On the Raşa Inequality for Higher Order Convex Functions II

Andrzej Komisarski and Teresa Rajba

Abstract. We give necessary and sufficient conditions for Borel measures \( \mu \) and \( \nu \) to satisfy the following \((q-1)\)th convex ordering relation for \( q \)th convolution power of the difference of \( \mu \) and \( \nu \)

\[
(\nu - \mu)^* \geq (q-1) - cx, \quad q \geq 2,
\]

which was introduced by Komisarski and Rajba (Results Math 76(2):103–115, 2021), and we gave in Komisarski and Rajba (Results Math 76(2):103–115, 2021) a useful sufficient condition for its verification. We give also necessary and sufficient conditions for discrete probability distributions. This inequality is a generalization of the inequality given recently by Abel and Leviatan (Results Math 75(4):181–193, 2020) and the Raşa type inequalities given in Komisarski and Rajba (Math Anal Appl 478:182–194, 2019).

Mathematics Subject Classification. 26D05, 39B62, 60E15.

Keywords. Inequalities related to stochastic convex orderings, higher order stochastic convex orderings, functional inequalities related to convexity.

1. Introduction

The Bernstein operator \( B_n \) associated with a continuous function \( \varphi : [0,1] \to \mathbb{R} \) (see [10]) is defined by

\[
B_n(\varphi)(x) = \sum_{i=0}^{n} p_{n,i}(x) \varphi \left( \frac{i}{n} \right), \quad x \in [0,1],
\]
where
\[ p_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad 0 \leq j \leq n. \]

Mrowiec et al. [11] proved the following theorem on inequality for Bernstein operators.

**Theorem 1.** Let \( n \in \mathbb{N} \) and \( x, y \in [0,1] \). Then
\[
\sum_{i=0}^{n} \sum_{j=0}^{n} (p_{n,i}(x)p_{n,j}(x) + p_{n,i}(y)p_{n,j}(y) - 2p_{n,i}(x)p_{n,j}(y)) \varphi \left( \frac{i+j}{2n} \right) \geq 0 \quad (1)
\]
for all convex functions \( \varphi \in C([0,1]) \).

This inequality was stated by Ioan Raşă as an open problem about thirty years ago. During the Conference on Ulam’s Type Stability (Rytro, Poland, 2014), Raşă [15] recalled his problem. Theorem 1 affirms the conjecture (see also [1–4,6,7,16] for further results on the I. Rašă problem).

The proof given by Mrowiec et al. [11] makes use of probability theory. Let \( \mu \) and \( \nu \) be two finite Borel measures on \( \mathbb{R} \) such that
\[
\int_{\mathbb{R}} \varphi(x) \mu(dx) \leq \int_{\mathbb{R}} \varphi(x) \nu(dx) \quad \text{for all convex functions } \varphi: \mathbb{R} \to \mathbb{R}
\]
promvided the integrals exist. Then \( \mu \) is said to be smaller then \( \nu \) in the convex order (denoted as \( \mu \leq_{cx} \nu \)). In [11], the authors proved the following stochastic convex ordering relation for convolutions of binomial distributions \( B(n,x) \) and \( B(n,y) \) (\( n \in \mathbb{N}, x, y \in [0,1] \)):
\[
B(n,x) * B(n,y) \leq_{cx} \frac{1}{2} (B(n,x) * B(n,x) + B(n,y) * B(n,y)), \quad (2)
\]
which is a probabilistic version of inequality (1) and is equivalent to (1).

In [5], we gave a generalization of inequality (2). We introduced and studied the following convex ordering relation
\[
\mu * \nu \leq_{cx} \frac{1}{2} (\mu * \mu + \nu * \nu), \quad (3)
\]
where \( \mu \) and \( \nu \) are two probability distributions on \( \mathbb{R} \). We note, that inequality (3) can be regarded as the Raşă type inequality. In [5], we proved Theorem 2.3 providing a very useful sufficient condition for verification that \( \mu \) and \( \nu \) satisfy (3). We applied Theorem 2.3 for \( \mu \) and \( \nu \) from various families of probability distributions. In particular, we obtained a new proof for binomial distributions, which is significantly simpler and shorter than that given in [11].

In [7], we gave also necessary and sufficient condition for verification that \( \mu \) and \( \nu \) satisfy (3).
Recently, Abel and Leviatan [2] gave a generalization of the Raşa inequality (1) to \( q \)-monotone functions. Given a function \( f : I \to \mathbb{R} \), denote
\[
\Delta^1_h f(x) = \Delta_h f(x) = f(x + h) - f(x),
\]
\[
\Delta^{n+1}_h f(x) = \Delta^n_h (\Delta_h f(x)),
\]
\[
\Delta_{h_1 \ldots h_n h_{n+1}} f(x) = \Delta_{h_1} \ldots \Delta_{h_n} \Delta_{h_{n+1}} f(x) = \Delta_{h_1 \ldots h_n} (\Delta_{h_{n+1}} f(x))
\]
for \( n \in \mathbb{N}, x \in I \) and \( h, h_1, \ldots, h_n, h_{n+1} \geq 0 \) with all needed arguments belonging to \( I \). Let \( q \geq 1 \). A function \( f : I \to \mathbb{R} \) is \( q \)-monotone if \( \Delta^q_h f(x) \geq 0 \) for all \( h \geq 0 \) and \( x \in \mathbb{R} \) such that \( x, x + qh \in I \).

**Theorem 2.** ([2]) Let \( q, n \in \mathbb{N} \). If \( f \in C([0,1]) \) is a \( q \)-monotone function, then
\[
\text{sgn}(x - y)^q \sum_{\nu_1, \ldots, \nu_q = 0}^{n} (-1)^{q-j} {q \choose j} \left( \prod_{i=1}^{j} p_{n, \nu_i}(x) \right) \left( \prod_{i=j+1}^{q} p_{n, \nu_i}(y) \right) \geq 0.
\]

In [8], we give a probabilistic version of inequality (4) and consider some generalization of (4) in terms of higher order convex orders. Let us review some notations.

In the classical theory of convex functions their natural generalization are convex functions of higher-order.

The convexity of \( n \)-th order (or \( n \)-convexity) was defined in terms of divided differences by Popoviciu [12,13], however, we will not state it here. Instead we list some properties of \( n \)-th order convex functions (see [9]).

**Proposition 1.** If \( I \subset \mathbb{R} \) is an open interval, then a function \( f : I \to \mathbb{R} \) is \( n \)-convex on \( I \) (\( n \geq 1 \)) if, and only if, its derivative \( f^{(n-1)} \) exists and is convex on \( I \) (with the convention \( f^{(0)}(x) = f(x) \)).

**Proposition 2.** Let \( f \in C(I) \) and \( n \geq 1 \). Then the following statements are equivalent:
(a) \( f \) is \( n \)-convex on \( I \).

(b) \( f \) is \( (n+1) \)-monotone on \( I \).

(c) \( \Delta^{n+1}_h f(x) \geq 0 \)
for all \( x \in I \) and \( h \geq 0 \) with \( x + (n+1)h \in I \).

(d) \( \Delta_{h_1} \ldots \Delta_{h_n} \Delta_{h_{n+1}} f(x) \geq 0 \)
for all \( x \in I \) and \( h_1, \ldots, h_n, h_{n+1} \geq 0 \) with \( x + h_1 + \cdots + h_{n+1} \in I \).

Recall the definition of \( n \)-convex orders ([17]).

**Definition 1.** Let \( n \geq 1 \). Let \( \mu \) and \( \nu \) be two finite signed Borel measures on \( I \) such that
\[
\int_I \varphi(x) \mu(dx) \leq \int_I \varphi(x) \nu(dx)
\]
for all \( n \)-convex functions \( \varphi : I \to \mathbb{R} \).
provided the integrals exist. Then \( \mu \) is said to be smaller then \( \nu \) in the \( n \)-convex order (denoted as \( \mu \leq_{n-\text{cx}} \nu \ ) .

In particular, \( \mu \leq_{1-\text{cx}} \nu \) coincides with \( \mu \leq_{\text{cx}} \nu \). In [8], we study the following generalization of (3)

\[
(\nu - \mu)^q \geq_{(q-1)-\text{cx}} 0, \quad q \geq 2,
\]

where \( \mu, \nu \) are probability measures. Inequality (5) can be regarded as the Raşa type inequality.

Let \( \mu \) be a probability distribution on \( \mathbb{R} \). For \( x \in \mathbb{R} \) the delta symbol \( \delta_x \) denotes one-point probability distribution satisfying \( \delta_x(\{x\}) = 1 \). As usual, \( F_\mu(x) = \mu((-\infty, x]) \ (x \in \mathbb{R}) \) stands for the cumulative distribution function of \( \mu \). If \( \mu \) and \( \nu \) are two probability distributions such that \( F_\mu(x) \geq F_\nu(x) \) for all \( x \in \mathbb{R} \), then \( \mu \) is said to be smaller than \( \nu \) in the usual stochastic order (denoted by \( \mu \leq_{\text{st}} \nu \ )) .

In the following theorem, we give a very useful sufficient condition that will be used for verification of (5).

**Theorem 3** ([8]). Let \( q \geq 2 \). Let \( \mu \) and \( \nu \) be two probability distributions on \( I \), such that \( \mu \leq_{\text{st}} \nu \). Then

\[
(\nu - \mu)^q \geq_{(q-1)-\text{cx}} 0.
\]

In particular, by Theorem 3, taking binomial distributions, we obtain

\[
[s\text{gn}(x - y)]^q (B(n, x) - B(n, y))^q \geq_{(q-1)-\text{cx}} 0, \quad q \geq 2,
\]

which is equivalent to inequality (4). Consequently, we obtain a new proof of inequality (4) given by Abel and Leviatan [2], which is significantly simpler and shorter than that given in [2].

In [8], we apply Theorem 3 for \( \mu \) and \( \nu \) from various families of probability distributions. Using inequality (5), we can also obtain inequalities related to some approximation operators associated with \( \mu \) and \( \nu \) (such as Bernstein-Schnabl operators, Mirakyan-Szász operators, Baskakov operators and others). We proved also the following generalization of (6).

**Theorem 4.** ([8]) Let \( q \geq 2 \). Let \( \mu_1, \ldots, \mu_q \) and \( \nu_1, \ldots, \nu_q \) be probability distributions on \( I \), such that \( \mu_i \leq_{\text{st}} \nu_i \) for \( i = 1, \ldots, q \). Then

\[
(\nu_1 - \mu_1) \ast \ldots \ast (\nu_q - \mu_q) \geq_{(q-1)-\text{cx}} 0.
\]

In this paper, we give necessary and sufficient conditions for verification of (7). We give also necessary and sufficient conditions for verification of (6) for discrete probability distributions.
2. Main Results

We will use the following characterization of $n$-convex orders for signed measures.

**Proposition 3** (Corollary 2.1 [14]). Let $\gamma$ be a signed measure on $\mathbb{R}$, which is concentrated on the interval $(a, b)$ (bounded or unbounded) and such that $\int_a^b |x^n| |\gamma|(dx) < \infty$. Then in order that

$$\int_a^b f(x) \gamma(dx) \geq 0$$

for all $n$-convex functions $f : (a, b) \to \mathbb{R}$, it is necessary and sufficient that $\gamma$ verify the following conditions:

$$\gamma((a, b)) = 0,$$

$$\int_a^b x^k \gamma(dx) = 0 \quad \text{for} \quad k = 1, \ldots, n,$$

$$\int_a^b (x - A)^n_+ \gamma(dx) \geq 0 \quad \text{for all} \quad A \in (a, b).$$

First we prove the following lemma on the moments of the convolutions of signed measures. Let $\gamma$ be a signed measure on $\mathbb{R}$ such that $\int_{-\infty}^{\infty} |x|^{n-1} |\gamma|(dx) < \infty$. We denote by $m_n(\gamma)$ the $n$-th moment of $\gamma$, $m_n(\gamma) = \int_{-\infty}^{\infty} x^n \gamma(dx)$.

**Lemma 5.** Let $\gamma_1, \ldots, \gamma_n$ be signed measures on $\mathbb{R}$ such that $\gamma_i(\mathbb{R}) = 0$ and $\int_{-\infty}^{\infty} |x|^{n-1} |\gamma_i|(dx) < \infty$, $i = 1, \ldots, n$. Then

(a) $\gamma_1 \ast \ldots \ast \gamma_k(\mathbb{R}) = 0$, $k = 1, \ldots, n$,

(b) $\int_{-\infty}^{\infty} x^k \gamma_1 \ast \ldots \ast \gamma_m(dx) = 0$ for all integers $0 < k < m \leq n$.

**Proof.**

(a) Let $k = 1, \ldots, n$

$$\gamma_1 \ast \ldots \ast \gamma_k(\mathbb{R}) = \int_{-\infty}^{\infty} \gamma_1 \ast \ldots \ast \gamma_k(dx)$$

$$= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \gamma_1(du_1) \ldots \gamma_k(du_k) = 0.$$
(b) Let $0 < k < m \leq n$. We have
\[
\int_{-\infty}^{\infty} x^k \gamma_1 \ast \ldots \ast \gamma_m (dx)
= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} (u_1 + \ldots + u_m)^k \gamma_1 (du_1) \ldots \gamma_m (du_m)
= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \sum_{n_1 + \ldots + n_m = k} \binom{k}{n_1, \ldots, n_m} u_1^{n_1} \ldots u_m^{n_m} \gamma_1 (du_1) \ldots \gamma_m (du_m)
= \sum_{n_1 + \ldots + n_m = k} \binom{k}{n_1, \ldots, n_m} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} u_1^{n_1} \ldots u_m^{n_m} \gamma_1 (du_1) \ldots \gamma_m (du_m).
\]

Let us consider one of the components of the sum above
\[
\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} u_1^{n_1} \ldots u_m^{n_m} \gamma_1 (du_1) \ldots \gamma_m (du_m),
\]
where $n_1 + \ldots + n_m = k$. Without loss of generality we may assume that $n_1 \geq n_2 \geq \ldots \geq n_m$. Then there exists $1 \leq j \leq k$ such that $n_j > 0$ and $n_{j+1} = \ldots = n_m = 0$. Taking into account that $\gamma_{j+1} (\mathbb{R}) = \ldots = \gamma_m (\mathbb{R}) = 0$, we obtain
\[
\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} u_1^{n_1} \ldots u_m^{n_m} \gamma_1 (du_1) \ldots \gamma_m (du_m)
= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} u_1^{n_1} \ldots u_j^{n_j} \gamma_1 (du_1) \ldots \gamma_m (du_m)
= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} u_2^{n_2} \ldots u_j^{n_j} \int_{-\infty}^{\infty} u_1^{n_1} \gamma_1 (du_1) \gamma_2 (du_2) \ldots \gamma_m (du_m)
= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} u_2^{n_2} \ldots u_j^{n_j} m_{n_1} (\gamma_1) \gamma_2 (du_2) \ldots \gamma_m (du_m)
= m_{n_1} (\gamma_1) \ldots m_{n_j} (\gamma_j) \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \gamma_{j+1} (du_{j+1}) \ldots \gamma_m (du_m) = 0.
\]

This completes the proof of (b). \qed

In the following theorem we give necessary and sufficient conditions for verification of (7).

**Theorem 6.** Let $q \geq 2$. Let $\mu_1, \ldots, \mu_q$ and $\nu_1, \ldots, \nu_q$ be finite Borel measures $\mathbb{R}$, such that $\int_{-\infty}^{\infty} |x|^{q-1} \mu_i (dx) < \infty$, $\int_{-\infty}^{\infty} |x|^{q-1} \nu_i (dx) < \infty$ and $\mu_i (\mathbb{R}) = \nu_i (\mathbb{R})$, with $F_i (x) = \mu_i ((-\infty, x])$ and $G_i (x) = \nu_i ((-\infty, x])$, for $i = 1, \ldots, q$. Then the following conditions are equivalent:

a) $(\nu_1 - \mu_1) \ast \ldots \ast (\nu_q - \mu_q) \geq (q-1)-cx 0.$
b) $[(F_1 - G_1) \ast \ldots \ast (F_q - G_q)] (x) \geq 0$ for all $x \in \mathbb{R}$.
Proof. Let $\tau_i = \nu_i - \mu_i$, $i = 1, \ldots, q$. Then $\tau_i$ is a signed measure on $\mathbb{R}$ such that $\tau_i(\mathbb{R}) = 0$ and $\int_{-\infty}^{\infty} |x|^{q-1} |\tau_i|(dx) < \infty$, $i = 1, \ldots, q$. By Lemma 5, we obtain
\begin{align}
\tau_1 \ast \ldots \ast \tau_q(\mathbb{R}) &= 0, \\
\int_{-\infty}^{\infty} x^k \tau_1 \ast \ldots \ast \tau_q(dx) &= 0 \quad \text{for } k = 1, \ldots, q - 1.
\end{align}

By Proposition 3, taking into account (8) and (9), in order that
\begin{align}
\int_{-\infty}^{\infty} f(x) \tau_1 \ast \ldots \ast \tau_q(dx) \geq 0
\end{align}

for all $(q - 1)$-convex functions $f: \mathbb{R} \to \mathbb{R}$, it is necessary and sufficient that
\begin{align}
\int_{-\infty}^{\infty} (x - A)^{q-1} \tau_1 \ast \ldots \ast \tau_q(dx) \geq 0 \quad \text{for all } A \in \mathbb{R}.
\end{align}

Taking into account that for $x \geq A$
\begin{align}
(x - A)^{q-1} = (q-1)! \int_A^x \int_A^{s_{q-1}} \int_A^{s_{q-2}} \ldots \int_A^{s_2} 1 \lambda(ds_1)\lambda(ds_2)\ldots\lambda(ds_{q-1}),
\end{align}

we obtain
\begin{align}
((q-1)!)^{-1} \int_{-\infty}^{\infty} (x - A)^{q-1} \tau_1 \ast \ldots \ast \tau_q(dx) &= \int \ldots \int \int_{A \to t_1}^{u_1 + \ldots + u_q \geq A} \int_A^{s_{q-1}} \int_A^{s_{q-2}} \ldots \int_A^{s_2} 1 \lambda(ds_1)\lambda(ds_2)\ldots\lambda(ds_{q-1})
\tau_1(du_1)\tau_2(du_2)\ldots\tau_q(du_q).
\end{align}

The variables satisfy the inequalities $A < s_1 < s_2 < \ldots < s_{q-1} < u_1 + \ldots + u_q$, which after the substitution $t_i = s_i - \sum_{j=1}^{i} u_j$, $i = 1, \ldots, q - 1$ are equivalent to $A - t_1 = A - s_1 + u_1 < t_1$, $t_{i-1} - t_i = s_{i-1} - s_i + u_i < u_i$ for $i = 2, 3, \ldots, q - 1$ and $t_{q-1} = s_{q-1} - \sum_{j=1}^{q} u_j + u_q < u_q$. After changing the order of integration, we obtain
\begin{align}
\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \int_{A-t_1}^{t_1-t_2} \int_{t_{q-2}-t_{q-1}}^{t_{q-1}} 1 \tau_q(du_q)\tau_{q-1}(du_{q-1}) \ldots \tau_1(du_1) \lambda(dt_1)\ldots\lambda(dt_{q-1})
&= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \tau_1((A - t_1, +\infty))\tau_2((t_1 - t_2, +\infty))\tau_3((t_2 - t_3, +\infty))\ldots\tau_{q-1}((t_{q-2} - t_{q-1}, +\infty))\tau_q((t_{q-1}, +\infty))\lambda(dt_1)\ldots\lambda(dt_{q-1})
&= \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} (F_1 - G_1)(A - t_1)(F_2 - G_2)(t_1 - t_2) \ldots (F_{q-1} - G_{q-1})(t_{q-2} - t_{q-1})(F_q - G_q)(t_{q-1})\lambda(dt_1)\ldots\lambda(dt_{q-1})
&= (F_1 - G_1) \ast (F_2 - G_2) \ast \ldots \ast (F_q - G_q)(A).
\end{align}
By the above equalities, we obtain that
\[
\int_{-\infty}^{\infty} (x - A)^{q - 1} \tau_1 \ast \ldots \ast \tau_q(dx) \geq 0 \text{ iff } (F_1 - G_1) \ast (F_2 - G_2) \ast \ldots \ast (F_q - G_q)(A) \geq 0.
\]
The theorem is proved. \(\square\)

**Corollary 7.** Theorem 6 is a generalization of Theorem 4. Indeed, if \(\mu_i \leq_{st} \nu_i\) for \(i = 1, \ldots, q\), then obviously the condition (b) in Theorem 6 is satisfied.

**Corollary 8.** Let \(\mu\) and \(\nu\) be two finite Borel measures on \(\mathbb{R}\) with finite \((q - 1)\)-th moments such that \(\mu(\mathbb{R}) = \nu(\mathbb{R})\), with \(F(x) = \mu((\infty, x])\) and \(G(x) = \nu((\infty, x])\). Then the following conditions are equivalent:

(a) \((\nu - \mu)^{*q} \geq (q - 1)_{-cx} 0\).

(b) \((F - G)^{*q} (x) \geq 0\) for all \(x \in \mathbb{R}\).

**Example 1.** If \(\mu = \frac{1}{2} \delta_{-3} + \frac{1}{2} \delta_1\) and \(\nu = \frac{3}{4} \delta_0 + \frac{1}{4} \delta_4\), then \((\nu - \mu)^*4 \geq_{3-cx} 0\), although \(\mu \leq_{st} \nu\) is not satisfied (we leave the proof to the reader).

**Corollary 9.** Let \(\tau_1, \ldots, \tau_q\) be signed measures on \(\mathbb{R}\) with finite \((q - 1)\)-th moments such that \(\tau_i(\mathbb{R}) = 0\), \(F_{\tau_i}(x) = \tau_i((x, \infty))\), \(i = 1, \ldots, q\). Then the following conditions are equivalent:

(a) \(\tau_1 \ast \ldots \ast \tau_q \geq_{(q - 1)_{-cx}} 0\).

(b) \((F_{\tau_1} \ast \ldots \ast F_{\tau_q})(x) \geq 0\) for all \(x \in \mathbb{R}\).

**Definition 2.** Let \(n \geq 1\). Let \(\mu\) and \(\nu\) be two finite signed Borel measures on \(\mathbb{R}\) such that
\[
\int_I \varphi(x) \mu(dx) \leq \int_I \varphi(x) \nu(dx) \quad \text{for all } n\text{-monotone functions } \varphi : \mathbb{R} \rightarrow \mathbb{R} \quad (10)
\]
provided the integrals exist. Then \(\mu\) is said to be smaller then \(\nu\) in the \(n\)-monotone order (denoted as \(\mu \prec_{n\text{-monot}} \nu\)).

Then \(\mu \prec_{n\text{-monot}} \nu \equiv \mu \leq_{(n-1)_{-cx}} \nu \) for \(n \geq 2\), and \(\mu \prec_{1\text{-monot}} \nu \equiv \mu \leq_{st} \nu\) (we recall that \(\mu \leq_{st} \nu \equiv \mu((\infty, x]) \geq \nu((\infty, x])\) for all \(x \in \mathbb{R}\)). Note that Theorem 4 on sufficient condition can be written equivalently in terms of signed measures as follows.

**Theorem 10** ([8]). Let \(q \geq 2\). Let \(\tau_1, \ldots, \tau_q\) be signed measures on \(\mathbb{R}\), such that \(\tau_i(\mathbb{R}) = 0\) and \(\tau_i \succ_{1\text{-monot}} 0\) for \(i = 1, \ldots, q\). Then
\[
\tau_1 \ast \ldots \ast \tau_q \succ_{q\text{-monot}} 0.
\]

We prove some generalization of Theorem 10.

**Theorem 11.** Let \(q_1, q_2 \geq 1\). Let \(\tau_1, \tau_2\) be signed measures on \(\mathbb{R}\), such that \(\tau_i(\mathbb{R}) = 0\) and \(\tau_i \succ_{q_i\text{-monot}} 0\) for \(i = 1, 2\). Then
\[
\tau_1 \ast \tau_2 \succ_{(q_1 + q_2)\text{-monot}} 0.
\]
Proof. Let \( q_1, q_2 \geq 1 \). Let \( \tau_1, \tau_2 \) be signed measures on \( \mathbb{R} \), such that \( \tau_i(\mathbb{R}) = 0 \) and \( \tau_i \succ_{q_i-\text{monot}} 0 \) for \( i = 1, 2 \). Let \( f : \mathbb{R} \to \mathbb{R} \) be a \((q_1 + q_2)\)-monotone function such that \( \int_{\mathbb{R}} f(x) \tau_1 \tau_2(dx) \) exists. Then for any \( y \in \mathbb{R} \) the function \( x \mapsto f(x+y) \) \( (x \in \mathbb{R}) \) is \((q_1 + q_2)\)-monotone, which implies that
\[
\Delta_{h_1}^{q_1} \Delta_{h_2}^{q_2} f(x+y) \geq 0 \quad \text{for all } h_1, h_2 \geq 0.
\]
This implies that the function \( x \mapsto \Delta_{h_2}^{q_2} f(x+y) \) is \( q_1 \)-monotone. Taking into account that \( \tau_1 \succ_{q_1-\text{monot}} 0 \), we have
\[
\int_{\mathbb{R}} \Delta_{h_2}^{q_2} f(x+y) \tau_1(dx) \geq 0,
\]
consequently, we obtain
\[
\Delta_{h_2}^{q_2} \int_{\mathbb{R}} f(x+y) \tau_1(dx) \geq 0 \quad (11)
\]
for all \( h_2 \geq 0 \) and \( y \in \mathbb{R} \). By (11), we conclude that the function \( y \mapsto \int_{\mathbb{R}} f(x+y) \tau_1(dx) \) is \( q_2 \)-monotone. Since \( \tau_2 \succ_{q_2-\text{monot}} 0 \), it follows
\[
\int_{\mathbb{R}} \int_{\mathbb{R}} f(x+y) \tau_1(dx) \tau_2(dy) \geq 0,
\]
which implies
\[
\int_{\mathbb{R}} f(x) \tau_1 \tau_2(dx) \geq 0. \quad (12)
\]
Since inequality (12) is satisfied for all \((q_1 + q_2)\)-monotone functions (for which the integral exists), it follows that \( \tau_1 \tau_2 \succ_{(q_1 + q_2)-\text{monot}} 0 \), which completes the proof of the theorem.

Similarly, one can prove a generalization of Theorem 11.

**Theorem 12.** Let \( n \geq 2 \). Let \( q_1, \ldots, q_n \geq 1 \). Let \( \tau_1, \ldots, \tau_n \) be signed measures on \( \mathbb{R} \), such that \( \tau_i(\mathbb{R}) = 0 \) and \( \tau_i \succ_{q_i-\text{monot}} 0 \) for \( i = 1, \ldots, n \). Then
\[
\tau_1 \cdots \tau_n \succ_{(q_1+\ldots+q_n)-\text{monot}} 0.
\]

**Corollary 13.** Let \( n \geq 2 \). Let \( q_1, \ldots, q_n \geq 1 \). Let \( \mu_1, \ldots, \mu_n \) and \( \nu_1, \ldots, \nu_n \) be probability distributions on \( \mathbb{R} \), such that \( \mu_i \succ_{q_i-\text{monot}} \nu_i \) for \( i = 1, \ldots, n \). Then
\[
(\mu_1 - \nu_1) \cdots (\mu_n - \nu_n) \succ_{(q_1+\ldots+q_n)-\text{monot}} 0.
\]

Consider now the discrete probability distribution \( \mu \) concentrated on the set of non-negative integers \( \{0, 1, 2, \ldots\} \), with \( a_k = \mu(\{k\}) \) \( (k = 0, 1, 2, \ldots) \). Then the probability generating function corresponding to \( \mu \) is given by the formula \( f(z) = \sum_{k=0}^{\infty} a_k z^k \).

**Theorem 14.** Let \( \mu \) and \( \nu \) be discrete probability distributions concentrated on the set of non-negative integers \( \{0, 1, 2, \ldots\} \), with \( a_k = \mu(\{k\}) \) and \( b_k = \nu(\{k\}) \) \( (k = 0, 1, 2, \ldots) \). Assume that \( \mu \) and \( \nu \) have finite \( q \)-th moments. Let \( F, f \) and \( G, g \) be the distribution functions and the generating functions corresponding to \( \mu \) and \( \nu \), respectively. Then the following conditions are equivalent:
(1) For all $(q - 1)$-convex functions $\varphi : \mathbb{R} \to \mathbb{R}$

$$\sum_{i_1=0}^{\infty} \cdots \sum_{i_q=0}^{\infty} \left( b_{i_1} - a_{i_1} \right) \cdots \left( b_{i_q} - a_{i_q} \right) \varphi(i_1 + \cdots + i_q) \geq 0.$$  \hfill (13)

(2) $(F - G)^* q(x) \geq 0$ for all $x \in \mathbb{R}$.

(3) \[ \frac{d^k}{dz^k} \left( \frac{g(z) - f(z)}{z - 1} \right)^q \bigg|_{z=0} \geq 0 \quad \text{for } k = 0, 1, \ldots \]

Proof. Note that (13) is equivalent to the relation $(\nu - \mu)^* q \geq (q - 1) - c x 0$. Indeed, let $\varphi : \mathbb{R} \to \mathbb{R}$ be a $(q - 1)$-convex function. Then, we have

$$\int_{-\infty}^{\infty} \varphi(x)(\nu - \mu)^* q(dx)$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \varphi(u_1 + \cdots + u_q)(\nu - \mu)(du_1) \cdots (\nu - \mu)(du_q)$$

$$= \sum_{i_1=0}^{\infty} \cdots \sum_{i_q=0}^{\infty} \varphi(i_1 + \cdots + i_q)(\nu - \mu)(\{i_1\}) \cdots (\nu - \mu)(\{i_q\})$$

$$= \sum_{i_1=0}^{\infty} \cdots \sum_{i_q=0}^{\infty} (b_{i_1} - a_{i_1}) \cdots (b_{i_q} - a_{i_q}) \varphi(i_1 + \cdots + i_q).$$

Consequently, the equivalence of (1) and (2) clearly follows from Corollary 8. It suffices to prove the equivalence of (2) and (3).

In the following calculations, we use the existence and finiteness of the $q$th moments of the probability distributions $\mu$ and $\nu$, which implies that all the following series are absolutely convergent for $z \in [-1, 1]$ and we can change the summation order. By the equality $\sum_{k=0}^{\infty} a_k = 1$, we have

$$f(z) - 1 \over z - 1 = \sum_{k=0}^{\infty} a_k \frac{z^k - 1}{z - 1} = \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} a_k z^i = \sum_{i=0}^{\infty} \sum_{k=i+1}^{\infty} a_k z^i = \sum_{i=0}^{\infty} F(i) z^i$$

for every $z \in [-1, 1)$, where $F$ is the tail distribution of $\mu$. Note that $\sum_{i=0}^{\infty} F(i) = \int_{\mathbb{R}} x \mu(dx) < \infty$. Similarly, we obtain

$$g(z) - 1 \over z - 1 = \sum_{i=0}^{\infty} G(i) z^i,$$

where $G$ is the tail distribution of $\nu$. Therefore, for every $z \in [-1, 1)$ we have

$$\left( \frac{g(z) - f(z)}{z - 1} \right)^q = \left( \sum_{i=0}^{\infty} (G - F)(i) z^i \right)^q = \left( \sum_{i=0}^{\infty} (F - G)(i) z^i \right)^q$$

$$= \sum_{i=0}^{\infty} (F - G)^* q(i) z^i.$$
Here, $\ast$ denotes the discrete convolution (the Euler product) of sequences. Condition (3) is equivalent to the non-negativity of all the coefficients in the above series. Note that $(F - G)^\ast q(i) = (F - G)^\ast q(i)$, $i = 0, \ldots, \infty$ (cf. the proof of Theorem 2.6 in [7]). Therefore, the non-negativity of all the terms $(F - G)^\ast q(i)$ is equivalent to (2). The theorem is proved. $\Box$

Theorem 14 is a generalization of results [4,7] on the Raša inequality for convex functions.

Acknowledgements
The authors thank the reviewer for a very detailed and insightful review as well as valuable suggestions and comments.

Author contributions Andrzej Komisarski and Teresa Rajba contributed equally to this work.

Funding No extra funding for this research.

Declarations

Conflict of interest The author declares that they have no conflict of interest.

References
[1] Abel, U.: An inequality involving Bernstein polynomials and convex functions. J. Approx. Theory. 222, 1–7 (2017)
[2] Abel, U., Leviatan, D.: An extension of Raša’s conjecture to q-Monotone functions. Results Math. 75(4), 181–193 (2020)
[3] Abel, U., Raša, I.: A sharpening problem on Bernstein polynomials and convex functions. Math. Inequal. Appl. 21(3), 773–777 (2018)
[4] Gavrea, B.: On a convexity problem in connection with some linear operators. J. Math. Anal. Appl. 461, 319–332 (2018)
[5] Komisarski, A., Rajba, T.: Muirhead inequality for convex orders and a problem of I. Raša on Bernstein polynomials. J. Math. Anal. Appl. 458, 821–830 (2018)
[6] Komisarski, A., Rajba, T.: A sharpening of a problem on Bernstein polynomials and convex functions and related results. Math. Inequal. Appl. 21(4), 1125–1133 (2018)
[7] Komisarski, A., Rajba, T.: Convex order for convolution polynomials of Borel measures. J. Math. Anal. Appl. 478, 182–194 (2019)
[8] Komisarski, A., Rajba, T.: On the Raša inequality for higher order convex functions. Results Math. 76(2), 103–115 (2021)
[9] Kuczma, M.: An introduction to the theory of functional equations and inequalities. Prace Naukowe Uniwersytetu Śląskiego w Katowicach, vol. 489, Państwowe Wydawnictwo Naukowe—Uniwersytet Śląski, Warszawa, Kraków, Katowice (1985)

[10] Lorentz, G.G.: Bernstein polynomials. Mathematical Expositions. No. 8., University of Toronto Press. Toronto (1953)

[11] Mrowiec, J., Rajba, T., Wąsowicz, S.: A solution to the problem of Raşa connected with Bernstein polynomials. J. Math. Anal. Appl. 446, 864–878 (2017)

[12] Popoviciu, T.: Sur quelques proprietes des fonctions d’une ou de deux variables reelles. Mathematica 8, 1–85 (1934)

[13] Popoviciu, T.: Les Fonctions Convexes. Hermann, Paris (1944)

[14] Rajba, T.: On a generalization of a theorem of Levin and Steckin and inequalities of the Hermite-Hadamard type. Math. Inequal. Appl. 20(2), 363–375 (2017)

[15] Raşa, I.: 2. Problem, p. 164. In: Report of Meeting Conference on Ulam’s Type Stability, Rytro, Poland, June 2–6, 2014 (2014) (Ann. Univ. Paedagog. Crac. Stud. Math. 13, 139–169). https://doi.org/10.2478/aupcsm-2014-0011

[16] Raşa, I.: Bernstein polynomials and convexity: recent probabilistic and analytic proofs. The Workshop “Numerical Analysis, Approximation and Modeling”, T. Popoviciu Institute of Numerical Analysis, Cluj-Napoca (2017). http://ictp.acad.ro/zileleacademice-clujene-2017/

[17] Shaked, M., Shanthikumar, J.G.: Stochastic Orders. Springer, Berlin (2007)

Teresa Rajba
Department of Mathematics
University of Bielsko-Biała
ul. Willowa 2
43-309 Bielsko-Biała
Poland
e-mail: trajba@ath.bielsko.pl

Andrzej Komisarski
Faculty of Mathematics and Computer Science
University of Łódź
ul. Banacha 22
90-238 Łódź
Poland
e-mail: andkom@math.uni.lodz.pl

Received: November 15, 2021.
Accepted: February 6, 2022.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.