Nonlinear physical modeling sound synthesis of cymbals involving dynamics of washers and sticks/mallets

Toshiya Samejima*
Faculty of Design, Kyushu University,
4–9–1 Shiobaru, Minami-ku, Fukuoka, 815–8540 Japan
(Received 14 December 2020, Accepted for publication 9 July 2021)

Abstract: This paper is concerned with the extension of the existing nonlinear physical modeling sound synthesis of cymbals to that involving the dynamics of washers supporting the center of cymbals and sticks/mallets striking the cymbals. The body of a cymbal is physically modeled as a shallow spherical shell and its governing equation is discretized in space using the finite difference method, as was implemented in existing research. In addition, a washer, related to the support conditions of the cymbal, is modeled as a single-degree-of-freedom vibration system and involved in the physical model of the cymbal. Furthermore, a stick/mallet striking the cymbal is also involved by modeling it as a multi-degree-of-freedom system using the finite element method and coupled with the cymbal vibration. The time-domain differential of the total system is discretized using implicit finite difference schemes. Trial numerical calculations demonstrate that the developed method is effective in the sound synthesis of the cymbal, grasping the change of timbre due to the dynamics of washers and sticks/mallets.

Keywords: Cymbal, Shallow spherical shell, Nonlinearity, Finite difference method, Finite element method, Physical modeling

1. INTRODUCTION

The cymbal is one of the idiophones and a percussion instrument, as shown in Fig. 1. This figure includes not only the body of a cymbal but also a washer and stick/mallet as a total cymbal system. Since the essence of the cymbal is nonlinearity from a physical point of view, cymbals are also called nonlinear percussion instruments [1]. The synthesis of its sound based on physical models has been carried out only infrequently in the literature owing to its difficulties. One of the difficulties is in dealing with the strong nonlinearity exhibited by thin structures when struck. This kind of nonlinearity leads to crucial perceptual effects, such as pitch glides, crashes, and the slow buildup of high-frequency energy. These are not adequately captured by linear approximation analyses; hence, the nonlinear physical modeling of a cymbal cannot be avoided.

Bilbao has physically modeled the cymbal unit as a shallow, elastic, spherical shell and performed a numerical analysis using time/space domain finite difference methods to solve the governing equation over a nonlinear vibration field [2]. The numerical results generated various perceptual effects due to the nonlinearity mentioned above. However, the physical model presented was rather basic, especially for the boundary and excitation conditions of the vibration field: the implemented boundary condition, i.e., the mathematical expression of the supporting system of the cymbal, was a clamped center condition; the implemented excitation condition was a useful synthesis shortcut of simply specifying the excitation function as a given function, such as a Hanning pulse (one period of a raised cosine). For a more realistic physical model of a cymbal, not only the interaction between the shell and the support structure, such as a ring-shaped damper, but also a striking mechanism such as a stick/mallet, should be introduced.

In this work, we try to extend the existing physical modeling of cymbals to a more realistic one. This includes modeling washers, related to the support conditions of the cymbal, as single-degree-of-freedom (SDOF) vibration systems. Additionally, a stick/mallet striking the cymbal is modeled as a multi-degree-of-freedom (MDOF) system using the finite element method (FEM) and associated with the cymbal. Trial calculations using the proposed method are carried out to demonstrate the effects of the support condition at the washer and the grip condition of the stick/mallet. As a result, we expect that the nonlinear physical
modeling of cymbals can be achieved with more realistic dynamic conditions.

2. PHYSICAL MODEL OF A CYMBAL AND ITS NUMERICAL ANALYSIS

2.1. Nonlinear Vibration Equation of a Shallow Spherical Shell

The body of a cymbal is modeled as a shallow spherical shell, as shown in Fig. 2. Precisely, the shape of this shallow spherical shell is not the same as the shape of the body of a real cymbal, which comprises a cup and bow whose curvatures are different. For the analysis of a vibration field having irregular geometries, the FEM, which can express such irregular shapes precisely, is suitable [2]. However, both dealing with the strong nonlinearity and expressing irregular shapes finely in a numerical analysis at the same time lead to relatively high computational complexity, which is unfavorable for physical modeling sound synthesis. In the case of cymbals, their nonlinear dynamics are well described using a shallow spherical shell, which is a relatively simple alternative and a good starting point [2]. A nonlinear equation of motion for the vibrational displacement $u$ in the normal direction of a shallow spherical shell, which is based on the von Kármán system, is expressed as [2]

$$
\frac{\partial^2 u}{\partial t^2} = -D \nabla^4 u + L(\phi, u) - \frac{1}{R_s} \nabla^2 \phi 
$$

$$
- 2\sigma_0 \rho \frac{\partial u}{\partial t} + 2\sigma_1 \rho \nabla^2 \frac{\partial u}{\partial t} + \epsilon f, \quad (1)
$$

$$
\nabla^4 \phi = -\frac{Eh}{2} L(u, u) + \frac{Eh}{R_s} \nabla^2 u, \quad (2)
$$

with

$$
D = \frac{Eh^3}{12(1 - \nu^2)},
$$

$$
L(\phi, u) = \frac{1}{r} \frac{\partial^2 \phi}{\partial r^2} \left( \frac{\partial u}{\partial r} + \frac{1}{r} \frac{\partial^2 u}{\partial \theta^2} \right) + \frac{1}{r} \frac{\partial^2 u}{\partial r^2} \left( \frac{\partial \phi}{\partial r} + \frac{1}{r} \frac{\partial^2 \phi}{\partial \theta^2} \right) - 2 \left( \frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial \phi}{\partial \theta} \right) \left( \frac{1}{r} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial u}{\partial \theta} \right)
$$

where $\phi$ is the Airy stress function, $f$ is the exciting force, $\rho$ is the density, $E$ is Young’s modulus, $\nu$ is Poisson’s ratio, $\sigma_0$ and $\sigma_1$ are loss parameters, and $\epsilon$ is the spatial distribution of the force. The operator $\nabla^2$ denotes the Laplacian, defined in radial coordinates. The operator $\nabla^4$ indicates a double application of the Laplacian operator, which is referred to as a bi-Laplacian or biharmonic operator. In Eqs. (1) and (2), if dimensionless variables

$$
\bar{r} = \frac{r}{a}, \quad \bar{\theta} = \theta, \quad \bar{u} = \frac{u}{h}, \quad \bar{\phi} = \frac{\phi}{D}
$$

are introduced, the equation of motion can be simplified as

$$
\frac{\partial^2 \bar{u}}{\partial \bar{t}^2} = -\kappa^2 \left( \nabla^4 \bar{u} - L(\bar{\phi}, \bar{u}) + q \nabla^2 \bar{\phi} \right) - 2\sigma_0 \frac{\partial \bar{u}}{\partial \bar{t}} + 2\sigma_1 \nabla^2 \frac{\partial \bar{u}}{\partial \bar{t}} + \bar{b} \bar{f}, \quad (4)
$$

$$
\nabla^2 \bar{\phi} = -L(\bar{\phi}, \bar{u}) + 2q \nabla^2 \bar{u}, \quad (5)
$$

with

$$
\kappa^2 = \frac{D}{\rho h a^4}, \quad q = a^2 \sqrt{6(1 - \nu^2) / h R_s}, \quad b = \sqrt{6(1 - \nu^2) / \rho h^2}.
$$

2.2. Boundary Condition at the Periphery: Free

When the peripheral part of the spherical shell ($r = a$) is free, with the bending moment of the spherical shell and the shear stress as 0, the governing equation is expressed as [2,3]

$$
-D \left( \frac{\partial^2 \bar{u}}{\partial \bar{r}^2} + \frac{\nu \partial \bar{u}}{\partial \bar{r}} + \frac{\nu}{r^2} \frac{\partial^2 \bar{u}}{\partial \bar{\theta}^2} \right) \bigg|_{r=a} = 0, \quad (6)
$$

$$
D \left( \frac{\partial^2 \bar{u}}{\partial \bar{\theta}^2} + \frac{1}{r^2} \frac{\partial \bar{u}}{\partial \bar{r}} - \frac{1}{r^2} \frac{\partial \bar{u}}{\partial \bar{r}} + \frac{\nu - 3}{r^3} \frac{\partial^2 \bar{u}}{\partial \bar{\theta}^2} \right)
$$
Thus, one can construct the formula for the boundary shear stress balancing with the mechanical impedance.

\[
\phi(\tilde{r})_{\tilde{r}=0} = \frac{\partial \phi}{\partial \tilde{r}} |_{\tilde{r}=0} = 0.
\]  

In these boundary condition expressions, the dimensionless variables given in Eq. (3) are also introduced:

\[
-D \left( \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2} + \frac{\nu \partial \tilde{u}}{\tilde{r}} + \frac{\nu}{\tilde{r}^2} \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} \right)|_{\tilde{r}=1} = 0,
\]

\[
D \left( \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2} + \frac{\nu \partial \tilde{u}}{\tilde{r}} + \frac{\nu}{\tilde{r}^2} \frac{\partial^2 \tilde{u}}{\partial \tilde{t}^2} \right) \left|_{\tilde{r}=1} + \frac{2 - \nu}{\tilde{r}^2} \left( \frac{\partial^2 \tilde{u}}{\partial \tilde{r} \partial \tilde{t}} \right) \right|_{\tilde{r}=1} = 0,
\]

\[
\tilde{\phi}(\tilde{r})_{\tilde{r}=1} = \frac{\partial \tilde{\phi}}{\partial \tilde{r}} |_{\tilde{r}=1} = 0.
\]

2.3. Boundary Condition at the Center: Clamped

If the shell center is attached to a supporting structure, a boundary condition that expresses the support is required near \( r = 0 \). A simple assumption is that the shell is clamped over an inner circumference of radius \( c \ll 1 \). In this case, the boundary conditions are given by \[2,3\]

\[
u \left( \frac{\partial \tilde{u}}{\partial \tilde{r}} \right) |_{\tilde{r}=0} = 0,
\]

\[
\frac{\partial \phi}{\partial \tilde{r}} |_{\tilde{r}=0} = 0.
\]

2.4. Boundary Condition at the Center: Impedance-supported

The assumption that the shell is clamped near the center, which was implemented in an existing work \[2\], is not necessarily realistic. The support may allow the shell to tilt and may also involve a distributed connection to an annular support structure, such as washers. In this paper, a washer with \( r = c \) is modeled as the mechanical impedance of an SDOF system connected to the spherical shell, as shown in Fig. 3. If we consider only the mechanical impedance in the normal direction of the spherical shell, we can apply the left-hand side of Eq. (7) to express the shear stress balancing with the mechanical impedance. Thus, one can construct the formula for the boundary conditions as

\[
D \left( \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2} + \frac{\nu \partial \tilde{u}}{\tilde{r}} + \frac{\nu}{\til{r}^2} \frac{\partial^2 \tilde{u}}{\partial \til{t}^2} \right) |_{\til{r}=1} = 0,
\]

\[
-D \left( \frac{\partial^2 \tilde{u}}{\partial \til{r}^2} + \frac{\nu \partial \til{u}}{\til{r}} - \frac{\nu}{\til{r}^2} \frac{\partial^2 \til{u}}{\partial \til{t}^2} \right) |_{\til{r}=1} + \frac{2 - \nu}{\til{r}^2} \left( \frac{\partial^2 \til{u}}{\partial \til{r} \partial \til{t}} \right) |_{\til{r}=1} = 0,
\]

\[
\tilde{\phi}(\tilde{r})_{\til{r}=1} = \frac{\partial \til{\phi}}{\partial \til{r}} |_{\til{r}=1} = 0.
\]

2.5. Application of the Finite Difference Method

In the \((\tilde{r}, \til{\theta})\) space, we set an equally spaced grid of \( N_r \) in the \( \tilde{r} \)-direction by \( N_\theta \) in the \( \til{\theta} \)-direction. Figure 4 indicates the grid condition when \( N_r = 5 \) and \( N_\theta = 6 \) as an example. In this grid, the index number \( l = 3 \) in the \( \til{r} \)-direction corresponds to the inner circumference of radius \( r = c \), and \( l = N_r \) corresponds to the periphery \( r = a \). Using this grid, the nonlinear equation of motion of the shell, given by Eqs. (4) and (5), can be approximated using the central difference of the spatial differential,

\[
\frac{\partial^2 \tilde{u}}{\partial \til{r}^2} = -\kappa^2 [D_{\til{r}\til{r}} \til{u} - L(\phi, \til{u}) + qD_{\til{r}\til{r}} \phi]
\]

\[
-2\sigma_1 \frac{\partial \til{u}}{\partial \til{t}} + 2\sigma_1 D_{\til{r}\til{r}} \frac{\partial \til{u}}{\partial \til{t}} + \beta \epsilon f,
\]

\[
D_{\til{r}\til{r}} \phi = -L(\til{u}, \til{u}) + 2q D_{\til{r}\til{r}} \til{u},
\]

where \( \til{u} \) is a vector consisting of vibrational displacement.
\[ \begin{align*}
\vec{u}_n & \text{ at the global node number } n, \text{ and } \vec{\phi} \text{ is a vector consisting} \\
& \text{of a value of the Airy stress function } \vec{\phi}_n. \ D_{v2} \text{ and } D_{v4} \text{ express differential matrices} \\
& \text{applying } \nabla^2 \text{ and } \nabla^4, \text{ respectively. Detailed matrix representations} \\
& \text{of the differential matrices } D_{v2} \text{ and } D_{v4}, \text{ and other differential matrices} \\
& \text{appearing in this paper are given in Appendix. In the} \\
& \text{}\hat{r}-\text{direction, only real nodes, which have the index numbers} \\
& l = 3, 4, \ldots, N_r, \text{ are associated with the vibrational} \\
& \text{displacement } \vec{u} \text{ on the shell. Virtual nodes, which have the} \\
& \text{index numbers } l = 1, 2, N_r + 1, \text{ and } N_r + 2, \text{ are used for} \\
& \text{calculating the central difference at the boundaries, i.e.,} \\
& l = 3 \text{ and } N_r. \text{ Note that the variables } \vec{u}_n \text{ at these virtual} \\
& \text{nodes are to be removed using the equations for the} \\
& \text{boundary conditions.}
\end{align*} \]

For the time differential, an implicit difference method [4] with a free parameter \( g \) is applied. As a result, for a discrete time \( k \), we can construct an update formula describing the development of vibrational displacement vector \( \vec{u}_k \) with time \( k \):

\[\begin{align*}
A \vec{u}_{k+1} &= \vec{B} \vec{u}_k + C \vec{u}_{k-1} + \kappa^2 h_i^2 L(\vec{\phi}_k, \vec{u}_k) \\
& \quad - \kappa^2 h_i^2 q D_{v2} \vec{\phi}_k + h_i^2 b e f_k, \quad (20) \\
D_{v4} \vec{\phi}_k &= -L(\vec{u}_k, \vec{u}_k) + 2 q D_{v2} \vec{u}_k, \quad (21)
\end{align*} \]

with

\[\begin{align*}
A &= (1 + \sigma_0 h_k) I + g \kappa^2 h_i^2 D_{v4} - \sigma_1 h_l D_{v2}, \\
B &= 2 I + g \kappa^2 h_i^2 (2g - 1) D_{v4}, \\
C &= (\sigma_0 h_1 - 1) I - g \kappa^2 h_i^2 D_{v4} - \sigma_1 h_l D_{v2},
\end{align*} \]

where \( h_l \) is the discrete time step.

The boundary conditions at the periphery, given by Eqs. (9), (10), and (11), can also be approximated using the central difference of the spatial differential. The time differential is not required for these conditions; thus, we construct the formula describing the development of \( \vec{u}_k \) and \( \vec{\phi}_k \) with time \( k + 1 \):

\[\begin{align*}
\frac{-D}{\alpha^2} [D_{rr} + vD_{r}^{(R)}] \vec{u}_{k+1}|_{t=N_t} &= 0, \quad (22) \\
\frac{D}{\alpha^3} [D_{rr} + D_{r}^{(R)} - D_{r}^{RR} + (v - 3)D_{00}^{RRR}] \\
& + (2 - v)D_{00}^{RRR}] \vec{u}_{k+1}|_{t=N_t} = 0, \quad (23) \\
I \vec{\phi}_{k+1}|_{t=N_t} &= 0, \quad (24) \\
D_{r} \vec{\phi}_{k+1}|_{t=N_t} &= 0. \quad (25)
\end{align*} \]

The boundary conditions at the center, given by Eqs. (16) and (17), are approximated using the central difference of the spatial differential:

\[\begin{align*}
\frac{D}{\alpha^3} [D_{rr} + vD_{r}^{(R)}] \vec{u}_{k+1}|_{t=0} &= 0, \quad (26) \\
\frac{-D}{\alpha^2} [D_{rr} + D_{r}^{(R)} - D_{r}^{RR} + (v - 3)D_{00}^{RRR}] \\
& + (2 - v)D_{00}^{RRR}] \vec{u}_{k+1}|_{t=0} = 0, \quad (27) \\
M_{W} \frac{\partial^2 \vec{u}_l}{\partial t^2} + R_{W} \frac{\partial \vec{u}_l}{\partial t} + K_{W} \vec{u}_l|_{t=0} &= 0, \quad (28) \\
D_{r} \vec{\phi}_{k+1}|_{t=0} &= 0. \quad (29)
\end{align*} \]

In this case, Eq. (27) requires the time differential; thus, we apply another implicit difference method [5] with free parameter \( \phi \). As a result, we obtain an update formula to calculate \( \vec{u}_k \) and \( \vec{\phi}_k \) at the center:

\[\begin{align*}
A_W \vec{u}_{k+1}|_{t=0} &= B_W \vec{u}_k|_{t=0} + C_W \vec{u}_{k-1}|_{t=0}. \quad (30)
\end{align*} \]

with

\[\begin{align*}
A_W &= M_W + h_i^2 R_W + \frac{1 - \phi}{2} h_i^2 K_W, \\
B_W &= 2M_W - \phi h_i^2 K_W \\
& + h_i^2 \frac{-D}{\alpha^3} [D_{rr} + D_{r}^{(R)} - D_{r}^{RR} + (v - 3)D_{00}^{RRR}]|_{t=0}, \\
C_W &= -M_W + h_i^2 R_W - \frac{1}{2} - \phi \frac{1}{2} h_i^2 K_W.
\end{align*} \]

3. PHYSICAL MODEL OF A STICK/MALLET AND ITS NUMERICAL ANALYSIS

In general, a cymbal is struck with a stick or mallet that is gripped by a player. Sticks can be divided into two parts: a stick and tip. Similarly, mallets are composed of a mallet stick and mallet head. These components can be modeled as the same physical system, as shown in Fig. 5. Thus, from now on, we refer to only a mallet as the striking mechanism.
3.1. Physical Model of a Mallet Head and Its Coupling with a Cymbal

For the physical model of a mallet head, we adopt the analytical method for hammer–string interaction developed in the sound synthesis of pianos [6] and also used in the sound synthesis of kettledrums as mallet–drumhead collision vibration [7].

3.1.1. Equation of motion of an SDOF system

The motion of a mallet head is described by the displacement $u_M$ of its center of gravity. During the contact, the mallet head is subjected to the interaction force $f_{M1}$ between the mallet head and the cymbal due to the compression of the mallet head. The mallet head is also subjected to the interaction force $f_{M2}$ between the mallet head and the mallet stick. Thus, the displacement $u_M$ is governed by Newton’s second law:

$$M_M \frac{d^2 u_M}{dt^2} = f_{M1} - f_{M2}.$$  \hspace{1cm} (31)

This means that the mallet head is modeled as an SDOF system. From the positional relationships shown in Fig. 6, the excitation force $f_{M1}$ applied to the mallet head can be given as

$$f_{M1} = K_M [(u_0 - (u_M - d_M))^+]^\alpha,$$  \hspace{1cm} (32)

where $K_M$ is the coefficient of mallet stiffness, $\alpha$ is the stiffness nonlinear exponent, and $[.]^+$ is an operator that only takes positive values. The counteracting force $f$ applied to the cymbal is clearly given by

$$f = -f_{M1}.$$  \hspace{1cm} (33)

3.1.2. Application of the finite difference method

For the time differential of Eq. (31), we adopt a simple approximation of the central difference scheme. The resultant update formula describing the development of $u_{M,k}$ with time $k$ is expressed as

$$u_{M,k+1} = 2u_{M,k} + h_1^2 \frac{K_M}{M_M} [(u_k - (u_{M,k} - d_M))^+]^\alpha$$

$$- u_{M,k-1} - h_1^2 f_{M2,k}.$$  \hspace{1cm} (34)

3.2. Physical Model of a Mallet Stick and Its Coupling with the Mallet Head

3.2.1. Vibration equation of an Euler–Bernoulli beam

As shown in Fig. 5, a mallet stick is physically modeled as an Euler–Bernoulli beam supported by a rotational spring with rotational spring constant $K_R$ on one edge. For the transverse vibrational displacement $\xi_S$ of the beam with the length along the $x$-axis, its governing equation can be expressed as

$$\rho_S S_S \frac{d^2 \xi_S}{dt^2} = -E_S I_S \frac{d^2 \xi_S}{dx^4} - R_S \frac{\partial \xi_S}{\partial t} + \epsilon_S f_S,$$  \hspace{1cm} (35)

where $S_S$ is the cross-sectional area, $E_S$ is Young’s modulus, $I_S$ is the cross-sectional second moment of inertia, $R_S$ is the resistance coefficient per unit length, $f_S$ is the excitation force, and $\epsilon_S$ is the spatial distribution of the force.

Now, let $u_S$ be the value of $\xi_S$ at the end of the beam, where the mallet head is attached, as shown in Fig. 6. If we suppose the simplest type of connection between the mallet stick and the mallet head, the following conditions are required:

$$u_M = u_S,$$  \hspace{1cm} (36)

$$f_{M2} = -f_S.$$  \hspace{1cm} (37)

3.2.2. Application of the finite element and finite difference method

In Eq. (35), if we apply the FEM [8,9] to the space differential, the following equation of motion of the MDOF system arises:

$$M_S \frac{d^2 \delta_S}{dt^2} + R_S \frac{\partial \delta_S}{\partial t} + K_S \delta_S = \epsilon_S f_S,$$  \hspace{1cm} (38)

where $\delta_S$ is a vector consisting of the vibrational nodal values. The implicit difference method [5] with free parameter $\phi$ is applied to the time differential. As a result, we can obtain an update formula describing the time development of $\delta_{S,k}$ with time $k$:

$$A_S \delta_{S,k+1} = B_S \delta_{S,k} + C_S \delta_{S,k-1} + h_1^2 \epsilon_S f_{S,k},$$  \hspace{1cm} (39)

with

318
4. NUMERICAL EXAMPLE: EFFECT OF THE SUPPORTED CONDITION AT THE WASHER

4.1. Conditions of the Calculation

The physical values shown in Table 1 based on the actual measurement [10] of a cymbal (A Zildjian 18” 45 cm MEDIUM CRASH) were used, and the vibrational displacement time history was calculated numerically through the developed physical model. The collision point with the mallet head was assumed to be (\(\vec{r} = 0.6\), \(\vec{\theta} = \pi\)). The damping coefficients were \(\sigma_0 = 1.34\) and \(\sigma_1 = 1.20 \times 10^{-3}\). The mesh conditions were set to \(N_r = 24\), \(N_\theta = 60\), and \(1/h = 88.2\) kHz. Additionally, the free parameter was set to \(g = 0.2498\). The discretization of the mallet stick involved third-order beam elements with 21 nodes and 20 elements. The free parameter was assumed as \(\phi = 0.7\). In order to explore the changes in the timbre due to the method of the supported condition at the washer, two boundary conditions at the center of the shell were chosen: fully fixed support (clamped) and support using a washer (impedance-supported). As a preliminary study, only the stiffness of the washer, \(K_W\), was set to its measured value, while the mass and resistance were set to 0. At the beam element at the edge, corresponding to the part gripped by hand, the boundary condition of fully fixed support (clamped hand) was introduced. In addition, the acceleration \(a_H(t)\) at the clamped hand section (the part gripped by hand) was fixed to the initial acceleration \(a_{H0} = 1 \times 10^4\) m/s² and the velocity \(v_H(t) = a_{H0}t\) was introduced, as boundary conditions. Once the velocity \(v_H(t)\) reached \(v_{H\text{limit}} = 10\) m/s, \(a_H(t)\) was fixed to zero. The vibrational acceleration at \((\vec{r} = 0.6, \vec{\theta} = 0)\) was the signal output as a physical quantity, approximating the sound radiated from the cymbal.

4.2. Numerical Results

The time histories of the calculated vibrational acceleration as the output signal of the cymbal are shown in Fig. 7. One can see that when the boundary condition is impedance-supported, the signal is amplified compared with that when the boundary condition is clamped. Figure 8 shows the frequency response functions of the vibrational acceleration. It can be seen that the low-order resonant frequencies are particularly changed by considering the support condition of the cymbal as a mechanical impedance.

As a rough reference, the low-order eigenfrequencies and nodal contours of the cymbal, which are calculated

\[
A_S = M_S + \frac{h_l}{2} R_S + \frac{1 - \phi}{2} h_l^2 K_S, \\
B_S = 2M_S - \phi h_l^2 K_S, \\
C_S = -M_S + \frac{h_l}{2} R_S - \frac{1 - \phi}{2} h_l^2 K_S.
\]

Table 1 Geometric and material properties of the cymbal and mallet used in the calculation.

| Parameters                  | Values          |
|-----------------------------|-----------------|
| For cymbal                  |                 |
| Outer diameter: \(2a\)      | 0.230 × 2 m     |
| Rise at the center: \(r_{\text{sic}}\) | 0.040 m       |
| Thickness: \(h\)            | 0.001 m         |
| Young’s modulus: \(E\)      | 1.1 × 10¹¹ Pa   |
| Poisson’s ratio: \(v\)      | 0.33            |
| Density: \(\rho\)          | 8,860 kg/m³     |
| Stiffness of washer: \(K_W\) | 1.65 × 10⁵ N/m/m |
| Loss parameter: \(\sigma_0\) | 1.34            |
| Loss parameter: \(\sigma_1\) | 1.20 × 10⁻³    |
| For mallet head             |                 |
| Radius of mallet: \(d_M\)   | 0.025 m         |
| Effective mass of mallet: \(M_M\) | 0.028 kg       |
| Coefficient of mallet stiffness: \(K_M\) | 1.6 × 10⁸ Nm⁻α |
| Stiffness nonlinear exponent: \(\alpha\) | 2.54            |
| For mallet stick            |                 |
| Length: \(L\)              | 0.390 m         |
| Cross section area: \(S_S\) | 1.43139 × 10⁻⁴ m² |
| Young’s modulus: \(E_S\)    | 8.8 × 10⁶ Pa    |
| Poisson’s ratio: \(v_S\)    | 0.30            |
| Density: \(\rho_S\)        | 400 kg/m³       |
| Cross-sectional moment of inertia: \(I_S\) | 1.63044 × 10⁻⁹ m⁴ |
| Rotational spring stiffness: \(K_R\) | 140 Nm/rad  |
| Damping matrix: \(R_S\)     | 10 M_S          |
| Initial acceleration of stick: \(a_{H0}\) | -1 × 10⁴ m/s²  |
| Initial velocity of stick: \(v_{H0}\) | 0 m/s         |
| Limit of velocity of stick: \(v_{H\text{limit}}\) | -10 m/s }
with the linear FEM, i.e., ignoring the nonlinear terms, are given in Fig. 9. Using the planer shell elements \([10,11]\), the equation of motion of the MDOF system was constructed, then its eigenvalue problem was solved. The figure indicates nodal contours for normal components of the displacement to the shell surface. For the clamped condition, the eigenfrequency of the \((1,1)\) mode is higher than that of the \((2,1)\) mode, though the mode number \((1,1)\) is lower than the mode number \((2,1)\). This is because the shell center is clamped. When the number of nodal diameters is two, the tilt of the shell surface at the center is inherently zero regardless of whether the center is clamped or not. Thus, the clamped condition at the center does not affect the \((2,1)\) mode shape. In contrast, when the number of nodal diameters is one, the clamped condition at the center contributes to increasing the stiffness at the center, thereby the eigenfrequency of the \((1,1)\) mode increases. The \((0,0)\) mode of the impedance-supported condition has no nodal lines; in this mode, the cymbal acts as an SDOF system. For both the \((2,1)\) and \((1,1)\) modes of the clamped condition and the \((1,0)\) and \((2,0)\) modes of the impedance-supported condition, the resonant frequencies observed in the frequency response functions are lower than the eigenfrequencies calculated with the linear FEM. This may be attributed to the nonlinear effect as well as the damping effect.

Now, suppose that a mesh whose characteristic length is smaller than \(1/10\) of the wavelength provides sufficient computational accuracy. Then, under the present mesh conditions, the highest mode number ensuring the accuracy is roughly estimated to be \((6,5)\). Its eigenfrequency calculated with the linear FEM for the impedance-supported condition is 2,047 Hz; below this frequency, the calculated data could be reliable, although not for all the modes. In a high-frequency range, in general, individual harmonics of the cymbal undergo bifurcations, leading to a noise-like sound having random and continuous spectral characteristics. More accurate calculations in such a high-frequency range should be carried out in the next stage.

In the frequency range from about 100 to 500 Hz, the output level of the impedance-supported condition is boosted compared with that of the clamped condition. These characteristics show qualitative agreement with the measured results and the results calculated with the linear FEM given in the author’s previous work [12].

Also as a rough reference, a measured frequency response function of the vibrational acceleration of the real cymbal used for the calculation model is given in Fig. 8(c). The real cymbal was struck with an impulse hammer (PCB 086C01) in an anechoic chamber, and the output signal of an accelerometer (PCB 352B10) attached to the cymbal was recorded in a PC through an audio interface. Here, since experimental conditions such as the strength of the striking and the tightness of the binding of the cymbal washer were not strictly controlled, we do not strictly compare the calculated data with the measured data. Nevertheless, rough frequency characteristics, such as...
several clear peaks in the low frequency range and the build-up of high-frequency energy over around 700 Hz, can be seen in both the calculated and measured data; this demonstrates that the developed method can grasp, in part, the dynamics of a real cymbal system. Further detailed comparison will be left for future work.

In Fig. 10, the spectrograms of the vibrational acceleration are shown. One of the primary special features of the cymbal sound called a crash sound, which is a slow build-up of high-frequency components after a strike, is clearly demonstrated by the proposed physical modeling of the cymbal and its numerical calculation. In real cymbals, this slow build-up of high-frequency components is often over several hundred milliseconds [1,2]. The proposed physical model can reproduce this nature with acceptable accuracy. When these output signals were played as audio signals, we could indeed hear satisfactory cymbal-like sounds. When the boundary condition is impedance-supported, the high-frequency components remain for a rather longer time than those when the boundary condition is clamped. Note that, in these numerical examples, the mass and resistance components of the washer were not used. If the shell center was supported by a spring, its boundary condition would become rather close to that of pin support. In general, vibrational systems with pin support will have lower eigenfrequencies than those with a clamped end, and lower frequency components will have a relatively long time constant. The crash sound mentioned above, which is a noise-like sound, is generated since individual harmonics of the shell undergo bifurcations. It is predicted that these effects will result in a longer sustainment of the high-frequency components for the impedance-supported cymbal.

Figure 11 shows snapshots of the time development of the deformation of the spherical shell. Contrary to setting the center section of the spherical shell to the clamped condition, setting it to the impedance-supported condition can exhibit the vibrational displacement with a tilt at the center section, which is a physically reasonable result. Through the developed physical model of cymbals, we can

Fig. 10  Spectrograms of the vibrational acceleration.

(a) Clamped.  
(b) Impedance-supported.

Fig. 11  Snapshots of the time evolution of the shape of the spherical shell.

(a) Clamped.  
(b) Impedance-supported.
estimate the change in the vibration state of cymbals according to their support conditions.

5. NUMERICAL EXAMPLE: EFFECT OF THE MALLET DYNAMICS

5.1. Conditions of the Calculation

The physical values of the cymbal and mallet, mesh conditions, and other numerical parameters were set to be the same as those in the preceding section. For the boundary condition at the center of the shell, support using the washer (impedance-supported condition) was adopted. In order to explore the changes in the timbre due to the method of holding the mallet stick, we considered two patterns: fully fixed support (clamped hand) and support using a rotational spring (rotational spring hand) for the beam element at the edge corresponding to the part gripped by hand. Takashima et al. investigated the tone change of a cymbal due to the difference in grip tightness through an experimental approach [13]. They assumed two kinds of grip tightness: a tight grip and loose grip. We considered that the tight grip could be expressed by a clamped hand and the loose grip by a rotational spring hand. The rotational spring stiffness used for expressing the loose grip was estimated through the actual measurement of a mallet gripped by hand. As a preliminary and trial examination, an approximate value was obtained as follows. A mallet was pressed against a weight scale, then the relationship between the load and the rotational displacement was roughly measured. From the relationship, we identified the rotational spring stiffness $K_R$, which is given in Table 1. For the conditions of the acceleration $a_H(t)$ and the velocity $v_H(t)$ at the edge corresponding to the part gripped by hand, the same conditions as those in the preceding section were introduced.

5.2. Numerical Results

The time histories of the excitation force are shown in Fig. 12. First, it is observed that the complexity of the interaction force profiles, which exhibit multiple peaks (in this case, two peaks) due to the return of reflected waves to the mallet head can be reproduced. This feature is one of the primary aspects with the consideration of the coupling musical instruments with hammer/mallet models [14,15]. The proposed physical modeling of the cymbal and its numerical calculation can reproduce such behavior. When modeling the mallet stick as an Euler–Bernoulli beam with a rotational spring support, the excitation force between the mallet head and the cymbal is slightly less than that with a fully fixed support. This leads to a decrease in amplitude in the time history of the vibrational acceleration, as shown in Fig. 13. Examining the waveforms of the excitation forces in Fig. 12 in detail, it is notable that the second peak in (b) (rotational spring hand) is slightly smaller than that in (a) (clamped hand). The method of holding the mallet stick also has an effect on the vibrational acceleration spectrogram, as shown in Fig. 14. In both (a) and (b), the spectrogram patterns show growth at higher frequencies with time delay, which indicates the crash sound. With the rotational spring support, the time delay of the crash sound is slightly longer than that in the case of the fully fixed support. Figure 15 shows snapshots of the time development of the deformation in the beam. If we investigated the change in the vibration state of a cymbal by including its
striking mechanism through the proposed physical model of cymbal systems, we could reveal how the timbre of cymbals alters according to the mallet properties and grip conditions.

6. CONCLUSIONS AND FUTURE WORK

The existing nonlinear vibration analysis of cymbals was altered to fit both the support condition of the cymbal and the striking mechanism in actual situations more realistically. One alteration is that a physical model of washers supporting a cymbal at its center was incorporated. The washer was modeled as the mechanical impedance of an SDOF system, which was dealt with as the boundary condition of the spherical shell, which is the physical model of the cymbal. The other alteration is that the dynamics of a stick/mallet striking a cymbal were taken into account in the physical model. The mallet head was modeled as an SDOF system having a mass and nonlinear stiffness. The mallet stick was modeled as an Euler–Bernoulli beam and treated as an MDOF system using the FEM. To express the difference in grip condition by hand, a support condition modeled by a rotational spring was introduced as a boundary condition to the finite element model.

For the support condition with the cymbal washer, the results of numerical calculation using the proposed method demonstrated, in part, the real characteristics of a cymbal observed in the author’s previous measurement. The other numerical calculation results for the dynamics of a stick/mallet suggested that the difference in grip condition, such as a tight grip or loose grip, has an effect on the timbre of a cymbal.

In future studies, it is necessary:
- to further validate the proposed physical modeling and its numerical analysis by comparing numerical predictions with data measured in experiments on real cymbal systems for many kinds of configuration,
- to investigate the characteristics of the proposed numerical analysis, i.e., to verify the numerical process employed in the proposed method so that one can obtain correct and reliable numerical solutions of the proposed physical modeling,
- to build a cymbal–acoustics system, i.e., to take into account the interaction between shell vibration and its

![Fig. 14 Spectrograms of the vibrational acceleration.](image)

![Fig. 15 Snapshots of the time evolution of the shape of the beam.](image)
radiated sound, using acoustic scattering analysis around a thin obstacle,
• to give guidelines for designing and playing cymbals to produce their desired timbre through the proposed numerical analysis.

ACKNOWLEDGMENT

This work was supported by JSPS KAKENHI Grant Number JP19H04126.

REFERENCES

[1] A. Chaigne, C. Touzé and O. Thomas, “Nonlinear vibrations and chaos in gongs and cymbals,” Acoust. Sci. & Tech., 26, 403–409 (2005).

[2] S. Bilbao, “Percussion synthesis based on models of nonlinear shell vibration,” IEEE Trans. Audio Speech Lang. Process., 18, 872–880 (2010).

[3] C. Touzé, O. Thomas and A. Chaigne, “Asymmetric non-linear forced vibrations of free-edge circular plate. Part 1: theory,” J. Sound Vib., 258, 649–676 (2002).

[4] S. Bilbao, Numerical Sound Synthesis — Finite Difference Schemes and Simulation in Musical Acoustics— (John Wiley & Sons, West Sussex, 2009), Sect. 13.3.1.

[5] S. Bilbao, Numerical Sound Synthesis — Finite Difference Schemes and Simulation in Musical Acoustics— (John Wiley & Sons, West Sussex, 2009), Sect. 3.3.2.

[6] A. Chaigne and A. Askenfelt, “Numerical simulations of piano strings. I. A physical model for a struck string using finite difference methods,” J. Acoust. Soc. Am., 95, 1112–1118 (1994).

[7] L. Rhaouti, A. Chaigne and P. Joly, “Time-domain modeling and numerical simulation of a kettledrum,” J. Acoust. Soc. Am., 105, 3545–3562 (1999).

[8] K. Washizu, H. Miyamoto, Y. Yamada, Y. Yamamoto and T. Kawai, Yugen yousohou hand book I kiso hen (Baihukan, Tokyo, 1981), Chap. 3, Sect. 2.7 (in Japanese).

[9] Y. Kagawa, Yugen yousohou ni yoru shindo-onkyo kougaku/ Yugen yousohou hand book I kiso hen (Baihukan, Tokyo, 1981), Chap. 3, Sect. 5.2 (in Japanese).

[10] O. C. Zienkiewicz, The Finite Element Method in Engineering Science (McGraw-Hill, New York, 1971), Chap. 10.

[11] K. Washizu, H. Miyamoto, Y. Yamada, Y. Yamamoto and T. Kawai, Yugen yousohou hand book I kiso hen (Baihukan, Tokyo, 1981), Chap. 3, Sect. 5.2 (in Japanese).

[12] A. Takahashi and T. Samejima, “Finite element analysis of cymbal vibration using axisymmetric thin shell element,” Proc. Autumn Meet. Acoust. Soc. Jpn., pp. 685–686 (2016.9) (in Japanese).

[13] S. Takashima, N. Wakatsuki and K. Mizutani, “Measurement of tone change of cymbal depending on grip tightness,” Proc. Musical Acoustics Acoust. Soc. Jpn., 37(6), pp. 71–74 (2018) (in Japanese).

[14] A. Chaigne and A. Askenfelt, “Numerical simulations of piano strings. II. Comparisons with measurements and systematic exploration of some hammer-string parameters,” J. Acoust. Soc. Am., 95, 1631–1640 (1994).

[15] S. Bilbao, Numerical Sound Synthesis — Finite Difference Schemes and Simulation in Musical Acoustics— (John Wiley & Sons, West Sussex, 2009), Sect. 7.5.

APPENDIX: DETAILS OF THE DIFFERENTIAL MATRICES

In this Appendix, the differential matrices used in this paper are given with their details. First, matrix representations of the ordinary central difference for approximating the 1-D spatial differential are prepared:

\[
\begin{bmatrix}
0 & 1 \\
-1 & 0 & 1 \\
& & \ddots \\
& & & -1 & 0 & 1 \\
\end{bmatrix}
\]

\[
D_r^{(1)} = \frac{1}{2h_r}
\]

\[
\begin{bmatrix}
-2 & 1 \\
1 & -2 & 1 \\
& & \ddots \\
& & & 1 & -2 & 1 \\
\end{bmatrix}
\]

\[
D_{rr}^{(1)} = \frac{1}{h_r^2}
\]

\[
\begin{bmatrix}
0 & -2 & 1 \\
2 & 0 & -2 & 1 \\
& & \ddots \\
& & & -1 & 2 & 0 & -2 \\
\end{bmatrix}
\]

\[
D_{rrr}^{(1)} = \frac{1}{h_r^3}
\]

\[
\begin{bmatrix}
6 & -4 & 1 \\
-4 & 6 & -4 & 1 \\
1 & -4 & 6 & -4 & 1 \\
& & \ddots \\
1 & -4 & 6 & -4 & 1 \\
\end{bmatrix}
\]

where \(D_r^{(1)}\) is an approximation to \(\partial/\partial r\), \(D_{rr}^{(1)}\) is an approximation to \(\partial^2/\partial r^2\), and so forth. Needless to say, the blank space indicates elements of 0. \(h_r\) is the grid spacing in the \(r\)-direction, as shown in Fig. 4. For the \(\theta\)-direction, although the matrix representations can be written in the same manner, one must consider periodicity in this direction. As a result, the matrix representations are expressed as
where \( h_0 \) is the grid spacing in the \( \theta \)-direction, also as shown in Fig. 4.

In addition to \( D^{(1)}_{\theta \theta} \), etc., the \( N_0 \times N_0 \) identity matrix \( I_\theta \) and the matrix \( R \) given below are also prepared:

\[
R = \text{diag}\left[ \frac{1}{r_i} \right], \quad i = 1, 2, \ldots, N_r + 2.
\]

Using the matrices prepared above, the differential matrices appearing in the discretized forms Eqs. (18) and (19) of the governing equations can be constructed as

\[
D_{\theta^2} = D^{(1)}_{rr} \otimes I_\theta + [R D^{(1)}_{rr}] \otimes I_\theta + R^2 \otimes D^{(1)}_{00},
\]

\[
D_{\theta^4} = D^{(1)}_{rrr} \otimes I_\theta + 2[R^2 D^{(1)}_{rr}] \otimes D^{(1)}_{00} + R^2 \otimes D^{(1)}_{0000} + 2[RD^{(1)}_{rr}] \otimes I_\theta - 2[R^2 D^{(1)}_{rr}] \otimes D^{(1)}_{00} + 4R^4 \otimes D^{(1)}_{00} + [R^3 D^{(1)}_{rr}] \otimes I_\theta,
\]

where the symbol \( \otimes \) indicates the Kronecker product of two matrices.

The differential matrices used in the governing equations of the boundary conditions and those appearing in the nonlinear operator \( L(\phi, u) \) are given below:

\[
D_r = D^{(1)}_r \otimes I_\theta, \quad D_{rr} = D^{(1)}_{rr} \otimes I_\theta
\]

\[
D_{rrr} = D^{(1)}_{rrr} \otimes I_\theta
\]

\[
D^{(R)} = [R D^{(1)}_r] \otimes I_\theta, \quad D^{(RR)} = [R^2 D^{(1)}_r] \otimes I_\theta
\]

\[
D^{(R)}_{rr} = [R D^{(1)}_{rr}] \otimes I_\theta, \quad D^{(RR)}_{rr} = [R^2 D^{(1)}_{rr}] \otimes I_\theta
\]

Toshiya Samejima received his B.E. and M.E. degrees from the Department of Architecture and Architectural Engineering, Waseda University, Japan, in 1992 and 1994, respectively. He received his Ph.D. degree from the Department of Architecture, the University of Tokyo, Japan, in 1997. After a period as a Research Associate with the Advanced Research Institute for Science and Engineering, Waseda University, he is currently an Associate Professor in the Faculty of Design, Kyushu University, Japan. His research interests are theoretical/numerical analysis and control of sound and vibration. He is a member of ASJ, AIJ, INCE/I, and MASJ.