AV-differential geometry and Newtonian mechanics

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Abstract

A frame independent formulation of analytical mechanics in the Newtonian space-time is presented. The differential geometry of affine values (AV-differential geometry) i.e., the differential geometry in which affine bundles replace vector bundles and sections of one dimensional affine bundles replace functions on manifolds, is used. Lagrangian and hamiltonian generating objects, together with the Legendre transformation independent on inertial frame are constructed.

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1 Introduction

Mathematical formulation of analytical mechanics is usually based on objects that have vector character. So is the case of the most of mathematical physics. We use tangent vectors as infinitesimal configurations, cotangent vectors as momenta, we describe dynamics using forms (symplectic form) and multivectors (Poisson bracket) and finally we use an algebra of smooth functions. However, there are cases where we find difficulties while working with vector-like objects. For example, in the analytical mechanics of charged particles we have a problem of gauge dependence of lagrangians. In Newtonian mechanics there is a strong dependence on inertial frame, both in lagrangian and hamiltonian formulation. In the mechanics of non-autonomous system we are forced to choose a reference vector field on the space-time that fulfills certain conditions or we cannot write the dynamics at all. In all those cases the traditional language of differential geometry seems to introduce too much mathematical structure. In other words, there is too much structure with comparison to what is really needed to define and describe the behavior of the system. As a consequence we have to put in an additional information to the system such as gauge or reference frame.

As it is well known, Newton’s equations do not depend on the inertial frame chosen. Therefore, a geometric formulation of Newtonian mechanics in Newtonian space-time is possible. On the other hand, analytical mechanics in Newtonian space-time is not possible in a standard framework. Different lagrangians and different hamiltonians are used for different inertial frames. The same is true for Hamilton-Jacobi theory and Schrödinger wave mechanics. Frame independence of the Lagrangian formulation of Newtonian dynamics can be achieved by increasing the dimension of the configuration space of the particle. The four dimensional space-time of general relativity is replaced by the five dimensional manifold (as in the Kaluza theory) [9], [10], [3]. An alternate approach is proposed in the present note. The four dimensional space-time is used as the configuration space. The phase space is no longer a cotangent bundle and not even a vector bundle. It is an affine bundle modelled on the cotangent bundle of the space-time manifold. The Lagrangian is a section of an affine line

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bundle over the tangent bundle of the space-time manifold. The proper geometric tools are provided by the geometry of affine values (AV-differential geometry). We call the geometry of affine values the differential geometry that is built using sections of one-dimensional affine bundle over the manifold instead of functions on the manifold. The affine bundle we use is equipped with the fiber action of the group \((\mathbb{R},+)\), so we can add reals to elements of fibers and real functions to sections, but there is no distinguished "zero section". Those elements of the geometry of affine values that are needed in the Newtonian mechanics are described in section 4.1, an extended presentation of the theory can be found in \([1,2]\). In fact the geometry of affine values appeared much earlier in works of W.M. Tulczyjew (see e.g. \([14]\)) and has been applied to the description of the dynamics of charged particles in \([12]\). Affine phase bundle for analytical mechanics in the Newtonian space-time was discussed in papers of Tulczyjew \([9]\) and Pidello \([7]\). Some problems concerning analytical mechanics on affine bundles are discussed in \([6]\) and \([8]\).

2 Newtonian space-time

The Newtonian space-time (some authors prefer to call it Galilean space-time, but we follow the notation of Benenti and Tulczyjew) is a system \((N,\tau,g)\) where \(N\) is a four-dimensional affine space with the model vector space \(V\), \(\tau\) is a non-zero element of \(V^*\) and \(g: E_0 \to E_0^*\) represents an Euclidean metric on \(E_0 = \ker \tau\). The elements of the space \(N\) represent events. The time elapsed between two events is measured by \(\tau\):

\[
\Delta t(x, x') = \langle \tau, x - x' \rangle.
\]

The distance between two simultaneous events is measured by \(g\):

\[
d(x, x') = \sqrt{g(x - x', x - x')}.
\]

The space-time \(N\) is fibred over the time \(T = N/E_0\) which is one-dimensional affine space modelled on \(\mathbb{R}\). By \(\eta\) we will denote the canonical projection

\[
\eta: N \to T,
\]

by \(\iota\) the canonical embedding

\[
\iota: E_0 \to V,
\]

and by \(\iota^*\) the dual projection

\[
\iota^*: V^* \to E_0^*.
\]

By means of \(\iota\) and \(\iota^*\) we can define a contravariant tensor \(g'\) on \(V^*\):

\[
g' = \iota \circ g^{-1} \circ \iota^*.
\]

The kernel of \(g'\) is a one-dimensional subspace of \(V^*\) spanned by \(\tau\).

Let \(E_1\) be an affine subspace of \(V\) defined by the equation \(\langle \tau, v \rangle = 1\). The model vector space for this subspace is \(E_0\). An element of \(E_1\) can represent velocity of a particle. The affine structure of \(N\) allows us to associate to an element \(u\) of \(E_1\) the family of inertial observers that move in the space-time with the constant velocity \(u\). This way we can interpret an element of \(E_1\) as an inertial reference frame.

For a fixed inertial frame \(u\), we define the space \(Q\) of world lines of all inertial observers. It is the quotient affine space \(N/\{u\}\). The space-time \(N\) becomes the product of affine spaces

\[
N = Q \times T.
\]

The model vector space for \(Q\) is the quotient vector space \(V/\{u\}\) that can be identified with \(E_0\). The corresponding canonical projection is

\[
\iota_u: V \to E_0: v \mapsto \iota_u(v) = v - \langle \tau, v \rangle u
\]

and the splitting \(V = E_0 \times \mathbb{R}\) is given by

\[
V \ni v \mapsto (\iota_u(v), \langle \tau, v \rangle) \in E_0 \times \mathbb{R}.
\]
The dual splitting is given by
\[ V^* \ni p \mapsto (i^*(p), \langle p, u \rangle) \in E_0^* \times \mathbb{R}. \]

The tangent bundle \( TN \) we identify with the product \( N \times V \) and the subbundle \( VN \) of vectors vertical with respect to the projection on time, with \( N \times E_0 \). Consequently, the bundle \( V^1N \) of infinitesimal configurations (positions and velocities) of particles moving in the space-time \( N \) is identified with \( N \times E_1 \). When the inertial frame \( u \) is chosen, \( E_1 \) is identified with \( E_0 \) and \( V^1N \) is identified with \( VN \).

The vector dual \( V^*N \) for \( VN \) is a quotient bundle of \( N \times V^* \) by the one-dimensional subbundle \( N \times \langle \tau \rangle \). We can identify it with \( N \times E^*_0 \). Using the inertial frame we can make it a subbundle of \( T^*N \).

3 Analytical mechanics in a fixed inertial frame

In the following section we will present the analytical mechanics of one particle in the Newtonian space-time in the fixed inertial frame \( u \). Before we start working on physics we shall recall basic constructions and facts about generating objects for lagrangian submanifolds in \( T^*M \). (In the homogeneous formulation of the dynamics we will be using more general generating objects than just a function on the manifold \( M \).) The details can be found in [11] and [5]. Later we concentrate on the inhomogeneous formulation of the analytical mechanics of one particle, suitable for trajectories parameterized by the time, then we pass to the homogeneous one. The homogeneous formulation accepts all parameterizations.

3.1 Generating objects for lagrangian submanifolds in the cotangent bundle

In many cases the dynamics of a mechanical system is obtained as an inverse image by a symplectomorphism of a certain lagrangian submanifold. It can be lagrangian submanifold of the cotangent bundle to the phase space generated by a hamiltonian or the lagrangian submanifold of the cotangent bundle to the space of infinitesimal configurations generated by a lagrangian. In some cases however one needs more general generating object than just a function on a manifold.

The most general generating object for a lagrangian submanifold of a cotangent bundle is a family of functions i.e. a function on the total space of a fibration over the manifold. In the following we will recall some definitions and constructions that will be used later in the context of hamiltonian and lagrangian dynamics.

Let \( \rho : N \to M \) be a differential fibration and \( F : N \to \mathbb{R} \) be a smooth function. The pair \( (F, \rho) \) will be called a family of functions in a sense that it is a family of functions on the fibers of \( \rho \) parameterized by points of \( M \). We will need the following definition: The set
\[ S(F, \rho) = \{ n \in N : \forall v \in V_nN \langle dF(n), v \rangle = 0 \} \]
is called the critical set of the family \( (F, \rho) \).

The process of generating a lagrangian submanifold can be described in two equivalent ways:

1. The function \( F \) generates the lagrangian submanifold \( dF(N) \subset T^*N \). Then we use the canonical projection \( \hat{\rho} : V^0 N \to T^*M \), where \( V^0N \) is the anihilator of the vertical bundle \( VN \), to obtain a subset \( L \) of \( TM \):
\[ L = \hat{\rho}(V^0N \cap dF(N)). \]

In another words we apply the symplectic reduction with respect to a coisotropic submanifold \( V^0N \) to \( dF(N) \). We have a theorem:

**Theorem 1** ([5]) If \( V^0N \) and \( dF(N) \) have clear intersection than the set \( L \) is an immersed lagrangian submanifold of \( T^*M \).
The proof and other details of the construction can be found in [5].

2. The generated set $L$ is the image of the critical set $S(F, \rho)$ by the mapping

$$\kappa : S(F, \rho) \rightarrow T^* M$$

defined as follows

$$\forall v \in T_m M \quad \forall w \in T_n N : T\rho(w) = v \quad \langle \kappa(n), v \rangle = \langle dF(n), w \rangle.$$ 

The conditions for $F$ to be a generating object of a lagrangian submanifold are formulated in a language of Hessian which may be more familiar for people working in analytical mechanics. In the following we will concentrate on the second approach.

Let $n$ be a critical point of the family $(F, \rho)$, let $O$ be a neighbourhood of $(0, 0)$ in $\mathbb{R}^2$ and $\chi : O \rightarrow N$ be a smooth mapping such that $\chi(0, 0) = n$ and $\rho(\chi(s, t)) = \rho(\chi(0, t))$ for all $s$. It means that the parameter $s$ changes along the fibre of $\rho$. It is easy to show that

$$\frac{\partial^2}{\partial t \partial s}|_{t=0, s=0} F \circ \chi$$

depends only on the vectors tangent to the curves $s \mapsto \chi(s, 0)$, $t \mapsto \chi(0, t)$ in points $s = 0$, $t = 0$ respectively. Since it is always possible to construct $\chi$ for a given pair of vectors we can define the mapping

$$H(F, \rho, n) : \mathcal{V}_n N \times T_n N \rightarrow \mathbb{R}, \quad H(F, \rho, n)(v, w) = \frac{\partial^2}{\partial t \partial s}|_{t=0, s=0} F \circ \chi.$$ 

One can show that $H(F, \rho, n)$ is linear in both arguments. Moreover if $v$ and $w$ are both vertical it is symmetric. The mapping $H(F, \rho, n)$ is called a Hessian of the family $(F, \rho)$ in the point $n$.

We have the following definition: A family $(F, \rho)$ is called a Morse family if the rank of the Hessian is maximal in every point of the critical set.

**Theorem 2** The set generated by a Morse family is an immersed lagrangian submanifold of $T^* M$.

Let us look at the mechanical example:

**Example:** Let $Q$ be a space of configurations of an autonomous mechanical system with first-order lagrangian $L : TQ \rightarrow \mathbb{R}$. Let $M = T^* Q$ denote the phase space. We take $N$ to be $T^* Q \times_Q TQ$ and $\rho : N \ni (p, v) \mapsto p \in M$. The critical set of the family

$$F(p, v) = L(v) - \langle p, v \rangle$$

is given by

$$S(F, \rho) = \{(p, v) : d_V L(v) = p\}.$$ 

By $d_V$ we denote the vertical differential, i.e. the differential along the fibre of $TQ \rightarrow Q$. In local coordinates the matrix of Hessian assumes the form

$$H(F, \rho, n) = \begin{bmatrix}
\frac{\partial^2 L}{\partial v^i \partial v^j} \\
\frac{\partial^2 L}{\partial v^i \partial d_v} \\
\frac{\partial^2 L}{\partial d_v^i} \\
\delta_{ij}
\end{bmatrix}$$

and is clearly of the maximal rank equal $\text{dim } Q$. Therefore $(F, \rho)$ is a Morse family. It generates a lagrangian submanifold of $T^* T^* Q$. The inverse image of the submanifold by the symplectomorphism $\beta_Q$ associated to the symplectic form on $T^* Q$ is the dynamics of the system.

In some cases a generating object can be simplified. Suppose that $\rho$ is a composition of two projections $\rho = \rho_2 \circ \rho_1$:

$$N \xrightarrow{\rho_1} P \xrightarrow{\rho_2} M$$
and the critical set \( S(F, \rho_1) \) is contained in an image of some section \( \sigma : P \to N \). In another words \( \rho_1 \) establishes one-to-one correspondence between \( S(F, \rho_1) \) and \( \rho_1(S(F, \rho_1)) \). In this situation the family \((F, \rho)\) can be simplified. The same set \( L \) is generated by the family \( F_1 : P \to \mathbb{R}, F_1(p) = F(\sigma(p)) \).

**Example.** Let us consider the family from the previous example: \( F(v, p) = L(v) - \langle p, v \rangle \). The critical set \( S(f, \rho) \) is given by the equation

\[
\frac{\partial L}{\partial v^i} = p_i.
\]

The condition for \( S(F, \rho) \) to be locally an image of the section is given by the implicit function theorem. In case of regular lagrangian, i.e. if the matrix \( \frac{\partial^2 L}{\partial v^i \partial v^j} \) is invertible we can locally have one-to-one correspondence between velocities and momenta. We can therefore locally simplify the family \( F \) obtaining hamiltonian function. We call a lagrangian hyperregular if there is a global diffeomorphism between \( TM \) and \( T^*M \) given by the equation

\[
\frac{\partial L}{\partial v^i} = p_i.
\]

In this case family \((F, \rho)\) can be globally reduced to the hamiltonian function.

### 3.2 Inhomogeneous dynamics described in a fixed inertial frame

Let \( u \in E_1 \) represent an inertial frame. For a fixed time \( t \in T \), the phase space for a particle with mass \( m \) with respect to the inertial frame \( u \) is \( T^*N_t \simeq N_t \times E^*_0 \), where \( N_t = \eta^{-1}(t) \). The collection of phase spaces form a phase bundle \( V^*N \simeq N \times E^*_0 \). Phase space trajectories of the system are solutions of the well-known equations of motion:

\[
\dot{p} = -d_s \varphi(x) \quad \dot{x} = g^{-1}\left(\frac{p}{m}\right) + u,
\]

where \((x, p, \dot{x}, \dot{p}) \in \mathcal{V}^1 V^*N \subset \mathcal{V}^*N \simeq N \times E^*_0 \times V \times E^*_0 \) and \( \varphi : N \to \mathbb{R} \) is a potential. Subscript \( s \) in \( d_s \) means that we differentiate only in spatial directions i.e. the directions vertical with respect to the projection on time, therefore \( d_s \varphi(x) \in E^*_0 \). The equations define a vector field on \( V^*N \) with values in \( \mathcal{V}^1V^*N \), i.e. a section of the bundle \( \mathcal{V}^1V^*N \to V^*N \). The image of the vector field \( \ell^{1,u} \) we will call the *inhomogeneous dynamics* and denote by \( D_{i,u} \). It can be generated directly by the lagrangian

\[
\ell_{i,u} : \mathcal{V}^1N \to \mathbb{R} : (x, w) \mapsto \frac{m}{2} \langle g(w - u), w - u \rangle - \varphi(x).
\]

With this lagrangian we associate the Legendre mapping

\[
\mathcal{L}_{i,u} : \mathcal{V}^1N \to V^*N : (x, w) \mapsto g \circ i_u(v),
\]

i.e. the vertical derivative of \( \ell_{i,u} \) with respect to the projection \( \nabla N \to N \).

The procedure of generating the inhomogeneous dynamics from the lagrangian \( \ell_{i,u} \) is as follows. The image of the vertical derivative \( d_s \ell_{i,u} \) is a submanifold of

\[
\mathcal{V}^*\mathcal{V}^1N \simeq N \times E_1 \times E^*_0 \times E^*_0.
\]

which is canonically isomorphic (as an affine space) to

\[
\mathcal{V}^1\mathcal{V}^*N \simeq N \times E^*_0 \times E_1 \times E^*_0.
\]

This isomorphism we obtain by a reduction of the canonical diffeomorphism

\[
\alpha_M : \mathcal{T}T^*M \to T^*T\mathcal{M}
\]

valid for any differential manifold \( M \) (for the definition of \( \alpha_M \) see [1]), which for the affine space \( N \) assumes the form

\[
\alpha_N(x, a, v, b) = (x, v, b, a).
\]
After the reduction, we get
\[ \alpha_N^i: N \times E_0^\ast \times E_1 \times E_k^0 \to N \times E_1 \times E_0^\ast \times E_k^0. \]

Using \( \alpha_N^i \) we can obtain \( D_{i,u} \) from \( d_s\ell_{i,u} \) by taking an inverse-image:
\[ D_{i,u} = (\alpha_N^i)^{-1}(d_s\ell_{i,u}(V^1 N)). \]

The dynamics \( D_{i,u} \) cannot be generated directly from a hamiltonian by means of the canonical Poisson structure on \( V^*N \), which is the reduced canonical Poisson (symplectic) structure of \( T^*N \). In the coordinates adapted to the structure of the bundle \((t,x^i,p_i)\) the Poisson bi-vector is given by
\[ \Lambda = \partial p_i \wedge \partial x^i. \]

Symplectic leaves for this Poisson structure are cotangent bundles \( T^*N_t \), where \( N_t = \eta^{-1}(t) \). It follows that every hamiltonian vector field is vertical with respect to the projection on time. However, using reference frame \( u \), we can generate first the vertical part of the dynamics (4), i.e. the equations
\[ \dot{p} = -d_s\varphi(x) \quad \dot{x} = g^{-1}(\frac{p}{m}), \]

and add the reference vector field \( u \). The hamiltonian function for the problem reads
\[ h_{i,u}(x,p) = \frac{1}{2m}(p,g^{-1}(p)) + \varphi(x), \]
where \((x,p) \in V^*N\).

The system (4) can be generated also from lagrangian function defined on \( VN \) by the formula:
\[ \ell_{i,u}(x,w) = \frac{m}{2}\langle g(w),w \rangle - \varphi(x). \]

We identify the fiber over \( t \) of \( VN \) with \( TN_t \) and use the standard procedure to generate a submanifold \( D_{u,t} \) in \( TT^*N_t \). The collection of these submanifolds give us the system (4).

The dynamics (4) and the generating procedures depend strongly on the choice of the reference frame. In particular, the relation velocity-momenta is frame-dependent which means that we have to redefine the phase manifold for the particle to obtain frame-independent dynamics. Also the hamiltonian formulation will be possible if we replace the canonical Poisson tensor by a more adapted object.

### 3.3 Homogeneous dynamics described in a fixed inertial frame

In the homogeneous formulation of the dynamics infinitesimal configurations are pairs \((x,v) \in N \times V^+ \subset N \times V \simeq TN \) where \( V^+ \) is an open set of vectors such that \( \langle \tau,v \rangle > 0 \).

The homogeneous lagrangian is an extension by homogeneity of the \( \ell_{i,u} \) from (3.2) and is given by the formula:
\[ \ell_{h,u}(x,v) = \frac{m}{2\langle \tau,v \rangle} (g(\iota_u(v)), \iota_u(v)) - \langle \tau,v \rangle \varphi(x). \]

This choice guaranties that the action calculated for a piece of the world line, which is one dimensional oriented submanifold of the space-time, does not depend on its parametrization. However, we still have to use the fixed inertial frame \( u \).

The image of the differential of \( \ell_{h,u} \) is a lagrangian submanifold of \( T^*TN \simeq N \times V \times V^* \times V^* \). An element \((x,v,a_x,a_v)\) is in the image of \( d\ell_{h,u} \) if it satisfies the following equations
\[ \begin{aligned}
  v &\in V^+, \\
  a_x &= -\langle \tau,v \rangle d\varphi(x), \\
  a_v &= \frac{\partial}{\partial v} (g \circ \iota_u)(v) - \frac{m}{2\langle \tau,v \rangle} (g(\iota_u(v)), \iota_u(v)) \tau - \varphi(x)\tau.
\end{aligned} \]
The image of \( d\ell_{h,u}(N \times V^+) \) by the mapping \( \alpha_{N}^{-1} \) is a lagrangian submanifold of \( T^*N \). This submanifold we will call the homogeneous dynamics and denote by \( D_{h,u} \). An element \((x,p,\dot{x},\dot{p})\) of \( T^*N \simeq N \times V^* \times V \times V^* \) is in \( D_{h,u} \) if

\[
\begin{cases}
    \dot{x} = v, \\
    \dot{p} = -(\tau, v) d\varphi(x), \\
    p = \frac{m}{(\tau, v)} i_u^*g \circ \iota_u(x, \tau, \iota_u(v)) - \frac{m}{2(\tau, v)} (g(\iota_u(v)), \iota_u(v)) \tau - \varphi(x) \tau
\end{cases}
\]  

(8)

for some \( v \in V^+ \), i.e. \( \langle \tau, v \rangle > 0 \). We observe that \( D_{h,u} \) does not project on the whole \( T^*N \), but \((x,p)\) must satisfy the following equation:

\[
\frac{1}{2m} \langle p, g'(p) \rangle + \langle p, u \rangle + \varphi(x) = 0. 
\]  

(9)

The equation (9) is the analog of the mass-shell equation \( p^2 = m^2 \) in the relativistic mechanics. Since there is the difference in signature of \( g' \) between the Newtonian and the relativistic case, we obtain here a paraboloid of constant mass instead of relativistic hyperboloid. The mass-shell will be denoted by \( K_{m,u} \).

It is possible to generate the dynamics \( D_{h,u} \) directly by a generalized hamiltonian system. The hamiltonian generating object (see [13]) is the family

\[
N \times V^* \times V^+ \xrightarrow{-H_{h,u}} \mathbb{R}, \quad \zeta\downarrow \quad N \times V^* \quad (10)
\]

where

\[
H_{h,u}(x, p, v) = \langle p, v \rangle - \ell_{h,u}(x, v) \in \mathbb{R}. 
\]

(11)

This family can be simplified. The fibration \( \zeta \) can be represented as a composition \( \zeta'' \circ \zeta' \), where

\[
\zeta': N \times V^* \times V^+ \to N \times V^* \times \mathbb{R}^+: (x, p, v) \mapsto (x, p, \langle \tau, v \rangle), 
\]

and

\[
\zeta'': N \times V^* \times \mathbb{R}^+ \to N \times V^* : (x, p, r) \mapsto (x, p). 
\]

Equating to zero the derivative of \( H_{h,u} \) along the fibres of \( \zeta' \) we obtain the relation

\[
v = \frac{\langle \tau, v \rangle}{m} g^{-1} \circ i^*(p) + \langle \tau, v \rangle u. 
\]  

(12)

It follows that the family (10) is equivalent (generates the same object) to the reduced family

\[
N \times V^* \times \mathbb{R}^+ \xrightarrow{-\bar{H}_{h,u}} \mathbb{R}, \quad \zeta''\downarrow \quad N \times V^* \quad (13)
\]

where

\[
\bar{H}_{h,u}(x, p, r) = r \left( \frac{1}{2m} \langle p, g'(p) \rangle + \langle p, u \rangle + \varphi(x) \right). 
\]  

(14)

No further simplification is possible.

The critical set \( S(\bar{H}_{h,u}, \zeta'') \) is the submanifold

\[
\left\{ (x, p, r) \in N \times V^* \times \mathbb{R}^+ : \frac{1}{2m} \langle p, g'(p) \rangle + \langle p, u \rangle + \varphi(x) = 0 \right\}
\]

and its image \( \zeta''(S(\bar{H}_{h,u}, \zeta'')) \) is the mass shell \( K_{m,u} \).
The function $H_{h,u}$ is zero on $S(\bar{H}_{h,u}, c''')$ and projects to the zero function on $K_{m,u}$. However, a Dirac system with the zero function on the constraints $K_{m,u}$ does not generate $D_{h,u}$. The lagrangian submanifold $\tilde{D}_{h,u} \subset T^*N$ generated by this system is exactly the characteristic distribution of $K_{m,u}$, i.e.

$$\tilde{D}_{h,u} = (TK_{m,u})$$

and does not respect the condition $\langle \tau, v \rangle > 0$. We have only $D_{h,u} \subset \tilde{D}_{h,u}$.

4 The dynamics independent on inertial frame

4.1 Special vector and affine spaces

A vector space $W$ with distinguished non-zero element $v$ we will call a special vector space. A canonical example of a special vector space is $(\mathbb{R}, 1)$. It will be denoted by $I$. If $A$ is an affine space then $\text{Aff}(A, \mathbb{R})$ – the vector space of all affine functions with real values on $A$ – is a special vector space with distinguished element $1_A$ being a constant function on $A$ equal to 1. The space $\text{Aff}(A, \mathbb{R})$ will be denoted by $A^I$ and called a vector dual for $A$. Having a special vector space $(W, v)$ we can define its affine dual by choosing a subspace in $W^*$ of those linear functions that take the value 1 on $v$:

$$W^\sharp = \{ \varphi \in W^* : \varphi(v) = 1 \}.$$ 

We have that

**Theorem 3 ([2])** For $(W, v)$ and $A$ such that $\dim W < \infty$ and $\dim A < \infty$

\[
\begin{align*}
(W^\sharp)^I, 1_{W^\sharp} &= V, \\
(A^I)^I &= A.
\end{align*}
\]

An affine space modelled on a special vector space will be called a special affine space. Similar definitions we can introduce for bundles: a special vector bundle is a vector bundle with distinguished non-vanishing section and a special affine bundle is an affine bundle modelled on a special vector bundle.

4.2 The geometry of affine values

The geometry of affine values is, roughly speaking, the differential geometry built on the set of sections of one-dimensional special affine bundle $\zeta : Z \rightarrow M$ modelled on $M \times I$, instead of just functions on $M$. The bundle $Z$ will be called a bundle of affine values. Since $Z$ is modelled on $M \times I$ we can add reals in each fiber of $Z$, i.e. $Z$ is an $\mathbb{R}$-principal bundle. The vertical vector field on $Z$ which is the fundamental vector field for the action of $\mathbb{R}$ will be denoted by $X_Z$. Let us now consider an example of a bundle of affine values: If $(A, v)$ is a special affine space modelled on $(W, v)$, then we have the quotient affine space $A = A/\langle v \rangle$. The affine spaces $A$ and $\bar{A}$ together with the canonical projection form an example of a bundle $A$ of affine values. The appropriate action of $\mathbb{R}$ in the fibers is given by

$$A \times \mathbb{R} \ni (a, r) \mapsto (a - rv) \in A$$

and the fundamental vector field $X_A$ is a constant vector field equal to $v$ on $A$.

The affine analog of the cotangent bundle $T^*M$ in the geometry of affine values is called a phase bundle and denoted by $PZ$. We define an equivalence relation in the set of pairs of $(m, \sigma)$, where $m \in M$ and $\sigma$ is a section of $Z$. We say that $(m, \sigma)$, $(m', \sigma')$ are equivalent if $m = m'$ and $d(\sigma - \sigma')(m) = 0$, where we have identified the difference of sections of $Z$ with a function on $M$. The equivalence class of $(m, \sigma)$ is denoted by $\sigma(m)$. The set of equivalence classes is denoted by $PZ$ and called the phase bundle for $Z$. It is, of course, the bundle over $M$ with the projection $d\sigma(m) : m$. It is obvious that

$$P\zeta : PZ \rightarrow M : d\sigma(m) \mapsto m$$

is an affine bundle modelled on the cotangent bundle $T^*M \rightarrow M$.
The structure of $\mathbb{P}Z$ is similar to the structure of the cotangent bundle $T^*M$. In particular on $\mathbb{P}Z$ there is a canonical symplectic form defined as follows. Any section $\sigma$ of the bundle $Z \rightarrow M$ gives the trivialization

$$I_{\sigma} : Z \rightarrow M \times \mathbb{R}, \quad I_{\sigma}(z) = (\zeta(z), z - \sigma(\zeta(z)))$$

and further

$$I_{d\sigma} : \mathbb{P}Z \rightarrow T^*M, \quad I_{d\sigma}(\alpha) = \alpha - d\sigma(P\zeta(\alpha)).$$

For two sections $\sigma, \sigma'$ the mappings $I_{d\sigma} \cdot I_{d\sigma'}$ differ by the translation by the affine form $d(\sigma - \sigma')$, i.e.

$$I_{d\sigma'} \circ I_{d\sigma}^{-1} : T^*M \rightarrow T^*M, \quad \phi \mapsto \phi + d(\sigma - \sigma')(\phi).$$

Using the well-known property of the canonical symplectic form $\omega_M$ on $T^*M$ that translation by closed forms are symplectomorphisms we conclude that $I_{d\sigma}^*\omega_M$ does not depend on the choice of $\sigma$ and therefore it is a canonical symplectic form on $\mathbb{P}Z$. It will be denoted by $\omega_Z$. More information about the canonical symplectic form $\omega_Z$ and about the structure of $\mathbb{P}Z$ can be found in \cite{2} and \cite{15}.

As an example we construct a phase bundle for the bundle of affine values built out of a special affine space $(A, v)$. In the set of all sections of the bundle $A$ there is a distinguished set of affine sections, since $A$ and $A$ are affine spaces. We observe that there are affine representatives in every equivalence class $d\sigma(m)$ that differ by a constant function. Every choice of a reference point in $A$ defines a one-to-one correspondence between affine sections of $A$ and affine sections of the model bundle $W$. This correspondence projects to the bijection between corresponding phase spaces, which does not depend on the choice of the reference point. There is also one linear representative for each element in $P^*W$, i.e. such an affine section that takes value 0 at the point $0 \in W$. The set of elements of a phase bundle can be therefore identified with a set of pairs: point in $m$ and a linear injection from $W^*$ to $W$. Moreover, we observe that such linear injections are in one-to-one correspondence with linear functions on $W$ that they take value 1 on $v$ (or the canonical vector field $X_W$ evaluated on the function gives 1). The image of a linear section is a level-0 set for the corresponding function. The functions that correspond to linear sections form the affine dual $W^\dagger$, therefore we have

$$P^*W \simeq W^* \times W^\dagger, \quad P^*A \simeq A \times W^\dagger.$$  \hfill (15)

Let us now see what is a canonical symplectic form on $P^*A$. For it we take $\sigma$ – an affine section of $A$ and $F_\sigma$ – a corresponding affine function on $A$ such that it’s linear part $dF_\sigma$ takes value 1 on $v$. The image of $\sigma$ is a level-zero set for $F_\sigma$. We see that

$$I_{\sigma} : A \rightarrow A \times \mathbb{R}, \quad a \mapsto (a, a - \sigma(a)),$$  

and further

$$I_{d\sigma} : A \times W^\dagger \rightarrow T^*A \simeq A \times W_*^*, \quad (a, f) \mapsto (a, f - dF_\sigma).$$

Identifying $TT^*A$ with $A \times W_*^* \times W \times W^*$ we get

$$\omega_A((a, \varphi, x, \psi), (a, \varphi, x', \psi')) = \psi'(x) - \psi(x')$$

and therefore the same expression for $\omega_A$ reads

$$\omega_A((a, f, x, \psi), (a, f, x', \psi')) = \psi'(x) - \psi(x'),$$

where $T^*P^*A$ is identified with $A \times W^\dagger \times W \times W^*$.  

### 4.3 Frame independent lagrangian

Now, we will collect all the homogeneous lagrangians for all inertial frames and construct for them a universal object which does not depend on an inertial frame. It is convenient to treat a lagrangian as a section of the trivial bundle $N \times V \times \mathbb{R} \rightarrow N \times V$ rather than as a function.

For two reference frames $\alpha$ and $\alpha'$, we have the difference

$$\ell_{h, \alpha}(x, u) - \ell_{h, \alpha'}(x, v) = m \langle g(u' - u), \frac{d}{dt} \psi_{\alpha'}(v) \rangle.$$
Let us denote $\ell_{h,u} g(u' - u)$ by $\sigma(u', u)$. With this notation

$$\ell_{h,u}(x, v) - \ell_{h,u'}(x, v) = m(\sigma(u', u), v). \quad (16)$$

For $\sigma$ we have the following equalities

$$\sigma(u', u) = -\sigma(u, u'), \quad (17)$$

$$\sigma(u'', u') + \sigma(u', u) = \sigma(u'', u). \quad (18)$$

In the $E_1 \times N \times V \times \mathbb{R}$, we introduce the following relation:

$$(u, x, v, r) \sim (u', x', v', r') \iff \begin{cases} x = x', \\ v = v', \\ r = r' + m(\sigma(u', u), v). \end{cases} \quad (19)$$

From (16) we obtain that $\sim$ is symmetric and reflexive, from (17) that it is transitive, therefore it is an equivalence relation. Since the relation does not affect $N$ at all, it is obvious that in the set of equivalence classes we have a cartesian product structure $N \times W$. In $W$ we distinguish two elements: $w_0 = [u, 0, 0]$ and $w_1 = [u, 0, -1]$,

$$w_0 = \{(u, 0, 0) : u \in E_1\}, \quad w_1 = \{(u, 0, -1) : u \in E_1\},$$

and two natural operations:

$$+: W \times W \to W \quad \circ : \mathbb{R} \times W \to W$$

$$[u, v, r] + [u', v', r'] = [u + \frac{u' + v'}{2}, v + v', r + r' + m(\sigma(u, \frac{u' + v'}{2}), v) + \langle \sigma(u', \frac{u' + v'}{2}), v' \rangle], \quad \lambda \circ [u, v, r] = [u, \lambda v, \lambda r]. \quad (20)$$

The above operations are well defined that can be checked by direct calculation. Some more calculation one needs to show that

**Proposition 1** $(W, +, \circ)$ is a vector space with $w_0$ as the zero-vector. Moreover $(W, w_1)$ is a special vector space such that $W/ <w_1> \simeq V$

The canonical projection $W \to V$ will be denoted by $\zeta$. It follows from (16) that quadruples $(u, x, v, \ell_{h,u}(x, v))$ and $(u', x, v, \ell_{h,u'}(x, v))$ are equivalent. Consequently, frame dependent lagrangian defines a section $\ell_h$ over $N \times V^+$ of the one-dimensional special affine bundle (a bundle of affine values) $N \times W \to N \times V$ which does not depend on the inertial frame. The section $\ell_h$ will be called an affine lagrangian for the homogeneous lagrangian independent on the choice of inertial frame. In the following we show that the bundle $N \times W \to N \times V$ carries a structure, which can be used for generating the frame-independent dynamics. We begin with the construction of the phase space.

### 4.4 Phase space

In the frame dependent formulation of the dynamics, the phase space for the massive particle is $T^* N \simeq N \times V^*$. For each frame $u$ we have the Legendre mapping

$$\mathcal{L}_u : TN \supset N \times V^+ \to T^* N$$

$$\mathcal{L}_u : (x, v) \mapsto \frac{m}{2(\tau, v)} \tau, \quad (21)$$

i.e. the vertical derivative of $\ell_{h,u}$ with respect to the projection $TN \to N$.

Since $\ell_{h,u}(x, v) - \ell_{h,u'}(x, v) = m(\sigma(u', u), v)$, we have also

$$\mathcal{L}_u(v) - \mathcal{L}_{u'}(v) = m(\sigma(u', u), v). \quad (22)$$
Proposition 2 A mapping $\Phi_{u',u} : T^*N \to T^*N$ defined by

$$\Phi_{u',u}(x,p) = (x, p + m\sigma(u', u))$$

has the following properties

1. $\Phi_{u',u}(K_{m,u'}) = K_{m,u}$,
2. it is a symplectomorphism of the canonical symplectic structure on $T^*N$,
3. $T\Phi_{u',u}(D_{h,u'}) = D_{h,u}$.

Proof. The image of $L_u$ is $K_{m,u}$, so the first property is an immediate consequence of (22) and the definition of $\Phi_{u',u}$. The mapping $\Phi_{u',u}$ is a translation by a constant vector. It follows that it is a symplectomorphism. Consequently,

$$T\Phi_{u',u}((TK_{m,u'})^\text{§}) = (TK_{m,u})^\text{§}$$

and

$$T\Phi_{u',u}(D_{h,u'}) = D_{h,u}.$$  

Since $\Phi_{u',u}$ respects the time orientation, we have also

$$T\Phi_{u',u}(D_{h,u'}) = D_{h,u}.$$  

The above observation suggests the following equivalence relation in $E_1 \times N \times V^*$:

$$(u, x, p) \sim (u', x', p') \iff \begin{cases} x = x', \\ p = p' + m\sigma(u', u). \end{cases} \quad (23)$$

Again, we have the obvious structure of the cartesian product in the set of equivalence classes: $N \times P$. The set $N \times P$ will be called an affine phase space. The set $P$ is an affine space modelled on $V^*$:

$$[u, p] + \pi = [u, p + \pi]$$

for $\pi \in V^*$.

An element of $P$ will be denoted by $p$.

It follows from Proposition 4.4 that $N \times P$ is a symplectic manifold and the isomorphism of tangent and cotangent bundles assumes the form

$$\beta : T(N \times P) \simeq N \times P \times V \times V^* \to T^*(N \times P) \simeq N \times P \times V^* \times V$$

$$(x, p, v, a) \mapsto (x, p, a, -v) \quad (24)$$

Moreover, the equivalence classes of the elements of mass-shells form the universal mass shell $K_m$ and the elements of frame dependent dynamics form the universal dynamics $D_h$ which is contained in $(TK_m)^\text{§}$.

A straightforward calculation shows that the function

$$E_1 \times N \times V^* \ni (u, x, p) \mapsto \frac{1}{2m} \langle p, g'(p) \rangle + \langle p, u \rangle$$

is constant on equivalence classes and projects to a function on $N \times P$. We denote this function by $\Psi_m$. It follows that the generating object (13) of the dynamics $D_{h,u}$ defines a generating object

$$N \times P \times \mathbb{R}_+ \xrightarrow{-\bar{H}_h} \mathbb{R},$$

$$\zeta'' \downarrow$$

$$N \times V^*$$

of the dynamics $D_h$, where

$$\bar{H}_h(x, p, r) = r(\Psi_m + \varphi(x)). \quad (26)$$

11
4.5 Lagrangian as a generating object

In the previous section we have constructed the frame independent dynamics $D_h$ and a hamiltonian generating object. Now, we show that the frame independent affine lagrangian $\ell_h$ is also a generating object of $D_h$. $\ell_h$ is a section of a bundle of affine values $N \times W \rightarrow N \times V$ and its differential is a section of $P(N \times W) \rightarrow N \times V^+$. For a given frame $u$, we identify $N \times W$ with $N \times V \times \mathbb{R}$ and a section of $\zeta$ with a function on $N \times V$. Consequently, an affine covector $a \in P(N \times W)$ is represented by a covector $a_u \in T^*(N \times V) = N \times V \times V^* \times V^+$. It follows from (19) that $a_u = (x, v, a, b)$ and $a_{u'} = (x, v, a + m\sigma(u, u'))$ represent the same element of $P(N \times W)$. In the process of generation of frame dependent dynamics we use the canonical isomorphism $\alpha_N : TT^*N \rightarrow T^*TN$ (3.3). We observe that

$$\alpha_N(x, a + m\sigma(u, u'), v) = (x, v, b, a + m\sigma(u, u')),$$

hence $\alpha_N$ defines an isomorphism

$$\alpha : T(N \times P) \rightarrow P(N \times W)$$

and $(D_h)$ is the inverse image of $d_h(N \times V^+)$ by $\alpha$.

Now, we can summarize our constructions. We have canonical symplectic structure on $N \times P$ with the corresponding mapping

$$\beta : T(N \times P) \rightarrow T^*(N \times P),$$

which forms the basis for the hamiltonian formulation of the dynamics. Together with $\alpha$ it gives rise to the following diagram (Tulczyjew triple):

$$\begin{array}{ccc}
(T^*(N \times P), \omega_{N \times P}) & \xleftarrow{\beta} & (T(N \times P), d_T\omega_P) \\
\downarrow \alpha & & \downarrow \alpha \\
N \times P & \xrightarrow{} & N \times V
\end{array}$$

4.6 The Legendre transformation

The Legendre transformation is the passage from lagrangian to hamiltonian generating object. In previous sections it was done with the knowledge of the Legendre transformation for the frame dependent dynamics. In that case we make use of the canonical symplectomorphism $\gamma_M : T^*T^*M \rightarrow T^*TM$ generated by $(\cdot, \cdot) : TM \times_M T^*M \rightarrow \mathbb{R}$, where $M$ is a manifold and $(\cdot, \cdot)$ is the canonical pairing between vectors and covectors. It follows that the inverse image $\gamma_M^{-1}(L)$ of a lagrangian submanifold $L$ generated by a lagrangian $\ell$ is generated by a Morse family

$$\ell - (\cdot, \cdot) : TM \times_M T^*M \rightarrow \mathbb{R},$$

where $TM \times_M T^*M$ is considered a fibration over $T^*M$ (see [13] for details).

Now, we show that analogous procedure can be applied in the case of the affine framework. First, we observe that every element $w \in W$ defines, in natural way, an affine function on $P$:

$$f_w(p) = \langle p, v \rangle - r, \quad \text{where} \quad w = [u, v, r], \quad p = [u, p]. \quad (27)$$

Indeed, when we take another representative of $w$ and $p$, e.g. $(u', v, r')$ and $(u', p')$ respectively, then we obtain

$$\langle p', v \rangle - r' = \langle p - m\sigma(u', u), v \rangle - r + m\sigma(u', u), v \rangle = \langle p, v \rangle - r.$$  

The element $w_1$ defines the constant function equal to 1 on $P$:

$$f_{w_1}(p) = \langle p, 0 \rangle - (-1) = 1.$$  

This implies the following:
\textbf{Proposition 3} There is a natural isomorphism between $P^\dagger$ and $(W, w_1)$ given by

$$f_{[u,v,r]}([u,p]) = (p,v) - r.$$ 

It means that also $W^\dagger \simeq P$. 

With this isomorphism we have (see \textbf{(1)} \textbf{)} $P(N \times W) \simeq N \times V \times V^* \times P$ and $\alpha: T(N \times P) \rightarrow P(N \times W)$ assumes the form

$$\alpha: (x,p,v,a) \mapsto (x,v,a,p).$$ \hspace{1cm} (28) 

$(W, w_1)$, being a special vector space, has a structure of a one-dimensional affine bundle modelled on $V \times I$. The action of the group $(\mathbb{R}, +)$ in the fiber over $V$ comes from the natural action in the fiber of $E_1 \times V \times \mathbb{R} \rightarrow E_1 \times V$. The fundamental vector field $X_W$ for this action is a constant vector field with value $w_1$ at every point.

Now, we need a pairing between $P$ and $V$, which reduces to $\langle \cdot, \cdot \rangle$ (as a section of the trivial bundle $V \times V^* \times \mathbb{R}$) in the vector case. The pairing is a section of $P \times W$ over $P \times V$ defined by

$$P \times V \ni (p,v) \mapsto \langle p,v \rangle = [u,v,(p,v)] \in W, \text{ where } p = [u,p].$$ \hspace{1cm} (29) 

The above definition is correct, i.e. does not depend on the choice of representatives:

$$[u,v,\langle p,v \rangle] = [u',v,\langle p,v \rangle - m\sigma(u',u),v] = [u',v,\langle p',v \rangle].$$ \hspace{1cm} (30) 

It remains to show that the pairing \textbf{(30)} generates an isomorphism between $T(N \times P)$ and $P(V \times W)$.

\textbf{Proposition 4} There is a natural symplectomorphism between $P((N \times W) \times (N \times P))$ and $P(N \times W) \ominus T^*(N \times P)$.

\textbf{Proof.} It is enough to check that any section of $(N \times W) \times (N \times P)$ over $(N \times V) \times (N \times P)$ is equivalent to a section $\sigma$ of the form

$$\sigma(x,y,z) = \sigma_0(x) + f_1(x) - f_2(z) - f_3(y),$$

where $\sigma_0$ is a linear section of $W \rightarrow V$ and functions $f_i$ are affine. 

Similar arguments show that $P(N \times W) \simeq N \times V \times V^* \times P$.

The canonical diagonal inclusion $N \subset N \times N$ implies the projection

$$V^* \times V^* \rightarrow V^*: (a,b) \mapsto a + b$$

and consequently a relation between $P(N \times W \times P)$ and $P((N \times W) \times (N \times P))$. With this relation a section of $P(N \times W \times P)$ over $N \times V \times P$ defines a submanifold of

$$P((N \times W) \times (N \times P)) = P(N \times W) \ominus T^*(N \times P)$$

i.e., a symplectic relation

$$T^*(N \times P) \hookrightarrow P(N \times W).$$

In particular, the differential of the pairing $\langle \cdot, \cdot \rangle$ generates a relation

$$\gamma: T^*(N \times P) \simeq N \times P \times V^* \times V \rightarrow P(N \times W) \simeq N \times V \times V^* \times P$$ \hspace{1cm} (31) 

It easy task to verify that this relation has the following representation

$$N \times P \times V^* \times V \ni (x,p,a,v) \mapsto (x,-v,a,p) \in N \times V \times V^* \times P.$$ 

We see from \textbf{(25) and (24)} that $\gamma = \alpha \circ \beta^{-1}$, and consequently $\gamma \circ \alpha(D_h) = \beta(D_h)$. Following the general rule for composing of generating objects (see \textbf{(3)}), we conclude that $\beta(D_h)$ is generated by the Morse family

\begin{equation}
\begin{array}{c}
\begin{array}{c}
N \times P \times V^+ \xrightarrow{-H_h} \mathbb{R}, \\
\downarrow \\
N \times P
\end{array}
\end{array}
\end{equation}

\[13\]
where

\[ H_h(x, v, p) = \langle p, v \rangle - \ell_h(x, v). \]  

(33)

As in the frame-dependent case, this family can be reduced to the family \( \Xi \).

### 4.7 Inhomogeneous formulation of the dynamics

The inhomogeneous formulation of the dynamics is obtained by the reduction of the homogeneous one with respect to the canonical injection

\[ \iota: W \supset W_1 \rightarrow W, \]

where \( W_1 = \zeta^{-1}(E_1) \) is an affine subspace of \( W \). The projection \( \zeta \), restricted to \( W_1 \), will be denoted by \( \zeta_1 \). \( W_1 \) is a special affine space modelled on a special vector space \( W_0 = \zeta^{-1}(E_0) \). As in Section 4.2 we prove that

\[ \mathcal{P}W_1 = E_1 \times W_0^\dagger \]

and consequently,

\[ \mathcal{P}(N \times W_1) = N \times E_1 \times V^* \times W_0^\dagger. \]

The affine space \( W_0^\dagger \), which we denote by \( P_0 \), is the quotient of \( P = W^\dagger \) (Proposition 3) by the one-dimensional subspace of \( V^* \), spanned by \( \tau \). The canonical isomorphism

\[ \alpha: T(N \times P) = N \times P \times V \times V^* \rightarrow \mathcal{P}(N \times W) = N \times V \times V^* \times P \]

(see Section 4.3) projects to

\[ \alpha^1: V^1(N \times P_0) = N \times P_0 \times E_1 \times E_0^* \rightarrow \mathcal{P}_v(N \times W_1) = N \times E_1 \times E_0^* \times P_0, \]

where \( \mathcal{P}_v \) is an affine analogue of \( V^* \) (Section 3.1), i.e. \( \mathcal{P}_v(N \times W_1) \) contains affine differentials in vertical (with respect to the projection on \( T \)) directions only. We have obvious canonical projection

\[ \mathcal{P}(N \times W_1) = N \times E_1 \times V^* \times P_0 \rightarrow N \times E_1 \times E_0^* \times P_0 \]

(Proposition 3) by the

\[ \mathcal{P}(N \times W_1) = N \times E_1 \times V^* \times P_0 \rightarrow N \times E_1 \times E_0^* \times P_0 = \mathcal{P}_v(N \times W_1) \]

With the isomorphism \( \alpha^1 \) the restriction \( \ell_1 \) of \( \ell_h \) to \( V^1N \) generates a submanifold \( D_t \) of \( V^1(N \times P_0) \subset T(N \times P_0) \).

The Hamiltonian formulation of the dynamics requires generalization of standard concepts like Poisson structure. The hamiltonian is, like lagrangian, not a function, but a section of a bundle of affine values. Let us first notice that the pair \( (P, \tau) \) is a special affine space, hence \( N \times P \) is a bundle of affine values over \( N \times P_0 \). This is the bundle of a hamiltonian. The corresponding phase bundle \( \mathcal{P}(N \times P) \) is canonically mapped to \( T(N \times P_0) \). (The detailed discussion of this mapping and of the involved geometric structures, like Lie affgebroid and affpoisson structure, we postpone to a separate publication, see also \( 2 \). \( 10 \). \( 43 \).) It follows that a section of \( N \times P \rightarrow N \times P_0 \) generates a vector field on \( N \times P_0 \). For the dynamics of a massive particle, the hamiltonian section is given by the equation \( \Psi_m + \varphi(x) = 0 \) (Section 4.4).

### 4.8 Affine Newtonian lagrangian bundle

As we have seen in \( 4 \) the metric \( g \) which has been used in the definition of the lagrangian appears (multiplied by the mass) as the Legendre map in the inhomogeneous, frame-dependent formulation of the dynamics. In the frame-independent formulation, the Legendre map is a mapping of the form

\[ \mathcal{L}_i: V^1N = N \times E_1 \rightarrow N \times P_0 \]

\[ : (x, w) \mapsto \mathfrak{g}. \]  

(34)

where \( \mathfrak{g} \) is an affine mapping with the linear part \( mg \). This observation justifies the following definition. For a special affine space \( A = (A, v) \) with the model special vector space \( (V, v) \), an affine metric is a mapping

\[ h: A \rightarrow V^\dagger. \]

14
such that its linear part $h$ is a metric. The relation between affine metrics and kinetic energy part of lagrangians is established in the following proposition.

**Proposition 5** Let $h : A \rightarrow V^\dagger$ be an affine metric. There exist unique, up to a constant, section $\ell$ of the affine value bundle $A \rightarrow A$ such that $d\ell = h$.

**Proof.** Let $a$ be an arbitrary element in $A$ an let $\overline{a}$ be its projection onto $A$. We define a section $\ell$ by the formula

$$A \ni b \mapsto a + \langle h(a), b - \overline{a} \rangle + \frac{1}{2} \langle h(b - \overline{a}), b - \overline{a} \rangle,$$

where we interpret an element of $V^\dagger$ as a linear section of the bundle $V \rightarrow V$, and $\langle h(a), b - \overline{a} \rangle$ is the value of the section $h(b)$ at the point $b - \overline{a}$.

We have

$$d\ell(b) = h(a) + h(b - \overline{a}) = h(b)$$

4.9 Concluding remarks

Frame independent inhomogeneous formulation of the dynamics requires affine bundles, affine values and an affine version of a metric tensor. The next step is to build a frame independent framework for Hamilton-Jacobi theory and the wave mechanics (Schrödinger equation).

**References**

[1] K. Grabowska, J. Grabowski and P. Urbański: Lie brackets on affine bundles, Ann. Global Anal. Geom. 24 (2003), 101-130.

[2] K. Grabowska, J. Grabowski and P. Urbański: $AV$-differential geometry: Poisson and Jacobi structures, J. Geom. Phys. 52 (2004) no. 4, 398-446.

[3] C. Duval, G. Burdet, H.P. Künzle, M. and Perrin, Bargmann structures and Newton-Cartan theory, Phys. Rev. D, 31, 1841–1853.

[4] D. Iglesias, J.C. Marrero, E. Padrón, D. Sosa, Lagrangian submanifolds and dynamics on Lie affgebroids, preprint [arXiv:math.DG/0505117](http://arxiv.org/abs/math.DG/0505117) to appear in Reports on Math. Phys.

[5] P. Libermann, Ch. Marle, Symplectic geometry and Analytical Mechanics, Math. Appl. 35, Reidel Dordrecht, 1987.

[6] E. Martínez, T. Mestdag and W. Sarlet, Lie algebroid structures and Lagrangian systems on affine bundles, J. Geom. Phys. 44 (2002), no. 1, 70–95.

[7] G. Pidello, Una formulazione intrinseca della meccanica newtoniana, Tesi di dottorato di Ricerca in Matematica, Consorzio Interuniversitario Nord - Ovest, 1987/1988.

[8] W. Sarlet, T. Mestdag and E. Martínez, Lie algebroid structures on a class of affine bundles, J. Math. Phys. 43 (2002), no. 11, 5654–5674.

[9] W.M. Tulczyjew, Frame independence of analytical mechanics, Atti Accad. Sci. Torino, 119 (1985), 273–279.

[10] W.M. Tulczyjew, Mécanique ondulatoire dans l’espace-temps newtonien, C. R. Acad. Sc. Paris, 301 (1985), 419–421.

[11] W.M. Tulczyjew, Geometric Formulations of Physical Theories, Monographs and Textbooks in Physical Science, Lecture Notes 11, Bibliopolis, Naples, 1989.
[12] W.M. Tulczyjew, P. Urbański, *An affine framework for the dynamics of charged particles*, Atti Accad. Sci. Torino Suppl. n. 2, **126** 1992, 257–265.

[13] W.M. Tulczyjew, P. Urbański, *Homogeneous Lagrangian systems*, in ”Gravitation, Electromagnetism and Geometric Structures” ed. G. Ferrarese, Pitagora Editrice 1996, 91–136.

[14] W.M. Tulczyjew, P. Urbański, S. Zakrzewski, *A pseudocategory of principal bundles*, Atti Accad. Sci. Torino, **122** (1988), 66–72.

[15] P. Urbański, *Affine framework for analytical mechanics*, in ”Classical and Quantum Integrability”, Grabowski, J., Marmo, G., Urbański, P. (eds.), Banach Center Publications, vol. **59** (2003), 257–279.

[16] P. Urbański, *Affine Poisson structure in analytical mechanics*, in ”Quantization and Infinite-Dimensional Systems”, J-P. Antoine and others (eds), Plenum Press, New York and London, 1994, 123–129.