Jordan form, parabolicity and other features of change of type transition for hydrodynamic type systems

B G Konopelchenko\textsuperscript{1} and G Ortenzi\textsuperscript{2}

\textsuperscript{1} Dipartimento di Matematica e Fisica ‘Ennio de Giorgi’, Università del Salento INFN, Sezione di Lecce, 73100 Lecce, Italy
\textsuperscript{2} Dipartimento di Matematica Pura e Applicazioni, Università di Milano Bicocca, 20125 Milano, Italy

E-mail: konopel@le.infn.it and giovanni.ortenzi@unimib.it

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Abstract
Changes of type transitions for two-component hydrodynamic type systems are discussed. It is shown that these systems generically assume the Jordan form (with $2 \times 2$ Jordan block) on the transition line with hodograph equations becoming parabolic. Conditions which allow or forbid the transition from the hyperbolic domain to elliptic one are discussed. Hamiltonian systems and their special subclasses and equations, such as dispersionless nonlinear Schrödinger, dispersionless Boussinesq, one-dimensional isentropic gas dynamics equations, and nonlinear wave equations are studied. Numerical results concerning the crossing of transition line for the dispersionless Boussinesq equation are also presented.

Keywords: hyperbolic-elliptic transition, parabolic systems, Hamiltonian hydrodynamic systems

(Some figures may appear in colour only in the online journal)

1. Introduction

Differential equations and systems of mixed types have always attracted great interest due to the presence of both hyperbolic and elliptic regimes, the possibility of transition from one to the other, and the numerous interpretations of such transitions in various fields of mathematics, physics and applied science (see e.g. [1–7]).

The study of the properties of the systems of quasi-linear equations near the transition line (also referred to as sonic line, parabolic line or hyperbolic-elliptic boundary) is of particular interest due to the connection with the problem of nonlinear stability of systems of mixed
The results obtained in the recent papers [8–10] contributed significantly to understanding and clarifying the situation. On the other hand, the calculations made, for instance, in [10] are based on the assumption that the matrix $V$ for the hydrodynamic type systems

$$\tilde{u}_t = V(\tilde{u})\tilde{u}_x,$$

‘is degenerated yet diagonalizable’ on the sonic line [10].

However, there are number of systems for which this is not the case. The simplest example is provided by the well-known one-layer Benney system (or dispersionless nonlinear Schrödinger equation (dNLS))

$$u_t = uu_x + v_x,$$
$$v_t = vv_x + uv_x.$$  \tag{2}

Characteristic speeds for (2) are $\lambda_{\pm} = u \pm \sqrt{v}$ and the transition line is given by equation $v(x, t) = 0$. On the transition line the matrix $V$ takes the form

$$V_0 = \begin{pmatrix} u & 1 \\ 0 & u \end{pmatrix} \tag{3}$$

which is obviously non-diagonalizable. Another example is provided by the dispersionless Boussinesq (dB) equation

$$u_{tt} = \frac{1}{2}(u^2)_{xx}.$$  \tag{4}

or the system

$$u_t = v_x,$$
$$v_t = uu_x.$$  \tag{5}

In this case the matrix $V$ on the transition line is

$$V_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$  \tag{6}

i.e. the Jordan block with zero eigenvalue.

The appearance of Jordan blocks in the these examples is a clear manifestation of the generic structure for the hydrodynamics type systems on the transition line.

These two systems represent two different classes of hydrodynamic systems of mixed type. For dNLS equation (2) the hyperbolic domain ($v > 0$) is separated from the elliptic one ($v < 0$). For the dB equation (4) the transition from the hyperbolic domain ($u > 0$) to the elliptic domain ($u < 0$) is allowed.

In the present paper these phenomenona are studied for the two-component systems (1) of mixed type. It is shown that generically a two-component system (1) at the transition line is of Jordan form. Hodograph equations are manifestly parabolic on the transition line. This parabolic regime separates the hyperbolic domain describing wave propagation and the elliptic domain containing quasi-conformal mapping. Conditions under which solutions of the two-component system (1) may belong to both the hyperbolic and elliptic domain or avoid the crossing of the transition line, are discussed.

Hamiltonian systems are considered in detail as illustrative examples. It is shown that the presence of the Jordan block on the transition line is a typical behavior of Hamiltonian systems.

It is also shown that in the generic case the characteristics in the $(u, v)$ plane (simple waves) have universal behavior.
near the point \((u_0, v_0)\) of contact with the transition line. The dB equation is characteristically representative of such a behavior. Particular classes of Hamiltonian systems, including gasdynamics equations and nonlinear wave equations, are considered.

Numerical results for the dB equation showing the particularities of crossing of the transition line are presented too.

The paper is organized as follows. Some basic well-known results for the \(2 \times 2\) system (8), including hodograph equations, are given in section 2. Behavior of the system (8) on the transition line, its Jordan form, and parabolic character are considered in section 3. Section 4 shows the behavior of the system near the transition line from the elliptic side. Necessary and sufficient conditions which allow or forbid the crossing of the transition line are discussed in section 5. These results applied to general Hamiltonian systems are presented in section 6. Special classes of Hamiltonian systems and, in particular, equations of motion for isentropic gas equations and nonlinear wave equations are considered in sections 7 and 8. Some numerical results for the dB equation near to the transition line are presented in section 9.

2. General formulae

We will consider two component quasi-linear system of mixed type of first order

\[
\begin{pmatrix}
u \\ v_t \\
\end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} u \\ v_t \\
\end{pmatrix}
\]

where \(A, B, C, D\) are certain real functions of \(u\) and \(v\) and the subscripts denote derivatives. For convenience we will recall here some basic known facts (see e.g. [1, 11, 12]). Generically the matrix

\[
V = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

has two distinct eigenvalues given by

\[
\lambda_{\pm} = \frac{A + D \pm \sqrt{(A - D)^2 + 4BC}}{2}.
\]

If \(\Omega \equiv (A - D)^2 + 4BC > 0\) the system is hyperbolic, while at \(\Omega < 0\) it is elliptic. In this paper we will assume that \(\Omega(u, v)\) is a smooth function of \(u\) and \(v\). So, the hyperbolic and elliptic domains are separated by the transition line give by the equation

\[
\Omega(u(x, t), v(x, t)) = 0.
\]

Classical hodograph equation for the system (8) is

\[
\begin{pmatrix} x_u \\ x_v \\
\end{pmatrix} = \begin{pmatrix} -D & C \\ B & -A \end{pmatrix} \begin{pmatrix} t_u \\ t_v \\
\end{pmatrix}.
\]

As a consequence, the variables \(t\) and \(x\) obey the second order equations

\[
Ct_{vv} + 2A - D - Bt_{tu} - (B_u + D_v)t_u + (A_u + C_v)t_v = 0
\]

and
\[
\left( \frac{Ax_u + Cx_v}{AD - BC} \right)_v - \left( \frac{Bx_u + Dx_v}{AD - BC} \right)_u = 0. \tag{14}
\]

If the system (8) has a conservation equation \( Q_t = P_x \) then \( Q \) obeys the equation
\[
CQ_{vv} + (A - D)Q_{uv} - BQ_{uu} - (B_u - A_v)Q_u + (-D_u + C_v)Q_v = 0. \tag{15}
\]
In the hyperbolic domain there are two real Riemann invariants \( r_+ \) and \( r_- \) such that the system (8) is equivalent to

\[
r_{\pm t} = \lambda_{\pm} r_{\pm x} \tag{16}
\]
with two distinct characteristic speeds \( \lambda_+ \) and \( \lambda_- \). In the elliptic domain \( \lambda_+ \) and \( \lambda_- \) are complex-conjugate to each-other and one has the single complex equation

\[
r_{+ t} = \lambda_+ r_{+ x} \tag{17}
\]
with \( r_- \) being the complex conjugate to \( r_+ \). The Riemann invariants obey the system

\[
(A - \lambda_{\pm})r_{\pm u} + Cr_{\pm v} = 0, \\
B r_{\pm u} + (D - \lambda_{\pm}) r_{\pm v} = 0. \tag{18}
\]
Only two among these equations are independent, say

\[
B r_{\pm u} + (D - \lambda_{\pm}) r_{\pm v} = 0 \tag{19}
\]
or

\[
r_{\pm u} = \frac{A - D \pm \sqrt{\Omega}}{2B} r_{\pm v}. \tag{20}
\]

Equation (20) are the diagonal form of the hodograph equation (12) rewritten as

\[
\begin{pmatrix} tu \\ x_u \end{pmatrix} = \frac{1}{B} \begin{pmatrix} A & 1 \\ BC - AD & -D \end{pmatrix} \begin{pmatrix} tv \\ x_v \end{pmatrix}. \tag{21}
\]
Indeed the eigenvalues \( \mu_{\pm} \) of the matrix present in (21) are

\[
\mu_{\pm} = \frac{A - D \pm \sqrt{\Omega}}{2B}. \tag{22}
\]

The characteristics for equation (20) are defined by the equation

\[
\left( \frac{dv}{du} \right)_{\pm} = -\mu_{\pm}. \tag{23}
\]

The Riemann invariants are constants along these characteristics \( v_{\pm} \) in the hodograph space.

Equation (23) are those which define simple waves for the system (8). The simple waves are the hodograph counterpart of the usual characteristics \( \left( \frac{dv}{dt} \right)_{\pm} = -\lambda_{\pm} \) in the space \((x, t)\).

We note also that the components \( y_1 \) and \( y_2 \) of the eigenvector \( y \) of the matrix \( V \) in (9) on the transition line obey the equation

\[
(A - D)y_1 + 2By_2 = 0. \tag{24}
\]

Hydrodynamic type systems and, in particular, the system (8) exhibit one more important phenomenon, the so-called gradient catastrophe, i.e. unboundedness of derivatives of \( u \) and \( v \) at finite \( x \) and \( t \) while \( u(x, t) \) and \( v(x, t) \) remain bounded (see e.g. [11]). Interference of the gradient catastrophe and crossing the transition line is a rather complicated problem. To simplify
the analysis we will assume in the rest of the paper that the solutions of the system (8) avoid gradient catastrophe, at least, before the crossing of the transition line (if so).

3. Transition line and Jordan form

Study of the behavior of systems of mixed type on the transition line and nearby is fundamental for understanding their properties. The system (8) can be of mixed type only when $BC < 0$. In the case $BC \geq 0$ (including symmetric matrices) it is hyperbolic except the degeneration at the set of points defined by the equation $A = D$ and $B = 0$ (or $C = 0$) in generic case or at the line in the case $A = D$ and $B = C = 0$.

It was already shown in the Introduction that on the transition lines the dNLS and dB equations assume the special form with Jordan blocks. These results can be obtained as the limit, performed accurately, of the equation (16) for Riemann invariants when a solution $(u, v)$ approaches the transition line. In the dNLS equation (2) case

$$\lambda_{\pm} = u \pm \sqrt{v}, \quad r_{\pm} = u \pm 2\sqrt{v}$$

and the transition line is given by the equation $v(x, t) = 0$. Equations for the Riemann invariants in terms of $u$ and $v$

$$(u \pm 2\sqrt{v})_t = (u \pm \sqrt{v})(u \pm 2\sqrt{v})_x$$

or

$$u_t \pm \frac{1}{\sqrt{v}} v_t = uu_t \pm \sqrt{v} u_x \pm \frac{1}{\sqrt{v}} uv_x + v_x.$$  

In the limit $v \to 0$ the two leading order terms $v^{-1/2}$ and $\sqrt{v}$ give the system

$$u_t = uu_x + v_x, \quad v_t = uv_x.$$  

This system has matrix $V$ given by (3), that is the Jordan form with $\lambda = u$.

In the dB equation (5) case

$$\lambda_{\pm} = \pm u^{1/2}, \quad r_{\pm} = v \pm \frac{2}{3}u^{3/2},$$

and the transition line is given by the equation $u(x, t) = 0$. So equation (16) in terms of $u$ and $v$

$$(v \pm \frac{2}{3}u^{3/2})_t = \pm u^{1/2}(v \pm \frac{2}{3}u^{3/2})_x$$

or

$$v_t \pm u^{1/2}u_t = \pm u^{1/2}v_x + uu_x.$$  

When $x$ and $t$ approaches the transition line $u(x, t) = 0$ in the leading orders $u^0$ and $u^{1/2}$ one gets

$$v_t = 0, \quad u_t = v_x,$$

i.e. the Jordan form with $\lambda = 0$.

In the general case the matrix $V$ for the system (8) apparently is not of the Jordan block form on the transition line $\Omega$ (11). It can be parameterized at $C \neq 0$ as
where \( \lambda = \lambda_+ = \lambda_- = (A + D)/2 \). Such a matrix has the form

\[
V_0 = \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + N
\]

(34)

where \( N \) is the general \( 2 \times 2 \) nilpotent matrix.

It is straightforward to check that there exists a two parameter family of invertible matrices \( P \) such that

\[
PV_0P^{-1} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}
\]

(35)

The family \( P \) at \( C \neq 0 \) is given by

\[
P = a \begin{pmatrix} -b & 1 + \sqrt{-BC}b \\ C & -\sqrt{-BC} \end{pmatrix}
\]

(36)

where \( a = a(u, v) \) and \( b = b(u, v) \) are arbitrary functions. In the case of \( C = 0 \) and \( B \neq 0 \) the matrix \( P \) becomes

\[
P = a \begin{pmatrix} 1 & b \\ 0 & B \end{pmatrix}
\]

(37)

where \( a = a(u, v) \) and \( b = b(u, v) \) are still arbitrary functions. Thus in all non diagonal cases the matrix \( V_0 \) is equivalent to a Jordan block, and the system (8) is equivalent to

\[
P \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} P \begin{pmatrix} u_x \\ v_x \end{pmatrix}
\]

(38)

In our construction the systems in the Jordan forms or the system (38) arise on the transition line only. In the paper [13] the system (8) with matrix \( V \) given by (3) on the whole plane \((x, t)\) and its multi-component analogs with Jordan blocks has been derived via the confluence process for the Lauricella-type functions associated with Grassmannians \( \text{Gr}(2, 5) \) and \( \text{Gr}(2, n) \).

Let us consider the system (8) with the matrix \( V = V_0 \) given by (33). It is parabolic on the plane \((x, t)\). In this case there are variables \( u^*, v^* \) such the system (38) takes the form

\[
\begin{pmatrix} u^*_t \\ v^*_t \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} u^*_x \\ v^*_x \end{pmatrix}
\]

(39)

which will be referred to as the Jordan form. The relation between the Jordan variables \( u^*, v^* \) and the original ones \( u, v \) can be found by solving the equations

\[
\begin{pmatrix} u^*_t \\ v^*_t \end{pmatrix} = P \begin{pmatrix} u_t \\ v_t \end{pmatrix}, \quad \begin{pmatrix} u^*_x \\ v^*_x \end{pmatrix} = P \begin{pmatrix} u_x \\ v_x \end{pmatrix}
\]

(40)

These equations can be solved only if one finds the suitable integrating factors \( a \) and \( b \) which must be chosen such that

\[
da^* = ab \, du + a \left( 1 + \frac{\sqrt{-BC}}{C} b \right) \, dv, \quad dv^* = -a \, du - a\sqrt{-BC} \, dv,
\]

(41)
which fix $a$ and $b$ thanks to the compatibility conditions

$$
(ab)_v = \left( a + \frac{a\sqrt{-BC}}{C} b \right)_u, \quad a_v = (a\sqrt{-BC})_u.
$$

(42)

In case when $C = 0$ the previous relations become

$$
du^* = a\, du + ab\, dv, \quad dv^* = aB\, dv,
$$

(43)

and

$$
a_v = (ab)_u, \quad (aB)_u = 0.
$$

(44)

Therefore the Jordan variables in parabolic systems play a role similar to the Riemann invariants for the standard diagonalizable case.

### 4. Elliptic domain

At the hyperbolic domain solution of the system (8) describe wave motions. Properties of solutions of the system (8) in the elliptic domain are quite different. Their treatment as the function defining quasi-conformal mappings is one of the possible interpretations [14]. Indeed equation (16) can be rewritten as [15]

$$
r_z = \frac{1 + i\lambda}{1 - i\lambda} z,
$$

(45)

where $\lambda = \lambda_+ = (A + D + \sqrt{\Omega})/2$, $r = r_+$, $z = x + it$ and the overline stands for complex conjugate. Such equations are known as Beltrami nonlinear equations [16, 17]. A solution of (45) defines a quasi-conformal mapping $r : (z, \bar{z}) \to (r, \bar{r})$ if the complex dilation $\mu = \frac{1 + i\lambda}{1 - i\lambda}$ obeys the condition

$$
|\mu| = \left| \frac{1 + i\lambda}{1 - i\lambda} \right| < 1.
$$

(46)

Using the explicit form of $\lambda$, in the region $\Omega < 0$ we have

$$
\mu = \frac{2 + i(A + D) - \sqrt{-\Omega}}{2 - i(A + D) + \sqrt{-\Omega}}.
$$

(47)

It is easy to see that condition (46) is always satisfied when $\Omega < 0$. So any solution of the system (8) in the elliptic domain defines quasi-conformal mapping.

At the transition line $\Omega = 0$

$$
|\mu| = \left| \frac{2 + i(A + D)}{2 - i(A + D)} \right| = 1
$$

(48)

and the quasi conformal mappings degenerates. For instance it maps the unit circle in the plane $(z, \bar{z})$ in the degenerate ellipse in the plane $(r, \bar{r})$ with ratio of major and minor axes going to infinity (see e.g. [16]).

In the hodograph space, equation (20) are equivalent to the linear Beltrami equation

$$
r_\pi = \frac{1 + i\lambda(w, \bar{w})}{1 - i\lambda(w, \bar{w})} r_w.
$$

(49)
where $w = v + iu$ and $\tilde{\lambda} = \frac{A - D + i\sqrt{\Omega}}{2B}$. Similar to the calculations presented before, one shows that the condition (46) is always satisfied in the elliptic domain and so any solution of the equation (49) defines a quasi-conformal mapping $(w, \overline{w}) \to (r, \overline{r})$. On the transition line $\Omega = 0$ again $\frac{1 + i\tilde{\lambda}}{1 - i\tilde{\lambda}} = 1$ and quasi-conformal mappings become singular.

So, both in the elliptic and hyperbolic domains solutions of the mixed system (8) exhibit particular behavior when they approach the transition line $\Omega = 0$. Namely, approaching the transition line from the hyperbolic side, waves become unstable (see e.g. [18]) converting into a diffusion-type process, governed by parabolic equations and transforming into quasi-conformal mappings dynamics beyond the transition line. Approaching the transition line from the elliptic side, the quasi-conformal mappings degenerate into singular ones with $|\mu| = 1$ which maps the two-dimensional domains in $\mathbb{C}$ into quasi one-dimensional ones. Beyond the transition line these quasi one-dimensional objects are transformed into moving waves.

5. Transition line and its crossing

Since Riemann invariants are constants along characteristics (real or complex) the problem of whether or not a transition line can be crossed is reduced to the study of the respective properties of characteristics and transition lines (see e.g. [8–10]). Comparison of the formulae for characteristics and transition lines in the original variables $(t, x)$ and hodograph variables $(u, v)$ (see e.g. formulae (11) and (23) clearly indicates that the latter are more appropriate for our purpose. The use of simple waves in [9] provides us further support for such an observation.

In the hodograph space $(u, v)$ the characteristics and transition lines are given by formula (23) and (11) respectively. Let us begins with the hyperbolic domain and let us assume that the derivatives involved are bounded. Thus we have two families of plane characteristic lines (ChL) in the hodograph space $(B \neq 0)$

$$\text{ChL : } \pm \left( \frac{dv}{du} \right) = -\frac{A - D \pm \sqrt{\Omega}}{2B},$$

and a single transition line (TL)

$$\text{TL : } \Omega(u, v) = 0.$$

The two simplest cases are: (1) the two families (50) do not have common points with (51) and (2) they coincide at least on some interval. In the latter case, on the transition line one has the equations

$$\frac{dv}{du} = \frac{D - A}{2B}, \quad \Omega = 0,$$

which should be equivalent to each other. Since on the transition line

$$d\Omega = \Omega_u du + \Omega_v dv = 0,$$

the necessary condition for this is given by (if $\Omega_v \neq 0$)

$$\frac{D - A}{2B} + \frac{\Omega_u}{\Omega_v} = 0.$$

Obviously in both cases the transition from the hyperbolic domain to the elliptic one is impossible.

Another simple case corresponds to the transversal intersections of ChL and TL. To derive the corresponding condition it is sufficient to consider these lines at points of intersection. Two
characteristics touch each other and at the point on TL their tangents are (assuming that both curves are smooth)

\[ \frac{dv}{du} \bigg|_{\text{ChL}} = \frac{D - A}{2B}. \tag{55} \]

The tangent to the TL at the same point is given by (at \( \Omega_v \neq 0 \))

\[ \frac{dv}{du} \bigg|_{\text{TL}} = -\frac{\Omega_u}{\Omega_v} \bigg|_{\text{TL}}. \tag{56} \]

The characteristic and transition line cross transversally (with angle \( \neq 0 \)) if

\[ \frac{dv}{du} \bigg|_{\text{ChL}} \neq \frac{dv}{du} \bigg|_{\text{TL}}, \tag{57} \]

i.e.

\[ \left( \frac{D - A}{2B} + \frac{\Omega_u}{\Omega_v} \right) \bigg|_{\text{TL}} \neq 0. \tag{58} \]

Thus, if condition (58) is satisfied, the transition from the hyperbolic domain to the elliptic one is not forbidden.

There are eight other possibilities. The first four are given by the figure 1 and its reflections on each of the two curves with respect to the straight line of the common tangent at the point \((u_0, v_0)\). In these four cases characteristic lines touch the transition lines at the point \((u_0, v_0)\) and then turn back to the hyperbolic domain. So the transition is forbidden. At the point \((u_0, v_0)\) the tangents of both sides coincide and so

\[ \left( \frac{dv}{du} \bigg|_{\text{ChL}} \right)_{(u_0,v_0)} = \left( \frac{dv}{du} \bigg|_{\text{TL}} \right)_{(u_0,v_0)} = \left( \frac{D - A}{2B} + \frac{\Omega_u}{\Omega_v} \right)_{(u_0,v_0)} = 0. \tag{59} \]

The fact of non-crossing is invariant under the transformation of coordinates. Choosing the coordinates \((u, v)\) near to the point \((u_0, v_0)\) in such a way that the axes \( v = 0 \) coincide with the common tangent, it is not difficult to show in all four cases that the difference

\[ \Delta T \equiv \left( \frac{dv}{du} \bigg|_{\text{ChL}} \right) - \left( \frac{dv}{du} \bigg|_{\text{TL}} \right) \]

changes sign passing the point \((u_0, v_0)\).

The second four cases are given by figure 2 and three other possible cases are obtained by reflection of each of two curves with respect to the straight lines of the common tangent at the point \((u_0, v_0)\). In these cases the characteristic line touches the transition line at the point \((u_0, v_0)\) and then passes into the elliptic domain with characteristic speeds becoming complex. Thus, the transition is not forbidden.

Thus in the eight cases considered above the behavior of \(\Delta T\) near the point of touch \((u_0, v_0)\) distinguishes the cases of crossing and non-crossing. So we conclude that the transition from the hyperbolic to the elliptic domain is not possible if either \(\Delta T|_{\Omega = 0} = 0\) on some interval of the transition line or \(\Delta T|_{\Omega(u_0,v_0)=0} = 0\) at some point on TL and \(\Delta T\) changes sign passing from one side of the touch point \((u_0, v_0)\) to another. In particular, the comparison of the condition (59) rewritten as

\[ \left( (D - A)\Omega_v + 2B\Omega_u \right) \bigg|_{(u_0,v_0)} = 0 \tag{61} \]

and the relation (24) shows that the eigenvector of the matrix \(V\) corresponding to the double eigenvalue \(\lambda\) is tangent to the transition line at the point \((u_0, v_0)\), in agreement with the
necessary condition of non-crossing (nonlinear stability) proposed in [8]. On the other hand if
\[ \Delta T_{|\Omega} \neq 0 \text{ or } \Delta T_{|(u_0,v_0)} = 0 \]
and \( \Delta T \) does not change sign at the touching point \( (u_0,v_0) \) the conditions of non-crossing are not satisfied and the transition from the hyperbolic domain to the elliptic one is not forbidden.

In the analysis presented above it was assumed that all derivatives including \( \frac{d^2 v}{du^2} \) are bounded. The cases of possible unboundedness require special consideration. To clarify the point let us consider the system
\[
\begin{pmatrix}
  u_t \\
  v_t
\end{pmatrix} = \begin{pmatrix}
  0 & 1 \\
  f(v) & 0
\end{pmatrix} \begin{pmatrix}
  u_x \\
  v_x
\end{pmatrix}
\]  

(62)

where \( f(v) = v^\alpha \) and \( \alpha = 2n + 1 \) or \( \alpha = 1/(2n + 1) \) with \( n = 0, 1, 2, 3, \ldots \). In this case the transition line is given by \( v = 0 \) and the characteristics in the hodograph space are defined by the equation
\[
\frac{dv}{du} = \mp v^{\alpha/2}.
\]  

(63)
So, the characteristics are given by the lines

\[ u \pm \frac{2}{2-\alpha} \sqrt{\frac{2-\alpha}{2}} u^{(2-\alpha)/2} = u_0 = \text{const}, \quad v \neq 0 \]

with arbitrary \( u_0 \) and by the straight line \( v = 0 \). The transition line \( v = 0 \) is then (degenerate) characteristic. The behavior of other characteristics is quite different for \( \alpha > 2 \) and \( 0 < \alpha < 2 \).

In the case \( \alpha = 2n + 1 \) and \( n \geq 1 \), \( v(u) \) has a singularity at \( u = u_0 \) and it may touch the transition line only at the infinity \( u \to \infty \). So the transition from hyperbolic to elliptic domain is not possible. For \( \alpha = \frac{1}{2n+1} \), the characteristics (64) touch the transition line at finite point \( (u_0, 0) \) where \( \frac{dv}{du} \bigg|_{u=u_0} = 0 \).

Since

\[ \frac{d^2v}{du^2} = \left( \frac{2 - \alpha}{2} (u - u_0) \right)^\frac{2(\alpha-1)}{2(\alpha+1)} = \frac{\alpha}{2} v^{\alpha-1} \]

the characteristics have completely different behavior in the cases \( \alpha = 1 \) and \( \alpha < 1 \). For \( \alpha = 1 \) one has

\[ \frac{d^2v}{du^2} \bigg|_{(u_0, 0), \alpha=1} = \text{const}. \]

So characteristics smoothly approach the transition line and, hence, the transition is not possible. Note that at \( \alpha = 1 \) the system (62) represents the dispersionless Toda chain

\[ (\log v)_t = v_x. \]

In contrast, for \( \alpha < 1 \) \((n = 1, 2, 3, \ldots)\) the normal to the characteristic, i.e. velocity \( \frac{dx}{dt} \) with which the characteristic approaches (in normal direction) the transition line \( v = 0 \) grows to infinity (at the point \( (u_0, 0) \)). Such a behavior allows us to suggest that the characteristics may jump across the line \( v = 0 \) and transition from the hyperbolic domain to the elliptic one \( (v < 0) \) could be possible. The numerical results of the system (62) with \( f = v^{1/3} \ (\alpha = 1/3) \) presented in the paper [8] support this observation. Another indication that the system (62) with \( f = v^\alpha, \alpha < 1 \) has rather special properties is provided by equations (13) and (14), i.e.

\[ v^\alpha t_u - t_{uu} + \alpha v^{\alpha-1} t_v = 0 \]

and similar equation for \( x \). At the transition line \( v = 0 \) they are singular equations of parabolic type.

Analysis of the systems which have properties similar to those of the system (62) with \( f = v^\alpha, \alpha < 1 \) requires a separate study which will be performed elsewhere. The possibility of transition from the elliptic to hyperbolic domain, corresponding conditions and associated quasi-conformal mapping are of interest too. These problems will be considered in a separate publication.

6. Hamiltonian systems

In this and the next sections we will consider some concrete classes of equation (8). We begin with the Hamiltonian systems which for the two component case can always be locally put in the form [19, 20]

\[ \begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} 0 & \partial_t \alpha \\ \partial_x \alpha & 0 \end{pmatrix} \begin{pmatrix} u_x \\ h_t \end{pmatrix} \]

(69)
with Hamiltonian \( H = \int h(u, v) \, dx \). So, the system (8) is
\[
\begin{pmatrix}
u_t \\
v_x
\end{pmatrix} = \begin{pmatrix} h_{uv} & h_{uv} \\ h_{uu} & h_{vv} \end{pmatrix} \begin{pmatrix} u_t \\
v_x
\end{pmatrix}.
\]
(70)
In this case \( \Omega = 4h_{uu}h_{vv} \). Equation (13) is
\[
h_{uu}v_t - h_{vv}u_t - 2h_{uvv}u_t + 2h_{uvv}v_t = 0,
\]
(71)
while characteristics in the hodograph space (simple waves) are defined by the equation
\[
\left( \frac{dv}{du} \right)_\pm = \mp \sqrt{\frac{h_{uu}}{h_{vv}}}.
\]
(72)
The transition line is given by
\[
h_{uu}h_{vv} = 0,
\]
(73)
assuming that \( h_{uu}h_{vv} \) may change sign.

In order to deal with the generic non-diagonalizable case we defined the transition line as
\[
h_{uu} = 0, \quad h_{vv}|_{h_{uu}=0} \neq 0.
\]
(74)
On the transition line the matrix \( V \) is equivalent to the Jordan block
\[
\begin{pmatrix}
\lambda & 1 \\
0 & \lambda
\end{pmatrix}
\]
with \( \lambda = h_{uv}|_{h_{uu}=0} \).

The transformation matrix \( P \) is given by (37) with \( B = h_{vv}|_{h_{uu}=0} \). We note that at points where \( h_{vv} = 0 \) and \( h_{uu} = 0 \) on the transition line the matrix \( V \) is degenerated to a constant diagonal matrix.

When it holds (74) one also has
\[
\Delta T = \left. \frac{h_{uu}}{h_{vv}|_{h_{uu}=0}} \right|_{h_{uu}=0}.
\]
(75)
Thus in the generic case with \( h_{uu}|_{h_{uu}=0} \neq 0 \) we have
\[
\Delta T \neq 0
\]
(76)
and transition from the hyperbolic domain \( h_{uu}h_{vv} > 0 \) to the elliptic one is not forbidden.

Both dNLS and dB equations are Hamiltonian ones (see e.g. [11, 19, 20] ) with respectively
\[
h_{\text{dNLS}} = \frac{1}{2} v^2 + \frac{1}{2} u^2,
\]
(77)
and
\[
h_{\text{dB}} = \frac{1}{2} v^2 + \frac{1}{6} u^3.
\]
(78)
Now let us study the behavior of characteristics in the \((u, v)\) plane (simple waves) near the point \((u_0, v_0)\) of contact with the transition line \( h_{uu} = 0 \) and \( h_{vv} \neq 0 \) for general Hamiltonian systems (70).

Expanding the right hand side of (72) near the point \((u_0, v_0)\) and assuming that \( h_{uuu} \neq 0 \) and the derivatives involved are finite, one obtains
\[
\left( \frac{dv}{du} \right)_\pm = \pm \sqrt{a(u - u_0) + b(v - v_0)}, \quad \text{with} \quad a = \frac{h_{uuu}^0}{h_{vv}^0} \neq 0, \quad b = \frac{h_{uuv}^0}{h_{vv}^0},
\]
(79)
where \( f^0 \equiv f(u_0, v_0) \). For infinitesimal \( \delta u = u - u_0 \) and \( \delta v = v - v_0 \) equation (79) takes the form
\[
\left( \frac{\delta v}{\delta u} \right)^2 = a \delta u + b \delta v.
\] (80)

Hence
\[
\delta v = \frac{1}{2} b (\delta u)^2 \pm \sqrt{\frac{b^4}{2} (\delta u)^4 + a (\delta u)^3}.
\] (81)

So at \( \delta u \to 0, \delta v \to 0 \) one has at the leading order
\[
v - v_0 \simeq \pm \sqrt{\frac{h_0}{h_{vv}} u_0 (u - u_0)^{3/2}}.
\] (82)

This formula gives us the universal behavior of \((u, v)\) characteristics near the transition line for general Hamiltonian system (70) in the generic case \(h_0^{uuv} \neq 0\). The simplest and characteristic example of such a behavior is provided by the dB equation for which \(h_0^{uuv} = 1\). The fact that for the general stationary plane motion of compressible gas (described by Chaplygin equation), the behavior of characteristics near sonic line given by the formula (82), has been known for a long time (see e.g. [12], §118) should also be noted.

In particular cases the behavior of characteristics near to the transition line can be quite different. If at the transition point \((u_0, v_0)\) also \(h_0^{uuv} = 0\), then, instead of equation (80) one has
\[
\left( \frac{\delta v}{\delta u} \right)^2 = b \delta v + c (\delta u)^2 + d \delta u \delta v + f (\delta v)^2
\] (83)

where \(c, d\) and \(f\) are certain constants depending on \(h\) and its derivatives evaluated at \((u_0, v_0)\) given by
\[
b = \frac{h_0^{uuv}}{h_{vv}}, \quad c = \frac{h_0^{uuv}}{2h_{vv}}, \quad d = \frac{h_0^{vuv} h_0^{uuv} - h_0^{uuv} h_0^{uvu}}{(h_{vv})^2}, \quad f = \frac{h_0^{uvu} h_0^{uuv} - 2h_0^{vuv} h_0^{uuv}}{2(h_{vv})^2}. \] (84)

Solving this equation and considering the limit of infinitesimal \(\delta u\) and \(\delta v\) one obtains
\[
v - v_0 \simeq \sqrt{\frac{h_0^{uuv}}{2h_{vv}} (u - u_0)^2},
\] (85)

and in this case
\[
\Delta T \simeq (u - u_0).
\] (86)

So \(\Delta T|_{u_0v_0} = 0\) and \(\Delta T\) changes sign at the point \(u_0\) and, hence, transition is forbidden.

In a similar way one can show that in the case when the derivative
\[
\left. \frac{\partial^k h}{\partial \epsilon^k} \right|_{u=0} = 0 \quad \text{for} \quad k = 3, 4, 5, \ldots, n
\] (87)

the behavior of near the transition line is of the type
\[
v - v_0 \simeq (u - u_0)^{(n+1)/2}.
\] (88)

Again \(\Delta T|_{u_0v_0} = 0\) and since
\[
\Delta T \simeq (u - u_0)^{(n-1)/2},
\] (89)

the transition is not allowed in the cases when \(n = 4m + 3\) for \(m = 0, 1, 2, \ldots\), while it is not forbidden in other cases.
Finally if the transition line is given by the equation \( h_{vv} = 0 \) with \( h_{uv} \big|_{v=0} \neq 0 \) then one has the results presented above with exchange \( u \leftrightarrow v \).

For the dNLS equation the transition line \( v = 0 \) is a characteristic and \( \frac{dv}{du} \big|_{v=0} = 0 \). Hence, the transition is not possible. We remark that this case is in some sense degenerate because the Hamiltonian is quadratic in \( u \) and all the partial derivatives of \( \frac{\partial^k h}{\partial u^k} = 0, \ k \geq 3 \) are zero. For the dB equation, in contrast, the transition line is \( u = 0 \), and the characteristic crosses the transition line orthogonally \( \frac{dv}{du} \big|_{u=0} = 0 \) and \( h_{uv} = 0 \). Therefore

\[
\Delta T = \frac{h_{uuu}}{h_{uv}} \big|_{u=0} \to \infty, \tag{90}
\]

and consequently the transition from the hyperbolic domain to elliptic one is not forbidden.

7. Special classes of Hamiltonian systems: gas dynamics equations

Expressions (77) and (78) suggest us to consider two special classes of systems with Hamiltonian densities

\[
h_1 = F_1(v) + \frac{1}{2}vu^2,
\]

\[
h_2 = F_2(v) + F_3(u), \tag{91}
\]

where \( F_1, F_2, F_3 \) are functions of a single variable. For the Hamiltonian density of the form \( h_1 \) one has the system

\[
\begin{pmatrix}
u_t \\
u_v \\

\end{pmatrix} = \begin{pmatrix} u & F_{1vv} \\
v & u \\

\end{pmatrix} \begin{pmatrix} u_t \\
v_t \\

\end{pmatrix}, \tag{92}
\]

and \( \Omega = 4vF_{1vv} \). Simple waves are defined by equation

\[
\sqrt{\frac{F_{1vv}}{v}} dv = \pm du, \tag{93}
\]

The relations \( h_{1uuu} = 0 \) and \( h_{1uvv} = 1 \) imply \( \Delta T = 0 \).

There are two quite different situations. The first corresponds to the case when the transition line is given by \( v = 0 \) and \( F_{1vv} > 0 \) near \( v = 0 \). For the system of mixed type, one has

\[
F_1 = \frac{1}{2n(n-1)} v^{2n}, \quad n = 1, 2, 3, \ldots, \tag{94}
\]

the characteristic in \((u, v)\)-plane are given by the equation

\[
\frac{dv}{du} = \pm \sqrt{2n(2n-1)} v^{\frac{1-2n}{2n-1}} \tag{95}
\]

and so

\[
\frac{d^2v}{du^2} \sim v^{2(1-n)}. \tag{96}
\]

Thus if \( n \geq 2 \) the acceleration \( \frac{d^2v}{du^2} \) diverges as the characteristic approaches the transition line \( v = 0 \) (which is also a characteristic) and, hence, the transition is not forbidden.
If \( v \neq 0 \) everywhere the properties of the system (92) are quite different. Introducing the function \( P(v) \) defined by the equation
\[
F_{1v} = \frac{P'(v)}{v},
\]
one rewrites the system (92) as
\[
\begin{align*}
  u_t &= u u_x + \frac{P_x}{\rho}, \\
  \rho_t &= (u\rho)_x.
\end{align*}
\]
(98)

It is the general isentropic one-dimensional gas-dynamic equation with \( u \) being velocity, \( v \) being density \( \rho \), \( P(x) \) is the pressure and \( t \rightarrow -t \) (see e.g. [11, 12]). The TL-line is defined by
\[
\Omega = 4\rho F_{1v} = 4P'(\rho) = 0
\]
(99)
and the characteristics in the space \((u, v)\) are defined by the equation
\[
\frac{\rho}{u} = \pm \sqrt{\frac{\rho^2}{P'(\rho)}}.
\]
(100)
For ordinary media
\[
\frac{dP}{d\rho} \bigg|_S = c^2,
\]
(101)
i.e. squared sound speed \( c \) in the medium. Thus, for normal cases the system is hyperbolic everywhere. So the system (98) is of mixed type for particular macroscopic systems for which the derivative \( \frac{dP}{d\rho} \bigg|_S \) can vanish at some value of density \( \rho_0 \) (zero sound speed point) and change sign passing through this value (see e.g. [21–25]).

Such a situation is realized, for instance, for the functions \( P \) which for small \( \rho - \rho_0 \) are of the form
\[
\begin{align*}
  (a) & : P \sim (\rho - \rho_0)^{2n+2}, \\
  (b) & : P \sim (\rho - \rho_0)^{\frac{2n+2}{2n+1}},
\end{align*}
\]
where \( n = 0, \pm 1, \pm 2, \ldots \). Near the transition line \( \rho = \rho_0 \) one has
\[
\frac{d^2 \rho}{d\rho^2} \sim (\rho - \rho_0)^{-(1+\gamma)}
\]
(103)
where \( \gamma = 2n + 2 \), or \( \gamma = \frac{2n+2}{2n+1} \). In both cases \( \frac{d^2 \rho}{d\rho^2} \rightarrow \infty \) as \( \rho \rightarrow \rho_0 \) and, hence, the transition is not forbidden.

8. Nonlinear wave type equations

The system with Hamiltonian density \( h_2 (91) \) is of the form
\[
\begin{pmatrix}
  u_t \\
  v_t
\end{pmatrix} =
\begin{pmatrix}
  0 & F_2'(v) \\
  F_2'(u) & 0
\end{pmatrix}
\begin{pmatrix}
  u_x \\
  v_x
\end{pmatrix}.
\]
(104)
This system is, in fact, the system of two conservation laws
\[
\begin{align*}
  u_t &= (F_2'(v))_x, \\
  v_t &= (F_2'(u))_x
\end{align*}
\]
(105)
and it is equivalent to the single equation
\[ A(w_x) = B(w_t), \]

where \( w_x = u, \ w_t = F_2'(v) \), \( B(y) = F_4'(y) \) and \( A(y) = (F_2')^{-1}(y) \). In the particular case \( F_2(v) = v^2/2 \) equation (106) takes the form

\[ u_{tt} = (F_3'(u_x))_x \]

or

\[ u_{tt} = (F_3'(u_u))_x, \]

i.e. the standard form of the nonlinear wave equations (see e.g. [20] with \( F_3 = P \)). For the dB equation

\[ F_3^{\text{dB}} = \frac{1}{6}u^3 + \frac{1}{2}c^2u^2. \]

For the system (104) one has

\[ \Omega = 4F_3'(u)F_2''(v) \]

and equations for \((u, v)\) characteristics are given by

\[ \sqrt{F_2''(v)}dv = \pm \sqrt{F_3'(u)}du. \]

First we consider the case when the transition line is given by

\[ F_3'(u_0) = 0, \quad F_2''(v) \neq 0, \]

which includes the nonlinear wave case (108) where \( F_2''(v) = 1 \). For the system of mixed type near the point \( u_0 \) the function \( F_3(u) \) should be of the form

\[ F_3(u) \sim \text{const} (u - u_0)^{2n+1}, \quad n = 1, 2, 3, \ldots. \]

For \((u, v)\) characteristics near the transition line \( u = u_0 \) one has

\[ \begin{align*}
(A(w_x))_x = (B(w_t))_x
\end{align*} \]
So $\frac{dv}{du} \to 0$ as $u \to u_0$ and, hence, the characteristic approaches the transition line orthogonally. Thus the change of type is not forbidden. It is clearly so for nonlinear wave equations with $F_3(u) = u^{2n+1}$, $n = 1, 2, 3, \ldots$. Note that nonlinear wave equations with $F_3(u) = u^{2n}$, $n = 1, 2, 3, \ldots$ are hyperbolic everywhere. The same is valid for the dispersionless Toda equation $u_t = \exp(u)_{xx}$ for which $F_3(u) = \exp(u)$ (see e.g. [20]).
If, instead of (112), the transition line is given by
\[ F''_2(u) \neq 0, \quad F''_3(v_0) = 0, \] (115)
one has the same results with the exchange \( u \leftrightarrow v \).

9. Numerical example of transitions for the dB equation

Here we present some numerical results for the dB equation as the characteristic representative of the generic class of Hamiltonian systems.

Let us consider the class of periodic solutions with fixed boundary values and with initial conditions
\[ (u(x,0) - c)^2 + v^2 = 1, \quad u(x,0) = c + \sin(x) \] (116)
where \( c > 1 \) is assumed in order to start in the hyperbolic sector. The simple waves for dB equation are (see figure 3)

\[
v \pm \frac{2}{3} u^{3/2} = k, \quad k \in \mathbb{R}.
\]  

(117)

For every value of \( c \) there are four simple waves tangent to the circle. The two lower ones satisfy the system

\[
\left( \frac{3}{2} (k \pm v_c) \right)^{2/3} = c - \sqrt{1 - v_c^2},
\]

\[
\frac{1}{(k - v_c)^{1/3}} = \frac{v_c}{\sqrt{1 - v_c^2}}.
\]

(118)

Figure 7. Circle evolution with \( c = 1.4 < c_{\text{crit}} \), \( t \) from zero to \( t = 2 \) with equispaced steps. The curve can cross the transition line \( u = 0 \).

Here \( v_c \) is the value of \( v \) at contact point. For every value of \( c > 1 \) the previous systems admits two solutions corresponding to two different simple waves symmetric with respect to the \( u \) axis. Only above a minimum value of \( c \), called \( c_{\text{crit}} \), the couple of tangent simple wave intersect each other. Therefore only above \( c_{\text{crit}} \) the hyperbolic elliptic transition is forbidden.
because the initial condition (see e.g. [9]) is separated from the transition line by the two tangent simple waves.

The critical value $c_{\text{crit}}$ can be estimated as follows. At $c_{\text{crit}}$ the intersection of the tangent simple waves, because of the problem symmetry, is at the origin $u = 0, v = 0$, i.e. with $k = 0$ in (118). Solving therefore (118) with $k = 0$ we obtain the value of this critical constant (in case (116)) which is $c_{\text{crit}} \simeq 1.7472$.

In figure 4 the possibility of the hyperbolic elliptic transition for different values of the parameter $c$ in the family of initial conditions (116) is shown. The solid circle is the initial data at $c = 3$ which is greater than the critical value. In this case the initial data are bounded from below by two simple waves (two solid open curves) which are tangent to the initial data and intersect each other above the transition line $u = 0$. This behavior prevents the transition. The dashed circle (initial conditions with $c = c_{\text{crit}}$) is the critical conditions in the family which forbid the transition. Actually the two tangent simple waves (dashed open curves) intersect exactly at the transition line. Finally the dotted circle is the initial data at $c = 1.4$ which is below the critical value: the tangent simple waves (two dotted curves) have no intersection and the initial data could reach the transition line. In figures 5 (left) and 6 it is shown the evolution of $u$ in dependence of $(x, t)$ and $v$ respectively in the non transition case of $c = 3 > c_{\text{crit}}$. In figures 5 (right) and 7 is shown the evolution of $u$ in dependence of $(x, t)$ and $v$ respectively in the transition case of $c = 1.4 < c_{\text{crit}}$. At the second to last step the curve is tangent to the transition line. However this line is not a characteristic in the $(u, v)$ space and the curve can cross the transition line as can be seen in the last plot.

Figure 8. Comparison of the evolution of $u$ with a slightly subcritical ($c = 1.74$, gray) and supercritical ($c = 1.76$, black) parameter in the initial conditions (116). Left figure: initial conditions $t = 0$; center figure $t \simeq 2.73$: the last time step where the computer give the evolution of both the initial conditions; right figure $t = 2.9$: later time step for the supercritical case.

Figure 9. A zoom at the transition time $t \simeq 2.73$ of $u$ with a slightly subcritical ($c = 1.74$, gray) and supercritical ($c = 1.76$, black) parameters in the initial conditions (116).
In order to better characterize the behavior near to critical value \( c_{\text{crit}} \approx 1.7472 \) we will numerically compare the evolutions of the \( u \) field with a slightly subcritical \( (c = 1.74) \) and supercritical \( (c = 1.76) \) parameter. As one can see in figure 8 the motion with these initial data is almost indistinguishable until the time \( t \approx 2.73 \) when the subcritical solution stops because it is no more strictly hyperbolic. In the supercritical case \( u(x, t) \) evolves smoothly as we show in the last plot. A zoom of \( u \) near to the transition line at the transition time is shown in the figure 9.

All the numerical evolutions are obtained using Mathematica 9.

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