Abstract. This paper introduces the notion of orbit coherence in a permutation group. Let $G$ be a group of permutations of a set $\Omega$. Let $\pi(G)$ be the set of partitions of $\Omega$ which arise as the orbit partition of an element of $G$. The set of partitions of $\Omega$ is naturally ordered by refinement, and admits join and meet operations. We say that $G$ is join-coherent if $\pi(G)$ is join-closed, and meet-coherent if $\pi(G)$ is meet-closed.

Our central theorem states that the centralizer in $\text{Sym}(\Omega)$ of any permutation $g$ is meet-coherent, and subject to a certain finiteness condition on the orbits of $g$, also join-coherent. In particular, if $g$ is a permutation of degree $n$, then $\pi(\text{Cent}_{S_n}(g))$ is a lattice of partitions.

A related result states that the intransitive direct product and the imprimitive wreath product of two finite permutation groups are join-coherent if and only if each of the groups is join-coherent. We classify the groups $G$ such that $\pi(G)$ is a chain; also the primitive join-coherent groups of finite degree; and also the join-coherent groups of degree $n$ containing a normal $n$-cycle.

1. Introduction

Let $G$ be a group acting on a set $\Omega$. Each element $g$ of $G$ has associated with it a partition $\pi(g)$ of $\Omega$, whose parts are the orbits of $g$. We define $\pi(G)$ to be the set $\{\pi(g) \mid g \in G\}$.

If $\mathcal{P}, \mathcal{Q}$ are two partitions of $\Omega$ then we say that $\mathcal{P}$ is a refinement of $\mathcal{Q}$, and write $\mathcal{P} \preceq \mathcal{Q}$, if every part of $\mathcal{P}$ is contained in a part of $\mathcal{Q}$. The set of all partitions of $\Omega$ forms a lattice under $\preceq$. That is to say, any two partitions $\mathcal{P}$ and $\mathcal{Q}$ have a supremum with respect to refinement, denoted $\mathcal{P} \lor \mathcal{Q}$, and an infimum with respect to refinement, denoted $\mathcal{P} \land \mathcal{Q}$. The parts of $\mathcal{P} \land \mathcal{Q}$ are precisely the non-empty intersections of the parts of $\mathcal{P}$ and $\mathcal{Q}$; for a description of $\mathcal{P} \lor \mathcal{Q}$ see Section 2 below. The lattice of all partitions of $\Omega$ is called the congruence lattice on $\Omega$, and is denoted $\text{Con}(\Omega)$.

The object of this paper is to prove a number of interesting structural and classification results on the permutation groups possessing one or both of the following properties.

Definition. Let $G$ be a group acting on a set.

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(1) We say that $G$ is join-coherent if $\pi(G)$ is closed under $\lor$.
(2) We say that $G$ is meet-coherent if $\pi(G)$ is closed under $\land$.

We refer to these properties collectively as orbit coherence properties. Our first main theorem describes the groups $G$ such that $\pi(G)$ is a chain. It is clear that any such groups is both join- and meet-coherent.

**Theorem 1.** Let $\Omega$ be a set, and let $G \leq \text{Sym}(\Omega)$ be such that $\pi(G)$ is a chain. There is a prime $p$ such that the length of any cycle of any element of $G$ is a power of $p$. Furthermore, $G$ is abelian, and either periodic or torsion-free.

(1) If $G$ is periodic then $G$ is either a finite cyclic group of $p$-power order, or else isomorphic to the Prüfer $p$-group.

(2) If $G$ is torsion-free then $G$ is isomorphic to a subgroup of the $p$-adic rational numbers $\mathbb{Q}_p$. In this case $G$ has infinitely many orbits on $\Omega$, and the permutation group induced by its action on a single orbit is periodic.

Our second theorem determines when a direct product or wreath product of permutation groups inherits join coherence from its factors. The actions of these groups referred to in this theorem are defined in Sections 4 and 5 below.

**Theorem 2.** Let $X$ and $Y$ be sets and let $G \leq \text{Sym}(X)$ and $H \leq \text{Sym}(Y)$ be permutation groups.

(a) If $G$ and $H$ are finite then $G \times H$ is join-coherent in its product action on $X \times Y$ if and only if $G$ and $H$ are join-coherent and have coprime orders;

(b) If $Y$ is finite then $G \wr H$ is join-coherent in its imprimitive action on $X \times Y$ if and only if $G$ and $H$ are join-coherent.

Our third main theorem, on centralizers in a symmetric group, is the central result of this paper.

**Theorem 3.** Let $\Omega$ be a set, and let $g \in \text{Sym}(\Omega)$. For $k \in \mathbb{N} \cup \{\infty\}$ let $m_k$ be the number of orbits of $g$ of length $k$.

(a) $\text{Cent}_{\text{Sym}(\Omega)}(g)$ is meet-coherent.

(b) If $m_k$ is finite for all $k \neq 1$, including $k = \infty$, then $\text{Cent}_{\text{Sym}(\Omega)}(g)$ is join-coherent.

We also show that $\text{Cent}_{\text{Sym}(\Omega)}(g)$ is not join-coherent for any $g$ for which the condition on the values of $m_k$ in the second part of the theorem fails, and so this condition is necessary.
Theorem 3 implies, in particular, that any centralizer in a finite symmetric group is both join- and meet-coherent. This is a remarkable fact, and the starting point of our investigation, at least chronologically. The observation that this important class of groups exhibits orbit coherence justifies our study of these properties, and motivates the search for further examples.

The second part of the paper contains a partial classification of finite transitive join-coherent permutation groups. Our analysis depends on the fact that such a group necessarily contains a full cycle, since the join of the orbit partitions of its elements is the trivial one-part partition. The primitive permutation groups containing full cycles are known; we use this classification to prove the following theorem.

**Theorem 4.** A primitive permutation group of finite degree is join-coherent if and only if it is a symmetric group or a subgroup of $AGL_1(p)$ in its action on $p$ points, where $p$ is prime.

We also give a complete classification of the finite transitive join-coherent groups in which the subgroup generated by a full cycle is normal.

**Theorem 5.** Let $G$ be a permutation group on $n$ points, containing a normal cyclic subgroup of order $n$ acting regularly. Let $n$ have prime factorization $\prod p_i^{a_i}$. Then $G$ is join-coherent if and only if there exists for each $i$ a transitive permutation group $H_i$ on $p_i^{a_i}$ points, such that:

- if $a_i > 1$ then $H_i$ is either cyclic or the extension of a cyclic group of order $p_i^{a_i}$ by the automorphism $x \mapsto x^r$ where $r = p_i^{a_i-1} + 1$,
- if $a_i = 1$ then $H_i$ is a subgroup of the Frobenius group of order $p(p-1)$,
- the orders of the groups $H_i$ are mutually coprime,
- $G$ is permutation isomorphic to the direct product of the groups $H_i$ in its product action.

Note that the permutation groups classified by the Theorem 5 are always imprimitive, unless $n$ is prime. It would be interesting, but we believe difficult, to extend our results to a complete classification of all finite join-coherent permutation groups. The principal obstruction to such a result is the apparently hard problem of classifying those join-coherent transitive imprimitive groups that are not reducible as direct products or as wreath products, in the manner described in Theorem 2, and which do not normalize a full cycle. One example of such a permutation groups is the permutation group of degree 12 generated by

$$(1\ 7)(4\ 10), \quad (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12).$$
It is not hard to check that this group is an imprimitive join-coherent subgroup of index 4 in $C_4 \wr C_3$, and that it does not admit a non-trivial factorization as a direct product or a wreath product.

In smaller degrees our results do yield a complete classification: every join-coherent permutation group of degree at most 11 is either a cyclic group acting regularly, a symmetric group, one of the groups described in Theorem 5, or an imprimitive wreath product of join-coherent groups of smaller degree, as constructed in Theorem 2(b).

The fact that there are no further join-coherent groups of degree at most 11, and also the join-coherence of the group of degree 12 presented above, have been verified computationally. In Section 8, and again in Section 9, we require computer calculation to verify that particular groups are not join-coherent. All of our computations have been performed using Magma [7]. The code for these computations is available from the second author’s website\(^1\).

**Further remarks and background.** It will be clear from the statement of our main results above that the majority of them concern join-coherence rather than meet-coherence. In part this is because a finitely generated join-coherent permutation group must contain a full-cycle, and the restriction on the structure of the group that this imposes is very useful. However an alternative characterization of join-coherence suggests that it is a particularly natural property to study: a permutation group $G$ is join-coherent if and only if for every finitely-generated subgroup $H$ of $G$, there exists an element $h \in G$ whose orbits are the same as the orbits of $H$. There is no similar characterization of meet-coherence in terms of subgroups.

There are groups which exhibit any combination of the properties of join- and meet-coherence. Any symmetric group is both join- and meet-coherent, but any non-cyclic alternating group is neither. The group $C_2 \times C_2$ acting regularly on itself is meet- but not join-coherent. Examples of groups that are join- but not meet-coherent are less easy to find, but one can check that the non-cyclic group of order 21, in its action as a Frobenius group on 7 points, is such an example (see Section 7 for our general results on Frobenius groups).

In the context of lattices, the operations $\lor$ and $\land$ are dual to one another. This duality is not inherited to any great extent by the notions of join- and meet-coherence of permutation groups. An asymmetry can be observed even in the congruence lattice of all partitions of a set: compare for example the two parts of Lemma 2.1 below. For this reason, while there are some parts

\(^1\)See [www.ma.rhul.ac.uk/~uvah099](http://www.ma.rhul.ac.uk/~uvah099).
of the paper, for example Section 4, where join- and meet-coherence admit a common treatment, it is usually necessary to treat each property separately.

Literature on the orbit partitions of permutations is surprisingly sparse. As we hope that this paper shows, there are interesting general properties that remain to be discovered, and we believe that further study is warranted. One earlier investigation which perhaps has something of the same flavour is that of Cameron [3] into cycle-closed permutation groups. If $G$ is a permutation group on a finite set then the cycle-closure $C(G)$ is the group generated by all of the cycles of elements of $G$. Cameron proves that any group which is equal to its cycle-closure is isomorphic as a permutation group to a direct product of symmetric groups in their natural action and cyclic groups acting regularly, with the factors acting on disjoint sets. He also shows that if $G = G_0$, and $G_{i+1} = C(G_i)$, then $G_4 = G_3$ and that there exist groups for which $G_2 \neq G_3$.

There is an extensive literature on the lattice of subspaces of a vector space invariant under a group of linear transformations. In this context both the lattice elements and the lattice operations differ from ours, so there is no immediate connection to our situation. Indeed, we show in Proposition 8.7 below that the general linear group $GL(V)$ is never join-coherent, except in cases where it acts on $V \setminus \{0\}$ as a full symmetric group. Nonetheless, there are certain parallels that it is interesting to observe. For instance, if $T$ is an invertible linear map on a vector space $V$ then, by [2, Theorem 2], the lattice of invariant subspaces of $V$ is a chain if and only if $T$ is cyclic of primary type. Correspondingly, it follows from our classification of groups $G$ such that $\pi(G)$ is a chain in Theorem 1 that if $G$ is a finite permutation group then $\pi(G)$ is a chain if and only if $G$ is a cyclic of prime-power order. Thus if we regard $T$ as a permutation of $V$ then $\pi(\langle T \rangle)$ is a chain if and only if the stronger condition holds that there is a prime $p$ such that $V$ is a direct sum of subspaces on which $T$ acts as a Singer element of $p$-power order. Such elements exist, for example, in $GL_d(F_{2^a})$ whenever $2^ad - 1$ is a prime power. For an introduction to the theory of invariant subspaces we refer the reader to [2].

There are various areas of group theory in which lattices have previously arisen which are not directly related to orbit partitions. Subgroup lattices, for example, have been well studied. A well-known theorem of Ore [9] states that the subgroup lattice of a group $G$ is distributive if and only if $G$ is locally cyclic. Locally cyclic groups also play a part in this paper. In Proposition 3.3 we show that they are precisely the groups that are join-coherent in their regular action. Any locally cyclic $p$-group is isomorphic to a subgroup of
the Prüfer $p$-group; these groups appears also in Theorem 1, as the class of transitive permutation groups $G$ such that $\pi(G)$ is a chain.

**Outline.** The outline of this paper is as follows. In Section 2 we prove some general results with a lattice-theoretic flavour that will be used throughout the paper. We begin our structural results in Section 3 where we determine when a group acting regularly on itself is join- or meet- coherent. This section also contains an account of the permutation groups $G$ for which $\pi(G)$ is a chain; any such group is obviously both join- and meet-coherent.

Theorem 2 is proved for direct products in Proposition 4.3 and for wreath products in Proposition 2.4 and in Section 5. Theorem 3 on centralizers is proved in Section 6. We have chosen to offer logically independent arguments in Section 5 and Section 6, even though they are quite closely connected. This is partly so that they may be read independently, and partly because the two lines of approach appear to offer different insights.

The second part of the paper, which focusses on classification results, begins in Section 7 where we classify join-closed Frobenius groups of prime degree. In Section 8 we determine when a linear group has a join-coherent action on points or lines; these results are used in the proof of Theorem 4 in Section 9. Finally, Theorem 5 is proved in Section 10.

To avoid having to specify common group actions every time they occur, we shall adopt the following conventions. Any group mentioned as acting on a set at its first appearance will be assumed always to act on that set, unless another action is explicitly given. In particular, Sym($\Omega$) always acts naturally on the set $\Omega$, and the finite symmetric group $S_n$ always acts on $n$ points. The cyclic group $C_k$, for $k \in \mathbb{N} \cup \{\infty\}$, acts on itself by translation.

### 2. Partitions and imprimitive actions

In this section we collect some facts about lattices of partitions that will be useful in later parts of the paper. For an introduction to the general theory, see for instance [5].

Given a set partition $P$ of a set $\Omega$ we define a corresponding relation $\equiv_P$ on $\Omega$ in which $x \equiv_P y$ if and only if $x$ and $y$ lie in the same part of $P$. If $P$ and $Q$ are set partitions of $\Omega$ then it is not hard to see that $P \vee Q$ is the set partition $R$ such that $\equiv_R$ is the transitive closure of the relation $\equiv$ defined by

$$x \equiv y \iff x \equiv_P y \text{ or } x \equiv_Q y$$

for $x, y \in \Omega$. Similarly $P \wedge Q$ is the set partition $R$ such that

$$x \equiv_R y \iff x \equiv_P y \text{ and } x \equiv_Q y.$$
Equivalently, as we have already remarked,
\[ P \wedge Q = \{ P \cap Q \mid P \in P, Q \in Q, P \cap Q \neq \emptyset \}. \]

The congruence lattice \( \text{Con}(\Omega) \) of all partitions of \( \Omega \) is distributive, i.e. the lattice operations \( \lor \) and \( \land \) distribute over one another. In the language of lattice theory, a permutation group \( G \leq \text{Sym}(\Omega) \) is join-coherent if and only if \( \pi(G) \) is an upper subsemilattice of \( \text{Con}(\Omega) \), meet-coherent if and only if \( \pi(G) \) is a lower subsemilattice of \( \text{Con}(\Omega) \), and both join- and meet-coherent if and only if \( \pi(G) \) is a sublattice of \( \text{Con}(\Omega) \).

**Lemma 2.1.** Let \( L \) be a distributive lattice with respect to \( \preceq \), and let \( x \in L \).

1. Define
   \[ \text{Up}(x) = \{ y \in L \mid x \preceq y \}, \]
   \[ \text{Dn}(x) = \{ y \in L \mid y \preceq x \}. \]
   Then \( \text{Up}(x) \) and \( \text{Dn}(x) \) are sublattices of \( L \).
2. The maps \( \varphi^x : L \to \text{Up}(x) \) and \( \varphi_x : L \to \text{Dn}(x) \) given by
   \[ y \varphi^x = y \lor x, \quad y \varphi_x = y \land x, \]
   are lattice homomorphisms.

**Proof.** The first part follows directly from the definitions of \( \lor \) and \( \land \), and the second part from the definition of distributivity. \( \square \)

In part (1) of the following lemma, \( \text{Con}(\mathcal{B}) \) is the congruence lattice on the set \( \{ B \in \mathcal{B} : \mathcal{B} \} \) of parts of a partition \( \mathcal{B} \).

**Lemma 2.2.** Let \( \Omega \) be a set and let \( \mathcal{B} \in \text{Con}(\Omega) \).

1. \( \text{Up}(\mathcal{B}) \cong \text{Con}(\mathcal{B}) \).
2. \( \text{Dn}(\mathcal{B}) \cong \prod_{B \in \mathcal{B}} \text{Con}(B) \).

**Proof.** If \( \mathcal{B} \preceq \mathcal{A} \) then each part of \( \mathcal{A} \) is a union of parts of \( \mathcal{B} \). Hence \( \mathcal{A} \) determines and is determined by a partition of \( \mathcal{B} \); this gives a bijection between \( \text{Up}(\mathcal{B}) \) and \( \text{Con}(\mathcal{B}) \) that is a lattice isomorphism. For the second part we note that whenever \( \mathcal{A} \preceq \mathcal{B} \), each part \( B \) of \( \mathcal{B} \) is a union of parts of \( \mathcal{A} \), and so a subset of the parts of \( \mathcal{A} \) form a subpartition of \( \mathcal{B} \). Clearly \( \mathcal{A} \) itself is determined by these subpartitions of the parts of \( \mathcal{B} \), and thus \( \mathcal{A} \) determines and is determined by an element of \( \prod_{B \in \mathcal{B}} \text{Con}(B) \). Again it is easy to see that this bijection is a lattice isomorphism. \( \square \)

The next proposition is a straightforward consequence of Lemma 2.2. As a standing convention, we avoid the use of the word ‘respectively’ when the same short statement or proof works for either a join- or a meet-coherent group.
Proposition 2.3. Let $G$ be a join- or meet-coherent permutation group on $\Omega$.

(1) Let $\mathcal{B}$ be a partition of $\Omega$, and let $H$ be the group of permutations which fix every part of $\mathcal{B}$ set-wise. Then $G \cap H$ is join- or meet-coherent.

(2) Let $X \subseteq \Omega$, and let $H$ be the set-stabilizer of $X$ in $G$. Then $H$ is join- or meet-coherent.

(3) Any point-stabilizer in $G$ is join- or meet-coherent.

Proof. Note that $\pi(h) \triangleleft \mathcal{B}$ if and only if $Bh = B$ for all $B \in \mathcal{B}$. Thus $\pi(H) = \text{Dn}(B)$, which is closed under $\lor$ and $\land$. The first part of proposition follows, since the intersection of two join- or meet-closed sets is join- or meet-closed. The second part follows from the first by taking $\mathcal{B} = \{X, \Omega \setminus X\}$, and the third part follows from the second by taking $X$ to be a singleton set. \qed

Recall that a transitive permutation group $G$ on $\Omega$ is said to be imprimitive if it stabilizes a non-trivial partition $\mathcal{B}$ of $\Omega$, in the sense that $xg \equiv B \iff yg \equiv B$ for all $x, y \in \Omega$ and $g \in G$. In this case, one says that $\mathcal{B}$ is a system of imprimitivity for the action of $G$ on $\Omega$. Otherwise $G$ is primitive. If $\mathcal{B}$ is a $G$-invariant partition of $\Omega$ then $\mathcal{B}$ inherits an action of $G$, since for $B \in \mathcal{B}$ and $g \in G$ we have $Bg \in \mathcal{B}$.

The next proposition gives the easier direction in Theorem 2(b). The proof of this theorem is completed in Proposition 5.2.

Proposition 2.4. Let $G$ be join- or meet-coherent on $\Omega$, and let $\mathcal{B}$ be a system of imprimitivity for the action.

(1) The action of $G$ on $\mathcal{B}$ is join- or meet-coherent.

(2) The set-stabilizer in $G$ of a block $B$ of $\mathcal{B}$ acts join- or meet-coherently on $B$.

Proof. The second part is immediate from Proposition 2.3(2). For the first part, let $\vartheta : \text{Up}(\mathcal{B}) \rightarrow \text{Con}(\mathcal{B})$ be the isomorphism in Lemma 2.2(1). By Lemma 2.1(2), the composite map $\varphi^B \vartheta : \text{Con}(\Omega) \rightarrow \text{Con}(\mathcal{B})$ is a homomorphism. It is easy to check that if $g \in G$ has orbit partition $\mathcal{P}$ on $\Omega$ then $g$ has orbit partition $\mathcal{P} \varphi^B \vartheta = (\mathcal{P} \lor \mathcal{B}) \vartheta$ on $\mathcal{B}$. The result now follows from the fact that $\text{Con}(\mathcal{B})$ is distributive. \qed

If $G$ is a group acting on a set $\Omega$, then we may consider the action of $G$ on $\text{Con}(\Omega)$, defined for $g \in G$ and $\mathcal{P} \in \text{Con}(\Omega)$ by $\mathcal{P}^g = \{P^g \mid P \in \mathcal{P}\}$. We use this action in the critical Lemma 6.2 below. In connection with the previous proposition, we note that if $G$ is transitive then $\mathcal{B}$ is a system of imprimitivity for $G$ if and only if $\mathcal{B}^g = G$ for each $g \in G$. 
3. Regular representations and chains

Let $G$ be a group of permutations of a set $\Omega$. We say that $G$ acts semiregularly if every element of $\Omega$ has a trivial point stabilizer in $G$. This is equivalent to the condition that for every element $g \in G$, all of the parts of the orbit partition $\pi(g)$ of $\Omega$ are of the same size. We say that the action of $G$ is regular if it is semiregular and transitive.

**Proposition 3.1.** A group $G$ acting semiregularly is meet-coherent.

**Proof.** Suppose that $G$ acts semiregularly on $\Omega$. Let $x, y \in G$, and let $z$ be a generator for the cyclic group $\langle x \rangle \cap \langle y \rangle$. We shall show that $\pi(z) = \pi(x) \wedge \pi(y)$.

Let $u \in \Omega$, and let $P$ and $Q$ be the parts of $\pi(x)$ and $\pi(y)$ respectively which contain $u$. Then

$$P = \{ux^i \mid i \in \mathbb{Z}\} \quad \text{and} \quad Q = \{uy^j \mid j \in \mathbb{Z}\}.$$  

Let $v \in P \cap Q$, and let $i, j \in \mathbb{Z}$ be such that $v = ux^i = uy^j$. By the semiregularity of $G$, we have $x^i = y^j$, and so $x^i, y^j \in \langle z \rangle$. It follows that $P \cap Q = \{uz^k \mid k \in \mathbb{Z}\}$, and hence that $P \cap Q$ is a part of $\pi(z)$. \hfill $\square$

We recall that a group $G$ is said to be locally cyclic if any pair of elements of $G$ generate a cyclic subgroup. The following lemma states a well-known fact.

**Lemma 3.2.** A group is locally cyclic if and only if it is isomorphic to a section of $Q$.

**Proof.** It is clear that a locally cyclic group is either periodic or torsion-free. If it is periodic then it has at most one subgroup of order $n$ for each $n \in \mathbb{N}$, and it is easy to see that it embeds into the quotient group $Q/\mathbb{Z}$; for torsion-free groups the result was first proved in [1]\(^2\).

**Proposition 3.3.** A group $G$ acting regularly is join-coherent if and only if it is locally cyclic.

**Proof.** We may assume without loss of generality that $G$ is transitive, and so we may suppose that it acts on itself by translation. Let $x, y \in G$, and let $H = \langle x, y \rangle$. The partition $\pi(x) \lor \pi(y)$ is precisely the partition of $G$ into cosets of $H$. If $H = \langle z \rangle$ then we have $\pi(x) \lor \pi(y) = \pi(z) \in \pi(G)$. On the other hand, if $H$ is not cyclic then it cannot be a part of $\pi(z)$ for any $z \in G$, and so $\pi(x) \lor \pi(y) \notin \pi(G)$. \hfill $\square$

We define a locally cyclic group of particular importance to us.

\(^2\)We thank Mark Sapir for this reference.
**Definition.** The Prüfer \(p\)-group, \(P\), is the subgroup of \(\mathbb{Q}/\mathbb{Z}\) generated by the cosets containing \(1/p^i\) for \(i \in \mathbb{N}\).

The group \(P\) appears in the statement of Theorem 1, whose proof occupies the remainder of this section. We begin with the following proposition.

**Proposition 3.4.** Let \(\Omega\) be a set, and let \(G \leq \text{Sym}(\Omega)\) be such that \(\pi(G)\) is a chain.

1. No element of \(G\) has an infinite cycle.
2. Let \(O\) be an orbit of \(G\), and let \(G_O \leq \text{Sym}(O)\) be the permutation group induced by the action of \(G\) on \(O\). Then \(G_O\) is regular on \(O\).
3. There is a prime \(p\) such that every cycle of every element of \(g\) has \(p\)-power length.
4. If \(G\) acts transitively, then there is a prime \(p\) such that \(G\) is isomorphic to a subgroup of \(P\).

**Proof.** It is clear that if \(g \in G\) has an infinite cycle then \(\pi(g^a)\) and \(\pi(g^b)\) are incomparable whenever \(a\) and \(b\) are natural numbers such that neither divides the other. Clearly this implies (1).

For (2), let \(g\) be an element of \(G\) which acts non-trivially on \(O\), and suppose that \(g\) has a fixed point \(x \in O\). Let \(z \in O\) be such that \(zg \neq z\), and let \(h \in G\) be such that \(xh = z\). Then \(g^h\) has \(z\) as a fixed point, and \(xg^h \neq x\). Hence the partitions \(\pi(g)\) and \(\pi(g^h)\) are incomparable, a contradiction. Therefore \(g\) has no fixed points on \(O\) and it follows that the action of \(G_O\) is regular.

For (3), suppose that there exist distinct primes \(q\) and \(r\), such that \(\pi(g)\) has a part \(Q\) of size divisible by \(q\), and another part \(R\) of size divisible by \(r\). If \(g\) has order \(m\) then \(\pi(g^{m/r})\) has singleton parts corresponding to the elements of \(Q\), and \(R\) is a union of parts of \(\pi(g^{m/r})\) of size at least 2. A similar remark holds for \(\pi(g^{n/q})\) with \(Q\) and \(R\) swapped, and so \(\pi(g^{m/r})\) and \(\pi(g^{n/q})\) are not comparable; again this is a contradiction. It follows that for each \(g \in G\), there is a prime \(p_g\) such that every cycle of \(g\) has size a power of \(p_g\).

We now show that \(p_g = p_h\) for all non-identity permutations \(g, h \in G\). We may suppose without loss of generality that \(\pi(h) \preceq \pi(g)\), and so each orbit of \(\langle g \rangle\) is a union of orbits of \(\langle h \rangle\). But by (2), \(h\) acts regularly on each of its orbits. There exists a \(\langle g \rangle\)-orbit \(O\) on which \(h\) acts non-trivially, and so \(p_h\) divides \(|O|\). But \(|O|\) is a power of \(p_g\), and so \(p_h = p_g\).

For (4), we note that if \(G\) is transitive, then by (2) it acts regularly. Now by Proposition 3.3, \(G\) is locally cyclic. By Lemma 3.2 it follows that \(G\) is isomorphic to a section of \(\mathbb{Q}\). Now an element \(g \in G\) has only finite cycles.
by (1), and since $G$ acts regularly, all of the cycles of $g$ have the same length. Therefore $g$ has finite order, and so $G$ is not torsion-free. So $G$ is isomorphic to a subgroup of $\mathbb{Q}/\mathbb{Z}$. Write $g$ as $a/b + \mathbb{Z}$, where $a$ and $b$ are coprime. Since every cycle of $g$ has $p$-power length by (3), the denominator $b$ must be a power of $p$; hence $G$ is isomorphic to a subgroup of the Prüfer $p$-group. □

In particular, we note that (4) implies that any finite group whose orbit partitions form a chain is cyclic.

There are interesting examples of groups $G \leq \text{Sym}(\mathbb{N})$ acting intransitively, such that $G$ is not locally cyclic, but $\pi(G)$ is a chain. Let $\alpha$ be an irrational element of the $p$-adic integers $\mathbb{Z}_p$, such that $p$ does not divide $\alpha$, and define $\alpha_i \in \{0, 1, \ldots, 2^i - 1\}$ by $\alpha_i = \alpha \mod 2^i$. For instance, we may take $p = 2$ and $\alpha = 101001000100001 \ldots \in \mathbb{Z}_2$; the sequence $(\alpha_i)$ here is $(1, 1, 5, 5, 37, 37, 37, 549, \ldots)$. Let $\{c_i \mid i \in \mathbb{N}\}$ be a set of mutually disjoint cycles in $\text{Sym}(\mathbb{N})$, such that $c_i$ has length $2^i$. Let $G \leq \text{Sym}(\Omega)$ be generated by $g$ and $h$, where $g = \prod_{i=1}^{\infty} c_i$ and $h = \prod_{i=1}^{\infty} c_i^{\alpha_i}$. Then $g$ and $h$ have the same orbit partition, but it is easily seen that they are not powers of one another. Hence $G$ is not cyclic. However, the permutation group induced by $G$ on any finite set of its orbits is cyclic, since for any given $i$ there exists $\beta_i \in \mathbb{N}$ such that $\alpha_i \beta_i \equiv 1 \mod p^i$, and so $h^{\beta_i}$ agrees with $g$ on the orbits of all of the cycles $c_j$ for $j \leq i$. It follows easily that $\pi(G)$ is a chain.

The group in this example falls under (2) in Theorem 1; to prove this theorem we shall use Proposition 3.4 and the following technical lemma.3

**Lemma 3.5.** Let $p$ be a prime and let $I$ be a totally ordered set. For each $i \in I$ let $P_i$ be an isomorphic copy of the Prüfer $p$-group $P$. Let

$$\{f_{ji} : P_j \to P_i \mid i, j \in I, i \leq j\}$$

be a set of non-zero homomorphisms, with the property that $f_{kj}f_{ji} = f_{ki}$ whenever $i \leq j \leq k$. Let $M$ be the inverse limit $\lim \leftarrow P_i$, taken with respect to the totally ordered set $I$ and the homomorphisms $f_{ji}$. If all but finitely many of the $f_{ji}$ are isomorphisms then $M \cong P$, and otherwise $M \cong \mathbb{Q}_p$, the additive group of $p$-adic rational numbers.

**Proof.** If $f : P \to P$ is an endomorphism, then for each $i \in \mathbb{N}$, there exists a unique $a_i \in \{0, 1, \ldots, p^i - 1\}$ such that

$$\quad (x/p^i + \mathbb{Z})f = a_i(x/p^i + \mathbb{Z}) \quad \text{for all } x \in \mathbb{Z}. \quad (\ast)$$

3The authors would like to thank John MacQuarrie for a helpful conversation on this subject, and Benjamin Klopsch for suggesting the final step in the proof.
It is easily seen that if \( i \leq j \) then \( a_j \equiv a_i \mod p^i \). Therefore if \( a \in \mathbb{Z}_p \) is the \( p \)-adic integer such that \( a \equiv a_i \mod p^i \) for each \( i \in \mathbb{N} \), then \( f \) is the map \( \mu(a) : P \to P \) defined by \((*)\) above. It follows that at most a countable infinity of the maps \( f_{ji} \) are non-isomorphisms.

Observe that \( f_{a} \) is surjective unless \( a = 0 \), and an isomorphism if and only if \( a \) is not divisible by \( p \). If all but finitely many of the \( f_{ji} \) are isomorphisms, then it is clear that \( M \) is isomorphic to \( P \). Otherwise there exists an infinite increasing sequence \((i_k)\) of elements of \( I \), such that if we set \( \mathcal{R}_k = P_{i_k} \) and

\[
g_k = f_{i_k+1} : \mathcal{R}_{k+1} \to \mathcal{R}_k
\]

for \( k \in \mathbb{N} \), then each \( g_k \) is a non-isomorphism, and \( M \cong \varprojlim_k \mathcal{R}_k \).

For each \( k \in \mathbb{N} \), let \( a_k \in \mathbb{Z}_p \) be such that \( g_k = \mu(a_k) \). Let \( a_k = p^{e_k} b_k \) where \( p \) does not divide \( b_k \); by assumption \( e_k \geq 1 \) for each \( k \). In the commutative diagram

\[
\begin{array}{cccccc}
R_1 & \xleftarrow{\mu(a_1)} & R_2 & \xleftarrow{\mu(a_2)} & R_3 & \xleftarrow{\mu(a_3)} & R_4 & \xleftarrow{\mu(a_4)} & \cdots \\
| & & | & & | & & | & & |
\mu(b_1) & \mu(b_2) & \mu(b_3) & & & & & & \\
R_1 & \xleftarrow{\mu(p^{e_1})} & R_2 & \xleftarrow{\mu(p^{e_2})} & R_3 & \xleftarrow{\mu(p^{e_3})} & R_4 & \xleftarrow{\mu(p^{e_4})} & \cdots
\end{array}
\]

all of the vertical arrows are isomorphisms. It follows easily that the inverse limits constructed with respect to the top and bottom rows are isomorphic. Observe that the inverse system

\[
P_1 \xleftarrow{\mu(p)} P_2 \xleftarrow{\mu(p)} P_3 \xleftarrow{\mu(p)} P_4 \xleftarrow{\mu(p)} \cdots
\]

in which all of the maps are multiplication by \( p \), is a refinement of the bottom row of the diagram \((\dagger)\), and so it defines the same inverse limit.

Finally we note that \( P \cong \mathbb{Q}_p/\mathbb{Z}_p \), and that after applying this isomorphism, the map \( \mu(p) : P \to P \) is induced by multiplication by \( p \) in \( \mathbb{Q}_p \). Hence \( M \cong \varprojlim \mathbb{Q}_p/p^k \mathbb{Z}_p \). Since \( p^k \mathbb{Z}_p \) is an open subgroup of \( \mathbb{Q}_p \), and since \( \bigcap_k p^k \mathbb{Z}_p = \{0\} \), it follows that \( M \cong \mathbb{Q}_p \) as required. \( \square \)

We are now ready to prove Theorem 1, which we restate below for convenience.

**Theorem 1.** Let \( \Omega \) be a set, and let \( G \leq \text{Sym}(\Omega) \) be such that \( \pi(G) \) is a chain. Let \( p \) be the unique prime dividing the cycle lengths of elements of \( G \). Then \( G \) is abelian, and either periodic or torsion-free.

1. If \( G \) is periodic then \( G \) is either a finite cyclic group of \( p \)-power order, or else isomorphic to the Prüfer \( p \)-group \( P \).
(2) If $G$ is torsion-free then $G$ is isomorphic to a subgroup of the $p$-adic rational numbers $\mathbb{Q}_p$. In this case $G$ has infinitely many orbits on $\Omega$, and the permutation group induced by its action on a single orbit is periodic.

Proof. By Proposition 3.4(4), $G$ acts as a subgroup of $\mathbb{P}$ on each of its orbits. It follows that $G$ is abelian. Suppose that $g$ has infinite order, and $h$ finite order $p^a$. Then clearly $\pi(h) \nsucc \pi(g^i)$ for any $i \in \mathbb{N}$, since $g$ has cycles longer than $ip^a$. But each cycle of $g$ is finite, and so every element of $\Omega$ is fixed by some power of $g$. Hence $h$ is the identity of $G$. This shows that $G$ is either torsion-free or periodic.

Suppose that $K_1$ and $K_2$ are the kernels of the action of $G$ on distinct orbits. If there exists $k_1 \in K_1 \setminus K_2$ and $k_2 \in K_2 \setminus K_1$ then the orbit partitions $\pi(k_1)$ and $\pi(k_2)$ are clearly incomparable. Hence the kernels of the action of $G$ on its various orbits form a chain of subgroups of $G$. Since $G$ acts faithfully on $\Omega$, the intersection of all these kernels is trivial.

Suppose that $G$ is periodic. Let $g_1, \ldots, g_r \in G$ be a finite collection of elements, and let $H = \langle g_1, \ldots, g_r \rangle$. Then $H$ is finite. Since $H$ satisfies the descending chain condition on subgroups, there is an orbit $\mathcal{O}$ of $G$ for which the kernel $K_\mathcal{O}$ of $G$ acting on $\mathcal{O}$ intersects trivially with $H$. Therefore $H$ is isomorphic to a subgroup of $G/K_\mathcal{O}$. It now follows from Proposition 3.4(4) that $H$ is isomorphic to a subgroup of $\mathbb{P}$. Hence any finitely generated subgroup of $G$ is cyclic, and so $G$ is locally cyclic. Now by Lemma 3.2 we see that $G$ itself is isomorphic to a subgroup of $\mathbb{P}$.

The remaining case is when $G$ is torsion-free. Let $\mathcal{O}_i$ for $i \in I$ be the set of orbits of $G$, where $I$ is a suitable indexing set, and let $K_i$ be the kernel of $G$ acting on $\mathcal{O}_i$. Order $I$ so that, for $i, j \in I$ we have $i \leq j$ if and only if $K_i \geq K_j$. For $i \leq j$ let $f_{ji} : G/K_j \rightarrow G/K_i$ be the canonical surjection. Fix for each $i \in I$ an isomorphism $G/K_i \cong P_i$ where $P_i \cong \mathbb{P}$. Since $\bigcap_{i \in I} K_i = \{1\}$ and each $G/K_{\mathcal{O}_i}$ is isomorphic to a subgroup of $\mathbb{P}$, we see that $G$ is isomorphic to a subgroup of the inverse limit $\varprojlim P_i$, taken with respect to the totally ordered set $I$ and the homomorphisms $f_{ji}$. The theorem now follows from Lemma 3.5. $\Box$

We remark that if $G \in \text{Sym}(\Omega)$ is isomorphic to a subgroup of $\mathbb{P}$, then it has an orbit on $\Omega$ on which it acts faithfully and regularly. For if $G$ is non-trivial then the intersection of the non-trivial subgroups of $G$ has order $p$; now since $G$ acts faithfully on $\Omega$, it follows that there is an orbit $\mathcal{O}$ on which $G$ acts with trivial kernel. Since $G$ is abelian, its action on $\mathcal{O}$ is regular.
4. Direct products

Suppose that $G$ and $H$ are groups acting on sets $X$ and $Y$ respectively. There are two natural actions of the direct product $G \times H$, namely the \textit{intransitive action} on the disjoint union $X \cup Y$ and the \textit{product action} on $X \times Y$. For $(g, h) \in G \times H$ the intransitive action is defined by

$$z(g, h) = \begin{cases} zg & \text{if } z \in X, \\ zh & \text{if } z \in Y. \end{cases}$$

where $z \in X \cup Y$, and the product action by $(x, y)(g, h) = (xg, yh)$ where $(x, y) \in X \times Y$. The product action is the subject of Theorem 2(a), which we prove in this section. Both of these actions also arise in later parts of the paper.

Lemma 4.1. Let $g \in \text{Sym}(X)$ and $h \in \text{Sym}(Y)$, and let $k = (g, h) \in \text{Sym}(X) \times \text{Sym}(Y)$.

1. In its action on $X \cup Y$ we have $\pi(k) = \pi(x) \cup \pi(y)$.
2. Suppose that $g$ and $h$ have finite coprime orders. Then in its action on $X \times Y$ we have $\pi(k) = \{P \times Q \mid P \in \pi(g), Q \in \pi(h)\}$.

Proof. Both parts are straightforward. □

The intransitive direct product action is dealt with in the next proposition.

Proposition 4.2. Let $G$ and $H$ be groups acting on sets $X$ and $Y$ respectively.

1. The action of $G \times H$ on $X \cup Y$ is join-coherent if and only if the actions of $G$ and $H$ are join-coherent.
2. The action of $G \times H$ on $X \times Y$ is meet-coherent if and only if the actions of $G$ and $H$ are meet-coherent.

Proof. Suppose that $g_1, g_2, g_3 \in G$ and $h_1, h_2, h_3 \in H$. It is clear from Lemma 4.1(1) that

$$\pi(g_1, h_1) \lor \pi(g_2, h_2) = \pi(g_3, h_3)$$

if and only if $\pi(g_1) \lor \pi(g_2) = \pi(g_3)$ and $\pi(h_1) \lor \pi(h_2) = \pi(h_3)$, and similarly

$$\pi(g_1, h_1) \land \pi(g_2, h_2) = \pi(g_3, h_3)$$

if and only if $\pi(g_1) \land \pi(g_2) = \pi(g_3)$ and $\pi(h_1) \land \pi(h_2) = \pi(h_3)$. The result follows. □

Turning to the product action, we prove Theorem 2(a). For convenience we restate this result as the following proposition.
Proposition 4.3. Let $X$ and $Y$ be sets and let $G \leq \text{Sym}(X)$ and $H \leq \text{Sym}(Y)$ be join-coherent permutation groups. Then $G \times H$ is join-coherent on $X \times Y$ if and only if $G$ and $H$ have coprime orders.

Proof. We suppose that $G$ and $H$ act join-coherently. If $|G|$ and $|H|$ are coprime then the join-coherence of $G \times H$ on $X \times Y$ follows from Lemma 4.1(2).

Suppose conversely that there is a prime $p$ which divides both $|G|$ and $|H|$. Let $g \in G$ have order $p^a$, and let $h \in H$ have order $p^b$, where these are the largest orders of $p$-elements in each group. Then it is clear that $\pi(g)$ has a part of size $p^a$, and $\pi(h)$ a part of size $p^b$. Now $\pi(g,1) \lor \pi(1,h)$ has a part of size $p^{a+b}$, and it follows that it cannot be in $\pi(G \times H)$, since the greatest order of a $p$-element in $G \times H$ is $\max(p^a,p^b)$. Therefore $G \times H$ is not join-coherent.

The following result may be viewed as a partial converse to Proposition 4.3.

Proposition 4.4. Suppose that $G$ acts on finite sets $X$ and $Y$, and that these actions have kernels $K_X$ and $K_Y$ respectively, where $K_X \cap K_Y = 1$. If $G$ is join-coherent on $X \times Y$ then

1. $K_Y$ is join-coherent on $X$ and $K_X$ is join-coherent on $Y$,
2. $|K_X|$ and $|K_Y|$ are coprime,
3. $G = K_Y K_X \cong K_Y \times K_X$.

Proof. Consider the partition $\mathcal{B}$ of $X \times Y$ into parts $X \times \{y\}$ for $y \in Y$. Clearly $K_Y$ is the largest subgroup of $G$ which stabilizes the parts of $\mathcal{B}$, and its action on each part is that of $K_Y$ on $X$. By Proposition 2.3(1), this action is join-coherent, and (1) follows.

Let $p$ be a prime, and let $p^a$ and $p^b$ be the largest $p$-power divisors of $G/K_X$ and $G/K_Y$ respectively. If $P$ is a Sylow $p$-subgroup of $G$, then $P$ contains elements $g_X$ and $g_Y$ such that $g_X K_X$ has order $p^a$ in $G/K_X$ and $g_Y K_Y$ has order $p^b$ in $G/K_Y$. It follows that $g_X$ has an orbit $O_X$ on $X$ of size $p^a$, and that $g_Y$ has an orbit $O_Y$ on $Y$ of size $p^b$. Now $\langle g_X, g_Y \rangle$ is a subgroup of $P$, and so its orbits on $X \times Y$ have $p$-power order. One of these orbits contains $O_X \times O_Y$, and so has order $p^c$ for some $c \geq a + b$.

Since $G$ is join-coherent on $X \times Y$, it must contain an element $g$ whose orbits are those of $\langle g_X, g_Y \rangle$. It is clear that $g$ must be a $p$-element of order at least $p^{a+b}$. Since $G/K_X$ and $G/K_Y$ have $p$-exponent $p^a$ and $p^b$ respectively, it follows that $g^a \in K_X$, $g^b \in K_Y$ and so

$$g^{\max(a,b)} \in K_X \cap K_Y = 1.$$
Hence one of \(a\) or \(b\) is 0 and \(G/K_X\) and \(G/K_Y\) have coprime orders.

Suppose that \(p^r\) divide \(|G|\). Since at least one of \(G/K_X\) or \(G/K_Y\) has order coprime with \(p\), we see that \(p^r\) must divide one of \(|K_X|\) or \(|K_Y|\), and it follows that \(|G|/|K_X|\cdot|K_Y|\). But since \(K_X \cap K_Y = 1\) we have \(|K_YK_X| = |K_Y| \cdot |K_X|\), and hence \(G = K_Y K_X\); since \(K_X\) and \(K_Y\) are both normal, this implies that \(G \cong K_Y \times K_X\), as stated in (3). Now (2) follows from the final sentence of the previous paragraph.

Necessary and sufficient criteria for the meet-coherence of \(G \times H\) in its product action are harder to obtain. Since we shall not need any such results in later parts of the paper, we merely offer the following partial result.

**Proposition 4.5.** Let \(X\) and \(Y\) be sets and let \(G \leq \text{Sym}(X)\) and \(H \leq \text{Sym}(Y)\) be meet-coherent permutation groups. If \(G\) and \(H\) are finite of coprime order, then \(G \times H\) is meet-coherent on \(X \times Y\).

**Proof.** This follows easily from Lemma 4.1(2). \(\square\)

There are examples of groups \(G \leq \text{Sym}(X)\) and \(H \leq \text{Sym}(Y)\), not of coprime order, such that \(G \times H\) is meet-coherent on \(X \times Y\). For instance if both \(G\) and \(H\) act semiregularly, then so does \(G \times H\), and so by Proposition 3.1 we see that \(G \times H\) is meet-coherent regardless of the orders of \(G\) and \(H\).

5. Wreath products

Given sets \(S\) and \(T\), we write \(S^T\) for the set of maps from \(T\) to \(S\). As usual, we shall write all maps on the right. If \(S\) is a group, then \(S^T\) inherits a group structure as the direct product of \(|T|\) copies of \(S\); here \(|T|\) may be infinite.

Throughout this section let \(G\) and \(H\) be groups acting on sets \(X\) and \(Y\) respectively. The unrestricted wreath product \(G \wr Y H\) is defined to be the semidirect product \(G^Y \rtimes H\), where the action of \(H\) on \(G^Y\) is given by \(fh = h^{-1} \circ f\) for \(f \in G^Y\) and \(h \in H\). We shall often write \(G \wr Y H\) for \(G \wr Y H\); this abuse of notation is harmless when the set \(Y\) on which \(H\) acts is unambiguous.

There are two natural actions for \(G \wr H\). In the first \(G \wr H\) acts on \(X^Y\) (see [4, Section 4.3]); in this action the wreath product does not generally inherit join- or meet-coherence from \(G\) and \(H\); for instance \(C_2 \wr C_3\) is neither join- nor meet-coherent in this action. For this reason we shall not discuss it any further here.

The second action of the wreath product is the imprimitive action on \(X \times Y\). If \(f : Y \to G\) and \(h \in H\) then this action is given by

\[(x, y)fh = (x(yf), yh) \quad \text{for } x \in X, y \in Y.\]
For $y \in Y$ let $B_y = X \times \{y\}$. Then $\{B_y \mid y \in Y\}$ is a system of imprimitivity for the action, in the sense defined after Proposition 2.3. In general $G \wr H$ does not inherit meet-coherence from $G$ and $H$ in the imprimitve action; for instance $S_3 \wr C_2$ is not meet-coherent. Join-coherence, however, is inherited in the case that $Y$ is finite, and the proof of this fact is the object of this section.

**Definition.** Let $\mathcal{P}$ be a partition of $X \times Y$.

1. We write $\tilde{\mathcal{P}}$ for the partition of $Y$ given by $y_1 \equiv_{\tilde{\mathcal{P}}} y_2 \iff (x_1, y_1) \equiv_{\mathcal{P}} (x_2, y_2)$ for some $x_1, x_2 \in X$.

2. For $y \in Y$ we write $\mathcal{P}[y]$ for the partition of $X$ given by $x_1 \equiv_{\mathcal{P}[y]} x_2 \iff (x_1, y) \equiv_{\mathcal{P}} (x_2, y)$.

The following lemma provides a characterization of the orbit partitions of the elements of a wreath product in its imprimitive action. For the application to Theorem 2(b) we only need the case where $Y$ is finite, which permits some simplifications to the statement and proof.

**Lemma 5.1.** Let $\mathcal{P}$ be a partition of $X \times Y$. Then $\mathcal{P}$ is the orbit partition for an element of $G \wr H$ if and only if the following conditions hold.

1. There exists an element $h \in H$ with orbit partition $\tilde{\mathcal{P}}$ on $Y$.

2. For every $y \in Y$, there exists an element $g \in G$ with orbit partition $\mathcal{P}[y]$ on $X$.

3. If $y$ lies in an infinite part of $\tilde{\mathcal{P}}$ then $\mathcal{P}[y]$ is the partition of $X$ into singleton sets.

4. Whenever elements $y, z \in Y$ lie in the same part of $\tilde{\mathcal{P}}$, there exists $c \in G$ such that $(x, y) \equiv_{\mathcal{P}} (xc, z)$ for all $x \in X$.

**Proof.** Let $f \in G^Y$, let $h \in H$, and let $k = fh \in G \wr H$. It is clear that if $\mathcal{P}$ is the orbit partition of $k$ on $X \times Y$, then $\tilde{\mathcal{P}}$ is the orbit partition of $h$ on $Y$, and so (1) is necessary. Let $t$ be a positive integer. A simple calculation shows that $k^t = f_t h^t$ where $f_t \in G^Y$ is defined by

$$f_t : y \mapsto (yf)(yh)f \cdots ((yh^{t-1})f)$$

Hence

$$(x, y)k^t = (xf, zh^t)$$

for all $x \in X$, and it easily follows that (4) is necessary.

Suppose that $y \in Y$ lies in a finite $h$-orbit of length $m$. In this case it is not hard to see that $yf_m$ has the orbit partition $\mathcal{P}[y]$ on $X$, since $yf_m = (yf_m)^a$.
for all \( a \in \mathbb{Z} \). On the other hand, if \( y \) lies in an infinite orbit of \( h \) then \( \mathcal{P}_{[y]} \) is the partition of \( X \) into singleton parts, which is the orbit partition of the identity of \( G \). So (2) and (3) are necessary.

Now suppose that the stated conditions hold. We shall construct a permutation \( k \in G \wr H \) such that \( \pi(k) = \mathcal{P} \). By (1) there exists an element \( h \) of \( H \) whose orbit partition is \( \mathcal{P} \). Let \( \{y_i \mid i \in I\} \) be a set of representatives for the orbits of \( h \), where \( I \) is a suitable indexing set. Let \( m_i \in \mathbb{N} \cup \{\infty\} \) be the length of the orbit containing \( y_i \). By (4) there exists \( c_{(i,t)} \in G \) for \( i \in I \) such that

\[(\ast) \quad (x, y_i h^t) \equiv_p (x c_{(i,t)}, y_i h^{t+1})
\]

for all \( x \in X \) and \( t \in \mathbb{Z} \).

Suppose that \( m_i < \infty \). Then the element

\[b_i = c_{(i,0)}c_{(i,1)} \cdots c_{(i,m_i-1)} \]

of \( G \) stabilizes the partition \( \mathcal{P}_{[y_i]} \) of \( X \). It is possible that \( \pi(b_i) \) is a strict refinement of \( \mathcal{P}_{[y_i]} \). However, by condition (2), there exists \( g_i \in G \) such that \( \pi(g_i) = \mathcal{P}_{[y_i]} \); if we replace \( c_{(i,0)} \) with \( g_i b_i^{-1} c_{(i,0)} \) we then have \( b_i = g_i \) and so \( \pi(b_i) = \mathcal{P}_{[y_i]} \). We may therefore suppose that the \( c_{(i,t)} \) have been chosen so that \( \pi(b_i) = \mathcal{P}_{[y_i]} \) for all \( i \) such that \( m_i < \infty \).

Let \( f \in G^Y \) be defined by \((y_i h^t)f = c_{(i,t)}\) for each \( i \in I \), where \( t \in \{0, \ldots, m_i - 1\} \) if \( m_i < \infty \) and \( t \in \mathbb{Z} \) otherwise. Let \( k = fh \in G \wr H \). We shall show that \( \pi(k) = \mathcal{P} \).

We define

\[b_{(i,t)} = \begin{cases} c_{(i,0)} \cdots c_{(i,t-1)} & \text{if } t \geq 0 \\ c_{(i,-1)}^{-1} \cdots c_{(i,t-1)}^{-1} & \text{if } t < 0 \end{cases} \]

and observe that

\[(x, y_i) k^t = (xb_{(i,t)}, y_i h^t)\]

for all \( t \in \mathbb{Z} \). In particular, if \( m_i < \infty \) then, since \( b_{(i,m_i)} = g_i \), we have \((x, y_i) k^{m_i} = (xy_i, y_i)\). Observe also that, by (\ast), we have

\[(x, y_i) \equiv_p (xb_{(i,t)}, y_i h^t).\]

Suppose that \((x_1, y_i h^s) \equiv_p (x_2, y_i h^t)\). Then by the observation immediately above, we have

\[(x_1, y_i h^s) \equiv_p (x_1 b_{(i,s)}^{-1}, y_i)\]
\[(x_2, y_i h^t) \equiv_p (x_2 b_{(i,t)}^{-1}, y_i).\]

Hence

\[x_1 b_{(i,s)}^{-1} \equiv_{[y_i]} x_2 b_{(i,t)}^{-1}.\]
If \( m_i < \infty \) then, since \( \pi(b_{(i,m_i)}, y_i) = \mathcal{P}_{[y_i]} \), the elements \( (x_1b_{(i,s)}^{-1}, y_i) \) and \( (x_2b_{(i,t)}^{-1}, y_i) \) lie in the same orbit of \( k^{m_i} \). Therefore

\[(†) \quad (x_1, y_i h^s)k^{-s} = (x_1b_{(i,s)}^{-1}, y_i) \equiv_{\pi(k)} (x_2b_{(i,t)}^{-1}, y_i) = (x_2, y_i h^t)k^{-t}.\]

On the other hand, if \( m_i = \infty \) then, by (3), \( \mathcal{P}_{[y_i]} \) is the partition of \( X \) into singleton sets and \( x_1b_{(i,s)}^{-1} = x_2b_{(i,t)}^{-1} \). Therefore (†) also holds in this case. It follows that

\[(x_1, y_i h^s) \equiv_{\pi(k)} (x_2, y_i h^t),\]

and so \( \mathcal{P} \not\preceq \pi(k) \).

The argument of the previous paragraph in reverse, again using (†), implies that \( \pi(k) \not\preceq \mathcal{P} \), and so we have equality as required.

With Lemma 5.1, we are now in a position to prove the following result, which clearly implies Theorem 2(b).

**Proposition 5.2.** Suppose that \( G \) and \( H \) are join-coherent on \( X \) and \( Y \) respectively, and that \( Y \) is finite. Then \( G \wr H \) is join-coherent in its imprimitive action on \( X \times Y \).

**Proof.** Let \( f_1h_1 \) and \( f_2h_2 \) be elements of \( G \wr H \), and let \( K \) be the subgroup they generate. Let \( \mathcal{P} \) be the orbit partition of \( K \) on \( X \times Y \). It suffices to show that the conditions stated in Lemma 5.1 are satisfied by \( \mathcal{P} \). Notice that condition (3) is satisfied vacuously, since \( Y \) is supposed to be finite.

Since \( H \) is join-coherent, it has an element \( h \) whose orbit partition \( \mathcal{S} \) on \( Y \) is the same as that of the subgroup \( \langle h_1, h_2 \rangle \). It is easy to see that \( \mathcal{P} = \mathcal{S} \), and so (1) is satisfied.

For \( y \in Y \) let \( B_y = X \times \{y\} \); as noted at the start of this section, \( \{B_y \mid y \in Y\} \) is a system of imprimitivity for \( G \wr H \) on \( X \times Y \). We write \( R_y \) for the set-wise stabilizer of \( B_y \) in \( G \wr H \). Then there is an action of \( R_y \) on \( X \) inherited from its action on \( B_y \); in fact the map \( fh \mapsto yf \) is a homomorphism from \( R_y \) to \( G \). Under the action given by this homomorphism, the intersection \( K \cap R_y \) has orbit partition \( \mathcal{P}_{[y]} \). Now \( K \cap R_y \) is finitely generated, since it has finite index in \( K \). Since \( G \) is join-coherent, it follows that there exists an element \( c_y \) of \( G \) whose orbit partition is \( \mathcal{P}_{[y]} \). This gives us (2).

Finally suppose that \( y \) and \( z \) lie in the same part of \( \hat{\mathcal{P}} \). Then there exist \( f \in G^Y \) and \( h \in H \) such that \( yf = z \), and such that \( fh \in K \). Let \( c = yf \).

We see that \((x, y) \equiv_{\mathcal{P}} (xc, z)\) for all \( x \in X \), and so (4) is satisfied. This completes the proof. \( \Box \)
6. Centralizers

The first part of this paper ends with the proof of Theorem 3. The structure of centralizers in symmetric groups is well known.

Lemma 6.1. Let $\Omega$ be a set, let $G = \text{Sym}(\Omega)$, and let $g \in G$. For $k \in \mathbb{N} \cup \{\infty\}$, let $\Delta_k$ be the set of orbits of $g$ of length $k$. Then as permutation groups, we have

$$\text{Cent}_G(g) = \prod_{k \in \mathbb{N} \cup \{\infty\}} (C_k \wr \text{Sym}(\Delta_k)),$$

where the wreath products take the imprimitive action, and the factors in the direct product act on disjoint sets.

Suppose that $g \in \text{Sym}(\Omega)$ has finitely many orbits of all lengths $k \geq 2$, including $k = \infty$. Then the join-coherence of $\text{Cent}_G(g)$ follows immediately from Propositions 4.2(1) and 5.2. This is sufficient to establish Theorem 3 so far as join-coherence is concerned. However it tells us nothing about meet-coherence, for which we require an account of orbit partitions in the centralizers of semiregular permutations. The analysis given below allows a straightforward common treatment of join- and meet-coherence, and so this section stands independently of the results of Section 5.

We begin with a criterion for a partition of $\Omega$ to be the orbit partition of an element of the centralizer of a semiregular permutation $g \in \text{Sym}(\Omega)$.

Lemma 6.2. Let $\Omega$ be a set and let $G = \text{Sym}(\Omega)$. Let $\mathcal{P}$ be a partition of $\Omega$, and let $g \in G$ be an element such that $\langle g \rangle$ acts semiregularly on $\Omega$. There exists $h \in \text{Cent}_G(g)$ with orbit partition $\mathcal{P}$ if and only if the following conditions hold:

1. $\mathcal{P}^g = \mathcal{P}$;
2. every part of $\mathcal{P}$ is countable;
3. if $P$ is an infinite part of $\mathcal{P}$, then either $P$ meets only finitely many $g$ orbits, or else the elements of $P$ lie in distinct $g$ orbits.

If $m \in \mathbb{N} \cup \{\infty\}$ is the order of the semiregular permutation $g$, and $\Delta$ is the set of orbits of $g$ then, by Lemma 6.1, we have $\text{Cent}_G(g) \cong C_m \wr \text{Sym}(\Delta)$, in its imprimitive action. Using this, it is possible to prove Lemma 6.2 by showing the equivalence of its conditions with those of Lemma 5.1. However we prefer to give an independent and more illuminating proof in which we explicitly construct an element of $\text{Cent}_G(g)$ whose orbit partition is $\mathcal{P}$.

Proof. It is clear that the first two conditions are necessary. To see that the third condition is also necessary, suppose that $\mathcal{P}$ is the orbit partition of
Let $h \in \text{Cent}_G(g)$. Let $P$ be a part of $\mathcal{P}$, and let $x \in P$. Since $h$ stabilizes the orbit partition $\rho$ of $g$, the set $A = \{n \in \mathbb{Z} \mid xh^n \equiv \rho x\}$ is a subgroup of $\mathbb{Z}$. The index $|\mathbb{Z} : A|$ is equal to the number of $g$-orbits represented in $P$; if this number is infinite then $A = \{0\}$, since $\mathbb{Z}$ has no other subgroup of infinite index.

Now suppose that the stated conditions hold. We shall construct an element $h \in \text{Cent}_G(g)$ whose orbit partition is $\mathcal{P}$. Let $\{P_i \mid i \in I\}$ be a set of orbit representatives for the action of $g$ on the parts of $\mathcal{P}$, where $I$ is a suitable indexing ordinal. Let $S_i = \bigcup_j P_ig^j$. It is clear that the sets $S_i$ form a partition $\sigma$ of $\Omega$. In fact it is easy to see that $S = \mathcal{P} \vee \rho$, where $\rho$ is the orbit partition of $g$.

Since $h$ may be defined separately on the distinct parts of $\mathcal{S}$, we may suppose without loss of generality that $S$ is the trivial partition of $\Omega$ into a single part.

Let $P$ be a part of $\mathcal{P}$, and let $X = \{x_j \mid j \in J\} \subseteq P$ be a set of representatives for the orbits of $g$ on $\Omega$. Here the indexing ordinal $J$ may be finite, or it may be taken to be the smallest infinite ordinal $\omega$. By assumption $P$ is countable, and either $J$ is finite or else $X = P$. Let $t$ be the least positive integer such that $Pg^t = P$, or 0 if no such integer exists. It is clear that if $t = 0$ then $X = P$.

We have assumed also that $g$ is a semiregular permutation. Let $m$ be the length of a cycle of $g$; here $m$ may be infinite. For convenience we define $M = \mathbb{Z}/m\mathbb{Z}$ if $m$ is finite, and $M = \mathbb{Z}$ if $m$ is infinite. We shall allow $g$ to take exponents from $M$. Every element of $\Omega$ has a unique representation as $x_jg^k$ for some $j \in J$ and $k \in M$.

Observe that $x_jg^k$ lies in $P$ if and only if $k$ is a multiple of $t$. Define $h$ on $\Omega$ by

$$x_jg^kh = \begin{cases} x_{j+1}g^k & \text{if } j + 1 < J, \\ x_0g^{k+t} & \text{if } j + 1 = J. \end{cases}$$

(The second line of the definition, of course, arises only when $J$ is finite.) Thus $h$ fixes each part of $\mathcal{P}$, and acts cyclically on the orbits of $g$.

We now show that $h$ commutes with $g$. It suffices to show that $x_jg^k$ has the same image under $gh$ and under $hg$, for $j \in J$ and $k \in M$. Suppose that $j + 1 < J$; then

$$(x_jg^k)gh = x_jg^{k+1}h = x_{j+1}g^{k+1} = x_{j+1}g^kg = (x_jg^k)hg.$$ 

For the remaining case, we have

$$(x_{j-1}g^k)gh = x_{j-1}g^{k+1}h = x_{1}g^{k+1+t} = x_{1}g^{k+t}g = (x_{j-1}g^k)hg.$$
It is clear that the points $x_j$ lie in a single orbit $O$ of $h$ and so $X \subseteq O \subseteq P$. If $J$ is infinite then clearly $O = P$, since $X = P$. If $J$ is finite then $O$ contains every element of the form $x_j g^k$ for $j \in J$ and for a multiple $k$ of $t$; but since every element of $P$ is of this form, we have $O = P$ in this case too. Now since $g$ and $h$ commute, every other part of $P$ is also an orbit of $h$, and so $\pi(h) = P$ as required. 

We are now in a position to prove Theorem 3, which we restate for convenience.

**Theorem 3.** Let $\Omega$ be a set, and let $g \in \text{Sym}(\Omega)$. For $k \in \mathbb{N} \cup \{\infty\}$ let $m_k$ be the number of orbits of $g$ of length $k$.

(a) $\text{Cent}_{\text{Sym}(\Omega)}(g)$ is meet-coherent.

(b) If $m_k$ is finite for all $k \neq 1$, including $k = \infty$, then $\text{Cent}_{\text{Sym}(\Omega)}(g)$ is join-coherent.

**Proof.** Let $G = \text{Sym}(\Omega)$. By Lemma 6.1, we see that $\text{Cent}_G(g)$ is the direct product of centralizers of semiregular permutations. By the parts of Proposition 4.2 it will be sufficient to prove the result in the case that $g$ acts semiregularly on $\Omega$. We therefore suppose that $g$ is a product of cycles of length $k$, where $k \in \mathbb{N} \cup \{\infty\}$.

Let $P$ and $Q$ be partitions in $\text{Cent}_G(g)$. By Lemma 6.2 we have $P^g = P$ and $Q^g = Q$, from which it follows easily that $(P \lor Q)^g = P \lor Q$ and $(P \land Q)^g = P \land Q$. It is clear that $P \land Q$ also satisfies conditions (2) and (3) of Lemma 6.2, and so $P \land Q \in \pi(\text{Cent}_G(g))$. Hence $\pi(\text{Cent}_G(g))$ is meet-coherent.

Each part of $P$ and each part of $Q$ is countable, and so the parts of $P \lor Q$ are countable. If $k > 1$, then by hypothesis $g$ has only finitely many orbits; therefore the only case in which a part $S$ of $P \lor Q$ can meet an infinite number of orbits is when $k = 1$, and clearly in this case $S$ meets each orbit at most one point. Hence $P \lor Q$ also satisfies conditions (2) and (3) of Lemma 6.2, and so $P \lor Q \in \pi(\text{Cent}_G(g))$. Hence $\pi(\text{Cent}_G(g))$ is join-coherent. 

We have seen that the condition that the orbit multiplicities $m_k$ are finite for $k > 1$ is a sufficient condition for $\text{Cent}_{\text{Sym}(\Omega)}(g)$ to be join-coherent. Our next proposition is that this is also a necessary condition.

**Proposition 6.3.** Let $g$ be a permutation of a set $\Omega$ which has an infinite number of cycles of length $k$, for some $k > 1$, possibly infinite. Then $\text{Cent}_{\text{Sym}(\Omega)}(g)$ is not join-coherent.
Proof. We may assume that $g$ is semiregular, and that all of its orbits have length $k$. We suppose for simplicity that $\Omega$ is countable; the generalization to higher cardinalities is straightforward. Let $S$ be a set of representatives for the orbits of $g$. We define $S$ by

$$S = \{ S^g \mid i \in K \}$$

where $K = \{0, \ldots, k - 1\}$ if $k < \infty$ and $K = \mathbb{Z}$ if $k = \infty$.

It is easy to see that $S^g = S$. By assumption each part of $S$ is countable, and it is clear that each part of $S$ has a single element in common with each orbit of $g$. Hence $S$ satisfies the conditions of Lemma 6.2, and so $S \in \pi(\text{Cent}_G(g))$. However $\pi(g) \lor S$ is the trivial partition of $\Omega$ into a single part. Since $k > 1$, this partition does not satisfy condition (3) of Lemma 6.2, and so $\pi(\text{Cent}_G(g))$ is not join-coherent. \qed

7. Frobenius groups

We recall that a Frobenius group is a transitive permutation group $G$ on a finite set $\Omega$, such that each point stabilizer in $G$ is non-trivial, but the intersection of the stabilizers of distinct points is trivial. The fixed-point free elements of $G$, together with the identity of $G$, form a normal subgroup $K$, called the Frobenius kernel. The Frobenius kernel acts regularly, and it is often useful to identify $K$ with $\Omega$ by fixing an element $\omega \in \Omega$ and mapping $k$ to $\omega k$. The stabilizer $H$ of a point $\omega \in \Omega$ is called a Frobenius complement, and acts semiregularly on $\Omega \setminus \{\omega\}$. Identifying $\Omega$ with $K$ one finds that any complement $H$ embeds into $\text{Aut}(K)$, and so $G$ is isomorphic as a permutation group to $K \rtimes H$, where $K$ has the right regular action on itself. (For further results on Frobenius groups the reader is referred to Theorems 2.7.5 and 2.7.6 of [6].)

We give a complete account of join- and meet-coherence in Frobenius groups. As well as being a natural object of study, these results will be important in subsequent sections.

Proposition 7.1. A Frobenius group is meet-coherent if and only if it is dihedral of prime degree.

Proof. Let $D$ be the dihedral group of degree $p$ acting on a $p$-gon $\Pi$. For any vertices $\alpha$ and $\beta$ of $\Pi$, there is a unique reflection mapping $\alpha$ to $\beta$. It follows that if $\mathcal{P}_1$ and $\mathcal{P}_2$ are the orbit partitions of distinct reflections then $\mathcal{P}_1 \land \mathcal{P}_2$ is the discrete partition. Since the only non-identity elements of $D$ are reflections and full cycles, we see that $D$ is meet-coherent.

For the converse, let $G$ be a meet-coherent Frobenius group acting on $\Omega$ with kernel $N$. We observe that if $h$ and $k$ lie in different point stabilizers
then, since \( \pi(h) \wedge \pi(k) \) has two singleton parts, \( \pi(h) \wedge \pi(k) \) is the orbit partition of the identity. Hence if \( O_h \) is an orbit of \( h \) and \( O_k \) is an orbit of \( k \) then \( |O_h \cap O_k| \leq 1 \).

Suppose that \( |G : N| > 2 \). Let \( \alpha, \beta \in \Omega \) be distinct points and choose \( h \in \text{Stab}(\alpha), k \in \text{Stab}(\beta) \) such that \( hk \notin N \). (This is possible because each point stabilizer meets every coset of \( N \).) By our choice, there is a further point \( \gamma \in \Omega \) such that \( \gamma hk = \gamma \). But now \( \{\gamma, \gamma h\} \) is part of an \( h \)-orbit, and \( \{\gamma h, \gamma hk\} \) is part of a \( k \)-orbit, in contradiction to the observation in the previous paragraph. Therefore \( |G : N| = 2 \) and, taking \( h \in G \setminus N \), we see that \( g^h = g^{-1} \) for all \( g \in N \).

It is easy to show that if a group admits a fixed-point free automorphism sending each element to its inverse then it is an abelian group of odd order. Suppose that \( N \) has a proper subgroup, \( M \) say. Since \( M \) has odd order, we have \( [N : M] \geq 3 \). If \( g \in N \) then the orbits of \( g \) are the cosets of \( N \), whereas \( h \) preserves \( N \) and acts non-trivially on the cosets \( N/M \). It easily follows that \( \pi(g) \wedge \pi(h) \) is not the orbit partition of any permutation in \( G \). Therefore \( N \) is cyclic of prime order and \( G \) is a dihedral group of order \( 2p \). □

Since a transitive join-coherent permutation group \( G \) of finite degree contains a full cycle, it is clear that if \( G \) is a Frobenius group then its kernel is cyclic. For this reason, we provide a description of Frobenius groups with a cyclic kernel.

**Lemma 7.2.** Let \( n \in \mathbb{N} \), and let \( H \) be a non-trivial finite group. There exists a Frobenius group with cyclic kernel \( K \cong C_n \) and complement \( H \) if and only if \( H \cong C_r \), where \( r \) divides \( p - 1 \) for each prime divisor \( p \) of \( n \).

**Proof.** Suppose that there exists such a Frobenius group. As mentioned at the start of this section, we may consider \( H \) as a group of automorphisms of \( K \). Note that any non-identity element of \( H \) acts without fixed points on the non-identity elements of \( K \). It follows that \( n \) is odd, since an even cyclic group has a unique element of order 2. Let \( p \) be a prime divisor of \( n \) and let \( L \cong C_p \) be the unique subgroup of \( K \) of order \( p \). Then \( H \) acts faithfully on \( L \) and since \( \text{Aut}(L) \cong C_{p-1} \), it follows that \( H \) is cyclic of order dividing \( p - 1 \). Hence the order of \( H \) divides \( p - 1 \) for each prime divisor \( p \) of \( n \).

Conversely, let \( n = p_1^{a_1} \ldots p_t^{a_t} \), let \( K = \langle x \rangle \cong C_n \), and let \( r \) divide \( p_i - 1 \) for all \( i \). The Chinese Remainder Theorem allows us to choose \( d \in \mathbb{N} \) such that \( d \) has multiplicative order \( r \) modulo \( p_i^{a_i} \) for all \( i \). The map \( h : x \mapsto x^d \) is an automorphism of \( K \) of order \( r \), and it is easy to check that \( H = \langle h \rangle \) is a Frobenius complement for \( K \). □

We are now in a position to prove the following.
Proposition 7.3. A Frobenius group is join-coherent if and only if it has prime degree.

Proof. Let $G$ be a Frobenius group with kernel $K$ and complement $H$. We identify the set on which $G$ acts with $K$.

Suppose that $G$ has prime degree $p$, so $K \cong C_p$. Let $X$ be a subgroup of $G$. If $K \leq X$ then $X$ is transitive, and its orbit partition is that of a generator of $K$. Otherwise $X \cap K$ is trivial, and since $G/K$ is abelian by Lemma 7.2, it follows that $X$ is also abelian. If $x$ is a non-identity element of $X$, then $x$ fixes a unique element $a$ of $K$; since $X$ centralizes $x$ we have $X \leq \text{Stab}_G(a)$. By Proposition 2.3(3) the point stabilizers in $G$ act join-coherently on $K$, and so there exists $h \in \text{Stab}_G(a)$ such that the orbit partition of $X$ is $\pi(h)$. Therefore $G$ itself is join-coherent, as required.

For the converse implication, we observe that the non-identity elements of $K$ are precisely the elements of $G$ whose orbit partition has no singleton parts. Hence if $G$ is join-coherent then $\{\pi(k) \mid k \in K\}$ is closed under taking joins. Since the action of $G$ on $K$ is regular, it follows from Proposition 3.3 that $K$ is cyclic.

Suppose for a contradiction that $|K|$ is composite. Then $K$ has a characteristic non-trivial proper cyclic subgroup $\langle k \rangle$. The orbit partition $\pi(k)$ of the element $k$ is simply the partition of $K$ into the cosets of $\langle k \rangle$. Let $h$ be a non-identity element of $H$. By Lemma 7.2 we see that $|H|$ is coprime with $|K|$, and hence with $|\langle k \rangle|$. Since $H$ acts semiregularly on $K \setminus \{1\}$, no proper coset of $\langle k \rangle$ in $K$ can be a union of orbits of $h$. Now consider the partition $\pi(k) \vee \pi(h)$. It has one part equal to $\langle k \rangle$ itself, and every other part is a union of two or more cosets of $\langle k \rangle$. But this partition cannot be in $\pi(G)$, since it has parts of different sizes, but no singleton parts. Hence $G$ is not join-coherent. \qed

8. Join-coherence in linear groups

In this section we let $G$ be a group in the range $\text{SL}(V) \leq G \leq \text{GL}(V)$, where $V$ is a vector space of dimension $d$ over a field $K$. Such a group has a natural action on the non-zero points of the space $V$, and the quotient $G/Z(G)$ acts on the lines of $V$. Since the lines of $V$ form a system of imprimitivity for the action of $G$, we see by Proposition 2.4(1) that if $G$ acts join-coherently then so does $G/Z(G)$. The main results of this section are Proposition 8.6 and Proposition 8.7 show that if $d > 1$ then these actions are almost never join-coherent. The results from this section will be needed in our classification of join-coherent primitive groups in Section 9.
When $d = 1$ we see that $G$ is a cyclic group acting semiregularly, and so the action is join-coherent; of course the group $G/Z(G)$ is trivial in this case. When $d > 1$ it is simple to reduce to the case when $V$ is a 2-dimensional space; in this case we shall write $\Lambda$ for the set of lines in $V$. We identify $\Lambda$ with the set $K \cup \{\infty\}$, by associating the line through $(a, b)$ with $b/a$ when $a \neq 0$, and with $\infty$ when $a = 0$. The group $\text{PGL}_2(K)$ acting on $\Lambda$ may then be identified with the group of fractional linear transformations $\alpha \mapsto \frac{a\alpha + b}{c\alpha + d}$, $a, b, c, d \in K$, $ad - bc \neq 0$.

The following fact is well known.

**Lemma 8.1.** The action of the group $\text{PGL}_2(K)$ on $\Lambda$ is sharply 3-transitive.

Let $L$ be a subfield of $K$, and let $V_L \subseteq V$ be a $d$-dimensional vector space over $L$ such that $V_L \otimes_L K = V$. Then the set stabilizer of $V_L$ in $\text{GL}(V)$ is isomorphic to $\text{GL}(V_L)$. If $\text{PGL}(V)$ is join-coherent on the lines of $V$ then $\text{PGL}(V_L)$ is join-coherent on the lines of $V_L$, by Proposition 2.3(2). We shall write $K_0$ for the characteristic subfield of $K$, and $\Lambda_0$ for the subset $K_0 \cup \{\infty\} \subseteq \Lambda$.

Our next proposition establishes that $\text{PGL}_2(K)$ and $\text{PSL}_2(K)$ are join-coherent only in a few small cases.

**Lemma 8.2.** Let $K$ be a field, and let $G$ be in the range $\text{SL}_2(K) \leq G \leq \text{GL}_2(K)$. Then $G/Z(G)$ is join-coherent in its action on $\Lambda$ if and only if $G$ is $\text{GL}_2(F_2)$ or $\text{GL}_2(F_3)$.

**Proof.** We suppose first that $\text{char} \ K > 2$. It is not hard to show that if $G$ contains no elements with determinant $-1$ then $G/Z(G)$ contains no elements which act as full cycles on $\Lambda_0$; since the action of $G/Z(G)$ is transitive, it follows that it cannot be join-coherent. We shall assume, therefore, that elements of $\text{GL}_2(K)$ with determinant $-1$ lie in $G$. We observe that if $|K| \leq 3$ then $G = \text{GL}_2(K)$; furthermore, the groups $\text{PGL}_2(2)$ and $\text{PGL}_2(3)$ are isomorphic to the symmetric groups $S_3$ and $S_4$ respectively, in their natural actions. Both groups are therefore join-coherent.

Let $g, h \in \text{PGL}_2(K)$ be given by $g : \alpha \mapsto -\alpha$ and $h : \alpha \mapsto 1/\alpha$. The parts of $\pi(g) \lor \pi(h)$ have the form

$$\{\alpha, -\alpha, 1/\alpha, -1/\alpha\}.$$  

There are two parts $\{0, \infty\}$ and $\{1, -1\}$ of size 2. If $-1$ has square-roots in $K$, then they form another part of size 2. The other parts all have size 4. If $k \in \text{PGL}_2(K)$ has such an orbit partition then $k^2$ has at least 4 fixed points; since $\text{PGL}_2(K)$ is sharply 3-transitive, it follows that $k^2 = 1$. But
this implies that \( k \) has no orbits of length 4, and hence \( K \) is either \( F_3 \) or \( F_5 \).

The group \( \text{PGL}_2(F_5) \) is isomorphic to \( S_5 \), in its non-standard action on 6 points. It is not hard to show that if \( t \) and \( t' \) are any two distinct commuting triple transpositions in \( \text{PGL}_2(F_5) \) then \( \pi(t) \lor \pi(t') \) has one part of size 2 and one part of size 4, and hence \( \pi(t) \lor \pi(t') \) is not the orbit partition of any element of \( \text{PGL}_2(F_5) \). Therefore \( \text{PGL}_2(F_5) \) is not join-coherent.

Next suppose that \( \text{char } K = 2 \). Then \( \text{PSL}_2(K) \) contains the elements \( g \) and \( h \) given by \( g : \alpha \mapsto \alpha + 1 \) and \( h : \alpha \mapsto \alpha / (\alpha + 1) \). The parts of \( \pi(g) \lor \pi(h) \) have the form \( \{ \alpha, 1/\alpha, \alpha + 1, 1/(\alpha + 1), \alpha/(\alpha + 1), (\alpha + 1)/\alpha \} \).

One part \( \{ 0, 1, \infty \} \) has size 3; if \( K \) contains primitive cube-roots of 1 then they form a part of size 2. The rest of the parts have size 6. If an element \( k \) of \( \text{PGL}_2(K) \) has this orbit partition, then \( k^3 \) fixes 3 elements, and is therefore the identity. It follows that \( k \) has no parts of size 2 or 6, and hence \( K = F_2 \).

Finally, suppose that \( \text{char } K = 0 \). Then \( K_0 \) is the rational field \( \mathbb{Q} \). Let \( g, h \in \text{PSL}_2(\mathbb{Q}) \) be given by \( g : \alpha \mapsto \alpha + 1 \) and \( h : \alpha \mapsto \alpha / (\alpha + 1) \). Then it is not hard to show that \( \pi(g) \lor \pi(h) \) has a part equal to \( \Lambda_0 = \mathbb{Q} \cup \{ \infty \} \).

Suppose that \( k \in \text{PSL}_2(K) \) has an orbit equal to \( \Lambda_0 \). It is easy to show that \( k \in \text{PSL}_2(\mathbb{Q}) \). Thinking of \( k \) as a fractional linear transformation of \( \mathbb{C} \cup \{ \infty \} \) it is well known that for each \( z \in \mathbb{C} \cup \{ \infty \} \), the orbit \( \{ zk^i \mid i \in \mathbb{Z} \} \) has at most 2 limit points. Hence the orbit of 0 under \( k \) cannot be \( \Lambda_0 \), and therefore \( \Lambda_0 \) is not a part of \( k \).

Lemma 8.2 is the basis for the following more general statements.

**Proposition 8.3.** Let \( V \) be a vector space of dimension \( d \) over the field \( K \), where \( d > 1 \). Let \( G \) be a group such that \( \text{SL}(V) \leq G \leq \text{GL}(V) \). Then the action of \( G/Z(G) \) on the lines of \( V \) is not join-coherent unless \( d = 2 \) and \( |K| \leq 3 \).

**Proof.** Let \( W \) be a proper subspace of \( V \), and let \( G_W \) be the set-stabilizer of \( W \) in \( G \). Let \( H \) be the permutation group corresponding to the action of \( G_W \) on \( W \); then \( H \cong \text{GL}(W) \). Now by Lemma 2.3(2), if \( G/Z(G) \) is join-coherent on the lines of \( V \) then \( \text{PGL}(W) \) acts join-coherently on the lines of \( W \).

If \( d > 2 \) then we may take \( W \) to be a 2-dimensional subspace. Now by Lemma 8.2 we see that \( G \) cannot be join-coherent unless \( |K| \leq 3 \). If \( d > 3 \), then we may take \( W \) to be 3-dimensional; a straightforward computation shows that neither \( \text{PGL}_3(F_2) \) nor \( \text{PGL}_3(F_3) \) is join-coherent, and so \( G \) cannot be join-coherent in this case either. \( \square \)
Let \( \Phi \) be a non-trivial group of automorphisms of the field \( K \), and let \( G \) be a group such that \( \text{SL}(V) \leq G \leq \text{GL}(V) \), where \( V \) is a space of dimension \( d \) over \( K \). Then \( G \cdot \Phi \) acts on the non-zero points of \( V \), and \( (G \cdot \Phi)/Z(G) \) acts on the lines.

**Lemma 8.4.** Let \( \Phi \) be a non-trivial group of automorphisms of a finite field \( K \). Let \( G \) be in the range \( \text{SL}_2(K) \leq G \leq \text{GL}_2(K) \). If \( (G \cdot \Phi)/Z(G) \) is join-coherent on \( \Lambda \), then \( K = F_4 \), \( G = \text{GL}_2(F_4) \), and \( \Phi = \text{Gal}(F_4:F_2) \).

**Proof.** Suppose that \(|K| = p^r \) where \( p \) is prime. Since \( K \) admits non-trivial automorphisms, we must have \( r > 1 \). Let \( H \) be a point-stabilizer in \( G \), where \( G \) acts on \( \Lambda \). Since \( \text{PSL}_2(K) \) is 2-transitive on \( \Lambda \), the action of \( H \) on \( p^r \) points is transitive. If \( G \) is join-coherent, then, by Proposition 2.3(3), so is \( H \). It follows that \( H \) must contain an element of order \( p^r \). Let \( r = p^a m \) where \( p \) does not divide \( m \). Let \( g \in G \) be a \( p \)-element. Since the full automorphism group of \( K \) has order \( r \), we see that \( g^r \in \text{PGL}_2(K) \). But a non-trivial unipotent element of \( \text{GL}_2(K) \) has order \( p \), and so \( g^{rp} = 1 \). Hence the order of \( g \) is at most \( p^a + 1 \), which is less than \( p^r \) except in the case when \( a = 1, r = 2 \) and \( p = 2 \), so \( p^r = 4 \).

When \( K = F_4 \) the group \( \text{PGL}_2(F_4) \cdot \text{Gal}(F_4:F_2) \) is isomorphic to the symmetric group \( S_5 \), in its standard action on 4 points, and is therefore join-coherent. However its subgroup \( \text{SL}_2(K) \cdot \text{Gal}(F_4:F_2) \) is isomorphic to the alternating group \( A_5 \), which is not join-coherent. \( \square \)

The principal difficulty in extending Lemma 8.4 to general fields comes from simple transcendental extensions of \( F_p \) for small primes \( p \). The following lemma isolates the result we need.

**Lemma 8.5.** Let \( p \) be prime and let \( K = F_p(x) \) where \( x \) is a transcendental element. Let \( \Phi = \text{Gal}(K:F_p) \) and let \( H = \text{PGL}_2(K) \). Representing elements of \( K \cup \{\infty\} \) as rational expressions \( P(x)/Q(x) \) where \( P, Q \) are polynomials (and taking \( Q = 0 \) to represent \( \infty \)), there is an action of \( H \cdot \Phi \) on \( K \cup \{\infty\} \) defined by

\[
h \varphi : \frac{P(x)}{Q(x)} \rightarrow \left( \frac{P(x \varphi)}{Q(x \varphi)} \right) h.
\]

In this action the orbit of \( x^{p+1} + x^p \) is regular.

**Proof.** Let \( P(x) = x^{p+1} + x^p \). It suffices to show that if \( Ph = P(x \varphi) \) for \( h \in H \) and \( \varphi \in \text{Gal}(K:F_p) \), then \( h = 1 \) and \( \varphi = 1 \). Any element of \( \text{Gal}(K:F_p) \) is determined by its effect on \( x \); let

\[
x \varphi = \frac{ax + b}{cx + d}
\]
where $a, b, c, d \in \mathbb{F}_p$ and $ad - bc \neq 0$. Let

$$h(t) = \frac{At + B}{Ct + D} \quad \text{for all } t \in \mathbb{F}_p(x)$$

where $A, B, C, D \in \mathbb{F}_p$ and $AD - BC \neq 0$.

Now suppose that

$$P(x) = \frac{(ax + b)^p + 1}{(cx + d)^{p+1}} + \frac{(ax + b)^p}{(cx + d)^p} = \frac{A(x^{p+1} + xp) + B}{C(x^{p+1} + xp) + D} = Ph.$$ 

Then using the fact that $(rx + s)^p = rx^p + s$ for any $r, s \in K_0$, we have

$$(ax^p + b)((a+c)x + (b+d))(Cx^{p+1} + Cx^p + D) = (cx^p + d)(cx + d)(Ax^{p+1} + Ax^p + B).$$

Since $x$ is transcendental over $\mathbb{F}_p$, the coefficients on the two sides of this equation must be the same. Comparing the constant and linear terms, we see that $bD(b + d) = Bd^2$ and $bD(a + c) = Bdc$. Since $ad - bc \neq 0$, it follows that $Bd = 0$; similarly we have $bD = 0$. This implies that $B = b = 0$ or $D = d = 0$, since we cannot have $B = D = 0$ or $b = d = 0$.

If $D = d = 0$ then we may assume $C = c = 1$, and so obtain

$$(ax^p + b)((a+1)x + b)(x + 1) = x^2(Ax^{p+1} + Ax^p + B).$$

Now $x + 1$ must divide the right-hand side; but that implies that $B = 0$, which is impossible. Therefore we have $B = b = 0$. We may assume $A = a = 1$, and so obtain

$$((c + 1)x + d)(Cx^{p+1} + Cx^p + D) = (cx^p + d)(cx + d)(x + 1).$$

Now $(x + 1)$ must divide the left-hand side, which since $D \neq 0$ implies that $c + 1 = d$. We have

$$dCx^{p+1} + dCx^p + dD = c^2x^{p+1} + dcx^p + dcx + d^2.$$ 

But now $dc = 0$ and so $c = 0$. But this implies that $C = 0$ and $d = D = 1$. So we have shown that both $h$ and $\varphi$ are the identity, as required. $\square$

We are now ready to prove our main result on the action of linear groups on lines.

**Proposition 8.6.** Let $\Phi$ be a non-trivial group of automorphisms of a field $K$. Let $V$ be a $d$-dimensional space over $K$, and let $G$ be in the range $\text{SL}(V) \leq G \leq \text{GL}(V)$. If $(G \cdot \Phi)/Z(G)$ is join-coherent on the lines of $V$, then $K = \mathbb{F}_4$, $G = \text{GL}(V)$ and $\Phi = \text{Gal}(\mathbb{F}_4 : \mathbb{F}_2)$.

**Proof.** Let $W_0 \subseteq V$ be a 2-dimensional space over $K_0$, and let $W = W_0 \otimes_{K_0} K$. Then $\Phi$ stabilizes $W$ as a set. By Proposition 2.3(2), if $(G \cdot \Phi)/Z(G)$ is join-coherent on the lines of $V$, then the set stabilizer of $W$ is join-coherent on the lines of $W$. Therefore, provided $K \neq \mathbb{F}_4$, it is sufficient to prove the
theorem in the case \( d = 2 \). By a similar argument, it is sufficient for the case \( K = \mathbb{F}_4 \) to deal with a single group, \( \text{PGL}_3(4) \cdot \text{Gal}(\mathbb{F}_4 : \mathbb{F}_2) \). A straightforward computation establishes that this group is not join-coherent on lines; we omit the details here.

We shall therefore assume that \( V \) is 2-dimensional. Since \( K \) admits a non-trivial automorphism, it is not equal to its characteristic subfield \( K_0 \). Without loss of generality, we may suppose that \( K \) is a simple extension of \( K_0 \) by an element \( x \). If \( x \) is algebraic over \( K_0 \) then \( K \) is finite, and then by Lemma 8.4 we must have \( |K| = 4 \).

Suppose, then, that \( x \) is transcendental over \( K_0 \). An element \( \varphi \) of \( \text{Gal}(K : K_0) \) is determined by the image of \( x \), and satisfies

\[
x \varphi = \frac{ax + b}{cx + d}, \quad a, b, c, d \in K_0, \quad ad - bc \neq 0.
\]

It follows that \( \text{Gal}(K : K_0) \cong \text{PSL}_2(K_0) \).

Let \( H \) be the set stabilizer of \( \Lambda_0 \) in \( G/Z(G) \). Since \( \Phi \) acts trivially on \( \Lambda_0 \), we see that the action of \( H \) on \( \Lambda_0 \) is join-coherent. By Lemma 8.2 it follows that \( |K_0| \leq 3 \), and so \( K_0 = \mathbb{F}_p \) where \( p \leq 3 \). Identifying \( \Lambda \) with fractions \( P(x)/Q(x) \in K \), as in Lemma 8.5, we obtain a join coherent action of \( H \cdot \Phi \) on \( K \cup \{\infty\} \). By Lemma 8.5 this group has a regular orbit in this action. However \( H \cdot \Phi \) is not locally cyclic, so by Proposition 3.3 the action of \( H \cdot \Phi \) is not join-coherent.

Finally, we extend the results to the action of \( \text{GL}(V) \) on the non-zero points of \( V \).

**Proposition 8.7.** Let \( V \) be a \( d \)-dimensional vector space over a field \( K \), where \( d > 1 \). let \( \Phi \) be a group (possibly trivial) of automorphisms of \( K \), and let \( G \) be such that \( \text{SL}(V) \leq G \leq \text{GL}(V) \). If \( G \cdot \Phi \) is join-coherent in its action on \( V \setminus \{0\} \) then \( K = \mathbb{F}_2 \) and \( d = 2 \).

**Proof.** The lines of \( V \) form a system of imprimitivity, and so by Proposition 2.4(1), if \( G \cdot \Phi \) is join-coherent on points, then \( (G \cdot \Phi)/Z(G) \) is join-coherent on lines. It follows that \( G \cdot \Phi \) from Propositions 8.3 and 8.6 that \( G \cdot \Phi \) is one of \( \text{GL}_2(\mathbb{F}_2) \), \( \text{GL}_2(\mathbb{F}_3) \) or \( \text{GL}_2(\mathbb{F}_4) \cdot \text{Gal}(\mathbb{F}_4 : \mathbb{F}_2) \). A straightforward computation shows that the only one of these groups which is join-coherent on points is \( \text{GL}_2(\mathbb{F}_2) \). \( \square \)

**9. Primitive join-coherent groups of finite degree**

The following lemma is [8, Theorem 3], which draws together results from various sources. We shall write \( \text{PTL}_d(q) \) for the group \( \text{PGL}_d(q) \cdot \Phi \), where \( \Phi \) is the Galois group of \( \mathbb{F}_q \) over its prime subfield.
Lemma 9.1. Let $G$ be a primitive permutation group on $n$ points containing an $n$-cycle. Then one of the following holds.

1. $G$ is $S_n$ or $A_n$.
2. $n = p$ is a prime, and $G \leq AGL_1(p)$.
3. $\text{PGL}_d(q) \leq G \leq \text{PGL}_d(q)$ for $d > 1$, where $n = (q^d - 1)/(q - 1)$, the action being either on projective points or on hyperplanes.
4. $G$ is $\text{PSL}_2(11)$ or $M_{11}$ acting on 11 points, or $M_{23}$ acting on 23 points.

Let $G$ be a primitive join-coherent group of finite degree. Since $G$ is join-coherent, it contains an $n$-cycle, and so it is one of those classified in Lemma 9.1. In fact we will establish Theorem 4, that a primitive permutation group of finite degree is join coherent if and only if it is a symmetric group or a subgroup of $AGL_1(p)$ in its action on $p$ points.

Proof. We work through the list of groups in Lemma 9.1. Certainly $S_n$ is join-coherent, and we have seen that $A_n$ is not join-coherent when $n > 3$. A transitive subgroup of $AGL_1(p)$ is a Frobenius group, and is therefore join-coherent by Proposition 7.3.

Suppose that $\text{PGL}_d(q) \leq G \leq \text{PGL}_d(q)$. The actions on points and on hyperplanes are dual to one another, and it therefore suffices to rule out join-coherence for one of them. By Propositions 8.3 and 8.6, the only join-coherent examples in the action on points are $\text{PGL}_2(F_2)$, $\text{PGL}_2(F_3)$ and $\text{PGL}_2(F_4)$. But these are isomorphic as permutation groups to $S_3$, $S_4$ and $S_5$ respectively, in their natural actions.

We have therefore reduced the proof to a small number of low degree groups, namely $\text{PSL}_2(11)$ in its action on 11 points, and the Mathieu groups $M_{11}$ and $M_{23}$. (Certain other small examples requiring individual computation have been mentioned in Section 8.) Establishing that none of these is join-coherent is a straightforward computational task. □

10. Groups containing a proper normal cyclic subgroup acting regularly

We have observed that a join-coherent permutation group on a finite set must contain a full cycle. We end this paper by investigating the situation when this cycle generates a normal subgroup.

Let $G$ act on $\Omega$, a set of size $n$. Suppose that $K$ is a transitive normal cyclic subgroup of $G$ of order $n$. Let $H$ be the stabilizer of a point $\omega \in \Omega$. Then clearly $G = K \rtimes H$, and by the argument indicated at the start of Section 7, we may identify $\Omega$ with $K$ by the bijection sending $\omega k \in \Omega$ to $k \in K$. The action of $H$ on $\Omega$ then defines an embedding of $H$ into $\text{Aut}(K)$. 

Every subgroup of $K$ is characteristic in $K$, and therefore invariant under the action of $H$.

Suppose that $G$ is join-coherent. If $n = ab$ for coprime $a, b$ then $K \cong C_a \times C_b$, and we see from Proposition 4.4 that $G$ factorizes as $G_1 \times G_2$, where $G_1$ is join-coherent on $C_a$, $G_2$ is join-coherent on $C_b$, and the factors $G_1$ and $G_2$ have coprime orders. Therefore, to obtain a complete classification, it suffices to consider the case that $n$ is a prime power.

One trivial possibility is that $G = K$; in this case the action of $G$ is semiregular, and join-coherent by Proposition 3.3. If $K$ is assumed to be a proper subgroup, then it turns out that the classification divides into two cases: the case that $n = p$ is prime, and the case that $n = p^a$ for $a > 1$.

**Proposition 10.1.** Let $p$ be prime, and $a > 1$. Let $\Gamma(p^a)$ be the extension of the additive group $\mathbb{Z}/p^a\mathbb{Z}$ by the automorphism $f : x \mapsto rx$, where $r = p^a - 1 + 1$. Then $\Gamma(p^a)$ is join-coherent.

**Proof.** An element $g$ of $\Gamma(p^a)$ may be represented as $x \mapsto r^jx + i$ for non-negative integers $i < p^a$ and $j < p$. A straightforward calculation shows that

$$xg^t - x = tjp^{a-1}x + \frac{t(t-1)}{2}ijp^{a-1} + ti.$$ 

It follows that $xg = x$ if and only if $jp^{a-1}x + i = 0$. Moreover, if $c \geq 1$ then

$$xg^{p^c} - x = \begin{cases} 
  p^ci & \text{if } p \text{ is odd} \\
  2^ci(2^{2^c-2}(2^c - 1)j + 1) & \text{if } p = 2.
\end{cases}$$

Clearly $xg^t - x = 0$ if and only if the $g$-orbit containing $x$ has size dividing $t$. Since $\Gamma(p^a)$ is a $p$-group, the preceding equation allows us to describe the orbit partitions occurring in $\Gamma(p^a)$. Let $p^b$ be the highest power of $p$ dividing $i$ if $i \neq 0$, and let $b = a$ if $i = 0$.

1. If $b < a - 1$, or if $j = 0$, then the orbits of $g$ are the cosets of $\langle p^b \rangle$ in $\mathbb{Z}/p^a\mathbb{Z}$.
2. If $b \geq a - 1$ and $j \neq 0$, then $x$ is a fixed point of $g$ if and only if $p$ divides $jx + k$ where $i = p^{a-1}k$. Thus the fixed points of $g$ form a coset of $\langle p \rangle$ in $\mathbb{Z}/p^a\mathbb{Z}$. The remaining orbits have size $p$ and are cosets of $\langle p^{a-1} \rangle$ in $\mathbb{Z}/p^a\mathbb{Z}$.

From this description it is clear that $\pi(G)$ is closed under the join operation. 

We remark that when $p$ is odd, the group $\Gamma(p^a)$ is the unique extension of $\mathbb{Z}/p^a\mathbb{Z}$ by an automorphism of order $p$. When $p = 2$ there are three such extensions (for $a \geq 3$), of which $\Gamma(p^a)$ is the one which is neither dihedral
nor quasidihedral. (There appears to be no widely accepted name for this group.)

**Proposition 10.2.** Let $p$ be prime, let $a > 1$, and let $C$ be the additive group $\mathbb{Z}/p^a\mathbb{Z}$. Let $T$ be a non-trivial group of automorphisms of $C$. The group $C \rtimes T$ is join-coherent if and only if it is the group $\Gamma(p^a)$ from Proposition 10.1.

**Proof.** Let $G$ be the full group of affine transformations of $\mathbb{Z}/p^a\mathbb{Z}$. Then $C \rtimes T \leq G$. Let $H$ be the unique subgroup of $C$ of order $p$.

We describe the elements of $G$ which have an orbit equal to $H$. Suppose that $g : x \mapsto rx + s$ is such an element. Then it is easy to see that $s = mp^{a-1}$ for some $m$ not divisible by $p$, since the image of 0 under $g$ is a non-identity element of $H$. Furthermore, since $g$ has no fixed points in $H$, we see that $r \equiv 1 \mod p$. But these restrictions on $r$ and $s$ imply that the equation $x = rx + s$ has a solution in $\mathbb{Z}/p^a\mathbb{Z}$, except in the case that $r = 1$, when $g \in H$. Hence $g$ must have a fixed point in $C \setminus H$, except in the case that its orbits are precisely the cosets of $H$ in $C$.

Let $h$ be a generator of $H$. The orbits of $h$ are the cosets of $H$, and the automorphism group $T$ clearly stabilizes $H$ set-wise. It follows that for any $t \in T$, the join $\pi(h) \vee \pi(t)$ has $H$ as a part, and that every part is a union of cosets of $H$. But such a partition has no singleton part, and so cannot be in $\pi(G)$ unless each of its parts is a single coset; this implies that if $C \rtimes T$ is join-coherent, then the action of $T$ on the cosets of $H$ is trivial. It is easy to see that this is the case only if $T = \langle f \rangle$, where $f$ is as in Proposition 10.1. □

We are now in a position to prove Theorem 5. For convenience we restate the theorem below.

**Theorem 5.** Let $G$ be a permutation group on $n$ points, containing a normal cyclic subgroup of order $n$ acting regularly. Let $n$ have prime factorization $\prod_i p_i^{a_i}$. Then $G$ is join-coherent if and only if there exists for each $i$ a transitive permutation group $H_i$ on $p_i^{a_i}$ points, such that:

- if $a_i > 1$ then $H_i$ is either cyclic or the extension of a cyclic group of order $p_i^{a_i}$ by the automorphism $x \mapsto x^r$ where $r = p_i^{a_i-1} + 1$,
- if $a_i = 1$ then $H_i$ is a subgroup of the Frobenius group of order $p(p-1)$,
- the orders of the groups $H_i$ are mutually coprime,
- $G$ is permutation isomorphic to the direct product of the groups $H_i$ in its product action.

**Proof.** Let $n = \prod_i p_i^{a_i}$, and suppose that $G$ is a join-coherent permutation group on $n$ points containing a regular normal cyclic subgroup $C$ of order $n$. 
Then we can regard $G$ as acting on $C$. Let $K_i$ be the unique subgroup of $C$ of order $p_i^{a_i}$. Since $C \cong \prod_i K_i$, it follows easily from Proposition 4.4 that $G \cong \prod_i H_i$, where $H_i$ is the kernel of $G$ in its action on the complement $\prod_{j \neq i} K_j$ of $K_i$. Moreover Proposition 4.4 implies that the groups $H_i$ and $H_j$ have coprime orders whenever $i \neq j$, and that $H_i$ acts join-coherently on $K_i$ for all $i$. If $a_i > 1$ then, by Proposition 10.2, either $H_i$ is cyclic of order $p_i^{a_i}$ or $H_i$ is isomorphic to $\Gamma(p_i^{a_i})$, while if $a_i = 1$ then $H_i$ is a subgroup of the normalizer in $S_p$ of a $p$-cycle, and so is a subgroup of the Frobenius group of order $p(p-1)$. This completes the proof in one direction.

For the converse, suppose that we have for each $i$ a permutation group $H_i$ on $p_i^{a_i}$ points, containing a regular normal cyclic subgroup, and such that if $a_i > 1$ then $H_i$ is either cyclic or isomorphic to $\Gamma(p_i^{a_i})$. If $a_i > 1$ then Proposition 10.1 tells us that $H_i$ join-coherent. If $a_i = 1$ on the other hand, then $H_i$ is either cyclic or else a Frobenius group of prime degree, and so it is join-coherent by Proposition 7.3. If the orders of the groups $H_i$ are coprime, then their direct product is join-coherent by Proposition 4.3, and this completes the proof. $\square$

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