A family of scalar-Einstein and vacuum Brans-Dicke axisymmetric solutions

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Abstract

A new method is proposed, that establishes a one to one correspondence between the whole set of static axially symmetric vacuum GR solutions and a specific class of stationary axially symmetric scalar-Einstein ($R_{ab} = \partial_a \phi \partial_b \phi$) solutions having a given mass and a given angular momentum. The method explicitly takes advantage of the Kerr metric Ricci flatness. This also results in a class of stationary axially symmetric vacuum, i.e. Kerrlike, Brans-Dicke solutions. A particular solution, that is asymptotically flat, is more closely considered. It converges to Kerr for a vanishing scalar charge, but fails to converge to the Fisher-Janis-Newman-Winicour solution for a vanishing "rotation parameter". This solution exhibits a naked singularity having a ringlike structure.

I- Introduction

Obtaining exact solutions of a relativistic gravity theory is far from being an easy task, due to the highly non-linear character of the involved field equations. The most known solutions are Schwarzschild, Kerr, and Friedmann-Lemaître-Robertson-Walker (FLRW), that solve the general relativity (GR) field equation in vacuum for the two first ones. These three solutions (or family of solutions for FLRW) are of obvious usefulness in the astronomical framework. Some other exact GR solutions are known, that are also very useful for astronomical purposes [1][2][3]. On the other hand, an incredibly huge number of solutions, the usefulness of which is not obvious a priori, have been obtained so far [4]. However, despite the non immediate relevance of a solution, its usefulness should not be underestimated. Indeed, any solution may reveal some unexpected property of the considered theory. It may also happen that a solution only later turns out to be of genuine astronomical relevance: the emergence of the black hole (BH) concept from the Schwarzschild solution is probably the most obvious example.

Since many attempts to quantize gravity, and/or to unify gravity with other interactions, return a classical gravitational sector having not a GR, but a scalar-tensor (ST) structure [5][6][7], Brans-Dicke (BD) and ST gravity theories are considered as valuable alternatives to GR, despite the fact that the latter successfully passes solar system tests up to now [8]. (Let us also point out that many ST theories are driven to mimic the GR behaviour as a consequence of the cosmic expansion [9][10].) Thence the interest in ST theories, and specifically in BD, that
just involves a constant parameter $\omega$ instead of an arbitrary function $\omega(\Phi)$. In this context, looking for exact BD/ST solutions is particularly appropriate to point out relevant qualitative features with respect to GR. For instance, the spherical Brans class I solution [11][12] generally exhibits a naked singularity (NakS) or wormhole structure [13] (see also [14] for pioneering works on the detailed significance of the Class I, II, III and IV Brans solutions), such features being absent from the Schwarzschild GR solution. Besides, it has been recently shown that a particle orbiting a large $\omega$ Brans class I solution results in an observed (by a far observer) unbound orbital frequency, depending on how much the solution is scalarized [15]. One may then suspect striking qualitative differences in extreme mass ratio binaries BD/ST gravitational radiation, with respect to GR, since GR orbital frequencies cannot exceed the innermost circular orbital value. Nevertheless, let us remind that under some conditions (mainly regularity, asymptotic flatness and finite area horizon), vacuum and stationary BH like solutions are the same in BD/ST as in GR [16][17].

It is known from long that, by the means of a conformal transformation, any vacuum BD solution is associated to a massless scalar filled GR solution [18]. Thence seeking vacuum BD solutions can be reformulated as a scalar-GR problem. From the Hawking’s theorem [16], it is clear that any stationary axisymmetric (SAS) vacuum BD, but non Kerr, solution should exhibit a NakS structure, unless exhibiting some peculiar feature that allow it to evade the theorem (like being not asymptotically flat).

A method has been proposed by [19], that allows to generate a BD vacuum SAS solution from a GR vacuum SAS seed one, provided one is able to solve a given non linear PDE system. A class of electrovacuum BD solutions has been obtained by [20], that generalises the Majumdar-Papapetrou GR solution. The same authors later proposed a method that allows to generate SAS vacuum BD solutions from SAS vacuum GR ones (and also from static axisymmetric vacuum BD ones) [21], but the solutions then obtained are generally not asymptotically flat. The SAS, but only one coordinate dependent, case has been solved by [22]. The case of ST SAS solutions has been considered by [23], and explicit solutions are given in a few case of very specific ST (not BD) theories. Along the same lines as [19], it is claimed in [24] that any SAS vacuum BD solution can be obtained by non linearly combining any SAS vacuum GR solution and any vacuum solution of the Weyl class. A Kerr-type BD solution has been obtained in [25], that was built in such a way that it reduces to the Kerr solution in the $\omega \rightarrow \infty$ limit. Inspired by [20] and [22] works, a BD BH-like solution is proposed by [26], but this solution is not asymptotically flat. Motivated by particle collisions near a Kerr-like BD BH, another non asymptotically flat solution is obtained in [27], that is also derived using the [20] method. Starting from an unusual formulation of the Lewis metric, a two parameters extension of the method initiated in [21] allowed [28] to derive a BD version of the Ernst equation. The method is applied to some examples, but here again, the obtained spacetimes are generally not asymptotically flat. The matter filled case has been considered by [29] in self interacting (ie with a potential $V(\Phi)$) BD, but only static axisymmetric spacetimes were considered. Very recently, a way to generate new solutions starting from known ones has been proposed by [30]. The technique makes use of a BD symmetry in the traceless case ($g^{ab}T_{ab} = 0$).

It is worth reminding that a scalar-metric was proposed by [31] as an SAS vacuum BD solution. Its conformally related scalar-GR version is given in [32]. This metric, or its [32] form, has been used in several papers to characterize physics in a NakS (versus BH) field [32][33][34][35], and also to suggest rotating antiscalar solutions as alternatives to Kerr BHs.
However, the [31] scalar-metric is actually not a vacuum BD solution, as it is explicitly shown in [37]. The reason is that the authors of [31] applied, without justification, to a BD spherical vacuum solution (the Brans Class I) the Newman-Janis (NJ) algorithm, the authors (NJ) having just noticed it to be an unexpected way to recover Kerr from Schwarzschild [38]. Let us stress that, even in the GR framework, determining the ability of the NJ algorithm (or some equivalent reformulations) to generate new solutions from a seed one is not an easy task [39]. The NJ algorithm, as well as some modified versions, nevertheless received continuous interest, not only as a way to suggest new (generally non perfect fluid filled) solutions, but also in the context of other theories, like supergravity. See for instance [40] for a recent review. Related to the NJ finding [38], let us also mention that another (but fully justified) way to derive Kerr from Schwarzschild was obtained in [41].

In this paper, a new one to one correspondence between static axially symmetric vacuum GR solutions and SAS massless scalar GR solutions, with metrics having some prior form, is established. This prior form is inspired from the metric found in [27], referred to as the SB solution in the following. It is defined as a modification of the Kerr metric, with given \((m, a)\) parameters, by inserting two unknown metric functions in a suitable manner. Thence these functions are demanded to be such that the scalar-Einstein field equations are satisfied. Thence, it uses Kerr as a seed in some sense, but in a way that differs from the generating techniques previously reviewed. The definition of some well suited \((\alpha, \beta)\) coordinates then allows to establish the correspondence. Let us stress that, unlike the NJ algorithm, the method is completely justified since it involves an explicit solving of the relevant field equations. The SB solution is recovered as a special case, and other explicit solutions are built, one of them being asymptotically flat. In some sense, the introduction of the \((\alpha, \beta)\) coordinates cures the problem of solving the non linear PDE system obtained in [19], for the considered prior metric form.

Outline of the paper

The prior form of the metric and the related index notations are defined in section II. The first half of the Einstein equations is considered in II-1. The Klein-Gordon (KG) equation is used in II-2, that suggests the definition of the \((\alpha (r, \theta), \beta (r, \theta))\) coordinates. In II-3, the second half of the Einstein equations is considered. The correspondence with the general static axisymmetric vacuum GR case is explicit in II-4. A particular asymptotically flat solution is then considered in section III. The links with the BD theory is discussed in section IV, while section V is dedicated to a brief conclusion.

II- The considered set of axisymmetric metrics

We consider in this paper metrics having the form

\[
\begin{align*}
 g_{pq} &= e^{A}k_{pq} \\
 g_{uv} &= e^{B}k_{uv}
\end{align*}
\]  

where \(k_{ab}\) is the Kerr metric

\[
\begin{pmatrix}
 k_{00} & k_{03} & 0 & 0 \\
 k_{03} & k_{33} & 0 & 0 \\
 0 & 0 & k_{11} & 0 \\
 0 & 0 & 0 & k_{22}
\end{pmatrix}
= 
\begin{pmatrix}
 -V & -w(1-V) & 0 & 0 \\
 -w(1-V) & 2w^2 - w^2 V + \Sigma \sin^2 \theta & 0 & 0 \\
 0 & 0 & 0 & \Sigma \\
 0 & 0 & 0 & \Sigma
\end{pmatrix}
\]
in Boyer-Lindquist coordinates, and where \( A \) and \( B \) are \((r, \theta)\) dependent functions. One has introduced the usual quantities

\[
\begin{align*}
  w(\theta) &= a \sin^2 \theta \\
  \Sigma(r, \theta) &= r^2 + a^2 \cos^2 \theta \\
  \Delta(r) &= r^2 - 2mr + a^2 \\
  V(r, \theta) &= 1 - \frac{2mr}{\Sigma}.
\end{align*}
\]

Besides the usual index convention \((x^0, x^1, x^2, x^3) = (t, r, \theta, \phi)\), let us also make the convention \((p, q, r, s)\) indexes \(\in \{0, 3\}\) while the \((a, b, c, d, e)\) indexes take the four spacetime values. The ordering in (0.2) explicits the block diagonal structure of Kerr’s metric. The metric (0.1a-0.1b) is also block diagonal, each block being ”conformally” related to the Kerr corresponding one, but with different ”conformal” factors. The requirement (0.1a-0.1b) implicitly means imposing two prior relations between the four metric functions describing the general form of an SAS metric (see for instance eq. (1) of [42]).

The metric (0.1a-0.1b) is required to solve the scalar-Einstein equation

\[
R_{ab} = \partial_a \varphi \partial_b \varphi
\]

where \( \varphi \) is an \((r, \theta)\) dependent scalar field. The special case \( A = B = 0 \) solves (0.4) for \( \varphi = 0 \), since the Kerr metric (0.2) is Ricci flat.

The SB solution [27] corresponds to \( A = 0 \) and \( e^B = (\Delta \sin^2 \theta)^{\frac{1}{2}} \). Of course it should be recovered as a special solution of (0.4) with the corresponding scalar, that reads \( \varphi_{SB} = \sqrt{\frac{2}{\Xi}} \ln (\Delta \sin^2 \theta) \). This will be checked later.

From the \((r, \theta)\) dependence of the considered scalar-metric, one has \( \partial_p (A, B, \varphi) = 0 \). The non zero connexion components then read

\[
\begin{align*}
  \Gamma^p_{qu} &= K^p_{qu} + \frac{1}{2} \delta^p_q \partial_u A \\
  \Gamma^u_{pq} &= e^{A-B} \left( K^u_{pq} - \frac{1}{2} k^{ux}_{pq} k_{ux} \partial_x A \right) \\
  \Gamma^u_{vw} &= K^u_{vw} + \gamma^u_{vw}
\end{align*}
\]

where the \( K^c_{ab} \) quantities are the Kerr connexion components

\[
\begin{align*}
  K^p_{qu} &= \frac{1}{2} k^{pr} \partial_u k_{qr} \\
  K^u_{pq} &= -\frac{1}{2} k^{ux} \partial_u k_{pq} \\
  K^w_{vw} &= \frac{1}{2} k^{ux} (\partial_v k_{wx} + \partial_w k_{vx} - \partial_x k_{vw})
\end{align*}
\]
and where one has defined
\[
\gamma^u_{vw} = \frac{1}{2} \left( \delta^u_v \partial_w B + \delta^u_w \partial_v B - k_{vw} k^u_z \partial_z B \right). \tag{0.7}
\]

For convenience, let us introduce the following notation
\[
(c, s) = (\cos \theta, \sin \theta). \tag{0.8}
\]

II-1- The \((pq)\) Einstein equation components

From (0.4), one has \(R_{pq} = 0\), that writes
\[
\frac{1}{\sqrt{-g}} \partial_w \left( \sqrt{-g} \Gamma^w_{pq} \right) - \Gamma^d_{pc} \Gamma^e_{qd} = 0. \tag{0.9}
\]

Using (0.5a), (0.5b), but also \(R_{pq} (k_{ab}) = 0\) (Kerr’s metric being Ricci flat) and \(\partial_x k_{pq} = K^r_{pq} k_{pr} + K^r_{pq} k_{qr}\) (from the Ricci identity on Kerr’s metric), (0.9) yields an equation that is only \(A\) dependent
\[
2K^w_{pq} \partial_w A - k_{pq} \left[ k^{wx} \partial_w A \partial_x A + \frac{1}{\sqrt{-k}} \partial_w \left( \sqrt{-k} k^{wx} \partial_x A \right) \right] = 0. \tag{0.10}
\]

Eliminating the bracket term thanks to the contraction by \(k^{pq}\), one obtains (since \(k^{pq} k_{pq} = 2\))
\[
(2K^w_{pq} - k_{pq} k^{rs} K^w_{rs}) \partial_w A = 0. \tag{0.11}
\]

Using the Kerr metric (0.2) and the connexion components (0.5b), (0.11) yields
\[
k^{wx} \partial_x \left( \frac{k_{pq}}{\sqrt{k}} \right) \partial_w A = 0. \tag{0.12}
\]

This equation has to be satisfied by both \(k_{00}\), \(k_{03}\) and \(k_{33}\). Writing out the two equations for \(k_{00}\) and \(k_{03}\), one obtains two homogeneous equations on \(\partial_1 A\) and \(\partial_2 A\). One then shows that having a non trivial solution requires \(a = 0\). Thence, \(A\) is necessarily constant for \(a \neq 0\). One can then set \(A = 0\) by redefining \(B\) and the \(ds^2\) units.

The fact that \(g_{pq} = k_{pq}\) has two straightforward consequences: (1) the equatorial (\(\theta = \pi/2\)) circular orbits and their linear planar stability (both do not involve \(g_{11}\)) are obtained solving the same equations as the Kerr’s case, and (2) the horizon and ergosphere are the “same” as Kerr’s
\[
\begin{align*}
    r_h &= m + \sqrt{m^2 - a^2} \quad \text{(0.13)} \\
    r_e (\theta) &= m + \sqrt{m^2 - a^2 c^2}. \quad \text{(0.14)}
\end{align*}
\]

To be precise, let us point out that these claims just concern the functions that describe these orbits and surfaces in terms of \(\theta\) and of the \((m, a)\) parameters. Indeed, the metric components \(g_{uv}\) enter their geometric and relative properties. For instance, the geometric radial distance between two circular orbits having circumferences \(C\) and \(C'\) depends on \(g_{11}\).

Since \(A = 0\), the metric (0.1a, 0.1b) achieves the form
\[
ds^2 = k_{00} dt^2 + 2k_{03} dt d\phi + k_{33} d\phi^2 + e^B (k_{11} dr^2 + k_{22} d\theta^2). \tag{0.15}
\]
II-2- The Klein-Gordon equation

To pursue the integration, it would be sufficient to solve the \((uv)\) components of \((0.4)\), since the KG equation is a direct consequence of \((0.4)\). It is nevertheless useful to write out the KG equation

\[
\partial_a \left( \sqrt{-g} g^{ab} \partial_b \phi \right) = 0.
\]  

(0.16)

Using \((0.15)\), it reads

\[
\partial_1 (s \Delta \partial_1 \phi) + \partial_2 (s \partial_2 \phi) = 0.
\]

(0.17)

This form suggests defining the alternative radial coordinate

\[
\rho \equiv \ln \left( \frac{r - m + \sqrt{\Delta}}{\sqrt{m^2 - a^2}} \right)
\]

(0.18)

that yields

\[
\partial_\rho = \sqrt{\Delta} \partial_1
\]

(0.19a)

\[
r - m = \sqrt{m^2 - a^2} \cosh \rho
\]

(0.19b)

\[
\Delta = (m^2 - a^2) \sinh^2 \rho.
\]

(0.19c)

This allows rewriting \((0.17)\) in the form

\[
\partial_\rho (sS \partial_\rho \phi) + \partial_2 (sS \partial_2 \phi) = 0.
\]

(0.20)

For convenience, one has introduced the following notation

\[(C, S) = (\cosh \rho, \sinh \rho).\]

(0.21)

One sees that this form returns two obvious solutions

\[
\phi_{SB} = \Lambda \ln (Ss)
\]

(0.22a)

\[
\phi_2 = \Lambda C c
\]

(0.22b)

where \(\Lambda\) is any constant, and \(\phi_{SB}\) the scalar entering the SB solution [27]. The form of these two solutions suggests defining new \((\alpha, \beta)\) coordinates by

\[(\alpha, \beta) = (Ss, Cc)\].

(0.23)

In terms of the initial Boyer-Lindquist coordinates, these coordinates read

\[
\alpha = \frac{\sqrt{r^2 - 2mr + a^2}}{\sqrt{m^2 - a^2}} \sin \theta
\]

(0.24a)

\[
\beta = \frac{r - m}{\sqrt{m^2 - a^2}} \cos \theta
\]

(0.24b)
and behave as \((r \sin \theta, r \cos \theta)\) for \(r \rightarrow \infty\), up to the \((m^2 - a^2)^{-1/2}\) factor. From (0.28) and the trigonometric identities \(c^2 + s^2 = 1\) and \(C^2 - S^2 = 1\), one obtains the following relations

\[
\left[(Cs)^2 + (Sc)^2\right] \partial_\alpha (C, S) = Cs(S, C) \tag{0.25}
\]

\[
\left[(Cs)^2 + (Sc)^2\right] \partial_\beta (C, S) = Sc(S, C) \tag{0.26}
\]

that turn out to be useful in the calculations to do. The derivation operators transform as

\[
\partial_\rho = Cs \partial_\alpha + Sc \partial_\beta \tag{0.26}
\]

Reinserting in (0.20) returns, after a lengthy calculation, and using (0.25), the KG equation in the nice form

\[
\partial_\alpha (\alpha \partial_\alpha \phi) + \partial_\beta (\partial_\beta \phi) = 0. \tag{0.27}
\]

This form points out two other obvious solutions, besides (0.22a) and (0.22b)

\[
\phi_3 = \Lambda \beta \ln \alpha \tag{0.28a}
\]

\[
\phi_N = \frac{\Lambda}{\sqrt{\alpha^2 + \beta^2}}. \tag{0.28b}
\]

The solution (0.28a) is nothing but the product of (0.22a) and (0.22b), that turns out to be a solution too. The solution (0.28b) results from the fact that (0.27) is the classical Laplacian equation written in cylindrical coordinates, in the case of an axisymmetric potential. For this reason, we will refer to (0.28b) as being the Newtonian scalar.

It may be worth spotting that if \(\phi\) solves (0.27), so do its successive derivatives with respect to \(\beta\). (Remark that \(\phi_{SB} = \partial_\beta \phi_3\).) Along these lines, new solutions can also be obtained by integration with respect to \(\beta\). For instance

\[
\phi_5 = \Lambda \ln \left(\beta + \sqrt{\alpha^2 + \beta^2}\right) \tag{0.29a}
\]

\[
\phi_6 = \Lambda \left[\beta \ln \left(\beta + \sqrt{\alpha^2 + \beta^2}\right) - \sqrt{\alpha^2 + \beta^2}\right] \tag{0.29b}
\]

also solve (0.27), and are related to \(\phi_N\) by \(\phi_N = \partial_\beta \phi_5 = \partial_\beta \partial_\beta \phi_6\).

See the appendix for a quicker demonstration of (0.27), using the \((\alpha, \beta)\) coordinates.

II-3- The \((uv)\) Einstein equation components

From (0.4), one has \(R_{uv} = \partial_u \phi \partial_v \phi\), that writes

\[
\frac{1}{\sqrt{-g}} \partial_v \left(\sqrt{-g} G^{uv}\right) - \partial_u \partial_v \ln \sqrt{-g} - \Gamma^u_{vp} \Gamma^p_v - \Gamma^v_{ux} \Gamma^x_v = \partial_u \phi \partial_v \phi. \tag{0.30}
\]
Using (0.5a), (0.5c) and (0.7), with $A = 0$, but also $R_{uv}(k_{ab}) = 0$ (Kerr’s metric is Ricci flat) and $\partial_y k_{uv} = k_{vy} k_{cv} + k_{vy} k_{ux}$ (from the Ricci identity on Kerr’s metric), (0.30) yields

$$K_{uv}^p \partial_u B + K_{up}^v \partial_v B = \frac{1}{\sqrt{-k}} k_{uv} \partial_w \left( \sqrt{-k} k_{wz} \partial_z B \right) = 2 \partial_u \varphi \partial_v \varphi. \quad (0.31)$$

Expliciting the (11), (12) and (22) components yields, using the $\rho$ radial coordinate and (0.19a)

$$Cs \partial_\rho B - Ss \partial_\rho \partial_\rho B - So (s \partial_\rho B) = 2SS (\partial_\rho \varphi)^2 \quad (0.32a)$$

$$Sc \partial_\rho B + Cs \partial_\rho B = 2SS \partial_\rho \varphi \partial_\rho \varphi \quad (0.32b)$$

$$2Sc \partial_\rho B - So (s \partial_\rho B) - Ss \partial_\rho \partial_\rho B - Cs \partial_\rho B = 2SS (\partial_\rho \varphi)^2. \quad (0.32c)$$

Let us now rewrite these equations in $(\alpha, \beta)$ coordinates. Combining the equation (0.32b), and the difference of (0.32a) and (0.32c), returns

$$\partial_\alpha B = \alpha \left[ (\partial_\alpha \varphi)^2 - (\partial_\beta \varphi)^2 \right] \quad (0.33a)$$

$$\partial_\beta B = 2 \alpha \partial_\alpha \varphi \partial_\beta \varphi. \quad (0.33b)$$

It appears that the integrability condition of (0.33a) and (0.33b) is ensured by the KG equation (0.27). An $(uv)$ equation remains to be written, that can be built from the sum of (0.32a) and (0.32c). It turns out that this equation is solved thanks to (0.27).

See the appendix for a quicker demonstration of these results, using the $(\alpha, \beta)$ coordinates from the start.

II-4- Generating solutions

From the previous sections, a $(\varphi, B)$ solution can be obtained by (1) solving first the KG equation (0.27), and then (2) integrating the system (0.33a)-(0.33b). There is then a direct correspondence with the issue of seeking a static axisymmetric solution of vacuum GR. Indeed, such a metric can be written

$$ds^2 = -e^{2U} dt^2 + e^{-2U} \left[ e^{2\gamma} (d\rho^2 + dz^2) + \rho^2 d\phi^2 \right] \quad (0.34)$$

where the metric functions $U(\rho, z)$ and $\gamma(\rho, z)$ have to solve

$$\partial_\rho (\rho \partial_\rho U) + \rho \partial_\gamma \partial_\rho U = 0 \quad (0.35a)$$

$$\partial_\rho \gamma = \rho \left[ (\partial_\rho U)^2 - (\partial_\gamma U)^2 \right] \quad (0.35b)$$

$$\partial_\gamma = 2 \rho \partial_\rho U \partial_\rho U. \quad (0.35c)$$

Thence, any $(U(\rho, z), \gamma(\rho, z))$ solution of (0.35a)-(0.35c) immediately results in a $(\varphi(\alpha, \beta), B(\alpha, \beta))$ solution of the problem considered in this paper, by just making the changes $(\rho, z) \to (\alpha, \beta)$ and $(U, \gamma) \to (\varphi, B)$. There is then a one to one correspondence between these two problems. Let us stress that the $(\alpha, \beta)$ definition depends on the Kerr’s mass and angular momentum, in such a way that the correspondence works for any $(m, a)$ parameters. Note that while both problems are GR issues, the former corresponds to a static field in a vacuum spacetime, while the latter to a rotating field in a not vacuum (but massless scalar filled) spacetime.
Considering the four first solutions of the KG equation obtained in II-2, one obtains

\[ \varphi_{SB} = \Lambda \ln \alpha \quad \Rightarrow \quad B_{SB} = \Lambda^2 \ln \alpha \]  
(0.36a)

\[ \varphi_2 = \Lambda \beta \quad \Rightarrow \quad B_2 = -\frac{\Lambda^2 \alpha^2}{2} \]  
(0.36b)

\[ \varphi_3 = \Lambda \beta \ln \alpha \quad \Rightarrow \quad B_3 = \Lambda^2 \left( \beta^2 \ln \alpha - \frac{1}{4} \alpha^2 \left[ 1 - 2 \ln \alpha + 2 \left( \ln \alpha \right)^2 \right] \right) \]  
(0.36c)

\[ \varphi_N = \frac{\Lambda}{\sqrt{\alpha^2 + \beta^2}} \quad \Rightarrow \quad B_N = -\frac{\Lambda^2 \alpha^2}{2 \left( \alpha^2 + \beta^2 \right)^2} \]  
(0.36d)

The \((\varphi_{SB}, B_{SB})\) solution is the SB solution \([27]\). It is clear from \((0.24a)-(0.24b)\) and \((0.15)\) that the SB, the \((\varphi_2, B_2)\) and the \((\varphi_3, B_3)\) solutions are not asymptotically flat, a point that makes their astrophysical usefulness debatable. On the other hand, the Newtonian solution \((\varphi_N, B_N)\) is asymptotically flat.

From the previously pointed out correspondence, the SB solution is associated to the Minkowski spacetime (in some non cartesian coordinates), while the Newtonian one is associated to the Curzon-Chazy spacetime \([2]\).

It is worth also having a look on the fifth solution of the KG equation obtained in II-2. Integrating for \(B_5\) yields

\[ B_5 = 2\Lambda^2 \ln \left( 1 + \frac{\beta}{\sqrt{\alpha^2 + \beta^2}} \right) \]  
(0.37)

ie, in terms of \((r, \theta)\) coordinates, using \((0.23)\)

\[ e^{B_5} = \left( 1 + \frac{(r - m)c}{\sqrt{r^2 - 2mr + a^2 + (m^2 - a^2)c^2}} \right)^{2\Lambda^2} \]  
(0.38)

This, with \((0.15)\), suggests that the spacetime is not asymptotically flat. From the previous correspondence, the \((\varphi_5, B_5)\) solution is associated to the Gautreau-Hoffman spacetime \([2]\).

The \((\varphi_6, B_6)\) solution is obviously not asymptotically flat. Indeed, \((0.4)\) shows that it is even not Ricci flat at infinity, since \(\partial_\alpha \varphi_6 = -\Lambda \alpha \left( \beta + \sqrt{\alpha^2 + \beta^2} \right)^{-1}\) and \(\partial_\beta \varphi_6 = \varphi_5 = \Lambda \ln \left( \beta + \sqrt{\alpha^2 + \beta^2} \right)\) do not vanish at infinity.

III- The Newtonian solution

Since it is asymptotically flat, let us have a closer look on the Newtonian solution. Its metric \((0.36d)\) explicitly reads

\[ ds_N^2 = g_{00} dt^2 + 2 k_{03} dt d\phi + k_{33} d\phi^2 + \exp \left( -\frac{\Lambda^2 \left( m^2 - a^2 \right) \left( r^2 - 2mr + a^2 \right) s^2}{2 \left[ r^2 - 2mr + a^2 + (m^2 - a^2)c^2 \right]^2} \right) \left( k_{11} dr^2 + k_{22} d\theta^2 \right) . \]  
(0.39)

The \(g_{00}\) and \(g_{03}\) metric tensor components being identical to Kerr, the \(m\) and \(a\) constants have the same ADM mass and angular momentum meanings. As already mentionned, the Kerr
horizon (0.13) and ergosphere (0.14) surfaces are recovered, since they just depend on the $g_{pq}$ metric components, which are the same as Kerr.

The scalar associated to (0.39) reads, in terms of $(r, \theta)$ coordinates

$$\varphi_N = \frac{\Lambda \sqrt{m^2 - a^2}}{\sqrt{r^2 - 2mr + (m^2 - a^2)c^2}}.$$ 

(0.40)

The explicit $\theta$ dependence of $\varphi_N$ shows that the (0.39)-(0.40) solution is not the BD-Kerr solution obtained in [19] (equations (24)-(26) of [19]).

III-1- Singularities

One knows that the Kerr metric is singularity free outside its (external) horizon (0.13), and that the horizon itself is a regular surface. However, from the Hawking theorem [16], this should not be true for the spacetime (0.39) since $\partial \varphi_N \neq 0$ while (1) it is asymptotically flat and, (2) its horizon surface is finite. Indeed, the scalar curvature reads, from (0.4)

$$R = g^{uv} \partial_u \varphi_N \partial_v \varphi_N$$

(0.41)

which, from (0.36d), yields

$$R = \frac{16\Lambda^2}{\Sigma} \frac{C^2 S^2 + c^2 s^2}{(S^2 s^2 + C^2 c^2)^2} \exp \left( \frac{\Lambda^2 S^2 s^2}{2(S^2 s^2 + C^2 c^2)^2} \right).$$

(0.42)

One sees that (0.42) diverges whatever the way $s^2 S^2 + c^2 C^2 \to 0$, i.e. the way $(c, S) \to (0, 0)$ (and only in this case for $r \geq r_h$). The points having $\theta = \pi/2$ and $\rho = 0$ are then NakS points. Thence, the horizon is not regular everywhere, since its equator $\theta = \pi/2$, and the equator only (considering points having $r \geq r_h$), is scalar curvature singular.

However, the fact that the scalar curvature does not diverge on the points having $(r \geq r_h, \theta) \neq (r_h, \pi/2)$ is not sufficient to prove that none of these points is singular. A quantity that is often regarded as a reliable singularity indicator is the Kretschmann scalar

$$\tilde{K} \equiv R_{abcd} R^{abcd}$$

(0.43)

where $R_{abcd}$ is the Riemann-Christoffel curvature tensor. From (0.5a)-(0.5d), one finds that

$$R_{pqrs} = e^{-B} Q_{pqrs}$$
$$R_{pquv} = Q_{pquv}$$
$$R_{pauv} = Q_{pauv} + q_{pauv}$$
$$R_{uwuv} = e^B (Q_{uwuv} + q_{uwuv})$$
$$R_{pqru} = R_{pquv} = 0$$

where $Q_{abcd}$ is the Kerr’s metric Riemann-Christoffel curvature tensor, and

$$q_{pauv} = k_{pu} K^{r}_{qw} \gamma_{uw}$$
$$q_{uwuv} = k_{uy} \left( \partial_w \gamma_{ux} - \partial_x \gamma_{uw} + K^z_{wx} \gamma_w^z + K^y_{wx} \gamma_w^y - K^z_{wx} \gamma_z^w - K^y_{wx} \gamma_z^w + \gamma_w^z \gamma_x^y - \gamma_x^z \gamma_w^y \right).$$

(0.45)
If $B = 0$, which implies $\gamma^v_w = 0$ from (0.7), $\hat{K}$ is finite since the Kerr’s metric is regular in the considered region. Any divergence of $\hat{K}$ can then only appear from the $\gamma^v_w$ and $\partial \gamma^v_w$, ie from the $\partial B_N$ and $\partial \partial B_N$, quantities entering the $q_{abcd}$ terms in (0.44). It is then easy to see from (0.36d) that a divergence of $\hat{K}$ can only occur at points where $c = S = 0$. The set of Kretschmann singularity points is then included in the scalar curvature points set. This strongly suggests that the horizon is regular at points not belonging to the equatorial NakS ring $(r, \theta) = (r_h, \pi/2)$. Let us stress that it was recently found by [43] that the (not asymptotically flat) SB solution, ie (0.36a), also exhibits such a ringlike NakS structure.

### III-2. The $a = 0$ subcase (static case)

Let us now specify the Newtonian solution (0.36d) to the $a = 0$ case. The vacuum Kerr’s solution returns the (vacuum) spherical Schwarzschild solution for $a = 0$. In the non vacuum, but massless scalar filled, case, a spherical solution is known, often named the JNW metric, referring to the Janis-Newman-Winicour 1968’s paper [44]. It is worth to point out that this solution was in fact earlier discovered by Fisher in his 1948 paper [45], but using an areal radial coordinate. It seems then fair to use the FJNW acronym when referring to this solution, and so will I do in this paper. The FJNW solution includes Schwarzschild as a limit (non scalarized) case. It is then natural to suspect the existence of a massless scalar filled GR solution, that would depend on a "rotation parameter" $a$, and that would return (1) the Kerr spacetime for a vanishing scalar, and (2) the FJNW solution for $a = 0$.

The solution (0.36d) indeed returns the Kerr spacetime for a vanishing scalar, ie for $\Lambda = 0$. On the other hand, making $a = 0$ in (0.39) does not return the FJNW solution, but

$$
\begin{align*}
 ds^2 &= - \left(1 - \frac{2m}{r}\right) dt^2 + \exp \left( - \frac{\Lambda^2 m^2 \left(r^2 - 2mr + m^2 c^2\right)}{2 \left(r^2 - 2mr + m^2 c^2\right)^2} \right) \left[ 1 - \frac{2m}{r} \right]^{-1} dr^2 + r^2 d\theta^2 + r^2 s^2 d\phi^2. 
\end{align*}
$$

(0.46)

This static but non spherical solution is known from long, see equations (17)-(18) of Penney’s paper [46]. From (0.40), the scalar field reads

$$
\varphi = \frac{m\Lambda}{\sqrt{r^2 - 2mr + m^2 c^2}}. 
$$

(0.47)

The metric (0.46) has an horizon, that reads

$$
r = 2m 
$$

(0.48)

and whose equator’s points $\theta = \pi/2$, and only these points, are singular, as it can be directly checked from the $(S^2 s^2 + C^2 c^2)$ expression entering (0.42), that reads for $a = 0$

$$
S^2 s^2 + C^2 c^2 = \left(\frac{r}{m} - 1\right)^2 - s^2. 
$$

(0.49)

1 Extending to the $r < r_h$ region, one sees on (0.42) that $\Sigma = 0$ is a scalar curvature singularity of the considered spacetime (it is a singularity, but not a scalar curvature one, in Kerr’s spacetime). The spacetime has then two ring singularities: one is naked, the second being hidden (the latter being the counterpart of the usual Kerr’s, in some sense).
This agrees with the fact that the metric, while static since $g_{00} = 0$, is not spherical because of the presence of the \( \theta \) dependent exponential in front of the bracket in (0.46). Incidentally, this also shows that (0.46) can not be FJNW in disguise. The ring character of the NakS, stuck on the horizon, is obviously a property inherited from the general \( a \neq 0 \) case.

The fact that the FJNW metric does not appear as a subcase of (0.46) means that (0.39) can not be interpreted as a "rotating version of FJNW". It also strongly suggests the existence of asymptotically flat Kerr like solutions of (0.4), that do not fulfil (0.1a). Indeed, recovering FJNW as a rotationless limit case, with its properties ([15]), is incompatible with the \( A = 0 \) conclusion obtained in II-1.

III-3- Orbits in the equatorial plane of the static solution

As already mentioned, the presence of the scalar field affects neither the existence condition of equatorial circular orbits nor their linear stability (using the radial coordinate defined by the form of the metric (0.15)). The situation is very similar to the case of a scalarized version of the \( \gamma \)-metric reported in [47]. However, this does not mean that the physics is unaffected by the scalar, even in the equatorial plane. The aim of this subsection is to illustrate this point. For convenience, we specify to the static case, that makes all the calculations easily tractable.

In the \( \theta = \pi/2 \) plane, the metric (0.46) simplifies into

\[
\text{ds}^2 = - \left( 1 - \frac{2m}{r} \right) dt^2 + \exp \left( - \frac{\Lambda^2 m^2}{2 (r^2 - 2mr)} \right) \left( 1 - \frac{2m}{r} \right)^{-1} dr^2 + r^2 d\phi^2. \tag{0.50}
\]

One has already pointed out that the exponential term in \( g_{11} \) impacts the distance between close circular orbits. The effect is mostly important near the \( r = 2m \) NakS, where the distance between two close orbits of radii \( r \) and \( r + dr \) goes to zero, because of the vanishing of the exponential factor. Related to this, the (coordinate) time needed for a photon to radially propagate from \( r = 2m \) to an external observer at coordinate \( r_{\text{obs}} \), that reads

\[
\Delta t (r = 2m \to r_{\text{obs}}) = \int_{2m}^{r_{\text{obs}}} \frac{r dr}{r - 2m} \exp \left( - \frac{\Lambda^2 m^2}{4 (r^2 - 2mr)} \right), \tag{0.51}
\]

converges for \( \Lambda \neq 0 \), unlike what happens in the Schwarzschild case. On the other hand, the convergence does not occur for a radial photon \( (\theta = 0 \text{ or } \pi) \), since the argument of the exponential term in (0.46) cancels then. More generally, the convergence does not occur for any photon crossing the Penney’s \( r = 2m \) sphere at any non equatorial point, since then \( e \neq 0 \), so that the exponential goes to 1 when \( r \to 2m \). However, let us also spot that despite the convergence of (0.51), the far observed behaviour of a clock at rest close to the Penney’s sphere is frozen, even in the equatorial plane, since \( g_{00} (r = 2m) = 0 \) for any \( \theta \), the \( \pi/2 \) case included. The consequence is that in the equatorial plane, while circular close to the NakS (non geodesic) motions are frozen (the areal NakS circumference being \( 4\pi m \)), radial infallings are not.

Let us point out that the same observations, ie (1) (coordinate) time convergence for a radial photon propagating from any NakS point, but (2) frozen rest clock behaviour near any NakS point, also occurs in the FJNW spherical metric [44][45]. This is obvious from the isotropic form of this metric [15]

\[
\text{ds}^2 = - \left( \frac{r - k}{r + k} \right)^{2\lambda} dt^2 + \left( \frac{r + k}{r} \right)^4 \left( \frac{r - k}{r + k} \right)^{2-2\lambda} (dr^2 + r^2 d\Omega^2) \tag{0.52}
\]
where \( \lambda \in [0, 1] \) (\( \lambda = 1 \) corresponding to the GR Schwarzschild metric). At least for these two metrics, the time convergence for a photon reaching the horizon only concerns orbits approaching points belonging to the NakS location.

### IV- Brans-Dicke vacuum solutions

It is well-known [48] that in a four-dimensional spacetime, the conformal transformation

\[
g_{ab} = \Phi g_{ab} \tag{0.53}
\]
yields, for any constant \( \omega \) (supposed to be \( > -3/2 \))

\[
\int \left[ \Phi R - \frac{\omega}{\Phi} (\partial \Phi)^2 \right] \sqrt{-g} d^4x = \int \left( R - \frac{1}{2} (\partial \varphi)^2 \right) \sqrt{-g} d^4x \tag{0.54}
\]

where \( \Phi \) is any positive scalar function and

\[
\varphi = \sqrt{2\omega + 3} \ln \Phi. \tag{0.55}
\]

This means that the vacuum BD action of the BD gravitational field \((\Phi, g_{ab})\) identifies with the GR action, with gravitational field \(g_{ab}\), but filled by the (matter source) massless scalar \(\varphi\). Thence, any solution of (0.4) is conformally associated to a vacuum BD solution [18]. This \((\omega)BD\) solution reads

\[
\Phi = \exp \left( \frac{\varphi}{\sqrt{2\omega + 3}} \right) \tag{0.56a}
\]

\[
\bar{g}_{ab} = \exp \left( -\frac{\varphi}{\sqrt{2\omega + 3}} \right) g_{ab}. \tag{0.56b}
\]

It is then possible to built a vacuum SAS BD solution from any vacuum static axisymmetric GR solution, using first the correspondance reported in II-4. Let us mention that the Kerr-like BD solution built this way from the Newtonian solution (0.36d), or (0.39)-(0.40), is also asymptotically flat, since \(\varphi_N\) vanishes in far regions.

Experiments strongly constrain BD/ST theories to satisfy \(\omega_0 > 4.10^5\), where \(\omega_0\) is the BD parameter, or the present value of \(\omega(\Phi)\) in the ST case [8]. In such circumstances, vacuum BD gravity, and also to some extent ST gravity, is asymptotically equivalent to massless scalar filled GR [15][28][49]. In other words, the solutions obtained solving (0.27) and (0.33a)-(0.33b) can directly serve as vacuum BD/ST solutions in the large \(\omega\) case, without having to explicitly consider the conformal correspondance (0.56a)-(0.56b). Linked to this, let us remind that for any scalar function \(\varphi\) chosen "independently on \(\omega\)”, (0.56a) yields

\[
\Phi = 1 + \frac{\varphi}{\sqrt{2\omega}} + O \left( \frac{1}{\omega} \right). \tag{0.57}
\]

Despite that \(\Phi\) goes to a constant value, the \(\frac{\partial_x \Phi \partial_y \Phi}{\sqrt{2\omega}}\) term entering the full BD equation does then not vanish in the large \(\omega\) limit, but results in a \(\partial_x \varphi \partial_y \varphi\) contribution. This is coherent with

\(^2\)Let us point out that in the FJNW case, the fact that close to the NakS rest clocks are frozen does not mean that circular motions are frozen, since the NakS circumference is zero. Indeed, close to NakS circular geodesics (that exist for \(\lambda < 1/2\)) return a divergent far observed frequency [15].
the fact that a vacuum BD solution does not reduce to a GR vacuum solution (that would have been Kerr in the SAS case) in the \( \omega \to \infty \) limit [49].

Let us remark that while the massive geodesics are not the same in the scalar-Einstein and in the corresponding BD solutions, they are asymptotically identical in the \( \omega \to \infty \) limit, since the conformal factor is constant in this limit.

V- Conclusion and outlook

One has established a non trivial one to one correspondence that allows to built an SAS massless scalar-GR solution from any static axisymmetric vacuum GR solution. While this in turn also results in a way to obtain an SAS vacuum BD solution, one has pointed out that the so obtained massless scalar-GR solutions can also directly serve as a (large \( \omega \)) vacuum BD/ST solutions for practical purposes.

It was claimed in [25] that a Kerr-like BD solution should: (1) only depend on the three \((m, a, \omega)\) parameters, (2) go to Kerr spacetime in the \(\omega \to \infty\) limit, and (3) return Schwarzschild’s spacetime for \(a = 0\). From the results obtained in this paper, it seems that the truth is by far more complex. The solutions explicitly presented in this paper can be interpreted as limit of BD solutions for \(\omega \to \infty\), but they differ from Kerr spacetime. The particular case (0.46) of the Newtonian solution has \(a = 0\) but differs from Schwarzschild. The fact that the Newtonian solution (0.39), for instance, depends on a parameter \(\Lambda\), besides \((m, a)\), shows that the \((\omega)BD\) solution built from it using (0.56a)-(0.56b) results in a family of Kerr-like solutions that depends on more than the three \((m, a, \omega)\) parameters.

Let us mention that a lot of authors explicitly or implicitly still suppose that (2) is true (see for instance [26] or [27]). Such a presupposition results in a drastic impoverishment of the (large \(\omega\)) ST potential predictions. Among these the drastically different behaviour of far observed orbital frequencies in the spherical case with respect to GR [15]. It would be of interest to know whether such a different from GR behaviour could also occur in the rotating case. This requires exploring more general SAS solutions of (0.4) with \(g_{pq} \neq k_{pq}\), which means ruling out the restrictive hypothesis (0.1a).

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Appendix: calculations using the \((\alpha, \beta)\) coordinates

Using (0.18) and (0.23), one obtains, in the \((\tilde{x}^a) \equiv (t, \alpha, \beta, \phi)\) coordinates

\[
ds^2 = g_{ab} \tilde{x}^a d\tilde{x}^b = k_{00} dt^2 + 2k_{03} dt d\phi + k_{33} d\phi^2 + G (d\alpha^2 + d\beta^2)
\]

(0.58)

where

\[
G = \frac{\Sigma e^B}{S^2 c^2 + C^2 s^2}.
\]

(0.59)

From (0.23) and the trigonometric identities, one could explicitly obtain \((c, s, C, S)\) in terms of \((\alpha, \beta)\), but this is not needed here. Indeed, since \(\sqrt{-g} = G s \sqrt{\Sigma} = \sqrt{m^2 - a^2 c G}\), the
Dalambertian operator reads

$$\Box \varphi = \frac{1}{\sqrt{-\tilde{g}}} \left[ \partial_\alpha \left( \sqrt{-\tilde{g}} g^{\alpha\beta} \partial_\beta \varphi \right) + \partial_\beta \left( \sqrt{-\tilde{g}} g^{\alpha\beta} \partial_\beta \varphi \right) \right] \tag{0.60}$$

from which one directly obtains (0.27).

Writing now (0.58) in the form

$$ds^2 = k_{00} dt^2 + 2 k_{03} dt d\phi + k_{33} d\phi^2 + H e^B \left( d\alpha^2 + d\beta^2 \right) \tag{0.61}$$

(the $B = 0$ version of which being Kerr in $(\tilde{\varphi}, \tilde{x})$ coordinates), the $\tilde{\Gamma}_{\alpha\beta}^\gamma$ connexion components achieve a form like (0.5a)-(0.5c) with $A = 0$, but with $(\alpha, \beta)$ instead of $(r, \theta)$. Thence, (0.31) is replaced by

$$\tilde{K}_{\alpha\beta}^\gamma \partial_\alpha B + \tilde{K}_{\alpha\beta}^\gamma \partial_\beta B - \frac{1}{\alpha H e^B} k_{uv} \left[ \partial_\alpha (\alpha \partial_\alpha B) + \alpha \partial_\beta \partial_\beta B \right] = 2 \partial_\alpha \varphi \partial_\alpha \varphi \tag{0.62}$$

with $\tilde{K}_{\alpha\beta}^\gamma = \partial_\alpha \ln \alpha$. Writing out the $(uv) = (\alpha\beta)$ component directly returns (0.33b). The $(uv) = (\alpha\alpha)$ and $(uv) = (\beta\beta)$ components read

$$2 \partial_\alpha B - \partial_\alpha (\alpha \partial_\alpha B) - \alpha \partial_\beta \partial_\beta B = 2 \alpha \partial_\alpha \varphi \partial_\alpha \varphi \tag{0.63a}$$

$$\partial_\alpha (\alpha \partial_\alpha B) + \alpha \partial_\beta \partial_\beta B = -2 \alpha \partial_\beta \varphi \partial_\beta \varphi. \tag{0.63b}$$

Summing (0.63a) and (0.63b) returns (0.33a). Inserting $\partial B$ from (0.33a)-(0.33b), the remaining equation, for instance (0.63b), is satisfied thanks to (0.27).

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