New $N = 1$ Supersymmetric 3-dimensional Superstring Vacua from U-manifolds

Gottfried Curio$^1$ and Dieter Lüst$^2$\footnote{curio@ias.edu, luest@qft1.physik.hu-berlin.de}

(1) School of Natural Sciences, Institute for Advanced Study, Princeton, NJ 08540
(2) Humboldt-Universität, Institut für Physik, Invalidenstraße 110, 10115 Berlin, Germany

Making use of non-perturbative U-duality symmetries of type II strings we construct new ‘superstring’ vacua in three dimensions with $N=1$ supersymmetry. This has an interpretation as compactifying formally from 13 dimensions (S-theory) on Calabi-Yau 5-folds possessing a $T^3 \times T^2$ fibration. We describe some part of the massless multiplets, given by the Hodge spectrum, and point to a corresponding 5-brane configuration.

\footnote{The first author is partially supported by NSF grant DMS 9627351}
1 S-Theory

The F-theory construction \[1, 2\] can be generalized by considering the type IIB string compactified (on a torus) to lower \((d')\) dimensions \[3\]. If one allows the scalar fields (U-fields) to vary over some part of the \(d'\)-dimensional space and allows them to jump, consistent with the U-dualities, this data will translate to a (complex) \(n\)-dimensional manifold \(K^n\) whose (real) \(b\)-dimensional base \(B^b\) is the visible space from \(d'\) to \(d' - b\) dimensions and the (real) \(2n - b\) dimensional fibre being the geometrization of the U-duality. Hence this construction leads to type IIB string vacua with \(d = d' - b\) flat, uncompactified dimensions. As an example consider S-theory \[3, 4\] with \(d' = 8\). In this case the scalar moduli space is given by the coset

\[
\text{Sl}(3, \mathbb{Z}) \times \text{Sl}(2, \mathbb{Z}) \backslash \text{Sl}(3, \mathbb{R}) \times \text{Sl}(2, \mathbb{R}) / \text{SO}(3) \times \text{SO}(2),
\]

and the U-duality group is \(\text{Sl}(3, \mathbb{Z}) \times \text{Sl}(2, \mathbb{Z})\). In S-theory the seven scalar fields, which parametrize the above coset, are allowed to vary over the 5-dimensional base \(B^5\). The U-duality group arises as a combination of two contributions: on the one hand one has the \(\text{Sl}(2, \mathbb{Z})\) which exists already in 10 dimensions (the ‘S-duality’ of the type IIB string), and is used there to append the F-theory elliptic torus, leading to a theory living formally in 12 dimensions. This \(\text{Sl}(2, \mathbb{Z})_S\) is united with the \(\text{Sl}(2, \mathbb{Z})_T \times \text{Sl}(2, \mathbb{Z})_U\) which arises on the other hand after compactification of the type IIB theory to 8 dimensions on a \(T^2\); to be precise, \(\text{Sl}(2, \mathbb{Z})_T\) combines with the \(\text{Sl}(2, \mathbb{Z})_S\) to the \(\text{Sl}(3, \mathbb{Z})\), whereas the \(\text{Sl}(2, \mathbb{Z})_U\) remains giving the \(\text{Sl}(2, \mathbb{Z})\) factor of \(\text{Sl}(3, \mathbb{Z}) \times \text{Sl}(2, \mathbb{Z})\). Note that after compactification on the \(T^2\) to 8 dimensions the theory becomes equivalent to type IIA and thereby to \(M\) theory on \(T^3\), which gives a further view on the U-duality group \(\text{Sl}(3, \mathbb{Z}) \times \text{Sl}(2, \mathbb{Z})\). So the (possibly reducible) Calabi-Yau 5-fold \(K^5\) must be a \(T^3 \times T^2\) fibration over \(B^5\):

\[
K^5 \rightarrow_{T^3 \times T^2} B^5.
\]

Since one has appended in 8 dimensions a 5-dimensional torus \(T^3 \times T^2\), S-theory can be regarded as a 13-dimensional theory.

Next we discuss what kind of 5-folds \(K^5\) with \(T^3 \times T^2\) fibration can be constructed. First consider splitting (reducible) 5-folds which lead to \(N = 2\) supersymmetry in 3 dimensions. One possible choice \[3\] is a product space where one factor is a Calabi-Yau 3-fold \(CY^3\) with \(T^3\) fibration and the other factor is an elliptic \(K^3\), i.e.

\[
K^5 = CY^3 \times K^3, \quad CY^3 \rightarrow_{T^3} B^3, \quad K^3 \rightarrow_{T^2} S^2.
\]

Therefore the total 5-dimensional base \(B^5\) is given by

\[
B^5 = B^3 \times S^2.
\]

We will always assume that \(B^3 = S^3\) (cf. also \[4\]; this is connected with mirror symmetry on Calabi-Yau 3-folds \[3, 4\]). Another class of vacua with \(N = 2\) supersymmetry is given by

\[
K^5 = CY^4 \times T^2, \quad CY^4 \rightarrow_{T^3} B^5.
\]
The CY$^4$ is assumed to be CY$^3$ fibered over $P^1$. Therefore $B^5$ is a $S^3$ fibration over $S^2$.

In this paper we will see (section 3) that Calabi-Yau 5-folds exist which are $T^3 \times T^2$ fibrations but not product spaces. This leads to $N = 1$ supersymmetry in 3 dimensions. Before, this was, in string compactification, possible to be reached only using the somewhat difficult to handle spaces of exceptional holonomy (heterotic string on $G_2$-manifold, M-theory on Spin(7)-manifold). By contrast it is realised here still in the framework of the well-suited Calabi-Yau spaces.

The new three-dimensional superstring vacua described in this paper might also lead to a geometric understanding of (non-perturbative) effects in 3D, $N = 1$ supersymmetric field theory [7], possibly very much like the realisation of instantons, contributing to the superpotential, via internal geometrical cycles in the context of M-theory resp. F-theory on a Calabi-Yau four-fold (leading to $N = 2$ in 3D resp. $N = 1$ in 4D).

Finally in view of Witten’s szenario [8] relating $N = 1$ supersymmetric theories in three dimensions to non-supersymmetric theories in four dimensions with vanishing cosmological constant one can try to relate the described 3D vacua to 4D vacua of $N = 0$ like it was tried [1] for M-theory on Spin(7) manifolds.

2 Some dualities

Let us first recall [3] the duality symmetries between S-theory on the one hand and F-theory and the heterotic string on the other hand. One derives first that S-theory on $K^5 \times S^1$ is dual to F-theory on $K^5$:

\[ d = 2 : \quad S|_{K^5 \times S^1} \leftrightarrow F|_{K^5}, \]

Here in $F|_{K^5}$ - i.e. as soon as one has left S-theory, which is in a sense a type IIB theory with additional structure, and has reached F-theory, which is a type IIB theory with a different additional structure - the $T^2$ fibre of $K^5$ corresponds to the elliptic fibre used in F-theory to codify the type IIB complex coupling constant. On the other hand, the volume of the $T^3$ fibre of $K^5$ in F-theory corresponds to the inverse radius of $S^1$. One can go further down in dimensions and arrives at the following chain of dualities

\[ d = 1 : \quad S|_{K^5 \times T^2} \leftrightarrow F|_{K^5 \times S^1} \leftrightarrow M|_{K^5}, \]

where the inverse radius of the extra circle is related to the volume of the $T^2$ fibre in M-theory.

Before we discuss S-theory on CY$^5$ in greater detail let us consider the reducible cases of S-theory on $K^5 = CY^3 \times K3$ with an elliptically fibered K3 resp. on $K^5 = CY^4 \times T^2$ with a CY$^3$ fibered CY$^4$. In case of S-theory on $K^5 = CY^3 \times K3$ the above chain of dualities can be extended [3]. As, upon compactification on $S^1$, this is dual to F-theory
on \( CY^3 \times K3 \) and F-theory on the elliptic \( K3 \) is dual to the heterotic string on \( T^2 \); this means that in 2 dimensions we have a duality with a heterotic string on \( CY^3 \times T^2 \):

\[
d = 2 : \quad S_{\mid CY^3 \times K3 \times S^1} \leftrightarrow H_{\mid CY^3 \times T^2},
\]

(8)

The other reducible case of S-theory on \( K^5 = CY^4 \times T^2 \) with a \( CY^3 \) fibered \( CY^4 \) is also interesting to consider and not already covered (via duality) by some other theory known before, especially it is not dual to \( M \)-theory on \( CY^4 \). For this note that after compactification on \( S^1 \) the \( T^2 \) factor is now the \( F \)-theory elliptic fibre, i.e. this is simply type IIB on \( CY^4 \) of \((0,4)\) spacetime supersymmetry in 2 dimensions. By contrast \( M \)-theory on \( CY^4 \) is the lifting to 3 dimensions of the non-chiral theory of \((2,2)\) spacetime supersymmetry in 2 dimensions given by type IIA on \( CY^4 \); in other words the 13-dimensional S-theory on \( CY^4 \times T^2 \) can be viewed as the lift of type IIB on a \( CY^4 \) from 2 to 3 dimensions just as the 11-dimensional \( M \)-theory does the corresponding thing for type IIA.

Let us come now to the 3-dimensional theories with \( N = 1 \) supersymmetry like S-theory on \( CY^5 \) or \( M \)-theory on a \( Spin(7) \) manifold (or the heterotic string on a \( G_2 \) manifold). Let us consider, in view of the observation just made for the reducible \( CY^4 \times T^2 \) case, again first the situation in 2 dimensions. There one has again the non-chiral theory given by type IIA on a \( Spin(7) \) manifold and the chiral one given by F-theory on \( CY^5 \). Then \( M \) theory on \( Spin(7) \) lifts the first, non-chiral, theory to 3 dimensions, whereas S-theory on \( CY^5 \) lifts the chiral theory to 3 dimensions.

### 3 Some Calabi-Yau 5-folds

For the compactification of S-theory we will now construct (complex) Calabi-Yau 5-folds which have a \( T^3 \times T^2 \) fibration. The \( T^3 \) part we will always get from a \( CY^3 \) (which is assumed to have a mirror, and so a \( T^3 \) fibration over a real 3-dimensional base \( B^3 = S^3 \), cf. [3]). We concentrate on spaces which are true (irreducible, non-splitting) \( CY^5 \). We will discuss below examples of the form \( X^4 \times _1 dP_9 \), where \( X^4 \) is a (non Calabi-Yau) fourfold which has a \( CY^3 \) fibration over \( P^1 \) (whose fibration structure is also part of the ‘input data’ structure and not given by the \( CY^3 \) alone, will be described below) and \( dP_9 := \left[ \frac{P^3}{P^1} \right] \) is a surface having an elliptic fibration over \( P^1 \). Clearly, the very idea of this fibre product is that the splitting of the fibration does not imply the splitting of the total space. The real 5-dimensional base \( B^5 \) for the \( T^3 \times T^2 \) fibration is a \( S^3 \) fibration over \( S^2 \), which will be described in more detail in the next section.

\footnote{For example one can have a description derived from a representation \( CY^5 = (CY^4 \times T^2)/\mathbb{Z}_2 \) which is to be understood in the same sense as the construction of the \( CY^{19,19} = dP_9 \times _1 dP_9 \) from the quotient \((K3 \times T^2)/\mathbb{Z}_2 \) (cf. appendix and [1]); similarly one could study a \( CY^5 = (CY^3 \times K3)/\mathbb{Z}_2 \) version, this time with the \( dP_9 \)-fibrefactor appearing by ‘reduction’ from \( K3 \).}

\footnote{cf. the appendix and [11, 12, 13]
For example one can build out of the Calabi-Yau 3-fold $CY^{19,19}$, by fibering it over a further $P^1$, the 4-fold $X^4 = \begin{pmatrix} p^1 & 3 & 0 \\ p^1 & 1 & 1 \\ p^2 & 0 & 3 \\ p^1 & 0 & 1 \end{pmatrix} = dP_9 \times P_1$ with $K := \begin{pmatrix} p^1 \\ p^2 \\ p^1 \end{pmatrix}$ and finally the Calabi-Yau 5-fold $CY^5 = \begin{pmatrix} p^1 & 3 & 0 & 0 \\ p^1 & 1 & 1 & 0 \\ p^2 & 0 & 3 & 0 \\ p^1 & 0 & 1 & 1 \\ p^2 & 0 & 0 & 3 \end{pmatrix} = dP_9 \times P_1 \times P_1 \times P_1$. We will analyse that example further below.

Let us analyse now the cohomology of a genera (i.e. at first not necessarily of the fibre-product form) Calabi-Yau 5-fold. Its Hodge diamond looks like

\[
\begin{array}{cccccccc}
1 & & & & & & & \\
& 0 & & & & & & \\
& & 0 & & h^{11} & 0 & & \\
& & & 0 & & h^{21} & h^{21} & 0 \\
& & & & & & h^{31} & h^{32} & h^{32} & h^{41} & 1 \\
1 & & & & & & & \\
\end{array}
\]

Below we will compute the Hodge numbers which have the most immediate interpretation, $h^{11}$ as Kähler parameters and $h^{41}$ as complex deformations, from our input data in the class of $X^4 \times P_1 \times dP_9$ spaces. Furthermore from the $CY^3 \times T^2$ fibration over $P^1$ of these spaces one gets for the Euler number $e$ of the 5-fold $e = 12 \cdot e(CY^3)$ as $e(dP_9) = 12$. So we will ‘know’ 3 of the 6 unknown numbers. In the Calabi-Yau 4-fold case one gets one further information from a relation derived in \[13\]; this is enough for the 4 unknowns in the 4-fold case, in our case 2 unknowns remain. Let us see in detail how this happens.

The index of the (1,0)-forms-valued $\bar{\partial}$ operator, $\text{ind}\, \bar{\partial} = \sum_{q=0}^{5} (-1)^q h^{q,1}$, is according to the index theorem given by

\[
\text{ind}\, \bar{\partial} = \int_X Td(X)ch(T^*X),
\]

where for the Calabi-Yau 5-fold $X$ one has

\[
Td(X^5_{CY}) = 1 + \frac{c_2}{12} + \frac{3c_2^2 - c_4}{720},
\]

\[
ch(T^*X^5_{CY}) = 5 - c_2 + \frac{-c_3}{2} + \frac{c_2^2 - 2c_4}{12} + \frac{-c_5 - c_2 \cdot (-c_3)}{24}.
\]

\[d\text{Note that one has actually an }T^2_0 \times T^2_0 \times T^2_0 \text{ fibration over }P^1 \times P^1; \text{ in the quotient description here the } CY^4 = dP_9 \times P_1 B \text{ (cf. appendix) with } E_8 \text{ superpotential of } E_8 \text{ occurs.}
\]

\[e\text{assumed to be non-splitting; the Hodge diamonds for the reducible cases } CY^4 \times T^2 \text{ and } CY^3 \times K3 \text{ are of course trivially computed.}
\]
so that one gets finally the relation
\[
- \frac{e}{24} = h^{41} - h^{31} + h^{21} - h^{11}. \tag{9}
\]

Taken together with the obvious relation
\[
\frac{e}{2} = h^{11} - h^{41} + 2(h^{31} - h^{21}) + h^{22} - h^{32} \tag{10}
\]
one can now express the cohomology completely in terms of the known numbers \(e, \delta := h^{41} - h^{11}\) and the remaining unknowns \(h^{21}\) and \(h^{22}\)
\[
h^{31} = h^{21} + \delta + \frac{e}{24}
\]
\[
h^{32} = h^{22} + \delta - \frac{5}{12}e \tag{11}
\]

Also one has now for the 5-folds of the special form \(CY^5 = X^4 \times P^1\) that (cf. appendix; \(CY^3\) denotes the fibre of \(X^4\))
\[
h^{21} = h^{21}(X^4),
\]
\[
h^{22} = 10h^{11}(CY^3) + 2h^{21}(X^4) + h^{22}(X^4) + 1. \tag{12}
\]

Now let us come back to our example \(CY^5 = dP_9 \times P^1 K \times P^1 dP_9\) of \(e = 0\). This decomposition allows one to find for the Kähler classes \(h^{11} = 10 - 1 + 3 + 10 - 1 = 21\). On the other hand one has (with \(#defdP_9 = 8\) for the complex deformations that \(h^{41} = 8 + 3 + #defK + 8 + 3\), so with \(#defK = 2 \cdot 10 \cdot 2 - (3 + 8 + 3) - 1 = 25\) one gets \(h^{41} = 47\). Furthermore the decomposition shows (cf. the appendix) that \(h^{21} = h^{21}(K) = #defK - (#defdP_9 - 1) = 2 \cdot 10 \cdot 2 - (3 + 8 + 3) - 1 - (8 - 1) = 18\) and \(h^{22} = 10 \cdot 19 + 2 \cdot 18 + h^{22}(X^4) + 1\) which gives with \(h^{22}(X^4) = 10 \cdot 10 + 2h^{21}(X^4) + h^{11}(K) + 1 = 140\) that \(h^{22} = 367\), so

\[
\begin{array}{ccccccc}
1 & 0 & 0 & 0 & 0 & 1 & 47 \\
0 & 21 & 0 & 0 & 44 & 393 & 393 \\
0 & 18 & 18 & 0 & 0 & 47 & 1
\end{array}
\]

Similarly one can use any of the existing lists of Calabi-Yau 4-folds (for example with the \(STU\)-Calabi-Yau \(P_{1,1,2,8,12}(24)\) as 3-fold fibre), go to the correspondingly reduced \(X^4\) (model \(X^4_A\) for the example just mentioned, cf. \([11]\)) and describe a \(CY^5\).
4 The brane point of view

Just as one can interpret an F-theory vacuum either as a Calabi-Yau compactification of a formally twelve-dimensional theory or as a type IIB vacuum with varying dilaton and 7-branes one can use the alternative brane point of view for S-theory as well. This was studied especially for the $T^3$ part in [4]; let us recall this point of view first and then interpret our example that way.

So what is given according to this point of view is really an 8D vacuum configuration with varying moduli consistent with the U-duality group $SL(3, \mathbb{Z}) \times SL(2, \mathbb{Z})$. The relevant moduli space eq.(1) leads as described to the idea of U-manifolds admitting a $T^3 \times T^2$ fibration. $T^2$ fibrations were studied in F-theory so let us focus on the five-dimensional piece of the moduli space parametrizing a three-torus of constant volume. This translates to a family of 5-branes that transform consistently with $SL(3, \mathbb{Z})$, living on $S^3$, where each individual member lives on a ‘line’ (set of real codimension two) in the base $S^3$. (In addition the 5-branes are also wrapped around the $S^2$ part of the $B^5$ base.) Phrased differently (and making it comparable to the cosmic string viewpoint in F-theory) the solution may be viewed as a mapping of the base into the five-dimensional moduli space, which is locally $SL(3, \mathbb{R})/SO(3)$ and actually an orbifold from the identification under the action of $SL(3, \mathbb{Z})$ U-duality. The pullback of the orbifold singularities leads then to the 5-brane configuration wrapping the singular lines (compare the F-theory picture relating the degenerate elliptic curves with the 7-brane locations; here the $T^3$'s are expected to degenerate along the singular lines, which correspond to the one-dimensional compact part of the world volume of the 5-branes; but note that in the F-theoretic case with the $K3$ there are only parallel branes involved).

Let us describe the real 5-dimensional base space $B^5$ for the $T^3 \times T^2$ fibration of $CY^5 = X^4 \times \mathbb{P}_1$ dP$_9$ more fully. It is as already remarked a $S^3$ fibration over $S^2$. This simply connected space has the one non-trivial Betti number $b_2 = 1 (= b_3)$. Because of $\pi_1(SO(4)) = \mathbb{Z}_2$ there exist actually only two possibilities for $B^5$, either the product $S^2 \times S^3$ or the twisted version. Furthermore, taking into account the possibility to represent the $S^3$ fibre itself as a $S^1$ fibration over $S^2_f$ (the Hopf fibration), the mentioned ambiguity for the $B^5$ space can, because of $\pi_1(SO(3)) = \mathbb{Z}_2$, already be read of from the 4-manifold consisting of the $S^2_f$ fibration over the base $S^2$. Note that, in the case of a complex structure for this 4-manifold, the ‘fibre-type’ ($\in \mathbb{Z}_2$) ambiguity of the Hirzebruch surface $F_n$ (being a $P^1 = S^2_f$ fibration over the base $P^1 = S^2$) is described by $n$ being even/odd (note the deformation $F_2 \rightarrow F_0$).

The $T^3$ fibration over $S^3$ of the $CY^3$ fibre will be combined with the $T^2_e$ fibration over $P^1_d$ of the $dP^d_9$. The singular lines in the $S^3$ are now replaced by singular loci of real codimension two in the base $B^5$ (which itself is a $S^3$ fibration over $S^2_f$). Note that we get a 5-brane picture in total as the $T^2$ fibration part gives also 5-branes: namely 7-branes (cf. F-theory) compactified on the $T^2$ which brought us from 10 to 8 dimensions; note that here the 7-branes have their locus not on a $P^1$, compactifying 10 dimensions to 8
dimensions, but on the $P^1_d$, compactifying from 5 dimensions to 3 dimensions. Of course the relevant singular loci consist now in the singular lines of the three-fold with their parameters running over the $S^2_b$ base of $B^5$ on the one hand, and furthermore in the $S^3$ fibers over the 12 singular points (for the $dP^d_{ge}$) on $S^2_d$. So one gets in both cases subspaces of real dimension three in $B^5$, i.e. the loci of the 5-branes are of real codimension 2 in the base.

Finally let us also consider the analogue of the (now not internal but spacetime-filling) 3-branes which have to be turned on for F-theory on a 4-fold. These are, as already mentioned in [3], (spacetime-filling) 4-branes in the case of S-theory on a 4-fold. Now our $CY^5$ is $CY^4$ fibered over $P^1$; so in this further compactification process the 4-branes wrap the $P^1$ and become (spacetime-filling) membranes in three dimensions. But as our $CY^4$ fibre had to be the reducible $CY^3 \times T^2$ of Euler number zero, the mentioned branes do not actually occur.

5 The Spectrum

Let us now read of from the Hodge diamond some part of the spectrum of massless multiplets. We will do this by the same strategy which is used to get a corresponding part of the F-theory spectrum from information about type II compactifications [3, 11]. This uses that F-theory on $X \times T^2$ is type IIA on X and furthermore that F-theory, being partly simply type IIB on the basis B of the relevant Calabi-Yau space in question, shows a sensitivity on the Hodge numbers of B (which is not seen any longer - after further compactification on $T^2$ - in the type IIA description). Now in our case here we will use the same procedure and relate S-theory after further compactification on $T^2$ to M-theory. Two special features appear in our setup: first in three dimensions both the possible multiplets, the scalars and the vectors, are actually, because of duality, in some sense indistinguishable; secondly, as we followed quite strictly the adiabatic strategy (splicing a $CY^3$ and a $T^2$ together over the new $P^1$), we are actually in a case which for F-theory on Calabi-Yau 3-folds would correspond to having only Hirzebruch surfaces as bases and no further birational transformations (blowings up and down) made on the base (i.e. no non-trivial tensor multiplets); this is reflected here in the property of $B_3$ being a $S^3$ fibration over $S^2$, so its cohomological data relevant here (Betti numbers) are fixed (cf. sect. 2).

Now, the searched for spectrum is that of S-theory on $CY^5$. This leads to $N = 1$ supersymmetry in three dimensions. Beside the $N = 1$ supergravity multiplet, which contains as its bosonic degree of freedom the metric $g_{\mu \nu}$ ($\mu, \nu = 0, 1, 2$), there will be $S_3$ real $N = 1$ scalar multiplets plus $V_3$ real $N = 1$ vector multiplets. The on-shell degrees of freedom of each $N = 1$ scalar multiplet are given by one real scalar field plus one real Majorana spinor; the $N = 1$ vector multiplets in three dimensions contain, on-shell, one vector field plus one real Majorana spinor. Since in three dimensions a vector is dual to a scalar, there is a (supersymmetric) Poincare duality between the scalar and the vector multiplets.
To obtain the spectrum of $S$-theory on $CY^5$ we start with the consideration of the type IIB superstring in ten dimensions. Its massless bosonic fields are

$$g_{MN}, B_{MN}, \phi, \phi', A_{MN}, A_{MNP}^+. \quad (13)$$

To obtain $S$-theory vacua we first have to compactify the type IIB superstring on a two-dimensional torus $T^2$ to eight dimensions. This leads to non-chiral eight-dimensional $N = 2$ supergravity (like the type IIA compactification on $T^2$). The only massless supermultiplet is the supergravity multiplet. From eq.(13) it is easy to see that the eight-dimensional $N = 2$ supergravity multiplet contains the following massless bosonic fields:

$$g_{MN}, 7\phi, 6A_M, 3A_{MN}, A_{MNP}. \quad (14)$$

(Now the indices $M, N, \ldots$ run over 0, \ldots, 7; note that a possible 4-form is dual to the 2-form coming from the 10D 4-form which (i.e. its field-strength) is self-dual.) The seven scalar fields parametrize the non-compact coset space eq.(1).

At the next step we compactify this eight-dimensional theory down to three dimensions on the $S$-theory base space $B^5$ to obtain the base-sensitive part of the three-dimensional spectrum. One performs the harmonic analysis on $B^5$ deriving from eq.(14) the following contributions $s_3$ and $v_3$ to the number of scalar and $U(1)$ vector fields (as $b_1 = 0, b_2 = b_3 = 1$):

$$s_3 = 7 + 6b_1 + 3b_2 + 1b_3 = 11, \quad v_3 = 6 + 3b_1 + b_2 = 7. \quad (15)$$

Now we consider $M$-theory on $CY^5$, which leads to $N = 2$ supergravity in one dimension. There are $V^M_1$ vector multiplets with one real physical scalar and one non-propagating vector (plus one non-propagating scalar). In addition we will have $S^M_1$ scalar multiplets with each one physical scalar field. The internal metric of $CY^5$ provides $h^{1,1} + 2h^{4,1}$ real scalars. The 11-dimensional field $A_{MNP}$ will contribute in addition $h^{1,1}$ $U(1)$ vectors plus $2h^{2,1}$ scalars. So in total we derive:

$$V^M_1 = h^{1,1}, \quad S^M_1 = 2h^{2,1} + 2h^{4,1}. \quad (16)$$

This $M$-theory spectrum can be directly compared with the $S$-theory spectrum on $CY^5 \times T^2$. The three-dimensional massless spectrum of $S$-theory, denoted by $S_3$ and $V_3$ is related to the one dimensional $S$-theory spectrum as follows:

$$V^S_1 = V_3 + 2, \quad S^S_1 = S_3. \quad (17)$$

So with eq.(16) the $S$-theory/M-theory duality in one dimension leads to the constraint $V^S_1 + S^S_1 = V^M_1 + S^M_1$, and hence we derive

$$V_3 + S_3 = h^{1,1} + 2h^{2,1} + 2h^{4,1} - 2. \quad (18)$$
Including the base-sensitive part one gets

\[
S_3 = 2h^{2,1} + 2h^{4,1} + s_3 - v_3 = 2h^{2,1} + 2h^{4,1} + 4,
\]
\[
V_3 = h^{1,1} + v_3 - s_3 - 2 = h^{1,1} - 6. \tag{19}
\]

Note that the \( V_3 \) only counts the massless abelian vector fields; at special loci in the moduli space additional non-abelian gauge bosons together with charged matter fields are expected to become massless.

We like to thank A. Miemiec, R. Minasian, C. Vafa and E. Witten for useful discussions.

A Appendix

5-folds as fibre products

For more details on many of the mentioned spaces cf. for example [11, 10].

The surface \( dP_9 = \begin{bmatrix} p^2 \\ p^1 \\ 3 \\ 1 \end{bmatrix} \) has the one nontrivial Hodge number \( h^{11} = 10 \) and \( 8 = 10 \cdot 2 - (8 + 3) - 1 \) complex deformations. Note that you can visualize the 10 classes on the one hand by viewing the surface \( P^2 \) blown up in the 9 intersection points of the two cubics (in this sense it is a generalization of the del Pezzo surfaces \( dP_i \) for \( i = 1, \ldots , 8 \)); on the other hand you can understand their appearance topologically in the elliptic fibration picture via the fact that an \( S^1 \) of the fibre moving between two vanishing points traces out an \( S^2 = P^1 \) (cf. [13]).

The Calabi-Yau three-fold \( CY^{19,19} = \begin{bmatrix} p^2 \\ p^1 \\ 3 \\ 0 \\ 1 \\ 1 \\ 0 \\ 3 \end{bmatrix} \) \( = dP_9 \times_{p^1} dP_9 \) has obviously \( h^{11} = 10 + 10 - 1 \) and so from \( e = 0 \) also \( h^{21} = 19 \), which you can also count directly as \( 8 + 3 + 8 \) (as one can use the reparametrization freedom on the \( P^1 \) only once). Note that also \( CY^{19,19} = (K3 \times T^2)/\mathbb{Z}_2 \) (here the first of the two \( dP_9 \)-fibre factors is appearing ‘by reduction’ from the former \( K3 = \begin{bmatrix} p^2 \\ p^1 \\ 3 \\ 2 \end{bmatrix} \)-factor, the second one is ‘emerging’ from the constant \( T^2 \)-factor in the process of smoothing out the quotient).

The Calabi-Yau four-fold \( CY^4 = \begin{bmatrix} p^2 \\ p^1 \\ 3 \\ 0 \\ 1 \\ 1 \\ 0 \\ 2 \\ 0 \\ 3 \end{bmatrix} \) \( = dP_9 \times_{p^1} \mathcal{B} \) with \( \mathcal{B} := \begin{bmatrix} p^1 \\ p^2 \\ 1 \\ 2 \\ 3 \end{bmatrix} \) has the
following Hodge diamond (model A, cf. [11])

\[
\begin{array}{cccccccc}
A & 1 &  &  &  &  &  &  \\
0 & 0 &  &  &  &  &  & \\
0 & 12 & 0 &  &  &  &  & \\
0 & 28 & 28 & 0 &  &  &  & \\
1 & 56 & 260 & 56 & 1 &  &  & \\
\end{array}
\quad
\begin{array}{cccccccc}
A' & 1 &  &  &  &  &  &  \\
0 & 0 &  &  &  &  &  & \\
0 & 12 & 0 &  &  &  &  & \\
0 & 28 & 28 & 0 &  &  &  & \\
1 & 56 & 260 & 56 & 1 &  &  & \\
\end{array}
\]

where also \( h_{21} = h_{21}(B) \) (cf. for this numerically [11]); it can be seen also from the topological ‘tracing out’ argument above that no new classes appear: just as the \( K3 \) fibration of \( B \) over \( P^1 \) shows how a \( S^2 \) moving over the base between two vanishing points traces out a \( S^3 \) for \( h_{21}(B) \), the corresponding \( T^2 \times K3 \) fibration over \( P^1 \) of \( CY^4 = dP_9 \times_{P^1} B \) shows that no new classes appear). Note also that \( CY^4 = (T^2 \times CY^3)/\mathbb{Z}_2 \). There is an ‘alternative version’ of this space which uses instead of the \( CY^3 = \left[ \begin{array}{ccc} p^1 & \ast & \ast \\ \ast & p^2 & \ast \\ \ast & \ast & \ast \end{array} \right] \), from which \( B \) is derived (by quadratic base change in the one \( P^1 \)), the well-known \( CY^{3,243} \); this leads to a \( CY^4 \) with Hodge-diamond shown above as model \( A' [11] \).

So one has in both cases

\[
h_{22} = 204 + 2h_{21}. \tag{20}
\]

This relation which of course is not accidental has two explanations: a numerical one and a geometrical one. The latter will be of relevance for our understanding of \( h_{22} \) of a \( CY^5 \).

Now first the numerical argument: one has

\[
h_{22} = e - 4 - 2(h_{11} - 2h_{21} + h_{31}) = \frac{2}{3} e + 12 + 2h_{21} \tag{21}
\]

(using \( h_{11} - h_{21} + h_{31} = \frac{e}{6} - 8 \) (cf. [11])) and \( e = 12 \cdot 24 = 288 \) shows the relation asserted above. Secondly the geometrical interpretation makes visible the classes (of the relevant Hodge type) from the following four topological sources of 4-cycles:

\[
\begin{align*}
S^2 \times S^2: & \quad 20 \cdot 10 \\
S^1 \times 3\text{-cycle}: & \quad 2h_{21} \\
4\text{-cycle}\times\text{point}: & \quad h_{22}(B) = h_{11}(B) = 3 \\
\text{point}\times dP_9: & \quad 1.
\end{align*}
\]

So if we come now to the Calabi-Yau five-fold \( CY^5 = X^4 \times_{P^1} dP_9 \) we have again \( h_{21} = h_{21}(X^4) \) and in the case of \( X^4 = dP_9 \times_{P^1} K \) further \( h_{21} = h_{21}(K) \). Also we have from the geometrical arguments showing how \( h_{22} \) arises that (let \( CY^3 \) be the fibre of \( X^4 \) over \( P^1 \))

\[
h_{22} = 10h_{11}(CY^3) + 2h_{21}(X^4) + h_{22}(X^4) + 1. \tag{22}
\]
References

[1] C. Vafa, Evidence for F-Theory, Nucl. Phys. B469 (1996) 493, [hep-th/9602022].

[2] D. R. Morrison and C. Vafa, Compactification of F-theory on Calabi-Yau Threefolds I,II, Nucl. Phys. B 473 (1996) 74, [hep-th/9602114]; Nucl. Phys. B 476 (1996) 437, [hep-th/9603161].

[3] A. Kumar and C. Vafa, U-manifolds, Phys. Lett. B 396 (1997) 85, [hep-th/9611007].

[4] J. Liu and R. Minasian, U-branes and T³ fibrations, [hep-th/9707125].

[5] A. Strominger, S.-T. Yau and E. Zaslow, Mirror Symmetry is T-Duality, Nucl. Phys. B 479 (1996) 243, [hep-th/9606040].

[6] M. Gross and P.M.H. Wilson, Mirror Symmetry via 3-tori for a class of Calabi-Yau Threefolds, [alg-geom/9608004].

[7] I. Affleck, J.A. Harvey and E. Witten, Instantons and (super)symmetry breaking in 2+1 dimensions, Nucl. Phys. B 206 (1982) 413.

[8] E. Witten, Strong Coupling and the Cosmological Constant, Mod. Phys. Lett. A 10 (1995) 2153, [hep-th/9506101].

[9] M. Bershadsky and V. Sadov, F Theory on K3 x K3 and Instantons on 7-Branes, Nucl. Phys. B 510 (1998) 232, [hep-th/9703194].

[10] R. Donagi, A. Grassi and E. Witten, A Nonperturbative Superpotential with E8 Symmetry, Mod. Phys. Lett. A11 (1996) 2199, [hep-th/9607091].

[11] G. Curio and D. Lüst, A Class of N = 1 Dual String Pairs and its Modular Superpotential, Int. J. Mod. Phys. A12 (1997) 5847, [hep-th/9703007]; B. Andreas, G. Curio and D. Lüst, N=1 Dual String Pairs and their Massless Spectra, Nucl. Phys. B 507 (1997) 175, [hep-th/9705174].

[12] D.D. Joyce, Compact 8-manifolds with holonomy Spin(7), Invent. Math. 123 (1996) 507.

[13] O. Ganor, A Test Of The Chiral E8 Current Algebra On A 6D Non-Critical String, Nucl. Phys. B479 (1996) 197, hepth/9607020.

[14] K. Mohri, F- Theory Vacua in Four Dimensions And Toric Threefolds, [hep-th/9701147].

[15] S. Sethi, C. Vafa and E. Witten, Constraints on Low-Dimensional String Compactifications, Nucl. Phys. B 480 (1996) 213, [hep-th/9606122].