A resolution of non-uniqueness puzzle of periodic orbits in the 2-dim anisotropic Kepler problem: bifurcation $U \to S + U'$

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Abstract. Using binary coding of orbit we introduce a finite level ($N$) surface over the initial value domain $D$ of 2-dim AKP. It gives a tiling of $D$ by base ribbons. The scheme of the one-time map is studied and the properness of the tiling is proved. This analysis in turn resolves the long standing puzzle in AKP—the non-uniqueness issue of a PO for a given code. We argue that the unique existence of a periodic orbit (PO) for a given binary code generally holds (for inverse anisotropy parameter $\gamma < 8/9$) but there is a remarkable exception in which a ribbon with a certain code escapes from shrinking at large $N$ and embodies the Broucke-type stable PO ($S$). It comes along the bifurcation of an unstable PO ($U$): $U(R) \to S(R) + U'(NR)$ (R for retracing and NR for non-retracing). An analysis based on orbit topology clarifies the pattern of the bifurcation; we give a conjecture that it occurs among odd rank $Y$-symmetric POs.

1. Introduction

Anisotropic Kepler Problem (AKP), cultivated by Gutzwiller [1, 2, 3, 4], is a vital testing ground of quantum chaos; we can control strength of chaos by a single tuning a parameter—the mass anisotropy. (We use inverse of anisotropy, $\gamma$). Here we focus on 2D-AKP, where a binary coding of the orbit is possible. It is proved by Devaney [5] mathematically (and showed in physical terms by Gutzwiller [2]) that there exist at least one periodic orbit (PO) for a given rational code in the high anisotropy region ($\gamma < 8/9$). Gutzwiller also conjectured from numerical results that the PO is unique. On the other hand, Devaney instigated Broucke to find a counterexample in the form of a stable PO [4]. Broucke reported two examples of stable PO [6], and more recently Contopoulos and Harsoula have reported rather many stable orbits, which may defy the belief that AKP is an Anosov system [7].

In this report, we focus on the coarse-grained level $N$ Devil’s Staircase Surface (DSS), which defines a tiling of the properly compactified initial value domain on the Poincaré surface of section. (Let us call it Gutzwiller’s rectangle $D$). We give a proof that the tiling is proper—that is, the DSS is monotonic—by an investigation of the generating mechanism of a level $N + 1$ tiling from level $N$ one by an investigation of the action of one-time map on $D$. It in turn shows that...
there is a special case that a ribbon with certain code does not shrink at large $N$, and this case occurs along with a transition $U \to S + U'$ with decreasing mass anisotropy. The transition is found to occur among $Y$-symmetric orbits, which comes from the fact that non-shrinking overlap of ribbons is produced when future (F) and past (P) ribbons become tangent each other. A topology consideration explains the detailed pattern of the bifurcation.

2. Devil’s Staircase Surface and the One-time Map

2.1. Gutzwiller’s rectangle

The Hamiltonian of two-dimensional AKP is given by

$$H = \frac{u^2}{2\mu} + \frac{v^2}{2\nu} - \frac{1}{r}, \quad (r \equiv \sqrt{x^2 + y^2}, \ \mu > \nu)$$

(1)

where $u \equiv p_x, v \equiv p_y$ and the mass anisotropy is proportional to $1 - \gamma$ with $\gamma \equiv \nu/\mu < 1$. Because $y/x = (\mu/\nu)y/x > y/x$, the orbit tends to cross the heavy $x$-axis more frequently. Thus, the Poincaré surface of section (PSS) is specified by the condition $y = 0$, and we can encode the orbit by the sequence of $a_i = \pm 1$, the sign of $x_i$ at the $i$-th crossing of the orbit with the $x$-axis.

The future sequence is $(a_0, a_1, \cdots)$ and the past is $(a_0, a_{-1}, \cdots)$.

The constant energy condition determines the kinematically allowed region on the PSS (the physical region for short). As the potential is a homogeneous in coordinates, the system has a scaling property. Any value for the energy is equivalent and we take $H = -1/2$ by convention.

The physical region of PSS is compactified into a rectangle

$$D = \left\{ (X, U) \left| |X| \leq 2 \text{ and } |U| \leq B \equiv \frac{\sqrt{\mu \pi}}{2} \right. \right\}$$

(2)

by an area preserving map;

$$X = x \left(1 + u^2/\mu\right), \quad U = \sqrt{\mu} \arctan \left(u/\sqrt{\mu}\right).$$

(3)

The collision manifold is the I-shaped backbone of $D$:

$$I = \left\{ (X, U) \left| |X| \leq 2 \text{ and } U = \pm B \right. \right\} \cup \left\{ (X, U) \left| X = 0 \text{ and } |U| \leq B \right. \right\}.$$  

(4)

2.2. Symbolic coding of orbits and Devil’s staircase surface

The future and past surfaces over $D$ at level $N$ are respectively described by the height functions

$$\zeta_N^F(X_0, U_0) \equiv \sum_{j=0}^{N} \frac{1}{2j+1} a_j(X_0, U_0), \quad \zeta_N^P(X_0, U_0) \equiv \sum_{j=0}^{N} \frac{1}{2j+1} a_{-j}(X_0, U_0).$$

(5)

As Fig. 1 shows, the surface defined by (5) has rather simple structure. The level $N$ surface by definition consists of $2N+1$ distinct heights and the base of it is divided into ribbons. Furthermore, the height of the step is monotonously increasing, if we traverse ribbons from left to right. We express that the ribbons are tiling $D$ properly.\(^1\)

\(^1\) The edge, separating two adjacent ribbons, can be worked out by Gutzwiller’s technique of the collision parameter analysis. They are the unstable and stable manifolds of the collision for the future and past DSS respectively [2]. See related discussion for Fig. 3.
Now, the code of a PO is cyclic; $a_{PO}^F = (a_0, a_1, \ldots, a_{2n-1}; a_{2n} = a_0)$; $n$ is the rank of the PO.\(^2\) If the initial point of the PO is $(X_0^*, U_0^*) \in D$, its height is naturally defined by

$$\zeta_{PO}^F(X_0^*, U_0^*) = \frac{1}{1 - \frac{1}{2^n}} \left\{ \sum_{j=0}^{2n-1} \frac{1}{2^{j+1}} a_j(X_0^*, U_0^*) \right\}.$$ \hspace{1cm} (6)

The following holds:

\textit{A step of the level $N$ DSS whose height is given by the first $N+1$ bits of the string (the primary part or its repetition) is the closest to the PO in height. See Fig. 1(c).}

To see this, consider the amount of misfit between (5) and (6);

$$\delta \equiv \zeta_{PO}^F - \zeta_N^F = \sum_{j=N+1}^{\infty} \frac{1}{2^{j+1}} a_j(X_0^*, U_0^*)$$ \hspace{1cm} (7)

We find $|\delta| \leq 1/2^{N+1}$. On the other hand, the difference of height between neighboring steps is $\Delta_N = 1/2^N$. Therefore, the selected step is the closest\(^3\). At the large $N$ limit, the misfit $\delta$ vanishes and the PO asymptotically sits on the selected step.

2.3. One-time map

We now discuss the one-time map $F$:

$$F : \quad D_0 \quad \rightarrow \quad D_1$$

$$(X_0, U_0) \quad \rightarrow \quad (X_1, U_1).$$ \hspace{1cm} (8)

Because the AKP flow involves both hyperbolic and elliptic singularities, separation as well as blow up inevitably occur. We have to back up the numerical calculation by the map restricted to the collision manifold $I$.

$$\begin{align*}
D_0 \cup D_1 \\
F|_I : \quad I \quad \rightarrow \quad I.
\end{align*}$$ \hspace{1cm} (9)

\(^2\) The length of PO must be even (2n), because it must cross the x-axis even times to complete its period.

\(^3\) This statement holds irrespective of whether the tiling by ribbons is proper or not.
The equation of motion is reduced to an autonomous form
\[ \frac{d\psi}{dt'} = - (\sqrt{\mu} \cos \psi - \sqrt{\nu} \sin \psi \cos \psi), \quad \frac{d\psi}{dt'} = -2 \left( \sqrt{\mu} \cos \psi \sin \psi - \sqrt{\nu} \sin \psi \cos \psi \right). \] (11)

This gives the map \( \mathcal{M} : (\psi_0, \psi_1) \to (\psi_1, \psi_1) \), where \((\psi_0, \psi_0)\) and \((\psi_1, \psi_1)\) parameterize respectively the initial and final \(I\) in (9). This invokes a new critical angle \(\psi_c\), and region 2 is divided into three sub-regions \(2A, 2B, 2C\).

| \(X_0\) | \(U_0\) | \(X_1\) | \(U_1\) |
|-------|-------|-------|-------|
| \(0\) | \(\psi_0\) | \(1\) | \(2\) \(\rightarrow\) \(X_e\) | \(B\) | \(1'\) | \(2\) \(\rightarrow\) \(X_h\) | \(-B\) |
| \(\psi_0\) | \(\pi\) | \(\psi_0\) | \(2A\) | \(X_e\) | \(\rightarrow\) \(0\) | \(B\) | \(2A'\) | \(-X_h\) | \(\rightarrow\) \(-X_c\) | \(-B\) |
| \(\pi/2\) | \(\pi - \psi_0\) | \(2B\) | \(X_e\) | \(\rightarrow\) \(0\) | \(-B\) | \(-X_e\) | \(\rightarrow\) \(0\) | \(-B\) |
| \(\pi - \psi_0\) | \(\pi - \psi_h\) | \(2C\) | \(X_e\) | \(\rightarrow\) \(2\) | \(-B\) | \(-X_c\) | \(\rightarrow\) \(0\) | \(-B\) |
| \(\pi - \psi_h\) | \(\pi\) | \(3\) | \(X_h\) | \(\rightarrow\) \(2\) | \(-B\) | \(3'\) | \(X_v\) | \(\rightarrow\) \(2\) | \(-B\) |

Table 1. Action of \(\mathcal{F}_I\) on five regions. \(X_v = 2\cos^2 \psi_0\), \(X_h = 2\cos^2 \psi_0\), \(X_c = 2\cos^2 \psi_c\). \((B = \pi \sqrt{\mu}/2)\). Arrow represents the direction of increasing \(\psi_0\). At the separation (solid line) by \(H_v (H_h)\), the signs of both \(X_1\) and \(U_1\) flip, while at the sub-criticality (dashed line) by \(\psi_0 (\psi_1)\) passing through \(\pi/2 (3\pi/2)\), only the \(U_1\) sign is flipped.

The scheme of \(\mathcal{F}_I\) is shown in Table 1 and it is combined with the numerically calculated interior map in Fig. 2.

![Figure 2](image-url)  
**Figure 2.** The scheme of one-time map. Points that blows up \((V^0_{+\pm} \rightarrow C^1_{+\pm})\) (or focused in \(C^0_{+\pm} \rightarrow V^1_{+\pm}\)), are depicted by quarter-circles.

![Figure 3](image-url)  
**Figure 3.** Inset shows orbit from collision. Dashed line is projection to \((X, U)\) plane.

\(^4\) The collision occurs at \(\chi = \infty\). This is equivalent to the blow up technique used by Devaney [5] to remove the singularity due to the collision. See also McGehee [8].
The boundary information is faithfully reflected in the interior result. Most important is the following: the separator curve $C^0$ is double sided, one side is mapped to $v^+_{1-}$ and the other to $v^+_{1+}$. The reason why a curve is contracted into a point and a point is blown up to a curve can be understood as follows. In Fig. 3, we show the result of collision parameter analysis which follows [9]. Each orbit emanating from the collision is labeled by a parameter (corresponding to the emission angle) and the position of its first arrival on the Poincaré surface forms a curve in the $(X_1, U_1)$–Gutzwiller’s rectangle. This is nothing but the core of the double line $C^1_{++}$ and $C^1_{+-}$.

3. Generation of Ribbon-tiling and a Proof of its Properness

Let us prove the following:

For any $N$, the ribbons are tiled properly.

Firstly $N = 1$. It is just determined by the one-time map by itself (Fig. 2) and the heights of (future) ribbons by (5) distribute from left to right as

$$\zeta^{N=1} = \begin{array}{cccc}
\frac{3}{4} & \frac{1}{3} & \frac{1}{3} & \frac{3}{4} \\
\uparrow & \uparrow & \uparrow & \\
\text{Ribbon:} (a_0, a_1) : 1 : (--) & 2 : (+-) & 3 : (+-) & 4 : (++) \\
L : \rightarrow & \rightarrow & \rightarrow : R
\end{array} \quad (12)$$

Clearly the ribbons are properly tiled at $N = 1$.

Now let us assume the level $N$ tiling is proper and show that the properness of the level $N + 1$ tiling follows from it. It is a crack of nutshell as shown in Fig. 4: place $D_{-1}$ in front of $D_0$, and project the level $N$ tiling on $D_0$ by $F^{-1}$ back to $D_{-1}$. Then, $F^{N+1}$ induces on $D_{-1}$ the level $N + 1$ tiling. Remarkably this way keeps the data from $D_0$ up to $D_N$ intact. (By time translation symmetry, it is equivalent to the level $N + 1$ on $D_0$).

![Figure 4. Construction of the level $N + 1$ tiling on $D_{-1}$.](image)

In Fig. 5 the scheme\(^5\) of $F^{-1}$ is laid over the level $N$ tiling (by $2^{N+1}$ ribbons) of $D_0$. It shows succinctly how $F^{-1}$ creates the tiling of $D_{-1}$. Most important is the contraction of the separator lines

$$C^0_{++} \xrightarrow{F^{-1}} v^+_{1+}, \quad C^0_{+-} \xrightarrow{F^{-1}} v^+_{1-}, \quad (13)$$

where the focus points $v^+_{1+}$ and $v^+_{1-}$ are respectively on the right-top and left-down boundary of $D_{-1}$. Thus, ribbons in the right-half (left-half) of $D_0$ are transversely separated into two groups $R_I$ and $R_{II}$ ($R_{III}$ and $R_{IV}$), and they create new tiling of $D_{-1}$. Here it should be noted that

\(^5\) The scheme of $F^{-1}$ is the reverse of $F$ in Fig. 2, except that the superscript of all parts in the scheme are one step back in time.
the order of \( R_{II} \) and \( R_{III} \) is left-right switched by the action of \( \mathcal{F}^{-1} \). Note that the number of ribbons are doubled and each becomes full-height extending from \( U_{-1} = -B \) to \( B \). Thus, they constitute in general finer tiling after \( \mathcal{F}^{-1} \).

\[
\mathcal{F}^{-1} : D_0 \to D_{-1}. \text{ Regions } R_K, K = I, \cdots, IV \text{ correspond to respectively } (++)', (-+)', (+-)', (--)' \text{ in Fig. 2}
\]

Now, from \( a_j (X_{-1}, U_{-1}) = a_{j-1}(X_0, U_0) \), we find

\[
\zeta^{N+1} (X_{-1}, U_{-1}) = \frac{1}{2} \text{sign} (X_{-1}) + \frac{1}{2} \zeta^N (X_0, U_0),
\]

which calculates \( \zeta^{N+1}(X_{-1}, U_{-1}) \). It is just half of previous heights except for an overall shift by the first term. Now, because the \( \mathcal{F}^{-1} \) is orientation preserving, this tells that the properness of tiling in each region is preserved via \( \mathcal{F}^{-1} \). Taking into account of the first term, which is \((+1/2, -1/2, +1/2, -1/2)\) for \( I, II, III, IV \) (due to the left-right switch mentioned above), we find the height distribution of \( N + 1 \) tiling is

\[
\zeta^{N+1} \subset (-1, -\frac{1}{2}) \quad (-\frac{1}{2}, 0) \quad (0, \frac{1}{2}) \quad (\frac{1}{2}, 1)
\]

\[
\text{Region in } D_{-1} : \quad \mathcal{F}^{-1}(R_{IV}) \quad \mathcal{F}^{-1}(R_{II}) \quad \mathcal{F}^{-1}(R_{III}) \quad \mathcal{F}^{-1}(R_I). \quad \mathcal{F}^{-1}(R).
\]

Apparantly, the tiling is not-only within each region proper but also globally proper. ■

4. Stable and Unstable Periodic Orbits in AKP

4.1. Use of Ribbons to Locate the Initial Point of a Periodic Orbit

Now, we are ready to resolve the long standing issue on the non-uniqueness puzzle of PO at a given code. In short, there are two cases: (A) Normally ribbons shrink roughly \( \propto 1/2^{N+1} \), and the corresponding PO uniquely singled out at a cross junction of \( F \) and \( P \) asymptotic curves. See Fig. 6.
Figure 6. Longitudinal ribbon splitting process at $\gamma = 0.2$. Circle dot; PO with code (+ +−−), rank $n = 2$. (Id=3 in [3]). $(\zeta^F, \zeta^P) = (3/5, 1/5)$ by (6) and enclosed in the junction of F and P ribbons with step heights $(5/8, 1/8)$ ($(9/16, 3/16)$) at $N = 2$ ($N = 3$) as calculated by (5).

(B) Exceptional case: a certain ribbon escapes from shrinking. The best example is the so called Broucke’s Island; let us start from a case study of PO3-6.

4.2. Example: Broucke stable PO
It is rank $n = 3$ with the code (+ −−−−). In Fig. 7, the evolution of a ribbon which encloses the initial point of $(X^*_0, U^*_0)$ of the Broucke’s PO (call it Broucke’s ribbon and give a mark ‘B’) along with two neighboring ribbons is tracked by successively acting $F^{-1}$ on them. Remarkably, only the Broucke’s ribbon survives, while others (even the adjacent ones) diminish rapidly.

Figure 7. Evolution of future ribbons in the case of Broucke’s PO3. Snap shots $T_N$ at $N = (6, 7, 8; 12, 13, 14; 42, 43, 44)$. $\gamma = 0.611$. Circle-dots: Broucke’s stable (S) and associated unstable (U’) PO. ‘B’: the ribbon enclosing them.

Why can it survive under the chopping by the separator? It is in a very intriguing way:
Broucke’s ribbon is protecting itself from shrinkage by changing at an early stage its tail to a line, so that the chopping becomes immaterial.

Now let us follow the decrease of the anisotropy by increasing and investigate how the Broucke stable PO3-6 comes out. As shown in Fig. 8, there is a threshold where the unstable PO (U) changes into the stable one (S) following the advent of non-shrinking ribbons enclosing S. At the same time, a new unstable PO (U’ is born. The overlap of non-shrinking F and P ribbons is a region where the orbit from it repeats infinitely the codes, and it is natural that the stable PO is in the center of it. On the other hand, the unstable PO locates at the corner of the overlap. It must be inside the overlap, because the code must be repeated to be periodic, but it must be at the edge, because a slight shift of the initial position must lead to an exponential blow up of the shift for the PO to be unstable. Summing up, the stable PO emerges in a bifurcation process $U \rightarrow S + U'$. 

![Figure 8](image)

**Figure 8.** Broucke PO3-6 ($N = 48$) $U \rightarrow S + U'$. $\gamma_{th} = 0.572350895$.

![Figure 9](image)

**Figure 9.** Broucke PO3-6: Lyapunov exponents and Period.

It is noteworthy that all of the POs ($U, S, U'$) are symmetric under the $Y$ transformation:

$Y: x \rightarrow x, \ y \rightarrow -y$.

The non-shrinking ribbons emerge just when the F and P ribbons become tangent each other at $U_0 = 0$. $U_0 = 0$ implies $p_x = 0$, therefore we consider the $Y$-symmetry is essential feature of the process. Also, we note that $U$ and $S$ are self-retracing ($R$) and $U'$ is self-non-retracing ($NR$); this bifurcation is actually

$U(R) \rightarrow S(R) + U'(NR)$, \hspace{1cm} (16)

Finally, the bifurcation diagram is shown in Fig. 9. We observe the followings.

1. Near the threshold, $\lambda_{max} \propto |\gamma - \gamma_{th}|^{1/2}$ gives a good description.
2. $U_0^*$ for unstable ones also exhibits the typical threshold behaviour $U_0^* \propto (\gamma - \gamma_{th})^{1/2}$.
3. The periods of the periodic orbits are insensitive to the transition. The period of $U$ as a function of $\gamma$ below $\gamma_{th}$ smoothly continues to that of $S$ above $\gamma_{th}$. This itself may be natural since both POs are of the same pattern (self-retracing), but, curiously, the period the NR PO ($U'$) has also degenerate period in very good approximation.
4.3. Symmetry-consideration

Since all POs ($U$, $S$ and $U'$) are $Y$-symmetric in the transition of the Broucke type, let us now contemplate on the $Y$-symmetric POs. By a simple topology consideration, we can explain that the pattern $U(R) \rightarrow S(R) + U'(NR)$ is natural. We prepare two concepts. Firstly, in order to distinguish $R$ and $NR$ orbits, it is useful to extend the homotopy idea. We consider that a $NR$ PO is as usual homotopic to $S^1$, but that $R$ PO, where the particle is going back and forth between two turning points, is homotopic to a squashed $S^1$. See Fig. 10.

![Figure 10](image1)

**Figure 10.** A NR PO (such as $U'$ in PO3-6) is homotopic to $S^1$, while a R PO ($U$ and $S$) to a squashed $S^1$.

Secondly, Let $n_\perp$ the number of perpendicular crossing of a $Y$-symmetric PO with the heavy x-axis. Then, $n_\perp$ must be either 2 or 0, because an orbit with odd $n_\perp$ cannot be closed while $n_\perp = 4, 6, \cdots$ can close but in disconnected loops.

Now we are ready to prove a remarkable fact:

*Any $Y$-symmetric periodic orbit is subject to one of the following three classes;*

(a) $R$ with $n_\perp = 2$,
(b) $NR$ with $n_\perp = 0$,
(c) $NR$ with $n_\perp = 2$.

To see this, it is sufficient to consider how to realize an appropriate $n_\perp$ for the topology of $R$ PO and $NR$ PO respectively. See Fig. 11.

![Figure 11](image2)

**Figure 11.** Left : three classes of $Y$-symmetric PO. Right upper diagram; Broucke PO3-6 transition $U(R) \rightarrow S(R) + U'(NR)$. Right lower; sample orbits in class (c).
(i) For R PO (a squashed $S^1$) to be $Y$-symmetric, a perpendicular crossing of the $x$-axis is needed. But just a single perpendicular crossing already saturates $n_1 = 2$; the crossing is multiplicity 2 in itself. Other crossings are X-type junction each with multiplicity 4. This is class (a).

(ii) On the other hand, for a NR PO, it can be $Y$-symmetric without perpendicular crossing; either $n_1 = 0$ (b) or $n_1 = 2$ (c).

In (b), all the crossings are X-type junction with multiplicity 2. In (c), all the crossings are X-type except for two distinct perpendicular crossings each with multiplicity one.

Having proved the classification, let us reconsider the Broucke transition $U \rightarrow S + U'$ in the light of it. The PO ($U$) at high anisotropy ($\gamma < \gamma_{th}$) is self-retracing and in class (a). Under the decrease of the anisotropy, it gradually changes its initial value $(X_0, U_0) = (X_0^*, 0)$ to remain closed. When $\gamma$ exceeds $\gamma_{th}$, the PO starts transition. It can proceed in three routes: (a) $\rightarrow$ (a), (b), (c), call them type (I), (II), (III) respectively. Routes II and III involve a topology change from squashed $S^1$ to $S^1$ as shown in Fig. 12. That is, a small shift at the crossing implies global deformation of the PO.

Figure 12. Class-(a)-PO can change in three routes (I), (II) and (III). Note in route (I), PO remains in class (a) and only the orbit profile changes $U(R) \rightarrow S(R)$.

Broucke-type transition occurs when the F and P ribbons become tangent each other at $U_0^* = 0$. As we saw in Fig. 8, PO (S) remains with $U_0^* = 0$ and with small $\Delta X_0^*$ enveloped by the overlap of F and P ribbons, while PO ($U'$) is subject to rapid increase of $U_0^*$ ($U_0^* \propto (\gamma - \gamma_{th})^{1/2}$). This means $U \rightarrow S$ proceed in route (I), while $U \rightarrow U'$ in route (II). And this is exactly what is observed. See the Broucke PO3-6 transition in right diagram in Fig. 11. We can now tell from the classification why PO (S) is R and PO ($U'$) is NR in (16).

For a PO in class (a), the total number of crossings of the $x$-axis (i.e. the length) is $4n + 2$, $n$ is the number of cross-junction with multiplicity 4 and 2 from a single perpendicular crossing with multiplicity 2. The rank $n$ of a PO is half of its length. Thus, the rank of class (a) PO is $2n + 1$ and we conjecture that the Broucke-type transition, associated with the non-shrinking ribbon, will occur in odd rank $Y$-symmetric PO.  

4.4. The case of PO-15

Now we apply the topology analysis to a PO at rank 15 (length 30) whose code is $+-+-+-++-++-+$. We have chosen this PO as the longest one among stable POs recently discovered by Contopoulos and Harsoula [7]. We note that the analysis in [7] is based on one-dimensional shooting ($p_x = 0$ assumed). Therefore, it cannot catch up the bifurcation $U \rightarrow U' + S$, where $U'$ has non-vanishing $U_0^*$. The bifurcation diagram is shown in Fig. 13.

6 Precisely, the tangency of the F and P ribbons at $U_0 = 0$ implies $Y$-symmetry of involved POs, and $n_1 = 2$, but the possibility of the pre-PO ($U$ in the Broucke case) being in class (c) is not logically excluded.

7 We have read the $(x_0, y_0 = 0)$ by eye from their Figs. 4 and 6, and refined it by 2-dimensional shooting for higher accuracy.
Amazingly the first bifurcation $U \to U' + S$ is followed by a second one $S \to S' + S''$. (It occurs in near integrable limit $\gamma > 8/9$). The location of POs inside the relevant ribbons are depicted in Fig. 14. The orbit profiles are given in Fig. 15; remarkably, the multiplicity characteristics are just in accord with our topology analysis.

**Figure 13.**

**Figure 14.**

**Figure 15.** Fine structure of orbit profile. From the number of crossings we determine its topology. The region marked by an asterisk are resolved in the doubly magnified diagram.

### 5. Conclusion

The key of this work is an introduction of *tiling by ribbons* of the initial value domain $D$. This is a switch from a dual picture (Gutzwiller’s collision parameter analysis). By the use of tiling
defined at finite $N$, it has become tractable to track the generation of higher level DSS. This in particular flags the advent of non-shrinking ribbons. Our resolution of the non-uniqueness puzzle of POs in AKP is quite simple. When the ribbons shrink at large $N$, a unique PO is singled out for a given code; if the ribbons escape from shrinking, then a stable PO is associated with an unstable PO in the bifurcation process $U(R) \to S(R) + U'(NR)$. A topology argument supports the pattern of this bifurcation. A conjecture is drawn that such a code-preserving bifurcation occurs among $Y$-symmetric rank odd POs.

Where to proceed? First, we need a better analytic understanding of the condition for the advent of the non-shrinking ribbons. To this end, a local bifurcation analysis using the normal form will be helpful. We envisage another interesting use of it. We have successfully traced the quantum-classical correspondence in AKP in terms of inverse chaology [10]. However, if one wishes to apply the Gutzwiller trace formula directly to the bifurcation process, one instantly fails because either the Lyapunov exponent (for $U$) or the rotation index ($S$) vanishes at the bifurcation threshold. We are planning to apply the inverse chaology to this challenging bifurcation process by stepping forward to the next order in $\hbar$ and by including the satellite contribution via the normal form analysis ([11], [12] and [13]).

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