On multipliers on compact Lie groups

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Abstract

In this note we announce $L^p$ multiplier theorems for invariant and non-invariant operators on compact Lie groups in the spirit of the well-known Hörmander-Mikhlin theorem on $\mathbb{R}^n$ and its variants on tori $\mathbb{T}^n$. Applications are given to the mapping properties of pseudo-differential operators on $L^p$-spaces and to a-priori estimates for non-hypoelliptic operators.

1. Introduction.

Let $G$ be a compact Lie group of dimension $n$, with identity $1$ and the unitary dual $\hat{G}$. The following considerations are based on the group Fourier transform

$$\mathcal{F}\phi = \hat{\phi}(\xi) = \int_G \phi(x)\xi(x)^*x, \quad \phi(x) = \sum_{[\xi] \in \hat{G}} d_\xi \text{Tr}(\xi(x)\hat{\phi}(\xi)) = \mathcal{F}^{-1}[\hat{\phi}]$$

(1)

defined in terms of equivalence classes $[\xi]$ of irreducible unitary representations $\xi : G \to U(d_\xi)$ of dimension $d_\xi$. The Peter–Weyl theorem on $G$ implies in particular that this pair of transforms is inverse to each other and that the Plancherel identity

$$\|\phi\|_2^2 = \sum_{[\xi] \in \hat{G}} d_\xi \|\hat{\phi}(\xi)\|_{HS}^2 =: \|\hat{\phi}\|_{L^2(\hat{G})}$$

(2)

holds true for all $\phi \in L^2(G)$. Here $\|\hat{\phi}(\xi)\|_{HS}^2 = \text{Tr}(\hat{\phi}(\xi)\hat{\phi}(\xi)^*)$ denotes the Hilbert–Schmidt norm of matrices. The Fourier inversion statement (1) is valid for all $\phi \in \mathcal{D}'(G)$ and the Fourier series converges in $C^\infty(G)$ provided $\phi$ is smooth. It is further convenient to denote $\langle \xi \rangle = \max\{1, \lambda_\xi\}$, where $-\lambda_\xi^2$ is the eigenvalue of the Laplace-Beltrami (Casimir) operator acting on the matrix coefficients associated to

*The research was supported by the EPSRC grant EP/G007233/1.

1Keywords: multipliers, pseudo-differential operators, Lie groups
the representation ξ. The Sobolev spaces can be characterised by Fourier coefficients as

$$\phi \in H^s(G) \iff \langle \xi \rangle^s \hat{\phi}(\xi) \in \ell^2(\hat{G}),$$

where $\ell^2(\hat{G})$ is defined as the space of matrix-valued sequences such that the sum on the right-hand side of (2) is finite.

In the following we consider continuous linear operators $A : C^\infty(G) \to D'(G)$, which can be characterised by their symbol

$$\sigma_A(x, \xi) = \xi(x)^*(A\xi)(x)$$

which is a function on $G \times \hat{G}$ taking matrices from $\mathbb{C}^{d_\xi \times d_\xi}$ as values. As a consequence of (1) we obtain that for any given $\phi \in C^\infty(G)$ the distribution $A\phi \in D'(G)$ satisfies

$$A\phi(x) = \sum_{\xi \in \hat{G}} d_\xi \text{Tr}(\xi(x)\sigma_A(x, \xi)\hat{\phi}(\xi)).$$

(4)

We denote the operator $A$ defined by a symbol $\sigma_A$ as $\text{op}(\sigma_A)$. This quantisation and its properties have been consistently developed in [7] and we refer to it for details.

We speak of a Fourier multiplier if the symbol $\sigma_A(x, \xi)$ is independent of the first argument. This is equivalent to requiring that $A$ commutes with left translations.

It is evident from the Plancherel identity that such an operator is $L^2$-bounded if and only if $\sup_{\xi \in \hat{G}} \|\sigma_A(\xi)\|_{\text{op}} < \infty$, where $\| \cdot \|_{\text{op}}$ denotes the operator norm on the inner-product space $\mathbb{C}^{d_\xi}$.

In the book [7], as well as in the paper [8] the authors gave a characterisation of Hörmander type pseudo-differential operators on $G$ in terms of their matrix-valued symbols. The symbol classes, as well as the multiplier theorems given below, depend on the so-called difference operators acting on moderate sequences of matrices, i.e., on elements of

$$\Sigma(\hat{G}) = \{ \sigma : \xi \mapsto \sigma(\xi) \in \mathbb{C}^{d_\xi \times d_\xi} : \|\sigma(\xi)\|_{\text{op}} \lesssim \langle \xi \rangle^N \text{ for some } N \}. $$

2
A difference operator $Q$ of order $\ell$ is defined in terms of a corresponding function $q \in C^\infty(G)$, which vanishes to (at least) $\ell$th order in the identity element $1 \in G$ via

$$Q\sigma = \mathcal{F} (q(x)\mathcal{F}^{-1}\sigma).$$

Note, that $\sigma \in \Sigma(\hat{G})$ implies $\mathcal{F}^{-1}\sigma \in \mathcal{D}'(G)$ and therefore the multiplication with a smooth function is well-defined. The main idea of introducing such operators is that applying differences to symbols of Calderon–Zygmund operators brings an improvement in the behaviour of $\text{op}(Q\sigma)$ since we multiply the integral kernel of $\text{op}(\sigma)$ by a function vanishing on its singular set.

Different collections of difference operators have been explored in \cite{8} in the pseudodifferential setting. Difference operators of particular interest arise from matrix-coefficients of representations. For a fixed irreducible representation $\xi_0$ we define the (matrix-valued) difference operator $\xi_0 \mathcal{D} = (\xi_0 \mathcal{D}_{ij})_{i,j=1,\ldots,d_\xi_0}$ corresponding to the matrix elements of the matrix-valued function $\xi_0(x) - I$, with $q_{ij}(x) = \xi_0(x)_{ij} - \delta_{ij}$ in \eqref{5}, $\delta_{ij}$ the Kronecker delta. If the representation is fixed, we omit the index $\xi_0$. For a sequence of difference operators of this type, $\mathcal{D}_1 = \xi_1 \mathcal{D}_{i_1j_1}, \mathcal{D}_2 = \xi_2 \mathcal{D}_{i_2j_2}, \ldots, \mathcal{D}_k = \xi_k \mathcal{D}_{i_kj_k}$, with $[\xi_m] \in \hat{G}$, $1 \leq i_m, j_m \leq d_{\xi_m}, 1 \leq m \leq k$, we define $\mathcal{D}^\alpha = \mathcal{D}_1^{a_1} \cdots \mathcal{D}_k^{a_k}$. In the sequel we will work with a collection $\Delta_0$ of representations chosen as follows. Let $\tilde{\Delta}_0$ be the collection of the irreducible components of the adjoint representation, so that $\text{Ad} = (\dim Z(G))1 \oplus \bigoplus_{\xi \in \tilde{\Delta}_0} \xi$, where $\xi$ are irreducible representations and $1$ is the trivial one-dimensional representation. In the case when the centre $Z(G)$ of the group is nontrivial, we extend the collection $\tilde{\Delta}_0$ to some collection $\Delta_0$ by adding to $\tilde{\Delta}_0$ a family of irreducible representations such that their direct sum is nontrivial on $Z(G)$, and such that the function

$$\rho^2(x) = \sum_{[\xi] \in \Delta_0} (d_\xi - \text{Tr} \xi(x)) \geq 0$$

(which vanishes only in $x = 1$) would define the square of some distance function on $Z(G)$, and such that the function
G near the identity element. Such an extension is always possible, and we denote by \( \Delta_0 \) any such extension; in the case of the trivial centre we do not have to take an extension and we set \( \Delta_0 = \tilde{\Delta}_0 \). We denote further by \( \triangle^* \) the second order difference operator associated to \( \rho^2(x) \), \( \triangle^* = F \rho^2(x) F^{-1} \). In the sequel, when we write \( D^\alpha \), we can always assume that it is composed only of \( \xi_m D_{i_m j_m} \) with \( [\xi_m] \in \Delta_0 \).

2. Main results.

The following condition (6) is a natural relaxation from the \( L^p \)-boundedness of zero order pseudo-differential operators to a multiplier theorem and generalises the Hörmander–Mikhlin ([5, 6], [4]) theorem to arbitrary groups.

**Theorem 1** Denote by \( \kappa \) be the smallest even integer larger than \( \frac{1}{2} \dim G \). Let \( A : C^\infty(G) \to \mathcal{D}'(G) \) be left-invariant. Assume that its symbol \( \sigma_A \) satisfies

\[
\| \triangle^{\kappa/2} \sigma_A(\xi) \|_{op} \leq C(\xi)^{-\kappa} \quad \text{and} \quad \| D^\alpha \sigma_A(\xi) \|_{op} \leq C_\alpha(\xi)^{-|\alpha|}
\]

for all multi-indices \( \alpha \) with \( |\alpha| \leq \kappa - 1 \), and for all \( [\xi] \in \hat{G} \). Then the operator \( A \) is of weak type \((1,1)\) and \( L^p \)-bounded for all \( 1 < p < \infty \).

We now give some particular applications of Theorem 1. The selection is not complete and indicates a few applications which could be derived from the main result. Full proofs can be found in [9].

**Theorem 2** Assume that \( \sigma_A \in \mathcal{S}_\rho^0(G) \), i.e., by definition, it satisfies inequalities

\[
\| D^\alpha \sigma_A(\xi) \|_{op} \leq C_\alpha(\xi)^{-\rho|\alpha|},
\]

for some \( \rho \in [0,1] \) and all \( \alpha \). Then \( A \) defines a bounded operator from \( W^{r,p}(G) \) to \( L^p(G) \) for \( r = \kappa(1 - \rho)|\frac{1}{p} - \frac{1}{2}|, \kappa \) as in Theorem 1 and \( 1 < p < \infty \).

The operator \( A \) is said to be of weak type \((1,1)\) if there exists a constant \( C > 0 \) such that for all \( \lambda > 0 \) and \( u \in L^1(G) \) the inequality \( \mu \{ x \in G : |Au(x)| > \lambda \} \leq C\|u\|_{L^1(G)}/\lambda \) holds true, where \( \mu \) is the Haar measure on \( G \).

Here \( W^{r,p}(G) \) stands for the Sobolev space consisting of all distributions \( f \) such that \( (I - \mathcal{L})^{r/2} f \in L^p(G) \), where \( \mathcal{L} \) is a Laplacian (Laplace-Betrami operator, Casimir element) on \( G \).
The previous statement applies in particular to the parametrices constructed in \[8\]. We will give two examples on the group \( SU(2) \cong S^3 \). Let \( D_1, D_2 \) and \( D_3 \) be an orthonormal basis of \( \mathfrak{su}(2) \). Then both, the sub-Laplacian \( L_s = D_1^2 + D_2^2 \) as well as the 'heat' operator \( H = D_3 - D_1^2 - D_2^2 \) have a parametrix from \( \text{op}\mathcal{S}^{-1}(S^3) \) and therefore the sub-elliptic estimates
\[
\|u\|_{W^{p,1-\frac{1}{p},\frac{1}{2}}(S^3)} \leq C_p \|L_s u\|_{L^p(S^3)} \quad \text{and} \quad \|u\|_{W^{p,1-\frac{1}{p},\frac{1}{2}}(S^3)} \leq C_p \|H u\|_{L^p(S^3)}
\]
are valid for all \( 1 < p < \infty \). The following statement concerns operators which are neither locally invertible nor locally hypoelliptic.

**Corollary 3** Let \( X \) be a left-invariant real vector field on \( G \). Then there exists a discrete exceptional set \( \mathcal{C} \subset \mathbb{i}\mathbb{R} \), such that for any complex number \( c \not\in \mathcal{C} \) the operator \( X + c \) is invertible with inverse in \( \text{op}\mathcal{S}^0(G) \). Consequently, the inequality
\[
\|f\|_{L^p(G)} \leq C_p \|(X + c)f\|_{W^{p,1-\frac{1}{p},\frac{1}{2}}(G)}
\]
holds true for all \( 1 < p < \infty \) and all functions \( f \) from that Sobolev space, with \( \kappa \) as above.

For the particular case \( G = SU(2) \), the exceptional set coincides with the spectrum of the skew-selfadjoint realisation of \( X \) suitably normalised with respect to the Killing norm, e.g., \( \mathcal{C} = \mathbb{i}\frac{1}{2}\mathbb{Z} \) if \( X = D_3 \).

The Hörmander multiplier theorem \[4\], although formulated in \( \mathbb{R}^n \), has a natural analogue on the torus \( T^n \). The assumptions in Theorem \[1\] on the top order difference brings a refinement of the toroidal multiplier theorem, at least for some dimensions. If \( G = T^n = \mathbb{R}^n/\mathbb{Z}^n \), the set \( \Delta_0 \) can be chosen to consist of \( 2n \) functions \( e^{\pm 2\pi i x_j} \), \( 1 \leq j \leq n \). Consequently, we have that \( \rho^2(x) = 2n - \sum_{j=1}^n (e^{2\pi i x_j} + e^{-2\pi i x_j}) \) in \( G \).

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\[4\] The class \( \text{op}\mathcal{S}^{-1}(S^3) \) is defined as the class of operators with symbols \( \sigma_A \) satisfying the inequalities \( \|D^\alpha \sigma_A(\xi)\|_{\text{op}} \leq C_\alpha(\xi)^{-1-|\alpha|/2} \).
and hence \( \Delta \sigma(\xi) = 2n\sigma(\xi) - \sum_{j=1}^{n} (\sigma(\xi + e_j) + \sigma(x - e_j)) \), where \( \xi \in \mathbb{Z}^n \) and \( e_j \) is its \( j \)th unit basis vector in \( \mathbb{Z}^n \).

A (translation) invariant operator \( A \) and its symbol \( \sigma_A \) are related by \( \sigma_A(k) = e^{-2\pi i x \cdot k} (\mathcal{A}e^{2\pi i x \cdot k})_{|x=0} \) and \( A\phi(x) = \sum_{k \in \mathbb{Z}^n} e^{2\pi i x \cdot k} \sigma_A(k) \hat{\phi}(k) \). Thus, it follows from Theorem 1 that, for example on \( \mathbb{T}^3 \), a translation invariant operator \( A \) is bounded on \( L^p(\mathbb{T}^3) \) provided that there is a constant \( C > 0 \) such that \( |\sigma_A(k)| \leq C \), \( |k| |\sigma_A(k + e_j) - \sigma_A(k)| \leq C \) and

\[
|k|^2 |\sigma_A(k)| - \frac{1}{6} \sum_{j=1}^{3} (|\sigma_A(k + e_j) + \sigma_A(k - e_j)| \leq C, \tag{8}
\]

for all \( k \in \mathbb{Z}^3 \) and all (three) unit vectors \( e_j, j = 1, 2, 3 \). Here we do not make assumptions on all second order differences in (8), but only on one of them.

Finally, Theorem 1 also implies a boundedness statement for operators of form (4). Let for this \( \partial_{x_j}, 1 \leq j \leq n \), be a collection of left invariant first order differential operators corresponding to some linearly independent family of the left-invariant vector fields on \( G \). As usual, we denote \( \partial^\beta_x = \partial^\beta_{x_1} \cdots \partial^\beta_{x_n} \).

**Theorem 4** Denote by \( \sigma \) be the smallest even integer larger than \( \frac{n}{2} \), \( n \) the dimension of the group \( G \). Let \( 1 < p < \infty \) and let \( l > \frac{n}{p} \) be an integer. Let \( A : C^\infty(G) \rightarrow \mathcal{D}'(G) \) be a linear continuous operator such that its matrix symbol \( \sigma_A \) satisfies

\[
\| \partial_x^\beta \mathcal{D}^\alpha \sigma_A(x, \xi) \|_{op} \leq C_{\alpha, \beta} (\xi)^{-|\alpha|}
\]

for all multi-indices \( \alpha, \beta \) with \( |\alpha| \leq \sigma \) and \( |\beta| \leq l \), for all \( x \in G \) and \( [\xi] \in \hat{G} \). Then the operator \( A \) is bounded on \( L^p(G) \).

3. **Discussion.** 1. The conditions are needed for the week type \((1,1)\) property. Interpolation allows to reduce assumptions on the number of differences for \( L^p \)-boundedness. The result generalises the corresponding statements in the case of the group \( \text{SU}(2) \) in [1], [2], also presented in [3].
2. Examples similar to (7) can be given for arbitrary compact Lie groups. The assumptions of Theorem 2 concerning the numbers of difference operators can be relaxed to the same as those in Theorem 1.

3. If the operator \( A \in \Psi^0(G) \) is the usual pseudo-differential operator of Hörmander type of order 0 on \( G \) (i.e. in all local coordinate it belongs to Hörmander class \( \Psi^0(\mathbb{R}^n) \)), it was shown in [7] the estimates (1) hold for all \( \alpha, \beta \). The converse is also true. Namely, if estimates (1) hold for all \( \alpha, \beta \), then \( A \in \Psi^0(G) \), cf. [8].

4. Noncommutative matrix quantisation (3)–(4) has a full symbolic calculus (compositions, adjoints, parametrix, etc.), which have been established in the monograph [7].

5. On SU(2) the operators corresponding to our difference operators but defined explicitly in terms of the Clebsch-Gordan coefficients have been used in [2, 3]. The general definition (3), the main tool in the present investigation, has been introduced and analysed in [7] and [8].

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