Exact Sample Size Methods for Estimating Parameters of Discrete Distributions *

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Abstract

In this paper, we develop an approach for the exact determination of the minimum sample size for estimating the parameter of an integer-valued random variable, which is parameterized by its expectation. Under some continuity and unimodal property assumptions, the exact computation is accomplished by reducing infinite many evaluations of coverage probability to finite many evaluations. Such a reduction is based on our discovery that the minimum of coverage probability with respect to the parameter bounded in an interval is attained at a discrete set of finite many values.

1 Introduction

Let $X$ be an integer-valued discrete random variable defined in a probability space $(\Omega, \mathcal{F}, \Pr)$ such that the probability mass function is parameterized by its expectation

$$\mathbb{E}[X] = \theta \in \Theta \subseteq (0, \infty),$$

where $\Theta$ is the parameter space. It is a frequent problem to construct an estimator $\hat{\theta}_n$ for $\theta$ based on $n$ identical and independent samples $X_1, \cdots, X_n$ of $X$. An unbiased estimate of $\theta$ is conventionally taken as

$$\hat{\theta}_n = \frac{Y_n}{n},$$

where

$$Y_n = \sum_{i=1}^{n} X_i.$$  

A crucial question in the estimation is as follows:

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Given the knowledge that $\theta$ belongs to interval $[a, b]$, what is the minimum sample size $n$ that guarantees the difference between $\hat{\theta}_n$ and $\theta$ be bounded within some prescribed margin of error with a confidence level higher than a prescribed value?

In this paper, we shall address this question based the assumption that for any interval $\mathcal{I}$, the probability $\Pr\{Y_n \in \mathcal{I} \mid \theta\}$ is a continuous unimodal function of $\theta \in \Theta$, where the notation $\Pr\{E \mid \theta\}$ denotes the probability of event $E$ which is associated with the parameter $\theta \in \Theta$. This notation will be used throughout the paper. Clearly, the assumption is satisfied if $X$ is a Bernoulli or Poisson random variable.

In this paper, the notion of a unimodal function is as follows: A function $f(x)$ is said to be a unimodal function of $x \in [u, v]$ if there exists a number $x^* \in [u, v]$ such that for any $x_1, x_2, x_3, x_4$ with $u \leq x_1 \leq x_2 \leq x^* \leq x_3 \leq x_4 \leq v$,

$$f(u) \leq f(x_1) \leq f(x_2) \leq f(x^*) , \quad f(x^*) \geq f(x_3) \geq f(x_4) \geq f(v).$$

The paper is organized as follows. In Section 2, the techniques for computing the minimum sample size is developed with the margin of error taken as a bound of absolute error. In Section 3, we derive corresponding sample size method by using relative error bound as the margin of error. In Section 4, we develop techniques for computing minimum sample size with a mixed error criterion. In Section 5, we consider the sample size problem in the context of range-preserving estimator. Section 6 is the conclusion. The proofs are given in Appendices. This work is an extension of the recent works [1, 2] and [3].

Throughout this paper, we shall use the following notations. The set of integers is denoted by $\mathbb{Z}$. The ceiling function and floor function are denoted respectively by $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ (i.e., $\lceil x \rceil$ represents the smallest integer no less than $x$; $\lfloor x \rfloor$ represents the largest integer no greater than $x$). For integers $k \leq l$, the probability $\Pr\{k \leq Y_n \leq l \mid \theta\}$ is denoted by $S(n, k, l, \theta)$. The left limit as $\eta$ tends to 0 is denoted as $\operatorname{lim}_{\eta \downarrow 0}$. The other notations will be made clear as we proceed.

## 2 Control of Absolute Error

Let $\varepsilon \in (0, 1)$ be the margin of absolute error and $\delta \in (0, 1)$ be the confidence parameter. In many applications, it is desirable to find the minimum sample size $n$ such that

$$\Pr\{\lvert \hat{\theta}_n - \theta \rvert < \varepsilon \mid \theta\} > 1 - \delta$$

for any $\theta \in [a, b]$. Here $\Pr\{\lvert \hat{\theta}_n - \theta \rvert < \varepsilon \mid \theta\}$ is referred to as the coverage probability. The interval $[a, b]$ is introduced to take into account the knowledge of $\theta$. The exact determination of minimum sample size is readily tractable with modern computational power by taking advantage of the behavior of the coverage probability characterized by Theorem 1 as follows.
Theorem 1 Let $\varepsilon > 0$ and $0 \leq a < b$ such that $[a, b] \subseteq \Theta$. Assume that for any interval $\mathcal{I}$, the probability $\Pr \{Y_n \in \mathcal{I} \mid \theta\}$ is a continuous unimodal function of $\theta \in \Theta$. Then, the minimum of $\Pr \{|\hat{\theta}_n - \theta| < \varepsilon \mid \theta\}$ with respect to $\theta \in [a, b]$ is achieved at the finite set $\{a, b\} \cup \{\ell_n + \varepsilon \in (a, b) : \ell \in \mathbb{Z}\} \cup \{\ell_n - \varepsilon \in (a, b) : \ell \in \mathbb{Z}\}$, which has less than $2n(b - a) + 4$ elements.

See Appendix A for a proof. The application of Theorem 1 in the computation of minimum sample size is obvious. For a fixed sample size $n$, since the minimum of coverage probability with $\theta \in [a, b]$ is attained at a finite set, it can determined by a computer whether the sample size $n$ is large enough to ensure $\Pr \{|\hat{\theta}_n - \theta| < \varepsilon \mid \theta\} > 1 - \delta$ for any $\theta \in [a, b]$. Starting from $n = 2$, one can find the minimum sample size by gradually incrementing $n$ and checking whether $n$ is large enough.

3 Control of Relative Error

Let $\varepsilon \in (0, 1)$ be the margin of relative error and $\delta \in (0, 1)$ be the confidence parameter. It is interesting to determine the minimum sample size $n$ so that

$$\Pr \left\{ \left| \frac{\hat{\theta}_n - \theta}{\theta} \right| < \varepsilon \mid \theta \right\} > 1 - \delta$$

for any $\theta \in [a, b]$. As has been pointed out in Section 2, an essential machinery is to reduce infinite many evaluations of the coverage probability $\Pr \{|\hat{\theta}_n - \theta| < \varepsilon \mid \theta\}$ to finite many evaluations. Such reduction can be accomplished by making use of Theorem 1 as follows.

Theorem 2 Let $0 < \varepsilon < 1$ and $0 < a < b$ such that $[a, b] \subseteq \Theta$. Assume that for any interval $\mathcal{I}$, the probability $\Pr \{Y_n \in \mathcal{I} \mid \theta\}$ is a continuous unimodal function of $\theta \in \Theta$. Then, the minimum of $\Pr \left\{ \left| \frac{\hat{\theta}_n - \theta}{\theta} \right| < \varepsilon \mid \theta \right\}$ with respect to $\theta \in [a, b]$ is achieved at the finite set $\{a, b\} \cup \{\ell_n \in (a, b) : \ell \in \mathbb{Z}\}$, which has less than $2n(b - a) + 4$ elements.

See Appendix B for a proof.

4 Control of Absolute Error or Relative Error

Let $\varepsilon_a > 0$ and $\varepsilon_r \in (0, 1)$ be respectively the margins of absolute error and relative error. Let $\delta \in (0, 1)$ be the confidence parameter. In many situations, it is desirable to find the smallest sample size $n$ such that

$$\Pr \left\{ \left| \hat{\theta}_n - \theta \right| < \varepsilon_a \text{ or } \left| \frac{\hat{\theta}_n - \theta}{\theta} \right| < \varepsilon_r \mid \theta \right\} > 1 - \delta \tag{1}$$

for any $\theta \in [a, b]$. To make it possible to compute exactly the minimum sample size associated with (1), we have Theorem 3 as follows.
Theorem 3 Let \( \varepsilon_a > 0 \), \( 0 < \varepsilon_r < 1 \) and \( 0 \leq a < \frac{\varepsilon_a}{\varepsilon_r} < b \) such that \([a,b] \subseteq \Theta\). Assume that for any interval \( \mathcal{I} \), the probability \( \Pr \{ Y_n \in \mathcal{I} \mid \theta \} \) is a continuous unimodal function of \( \theta \in \Theta \). Then, the minimum of \( \Pr \{ |\hat{\theta}_n - \theta| < \varepsilon_a \text{ or } |\frac{\hat{\theta}_n - \theta}{\theta}| < \varepsilon_r \mid \theta \} \) with respect to \( \theta \in [a,b] \) is achieved at the finite set \( \{a,b, \frac{a}{\varepsilon_r} \} \cup \{ \frac{k}{n} + \varepsilon_a \in (a, \frac{a}{\varepsilon_r}) : k \in \mathbb{Z} \} \cup \{ \frac{k}{n} - \varepsilon_a \in (\frac{a}{\varepsilon_r}, b) : k \in \mathbb{Z} \} \cup \{ \frac{k}{n(1+\varepsilon_r)} + \varepsilon_r \in (\frac{a}{\varepsilon_r}, b) : k \in \mathbb{Z} \} \), which has less than \( 2n(b-a) + 7 \) elements.

Theorem 3 can be shown by applying Theorem 1 and Theorem 2 with the observation that

\[
\Pr \left\{ |\hat{\theta}_n - \theta| < \varepsilon_a \text{ or } |\frac{\hat{\theta}_n - \theta}{\theta}| < \varepsilon_r \right\} = \left\{ \begin{array}{ll}
\Pr \{ |\hat{\theta}_n - \theta| < \varepsilon_a \} & \text{for } \theta \in \left[ a, \frac{a}{\varepsilon_r} \right], \\
\Pr \{ |\frac{\hat{\theta}_n - \theta}{\theta}| < \varepsilon_r \} & \text{for } \theta \in \left( \frac{a}{\varepsilon_r}, b \right).
\end{array} \right.
\]

5 Error Control for Range-Preserving Estimator

In many situations, it may be appropriate to use the range-preserving estimator \( \tilde{\theta}_n \) for \( \theta \), which is defined as

\[
\tilde{\theta}_n = \left\{ \begin{array}{ll}
\hat{\theta}_n & \text{for } \hat{\theta}_n \in [a,b], \\
a & \text{for } \hat{\theta}_n < a, \\
b & \text{for } \hat{\theta}_n > b.
\end{array} \right.
\]

See [3] and the references therein. Recently, Gamrot [3] has established an exact sample size method for estimating a binomial parameter using the range preserving estimator. Inspired by the work of Gamrot [3], we have developed an exact approach for the general sample size problem of estimating the parameter of a discrete distribution under certain unimodal property and continuity assumptions. In the sequel, let \((u,v)\) denote an open interval if \(u < v\). In the case that \(u \geq v\), \((u,v)\) is an empty set.

To determine the minimum sample size for controlling absolute error in the context of using the range-preserving estimator, we have the following results.

Theorem 4 Assume that \( \varepsilon > 0 \) and \( 0 < a < b < \infty \) such that \([a,b] \subseteq \Theta\). Assume that for any interval \( \mathcal{I} \), the probability \( \Pr \{ Y_n \in \mathcal{I} \mid \theta \} \) is a continuous unimodal function of \( \theta \in \Theta \). Then, the minimum of \( \Pr \{ |\tilde{\theta}_n - \theta| < \varepsilon \mid \theta \} \) with respect to \( \theta \in [a,b] \) is attained at the finite set \( A \cap [a,b] \), where

\[
A = \{a, b, a+\varepsilon, b-\varepsilon\} \cup \left\{ \frac{k}{n} - \varepsilon \in (a,b-\varepsilon) : k \in \mathbb{Z} \right\} \cup \left\{ \frac{k}{n} + \varepsilon \in (a+\varepsilon, b) : k \in \mathbb{Z} \right\},
\]

which has less than \( 2n(b-a-\varepsilon) + 6 \) elements.

See Appendix C for a proof of Theorem 4.

To determine the minimum sample size for controlling relative error in the context of using the range-preserving estimator, we have the following results.
Let of using the range-preserving estimator, we have the following results. I

\[ A = \left\{ a, b, \frac{a}{1-\varepsilon}, \frac{b}{1+\varepsilon} \right\} \cup \left\{ \frac{k}{n(1+\varepsilon)} \in \left( a, \frac{b}{1+\varepsilon} \right) : k \in \mathbb{Z} \right\} \cup \left\{ \frac{k}{n(1-\varepsilon)} \in \left( a, \frac{b}{1-\varepsilon} \right) : k \in \mathbb{Z} \right\}, \quad (3) \]

which has less than \( 2n(b-a) - n\varepsilon(a+b) + 6 \) elements.

See Appendix I for a proof. It should be noted that Theorem 5 is a generalization of Theorems 4 and 5 of Gamrot [3].

To determine the minimum sample size for controlling absolute and relative error in the context of using the range-preserving estimator, we have the following results.

**Theorem 6** Let \( \varepsilon_a > 0 \), \( 0 < \varepsilon_r < 1 \) and \( 0 \leq a < \frac{\varepsilon_a}{\varepsilon_r} < b \) such that \( [a,b] \subseteq \Theta \). Assume that for any interval \( \mathcal{I} \), the probability \( \Pr \{ Y_n \in \mathcal{I} \mid \theta \} \) is a continuous unimodal function of \( \theta \in \Theta \). Then, the minimum of \( \Pr \{ |\theta_n - \theta| < \varepsilon \mid \theta \} \) with respect to \( \theta \in [a,b] \) is achieved at the finite set \( A \cap [a,b] \), where

\[
A = \left\{ a, a + \varepsilon_a, \frac{\varepsilon_a}{\varepsilon_r} - \varepsilon_a, \frac{\varepsilon_a}{\varepsilon_r} \right\} \cup \left\{ \frac{k}{n} - \varepsilon_a \in \left( a, \frac{\varepsilon_a}{\varepsilon_r} - \varepsilon_a \right) : k \in \mathbb{Z} \right\} \cup \left\{ \frac{k}{n} + \varepsilon_a \in \left( a + \varepsilon_a, \frac{\varepsilon_a}{\varepsilon_r} \right) : k \in \mathbb{Z} \right\},
\]

\[
B = \left\{ b, b - \varepsilon_r, \frac{\varepsilon_a}{1-\varepsilon_r} \right\} \cup \left\{ \frac{k}{n(1+\varepsilon_r)} \in \left( \frac{\varepsilon_a}{\varepsilon_r}, \frac{b}{1+\varepsilon_r} \right) : k \in \mathbb{Z} \right\} \cup \left\{ \frac{k}{n(1-\varepsilon_r)} \in \left( \frac{\varepsilon_a}{1-\varepsilon_r}, \frac{b}{1-\varepsilon_r} \right) : k \in \mathbb{Z} \right\}.
\]

Moreover, \( A \cup B \) has less than \( 2n(b-a-\varepsilon_a) - n(\varepsilon_a + b\varepsilon_r) + 11 \) elements.

**Theorem 6** can be shown by applying Theorem 4 and Theorem 5 with the observation that

\[
\Pr \left\{ \left| \frac{\theta_n - \theta}{\theta} \right| < \varepsilon_r \right\} = \frac{\Pr \{ |\theta_n - \theta| < \varepsilon \} \mid \theta \in \left( \frac{\varepsilon_a}{\varepsilon_r}, b \right)}{\Pr \{ |\theta_n - \theta| < \varepsilon_r \} \mid \theta \in \left( \frac{\varepsilon_a}{\varepsilon_r}, b \right)}.
\]

6 Conclusion

We have developed an exact method for the computation of minimum sample size for estimating the parameter of an integer-valued discrete random variable, which only requires finite many evaluations of the coverage probability. Our sample size method permits rigorous control of statistical error for estimating parameters of common distributions such as binomial and Poisson distributions.

A Proof of Theorem 1

Define

\[
C(\theta) = \Pr \left\{ \left| \frac{Y_n}{n} - \theta \right| < \varepsilon \mid \theta \right\} = \Pr \{ g(\theta) \leq Y_n \leq h(\theta) \mid \theta \}
\]
where
\[ g(\theta) = \lfloor n(\theta - \varepsilon) \rfloor + 1, \quad h(\theta) = \lceil n(\theta + \varepsilon) \rceil - 1. \] (4)

It should be noted that \( C(\theta) \), \( g(\theta) \) and \( h(\theta) \) are actually multivariate functions of \( \theta, \varepsilon \) and \( n \). For simplicity of notations, we drop the arguments \( n \) and \( \varepsilon \) throughout the proof of Theorem 1.

We need some preliminary results.

**Lemma 1** Let \( \theta = \frac{\ell}{n} - \varepsilon \) where \( \ell \in \mathbb{Z} \). Then, \( h(\theta) = h(\theta + 1) = \ell \) for any \( \theta \in (\theta, \theta + 1) \).

**Proof.** For \( \theta \in (\theta, \theta + 1) \), we have \( 0 < n(\theta - \theta - \varepsilon) < 1 \) and
\[
 h(\theta) &= \left\lfloor n(\theta + \varepsilon) \right\rfloor - 1 \\
 &= \left\lfloor n(\theta + \varepsilon + \theta - \theta - \varepsilon) \right\rfloor - 1 \\
 &= \left\lfloor n\left(\frac{\ell}{n} - \varepsilon + \varepsilon + \theta - \theta - \varepsilon\right)\right\rfloor - 1 \\
 &= \ell - 1 + \left\lfloor n(\theta - \theta)\right\rfloor \\
 &= \ell \\
 &= \left\lfloor n\left(\frac{\ell + 1}{n} - \varepsilon + \varepsilon\right)\right\rfloor - 1 = h(\theta + 1).
\]

\[ \Box \]

**Lemma 2** Let \( \theta = \frac{\ell}{n} + \varepsilon \) where \( \ell \in \mathbb{Z} \). Then, \( g(\theta) = g(\theta + 1) = \ell + 1 \) for any \( \theta \in (\theta, \theta + 1) \).

**Proof.** For \( \theta \in (\theta, \theta + 1) \), we have \( -1 < n(\theta - \theta - 1) < 0 \) and
\[
 g(\theta) &= \left\lfloor n(\theta - \varepsilon) \right\rfloor + 1 \\
 &= \left\lfloor n(\theta + 1 - \varepsilon + \theta - \theta - 1) \right\rfloor + 1 \\
 &= \left\lfloor n\left(\frac{\ell + 1}{n} + \varepsilon - \varepsilon\right)\right\rfloor + \left\lfloor n(\theta - \theta + 1)\right\rfloor + 1 \\
 &= \left\lfloor n\left(\frac{\ell + 1}{n} + \varepsilon - \varepsilon\right)\right\rfloor - 1 + 1 \\
 &= \ell + 1 \\
 &= \left\lfloor n\left(\frac{\ell}{n} + \varepsilon - \varepsilon\right)\right\rfloor + 1 = g(\theta).
\]

\[ \Box \]

**Lemma 3** Let \( \alpha < \beta \) be two consecutive elements of the ascending arrangement of all distinct elements of \( \{a, b\} \cup \left\{\frac{\ell}{n} + \varepsilon \in (a, b) : \ell \in \mathbb{Z}\} \cup \left\{\frac{\ell}{n} - \varepsilon \in (a, b) : \ell \in \mathbb{Z}\right\}. Then, both \( g(\theta) \) and \( h(\theta) \) are constants for any \( \theta \in (\alpha, \beta) \).
Proof. Since $\alpha$ and $\beta$ are two consecutive elements of the ascending arrangement of all distinct elements of the set, it must be true that there is no integer $\ell$ such that $\alpha < \frac{\ell}{n} + \varepsilon < \beta$ or $\alpha < \frac{\ell}{n} - \varepsilon < \beta$. It follows that there exist two integers $\ell$ and $\ell'$ such that $(\alpha, \beta) \subseteq \left(\frac{\ell}{n} + \varepsilon, \frac{\ell}{n} + \varepsilon + \frac{1}{n}\right)$ and $(\alpha, \beta) \subseteq \left(\frac{\ell}{n} - \varepsilon, \frac{\ell}{n} + \varepsilon - \varepsilon\right)$. Applying Lemma 1 and Lemma 2, we have $g(\theta) = g\left(\frac{\ell}{n} + \varepsilon\right)$ and $h(\theta) = \frac{\ell}{n} - \varepsilon$ for any $\theta \in (\alpha, \beta)$.

\[ \square \]

Lemma 4 For any $\theta \in (0, 1)$, $\lim_{\eta \downarrow 0} C(\theta + \eta) \geq C(\theta)$ and $\lim_{\eta \downarrow 0} C(\theta - \eta) \geq C(\theta)$.

Proof. Observing that $h(\theta + \eta) \geq h(\theta)$ for any $\eta > 0$ and that

\[
g(\theta + \eta) = \left\lfloor n(\theta + \eta - \varepsilon) \right\rfloor + 1 = \left\lfloor n(\theta - \varepsilon) \right\rfloor + 1 + \left\lfloor n(\theta - \varepsilon) - \left\lfloor n(\theta - \varepsilon) \right\rfloor \right\rfloor + n\eta = \left\lfloor n(\theta - \varepsilon) \right\rfloor + 1 = g(\theta)
\]

for $0 < \eta < \frac{1 + \left\lfloor n(\theta - \varepsilon) \right\rfloor - \left\lfloor n(\theta - \varepsilon) + 1 \right\rfloor}{n}$, we have

\[
S(n, g(\theta + \eta), h(\theta + \eta), \theta + \eta) \geq S(n, g(\theta), h(\theta), \theta + \eta) \tag{5}
\]

for $0 < \eta < \frac{1 + \left\lfloor n(\theta - \varepsilon) \right\rfloor - \left\lfloor n(\theta - \varepsilon) + 1 \right\rfloor}{n}$. Since

\[
h(\theta + \eta) = \left\lfloor n(\theta + \varepsilon) \right\rfloor - 1 = \left\lfloor n(\theta + \varepsilon) \right\rfloor - 1 + \left\lfloor n(\theta + \varepsilon) - \left\lfloor n(\theta + \varepsilon) \right\rfloor \right\rfloor + n\eta,
\]

we have

\[
h(\theta + \eta) = \begin{cases} 
\left\lfloor n(\theta + \varepsilon) \right\rfloor & \text{for } n(\theta + \varepsilon) = \left\lfloor n(\theta + \varepsilon) \right\rfloor \text{ and } 0 < \eta < \frac{1}{n}, \\
\left\lfloor n(\theta + \varepsilon) \right\rfloor - 1 & \text{for } n(\theta + \varepsilon) \neq \left\lfloor n(\theta + \varepsilon) \right\rfloor \text{ and } 0 < \eta < \frac{\left\lfloor n(\theta + \varepsilon) \right\rfloor - n(\theta + \varepsilon)}{n}.
\end{cases}
\]

It follows that both $g(\theta + \eta)$ and $h(\theta + \eta)$ are independent of $\eta$ if $\eta > 0$ is small enough. Since $S(n, g, h, \theta + \eta)$ is continuous with respect to $\eta$ for fixed $g$ and $h$, we have that $\lim_{\eta \downarrow 0} S(n, g(\theta + \eta), h(\theta + \eta), \theta + \eta)$ exists. As a result,

\[
\lim_{\eta \downarrow 0} C(\theta + \eta) = \lim_{\eta \downarrow 0} S(n, g(\theta + \eta), h(\theta + \eta), \theta + \eta)
\]

\[
\geq \lim_{\eta \downarrow 0} S(n, g(\theta), h(\theta), \theta + \eta) = S(n, g(\theta), h(\theta), \theta) = C(\theta)
\]

where the inequality follows from (5).

Observing that $g(\theta - \eta) \leq g(\theta)$ for any $\eta > 0$ and that

\[
h(\theta - \eta) = \left\lfloor n(\theta - \eta + \varepsilon) \right\rfloor - 1 = \left\lfloor n(\theta + \varepsilon) \right\rfloor - 1 + \left\lfloor n(\theta + \varepsilon) - \left\lfloor n(\theta + \varepsilon) \right\rfloor \right\rfloor - n\eta = \left\lfloor n(\theta + \varepsilon) \right\rfloor - 1 = h(\theta)
\]
for $0 < \eta < \frac{1 + n(\theta + \varepsilon) - [n(\theta + \varepsilon)]}{n}$, we have
\[ S(n, g(\theta - \eta), h(\theta - \eta), \theta - \eta) \geq S(n, g(\theta), h(\theta), \theta - \eta) \tag{6} \]
for $0 < \eta < \min \left\{ \theta, \frac{1 + n(\theta + \varepsilon) - [n(\theta + \varepsilon)]}{n} \right\}$. Since
\[ g(\theta - \eta) = [n(\theta - \eta - \varepsilon)] + 1 \]
\[ = [n(\theta - \varepsilon)] + 1 + [n(\theta - \varepsilon) - [n(\theta - \varepsilon)]] - n\eta, \]
we have
\[ g(\theta - \eta) = \begin{cases} 
[n(\theta - \varepsilon)] & \text{for } n(\theta - \varepsilon) = [n(\theta - \varepsilon)] \text{ and } 0 < \eta < \frac{1}{n}, \\
[n(\theta - \varepsilon)] + 1 & \text{for } n(\theta - \varepsilon) \neq [n(\theta - \varepsilon)] \text{ and } 0 < \eta < \frac{n(\theta - \varepsilon) - [n(\theta - \varepsilon)]}{n}. 
\end{cases} \]

It follows that both $g(\theta - \eta)$ and $h(\theta - \eta)$ are independent of $\eta$ if $\eta > 0$ is small enough. Since $S(n, g, h, \theta - \eta)$ is continuous with respect to $\eta$ for fixed $g$ and $h$, we have that $\lim_{\eta \downarrow 0} S(n, g(\theta - \eta), h(\theta - \eta), \theta - \eta)$ exists. Hence,
\[ \lim_{\eta \downarrow 0} C(\theta - \eta) = \lim_{\eta \downarrow 0} S(n, g(\theta - \eta), h(\theta - \eta), \theta - \eta) \]
\[ \geq \lim_{\eta \downarrow 0} S(n, g(\theta), h(\theta), \theta - \eta) = S(n, g(\theta), h(\theta), \theta) = C(\theta), \]
where the inequality follows from (6).

\[ \square \]

Lemma 5 Let $\alpha < \beta$ be two consecutive elements of the ascending arrangement of all distinct elements of $\{a, b\} \cup \{\frac{\ell}{n} + \varepsilon \in (a, b) : \ell \in \mathbb{Z}\} \cup \{\frac{\ell}{n} - \varepsilon \in (a, b) : \ell \in \mathbb{Z}\}$. Then, $C(\theta) \geq \min\{C(\alpha), C(\beta)\}$ for any $\theta \in (\alpha, \beta)$.

\textbf{Proof.} By Lemma 3 both $g(\theta)$ and $h(\theta)$ are constants for any $\theta \in (\alpha, \beta)$. Hence, we can drop the argument and write $g(\theta) = g$, $h(\theta) = h$ and $C(\theta) = S(n, g, h, \theta)$.

For $\theta \in (\alpha, \beta)$, define interval $[\alpha + \eta, \beta - \eta]$ with $0 < \eta < \min \left( \theta - \alpha, \beta - \theta, \frac{\beta - \alpha}{2} \right)$. Then,
\[ C(\theta) \geq \min_{\mu \in [\alpha + \eta, \beta - \eta]} C(\mu). \]

From the assumption that for any interval $\mathcal{I}$, the probability $\Pr \{ Y_\alpha \in \mathcal{I} \mid \theta \}$ is a continuous unimodal function of $\theta \in \Theta$, we can see that, for $0 < \eta < \min \left( \theta - \alpha, \beta - \theta, \frac{\beta - \alpha}{2} \right)$, one of the following three cases must be true: (1) $C(\mu)$ decreases monotonically for $\mu \in [\alpha + \eta, \beta - \eta]$; (2) $C(\mu)$ increases monotonically for $\mu \in [\alpha + \eta, \beta - \eta]$; (3) there exists a number $\theta \in (\alpha + \eta, \beta - \eta)$ such that $C(\mu)$ increases monotonically for $\mu \in [\alpha + \eta, \theta]$ and decreases monotonically for $\mu \in (\theta, \beta - \eta]$.

It follows that
\[ C(\theta) \geq \min_{\mu \in [\alpha + \eta, \beta - \eta]} C(\mu) = \min\{C(\alpha + \eta), C(\beta - \eta)\} \]

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for $0 < \eta < \min\left(\theta - \alpha, \beta - \theta, \frac{2\alpha}{\alpha + \beta}\right)$. By Lemma 4, both $\lim_{\eta \downarrow 0} C(\alpha + \eta)$ and $\lim_{\eta \downarrow 0} C(\beta - \eta)$ exist and

$$C(\theta) \geq \lim_{\eta \downarrow 0} \min\{C(\alpha + \eta), C(\beta - \eta)\} = \min\left\{\lim_{\eta \downarrow 0} C(\alpha + \eta), \lim_{\eta \downarrow 0} C(\beta - \eta)\right\} \geq \min\{C(\alpha), C(\beta)\}$$

for any $\theta \in (\alpha, \beta)$.

Finally, to show Theorem 1, note that the statement about the coverage probability follows immediately from Lemma 5. The number of elements of the finite set can be calculated by using the property of the ceiling and floor functions.

**B Proof of Theorem 2**

Define

$$C(\theta) = \Pr\left\{\left|\frac{Y_n}{n} - \theta\right| < \varepsilon \theta \mid \theta\right\} = \Pr\{g(\theta) \leq Y_n \leq h(\theta) \mid \theta\}$$

where

$$g(\theta) = \lceil n\theta(1 - \varepsilon) \rceil + 1, \quad h(\theta) = \lceil n\theta(1 + \varepsilon) \rceil - 1. \quad (7)$$

It should be noted that $C(\theta)$, $g(\theta)$ and $h(\theta)$ are actually multivariate functions of $\theta$, $\varepsilon$ and $n$. For simplicity of notations, we drop the arguments $n$ and $\varepsilon$ throughout the proof of Theorem 2.

We need some preliminary results.

**Lemma 6** Let $\theta_{\ell} = \frac{\ell}{n(1 + \varepsilon)}$ where $\ell \in \mathbb{Z}$. Then, $h(\theta) = h(\theta_{\ell+1}) = \ell$ for any $\theta \in (\theta_{\ell}, \theta_{\ell+1})$.

**Proof.** For $\theta \in (\theta_{\ell}, \theta_{\ell+1})$, we have $0 < n(1 + \varepsilon)(\theta - \theta_{\ell}) < 1$ and

$$h(\theta) = \lceil n\theta(1 + \varepsilon) \rceil - 1 = \lceil n\theta_{\ell}(1 + \varepsilon) + (1 + \varepsilon)(\theta - \theta_{\ell}) \rceil - 1 = \lceil n\left(\frac{\ell}{n} + (1 + \varepsilon)(\theta - \theta_{\ell})\right) \rceil - 1 = \ell - 1 + \lceil n(1 + \varepsilon)(\theta - \theta_{\ell}) \rceil = \ell = \lceil n\left(\frac{\ell + 1}{n(1 + \varepsilon)} \times (1 + \varepsilon)\right) \rceil - 1 = h(\theta_{\ell+1})$$

**Lemma 7** Let $\theta_{\ell} = \frac{\ell}{n(1 - \varepsilon)}$ where $\ell \in \mathbb{Z}$. Then, $g(\theta) = g(\theta_{\ell}) = \ell + 1$ for any $\theta \in (\theta_{\ell}, \theta_{\ell+1})$.  


**Proof.** For \( \theta \in (\theta_{\ell}, \theta_{\ell+1}) \), we have \(-1 < n(1 - \varepsilon)(\theta - \theta_{\ell+1}) < 0\) and

\[
g(\theta) = \lfloor n\theta(1 - \varepsilon) \rfloor + 1
= \left\lfloor n[\theta_{\ell+1}(1 - \varepsilon) + (1 - \varepsilon)(\theta - \theta_{\ell+1})] \right\rfloor + 1
= \left\lfloor n \times \frac{\ell + 1}{n(1 - \varepsilon)} \times (1 - \varepsilon) \right\rfloor + \left\lfloor n(1 - \varepsilon)(\theta - \theta_{\ell+1}) \right\rfloor + 1
= \left\lfloor n \times \frac{\ell + 1}{n(1 - \varepsilon)} \times (1 - \varepsilon) \right\rfloor + 1
= \ell + 1
= \left\lfloor n \times \frac{\ell}{n(1 - \varepsilon)} \times (1 - \varepsilon) \right\rfloor + 1 = g(\theta_{\ell}).
\]

\[\square\]

**Lemma 8** Let \( \alpha < \beta \) be two consecutive elements of the ascending arrangement of all distinct elements of \( \{a, b\} \cup \{\frac{\ell}{n(1-\varepsilon)} \in (a, b) : \ell \in \mathbb{Z}\} \cup \{\frac{\ell+1}{n(1-\varepsilon)} \in (a, b) : \ell \in \mathbb{Z}\} \). Then, both \( g(\theta) \) and \( h(\theta) \) are constants for any \( \theta \in (\alpha, \beta) \).

**Proof.** Since \( \alpha \) and \( \beta \) are two consecutive elements of the ascending arrangement of all distinct elements of the set, it must be true that there is no integer \( \ell \) such that \( \alpha < \frac{\ell}{n(1-\varepsilon)} < \beta \) or \( \alpha < \frac{\ell+1}{n(1-\varepsilon)} < \beta \). It follows that there exist two integers \( \ell \) and \( \ell' \) such that \((\alpha, \beta) \subseteq \left(\frac{\ell}{n(1-\varepsilon)}, \frac{\ell+1}{n(1-\varepsilon)}\right)\) and \((\alpha, \beta) \subseteq \left(\frac{\ell'}{n(1-\varepsilon)}, \frac{\ell'+1}{n(1-\varepsilon)}\right)\). Applying Lemma 5 and Lemma 7, we have \( g(\theta) = g\left(\frac{\ell}{n(1-\varepsilon)}\right) \) and \( h(\theta) = h\left(\frac{\ell+1}{n(1-\varepsilon)}\right) \) for any \( \theta \in (\alpha, \beta) \).

\[\square\]

**Lemma 9** For any \( \theta \in (0, 1) \), \( \lim_{\eta \downarrow 0} C(\theta + \eta) \geq C(\theta) \) and \( \lim_{\eta \downarrow 0} C(\theta - \eta) \geq C(\theta) \).

**Proof.** Observing that \( h(\theta + \eta) \geq h(\theta) \) for any \( \eta > 0 \) and that

\[
g(\theta + \eta) = \lfloor n(\theta + \eta)(1 - \varepsilon) \rfloor + 1
= \lfloor n\theta(1 - \varepsilon) \rfloor + 1 + \lfloor n\eta(1 - \varepsilon) \rfloor + n\eta(1 - \varepsilon)
= \lfloor n\theta(1 - \varepsilon) \rfloor + 1 = g(\theta)
\]

for \( 0 < \eta < \frac{1 + \lfloor n\theta(1 - \varepsilon) \rfloor + n\eta(1 - \varepsilon)}{n(1 - \varepsilon)} \), we have

\[
S(n, g(\theta + \eta), h(\theta + \eta), \theta + \eta) \geq S(n, g(\theta), h(\theta), \theta + \eta)
\]

(8)

for \( 0 < \eta < \frac{1 + \lfloor n\theta(1 - \varepsilon) \rfloor + n\eta(1 - \varepsilon)}{n(1 - \varepsilon)} \). Since

\[
h(\theta + \eta) = \lfloor n(\theta + \eta)(1 + \varepsilon) \rfloor - 1
= \lfloor n\theta(1 + \varepsilon) \rfloor - 1 + \lfloor n\eta(1 + \varepsilon) \rfloor + n\eta(1 + \varepsilon),
\]

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we have
\[
h(\theta + \eta) = \begin{cases} 
[n\theta(1 + \varepsilon)] & \text{for } n\theta(1 + \varepsilon) = [n\theta(1 + \varepsilon)] \text{ and } 0 < \eta < \frac{1}{n(1+\varepsilon)}, \\
[n\theta(1 + \varepsilon)] - 1 & \text{for } n\theta(1 + \varepsilon) \neq [n\theta(1 + \varepsilon)] \text{ and } 0 < \eta < \frac{[n\theta(1 + \varepsilon)] - n\theta(1 + \varepsilon)}{n(1+\varepsilon)}. 
\end{cases}
\]

It follows that both \(g(\theta + \eta)\) and \(h(\theta + \eta)\) are independent of \(\eta\) if \(\eta > 0\) is small enough. Since \(S(n, g, h, \theta + \eta)\) is continuous with respect to \(\eta\) for fixed \(g\) and \(h\), we have that \(\lim_{\eta \downarrow 0} S(n, g(\theta + \eta), h(\theta + \eta), \theta + \eta)\) exists. As a result,
\[
\lim_{\eta \downarrow 0} C(\theta + \eta) = \lim_{\eta \downarrow 0} S(n, g(\theta + \eta), h(\theta + \eta), \theta + \eta) \geq \lim_{\eta \downarrow 0} S(n, g(\theta), h(\theta), \theta + \eta) = S(n, g(\theta), h(\theta), \theta) = C(\theta),
\]
where the inequality follows from [5].

Observing that \(g(\theta - \eta) \leq g(\theta)\) for any \(\eta > 0\) and that
\[
h(\theta - \eta) = [n(\theta - \eta)(1 + \varepsilon)] - 1 = [n\theta (1 + \varepsilon)] - 1 + \left[ n\theta(1 + \varepsilon) - [n\theta(1 + \varepsilon)] \right] - n\eta(1 + \varepsilon) = \left[ n\theta (1 + \varepsilon) \right] - 1 = h(\theta)
\]
for \(0 < \eta < \frac{1 + n\theta(1 + \varepsilon) - [n\theta(1 + \varepsilon)]}{n(1+\varepsilon)}\), we have
\[
S(n, g(\theta - \eta), h(\theta - \eta), \theta - \eta) \geq S(n, g(\theta), h(\theta), \theta - \eta)
\]
for \(0 < \eta < \min \left\{ \theta, \frac{1 + n\theta(1 + \varepsilon) - [n\theta(1 + \varepsilon)]}{n(1+\varepsilon)} \right\}\). Since
\[
g(\theta - \eta) = [n(\theta - \eta)(1 - \varepsilon)] + 1 = [n\theta (1 - \varepsilon)] + 1 + \left[ n\theta(1 - \varepsilon) - [n\theta(1 - \varepsilon)] \right] - n\eta(1 - \varepsilon)
\]
we have
\[
g(\theta - \eta) = \begin{cases} 
[n\theta(1 - \varepsilon)] & \text{for } n\theta(1 - \varepsilon) = [n\theta(1 - \varepsilon)] \text{ and } 0 < \eta < \frac{1}{n(1-\varepsilon)}, \\
[n\theta(1 - \varepsilon)] + 1 & \text{for } n\theta(1 - \varepsilon) \neq [n\theta(1 - \varepsilon)] \text{ and } 0 < \eta < \frac{[n\theta(1 - \varepsilon)] - n\theta(1 - \varepsilon)}{n(1-\varepsilon)}. 
\end{cases}
\]

It follows that both \(g(\theta - \eta)\) and \(h(\theta - \eta)\) are independent of \(\eta\) if \(\eta > 0\) is small enough. Since \(S(n, g, h, \theta - \eta)\) is continuous with respect to \(\eta\) for fixed \(g\) and \(h\), we have that \(\lim_{\eta \downarrow 0} S(n, g(\theta - \eta), h(\theta - \eta), \theta - \eta)\) exists. Hence,
\[
\lim_{\eta \downarrow 0} C(\theta - \eta) = \lim_{\eta \downarrow 0} S(n, g(\theta - \eta), h(\theta - \eta), \theta - \eta) \geq \lim_{\eta \downarrow 0} S(n, g(\theta), h(\theta), \theta - \eta) = S(n, g(\theta), h(\theta), \theta) = C(\theta),
\]
where the inequality follows from [5].

\[
\square
\]

By a similar argument as that of Lemma [5] we have
Lemma 10 Let $\alpha < \beta$ be two consecutive elements of the ascending arrangement of all distinct elements of $\{a, b\} \cup \{\frac{\ell}{m(1 + \varepsilon)} \in (a, b) : \ell \in \mathbb{Z}\}$. Then, $C(\theta) \geq \min\{C(\alpha), C(\beta)\}$ for any $\theta \in (\alpha, \beta)$.

Finally, to show Theorem 2, note that the statement about the coverage probability follows immediately from Lemma 10. The number of elements of the finite set can be calculated by using the property of the ceiling and floor functions.

C Proof of Theorem 4

We need some preliminary results.

Lemma 11 Assume that $a + \varepsilon \leq b - \varepsilon$. Then,

$$\{\hat{\theta}_n - \theta | \geq \varepsilon, \theta \in [a + \varepsilon, b - \varepsilon]\} \subseteq \{\hat{\theta}_n - \theta | \geq \varepsilon, \theta \in [a, b]\} \subseteq \{\theta \in [a + \varepsilon, b - \varepsilon] \cap (a, b) \subseteq \{\theta \in [a, b] \cap (a, b) \}

(10)

for $\theta \in \Theta$. For $\theta \in [a + \varepsilon, b - \varepsilon]$, we have $\theta + \varepsilon > a$ and

$$\{\hat{\theta}_n \geq \theta + \varepsilon, \hat{\theta}_n < a\} \subseteq \{\hat{\theta}_n > a\} \cap \{\hat{\theta}_n < a\} = \emptyset,$$

$$\{\hat{\theta}_n \geq \theta + \varepsilon, \hat{\theta}_n < a\} \subseteq \{\hat{\theta}_n > a\} \cap \{\hat{\theta}_n < a\} = \emptyset,$$

which implies that

$$\{\hat{\theta}_n \geq \theta + \varepsilon, \hat{\theta}_n < a\} = \{\theta \in [a + \varepsilon, b - \varepsilon] \cap (a, b) \}

(14)

for $\theta \in [a + \varepsilon, b - \varepsilon]$. On the other hand, for $\theta \in [a + \varepsilon, b - \varepsilon]$, we have $\theta - \varepsilon < b$ and

$$\{\hat{\theta}_n \leq \theta - \varepsilon, \hat{\theta}_n > b\} \subseteq \{\hat{\theta}_n > b\} \cap \{\hat{\theta}_n > b\} = \emptyset,$$

$$\{\hat{\theta}_n \leq \theta - \varepsilon, \hat{\theta}_n > b\} \subseteq \{\hat{\theta}_n > b\} \cap \{\hat{\theta}_n > b\} = \emptyset,$$

which implies that

$$\{\hat{\theta}_n \leq \theta - \varepsilon, \hat{\theta}_n > b\} = \{\theta \in [a + \varepsilon, b - \varepsilon] \cap (a, b) \}

(15)

for $\theta \in [a + \varepsilon, b - \varepsilon]$. Combining (13), (14), and (15) completes the proof of (10).

To prove (11), note that, for $\theta \in [a, a + \varepsilon)$, we have $\theta - \varepsilon < a$ and

$$\{\hat{\theta}_n \leq \theta - \varepsilon\} \subseteq \hat{\theta}_n < a = \emptyset.$$
On the other hand, for $\theta \in [a, a + \varepsilon)$, we have $a < \theta + \varepsilon < b$ and

$$\{\tilde{\theta}_n \geq \theta + \varepsilon\} = \{\tilde{\theta}_n \geq \theta + \varepsilon\}.$$  

This proves (11).

To prove (12), note that, for $\theta \in (b - \varepsilon, b]$, we have $\theta + \varepsilon > b$ and

$$\{\tilde{\theta}_n \geq \theta + \varepsilon\} \subseteq \{\tilde{\theta}_n > b\} = \emptyset.$$  

On the other hand, for $\theta \in (b - \varepsilon, b]$, we have $a < \theta - \varepsilon < b$ and

$$\{\tilde{\theta}_n \leq \theta - \varepsilon\} = \{\tilde{\theta}_n \leq \theta - \varepsilon\}.$$  

This proves (12). The proof of the lemma is thus completed.

Lemma 12 Assume that $a + \varepsilon > b - \varepsilon$. Then,

$$\{\tilde{\theta}_n - \theta \geq \varepsilon\} = \emptyset \quad \text{for} \ \theta \in (b - \varepsilon, a + \varepsilon),$$  

(16)

$$\{\tilde{\theta}_n - \theta \geq \varepsilon\} = \{\tilde{\theta}_n \geq \theta + \varepsilon\} \quad \text{for} \ \theta \in [a, b - \varepsilon],$$  

(17)

$$\{\tilde{\theta}_n - \theta \geq \varepsilon\} = \{\tilde{\theta}_n \leq \theta - \varepsilon\} \quad \text{for} \ \theta \in [a - \varepsilon, b].$$  

(18)

Proof. To show (16), note that, for $\theta \in (b - \varepsilon, a + \varepsilon)$, we have $\theta - \varepsilon < a$ and

$$\{\tilde{\theta}_n \leq \theta - \varepsilon\} \subseteq \{\tilde{\theta}_n < a\} = \emptyset.$$  

On the other hand, for $\theta \in (b - \varepsilon, a + \varepsilon)$, we have $\theta + \varepsilon > b$ and

$$\{\tilde{\theta}_n \geq \theta + \varepsilon\} \subseteq \{\tilde{\theta}_n > b\} = \emptyset.$$  

This proves (16).

To show (17), note that for $\theta \in [a, b - \varepsilon]$, we have $\theta - \varepsilon < a$ and

$$\{\tilde{\theta}_n \leq \theta - \varepsilon\} \subseteq \{\tilde{\theta}_n < a\} = \emptyset.$$  

On the other hand, for $\theta \in [a, b - \varepsilon]$, we have $a < \theta + \varepsilon \leq b$ and

$$\{\tilde{\theta}_n \geq \theta + \varepsilon\} = \{\tilde{\theta}_n \geq \theta + \varepsilon\}.$$  

This proves (17).

To show (18), note that for $\theta \in [a + \varepsilon, b]$, we have $\theta + \varepsilon > b$ and

$$\{\tilde{\theta}_n \geq \theta + \varepsilon\} \subseteq \{\tilde{\theta}_n > b\} = \emptyset.$$  

On the other hand, for $\theta \in [a + \varepsilon, b]$, we have $a \leq \theta - \varepsilon < b$ and

$$\{\tilde{\theta}_n \leq \theta - \varepsilon\} = \{\tilde{\theta}_n \leq \theta - \varepsilon\}.$$  

This proves (18). The proof of the lemma is thus completed.
Lemma 13 Assume that $a + \varepsilon \leq b - \varepsilon$. Then, the minimum of $\Pr\{\tilde{\theta}_n - \theta < \varepsilon \mid \theta\}$ with respect to $\theta \in [a + \varepsilon, b - \varepsilon]$ is attained at the finite set

$$A_0 = (a + \varepsilon, b - \varepsilon) \bigcup \left\{ \frac{k}{n} - \varepsilon \in (a + \varepsilon, b - \varepsilon) : k \in \mathbb{Z} \right\} \bigcup \left\{ \frac{k}{n} + \varepsilon \in (a + \varepsilon, b - \varepsilon) : k \in \mathbb{Z} \right\}$$

Proof. From $\theta \in [a + \varepsilon, b - \varepsilon]$, it follows from (10) of Lemma 11 that

$$\Pr\{\tilde{\theta}_n - \theta < \varepsilon \mid \theta\} = \Pr\{\tilde{\theta}_n - \theta < \varepsilon \mid \theta\}.$$

Hence, by Theorem 1, the lemma follows.

Lemma 14 Assume that $a + \varepsilon \leq b - \varepsilon$. Then, the minimum of $\Pr\{\tilde{\theta}_n - \theta < \varepsilon \mid \theta\}$ with respect to $\theta \in [a, a + \varepsilon]$ is attained at the finite set

$$A_1 = (a, a + \varepsilon) \bigcup \left\{ \frac{k}{n} - \varepsilon \in (a, a + \varepsilon) : k \in \mathbb{Z} \right\}.$$

Proof. For $\theta \in [a, a + \varepsilon]$, it follows from (11) of Lemma 11 that $\Pr\{\tilde{\theta}_n - \theta < \varepsilon \mid \theta\} = \Pr\{\tilde{\theta}_n < \theta + \varepsilon \mid \theta\}$. Let $\alpha < \beta$ be two consecutive elements of $A_1$. It follows from Lemma 1 that

$$\Pr\{\tilde{\theta}_n - \theta < \varepsilon \mid \theta\} = \Pr\{\tilde{\theta}_n < \theta + \varepsilon \mid \theta\} = \Pr\{\tilde{\theta}_n < \theta^* + \varepsilon \mid \theta\}$$

for $\theta \in (\alpha, \beta)$, where $\theta^* = \frac{\alpha + \beta}{2}$. Now let $\eta \in (0, \frac{\alpha - \beta}{2})$. Recalling the assumption that for any interval $\mathcal{I}$, the probability $\Pr\{Y_n \in \mathcal{I} \mid \theta\}$ is a unimodal function of $\theta \in [a, b]$, we have that

$$\Pr\{\tilde{\theta}_n < \theta + \varepsilon \mid \theta\} \geq \min \left[ \Pr\{\tilde{\theta}_n < (\alpha + \eta) + \varepsilon \mid \alpha + \eta\}, \Pr\{\tilde{\theta}_n < (\beta - \eta) + \varepsilon \mid \beta - \eta\} \right]$$

for any $\theta \in (\alpha, \beta)$, where the function $h(.)$ is defined by (4). From Lemma 1, we know that

$$h(\alpha + \eta) = h(\beta - \eta) = h(\beta).$$

Hence,

$$\Pr\{\tilde{\theta}_n < \theta + \varepsilon \mid \theta\} \geq \min \left[ \Pr\{Y_n \leq h(\beta) \mid \alpha + \eta\}, \Pr\{Y_n \leq h(\beta) \mid \beta - \eta\} \right]$$

for any $\theta \in (\alpha, \beta)$. Recalling the assumption that for any interval $\mathcal{I}$, the probability $\Pr\{Y_n \in \mathcal{I} \mid \theta\}$ is a continuous function of $\theta \in [a, b]$, we have

$$\Pr\{\tilde{\theta}_n - \theta < \varepsilon \mid \theta\} = \Pr\{\tilde{\theta}_n < \theta + \varepsilon \mid \theta\} \geq \lim_{\eta \downarrow 0} \min \left[ \Pr\{Y_n \leq h(\beta) \mid \alpha + \eta\}, \Pr\{Y_n \leq h(\beta) \mid \beta - \eta\} \right]$$

$$\geq \min \left[ \lim_{\eta \downarrow 0} \Pr\{Y_n \leq h(\beta) \mid \alpha + \eta\}, \lim_{\eta \downarrow 0} \Pr\{Y_n \leq h(\beta) \mid \beta - \eta\} \right]$$

$$= \min \left[ \Pr\{Y_n \leq h(\beta) \mid \alpha\}, \Pr\{Y_n \leq h(\beta) \mid \beta\} \right]$$

$$= \min \left[ \Pr\{\tilde{\theta}_n < \alpha + \varepsilon \mid \alpha\}, \Pr\{\tilde{\theta}_n < \beta + \varepsilon \mid \beta\} \right]$$

$$= \min \left[ \Pr\{\tilde{\theta}_n - \alpha < \varepsilon \mid \alpha\}, \Pr\{\tilde{\theta}_n - \beta < \varepsilon \mid \beta\} \right].$$
for any $\theta \in (\alpha, \beta)$. This immediately leads to the result of the lemma.

Lemma 15 Assume that $a + \varepsilon \leq b - \varepsilon$. Then, the minimum of $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \mid \theta\}$ with respect to $\theta \in [b - \varepsilon, b]$ is attained at the finite set

$$A_2 = \{b, b - \varepsilon\} \bigcup \left\{ \frac{k}{n} + \varepsilon \in (b - \varepsilon, b) : k \in \mathbb{Z} \right\}.$$  

Proof. For $\theta \in (b - \varepsilon, b]$, it follows from (12) of Lemma 11 that $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \mid \theta\} = \Pr\{\tilde{\theta}_n > \theta - \varepsilon \mid \theta\}$. Let $\alpha < \beta$ be two consecutive elements of $A_2$. It follows from Lemma 2 that

$$\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \mid \theta\} = \Pr\{\tilde{\theta}_n > \theta - \varepsilon \mid \theta\} = \Pr\{\tilde{\theta}_n > \theta^* - \varepsilon \mid \theta\}$$

for $\theta \in (\alpha, \beta)$, where $\theta^* = \frac{\alpha + \beta}{2}$. Now let $\eta \in (0, \frac{\beta - \alpha}{2})$. Invoking the assumption that for any interval $\mathcal{I}$, the probability $\Pr\{Y_n \in \mathcal{I} \mid \theta\}$ is a unimodal function of $\theta \in [a, b]$, we have

$$\Pr\{\tilde{\theta}_n > \theta - \varepsilon \mid \theta\} \geq \min \left[ \Pr\{\tilde{\theta}_n > (\alpha + \eta) - \varepsilon \mid \alpha + \eta\}, \Pr\{\tilde{\theta}_n > (\beta - \eta) - \varepsilon \mid \beta - \eta\} \right]$$

for any $\theta \in (\alpha, \beta)$, where the function $g(\cdot)$ is defined by (14). From Lemma 2, we know that

$$g(\alpha + \eta) = g(\beta - \eta) = g(\alpha).$$

Hence,

$$\Pr\{\tilde{\theta}_n > \theta - \varepsilon \mid \theta\} \geq \min \left[ \Pr\{Y_n \geq g(\alpha) \mid \alpha + \eta\}, \Pr\{Y_n \geq g(\alpha) \mid \beta - \eta\} \right]$$

for any $\theta \in (\alpha, \beta)$. Recalling the assumption that for any interval $\mathcal{I}$, the probability $\Pr\{Y_n \in \mathcal{I} \mid \theta\}$ is a continuous function of $\theta \in [a, b]$, we have

$$\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \mid \theta\} = \Pr\{\tilde{\theta}_n > \theta - \varepsilon \mid \theta\} \geq \lim_{\eta \downarrow 0} \min \left[ \Pr\{Y_n \geq g(\alpha) \mid \alpha + \eta\}, \Pr\{Y_n \geq g(\alpha) \mid \beta - \eta\} \right]$$

for any $\theta \in (\alpha, \beta)$. This immediately leads to the result of the lemma.

By virtue of (16) of Lemma 12, we have the following result.

Lemma 16 Assume that $a + \varepsilon > b - \varepsilon$. Then, $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \mid \theta\} = 1$ for $\theta \in (b - \varepsilon, a + \varepsilon)$. 

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By virtue of (17) of Lemma 12 and a similar argument as that of Lemma 14, we have the following result.

**Lemma 17** Assume that $b \geq a + \varepsilon > b - \varepsilon$. Then, the minimum of $\Pr\{ |\tilde{\theta}_n - \theta| < \varepsilon \mid \theta \}$ with respect to $\theta \in [a, b - \varepsilon]$ is attained at the finite set

$$A_3 = \{a, b - \varepsilon\} \bigcup \left\{ \frac{k}{n} - \varepsilon \in (a, b - \varepsilon) : k \in \mathbb{Z} \right\}.$$

By virtue of (18) of Lemma 12 and a similar argument as that of Lemma 15, we have the following result.

**Lemma 18** Assume that $b \geq a + \varepsilon > b - \varepsilon$. Then, the minimum of $\Pr\{ |\tilde{\theta}_n - \theta| < \varepsilon \mid \theta \}$ with respect to $\theta \in [a + \varepsilon, b]$ is attained at the finite set

$$A_4 = \{b, a + \varepsilon\} \bigcup \left\{ \frac{k}{n} + \varepsilon \in (a + \varepsilon, b) : k \in \mathbb{Z} \right\}.$$

We are now in a position to prove the theorem. We need to consider three cases as follows.

In the case of $a + \varepsilon \leq b - \varepsilon$, it follows from Lemmas 13–15 that, the minimum of $\Pr\{ |\tilde{\theta}_n - \theta| < \varepsilon \mid \theta \}$ with respect to $\theta \in [a, b]$ is attained at the finite set $(A_0 \cup A_1 \cup A_2) \cap [a, b] = A \cap [a, b]$, where $A$ is the set defined by (2).

In the case of $b \geq a + \varepsilon > b - \varepsilon$, it follows from Lemmas 16–18 that, the minimum of $\Pr\{ |\tilde{\theta}_n - \theta| < \varepsilon \mid \theta \}$ with respect to $\theta \in [a, b]$ is attained at the finite set $(A_3 \cup A_4) \cap [a, b] = A \cap [a, b]$.

In the case of $a + \varepsilon > b$, we have $b - \varepsilon < a < b - a + \varepsilon$. It follows from Lemma 16 that $\Pr\{ |\tilde{\theta}_n - \theta| < \varepsilon \mid \theta \} = 1$ for $\theta \in [a, b]$. Thus, the minimum of $\Pr\{ |\tilde{\theta}_n - \theta| < \varepsilon \mid \theta \}$ with respect to $\theta \in [a, b]$ is equal to 1, which is attained at $\{a, b\} = A \cap [a, b]$.

The number of elements of the finite set can be calculated by using the property of the ceiling and floor functions. This completes the proof of the theorem.

### D Proof of Theorem 5

We need some preliminary results. The following Lemmas 19–21 are more general but similar to the results of [3]. To justify these results, we also follow similar arguments as that of [3].

**Lemma 19** Assume that $\frac{a}{1 - \varepsilon} \leq \frac{b}{1 + \varepsilon}$. Then,

\[
\{ |\tilde{\theta}_n - \theta| \geq \varepsilon \mid \theta \} = \{ |\tilde{\theta}_n - \theta| \geq \varepsilon \mid \theta \} \quad \text{for} \quad \theta \in \left[ \frac{a}{1 - \varepsilon}, \frac{b}{1 + \varepsilon} \right],
\]

\[
\{ |\tilde{\theta}_n - \theta| \geq \varepsilon \mid \theta \} = \{ \tilde{\theta}_n \geq (1 + \varepsilon) \mid \theta \} \quad \text{for} \quad \theta \in \left[ a, \frac{a}{1 - \varepsilon} \right],
\]

\[
\{ |\tilde{\theta}_n - \theta| \geq \varepsilon \mid \theta \} = \{ \tilde{\theta}_n \leq (1 - \varepsilon) \mid \theta \} \quad \text{for} \quad \theta \in \left( \frac{b}{1 + \varepsilon}, b \right].
\]

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Proof. To prove (19), recalling the definition of the range-preserving estimator, we have that
\[
\{\tilde{\theta}_n - \theta \geq \varepsilon \theta, \tilde{\theta}_n \in [a, b]\} = \{|\theta_n - \theta| \geq \varepsilon \theta, \theta_n \in [a, b]\}
\] (22)
for \(\theta \in \Theta\). For \(\theta \in \left[\frac{a}{1+\varepsilon}, \frac{b}{1+\varepsilon}\right]\), we have \((1+\varepsilon)\theta > a\) and
\[
\{\tilde{\theta}_n \geq (1+\varepsilon)\theta, \tilde{\theta}_n < a\} \subseteq \{\tilde{\theta}_n > a, \tilde{\theta}_n < a\} = \emptyset,
\]
\[
\{\tilde{\theta}_n \geq (1+\varepsilon)\theta, \tilde{\theta}_n < a\} \subseteq \{\tilde{\theta}_n > a, \tilde{\theta}_n < a\} = \emptyset,
\]
which implies that
\[
\{\tilde{\theta}_n \geq (1+\varepsilon)\theta, \tilde{\theta}_n < a\} = \{\tilde{\theta}_n \geq (1+\varepsilon)\theta, \tilde{\theta}_n < a\}
\] (23)
for \(\theta \in \left[\frac{a}{1+\varepsilon}, \frac{b}{1+\varepsilon}\right]\). On the other hand, for \(\theta \in \left[\frac{a}{1+\varepsilon}, \frac{b}{1+\varepsilon}\right]\), we have \((1-\varepsilon)\theta < b\) and
\[
\{\tilde{\theta}_n \leq (1-\varepsilon)\theta, \tilde{\theta}_n > b\} \subseteq \{\tilde{\theta}_n < b, \tilde{\theta}_n > b\} = \emptyset,
\]
\[
\{\tilde{\theta}_n \leq (1-\varepsilon)\theta, \tilde{\theta}_n > b\} \subseteq \{\tilde{\theta}_n < b, \tilde{\theta}_n > b\} = \emptyset,
\]
which implies that
\[
\{\tilde{\theta}_n \leq (1-\varepsilon)\theta, \tilde{\theta}_n > b\} = \{\tilde{\theta}_n \leq (1-\varepsilon)\theta, \tilde{\theta}_n > b\}
\] (24)
for \(\theta \in \left[\frac{a}{1-\varepsilon}, \frac{b}{1+\varepsilon}\right]\). Combining (22), (23), and (24) completes the proof of (19).

To prove (20), note that, for \(\theta \in \left[a, \frac{a}{1+\varepsilon}\right]\), we have \((1-\varepsilon)\theta < a\) and
\[
\{\tilde{\theta}_n \leq (1-\varepsilon)\theta\} \subseteq \{\tilde{\theta}_n < a\} = \emptyset.
\]
On the other hand, for \(\theta \in \left[a, \frac{a}{1+\varepsilon}\right]\), we have \(a < (1+\varepsilon)\theta < b\) and
\[
\{\tilde{\theta}_n \geq (1+\varepsilon)\theta\} = \{\tilde{\theta}_n \geq (1+\varepsilon)\theta\}.
\]
This proves (20).

To prove (21), note that, for \(\theta \in \left(\frac{b}{1+\varepsilon}, b\right]\), we have \((1+\varepsilon)\theta > b\) and
\[
\{\tilde{\theta}_n \geq (1+\varepsilon)\theta\} \subseteq \{\tilde{\theta}_n > b\} = \emptyset.
\]
On the other hand, for \(\theta \in \left(\frac{b}{1+\varepsilon}, b\right]\), we have \(a < (1-\varepsilon)\theta < b\) and
\[
\{\tilde{\theta}_n \leq (1-\varepsilon)\theta\} = \{\tilde{\theta}_n \leq (1-\varepsilon)\theta\}.
\]
This proves (21). The proof of the lemma is thus completed. \(\square\)

Lemma 20 Assume that \(\frac{a}{1-\varepsilon} > \frac{b}{1+\varepsilon}\). Then,
\[
\{\bar{\theta}_n - \theta \geq \varepsilon \theta\} = \emptyset \quad \text{for} \quad \theta \in \left(\frac{b}{1+\varepsilon}, \frac{a}{1-\varepsilon}\right),
\] (25)
\[
\{\bar{\theta}_n - \theta \geq \varepsilon \theta\} = \{\bar{\theta}_n \geq (1+\varepsilon) \theta\} \quad \text{for} \quad \theta \in \left[a, \frac{b}{1+\varepsilon}\right],
\] (26)
\[
\{\bar{\theta}_n - \theta \geq \varepsilon \theta\} = \{\bar{\theta}_n \leq (1-\varepsilon) \theta\} \quad \text{for} \quad \theta \in \left[\frac{a}{1-\varepsilon}, b\right],
\] (27)
Proof. To show (25), note that, for \( \theta \in \left( \frac{b}{1+\varepsilon}, \frac{a}{1-\varepsilon} \right) \), we have \((1 - \varepsilon)\theta < a\) and

\[ \{ \hat{\theta}_n \leq (1 - \varepsilon)\theta \} \subseteq \{ \hat{\theta}_n < a \} = \emptyset. \]

On the other hand, for \( \theta \in \left( \frac{b}{1+\varepsilon}, \frac{a}{1-\varepsilon} \right) \), we have \((1 + \varepsilon)\theta > b\) and

\[ \{ \hat{\theta}_n \geq (1 + \varepsilon)\theta \} \subseteq \{ \hat{\theta}_n > b \} = \emptyset. \]

This proves (25).

To show (26), note that for \( \theta \in \left[ a, \frac{b}{1+\varepsilon} \right] \), we have \((1 - \varepsilon)\theta < a\) and

\[ \{ \hat{\theta}_n \leq (1 - \varepsilon)\theta \} \subseteq \{ \hat{\theta}_n < a \} = \emptyset. \]

On the other hand, for \( \theta \in \left[ a, \frac{b}{1+\varepsilon} \right] \), we have \(a < (1 + \varepsilon)\theta \leq b\) and

\[ \{ \hat{\theta}_n \geq (1 + \varepsilon)\theta \} = \{ \hat{\theta}_n \geq (1 + \varepsilon)\theta \}. \]

This proves (26).

To show (27), note that for \( \theta \in \left[ \frac{a}{1-\varepsilon}, b \right] \), we have \((1 + \varepsilon)\theta > b\) and

\[ \{ \hat{\theta}_n \geq (1 + \varepsilon)\theta \} \subseteq \{ \hat{\theta}_n > b \} = \emptyset. \]

On the other hand, for \( \theta \in \left[ \frac{a}{1-\varepsilon}, b \right] \), we have \(a \leq (1 - \varepsilon)\theta < b\) and

\[ \{ \hat{\theta}_n \leq (1 - \varepsilon)\theta \} = \{ \hat{\theta}_n \leq (1 - \varepsilon)\theta \}. \]

This proves (27). The proof of the lemma is thus completed.

\[ \blacksquare \]

Lemma 21 Assume that \( \frac{a}{1 - \varepsilon} \leq \frac{b}{1 + \varepsilon} \). Then, the minimum of \( \Pr \{ |\hat{\theta}_n - \theta| < \varepsilon\theta | \theta \} \) with respect to \( \theta \in \left[ \frac{a}{1 - \varepsilon}, \frac{b}{1 + \varepsilon} \right] \) is attained at the finite set

\[ A_0 = \left\{ \frac{a}{1 - \varepsilon}, \frac{b}{1 + \varepsilon} \right\} \cup \left\{ \frac{k}{n(1 + \varepsilon)} \in \left( \frac{a}{1 - \varepsilon}, \frac{b}{1 + \varepsilon} \right) : k \in \mathbb{Z} \right\} \cup \left\{ \frac{k}{n(1 - \varepsilon)} \in \left( \frac{a}{1 - \varepsilon}, \frac{b}{1 + \varepsilon} \right) : k \in \mathbb{Z} \right\} \]

Proof. From \( \theta \in \left[ \frac{a}{1 - \varepsilon}, \frac{b}{1 + \varepsilon} \right] \), it follows from (19) of Lemma 19 that

\[ \Pr\{ |\hat{\theta}_n - \theta| < \varepsilon\theta | \theta \} = \Pr\{ |\hat{\theta}_n - \theta| < \varepsilon\theta | \theta \}. \]

Hence, by Theorem 2 the lemma follows.

\[ \blacksquare \]

The following Lemmas 22 and 23 are more general but similar to the results of [3]. To justify these results, an analysis of discontinuity is necessary.

Lemma 22 Assume that \( \frac{a}{1 - \varepsilon} \leq \frac{b}{1 + \varepsilon} \). Then, the minimum of \( \Pr \{ |\hat{\theta}_n - \theta| < \varepsilon\theta | \theta \} \) with respect to \( \theta \in \left[ a, \frac{a}{1 - \varepsilon} \right] \) is attained at the finite set

\[ A_1 = \left\{ a, \frac{a}{1 - \varepsilon} \right\} \cup \left\{ \frac{k}{n(1 + \varepsilon)} \in \left( a, \frac{a}{1 - \varepsilon} \right) : k \in \mathbb{Z} \right\}. \]
Lemma 23 Assume that \( \frac{a}{1 - \varepsilon} \leq \frac{b}{1 + \varepsilon} \). Then, the minimum of \( \Pr\{\bar{\theta}_n - \theta < \varepsilon \theta \mid \theta \} \) with respect to \( \theta \in \left[ \frac{b}{1 + \varepsilon}, b \right] \) is attained at the finite set

\[
A_2 = \left\{ b, \frac{b}{1 + \varepsilon} \right\} \cup \left\{ \frac{k}{n(1 - \varepsilon)} \in \left( \frac{b}{1 + \varepsilon}, b \right) : k \in \mathbb{Z} \right\}.
\]

Proof. For \( \theta \in \left( \frac{b}{1 + \varepsilon}, b \right] \), it follows from (21) of Lemma 19 that \( \Pr\{\bar{\theta}_n - \theta < \varepsilon \theta \mid \theta \} = \Pr\{\bar{\theta}_n > (1 - \varepsilon)\theta \mid \theta \} \). Let \( \alpha < \beta \) be two consecutive elements of \( A_2 \). It follows from Lemma 6 that

\[
\Pr\{\bar{\theta}_n - \theta < \varepsilon \theta \mid \theta \} = \Pr\{\bar{\theta}_n < (1 + \varepsilon)\theta \mid \theta \} = \Pr\{\bar{\theta}_n < (1 + \varepsilon)\theta^* \mid \theta \}
\]

for \( \theta \in (\alpha, \beta) \), where \( \theta^* = \frac{\alpha + \beta}{2} \). Now let \( \eta \in (0, \frac{\beta - \alpha}{2}) \). Recalling the assumption that for any interval \( \mathcal{I} \), the probability \( \Pr\{Y_n \in \mathcal{I} \mid \theta \} \) is a unimodal function of \( \theta \in [a, b] \), we have that

\[
\Pr\{\bar{\theta}_n < (1 + \varepsilon)\theta \mid \theta \} \geq \min \left[ \Pr\{\bar{\theta}_n < (1 + \varepsilon)(\alpha + \eta) \mid \alpha + \eta \}, \Pr\{\bar{\theta}_n < (1 + \varepsilon)(\beta - \eta) \mid \beta - \eta \} \right]
\]

\[
= \min \left[ \Pr\{Y_n \leq h(\alpha + \eta) \mid \alpha + \eta \}, \Pr\{Y_n \leq h(\beta - \eta) \mid \beta - \eta \} \right]
\]

for any \( \theta \in (\alpha, \beta) \), where the function \( h(.) \) is defined by (7). From Lemma 6, we know that \( h(\alpha + \eta) = h(\beta - \eta) = h(\beta) \).

Hence,

\[
\Pr\{\bar{\theta}_n < (1 + \varepsilon)\theta \mid \theta \} \geq \min \left[ \Pr\{Y_n \leq h(\beta) \mid \alpha + \eta \}, \Pr\{Y_n \leq h(\beta) \mid \beta - \eta \} \right]
\]

for any \( \theta \in (\alpha, \beta) \). Recalling the assumption that for any interval \( \mathcal{I} \), the probability \( \Pr\{Y_n \in \mathcal{I} \mid \theta \} \) is a continuous function of \( \theta \in [a, b] \), we have

\[
\Pr\{\bar{\theta}_n - \theta < \varepsilon \theta \mid \theta \} = \Pr\{\bar{\theta}_n < (1 + \varepsilon)\theta \mid \theta \}
\]

\[
\geq \lim_{\eta \downarrow 0} \min \left[ \Pr\{Y_n \leq h(\beta) \mid \alpha + \eta \}, \Pr\{Y_n \leq h(\beta) \mid \beta - \eta \} \right]
\]

\[
= \min \left[ \lim_{\eta \downarrow 0} \Pr\{Y_n \leq h(\beta) \mid \alpha + \eta \}, \lim_{\eta \downarrow 0} \Pr\{Y_n \leq h(\beta) \mid \beta - \eta \} \right]
\]

\[
= \min \left[ \Pr\{Y_n \leq h(\beta) \mid \alpha \}, \Pr\{Y_n \leq h(\beta) \mid \beta \} \right]
\]

\[
\geq \min \left[ \Pr\{Y_n \leq h(\beta) - \eta \mid \alpha \}, \Pr\{Y_n \leq h(\beta) \mid \beta \} \right]
\]

\[
= \min \left[ \Pr\{\bar{\theta}_n < (1 + \varepsilon)\alpha \mid \alpha \}, \Pr\{\bar{\theta}_n < (1 + \varepsilon)\beta \mid \beta \} \right]
\]

\[
= \min \left[ \Pr\{\bar{\theta}_n - \alpha < \varepsilon \alpha \mid \alpha \}, \Pr\{\bar{\theta}_n - \beta < \varepsilon \beta \mid \beta \} \right]
\]

for any \( \theta \in (\alpha, \beta) \). This immediately leads to the result of the lemma. \( \square \)
for $\theta \in (\alpha, \beta)$, where $\theta^* = \frac{a + \beta}{2}$. Now let $\eta \in (0, \frac{\beta - \alpha}{2})$. Invoking the assumption that for any interval $\mathcal{I}$, the probability $\Pr\{Y_n \in \mathcal{I} \mid \theta\}$ is a unimodal function of $\theta \in [a, b]$, we have

$$\Pr\{\tilde{\theta}_n > (1 - \varepsilon)\theta \mid \theta\} \geq \min \left[ \Pr\{\tilde{\theta}_n > (1 - \varepsilon)(\alpha + \eta) \mid \alpha + \eta\}, \Pr\{\tilde{\theta}_n > (1 - \varepsilon)(\beta - \eta) \mid \beta - \eta\} \right]$$

$$= \min \left[ \Pr\{Y_n \geq g(\alpha + \eta) \mid \alpha + \eta\}, \Pr\{Y_n \geq g(\beta - \eta) \mid \beta - \eta\} \right]$$

for any $\theta \in (\alpha, \beta)$, where the function $g(.)$ is defined by (7). From Lemma 7 we know that

$$g(\alpha + \eta) = g(\beta - \eta) = g(\alpha).$$

Hence,

$$\Pr\{\tilde{\theta}_n > (1 - \varepsilon)\theta \mid \theta\} \geq \min \left[ \Pr\{Y_n \geq g(\alpha) \mid \alpha + \eta\}, \Pr\{Y_n \geq g(\alpha) \mid \beta - \eta\} \right]$$

for any $\theta \in (\alpha, \beta)$. Recalling the assumption that for any interval $\mathcal{I}$, the probability $\Pr\{Y_n \in \mathcal{I} \mid \theta\}$ is a continuous function of $\theta \in [a, b]$, we have

$$\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon\theta \mid \theta\} = \Pr\{\tilde{\theta}_n > (1 - \varepsilon)\theta \mid \theta\}$$

$$\geq \lim_{\eta \downarrow 0} \min \left[ \Pr\{Y_n \geq g(\alpha) \mid \alpha + \eta\}, \Pr\{Y_n \geq g(\alpha) \mid \beta - \eta\} \right]$$

$$= \min \left[ \lim_{\eta \downarrow 0} \Pr\{Y_n \geq g(\alpha) \mid \alpha + \eta\}, \Pr\{Y_n \geq g(\alpha) \mid \beta\} \right]$$

$$= \min \left[ \Pr\{Y_n \geq g(\alpha) \mid \alpha\}, \Pr\{Y_n \geq g(\beta) \mid \beta\} \right]$$

$$= \min \left[ \Pr\{\tilde{\theta}_n > (1 - \varepsilon)\alpha \mid \alpha\}, \Pr\{\tilde{\theta}_n > (1 - \varepsilon)\beta \mid \beta\} \right]$$

$$= \min \left[ \Pr\{|\tilde{\theta}_n - \alpha| < \varepsilon\alpha \mid \alpha\}, \Pr\{|\tilde{\theta}_n - \beta| < \varepsilon\beta \mid \beta\} \right]$$

for any $\theta \in (\alpha, \beta)$. This immediately leads to the result of the lemma. 

By virtue of (25) of Lemma 20 we have the following result.

**Lemma 24** Assume that $\frac{a}{1 - \varepsilon} > \frac{b}{1 + \varepsilon}$. Then, $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon\theta \mid \theta\} = 1$ for $\theta \in \left( \frac{b}{1 + \varepsilon}, \frac{a}{1 - \varepsilon} \right)$.

By virtue of (26) of Lemma 20 and a similar argument as that of Lemma 22 we have the following result.

**Lemma 25** Assume that $\frac{a}{1 - \varepsilon} > \frac{b}{1 + \varepsilon} \geq a$. Then, the minimum of $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon\theta \mid \theta\}$ with respect to $\theta \in \left[ a, \frac{b}{1 + \varepsilon} \right]$ is attained at the finite set

$$A_3 = \left\{ a, \frac{b}{1 + \varepsilon} \right\} \bigcup \left\{ \frac{k}{n(1 + \varepsilon)} \in \left( a, \frac{b}{1 + \varepsilon} \right) : k \in \mathbb{Z} \right\}.$$
Lemma 26 Assume that $b \geq \frac{a}{1-\varepsilon} > \frac{b}{1+\varepsilon}$. Then, the minimum of $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \theta | \theta\}$ with respect to $\theta \in \left[\frac{a}{1-\varepsilon}, b\right]$ is attained at the finite set

$$A_4 = \left\{ b, \frac{a}{1-\varepsilon} \right\} \cup \left\{ \frac{k}{n(1-\varepsilon)} \in \left(\frac{a}{1-\varepsilon}, b\right) : k \in \mathbb{Z} \right\}.$$  

We are now in a position to prove the theorem. We need to consider five cases as follows.

In the case of $a_1 - \varepsilon \leq b_1 + \varepsilon$, it follows from Lemmas 21–23 that, the minimum of $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \theta | \theta\}$ with respect to $\theta \in [a, b]$ is attained at the finite set $(A_0 \cup A_1 \cup A_2) \cap [a, b] = A \cap [a, b]$, where $A$ is the set defined by (3).

In the case of $b \geq \frac{a}{1-\varepsilon} > \frac{b}{1+\varepsilon} \geq a$, it follows from Lemmas 24–26 that, the minimum of $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \theta | \theta\}$ with respect to $\theta \in [a, b]$ is attained at the finite set $(A_3 \cup A_4) \cap [a, b] = A \cap [a, b]$.

In the case of $a_1 - \varepsilon > b > a_1 + \varepsilon \geq a$, it follows from Lemma 24 that $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \theta | \theta\} = 1$ for $\theta \in \left(\frac{b}{1+\varepsilon}, b\right)$. Using this result and Lemma 26, we have that the minimum of $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \theta | \theta\}$ with respect to $\theta \in [a, b]$ is attained at the finite set $(A_3 \cup \{b\}) \cap [a, b] = A \cap [a, b]$.

In the case of $b \geq \frac{a}{1-\varepsilon} > a > \frac{b}{1+\varepsilon}$, it follows from Lemma 24 that $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \theta | \theta\} = 1$ for $\theta \in \left[\frac{a}{1-\varepsilon}, a\right)$. Using this result and Lemma 26, we have that the minimum of $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \theta | \theta\}$ with respect to $\theta \in [a, b]$ is attained at the finite set $(A_4 \cup \{a\}) \cap [a, b] = A \cap [a, b]$.

In the case of $\frac{b}{1+\varepsilon} > b > a > \frac{b}{1+\varepsilon}$, it follows from Lemma 24 that $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \theta | \theta\} = 1$ for $\theta \in [a, b]$. Thus, the minimum of $\Pr\{|\tilde{\theta}_n - \theta| < \varepsilon \theta | \theta\}$ with respect to $\theta \in [a, b]$ is attained at the finite set $\{a, b\} \cap [a, b] = A \cap [a, b]$.

The number of elements of the finite set can be calculated by using the property of the ceiling and floor functions. This completes the proof of the theorem.

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