On a general stochastic differential equation for SIS epidemic models

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Abstract

We propose a general method for studying existence and uniqueness of global strong solutions, as well as conditions for their extinction or persistence, for a large class of models, which includes the susceptible-infected-susceptible stochastic differential equation presented in [6]. Our method allows for a great flexibility in the choice of the coefficients of the equation, while preserving the essential features of several models from mathematical epidemiology. The approach presented here relies on two classical theorems of the theory of stochastic differential equations: the first one establishing that the solution of an SDE never visits the region where both drift and diffusion coefficients vanish simultaneously, and the second one a comparison result for solution of SDEs having the same diffusion coefficient. The techniques utilized allow for future extensions to multidimensional problems and equations with more irregular coefficients. Simulations of some representative examples are also presented.

Key words and phrases: SIS epidemic model, Brownian motion, stochastic differential equations, extinction, persistence.

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1 Introduction

The susceptible-infected-susceptible (SIS) model is a simple mathematical model that describes, under suitable assumptions, the spread of diseases with no permanent

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immunity (see e.g. [3], [8]). In such models an individual starts being susceptible to a disease, at some point of time gets infected and then recovers after some other time interval, becoming susceptible again. If $S(t)$ and $I(t)$ denote the number of susceptibles and infecteds at time $t$, respectively, then the differential equations describing the spread of the disease are

\[
\begin{align*}
\frac{dS(t)}{dt} &= \mu N - \beta S(t)I(t) + \gamma I(t) - \mu S(t), \quad S(0) = s_0 > 0; \\
\frac{dI(t)}{dt} &= \beta S(t)I(t) - (\mu + \gamma)I(t), \quad I(0) = i_0 > 0.
\end{align*}
\]

(1.1)

Here, $N := s_0 + i_0$ is the initial size of the population amongst whom the disease is spreading, $\mu$ denotes the per capita death rate, $\gamma$ is the rate at which infected individuals become cured and $\beta$ stands for the disease transmission coefficient. If we impose the size of the population to be constant in time, i.e.

\[
\frac{d}{dt}(S(t) + I(t)) = 0,
\]

that means,

\[
S(t) + I(t) = S(0) + I(0) = s_0 + i_0 = N, \quad \text{for all } t \geq 0,
\]

then system (1.1) reduces to the differential equation

\[
\frac{dI(t)}{dt} = \beta(N - I(t))I(t) - (\mu + \gamma)I(t), \quad I(0) = i_0 \in ]0, N[,
\]

(1.2)

with $S(t) := N - I(t)$, for $t \geq 0$. Equation (1.2) can be solved explicitly and one finds that the ratio

\[
R_0 := \frac{\beta N}{\mu + \gamma},
\]

known as reproduction number of the infection, determines whether the disease will become extinct, i.e. $I(t)$ will tend to zero as $t$ goes to infinity, or will be persistent, i.e. $I(t)$ will tend to a positive limit as $t$ increases.

Several stochastic generalizations of the deterministic model (1.1) have been proposed in the literature. One popular approach is to model the random behaviour of $I(t)$ via discrete/continuous time Markov chains/processes, whose transition probabilities mimic the logistic-type dynamic described by (1.2). For a detailed overview of these methods we refer the reader to Chapter 3 in [3], Chapter 5 in [1] and the references quoted there.

A different point of view for examining the effect of environmental stochasticity is offered by the perturbation of the parameters specifying the model (1.2). Namely, writing equation (1.2) in the differential form

\[
dI(t) = \beta(N - I(t))I(t)dt - (\mu + \gamma)I(t)dt, \quad I(0) = i_0 \in ]0, N[,
\]

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one replaces the deterministic increment $\beta dt$ with the stochastic increment $\beta dt + \sigma_1 dB(t)$, where $\sigma_1$ is a new positive parameter and $\{B_1(t)\}_{t \geq 0}$ stands for a standard one-dimensional Brownian motion. This perturbation transforms the deterministic differential equation (1.2) into the stochastic differential equation

$$dI(t) = [\beta(N - I(t))I(t) - (\mu + \gamma)I(t)]dt + \sigma_1(N - I(t))I(t)dB_1(t),$$

(1.3)

with $I(0) = i_0 \in ]0, N[$. This is the model proposed in [6]. In this paper the authors first prove the existence of a unique global strong solution for (1.3), which lives in the interval $]0, N[$ with probability one for all $t \geq 0$; then, they define a new reproduction number, which involves the additional parameter $\sigma_1$, and establish conditions for extinction and persistence of the infection and existence of a stationary distribution for the stochastic process $\{I(t)\}_{t \geq 0}$. A key role in the proofs is played by a suitable Lyapunov function (see e.g. Chapter 4 in [12]), which is constructed ad hoc for the equation under investigation (see also the paper [5], which utilizes an analogous scheme for proposing a stochastic model for AIDS). In the recent paper [4] the model (1.3) is further generalized: the authors transform the deterministic increment $(\mu + \gamma)dt$ into $(\mu + \gamma)dt + \sigma_2 \sqrt{N - I(t)}dB_2(t)$, where $\sigma_2$ is a new positive parameter and $\{B_2(t)\}_{t \geq 0}$ is a standard one-dimensional Brownian motion independent of $\{B_1(t)\}_{t \geq 0}$. This procedure now leads to the stochastic differential equation

$$dI(t) = [\beta(N - I(t))I(t) - (\mu + \gamma)I(t)]dt + \sigma_1(N - I(t))I(t)dB_1(t) - \sigma_2 \sqrt{N - I(t)}I(t)dB_2(t),$$

(1.4)

with $I(0) = i_0 \in ]0, N[$. Also for this equation the authors are able to find a Lyapunov function entailing the existence of a unique global strong solution, which lives in the interval $]0, N[$ with probability one for all $t \geq 0$.

Aim of the present paper is to propose a general method for studying existence and uniqueness of global strong solutions, as well as conditions for their extinction or persistence, for a large class of models, which includes (1.3) as a particular case. Namely, we consider stochastic differential equations of the form

$$\begin{cases} 
  dI(t) = [f(I(t)) - h(I(t))]dt + \sum_{i=1}^{m} g_i(I(t))dB_i(t), & t > 0; \\
  I(0) = i_0 \in ]0, N[, 
\end{cases}$$

(1.5)

where the coefficients satisfy only those fairly general assumptions needed to derive the desired properties (see Theorem 2.3 below for the detailed assumptions). Our method allows for a great flexibility in the choice of the model while preserving the essential features of (1.3). In particular, we allow the diffusion coefficients to vanish on arbitrary intervals, thus ruling out the techniques based on Feller’s test for explosions (see for instance Chapter 5 in [10]). Also the method based on the Lyapunov function, which is successfully applied in [6], [5] and [4], doesn’t seem to be appropriate for the great
generality considered here. Our approach relies instead on two classical theorems of the theory of stochastic differential equations, which we restate for the readers’ convenience at the beginning of the next section (see Theorem 2.1 and Theorem 2.2 below). These results have suitable extensions to multidimensional equations with even more irregular coefficients; we are therefore confident on the potential extension of our analysis to more complex models, like for instance those considered in [7] and [2].

The paper is organized as follows: in Section 2 we start proving the existence of a unique global strong solution for equation (1.5) and that such solution lives in the interval $[0, N]$, for all $t \geq 0$ with probability one; then, we give sufficient conditions for extinction and persistence of the solution in the sense of the paper [6]; several simulations of numerical examples are also presented.

2 Main results

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with an $m$-dimensional standard Brownian motion $\{(B_1(t), ..., B_m(t))\}_{t \geq 0}$ and denote by $\{\mathcal{F}^B_t\}_{t \geq 0}$ its augmented natural filtration. In the sequel we will be working with one dimensional Itô’s type stochastic differential equations driven by the $m$-dimensional Brownian motion $\{(B_1(t), ..., B_m(t))\}_{t \geq 0}$. Our method is essentially based on the use of two general theorems from the theory of stochastic differential equations; for the readers’ convenience we now state a version of these results adapted to the current context.

Theorem 2.1. Let $\{X(t)\}_{t \geq 0}$ be the unique global strong solution of the stochastic differential equation

$$
\begin{cases}
  dX(t) = \mu(X(t))dt + \sum_{i=1}^{m} \sigma_i(X(t))dB_i(t), & t > 0; \\
  X(0) = x_0 \in \mathbb{R},
\end{cases}
$$

where the coefficients $\mu, \sigma_1, ..., \sigma_m : \mathbb{R} \to \mathbb{R}$ are assumed to be globally Lipschitz continuous. If we set

$$
\Lambda := \{x \in \mathbb{R} : \mu(x) = \sigma_1(x) = \cdots = \sigma_m(x) = 0\}
$$

and assume $x_0 \notin \Lambda$, then

$$
\mathbb{P}(X(t) \notin \Lambda, \text{ for all } t \geq 0) = 1.
$$

Proof. See the theorem in [11].

Theorem 2.2. Let $\{X(t)\}_{t \geq 0}$ be the unique global strong solution of the stochastic differential equation

$$
\begin{cases}
  dX(t) = \mu_1(X(t))dt + \sum_{i=1}^{m} \sigma_i(X(t))dB_i(t), & t > 0; \\
  X(0) = z \in \mathbb{R},
\end{cases}
$$
and \(\{Y(t)\}_{t \geq 0}\) be the unique global strong solution of the stochastic differential equation

\[
\begin{aligned}
dY(t) &= \mu_2(Y(t))dt + \sum_{i=1}^{m_i} \sigma_i(Y(t))dB_i(t), \quad t > 0; \\
Y(0) &= z \in \mathbb{R},
\end{aligned}
\]

where the coefficients \(\mu_1, \mu_2, \sigma_1, ..., \sigma_m : \mathbb{R} \to \mathbb{R}\) are assumed to be globally Lipschitz continuous. If \(\mu_1(z) \leq \mu_2(z)\), for all \(z \in \mathbb{R}\), then

\[
P(X(t) \leq Y(t), \text{ for all } t \geq 0) = 1.
\]

**Proof.** See Proposition 2.18, Chapter 5 in [10], where the proof is given for \(m = 1\). The extension to several Brownian motions is immediate. See also Theorem 1.1, Chapter VI in [9]. \(\square\)

### 2.1 Existence, uniqueness and support

We are now ready to state and prove the first main result of our paper.

**Theorem 2.3.** For \(i \in \{1, ..., m\}\), let \(f, g_i, h : \mathbb{R} \to \mathbb{R}\) be locally Lipschitz-continuous functions such that

1. \(f(0) = g_i(0) = 0\) and \(f(N) = g_i(N) = 0\), for some \(N > 0\);
2. \(h(0) = 0\) and \(h(x) > 0\), when \(x > 0\).

Then, the stochastic differential equation

\[
\begin{aligned}
dI(t) &= [f(I(t)) - h(I(t))]dt + \sum_{i=1}^{m_i} g_i(I(t))dB_i(t), \quad t > 0; \\
I(0) &= i_0 \in [0, N],
\end{aligned}
\]

admits a unique global strong solution, which satisfies \(P(0 < I(t) < N) = 1\), for all \(t \geq 0\).

**Proof.** The local Lipschitz-continuity of the coefficients entails pathwise uniqueness for equation (2.1), see for instance Theorem 2.5, Chapter 5 in [10]. Now, we consider the modified equation

\[
\begin{aligned}
d\tilde{I}(t) &= [\tilde{f}(\tilde{I}(t)) - \tilde{h}(\tilde{I}(t))]dt + \sum_{i=1}^{m_i} \tilde{g}_i(\tilde{I}(t))dB_i(t), \quad t > 0; \\
\tilde{I}(0) &= i_0 \in [0, N],
\end{aligned}
\]

where

\[
\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [0, N]; \\ 0, & \text{if } x \notin [0, N]\end{cases}\quad \text{and} \quad \tilde{g}_i(x) = \begin{cases} g_i(x), & \text{if } x \in [0, N]; \\ 0, & \text{if } x \notin [0, N]\end{cases}
\]
while
\[
\hat{h}(x) = \begin{cases} 
0, & \text{if } x < 0; \\
h(x), & \text{if } x \in [0, N]; \\
h(N), & \text{if } x > N.
\end{cases}
\]

The coefficients of equation (2.2) are bounded and globally Lipschitz-continuous; this implies the existence of a unique global strong solution \(\{I(t)\}_{t \geq 0}\) for (2.2). Moreover, the drift and diffusion coefficients vanish at \(x = 0\). Therefore, according to Theorem 2.1, the solution never visits the origin, unless it starts from there. Since \(I(0) = i_0 \in ]0, N[\), we deduce that \(I(t) > 0\), for all \(t \geq 0\), almost surely. Recalling the assumption \(h(x) > 0\) for \(x > 0\), we can rewrite equation (2.2) as
\[
\begin{align*}
\frac{dI(t)}{dt} &= [\bar{f}(I(t)) - \hat{h}(I(t)) + \sum_{i=1}^{m} \bar{g}_i(I(t))dB_i(t), \quad t > 0; \\
I(0) &= i_0 \in ]0, N[,
\end{align*}
\]
(2.3)

where \(x^+ := \max\{x, 0\}\). We now compare the solution of the previous equation with the one of
\[
\begin{align*}
\frac{dJ(t)}{dt} &= \bar{f}(J(t))dt + \sum_{i=1}^{m} \bar{g}_i(J(t))dB_i(t), \quad t > 0; \\
J(0) &= i_0 \in ]0, N[,
\end{align*}
\]
(2.4)

which also possesses a unique global strong solution \(\{J(t)\}_{t \geq 0}\). Systems (2.3) and (2.4) have the same initial condition and diffusion coefficients; moreover, the drift in (2.4) is greater than the drift in (2.3). By Theorem 2.2 we conclude that
\[
I(t) \leq J(t), \quad \text{for all } t \geq 0,
\]
almost surely. Moreover, both the drift and diffusion coefficients in (2.4) vanish at \(x = N\). Therefore, invoking once more Theorem 2.1, the solution never visits \(N\), unless it starts from there. Since \(J(0) = i_0 \in ]0, N[\), we deduce that \(J(t) < N\), for all \(t \geq 0\), almost surely. Combining all these facts, we conclude that
\[
0 < I(t) < N, \quad \text{for all } t \geq 0,
\]
almost surely. This in turn implies
\[
\bar{f}(I(t)) = f(I(t)), \quad \bar{g}_i(I(t)) = g_i(I(t)), \quad \hat{h}(I(t)) = h(I(t)),
\]
and that \(\{I(t)\}_{t \geq 0}\) solves equation
\[
\begin{align*}
\frac{dI(t)}{dt} &= [f(I(t)) - h(I(t)) + \sum_{i=1}^{m} g_i(I(t))dB_i(t), \quad t > 0; \\
I(0) &= i_0 \in ]0, N[,
\end{align*}
\]
which coincides with (2.1). The uniqueness of the solution completes the proof. ∎

**Remark 2.4.** It is immediate to verify that equation (1.3) is a particular case of (2.1). This shows that Theorem 2.3 generalizes the existence and uniqueness result proved in [6].
2.2 Extinction

We now investigate the asymptotic behaviour of the solution of (2.1); here we are interested in sufficient conditions for extinction.

**Theorem 2.5.** Under the same assumptions of Theorem 2.3 assuming in addition,

\[
\sup_{x \in [0,N]} \left\{ \frac{f(x) - h(x)}{x} - \frac{1}{2} \sum_{i=1}^{m} \frac{g_i^2(x)}{x^2} \right\} < 0, \tag{2.5}
\]

then the solution \( \{I(t)\}_{t \geq 0} \) of the stochastic differential equation (2.1) verifies

\[
\limsup_{t \to +\infty} \frac{\ln(I(t))}{t} < 0 \quad \text{almost surely},
\]

i.e. \( I(t) \) tends to zero exponentially, as \( t \) tends to infinity, almost surely.

**Proof.** First of all, we observe that the local Lipschitz-continuity of \( f \) implies the existence of a constant \( L_N \) such that

\[
|f(x) - f(0)| \leq L_N |x - 0|, \quad \text{for all } x \in [0,N].
\]

In particular, using the equality \( f(0) = 0 \), we can rewrite the previous condition as

\[
\left| \frac{f(x)}{x} \right| \leq L_N, \quad \text{for all } x \in [0,N].
\]

Since the same reasoning applies also to \( h \) and \( g_i \), for \( i \in \{1, ..., m\} \), we deduce that the supremum in (2.5) is always finite.

Now, let \( \{I(t)\}_{t \geq 0} \) be the unique global strong solution of equation (2.1). An application of the Itô formula gives

\[
\ln(I(t)) = \ln(i_0) + \int_0^t \left[ \frac{f(I(s)) - h(I(s))}{I(s)} - \frac{1}{2} \sum_{i=1}^{m} \frac{g_i^2(I(s))}{I(s)^2} \right] ds \tag{2.6}
\]

\[
+ \sum_{i=1}^{m} \int_0^t \frac{g_i(I(s))}{I(s)} dB_i(s).
\]

Note that the boundedness of the function \( x \mapsto \frac{g_i(x)}{x} \) on \( [0,N] \) mentioned above entails that the stochastic process

\[
t \mapsto \sum_{i=1}^{m} \int_0^t \frac{g_i(I(s))}{I(s)} dB_i(s), \quad t \geq 0,
\]
is an \((\{\mathcal{F}^B_t\}_{t \geq 0}, \mathbb{P})\)-martingale. Therefore, from the strong law of large numbers for martingales (see e.g. Theorem 3.4, Chapter 1 in [12]) we conclude that

\[
\lim_{t \to +\infty} \sum_{i=1}^{m} \frac{1}{t} \int_{0}^{t} \frac{g_i(I(s))}{I(s)} dB_i(s) = 0,
\]

almost surely. This fact, combined with (2.6) gives

\[
\limsup_{t \to +\infty} \frac{\ln(I(t))}{t} = \limsup_{t \to +\infty} \frac{1}{t} \int_{0}^{t} \left[ \frac{f(I(s)) - h(I(s))}{I(s)} - \frac{1}{2} \sum_{i=1}^{m} \frac{g_i^2(I(s))}{I(s)^2} \right] ds \\
\leq \limsup_{t \to +\infty} \sup_{x \in [0,N]} \left\{ \frac{f(x) - h(x)}{x} - \frac{1}{2} \sum_{i=1}^{m} \frac{g_i^2(x)}{x^2} \right\} < 0,
\]

almost surely. The proof is complete. \(\square\)

**Example 2.6.** Take

\[
m = 1, \quad N = 1, \quad f(x) = -x \ln(x), \quad h(x) = x, \quad g(x) = -x \ln(x).
\] (2.7)

With these choices,

\[
\gamma(x) := \frac{f(x) - h(x)}{x} - \frac{g^2(x)}{2x^2} = -\ln(x) - 1 - \frac{\ln(x)^2}{2}, \quad x \in [0, 1].
\] (2.8)

In this case the assumption for extinction is verified.

\[\text{Figure 1: Graph of } \gamma(x) \text{ in (2.8)}\]

\[\text{Figure 2: Simulations of the path } I(t) \text{ for the SDE (2.1) with (2.7) and its corresponding deterministic SIS model with } g = 0 \text{ with initial values (a) } I(0) = 0.1 \text{ and (b) } I(0) = 0.9\]
Example 2.7. Take

\[ m = 1, \quad N = 1, \quad f(x) = 3x^2(1 - x), \quad h(x) = x, \quad g(x) = x^2(1 - x). \tag{2.9} \]

With these choices,

\[ \gamma(x) := \frac{f(x) - h(x)}{x} - \frac{g^2(x)}{2x^2} = 3x(1 - x) - 1 - \frac{x^2(1 - x)^2}{2}, \quad x \in [0, 1]. \tag{2.10} \]

In this case the assumption for extinction is verified.

![Figure 3: Graph of \( \gamma(x) \) in (2.10)](image)

![Figure 4: Simulations of the path \( I(t) \) for the SDE (2.1) with (2.9) and its corresponding deterministic SIS model with \( g = 0 \) with initial values (a) \( I(0) = 0.1 \) and (b) \( I(0) = 0.9 \)](image)

2.3 Persistence

We now search for conditions ensuring the persistence, i.e. an oscillation behaviour, for the solution \( \{I(t)\}_{t \geq 0} \) of (2.1).

Theorem 2.8. Under the same assumptions of Theorem 2.3, if inequality

\[ \sup_{x \in [0, N]} \left\{ \frac{f(x) - h(x)}{x} - \frac{1}{2} \sum_{i=1}^{m} \frac{g_i^2(x)}{x^2} \right\} > 0, \tag{2.11} \]

holds and moreover the function

\[ x \mapsto \frac{f(x) - h(x)}{x} - \frac{1}{2} \sum_{i=1}^{m} \frac{g_i^2(x)}{x^2} \tag{2.12} \]
is strictly monotone decreasing on \([0, N]\), then the solution \(\{I(t)\}_{t \geq 0}\) of the stochastic differential equation (2.1) verifies
\[
\limsup_{t \to +\infty} I(t) \geq \xi \quad \text{and} \quad \liminf_{t \to +\infty} I(t) \leq \xi,
\]
almost surely. Here, \(\xi\) is the unique zero of the function (2.12) on the interval \([0, N]\). In words: \(I(t)\) oscillates around the level \(\xi\) infinitely often, as \(t\) tends to infinity, with probability one.

**Proof.** To ease the notation we set
\[
\gamma(x) := \frac{f(x) - h(x)}{x} - \frac{1}{2} \sum_{i=1}^{m} g_i(x) \frac{g_i(x)}{x^2}, \quad x \in [0, N].
\]
First of all, we note that \(\gamma(N) = -\frac{h(N)}{N} < 0\); this gives, in combination with (2.11) and the strict monotonicity of \(\gamma\), the existence and uniqueness of \(\xi\). Now, assume the first inequality in (2.13) to be false. This implies the existence of \(\varepsilon > 0\) such that
\[
P \left( \limsup_{t \to +\infty} I(t) \leq \xi - 2\varepsilon \right) > \varepsilon.
\]
In particular, for any \(\omega \in A := \{\limsup_{t \to +\infty} I(t) \leq \xi - 2\varepsilon\}\), there exists \(T(\omega)\) such that
\[
\limsup_{t \to +\infty} I(t, \omega) \leq I(t, \omega) \leq \xi - \varepsilon, \quad \text{for all} \ t \geq T(\omega),
\]
which implies
\[
\gamma(I(t, \omega)) \geq \gamma(\xi - \varepsilon) > 0, \quad \text{for all} \ \omega \in A \text{ and } t \geq T(\omega).
\]
Therefore, for \(\omega \in A\) and \(t > T(\omega)\) we can write
\[
\frac{\ln(I(t))}{t} = \frac{\ln(i_0)}{t} + \frac{1}{t} \int_{0}^{t} \gamma(I(s)) ds + \frac{1}{t} \sum_{i=1}^{m} \int_{0}^{t} \frac{g_i(I(s))}{I(s)} dB_i(s)
\]
\[
= \frac{\ln(i_0)}{t} + \frac{1}{t} \int_{0}^{T(\omega)} \gamma(I(s)) ds + \frac{1}{t} \int_{T(\omega)}^{t} \gamma(I(s)) ds
\]
\[
+ \frac{1}{t} \sum_{i=1}^{m} \int_{0}^{t} \frac{g_i(I(s))}{I(s)} dB_i(s)
\]
\[
\geq \frac{\ln(i_0)}{t} + \frac{1}{t} \int_{0}^{T(\omega)} \gamma(I(s)) ds + \frac{t - T(\omega)}{t} \gamma(\xi - \varepsilon)
\]
\[
+ \frac{1}{t} \sum_{i=1}^{m} \int_{0}^{t} \frac{g_i(I(s))}{I(s)} dB_i(s).
\]
Hence, recalling that the strong law of large numbers for martingales gives
\[
\lim_{t \to +\infty} \sum_{i=1}^{m} \frac{1}{t} \int_{0}^{t} \frac{g_i(I(s))}{I(s)} dB_i(s) = 0 \quad \text{almost surely},
\]
we conclude that
\[
\liminf_{t \to +\infty} \frac{\ln(I(t))}{t} \geq \gamma(\xi - \varepsilon) > 0, \quad \text{on the set } A,
\]
which implies
\[
\lim_{t \to +\infty} I(t) = +\infty.
\]
This contradicts (2.14) and hence prove the first inequality in (2.13). The second inequality in (2.13) is proven similarly; if the thesis is not true, then
\[
\mathbb{P} \left( \liminf_{t \to +\infty} I(t) \geq \xi + 2\varepsilon \right) > \varepsilon. \tag{2.15}
\]
for some positive \(\varepsilon\). In particular, for any \(\omega \in B := \{\liminf_{t \to +\infty} I(t) \geq \xi + 2\varepsilon\}\), there exists \(S(\omega)\) such that
\[
\liminf_{t \to +\infty} I(t, \omega) \geq I(t, \omega) \geq \xi + \varepsilon, \quad \text{for all } t \geq S(\omega),
\]
which implies
\[
\gamma(I(t, \omega)) \leq \gamma(\xi + \varepsilon) < 0, \quad \text{for all } t \geq S(\omega).
\]
Therefore, for \(\omega \in B\) and \(t > S(\omega)\) we can write
\[
\frac{\ln(I(t))}{t} = \frac{\ln(i_0)}{t} + \frac{1}{t} \int_{0}^{t} \gamma(I(s)) ds + \sum_{i=1}^{m} \frac{1}{t} \int_{0}^{t} \frac{g_i(I(s))}{I(s)} dB_i(s)
\]
\[
= \frac{\ln(i_0)}{t} + \frac{1}{t} \int_{0}^{S(\omega)} \gamma(I(s)) ds + \frac{1}{t} \int_{S(\omega)}^{t} \gamma(I(s)) ds
\]
\[
+ \sum_{i=1}^{m} \frac{1}{t} \int_{0}^{t} \frac{g_i(I(s))}{I(s)} dB_i(s)
\]
\[
\leq \frac{\ln(i_0)}{t} + \frac{1}{t} \int_{0}^{S(\omega)} \gamma(I(s)) ds + \frac{t - S(\omega)}{t} \gamma(\xi + \varepsilon)
\]
\[
+ \sum_{i=1}^{m} \frac{1}{t} \int_{0}^{t} \frac{g_i(I(s))}{I(s)} dB_i(s)
\]
Therefore,

\[
\limsup_{t \to +\infty} \frac{\ln(I(t))}{t} \leq \gamma(\xi + \varepsilon) < 0, \quad \text{on the set } B,
\]

which implies

\[
\lim_{t \to +\infty} I(t) = 0, \quad \text{on the set } B.
\]

This contradicts \((2.15)\) and hence proves the second inequality in \((2.13)\).

**Example 2.9.** Take

\[
m = 1, \quad N = 1, \quad f(x) = 3x(1 - x)^2, \quad h(x) = x, \quad g(x) = x(1 - x)^2. \tag{2.16}
\]

With these choices,

\[
\gamma(x) := \frac{f(x) - h(x)}{x} - \frac{g^2(x)}{2x^2} = 3(1 - x)^2 - 1 - \frac{(1 - x)^4}{2}, \quad x \in [0, 1]. \tag{2.17}
\]

In this case the assumption for persistence is verified.

![Figure 5: Graph of \(\gamma(x)\) in (2.17)](image)

![Figure 6: Simulations of the path \(I(t)\) for the SDE (2.1) with (2.16) and its corresponding deterministic SIS model with \(g = 0\) with initial values (a) \(I(0) = 0.1\) and (b) \(I(0) = 0.9\)](image)

**Remark 2.10.** Note that, in spite of the strong similarity between the coefficients of Example 2.7 and Example 2.9, the behaviour of the corresponding solutions differs considerably.
Example 2.11. Take

\[ m = 1, \quad N = 1, \quad f(x) = 3(e^x - 1)(1 - x), \quad h(x) = x, \quad g(x) = (e^x - 1)(1 - x). \] (2.18)

With these choices,

\[
\gamma(x) = \frac{f(x) - h(x)}{x} - \frac{g^2(x)}{2x^2} = \frac{3(e^x - 1)(1 - x)}{x} - 1 - \frac{(e^x - 1)^2(1 - x)^2}{2x^2}, \quad x \in [0, 1]. \tag{2.19}
\]

In this case the assumption for persistence is verified.

![Figure 7: Graph of \(\gamma(x)\) in (2.19)](image)

![Figure 8: Simulations of the path \(I(t)\) for the SDE (2.1) with (2.18) and its corresponding deterministic SIS model with \(g = 0\) with initial values (a) \(I(0) = 0.1\) and (b) \(I(0) = 0.9\)](image)
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