Non-linear partial differential equations with discrete state-dependent delays in a metric space

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Abstract. We investigate a class of non-linear partial differential equations with discrete state-dependent delays. The existence and uniqueness of strong solutions for initial functions from a Banach space are proved. To get the well-posed initial value problem we restrict our study to a smaller metric space, construct the dynamical system and prove the existence of a compact global attractor.

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1 Introduction

Theory of dynamical systems is a theory which describes qualitative properties of systems, changing in time. One of the oldest branches of this theory is the theory of delay differential equations. We refer to some classical monographs on the theory of ordinary (O.D.E.) delay equations [11, 8, 1]. A characteristic feature of any type of delay equations is that they generate infinite dimensional dynamical systems. The theory of partial (P.D.E.) delay equations is essentially less studied since such equations are simultaneously infinite dimensional in both time (as delay equations) and space (as P.D.E.s) variables, which makes the analysis more difficult. We refer to some works which are close to the present research [29, 4, 5, 3] and to the monograph [34].

Recently, much attention was paid to the investigations of a new class of delay equations - equations with a state-dependent delays (SDD) (see e.g. [30, 31, 32, 15, 16, 17, 18] and also the survey paper [12] for details and references). The study of these equations essentially differ from the ones of equations with constant or time-dependent delays. The main difficulty is that nonlinearities with SDDs are not Lipschitz continuous on the space of continuous functions - the main space of initial data, where the classical theory of delay equations is developed (see the references above). As a result, the corresponding initial value problem (IVP), in general, is not well-posed (in the sense of J. Hadamard...
An explicit example of the non-uniqueness of solutions to an ordinary equation with state-dependent delay (SDD) is given in [33] (see also [12] p.465). As noticed in [12, p.465] "typically, the IVP is uniquely solved for initial and other data which satisfy suitable Lipschitz conditions.”

First attempts to study P.D.E.s with SDDs have been made for different types of delays: for a distributed delay problem in [20, 21] (see also [22]) and for discrete SDDs in [13] (mild solutions, infinite delay) as well as in [21] (weak solutions, finite delay).

The following property of solutions of P.D.E.s (with or without delays) is very important for the study of equations with a discrete state-dependent delay. Considering any type of solutions (weak, mild, strong or classical) and having the property \( u \in C([a, b]; X) \), one cannot, in general, guarantee that the solution is a Lipschitz function \( u : [a, b] \rightarrow X \). This fact brings essential difficulties for the extension of the methods developed for O.D.E.s (see the discussion above). That is why in previous investigations we proposed alternative approaches i.e. approximations of a solution of a P.D.E. with a discrete SDD by a sequence of solutions of P.D.E.s with distributed SDDs [20, 21] or use an "ignoring condition" for a discrete SDD function [23].

The main goal of the present work is to make a step in extending the approach used for O.D.E.s with SDDs [30, 31, 12] to the case of P.D.E.s. Our idea is to look for a wider space \( Y \supset X \) such that a solution \( u : [a, b] \rightarrow Y \) be a Lipschitz function (with respect to the weaker norm of \( Y \)) and to construct a dynamical system on a subset of the space \( C([a, b]; Y) \). It is interesting to notice that, in contrast to the previous investigations, the dynamical system is constructed on a metric space which is not a linear space.

The article is organized as follows. Section 2 contains formulation of the model and the proof of the existence and uniqueness of strong solutions for initial functions from a Banach space. In Section 3 we construct an evolution operator \( S_t \) and study its asymptotic properties. Here we restrict the evolution operator to a smaller metric space to get the continuity of \( S_t \), which is not available in the initial Banach space. Section 4 deals with the particular case when the delay time is state-independent. Here we also compare the results with the state-dependent case.

## 2 Formulation of the model and basic properties

Our goal is to present an approach to study the following partial differential equation with state-dependent discrete delay

\[
\frac{\partial}{\partial t} u(t, x) + A u(t, x) + d u(t, x) = b (Bu(t - \eta(u_t), \cdot))(x) \equiv (F(u_t))(x), \quad x \in \Omega, \tag{1}
\]

where \( A \) is a densely-defined self-adjoint positive linear operator with domain \( D(A) \subset L^2(\Omega) \) and with compact resolvent, so \( A : D(A) \rightarrow L^2(\Omega) \) generates an analytic semigroup, \( \Omega \) is a smooth bounded domain in \( R^n \), \( B : L^2(\Omega) \rightarrow L^2(\Omega) \) is a bounded operator to be specified later, \( b : R \rightarrow R \) is a locally Lipschitz map and \( d \) is a non-negative constant. The function \( \eta(\cdot) : C([-r, 0]; L^2(\Omega)) \rightarrow [0, r] \subset R_+ \) represents the state-dependent discrete delay. We denote for short \( C \equiv C([-r, 0]; L^2(\Omega)) \). The norms in \( L^2(\Omega) \) and \( C \)
are denoted by $|| \cdot ||$ and $|| \cdot ||_C$ respectively. By $\langle \cdot , \cdot \rangle$ we denote the inner product in $L^2(\Omega)$. As usual for delay equations, we denote by $u_t$ the function of $\theta \in [-r, 0]$ by the formula $u_t \equiv u_t(\theta) \equiv u(t + \theta)$.

**Remark 1.** For example, the operator $B$ may be of the following forms (linear examples)

$$[Bv](x) \equiv \int_\Omega v(y) \bar{f}(x, y)dy, \quad x \in \Omega,$$

or even simpler

$$[Bv](x) \equiv \int_\Omega v(y) f(x - y)\ell(y)dy, \quad x \in \Omega,$$

where $f : \Omega - \Omega \rightarrow R$ is a smooth function, $\ell \in C_0^\infty(\Omega)$. In the last case the nonlinear term in (2) takes the form

$$(F(u_t))(x) \equiv b \left( \int_\Omega u(t - \eta(u_t), y) f(x - y)\ell(y)dy \right), \quad x \in \Omega,$$

We consider equation (1) with the following initial conditions

$$u|_{[-r,0]} = \varphi.$$  

(H.B) We will need the following Lipschitz property of the operator $B$

$$\exists L_B > 0 : \forall u, v \in L^2(\Omega) \Rightarrow ||Bu - Bv|| \leq L_B \cdot ||A^{1/2}(u - v)||.$$  

(H.\eta) The discrete delay function $\eta : \mathbb{C} \rightarrow [0, r]$ satisfies

$$\exists L_\eta > 0 : \forall \varphi, \psi \in \mathbb{C} \Rightarrow |\eta(\varphi) - \eta(\psi)| \leq L_\eta \cdot \max_{\theta \in [-r, 0]} ||A^{1/2}(\varphi(\theta) - \psi(\theta))||.$$  

**Remark 2.** For the term of the form (3), assuming that for all (almost all) $x \in \Omega \Rightarrow f(\cdot - x)\ell(\cdot) \in D(A^{1/2})$ and $u \in L^2(\Omega) \subset D(A^{-1/2})$ one gets $||\langle u,f(\cdot - x)\ell(\cdot) \rangle|| \leq ||A^{-1/2}u|| \cdot ||A^{1/2}f(\cdot - x)\ell(\cdot)||$ which implies

$$\left( \int_\Omega \left( \int_\Omega u(y)f(y-x)\ell(y)dy \right)^2 dx \right)^{1/2} \leq ||A^{-1/2}u|| \cdot \left( \int_\Omega ||A^{1/2}f(\cdot - x)\ell(\cdot)||^2 dx \right)^{1/2}.$$  

Hence, property (H.B) (see (6)) holds with $L_B \equiv \left( \int_\Omega ||A^{1/2}f(\cdot - x)\ell(\cdot)||^2 dx \right)^{1/2}$.

The same arguments hold (with $L_B \equiv \left( \int_\Omega ||A^{1/2}\bar{f}(x, \cdot)||^2 dx \right)^{1/2}$) for a more general term of the form (2).

Now we introduce the following

**Definition 1.** A vector-function $u(t) \in C([-r,T]; D(A^{-1/2})) \cap C([0,T]; D(A^{1/2})) \cap L^2(0,T; D(A))$ with derivative $\dot{u}(t) \in L^\infty(0,T; D(A^{-1/2}))$ is a strong solution of problem (1), (2) on an interval $[0,T]$ if
• \( u(\theta) = \varphi(\theta) \) for \( \theta \in [-r,0] \);

• for any function \( v \in L^2(0,T;L^2(\Omega)) \) such that \( \dot{v} \in L^2(0,T;D(A^{-1})) \) and \( v(T) = 0 \), one has

\[
- \int_0^T \langle u(t), \dot{v}(t) \rangle dt + \int_0^T \langle A^{1/2}u(t), A^{1/2}v(t) \rangle dt = \langle \varphi(0), v(0) \rangle + \int_0^T \langle F(u_t) - du(t), v(t) \rangle dt.
\]

(8)

Let us introduce the following space

\[
\mathcal{L} \equiv \left\{ \varphi \in C([-r,0];D(A^{-1/2})) | \sup_{s \neq t} \left\{ \frac{|A^{-1/2}(\varphi(s) - \varphi(t))|}{|s-t|} \right\} < +\infty; \ \varphi(0) \in D(A^{1/2}) \right\}
\]

with the natural norm

\[
||\varphi||_{\mathcal{L}} \equiv \max_{s \in [-r,0]} |A^{-1/2}\varphi(s)| + \sup_{s \neq t} \left\{ \frac{|A^{-1/2}(\varphi(s) - \varphi(t))|}{|s-t|} \right\} + |A^{1/2}\varphi(0)|.
\]

(9)

Now we prove the following theorem on the existence and uniqueness of solutions.

**Theorem 1.** Let assumptions (H.B) and (H.\( \eta \)) hold (see [31], [32]). Assume the function \( b : R^d \to R^d \) is locally Lipschitz and bounded \( (b(\cdot) \leq M_b) \).

Then for any initial function \( \varphi \in \mathcal{L} \) (the space \( \mathcal{L} \) is defined in [31], [32]) the problem (7), (8) has a unique strong solution on any time interval \( [0,T] \). The solution has the property \( \dot{u} \in L^2(0,T;L^2(\Omega)) \).

**Remark 3.** Let us notice that we do not assume that \( \varphi \in L^2(-r,0;D(A)) \), but the definition of strong solution above implies that

\[
u_t \in L^2(-r,0;D(A)), \ \forall t \geq r.
\]

(11)

**Proof of theorem 1.** Let us denote by \( \{e_k\}_{k=1}^\infty \) an orthonormal basis of \( L^2(\Omega) \) such that \( \lambda e_k = \lambda_k e_k, 0 < \lambda_1 < \ldots < \lambda_k \to +\infty \).

Consider Galerkin approximate solutions of order \( m \):

\[
u^m = \nu^m(t,x) = \sum_{k=1}^m g_{k,m}(t)e_k, \text{ such that } \left\{
\begin{array}{l}
\langle \ddot{\nu}^m + A\nu^m + du^m - F(\nu^m), e_k \rangle = 0, \\
\langle \nu^m(\theta), e_k \rangle = \langle \varphi(\theta), e_k \rangle, \ \forall \theta \in [-r,0]
\end{array}
\right.
\]

(12)

\forall k = 1,\ldots,m. \text{ Here } g_{k,m} \in C^1(0,T;R) \cap L^2(-r,T;R) \text{ with } \dot{g}_{k,m} \text{ being absolutely continuous.}

The system (12), is an (ordinary) differential equation in \( R^m \) with a concentrated state-dependent delay for the unknown vector function \( U(t) \equiv (g_{1,m}(t),\ldots,g_{m,m}(t)) \) (the corresponding theory is developed in [31], [32] see also a recent review [12]).

The key difference between equations with state-dependent and state-independent delays is that the first type of equations is not well-posed in the space of continuous (initial) functions. To get the well-posed initial value problem, the theory [31], [32], [12]
suggests to restrict considerations to a smaller space of Lipschitz continuous functions or even to a smaller subspace of $C^1([-r, 0]; R^m)$.

Condition $\varphi \in L$ implies that initial data $U(\cdot)[[-r, 0] \equiv P_m \varphi(\cdot)$ is Lipschitz continuous as a function from $[-r, 0]$ to $R^m$. Here $P_m$ is the orthogonal projection onto the subspace span $\{e_1, \ldots, e_m\} \subset L^2(\Omega)$. Hence we can apply the theory of O.D.E.s with state-dependent delay (see e.g. [12]) to get the local existence and uniqueness of solutions of (12).

Now we look for an a priori estimate to prove the continuation of solutions $u^m$ of (12) on any time interval $[0, T]$ and then use it for the proof (by the method of compactness, see [14]) of the existence of strong solutions to (1), (5).

We multiply the first equation in (12) by $\lambda g_{k,m}$ and sum for $k = 1, \ldots, m$ to get

$$\frac{1}{2} \frac{d}{dt} ||A^{1/2} u^m(t)||^2 + ||A u^m(t)||^2 + d \cdot ||A^{1/2} u^m(t)||^2 = \langle P_m F(u^m_t), Au^m(t) \rangle \leq \frac{1}{2} ||P_m F(u^m(t))||^2 + \frac{1}{2} ||Au^m(t)||^2.$$ 

Since the function $b$ is bounded ($b(\cdot) \leq M_b$), we have $||F(u^m_t)||^2 \leq M_b^2 |\Omega|$ (here $|\Omega| \equiv \int_\Omega 1 \, dx$) and, as a result, we conclude that

$$\frac{d}{dt} ||A^{1/2} u^m(t)||^2 + ||A u^m(t)||^2 \leq M_b^2 |\Omega|. \quad (13)$$

We integrate (13) with respect to $t$, use the properties $\varphi(0) \in D(A^{1/2})$, $u^m(0) = P_m \varphi(0) \in D(A^{1/2})$ and $||A^{1/2} u^m(0)|| = ||A^{1/2} P_m \varphi(0)|| \leq ||A^{1/2} \varphi(0)||$ to get an a priori estimate

$$||A^{1/2} u^m(t)||^2 + \int_0^t ||A u^m(\tau)||^2 d\tau \leq ||A^{1/2} \varphi(0)||^2 + M_b^2 |\Omega| \cdot T, \quad \forall m, \forall t \in [0, T]. \quad (14)$$

Estimate (14) means that

$$\{u^m\}_m \subset L^\infty(0, T; D(A)) \cap L^2(0, T; D(A)) \quad \forall m.$$ 

Using this and (12), we get that

$$\{\dot{u}^m\}_m \subset L^\infty(0, T; D(A^{-1/2})) \cap L^2(0, T; L^2(\Omega)).$$

Hence the family $\{(u^m; \dot{u}^m)\}_m \subset Z_1 \equiv \left(L^\infty(0, T; D(A^{1/2})) \cap L^2(0, T; D(A)) \right) \times \left(L^\infty(0, T; D(A^{-1/2})) \cap L^2(0, T; L^2(\Omega)) \right). \quad (15)$

Therefore there exist a subsequence $\{(u^k; \dot{u}^k)\}$ and an element $(u; \dot{u}) \in Z_1$ such that

$$\{(u^k; \dot{u}^k)\} \text{ *-weak converges to } (u; \dot{u}) \text{ in } Z_1. \quad (16)$$

The proof that any *-weak limit is a strong solution is standard.

Now we prove the uniqueness of strong solutions.
Using the properties $\varphi \in \mathcal{L}$, the definition of a strong solution $u$ and $\dot{v}(t) \in L^\infty(0,T;D(A^{-1/2}))$ (see (16)) we have that for any such a solution $u$ and any $T > 0$ there exists $L_{v,T} > 0$, such that

$$||A^{-1/2}(v(s^1) - v(s^2))|| \leq L_{v,T} \cdot |s^1 - s^2|, \quad \forall s^1, s^2 \in [-r, T].$$

(17)

Consider two strong solutions $u$ and $v$ of (1), (5) (not necessarily with the same initial function).

Assumption (H.B) (see (6)) and the Lipschitz property of $b$ imply

$$||F(u_{s^1}) - F(u_{s^2})||^2 = \int_\Omega |b(Bu(s^1 - \eta(u_{s^1}), x)) - b(Bv(s^2 - \eta(u_{s^2}), x))|^2 \, dx$$

$$\leq L_b^2 \int_\Omega |Bu(s^1 - \eta(u_{s^1}), x) - Bv(s^2 - \eta(u_{s^2}), x)|^2 \, dx$$

$$= L_b^2 \cdot ||[Bu](s^1 - \eta(u_{s^1}), \cdot) - [Bv](s^2 - \eta(u_{s^2}), \cdot)||^2$$

$$\leq L_b^2 L_B^2 \cdot ||A^{-1/2}(u(s^1 - \eta(u_{s^1})) - v(s^2 - \eta(u_{s^2})))||^2. \quad (18)$$

Now, for any two strong solutions, we have

$$F(u_t) - F(v_t) = b(Bu(t - \eta(u_t))) - b(Bv(t - \eta(v_t))) \pm b(Bv(t - \eta(u_t))).$$

Using the Lipschitz properties of $b$, $B$ and $\eta$ (see (6), (11), and also (17), (18), one gets

$$||F(u_t) - F(v_t)||$$

$$\leq L_b L_B \left( \max_{s \in [t-r,t]} ||A^{-1/2}(u(s) - v(s))|| + ||A^{-1/2}(v(t - \eta(u_t)) - v(t - \eta(v_t)))|| \right)$$

$$\leq L_b L_B \left( ||A^{-1/2}(u_t - v_t)||_{C} + L_{v,T} \cdot |\eta(u_t) - \eta(v_t)| \right)$$

$$\leq L_b L_B (1 + L_{v,T} \cdot L_\eta) \cdot ||A^{-1/2}(u_t - v_t)||_{C}. \quad (19)$$

We denote for short

$$C_{v,T} \equiv L_b L_B (1 + L_{v,T} \cdot L_\eta). \quad (20)$$

Now the standard variation-of-constants formula

$$u(t) = e^{-A t} u(0) + \int_0^t e^{-A(t-\tau)} F(u_\tau) \, d\tau$$

and (19) give

$$||A^{-1/2}(u_t - v_t)||_{C} \leq ||A^{-1/2}(u_0 - v_0)||_{C} + C_{v,T} \cdot \int_0^t e^{-\lambda_1(t-\tau)} ||A^{-1/2}(u_\tau - v_\tau)||_{C} \, d\tau.$$

The last estimate (by Gronwall’s lemma) implies

$$||A^{-1/2}(u_t - v_t)||_{C} \leq ||A^{-1/2}(u_0 - v_0)||_{C} \cdot \left[ 1 + \frac{C_{v,T}}{C_{v,T} - \lambda_1} \left( e^{(C_{v,T} - \lambda_1) t} - 1 \right) \right], \quad (21)$$

which gives the uniqueness of strong solutions of (1), (5).

The proof of theorem 1 is complete.
Remark 4. It is very important that the term \(1 + \frac{C_v T}{C_v T - \lambda_1} (e^{(C_v T - \lambda_1)t} - 1)\) in (21) tends to +\(\infty\), when \(L_v, T \to +\infty\), except the case \(L_\eta = 0\) (see (20)).

Let us get an additional estimate for strong solutions.

The standard variation-of-constants formula \(u(t) = e^{-A_t}u(0) + \int_0^t e^{-A(t-\tau)}F(u_\tau)\,d\tau\), \(\leq\), \(\geq\) and the estimate \(||A^\alpha e^{-tA}|| \leq (\frac{\alpha}{2})^\alpha e^{-\alpha}\) (see e.g. [6, (1.17), p.84]) give

\[
||A^{1/2}(u(t) - v(t))|| \leq e^{-\lambda t}||A^{1/2}(u(0) - v(0))|| + \int_0^t ||A^{1/2}e^{-A(t-\tau)}|| \cdot ||F(u_\tau) - F(v_\tau)||\,d\tau
\]

\[
\leq e^{-\lambda t}||A^{1/2}(u(0) - v(0))|| + 2t^{1/2}\left(\frac{1}{2}\right)^{1/2}e^{-1/2} \cdot C_v,T \cdot ||A^{-1/2}(u_0 - v_0)||_C.
\]

(22)

Here we used \(||A^{1/2}e^{-A(t-\tau)}|| \leq \left(\frac{1}{2}\right)^{1/2}e^{-1/2}\) and \(\int_0^t(t-\tau)^{-1/2}d\tau = 2t^{1/2}\).

Now estimates (21), (22) give

\[
||A^{1/2}(u(t) - v(t))|| + ||A^{-1/2}(u_t - v_t)||_C
\]

\[
\leq e^{-\lambda t}||A^{1/2}(u(0) - v(0))|| + D_{v,T} \cdot ||A^{-1/2}(u_0 - v_0)||_C.
\]

(23)

Here we denote

\[
D_{v,T} \equiv 2T^{1/2}\left(\frac{1}{2}\right)^{1/2}e^{-1/2} \cdot C_v,T + \left[1 + \frac{C_v T}{C_v T - \lambda_1} (e^{(C_v T - \lambda_1)t} - 1)\right].
\]

(24)

3 Asymptotic behavior

In this section we study long-time behavior of the strong solutions of the problem (1), (5).

Due to theorem 1, we define in the standard way the evolution semigroup \(S_t : \mathcal{L} \to \mathcal{L}\) (the space \(\mathcal{L}\) is defined in (2)) by the formula

\[
S_t \varphi \equiv u_t, \quad t \geq 0,
\]

(25)

where \(u(t)\) is the unique strong solution of the problem (1), (5).

Remark 5. We emphasize that the evolution semigroup \(S_t : \mathcal{L} \to \mathcal{L}\) is not a dynamical system in the standard sense (see e.g. [2, 28, 6]) since \(S_t\) is not a continuous mapping in the topology of \(\mathcal{L}\) i.e. the problem (1), (2) is not well-posed in the sense of J. Hadamard [2, 10].

Our first goal is to prove

Lemma 1. Let all the assumptions of theorem 1 be satisfied. Then for any \(\alpha \in (\frac{1}{2}, 1)\), there exists a bounded in the space \(C^1([-r, 0]; D(A^{-1/2})) \cap C([-r, 0]; D(A^{\alpha}))\) set \(\mathcal{B}_\alpha\), which absorbs any strong solution of the problem (1), (5) with any initial function \(\varphi \in \mathcal{L}\).

Proof of lemma 1. Using \(||A^{1/2}v||^2 \leq \lambda_1^{-1} \cdot ||Av||^2\), we get from (13) that

\[
\frac{d}{dt}||A^{1/2}u^m(t)||^2 + \lambda_1||A^{1/2}u^m(t)||^2 \leq M_0^2|\Omega|.
\]
We multiply the last estimate by $e^{\lambda t}$ and integrate over $[0, t]$ to obtain

$$||A^{1/2} u^m(t)||^2 \leq ||A^{1/2} u^m(0)||^2 e^{-\lambda t} + \lambda^{-1} M_b^2 \Omega$$

$$\leq ||A^{1/2} \varphi(0)||^2 e^{-\lambda t} + \lambda^{-1} M_b^2 \Omega.$$  \tag{26}

This and (12) give $||A^{-1/2} \dot{u}^m(t)||^2 \leq 2 ||A^{1/2} \varphi(0)||^2 e^{-\lambda t} + 2 \lambda^{-1} M_b^2 \Omega + M_b^2 \Omega$. The last two estimates imply

$$||A^{1/2} u^m(t)||^2 + ||A^{-1/2} \dot{u}^m(t)||^2 \leq 3 ||A^{1/2} \varphi(0)||^2 e^{-\lambda t} + (1 + 3 \lambda^{-1}) M_b^2 \Omega. \tag{27}$$

We get an analogous estimate for a strong solution of the problem (1), (5), using the well-known

**Proposition 1.** [35, Theorem 9]. Let $X$ be a Banach space. Then any *-weak convergent sequence $\{w_k\}_{n=1}^\infty \in X^*$ *-weak converges to an element $w_\infty \in X^*$ and $||w_\infty||_X \leq \liminf_{n \to \infty} ||w_n||_X$.

Now we consider the space $V \equiv C^1([-r, 0]; D(A^{-1/2})) \cap C([-r, 0]; D(A^{1/2}))$, fix any positive $\varepsilon_0$ and obtain that the ball $B_0$ of $V$

$$B_0 \equiv \{ v \in V : ||v||_V^2 \leq R_0^2 \equiv (1 + 3 \lambda^{-1}) M_b^2 \Omega + \varepsilon_0 \} \tag{28}$$

is absorbing for any strong solution of the problem (1), (5) (see (27)).

Now we are in a position to use the arguments presented in [3] Lemma 2.4.1, p.101 and get (for any $\frac{1}{2} < \alpha < 1$) the existence of the absorbing ball

$$B_{\alpha} \equiv \{ v \in C([-r, 0]; D(A^\alpha)) : ||v||_{C([-r, 0]; D(A^\alpha))} \leq R_{\alpha} \}, \tag{29}$$

where $R_{\alpha} \equiv (\alpha - 1/2)^{\alpha - 1/2} \cdot \left[ \lambda_1^{-1/2} M_b \sqrt{\Omega} + \varepsilon \right] + \frac{\alpha^2}{1 - \alpha} \cdot M_b \sqrt{\Omega}$ with any fixed $\varepsilon > 0$. More precisely, the standard variation-of-constants formula $u(t) = e^{-At} u(0) + \int_0^t e^{-A(t-\tau)} F(u_\tau) \, d\tau$ and the estimate $||A^\alpha e^{-tA}|| \leq \left( \frac{\alpha}{\tau} \right)^\alpha e^{-\alpha}$ (see e.g. [6] (1.17), p.84) give

$$||A^\alpha u(t + 1)|| \leq (\alpha - 1/2)^{\alpha - 1/2} ||A^{1/2} u(t)|| + \int_t^{t+1} \left( \frac{\alpha}{t + 1 - \tau} \right)^\alpha ||F(u_\tau)|| \, d\tau.$$ 

Let us consider any bounded in $\mathcal{L}$ set $\hat{B}$. Estimate (26) and Proposition 1 give $||A^{1/2} u(t)|| \leq \left[ \lambda_1^{-1/2} M_b \sqrt{\Omega} + \varepsilon \right]$ for all $t \geq t_{\hat{B}}$ (here $t_{\hat{B}}$ depends on $\hat{B}$ only). These and the estimate $||F(u_\tau)|| \leq M_b \sqrt{\Omega}$ imply (29).

The above estimates (28), (29) show that there exists a subset (a ball) of $V_\alpha \equiv C^1([-r, 0]; D(A^{-1/2})) \cap C([-r, 0]; D(A^\alpha))$ (here $\frac{1}{2} < \alpha < 1$)

$$BV_\alpha \equiv \{ v \in V_\alpha : ||v||_{V_\alpha} \leq \bar{R}_\alpha \}, \tag{30}$$

such that for any strong solution, starting in $\varphi$ from any bounded set $\hat{B} \subset \mathcal{L}$, there exists $t_{\hat{B}} \geq 0$ such that

$$S_t \varphi \in BV_\alpha, \quad \text{for all} \quad t \geq t_{\hat{B}}. \tag{31}$$
The proof of lemma 1 is complete. ■

We will use notation

\[ |||\varphi||| \equiv \sup_{s \neq t} \left\{ \frac{||A^{-1/2}(\varphi(s) - \varphi(t))||}{|s - t|} \right\} \text{ for } \varphi \in \mathcal{L}. \]

Let us fix \(R^0 > 0\) and consider the metric space \(\mathcal{L}_{R^0}\) which is the set \(\{ \varphi \in \mathcal{L} : |||\varphi||| \leq R^0 \}\) equipped with the metrics (c.f. (10))

\[ \rho(\varphi, \phi) \equiv \max_{s \in [-r,0]} ||A^{-1/2}(\varphi(s) - \phi(s))|| + ||A^{1/2}(\varphi(0) - \phi(0))||. \] (32)

One can check that \((\mathcal{L}_{R^0}; \rho)\) is a complete metric space and any set \(\{ \varphi \in \mathcal{L} : |||\varphi||| \leq R^0 \} < R^0 \) is closed.

We need the following (technical) assumption

(H.I) There exists \(R^0 > \tilde{R}_\alpha \) (\(\tilde{R}_\alpha\) is defined in (30)) such that the set \(\{ \varphi \in \mathcal{L} : |||\varphi||| \leq R^0 \}\) is positively invariant for the semigroup \(S_t\) i.e.

\[ \forall \varphi \in \mathcal{L} : |||\varphi||| \leq R^0 \Rightarrow |||S_t\varphi||| \leq R^0, \quad \forall t > 0. \] (33)

Our next result is the following

**Theorem 2.** Let (H.I) and all the assumptions of theorem 1 be satisfied. Then the evolution semigroup \(S_t : \mathcal{L}_{R^0} \rightarrow \mathcal{L}_{R^0}\), (see (23)), possesses a global attractor in the metric space \((\mathcal{L}_{R^0}; \rho)\).

Proof of theorem 2. Now we concentrate on the metric space \((\mathcal{L}_{R^0}; \rho)\) (here \(R^0 > \tilde{R}_\alpha\)). The reason for this is that the evolution semigroup \(S_t\) is not continuous on the whole space \(\mathcal{L}\) (see remark 5). On the other hand, we notice:

**Remark 6.** Estimate (23) implies that the evolution semigroup \(S_t\) is a continuous mapping in the topology of \((\mathcal{L}_{R^0}; \rho)\) i.e. \(\rho(S_t\varphi, S_t\phi) \leq D_{v,T} \cdot \rho(\varphi, \phi)\) for \(\varphi, \phi \in \mathcal{L}_{R^0}\), and \(t \in [0,T]\). Here \(D_{v,T}\) is defined by (24) (see also (20)) with \(L_{v,T} = R^0\) (c.f. (17)).

Corollary 4 from [24] implies that \(BV_\alpha\) is relatively compact in \(C([-r,0]; D(A^{-1/2}))\) (see also [24] lemma 1). This fact and the property \(|||A^0\varphi(0)||| \leq \tilde{R}_\alpha, \frac{1}{2} < \alpha < 1\) for all \(\varphi \in BV_\alpha\) gives that \(BV_\alpha\) is relatively compact in the topology of \((\mathcal{L}_{R^0}; \rho)\).

Let us consider the following set

\[ K \equiv Cl[BV_\alpha, \mathcal{L}_{R^0}; \rho], \]

where \(Cl[\cdot, \mathcal{L}_{R^0}; \rho]\) is the closure in the topology of \((\mathcal{L}_{R^0}; \rho)\). The above properties show that \(K\) is compact in \((\mathcal{L}_{R^0}; \rho)\).

We get (see (31)) that for any strong solution, starting in \(\varphi\) from any bounded set \(\tilde{B} \subset \mathcal{L}_{R^0}\), there exists \(t_B \geq 0\) such that

\[ S_t\varphi \in BV_\alpha \subset K, \quad \text{for all } t \geq t_B. \]

As a result, we conclude that the evolution semigroup \(S_t\) is asymptotically compact (and dissipative) in \((\mathcal{L}_{R^0}; \rho)\).

Finally, by the classical theorem on the existence of an attractor (see, for example, [2, 28, 6]) one gets that \((S_t; (\mathcal{L}_{R^0}; \rho))\) has a compact global attractor. The proof of Theorem 2 is complete. ■
3.1 Equation with a modified nonlinearity

Discussing the technical assumption (H.I), we notice that even in the case when (H.I) is not satisfied for the original system, Lemma 1 allows one to consider a modified system without modifying the long term dynamics of $S_t$ (see [7]). More precisely, one chooses [2] p.545] a $C^\infty$ function $\chi : [0, +\infty) \to [0, 1]$ such that

$$\begin{align*}
\chi(s) = 1, & \quad s \in [0, 1]; \\
\chi(s) = 0, & \quad s \in [2, +\infty); \\
0 \leq \chi(s) \leq 1, & \quad s \in [1, 2]
\end{align*}$$

and set

$$\bar{F}(\varphi) \equiv \chi \left( \frac{||\varphi||_{H,d}}{\bar{R}_\alpha} \right) \cdot F(\varphi).$$

Here we denoted for short $||\varphi||_{H,d} \equiv ||A^{1/2}\varphi(0)|| + d \cdot ||A^{-1/2}\varphi||_C$.

As a result, the modified system (1) (with $\bar{F}(\varphi)$ instead of $F(\varphi)$) has the same behavior inside of the (absorbing) set $BV_\alpha$ (in fact, the behavior is unchanged inside of a bigger set $\{ \varphi : ||\varphi||_{H,d} \leq \bar{R}_\alpha \}$).

Now, using $||\bar{F}(\varphi)|| \leq ||F(\varphi)|| \leq M_b\sqrt{\Omega}$ and the estimate $||A^{-1/2}\bar{u}(t)|| \leq ||A^{1/2}u(t)|| + d||A^{-1/2}u(t)|| + M_b\sqrt{\Omega}$, we conclude that the set

$$\mathcal{L}(\bar{R}_\alpha) \equiv \left\{ \varphi \in \mathcal{L} : ||\varphi||_{H,d} \leq 2\bar{R}_\alpha; \quad ||\varphi|| \leq 2\bar{R}_\alpha + M_b\sqrt{\Omega} \right\}$$

is positively invariant for the evolution operator, constructed by solutions of (1) with the modified nonlinearity $\bar{F}(\varphi)$. We notice that $BV_\alpha \subset \mathcal{L}(\bar{R}_\alpha) \subset \{ \varphi : ||\varphi||_{H,d} \leq 2\bar{R}_\alpha \}$.

This invariantness of the set $\mathcal{L}(\bar{R}_\alpha)$ gives the possibility to define (similar to (25)) an evolution operator $\bar{S}_t : \mathcal{L}(\bar{R}_\alpha) \to \mathcal{L}(\bar{R}_\alpha)$ by solutions of (1) with the modified nonlinearity $\bar{F}$.

Following the line of arguments presented in theorem 2, we prove the following analog to theorem 2

**Theorem 3.** Let (H.I) and all the assumptions of theorem 1 be satisfied. Then the evolution semigroup $\bar{S}_t : \mathcal{L}(\bar{R}_\alpha) \to \mathcal{L}(\bar{R}_\alpha)$, possesses a global attractor in the metric space $(\mathcal{L}(\bar{R}_\alpha); \rho)$. Here, as before, $\rho$ is the metrics defined by (32).

**Remark 7.** Discussing the restriction of our study from the linear space $\mathcal{L}$ to the metric spaces $(\mathcal{L}_{R^0}; \rho)$ or $(\mathcal{L}(\bar{R}_\alpha); \rho)$, we notice that it is a natural step even for ordinary differential equations with a discrete state-dependent delay. For example, in [30, Proposition 1 and Corollary 1] it is shown that maximal solutions of a scalar delay equation with a SDD constitute a semiflow on the set $\{ \phi : \text{Lip}(\phi) \leq k, ||\phi|| < w \} \subset C([-r, 0], R)$. Here $\text{Lip}(\phi) = \sup_{x \neq y} |\phi(x) - \phi(y)| \cdot |x - y|^{-1}$.

4 Particular case of a state-independent delay ($\eta = \text{const}$)

In this particular case, the assumption (H.\eta) (see (7)) is valid automatically with $L_\eta = 0$. Following the proof of theorem 1, one can see that the assumption
\[ \sup_{s \neq t} \left\{ \| A^{1/2}(\varphi(s) - \varphi(t)) \| \cdot |s - t|^{-1} \right\} < +\infty \]

is not needed in the case \( \eta = \text{const} \). This implies that for any initial function \( \varphi \in H \) (c.f. (2)),

\[
H \equiv \left\{ \varphi \in C([−r, 0]; D(A^{-1/2})) \mid \varphi(0) \in D(A^{1/2}) \right\}
\]

(34)

the problem (1), (5) has a strong solution. The uniqueness of a strong solution follows from (23) and the fact that \( L_\eta = 0 \) implies \( D_{v,T} \) (defined in (24)) is bounded for any \( \varphi \in H \) (c.f. remark 4 and (20)). This fact gives the continuity of \( S_t : H \to H \) (c.f. remark 5) and as a consequence, that the pair \( (S_t; H) \) is a dynamical system.

Following the proofs of lemma 1 and theorem 2 we have the following result.

**Theorem 4.** Assume \( \eta = \text{const} \). Let the assumption (H.B) hold and the function \( b : R \to R \) be locally Lipschitz and bounded.

Then for any initial function \( \varphi \in H \) the problem (1), (5) has a unique strong solution on any time interval \([0, T]\). The solution has the property \( \dot{u} \in L^2(0, T; L^2(\Omega)) \).

Moreover, the pair \( (S_t; H) \) constitutes a dynamical system which possesses a global attractor. The attractor is a bounded set in \( C^1([−r, 0]; D(A^{-1/2})) \cap C([−r, 0]; D(A^\alpha)) \) for any \( \alpha \in \left( \frac{1}{2}, 1 \right) \).

Now we can compare two cases (state-dependent and state-independent delays), assuming

- the assumption (H.B) holds and
- the function \( b : R \to R \) is locally Lipschitz and bounded.

| state-dependent delay \( \eta \) | state-independent delay |
|-----------------------------|------------------------|
| The existence and uniqueness of solutions | \( \varphi \in L \subset H \) and (H.\( \eta \)) | \( \varphi \in H \) |
| The continuity of \( S_t \) and existence of an attractor | \( S_t : (L_{R^0}; \rho) \to (L_{R^0}; \rho) \) | \( S_t : H \to H \) |
| | \( \tilde{S}_t : (L(\hat{R}_\alpha); \rho) \to (L(\hat{R}_\alpha); \rho) \) | |

**Remark 8.** We notice that \( L_{R^0} \subset L \subset H \) and the metrics \( \rho \) is the natural metrics of the space \( H \).

As an application (for both cases of state-dependent and state-independent delays) we can consider the diffusive Nicholson’s blowflies equation (see e.g. [27, 25]) with state-dependent delays. More precisely, we consider equation (11) where \(-A\) is the Laplace operator with the Dirichlet boundary conditions, \( \Omega \subset R^m \) is a bounded domain with a smooth boundary, the function \( f \) can be, for example, \( f(s) = \frac{1}{\sqrt{4\pi \alpha}} e^{-s^2/4\alpha} \), as in [26] (see remark 2), the nonlinear (birth) function \( b \) is given by \( b(w) = p \cdot we^{-w} \). Function \( b \) is bounded, so for any delay function \( \eta \), satisfying (H.\( \eta \)), the conditions of Theorems 1,2 are valid (in the case when the assumption (H.I) is satisfied). As a result, we conclude that the initial value problem (11), (5) is well-posed in \( (L_{R^0}; \rho) \) and the dynamical system \( (S_t, (L_{R^0}; \rho)) \) has a global attractor (Theorem 2).
If necessary, we modify the system according to subsection 3.1 and get the existence of a global attractor for the dynamical system \((\mathcal{S}; (\mathcal{L}(\bar{R}_a); \rho))\).

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