The quantum Casimir operators of $U_q(\mathfrak{gl}_n)$ and their eigenvalues

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Abstract
We show that the quantum Casimir operators of the quantum linear group constructed in early work of Bracken, Gould and Zhang together with one obvious central element generate the entire center of $U_q(\mathfrak{gl}_n)$. As a byproduct of the proof, we obtain intriguing new formulae for eigenvalues of these quantum Casimir operators, which are expressed in terms of the characters of a class of finite-dimensional irreducible representations of the classical general linear algebra.

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1. Introduction
Quantum groups originated from the theory of soluble models of the Yang–Baxter type in the mid-1980s. They have played important roles in various branches of mathematics and physics, most notably in two-dimensional soluble models in statistical mechanics and knot theory. The study of their structure and representation theory has been the focus of research in representation theory and continues to attract much attention. In particular, the discovery of the crystal basis and canonical basis [3, 5] was one of the most important achievements in representation theory in recent years.

In the early 1990s, a set of central elements of the quantum group $U_q(\mathfrak{gl}_n)$ was constructed in [8, 9]. We shall refer to them as the quantum Casimir operators of $U_q(\mathfrak{gl}_n)$. The reason for this terminology is the fact that these central elements of $U_q(\mathfrak{gl}_n)$ are the quantum analogs of the familiar Casimir operators of the universal enveloping algebra $U(\mathfrak{gl}_n)$ of $\mathfrak{gl}_n$ given by

$$\sum_{i=1}^{n} \sum_{i=1}^{n} E_{ij} E_{ij} \cdots E_{ij} (k = 1, 2, \ldots),$$

where $E_{ij}$ are the images of the matrix units under the canonical embedding of $\mathfrak{gl}_n$ in $U(\mathfrak{gl}_n)$. These Casimir operators of $U(\mathfrak{gl}_n)$ play...
an important role in the interacting boson model in nuclear physics. Their quantum analogs have also been applied in a similar way.

One obvious question was whether the quantum Casimir operators [8, 9] of \( U_q(\mathfrak{gl}_n) \) (supplemented with the obvious central element \( c \) given by (2.2)) generated the entire center of \( U_q(\mathfrak{gl}_n) \). The general expectation was that the answer should be affirmative, but no proof was ever given as far as we know. The main purpose of this paper is to give a rigorous proof. The result is described in theorem 4.1.

The proof of theorem 4.1 requires us to analyze the eigenvalues of the quantum Casimir operators of \( U_q(\mathfrak{gl}_n) \). A formula for the eigenvalues was obtained in [4] (in fact [4] treated \( U_q(\mathfrak{gl}_m|n) \), which included \( U_q(\mathfrak{gl}_n) \) as a special case). We cast the formula into a form readily usable for our purpose. This new formula is expressed in terms of the characters of a class of finite-dimensional irreducible representations of the classical general linear algebra. This result is rather intriguing, and we believe that it is interesting in its own right.

We should point out that the structure of the center of a quantum group is much studied at an abstract level. In particular, a quantum analog of the celebrated Harish-Chandra homomorphism in semi-simple Lie algebras has been established for quantum groups at generic \( q \). In the case of \( U_q(\mathfrak{gl}_n) \), a set of generators different from the quantum Casimir operators of [8] was constructed in [1].

2. The quantum general linear group

2.1. The quantum general linear group and its quantum Casimir operators

This section provides some basic materials on the general linear algebra \( \mathfrak{gl}_n \) and its quantum group \( U_q(\mathfrak{gl}_n) \). Let \( \varepsilon_i \), with \( i \in \mathbf{I} = \{1, 2, \ldots, n\} \), be a basis of an Euclidean space with the inner product \( (\varepsilon_i, \varepsilon_j) = \delta_{ij} \). Set \( \rho = \frac{1}{2} \sum_{i=1}^{n} (n - 2i + 1) \varepsilon_i \).

The quantized universal enveloping algebra \( U_q(\mathfrak{gl}_n) \) of the general linear algebra \( \mathfrak{gl}_n \) is a unital associative algebra over \( \mathbb{K} := \mathbb{C}(q) \) generated by \( K_i^{\pm 1}, E_i, F_i \), \( (i, i' \in \mathbf{I} := \mathbf{I} \setminus \{n\}) \), subject to the following relations:

\[
K_i K_i^{-1} = 1, \quad K_i^{\pm 1} K_j^{\pm 1} = K_j^{\pm 1} K_i^{\pm 1}, \quad K_i E_j K_i^{-1} = q^{(\varepsilon_i, \varepsilon_j)} E_j, \\
K_i F_j K_i^{-1} = q^{-(\varepsilon_i, \varepsilon_j)} F_j, \quad E_r E_s = E_s E_r, \quad F_r F_s = F_s F_r, \quad |r - s| \geq 2, \\
E_i F_j - F_j E_i = \delta_{ij} K_i K_i^{-1} - K_i^{-1} K_i, \quad S_{i, j \pm 1}^{(+)} = S_{i, j \pm 1}^{(-)} = 0,
\]

\[
S_{i, j \pm 1}^{(+)} = (E_i)^2 E_{i \pm 1} - (q + q^{-1}) E_i E_{i \pm 1} E_i + E_{i \pm 1}(E_i)^2, \\
S_{i, j \pm 1}^{(-)} = (F_i)^2 F_{i \pm 1} - (q + q^{-1}) F_i F_{i \pm 1} F_i + F_{i \pm 1}(F_i)^2.
\]

As we know, \( U_q(\mathfrak{gl}_n) \) possesses the structure of a Hopf algebra with the co-multiplication \( \Delta \), co-unit \( \varepsilon \) and antipode \( S \) respectively given by

\[
\Delta(E_i) = E_i \otimes K_i K_i^{-1} + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} K_i \otimes F_i, \\
\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i^{\pm 1}) = 1, \\
S(E_i) = -E_i K_i^{-1} K_i + 1, \quad S(F_i) = -K_i^{-1} F_i, \quad S(K_i^{\pm 1}) = K_i^{\mp 1}.
\]

The natural module \( V \) for \( U_q(\mathfrak{gl}_n) \) has the standard basis \( \{v_i \mid i \in \mathbf{I}\} \) such that

\[
E_i v_j = \delta_{i,j+1} v_i, \quad F_i v_j = \delta_{i,j-1} v_i, \quad K_i v_j = (1 + (q - 1) \delta_{i,j}) v_j.
\]
Denote by $\pi$ the $U_q(\mathfrak{gl}_n)$-representation relative to this basis; then $\pi(E_i) = E_{i,i+1}$, $\pi(F_i) = E_{i,i-1}$ and $\pi(K_i) = I + (q - 1)E_{i,i}$, where $E_{ij}$ are the matrices $(E_{ij})_{rs} = \delta_{ir}\delta_{js}$.

We now turn to the description of the center $Z$ of the quantum $U_q(\mathfrak{gl}_n)$. Let $U_q(\mathfrak{gl}_n)^- = U_q(\mathfrak{gl}_n)^+$ and $U_q(\mathfrak{gl}_n)^0$ be the subalgebras of $U_q(\mathfrak{gl}_n)$ generated by $F_i$ ($i' \in I'$), $E_i$ ($i' \in I'$) and $K_i^\pm$ ($i \in I$) respectively. Any element $z \in Z$ can be written as

$$z = z^{(0)} + \sum_x u_x^{(-)}u_x^{(0)}u_x^{(+)}$$

where $z^{(0)}$, $u_x^{(0)} \in U_q(\mathfrak{gl}_n)^0$, $u_x^{(+)} \in U_q(\mathfrak{gl}_n)^+$ and $u_x^{(-)} \in U_q(\mathfrak{gl}_n)^-$. The quantum Harish-Chandra homomorphism is an algebra homomorphism $\varphi : Z \rightarrow U_q(\mathfrak{gl}_n)^0$ such that $z \mapsto z^{(0)}$.

The dot action of the Weyl group $W$ of $\mathfrak{gl}_n$ on $U_q(\mathfrak{gl}_n)^0$ is given by the permutations of the elements $q^{-1}K_i$ ($i \in I$). Define

$$L_i = q^{(c_i)\varphi(K_i)}^2, \quad i \in I.$$ (2.1)

Then $L_i$ are permuted by the Weyl group. Let $U_q(\mathfrak{gl}_n)_{ev}$ be the subalgebra spanned by the elements $\Pi_{i,j}^n L_i^{l_i}$ for $l_i \in \mathbb{Z}$. We denote by $(U_q(\mathfrak{gl}_n)_{ev})^W$ the $W$-invariant subalgebra of $U_q(\mathfrak{gl}_n)_{ev}$. By using the quantum Harish-Chandra isomorphism for $U_q(\mathfrak{sl}_n)$ (see, e.g., [2, 7]), one can prove the following result.

**Lemma 2.1.** The Harish-Chandra homomorphism is an algebra isomorphism between the center $Z$ of $U_q(\mathfrak{gl}_n)$ and the subalgebra of $U_q(\mathfrak{gl}_n)^0$ generated by the elements of $(U_q(\mathfrak{gl}_n)_{ev})^W$ together with $c$, where

$$c = K_1^{-1}K_2^{-1}\cdots K_n^{-1}.$$ (2.2)

Note that $c$ is obviously $W$-invariant and $c^{\pm 2} \in (U_q(\mathfrak{gl}_n)_{ev})^W$. An equivalent description of this lemma can be found in [1]. In this paper, a set of generators for $Z$ was also given, which are different from the quantum Casimir operators of $[8, 9]$.

### 2.2. Quantum Casimir operators of $U_q(\mathfrak{gl}_n)$

The quantum Casimir operators of $U_q(\mathfrak{gl}_n)$ are the main objects for study in this paper, which we now briefly describe. As is well known, in the quantum group setting, we have neither a good quantum analog of tensor operators nor a procedure for ‘contracting tensors’ (however, see [11, 12]). Thus it is much harder to explicitly construct central elements for quantum groups. The construction of $[8, 9]$ was actually quite involved: it had to invoke the theory of [10] and also made use of the universal $R$-matrix of $U_q(\mathfrak{gl}_n)$. Thus for the sake of completeness and also clearness, we briefly explain the construction.

It is well known that $U_q(\mathfrak{gl}_n)$ is a quasi-triangular Hopf algebra, i.e. there exists an invertible element $R \in U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n)$ which is called the universal $R$-matrix of $U_q(\mathfrak{gl}_n)$, such that

$$R \Delta(x) = \Delta'(x)R, \quad \forall x \in U_q(\mathfrak{gl}_n),$$

$$(\Delta \otimes id)R = R_{13}R_{23}, \quad (id \otimes \Delta)R = R_{13}R_{12},$$

where $\Delta'$ is the opposite co-multiplication. Explicitly, $\Delta' = T \circ \Delta$, where $T : U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n) \rightarrow U_q(\mathfrak{gl}_n) \otimes U_q(\mathfrak{gl}_n)$ is the linear map defined for any $x, y \in U_q(\mathfrak{gl}_n)$ by $T(x \otimes y) = y \otimes x$. The $R$-matrix satisfies the celebrated Yang–Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.$$

Denote $R^T = T(R)$. Then $R^T R \Delta(x) = \Delta(x)R^T R \forall x \in U_q(\mathfrak{gl}_n)$.
For any $\alpha \in \mathbb{Z}$, let $\mathbb{Z}_{\geq \alpha} := \{ m \mid m \in \mathbb{Z}, m \geq \alpha \}$. The following quantum Casimir operators for $U_q(\mathfrak{gl}_n)$ were constructed in [8, 9]:

$$C_{n,k} = \text{Tr}_\pi \left( 1 \otimes q^{(2\rho)} \left( \frac{\Gamma - 1 \otimes 1}{q - q^{-1}} \right)^k \right), \quad k \in \mathbb{Z}_{\geq 0},$$

(2.3)

where $\text{Tr}_\pi$ represents the trace taken over $\pi$, and

$$\Gamma = (id \otimes \pi) R^T R, \quad q^{(2\rho)} = \prod_{i=1}^n q^{(\epsilon_i, 2\Lambda + 2\rho + \epsilon_i)}.$$

In [8, 9], the explicit formulae $C_{n,k}$ were given in terms of the generators of $U_q(\mathfrak{gl}_n)$. They indeed exhibit structural features resembling that of the Casimir operators of the classical general linear algebra.

Let $L_A$ be a finite-dimensional irreducible $U_q(\mathfrak{gl}_n)$-module with the highest weight $\Lambda \in \mathcal{H}^*$. Then each $C_{n,k}$ acts on $L_A$ by a scalar, which is given by the following formula [4]:

$$\chi_{\Lambda}(C_{n,k}) = (q - q^{-1} - 1)^k \sum_{i=1}^n q^{(\epsilon_i, 2\Lambda + 2\rho + \epsilon_i)} - C_{(\Lambda_0)} - 1)^k \prod_{j \neq i} q^{(\epsilon_i, 2\Lambda + 2\rho + \epsilon_j)} - q^{(\epsilon_j, 2\Lambda + 2\rho + \epsilon_j)},$$

(2.4)

where $\Lambda_0 = \epsilon_1$.

Since $\psi(z)$ belongs to $U_q(\mathfrak{gl}_n)^0$ for every $z \in \mathbb{Z}$, the right-hand side of the above formula must be a polynomial in $q^{(\epsilon_i, 2\Lambda + 2\rho)}$. Note that when applying $L_i$ to the highest weight vector $v_\Lambda$ of $L_A$, we have $L_i v_\Lambda = q^{(\epsilon_i, \Lambda + \rho)} v_\Lambda$. Then it follows that

$$C_{n,k} = \psi(C_{n,k}) = \sum_{i=1}^n \left( \frac{1}{q - q^{-1}} \right)^{n-k} \prod_{j \neq i} q^{L_j - q^{-1}} L_j - L_j.$$  

(2.5)

3. Analysis of $G_{n,k}$

Let us analyze the formula for $C_{n,k}$ to put it into a form which will be readily usable for the proof of theorem 4.1. Denote

$$G_{n,k} = \sum_{i=1}^n L_i^k P_{n,i}, \quad \text{where} \quad P_{n,i} = \prod_{j \neq i} q^{L_i - q^{-1}} L_j - L_j.$$  

(3.1)

Then for any $n \in \mathbb{Z}_{\geq 2}$ and $k \in \mathbb{Z}_{\geq 1}$, one can rewrite $C_{n,k}$ as

$$C_{n,k} = \frac{1}{(q^{-1} - q)^k} \sum_{j=0}^k \binom{k}{j} (-q^{-1})^j G_{n,j}.$$  

We want to prove that $G_{n,k} (k = 0, 1, \ldots, n)$ are polynomials in $L_i$.

**Lemma 3.1.** For any $n \geq 2$, the following two identities hold:

$$G_{n,0} = q^{n-1} + q^{n-3} + \cdots + q^{3-n} + q^{-n},$$

(3.2)

$$G_{n,1} = q^n (L_1 + L_2 + \cdots + L_n).$$

(3.3)
Proof. For \( i \neq n, n - 1 \), one has
\[
P_{n,i} = P_{n-2,i} \cdot \frac{q L_i - q^{-1} L_{n-1}}{L_i - L_{n-1}} = P_{n-2,i} \cdot \frac{q L_i - q^{-1} L_{n-1}}{L_i - L_{n-1}}.
\]

For any \( m, n \in \mathbb{Z}_{\geq 2} \) and \( i \in \mathbb{Z}_{\geq 0} \), denote \( P_{m,i} = \frac{q L_i - q^{-1} L_m}{L_i - L_m} \). Then \( P_{n,i} \) can be rewritten as
\[
P_{n,i} = P_{n-2,i,n-1} \cdot \frac{q L_i - q^{-1} L_n}{L_{n-1} - L_n}.
\]

For convenience, denote \( G_{n,k} = \sum_{j \neq i=1}^{n} L_j^k P_{n,i} \). Then we can rewrite \( G_{n,0} \) as follows:
\[
G_{n,0} = \sum_{j=1}^{n} L_j^1 P_{n,i} = \frac{q L_i - q^{-1} L_n}{L_{n-1} - L_n} + P_{n,n}.
\]

Meanwhile, for any \( n \in \mathbb{Z}_{\geq 2} \) and \( k \in \mathbb{Z}_{\geq 1} \), we have the following computations:
\[
G_{n,k} = L_n G_{n,k-1} = \sum_{i=1}^{n} L_i^{k-1} (L_i - L_n) P_{n,i}
\]

Thus
\[
G_{n,k} = q G_{n-1,k} - q^{-1} L_n G_{n-1,k-1} \quad \forall n \in \mathbb{Z}_{\geq 2}, \quad k \in \mathbb{Z}_{\geq 1}.
\]

It is easy to see that both (3.2) and (3.3) hold for the cases \( n = 2 \) and \( n = 3 \). Combining the identities (3.5) and (3.6), using induction on \( n \) in (3.2) and (3.3), one can get the formulae for \( G_{n,0} \) and \( G_{n,1} \). Hence we complete the proof of this lemma.

We introduce a generating function \( S_n(t) = \sum_{k=0}^{\infty} t^k G_{n,k} \). Then
\[
S_n(t) = G_{n,0} - t L_n S_n(t) = q S_{n-1}(t) - q^{-1} t L_n S_{n-1}(t) - q G_{n-1,0},
\]

which implies
\[
S_n(t) = \frac{q - q^{-1} t L_n}{1 - t L_n} S_{n-1}(t) + \frac{G_{n,0} - q G_{n-1,0}}{1 - t L_n}
\]

Thus
\[
G_{n,k} = q G_{n-1,k} + (q - q^{-1}) \sum_{i=0}^{k-1} t^i G_{n-1,i} + q^{-n-1} L_n^k.
\]
Note that it is by no means obvious from formula (2.5) itself that its right-hand side is a polynomial in $L_i$’s. To put our mind at peace, we observe the following result.

**Lemma 3.2.** For any $n \in \mathbb{Z}_{\geq 2}$ and $k \in \mathbb{Z}_{\geq 0}$, $G_{n,k}$ is a polynomial in $L_i$’s.

**Proof.** This lemma follows from lemma 3.1 and the identities (3.6) and (3.7). \[ \square \]

Form the formulae proved in lemma 3.1, we see that if $L_i$ is replaced by $e^{\epsilon_i}$ for $i = 1, 2, \ldots, n$, then the coefficient of $q^{n-1}$ in $G_{n,1}$ corresponds to the character of the basic irreducible representation of $U_q(\mathfrak{gl}_n)$. A natural problem is to understand all $G_{n,k}$ in similar terms, and we address this problem now.

Let us introduce a set of elements of $\lambda^j_k$ in $\sum_{i=1}^n \mathbb{Z}_{+} \epsilon_i$, which we write in terms of their coordinates relative to the basis $\epsilon_i$. We let $\lambda^1_k = (k, 0, 0, \ldots, 0), \lambda^2_k = (k - 1, 1, 0, \ldots, 0), \ldots, \lambda^n_k = (1, \ldots, 1, 0, \ldots, 0)$ in the case when $k < n$. We also set $\lambda^1_k = (k, 0, 0, \ldots, 0), \lambda^2_k = (k - 1, 1, 0, \ldots, 0), \ldots, \lambda^n_k = (k - n + 1, 1, \ldots, 1)$ in the case when $k \geq n$. Note that these weights respectively correspond to the following Young diagrams:

```
   k
  [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ]
   k-1
  [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ]
   k-2
  [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ]
     ...
   k-n+3
  [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ]
   k-n+2
  [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ] [ ]
   k-n+1
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     ...
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    n-1
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    n
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Denote by $ChL_{\lambda^j_k}$ the character of the irreducible $\mathfrak{gl}_n$-representation with the highest weight $\lambda^j_k$. Then

$$ChL_{\lambda^j_k} = \sum_{w \in W} \text{sign}(w) e^{w((k-n+1)\epsilon_1 + \epsilon_3 + \cdots + \epsilon_k) + \bar{\rho}},$$

where $\bar{\rho} = \rho + \frac{1}{2}(n-1) \sum_{i=1}^n \epsilon_i$. The formula is valid when $k - i \geq 0$. When $i > k$, the right-hand side vanishes identically.

**Lemma 3.3.** Let $Ch_{n,k}$ be the expression obtained from $G_{n,k}$ by replacing $L_i$ by $e^{\epsilon_i}$ for all $i = 1, 2, \ldots, n$. Then

$$\sum_{i=1}^n (-1)^{i-1} q^{n-2i+1} ChL_{\lambda^i_k} = Ch_{n,k} \quad \forall k = 1, 2, \ldots, \quad (3.9)$$

where $ChL_{\lambda^i_k} = 0$ if $k < i$.

**Proof.** Denote $W_{n-1} = \{ w \in W \mid w(\epsilon_1) = \epsilon_1 \}$. Then $W / W_{n-1} = \{ [w] \mid w \in W \mid w(\epsilon_j) = \epsilon_j, \forall j \neq 1 \}$. Denote $\bar{\rho}_{n-1} = \bar{\rho} - (n-1) \epsilon_1$. In the case when $k \geq n$, the left-hand side of
Then we obtain a new formula for the eigenvalues of the quantum Casimir operators of $U_q(\mathfrak{gl}_n)$ in the irreducible representation with the highest weight $\Lambda$.
Corollary 3.2.

$$\chi_\lambda(C_{n,k}) = (q - q^{-1})^{-2} \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} (q^{1-n})^l G_{n,l}.$$  

The advantage of this formula is that every term in \( G_{n,l} \) has a representation theoretical interpretation in terms of the general linear Lie algebra.

4. The main result

With the preparations in the previous sections, we can now prove the following theorem, which is the main result of the paper.

Theorem 4.1. The center \( Z \) of \( U_q(\mathfrak{gl}_n) \) is generated by \( c \) and the quantum Casimir operators \( C_{n,1}, \ldots, C_{n,n} \).

In order to prove the theorem, we need some basic results on symmetric polynomials [6], which we recall here. The complete homogeneous symmetric polynomial of degree \( k \) in \( n \) variables \( x_1, x_2, \ldots, x_n \), written as \( h_k \), for \( k = 0, 1, 2, \ldots \), is the sum of all monomials of total degree \( k \) in the variables. Formally,

$$h_k(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1}x_{i_2}\cdots x_{i_k}.$$  

It is well known that the set of complete homogeneous symmetric polynomials

$$h_1(x_1, x_2, \ldots, x_n), h_2(x_1, x_2, \ldots, x_n), \ldots, h_n(x_1, x_2, \ldots, x_n)$$

generates the ring of symmetric polynomials in the \( n \) variables \( x_1, x_2, \ldots, x_n \).

Proof of theorem 4.1. Denote by \( \mathbb{K}[L_1, L_2, \ldots, L_n]^W \) the algebra of symmetric polynomials in the polynomial ring \( \mathbb{K}[L_1, L_2, \ldots, L_n] \). Given any element in \( (U_q(\mathfrak{gl}_n)^{ev})^W \), we can always express it in terms of the elements of \( \mathbb{K}[L_1, L_2, \ldots, L_n]^W \) and \( c \) algebraically. Note that \( c^{-2} \in \mathbb{K}[L_1, L_2, \ldots, L_n]^W \). Therefore, in order to prove theorem 4.1, it suffices to show that \( G_{n,k} \) (\( k = 1, \ldots, n \)) generate \( \mathbb{K}[L_1, L_2, \ldots, L_n]^W \).

Note that \( \Gamma_{k,1} \) in \( G_{n,k} \) is a complete symmetric polynomial in \( L_1, \ldots, L_n \). Thus \( \Gamma_{1,1}, \Gamma_{2,1}, \ldots, \Gamma_{n,1} \) are a set of generators of the ring \( \mathbb{K}[L_1, L_2, \ldots, L_n]^W \) of the symmetric polynomial.

Now \( \Gamma_{1,1} \) is equal to \( q^{1-n}G_{n,1} \), and we can easily express \( \Gamma_{2,1} \) in terms of \( G_{n,1} \) and \( G_{n,2} \). Inductively we can show that \( \Gamma_{k,1} \) can always be expressed in terms of \( G_{n,1}, G_{n,2}, \ldots, G_{n,k} \). Thus \( G_{n,1}, G_{n,2}, \ldots, G_{n,n} \) are also a set of generators of the symmetric polynomial ring \( \mathbb{K}[L_1, L_2, \ldots, L_n]^W \). Since the elements of \( \mathbb{K}[L_1, L_2, \ldots, L_n]^W \) and \( c \) together generate \( \varphi(Z) \), we complete the proof of the theorem. \( \square \)

5. Conclusion

We have shown in this paper that the quantum Casimir operators for \( U_q(\mathfrak{gl}_n) \) constructed in [8, 9] together with the obvious central element \( c \) (defined by (2.2)) generate the entire center of \( U_q(\mathfrak{gl}_n) \). Our proof relied in an essential way on the representation theory of the general linear algebra and made use of results on symmetric polynomials. As a by-product, the proof also yielded new formulae for eigenvalues of these quantum Casimir operators with a representation theoretical interpretation.
We wish to mention that the analogs of the quantum Casimir operators for the quantum
general linear supergroup $U_q(gl_{m|n})$ were constructed by Zhang in the early 1990s. We expect
that these quantum Casimir operators supplemented with the analog of $c$ also generate the
entire center of $U_q(gl_{m|n})$. We shall investigate this matter in a future publication.

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