OBSERVATIONS ON THE POSITIVE PELL EQUATION

$y^2 = 20(x^2 + 1)$

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Abstract

This paper concerns with the problem of obtaining non-zero distinct integer solutions to the positive pell equation represented by the binary quadratic equation $y^2 = 20(x^2 + 1)$. A few interesting relations among the solutions are presented. Employing the solutions of the considered pell equation, a few relations between the figurate numbers are obtained. Further, by considering suitable linear combinations among the solutions of the considered hyperbola, the other choices of hyperbolas, parabolas and special Pythagorean triangle are exhibited along with their corresponding solutions.
1 INTRODUCTION

One of the areas of Number theory that has attracted many mathematicians since antiquity is the subject of diophantine equations. A diophantine equation is a polynomial equation in two or more unknowns such that only the integer solutions are determined. No doubt that diophantine equation possess supreme beauty and it is the most powerful creation of the human spirit. A pell equation is a type of non-linear diophantine equation in the form \( y^2 - Dx^2 = \pm 1 \) where \( D > 0 \) and square-free. The above equation is also called the Pell-Fermat equation. In Cartesian co-ordinates, this equation has the form of a hyperbola. The binary quadratic diophantine equation having the form

\[ y^2 = Dx^2 + N \quad (N > 0; D > 0, \text{ a non-square integer}) \]

is referred to as the positive form of the pell equation and the form

\[ y^2 = Dx^2 - N \quad (N > 0; D > 0, \text{ non-square integer}) \]

is called the negative form of the pell equation or related pell equation. It is worth to remind that the negative form of the pell equation is solvable only for certain values of \( D \) but always in the case of the positive form of the pell equation. An obvious generalizations to the Pell-like equation is the equation \( ax^2 - by^2 = N; a, b > 0, N \neq 0 \). In this context, one may refer [1, 8-15].

Pell equations arise in the investigation of numbers which are figurate in more than one way, for example, simultaneously square triangular and as such they are extremely important in Number theory [2-7]. In the solution of cubic equation and in certain other
situations it is desirable to have a method for extracting the cube root of a binomial surd. This may be accomplished by the aid of the pell equation. We use pell equation to solve Archimedes’ Cattle problem. Pell’s equation is connected to algebraic number theory, Chebyshev polynomials and continued fractions. Other applications include solving problems involving double equations, rational approximations to square roots, sums of consecutive integers, Pythagorean triangles with consecutive legs, consecutive Heronian triangles, sums of and consecutive squares and so on.

While searching for the equality among special polygonal numbers namely, Triangular and Dodecagonal numbers, we came across the positive pell equation given by $y^2 = 20(x^2 + 1)$ which is the motivation for our present work. In this communication, we obtain infinitely many integer solutions to the positive pell equation under consideration. Making use of the solutions of the considered equation, we have obtained different relations among the solutions, relations among figurate numbers, different choices of hyperbolas, parabolas and Pythagorean triangle together with their solutions.

2 NOTATIONS

- Polygonal number of rank $n$ with size $m$

$$t_{m,n} = n \left[ 1 + \frac{(n-1)(m-2)}{2} \right]$$

- Pyramidal number of rank $n$ with size $m$

$$P^m_n = \frac{1}{6} \left[ n(n+1) \right] \left[ (m-2)n + (5-m) \right]$$

3 METHOD OF ANALYSIS

The binary quadratic equation to be solved for its non-zero distinct integral solution is

$$y^2 = 20(x^2 + 1) \ldots \ldots (1)$$
whose smallest positive integer solution is

\[ x_0 = 2, \ y_0 = 10 \]

To obtain the other solutions of (1), consider the Pell equation

\[ y^2 = 20x^2 + 1 \ldots (2) \]

whose smallest positive integer solution is \((x_0, y_0) = (2, 9)\)

The general solution of (2) is given by

\[ \bar{y}_n = \frac{1}{2}f_n, \ \bar{x}_n = \frac{1}{4\sqrt{5}}g_n \]

where,

\[ f_n = (9 + 4\sqrt{5})^{n+1} + (9 - 4\sqrt{5})^{n+1}, \ g_n = (9 + 4\sqrt{5})^{n+1} + (9 - 4\sqrt{5})^{n+1} \]

Applying Brahmagupta lemma between \((x_0, y_0)\) and \((\bar{x}_n, \bar{y}_n)\), the other integer solutions of (1) are given by

\[ x_{n+1} = f_n + \sqrt{5}g_n \]
\[ y_{n+1} = 5f_n + 2\sqrt{5}g_n \]

The recurrence relations satisfied by \(x\) and \(y\) are given by

\[ x_{n+3} - 18x_{n+2} + x_{n+1} = 0 \]
\[ y_{n+3} - 18y_{n+2} + y_{n+1} = 0 \]

Some numerical examples of \(x\) and \(y\) satisfying (1) are given in the Table: 1 below:
From the above table, we observe some interesting relations among the solutions which are presented below:

- Both the $x$ and $y$ values are even
- Each of the following expressions is a nasty number

\[
\begin{align*}
3x_{n+3} - 51x_{n+2} + 12 \\
\frac{1}{6}(x_{n+4} - 305x_{n+3} + 72) \\
6y_{n+3} - 24x_{n+2} + 12 \\
\frac{1}{3}(2y_{n+3} - 152x_{n+2} + 36) \\
\frac{1}{161}(6y_{n+4} - 8184x_{n+3} + 1932) \\
51x_{n+4} - 915x_{n+3} + 12 \\
\frac{1}{3}(34y_{n+4} - 804x_{n+3} + 36) \\
102y_{n+4} - 456x_{n+3} + 12 \\
\frac{1}{3}(34y_{n+4} - 728x_{n+3} + 36) \\
\frac{1}{3}(610y_{n+4} - 152x_{n+3} + 36) \\
1830y_{n+4} - 8184x_{n+3} + 12 \\
\frac{1}{5}(57y_{n+4} - 3y_{n+3} + 60) \\
\frac{1}{30}(341y_{n+4} - 3y_{n+3} + 360) \\
\frac{1}{5}(1023y_{n+4} - 57y_{n+3} + 60)
\end{align*}
\]
• Each of the following expressions is a cubical integer

\[ \frac{1}{2} (x_{3|n+1} - 17x_{3|n+1} + 3x_{n+1} - 51x_{n+1}) \]
\[ \frac{1}{36} (y_{3|n+1} - 305y_{3|n+1} + 3y_{n+1} - 915y_{n+1}) \]
\[ y_{3|n+1} - 4y_{3|n+1} + 3y_{n+1} - 12y_{n+1} \]
\[ \frac{1}{9} (y_{3|n+1} - 76y_{3|n+1} + 3y_{n+1} - 228y_{n+1}) \]
\[ \frac{1}{2} (17x_{3|n+2} - 305x_{3|n+2} + 51x_{n+2} - 915x_{n+2}) \]
\[ \frac{1}{9} (17y_{3|n+2} - 4y_{3|n+2} + 5y_{n+2} - 12y_{n+2}) \]
\[ 17y_{3|n+2} - 76y_{3|n+2} + 51y_{n+2} - 228y_{n+2} \]
\[ \frac{1}{164} (305y_{3|n+3} - 4y_{3|n+3} + 915y_{n+3} - 12y_{n+3}) \]
\[ 305y_{3|n+3} - 1384y_{3|n+3} + 915y_{n+3} - 4092y_{n+3} \]
\[ \frac{1}{180} (341y_{3|n+3} - 3y_{3|n+3} + 1023y_{n+3} - 3y_{n+3}) \]
\[ \frac{1}{10} (341y_{3|n+3} - 19y_{3|n+3} + 1023y_{n+3} - 57y_{n+3}) \]

• Each of the following expressions is a bi-quadratic integer

\[ \frac{1}{80} (25y_{3|n+3} - 425y_{3|n+3} + 25y_{n+3} - 85y_{n+3})^2 - 100 \]
\[ \frac{1}{1600} (450y_{3|n+3} - 137250y_{3|n+3} + 255y_{n+3} - 1525y_{n+3})^2 - 32000 \]
\[ \frac{1}{25} (25y_{3|n+3} - 100y_{3|n+3} + 35y_{n+3} - 50y_{n+3})^2 - 500 \]
\[ \frac{1}{2025} (25y_{3|n+3} - 17100y_{3|n+3} + 445y_{n+3} - 1800y_{n+3})^2 - 40500 \]
\[ \frac{1}{20} (255y_{3|n+3} - 38550y_{3|n+3} + 205y_{n+3} - 1525y_{n+3})^2 - 100 \]
\[ \frac{1}{203} (255y_{3|n+3} - 4900y_{3|n+3} + 405y_{n+3} - 20y_{n+3})^2 - 40500 \]
\[ \frac{1}{25} (255y_{3|n+3} - 1900y_{3|n+3} + 85y_{n+3} - 30y_{n+3})^2 - 50 \]
\[ \frac{1}{6400} (22225y_{3|n+3} - 16100y_{3|n+3} + 6825y_{n+3} - 20y_{n+3})^2 - 1296000 \]
\[ \frac{1}{2025} (68625y_{3|n+3} - 171000y_{3|n+3} + 4450y_{n+3} - 3600y_{n+3})^2 - 405000 \]
\[ \frac{1}{203} (25625y_{3|n+3} - 34100y_{3|n+3} + 44525y_{n+3} - 6200y_{n+3})^2 - 80 \]
\[ \frac{1}{10000} (69000y_{3|n+3} - 10000y_{3|n+3} + 10y_{n+3} - 20y_{n+3})^2 - 20000 \]
\[ \frac{1}{3240000} (61300y_{3|n+3} - 61600y_{3|n+3} + 8620y_{n+3} - 50y_{n+3})^2 \]
\[ \frac{1}{10000} (6410000y_{3|n+3} - 19000y_{n+3} + 8520y_{n+3} - 30y_{n+3})^2 - 20000 \]
• Each of the following expressions is a quintic integer

\[
\frac{1}{2}(x_{n+1} - 17x_n) + 30P_1^2, \quad f_n = \frac{1}{2}(x_{n+1} - 17x_n)
\]

\[
\frac{1}{36}(-305x_{n+1} + x_n) + 30P_1^2, \quad f_n = \frac{1}{36}(x_{n+1} - 305x_n)
\]

\[
\frac{1}{2}(17x_{n+1} - 305x_n) + 30P_1^2, \quad f_n = \frac{1}{2}(17x_{n+1} - 305x_n)
\]

\[
\frac{1}{10}(19x_{n+1} - x_n) + 30P_1^2, \quad f_n = \frac{1}{10}(19x_{n+1} - x_n)
\]

\[
\frac{1}{10}(341x_{n+1} - 19x_n) + 30P_1^2, \quad f_n = \frac{1}{10}(341x_{n+1} - 19x_n)
\]

\[
\frac{1}{180}(341x_{n+1} - 19x_n) + 30P_1^2, \quad f_n = \frac{1}{180}(341x_{n+1} - 19x_n)
\]

\[
\frac{1}{9}(x_{n+1} - 76x_n) + 30P_1^2, \quad f_n = \frac{1}{9}(x_{n+1} - 76x_n)
\]

\[
\frac{1}{161}(x_{n+1} - 1364x_n) + 30P_1^2, \quad f_n = \frac{1}{161}(x_{n+1} - 1364x_n)
\]

\[
\frac{1}{9}(17x_{n+1} - 4x_n) + 30P_1^2, \quad f_n = \frac{1}{9}(17x_{n+1} - 4x_n)
\]

\[
17x_{n+1} - 76x_n + 30P_1^2, \quad f_n = 17x_{n+1} - 76x_n
\]

\[
\frac{1}{9}(17x_{n+1} - 1364x_n) + 30P_1^2, \quad f_n = \frac{1}{9}(17x_{n+1} - 1364x_n)
\]

\[
\frac{1}{161}(305x_{n+1} - 4x_n) + 30P_1^2, \quad f_n = \frac{1}{161}(305x_{n+1} - 4x_n)
\]

\[
\frac{1}{9}(305x_{n+1} - 76x_n) + 30P_1^2, \quad f_n = \frac{1}{9}(305x_{n+1} - 76x_n)
\]

\[
305x_{n+1} - 1364x_n + 30P_1^2, \quad f_n = 305x_{n+1} - 1364x_n
\]
• Relations among the solutions

\[ x_{n+2} - 18x_{n+1} + x_n = 0 \]
\[ 9x_{n+1} - x_{n+2} + 2y_{n+1} = 0 \]
\[ x_{n+1} - 9x_{n+2} + 2y_{n+1} = 0 \]
\[ y_{n+1} - 16y_{n+2} + 2y_{n+3} = 0 \]
\[ 36y_{n+1} - x_{n+3} + 16y_{n+3} = 0 \]
\[ 4y_{n+2} - x_{n+3} + x_{n+1} = 0 \]
\[ 36y_{n+2} - 16y_{n+3} + x_{n+1} = 0 \]
\[ y_{n+2} - 9y_{n+1} - 40x_{n+1} = 0 \]
\[ y_{n+3} - 16y_{n+1} - 720x_{n+1} = 0 \]
\[ 16y_{n+2} + 40x_{n+1} - 9y_{n+2} = 0 \]
\[ 9x_{n+1} - 16y_{n+1} - 2y_{n+2} = 0 \]
\[ x_{n+3} - 9x_{n+2} - 2y_{n+1} = 0 \]
\[ 9x_{n+3} - x_{n+2} - 2y_{n+3} = 0 \]
\[ y_{n+1} + 40x_{n+2} - 9y_{n+2} = 0 \]
\[ 9y_{n+1} + 720x_{n+1} - 9y_{n+3} = 0 \]
\[ 9y_{n+2} + 40x_{n+1} - y_{n+3} = 0 \]
\[ y_{n+3} - 80x_{n+2} - y_{n+2} = 0 \]
\[ 9y_{n+3} + 40x_{n+3} - 16y_{n+2} = 0 \]
\[ y_{n+1} + 720x_{n+1} - 16y_{n+3} = 0 \]
\[ y_{n+2} + 40x_{n+3} - 9y_{n+2} = 0 \]
\[ 16y_{n+2} - 9y_{n+1} - 40x_{n+3} = 0 \]
\[ 18y_{n+2} - y_{n+3} - y_{n+3} = 0 \]

3.1 Remarkable observations

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbolas which are presented in the Table: 2
2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabolas which are presented in the Table: 3

### Table: 3 PARABOLAS

| S.No | Parabola | \((x, y)\)                  |
|------|----------|----------------------------|
| 1    | \(200X^2 - Y^2 = 8000\) | \((100, -80), (-100, 80)\) |
| 2    | \(400X^2 + 2000Y^2 = 20000\) | \((50, -50), (-50, 50)\) |
| 3    | \(1600X^2 - Y^2 = 320000\) | \((80, -80), (-80, 80)\) |
| 4    | \(1600X^2 - 2000Y^2 = 400000\) | \((80, 80), (-80, -80)\) |
| 5    | \(1600X^2 - 2000Y^2 = 512000\) | \((80, 80), (-80, -80)\) |
| 6    | \(200X^2 - Y^2 = 8000\) | \((100, -80), (-100, 80)\) |
| 7    | \(400X^2 - 2000Y^2 = 160000\) | \((80, -80), (-80, 80)\) |
| 8    | \(800X^2 - 2000Y^2 = 320000\) | \((80, -80), (-80, 80)\) |
| 9    | \(1600X^2 - 2000Y^2 = 640000\) | \((80, -80), (-80, 80)\) |
| 10   | \(1600X^2 - 2000Y^2 = 800000\) | \((80, -80), (-80, 80)\) |
| 11   | \(1600X^2 - 2000Y^2 = 960000\) | \((80, -80), (-80, 80)\) |
| 12   | \(1600X^2 - 2000Y^2 = 1120000\) | \((80, -80), (-80, 80)\) |
| 13   | \(1600X^2 - 2000Y^2 = 1280000\) | \((80, -80), (-80, 80)\) |
| 14   | \(1600X^2 - 2000Y^2 = 1440000\) | \((80, -80), (-80, 80)\) |
| 15   | \(1600X^2 - 2000Y^2 = 1600000\) | \((80, -80), (-80, 80)\) |
3. Let \( p, q \) be two non-zero distinct integers such that \( p > q > 0 \). Treat \( p, q \) as the generators of the Pythagorean triangle \( T(\alpha, \beta, \gamma) \) where \( X = 2pq \), \( Y = p^2 - q^2 \), \( Z = p^2 + q^2 \), \( p > q > 0 \)

Taking \( p = x_{n+1} + y_{n+1}, q = x_{n+1} \), it is observed that \( T(\alpha, \beta, \gamma) \) is satisfied by the following relations:

- \( X - 10Y + 9Z = -20 \)
- \( 3(\tfrac{X}{\alpha} - \tfrac{4\beta}{\gamma}) \) is a Nasty number.
- \( \tfrac{2\alpha}{\beta} = x_{n+1}y_{n+1} \)

4. Relations between solutions and special polygonal numbers

- \( 9(P_3^y* t_{3,x})^2 = 20(P_5^y* t_{3,y+1})^2 + 20(t_{3,x} * t_{3,y+1})^2 \)
- \( 9(P_3^y* t_{3,x})^2 = 20(P_5^y* t_{3,y-2})^2 + 20(t_{3,x} * t_{3,y-2})^2 \)
- \( 9(P_5^z* t_{3,2y-2})^2 = 5(P_5^x* t_{3,2y-2})^2 + 5(t_{3,2y-2} * t_{3,x})^2 \)
- \( 9(P_5^x* t_{3,2x-2})^2 = 720(P_3^y* t_{3,y-2})^2 + 20(t_{3,2x-2} * t_{3,y-2})^2 \)
- Let \( \{n_{2s+1}\} \) and \( \{m_{2s+1}\} \) be sequences of positive integers defined by

\[
n_{2s+1} = \frac{1}{20}(y_{2s+1} - 10), \quad m_{2s+1} = \frac{1}{5}(x_{2s+1} + 2), \quad s = 0, 1, 2, \ldots
\]

**Observations**

(i) \( 8t_{3, n_{2s+1}} = t_{12, m_{2s+1}} \)
(ii) \( 240t_{3, n_{2s+1}} + 24 \) is a nasty number.

- Let \( \{n_{s+1}\} \) and \( \{m_{s+1}\} \) be sequences of positive integers defined by

\[
n_{s+1} = \frac{1}{20}(y_{s+1} - 10), \quad m_{s+1} = \frac{1}{5}(x_{s+1} - 2), \quad s = 0, 1, 2, \ldots
\]

**Observations**

(i) \( 10t_{3, n_{s+1}} = 2t_{3, m_{s+1}} + m_{s+1} \)
(ii) \( 40t_{3, n_{s+1}} - t_{0, m_{s+1}} \equiv 0 (mod 11) \)
(iii) \( 60t_{3, n_{s+1}} + 6 \) is a Nasty number.
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