Wannier band transitions in disordered $\pi$-flux ladders

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Boundary obstructed topological insulators are an unusual class of higher-order topological insulators with topological characteristics determined by the so-called Wannier bands. Boundary obstructed phases can harbor hinge/corner modes, but these modes can often be destabilized by a phase transition on the boundary instead of the bulk. While there has been much work on the stability of topological insulators in the presence disorder, the topology of a disordered Wannier band, and disorder-induced Wannier transitions have not been extensively studied. In this work, we focus on the simplest example of a Wannier topological insulator: a mirror-symmetric $\pi$-flux ladder in 1D. We find that the Wannier topology is robust to disorder, and derive a real-space renormalization group procedure to understand a new type of strong disorder-induced transition between non-trivial and trivial Wannier topological phases. We also establish a connection between the Wannier topology of the ladder and the energy band topology of a related system with a physical boundary cut, something which has generally been conjectured for clean models, but has not been studied in the presence of disorder.

I. INTRODUCTION

Topological insulators are phases of matter that cannot be deformed to a trivial atomic limit without closing the energy gap or breaking a protecting symmetry.\cite{1,2} Topological insulators can be protected by internal symmetries\cite{3,4}, which results in the periodic classification table of topological insulators and superconductors\cite{5,6}. They can also be protected by spatial symmetries such as reflection or rotation, leading to topological crystalline insulators (TCIs)\cite{7,8,9}. One important characteristic of topological insulators is the spectroscopy of their boundary states. TIs protected by internal symmetries display protected surface modes on any boundary, while TIs can typically display protected surface modes on only boundaries that respect the spatial symmetry.

Recently, there has been interest in so-called higher-order topological insulators (HOTIs)\cite{10,11,12,13}. HOTIs are crystalline insulators that display protected modes on certain corners or hinges determined by spatial symmetries. Some of these HOTIs\cite{11,12} are TCIs in the usual sense, in that they cannot be deformed to a trivial atomic limit without closing the energy gap or breaking the spatial symmetry. However, there also exist HOTIs that can be symmetrically deformed to a trivial atomic limit without closing the bulk gap\cite{13,14}. These HOTIs have been dubbed boundary obstructed topological insulators (BOTIs) in Ref. \cite{17}. A key characteristic of a BOTI is that, rather than having properties protected by an energy gap, they have properties protected by a Wannier gap (see Sec. \ref{sec:Wannier}).

An important question is if topological properties are robust to disorder. In the case of TIs protected by an internal symmetry, one can generally define topological invariants that are robust to symmetry-preserving disorder, which can change only when delocalized states appear at the Fermi level\cite{18,19,20,21,22}. Naively, TCIs and BOTIs should not be robust to disorder, since any disorder breaks the spatial symmetry. However, provided the disorder respects the spatial symmetry on average, one can still define robust topological invariants for TCIs\cite{26,27,28,29,30} which are stable provided the bulk gap remains open\cite{26}. However, there has not been any study of the effects of disorder on the BOTIs.

In this article we take the first approach at characterizing BOTIs in the presence of disorder. To this end we study a minimal model of a BOTI, what we might call a topological Wannier insulator. Our model is a 1D $\pi$-flux ladder having Wannier band topology protected by reflection symmetries $M_x, M_y$. This model originated in Ref. \cite{15} where it was used as a building block for the 2D quadrupole BOTI phase. Since our model is one-dimensional, the Wannier topology does not indicate the existence of corner or hinge modes. However, the simplicity of the model allows us to study a new type of phase transition, i.e., a disorder-induced Wannier transition, in detail. Furthermore, since this model is a single layer of the 2D quadrupole BOTI, our work can serve as a starting point for more computationally intensive studies of disordered 2D BOTIs.

As a summary of our results, we find that the Wannier topology is stable to symmetry-breaking disorder provided the symmetries are obeyed on average, and there is a sharp Wannier transition at a critical value of the disorder at which the Wannier gap closes. This transition is distinct from the usual disorder-induced topological transitions, in that it occurs without delocalized states crossing the Fermi level; in fact, the system remains gapped throughout. In addition, we find that the Wannier topology is connected to the energy band topology of an artificial “edge” introduced by cutting the $\hat{y}$ bonds of the ladder to separate one leg of the ladder from the other. This represents the first evidence that the Wannier topology gives a robust signature for the boundary topology in the presence of disorder. Finally, we introduce a real-space renormalization group approach that explains the robustness of the Wannier topology, and the connection...
between the Wannier gap closing and the Wannier transition. Importantly, our model can be realized in a number of experimental systems including mechanical resonator arrays, microwave resonator arrays, circuit resonators, and cold-atoms. Hence we anticipate that our results will be immediately relevant for experiments in higher order topology.

II. THE CLEAN $\pi$-FLUX LADDER

The $\pi$-flux ladder model we use is shown in Fig. 1b. The model has four sites per unit cell, and a magnetic flux of $\pi$ through each plaquette. In terms of the intercell hopping $\lambda$ and intracell hoppings $\gamma_x, \gamma_y$, the Bloch Hamiltonian is given by

$$
\hat{H}(k) = [\gamma_x + \lambda \cos(k)] \tau_0 \sigma_1 + \lambda \sin(k) \tau_0 \sigma_2 + \gamma_y \tau_1 \sigma_3, \quad (1)
$$

where $\tau_i, \sigma_i$ are the Pauli matrices acting on the vertical and horizontal degrees of freedom respectively. This model has anticommuting reflection symmetries $\hat{M}_x = \tau_3 \sigma_1$, $\hat{M}_y = \tau_1 \sigma_0$, and the energy spectrum is given by

$$
E_k = \pm \sqrt{\gamma_x^2 + 2 \gamma_x \lambda \cos(k) + \lambda^2 + \gamma_y^2}, \quad (2)
$$

where each energy level is doubly degenerate. The spectrum is gapped provided $\gamma_y \neq 0$ or $|\gamma_x| \neq |\lambda|$. The gapped phases of the model maintain the reflection symmetry $\hat{M}_x$, and are classified (in 1D) by an index $\nu \in \mathbb{Z}$. Furthermore they display a quantized electric polarization $p$ given by $2p = \nu \mod 2$ [11,29]. However, all gapped phases of our model are connected by some gapped path in $(\gamma_x, \gamma_y, \lambda)$, thus $H$ can only describe a single topological phase. Indeed, the index $\nu$ of this model is zero, and $p = 0$, for any values of the parameters. We would thus classify this model as a trivial TCI.

To uncover the topological properties of this model we need to refine our approach. Refs. [15,16] recently introduced the idea of Wannier band topology, a topological characterization that can change even while the energy gap remains open. The idea is to divide the occupied subspace of the Hamiltonian into two separate subspaces, each of which is localized in a different part of the unit cell; these individual subspaces can then have their own topological invariants. Concretely, we define the projection operator onto the occupied states, and the $y$-position operator:

$$
\hat{P}_k = \sum_i |u_{ki}^\dagger \rangle \langle u_{ki}^{\prime}|, \quad \hat{Y} \langle a | \{ y_0 | a \} = \left\{ \begin{array}{ll} y_0 | a & \text{if } a = 1,2 \\ -y_0 | a & \text{if } a = 3,4 \end{array} \right., \quad (3)
$$

where $|u_{ki}^{1,2}\rangle$ are the two occupied eigenstates of $\hat{H}_k$ at half-filling, and $\pm y_0$ is the vertical position of the upper/lower sites. We find $\hat{P}_k \hat{Y} \hat{P}_k$ has two nonzero eigenvectors for each value of $k$, $|w^\pm_k\rangle$, with eigenvalues $\nu_k^\pm$. We call $\hat{P}_k \hat{Y} \hat{P}_k$ the Wannier Hamiltonian, $|W^\pm_k\rangle$ the Wannier bands, and $g = \min_k \{ \nu_k^+ - \nu_k^- \}$ the Wannier gap. Provided $g > 0$, $|W^\pm_k\rangle$ is a smooth function of $k$. We can then associate a polarization to each Wannier band:

$$
p^\pm = i \int \frac{dk}{2\pi} \langle w^\pm_k | \partial_k w^\pm_k \rangle, \quad (4)
$$

Due to $\hat{M}_y$ symmetry, $p^\pm$ is quantized to 0 or 1/2 [15,16], and thus serves as a topological invariant for the Wannier bands. Note that we must also have $p^+ = p^-$ due to $\hat{M}_y$ symmetry. In terms of the parameters $(\gamma_x, \gamma_y, \lambda)$ of our model, we have $p^\pm = 1/2$ for $|\lambda| > |\gamma_x|$, and $p^\pm = 0$ for $|\lambda| < |\gamma_x|$. Exactly at $|\lambda| = |\gamma_x|$, the Wannier gap closes. We therefore see that while our model is topologically trivial with respect to the usual topological invariant, it describes two topological Wannier phases separated by a Wannier gap closing.

In higher-dimensional models, the Wannier bands have been related to the edge[14,17,36] or entanglement[14,37] Hamiltonian for a system with boundary. For our system, we can give a similar interpretation. First, we note that cutting the $y$ bonds of the chain results in two isolated Su-Schrieffer-Heeger (SSH) chains [38] (Fig. 1b). Then the topology of the upper/lower Wannier band is identical to the topology of the upper/lower SSH chain that results when $\gamma_y$ is set to zero. This can trivially be seen by noting that tuning $\gamma_y$ to zero cannot close the Wannier gap or the energy gap, thus the Wannier polarizations cannot change during the transition. When $\gamma_y = 0$, the upper/lower Wannier bands are precisely the ground states of the upper/lower SSH chains. Therefore, the polarization of the Wannier bands are identical to the polarization of the corresponding SSH chains of the cut system.

III. DISORDER-INDUCED TRANSITIONS IN THE $\pi$-FLUX LADDER

To study the effect of disorder on Wannier band topology, we perturb each link in our Hamiltonian by a random amount

$$
\gamma_{x,y}^n = \gamma_{x,y} + W_n \omega_{x,y}^n, \quad \lambda^n = \lambda + W \omega^n, \quad (5)
$$

where $W_n \leq 0$ is a random number, which we add to the hopping and chemical potentials, respectively.
where the $\omega_n^{\alpha} \in [0, 1]$ are uniformly distributed random variables, and $(W_\gamma, W_\lambda)$ parameterize the disorder strength. Note that we choose the link disorder to be positive rather than symmetric about zero; this has the effect of ensuring the disordered model still has $\pi$-flux through each plaquette. While maintaining $\pi$-flux is not essential for our conclusions, this choice separates transitions where the Wannier gap closes from the ones where the energy gap closes, thus allowing us to isolate the Wannier topology. While our disorder breaks $M_z$ and $M_y$ symmetries, there is still a sense in which these symmetries on average. We say a symmetry $\hat{S}$ is respected on average if a disordered Hamiltonian $H$ occurs with equal probability as $\hat{S}^\dagger \hat{H} \hat{S}[30].$ We see that our disordered Hamiltonians indeed respect these symmetries on average.

In the presence of disorder we can still define Wannier topology by introducing the analogous operators

$$\hat{P} = \sum_n |\psi_n\rangle \langle \psi_n|, \quad \hat{Y} |x, a\rangle = \left\{ \begin{array}{ll} y_0 |x, a\rangle & a = 1, 2 \\
\gamma y_0 |x, a\rangle & a = 3, 4 \end{array} \right. ,$$

where $|\psi_n\rangle$ are the occupied states of $H$. The upper/lower Wannier bands are the eigenstates $|w_n^\pm\rangle$ of the Wannier Hamiltonian $\hat{P} \hat{Y} \hat{P}$ with corresponding Wannier values $v_n^\pm$. We can then define the polarization of the upper/lower Wannier band $p^\pm$ either by the method of Refs. [26 and 39] or the method of Ref. [40]. For simplicity, we use the method of Ref. [40] and define

$$z^\pm \equiv \text{det} (\hat{P} e^{i\alpha \hat{X}} \hat{P}^\pm), \quad p^\pm = \frac{1}{2\pi} \text{arg}(z^\pm),$$

where $\hat{P}^\pm$ is the projector onto the upper/lower Wannier band, $\alpha = 2\pi/L_x$, and $L_x$ is the length of the system in the $x$-direction. Using these quantities we can also define the localization length of states in the Wannier bands [41]:

$$\Lambda^\pm = \frac{1}{2\pi} \sqrt{-L_x \log |z^\pm|^2}.$$

It has been proven that as long as a Hamiltonian is local, gapped, and respects $M_z$ and translation symmetry on average, the $x$-polarization is self-averaging and quantized to 0 or $1/2[26].$ This result, applied to the Wannier Hamiltonian $\hat{P} \hat{Y} \hat{P}$, implies that the Wannier polarizations are self-averaging and quantized to 0 or 1/2 in the presence of a Wannier gap. To see this, we note that the Wannier Hamiltonian $\hat{P} \hat{Y} \hat{P}$ is local provided our original Hamiltonian is gapped, and that a projector $\hat{P}$ is equally likely to occur as its symmetry-related counterpart $M_\alpha \hat{P} M_\alpha$. Finally, since $\hat{P} \hat{Y} \hat{P}$ respects $M_z$ symmetry on average, we can also conclude that $p^+ = p^-$ even with disorder.

An example of this characteristic behavior is shown in Fig. 2. For those calculations we set $(\gamma_x, \gamma_y, \lambda) = (0.5, 1, 1)$, and tune only $W_\gamma$. In Fig. 2a, we plot $p^\pm$ as a function of $W_\gamma$. In Fig. 2b, we plot the Wannier gap as a function of $W_\gamma$. We see that, as predicted, $p^\pm$ are quantized to 0 or 1/2, and they change at only the point where the Wannier gap closes. For comparison, we show the energy gap in Fig. 2c and find that the energy gap remains open during this process.

Intriguingly, we also find that the Wannier band polarization and the polarization of the isolated upper/lower SSH chains agree even with disorder, as shown in Fig. 2d where we plot the polarization of the upper chain as a function of $W_\gamma$. To be clear, we are comparing the polarization of the Wannier bands with the polarization of our model at the same parameter values, but with the vertical bonds (i.e., the rungs) turned off. For our model, with only bond disorder, we can use the results of [23] to analytically predict the phase diagram of the decoupled upper/lower SSH chains, and compare it to the calculated nested Wannier polarizations. The result is shown in Fig. 3. Here, the line denotes the exact phase boundary for a single SSH chain, while the color indicates the nested polarization for the $\pi$-flux ladder. We see that the transition of the upper SSH chain exactly agrees with the Wannier transition, and thus we can analytically predict the location of the Wannier transition (i.e., even when the chains are coupled). We find this result actually holds for more general Hamiltonians, such as those including random on-site disorder or next-nearest-neighbor hoppings that preserve the symmetries on average, though we cannot determine the phase diagram of the decoupled chains analytically in these cases. From this we can conclude that, even in the presence of disorder, the Wannier Hamiltonian describes the topological properties of the physical system with a spatial cut.

IV. RENORMALIZATION GROUP PICTURE OF THE TRANSITION

We can gain additional insight into the nature of the Wannier transition through a real-space renormalization group (RG) procedure. Our RG procedure is illustrated...
direct calculation of \( p^+ \), the strength to construct the RG procedure we first need to define insulators (SSH chains with chiral symmetry) in Ref. 23. A similar method has been applied to disordered 1D topological insulators in Ref. 42 to study random spin chains. Indeed, a similar procedure does not generate diagonal couplings. In addition, the signs of the effective couplings preserve the sublattice symmetry of the original model, our RG procedure; the phase transition happens when the upper SSH chain has a topological transition.

FIG. 3. \( p^+ \) as a function of \((W_x, W_y)\). The color denotes the direct calculation of \( p^+ \), while the overlaid line is the analytically predicted topological transition of the corresponding upper SSH chain\(^{[23]}\). We see that the Wannier transition occurs exactly when the upper SSH chain has a topological transition.

in Fig. 4a, and is based on the real space RG introduced in Ref. 42 to study random spin chains. We focus on the strongest plaquette of the ladder, and project into its local ground state. This generates new effective couplings between previously unconnected sites in a way that preserves the structure of the ladder. (b) At the end of the RG procedure, the system is approximated by disconnected \( \pi \)-flux plaquettes. We have seen that the \( \pi \)-flux ladder possesses a robust Wannier topological invariant, \( p^\pm \), protected by the unit cell \( a \) and unit cell \( b \) contributes a phase of \( e^{in(a+b)/2} \) to \( z^\pm \). In Fig. 5a, we schematically illustrate the plaquettes for a clean system in the topological phase, and the contribution of each plaquette to \( \arg(z^\pm) \). Introducing disorder can locally rearrange the plaquettes, as in Fig. 5b, but these local rearrangements do not change \( \arg(z^\pm) \). The RG demonstrates that the critical point is characterized by a diverging Wannier localization length \( \Lambda^\pm \). From Eq. 8 if our plaquettes span sites \( a_j \) and \( b_j \), the localization length is given by

\[
\Lambda^\pm = \frac{1}{2\pi} \sqrt{L_x \sum_j \log\csc^2(\frac{\pi a_j - b_j}{L_x})}
\]

(9)

which diverges when one of the \( (a_j - b_j) = L_x/2 \). Thus, the only way for a rearrangement to change \( p^\pm \) is for a plaquette to grow to length \( L_x/2 \), which can change the sign of \( p^\pm \) and thus change \( p^\pm \) by 1/2, as in Fig. 5b.

Heuristically we see that, because of the nature of the RG, plaquettes spanning a larger width have weaker bonds in the \( x \)-direction. In the thermodynamic limit, a plaquette of width \( L_x/2 \) will have vanishing hopping in the \( x \)-direction, and thus the plaquette will effectively have only vertical bonds. This configuration is exactly what is required for the Wannier gap to vanish (electrons are exactly half way between upper chain and lower chain). We thus see a clear connection between the Wannier gap closing and the topological phase transition from the RG procedure; the phase transition happens when long-range plaquettes form, which have arbitrarily small Wannier gap.

V. DISCUSSION

We have seen that the \( \pi \)-flux ladder possesses a robust Wannier topological invariant, \( p^\pm \), protected by the
mirror symmetries $\hat{M}_x, \hat{M}_y$. In the presence of disorder, the Wannier topological invariant remains quantized so long as the energy and Wannier gaps remain open. The physical interpretation of the Wannier topological invariant in 1D is subtle, since, for example, there are no corners to display robust mid-gap modes or fractional charges. However, we have seen that introducing an artificial “edge” into the system connects the Wannier topology and the edge topology for both clean and disordered systems. This conclusion holds numerically for general local symmetry-preserving disorder, and not just the link disorder we’ve studied here. We conjecture that this is because tuning the $y$-bonds to zero cannot close the energy or Wannier gap provided the Hamiltonian obeys $M_x$ and $M_y$ symmetries on average. This work thus provides the first evidence of a robust connection between Wannier and edge topology in the presence of disorder. Additionally, we anticipate that our results can be immediately tested in metamaterial or cold atom experiments.[31–34]

Because of the simplified nature of our model, we can directly predict the behavior of the “edge” from first principles and thus confirm the connection between Wannier topology and edge topology for this model. We can also gain additional insight into the nature of the Wannier transition through a real-space RG, which offers a qualitative explanation of the local stability to disorder and the connection between the Wannier transition, Wannier delocalization, and the Wannier gap closing.

While these results form the first example of the stability of Wannier topology to disorder, ultimately it will be useful to generalize these results to higher-dimensional models, where nontrivial Wannier topology implies protected corner or hinge modes. It will be especially interesting to see if the connection between Wannier topology and edge topology remains for these higher-dimensional models.

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In this Supplement, we give the details of the local projection involved in the RG, and prove that the resulting effective Hamiltonian is again a π-flux ladder. We follow the procedure outlined in Ref. [42] for finding the effective Hamiltonian as a perturbation and work to second order in \( \hat{V} \); this is valid provided the energy gap is much larger than the hoppings. Then the general formula for the effective Hamiltonian is given by:

\[
\hat{H}_{\text{eff}} = E_g + \hat{P}\hat{V}\hat{P} + \hat{P}\hat{V}(E_g - \hat{Q}\hat{H}_0\hat{Q})^{-1}\hat{Q}\hat{V}\hat{P}
\]

(S1)

where \( E_g \) is the ground state energy of \( \hat{H}_0 \), \( \hat{P} \) is the projector onto the degenerate ground state subspace of \( \hat{H}_0 \), and \( \hat{Q} = 1 - \hat{P} \). In our case, we can make a few simplifications. First, since we are only interested in the ground state wavefunctions and not their energies, we can neglect the constant \( E_g \). Second, our particular \( \hat{V} \) satisfies \( \hat{P}\hat{V}\hat{P} = 0 \).
Now, when acting on a state of the form $|\cdot\rangle$ since such terms take us to states orthogonal to $|\vec{\gamma}\rangle$. The equation becomes since every state in the range of $\hat{P}$ has two electrons in the central plaquette, and $\hat{V}$ always changes the number of electrons in the central plaquette by one. For the same reason, we have $Q\hat{V}\hat{P} = \hat{V}\hat{P}$. Thus in total, our simplified equation becomes

$$\hat{H}_{\text{eff}} = \hat{P}\hat{V}(E_g - \hat{Q}\hat{H}_0\hat{Q})^{-1}\hat{V}\hat{P}. \quad (S2)$$

We need to prove three things about $H_{\text{eff}}$:

1. $H_{\text{eff}}$ is a non-interacting Hamiltonian, i.e. it contains only hopping and on-site terms.
2. $H_{\text{eff}}$ has the same sublattice symmetry as the original Hamiltonian, so that there are no on-site terms or hoppings from (a) to (c) or from (b) to (d).
3. $H_{\text{eff}}$ has $\pi$-flux through the plaquette spanned by (a)-(d) and combines with the rest of the ladder to create another $\pi$-flux ladder.

To prove these results, we’ll derive an explicit formula for $H_{\text{eff}}$ in terms of $\hat{V}$ and $\hat{H}_0$ that makes no reference to $\hat{P}$ or $\hat{Q}$. We start by rewriting Eq. S2 in second-quantized notation. We’ll define operators $\hat{c}_{\alpha}^\dagger, \alpha \in \{a, b, c, d\}$ to be the annihilation (creation) operators for the sites outside the central plaquette, and $\hat{\gamma}_{i}^{\dagger}, i \in \{1, 2, 3, 4\}$ to be the annihilation (creation) operators for the $i$th energy mode on the central plaquette with energy $E_i$. We can write a second quantized basis of states as $|abcdn_1n_2n_3n_4\rangle$ with $a-d$ being the occupation numbers of sites (a)-(d), and $n_i$ being the occupation number of the $i$th energy mode on the central plaquette. In terms of these operators, we can write $\hat{V}$ as

$$\hat{V} = \sum_{\alpha, i} \left[ \hat{c}_{\alpha}^\dagger V_{\alpha i} \hat{\gamma}_{i} + \hat{\gamma}_{i}^\dagger V_{i\alpha}^\dagger \hat{c}_{\alpha} \right] \quad (S3)$$

where $V_{\alpha i}$ is some matrix. In this notation, $[\hat{c}_{\alpha}^\dagger, \hat{P}] = 0$, and the effect of $\hat{P}$ is to project to the orthonormal basis $\{|abcd1100\rangle|a, b, c, d = 0, 1\}$. We can then write the action of $H_{\text{eff}}$ on any state in the range of $\hat{P}$ as (we ignore terms that involve $\hat{\gamma}_{i}^\dagger \hat{\gamma}_{j}$ or $\hat{\gamma}_{i} \hat{\gamma}_{j}^\dagger$ since such terms take us to states orthogonal to $\hat{P}$)

$$\hat{H}_{\text{eff}} = \sum_{i, j, \alpha, \beta} \left[ \hat{c}_{\alpha}^\dagger V_{\alpha i} \hat{P} \hat{\gamma}_{i}^{\dagger}(E_g - \hat{Q}\hat{H}_0\hat{Q})^{-1}\hat{\gamma}_{j} \hat{V}_{j\beta}^\dagger \hat{P} \hat{\gamma}_{j}^{\dagger}(E_g - \hat{Q}\hat{H}_0\hat{Q})^{-1}\hat{\gamma}_{i} V_{i\alpha} \hat{c}_{\alpha}^\dagger \right]. \quad (S4)$$

Now, when acting on a state of the form $|abcd1100\rangle$, the operator $\hat{P} \hat{\gamma}_{i}^{\dagger}(E_g - \hat{Q}\hat{H}_0\hat{Q})^{-1}\hat{\gamma}_{j} \hat{V}_{j\beta}^\dagger \hat{P} \hat{\gamma}_{j}^{\dagger}(E_g - \hat{Q}\hat{H}_0\hat{Q})^{-1}\hat{\gamma}_{i}$ has a simple action:

$$\hat{P} \hat{\gamma}_{i}^{\dagger}(E_g - \hat{Q}\hat{H}_0\hat{Q})^{-1}\hat{\gamma}_{j} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\epsilon_i} & 0 \\ 0 & 0 & 0 & -\frac{1}{\epsilon_j} \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij}. \quad (S5)$$

Similarly,

$$\hat{P} \hat{\gamma}_{j}^{\dagger}(E_g - \hat{Q}\hat{H}_0\hat{Q})^{-1}\hat{\gamma}_{i} = \begin{pmatrix} \frac{1}{\epsilon_i} & 0 & 0 & 0 \\ 0 & \frac{1}{\epsilon_j} & 0 & 0 \\ 0 & 0 & \frac{1}{\epsilon_i} & 0 \\ 0 & 0 & 0 & \frac{1}{\epsilon_j} \end{pmatrix}_{ij}. \quad (S6)$$
Thus in total, we can write
\[
\hat{H}_{\text{eff}} = \sum_{i,j,\alpha,\beta} \left[ \hat{c}_\alpha^\dagger V_{\alpha i} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix} \right] V_{j\beta}^\dagger \hat{c}_\beta + \hat{c}_\beta^\dagger V_{j\beta} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \hat{c}_\alpha V_{\alpha i}^\dagger \right] 
\] (S7)

\[
= \sum_{i,j,\alpha,\beta} \left[ \hat{c}_\alpha^\dagger V_{\alpha i} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix} \right] V_{j\beta}^\dagger \hat{c}_\beta + \hat{c}_\beta^\dagger V_{j\beta} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \hat{c}_\alpha V_{\alpha i}^\dagger \right] + \delta_{\alpha\beta} V_{\alpha i}^\dagger \begin{pmatrix} 0 & 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V_{j\beta}^\dagger \right] 
\] (S8)

\[
= -\sum_{i,\alpha,\beta} \left[ \hat{c}_\alpha^\dagger V_{\alpha i} \frac{1}{E_i} V_{j\beta}^\dagger \hat{c}_\beta \right] 
\] (S9)

where in the last line, we have dropped an irrelevant constant term. We thus clearly see that $H_{\text{eff}}$ is non-interacting.

Since we now know all our operators $\hat{H}_0$, $\hat{V}$, and $\hat{H}_{\text{eff}}$ are non-interacting, it is more convenient going forward to work with one-body operators rather than the many-body, second-quantized operators. We’ll denote one-body operators using the same symbols as our many-body operators, with the understanding that in what follows we are working only with the one-body operators. We divide our full Hilbert space into two subspaces, $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_p$, where $\mathcal{H}_s$ is the Hilbert space of sites (a)-(d), and $\mathcal{H}_p$ is the Hilbert space of the central plaquette. Then we define $\hat{V} : \mathcal{H}_s \to \mathcal{H}_p$ to be all hoppings from sites (a)-(d) to the central plaquette, so that $\hat{V}^\dagger : \mathcal{H}_p \to \mathcal{H}_s$ consists of all hoppings in the reverse direction. We define $\hat{H}_0 : \mathcal{H}_p \to \mathcal{H}_p$ to be all hoppings within the central plaquette. Note that by restricting $\hat{H}_0$ to $\mathcal{H}_p$ rather than the full Hilbert space, we’ve ensured $\hat{H}_0$ is invertible. Finally, we define the one-body $\hat{H}_{\text{eff}} : \mathcal{H}_s \to \mathcal{H}_s$ to be the restriction of the effective Hamiltonian to $\mathcal{H}_s$. In terms of these one-body operators, Eq. (S9) becomes

\[
\hat{H}_{\text{eff}} = -\hat{V}^\dagger \hat{H}_0^{-1} \hat{V} - \hat{V} \hat{H}_0^{-1} \hat{V}^\dagger. 
\] (S10)

To prove that $\hat{H}_{\text{eff}}$ has sublattice symmetry, we note that $\hat{V}$, $\hat{V}^\dagger$, and $\hat{H}_0^{-1}$ all have sublattice symmetry, and the product of three sublattice-symmetric operators is also sublattice-symmetric. Thus from Eq. (S10) we immediately see $\hat{H}_{\text{eff}}$ has sublattice symmetry.

Finally, we need to prove that the RG preserves the $\pi$-flux structure. We first note that $\hat{H}_0$ has the form

\[
\hat{H}_0 = \begin{pmatrix} 0 & 0 & + & - \\ 0 & 0 & + & + \\ + & 0 & 0 & + \\ + & + & 0 & 0 \end{pmatrix} 
\] (S11)

Inspection shows that $\hat{H}_0^{-1}$ also has the same sign structure:

\[
\hat{H}_0^{-1} = \begin{pmatrix} 0 & 0 & + & - \\ 0 & 0 & + & + \\ + & 0 & 0 & + \\ + & + & 0 & 0 \end{pmatrix} 
\] (S12)

Then from Eq. (S10) and the fact that all the hoppings in $\hat{V}$ are positive we immediately see the signs of the hoppings in $H_{\text{eff}}$ are as shown in Fig. SI.

This sign structure ensures that the new induced hoppings reinforce the existing couplings to maintain a $\pi$-flux structure in the rest of the ladder, as shown in Fig. SI. As the RG progresses, the central plaquette and neighboring bonds can have other sign structures, two examples of which are shown in Figs. SI.b,c. The sign structures are always real and symmetric with respect to $\hat{M}_p$. A similar analysis shows that in all cases, the resulting induced hoppings always reinforce the existing couplings and maintain the overall $\pi$-flux structure. This can be seen just by inspecting the figures, using Eq. (S10) and the fact that $\hat{H}_0^{-1}$ always has the same sign structure as $\hat{H}_0$. 


FIG. S2. The RG generates $\pi$-flux ladders with all sign structures compatible with $\pi$-flux that are both real and $\hat{M}_y$ symmetric. For each of these sign structures, the RG results in induced couplings that reinforce existing couplings and result in a new $\pi$-flux ladder.