Dynamical systems with a parallel tensor

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Abstract

We classify those rational maps \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) for which there exists a contravariant tensor \( q \) which is parallel, i.e. such that \( f^* q \sim q \), by proving that such maps preserve a parabolic orbifold.

1 Introduction

A holomorphic dynamical system on the Riemann sphere \( \mathbb{P}^1 \) is the data of a rational map \( f : \mathbb{P}^1 \to \mathbb{P}^1 \). From the viewpoint of Dynamics, the principal object of interest is the study of the space of orbits \( \mathbb{P}^1/f \) under the equivalence relation generated by \( f \), namely \( z \sim w \) if and only if there exist nonnegative integers \( n \) and \( m \) such that \( f^n(z) = f^m(w) \).

As one can imagine, this problem is not easily solved since the quotient space \( \mathbb{P}^1/f \) does not exist in general, at least not in a classic sense. The fact that the dynamical system is not given by the action of a group, unless the degree of the map \( f \) is 1, is one of the main obstacles in the matter.

However we can always consider the simplicial object, \( \mathbb{P}^\bullet f \), associated to the dynamical system which in degrees 0 and 1 is given by,

\[
\mathbb{P}^1 \equiv \coprod_{n \in \mathbb{N}} \Gamma_{f^n} := \coprod_{n \in \mathbb{N}} \{(x, f^n(x)) : x \in \mathbb{P}^1\}
\]

and we can identify sheaves on \( \mathbb{P}^1/f \) with a simplicial sheaf on \( \mathbb{P}^\bullet f \) in the sense of \cite{2} 5.I.6. Consequently to give a sheaf on \( \mathbb{P}^\bullet f \) is equivalent to giving a sheaf \( \mathcal{F} \) on \( \mathbb{P}^1 \) together with a map of sheaves,

\[
A_f : f^* \mathcal{F} \to \mathcal{F}
\]

One question that arises naturally in the study of dynamical systems is whether we are able to define “objects” that are invariant for the dynamics (i.e. global sections of a simplicial sheaf) and if possible, understand their nature.

Specifically given a simplicial sheaf \( \mathcal{F} \) as above on \( \mathbb{P}^\bullet f \), a global section is, by definition, an element \( q \in H^0(\mathbb{P}^1, \mathcal{F}) \) which is invariant for the action of \( f \), i.e.

\[
A_f(f^*q) = q,
\]
and we write $q \in H^0(\mathbb{P}^1/f, \mathcal{F})$ as a shorthand for $H^0(\mathbb{P}^1, \mathcal{F})$.

The main purpose of our work is to investigate the existence of $k$-th differentials on $\mathbb{P}^1$ which, after twisting by a locally constant simplicial line bundle, are invariant for the dynamical system generated by $f$, and, to classify them. We have been introduced to this problem while we were studying the work of Adam L. Epstein [4]. In his extension of *Infinitesimal Thurston rigidity* he shows that a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d > 1$, for which there exists a meromorphic quadratic differential $q$ with $f^*_q = dq$ is a “Lattès map” [4]. This led us to ask for which maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ does there exist a non-zero meromorphic global section $q$ of $\Omega^k_{\mathbb{P}^1}$ and a constant $\lambda \in \mathbb{C}^*$ such that:

$$f^*_q = \lambda q$$

wherein we employ the standard convention, [2], of identifying $A_f$ of a differential with its image.

To interpret (1) in the simplicial language of [2], observe that we have a simplicial local system $L_\lambda$ given by way of the action on the trivial sheaf,

$$A_f(f^*1) := \lambda$$

and we define $\Omega^k_{\mathbb{P}^1}(\lambda) := \Omega^k_{\mathbb{P}^1} \otimes \mathcal{O}_{\lambda^{-1}}$, so that a meromorphic differential satisfies (1) if and only if,

$$q \in H^0(\mathbb{P}^1/f, \Omega^k_{\mathbb{P}^1}(\lambda)).$$

In any case, whether in the simplicial language or the more elementary (1) our main theorem is:

**Theorem.** All the rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d > 1$ for which there exists a nonzero holomorphic section $q \in H^0(\mathbb{P}^1/f, \Omega^k_{\mathbb{P}^1}(\lambda))$ are (modulo at worst an element of $PGL_2(\mathbb{C})$ of order 2 or 3) equivalent to the action of an endomorphism of elliptic curves, and thus the action of $f$ comes from the action of a group of automorphisms of $\mathbb{C}$.

They are all listed in Table 1.

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## 2 A simple case

The space $Rat(d)$ of all rational maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ with $deg(f) = d$, is never a group unless $d = 1$, i.e. the group of automorphisms of the complex projective line $PGL_2(\mathbb{C})$.

The subgroup generated by $f \in PGL_2(\mathbb{C})$ is clearly isomorphic to $\mathbb{Z}$, and it acts on $\mathbb{P}^1$ through
Recall the Jordan decomposition of $2 \times 2$ matrices, to wit:

**Revision 2.1.** Any $f \in PGL_2(\mathbb{C})$ is conjugated by an element of $PGL_2(\mathbb{C})$ to one of the following:

1) $f(z) = z + \beta$, $\beta \in \mathbb{G}_a$ (if and only if $f$ has only one fixed point);

2) $f(z) = \alpha z$, $\alpha \in \mathbb{G}_m$ (if and only if $f$ has two distinct fixed points).

We want to characterize all meromorphic global sections $q$ of $\Omega$\(\otimes_k \mathbb{P}^1\) satisfying $f^*q = \lambda q$. Since $f$ is an automorphism, for any $x \in \mathbb{P}^1$ we have $ord_x(q) = ord_{f(x)}(q)$, hence for any $k \in \mathbb{Z}$ the sets $S_k = \{x \in \mathbb{P}^1 : ord_x(q) = k\}$ are completely invariant for the dynamics, i.e. $f^{-1}(S_k) = S_k$.

From [24] we deduce easily that in Case 1) the only finite set which is completely invariant for $f$ is the fixed point $\infty$. We conclude that $\infty$ is the unique pole of $q$, hence $q(z) = const \cdot dz^k$. In Case 2) $q$ may have both poles and zeroes. If they are contained in $\{0, \infty\}$ then $q(z) = const \cdot z^a (dz/z)^k$, and the coefficient $a$ is determined by $ord_0(q)$. Suppose that for some $k \in \mathbb{Z}$ we have

$$S_k \neq \emptyset, \text{ and } S_k \not\subset \{0, \infty\}$$

(2)

Since for any $x \in S_k$ we have $\alpha^n x = x$ for some $n > 1$, there exists some minimal $n$ such that $f^n = id$. It follows that $\alpha$ is a primitive $n$-th root of unity, and as $f^*q = \lambda q$ we see easily that $\lambda = \alpha^j$ for some $j < n$.

Note that the action of $f : \mathbb{G}_m \to \mathbb{G}_m$ is a free action, so the quotient map $p : \mathbb{G}_m \to \mathbb{G}_m/f$ is canonically a $\mu_n$-torsor.

Define

$$Q(z) = z^j \left(\frac{dz}{z}\right)^k.$$  (3)

and note that $Q \in H^0(\mathbb{G}_m, \mathbb{G}_m^{\otimes k})$.

Moreover $f^*Q = \lambda Q$, i.e. $Q \in H^0(\mathbb{G}_m/f, \mathbb{G}_m^{\otimes k} \otimes L_{\lambda}^{-1})$ where $L_{\lambda}$ denotes the sheaf on $\mathbb{G}_m/f$ given by the action on the trivial sheaf $A_f(f^*1) = \lambda$.

Thus “multiplication by $Q$” yields an $f$-invariant isomorphism of sheaves on $\mathbb{G}_m$

$$\mathcal{O} \isom \mathbb{G}_m^{\otimes k} \otimes L_{\lambda}^{-1}$$

We deduce the following.

**Fact 2.2.** Let $f$ satisfy condition (2), then a meromorphic global section $q \in H^0(\mathbb{G}_m/f, \mathbb{G}_m^{\otimes k} \otimes L_{\lambda}^{-1})$, i.e. a meromorphic $k$-th differential $q$ with $f^*q = \lambda q$, is necessarily of the form $q(z) = g(z^n)Q(z)$, where $g$ is a meromorphic function and $Q$ is given by (3).
3 Dynamical systems on the Riemann sphere with a parallel tensor

In the holomorphic category, a non-unit endomorphism of $\mathbb{P}^1$ is a rational map $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d > 1$. We denote by $\Omega_{\mathbb{P}^1}$ the sheaf of holomorphic differential forms on $\mathbb{P}^1$, given by the canonical action $dz \to f'(z)dz$, and by $\Omega_{\mathbb{P}^1}^k$ its $k$-th tensor power.

Let us suppose from now on that $f$ verifies the following Assumption:

Assumption 3.1. There exists $k \in \mathbb{N}^*$ and a global meromorphic section of $\Omega_{\mathbb{P}^1}^k$, which we will denote by $q$, such that $f^*q = \lambda q$, for some $\lambda \in \mathbb{C}^*$. Let 'z' be a local coordinate around a point $x \in \mathbb{P}^1$, we can write $q$ in the form $q(z)dz^k$, where $q(z)$ is a meromorphic function of $z$. For any $y \in f^{-1}(x)$, let 's' be the local coordinate around $y$ such that the map $f$ in this coordinates takes the form $s \mapsto s^n$, where $n := \deg_y(f)$. Within this notation we have $f^*q = q(s^n)(ns^{n-1}ds)^k$ and it follows easily that

\[ \text{ord}_y(f^*q) = \deg_y(f)(\text{ord}_x(q) + k) - k \]  

Now Assumption 3.1 clearly implies that $\text{ord}_y(f^*q) = \text{ord}_y(q)$, so we obtain

\[ \forall x \in \mathbb{P}^1, \quad \text{ord}_x(q) = \deg_x(f)(\text{ord}_f(x) + k) - k \]  

3.1 Considerations on zeroes and poles

In this section we are going to show that (5) constrains the number of zeroes and poles of $q$.

Lemma 3.2. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d > 1$ which satisfies Assumption 3.1. Then $q$ has no zeroes, that is the set $Z := \{x \in \mathbb{P}^1 : \text{ord}_x(q) > 0\}$ is the empty set.

Proof. Let us define the divisor $Z := \sum_{x \in Z} \text{ord}_x(q)x$ on $\mathbb{P}^1$, supported on the zeroes of $q$. Note that the statement of our Lemma is equivalent to

\[ \deg(Z) = 0 \]  

We claim that $Z$ is a backward invariant set for the dynamics, i.e. $f^{-1}(Z) \subset Z$. Given a zero $x \in Z$ of $q$ with $\text{ord}_x(q) = e > 0$, we see from (4) that for any $y \in f^{-1}(x)$, setting $n := \deg_y(f)$, we have $\text{ord}_y(q) = ne + k(n-1) > 0$, i.e. $y \in Z$. Observe now that, from the definition of $f^*Z$, we have

\[ \text{ord}_y(f^*q) = \text{ord}_y(f^*Z) + k(n-1) \geq \text{ord}_y(f^*Z) \]  

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Thus summing (7) over all $y \in f^{-1}(Z)$, we obtain
\[
\sum_{y \in f^{-1}(Z)} \text{ord}_y(f^*q) \geq \sum_{y \in f^{-1}(Z)} \text{ord}_y(f^*Z) = \text{deg}(f^*Z) \tag{8}
\]

The left hand side of (8) is obviously less than or equal to
\[
\sum_{y \in Z} \text{ord}_y(f^*q) = \sum_{y \in Z} \text{ord}_y(q) = \text{deg}(Z)
\]
since we are summing nonnegative numbers over a smaller set.
Finally we obtain $\text{deg}(Z) \geq \text{deg}(f^*Z) = d \cdot \text{deg}(Z)$, which implies (6) as we assumed $d > 1$.

**Lemma 3.3.** Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d > 1$, which satisfies **Assumption 3.1.**
If $x \in \mathbb{P}^1$ is a pole of $q$, then
\[
-k \leq \text{ord}_x(q) \leq -\frac{k}{2} \tag{9}
\]

**Proof.** We first prove the left inequality which is equivalent to
\[
\{ x \in \mathbb{P}^1 : \text{ord}_x(q) < -k \} = \emptyset \tag{10}
\]
In order to show this let us consider the following divisor on $\mathbb{P}^1$ supported on $P_k = \{ x \in \mathbb{P}^1 : \text{ord}_x(q) \leq -k \}$
\[
P_k := \sum_{x \in P_k} (-\text{ord}_x(q) - k) x
\]
We claim that $\text{deg}(P_k) = 0$ from which (10) follows immediately.
Note that equation (4) implies that $P_k$ is a backward invariant set. Moreover we have $-\text{ord}_y(q) - k = \text{deg}_y(f)(-\text{ord}_x(q) - k) = \text{ord}_y(f^*P_k)$, so that,
\[
\text{deg}(P_k) = \sum_{x \in P_k} (-\text{ord}_x(q) - k) \geq \sum_{y \in f^{-1}(P_k)} (-\text{ord}_y(q) - k) = \sum_{y \in f^{-1}(P_k)} \text{ord}_y(f^*P_k) = \text{deg}(f^*P_k) = d \cdot \text{deg}(P_k)
\]

Thus just as $\text{deg}(P_m) \geq 0$ and we assumed $d > 1$, it follows that $\text{deg}(P_k) = 0$.

Let us prove now the right hand side of (9), i.e.
\[
\{ x \in \mathbb{P}^1 : -\frac{k}{2} < \text{ord}_x(q) < 0 \} = \emptyset \tag{11}
\]

Let $x \in \mathbb{P}^1$ be a pole of $q$ of order $m := -\text{ord}_x(q) \neq k$ and let $y \in f^{-1}(x)$. Note that from (10) we have $m < k$ and also, from 3.1.2 that $\text{ord}_y(q) \leq 0$.
Consequently, from (10) we obtain
\[
\text{ord}_y(f) \leq \frac{k}{k-m} \tag{12}
\]

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Now, as \( \frac{k}{k-m} < 2 \), \( \forall \ 0 < m < \frac{k}{2} \), we deduce that \( \deg_y(f) = 1 \). Thus we see again from (5) that the set of poles of \( q \) of order \( m \), with \( 0 < m < \frac{k}{2} \), is backward invariant.

For any such integer \( m \) let us consider the divisor on \( \mathbb{P}^1 \) given by

\[
P_m := \sum_{x: \text{ord}_x(q) = -m} m x
\]

It follows from the considerations above that \( f^*P_m \subset P_m \), and therefore \( \deg(f^*P_m) \leq \deg(P_m) \). From \( \deg(f^*P_m) = d \deg(P_m) \) and \( d > 1 \), we deduce that \( \deg(P_m) = 0 \), which implies (11).

\[ \square \]

### 3.2 Main Lemma: the dynamical system preserves a parabolic orbifold

In this section we discuss the main consequence of Assumption 3.1, i.e. the existence of an orbifold or eventually an orbifold with boundary that is preserved by \( f \). We refer to [8] and to [3] for a formal definition of an orbifold. Observe that under our hypothesis, given any ramification point \( x \in \text{Ram}_f \), its image \( f(x) \) is a pole of \( q \), since Lemma 3.2 and (5) imply that \( \text{ord}_{f(x)}(q) < \text{ord}_x(q) \leq 0 \), i.e. the order is decreasing.

Thus a map \( f \) satisfying Assumption 3.1 must necessarily be a post-critically finite map, i.e. the post-critical set of \( f \), \( \mathcal{P}_f = \bigcup_{n>0} f^n(\text{Ram}_f) \) is finite.

**Definition 3.4.** Let \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) be a rational map of degree \( d > 1 \). We say that \( f \) preserves an orbifold, or also that \( f \) lifts to a map of orbifolds, if there exists a function \( \nu : \mathbb{P}^1 \to \mathbb{N}^* \cup \{\infty\} \) satisfying the following conditions:

1) \( \nu(x) = 1 \) if \( x \notin P_f \)
2) \( \forall x \in \mathbb{P}^1, \ \nu(x) \text{ divides } \nu(y) \deg_y(f) \) whenever \( y \in f^{-1}(x) \) \hspace{1cm} (13)

We shall denote by \( \nu_f \) the smallest among all functions \( \nu \) satisfying condition (13). Also, we denote by \( \mathcal{O} = (\mathbb{P}^1, \nu_f) \) the orbifold preserved by \( f \) and we shall refer to it, for brevity’s sake, through the string \( (\nu_f(x))_{x \in P_f} \).

By definition if \( \nu_f \) takes the value \( \infty \) we will say that \( \mathcal{O} \) is an orbifold with boundary, which we will denote by \( (\mathcal{O}, D) \) i.e. \( D \subset \mathcal{O} \) is the set of singular points of weight \( \infty \). We can associate to \( \mathcal{O} \) its Euler characteristic

\[
\chi(\mathcal{O}) = 2 - \sum_{x \in P_f} \left( 1 - \frac{1}{\nu_f(x)} \right)
\]

which is well defined and extended in the obvious sense if \( \nu_f \) takes value \( \infty \). We shall call an orbifold \( \mathcal{O} \) **hyperbolic** if \( \chi(\mathcal{O}) < 0 \), **parabolic** if \( \chi(\mathcal{O}) = 0 \) and **elliptic** otherwise. We enunciate here the main result of our work:
Lemma 3.5. Let \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) be a rational map of degree \( d > 1 \) which satisfies Assumption \( 3.4 \). Then \( f \) preserves one of the following parabolic orbifolds (resp. orbifold with boundary):

(i) \((\infty, \infty)\)
(ii) \((2, 2, \infty)\)
(iii) \((2, 2, 2)\)
(iv) \((3, 3, 3)\)
(v) \((2, 4, 4)\)
(vi) \((2, 3, 6)\)

Proof. It is a fact that, for \( q \) a meromorphic global section of the sheaf \( \Omega_{\mathbb{P}^1}^{\otimes k} \), we have

\[
\sum_{x \in \mathbb{P}^1} \operatorname{ord}_x(q) = \deg(\Omega_{\mathbb{P}^1}^{\otimes k}) = -2k
\] (14)

We shall call \( n_i \) the number of poles of \( q \) of order \( i \geq 0 \), i.e. \( n_i = \#\{x \in \mathbb{P}^1 : \operatorname{ord}_x(q) = -i\} \). Recall that in Lemma 3.2 and 3.3 we proved that \( n_i = 0 \forall i < \frac{k}{2} \), hence setting \( \alpha_i = \frac{i}{k} \), equation (14) becomes:

\[
\sum_{i=i_0}^{k} n_i \alpha_i = 2
\] (15)

where \( i_0 = \left\lceil \frac{k}{2} \right\rceil \) is the smallest integer greater than \( \frac{k}{2} \).

In the first pace we discuss the only two solutions of (15) with \( n_k \neq 0 \). One is \( n_k = 2 \), which implies \( n_i = 0 \forall i \neq k \), while the other, which may occur only if \( k \) is even, is \( n_k = 1 \) and \( n_i = 2 \). We proved in 3.3 that the set of poles of \( q \) of order \( k \) is a complete invariant set for the dynamics, hence in the first case if we define

\[
\nu_f(x) = \begin{cases} 
\infty, & \text{if } \operatorname{ord}_x(q) = k; \\
1, & \text{otherwise.}
\end{cases}
\]

it is clear that \( \nu_f \) satisfies condition (13), i.e. \( f \) preserves an orbifold with boundary of type (i).

If the second case occurs, let \( \infty \) be the pole of \( q \) of order \( k \), which is a fixed point of \( f \) and let \( P = \{p_1, p_2\} \) be the other poles of \( q \), necessarily of order \( \frac{k}{2} \).

Recall from (12) that given a pole \( x \) of \( q \) with \( \operatorname{ord}_x(q) = -n \) and given any \( y \in f^{-1}(x) \), we have

\[
\deg_y(f) \leq \frac{1}{1 - \alpha_n}
\] (16)
Thus for any $y \in f^{-1}(p_i)$, $i = 1, 2$ we have $\text{deg}_y(f) \leq 2$. From (5) we see that if $\text{deg}_y(f) = 1$ then $y \in P$, and if $\text{deg}_y(f) = 2$ then $\text{ord}_y(q) = 0$. We define

$$\nu_f(x) = \begin{cases} 2, & \text{if } x \in P; \\ \infty, & \text{if } x = \infty; \\ 1, & \text{otherwise.} \end{cases} \quad (17)$$

It follows from the discussion above that $\nu_f$ satisfies condition (13), so $f$ preserves an orbifold with boundary of type (ii).

Our aim is now to show that any solution of (15) with $n = 1$ or 2, meaning that respectively, $y \in P$, and we have $\nu_f(x) = 1$ for all $x$ in $\mathbb{C}$. From (16) we have $\text{ord}_y(q) = 0$. Hence $\nu_f$ satisfies condition (13), i.e. $f$ preserves the orbifold (iii).

Suppose now that in equation (15) we have $n = 3$ for some $i$, then necessarily $\frac{1}{2} < \alpha_i \leq \frac{3}{2}$. Note that the latter inequalities cannot be strict, since otherwise there would exist an index $j \neq i$ such that $n_j \neq 0$ and consequently $0 < n_j \alpha_j = 2 - 3\alpha_i < \frac{1}{2}$ which is impossible. Therefore $n_i = 3$ for some $i$ if and only if $\alpha_i = \frac{2}{3}$ (note that it makes sense only if $k \equiv 0 \pmod{3}$). In this case let $P = \{p_1, p_2, p_3, p_4\}$ be the set of poles of $g$, each pole having order $\frac{1}{2}$ and let us define

$$\nu_f(x) = \begin{cases} 2, & \text{if } x \in P; \\ 1, & \text{otherwise.} \end{cases} \quad (18)$$

As we have seen before, for any $y \in f^{-1}(P)$ the only possibilities for $\text{deg}_y(f)$ are 1 or 2, meaning that respectively, $y \in P$ or $\text{ord}_y(q) = 0$. Hence $\nu_f$ satisfies condition (13), i.e. $f$ preserves the orbifold (iii).

Suppose now that in equation (15) we have $n_i = 3$ for some $i$, then necessarily $\frac{1}{2} < \alpha_i \leq \frac{3}{2}$. Note that the latter inequalities cannot be strict, since otherwise there would exist an index $j \neq i$ such that $n_j \neq 0$ and consequently $0 < n_j \alpha_j = 2 - 3\alpha_i < \frac{1}{2}$ which is impossible. Therefore $n_i = 3$ for some $i$ if and only if $\alpha_i = \frac{2}{3}$ (note that it makes sense only if $k \equiv 0 \pmod{3}$). In this case let $P = \{p_1, p_2, p_3\}$ be the set of poles of $g$, each of order $\frac{1}{2}$ and let us define

$$\nu_f(x) = \begin{cases} 3, & \text{if } x \in P; \\ 1, & \text{otherwise.} \end{cases} \quad (19)$$

It follows from (16) that for any $y \in f^{-1}(P)$, $\text{deg}_y(f) = 3$, but $\text{deg}_y(f) = 2$ implies $\text{ord}_y(q) = -\frac{k}{3}$ which is impossible, hence the only possibilities for the local degree of $f$ at $y$ are 1 or 3, implying that respectively, $y \in P$ or $\text{ord}_y(q) = 0$. We conclude that $\nu_f$ satisfies condition (13), i.e. $f$ preserves an orbifold of type (iv).

Suppose now that $n_i = 2$ for some $i \neq k, \frac{k}{2}$, then we have necessarily $\frac{1}{2} < \alpha_i \leq \frac{3}{2}$. In fact, note that if $n_i = 2$ then there is only another nonzero $n_j$ solving (16), hence $n_j \alpha_j = 2(1 - \alpha_i)$ and consequently, if $\alpha_i > \frac{1}{2}$, we should have $n_j \alpha_j < \frac{1}{2}$ which is impossible. Note that $\alpha_i \neq \frac{2}{3}$, otherwise we would have $n_i = 3$.

We claim that there are no solutions of (15) with $n_i = 2$ and $\frac{1}{2} < \alpha_i < \frac{3}{2}$.

In fact, in this case (15) implies that $g$ has exactly three poles, two of which having order $\alpha_i k$ and one having order $\alpha_j k = 2(1 - \alpha_i)k$. However from (16) we deduce that $\text{deg}_y(f) \leq 3$ for any $y$ in the fiber of the poles of order $\alpha_i k$. Clearly $\text{deg}_y(f)$ cannot be equal to 2 or 3, otherwise we would have, respectively, $-\text{ord}_y(q)/k = 2\alpha_i - 1 < \frac{1}{2}$ or $-\text{ord}_y(q)/k = 3\alpha_i - 2 < \frac{1}{2}$, which is impossible. Therefore we obtain $\text{deg}_y(f) = 1$, from which we deduce that the image of
the ramification of \( f \) consists of the other pole of \( q \) (recall that \( f \) maps any ramification point to a pole of \( q \)). The following simple computation shows that the image of the ramification of a rational map of degree \( d > 1 \) cannot consist of one point. Let \( p \) be this point and suppose that \( f^{-1}(p) = e_1x_1 + \cdots + e_rx_r + y_1 + \cdots + y_s \), with \( e_1 + \cdots + e_r + s = d \). By the Riemann-Hurwitz formula, the ramification of a rational map of degree \( d \) has order \( 2d - 2 \), hence we obtain
\[ 2d - 2 = (e_1 - 1) + \cdots + (e_r - 1) = d - (r + s), \]
which is absurd since \( d > 1 \). Therefore we conclude that \( n_i = 2 \) for some \( i \neq k, \frac{2}{3} \) if and only if \( \alpha = \frac{3}{2} \) (note that it makes sense only if \( k \equiv 0 \pmod{4} \)).

In this case let \( P = \{ p_1, p_2 \} \) be the set poles of \( q \) of order \( \frac{2}{3}k \) and let \( p \) be the other pole of \( q \) necessarily of order \( \frac{1}{2} \). It is natural to define
\[
\nu_f(x) = \begin{cases} 
2, & \text{if } x = p; \\
4, & \text{if } x \in P; \\
1, & \text{otherwise.} 
\end{cases}
\]

We already know that \( \nu_f \) satisfies condition (13) for \( x = p \), so we are left to show it holds also for \( x \in P \). In view of (3), given any \( y \in f^{-1}(P) \), the possible values for \( \deg_y(f) \) are 1, 2 or 4, meaning that \( \nu_f(y) \) is, respectively, equal to 4, 2, 1. Thus we have \( \nu_f(y)\deg_y(f) = 4 \) in each case, so we conclude that \( f \) preserves an orbifold of type \((v)\).

Finally we suppose that \( n_i \leq 1 \), \( \forall i \). It is clear that in this case we must have \( \#\{ i : n_i \neq 0 \} = 3 \), so we can rewrite equation (15) in the form \( \alpha + \beta + \gamma = 2 \), with \( \alpha, \beta, \gamma \in \mathbb{Q} \) satisfying \( \frac{1}{2} \leq \alpha < \beta < \gamma < 1 \). We claim that this equation has only one solution, which is \( \alpha = \frac{2}{3}, \beta = \frac{3}{4}, \gamma = \frac{5}{6} \) (note that it makes sense only if \( k \equiv 0 \pmod{6} \)). Suppose that \( \alpha \neq \frac{2}{3} \) and observe that \( \alpha < \frac{2}{3} \). Thus, from (10), we deduce that for any \( y \) in the fiber of the pole of order \( \alpha k \) we have \( \deg_y(f) < 3 \). Nevertheless, \( \deg_y(f) \) cannot be 2, since otherwise we should have \( -\ord_y(q)/k < \frac{1}{2} \), which is impossible. We conclude that the fiber of this pole must consist of exactly \( d \) non-ramified different points, say \( \{ y_1, \ldots, y_d \} \), and for each of these points we should have \( \ord_{y_i}(q) = -\alpha k \), but this leads to an absurd, since there is only one pole of such order. We have reduced our equation to \( \beta + \gamma = \frac{3}{2} \), with \( \frac{1}{2} < \beta < \gamma < 1 \), but now the same argument used for \( \alpha \) shows that this is possible if and only if \( \beta = \frac{2}{3} \), since otherwise the pole of order \( \beta k \) would not be a branched point, which leads us to an absurd. Calling \( p_1, p_2, p_3 \) the poles of \( q \) of order, respectively, \( \alpha k, \beta k, \gamma k \), we define
\[
\nu_f(x) = \begin{cases} 
2, & \text{if } x = p_1; \\
3, & \text{if } x = p_2; \\
6, & \text{if } x = p_3; \\
1, & \text{otherwise.} 
\end{cases}
\]

We already know that \( \nu_f \) satisfies condition (13) for \( x = p_1, p_2 \), so we need only to show that it holds also for \( p_3 \). For any \( y \in f^{-1}(p_3) \) the possible values for \( D = \deg_y(f) \) are \( 1, 2, 3, 6 \) since from equation (16) we have \( -\ord_y(q)/k = \cdots \)
1 - \frac{2}{5}, and this can only be equal to 0, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}. It follows that \( \nu_f(y) \text{deg}_y(f) = 6 \)
in each case, so \( \nu_f \) satisfies condition (iii), i.e. \( f \) preserves an orbifold of type (vi).

**Remark 3.6.** We have shown that every solution of (15) with \( \alpha_i \neq 1 \) is such that \( \alpha_i = \left(1 - \frac{1}{n}\right) \) for some \( n = 2, 3, 4 \) or 6, i.e. \( q \) may only have poles of order \( k \) or \( (1 - \frac{1}{n}) k \).

In the latter case, observe that we can write equation (15) as

\[
\sum_{i \in I} \left(1 - \frac{1}{e_i}\right) = 2
\]

where \( e_i \) are not necessarily distinct integers.

Moreover observe that we have defined in each case

\[
\nu_f(x) = n \text{ whenever } \text{ord}_x(q) = - \left(1 - \frac{1}{n}\right)k
\]

for any \( n = 1, 2, 3, 4, 6, \) in such a way that

\[
\sum_{x \in \mathbb{P}^1} \left(1 - \frac{1}{\nu_f(x)}\right) = 2
\]

We conclude that the orbifolds (iii)-(vi) of Lemma 3.5 are also all the possible parabolic orbifolds on \( \mathbb{P}^1 \).

## 4 Maps preserving a parabolic orbifold

In this chapter we will discuss the main consequences of Lemma 3.5. We have a rational map \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) that has an invariant orbifold \( O \), i.e. there exists a map \( \tilde{f} : O \to O \) such that the following diagram commutes

\[
\begin{array}{ccc}
O & \xrightarrow{\tilde{f}} & O \\
\downarrow p & & \downarrow p \\
\mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1
\end{array}
\]

(here \( p \) denotes the natural projection map), with \( \chi(O) = 0 \).

We will show that \( O \) is the quotient of an elliptic curve \( E \) by the action of a (finite) group \( G \) of automorphisms of \( E \) and that \( f \) lifts to a morphism of
G-torsors $F : E \to E$, such that the following diagram commutes:

$$
\begin{array}{ccc}
E & \xrightarrow{F} & E \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathcal{O} & \xrightarrow{\tilde{f}} & \mathcal{O} \\
\downarrow{p} & & \downarrow{p} \\
p^1 & \xrightarrow{f} & p^1
\end{array}
$$

(24)

4.1 Construction of a torsor associated to a torsion line bundle

We begin recalling some general facts concerning line bundles over an orbifold $\mathcal{O}$. We say that $\pi : L \to \mathcal{O}$ is a torsion line bundle of order $n$ if $L^{\otimes n}$ is trivial, i.e. if there exists an isomorphism of line bundles on $\mathcal{O}$

$$
L^{\otimes n} \xrightarrow{\sim} \mathcal{O} \times \mathbb{C}
$$

(25)

where $p_1$ denotes the projection on the first factor.

It is well known that a torsion line bundle over a compact manifold $X$ defines a $\mu_n$-torsor over $X$, which is unique up to an almost unique isomorphism, i.e. rather than being unique, the isomorphisms between any two such torsors form a principal homogeneous space under $\mu_n$. We are going to prove that this property still holds if $X$ is an orbifold whose underlying space is $\mathbb{P}^1$.

Let us consider an orbifold $\mathcal{O}$ modelled on $\mathbb{P}^1$ whose set of singular points is $\{x_1, \ldots, x_r\} \subset \mathbb{P}^1$, each $x_i$ having finite weight $n_i$. This means that the monodromy group of each $x_i$ is the group of $n_i$-th roots of unity $\mu_{n_i}$ and arbitrarily small (non-space like) neighborhood $U_i$ of $x_i$ in $\mathcal{O}$ are described as follows: we can choose a disk $\Delta_i$ centered at the origin such that $U_i$ is the classifying champ, $[\Delta/\mu_{n_i}]$, [5 2.4.2], for the action $\Delta_i \cong \Delta_i \times \mu_{n_i}$ of $\mu_{n_i}$ by rotations on $\Delta_i$.

Let us recall that a line bundle $L$ on $\mathcal{O}$ can be described on $U_i = [\Delta_i/\mu_{n_i}] \ni x_i$ as follows: we have that $L|_{\Delta_i}$ is the trivial bundle, with action determined by a representation $\rho_i : \mu_{n_i} \to \mathbb{C}^\times$. Note that $L^{\otimes n_i}|_{U_i}$ is the trivial bundle on $U_i$ since such a representation has order dividing $n_i$.

If we set $Z = \bigsqcup_i p^{-1}(x_i)$ and $U = \mathcal{O} \setminus Z$, then $L$ is completely determined by the triple

$$(L|_Z, L|_U, \phi)$$

(26)

where we have $L|_Z = \bigsqcup_i (\rho_i : \mu_{n_i} \to \mathbb{C}^\times)$ and $\phi = \bigsqcup_i \phi_i$, each $\phi_i : L|_{U_i \setminus \{x_i\}} \to L|_U$ being the gluing map of the bundle $L$. 
We have that \( L \) is torsion of order \( n \) if and only if \( L^\otimes n \) defines a line bundle on \( \mathbb{P}^1 \) of degree 0, since the following sequence
\[
0 \longrightarrow Pic(\mathbb{P}^1) \longrightarrow Pic(\mathcal{O}) \longrightarrow \prod_i (\rho_i : \mu_{n_i} \to \mathbb{C}^*) \tag{27}
\]
is exact. Indeed if each representation \( \rho_i \) is trivial, then \( L|_{U_i} \) is trivial so the maps \( \phi \) are just gluing with the trivial line bundle on the moduli of \( U_i \), i.e. the naive quotient of the \( \mu_{n_i} \) action, identified with open subset of \( \mathbb{P}^1 \), and so the kernel is a line bundle on \( \mathbb{P}^1 \). We know, however, that \( deg : Pic(\mathbb{P}^1) \cong \mathbb{Z} \) is an isomorphism, so if \( deg(L) = 0 \) and the order of each representation \( \rho_i \) divides \( n \), then from (26) and (27) it follows that \( L \) is torsion of order dividing \( n \).

**Lemma 4.1.** Let \( \pi : L \to \mathcal{O} \) be a \( n \)-torsion line bundle over an orbifold \( \mathcal{O} \) whose underlying space is compact, then associated to \( L \) there is a unique, up to isomorphism, \( \mu_n \) torsor \( E \subset L \xrightarrow{\sim} \mathcal{O} \). Better still the singular points of \( E \) lie over those of \( \mathcal{O} \), and if the underlying space of \( \mathcal{O} \) is \( \mathbb{P}^1 \) with \( y \) a singular point of \( E \) lying, in the above notation (26), over \( x_i \) then the local monodromy of \( E \) at \( y \) is the kernel of \( \rho_i \).

**Proof.** From the exact sequence of sheaves on \( \mathcal{O} \)
\[
0 \to \mu_n \to \mathbb{C}_m \xrightarrow{\cong} \mathbb{C}_m \to 0 \tag{28}
\]
the long exact sequence in co-homology, since the underlying space is compact, reads
\[
0 \to \mu_n \to \mathbb{C}^* \xrightarrow{\cong} \mathbb{C}^* \to H^1(\mathcal{O}, \mu_n) \to Pic(\mathcal{O}) \xrightarrow{\cong} Pic(\mathcal{O}) \tag{29}
\]
so, isomorphism classes of \( \mu_n \) torsors are exactly \( n \)-torsion line bundles. In order to compute the singular points of \( E \), we recall how to construct the torsor starting from the bundle.

In the first pace, given a vector bundle \( p : E \to \mathcal{O} \) we can form the tensor power of \( E \), \( p' : E^\otimes n \to \mathcal{O} \) and we have a canonical map of bundles
\[
\begin{array}{ccc}
E & \xrightarrow{F_n} & E^\otimes n \\
\downarrow p & & \downarrow p' \\
\mathcal{O} & \quad & \mathcal{O}
\end{array} \tag{30}
\]
where the map \( F_n \) sends an element \( e \in E \) to its tensor power \( e^\otimes n \in E^\otimes n \). Therefore if \( L \) is a torsion line bundle of order \( n \) we have the following commutative diagram:
\[
\begin{array}{ccc}
L & \xrightarrow{F_n} & \mathcal{O} \times \mathbb{C} \\
\downarrow \pi & & \downarrow p_1 \\
\mathcal{O} & \quad & \mathcal{O}
\end{array} \tag{31}
\]
and for any \( \lambda \in \mathbb{C}^* \) our \( \mu_n \) torsor is isomorphic to \( E = F_n^{-1}(\mathcal{O} \times \lambda) \).
To compute the monodromy at \( y \to x_i \) should \( \mathcal{O} \) have underlying space \( \mathbb{P}^1 \), observe that around \( x_i \), (31) corresponds to the map of groupoids,

\[
(x, n(v)) := (x, v^n) \quad \Delta \times \mathbb{C} \overset{\text{trivial action}}{\longrightarrow} \Delta \times \Gamma_i \times \mathbb{C}
\]

\[
(x, v) \quad \Delta \times \mathbb{C} \overset{\text{diagonal action}}{\longrightarrow} \Delta \times \Gamma_i \times \mathbb{C}
\]

in which the diagonal action is \((x, v) \to (x^\gamma, \rho_i(\gamma)v)\).

Hence we see that \( F_{-1}(\Delta \times \lambda) \) is isomorphic to the \( \mu_n \) torsor \( \Delta \times n^{-1}(\lambda) \) with the stabilizer \( \Gamma_i \) of \( x_i \) acting diagonally, i.e. \((x, l) \to (x^\gamma, \rho_i(\gamma)l)\), where \( x \in \Delta \) and \( l \in n^{-1}(\lambda) \).

We have that \( x \in \text{Stab}(x \times l) \iff x = x^\gamma = x \) and \( \rho_p(\gamma)l = l \), which gives \( x \in \Gamma_i \) and \( \gamma \in \ker(\rho_p) \).

4.2 Holomorphic differentials on a parabolic orbifold

In this section we show how the construction of Lemma 4.1 applies to the line bundle \( \Omega_{\mathcal{O}} \) of holomorphic differential forms over a parabolic orbifold.

Around a non-space like point \( x_i \), in the above notation (26), we have that \( \Omega_{\mathcal{O}} \mid U_i \) is the \( \mu_n \) module \( O \mid \Delta_i dz \), where \( O \mid \Delta_i \) denotes the sheaf of holomorphic functions on the disk, with action given by

\[
f(z)dz \to f(\theta z)\theta dz, \quad \theta \in \mu_n_i
\]

Around a boundary point of \( \mathcal{O} \), i.e. a singular point with weight \( \infty \), there is no orbifold structure, but still morally, if not mathematically, the monodromy group of such a point is isomorphic to \( \mathbb{Z} \).

We will denote by \( (\mathcal{O}, D) \) an orbifold \( \mathcal{O} \) with boundary \( D \), i.e. \( D \subset \mathcal{O} \) is the set of singular points of weight \( \infty \), and by \( \Omega_{\mathcal{O}}(\log D) \) the sheaf of holomorphic differential forms on \( \mathcal{O} \) with logarithmic poles on the boundary \( D \) (see [1]). Moreover we will use the notation \( f : (\mathcal{O}_1, D_1) \to (\mathcal{O}_2, D_2) \) for a map between orbifolds with boundary, meaning as usually that \( f^{-1}D_2 \subseteq D_1 \).

Lemma 4.2. Let \( \mathcal{O} = (\mathbb{P}^1, \nu_f) \) (resp. \( \mathcal{O}, D = (\mathbb{P}^1, \nu_f) \)) be a parabolic orbifold (resp. orbifold with boundary) invariant for \( f \) as in Lemma 4.1, and let \( q \) the meromorphic section of \( \Omega_{\mathcal{O}}^{\otimes k} \) such that \( f^*q / q \). We have:

1. \( \Omega_{\mathcal{O}} \) (resp. \( \Omega_{\mathcal{O}}(\log D) \)) is a torsion line bundle of order \( n := \text{lcm}\{\nu_f(x) : x \notin D\} \).

2. If we denote by \( p : \mathcal{O} \to \mathbb{P}^1 \) the natural projection, then \( \tilde{q} := p^*q \in H^0(\Omega_{\mathcal{O}}^{\otimes k}) \) (resp. \( H^0(\Omega_{\mathcal{O}}(\log D)^{\otimes k}) \))

Proof. 1) From the definition of \( n \) all the local representations (27) have order that divides \( n \), so we need only to show that \( \text{deg}(\Omega_{\mathcal{O}}) = 0 \) (resp. \( \text{deg}(\Omega_{\mathcal{O}}(\log D)) = 0 \), but this follows from the fact that \( \text{deg}(\Omega_{\mathcal{O}}) = -\chi(\mathcal{O}) \) (resp. \( \text{deg}(\Omega_{\mathcal{O}}(\log D)) = \chi(\mathcal{O}) \))
−χ(\mathcal{O}, D)).

2) Let us consider a weighted point \( x \in \mathcal{O} \) of order \( n \). The projection map \( p : \mathcal{O} \to \mathbb{P}^1 \) in the orbifold coordinate 's' around \( x \) takes the form \( s \to s^n \).

We have seen in 3.6 that \( p(x) \) is a pole of \( q \) of order \((1 - \frac{1}{n}) k\), so we have that

\[
\tilde{q}(z) = p^* \left( \text{const} \cdot z^k \left( \frac{dz}{z} \right)^k \right) = \text{const} \cdot ds^k
\]

Thus \( \tilde{q} = p^* q \) is a holomorphic section of \( \Omega^{\otimes k} \) (resp. \( \Omega_{\mathcal{O}}(\log D)^{\otimes k} \)) satisfying \( \tilde{f}^*(\tilde{q}) / \tilde{q} \).

\[\text{Lemma 4.3.}\]

Let \( \mathcal{O} = (\mathbb{P}^1, \nu_f) \) (resp. \( (\mathcal{O}, D) = (\mathbb{P}^1, \nu_f) \)) be an orbifold (resp. orbifold with boundary) invariant for \( f \) as in Lemma 3.3. There exists an elliptic curve \( E \) (resp. the multiplicative group \( \mathbb{G}_m \)) which is a \( \mu_n \)-torsor over \( \mathcal{O} \) (resp. \( \mathcal{O} \setminus D \)) where \( n \) is the order of torsion of \( \Omega_{\mathcal{O}} \) (resp. \( \Omega_{\mathcal{O}}(\log D) \)). Moreover \( E \) (resp. the multiplicative group \( \mathbb{G}_m \) viewed as the manifold \( (\mathbb{P}^1, 0 + \infty) \) with boundary) is invariant for \( f \), i.e. there exists a morphism of torsors \( F : E \to E \) (resp. \( F : (\mathbb{P}^1, 0 + \infty) \to (\mathbb{P}^1, 0 + \infty) \)) such that the following diagram commutes:

\[
\begin{array}{ccc}
(X, \partial) & \xrightarrow{F} & (X, \partial) \\
\pi \downarrow & & \pi \downarrow \\
(O, D) & \xrightarrow{f} & (O, D)
\end{array}
\]

where \( (X, \partial) = (E, \emptyset) \) (resp. \( (X, \partial) = (\mathbb{P}^1, 0 + \infty) \)).

\[\text{Proof.}\]

We have seen in 4.2 that if \( n = \text{lcm}\{n_1, \ldots, n_r\} \), then \( \Omega_{\mathcal{O}} \) (resp. \( \Omega_{\mathcal{O}}(\log D) \)) is a torsion line bundle of order \( n \) and we know from Lemma 4.1 that it defines a unique \( \mu_n \)-torsor. On the other hand, the representation defining \( \Omega_{\mathcal{O}} \) (resp. \( \Omega_{\mathcal{O}}(\log D) \)) in a neighborhood of each \( x_i \) is given by the action of \( \mu_{n_i} \) on differential forms which is a faithful representation so by the second part of 4.1 it follows that \( E := E \) is a Riemann surface (resp. Riemann surface with boundary) with Euler characteristic 0 i.e. \( E \) is an elliptic curve if \( \mathcal{O} \) is compact, resp. \( \mathbb{G}_m \) identified with \( (\mathbb{P}^1, 0 + \infty) \) should the boundary be non-empty.

It remains to show that this implies the existence of a map of torsors \( F : E \to E \), resp. with boundary, which makes the diagram (33) commute. This follows from the fact that the isomorphism \( f^* \Omega_{\mathcal{O}} \cong \Omega_{\mathcal{O}} \) (resp. \( f^* \Omega_{\mathcal{O}}(\log D) \cong \Omega_{\mathcal{O}}(\log D) \)) affords an isomorphism of \( \mu_n \)-torsors \( f^* E \cong E \), resp. with boundary.

To see this, from (29) we have a commutative diagram with exact rows:

\[
\begin{array}{cccc}
0 & \to & H^1(\mathcal{O}, \mu_n) & \to & H^1(\mathcal{O}, \mathbb{G}_m) \\
\downarrow f^* & & \downarrow f^* & & \downarrow f^* \\
0 & \to & H^1(\mathcal{O}, \mu_n) & \to & H^1(\mathcal{O}, \mathbb{G}_m)
\end{array}
\]
Thus just as $[\Omega \Omega] \ (\text{resp. } [\Omega \Omega (\log D)]) \in H^1(\mathcal{O}, \mathbb{G}_m)$ is fixed by $f^*$, so is $[E] \in H^1(\mathcal{O}, \mu_n)$, hence we have an isomorphism of $\mu_n$-torsors $E \sim f^* E$. Consequently we obtain the following commutative diagram,

$$
\begin{array}{ccc}
E & \sim & f^* E \\
\downarrow & & \downarrow \\
\mathcal{O} & \sim & \mathcal{O}
\end{array}
$$

and hence by composition we obtain a $\mu_n$-equivariant map $F : E \to E$.

\[\square\]

**Corollary 4.4.** Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d > 1$ which satisfies **Assumption 3.1**. Then $f$ preserves a parabolic orbifold (resp. a parabolic orbifold with boundary) which can be realized as a quotient of $\mathbb{C}$ (resp. of $\mathbb{C} \cup \{\infty\}$) by the action of a discrete subgroup of $\text{Aut}(\mathbb{C})$.

The following table illustrates all the possibilities:

| Orbifold $\mathcal{O}$ preserved by $f$ | $\Gamma \subset \text{Aut}(\mathbb{C})$ defining $\mathcal{O}$ |
|----------------------------------------|------------------------------------------|
| $(\infty, \infty)$ | $\mathbb{Z}$, acting by translations |
| $(2, 2, \infty)$ | $< \mathbb{Z}, z \mapsto -z >$ |
| $(2, 2, 2)$ | $< \mathbb{Z} \oplus \mathbb{Z} \tau, z \mapsto -z >; \ \tau \text{ s.t. } \Im(\tau) > 0$ |
| $(3, 3, 3)$ | $< \mathbb{Z} [\zeta], z \mapsto \zeta z >; \ \zeta \text{ s.t. } \zeta^2 + \zeta + 1 = 0$ |
| $(2, 4, 4)$ | $< \mathbb{Z} [i], z \mapsto iz >, i \text{ s.t. } i^2 + 1 = 0$ |
| $(2, 3, 6)$ | $< \mathbb{Z} [\zeta], z \mapsto -\zeta z >, \ \zeta \text{ as above}$ |

\[\text{Proof.}\] We have seen that there exists a covering map $\pi : E \to \mathcal{O}$ which can be viewed as the quotient map $E \to [E/\mu_n]$, for $n = 2, 3, 4, 6$, (resp. we have an isomorphism $\pi : \mathbb{C}^* \to \mathcal{O} \setminus \text{D}$ in the case $(\infty, \infty)$, or a double cover $\pi : \mathbb{C}^* \to \mathcal{O} \setminus \text{D}$ in the case $(2, 2, \infty)$). Consequently we have $\mu_n \hookrightarrow \text{Aut}(E)$. Every such automorphism can be lifted to a linear map ‘$z \mapsto \alpha z$’ on $\mathbb{C}$, the universal covering space of $E$, which must satisfy $\alpha \Lambda = \Lambda$, where $\Lambda$ is the lattice defining $E$. When $\alpha \in \mu_n$ a simple computation shows that $\Lambda = \mathbb{Z} [\mu_n]$ for $n = 3, 4, 6$. In the case $n = 2$ the condition above is empty, hence $\Lambda$ is generic.

\[\text{Remark 4.5.}\] The orbifolds listed in Corollary 4.4 are the only one which can be realized as quotients of an elliptic curve $E$ for the action of a group of automorphisms of $E$. In fact it is well known that the group of automorphisms of an elliptic curve $E$ is a finite cyclic group $G$ of order 2, 4 or 6 (see [7]).

Consider the following exact sequence of algebraic groups:

$$
0 \longrightarrow \text{Aut}^0(E) \longrightarrow \text{Aut}(E) \longrightarrow G \longrightarrow 0
$$
where $\text{Aut}^0(E)$ is the connected component of the identity in $\text{Aut}(E)$. We have that $E \xrightarrow{} \text{Aut}^0(E)$, where we identify $E$ with the subgroup of translations of $\text{Aut}(E)$.

Consider the action of $G$ on the elliptic curve, with quotient map $f : E \to E/G$. It is a fact that the naive quotient $E/G$ is isomorphic to $\mathbb{P}^1$, as the map $f$ is necessarily ramified and the Riemann-Hurwitz formula gives $\chi(E/G) > 0$.

It follows that $\#\text{Ram}_f = 2\#G$, so if the fiber of each $p \in E/G$ consists of $n_p$ distinct elements, each of order $e_p = \#\text{Stab}_G(p)$, we can write Riemann-Hurwitz as follows (note that we have $n_p e_p = \#G$),

$$\sum_{p \in \mathbb{P}^1} \left(1 - \frac{1}{e_p}\right) = 2 \quad (34)$$

As discussed in 3.6, the only integer solutions of (34) are given by $(iii) - (vi)$ of Lemma 3.5. Thus we conclude that $E/G$ can be endowed with the structure of a parabolic orbifold i.e. an orbifold of type $(iii) - (vi)$.

### 4.3 Conclusions

We conclude this chapter formulating at first a structural theorem for the maps $f$ with a parallel tensor $q$ and finally discussing the invariant nature of $q$.

**Theorem 4.6.** Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d > 1$ which satisfies Assumption 3.4. Then, up to conjugation with an element of $\text{PGL}_2(\mathbb{C})$, $f$ is induced by one of the automorphisms of $\mathbb{C}$ contained in Table 1, i.e. $f$ is obtained as quotient of such automorphisms under the action of a discrete group of automorphisms of the complex plane.

| Orbifold $\mathcal{O}$ preserved by $f$ | Automorphism of $\mathbb{C}$ inducing $f$ |
|----------------------------------------|------------------------------------------|
| $(\infty, \infty)$                     | $z \mapsto nz; n \in \mathbb{Z}$ s.t. $|n| > 1$ |
| $(2, 2, \infty)$                       | $z \mapsto nz + \beta; n$ as above, $\beta = 0, \frac{1}{2}$ |
| $(2, 2, 2)$                            | $z \mapsto \alpha z + \beta; \alpha$ an integer in an imaginary quadratic field, $\beta \in \mathbb{E}[2]$ |
| $(3, 3, 3)$                            | $z \mapsto \alpha z + \beta; \alpha \in \mathbb{Z}[\zeta], \beta = 0, \frac{1}{2}(\zeta + 1), \frac{1}{2}i\sqrt{3}$ |
| $(2, 4, 4)$                            | $z \mapsto \alpha z + \beta; \alpha \in \mathbb{Z}[i], \beta = 0, \frac{1}{2}(i + 1)$ |
| $(2, 3, 6)$                            | $z \mapsto \alpha z; \alpha \in \mathbb{Z}[\zeta]$ |

Table 1: Dynamical systems admitting a parallel tensor

**Proof.** This classification is originally due to the work of A. Douady and J. H. Hubbard [3].

From 4.4 we know that $f$ preserves one of the orbifolds listed above, which is isomorphic to the quotient space $[E/\mu_n]$, for $n = 2, 3, 4, 6$. Moreover, as we have seen in 4.3, $f$ lifts to a map $F : E \to E$ which commutes with the action of $\mu_n$. 

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It is well known, [7], that $F$ is the composition of a translation with an endomorphism of $E$, so, as we have $\text{End}(\mathbb{C}/\mathbb{Z}[\mu_n]) \cong \mathbb{Z}[\mu_n]$, all endomorphisms are allowed since they commute with the action of $\mu_n$. However, the translation by any $Q \in E$ commutes with the action if and only if $Q$ is fixed by $\mu_n$. Consequently the only translations allowed are the solutions of

$$\theta z \equiv z \pmod{\mathbb{Z}[\theta]}$$

where $\theta$ is a primitive $n$-th root of unity. Table 1 contains the solutions of (35) lying in the fundamental domain of the elliptic curve. For $n = 2$ they consist of the subgroup $E[2]$ of 2-torsion, while there are only two non-zero fixed points for $n = 4$ and there are no fixed points different from zero for $n = 6$, since the order of the stabilizer at each point is given by its weight as a singular point in the orbifold structure.

Note that, modulo composition with a translation, $f$ lifts to an automorphism of the complex plane of the form $z \mapsto \alpha z$. Recall that when $\alpha \notin \mathbb{Z}$ we say that the elliptic curve $E$ has complex multiplication and if multiplication by a complex $\alpha$, (which must be an integer in some imaginary quadratic field, as $\alpha \Lambda \subset \Lambda$, see [3]) is allowed in $E$, then the complex structure of $E$ is completely determined, i.e. we have $\Lambda \otimes \mathbb{Z} \mathbb{Q} = \mathbb{Q}[[\alpha]]$.

Observe that we can compute explicitly the degree of $f$ in each case. In fact, from (33) we deduce that $d = \text{deg}(f) = \text{deg}(F)$ and we know that, given $m \in \mathbb{Z}$, the “multiplication by $m$” $[m] : E \to E$, has degree $|m|$ if $E \cong \mathbb{C}^*$, while it has degree $m^2$ in the other cases, since it is equal to $\#(\Lambda[\frac{1}{m}] / \Lambda)$. If $E$ has complex multiplication $[\alpha] : z \mapsto \alpha z$, then as $\text{deg}([\alpha] \circ [\alpha]) = |\alpha|^2$, we obtain $\text{deg}([\alpha]) = |\alpha|^2$ (see also [3]).

**Remarks 4.7.**

- Note that in each case we have written the maps $f : \mathbb{P}^1 \to \mathbb{P}^1$ in the form $f = A \circ e$, where $A \in \text{PGL}_2(\mathbb{C})$ corresponds to the translation by a fixed point on $E$ and $e$ is a rational map with the same degree of $f$, corresponding to the endomorphism $z \mapsto \alpha z$ on $E$. Observe that the automorphism $A$ acts by permuting the poles of $q$.
  A simple computation shows that $A$ has order 2 or 3. In fact if $E = \mathbb{C}/\mathbb{Z}[i]$ such a translation permutes the two fixed points of $z \mapsto iz$, hence the corresponding map on $\mathbb{P}^1$ has the form $z \mapsto 1/z$. If $E = \mathbb{C}/\mathbb{Z}[\zeta]$, the three fixed points are permuted cyclically by the translation, hence the corresponding map on $\mathbb{P}^1$ has the form $z \mapsto \zeta z$.

- Cases (1) and (2) of Table 1 are the most explicit:
  In case (1) they are the maps $z \mapsto z^n$, since the exponential function is the universal covering map of $\mathbb{C}^*$.
  In case (2) they are (up to sign) the Tchebycheff polynomials $P_n(z)$ defined by $P_n(cosz) = cos(nz)$, since the cosine function is the universal covering map of $\mathbb{C}^*$ which commutes with $z \mapsto -z$. 

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Note that in each case we have shown that (modulo multiplication by an element of $\text{PGL}_2(\mathbb{C})$) the action of $f$ on $\mathbb{P}^1$, which a priori is given by a semigroup, is globally equivalent to the action of some discrete group $G$ on $\mathbb{C}$, given by an extension of $\mathbb{Z}$ by $\Lambda$

\[ 0 \longrightarrow \Lambda \longrightarrow G \longrightarrow \mathbb{Z} \longrightarrow 0 \]

i.e. $G$ is the semidirect product $\mathbb{Z} \rtimes \Lambda$, with the obvious action.

We have seen in [4.2] that $\tilde{q} = p^*q$ is a holomorphic section of $\Omega^k_\mathcal{O}$, hence $\pi^*\tilde{q}$ is a constant multiple of $dz^k$. It follows that the eigenvalue $\lambda$ such that $f^*q = \lambda q$ can be computed explicitly. Referring to Table 1, in the first two cases we have simply $|\lambda| = \deg(f)^k = d^k$ since the lifted map is multiplication by $m$ on $\mathbb{C}^*$, which has degree $|m|$, and clearly $|m|^k dz^k = m^k dz^k$. Finally, in the other cases, we have $d = |\alpha|^2$ from which we deduce $|\lambda| = d^{k/2}$.

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