma 13. If \(\zeta \geq C_0/2C_1\), the result follows from Lemma 12. Otherwise, note that \(|\mu_1 - \mu_{\tau+1}| \geq \tau \delta\), and therefore by Assumption 1;

\[
\sigma_1 + \sigma_{\tau+1} \geq \min \left\{ \frac{1}{C_1}, \frac{\tau \delta}{C_1 \zeta} \right\}.
\]

Now, assume w.l.o.g. that \(\sigma_1 \geq \sigma_{\tau+1}\), then again by Assumption 1 we have that for all \(t \in \{2, \ldots, \tau\}\); \(\sigma_t \geq \sigma_1\), or \(\sigma_t \geq 1\), or

\[
\zeta \geq \frac{C_0}{C_1} - \frac{\sigma_t}{\sigma_1} \iff \frac{\sigma_t}{\sigma_1} \geq \frac{C_0}{C_1} - \zeta \geq \frac{C_0}{2C_1}.
\]

In any case, it follows that

\[
\sigma_t \geq \min \left\{ 1, \frac{C_0}{2C_1} \sigma_1 \right\} \geq \frac{C_0}{2C_1^2} \min \left\{ 1, \frac{\tau \delta}{\zeta} \right\}.
\]
Therefore by Lemma 11;
\[ \sum_{t=1}^{\tau} \mathbb{E}[f_t(w_t^*) - f_t(w_t^*)] \geq \sum_{t=2}^{\tau} \frac{\sigma_t^2}{2} \geq (\tau - 1) \frac{C_0^2}{2} \min \left\{ 1, \frac{(\tau \delta)^2}{\zeta^2} \right\} \geq \tau \frac{C_0^2}{16C_1^4} \min \left\{ 1, \frac{(\tau \delta)^2}{\zeta^2} \right\}, \]
and we are done. \hfill \Box

**Proof of Lemma 14.** Since the problem involves only linear equality constraints, any optimal solution must satisfy the KKT conditions. Consider the Lagrangian of the problem;
\[ \mathcal{L}(\alpha, \beta, \gamma_1, \gamma_2) = \sum_{k=1}^{K} \frac{\alpha_k^3}{\beta_k} + \gamma_1 \left( \sum_{k=1}^{K} \alpha_k - 1 \right) + \gamma_2 \left( \sum_{k=1}^{K} \beta_k - 1 \right). \]
Let \( \alpha^*, \beta^* \) be any optimal solution, then by the KKT conditions it follows that for all \( k \leq K \)
\[ 0 = \frac{\partial \mathcal{L}}{\partial \beta_k} = -2 \frac{(\alpha_k^*)^3}{(\beta_k^*)^3} + \gamma_2 \implies \alpha_k^* = (\gamma_2/2)^{1/3} \beta_k^*. \]
By the problem constraints, this further implies \( \gamma_2 = 2 \) and therefore \( \alpha_k^* = \beta_k^* \) for all \( k \leq K \). Hence
\[ \sum_{k=1}^{K} \frac{(\alpha_k^*)^3}{(\beta_k^*)^3} = \sum_{k=1}^{K} \alpha_k = 1, \]
which completes the proof. \hfill \Box

**A.3.1 Discussion of the applicability of Assumption 1 to common distributions**

In this section, we show Assumption 1 holds for a general class of “nice” distributions. Algorithms in the spirit of Algorithm 2 we have presented in this work make use of such “nice” distributions as defined below, followed by an orthogonal projection onto the decision set. (Note, however, that the arguments in this section do not establish Assumption 1 holds after the projection operation.) Notably though, an immediate implication of the proof of Theorem 6 is that any follow-the-leader algorithm (that may not satisfy Assumption 1), is subject to the \( \Omega(T/S^2) \) lower bound, and therefore in particular Algorithm 2. We proceed now with the formal definition of “nice” distributions and follow with arguments linking them to Assumption 1.

**Definition 1.** We say \( \mathcal{P} = \mathcal{P}(C_\mu, C_\sigma) \) is a family of “nice” distributions with constants \( C_\sigma \geq C_\mu > 0 \), if the following holds. Let \( p \in \mathcal{P} \), denote \( X \sim p, \mu_p := \mathbb{E}X, \sigma_p^2 := \mathbb{E}(X - \mu)^2 \), then

(i) \( p \) is symmetric; \( p(\mu_p - x) = p(\mu_p + x) \) \( \forall x > 0 \).

(ii) For \( x \in [\mu_p - \sigma_p, \mu_p + \sigma_p] \), it holds that \( \frac{1}{C_\sigma \sigma_p} \leq p(x) \leq \frac{1}{C_\mu \sigma_p} \).

It is not hard to see the Normal, Laplace, and Uniform are examples of families that satisfy the above assumptions with an appropriate choice of constants \( C_\mu, C_\sigma \). Proceeding, let \( \mathcal{P} = \mathcal{P}(C_\mu, C_\sigma) \) be a family of “nice” distributions. Our first lemma establishes the first part of Assumption 1.

**Lemma 15.** For any \( p_1, p_2 \in \mathcal{P} \), it holds that
\[ \|p_1 - p_2\|_{TV} \geq \frac{1}{2C_\sigma} \min \left\{ 1, \frac{\|\mu_1 - \mu_2\|}{\sigma_1 + \sigma_2} \right\}. \]

**Proof.** Let \( p_1, p_2 \in \mathcal{P} \), and assume w.l.o.g. \( \mu_1 \leq \mu_2 \). We have
\[ 2 \|p_1 - p_2\|_{TV} \geq \int_{\mu_1}^{\infty} |p_1 - p_2| \geq \int_{\mu_1}^{\infty} p_2 - \int_{\mu_1}^{\infty} p_1 = \int_{\mu_1}^{\mu_2} p_2, \]

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where the last equality follows by the symmetry assumption \((i)\). In addition, by property \((ii)\) we have that
\[
\int_{\mu_1}^{\mu_2} p_2 \geq \min \left\{ \frac{\mu_2 - \mu_1}{C_\sigma \sigma_2}, \frac{1}{C_\sigma} \right\},
\]
and the result follows.

The second part of Assumption 1 lower bounds the total variation with relation to the variances, regardless of the distance between means. The below lemma establishes it holds for \(\Psi\) with the appropriate choice of constants.

**Lemma 16.** For any \(p_1, p_2 \in \Psi\) with variances \(\sigma_1^2 \leq \sigma_2^2\), it holds that
\[
\|p_1 - p_2\|_{TV} \geq \frac{1}{2C_\mu C_\sigma} \left( \frac{(C_\mu/C_\sigma) - \sigma_1}{\sigma_2} \right).
\]

**Proof.** Let \(p_1, p_2 \in \Psi\), with means \(\mu_1, \mu_2 \in \mathbb{R}\) and variances \(\sigma_1^2 \leq \sigma_2^2\). By property \((ii)\) of Definition 1, we have
\[
2 \|p_1 - p_2\|_{TV} \geq \int_{\mu_1}^{\mu_1+\sigma_1} |p_1 - p_2| \geq \sigma_1 \left( \frac{1}{C_\sigma \sigma_1} - \frac{1}{C_\mu \sigma_2} \right)
\]
\[
= \sigma_1 \left( \frac{C_\mu \sigma_2 - C_\sigma \sigma_1}{C_\mu C_\sigma \sigma_1 \sigma_2} \right) = \frac{1}{C_\mu C_\sigma} \left( \frac{(C_\mu/C_\sigma) - \sigma_1}{\sigma_2} \right),
\]
which proves the result. \(\square\)

**B Lazy switching, costs, and budgets**

In this section, we discuss the relation between three notions of switching related OCO; \(S\)-lazy, \(S\)-switching-budget, and \(c\)-switching-cost. Similar to lazy OCO studied in this paper, the switching-budget setting explored by Altschuler and Talwar (2018) also limits the player to a given number of switches \(S\), though the limit is applied to the actual number of switches and not the expected. The \(c\)-switching-cost variant on the other hand, charges the player a unit cost of \(c\) for every switch. Lemma 17 below establishes this is in fact equivalent to limiting the number of expected switches. Table 2 lists results of prior work in both forms - as a function of the switching cost \(c\) and as a function of the number of switches \(S\).

Lazy vs switching-cost. There is a natural correspondence between switching-cost and lazy OCO algorithms, as long as their guarantees are given in reasonable parametric forms. This is summarized by the following lemma.

**Lemma 17.** Denote the \(c\)-switching-cost regret by
\[
\mathcal{R}_T^{(c)} := \sum_{t=1}^{T} f_t(w_t) + c \cdot \mathbf{1}\{w_t \neq w_{t-1}\} - \min_{w \in W} \sum_{t=1}^{T} f_t(w)
\]

1. If \(A_c\) guarantees \(\mathbb{E} \mathcal{R}_T^{(c)} = (Tc)^{\alpha}\), then it may be converted to a \(S\)-lazy player \(A_S'\) with \(\mathbb{E} \mathcal{R}_T = \left( \frac{T}{S} \right)^{\frac{\alpha}{1+\alpha}}\).

2. If \(A_S\) is an \(S\)-lazy player with \(\mathbb{E} \mathcal{R}_T = \frac{T}{S^{\gamma}}\), then it may be converted to a \(c\)-switching-cost player \(A_c'\) with \(\mathbb{E} \mathcal{R}_T^{(c)} = T^{\frac{1}{1+\gamma}} c^{\frac{\gamma}{1+\gamma}}\).
Proof. Assume an online player \(A_S\) is a \(S\)-lazy algorithm with an expected regret guarantees of \(R(T, S)\). Then reducing \(A_S\) to a \(c\)-switching-cost algorithm \(A'_c\) involves choosing \(S = S(c)\) so as to minimize
\[
\mathbb{E} \left[ R^{(c)}(A'_c) \right] = R(T, S) + cS
\]
To that end choose \(S = S(c)\) to satisfy \(S = \frac{R(T, S)}{c}\), which gives
\[
R(T, c) = 2 \cdot c \cdot S(c).
\]
For the other direction, assume \(A_c\) has \(R(T, c)\) expected regret guarantee in the \(c\)-switching-cost setting. We reduce \(A_c\) to a \(S\)-lazy algorithm \(A'_S\) by solving \(cS = R(T, c)\) for \(c = c(S)\) and running \(A_c(c(S))\). This guarantees
\[
\mathbb{E} \left[ S(A'_S) \right] \leq S \quad \text{and} \quad \mathbb{E} \left[ R_T(A'_S) \right] \leq c(S) \cdot S.
\]
In particular, if \(A_c\) has a \(R(T, c) = (Tc)^\alpha\) guarantee then
\[
cS = T^\alpha \cdot c^\alpha \implies c = \left( \frac{T^\alpha}{S} \right)^{\frac{1}{1-\alpha}},
\]
and the reduction gives \(A'_S\) with
\[
\mathbb{E} \left[ S(A'_S) \right] \leq S \quad \text{and} \quad \mathbb{E} \left[ R_T(A'_S) \right] \leq S \left( \frac{T^\alpha}{S} \right)^{\frac{1}{1-\alpha}} = \left( \frac{T}{S} \right)^{\frac{\alpha}{1-\alpha}}.
\]
Similarly, if \(A_S\) has a \(R(T, S) = T^p\) guarantee, the reduction gives \(A'_c\) with
\[
\mathbb{E} \left[ R^{(c)}(A'_c) \right] = T^\frac{p}{1+p} \cdot c^\frac{p}{1+p}.
\]

Lazy vs switching-budget. Recently studied in the work of Altschuler and Talwar (2018), switching-budget regret is defined as the regret guarantee achievable by an algorithm under a hard cap budget of \(S \in [T]\) switches, a limit that should be met on every game execution. In general, while an \(S\)-switching-budget player is of course also an \(S\)-lazy one, the converse is not necessarily true. However, when the adversary is adaptive, the two notions are equivalent. To see this, consider the adaptive adversary given in Chen et al. (2019, Proposition 6). This adversary ensures \(\Omega(T/\sqrt{S_T})\) regret against any player that makes \(S_T\) switches, and furthermore, does not need to know the number of switches in advance. In particular, this adversary is completely unaffected by any randomness employed by the player. Therefore, By Jensen’s inequality and convexity of \(x \mapsto 1/\sqrt{x}\), we have that \(\mathbb{E} R_T \geq T/\sqrt{S}\) when \(\mathbb{E} S_T = S\). In addition, a deterministic player employing the blocking technique ensures \(R_T = O(T/\sqrt{S})\). Thus, we have that the minimax switching regret of both \(S\)-switching-budget and \(S\)-lazy is \(\Theta(T/\sqrt{S})\) in the adaptive adversary setting.

C Sampling from maximal couplings

In this section, we provide proofs for Section 2.3.
Table 2: Lazy OCO / switching-cost bounds from previous work ($c \geq 1$).

| Setting | Adversary | S-Lazy OCO | c-Switching Cost | Reference |
|---------|-----------|------------|------------------|-----------|
| MAB     | Oblivious | $O\left(\frac{T}{\sqrt{S}}\right)$ \(^b\) | $O\left(T^{2/3}c^{1/3}\right)$ | Arora et al. (2012) |
| MAB     | Oblivious | $\tilde{\Omega}\left(\frac{T}{\sqrt{S}}\right)$ \(^b\) | $\tilde{\Omega}\left(T^{2/3}c^{1/3}\right)$ | Dekel et al. (2014) |
| Experts | Oblivious | $O\left(\frac{T}{S}\right)$ \(^a\) | $O\left(\sqrt{Tc}\right)$ | Kalai and Vempala (2005) |
|         |           |            |                  | Geulen et al. (2010) |
|         |           |            |                  | Devroye et al. (2013) |
| Experts | Oblivious | $\tilde{\Theta}\left(\frac{T}{S}\right)$ \(^a, b\) | - | Altschuler and Talwar (2018) |
| OCO     | Oblivious | $O\left(\frac{dT}{S}\right)$ \(^a\) | $O\left(d\sqrt{Tc}\right)$ | Anava et al. (2015) |
| OCO     | Adaptive  | $\Theta\left(\frac{T}{\sqrt{S}}\right)$ \(^b\) | $\Theta\left(T^{2/3}c^{1/3}\right)$ | Chen et al. (2019) |

\(^a\) For $S = O(\sqrt{T})$.  
\(^b\) Also apply in the switching-budget setting.

\textbf{Proof of Lemma 1.} The algorithm samples from $\mathcal{P}$ when it reaches the loop, which happens if $\mathcal{P}(x) < z$ for $z \sim \mathcal{U}[0, \mathcal{Q}(x)]$. Denote

$$B := \left\{ w \in \mathbb{R}^d \mid \mathcal{P}(w) > \mathcal{Q}(w) \right\},$$

and observe

$$\Pr(\mathcal{P} \text{ is sampled from}) = \Pr(\mathcal{P}(x) < z)$$

\begin{align*}
&= \int_{x \notin B} \mathcal{Q}(x) \left(1 - \frac{\mathcal{P}(x)}{\mathcal{Q}(x)}\right) dx \\
&= \int_{x \notin B} \mathcal{Q}(x) - \mathcal{P}(x) dx \\
&= \|\mathcal{Q} - \mathcal{P}\|_{TV} \qedhere
\end{align*}

\textbf{Proof of Lemma 2.} Set

$$\mathbf{W} := \text{LazySample}(\mathbf{X}, \mathcal{Q}, \mathcal{P}); \quad \mathbf{X} \sim \mathcal{Q},$$

and we aim to prove that

$$\Pr(\mathbf{W} = w) = \mathcal{P}(w).$$

We analyse each of the two cases $P(w) > Q(w)$ and $P(w) < Q(w)$ separately. Note $w$ is not random, it is just the sample point at which we want to show the two densities are equal. Proceeding, denote

$$B := \left\{ w \in \mathbb{R}^d \mid \mathcal{P}(w) > \mathcal{Q}(w) \right\},$$

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and first assume \( w \notin B \). In this case it must be that \( w \) was returned by the first return statement, and thus equals \( x \). We have

\[
\Pr(W = w) = \Pr(W = w \land X \in B) + \Pr(W = w \land X \notin B)
\]
\[
= \Pr(W = w \land X \notin B)
\]
\[
= \Pr(X = w \land \mathcal{P}(w) < z) \quad \text{(where } z \sim \mathcal{U}[0, Q(x)])
\]
\[
= \frac{Q(w)}{Q(w)}
\]
\[
= \mathcal{P}(w).
\]

In addition, the running time is clearly \( O(1) \). Now consider the case that \( w \in B \). As before, we have

\[
\Pr(W = w) = \Pr(W = w \land X \in B) + \Pr(W = w \land X \notin B),
\]
and

\[
\Pr(W = w \land X \in B) = \Pr(X = w) = Q(w).
\]

First, observe that on each iteration the probability to exit the loop is

\[
\int_B \mathcal{P}(t) \left(1 - \frac{Q(t)}{\mathcal{P}(t)}\right) dt = \|\mathcal{P} - Q\|_{TV}.
\]

Therefore, we are expected to exit it in time \( O(\|\mathcal{P} - Q\|_{TV}^{-1}) \), and together with Lemma 1 this implies the \( O(1) \) running time. To compute \( \Pr(W = w \land X \notin B) \), denote the density of the value produced with the loop by \( \nu(y) \), and observe that for \( y \in B \)

\[
\nu(y) = \frac{\mathcal{P}(y) \left(1 - \frac{Q(y)}{\mathcal{P}(y)}\right)}{\int_B \mathcal{P}(t) \left(1 - \frac{Q(t)}{\mathcal{P}(t)}\right) dt} = \frac{\mathcal{P}(y) - Q(y)}{\mathcal{P}(B) - Q(B)}.
\]

Therefore

\[
\Pr(W = w \land X \notin B) = \int_{x \in B} Q(x) \Pr(W = w \mid X = x) dx
\]
\[
= \int_{x \in B} Q(x) \left(1 - \frac{\mathcal{P}(x)}{Q(x)}\right) \nu(x) dx
\]
\[
= \frac{\mathcal{P}(x) - Q(x)}{\mathcal{P}(B) - Q(B)} \int_{x \in B} Q(x) dx - \mathcal{P}(x) dx
\]
\[
= \frac{\mathcal{P}(x) - Q(x)}{\mathcal{P}(B) - Q(B)} (\mathcal{P}(B) - Q(B))
\]
\[
= \mathcal{P}(w) - Q(w).
\]

All in all, we have obtained

\[
\Pr(W = w) = Q(w) + \mathcal{P}(w) - Q(w) = \mathcal{P}(w),
\]

which concludes the case of \( w \in B \) and therefore completes the proof.  \( \square \)
D Proof of Lemma 3

We begin with establishing a closed form expression for the density function $Q_t$. The proofs below are based on arguments given in Agarwal et al. (2023).

**Lemma 18.** For any $w \in W$, we have

$$Q_t(w) = \nu_t(-\nabla \phi_t(w)) \det \left( -\nabla^2 \phi_t(w) \right),$$

where $\phi_t(w) := \phi_t(w; 0)$, and $\nu_t$ denotes the density function of $p_t \sim \text{Lap}(\sigma_t)$ (see Eq. (1)).

**Proof.** By the fact that the objective $\phi_t$ includes the Legendre function $B$, we have that $w_t(p) \in \text{int}(W)$ for all $p \in \mathbb{R}^d$. Further, by differentiability of $\phi_t$ and optimality conditions, we have;

$$0 = \nabla \phi_t(w_t(p); p) = \nabla \phi_t(w_t(p)) + p \implies -\nabla \phi_t(w_t(p)) = p.$$

By strict convexity of $\phi_t$, the mapping $w \mapsto -\nabla \phi_t(w)$ is one-to-one, with its inverse given by $w_t(p) = \nabla \phi^*_t(-p)$ where $\phi^*_t$ denotes the Fenchel conjugate of $\phi_t$. The result now follows by the change of variables formula (e.g., Bogachev and Ruan, 2007).

**Proof of Lemma 3.** First, we establish a uniform bound on the density ratio $\frac{Q_{t+1}(w)}{Q_t(w)}$. By Lemma 18, we have

$$\frac{Q_{t+1}(w)}{Q_t(w)} = \frac{\nu_{t+1}(-\nabla \phi_{t+1}(w))}{\nu_t(-\nabla \phi_t(w))} \left| \frac{\det(-\nabla^2 \phi_{t+1}(w))}{\det(-\nabla^2 \phi_t(w))} \right|.$$

To bound the second term, note that since $f_t$ is $\beta$-smooth,

$$\|\nabla^2 \phi_t(w) - \nabla^2 \phi_{t+1}(w)\| = \|\nabla^2 f_t(w)\| \leq \beta.$$

Further, letting $\lambda_{t,i}$ denote that $i$’th largest eigenvalue of $\nabla^2 \phi_t(w)$, this implies $|\lambda_{t,i} - \lambda_{t+1,i}| \leq \beta$ for all $i \in [d]$ (Bhatia, 2013). In addition, using that $\phi_t$ is $(1/\eta_t)$-strongly convex, we have $\lambda_{t,i} \geq 1/\eta_t$, hence;

$$\left| \frac{\nu_{t+1}(-\nabla \phi_{t+1}(w))}{\nu_t(-\nabla \phi_t(w))} \right| \leq \prod_{i=1}^d \frac{\lambda_{t,i} + \beta}{\lambda_{t,i}} \leq \prod_{i=1}^d (1 + \eta_t \beta) \leq e^{\eta_t \beta d}.$$

For the first term,

$$\frac{\nu_{t+1}(-\nabla \phi_{t+1}(w))}{\nu_t(-\nabla \phi_t(w))} = \exp \left( -\frac{\|\nabla \phi_{t+1}(w)\|_1}{\sigma_t} \right) = \exp \left( (\|\nabla \phi_t(w)\|_1 - \|\nabla \phi_{t+1}(w)\|_1)/\sigma_t \right) \leq \exp \left( (\|\nabla \phi_t(w) - \nabla \phi_{t+1}(w)\|_1)/\sigma_t \right) = \exp \left( (\|f_t(w)\|_1)/\sigma_t \right) \leq e^{\sqrt{d}G/\sigma_t}.$$

Proceeding, we let $\delta_t := \eta_t \beta d + \sqrt{d}G/\sigma$, and note the above implies, for all $w$;

$$\frac{Q_{t+1}(w)}{Q_t(w)} \leq e^{\delta_t}$$

$$\iff \frac{Q_t(w)}{Q_{t+1}(w)} \geq e^{-\delta_t} \geq 1 - \delta_t$$

$$\iff Q_t(w) \geq Q_{t+1}(w) - \delta_t Q_{t+1}(w)$$
Thus, for any set $E \subseteq \text{int}(W)$;

$$Q_{t+1}(E) - Q_{t}(E) \leq \delta_t Q_{t+1}(E) \leq \delta_t.$$

The other direction follows from identical symmetric arguments, and establishes

$$\max_{E \in \text{int}(W)} |Q_{t+1}(E) - Q_{t}(E)| \leq \delta_t = \eta \beta d + \sqrt{dG}/\sigma,$$

which completes the proof.

## E Technical lemmas

### Proof of Lemma 10.

We have

$$(\mu_p - \mu_q)^2 = \left( \sum_x (p(x) - q(x))x \right)^2$$

$$= \left( \sum_x (p(x) - q(x))(x - \mu_p) \right)^2$$

$$\leq \left( \sum_x |p(x) - q(x)|(x - \mu_p) \right)^2$$

$$= \left( \sum_x \sqrt{|p(x) - q(x)|} \sqrt{|p(x) - q(x)|(x - \mu_p)} \right)^2$$

$$\leq \left( \sum_x |p(x) - q(x)| \right) \left( \sum_x |p(x) - q(x)|(x - \mu_p)^2 \right)$$

$$= 2 \|p - q\|_{TV} \left( \sum_x |p(x) - q(x)|(x - \mu_p)^2 \right).$$

In addition;

$$\sum_x |p(x) - q(x)|(x - \mu_p)^2$$

$$= \sum_{x:p(x)>q(x)} (p(x) - q(x))(x - \mu_p)^2 + \sum_{x:q(x)>p(x)} (q(x) - p(x))(x - \mu_p)^2$$

$$\leq \sum_x p(x)(x - \mu_p)^2 + \sum_x q(x)(x - \mu_p)^2$$

$$= \sigma_p^2 + \sum_x q(x)(x - \mu_q)^2 + \sum_x q(x)(\mu_p - \mu_q)^2$$

$$= \sigma_p^2 + \sigma_q^2 + (\mu_p - \mu_q)^2.$$

To conclude we have shown that

$$\frac{(\mu_p - \mu_q)^2}{2 \|p - q\|_{TV}} \leq \sigma_p^2 + \sigma_q^2 + (\mu_p - \mu_q)^2.$$
Proof of Lemma 9. Let $x \in \Omega$, and observe
\[
\Pr(Y = x) \geq \Pr(Y = x \land X = x) = \Pr(X = x) \Pr(Y = x \mid X = x) = \Pr(X = x)(1 - \Pr(Y \neq x \mid X = x)).
\]
Rearranging the above, we obtain
\[
\Pr(X = x) \Pr(Y \neq x \mid X = x) \geq \Pr(X = x) - \Pr(Y = x),
\]
meaning $\Pr(X = x \land Y \neq x) \geq p(x) - q(x)$. Now
\[
\Pr(X \neq Y) = \sum_{x \in \Omega} \Pr(X = x \land Y \neq x)
\geq \sum_{x : p(x) > q(x)} \Pr(X = x \land Y \neq x)
\geq \sum_{x : p(x) > q(x)} p(x) - q(x)
= \|p - q\|_{TV}.
\]
□

Proof of Lemma 7 (based on Slivkins, 2019). By Pinsker’s Inequality and chain rule of the KL-divergence we have
\[
\|\text{Ber}_p^j - \text{Ber}_q^j\|_{TV}^2 \leq \frac{1}{2} D_{KL}(\text{Ber}_p^j \parallel \text{Ber}_q^j) = \frac{j}{2} D_{KL}(\text{Ber}_p \parallel \text{Ber}_q).
\]
In addition;
\[
D_{KL}(\text{Ber}_p \parallel \text{Ber}_q) = \frac{1 + \epsilon}{2} \log \frac{1 + \epsilon}{1 - \epsilon} + \frac{1 - \epsilon}{2} \log \frac{1 - \epsilon}{1 + \epsilon}
= \frac{1}{2} \left( \log \frac{1 + \epsilon}{1 - \epsilon} + \log \frac{1 - \epsilon}{1 + \epsilon} \right) + \frac{\epsilon}{2} \left( \log \frac{1 + \epsilon}{1 - \epsilon} - \log \frac{1 - \epsilon}{1 + \epsilon} \right)
= \frac{\epsilon}{2} \log \frac{(1 + \epsilon)^2}{(1 - \epsilon)^2}
= \frac{\epsilon}{2} \log \frac{(1 - \epsilon)^2 + 4\epsilon(1 - \epsilon) + 4\epsilon^2}{(1 - \epsilon)^2}
= \frac{\epsilon}{2} \log \left( 1 + \frac{4\epsilon}{(1 - \epsilon)^2} \right)
\leq \frac{\epsilon}{2} \log (1 + 8\epsilon)
\leq 4\epsilon^2.
\]
□

Lemma 19. Let $\phi : W \to \mathbb{R}$ be $\lambda$-strongly convex, $h : W \to \mathbb{R}$ differentiable, and set $x^* = \arg\min_{x \in W} \{ \phi(x) + h(x) \}$, $x' = \arg\min_{x \in W} \phi(x)$. It holds that
\[
\|x^* - x'\| \leq \frac{2 \|\nabla h(x')\|}{\lambda}.
\]

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Proof. Denote $\psi(x) = \phi(x) + h(x)$. We have
\[
\frac{\lambda}{2} \|x' - x^*\|^2 \leq \psi(x') - \psi(x^*) \leq \nabla \psi(x')^T (x' - x^*).
\]
In addition, $x'$ minimizes $\psi - h$ over $W$, therefore
\[
0 \leq (\nabla \psi(x') - \nabla h(x'))^T (x^* - x').
\]
\[
\iff \nabla \psi(x')^T (x' - x^*) \leq \nabla h(x')^T (x' - x^*) \leq \|\nabla h(x')\| \|x' - x^*\|.
\]
Combining the above inequalities we obtain
\[
\frac{\lambda}{2} \|x' - x^*\|^2 \leq \|\nabla h(x')\| \|x' - x^*\|,
\]
and the result follows after dividing both sides by $(\lambda/2) \|x' - x^*\|$. \qed