Entangling two high-Q microwave resonators assisted by a resonator terminated with SQUIDs

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Abstract

We propose a superconducting circuit for quantum information processing (QIP) on high-quality (high-Q) superconducting resonators (SRs). In the circuit, two high-Q SRs are coupled to a high-frequency SR (acts as a quantum bus) assisted by superconducting quantum interference devices (SQUIDs) terminate in both ends of the high-frequency resonator. Each coupling strength between each high-Q resonator and the high-frequency resonator can be tuned independently from zero to the strong-coupling regime via the external flux threading through the SQUID. In the circuit, the frequencies of the two high-Q resonators are far detuned from the high-frequency resonator. That is, quantum information stored in high-Q resonators cannot be populated in the high-frequency resonator, which lets the bus can be designed to link lots of high-Q resonators for the large-scale QIP. To show the circuit can be used to achieve the QIP, we present a high-fidelity scheme to generate Bell state on the two high-Q resonators. The scheme shows that, to achieve the entanglement operation on high-Q resonators, fast tuning on the coupling is no longer mandatory and the coupling strengths are not required to be turned on or off simultaneously.

1. Introduction

Quantum information processing (QIP) plays an important role in quantum mechanics [1]. Many interesting systems have been proposed to realize QIP, such as photons in one degree of freedom [2–4] and photons in multi-degrees of freedom [5–7], quantum dots [8–12], nuclear magnetic resonance [13–16], diamond nitrogen-vacancy centers [17–21], and circuit quantum electrodynamics (QED) [22, 23]. Among these quantum systems, circuit QED composed of the superconducting transmission line resonator and the Josephson-junction-based superconducting qubit has emerged as a highly promising candidate for quantum computation as its good ability for integration, controllability, and scalability [24–28].

Superconducting resonator (SR) has been studied for completing basic tasks in QIP because of its high-quality factor and large zero-point electric fields, such as implementing quantum algorithms [29, 30], coupling superconducting qubits [31–33], entangling remote qubits [34–38], and detecting the motion of a mechanical oscillator near the ground state [39]. In these studies, the SR is fixed with static boundaries, which makes the dynamic of the resonator behaves like a harmonic oscillator. Besides, embedded with Josephson junctions or superconducting quantum interference devices (SQUIDs) to add nonlinearity to the dynamics, the resonator can also exhibit the other characters, such as kerr effect [40] and tunable eigenfrequencies [41–43] which can be used as the measurement and the control of the superconducting qubit [44–47], parameter amplifiers [48–50], and parameter converters [51–53]. SQUIDs located at the ends of the resonator can also result in tunable boundary conditions controlled by external flux which threads the SQUIDs. This property has been used to create the phenomena like Hawking radiation [54], dynamical Casimir effect [55–59], and twin paradox [60]. Further more, resonators can be nonlinearly coupled to each other by modulating one resonator’s boundary.
with flux created by other resonators to achieve optomechanical or quadratic optomechanical-like coupling between resonators [61, 62].

QIP based on SRs has been studied a lot recently both in theory and in experiment, such as the generation of quantum entanglement [63–67], the construction of universal quantum logical gates [25, 68–70], and the non-demolition detection of single microwave photon in a SR [71]. To realize large-scale QIP, lots of resonators should be coupled together, which leads to the crosstalk among them. To overcome the crosstalk, one method is to turn the frequencies of resonators to detune with each other largely. Another one is to turn off the coupling between two resonators. There have been many studies attending to achieve the two methods assisted by the flux qubit [72–75] or the SQUID [76–80].

After embedding with a qubit or a SQUID in a resonator, its quality factor will be reduced largely. This kind of resonators is hard to be treated as a good carrier of quantum information for QIP. Here, we focus on the tunability of the coupling among SRs and propose a three-resonator superconducting circuit to achieve QIP on two high-quality factor resonators. The circuit is composed of a resonator (acts as a quantum bus) inserted with SQUIDs and two resonators with high-quality (high-Q) factors act as carriers of quantum information. These resonators are specially arranged and nonlinearly coupled by using the interaction between the SQUID and the flux produced by the high-Q resonator. The coupling strength between each high-Q resonator and the bus can be tuned from non-coupling to the strong coupling regime independently. To show the circuit is suitable for QIP, we give a simple scheme to generate a high-fidelity Bell state on the two high-Q SRs. The scheme is achieved in the dispersive regime when the frequencies of the high-Q resonators are detuned from the quantum bus, which allows the information not to be populated in the low-quality bus. Furthermore, the tunable coupling strength between the quantum bus and high-Q resonators lets our circuit suitable for the large-scale QIP as the crosstalk can be overcome robustly.

This paper is organized as follows: in section 2, we introduce the configuration of the circuit and change it into an alternative lumped model which can be described in mathematical language. This circuit can be expressed in several Lagrangian formalisms and described by a two-mechanical-oscillator cavity optomechanical-like Hamiltonian after the canonical quantization. The coupling strengths between resonators are tunable and evaluated detailedly. In section 3, we construct a Bell state on the two high-Q resonators with fidelity about 99.2% [81] by considering the dissipation of resonators. Finally, we summarize our result in section 4.

2. Circuit model

Here, we analyze the circuit composed of a high-frequency resonator (C) coupled to two high-Q resonators (A and B) assisted by two SQUIDs as shown in figure 1(a) with quantum mechanics in detail and derive an effective Hamiltonian of the circuit with large range tunable coupling strength between resonator A (B) and C. First, we decompose the circuit into several lumped-circuit elements in order to write the Lagrangian and boundary conditions of the circuit. Second, with separation of variable method, we derive the mode frequencies of resonators analytically. Finally, we derive the Hamiltonian of the system with canonical quantization procedure.

2.1. Lagrangian and boundary conditions

Considering a SR with high quality factor, the capacitive coupling of the resonator to external transmission lines can be neglected i.e. the capacitances located at the two ends of the resonator tend to zero \((C_{st} \rightarrow 0)\), which means resonators A and B can be considered as open ended resonators. The flux field \(\Phi_\alpha (x, t)\) at position \(x\) (a coordinate of coordinate axis \(X\) as shown in figure 1(a)) of the resonator is related to the time-integral of the voltage as \(\Phi_\alpha (x, t) \equiv \int_{-\infty}^{t} V_\alpha (x, t') dt'\) (where \(\alpha = A\) and \(B\)). The superconducting phase of the macroscopic wave function which describes the superconductors is \(\phi_\alpha (x, t) = 2\pi \frac{\Phi_\alpha (x, t)}{\Phi_0}\). Assuming that resonators A and B are uniform and the characteristic capacitances and inductances per unit length at any position are constants, i.e. \(c^A (x) = c^B (x) = c_0\) and \(l^A (x) = l^B (x) = l_0\), the Lagrangian density of the resonators can be expressed as [82, 83]

\[
\mathcal{L}_\alpha = \frac{c_0}{2} \left[ \frac{\partial \Phi_\alpha (x, t)}{\partial t} \right]^2 - \frac{1}{2l_0} \left[ \frac{\partial \Phi_\alpha (x, t)}{\partial x} \right]^2.
\]  

(1)

In the limit \(C_{st} \rightarrow 0\), resonators A and B are open ended at both \(x = 0\) and \(x = l\) (see figure 1(b)), which leads to the boundary conditions of the two resonators as \(\frac{\partial}{\partial x} \Phi_\alpha (0, t) = 0\) and \(\frac{\partial}{\partial x} \Phi_\alpha (l, t) = 0\). \(l\) is the length of the resonators.

We now consider the resonator C which is placed in the middle of the circuit and terminated with two SQUIDs as shown in figure 1(a). The flux threading SQUIDs generated by high-Q resonators refresh the boundary conditions of the resonator C. We make the assumption that both SQUIDs are identical and
symmetric, i.e. $C_{j1} = C_{j2} = C_{j1}' = C_{j2}' = C/2$ and $E_{j1} = E_{j2} = E_{j1}' = E_{j2}' = E_j$, where subscripts $l$ and $r$ label the left SQUID and the right SQUID respectively. The characteristic capacitance and inductance of resonator $C$ are treated as constants ($C'(y) = C_0$ and $l'(y) = l_0$). The flux across each junction of the SQUID is $\Phi_{j\beta}(\beta = l, r$ and $i = 1, 2)$. The external flux can be written as $\Phi_{ex} = \Phi_{j1} - \Phi_{j2}$ and $\Phi_{ex}' = \Phi_{j1}' - \Phi_{j2}'$. The Lagrangian of the resonator $C$ consists of three parts

$$L_C = \int dy L_C + L_C' + L_C''$$

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where
\[
\mathcal{L}_C = \frac{c_0}{2} \left[ \frac{\partial \Phi(y, t)}{\partial t} \right]^2 - \frac{1}{2l_0} \left[ \frac{\partial \Phi(y, t)}{\partial y} \right]^2,
\]
\[
\mathcal{L}_L = \frac{c_1(0)}{2} \left[ \frac{\partial \Phi(0, t)}{\partial t} \right]^2 + E_j(\Phi_{\text{ext}}) \cos \left( 2\pi \frac{\Phi(0, t)}{\Phi_0} \right),
\]
\[
\mathcal{L}_S = \frac{c_1(d_s)}{2} \left[ \frac{\partial \Phi(d_s, t)}{\partial t} \right]^2 + E_j(\Phi'_{\text{ext}}) \cos \left( 2\pi \frac{\Phi(d_s, t)}{\Phi_0} \right).
\]

Here, \( \mathcal{L}_C, \mathcal{L}_L, \) and \( \mathcal{L}_S \) are the Lagrangian densities of resonator \( C \), left SQUID, and right SQUID, respectively. The flux field generated by the resonator \( C \) at the position \( y \) (a coordinate of coordinate axis \( Y \) as shown in figure 1(a)) is \( \Phi(y, t) \).

The Lagrangian of the SQUID is written in the form of an effective Josephson junction with Josephson energy \( E_j(\Phi_{\text{ext}}) = 2E_j |\cos(\pi \Phi_{\text{ext}}/\Phi_0)| [61, 62] \). This energy induces an effective inductor:

\[
L_j^{(\nu)} = \left( \frac{\Phi_0}{2\pi} \right)^2 \left( \frac{1}{E_j(\Phi_{\text{ext}})} \right) = L_{j0} \left[ \sec \left( \pi \frac{\Phi_{\text{ext}}}{\Phi_0} \right) \right],
\]

which can be adjusted by the external flux. Here \( L_{j0} = \left( \frac{\Phi_0}{2\pi} \right)^2 \left( \frac{1}{E_j} \right) \). With the method of Euler–Lagrange equation of motion, two effective boundary conditions introduced by the SQUIDs of the resonator \( C \) can be expressed as [84]:

\[
C_i \frac{\partial^2 \Phi_i(0, t)}{\partial t^2} - \frac{1}{l_0} \frac{\partial}{\partial y} \Phi_i(0, t) + \frac{1}{L_j} \Phi_i(0, t) = 0,
\]
\[
C_i \frac{\partial^2 \Phi_i(d_i, t)}{\partial t^2} + \frac{1}{l_0} \frac{\partial}{\partial y} \Phi_i(d_i, t) + \frac{1}{L_j} \Phi_i(d_i, t) = 0.
\]

\( d_i \) is the length of resonator \( C \). Here SQUIDs are assumed in the phase regime, i.e. the charge energy is small compared to the Josephson energy and the quantum fluctuations of the phase across the SQUIDs are small enough (\( 2\pi |\Phi(0, t)|/\Phi_0, 2\pi |\Phi(d_i, t)|/\Phi_0 \ll 1 \)). Here we have expanded the cosine function in equation (2) and omitted higher order items.

2.2. Modes

From equation (1), equations of motion for flux field obey the one-dimensional massless Klein–Gordon wave equation

\[
\frac{\partial^2}{\partial x^2} \Phi_\nu(x, t) - \omega^2 \frac{\partial^2}{\partial t^2} \Phi_\nu(x, t) = 0,
\]

where \( v = \frac{1}{\sqrt{l_0 c_0}} \) is the velocity of photons in resonators. \( \Phi_\nu(x, t) \) can be solved with the separation of variables method by assuming \( \Phi_\nu(x, t) = u_\nu(x)\psi_\nu(t) \). Then, the wave equation yields two independent ordinary differential equations

\[
\frac{\partial^2}{\partial x^2} u_\nu(x) + k_\nu^2 u_\nu(x) = 0,
\]
\[
\frac{\partial^2}{\partial t^2} \psi_\nu(t) + \omega_\nu^2 \psi_\nu(t) = 0,
\]

where \( k_\nu(\omega_\nu = k_\nu v) \) is the vector number (frequency) of photons. The general solution for \( u_\nu(x) \) is a linear combination of sine and cosine functions as

\[
u_\nu(x) = \xi_1 \cos(k_\nu x) + \xi_2 \sin(k_\nu x),
\]

where \( \xi_1 \) and \( \xi_2 \) are constants determined by boundary conditions. By taking equation (9) into the boundary conditions of the open resonators, we can get discrete mode functions as

\[
u_\nu(x) = \sum_n u_{\nu,n},
\]
\[
u_{\nu,n}(x) = \sqrt{\frac{n\pi}{d}} \cos\left( \frac{n\pi}{d} x \right),
\]
which satisfies the orthonormality conditions as
\[\int_0^d \frac{1}{d} \frac{d\psi_{n,m}}{dx} dx = \delta_{mn}\]
\[\int_0^d \frac{\partial \psi_{n,m}}{\partial x} \frac{\partial \psi_{n,m}}{\partial x} dx = \frac{1}{d} \delta_{mn},\]
with normal frequencies
\[\omega_{n,m} = \frac{n \pi}{d}v_u.\] (13)

Here \(n\) and \(m\) label the discrete modes of resonator \(\alpha\).

The normal frequencies of resonator \(C\) can be obtained by combining equations (3) and (5), and can be expressed as
\[
\tan(k_{c,d_j}) = \frac{1 - \frac{\omega_j^2}{\omega_p^2} \frac{l_0}{l_{k,0}^2}}{1 - \left(1 - \frac{\omega_j^2}{\omega_p^2} \frac{l_0}{l_{k,0}^2} \frac{l_0}{l_{k,0}^2} \frac{l_0}{l_{k,0}^2}\right)},
\] (14)

where \(\omega_j^{(r)} = 1/\sqrt{L_j^{(r)}C_j}\) is the plasma frequency of the left (right) SQUID. We assume that the plasma frequency is larger than the normal mode frequency \(\omega_{n,m}^{(r)} > \omega_{c,n}\) and treat \(l_{k,0}/l_0\) as virtual length which satisfies \(k_{c,0}d_{j(l)} = k_{c,0}l_{j(l)}^0 \ll 1\). By making approximations as
\[
\frac{1}{\tan(k_{c,n}d_{j(l)})} = \left(1 - \frac{\omega_j^2}{\omega_p^2} \frac{l_0}{l_{k,0}^2} \frac{l_0}{l_{k,0}^2} \right) \approx \frac{l_0}{l_{k,0}^2} \frac{l_0}{l_{k,0}^2} \frac{l_0}{l_{k,0}^2},
\] (15)

Equation (14) can be rewritten as
\[
\tan(k_{c,n}(d_c + \Delta d_r + \Delta d_l)) = O(\rho^3),
\] (16)

which leads to an analytical mode frequency of resonator \(C\) as
\[\omega_{c,n} = \frac{n \pi}{d}v_u.\] (17)

The additional virtual length \(\Delta d_{j(l)}\) induced by the SQUIDs of resonator \(C\) is much shorter than the mode wavelength and can be adjusted by the flux threading the SQUIDs. It is the adjustable virtual length that make the frequency of resonator \(C\) tunable.

The external applied flux can be decomposed in a static bias \(\Phi_{0A}^0\) (the static bias threading left SQUID and right SQUID are equal) and a small variation \(\delta \Phi_{0A}^{(r)}\) as
\[\Phi_{0A}^{(r)} = \Phi_{0A}^0 + \delta \Phi_{0A}^{(r)},\] (18)

where \(\delta \Phi_{0A}^{(r)}\) is the flux generated by resonator \(A\) \((B)\). The additional length \(\Delta d_{j(l)}\) can be expanded in the form of a static term corresponding to the static bias and a deviation term corresponding to the induced quantum flux as
\[
\Delta d_{j(l)} = \Delta d^0 + \delta d \delta \Phi_{0A}^{(r)},
\] (19)

with \[61, 62\]
\[
\delta d = \frac{1}{2} \frac{\Phi_0}{2\pi} \frac{1}{L_0 E_f(\Phi_{0A}^0)} \tan \left(\pi \frac{\Phi_0}{\Phi_0}\right),
\] (20)
\[
\Delta d^0 = \frac{\Phi_0}{2\pi} \frac{1}{L_0 E_f(\Phi_{0A}^0)}.
\] (21)

Then, the mode frequency \(\omega_{c,n}\) (equation (17)) can be expanded as
\[
\omega_{c,n} = \frac{n \pi}{d}v_u \left(1 - \frac{\delta d}{d_{eff}} \delta \Phi_{0A}^0 - \frac{\delta d}{d_{eff}} \delta \Phi_{0A}^{(r)}\right) = \omega_{c,n}' - \omega_{c,n}' \frac{\delta d}{d_{eff}} \delta \Phi_{0A}^0 - \omega_{c,n}' \frac{\delta d}{d_{eff}} \delta \Phi_{0A}^{(r)},\] (22)
where \( d_{\text{eff}} \) is the effective length of resonator \( C \) with respect to the static flux and has the form as [61, 62]

\[
d_{\text{eff}} = d_e + 2\Delta d^0. \tag{23}
\]

In addition, the mode functions of resonator \( C \) should obey the orthogonality conditions by considering the

\[
\int_0^d u_n(y)u_m(y)\,dy + C_{J}u_n(0)u_m(0) + C_{J}u_n(d_e)u_m(d_e) = C_\Sigma \delta_{nm},
\]

\[
\frac{1}{\hbar} \int_0^d \frac{\partial u_n(y)}{\partial y} \frac{\partial u_m(y)}{\partial y}\,dy + \frac{1}{L_j}u_n(0)u_m(0) + \frac{1}{L_j}u_n(d_e)u_m(d_e) = \frac{1}{L_n} \delta_{nm}, \tag{24}
\]

Here \( C_\Sigma = c_0 d_e + 2\Delta G \) is the total capacitance of resonator \( C \), \( L_n = (\omega_n^2 C_\Sigma)^{-1} \) is the effective inductance of resonator \( n \).

### 2.3. Hamiltonian of the system

With the orthonormality conditions in equation (24), the Lagrangian of resonator \( C \) becomes

\[
L_C = \sum_n \left[ \frac{1}{2} C_\Sigma \dot{\psi}_n^2 - \frac{1}{2L_n} \psi_n^2 \right].
\]

By using the Legendre transformation method [84], equation (25) leads to the harmonic Hamiltonian

\[
H_C = \sum_n \left[ \frac{1}{2} \theta_n^2 + \frac{C_\Sigma \omega_{C,n}^2}{2} \right].
\]

Here \( \theta_n = \frac{\partial L}{\partial \psi_n} = C_\Sigma \dot{\psi}_n \) is the momentum conjugate to \( \psi_n \). Canonical variables \((\theta_n, \psi_n)\) are treated as quantum operators which satisfy the canonical commutation relation \([\psi_n, \dot{\psi}_m] = i\hbar \delta_{nm}\). The creation and annihilation operators of excitations in mode \( n \) are defined as \( \dot{\psi}_n = \sqrt{\frac{\hbar}{2\omega_{C,n}C_\Sigma}} (\hat{c}_n^+ + \hat{c}_n) \) and \( \hat{\theta}_n = i \sqrt{\frac{\hbar\omega_{A,n}C_\Sigma}{2}} (\hat{c}_n^+ - \hat{c}_n) \), respectively. Then, Hamiltonian \( H_C \) can be quantized as

\[
H_C = \sum_n \hbar \omega_{C,n} (\hat{c}_n^+ \hat{c}_n + \frac{1}{2}). \tag{27}
\]

The creation operator \( \hat{c}_n^+ \) creates a microwave photon in resonator \( C \) with frequency \( \omega_{C,n} \) while annihilation operator \( \hat{c}_n \) destroys one. The annihilation operators satisfy the commutation relations \([\hat{c}_n, \hat{c}_m] = [\hat{c}_n^+, \hat{c}_m^+] = 0 \) and \([\hat{c}_n, \hat{c}_m^+] = [\hat{c}_n^+, \hat{c}_m] = \delta_{nk} \).

Similarly, we can get Hamiltonians of resonator \( A \) and resonator \( B \) as [61]

\[
H_A = \sum_n \hbar \omega_{A,n} \hat{\alpha}_n^+ \hat{\alpha}_n + \frac{1}{2}, \tag{28}
\]

\[
H_B = \sum_n \hbar \omega_{B,n} \hat{b}_n^+ \hat{b}_n + \frac{1}{2}. \tag{29}
\]

Here the annihilation operator \( \hat{\alpha}_n (\hat{b}_n) \) destroys a microwave photon with frequency \( \omega_{A,n} (\omega_{B,n}) \) in resonator \( A \) (resonator \( B \)) and the creation operator \( \hat{\alpha}_n^+ (\hat{b}_n^+) \) creates one in resonator \( A \) (resonator \( B \)). These operators satisfy the commutation relations \([\hat{\alpha}_n, \hat{\alpha}_m] = [\hat{\alpha}_n^+, \hat{\alpha}_m^+] = [\hat{b}_n, \hat{b}_m] = [\hat{b}_n^+, \hat{b}_m^+] = 0 \) and \([\hat{\alpha}_n, \hat{\alpha}_m^+] = [\hat{b}_n, \hat{b}_m^+] = \delta_{nk} \). We assume the external flux \( \delta \Phi_{\text{ext}}^{(l)} \) generated by resonator \( A \) and resonator \( B \) have the form

\[
\delta \Phi_{\text{ext}}^{(l)} = \sum_m G_{A,m} (\hat{b}_m + \hat{b}_m^+),
\]

\[
\delta \Phi_{\text{ext}}^{(l)} = \sum_m G_{B,m} (\hat{a}_m + \hat{a}_m^+), \tag{30}
\]

where \( G_{A,m} \) and \( G_{B,m} \) are geometrical factors which are related to the spatial arrangement of the resonators. By combining equation (22) and equations (27)–(30), the Hamiltonian of the whole system becomes

\[
H = H_A + H_B + H_C = \hbar \sum_n \omega_{A,n} \hat{\alpha}_n^+ \hat{\alpha}_n + \hbar \omega_{B,n} \hat{b}_n^+ \hat{b}_n + \hbar \omega_{C,n} \hat{c}_n^+ \hat{c}_n - \hbar \sum_{nm} G_{A,m} (\hat{a}_m + \hat{a}_m^+) \hat{c}_n^+ \hat{c}_n - \hbar \sum_{nm} G_{B,m} (\hat{b}_m + \hat{b}_m^+) \hat{c}_n^+ \hat{c}_n, \tag{31}
\]

where \( G_{\alpha,m} = \omega_{\alpha,n} \frac{\delta d}{\delta (G_{\alpha,n})} \) is the coupling strength between the \( n \)th mode of resonator \( \alpha \) and the \( m \)th mode of resonator \( C \). \( \delta d \) and \( \delta \text{eff} \) are the functions of external flux \( \Phi_{\text{ext}}^{(l)} \) while \( G_{A,m} \) and \( G_{B,m} \) are determined by the geometrical arrangement of the circuit. Hence, \( G_{\alpha,m} \) is the function of external flux \( \Phi_{\text{ext}}^{(l)} \) and the geometrical
arrangement. By considering the symmetric arrangement of the circuit \( G_{A,m} = G_{B,m} = G_{m} \) and the fundamental mode \( (m = 1) \) of resonator \( A (B) \), the Hamiltonian \( H \) reduces to

\[
H_{1m} = \hbar \omega_0 \hat{a}^\dagger \hat{a} + \hbar \omega_1 \hat{b}^\dagger \hat{b} + \hbar \omega_2 \hat{c}^\dagger \hat{c} - \hbar g_1 (\hat{b}^\dagger \hat{a} + \hat{a}^\dagger \hat{b}) \hat{c}^\dagger \hat{c} - \hbar g_0 (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) \hat{c}^\dagger \hat{c},
\]

where \( \omega_{1} = \frac{\omega_{0}}{\sqrt{2}} \) is the fundamental modes of resonator \( A \) and \( B \), \( \omega_{i} \) is the eigenfrequency of resonator \( C \), and 

\[ g_{i} = \omega_{i} \frac{d_{eff}}{a_{eff} G_{i}} \] is the coupling strength between resonator \( A (B) \) and resonator \( C \). Some subscripts of operators have been neglected here for convenience.

\subsection*{2.4. Coupling strength}

The coupling strength \( g_{i} = \omega_{i} \frac{d_{eff}}{a_{eff} G_{i}} \) is determined by the eigenfrequency \( \omega_{i} \), the additional length \( \delta d \), and the geometry factor \( G_{i} \). \( \omega_{i} \) and the length of resonator \( C \) have an inverse relationship and can be enhanced by tuning the external flux away from the \( 0.5 \Phi_{0} \) [41–43] while other parameters are fixed. This property has a positive impact on our scheme for achieving the entanglement on high-Q SRs in section 3. Here \( \omega_{i} \) is treated as a fixed frequency to analyze the coupling strength \( g_{i} \) for convenience [61]. The additional length \( \delta d \) is the function of the external flux and can be adjusted continuously. The geometry factor \( G_{m} \) depends on the arrangement of the circuit. Here, we only take the geometry factor between resonator \( B \) and resonator \( C \) as an example because the geometry factor between resonator \( B \) and resonator \( C \) is the same as the one between resonator \( A \) and resonator \( C \). An analytical expression of \( G_{m} \) can be obtained by assuming that the magnetic field \( B(x, r) \) generated by the resonator \( B \) has the form [61, 62]

\[
B(x, r) = \frac{\mu_{0} B_{0}(x)}{2\pi r},
\]

where \( r \) is the radial distance from the resonator \( B \), \( \mu_{0} \) is the permeability of free space. \( B_{0}(x) \) is the current in resonator \( B \) at position \( x \). We evaluate the current by using the derivative of the magnetic field \( \Phi_{B}(x, t) \) at position \( x_{0} \) as

\[
B_{0}(x_{0}) = -\frac{1}{\mu_{0} I_{B}(x, t)} \frac{\partial}{\partial x} \Phi_{B}(x, t) |_{x=x_{0}}.
\]

By comparing the integral magnetic flux through the SQUID area in equation (30), the geometry factor of the fundamental modes of resonators \( B \) and \( C \) can be expressed as

\[
G_{1} = \frac{\mu_{0} \Delta}{2\pi \mu_{0} d} \sqrt{\pi hZ} \ln \frac{r_{2}}{r_{1}}.
\]

Here \( Z = \sqrt{\mu_{0} / \rho_{0}} \) is the characteristic impedance of the resonator. \( r_{2} \) and \( r_{1} \) are the distance from the axle wire of resonator \( B \) to the left and right boundary of the right SQUID as shown in figure 1(a). We have set the midpoint of resonator \( B \) as \( x_{0} \) and \( r_{2} / r_{1} = 2 \) for convenience.

The coupling strength [61] between the fundamental modes of resonator \( B \) and \( C \) can be expressed as

\[
g_{i} = \omega_{i} \frac{\delta d}{a_{eff} G_{i}} \frac{\mu_{0} \Delta}{2\pi \mu_{0} d} \sqrt{\pi hZ} \ln 2.
\]

Figure 2 shows the normalized coupling strength \( g_{i} / \omega_{0} \) as a function of the external flux and the frequency ratio \( \omega_{0} / \omega_{1} \). The coupling strength grows with the increase of the external flux and the decrease of the ratio \( \omega_{0} / \omega_{1} \). A small \( \omega_{0} / \omega_{1} \) and a large external flux take the coupling between the resonator \( A (B) \) and the resonator \( C \) into the strong coupling regime. When there is no flux threading the SQUIDs, i.e. \( \Phi_{C,0}^{\text{int}} = 0 \), the interactions between resonators are vanished (as shown in the inset of figure 2) and the frequency of resonator \( C \) increases which enhances the frequency detuning between the quantum bus and high-Q resonators.

\subsection*{3. Entanglement generation}

To generate the entanglement on resonator \( A \) and resonator \( B \), we consider the whole system (shown in figure 1) is in the dispersive regime where the eigenfrequency of resonator \( C \) is at least 10 times of the eigenfrequencies of the high-Q resonators \( \left( \frac{\omega_{2}}{\omega_{0}} > 10 \right) \). The coupling strength satisfies the relation \( \frac{\omega_{0}}{\omega_{2}} < 1 \). By applying a driving field on resonator \( C \), the Hamiltonian of the system follows

\[
H_{i} / \hbar = \omega_{0} \hat{a}^\dagger \hat{a} + \omega_{1} \hat{b}^\dagger \hat{b} + \omega_{2} \hat{c}^\dagger \hat{c} - g_{1} (\hat{b}^\dagger \hat{a} + \hat{a}^\dagger \hat{b}) \hat{c}^\dagger \hat{c} - g_{0} (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) \hat{c}^\dagger \hat{c} + \beta (\hat{c} e^{i\omega_{2}t} + \hat{c}^\dagger e^{-i\omega_{2}t}).
\]
After an unitary transformation $U_1 = \exp(-i \omega t \hat{c}^\dagger \hat{c})$, Hamiltonian $H_1$ becomes

$$H_2/\hbar = \omega_0 \hat{a}^\dagger \hat{a} + \omega_0 \hat{b}^\dagger \hat{b} + \omega \hat{c}^\dagger \hat{c} - g_1 (\hat{b} + \hat{b}^\dagger) \hat{c}^\dagger \hat{c} - g_2 (\hat{a} + \hat{a}^\dagger) \hat{c}^\dagger \hat{c} + \hbar \beta (\hat{c} + \hat{c}^\dagger).$$

Here $\omega = \omega_c - \omega_d$. Another unitary operation $U_2 = \exp \left( -\frac{\hat{b}^\dagger \hat{b} + \hat{a}^\dagger \hat{a}}{\omega_0} \right)$ transforms the Hamiltonian of the system from $H_2$ to

$$H_3/\hbar = \omega_0 \hat{a}^\dagger \hat{a} + \omega_0 \hat{b}^\dagger \hat{b} + \omega \hat{c}^\dagger \hat{c} + \beta \hat{c}^\dagger \exp \left( -\frac{g_1}{\omega_0} (\hat{b}^\dagger \hat{b} + \hat{a}^\dagger \hat{a}) + \hbar \text{c.c.} \right).$$

Considering the Lamb–Dicke approximation and the rotating wave approximation, Hamiltonian $H_3$ can be reduced to

$$H_4/\hbar = \omega_0 (\hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}) + \omega \hat{c}^\dagger \hat{c} + \frac{\beta g_1}{\omega_0} (\hat{a}^\dagger \hat{c} + \hat{c}^\dagger \hat{a} + \hat{b}^\dagger \hat{c} + \hat{c}^\dagger \hat{b}).$$

With the effective Hamiltonian theory [85], the indirect interaction between high-Q resonators emerges. Then, Hamiltonian $H_4$ can be reduced as

$$H_{\text{eff}}/\hbar = \chi (\hat{b}^\dagger \hat{b} + \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b} - 2 \hat{c}^\dagger \hat{c}),$$

where $\chi = \frac{\beta g_1}{\omega_0}$ and $\bar{\omega} = \omega - \omega_0$. Therefore, the original Hamiltonian has the relation $H_1 \approx U_1 U_2 H_{\text{eff}} U_2^\dagger U_1^\dagger$ with the effective Hamiltonian. $H_{\text{eff}}$ can be used to construct a Bell state on resonator $A$ and resonator $B$. By assuming the initial state of the system is $|\psi(0)\rangle = |1\rangle_A |0\rangle_B |0\rangle_C$, the evolution of the system is given by

$$|\psi(t)\rangle \sim [\cos(\chi t)|1\rangle_A |0\rangle_B - i \sin(\chi t)|0\rangle_A |1\rangle_B |0\rangle_C],$$

where $|0\rangle_A$ ($|1\rangle_A$) is the Fock state of resonator $A$ ($\alpha = A, B, C$). The excitation of the high-Q resonator can be achieved by coupling a superconducting qubit (not drawn in figure 1 for convenience) to the bus in the dispersive regime and letting the frequency of the qubit equal to the one of the high-Q resonator to exchange the energy from the qubit to the resonator. After an operation time of $\chi t = \frac{\pi}{4}$, the state of the system will evolve from $|\psi(0)\rangle = |1\rangle_A |0\rangle_B |0\rangle_C$ to

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2}} (|1\rangle_A |0\rangle_B - i |0\rangle_A |1\rangle_B |0\rangle_C).$$

To show the feasibility of our scheme, we numerically simulate the fidelity of the state and the population of a microwave in resonator $\alpha$ with the Lindblad–Kossakowski master equation.
Here, $\kappa_\alpha$ is the dissipation rate of resonator $\alpha$, $\hat{\rho}$ is the density operator of the system. The population of a microwave photon in resonator $\alpha$ and the fidelity of the state are defined as

$$P_\alpha = \langle \psi_{\alpha}\hat{\rho}(t)\psi_{\alpha}^\dagger \rangle, \quad F = \langle \psi_{\alpha}\hat{\rho}(t)\psi_{\alpha}^\dagger \rangle.$$  

Here, $|\psi_\alpha\rangle = |1\rangle_a|0\rangle_b|0\rangle_c$, $|\psi_b\rangle = |0\rangle_a|1\rangle_b|0\rangle_c$, $|\psi_C\rangle = |0\rangle_a|0\rangle_b|1\rangle_c$, and $\hat{\rho}(t)$ is the real density operator of the system.

Figure 3(a) shows the occupation probabilities of a photon in three resonators varies with the time $t$. Parameters taken here are the ones indicated by the carmine marker in figure 2 with the coupling strength $g_1 / \omega_b = 0.3$ [61], the external flux $\Phi_{\text{ext}} = 0.44\Phi_0$, the frequency $\omega_b/(2\pi) = 1$ GHz, the frequency ratio $\omega_d / \omega_b = 0.031$, $\beta = 0.12\omega_b$, and $\omega_d = 30.8 \omega_b$. The Rabi-oscillation appears between resonator $A$ and resonator $B$ as expected by equation (42). The maximum value of the photon populated in resonator $C$ is about 0.1, which indicates that the resonator $C$ is only virtually excited. Figure 3(b) shows the fidelity of the state evolves with the operation time $t$ with the dissipation rates $\kappa_A^{-1} = \kappa_B^{-1} = 20 \mu s$, and $\kappa_C^{-1} = 300$ ns. In our scheme, coupling strength $g_1$ should equal to $g_2$, which is described by the black solid line in the figure 3(b). Here, the system evolves to the Bell state at $46.17$ ns with the fidelity of 99.26%. In experiments, parameters can not meet the requirement of the scheme perfectly. So, in figure 3(b), we give the fidelity of the state varies with $\delta_0 = (g_1 - g_2)/g_2 = \pm 0.1$ ($g_1$ is fixed here) described by the blue dotted line and the red dashed–dotted line with maximal fidelities of 98.62% and 98.65%, respectively. That is, our scheme can work well when the detuning between $g_1$ and $g_2$ reaches 0.1$g_1$. Figure 3(c) shows the fidelity of the Bell state varies with different dissipation rates of resonator $A$ and resonator $B$ while the dissipation rate of resonator $C$ is fixed. The dissipation rates of the high-Q resonators are $\kappa_A^{-1} = \kappa_B^{-1} = \kappa^{-1}$. Dissipations of the high-Q resonators can reduce the fidelity of the entangled-state a little within a short operation time. It is worth noticing that the quality factor of a SR has been demonstrated with a
4. Conclusion

In conclusion, we have proposed a three-resonator circuit to achieve a high fidelity entanglement on two high-Q resonators for QIP. In the circuit, each high-Q resonator is coupled to the high-frequency resonator (acts as a quantum bus) with tunable coupling assisted by a SQUID terminates the bus. The flux-sensitive boundary condition constructed by the SQUID is the key element to couple different resonators. Special arrangement of the circuit leads to the optomechanical like Hamiltonian, in which the coupling strength between each high-Q resonator and the bus can be tuned independently by applying a external-controlled flux through the SQUID. This property helps us to overcome the crosstalk among high-Q resonators in a quantum processor composed of lots of high-Q resonators coupled to a quantum bus. For the achievement of the entanglement, the fast operation on the tuning of the coupling is unnecessary as one can turn on the coupling first and then apply the drive field to achieve the entanglement on high-Q resonators. Moreover, coupling strengths need not to be turned on or off simultaneously.

To show the feasibility of our circuit for the QIP, we propose a high-fidelity scheme to construct Bell state on high-Q resonators. Resonator embedded with the SQUID suffers a decrease on its quality factor and may not suitable for acting a quantum information carrier. So, resonator embedded with two SQUIDs is chosen as a quantum bus which should be virtually excited as the scheme works in the dispersive regime where the frequency of the resonator C is far detuned from the ones of resonators A and B. The fidelity of the scheme achieves a high fidelity of 99.2% by considering the possible dissipation rates of three resonators.

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References

[1] Nielsen M A and Chuang I L 2000 Quantum Computation and Quantum Communication (Cambridge: Cambridge University Press)
[2] Knill E, Laflamme R and Milburn G J 2001 Nature 409 46
[3] Nemoto K and Munro W J 2004 Phys. Rev. Lett. 93 250502
[4] Beenakker C W J, DiVincenzo D P, Emary C and Kindermann M 2004 Phys. Rev. Lett. 93 020501
[5] Ren B C and Deng F G 2014 Sci. Rep. 4 4623
[6] Ren B C, Wang G Y and Deng F G 2015 Phys. Rev. A 91 032328
[7] Li T and Long G L 2016 Phys. Rev. A 94 022343
[8] Li X, Wu Y, Steed D, Gammon D, Stievater T H, Katzer D S, Park D, Piermarocchi C and Sham L J 2003 Science 301 809
[9] Hu C Y, Young A, O’Brien J L, Munro W J and Rarity J G 2008 Phys. Rev. B 78 085307
[10] Wei H R and Deng F G 2013 Phys. Rev. A 87 022305
[11] Wang C 2012 Phys. Rev. A 86 012323
[12] Wang T J and Wang C 2015 IEEE J. Sel. Top. Quantum Electron. 21 91–7
[13] Gershfenfeld N A and Chuang I L 1997 Science 275 350
[14] Jones J A, Mosca M and Hansen R H 1998 Nature 393 344
[15] Long G L and Xiao L 2003 J. Chem. Phys. 119 8473
[16] Feng G, Xu G and Long G 2013 Phys. Rev. Lett. 110 190501
[17] Togan E et al 2010 Nature 466 730
[18] Neumann P et al 2010 Nat. Phys. 6 249
[19] Song X K, Ai Q, Qiu J and Deng F G 2016 Phys. Rev. A 93 052324
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[20] Song X K, Zhang H, Ai Q, Qiu J and Deng F G 2016 New J. Phys. 18 023001
[21] Wei H R and Deng F G 2013 Phys. Rev. A 88 042323
[22] Scully M O and Zubairy M S 1997 Quantum Optics (Cambridge: Cambridge University Press)
[23] Liao J, Wu Q Q and Nori F 2014 Phys. Rev. A 89 014302
[24] Blais A, Huang R S, Wallraff A, Girvin S M and Schoelkopf R J 2004 Phys. Rev. A 69 062320
[25] Wallraff A, Schuster D I, Blais A, Frunzio L, Huang R S, Majer J, Kumar S, Girvin S M and Schoelkopf R J 2004 Nature 431 162
[26] Barends R et al 2014 Nature 508 500
[27] Wang W, Hu L, Xu Y, Liu K, Ma Y, Zheng S B, Vijay R, Song Y P, Duan L M and Sun L 2017 Phys. Rev. Lett. 118 232604
[28] McKay D C, Filipp S, Mezzacapo A, Magesan E, Chow J M and Gambetta J M 2016 Phys. Rev. Appl. 6 064007
[29] DiCarlo L et al 2009 Nature 460 240
[30] Reed M D, DiCarlo L, Nigg S E, Sun L, Frunzio L, Girvin S M and Schoelkopf R J 2012 Nature 482 382
[31] You J Q and Nori F 2003 Phys. Rev. B 68 064509
[32] Blais A, Huang R S, Wallraff A, Girvin S M and Schoelkopf R J 2004 Phys. Rev. A 69 062320
[33] Schoelkopf R J and Girvin S M 2008 Nature 451 664
[34] Sillanpaa M A, Park J I and Simmonds R W 2007 Nature 449 438
[35] Majer J et al 2007 Nature 449 443
[36] DiCarlo L, Reed M D, Sun L, Johnson B R, Chow J M, Gambetta J M, Frunzio L, Girvin S M, Devoret M H and Schoelkopf R J 2010 Nature 467 574
[37] Neeley M et al 2010 Nature 467 570
[38] Liu T, Xiong S J, Cao X Z, Su Q P and Yang C P 2015 Opt. Lett. 40 5602
[39] Rocheleau T, Ndukum T, Macklin J B, Clerk A A and Schwab K C 2010 Nature 463 72
[40] Bourassa J, Beaudoin F, Gambetta J M and Blais A 2012 Phys. Rev. A 86 012314
[41] Wallquist M, Shumeiko V S and Wendin G 2006 Phys. Rev. B 74 224506
[42] Sandberg M, Wilson C M, Persson F, Bauch T, Johansson G, Shumeiko V, Duty T and Delsing P 2008 Appl. Phys. Lett. 92 203501
[43] Healey J E, Lindstrom T, Colcough M S, Muirhead C M and Tzalenchuk A Y 2008 Appl. Phys. Lett. 93 043513
[44] Siddiqi I, Vijay R, Metcalfe M, Boaknin E, Frunzio L, Schoelkopf R J and Devoret M H 2006 Phys. Rev. B 73 054510
[45] Lupsu F, Driessen E F C, Roos C, Marzec E and Behrends M 2008 Phys. Rev. Lett. 101 120501
[46] Metcalfe M B, Boaknin E, Manucharyan V, Vijay R, Siddiqi I, Rigetti C, Frunzio L, Schoelkopf R J and Devoret M H 2007 Phys. Rev. B 76 174516
[47] Mallet F, Ong F R, Palacios-Laloy A, Nguyen F, Bertet P, Vion D and Esteve D 2009 Nat. Phys. 5 791
[48] Yurke B, Kaminsky P G, Miller R E, Whittaker E A, Smith A D, Silver A H and Simon R W 1988 Phys. Rev. Lett. 60 7664
[49] Yamamoto T, Inomata K, Watanabe M, Matsuoka K, Miyazaki T, Oliver W D, Nakamura Y and Tsai J S 2008 Appl. Phys. Lett. 93 043510
[50] Castellanos-Beltran M, Irwin K D, Hilton G C, Vae I R and Lehnert K W 2008 Nat. Phys. 4 929
[51] Bergel N, Vijay R, Manucharyan V, Siddiqi I, Schoelkopf R J, Girvin S M and Devoret M H 2010 Nat. Phys. 6 296
[52] Bergel N, Schacker T, Metcalfe M, Vijay R, Manucharyan V, Frunzio L, Prober D, Schoelkopf R J, Girvin S and Devoret M 2010 Nature 465 64
[53] Zakka-Bajani E, Nguyen F, Lee M, Vale L R, Simmonds R W and Aumentado J 2011 Nat. Phys. 7 599
[54] Nation P D, Blencowe M P, Rimberg A J and Buek E 2009 Phys. Rev. Lett. 103 087004
[55] Johansson J R, Johansson G, Wilson C M and Nori F 2009 Phys. Rev. Lett. 103 147403
[56] Johansson J R, Johansson G, Wilson C M and Nori F 2010 Phys. Rev. A 82 052509
[57] Wilson C M, Johansson G, Pourkabirian A, Simoen M, Johansson J R, Duty T, Nori F and Delsing P 2011 Nature 479 376
[58] Johansson J R, Johansson G, Wilson C M, Delsing P and Nori F 2013 Phys. Rev. A 87 043804
[59] Felicetti S, Sanz M, Lamata L, Romero G, Johannson G, Delsing P and Solano E 2014 Phys. Rev. Lett. 113 090602
[60] Lindkvist J, Sabin C, Fuentes I, Dragan A, Svenssm I M, Delsing P and Johannson G 2014 Phys. Rev. A 90 052113
[61] Johansson J R, Johannson G and Nori F 2014 Phys. Rev. A 90 053833
[62] Kim E J, Johansson J R and Nori F 2015 Phys. Rev. A 91 033835
[63] Wang H et al 2011 Phys. Rev. Lett. 106 060401
[64] Yang C P, Su Q P, Zheng S B and Han S 2013 Phys. Rev. A 87 022320
[65] Yang C P, Su Q P, Zheng S B, Nori F and Han S 2017 Phys. Rev. A 95 052341
[66] Strach F W, Onyango D, Jacobs K and Simmonds R W 2012 Phys. Rev. A 85 022335
[67] Sharma R and Strach F W 2016 Phys. Rev. A 93 012342
[68] Hua M, Tao M J and Deng F G 2015 Sci. Rep. 5 9274
[69] Hua M, Tao M J and Deng F G 2014 Phys. Rev. A 90 012328
[70] Du L H, Hu Y, Zhou Z W, Guo G C and Zhou X X 2010 New J. Phys. 12 063015
[71] Johnson B R et al 2010 Nat. Phys. 6 663
[72] Mariantoni M, Deppe F, Marx A, Gross R, Wilhelm F K and Solano E 2008 Phys. Rev. B 78 104508
[73] Reuther G M et al 2010 Phys. Rev. B 81 144510
[74] Baust A et al 2015 Phys. Rev. B 91 014515
[75] Baust A et al 2016 Phys. Rev. B 93 214501
[76] Tian L, Allman M S and Simmonds R W 2008 New J. Phys. 10 115001
[77] Chiroli L, Burikhat G, Kumar S and D'Innocenzo D P 2010 Phys. Rev. Lett. 104 230502
[78] Peropadre B, Zueco D, Walschner F, Deppe F, Marx A, Gross R and Garcia-Ripoll J J 2013 Phys. Rev. B 87 134504
[79] Yin Y et al 2013 Phys. Rev. Lett. 110 170501
[80] Walschner F et al 2016 Eur. Phys. J. Quantum Technol. 3 10
[81] Johansson J, Nation P and Nori F 2012 Comput. Phys. Commun. 183 1760
[82] Yurke B and Denker J S 1984 Phys. Rev. A 29 1419
[83] Devoret M 1997 Quantum Fluctuations, Les Houches LXIII, 1995 (Amsterdam: Elsevier) pp 351–86
[84] Goldstein H 1980 Classical Mechanics 2nd edn (Reading, MA: Addison-Wesley)
[85] James D F V and Jerke J 1997 Can. J. Phys. 85 625
[86] MeGrant A et al 2012 Appl. Phys. Lett. 100 113510