Node-Balancing by Edge-Increments

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Abstract. Suppose you are given a graph $G = (V, E)$ with a weight assignment $w : V \to \mathbb{Z}$ and that your objective is to modify $w$ using legal steps such that all vertices will have the same weight, where in each legal step you are allowed to choose an edge and increment the weights of its end points by 1.

In this paper we study several variants of this problem for graphs and hypergraphs. On the combinatorial side we show connections with fundamental results from matching theory such as Hall’s Theorem and Tutte’s Theorem. On the algorithmic side we study the computational complexity of associated decision problems.

Our main results are a characterization of the graphs for which any initial assignment can be balanced by edge-increments and a strongly polynomial-time algorithm that computes a balancing sequence of increments if one exists.

1 Introduction

The following puzzle is often used as an introductory puzzle for the method of invariance and potential functions: Six boxes numbered 1 to 6 are arranged in a cycle. For every $1 \leq i \leq 6$, we start with $i$ oranges in box number $i$. At each step we are allowed to add one orange to each of two adjacent boxes. Prove that we will never be able to make all boxes contain the same number of oranges.
One of the simple solutions to this puzzle is to observe that the total number of oranges in boxes 1, 3, 5 is always smaller than the total number of oranges in boxes 2, 4, 6 and this never changes through each step of the game.

In this paper we consider the natural generalization of the puzzle above to arbitrary graphs. Let $G = (V, E)$ be a finite graph, let $w : V \rightarrow \mathbb{N}$ be a non-negative integer weight function on its vertices and let $e = \{u, v\} \in E$. A positive step on $e$ modifies $w$ by increasing the weights of $u, v$ by 1 unit. We say that $w$ is equatable in $G$ if there exists a sequence of only positive steps, $S = s_1, \ldots, s_m$, after which all vertices have the same weight. We also say that the sequence $S$ positively equates $w$.

Our main results are the following.

i) We characterize those graphs $G = (V, E)$ for which any initial assigmnet $w : V \rightarrow \mathbb{N}_0$ is equatable. These are the connected graphs with an odd number of nodes for which $G - U$ has less than $|U|$ isolated vertices for any $U \subset V$. Here $G - U$ is the subgraph of $G$ that is induced by $V \setminus U$. (Theorem 1)

ii) We show that the following problem can be solved in strongly polynomial time. Given a graph $G = (V, E)$ and an initial assignment $w : V \rightarrow \mathbb{N}_0$, decide whether $w$ is equatable and compute an equating multiset of edges. (Theorem 3)

iii) An initial assignment $w$ of the nodes of a bipartite graph $G = (L + R, E)$ is not equatable if $w(L) \neq w(R)$, the difference $w(L) - w(R)$ is invariant under edge-increments. However, each balanced assignment with $w(L) = w(R)$ is equatable if and only if the strict Hall condition holds: For any nonempty set of vertices $X$ that is properly contained in $L$ or in $R$, one has $|X| < |N(X)|$. Here $N(X)$ denotes the neighborhood of $X$. (Theorem 4)

iv) Finally we show that the analog of the decision problem ii) is NP-hard for hypergraphs. (Theorem 5).

Related Work. The problem of equating the node-weights is closely related to perfect $b$-matchings, [15]. Let $b \in \mathbb{N}_0^{|V|}$ be a vector of non-negative node-weights. A $b$-matching of a graph $G = (V, E)$ is a vector $x \in \mathbb{N}_0^{|E|}$ that satisfies

$$\sum_{e \in \delta(v)} x_e \leq b_v, \quad (1)$$

where $\delta(v)$ denotes the set of edges of $G$ that are incident to $v$. A $b$-matching is perfect, if the inequality in (1) can be replaced by equality. Thus $b$-matchings are a generalization of matchings, where $b$ is the all ones vector.

What is the relationship between $b$-matchings and the process of equating positive weights in graphs by edge-increments? Suppose that the given initial weight assignment $w \in \mathbb{N}_0^{|V|}$ is equatable and that the resulting equated node-weight is $\beta \in \mathbb{N}$. Then, the edge-increments that lead to the balanced node-weight $\beta$ are a $b$-matching $x \in \mathbb{N}_0^{|E|}$ with $b_v = \beta - w_v$ for each vertex $v$. By