Research Article

Unification of Two-Variable Family of Apostol-Type Polynomials with Applications

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In this paper, the two-variable unified family of generalized Apostol-type polynomials is introduced, and some implicit forms and general symmetry identities are derived. Also, we obtain new degenerate Apostol-type numbers and polynomials constructed from the new 2-variable unified family. We derive explicit formulae of polynomials and identities that include some special numbers and polynomials. In addition, a probabilistic representation of the new family and some statistical properties are obtained.

1. Introduction

Khan and Raza [1] defined the 2-variable general polynomials (2VGP) $p_n(x, y)$ by means of the following generating function:

$$e^{xt}\varphi(y, t) = \sum_{n=0}^{\infty} p_n(x, y) \frac{t^n}{n!}, \quad p_0(x, y) = 1,$$  \hspace{1cm} (1)

where $\varphi(y, t)$ has series expansion

$$\varphi(y, t) = \sum_{n=0}^{\infty} \varphi_n(y) \frac{t^n}{n!}, \quad \varphi_0(y) \neq 0.$$  \hspace{1cm} (2)

The 2-variable general polynomials $p_n(x, y)$ contain a number of important special polynomials of two variables.

Generating functions for certain members that belong to the 2VGP are given as follows:

The higher-order Hermite polynomials, sometimes called the Gould–Hopper polynomials $H_n^{(m)}(x, y)$ are defined by [2].

$$e^{xt+yt} = \sum_{n=0}^{\infty} H_n^{(m)}(x, y) \frac{t^n}{n!}.$$  \hspace{1cm} (3)

The 2-variable Hermite polynomials $H_n(x, y)$ are defined by [3].

$$e^{x+y} = \sum_{n=0}^{\infty} H_n(x, y) \frac{t^n}{n!}. \hspace{1cm} (4)$$

The 2-variable generalized Laguerre polynomials $mL_n(y, x)$ are defined by the following generating function [4]:

$$e^{xt}C_0(\frac{yt}{m}) = \sum_{n=0}^{\infty} mL_n(y, x) \frac{t^n}{n!},$$  \hspace{1cm} (5)

where $C_0(y)$ is the 0-th order Tricomi function [5].

$$C_0(y) = \frac{\sum_{r=0}^{\infty} (-1)^r y^r}{(r!)^2}. \hspace{1cm} (6)$$

The 2-variable Laguerre polynomials $L_n(y, x)$ are defined by [6]

$$e^{yt}C_0(\frac{x}{(r!)^2}) = \sum_{n=0}^{\infty} L_n(y, x) \frac{t^n}{n!}.$$  \hspace{1cm} (7)

The 2-variable truncated exponential polynomials of order $r e_n^{(r)}(x, y)$ are defined [7]
\[
e^{x t} = \sum_{n=0}^{\infty} e^{(r)}_n(x, y) \frac{t^n}{n!}
\]

(8)

In particular, we note that

\[
e^{(2)}_n(x, y) = n! [2]_n e_n(x, y),
\]

(9)

where \([2]_n e_n(x, y)\) denotes the 2-variable truncated exponential polynomials [8].

The 2-variable truncated exponential polynomials \([2]_n e^{(r)}_n(x, y)\) are defined by [8].

\[
e(x, y) = \sum_{n=0}^{\infty} [2]_n e_n(x, y) \frac{t^n}{n!}
\]

(10)

Khan et al. [9] introduced the 2-variable Apostol type polynomials of order \(\alpha\), by

\[
\sum_{n=0}^{\infty} p^{(\alpha)}_n(x, y; \lambda; \mu) \frac{t^n}{n!} = \left(\frac{2^n x^r}{\lambda e^t + 1}\right) e^{x t} \varphi(y, t), \quad |t| < |\log(-\lambda)|.
\]

(11)

Araci et al. [10] introduced and investigate a new unified class of generalized Apostol type polynomials by the following generating function:

\[\sum_{n=0}^{\infty} H^{(\alpha)}_n(x, y; a, b, c; \mu, \nu, \lambda) \frac{t^n}{n!} = \left(\frac{2^n x^r}{\lambda b^r + a^r}\right) c^{\mu + \nu r}, \quad c > 1; j > 2; |t| < \frac{\log - \lambda}{\log(c/b)}.\]

Section 5, we obtain the generating functions for new special polynomials that belong to the new version of 2-variable unified Apostol-type polynomials. In Section 6, the probabilistic representation of the new family and its statistical properties are presented.

2. Unification of Two-Variable Apostol-Type Polynomials

The two-variable unified family of generalized Apostol-type polynomials of order \(r\), denoted by \(p^{(\alpha)}_n(x, y; a, b, c, \nu, \mu; \tau)\) is defined as the Apostol type convolution of the 2-variable general polynomials \(p_n(x, y)\).

\[\sum_{n=0}^{\infty} \mathcal{U}^{(\alpha)}_n(x, y; a, b, c, \nu, \mu; \tau) \frac{t^n}{n!} = \prod_{i=0}^{r-1} \left(\frac{\log(\alpha_i)}{\log(b/a)}\right)^{\nu_i, \mu_i} e^{x t} \varphi(y, t),
\]

(13)

where \(\tau = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1})\) is a sequence of complex numbers.

Setting \(c = e\) and \(\varphi(y, t) = 1\) in (13), we get the following definition.

\[\mathcal{U}^{(\alpha)}_n(x; a, b, \nu, \mu; \tau) \frac{t^n}{n!} = \prod_{i=0}^{r-1} \left(\frac{\log(\alpha_i)}{\log(b/a)}\right)^{\nu_i, \mu_i} e^{x t}.\]

(14)

Remark 1. Setting \(x = 0\) in (14), then we obtain the new unified family of generalized Apostol-type numbers, which is defined as
\[ \sum_{n=0}^{\infty} \mathcal{U}_n^{(r)}(a, b, \nu, \mu, \overline{\nu}) \frac{t^n}{n!} = \frac{(-1)^r t^r 2^\mu}{\prod_{i=0}^{r-1} (\alpha_i b_i^2 - a_i^2)} \]  \hspace{1cm} (15)

Also, setting \( \epsilon = e \) in (13), we get the following definition.

Definition 3. The two-variable unified family of generalized Apostol-type polynomials of order \( r \)
\( p\mathcal{U}_n^{(r)}(x, y; a, b; \nu, \mu, \overline{\nu}) \) is defined by the following generating function
\[ \sum_{n=0}^{\infty} p\mathcal{U}_n^{(r)}(x, y; a, b; \nu, \mu, \overline{\nu}) \frac{t^n}{n!} = \frac{(-1)^r t^r 2^\mu}{\prod_{i=0}^{r-1} (\alpha_i b_i^2 - a_i^2)} e^{x t} \varphi(y, t). \]  \hspace{1cm} (16)

We obtain the series definition of \( p\mathcal{U}_n^{(r)}(x, y; a, b; \nu, \mu, \overline{\nu}) \) by the following theorem.

Theorem 1. The two-variable unified family of generalized Apostol-type of order \( r \) \( p\mathcal{U}_n^{(r)}(x, y; a, b; \nu, \mu, \overline{\nu}) \) is defined by the following series:
\[ p\mathcal{U}_n^{(r)}(x, y; a, b; c; \mu, \overline{\nu}) = 2^r \nu^r \sum_{m_1, m_2, \ldots, m_r \geq 0} \left( \frac{n}{m} \right) \prod_{i=1}^{r} (\alpha_{i-1})^{m_i} \left( \log \left( \frac{b_i}{a_i} \right) \sum_{i=1}^{r} m_i \frac{c_i}{a_i^2} \right)^n \varphi_m(y). \]  \hspace{1cm} (18)

Proof. The left-hand side of (13) is equal to
\[ \frac{(-1)^r 2^\mu}{\prod_{i=0}^{r-1} (\alpha_i b_i^2 - a_i^2)} e^{x t} \varphi(y, t) = \frac{2^r \nu^r e^{(\log^2/a^2)}}{\prod_{i=0}^{r-1} (1 - \alpha_i e^2 \log(b/a))} \varphi(y, t) \]
\[ = \sum_{m_1, m_2, \ldots, m_r \geq 0} \prod_{i=0}^{r-1} (\alpha_i) m_i \left( \log \left( \frac{b_i}{a_i} \right) \sum_{i=1}^{r} m_i \frac{c_i}{a_i^2} \right)^n \varphi_m(y). \]  \hspace{1cm} (19)

By (13), we get
\[ \sum_{n=0}^{\infty} p\mathcal{U}_n^{(r)}(x, y; a, b, c; \mu, \overline{\nu}) \frac{t^n}{n!} = \sum_{n=0}^{\infty} 2^r \nu^r \sum_{m_1, m_2, \ldots, m_r \geq 0} \left( \frac{n}{m} \right) \prod_{i=0}^{r-1} (\alpha_{i-1})^{m_i} \left( \log \left( \frac{b_i}{a_i} \right) \sum_{i=1}^{r} m_i \frac{c_i}{a_i^2} \right)^n \varphi_m(y) \frac{t^n}{n!}. \]  \hspace{1cm} (20)

By comparing the coefficients on both sides in the last equation, we obtain (18). \( \square \)

Proof. Using equation (13) and the Cauchy-product rule, we can easily yield (17).

Now, by taking certain values of parameters in equations (13) and (16), we can find the generating functions and other results for the mixed special polynomials related to \( p\mathcal{U}_n^{(r)}(x, y; a, b, \nu, \mu, \overline{\nu}) \). We present the generating function and series definitions for these polynomials in Table 1. \( \square \)

Theorem 2. For \( a, b \in \mathbb{R}^+ \) and \( \nu = 0 \), the explicit formula of \( p\mathcal{U}_n^{(r)}(x, y; a, b, c; \mu, \overline{\nu}) \) can be expressed as
Table 1: Special cases of the $p\mathcal{U}^{(r)}_n(x, y; a, b, v, \mu; \tau_r)$.

| No. | Values of the parameters | Relation between $p\mathcal{U}^{(r)}_n(x, y; a, b, v, \mu; \tau_r)$ and its special cases | Name of the results special polynomials |
|-----|-------------------------|--------------------------------------------------------------------------------|--------------------------------------|
| 1   | $\alpha = -\lambda$ and $\varphi(y, t) = 1$ in (13) | $p\mathcal{U}^{(r)}_n(x; a, b, v, \mu; -\lambda) = \bar{F}^{(r)}(x; a, b, v, \mu; \lambda)$ | The generalized Apostol-type Gould-Hopper polynomials, [10] |
| 2   | $\alpha = -\lambda$ and $\varphi(y, t) = e^{ct}$, $j > 2$ in (13) | $p\mathcal{U}^{(r)}_n(x; a, b, v, \mu; -\lambda) = \bar{F}^{(r)}(x; a, b, v, \mu; \lambda)$ | The generalized Apostol type polynomials of order $r$, [10] |
| 3   | $\alpha = -\lambda$, $b = e$ and $a = 1$ in (16) | $p\mathcal{U}^{(r)}_n(x; y; 1, e, v, \mu; -\lambda) = P_F^{(r)}(x; y; 1, e, v, \mu; \lambda)$ | The 2-variable Apostol type polynomials of order $r$, [9] |
| 4   | $\alpha = \lambda$, $b = e$, $a = 1$ and $v = 0$ in (16) | $p\mathcal{U}^{(r)}_n(x; y; 1, 1, 0; \lambda) = \bar{P}_F^{(r)}(x; y; 1, 1, 0; \lambda)$ | The 2-variable Apostol-Bernoulli polynomials of order $r$, [9] |
| 5   | $\alpha = -\lambda$, $b = e$, $a = 1$ and $\mu = 0$ in (16) | $p\mathcal{U}^{(r)}_n(x; y; 1, e, 0; \lambda) = \bar{P}_F^{(r)}(x; y; 1, e, 0; \lambda)$ | The 2-variable Apostol-Euler polynomials of order $r$, [9] |
| 6   | $\alpha = -\lambda$, $b = e$, $a = 1$ and $\mu = 1$ in (16) | $p\mathcal{U}^{(r)}_n(x; y; 1, e, 1; \lambda) = \bar{P}_F^{(r)}(x; y; 1, e, 1; \lambda)$ | The 2-variable Apostol-Genocchi polynomials of order $r$, [10] |
| 7   | $\alpha = -\lambda$, $b = e$, $a = 1$ and $\varphi(y, t) = 1$ in (16) | $p\mathcal{U}^{(r)}_n(x; y; 1, e, v, \mu; -\lambda) = F^{(r)}(x; y; 1, e, v, \mu; \lambda)$ | The generalized Apostol type polynomials of order $r$, [20] |
| 8   | $\alpha = 1$, $b = e$, $\mu = (1-k)$, $v = k$ in (14) | $p\mathcal{U}^{(r)}_n(x; y; 1, e, v, \mu; \tau_r) = \bar{M}^{(r)}_t(x; k; \tau_r)$ | The unified family of generalized Apostol-Euler, Bernoulli and Genocchi polynomials, [21] |
| 9   | $\alpha = (1-k)$, $\mu = k$, $\varphi(y, t) = e^{ct}$ in (13) | $p\mathcal{U}^{(r)}_n(x; y; a, b, e, \mu; \tau_r) = H^{(r)}(x; y; a, b, e, \mu; \tau_r)$ | The generalization of Apostol-Hermite Genocchi polynomials, [22] |

3. Implicit Summation Formulas for the Two-Variable Unified Family of Generalized Apostol-Type Polynomials

Theorem 3. Let $a > b$ and $a \neq b$. Then for $x, y, z \in R$ and $n \geq 0$. The following implicit summation formula for $p\mathcal{U}^{(r)}_n(x, y; a, b, v, \mu; \tau_r)$ holds true as follows:

$$\sum_{n=0}^{\infty} p\mathcal{U}^{(r)}_n(x, y; a, b, v, \mu; \tau_r) z^n = \frac{\sum_{n=0}^{\infty} p\mathcal{U}^{(r)}_n(x, y; a, b, v, \mu; \tau_r) z^n}{n!}.$$  

Proof. Replacement of $x$ by $x + z$ in (16) gives

$$\sum_{n=0}^{\infty} p\mathcal{U}^{(r)}_n(x, y; a, b, v, \mu; \tau_r) z^n = \frac{(-1)^r t^{2n} \varphi(y, t)}{n!} \prod_{i=0}^{r-1} (a_i b^i - a^i) e^{(x+z)t} \varphi(y, t) \tag{22}$$

We get

$$\sum_{n=0}^{\infty} p\mathcal{U}^{(r)}_n(x, y; a, b, v, \mu; \tau_r) z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \mathcal{U}^{(r)}_n(x; a, b, v, \mu; \tau_r) p_{n-k}(z, y) \frac{t^n}{n!} \tag{23}$$

Replacing $n$ with $m$ in the right-hand side, hence equating the coefficients of $t$ in both sides of the last equation yields (21).

Theorem 4. The next implicit summation formula for $p\mathcal{U}^{(r)}_n(x, y; a, b, \mu; \tau_r)$ in terms of generalized Apostol type polynomial $\mathcal{U}^{(r)}_n(x; a, b, v; \tau_r)$ is obtained as follows:

$$p\mathcal{U}^{(r)}_n(x, y; a, b, v, \mu; \tau_r) \frac{t^n}{n!} = \sum_{k=0}^{n} \sum_{m=0}^{n-k} \binom{n}{k} \mathcal{U}^{(r)}_n(x; a, b, v, \mu; \tau_r) p_{n-k}(z, y) \frac{t^n}{n!} \tag{24}$$

Proof. Replacing $x$ by $x + z$ in (16) gives

$$\sum_{n=0}^{\infty} p\mathcal{U}^{(r)}_n(x, y; a, b, v, \mu; \tau_r) z^n = \frac{(-1)^r t^{2n} \varphi(y, t)}{n!} \prod_{i=0}^{r-1} (a_i b^i - a^i) e^{(x+z)t} \varphi(y, t) \tag{22}$$

We get

$$\sum_{n=0}^{\infty} p\mathcal{U}^{(r)}_n(x, y; a, b, v, \mu; \tau_r) z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \mathcal{U}^{(r)}_n(x; a, b, v, \mu; \tau_r) p_{n-k}(z, y) \frac{t^n}{n!} \tag{25}$$

Equating the coefficients of $t^n$ on both sides, yields (23).
Theorem 5. Let $a, b > 0$ and $a \neq b$. Then for $x, y, z \in \mathbb{R}$ and $n \geq 0$, the following implicit summation formula for $p \mathcal{U}_n^{(r)} (x, y; a, b; \mu; \overline{\alpha})$ holds true as follows:

$$p \mathcal{U}_{nm}^{(r)} (z; y; a, b, \nu; \overline{\alpha}) = \sum_{p,q=0}^{nm} \binom{n}{p} \binom{m}{q} (z-x)^{p+q} p \mathcal{U}_{nm-p-q}^{(r)} (x, y; a, b, \nu; \overline{\alpha}).$$

(26)

Proof. Replacing $t$ by $t+u$ in the generating function (16) and using the following rule [23]:

$$\sum_{n,m=0}^{\infty} f (N) \frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f (m+n) \frac{x^n y^m}{n! m!},$$

(27)

Replacing $x$ by $z$ in the previous equation and then equal both sides, we get

$$e^{-x(t+u)} \sum_{n,m=0}^{\infty} p \mathcal{U}_{nm}^{(r)} (x, y; a, b, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!} = \sum_{n,m=0}^{\infty} p \mathcal{U}_{nm}^{(r)} (z, y; a, b, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!} \]$$

(28)

On expanding exponential function (29) gives

$$\sum_{N=0}^{\infty} \frac{[(z-x)(t+u)]^N}{N!} \sum_{n,m=0}^{\infty} p \mathcal{U}_{nm}^{(r)} (x, y; a, b, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!} = \sum_{n,m=0}^{\infty} p \mathcal{U}_{nm}^{(r)} (z, y; a, b, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!} \]$$

(30)

Using equation (27) in the left-hand side of equation (30), we find

$$\sum_{p,q=0}^{\infty} \frac{(z-x)^{p+q} t^p u^q}{p! q!} \sum_{n,m=0}^{\infty} p \mathcal{U}_{nm}^{(r)} (x, y; a, b, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!} = \sum_{n,m=0}^{\infty} p \mathcal{U}_{nm}^{(r)} (z, y; a, b, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!} \]$$

(31)

Now replacing $n$ by $n-p$, $m$ by $m-q$, and using the Cauchy-product rule in the left-hand side of (31), we get

$$\sum_{n,m=0}^{\infty} \sum_{p,q=0}^{nm} \frac{(z-x)^{p+q}}{p! q!} \mathcal{U}_{nm-p-q}^{(r)} (x, y; a, b, \nu; \overline{\alpha}) \frac{t^n u^m}{(n-p)! (m-q)!} = \sum_{n,m=0}^{\infty} p \mathcal{U}_{nm}^{(r)} (z, y; a, b, \nu; \overline{\alpha}) \frac{t^n u^m}{n! m!} \]$$

(32)

Finally, on equating the coefficients of the like powers of $t$ and $u$ in the above equation, yields (26).

Theorem 6. The following implicit summation formula for $p \mathcal{U}_n^{(r)} (x, y; a, b; \nu; \overline{\alpha})$ holds true as follows:
\begin{equation}
pU_n^{(r)}(x + 1, y, a, b, v, \mu; \overline{a}) = \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) pU_{n-m}^{(r)}(x, y, a, b, v; \overline{a}). \tag{33}
\end{equation}

(ii)

\begin{equation}
pU_n^{(r)}(x + z, y; a, b, v, \mu; \overline{a}) = \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \overline{pU}_{n-m}^{(r)}(z, a, b, v, \mu; \overline{a}) p_m(x, y). \tag{34}
\end{equation}

**Proof.** Using equation (16), we can easily obtain (33) and (34). \hfill \Box

**Theorem 7.** Let \(c, d > 0\) and \(n \geq 0\). For \(x, y \in \mathbb{R}\), then the following identity holds true

\begin{equation}
\sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) a^m c^{n-m} pU_{n-m}^{(r)}(dx, y, a, b, v, \mu; \overline{a}) pU_m^{(r)}(cx, y, a, b, v, \mu; \overline{a})
= \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) c^m d^{n-m} pU_{n-m}^{(r)}(cx, y, a, b, v, \mu; \overline{a}) \overline{pU}_m^{(r)}(dx, y, a, b, v, \mu; \overline{a}). \tag{35}
\end{equation}

**Proof.** Let

\begin{equation}
G(t) = \frac{(-1)^{\gamma}2^{2\mu}(ct)^r}{\left( \prod_{i=0}^{r-1}(a_i b^i - a^i) \right) \left( \prod_{i=0}^{r-1}(a_i b^i - a^i) \right)^{\gamma}} e^{2\gamma dx} \varphi(y, dt) \varphi(y, ct). \tag{36}
\end{equation}

Then the expression for \(G(t)\) is symmetric in \(c\) and \(d\) and we can expand \(G(t)\) into series in two ways.

Firstly

\begin{equation}
G(t) = \frac{1}{(cd)^r} \left( \frac{(-1)^{\gamma}2^{2\mu}(dt)^r}{\left( \prod_{i=0}^{r-1}(a_i b^i - a^i) \right) \left( \prod_{i=0}^{r-1}(a_i b^i - a^i) \right)^{\gamma}} \right) e^{cx dt} \left( \frac{(-1)^{\gamma}2^{2\mu}(ct)^r}{\left( \prod_{i=0}^{r-1}(a_i b^i - a^i) \right) \left( \prod_{i=0}^{r-1}(a_i b^i - a^i) \right)^{\gamma}} \right) e^{dx} \varphi(y, dt) \varphi(y, ct).
\end{equation}

\begin{equation}
G(t) = \frac{1}{(ab)^r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) a^m c^{n-m} pU_{n-m}^{(r)}(dx, y, a, b, v, \mu; \overline{a}) \overline{pU}_m^{(r)}(dx, y, a, b, v, \mu; \overline{a}) \frac{(ct)^m}{m!} \tag{37}
\end{equation}

Secondly

\begin{equation}
G(t) = \frac{1}{(cd)^r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) c^m d^{n-m} \overline{pU}_{n-m}^{(r)}(dx, y, a, b, v, \mu; \overline{a}) pU_m^{(r)}(cx, y, a, b, v, \mu; \overline{a}) \frac{(ct)^m}{m!} \tag{38}
\end{equation}

Form equations (37) and (38), by comparing the coefficients of \(t^n\) on both sides, yields (35). \hfill \Box
4. Applications

The 2VGP family $p_n(x, y)$ contains a number of important special polynomials of two variables. Some members belonging to the 2VGP family are considered in Section 1. We notice that for every member belonging to the 2VGP, there is a new special polynomial that belongs to the $p U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$ family. Thus, by selecting a suitable choice for the function $\varphi(y, t)$ in equation (16), the generating function for the corresponding member belongs to $p U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$ is a family that can be obtained.

**Example 1.** Setting $\varphi(y, t) = e^{yt}n$ in the left-hand side of generating function (16), gives Gould–Hopper type polynomials (GHATP), denoted by $H^{(m)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$, are defined by

$$H^{(m)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r) = (-1)^n t^n n^r \left( \frac{e^{x+yt}}{n!} \right)^{x+y}.$$

(39)

Setting suitable values of the parameters in the results of the GHATP $H^{(m)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$, we obtain the following results:

1. If $\alpha_i = \lambda, b = e, a = 1, v = 1, u = 0$, then $H^{(m)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r) = (-1)^n H^{(m)}W_n^{(r)}(x, y, \lambda)$ (Gould–Hopper–Apostol–Bernoulli polynomials of order $r$) (see [9]).
2. If $\alpha_i = \lambda, b = e, a = 1, v = 0, u = 1$, then $H^{(m)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r) = H^{(m)}W_n^{(r)}(x, y, \lambda)$ (Gould–Hopper–Apostol–Euler polynomials of order $r$), see [9].
3. If $\alpha_i = \lambda, b = e, a = 1, v = 1, u = 1$, then $H^{(m)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r) = H^{(m)}W_n^{(r)}(x, y, \lambda)$ (Gould–Hopper–Apostol–Genocchi polynomials of order $r$), [9].

The series definitions and other results for the GHATP $H^{(m)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$ are given in Table 2.

Remark 2. For $m = 2$, the $H^{(m)}U_n^{(r)}(x, y)$ reduce to $H_n(x, y)$. Therefore, setting $m = 2$ in equation (39), we obtain the following generating function for the 2-variable Hermite Apostol type polynomials, denoted by $H^{(r)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$ as follows:

$$H^{(r)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r) = (-1)^n t^n \left( \frac{e^{x+yt}}{n!} \right)^{x+y}.$$

(40)

The series definitions and some results for the 2-variable Hermite Apostol type polynomials $H^{(r)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$ can be deduced by setting $m = 2$ in the results given in Table 2.

**Example 2.** Setting $\varphi(y, t) = C_0 (-yt^n)$ (for which the $p_n(x, y)$ reduce to the $mL_n(x, y)$) in the left-hand side of generating function (16), we find that the resultant 2-variable generalized Laguerre Apostol type polynomials (2VLAGATP), denoted by $mL^{(r)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$ are defined by the following generating function:

$$mL^{(r)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r) = \left( \frac{(-1)^n t^n e^{xt}}{n!} \right)^{x+y}.$$

(41)

The series definitions and other results for the 2VLAGATP $mL^{(r)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$ are given in Table 3.

Remark 3. Since for $m = 1$ and $y \rightarrow -y$, then $mL_n(x, y)$ reduce to the $L_n(x, y)$. Therefore, setting $m = 1$ and $y \rightarrow -y$ in equation (41), we obtain the generating function for the 2-variable Laguerre Apostol type polynomials, denoted by $L^{(r)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$ as

$$L^{(r)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r) = \left( \frac{(-1)^n t^n e^{xt}}{n!} \right)^{x+y}.$$

(42)

The series definitions and other results for the GHATP $L^{(r)}U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$ can be obtained by setting $m = 1$ and $y \rightarrow -y$ in the results given in Table 3.

Remark 4. Since for $x = 1$, the $L_n(x, y)$ reduce to the classical Laguerre polynomials $L_n(y)$. Therefore, setting $x = 1$ in equation (42), we obtain the following generating function for the Laguerre Apostol type polynomials, denoted by $L^{(r)}U_n^{(r)}(y, a, b, v, u; \alpha_r)$:

$$L^{(r)}U_n^{(r)}(y, a, b, v, u; \alpha_r) = \left( \frac{(-1)^n t^n e^{xt}}{n!} \right)^{y}.$$

(43)

The series definitions and other results for the GHATP $L^{(r)}U_n^{(r)}(y, a, b, v, u; \alpha_r)$ can be obtained by setting $m = 1$, $y \rightarrow -y$, and $x = 1$ in the results given in Table 3.

Setting suitable values of the parameters in the results of the GHATP $L^{(r)}U_n^{(r)}(y, a, b, v, u; \alpha_r)$, we obtain results for 2-variable generalized Laguerre–Apostol polynomials related to the 2VTEATP $e(\beta)U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$ (for more details see [9]).

**Example 3.** Setting $\varphi(y, t) = 1 - yt^\beta$ (for which the $p_n(x, y)$ reduce to the $C_{\beta}(x, y)$) in the left-hand side of generating function (16), we obtain the 2-variable truncated exponential Apostol type polynomials of order $\beta$ (2VTEATP), denoted by $e(\beta)U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$ are defined by

$$e(\beta)U_n^{(r)}(x, y, a, b, v, u; \alpha_r) = \left( \frac{(-1)^n t^n e^{xt}}{n!} \right)^{1 - yt^\beta}.$$

(44)

The series definitions and other results for the 2VTEATP $e(\beta)U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$ are given in Table 4.

Setting suitable values of the parameters in the results of the 2VTEATP $e(\beta)U_n^{(r)}(x, y, a, b, v, u; \alpha_r)$, we obtain results for 2-variable
Note. We note that for $\lambda = 1$ in the results derived above, we obtain the corresponding results for the Gould–Hopper–Bernoulli polynomials (GHBP) of order $r$, Gould–Hopper–Euler polynomials (GHEP) of order $r$, and Gould–Hopper–Genocchi polynomials (GHGP) of order $r$, respectively.

Table 2: Results for $H^{(m)}\chi_r^n (x, y, a, b, v, \mu, \pi_r)$.

| S.No. | Results | Expressions |
|-------|---------|-------------|
| 1     | Series definition | $H^{(m)}\chi_r^n (x, y, a, b, v, \mu, \pi_r) = \sum_{z=-\infty}^n \n (z) \chi_r^n (a, b, v, \mu, \pi_r) H_z^{(m)} (x, y)$ |
| 2     | Summation formulae | $H^{(m)}\chi_r^n (x + w, a, b, v, \mu, \pi_r) = \sum_{z=-\infty}^n \n (z) \chi_r^n (x, y, a, b, v, \mu, \pi_r) u^{m-z}$ |

Table 3: Results for mL $\chi_r^n (x, y, a, b, v, \mu, \pi_r)$.

| S.No. | Results | Expressions |
|-------|---------|-------------|
| 1     | Series definition | mL $\chi_r^n (x, y, a, b, v, \mu, \pi_r) = \sum_{z=-\infty}^n \n (z) \chi_r^n (a, b, v, \mu, \pi_r) mL_z (y, x)$ |
| 2     | Summation formulae | mL $\chi_r^n (x + w, a, b, v, \mu, \pi_r) = \sum_{z=-\infty}^n \n (z) \chi_r^n (x, y, a, b, v, \mu, \pi_r) u^{m-z}$ |

Table 4: Results for $e (\beta)\chi_r^n (x, y, a, b, v, \mu, \pi_r)$.

| S.No. | Results | Expressions |
|-------|---------|-------------|
| 1     | Series definition | $e (\beta)\chi_r^n (x, y, a, b, v, \mu, \pi_r) = \sum_{z=-\infty}^n \n (z) \chi_r^n (a, b, v, \mu, \pi_r) e^{\beta} (x, y)$ |
| 2     | Summation formulae | $e (\beta)\chi_r^n (x + w, a, b, v, \mu, \pi_r) = \sum_{z=-\infty}^n \n (z) \chi_r^n (x, y, a, b, v, \mu, \pi_r) e^{\beta} (w, y)$ |

truncated exponential Apostol polynomials related to mL $\chi_r^n (x, y, a, b, v, \mu, \pi_r)$ (for more details see [9]).

Remark 5. Since for $\beta = 2$, the $e_n^{(2)} (x, y)$ of order $\beta$ reduce to the $[2]e_n (x, y)$. Therefore, taking $\beta = 2$ in equation (44), we obtain the following generating function for the 2-variable truncated exponential Apostol type polynomials, denoted by $[2]e^{(2)}_n (x, y, a, b, v, \mu, \pi_r)$ as

$$\sum_{n=0}^{\infty} [2]e^{(2)}_n (x, y, a, b, v, \mu, \pi_r) \frac{t^n}{n!} = \frac{e^{xt}}{\prod_{i=0}^{\infty} (a_i b_i - a_i)} \left( 1 - yt \right)$$

(45)

The series definitions and other results for the 2VTEATP $[2]e^{(2)}_n (x, y, a, b, v, \mu, \pi_r)$ can be obtained by taking $\beta = 2$ in the results given in Table 4.

Remark 6. Since for $y = 1$, the $[2]e_n (x, y)$ of order $\beta$ reduce to the truncated exponential polynomials $[2]e_n (x)$. Therefore, taking $y = 1$ in equation (45), we get the following generating function for the truncated exponential polynomials, denoted by $[2]e^{(2)}_n (x, a, b, v, \mu, \pi_r)

$$\sum_{n=0}^{\infty} [2]e^{(2)}_n (x, a, b, v, \mu, \pi_r) \frac{t^n}{n!} = \frac{(-1)^n e^{t^n} t^n n!}{\prod_{i=0}^{\infty} (a_i b_i - a_i)} \left( 1 - t_i \right)$$

(46)

5. Two-Variable Degenerate Apostol-type Polynomials

In this section, replacing $e^r \rightarrow (1 + \lambda t)^{1/\lambda}$, and $a = 1$, $a = e = e$ in (13), a new class of two-variable degenerate Apostol
type polynomials $pD \mathcal{U}_n^{(r)}(x, y; \alpha; \beta, \lambda, \mu)$ is given. Some identities and properties are obtained.

\[ \sum_{n=0}^{\infty} pD \mathcal{U}_n^{(r)}(x, y; \alpha; \beta, \lambda, \mu) \frac{t^n}{n!} = \frac{t^r2^\mu}{\prod_{i=0}^{r-1}(1 + \lambda t)^{1/\lambda} - \alpha} \varphi(y, t). \] (47)

5.1. Special Cases

(1) If $\alpha_i = 1, r = 1, v = 1, \mu = 0$ and $\varphi(y, t) = 1$ in (47), we obtain

\[ pD \mathcal{U}_n^{(1)}(x, y; 1, 0; \lambda, 1) = B_{n\lambda}(x). \] (48)

(Degenerate Bernoulli polynomials, see [13]).

(2) If $\alpha_i = -1, r = 1, v = 0, \mu = 1$ and $\varphi(y, t) = 1$ in (47), we obtain

\[ pD \mathcal{U}_n^{(1)}(x; -1, 1; \lambda, 0) = E_{n\lambda}(x). \] (49)

(Degenerate Euler polynomials, see [13]).

(3) If $\alpha_i = 1, i = 0, 1, \ldots, r - 1, v = \mu = 1$ and $\varphi(y, t) = 1$ in (47), we obtain

\[ pD \mathcal{U}_n^{(r)}(\alpha, \beta, \lambda; \lambda, \mu) = \sum_{x_1, \ldots, x_n=0}^{\infty} \prod_{i=1}^{r} (\alpha_{i-1})^{x_i} \sum_{k=0}^{[n/2]} \frac{\beta^k n!}{k!(n-2k)!} \prod_{i=0}^{n-2k-1} \left( \alpha - \sum_{i=0}^{r} x_i - r - i \lambda \right). \] (52)

**Proof.** Putting $x = \alpha, y = \beta$, and $\varphi(\beta, t) = e^{\beta t}$ in (47), we have

\[ \sum_{n=0}^{\infty} pD \mathcal{U}_n^{(r)}(\alpha, \beta, \lambda; \lambda, \mu) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \sum_{x_1, \ldots, x_n=0}^{\infty} (\alpha_{i-1})^{x_i} (1 + \lambda t) \left( e^{-\sum_{i=1}^{r} x_i + r \lambda} \right) e^{\beta t}. \] (53)

Then, the generating function of degenerate generalized Hermite polynomials is given by [12].

\[ (1 + \lambda t)^{\alpha/\lambda} \frac{e^{\beta t}}{n!} = \sum_{n=0}^{\infty} H_n(\alpha, \beta, \lambda) \frac{t^n}{n!}. \] (54)

\[ \sum_{n=0}^{\infty} pD \mathcal{U}_n^{(r)}(\alpha, \beta, \lambda; \lambda, \mu) \frac{t^n}{n!} = \sum_{m=0}^{\infty} \sum_{x_1, \ldots, x_n=0}^{\infty} (\alpha_{i-1})^{x_i} H_n \left( \alpha - \sum_{i=1}^{r} x_i - r, \beta, \lambda \right) \frac{t^n}{n!}. \] (55)
Equating the coefficients of $t^n$ on both sides in the last equation, yields (52).

**Theorem 9.** For $n \geq 0$, we have

\[
\sum_{m=0}^{n} pD U_n^{(r)}(x, y; n) S(n, m) = \sum_{k=0}^{\lfloor n r \rfloor} \binom{n}{k} D U_n^{(r)}(x; y, \mu; \lambda, \alpha_r) S(k, m) \text{Bel}_{n-k, \lambda}(y),
\]

where Bel_{n-k, \lambda}(x) are degenerate Bell polynomials, see ([14]) and $S(n, m)$ are Stirling numbers of the second kind, see ([24]).

**Proof.** Replacing $t$ by $(e^x - 1)$ in (47), we get

\[
\frac{(e^x - 1)^m}{m!} = \sum_{n=0}^{\infty} S(n, m) \frac{t^n}{n!}.
\]

It is well known that the Stirling numbers of the second kind are defined by (see [24])

\[
\frac{(e^x - 1)^m}{m!} = \sum_{n=0}^{\infty} S(n, m) \frac{t^n}{n!}.
\]

We get

\[
\frac{(e^x - 1)^m}{m!} = \sum_{n=0}^{\infty} \sum_{m=0}^{n} pD U_n^{(r)}(x, y; n, m) S(n, m) \frac{t^n}{n!}.
\]

Let us put, [14].

\[
\phi(y, (e^x - 1)) = (1 + \lambda(e^x - 1))^y = \sum_{\ell=0}^{\infty} \text{Bel}_{\ell, \lambda}(y) \frac{t^\ell}{\ell!}.
\]

in the left-hand side of (59), we have

\[
\phi(y, (e^x - 1)) = (1 + \lambda(e^x - 1))^y = \sum_{\ell=0}^{\infty} \text{Bel}_{\ell, \lambda}(y) \frac{t^\ell}{\ell!}.
\]

Therefore, by comparing the coefficients of $t^n$ on both sides of (59) and (61), we obtain (56).
where \( s(n,m) \) are the Stirling numbers of the first kind, see [24].

\[
\frac{t^{r}2^{\mu}}{\prod_{i=0}^{r-1} ((1 + \lambda t)^{1/\lambda} - \alpha_i)} (1 + \lambda t)^{\mu/\lambda} \varphi(y,t) = \sum_{n=0}^{\infty} pD \mathcal{H}^{(r)}_{m}(y; \nu, \mu; \lambda, \pi_r) \frac{t^{n}}{n!} \sum_{m=0}^{\infty} \frac{(x^{\mu})_{m}}{m!} (\lambda t)^{m}
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left( \begin{array}{c} n \\ m \end{array} \right) \frac{D \mathcal{H}^{(r)}_{n-m}(y; \nu, \mu; \lambda, \pi_r)}{\lambda^{m}} \sum_{\ell=0}^{m} s(m, \ell) x^{\ell} \lambda^{m-\ell} t^{\ell}
\]

(63)

Therefore, by equating the right-hand side of (47) and the last equation, we obtain (62).

\[ \square \]

6. Probabilistic Application

\[ \text{Definition 11.} \] Let \( X_1, X_2, \ldots, X_r \) be a nonnegative random variable. Then \( \bar{X} = (X_1, X_2, \ldots, X_r) \) is said to be generalized degenerate Hermite distribution (GDHD), if its probability mass function is

\[
P(X) = \frac{\prod_{i=1}^{r} (\alpha_{i-1})^y H_n(y - \sum_{i=1}^{r} x_i - r, \beta, \lambda)}{D(\pi, \gamma, \beta, \lambda)},
\]

(64)

where

\[
D(\pi, \gamma, \beta, \lambda) = \sum_{\ell_1, \ell_2, \ldots, \ell_i=0}^{n} (\alpha_{i-1})^{\ell_i} H_n\left( y - \sum_{i=1}^{r} \ell_i - r, \beta, \lambda \right),
\]

(65)

and

\[
D(\pi, \gamma, \beta, \lambda) \text{ is convergent and positive for } \pi_r = (\alpha_0, \alpha_1, \ldots, \alpha_{r-1}), \quad 0 < \alpha_r < 1.
\]

6.1. Statistical Properties of GDHD Model

6.1.1. Cumulative Distribution Function. The cumulative distribution function (CDF) of GDHD is given by

\[
P(X \leq x_i) = 1 - \alpha_i^{x_i+1} D(\pi, \gamma + x_i + 1, \beta, \lambda) D(\pi, \gamma, \beta, \lambda).
\]

(67)

6.1.2. Moments and Related Measures. The moment generating function is given by

\[
E(X) = \frac{\sum_{x_1, x_2, \ldots, x_r = 0}^{\infty} x_i \prod_{i=1}^{r} (\alpha_{i-1})^y H_n(y - \sum_{i=1}^{r} x_i - r, \beta, \lambda)}{D(\pi, \gamma, \beta, \lambda)}.
\]

(70)

\[
\text{Var}(X) = \left[ \frac{\sum_{x_1, x_2, \ldots, x_r = 0}^{\infty} x_i^2 \prod_{i=1}^{r} (\alpha_{i-1})^y H_n(y - \sum_{i=1}^{r} x_i - r, \beta, \lambda)}{D(\pi, \gamma, \beta, \lambda)} \right] - \left[ \frac{\sum_{x_1, x_2, \ldots, x_r = 0}^{\infty} x_i \prod_{i=1}^{r} (\alpha_{i-1})^y H_n(y - \sum_{i=1}^{r} x_i - r, \beta, \lambda)}{D(\pi, \gamma, \beta, \lambda)} \right]^2.
\]

(71)
Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest with this study.

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