ON THE RELATIVE MORRISON-KAWAMATA CONE CONJECTURE

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Abstract. We relate the Morrison-Kawamata cone conjecture for Calabi-Yau fiber spaces to the existence of Shokurov polytopes. For K3 fibrations, the existence of (weak) fundamental domains for movable cones is established. The relationship between the relative cone conjecture and the cone conjecture for its geometric or generic fibers is studied.

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1. Introduction

The purpose of this paper is to study the following (relative) Morrison-Kawamata cone conjecture [Mor93, Mor96, Kaw97, Tot09].

Conjecture 1.1. Let $(X, \Delta) \to S$ be a klt Calabi-Yau fiber space. Let $\Gamma_B, \Gamma_A$ be the images of the pseudo-automorphism group $\text{PsAut}(X/S, \Delta)$ and the automorphism group $\text{Aut}(X/S, \Delta)$ under the natural group homomorphism $\text{PsAut}(X/S, \Delta) \to \text{GL}(N^1(X/S)_\mathbb{R})$ respectively.

1. The cone $\overline{\text{Mov}}^c(X/S)$ has a (weak) rational polyhedral fundamental domain under the action of $\Gamma_B$.
2. The cone $\overline{\text{Amp}}^c(X/S)$ has a (weak) rational polyhedral fundamental domain under the action of $\Gamma_A$.

Relevant notions in Conjecture 1.1 are explained in Section 2 and Section 3. There are different choices of cones in the cone conjecture, see Remark 5.6 for the reason of the above choice.

At the expense of some ambiguity, for simplicity, we call Conjecture 1.1 (1) and (2) the (weak) cone conjecture for movable cones and the (weak) cone conjecture for ample cones respectively. Although our primary interest is in complex varieties, we need to work with
non-algebraically closed fields. When $X$ is a smooth Calabi-Yau variety over a field $K$, the analogous cone conjecture still makes sense, and we also call it the cone conjecture.

The cone conjecture is beyond merely predicting the shape of cones of Calabi-Yau varieties. In fact, for an arbitrary klt pair, if $(X, \Delta)$ is its minimal model and $(X, \Delta) \to S$ is the morphism to its canonical model, then the cone conjecture for movable cones predicts finiteness of minimal models (see Proposition 5.3 for the precise statement). Moreover, compared with the weaker predictions of finitely many $\text{PsAut}(X/S, \Delta)$-or $\text{Aut}(X/S, \Delta)$-equivalence classes, it seems that the existence of (weak) fundamental domains is crucial in the proof of the cone conjecture (see the proof of Proposition 6.4).

When $X \to S$ is a birational morphism, [BCHM10] established the finiteness of $\text{PsAut}(X/S)$-equivalence classes. Finiteness of $\text{PsAut}(X/S)$-equivalence classes is also known when $\dim X \leq 3$, $\dim S > 0$ ([Kaw97]) and elliptic fibrations ([FHS21]). When $S$ is a point, Conjecture 1.1 is known for surfaces ([Tot09]), abelian varieties ([PS12]) and large classes of Calabi-Yau manifolds with Picard number 2 ([Ogu14, LP13]). Analogous cone conjecture of $\text{Mov}(X/C)_+$ (see Definition 3.1) is also known for a projective hyperkähler manifold $X$ ([Mar11]). Over arbitrary fields of characteristic $\neq 2$, the cone conjecture is known for K3 surfaces [BLvL20]. Analogous cone conjecture of $\text{Mov}(X/K)_+$ is also known for a hyperkähler variety $X$ over a field $K$ with characteristic 0 ([Tak21], cf. Remark 6.3). On the other hand, it is known that Conjecture 1.1 no longer holds true for lc pairs (see [Tot09]). We recommend [LOP18] for a survey of relevant results.

The new ingredient of the current paper is to study the cone conjecture from the perspective of Shokurov polytopes. We propose the following conjecture which seems to be more tractable.

**Conjecture 1.2.** Let $f : (X, \Delta) \to S$ be a klt Calabi-Yau fiber space.

1. There exists a polyhedral cone $P_M \subset \text{Eff}(X/S)$ such that

   $$\bigcup_{g \in \text{PsAut}(X/S, \Delta)} g \cdot P_M \supset \text{Mov}(X/S).$$

2. There exists a polyhedral cone $P_A \subset \text{Eff}(X/S)$ such that

   $$\bigcup_{g \in \text{Aut}(X/S, \Delta)} g \cdot P_A \supset \text{Amp}(X/S).$$

Using results of [Loo14] and assuming standard conjectures of log minimal model program (LMMP), we show that Conjecture 1.2 is nearly equivalent to the cone conjecture (when $S$ is a point, they are indeed equivalent).

**Theorem 1.3.** Let $f : (X, \Delta) \to S$ be a klt Calabi-Yau fiber space. Assume that good minimal models exist for effective klt pairs in dimension $\dim(X/S)$.

1. If $R_1 f_* \mathcal{O}_X = 0$, then the weak cone conjecture for $\overline{\text{Mov}}(X/S)$ is equivalent to the Conjecture 1.2 (1).

2. If $\overline{\text{Mov}}(X/S)$ is non-degenerate, then the cone conjecture for $\overline{\text{Mov}}^c(X/S)$ is equivalent to the Conjecture 1.2 (1);

3. The cone conjecture for $\overline{\text{Amp}}^c(X/S)$ is equivalent to the Conjecture 1.2 (2).
It seems that Conjecture 1.2 is more fundamental as it incorporates in both the finiteness of models or contractions, and the existence of fundamental domains.

Using this circle of ideas, we study a Calabi-Yau fiber space $X \to S$ fibered by K3 surfaces. This means that for a general closed point $t \in S$, its fiber $X_t$ is a smooth K3 surface. We establish the (weak) cone conjecture of $\overline{\text{Mov}}(X/S)$ for K3 fibrations.

**Theorem 1.4.** Let $f : X \to S$ be a Calabi-Yau fiber space such that $X$ has terminal singularities.

If $f$ is fibered by K3 surfaces, then the weak cone conjecture of $\overline{\text{Mov}}(X/S)$ holds true.

Moreover, if $\overline{\text{Mov}}(X/S)$ is non-degenerate, then the cone conjecture holds true for $\overline{\text{Mov}}(X/S)$.

In particular, if $S$ is $\mathbb{Q}$-factorial, then the cone conjecture holds true for $\overline{\text{Mov}}(X/S)$.

We discuss the contents of the paper. Section 2 gives necessary background materials and fixes notation. Section 3 develops the geometry of convex cones following [Loo14]. Section 4 establishes properties of generic and geometric fibers which will be used in Section 6. Section 5 studies the relationship between the cone conjecture and Conjecture 1.2. In particular, Theorem 1.3 is proven. Section 6 studies the cone conjecture by assuming that it holds true for geometric or generic fibers. Theorem 1.4 is shown in Section 6.1.

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## 2. Preliminaries

Let $f : X \to S$ be a projective morphism between normal quasi-projective varieties over $\mathbb{C}$. Then $f$ is called a fibration if $f$ has connected fibers. We write $X/S$ to mean that $X$ is over $S$.

By divisors, we mean Weil divisors. For $\mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and two $\mathbb{K}$-divisors $A, B$ on $X$, $A \sim_{\mathbb{K}} B/S$ means that $A$ and $B$ are $\mathbb{K}$-linearly equivalent over $S$. If $A, B$ are $\mathbb{R}$-Cartier divisors, then $A \equiv B/S$ means that $A$ and $B$ are numerically equivalent over $S$.

We use $\text{Supp} E$ to denote the support of the divisor $E$. A divisor $E$ on $X$ is called a vertical divisor (over $S$) if $f(\text{Supp} E) \neq S$. A vertical divisor $E$ is called a very exceptional divisor if for any prime divisor $P$ on $S$, over the generic point of $P$, we have $\text{Supp} f^* P \notin \text{Supp} E$ (see [Bir12, Definition 3.1]). If $f$ is a birational morphism, then the notion of very exceptional divisor coincides with that of exceptional divisor.

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1 Using these ideas, [Xu24, Theorem 14] shows that the cone conjecture for ample cones follows from the cone conjecture for movable cones. [GLSW24] pushes this further to cone conjecture for effective cone.

2 In the subsequent paper [LI23], we establish the weak cone conjecture for movable cones of terminal Calabi-Yau fibrations in relative dimension $\leq 2$. This is partially extended to klt Calabi-Yau fibrations in relative dimension two by [MS24].
Let $X$ be a normal complex variety and $\Delta$ be an $\mathbb{R}$-divisor on $X$, then $(X, \Delta)$ is called a log pair. We assume that $K_X + \Delta$ is $\mathbb{R}$-Cartier for a log pair $(X, \Delta)$. Then $f : (X, \Delta) \to S$ is called a Calabi-Yau fibration/fiber space if $X \to S$ is a fibration, $X$ is $\mathbb{Q}$-factorial and $K_X + \Delta \sim_{\mathbb{R}} 0/S$. When $(X, \Delta)$ has lc singularities (see Section 2.2), then $K_X + \Delta \sim_{\mathbb{R}} 0/S$ is equivalent to the weaker condition $K_X + \Delta \equiv 0/S$ by [HX16, Corollary 1.4].

2.1. **Movable cones and ample cones.** Let $V$ be a finite-dimensional real vector space with a rational structure, that is, a $\mathbb{Q}$-vector subspace $V(\mathbb{Q})$ of $V$ such that $V = V(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$. A set $C \subset V$ is called a cone if for any $x \in C$ and $\lambda \in \mathbb{R}_{>0}$, we have $\lambda \cdot x \in C$. We use $\text{Int}(C)$ to denote the relative interior of $C$ and call $\text{Int}(C)$ the open cone. By convention, the origin is an open cone. A cone is called a polyhedral cone (resp. rational polyhedral cone) if it is a closed convex cone generated by finite vectors (resp. rational vectors). If $S \subset V$ is a subset, then $\text{Conv}(S)$ denotes the convex hull of $S$, and $\text{Cone}(S)$ denotes the closed convex cone generated by $S$. As we are only concerned about convex cones in this paper, we also call them cones.

Let $\text{Pic}(X/S)$ be the relative Picard group. Let

$$N^1(X/S) := \text{Pic}(X/S)/\equiv$$

be the lattice. Set $\text{Pic}(X/S)_{\mathbb{K}} := \text{Pic}(X/S) \otimes_{\mathbb{Z}} \mathbb{K}$ and $N^1(X/S)_{\mathbb{K}} := N^1(X/S) \otimes_{\mathbb{Z}} \mathbb{K}$ for $\mathbb{K} = \mathbb{Q}$ or $\mathbb{R}$. If $D$ is an $\mathbb{R}$-Cartier divisor, then $[D] \in N^1(X/S)_{\mathbb{R}}$ denotes the corresponding divisor class. To abuse the terminology, we also call $[D]$ an $\mathbb{R}$-Cartier divisor.

Recall that a Cartier divisor $D$ is called movable if the base locus of the linear system $|D|$ has codimension $> 1$. We list relevant cones inside $N^1(X/S)_{\mathbb{R}}$ which appear in the paper:

1. $\text{Eff}(X/S)$: the cone generated by effective Cartier divisors;
2. $\overline{\text{Eff}}(X/S)$: the closure of $\text{Eff}(X/S)$;
3. $\text{Mov}(X/S)$: the cone generated by movable divisors;
4. $\overline{\text{Mov}}(X/S)$: the closure of $\text{Mov}(X/S)$;
5. $\overline{\text{Mov}}(X/S) := \overline{\text{Mov}}(X/S) \cap \text{Eff}(X/S)$;
6. $\text{Mov}(X/S)_{+} := \text{Conv}(\overline{\text{Mov}}(X/S) \cap N^1(X/U)_{\mathbb{Q}})$ (see Definition 3.1);
7. $\text{Amp}(X/S)$: the cone generated by ample divisors;
8. $\overline{\text{Amp}}(X/S)$: the closure of $\text{Amp}(X/S)$;
9. $\overline{\text{Amp}}(X/S) := \overline{\text{Amp}}(X/S) \cap \text{Eff}(X/S)$;
10. $\text{Amp}(X/S)_{+} := \text{Conv}(\overline{\text{Amp}}(X/S) \cap N^1(X/U)_{\mathbb{Q}})$.

If $K$ is a field of characteristic zero and $X$ is a variety over $K$, then the above cones still make sense for $X$. We use $\text{Mov}(X/K)$, $\text{Amp}(X/K)$, etc. to denote the corresponding cones.

Recall that for a birational map $g : X \dashrightarrow Y/S$, if $D$ is an $\mathbb{R}$-Cartier divisor on $X$, then the pushforward of $D$, $g_*D$, is defined as follows. Let $p : W \to X, q : W \to X$ be birational morphisms such that $g \circ p = q$, then $g_*D := q_*(p^*D)$. This is independent of the choice of $p$ and $q$.

Let $\Delta$ be a divisor on a $\mathbb{Q}$-factorial variety $X$. We use $\text{Bir}(X/S, \Delta)$ to denote the birational automorphism group of $(X, \Delta)$ over $S$. To be precise, $\text{Bir}(X/S, \Delta)$ consists of birational maps $g : X \dashrightarrow X/S$ such that $g_* \text{Supp} \Delta = \text{Supp} \Delta$. A birational map is called a pseudo-automorphism if it is isomorphic in codimension 1. Let $\text{PsAut}(X/S, \Delta)$ be the subgroup of $\text{Bir}(X/S, \Delta)$ consisting of pseudo-automorphisms. Let $\text{Aut}(X/S, \Delta)$ be the subgroup of $\text{Bir}(X/S, \Delta)$ consisting of automorphisms of $X/S$. For a field $K$, if $X$ is a variety over $K$ and $\Delta$
is a divisor on $X$, then we still use $\text{Bir}(X/K, \Delta)$, $\text{PsAut}(X/K, \Delta)$ and $\text{Aut}(X/K, \Delta)$ to denote the birational automorphism group, the pseudo-automorphism group and the automorphism group of $X/K$ respectively.

Let $g \in \text{Bir}(X/S, \Delta)$ and $D$ be an $\mathbb{R}$-Cartier divisor on a $\mathbb{Q}$-factorial variety $X$. Because the pushforward map $g_{\ast}$ preserves numerical equivalence classes, there is a linear map

$$g_{\ast} : N^1(X/S)_{\mathbb{R}} \to N^1(X/S)_{\mathbb{R}}, \quad [D] \mapsto [g_{\ast}D].$$

However, if $g \in \text{Bir}(X/S)$ is not isomorphic in codimension 1, then for $[D] \in \text{Mov}(X/S)$, $[g_{\ast}D]$ may not be in $\text{Mov}(X/S)$. Moreover, $(g, [D]) \mapsto [g_{\ast}D]$ is not a group action of $\text{Bir}(X/S, \Delta)$ on $N^1(X/S)_{\mathbb{R}}$. For one thing, if $D$ is a divisor contracted by $g$, then $g_{\ast}^{-1}(g_{\ast}[D]) = 0 \neq (g^{-1} \circ g)_{\ast}[D]$.

On the other hand, it is straightforward to check that

$$\text{PsAut}(X/S, \Delta) \times N^1(X/S)_{\mathbb{R}} \to N^1(X/S)_{\mathbb{R}}$$

$$(g, [D]) \mapsto [g_{\ast}D],$$

is a group action. We use $g \cdot D, g \cdot [D]$ to denote $g_{\ast}D, [g_{\ast}D]$ respectively. Let $\Gamma_B$ and $\Gamma_A$ be the images of $\text{PsAut}(X/S, \Delta)$ and $\text{Aut}(X/S, \Delta)$ under the natural group homomorphism

$$\iota : \text{PsAut}(X/S, \Delta) \to \text{GL}(N^1(X/S)_{\mathbb{R}}).$$

Because $\Gamma_B, \Gamma_A \subset \text{GL}(N^1(X/S))$, $\Gamma_B$ and $\Gamma_A$ are discrete subgroups. By abusing the notation, we also write $g$ for $\iota(g) \in \Gamma_B$, and denote $\iota(g)([D])$ by $g \cdot [D]$. Then the cones $\text{Mov}(X/S), \text{Mov}(X/S), \text{Mov}(X/S)$ and $\text{Mov}(X/S)_+$ are all invariant under the action of $\text{PsAut}(X/S, \Delta)$. Similarly, $\text{Amp}(X/S), \text{Amp}(X/S), \text{Amp}(X/S)$ and $\text{Amp}(X/S)_+$ are all invariant under the action of $\text{Aut}(X/S, \Delta)$.

When $X$ is not a $\mathbb{Q}$-factorial variety, if $X$ admits a small $\mathbb{Q}$-factorization $\tilde{X} \to X$, then the (weak) cone conjecture of $\text{Mov}(X/S)$ is referred to as the (weak) cone conjecture of $\text{Mov}(\tilde{X}/S)$. The validity of the conjecture is independent of the choice of $\tilde{X}$.

The following example gives a birational map which is not a pseudo-automorphism.

**Example 2.1.** Let $f(x, y, z)$ be a general homogenous cubic polynomial. Let $D := \{f(x, y, z) = 0\} \subset \mathbb{P}^2$ and $B := \{f(-x, y, z) = 0\} \subset \mathbb{P}^2$. Then $(\mathbb{P}^2, \frac{1}{2}D + \frac{1}{2}B)$ is a klt Calabi-Yau pair. Let

$$p_1 = [a : b : c], p_2 = [-a : b : c] \in D \cap B$$

be two distinct points. Let $\pi_i : X_i \to \mathbb{P}^2$, $i = 1, 2$ be the blowing up of $p_i$ such that $E_i, i = 1, 2$ are corresponding exceptional divisors. If $D_i, B_i$ are the strict transforms of $D, B$ on $X_i$, then

$$K_{X_i} + \frac{1}{2}D_i + \frac{1}{2}B_i = \pi_i^\ast(K_{\mathbb{P}^2} + \frac{1}{2}D + \frac{1}{2}B).$$

Therefore, each $(X_1, \frac{1}{2}D_1 + \frac{1}{2}B_1)$ is a klt Calabi-Yau pair. Moreover, $(X_1, \frac{1}{2}D_1 + \frac{1}{2}B_1)$ is isomorphic to $(X_2, \frac{1}{2}D_2 + \frac{1}{2}B_2)$ through $\pi_2^{-1} \circ \tau \circ \pi_1$, where $\tau : \mathbb{P}^2 \to \mathbb{P}^2$ is given by $[x : y : z] \mapsto [-x : y : z]$. However, the birational map

$$\pi_2^{-1} \circ \pi_1 : (X_1, \frac{1}{2}D_1 + \frac{1}{2}B_1) \dashrightarrow (X_2, \frac{1}{2}D_2 + \frac{1}{2}B_2)$$

is not isomorphic in codimension 1. In fact, this map contracts $E_1$ and extracts $E_2$. 
2.2. Minimal models of varieties. Let \((X, \Delta)\) be a log pair. For a divisor \(D\) over \(X\), if \(f : Y \to X\) is a birational morphism from a smooth variety \(Y\) such that \(D\) is a prime divisor on \(Y\), then the log discrepancy of \(D\) with respect to \((X, \Delta)\) is defined to be
\[
a(D; X, \Delta) := \text{mult}_D(K_Y - f^*(K_X + \Delta)) + 1.
\]
This definition is independent of the choice of \(Y\). A log pair \((X, \Delta)\) (or its singularity) is called sub-klt (resp. sub-lc) if the log discrepancy of any divisor over \(X\) is \(\geq 0\) (resp. \(> 0\)). If \(\Delta \geq 0\), then a sub-klt (resp. sub-lc) pair \((X, \Delta)\) is called klt (resp. lc). If \(\Delta = 0\) and the log discrepancy of any divisor over \(X\) is \(> 1\), then \(X\) is said to have terminal singularities. A fibration/fiber space \((X, \Delta) \to S\) is called a klt (resp. terminal) fibration/fiber space if \((X, \Delta)\) is klt (resp. terminal).

Let \(X \to S\) be a projective morphism of normal quasi-projective varieties. Suppose that \((X, \Delta)\) is klt. Let \(\phi : X \dasharrow Y/S\) be a birational contraction (i.e. \(\phi\) does not extract divisors) of normal quasi-projective varieties over \(S\), where \(Y\) is projective over \(S\). We write \(\Delta_Y := \phi_*\Delta\) for the strict transform of \(\Delta\). Then \((Y/S, \Delta_Y)\) is a weak log canonical model of \((X/S, \Delta)\) if \(K_Y + \Delta_Y\) is nef/sad and \(a(D; Y, \Delta_Y) \geq a(D; X, \Delta)\) for any divisor \(D\) over \(X\).

**Lemma 2.2.** Let \((X/S, \Delta)\) be a klt pair with \([K_X + \Delta] \in \overline{\text{Mov}}(X/S)\). Suppose that \(g : (X/S, \Delta) \dasharrow (Y/S, \Delta_Y)\) is a weak log canonical model of \((X/S, \Delta)\). Then \((X/S, \Delta)\) admits a weak log canonical model \((Y'/S, \Delta_{Y'})\) such that
\[
\begin{align*}
(1) & \ Y' \text{ is } \mathbb{Q}-\text{factorial}, \\
(2) & \ X, Y' \text{ are isomorphic in codimension } 1, \text{ and} \\
(3) & \ \text{there exists a morphism } \nu : Y' \to Y/S \text{ such that } K_{Y'} + \Delta_{Y'} = \nu^*(K_Y + \Delta_Y).
\end{align*}
\]

**Proof.** If \(E\) is a prime divisor on \(X\) which is exceptional over \(Y\) and
\[
a(E; X, \Delta) = a(E; Y, \Delta_Y),
\]
then by \(a(E; X, \Delta) \leq 1\), we have \(a(E; Y, \Delta_Y) \leq 1\). By [BCHM10, Corollary 1.4.3], there exist a \(\mathbb{Q}\)-factorial variety \(Y'\) and a birational morphism \(\nu : Y' \to Y\) which extracts all such divisors. Hence \(K_{Y'} + \Delta_{Y'} = \nu^*(K_Y + \Delta_Y)\) where \(\Delta_{Y'}\) is the strict transform of \(\Delta\) on \(Y'\). Moreover, if \(E\) is an exceptional divisor for \(g \circ \nu^{-1}\), then
\[
a(E; X, \Delta) < a(E; Y', \Delta_{Y'}). \tag{2.2.1}
\]
It suffices to show that \(X \dasharrow Y'\) is isomorphic in codimension 1. Let \(p : W \to X\) and \(q : W \to Y'\) be birational morphisms such that \(q \circ p^{-1} = g \circ \nu^{-1}\). Then we have
\[
p^*(K_X + \Delta) = q^*(K_{Y'} + \Delta_{Y'}) + E + F,
\]
where \(F \geq 0\) is a \(p\)-exceptional divisor and \(E \geq 0\) is a \(q\)-exceptional divisor but not \(p\)-exceptional. By (2.2.1), \(\text{Supp}(p(E)) = \text{Exc}(g \circ \nu^{-1})\). Therefore, it suffices to show \(E = 0\).

Suppose that \(E > 0\) and \(\Gamma\) is an irreducible component of \(E\). As \(K_{Y'} + \Delta_{Y'}\) is nef/sad,
\[
\sigma_{\Gamma}(q^*(K_{Y'} + \Delta_{Y'}) + E + F; W/S) = \text{mult}_{\Gamma} E > 0,
\]
where \(\sigma_{\Gamma}(q^*(K_{Y'} + \Delta_{Y'}) + E + F; W/S)\) is the coefficient of \(\Gamma\) in the relative \(\sigma\)-decomposition of \(q^*(K_{Y'} + \Delta_{Y'}) + E + F\) (see [Nak04, Chapter III]). On the other hand, as \([K_X + \Delta] \in \overline{\text{Mov}}(X/S)\), if \(\sigma_{\Gamma}(p^*(K_X + \Delta)) > 0\), then \(\Gamma\) must be \(p\)-exceptional. This contradicts with the choice of \(\Gamma\).  \(
\square
\)
A weak log canonical model \((Y/S, \Delta_Y)\) of \((X/S, \Delta)\) is called a good minimal model of \((X/S, \Delta)\) if \(K_Y + \Delta_Y\) is semi-ample/\(S\). It is well-known that the existence of a good minimal model of \((X/S, \Delta)\) implies that any weak log canonical model of \((X/S, \Delta)\) is a good minimal model (for example, see [Bir12, Remark 2.7]).

By saying that “good minimal models of effective klt pairs exist in dimension \(n\)”, we mean that for any projective variety \(X\) of dimension \(n\) over \(\mathbb{C}\), if \((X, \Delta)\) is klt and the Kodaira dimension \(\kappa(K_X + \Delta) \geq 0\), then \((X, \Delta)\) has a good minimal model.

**Theorem 2.3** ([HX13, Theorem 2.12]). Let \(f : X \to S\) be a surjective projective morphism and \((X, \Delta)\) a klt pair such that for a very general closed point \(s \in S\), the fiber \((X_s, \Delta_s = \Delta|_{X_s})\) has a good minimal model. Then \((X, \Delta)\) has a good minimal model over \(S\).

[HX13, Theorem 2.12] states for a \(\mathbb{Q}\)-divisor \(\Delta\). However, it still holds for an \(\mathbb{R}\)-divisor \(\Delta\): in the proof of [HX13, Theorem 2.12], one only needs to replace \(\text{Proj}_S \oplus_{m \in \mathbb{Z}_{\geq 0}} R^0 f_* O_X(m(K_X + \Delta))\) by the canonical model of \((X/S, \Delta)\) whose existence is known for effective klt pairs by [Li22]. Indeed, because \(\kappa(K_{X_s} + \Delta_s) \geq 0\) for a very general \(s \in S\) by assumption, \(K_X + \Delta \sim_{\mathbb{R}} E/S\) with \(E \geq 0\) by [Li22, Theorem 3.15].

2.3. **Shokurov polytopes.** Let \(V\) be a finite-dimensional \(\mathbb{R}\)-vector space with a rational structure. A polytope (resp. rational polytope) \(P \subset V\) is the convex hull of finite points (resp. rational points) in \(V\). In particular, a polytope is always closed and bounded. We use \(\text{Int}(P)\) to denote the relative interior of \(P\) and call \(\text{Int}(P)\) the open polytope. By convention, a single point is an open polytope. Therefore, \(\mathbb{R}_{>0} \cdot P\) is an open polyhedral cone iff \(P\) is an open polytope.

**Theorem 2.4** ([SC11, Theorem 3.4]). Let \(X\) be a \(\mathbb{Q}\)-factorial variety and \(f : X \to S\) be a fibration. Assume that good minimal models exist for effective klt pairs in dimension \(\dim(X/S)\). Let \(D_i, i = 1, \ldots, k\) be effective \(\mathbb{Q}\)-divisors on \(X\). Suppose that \(P \subset \bigoplus_{i=1}^k [0, 1) D_i\) is a rational polytope such that for any \(\Delta \in P\), \((X, \Delta)\) is klt and \(\kappa(K_F + \Delta|_F) \geq 0\), where \(F\) is a general fiber of \(f\).

Then \(P\) can be decomposed into a disjoint union of finitely many open rational polytopes \(P = \bigcup_{i=1}^n Q_i^0\) such that for any \(B, D \in Q_i^0\), \((Y/S, B_Y)\) is a weak log canonical model of \((X/S, B)\), then \((Y/S, D_Y)\) is also a weak log canonical model of \((X/S, D)\).

For the convenience of the reader, we give the proof of Theorem 2.4. The argument essentially follows from [BCHM10, Lemma 7.1]. However, we need to take care of the weaker assumption on the existence of weak log canonical models, as opposed to log terminal models.

**Proof of Theorem 2.4.** We proceed by induction on the dimension of \(P\). Note that by Theorem 2.3, \((X/S, \Delta)\) has a good minimal model/\(S\) for any \(\Delta \in P\).

**Step 1.** If there exists a \(\Delta_0 \in \text{Int}(P)\) such that \(K_X + \Delta_0 \equiv 0/S\), then we show the claim. In fact, let \(P'\) be a face of \(P\). By the induction hypothesis, \(P' = \bigcup Q_j^0\) such that each \(Q_j^0\) is an open rational polytope, and for \(B, D \in Q_j^0\), \((Y/S, B_Y)\) is a weak log canonical model of \((X/S, B)\), then \((Y/S, D_Y)\) is also a weak log canonical model of \((X/S, D)\). For \(t \in [0, 1]\),

\[
K_X + tB + (1-t)\Delta_0 = t(K_X + B) + (1-t)(K_X + \Delta_0) \equiv t(K_X + B)/S, \\
K_X + tD + (1-t)\Delta_0 = t(K_X + D) + (1-t)(K_X + \Delta_0) \equiv t(K_X + D)/S.
\]
Hence, \((Y/S, tB_Y + (1 - t)\Delta_{0,Y})\) is a weak log canonical model of \((X/S, tB + (1 - t)\Delta_0)\) iff \((Y/S, B_Y)\) is a weak log canonical model of \((X/S, B)\) iff \((Y/S, D_Y)\) is a weak log canonical model of \((X/S, D)\) iff \((Y/S, tD_Y + (1 - t)\Delta_{0,Y})\) is a weak log canonical model of \((X/S, tD + (1 - t)\Delta_0)\). Therefore,

\[
P = \left( \bigsqcup_j \text{Int(Conv}(\bar{Q}_j, \Delta_0)) \right) \sqcup \{\Delta_0\}
\]

satisfies the claim.

Step 2. Next, we show the general case. It suffices to show the result locally around any point \(\Delta_0 \in P\). We can assume that \(\Delta_0\) is a rational point and \(P\) is a small rational polytope containing \(\Delta_0\). During the argument, by saying that shrinking \(P\), we mean that replacing \(P\) by a sufficiently small rational polytope \(P' \subset P\) such that \(P' \supset P \cap \mathbb{B}(\Delta_0, \epsilon)\), where \(\mathbb{B}(\Delta_0, \epsilon) \subset \bigoplus_{i=1}^k \mathbb{R} \cdot D_i\) is the ball centered at \(\Delta_0\) with radius \(\epsilon \in \mathbb{R}_{>0}\).

Let \((X'/S, \Delta'_0)\) be a weak log canonical model of \((X/S, \Delta_0)\). By Theorem 2.3, there exist a contraction \(\pi : X' \to Z'/S\) and an ample/S \(\mathbb{Q}\)-Cartier divisor \(A\) on \(Z'\) such that \(K_{X'} + \Delta'_0 \sim_{\mathbb{Q}} \pi^*A/S\).

Let \(p : W \to X, q : W \to X'\) be birational morphisms such that \(q \circ p^{-1}\) is the natural map \(X \dashrightarrow X'\). Moreover, we assume that \(p\) is a log resolution of \((X, \sum_{i=1}^k D_i)\). Let \(\tilde{D}_i, i = 1, \ldots, k\) be the strict transforms of \(D_i\) on \(W\), and \(E_j, j = 1, \ldots, l\) be prime \(q\)-exceptional divisors. Shrinking \(P\), there is a linear bijective map defined over \(\mathbb{Q}\),

\[
L : P \to P_W, \quad \Delta \mapsto L(\Delta),
\]
such that \(P_W \subset (\bigoplus_{i=1}^k \tilde{D}_i) \oplus (\bigoplus_{j=1}^l E_j)\) is a rational polytope, and

\[
(2.3.1) \quad K_W + L(\Delta) = p^*(K_X + \Delta) + E(\Delta),
\]
such that \(E(\Delta) \geq 0\) is \(p\)-exceptional and \((W, L(\Delta))\) is still klt for each \(\Delta \in P\). Let \(\Delta_{W,0} := L(\Delta_0)\). Run a \((K_W + \Delta_{W,0})\)-LMMP with scaling of an ample divisor over \(X'\), then it terminates with \((W'/X', \Delta_{W',0})\) by [BCHM10, Corollary 1.4.2]. As \((X'/S, \Delta'_0)\) is a weak log canonical model of \((X/S, \Delta_0)\), there exists a \(q\)-exceptional divisor \(E_0 \geq 0\) such that

\[
p^*(K_X + \Delta_0) = q^*(K_{X'} + \Delta'_0) + E_0.
\]

Hence

\[
K_W + \Delta_{W,0} = q^*(K_{X'} + \Delta'_0) + E(\Delta_0) + E_0.
\]

Because \(E(\Delta_0) + E_0 \geq 0\) is \(q\)-exceptional, we have

\[
K_{W'} + \Delta_{W',0} = q^*(K_{X'} + \Delta'_0),
\]
where \( q' : W' \to X' \) is the natural morphism. In particular,

\[
K_{W'} + \Delta_{W',0} \equiv 0/Z'.
\]

Let \( \theta : W \to W'/X \) be the natural map. Shrinking \( P \), we can assume that \( \theta \) is \((K_W + L(\Delta))\)-negative (see [BCHM10, Definition 3.6.1]) for each \( \Delta \in P \).

Step 3. By Step 1, \( P_W = \sqcup Q_i^o \) can be decomposed into a disjoint union of finitely many open rational polytopes such that for any \( B', D' \in Q_i^o \), if \( (W_i''/Z', B'') \) is a weak log canonical model of \((W'/Z', B')\) then \((W_i''/Z', D'')\) is a weak log canonical model of \((W'/Z', D')\), where \( B'', D'' \) are the strict transforms of \( B', D' \) respectively. In the sequel, we fix a \( W_i'' \) for each \( Q_i^o \).

We claim that after shrinking \( P_W \), for any \( \Delta_i \in Q_i^o \), \( K_{W''} + \Delta_i'' \) is nef over \( S \), where \( \Delta_i'' \) is the strict transform of \( \Delta_i \). Let \( \Delta_i \) be a vertex of \( Q_i^o \). By Theorem 2.3, \( (W_i''/Z', \Delta_i'') \) is semi-ample/\( Z' \). Let \( \tau : W_i'' \to T_i/Z' \) be the morphism such that \( K_{W_i''} + \Delta_i'' \sim_{\mathbb{Q}} \tau^*H_i/Z' \), where \( H_i \) is an ample/\( Z' \) \( \mathbb{Q} \)-Cartier divisor on \( T_i \). Hence, there is a \( \mathbb{Q} \)-Cartier divisor \( \Theta \) on \( Z' \) such that \( K_{W_i''} + \Delta_i'' = \tau^*H_i + (\mu \circ \tau)^*\Theta \), where \( \mu : T_i \to Z' \). Then \( t(H_i + \mu^*\Theta) + (1-t)\mu^*A \) is nef over \( S \) when \( t \in [0, t_0] \) for some rational number \( 0 < t_0 \ll 1 \). Note that

\[
t(K_{W_i''} + \Delta_i'') \sim E + (1-t)(K_{W_i''} + \Delta_i'') \sim R \tau^*(t(H_i + \mu^*\Theta) + (1-t)\mu^*A)/S,
\]

where \( \Delta_{W,0} \) is the strict transform of \( \Delta_{W,0} \) on \( W_i'' \). Replacing \( \Delta_i \) by \( t_0\Delta_i + (1-t_0)\Delta_{W,0} \) and repeating this process for each vertex of \( Q_i^o \), we obtain a polytope satisfying the desired claim.

Step 4. Let \( P := L^{-1}(P_W) \) be the polytope corresponding to \( P_W \) under the map \( L \). Let \( Q_i^o := L^{-1}(Q_i^o) \) be the corresponding open rational polytope. To complete the proof, it suffices to show that for any \( B, D \in Q_i^o \), if \( (Y/S, B_Y) \) is a weak log canonical model of \((X/S, B)\) then \((Y/S, D_Y) \) is a weak log canonical model of \((X/S, D)\).

Let \( r : V \to W, s : V \to W', \tilde{p} : V \to Y, \tilde{q} : V \to W'' \) be birational morphisms which commute with the existing maps. Moreover, \( r, s, \tilde{p}, \tilde{q} \) can be assumed to be log resolutions. By (2.3.1), for \( \Delta \in Q_i^o \),

\[
(2.3.2) \quad r^*(K_W + L(\Delta)) = r^*p^*(K_X + \Delta) + r^*(E(\Delta)).
\]

As \( \theta : W \to W' \) is \((K_W + L(\Delta))\)-negative, \((W_i''/S, L(\Delta_i''))\) is also a weak log canonical model of \((W/S, L(\Delta))\), where \( L(\Delta') \) is the strict transform of \( L(\Delta) \). Then there is a \( \tilde{q} \)-exceptional divisor \( F(\Delta) \geq 0 \) such that

\[
r^*(K_W + L(\Delta)) = \tilde{q}^*(K_{W''} + L(\Delta'')) + F(\Delta).
\]

Combining with (2.3.2), we have

\[
\tilde{q}^*(K_{W''} + L(\Delta'')) + F(\Delta) = r^*p^*(K_X + \Delta) + r^*(E(\Delta)).
\]

Hence \(-F(\Delta) + r^*(E(\Delta)) \) is nef over \( X \). As \( (p \circ r)_*(-F(\Delta) + r^*(E(\Delta))) \) is nef over \( X \), we have

\[
-F(\Delta) + r^*(E(\Delta)) \leq 0
\]

by the negativity lemma (see [KM98, Lemma 3.39]). Let

\[
(2.3.3) \quad r^*p^*(K_X + \Delta) = \tilde{p}^*(K_Y + \Delta_Y) + \Theta(\Delta),
\]
where $\Theta(\Delta)$ is $\check{p}$-exceptional. Hence
\[
\check{q}^*(K_{W'} + L(\Delta)'') + F(\Delta) = \check{p}^*(K_Y + \Delta_Y) + \Theta(\Delta) + r^*(E(\Delta)) .
\]
As $-F(\Delta) + \Theta(\Delta) + r^*(E(\Delta))$ is nef over $Y$ and
\[
\check{p}_*( -F(\Delta) + \Theta(\Delta) + r^*(E(\Delta)) ) = \check{p}_*(-F(\Delta)) \leq 0 ,
\]
we have $-F(\Delta) + \Theta(\Delta) + r^*(E(\Delta)) \leq 0$ by the negativity lemma.

Now we use that $(Y/S, B_Y)$ is a weak log canonical model of $(X/S, B)$. As $K_Y + B_Y$ is nef/S, $F(B) - \Theta(B) - r^*(E(B))$ is nef over $W''$. As $F(B)$ is $\check{q}$-exceptional, we have
\[
\check{q}_*(F(B) - \Theta(B) - r^*(E(B))) \leq 0 .
\]
By the negativity lemma again, we have $F(B) - \Theta(B) - r^*(E(B)) \leq 0$. Therefore, $F(B) - \Theta(B) - r^*(E(B))$ is linear for $\Delta \in Q_i^\circ$. Because $Q_i^\circ$ is open and $F(\Delta) - \Theta(\Delta) - r^*(E(\Delta)) \geq 0$ for each $\Delta \in Q_i^\circ$, we have $F(\Delta) - \Theta(\Delta) - r^*(E(\Delta)) = 0$ for each $\Delta \in Q_i^\circ$. Thus $\Theta(\Delta) = F(\Delta) - r^*(E(\Delta)) \geq 0$ and
\[
\check{q}^*(K_{W'} + L(\Delta)'') = \check{p}^*(K_Y + \Delta_Y)
\]
is nef/S for any $\Delta \in Q_i^\circ$. As $X \dashrightarrow Y$ does not extract divisors, $(Y, D_Y)$ is also a weak log canonical model of $(X, D)$ by (2.3.3). \qed

**Remark 2.5.** In the proof of Theorem 2.4, we use the existence of good minimal models in Step 1 and Step 3. In Step 1, this is needed to ensure that the statement of Theorem 2.4 holds true for lower dimensional polytopes. In Step 3, let $L(\Delta)'$ be the strict transform of $L(\Delta)$ on $W'$ (see (2.3.1)), then we need that $(F', \Delta'|_{F'})$ has a good minimal model, where $F'$ is a general fiber of $\pi \circ q' : W' \to Z'$. Therefore, the assumption that “good minimal models exist for effective klt pairs in dimension $\text{dim}(X/S)$” can be replaced by the following more precise form:

Suppose that $h : X \dashrightarrow X'/S$ is a birational contraction and $p : W \to X/S, q : W \to X'/S$ are birational morphisms such that $h = q \circ p^{-1}$. Suppose that $X'$ factors through a projective morphism $\pi : X' \to Z'/S$ between normal varieties. Let $\theta : W \dashrightarrow W'$ be a birational contraction and $q' : W' \to X'$ be a morphism such that $q' = \theta \circ q^{-1}$. For each $\Delta \in P$, let
\[
K_W + \check{\Delta} = p^*(K_X + \Delta) + E
\]
such that $(W, \check{\Delta})$ is klt and $E \geq 0$ is $\check{p}$-exceptional. Let $\check{\Delta}'$ be the strict transform of $\check{\Delta}$ on $W'$ and $F'$ be a general fiber of $\pi \circ q' : W' \to Z'$. Then $(F', \check{\Delta}'|_{F'})$ has a good minimal model.

This remark will be needed in the proof of Proposition 4.5 (3) and (4).

**Theorem 2.6.** Let $(X, \Delta) \to S$ be a klt Calabi-Yau fiber space. Assume that good minimal models of effective klt pairs exist in dimension $\text{dim}(X/S)$. Let $P \in \text{Eff}(X/S)$ be a rational polyhedral cone. Then $P$ is a finite union of open rational polyhedral cones $P = \cup_{i=0}^{m} P_i^\circ$ such that whenever

1. $B, D$ are effective divisors with $[B], [D] \in P_i^\circ$, and
2. $(X, \Delta + \epsilon B), (X, \Delta + \epsilon D)$ are klt for some $\epsilon \in \mathbb{R}_{>0}$,

then if $(Y/S, \Delta_Y + \epsilon B_Y)$ is a weak log canonical model of $(X/S, \Delta + \epsilon B)$, then $(Y/S, \Delta_Y + \epsilon D_Y)$ is a weak log canonical model of $(X/S, \Delta + \epsilon D)$. 
Proof. Let $\Delta = \sum_{i=1}^m c_i \Delta_i$ be the decomposition into irreducible components. Then

$$\{ \Theta \in \oplus_{i=1}^m \mathbb{R} \cdot \Delta_i \mid K_X + \Theta \equiv 0/S \}$$

is a subspace of $\oplus_{i=1}^m \mathbb{R} \cdot \Delta_i$ defined over $\mathbb{Q}$. Hence, there exits a $\mathbb{Q}$-Cartier divisor $\tilde{\Delta}$ such that $(X, \tilde{\Delta}) \to S$ is a klt Calabi-Yau fiber space. Let $\tilde{\epsilon} \in \mathbb{R}_{>0}$ such that $(X, \Delta + \tilde{\epsilon}B)$ is klt. By

$$K_X + \Delta + \epsilon B \equiv \frac{\epsilon}{\tilde{\epsilon}}(K_X + \tilde{\Delta} + \tilde{\epsilon}B)/S,$$

$(Y/S, \Delta_Y + \epsilon B_Y)$ is a weak log canonical model of $(X/S, \Delta + \epsilon B)$ iff $(Y/S, \tilde{\Delta}_Y + \tilde{\epsilon}B_Y)$ is a weak log canonical model of $(X/S, \tilde{\Delta} + \tilde{\epsilon}B)$. Therefore, replacing $\Delta$ by $\tilde{\Delta}$, we can assume that $\Delta$ is a $\mathbb{Q}$-Cartier divisor.

Let $\tilde{P} \subset \text{Eff}(X/S)$ be a rational polytope such that $\mathbb{R}_{>0} \cdot \tilde{P} = P$. We can choose $\tilde{P}$ such that $0 \notin \tilde{P}$. Let $\Delta_j \geq 0, j = 1 \ldots k$ be effective $\mathbb{Q}$-Cartier divisors such that $\tilde{P} = \text{Conv}(\Delta_j \mid j = 1 \ldots k)$. Replacing $\Delta_j$ by $\epsilon \Delta_j$ for some $\epsilon \in \mathbb{R}_{>0}$, we can assume that $(X, \Delta + \Delta_j)$ is klt for each $j$. Let

$$\tilde{P} + \Delta = \bigcup_{i=1}^m (Q_i^\circ + \Delta)$$

be the decomposition as in Theorem 2.4. For each open rational polytope $Q_i^\circ + \Delta$, and $\Theta_1 + \Delta, \Theta_2 + \Delta \in Q_i^\circ + \Delta$, if $(Y/S, \Theta_1Y + \Delta_Y)$ is a weak log canonical model of $(X, S, \Theta_1 + \Delta)$, then $(Y/S, \Theta_2Y + \Delta_Y)$ is a weak log canonical model of $(X, S, \Theta_2 + \Delta)$. Let $P_0^\circ$ be the image of the open rational polyhedral cone $\mathbb{R}_{>0} \cdot Q_i^\circ$ in $N^1(X/S)$. Set $P_0^\circ = \{0\}$, then $P = \bigcup_{i=0}^m P_i^\circ$. Note that this union may not be disjoint.

The claim certainly holds true for $P_0^\circ$. For effective divisors $B, D$ with $[B], [D] \in P_i^\circ, i > 0$, there exist $\Delta_B, \Delta_D \in Q_i^\circ$ such that

$$[B] = t[\Delta_B], \quad [D] = s[\Delta_D] \quad \text{for some} \ t, s \in \mathbb{R}_{>0}.$$ 

By $K_X + \Delta \equiv 0/S$,

$$K_X + \Delta + \Delta_B \equiv \frac{1}{\epsilon t}(K_X + \Delta + \epsilon B)/S$$

(2.3.4)

$$K_X + \Delta + \Delta_D \equiv \frac{1}{\epsilon s}(K_X + \Delta + \epsilon D)/S.$$ 

Therefore, $(Y/S, \Delta_Y + \epsilon B_Y)$ is a weak log canonical model of $(X/S, \Delta + \epsilon B)$ iff $(Y/S, \Delta_Y + \Delta_B, \Delta_Y)$ is a weak log canonical model of $(X/S, \Delta + \Delta_B)$. By Theorem 2.4, this implies that $(Y/S, \Delta_Y + \Delta_D, \Delta_Y)$ is a weak log canonical model of $(X/S, \Delta + \Delta_D)$. Hence $(Y/S, \Delta_Y + \epsilon D_Y)$ is a weak log canonical model of $(X/S, \Delta + \epsilon D)$ by (2.3.4) again. \hfill \Box

Theorem 2.7 ([Sho96, §6.2. First main theorem]). Let $(X, \Delta) \to S$ be a klt Calabi-Yau fiber space. Let $P \subset \text{Eff}(X/S)$ be a rational polyhedral cone. Then

$$P_N := P \cap \overline{\text{Amp}(X/S)}$$

is a rational polyhedral cone.

Proof. Let $D_j, 1 \leq j \leq k$ be effective $\mathbb{Q}$-Cartier divisors on $X$ such that $P = \text{Cone}([D_j] \mid 1 \leq j \leq k)$. Replacing $D_j$ by $\epsilon D_j$ for some $\epsilon \in \mathbb{Q}_{>0}$, we can assume that $(X, \Delta + D_j)$ is klt for each $j$. Then

$$N = \{ D \in \oplus_{j=1}^k [0, 1] D_j \mid D \text{ is nef over } S \}.$$
is a rational polytope by [Sho96, §6.2. first main theorem] (also see [Bir11, Proposition 3.2 (3)]). The image $\mathcal{N}$ of $\mathcal{N}$ in $N^1(X/S)_{\mathbb{R}}$ is still a rational polytope. By the construction, $P_{\mathcal{N}} = \text{Cone}(\mathcal{N})$.

Thus $P_{\mathcal{N}}$ is a rational polyhedral cone.

\section{3. Geometry of convex cones}

Let $V(\mathbb{Z})$ be a lattice and $V(\mathbb{Q}) := V(\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$, $V := V(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$. A cone $C \subset V$ is non-degenerate if it does not contain an affine line. This is equivalent to say that its closure $\overline{C}$ does not contain a non-trivial vector space.

In the following, we assume that $\Gamma$ is a group and $\rho : \Gamma \to \text{GL}(V)$ is a group homomorphism. The group $\Gamma$ acts on $V$ through $\rho$. For $\gamma \in \Gamma$ and $x \in V$, we write $\gamma \cdot x$ or $\gamma x$ for the action. For a set $S \subset V$, set $\Gamma \cdot S := \{ \gamma \cdot x \mid \gamma \in \Gamma, x \in S \}$. Suppose that this action leaves a convex cone $C$ and some lattice in $V(\mathbb{Q})$ invariant. We assume that $\dim C = \dim V$. The following definition slightly generalizes [Loo14, Proposition-Definition 4.1].

\begin{definition}
Under the above notation and assumptions.

(1) Suppose that $C \subset V$ is an open convex cone (may be degenerate). Let $C_+ := \text{Conv}(\overline{C} \cap V(\mathbb{Q}))$ be the convex hull of rational points in $\overline{C}$.

(2) We say that $(C_+, \Gamma)$ is of polyhedral type if there is a polyhedral cone $\Pi \subset C_+$ such that $\Gamma \cdot \Pi \supset C$.

\end{definition}

\begin{remark}
Recall that a polyhedral cone is closed by definition (see Section 2.1).

\end{remark}

\begin{proposition}[[Loo14, Proposition-Definition 4.1]].
Under the above notation and assumptions. If $C$ is non-degenerate, then the following conditions are equivalent:

(1) there exists a polyhedral cone $\Pi \subset C_+$ with $\Gamma \cdot \Pi = C_+$;

(2) there exists a polyhedral cone $\Pi \subset C_+$ with $\Gamma \cdot \Pi \supset C$.

Moreover, in case (2), we necessarily have $\Gamma \cdot \Pi = C_+$.

\end{proposition}

\begin{definition}
Let $\rho : \Gamma \hookrightarrow \text{GL}(V)$ be an injective group homomorphism and $C \subset V$ be a cone (may not necessarily be open). Let $\Pi \subset C$ be a (rational) polyhedral cone. Suppose that $\Gamma$ acts on $C$. Then $\Pi$ is called a weak (rational) polyhedral fundamental domain for $C$ under the action $\Gamma$ if

(1) $\Gamma \cdot \Pi = C$, and

(2) for each $\gamma \in \Gamma$, either $\gamma \Pi = \Pi$ or $\gamma \Pi \cap \text{Int}(\Pi) = \emptyset$.

Moreover, for $\Gamma_{\Pi} := \{ \gamma \in \Gamma \mid \gamma \Pi = \Pi \}$, if $\Gamma_{\Pi} = \{ \text{id} \}$, then $\Pi$ is called a (rational) polyhedral fundamental domain.

\end{definition}

\begin{lemma}[[Loo14, Theorem 3.8 & Application 4.14]].
Under the notation and assumptions of Definition 3.1, suppose that $\rho : \Gamma \hookrightarrow \text{GL}(V)$ is injective. Let $(C_+, \Gamma)$ be of polyhedral type with $C$ non-degenerate. Then under the action of $\Gamma$, $C_+$ admits a rational polyhedral fundamental domain.

\end{lemma}
Lemma 3.7. \( W \subset V \) be the maximal \( \Gamma \)-invariant, and thus \( \dim \) \( \text{Int}(C^*) = \dim V \).

The group \( \Gamma \) naturally acts on \( V^* \). In fact, for \( x, y \in V^* \), \( \gamma \cdot x, y \) is defined by the relation \( \langle x, \gamma \cdot y \rangle = \langle \gamma \cdot x, y \rangle \) for all \( x \in V \). It is straightforward to check that this action gives an injective group homomorphism \( \Gamma \rightarrow \text{GL}(V^*) \) which leaves \( C^* \) and a lattice in \( V^*(\mathbb{Q}) \) invariant. Therefore, by [Loo14, Theorem 3.8], \( \Gamma \) acts properly discontinuously on \( \text{Int}(C^*) \).

By [Loo14, Application 4.14], for each \( \xi \in \text{Int}(C^*) \cap V(\mathbb{Q})^* \), there is a rational polyhedral cone \( \sigma \) associated with \( \xi \), such that \( \sigma \) is a rational polyhedral fundamental domain for the action of \( \Gamma \) on \( C_+ \) whenever the stabilizer subgroup \( \Gamma_\xi = \{1\} \). It suffices to find such \( \xi \) to complete the proof. As \( \Gamma \) acts properly discontinuously on \( \text{Int}(C^*) \), for any polyhedral cone \( P \subset \text{Int}(C^*) \) such that \( \dim P = \dim \text{Int}(C^*) = \dim V \), the set

\[
\{ \gamma \in \Gamma \mid \gamma P^0 \cap P^0 \neq \emptyset \}
\]

is a finite set. Then a general \( \xi \in P^0 \cap V^*(\mathbb{Q}) \) satisfies \( \Gamma_\xi = \{1\} \).

The following consequence of having a polyhedral fundamental domain is well-known (see [Loo14, Corollary 4.15] or [Mor15, (4.7.7) Proposition])

**Theorem 3.6.** Let \( \rho : \Gamma \rightarrow \text{GL}(V) \) be an injective group homomorphism and \( C \subset V \) be a cone. Suppose that \( C \) is \( \Gamma \)-invariant. If \( C \) admits a polyhedral fundamental domain under the action of \( \Gamma \), then \( \Gamma \) is finitely presented.

For a possibly degenerate open convex cone \( C \), let \( W \subset \tilde{C} \) be the maximal \( \mathbb{R} \)-linear vector space. We say that \( W \) is defined over \( \mathbb{Q} \) if \( W = W(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \) where \( W(\mathbb{Q}) = W \cap V(\mathbb{Q}) \). In this case, \( V/W = (V(\mathbb{Q})/W(\mathbb{Q})) \otimes_{\mathbb{Q}} \mathbb{R} \) has a nature lattice structure, and we denote everything in \( V/W \) by \( \langle \cdot, \cdot \rangle \). For example, \( (C_+) \) is the image of \( C_+ \) under the projection \( p : V \rightarrow V/W \). By the maximality, \( W \) is \( \Gamma \)-invariant, and thus \( V/W, \tilde{C} \) admit natural \( \Gamma \)-actions.

**Lemma 3.7.** Under the above notation and assumptions,

1. \( \tilde{\tilde{C}} = \tilde{\tilde{C}} \),
2. \( (\tilde{C})_+ = (\tilde{C}_+) \), which is denoted by \( \tilde{C}_+ \), and
3. if \( (C_+, \Gamma) \) is of polyhedral type, then \( (\tilde{C}_+, \Gamma) \) is still of polyhedral type. More precisely, if \( \Pi \subset (\tilde{C}_+) \) is a polyhedral cone with \( \Gamma \cdot \Pi \supset C \), then \( \Pi \subset \tilde{C}_+ \) and \( \Gamma \cdot \Pi \supset \tilde{C} \).

**Proof.** For (1), \( \tilde{\tilde{C}} \supset \tilde{\tilde{C}} \) trivially holds. The converse does not hold for an arbitrary linear projection (see [Loo14, Remark 2.5]). In our case, let \( j : V/W \rightarrow V \) be a splitting of \( p : V \rightarrow V/W \). For \( x \in \tilde{\tilde{C}} \), let \( \tilde{a}_n \rightarrow x \) with \( a_n \in \tilde{C} \). Then \( a_n - j(\tilde{a}_n) \in W \) as \( p(a_n - j(\tilde{a}_n)) = 0 \). By \( W \subset \tilde{C} \), \( j(\tilde{a}_n) \in \tilde{C} \). Moreover, as \( \{\tilde{a}_n\}_{n \in \mathbb{N}} \) converges and \( j \) is continuous, \( \{j(\tilde{a}_n)\}_{n \in \mathbb{N}} \) converges to \( \alpha \in \tilde{C} \). Thus \( \tilde{a}_n = p(j(\tilde{a}_n)) \rightarrow p(\alpha) \). Hence \( x = p(\alpha) \in \tilde{\tilde{C}} \).
For (2), we first show
\[ \tilde{C} \cap (V/W)(\mathbb{Q}) = (\tilde{C} \cap V(\mathbb{Q})). \]
The “\( \supset \)” follows from definition. For the converse, let \( \tilde{a} \in \tilde{C} \) be a rational point in \( V/W \). Then by \( W \subset \tilde{C} \), we can assume that \( a \in \tilde{C} \) is a rational point in \( V \). This gives “\( \subset \)”. Next, we show
\[ \text{Conv} \left( \tilde{C} \cap V(\mathbb{Q}) \right) = \text{Conv}(\tilde{C} \cap V(\mathbb{Q})). \]
As the image of a convex set is still convex, we have “\( \subset \)”.

For a set \( S \subset V \), we have
\[ \text{Conv}(S) = \left\{ \sum_{i \in I} \lambda_i s_i \mid \sum_{i \in I} \lambda_i = 1, \lambda_i > 0, |I| < \infty, s_i \in S \right\}. \]
For \( a \in \text{Conv}(\tilde{C} \cap V(\mathbb{Q})) \), take finitely many \( s_i \in \tilde{C} \cap V(\mathbb{Q}) \), and \( \lambda_i > 0, \sum_{i \in I} \lambda_i = 1 \), so that \( a = \sum \lambda_i s_i \). Thus \( \tilde{a} = \sum \lambda_i \tilde{s}_i \in \text{Conv} \left( \tilde{C} \cap V(\mathbb{Q}) \right) \). This shows the converse inclusion.

Finally, by (1),
\[ (\tilde{C})_+ = \text{Conv}(\tilde{C} \cap (V/W)(\mathbb{Q})) = \text{Conv}(\tilde{C} \cap (V/W)(\mathbb{Q})). \]
Then (2) follows from
\[ \text{Conv}(\tilde{C} \cap (V/W)(\mathbb{Q})) = \text{Conv} \left( \tilde{C} \cap V(\mathbb{Q}) \right) = \text{Conv}(\tilde{C} \cap V(\mathbb{Q})) = (\tilde{C}_+). \]

For (3), let \( \Pi \subset C_+ \) be a polyhedral cone such that \( \Gamma \cdot \Pi \supset C \). By (2), \( \tilde{\Pi} \subset \tilde{C}_+ \). Moreover, \( \Gamma \cdot \tilde{\Pi} \supset \tilde{C} \).

**Proposition 3.8.** Let \( (C_+, \Gamma) \) be of polyhedral type. Let \( W \subset \tilde{C} \) be the maximal vector space. Suppose that \( W \) is defined over \( \mathbb{Q} \). Then there is a rational polyhedral cone \( \Pi \subset C_+ \) such that \( \Gamma \cdot \Pi = C_+ \), and for each \( \gamma \in \Gamma \), either \( \gamma \Pi \cap \text{Int}(\Pi) = \emptyset \) or \( \gamma \Pi = \Pi \). Moreover,
\[ \{ \gamma \in \Gamma \mid \gamma \Pi = \Pi \} = \{ \gamma \in \Gamma \mid \gamma \text{ acts trivially on } V/W \}. \]

**Proof.** By Lemma 3.7 (3), \( (\tilde{C}_+, \Gamma) \) is still of polyhedral type. By Lemma 3.5, there is a rational polyhedral cone \( \tilde{\Pi} \) as a fundamental domain of \( \tilde{C}_+ \) under the action of \( \tilde{\rho}(\Gamma) \), where \( \tilde{\rho} : \Gamma \rightarrow \text{GL}(V/W) \) is the natural group homomorphism. By Lemma 3.7 (2), let \( \Pi' \subset C_+ \) be a rational polyhedral cone such that \( p(\Pi') = \tilde{\Pi} \), where \( p : V \rightarrow V/W \). Let \( \Pi := \Pi' + W \) which is a rational polyhedral cone. As \( \gamma(\Pi' + W) = (\gamma b + w) + W \), by Lemma 3.7 (2), we have \( \Gamma \cdot \Pi = C_+ \).

If \( \gamma \tilde{\Pi} \cap \text{Int}(\tilde{\Pi}) = \emptyset \), then \( \gamma \Pi \cap \text{Int}(\Pi) = \emptyset \) as \( \text{Int}(\Pi) \) maps to \( \text{Int}(\tilde{\Pi}) \). If \( \gamma \tilde{\Pi} = \tilde{\Pi} \), then we claim that \( \gamma \Pi = \Pi \). In fact, for some \( a \in \Pi' \), we have \( \gamma \cdot a = b + w \) for some \( b \in \Pi' \), \( w \in W \). Thus \( \gamma \Pi \subset \Pi \). Similarly, \( \gamma^{-1} \Pi \subset \Pi \). This shows the claim. Moreover, \( \gamma \Pi = \Pi \) iff \( \gamma \) acts trivially on \( \tilde{\Pi} \) if \( \gamma \) acts trivially on \( V/W \) because \( \tilde{\Pi} \) is a fundamental domain under the action of \( \tilde{\rho}(\Gamma) \). \[ \square \]
4. Generic properties of fibrations and structures of cones

Let \((X, \Delta) \to S\) be a fiber space. Let \(K := K(S)\) be the field of rational functions on \(S\) and \(\bar{K}\) be the algebraic closure of \(K\). Then \(X_\bar{K} := X \times_S \text{Spec } \bar{K}\) is the geometric fiber of \(f\). Set \(\Delta_\bar{K} := \Delta \times_S \text{Spec } \bar{K}\).

**Proposition 4.1.** Let \(f : X \to S\) be a fibration.

1. If \((X, \Delta)\) has klt singularities, then \((X_\bar{K}, \Delta_\bar{K})\) still has klt singularities. Moreover, if \(f : (X, \Delta) \to S\) is a klt Calabi-Yau fiber space, then \((X_\bar{K}, \Delta_\bar{K})\) is a klt Calabi-Yau pair over \(\text{Spec } \bar{K}\).

2. For a finite base change \(h : T \to S\) between varieties, let \(U \subset S\) be a non-empty open set and \(V = h^{-1}(U)\). Then we can shrink \(U\) such that \(X_V := X \times_U V\) satisfies the following properties.

   If \((X, \Delta)\) has klt singularities, then \((X_V, \Delta_V)\) still has klt singularities, where \(\Delta_V := \Delta \times_T V\). Moreover, if \(f : (X, \Delta) \to S\) is a klt Calabi-Yau fiber space, then \((X_V, \Delta_V)\) has klt singularities and \(K_{X_V} + \Delta_V \sim_R 0/V\).

**Proof.** For (1), we first show that \(X_\bar{K}\) is normal. This is a local statement for both source and target, hence we can assume that \(f : \text{Spec } A \to \text{Spec } B\). The collection of affine open sets \(\{\text{Spec } B_i \subset \text{Spec } B \mid i\}\) forms a direct system such that \(\varinjlim B_i = K\). Then \(A \otimes_B \varinjlim B_i = \varinjlim(A \otimes_B B_i)\). As \(A \otimes_B B_i\) is normal, by [Sta22, Lemma 037D], \(A \otimes_B K\) is also normal. Then \(X_\bar{K} = X_K \otimes_K \bar{K}\) is normal by [Sta22, Lemma 0C3M]. Let \(v : \text{Spec } \bar{K} \to S, u : X_\bar{K} \to X\) and \(\bar{f} : X_\bar{K} \to \text{Spec } \bar{K}\) be natural morphisms. Then \(\bar{f}_*(u^*O_X) = v^*(f_*O_X)\). By \(u^*O_X = O_{X_\bar{K}}\) and \(f_*O_X = O_S\), we see that \(X_\bar{K}\) is connected. Hence \(X_\bar{K}\) is an irreducible normal variety over \(\bar{K}\).

Next, we show that \(X_\bar{K}\) has klt singularities. Let \(X_{\text{reg}}\) be the smooth part of \(X\). Shrinking \(S\), we can assume that \(S\) is smooth and \(X_{\text{reg}} \to S\) is a smooth morphism. Then the sequence
\[
0 \to f^*\Omega_S \to \Omega_{X_{\text{reg}}} \to \Omega_{X_{\text{reg}}/S} \to 0
\]
is exact. Let \(r = \dim(X/S)\). By
\[
(\Omega_{X_{\text{reg}}/S})_\bar{K} = (\Omega_{(X_{\text{reg}})_\bar{K}}) \text{ and } O_X(K_{X_{\text{reg}}/S}) = \wedge^r\Omega_{X_{\text{reg}}/S},
\]
we have
\[
(K_{X_{\text{reg}}/S})_\bar{K} \sim K_{(X_{\text{reg}})_\bar{K}}.
\]
By \(\text{codim}(X \setminus X_{\text{reg}}) \geq 2\), we have
\[
(K_{X/S})_\bar{K} - K_{X_\bar{K}} \sim 0.
\]
Take a log resolution \(g : \tilde{X} \to X\), then
\[
K_{\tilde{X}/S} + \tilde{\Delta} = g^*(K_{X/S} + \Delta)
\]
with coefficients of \(\tilde{\Delta} < -1\). The natural morphism \(\bar{g} : \tilde{X}_\bar{K} \to X_\bar{K}\) is also a log resolution and the above argument implies that
\[
K_{\tilde{X}_\bar{K}} + \tilde{\Delta}_\bar{K} = \bar{g}^*(K_{X_\bar{K}} + \Delta_\bar{K}).
\]
As coefficients of \(\tilde{\Delta}_\bar{K} < -1\), \((X_\bar{K}, \Delta_\bar{K})\) has klt singularities.

When \(f : (X, \Delta) \to S\) is a klt Calabi-Yau fiber space. We only need to note that \(K_X + \Delta \sim_R 0/S\) implies that \(K_{X_\bar{K}} + \Delta_\bar{K} \sim_R 0\).
For (2), shrinking $U$, we can assume that $V \to S$ is étale. We first show that $X_V$ is normal. Note that $\phi : X_V \to X$ is also étale. Let $x \in X_V$ be a point (not necessarily a closed point) and $y = \phi(x)$. Set $\mathcal{O}_x := \mathcal{O}_{X_V,x}$ (resp. $\mathcal{O}_y := \mathcal{O}_{X_U,y}$). Let $\hat{\mathcal{O}}_x$ (resp. $\hat{\mathcal{O}}_y$) be the completion with respect to the maximal ideal. By [Har77, III, Exercise 10.4],

$$\hat{\mathcal{O}}_y \otimes_{k(y)} k(x) \simeq \hat{\mathcal{O}}_x,$$

where $k(y) \subset \hat{\mathcal{O}}_y$ and $k(x) \subset \hat{\mathcal{O}}_x$ are fields of representatives. Note that $k(y)$ and $k(x)$ are of characteristic zero. We claim that $\hat{\mathcal{O}}_x$ is normal. In fact, as $X$ has klt singularities, $X$ is Cohen-Macaulay. Hence $\hat{\mathcal{O}}_y$ is Cohen-Macaulay by [Sta22, Lemma 07NX]. In particular, it satisfies Serre’s condition $S_2$. As $\hat{\mathcal{O}}_y$ is certainly regular in codimension 1, it is normal. Then [Sta22, Lemma 0C3M] shows that $\hat{\mathcal{O}}_y \otimes_{k(y)} k(x)$ is normal. Thus $\mathcal{O}_x$ is normal by [Sta22, Lemma 0FZ]. This shows that $X_V$ is normal.

Let $f_V : X_V \to V$ be the natural map. By $V \to U$ flat and $f_*\mathcal{O}_X = \mathcal{O}_S$, $(f_V)_*\mathcal{O}_{X_V} = \mathcal{O}_V$, and thus $H^0(X_V, \mathcal{O}_{X_V}) = H^0(V, \mathcal{O}_V)$ is an integral domain. This shows that $X_V$ is an irreducible normal variety.

Let $\pi : W \to X$ be a log resolution of $(X, \Delta)$ with natural morphisms $\pi_V : W_V \to X_V$ and $\tilde{\phi} : W_V \to W$. Set $K_W + \Delta_W := \pi^*(K_X + \Delta)$ and $K_{W_V} + \Delta_{W_V} := \pi_V^*(K_{X_V} + \Delta_V)$. As $\tilde{\phi}$ is étale, we have

$$K_{X_V} + \Delta_V = \tilde{\phi}^*(K_X + \Delta).$$

Therefore, $K_{W_V} + \Delta_{W_V} = \tilde{\phi}^*(K_W + \Delta_W)$. By $(X, \Delta)$ klt, coefficients of $\Delta_W$ are $< -1$. As $\tilde{\phi}$ is étale, $\Delta_W = \phi^*\Delta_W$, and the coefficients of $\Delta_{W_V}$ are $< -1$. As $W_V \to X_V$ is a log resolution of $(X_V, \Delta_V)$, $(X_V, \Delta_V)$ is still klt.

When $f : (X, \Delta) \to S$ is a klt Calabi-Yau fiber space, then $K_X + \Delta \sim_R 0/S$ implies that $K_{X_V} + \Delta_V = \phi^*(K_X + \Delta) \sim_R 0/V$. \hfill $\square$

**Remark 4.2.** Even if $X$ is $\mathbb{Q}$-factorial, $X_K$ and $X_V$ may not be $\mathbb{Q}$-factorial.

**Proposition 4.3.** Let $f : X \to S$ be a fibration. For any sufficiently small open set $U \subset S$, there exists a natural inclusion

$$N^1(X_U/U)_{\mathbb{R}} \hookrightarrow N^1(X_K)_{\mathbb{R}}, \quad [D] \mapsto [D_K].$$

**Proof.** First, we show that there is a natural map

$$N^1(X/S)_{\mathbb{R}} \to N^1(X_K)_{\mathbb{R}}, \quad [D] \mapsto [D_K].$$

Because $\{D \in \text{Pic}(X/S)_{\mathbb{R}} \mid D \equiv 0/S\}$ is defined over $\mathbb{Q}$, we only need to show that if a Cartier divisor $D \equiv 0/S$, then $D_K \equiv 0$. Replacing $X$ by a resolution $\tilde{X} \to X$ and $D, D_K$ by their pullbacks on $\tilde{X}, \tilde{X}_K$ respectively, we can assume that $X$ is smooth.

Let $C_K \to X_K$ be a smooth curve, we will show $D_K \cdot C_K = 0$. By definition, this is to show that the coefficient of $m$ in the polynomial $\chi(C_K, mD_K)$ is 0. Let $C$ be a spreading out of $C_K$ over a variety $T$ such that $h : T \to S$ is a finite morphism (see [Poo17, Chapter 3.2]). We can assume that $C$ is smooth over $T$. Shrinking $S$, we can assume that $S$ is smooth and $h$ is étale. By Proposition 4.1 (2), we can assume that $X_T$ is normal. Shrinking $T$ further, we may assume that $T = \text{Spec} A$ is affine. Moreover, as $\text{Spec} \tilde{K} \to T$ is flat, [Har77, III Prop 9.3] implies that

$$H^i(C, mD_T) \otimes_A \tilde{K} \simeq H^i(C_K, mD_K).$$
Thus
\[
\chi(C, mD_K) = \sum (-1)^k \dim_K H^i(C, mD_T) \otimes_A K.
\]
Shrinking $T$, by [Har77, III Prop 12.9], we have
\[
H^i(C, mD_T) \otimes_A k(t) \simeq H^i(C_t, mD_t),
\]
where $t \in T$ is a closed point. [Har77, III Prop 12.9] also implies that $H^i(C, mD_T)$ is a free $A$-module. Thus
\[
\dim H^i(C, mD_T) \otimes_A K = \dim H^i(C, mD_T) \otimes_A k(t) = \dim H^i(C_t, mD_t).
\]
Let $\phi : X_T \to X$ be the natural morphism. Then $D_T \cdot C_t = \phi^* D \cdot C_t = D \cdot \phi_* C_t = 0$. Therefore, the coefficient of $m$ in
\[
\chi(C, mD_K) = \chi(C_t, mD_t)
\]
is 0. This shows that $D_K \equiv 0$.

Next, to get the desired inclusion for any sufficiently small open set, it suffices to find one such open set.

Suppose that $D_K \equiv 0$, we want to find $U$ such that $D \equiv 0/U$ (this $U$ may depend on $D$). Let $X \to X$ be a resolution, and $\tilde{D}$ be the pullback of $D$. We have $\tilde{D}_K \equiv 0$ on $\tilde{X}_K$. If $\tilde{D} \equiv 0/U$, then $D \equiv 0/U$. Therefore, we can assume that $X$ is smooth.

By [Kle05, Theorem 9.6.3 (a) and (b)], there exists an $m \in \mathbb{Z}_{>0}$ such that $mD_K$ is algebraically equivalent to $O_{X_K}$. That is, there exist connected $K$-schemes of finite type $\tilde{B}_i, 1 \leq i \leq n$, invertible sheaves $\tilde{M}_i$ on $X_{\tilde{B}_i}$ and closed points $s_i, t_i$ of $\tilde{B}_i$ such that
\[
O(mD_{\tilde{B}_i})_{s_i} \simeq \tilde{M}_{1,s_i}, \tilde{M}_{1,t_i} \simeq \tilde{M}_{2,s_2}, \ldots, \tilde{M}_{n-1,t_{n-1}} \simeq \tilde{M}_{n,s_n}, \tilde{M}_{n,t_n} \simeq O_{X_K,t_n}
\]
(see [Kle05, Definition 9.5.9]). Moreover, connecting $s_i, t_i$ by the image of a smooth curve, we can further assume that $\tilde{B}_i$ is a smooth curve. All the above schemes, sheaves, and isomorphisms are defined in a finite extension of $K$.

By generic smoothness, shrinking $S$ to $U$, we can assume that $X_U \to U$ is smooth. There is a finite morphism $T \to U$ such that every objects and relations mentioned above over $\tilde{K}$ is defined on $X_T \to T$. In particular, there are dominant morphisms of finite type $B_i \to T, 1 \leq i \leq n$ with sections $\tilde{s}_i, \tilde{t}_i : T \to B_i$, and invertible sheaves $M_i$ on $X_{B_i}$ such that
\[
O(mD)_{\tilde{s}_i(T)} \simeq M_{1,\tilde{s}_i(T)}, M_{1,\tilde{t}_i(T)} \simeq M_{2,\tilde{s}_2(T)}, \ldots
\]
\[
\cdots, M_{n-1,\tilde{t}_{n-1}(T)} \simeq M_{n,\tilde{s}_n(T)}, M_{n,\tilde{t}_n(T)} \simeq O_{\tilde{t}_n(T)},
\]
where $O(mD)_{\tilde{s}_i(T)} = \tau_i^* O(mD)|_{\theta_i^{-1}(\tilde{s}_i(T))}$ with $\tau_i : X_{B_i} \to X_T \to X_U, \theta_1 : X_{B_1} \to B_1$ (other sheaves are defined similarly). Note that $X_T \to T$ is isomorphic to both $\theta_i^{-1}(\tilde{s}_i(T)) \to \tilde{s}_i(T)$ and $\theta_i^{-1}(\tilde{t}_i(T)) \to \tilde{t}_i(T)$, where $\theta_i : X_{B_i} \to B_i$. Shrinking $U, T$ further, we can assume that each $B_i$ is also smooth.

We will show that $mg^* D \equiv 0/T$, where $g : X_T \to X_U$. Because the intersection is taken in the singular cohomology groups, this can be checked in the analytic topology. First, as $B_i$ is smooth, shrinking $T$ (hence also $U$), we can assume that $B_i \to T$ is smooth. As $X_T \to T$ is a smooth morphism, $X_{B_i} \to B_i$ is also a smooth morphism between smooth varieties. Thus $X_{B_i} \to B_i$ is locally trivial in the analytic topology by Ehresmann’s theorem. Let
Let \( \ell \in \theta^{-1}_1(\tilde{s}_1(T)) \) be a curve which maps to a point on \( \tilde{s}_1(T) \). Let \( \ell' \subset \theta^{-1}_1(\tilde{t}_1(T)) \) be a manifold which is a deformation of \( \ell \) in the analytic topology (we do not need \( \ell' \) to be an algebraic curve). By induction on \( i \), it is enough to show

\[
M_{1,\tilde{s}_1(T)} \cdot \ell = M_{1,\tilde{s}_1(T)} \cdot \ell'.
\]

As \( M_{1,\tilde{s}_1(T)} \cdot \ell = M_1 \cdot \ell \) and \( M_{1,\tilde{s}_1(T)} \cdot \ell' = M_1 \cdot \ell' \), the desired result follows. Hence \( D \equiv 0/U \).

To obtain an open set \( U \) which is independent of divisors, we can use one of the following two approaches:

(A) By \( \dim N^1(X/S)_{\mathbb{R}} < \infty \), we have

\[
\ker(N^1(X/S)_{\mathbb{R}} \rightarrow N^1(X_{\bar{K}})_{\mathbb{R}}) < \infty.
\]

Let \([D_1], \ldots, [D_e]\) be a basis of \( \ker(N^1(X/S)_{\mathbb{R}} \rightarrow N^1(X_{\bar{K}})_{\mathbb{R}}) \). By the above construction, there exists an open set \( U_i \) such that \( D_i \equiv 0/U_i \). Then \( U := \cap_{i=1}^e U_i \) satisfies the desired property.

(B) Replacing \( X \) by a resolution, it is enough to show the claim for smooth \( X \). Shrinking \( U \), we can assume that \( X_U \rightarrow U \) is smooth. We show that \( U \) satisfies the desired property. Let \( D \) be any divisor such that \( D_{\bar{K}} \equiv 0 \). By the above construction, there exists an open set \( V \subset U \) such that \( D_V \equiv 0/V \). We claim that \( D \equiv 0/U \). It is enough to show that for any curve \( \ell \) such that \( \ell \) maps to a point in \( U - V \), we have \( D \cdot \ell = 0 \). By Ehresmann’s theorem, \( \ell \) can be deformed to a complex manifold \( \ell' \) in the analytic topology such that \( \ell' \) maps to a point in \( V \) under \( f \). Thus \( D \cdot \ell' = D \cdot \ell \). By the dual form of the Lefschetz theorem on \((1,1)\)-classes, there exists an algebraic curve \( \ell'' \) such that \( \ell'' \) maps to a point in \( V \) under \( f \) and \( D \cdot \ell'' = D \cdot \ell'' \). Therefore, \( D \cdot \ell = D \cdot \ell'' = 0 \).

**Proposition 4.4** ([Li23, Proposition 3.8]). Let \((X, \Delta) \rightarrow S\) be a klt Calabi-Yau fiber space. Let \( W \) and \( W' \) be the maximal vector spaces in \( \text{Eff}(X/S) \) and \( \text{Mov}(X/S) \), respectively. Then \( W \) and \( W' \) are defined over \( \mathbb{Q} \).

We thank Chen Jiang for pointing out that \( W \) is defined over \( \mathbb{Q} \).

We can describe \( W \) concretely when \( R^1 f_* \mathcal{O}_X = 0 \).

**Proposition 4.5.** Let \( f : (X, \Delta) \rightarrow S \) be a klt Calabi-Yau fiber space. Assume that \( R^1 f_* \mathcal{O}_X = 0 \). Then the following results hold true.

1. There is a natural surjective linear map

\[
r : N^1(X/S)_{\mathbb{R}} \rightarrow N^1(X_{U}/U)_{\mathbb{R}} \quad [D] \mapsto [D|_U].
\]

When \( U \) is sufficiently small, we have

\[
\ker(r) = \text{Span}_{\mathbb{R}} \{ [D] \mid \text{Supp} D \subset \text{Supp}(X - X_U) \}.
\]

(4.0.1)

2. The maximal vector space \( W \subset \text{Eff}(X/S) \) is generated by divisors in \( \ker(r) \). In particular, \( W \subset \text{Eff}(X/S) \).

3. If \( S \) is \( \mathbb{Q} \)-factorial, then \( \text{Mov}(X/S) \) is non-degenerate.

**Proof.** (1) Note that \( r([D]) = [D|_U] \) is well-defined. If \( D_U \) is a divisor on \( X_U \) such that \( D_U = \sum c_i B_i \) is the decomposition into irreducible components, then \( \overline{D_U} := \sum c_i B_i \) is a divisor on \( X \) such that \( (\overline{D_U})|_U = D_U \). Hence \( r \) is surjective.
Let \( \bar{f} : X_{\bar{K}} \to \text{Spec} \bar{K} \). As \( \text{Spec} \bar{K} \to S \) is flat, \( R^1\bar{f}_*\mathcal{O}_{X_{\bar{K}}} = (R^1f_*\mathcal{O}_X)_\bar{K} = 0 \). Thus \( N^1(X_{\bar{K}})_Q \cong \text{Pic}(X_{\bar{K}})_Q \). As \( r \) is defined over \( Q \), \( \text{Ker}(r) \) is also defined over \( Q \). It is enough to show (4.0.1) for Cartier divisors. Take \( D \) to be a Cartier divisor such that \([D] \in \text{Ker}(r)\). Shrinking \( S \) to \( U \) as in Proposition 4.3, then by Proposition 4.3, we have \( D_K \equiv 0 \). Possibly replacing \( D \) by a multiple, we can assume \( D_K \sim 0 \). Thus \( \tau = \text{div}(\bar{\alpha}) \) for some \( \bar{\alpha} \in K(X_{\bar{K}}) \). Shrinking \( U \) further, we can assume that there is a finite Galois morphism \( T \to U \) such that the above relation is defined on \( X_T/U \). In particular, \( D_T := D|_T = \text{div}(\alpha) \) for some \( \alpha \in K(X_T) \). As \( D_T \) is \( \text{Gal}(X_T/X_U) \)-invariant, we have

\[
mD_T = \text{div}(\tau) \quad \text{with} \quad \tau := \prod_{\theta \in \text{Gal}(X_T/X_U)} \theta(\alpha),
\]

where \( m = |\text{Gal}(X_T/X_U)| \). As \( \tau \) is \( \text{Gal}(X_T/X_U) \)-invariant, there exists a \( \beta \in K(X) \) whose pullback is \( \tau \) under the morphism \( X_T \to X_U \). Thus \( mD_U = \text{div}(\beta) \) on \( X_U \). Therefore,

\[
\text{Supp}(mD - \text{div}(\beta)) \subset X - X_U.
\]

This shows “\( \subset \)” in (4.0.1). The converse inclusion is trivial.

(2) For any \([D] \in \text{Eff}(X/S)\), we know \( r([D]) \in \text{Eff}(X_U/U) \). We claim that if \( r([D]) \neq 0 \), then \([D] \notin W \). Otherwise, \( r([D]) \neq 0 \) implies that \([D_K] \neq 0 \) by Proposition 4.3. If \([D] \in W \), then \([D_K], -[D_K] \in \text{Eff}(X_{\bar{K}})\). Hence \( \text{Eff}(X_{\bar{K}}) \) is degenerate. This is a contradiction as \( X_{\bar{K}} \) is projective. Therefore, \([D] \in W \) implies that \([D] \in \text{Ker}(r)\).

Conversely, let \( D \) be an \( \mathbb{R} \)-Cartier divisor such that

\[
\text{Supp} D \subset \text{Supp}(X - X_U).
\]

Then \( f(\text{Supp} D) \neq S \). There is an ample divisor \( H > 0 \) on \( S \) such that \( f(\text{Supp} D) \subset \text{Supp} H \). Thus \( D + kf^*H > 0 \) for some \( k \gg 1 \) and \([D + kf^*H] = [D] \in \text{Eff}(X/S)\).

(3) Assume that \( S \) is \( \mathbb{Q} \)-factorial. Let \( W' \subset \text{Mov}(X/S) \) be the maximal vector space. We claim that if \( 0 \neq [D] \in W' \), then there exists a family of curves \( \{C_t \mid t \in R\} \) which covers a divisor such that \([C_t] \in N_1(X/S)\) and \( D \cdot C_t \neq 0 \). By (2), we can assume that \( D \) is vertical over \( S \). Replacing \( D \) by \( D - f^*B \) for some \( \mathbb{R} \)-Cartier divisor \( B \) on \( S \) and shrinking \( S \), we can assume that \( D \) is a very exceptional divisor. Write \( D = D^+ - D^- \) such that \( D^+, D^- \geq 0 \) do not have common components. If \( D^+ \neq 0 \) (resp. \( D^- \neq 0 \)), then by the standard reduce-to-surface argument (for example, see [Bir12, Lemma 3.3]), there exists a family of curves \( \{C_t \mid t \in R\} \) covering an irreducible component of \( \text{Supp} D^+ \) (resp. \( \text{Supp} D^- \)) such that \([C_t] \in N_1(X/S)\) and \( D^+ \cdot C_t < 0 \) (resp. \( D^- \cdot C_t < 0 \)). This shows the claim.

Possibly replacing \( D \) by \(-D \in W' \), we can assume that \( D \cdot C_t < 0 \). This contradicts with \( D \in \text{Mov}(X/S) \).

\[
\text{Remark 4.6.} \quad \text{As} \ X \text{ and} \ S \text{ have rational singularities, by [Kol86, Corollary 7.8],} \ R^1f_*\mathcal{O}_X \text{ is torsion free.}
\]

\[
\text{Question 4.7.} \quad \text{Do the claims in Proposition 4.5 still hold true for an arbitrary fibration} \ f : X \to S? \]

Recall that \( \Gamma_B \) is the image of \( \text{PsAut}(X/S, \Delta) \) under the natural group homomorphism \( \iota : \text{PsAut}(X/S, \Delta) \to \text{GL}(N^1(X/S)_{\mathbb{R}}) \).
Lemma 4.8. Let $f : X \to S$ be a Calabi-Yau fiber space such that $X$ has terminal singularities. Assume that $R^1 f_* \mathcal{O}_X = 0$. Let $W \subset \overline{\text{Mov}(X/S)}$ be the maximal vector space. Then
\[ \Gamma_W := \{ \gamma \in \Gamma_B \mid \gamma \text{ acts trivially on } N^1(X/S)_{\mathbb{R}}/W \} \]
is a finite group.

Proof. As $R^1 f_* \mathcal{O}_X = 0$, we have $H^1(X_K, \mathcal{O}_{X_K}) = 0$ and thus $\text{Pic}(X)_{\mathbb{Q}} \simeq N^1(X_K)_{\mathbb{Q}}$. Let $G := \{ g \in \text{PsAut}(X/S) \mid \iota(g) \in \Gamma_W \}$. It suffices to show that $G$ is a finite set. By Proposition 4.5 (1) and (2), there exists an open set $U \subset S$ such that $N^1(X/S)_{\mathbb{R}}/W \to N^1(X_U/U)_{\mathbb{R}}$ is surjective. Let $H$ be an ample $S$ divisor on $X_U$. Then $g \cdot H \equiv H/U$ for any $g \in G$. Thus $g_K \cdot H_K \equiv H_K$ in $N^1(X_K)$, where $g_K$ and $H_K$ correspond to $g$ and $H$ respectively after the base change. Replacing $H$ by a multiple, we can assume that $g_K \cdot H_K \sim H_K$. The same argument as [Ogu14, Proposition 2.4] shows that $\{ g_K \mid g \in G \}$ is a finite set. Note that the smoothness assumption on $X_K$ in [Ogu14, Proposition 2.4] can be relaxed to the assumption that $X_K$ has terminal singularities by [Han87, (3.14) Theorem]. For $g, h \in G$, if $g_K = h_K$, then $g = h$. Thus $G$ is also a finite set. \qed

Remark 4.9. The group $\Gamma_W$ may not be trivial. There exists a sequence of flops which is also a sequence of birational automorphisms (see [Kaw97, Example 3.8 (4)]). Then for each $X \to S$, we have $N^1(X/S)_{\mathbb{R}} = \text{Mov}(X/S) = W = \mathbb{R}$. $\Gamma_W = \{ \pm 1 \}$ which acts trivially on $N^1(X/S)_{\mathbb{R}}/W$.

5. A VARIANT OF THE CONE CONJECTURE

In this section, we study the relationship between the cone conjecture and Conjecture 1.2. Note that in Conjecture 1.2, by enlarging $P_M$ and $P_A$, we can always assume that $P_M$ and $P_A$ are rational polyhedral cones. Recall that a polyhedral cone is closed by definition and $\Gamma_B$ (resp. $\Gamma_A$) is the image of $\text{PsAut}(X/S, \Delta)$ (resp. $\text{Aut}(X/S, \Delta)$) under the group homomorphism $\text{PsAut}(X/S, \Delta) \to \text{GL}(N^1(X/S)_{\mathbb{R}})$. By Definition 3.1, we set
\[ \text{Mov}(X/S)_+ := \text{Conv}(\overline{\text{Mov}(X/S) \cap N^1(X/S)_{\mathbb{Q}}}), \]
\[ \text{Amp}(X/S)_+ := \text{Conv}(\overline{\text{Amp}(X/S) \cap N^1(X/S)_{\mathbb{Q}}}). \]

Lemma 5.1. Let $f : (X, \Delta) \to S$ be a klt Calabi-Yau fiber space.

1. We have $\overline{\text{Amp}}^{\vee}(X/S) \subset \text{Amp}(X/S)_+.$
2. Assume the existence of good minimal models for effective klt pairs in $\text{dim}(X/S)$, then $\overline{\text{Mov}}^{\vee}(X/S) \subset \text{Mov}(X/S)_+.$

Proof. For $[D] \in \text{Eff}(X/S)$, replacing $D$ by a divisor which is numerically equivalent to $D$, we can assume that the irreducible decomposition of $D = \sum_{i=1}^k a_i D_i$ with $a_i > 0$. Let $P := \text{Cone}([D_i] \mid i = 1, \ldots, k) \subset \text{Eff}(X/S)$ be a rational polyhedral cone.

For (1), assume that $[D] \in \overline{\text{Amp}}^{\vee}(X/S)$. By Theorem 2.7, $P_N = P \cap \overline{\text{Amp}}(X/S)$ is a rational polyhedral cone. Thus
\[ [D] \in P_N \subset \text{Amp}(X/S)_+. \]

For (2), assume that $[D] \in \overline{\text{Mov}}^{\vee}(X/S)$. Let $P = \bigcup_{i=1}^m P_i$ be the union of finitely many open rational polyhedral cones satisfying the claim of Theorem 2.6. Suppose that $[D] \in P_i$. Let $[B] \in P_i$ such that $B \geq 0$. Choose $\epsilon \in \mathbb{Q}_{>0}$ such that both $(X, \Delta + \epsilon D)$ and
Let \( f : (X, \Delta) \to S \) be a klt Calabi-Yau fiber space. Assume the existence of good minimal models for effective klt pairs in \( \dim(X/S) \). If there exists a rational polyhedral cone \( P_M \subset \operatorname{Eff}(X/S) \) satisfying Conjecture 1.2 (2), then there is a rational polyhedral cone \( Q_M \subset \operatorname{Mov}(X/S) \cap P_M \) such that

\[
\bigcup_{g \in \operatorname{PsAut}(X/S,\Delta)} g \cdot Q_M = \operatorname{Mov}(X/S).
\]

Similarly, if there exists a rational polyhedral cone \( P_A \subset \operatorname{Eff}(X/S) \) satisfying Conjecture 1.2 (2), then there is a rational polyhedral cone \( Q_A \subset \overline{\operatorname{Amp}(X/S)} \cap P_A \) such that

\[
\bigcup_{g \in \operatorname{Aut}(X/S,\Delta)} g \cdot Q_A = \overline{\operatorname{Amp}(X/S)}.
\]

Proof. For (5.0.1), by Theorem 2.6, \( P_M = \bigcup_{i=0}^{m_i} P_i^o \) is a union of finitely many open rational polyhedral cones. Let \( P_i^o, \ldots, P_k^o \) be the polyhedral cones such that \( P_j^o \cap \overline{\operatorname{Mov}(X/S)} \neq \emptyset \).

We claim that \( P_j \coloneqq \overline{P_j^o} \subset \operatorname{Mov}(X/S) \). Let \( D \geq 0 \) such that \( [D] \in P_j^o \cap \overline{\operatorname{Mov}(X/S)} \). Assume that \((Y/S, \Delta_Y + \epsilon D_Y)\) is a weak log canonical model of \((X/S, \Delta + \epsilon D)\) for some \( \epsilon \in \mathbb{Q}_{>0} \). By Lemma 2.2, we can assume that \( X, Y \) are isomorphism in codimension 1. Take \( [B'] \in P_j \), by \( P_M \subset \operatorname{Eff}(X/S) \), there exists a sequence \( \{[B_i]\}_{i \in \mathbb{N}} \subset P_i^o \) with \( B_i \geq 0 \) such that \( \lim[B_i] = [B'] \) and \( \lim B_i = B \) as the limit of Weil divisors. Thus \( [B'] = [B] \). By Theorem 2.6, there exists a \( \delta \in \mathbb{Q}_{>0} \) such that \((Y/S, \Delta_Y + \delta B_i)_Y\) is a weak log canonical model of \((X/S, \Delta + \delta B_i)\) for each \( l \). Thus \((Y/S, \Delta_Y + \delta B)\) is also a weak log canonical model of \((X/S, \Delta + \delta B)\). By Theorem 2.3, \( B_Y \) is semiample.\( /S \). Hence \( B \) is movable, and thus \( [B] \in \operatorname{Mov}(X/S) \).

Let \( Q_M \coloneqq \operatorname{Cone}(P_1,\ldots,P_k) \) be the cone generated by \( P_j, 1 \leq j \leq k \). Then \( Q_M \subset \operatorname{Mov}(X/S) \). For \( [M] \in \operatorname{Mov}(X/S) \), there exists a \( g \in \operatorname{PsAut}(X/S,\Delta) \) such that \( g \cdot [M] \in P_M \). Thus \( g \cdot [M] \in P_j \) for some \( j \) and hence \( g \cdot [M] \in Q_M \). This shows (5.0.1).

For (5.0.2), Theorem 2.7 shows that \( Q_A \coloneqq P_A \cap \overline{\operatorname{Amp}(X/S)} \) is a rational polyhedral cone. By \( P_A \subset \operatorname{Eff}(X/S) \), we have \( Q_A \subset \overline{\operatorname{Amp}(X/S)} \). For any \( [H] \in \operatorname{Amp}(X/S) \), there exist an \( [H'] \in P_A \) and a \( g \in \operatorname{Aut}(X/S,\Delta) \) such that \( g \cdot [H'] = [H] \). Hence \( [H'] \in Q_A \). Thus \( \Gamma_A : Q_A \supset \operatorname{Amp}(X/S) \). As \( \operatorname{Amp}(X/S) \) is non-degenerate and \( Q_A \subset \operatorname{Amp}(X/S)_+ \) by Lemma 5.1, Proposition 3.3 implies that \( \Gamma_A : Q_A = \operatorname{Amp}(X/S)_+ \supset \overline{\operatorname{Amp}(X/S)} \). The “ \( \subset \) ” of (5.0.2) follows from definition. \( \square \)

Proposition 5.3. Let \( f : (X, \Delta) \to S \) be a klt Calabi-Yau fiber space. Let \( W \subset \overline{\operatorname{Mov}(X/S)} \) be the maximal vector space. Assume that good minimal models exist for effective klt pairs in dimension \( \dim(X/S) \). Suppose that there is a polyhedral cone \( P \subset \operatorname{Mov}(X/S) \) such that

\[ \operatorname{PsAut}(X/S,\Delta) \cdot P = \operatorname{Mov}(X/S) \]

We have the following results.
(1) If either $R^1f_*\mathcal{O}_X = 0$ or $W = 0$, then we have
\[\operatorname{Mov}(X/S) = \overline{\operatorname{Mov}}(X/S) = \operatorname{Mov}(X/S)^+.\]

(2) There are finitely many varieties $Y_j/S, j \in J$ such that if $X \dashrightarrow Y/S$ is isomorphic in codimension 1 with $Y$ a $\mathbb{Q}$-factorial variety, then $Y \simeq Y_j/S$ for some $j \in J$.

(3) If $\overline{\operatorname{Mov}}(X/S)$ is non-degenerate, then $\overline{\operatorname{Mov}}(X/S)$ has a rational polyhedral fundamental domain under the action of $\Gamma_B$.

(4) If $R^1f_*\mathcal{O}_X = 0$, then $\overline{\operatorname{Mov}}(X/S)$ has a weak rational polyhedral fundamental domain (maybe degenerate) under the action of $\Gamma_B$.

Proof. Possibly enlarging $P$, we can assume that $P$ is a rational polyhedral cone.

For (1), we have “⊂” for the above three cones by Lemma 5.1. By Proposition 4.4, $W$ is defined over $\mathbb{Q}$. By definition, $\operatorname{Mov}(X/S) \supset \operatorname{Int}(\overline{\operatorname{Mov}}(X/S))$. Then $\Gamma_B \cdot P \supset \operatorname{Int}(\overline{\operatorname{Mov}}(X/S))$.

Thus $(\operatorname{Mov}(X/S)^+, \Gamma_B)$ is of polyhedral type. We follow the notation of Lemma 3.7. By Lemma 3.7 (3) and Proposition 3.3, we have
\[(5.0.3) \quad \Gamma_B \cdot \bar{P} = (\overline{\operatorname{Mov}}(X/S))^+ = (\operatorname{Mov}(X/S))^+\bar{P},\]

where the last equality follows from Lemma 3.7 (2).

We claim that $W \subset \operatorname{Mov}(X/S)$. By Proposition 4.5 (1) and (2), $W \subset \operatorname{Eff}(X/S)$. Let $[D] \in W$ be a rational point such that $D \geq 0$. Then for a sufficiently small $\epsilon \in \mathbb{Q}_{>0}$, $(X/S, \Delta + \epsilon D)$ has a weak log canonical model $(Y/S, \Delta + \epsilon D_Y)$. Because $[D] \in \overline{\operatorname{Mov}}(X/S)$, by Lemma 2.2, we can assume that $X, Y$ are isomorphic in codimension 1. Note that $D_Y$ is semi-ample/$S$ by Theorem 2.3. Thus $[D] \in \operatorname{Mov}(X/S)$. As $W$ is $\Gamma$-invariant and $\Gamma_B \cdot (P + W) = \operatorname{Mov}(X/S)^+$ by (5.0.3), we have $\operatorname{Mov}(X/S) = \operatorname{Mov}(X/S)^+$.

For (2), there exists a decomposition $P = \bigcup_{i=1}^k P_k$ as in Theorem 2.6. For each $j$, by Lemma 2.2 and Theorem 2.3, we can choose a $f_j : X \dashrightarrow Y_j/S$ which is isomorphic in codimension 1 such that if $[D] \in P_k$ with $D \geq 0$, then $(Y_j/S, \Delta_Y + \epsilon D_Y)$ is a $\mathbb{Q}$-factorial weak log canonical model of $(X/S, \Delta + \epsilon D)$ for some $\epsilon \in \mathbb{Q}_{>0}$. We claim that if $g : Y \dashrightarrow X/S$ is isomorphic in codimension 1, then $Y \simeq Y_j/S$ for some $j$. In fact, let $A \geq 0$ be an ample/$S$ divisor on $Y$. Then $g_*A \in \operatorname{Mov}(X/S)$. Let $\sigma \in \operatorname{PsAut}(X/S, \Delta)$ such that $\sigma \cdot g_*A \in P_j$. Then $\sigma \cdot g_*A \in P^\circ_j$ for some $j$. Note that $Y, Y_j$ are $\mathbb{Q}$-factorial varieties. Because $(\sigma \cdot g_*A)_{Y_j} = (f_j \circ \sigma \circ g)_*A$ is nef/$S$ and
\[f_j \circ \sigma \circ g : Y \dashrightarrow X \dashrightarrow X \dashrightarrow Y_j/S\]
is isomorphic in codimension 1, we have $Y \simeq Y_j/S$.

For (3) and (4), note that $(\operatorname{Mov}(X/S)^+, \Gamma_B)$ is of polyhedral type. By Proposition 4.4 and Proposition 3.8, there is a rational polyhedral cone $\Pi$ such that $\Gamma_B \cdot \Pi = \operatorname{Mov}(X/S)^+$, and for each $\gamma \in \Gamma_B$, either $\gamma \Pi \cap \operatorname{Int}(\Pi) = \emptyset$ or $\gamma \Pi = \Pi$. Moreover,
\[\{ \gamma \in \Gamma_B \mid \gamma \Pi = \Pi \} = \{ \gamma \in \Gamma_B \mid \gamma \text{ acts trivially on } N^1(X/S) \mathbb{R}/W \}.
\]

Hence $\Pi$ is a weak rational polyhedral fundamental domain. In particular, if $W = 0$, then $\Pi$ is a rational polyhedral fundamental domain.

\begin{remark}
The assumption in Proposition 5.3 (1) is necessary. \cite[Example 3.8 (2)]{Kaw97} gives an elliptic fibration (hence $R^1f_*\mathcal{O}_X \neq 0$) with $W \neq 0$ such that $\operatorname{Mov}(X/S) = \overline{\operatorname{Mov}}(X/S) \neq \operatorname{Mov}(X/S)^+$. In this example, $W$ is defined over $\mathbb{Q}$ but $W \not\subset \operatorname{Mov}(X/S)$.
\end{remark}
Proposition 5.5. Let $f : (X, \Delta) \to S$ be a klt Calabi-Yau fiber space. Suppose that there is a polyhedral cone $P \subset \overline{\text{Amp}}(X/S)$ such that $\text{Aut}(X/S, \Delta) \cdot P = \overline{\text{Amp}}(X/S)$. We have the following results.

1. There are finitely many varieties $Y_j/S, j \in J$ such that if $X \to Z/S$ is a surjective fibration to a normal variety $Z$, then $Y_j \simeq Z/S$ for some $j \in J$.

2. The cone $\overline{\text{Amp}}(X/S)$ has a rational polyhedral fundamental domain.

This result can be shown analogously as Proposition 5.3 and thus we only sketch the proof.

Sketch of the Proof. For (1), let $A$ be an ample $S$ divisor on $Z$. Then for a morphism $g : X \to Z/S$, $g^*A$ lies in $\overline{\text{Amp}}(X/S)$. There exists $\theta \in \text{Aut}(X/S, \Delta)$ such that $[\theta \cdot g^*A]$ lies in the interior of a face $F \subset P$. The morphism $g \circ \theta^{-1} : X \to Z$ corresponds to the contraction of $F$. As $P$ is a polyhedral cone, there are only finitely many faces.

(2) follows from Lemma 3.5 as $\text{Amp}(X/S)$ is non-degenerate. \qed

We have the following remark regarding the cones chosen in the statement of the cone conjecture (cf. [LOP18, Section 3]):

Remark 5.6. Let $f : (X, \Delta) \to S$ be a klt Calabi-Yau fiber space. Assuming that good minimal models of effective klt pairs exist in dimension $\dim(X/S)$ and either $R^1f_*\mathcal{O}_X = 0$ or $\overline{\text{Mov}}(X/S)$ is non-degenerate, Lemma 5.2 and Proposition 5.3 (1) imply that $\overline{\text{Mov}}(X/S)$ has a (weak) rational polyhedral fundamental domain iff $\overline{\text{Mov}}(X/S)$ has a (weak) rational polyhedral fundamental domain.

Therefore, at least when $S$ is a point, modulo the standard conjectures in the minimal model program, there is no difference to state the cone conjecture of movable cones for either $\overline{\text{Mov}}(X/S)$ or $\overline{\text{Mov}}(X/S)$.

If $\overline{\text{Amp}}(X/S)$ has a (weak) rational polyhedral fundamental domain, then $(\overline{\text{Amp}}(X/S)_+, \Gamma_A)$ is of polyhedral type. Proposition 3.3 and Lemma 5.1 imply that

$$\overline{\text{Amp}}(X/S) = \overline{\text{Mov}}(X/S) = \overline{\text{Amp}}(X/S)_+.$$  

Therefore, $\overline{\text{Amp}}(X/S)$ has a rational polyhedral fundamental domain by Lemma 3.5. In summary, the cone conjecture for $\overline{\text{Amp}}(X/S)$ implies that for $\overline{\text{Amp}}(X/S)$.

However, a priori, $\overline{\text{Mov}}(X/S)_+$ (resp. $\overline{\text{Amp}}(X/S)_+$) has a rational polyhedral fundamental domain $\Pi$ may not imply that $\overline{\text{Mov}}(X/S)$ (resp. $\overline{\text{Amp}}(X/S)$) has a rational polyhedral fundamental domain. More importantly, under this assumption, we only know $\Pi \subset \overline{\text{Eff}}(X/S)$, hence Theorem 2.6 and Theorem 2.7 do not apply in this setting. Therefore, the argument of finiteness of birational models which are isomorphic in codimension 1 (resp. finiteness of contraction morphisms) breaks. It is for this reason that we do not state the cone conjectures for $\overline{\text{Mov}}(X/S)_+$ and $\overline{\text{Amp}}(X/S)_+$.

The above discussions lead to the proof of Theorem 1.3.

Proof of Theorem 1.3. The (1) and (2) follow from Lemma 5.2 and Proposition 5.3 (4) and (3). The (3) follows from Lemma 5.2 and Proposition 5.5 (2). \qed
6. Generic and Geometric cone conjectures

6.1. Generic cone conjecture. For a Calabi-Yau fiber space, we study the relationship between the relative cone conjecture and the cone conjecture of its generic fiber. Conjecture 1.2 is especially convenient to study movable cones in the relative setting. Hence we only focus on the cone conjecture for movable cones in this section.

Let \( f : X \to S \) be a Calabi-Yau fiber space. Recall that \( K := K(S) \) is the field of rational functions of \( S \), and \( X_K := X \times_S \text{Spec} \, K \).

**Theorem 6.1.** Let \( f : X \to S \) be a Calabi-Yau fiber space such that \( X \) has terminal singularities. Suppose that good minimal models of effective klt pairs exist in dimension \( \dim(X/S) \).

Assume that \( R^1 f_* \mathcal{O}_X = 0 \).

If the weak cone conjecture holds true for \( \overline{\text{Mov}}(X_K/K) \), then the weak cone conjecture holds true for \( \overline{\text{Mov}}^e(X/S) \).

Moreover, if \( \text{Mov}(X/S) \) is non-degenerate, then the cone conjecture holds true for \( \overline{\text{Mov}}^e(X/S) \). In particular, if \( S \) is \( \mathbb{Q} \)-factorial, then the cone conjecture holds true for \( \overline{\text{Mov}}^e(X/S) \).

**Proof.** Let \( \Pi_K \subset \overline{\text{Mov}}(X_K/K) \) be a polyhedral cone such that

\[
\text{PsAut}(X_K/K) \cdot \Pi_K = \text{Mov}(X_K/K).
\]

Let \( \Pi \subset \text{Eff}(X/S) \) be a polyhedral cone which is a lift of \( \Pi_K \). In other words, \( \Pi \) maps to \( \Pi_K \) under \( N^1(X/S)_\mathbb{R} \to N^1(X_K/K)_\mathbb{R} \).

If \( g_K \in \text{PsAut}(X_K/K) \), then \( g_K \) can be viewed as a birational morphism \( g \) of \( X \) over \( S \). Then \( g \in \text{Bir}(X/S) = \text{PsAut}(X/S) \) as \( K_X \) is nef/\( S \) and \( X \) has terminal singularities.

Let \( W \subset \overline{\text{Mov}}(X/S) \) be the maximal vector space. We claim that for \( P := \text{Cone}(\Pi \cup W) \),

\[
P \subset \text{Eff}(X/S) \quad \text{and} \quad \text{PsAut}(X/S) \cdot P \supset \text{Mov}(X/S).
\]

(6.1.1)

By Proposition 4.5 (2), \( W \) is generated by vertical divisors and thus \( W \subset \text{Eff}(X/S) \). This shows \( P \subset \text{Eff}(X/S) \). Next, for any \([M] \in \text{Mov}(X/S)\) such that \( M \) is an \( \mathbb{R} \)-Cartier divisor. There exist an \( \mathbb{R} \)-Cartier divisor \( D \) on \( X \) and a \( g \in \text{PsAut}(X/S) \) such that \([D] \in \Pi \) and \( g_K \cdot [D_K] = [M_K] \). As \( R^1 f_* \mathcal{O}_X = 0 \), \( g_K \cdot D_K \sim_\mathbb{R} M_K \). Therefore, there exists a vertical divisor \( B \) on \( X \) such that \( g \cdot D + B \sim_\mathbb{R} M/S \). Thus \( D + g^{-1} \cdot B \in P \) and \( g \cdot [D + g^{-1} \cdot B] = [M] \). This shows \( \text{PsAut}(X/S) \cdot P \supset \text{Mov}(X/S) \).

The (6.1.1) shows that Conjecture 1.2 (1) is satisfied. Then Theorem 1.3 (1) and (2) imply the desired claim. Note that by Proposition 4.5 (4), if \( S \) is \( \mathbb{Q} \)-factorial, then \( \overline{\text{Mov}}(X/S) \) is non-degenerate. \( \square \)

**Remark 6.2.** The above argument does not work for a log pair \((X/S, \Delta)\) because each \( g \in \text{PsAut}(X_K/K, \Delta_K) \) may not lift to \( \text{PsAut}(X/S, \Delta) \).

Now Theorem 1.4 follows from Theorem 6.1 and the cone conjecture of K3 surfaces over arbitrary fields with characteristic \( \neq 2 \) ([BLvL20]).

**Proof of Theorem 1.4.** We have \( R^1 f_* \mathcal{O}_X \otimes k(t) \simeq H^1(X_t, \mathcal{O}_{X_t}) = 0 \), where \( t \in S \) is a general closed point. Hence \( R^1 f_* \mathcal{O}_X \) is a torsion sheaf and thus \( R^1 f_* \mathcal{O}_X = 0 \) by Remark 4.6.
We claim that $X_K$ is a smooth K3 surface. Let $U \subset S$ be a smooth open set such that $X_U \to U$ is flat and for any closed point $t \in U$, $X_t$ is a K3 surface. By [Sta22, Lemma 01V8], $f_U : X_U \to U$ is a smooth morphism. Thus $X_K/K$ is smooth. Note that

$$\text{Spec } K(S) \to U, \quad \text{Spec } k(t) \to U$$

are flat morphisms, where $t \in U$ is a closed point. Then [Har77, III Prop 9.3] implies that for a quasi-coherent sheaf $\mathcal{F}$ on $X_U$ and $i \geq 0$,

$$H^i(X_U, \mathcal{F}) \otimes_U K \simeq H^i(X_K, \mathcal{F}_{X_K}),$$

$$H^i(X_U, \mathcal{F}) \otimes_U k(t) \simeq H^i(X_t, \mathcal{F}_{X_t}).$$

(6.1.2)

First, applying (6.1.2) to $\omega_{X_U/U}$ and $i = 0$, we have $\mathcal{O}_{X_K}(K_{X_K}) \simeq \mathcal{O}_{X_K}$. Next, applying (6.1.2) to $\mathcal{O}_{X_U}$ and $i = 1$, we have $H^1(X_K, \mathcal{O}_{X_K}) = 0$. This shows that $X_K/K$ is a K3 surface.

We claim that $\text{Amp}(X_K/K) = \text{Amp}(X_K/K)$. It suffices to show that $\text{Amp}(X_K/K) \subset \text{Eff}(X_K/K)$. Let $D_K$ be a Cartier divisor on $X_K$ such that $[D_K] \in \text{Amp}(X_K/K) \cap N^1(X_K/K)_{\mathbb{Q}}$. A similar argument as above shows that $X_K$ is a K3 surface over $\bar{K}$. An application of Riemann-Roch shows that there exists an effective divisor $E_{\bar{K}}$ such that $D_{\bar{K}} \sim E_{\bar{K}}$. To see $[D_K] \in \text{Eff}(X_K/K)$, we can argue similarly as Proposition 4.5 (1). Hence, we only sketch the argument. Shrinking $S$, there is a finite Galois morphism $T \to S$ such that everything above can be lifted to $T$. Let $\mu : X_T \to X_S$. In particular, if $D$ is a divisor on $X$ which is $D_K$ after base change, then there is a divisor $E > 0$ on $X_T$ which is $E_{\bar{K}}$ after base change, and $D_T \sim E/T$. Then

$$m\mu^*D = mD_T \sim \sum_{[\theta] \in \text{Gal}(X_T/X)} \theta \cdot E /S,$$

where $m = |\text{Gal}(X_T/X)|$. Hence $[D] \in \text{Eff}(X/S)$ and thus $[D_K] \in \text{Eff}(X_K/K)$.

By [BLvL20, Corollary 3.15], there is a rational polyhedral cone $\Pi \subset \text{Amp}(X_K/K)$ which is a fundamental domain of $\text{Amp}(X_K/K)$ under the action of $\text{Aut}(X_K/K)$. By $\text{Amp}(X_K/K) = \text{Amp}(X_K/K) = \text{Mov}(X_K/K)$ and $\text{Aut}(X_K/K) = \text{PsAut}(X_K/K)$, Theorem 6.1 implies the desired result. \qed

Remark 6.3. For a projective hyperkähler manifold $X$ over a characteristic zero field $k$, [Tak21, Theorem 1.0.5] showed that $\text{Mov}(X/k)_+$ has a rational polyhedral fundamental domain $\Pi$ under the action of $\text{Bir}(X/k)$. However, this is not sufficient to deduce the cone conjecture for movable cones of hyperkähler fibrations. Indeed, we do not know that $\Pi \subset \text{Eff}(X/k)$.

On the other hand, when $k = \mathbb{C}$, it is conjectured that $\text{Mov}(X/k)_+ \subset \text{Eff}(X/k)$ (see [BM14, Conjecture 1.4]). We refer to [HPX24] for the latest developments on the cone conjecture for irreducible holomorphic symplectic manifolds.

6.2. Geometric cone conjecture. For a klt Calabi-Yau fiber space, we study the relationship between the relative cone conjecture and the cone conjecture of its geometric fiber. We show that the cone conjecture for the movable cone of the geometric fiber implies the relative cone conjecture for the movable cone of a finite Galois base change.

Recall that if $X$ is a non-$\mathbb{Q}$-factorial variety over $S$ and $\bar{X} \to X$ is a small $\mathbb{Q}$-factorization, then the (weak) cone conjecture for $\text{Mov}(X/S)$ is referred to that for $\text{Mov}(X/S)$. By [BCHM10, Corollary 1.4.3], such small $\mathbb{Q}$-factorization exists if there is a divisor $\Delta$ such that $(X, \Delta)$ is klt.
For a klt Calabi-Yau fiber space $f : (X, \Delta) \to S$. Let $K = K(S)$ and $\bar{K}$ be the algebraic closure of $K$. For $g \in \text{Bir}(X/S)$, let $g_{\bar{K}} \in \text{Bir}(X_{\bar{K}}/\bar{K})$ be the extension of $g$ under the base change $\text{Spec} \bar{K} \to S$. Let $X_{\bar{K}} := X \times_S \text{Spec} \bar{K}$ be the geometric fiber of $f$. Set $\Delta_{\bar{K}} := \Delta \times_S \text{Spec} \bar{K}$. By Proposition 4.1 (1), $(X_{\bar{K}}, \Delta_{\bar{K}})$ is still klt. Let $\bar{\pi} : \bar{X}_{\bar{K}} \to X_{\bar{K}}$ be a small $\mathbb{Q}$-factorization. Set $\Delta_{\bar{K}}$ be the strict transform of $\Delta_{\bar{K}}$. Let $\tilde{\Gamma}_B$ be the image of $\text{PsAut}(\bar{X}_{\bar{K}}/\bar{K}, \Delta_{\bar{K}}) (= \text{PsAut}(X_{\bar{K}}/\bar{K}, \Delta_{\bar{K}}))$ under the group homomorphism

\[(6.2.1) \quad \iota_K : \text{PsAut}(\bar{X}_{\bar{K}}/\bar{K}, \Delta_{\bar{K}}) \to \text{GL}(N^1(\bar{X}_{\bar{K}}/\bar{K})_\mathbb{R}).\]

**Proposition 6.4.** Under the above notation and assumptions.

1. If the weak cone conjecture of $\text{Mov}(X_{\bar{K}}/\bar{K})$ holds true, then there is a dominant étale Galois morphism $T \to S$ such that for any $\bar{g} \in \text{PsAut}(X_{\bar{K}}/\bar{K}, \Delta_{\bar{K}})$, there exists a $g \in \text{PsAut}(\bar{X}_T/T, \Delta_T)$ such that $g_{\bar{K}}$ and $\bar{g}$ induce the same action on $N^1(\bar{X}_{\bar{K}}/\bar{K})_\mathbb{R}$. Here $\bar{X}_T$ is a small $\mathbb{Q}$-factorization of $X_T$ and $\Delta_T$ is the strict transform of $\Delta_T$.

2. If the weak cone conjecture of $\text{Amp}(X_{\bar{K}}/\bar{K})$ holds true, then there is a dominant étale Galois morphism $T \to S$ such that for any $\bar{g} \in \text{Aut}(X_{\bar{K}}/\bar{K}, \Delta_{\bar{K}})$, there exists a $g \in \text{Aut}(X_T/T, \Delta_T)$ such that $g_{\bar{K}}$ and $\bar{g}$ induce the same action on $N^1(X_{\bar{K}}/\bar{K})_\mathbb{R}$.

**Proof.** We only show (1) as (2) can be shown analogously.

By assumption, Conjecture 1.2 (1) is satisfied for $\text{Mov}(\bar{X}_{\bar{K}}/\bar{K})$. As $\text{Mov}(\bar{X}_{\bar{K}}/\bar{K})$ is non-degenerate, Theorem 1.3 (2) shows that the cone conjecture holds true for $\text{Mov}(\bar{X}_{\bar{K}}/\bar{K})$. By Theorem 3.6, $\tilde{\Gamma}_B$ is finitely presented (the following argument only needs it to be finitely generated). Choose

$$\bar{g}_1, \ldots, \bar{g}_m \in \text{PsAut}(\bar{X}_{\bar{K}}/\bar{K}, \Delta_{\bar{K}})$$

such that $\iota_K(\bar{g}_1), \ldots, \iota_K(\bar{g}_m)$ (see (6.2.1)) are generators of $\tilde{\Gamma}_B$. As $N^1(\bar{X}_{\bar{K}}/\bar{K})_\mathbb{R}$ is of finite dimension, let $\bar{D}_1, \ldots, \bar{D}_\rho$ be divisors such that $[\bar{D}_1], \ldots, [\bar{D}_\rho]$ is a basis. There is a finite base change $T \to S$ such that $\bar{g}_j$ and $\bar{D}_i := \bar{\pi}_* D_i$ can be defined on $X_T \to T$. In other words, there exist a $g_j \in \text{Bir}(X_T/T, \Delta_T)$ and a $D_i$ on $X_T$, such that $(g_j)_K = \bar{g}_j$ and $(D_i)_K = \bar{D}_i$. Shrink $T$, $(X_T, \Delta_T)$ has klt singularities by Proposition 4.1 (2). Let $\mu : \bar{X}_T \to X_T$ be a small $\mathbb{Q}$-factorization. Let $\bar{D}_i := \mu^{-1}_* D_i$. Shrinking $T$ further, there exists a natural inclusion $N^1(\bar{X}_T/T)_\mathbb{R} \hookrightarrow N^1((\bar{X}_T)_K/\bar{K})_\mathbb{R}$ by Proposition 4.3. Because $((\bar{X}_T)_K, (\Delta_T)_K)$ has klt singularities by Proposition 4.1 (1) and $(\bar{X}_T)_K \to (X_T)_{\bar{K}} = X_K$ is a small morphism, a small $\mathbb{Q}$-factorization $Y_K \to (\bar{X}_T)_K$ is still a small $\mathbb{Q}$-factorization of $X_K$. Thus

$$N^1(\bar{X}_T/T)_\mathbb{R} \hookrightarrow N^1((\bar{X}_T)_K/\bar{K})_\mathbb{R} \hookrightarrow N^1(Y_K/\bar{K})_\mathbb{R} \cong N^1(\bar{X}_K/\bar{K})_\mathbb{R}.$$ 

By the choice of $T$, this is also a surjective map. Hence

$$N^1((\bar{X}_T)_K/\bar{K})_\mathbb{R} \cong N^1(Y_K/\bar{K})_\mathbb{R}$$

and thus $(\bar{X}_T)_K$ is $\mathbb{Q}$-factorial. As $(\bar{X}_T)_K \to X_K$ is a small $\mathbb{Q}$-factorization, it suffices to show the claim for $N^1((\bar{X}_T)_K/\bar{K})_\mathbb{R}$. Possibly taking a Galois cover of $T$ and shrinking $T$, we can assume that $T \to S$ is a dominant étale Galois morphism.
We claim that after shrinking $T$, we have $g_j \in \text{PsAut}(\bar{X}_T/T, \Delta_T)$ for each $j$. If $g_j \in \text{Bir}(X_T/T, \Delta_T) \setminus \text{PsAut}(X_T/T, \Delta_T)$, then there are finitely many divisors $B_l, l \in J$ which are contracted by $g_j$ or $g_j^{-1}$. As $(g_j)_\mathbb{K}$ and $(g_j^{-1})_\mathbb{K}$ do not contract $(B_l)_\mathbb{K}$, $B_l$ is vertical over $T$. Therefore, shrinking $T$, we can assume that $g_j$ and $g_j^{-1}$ do not contract divisors for each $j$. This shows the claim.

Finally, let $\bar{D}_i := (\bar{D}_i)_\mathbb{K}$, and for $g, h \in \{g_j \mid 1 \leq j \leq m\}$, let $\bar{g} := g_\mathbb{K}, \bar{h} := h_\mathbb{K}$. Then for each $i$, 

$$\bar{g}_*(\bar{h}_*(\bar{D}_i)) = (g \circ h)_*(\bar{D}_i).$$

This implies that 

$$\iota_\mathbb{K}(\bar{g})(\iota_\mathbb{K}(\bar{h}) \cdot [\bar{D}_i]) = \iota_\mathbb{K}(g \cdot h) \cdot [\bar{D}_i].$$

As $[\bar{D}_i], i = 1, \ldots, \rho$ is a basis of $N^1((\bar{X}_T)_\mathbb{K}/\bar{K})_\mathbb{R}$, we have 

$$\iota_\mathbb{K}(\bar{g})\iota_\mathbb{K}(\bar{h}) = \iota_\mathbb{K}(g \cdot h).$$

Now the desired result follows as $\iota_\mathbb{K}(\bar{g}_j), 1 \leq j \leq m$ generate $\Gamma_B$. \hfill $\square$

**Remark 6.5.** Let $T' \to S$ be a dominant étale Galois morphism which factors through $T \to S$. By the proof of Proposition 6.4, after shrinking $T'$, the claims in Proposition 6.4 still hold true for $T' \to S$.

**Theorem 6.6.** Let $f : (X, \Delta) \to S$ be a klt Calabi-Yau fiber space. Assume that good minimal models of effective klt pairs exist in dimension $\dim(X/S)$.

1. Assume that the weak cone conjecture holds true for $\overline{\text{Mov}}^e(X_\mathbb{K}/\bar{K})$. Then there is a dominant étale Galois morphism $T \to S$ such that the cone conjecture holds true for $\overline{\text{Mov}}(X_T/T)$.

2. Assume that the weak cone conjecture holds true for $\overline{\text{Amp}}^e(X_\mathbb{K}/\bar{K})$. Then there is a dominant étale Galois morphism $T \to S$ such that the cone conjecture holds true for $\overline{\text{Amp}}^e(X_T/T)$.

**Proof.** We only show (1) as (2) can be shown analogously.

By the proof of Proposition 6.4, there exist a dominant étale Galois morphism $T \to S$ and a small $\mathbb{Q}$-factorization $\bar{X}_T \to X_T$ such that $(\bar{X}_T, \bar{\Delta}_T) \to T$ is a klt Calabi-Yau fiber space and $(\bar{X}_T)_\mathbb{K} \to X_\mathbb{K}$ is a small $\mathbb{Q}$-factorization. Replacing $(X, \Delta) \to S$ by $(\bar{X}_T, \bar{\Delta}_T) \to T$, we can assume that $X_\mathbb{K}$ is $\mathbb{Q}$-factorial.

Let $\Pi_\mathbb{K} \subset \overline{\text{Mov}}^e(X_\mathbb{K}/\bar{K})$ be a rational polyhedral cone such that 

$$\text{PsAut}(X_\mathbb{K}/\bar{K}, \Delta_\mathbb{K}) \cdot \Pi_\mathbb{K} = \overline{\text{Mov}}^e(X_\mathbb{K}/\bar{K}).$$

There exist a finite morphism $T \to S$ and finitely many effective divisors $D_j, j \in J$ on $X_T$ such that $\text{Cone}([(D_j)_\mathbb{K}] \mid j \in J) = \Pi_\mathbb{K}$. Replacing $T$ by a higher finite morphism and shrinking $T$, we can further assume that $T \to S$ is a dominant étale Galois morphism which satisfies Proposition 6.4 (1) (see Remark 6.5). We can further assume that $(X_T, \Delta_T)$ has klt singularities with $K_{X_T} + \Delta_T \sim_{\mathbb{R}} 0/T$ by Proposition 4.1 (2).

Let $\mu : \bar{X}_T \to X_T$ be a small $\mathbb{Q}$-factorization and $\bar{D}_j := \mu_\ast^{-1}D_j, j \in J$. Shrinking $T$, there is a natural inclusion $N^1(\bar{X}_T/T)_{\mathbb{R}} \hookrightarrow N^1((\bar{X}_T)_\mathbb{K}/\bar{K})_{\mathbb{R}}$. Because $(\bar{X}_T)_\mathbb{K} \to (X_T)_\mathbb{K} = X_\mathbb{K}$ is
a small morphism and \((\widetilde{X}_T)_K, (\Delta_T)_K)\) is klt by Proposition 4.1 (1), there exists a small \(\mathbb{Q}\)-factorization \(Y_K \rightarrow (\widetilde{X}_T)_K\) which is also a small \(\mathbb{Q}\)-factorization of \(X_K\). By \(X_K\) \(\mathbb{Q}\)-factorial, we have \(Y_K = X_K\) and thus \((\widetilde{X}_T)_K = X_K\). Hence
\[
(6.2.3) \quad \text{Mov}(\widetilde{X}_T/T) \hookrightarrow \text{Mov}(X_K/K).
\]

Let \(\Pi := \text{Cone}(\{\widetilde{D}_j \mid j \in J\}) \subset \text{Eff}(\widetilde{X}_T/T)\). We claim that
\[
\text{PsAut}(\widetilde{X}_T/T, \Delta_T) \cdot \Pi \supset \text{Mov}(\widetilde{X}_T/T).
\]
In fact, let \([D] \in \text{Mov}(\widetilde{X}_T/T)\). Then there exist a \(\tilde{g} \in \text{PsAut}(X_K/K, \Delta_K)\) and a \([\tilde{B}] \in \Pi_K\) such that \(\tilde{g} \cdot [\tilde{B}] = [D_K]\). By the construction of \(\widetilde{X}_T\) and Proposition 6.4 (1), there exist a \(g \in \text{PsAut}(X_T/T, \Delta_K)\) and a \(\Theta \in \Pi\) such that \([\Theta_K] = [\tilde{B}]\) and
\[
([g_\ast \Theta])_K = g_K \cdot [\Theta_K] = \tilde{g} \cdot [\tilde{B}] = [D_K].
\]
By (6.2.3), \(g \cdot [\Theta] = [g_\ast \Theta] = [D] \in \text{Mov}(\widetilde{X}_T/T)\).

Therefore, Conjecture 1.2 (1) is satisfied. As \(\overline{\text{Mov}}(X_K/K)\) is non-degenerate, \(\overline{\text{Mov}}(X_T/T)\) is non-degenerate by (6.2.3). Hence, (1) follows from Theorem 1.3 (2). \(\square\)

It is desirable to deduce the cone conjecture of the Calabi-Yau fiber space \((X, \Delta) \rightarrow S\) from \((X_T, \Delta_T) \rightarrow T\), where \(T \rightarrow S\) is a dominant étale Galois morphism. This seems to be a difficult problem. The main obstacle is to descend elements from \(\text{PsAut}(X_T/T, \Delta_T)\) and \(\text{Aut}(X_T/T, \Delta_T)\) to \(\text{PsAut}(X/S, \Delta)\) and \(\text{Aut}(X/S, \Delta)\). We propose the following question.

**Question 6.7.** Let \(f : X \rightarrow S\) be a terminal Calabi-Yau fiber space. Let \(T \rightarrow S\) be a dominant étale Galois morphism. Possibly shrinking \(T\), there is a natural group homomorphism
\[
\text{PsAut}(X/S) \hookrightarrow \text{PsAut}(X_T/T).
\]
Let \(\Gamma_S\) and \(\Gamma_T\) be the images of \(\text{PsAut}(X/S)\) and \(\text{PsAut}(X_T/T)\) under the group homomorphism \(\text{PsAut}(\widetilde{X}_T/T) \rightarrow \text{GL}(N^1(\widetilde{X}_T/T)_\mathbb{R})\). Is \(\Gamma_S\) a finite index subgroup of \(\Gamma_T\)?

A positive answer to Question 6.7 would give that the weak cone conjecture for \(\overline{\text{Mov}}(X_K/K)\) implies that for \(\overline{\text{Mov}}(X/S)\).

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