RELATIVE NASH-TYPE AND $L^2$-SOBOLEV INEQUALITIES FOR DUNKL OPERATORS AND APPLICATIONS

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ABSTRACT. We investigate local variants of Nash inequalities in the context of Dunkl operators. Pseudo-Poincaré inequalities are first established using pointwise gradient estimates of the Dunkl heat kernel. These inequalities allow to obtain relative Nash-type inequalities which are used to derive mean value inequalities for subsolutions of the heat equation on orbits of balls not necessarily centered on the origin.

1. Introduction and main results

The Nash inequality [9] states the existence of a constant $C > 0$, such that
\[ \|f\|_{L^2}^{1+\frac{2}{N}} \leq C \|\nabla f\|_{L^2} \|f\|_{L^2}^{\frac{2}{N}}, \]
for any $f \in C_0^\infty(\mathbb{R}^N)$. This inequality is of fundamental importance because it accounts in a very simple interpolative way how a control of the $L^2$-norm of a function, under a normalization condition, results in a lower bound of the $L^2$-norm of its gradient. It was introduced by Nash in 1958 to obtain regularity properties of the solutions to parabolic partial differential equations.

Inequality (1.1) generalizes to the context of Dunkl operators in the following form [14]
\[ \|f\|_{L^2,\kappa}^{1+\frac{2}{N+2\gamma}} \leq C \|\nabla_{\kappa} f\|_{L^2,\kappa} \|f\|_{L^2}^{\frac{2}{N+2\gamma}}, \quad f \in C_0^\infty(\mathbb{R}^N). \]
The number $N + 2\gamma$ is the homogeneous dimension and $\nabla_{\kappa}$ is the Dunkl gradient built from the Dunkl operators. The norms $\|\cdot\|_{k,q}$ are computed with respect to the weighted measure

\[ d\mu_\kappa(x) = \omega_\kappa(x) dx = \prod_{\alpha \in \mathcal{R}_+} |<\alpha, x>|^{2\kappa_\alpha} dx, \]

where $\mathcal{R}_+$ is a fixed positive root system and $\kappa$ is a nonnegative multiplicity function $\alpha \to \kappa_\alpha$ defined on $\mathcal{R}_+$ (see Sect. 2). The weight $\omega_\kappa$ is homogeneous of degree $2\gamma$. For $\kappa = 0$, Dunkl operators reduce to the usual partial derivatives and $d\mu_0(x)$ is the Lebesgue measure.

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In a recent paper [14] Velicu established (1.2) and used it to obtain an elementary proof of the Sobolev inequality in the case \( p = 2 \). Nash’s inequality can be seen as a weaker form of the Sobolev inequality, since it can be deduced from it using Hölder’s inequality, but in fact these two inequalities are equivalent to each other and equivalent to the ultracontractive bound on the Markov semigroup associated to the Dunkl Laplacian \( \Delta_\kappa \) (see [14]).

The aim of this article is to investigate scale-invariant local variants of Nash-Dunkl inequality (1.2). Our main result is the following family of Nash-Dunkl inequalities on balls.

**Theorem 1.1.** (Relative Nash-Dunkl inequality). Let \( B \subset \mathbb{R}^N \) be an Euclidean ball of radius \( r(B) > 0 \). Then for any \( p > 1 \) there exists a constant \( C > 0 \) independent of \( B \) such that for any function \( f \in C^\infty_0(B) \),

\[
\left( \int_B |f|^p d\mu_\kappa \right)^{1 + \frac{1}{p'}} \leq C \frac{r(B)^p}{v_\kappa(B)^{\frac{p}{N+2\gamma}}} \left[ \int \left( |\nabla_\kappa f|^p + \frac{|f|^p}{r(B)^p} \right) d\mu_\kappa \right] \|f\|_{L^{\frac{p}{p'}}(\kappa)}^{\frac{p}{p'}},
\]  

(1.4)

where \( p' \) denotes the Hölder conjugate exponent of \( p \) and where the volume \( v_\kappa(B) \) is computed with respect to the Dunkl measure (1.3).

For \( p = 1 \), the previous inequality loses its meaning. A substitute is given by the following weak Nash-type inequality.

**Theorem 1.2.** (Weak relative Nash-Dunkl inequality). Let \( B \subset \mathbb{R}^N \) be an Euclidean ball of radius \( r(B) > 0 \). Then, there exists a constant \( C > 0 \) independent of \( B \) such that for any function \( f \in C^\infty_0(B) \) and \( \lambda > 0 \),

\[
\lambda^{1 + \frac{1}{N+2\gamma}} \mu_\kappa \{ |f| \geq \lambda \} \leq C \frac{r(B)}{v_\kappa(B)^{1 + 2\gamma}} \left[ \int \left( |\nabla_\kappa f| + \frac{|f|}{r(B)} \right) d\mu_\kappa \right] \|f\|_{L^{1/N}(\kappa)}^{1/N}. \]

(1.5)

It is well known (see [5], [12]) that for \( p = 2 \) inequality (1.4) is equivalent to the following Sobolev-type inequality.

**Theorem 1.3.** (Relative Sobolev-Dunkl inequality). Let \( B \subset \mathbb{R}^N \) be an Euclidean ball of radius \( r(B) > 0 \). Then there exists a constant \( C > 0 \) independent of \( B \) such that for any function \( f \in C^\infty_0(B) \),

\[
\left( \int_B |f|^{2(N+2\gamma) \frac{N+2\gamma-2}{N+2\gamma}} d\mu_\kappa \right)^{\frac{N+2\gamma}{N+2\gamma-2}} \leq C \frac{r(B)^2}{v_\kappa(B)^{\frac{2}{N+2\gamma}}} \left[ \int \left( |\nabla_\kappa f|^2 + \frac{|f|^2}{r(B)^2} \right) d\mu_\kappa \right].
\]

(1.6)

We qualify these inequalities as relative to refer to the ball \( B \) where they are considered. The important point is their invariance by scaling and the fact that the constant \( C \) is independent of the ball \( B \).

Notice that letting \( p = 2 \) and \( r(B) \to \infty \) in (1.4) (resp, in (1.6) yields (1.2) (resp. the Dunkl-Sobolev inequality). This results follow from the fact that the Dunkl volume \( v_\kappa(B) \) satisfies the lower bound (see Sect. 2):

\[
v_\kappa(B) \geq cr(B)^{N+2\gamma}.
\]
Our method therefore offers an alternative approach to establishing Nash and Sobolev inequalities in the context of Dunkl operators allowing to generalize some results of [14].

The inequality of Nash (1.2) is easily demonstrated by an adaptation of the original approach of [9] thanks to the Dunkl transform. As Nash points out in [9], this inequality was in fact demonstrated, at his request, by E. Stein and the proof is based on the use of the Fourier transform.

The local variant (1.4) is more difficult to establish. One possible approach is to use pseudo-Poincaré inequalities. Such Inequalities were established by S. Adhikari, V. P. Anoop and S. Parui in [1] for \( p = 2 \) by Velicu in [14] for \( 1 \leq p \leq 2 \). The proofs developed in [1] and [14] are different in nature. The \( L^2 \) nature of the inequality established in [1] allows the Dunkl transform to be used and the Velicu result is based on the use of the carré-du-champ operator and semi-group techniques.

The main contribution of this paper is to note that the gradient heat kernel estimates recently established by Anker et al in [3] allow to remove the restrictive hypothesis \( 1 \leq p \leq 2 \) and to derive pseudo-Poincaré inequalities for all \( p \geq 1 \). Once we have these inequalities we are able to adapt the general approach developed in [4], [12], [13] and derive the relative Nash and Sobolev inequalities (1.4), (1.5) and (1.6). Finally, as an application of the \( L^2 \)-Sobolev inequality (1.6) we derive mean value inequalities for subsolutions of the heat equation using Moser’s iteration argument.

2. Background and preliminaries

In this section we recall some important properties of Dunkl operators and collect some preliminary assertions which are necessary in the proof of our main results. For more details see [2], [8], [6], [7] and [10] for an overview of Dunkl theory.

Throughout the remainder of this paper, \( C \) will denote a positive constant which can differ from one occurrence to another, even in the same formula and we will use \( A \approx B \) to say that the ratio \( A/B \) is bounded between two positive constants.

We consider \( \mathbb{R}^N \) with the Euclidean scalar product \( \langle \cdot, \cdot \rangle \) and its associated norm \( |x| = \sqrt{\langle x, x \rangle} \). For \( \alpha \in \mathbb{R}^N \setminus \{0\} \), the reflection \( \sigma_\alpha \) with respect to the hyperplan \( H_\alpha \) orthogonal to \( \alpha \) is given by

\[
\sigma_\alpha(x) = x - 2\frac{\langle x, \alpha \rangle}{|\alpha|^2} \alpha, \quad x \in \mathbb{R}^N.
\]

A finite set \( \mathcal{R} \subset \mathbb{R}^N \setminus \{0\} \) is called a reduced root subsystem if \( \mathcal{R} \cap \mathbb{R}_\alpha = \{ \pm \alpha \} \) and \( \sigma_\alpha \mathcal{R} = \mathcal{R} \) for all \( \alpha \in \mathcal{R} \). The finite group \( G \) generated by the reflections \( \sigma_\alpha, \alpha \in \mathcal{R} \) is called the Coxeter-Weyl group of \( \mathcal{R} \).

Then, we fix a \( G \)-invariant function \( \kappa : \mathcal{R} \to \mathbb{C} \) called the multiplicity function of the root system. We assume in this article that \( \kappa \) takes its values in \([0, +\infty[\) and that the root system is reduced and normalized so that \( |\alpha|^2 = 2 \), \( \alpha \in \mathcal{R} \).
The Dunkl operators $T_j$ ($j = 1, \ldots, N$), introduced in [6], are the following $\kappa$-deformations of the usual directional derivatives $\partial/\partial x_j$ by reflections

$$T_j f(x) = \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha \in \mathcal{R}^+} \kappa(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle x, \alpha \rangle},$$

where $\mathcal{R}^+$ is a positive subsystem of $\mathcal{R}$. The definition is of course independent of the choice of the positive subsystem since $\kappa$ is $G$-invariant. The Dunkl operators $T_j$ are skew-symmetric with respect to the $G$-invariant measure

$$d\mu_\kappa(x) = \omega_\kappa(x) dx = \prod_{\alpha \in \mathcal{R}^+} |\langle \alpha, x \rangle|^{2k_\alpha} dx.$$  

A fundamental property of these differential-difference operators is their commutativity, that is to say $T_k T_l = T_l T_k$. Closely related to them is the so-called intertwining operator $V_\kappa$ which is the unique linear isomorphism of $\bigoplus_{n \geq 0} P_n$ such that

$$V_\kappa(P_n) = P_n, \quad V_\kappa(1) = 1, \quad T_j V_\kappa = V_\kappa \partial_j, \quad \text{for} \quad j = 1, \ldots, N,$$

with $P_n$ the subspace of homogeneous polynomials of degree $n$ in $N$ variables. Even if the positivity of the intertwining operator has been established by M. Rösler, an explicit formula of $V_\kappa$ is not known in general. However, the operator $V_\kappa$ possesses the integral representation

$$V_\kappa f(x) = \int_{\mathbb{R}^N} f(y) d\mu_x(y),$$

where $\mu_x$ is a probability measure on $\mathbb{R}^N$ with support in the Euclidean ball of center 0 and radius $|x|$. The function $E(x, y) = V_\kappa^x[e^{i<x,y>}]$, where the superscript means that $V_\kappa$ is applied to the $x$ variable, plays an important role in the development of the Dunkl transform. In particular, the function

$$E(x, iy) = V_\kappa^x[e^{i<x,y>}], \quad x, y \in \mathbb{R}^N,$$

plays the role of $e^{i<x,y>}$ in the ordinary Fourier analysis. The Dunkl transform is defined in terms of it by

$$\mathcal{F}(f)(y) = c_\kappa \int_{\mathbb{R}^d} f(x) E(x, -iy) d\mu_\kappa(x) dx, \quad y \in \mathbb{R}^N.$$  

If $\kappa = 0$, then $V_\kappa = id$ and the Dunkl transform coincides with the usual Fourier transform. As in the classical case, the Dunkl transform defines a topological automorphism of $\mathcal{S}(\mathbb{R}^N)$ and extends to an isometry of $L^2(\mathbb{R}^N, d\mu_\kappa)$.

Let $\gamma = \sum_{\alpha \in \mathcal{R}^+} \kappa(\alpha)$. The number $N + 2\gamma$ is called the homogeneous dimension, because of the obvious scaling property

$$V_\kappa(ta, tr) = t^{N+2\gamma} V_\kappa(a, r), \quad t > 0,$$

where $V_\kappa(a, r) = \mu_\kappa(B_r(a))$, $B_r(a)$ being the Euclidean ball of radius $r$ and centered at $a \in \mathbb{R}^N$. 


We will also need to use the distance \( d(a,x) = \min_{\sigma \in G} |x - \sigma a| \) (the distance between the \( G \)-orbits \( O(a) \) and \( O(x) \)). Obviously, the corresponding balls
\[
B^G_r(a) = \{ x \in \mathbb{R}^N, \quad d(a,x) < r \} = O(B_r(a)), \quad a \in \mathbb{R}^N, \quad r > 0,
\]
satisfy
\[
V_\kappa(a,r) \leq \mu_\kappa \left( B^G_r(a) \right) \leq |G| V_\kappa(a,r). \tag{2.2}
\]

We will denote by \( \nabla_\kappa = (T_1, \ldots, T_N) \) the Dunkl gradient and \( \Delta_\kappa = \sum_{j=1}^N T_j^2 \) the Dunkl-Laplacian operator. The Dunkl-Laplacian acts on \( C^2 \)-functions as
\[
\Delta_\kappa f(x) = \Delta f(x) + 2 \sum_{\alpha \in \mathbb{R}^+} \kappa(\alpha) \left( <\nabla f(x), \alpha > f(x) - f(\alpha x) <x, \alpha>^2 \right),
\]
where \( \Delta \) is the classical Laplacian operator on \( \mathbb{R}^N \) and \( \nabla \) the associated gradient. The operator \( \Delta_\kappa \) is essentially self-adjoint on \( L^2(\mathbb{R}^N, d\mu_\kappa) \) and generates the heat semigroup \( T_t = e^{-t\Delta_\kappa}, \ (t > 0) \). Via the Dunkl transform, the heat semigroup is given by
\[
T_t f(x) = \mathcal{F}^{-1} \left( e^{-\frac{|\xi|^2}{4t}} \mathcal{F}(\xi) \right)(x).
\]
Alternately \([11]\)
\[
T_t f(x) = f * h_t(x) = \int_{\mathbb{R}^N} h_t(x,y) f(y) d\mu_\kappa(y), \quad t > 0, \quad x \in \mathbb{R}^N, \tag{2.3}
\]
where the heat kernel \( h_t(x,y) \) is given by a smooth positive radial convolution kernel. Notice that (2.3) defines a strongly continuous semigroup of linear contractions on \( L^p(\mathbb{R}^N, d\mu_\kappa) \), for every \( 1 \leq p < \infty \).

The heat kernel satisfies the following Gaussian upper bounds for the orbit distance \( d(x,y) \) (see \([3]\)):
\[
h_t(x,y) \leq \frac{C}{\max\{V_\kappa(x,\sqrt{t}), V_\kappa(y,\sqrt{t})\}} e^{-\frac{d(x,y)^2}{ct}}, \quad t > 0, \quad x,y \in \mathbb{R}^N. \tag{2.4}
\]

Another estimate which plays a fundamental role in our analysis is the Gaussian upper estimate of the spatial gradient of the heat kernel \([3]\):
\[
|\nabla_{\kappa,x} h_t(x,y)| \leq \frac{C}{\sqrt{t} \max\{V_\kappa(x,\sqrt{t}), V_\kappa(y,\sqrt{t})\}} e^{-\frac{d(x,y)^2}{ct}}, \quad t > 0, \quad x,y \in \mathbb{R}^N. \tag{2.5}
\]

Finally, the following volume estimates will be important in all subsequent proofs. The first two assertions are well known \([4, \text{Sect.} \ 3]\). We include detailed proofs for the reader’s convenience.

**Proposition 2.1.**

i) Let \( a \in \mathbb{R}^N \) and \( r > 0 \). Then
\[
V_\kappa(a,r) \approx r^N \prod_{\alpha \in \mathbb{R}^+} (|<\alpha,a>| + r)^{2\kappa(\alpha)}. \tag{2.6}
\]
There exists a constant $C \geq 1$ such that for every $a \in \mathbb{R}^N$ and for every $r \geq s > 0$,

$$C^{-1} \left( \frac{r}{s} \right)^N \leq \frac{V_\kappa(a, r)}{V_\kappa(a, s)} \leq C \left( \frac{r}{s} \right)^{N+2\gamma}. \tag{2.7}$$

There exists a constant $C \geq 1$ such that for every $a \in \mathbb{R}^N$, $0 < s \leq r$ and $y \in B_r(a),$

$$V_\kappa(a, r)s^{N+2\gamma} \leq CV_\kappa(y, s)r^{N+2\gamma}. \tag{2.8}$$

**Proof.** To prove (2.6) fix $a \in \mathbb{R}^N$ and $r > 0$, then using the change of variable $x = a + tu$, $0 < t < r$ and $u \in S_{N-1}$, we obtain that the volume $V_\kappa(a, r)$ is equal to

$$\int_0^r \int_{S_{N-1}} \prod_{\alpha \in \mathcal{R}_+} |<\alpha, a > + t < \alpha, u>|^{2\kappa(\alpha)} t^{N-1} dtd\sigma(u), \tag{2.9}$$

where $d\sigma$ is the induced Euclidean measure on the unit sphere $S_{N-1}$. Thus using the elementary estimate $|<\alpha, a > | + \sqrt{2}r \leq \sqrt{2}(|<\alpha, a > | + r)$, we obtain

$$V_\kappa(a, r) \leq Cr^N \prod_{\alpha \in \mathcal{R}_+} (|<\alpha, a > | + r)^{2\kappa(\alpha)}. \tag{2.10}$$

Let now establish a similar lower estimate for $V_\kappa(a, r)$. Using the fact that $\mathcal{R}$ is invariant with respect to the action of the Weyl group $G$, we obtain

$$V_\kappa(ga, r) = V_\kappa(a, r), \quad g \in G. \tag{2.11}$$

Property (2.11) combined with the scaling property (2.1) show that it suffices to estimate $V_\kappa(a, 1)$. Let $\mathcal{C} = \{x \in \mathbb{R}^N : <\alpha, x > > 0, \alpha \in \mathcal{R}_+\}$ denote the positive Weyl chamber associated to the root system $\mathcal{R}$. According to (2.11) we can suppose that $a \in \mathcal{C}$. So by (2.9) we obtain

$$V_\kappa(a, 1) \geq \int_0^1 \int_{\Sigma} \prod_{\alpha \in \mathcal{R}_+} (|<\alpha, a > + t < \alpha, u >|^{2\kappa(\alpha)} t^{N-1} dtd\sigma(u),$$

where $\Sigma \subset S_{N-1}$ is chosen such that $\gamma_0 = \min\{<\alpha, u >, \alpha \in \mathcal{R}_+, u \in \Sigma\} > 0$, so that

$$V_\kappa(a, 1) \geq \int_0^1 \int_{\Sigma} \prod_{\alpha \in \mathcal{R}_+} (|<\alpha, a > + t\gamma_0|^{2\kappa(\alpha)} t^{N-1} dtd\sigma(u)$$

$$\geq \frac{1}{C} \prod_{\alpha \in \mathcal{R}_+} (|<\alpha, a > + 1|^{2\kappa(\alpha)}). \tag{2.12}$$

Combining (2.10) and (2.12) we deduce (2.6). It follows in particular that for $0 < s \leq r$, we have

$$\frac{V_\kappa(a, r)}{V_\kappa(a, s)} \approx \left( \frac{r}{s} \right)^N \prod_{\alpha \in \mathcal{R}_+} \left( \frac{|<\alpha, a > | + r}{|<\alpha, a > | + s} \right)^{2\kappa(\alpha)}.$$
The claim ii) follows from the fact that for \( \lambda \geq 0, \lambda + s < \lambda + r < \frac{r}{s}(\lambda + s) \). Choosing \( r = 2s \) in (2.7) shows that the measure \( \mu \) satisfies the doubling property, i.e., there exists a positive constant \( C \) such that for every \( a \in \mathbb{R}^N \) and for every \( r > 0 \)

\[
V_\kappa(a, 2r) \leq CV_\kappa(a, r). \tag{2.13}
\]

Finally, let us prove (2.8). Using (2.6) we see that (2.8) is equivalent to

\[
s^{2\gamma} \prod_{\alpha \in \mathbb{R}^+} (|<\alpha, a> + r)^{2\kappa(\alpha)} \leq C_r^{2\gamma} \prod_{\alpha \in \mathbb{R}^+} (|<\alpha, y> + s)^{2\kappa(\alpha)}
\]

which is to show that

\[
\prod_{\alpha \in \mathbb{R}^+} \left( |<\alpha, a> + r|^{2\kappa(\alpha)} / |<\alpha, y> + s|^{2\kappa(\alpha)} \right) \leq C \left( \frac{T}{s} \right)^{2\gamma}.
\]

The assertion (2.8) follows from the fact that \( |<\alpha, a> + s| \leq \sqrt{2}r + |<\alpha, y> + s| \) and that \( r \geq s \). \( \square \)

## 3. Pseudo-Poincaré and Nash-type inequalities

In this section, we establish pseudo-Poincaré inequalities and we make a detailed study of Nash-type inequalities (1.4) and (1.5). For this purpose we also get operator norm estimates for the family \( \{T_t^2, 0 < t \leq r\} \) acting from \( L^1(B, d\mu_\kappa) \rightarrow L^p(R^N, d\mu_\kappa) \), for \( B \subset \mathbb{R}^N, p \geq 1 \).

**Proposition 3.1.** (Pseudo-Poincaré inequality). For any \( 1 \leq p < \infty \), there exists a constant \( C > 0 \) such that for all \( f \in C^\infty_0(\mathbb{R}^N), t > 0 \), we have

\[
\|f - T_tf\|_{p,\kappa} \leq C\sqrt{t}\|\nabla_\kappa f\|_p. \tag{3.1}
\]

For the proof of Proposition 3.1 we need the following lemma

**Lemma 3.1.** (Schur’s test) Assume that \( k \) is a measurable function on \( \mathbb{R}^N \) that satisfies the mixed-norm conditions:

\[
C_1 = \sup_{x \in \mathbb{R}^N} \int |k(x, y)|d\mu_\kappa(y) < \infty, \quad C_2 = \sup_{y \in \mathbb{R}^N} \int |k(x, y)|d\mu_\kappa(x) < \infty.
\]

Then the integral operator induced by the kernel \( k(x, y) \) (i.e. the operator defined by

\[
T_kf(x) = \int k(x, y)f(y)d\mu_\kappa(y)
\]

defines a bounded mapping of \( L^p(\mathbb{R}^N, d\mu_\kappa) \) into itself for every \( 1 \leq p \leq \infty \), with

\[
\|T_k\|_{L^p(\mathbb{R}^N, d\mu_\kappa) \rightarrow L^p(\mathbb{R}^d, d\mu_\kappa)} \leq C_1^{-\frac{1}{p}}C_2^{\frac{1}{p}}.
\]

**Proof.** We note that

\[
T_t f - f = \int_0^t \frac{\partial}{\partial s} T_s f ds = \int_0^t \Delta_\kappa T_s f ds. \tag{3.2}
\]
Fix $p \geq 1$. To estimate the $L^p(d\mu_\kappa)$-norm of the difference (3.2), fix $g \in C^\infty_0(\mathbb{R}^N)$ satisfying $\|g\|_{p',\kappa} = 1$, where $p'$ denotes the Hölder conjugate exponent of $p$. Integrating (3.2) against $g$ and using the symmetry of the semi-group $(T_t)_{t \geq 0}$ yield

$$\int (T_t f - f) \, g d\mu_\kappa = \int (T_t g - g) \, f d\mu_\kappa = \int_0^t \langle \nabla_\kappa T_s g, \nabla_\kappa f \rangle \, d\mu_\kappa ds.$$ 

It follows that

$$\left| \int (T_t f - f) \, g d\mu_\kappa \right| \leq \int_0^t \|\nabla_\kappa T_s g\|_{p',\kappa} \|\nabla_\kappa f\|_{p,\kappa} \, ds.$$ 

Thanks to Gaussian estimates (2.5) we know that, for any $x \in \mathbb{R}^N$ and $s > 0$,

$$\|\nabla_\kappa h_s(x,\cdot)\|_{1,\kappa} \leq C_s \int_{\mathbb{R}^N} \frac{1}{V_\kappa(x,\sqrt{s})} \exp \left( -\frac{cd(x,y)^2}{s} \right) \, d\mu_\kappa(y).$$

A dyadic decomposition on the annulus

$$B^G_{2^j+1}\sqrt{s}(a) \setminus B^G_{2^j}\sqrt{s}(a) = \{ 2^j \sqrt{s} \leq d(x,y) \leq 2^{j+1} \sqrt{s} \}, j \in \mathbb{N},$$

shows that

$$\|\nabla_\kappa h_s(x,\cdot)\|_{1,\kappa} \leq C s^{-1/2} \left[ 1 + \sum_{j=0}^{\infty} \frac{V_\kappa(x,2^{j+1}\sqrt{s})}{V_\kappa(x,2^j\sqrt{s})} e^{c2^j} \right] \leq C s^{-1/2},$$

because of (2.2) and the doubling volume property (2.13). Using Schur’s test and taking the supremum over all functions $g$ satisfying $\|g\|_{p',\kappa} = 1$ give (3.1).

Let us now estimate the operator norm of each of the elements of the family of operators \( \{T_t^2, \ 0 < t \leq r \} \) acting from $L^1 \left( B_r(a), d\mu_\kappa \right) \to L^p \left( \mathbb{R}^N, d\mu_\kappa \right)$, for $a \in \mathbb{R}^N$, $r > 0$ and $p \geq 1$ fixed.

**Proposition 3.2.** For any $1 \leq p < \infty$, there exists a constant $c > 0$ such that for all $a \in \mathbb{R}^N$, $r > 0$, we have

$$\|T_t^2\|_{L^1(B_r(a),d\mu_\kappa) \to L^p(\mathbb{R}^N,d\mu_\kappa)} \leq \frac{C}{\sqrt[3]{V_\kappa(a,r)^{N+2\gamma}}} \left( \frac{r}{t} \right)^{N+2\gamma}, \quad 0 < t \leq r,$$  

(3.3)

where $p'$ denotes the Hölder conjugate exponent of $p$.

**Proof.** An obvious interpolation argument shows that

$$\|T_t^2\|_{L^1(B_r(a),d\mu_\kappa) \to L^p(\mathbb{R}^N,d\mu_\kappa)} \leq \left( \sup \left\{ h_{t^2}(x,y), \ x \in \mathbb{R}^N, y \in B_r(a) \right\} \right)^{1/p'}.$$  

(3.4)

Let $x \in \mathbb{R}^N$, $y \in B_r(a)$ and $0 < t \leq r$. Thanks to (2.4) and (2.8)

$$h_{t^2}(x,y) \leq \frac{C}{V_\kappa(y,t)} \leq \frac{C}{V_\kappa(a,r)} \left( \frac{r}{t} \right)^{N+2\gamma}.$$  

Hence it follows that:

$$\sup \left\{ h_{t^2}(x,y), \ x \in \mathbb{R}^N, y \in B_r(a) \right\} \leq \frac{C}{V_\kappa(a,r)} \left( \frac{r}{t} \right)^{N+2\gamma}.$$  

(3.5)

The proof of (3.3) is complete, by applying (3.4).
Now, we will show how pseudo-Poincaré inequality (3.1) and the estimation (3.3) lead to Nash-type inequalities (1.4) and (1.5).

**Proof of Theorem 1.1.** Fix $1 < p < \infty$, $a \in \mathbb{R}^N$, $r > 0$ and $f \in C_0^\infty(\mathbb{R}^N)$. Let $0 < t \leq r$. Write
\[
\|f\|_{p,\kappa} \leq \|f - T_{t^2}f\|_{p,\kappa} + \|T_{t^2}f\|_{p,\kappa}.
\]
Using (3.1) and (3.3) we obtain
\[
\|f\|_{p,\kappa} \leq Ct\|\nabla f\|_{p,\kappa} + C\left(\frac{r^{N+2\gamma}}{V_\kappa(a, r)}\right)^\frac{1}{p'} t^{-\frac{N+2\gamma}{p'}}\|f\|_{1,\kappa},
\]
where $p'$ denotes the Hölder conjugate exponent of $p$. Combining with the obvious estimate $\|f\|_{p,\kappa} \leq t\|f\|_{p,\kappa}/r$ which is valid for any $t > r$, we deduce that for any $t > 0$
\[
\|f\|_{p,\kappa} \leq Ct\left[\|\nabla f\|_{p,\kappa} + \frac{1}{r}\|f\|_{p,\kappa}\right] + C\left(\frac{r^{N+2\gamma}}{V_\kappa(a, r)}\right)^\frac{1}{p'} t^{-\frac{N+2\gamma}{p'}}\|f\|_{1,\kappa}.
\]
Optimizing over $t > 0$ yields
\[
\|f\|_{p,\kappa}^{1 + \frac{p'}{N+2\gamma}} \leq \frac{Ct}{(V_\kappa(a, r))^{\frac{1}{N+2\gamma}}} \left[\|\nabla f\|_{p,\kappa} + \frac{1}{r}\|f\|_{p,\kappa}\right] \|f\|_{1,\kappa}^{\frac{p'}{N+2\gamma}}.
\]

**Proof of Theorem 1.2.** Fix $C_0^\infty(\mathbb{R}^N)$ and a real $\lambda > 0$. For any $0 < t \leq r$, write
\[
\mu_\kappa\{|f| \geq \lambda\} \leq \mu_\kappa\{|f - T_{t^2}f| \geq \lambda/2\} + \mu_\kappa\{T_{t^2}|f| \geq \lambda/2\}.
\]
Assume that $\lambda \geq 4C\|f\|_{1,\kappa}V_\kappa(a, r)^{-1}$ (where $C$ denotes the constant that appears in (3.5) and pick $t \leq r$ so that
\[
\lambda = 4Ct^{-N+2\gamma}\|f\|_{1,\kappa}V_\kappa(a, r)^{-1}r^{N+2\gamma}.
\]
Then it follows from (3.5) that $\|T_{t^2}|f|\|_\infty$ is dominated by $\lambda/4$. Thus
\[
\mu_\kappa\{|f| \geq \lambda\} \leq \mu_\kappa\{|f - T_{t^2}f| \geq \lambda/2\} \leq \frac{2}{\lambda}\|f - T_{t^2}f\|_{1,\kappa,\lambda} \leq 2Ct\|\nabla f\|_{1,\kappa}/\lambda,
\]
where the last inequality is obtained by applying (3.1). It follows that
\[
\mu_\kappa\{|f| \geq \lambda\} \leq CrV_\kappa(a, r)^{-\frac{1}{N+2\gamma}}\|f\|_{1,\kappa}^{\frac{1}{N+2\gamma}}\|\nabla f\|_{1,\kappa,\lambda}^{-1}\|\nabla f\|_{1,\kappa,\lambda}^{-\frac{1}{N+2\gamma}}. \tag{3.6}
\]
On the other hand if $\lambda < 4C\|f\|_{1,\kappa}V_\kappa(a, r)^{-1}$, we simply write
\[
\mu_\kappa\{|f| \geq \lambda\} \leq \|f\|_{1,\kappa,\lambda}^{-1},
\]
which implies
\[
\mu_\kappa\{|f| \geq \lambda\} \leq CV_\kappa(a, r)^{-\frac{1}{N+2\gamma}}\|f\|_{1,\kappa}^{\frac{1}{N+2\gamma}}\lambda^{-1}\|\nabla f\|_{1,\kappa,\lambda}^{-\frac{1}{N+2\gamma}}.
\]
Combining with (3.6) we deduce the weak Nash inequality
\[
\lambda^{1 + \frac{1}{N+2\gamma}}\mu_\kappa\{|f| \geq \lambda\} \leq \frac{Cr}{V_\kappa(a, r)^{\frac{1}{N+2\gamma}}}\left(\|\nabla f\|_{1,\kappa} + \frac{1}{r}\|f\|_{1,\kappa}\right)\|f\|_{1,\kappa}^{\frac{1}{N+2\gamma}}, \quad \lambda > 0.
\]
Remark. It is easy to see that the considerations of this section immediately generalize to balls $B_r^G(a)$. In particular they lead to the following Sobolev inequality which will be crucial for the applications of the following section.

**Theorem 3.1.** Let $a \in \mathbb{R}^N$ and $r > 0$. Then there exists a constant $C > 0$ independent of $a$ and $r$ such that for all $f \in C_0^\infty(B_r^G(a))$:

$$
\left( \int_{B_r^G(a)} |f|^{\frac{2(N+2\gamma)}{N+2\gamma-2}} d\mu_\kappa \right)^{\frac{N+2\gamma-2}{N+2\gamma}} \leq \frac{Cr^2}{(V_\kappa(a,r))^{\frac{2}{N+2\gamma}}} \int_{B_r^G(a)} \left( |\nabla_\kappa(f)|^2 + \frac{|f|^2}{r^2} \right) d\mu_\kappa. \quad (3.7)
$$

4. **Mean value inequalities**

In this section we shall derive $L^p$-mean value inequalities using Moser’s iterative technique. These inequalities concern subsolutions of the heat equation on orbits of balls not necessarily centered on the origin and are only based on the Sobolev inequality stated in $(3.7)$.

Let us fix some notations. For $a \in \mathbb{R}^N$, $r > 0$, $s \in \mathbb{R}$ and $0 < \delta < 1$, set

$$Q = [s, s + r^2][B_r^G(a)]$$
$$Q_\delta = [s + \delta r^2, s + r^2][B_{(1-\delta)r}(a)]$$

and for a function $u : Q \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{N}$, and $p \geq 1$, set

$$\|u\|_{p,Q}^p = \int_s^{s+r^2} \int_{B_r^G(a)} |u(x,t)|^p d\mu_\kappa(x) dt.$$

Let us prove some auxiliary results.

**Lemma 4.1.** Let $u$ be a non-negative parabolic subsolution in $Q$, i.e., $u$ satisfies

$$\left( \frac{\partial}{\partial t} - \Delta_\kappa \right) u \leq 0$$

in $Q$. Then for all $p \geq 2$, $(x,t) \in Q \to u^p(x,t)$ is also a non-negative subsolution.

**Proof.** One has
\[ \frac{\partial}{\partial t} u^p - \Delta \kappa u^p \]
\[ = p u^{p-1} \frac{\partial}{\partial t} u - \Delta u^p - 2 \sum_{\alpha \in \mathbb{R}_+} \kappa(\alpha) \left[ p u^{p-1} \frac{\nabla u(x)}{x}, \alpha > - \frac{u^p(x) - u^p(\sigma \alpha x)}{x, \alpha^2} \right] \]
\[ = p u^{p-1} \frac{\partial}{\partial t} u - pu^{p-1} \Delta u^p - p(p - 1)u^{p-2}|\nabla u|^2 \]
\[ - 2 \sum_{\alpha \in \mathbb{R}_+} \kappa(\alpha) \left[ p u^{p-1} \frac{\nabla u(x)}{x}, \alpha > - \frac{u^p(x) - u^p(\sigma \alpha x)}{x, \alpha^2} \right] \]
\[ = p u^{p-1} \left( \frac{\partial}{\partial t} - \Delta \kappa \right) u - p(p - 1)|\nabla u|^2 \]
\[ - 2 \sum_{\alpha \in \mathbb{R}_+} \frac{\kappa(\alpha)}{x, \alpha^2} \left[ p u^{p-1}(x) \left( u(x) - u(\sigma \alpha x) - u^p(x) - u^p(\sigma \alpha x) \right) \right]. \]

Using the fact that \( u^p(\sigma \alpha x) \geq u^p(x) + pu^{p-1}(x) \left( u(\sigma \alpha x) - u(x) \right) \) (\( p \) is greater than 1), we deduce
\[ \left( \frac{\partial}{\partial t} - \Delta \kappa \right) u^p(x, t) \leq 0, \quad (x, t) \in Q. \]

**Proposition 4.1.** Let \( 0 < \delta < 1 \) and let \( Q, Q_{\delta} \) and \( u \) be as above. Then there exists a positive constant \( C' \), such that for any \( 0 < \lambda < \eta \) and \( p \geq 2 \)
\[ \int_{Q_\eta} u^{2\theta} d\mu_\kappa dt \leq \frac{C' r^{2(1-\theta)}}{\tau^{2(1-\theta)} (V_\kappa(a,r))^{\frac{2}{N+2\gamma}}} \left[ \int_{Q_\lambda} u^{2p} d\mu_\kappa dt \right]^\theta. \] (4.1)

where \( \theta = 1 + \frac{2}{N + 2\gamma} \) and \( \tau = \eta - \lambda \).

**Proof.** We observe first that for any non-negative function \( \phi \in C_0^\infty(B_r^G(a)) \), we have
\[ \int \left( \phi \frac{\partial}{\partial t} u + \nabla \kappa \phi, \nabla u \right) d\mu_\kappa = \int \phi \left( \frac{\partial}{\partial t} - \Delta \kappa \right) u d\mu_\kappa \leq 0. \] (4.2)

Set
\[ \Gamma_\kappa(\phi, u) = \left< \nabla \phi, \nabla u \right> + \sum_{\alpha \in \mathbb{R}_+} \kappa(\alpha) \frac{(\phi(x) - \phi(\sigma \alpha x))(u(x) - u(\sigma \alpha x))}{x, \alpha^2}. \]

With \( \phi = \psi^2 u \), we obtain
\[ \Gamma_\kappa(\psi^2 u, u) = \left[ 2\psi \nabla \psi u + \psi^2 \nabla u \right] \nabla u \]
\[ + \sum_{\alpha \in \mathbb{R}_+} \kappa(\alpha) \frac{((\psi^2 u)(x) - (\psi^2 u)(\sigma \alpha x))(u(x) - u(\sigma \alpha x))}{x, \alpha^2}. \]
Since

\[ [2\psi \nabla u + \psi^2 \nabla u] \nabla u = 2\psi \nabla \psi \nabla u + \psi^2 (\nabla u)^2 \]

\[ = (u \nabla \psi + \psi^2 \nabla u)^2 - u^2 (\nabla \psi)^2 \]

\[ = (\nabla (\psi u))^2 - u^2 (x)(\nabla \psi)^2 (x). \tag{4.3} \]

Otherwise \(((\psi^2 u)(x) - (\psi^2 u)(\sigma_\alpha x))(u(x) - u(\sigma_\alpha x))\)

\[ = [\psi^2 (x)u^2 (x) + \psi^2 (\sigma_\alpha x)u^2 (\sigma_\alpha x)] \]

\[ - [\psi^2 (x)u(x)u(\sigma_\alpha x) - \psi^2 (\sigma_\alpha x)u(x)u(\sigma_\alpha x)] \]

\[ = (\psi u(x) - \psi u(\sigma_\alpha x))^2 - u(x)u(\sigma_\alpha x)(\psi(x) - \psi(\sigma_\alpha x))^2. \tag{4.4} \]

(4.3) and (4.4) lead to

\[ \Gamma_\kappa (\psi^2 u, u) = \Gamma_\kappa (\psi u) - u^2 (x)(\nabla \psi)^2 (x) - u(x) \sum_{\alpha \in \mathcal{R}_+} \kappa (\alpha) u(\sigma_\alpha x) \frac{(\psi(x) - \psi(\sigma_\alpha x))^2}{\psi(x) - \psi(\sigma_\alpha x)} < \alpha, x >^2, \]

where we use the notation \(\Gamma_\kappa (v) = \Gamma_\kappa (v, v)\). Applying (4.2) to \(\psi^2 u\), where \(\psi \in C_0^\infty (B_r^c (a))\), it follows from the previous computation that

\[ \int_{B_r^c (a)} \left( \psi^2 u \frac{\partial u}{\partial t} + \Gamma_\kappa (\psi u) \right) d\mu_\kappa \]

\[ \leq \int_{B_r^c (a)} u^2 (x)(\nabla \psi)^2 (x) d\mu_\kappa (x) \]

\[ + \int_{B_r^c (a)} u(x) \sum_{\alpha \in \mathcal{R}_+} \kappa (\alpha) u(\sigma_\alpha x) \frac{(\psi(x) - \psi(\sigma_\alpha x))^2}{\psi(x) - \psi(\sigma_\alpha x)} < \alpha, x >^2 d\mu_\kappa (x). \]

Assuming \(\psi\) invariant under the action of \(G\) we deduce then that

\[ \int_{B_r^c (a)} \left( \psi^2 u \frac{\partial u}{\partial t} + \Gamma_\kappa (\psi u) \right) d\mu_\kappa \leq \| \nabla \psi \|_\infty^2 \int_{\supp \psi} u^2 d\mu_\kappa. \tag{4.5} \]

Let \(\chi\) denotes a non-negative smooth function of the time variable. We have

\[ \frac{\partial}{\partial t} \int_{B_r^c (a)} (\chi \psi u)^2 d\mu_\kappa = 2 \int_{B_r^c (a)} \left( \frac{d\chi}{dt} \psi^2 u^2 + \frac{\partial u}{\partial t} u^2 \psi^2 \chi \right) d\mu_\kappa \]

\[ \leq 2\chi \| \chi' \|_\infty \int_{B_r^c (a)} \psi^2 u^2 d\mu_\kappa + 2\chi^2 \int_{B_r^c (a)} \psi^2 u^2 \frac{\partial u}{\partial t} d\mu_\kappa. \tag{4.6} \]

Combining (4.5) and (4.6) we deduce that

\[ \frac{\partial}{\partial t} \int_{B_r^c (a)} (\chi \psi u)^2 d\mu_\kappa + \chi^2 \int_{B_r^c (a)} \Gamma_\kappa (\psi u) d\mu_\kappa \leq 2\chi \| \chi' \|_\infty \| \psi \|_\infty^2 \int_{\supp \psi} u^2 d\mu_\kappa \]

\[ + 2\chi^2 \| \nabla \psi \|_\infty^2 \int_{\supp \psi} u^2 d\mu_\kappa. \]
Hence
\[ \frac{\partial}{\partial t} \int_{B^G(a)} (\chi \psi u)^2 \, d\mu_\kappa + \chi^2 \int_{B^G(a)} \Gamma_\kappa(\psi u) \, d\mu_\kappa \leq 2\chi \, \text{supp} \psi \left[ \chi \|\nabla \psi\|_\infty^2 + \|\chi'\|_\infty \|\psi\|_\infty^2 \right] \int_{\text{support } \psi} u^2 \, d\mu_\kappa. \] (4.7)

Let \( \psi \) satisfying
\[
\begin{cases}
0 \leq \psi \leq |G| \\
\text{supp } \psi \subset B^G_{(1-\lambda)r}(a) \\
\psi = |G| \quad \text{on } B^G_{(1-\eta)r}(a) \\
|\nabla \psi| \leq \frac{|G|}{\tau r}.
\end{cases}
\]

To construct such \( \psi \) it suffices to choose a function \( \psi_a \) such that
\[
\begin{cases}
0 \leq \psi_a \leq 1 \\
\text{supp} \psi_a \subset B_{(1-\lambda)r}(a) \\
\psi_a = 1, \quad \text{on } B_{(1-\eta)r}(a) \\
|\nabla \psi_a| \leq (\tau r)^{-1}
\end{cases}
\]

and choose
\[
\psi(x) = \sum_{\sigma \in G} \psi_a(\sigma x), \quad x \in \mathbb{R}^N.
\]

Fix \( s \in \mathbb{R} \) and \( \chi \) such that
\[
\begin{cases}
0 \leq \chi \leq 1 \\
\chi = 0 \quad \text{on } ]-\infty, s + \lambda r^2[ \\
\chi = 1 \quad \text{on } ]s + \eta r^2, +\infty[ \\
|\chi'| \leq \frac{1}{\tau r^2}
\end{cases}
\]

Integrating (4.7) over \( [s, t] \) with \( t \in ]s + \lambda r^2, s + \tau r^2[ \), we obtain
\[
\sup_{s + \eta r^2 < t < s + \tau r^2} \left\{ \int_{B^G_{(1-\eta)r}(a)} u^2 \, d\mu_\kappa \right\} + \int_{s + \eta r^2}^{s + \tau r^2} \int_{B^G_{(1-\eta)r}(a)} \Gamma_\kappa(u) \, d\mu_\kappa \, dt \leq \frac{4|G|}{\tau^2 r^2} \int_{s + \lambda r^2}^{s + \tau r^2} \int_{B_{(1-\lambda)r}G(a)} u^2 \, d\mu_\kappa \, dt.
\] (4.8)

Thanks to Hölder’s inequality, we have
\[
\int |f|^{2\theta} \, d\mu_\kappa \leq \left( \int |f|^{\frac{2(N+2)}{N+2\gamma}} \, d\mu_\kappa \right)^{\frac{N+2\gamma-2}{N+2\gamma}} \left( \int |f|^2 \, d\mu_\kappa \right)^{\frac{2}{N+2\gamma}}.
\]

Combining with Sobolev’s inequality (3.7) gives
\[
\int |f|^{2\theta} \, d\mu_\kappa \leq \frac{C r^2}{(V_\kappa(a, r))^{\frac{N+2\gamma}{N+2\gamma}}} \left( \int \left( \Gamma_\kappa(f) + \frac{|f|^2}{r^2} \right) \, d\mu_\kappa \right)^{\frac{2}{N+2\gamma}}\]
for all \( f \in C_0^\infty(B_r^G(a)) \). The above inequality gives for a subsolution \( u \)

\[
\int_{Q_\eta} u^{2\theta} d\mu_\kappa dt \leq \frac{C(1-\eta)^2 r^2}{(V_\kappa(a, (1-\eta)r))^{\frac{2}{N+2\gamma}}} \left[ \int_{Q_\eta} \left( \Gamma_\kappa(u) + \frac{u^2}{r^2} \right) d\mu_\kappa dt \right] \times \sup_{s+\eta^2 < t < s+r^2} \left( \int_{B_G(a, (1-\eta)r)} u^2 d\mu_\kappa \right)^\frac{2}{N+2\gamma}.
\]  

(4.9)

Combining (4.8) with (4.9) we deduce

\[
\int_{Q_\eta} u^{2\theta} d\mu_\kappa dt \leq \frac{C(1-\eta)^2 r^2}{(V_\kappa(a, (1-\eta)r))^{\frac{2}{N+2\gamma}}} \left[ \int_{Q_\lambda} u^2 d\mu_\kappa dt \right]^{\theta}.
\]  

(4.10)

Using Lemma 4.1 and applying (4.10) to \( u \) completes the proof of Proposition 4.1.

\[\Box\]

Our next step is to prove the following \( L^p \) mean value inequality.

**Theorem 4.1.** Let \( 0 < \delta < 1 \) and let \( Q, Q_\delta \) and \( u \) be as above. Then there exists a constant \( C > 0 \) such that for \( p \geq 2 \) and any non-negative subsolution in \( Q \),

\[
\sup_{Q_\delta} u \leq C \left( \frac{\delta-(N+2\gamma+2)}{r^2 V_\kappa(a, r)} \right)^\frac{1}{p} \| u \|_{p, Q}.
\]

**Proof.** We resume the notation of the proof of Proposition 4.1. Set for \( i \in \mathbb{N} \)

\[
\lambda_0 = 0, \quad \lambda_i = \delta \sum_{j=1}^{i} 2^{-j}, i \geq 1.
\]

Applying Proposition 4.1 with \( p = p_i = p\theta^i \), \( \lambda = \lambda_i \), \( \eta = \lambda_{i+1} \), then \( \tau_i = \lambda_{i+1} - \lambda_i = \delta 2^{-1-i} \),

\[
\int_{Q_{\lambda_{i+1}}} u^{2\theta^{i+1}} d\mu_\kappa dt \leq \frac{r^2}{(V_\kappa(a, r))^{\frac{2}{N+2\gamma}}} \left[ \frac{(4|G|^2 + 1)4^{i+1}}{r^2 \delta^2} \int_{Q_{\lambda_i}} u^{2\theta^i} d\mu_\kappa dt \right]^{\theta}.
\]

Hence

\[
\left[ \int_{Q_{\lambda_{i+1}}} u^{2\theta^{i+1}} d\mu_\kappa dt \right]^{\frac{1}{p_i+1}} \leq \left( \frac{C_2(i)^2}{(V_\kappa(a, r))^{\frac{2}{N+2\gamma}}} \right)^\frac{1}{p_i+1} \left( \frac{(4|G|^2 + 1)4^{i+1}}{r^2 \delta^2} \right)^\frac{1}{p_i} \left[ \int_{Q_{\lambda_i}} u^{2\theta^i} d\mu_\kappa dt \right]^{\frac{1}{p_i}} \leq \left( 4C_2(i) \frac{r^2}{(V_\kappa(a, r))^{\frac{2}{N+2\gamma}}} \right)^{C_1(i+1)-1} (C r \delta)^{-2C_1(i)} \int_{Q} u^2 d\mu_\kappa dt \right]^{\frac{1}{p}}.
\]

(4.11)
where
\[ C_1(i) = \sum_{j=0}^i \theta^{-j}; \quad C_2(i) = \sum_{j=0}^i (j+1)\theta^{-j}. \]

Observe that \( \lambda_i \to \delta \) as \( i \to +\infty \)
\[ \sum_{j=1}^\infty \theta^{-j} = \frac{N + 2\gamma}{2}. \]

and
\[ \lim_{q \to +\infty} \|f\|_{q,\kappa} = \|f\|_{q,\infty}. \]

Thus, letting \( i \to +\infty \), we obtain
\[ \sup_{Q_\delta} u \leq C(N,\gamma) \left( \frac{\delta^{-(N+2\gamma+2)}}{r^2V_\kappa(a,r)} \right)^{\frac{1}{p}} \|u\|_{p,Q}, \]
where \( C(N,\gamma) = \left[ 4^{(N+2\gamma+2)^2} C^{N+2\gamma+2} \right]^{\frac{1}{p}}. \) This completes the proof of Theorem 4.1.

Finally we extend Theorem 4.1 to \( 0 < p < 2 \).

**Corollary 4.1.** Fix \( 0 < p < 2 \) and let \( \delta \in ]0,1[. \) Then, any non-negative \( u \) such that
\[ \left( \frac{\partial}{\partial t} - \Delta_\kappa \right) u \leq 0 \quad \text{in } Q = ]s,s + r^2[ \times B_r^G(a); \]
a \( a \in \mathbb{R}^N, r > 0, \) satisfies
\[ \sup_{Q_\delta} u \leq C\delta^{-(N+2\gamma+2)} \left( r^2V_\kappa(a,r) \right)^{-\frac{1}{p}} \|u\|_{p,Q} \]
where the constant \( C > 0 \) is independent of \( u, \delta, a, r \) and \( s \).

**Proof.** Let \( 0 < p < 2 \). Fix \( 0 < \rho < \frac{1}{2} \) and set \( \tau = \frac{\rho}{4} \). Theorem 4.1 yields
\[ \sup_{Q_\rho} u \leq C\tau^{-(N+2\gamma+2)} \left( r^2V_\kappa(a,r) \right)^{-\frac{1}{p}} \|u\|_{2,Q_{\tau}}. \]

Now, as \( \|u\|_{2,Q_{\tau}} \leq \|u\|_{p,Q}^{\frac{p}{2}} \left( \sup_{Q_{\tau}} u \right)^{\frac{1-p}{2}}, \) we get
\[ \sup_{Q_\rho} u \leq C\tau^{-(N+2\gamma+2)} \left( r^2V_\kappa(a,r) \right)^{-\frac{1}{p}} \|u\|_{p,Q}^{\frac{p}{2}} \left( \sup_{Q_{\tau}} u \right)^{1-\frac{p}{2}}. \] (4.11)
Let $0 < \delta < \frac{1}{2}$, $\rho_0 = \delta$ and set for $i = 0, 1 \ldots \tau_{i+1} = \frac{\rho_i}{4}$ and $\rho_{i+1} = \rho_i - \tau_{i+1}$. Applying (4.11) for each $i$ yields

$$\sup_{Q_{\rho_{i-1}}} u \leq C \frac{\delta^{N+2\gamma+2}}{2} \left( r^2 V_\kappa(a,r) \right)^{-\frac{\delta}{2}} \|u\|_{p,Q}^{\frac{p}{2} \left( \sup_{Q_{\rho_i}} u \right)^{1-\frac{\delta}{2}}}.$$ 

Integrating gives

$$\sup_{Q_\delta} u \leq C_i \left( \delta^{N+2\gamma+2} \left( r^2 V_\kappa(a,r) \right)^{-\frac{\delta}{2}} \|u\|_{p,Q}^{\frac{p}{2} \left( \sup_{Q_\delta} u \right)^{1-\frac{\delta}{2}}} \right)^{\frac{1}{2} \sum_{j=0}^{i-1} \left(1 - \frac{p}{2}\right)^j (j+1)}.$$ 

where $C_i = C$. When $i$ tends to infinity, this yields

$$\sup_{Q_\delta} u \leq C \frac{N+2\gamma+2}{p^2} \left( \delta - \frac{N+2\gamma+2}{2} \left( r^2 V_\kappa(a,r) \right)^{-\frac{\delta}{2}} \right)^{\frac{p}{2}} \|u\|_{p,Q},$$

which implies the desired inequality. 

Conflict of interest

The authors declare that they have no conflict of interest.

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