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A strategy-based proof of the existence of the value in zero-sum differential games

Pablo Maldonado and Miquel Oliu-Barton *

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Abstract

The value of a zero-sum differential games is known to exist, under Isaacs condition, as the unique viscosity solution of a Hamilton-Jacobi-Bellman equation. In this note we provide a new proof via the construction of $\varepsilon$-optimal strategies, which is inspired in the “extremal aiming” method from [3].

1 Introduction

Let $U$ and $V$ be compact subsets of some euclidean space, let $\| \cdot \|$ be the euclidean norm in $\mathbb{R}^n$, and let $f : [0, 1] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n$.

**Assumption 1:**

1a. $f$ is uniformly bounded, i.e. $\|f\| := \sup_{(t, x, u, v)} \|f(t, x, u, v)\| < +\infty$,

1b. $\exists c \geq 0$ such that $\forall (u, v) \in U \times V, \forall s, t \in [0, 1], \forall x, y \in \mathbb{R}^n$:

$$\|f(t, x, u, v) - f(s, y, u, v)\| \leq c(|t - s| + \|x - y\|),$$

The directional game For any $(t, x) \in [0, 1] \times \mathbb{R}^n$ and any $\xi \in \mathbb{R}^n$, consider the one-shot game $\Gamma(t, x, \xi)$, with actions sets $U$ and $V$ and payoff function:

$$(u, v) \mapsto \langle \xi, f(t, x, u, v) \rangle.$$

Let $H^-(t, x, \xi)$ and $H^+(t, x, \xi)$ be its maxmin and minmax respectively:

$$H^-(t, x, \xi) := \max_{u \in U} \min_{v \in V} \langle \xi, f(t, x, u, v) \rangle,$$

$$H^+(t, x, \xi) := \min_{v \in V} \max_{u \in U} \langle \xi, f(t, x, u, v) \rangle.$$
These functions satisfy $H^- \leq H^+$. If the equality $H^+(t, x, \xi) = H^-(t, x, \xi)$ holds, the game $\Gamma(t, x, \xi)$ has a value.

**Assumption 2:** $\forall (t, x, \xi) \in [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n$, the game $\Gamma(t, x, \xi)$ has a value $H(t, x, \xi)$.

### 1.1 An important Lemma

Introduce the sets of controls:

$$U = \{ u : [0, 1] \to U, \text{ measurable} \}, \quad V = \{ v : [0, 1] \to V, \text{ measurable} \}.$$ 

Let $(u, v) \in U \times V$, $t_0 \in [0, 1]$, $(x_0, w_0) \in (\mathbb{R}^n)^2$ and let $(u^*, v^*)$ be a couple of optimal actions in $\Gamma(t_0, x_0, x_0 - w_0)$. Define two continuous trajectories in $\mathbb{R}^n$, $x : [t_0, 1] \to \mathbb{R}^n$ and $w : [t_0, 1] \to \mathbb{R}^n$, by:

$$x(t_0) = x_0, \quad \text{and} \quad \dot{x}(t) = f(t, x(t), u(t), v^*), \ a.e.$$  

$$w(t_0) = w_0, \quad \text{and} \quad \dot{w}(t) = f(t, w(t), u^*, v(t)), \ a.e.$$ 

The following lemma is inspired by Lemma 2.3.1 in [3].

**Lemma 1.** Under Assumptions 1 and 2, there exists $A, B \geq 0$ such that $\forall t \in [t_0, 1]$:

$$\|x(t) - w(t)\|^2 \leq (1 + (t - t_0)A)\|x_0 - w_0\|^2 + B(t - t_0)^2.$$ 

**Proof.** Notation: let $d_0 := \|x_0 - w_0\|$ and $d(t) := \|x(t) - w(t)\|$. Then:

$$d^2(t) = \|(x_0 - w_0) + \int_{t_0}^{t} f(s, x(s), u(s), v^*) - f(s, w(s), u^*, v(s)) ds\|^2. \ (1.1)$$

The boundedness of $f$ implies that

$$\|\int_{t_0}^{t} f(s, x(s), u(s), v^*) - f(s, w(s), u^*, v(s)) ds\|^2 \leq 4\|f\|^2(t - t_0)^2. \ (1.2)$$

**Claim:** For all $s \in [t_0, 1]$, and for all $(u, v) \in U \times V$:

$$(x_0 - w_0, f(s, x(s), u, v^*) - f(s, w(s), u^*, v(s)) ds) \leq 2C(s)d_0 + cd_0^2, \quad (1.3)$$

where $C(s) := c(1 + \|f\|(s - t_0)).$

Let us prove this claim. Assumption 1 implies $\|x(s) - x_0\| \leq (s - t_0)\|f\|$, and then:

$$\|f(s, x(s), u, v^*) - f(t_0, x_0, u, v^*)\| \leq c((s - t_0) + \|f\|(s - t_0)) = C(s).$$

Then, using Cauchy-Schwartz inequality, and the optimality of $v^*$:

$$\langle x_0 - w_0, f(s, x(s), u, v^*) \rangle \leq \langle x_0 - w_0, f(t_0, x_0, u, v^*) \rangle + C(s)d_0,$$

$$\leq H^+(t_0, x_0, x_0 - w_0) + C(s)d_0.$$ 

Similarly, Assumption 1 implies $\|w(s) - x_0\| \leq d_0 + (s - t_0)\|f\|$, and then:

$$\|f(s, w(s), u^*, v) - f(t_0, x_0, u^*, v)\| \leq C(s) + cd_0.$$
Using Cauchy-Schwartz inequality, and the optimality of $u^*$:

$$
\langle x_0 - w_0, f(s, x(s), u^*, v) \rangle \geq \langle x_0 - w_0, f(t_0, x_0, u^*, v) \rangle - (C(s) + cd_0)d_0,
\quad \geq H(t_0, x_0 - w_0) - C(s)d_0 - cd_0^2.
$$

The claim follows from Assumption 2. In particular, it holds for $(u, v) = (u(s), v(s))$.

Note that $\int_{t_0}^t 2C(s)ds = (t - t_0)C(t)$. Thus, integrating (1.3) over $[t_0, t]$ yields:

$$
\int_{t_0}^t \langle x_0 - w_0, f(s, x(s), u(s), v^*) - f(s, w(s), u^*, v(s)) \rangle ds \leq (t - t_0)(C(t)d_0 + cd_0^2).
$$

(1.4)

Go back to (1.1) using the estimates (1.2) and (1.4). We have proved:

$$
d^2(t) \leq d_0^2 + 4\|f\|^2(t - t_0)^2 + 2(t - t_0)C(t)d_0 + 2c(t - t_0)d_0^2.
$$

Finally, use the relations $d_0 \leq 1 + d_0^2$, $C(t) \leq c(1 + \|f\|)$ and $(t - t_0)C(t) = c(1 + \|f\|)(t - t_0)^2$ to obtain the result, with $A = 3c + 2\|f\|$ and $B = 4\|f\|^2 + 2c(1 + \|f\|)$. 

\section*{1.2 Consequences}

In this section, we give three direct consequences of Lemma 1. Let $d : \mathbb{R}^n \times \mathcal{P}(\mathbb{R}^n) \to \mathbb{R}$ denote the usual distance to a set in $\mathbb{R}^n$.

1. Consider some sequence of times $\Pi = \{t_0 < t_1 < \cdots < t_N\}$ in $[0, 1]$, and let $\|\Pi\| := \max\{t_m - t_{m-1}, m = 1, \ldots, N\}$. Let $(u, v) \in \mathcal{U} \times \mathcal{V}$ be a fixed pair of controls. Define the trajectories $x$ and $w$ on $[t_0, t_N]$ inductively. Let $x(t_0) = x_0$, $w(t_0) = w_0$ and suppose that $x(t)$ and $w(t)$ are already defined on $[t_0, t_m]$. Let $(u_m^*, v_m^*) \in U \times V$ be a couple of optimal actions in $\Gamma(t_m, x(t_m), x(t_m) - w(t_m))$. Then, on $[t_m, t_{m+1}]$, let $x$ and $w$ be the unique absolutely continuous solutions of:

$$
\begin{align*}
\dot{x}(t) &= f(t, x(t), u(t), v_m^*), \\
\dot{w}(t) &= f(t, w(t), u_m^*, v(t)).
\end{align*}
$$

Corollary 1.1. \textit{Under Assumptions 1 and 2:}

$$
\|x(t_N) - w(t_N)\| \leq e^{\|\Pi\|}(\|x_0 - w_0\|^2 + B\|\Pi\|).
$$

\textit{Proof.} For any $0 \leq m \leq N$, let $d_m := \|x(t_m) - w(t_m)\|$. Lemma 1 yields:

$$
d_m^2 \leq (1 + (t_m - t_{m-1})A)d_{m-1}^2 + B(t_m - t_{m-1})^2.
$$

Then, by induction: $d_N^2 \leq \exp(A \sum_{m=1}^N (t_m - t_{m-1})d_{m-1}^2 + B \sum_{m=1}^N (t_m - t_{m-1})^2)$. The result follows, since $t_N - t_0 \leq 1$ and $\sum_{m=1}^N (t_m - t_{m-1})^2 \leq \|\Pi\|$. 

2. For any $(t_0, x_0) \in [0, 1] \times \mathbb{R}^n$ and $(u, v) \in \mathcal{U} \times \mathcal{V}$, let $x = x[t_0, x_0, u, v]$ be the unique absolutely continuous solution in $[t_0, 1]$ of:

$$
\begin{align*}
x(t_0) &= x_0, \\
\dot{x}(t) &= f(t, x(t), u(t), v(t)), \text{ a.e.}
\end{align*}
$$

That is, $x[t_0, x_0, u, v]$ is the trajectory induced by the initial position $(t_0, x_0)$ and the controls $(u, v)$. For any $u \in U$, let $x[t_0, x_0, u, v]$ be the trajectory induced by $(t_0, x_0, v)$ and the constant control $u \equiv u$.

Define two properties for sets $W \subset [t_0, 1] \times \mathbb{R}^n$. 

\[3\]
• **P1:** For any \( t \in [t_0, 1] \), \( W(t) := \{ x \in \mathbb{R}^n \mid (t, x) \in W \} \) is closed and nonempty.

• **P2:** For any \( (t, x) \in W \) and any \( t_1 \in [t, 1] \):

\[
\sup_{u \in U} \inf_{v \in V} d(x(t, x, u, v)(t_1), W(t_1)) = 0,
\]

where \( d \) is the usual distance in \( \mathbb{R}^n \).

**Corollary 1.2.** Let \( W \subset [t_0, 1] \times \mathbb{R}^n \) satisfy \( P1 \) and \( P2 \). Under Assumptions 1 and 2, there exists \( v^* \in V \) such that, \( \forall t \in [t_0, 1], \forall u \in U \):

\[
d^2(x(t_0, x_0, u, v^*)(t), W(t)) \leq (1 + (t - t_0)A)d^2(x_0, W(t_0)) + B(t - t_0)^2.
\]

**Proof.** Let \( w_0 \in \arg\min_{w \in \gamma_{(t_0, x_0, x_0 - w)}} \|x_0 - w\| \) be some closest point (which exists by \( P1 \)). Let \( (u^*, v^*) \) be optimal in \( \Gamma(t_0, x_0, x_0 - w_0) \). By \( P2 \), \( \forall \varepsilon > 0, \exists v_\varepsilon \) such that \( w_\varepsilon(t) := x(t_0, w_0, u^*, v_\varepsilon)(t) \) satisfies \( d(w_\varepsilon(t), W(t)) \leq \varepsilon \). The triangular equality implies \( d(x(t), W(t)) \leq \|x(t) - w_\varepsilon(t)\| + \varepsilon \). Taking the limit, as \( \varepsilon \to 0 \):

\[
d^2(x(t), W(t)) \leq \lim_{\varepsilon \to 0} \|x(t) - w_\varepsilon(t)\|^2,
\]

where \( \|x(t) - w_\varepsilon(t)\|^2 \leq (1 + (t - t_0)A)\|x_0 - w_0\|^2 + B(t - t_0)^2 \) for any \( \varepsilon > 0 \), by Lemma 1, and where \( \|x_0 - w_0\| = d(x_0, W(t_0)) \) by definition. \( \square \)

3. Putting Corollaries 1.1 and 1.2 together, one obtains the following result.

**Corollary 1.3.** Let \( W \subset [t_0, 1] \times \mathbb{R}^n \) satisfy \( P1 \) and \( P2 \), let \( \Pi = \{t_0 < \cdots < t_N\} \) be a sequence of times, and let \( x_0 \in W(t_0) \). Under Assumptions 1 and 2, there exist \( v_0^*, \ldots, v_{N-1}^* \in V \) such that, for \( v \equiv v_m^* \), on \( [t_m, t_{m+1}] \), and for all \( u \in U \):

\[
d^2(x(t_0, x_0, u, v(t_N)), W(t_N)) \leq e^A B \|\Pi\|.
\]

## 2 Differential Games

For any \( (t_0, x_0) \in [0, 1] \times \mathbb{R}^n \), consider now the zero-sum differential with the following two-controlled dynamic

\[
x(t_0) = x_0, \quad \dot{x}(t) = f(t, x(t), u(t), v(t)), \text{ a.e. on } [t_0, 1].
\]

**Definition 2.1.** A strategy for player 2 is a map \( \beta : U \to V \) such that, for some finite partition \( t_0 < t_1 < \cdots < t_N = 1 \) of \([t_0, 1], \forall u_1, u_2 \in U : \)

\[
u_1 \equiv u_2 \text{ a.e. on } [t_0, t_m] \implies \beta(u_1) \equiv \beta(u_2) \text{ a.e. on } [t_0, t_{m+1} \land 1].
\]

These strategies are called nonanticipative strategies with delay (NAD) in [1], in contrast to the classical nonanticipative strategies. The strategies for player 1 are defined in a dual manner. Let \( B \) (resp. \( A \)) the set of strategies for Player 2 (resp. 1). For any pair of strategies \((\alpha, \beta) \in A \times B, [1] \) establishes the following crucial result: there exists a unique pair \((u, v) \in U \times V \) such that \( \alpha(v) = u \), and \( \beta(u) = v \). Denote by \( x[t_0, x_0, u, v] \) the trajectory induced by the pair \((u, v)\).
Let $g : \mathbb{R}^n \to \mathbb{R}$ some function. The differential game with initial time $t_0$, initial state $x_0$, and terminal payoff $g$ is denoted by $\mathcal{G}(t_0, x_0)$. Introduce the upper and lower value functions:

$$
V^-(t_0, x_0) := \sup_{\alpha \in A} \inf_{\beta \in B} g(x[t_0, x_0, \alpha, \beta](1)),
$$

$$
V^+(t_0, x_0) := \inf_{\beta \in B} \sup_{\alpha \in A} g(x[t_0, x_0, \alpha, \beta](1)).
$$

The inequality $V^- \leq V^+$ holds everywhere. If $V^-(t_0, x_0) = V^+(t_0, x_0)$, the game $\mathcal{G}(t_0, x_0)$ has a value. Notice that its lower and upper Hamiltonian of are precisely the maxmin and the minmax of the directional games defined in Section 1. Consequently, Assumption 2 is precisely Isaacs’ condition.

**Assumption 3:** $g$ is $\alpha$-Lipschitz continuous, i.e. $|g(x) - g(y)| \leq c \|x - y\|$, $\forall x, y \in \mathbb{R}^n$.

### 2.1 Existence and characterization of the value

Let $\phi : [t_0, 1] \times \mathbb{R}^n \to \mathbb{R}$ be a real function satisfying the following properties:

(i) $x \mapsto \phi(t, x)$ is lower semicontinuous, $\forall t \in [t_0, 1]$,

(ii) $\forall (t, x) \in [t_0, 1] \times \mathbb{R}^n$, $\forall t_1 \in [t, 1]$:

$$
\phi(t, x) \geq \sup_{u \in U} \inf_{v \in V} \phi(t_1, x[t, x, u, v](t_1)),
$$

(iii) $\phi(1, x) \geq g(x)$, $\forall x \in \mathbb{R}^n$.

For any $\ell \in \mathbb{R}$, define the $\ell$-level set of $\phi$ by:

$$
W^\phi_\ell = \{(t, x) \in [t_0, 1] \times \mathbb{R}^n \mid \phi(t, x) \leq \ell\},
$$

(2.1)

**Lemma 2.** For any $\ell \geq \phi(t_0, x_0)$, the $\ell$-level set of $\phi$ satisfies $P1$ and $P2$.

**Proof.** Note that $W^\phi_\ell(t_0)$ is nonempty, since $x_0 \in W^\phi_\ell(t_0)$. (i) implies that $W^\phi_\ell(t)$ is a closed set, $\forall t \in [0, 1]$. On the other hand, by (ii) for all $(t, x) \in [t_0, 1] \times \mathbb{R}^n$, $t_1 \in [t, 1]$, $u \in U$, and $n \in \mathbb{N}^*$, there exists $v_n \in V$ such that:

$$
\phi(t, x) \geq \phi(t_1, x[t, x, u, v_n](t_1)) - \frac{1}{n}. \quad (2.2)
$$

The boundedness of $f$ implies that $x_n := x[t, x, u, v_n](t_1)$ belongs to some compact set. Consider a subsequence $(x_n)_n$ such that $\lim_{n \to \infty} \phi(t_1, x_n) = \lim_{n \to \infty} \phi(t_1, x_n)$, and such that $(x_n)_n$ converges to some $\bar{x} \in \mathbb{R}^n$. Then, taking the limit, as $n \to \infty$, in (2.2) implies, using (i) and $\ell \geq \phi(t, x)$:

$$
\phi(t_1, \bar{x}) \leq \lim_{n \to \infty} \phi(t_1, x_n) \leq \phi(t, x) \leq \ell.
$$

Hence $\bar{x} \in W^\phi_\ell(t_1)$, and $\inf_{n \in \mathbb{N}} d(x[t, x, u, v_n](t_1), W^\phi_\ell) = 0$. In particular, $W^\phi_\ell(t_1)$ is nonempty, and $P1$ and $P2$ hold.

\qed
2.1.1 Extremal strategies in \( G(t_0, x_0) \)

Let \( \Pi = \{t_0 < \cdots < t_N = 1\} \) be partition of \([t_0, 1]\), let \( \|\Pi\| = \max\{t_m - t_{m-1}, m = 1, \ldots, N\} \), and let \( W^\phi \subset [t_0, 1] \times \mathbb{R}^n \) be the \( \phi(t_0, x_0) \)-level set of \( \phi \).

**Definition 2.2.** An extremal strategy \( \beta = \beta(\phi, \Pi) \) is defined inductively: suppose \( \beta \) is already defined on \([t_0, t_m]\) and let \( x_m = x[t_0, x_0, u, \beta(t_m)] \). Then, \( \forall u \in U \):

- If \( x_m \in W^\phi(t_m) \), set \( \beta(u)(s) = v \), for any \( v \in V \), \( \forall s \in [t_m, t_{m+1}] \).
- If \( x_m \not\in W^\phi(t_m) \), let \( w_m = \arg\min_{w \in W^\phi(t_m)} \|x_m - w\| \) be some closest point, and let \( v^*_m \) be some optimal action in the directional game \( \Gamma(t_m, x_m, x_m - w_m) \). Set \( \beta(u)(s) = v^*_m \), \( \forall s \in [t_m, t_{m+1}] \).

These strategies are inspired by the *extremal aiming* method of Krasovskii and Subbotin (see Section 2.4 in [3]). Notice that \( \beta \) is defined up to some selection rule since \( V \), the set of closest points and the set of minimizers may have more than one element.

**Proposition 2.1.** Under Assumptions 1, 2 and 3, \( \exists C \geq 0 \) such that:

\[
g(x[t_0, x_0, u, \beta(u)](1)) \leq \phi(t_0, x_0) + C \sqrt{\|\Pi\|}, \quad \forall u \in U,
\]

for any extremal strategy \( \beta = \beta(\phi, \Pi) \).

**Proof.** \( W^\phi \) satisfies P1 and P2 by Lemma 2. Applying Corollary 1.3:

\[
d^2(x_N, W^\phi(t_N)) \leq e^A B \|\Pi\|.
\]

Now, by (iii), and since \( t_N = 1 \):

\[
W^\phi(t_N) = \{x \in \mathbb{R}^n \mid \phi(1, x) \leq \phi(t_0, x_0)\} \subset \{x \in \mathbb{R}^n \mid g(x) \leq \phi(t_0, x_0)\}.
\]

Let \( w_N = \arg\min_{w \in W^\phi(1)} \|x_N - w\| \) be some closest point. By Assumption 3:

\[
g(x_N) \leq g(w_N) + c \|x_N - w_N\| \leq \phi(t_0, x_0) + cd(x_N, W^\phi(t_N)).
\]

The result follows, recalling that \( x_N = x[t_0, x_0, u, \beta(u)](1) \). Explicitly, \( C = ce^{aB} \).

\( \square \)

Proposition 2.1 applies to any function satisfying (i), (ii) and (iii). Consequently, under Assumptions 1, 2 and 3:

\[
V^+(t_0, x_0) \leq \inf\{\phi(t_0, x_0) \mid \phi : [t_0, 1] \times \mathbb{R}^n \to \mathbb{R} \text{ satisfying (i), (ii), (iii)}\}. \quad (2.3)
\]

**Theorem 2.3.** Under Assumptions 1, 2 and 3, the differential game \( G(t_0, x_0) \) has a value, characterized as:

\[
V(t_0, x_0) = \min_{\phi \text{ satisfying (i), (ii), (iii)}} \phi(t_0, x_0).
\]

The strategies \( \beta(V, \Pi) \) are asymptotically optimal for player 2, as \( \|\Pi\| \to 0 \).
Proof. By (2.3), it is enough to prove that $V^-$ satisfies (i), (ii) and (iii), where (iii) is immediate. Assumption 1, and Gronwall’s lemma imply that $\forall t \in [t_0, 1]$, $\forall (u, v) \in U \times V$, and $\forall x, y \in \mathbb{R}^n$:

$$\|x[t_0, x, u, v](t) - x[t_0, y, u, v](t)\| \leq e^{c(t-t_0)}\|x - y\|.$$ 

Assumption 2 gives then, $\forall (u, v) \in U \times V$, and $\forall x, y \in \mathbb{R}^n$:

$$|g(x[t_0, x, u, v](1)) - g(x[t_0, y, u, v](1))| \leq ce^{c(1-t_0)}\|x - y\|.$$ 

Thus, by standard arguments, $x \mapsto V^-(t, x)$ is $ce^c$-Lipschitz continuous $\forall t \in [t_0, 1]$ and, in particular, $V^-$ satisfies (i). On the other hand, (ii) is a weak version of the classical dynamic programming principle (see [2], for nonanticipative strategies, and [1] for NAD strategies, defined above): $\forall (t, x) \in [t_0, 1] \times \mathbb{R}^n$, $\forall t_1 \in [t, 1]$: 

$$V^-(t, x) = \sup_{\alpha \in A} \inf_{v \in V} V^-(t_1, x[t, x, \alpha(v), v](t_1)).$$ 

Finally, let $\beta(V, \Pi)$ be an extremal strategy. By Corollary 2.1:

$$g(x[t_0, x_0, u, \beta(V, \Pi)(u)](1)) \leq V(t_0, x_0) + C\sqrt{\|\Pi\|}, \quad \forall u \in U.$$ 

Consequently, for any $\varepsilon > 0$, $\beta(V, \Pi)$ is $\varepsilon$-optimal for sufficiently small $\|\Pi\|$.

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