THE ARGUMENT PRINCIPLE AND HOLOMORPHIC EXTENDIBILITY
TO FINITE RIEMANN SURFACES

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ABSTRACT Let $M$ be a finite Riemann surface and let $A(M)$ be the algebra of all continuous functions on $M \cup bM$ which are holomorphic on $M$. We prove that a continuous function $\Phi$ on $bM$ extends to a function in $A(M)$ if and only if for every $f, g$ in $A(M)$ such that $f\Phi + g \neq 0$ on $bM$, the change of argument of $f\Phi + g$ is nonnegative.

1. Introduction and the main result

Let $M$ be a finite Riemann surface, that is, a region in a Riemann surface whose boundary $bM$ consists of finitely many pairwise disjoint real analytic simple closed curves such that $M \cup bM$ is compact. We give $bM$ the standard orientation. Denote by $A(M)$ the algebra of all continuous functions on $M \cup bM$ which are holomorphic on $M$. We denote by $\Delta$ the open unit disc in $\mathbb{C}$.

The present paper deals with the problem of characterizing the boundary values of functions in $A(M)$ in terms of the argument principle. Our main result is

THEOREM 1.1 Let $M$ be a finite Riemann surface. A continuous function $\Phi$ on $bM$ extends holomorphically through $M$ if and only if for every $f, g$ in $A(M)$ such that $f\Phi + g \neq 0$ on $bM$, the change of argument of $f\Phi + g$ along $bM$ is nonnegative.

In [Gl1] it was shown that $\Phi$ extends holomorphically through $M$ if and only if for every polynomial $P$ with coefficients in $A(M)$ such that $P(\Phi) \neq 0$ on $bM$, the change of argument of $P(\Phi)$ along $bM$ is nonnegative. This is an easy consequence of the fact that the algebra $A(M)|bM$ is a maximal subalgebra of $C(bM)$ [R]. Theorem 1.1 sharpens this by saying that in this characterization it suffices to take the polynomials of degree one.

The only if part of the theorem is an obvious consequence of the argument principle. In fact, if $\Phi$ admits the extension $\tilde{\Phi} \in A(M)$ then the change of the argument of $f\tilde{\Phi} + g$ along $bM$ equals $2\pi$ times the number of zeros of $f\tilde{\Phi} + g$ on $M$.

2. Preliminaries

Let $g \geq 0$ be the genus of $M$ and let $m$ be the number of boundary components. Let $\nu = 2g + m - 1$ and let simple closed curves $\gamma_1, \gamma_2, \ldots, \gamma_\nu \subset M$ form a canonical basis for $M$ which, together with $\gamma'_1, \gamma'_2, \ldots, \gamma'_\nu$ forms a symmetric canonical basis for the double $M^*$ of $M$ a closed Riemann surface of genus $2\nu$ [K].
Let \( U \subset M^* \) be a small open connected neighbourhood of \( M \cup bM \) and let \( W \subset M^* \backslash U \) be an open set which does not meet any of the curves \( \gamma'_1, \gamma'_2, \cdots, \gamma'_\nu \). It is known that there is a meromorphic differential on \( M^* \) with simple poles all contained in \( W \) and with prescribed periods along the curves of the symmetric canonical basis of \( M^* \) [K, p.19]. In particular, for each \( j, 1 \leq j \leq \nu \), there is such a differential whose period along \( \gamma_j \) equals \( i \) and whose periods along all \( \gamma_k, k \neq j \), are 0. On \( U \) the real part of such a differential is the differential of a single valued harmonic function \( u_j \) on \( U \) whose conjugate differential \(*du_j \) has period one along \( \gamma_j \) and period 0 along each \( \gamma_k, k \neq j \). (See [CF] for a different construction of such functions). It follows that given a complex valued harmonic function \( u \) on \( M \) there are unique constants \( c_1(u), c_2(u), \cdots, c_\nu(u) \) such that \( u - \sum_{j=1}^\nu c_j(u)u_j \) has a single valued conjugate, that is,

\[
u(z) - \sum_{j=1}^\nu c_j(u)u_j(z) = F(z) + \overline{G(z)} \quad (z \in M)\]

where \( F \) and \( G \) are (single valued) holomorphic functions on \( M \). The constants \( c_j(u) \) are the periods of the conjugate differential \(*du \) and it is easy to see that they depend continuously on \( u \) in the sup norm. If the function \( u \) extends smoothly to \( M \cup bM \) then the same holds for the functions \( F \) and \( G \), so in this case \( F \) and \( G \) belong to \( A(M) \) [B, p.91].

### 3. Functions with single valued conjugates

If \( \Phi \) is a continuous function on \( bM \) there is a unique continuous extension \( \mathcal{H}(\Phi) \) of \( \Phi \) to \( M \cup bM \) which is harmonic on \( M \). We shall say that \( \Phi \) has a single valued conjugate if \( \mathcal{H}(\Phi) \) has a single valued conjugate on \( M \). If this is the case then \( \mathcal{H}(\Phi) = F + \overline{G} \) where \( F \) and \( G \) are holomorphic functions on \( M \). In the special case when \( \Phi \) is smooth the functions \( F \) and \( G \) belong to \( A(M) \).

**Proposition 3.1** Let \( \Phi \) be a continuous function on \( bM \). There is a nonconstant function \( g, \) holomorphic on \( U \), such that the function \( z \mapsto g(z)\Phi(z) \) \( (z \in bM) \) has a single valued conjugate.

**Proof.** Let \( \omega \) be a nonconstant holomorphic function on \( U \). For each \( j, 1 \leq j \leq \nu + 1 \), there are constants \( c_{j1}, c_{j2}, \cdots, c_{j\nu} \) such that the function

\[
\zeta \mapsto \omega(\zeta)^j\Phi(\zeta) + \sum_{k=1}^\nu c_{jk}u_k(\zeta) \quad (\zeta \in bM)
\]

has a single valued conjugate. Since \( \nu + 1 \) rows \([c_{j1}, c_{j2}, \cdots, c_{j\nu}] \) \((1 \leq j \leq \nu + 1)\) are linearly dependent there are numbers \( \lambda_1, \lambda_2, \cdots, \lambda_{\nu+1} \), not all equal to zero, such that \( \sum_{j=1}^{\nu+1} \lambda_j[c_{j1}, c_{j2}, \cdots, c_{j\nu}] = [0, 0, \cdots, 0] \) which means that the function \( \zeta \mapsto \lambda_1\omega(\zeta)\Phi(\zeta) + \cdots + \lambda_{\nu+1}\omega(\zeta)^{\nu+1}\Phi(\zeta) \) \( (\zeta \in bM) \) has a single valued conjugate. Thus, there is a polynomial \( P \) with complex coefficients of degree at least one such that the function \( \zeta \mapsto P(\omega(\zeta))\Phi(\zeta) \) \( (\zeta \in bM) \) has a single valued conjugate. Assume for a moment that there is a constant \( c \) such that

\[
P(\omega(\zeta)) + c \equiv 0 \quad (\zeta \in bM).
\]

(3.1)
Since $P$ has degree at least one it follows that $P + c$ has degree at least one so there are $k \geq 1$ and complex numbers $\alpha \neq 0, \omega_1, \omega_2, \ldots, \omega_k$ such that $P(z) + c = \alpha(z - \omega_1) \cdots (z - \omega_k)$ so (3.1) implies that
\[
(\omega(\zeta) - \omega_1) \cdots (\omega(\zeta) - \omega_k) \equiv 0 \quad (\zeta \in bM).
\]
(3.2)
Since $\omega$ is holomorphic and nonconstant on $U$ it follows that each factor in (3.2) has at most finitely many zeros on $bM$. This shows that (3.2) and hence (3.1) is impossible. This proves that $g = P(\omega)$ is a nonconstant function, holomorphic on $U$ and such that the function $\zeta \mapsto g(\zeta) \Phi(\zeta) \ (\zeta \in bM)$ has a single valued conjugate. The proof is complete.

**Proposition 3.2** Let $\Phi$ be a continuous function on $bM$ which does not extend holomorphically through $M$. Given a nonconstant holomorphic function $g$ on $U$ there is an $a \in M$ such that $\mathcal{H}((g - g(a)\Phi)(a)) \neq 0$.

**Proof.** Suppose that $\mathcal{H}((g - g(a)\Phi)(a)) \equiv 0 \ (a \in M)$. It follows that $\mathcal{H}(g(\Phi))(a) = g(a)\mathcal{H}(\Phi)(a) \ (a \in M)$ which, in particular, implies that $a \mapsto g(a)\mathcal{H}(\Phi)(a)$ is harmonic on $M$. Let $a \in M$ and let $\varphi: \Delta \to M$ be a parametric disc, $\varphi(0) = a$. Then $\zeta \mapsto g(\varphi(\zeta))\mathcal{H}(\Phi)(\varphi(\zeta))$, a product of $g \circ \varphi$ a nonconstant holomorphic function on $\Delta$ and $\mathcal{H}(\Phi) \circ \varphi$, a harmonic function on $\Delta$ is harmonic on $\Delta$ which implies $[\text{GL2}]$ that $\mathcal{H}(\Phi) \circ \varphi$ is holomorphic on $\Delta$. Thus, $\mathcal{H}(\Phi)$ is holomorphic in a neighbourhood of $a$ and since $\mathcal{H}(\Phi)$ is harmonic on $M$ it follows that $\mathcal{H}(\Phi)$ is holomorphic on $M$ which is impossible since $\Phi$ does not extend holomorphically through $M$. This completes the proof.

4. The proof in the case when $\Phi$ has a single valued conjugate

**Proposition 4.1** Suppose that $\Phi$ is a continuous function on $bM$ which has a single valued conjugate and which does not extend holomorphically through $M$. There are functions $P,Q \in A(M)$ such that $P\Phi + Q \neq 0$ on $bM$ and such that the change of argument of $P\Phi + Q$ along $bM$ is negative.

**Proof.** By Proposition 3.1 there is a function $g$, holomorphic and nonconstant on $U$, such that the function $\zeta \mapsto g(\zeta)\Phi(\zeta) \ (\zeta \in bM)$ has a single valued conjugate. Since $\Phi$ has a single valued conjugate it follows that for each $a \in M$ the function $\zeta \mapsto [g(\zeta) - g(a)]\Phi(\zeta) \ (\zeta \in bM)$ has a single valued conjugate. Since $\Phi$ does not extend holomorphically through $M$, Proposition 3.2 implies that $\mathcal{H}((g - g(a))\Phi)(a) \neq 0$ for some $a \in M$. With no loss of generality assume that
\[
\mathcal{H}((g - g(a))\Phi)(a) = 5\eta > 0. \quad (4.1)
\]
To make the proof easier to understand we first show how we complete the proof in the special case when $\Phi$ is smooth. In this case there are $F,G \in A(M), \ F(a) = G(a) = 0$, such that
\[
\mathcal{H}((g - g(a))\Phi) = F + G + 5\eta. \quad (4.2)
\]
It follows that
\[
(g(\zeta) - g(a))\Phi(\zeta) - F(\zeta) - G(\zeta) \in 5\eta + i\mathbb{R} \quad (\zeta \in bM)
\]
which implies that
\[(g(\zeta) - g(a))\Phi(\zeta) - F(\zeta) - G(\zeta) \neq 0 \quad (\zeta \in bM)\]

and that the change of argument of the function \(\zeta \mapsto (g(\zeta) - g(a))\Phi(\zeta) - F(\zeta) - G(\zeta)\) along \(bM\) is zero.

Let \(p_a\) be a holomorphic function on \(U\) whose only zero on \(U\) is a single zero at \(a\) [BS, p.566]. Note that \(P = (g - g(a))/p_a \in A(M)\) and, since \(F(a) = G(a) = 0\) it follows also that \(Q = -(F + G)/p_a \in A(M)\). Since \((g - g(a))\Phi - F - G = p_a(P\Phi + Q)\) on \(bM\), the argument principle implies that the change of argument of \(P\Phi + Q\) along \(bM\) equals \(-2\pi\). This completes the proof in the case when \(\Phi\) is smooth.

We now proceed to the proof for general continuous \(\Phi\). Given a continuous function \(\Psi\) on \(M\) there are constants \(c_1, c_2, \cdots c_\nu\) such that \(\zeta \mapsto (g(\zeta) - g(a))\Psi(\zeta) + \sum_{j=1}^\nu c_ju_j(\zeta)\) (\(\zeta \in bM\)) has a single valued conjugate. The constants \(c_j\), \(1 \leq j \leq \nu\) depend continuously on \(\Psi\) and they all vanish if \(\Psi = \Phi\). So there is a \(\delta > 0\) such that
\[
|\sum_{j=1}^\nu c_ju_j(z)| < \eta \quad (z \in M \cup bM) \quad (4.3)
\]

and
\[
|(g(z) - g(a))(\Psi(z) - \Phi(z))| < \eta \quad (z \in bM) \quad (4.4)
\]

provided that \(|\Psi - \Phi| < \delta\) on \(bM\). Let \(\Psi\) be a smooth function on \(bM\) such that \(|\Psi - \Phi| < \delta\) on \(bM\). The function \((g - g(a))\Psi + \sum_{j=1}^\nu c_ju_j\) is smooth on \(bM\) and has a single valued conjugate, so there are \(F, G \in A(M)\), \(\bar{F}(a) = G(a) = 0\), and a constant \(\gamma\) such that

\[
\mathcal{H}((g - g(a))\Psi) + \sum_{j=1}^\nu c_ju_j = F + \bar{G} + \gamma \quad \text{on } M \cup bM
\]

so

\[
\gamma = \mathcal{H}((g - g(a))\Psi)(a) + \sum_{j=1}^\nu c_ju_j(a).
\]

By the maximum principle (4.4) implies that \(|\mathcal{H}((g - g(a))\Psi)(a)| < \eta\) so (4.2) and (4.3) imply that
\[
|\gamma - 5\eta| < 2\eta \quad (4.5)
\]

On \(bM\) we have \((g - g(a))\Phi - F - G = (g - g(a))(\Phi - \Psi) + (g - g(a))\Psi - F - G = (g - g(a))\Phi - \Psi + \gamma - \sum_{j=1}^\nu c_ju_j + G - \bar{G}\) which, by (4.3), (4.4) and (4.5) implies that
\[
|5\eta - [(g - g(a))\Phi - \Psi + \gamma - \sum_{j=1}^\nu c_ju_j]| < 4\eta \quad \text{on } bM
\]

which implies that
\[
(g(\zeta) - g(a))\Phi(\zeta) - F(\zeta) - G(\zeta) \in [\eta, 9\eta] + i\mathbb{R} \quad (\zeta \in bM).
\]
So \((g - g(a))\Phi - F - G \neq 0\) on \(bM\) and the change of argument of \((g - g(a))\Phi - F - G\) along \(bM\) is zero. We now complete the proof in the same way as in the case when \(\Phi\) was smooth. The proof is complete.

5. Completion of the proof of Theorem 1.1

**Proposition 5.1** Let \(f, g \in A(\Delta)\). Assume that \(g \neq 0\) on \(b\Delta \setminus \{1\}\), \(g(1) = 0\), and that \(g\) extends holomorphically into a neighbourhood of \(1\). Assume that the function \(\zeta \mapsto f(\zeta)/g(\zeta) (\zeta \in b\Delta \setminus \{1\})\) extends continuously to \(b\Delta\). Then it extends to a function from \(A(\Delta)\).

**Proof.** We first prove the proposition in the case when \(g(\zeta) = \zeta - 1\). The following proof was shown to the author by Miran Černe. Denote by \(p\) the continuous extension of \(\zeta \mapsto f(\zeta)/(\zeta - 1) (\zeta \in b\Delta \setminus \{1\})\) to \(b\Delta\) and let \(\sum a_n e^{in\theta}\) be the Fourier series of \(p\). Let \(\sum b_n e^{in\theta}\) be the Fourier series of \(\zeta \mapsto f(\zeta) = (\zeta - 1)p(\zeta)\). Clearly \(b_n = a_{n-1} - a_n (n \in \mathbb{Z})\). Since \(f\) belongs to \(A(\Delta)\) we have \(b_n = 0 (n < 0)\) which implies that \(a_{n-1} = a_n (n < 0)\). Since \(\lim_{n \to \pm \infty} a_n = 0\) it follows that \(a_n = 0 (n < 0)\) which implies that \(p\) extends to a function in \(A(\Delta)\). This completes the proof in the special case when \(g(\zeta) = \zeta - 1\). The repeated application of the special case proves the proposition in the case when \(g(\zeta) = (\zeta - 1)^n, n \in \mathbb{N}\). In the general case write \(g(\zeta) = (\zeta - 1)^n h(\zeta)\) where \(h \in A(\Delta)\), \(h(\zeta) \neq 0 (\zeta \in \overline{\Delta})\) and \(n \in \mathbb{N}\). The preceding discussion implies that the function \(\zeta \mapsto (f/h)(\zeta)/(\zeta - 1)^n (\zeta \in b\Delta \setminus \{1\})\) extends to a function from \(A(\Delta)\) which shows that \(\zeta \mapsto f(\zeta)/g(\zeta) (\zeta \in b\Delta \setminus \{1\})\) extends to a function from \(A(\Delta)\). This completes the proof.

**Proposition 5.2** Let \(\Phi\) be a continuous function on \(bM\) and let \(g\) be a holomorphic function on \(U\), \(g \neq 0\). Assume that the function \(\zeta \mapsto g(\zeta)\Phi(\zeta)\) extends holomorphically through \(M\). Then there is a holomorphic function \(h\) on \(U\) without a zero on \(bM\) such that the function \(\zeta \mapsto h(\zeta)\Phi(\zeta)\) extends holomorphically through \(M\).

**Proof.** By the assumption there is a function \(G \in A(M)\) such that \(g\Phi = G\) on \(bM\). If \(g\) is a constant there is nothing to prove so assume that \(g\) is not a constant and \(g(b) = 0\) for some \(b \in bM\). The point \(b\) is an isolated zero of \(g\), let its degree be \(k\). Let \(p_b\) be a holomorphic function on \(U\) whose only zero on \(U\) is a single zero at \(b\). The function \(g_1 = g/p_b^k\) is holomorphic on \(U\) and has no zero at \(b\) so the function \(g_1\Phi\) is continuous on \(bM\). This means that the function \(\zeta \mapsto G(\zeta)/p_b(\zeta)^k (\zeta \in bM \setminus \{b\})\) extends continuously to \(bM\). Using Proposition 5.1 we see that the function \(\zeta \mapsto G(\zeta)/p_b(\zeta)^k\) extends to a function \(G_1 \in A(M)\). So, on \(bM\) we have \(g_1\Phi = G_1\) where \(G_1 \in A(M)\) and where \(g_1\) is holomorphic on \(U\) and has the same zeros as \(g\) except at \(b\) where \(g_1\) is different from 0. Repeating the process at each zero of \(g\) contained in \(bM\) (there are only finitely many of these) we arrive at a function \(h\) holomorphic on \(U\) with no zero on \(bM\) such that \(\zeta \mapsto h(\zeta)\Phi(\zeta)\) extends to a function from \(A(M)\). This completes the proof.

**Proof of Theorem 1.1 continued.** Suppose that \(\Phi\) does not extend holomorphically through \(M\). In the case when \(\Phi\) has a single valued conjugate Proposition 4.1 implies that there are \(P, Q \in A(M)\) such that \(P\Phi + Q \neq 0\) on \(bM\) and such that the change of argument of \(P\Phi + Q\) along \(bM\) is negative. Suppose now that \(\Phi\) does not have a single valued
conjugate. By Proposition 3.1 there is a nonconstant function \( g \) holomorphic on \( U \) such that \((g|bM)\Phi\) has a single valued conjugate. If \((g|bM)\Phi\) does not extend holomorphically through \( M \) Proposition 4.1 implies that there are \( P_1, Q \in A(M) \) such that \( P_1g\Phi + Q \neq 0 \) on \( bM \) and such that the change of argument of \( P_1g\Phi + Q \) along \( bM \) is negative. Putting \( P = P_1g \) completes the proof in the case when \((g|bM)\Phi\) does not extend holomorphically through \( M \).

Suppose now that there is a \( G \in A(M) \) such that \( g(\zeta)\Phi(\zeta) = G(\zeta) \) \( (\zeta \in bM) \). By Proposition 5.2 there are \( H \in A(M) \) and a holomorphic function \( h \) on \( U \) having no zero on \( bM \) such that \( h(\zeta)\Phi(\zeta) = H(\zeta) \) \( (\zeta \in bM) \). Dividing both sides with powers of functions \( p_b, \ b \in M \) where \( p_b \) is a holomorphic function on \( U \) whose only zero on \( U \) is a single zero at \( b \), we may assume with no loss of generality that \( h \) and \( H \) have no common zero on \( M \). Since \( h \) has no zero on \( bM \) and since \( \Phi \) does not extend holomorphically through \( M \) it follows that there is an \( a \in M \) which is a zero of \( h \) and not a zero of \( H \). On \( bM \) we have

\[
\frac{h}{p_a} \Phi = \frac{H}{p_a} = \frac{H - H(a)}{p_a} + \frac{H(a)}{p_a}
\]

where \( H(a) \neq 0 \). The functions \( P = h/p_a \) and \( Q = -(H - H(a))/p_a \) both belong to \( A(M) \) and we have \( P\Phi + Q = H(a)/p_a \) on \( bM \) which implies that \( P\Phi + Q \neq 0 \) on \( bM \) and, by the argument principle, the change of argument of \( P\Phi + Q \) along \( bM \) is negative. The proof of Theorem 1.1 is complete.

6. A remark

If \( M \) is a finitely connected domain in \( \mathbb{C} \) then \( \Phi \in C(bM) \) extends holomorphically through \( M \) provided that for each \( Q \in A(M) \) such that \( \Phi + Q \neq 0 \) on \( bM \), the change of argument of \( \Phi + Q \) along \( bM \) is nonnegative (that is, if \( M \) is a finitely connected domain in the plane then in Theorem 1.1 it suffices to take \( P \equiv 1 \)). The question whether the same holds for general finite Riemann surfaces remains open.

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