Massless Neutrino Oscillations via Quantum Tunneling

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Abstract: In the current theory, neutrino oscillations require that the masses of neutrinos are non-vanishing. By analogy with the oscillation of quantum two-state system, we assume that neutrino oscillations may be regarded as quantum tunneling process. The difference of the quantum numbers of two particles may be regarded as a barrier between them. Thus neutrinos with vanishing mass can also oscillate. The hypothesis can also be applied to quark mixing, flavor violation of charged leptons, CP violation of neutral kaons and electroweak mixing.

1 Introduction

The experiments with solar, atmospheric and reactor neutrinos have provided compelling evidences for the existence of neutrino oscillations [1–11]. According to the current theory, neutrino oscillations are due to non-vanishing masses. Neutrino oscillations include oscillations in vacuum and in matter. The Seesaw mechanism has been proposed to explain why the neutrino masses are so small [12, 13]. The oscillation length can be derived by the current theory [14]. But some of the assumptions, such as equal-energy or equal-momentum, are controversial. This has led to the wave packet description of neutrinos [15–25]. However, both the two descriptions face a problem that the mixing will become incoherent and neutrinos will cease to oscillate after a long distance of flight.

In order to realize neutrino oscillations within the framework of standard model, we assume that neutrino oscillations are analogous to the oscillation of quantum two-state system, then massless neutrinos can also oscillate.

2 The current theory of neutrino oscillations

As a plane wave, neutrino can oscillate via four assumptions: equal-energy, equal-momentum, energy-momentum conservation and equal-velocity. For a review of these assumptions one may see [24, 25]. We mainly discuss the former two assumptions, which can be found in [14]. For the latter two assumptions, one may see [24, 25] and the references therein.

For convenience we work in the natural units, where $\hbar = c = 1$. For simplicity, we leave the tauon neutrino $\nu_\tau$ out of the following and assume that only the electron neutrino $\nu_e$ and muon neutrino $\nu_\mu$ mix with each other. As $\nu_e$ and $\nu_\mu$ are not the energy eigenstates, we denote the eigenstates of the Hamiltonian with $\nu_1$ and $\nu_2$, respectively, and for which we make the following ansatz

$$
\begin{pmatrix}
\nu_1 \\
\nu_2
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\nu_e \\
\nu_\mu
\end{pmatrix},
$$

(1)

The inversion of Eq. (1) is
\[
\begin{pmatrix}
    v_e \\
    v_{\mu}
\end{pmatrix} =
\begin{pmatrix}
    \cos \theta & \sin \theta \\
    -\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
    v_1 \\
    v_2
\end{pmatrix},
\]

(2)

First we assume that an electron neutrino \( v_e \) is produced with definite momentum \( p \) at point \( x \) and at time \( t = 0 \). In the energy representation it holds that, for the evolution of the state

\[
\begin{pmatrix}
    v_1(x,t) \\
    v_2(x,t)
\end{pmatrix} =
\begin{pmatrix}
    e^{-iE_1 t} & 0 \\
    0 & e^{-iE_2 t}
\end{pmatrix}
\begin{pmatrix}
    v_1(0) \\
    v_2(0)
\end{pmatrix} e^{i p \cdot x},
\]

(3)

where \( E_1 \) and \( E_2 \) are the energy eigenvalues of \( v_1 \) and \( v_2 \), respectively, and we have

\[
E_1 = \sqrt{p^2 + m_1^2}, \quad E_2 = \sqrt{p^2 + m_2^2},
\]

(4)

where \( m_1 \) and \( m_2 \) are the masses of the eigenstates \( v_1 \) and \( v_2 \), respectively. With the help of above equations, we obtain

\[
\begin{pmatrix}
    v_e(x,t) \\
    v_\mu(x,t)
\end{pmatrix} =
\begin{pmatrix}
    \cos \theta & \sin \theta \\
    -\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
    e^{-iE_1 t} & 0 \\
    0 & e^{-iE_2 t}
\end{pmatrix}
\begin{pmatrix}
    \cos \theta & -\sin \theta \\
    \sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
    v_e(0) \\
    v_\mu(0)
\end{pmatrix} e^{i p \cdot x}.
\]

(5)

According to the assumption that purely neutrinos of the electronic type are emitted at the source, we have \( v_e(0) = 1 \) and \( v_\mu(0) = 0 \). Then the probability of finding a muon neutrino \( v_\mu \) at time \( t \) is

\[
|v_\mu(x,t)|^2 = |\sin \theta \cos \theta (e^{-iE_1 t} - e^{-iE_2 t})|^2 = \sin^2 (2\theta) \sin^2 \frac{(E_2 - E_1)t}{2}.
\]

(6)

If \( m_1 = m_2 = 0 \), we have \( E_1 = E_2 \), then the above expression is equal to zero and neutrinos cannot oscillate. In relativistic limit, we have

\[
E_2 - E_1 = \sqrt{m_2^2 + p^2} - \sqrt{m_1^2 + p^2} \approx \frac{(m_2^2 - m_1^2)}{2p}.
\]

(7)

Let \( \Delta m^2_{21} = (m_2^2 - m_1^2) \) and \( t \approx l \), the oscillation length \( l \) can be written as

\[
l = \frac{4\pi p}{\Delta m^2_{21}}.
\]

(8)

We then suppose that neutrinos are produced with a definite energy \( E \), and make the ansatz

\[
\begin{pmatrix}
    v_1(x,t) \\
    v_2(x,t)
\end{pmatrix} =
\begin{pmatrix}
    e^{i p_1 x} v_1(0) \\
    e^{i p_2 x} v_2(0)
\end{pmatrix},
\]

(9)

where \( p_1 = \sqrt{E^2 - m_1^2} \), \( p_2 = \sqrt{E^2 - m_2^2} \). And again set \( v_e(0) = 1 \) and \( v_\mu(0) = 0 \) for \( x = 0 \). It results that
In relativistic limit, we have

\[ \gamma_i \gamma_j \approx \frac{m^2_i - m^2_j}{2E} \]

Since \( E \approx p \), this expression is identical to Eq. (7). Besides the assumptions of equal–momentum and equal–energy, there are also assumptions of energy–momentum conservation and equal–velocity. Although the four assumptions lead to the same result in the relativistic limit, no one gives arguments for their correctness. A first look at the four assumptions shows that they are incompatible. For example, if the neutrinos with different masses had the same energy they could not have the same momentum and vice versa. The assumption of energy–momentum conservation seems to be the most satisfactory one, while the equal–velocity assumption is the most unlikely one. An explanation in [26] was proposed to account for why equal–velocity assumption can be ruled out. Assuming the two mass eigenstates have a same velocity \( u \), we immediately arrive at \( \gamma_1 = \gamma_2 = \sqrt{1 - u^2} \), then we get \( E_1 / E_2 = m_1 / m_2 \). This equality cannot hold because \( E_1 / E_2 \approx 1 \), while \( m_1 / m_2 \) may be extremely small or extremely large.

Since the plane wave description of neutrino is unsatisfactory, it’s natural to describe it with wave packet. The detailed discussion may refer to [15–25]. But a problem remains for the two descriptions, that is, since the equal–velocity assumption is not correct, different neutrinos will travel with a different velocity. Then after a long distance of flight (e.g. the neutrinos reaching the earth from supernovas), the neutrino mixing will become incoherent and neutrinos will cease to oscillate. This difficulty will be swept away with massless neutrino oscillations model in the following.

The above analysis of two–flavor neutrino mixing can be easily extended to the instance of three–flavor mixing, which is described with PMNS matrix [27].

3 Neutrino oscillations via barrier tunneling

3.1 Oscillation of quantum two-state system

There is a two-state system in quantum physics, which has two quantum states with a symmetric structure. The system will oscillate between the two states. The following discussion may refer to [28]. Taking ammonia molecule as an example. There are two possible positions for the nitrogen atom, which may be on one side of the plane or on the other, as shown in Fig. 1. We denote the two quantum states by \( |C_1\rangle \) and \( |C_2\rangle \), respectively. In Fig. 1, the nitrogen atom must penetrate a barrier when flipping to the other side. Even if its energy is not high enough to traverse the barrier from the classical point of view, there is a certain probability for the nitrogen atom to tunnel through the barrier. So we suppose the Schrödinger equation for the flipping of the nitrogen atom to be

\[ i\hbar \frac{\partial}{\partial t} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} E_0 - A & A \\ -A & E_0 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \]
It’s not difficult to solve this differential equation. The detailed calculations may refer to \cite{28}. We only give the result here. Suppose the system is in state $|C_1\rangle$ at $t=0$, i.e., $C_1(0)=1$, $C_2(0)=0$. Then the probabilities of the system in states $|C_1\rangle$ and $|C_2\rangle$ are

\[
\begin{align*}
|C_1(t)|^2 &= \cos^2 At \\
|C_2(t)|^2 &= \sin^2 At
\end{align*}
\]  

(13)

The eigenvalues of the energy of the system are $E_1 = E_0 - A$ and $E_2 = E_0 + A$, respectively.

3.2 Neutrino oscillations via barrier tunneling

We see that Eqs. (6) and (13) are identical in the case of $\theta = \pi/4$ and $A = (E_2 - E_1)/2$. So if we wish to let neutrinos oscillate with vanishing mass, we may simply think that neutrino oscillations are the consequence of barrier tunneling. In this case, even if $m_1 = m_2 = 0$, we still have $E_1 \neq E_2$, where $E_1$ and $E_2$ are the eigenvalues of the energy of the system.

A problem with quantum barrier tunneling is that we only know that the difference of the eigenvalues of energy of the system is caused by the existence of barrier, but we can not determine the expression of $A$. We need to find out the relationship between $A$ and the energy of the neutrino. Because the energy splitting is very small, we may use the average energy $E$ to denote $E_1$ or $E_2$. Since there is no prior guideline to follow, we may use Bohr’s correspondence principle to seek some clues. This principle states that quantum mechanics reduces to classical mechanics in the limit of large quantum numbers, while large quantum numbers implies high system energy, therefore when the neutrino energy is very high, the neutrino is more like a classical particle that is in the superposition of two states rather than oscillation between them, that is, the energy splitting will tend to decreases with the increase of neutrino energy. Then we may assume that

\[A = \frac{1}{2} \Delta E = \frac{k}{E^n} \]  

(14)

From the fitting of experimental results we have $n \approx 1.14$ \cite{29}, so a reasonable assumption is $n=1$. Then we get the same result as Eq. (8), and $k$ is equivalent to $\Delta m^2_{21}$ in Eq. (8).

We have pointed out earlier that the assumption of equal-energy or equal-momentum is controversial when using the plane wave model. This difficulty is overcome when quantum tunneling hypothesis is adopted. In our theory, the neutrino mass is zero, and the energies of the
three flavor neutrinos are the same, but due to the existence of barriers, the eigenvalues of energy of the system are different, so the system can still oscillate. Our model can also explain the phenomenon of neutrino oscillation enhancement in matter. Since only $\nu_e$ can interact with matter, which is equivalent to that there is a potential energy in addition to the kinetic energy when $\nu_e$ passes through matter and its total energy increases, it is easier for $\nu_e$ to transit to $\nu_\mu$ or $\nu_\tau$.

There are also other massless neutrino oscillations models. But they all need some additional assumptions. For example, residual symmetry in [30], open system in [31] and modified Dirac equations in [32], are respectively introduced in order to keep neutrinos massless. By contrast, our model is the simplest, and it is natural to extend the oscillation of quantum two-state system to neutrino oscillations.

3.3 Formulation mechanism of barrier

We have adopted the hypothesis of barrier tunneling to explain the neutrino oscillations, then the next question is: How are the barriers between neutrinos formed? Let’s first see some tunneling phenomena. In the case of scanning tunneling microscope, electrons tunnel through air from one metal to another metal, the barrier is the air between the two metals. In the Josephson junction, the barrier is the insulator sandwiched between the superconductors. In the photon tunneling experiment, the barrier is the air between the two prisms [33, 34]. These tunneling phenomena are quantum effects, and they can be explained with quantum mechanics without introducing new interaction. The only requirement for quantum tunneling is the conservation of energy and momentum of the particles before and after tunneling.

Now let’s see the electron transitions outside the nucleus between different orbitals, which may also be regarded as a quantum tunneling phenomenon. The barrier here is not a tangible substance, but the difference of quantum numbers ($n$, $l$, $m$) of the electron in different orbitals. Due to the different energies of the electron in different orbitals, the electron must emit or absorb a photon during the transition process. For barrier between neutrinos, the difference of the quantum number is flavor violation. The barriers ensure that the neutrino is in a pure state of flavor rather than in the superposition state at any time, so that we can distinguish between different generations (flavors) of neutrinos.

What if there are no barriers between different flavors of neutrino? In this case, neutrino cannot maintain the independence of flavor, it will become a simple mixture of three flavors. Its property will be different from any flavor of the neutrino. To illustrate this point, we take the polarization of light as an example. The linear polarization can be considered as the superposition of the left-handed and right-handed circular polarizations. There is no barrier between the two eigenstates. Thus the property of the linearly polarized light is different from that of the circularly polarized light. We see that the existence of the barrier of neutrino not only is the theoretical requirement but also has physical significance.

Let’s consider another question: Why do neutrinos oscillate? We know that there are three generations of quarks in nature, and their masses are different. So are the three generations (flavors) of charged leptons. Heavier quarks and charged leptons will decay into lighter ones.
That is, flavor violation exists between them. But for the three generations (flavors) of neutrinos with equal zero mass, how to realize flavor violation? So neutrinos must oscillate, which is the consequence of their vanishing masses.

We might also understand neutrino oscillations from the properties of microscopic particles. Microscopic particles are never satisfied with staying at the same place. They will appear in every place in the coherent volume simultaneously. Similarly, a microscopic particle is not satisfied to appear as only character. If possible, it will change its character from one to another, which is the intrinsic cause of neutrino oscillations. As flavor oscillation is an intrinsic property of fundamental particles, not only neutrinos can oscillate, but also other particles will oscillate. We shall see some examples in the following.

4 Oscillations of other particles

4.1 Quark mixing

We have discussed neutrino oscillations via barrier tunneling. One may wonder whether quarks can oscillate in this way. We think that quark mixing is actually quark oscillation. By convention, the up-type quarks ($u$, $c$, $t$) are chosen to be pure states. The mixing of three generations of down-type quarks ($d$, $s$, $b$) is represented by CKM matrix. For simplicity, we only consider the mixing between $d$ and $s$ quarks. Unlike neutrinos, both $d$ and $s$ quarks are massive. It is impossible for two free particles with different masses to have the same energy and momentum. We can see this point from the following expressions:

$$m_1 u_1 = m_2 u_2$$  \hspace{1cm} (15)

$$\frac{m_1 u_1}{\sqrt{1-u_1^2 / c^2}} = \frac{m_2 u_2}{\sqrt{1-u_2^2 / c^2}}$$  \hspace{1cm} (16)

If Eq. (15) holds, we arrive at $u_1 = u_2$ from Eq. (16), then Eq. (15) will not be true. However, free quarks do not exist. Quarks in hadron interact with other quarks at all times, that is, they also have potential energies. Then it is possible for quarks in hadrons to maintain the conservation of energy and momentum during oscillation.

The current theory only considers the mixing of d-type quarks. There is no reason that only d-type quarks can oscillate, theoretically u-type quarks may also oscillate. However, the mass differences between the three generations of $u$, $c$ and $t$ quarks are too large, while the mixing angle is related to the mass ratio between different quarks, for example, we have $\theta_C^2 \approx m_d / m_s$, so the mixing angles of u-type quarks are very small and can be ignored.

It should be pointed out that, for the origin of quark mixing, the current theory attributes it to the coupling between quarks of different generations. For example, $u$ and $\bar{d}$ quark will form a $W^+$ boson, while $u$ and $\bar{s}$ quarks will also form a $W^+$ boson. The current theory does not explain why quarks of different generations can be coupled in the process of weak interaction, so it is just another equivalent statement of quark mixing. Our above theory attributed it to the oscillation between quarks of different generations, thus explaining the origin of quark mixing at
a deep level.

4.2 Forbidden transition of charged leptons

Unlike quark confinement, leptons can exist freely. However, due to the conservation of energy and momentum, massive leptons cannot oscillate freely. For example, a free muon cannot directly transform into an electron, but it can do so by emitting a photon, that is, the following process may occur:

$$\mu^- \rightarrow e^- + \gamma$$

(17)

This is actually electromagnetic decay. However, this process is realized by barrier tunneling rather than electromagnetic interaction. This process is analogous to the transition of electron between different orbitals. As the mass difference between electron and muon is very large, this transition probability is very small. Besides, this decay will be greatly compressed by the muon weak decay, as the following process are more likely to occur:

$$\mu^- \rightarrow e^- + \nu_\mu + \bar{\nu}_e$$

(18)

The reaction of Eq. (17) has not been observed experimentally so far. As the photon emitted in this reaction has a very large energy, it may also transformed into a positron and an electron, so two electrons and a positron will be observed in the experiment. The aim of the Mu3e experiment being planned is to observe the above result. This experiment is a project of the Paul Scherrer Institute in Switzerland. The experiment will produce $10^9$ muons per second and observe more than $10^{16}$ muon decay events. The experiment is expected to officially run in 2028. One may visit https://www.psi.ch/en/mu3e to track the experimental progress.

It is worth pointing out that in order to realize the reaction of Eq. (17), an undiscovered particle X is needed in the current theory. The mass of X particle is of the order of $10^3$ TeV. The Feynman diagram of the reaction is shown in Fig. 2. If the future experiment confirms the result of Eq. (17), it implies new physics beyond the standard model. However, according to our theory, the reaction of Eq. (17) can be explained with quantum tunneling without introducing X particle.

4.3 Oscillation of neutral kaons

We now see another oscillation phenomenon—the oscillation of neutral kaons. In the oscillation process, the CP quantum number of the particle is violated. The oscillation between $K^0 (d\bar{s})$ and $\bar{K}^0 (\bar{d}s)$ is realized by weak interaction [14]. This process can be described as
The $d$ quark in the $K^0$ meson captures a virtual $\bar{u}$ quark in the vacuum and becomes a $W^-$ boson, which then decays into $s$ and $\bar{u}$ quarks; in the meanwhile, the $\bar{s}$ in $\bar{K}^0$ captures a virtual $u$ quark in the vacuum and becomes a $W^+$ boson, which then decays into $d$ and $u$ quarks. $U$ quark and $\bar{u}$ quark annihilate into vacuum, and $d\bar{s}$ is left. By this way, a $K^0$ meson will transform into a $\bar{K}^0$ meson. The process can be written as

$$K^0 \rightarrow \begin{cases} \text{d} + \bar{u} \rightarrow W^- \rightarrow s + \bar{u} \\ \bar{s} + u \rightarrow W^+ \rightarrow d + u \end{cases} \rightarrow \bar{K}^0$$

With $K^0$ and $\bar{K}^0$ we can construct two quantum states, namely

$$K_1 = \frac{1}{\sqrt{2}}(K^0 + \bar{K}^0), \quad K_2 = \frac{1}{\sqrt{2}}(K^0 - \bar{K}^0)$$

As $K_1$ and $K_2$ have the same mass but different CP quantum numbers, i.e., $\text{CP}K_1 = K_1$ and $\text{CP}K_2 = -K_2$, there will be an oscillation between $K_1$ and $K_2$. The consequence of the oscillation is that neither $K_1$ nor $K_2$ is the eigenstate of the energy. What we observe experimentally are the mixture of $K_1$ and $K_2$, i.e., $K^0_S = K_1 + eK_2$ and $K^0_L = eK_1 + K_2$, where $e = 2.22 \times 10^{-7} e^\theta$, $\theta = 43.5^\circ$ [35]. $K^0_S$ and $K^0_L$ are the eigenstates of energy, and there is a small energy difference of $\Delta m e^2 = 0.529 \times 10^{10} \text{eV}$ between them [36].

The oscillation between $K^0$ and $\bar{K}^0$ is realized by virtual particles, while the oscillation between $K_1$ and $K_2$ is realized by barrier tunneling. It can be vividly imagined that there is a bridge between $K^0$ and $\bar{K}^0$; while there is no bridge between $K_1$ and $K_2$, and the gap can be crossed by one large step.

When neutral kaons are born, half of them are in $K_1$ state, and half of them are in $K_2$ state. As the value of $e$ is very small, it can be approximately thought that $K_1$ is the short-lived $K^0_S$ ($0.89 \times 10^{-10}$ s), and $K_2$ is long-lived $K^0_L$ ($5.1 \times 10^{-8}$ s). Because the CP value of pion is -1, the CP value of two pions is $(-1)^2 = 1$, while the CP value of three pions is $(-1)^3 = -1$. According to CP conservation, $K_1$ ($K^0_S$) will decay into two pions, and $K_2$ ($K^0_L$) will decay into three pions. Due to the oscillation between $K_1$ and $K_2$, a small part (about two thousandths) of $K_2$ will first transform into $K_1$ before decaying into two pions, resulting in CP violation.

It should be pointed out that the current theory attributes the phenomenon of $K^0_L$ decaying into two pions to the CP violation, and does not explain how this process occurs. Our theory divides this process into two stages: The first is $K_2$ oscillating to $K_1$, and the second is $K_1$ decaying into two pions. Therefore, CP non-conservation of neutral kaons can be attributed to the oscillation between $K_1$ and $K_2$; while CP is still conserved during the decay process.

### 4.4 Electroweak mixing

Beyond the electroweak scale, the properties of photon and $Z^0$ boson are almost identical: They are electrically neutral; they have vanishing zero mass and 1 spin. On the other hand, there are differences between them. For example, photons can only interact with charged particles, while $Z^0$ boson can interact with both charged and neutral particles. This difference constitutes
the barrier between photon and $Z^0$ boson, so there exists oscillation between them (In fact, it is the oscillation between $W^0$ and $B$ particles, while photon and $Z^0$ bosons are the mixed products of $W^0$ and $B$ particles.), which is the origin of electroweak mixing, and the mixing angle is Weinberg angle $\theta_W$. Below the electroweak scale, there is no oscillation between photon and $Z^0$ boson due to their different masses, and there is only the decay of $Z^0$ boson, which can transform into a pair of photons, a pair of charged particles or other particles.

5 Conclusion

Neutrino oscillations with vanishing mass lead to a simple version of standard model, and we have already had an appropriate equation to describe the behavior of neutrinos, i.e., Weyl equation. Barrier tunneling provides a simple explanation for neutrino oscillations, and the difficulty with the plane wave and wave packet descriptions will not exist. The barrier tunneling hypothesis can be easily extended to quark mixing, forbidden transition of charged leptons, neutral kaon oscillation and electroweak mixing. Such a hypothesis can explain a number of phenomena, indicating that it is reasonable.

As long as the neutrino mass is zero and there are three generations of neutrinos, neutrino is bound to oscillate, which is determined by the property of the microscopic particle itself. If a microscopic particle has multiple states and the energies of these states are the same, then the particle will oscillate between different states. It will never be satisfied with only existing in one state, it will try all possible states.

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