Polyline Simplification under the Local Fréchet Distance Has Almost-Quadratic Runtime in 2D

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Abstract
Given a polyline on \( n \) vertices, the polyline simplification problem asks for a minimum-size subsequence of these vertices defining a new polyline whose distance to the original polyline is at most a given threshold under some distance measure, usually the local Hausdorff or the local Fréchet distance. Here, \( \text{local} \) means that, for each line segment of the simplified polyline, only the distance to the corresponding sub-curve in the original polyline is measured.

This minimization problem is polynomial-time solvable (for the standard distance measures) as first shown by Imai and Iri for the local Hausdorff distance by a cubic-time algorithm. Later, Chan and Chin improved this running time to \( O(n^2) \). For the local Fréchet distance, the situation has been more intricate for the last decades. Doubtlessly, cubic running time is possible as shown by Godau adjusting the Imai–Iri algorithm to the Fréchet metric. This running time has been cited as state-of-the-art many times for the last 30 years also in influential articles. Very recently Buchin et al. [ESA ’22] showed how to improve this running time to \( O(n^{5/2+\varepsilon}) \) for any \( \varepsilon > 0 \) as an application of a sophisticated data structure to store polylines for Fréchet distance queries.

But there are actually older techniques and concepts in literature capable of solving this problem in almost-quadratic time. Namely, Melkman and O’Rourke [Computational Morphology ’88] introduced a geometric data structure to solve polyline simplification under the local Hausdorff distance in \( O(n^2 \log n) \) time, and Guibas, Hershberger, Mitchell, and Snoeyink [Int. J. Comput. Geom. Appl. ’93] considered polyline simplification under the Fréchet distance as \( \text{ordered stabbing} \) and provided an algorithm with a running time of \( O(n^2 \log^2 n) \), but they did not restrict the simplified polyline to use only vertices of the original polyline.

We show that their techniques can be adjusted to solve polyline simplification under the local Fréchet distance in \( O(n^2 \log n) \) time. This algorithm may serve as a more efficient subroutine for multiple other algorithms. We provide a simple algorithm description as well as rigorous proofs to substantiate this theorem. We also investigate the geometric data structure introduced by Melkman and O’Rourke, which we refer to as \( \text{wavefront} \), in more detail and feature some interesting properties. As a result, we can prove that under the \( L_1 \) and the \( L_\infty \) norm, the algorithm can be significantly simplified and then only requires a running time of \( O(n^2) \). We also define a natural class of polylines where our algorithm always achieves this running time also in the Euclidean norm \( L_2 \).

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1 Introduction
Polyline simplification has a long history in computational geometry, where it has also been known as \( \text{polygonal approximation} \), \( \text{line generalization} \), or \( \varepsilon \)-\( \text{simplification} \). It owes its relevance – also beyond computational geometry – to a large variety of applications, such as processing of vector graphics [32, 35], robotics [29, 15], trajectory clustering [8], shape analysis [27], data compression [26], curve fitting [30], and map visualization [2, 16, 18, 24, 33].

The task of polyline simplification is to replace a given polyline on \( n \) vertices with a minimum-size subsequence of its vertices while ensuring that the input and the output polyline are sufficiently similar. The similarity is governed by a given distance threshold \( \delta \). Line segments
between vertices in the output polyline are called shortcuts. To determine the similarity of the input and output polyline, the Hausdorff and the Fréchet distance are the most commonly used measures. Both can be applied either globally or locally. In the global version, the distance between the entire input and output polyline is measured and must not exceed $\delta$. In the local version, the distance between each shortcut and the part of the input polyline it bridges must not exceed $\delta$.

For many applications, local similarity is more sensible and intuitive. Simplifications with global similarity have only been studied recently. For the global (undirected) Hausdorff distance, computing a simplified polyline with the smallest number of shortcuts yields an NP-hard problem \([32]\). For the global Fréchet distance, an $O(n^3)$ time algorithm was designed by Bringmann and Chaudhury \([9]\). For the more extensively studied and more classical problem of simplification with the local Hausdorff distance, the Imai–Iri algorithm \([23]\), published in 1988, guarantees a running time of $O(n^3)$ by reducing the simplification problem to a graph problem. Essentially, the algorithm constructs a graph where the polyline vertices are the nodes and where there is an edge between a pair of vertices if they can be connected with a shortcut respecting the $\delta$ distance bound. For the local Hausdorff distance, Melkman and O’Rourke \([25]\) showed already in 1988 that the Imai–Iri algorithm can be improved to run in $O(n^2 \log n)$ time by making the graph construction phase more efficient. In 1996, Chan and Chin \([12]\) further reduced the running time to $O(n^2)$.

For the local Fréchet distance, though, the cubic running time of the Imai–Iri algorithm, which was shown by Godau \([17]\) in 1991, was a longstanding bound and used as a building block or reference also in recent publications \([9, 32]\). Agarwal, Har-Peled, Mustafa and Wang \([1]\) explicitly posed the problem of whether there exists a subcubic algorithm for polyline simplification under the local Fréchet distance as an open question in 2005. The question was answered positively very recently by Buchin, van der Hoog, Ophelders, Schlipf, Silveira, and Staals \([11]\). They describe a data structure that outputs the Fréchet distance between any line segment and any subpolyline of a preprocessed input polyline in $O(\sqrt{n} \log^2 n)$ time. Using this data structure to check whether there is a valid shortcut for every vertex pair, a polyline simplified optimally can be computed in $O(n^{5/2+\varepsilon})$ time (and space) for any $\varepsilon > 0$. We remark that this data structure is quite sophisticated and actually more powerful than required for polyline simplification. To check whether there is a valid shortcut between a vertex pair with respect to the Fréchet distance, it suffices to be able to decide whether the distance of the shortcut to its subpolyline is at most $\delta$. However, the data structure always returns the exact distance value and it can accomplish this for arbitrary line segments and not only potential shortcuts.

### 1.1 Related Work

The most practically relevant setting for polyline simplification is to consider two-dimensional input curves in the Euclidean plane (i.e., under the $L_2$ norm). However, the problem was also studied in higher dimensions $d > 2$ and under different norms.

The $O(n^3)$ time algorithm by Imai, Iri, and Godau \([23, 17]\) for polyline simplification under the local Hausdorff and Fréchet distance as well as the $O(n^3)$ time algorithm by Bringmann and Chaudhury \([9]\) for the global Fréchet distance can be generalized to work in $\mathbb{R}^{d+2}$ with the running time only increasing by a polynomial factor in $d$. For the local Hausdorff distance, Chan and Chin \([12]\) showed that the Imai–Iri algorithm can be improved to run in $O(n^2)$ time for $L_1$, $L_2$ and $L_\infty$ (the concept can also be applied to any $L^p$ up to possible numerical issues that are further discussed in Section \([3]\)). Furthermore, Barequet et al. \([5]\) proposed an $O(n^2 \log n)$ time algorithm for the local Hausdorff distance under the
For example, polyline simplification under the discrete Fréchet distance (where only the distance between the vertices but not the points on the line segments in between matters) can be solved to optimality in $O(n^2)$ time [6]. However, the discrete Fréchet distance heavily

| L_1 | $\tilde{O}(n^{3-\varepsilon})$ [9] | $\tilde{O}(2^d n^2)$ $\tilde{O}(n^3)$ | $O(n^2)$ $d = 2$ [Thm. 2] | $\tilde{O}(n^3)$ [9] |
| L_p∈(1,∞) $\forall p \neq 2$ | $\tilde{O}(n^{3-\varepsilon})$ [9] | $O(n^2)$ $d = 2$ [12] $\tilde{O}(n^3)$ $d = 3$ [5] | $O(n^2 \log n)$ $d = 2$ [Cor. 17] | $\tilde{O}(n^3)$ [9] |
| L_2 | $\tilde{O}(n^{2-\varepsilon})$ [10] | $O(n^2)$ $d = 2$ [12] $\tilde{O}(n^3)$ $d = 3$ [5] | $O(n^2 \log n)$ $d = 2$ [Thm. 1] | $\tilde{O}(n^3)$ [9] |
| L_∞ | $\tilde{O}(n^{2-\varepsilon})$ [10] | $O(n^2)$ $d = 2$ [12] $\tilde{O}(n^3)$ $d = 3$ [5] | $O(n^2)$ $d = 2$ [Thm. 2] | $\tilde{O}(n^3)$ [9] |

Table 1: Conditional lower bounds (presented as running times that are excluded) and algorithmic upper bounds for polyline simplification in $\mathbb{R}^d$ under different similarity measures and $L_p$ metrics. Here, $n$ is the number of vertices, $d$ the number of dimensions and $\varepsilon$ is any constant $> 0$. The $\tilde{O}$-notation hides polynomial factors in $d$. Blue entries mark the results presented in this manuscript.

L_2 norm in $\mathbb{R}^3$, as well as an $O(d2^d n^2)$ algorithm for L_1 and an $O(d^5 n^2)$ algorithm for L_∞.

Bringmann and Chaudhury [9] have also proven a conditional lower bound for simplification in $\mathbb{R}^d$ under the local Hausdorff distance as well as under the local and global Fréchet distance. More precisely, for $L_p$ with $p \in [1, \infty), p \neq 2$, algorithms with a running time subcubic in $n$ and polynomial in $d$ were ruled out (unless the $\forall \exists$-OV hypothesis fails). For large dimensions $d$, the algorithmic upper bounds of $O(n^3 \cdot \text{poly}(d))$ for polyline simplification discussed above are hence tight. However, the lower bound still allows the existence of simplification algorithms with a running time in $O(n^k \cdot \exp(d))$ with $k < 3$. Hence, for small values of $d$ (which are of high practical relevance), faster algorithms are possible, as evidenced by the $O(d2^d n^2)$ time algorithm for the local Hausdorff distance under the L_1 norm [5].

For L_2 and L_∞, the best currently known conditional lower bound for the three similarity measures, local Hausdorff distance, local Fréchet distance, and global Fréchet distance, was proven by Buchin et al. [10]. It rules out algorithms with a subquadratic running time in $n$ and polynomial running time in $d$ (unless SETH fails). Here again, better running times for simplification problems in a low-dimensional space with $d \in o(\log n)$ are still possible. Table 1 provides an overview of known lower and upper bounds for optimal polyline simplification.

As the cubic running time of the Inai–Iri algorithm and the quadratic running time of the Chan–Chin algorithm may be prohibitive for processing long polylines even for $d = 2$, heuristics and approximation algorithms have been investigated. The Douglas–Peucker algorithm [13], one of the most simple and widely used heuristics, computes a simplified polyline under the local Hausdorff distance in $O(n \log n)$ time [21] and under the local Fréchet distance in $O(n^2)$ time [52] – but without any guarantee regarding the solution size. Agarwal et al. [1] presented an approximation algorithm with a running time of $O(n \log n)$ that works for any $L_p$ norm and generalizes to $\mathbb{R}^d$. It computes a simplification under the local Fréchet distance for $\delta$, where the simplification size does not exceed the optimal simplification size for $\delta/2$. There are other heuristics neither based on the Hausdorff nor the Fréchet distance like the algorithm by Visvalingham and Whyatt [54], which measures the importance of a vertex by the triangular area it adds.

There are also variants of the Fréchet distance which allow for faster polyline simplification. For example, polyline simplification under the discrete Fréchet distance (where only the distance between the vertices but not the points on the line segments in between matters) can be solved to optimality in $O(n^2)$ time [6]. However, the discrete Fréchet distance heavily
depends on the density of vertices on the polyline, and hence for many applications the continuous Fréchet distance studied in this paper constitutes a more meaningful measure.

The problem variant where the requirement is dropped that all vertices of the simplification must be vertices of the input polyline is called a weak simplification. Guibas et al. [19] showed that an optimal weak simplification under the (global) Fréchet distance can be computed in $O(n^2 \log^2 n)$ time. Later Agarwal et al. [11] gave an $O(n \log n)$-time approximation algorithm for a weak simplification violating the distance threshold $\delta$ by a factor of at most 8; see also an overview by Van de Kerkhof et al. [31].

Regarding our techniques, we remark that the concept of a wavefront being comprised of (circular) arcs is well-established in computational geometry. Mitchell, Mount, and Papadimitriou [28] have introduced the continuous Dijkstra method in 1987, where a wavefront is expanded along a surface of a polyhedron to allow shortest path computations. This method was also used by Hershberger and Suri [22] to compute a shortest path in the plane given a set of polygonal obstacles. Also, many sweep-line algorithms maintain a kind of a wavefront. For example, the famous algorithm by Fortune [16] for computing a Voronoi diagram maintains a wavefront made up of hyperbolic curves. Another approach computes a (weighted) Voronoi diagram directly by a wavefront expanding around the input points [20]. Apart from a few similarities regarding the computation of intersection points of (circular) arcs, line segments, etc., we use a different type of wavefront. In all of the aforementioned examples, the wavefront is a kinetic data structure where things move or expand continuously (although it suffices to consider a few discrete events). In contrast, the structure by Melkman and O’Rourke [25] that we call a wavefront is more static and simple: it is a collection of (circular) arcs that we update iteratively with a new unit circle. There is no expansion of existing arcs or “time in between”.

1.2 Contribution

We present an algorithm for polyline simplification under the local Fréchet distance in two dimensions and for several $L_p$ norms with a (near)-quadratic running time; see Section 3.

Our algorithm heavily builds upon the Melkman–O’Rourke algorithm [25]. They exploit the geometric properties of the local Hausdorff distance using cone-shaped wedges and a wavefront to accelerate the shortcut graph construction. We adapt both of these concepts to the local Fréchet distance. We carefully study the properties of the resulting wavefront and explain how to maintain and efficiently update a wavefront data structure which at its core is a simple balanced binary search tree. As our main result, we prove that the asymptotic running time of the Melkman–O’Rourke algorithm does not increase with our modifications; see Section 4.

▶ Theorem 1. A two-dimensional $n$-vertex polyline can be simplified optimally under the local Fréchet distance in the $L_2$ norm (the Euclidean norm) in $O(n^2 \log n)$ time and $O(n)$ space.

This is a large improvement compared to the cubic running time by Imai and Iri and by Godau, and also to the best currently claimed running time bound of $O(n^{5/2+\varepsilon})$ [11], compared to which it is also simpler. It is also faster than the $O(n^2 \log^2 n)$ running time of the weak simplification algorithm by Guibas et al. [19], Theorem 14) by a logarithmic factor. However, we remark that parts of their algorithm (Def. 4, Theorem 7, Lemma 8, Lemma 9) can be used to obtain the same result as described in their article and we partially

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1 This expansion is realized step-wise by discrete events.
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Figure 1 A polyline $L$ (dashed gray) and a simplification $S$ of $L$ (solid blue) according to a distance measure with distance parameter $\delta$.

re-use their techniques. Yet, their procedure is more complicated since it maintains more geometric information only needed for weak simplifications. Our algorithm is hence more straightforward for the setting of polyline simplification under the local Fréchet distance, which makes it conceptually easier to understand.

Furthermore, we show that under the $L_1$ and $L_\infty$ norm, the wavefront has constant complexity which improves the running time to $O(n^2)$. We argue that for a natural class of polylines, a quadratic running time can be achieved as well. To this end, we introduce the concept of $\nu$-light polylines; see Section 5.

Theorem 2. A two-dimensional $n$-vertex polyline can be simplified optimally under the local Fréchet distance in the $L_1$ and $L_\infty$ norm in $O(n^2)$ time and $O(n)$ space.

Besides this new application for the local Fréchet distance, investigating the structure and implementation of wedges and wavefronts is of independent interest and may also help to understand better the work by Melkman and O’Rourke [25] from 1988 and the algorithm by Guibas et al. [19], who both employ these data structures but give little detail on its structural properties and on how to perform operations with the wavefront. To this end, we have an extensive appendix with detailed proofs, which we have separated from the main part to keep the algorithm description more compact and understandable on its own, while the interested reader can find the additional content in the appendix. Statements whose proofs can be found in the appendix are marked with “⋆”.

2 Preliminaries

We start with some basic definitions in the field of polyline simplification, then recapitulate the important ingredients of the polyline simplification algorithms by Imai and Iri [23], Melkman and O’Rourke [25], as well as Guibas et al. [19], and finally specify the notation we use thereupon.

2.1 Basic Definitions

A polyline is a series of line segments that are defined by a sequence of $d$-dimensional points $L = (p_1, p_2, \ldots, p_n)$, which we call vertices. By $n$, we denote the length of a polyline. For $1 \leq i \leq j \leq n$, we let $L[p_i, p_j] := (p_i, p_{i+1}, \ldots, p_j)$, that is, the subpolyline of $L$ starting at vertex $p_i$, ending at vertex $p_j$, and including all vertices in between in order. The continuous (but not smooth) curve induced by the vertices of a polyline $L$ is denoted as $c_L : [1, n] \to \mathbb{R}^d$ with $c_L : x \mapsto ((|x| + 1 - x)p_{\lfloor x \rfloor} + (x - |x|)p_{\lceil x \rceil})$. The polyline simplification problem is defined as follows; for an example see Figure 1.
Definition 3 (Polyline Simplification). Given a polyline \( L = \langle p_1, p_2, \ldots, p_n \rangle \), a distance measure \( d_X \) for determining the distance between two polylines, and a distance threshold parameter \( \delta \), the objective is to obtain a minimum-size subsequence \( S \) of \( L \) such that \( p_1, p_n \in S \), and \( d_X(L, S) \leq \delta \). We refer to \( S \) as a simplification of the (original) polyline \( L \).

Next, we discuss typical candidates for such a distance measure \( d_X \), namely the Hausdorff and the Fréchet distance in their local and their global variant.

Definition 4 (Hausdorff Distance). Given two polylines \( L = \langle p_1, \ldots, p_n \rangle \) and \( L' = \langle q_1, \ldots, q_m \rangle \), the (undirected) Hausdorff distance \( d_H(L, L') \) is defined as

\[
d_H(L, L') := \sup_{p \in c_L} \inf_{q \in c_{L'}} d(p, q), \sup_{q \in c_{L'}} \inf_{p \in c_L} d(p, q),
\]

where \( \sup \) is the supremum, \( \inf \) is the infimum and \( d(p, q) \) is the distance between the points \( p \) and \( q \) under some norm (e.g., the \( L_2 \) norm).

An often raised criticism concerning the use of the Hausdorff distance is that it does not reflect the similarity of the courses of two polylines. In contrast, the Fréchet distance measures the maximum distance between two polylines while traversing them in parallel and is therefore often regarded as the better suited measure for polyline similarity.

Definition 5 (Fréchet Distance). Given two polylines \( L = \langle p_1, \ldots, p_n \rangle \) and \( L' = \langle q_1, \ldots, q_m \rangle \), the Fréchet distance \( d_F(L, L') \) is defined as

\[
d_F(L, L') := \inf_{\alpha, \beta \in [0,1]} \max_{t \in [0,1]} d(c_L(\alpha(t)), c_{L'}(\beta(t))),
\]

where \( \alpha : [0,1] \to [1, n] \) and \( \beta : [0,1] \to [1, m] \) are continuous and non-decreasing functions such that \( \alpha(0) = \beta(0) = 1, \alpha(1) = n, \) and \( \beta(1) = m \).

Observe that the Hausdorff distance is a lower bound for the Fréchet distance. Later, when considering the Fréchet distance, we may say that the distance threshold \( \delta \) is respected or exceeded already in the Hausdorff distance since not exceeding \( \delta \) for the Hausdorff distance is a necessary condition to not exceed \( \delta \) for the Fréchet distance.

Conventionally, in the context of polyline simplification, the local Hausdorff and local Fréchet distance is used, which only measures the Hausdorff or Fréchet distance between a line segment \( \langle p_i, p_j \rangle \) of the simplification and its corresponding subpolyline \( L[p_i, p_j] \) in the original polyline. Similar to the advantage of the Fréchet distance over the Hausdorff distance, the advantage of a local over a global distance measure is that the we compare only related parts of a polyline and its simplification. Hence, using the local Fréchet distance for polyline simplification is an arguably sensible choice.

When using a local distance measure, we can tell for each pair of vertices \( \langle p_i, p_j \rangle \) (for \( 1 \leq i < j \leq n \)) in the original polyline independently whether a simplification may contain the line segment \( \langle p_i, p_j \rangle \) or not by only considering the distance between the line segment \( \langle p_i, p_j \rangle \) and its corresponding subpolyline. When considering such a pair \( \langle p_i, p_j \rangle \) as a line segment for a simplification, we call it a shortcut. If the distance between a segment \( \langle p_i, p_j \rangle \) and its corresponding subpolyline does not exceed the distance threshold \( \delta \), we call it a valid shortcut. Note that trivially \( \langle p_i, p_{i+1} \rangle \) is always a valid shortcut for any \( i \in \{1, \ldots, n-1\} \).

In the definition of the Hausdorff and Fréchet distance, we can choose how the distance between two \( d \)-dimensional points, or vectors, is determined. Typically, a vector norm is used for this purpose. For \( p \in [1, \infty) \), the \( L_p \) norm of a vector \( x \in \mathbb{R}^d \) is defined as
Figure 2 Unit circles in $L_p$ norms for selected values of $p$. As their radius, we use $\delta$ instead of 1.

$\|x\|_p := \left( \sum_{i=1}^{d} |x_i|^p \right)^{1/p}$. For $p = 1$, it is called the Manhattan norm, for $p = 2$, the Euclidean norm. For $p \to \infty$, $L_\infty$ is called the maximum norm and it is defined as $\max_{i=1,...,n} |x_i|$. The unit sphere $S^p_d$ is the set of points in $\mathbb{R}^d$ within unit distance to the origin. While this unit is conventionally set to 1, we use $\delta$ here instead as this allows for easier integration to polyline simplification with error bound $\delta$. We hence define $S^p_d := \{ x \in \mathbb{R}^d | \|x\|_p \leq \delta \}$. For $d = 2$, $S^2_p$ is also called the unit circle in $L_p$; we illustrate some examples in Figure 2. For $L_1$ and $L_\infty$, it actually forms a square with side length $\sqrt{2}\delta$ and $2\delta$, respectively. For $L_2$, it is indeed a circle with radius $\delta$. For $p$ between 2 and $\infty$, it forms a supercircle which for larger $p$ resembles more and more a square. We refer to a contiguous subset of the boundary of a unit circle in $L_p$ as an arc.

2.2 Imai–Iri Algorithm

Given an $n$-vertex polyline $L = \langle p_1, \ldots, p_n \rangle$, the polyline simplification algorithm by Imai and Iri [23] proceeds in two phases. In the first phase, the shortcut graph is constructed. This graph has a node for each vertex of $L$ and it has an edge between two nodes if and only if there is a valid shortcut between the corresponding two vertices of $L$. For the Hausdorff and the Fréchet distance, it can be checked in $O(n)$ time whether the distance between a line segment and a polyline having $O(n)$ vertices exceeds $\delta$ [4]. Hence, the total running time of the first phase amounts to $O(n^3)$. In the second phase, a shortest path from the first node $p_1$ to the last node $p_n$ is computed in the shortcut graph, which can be accomplished in $O(n)$ time. In a naive implementation, the space consumption is in $O(n^2)$. However, it is not necessary to first construct the full shortcut graph and to compute the shortest path subsequently. Instead, the space consumption can be reduced to $O(n)$ by interleaving the two phases as follows: For $p_i$, the shortest path distance $d_i$ from $p_i$ to $p_n$ via shortcuts can be computed in linear time by considering all valid shortcuts $\langle p_i, p_j \rangle$ to vertices $p_j$ with $j > i$ and setting $d_i = 1 + \min_{\langle p_i, p_j \rangle} d_j$. Hence, if the vertices are traversed in reverse order, only the distance values for already processed vertices and the shortcuts of the currently considered vertex need to be kept in memory to compute the correct solution without increasing the asymptotic running time.

2.3 Melkman–O’Rourke Algorithm

Since in the Imai–Iri algorithm the construction of the shortcut graph dominates the runtime, accelerating this first phase also leads to an overall improvement. Melkman and O’Rourke [25] introduced a faster technique to compute the shortcut graph for the local Hausdorff distance.
Starting once at each vertex \( p_i \) for \( i \in \{1, \ldots, n\} \), they traverse the rest of the polyline vertex by vertex in \( O(n \log n) \) time to determine all valid shortcuts originating at \( p_i \).

To this end, they maintain a cone-shaped region called \textit{wedge} in which all valid shortcuts are required to lie. When traversing the polyline, the wedge may become narrower iteratively. Moreover, they maintain a \textit{wavefront} which is a sequence of circular arcs of unit circles. The wavefront subdivides the wedge into two regions – a valid shortcut \( \langle p_i, p_j \rangle \) has the endpoint \( p_j \) in the region not containing \( p_i \). In other words, a valid shortcut needs to cross the wavefront. The wavefront has size in \( O(n) \) and is stored in a balanced search tree (in the original article an augmented 2-3-tree) such that querying and updating operations can be performed in amortized \( O(\log n) \) time.

Containment in the wedge can be checked in constant time and the position of a vertex relative to the wavefront can be determined in \( O(\log n) \) time. Updating the wedge can be done in constant time. Updating the wavefront may involve adding an arc and removing several arcs. Here, the crucial observation \cite{25} is that the order of arcs on the wavefront is reverse to the order of the corresponding unit circle centers – all with respect to the angle around \( p_i \). This allows for binary search in \( O(\log n) \) time to locate a new arc within the wavefront. Although a linear number of arcs may be removed from the wavefront in a single step, over all steps any arc is removed at most once. Amortized, this results in a running time of \( O(n \log n) \) per starting vertex \( p_i \) and \( O(n^2 \log n) \) in total.

2.4 Algorithm by Guibas, Hershberger, Mitchell, and Snoeyink

Guibas et al. \cite{19} study weak polyline simplification. There, given an \( n \)-vertex polyline \( L = \langle p_1, \ldots, p_n \rangle \) and a distance threshold \( \delta \), the objective is to compute any polyline \( S = \langle q_1, \ldots, q_m \rangle \) of smallest possible length \( m \) that hits all unit circles around the vertices in \( L \) in the given order, which they call \textit{ordered stabbing}. To additionally have Fréchet distance at most \( \delta \) between \( L \) and \( S \), each vertex \( q_j \) of \( S \) needs to be in distance \( \leq \delta \) to some point of \( c_i \). They describe a 2-approximation algorithm running in \( O(n^2 \log n) \) time and a dynamic program solving this problem exactly in \( O(n^2 \log^2 n) \) time.

Both algorithms essentially rely on a subroutine to decide whether there exists a line \( \ell \) (a \textit{stabbing line}) that intersects a given set of \( n \) ordered unit circles \( \langle C_1, \ldots, C_n \rangle \) such that \( \ell \) hits some points \( (r_1, \ldots, r_n) \) with \( r_i \in C_i \), for \( i \in \{1, \ldots, n\} \) in order \cite{19}, Def. 4). This subroutine runs in \( O(n \log n) \) time \cite{19}, Lemma 9). It is based on an algorithm computing iteratively two hulls and two limiting lines through the unit circles that describe all stabbing lines \cite{19}, Algorithm 1). They also maintain the wavefront as described in the Melkman–O’Rourke algorithm. However, they add an update step to ensure that the stabbing line respects the order of the unit circles. We use conceptually the same update step and explain it in more detail in Sections \ref{sec: Yuan} and \ref{sec: Yuan2}.

To compute an optimal weak simplification \( S \) with the dynamic program, this subroutine is called once per unit circle induced by vertices in \( L \). Guibas et al. remark that the wavefront might have non-constant complexity. Hence they refrain from storing it explicitly. They only keep the wedge and support information in memory and construct the remaining information

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\(^2\) Melkman and O’Rourke \cite{25} use the term \textit{frontier} instead of wavefront. Within the cone, they only call the region on the other side of the frontier \textit{wedge} and they call the associated data structure \textit{wedge data structure}. Our notation to call the whole cone \textit{wedge} is in line with the algorithm by Chan and Chin \cite{12}.

\(^3\) This is necessary because here the Fréchet distance is considered, while Melkman and O’Rourke only considered the Hausdorff distance.
necessary to perform the update steps on demand. This further complicates the algorithm and adds a logarithmic factor per unit circle to the overall running time, which then is in $O(n^2 \log^2 n)$.

2.5 Definitions and Notation

The following definitions are illustrated in Figure 3. When starting at $p_i$ and encountering $p_j$ during the traversal, we denote by $D_{i,j}$ the local wedge of $p_i$ and $p_j$ that is the area between the two tangential rays of the unit circle around $p_j$ emanating at $p_i$. The (global) wedge $W_{i,j}$ is an angular region having its origin at $p_i$. We define $W_{i,i}$ to be the whole plane and each $W_{i,j}$ for $j > i$ is essentially the intersection of all local wedges up to $D_{i,j}$. We remark that, as mentioned in Section 2.4, we apply an extra update step described in Section 3.1 specific to the Fréchet distance, which may narrow the wedge when obtaining $W_{i,j}$ from $W_{i,j-1}$. Therefore, $W_{i,j} \subseteq \bigcap_{k \in \{i+1, i+2, \ldots, j\}} D_{i,k}$ holds. We give a precise inductive definition of the wedge $W_{i,j}$ when we describe the algorithm in Section 3.1.

Let $C_j$ be the unit circle around $p_j$ and let $l_j$ ($r_j$) be the left (right) tangential point of $C_j$ and $D_{i,j}$. Between $l_j$ and $r_j$, there are two arcs of $C_j$ — the bottom arc and the top arc.

Clearly, any ray emanating at $p_i$ intersects the bottom and the top arc at most once each. We call the bottom arc of $C_j$ between $l_j$ and $r_j$ the wave of $D_{i,j}$. We call the region within $D_{i,j}$ and above and on its wave the local valid region of $D_{i,j}$. We call the region within $W_{i,j}$ and above and on the wavefront the valid region of $W_{i,j}$ (for $W_{i,i}$ the whole plane).

The wavefront itself is defined inductively. The wavefront of $W_{i,j}$ (for $j > i$) is the boundary of the intersection of the valid region of $W_{i,j-1}$ and the local valid region of $D_{i,j}$ within $W_{i,j}$ and excluding the boundary of $W_{i,j}$. Intuitively, it is the wavefront of $W_{i,j-1}$ within $W_{i,j}$ where we cut out the bottom arc of $C_j$.

3 Local-Fréchet Simplification Algorithm in Near-Quadratic Time

In this section, we describe how to obtain our algorithm for polyline simplification under the local Fréchet distance running in near-quadratic time by means of Melkman and O’Rourke [25] and integrating ideas from Guibas et al. [19].

3.1 Outline

As all Imai–Iri based algorithms, we build the shortcut graph by traversing the given polyline $n$ times — starting once from each vertex $p_i$ for $i \in \{1, \ldots, n\}$ and determining all shortcuts starting at $p_i$. For each $p_i$, we construct a wedge with a wavefront, whose properties are analyzed in more detail in Section 4.

Next, we describe how to determine, for each vertex $p_i$, the set of subsequent vertices to which $p_i$ has a valid shortcut. We traverse the polyline in order $p_{i+1}, p_{i+2}, \ldots, p_n$. During this traversal, we maintain the wedge in which all valid shortcuts need to lie. This would, as in the algorithm by Chan and Chin [12], suffice to assure that the Hausdorff distance threshold is not violated (which is a lower bound for the Fréchet distance). To also not exceed

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4 W.l.o.g., we assume hereunder that $p_i$ is below $p_{i+1}$ and therefore at the bottom of a wedge. Moreover, w.l.o.g., we assume that $p_{i+1}$ has a distance of at least $\delta$ to $p_i$ because otherwise, we could ignore all vertices following $p_i$ and having distance $\leq \delta$ to $p_i$ since they are in $\delta$-distance to any shortcut $(p_i, p_j)$. Note, though, that a vertex $p_j$ with $j > i + 1$ could have distance $\leq \delta$ to $p_i$. Then, we define $D_{i,j}$ as the whole plane and the whole boundary of $C_j$ as its top arc.
Figure 3 Iterative construction of the wedge in the $L_1$, $L_2$ and $L_\infty$ norm: The local wedges $D_{1,2}$, $D_{1,3}$, and $D_{1,4}$ are visualized in pink. Here, the intersections of local wedges define the wedges $W_{1,2}$, $W_{1,3}$, and $W_{1,4}$. The wavefront is a sequence of unit circle arcs and is depicted in blue. Within the wedge and above the wavefront, there is the valid region (depicted in hatched green). This is the area, where a subsequent vertex $p_j$ needs to lie if there is a valid shortcut $\langle p_1, p_j \rangle$. For example, $\langle p_1, p_5 \rangle$ is a valid shortcut in the $L_\infty$ norm, whereas in the $L_1$ and $L_2$ norm it is not.
When encountering $p_j$, we update the wedge in two steps – even if $p_j$ lies outside the wedge.

**Figure 4** Updating the wedge and its wavefront in the $L_2$ norm.

the Fréchet distance threshold, we use the wavefront. As in the algorithm by Melkman and O’Rourke, the invariant maintained is that for a valid shortcut from $p_i$ to $p_j$ with $j > i$, the vertex $p_j$ has to be within the valid region of the wedge $W_{i,j-1}$. In this case, we add the directed edge $p_ip_j$ to the shortcut graph.

Then, regardless of whether $(p_i, p_j)$ is a valid shortcut or not, we first update the wedge $W_{i,j-1}$ to an intermediate wedge $W'_{i,j}$ by computing the intersection between $W_{i,j-1}$ and the local wedge $D_{i,j}$. Afterwards, we update the intermediate wedge $W'_{i,j}$ to the wedge $W_{i,j}$ and we update the wavefront.

This update step is illustrated for the $L_2$ norm in Figure 4 and for multiple steps and multiple norms in Figure 3. For the $L_2$ norm and for the $L_1$ and $L_{\infty}$ norms, we give more detail on this update step in Sections 4.2 and 5.1, respectively. It works as follows. A valid shortcut $(p_i, p_k)$ with $k > j$ in the Fréchet distance needs to go through the intersection region $I$ between the current valid region and the unit circle $C_j$ around $p_j$. Otherwise, the vertices of the subpolyline from $p_i$ to $p_k$ would be encountered in the wrong order contradicting the definition of the Fréchet distance. Hence, we narrow the intermediate wedge $W'_{i,j}$ such that the rays $R_l$ and $R_r$ emanating at $p_i$ and enclosing $I$ constitute the wedge $W_{i,j}$; see Figure 4a. This extra narrowing step is also applied by Guibas et al. in their line stabbing algorithm, but not by Melkman and O’Rourke. For the Hausdorff distance, it is irrelevant in which order the intermediate points of a shortcut are encountered by the shortcut segment.

Thereafter, we update the wavefront as Melkman and O’Rourke do. The part of the bottom arc of the unit circle $C_j$ around $p_j$ that is above the current wavefront is included into the new wavefront. Pictorially, the wavefront is moving upwards. For an example see Figure 4b. There, we compute the intersection point $s$ between $C_j$ and the wavefront and replace the arcs $a_i$ and $a_{i+1}$ of the wavefront by the arcs $a_t$ (which is a part of $a'_i$) and $a_{t+1}$ (which is a part of $C_j$). There can be up to two intersection points between $C_j$ and the wavefront.

If the valid region becomes empty, we abort the search for further shortcuts from $p_i$.  

![Figure 4a](image)

(a) When encountering $p_j$, we update the wedge in two steps – even if $p_j$ lies outside the wedge.

![Figure 4b](image)

(b) Vertex $p_j$ contributes an arc to the new wavefront. Here, $(p_i, p_j)$ is also a valid shortcut.
3.2 Correctness

To show that the algorithm works correctly, we prove two things: that all shortcuts the algorithm finds are valid (Lemma 11) and that the algorithm finds all valid shortcuts (Lemma 12). As it is more difficult to show this directly, we first state five helpful lemmas, which we use throughout the remainder of this manuscript.

- **Lemma 6.** Given two unit circles and a point \( p \) outside of the unit circles. If the two bottom arcs (with respect to \( p \)) intersect, then the second intersection point is between their top arcs.

- **Lemma 7.** If a unit circle \( C \) intersects the wavefront more than once, then on the left side of the leftmost intersection point \( s_1 \) (relative to rays originating in \( p \)) and on the right side of the rightmost intersection point \( s_2 \), \( C \) is below the wavefront. In other words, the intersection pattern depicted in Figure 7a cannot occur.

- **Lemma 8.** Consider the wavefront of \( W_{i,k} \). For every vertex \( p_j \) \((i < j \leq k)\) whose unit circle \( C_j \) contributes an arc of the wavefront of \( W_{i,k} \), the wavefront of \( W_{i,k} \) lies completely inside \( C_j \).

Lemma 8 directly implies the following lemma.

- **Lemma 9.** Let \( q \) be a point lying on the wavefront of the wedge \( W_{i,j} \). Then, \( d(p_i, q) \leq \delta \).

- **Lemma 10.** Let \( R \) be a ray emanating at \( p_i \) and lying inside the wedges \( W_{i,j} \) and \( W_{i,k} \) for some \( i < j < k \). Moreover, let \( q_j \) and \( q_k \) be the intersection points between \( R \) and the wavefronts of \( W_{i,j} \) and \( W_{i,k} \), respectively. Then, \( d(p_i, q_j) \leq d(p_i, q_k) \).

**Proof.** Assume for contradiction that \( d(p_i, q_j) > d(p_i, q_k) \). Then, \( q_k \) is below the wavefront of \( W_{i,j} \) and, hence, \( q_k \) does not lie in the valid region of \( W_{i,j} \) but in the valid region of \( W_{i,k} \). However, the valid region of \( W_{i,k} \) is the intersection of the local valid region of \( D_{i,k} \) and all previous valid regions including \( W_{i,j} \) and, thus, the valid region of \( W_{i,k} \) is a subset of \( W_{i,j} \).

Putting Lemma 10 in other words, the wavefront may only move away but never towards \( p_i \) during the execution of the algorithm. We are now ready to prove the correctness of the algorithm by the following two lemmas.

- **Lemma 11.** Any shortcut found by the algorithm is valid under the local Fréchet distance and any \( L_p \) norm with \( p \in [1, \infty] \).

**Proof.** Let \( (p_i, p_k) \) be a shortcut found by the algorithm. We show that there is a mapping of the vertices \( (p_{i+1}, p_{i+2}, \ldots, p_{k-1}) \) onto points \( (m_{i+1}, m_{i+2}, \ldots, m_{k-1}) \), such that \( m_j \in \overline{p_ip_k} \) and \( d(p_j, m_j) \leq \delta \) for every \( j \in \{i + 1, \ldots, k - 1\} \), and \( m_j \) precedes or equals \( m_{j+1} \) for every \( j \in \{i + 1, \ldots, k - 2\} \) when traversing \( \overline{p_ip_k} \) from \( p_i \) to \( p_k \). Clearly, this implies that also the Fréchet distance between each pair of line segments \( \overline{p_ip_{j+1}} \) and \( \overline{m_jm_{j+1}} \) is at most \( \delta \) and, hence, \( (p_i, p_k) \) is a valid shortcut. In the remainder of this proof, we describe how to obtain \( m_{i+1}, m_{i+2}, \ldots, m_{k-1} \in \overline{p_ip_k} \). To this end, we consider the wedge \( W_{i,j} \) and the corresponding wavefront for each \( j \in \{i + 1, \ldots, k - 1\} \), i.e., for each intermediate step when executing the algorithm. By construction of the algorithm, \( \overline{p_ip_k} \) lies inside the wedge \( W_{i,j} \) and \( p_k \) lies above its wavefront (since \( p_k \) lies in the valid region of \( W_{i,k-1} \) and, by Lemma 10, the wavefront has never moved towards \( p_i \)). Let \( m_j \) be the intersection point of \( \overline{p_ip_k} \) and the wavefront of \( W_{i,j} \). By Lemma 9 \( d(p_j, m_j) \leq \delta \). Moreover, by Lemma 10 \( m_j \) precedes or equals \( m_{j+1} \) for any \( j \in \{i + 1, \ldots, k - 2\} \) when traversing \( \overline{p_ip_k} \) from \( p_i \) to \( p_k \).
We first prove that the wavefront always has a size in $O(n^2 \log n)$ time, we next analyze the properties of the wavefront and discuss how to store and update it using a suitable (simple) data structure.

4.1 Size of the Wavefront

We first prove that the wavefront always has a size in $O(n)$. This insight is based on two properties proven in the following lemmas.

Lemma 13. Any ray emanating at $p_i$ intersects the wavefront at most once.

Proof. We prove this statement inductively. As $W_{i,i+1} = D_{i,i+1}$, consider the wave of $D_{i,i+1}$. Since the unit circle in any $L_p$ norm for $p \in [1, \infty]$ is convex, any ray emanating at $p_i$ intersects a unit circle at most twice. The first intersection is with the bottom arc of the unit circle $C_{i+1}$ and the second intersection is with the top arc of $C_{i+1}$. As the wave of $D_{i,i+1}$ is defined as the bottom arc of $C_{i+1}$, any ray emanating at $p_i$ intersects the wave of $D_{i,i+1}$ at most once.

It remains to show the induction step for all $j > i + 1$. By the induction hypothesis, we know that any ray emanating at $p_i$ intersects the wavefront of $W_{i,j-1}$ at most once. The

\begin{align*}
\text{Lemma 12. All valid shortcuts under the local Fréchet distance and any } L_p \text{ norm with } p \in [1, \infty] \text{ are found by the algorithm.}
\end{align*}

Proof. Suppose for the sake of a contradiction that there is a valid shortcut $\langle p_i, p_k \rangle$ that was not found by the algorithm.

If $p_k$ lay outside of $\bigcap_{j \in \{i+1,i+2,...,k-1\}} D_{i,j}$, there would be some $p_{j'}$ with $i < j' < k$ such that $d(p_{j'}, \overline{p_ip_k}) > \delta$. So, as in the algorithm by Chan and Chin [12], already the Hausdorff distance requirement would be violated and $\langle p_i, p_k \rangle$ would be no valid shortcut.

Hence, $p_k$ lies inside $\bigcap_{j \in \{i+1,i+2,...,k-1\}} D_{i,j}$.

Suppose now that $p_k$ lies inside $\bigcap_{j \in \{i+1,i+2,...,k-1\}} D_{i,j}$ but outside $W_{i,k-1}$. W.l.o.g. $p_k$ lies to the left of the wedge $W_{i,k-1}$. We know that there is some $p_j$ with $i < j < k$ for which the extra narrowing step from Section 3.1 has been applied such that $p_k$ lies to the left of $W_{i,j}$. For constructing $W_{i,j}$, we have considered the intersection area $I$ between $C_j$ and the wavefront of $W_{i,j-1}$. The left endpoint of $I$ lies on the boundary of $W_{i,j}$ and is the intersection point between $C_j$ and an arc of the wavefront of $W_{i,j-1}$ belonging to a vertex $p_{j'}$ with $i < j' < j$. Now consider the ray $R$ that we obtain by extending $\overline{p_ip_k}$ at $p_k$. When traversing $R$, we first enter and leave the interior of $C_j$ before we enter the interior of $C_{j'}$. Hence, the Fréchet distance between $\overline{p_ip_k}$ and $L[p_i,p_k]$ is greater than $\delta$ due to the order of $p_j$ and $p_j$ within $L[p_i,p_k]$. Therefore, $p_k$ lies inside $W_{i,k-1}$.

Finally, suppose that $p_k$ lies inside $W_{i,k-1}$ but not in the valid region, i.e., $p_k$ lies below the wavefront of $W_{i,k-1}$. Since $p_k$ is below the wavefront, the line segment $\overline{p_ip_k}$ does not intersect the wavefront (otherwise, we would violate Lemma 13 see below). Again, consider the ray $R$ that we obtain by extending $\overline{p_ip_k}$ at $p_k$. Let the intersection point of $R$ and the wavefront of $W_{i,k-1}$ be $w$. The point $w$ lies on an arc of the wavefront. This arc is part of the bottom arc of a unit circle $C_j$ belonging to some $p_j$ with $i < j < k$. Since it is the bottom arc, $p_k$ lies outside $C_j$ and $d(p_k,p_j) > \delta$.

Therefore, $p_k$ lies inside the valid region of $W_{i,k-1}$. However, these are precisely the vertices for which the algorithm adds a shortcut. ▶

4 The Wavefront Data Structure

At the heart of the algorithm lies the maintenance of the wavefront. To show that the algorithm can be implemented to run in $O(n^2 \log n)$ time, we next analyze the properties of the wavefront and discuss how to store and update it using a suitable (simple) data structure.
wavefront of \( W_{i,j} \) is the boundary of the intersection of the valid region of \( W_{i,j-1} \) and the local valid region of \( D_{i,j} \). Consider a ray \( R \) originating at \( p_i \). The ray \( R \) enters the valid region of \( W_{i,j-1} \) at most at one point \( q \) where it also intersects the wavefront of \( W_{i,j-1} \), and it enters the local valid region of \( D_{i,j} \) at most at one point \( q' \) where it also intersects the wave of \( D_{i,j} \). Hence, \( R \) enters the intersection of the valid region of \( W_{i,j-1} \) and the local valid region of \( D_{i,j} \) at most at one point — namely either at \( q \) or at \( q' \) (or \( q = q' \)). This is the only point of the wavefront of \( W_{i,j} \) that is shared with \( R \). ▶

We can make a similar statement for unit circles. The number of intersection points between a unit circle and the wavefront is important for updating the wavefront.

**Lemma 14**. Any unit circle of radius \( \delta \) intersects the wavefront at most twice.

From Lemma 14 it follows that in each step, the size of the wavefront increases at most by 2. This leads us to the following lemma.

**Lemma 15.** The wavefront consists of at most \( O(n) \) arcs under any \( L_p \in (1, \infty) \) norm.

**Proof.** According to the inductive definition, we start with a wavefront consisting of one arc. Now in each step where we extend the wavefront, we consider the intersection between the current valid region and a local valid region — one is defined by the current wavefront, the other is defined by a single arc \( a \). This is the intersection between the current wavefront and the unit circle on which \( a \) lies. By Lemma 14 we know that there are at most two intersection points. This means, the number of arcs on the wavefront increases by at most two. In the worst case, we start at vertex \( p_1 \) and adjust the wavefront \( n-1 \) times until we have created the wavefront of \( W_{1,n} \). Therefore, any wavefront consists of at most \( 2n - 3 \in O(n) \) arcs. ▶

### 4.2 Wavefront Maintenance under the \( L_2 \) Norm (Euclidean Norm)

As there might be a linear number of arcs on the wavefront, we cannot simply iterate over all arcs in each step of the algorithm since this would require cubic time in total. Therefore, we employ a data structure that allows for querying, inserting, and removing an object in logarithmic time. Similar to Melkman and O’Rourke, we use a balanced search tree (e.g., a red-black tree) where we store the circular arcs of the wavefront. The keys according to which the circular arcs are arranged in the search tree are the angles of their starting points with respect to \( p_i \). These angles cover a range of less than \( \pi \), hence, we may rotate the drawing when computing the wavefronts of a vertex \( p_i \) to avoid “jumps” from \( 2\pi \) to 0. We can then locate a point \( p_j \) relative to the wavefront and add or remove an arc on the wavefront in logarithmic time.

Note that, different from Melkman and O’Rourke and similar to Guibas et al., we have an additional update step where we determine the intersection region \( I \) and potentially make the wedge narrower. We show that we can update the wavefront in amortized logarithmic time using a simple case distinction. We compute the intersection area \( I \) only implicitly. For an overview, see Figure 5.

Say we are computing all valid shortcuts starting at \( p_1 \) and we are currently processing a vertex \( p_j \), which is the center of the unit circle \( C_j \). We have already constructed the

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5 One can further observe that, by Lemma 7, the number of arcs on the wavefront increases actually by at most one per vertex \( p_j \) (\( j \in \{2, \ldots, n\} \)). This means any wavefront consists of at most \( n-1 \) arcs.

6 To represent a circular arc, we store its corresponding unit circle center and the points where the arc starts and ends.
Figure 5  Cases for updating the wavefront when a vertex $p_j$, whose unit circle is $C_j$, is added. The wavefront of the previous step is illustrated as a blue wavy line. Green background color (everything inside $C_j$) highlights the parts that remain part of the wavefront and red and orange background color (everything outside $C_j$) highlight the parts that are removed from the wavefront. The part of the bottom arc of $C_j$ that becomes part of the wavefront is highlighted by blue background color.
intermediate wedge $W'_{i,j}$ and clipped the wavefront of $W_{i,j-1}$ along the left and the right bounding rays $R_l$ and $R_r$ of $W'_{i,j}$. For this clipping, we may have removed a linear number of arcs, however, over all iterations we remove every arc at most once. Now, both $R_l$ and $R_r$ intersect $C_j$ twice or touch $C_j$. Let $q_1$ and $q_2$ denote the intersection points between $R_l$ and $C_j$ (where $q_1$ is on the bottom arc of $C_j$). Similarly, let $q_3$ and $q_4$ denote the intersection points between $R_r$ and $C_j$ (where $q_3$ is on the bottom arc of $C_j$). Moreover, let $l$ and $r$ denote the intersection point between the wavefront and $R_l$ and between the wavefront and $R_r$, respectively.

The relative positions of $l$, $q_1$, and $q_2$ on $R_l$ and the relative positions of $r$, $q_3$, and $q_4$ on $R_r$ determine where the intersection points $s_1$ and $s_2$ (if they exist) between $C_j$ and the wavefront of $W_{i,j-1}$ can lie. (Recall that there are at most two intersection points by Lemma 13.) In the following, we write $a < b$ if $a$ is below $b$ along the ray $R_l$ or $R_r$. If $q_1 = l$ or $q_2 = l$, then we proceed as if $R_l$ was moved to the right by a tiny bit (symmetrically as if $R_r$ was moved to the left). Thus, at such a point, the angle of the incident arc of the wavefront and $C_j$ matters for the order. For the degenerate case $q_1 = l = q_2$, which includes a touching point between $R_l$ and $C_j$, we hence assume $q_1 < l < q_2$.

Next, we consider all orderings of $l$, $r$, $q_1$, $q_2$, $q_3$, and $q_4$. This gives rise to the following nine cases. We remark that two of them (Case TB and Case BT) cannot occur and two pairs of the remaining cases are symmetric, which leaves essentially five different cases.

(Case TB:) $q_1 < q_2 < l$ and $r < q_3 < q_4$; see Figure 5a. This case cannot occur. Suppose for a contradiction that we have this configuration. Then, there are precisely two intersection points $s_1$ and $s_2$ between the wavefront and $C_j$. The left intersection point $s_1$ is between the top arc of $C_j$ and an arc $a_k$ of the wavefront, which is part of the bottom arc of a unit circle $C_k$ belonging to a vertex $p_k$ with $i < k < j$. By Lemma 8, $C_k$ contains the whole wavefront, thus including $s_2$, which means $C_k$ and $C_j$ intersect a second time such that this intersection point is to the left of $s_2$. Then, however, $C_k$ and $C_j$ intersect once with both bottom arcs and once with a bottom and a top arc — a contradiction to Lemma 8. For more details on this argument, see in Appendix A the proof of Lemma 8 and Figure 8.

Case TM: $q_1 < q_2 < l$ and $q_3 < r < q_4$; see Figure 5b. There is an intersection point $s_1$ between $C_j$ and the wavefront. We traverse the arcs in the balanced search tree representing the wavefront starting at the leftmost arc, which in turn starts at point $l$, and remove all arcs that we encounter until we find the intersection point $s_1$ between $C_j$ and an arc $a$ of the wavefront. We update $a$ to start at $s_1$ and the left bounding ray of $W_{i,j}$ to go through $s_1$. There cannot be a second intersection point because otherwise there would also be a third intersection point between a unit circle and the wavefront.

Case TT: $q_1 < q_2 < l$ and $q_3 < q_4 < r$; see Figure 5c. In this case, we either have two or no intersection points between the wavefront and $C_j$. We traverse the wavefront starting at $l$ and remove all arcs that we encounter and that do not intersect $C_j$. If we do not find any intersection point but reach $r$, then there cannot be any further valid shortcut starting at $v_1$ and we abort. Otherwise, we have found $s_1$ and proceed symmetrically at $r$ to find $s_2$.

Case MB: $q_1 < l < q_2$ and $r < q_3 < q_4$; see Figure 5d. There is precisely one intersection point $s_1$ between the wavefront and the bottom arc of $C_j$. We traverse the wavefront starting at $r$ and remove all arcs that we encounter and that do not intersect $C_j$ until we have

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7 In the following we just say for short, “we traverse the wavefront starting at $l$.”
found $s_1$. We clip the arc of the wavefront at $s_1$ and append the bottom arc of $C_j$ between $s_1$ and $r$ to the wavefront.

**Case MM:** $q_1 \prec l \prec q_2$ and $q_3 \prec r \prec q_4$; see Figure 5e. There are either two or no intersection points between the wavefront and $C_j$. If there are two intersection points, then they are on the bottom arc of $C_j$ as otherwise, it would contradict Lemma 7. The order of the arcs on the wavefront around $p_i$ is reverse to the order of the corresponding unit circle centers around $p_i$ [25]. By binary search, we determine the arcs $a_k$ and $a_{k+1}$ such that $p_j$ lies in between the corresponding unit circle centers of $a_k$ and $a_{k+1}$ with respect to the angle around $p_i$. If $a_k$ and $a_{k+1}$ are completely contained inside $C_j$, then there is no intersection point and the wavefront remains unchanged.

Otherwise, we traverse the wavefront starting at $a_k$ to the left until we have found an arc intersecting $C_j$, which gives us $s_1$. We remove all arcs along the way and split the arc containing $s_1$ at $s_1$. Symmetrically, we traverse the wavefront starting at $a_{k+1}$ to the right to find $s_2$. Finally, at the resulting gap, we insert the arc of $C_j$ between $s_1$ and $s_2$ into the wavefront.

**Case MT:** $q_1 \prec l \prec q_2$ and $q_3 \prec q_2 \prec r$; see Figure 5f. This case is symmetric to Case TM.

**Case BB:** $l \prec q_1 \prec q_2$ and $r \prec q_3 \prec q_4$; see Figure 5g. There is no intersection point between the wavefront and $C_j$. There cannot be a single intersection point and if there were two intersection points, it would contradict Lemma 7. We replace the whole wavefront by the arc of $C_j$ from $q_1$ to $q_3$.

**Case BM:** $l \prec q_1 \prec q_2$ and $q_3 \prec r \prec q_4$; see Figure 5h. This case is symmetric to Case MB.

(Case BT) $l \prec q_1 \prec q_2$ and $q_3 \prec q_4 \prec r$; see Figure 5i. Since this configuration is symmetric to Case TB, this case also cannot occur.

Note that we only do binary search in logarithmic time or if we traverse multiple arcs, we remove them from the wavefront. During the whole process, we add, for any vertex $p_j$ with $j > i$, at most one arc to the wavefront. Therefore, we conclude Lemma 16.

**Lemma 16.** Given a two-dimensional $n$-vertex polyline $L$ and a vertex $p \in L$, we can find all valid shortcuts under the local Fréchet distance starting at $p$ in $O(n \log n)$ time.

This update process in amortized logarithmic time per vertex is tailored specifically for this polyline simplification algorithm. We remark, however, that the more general problem of determining the two, one, or zero intersection points between a unit circle and the wavefront can also be accomplished in logarithmic time by a recursive case distinction.

Now we have all ingredients to prove our main theorem.

**Theorem 1.** A two-dimensional $n$-vertex polyline can be simplified optimally under the local Fréchet distance in the $L_2$ norm (the Euclidean norm) in $O(n^2 \log n)$ time and $O(n)$ space.

**Proof.** According to Lemmas 11 and 12, the algorithm we describe in Section 3.1 finds all valid shortcuts. It remains to analyze the runtime. We consider each of the $n$ vertices as potential shortcut starting point $p_i$. When we encounter a vertex $p_j$ with $j > i$, we determine in logarithmic time whether it is in the valid region. We do this by computing the ray emanating at $p_i$ and going through $p_j$, and querying the arc it intersects in the wavefront. Then, using the case distinction of Lemma 16, we update the wavefront and the wedge. This needs amortized logarithmic time and over all steps $O(n \log n)$ time.
Consequently, we construct the shortcut graph in $O(n^2 \log n)$ time. In the resulting shortcut graph, we can find an optimal polyline simplification by finding a shortest path in $O(n^2)$ time.

Regarding space consumption, we observe that the wavefront maintenance only requires linear space at any time. As we can compute the set of outgoing shortcuts of each vertex $p_i$ individually, we can also easily apply the space reduction approach described for the Imai–Iri algorithm in Section 2.2 to get an overall space consumption in $O(n)$.

4.3 Extension to General $L_p$ Norms

We can use our data structure for the $L_2$ norm also for the $L_p$ norm for $p \in (1, \infty)$. However, we should take this with a grain of salt as computing the intersection points between lines and unit circles and between pairs of unit circles in the $L_p$ norm for $p \in (1, \infty) \setminus \{2\}$ may involve solving equations of degree $p$, which may raise numerical questions for the required precision. To avoid this, we assume for the following corollary that we can determine intersection points between unit circles and lines (or another unit circle) in all $L_p$ norms in constant time.

$\triangleright$ Corollary 17. A two-dimensional $n$-vertex polyline can be simplified optimally under the local Fréchet distance in the $L_p$ norm for $p \in (1, \infty)$ in $O(n^2 \log n)$ time and $O(n)$ space, given that we can compute intersection points between a unit circle and a line and between two unit circles in constant time.

5 Use Cases with Small Wavefronts

The most complicated part of the algorithm is the efficient maintenance of the wavefront. But in case the wavefront has low complexity, we do not need any dynamic binary search data structure to make updates, but we can simply iterate over a linked list representing the whole wavefront to determine the relevant intersection points and to perform the dynamic changes. We next discuss use cases where the wavefront complexity is provably small.

5.1 $L_1$ and $L_{\infty}$ Norm (Manhattan and Maximum Norm)

In the $L_1$ and $L_{\infty}$ norm, the unit circles are actually square-shaped. Thus, the wavefront consists of a sequence of orthogonal line segments. As we show in the next lemma, this reduces the potential size of the wavefront significantly.

$\triangleright$ Lemma 18. In the $L_1$ norm and the $L_{\infty}$ norm, the wavefront always consists of either one or two (orthogonal) straight line segments. These straight line segments are horizontal or vertical in the $L_{\infty}$ norm and rotated by 45 degrees in the $L_1$ norm.

Proof. We show this claim inductively. For $W_{i,i+1} = D_{i,i+1}$ it is just the bottom arc of a square (the unit circle in $L_1$ or $L_{\infty}$). Clearly, this is either one or two orthogonal line segments – horizontal or vertical line segments in the $L_{\infty}$ norm and line segments rotated by 45 degrees in the $L_1$ norm.

When we compute the wavefront of $W_{i,j}$, we compute the intersection of the valid region of $W_{i,j-1}$ (which is bounded by one or two orthogonal line segments by the induction hypothesis) and the local valid region of $D_{i,j}$ (which is bounded by one or two line segments parallel to the ones of $W_{i,j-1}$). Computing the boundary of this intersection in the $L_{\infty}$ norm can be done by computing the intersection of two axis-parallel rectangles. The intersection of two axis-parallel rectangles is again an axis-parallel rectangle. In the $L_1$ norm, the situation is the same but rotated by 45 degrees.

$\triangleright$
We hence obtain the following theorem.

**Theorem 2.** A two-dimensional $n$-vertex polyline can be simplified optimally under the local Fréchet distance in the $L_1$ and $L_\infty$ norm in $O(n^2)$ time and $O(n)$ space.

### 5.2 Light Polylines

In Lemma 8, we have observed that for any vertex $p$ whose unit circle $C$ contributes to the current wavefront, $C$ actually contains the complete wavefront. Thus, if two vertices contribute an arc to the current wavefront, they are within a distance of $2\delta$, i.e., inside a unit circle. To end up with a complex wavefront, there hence need to be many vertices in close proximity (and they also need to occur in a specific pattern for all of their unit circles contributing to the wavefront simultaneously). Accordingly, if we consider polylines with bounded vertex density, the wavefront complexity is bounded as well. To formalize this, we introduce the natural class of $\nu$-light polylines.

**Definition 19.** A polyline $L$ in $d$ dimensions is $\nu$-light if for any $k \in \{2, \ldots, n\}$, no set (in particular not the closest set) of $k$ vertices of $L$ lies in a ball of radius less than $(k/\nu)^{1/d}$.

Before we exploit the properties of $\nu$-lightness in the context of polyline simplification, we want to gain some more intuition behind this definition. If a polyline in two dimensions is $\nu$-light, this guarantees that the vertices are somewhat well distributed: the closest pair of vertices has a distance of at least $2 \cdot \sqrt{2}/\nu$, the closest triplet of vertices has a surrounding circle of diameter at least $2 \cdot \sqrt{3}/\nu$ and so on. An alternative (and maybe more intuitive) definition is that a polyline $L$ is $\nu$-light if for any point $p \in \mathbb{R}^d$ and any radius $r > 0$, the number of vertices of $L$ inside the ball $B_r(p)$ of radius $r$ centered at $p$ is at most $\max\{\nu r^d, 1\}$.

This shows the connection to the related concepts of $c$-packed curves, $\phi$-low density curves, and $\kappa$-bounded curves studied in previous work to show improved bounds, e.g., for computing the (approximate) Fréchet distance between two curves [14]. The main difference is that these classifications do not distinguish between polyline vertices and points on the straight line segments in between, which, however, is important in our scenario.

Now let us revisit our algorithm for polyline simplification with distance threshold $\delta$ for the local Fréchet distance. In a ball of radius $r = \delta$ (i.e., a unit circle), a two-dimensional $\nu$-light polyline has $O(\nu \delta^2)$ vertices. Hence, for any constant choice of $\delta$, the wavefront complexity is $O(\nu)$. The running time of the algorithm is then in $O(n^2 \log \nu)$, or in $O(n^2 \nu)$ when omitting the tree data structure. So if $\nu \in O(1)$, the resulting running time is quadratic even without using a dedicated dynamic data structure.

**Theorem 20.** A two-dimensional $\nu$-light $n$-vertex polyline can be simplified optimally under the local Fréchet distance in the $L_p$ norm for $p \in [1, \infty]$ in $O(n^2)$ time and $O(n)$ space, given that we can compute intersection points between a unit circle and a line and between two unit circles in constant time.

### 6 Conclusions and Future Work

We have identified and closed a seeming gap in literature concerning a natural problem in computational geometry. Namely, the question of whether there is a subcubic-time algorithm that computes for a given polyline an optimal simplification under the local Fréchet distance. Simultaneous to us, Buchin et al. [11] have answered this question positively by providing an $O(n^{5/2+\varepsilon})$ time algorithm as an application of a new data structure. We have described an algorithm with a running time in $O(n^2 \log n)$, which is worst-case optimal up to a logarithmic...
factor for any algorithm that explicitly or implicitly constructs the whole shortcut graph. Our algorithm is simpler and faster than the one of Buchin et al.

Our algorithm does not provide new techniques, but we use, modify, and combine existing approaches for similar polyline simplification algorithms, which have not been described for this precise setting yet. Although the Chan–Chin algorithm for the local Hausdorff distance is asymptotically faster, it requires two sweeps over the polyline to identify the set of valid shortcuts, while we maintain the elegance of the Melkman–O’Rourke algorithm by requiring only a single sweep. Moreover, our algorithm is simpler than the line stabbing algorithm by Guibas et al.

As polyline simplification is a building block, for example, for trajectory clustering algorithms [8] or simplification algorithms for more generalized structures [7], our result may also help to improve running times of such approaches. Due to its basic nature, there are many more use cases conceivable where this algorithm can serve as a black-box subroutine. Also, note that our running time is only a logarithmic factor slower than the widely used Douglas–Peucker heuristic for the local Fréchet distance [32], but we compute the optimal simplification.

Moreover, we conjecture that in practice, the running time of the algorithm we describe should be quadratic even if one omits the tree data structure and simply uses a linked list and linear searches to maintain and update the wavefront. For a large wavefront to arise, the polyline vertices need to form a specific pattern, which is unlikely to occur naturally. It would be interesting to validate this claim empirically – maybe even including the concept of $\nu$-light polylines.

Furthermore, the investigation of lower bounds could shed light on the question of whether our upper bounds are tight. Existing lower bounds only apply to simplification of polylines in high dimensions. For the practically most relevant use case of two dimensions no (conditional) lower bounds are known, though. Another direction for future work would be to generalize the algorithm to work in higher dimensions, which implies a more complex wavefront. Finally, one could also consider further distance measures, as e.g. the Fréchet distance under the $L_p$ metric for $p \in (0,1)$, where the respective unit circles are not convex anymore, which could make updating the wavefront data structure more expensive.

In some applications, it can be a drawback that in the classical definition of a polyline simplification algorithm, the two endpoints of a polyline always need to be kept. One could relax this constraint and investigate the setting that also the endpoints may be removed as long as this removal does not violate the distance constraint. This may be done in a (user) study using real-world examples.

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We start the appendix with a structural lemma, which we employ for the proofs of Lemmas 7, 8, 14, and 16. It does not yet use the wavefront.

▶ **Lemma 6 (⋆).** Given two unit circles and a point \( p \) outside of the unit circles. If the two bottom arcs (with respect to \( p \)) intersect, then the second intersection point is between their top arcs.

**Proof.** For an illustration of this proof see Figure 6. Let \( C \) and \( C' \) be the two unit circles with the center of \( C \) being left of the center of \( C' \) w.r.t. \( p \). Now the cone between the right tangential from \( p \) on \( C \) and the left tangential from \( p \) on \( C' \) contains all of the intersection area of \( C \) and \( C' \), and hence also both intersection points. We call the tangential points \( r_C \) and \( l_C' \), respectively. Note that \( r_C = l_C' \) is excluded as then \( C \) and \( C' \) would only have a single intersection point. For the intersection point \( s \) between the bottom arcs of \( C \) and \( C' \), we know that the line segment \( ps \) does not intersect the inner part of any of the two circles by definition of the bottom arc. Hence the ray elongating this line segment has to go through the intersection area of \( C \) and \( C' \) above \( s \). Therefore, the partial bottom arc of \( C \) from \( s \) to \( r_C \) and the partial bottom arc of \( C' \) from \( s \) to \( l_C' \) are both on the boundary of the intersection area. As the intersection area is convex, it means that the line segment \( l_{C'}r_C \) is fully contained in the intersection area, and the intersection points have to be on opposite sites of the line through \( l_{C'}r_C \) and \( r_C \). Accordingly, the second intersection point \( s' \) of \( C \) and \( C' \) then has to lie above \( l_{C'}r_C \) and is therefore on the respective top arcs of \( C \) and \( C' \). ◀

We continue with another structural lemma, which seems rather special at first glance, but we employ it several times here in the appendix and in the main part, e.g., in Section 4.2 where we analyze the cases for the wavefront maintenance.

▶ **Lemma 7 (⋆).** If a unit circle \( C \) intersects the wavefront more than once, then on the left side of the leftmost intersection point \( s_1 \) (relative to rays originating in \( p_i \)) and on the right side of the rightmost intersection point \( s_2 \), \( C \) is below the wavefront. In other words, the intersection pattern depicted in Figure 7a cannot occur.
**Figure 7** Sketch for Lemma 7: a unit circle $C$ intersects the wavefront (blue wavy line) twice.

**Proof.** Clearly, if at $s_1$ the top arc of $C$ intersects the wavefront, then on the left side of $s_1$, $C$ is below the wavefront. Symmetrically, the same holds for $s_2$.

Now assume that at $s_1$ and at $s_2$, the bottom arc of $C$ intersects the arcs $a_j$ and $a_k$ of the wavefront, respectively. We denote their unit circles by $C_j$ and $C_k$. W.l.o.g. let $C$ on the left side of $s_1$ be above the wavefront. By Lemma 5, $C_j$ contains the rest of the wavefront including all of $a_k$. This means, that $C$ intersects $C_j$ at $s_3$ in between $s_1$ and $s_2$ (potentially $s_2 = s_3$ if $C_j = C_k$); see Figure 7b. Because the intersection of $C$ at $s_2$ is with the bottom arc of $C$, the intersection of $C$ and $C_j$ at $s_3$ is also with the bottom arc of $C$. This contradicts Lemma 6.

Finally, assume w.l.o.g. that at $s_1$ the top arc of $C$ intersects the arc $a_j$ of the wavefront and at $s_2$ the bottom arc of $C$ intersects the arc $a_k$ of the wavefront; see Figure 7c. Again by Lemma 5 the unit circle $C_j$ of $a_j$ contains the wavefront including the whole arc $a_k$. Hence, there is an intersection point $s_3$ of $C$ and $C_j$ in between $s_1$ and $s_2$ (where $s_2 \neq s_3$ and $C_j \neq C_k$ as otherwise $C$ and $C_j$ would have an intersection between their bottom arcs and between a bottom and a top arc). At $s_3$ there is the bottom arc of $C$ (since later at $s_2$, there is also the bottom arc of $C$ involved). If $C_j$ also would have its bottom arc at $s_3$, it would contradict Lemma 6. Therefore, at $s_3$, there is the top arc of $C_j$. This however means that $s_3$ is outside $D_{i,j} - a$ contradiction.

**Lemma 8**. Consider the wavefront of $W_{i,k}$. For every vertex $p_j$ ($i < j \leq k$) whose unit circle $C_j$ contributes an arc of the wavefront of $W_{i,k}$, the wavefront of $W_{i,k}$ lies completely inside $C_j$.

**Proof.** We argue that, for all $j \in \{i+1, \ldots, k\}$, the claim is true by considering first all arcs that had been added before and then all arcs that had been added after the arc of $C_j$ had been added to the wavefront.

All arcs $a_{j'}$ on the wavefront belonging to a vertex $p_{j'}$ with $j' < j$ are inside $C_j$ because when the wavefront of $W_{i,j}$ has been constructed, the wavefront of $W_{i,j}$ consisted of arcs of the wave of $D_{i,j}$, i.e., arcs of $C_j$, and it consisted of arcs of the wavefront of $W_{i,j-1}$ lying inside $I$, i.e., the intersection between $C_j$ and the valid region of $W_{i,j-1}$.

---

8 We remark that even without the extra narrowing step using $I$, if $C_j$ contributed an arc of the wavefront of $W_{i,j}$, $C_j$ contained the whole wavefront of $W_{i,j}$. 
Lemma 14 (⋆). Any unit circle of radius $\delta$ intersects the wavefront at most twice.

Proof. We prove this statement inductively. Say $p_i$ is our start vertex and we consider the wavefront of $W_{i,i+1}$, which is the same as the wave of $D_{i,i+1}$, which is part of the boundary of a unit circle. Since each pair of unit circles in the $L_p$ norm for $p \in [1,\infty]$ intersects at most twice, we know that any unit circle intersects the wavefront of $W_{i,i+1}$ at most twice.

It remains to show the induction step for all $j > i + 1$. Assume for a contradiction that a unit circle $C$ intersects the wavefront of $W_{i,j}$ more than twice. Observe that the wavefront of $W_{i,j}$ is a subset of the wavefront of $W_{i,j-1}$ and the wave of $D_{i,j}$. Say $C$ intersects the wavefront of $W_{i,j-1}$ at $q_1$ and $q_2$ and $C$ intersects the wave of $D_{i,j}$ at $q_3$ and $q_4$ (maybe one of these points does not exist.) Next, we argue topologically that at most two points of $\{q_1, q_2, q_3, q_4\}$ lie on the wavefront of $W_{i,j}$, which is a contradiction.

By the induction hypothesis, the wavefront of $W_{i,j-1}$ and the wave of $D_{i,j}$ intersect at most twice. Let these intersection points from left to right be $s_1$ and $s_2$; see Figure 9. Let
the subdivisions of the wavefront of $W_{i,j-1}$ and the wave of $D_{i,j}$ induced by $s_1$ and $s_2$ be $A_1, A_2, A_3$ and $a_1, a_2, a_3$, respectively. Some of them may be empty. Clearly, the wavefront of $W_{i,j}$ is either $A_1-a_2-A_3$ or $a_1-A_2-a_3$. By Lemma 7, we know that it cannot be $a_1-A_2-a_3$, therefore, it is $A_1-a_2-A_3$.

Next, we analyze the intersection points $q_3$ and $q_4$ (maybe $q_4$ does not exist). Either one or two of them lies on $a_2$ as otherwise there are no more than two intersection points of $C$ with the new wavefront.

**Case A:** The intersection points $q_3$ and $q_4$ lie on $a_2$; see Figure 9a. As both intersection points are between the unit circle $C$ and the wave of $D_{i,j}$, i.e., a bottom arc of another unit circle, we know by Lemma 6 that $q_3$ and $q_4$ are contained in the top arc of $C$. Thus, there is no ray $R$ to the left of $q_3$ or to the right of $q_4$ originating at $p_i$ and intersecting the arc of $C$ between $q_3$ and $q_4$ as otherwise $R$ would intersect the top arc of $C$ twice. Therefore, the arc of $C$ between $q_3$ and $q_4$ lies in the valid region (hatched orange in Figure 9a) without reaching $A_1$ or $A_3$. When $C$ passes through $q_3$ and $q_4$, it reaches the region between $a_2$ and $A_2$. If there are intersections between $W_{i,j-1}$ and $C$, they both lie on $A_2$.

**Case B:** Only one intersection point, let it be $q_3$, lies on $a_2$; see Figure 9b. If it is a touching point, then $C$ lies in the region between $a_2$ and $A_2$ before and after reaching $q_3$ (because we can assume that both unit circles are non-identical). If it is an intersection point, then $C$ passes through $q_3$ into the region between $a_2$ and $A_2$ (hatched orange in Figure 9b). To leave this region, $q_1$ (or $q_2$) lies on $A_2$. Hence, there are at most two points of $\{q_1, q_2, q_3, q_4\}$ on the new wavefront.