Mathematical modeling of axisymmetric flow of granular materials

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Abstract. A mathematical model of the axisymmetric flow of cohesive granular materials is considered. A plastically compressible material is considered as a mathematical model. This material has an additional characteristic - its own rotation of particles. The existence of this characteristic leads to the asymmetry of the stress tensor and the presence of torque stresses. Asymmetric components of the stress tensor are balanced by normal pressure due to the presence of rolling friction. It is shown that the distinctive characteristic of the axisymmetric state is the layering of its flow. The condition for the flow of cohesive bulk material is obtained.

1. Introduction

Models of plastically incompressible material which yield condition depends on the average stress, are used mainly in the mechanics of soils and granular materials [1–3]. Various models describing granular materials are considered in [4–7], and methods for determining the behavior of such materials are proposed. In the classical Sokolovsky model, for deformation of soils [2] plasticity condition is closed and represented by the surface of a cone in the space of principal stresses.

In the space of the first and second invariants of the stress tensors, the Sokolovsky plasticity condition is a triangle (figure 1), with the shear rate and the expansion rate being related by ratio \( \dot{\gamma} = \gamma \tan \alpha \). For the case of small internal friction angles \( \alpha \), the incompressibility of the material at an uncertain shear rate \( \dot{\gamma} \) takes place. However, when deforming materials with pronounced granularity or fragmentation, the expansion rate can not be neglected. The flow surfaces constructed on the basis of experimental data are closed. This means that there is a stress state, in which the shear rates are absent, and the expansion rates are different from zero [3, 8, 9].
The plasticity condition in the space of the second invariant of the symmetric stress tensor and the first invariant of the stress tensor is family of curves: hyperboles at $\alpha > f$ (f is a coefficient of rolling friction); parabolas at $\alpha = f$; ellipses at $\alpha < f$ (see figure 2). Under the condition of a small rolling friction coefficient $f \approx 0$, the closed plasticity condition passes into the Sokolovsky plasticity condition, which at a small internal friction angle $\alpha \approx 0$ passes into the Mises plasticity condition.

The closeness of the plasticity condition in the space of principal stresses can be explained by the fact that classical soil mechanics [2] assumes the stress tensor to be symmetric (it is a special case of the General stress state). Taking into account the asymmetric parts of the second invariant of the stress tensor leads to a closed plasticity condition [8, 10].

The fact of the simultaneous presence of expansion and shear rates of deformation can be caused by partial destruction of solid inclusions and their rearrangement. Rearrangement of particles is the main factor of loosening or shrinking of granular materials. In this case, individual particles of granular material not only move in space but also rotate, as one particle is pressed against another by a force corresponding to the pressure inside the array.
Particles or individual granules have a non-spherical nature. They move due to rotation accompanied by loosening, and the relative movement will be accompanied by the presence of rolling friction.

For granules with a significant difference from the spherical shape, the rolling friction coefficient can be large \( f \delta/R = \{10...100\} \), \( \delta \) is the classical rolling friction coefficient measured in units of length, and \( R \) is the reduced radius of the particles in the form of a ball).

The account of rolling friction at the deformation of particles due to their relative rotation is possible at the equality of driving forces and rolling resistance forces. In space of stresses, it can be formulated in the form [10]

\[
\alpha_i (I^2_{2m} - M^2_0) + I^2_{2[\sigma]} - f^2 I^2_{1\sigma} = 0,
\]

where \( I^2_{2[\sigma]} = 1/2 \cdot \sigma'_{ij} \sigma'_{ij} \), \( \alpha_i \) is the coefficient determining the effect of torque stresses on the rotation of particles, \( M_0 \) is the limit moment of plasticity during the rotation of particles.

Figure 3 shows a schematic representation of the element subjected to pressure \( I_{1\sigma} \), which occurs when the moment of resistance \( f I_{1\sigma} \) and the moment of forces move from the antisymmetric part of the stress tensor \( I'_{2[\sigma]} \).

For the case of a small effect of torque stresses on rotation \( (\alpha_i \approx 0) \), the plasticity condition takes the form:

\[
I^2_{2[\sigma]} - f^2 I^2_{1\sigma} = 0.
\]

![Figure 3](image)

**Figure 3.** The rolling of the element.

Thus, the condition of the ultimate equilibrium of granular particles requires knowledge of the structural parameter of the material, namely the coefficient of rolling friction, i.e. the relative deviation of the shape of the granules from their averaged size in the form of a ball.

### 2. The axisymmetric problem statement

In order to account for the rotation of individual particles in the axisymmetric flow of cohesive granular materials, we consider a refined mathematical model of Sokolovsky in a cylindrical coordinate system.

The axisymmetric state of the medium flow is characterized by the following elements of the stress tensor, components of the velocity displacement along the axes \( r, \theta, z \) and the strain velocity tensor [1]:

\[
\sigma_r = \sigma_r(r, z), \quad \sigma_\theta = \sigma_\theta(r, z), \quad \sigma_z = \sigma_z(r, z), \quad \sigma_{rz} = \sigma_{rz}(r, z),
\]

\[
\sigma_{\theta r}(r, z) \neq \sigma_{\theta z}(r, z), \quad \sigma_{\theta\theta} = \sigma_{\theta\theta} = \sigma_{\theta z} = 0.
\]
The axisymmetric stress state of a microstructural cohesive bulk medium in a homogeneous gravity field is determined by two equilibrium equations, the plasticity condition, and the stress-dilatancy relation.

To close the system of equations in stresses, we use the associated law of plastic flow, which allows us to connect the strain rate with the stress state. As a result, we obtain a system of equations, which is redefined for the components of the velocity displacement \( rU \) and \( zU \). The uniqueness condition of the solution of the system is the missing equation for the closure of the system in stresses. Two equations of the system allow determining the velocity components \( rU \) and \( zU \). The third equation, which characterizes the relationship between shear rate and expansion rate, is the equation of compatibility of velocities displacement, i.e. it is one of the forms of the dilatancy equation.

The dilatancy equation contains the components of the displacement rate and stress components. We can eliminate the displacement rate using the first two equations and obtain the equation of dilatancy in stresses to close the system in stresses.

3. The stress state of the loose medium
The dilatancy condition \( \dot{\varepsilon}/\dot{\gamma} = \Delta \) connecting expansion and shear rates of deformation, presented in terms of stresses, has the form [10, 11]

\[
\sigma_r = \sigma_\theta.
\] (1)

Based on (1), the equilibrium equations and the plasticity condition will take the form

\[
\sigma_{r,z} + \left( \tau + \frac{2f\sigma_r + f\sigma_z}{\sqrt{2}} \right)_{z} = 0,
\]

\[
\left( \tau - \frac{2f\sigma_r + f\sigma_z}{\sqrt{2}} \right)_{r} + \frac{f(2\sigma_r + \sigma_z)}{r\sqrt{2}} + \frac{\tau}{r} = \gamma,
\] (2)

\[
2\sigma^2 + \sigma^2 + 2\tau^2 - \frac{1}{3}(2\sigma_r + \sigma_z)^2 - Y^2 = 2\alpha Y(2\sigma_r + \sigma_z) + (f^2 - \alpha^2)(2\sigma_r + \sigma_z)^2 = 0,
\]

where \( \tau = \sigma_{r,z} \) is the symmetric part of the tangent stresses, \( \gamma = \rho g \) is the gravity, \( Y \) is the adhesion.

The system of equations (1), (2) contains four unknown variables \( \sigma_r, \sigma_\theta, \sigma_z, \tau \) and is closed. Let \( \sigma = \sigma_r + \sigma_\theta + \sigma_z = 2\sigma_r + \sigma_z \) be tripled average stress, then

\[
\frac{\sigma_{r,z} + \tau}{\sqrt{2}} + \frac{f}{\sqrt{2}}\sigma_{r,z} = 0,
\]

\[
\tau_{r} - \frac{f}{\sqrt{2}}\sigma_{r} + \sigma_{z} = -\gamma,
\]

\[
2\frac{\tau^2}{3} + \sigma^2 = Y^2 = 2\alpha Y(2\sigma_r + \sigma_z) + (f^2 - \alpha^2)(2\sigma_r + \sigma_z)^2 - 2\sigma^2 - 4\sigma, \sigma_z = 0.
\]
The system contains four unknown variables $\sigma_r, \sigma_z, \tau_r$ and $\tau_z$, but $\sigma_r, \sigma_z$ and $\tau_r$ are related. Since $\sigma_r = \sigma - 2\sigma_z$, then substituting $\sigma_z$ in the first and third equations and expressing $\tau$ from the plasticity condition, we obtain

$$\sigma_r + \tau_z + \frac{f}{\sqrt{2}} \sigma = 0,$$

$$\tau_r - \frac{f}{\sqrt{2}} \sigma + \frac{r}{r} \frac{\sigma}{\sigma} + \tau = -\gamma,$$

$$|r| = \frac{1}{\sqrt{2}} \left( -\frac{2}{3} f^2 + \alpha^2 \right) \sigma^2 + 2\alpha Y\sigma + Y^2 + 4\sigma r - 6\sigma_r. $$

To exclude the tangent stress $\tau$ from the first two equations of the system we present $\tau_r$ and $\tau_z$ as

$$\tau_r = \tau_r \sigma_r + \tau_r \sigma_z \quad \tau_z = \tau_z \sigma_r + \tau_z \sigma_z.$$

After the elimination of $\sigma_z, \tau_r$ and $\tau_z$ we obtain a system of two partial differential equations and one finite equation

$$\sigma_r + \sigma_z \left( \tau_r + \frac{f}{\sqrt{2}} \right) + \sigma_r \left( \tau_z \right) = 0,$$

$$\sigma_r \left( \tau_r - \frac{f}{\sqrt{2}} \right) + \sigma_z \left( \tau_z \right) - 2\sigma_r - \frac{r}{r} \frac{\sigma}{\sigma} + \tau = -\gamma,$$

$$|r| = \frac{1}{\sqrt{2}} \left( -\frac{2}{3} f^2 + \alpha^2 \right) \sigma^2 + 2\alpha Y\sigma + Y^2 + 4\sigma r - 6\sigma_r. $$

Equations of the system (3) together with equations

$$d\sigma = \sigma_r dr + \sigma_z dz,$$

$$d\sigma_r = \sigma_r dr + \sigma_z dz$$

form a linear system of partial differential equations with respect to the vector $\Sigma = (\sigma_r, \sigma_z, \tau_r, \tau_z)^T$.

$$A\Sigma = a,$$

where $a = \left( 0; \frac{f}{\sqrt{2}} \sigma - \frac{\tau}{r}; -\gamma; d\sigma, d\sigma_r, d\sigma_z \right)^T, \quad A = \begin{pmatrix} 0 & \frac{f}{\sqrt{2}} & 1 & \tau_r \\ \tau_r & -\frac{f}{\sqrt{2}} & 1 & \tau_z \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To consider the flow, we introduce a small parameter $\delta$ and make a replacement $r = \xi \delta$. Matrix $A$ of the system of equations takes the form
\[
A^* = \begin{pmatrix}
0 & \tau_{\sigma} + \frac{f}{\sqrt{2}} & \delta & \tau_{\sigma,\epsilon} \\
\left(\tau_{\sigma} - \frac{f}{\sqrt{2}}\right)\delta & 1 & \delta\left(\tau_{\sigma,\epsilon}\right) & -2 \\
\delta d\xi & d\xi & 0 & 0 \\
0 & 0 & \delta d\xi & d\xi
\end{pmatrix}.
\]

In the zone \( r \approx 0 \), i.e. \( \delta = 0 \), equation (4) in the zero approximation gives two equations

\[ \frac{f \sigma}{\sqrt{2}} - \tau = 0, \quad \left(\tau_{\sigma} + \frac{f}{\sqrt{2}}\right)\sigma_{\zeta,\epsilon} + \tau_{\sigma,\epsilon}\sigma_{r,\zeta} = 0. \]

From the last equations, it follows that

\[ \sigma = \frac{\sqrt{2}}{f} \tau, \quad \tau_{\epsilon} = 0, \quad \tag{5} \]

hence \( \tau = \tau(r, z_0) \).

The equilibrium equations and the plasticity condition taking into account (5) will have the form

\[ \sigma_{r,\epsilon} = 0, \quad \sigma_{z,\epsilon} = -\gamma', \quad \tag{6} \]

\[ 2\sigma_r^2 + \sigma_{\zeta}^2 - Y^2 + \tau^2\left[4 - \frac{2}{f^2}\left(\frac{1}{3} + \alpha^2\right)\right] \frac{2\sqrt{2}\alpha Y}{f} \tau = 0. \quad \tag{7} \]

From the first equilibrium equation, it follows that \( \sigma_r = \sigma_r(r_0, z) \), i.e. \( \sigma_r \) is constant for every \( z \). From the second equilibrium equation, it follows that

\[ \sigma_{\zeta} = \sigma_{\zeta}^0 - \gamma z, \quad \tag{8} \]

where \( \sigma_{\zeta}^0 = \text{const} \).

From equations (5), it follows that \( \sigma_{\zeta} = 0 \), i.e. \( \sigma \) does not depend on \( z \). From equations (6) and (7), it follows that \( \sigma_r, \sigma_{\zeta} \) do not depend on \( r \), and since \( \sigma = 2\sigma_r + \sigma_{\zeta} \), then \( \sigma = \text{const} \), therefore

\[ \sigma_r = \frac{1}{2} \gamma z. \quad \tag{9} \]

The above reasoning concerns the symmetric part of the stress tensor. The full stress tensor \( \tilde{\sigma} \) has the form

\[ \tilde{\sigma} = \sigma_{(1)} + \sigma_{[1]}, \]

where \( \sigma_{(1)} = \begin{pmatrix} \sigma_r & 0 & \tau \\ 0 & \sigma_\theta & 0 \\ \tau & 0 & \sigma_z \end{pmatrix} \), \( \sigma_{[1]} = \begin{pmatrix} 0 & 0 & q \\ 0 & 0 & 0 \\ -q & 0 & 0 \end{pmatrix} \), \( q = \sigma_{(2)} \).

From the plasticity condition we have

\[ q = \frac{f \sigma}{\sqrt{2}} \]

or taking into account (5)
\[ q = \tau. \]  

The full stress tensor \( \tilde{\sigma} \) will have the form

\[
\tilde{\sigma} = \begin{pmatrix} \sigma_r & 0 & 2\tau \\ 0 & \sigma_\theta & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}.
\]

\[ \text{Figure 4.} \text{ An element of medium with the stresses indicated on it.} \]

Figure 4 shows the stress state of the cohesive bulk medium. It is characterized by the fact that tangential stresses are presented only on the vertical planes. This explains the so-called “arching effect" or, in other words, the effect of “bridging” in cohesive bulk materials. When a certain load is applied to the material sample (in figure 4 it is \( \sigma_r \)), then the sample particles begin to flow or fall, but due to rolling friction, which is directed opposite to the movement, the flow stops on some surface (arch).

The plasticity condition (7) is a quadratic equation and for the flow of the material, the discriminant of this equation must be nonnegative:

\[
D = \frac{3}{2} \gamma^2 z^2 \left( \frac{1}{3f^2} + \frac{\alpha^2}{f^2} - 2 \right) + 2Y^2 - \frac{Y^2}{3f^2} \geq 0. \tag{11}
\]

Equation (11) is a flow condition of cohesive bulk material. From (11) we can obtain the restriction on the flow region

\[
z \geq \frac{Y}{\gamma} \sqrt{\frac{2(1-6f^2)}{3(1+3\alpha^2 - 6f^2)}} = z^*. \tag{12}
\]

When \( z < z^* \) the plastic flow of the bulk medium is absent, and the material behaves as a rigid body (figure 3).

Figure 5 shows the area of plastic flow of a cohesive bulk material under the action of its weight. It follows from (12) that with an increase in adhesion and a decrease in the specific gravity of the material, the area of rigid behavior of the material increases.
When loading the half-space with an axisymmetric load $\sigma^0 \neq 0$, the condition (11) will take the form:

$$D = 4z^2r^2\left(-6 + \frac{1}{f^2} + \frac{3\alpha^2}{f^2}\right) + 16z\gamma\sigma^0\left(2 - \frac{1}{3f^2} - \frac{\alpha^2}{f^2}\right) +$$

$$+8\left[2(Y^2 - \sigma^0) - \frac{1}{3f^2}(Y^2 - \sigma^0) + \frac{\alpha^2\sigma^0}{f^2}\right] \geq 0.$$  

Tangential stress $\tau$, corresponding to the load $\sigma^0$ of pressure or take-out takes the form:

$$\tau_{1,2} = \frac{\sqrt{2}\alpha Y}{2f\left(2 - \frac{1}{3f^2} - \frac{\alpha^2}{f^2}\right)} \pm \frac{\sqrt{D}}{2\left[4 - \frac{2}{f^2}\left(\frac{1}{3} + \alpha^2\right)\right]}.$$

Thus, equations (10) and (13) determine the stress state of a weighty bulk medium.

4. **The velocity field of the axisymmetric problem**

The components of the displacement velocity vector are determined by the equations [11]

$$dU_r = \frac{1}{D_1} \frac{U_r}{r} dr, \quad dU_z = \frac{1}{D_2} \frac{U_z}{r} dz.$$  

where $D_1 = \frac{\sigma_\theta - \beta I_{1\sigma} - \alpha Y}{\sigma_\theta - \beta I_{1\sigma} - \alpha Y}$, $D_2 = \frac{\sigma_\theta - \beta I_{1\sigma} - \alpha Y}{\sigma_\theta - \beta I_{1\sigma} - \alpha Y}$, $\beta = \sqrt{3} + \alpha^2 - f^2$.

Equations (14) are ordinary differential equations along with the characteristics $r = \text{const}$ and $z = \text{const}$, and can be integrated.

Let us find the values of the components of the displacement velocity vector at $\sigma_\theta = \sigma_\theta$, i.e. for $D_1 = 1$. The first equation of the system (14) will take the form

$$\frac{dU_r}{dr} = \frac{U_r}{r},$$

and we get the value of the radial component of the velocity:

$$U_r = \frac{U_r(r_0(z), z)}{r_0(z)} r.$$  

\[\text{Figure 5. Schematic representation of a half-space loaded with mass forces (1-rigid body, 2-plastic flow region).}\]
Since the stress state is known, taking into account (8) and (9) \( D_z \) will take the form:

\[
D_z = 1 - \frac{3\gamma z + 2\sigma_z^0}{2\gamma z - 2\sigma_z^0 + 2\beta\sigma_z^0 + 2\alpha Y}.
\]

(Integrating the second equation of the system (14), taking into account (15) and (16) for the case \( U,(r_0,z) = \text{const} \) and \( r_0(z) = \text{const} \), we obtain the axial component of the velocity:

\[
U_z = U_z(r, z_0(r)) - 2 \frac{U_z(r_0(z), z)}{r_0(z)} \left[ z - z_0 + \frac{(3\beta - 1)\sigma_z^0 + 3\alpha Y}{\gamma} \ln \frac{\gamma z - 2\beta\sigma_z^0 - 2\alpha Y}{\gamma z_0 - 2\beta\sigma_z^0 - 2\alpha Y} \right].
\]

Thus, expressions (15) and (17) define the velocity field of the cohesive bulk medium.

Figure 6 shows the dependences of components of the displacement velocity vector on coordinates.

Let us get the current lines of particles of a loose medium for a stationary flow, using the expression of the particle velocity

\[
\frac{dr}{dt} = U_r, \quad \frac{dz}{dt} = U_z.
\]

The differential equation for the current line of particles of bulk material is obtained from (15, 17, 18) excluding time \( t \)

\[
\frac{dz}{dr} = \frac{U_z(r, z_0(r)) - 2 \frac{U_z(r_0(z), z)}{r_0(z)} \left[ z - z_0 + \frac{(3\beta - 1)\sigma_z^0 + 3\alpha Y}{\gamma} \ln \frac{\gamma z - 2\beta\sigma_z^0 - 2\alpha Y}{\gamma z_0 - 2\beta\sigma_z^0 - 2\alpha Y} \right]}{U_z(r_0(z), z) / r_0(z)}.
\]

After the transform we get

\[
\frac{dz}{dr} = \frac{1}{r} \left[ \frac{U_z(r, z_0(r))}{U_z(r_0(z), z)} \right] r_0(z) + 2z_0 + 2 \frac{(3\beta - 1)\sigma_z^0 + 3\alpha Y}{\gamma} \times \ln \frac{\gamma z_0 - 2\beta\sigma_z^0 - 2\alpha Y}{\gamma z_0 - 2\beta\sigma_z^0 - 2\alpha Y}.
\]
At \( U, r_0(z), z = \text{const} \), \( U, r, z_0(r) = \text{const} \) and \( r_0(z) = \text{const} \) the solution of the last equation has the form

\[
0 = \frac{c}{(b + (1 + a)\gamma) r^{2(b + (1 + a)\gamma)\gamma}} \left[ a(\gamma + b)\ln(\gamma + b) \frac{U, r, z_0(r)}{U, r_0(z), z} r_0(z) - 2z_0 - 2a\ln|\gamma z_0 + b| \right]. \tag{19}
\]

where \( a = \frac{(3\beta - 1)\sigma_0^\alpha + 3\alpha Y}{\gamma}, b = -2\beta\sigma_0^\alpha - 2\alpha Y, c = \text{const} \).

The last expression defines a family of current lines of weighty particles (figure 7).

\[\text{Figure 7. Current lines near theaxis of symmetry.}\]

The current lines of weighty particles of bulk material are a family of hyperboles having horizontal asymptotes \( z = \text{const} \). Thus, the particles of bulk material near the planes \( z = \text{const} \) move almost horizontally, and only near the axis of symmetry, there is a vertical movement. As an illustration of the flow of cohesive bulk material, consider a vertically arranged cylindrical container of large radius, in the lower base of which there is a hole filled with an ideal liquid. The flow of liquid will occur approximately in radial directions to the hole. For the granular material, the outflow occurs in the following order: the top layer located near the hole flows, and then the layers located below the current lines flow, as shown in figure 4. If we assume the presence of an upper marked loose layer, then when the particles of this upper layer flow through the hole earlier than the particles of the underlying layers.

5. Conclusion

A mathematical model of the ultimate stress-strain state of bulk materials with closed plasticity condition is considered. The closed plasticity condition assumes the asymmetry of the stress tensor and is characterized by the simultaneous presence of expansion and shear deformation.

For the axisymmetric problem, the stress state and the restriction on the plastic flow region of the material under consideration are obtained. Expressions specifying the velocity field of particle displacements of bulk material are obtained and current lines for stationary flow are constructed.

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