Deformed Conformal and Supersymmetric Quantum Mechanics

Vyacheslav Spiridonov

Laboratoire de Physique Nucléaire, Université de Montréal, C.P. 6128, succ. A, Montréal, Québec, H3C 3J7, Canada

Abstract

Within the standard quantum mechanics a $q$-deformation of the simplest $N = 2$ supersymmetry algebra is suggested. Resulting physical systems do not have conserved charges and degeneracies in the spectra. Instead, superpartner Hamiltonians are $q$-isospectral, i.e. the spectrum of one can be obtained from another (with possible exception of the lowest level) by the $q^2$-factor scaling. A special class of the self-similar potentials is shown to obey the dynamical conformal symmetry algebra $su_q(1,1)$. These potentials exhibit exponential spectra and corresponding raising and lowering operators satisfy the $q$-deformed harmonic oscillator algebra of Biedenharn and Macfarlane.

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1On leave of absence from the Institute for Nuclear Research, Moscow, Russia
2e-mail address: spiridonov@lps.umontreal.ca
1. Introduction

As a rule, group-theoretical methods alleviate description of the complicated physical systems. In this respect Lie algebras are of a special interest. Very elegant examples of their applications were found in quantum mechanics within the concept of the spectrum generating, or, dynamical (super)symmetry algebras [1]. Not long ago a wide attention was drawn to the deformations of the Lie algebras known nowadays under the name of quantum algebras, or, quantum groups [4]. Physical models where a coupling constant is related to the deformation parameter $q$ and where Hamiltonian commutes with the generators of the quantum algebra $su_q(2)$ were found on the linear lattice [3]. Thus, some kind of equivalence of the perturbation of the interaction between "particles" to the deformation of the symmetry algebras governing the dynamics was demonstrated.

Biedenharn and Macfarlane introduced $q$-deformed harmonic oscillator as a building block of the quantum algebras [4, 5]. Many mathematical applications of the $q$-oscillators appeared since that time [3, 7] (an overview of the algebraic aspects of the $q$-analysis can be found in Ref.[8]). Physical models of $q$-oscillators can be divided into the three classes. The first class, inspired by Ref.[3], is related to the lattice systems [7, 9]. In the second class dynamical quantities are defined on the "quantum planes" – the spaces with non-commutative coordinates [10]. Although Schrödinger equation in this approach looks similar to the standard one, an explicit representation of it in terms of the normal calculus results in the non-local finite-difference equation. Parameter $q$ responsible for the non-commutativity of quantum space coordinates serves as some non-local scale on the continuous manifolds and, therefore, the basic physical principles are drastically changed in this type of deformation. We shall not pursue here the routes of these two groups of models.

The third – dynamical symmetry realization class – is purely phenomenological: one deforms already known spectra by choosing the Hamiltonian as some combination of the quantum algebra generators [11], or, as an anticommutator of the formal $q$-oscillator creation and annihilation operators [4, 5]. This application, in fact, does not have straightforward physical meaning because of the non-uniqueness of the deformation procedure. Even within the standard physical concepts exact knowledge of the spectrum is not enough for the reconstruction of the potential. For a given potential with infinite number of bound states one can associate another potential with infinitely many independent parameters and the same spectrum [12]. Therefore the physics behind such deformations is not completely fixed. One should precisely describe what kind of interaction between the excitations leads to peculiar change of the spectrum. Some analysis of the inverse problems can be found in Ref.[13], where simulations of periodic potentials with the prescribed band structure was performed, and in Ref.[14], where a reconstruction of artificial symmetric non-oscillating potential generating prescribed discrete spectrum in the WKB-approximation was considered. $q$-Analogs of the harmonic oscillators were also used for the description of small violation of the statistics of identical particles [15, 16].

Recently the author proposed new approach to the problem of the quantum algebra symmetries in physical models [17], namely, to take exactly solvable Schrödinger potentials and deform their shape (e.g., by changing the Taylor series expansion coefficients) in such a way that the problem remains to be exactly solvable but the spectrum acquires complicated functional character. This idea was stimulated by the Shabat’s one-dimensional reflectionless potential showing peculiar self-similar behavior and describing an infinite number soliton system [18]. The latter was identified in Ref.[17] as a representative of a general two parameter potential unifying via the $q$-deformation
conformally invariant harmonic and Coulomb, Rosen-Morse, and Pöschl-Teller potentials. The hidden quantum algebraic symmetry was claimed to be responsible for the exponential character of the spectrum. In comparison with the discussed above third group of models present approach is the direct one – physical interaction is fixed first and the question on the quantum algebra behind prescribed rule of \( q \)-deformation is secondary.

In this paper we extend further the results of Ref. [17] and propose general deformation of the supersymmetric (SUSY) quantum mechanics [19]. We define \( q \)-SUSY algebra and provide explicit realization of that on the Hilbert space of square integrable functions. The degeneracies of standard SUSY models are lifted. The set of self-similar potentials naturally appears as a particular example of \( q \)-SUSY system obeying dynamical symmetry algebra \( su_q(1, 1) \). In particular, the raising and lowering operators entering the definition of the supercharges are shown to generate \( q \)-oscillator algebra of Biedenharn and Macfarlane. The whole construction is based on the commutative analysis and has many physical applications.

2. SUSY quantum mechanics

The simplest \( N = 2 \) SUSY quantum mechanics is fixed by the following algebraic relations between the Hamiltonian of a system \( H \) and supercharges \( Q, Q^\dagger \) [19]:

\[
\{Q, Q^\dagger\} = H, \quad Q^2 = (Q^\dagger)^2 = 0, \quad [H, Q] = [H, Q^\dagger] = 0. \tag{1}
\]

All operators are supposed to be well defined on the relevant Hilbert space. Then, independently on the explicit realizations the spectrum is two-fold degenerate and the ground state energy is semipositive, \( E_{\text{vac}} \geq 0 \).

Let us consider a particle moving in the one-dimensional space. Below, the coordinate \( x \) is tacitly assumed to cover the whole line, \( x \in \mathbb{R} \), if it is not explicitly stated that it belongs to some cut. Standard representation of the algebra (1) contains one free superpotential \( W(x) \) [20]:

\[
Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \quad A = (p + iW(x))/\sqrt{2}, \quad [x, p] = i, \tag{2}
\]

\[
H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} A^\dagger A & 0 \\ 0 & AA^\dagger \end{pmatrix} = \frac{1}{2}(p^2 + W^2(x) + W'(x)\sigma_3), \tag{3}
\]

\[
W'(x) \equiv \frac{d}{dx}W(x), \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

It describes a particle with two-dimensional internal space the basis vectors of which can be identified with the spin “up” and “down” states.

The subhamiltonians \( H_\pm \) are isospectral as a result of the intertwining relations

\[
H_+ A^\dagger = A^\dagger H_-, \quad AH_+ = H_- A. \tag{4}
\]

The only possible difference concerns the lowest level. Note that the choice \( W(x) = -x \) corresponds to the harmonic oscillator problem and then \( A^\dagger, A \) coincide with the bosonic creation and annihilation operators \( a^\dagger, a \) which satisfy the algebra

\[
[a, a^\dagger] = 1, \quad [N, a^\dagger] = a^\dagger, \quad [N, a] = -a, \tag{5}
\]

where \( N \) is the number operator, \( N = a^\dagger a \). This, and another particular choice, \( W(x) = \lambda/x \), correspond to the conformally invariant dynamics [21].
3. \( q \)-Deformed SUSY quantum mechanics

Now we shall introduce the tools needed for the quantum algebraic deformation of the above construction. Let \( T_q \) be smooth \( q \)-scaling operator defined on the continuous functions

\[
T_q f(x) = f(qx),
\]

where \( q \) is a real non-negative parameter. Evident properties of this operator are listed below

\[
T_q f(x)g(x) = [T_q f(x)][T_q g(x)], \quad T_q \frac{d}{dx} = q^{-1} \frac{d}{dx} T_q,
\]

\[
T_q T_p = T_{qp}, \quad T_q^{-1} = T_q^{-1}, \quad T_1 = 1.
\]

On the Hilbert space of square integrable functions \( \mathcal{L}_2 \) one has

\[
\int_{-\infty}^{\infty} \phi^*(x)\psi(qx)dx = q^{-1} \int_{-\infty}^{\infty} \phi^*(q^{-1}x)\psi(x)dx,
\]

where from the hermitian conjugate of \( T_q \) can be found

\[
T_q^\dagger = q^{-1}T_q^{-1}, \quad (T_q^\dagger)^\dagger = T_q.
\]

As a result, \( \sqrt{q} T_q \) is a unitary operator. Because we take wave functions to be infinitely differentiable, an explicit realization of \( T_q \) is provided by the operator

\[
T_q = e^{\ln q x d/dx} = q^{x d/dx}.
\]

Expanding \((14)\) into the formal series and using integration by parts one can prove relations \((3)\) on the finite coordinate cut as well because wave functions vanish on the boundaries.

Let us define the \( q \)-deformed factorization operators

\[
A^\dagger = \frac{1}{\sqrt{2}}(p - iW(x)) T_q, \quad A = \frac{q^{-1}}{\sqrt{2}} T_q^{-1}(p + iW(x)),
\]

where \( W(x) \) is arbitrary function and for the convinience we use the same notations as in the undeformed case \((3)\). \( A \) and \( A^\dagger \) are hermitian conjugates of each other on the \( \mathcal{L}_2 \). Now one has

\[
A^\dagger A = \frac{1}{2} q^{-1}(p^2 + W^2(x) + W'(x)) \equiv q^{-1}H_+, \quad (12)
\]

\[
AA^\dagger = \frac{1}{2} q^{-1}T_q^{-1}(p^2 + W^2(x) - W'(x))T_q
\]

\[
= \frac{1}{2} q (p^2 + q^{-2} W^2(q^{-1}x) - q^{-1}W'(q^{-1}x)) \equiv qH_-.
\]

We define \( q \)-deformed SUSY Hamiltonian and supercharges to be

\[
H = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} = \begin{pmatrix} qA^\dagger A & 0 \\ 0 & q^{-1}AA^\dagger \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}.
\]

These operators satisfy the following \( q \)-deformed version of the \( N = 2 \) SUSY algebra

\[
\{Q^\dagger, Q\}_q = H, \quad \{Q, Q\}_q = \{Q^\dagger, Q^\dagger\}_q = 0, \quad [H, Q]_q = [Q^\dagger, H]_q = 0,
\]

\[(15)\]
where we introduced $q$-brackets
\[ [X, Y]_q \equiv qXY - q^{-1}YX, \quad [Y, X]_q = -[X, Y]_{q^{-1}}, \tag{16} \]
\[ \{X, Y\}_q \equiv qXY + q^{-1}YX, \quad \{Y, X\}_q = \{X, Y\}_{q^{-1}}. \tag{17} \]
Note that the supercharges are not conserved because they do not commute with the Hamiltonian (in this respect our algebra principally differs from the formal construction of Ref.\[22\]). An interesting property of the algebra (15) is that it shares with (1) the semipositiveness of the ground state energy which follows from the observation that $Q^\dagger, Q$ and the operator $q^{-\sigma_3}H$ satisfy ordinary SUSY algebra (1). Evidently, in the limit $q \to 1$ one recovers conventional SUSY quantum mechanics.

For the subhamiltonians $H_{\pm}$ the intertwining relations look as follows
\[ H_+A^\dagger = q^2 A^\dagger H_-, \quad AH_+ = q^2 H_- A. \tag{18} \]
Hence, $H_{\pm}$ are not isospectral but rather $q$-isospectral, i.e. the spectrum of $H_+$ can be obtained from the spectrum of $H_-$ just by the $q^2$-factor scaling:
\[ H_+ \psi^{(+)} = E^{(+)} \psi^{(+)}, \quad H_- \psi^{(-)} = E^{(-)} \psi^{(-)}, \tag{19} \]
\[ E^{(+)} = q^2 E^{(-)}, \quad \psi^{(+)} \propto A^\dagger \psi^{(-)}, \quad \psi^{(-)} \propto A \psi^{(+)}. \tag{20} \]
Possible exception concerns only the lowest level in the same spirit as it was in the undeformed SUSY quantum mechanics. If $A^\dagger, A$ do not have zero modes then there is one-to-one correspondence between the spectra. We name this situation as a spontaneously broken $q$-SUSY because for it $E_{\text{vac}} > 0$. If $A$ (or, $A^\dagger$) has zero mode then $q$-SUSY is exact, $E_{\text{vac}} = 0$, and $H_-$ (or, $H_+$) has one level less than its superpartner $H_+$ (or, $H_-$).

As a simplest physical example let us consider the case $W(x) = -x$. The Hamiltonian takes the form
\[ 4H = 2p^2 + (1 + q^{-4})x^2 + q^{-2} - 1 + ((1 - q^{-4})x^2 - 1 - q^{-2})\sigma_3, \tag{21} \]
and describes a spin-1/2 particle in the harmonic well and related magnetic field along the third axis. The physical meaning of the deformation parameter $q$ is analogous to that in the XXZ-model \[3\] – it is a specific interaction constant in the standard physical sense. This model has exact $q$-SUSY and if $q^2$ is equal to the rational number the spectrum exhibits accidental degeneracies.

### 4. General deformation of the SUSY quantum mechanics

Described above $q$-deformation of the SUSY quantum mechanics is by no means unique. If one choses in the formulae (11) $T_q$ to be not $q$-scaling operator but, instead, the shift operator
\[ T_q f(x) = f(x + q), \quad T_q = e^{q \partial / \partial x}, \tag{22} \]
then SUSY algebra will not be deformed at all. The superpartner Hamiltonians will be isospectral and the presence of the $T_q$-operator results in the very simple deformation of the standard superpartner potential $U_-(x) \to U_-(x - q)$ (kinetic term is invariant). Evidently such deformation does not change the spectrum of $U_-$ and that is why SUSY algebra remains intact but physically
this creates new SUSY quantum mechanical models. The crucial point in generating of them was the use of the essentially infinite order differential operators as the intertwining operators.

The most general choice of $T_q$ would be the shift operator in the arbitrarily chosen change of the variable function $z = z(x)$

$$T_q f(z(x)) = f(z(x) + q), \quad T_q = e^{q d/dz(x)}, \quad \frac{d}{dz} = \frac{1}{z'(x)} \frac{d}{dx}. \quad (23)$$

The choices $z = \ln x$ and $z = x$ were already discussed above. In general, operator $T_q$ will not preserve the form of the kinetical term in the $H$-Hamiltonian. Physically such change would correspond to the transition from the motion of the particle on the flat space to the curved space dynamics. Application of the described construction to the spherically symmetric potentials is straightforward. The general higher dimensional SUSY models are more complicated but can also be "deformed". It is evident that all quantum mechanical problems discussed within the SUSY approach can be considered in the suggested fasion. We leave detailed discussion of the approach and physical applications for the future publications.

5. $q$-Deformed conformal quantum mechanics

Explicit form of the $su(1,1)$ dynamical symmetry generators can be read off from the harmonic oscillator $[3]$ realization

$$K_+ = \frac{1}{2}(a^+)^2, \quad K_- = \frac{1}{2}a^2, \quad K_0 = \frac{1}{2}(N + \frac{1}{2}), \quad (24)$$

$$[K_0, K_{\pm}] = \pm K_{\pm}, \quad [K_+, K_-] = -2K_0. \quad (25)$$

Let us show that the potentials considered in Refs.$[18, 17]$ obey the quantum conformal symmetry algebra $su_q(1,1)$.

First, we shall derive those potentials within another physical situation with the help of $q$-SUSY. Let us consider the Hamiltonian of a spin-1/2 particle in the external potential $\frac{1}{2}U(x)$ and the magnetic field $\frac{1}{2}B(x)$ along the third axis

$$H = \frac{1}{2}(p^2 + U(x) + B(x)\sigma_3) \quad (26)$$

and impose two conditions: we take magnetic field to be homogeneous

$$B = -\beta^2 q^{-2} = const \quad (27)$$

and require the presence of $q$-SUSY $[15]$. Equating (20) and (14) we arrive at the potential

$$U(x) = W^2(x) + W'(x) + \beta^2 q^{-2}, \quad (28)$$

where $W(x)$ satisfies the following mixed finite-difference and differential equation

$$W'(x) - W^2(x) + qW'(qx) + q^2W^2(qx) + 2\beta^2 = 0. \quad (29)$$
This is the condition of a self-similarity \cite{18} which bootstraps the potential in different points (in Ref.\cite{17} $\beta^2 = \gamma^2(1 + q^2)/2$ parametrization was used). Smooth solution of (23) for symmetric potentials $U(-x) = U(x)$ is given by the following power series

$$W(x) = \sum_{i=1}^{\infty} c_i x^{2i-1}, \quad c_i = \frac{1 - q^{2i} - \frac{1}{1 + q^{2i} - 1}}{\sum_{m=1}^{i-1} c_{i-m}c_m}, \quad c_1 = -\frac{2\beta^2}{1 + q^2}. \quad (30)$$

In different limits of the parameters several well known exactly solvable problems arise: 1. Rosen-Morse – at $q \to 0$; 2. Pöschl-Teller – at $\beta \propto q \to \infty$; 3. Harmonic well – at $q \to 1$; 4. Coulomb potential – at $q \to 0$ and $\beta = 0$. Note that at $q > 1$ the range of the coordinate should be restricted to the finite cut. Soliton solution of Shabat \cite{17} corresponds to the range $0 < q < 1$ at fixed $\beta$. Within the taken physical situation the potential (28) describes $q$-deformation of the Landau-level problem.

We already know that the spectra of $H_{\pm}$ subhamiltonians are related via the $q^2$-scaling

$$E_{n+1}^{(+)} = q^2 E_n^{(-)}, \quad (31)$$

where the number $n$ numerates levels from below for both spectra. Because $q$-SUSY is exact in this model the lowest level of $H_{-}$ corresponds to the first excited state of $H_{+}$. But at the restriction (27) the spectra differ only by a constant,

$$E_{n}^{(+)} = E_{n}^{(-)} - \beta^2 q^{-2}, \quad (32)$$

Conditions (31) and (32) give us the spectrum

$$E_{n,m} = \beta^2 \left( q^{-2m} - q^{2n} \right) \frac{1}{1 - q^2}, \quad m = 0, 1; \quad n = 0, 1, \ldots, \infty. \quad (33)$$

At $q < 1$ there are two finite accumulation points, i.e. (33) somehow approximates the two-band spectrum. At $q > 1$ energy eigenvalues exponentially grow to the infinity.

Not more difficult is the derivation of the dynamical symmetry algebra. To find that we rewrite relations (12), (13) for the superpotential (30)

$$A^\dagger A = q^{-1}H + \frac{\beta^2 q^{-1}}{1 - q^2}, \quad AA^\dagger = q H + \frac{\beta^2 q^{-1}}{1 - q^2}, \quad (34)$$

where $H$ is the Hamiltonian with purely exponential spectrum

$$H = \frac{1}{2}(p^2 + W^2(x) + W'(x)) - \frac{\beta^2}{1 - q^2}, \quad E_n = -\frac{\beta^2}{1 - q^2} q^{2n}. \quad (35)$$

Evidently,

$$AA^\dagger - q^2 A^\dagger A = \beta^2 q^{-1}. \quad (36)$$

Normalization of the r.h.s. of (36) to unity results in the algebra used in Ref.\cite{16} for the description of small violation of the Bose statistics.

The shifted Hamiltonian (33) and $A^\dagger$, $A$ operators $q$-commute

$$[A^\dagger, H]_q = [H, A]_q = 0,$$
or,

\[ H A^\dagger = q^2 A^\dagger H, \quad A H = q^2 H A. \]  

(37)

These are typical braid-type commutation relations. Energy eigenfunctions \(|n\rangle\) can be uniquely determined from the ladder operators action

\[ A^\dagger |n\rangle = \beta q^{-1/2} \sqrt{\frac{1 - q^{2(n+1)}}{1 - q^2}} (n + 1), \quad A |n\rangle = \beta q^{-1/2} \sqrt{\frac{1 - q^{2n}}{1 - q^2}} |n - 1\rangle. \]  

(38)

It is convenient to introduce the formal number operator

\[ N = \frac{\ln[(q^2 - 1)H/\beta^2]}{\ln q^2}, \quad N |n\rangle = n |n\rangle, \]  

(39)

which is well defined only on the eigenstates of the Hamiltonian. Now one can check that the operators

\[ a_q = \frac{q}{\beta} A q^{-N/2}, \quad a_q^\dagger = \frac{q}{\beta} q^{-N/2} A^\dagger \]  

(40)

satisfy the \(q\)-deformed harmonic oscillator algebra of Biedenharn and Macfarlane [4, 5]

\[ a_q a_q^\dagger - q a_q^\dagger a_q = q^{-N}, \quad [N, a_q^\dagger] = a_q^\dagger, \quad [N, a_q] = -a_q. \]  

(41)

Substituting \(a_q^\dagger\) and \(a_q\) into the definitions (24) and renormalizing \(2K_\pm/(q + q^{-1}) \rightarrow K_\pm\) we get commutation relations of the quantum algebra \(su_q(1, 1)\)

\[ [K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -\frac{\mu^{2K_0} - \mu^{-2K_0}}{\mu - \mu^{-1}}, \quad \mu = q^2. \]  

(42)

Therefore the dynamical symmetry algebra of the model is \(su_q(1, 1)\).

Let us compare our deformed (super)conformal quantum mechanics with the construction of Ref.[23]. Kalnins, Levine, and Miller called as the conformal symmetry generator any differential operator \(L(t)\) which maps solutions of the time-dependent Schrödinger equation to the solutions, i.e. which satisfies the relation

\[ i \frac{\partial}{\partial t} L - [H, L] = R (i \frac{\partial}{\partial t} - H), \]  

(43)

where \(R\) is some operator. On the shell of Schrödinger equation solutions \(L(t)\) is conserved and all higher powers of the space derivative, entering the definition of \(L(t)\), can be replaced by the powers of \(\partial/\partial t\) and linear in \(\partial/\partial x\) term. But any analytical function of \(\partial/\partial t\) is replaced by the function of energy when applied to the stationary states. This trick allows to simulate any infinite order differential operator by the one linear in space derivative and to prove that a solution with energy \(E\) can always be mapped to the not-necessarily normalizable solution with the energy \(E + f(E)\) where \(f(E)\) is arbitrary analytical function. "On-shell" raising and lowering operators always can be found if one knows the basis solutions of the Schrödinger equation but sometimes it is easier to find symmetry generators and use them in search of the spectrum.

In our construction we have "off-shell" symmetry generators, which satisfy quantum algebraic relations in the operator sense. In this respect our results are complimentary to those of the Ref[23]. Indeed, time-dependent factorization operators

\[ A^\dagger(t) = e^{i(q^2 - 1)tH} A^\dagger, \quad A(t) = e^{i(q^2 - 1)tH} A \]  

(44)
satisfy (43) with $R \equiv 0$ and, so, are real conserved quantities. On the stationary states the exponential prefactors in (44) coincide with the time shift operators. One can reduce operators $A^\dagger(t), A(t)$ to the linear in $\partial/\partial x$ ”on-shell”-form and then they, probably, shall coincide with the ladder operators of Ref.[23] corresponding to the Hamiltonian (35).

6. Conclusions

To conclude, in this paper we have suggested a deformation of the SUSY quantum mechanics. The main feature of the construction is that the superpartner Hamiltonians satisfy non-trivial braid-type intertwining relations which remove degeneracies of the original SUSY spectra. Deformed SUSY algebra preserves semipositiveness of the vacuum energy. Peculiar set of $q$-SUSY potentials arising within the Landau-level-like problem obey $q$-deformed dynamical conformal symmetry algebra $su_q(1, 1)$. Corresponding raising and lowering operators satisfy $q$-deformed oscillator algebra of Biedenharn and Macfarlane. A more general type of potential deformations applicable in any dimensional space is outlined.

It is clear that $q$-scaling is a particular example of the possible transformations of the spectra. In general one should be able to analytically describe the map of a given potential with spectrum $E_n$ to the particular potential with the spectrum $f(E_n)$ for any analytical function $f(E)$. A problem of arbitrary non-linear deformation of the Lie algebras was treated in Ref.[24] using the symbols of operators without well defined coordinate representation on the ordinary Hilbert space $L_2$. Certainly, the method of Ref.[23] should be helpful in the analysis of this interesting problem and the model with exponential spectrum given in Sect. 5 shows that sometimes one can even find well defined ”off-shell” spectrum generating algebra.

The Hopf algebra structure of the quantum groups was not mentioned because its physical meaning within the standard quantum mechanical context is unknown to the author. Perhaps the many identical body problems shall elucidate this point. Another speculative conjecture is that the results of this and [17, 18] papers may be useful in seeking for the $q$-deformations of the non-linear integrable evolution equations, like KdV, $\sin$-Gordon, etc. We end by the remark that presented type of $q$-deformation can also be developed for the parasupersymmetric quantum mechanics [25] where higher (odd and even) dimensional internal spaces are involved.

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