2D Quantum Gravity and the Miura Map

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ABSTRACT

We study the $sL(3, C)$ mKDV string theories. We obtain the flows and the string equations. Using the generalized Miura map, we show that we have an "unification" of these models with the $[\bar{P}, Q] = Q sL(3, C)$ KDV ones in the framework of open-closed string theories in minimal models backgrounds.

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1. INTRODUCTION

Recently important progress has been made in the description of theories which look very similar to two-dimensional gravity coupled to matter. Starting with Hermitian matrix models in the simplest phase, one finds that the KDV integrable hierarchy supplied by the condition $[P, Q] = 1$ leads to a possible candidate for 2D gravity coupled to minimal conformal models\cite{1,2,3,4}. However this approach doesn’t give a successful non-perturbative description for all the models; in particular it’s not the case for the simplest one: pure gravity\cite{5}.

Another possible definition which is based on the assumption that the KDV flows hold non-perturbatively has been proposed\cite{6}. These theories are described by the KDV hierarchy supplied by the condition $[\tilde{P}, Q] = Q$ and can be seen as arising from complex matrix models\cite{7}. This approach leads to a successful non-perturbative description for all the models.

Apart from the theories based on the KDV integrable hierarchy, there exists models giving rise in the double scaling limit to a description in terms of the mKDV hierarchy. They were first found by considering unitary matrix models\cite{8} then they were obtained in the double scaling limit of usual Hermitian matrix models in the 2 cuts phase\cite{9}. Many conjectures\cite{10} have been advanced in order to identify physically these theories.

Recently some authors\cite{11} argued that the mKDV theories and the KDV ones supplied by the $[\tilde{P}, Q] = Q$ condition are two descriptions of the same 2D gravitational system. They showed by studying the so-called Miura map that the $sL(2, C)$ mKDV string equation maps to the $[\tilde{P}, Q] = Q$ one with a non-zero open string coupling constant.

This letter is concerned with a generalization of this identification. We study the mKDV theories associated with the $sL(3, C)$ algebra (noted mKDV(3)) in the Zakharov-Shabat(ZS) formalism\cite{12}. We obtain the corresponding flows and string equations. After that using the generalized Miura map we show that the mKDV(3) string equations are mapped onto the $[\tilde{P}, Q] = Q$ string equations.
corresponding to the Boussinesque hierarchy with a non-zero open string coupling constant.

This paper is organized as follows: section 2 deals with a review of the known results in the $sL(2, C)$ case. In section 3, we study the $sL(3, C)$ generalization. Finally, in section 4, we discuss the results.

2. THE $sL(2, C)$ THEORIES

In this section we review the connections existing between the $mKDV$ string models and the $KDV$ ones via the Miura map. In the ZS scheme the $mKDV$ flows are associated with the following first order operator\cite{12}:

$$L_1 = \partial_x + \frac{f}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & \xi \\ 1 & 0 \end{pmatrix}$$

(2.1)

Using the transformation $L_2 = S L_1 S^{-1}$ where:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & \xi^{\frac{1}{2}} \end{pmatrix}$$

(2.2)

we can also describe the $mKDV$ flows with:

$$L_2 = \partial_x + f \sigma_3 + \lambda \sigma_1$$

(2.3)

where we have $\lambda^2 = \xi$.

This is the operator $L$ we get when we study the 2 cuts Hermitian matrix models characterized by even potentials\cite{13,10,14,15}. In that context the physical specific heat is given by: $(logZ)'' = -\frac{1}{4} f^2$.

The flows compatible with the reduction (2.3) can be calculated by recurrence and
are given by\textsuperscript{[10,15]}:

\[ \frac{df}{dt_k} = F_{2k+1} \quad (2.4) \]

where:

\[ H'_k + f F_k = 0 \]
\[ G_{k+1} = F'_k + f H_k \quad (2.5) \]
\[ F_{k+1} = G t_k \]

with the initial conditions: \( F_0 = 0 \quad G_0 = f \quad H_0 = 0 \) and the identification \( t_0 = x \).

The mKDV massive string equation determined by the compatibility\textsuperscript{[13]} between the flows, \( L_2 \) and the operator \( \frac{d}{dx} - M \) is:

\[ \sum_{k=1}^{\infty} t_k (2k + 1) G_{2k} + x f = 0 \quad (2.6) \]

The equations (2.4),(2.6) characterize completely the hierarchy of the mKDV models.

With the conventions choosen here the Miura map, which transforms the mKDV hierarchy into the KDV one, is:

\[ u = \frac{f^2}{4} + \frac{ft}{2} \quad (2.7) \]

Under (2.7) the flows (2.4) become:

\[ \frac{du}{dt_k} = \partial_x \left[ \frac{1}{2} (F_{2k+1} - H_{2k+1}) \right] \quad (2.8) \]

Defining

\[ R_{k+1}[u] = \frac{1}{2} (F_{2k+1} - H_{2k+1}) \quad (2.9) \]

Using (2.5), it’s easy to check that:

\[ R'_{k+1} = R''_k - 4uR'_k - 2u'R_k \quad (2.10) \]
Thus the \( mKDV \) flows are indeed mapped onto the \( KDV \) ones:

\[
\frac{du}{dt_k} = \partial_x R_{k+1}[u]
\]

(2.11)

where the \( R_k \) are the usual Gelfand Dikii potential\(^{[16]}\) (with \( R_0 = -\frac{1}{2} \)).

We now turn to the string equation. Using (2.9) and (2.5) we have:

\[
G_{2k} = 2D^* R_k
\]

(2.12)

where : \( D^* = \partial_x - f \)

We can thus rewrite (2.6) as:

\[
2D^*(\mathcal{R}) + 1 = 0
\]

(2.13)

where:

\[
\mathcal{R} = \sum_{k=0} t_k (2k+1) R_k[u]
\]

(2.14)

For a fixed critical model characterized by \( k \) we have : \( \mathcal{R} = R_k - \frac{\pi}{2} \).

It’s now possible to extract \( f \):

\[
f = \frac{\mathcal{R}'}{\mathcal{R}} + \frac{1}{\mathcal{R}}
\]

(2.15)

Using the Miura map(2.7) we finally find:

\[
(\mathcal{R}')^2 + 4u\mathcal{R}^2 - 2\mathcal{R}\mathcal{R}'' = \frac{1}{4}
\]

(2.16)

which is the \( \tilde{P}, Q = Q \ KDV \) string equation\(^{[6]}\) with a non-vanishing open string coupling constant\(^{[17]}\).

This result argues thus for an unification of the \( KDV \) and the \( mKDV \ sL(2, C) \) theories in the framework of open-closed string theory in the \( (2, 2k - 1) \) minimal models backgrounds\(^{[11]}\).
3. The \( sL(3, C) \) Generalized \( mKDV \) MODELS

We consider the \( sL(3, C) \) algebra given by the following \( 3 \times 3 \) matrices:

\[
S_1 = \delta_{1,2}, \quad S_2 = \delta_{1,3}, \quad S_3 = \delta_{2,1}, \quad S_4 = \delta_{2,3},
\]

\[
S_5 = \delta_{3,1}, \quad S_6 = \delta_{3,2}
\]

\[
H_1 = \text{diag}(1, -1, 0) \quad H_2 = \text{diag}(1, 0, -1)
\]

The generalization of (2.1) is given by:

\[
L_1 = \partial_x + \begin{pmatrix} q_1 & 0 & 0 \\ 0 & q_2 & 0 \\ 0 & 0 & q_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \xi \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

(3.1)

where:

\[
\sum q_i = 0 \quad (3.2)
\]

Performing the transformation \( L_2 = S L_1 S^{-1} \) where \( S = \text{diag}(1, \xi^1, \xi^2) \) and using the condition (3.2) we write:

\[
L_2 = \partial_x + f H_1 + g H_2 + \lambda A_1
\]

(3.3)

where:

\[
A_1 = S_2 + S_3 + S_6
\]

(3.4)

and \( \lambda^3 = \xi \)

In analogy with (2.3) we are going to study the ”string theory” associated with the reduction (3.3).

We have first to find the flows coherent with this reduction. By the ZS method for lie algebras \(^{[12]}\), we know that all the possible candidates are:

\[
\frac{d}{d\tau_{i,k}} L_2 = -[(M_{i,k})_+, L_2] = [\text{res}(M_{i,k}), A_1]
\]

(3.5)

with \( k = 1, 2, \ldots \) and \( i = 1, 2 \);

where we have \( M_{i,k} = e^{-adU}(A_1 \lambda^k) \), the subscript + stands for the part of the series

6
with positive powers of $\lambda$, and the transformation $e^{adU}$ is defined as usually$^{[12]}$ by the fact that:

$$e^{adU}(L_2) = L_2 + [u, L_2] + \frac{1}{2}[u, [u, L_2]] + \ldots$$

$$= \partial_x + \lambda A_1 + \sum_{i=0}^{\infty} (B_i A_1 + C_i A_2) \lambda^{-i}$$

(3.6)

where $u$ is a series of negative powers of $\lambda$ with coefficients being functions of $x$ with values in $sL(3, C)$, and $A_2$ is the second element of $\text{Ker}A_1$:

$$A_2 = S_1 + S_4 + S_5$$

(3.7)

Writing:

$$\text{Res}(M_{i,k}) = \sum_{j=1}^{6} a_{i,j,k} S_j + b_{i,k} H_1 + c_{i,k} H_2$$

(3.8)

we have the following recursion relations:

$$a_{i,3,k+1} - a_{i,2,k+1} = a_{i,4,k}' + (g - f)a_{i,4,k}$$

$$a_{i,6,k+1} - a_{i,3,k+1} = a_{i,5,k}' - (f + 2g)a_{i,5,k}$$

$$b_{i,k+1} + 2b_{i,k+1} = a_{i,2,k}' + (f + 2g)a_{i,2,k}$$

$$b_{i,k+1} - b_{i,k+1} = a_{i,6,k}' + (f - g)a_{i,6,k}$$

$$a_{i,1,k+1} - a_{i,4,k+1} = b_{i,k}'$$

$$a_{i,4,k+1} - a_{i,5,k+1} = c_{i,k}'$$

$$a_{i,1,k}' + a_{i,4,k}' + a_{i,5,k}' + (2f + g)a_{i,1,k} + (g - f)a_{i,4,k} - (f + 2g)a_{i,5,k} = 0$$

$$a_{i,2,k}' + a_{i,3,k}' + a_{i,6,k}' + (f + 2g)a_{i,2,k} - (2f + g)a_{i,3,k} + (f - g)a_{i,6,k} = 0$$

(3.9)

with the following non-zero initial values:

$$b_{1,0} = f \quad c_{1,0} = g \quad a_{2,2,0} = f \quad a_{2,3,0} = g \quad a_{2,6,0} = -f - g$$

(3.10)
Since the residues coherent with the reduction we are considering must satisfy:

\[ b_{i,k} = c_{i,k} = a_{i,2,k} = a_{i,3,k} = a_{i,6,k} = 0 \]

we have to restrict ourself to the following flows: \( t_{i,k} \equiv \tau_{i,i+3k} \) where \( k = 0, 1, 2, \ldots \) and \( i = 1, 2 \). These ones are thus given by:

\[
\begin{align*}
\frac{df}{dt_{i,k}} &= \partial_x b_{i,i+3k-1} \\
\frac{dg}{dt_{i,k}} &= \partial_x c_{i,i+3k-1}
\end{align*}
\]

(3.11)

where we have \( t_{1,0} = x \).

We now turn to the determination of the string equations by flatness conditions\cite{13}.

The compatibility condition between \( L_2 \) and the operator \( \frac{d}{dx} - P \) gives: \([P, L_2] = A_1\).

The solutions are given by \( P_{i,k} = (M_{i,i+3k-1})_+ - xA_1 \) which lead to:

\[
\begin{align*}
b_{i,i+3k-1} &= xf \\
c_{i,i+3k-1} &= xg
\end{align*}
\]

(3.12)

The \( mKDV(3) \) “string theories” are thus characterized by the flows (3.11) and the string equations (3.12).

We now study the generalized Miura transformation. This one is defined by the action of an upper triangular matrix \( S_m \) on the \( L_1 \) operator so that we have:

\[
L_{kdv} = S_m^{-1} L_1 S_m = \partial_x + \begin{pmatrix}
0 & 0 & -\left(\frac{3}{4}u_2 + u_3\right) \\
0 & 0 & -\frac{3}{2}u_2 \\
0 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & \xi \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}
\]

(3.13)

which is the ZS formulation of the usual \( sL(3,C) \) operator (A.1).
The unique \( S_m \) leading to (3.13) gives us the following relations:

\[
\begin{align*}
  u_2 &= -\frac{2}{3}(f' + 2g' + f^2 + g^2 + fg) \\
  u_3 &= -(\frac{f''}{2} + ff' + fg(f + g) + \frac{3}{2}f'g + fg')
\end{align*}
\]  

(3.14)

under the Miura map the flows become:

\[
\frac{d}{dt_{k,i}} \left( \frac{3}{2}u_2 \right) = B \begin{pmatrix} f_k \\ g_k \end{pmatrix} = \begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} \begin{pmatrix} R^2_{i,k+1} \\ R^3_{i,k+1} \end{pmatrix}
\]

(3.15)

where we have defined \( R^2_{i,k} \) and \( R^3_{i,k} \) and \( B \) is the “Frechet Jacobian”:

\[
B = \begin{pmatrix} \partial \partial + 2f + g & \partial \partial + 2f + 2g \\ \frac{\partial^2}{2} + f \partial + \frac{3}{2}g \partial + 2fg + g^2 + f' + \frac{f}{2} + \frac{3}{2}f' + 2fg + f^2 \end{pmatrix}
\]

(3.16)

Using (3.11), (3.15) and (3.9) we get:

\[
\begin{align*}
  R^2_{i,k+1} &= \frac{3}{2}f(a_{i,5,i+3k} + \frac{3}{2}(a_{i,6,i+3k+1} + a_{i,3,i+3k+1}) \\
  R^3_{i,k+1} &= 3a_{i,5,i+3k}
\end{align*}
\]  

(3.17)

By (3.10) we have the initial values:

\[
R^3_{1,1} = \frac{3}{2}u_2 \quad R^2_{1,1} = 2u_3 \quad R^1_{1,1} = u_3 \quad R^2_{2,1} = -\frac{1}{4}(u''_2 + 3u^2_2)
\]

Using (3.9) it’s easy to show that the functions (3.17) satisfy the recursion relations (A.2) of the Boussinesque hierarchy given in the appendix. The transformation (3.14) maps thus, as expected\(^{[18]}\), the \( mKDV(3) \) flows onto the \( KDV(3) \) ones.

We now consider the string equation (3.12). Taking its derivative, using (3.11) and (3.15) we get:

\[
B \begin{pmatrix} (xf)' \\ (xg)' \end{pmatrix} = D_2 \begin{pmatrix} R^2_{i,k} \\ R^3_{i,k} \end{pmatrix}
\]

(3.18)

where we have used \( D_2 \), the second hamiltonian structure of the Boussinesque hierarchy (see appendix).
Remarking that:

\[ B \begin{pmatrix} (xf)' \\ (xg)' \end{pmatrix} = \begin{pmatrix} 3u_2 + \frac{3}{2}xu'_2 \\ 3u_3 + xu'_3 \end{pmatrix} \] (3.19)

The string equation becomes:

\[ D_2 \begin{pmatrix} \mathcal{R}_{2,i,k} \\ \mathcal{R}_{3,i,k} \end{pmatrix} = 0 \] (3.20)

where we have: \( \mathcal{R}_{2,i,k} = R_{i,k}^2 - 3x \) and \( \mathcal{R}_{3,i,k} = R_{i,k}^3 \)

Now by multiplying on the left (3.20) by \( (\mathcal{R}_2, \mathcal{R}_3) \) and integrating once we finally get:

\[ \frac{1}{3}(\mathcal{R}_2')^2 - \frac{1}{2}u_2\mathcal{R}_2^2 - \frac{2}{3}\mathcal{R}_2\mathcal{R}_2' - u_3\mathcal{R}_2\mathcal{R}_3 + \frac{1}{18}(\mathcal{R}_3^4) - \mathcal{R}_3'\mathcal{R}_3^{(3)} - \frac{1}{2}(\mathcal{R}_3'')^2 \]
\[ + \frac{5}{12}(u_2\mathcal{R}_3\mathcal{R}_3 - \frac{1}{2}u_2(\mathcal{R}_3')^2 + \frac{1}{2}u_2\mathcal{R}_3\mathcal{R}_3') + \frac{1}{12}(3u_2^2 + u_2'')\mathcal{R}_3^2 = 3 \] (3.21)

The constant on the rhs of (3.21) is determined by the scaling property of the string equation (3.12). Indeed, defining \( f = \alpha \tilde{f}, \ g = \alpha \tilde{g} \) and \( \tilde{x} = \alpha x \) we have in the \( \alpha = 0 \) limit \( \tilde{f} = \tilde{g} = 0 \). Thus we have \( \tilde{u}_2 = \alpha^{-2}u_2 = 0 \) and \( \tilde{u}_3 = \alpha^{-3}u_3 = 0 \) which fix the constant to be 3.

We have thus shown that the Miura map transforms the \( mKDV(3) \) string equations (3.12) onto (3.21) which are the \([\tilde{P}, Q] = Q\) equations for the \((p, 3) \ KDV\) models\(^{[19]}\) with a non-vanishing constant which can play the role of an open string coupling\(^{[17,14]}\).
4. DISCUSSION

We have studied the $sL(3, C)$ generalization of the $mKDV$ string models deriving the flows and the string equations. We have shown, generalizing the results of Dalley et al.\cite{11}, that the $KDV(3)$ and the $mKDV(3)$ theories are unified by the Miura map in the picture of open-closed string theories in $(p, 3)$ minimal backgrounds: the $mKDV(3)$ closed string equations being mapped onto the $KDV(3)$ open ones. It’s natural to conjecture that we should obtain the same results for all $sL(n, C)$ models.

Finally there is an important question we would like to emphasize. For the $sL(2, C)$ $mKDV$ theory – the only one up to now which possesses an underlying matrix model leading to an identification of the physical specific heat – we see that the quantity which is solution of the $KDV$ $[\tilde{P}, Q] = Q$ equation after having performed the Miura map is not the specific heat $-f^2 + f^2$ but $-f^2 - f'$. It would be interesting to understand this discrepancy which could be related to the fact that we are mapping a closed string theory onto an open one. We hope to address this problem elsewhere.

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APPENDIX

The Boussinesque hierarchy is the one associated with:

\[ Q = d^3 + \frac{3}{4}\{u_2, d\} + u_3 \]  \hspace{1cm} (A.1)

where \( d \equiv \partial_x \)

The corresponding flows are:

\[ \alpha_i \frac{\partial}{\partial t_{l,k}} u_i = D_{ij}^l R_{l,k+1}^j = D_{ij}^l R_{l,k}^j \]  \hspace{1cm} (A.2)

where \( i, j = 2, 3 \), \( \alpha_2 = \frac{3}{2} \), \( \alpha_3 = 1 \) and \( D_1 \) (resp. \( D_2 \)) is associated with the first (resp. second) Hamiltonian structure:

\[
\begin{align*}
D_1^{22} &= D_1^{33} = 0 & D_1^{32} &= D_1^{23} = d \\
D_2^{22} &= \frac{2}{3}d^3 + u_2 d + \frac{1}{2} u_2' \\
D_2^{23} &= u_3 d + \frac{2}{3} u_3' \\
D_2^{32} &= u_3 d + \frac{1}{3} u_3'
\end{align*}
\]

\[
D_3^{22} = \frac{1}{18}d^5 - \frac{5}{12} u_2 d^3 - \frac{5}{8} u_2' d^2 - \left( \frac{1}{2} u_2^2 + \frac{3}{8} u_2'' \right) d - \left( \frac{1}{2} u_2' u_2 + \frac{1}{12} u_2''' \right)
\]

And the first \( R \) are:

\[
R_{1,0}^2 = 3 \hspace{1cm} R_{2,0}^2 = 0 \hspace{1cm} R_{1,0}^3 = 0 \hspace{1cm} R_{2,0}^3 = 3
\]
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