Abstract

Alon, Shpilka and Umans considered the following version of usual sunflower-free subset: a subset \( F \subseteq \{1, \ldots, D\}^n \) for \( D > 2 \) is sunflower-free if for every distinct triple \( x, y, z \in F \) there exists a coordinate \( i \) where exactly two of \( x_i, y_i, z_i \) are equal. Combining the polynomial method with character theory Naslund and Sawin proved that any sunflower-free set \( F \subseteq \{1, \ldots, D\}^n \) has size

\[
|F| \leq c_D^n,
\]

where \( c_D = \frac{3}{2^{2/3}} (D - 1)^{2/3} \).

In this short note we give a new upper bound for the size of sunflower-free subsets of \( \{1, \ldots, D\}^n \).

Our main result is a new upper bound for the size of sunflower-free \( k \)-uniform subsets.

More precisely, let \( k \) be an arbitrary integer. Let \( F \) be a sunflower-free \( k \)-uniform set system. Consider \( M := | \bigcup_{F \in \mathcal{F}} F | \). Then

\[
|\mathcal{F}| \leq 3 \left( \frac{2k}{3} \right)^k (2^{1/3} \cdot 3^e)^k \left( \left\lfloor \frac{M}{k} \right\rfloor - 1 \right)^{\frac{2k}{3}}.
\]

In the proof we use Naslund and Sawin’s result about sunflower-free subsets in \( \{1, \ldots, D\}^n \).
1 Introduction

Let \([n]\) stand for the set \(\{1, 2, \ldots, n\}\). We denote the family of all subsets of \([n]\) by \(2^{[n]}\).

Let \(X\) be a fixed subset of \([n]\). Let \(0 \leq k \leq n\) integers. We denote by \(\binom{X}{k}\) the family of all \(k\) element subsets of \(X\).

We say that a family \(\mathcal{F}\) of subsets of \([n]\) \(k\)-uniform, if \(|F| = k\) for each \(F \in \mathcal{F}\).

Recall that a family \(\mathcal{F} = \{F_1, \ldots, F_m\}\) of subsets of \([n]\) is a sunflower (or \(\Delta\)-system) with \(t\) petals if

\[ F_i \cap F_j = \bigcap_{s=1}^{t} F_s \]

for each \(1 \leq i, j \leq t\).

The kernel of a sunflower is the intersection of the members of this sunflower.

By definition a family of disjoint sets is a sunflower with empty kernel.

Erdős and Rado gave a remarkable upper bound for the size of a \(k\)-uniform family without a sunflower with \(t\) petals (see [7]).

**Theorem 1.1** (Sunflower theorem) If \(\mathcal{F}\) is a \(k\)-uniform set system with more than

\[ k!(t - 1)^k(1 - \sum_{s=1}^{k-1} \frac{s}{(s+1)!(t-1)^s}) \]

members, then \(\mathcal{F}\) contains a sunflower with \(t\) petals.

Later Kostochka improved this upper bound in [12].

**Theorem 1.2** Let \(t > 2\) and \(\alpha > 1\) be fixed integers. Let \(k\) be an arbitrary integer. Then there exists a constant \(D(t, \alpha)\) such that if \(\mathcal{F}\) is a \(k\)-uniform set system with more than

\[ D(t, \alpha)k! \left( \frac{\log \log \log k}{\alpha \log \log k} \right)^k \]

members, then \(\mathcal{F}\) contains a sunflower with \(t\) petals.

The following statement is conjectured by Erdős and Rado in [7].
**Conjecture 1** For each $t$, there exists a constant $C(t)$ such that if $\mathcal{F}$ is a $k$-uniform set system with more than $C(t)^k$ members, then $\mathcal{F}$ contains a sunflower with $t$ petals.

It is well-known that Erdős offered 1000 dollars for the proof or disproof of this conjecture for $t = 3$ (see [4]).

Naslund and Sawin proved the following upper bound for the size of a sunflower-free family in [13]. Their argument based on Tao’s slice–rank bounding method (see the blog [14]).

**Theorem 1.3** Let $\mathcal{F}$ be a family of subsets of $[n]$ without a sunflower with 3 petals. Then

$$|\mathcal{F}| \leq 3(n + 1) \sum_{i=0}^{[n/3]} \binom{n}{i}.$$  

Alon, Shpilka and Umans considered the following version of usual sunflowers in [1]: Let $D > 2$, $n \geq 1$ be integers. Then $k$ vectors $v_1, \ldots, v_k \in \mathbb{Z}_D^n$ form a $k$-sunflower if for every coordinate $i \in [n]$ it holds that either $(v_1)_i = \ldots = (v_k)_i$ or they all differ on that coordinate.

In the following the ’sunflower’ term means always a 3-sunflower.

Naslund and Sawin gave the following upper bounds for the size of sunflower-free families in [13] Theorem 2. Their proof worked only for 3-sunflowers.

**Theorem 1.4** Let $D > 2$, $n \geq 1$ be integers. Let $\mathcal{F} \subseteq \mathbb{Z}_D^n$ be a sunflower-free family in $\mathbb{Z}_D^n$. Then

$$|\mathcal{F}| \leq c_D^n,$$

where $c_D = \frac{3}{2^{2/3}}(D - 1)^{2/3}$.

Let $D > 2$, $n \geq 1$ be integers. Let $s(D, n)$ denote the maximum size of a sunflower-free family in $\mathbb{Z}_D^n$.

Define $J(q) := \frac{1}{q} \left( \min_{0 < x < 1} \left( \frac{1-x^q}{1-x} e^{-\frac{n+1}{3}} \right) \right)$ for each $q > 1$.

This $J(q)$ constant appeared in Ellenberg and Gijswijt’s bound for the size of three-term progression-free sets (see [5]). Blasiak, Church, Cohn,
Grochow and Umans proved in [2] Proposition 4.12 that $J(q)$ is a decreasing function of $q$ and

$$\lim_{q \to \infty} J(q) = \inf_{z>3} \frac{z - z^{-2}}{3 \log(z)} = 0.8414 \ldots.$$ 

It is easy to verify that $J(3) = 0.9184$, consequently $J(q)$ lies in the range

$$0.8414 \leq J(q) \leq 0.9184$$

for each $q \geq 3$.

Since a sunflower-free family in $\mathbb{Z}_n^D$ can not contain a three-term arithmetic progression, hence the Ellenberg and Gijswijt’s striking result (see [5]) implies the following upper bound.

**Theorem 1.5** Let $n \geq 1$ be an integer, $p^\alpha > 2$ be a prime power. Let $\mathcal{F} \subseteq \mathbb{Z}_n^D$ be a sunflower-free family in $\mathbb{Z}_n^D$. Then

$$|\mathcal{F}| \leq (J(p^\alpha)p^\alpha)^n.$$ 

Now we give some new bounds for the size of sunflower-free families in $\mathbb{Z}_n^D$.

The Chinese Remainder Theorem implies immediately the following result.

**Theorem 1.6** Let $m = p_1^{\alpha_1} \cdot \ldots \cdot p_r^{\alpha_r}$, where $p_i$ are different primes. Then

$$s(m,n) \leq s(p_1^{\alpha_1},n) \cdot \ldots \cdot s(p_r^{\alpha_r},n) \leq \left(\prod_{i=1}^r J(p_i^{\alpha_i})m\right)^n.$$ 

**Proof.**

By the Chinese Remainder Theorem there exists a bijection

$$\phi : \mathbb{Z}_m \to \mathbb{Z}_{p_1^{\alpha_1}} \times \ldots \times \mathbb{Z}_{p_r^{\alpha_r}}.$$ 

We can extend this bijection in a natural way to $(\mathbb{Z}_m)^n$ and we get the bijection

$$\phi^* : (\mathbb{Z}_m)^n \to (\mathbb{Z}_{p_1^{\alpha_1}})^n \times \ldots \times (\mathbb{Z}_{p_r^{\alpha_r}})^n.$$ 

Let $\mathcal{F} \subseteq (\mathbb{Z}_m)^n$ be a sunflower-free family in $(\mathbb{Z}_m)^n$. 

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Then it is easy to check that $\phi^*(F)$ is a sunflower-free family in $(\mathbb{Z}_{p_1})^n \times \ldots \times (\mathbb{Z}_{p_r})^n$. Hence

$$|\mathcal{F}| \leq |\phi^*(\mathcal{F})| \leq s(p_1^{\alpha_1}, n) \cdot \ldots \cdot s(p_r^{\alpha_r}, n).$$

Next we give an other new upper bound for the size of sunflower-free families in $\mathbb{Z}_D^n$, which is independent of $D$.

**Theorem 1.7** Let $D > 2$, $n \geq 1$ be integers, $\alpha > 1$ be a real number. Let $\mathcal{F} \subseteq \mathbb{Z}_D^n$ be a sunflower-free family in $\mathbb{Z}_D^n$. Then there exists a constant $K(\alpha) > 0$ such that

$$|\mathcal{F}| \leq K(\alpha) n! \left( \frac{(\log \log n)^2}{\alpha \log \log n} \right)^n.$$

Our main result is a new upper bound for the size of $k$-uniform sunflower-free families. In the proof we use Theorem 1.4 and Erdős and Kleitman’s famous result about $k$-partite hypergraphs.

**Theorem 1.8** Let $k$ be an arbitrary integer. Let $\mathcal{F}$ be a sunflower-free $k$-uniform set system. Let $M := | \bigcup_{F \in \mathcal{F}} F |$. Then

$$|\mathcal{F}| \leq 3 \left( \left\lceil \frac{2k}{3} \right\rceil + 1 \right) (2^{1/3} \cdot 3^e)^k \left( \left\lceil \frac{M}{k} \right\rceil - 1 \right)^{\left\lceil 2k \right\rceil}.$$

**Corollary 1.9** Let $k$ be an arbitrary integer. Let $\epsilon > 0$ be a fixed real number. Let $\mathcal{F}$ be a sunflower-free $k$-uniform set system. Suppose that

$$| \bigcup_{F \in \mathcal{F}} F | \leq k^{2.5-\epsilon}.$$

Then

$$|\mathcal{F}| \leq 3 \left( \left\lceil \frac{2k}{3} \right\rceil + 1 \right) (2^{1/3} \cdot 3^e)^k k^{\left( 2k - \frac{\epsilon}{3} \right)}.$$

**Proof.**

Define $M := | \bigcup_{F \in \mathcal{F}} F |$. Then

$$\frac{M}{k} \leq k^{1.5-\epsilon}.$$

Theorem 1.8 gives us the desired result.

We present our proofs in Section 2.
2 Proofs

Proof of Theorem 1.7:

Let $F \subseteq (\mathbb{Z}_D)^n$ be a sunflower-free family in $(\mathbb{Z}_D)^n$. We define first a hypergraph corresponding to $F$.

Let $U := [D] \times [n]$ denote the universe of this hypergraph.

Then for each vector $v \in \mathbb{Z}_D^n$ we can define the set

$$M(v) := \{(v_1 + 1, 1), \ldots, (v_n + 1, n)\} \subseteq U.$$

It is clear that $M(v)$ are $n$-sets. Consider the hypergraph

$$M(F) := \{M(v) : v \in F\}.$$

Then $M(F)$ is an $n$-uniform set family.

It is easy to check that $M(F)$ is a sunflower-free hypergraph, since $F$ is sunflower-free. Consequently we can apply Theorem 1.2 to the hypergraph $M(F)$ and we get our result.

Suppose that $K \subseteq \binom{X}{k}$ and that for some disjoint decomposition

$$X = X_1 \oplus \ldots \oplus X_m,$$

$K$ satisfies the equality $|F \cap X_i| = 1$ for all $F \in K$ and $1 \leq i \leq m$. Then $K$ is an $m$-partite hypergraph.

Erdős and Kleitman proved in [6] the following well-known result using an averaging argument.

**Theorem 2.1** Suppose that $K \subseteq \binom{X}{k}$. Then there exists a subfamily $G \subseteq F$ such that $G$ is $k$-partite and satisfies

$$|G| \geq \frac{k!}{k^k} |K|.$$

(1)

We use also in our proof the following generalization of Theorem 1.4.
**Theorem 2.2** Let $D_i \geq 3$ be integers for each $i \in [n]$. Let $\mathcal{H} \subseteq \mathbb{Z}_{D_1} \times \ldots \times \mathbb{Z}_{D_n}$ be a sunflower-free family in $\mathbb{Z}_{D_1} \times \ldots \times \mathbb{Z}_{D_n}$. Then

$$|\mathcal{H}| \leq 3 \sum_{I \subseteq [n], 0 \leq |I| \leq 2n} \prod_{i \in I} (D_i - 1).$$

**Proof.**
A simple modification of the argument appearing the proof of Theorem 1.4 works as a proof of Theorem 2.2.

It is easy to verify the following Proposition.

**Proposition 2.3** Let $D_i \geq 3$ be integers for each $i \in [n]$. Define $M := \sum_i D_i$. Then

$$\sum_{I \subseteq [n], 0 \leq |I| \leq 2n} \prod_{i \in I} (D_i - 1) \leq \sum_{j=0}^{\frac{2n}{3}} \binom{n}{j} \left(\left\lceil \frac{M}{n} \right\rceil - 1\right)^j.$$

**Proof of Theorem 1.9**

Let $\mathcal{F}$ be a sunflower-free $k$-uniform set system. Define $M := |\bigcup_{F \in \mathcal{F}} F|$ and $X := \bigcup_{F \in \mathcal{F}} F$.

By Theorem 2.1 there exists a subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that $\mathcal{G}$ is $k$-partite and satisfies

$$|\mathcal{G}| \geq \frac{k!}{k^k} |\mathcal{F}| \geq \frac{|\mathcal{F}|}{e^k}.$$

Consider the disjoint decomposition into classes

$$X = C_1 \oplus \ldots \oplus C_k,$$

where $\mathcal{G}$ satisfies the equality $|G \cap C_i| = 1$ for all $G \in \mathcal{G}$ and $1 \leq i \leq k$.

We can suppose that $C_1, \ldots, C_t$ are the classes with $|C_i| = 2$ for each $1 \leq i \leq t$ and $|C_i| \geq 3$ for each $i > t$. 

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Let $C_i = \{x_i, y_i\}$ for each $1 \leq i \leq t$. Define for each $N \subseteq [t]$ the following subfamily of $\mathcal{G}$:

$$\mathcal{G}(N) := \{G \in \mathcal{G} : \{x_i : i \in N\} \cup \{y_i : i \in [t] \setminus N\} \subseteq G\}.$$ 

Denote by $L \subseteq [t]$ the subset with

$$|\mathcal{G}(L)| = \max_{N \subseteq [t]} |\mathcal{G}(N)|.$$

Consider the set system

$$\mathcal{H} := \{F \setminus L : F \in \mathcal{G}(L)\}.$$

Clearly here $X \setminus L = \bigcup_{H \in \mathcal{H}} H$.

Then $\mathcal{H}$ is a $(k-t)$-uniform, $(k-t)$-partite set system with the disjoint decomposition into classes

$$X \setminus L = B_1 \oplus \ldots \oplus B_{k-t},$$

where $\mathcal{H}$ satisfies the equality $|H \cap B_i| = 1$ for all $H \in \mathcal{H}$ and $1 \leq i \leq k-t$. Our construction of the set system $\mathcal{H}$ shows that $|B_i| \geq 3$ for each $1 \leq i \leq k-t$.

On the other hand it follows from the equality

$$|\mathcal{G}(L)| = \max_{N \subseteq [t]} |\mathcal{G}(N)|$$

that

$$|\mathcal{G}| \leq \sum_{N \subseteq [t]} |\mathcal{G}(N)| \leq 2^t|\mathcal{G}(L)| = 2^t|\mathcal{H}| \leq 2^k|\mathcal{H}|.$$  \hspace{1cm} (2)

In the following we consider only the case when $t = 0$. The $t > 0$ cases can be treated in a similar way.

We use the following Proposition in our proof.

**Proposition 2.4** Let $D_i \geq 3$ be integers for each $i \in [n]$. Define $M := \sum_i D_i$. Then there exists an injection $\psi : \mathbb{Z}_{D_1} \times \ldots \times \mathbb{Z}_{D_n} \to \binom{[M]}{n}$ such that each $n$-uniform, $n$-partite set system with classes $C_i$, where $|C_i| = D_i$ for each $1 \leq i \leq n$ is precisely the image set of the map $\psi$ and each $n$-uniform, $n$-partite and sunflower-free family with classes $C_i$, where $|C_i| = D_i$ for each $1 \leq i \leq n$ corresponds to a sunflower-free family in $\mathbb{Z}_{D_1} \times \ldots \times \mathbb{Z}_{D_n}$. 

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We can apply Proposition 2.4 with the choices $D_i := |B_i|$ and we get that $T := \psi^{-1}(\mathcal{H})$ is a sunflower-free family in the group $\mathbb{Z}_{D_1} \times \ldots \times \mathbb{Z}_{D_k}$.

It follows from Theorem 2.2 that

$$|\mathcal{H}| = |\mathcal{T}| \leq 3 \cdot \sum_{I \subseteq \{n\}, 0 \leq |I| \leq \frac{2k}{2}} \prod_{i \in I} (D_i - 1).$$

But we get from Proposition 2.3 that

$$3 \cdot \sum_{I \subseteq \{n\}, 0 \leq |I| \leq \frac{2k}{3}} \prod_{i \in I} (D_i - 1) \leq 3 \sum_{j=0}^{2k} \binom{k}{j} \left( \left\lceil \frac{M}{k} \right\rceil - 1 \right)^j.$$

Hence

$$3 \sum_{j=0}^{2k} \binom{k}{j} \left( \left\lceil \frac{M}{k} \right\rceil - 1 \right)^j \leq 3 \left( \left\lceil \frac{2k}{3} \right\rceil + 1 \right) \left( \frac{3}{2^{2/3}} \right)^k \left( \frac{M}{k} - 1 \right)^{\frac{2k}{3}}.$$

The desired upper bound follows from equations (1) and (2):

$$|\mathcal{F}| \leq e^k |\mathcal{G}| \leq 2^k e^k |\mathcal{H}| \leq 3 \left( \left\lceil \frac{2k}{3} \right\rceil + 1 \right) \left( 2^{1/3} \cdot 3e \right)^k \left( \frac{M}{k} - 1 \right)^{\frac{2k}{3}}.$$

\section{Concluding remarks}

The following conjecture implies an unconditional, strong upper bound for the size of any sunflower-free $k$-uniform set system.

\textbf{Conjecture 2} There exists a $D > 0$ constant such that if $\mathcal{F}$ is any sunflower-free $k$-uniform set system, then

$$|\bigcup_{F \in \mathcal{F}} F| \leq D k^2.$$

We give here a weaker version of Conjecture 2.

\textbf{Conjecture 3} Let $\mathcal{F}$ be a sunflower-free $k$-uniform set system. Then there exist $F_1, \ldots, F_{2k} \in \mathcal{F}$ such that

$$\bigcup_{F \in \mathcal{F}} F = \bigcup_{j=1}^{2k} F_j.$$
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