A 9-dimensional algebra which is not a block of a finite group

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Abstract

We rule out a certain 9-dimensional algebra over an algebraically closed field to be the basic algebra of a block of a finite group, thereby completing the classification of basic algebras of dimension at most 12 of blocks of finite group algebras.

1 Introduction

Basic algebras of block algebras of finite groups over an algebraically closed field of dimension at most 12 have been classified in [13], except for one 9-dimensional symmetric algebra over an algebraically closed field $k$ of characteristic 3 with two isomorphism classes of simple modules for which it is not known whether it actually arises as a basic algebra of a block of a finite group algebra. The purpose of this paper is to show that this algebra does not arise in this way. It is shown in [13, Section 2.9] that if $A$ is a 9-dimensional basic algebra over an algebraically closed field $k$ of prime characteristic $p$ with two isomorphism classes of simple modules such that $A$ is isomorphic to a basic algebra of a block $B$ of $kG$ for some finite group $G$, then the algebra $A$ has the Cartan matrix

$$ C = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix}, $$

Since the elementary divisors of $C$ are 9 and 1, it follows that $p = 3$ and that a defect group $P$ of $B$ is either cyclic (in which case $A$ is a Brauer tree algebra) or $P$ is elementary abelian of order 9. We will show that the second case does not arise.

Theorem 1.1. Let $k$ be an algebraically closed field of prime characteristic $p$. Let $G$ be a finite group and $B$ a block of $kG$ with Cartan matrix $C$ as above. Then $p = 3$, the defect groups of $B$ are cyclic of order 9, and $B$ is Morita equivalent to the Brauer tree algebra of the tree with two edges, exceptional multiplicity 4 and exceptional vertex at the end of the tree.

The proof of Theorem 1.1 proceeds in the following stages. We first identify in Theorem 2.1 any hypothetical basic algebra $A$ of a block with Cartan matrix $C$ as above and a noncyclic defect group. It turns out that there is only one candidate algebra, up to isomorphism. In Section 3 we give a description of the structure of this candidate $A$, and we show in Theorem 5.1 that $A$ is not isomorphic to a basic algebra of a block. The proof of Theorem 2.1 amounts essentially to filling in the details in [13, section 2.9]. For the proof of Theorem 5.1 we combine a stable equivalence of Puig [17], a result of Broué in [5] on the invariance of stable centres under stable equivalences...
of Morita type, results of Kiyota [9] on blocks with an elementary abelian defect group of order 9, and properties of blocks with symmetric stable centres from [8]. A slightly different approach to proving Theorem 5.1 is outlined in the last section, first showing in Proposition 6.1 a more precise result on the stable equivalence class of $A$, and then using Rouquier’s stable equivalences for blocks with elementary abelian defect groups of rank 2 from [18].

Sambale [19] recently extended the classification of blocks with a low-dimensional basic algebra to the dimensions 13 and 14, and in dimension 15 the only open question is whether a certain Brauer tree algebra does arise as a block algebra.

For background material on describing finite-dimensional algebras in terms of their quivers and relations, see [2, Chapter III, Section 1], and for Brauer tree algebras, as part of the theory of blocks with cyclic defect groups, see [1] Chapter 5, Section 17 and [15] Sections 11.7 and 11.8.

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2 The basic algebra $A$ of a noncyclic block with Cartan matrix $C$

The following result is stated in [6] without proof; for the convenience of the reader we give a detailed proof, following in part the arguments in [13, Section 2.9].

Theorem 2.1. Let $k$ be an algebraically closed field of prime characteristic $p$. Let $A$ be a basic algebra with Cartan matrix

$$C = \begin{pmatrix} 5 & 1 \\ 1 & 2 \end{pmatrix},$$

such that $A$ is Morita equivalent to a block $B$ of $kG$, for some finite group $G$, with a noncyclic defect group $P$. Then $p = 3$, we have $P \cong C_3 \times C_3$, and $A$ is isomorphic to the algebra given by the quiver

$$\begin{array}{c}
\gamma \\
\delta \\
\alpha \\
\beta \\
\delta \\
\gamma \\
\alpha \\
\beta \\
\gamma \\
\delta
\end{array}$$

with relations $\delta^2 = \gamma^3 = \alpha \beta$, $\delta \gamma = \gamma \delta = 0$, $\delta \alpha = \gamma \alpha = 0$, and $\beta \delta = \beta \gamma = 0$. In particular, we have $|\text{Irr}(B)| = \dim_k(Z(A)) = 6$, and the decomposition matrix of $B$ is equal to

$$D = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}.$$
Proof. As mentioned above, since the elementary divisors of the Cartan matrix $C$ are 1 and 9, it follows that $p = 3$ and that $B$ has a defect group $P$ of order 9. Since $P$ is assumed to be noncyclic, it follows that $P \cong C_3 \times C_3$.

Let $\{i, j\}$ be a primitive decomposition of 1 in $A$. Set $S = Ai/J(A)i$ and $T = Aj/J(A)j$. It follows from the entries of the Cartan matrix that we may choose notation such that $Ai$ has composition length 6 and $Aj$ has composition length 3. Since the top and bottom composition factor of $Aj$ are both isomorphic to $T$, it follows that $Aj$ is uniserial, with composition factors $T$, $S$, $T$ (from top to bottom). In what follows, we tend to use the same notation for generators in $A$ corresponding to homomorphisms between projective indecomposables; this reverses the order in relations since $\text{End}_A(A)$ is isomorphic to the opposite algebra $A^{op}$.

We label the two vertices of the quiver of $A$ by $i$ and $j$. The quiver of $A$ contains a unique arrow from $i$ to $j$ and no loop at $j$ because $J(A)i/J(A)j^2 \cong S$. Thus there is an $A$-homomorphism 

$$\alpha : Ai \rightarrow Aj$$

with image $\text{Im}(\alpha) = V$ uniserial of length 2, with composition factors $S$, $T$. Since $Aj$ is uniserial of length 3, it follows that $V = J(A)j$ is the unique submodule of length 2 in $Aj$.

The symmetry of $A$ implies that the quiver of $A$ contains a path from $j$ to $i$. This forces that the quiver of $A$ has an arrow from $j$ to $i$. Since the Cartan matrix of $A$ implies that $Ai$ has exactly one composition factor $T$, it follows that the quiver of $A$ contains exactly one arrow from $j$ to $i$. This arrow corresponds to an $A$-homomorphism 

$$\beta : Aj \rightarrow Ai$$

which is not injective as $Aj$ is an injective module. Thus $U = \text{Im}(\beta)$ is a submodule of $Ai$ of length at most 2. The length of $U$ cannot be 1, because the top composition factor of $U$ is $T$, but the unique simple submodule of $Ai$ is isomorphic to $S$. Thus $U$ is a uniserial submodule of length 2 of $Ai$, with composition factors $T$, $S$. It follows that $\beta \circ \alpha$ is an endomorphism of $Ai$ with image $\text{soc}(Ai)$.

Since $U = \text{Im}(\beta)$ and $\beta$ corresponds to an arrow in the quiver of $A$, it follows that $U$ is not contained in $J(A)^2i$. Thus the simple submodule $U/\text{soc}(Ai)$ of $J(A)i/\text{soc}(Ai)$ is not contained in the radical of $J(A)i/\text{soc}(Ai)$, and therefore must be a direct summand. Let $M$ be a submodule of $Ai$ such that $M/\text{soc}(Ai)$ is a complement of $U/\text{soc}(Ai)$ in $J(A)i/\text{soc}(Ai)$. Then

$$J(A)i = U + M$$

$$\text{soc}(Ai) = U \cap M$$

and, by the Cartan matrix, $M$ has composition length 4, and all composition factors of $M$ are isomorphic to $S$, and $\text{soc}(M) = \text{soc}(Ai)$. Equivalently, $M/\text{soc}(Ai)$ has length 3, with all composition factors isomorphic to $S$. We rule out some cases.

(1) $M/\text{soc}(Ai)$ cannot be semisimple. Indeed, if it were semisimple, then $J(A)i/\text{soc}(Ai) = U/\text{soc}(Ai) \oplus M/\text{soc}(Ai)$ would be semisimple. This would imply that $J(A)^3i = \{0\}$. Since also $J(A)^3j = \{0\}$, it would follow that $\ell\ell(A) = 3$. But a result of Okuyama in [10] rules this out. Thus $M/\text{soc}(Ai)$ is not semisimple.

(2) $M/\text{soc}(Ai)$ cannot be uniserial. Indeed, if it were, then the quiver of $A$ would have a unique loop at $i$, corresponding to an endomorphism $\gamma$ of $Ai$ mapping $Ai$ onto $M$ (with kernel necessarily
equal to $U$ because $M$ has no composition factor isomorphic to $T$). Then $\gamma^5 = 0$ and $\gamma^4$ has image $\text{soc}(Ai) \cong S$.

By construction, $\alpha$ maps $U$ to $\text{soc}(Aj)$ and $\beta$ maps $V$ to $\text{soc}(Ai)$. Thus $\beta \circ \alpha$ sends $Ai$ onto $\text{soc}(Ai)$. Thus $\gamma^4$ and $\beta \circ \alpha$ differ at most by nonzero scalar. We may choose $\alpha$ such that $\gamma^4 = \beta \circ \alpha$.

The homomorphism $\alpha$ sends $M$ to zero, because $Aj$ contains no simple submodule isomorphic to $S$. Thus $\alpha \circ \gamma = 0$. Also, since $U$ is the kernel of $\gamma$, we have $\gamma \circ \beta = 0$. Using the same letters $\alpha$, $\beta$, $\gamma$ for the elements in $iAj$, $jAi$, $iAi$, respectively, it follows that $A$ is generated by $\{i, j, \alpha, \beta, \gamma\}$ with the (now opposite) relations $\gamma^4 = \alpha \beta$, $\gamma \alpha = 0 = \beta \gamma$, and all the obvious relations using that $i$, $j$ are orthogonal idempotents whose sum is 1.

We will show next that these relations that $A$ is a Brauer tree algebra, of a tree with two edges, exceptional multiplicity 4, and exceptional vertex at an end of the Brauer tree. By [15, Theorem 11.8.1] and its proof, such a Brauer tree algebra is generated by two orthogonal idempotents $i$, $j$ whose sum is 1, and two elements $r$, $s$ satisfying $ir = ri$, $jr = rj$, $is = sj$, $js = si$, $ir^4 + is^2 = 0$ and $jr + js^2 = 0$. Since $p = 3$ and $k$ is algebraically closed, we may multiply $s$ by a fourth root of unity, so that the latter two relations become $ir^4 = is^2$ and $jr = js^2$. One verifies that the assignment $r \mapsto \gamma + \beta \alpha$ and $s \mapsto \alpha + \beta$, together with the obvious assignments on the primitive idempotents, induces a surjective algebra homomorphism from this Brauer tree algebra to $A$. To see this, one first needs to verify that the above images of $r$ and $s$ satisfy the relations in $A$ corresponding to those involving $r$ and $s$ in the Brauer tree algebra. This follows easily from the given relations for the generating set of $A$. For the surjectivity one needs to observe that $\alpha$, $\beta$, $\gamma$ are in the image of this map. This follows from multiplying $r$, $s$ and their images by the primitive idempotents in the two algebras. Since both the Brauer tree algebra and $A$ have dimension 9, it follows that they are isomorphic.

This, however, would force $P$ to be cyclic, contradicting the current assumption that $P \cong C_3 \times C_3$.

(3) $M/\text{soc}(Ai)$ cannot be indecomposable. Indeed, if it were, then it would have Loewy length 2 because it has composition length 3, but is neither of length 1 (because it is not semisimple) nor of length 3 (because it is not uniserial). But then either its socle or its top is simple, and therefore it would have to be either a quotient of $Ai$, or a submodule of $Ai$. We rule out both cases.

Suppose first that $M/\text{soc}(Ai)$ is a quotient of $Ai$. Note that then $M$ itself has a simple top, isomorphic to $S$, hence is a quotient of $Ai$ because $Ai$ is projective. Comparing composition lengths yields $M \cong Ai/U$. But also $U + M = J(A)i$, so the image of $M$ in $Ai/U$ is the unique maximal submodule $J(A)i/U$ of $Ai/U \cong M$. Thus $J(A)M$ is the unique maximal submodule of $M$, and that maximal submodule is isomorphic to a quotient of $M$, hence has itself a unique maximal submodule. This however would imply that $M/\text{soc}(Ai)$ is uniserial of length 3, which was ruled out earlier.

Suppose finally that $M/\text{soc}(Ai)$ is a submodule of $Ai$. Then it must be a submodule of $M$, because it does not have a composition factor $T$. Moreover, $M$ and the image of $M/\text{soc}(Ai)$ in $M$ both have the same simple socle $\text{soc}(Ai)$. Thus $M/\text{soc}(Ai)$ divided by its socle (which is simple) is a submodule of $M/\text{soc}(Ai)$, which has a simple socle. Thus the first and second socle series quotients are both simple, again forcing $M/\text{soc}(Ai)$ to be uniserial, which is not possible.

(4) Combining the above, it follows that $M/\text{soc}(Ai)$ is a direct sum of $S$ and a uniserial module of length 2 with both composition factors $S$. That is, we have

$$M = M_1 + M_2$$
for some submodules $M_i$ of $M$ with

$$M_1 \cap M_2 = \text{soc}(Ai) = \text{soc}(M)$$

$$M_1/\text{soc}(Ai) \cong S$$

and $M_2/\text{soc}(Ai)$ uniserial of length 2. It follows that $M_1$ and $M_2$ are uniserial, of lengths 2 and 3, respectively.

We choose now $M_2$ as follows. By construction, we have a direct sum

$$J(A)/\text{soc}(Ai) = U/\text{soc}(Ai) \oplus M_1/\text{soc}(Ai) \oplus M_2/\text{soc}(Ai)$$

Thus we have

$$J(A)/\text{soc}(Ai) \cong (J(A)/\text{soc}(Ai))/\text{soc}(Ai) \cong M_2/\text{soc}(Ai).$$

This is a uniserial module with two composition factors isomorphic to $S$. Thus $Ai/(U + M_1)$ is uniserial with three composition factors isomorphic to $S$, because $Ai/J(A)i \cong S$. Since in particular its socle is simple, isomorphic to $S$, this module is isomorphic to a submodule of $Ai$. Choose an embedding $A/(U + M_1) \to Ai$ and replace $M_2$ by the image of this embedding. Then the composition of canonical maps

$$\gamma : Ai \to A/(U + M_1) \to Ai$$

is an $A$-endomorphism of $Ai$ with kernel $U + M_1$ and uniserial image $M_2$ of length three. Note that $M_1$ is uniserial of length two, so both a quotient and a submodule of $Ai$. Thus there is an endomorphism

$$\delta : Ai \to Ai$$

with image $M_1$. Since $M_1 \subseteq \ker(\gamma)$, we have

$$\gamma \circ \delta = 0.$$

We show next that we also have

$$\delta \circ \gamma = 0.$$

One way to see this is to observe that this is a calculation in the split local 5-dimensional symmetric algebra $\text{End}_A(Ai) \cong (iAi)^{op}$, which as a consequence of [10, B. Theorem], is commutative.

There is a (slightly more general) argument that works in this case. Since the $A$-module $Ai$, and hence also the image of $\gamma$, is generated by $i$, it suffices to show that $\delta(\gamma(i)) = 0$. Now since $\gamma \circ \delta = 0$, we have

$$0 = \gamma(\delta(i)) = \gamma(\delta(i)i) = \delta(i)\gamma(i)$$

Note that $\delta(i) = \delta(i^2) = i\delta(i) \in iAi$, and similarly, $\gamma(i) \in iAi$. Since $\text{Im}(\delta) = M_2$ has length 2, we have $\text{Im}(\delta) \subseteq \text{soc}^2(A)$. Thus $\delta(i) \in \text{soc}^2(A) \cap iAi \subseteq \text{soc}^2(iAi)$, and since $iAi$ is symmetric, we have $\text{soc}^2(iAi) \subseteq \text{Z}(iAi)$. It follows that

$$\delta(i)\gamma(i) = \gamma(i)\delta(i) = \delta(\gamma(i)i) = \delta(\gamma(i))$$
whence $\delta(\gamma(i)) = 0$, and so $\delta \circ \gamma = 0$ by the previous remarks. Thus $M_2 \subseteq \ker(\delta)$. Since $\text{Im}(\delta) = M_1$ has no composition factor $T$, it follows that $U \subseteq \ker(\delta)$. Together we get that $U + M_2 \subseteq \ker(\delta)$. Comparing composition lengths yields

$$\ker(\delta) = U + M_2.$$  

This implies that

$$\ker(\delta) \cap \text{Im}(\delta) = \text{soc}(Ai)$$

and hence the endomorphisms $\delta^2$ and $\gamma^3$ both map $Ai$ onto $\text{soc}(Ai)$. Thus they differ by a nonzero scalar. Up to adjusting $\delta, \beta$, we may therefore assume that

$$\delta^2 = \gamma^3 = \beta \circ \alpha$$

Since $\ker(\alpha)$ contains $M_1 + M_2$, it follows that

$$\alpha \circ \delta = \alpha \circ \gamma = 0.$$  

By taking these relations into account, it follows that $\text{End}_A(A)$ is spanned $k$-linearly by the set

$$\{i, j, \alpha, \beta, \gamma, \gamma^2, \delta, \delta^2, \alpha \circ \beta\}$$

so this is a basis of $\text{End}_A(A)$. We have identified here $i, j$ with the canonical projections of $A$ onto $Ai$ and $Aj$. Note that $\text{End}_A(A)$ is the algebra opposite to $A$. This accounts for the reverse order in the relations of the generators in $A$ (denoted abusively by the same letters). This shows that the quiver with relations of $A$ is as stated. The equation $C = (D^t)D$ implies that the second column of $D$ has exactly two nonzero entries and that these are equal to 1. The first row has either five entries equal to 1, which yields $|\text{Irr}(B)| = 6$ and the decomposition matrix $D$ as stated. Or the first row has one entry 2 and one entry 1. This would lead to a decomposition matrix of the form

$$D = \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}.$$  

In particular, this would yield $|\text{Irr}(B)| = 3$. But this is not possible, since $\dim_k(Z(A))$ is clearly greater than 3; indeed, $Z(A)$ contains the linearly independent elements $1, \delta, \gamma, \gamma^2$. This concludes the proof.

\section{The structure of the algebra $A$}

Let $k$ be an algebraically closed field. Throughout this section we denote by $A$ the $k$-algebra given in Theorem \ref{thm:structure}. We keep the notation of this theorem and identify the generators $i, j, \alpha, \beta, \gamma, \delta$ with their images in $A$.

\begin{lemma} \label{lem:structure}

(i) The set $\{i, j, \alpha, \beta, \beta \alpha, \gamma, \gamma^2, \delta, \delta^2\}$ is a $k$-basis of $A$.

\end{lemma}
(ii) The set \( \{ \alpha, \beta, \alpha \beta - \beta \alpha \} \) is a \( k \)-basis of \([A, A]\).

(iii) The set \( \{ 1, \gamma, \gamma^2, \delta, \delta^2, \beta \alpha \} \) is a \( k \)-basis of \( Z(A) \).

(iv) The set \( \{ \alpha \beta, \beta \alpha \} \) is a \( k \)-basis of \( \text{soc}(A) \).

**Proof.** This follows immediately from the relations of the quiver of \( A \). \( \square \)

**Lemma 3.2.** There is a unique symmetrising form \( s: A \to k \) such that

\[ s(\alpha \beta) = s(\beta \alpha) = 1 \]

and such that

\[ s(i) = s(j) = s(\alpha) = s(\beta) = s(\gamma) = s(\gamma^2) = s(\delta) = 0 \]

The dual basis with respect to the form \( s \) of the basis

\( \{ i, j, \alpha, \beta, \beta \alpha, \gamma, \gamma^2, \delta, \delta^2 \} \)

is, in this order, the basis

\( \{ \alpha \beta, \beta \alpha, \beta, \alpha, j, \gamma^2, \gamma, \delta, i \} \)

**Proof.** Straightforward verification. \( \square \)

See [5, §5.B] or [14, Definition 2.16.10] for details regarding the definitions and some properties of the projective ideal \( \mathcal{Z}^{pr}(A) \) in \( Z(A) \) and the stable centre \( \mathcal{Z}(A) = Z(A)/\mathcal{Z}^{pr}(A) \).

**Lemma 3.3.** Let \( \text{char}(k) = 3 \). The projective ideal \( \mathcal{Z}^{pr}(A) \) is one-dimensional, with basis \( \{ \alpha \beta - \beta \alpha \} \), we have an isomorphism of \( k \)-algebras

\[ \mathcal{Z}(A) \cong k[x, y]/(x^3 - y^2, xy, y^3) \]

induced by the map sending \( x \) to \( \gamma \) and \( y \) to \( \delta \), and after identifying \( x \) and \( y \) with their images in the quotient, the following statements hold:

(i) The set \( \{ 1, x, x^2, y, y^2 \} \) is a \( k \)-basis of \( \mathcal{Z}(A) \), and in particular \( \dim_k(\mathcal{Z}(A)) = 5 \).

(ii) The set \( \{ x, x^2, y, y^2 \} \) is a \( k \)-basis of \( \mathcal{J}(\mathcal{Z}(A)) \).

(iii) The set \( \{ x^2, y^2 \} \) is a \( k \)-basis of \( \mathcal{J}(\mathcal{Z}(A))^2 \).

(iv) The set \( \{ y^2 \} \) is a \( k \)-basis of \( \text{soc}(\mathcal{Z}(A)) \), and \( \mathcal{J}(\mathcal{Z}(A))^3 = \text{soc}(\mathcal{Z}(A)) \).

(v) The \( k \)-algebra \( \mathcal{Z}(A) \) is a symmetric algebra.

**Proof.** It follows from lemma 3.2 that the relative trace map \( \text{Tr}_1^A \) from \( A \) to \( Z(A) \) is given by

\[ \text{Tr}_1^A(u) = iu \alpha \beta + ju \beta \alpha + \alpha u \beta + \beta u \alpha + \beta \alpha j + \gamma u \gamma^2 + \gamma^2 u \gamma + \delta u \delta + \delta^2 u i \]

for all \( u \in A \). One checks, using \( \text{char}(k) = 3 \), that

\[ \text{Tr}_1^A(i) = -\text{Tr}_1^A(j) = \beta \alpha - \alpha \beta \]

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and that $\text{Tr}_1^A$ vanishes on all basis elements different from $i, j$. Statement (i) then follows from the relations in the quiver of $A$ and Lemma 3.1. The algebra $\mathbb{Z}(A)$ is split local, proving statement (ii), whilst a straightforward computation shows both statement (iii) and (iv). Finally, a simple verification proves that the map $s : \mathbb{Z}(A) \to k$ such that

$$s(y^2) = 1$$

and such that

$$s(1) = s(x) = s(x^2) = s(y) = 0$$

is a symmetrising form on $\mathbb{Z}(A)$. One verifies also that the dual basis with respect to the form $s$ of the basis

$$\{1, x, y, x^2, y^2\}$$

is, in this order, the basis

$$\{y^2, x^2, y, x, 1\}.$$

This completes the proof. 

\begin{proof}

Remark 3.4. Note that by a result of Erdmann [7, I.10.8(i)], $A$ is of wild representation type.

4 The stable centre of the group algebra $k(P \rtimes C_2)$.

Let $k$ be a field of characteristic 3. Set $P = C_3 \times C_3$ and $E$ the subgroup of $\text{Aut}(P)$ of order 2 such that the nontrivial element $t$ of $E$ acts as inversion on $P$. Denote by $H = P \rtimes E$ the corresponding semidirect product; this is a Frobenius group. Denote by $r$ and $s$ generators of the two factors $C_3$ of $P$. The following Lemma holds in greater generality (see Remark 4.1 in [8]; we state only what we need in this paper.

\begin{lemma}

The projective ideal $Z^{pr}(kH)$ is one-dimensional, with $k$-basis $\{\sum_{x \in P} xt\}$, and we have an isomorphism of $k$-algebras

$$\mathbb{Z}(kH) \cong (kP)^E$$

induced by the map sending $x + x^{-1}$ in $(kP)^E$ to its image in $\mathbb{Z}(kH)$. In particular, we have $\dim_k(\mathbb{Z}(kH)) = 5$, and the image of the set $\{1, r + r^2, s + s^2, r^2s + rs^2, rs + r^2s^2\}$ is a $k$-basis of $\mathbb{Z}(kH)$.

\begin{proof}

The relative trace map $\text{Tr}_1^H$ from $kH$ to $Z(kH)$ satisfies $\text{Tr}_1^H = \text{Tr}^P_{kH} \circ \text{Tr}_1^P$. We calculate for all $a \in P$

$$\text{Tr}_1^P(a) = \sum_{g \in P} gag^{-1} = \sum_{|P|} a = 9 \cdot a = 0$$

Thus for every $c \in kP$ we have $\text{Tr}_1^H(c) = \text{Tr}^P_{kH}(\text{Tr}_1^P(c)) = 0$. On the other hand, for every element of the form $at$ in $H$, where $a \in P$, we have
The conjugacy classes of $G$ are given by \{1\}, \{r, r^2\}, \{s, s^2\}, \{r^2s, rs, r^2s^2\}, \{rs, r^2s^2\}$ and \{xt | x ∈ P\}. The last statement follows.

**Lemma 4.2.** There is an isomorphism of $k$-algebras

$$ Z(kH) \cong \left( k[x, y]/(x^3, y^3) \right)^E $$

with inverse induced by the map sending $x$ to $r - 1$ and $y$ to $s - 1$, where the nontrivial element $t$ of $E$ acts by $x^t = x^2 + 2x$ and $y^t = y^2 + 2y$. After identifying $x$ and $y$ with their images in $k[x, y]/(x^3, y^3)$, the following statements hold:

(i) The image of the set \{1, x^2, y^2, xy + x^2y + xy^2, x^2y^2\} is a $k$-basis of $Z(kH)$.

(ii) The set \{x^2, y^2, xy + x^2y + xy^2, x^2y^2\} is a $k$-basis of $J(Z(kH))$.

(iii) The set \{x^2y^2\} is a $k$-basis of $\text{soc}(Z(kH))$, and $J(Z(kH))^2 = \text{soc}(Z(kH))$. In particular, $\dim_k(J(Z(kH))^2) = 1$.

(iv) The $k$-algebra $Z(kH)$ is symmetric.

**Proof.** By Lemma 4.1 we have $Z(kH) \cong (kP)^E$. Since $k$ has characteristic 3, we have an isomorphism $kP \cong k[x, y]/(x^3, y^3)$ induced by the map given in the statement of the lemma. Under this isomorphism, the action of $t$ on $x$ and $y$ is given by $x^t = x^2 + 2x$ and $y^t = y^2 + 2y$ as stated. It is straightforward to then verify that this isomorphism gives

$$ r + r^t \mapsto x^2 + 2, $$
$$ r^2 + r^t \mapsto y^2 + 2, $$
$$ rs + (rs)^t \mapsto 2 + x^2 + y^2 + 2xy + 2x^2y + 2xy^2 + x^2y^2, $$
$$ r^2s + (r^2s)^t \mapsto 2 + x^2 + y^2 + xy + x^2y + xy^2. $$

This proves the statement (i) and (ii). A straightforward computation proves statement (iii). The final statement is given in general in [R, Corollary 1.3], with an explicit symmetrising form $s : Z(kH) \to k$ given by $s(x^2y^2) = 1$ and sending all other basis elements to 0.  

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5 Proof of Theorem 1.1

Theorem 1.1 will be an immediate consequence of Theorem 2.1 and the following result.

Theorem 5.1. Let $k$ be an algebraically closed field of prime characteristic $p$, and let $A$ be the algebra given in Theorem 2.1. Then $A$ is not isomorphic to a basic algebra of a block of a finite group algebra over $k$.

Proof. Arguing by contradiction, suppose that $A$ is isomorphic to a basic algebra of a block $B$ of $kG$, for some finite group $G$. Denote by $P$ a defect group of $B$. By Theorem 2.1, we have $p = 3$ and $P \cong C_3 \times C_3$. By Lemma 3.3, the stable centre $Z(A)$ is symmetric, hence so is $Z(B)$, as $A$ and $B$ are Morita equivalent. It follows from [8, Proposition 3.8] that we have an algebra isomorphism $Z(A) \cong (kP)^E$ where $E$ is the inertial quotient of the block $B$. Again by Lemma 3.3, we have $\dim_k((kP)^E) = 5$, or equivalently, $E$ has five orbits in $P$. The list of possible inertial quotients in Kiyota’s paper [9] shows that $E$ is isomorphic to one of $1, C_2, C_2 \times C_2, C_4, D_8, Q_8, SD_{16}$. In all cases except for $E \cong C_2$ is the action of $E$ on $P$ determined, up to equivalence, by the isomorphism class of $E$. Thus if $E$ contains a cyclic subgroup of order 4, then $E$ has at most 3 orbits, and if $E$ is the Klein four group, then $E$ has 4 orbits. Therefore we have $E \cong C_2$. If the nontrivial element $t$ of $E$ has a nontrivial fixed point in $P$ (or equivalently, if $t$ centralises one of the factors $C_3$ of $P$ and acts as inversion on the other), then $E$ has 6 orbits. Thus $t$ has no nontrivial fixed point in $P$, and the group $H = P \times E$ is the Frobenius group considered in the previous section. By a result of Puig [14, 6.8] (also described in [15, Theorem 10.5.1]), there is a stable equivalence of Morita type between $B$ and $kH$, hence between $A$ and $kH$. By a result of Broué [5, 5.4] (see also [14, Corollary 2.17.14]), there is an algebra isomorphism $Z(A) \cong Z(kH)$. This, however, contradicts the calculations in the Lemmas 3.3 and 4.2 which show that the dimension of $J(Z(A))^2$ and of $J(Z(kH))^2$ are different. This contradiction completes the proof.

6 Further remarks

Using the arguments of the proof of Theorem 5.1 it is possible to prove some slightly stronger statements about the stable equivalence class of the algebra $A$ from Theorem 2.1.

Proposition 6.1. Let $k$ be an algebraically closed field of prime characteristic $p$ and let $A$ be the algebra in Theorem 2.1. Let $P$ be a finite $p$-group, $E$ a $p'$-subgroup of Aut($P$), and $\tau \in H^2(E; k^\times)$. There does not exist a stable equivalence of Morita type between $A$ and the twisted group algebra $k_\tau(P \rtimes E)$.

Proof. Arguing by contradiction, suppose that there is a stable equivalence of Morita type between $A$ and $k_\tau(P \rtimes E)$. Note that $k_\tau(P \rtimes E)$ is a block of a central $p'$-extension of $P \rtimes E$ with defect group $P$, so its Cartan matrix has a determinant divisible by $|P|$. By [14, Proposition 4.14.13], the
Cartan matrices of the algebras $A$ and $k_r(P \rtimes E)$ have the same determinant, which is 9. Since $A$ is clearly not of finite representation type (cf. Remark 3.4), it follows that $P$ is not cyclic, hence $P \cong C_3 \times C_3$. Using as before Broué’s result [5, 5.4], we have an isomorphism $Z(A) \cong Z(k_r(P \rtimes E))$. Since $Z(A)$ is symmetric, so is $Z(k_r(P \rtimes E))$. Since $k_r(P \rtimes E)$ is a block of a central $p'$-extension of $P \rtimes E$ with defect group $P$ and inertial quotient $E$, it follows again from [5, Proposition 3.8] that $Z(A) \cong (kP)^E$. From this point onward, the rest of the proof follows the proof of Theorem 5.1, whence the result. 

Remark 6.2. By results of Rouquier [18, 6.3] (see also [12, Theorem A2]), for any block $B$ with an elementary abelian defect group of rank 2 there is a stable equivalence of Morita type between $B$ and its Brauer correspondent, which by a result of Külshammer [11], is Morita equivalent to a twisted semidirect product group algebra as in Proposition 6.1. Thus Theorem 5.1 can be obtained as a consequence of Proposition 6.1 and Rouquier’s stable equivalence.

Remark 6.3. A slightly different proof of Theorem 5.1 makes use of Broué’s surjective algebra homomorphism $Z(B) \to (kZ(P))^E$ from [4, Proposition III (1.1)], induced by the Brauer homomorphism $Br_P$, where here $P$ is a (not necessarily abelian) defect group of a block $B$ of a finite group algebra $kG$, with $k$ an algebraically closed field of prime characteristic $p$. If $P$ is normal in $G$, then it is easy to see that Broué’s homomorphism is split surjective, but this is not known in general. If $B$ is a block with $P$ nontrivial such that there exists a stable equivalence of Morita type between $B$ and its Brauer correspondent, then this implies the existence of at least some split surjective algebra homomorphism $Z(B) \to kZ(P)^E$.

Kiyota’s list in [9] shows that if $A$ were isomorphic to a basic algebra of a block with defect group $P \cong C_3 \times C_3$, then $E$ would be isomorphic to one of $C_2$ or $D_8$ (subcase (b) in Kiyota’s list). The case $C_2$ can be ruled out as above, and the case $D_8$ can be ruled out by using Rouquier’s stable equivalence, and by showing that if $E \cong D_8$, then $(kP)^E$ is uniserial of dimension 3, but $Z(A)$ admits no split surjective algebra homomorphism onto a uniserial algebra of dimension 3. Note that $Z(A)$ does though admit a surjective algebra homomorphism onto a uniserial algebra of dimension 3, so the splitting is an essential point in this argument, and may warrant further investigation.

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