QUANTUM AFFINE GELFAND-TSETLIN BASES AND QUANTUM TORDOIDAL ALGEBRA VIA K-THEORY OF AFFINE LAUMON SPACES

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Abstract. Laumon moduli spaces are certain smooth closures of the moduli spaces of maps from the projective line to the flag variety of $GL_n$. We construct the action of the quantum loop algebra $U_v(sl_n)$ in the K-theory of Laumon spaces by certain natural correspondences. Also we construct the action of the quantum toroidal algebra $\hat{U}_v(sl_n)$ in the K-theory of the affine version of Laumon spaces.

1. Introduction

1.1. This note is a sequel to [3, 4]. The moduli spaces $Q_d$ were introduced by G. Laumon in [8] and [9]. They are certain partial compactifications of the moduli spaces of degree $d$ based maps from $P^1$ to the flag variety $B_n$ of $GL_n$. The authors of [3, 4] considered the localized equivariant cohomology $R = \bigoplus_d H^*_T \times C^* (Q_d) \otimes H^*_D (pt)$ frac($H^*_D (pt)$) where $T$ is a Cartan torus of $GL_n$ acting naturally on the target $B_n$, and $C^*$ acts as "loop rotations" on the source $P^1$. They constructed the action of the Yangian $Y(sl_n)$ on $R$, the new Drinfeld generators acting by natural correspondences.

In this note we write (in style of [4]) the formulas for the action of "Drinfeld generators" of the quantum loop algebra in the localized equivariant $K$-theory $M = \bigoplus_d K^*_T \times C^* (Q_d) \otimes K^*_D (pt)$ frac($K^*_D (pt)$). In fact, the correspondences defining this action are very similar to the correspondences used by H. Nakajima [12] to construct the action of loop algebra in the equivariant K-theory of quiver varieties.

We prove the main theorem directly by checking all relations in the fixed points basis.

There is an affine version of the Laumon spaces, namely the moduli spaces $P^1$ of parabolic sheaves on $P^1 \times P^1$, a certain partial compactification of the moduli spaces of degree $d$ based maps from $P^1$ to the "thick" flag variety of the loop group $\hat{SL}_n$, see [5]. The similar correspondences give rise to the action of the quantum toroidal algebra $\hat{U}_v(\hat{sl}_n)$ on the sum of localized equivariant K-groups $V = \bigoplus_d K^*_T \times C^* \times C^* (P^1) \otimes K^*_D (pt)$ frac($K^*_D (pt)$) where the second copy of $C^*$ acts by the loop rotation on the second copy of $P^1$ (Theorem 4.13).

Since the fixed point basis of $M$ corresponds to the Gelfand-Tsetlin basis of the universal Verma module over $U_v(\hat{sl}_n)$ (Theorem 6.3 in [3]), we propose to call the fixed point basis of $V$ the affine Gelfand-Tsetlin basis. We expect that the specialization of the affine Gelfand-Tsetlin basis gives rise to a basis in the integrable $\hat{sl}_n$-modules (which we also propose to call the affine Gelfand-Tsetlin basis). We expect (see 4.17) that the action of $\hat{U}_v(\hat{sl}_n)$ on the integrable $\hat{sl}_n$-modules coincides with the action of Uglov and Takehura [14].

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2. Lauçon Spaces and Quantum Loop Algebra $U_q(\mathfrak{sl}_n)$

2.1. We recall the setup of [2, 3, 4]. Let $C$ be a smooth projective curve of genus zero. We fix a coordinate $z$ on $C$, and consider the action of $C^*$ on $C$ such that $v(z) = e^{-z}$. We have $C^* = \{0, \infty\}$.

We consider an $n$-dimensional vector space $W$ with a basis $w_1, \ldots, w_n$. This defines a Cartan torus $T \subset G = GL_n \subset Aut(W)$. We also consider its $2^n$-fold cover, the bigger torus $\tilde{T}$, acting on $W$ as follows: for $\tilde{T} \ni t = (t_1, \ldots, t_n)$ we have $\tilde{t}(w_i) = t_i^2 w_i$. We denote by $B$ the flag variety of $G$.

2.2. Given an $(n-1)$-tuple of nonnegative integers $\underline{d} = (d_1, \ldots, d_{n-1})$, we consider the Lauçon's quasiflags' space $\Omega_{\underline{d}}$, see [9], 4.2. It is the moduli space of flags of locally free subsheaves $0 \subset W_1 \subset \ldots \subset W_{n-1} \subset W = W \otimes O_C$ such that rank($W_k$) = $k$, and deg($W_k$) = $-d_k$.

It is known to be a smooth projective variety of dimension $2d_1 + \ldots + 2d_{n-1} + \dim B$, see [8], 2.10.

2.3. We consider the following locally closed subvariety $\Omega_{\underline{d}} \subset \Omega_{\underline{d}}$ (quasiflags based at $\infty \in C$) formed by the flags $0 \subset W_1 \subset \ldots \subset W_{n-1} \subset W = W \otimes O_C$ such that $W_i \subset W$ is a vector subbundle in a neighbourhood of $\infty \in C$, and the fiber of $W_i$ at $\infty$ equals the span $\langle w_1, \ldots, w_i \rangle \subset W$.

It is known to be a smooth quasiprojective variety of dimension $2d_1 + \ldots + 2d_{n-1}$.

2.4. Fixed points. The group $G \times C^*$ acts naturally on $\Omega_{\underline{d}}$, and the group $\tilde{T} \times C^*$ acts naturally on $\Omega_{\underline{d}}$. The set of fixed points of $\tilde{T} \times C^*$ on $\Omega_{\underline{d}}$ is finite; we recall its description from [9], 2.11.

Let $\underline{d}$ be a collection of nonnegative integers $(d_i)$, $i \geq j$, such that $d_i = \sum_{j=1}^i d_{ij}$, and for $i \geq k \geq j$ we have $d_{kj} \geq d_{ij}$. Abusing notation we denote by $\underline{d}$ the corresponding $\tilde{T} \times C^*$-fixed point in $\Omega_{\underline{d}}$:

\[
W_1 = O_C(-d_{11} \cdot 0) w_1, \\
W_2 = O_C(-d_{21} \cdot 0) w_1 \oplus O_C(-d_{22} \cdot 0) w_2, \\
\ldots, \\
W_{n-1} = O_C(-d_{n-1,1} \cdot 0) w_1 \oplus \ldots \oplus O_C(-d_{n-1,n-1} \cdot 0) w_{n-1}.
\]

2.5. For $i \in \{1, \ldots, n-1\}$, and $\underline{d} = (d_1, \ldots, d_{n-1})$, we set $\underline{d} + i := (d_1, \ldots, d_i + 1, \ldots, d_{n-1})$. We have a correspondence $E_{\underline{d}, i} \subset \Omega_{\underline{d}} \times \Omega_{\underline{d} + i}$ formed by the pairs $(W_j, W_{j'}')$ such that for $j \neq i$ we have $W_j = W_{j'}'$, and $W_{j'}' \subset W_i$, see [9], 3.1. In other words, $E_{\underline{d}, i}$ is the moduli space of flags of locally free sheaves $0 \subset W_1 \subset \ldots W_{i-1} \subset W_i' \subset W_i \subset W_{i+1} \ldots \subset W_{n-1} \subset W$ such that rank($W_k$) = $k$, and deg($W_k$) = $-d_k$, while rank($W_{ij}'$) = $i$, and deg($W_{ij}'$) = $-d_i - 1$.

According to [9], 2.10, $E_{\underline{d}, i}$ is a smooth projective algebraic variety of dimension $2d_1 + \ldots + 2d_{n-1} + \dim B + 1$.

We denote by $p$ (resp. $q$) the natural projection $E_{\underline{d}, i} \rightarrow \Omega_{\underline{d}}$ (resp. $E_{\underline{d}, i} \rightarrow \Omega_{\underline{d} + i}$). We also have a map $r : E_{\underline{d}, i} \rightarrow C$,

\[
(0 \subset W_1 \subset \ldots W_{i-1} \subset W_i' \subset W_i \subset W_{i+1} \ldots \subset W_{n-1} \subset W) \mapsto \text{supp}(W_i/W_{ij}').
\]

The correspondence $E_{\underline{d}, i}$ comes equipped with a natural line bundle $L_i$ whose fiber at a point

\[
(0 \subset W_1 \subset \ldots W_{i-1} \subset W_i' \subset W_i \subset W_{i+1} \ldots \subset W_{n-1} \subset W)
\]
equals \( \Gamma(C, W_i/W_i') \).

Finally, we have a transposed correspondence \( T_{\mathcal{E}_d} \subset \mathcal{O}_{d+i} \times \mathcal{O}_{d} \).

2.6. Restricting to \( \mathcal{O}_d \subset \mathcal{O}_d \) we obtain the correspondence \( \mathcal{E}_{d,i} \subset \mathcal{O}_{d} \times \mathcal{O}_{d+1} \) together with line bundle \( L_i \) and the natural maps \( p : E_{d,i} \rightarrow \mathcal{O}_d \), \( q : E_{d,i} \rightarrow \mathcal{O}_{d+i} \), \( r : E_{d,i} \rightarrow C - \infty \).

We also have a transposed correspondence \( T_{\mathcal{E}_d} \subset \mathcal{O}_{d+i} \times \mathcal{O}_{d} \). It is a smooth quasiprojective variety of dimension \( 2d_1 + \ldots + 2d_{n-1} + 1 \).

2.7. We denote by \( 'M \) the direct sum of equivariant (complexified) K-groups:

\[
'M = \bigoplus d \mathcal{K}^{\mathcal{T} \times C^*}(\mathcal{O}_d).
\]

It is a module over \( \mathcal{K}^{\mathcal{T} \times C^*}(pt) = \mathbb{C}[T \times C^*] = \mathbb{C}[x_1, \ldots, x_n, v] \). We define

\[
M = 'M \otimes_{\mathcal{K}^{\mathcal{T} \times C^*}(pt)} \text{Frac}(\mathcal{K}^{\mathcal{T} \times C^*}(pt))
\]

We have an evident grading

\[
M = \bigoplus d M_d, M_d = \mathcal{K}^{\mathcal{T} \times C^*}(\mathcal{O}_d) \otimes_{\mathcal{K}^{\mathcal{T} \times C^*}(pt)} \text{Frac}(\mathcal{K}^{\mathcal{T} \times C^*}(pt)).
\]

2.8. For the quantum universal enveloping algebra \( U_v(\mathfrak{gl}_n) \) we follow the notations of section 2 of [10]. Namely, \( U_v(\mathfrak{gl}_n) \) has generators \( t_{1}^{\pm 1}, \ldots, t_n^{\pm 1}, e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1} \) with the following defining relations (formulas (2.1) of loc. cit.):

\[
(1) \quad t_i t_j = t_j t_i, \quad t_i t_{i+1}^{-1} = t_{i+1}^{-1} t_i = 1
\]

\[
(2) \quad t_i e_j t_i^{-1} = e_j t_{i+j}^{\delta_{i,j+1}} - t_{i+j}^{\delta_{i,j+1}} t_i, \quad t_i f_j t_i^{-1} = f_j v^{-\delta_{i,j+1}} - t_{i+j}^{\delta_{i,j+1}} t_i
\]

\[
(3) \quad [e_i, f_j] = \delta_{i,j} \frac{v^i - v^{-i}}{v - v^{-1}} \quad \xi_i = t_i t_{i+1}^{-1}
\]

\[
(4) \quad [e_i, e_j] = [f_i, f_j] = 0 \quad (|i - j| > 1)
\]

\[
(5) \quad [e_i, [e_i, e_{i+1}]] = [f_i, [f_i, f_{i+1}]] = [a, b] = ab - ba
\]

The subalgebra generated by \( \xi_i, \xi_i^{-1}, e_i, f_i (1 \leq i \leq n - 1) \) is isomorphic to \( U_v(\mathfrak{sl}_n) \). We denote by \( U_v(\mathfrak{gl}_n)_{\leq 0} \) the subalgebra of \( U_v(\mathfrak{gl}_n) \) generated by \( t_i, t_i^{-1}, f_i \). It acts on the field \( \mathbb{C}(\mathcal{T} \times C^*) \) as follows: \( f_i \) acts trivially for any \( 1 \leq i \leq n - 1 \), and \( t_i \) acts by multiplication by \( t_i v^{-i} \). We define the universal Verma module \( \mathcal{M} \) over \( U_v(\mathfrak{gl}_n) \) as \( \mathcal{M} := U_v(\mathfrak{gl}_n \otimes U_v(\mathfrak{gl}_n)_{\leq 0} \mathbb{C}(\mathcal{T} \times C^*) \).

We define the following operators on \( M \):

\[
(6) \quad t_i = t_i v^{d_i - d_{i+1} - 1} : M_d \rightarrow M_{d+1}
\]

\[
(7) \quad e_i = t_{i+1}^{d_{i+1} - d_i} t_i^{-1} : M_d \rightarrow M_{d-1}
\]

\[
(8) \quad f_i = -t_i^{d_i - d_{i+1} - 1} q_i (L_i \otimes p_i^*) : M_d \rightarrow M_{d+1}
\]

The following theorem is Theorem 2.12 of [2].
Theorem 2.9. These operators satisfy the relations in $U_q(\mathfrak{gl}_n)$, i.e., they give rise to the action of $U_q(\mathfrak{gl}_n)$ on $M$. Moreover, there is a unique isomorphism $\psi : M \rightarrow \mathcal{M}$ carrying $[O_{\mathfrak{sl}_n}] \in M$ to the lowest weight vector $1 \in \mathbb{C}(\bar{T} \times \mathbb{C}^*) \subset \mathcal{M}$.

Remark 2.10. These notations coincide with those from [2] (see Theorem 2.12 and Conjecture 3.7 of loc. cit.) after Chevalley involution.

2.11. Gelfand-Tsetlin basis of the universal Verma module. The construction of Gelfand-Tsetlin basis for the representations of quantum $\mathfrak{sl}_n$ goes back to M. Jimbo [7]. We will follow the approach of [10]. To a collection $\vec{d} = (d_{ij})$, $n-1 \geq i \geq j$ we associate a Gelfand-Tsetlin pattern $\Lambda = \Lambda(\vec{d}) := (\lambda_{ij})$, $n \geq i \geq j$ as follows: $v_{\lambda_{ij}} := t_j v^{j-1}$, $n \geq j \geq 1$; $v_{\lambda_{ij}} := t_j v^{j-1-d_{ij}}$, $n-1 \geq i \geq j \geq 1$. Now we define $\xi_{\vec{d}} = \xi_{\Lambda} \in \mathcal{M}$ by the formula (5.12) of [11]. According to Proposition 5.1 of loc. cit., the set $\{\xi_{\vec{d}}\}$ (over all collections $\vec{d}$) forms a basis of $\mathcal{M}$.

According to the Thomason localization theorem, restriction to the $\bar{T} \times \mathbb{C}^*$-fixed point set induces an isomorphism

$$K^{\bar{T} \times \mathbb{C}^*}(\Omega_{\vec{d}}) \otimes_{K^{\bar{T} \times \mathbb{C}^*(pt)}} \text{Frac}(K^{\bar{T} \times \mathbb{C}^*}(\Omega_{\vec{d}})) \rightarrow K^{\bar{T} \times \mathbb{C}^*}(\Omega_{\vec{d}}) \otimes_{K^{\bar{T} \times \mathbb{C}^*(pt)}} \text{Frac}(K^{\bar{T} \times \mathbb{C}^*}(pt))$$

The structure sheaves $\mathcal{M}_{\vec{d}}$ of the $\bar{T} \times \mathbb{C}^*$-fixed points $\vec{d}$ (see [2,4]) form a basis in $\oplus_{\vec{d}} K^{\bar{T} \times \mathbb{C}^*}(\Omega_{\vec{d}}) \otimes_{K^{\bar{T} \times \mathbb{C}^*(pt)}} \text{Frac}(K^{\bar{T} \times \mathbb{C}^*}(pt))$. The embedding of a point $\vec{d}$ into $\Omega_{\vec{d}}$ is a proper morphism, so the direct image in the equivariant K-theory is well defined, and we will denote by $\{\mathcal{M}_{\vec{d}}\} \in M_{\vec{d}}$ the direct image of the structure sheaves of the point $\vec{d}$. The set $\{\mathcal{M}_{\vec{d}}\}$ forms a basis of $\mathcal{M}$.

The following theorem is Theorem 6.3 of [3] and Corollary 2.20 of [2].

Theorem 2.12. a) The isomorphism $\Psi : M \rightarrow \mathcal{M}$ of Theorem 2.9 takes $\{\mathcal{M}_{\vec{d}}\}$ to

$$(v^2 - 1)^{-\frac{1}{12}} \prod_j \sum_{i \geq j} d_{ij} v^{\sum_i i d_i - |d_{ij}| - \frac{1}{12} - \frac{d_{ij}^2}{2}} \xi_{\vec{d}}$$

b) The matrix coefficients of the operators $e_i, f_i$ in the fixed point basis $\{\mathcal{M}_{\vec{d}}\}$ of $M$ are as follows:

$$\xi_{\vec{d}(\vec{d}_{\vec{d}})} = -t_{i}^{-1} v^{d_{i}-d_{i-1}+1} t_{j}^{2} v^{-2d_{i,j}} \times$$

$$(1 - v^2)^{-1} \prod_{j \neq k \leq i} (1 - t_{j}^{2}k^{-2} v^{2d_{i,k}-2d_{i,j}}) \prod_{k \leq i-1} (1 - t_{j}^{2}k^{-2} v^{2d_{i-1,k}-2d_{i,j}})$$

if $\vec{d}_{i,j} = d_{i,j} + 1$ for certain $j \leq i$;

$$\xi_{\vec{d}(\vec{d}_{\vec{d}})} = t_{i+1}^{-1} v^{d_{i+1}-d_{i-1}+1} \times$$

$$(1 - v^2)^{-1} \prod_{j \neq k \leq i} (1 - t_{j}^{2}k^{-2} v^{2d_{i,j}-2d_{i,k}}) \prod_{k \leq i+1} (1 - t_{j}^{2}k^{-2} v^{2d_{i,j}-2d_{i+1,k}})$$

if $\vec{d}_{i,j} = d_{i,j} - 1$ for certain $j \leq i$;

All the other matrix coefficients of $e_i, f_i$ vanish.
2.13. Quantum loop algebra $U_v(L\mathfrak{sl}_n)$. Let $(\theta_{kl})_{1 \leq k,l \leq n-1} = A_{n-1}$ stand for the Cartan matrix of $\mathfrak{sl}_n$. For the quantum loop algebra $U_v(L\mathfrak{sl}_n)$ we follow the notations of \cite{12}. Namely, the quantum loop algebra $U_v(L\mathfrak{sl}_n)$ is an associative algebra over $\mathbb{Q}(v)$ generated by $e_{k,r}, f_{k,r}, v^h, h_{k,m} (1 \leq k,l \leq n-1, r \in \mathbb{Z}, m \in \mathbb{Z} \setminus \{0\})$ with the following defining relations:

\begin{align*}
\psi_i^+(z)\psi_i^+(w) &= \psi_i^+(w)\psi_i^+(z), \\
(z-v^{\pm ak_l}w)\psi_i^+(z)x_i^+(w) &= x_i^+(w)\psi_i^+(z)(v^{\pm ak_l}z-w), \\
[x_i^+(z), x_i^-(w)] &= \frac{\delta_{kl}}{v-v^{-1}}\{\delta(w/z)\psi_i^+(w) - \delta(z/w)\psi_i^-(z)\}, \\
(z-v^{\pm 2}w)x_i^+(z)x_i^+(w) &= x_i^+(w)x_i^+(z)(v^{\pm 2}z-w), \\
(z-v^{\pm ak_l}w)x_i^+(z)x_i^+(w) &= x_i^+(w)x_i^+(z)(v^{\pm ak_l}z-w), k \neq l, \\
\{x_i^+(z_1)x_i^+(z_2)x_i^+(w_1) - (v+v^{-1})x_i^+(z_1)x_i^+(z_2)x_i^+(w_1)x_i^+(z_2) + x_i^+(w_1)x_i^+(z_1)x_i^+(z_2)\} + \{z_1 \leftrightarrow z_2\} &= 0,
\end{align*}

where $s,s' = \pm$. Here $\delta(z), x_i^\pm(z), \psi_i^\pm(z)$ are generating functions defined as following

\begin{align*}
\delta(z) &:= \sum_{r=-\infty}^{\infty} z^r, x_i^+(z) := \sum_{r=-\infty}^{\infty} c_{k,r} z^{-r}, x_i^-(z) := \sum_{r=-\infty}^{\infty} f_{k,r} z^{-r}, \\
\psi_i^+(z) &:= v^{+h_k} \exp\left(\pm(v-v^{-1}) \sum_{m=1}^{\infty} h_{k,\pm m} z^{\pm m}\right).
\end{align*}

2.14. For any $0 \leq i \leq n$ we will denote by $\mathcal{W}_i$ the tautological $i$-dimensional vector bundle on $\mathbb{O}_\mathbb{P} \times \mathbb{C}$. By the Künneth formula we have $K^{T \times C^*}(\mathbb{O}_\mathbb{P} \times \mathbb{C}) = K^{T \times C^*}(\mathbb{O}_\mathbb{P}) \otimes 1 + K^{T \times C^*}(\mathbb{O}_\mathbb{P}) \otimes \tau$ where $\tau \in K^{T \times C^*}(\mathbb{C})$ is the class of $\mathcal{O}(1)$. Under this decomposition, for the exterior power $\Lambda^j(\mathcal{W}_i)$ we have $\Lambda^j(\mathcal{W}_i) =: \Lambda^j_{(j)}(\mathcal{W}_i) \otimes 1 + \Lambda^j_{(j-1)}(\mathcal{W}_i) \otimes \tau$ where $\Lambda^j_{(j)}(\mathcal{W}_i), \Lambda^j_{(j-1)}(\mathcal{W}_i) \in K^{T \times C^*}(\mathbb{O}_\mathbb{P})$.

For $0 \leq m \leq n$ we introduce the generating series $b_m(z)$ with coefficients in the equivariant $K$-theory of $\mathbb{O}_\mathbb{P}$ as follows: $b_m(z) := 1 + \sum_{1 \leq j \leq m} \left( \Lambda^j_{(j)}(\mathcal{W}_m) - v^j \Lambda^j_{(j-1)}(\mathcal{W}_m) \right) (-z)^{-j}$. In particular, $b_0(z) := 1$.

Let $v$ stand for the character of $T \times \mathbb{C}^* : (t,v) \mapsto v$. We define the line bundle $L'_k := v^k L_k$ on the correspondence $\mathbb{E}_{d,k}$, that is $L'_k$ and $L_k$ are isomorphic as line bundles but the equivariant structure of $L'_k$ is obtained from the equivariant structure of $L_k$ by the twist by the character $v^k$.

We also define the operators

\begin{align*}
e_{k,r} := t_{k+1}^{-1} v^{d_{k+1}-d_k+1-k} p_*((L'_k)^{\otimes r} \otimes q^*) : M_d \rightarrow M_{d-k}, \\
f_{k,r} := -t_{k}^{-1} v^{d_k-d_{k-1}+1+k} q_*(L_k \otimes (L'_k)^{\otimes r} \otimes p^*) : M_d \rightarrow M_{d+k}
\end{align*}

We consider the following generating series of operators on $M$:
(17) \[ x_k^+(z) = \sum_{r=-\infty}^{\infty} e_{k,r} z^{-r} : M_d \to M_{d-k}[[z,z^{-1}]] \]

(18) \[ x_k^-(z) = \sum_{r=-\infty}^{\infty} f_{k,r} z^{-r} : M_d \to M_{d+k}[[z,z^{-1}]] \]

(19) \[ \psi_k^+(z) |_{M_d} = \sum_{r=0}^{\infty} \psi_{k,r}^+ z^{-r} =: \\
= t_{k+1}^{-1} t_k v^{d_{k+1} - 2d_k + d_k - 1 - 1} (b_k(zv^{-k-2})^{-1} b_k(zv^{-k})^{-1} b_{k-1}(zv^{-k-2}))^+ \in M_d[[z]] ; \]

(20) \[ \psi_k^-(z) |_{M_d} = \sum_{r=0}^{\infty} \psi_{k,r}^- z^{-r} := \\
= t_{k+1}^{-1} t_k v^{d_{k+1} - 2d_k + d_k - 1 - 1} (b_k(zv^{-k-2})^{-1} b_k(zv^{-k})^{-1} b_{k-1}(zv^{-k-2}))^- \in M_d[[z]] ; \]

where \((\cdot)^\pm\) denotes the expansion at \(z = \infty, 0\) respectively.

**Theorem 2.15.** These generating series of operators \(\psi_k^\pm(z), x_k^\pm(z)\) on \(M\) defined in [27, 24] satisfy the relations in \(U_v(\mathfrak{sl}_n)\), i.e. they give rise to the action of \(U_v(\mathfrak{sl}_n)\) on \(M\).

**Remark 2.16.** For the quantum group \(U_v(\mathfrak{sl}_n)\) (generated by \(e_{k,0}, f_{k,0}, \psi_k^\pm\) in \(U_v(\mathfrak{sl}_n)\)) we get formulas [3, 8]. Formulas (17, 20) are very similar to those for equivariant cohomology in [4].

**Definition 2.17.** We assign to a collection \(\mathbf{d}\) a collection of weights \(s_{ij} := t^2 v^{-2d_{ij}}\).

**Proposition 2.18.** a) The matrix coefficients of the operators \(f_{i,r}, e_{i,r}\) in the fixed points basis \(\{\mathbf{d}\}\) of \(M\) are as follows:

\[ f_{i,v}^{\mathbf{d}} = -t_i^{-1} v^{d_i - d_i - 1 - 1 + s_{i,j}} (s_{i,j} v^i) (1 - v^2)^{-1} \prod_{j \neq k \leq i} (1 - s_{i,j} s_{i,k}^{-1}) \prod_{k \leq i - 1} (1 - s_{i,j} s_{i-1,k}^{-1}) \]

if \(d_{i,j} = d_{i,j} + 1\) for certain \(j \leq i\);

\[ e_{i,v}^{\mathbf{d}} = t_i^{-1} v^{d_i - d_i - 1 + i - s_{i,j} v^i} (1 - v^2)^{-1} \prod_{j \neq k \leq i} (1 - s_{i,k} s_{i,j}^{-1}) \prod_{k \leq i + 1} (1 - s_{i+1,k} s_{i,j}^{-1}) \]

if \(d_{i,j} = d_{i,j} - 1\) for certain \(j \leq i\);

All the other matrix coefficients of \(e_{i,r}, f_{i,r}\) vanish.

b) The eigenvalue of \(\psi_k^\pm(z)\) on \(\{\mathbf{d}\}\) equals

\[ t_{i+1}^{-1} t_i v^{d_{i+1} - d_i - 1 - 1} \prod_{j \leq i} (1 - z^{-1} v^{i+2} s_{i,j})^{-1} (1 - z^{-1} v^i s_{i,j})^{-1} \prod_{j \leq i - 1} (1 - z^{-1} v^{i+2} s_{i+1,j}) \prod_{j \leq i - 1} (1 - z^{-1} v^i s_{i-1,j}) \]

**Proof.** a) Follows directly from Theorem 2.12).

b) Follows from the exactness of functor \(\Lambda_{z}(L) := \sum_{j=0}^{\text{rank}(L)} z^j \Lambda^j L\) on the category of coherent sheaves and the fact that \(\{s_{i,j}\}_{j \leq i}\) is the set of characters of \(\mathfrak{T} \times \mathbb{C}^*\) in the stalk of \(\mathbf{W}\) at the fixed point \(\{\mathbf{d}, 0\} \in \Omega_\mathbf{d} \times \mathbb{C}^*\). \(\square\)
2.19. Now we formulate a corollary which will be used in Section 3.

For any $0 \leq m < i \leq n$ we will denote by $\mathcal{W}_m$, the quotient $\mathcal{W}/\mathcal{W}_m$ of the tautological vector bundles on $\Omega^\tau \times C$. Under the Künneth decomposition, for the exterior power $\Lambda^j(\mathcal{W}_m)$ we have

$$\Lambda^j(\mathcal{W}_m) =: \Lambda^j(\mathcal{W}_m) \otimes 1 + \Lambda^j(\mathcal{W}_m) \otimes \tau$$

where $\Lambda^j(\mathcal{W}_m), \Lambda^j(\mathcal{W}_m) \in K^T \otimes C(\Omega^\tau)$. We introduce the generating series $b_m(z) := 1 + \sum_{j=1}^{\infty} \left( \Lambda^j(\mathcal{W}_m) - v\Lambda^j(\mathcal{W}_m) \right)(-z)^{-j}$. 

**Corollary 2.20.** $\psi^\pm_i(z) |_{\mathcal{W}_m} = t_{i+1}^{-1} t_i v^{d_i+1-2d_i+d_i-1} (b_m(zv^{-i-2}) - b_m(zv^{-i-1}) b_{m-1}(zv^{-i-2}))^{\pm}$ for any $m < i$.

**Proof.** Let $\Lambda_z(L) := \sum_{j=0}^{\text{rank}(L)} z^j \Lambda^j L$. Then on the one hand

$$\Lambda_z(\mathcal{W}_j) = \Lambda_z(\mathcal{W}_m) \Lambda_z(\mathcal{W}_{mj}),$$

while on the other hand

$$\Lambda_{-1/z}(\mathcal{W}_j) \mid_{r=-v} = b_j(z), \quad \Lambda_{-1/z}(\mathcal{W}_{mj}) \mid_{r=-v} = b_{mj}(z), \quad \Lambda_{-1/z}(\mathcal{W}_m) \mid_{r=-v} = b_m(z).$$

$\square$

3. PROOF OF THEOREM 2.15

Let us check equation (12) firstly. We will prove it for $x_k^-$ (case $x_k^+$ is entirely analogous).

**Proof.** We need to check for any integers $a, b$: $f_{i,a+1} f_{i,b} - v^{-2} f_{i,a} f_{i,b+1} = v^{-2} f_{i,b} f_{i,a+1} - f_{i,b+1} f_{i,a}$. Let us compute both sides in the fixed points basis:

a) $\tilde{[\tilde{d}, \tilde{d}]} = [\tilde{d} + \delta_{i,j_1} + \delta_{i,j_2}] (j_1 \neq j_2)$.

$$
\left( f_{i,a+1} f_{i,b} - v^{-2} f_{i,a} f_{i,b+1} \right) [\tilde{d}] = P v^{(a+b+1)} 
\left[ s_{i,j_1}^{a+1} (1 - s_{i,j_1} s_{i,j_2})^{-1} (1 - v^2 s_{i,j_2} s_{i,j_1})^{-1} - v^{-2} s_{i,j_1} s_{i,j_2} (1 - s_{i,j_1} s_{i,j_2})^{-1} (1 - v^2 s_{i,j_2} s_{i,j_1})^{-1} + \{ j_1 \leftrightarrow j_2 \} \right]
$$

Similarly

$$
\left( v^{-2} f_{i,b} f_{i,a+1} - f_{i,b+1} f_{i,a} \right) [\tilde{d}] = P v^{(a+b+1)} 
\left[ (v^{-2} s_{i,j_1}^{a+1} - s_{i,j_1} s_{i,j_2}^{b-1})(1 - s_{i,j_1} s_{i,j_2})^{-1} (1 - v^2 s_{i,j_2} s_{i,j_1})^{-1} + \{ j_1 \leftrightarrow j_2 \} \right].
$$

Here

$$P = (-t_i)^{-2} v^{2d_i-2d_{i-1}+2i-1} s_{i,j_1} s_{i,j_2} \times
(1 - v^2)^{-2} \prod_{j_1,j_2 \neq k \leq i} (1 - s_{i,j_1} s_{i,k}^{-1})^{-1} (1 - s_{i,j_2} s_{i,k}^{-1})^{-1} \prod_{k \leq i-1} (1 - s_{i,j_1} s_{i-1,k}^{-1})(1 - s_{i,j_2} s_{i-1,k}^{-1}).$$

So we have to prove that

$$
(s_{i,j_1}^{a+1} - v^{-2} s_{i,j_1}^{a+1} s_{i,j_2} - v^{-2} s_{i,j_1} s_{i,j_2}^{b-1} + s_{i,j_1} s_{i,j_2}^{b-1})(1 - s_{i,j_1} s_{i,j_2})^{-1} (1 - v^2 s_{i,j_2} s_{i,j_1})^{-1} =
$$

$$(s_{i,j_1}^{a+1} s_{i,j_2} - s_{i,j_2}^{a-1})(s_{i,j_1} - v^{-2} s_{i,j_1}) s_{i,j_2} (s_{i,j_2} - v^{-2} s_{i,j_2}) s_{i,j_1} (s_{i,j_1} - v^{-2} s_{i,j_1})^{-1} (s_{i,j_2} - v^{-2} s_{i,j_2})^{-1} =
$$

$$
= \frac{s_{i,j_1} s_{i,j_2} (s_{i,j_1} s_{i,j_2}^{b-1} + s_{i,j_2} s_{i,j_1}^{b-1})}{v^2 (s_{i,j_1} - s_{i,j_2})}
$$

is antisymmetric with respect to $\{ j_1 \leftrightarrow j_2 \}$ which is obvious.

b) $\tilde{[\tilde{d}, \tilde{d}']} = \tilde{d} + 2\delta_{i,j_1}$.

In this case define

$$P' := (-t_i)^{-2} v^{2d_i-2d_{i-1}+2i-3} s_{i,j_1} s_{i,j_2} \times$$

...
\[
(1 - v^2)^{-2} \prod_{j_1 \neq k \leq i} (1 - s_{i,j_1} s_{i,k}^{-1})^{-1} (1 - s_{i,j_2} s_{i,k}^{-1})^{-1} \prod_{k \leq i-1} (1 - s_{i,j_1} s_{i,k}^{-1})(1 - s_{i,j_2} s_{i,k}^{-1}).
\]

Then:
\[
(f_{i,a+1} f_{i,b} - v^{-2} f_{i,a} f_{i,b+1}) (\widetilde{\mathbf{d}} - \mathbf{d}) = P' v^{i(a+b+1)} s_{i,j_1} s_{i,j_2}^{-1} \left( v^{-2(a+1)} - v^{-2b} \right) = 0 = (v^{-2} f_{i,b} f_{i,a+1} - f_{i,b+1} f_{i,a}) (\widetilde{\mathbf{d}} - \mathbf{d}).
\]

So the equality holds again. \(\square\)

Let us check equation (13) now. We will prove it only for \(x_k^-\) again.

**Proof.** If \(|k - l| > 1\) then it is obvious that in the fixed points basis the formulas are the same.

So let us check it for \(l = i + 1, k = i\), i.e. for any integers \(a, b\): \(f_{i,a+1} f_{i,b} - v f_{i,a} f_{i,b+1} = v f_{i+1,b} f_{i,a+1} - f_{i+1,b+1} f_{i,a}\).

Let us compute both sides in the fixed points basis:

a) \(\mathbf{d} = \mathbf{d} + \delta_{i,j_1} + \delta_{i+1,j_2} \) (\(j_1 \neq j_2\)).

\[
(f_{i,a+1} f_{i,b} - v f_{i,a} f_{i,b+1}) (\widetilde{\mathbf{d}} - \mathbf{d}) = P' v^{i(a+b+1)} s_{i,j_1} s_{i,j_2}^{-1} \left( v^{-i(a+1)+(i+1)b} s_{i+1,j_2} s_{i,j_1} - v^{i(a+1)+(i+1)(b+1)} s_{i+1,j_2} s_{i,j_1} \right),
\]

where
\[
P = \prod_{j_1 \neq k \leq i} (1 - s_{i,j_1} s_{i,k}^{-1})^{-1} \prod_{k \leq i-1} (1 - s_{i,j_2} s_{i,k}^{-1})^{-1} \prod_{j_1 \neq k \leq i} (1 - s_{i,j_2} s_{i,k}^{-1}).
\]

After dividing both right hand sides by \(P s_{i,j_1} s_{i+1,j_2} v^{i(a+1)b}\) we get an equality:
\[
v (v s_{i,j_1} - v^{i+2} s_{i+1,j_2}) (s_{i,j_2} - s_{i+1,j_2}) = (v^{i+1} s_{i,j_1} - v^{i+1} s_{i+1,j_2}) (s_{i,j_2} - v^2 s_{i+1,j_2}).
\]

b) \(\mathbf{d} = \mathbf{d} + \delta_{i,j_1} + \delta_{i+1,j_2}\).

Similarly we get
\[
(v f_{i+1,b} f_{i,a+1} - f_{i+1,b+1} f_{i,a}) (\widetilde{\mathbf{d}} - \mathbf{d}) = P' v^{i(a+b+1)} (1 - v^{-2d_{i,j} - 2d_{i+1,j}}) \times
\]
\[
\left[ v^{i(a+1)+(i+1)b} v^{-2d_{i,j} - 2a(i+1)d_{i,j}} - v^{i(a+1)(b+1)+1} v^{-2(b+1)d_{i,j} - 2a(i+1)d_{i,j}} \right] = P' v^{i(a+b+1)} v^{i(a+1)+(i+1)b} v^{-2d_{i,j} - 2d_{i+1,j}} (v^{-2d_{i,j}} - v^{-2d_{i+1,j}}) v(1 - v^{-2d_{i,j} - 2d_{i+1,j}}).
\]

Let us check equation (11) for the case \(k \neq l\).

**Proof.** We have to show for any integers \(a, b\) the equality \(e_{k,a} f_{i,b} = f_{i,b} e_{k,a}\) holds.

If \(|k - l| > 1\) then this equation is obvious, since matrix elements in the fixed points basis are the same. So let us check the only nontrivial case: \(k = 1, l = i + 1\) (the pair \(k = i + 1, l = i\) is analogous). We consider the pair of fixed points \(\mathbf{d} = \mathbf{d} + \delta_{i,j_1} + \delta_{i+1,j_2}\) (here \(j_1, j_2\) might be equal).

\[
e_{i,a} f_{i+1,b} (\widetilde{\mathbf{d}} - \mathbf{d}) = P (1 - s_{i+1,j_2} s_{i,j_1}^{-1}) (1 - v^{-2} s_{i+1,j_2} s_{i,j_1}^{-1}),
\]
\[ f_{i+1,b} e_{i,a} | \tilde{\mathcal{D}} \tilde{\mathcal{D}} | = P(1 - v^{-2}s_{i+1,j_2}s_{i,j_1}^{-1})(1 - s_{i+1,j_2}s_{i,j_1}^{-1}), \]

where

\[ P = -t^{i-1} t^{i+1} v^{2d_1 + d_2 - 2d_1 - 3s_{i+1,j_2}^{-1}} \prod_{j_1 \neq k \leq i} (1 - s_{i+1,j_2}s_{i,j_1}^{-1})^{-1} \prod_{j_1 \neq k \leq i} (1 - s_{i,j_1}^{-1}s_{i,k})^{-1} \prod_{j_2 \neq k \leq i+1} (1 - s_{i,j_1}s_{i+1,k}). \]

This completes the proof of equation (11). \hfill \Box

Let us check equation (14) fourthly. We will prove it only for \( x_k^- \) again.

**Proof.** We have to prove for any integers \( a, b, c \) and \( j = i \pm 1 \) the equality holds:

\[ \{f_i a f_i c f_i b - (v + v^{-1}) f_i a f_i c f_i b + f_i c f_i a f_i b\} + \{a \leftarrow b\} = 0. \]

Let us consider the case \( j = i + 1 \) (the second case is similar).

We will show that matrix elements in the fixed points basis of the first bracket is antisymmetric with respect to a change \( \{a \leftarrow b\} \).

a) \[ \tilde{f}_{i,a} \tilde{f}_{i,b} \tilde{f}_{i+1,c} | \tilde{\mathcal{D}} \tilde{\mathcal{D}} \mid = \]

\[ P v^2 [s_{i,j_1}^a s_{i,j_2}^b (1 - s_{i+1,j_3}s_{i,j_2}^{-1})(1 - s_{i+1,j_3}s_{i,j_1}^{-1})^{-1}(1 - v^2 s_{i,j_1}s_{i,j_2}^{-1})^{-1} + \{j_1 \leftarrow j_2\}], \]

\[ \tilde{f}_{i,a} \tilde{f}_{i+1,c} \tilde{f}_{i,b} | \tilde{\mathcal{D}} \tilde{\mathcal{D}} \mid = \]

\[ P v [s_{i,j_1}^a s_{i,j_2}^b (1 - v^2 s_{i+1,j_3}s_{i,j_2}^{-1})(1 - s_{i,j_2}^{-1}s_{i,j_1}^{-1})^{-1}(1 - v^2 s_{i,j_1}s_{i,j_2}^{-1})^{-1} + \{j_1 \leftarrow j_2\}], \]

\[ \tilde{f}_{i+1,c} \tilde{f}_{i,a} \tilde{f}_{i,b} | \tilde{\mathcal{D}} \tilde{\mathcal{D}} \mid = \]

\[ P [s_{i,j_1}^a s_{i,j_2}^b (1 - v^2 s_{i+1,j_3}s_{i,j_2}^{-1})(1 - s_{i,j_2}^{-1}s_{i,j_1}^{-1})^{-1}(1 - v^2 s_{i,j_1}s_{i,j_2}^{-1})^{-1} + \{j_1 \leftarrow j_2\}]. \]

Thus:

\[ (f_i a f_i c f_i b - (v + v^{-1}) f_i a f_i c f_i b + f_i + c f_i a f_i b) | \tilde{\mathcal{D}} \tilde{\mathcal{D}} \mid = P s_{i,j_1}^a s_{i,j_2}^b \times \]

\[ \frac{(v^2 (s_{i,j_2} - s_{i,j_1})(s_{i,j_1} - s_{i+1,j_3}))}{(s_{i,j_1} - s_{i,j_2})(s_{i,j_2} - v^2 s_{i,j_1})} + \]

\[ \frac{(s_{i,j_1} - s_{i,j_2})(s_{i+1,j_3} - s_{i,j_2})}{(s_{i,j_1} - s_{i,j_2})(s_{i,j_2} - v^2 s_{i,j_1})} \]

\[ \{j_1 \leftarrow j_2\} = P(1 - v^2) \frac{s_{i,j_3} s_{i,j_1} s_{i,j_2}^a}{s_{i,j_1} - s_{i,j_2}} + \{j_1 \leftarrow j_2\} = P s_{i+1,j_3}(1 - v^2) \frac{s_{i,j_3} s_{i,j_1} s_{i,j_2}^b}{s_{i,j_1} - s_{i,j_2}}, \]

where

\[ P = -t^{i-1} t^{i+1} v^{2d_1 + d_2 - 2d_1 - 3s_{i+1,j_2}^{-1}} \prod_{j_1 \neq k \leq i} (1 - s_{i+1,j_2}s_{i,j_1}^{-1})^{-1} \prod_{j_1 \neq k \leq i} (1 - s_{i,j_1}^{-1}s_{i,k})^{-1} \prod_{j_2 \neq k \leq i+1} (1 - s_{i,j_1}s_{i+1,k}). \]

We see that

\[ f_i a f_i c f_i b - (v + v^{-1}) f_i a f_i c f_i b + f_i c f_i a f_i b | \tilde{\mathcal{D}} \tilde{\mathcal{D}} \mid = P s_{i+1,j_3}(1 - v^2) \frac{s_{i,j_3} s_{i,j_1} s_{i,j_2}^b}{s_{i,j_1} - s_{i,j_2}} \]
is antisymmetric with respect to $a \leftrightarrow b$.

b) $[\bar{d}, \bar{d}] = \bar{d} + 2\delta_{i,j_2} + \delta_{i+j_2}$.

By the same calculation one gets:

$$(f_1,a f_1 b f_1 + (v + v^{-1}) f_1 c f_1, b + f_1 c f_1 b) = 0,$$

where

$$P' \left[ \omega^2 (1 - s_{i+j_2} s_{i,j_1}^{-1}) - (1 + v^2) (1 - v^2 s_{i+j_2} s_{i,j_1}^{-1}) + (1 - v^4 s_{i+j_2} s_{i,j_1}^{-1}) \right] = 0,$$

Finally by showing that the subalgebra generated by formulas are already known). Let us further omit

This completes the proof of equation (14).

We will show that $\varphi_k^\pm (z)$ are determined uniquely by the conditions $\varphi_k^\pm (z) |_{M_d} = t_{i-1} v^d, i = 1, \ldots, n, z \neq 0, w \neq 0, s_{i, j_2}$ (we get these formulas from the fact that the subalgebra generated by $v^\pm, E_k, F_k$ is a quantum group $U_v(s_{i,j_2})$, for which the formulas are already known). Let us further omit $|_{M_d}$ for brevity. Next we will check

$$\varphi_k^\pm (z) \varphi_k^\pm (w) = \varphi_k^\pm (w) \varphi_k^\pm (z),$$

Finally by showing that $\varphi_k^\pm (z) = \varphi_k^\pm (z)$ we will get equations (11) from equations (21, 23). And so Theorem 2.15 will be proved.

From Proposition 2.18 one gets that $(v - v^{-1}) [x_i^+(z), x_i^-(w)]$ is diagonalizable in the fixed points basis and moreover its eigenvalue at $\{\bar{d}\}$ equals to

$$\sum_{a, b \in \mathbb{Z}} z^{-a} w^{-b} \chi_{i,a+b},$$

where

$$\chi_{i,a} = -t_{i-1}^{i-1} v^d, i = 1, \ldots, n, z \neq 0, w \neq 0, s_{i, j_2}.$$
So as we want an equality \((v - v^{-1})[x^+_i(z), x^-_i(w)] = \delta \left( \frac{z}{w} \right) \varphi^+_i(w) - \delta \left( \frac{w}{z} \right) \varphi^-_i(z) = \sum_{a,b|a+b>0} z^{-a}w^{-b} \varphi^+_{i,a+b} - \sum_{a,b|a+b>0} z^{-a}w^{-b} \varphi^-_{i,a+b} + \sum_{a,b|a+b>0} z^{-a}w^{-b}(\varphi^+_i - \varphi^-_i)\)
to hold, we determine \(\varphi^+_{i,s>0}, \varphi^-_{i,s<0}, \varphi^+_{i,s=0} - \varphi^-_{i,s=0}\) uniquely as they are equal to the corresponding \(\chi_{i,s}\). But we know from [2] that for \(\varphi^+_{i,0} = t_i t_{i+1}^{-1} v^d_{i+1-2d_i+1-d_{i-1}}, \varphi^-_{i,0} = t_i^{-1} t_{i+1} v^{-d_{i+1}+2d_i-d_{i-1}+1}\) the above equality for \(\varphi^+_{i,0} - \varphi^-_{i,0}\) holds. So we have determined all coefficients of the series \(\varphi^+_i(z)\).

Let us note that as all operators \(\varphi^\pm_{i,s}\) are diagonalizable in the fixed points basis the equation \((22)\) holds automatically. So let us check the equation \((23)\), i.e.

\[(z - v'^{a_k}w)\varphi^+_i(z)x^+_k(w) = x^+_k(w)\varphi^+_i(z)(v'^{a_k}z - w).\]

**Proof.** We claim that for \(k \neq l\) it follows directly from the equation \((13)\) and the construction of \(\varphi^+_i(z)\). So let us now check it for the case \(k = l, s = +, s' = -\).

Now we are computing the matrix elements of both sides in the fixed points basis on the pair \(\{d, d'\} = d + \delta_{i,p} < d\).

Let us notice that \(f_{i,b+1} |(\delta_{i,d}) = f_{i,b} |(\delta_{i,d}) \cdot s_{i,p} v^i\). And as \(\varphi^+_{i,s>0}\) are diagonalizable in the fixed points basis the only thing we have to check is the following equality for any \(a \geq 0\):

\[a) a > 0. \text{ Here we use the notations of Proposition 2.21 (2).}\]

Namely \(q := v^2, s_j := s_{ij} = t_{i}^{-1} v^{-2d_{i+1}}, r_k := t_{i}^{-1} v^{-2d_{i+1}}, k, p_k := t_{i}^{-1} v^{-2d_{i+1}}.\)

Then

\[
\varphi^+_{i,a} \mid d = v^a \prod_{j \leq i} s_j \prod_{k \leq i-1} p_k^{-1} \left( \sum_{j \leq i} s_j^{-2} \prod_{k \leq i+1} (s_j - qr_k) \prod_{k \leq i-1} (p_k - s_j) \prod_{j \neq k \leq i} ((s_j - qs_k)^{-1} - (s_k - s_j)^{-1}) s_j^a \right) \\
q \sum_{j \leq i} s_j^{-2} \prod_{k \leq i+1} (s_j - r_k) \prod_{k \leq i} (p_k - s_j) \prod_{j \neq k \leq i} ((s_j - qs_k)^{-1} - (s_k - s_j)^{-1}) s_j^a \\
q \prod_{j \neq k \leq i} ((s_j - qs_k)^{-1} - (s_k - s_j)^{-1}) s_j^a \\
s_p^{-2} q \prod_{k \leq i+1} (q^{-1} s_p - qr_k) \prod_{k \leq i-1} (p_k - q^{-1} s_p) \prod_{p \neq k \leq i} ((q^{-1} s_p - qs_k)^{-1} - (s_k - q^{-1} s_p)^{-1}) s_p^a \\
s_p^{-2} q \prod_{k \leq i+1} (q^{-1} s_p - r_k) \prod_{k \leq i-1} (p_k - s_p) \prod_{p \neq k \leq i} ((q^{-1} s_p - s_p)^{-1} - (s_p - s_p)^{-1}) s_p^a \right) .
\]

Hence:
It is straightforward to check that these two expressions coincide.

b) \(a = 0\). In this case we know that \(\varphi^+_{i,0} = \chi_{i,0} + \varphi^-_{i,0}\). The argument of a) shows

\[
(\chi_{i,1} - v^{-2} s_{i,p} v^t \chi_{i,0}) |_{\tilde{d} + \delta_{i,p}} = (v^{-2} \chi_{i,1} - s_{i,p} v^t \chi_{i,0}) |_{\tilde{d}}.
\]

So in order to prove

\[
(\varphi^+_{i,1} - v^{-2} s_{i,p} v^t \varphi^+_{i,0}) |_{\tilde{d} + \delta_{i,p}} = (v^{-2} \varphi^+_{i,1} - s_{i,p} v^t \varphi^+_{i,0}) |_{\tilde{d}}
\]

it is enough to check \(v^{-2} \varphi^+_{i,0} |_{\tilde{d} + \delta_{i,p}} = \varphi^+_{i,0} |_{\tilde{d}}\), which follows directly from the formula \(\varphi^+_{i,0} = t_{i,1}^{-1} t_{i+1} v^{-d_{i+1}+2d_i} - d_{i-1}+1\).

Now we rewrite the formulas for \(\varphi^+_i(z)\). From equation (23), we get for any \(a > 0\):

\[
(\varphi^+_{i,a+1} - v^{-a_k} t_p^{2} v^{-2d_k} t_p^k \varphi^+_{i,a}) |_{\tilde{d} + \delta_{k,p}} = (v^{-a_k} t_p^{2} v^{-2d_k} t_p^k \varphi^+_{i,a}) |_{\tilde{d}}
\]

i.e.

\[
\varphi^+_i(z) \left(1 - t_p^{2} v^{-a_k} t_p^{2} v^{-2d_k} t_p^k z^{-1}\right) |_{\tilde{d} + \delta_{k,p}} = \varphi^+_i(z) \left(v^{-a_k} t_p^{2} v^{-2d_k} t_p^k z^{-1}\right) |_{\tilde{d}}.
\]

This equation is especially interesting whenever \(a_k t_p \neq 0\) giving the following information:

\[
\frac{\varphi^+_i(z) |_{\tilde{d} + \delta_{i+1,p}}}{\varphi^+_i(z) |_{\tilde{d}}} = v \frac{1 - z^{-1} t_p^{2} v^{-2d_{i+1}} t_p^k}{1 - z^{-1} t_p^{2} v^{-2d_{i+1}} t_p^k}.
\]
Let us compute $\varphi_i^+(z)|_{d_0}$, where $d_0 = (d_{i,j} = 0)_{i,j}$. In this case the expression for $\varphi_i^+(z)$ reads as:

$$
\varphi_i^+(z)|_{d_0} = t_{i+1}^{-1}t_i v^{-1} - t_{i+1}^{-1}t_i v^{-1} (v^2 - 1)^{-1} t_i^2 \times
\sum_{a \geq 1} \prod_{k \leq i-1} (1 - t_i^2 t_k^{-2})^{-1} \prod_{k \leq i-1} (1 - t_i^2 t_k^{-2}) \prod_{k \leq i+1} (1 - v^2 t_k^2 t_i^{-2})(t_i^2 v^i z^{-1})^a
\times t_{i+1}^{-1}t_i v^{-1} - t_{i+1}^{-1}t_i v^{-1} (v^2 - 1)^{-1} (1 - v^2)(1 - t_i^2 t^{-2} v^2) \frac{t_i^2 v^i z^{-1}}{1 - t_i^2 v^i z^{-1}} = t_{i+1}^{-1}t_i v^{-1} - t_{i+1}^{-1}t_i v^{-1} (v^2 - 1)^{-1} (1 - t_i^2 v^i z^{-1})^{-1}.
$$

So

$$
\varphi_i^+(z)|_{d_0} = t_{i+1}^{-1}t_i v^{-1} (1 - t_i^2 v^i z^{-1})^{-1}.
$$

From equations (24) and (27) we get a simple formula for $\varphi_i^+(z)$, explicitly:

$$
\varphi_i^+(z) = t_{i+1}^{-1}t_i v^{d_{i+1} - 2d_i - 1} \prod_{p \leq i} (1 - z^{-1} t_p^2 v^{-2d_i - 1}).
$$

Now from equations (28) and Proposition 2.18) we get $\varphi_i^+(z) = \psi_i^+(z)$. In the same way one gets $\varphi_i^-(z) = \psi_i^-(z)$.

**Theorem 2.15 is proved.**

4. Parabolic sheaves and quantum toroidal algebra

In this section we generalize the previous results to the affine setting.

4.1. **Parabolic sheaves.** We recall the setup of section 3 of [2]. Let $X$ be another smooth projective curve of genus zero. We fix a coordinate $y$ on $X$, and consider the action of $\mathbb{C}^*$ on $X$ such that $c(x) = c^{-2} x$. We have $X^{\mathbb{C}^*} = \{0, \infty\}$. Let $S$ denote the product surface $\mathbb{C} \times X$. Let $D_\infty$ denote the divisor $\mathbb{C} \times \infty \times \mathbb{C} \times X$. Let $D_0$ denote the divisor $\mathbb{C} \times 0_X$.

Given an $n$-tuple of nonnegative integers $d = (d_0, \ldots, d_{n-1})$, we say that a **parabolic sheaf** $F_*$ of degree $d$ is an infinite flag of torsion free coherent sheaves of rank $n$ on $S$:

$$
\ldots \subset F_{-1} \subset F_0 \subset F_1 \subset \ldots
$$

such that:

(a) $F_{k+n} = F_k(D_0)$ for any $k$;
(b) $ch_1(F_k) = k[D_0]$ for any $k$: the first Chern classes are proportional to the fundamental class of $D_0$;
(c) $ch_2(F_k) = d_i$ for $i \equiv k \pmod{n}$;
(d) $F_0$ is locally free at $D_0$ and trivialized at $D_\infty$ : $F_0|_{D_\infty} = W \otimes O_{D_\infty}$;
(e) For $-n \leq k \leq 0$ the sheaf $F_k$ is locally free at $D_\infty$, and the quotient sheaves $F_k/F_{-n}$, $F_0/F_k$ (both supported at $D_0 = \mathbb{C} \times 0_X \subset S$) are both locally free at the point.
of degree \( \omega_{C \times 0_X} \); moreover, the local sections of \( \mathcal{F}_k|_{\infty_C \times X} \) are those sections of \( \mathcal{F}_0|_{\infty_C \times X} = W \otimes \mathcal{O}_X \) which take value in \( \langle w_1, \ldots, w_{n-k} \rangle \subset W \) at \( 0_X \in X \).

The fine moduli space \( \mathcal{P}_d \) of degree \( d \) parabolic sheaves exists and is a smooth connected quasiprojective variety of dimension \( 2d_0 + \ldots + 2d_{n-1} \).

4.2. Fixed points. The group \( \tilde{T} \times \mathbb{C}^* \times \mathbb{C}^* \) acts naturally on \( \mathcal{P}_d \) and its fixed point set is finite. In order to describe it, we recall the well known description of the fixed point set of \( \mathbb{C}^* \times \mathbb{C}^* \) on the Hilbert scheme of \( (\mathbb{C} - \infty_C) \times (X - \infty_X) \). Namely, the fixed points are parametrized by the Young diagrams, and for a diagram \( \lambda = (\lambda_0 \geq \lambda_1 \geq \ldots) \) (where \( \lambda_N = 0 \) for \( N \gg 0 \)) the corresponding fixed point is the ideal \( J_\lambda = \mathbb{C} y^0 \mathbb{C} y^1 \mathbb{C} y^2 \mathbb{C} y^3 \mathbb{C} y^n \). We will view \( J_\lambda \) as an ideal in \( \mathcal{O}_{C \times X} \) coinciding with \( \mathcal{O}_{C \times X} \) in a neighborhood of infinity.

We say \( \lambda \supset \mu \) if \( \lambda_i \geq \mu_i \) for any \( i \geq 0 \). We say \( \lambda \equiv \mu \) if \( \lambda_i \equiv \mu_i + 1 \) for any \( i \geq 0 \).

We consider a collection \( \lambda = (\lambda_{kl})_{1 \leq k,l \leq n} \) of Young diagrams satisfying the following inequalities:

\[
(30) \quad \lambda^{11} \subset \lambda^{21} \subset \ldots \subset \lambda^{n1} \subset \lambda^1; \quad \lambda^{22} \subset \lambda^{32} \subset \ldots \subset \lambda^{12} \subset \lambda^{22}; \ldots; \quad \lambda^{nn} \subset \lambda^{1n} \subset \ldots \subset \lambda^{n-1,n} \subset \lambda^{nn}
\]

We set \( d_k(\lambda) = \sum_{l=1}^{n} |\lambda_{kl}| \), and \( d(\lambda) = (d_0(\lambda), \ldots, d_{n-1}(\lambda)) \).

Given such a collection \( \lambda \) we define a parabolic sheaf \( \mathcal{F}_\bullet = \mathcal{F}_\bullet(\lambda) \), or just \( \lambda \) by an abuse of notation, as follows: for \( 1 \leq k \leq n \) we set

\[
(31) \quad \mathcal{F}_{k-n} = \bigoplus_{1 \leq l \leq k} J_{\lambda_{kl}} w_l \oplus \bigoplus_{k < l \leq n} J_{\lambda_{kl}}(-D_0) w_l
\]

Lemma 4.3. The correspondence \( \lambda \mapsto \mathcal{F}_\bullet(\lambda) \) is a bijection between the set of collections \( \lambda \) satisfying (\ref{eq1}) such that \( d(\lambda) = d \), and the set of \( \tilde{T} \times \mathbb{C}^* \times \mathbb{C}^* \)-fixed points in \( \mathcal{P}_d \).

Proof. Evident. \( \square \)

4.4. We will now introduce a different realization of parabolic sheaves, and another parametrization of the fixed point set which is very closely related to this new realization. We first learned of this construction from A. Okounkov, though it is already present in the work of Biswas \cite{1}. Let \( \sigma : \mathbb{C} \times X \rightarrow \mathbb{C} \times X \) denote the map \( \sigma(z,y) = (z, y^n) \), and let \( G = \mathbb{Z}/n\mathbb{Z} \). Then \( G \) acts on \( \mathbb{C} \times X \) by multiplying the coordinate on \( X \) with the \( n \)-th roots of unity.

A parabolic sheaf \( \mathcal{F}_\bullet \) is completely determined by the flag of sheaves

\[
\mathcal{F}_0(-D_0) \subset \mathcal{F}_{-n+1} \subset \ldots \subset \mathcal{F}_0,
\]

satisfying conditions \( a-e \) above. To \( \mathcal{F}_\bullet \), we can associate a single, \( G \) invariant sheaf \( \tilde{\mathcal{F}} \) on \( \mathbb{C} \times X \):

\[
\tilde{\mathcal{F}} = \sigma^* \mathcal{F}_{-n+1} + \sigma^* \mathcal{F}_{-n+2}(-D_0) + \ldots + \sigma^* \mathcal{F}_0(-n+1)D_0).
\]

This sheaf will have to satisfy certain numeric and framing conditions that mimic conditions (\ref{eq1}) \( \text{b)-(e). } \) Conversely, any \( G \) invariant sheaf \( \tilde{\mathcal{F}} \) that satisfies those numeric and framing conditions will determine a unique parabolic sheaf.

If \( \mathcal{F}_\bullet \) is a \( \tilde{T} \times \mathbb{C}^* \times \mathbb{C}^* \) fixed parabolic sheaf corresponding to a collection \( \lambda \) as in the previous section, then we have

\[
(32) \quad \tilde{\mathcal{F}} = \bigoplus_{l=1}^{n} J_{\lambda^l}(-l+1)D_0) w_l,
\]
where \((\lambda^1, \ldots, \lambda^n)\) is a collection of partitions, given by
\[
\lambda_i^l = \lambda_{ni-n[i/n]}^{i-k-l} = \lambda_i^{kl}.
\]
Here \(\lfloor \frac{k-l}{n} \rfloor\) stands for the maximal integer smaller than or equal to \(\frac{k-l}{n}\).

For \(j \in \mathbb{Z}\), let \((j \bmod n)\) denote that element of \(\{1, \ldots, n\}\) which is congruent to \(j\) modulo \(n\). For \(i \geq j \in \mathbb{Z}\), if we denote
\[
d_{ij} = \lambda_{i-j}^{\bmod n}
\]
we obtain a collection \((d_{ij}) = \tilde{\mathcal{d}} = \tilde{\mathcal{d}}(\lambda)\) of non-negative integers with the properties that
\[
d_{kj} \geq d_{ij} \quad \forall i \geq j; \quad d_{i+n,j+n} = d_{ij} \quad \forall i \geq j; \quad d_{ij} = 0 \quad \text{for} \quad i-j \gg 0.
\]
For \(1 \leq k \leq n\), let us write
\[
d_k(d) = \sum_{j} d_{kj} = \sum_{l=1}^{n} \sum_{i \leq i \leq \lfloor \frac{k-l}{n} \rfloor} d_{l(i+n)} = \sum_{l=1}^{n} \sum_{i \geq 0} \lambda_{ni-n[i/n]}^{i-k-l} = \sum_{l=1}^{n} \sum_{i \geq 0} \lambda_{n,i}^{l} = d_k(\lambda).
\]

Summarizing the above discussion, we have:

**Lemma 4.5.** The correspondence \(\lambda \mapsto \tilde{\mathcal{d}}(\lambda)\) is a bijection between the set of collections \(\lambda\) satisfying (30), and the set \(D\) of collections \(\tilde{\mathcal{d}}\) satisfying (32). We have \(\tilde{\mathcal{d}}(\lambda) = \tilde{\mathcal{d}}(\tilde{\mathcal{d}}(\lambda))\).

By virtue of Lemmas 4.3 and 4.5 we will parametrize and sometimes denote the \(\bar{T} \times \mathbb{C}^* \times \mathbb{C}^*\)-fixed points in \(\mathcal{P}_d\) by collections \(\mathcal{d}\) such that \(\mathcal{d} = \tilde{\mathcal{d}}(\lambda)\).

4.6. **Correspondences.** If the collections \(\mathcal{d}\) and \(\mathcal{d}'\) differ at the only place \(i \in I := \mathbb{Z}/n\mathbb{Z}\), and \(d'_i = d_i + 1\), then we consider the correspondence \(E_{\mathcal{d},i} \subset \mathcal{P}_{\mathcal{d}} \times \mathcal{P}_{\mathcal{d}'}\) formed by the pairs \((F_{\bullet}, \mathcal{F}_{\bullet}')\) such that for \(j \neq i \pmod{n}\) we have \(F_j = F_j'\), and for \(j \equiv i \pmod{n}\) we have \(F_j' \subset F_j\).

It is a smooth quasi-projective algebraic variety of dimension \(2 \sum_{i \neq i} d_i + 1\).

We denote by \(\mathbf{p}\) (resp. \(\mathbf{q}\)) the natural projection \(E_{\mathcal{d},i} \rightarrow \mathcal{P}_{\mathcal{d}}\) (resp. \(E_{\mathcal{d},i} \rightarrow \mathcal{P}_{\mathcal{d}'}\)). For \(j \equiv i \pmod{n}\) the correspondence \(E_{\mathcal{d},i}\) is equipped with a natural line bundle \(L_j\) whose fiber at \((\mathcal{F}_{\bullet}, \mathcal{F}_{\bullet}')\) equals \(\Gamma(\mathcal{C}, \mathcal{F}_j\mathcal{F}_j')\). Finally, we have a transposed correspondence \(\mathbf{T}E_{\mathcal{d},i} \subset \mathcal{P}_{\mathcal{d}'} \times \mathcal{P}_{\mathcal{d}}\).

4.7. We denote by \(V\) the direct sum of equivariant (complexified) K-groups: \(V = \oplus_{\mathcal{d}} K^{T \times \mathbb{C}^* \times \mathbb{C}^*}(\mathcal{P}_{\mathcal{d}})\). It is a module over \(K^{\bar{T} \times \mathbb{C}^* \times \mathbb{C}^*}(pt)\) equal to \(\mathbb{C}[\bar{T} \times \mathbb{C}^* \times \mathbb{C}^*] = \mathbb{C}[x_1, \ldots, x_n, v, u]\). Here \(u\) correspond to a character \((x_1, \ldots, x_n, v, u) \mapsto u\). We define \(V = V \otimes K^{\bar{T} \times \mathbb{C}^* \times \mathbb{C}^*}(pt)\).

We have an evident grading \(V = \oplus_{\mathcal{d}} V_{\mathcal{d}}\), \(V_{\mathcal{d}} = K^{\bar{T} \times \mathbb{C}^* \times \mathbb{C}^*}(\mathcal{P}_{\mathcal{d}}) \otimes K^{\bar{T} \times \mathbb{C}^* \times \mathbb{C}^*}(pt)\).

4.8. The grading and the correspondences \(\mathbf{T}E_{\mathcal{d},i}, E_{\mathcal{d},i}\) give rise to the following operators on \(V\) (note that though \(\mathbf{p}\) is not proper, \(\mathbf{p}^*\) is well defined on the localized equivariant K-theory due to the finiteness of the fixed point set of \(\bar{T} \times \mathbb{C}^* \times \mathbb{C}^*\)):
\[
\mathbf{p}_{i} = t_{i+1}^{-1} t_i^{-1} u^{-\delta_{0,i}} v^{-2d_i + d_{i-1} + d_{i+1} - 1}; \quad V_{\mathcal{d}} \rightarrow V_{\mathcal{d}}
\]
\[
\mathbf{c}_{i} = t_{i+1}^{-1} t_i^{-1} v^{d_{i+1} - d_i + 1} \mathbf{p}_{i}; \quad V_{\mathcal{d}} \rightarrow V_{\mathcal{d} - i}
\]
\[
\mathbf{f}_{i} = -t_{i}^{-1} u^{\delta_{0,i}} v^{d_i - d_{i-1} + 1} \mathbf{c}_{i}(L_i \otimes \mathbf{p}^*) ; \quad V_{\mathcal{d}} \rightarrow V_{\mathcal{d} + i}
\]
According to the Conjecture 3.7 of [2] the following theorem holds

**Theorem 4.9.** For \( n > 2 \), these operators \( t_i, e_i, f_i (i \in \mathbb{Z}/n\mathbb{Z}) \) satisfy the relations in \( U_v(\widehat{\mathfrak{sl}}_n) \), i.e. they give rise to the action of quantum affine group on \( V \).

Since the fixed point basis of \( M \) corresponds to the Gelfand-Tsetlin basis of the universal Verma module over \( U_v(\widehat{\mathfrak{sl}}_n) \), we propose to call the fixed point basis of \( V \) the affine Gelfand-Tsetlin basis.

4.10. Quantum toroidal algebra. Let \((\alpha_k)_{1 \leq k, l \leq n} = \widehat{\mathfrak{a}}_{n-1}\) stand for the Cartan matrix of \( \widehat{\mathfrak{a}}_n \). The double affine loop algebra \( U'_v(\widehat{\mathfrak{a}}_n) \) is an associative algebra over \( \mathbb{Q}(v) \) generated by \( e_{k,l}, f_{k,l}, v^h, h_{k,m} (1 \leq k \leq n, r, m \in \mathbb{Z} \setminus \{0\}) \) with the relations \([9][10]\), where \( k, l \) are understood as residues modulo \( n \), so that if \( k = n \) then \( k + 1 = 1 \).

The quantum toroidal algebra \( \hat{U}_v(\widehat{\mathfrak{a}}_n) \) is an associative algebra over \( \mathbb{C}(u,v) \) generated by \( e_{k,l}, f_{k,l}, v^h, h_{k,m} (1 \leq k \leq n, r, m \in \mathbb{Z} \setminus \{0\}) \) with the same relations as in \( U'_v(\widehat{\mathfrak{a}}_n) \) except for relations \([10][13]\) for the pairs \((k,l) = (1,n), (n,1)\). These relations are modified as follows. We introduce the shifted generating series \( \hat{x}_n^\pm(z) := x_n^\pm(zv^n u^2), \hat{\psi}^\pm_n(z) = \psi^\pm_n(zv^n u^2) \).

Now the new relations read

\[
\hat{x}_n^\pm(z)x_1^\pm(w)(z - v^\mp w) = (v^\mp z - w)x_1^\pm(w)\hat{x}_n^\pm(z),
\]

\[
\hat{\psi}_n^\pm(z)x_1^\pm(w)(z - v^\mp w) = x_1^\pm(w)\hat{\psi}_n^\pm(z)(v^\mp z - w),
\]

\[
\psi_1^\pm(z)\hat{x}_n^\pm(w)(z - v^\mp w) = \hat{x}_n^\pm(w)\psi_1^\pm(z)(v^\mp z - w).
\]

Thus we have \( U'_v(\widehat{\mathfrak{a}}_n) = \hat{U}_v(\widehat{\mathfrak{a}}_n)/(v^nu^2 = 1) \).

Note that \( \hat{U}_v(\widehat{\mathfrak{a}}_n) \) coincides with \( \hat{U} \) modification of \( \hat{U} \) introduced in \([15]\), with \( d \) not specialized to a complex number and with central element \( c = 1 \). The isomorphism \( \Psi : \hat{U}_v(\widehat{\mathfrak{a}}_n) \rightarrow \hat{U} \) takes \( v \) to \( v \) and \( u \) to \( d^2v^{-2} \). It is defined on the generating series as

\[
\Psi(x_1^\pm(z)) = e_{i-1}^\pm((d^2)^{-1}z), \quad \Psi(x_1^\pm(z)) = f_{i-1}^\pm((d^2)^{-1}z), \quad \Psi(\psi_1^\pm(z)) = k_{i-1}^\pm((d^2)^{-1}z).
\]

4.11. For any \( m \leq i \in \mathbb{Z} \) we will denote by \( \mathcal{W}_{mi} \) the quotient \( \mathcal{Z}_j/\mathcal{Z}_m \) of the tautological vector bundles on \( \mathcal{P}_d \times \mathbb{C} \). Under the K"unneth decomposition, for the exterior power \( \Lambda^J(\mathcal{W}_{mi}) \) we have \( \Lambda^J(\mathcal{W}_{mi}) =: \Lambda^J_{(i)}(\mathcal{W}_{mi}) \otimes 1 + \Lambda^J_{(j)}(\mathcal{W}_{mi}) \otimes \tau \) where \( \Lambda^J_{(i)}(\mathcal{W}_{mi}), \Lambda^J_{(j)}(\mathcal{W}_{mi}) \in K^{T \times \mathbb{C}^*}(\mathcal{P}_d) \).

We introduce the generating series \( b_{mi}(z) := 1 + \sum_{j=1}^{\infty} \left( \Lambda^J_{(i)}(\mathcal{W}_{mi}) - v\Lambda^J_{(j)}(\mathcal{W}_{mi}) \right)(-z)^{-j} \).

**Corollary 4.12.** The expression \( b_{mi}(zv^{-i-2})^{-1}b_{mi}(zv^{-i-1})^{-1}b_{m,i-1}(zv^{-i})b_{m,i+1}(zv^{-i-2}) \) is independent of \( m < i \).

The proof is similar to the proof of Corollary \([2][20]\).

We will denote by \( \psi^\pm_i(z) \) the common value of the expressions

\[
t_{i+1}^{-1}t_i v^{d_i+1-2d_i-1} (b_{mi}(zv^{-i-2})^{-1}b_{mi}(zv^{-i})^{-1}b_{m,i-1}(zv^{-i})b_{m,i+1}(zv^{-i-2}))^\pm.
\]

Recall that \( v \) stands for the character of \( \mathbb{T} \times \mathbb{C}^* \times \mathbb{C}^* \) : \( (L, v, u) \rightarrow v \). We define the line bundle \( L'_k := v^k L_k \) on the correspondence \( E_d \times \mathbb{C} \times \mathbb{C}^* \), that is \( L'_k \) and \( L_k \) are isomorphic as line bundles but the equivariant structure of \( L'_k \) is obtained from the equivariant structure of \( L_k \) by the twist by the character \( v^k \).
For $0 \leq k \leq n-1$ we consider the following generating series of operators on $V$:

$$\psi^\pm_k(z) = \sum_{r=0}^{\pm \infty} \psi^\pm_{k,r} z^{-r} : V_d \to V_d[[z^{\pm 1}]],$$

$$x^+_k(z) = \sum_{r=-\infty}^{\infty} e_{k,r} z^{-r} : V_d \to V_{d-k}[[z,z^{-1}]],$$

$$x^-_k(z) = \sum_{r=-\infty}^{\infty} f_{k,r} z^{-r} : V_d \to V_{d+k}[[z,z^{-1}]],$$

$$e_{k,r} := t^{-1}_{k+1} u^{d_{k+1}-d_k+1-k} p_*((L_k')^{\otimes r} \otimes q^*) : V_d \to V_{d-k}$$

$$f_{k,r} := -t^{-1}_{k+1} u^{d_{k+1}-d_k+1+k} q_*((L_k \otimes (L_k')^{\otimes r} \otimes p^*) : V_d \to V_{d+k}$$

**Theorem 4.13.** These generating series of operators $\psi^\pm_k(z), x^\pm_k(z)$ on $V$ defined in (43)(47) satisfy the relations in $\tilde{U}_v(\frak{g}_n)$, i.e. they give rise to the action of $\tilde{U}_v(\frak{g}_n)$ on $V$.

Now we compute the matrix coefficients of operators $e_{i,r}, f_{i,r}$ and the eigenvalues of $\psi^\pm_i(z)$. For this goal we need to know the torus character in the tangent space to $E_{d,i}$ (and $P_d$) at the torus fixed point given by indices $\vec{d}, \underline{d}'$ (and $\vec{d}$ correspondingly). These characters are computed in [4] (see Propositions 4.15, 4.21 and Remark 4.17 of loc. cit.):

**Proposition 4.14.** a) The torus character in the tangent space to $E_{d,i}$ at the torus fixed point given by indices $\vec{d}, \underline{d}'$ equals

$$\sum_{k=1}^{n} \sum_{l \leq k} t^2_{l} v^{2d_{(k-1)i''} - 1 - (v-2d_{ki}) - 1} u^{2|\frac{i'}{n}| - 2|\frac{j}{n}|} + \sum_{k=1}^{n} \sum_{l \leq k} t^2_{l} v^{2d_{(k-1)i''} - 1 - (v-2d_{ki}) - 1} u^{2|\frac{i'}{n}| - 2|\frac{j}{n}|}$$

$$- \sum_{k=1}^{n} \sum_{l \leq k} t^2_{l} v^{2d_{ki''} - 1 - (2v-2d_{ki}) - 1} u^{2|\frac{i'}{n}| - 2|\frac{j}{n}|} + \sum_{k=1}^{n} \sum_{l \leq k} t^2_{l} v^{2d_{ki''} - 1 - (2v-2d_{ki}) - 1} u^{2|\frac{i'}{n}| - 2|\frac{j}{n}|}$$

$$+ v^2 - v^{2d_{ij} + 2d_{(i-1)}} + t^2_{i} v^{2d_{ij} + 2d_{ki}} u^{2|\frac{i'}{n}| - 2|\frac{j}{n}|} + \sum_{j \neq k \leq i-1} t^2_{k} u^{-2d_{ij} - 2d_{(i-1)}} u^{2|\frac{i'}{n}| - 2|\frac{j}{n}|}$$

if $d'_{ij} = d_{ij} + 1$ for certain $j \leq i$.

b) The torus character in the tangent space to $P_d$ at the torus fixed point $\vec{d}$ equals

$$\sum_{k=1}^{n} \sum_{l \leq k} t^2_{l} v^{2d_{(k-1)i''} - 1 - (v-2d_{ki}) - 1} u^{2|\frac{i'}{n}| - 2|\frac{j}{n}|} + \sum_{k=1}^{n} \sum_{l \leq k} t^2_{l} v^{2d_{(k-1)i''} - 1 - (v-2d_{ki}) - 1} u^{2|\frac{i'}{n}| - 2|\frac{j}{n}|}$$

$$- \sum_{k=1}^{n} \sum_{l \leq k} t^2_{l} v^{2d_{ki''} - 1 - (2v-2d_{ki}) - 1} u^{2|\frac{i'}{n}| - 2|\frac{j}{n}|} + \sum_{k=1}^{n} \sum_{l \leq k} t^2_{l} v^{2d_{ki''} - 1 - (2v-2d_{ki}) - 1} u^{2|\frac{i'}{n}| - 2|\frac{j}{n}|}$$

So analogously to Theorem 3.17 ([4]) we get the following proposition.
Proposition 4.15. Define $p_{i,j} := t_{ij}^2 (\mod n)^{v - 2d_i + u - 2(\frac{n}{d})} = t_{ij}^2 (\mod n)^{v - 2d_i + u}. $

a) The matrix coefficients of the operators $f_{i,r}, e_{i,r}$ in the fixed points basis $\{\tilde{d}\}$ of $V$ are as follows:

$$ f_{i,r}(\tilde{d}) = -t_i^{-1}v^{d_i - d_i - i - 1}p_{i,j}(p_{i,j}v)^r(1 - v^2)^{-1} \prod_{j \neq k \leq i} (1 - p_{i,j}p_{i,k}^{-1})^{-1} \prod_{k \leq i - 1} (1 - p_{i,j}p_{i-1,k}^{-1}) $$

if $d_{i,j} = d_{i,j} + 1$ for certain $j \leq i$;

$$ e_{i,r}(\tilde{d}) = t_{i+1}^{-1}v^{d_{i+1} - d_i - i}(p_{i,j}v^{i+2})^r(1 - v^2)^{-1} \prod_{j \neq k \leq i} (1 - p_{i,k}p_{i,j}^{-1})^{-1} \prod_{k \leq i + 1} (1 - p_{i+1,k}p_{i,j}^{-1}) $$

if $d_{i,j} = d_{i,j} - 1$ for certain $j \leq i$.

All the other matrix coefficients of $e_{i,r}, f_{i,r}$ vanish.

b) The eigenvalue of $\psi_{i,s}^{\pm}(z)$ on $\{\tilde{d}\}$ equals

$$ t_{i+1}^{-1}v^{d_{i+1} - 2d_i + d_i - 1} \prod_{j \leq i} (1 - z^{-1}v^{i+2}p_{i,j})^{-1} \prod_{j \leq i + 1} (1 - z^{-1}v^{i+2}p_{i+1,j}) \prod_{j \leq i - 1} (1 - z^{-1}v^{i}p_{i-1,j}). $$

Remark 4.16. These formulas are the same as in Proposition 4.14 except with the change $s_{i,j} \rightarrow p_{i,j}$.

Proof. For arbitrary $k \in \mathbb{Z}$ we define $x_k^\pm(z), \psi_k^\pm(z)$ by the same formulas as in Proposition 4.15.

The formula for the eigenvalue of $\psi_{i,s}^{\pm}(z)$ on $\{\tilde{d}\}$ from Proposition 4.15b) shows that $\psi_{i,s}^{\pm}(z) = \psi_{i,s}^{\pm}(zv^nu^2) = \psi_{i,s}^{\pm}(z)$.

From the formulas of Proposition 4.15a) we get $x_{k-n}^{\pm}(z) = x_k^{\pm}(zv^nu^2)$. In particular, $x_0^{\pm}(z) = x_0^{\pm}(zv^nu^2) = \hat{x}_0^{\pm}(z)$.

Now the relations follow again from Theorem 2.15 and the remark above. □

4.17. Specialization of Gelfand-Tsetlin basis. We fix a positive integer $K$ (a level). We consider an $n$-tuple $\mu = (\mu_{1-n}, \ldots, \mu_0) \in \mathbb{Z}^n$ such that $\mu_0 + K \geq \mu_{1-n} \geq \mu_{2-n} \geq \ldots \geq \mu_{-1} \geq \mu_0$. We view $\mu$ as a dominant (integrable) weight of $\tilde{g}_n$ of level $K$. We extend $\mu$ to a nonincreasing sequence $\tilde{\mu} = (\tilde{\mu}_i)_{i \in \mathbb{N}}$ setting $\tilde{\mu}_i := \mu_i (\mod n) + \lfloor \frac{\mu_i}{d} \rfloor K$.

We define a subset $D(\mu)$ (affine Gelfand-Tsetlin patterns) of the set $D$ of all collections $\tilde{d}$ satisfying the conditions (35) as follows:

$$ \tilde{d} \in D(\mu) \iff d_{i,j} - \tilde{\mu}_j \leq d_{i+1,j+1} - \tilde{\mu}_{j+1} \forall j \leq i, l \geq 0. $$

We specialize the values of $t_1, \ldots, t_n, v, u$ so that

$$ u = v^{-K-n}, \quad t_j = u^\tilde{\mu}_j - 1. $$

We define the renormalized vectors

$$ (\tilde{d}) := C_{\tilde{d}}^{-1}(\tilde{d}) $$

where $C_{\tilde{d}}$ is the product $\prod_{w \in T_2, p_{\tilde{d}} w}$ of the weights of $\tilde{T} \times C^* \times C^*$ in the tangent space to $P_{\tilde{d}}$ at the point $\tilde{d}$. The explicit formula for the multiset $\{w\}$ is given in Proposition 4.14b).
**Proposition 4.18.** The only nonzero matrix coefficients of the operators $f_{i,r}, e_{i,r}$ in the renormalized fixed points basis $\{ \tilde{d} \}$ of $V$ are as follows:

$$e_{i,r}(\tilde{d}, \tilde{d}) = t_{i+1}^{-1} v^{d_{i+1} - d_i - i} (p_{i,j} v^i)^r (1 - v^2)^{-\frac{1}{2}} \prod_{j < k \leq i} (1 - p_{i,j} p_{i,k})^{-1} \prod_{k = i-1} (1 - p_{i+1,k}^{-1})$$

if $d'_{i,j} = d_{i,j} + 1$ for certain $j \leq i$;

$$f_{i,r}(\tilde{d}, \tilde{d}) = -t_i^{-1} v^{d_i - d_{i-1} - 2 + i} (p_{i,j} v^i)^r (1 - v^2)^{-\frac{1}{2}} \prod_{j < k \leq i} (1 - p_{i,j} p_{i,k})^{-1} \prod_{k = i+1} (1 - p_{i+1,k}^{-1})$$

if $d'_{i,j} = d_{i,j} - 1$ for certain $j \leq i$;

**Proof.** According to Proposition 4.15 the matrix elements $e_{i,r}(\tilde{d}, \tilde{d})$ are nonzero only if $\tilde{d}' = \tilde{d} + \delta_{i,j}$ $(\tilde{d}' = \tilde{d} - \delta_{i,j})$ for some $j \leq i$. According to the Bott-Lefschetz fixed point formula:

$$e_{i,r}(\tilde{d}, \tilde{d}) = t_{i+1}^{-1} v^{d_{i+1} - d_i - i} (t_{j}^2 v^{2d_{ij} + \frac{1}{2}} | \tilde{d}'^j v^i) \prod_{w \in T_{\tilde{d}, \tilde{d}'}} w \prod_{w \in T_{\tilde{d}, \tilde{d}'}} w$$

$$f_{i,r}(\tilde{d}, \tilde{d}) = -t_i^{-1} v^{d_i - d_{i-1} - 2 + i} (t_{j}^2 v^{2d_{ij} + \frac{1}{2}} | \tilde{d}'^j v^i) (t_{j}^2 v^{2d_{ij} + \frac{1}{2}} | \tilde{d}'^j v^i) \prod_{w \in T_{\tilde{d}, \tilde{d}'}} w \prod_{w \in T_{\tilde{d}, \tilde{d}'}} w$$

So after renormalizing vectors according to (4), we have:

$$e_{i,r}(\tilde{d}, \tilde{d}) = -f_{i,r}(\tilde{d}, \tilde{d}) t_{i+1}^{-1} v^{d_{i+1} - 2d_i + d_{i-1} + 2 - 2i} f_{i,r}(\tilde{d}, \tilde{d}) = -f_{i,r}(\tilde{d}, \tilde{d})$$

Now the proposition follows from Proposition 4.15. \qed

We define $V(\mu)$ as the $\mathbb{C}$-linear span of the vectors $\langle \tilde{d} \rangle$ for $\tilde{d} \in D(\mu)$.

**Theorem 4.19.** The formulas of Theorem 4.13 give rise to the action of $\hat{U}_v(\hat{\mathfrak{sl}}_n)/(u - v^{-K-n})$ in $V(\mu)$.

**Proof.** The proof is exactly the same as the proof of Theorem 3.20 of [4].

1. We have to check two things:
   a) for $\tilde{d} \in D(\mu)$ the denominators of the matrix coefficients $e_{i,r}(\tilde{d}, \tilde{d})$, $f_{i,r}(\tilde{d}, \tilde{d})$ do not vanish;
   b) for $\tilde{d} \in D(\mu)$, $\tilde{d}' \notin D(\mu)$ the numerators of the matrix coefficients $e_{i,r}(\tilde{d}, \tilde{d})$, $f_{i,r}(\tilde{d}, \tilde{d})$ do vanish.

Both are straightforward. \qed

Restricting $V(\mu)$ to $U_v(\hat{\mathfrak{sl}}_n) \subset \hat{U}_v(\hat{\mathfrak{sl}}_n)$ we obtain the same named $U_v(\hat{\mathfrak{sl}}_n)$-module with the Gelfand-Tsetlin basis parameterized by $D(\mu)$. Recall that in the proof of Theorem 3.22 [3] there was constructed a bijection between $D(\mu)$ and Tingley’s crystal $\mathcal{B}_\mu$ of cylindric plane partitions model of section 4 [13]. This answers Tingley’s Question 1 ([13], p.38).

Finally we formulate a conjecture:

**Conjecture 4.20.** $\hat{U}(\hat{\mathfrak{sl}}_n)/(u - v^{-K-n})$-module $V(\mu)$ is isomorphic to Uglow-Takeura module, constructed in [13].
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