FURTHER RESULTS ON THE PERTURBATION ESTIMATIONS FOR THE DRAZIN INVERSE

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Abstract. For \( n \times n \) complex singular matrix \( A \) with \( \text{ind}(A) = k > 1 \), let \( A^D \) be the Drazin inverse of \( A \). If a matrix \( B = A + E \) with \( \text{ind}(B) = 1 \) is said to be an acute perturbation of \( A \), if \( \|E\| \) is small and the spectral radius of \( B_g B - A^D A \) satisfies
\[
\rho(B_g B - A^D A) < 1,
\]
where \( B_g \) is the group inverse of \( B \).

The acute perturbation coincides with the stable perturbation of the group inverse, if the matrix \( B \) satisfies geometrical condition:
\[
\mathcal{R}(B) \cap \mathcal{N}(A^k) = \{0\}, \quad \mathcal{N}(B) \cap \mathcal{R}(A^k) = \{0\}
\]
which introduced by Vélez-Cerrada, Robles, and Castro-González, (Error bounds for the perturbation of the Drazin inverse under some geometrical conditions, Appl. Math. Comput., 215 (2009), 2154–2161).

Furthermore, two examples are provided to illustrate the acute perturbation of the Drazin inverse. We prove the correctness of the conjecture in a special case of \( \text{ind}(B) = 1 \) by Wei (Acute perturbation of the group inverse, Linear Algebra Appl., 534 (2017), 135–157).

1. Introduction and preliminaries. The group inverse and Drazin inverse have been investigated from the solution of singular linear systems \([2, 4, 9, 20, 23]\). Recently several papers, \([5, 6, 7, 14, 19, 25, 28, 32, 34, 36, 29]\), provide explicit formulae for the Drazin inverse to present the 2-norm (or Frobenius norm) bounds and the perturbation estimations. For the spectral projectors, one can find many results in \([10, 22, 24]\). There is a recent monograph \([8]\) on the algebraic properties of the generalized inverse.

In this paper, \( \mathbb{C}^{m \times n} \) is the set of \( m \times n \) complex matrices. If \( m = n \), then the identity matrix of order \( n \) and the null matrix in \( \mathbb{C}^{n \times n} \) are denoted simply by \( I_n \) and \( 0 \), respectively. For \( A \in \mathbb{C}^{n \times n} \), we denote \( \mathcal{R}(A) \) for its range and \( \mathcal{N}(A) \) for its null space. \( A^* \) is the conjugate transpose of the matrix \( A \). \( \| \cdot \| \) denotes the spectral...
norm. The Drazin inverse of $A \in \mathbb{C}^{n \times n}$ is the unique matrix $A^D \in \mathbb{C}^{n \times n}$ satisfying three equations [1, 4]

$$A^D A = AA^D, \quad A^D AA^D = A^D, \quad A^{l+1} A^D = A^l \quad \text{for all} \ l \geq k,$$

(1)

where $k$ is the smallest nonnegative integer satisfying $\text{rank}(A^{k+1}) = \text{rank}(A^k)$, $k$ is called the Drazin index of $A$ and is denoted by $\text{ind}(A)$. Clearly, $\text{ind}(A) = 0$ if and only if $A$ is nonsingular. If $\text{ind}(A) = 1$, then the Drazin inverse is called the group inverse of $A$ and presented by $A_\varnothing$. Let $A^\gamma = I_n - AA^D = P_{N(A^k), \mathcal{R}(A^k)}$ be the spectral projector on $\mathcal{R}(A^\gamma) = N(A^k)$ along $N(A^\gamma) = \mathcal{R}(A^k)$.

Let $A$ be a singular matrix with $\text{ind}(A) = k > 1$, and $B = A + E$ be a perturbation of $A$ with $\text{ind}(B) = 1$. Campbell and Meyer [3] presented a necessary and sufficient condition for the continuity of the Drazin inverse,

$$B_g \rightarrow A^D, \quad \text{if and only if} \quad \text{rank}(B) = \text{rank}(A^k).$$

In the general case of the perturbation analysis of the group inverse $B_g$ and the spectral projector $B^\pi_x = I_n - BB_g$, a condition of $(C_1)$ is introduced in [6, 25] for the stable perturbation of $A$.

**Definition 1.1.** ([6, 25]) Let the singular matrix $A$ with $\text{ind}(A) = k$ and $B = A + E$, if

$$(C_1) \quad \mathcal{R}(B) \cap N(A^k) = \{0\} \quad \text{and} \quad N(B) \cap \mathcal{R}(A^k) = \{0\},$$

then $B$ is called the stable perturbation of $A$.

One formula for $B^\pi_x$ is given in [6, Theorem 4.4] under the condition $(C_1)$ is satisfied. It is not easy to judge that $\|B^n - A^x\| < 1$ with respect to 2-norm.

Wedin [26] and Stewart [21] presented the acute perturbation of the Moore-Penrose inverse $A \in \mathbb{C}^{n \times n}$ in 1970s, respectively. They say that the range spaces $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are acute, if $B = A + E$ and

$$\|BB^\dagger - AA^\dagger\| < 1,$$

if and only if

$$\mathcal{R}(A) \cap N(B^\pi_x) = \{0\} \quad \text{and} \quad \mathcal{R}(B) \cap N(A^k) = \{0\}.$$ 

Similarly, the range spaces $\mathcal{R}(A^\pi_x)$ and $\mathcal{R}(B^\pi_x)$ are acute, if $\|B^\dagger B - A^\dagger A\| < 1$. The matrices $A$ and $B$ are called acute, if $\mathcal{R}(A)$ and $\mathcal{R}(B)$ are acute and $\mathcal{R}(A^\pi_x)$ and $\mathcal{R}(B^\pi_x)$ are acute. In this case, they say that $B$ is an acute perturbation of $A$ [21, 26].

Recently, we extend the acute perturbation from the Moore-Penrose inverse to the weighted Moore-Penrose inverse [17].

This note is organized as follows. In Section 2, we introduce the definition of the acute perturbation of the Drazin inverse, which is equivalent to the condition $(C_1)$ for matrices [6], and present several characterizations of acute perturbations based on the results from [6, 31]. In Section 3 we present our new results on the spectral radius and the spectral norm for the difference of $BB_g - AA^D$ via two examples.

### 2. Geometrical conditions on the acute perturbation.

Let $A \in \mathbb{C}^{n \times n}$ be singular with $\text{ind}(A) = k > 1$ and $\text{rank}(A^k) = r$. Then $A$ has the Jordan canonical form [1, 4],

$$A = P \begin{pmatrix} D & O \\ O & N \end{pmatrix} P^{-1}$$

(3)

for the invertible transformation matrix $P$ such that

$$D \in \mathbb{C}^{r \times r} \text{ is nonsingular,} \quad N \in \mathbb{C}^{(n-r) \times (n-r)} \text{ is nilpotent.}$$

(4)
Let $C = P^{-1}AP \in \mathbb{C}^{n \times n}$ and $A^\pi = I_n - AA^D$.

It follows from [4, Theorem 7.2.1] that $A^D$ and $A^\pi$ are given by

$$ P^{-1}A^DP = \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix}, \quad P^{-1}A^\pi P = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix}. $$

(5)

Let

$$ P = (P_1 \; P_2), \quad P^{-1} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, $$

where $P_1$ and $Q_1^*$ have the same column dimensions as $D$.

It is obvious that [27, 28]

$$ AA^D = P_1Q_1, \quad I_n - AA^D = P_2Q_2, \quad Q_1P_1 = I_r, $$

and

$$ \mathcal{R}(P_1) = \mathcal{N}(Q_2) = \mathcal{R}(A^D), \quad \mathcal{R}(P_2) = \mathcal{N}(Q_1) = \mathcal{N}(A^D). $$

Motivated by the acute perturbation of the Moore-Penrose inverse [21, 26], weighted Moore-Penrose inverse [17] and the group inverse [31], we present the acute perturbation for the Drazin inverse with respect to the spectral norm, which extends the recent results on the group inverse by Wei in [31].

**Definition 2.1.** A matrix $B = A + E \in \mathbb{C}^{n \times n}$ with $\text{ind}(B) = 1$ is said to be an acute perturbation of $A$ with $\text{ind}(A) = k$, if the perturbation matrix $\|E\|$ is small and

$$ \rho(BB_g - AA^D) < 1. $$

We present the geometrical conditions for the stable perturbation for the Drazin inverse.

**Lemma 2.2.** ([5, 37]) The following statements on $B \in \mathbb{C}^{n \times n}$ with $\text{ind}(B) = 1$ are equivalent:

(a) $B$ is a stable perturbation of $A$;
(b) $I_n - (B^\pi - A^\pi)^2$ is nonsingular;
(c) $B$ satisfies condition $(G_i)$: $\mathcal{R}(B) \cap \mathcal{N}(A^k) = \{0\}$ and $\mathcal{N}(B) \cap \mathcal{R}(A^k) = \{0\}$;
(d) $\text{rank}(B) = \text{rank}(A^k) = \text{rank}(A^kBA^k)$.

**Lemma 2.3.** Let $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = 1$ and $\text{rank}(A) = \text{rank}(A_{11})$, and $A$ be partitioned by

$$ A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} I_r & S \\ \end{pmatrix} A_{11}(I_r \; T) = \begin{pmatrix} A_{11} & A_{11}T \\ SA_{11} & SA_{11}T \end{pmatrix}, \quad A_{11} \in \mathbb{C}^{r \times r}, $$

where $T = A_{11}^{-1}A_{12} \in \mathbb{C}^{r \times (n-r)}$ and $S = A_{21}A_{11}^{-1} \in \mathbb{C}^{(n-r) \times r}$.

Then $A$ is group invertible if and only if $I_r + TS$ is nonsingular. In this case, $A_g$ and $A^\pi = I_n - AA_g$ can be presented by

$$ A_g = \begin{pmatrix} I_r \\ S \end{pmatrix} (I_r + TS)^{-1}A_{11}^{-1}(I_r + TS)^{-1}(I_r \; T) $$

$$ = \begin{pmatrix} [I_r + TS]A_{11}(I_r + TS)^{-1} & [I_r + TS]A_{11}(I_r + TS)^{-1} \\ S[I_r + TS]A_{11}(I_r + TS)^{-1} & S[I_r + TS]A_{11}(I_r + TS)^{-1} \\ \end{pmatrix} (6) $$
and
\[ A^\pi = I_n - \left( \begin{array}{c} I_r \\ S \end{array} \right)(I_r + TS)^{-1}(I_r T) \]
\[ = \left( \begin{array}{ccc} I_r - (I_r + TS)^{-1} & -(I_r + TS)^{-1}T \\ -S(I_r + TS)^{-1} & I_{n-r} - S(I_r + TS)^{-1}T \end{array} \right). \] (7)

Now we study the expression \( B_g \) and \( BB_g \) for the perturbation of the Drazin inverse.

**Lemma 2.4.** ([16, Lemma 2.2, Theorem 2.3], [29, Theorem 3.4]) Let \( B = A + E \in \mathbb{C}^{n \times n} \) with \( \text{ind}(A) = k \) and \( \text{rank}(A^k) = \text{rank}(B) \). If the perturbation \( \|E\| \) satisfies \( \|A^D\|\|E\| < \frac{1}{1+\sqrt{A^D}} \). Then \( \text{ind}(B) = 1 \) with the group inverse,
\[ P^{-1}B_gP = (C + F)g \]
\[ = \left( \begin{array}{c} I_r \\ F_{21}(D + F_{11})^{-1} \end{array} \right) X^{-1}(D + F_{11})^{-1}X^{-1}(I_r(D + F_{11})^{-1}F_{12}) \]
and the spectral projection,
\[ P^{-1}BB_gP = (C + F)(C + F)_g = \left( \begin{array}{c} I_r \\ F_{21}(D + F_{11})^{-1} \end{array} \right) X^{-1}(I_r(D + F_{11})^{-1}F_{12}) \]
where \( X = I_r + (D + F_{11})^{-1}F_{12}F_{21}(D + F_{11})^{-1} \).

**Proof.** Let the perturbation matrix be \( E \) and
\[ F = P^{-1}EP = \left( \begin{array}{cc} F_{11} & F_{12} \\ F_{21} & F_{22} \end{array} \right) = \left( \begin{array}{cc} Q_1 & \star \\ Q_2 & \star \end{array} \right)E(P_1\ P_2) = \left( \begin{array}{cc} Q_1EP_1 & Q_1EP_2 \\ Q_2EP_1 & Q_2EP_2 \end{array} \right). \]
Let \( C = P^{-1}AP = \left( \begin{array}{cc} D & \star \\ O & N \end{array} \right) = \left( \begin{array}{cc} Q_1 & \star \\ Q_2 & \star \end{array} \right)A(P_1\ P_2) = \left( \begin{array}{cc} Q_1AP_1 & Q_1AP_2 \\ Q_2AP_1 & Q_2AP_2 \end{array} \right). \]
If \( \text{rank}(B) = \text{rank}(A^k) \), then it follows from \([15, 29]\) that
\[ N + F_{22} = F_{21}(D + F_{11})^{-1}F_{12} \]
and
\[ C + F = P^{-1}(A + E)P = P^{-1}BP \]
\[ = \left( \begin{array}{cc} D + F_{11} & F_{12} \\ F_{21} & N + F_{22} \end{array} \right) \]
\[ = \left( \begin{array}{c} I_r \\ F_{21}(D + F_{11})^{-1} \end{array} \right) (D + F_{11})(I_r(D + F_{11})^{-1}F_{12}) \]
\[ = \left( \begin{array}{c} I_r \\ S \end{array} \right) (D + F_{11})(I_r T), \]
where \( S = F_{21}(D + F_{11})^{-1} \) and \( T = (D + F_{11})^{-1}F_{12} \).
If the perturbation \( \|E\| \) satisfies \( \|A^D\|\|E\| < \frac{1}{1+\sqrt{A^D}} \leq 1/2 \), then both \( I_n + EA^D \) and \( I_n + A^D E \) are nonsingular and
\[ \|A^D\|\|E\|\sqrt{\|A^D\|} < 1 - \|A^D\|\|E\|. \]
Thus
\[ \|\left( I_n + A^D E \right)^{-1} \| \leq \frac{1}{1 - \|A^D\|\|E\|}, \quad \|\left( I_n + EA^D \right)^{-1} \| \leq \frac{1}{1 - \|A^D\|\|E\|}. \]
It is obvious that \( D + F_{11} = Q_1(A + E)P_1 \) and \( D^{-1} = Q_1A^D P_1 \) with
\[
(D + F_{11})^{-1} = (I_r + D^{-1}F_{11})^{-1}D^{-1} = Q_1(I_r + A^D)E^{-1}A^DP_1 = Q_1A^D(I_r + EA^D)^{-1}P_1,
\]
and
\[
(D + F_{11})^{-1}F_{12} = Q_1(I_n + A^D)E^{-1}A^DP_1Q_1EP_2 = Q_1(I_n + A^D)E^{-1}A^DP_2,
\]
or
\[
F_{21}(D + F_{11})^{-1} = Q_2EP_1Q_1A^D(I_n + A^D)E^{-1}P_1 = Q_2EA^D(I_n + EA^D)^{-1}P_1,
\]
with
\[
X = I_r + TS = I_r + (D + F_{11})^{-1}F_{12}F_{21}(D + F_{11})^{-1} = Q_1P_1 + Q_1A^D E(I_n + A^D)^{-1}P_2Q_2EA^D(I_n + EA^D)^{-1}P_1
\]
\[
= Q_1(I_n + A^D E(I_n + A^D)^{-1}(I_n - AA^D)(I_n + EA^D)^{-1}EAD
\]
\[
= (I_n + A^D E(I_n + A^D)^{-1}A^D E(I_n - AA^D)EAD(I_n + EA^D)^{-1}P_1
\]
\[
= Q_1(I_n + Y)P_1,
\]
where
\[
Y = A^D E(I_n + A^D)^{-1}(I_n - AA^D)(I_n + EA^D)^{-1}EAD
\]
\[
= (I_n + A^D E(I_n - AA^D)EAD(I_n + EA^D)^{-1},
\]
since \( Q_1P_1 = I_r \) and \( \|I_n - AA^D\| = \|AA^D\| \) (see [24] or [28, Lemma 2.3]) and consequently,
\[
\|Y\| \leq \frac{\|A^D\|^2\|E(I_n - AA^D)E\|}{(1 - \|A^D\|^2\|E\|^2)^2} \leq \frac{\|A^D\|^2\|E\|^2\|AA^D\|}{(1 - \|A^D\|^2\|E\|^2)^2} < 1,
\]
\( I_n + Y \) is nonsingular.

We obtain \( YAA^D = AA^DY = Y \), since \( \mathcal{R}(P_1) = \mathcal{R}(A^D), \mathcal{N}(Q_1) = \mathcal{N}(A^D) \), we shall prove that \( X^{-1} = Q_1(I_n + Y)^{-1}P_1 \).

It is easy to verify that
\[
[Q_1(I_n + Y)P_1][Q_1(I_n + Y)^{-1}P_1] = Q_1(I_n + Y)AA^D(I_n + Y)^{-1}P_1 = Q_1AA^D(I_n + Y)(I_n + Y)^{-1}P_1 = Q_1P_1 = I_r,
\]
similarly, we have \( [Q_1(I_n + Y)^{-1}P_1][Q_1(I_n + Y)P_1] = I_r \). Thus
\[
X^{-1} = Q_1(I_n + Y)^{-1}P_1.
\]

It follows from [4, Theorems 7.7.5 and 7.7.7] that
\[
\text{ind}(C + F) = \text{ind}[(D + F_{11})(I + TS)] + 1 = 1,
\]
i.e., \( \text{ind}(B) = \text{ind}(A + E) = \text{ind}(C + F) = 1 \).
By [4, Theorems 7.7.6], we have
\[ P^{-1}B_gP = (C + F)_g \]
\[ = \left( \begin{array}{c} I_r \\ F_{21}(D + F_{11})^{-1} \end{array} \right) X^{-1}(D + F_{11})^{-1}X^{-1}(I_r(D + F_{11})^{-1}F_{12}), \]
and by simple computations, we obtain
\[ P^{-1}BB_gP = P^{-1}BPP^{-1}B_gP = (C + F)(C + F)_g \]
\[ = \left( \begin{array}{c} I_r \\ F_{21}(D + F_{11})^{-1} \end{array} \right) X^{-1}(I_r(D + F_{11})^{-1}F_{12}). \]

We can present the estimation for spectral radius of \( BB_g - AA^D \), which sharpens the upper bound of \( BB_g - AA_g \) [31, Theorem 2.1 (a)].

**Theorem 2.5.** Let \( B = A + E \in \mathbb{C}^{n \times n} \) with \( \text{ind}(A) = k \) and \( \text{rank}(A^k) = \text{rank}(B) \). If the perturbation \( ||E|| \) satisfies \( ||A^D||E|| < \frac{1}{1 + \sqrt{2||A^D||}} \). Then \( \text{ind}(B) = 1 \) and the spectral radius of \( BB_g(I_n - AA^D) \) and \( AA^D(I_n - BB_g) \) are exactly the same, such that

1. \( \rho(BB_g(I_n - AA^D)) = \rho(AA^D(I_n - BB_g)) \leq \frac{\rho(Y)}{1-\rho(Y)} \),
2. \( [\rho(BB_g - AA^D)]^2 = \rho(BB_g(I_n - AA^D)) = \rho(AA^D(I_n - BB_g)) < 1 \),

where \( Y = A^D(E(I_n + A^DE)^{-1}(I_n - AA^D)(I_n + EA^D)^{-1}EA^D) \) and \( \rho(Y) < 1/2 \).

**Proof.** It follows from Lemma 2.3 that
\[ P^{-1}BB_g(I_n - AA^D)P = P^{-1}BB_gPP^{-1}(I_n - AA^D)P \]
\[ = \left( \begin{array}{c} I_r \\ F_{21}(D + F_{11})^{-1} \end{array} \right) X^{-1}(I_r(D + F_{11})^{-1}F_{12}) \left( \begin{array}{cc} O & O \\ O & I_{n-r} \end{array} \right) \]
\[ = \left( \begin{array}{c} O \\ F_{21}(D + F_{11})^{-1}F_{12} \\ O \\ F_{21}(D + F_{11})^{-1}X^{-1}(D + F_{11})^{-1}F_{12} \end{array} \right), \]
and
\[ P^{-1}AA^D(I_n - BB_g)P = P^{-1}AA^DPPP^{-1}(I_n - BB_g)P \]
\[ = \left( \begin{array}{c} I_r \\ O \\ O \\ O \end{array} \right) \left[ I_n - \left( \begin{array}{c} I_r \\ F_{21}(D + F_{11})^{-1} \end{array} \right) X^{-1}(I_r(D + F_{11})^{-1}F_{12}) \right] \]
\[ = \left( \begin{array}{c} I_r - X^{-1} \\ -X^{-1}(D + F_{11})^{-1}F_{12} \\ O \\ O \end{array} \right). \]

We obtain
\[ I_r - X^{-1} = -(I_r - X)X^{-1} = (D + F_{11})^{-1}F_{12}F_{21}(D + F_{11})^{-1}X^{-1}. \]

From the proof of Lemma 2.3, it is obvious that
\[ (D + F_{11})^{-1} = Q_1(I + A^DE)^{-1}A^DP_1 = Q_1A^D(I + EA^D)^{-1}P_1, \]
and
\[ (D + F_{11})^{-1}F_{12} = Q_1(I_n + A^DE)^{-1}A^DEP_2, \]
or
\[ F_{21}(D + F_{11})^{-1} = Q_2EA^D(I_n + EA^D)^{-1}P_1, \]
with
\[ X = Q_1(I_n + Y)P_1, \quad X^{-1} = Q_1(I_n + Y)^{-1}P_1. \]
Since \( ||E|| \) is small and \((I_n + Y)^{-1}AA^D = AA^D(I_n + Y)^{-1} \), then the spectral radius of \( I_r - X^{-1} \) satisfies
\[
\rho(I_r - X^{-1}) = \rho(Q_1(I_n + A^D E) - 1 A^D E P_2 Q_2 (I_n + E A^D)^{-1} E A^D P_1 X^{-1})
\]
\[
= \rho(Q_1 A^D E(I_n + A^D E)^{-1} (I_n - A A^D) (I_n + E A^D)^{-1} E A^D P_1 Q_1 (I_n + Y)^{-1} P_1)
\]
\[
= \rho(Y A A^D (I_n + Y)^{-1} P_1 Q_1)
\]
\[
= \rho(Y (I_n + Y)^{-1} A A^D)
\]
\[
= \rho(Y (I_n + Y)^{-1})
\]
\[
\leq \frac{\rho(Y)}{1 - \rho(Y)}.
\]

If \( ||A^D|| ||E|| < \frac{1}{1 + \sqrt{2} ||A A^D||} \) and \( ||I_n - A A^D|| = ||A A^D|| \), then
\[
\rho(Y) \leq ||Y|| = ||A^D E(I_n + A^D E)^{-1} (I_n - A A^D) (I_n + E A^D)^{-1} E A^D||
\]
\[
\leq \left( \frac{||A^D|| ||E||}{1 - ||A^D|| ||E||} \right)^2 ||I_n - A A^D||
\]
\[
= \left( \frac{||A^D|| ||E||}{1 - ||A^D|| ||E||} \right)^2 ||A A^D||
\]
\[
< 1/2,
\]

and we can estimate the spectral radius of \( BB_g(I_n - A A^D) \).

\[
\rho(BB_g(I_n - A A^D)) = \rho[A A^D(I_n - BB_g)]
\]
\[
= \rho(F_{21} (D + F_{11})^{-1} X^{-1} (D + F_{11})^{-1} F_{12}]
\]
\[
= \rho([D + F_{11}]^{-1} F_{12} F_{21} (D + F_{11})^{-1} X^{-1}]
\]
\[
= \rho(I_r - X^{-1})
\]
\[
\leq \frac{\rho(Y)}{1 - \rho(Y)} < 1.
\]

Next we reveal the relationship between \( \rho(BB_g - A A^D) \) and \( \rho(BB_g(I_n - A A^D)) \).

It follows from Lemma 2.3 that
\[
[P^{-1}(BB_g - A A^D)P]\^2
\]
\[
= [((C + F)_g (C + F) - CC_g)]^2
\]
\[
= \left( \begin{array}{cc}
X^{-1} - I_r & X^{-1} (D + F_{11})^{-1} F_{12} \\
F_{21} (D + F_{11})^{-1} X^{-1} & F_{21} (D + F_{11})^{-1} X^{-1} (D + F_{11})^{-1} F_{12}
\end{array} \right)^2.
\]

\[
= \left( \begin{array}{cc}
I_r - X^{-1} & O \\
O & F_{21} (D + F_{11})^{-1} X^{-1} (D + F_{11})^{-1} F_{12}
\end{array} \right).
\]

It is obvious that the spectral radius of \( F_{21} (D + F_{11})^{-1} X^{-1} (D + F_{11})^{-1} F_{12} \) are the same as
\[
(D + F_{11})^{-1} F_{12} F_{21} (D + F_{11})^{-1} X^{-1} = I_r - X^{-1}.
\]
Then
\[ |ho(BB_g - AA^D)|^2 = \rho(I_r - X^{-1}) = \rho(BB_g(I_n - AA^D)) \]
\[ = \rho(AA^D(I_n - BB_g)) < 1. \]

Now we can give a necessary or sufficient condition for the acute perturbation of the Drazin inverse.

**Corollary 1.** If \( \text{rank}(A^k) = \text{rank}(B) \) and \( \|A^D\|E < \frac{1}{1 + \sqrt{2}\|AA^D\|} \), then \( B \) is an acute perturbation of \( A \).

**Corollary 2.** If \( B \) is an acute perturbation of \( A \), then \( \text{rank}(A^k) = \text{rank}(B) \) and \( I_n - (BB_g - AA^D) \) is invertible.

**Proof.** Denote \( H = BB_g - AA^D \). For a sufficiently small \( \epsilon > 0 \), there exists a matrix norm \[ \|\cdot\|_* \) so that \( \|H\|_* < \rho(H) + \epsilon < 1 \). It is obvious that \( I_n - (BB_g - AA^D) \) is invertible.

Based on the well known result [10, Theorem 2.6.4]: if \( \|H\|_* < 1 \) for a matrix norm, then the spectral projection matrices \( BB_g \) and \( AA^D \) have the same rank, i.e.,
\[ \text{rank}(B) = \text{rank}(BB_g) = \text{rank}(AA^D) = \text{rank}(A^k). \]

Now we present a necessary and sufficient condition for the acute perturbation of the Drazin inverse, which coincides with the stable perturbation of the Drazin inverse [6].

**Theorem 2.6.** \( B \) is an acute perturbation of \( A \), if and only if the geometrical conditions are satisfied
\[ \mathcal{R}(B) \cap \mathcal{N}(A^k) = \{0\} \quad \text{and} \quad \mathcal{N}(B) \cap \mathcal{R}(A^k) = \{0\}. \]

**Proof.** If \( \mathcal{R}(B) \cap \mathcal{N}(A^k) \neq \{0\} \), then there exists a nonzero vector \( x \in \mathcal{R}(B) \cap \mathcal{N}(A^k) \) such that
\[ BB_g x = x, \quad \text{and} \quad AA^D x = 0. \]
Then
\[ (BB_g - AA^D)x = x, \quad \text{i.e.,} \quad \rho(BB_g - AA^D) \geq 1, \]
\( B \) is not an acute perturbation of \( A \).

If \( \mathcal{N}(B) \cap \mathcal{R}(A^k) \neq \{0\} \), then there exists a nonzero vector \( y \in \mathcal{N}(B) \cap \mathcal{R}(A^k) \) such that
\[ BB_g y = 0, \quad \text{and} \quad AA^D y = y. \]
Then
\[ (BB_g - AA^D)y = -y, \quad \text{i.e.,} \quad \rho(BB_g - AA^D) \geq 1, \]
\( B \) is not an acute perturbation of \( A \).

It follows [5, Theorem 2.1] that
\[ \mathcal{R}(B) \cap \mathcal{N}(A^k) = \{0\} \quad \text{and} \quad \mathcal{N}(B) \cap \mathcal{R}(A^k) = \{0\}, \]
which is equivalent to
\[ \text{rank}(B) = \text{rank}(A^k) = \text{rank}(A^k BA^k). \]
With the help of Theorem 2.5, we can prove that \( \rho(BB_g - AA^D) < 1 \).

**Corollary 3.** B is an acute perturbation of A, if and only if

\[
\text{rank}(B) = \text{rank}(A^k) = \text{rank}(A^k BA^k).
\]

**Remark 1.** Castro-González et al. [6] derived that, if B is a stable perturbation of A, then

1. \( \mathcal{R}(B) \cap \mathcal{N}(A^k) = \{0\} \) and \( \mathcal{N}(B) \cap \mathcal{R}(A^k) = \{0\} \);
2. \( I_n - (BB_g - AA^D)^2 \) is invertible.

If B is not acute perturbation of A and \( \text{rank}(B) \geq \text{rank}(A^k) \), then \( \|BB_g - AA^D\| \geq 1 \) (see [28, 35]).

3. **Examples.** In this section, we provide two examples to illustrate the difference between the acute perturbation for the Drazin inverse and the spectral norm.

**Example 3.1.** ([29]) Let

\[
A = \begin{pmatrix}
0 & 0.25 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 + \varepsilon
\end{pmatrix}
\]

for a positive \( \varepsilon \). Then

\[
A^D = AA^D = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad B_g = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 + \varepsilon
\end{pmatrix}, \quad BB_g = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
\]

so \( \text{ind}(A) = 2 \) and \( \text{ind}(B) = 1 \). Thus \( \text{rank}(A^2) = \text{rank}(B) = \text{rank}(A^2 BA^2) \),

\[
BB_g = AA^D,
\]

so B is an acute perturbation of A, and

\[
\rho(BB_g - AA^D) = 0, \quad \text{and} \quad \|B_g - A^D\| = \frac{\varepsilon}{1 + \varepsilon} \approx \varepsilon - \varepsilon^2.
\]

**Example 3.2.** ([16]) Let

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

for a positive \( \varepsilon \) (\(< 1/2\)).

Then

\[
A^D = AA^D = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad B_g = BB_g = B,
\]

so \( \text{ind}(A) = 2 \) and \( \text{ind}(B) = 1 \). Thus \( \text{rank}(A^2) = \text{rank}(B) = \text{rank}(A^2 BA^2) \), so B is an acute perturbation of A, and

\[
\rho(BB_g - AA^D) = 0, \quad \text{and} \quad \|BB_g - AA^D\| = \varepsilon.
\]

In this case, \( \rho(BB_g - AA^D) \) is very close to \( \|BB_g - AA^D\| \).
4. **Concluding remarks.** For the general case of $A \in \mathbb{C}^{n \times n}$ with $\text{ind}(A) = k \geq 1$, and $B$ is an acute perturbation of $A$ with $\text{ind}(B) = j \geq 1$, Wei [31] conjectured that

$$\rho(BB^D - AA^D) < 1$$

is the acute perturbation of the Drazin inverse, if and only if $B$ satisfies condition $(C_s)$ [5], i.e.,

$$\mathcal{R}(B^j) \cap \mathcal{N}(A^k) = \{0\} \quad \text{and} \quad \mathcal{N}(B^j) \cap \mathcal{R}(A^k) = \{0\},$$

which is equivalent to the rank condition [5],

$$\text{rank}(B^j) = \text{rank}(A^k) = \text{rank}(A^k B^j A^k).$$

We only prove the correctness of the conjecture in a special case of $\text{ind}(B) = 1$ in this paper.

It will be our future research work for the general case of $\text{ind}(B) > 1$, which will be reported in a forthcoming paper.

Very recently, Ji and Wei [12] extend the notion of the Drazin inverse of a square matrix to an even-order square tensor with Einstein product. It will be very interesting to investigate the perturbation bounds for the Drazin inverse in the tensor case.

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