NONLINEAR PROJECTIVE FILTERING II:
APPLICATION TO REAL TIME SERIES

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Abstract—We discuss applications of nonlinear filtering of time series by locally linear phase space projections. Noise can be reduced whenever the error due to the manifold approximation is smaller than the noise in the system. Examples include the real time extraction of the fetal electrocardiogram from abdominal recordings.

I. INTRODUCTION

For nonlinear signals with intrinsic instabilities, the tasks of noise reduction — or signal separation in general — is difficult to accomplish with the traditional spectral approach since the signals may exhibit broad band frequency spectra. The theory of nonlinear dynamical systems, or chaos theory, provides alternative methods for these purposes based on a phase space representation of the data. The theory behind these methods is outlined in Ref. [1] in this volume. For the derivation of time series methods within the framework of chaos theory, heavy use has been made of the theoretical properties of nonlinear deterministic systems. For most methods, this also severely restricts the scope of systems they can be applied to. Formally, this is also the case for nonlinear filtering procedures that exploit the peculiar structure generated by deterministic phase space dynamics. In this paper we will show that if these algorithms are used with care, they can also give superior results in situations when pure determinism cannot be assumed. The reason is that, when represented in a low dimensional phase space, also non-deterministic systems may exhibit structures suitable for filtering purposes. We will give some examples of the latter statement and also discuss practical issues that have so far hampered widespread use of nonlinear filters.

II. METHOD

A scalar time series \( \{ s_n \} \), \( n = 1, \ldots, N \) can be unfolded in a multi-dimensional effective phase space using time delay coordinates \( s_n = (s_{n-(m-1)}\tau, \ldots, s_n) \) (\( \tau \) is a delay time). If \( \{ s_n \} \) is a scalar observation of a deterministic dynamical system, it can be shown under certain genericity conditions [2] that the reconstructed point set is a one-to-one image of the original attractor of the dynamical system. We will not assume here that there is such an underlying deterministic system. Nevertheless, general serial dependencies among the \( \{ s_n \} \) will cause the delay vectors \( \{ s_n \} \) to fill the available \( m \)-dimensional space in an inhomogeneous way. Linearly correlated Gaussian random variates will for example be distributed according to an anisotropic multivariate Gaussian distribution. Linear geometric filtering [4] in phase space seeks to identify the principal directions of this distribution and project onto them. The present algorithm can be seen as a nonlinear generalisation of this approach that takes into account that nonlinear signals will form curved structures in delay space. In particular, noisy deterministic signals form smeared-out lower dimensional manifolds. Nonlinear phase space filtering seeks to identify such structures and project onto them in order to reduce noise.

Let us recall the three main steps involved in the noise reduction algorithm described in Ref. [1]: (1) Find a low dimensional approximation to the “attractor” described by the trajectory \( \{ s_n \} \). (2) Project each point \( s_n \) in the trajectory orthogonally onto the approximation to the attractor to produce a cleaned vector \( \hat{s}_n \). (3) Convert the sequence of cleaned vectors \( \hat{s}_n \) back into the scalar time domain to produce a cleaned time series \( \hat{s}_n \).

III. REAL TIME FILTERING

All of the methods that have been proposed in the literature are formulated as a posteriori filters. The whole signal has to be available before a cleaned version can be computed, which is then invariably quite computer time intensive. One class of methods uses a global nonlinear function to represent the dynamics (at least approximately). This function has to be determined by a delicate fitting procedure and the actual filtering scheme (for example Ref. [3]) consists of an iterative minimisation procedure. The other class of
algorithms approximates the dynamics in phase space, or phase space geometry, by locally linear mappings. Here the tessellation of phase space into small neighbourhoods is the most time consuming step, along with the need to solve a least squares problem in each of these neighbourhoods. With fairly low dimensional signals, fast neighbour search algorithms (see [6] for an overview) are very helpful in this regard. Let us introduce some modifications to the locally projective noise reduction scheme discussed in Ref. [3] that make its use in real time signal processing feasible. (1) The data base of local neighbours that is needed to approximate the dynamics is restricted to points in the past. As a side effect, the curvature correction can be carried out during the first sweep through the data. (2) The number of neighbours required for each point is limited to a number that is just sufficient for statistical stability. (3) The last modification uses the fact that the dynamics is supposed to vary smoothly in phase space. The full linearised dynamics is reduced to a collection of representative points which is stored together with their local linear structure. Consequently, the local linear problem has to be solved only for points that are not yet well approximated by such a representative.

Thus, we have turned the procedure into a causal filter by restricting neighbour search to points defined by measurements made in the past, \( k < n \). By further limiting the number of neighbours searched for to \( U_{\text{max}} \), we have sped up the formation of the local covariance matrices considerably. Still, as the algorithm stands, we have to solve an \((m \times m)\) eigenvalue problem for each point that is to be processed. A large fraction of this work can be avoided on the base of the assumption that the local linear structure changes smoothly over phase space. By making local linear approximations we have already assumed smoothness of the underlying manifold in the \( C_1 \) sense. In most physical systems, the additional assumption of \( C_2 \) smoothness is not less justified. We cannot, however expect that the vectors which span the principal directions vary slowly from point to point. The reason is that often some eigenvalues are nearly degenerate and change indices from point to point when they are ordered by their magnitude. We thus refrain from interpolating principal components between phase space points. Instead, we choose a length scale \( h \) in phase space which is small enough such that the linear subspaces spanned by the local principal components can be regarded as effectively the same. Now we successively build up a data base of representative points for which the local points of tangency \( \tilde{s}_n^{(n)} \), and the local principal directions \( e_q, \quad q = 1, \ldots, Q \) have been determined already. For each new point \( s_n \) that is to be processed, we go through this collection of points to determine whether a representative is available closer than \( h \). In that case, we use the stored tangent point and principal directions of the representative in order to perform the projections. If not, a neighbourhood is formed around \( s_n \) in which the eigenvalue problem is solved. The point \( s_n \) is then included in the list of representatives.

For real time application, this means that the corrected value \( \hat{s}_n \) cannot be available before \( s_{n+m-1} \) has been measured and processed. Usually, however, this delay window is a very short time, at least compared to the duration of the recording, and the procedure can be regarded as effectively on-line as long as the computations necessary to obtain \( \hat{s}_n \) can be carried out fast enough.

### IV. RESULTS

**NMR laser data** — As a first example, we show in Fig. 1 the result of applying the described procedure to a data set from an NMR laser experiment [7]. The same data has also been used in Ref. [6]. The laser is periodically driven and once per driving cycle the envelope of the laser output is recorded. The resulting sampling rate is 91 Hz. At this rate, the nonlinear noise reduction scheme can be easily carried out in real time on a Pentium II processor at 200 MHz. Since further iterations can be carried out after the time corresponding to one embedding window without interfering with the previous steps, we could perform up to three iterations on a dual Pentium II workstation at 300 MHz in real time. The figure shows the result after two iterations. Projections from \( m = 7 \) down to \( Q = 2 \) dimensions were used, at least 100 neighbours were requested at a neighbourhood size of 2 units. The history was limited to 20000 samples, or 220 s. This fairly large data base is needed since the initial noise level is already small (less than 2% [8]) and small neighbourhoods are required to avoid the dominance of curvature artefacts.

**Magneto-cardiogram data** — The actual acceleration resulting from the above modifications strongly

![Figure 1: Result of nonlinear filtering of an NMR laser time series. An enlargement of about one quarter of the total linear extent of the attractor is shown.](image-url)
depends on the situation and it is difficult to give general rules and benchmarks. Let us however study a realistic example in some detail to illustrate the main points. In electrophysiological research, it is quite attractive to augment the measurements of electric potentials with recordings of the magnetic field strength. The latter penetrate intervening tissue much more efficiently. Of particular interest are magnetoencephalographic (MEG) recordings which allow to access regions of the brain noninvasively which cannot be monitored electrically using surface electrodes. In cardiology, the magneto-cardiogram (MCG) provides additional information to the traditional electrocardiogram (ECG). A particular application is the noninvasive monitoring of the fetal heart which is otherwise complicated by shielding of the electric field by intervening tissue. A common problem with magnetic recordings, however, is that the fields are rather feeble and the measurements have to be carried out in a shielded room. Even then, noise remains a major challenge for this experimental technique. We will demonstrate in the following how nonlinear noise reduction could be used for continuous MCG monitoring.

We use an MCG recording of a normal human subject at rest. The data was kindly provided by Carsten Sternickel at the University of Bonn. The sampling rate was 1000 Hz, which is quite high for cardiac monitoring. For the signal processing task, any sampling rate above about 200 Hz would be sufficient. (Below 200 Hz, the spike representing the depolarisation of the ventricle might not be resolved properly). In order to cover a significant fraction of one cardiac cycle by an embedding window, we chose an embedding with delay $\tau = 10$ ms in $m = 10$ dimensions. Neighbourhoods were formed with a radius of 0.06 uncalibrated A/D units of the recording, about three times the estimated noise level. We quote computation times for the processing of 10 s of MCG on a Pentium processor at 133 MHz, determined on a laptop PC running the Linux operating system. The timing results are summarised in Table 1. The data base of representatives for (f) was formed by assuming that the local linear subspaces are equivalent on length scales of the order of 0.06 A/D units.

Fetal ECG extraction — Let us finally discuss the extraction of the fetal electrocardiogram (FECG) from non-invasive maternal recordings. Other very similar applications include the removal of ECG artefacts from electro-myogram (EMG) recordings (electric potentials of muscle) and spike detection in electro-encephalogram (EEG) data.

Fetal ECG extraction can be regarded as a three-way filtering problem since we have to assume that a maternal abdominal ECG recording consists of three main components, the maternal ECG, the fetal ECG, and exogenous noise, mostly from action potentials of intervening muscle tissue. All three components have quite similar broad band power spectra and cannot be filtered apart by spectral methods. The fetal component is detectable from as early as the eleventh week of pregnancy. After about the twentieth week, the signal becomes weaker since the electric potential of the fetal heart is shielded by the vernix caseosa forming on the skin of the fetus. It appears again towards delivery. In Refs. [9, 10], it has been proposed to use a nonlinear phase space projection technique for the separation of the fetal signal from maternal and noise artefacts. A typical example of output of this procedure is shown in Fig. 4. The assumption made about the nature of the

| Computation Time | Description |
|------------------|-------------|
| 490 s            | all neighbours |
| 94 s             | box assisted neighbour search |
| 246 s            | all neighbours in past |
| 191 s            | $n - k < 5$ s |
| 44 s             | $U_{\text{max}} < 200$ |
| 6 s              | (e) and reuse of representatives |

Table 1: Computation time for nonlinear noise reduction of 10 s of an MCG recording sampled at 1000 Hz. See text for details.
data is that the maternal signal is well approximated by a low-dimensional manifold in delay reconstruction space. After projection onto this manifold, the maternal signal is separated from the noisy fetal component. Now it is assumed that the fetal ECG is also approximated by a low-dimensional manifold and the noise is removed by projection. Since both manifolds are curved, the projections have to be made onto linear approximations. For technical details see Refs. [9, 10].

V. DISCUSSION

We have demonstrated that by certain modifications to the algorithms described in the literature, nonlinear projective noise reduction can be turned into a signal processing tool that can in many situations run in real time in a data stream. The class of problems that can be solved by this approach is much wider than initially assumed since strict determinism of the signal is not necessary. Signal and noise have to be distinguishable by their shape in a reconstructed phase space.

The algorithms used in this paper and in Ref. [9] are publicly available as part of the TISEAN software project [12]. All the examples were indeed carried out with these implementations.

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References

[1] H. Kantz and T. Schreiber, “Nonlinear projective filtering I: Background in chaos theory”, this volume (1998).
[2] F. Takens, “Detecting Strange Attractors in Turbulence”, Lecture Notes in Math. Vol. 898, Springer, New York (1981).
[3] T. Sauer, J. Yorke, and M. Casdagli, “Embedology”, J. Stat. Phys. 65 (1991) 579.
[4] D. Broomhead and G. P. King, “Extracting qualitative dynamics from experimental data”, Physica D 20 (1986) 217.
[5] M. E. Davies, “Noise reduction by gradient descent”, Int. J. Bifurcation and Chaos 3 (1992) 113.
[6] T. Schreiber, “Efficient neighbor searching in nonlinear time series analysis”, Int. J. Bifurcation and Chaos 5 (1995) 349.
[7] M. Finardi, L. Flepp, J. Parisi, R. Holzner, R. Badii, and E. Brun, “Topological and metric analysis of heteroclinic crises in laser chaos”, Phys. Rev. Lett. 68 (1992) 2989.
[8] H. Kantz, T. Schreiber, I. Hoffmann, T. Buzug, G. Pfister, L. G. Flepp, J. Simonet, R. Badii, and E. Brun, “Nonlinear noise reduction: a case study on experimental data”, Phys. Rev. E 48 (1993) 1529.
[9] T. Schreiber and D. T. Kaplan, “Signal separation by nonlinear projections: The fetal electrocardiogram”, Phys. Rev. E 53 (1996) 4326.
[10] M. Richter, T. Schreiber, and D. T. Kaplan, “Fetal ECG extraction with nonlinear phase space projections”, IEEE Trans. Bio-Med. Eng. 45 (1998) 133.
[11] J. F Hofmeister, J. C. Slocumb, L. M. Kottmann, J. B. Pichiodtino, and D. G. Ellis, “A noninvasive method for recording the electrical activity of the human uterus in vivo”, Biomed. Instr. Technol. 9 (1994) 391.
[12] The TISEAN software package is publicly available for download from either http://www.mpipks-dresden.mpg.de/~tsa/TISEAN/docs/welcome.html or http://wptu38.physik.uni-wuppertal.de/Chaos/DOCS/welcome.html.