Cutoff for Rewiring Dynamics on Perfect Matchings

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Abstract

We establish cutoff for a natural random walk (RW) on the set of perfect matchings (PMs), based on ‘rewiring’. An n-PM is a pairing of 2n objects. The k-PM RW selects k pairs uniformly at random, disassociates the corresponding 2k objects, then chooses a new pairing on these 2k objects uniformly at random. The equilibrium distribution is uniform over all n-PMs.

The 2-PM RW was first introduced by Diaconis and Holmes [DH98, DH02], seen as a RW on phylogenetic trees. They established cutoff in this case. We establish cutoff for the k-PM RW whenever 2 ≤ k ≪ n. If k ≫ 1, then the mixing time is \( \frac{1}{2} \log n \) to leading order.

Diaconis and Holmes [DH02] relate the 2-PM RW to the random transpositions card shuffle. Ceccherini-Silberstein, Scarabotti and Tolli [CST07, CST08] establish the same result using representation theory. We are the first to handle k > 2. We relate the PM RW to conjugacy-invariant RWs on the permutation group by introducing a ‘cycle structure’ for PMs, then build on work of Berestycki, Schramm, Şengül and Zeitouni [Sch05, BSZ11, BSŞ19] on such RWs.

Keywords: mixing time, cutoff, random walks, perfect matchings, coalescence–fragmentation, random transpositions, conjugacy-invariant random walks

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1 Introduction

1.1 Model Set-Up

We analyse a random walk (RW) on the set of perfect matchings (PMs) on 2n objects, for n ∈ N. We represent a PM on 2n objects by a collection of unordered pairs:

\[ \mathcal{M}_n := \{ \eta = \cup_{t=1}^{n} \{ \eta_{2t-1}, \eta_{2t} \} \mid \eta_i \in [2n] \forall i \in [2n], \cup_{t=1}^{n} \{ \eta_t \} = [2n] \} \quad \text{for} \quad n \in \mathbb{N}; \]

here, [m] := \{1, ..., m\} for m ∈ N. We refer to an element of \( \mathcal{M}_n \) as an n-perfect matching. Note the double-braces: the union is a set of n pairs. That is, an n-PM is a collection of n disjoint pairs.

**Definition 1.1: Perfect Matching Random Walk.** Let k ∈ \([2, n] \cap \mathbb{N}\). The k-perfect matching random walk (k-PM RW) on \( \mathcal{M}_n \) has discrete-time dynamics, a step of which is described as follows:

- choose k matched pairs, say \( \cup_{t=1}^{k} \{ i_t, j_t \} \);
- disassociate the pairs to give 2k unpaired elements, ie \( \cup_{t=1}^{k} \{ i_t, j_t \} \);
- uniformly re-pair these 2k elements.

That is, k matched pairs are chosen, the matches are broken and a new matching on these 2k elements is chosen uniformly. We denote this process by \( M_{n,k} := (M_{t,n,k})_{t \geq 0} \).

The PM RW was first introduced by Diaconis and Holmes [DH98, DH02] in the case k = 2. They originally introduced a RW on phylogenetic trees, a biological concept object, along with a bijection between these leaf-labelled trees and PMs. Our extension to larger k allows more general RWs on phylogenetic trees to be studied. More on the biological and other motivations, including randomised algorithms and coding theory, can be found in [DH98, DH02]. The 2-PM RW was later studied in the representation theory community, where it is known as the party model.

We give a full discussion on related work in §1.4. Prior to this, we give a few brief remarks below, then give precise mixing definitions in §1.2 and state the main theorem in §1.3.

The dynamics of the PM RW are clearly transitive for the space \( \mathcal{M}_n \). Thus, we may assume that

\[ M_{0,n,k} = \{ \{1, 2\}, \{3, 4\}, ..., \{2n-1, 2n\} \}, \]

without loss of generality. We refer to this PM as the ‘identity’ matching and denote

\[ \text{id}_\ell := \{ \{1, 2\}, \{3, 4\}, ..., \{2\ell - 1, 2\ell\} \} \in \mathcal{M}_\ell \quad \text{for} \quad \ell \in \mathbb{N}. \]

The dynamics are irreducible. Thus, another consequence of the transitivity is that the unique invariant distribution of the dynamics, which we denote \( \pi_{\mathcal{M}_n} \), is uniform on \( \mathcal{M}_n \), ie \( \pi_{\mathcal{M}_n} = \text{Unif}(\mathcal{M}_n) \).

The re-pairing process involves choosing a new matching on a 2k-size subset of \([2n]\) and leaving the remainder fixed. By transitivity, it suffices to be able to sample a PM with at most k ‘non-fixed points’ wrt \( \text{id}_n \), ie pairs \( \{ \eta_{2t-1}, \eta_{2t} \} \neq \{2\ell - 1, 2\ell\} \) for \( \ell \in [n] \). Indeed, given a general PM, one first applies a permutation to the labels to send it to the identity matching \( \text{id}_n \), then replaces this with the sampled PM. Finally, the inverse of the original permutation is applied.

We may drop subscripts, defaulting to \( n \) or \( (n,k) \), as appropriate. Eg, \( \mathcal{M} = \mathcal{M}_n \) but \( M = M_{n,k} \). We also use some abbreviations for frequently-occurring words or phrases. There are some usual ones: “RW”, “TV”, “uar”, “wrt” and “whp” abbreviating “random walk”, “total variation”, “uniformly at random”, “with respect to” and “with high probability”, respectively. We also use “CS” for “cycle structure” and “PM” for “perfect matching”: these are not standard, but together they appear close to 300 times throughout the paper, so we feel their abbreviation is legitimate.

We do introduce other abbreviations throughout the paper, but only use these ‘locally’—shortly after their definition. The reader is not expected to remember such abbreviations for more than a couple of paragraphs. Contrastingly, the reader should remember PM and CS throughout the paper.

1.2 Mixing and Cutoff Definitions

Let \( X = (X_t)_{t \geq 0} \) be an ergodic Markov chain on a finite state space \( \Omega \). Write \( \pi \) for its unique invariant distribution. We are interested in the distance between the law of \( X_t \) and \( \pi \).
**Definition 1.2:** Total Variation Distance. Let \( \mu \) and \( \nu \) be probability distributions on \( \Omega \). The total variation (TV) distance between \( \mu \) and \( \nu \) is defined to be
\[
\|\mu - \nu\|_{TV} := \sup_{A \subseteq \Omega} |\mu(A) - \nu(A)|.
\]
This is known to be equivalent to half the \( \ell_1 \) distance; see, eg, [LPW17, Proposition 4.2].

The mixing time is the time \( t \) at which the law of \( X_t \) is close to \( \pi \) in TV.

**Definition 1.3:** Mixing Time. Define the mixing time \( t_{\text{mix}}(\cdot) \) by
\[
t_{\text{mix}}(\varepsilon) := \inf\{t \geq 0 : d(t) := \max_{x \in \Omega} \|P^t_x(X_t) \cdot - \pi\|_{TV} \leq \varepsilon \} \quad \text{for} \quad \varepsilon \in (0, 1).
\]
We are interested in a sequence \( (X_n)_{n \in \mathbb{N}} = ((X_{t,n})_{t \geq 0})_{n \in \mathbb{N}} \) of finite, ergodic Markov chains. Write \( \Omega_n \) for the state space, \( \pi_n \) for the unique invariant distribution, \( d_n(\cdot) \) for the worst-case TV distance and \( t_{\text{mix}}^n(\cdot) \) for the mixing time of the \( n \)-th chain \( X^n \). We want to determine the asymptotic behaviour of \( t_{\text{mix}}^n(\varepsilon) \) as \( n \to \infty \) for each fixed \( \varepsilon \in (0, 1) \).

In some special cases, the leading order term of \( t_{\text{mix}}^n(\varepsilon) \) as \( n \to \infty \) does not depend on \( \varepsilon \). This is known as cutoff. It is conjectured to hold for many natural sequences of Markov chains.

**Definition 1.4:** Cutoff. A sequence \( (X_n)_{n \in \mathbb{N}} \) of finite, ergodic Markov chains exhibits cutoff if
\[
\lim_{n \to \infty} \sup_{\varepsilon > 0} \frac{t_{\text{mix}}^n(\varepsilon)}{t_{\text{mix}}^n(1 - \varepsilon)} = 1 \quad \text{for all} \quad \varepsilon \in (0, 1).
\]
An equivalent definition is that there exists a sequence \( (t_n)_{n \in \mathbb{N}} \) of times such that
\[
\lim_{n \to \infty} \inf_{\varepsilon > 0} d^n((1 - \varepsilon)t_n) = 1 \quad \text{and} \quad \lim_{n \to \infty} \sup_{\varepsilon > 0} d^n((1 + \varepsilon)t_n) = 0 \quad \text{for all} \quad \varepsilon \in (0, 1).
\]
The sequence is then said to exhibit cutoff at time \( (t_n)_{n \in \mathbb{N}} \).

**1.3 Main Theorem**

Our main results is that the \( k \)-PM RW exhibits cutoff if \( 2 \leq k \ll n \). Further, we find the leading order of the cutoff time; if \( k \to \infty \) as \( n \to \infty \), then the leading order is given by \( \frac{n}{k} \log n \).

**Theorem 1.5:** Cutoff for the PM RW. Let \( n \in \mathbb{N} \). Let \( k \in \mathbb{N} \setminus \{1\} \) with \( k/n \to 0 \) as \( n \to \infty \). Let
\[
t := \frac{n \log n}{k - k/(2k - 1)} = \frac{n \log n}{k \left(1 - \frac{1}{2k - 1}\right)}.
\]
Then, the \( k \)-PM RW \( M = M_{n,k} \) exhibits cutoff at time \( t \). In particular, if \( k \to \infty \) as \( n \to \infty \), then
\[
t = \frac{n}{k} \log n \cdot (1 + o(1)).
\]
[Officially, this is for a sequence \( (k_n)_{n \in \mathbb{N}} \) with \( k_n/n \to 0 \) as \( n \to \infty \) and PM RWs \( (M_{n,k_n})_{n \in \mathbb{N}} \).]

The upper bound on mixing is the primary focus of this paper. There are five main steps, analysed in §3–§7. These are pulled together to conclude in §8. The lower bound in a straightforward coupon-collector argument. This is done in §9, with various details omitted.

**1.4 Related Previous Work**

Establishing cutoff for \( k = 2 \) has received attention in the past, but we are the first to study \( k > 2 \). Not only this, but we handle any \( k \ll n \). The case \( k = 2 \) is known in representation-theoretic literature as the party model. The chain is described by the Gelfand pair \( (\mathfrak{S}_{2n}, \mathfrak{S}_2 \wr \mathfrak{S}_n) \). A very readable introduction to this field can be found in the book of Ceccherini-Silberstein, Scarabotti and Tolli [CST08]. The \( k \)-PM RW with \( k = 2 \) is covered in [CST08, §11]; see also [CST07, §8]. It
Figure 1.1. The rematchings have the following correspondences:

- the ‘cross’ (left) \[ \{1, 2\}, \{3, 4\} \rightarrow \{1, 4\}, \{2, 3\} \] to a ‘transposition’;
- the ‘bar’ (right) \[ \{1, 2\}, \{3, 4\} \rightarrow \{1, 3\}, \{2, 4\} \] to a ‘reversal’.

appears that this approach may extend beyond \( k = 2 \), but it would involve complicated estimation of \( (2k - 1)!! = (2k - 1) \cdot (2k - 3) \cdots 3 \cdot 1 \) eigenvalues. The complexity of the transition matrix thus rapidly gets out of hand as \( k \) grows. It is perhaps only feasible for small \( k \).

Diaconis and Holmes [DH02] are able to avoid the theory of Gelfand pairs by directly relating the transition matrix for the PM RW to that used in the random transpositions (RT) card shuffle; see [DH02, Proposition 1]. This only applies for \( k = 2 \). It becomes a mixture of transition matrices of different card shuffles for larger \( k \). Such matrices are not necessarily jointly diagonalisable, rendering their approach much trickier for larger \( k \). Our approach can be seen as relating the PM RW to another type of card shuffle—namely, conjugacy-invariant RWs on the permutation group. There are a significant number of challenges in our own approach which arise only when \( k > 2 \).

A Markov chain on the permutation group \( S_n \) is a conjugacy-invariant RW if some conjugacy class \( \Gamma \subseteq S_n \) generates the walk: a step involves choosing \( \sigma \sim \text{Unif}(\Gamma) \) and composing the current location (permutation) with \( \sigma \). A by-now standard approach to analysing card shuffles corresponding to conjugacy-invariant RWs, such as the RT shuffle or \( k \)-cycle RW, is to project from the RW to its conjugacy class, then from the conjugacy class to the corresponding integer partition and finally use a variant of a coupling due to Schramm [Sch05]; see, eg, [BSZ11, BS19, Bor11]. A similar projection for the PM RW can be defined; see, eg, [BKLM19, GUW11]. We use this.

The previous work to which our approach is most related is that of Berestycki and Şengül [BS19], which builds in part on work of Berestycki, Schramm and Zeitouni [BSZ11]. Cutoff for the \( k \)-cycle RW for any fixed \( k \), not depending on \( n \), is established in [BSZ11]. They strongly believe that their argument can be extended to consider general conjugacy classes \( \Gamma \) with bounded support: \( k = |\Gamma| \), independent of \( n \). This is extended to allow any \( k = |\Gamma| \ll n \) in [BS19], similarly to what we allow.

Parts of our argument are very similar to those of [BS19]. We feel that it is important to detail which ideas are our own and which parts are adjustments or extensions of [BS19]. We do so in §2.

Cutoff for the \( k \)-cycle RW with \( 2 \leq k \ll n \) has also been established independently by Hough [Hou16]. He introduces and uses an asymptotic estimation of the characters of \( S_n \) evaluated at cycles. Hough tentatively suggests that his method can be extended to some conjugacy classes. However, he believes that some new ideas are needed to obtain the full result of [BS19].

The limit profile was determined for the \( k \)-cycle case by the current author and Nestoridi [NO22], building on work of Hough [Hou16] and Teysier [Tey20]. Extending this argument to general conjugacy classes, even of bounded support, appears to be very technically challenging.

The other work most related to ours is by Björnberg, Kotowski, Lee and Milos [BKLM19] on an interchange process with reversals. It is related to stochastic representations of quantum spin systems, namely anti/ferromagnetic Heisenberg models. Very roughly, the ferromagnetic model has interactions between spins which behave like ‘transpositions’; antiferromagnetic models additionally have ‘reversals’. Figure 1.1 shows the two possible PM rematchings, known as ‘transpositions’ and ‘reversals’, when \( k = 2 \). This shows the corresponds between the 2-PM RW and Heisenberg models.

The 2-PM RW is briefly discussed by Caputo, Liggett and Richthammer [CLR10]. This is the paper in which Aldous’s famous spectral gap conjecture is proved. The conjecture—now a
Theorem—regards the interchange processes (IP), which is a generalisation of the RT shuffle. They show that the spectral gap $\lambda_{IP}^n$ of the IP equals the spectral gap $\lambda_{PM}^n$ of the RW. The 2-PM RW is a projection of the IP, in a precise sense, which implies that its spectral gap $\lambda_{PM}^n$ satisfies

$$\lambda_{PW}^n = \lambda_{IP}^n \leq \lambda_{PM}^n.$$

Diaconis and Holmes [DH02] completely characterise the spectrum of the 2-PM RW. In particular, their work shows that this inequality is not tight when $n = 2$, ie when considering PMs on 4 objects. We defer the reader to [CLR10, §4.2.2] for further details.

The history of the 2-PM RW goes back to Diaconis and Holmes [DH02], being studied later by Ceccherini-Silberstein, Scarabotti and Tolli [CST07, CST08] and Caputo, Liggett and Richthammer [CLR10], as detailed above. The k-PM RW appeared recently in work of Avena, Güldas, van der Hofstad and den Hollander [AGHH18a, AGHH18b], albeit in a slightly different set-up: they use the k-PM RW to drive a dynamic graph model via the configuration model.

The configuration model (CM), introduced in different forms by Bender and Canfield [BC78] and Bollobás [Bol80, Bol01], randomly samples a graph with a given degree sequence as follows.

- Let $d = (d_1, ..., d_m) \in \mathbb{N}^m$. Assume that $n := \frac{1}{2}(d_1 + \cdots + d_m) \in \mathbb{N}$.
- Place $m$ vertices and attach $2n$ ‘half-edges’ to the vertices: $d_v$ to vertex $v$ for each $v \in V$.
- Uniformly pair the half-edges to create a graph on $m$ vertices with $n$ edges.

The resulting graph has $m$ vertices, $n$ edges and degree sequence $d$. An excellent introduction to the configuration model, with multiple explanatory figures, is given by van der Hofstad [Hof17, §7].

The above viewpoint is of the CM as a projection of a PM. The k-PM RW induces a dynamic random graph process: simply select $k$ edges, cut them to produce $2k$ half-edges and randomly re-pair the half-edges. This drives a dynamic CM by keeping the half-edges attached to the same vertices throughout. A non-backtracking RW is placed on this in [AGHH18a, AGHH18b] and its mixing properties are studied. Focus is on properties of the walk. The graph is not studied in detail. In particular, the question of the mixing time of the dynamic random graph is left open.

The CM need not be a simple graph. It is known that the probability of being simple is bounded under some regularity conditions for vertex degrees; see [Hof17, §7.4] for precise details. Importantly, the law of the CM conditioned on being simple is uniform over all simple graphs, with the appropriate degree sequence. Analogously, the dynamic CM does not consist only of simple graphs.

The switch chain is defined to be the dynamic CM, but where transitions are rejected if they give rise to a non-simple graph. The invariant distribution of this chain is uniform over all simple graphs with the appropriate degree sequence. The purpose of the switch chain is to draw from such graphs uniformly at random. Control on the mixing time is naturally required for such sampling.

Analysis of the switch chain has a long and rich history; far too much to discuss in totality here. Recent overviews can be found in the introductions of the recent papers [AK19, AK20], [Erd+19] or [TY20]. It was introduced by Kannan, Tetali and Vempala [KTV97, KTV99] in the late 90s, making it over 20 years old. Even so, it is still an extremely active area of research.

1.5 Acknowledgements

This is a single-author project, but I would be remiss not to acknowledge the input of others.

The initial question arose out of a research visit of mine to the EURANDOM group in the Netherlands during my PhD, in early 2019. I discussed this question extensively with Güldas, as well as Avena, van der Hofstad and den Hollander. We also discussed their work [AGHH18a, AGHH18b] on the dynamic CM. I gratefully acknowledge their insights and comments at the start of this project, as well as the hospitality of the EURANDOM group more widely. I met with Berestycki a few months later to discuss his work [BS19] and how it might adapted to my set-up.

I subsequently started building on these ideas at the end of my PhD. I had a number of very fruitful discussions with my then PhD supervisor Perla Sousi around this time, in late 2020. I gratefully acknowledge her ideas, comments and contributions.

The helpful comments provided by the anonymous reviewer significantly improved the presentation and clarity of this paper. They have my sincere thanks.
2 Outline of Approach and Comparison with [BS19]

This section first outlines the underlying approach. There are five main steps for the upper bound on mixing, which we detail below. The lower bound is much more straightforward, via a coupon-collector argument. Establishing the upper bound is the primary focus of the article.

To close the section, we compare and contrast the methods used in the current article with those of developed by Berestycki and Şengül [BS19]. Related comments are made throughout the paper. We feel that it is important to be transparent regarding the similarities and differences between our work and theirs, so we include this summary to gather all relevant remarks are together.

§3: Projecting to Cycle Structure and Partitions. The first step involves projecting the PM RW to its cycle structure—a concept that we introduce below, akin to that for permutations—and then further to its corresponding partition. This idea has become a standard approach when analysis conjugacy-invariant RWs on groups by now, being mentioned at least as early as [DH98]; it is used by [DH98, Sch05, BSZ11, Bor11, BŞ19] and surely many more. The current article extends the 2-PM RW of [DH98] to the general k-PM RW via a decomposition of a k-PM into k − 1 2-PMs.

§4: Decomposing into Swaps. It is natural to break down permutations into products of transpositions. Doing so permits analysis of the above partition walk via an adjustment of a coupling due to Schramm [Sch05]; this was done first by Berestycki, Schramm and Zeitouni [BSZ11] and used in [BŞ19]. Constructing an analogous decomposition of a k-PM into a sequence of k − 1 2-PMs (‘swaps’) is one of the most fundamental parts of the paper, permitting analysis of the partition walk.

The decomposition is easy for permutations: (a1, ..., aℓ) = (a1, a2)(a2, a3) · · · (aℓ−1, aℓ). Unfortunately, such a natural idea does not apply for PMs: after pairs labelled (a1, a2) are swapped, there is no way of identifying a specific one of the new pairs with a1 and the other with a2; see Figure 4.1. One of the main inventions of the whole paper is an algorithmic approach for drawing a uniform ‘single-cycle’ ℓ-PM, roughly corresponding to an ℓ-cycle in permutation language, from the above decomposition into swaps. We are able to use the principle behind this approach in a variety of other scenarios.

§5: Analysis of Partition Walk. A partition can be seen as a tiling of (0, 1] by rescaling. A single step of the partition walk when k = 2 involves choosing markers u, v ∈ {1/n, ..., n/n} uniformly: if u and v are in different blocks, then the blocks are merged; if they are in the same block, then the block is split according to some simple rule. When k > 2, a single round is broken into k − 1 steps via the above decomposition into swaps. The markers are no longer uniform: one corresponds to a marker from a previous step and the other is uniform on what is yet to be chosen.

If k ≪ \sqrt{n}, then this sampling, without replacement can be well-approximated by sampling with replacement, somewhat decoupling the k − 1 steps. This idea originated in [BSZ11] where k = 1. Berestycki and Şengül [BŞ19] realised that if k ≪ n, then each individual draw is still a uniform draw from a collection of at least n − k ≍ n objects. So, marginally the two processes are similar, even conditional on what has come before. They show that this is sufficient. We use the same idea, combined with our new decomposition algorithm discussed immediately above.

A variant of the coupling of Schramm [Sch05] is introduced in [BSZ11] and used almost unchanged in [BŞ19]. The variation from [Sch05] is only fairly minor, but is crucial to make sure that blocks in the partition walk do not become unmanageably small. Adjusting the coupling from conjugacy-invariant RWs to PM RWs is not trivial. The first marker in one swap is the same as the second marker in the previous swap in [BSZ11, BŞ19]. Such a statement cannot hold for the PM RW, because of the lack of identifiability of the previous matched discussed above. It makes defining and controlling the coupling of [Sch05] in our case more challenging compared with in [BSZ11, BŞ19]. The subtlety does not arise in [BKLM19] where k = 2, as there swaps are completely independent.

§6: Path Coupling Structure. We use a path-coupling argument to couple two partition walks. The structure of this argument originated in [BŞ19]. The application to conjugacy-invariant RWs was new in [BŞ19], in particular inspecting the relative distance of the two walks after order \beta n/k steps and letting \beta \to \infty. This is markedly different to the usual inspection after just a single step. The justification for this time \beta n/k is outlined in the next part. Their general path-coupling structure requires only very minor adjustment to apply in our set-up.
§7: Auxiliary Graph Process. The mixing time of random transpositions (2-cycles) on $n$ cards is order $n \log n$. However, Schramm [Sch05] showed that the approximate structure of the large cycles relax to uniformity in time order $n$. His proof goes via an auxiliary graph process: if and $j$ are connected at time $t$ if transposition $(i, j)$ has been applied by this time. This gives precisely the usual Erdős–Rényi graph. If $\gamma n$ transpositions are applied with $\gamma > \frac{1}{2}$, then the graph has a giant component whp. This giant component is key in analysing the structure of the large cycles.

The same idea is used in both [BSZ11, BS19]. The former restrict to CSs with support $k \geq 1$, whilst the latter allow any $k \ll n$. Both generalise Schramm’s construction to a hyper-graph: hyper-edge $\{a_1, \ldots, a_r\}$ is added if the cycle $(a_1, a_2, \ldots, a_r)$ is applied as part of the cycle decomposition of the element of $\Gamma$. The analysis in [BS19] is far more tricky than in [BSZ11].

We use exactly the same ideas, once we have the correct viewpoint relating an $n$-PM to an $n$-permutation. In particular, we do not consider a graph on $2n$ objects and connect objects $x$ and $y$ if they are matched at some point. There is a slight difference in our set-up: [BS19] considers a fixed CS $\Gamma$, while our CS changes from step to step. Some non-trivial adjustments are needed.

§8: Combining Results. All the above is developed for the upper bound on mixing. From these, particular the path-coupling bounds, concluding an upper bound on the mixing time is not difficult.

§9: Lower Bound. The lower bound is a standard coupon-collector argument, as in [BSZ11, BS19]: the number of fixed points is used as a distinguishing statistic. The idea is not new to [BSZ11, BS19]; rather, it has been in a variety of related papers in the past. It is so standard that it is deferred to the appendix of [BS19] and omitted completely from [BSZ11].

Comparison with [BS19]. Our article is strongly inspired by the methods of [BS19]. However, a significant number of new ideas are required. It is not even clear a priori that the PM RW can be related so closely to a conjugacy-invariant RW. The natural approach of viewing an $n$-PM as a permutation on its $2n$ objects does not allow this: the corresponding Cayley graph is not generated by a union conjugacy class. It can be viewed as a Gelfand pair $(\mathbb{S}_{2n}, \mathbb{S}_2 \wr \mathbb{S}_n)$, but this moves far from the probabilistic approach of [BS19] towards the representation-theoretic of [CST07, CST08].

Key is to introduce the ‘cycle structure’ of a PM via cycle lengths in an induced graph. We use this to relate an $n$-PM to an $n$-permutation. We then adjust the techniques developed for conjugacy-invariant RWs to this PM viewpoint. Some of these adjustments are trivial, but many are far from easy. Multiple subtleties arise for the PM RW which are not present for conjugacy-invariant RWs.

We feel that the merit and contribution of this paper is not in the technical proficiency of the argument, but rather in developing the correct viewpoint. The underlying ideas can be found in [BS19]; being able to utilise them for PMs is the challenge. For example, the lack of consistency in the labelling of previously interacted with objects, discussed at length throughout the paper, is a constant source of difficulty: in decomposing a PM into a sequence of swaps in §4; in the tiling of §5; in the construction of the auxiliary graph process in §7. Overcoming such challenges is paramount.

3 Reductions and Realisations

3.1 Cycle Definitions and Reduction to Cycle Structure

Let $\eta \in \mathcal{M}$ be a PM. Consider the graph $([2n], \eta)$, which has $2n$ vertices and edge-set given by the pairs in the PM $\eta$. Every vertex is of degree 1: it is paired with one other vertex. Now consider the union of this graph with $([2n], \text{id})$, as a multigraph, written $([2n], \eta \cup \text{id})$. Every vertex is now of degree 2 and each vertex is contained in a unique cycle of even length—this is counting an isolated double-edge between two vertices as a cycle of length 2. We define the cycle structure of $\eta$ as the vector giving the number of 2-cycles, 4-cycles and so on; see Definition 3.1 for the formal definition.

We sometimes consider arbitrary PMs, not specifying the number of underlying objects: write

$$\mathcal{M}_\infty := \bigcup_{n=1}^\infty \mathcal{M}_n.$$

This is a disjoint union and $\eta \in \mathcal{M}_n$ implies that $|\eta| = n$. Here, $\eta$ is a set whose elements are disjoint, unordered pairs; so $|\eta| = n$ means that there are $n$ disjoint, unordered pairs in $\eta$. We emphasise that every element of $\mathcal{M}_\infty$ is a finite set, i.e., corresponds to a PM on $2n$ objects for some $n \in \mathbb{N}$. 

3 Reductions and Realisations
**Definition 3.1: Cycle Structure.** The cycle structure (CS) of a PM $\eta \in \mathcal{M}_\infty$ is the vector
\[
C(\eta) := (C_1(\eta), C_2(\eta), \ldots) \in \mathbb{N}_0^N,
\]
where $C_\ell(\eta)$ is the number of $2\ell$-cycles in the multigraph $([2|\eta|], \eta \cup \text{id}_{|\eta|})$ for $\ell \in \mathbb{N}$. Write
\[
\mathcal{C}_n := \{ c \in \mathbb{N}_0^N \mid \sum_{\ell=1}^{\infty} \ell c_\ell = n \} \quad \text{for} \quad n \in \mathbb{N};
\]
this is the set of possible CSs for an element of $\mathcal{M}_n$, ie an $n$-PM. Write
\[
\mathcal{C}_\infty := \cup_{n=1}^{\infty} \mathcal{C}_n.
\]

Some examples are given in Figure 4.2. These figures correspond to Example 4.7 in which other statistics—namely, support and swap distance—are analysed. We do not repeat the image here; rather, the reader can look ahead to Page 13 to see the graphs corresponding to different PMs $\eta$.

**Terminology 3.2: Cycle Terminology.** We drop the trailing 0s when writing out a CS, eg writing $(1,2)$ rather than $(1,2,0,0,\ldots)$. A non/fixed point is assumed to be wrt the appropriate identity—id$_{|\eta|}$ in the above case—if the PM to which it is to be compared is omitted.

We refer to $C_\ell(\eta)$ as the number of $\ell$-cycles in the PM $\eta \in \mathcal{M}_\infty$ for $\ell \in \mathbb{N}$, not $2\ell$-cycles. Then, 1-cycles in the PM, ie 2-cycles in the multigraph, are precisely fixed points. Thus, there are $C_1(\eta)$ fixed points in the PM $\eta \in \mathcal{M}_\infty$; analogously, we say that there are $c_1$ fixed points in the CS $c \in \mathcal{C}_\infty$.

This choice of terminology, specifically the $\ell$-cycle vs $2\ell$-cycle distinction, may seem somewhat peculiar at first. However, we shall soon see that it makes the definitions for $n$-PMs analogous to the standard definitions for $n$-permutations, ie permutations on $n$ objects.

We use this terminology to set up the following notation. Recall that $\mathcal{M} = \mathcal{M}_n$ and $\mathcal{C} = \mathcal{C}_n$.

**Notation 3.3.** Write $\mathcal{M}' \subseteq \mathcal{M}$, respectively $\mathcal{C}' \subseteq \mathcal{C}$, for those $\eta \in \mathcal{M}$, respectively $c \in \mathcal{C}$, with at most $k$ non-fixed points wrt the identity id. Write $\mathcal{M}_\infty(c) \subseteq \mathcal{M}_\infty$ for those $\eta \in \mathcal{M}_\infty$ with CS $C(\eta) = c \in \mathcal{C}_\infty$. In mathematical notation, make the following definitions for $c \in \mathcal{C}_\infty$:
\[
\mathcal{M}_\infty(c) := \{ \eta \in \mathcal{M}_\infty \mid C(\eta) = c \}, \quad \mathcal{C}' := \{ c \in \mathcal{C}_n \mid c_1 \geq n - k \};
\]
\[
\mathcal{M}' := \{ \eta \in \mathcal{M}_n \mid C(\eta) \in \mathcal{C}' \} = \{ \eta \in \mathcal{M}_n \mid C_1(\eta) \geq n - k \} = \cup_{c \in \mathcal{C}'} \mathcal{M}_\infty(c).
\]

We emphasise that this is an important definition which the reader should commit to memory: adding a prime (t) to $\mathcal{M}$ or $\mathcal{C}$, giving $\mathcal{M}'$ or $\mathcal{C}'$, indicates that there are at most $k$ non-fixed points.

The purpose of introducing this cycle structure is that the law of the PM RW given its CS is uniform over all PMs with this given CS. A completely analogous projection is often used when studying conjugacy-invariant RWs on the permutation group. This means that projecting from $\mathcal{M}$ to $\mathcal{C}$ does not decrease the TV distance from equilibrium. We use $L(\cdot)$ to denote the law of a random variable. Abbreviate $C_t := C(M_t) \in \mathcal{C}$. Denote by $\pi_{\mathcal{C}}$ the invariant distribution of $C := (C_t)_{t \geq 0}$.

**Lemma 3.4: TV-Preserving Projection to Cycle Structure.** The projection of the PM RW from the perfect-matching space $\mathcal{M}$ to the cycle-structure space $\mathcal{C}$ is TV-preserving:
\[
\|L(M_t) - \pi_{\mathcal{M}}\|_{TV} = \|L(C_t) - \pi_{\mathcal{C}}\|_{TV}.
\]

**Proof.** The pairs at each round of the PM RW are chosen uniformly and independently between rounds. The uniformity of the PM RW given its CS is thus an immediate consequence of this symmetry. The TV-preservation claim follows immediately from this, eg by a trivial coupling. \(\square\)

### 3.2 From Cycle Structures to Integer Partitions
We have reduced from PMs to CSs. We now explain how to reduce further: from CSs to partitions.

Recall that all the cycles in the graph $([2|\eta|], \eta \cup \text{id})$ have even lengths and are disjoint. They thus form an integer partition of $[2n] = \{1, \ldots, 2n\}$. The terminology we used divided these lengths...
by 2—eg a fixed point, or 1-cycle, of a PM or CS corresponded to a 2-cycle in the graph. These halved values are all integers and form an integer partition of \([n] = \{1, \ldots, n\}\).

We can further divide these values by \(n\) to get a partition of \([0,1]\) with block lengths in \(\{0,1/n,1\} = [0,1] \cap (\frac{1}{n} \mathbb{Z})\). We refer to this latter situation as a \(\frac{1}{n}\)-integer partition of \([0,1]\).

We tend to drop \(\frac{1}{n}\)-prefactor, including it only when there may be ambiguity. Write

\[
\mathcal{P}_n := \{(x_1, \ldots, x_n) \in \{0,1/n, \ldots, (n-1)/n, 1\}^n \mid x_1 \geq \cdots \geq x_n, \sum_{i=1}^{\infty} x_i = 1\}.
\]

We are not always concerned about the non-increasing order of the blocks in the partition; in this case, we write a partition \(x \in \mathcal{P}_n\) as an unordered multiset \([x_1, \ldots, x_n]\). Abbreviate \(\mathcal{P} := \mathcal{P}_n\).

We also define the limiting case, which we refer to as a continuous partition of \([0,1]\). Write

\[
\mathcal{P}_\infty := \{(x_1, x_2, \ldots) \in [0,1]^\mathbb{Z} \mid x_1 \geq x_2 \geq \cdots, \sum_{i=1}^{\infty} x_i = 1\}.
\]

Contrary to our previous notation, \(\mathcal{P}_\infty \neq \bigcup_{n=1}^{\infty} \mathcal{P}_n\): all partitions have finitely many blocks in the latter. Further, the entries \(x_i\) of \(x \in \mathcal{P}_\infty\) need not be rationals.

**Definition 3.5: Coalescence–Fragmentation Chain.** Define \(\mathcal{P}(\eta) \in \mathcal{P}\) to be the integer partition corresponding to the PM \(\eta \in \mathfrak{M}\), with blocks in non-increasing order of size. Abbreviate \(P_t := \mathcal{P}(M_t)\), where \((M_t)_{t \geq 0} \in \mathfrak{M}^{\mathbb{N}_0}\) is the k-PM RW. Write \(\pi_\eta\) for the invariant distribution of \(P := (P_t)_{t \geq 0}\). The chain \((P_t)_{t \geq 0}\) is a coalescence–fragmentation chain.

We now describe the evolution of the coalescence–fragmentation chain when \(k = 2\). We then comment on how it differs from the corresponding chain for the random transpositions shuffle.

**Lemma 3.6: Evolution of Coalescence–Fragmentation Chain.** Suppose that \(k = 2\). Suppose that the coalescence–fragmentation chain is at the integer partition \(\lambda = [\lambda_1, \ldots, \lambda_n] \in \mathcal{P}\). Suppose that the two pairs/matches chosen are indexed by \(i\) and \(j\) are chosen, respectively. There are two cases.

- If the two pairs are in different blocks, say \(i \in \lambda_1\) and \(j \in \lambda_2\), then these two blocks merge. The resulting partition is given by \([\lambda_1 \cup \lambda_2, \lambda_3, \ldots, \lambda_n]\).

- If the two pairs are in the same block, say \(i, j \in \lambda_1\), then with probability \(\frac{1}{2}\) nothing changes and with probability \(\frac{1}{2}\) the block splits, say into \(\lambda_1^-\) and \(\lambda_1^+\). If there is a split, then location of the split is uniform. The resulting partition is given by \([\lambda_1^-, \lambda_1^+, \lambda_2, \ldots, \lambda_n]\) when there is a split.

The evolution is similar for \(k > 2\): one generates the \(k\)-rematching via \(k - 1\) swaps in a certain manner, described in Algorithm 4.1 below, and applies the above cases to each of the \(k - 1\) swaps.

**Remark 3.7: Comparison with Random Transpositions.** The chain corresponding to the random transpositions shuffle is extremely similar. The only difference is that there is a split every time the cards are in the same block there. The split occurs with probability \(\frac{1}{2}\) for our chain corresponding to PMs. This is the only difference. We demonstrate this in Figure 3.1. Splits are rejected half the time in the antiferromagnetic Heisenberg model, studied in [BKL19]; too; see Figure 1.1. △

The next lemma shows why we introduced the coalescence–fragmentation chain. Analogous results are used in [Sch05, Bor11, BZ11, BS19] for conjugacy-invariant RWs and in [DH02, BKL19] for 2-PM RW. Related claims are proved via representation theory, eg in [CST07, CST08, Hou16].

**Lemma 3.8: TV-Preserving Projection to Coalescence–Fragmentation Chain.** The projection of the \(k\)-PM RW from the perfect-matching space \(\mathfrak{M}\) to the integer-partition space \(\mathcal{P}\) is TV-preserving:

\[
\|L(M_t) - \pi_{\mathfrak{M}}\|_{TV} = \|L(C_t) - \pi_\varepsilon\|_{TV} = \|L(P_t) - \pi_\mathcal{P}\|_{TV}.
\]

**Proof.** The first equality is precisely Lemma 3.4. The second equality follows from arguments analogous to those used in there. The cycles partition \([1, \ldots, n]\) into blocks and the integer partition records how many blocks of each size there are. It does not, however, record where the blocks are located. Eg, the partitions of \([1, 2, 3, 4, 5, 6]\) given by \([\{1,2\}, \{3,4,5,6\}\] and \([\{1,2,3,4\}, \{5\}\] are different partitions, yet have the same block sizes. By symmetry, the law of the CS given its integer partition is uniform over all CSs with this given integer partition. □
Remark 3.9: Limiting Distribution. The limiting invariant distribution as \( n \to \infty \) is known to be the so-called Poisson–Dirichlet distribution with parameter \( \theta = \frac{1}{2} \), denoted \( \text{PD}(\frac{1}{2}) \). More precisely, the joint law of the rescaled cycle sizes converge in distribution to \( \text{PD}(\frac{1}{2}) \); see [BKLM19, Theorem 1.1]. This is revisited in more detail in §5.2; see, in particular, Lemma 5.10 and the surrounding discussion. A related Poisson–Dirichlet limit is investigated by Pitman [Pit02]. \( \triangle \)

4 Decomposing a Perfect Matching into a Sequence of Swaps

4.1 Generating a \( k \)-PM via \( k - 1 \) Swaps

Let \( \eta \in \Omega \) be an \( n \)-PM. Consider a single step of the \( k \)-PM RW, starting from \( \eta \): \( k \) pairs are chosen uniformly at random (uar); the PM restricted to these \( 2k \) objects is resampled and the remaining \( 2(n-k) \) objects are left alone. The resampled object is, up to a permutation of the labels, a \( k \)-PM. Being able to sample a \( k \)-PM uniformly is then sufficient in order to run the dynamics. We now describe a way to sample a \( k \)-PM uar choosing only 2 matches at a time.

Choose an arbitrary cycle structure \( c \in \mathcal{C} \); this has at most \( k \) non-fixed points and corresponds to a \( k \)-rematching inside an \( n \)-PM. We show how to draw \( \eta \) uar conditional on having CS \( c \), ie on \( \mathcal{C}(\eta) = c \). Recall that the CS \( c \) implicitly partitions \([k]\) and, by symmetry, the partition is uniform amongst all partitions with appropriately sized parts. The relative matching inside different blocks of the partition is independent. Thus, it suffices to be able to sample a single cycle of arbitrary length, ie an \( \ell \)-PM with one \( \ell \)-cycle for any \( \ell \in \mathbb{N} \). This is analogous to sampling uniformly a permutation given its CS. PMs with a single \( \ell \)-cycle are elements of \( \delta_{\ell} := \mathcal{C}_\infty(\delta_{\ell}) \) where \( \delta_{\ell}(m) := 1 \{ m = \ell \} \) for \( \ell, m \in \mathbb{N} \). How to sample such a PM is described in Algorithm 4.1 below.

It is well-known and easy to generate an \( \ell \)-cycle permutation via \( \ell - 1 \) transpositions, ie 2-cycles. This does not generalise to PMs, however. This was a highly non-trivial obstacle for us.

We first describe the usual way to sample an \( \ell \)-cycle permutation uar. Generation of an \( \ell \)-cycle permutation is trivial for \( \ell = 1 \) as there is only one 1-cycle. Assume now that \( \ell > 1 \). We use an inductive construction. Choose independently \( a_1 \sim \text{Unif}([n]) \), then \( a_2 \sim \text{Unif}([n] \setminus \{a_1\}) \), then \( a_3 \sim \text{Unif}([n] \setminus \{a_1,a_2\}) \) and so on. Then a uniformly random \( \ell \)-cycle can be generated via

\[
\sigma := (a_1,a_2)(a_2,a_3) \cdots (a_{\ell-1},a_\ell) = (a_1,\ldots,a_\ell).
\]

This does not generalise well to PMs because there is no concept of “using the second element of the previous transposition as the first element of the current transposition”: which of the new matches is the ‘second’ one? See Figure 3.1 for a visualisation of this difficulty. This difficulty and highly related ones will rear its ugly head repeatedly throughout our arguments.

We can adjust this method for sampling an \( \ell \)-cycle permutation in a subtle way, which is then amenable to PMs. Let \( b_i \sim \text{Unif}\{\{a_1,\ldots,a_i\}\} \) independently for each \( i \in [\ell] \) and set

\[
\sigma := (b_1,b_2)(b_2,b_3) \cdots (b_{\ell-1},a_\ell).
\]

It is easy to check that \( \sigma \) is still a uniform \( \ell \)-cycle. We are “using a uniformly random previously-used element as the first element of the current transposition”. We can translate this into the realm of PMs since we do know the set of previously interacted with pairs, so can choose one uar.

Figure 3.1. The pair of vertical edges is picked and rematched into either a cross (left) or a bar (right). The bar splits the cycle in two, but the cross does not.
The first step interacts with only \( \{a, b\} \); we can thus associate \( c' = c \). The second step interacts with \( \{b', c' = c\} \); there is no natural way to say whether \( b' \) corresponds to \( a \) or to \( b \). Our algorithm only needs the equality \( a \cup b = a' \cup b' \) (as sets); it chooses \( U \sim \text{Unif}(\{a', b'\}) \) does a swap with \( \{U, c\} \).

**Algorithm 4.1:** Generating a Uniform Cycle via Swaps. Initialise \( \eta_0 = \{\eta_{0,1}, ..., \eta_{0,\ell}\} := \text{id}_\ell \). Choose \( i \sim \text{Unif}(\lfloor \ell \rfloor) \) and set \( S_0 := \{i\} \). Perform the following steps sequentially for \( s = 1, ..., \ell - 1 \).

- Choose \( i \sim \text{Unif}(S_{s-1}) \) and \( j \sim \text{Unif}(\lfloor \ell \rfloor \setminus S_{s-1}) \) independently.
- Choose uniformly a new matching on \( \eta_{s-1,i} \cup \eta_{s-1,j} \) conditional on not being equal to \( \{\eta_{s-1,i}, \eta_{s-1,j}\} \). Denote this new matching \( \{x, y\} \). Note that \( x \cup y = \eta_{s-1,i} \cup \eta_{s-1,j} \).
- Set \( \eta_{s,m} := \eta_{s-1,m} \) for \( m \notin \{i, j\} \), set \( \eta_{s,i} := x \) and set \( \eta_{s,j} := y \). Set \( S_s := S_{s-1} \cup \{i, j\} \).

Output \( H := \eta_{\ell-1} = \{\eta_{\ell-1,1}, ..., \eta_{\ell-1,\ell}\} \).

**Lemma 4.2:** Generating a Uniform Cycle via Swaps. Let \( \ell \in \mathbb{N} \). Let \( H \) denote the (random) output of Algorithm 4.1. Recall that \( \mathfrak{M}_\ell = \mathfrak{C}_\infty(\delta_\ell) \) is the set of single-cycle \( \ell \)-PMs. Then,

\[
H \sim \text{Unif}(\mathfrak{M}_\ell).
\]

**Proof.** We use induction on \( \ell \). The bases cases \( \ell \in \{1, 2\} \) are trivial. The case \( \ell = 3 \) is easy to check by hand. Now assume that the claim holds for \( \ell \). We establish it for \( \ell + 1 \).

The algorithm consists of \( \ell \) steps. We break it into two parts: the first \( \ell - 1 \) steps and the final step. We use the notation from Algorithm 4.1. Note that \( |S_s| = s + 1 \). Let \( i \) be the (random) unique element of \( \lfloor \ell + 1 \rfloor \setminus S_{\ell-1} \). By symmetry, \( i \sim \text{Unif}(\lfloor \ell + 1 \rfloor) \). The inductive hypothesis implies that

\[
\eta_{\ell-1, \lfloor \ell + 1 \rfloor \setminus \{i\}} \sim \text{Unif}(\lfloor \ell + 1 \rfloor \setminus \{i\}) \quad \text{given} \quad i, \quad \text{or equivalently} \quad S_{\ell-1}.
\]

That is, the restriction of \( \eta_{\ell-1} \) to \( \lfloor \ell + 1 \rfloor \setminus \{i\} \) is a uniformly random \( \ell \)-PM on its support, given \( i \). The nature of Algorithm 4.1 means that the remaining matched pair is untouched: \( \eta_{0,i} = \cdots = \eta_{\ell-1,i} \). The final step comprises a swap with the \( i \)-th pair and the \( j \)-th pair, where \( j \sim \text{Unif}(\lfloor \ell + 1 \rfloor \setminus \{i\}) \) and is independent of \( i \). This leads to a uniform, single-cycle \((\ell + 1)\)-PM, as desired.

We now use this to generate a uniform \( k \)-rematching in the space of \( n \)-PMs, ie an \( n \)-PM with at most \( k \) non-fixed points. We break down a PM into its individual cycles according to its CS.

**Algorithm 4.3:** Generating a Uniform \( k \)-Matching via Its Cycle Decomposition. Let \( k \in [2, n] \cap \mathbb{N} \).

- Draw \( C \sim \text{C}(\text{Unif}(\mathfrak{M}')) \).
- Draw partition \( P = (P_1, ..., P_S) \) uniformly, conditional on having block lengths given by \( C \).
- Draw \( H_s \sim \text{Unif}(\mathfrak{M}_{\lfloor P_s \rfloor}) \), which is a single \( |P_s| \)-cycle, independently for each \( s \in [S] \).
- Combine to create \( H \): place the \( s \)-th cycle \( H_s \) in the \( s \)-th block \( P_s \) for each \( s \in [S] \).

Output \( H \).

It is immediate from Lemma 4.2 and the cycle decomposition that Algorithm 4.3 gives rise to a uniform element of \( \mathfrak{M}' \), ie \( n \)-PM with at most \( k \) non-fixed points.

4 Decomposing a Perfect Matching into a Sequence of Swaps
Corollary 4.4: Generating a Uniform $k$-Matching via Its Cycle Decomposition. Let $k, n \in \mathbb{N}$ with $2 \leq k \leq n$. Let $H$ denote the (random) output of Algorithm 4.3. Then,

$$H \sim \text{Unif}(\mathcal{M}).$$

4.2 Support Size and Distance from Identity for a Uniform $k$-Rematch

We now know how to sample a $k$-PM uniformly at random given its CS. A priori, one may assume that we must now calculate the law $CS$ of a uniform $k$-PM. It turns out that our proof does not require this, however. The only information we need is the support of the CS—namely, the number of pairs interacted with. E.g., the support of the CS $(0, 2)$ is 4 and of $(0, 0, 1)$ is 3; see Definition 4.5.

We take inspiration from the work of Berestycki and Şengül [BS19] on conjugacy-invariant RWs. They show that the mixing time is inversely proportional to the support of the CS used.

Definition 4.5: Support. The (size of the) support of a CS $c \in \mathcal{C}_\infty$ and a PM $\eta \in \mathcal{M}_\infty$ is

$$\#c := \sum_{\ell=2}^{\infty} (c_\ell) \quad \text{and} \quad \#\eta := \#C(\eta),$$

respectively. This is the number of non-fixed points:

$$\text{if } C(\eta) = c \in c^k, \text{ then } \#\eta = \#c = k - c_1.$$

We can view a $k$-PM as an $n$-PM by padding the end with $n - k$ fixed points. More formally, view an element $\eta \in \mathcal{M}_k$ as an element $\eta' \in \mathcal{M}'_n \subseteq \mathcal{M}_n$ by setting $\eta'_i := i$ for $i \in \{2k + 1, \ldots, n\}$. Let $c := C(\eta)$ and $c' := C(\eta')$ denote the CSs. Then $c_1' = c_1 + (n - k)$ and $c_\ell' = c_\ell$ for $\ell \geq 2$. Thus,

$$\#c = \sum_{\ell=2}^{\infty} \ell c_\ell = \sum_{\ell=2}^{\infty} \ell c_\ell' = \#c' \quad \text{and} \quad \#c = k - c_1 = n - c_1' = \#c'.$$

Berestycki and Şengül [BS19] use $|\cdot|$ to denote the support. This already has an established meaning of “size” or “cardinality” for sets, which PMs are. We use $\#$ to avoid this notational clash.

We also define the swap distance. This is just the number of swaps required to reach the identity.

Definition 4.6: Swap Distance. For a CS $c \in \mathcal{C}_\infty$ and a PM $\eta \in \mathcal{M}_\infty$, define

$$d(c) := \sum_{\ell=2}^{\infty} (\ell - 1) c_\ell \quad \text{and} \quad d(\eta) := d(C(\eta)).$$

We refer to $d$ as the swap distance from the identity, often referred to as just distance for brevity. Equivalently, $d(\eta)$ is the minimal number of swaps required to reach $id_{\mathcal{M}}$. Indeed, each $\ell$-cycle needs precisely $\ell - 1$ swaps to resolve its disparities compared with the identity.

Analogously, define $d(\eta, \eta')$ to be the minimal number of swaps required to move from $\eta$ to $\eta'$ for $\eta, \eta' \in \mathcal{M}_\infty$ with $|\eta| = |\eta'|$. This is the usual distance in the graph which has PMs as vertices and edges between PMs which differ by a single swap. This graph is transitive. Given $(\eta, \eta') \in \mathcal{M} \times \mathcal{M}$, relabel the objects so that these become $(\tilde{\eta}, \tilde{\eta'}) \in \mathcal{M} \times \mathcal{M}$ with $\tilde{\eta}' = id$. Then, $d(\tilde{\eta}, \tilde{\eta'}) = d(\tilde{\eta}).$

Example 4.7: Some Examples. Consider PMs on 8 elements, shown in Figure 4.2.

(a) Take $\eta := \{(1, 3), \{2, 4\}, \{5, 7\}, \{6, 8\}\}$. The graph is shown in Figure 4.2 (a).

Then $c := C(\eta) = (0, 2, 0, 0)$. Hence $\#\eta = \#c = 4$ and $d(\eta) = d(c) = 2$.

(b) Take $\eta := \{(1, 4), \{2, 3\}, \{5, 8\}, \{6, 7\}\}$. The graph is shown in Figure 4.2 (b).

Then $c := C(\eta) = (0, 2, 0, 0)$. Hence $\#\eta = \#c = 4$ and $d(\eta) = d(c) = 2$.

(c) Take $\eta := \{(1, 4), \{3, 6\}, \{5, 8\}, \{7, 2\}\}$. The graph is shown in Figure 4.2 (c).

Then $c := C(\eta) = (0, 0, 0, 1)$. Hence $\#\eta = \#c = 4$ and $d(\eta) = d(c) = 3$.

(d) Take $\eta := \{(1, 4), \{2, 6\}, \{3, 5\}, \{7, 8\}\}$. The graph is shown in Figure 4.2 (d).

Then $c := C(\eta) = (1, 0, 1, 0)$. Hence $\#\eta = \#c = 3$ and $d(\eta) = d(c) = 2$. \(\triangle\)

The key information required from the law of the CS, which is that of a uniform $k$-PM, is its expected support and distance. The mixing time actually only depends on the expected support; the expected distance is a tool used in the proof which need not be calculated explicitly.
**Definition 4.8:** Expected Support and Distance of Uniform $k$-Rematching. Let $k \in \mathbb{N} \setminus \{1\}$. Let $H \sim \text{Unif}(\mathcal{M}_k)$. Define the expected support $\kappa_k$ and expected distance $\rho_k$ as follows:

\[
\kappa_k := \mathbb{E}(\#H) = \mathbb{E}\left(\sum_{\ell=2}^{\infty} \ell \mathbb{C}_\ell(H)\right) = |\mathcal{M}_k|^{-1} \sum_{\eta \in \mathcal{M}_k} \sum_{\ell=2}^{k} \ell \mathbb{C}_\ell(\eta);
\]

\[
\rho_k := \mathbb{E}(d(H)) = \mathbb{E}\left(\sum_{\ell=2}^{\infty} (\ell - 1) \mathbb{C}_\ell(H)\right) = |\mathcal{M}_k|^{-1} \sum_{\eta \in \mathcal{M}_k} \sum_{\ell=2}^{k} (\ell - 1) \mathbb{C}_\ell(\eta).
\]

We are always interested in the expected support of a $k$-PM. Somewhat unusually, we abbreviate $\kappa := \kappa_k$. Officially, $k$ is a function of $n$, so this is suppressing the $n$-dependence via abbreviating $\kappa_k$.

**Lemma 4.9:** Expected Support of Uniform $k$-Rematching. We have

\[
\kappa_k = \mathbb{E}(\#H) = k - \frac{k}{2k-1} = k - \frac{1}{2} - \frac{1}{2k} + O(k^{-2}).
\]

**Proof.** This follows by a simple counting argument. Indeed, $\#H$ is simply $k$ minus the number of fixed points. Thus, we just need to calculate the number of fixed points in expectation. Start with the pair \{1, 2\} matched. There are $2k - 1$ other vertices to which 1 can be matched. So the probability that it remains matched to 2 is $1/(2k - 1)$. The expected number of fixed points is then $k/(2k - 1)$, by linearity of expectation, as $k$ matches are made. The lemma follows. \[\square\]

**Remark 4.10:** Support and Its Relation to Mixing. One can think of the support as “the number of random choices”. We analogue with permutations: a 3-cycle $(a, b, c)$ can be written as $(a, b)(b, c)$ and there are three choices, namely $a$, $b$ and $c$; a double-transposition $(a, b)(c, d)$ has four choices.

Berestycki and Şengüll [BS19] analyse the mixing time of the RW on the Cayley graph of the permutation group generated by a preset CS $\Gamma$: a step comprises applying a uniform permutation given the CS. They show that the mixing time is inversely proportional to the support $\#\Gamma$.

The lower bound given by [BS19] finds the time it takes for all cards to be touched. Decomposing a permutation into its CS, the number of cards touched equals the support $\#\Gamma$. A coupon-collector argument can be applied when this support has size $o(n)$ to deduce a lower bound of $(\#\Gamma)^{-1} n \log n$ when there are $n$ cards. It had long been conjectured that this, i.e. the time at which all cards have been touched, is indeed the correct mixing time. Berestycki and Şengüll [BS19] establish this.

We adjust this heuristic to PMs. The number of pairs interacted with in a given rematching is the support of that rematching. Suppose $k_i$ are interacted with on the $t$-th step. We wait until all the original pairs have been interacted with. We want to apply a coupon-collector counting argument to estimate this time. We have to be careful, though. Suppose that $\{1, 2\}, \{3, 4\}, \{5, 6\}$ becomes $\{\{1, 3\}, \{2, 4\}, \{5, 6\}\}$ and then $\{\{1, 3\}, \{2, 5\}, \{4, 6\}\}$; the first two pairs are interacted.
with in the first step, but which are in the second step? Certainly the third pair is, but is the first or the second? It does not matter: the first and second pairs have already been ‘collected’ in the first step. All that matters is that the second step included the third pair. Any as-yet ‘uncollected’ pair is in its original position, by definition. This allows us to apply the coupon-collector argument it the usual way: the collection takes time approximately \( \inf\{t \geq 0 \mid k_1 + \cdots + k_t \geq n \log n\} \).

The steps are independent, so the law of large numbers says that \( k_1 + \cdots + k_t \approx t \kappa \). It is thus natural to conjecture a mixing time of \( \kappa^{-1} n \log n \). This is what our main theorem verifies. \( \triangle \)

## 5 Analysis of Coalescence–Fragmentation Chain

### 5.1 Conditional Uniformity

Recall that a general \( k \)-PM can be written as a product of disjoint, single-cycle PM. The different single-cycle PMs correspond to different blocks in the partition. The order of their application is thus inconsequential. The single-cycle PMs are broken down into swaps. See Algorithm 4.1 for more details. General permutations of a given CS have a similar independence property.

These properties lead us to the notion of a refresh time, corresponds to the start of a new block.

**Definition 5.1: Refresh Time.** Let \( c \in \mathcal{C}_\infty \). Recall that \( d(c) = \sum_{\ell=2}^{\infty} (\ell - 1) c_\ell \) is the swap distance from the identity. Sampling uniformly from \( \mathcal{M}_\infty(c) \) involves \( d(c) \) swaps, grouped together in different batches: the application of an \( \ell \)-cycle requires \( \ell - 1 \) swaps; each batch corresponds to a block in the associated partition. Call \( s \in \{1, \ldots, d(c)\} \) a refresh time for \( c \) if it is of the form \( s = \sum_{i=2}^{m} (i - 1) c_i + 1 \) for some \( m \in \mathbb{N} \). In particular, 1 is a refresh time, since the empty sum is 0.

Berestycki and Şengül [BS19, Definition 4.1] define refresh times similarly. They always apply a permutation with the same CS. This means that all the refresh times can be defined in advance. Our CS is not preset, but varies from step to step. Thus, the refresh times vary according to the CS of the PM chosen in a given step. It will be enough, however, to condition in advance on the sequence of CSs, from which we can define the refresh times.

We now describe Algorithms 4.1 and 4.3 in terms of these refresh times. The reformulation is given as Algorithm 5.2. Importantly, we distinguish the first and second markers in the choice of two matches for a swap: the first marker is \( i \) and the second \( j \) in Algorithm 4.1.

**Algorithm 5.2: Conditional Uniformity.** Let \( c \in \mathcal{C} \) be a CS. Let \( s \in \mathbb{N} \). Set \( S'_0 := \emptyset \) and \( S_0 := \emptyset \). If \( s > 1 \), then let \((i_1, j_1), \ldots, (i_{s-1}, j_{s-1})\) denote the pairs chosen in the first \( s - 1 \) steps and set

\[
S'_{s-1} := \cup_{r=s-1}^{s-1} \{i_r, j_r\} \quad \text{and} \quad S_{s-1} := \cup_{r=s-1}^{s-1} \{i_r, j_r\},
\]

where \( s_- := \sup\{s' \leq s \mid s' \text{ is a refresh time}\} \) is the most recent refresh time before \( s \). This way, \( S'_{s-1} \), respectively \( S_{s-1} \), is the set of indices used in the previous blocks, respectively current block.\(^1\)

Perform the following steps sequentially for \( s = 1, \ldots, d(c) \). There are two cases according to whether \( s \) is a refresh time for \( c \) or not. If \( s \) is a refresh time, then \( s_- = s \) and thus \( S_{s-1} = \emptyset \).

- If \( s \) is a refresh time for \( c \), then it corresponds to the start of a new cycle.
  - The first marker \( i_s \) is chosen uniformly on \([n] \setminus S'_{s-1}\).
  - The second marker \( j_s \) is chosen uniformly on \([n] \setminus (S_{s-1} \cup \{i_s\})\).
- If \( s \) is not a refresh time for \( c \), then it corresponds to the continuation of a cycle.
  - The first marker \( i_s \) is chosen uniformly on \( S_{s-1} \).
  - The second marker \( j_s \) is chosen uniformly on \([n] \setminus (S'_{s-1} \cup S_{s-1} \cup \{i_s\})\).
- Perform a uniform swap of the pairs \( i_s \) and \( j_s \). \( \triangle \)

**Lemma 5.3: Conditional Uniformity Algorithm.** Let \( c \in \mathcal{C} \). The output of Algorithm 5.2 is a uniform PM with cycle structure \( c \), ie is a uniform element of \( \mathcal{G}(c) \).

---

\(^1\)This definition of \((S_{s})_{s>0}\) is a natural extension of Algorithm 4.1 where only a single cycle is considered. Performing a swap destroys the two original matches and two new ones are created; there is no real way of associating the old matches with the new ones. However, the set union of the objects interacted with is always well-defined.
The introduction of this map was one of the main innovations of Schramm [Sch05].

Definition 5.5: Relaxed Conditional Uniformity

\[ \text{Algorithm 5.4: Relaxed Conditional Uniformity.} \]
Let \( c \in \mathcal{C} \) be a cycle structure. Use the same notation for \((S_{s-1}), s \geq 0\) as in Algorithms 4.1 and 5.2. Perform the following steps sequentially for \( s = 1, \ldots, d(c) \).

1. If \( s \) is a refresh time for \( c \), then sample \( i_s \sim \text{Unif}(\{n\}) \) and \( j_s \sim \text{Unif}(\{n\} \setminus \{i_s\}) \).
2. If \( s \) is not a refresh time for \( c \), then sample \( i_s \sim \text{Unif}(S_{s-1}) \) and \( j_s \sim \text{Unif}(\{n\} \setminus \{i_s\}) \).

Perform a uniform swap with \((i_s, j_s)\) in each step \( s \).

\[ \text{Lemma 5.6: Relaxed Conditional Uniformity.} \]
Suppose that \( \Delta \) steps are taken under relaxed conditional uniformity (Algorithm 5.4). Then the probability that any choice of \((i, j)\) violates the original conditional uniformity conditions (Algorithm 5.2) is at most \( 2\Delta k/n \).

\[ \text{Proof.} \]
The condition is violated in a given step if the marker falls in the set of those already chosen. This set has size at most \( k \). Two markers are chosen each time in a uniform manner.

We henceforth proceed using the relaxed version of conditional uniformity. This is convenient for adapting a coupling which is based on an idea of Schramm [Sch05, §3]. The original coupling of [Sch05, §3] is designed for random transpositions, which naturally satisfies the relaxed version.

5.2 Schramm’s Coupling for the Coalescence–Fragmentation Chain

Key to the analysis of Berestycki and Şengül [BSZ11] is their use of a coupling between two realisations of the coalescence–fragmentation chain; see [BS19, §4.2.2]. The same coupling had already been used by [BSZ11, §3]. The original idea is due to Schramm [Sch05, §3]. There is a crucial difference in [BSZ11, BS19] compared with [Sch05]: the introduction of the measure-preserving map \( \Phi \), given in Algorithm 5.11 below. The introduction of this map was one of the main innovations of [BSZ11]. A version of Schramm’s coupling has been used recently by Björnberg, Kotowski, Lees and Miloš [BKLM19, §5.2] in a set-up similar to ours, but without the adaptation of [BSZ11, BS19].

Our description follows closely that of [BSZ11, BS19]. Some changes are required to take into account the fact that, for us, a single block does not always split when both markers fall in it.

First we describe the marginal evolution of the partition. We view this as a tiling of \((0, 1]\).

Set-Up for Tiling. We describe how to simulate a single round, ie \( k \)-rematch, via individual steps, ie swaps (2-rematches). To extend to multiple rounds, the procedure is repeated independently.
Let $c \in \mathcal{C}'$. This corresponds to choosing a $k$-rematching amongst $n$ objects. We apply $d(c)$ swaps. This involves choosing markers for each $s \in \{1, \ldots, d(c)\}$. This choice is performed differently according to whether or not $s$ is a refresh time; recall Definition 5.1 and Algorithm 5.4.

We use $P = (P_t)_{t \geq 0}$ to denote the coalescence–fragmentation process. We denote it $\mathcal{P} = (\mathcal{P}_s)_{s \geq 0}$ when looking at a single step, broken down by swaps indexed by $s$. This implicitly assumes that the CSs have been conditioned on. The process on the swap-timescale is then well-defined. Both $s$ and $t$ here indicate time: $t$ in the sense of the number of rounds, whilst $s$ in the sense of swaps.

The set $\mathcal{P}$ comprises all $\frac{1}{n}$-integer partitions. Given $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{P}$, we tile the interval $(0, 1]$ using the intervals $\{(0, \lambda_1], \ldots, (0, \lambda_n]\}$—the specific rule does not matter. We choose markers $u$ and $v$ in $\{1/n, \ldots, n/n\} \subseteq (0, 1]$ below and use them, scaled by $n$, as markers in Algorithm 5.4. △

**Definition 5.7:** Distinguished Tile. Suppose that $s \geq 1$ and that $\mathcal{P}_0, \ldots, \mathcal{P}_{s-1} \in \mathcal{P}$ have been defined.

- If $s$ is a refresh time, which includes $s = 1$, then select $u \sim \text{Unif}\{\{1/n, \ldots, n/n\}\}$ and distinguish the tile containing $u$; use it as the first marker in Algorithm 5.4.
- If $s$ is not a refresh time, and hence $s \geq 2$, then distinguish the tile containing the first marker of the current swap, ie. $\mathcal{I}_s$ in Algorithm 5.4.

There is a distinguished tile containing the first marker for the step $s$ in either case.

Define $\mathcal{P}_s$ to be the new partition, written in non-increasing block size order.

**Remark 5.8:** Permutations vs PMs. Berestycki and Şengil [BS19, §4.2.2] use an analogous distinguished tile construction. It is variant on fundamental ideas introduced by Schramm [Sch05]. There is a key difference, however, in theirs compared with ours:

- they take the distinguished tile to be the second marker from the previous swap, ie. $\mathcal{I}_s = \mathcal{I}_{s-1}$;
- we must choose $\mathcal{I}_s$ uniformly from the already-used indices $\mathcal{I}_{s-1} \in \mathcal{I}_{s-1}$.

The marginal evolution of the distinguished tile is thus simpler in their setup.

Björnberg, Kotowski, Lees and Miloś [BKLM19, §5.2] use a variant on Schramm’s coupling for the PM RW. They do not face similar issues, however, because they only study the 2-PM RW. This means that each round involves choosing only a single pair to swap and rounds are independent.

We thus need to extend the content of [BS19, §4.2.2] and [BKLM19, §5.2] with new ideas. △

The next algorithm describes our marginal evolution under relaxed conditional uniformity.

**Algorithm 5.9:** Marginal Evolution of the Tiling. We now describe a single step of the evolution of the tiling. Multiple steps are obtained by repeating the single-step evolution.

The single-step evolution is given by a map $T: (\lambda; u, v; b) \rightarrow \lambda'$. This implicitly assumes that $\lambda$ is at the left and $u = 1/n$. We use $u$ and $v$ as the first and marker in Algorithm 5.4, respectively.

- If $I \neq I'$, then merge tiles $I$ and $I'$ into a single tile. The new tile has size $|I| + |I'|$.
- If $I = I'$, then propose a split into two fragments at $v - 1/n$. Split if and only if $b = 1$: the new left-hand tile has size $v - 1/n$ and the new right-hand tile has size $|I| - (v - 1/n)$.

The output tiling $\lambda'$ is the tiling after this change written in non-increasing order.

This is extended to multiple steps by drawing $(u, v)$ as in Algorithm 5.4, scaled by $n$, and letting $b \sim \text{Bern}(\frac{1}{2})$ independently at each step. △

A continuous version of these dynamics has also been studied. There, $u, v \sim \text{Unif}(\{0, 1\})$, $b \sim \text{Bern}(\theta)$ and $w = v, \theta := \frac{1}{n}$ corresponds to our set-up, but general $\theta$ can be studied. The following lemma was proved by Pitman [Pit02] for general $\theta$. The case $\theta = 1$ was proved by Tsilevich [Tsi99]. See [GUW11, Theorem 7.1] for another proof. The main result of Schramm [Sch05] is that a corresponding tiling for random transpositions converges weakly to this continuous limit. It is important for Björnberg, Kotowski, Lees and Miloś [BKLM19] too; see [BKLM19, Lemma 5.2].

**Lemma 5.10:** Invariant Distribution of Tiling. The PD($\theta$) distribution is invariant for the continuous version of the tiling dynamics in Algorithm 5.9 when splits are accepted with probability $\theta$. 

5 Analysis of Coalescence–Fragmentation Chain
We now describe a coupling of two tilings. The history of this coupling was described at the start of this subsection. Let \( P \) and \( Q \) denote two realisations which are to be coupled. We describe a single round, as for the marginal evolution in Algorithm 5.9. Multiple rounds are obtained by repeating the single round procedure as described in Algorithm 5.9, using Algorithm 5.4. We use the notation \( \bar{P} = (\bar{P}_s)_{s \geq 0} \) and \( \bar{Q} = (\bar{Q}_s)_{s \geq 0} \) for a single round, analogously to before.

Suppose that the current partitions are \( \lambda \) and \( \mu \). Create two tilings of \( (0,1] \) using \( \lambda \) and using \( \mu \). We differentiate between the blocks that are \textit{matched} versus those that are \textit{unmatched}:

two blocks from \( \lambda \) and \( \mu \) are \textit{matched} if they are of identical size.

It may be possible to match the blocks in multiple ways; we choose an arbitrary matching. Our goal is to match blocks as quickly as possible, but in a way that does not create small unmatched blocks. Blocks are chosen at rate proportional to their size according to the marginal evolution. Thus, large unmatched blocks are relatively easy to fix, but small blocks take longer to select. We place all matched parts to the right; the unmatched parts then occupy the left part.

**Algorithm 5.11: Coupling of Tilings.** Let \( c \in \mathcal{C} \) be a cycle structure. Let \( s \in \{1,\ldots,d(c)\} \).

Suppose that \( \bar{P}_s = \lambda \) and \( \bar{Q}_s = \mu \). Suppose that the chosen markers are \( (u_\lambda, v_\lambda) \) and \( (u_\mu, v_\mu) \), respectively, for step \( s \); these will be chosen in a coupled way. Let \( I \) and \( I_\mu \) be the tiles containing the first markers \( u_\lambda \) and \( u_\mu \), respectively. Assume that either \( I_\lambda \) and \( I_\mu \) are matched to each other or they are both unmatched. We verify that this property is preserved by the coupling in Lemma 5.14.

We apply the transformation \( T \) from the marginal evolution, Algorithm 5.9. We move the tiles \( I_\lambda \) and \( I_\mu \) to the front of their respective tilings and assume that \( u_\lambda = 1/n = u_\mu \), as in Algorithm 5.9. We use the same coin toss \( b \) for both \( \lambda \) and \( \mu \). We set \( \lambda' := T(\lambda; u_\lambda, v_\lambda; b) \) and \( \mu' := T(\mu; u_\mu, v_\mu; b) \).

It remains to construct \( v_\mu \) as a function of \( v_\lambda \). If \( I_\lambda \) is matched, and hence is matched to \( I_\mu \), by assumption, then set \( v_\mu := v_\lambda \). Now suppose that \( I_\lambda \) is unmatched, and hence \( I_\mu \) is also unmatched, by assumption. We obtain \( v_\mu \) by applying a measure-preserving map \( \Phi \) to \( v_\lambda \): set \( v_\mu := \Phi(v_\lambda) \).

It remains to define \( \Phi \). Let \( \alpha := |I_\lambda| \) and \( \beta := |I_\mu| \) be the respective lengths of \( I_\lambda \) and \( I_\mu \); assume that \( \alpha \leq \beta \), without loss of generality. Denote \( \gamma := \lfloor \alpha n/2 \rfloor/n \).

\[
\Phi : [0,1] \to [0,1] : v \mapsto \begin{cases} v & \text{if } v > \beta \text{ or } 2/n \leq v \leq \gamma + 1/n, \\ v - \gamma & \text{if } \alpha < v \leq \beta, \\ v + \beta - \alpha & \text{if } \gamma + 1/n < v \leq \alpha. \end{cases}
\]

The map \( \Phi \) is illustrated in Figure 1.1; cf [BS19, Figure 1]. \( \triangle \)

**Remark 5.12: Potential Difficulties Arising from Rejecting Splits.** Consider the scenario in which \( v_\lambda \notin I_\lambda \) but \( v_\mu \in I_\mu \). The tile containing \( v_\lambda \) is always merged with \( I_\lambda \) in \( P \). A split of \( I_\mu \) is proposed in \( Q \), but may be rejected, namely if \( b = 0 \). This scenario does not arise for random transpositions in [Sch05] or conjugacy-invariant RWs in [BS19]; there, splits are always accepted.

It does not cause any issues for us, though. Indeed, larger tiles are easier to manage, since they are selected faster. Thus, not splitting is not an issue. \( \triangle \)

**Remark 5.13: Weak Convergence vs Mixing.** One always takes \( v_\mu := v_\lambda \) in the original coupling of Schramm [Sch05]. This is the case for Björnberg, Kotowski, Lees and Mikoł [BKLM19] too, who adjust the coupling of Schramm [Sch05] to an application analogous to the 2-PM RW. Both of these articles study \textit{weak convergence}, rather than \textit{mixing}. The adaptation to include the map \( \Phi \) was one of the fundamental innovations introduced by Berestycki, Schramm and Zeitouni [BSZ11]; it was then used by Berestycki and Şengül [BS19]. It is crucial when studying mixing, as we explain now. If one takes \( \Phi \) to be the identity, then this leads to the undesirable property that arbitrarily small, unmatched blocks may appear. These small, unmatched blocks remain in the system for a long time—indeed, it takes a long time for them to even be selected. This prevents coalescence. The map \( \Phi \) rectifies this issue: the worst thing that can happen is for the smallest unmatched block to become smaller by a factor 2 and this only happens with small probability; see Lemma 5.15. So, if the unmatched blocks are large initially, then they all remain relatively large. The coupling quickly selects and matches large, unmatched blocks. Thus coalescence occurs quickly. This is precisely why this map \( \Phi \) was introduced by Berestycki, Schramm and Zeitouni [BSZ11].

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Figure 5.1. Two tilings $\lambda$ and $\mu$. The grey shaded tiles are the distinguished tiles, namely $I \in \lambda$ and $J \in \mu$. They have width $|I| = \alpha$ and $|J| = \beta$, respectively. The arrows represent the map $\Phi$ from Algorithm 5.11. Four example pairs $(v_i, \Phi(v_i))$ are given. 

Another important property of the coupling is that the total number of unmatched blocks, i.e., the number in $P$ plus the number in $Q$, never increases; see Lemma 5.15 again. \(\triangle\)

We now verify that the coupling is well-defined.

**Lemma 5.14: Coupling of Tilings.** Suppose that the two distinguished tiles are either matched to each other or both unmatched at the start of a step of Algorithm 5.11. Then, this is also the case at the end of the step.

The coupling is a genuine coupling, i.e., has the correct marginals, and is Markovian.

**Proof.** The first claim involves some routine case analysis; see [BSZ11, Lemma 4.3]. Merging two tiles in one system and not splitting in the other preserves these conditions; cf Remark 5.12.

The second claim follows from the construction using Algorithms 5.4 and 5.9.

**Lemma 5.15: Evolution of Unmatched Blocks.** Let $\lambda, \mu \in \Psi$ and let $\lambda', \mu' \in \Psi$ be the corresponding integer partitions after one step of the coupling, i.e., of Algorithm 5.11. Let $U$ and $U'$ be the sizes of the smallest unmatched block in the pair $(\lambda, \mu)$ and $(\lambda', \mu')$, respectively. The following hold:

- $U' \geq \frac{1}{4} U n$ and $\mathbb{P}(U' \leq 2^{\lfloor \log_2 U \rfloor}) \leq 4 U / n$;
- the total number of unmatched partitions in $(\lambda', \mu')$ vs $(\lambda, \mu)$ cannot increase.

**Proof.** The proof of this lemma is almost the same as [BSZ11, Lemma 19]; see also [BS19, Lemma 4.4], where the details were omitted. Merging two tiles in one system and not splitting in the other preserves these conditions; cf Remark 5.12. We omit the details here too.

We now compare properties of the coupling used in the current article with those of [BS19]. In essence, the key in [BS19] is that the tiles do not get too small. The coalescence-fragmentation processes are the same except that we have the additional property of rejecting some splits. Thus, it is simple to couple the two approaches so that blocks are larger in our process than in theirs. This means that an identical proof as given in [BS19] applies here, leading to Lemma 5.16 below.

Recall that $(P_s)_{s \geq 0}$ and $(Q_s)_{s \geq 0}$ denote two tilings, coupled in the above manner, on the swap-timescale. That is, incrementing $s$ to $s + 1$ corresponds to applying a single swap, not a full round.

For ease of presentation, assume that a single round involves choosing a divergent (in $n$) number of swaps. This allows us to define easily $P_{[\delta^{-n}]}$ and $Q_{[\delta^{-n}]}$ with $\delta > 0$ arbitrary but fixed. Otherwise, simply concatenate sufficiently many rounds so that a least $[\delta^{-n}]$ swaps are made. We are using the relaxed law and $[\delta^{-n}] = \Theta(1) = o(n)$, so this concatenation has no negative effects.
The next lemma shows that \( \mathcal{P}_{\lfloor \delta - n \rfloor} = \mathcal{Q}_{\lfloor \delta - n \rfloor} \) \( \text{whp} \) if the initial tilings \( \mathcal{P}_0 \) and \( \mathcal{Q}_0 \) start with few unmatched blocks and the smallest unmatched block is not very small.

**Lemma 5.16:** Tiling Coupling Probability; cf [BS19, Lemma 4.11]. Let \( \mathcal{P}_0, \mathcal{Q}_0 \in \Psi \) be two tilings. Assume that there are at most 3 unmatched blocks between \( \mathcal{P}_0 \) and \( \mathcal{Q}_0 \). Write \( A_\delta \) for the event that the size of the smallest unmatched block is at least \( \delta \), for \( \delta > 0 \). Then,

\[
\limsup_{\delta \to 0} \limsup_{n \to \infty} \mathbb{P}(\mathcal{P}_{\lfloor \delta - n \rfloor} \neq \mathcal{Q}_{\lfloor \delta - n \rfloor}) \mathbf{1}(A_\delta) = 0.
\]

**Proof.** This lemma follows in a completely analogous way to how [BS19, Lemma 4.11] does for the conjugacy-invariant RW on the symmetric group. Rejecting splits has no ill effects. \( \square \)

We now briefly justify why we need only consider initial tilings with few unmatched blocks and smallest unmatched block not very small. Rigorous analysis comes later in Lemma 6.5.

**Justification of Assumptions in Lemma 5.16.** We perform a path coupling approach. We start at swap distance 1. It is easy to couple the tilings so that the swap distance remains 1 via a simple relabelling. The associated tilings then have at most 3 unmatched blocks.

We use a ‘burn-in’ phase. This will be long enough so that the associated tilings look \textit{roughly} like they should in equilibrium. In particular, there will be few very small blocks. We are able to deduce that the smallest unmatched block has size order 1, i.e. not vanishing with \( n, \text{whp} \). \( \triangle \)

# 6 The Three-Stage Coupling of Two Systems

## 6.1 Definition of Coupling

The overall coupling has three stages; cf [BS19, §4.2]. It is trivial to couple two PM systems, so that their relative swap distance remains constant, even on the swap-timescale, via a simple relabelling; see Definition 6.1 and Lemma 6.2. We call this the \textit{distance-preserving coupling}. It is this simple coupling which is used in Stages 1 and 3. Stage 2 uses our adaptation of Schramm [Sch05]’s coupling.

**Stage 1** The first stage is a burn-in period. It uses the distance-preserving coupling. We wish the burn-in period to end in such a configuration that the two tilings have few unmatched blocks and any unmatched blocks are not vanishingly small; cf Lemma 5.16. The length of the burn-in period asymptotically dominates the other two stages.

**Stage 2** The next stage uses the adaptation of Schramm’s coupling described in §5.2. It will be run for time order 1 on the swap-timescale; it does not necessarily involve an integer number of steps on the PM-timescale, which involves approximately \( k \) swaps.

**Stage 3** The final stage simply finishes off PM initiated in the second stage so that an integer number of PMs have been applied. It uses the distance-preserving coupling.

We construct the distance-preserving coupling one swap at a time. The informal idea is simple:

(i) relabel in the two PMs so that each is at the identity;
(ii) draw a new matching and replace the identity with this new matching;
(iii) undo the relabelling in each PM.

**Definition 6.1:** Swap Distance-Preserving Coupling. Suppose that the two PM RWs are at \( \mu \) and \( \nu \), respectively. Choose relabellings \( \sigma \) and \( \tau \) which translate \( \mu \) and \( \nu \) to the identity, respectively:

\[
\mu_{\sigma(i)} = i = \nu_{\tau(i)} \quad \text{for all} \quad i \in [2n].
\]

Now choose a PM \( \eta = \cup_{i} \{ \eta_{2i-1}, \eta_{2i} \} \). Define \( \mu' \) and \( \nu' \) by ‘undoing’ the relabelling of \( \sigma \) and \( \tau \), but starting from \( \eta \) rather than the identity:

\[
\mu'_i := \eta_{\sigma^{-1}(i)} \quad \text{and} \quad \nu'_i := \eta_{\tau^{-1}(i)} \quad \text{for} \quad i \in [2n].
\]

A single swap, rather than the full PM, is obtained by decomposing the new PM \( \eta \) into individual swaps and applying one at a time, choosing the relabellings \( \sigma \) and \( \tau \) anew each time.
Lemma 6.2: Swap Distance-Preserving Coupling. The coupling of Definition 6.1 is a genuine coupling of the PM RWs, is Markovian and preserves the swap distance, even on the swap-timescale.

Proof. This is immediate from the construction.

Recall the parameters $\kappa$ and $\rho$ representing, respectively, the expected support and expected distance of a uniformly chosen $k$-PM from Definition 4.8. This gives rise to an ‘average’ PM- and a swap-timescale. The precise timescales are only well-defined if a sequence of C/Ss are prescribed in advance and the PMs or swaps are chosen conditional on this.

We used $[\delta^{-9}]$ swaps used with our adaptation of Schramm’s coupling. Coalescence is achieved with probability tending to 1 as $\delta \to 0$ on the event there are initially at most 3 unmatched tiles and the unmatched tiles have size at least $\delta$; see Lemma 5.16. If this event fails, then we use the distance-preserving coupling instead. Importantly, the number of unmatched tiles is non-increasing under Schramm’s coupling, so the relative distance of the PMs remains at most $2$.

Recall from Definition 4.8 that $\kappa = \mathbb{E}(\#H)$ where $H \sim \text{Unif}([0,\delta])$; it is the expected support of a uniform $k$-PM, or equivalently of a uniform $k$-rematching. Roughly, this is the number of uniform choices per round. It is thus natural for our times, such as the mixing time, to scale inversely in $\kappa$.

Definition 6.3: Three-Stage Coupling. We define the three coupling stages on the swap-timescale:

Stage 1 is $[0,s_1)$, Stage 2 is $[s_1,s_2)$ and Stage 3 is $[s_2,s_3)$,

where $s_1 := \lceil (\beta n - \delta^{-9})/\kappa \rceil \rho$, $s_2 := s_1 + \delta^{-9}$ and $s_3 := \lceil \beta n / \kappa \rceil \rho$.

We use the distance-preserving coupling of Definition 6.1 in Stages 1 and 3. If the smallest unmatched block has size at least $\delta$ at time $s_1$, then we use Schramm’s coupling in Stage 2, lifted to the PM chain; otherwise, we use the distance-preserving coupling.

6.2 Coupling Time from Neighbouring Perfect Matchings

Suppose that we start with two neighbouring PMs, i.e. ones which differ by a single swap. Monotonicity of the number of unmatched blocks in the tilings implies that there are always at most 3 unmatched blocks and thus that the relative distance of the PMs is always at most $2$.

Recall the definition of $A_5$: the smallest unmatched block in the tiling has size at least $\delta$. Our first aim is to estimate the probability that the chains jointly satisfy $A_5$ at time $s_1$, in the limit $\delta \to 0$. This is the content of Lemma 6.5. It requires an adaptation of a hyper-graph argument introduced by Berestycki and Şengül [BS19, §3]. Given that $A_5$ is satisfied at time $s_1$, we use Lemma 5.16 to coalesce the chains using Schramm’s coupling with probability tending to 1 as $\delta \to 0$. The monotonicity in the number of unmatched blocks in Schramm’s coupling means that the swap distance remains bounded by $2$, even if this coalescence fails.

The following result controls the contraction in the relative distance between two PMs. The definition of the contraction rate $\theta(\beta)$ and threshold $\beta_0 \in (0,\infty)$ in Proposition 6.4 below are given in [BS19, Lemma 2.1], but the precise definitions are not important. What is important is that $\theta(\beta)$ is the asymptotic proportion of vertices in the giant component of an auxiliary graph process discussed in the next chapter. This is discussed more in the following two results and proofs.

Proposition 6.4: Relative Distance Contraction. Let $(\overline{M}_s)_{s \geq 0}$ and $(\overline{N}_s)_{s \geq 0}$ be two $k$-PM RWs on the swap-timescale. Suppose that $d(\overline{M}_0,\overline{N}_0) = 1$. Let $\beta \in (\beta_0,\infty)$. Define $s := [\beta n / \kappa] \rho$. Then

$$\limsup_{\delta \to 0} \limsup_{n \to \infty} \mathbb{E}(d(\overline{M}_s,\overline{N}_s)) \leq 1 - \theta(\beta)^2.$$
• There are at most 3 unmatched tiles in $(\mathcal{P}_{s_1}, \mathcal{Q}_{s_1})$.

• Write $A_4$ for the event that smallest unmatched block has size at least $\delta$ in $(\mathcal{P}_{s_1}, \mathcal{Q}_{s_1})$. Then,

$$\liminf_{\delta \to 0} \liminf_{n \to \infty} P(A_4) \geq \theta(\beta)^2.$$ 

Proposition 6.4 follows relatively easily from Lemma 6.5, as we now show.

**Proof of Proposition 6.4.** We start $\mathcal{M}$ and $\mathcal{N}$ at swap-distance 1, ie $d(\mathcal{M}_0, \mathcal{N}_0) = 1$. We use the distance-preserving coupling until $s_1$, on the swap-timescale. If $A_4$ holds at this time, then we run for a further $[\delta^{-9}]$ units of time, now using Schramm’s coupling. Coalesce is achieved with probability tending to 1 as $\delta \to 0$, by Lemma 5.16. Further, the relative distance is always bounded 2. Thus the expected relative distance tends to 0 as $\delta \to 0$ on this event. If $A_4$ does not hold, then we use the distance-preserving coupling. We use the distance-preserving coupling in $[s_2, s_3 = s)$.

The result now follows from Lemma 6.5 which controls the probability of $A_4$.

The proof of Lemma 6.5 is an adaptation of that of [BS19, Lemma 4.2]. It requires the construction of an auxiliary graph process, similar to that in [BS19, §3], which we give in §7. We now explain how to deduce Lemma 6.5 assuming results on that graph process, referencing §7.

**Proof of Lemma 6.5.** Berestycki and Şengül [BS19, §3] introduce an auxiliary graph process to control the sizes of unmatched tiles. Lemma 6.5 will follow analogously to [BS19, Lemma 4.2] once we have constructed an auxiliary graph process in a suitably analogous way to [BS19, §3]. We construct such an auxiliary graph process in §7; see, in particular, Algorithm 7.1. Precisely, we use the argument of [BS19, Lemma 4.2] along with Theorem 7.3, which is analogous to [BS19, Theorem 3.1], and the relation between Algorithm 7.1 and that of [BS19, §3], described in Remark 7.2.

The key part of the proof of [BS19, Lemma 4.2] is the following (paraphrased).

Let $A_1$ [no relation to $A_4$] be the event that the four points comprising the two transpositions fall within the largest component of the associated graph at time $s_1 \approx 3\beta^n/k$.

The relative size of the giant component converges to $\theta(\beta)$. Thus $P(A_1) \to \theta(\beta)^4$.

There are some minor parity constraints in [BS19]. This is why they consider two transpositions. We need only consider a single swap, which corresponds to a single transposition. The symmetry of the problem implies that the two labels in this swap may be chosen uniformly at random without replacement. This is why the limiting probability is the product of the limiting probabilities that the individual labels are in the giant. We also replace $k$ with $\kappa$, as discussed in Remark 4.10.

There is one further part in the proof of [BS19, Lemma 4.2] which does not obviously transfer to our set-up and proof. It goes as follows (paraphrased).

The rescaled cycles sizes at time $s_1$ converge in distribution to a PD(1) random variable, multiplied by $\theta(\beta)$; see [BS19, Theorem 3.6]. This implies that, conditional on the event $A_1$ above, the relative size of the cycles containing the four points comprising the two transpositions can be thought of as the size of four independent samples from a PD(1) distribution, multiplied by $\theta(\beta)$.

The probability that any one of these four samples has a size smaller than $\delta/\theta(\beta)$ tends to 0 as $\delta \to 0$.

The cycle sizes in our set-up are somewhat different to those in [BS19]: splits are rejected half the time here, while they are never rejected there. This means that the cycle sizes in our set-up are stochastically larger than those in theirs. Thus, the same lower bound holds.

We expect our cycle sizes to follow a PD($\frac{1}{4}$) limiting distribution—see Remark 6.6—but such a refined statement is not required for the simple lower bound described above.

**Remark 6.6: Poisson–Dirichlet Convergence of Rescaled Cycle Sizes.** Convergence in distribution for the cycles of the $k$-PM RW to PD($\frac{1}{4}$) seems extremely likely to hold [BS19, Theorem 3.6] is described by the authors as a “simple adaptation of the proof of Schramm [Sch05]”; they provide some, but not all, of the details in their appendix. We have already seen how the “interchange

---

2 Formally, a PD(1) random variable corresponds to a tiling of (0,1] broken up into infinitely many blocks. Draw $U \sim \text{Unif}(0,1)$ and let $S$ be the size of the tile containing $U$. This $S$ is what Berestycki and Şengül mean by “size”
process with reversals” studied by Björnberg, Kotowski, Lees and Miłoś [BKLM19] is analogous to the 2-PM RW. Their main result [BKLM19, Theorem 1.1] is that the appropriately rescaled cycle sizes converge to $\text{PD}(\frac{1}{2})$. See also [BKLM19, Lemma 5.2]; cf Lemma 5.10. The relevant proofs in [BKLM19, §5.2, “Schramm’s coupling”], are “identical or nearly identical to the corresponding proofs in [Sch05], so we omit the details, but give comments where there are differences related to the rejection of splits” (paraphrased). There are, of course, further arguments in [BKLM19].

We emphasise that our proof does not need this convergence. Our cycles are at least as large as those used in [BS §19]. Those cycles satisfied the required lower bounds. Thus, ours do too.

We leave the question of convergence in distribution open. We conjecture that a combination of the ideas from [BS §19, Theorem 3.6], which are “a simple adaptation of the proof of Schramm”, the ideas in [BKLM19], particularly those which are “nearly identical to the corresponding proofs by Schramm” and our extension of Schramm’s coupling to $k > 2$ are sufficient to prove the claim.

7 Auxiliary Graph Process

Time has come to introduce and analyse the aforementioned auxiliary graph process, analogous to that introduced by Berestycki and Şengül [BS §19, §3]. There, the authors use a fixed CS with support $k$ and consider a number $t$ of rounds with $t \approx \beta n/k$, for some $\beta \in (0, \infty)$. Our set-up involves choosing the CS randomly for each round. The expected support $\kappa = \kappa_0$ satisfies $\kappa = k - \frac{1}{2} + O(\frac{1}{k})$; recall Definition 4.8 and Lemma 4.9. Thus, our number $t$ of rounds satisfies $t \approx \beta n / \kappa$.

7.1 Constructing the Auxiliary Graph Process

We describe how to construct a hyper-graph in a way analogous to [BS §19, §3]. There, they apply a permutations $(\gamma_t)_{t \geq 1}$, each with preset CS. Such a permutation can be broken down into cycles, say $\gamma_t = \gamma_{t,1} \circ \cdots \circ \gamma_{t,r}$. The hyper-edge $\{a_1, \ldots, a_\ell\}$ is present in their hyper-graph at time $T$ if and only if $\gamma_{t,s} = \{a_1, \ldots, a_\ell\}$ for some $1 \leq s \leq r$ and $1 \leq t \leq T$. The hyper-edge $\{a_1, \ldots, a_\ell\}$ is independent of the order of its entries. Thus, $a_1, \ldots, a_\ell$ may appear in order in the cycle $\gamma_{t,s}$.

This is equivalent to adding a clique with support $\{a_1, \ldots, a_\ell\}$, ie adding all edges between $a_1, \ldots, a_\ell$, in a normal, non-hyper, graph. The equivalence comes from the fact that we are only interested in the size of connected components. We find this second viewpoint more natural.

Onto PMs. If we choose $\ell$ pairs to rematch into a cycle, say with labels $a_1, \ldots, a_\ell$, then we add a clique $\{a_1, \ldots, a_\ell\}$. We need a well-defined and consistent way of relabelling the matches after the rematching. We explain precisely what we mean by this, since it is a key step.

Each PM on $2n$ objects involves $n$ matches, or pairs, $m_1, \ldots, m_n$, which are labelled $1, \ldots, n$ in some manner. Suppose that we interact with the first two pairs, $m_1 = \{a, b\}$ and $m_2 = \{c, d\}$, giving rise to new matches $(m'_1, m'_2)$ satisfying $m'_1 \cup m'_2 = \{a, b, c, d\}$ and $m'_1 \cap m'_2 = m_1 \cap m_2$. The quadruple $(a, b, c, d)$ was initially matched as $m_1 = \{a, b\}$ and $m_2 = \{c, d\}$. Suppose that the interaction changes this to $\{a, c\}$ to $\{b, d\}$. There is no natural way of choosing $m'_1 = \{a, c\}$ or $m'_2 = \{b, d\}$.

We were in exactly the same quandary when sampling a uniform cycle via swaps. We use here exactly the same solution as there: we choose the particular labelling in the new matching uniformly; cf Algorithm 4.1.1, which generates a uniform cycle via swaps.

Recall that if both objects in the same cycle are swapped, then we split components only half the time for the PM RW, whereas splits always occur in this scenario for the conjugacy-invariant RWs of [BS §19]. Importantly, this difference is irrelevant to the graph process since hyper-edges/cliques are only added, never removed. One can view this as merging two components of the graph when two cycles merge, unless they were already connected, but never splitting a component of the graph, even if a cycle splits. Thus, this accept/reject of splits does not play a role.

Algorithm 7.1: Auxiliary Graph Process for the PM RW. Let $(c_t)_{t \geq 1} \in \mathbb{C}^N$ be a sequence of CSs. We construct a random graph process $(G_t)_{t \geq 0}$. We use an inductive construction. Define $G_0 := ([n], \emptyset)$ to be the empty graph. Suppose that $t \geq 0$ and that $G_t$ has been defined. We now define $G_{t+1}$.

- Choose the next CS, ie $c_{t+1} \in \mathbb{C}$.
- Choose a $|c_{t+1}|$-subset of $[n]$ uar, say $\{b_1, \ldots, b_{|c_{t+1}|}\} \subseteq [n]$.
Choose an associated partition\(^3\) \(\lambda \in \mathcal{P}_{t+1}\) uar.

- Perform the following steps independently for each \(\ell\)-cycle in the decomposition \((c_{t+1}, \lambda)\).\(^4\)
  - Suppose that the labels of the \(\ell\)-cycle are \(\{a_1, \ldots, a_\ell\} \subseteq \{b_{t+1}, \ldots, b_{t+1}\} \subseteq [n]\).
  - Add the clique \(\{a_1, \ldots, a_\ell\}\), ie all edges between the vertices \(a_1, \ldots, a_\ell\).\(^5\)
  - Relabel the vertices \(a_1, \ldots, a_\ell\) uar.

\[\triangle\]

**Remark 7.2: Comparison with [BS19, §3].** We compare this graph process with the generalisation of that in [BS19, §3] for conjugacy-invariant RWs, where we allow different CSs to be picked at stage in a quenched sense. This generalisation makes the algorithm for constructing the conjugacy-invariant graph process identical to that used for PMs, ie Algorithm 7.1 above, with one exception: the labels in the subset \(\{a_1, \ldots, a_\ell\}\) are randomised for the PM version, but not for conjugacy-invariant version.

This relabelling is inconsequential. Indeed, the partition is chosen uniformly and independently each time. The relabelling is only needed in order to couple with the PM RW. This immediately gives a natural coupling between the PM and conjugacy-invariant versions of the graph process.

Furthermore, we are only interested in the sizes of components later; see Theorem 7.3. If two vertices are connected, then it does not matter to which of these two vertices other vertices are connected: the same connected component will be formed.

The relabelling is there only to circumnavigate the identifiability issue mentioned before. \[\triangle\]

We have thus reduced the problem to a situation similar to that in [BS19]. There, a CS is fixed and used forever; a \(c \in \mathcal{C}\) is chosen and \(c_t := c\) for all \(t \geq 1\). Now, the sequence \((c_t)_{t \geq 1}\) need not be constant. The particular application that we are interested in is when each CS is chosen independently and according to a uniform \(k\)-rematching in an \(n\)-PM, ie \((c_t)_{t \geq 1} \sim \mathcal{C}(\text{Unif}(\mathcal{M}'))^N\).

### 7.2 Approximating Variable Cycle Structures by a Fixed One

Our desire is to show that which particular CS is used is irrelevant: in essence, all that matters is the rate at which an \(\ell\)-cycle is applied for each \(\ell\). We think of the growth of the graph process though an independent approximation. The process involves breaking a \(k\)-PM into single-cycle PMs which are (weakly) correlated to previously applied single-cycle PMs. Ignore the correlations for the moment and just determine the law of the choice of single-cycle PMs when the \(k\)-PM is chosen uar.

If CS \(c\) is chosen, then \(c_\ell\) is the number of \(\ell\)-cycles which are applied, for each \(\ell\). We can view this as a “drawing balls from an urn” problem in the following sense. Place \(B := \sum_{\ell \geq 2} \ell c_\ell\) balls in an urn: \(c_\ell\) of colour \(\ell\) for each \(\ell \geq 2\). Set \(b := 0\) and \(S := 0\). Repeat the following steps until \(b = B\).

- If \(b < B\), then draw a ball uniformly. Suppose that it is of colour \(\ell\).
- Choose an ordered collection of \(\ell\) elements uar from \([n] \setminus S\).
- Apply an \(\ell\)-cycle with this ordered collection.
- Add these elements to \(S\). Do not return the ball to the urn.
- Increment \(b\) by 1.

This perfectly simulates the application of a uniform PM with CS \(c\). We approximate by returning the ball to the urn and not updating the set \(S\). Each of the \(B\) steps then has the same description.

- Draw \(\ell \in \mathbb{N} \setminus \{1\}\) proportional to \((c_\ell)_{\ell \geq 2}^\infty\).
- Apply a uniformly chosen \(\ell\)-cycle.

A random number of \(\ell\)-cycles are applied in a single round; the expected number is \(c_\ell\).

We extend this from always using the same CS to define the law of this random number to choosing a random CS for each round. Let \(\Gamma \sim \mathcal{C}(\text{Unif}(\mathcal{M}'))\), ie the CS of a uniform \(n\)-PM with at most \(k\) non-fixed points. Draw \(c_\ell = (c_\ell)_{\ell \geq 1}^\infty \sim \mathcal{C}(\text{Unif}(\mathcal{M}'))\) independently for each \(t \geq 1\). A single round now involves applying \(\gamma_\ell := \mathbb{E}(\Gamma_\ell)\) \(\ell\)-cycles independently on average. Then,

\[\#\gamma = \sum_{\ell=2}^\infty \ell \gamma_\ell = \mathbb{E}(\#\Gamma) = \kappa \quad \text{where} \quad \gamma := (\gamma_\ell)_{\ell \geq 1}^\infty.\]

\[\text{the partition decides which elements of } \{b_{t+1}, \ldots, b_{t+1}\} \text{ go into which sub-cycle}\]

\[\text{The different } \ell\text{-cycles in the decomposition are disjoint. Thus, the order they are considered in is inconsequential}\]

\[\text{Alternatively, if using the hyper-graph viewpoint, add the hyper-edge } \{a_1, \ldots, a_\ell\}\]
That is, \( \gamma \) is almost a CS with support \( \kappa \), i.e. the average support of a uniformly chosen \( k \)-PM. It is not quite, though, as each \( \gamma_t \) need not be a non-negative integer.

We would like to be able to say, “Instead of choosing a random CS in each step, just use the average \( \gamma \), then apply some concentration results. This is legitimate since the order in which the \( \ell \)-cycles are applied is irrelevant for the random graph process.” The fact that \( \gamma \not\in \mathbb{N}_0^\infty \) prohibits this. It turns out to be unimportant, though. We group together multiple steps and approximate those by a genuine CS: roughly, we replace \( \gamma_t \) with \( \gamma_t' := \lceil \gamma_t \epsilon / \varepsilon \rceil \in \mathbb{N} \), corresponding to \( 1/\varepsilon \) steps.

### 7.3 Size of the Largest Component of the Graph

The purpose of this section is to determine the proportion of vertices in the largest component of the auxiliary graph process, asymptotically as \( n \to \infty \). The following theorem is an adaptation of [BS19, Theorem 3.1] to our set-up. In it, there is a critical threshold \( \beta_0 \) which \( \beta \) must be above and a proportion \( \theta(\beta) \), which will be the asymptotic proportion of vertices in the giant. The particular values and definitions of these parameters is unimportant, but can be found in [BS19, Lemma 2.1].

Recall that we consider a number \( T \) of round satisfying \( T \approx \beta n / \kappa \). We make this precise now. We evaluate the graph process of Algorithm 7.1 after this many rounds.

**Theorem 7.3:** cf [BS19, Theorem 3.1]. There exists a critical threshold \( \beta_0 \in (0, \infty) \) and a function \( \theta : \mathbb{R}_+ \to (0, 1) \) with the following properties. Fix \( \beta \in (\beta_0, \infty) \) arbitrarily. Suppose that \( T_0 \) satisfies \( T_0 \kappa / n \to \beta \) as \( n \to \infty \). Consider the random graph process \( \{G_t\}_{t \geq 0} \) evaluated at \( T_0 \). Then, the proportion of vertices which lie in the largest component converges to \( \theta(\beta) \) in probability as \( n \to \infty \).

We sketch the ideas behind Theorem 7.3. Even just the sketch proof is relatively technical. We include the majority of the details, but suppress the explicit description of \( 1 + o(1) \) terms. Controlling these efficiently is more of a notational challenge than a mathematical one. We trust that the details provided are sufficient for a masochistic\(^6\) reader to construct a rigorous proof.

We expect that the sketch is more complicated than it needs to be, but we have not found a simplification. Indeed, we even conjecture that a ‘quenched’ version of the theorem holds; see §7.4.

The proof involves comparing our graph process with that of [BS19, §3] and applying [BS19, Theorem 3.1]. There are two key reductions. We describe these two independently, then conclude.

#### Truncating the Cycle Sizes and Applying a Law of Large Numbers

Let \( \Gamma_t = (\Gamma_t)_{t=1}^T \overset{\text{id}}{\sim} \mathcal{C}(\text{Unif}(\mathbb{N}^k)) \) for \( t \in \mathbb{N}_0 \), i.e. \( k \)-rematchings in the space of \( n \)-PMs. Use CS \( \Gamma_t \) in round \( t \in \mathbb{N} \). Recall that \( \gamma_t = \mathbb{E}(\Gamma_t, t) \) is the number of \( \ell \)-cycles applied on average per round.

If \( \gamma_t \) were an integer for each \( \ell \), then we could simply use the CS \( \Gamma = (\gamma_t)_{t=1}^T \) for each round and then conclude via a Law of Large Number (LLN). But alas, it is not. In fact, \( a_\ell := \gamma_t \ell \to 1 \) as \( \ell \to \infty \). This is known for uniformly random permutations with \( a_\ell := 1 \) for all \( \ell \). An analogous proof holds for PMs; we omit the details. If \( k \gg 1 \), for example, then we can apply a LLN to say that each \( \ell \)-cycle \( (\ell \in \{1, \ldots, k\}) \) is applied a typically number of times. However, if \( k \) is sufficiently large, i.e \( k \gg \sqrt{n \log n} \), then the number of times that a \( k \)-cycle is applied is actually \( o(1) \). For such a large \( k \), though, there is not significant difference between applying a \( k \)-cycle, a \( (k-1) \)-cycle, etc.

We thus group together indices and assume that each group is applied a typical number of times.

We now proceed more formally. Assume first that \( k \to \infty \) as \( n \to \infty \). We explain the easier \( k \)-bounded case after. We group together indices \( \ell \) which are ‘approximately equal’. Let \( \xi > 0 \) with \( \xi \to 0 \) as \( n \to \infty \), but vanishing as slowly as we desire. Asymptotically, all the mass of the support comes from \( \ell \)-cycles with \( \ell \approx \kappa k \). Indeed, this follows simply from the expectation \( \gamma_t \approx \ell \):

\[
\sum_{t=1}^{\lfloor \kappa k \rfloor} \ell \gamma_t \approx \sum_{\ell \leq \xi k} \ell \gamma_t \approx \frac{k(1-\xi)}{\kappa} \approx k \quad \text{and} \quad \kappa \approx k,
\]

where the “\( \approx \)” signs hide \( 1 + o(1) \) factors, including \( 1 \pm \xi = 1 + o(1) \) factors. We use the following grouping. Let \( \varepsilon > 0 \) with \( \varepsilon \to 0 \) as \( n \to \infty \), again vanishingly slowly. Let

\[
I_i^- := \xi k (1 + \varepsilon)^i \wedge k, \quad I_i^+ := \xi k (1 + \varepsilon)^{i+1} \wedge k \quad \text{and} \quad I_i := (I_i^-, I_i^+] \cap \mathbb{N} \quad \text{for} \quad i \in \mathbb{N}_0.
\]

---

\( ^6 \) **masochist**: a person who enjoys an activity that appears to be painful or tedious.
Let \( i_{\text{max}} := \inf \{ i \in \mathbb{N} | \xi k(1 + \varepsilon)^{i+1} \geq k \} \). Then, for \( 0 \leq i < i_{\text{max}} \), we have

\[
\sum_{\ell \in I} \gamma_i = \sum_{\ell = [I_{\max}]}^{I_{\max}} a/\ell \approx (\xi k(1 + \varepsilon)^{i+1} - \xi k(1 + \varepsilon)^{i}) / (\xi k(1 + \varepsilon)^{i}) = \varepsilon.
\]

The LLN along with a union bound over \( i \in \{0, \ldots, i_{\text{max}} - 1\} \) gives

\[
\mathbb{P}\left( \sum_{i=1}^T \sum_{\ell \in I} \Gamma_i,\ell \approx \varepsilon T \text{ uniformly } \forall i \in \{0, \ldots, i_{\text{max}} - 1\} \right) \approx 1,
\]

where \( T \approx \beta n/k \) is the number of rounds. This requires the “\( \approx \)" sign inside the probability to be sufficiently weak compared with the decay of \( \varepsilon \) and \( \xi \). A similar bound holds jointly for \( i_{\text{max}} \), but taking into account the fact that \( |I_{\max}| \) has a slightly different form, due to the truncation at \( k \).

We use the following approximation \( G' \) to the original graph process \( G \): if an \( \ell \)-cycle is applied in \( G \) with \( \ell > \xi k \), then find \( i \) with \( \ell \in I_i \) and apply an \( I_{\max} \)-cycle in \( G' \). The processes \( G \) and \( G' \) can easily be coupled so that \( G_I \geq G'_{I_i} \) for all \( \ell \geq 0 \). These \( \ell \) and \( I_i \) satisfy \( \ell \approx I_{\max} \) uniformly. Thus, by continuity of \( \theta \), it is still the case that \( G'_{I_i} \) has a giant containing a proportion \( \theta' (\beta) \approx \theta (\beta) \) of the vertices asymptotically. This allows us to analyse \( G' \) instead of \( G \).

Analysis of \( G' \) is still not trivial. We cannot apply “an \( \varepsilon \)-proportion of an \( I_{\max} \)-cycle" in a single step. We would like to simply "enlarge" the CS by a factor \( 1/\varepsilon \) and multiply the number of rounds by \( \varepsilon \). We explain this concept via the following analogous situation.

- Alternate between applying a 2- and 3-cycle; thus each is applied half the time.
- ‘Enlarge’ this by a factor 2: apply a 2- and 3-cycle every round.
- Divide the number of rounds by 2: replace \( T \) by \( T/2 \).

This does not give rise to the same graph: choosing a 2- and 3-cycle in the same round conditions them to be disjoint; this is not the case when they are chosen in different rounds. It is reasonable to suspect that this difference is minor, however. Indeed, \( [BS19, \text{Theorem 3.1}] \) implies that this is the case when a fixed CS is ‘doubled’, i.e., there are twice as many \( \ell \)-cycles for each \( \ell \geq 2 \), and the number of rounds is halved. We show below that an analogous result holds for our \( \varepsilon \)-application.

It remains to comment on the \( k \)-bounded case. We do not need any rounding for this case since \( \gamma \approx 1 \) uniformly and thus all \( \ell \)-cycles are applied a constant proportion of the time. We simply condition that the number of \( \ell \)-cycles applied is typical for each \( \ell \), of which there are \( k \approx 1 \) different values. We then rescale time by common denominator of \( \gamma_1 \cdots \gamma_k \), which is order 1.

We show next that these adjusted processes give rise to giants of the same size asymptotically.

**Approximating Sampling without Replacement by Sampling with Replacement for Fixed CS**

Suppose that an \( \ell \)-cycle is being applied and that indices \( a = \{a_1, \ldots, a_r\} \) have already been chosen this round, by the application of previous cycles. \( \ell \) indices \( b = \{b_1, \ldots, b_\ell\} \subset \{a_1, \ldots, a_r\} \) are chosen uniformly without replacement from the restricted set \( \{1, \ldots, n\} \setminus a \). The clique \( b = \{b_1, \ldots, b_\ell\} \) is added to the graph. Suppose, instead, that we draw the indices with replacement and from the entirety of \( \{1, \ldots, n\} \): \( b_1', \ldots, b_\ell' \sim \text{Unif}(\{1, \ldots, n\}) \); set \( b' := \{b_1'', \ldots, b_\ell''\} \). Certainly \( b'' \setminus a \subseteq b \) stochastically. Define the graph \( G'' \) via the indices \( b'' \setminus a \) at the application of each cycle.

The fact that \( \ell \leq k \ll n \) implies that \( |b'' \setminus a''| \approx |b| = \ell \) whp. In particular, for every \( \ell \in \{1, \ldots, k\} \), we can find an \( \ell'' \) such that \( \ell'' \approx \ell \) uniformly at least \( \ell'' \) distinct elements are chosen whp when an \( \ell \)-cycle is applied. We think of this as “rounding \( \ell \) down to account for double counting”.

We can couple \( G \) and \( G'' \) by adding an \( \ell'' \)-clique to \( G'' \) whenever an \( \ell \)-clique is added to \( G \). There is some small probability that the inequality fails, but only a uniformly \( o(1) \) probability, set \( T'' := \beta n/k'' \), where \( k'' \approx k \) is the support of this slightly reduced-size CS. Then, \( G''_{T''} \) has a giant containing an asymptotic proportion \( \theta' (\beta) \approx \theta (\beta) \) of the vertices, using continuity and uniformity.

Finally, we release the restriction of applying exactly \( \ell'' \) to account for double counting in \( b'' \) and removing the previously-considered indices of \( a'' \): we simply choose \( \ell \) uniformly with replacement and add this clique. This only increases the size of the giant.

All in all, we have shown that the giant of the graph in which the indices are sampled with replacement, rather than without replacement, contains a proportion \( \theta(\beta) \) in probability.

**Concluding Given the Above Reductions**

We conclude the sketch by combining the two reductions just established.
(i) Replace \((G, T)\) with \((G', T')\), where \(T' \approx T\). This is the ‘rounding down’ process, in which we apply an \(\ell'\)-cycle in \(G'\) whenever an \(\ell\)-cycle is applied in \(G\), where \(\ell' := I_\ell\) with \(\ell \in I_\ell\). We also condition that a typical number of each \(\ell'\)-cycles are applied and group these together.

(ii) Replace \((G', T')\) with \((G'', T'')\), where \(T'' \approx T'\). This replaces the “sampling without replacement” in each round with “sampling with replacement”.

Importantly, there is no longer a concept of “multiple disjoint cycles in a single round” when sampling without replacement. This means that the ‘enlargement’ described at the end of the first part does not actually change the process at all. Thus our random choice graph process does indeed correspond, asymptotically, to the average choice, encoded by \(\gamma\) and \(\theta\).

**Alternative Proof: Copying Berestycki and Şengül’s Argument from [BS19, §3]**

We believe that our Theorem 7.3 can also be proved by following closely Berestycki and Şengül’s proof of [BS19, Theorem 3.1] in [BS19, §3]. Doing so, one sees that the particular structure of \(\Gamma\) is unimportant for their proof. Indeed, this almost has to be the case since their argument works when \(\Gamma\) comprises \(\frac{1}{k}\) disjoint transpositions, a single \(k\)-cycle or anything in-between. Berestycki and Şengül give a helpful verbal summary of this lemma, which we lightly paraphrase.

*It is perhaps surprising that [BS19, Lemma 3.2] is sufficient for the proof of [BS19, Theorem 3.1]. The lemma essentially only records whether a cycle is microscopic (finite) or “more than microscopic”. In particular, whether the mass of the CS comes from many small mesoscopic or fewer big cycles makes no difference.*

We have not checked carefully every detail in this argument. Indeed, the reductions that we described above are sufficient for our annealed set-up, so there was no need. However, the LLNs we used would not be so amenable to the quenched set-up, described below. The best way to prove a quenched statement may be to simply go through [BS19, §3], making the appropriate adjustments. These are no doubt relatively easy conceptually, but likely challenging technically.

**Convergence of Cycle Structure to Independent Poisson Process**

We remark for the sake of interest, rather than the proof, that the full vector of cycle lengths for a uniform permutation converges to that of an independent Poisson process in TV if \(k = o(n)\); see [AT92, Bar90, DP86]. TV analyses the entire vector: it is stronger than the more common weak convergence, which only analyses finite-dimensional marginals. Barbour [Bar90] uses the Chen–Stein method, which approximates certain (weakly) dependent variables by independent Poisson random variables. We have not checked carefully all the details, but we strongly suspect that the same argument can be used to establish convergence in TV for a uniform PM too.

### 7.4 Conjectured Extension to General ‘Quenched’ Cycle Structures

We have done our best to leave the above description as general as possible. In particular, we could estimate the law of the CS of a uniform PM. The reasons for our not doing this are twofold.

First and most important, we do not need to. The important term to control is \(1 - \theta(\beta)^2\); see Proposition 6.4. This is always approximately \(e^{-\beta}\) in the limit \(\beta \to \infty\), regardless of the law; see [BS19, Lemma 2.4] or Lemma 8.2. In particular, if we draw the CSs according to a different law, then this approximation still holds. Proposition 6.4 is evaluated at \((\beta n/k)p\) on the swap-timescale, which is equivalent to \(\beta n/k\) on the PM-timescale. The \(\beta\) in the contraction \(1 - \theta(\beta)^2 \approx e^{-\beta}\) and the \(\beta\) in the time \(\beta n/k\) end up cancelling. This is all made clear and rigorous in §8 below.

Second and more abstractly, the current formulation leads itself more naturally towards extension. We do not really need anywhere the randomness in the choice of the CS at each round. For example, suppose that \(\Gamma_0\) and \(\Gamma_1\) are two fixed CSs—say all transpositions (2-cycles) and all 3-cycles, respectively. Use \(\Gamma_b\) in the \(t\)-th step if \(t \equiv b \mod 2\). All our arguments would go through putting the same results as if one of \(\Gamma_0\) and \(\Gamma_1\) were chosen uniformly and independently at each round. The former is a ‘quenched’ statement and the latter an ‘annealed’.

We believe that this can be extended even further. If there is some ‘average behaviour’ of the quenched sequence which manifests itself on the PM-timescale order \(n/k\), then we expect that this
‘average behaviour’ can be used to define \( \theta \) appropriately. Indeed, the graph process is insensitive to the order in which the different \( \ell \)-cycles are applied; it is ‘Abelian’ in this sense. The coupling decomposes cycles into products of transpositions; it does not care what order these are applied or whether the transposition came from an \( \ell \)-cycle or an \( \ell \)-cycle. All that needs controlling carefully is the size of the small cycles and of the giant component after order \( n \) swaps have been applied, however those swaps may arise; recall the proofs of Proposition 6.4 and Lemma 6.5.

A quenched version of the lower bound actually holds easily. We elaborate in Remark 9.2.

### 8 Upper Bound for Cutoff

The ideas in this concluding section follow closely those employed by Berestycki and Şengül [BS19], differing only very slightly. Nevertheless, we include almost all the details for concreteness.

We are going to use the path coupling technique of Bubley and Dyer [BD97]; see [LPW17, Theorem 14.6] for a modern description. The following proposition is a rephrasing of Proposition 6.4, which is on the swap-timescale; the proposition below is given on the PM-timescale.

**Proposition 8.1: Relative Distance Contraction.** Let \((M_t)_{t \geq 0}\) and \((N_t)_{t \geq 0}\) be two PMs chains on the PM-timescale. Suppose that \(d(M_0, N_0) = 1\). Fix \( \beta \in (\beta_0, \infty) \). Let \( T_\beta := [\beta n / \kappa] \). Recall the contraction rate \( \theta \) from Proposition 6.4. There exists a coupling of \((M_t)_{t \geq 0}\) and \((N_t)_{t \geq 0}\) such that

\[
\lambda_\beta := \limsup_{n \to \infty} E(d(M_{T_\beta}, N_{T_\beta})) \quad \text{satisfies} \quad \limsup_{n \to \infty} \lambda_\beta \leq 1 - \theta(\beta)^2.
\]

We first informally justify the upper bound of \( e^{-m n \log n} \) on the mixing time. The standard path coupling bound says that the TV distance after time \( mt \) decays exponentially as \( (1 - \theta(\beta)^2)^m \). There is a diameter pre-factor which is \( n \). This uses the fact that \( \theta \) contraction rate in Proposition 6.1.

Let \( (M_t)_{t \geq 0}\) and \((N_t)_{t \geq 0}\) be two PMs chains. Recall the distance to be at most \( \epsilon \). It thus suffices for \( m \) to satisfy

\[
m \geq m_{\beta, \epsilon} := (\log n + \log(1/\epsilon)) / \log(1/\lambda_\beta).
\]

### Lemma 8.2: cf [BS19, Lemma 2.4].

We have

\[
\lim_{\beta \to \infty} \beta / \log(1 - \theta(\beta)^2) = -1.
\]

**Proof.** The proof is elementary analysis. See [BS19, Lemma 2.4] for analogous details.

This lemma then tells us, for the above \( m \), that

\[
mT_\beta \approx -\frac{1}{\lambda_\beta} n \log n \cdot \beta / \log(1 - \theta(\beta)^2) \to \frac{1}{\lambda_\beta} n \log n \quad \text{as} \quad \beta \to \infty.
\]

This informally justifies the upper bound of \( e^{-m n \log n} \). We now proceed formally and rigorously.

**Proof of Upper Bound in Theorem 1.5.** Let \((M_t)_{t \geq 0}\) and \((N_t)_{t \geq 0}\) be two PMs chains. Recall the \( d \) denotes the swap-distance; in particular, \( d(\mu, \nu) \geq 1 (\mu \neq \nu) \). Thus,

\[
\|M_t - N_t\|_{TV} \leq E(d(M_t, N_t)) \quad \text{for all} \quad t \geq 0,
\]

for any coupling of \( M_t \) and \( N_t \). Let \( \beta \in (\beta_0, \infty) \) and \( m \in \mathbb{N} \). Recall that \( T_\beta = [\beta n / \kappa] \). Iterating as in the path coupling method and applying Proposition 8.1 at each iteration, we obtain

\[
\|M_{mT_\beta} - N_{mT_\beta}\|_{TV} \leq n \lambda_\beta^m = n \exp(m \log \lambda_\beta).
\]

This uses the fact that \( d(M_0, N_0) \leq \max_{\mu, \nu \in \mathcal{M}_n} d(\mu, \nu) = n - 1 \leq n \).

Let \( \varepsilon > 0 \). We want the TV distance to be at most \( \varepsilon \). It thus suffices for \( m \) to satisfy

\[
m \geq m_{\beta, \varepsilon} := (\log n + \log(1/\varepsilon)) / \log(1/\lambda_\beta).
\]
It thus suffices to consider \( t \) with \( t \geq m_{\beta, \varepsilon} T_{\beta} \). Let \( \delta > 0 \) be arbitrarily small but constant. Set
\[
t := (1 + \delta)(n \log n / \kappa).
\]

Proposition 8.1 and Lemma 8.2 imply that we can choose \( \beta, \delta \) and \( n_{\delta, \varepsilon} \) large enough so that
\[
t \geq m_{\beta, \delta, \varepsilon} T_{\beta} \quad \text{for all} \quad n \geq n_{\delta, \varepsilon}.
\]

This completes the upper bound in Theorem 1.5 as \( \varepsilon \) and \( \delta \) were arbitrary.

\[\square\]

9 Lower Bound for Cutoff

The lower bound is just a simple coupon-collector argument, using the number of fixed points as a distinguishing statistic. We omit the details of this calculation, referencing to analogous ones.

A uniform \( k \)-PM has \( k/(2k - 1) = \Theta(1) \) fixed points wrt the identity in expectation. The application of an \( \ell \)-cycle involves choosing \( \ell \) elements of \([n]\) uar without replacement. This is approximately the same as choosing with replacement since \( \ell \leq k = o(n) \). In fact, if one does draw uar with replacement, then the number \( \ell' \) of draws required to get \( \ell \) distinct elements satisfies \( \ell' / \ell = 1 + o(1) \) whp. A coupon-collector argument shows that if only \( (1 - \delta)n \log n \) uniform choices are made, then divergently many elements of \([n]\) will not have been selected. The resulting PM then has a divergent number of fixed points. The number of fixed points thus acts as a distinguishing statistic.

A formal and rigorous proof in the case of a fixed CS is given by Berestycki and Şengül in their appendix, specifically [BS\$19, Appendix A]. It can be adapted to prove the following result.

**Proposition 9.1: Fixed Points.** Let \( k \in \mathbb{N} \) satisfy \( k / n \to 0 \) as \( n \to \infty \). Let \((c_t)_{t \geq 1} \in (\mathbb{W})^\mathbb{N}\) be an arbitrary sequence of CSs, each corresponding to a \( k \)-rematching in an \( n \)-PM. Let \((M_t)_{t \geq 0}\) be the ‘quenched’ PM RW in which CS \( c_t \) is used in round \( t \geq 1 \) with \( M_0 = \text{id} \), the identity. Let
\[
t_{\lambda} := \inf\{t \geq 0 \mid \sum_{t' = 0}^{t'} |c_{t'}| \geq \lambda n \log n\} \quad \text{for} \quad \lambda \in (0, \infty).
\]
Fix \( \lambda \in (0, 1) \) and \( K \in \mathbb{N} \). Then the number of fixed points in \( M_t \) is at least \( K \) whp if \( t \leq t_{\lambda} \).

The lower bound on mixing follows easily from this.

**Proof of Lower Bound in Theorem 1.5.** Suppose that the sequence \((c_t)_{t \geq 1}\) of CSs is drawn. The corresponding \( k \)-PMs are chosen independently. Let \( \delta \in (0, 1) \), independent of \( n \); set
\[
t := (1 - \delta)(n \log n / \kappa).
\]
Recall that \( n/\kappa \geq n/k \to \infty \) as \( n \to \infty \). The Law of Large Numbers thus implies that
\[
\sum_{t' = 0}^{t} |c_{t'}| \leq (1 - \frac{\delta}{2})n \log n \quad \text{whp}, \quad \text{ie} \quad t \leq t_{1 - \delta/2} \quad \text{whp}.
\]
There are thus divergently many fixed points in the PM at time \( t \) whp, by Proposition 9.1. Contrastingly, the expected number of fixed points in a uniform PM is at most \( k/(2k - 1) \leq 1 \), for any \( k \); see Lemma 4.9. The number of fixed points in the PM is thus a distinguishing statistic.

This completes the upper bound in Theorem 1.5 as \( \delta \) was arbitrary.

\[\square\]

**Remark 9.2: Extension to ‘Quenched’ Cycle Structures.** This argument extends easily to quenched cycle structures, where the sequence \((c_t)_{t \geq 1}\) of CSs is prescribed in advance, provided the support \( \kappa_t := \#c_t \) is uniformly \( o(n) \). Define \( T \) to be the natural coupon-collector threshold, ie
\[
T := \inf\{t \geq 0 \mid \kappa_1 + \cdots + \kappa_t \geq n \log n\}.
\]
Then, there are divergently many fixed points at \( t := (1 - \delta)T \) whp if \( \delta > 0 \) is independent of \( n \). \( \triangle \)
Bibliography

[AK19] G. Amanatidis and P. Kleer (2019). Rapid Mixing of the Switch Markov Chain for Strongly Stable Degree Sequences and 2-Class Joint Degree Matrices. *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, SIAM, Philadelphia, PA (966–985) MR3909529 DOI

[AK20] G. Amanatidis and P. Kleer (2020). Rapid Mixing of the Switch Markov Chain for Strongly Stable Degree Sequences. *Random Structures & Algorithms*. 57.3 (637–657) MR4140478 DOI

[AT92] R. Arratia and S. Tavaré (1992). The Cycle Structure of Random Permutations. *Annals of Probability*. 20.3 (1567–1591) MR1175278

[AGHH18a] L. Avena, H. Güldas, R. van der Hofstad and F. den Hollander (2018). Mixing Times of Random Walks on Dynamic Configuration Models. *Ann. Appl. Probab*. 28.4 (1977–2002) MR3843821 DOI

[AGHH18b] L. Avena, H. Güldas, R. van der Hofstad and F. den Hollander (2018). Random Walks on Dynamic Configuration Models: A Trichotomy. *Stochastic Processes and their Applications*. DOI

[Bar90] A. D. Barbour (1990). Poisson Approximation and the Chen-Stein Method: Comment. *Statistical Science*. 5.4 (425–427) DOI

[BC78] E. A. Bender and E. R. Canfield (1978). The Asymptotic Number of Labeled Graphs with Given Degree Sequences. *J. Combinatorial Theory Ser. A*. 24.3 (296–307) MR0505796 DOI

[BSZ11] N. Berestycki, O. Schramm and O. Zeitouni (2011). Mixing Times for Random $k$-Cycles and Coalescence-Fragmentation Chains. *Ann. Probab*. 39.5 (1815–1843) MR2884874 DOI

[BS19] N. Berestycki and B. Şengül (2019). Cutoff for Conjugacy-Invariant Random Walks on the Permutation Group. *Probab. Theory Related Fields*. 173.3-4 (1197–1241) MR3936154 DOI

[BKLM19] J. E. Björnberg, M. Kotowski, B. Lees and P. Mílos (2019). The Interchange Process with Reversals on the Complete Graph. *Electron. J. Probab*. 24 (Paper No. 108, 43 pp.) MR4017126 DOI

[Bol80] B. Bollobás (1980). A Probabilistic Proof of an Asymptotic Formula for the Number of Labelled Regular Graphs. *European J. Combin*. 1.4 (311–316) MR595929 DOI

[Bol01] B. Bollobás (2001). *Random Graphs*. Second ed., Cambridge University Press, Cambridge 73 MR1864966 DOI

[Bor11] O. Bormashenko (2011). A Coupling Argument for the Random Transposition Walk. *arXiv:1109.3915*

[BD97] R. Bubley and M. Dyer (1997). Path Coupling: A Technique for Proving Rapid Mixing in Markov Chains. *Proceedings of the 38th Annual Symposium on Foundations of Computer Science*, FOCS ’97, IEEE Computer Society (223–) DOI

[CLR10] P. Caputo, T. M. Liggett and T. Richthammer (2010). Proof of Aldous’ Spectral Gap Conjecture. *Journal of the American Mathematical Society*. 23.3 (831–851) MR2629990 DOI

[CST07] T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli (2007). Finite Gel’fand Pairs and Their Applications to Probability and Statistics. *Journal of Mathematical Sciences*. 141.2 (1182–1229) DOI

[CST08] T. Ceccherini-Silberstein, F. Scarabotti and F. Tolli (2008). *Harmonic Analysis on Finite Groups*. Cambridge University Press, Cambridge 108 MR2389056 DOI

[DH98] P. Diaconis and S. P. Holmes (1998). Matchings and Phylogenetic Trees. *Proc. Natl. Acad. Sci. USA*. 95.25 (14600–14602) MR1665632 DOI
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[Bibliography]

[DH02] P. Diaconis and S. P. Holmes (2002). Random Walks on Trees and Matchings. *Electron. J. Probab.* 7 (Paper No. 6, 17 pp.) MR1887626 DOI

[DP86] P. Diaconis and J. Pitman (1986). Permutations, Record Values and Random Measures. *Unpublished Lecture Notes, Statistics Department, University of California, Berkeley.*

[Erd+19] P. L. Erdős, C. Greenhill, T. R. Mezei, I. Miklós, D. Soltész and L. Soukup (2019). The Mixing Time of the Switch Markov Chains: A Unified Approach. *arXiv:1903.06600*

[GUW11] C. Goldschmidt, D. Ueltschi and P. Windridge (2011). Quantum Heisenberg Models and Their Probabilistic Representations. *Entropy and the Quantum II, Contemp. Math.* Amer. Math. Soc., Providence, RI 552 (177–224) MR2868048 DOI

[Hof17] R. van der Hofstad (2017). *Random Graphs and Complex Networks. Volume 1.* Cambridge University Press, Cambridge 1 MR3617364 DOI

[Hou16] R. Hough (2016). The Random $k$ Cycle Walk on the Symmetric Group. *Probab. Theory Related Fields.* 165.1-2 (447–482) MR3500276 DOI

[KTV97] R. Kannan, P. Tetali and S. Vempala (1997). Simple Markov-Chain Algorithms for Generating Bipartite Graphs and Tournaments (Extended Abstract). *Proceedings of the Eighth Annual ACM-SIAM Symposium on Discrete Algorithms (New Orleans, LA, 1997),* ACM, New York (193–200) MR1447665

[KTV99] R. Kannan, P. Tetali and S. Vempala (1999). Simple Markov-Chain Algorithms for Generating Bipartite Graphs and Tournaments. *Random Structures & Algorithms.* 14.4 (293–308) MR1691976 DOI

[LPW17] D. A. Levin, Y. Peres and E. L. Wilmer (2017). *Markov Chains and Mixing Times.* Second ed., American Mathematical Society, Providence, RI, USA MR3726904 DOI

[NO22] E. Nestoridi and S. Olesker-Taylor (2022). Limit Profiles for Reversible Markov Chains. *Probability Theory and Related Fields.* 182.1-2 (157–188) MR4367947 DOI

[Pit02] J. Pitman (2002). Poisson–Dirichlet and GEM Invariant Distributions for Split-and-Merge Transformation of an Interval Partition. *Combin. Probab. Comput.* 11.5 (501–514) MR1930355 DOI

[Sch05] O. Schramm (2005). Compositions of Random Transpositions. *Israel J. Math.* 147 (221–243) MR2166362 DOI

[Tey20] L. Teyssier (2020). Limit Profile for Random Transpositions. *Annals of Probability.* 48.5 (2323–2343) MR4152644 DOI

[TY20] K. Tikhomirov and P. Youssef (2020). Sharp Poincaré and Log-Sobolev Inequalities for the Switch Chain on Regular Bipartite Graphs. *arXiv:2007.02729*

[Tsi99] N. V. Tsilevich (1999). Stationary Random Partitions of a Natural Series. *Rossiiskaya Akademiya Nauk.* 44.1 (55–73) MR1751188 DOI

Bibliography Page 30 of 30