On the Conservation of Information in Quantum Physics

Marco Roncaglia

Received: 8 February 2019 / Accepted: 4 October 2019 / Published online: 20 October 2019
© Springer Science+Business Media, LLC, part of Springer Nature 2019

Abstract
According to quantum mechanics, the informational content of isolated systems does not change in time. Considering composite systems, it would be very useful to identify suitable indicators able to quantify the informational content of the single parts and to describe their evolution through balance equations, as it happens in the case of energy. Reasoning on the basic concepts of quantum mechanics, we show that there is an intrinsic quantum information encoded in the coherence of quantum states. Such information is measured by a function, called here coherent entropy, which turns out to be complementary to the von-Neumann entropy. We show that the total quantum information of multipartite systems is determined by the coherent entropy of the single subsystems plus their mutual information. Interestingly, the coherent entropy is found to be equal to the information conveyed in the future by quantum states, providing a further inspiring interpretation to this quantity. The vision proposed in this paper also suggests a natural and simple definition of an indicator for nonlocal correlations.

1 Introduction

Every physicist is confident with the principle of energy conservation and aware about its importance and implications. During time evolution in isolated systems, energy is converted from one form to another or transferred between different subsystems, provided the total amount remains the same. However, when we consider information the picture is not so clear. In quantum physics, the conservation of information has been related to no-cloning theorems [1], but apparently it has not been associated to a suitable conserved quantity. In many situations, the information of a quantum states with density operator $\rho$ is measured by the von-Neumann entropy $S(\rho) = -\text{Tr}(\rho \log_2 \rho)$.
In fact, in any isolated quantum system \( S(\rho) \) does not change in time [2,3], as a consequence of the fact that it depends only on the spectrum of \( \rho \), and the unitarity of time evolution preserves the spectrum at the quantum level. In this sense, people say that any physical process governed by quantum mechanics information is never lost. However, this is a static vision that involves isolated quantum systems. The present work is motivated by the need of identifying the right indicators to describe the full informational content of quantum states, with the aim of building a complete theory able to clearly describe how quantum information “flows” between interacting systems, accounting for a correct balance at any time.

In this paper, we propose to treat separately the coherent and the incoherent contributions of the informational content of quantum states. Starting from the very basic principles of quantum mechanics, we will introduce the concept of coherent entropy, a quantity able to detect the information that quantum states convey in time. In this context, pure states contain more coherent information than mixed states, as the missing information has been converted into correlations with the environment. We will find that mutual information between subsystems plus their coherent entropies are indeed conserved under unitary processes.

2 Informational Content of Quantum States

A pure state is the eigenstate of some complete observable, whose measurement gives a fully predictable outcome with zero von-Neumann entropy. However, it appears reductive to attribute a zero informational content to pure states. In contrast with single deterministic classical states, the superposition principle endows quantum states with an intrinsic informational content, which should find a proper indicator to be measured.

Let us take the textbook example of one qubit, i.e. a pure state in dimension \( d = 2 \) of the Hilbert space [2,3]. Observables are represented by the set of Pauli matrices with eigenvalues \( \{+1, -1\} \), where \( \sigma^z \) has eigenstates \( \{|\uparrow\rangle, |\downarrow\rangle\} \), and \( \sigma^x \) has eigenstates \( \{|+\rangle, |-\rangle\} \). If our source emits quantum objects in the state \( |\uparrow\rangle = (|+\rangle + |-\rangle)/\sqrt{2} \), a measurement along \( \sigma^x \) gives a random sequence of values \( +1 \) and \( -1 \), with equal probability \( p_+ = p_- = 1/2 \). In this case the entropy of information is one bit, i.e. the maximum obtainable for a dichotomic variable. Differently, if the observer measures along \( \sigma^z \), he obtains the constant sequence of \( +1 \), with zero entropy. This property of detecting different entropies under different measurements is genuinely quantum, as it is ultimately due to interference: in the present example, distinct states \( \{|+\rangle, |-\rangle\} \) coherently recombine into a single state \( |\uparrow\rangle \), thanks to their well-defined relative phases.

In quantum optics, this example is realized by the Mach–Zehnder interferometer (see Fig. 1), where the path of a single-photon beam is split in two different directions by a 50% reflective mirror (the beam splitter BS1) and then constructively recombined into a single path by a second beam splitter BS2. A measure of the presence of the photon in the upper \( |\uparrow\rangle \) or lower branch \( |\downarrow\rangle \) after BS1 gives a random sequence of \( +1 \) and \( -1 \), with equal probability. At the output after BS2, the photon detection produces a steady sequence of \( +1 \)'s, which yield a signal with zero entropy. As the
role of the beam splitter is to rotate the basis of measurement, we deduce that the entropy of the detected signal depends in essence on the observable we choose.

In the case of input mixed states the effect of interference is reduced, so in every measurement basis we expect to have a residual randomness with a nonvanishing entropy; the limit case is the completely mixed state, where the entropy of the outcome is maximal for every measurement. Notice that an observer who detects the signal after BS1 is not able to distinguish between the pure case and the totally mixed one as they have the same statistics. Hence, in order to account for the information carried in the pure case, it is important to introduce a specific measure, different from the entropy $S(\rho)$ which only quantifies the incoherent information in the mixed case.

### 3 Coherent Entropy

Once the measurement basis is fixed, the probability of obtaining a given output is encoded in the diagonal elements $\rho_{ii}, i = 1, \ldots, d$ of the density operator in that basis. The information of the output measurements is quantified by the diagonal entropy of $\rho$, which we define as $S(\tilde{\rho})$, where $\tilde{\rho}_{ij} = \delta_{ij}\rho_{ij}$ is the density operator where all the off-diagonal entries have been set to zero. Now, we define the coherent entropy as

$$S_c(\rho) = \max_{\sigma \in \mathcal{U}_\rho} [S(\tilde{\sigma})] - \min_{\sigma \in \mathcal{U}_\rho} [S(\tilde{\sigma})],$$

where $\mathcal{U}_\rho = \{U \rho U^\dagger : U \in \mathcal{M}_{d \times d}, U U^\dagger = I\}$ is the set of all matrices which are unitarily equivalent to $\rho$. In other words, $S_c(\rho)$ measures the difference between the maximal and the minimal entropy of the outcomes obtained by measuring $\rho$ over any possible observable. This difference accounts for all the interference effects, so it has to be intended as a measure \cite{4,5} of the coherent informational content of $\rho$. As it
should be for a proper intrinsic property of a quantum state, $S_c(\rho)$ is independent of any choice of measurement made by the experimenter, i.e. it is invariant under local unitary transformations. The apparently hard optimization problem of evaluating Eq. (1) eventually leads to a very simple result:

$$S_c(\rho) = \log_2 d - S(\rho). \quad (2)$$

Proof For every density operator $\sigma \in \mathcal{U}_\rho$, we have $S(\tilde{\sigma}) = -\text{Tr}(\tilde{\sigma} \log \tilde{\sigma}) = -\text{Tr}(\sigma \log \tilde{\sigma})$, as $\tilde{\sigma}$ is the diagonal part of $\sigma$. The difference $S(\tilde{\sigma}) - S(\sigma) = \text{Tr}[\sigma (\log \sigma - \log \tilde{\sigma})]$ is non negative as it coincides with the relative entropy $S(\tilde{\sigma} \parallel \sigma)$, and the equality holds if $\tilde{\sigma} = \sigma$ [6]. Since $S(\sigma) = S(\rho)$, $\forall \sigma \in \mathcal{U}_\rho$, we get $\min_{\sigma \in \mathcal{U}_\rho} [S(\tilde{\sigma})] = S(\rho)$. Regarding the first term in Eq. (1), we can say it is equal to the entropy of the totally mixed state, namely $\log_2 d$. Indeed, under the most general unitary group, every density operator can be transformed into the matrix with diagonal elements uniformly equal to $1/d$. (The proof of this statement is illustrated in the Supplemental Material.)

The assignment of a purely quantum entropic measure $S_c(\rho)$ to a state through Eq. (2) says that the informational content of a pure state is all coherent, while its (von-Neumann) entropy is zero. On the opposite side, in a totally mixed state $S_c(\rho) = 0$. Notice that the sum of $S_c(\rho)$ and $S(\rho)$ is always equal to $\log_2 d$ for every state, meaning that every quantum state (at variance with classical ones) produces a constant unavoidable maximal randomness in the outcomes.

Though the expression in the r.h.s. of Eq. (2) has already appeared in the literature [7] as the amount of thermodynamic work that $\rho$ can extract from a heat bath or the number of pure state distillable from $\rho$ [8], it was not obtained and interpreted in the present way.

4 Time Correlations

In this section we want to show that the coherent entropy of a given a state $\rho$ expressed in Eq. (2) is exactly equal to the amount of information conveyed between past and future measurements, due to quantum self-correlations in time. We consider the scheme depicted in Fig. 2: the state of interest $\rho$ is prepared by the measurement of some observable on $\rho_1$ at time $t_1$ and a subsequent decoherence through interaction with the environment. At a later time $t_2$, the quantum state undergoes another measurement. The time correlation between the two measurement signals $s_1, s_2$, with probabilities $p(s_1)$ and $p(s_2)$, is estimated by their mutual information

$$I_{1:2} = \sum_{s_1,s_2} p(s_1, s_2) \log_2 \left( \frac{p(s_1, s_2)}{p(s_1)p(s_2)} \right) \quad (3)$$

where $p(s_1, s_2)$ is the joint probability. The quantum state can be viewed as a channel, whose capacity is obtained by maximizing $I_{1:2}$ over all inputs.
For the sake of clarity, we present here a detailed calculation of $I_{1:2}$ in the case of one qubit and a depolarizing channel as a model of decoherence. Assume that initially we have the state $\rho_1 = \frac{1}{2}(\mathbb{1}_2 + r_1 \cdot \sigma)$ represented by the vector $r_1$ inside the Bloch sphere. At time $t_1$, we decide to perform a projective measurement along the direction described by the unit vector $\hat{n}_1$. The outcome $s_1 = \pm 1$ will correspond to the state $P_{\hat{n}_1}^{s_1} = \frac{1}{2}(\mathbb{1}_2 + s_1 \hat{n}_1 \cdot \sigma)$ with probability $p(s_1) = \text{Tr}(\rho_1 P_{\hat{n}_1}^{s_1}) = \frac{1}{2}(1 + s_1 r_1 \cdot \hat{n}_1)$. The subsequent depolarizing channel will simply reduce the length of the Bloch vector $\hat{n}_1 \to \mathbf{n}_1$, without changing its direction. Finally, at time $t_2$, we perform a second projective measurement along $\hat{n}_2$. The outcome $s_2 = \pm 1$ will be related to the state $P_{\hat{n}_2}^{s_2} = \frac{1}{2}(\mathbb{1}_2 + s_2 \hat{n}_2 \cdot \sigma)$ with conditional probability $p(s_2|s_1) = \text{Tr}(P_{\hat{n}_1}^{s_1} P_{\hat{n}_2}^{s_2}) = \frac{1}{2}(1 + s_1 s_2 n_1 \cdot \hat{n}_2)$. Hence we get $p(s_1, s_2) = p(s_2|s_1) p(s_1)$ and $p(s_2) = \sum_{s_1} p(s_1, s_2)$, so the mutual information (3) becomes

$$I_{1:2} = H_2 \left( \frac{1 + (r_1 \cdot \mathbf{n}_1)(n_1 \cdot \mathbf{n}_2)}{2} \right) - H_2 \left( \frac{1 + n_1 \cdot \mathbf{n}_2}{2} \right),$$

where $H_2(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)$ is the binary entropy. The maximum of $I_{1:2}$ is achieved for $r_1 \cdot \mathbf{n}_1 = 0$ (i.e., the first measurement is orthogonal to the initial state, or simply the initial state is totally mixed with $r_1 = 0$) and $n_1 \cdot \mathbf{n}_2 = |n_1|$ (i.e., the second measurement is collinear to the first one). This gives the value

$$I_{1:2} = \log_2 2 - H_2 \left( \frac{1 + |n_1|}{2} \right) = 1 - S(\rho),$$

which is the $d = 2$ version of $S_c(\rho)$ as expressed in Eq. (2). Notice that the two possibilities $s_1 = \pm 1$ of intermediate quantum state $\rho = \frac{1}{2}(\mathbb{1}_2 + s_1 n_1 \cdot \sigma)$ are unitarily equivalent, so they have the same entropy $S(\rho)$. It is possible to prove the exact match between $S_c$ and the maximal $I_{1:2}$ also in arbitrary dimension. (The proof is illustrated in the Supplemental Material.)

### 5 Conservation of Quantum Information

Assuming that the whole universe is in a pure state, then $\rho$ and its environment can be written in Schmidt decomposition and the entanglement entropy between them is...
exactly $S(\rho)$ [2,3]. Hence, for every quantum state $\rho$ of dimension $d$ the sum of the mutual information sent in time—quantified by its coherent entropy $S_c(\rho)$ in Eq. (2)—and the entanglement entropy $S(\rho)$ with the rest of the universe turns out to be the constant $\log d$. As a consequence, during a unitary evolution any loss of coherence is compensated by an equal increase of entanglement with the environment, and vice versa. This fact constitutes the basic statement for a conservation law of quantum information. If we interpret $S_c(\rho)$ as a measure of coherence of $\rho$, we obtain that Eq. (2) is an exact relation between coherence and entanglement.

Let us now consider the case of two spatially separated systems $A$ and $B$, described by an overall pure state $\rho_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}|$ and reduced density operators $\rho_A = \text{Tr}_B(\rho_{AB})$ and $\rho_B = \text{Tr}_A(\rho_{AB})$. Subadditivity of entropy [2,3] implies that the state of a system is more coherent than the sum of its parts,

$$S_c(\rho_{AB}) = S_c(\rho_A) + S_c(\rho_B) + I_{A:B}$$

where the excess of coherent entropy amounts to the non-negative quantity $I_{A:B} = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$, which is the (spatial) mutual information between the two systems $A$ and $B$. Curiously, $S_c(\rho_{AB})$ exceeds the sum of the contributions coming from its parts $A$ and $B$ even when $I_{A:B}$ receives contribution only from classical correlations. At variance with the entropy $S$, the quantity $S_c$ obeys the monotonicity property, and it is a convex function in the space of density matrices.

If $A$ and $B$ are isolated from the rest, so that unitary operations do not change the value of $S_c(\rho_{AB})$, then we observe that any variation of the “space-like” mutual information $I_{A:B}$ is compensated by an opposite variation of the “time-like” mutual information quantified by the coherent entropy $S_c(\rho_A) + S_c(\rho_B)$.

Specifying further to the case of pure $\rho_{AB}$, we fall in the situation where $B$ is the environment of $A$, and vice versa. Now, the mutual information $I_{A:B} = 2S(\rho_A) = 2S(\rho_B)$ quantifies the entanglement between $A$ and $B$, and does not contain any contribution from classical correlations. The total quantum information consists of $S_c(\rho_A)$ bits localized in $A$, the same amount in $B$, while the remainder $I_{A:B}$ is encoded in the correlations between $A$ and $B$. Every unitary process will alter the balance of these quantities, without changing their sum, which is equal to the constant $\log d_A + \log d_B$, i.e. the coherent entropy of the overall pure state.

Let us assume that $A$ is localized in a well-defined region in space, delimited by a closed surface $\Sigma$. The information stored in $I_{A:B}$ can be assigned to virtual bond degrees of freedom connecting the real individual subsystems in $A$ and $B$. Since all these bonds cross the surface $\Sigma$, such information can be topologically located on it. In other words, during the decoherence of $A$, also $B$ decoheres, and the consequent lost information flows from both sides toward the surface: a sort of complementary of the holographic principle known in quantum gravity [9]. In this picture, a change in $I_{A:B}$ yields no net “flow of coherence” through the surface.

In 1D lattice models, a well-known realization of such mechanism occurs when we describe matrix-product states (MPS) where the mutual information between two bipartition $A$ and $B$ of a chain is encoded in the matrices which describe the bond variables at the border between $A$ and $B$ [10].
6 Multi-partitions

After having analyzed the case of two systems, it is interesting to understand how the quantum information carried by a quantum state of a given system is distributed when we consider its partition in several parts [11]. In the case of a tripartition $ABC$, the overall coherent entropy is given by

$$S_c(\rho_{ABC}) = S_c(\rho_A) + S_c(\rho_B) + S_c(\rho_C) + I_{A:B} + I_{A:B:C} \quad (6)$$

or cyclic permutations of subscripts $A$, $B$, $C$. The advantage of having an expression like Eq. (6), is that it involves only entropies and mutual informations, which are non-negative objects quantifying amounts of information. The generalization to $n$ partitions ordered from 1 to $n$ is

$$S_c(\rho_{1\ldots n}) = \sum_{k=1}^{n} S_c(\rho_k) + I_{1:2} + I_{12:3} + \cdots + I_{1\ldots(n-1):n}$$

which can be made symmetric with respect to any label ordering.

7 Locally Achievable Coherence

The result (2) is obtained when the optimization problem (1) is solved in the space of all the possible unitary transformations $U_\rho$. However, one may be interested to restrict the calculation to the family of local transformations with respect of a given partition. For a bipartite state $\rho_{AB}$ we can define

$$S_c^{loc}(\rho_{AB}) = \max_{\sigma \in U_{\rho_{AB}}^{loc}} \left[ \tilde{S}(\sigma) \right] - \min_{\sigma \in U_{\rho_{AB}}^{loc}} \left[ \tilde{S}(\sigma) \right], \quad (7)$$

where $U_{\rho_{AB}}^{loc}$ is the set of all matrices which are equivalent to $\rho_{AB}$ under local unitaries $U_A \otimes U_B$. The result of such an optimization is not guaranteed to give the same clean expression as in Eq. (2); instead we expect a lesser value which must be calculated numerically. It is appropriate to define the coherence gap $G(\rho_{AB}) = S_c(\rho_{AB}) - S_c^{loc}(\rho_{AB})$, namely the information which cannot be accessed by local operations. The quantity $G(\rho_{AB})$ accounts for nonlocal correlations between $A$ and $B$ (not necessarily the entanglement) in a similar fashion as the quantum discord [12], or the deficit [8]. The remaining local correlations between $A$ and $B$ are quantified by

$$L(\rho_{AB}) = I_{A:B} - G(\rho_{AB}).$$

8 Examples

In order to familiarize with the concepts discussed in this paper, we analyze the repartition of information in some quantum states.
• The Bell state $|\Psi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is a pure state with $d = 4$, i.e. $S_c(\rho_{AB}) = 2$, meaning 2 bits of information. After partial trace we get $\rho_A = \text{Tr}_B(|\Psi^+\rangle \langle \Psi^+|) = \frac{1}{2}(|00\rangle \langle 00| + |11\rangle \langle 11|)$, so both the subsystems are totally mixed, with $S_c(\rho_A) = S_c(\rho_B) = 0$. The two bits are stored in the mutual information $I_{A:B} = 2$, which we can figure out as localized on the whole system $AB$, while $A$ and $B$ are separately incoherent. Notice that one bit is due to entanglement entropy, $S(\rho_A) = 1$, while the remaining bit involves the classical parity correlations [3]. Remarkably, $G(\rho_{AB}) = 1$ is the same as the entanglement entropy.

• The three-site GHZ state $|\Psi_{\text{GHZ}}^+\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$, a paradigmatic example where tripartite entanglement is present, while the pairwise one is zero. The single subsystems $A$, $B$ and $C$ are all totally incoherent. Three qubits are stored in the mutual information between pairs $I_{A:B} = I_{A:C} = I_{B:C} = 1$, all made of local correlations $L$. Both the nonlocal indicators $G$ and the entanglement of formation $E_f$ (exactly computable for pairs of qubits [13]) are vanishing between all pairs. Interestingly, $G(\rho_{ABC})$ can be calculated also for the full tripartite case, resulting in one nonlocally achievable bit.

• The three-site W state $|\Psi_W^+\rangle = (|001\rangle + |010\rangle + |100\rangle)/\sqrt{3}$. The single quantities are summarized in the table: where we notice that some information is carried by single sites, while 0.918 bits is stored in pairwise correlations: 0.252 local and 0.667 nonlocal. The presence of nonlocal correlations is confirmed also by 0.550 bits as entanglement of formation $E_f$.

| $|\Psi_W\rangle$ | $\rho_A$ | $\rho_B$ | $\rho_C$ | $\rho_{AB}$ | $\rho_{AC}$ | $\rho_{BC}$ | $\rho_{ABC}$ |
|-----------------|---------|---------|---------|-----------|-----------|-----------|-----------|
| $S$             | 0.918   | 0.918   | 0.918   | 0.918     | 0.918     | 0.918     | 0          |
| $S_c$           | 0.082   | 0.082   | 0.082   | 1.082     | 1.082     | 1.082     | 3          |
| $G$             | 0.667   | 0.667   |         | 0.667     | 0.667     | 1.667     |
| $L$             | 0.252   | 0.252   |         | 0.252     | 0.252     | 0.252     |
| $I$             | 0.918   | 0.918   | 0.918   | 0.550     | 0.550     | 0.550     |
| $E_f$           | 0.550   | 0.550   | 0.550   |           |           |           |

9 Conclusions

This paper illustrates some arguments which lead to the definition of an entropic value coming from coherent information in quantum states. Such a quantity, here called coherent entropy, is indeed physical as it quantifies the (mutual) information conveyed in time by quantum states; so it is fundamental to give a complete description of their informational content. By means of this quantity and ordinary mutual information between different systems, it is possible to write equations of conservation of information in multipartite states, during unitary processes. Looking at a specific part of an interacting system, we observe that “time-like” information is transformed into “space-like” one: the overall information is conserved and the “flow” through a closed surface is governed by a holographic principle.
The space-time symmetric treatment of mutual information suggests a possible use in general relativity. For instance, it could help to shed some light in solving the famous paradox of information loss in black holes [9,14]. The change of metric signature after crossing the event horizon could be responsible of the transformation of space-like information into time-like, i.e. a purification of quantum states. This is notoriously connected with the interpretation of the measurement postulate in quantum mechanics which invokes a collapse of the wavefunction after extracting some information about the original state. On the contrary, in the present framework the consequence of a projective measurement is to inject quantum (coherent) information into a state, as the output quantum state is pure.

Finally, the constructive procedure used to infer the formula for the coherent entropy suggests a natural and simple definition of the indicator $G$, which detects and measures nonlocal correlations.

It would be interesting to explore other possible consequences of conservation of coherent information in foundations of quantum mechanics, field theories and statistical mechanics.

Acknowledgements Many thanks to Lorenzo Campos Venuti per very helpful discussions. This paper is dedicated to the memory of my friend Roberto Ghedini, who was the most sincere person I ever knew.

References

1. Horodecki, M., Horodecki, R., Sen(De), A., Sen, U.: Common origin of no-cloning and no-deleting principles conservation of information. Found. Phys. 35, 2041–2049 (2005)
2. Nielsen, M.A., Chuang, I.L.: Quantum Computation and Quantum Information. Cambridge University Press, New York (2000)
3. Preskill, J.: Quantum Information and Computation. Lecture Notes for Physics 229. California Institute of Technology, Pasadena (1998)
4. Åberg, J.: Quantifying superposition. arXiv:quant-ph/0612146 (2006)
5. Baumgratz, T., Cramer, M., Plenio, M.B.: Quantifying coherence. Phys. Rev. Lett. 113, 140401 (2014)
6. Wehrl, A.: General properties of entropy. Rev. Mod. Phys. 50, 221–260 (1978)
7. Oppenheim, J., Horodecki, M., Horodecki, P., Horodecki, R.: Thermodynamical approach to quantifying quantum correlations. Phys. Rev. Lett. 89, 180402 (2002)
8. Horodecki, M., Horodecki, K., Horodecki, P., Horodecki, R., Oppenheim, J., Sen, A., Sen, U.: Local information as a resource in distributed quantum systems. Phys. Rev. Lett. 90, 100402 (2003)
9. Susskind, L.: The world as a hologram. J. Math. Phys. 36, 6377–6396 (1995)
10. Fannes, M., Nachtergaele, B., Werner, R.F.: Finitely correlated states on quantum spin chains. Commun. Math. Phys. 144, 443–490 (1992)
11. Costa, A.C.S., Angelo, R.M., Beims, M.W.: Monogamy and backflow of mutual information in non-Markovian thermal baths. Phys. Rev. A 90, 012322 (2014)
12. Ollivier, H., Zurek, W.H.: Quantum discord: a measure of the quantumness of correlations. Phys. Rev. Lett. 88, 017901 (2001)
13. Wootters, W.K.: Entanglement of formation of an arbitrary state of two qubits. Phys. Rev. Lett. 80, 2245–2248 (1998)
14. Braunstein, S.L., Pati, A.K.: Quantum information cannot be completely hidden in correlations: implications for the black-hole information paradox. Phys. Rev. Lett. 98, 080502 (2007)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.