Computational method for acoustic wave focusing

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Abstract

Scattering properties of a material are changed when the material is injected with small acoustically soft particles. It is shown that its new scattering behavior can be understood as a solution of a potential scattering problem with the potential $q$ explicitly related to the density of the small particles. In this paper we examine the inverse problem of designing a material with the desired focusing properties. An algorithm for such a problem is examined from the theoretical as well as from the numerical perspective.

1 Introduction

Let $D \subset \mathbb{R}^3$ be a bounded connected domain with Lipschitz boundary $S$. Denote by $n_0(x)$ the refraction coefficient in $D$, $n_0(x) = 1$ in $D' := \mathbb{R}^3 \setminus D$. Then the scattering of a plane acoustic wave $u_0 = u_0(x) = e^{i k_0 x}$, incident upon $D$, is described by the system:

$$[\nabla^2 + k^2 n_0(x)] u(x) = 0 \quad \text{in} \quad \mathbb{R}^3,$$

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\[ u(x) = u_0(x) + v(x), \]  

(2)

\[ v(x) = A(\alpha', \alpha) \frac{e^{ikr}}{r} + o \left( \frac{1}{r^2} \right), \quad r := |x| \to \infty, \quad \frac{x}{r} := \alpha', \]  

(3)

where \( v(x) \) is the scattered field, \( \alpha \in S^2 \) is the direction of the incident plane wave, and \( \alpha' \) is the direction of the scattered wave. The coefficient \( A(\alpha', \alpha) \) is called the scattering amplitude, \( k > 0 \) is the wave number, which is assumed to be fixed throughout the paper. For this reason the dependence of \( A \) on \( k \) is not shown.

Let \( D_m, 1 \leq m \leq M \), be a small particle, i.e.,

\[ k_0 a \ll 1, \text{ where } a = \frac{1}{2} \max_{1 \leq m \leq M} \text{diam } D_m, \quad k_0 = k \max_{x \in D} |n_0(x)|. \]  

(4)

The geometrical shape of \( D_m \) is arbitrary, but we assume that each \( D_m \) has a Lipschitz boundary. Moreover, the Lipschitz constant is the same for every domain \( D_m \). This is a technical assumption which can be relaxed. It allows one to use the properties of the electrostatic potentials. Let

\[ d := \min_{m \neq j} \text{dist}(D_m, D_j). \]  

(5)

Assume that

\[ a \ll d. \]  

(6)

We do not assume that \( d \gg \lambda_0 \), that is, that the distance between the particles is much larger than the wavelength. Under our assumptions it is possible that there are many small particles within the distances of the order of magnitude of the wavelength.

The particles are assumed to be acoustically soft, i.e.,

\[ u|_{S_m} = 0 \quad 1 \leq m \leq M. \]  

(7)

As a result of the distribution of many small particles in \( D \), one obtains a new material. We would like this "smart" material to have some desired properties. Specifically, we want this material to scatter the incident plane wave according to an a priori given desired radiation pattern, for example to focus the incident wave within a given solid angle. Is this possible? If yes, then how does one distribute the small particles in order to create such a material? In mathematical terms the problem is

**Given an arbitrary function** \( f(\beta) \in L^2(S^2) \), **can one distribute small particles in** \( D \) **so that the resulting medium generates the radiation pattern** \( A(\beta) := A(\beta, \alpha) \), **at a fixed** \( k > 0 \) **and a fixed** \( \alpha \in S^2 \), **such that**

\[ \| f(\beta) - A(\beta) \|_{L^2(S^2)} \leq \varepsilon, \]  

(8)

**where** \( \varepsilon > 0 \) **is an arbitrary small fixed number?**

The answer is yes. It is contained in the following Theorem.
Theorem 1. For any \( f \in L^2(S^2) \), an arbitrary small \( \varepsilon > 0 \), any fixed \( \alpha = \alpha_0 \in S^2 \), any fixed \( k = k_0 > 0 \), and any bounded domain \( D \subset \mathbb{R}^3 \), there exists a (non-unique) potential \( q(x) \in L^2(D) \), such that (8) holds.

The relation between the particle distribution density and the potential \( q \) is explained in Section 2 and Section 3. In Section 4 we give an algorithm for calculating such a potential. Numerical results are presented in Section 5. Our solution of this problem is based on our earlier results on wave scattering by small bodies of arbitrary shapes, see Ramm (2005b), as well as Ramm (2006a,b).

2 Scattering by many small particles

If many small particles \( D_m, 1 \leq m \leq M \), are embedded in \( D \), \( u \mid_{S_m} = 0 \), where \( S_m \) is the boundary of \( D_m \), then the scattering problem is:

\[
[\nabla^2 + k^2 - q_0(x)]u = 0 \quad \text{in} \quad \mathbb{R}^3 \setminus \bigcup_{m=1}^M D_m,
\]

\[
u_m \mid_{S_m} = 0, \quad m = 1, \ldots, M,
\]

\[u(x) = u_0(x) + A(\beta, \alpha) \frac{e^{ikr}}{r} + o \left( \frac{1}{r} \right), \quad r = |x| \to \infty, \quad \beta = \frac{x}{r},\]

and the solution \( u(x) \) is called the scattering solution. Here \( n_0(x) \) is a given refraction coefficient, \( n_0(x) > 0 \) in \( \mathbb{R}^3 \), \( u \) is the acoustic pressure,

\[q_0(x) := k^2[1 - n_0(x)] = 0 \quad \text{in} \quad D'.\]

Under the assumptions of Section 1, we can have \( d \ll \lambda \). We also assume that the quantity \( a/d^3 \) has a finite non-zero limit as \( M \to \infty \) and \( a/d \to 0 \). More precisely, if \( C_m \) is the electrical capacitance of the conductor with the shape \( D_m \), then we assume the existence of a limiting density \( C(x) \) of the capacitance per unit volume around every point \( x \in D \):

\[
\lim_{M \to \infty} \sum_{D_m \subset \tilde{D}} C_m = \int_{\tilde{D}} C(x)dx,
\]

where \( \tilde{D} \subset D \) is an arbitrary subdomain of \( D \). Note that the density of the volume of the small particles per unit volume is

\[O \left( \frac{a^3}{d^3} \right) \to 0 \quad \text{as} \quad \frac{a}{d} \to 0.\]

One can prove (see Ramm (2005b), p.103) that in the limit \( M \to \infty \) the function \( u \) solves the equation

\[
[\nabla^2 + k^2 - q(x)]u = 0 \quad \text{in} \quad \mathbb{R}^3, \quad q(x) = q_0(x) + C(x),
\]

\[3\]
where $C(x)$ is defined in (12), and $A(\beta, \alpha)$ in (11) corresponds to the potential $q(x)$ (see also Marchenko and Kruslov (1974), where similar homogenization-type problems are discussed).

If all the small particles are identical, $C_0$ is the capacitance of a conductor in the shape of a particle, and $N(x)$ is the number of small particles per unit volume around point $x$, then, up to the quantity of higher order of smallness as $\frac{a}{d} \to 0$, we have:

$$C(x) = N(x)C_0.$$ 

Therefore,

$$q(x) = q_0(x) + N(x)C_0,$$ 

and

$$N(x) = \frac{q(x) - q_0(x)}{C_0}.$$ 

Thus, one has an explicit one-to-one correspondence between $q(x)$ and the density $N(x)$ of the embedded particles per unit volume.

**Remark 1.** If the boundary condition on $S_m$ is of impedance type:

$$u_N = \zeta u \quad \text{on} \quad S_m,$$

where $N$ is the exterior unit normal to the boundary $S_m$, and $\zeta$ is a complex constant, the impedance, then the capacitance $C_0$ in formula (9) should be replaced by

$$C_\zeta = \frac{C_0}{1 + \frac{C_\zeta}{\zeta|S|}},$$

where $|S|$ is the surface area of $S$, and the corresponding potential $q(x)$ will be complex-valued, see Ramm (2005b), p. 97.

### 3 Scattering solutions

To establish Theorem 1, recall that for a fixed $k > 0$ the scattering problem (11)–(3) is equivalent to the Schroedinger scattering problem for the potential $q(x)$:

$$u_q = u_0 - \int_D g(x, y)q(y)u_q(y)dy, \quad g(x, y) := \frac{e^{ik|x-y|}}{4\pi|x-y|},$$

for which the scattering solution $u = u_q$ is the unique solution. The corresponding scattering amplitude is

$$A(\alpha', \alpha) = -\frac{1}{4\pi} \int_D e^{-ik\alpha'x}q(x)u_q(x, \alpha)dx,$$

where the dependence on $k$ is dropped since $k > 0$ is fixed.
If \( q \) is known, then \( A := A_q \) is known. Let \( q \in L^2(D) \) be a potential and \( A_q(\alpha', \alpha) \) be the corresponding scattering amplitude. Fix \( \alpha \in S^2 \) and denote

\[
A(\beta) := A_q(\alpha', \alpha), \quad \alpha' = \beta.
\]

Then

\[
A(\beta) = -\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x) dx, \quad h(x) := q(x)u_q(x, \alpha).
\]

Our goal in Theorem 1 is to find a potential \( q \) for which (8) is satisfied. First, we find an \( h(x) \) that satisfies

\[
\| f(\beta) + \frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x) dx \|_{L^2(S^2)} < \varepsilon.
\]

The existence of such an \( h \) follows from the following Theorem.

**Theorem 2.** Let \( f(\beta) \in L^2(S^2) \) be arbitrary. Then

\[
\inf_{h \in L^2(D)} \left\| f(\beta) + \frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x) dx \right\|_{L^2(S^2)} = 0.
\]

**Proof of Theorem 2.** If (20) fails, then there is a function \( f(\beta) \in L^2(S^2), f \neq 0 \), such that

\[
\int_{S^2} d\beta f(\beta) \int_D e^{-ik\beta \cdot x} h(x) dx = 0 \quad \forall h \in L^2(D).
\]

This implies

\[
\varphi(x) := \int_{S^2} d\beta f(\beta)e^{-ik\beta \cdot x} = 0 \quad \forall x \in D.
\]

The function \( \varphi(x) \) is an entire function of \( x \). Therefore (22) implies

\[
\varphi(x) = 0 \quad \forall x \in \mathbb{R}^3.
\]

This and the injectivity of the Fourier transform imply \( f(\beta) = 0 \). Note that \( \varphi(x) \) is the Fourier transform of the distribution \( f(\beta)\delta(k - \lambda)\lambda^{-2} \), where \( \delta(k - \lambda) \) is the delta-function and \( \lambda \beta \) is the Fourier transform variable. The injectivity of the Fourier transform implies \( f(\beta)\lambda^{-2}\delta(k - \lambda) = 0 \), so \( f(\beta) = 0 \). Theorem 2 is proved.

To find an \( h \) that satisfies (19) one can proceed as follows. Let \( \{Y_{\ell}(\beta)\}_{\ell=0}^{\infty}, Y_\ell = Y_{\ell,m}, -\ell \leq m \leq \ell, \) be the orthonormal in \( L^2(S^2) \) spherical harmonics,

\[
Y_{\ell,m}(-\beta) = (-1)^{\ell}Y_{\ell,m}(\beta), \quad \overline{Y_{\ell,m}(\beta)} = (-1)^{\ell+m}Y_{\ell,m}(\beta),
\]

\[
\hat{j}_{\ell}(r) := \left(\frac{\pi}{2r}\right)^{1/2} J_{\ell+\frac{1}{2}}(r),
\]

\[5\]
where $J_\ell$ are the Bessel functions and the overbar stands for the complex conjugate. It is known that
\[
e^{-ik\beta x} = \sum_{\ell=0, -\ell \leq m \leq \ell} 4\pi (-i)^\ell j_\ell(kr)\overline{Y_{\ell,m}(x^0)}Y_{\ell,m}(\beta), \quad x^0 := \frac{x}{|x|}.
\] (26)

Let us expand $f$ into the Fourier series with respect to spherical harmonics:
\[
f(\beta) = \sum_{\ell=0, -\ell \leq m \leq \ell} f_{\ell,m} Y_{\ell,m}(\beta).
\] (27)

Choose $L = L(\varepsilon)$ such that
\[
\sum_{\ell > L} |f_{\ell,m}|^2 \leq \varepsilon^2.
\] (28)

With so fixed $L$, take $h_{\ell,m}(r)$, $0 \leq \ell \leq L$, $-\ell \leq m \leq \ell$, such that
\[
f_{\ell,m} = -(-i)^\ell \left(\frac{\pi}{2k}\right)^{1/2} \int_0^b r^{3/2} J_{\ell+\frac{1}{2}}(kr) h_{\ell,m}(r) dr,
\] (29)

where $b > 0$, the origin $O$ is inside $D$, the ball centered at the origin and of radius $b$ belongs to $D$, and $h_{\ell,m}(r) = 0$ for $r > b$. There are many choices of $h_{\ell,m}(r)$ which satisfy (29). If (28) and (29) hold, then the norm on the left-hand side of (20) is smaller than $\varepsilon$.

A possible explicit choice of $h_{\ell,m}(r)$ is
\[
h_{\ell,m} = \begin{cases} 
-(-i)^\ell \sqrt{g_{\ell+\frac{1}{2}}(k)} & \ell \leq L, \\
0 & \ell > L 
\end{cases}
\] (30)

where $g_{\mu,\nu}(k) := \int_0^1 x^{\mu+\frac{1}{2}} J_\nu(kx) dx$. This integral can be calculated analytically, see Bateman and Erdelyi (1954), formula 8.5.8. We have assumed that $h(x) = 0$ for $|x| > 1$, and $b = 1$ in (29). Finally, let
\[
h(x) = \sum_{\ell=0}^L h_{\ell,m}(r) Y_{\ell,m}(\alpha').
\] (31)

This function satisfies inequality (19) by the construction.

4 Reconstruction of the potential

In the previous section we have shown how to find a function $h \in L^2(D)$ that satisfies (13). In this section a potential $q$ satisfying the conditions of Theorem 1 is constructed from such an $h$. The possibility of such a reconstruction follows from the following result.
Theorem 3. Let \( h \in L^2(D) \) be arbitrary. Then

\[
\inf_{q \in L^2(D)} \|h - qu_q(x, \alpha)\| = 0.
\]  

(32)

Here \( \alpha \in S^2 \) and \( k > 0 \) are arbitrary, fixed. Moreover, if \( |h|_{L^2(D)} \) is sufficiently small, then there exists a potential \( q \) such that

\[
h(x) = q(x)u_q(x, \alpha).
\]

(33)

This Theorem follows from Lemma 1 and Lemma 2 stated and proved below. For convenience let us summarize the method for finding a potential \( q \) satisfying the conditions of Theorem 1.

Method for Potential Reconstruction.

Let \( \varepsilon > 0 \).

Step 1. Given an arbitrary function \( f(\beta) \in L^2(S^2) \) find \( h \in L^2(D) \) such that (19) holds. This can be done using (30). Let

\[
h(x) = \sum_{\ell=0}^{L} h_{\ell,m}(r)Y_{\ell,m}(\alpha'),
\]

where \( L = L(\varepsilon) \), see (30).

Step 2. Use \( h \), obtained in Step 1, to find a potential \( q \in L^2(D) \) satisfying

\[
\|h - qu_q(x, \alpha)\| < \varepsilon.
\]

For \( f \) with a sufficiently small norm \( |f(\beta)|_{L^2(S^2)} \) such a potential \( q \) can be found using formula (37), see below. Formula (37) can be used for any \( f \) for which condition (36) holds.

Step 3. This potential \( q \) generates the scattering amplitude \( A(\beta) \) at fixed \( \alpha \) and \( k \), such that

\[
|f(\beta) - A_q(\beta)|_{L^2(S^2)} \leq C\varepsilon
\]

holds for some constant \( C \), independent of \( \varepsilon \).

Indeed, let \( \| \cdot \| := \| \cdot \|_{L^2(S^2)} \). Then

\[
|f(\beta) - A_q(\beta)| = |f(\beta) + \frac{1}{4\pi} \int_D e^{-ik\beta \cdot x}q(x)u(x)dx|\]

\[
\leq |f(\beta) + \frac{1}{4\pi} \int_D e^{ik\beta \cdot x}hd\sigma| + \frac{|D|}{4\pi} \varepsilon \]

\[
\leq \varepsilon + \frac{|D|}{4\pi}, \quad |D| = \text{meas } D.
\]

(35)

This concludes the proof of Theorem 1.
Lemma 1. Assume that $\sup_{x \in D} \left| \int_D g h dy \right| < 1$, or, more generally, that

$$\sup_{x \in D} \left| u_0(x) - \int_D g(x, y) h(y) dy \right| > 0, \quad g = \frac{e^{i|x-y|}}{4\pi|x-y|}. \quad (36)$$

Then equation (33) has a unique solution:

$$q(x) = \frac{h(x)}{u_0(x) - \int_D g(x, y) h(y) dy}, \quad u_0(x) = e^{ik\alpha \cdot x}, \quad q \in L^2(D). \quad (37)$$

Remark 2. It follows from Theorem 2 and the discussion afterward that $f$ and $h$ are proportional, so that if $\|f\|_{L^2(S^2)}$ is sufficiently small, then $\|h\|_{L^2(D)}$ is small, and then condition (36) is satisfied.

Proof of Lemma 1. The scattering solution corresponding to a potential $q$ solves the equation

$$u = u_0 - \int_D g(x, y) q(y) u(y) dy, \quad u_0 := e^{ik\alpha \cdot x}. \quad (38)$$

If $h(x) = q(x) u_q(x, \alpha)$ holds, i.e., if $h$ corresponds to a $q \in L^2(D)$, then $u = u_0 - \int_D g h dy$. Multiply this equation by $q$ and get

$$q(x) u(x) = q(x) u_0(x) - q(x) \int_D g(x, y) h(y) dy.$$

Using (33) and solving for $q$, one gets (37), provided that (36) holds. Condition (36) holds if $\|h\|_{L^2(D)}$ is sufficiently small. One has

$$\left| \int_D g(x, y) h dy \right| \leq \frac{1}{4\pi} \sup_x \left\| \frac{1}{|x-y|} \right\|_{L^2(D)} \|h\|_{L^2(D)}, \quad (39)$$

and

$$\frac{1}{4\pi} \left( \int_D \frac{dy}{|x-y|^2} \right)^{\frac{1}{2}} \leq \frac{\sqrt{a}}{\sqrt{4\pi}},$$

where $a = 0.5diamD$. If, for example,

$$\frac{\sqrt{a}}{\sqrt{4\pi}} \|h\|_{L^2(D)} < 1,$$

then condition (36) holds, and formula (37) yields the corresponding potential. This explains the role of the "smallness" assumption. \qed

Remark 3. If (36) fail, then formula (37) may yield a $q \notin L^2(D)$. As long as formula (37) yields a potential $q \in L^p(D)$, $p \geq 1$, our arguments essentially remain valid. In
our presentation we have used \( p = 2 \) because the numerical minimization in \( L^2 \)-norm is simpler.

The difficulty arises when formula (37) yields a potential which is not locally integrable. Numerical experiments showed that this case did not occur in practice in several test examples in which the "smallness" condition was not satisfied.

We prove that a suitable small perturbation \( h_\delta \) of \( h \) in \( L^2(D) \)-norm yields by formula (37) a bounded potential \( q_\delta \). This means that the "smallness" restriction on the norm of \( f \) is not essential.

**Lemma 2.** Assume that \( h \) is analytic in \( D \) and bounded in the closure of \( D \). There exists a small perturbation \( h_\delta \) of \( h \), \( \| h - h_\delta \|_{L^2(D)} < \delta \), such that the function

\[
q_\delta := \frac{h_\delta(x)}{u_0(x) - \int_D g(x, y) h_\delta(y) dy}
\]

is bounded.

**Outline of proof.** Suppose that for a given \( h \in L^2(D) \) condition (36) is not satisfied. Let us approximate \( h \) by an analytic function \( h_1 \) in \( D \), for example, by a polynomial, so that

\[
\| f(\beta) + \frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h_1(x) dx \| < \varepsilon.
\]

Denoting \( h_1 \) by \( h \) again, we may assume that \( h \) is analytic in \( D \) and in a domain which contains \( D \). We prove that it is possible to perturb \( h \) slightly so that for the perturbed \( h \), denoted \( h_\delta \), condition (36) is satisfied, and formula (37) yields a potential \( q_\delta \in L^2(D) \), for which inequality (8) holds, see Ramm (2006c) for details.

Finally we make some remarks about ill-posedness of our algorithm for finding \( q \) given \( f \). This problem is ill-posed because an arbitrary \( f \in L^2(S^2) \) cannot be the scattering amplitude \( A_q(\beta) \) corresponding to a compactly supported potential \( q \). Indeed, it is proved in Ramm (1992), Ramm (2002), that \( A(\beta) \) is infinitely differentiable on \( S^2 \) and is a restriction to \( S^2 \) of a function analytic on the algebraic variety in \( \mathbb{C}^3 \), defined by the equation \( \beta \cdot \beta = k^2 \). Finding \( h \) satisfying (19) is an ill-posed problem if \( \varepsilon \) is small. It is similar to solving the first-kind Fredholm integral equation

\[
\frac{1}{4\pi} \int_D e^{-ik\beta \cdot x} h(x) dx = -f(\beta)
\]

whose kernel is infinitely smooth. Our solution (30) shows the ill-posedness of the problem because the denominator in (30) tends to zero as \( \ell \) grows. Methods for stable solutions of ill-posed problems (see Ramm (2005a)) should be applied to finding \( h \). If \( h \) is found, then \( q \) is found by formula (37), provided that (36) holds. If (36) does not hold, one perturbs slightly \( h \) according to Lemma 2 and get a potential \( q_\delta \) by formula (37) with \( h_\delta \) in place of \( h \).
5 Numerical results

In this section we present results of numerical experiments for a design of the material capable of focusing the incoming plane wave into a desired solid angle. First, let us note that a direct implementation of the algorithm presented in the previous sections produces potentials $q$ with large magnitudes (of the order of $10^5$) in our examples. This happens because of the ill-posedness of the inverse scattering problem. To remedy this situation we have introduced an additional step in the potential reconstruction algorithm stated in Section 4.

Step 1b. Let $h_{l,m}$ be the coefficients of $h$ obtained according to (30). Bound the magnitudes of the coefficients by a predetermined constant $T > 0$, that is

$$h'_{l,m} = \begin{cases} h_{l,m} & \text{if } |h_{l,m}| \leq T \\ \frac{h_{l,m}}{|h_{l,m}|} T & \text{if } |h_{l,m}| > T. \end{cases}$$

Let

$$h(x) = \sum_{\ell=0}^{L} h'_{\ell,m}(r)Y_{\ell,m}(\alpha').$$

(40)

The bound on the function $h$ has the effect of bounding the potential $q$. This procedure regularizes the ill-posedness of the reconstruction process as discussed at the end of Section 4. The ill-posedness manifests itself in the divergence of the series (34) with $L = \infty$, when the regularization we have used by introducing the bounding constant, is not applied. However, from the numerical observations, the series (40) practically did not change with the increase in $L$ for $L > 5$. As expected, an increase in the value of $T$ improves the precision of the approximation of the desired scattering amplitude $f$, but it also increases the magnitude of the potential $q$. A reduction in the value of $T$ leads to a deteriorating approximation.

In all the experiments the incident direction $\alpha = (0, 0, 1)$, $k = 1.0$ and $L = 6$. The domain $D$ is the ball of radius 1 centered in the origin. In our first numerical experiment the goal was to focus the incoming plane wave into the solid angle $0 \leq \theta \leq \pi/4$, where $\theta$ is the polar angle measured from the incident direction $\alpha = (0, 0, 1)$. Figure 1 shows the cross-section through the incident direction of the desired (dotted line) and the attained absolute value (solid line) of the scattering amplitude. Figure 2 shows the contour plot of the absolute value of the recovered potential $q$ in a cross-section through the $z$-axis. The darker colors correspond to the larger values of $|q|$. In this experiment the maximum of the absolute value of the potential $q$ was about 1290 corresponding to the bounding constant $T = 100$. This value of $T$ was found by examining numerical results with larger and smaller values for this constant. For smaller $T$ the resulting radiation pattern has smaller magnitudes, i.e. the plot of its absolute value is located closer to the origin. For larger values of $T$ the maximal value $|q|$ of the potential approaches the order of $10^5$. 
Similarly, Figures 3 and 4 show the results of the numerical experiment aimed at focusing the same incident plane wave into the solid angle $0.2\pi \leq \theta \leq 0.5\pi$. The maximum of $|q|$ was about 1840 in this case, corresponding to the bounding constant $T = 800$. This value of $T$ was found experimentally as above. For smaller values of $T$ the resulting radiation pattern has a significant component in the region $|\theta| \leq 0.2\pi$, i.e. it produces a poor approximation for the desired scattering amplitude.

6 Conclusions

A method is developed for finding the number $N(x)$ of small acoustically soft particles to be embedded per unit volume around every point $x$ in a bounded domain, filled with a known material, in order that the resulting new material has the desired radiation pattern. Any wave field, not necessarily acoustic wave field, which satisfies equations (9)-(11) is covered by our theory.

On the boundary of each acoustically soft particle the Dirichlet condition holds.

The method is justified theoretically. Numerical examples of its application are presented. The ill-posedness of our problem is discussed and a regularization method for its stable solution is proposed and successfully tested numerically.

The direct application of the derived formula (30) may lead to large values of $q$. To remedy this situation the coefficients $h_{l,m}$ are bounded. This is a way to handle the ill-posedness of the inverse problem. The resulting algorithm exhibits a stable behavior. It serves as a regularizing algorithm for solving the original ill-posed problem. Numerical
Figure 2: Contour plot of the potential $q$ in experiment 1.

Figure 3: Attained (solid line), and targeted (dotted line) scattering amplitude $f(\beta)$ in experiment 2.
Figure 4: Contour plot of the potential $q$ in experiment 2.
results show that the method can produce materials with the desired focusing properties under the limitation that the desired radiation pattern $f(\beta)$ is well approximated by a short series of spherical harmonics.

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