Tunnelling in Dante’s Inferno

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Abstract

We study quantum tunnelling in Dante’s Inferno model of large field inflation. Such a tunnelling process, which will terminate inflation, becomes problematic if the tunnelling rate is rapid compared to the Hubble time scale at the time of inflation. Consequently, we constrain the parameter space of Dante’s Inferno model by demanding a suppressed tunnelling rate during inflation. The constraints are derived and explicit numerical bounds are provided for representative examples. Our considerations are at the level of an effective field theory; hence, the presented constraints have to hold regardless of any UV completion.
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1 Introduction

The slow-roll inflation paradigm has been phenomenologically successful, initially solving the naturalness issues in Big Bang Cosmology, and later explaining the primordial density perturbations. General predictions of slow-roll inflation on primordial density perturbations agree very well with recent Cosmic Microwave Background (CMB) observations.

Nevertheless, slow-roll inflation has its own naturalness issue. Protecting the flatness of the inflaton potential against quantum corrections has been a long-standing challenge. This issue is particularly severe in large field inflation models in which the inflaton enjoys super-Planckian field excursion.

A standard approach to explain the flatness of a potential in an effective field theory is imposing a symmetry. For example, natural inflation [1] assumes a continuous shift of an axion field as an approximate symmetry. The comparison of this model with CMB data requires a super-Planckian axion decay constant. Naively, this indicates that the symmetry must be respected at the Planck scale. However, there are strong indications that continuous global symmetries are not respected in a quantum theory of gravity (see [2] for a recent discussion together with a review of earlier studies).
An approach to circumvent this problem was proposed under the name of extra-
natural inflation [3]. This model realises a super-Planckian axion decay constant in four
dimensions by means of an effective gauge field theory in higher dimensions. The super-
Planckian axion decay constant is achieved at the expense of a very small gauge coupling. However, it was immediately noticed that this model is difficult to realise in string theory
[3, 4]. The lasting difficulty in realising extra-natural inflation in string theory led to the Weak Gravity Conjecture [5], which limits the relative weakness of gauge forces compared
to the gravitational force. This conjecture may eventually forbid a super-Planckian axion
decay constant in effective field theories which can consistently couple to gravity, though
several logical steps need to be examined in more detail.

If a super-Planckian axion decay constant is forbidden in effective field theories which
are consistently coupled to gravity, then a new major obstacle for the realisation of large
field inflation via natural inflation arises. However, a possible way out may be axion
monodromy inflation [6–10]. In this class of models, the axion decay constant is sub-
Planckian, but the axion couples to an additional degree of freedom, which we call winding
number direction below. An effective super-Planckian excursion of the inflaton is achieved
by going through the axion direction multiple times, with a shift in the winding number
direction for each round. This appears to be a promising avenue for realising large field
inflation. Nevertheless, the validity of axion monodromy inflation should be examined
further, both at the level of an effective field theory as well as at the level of an UV
completion. In particular, it has been pointed out that quantum tunnelling through the
potential roughly in the winding number direction may terminate inflation before it lasts
long enough for solving the naturalness issues in Big Bang Cosmology [10]. It turned
out that the tunnelling rate is highly model dependent. Tunnelling in related models has
subsequently been studied in [11–17].

In this article, we study tunnelling in an axion monodromy model, namely Dante’s
Inferno model [9]. We limit our study to the level of an effective field theory.

Besides phenomenological interests in this promising model, there is an attractive
technical feature: The potential wall orthogonal to the inflaton direction is explicitly
given. This allows us to apply a standard calculation à la Coleman [18] in order to
estimate the tunnelling rate. In particular, one can estimate the tension of the surface of
the bubble, through which the false “vacuum” decays. This is in contrast to other axion
monodromy models for which the tension of the wall is treated as an input from a UV
theory [10, 11, 13–17].

We constrain the parameter space of Dante’s Inferno model by requiring a suppressed
tunnelling rate during inflation. In particular, we will show that in some regions of
the parameter space, the suppression of the tunnelling process yields a new constraint.

1 Precisely speaking, during inflation the state is not in a local minimum of the potential, but slowly
rolling in \( \phi \)-direction. This point has been investigated in [16]. In this article, we will loosely use
the term (false) “vacuum” for such configurations, because we will be mainly dealing with slices of constant
\( \phi \) of the potential in which the state is in a local minimum.
This constraint comes purely at the level of an effective field theory; hence, regardless of the UV completion of the theory, the constraint has to hold. For a fixed ratio $\Lambda/f_1$, where $\Lambda$ is the parameter controlling the height of the sinusoidal potential and $f_1$ is the smaller axion decay constant in Dante’s Inferno model, the condition that tunnelling is suppressed introduces a lower bound on $f_1$ in such a parameter region. We demonstrate this observation by providing explicit numerical bounds in a couple of representative examples.

The outline of this article is as follows: Dante’s Inferno model is briefly reviewed in Sec. 2. Thereafter, we discuss quantum tunnelling and suppression thereof in Sec. 3. We exemplify these considerations for the choice of a monomial inflaton potential in Sec. 4. Lastly, Sec. 5 concludes. Three appendices provide the necessary background and details for choosing constant field values during inflation, the bounce solution, and the thin-wall approximation.

2 Dante’s Inferno model

In this section, we review Dante’s Inferno model [9] and fix our notation. The dynamics of the model are governed by the following action:

$$S_{DI} = \int d^4x \sqrt{-g} \left[ \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - V_{DI}(\phi_1, \phi_2) \right],$$  

(2.1a)  

where the scalar potential is given by

$$V_{DI}(\phi_1, \phi_2) = V_1(\phi_1) + \Lambda^4 \left( 1 - \cos \left( \phi_1 f_1 - \phi_2 f_2 \right) \right).$$  

(2.1b)  

Fig. 1 displays the behaviour of the potential $V_{DI}(\phi_1, \phi_2)$ for some parameter values. It is convenient to perform the following rotation in the field space:

$$\begin{pmatrix} \chi \\ \phi \end{pmatrix} = \begin{pmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix},$$  

(2.2a)  

where

$$\sin \gamma := \frac{f_1}{\sqrt{f_1^2 + f_2^2}}, \quad \cos \gamma := \frac{f_2}{\sqrt{f_1^2 + f_2^2}}.$$  

(2.2b)  

In terms of the rotated fields, the potential (2.1b) becomes

$$V_{DI}(\chi, \phi) = V_1(\chi \cos \gamma + \phi \sin \gamma) + \Lambda^4 \left( 1 - \cos \frac{\chi}{f} \right),$$  

(2.3a)  

where

$$f := \frac{f_1 f_2}{\sqrt{f_1^2 + f_2^2}}.$$  

(2.3b)
Figure 1: The potential $V_{DI}(\phi_1, \phi_2)$ of Dante’s Inferno model for the exemplary parameters $\Lambda = 10^{-3}$, $f_1 = 10^{-4}$, and $f_2 = 20 \cdot f_1$, together with a monomial inflaton potential, c.f. Sec. 4 with parameters $p = 1$ and $N = 60$. The reference point $\phi_{1*}$ is defined around (2.7). The boundaries of the $\phi_2$-direction in the plot should be identified. Going around in $\phi_2$-direction through the valley of the potential results in a shift in $\phi_1$-direction.

According to [9], the following two conditions are required for Dante’s Inferno model:

\begin{align}
2\pi f_1 & \ll 2\pi f_2 \lesssim M_P , \\
\frac{\Lambda^4}{f} & \gg V'_1 .
\end{align}

Here, $M_P := (8\pi G)^{-1/2}$ is the reduced Planck mass with $G$ being Newton’s constant. For later convenience, we rewrite condition (2.4b) as

\begin{align}
s := \frac{\Lambda^4}{fV'_1} & \gg 1.
\end{align}

The last inequality in (2.4a) is expected to follow from the Weak Gravity Conjecture as we reviewed in the Introduction, which we assume in this article. Next, condition (2.4a) implies

\begin{align}
f & \simeq f_1 , \quad \cos \gamma \simeq 1 , \quad \sin \gamma \simeq \frac{f_1}{f_2} \ll 1 .
\end{align}

Now, let us take a closer look on the origin of condition (2.4a). In large field inflation, we typically have $\phi_* \gtrsim 10M_P$, where the suffix * indicates that it is the value when the pivot scale exited the horizon (see (4.5) in Sec. 4 for values of $\phi_*$ in explicit examples). This constrains $\sin \gamma$ via

\begin{align}
\sin \gamma = \frac{\phi_{1*}}{\phi_*} .
\end{align}
For the effective field theory description of the potential $V_1(\phi_1)$ to be valid, a natural expectation is that $\phi_{1*}$ is bounded from above by the reduced Planck scale $M_P$. This assumption together with (2.7) implies that $\sin \gamma \lesssim 0.1$ whenever $\phi_{*} \gtrsim 10M_P$. Note that the effective description may break down at a much smaller energy scale, such that the value of $\phi_{1*}$ decreases accordingly. For example, for a moderate model assumption $\phi_{1*} \lesssim 10^{-1}M_P$, then (2.7) imposes $\sin \gamma \lesssim 10^{-2}$.

Next, let us examine condition (2.4b), which implies that the field $\chi$ first settles down to the local minimum in a slice of constant $\phi$ before the field $\phi$, which plays the role of inflaton in Dante’s Inferno model, starts to slow-roll. Then, from (2.3a) the inflaton potential $V_I(\phi)$ is given by

$$V_I(\phi) = V_1(\phi \sin \gamma) .$$

(2.8)

We refer to Fig. 1 to illustrate that the inflaton rolls along the bottom of the valley. As one observes, there seem to be numerous valleys in the potential, but all of them are connected by the periodic identification in $\phi_2$-direction. As the inflaton rolls along the valley one period in $\phi_2$-direction, the bottom of the valley is shifted in $\phi_1$-direction. While the axion decay constant $f_2$ is sub-Planckian as in (2.4a), super-Planckian inflaton excursion can be achieved by going round in $\phi_2$-direction several times.

However, the slow-roll inflation may terminate if quantum tunnelling through the wall of the valley happens. Requiring that the tunnelling rate is sufficiently small compared to the Hubble time scale during inflation may impose further constraints on the parameter space of Dante’s Inferno model. We will explore the consequences of this requirement in the next section.

2.1 Dante’s Inferno model from higher dimensional gauge theories

Dante’s Inferno model can be obtained from higher dimensional gauge theories. In this circumstance there is an additional constraint on the parameters [19,20], which reads

$$\Lambda^4 \sim \frac{3c}{\pi^2(2\pi L_5)^4} ,$$

(2.9)

where the natural value of $c$ is $O(1)$. The axion decay constants $f_1$ and $f_2$ are given as

$$f_1 = \frac{1}{g_1(2\pi L_5)} , \quad f_2 = \frac{1}{g_2(2\pi L_5)} ,$$

(2.10)

where $g_1$ and $g_2$ are the gauge couplings in four-dimension. From (2.9) and (2.10), and assuming that the perturbative approximation is valid, i.e. $g_1 \lesssim 1$, we obtain

$$\Lambda \lesssim f .$$

(2.11)
3 Tunnelling in Dante’s Inferno model

It is well-known that a quantum field theory with two local minima, $\psi_{\pm}$, of the potential has two classically stable equilibrium states. However, assuming that $\psi_{-}$ is the unique state with lowest energy, the state $\psi_{+}$ is rendered unstable quantum mechanically, because of a non-vanishing tunnelling probability through the potential barrier into the so-called true vacuum state $\psi_{-}$. The decay of a false vacuum $\psi_{+}$ proceeds by nucleation of bubbles, inside which the true vacuum\(^2\) resides. The tunnelling rate per volume $\Gamma/\text{Vol}$ between true and false vacua, as discussed in [18, 21], can be parametrised by two quantities $A$ and $B$ (in leading order) via

$$\frac{\Gamma}{\text{Vol}} = A e^{-B/\hbar} \left[ 1 + \mathcal{O}(\hbar) \right]. \quad (3.1)$$

While the details of the coefficient $A$ are somewhat complicated, it is possible to provide a closed expression for $B$ solely from the semi-classical treatment. The relevant solution has been referred to as *bounce* and is reviewed in App. B. From (3.1) it is apparent that the tunnelling process is suppressed provided $B \gg \hbar$ and the pre-factor $A$ is well-behaved. A dimensional analysis of the pre-factor reveals $A \sim M^4$, where $M$ is a relevant mass scale in the model. (We refer, for example, to [22, 23] for numerical calculations of the coefficients in the case of a simple scalar field theory.) This estimate may be off by a few orders, but the error will still be small compared to the exponential suppression factor $e^{-B/\hbar}$. However, since $B$ is positive, there may be scenarios in which the tunnelling is not exponentially suppressed, i.e. $e^{-B/\hbar} \sim \mathcal{O}(1)$. For instance in inflation models, if $A$ induces a rapid rate compared to the Hubble time scale during inflation, then the tunnelling becomes potentially dangerous as it might terminate inflation too early. More precisely, this happens if $A \gtrsim H^4$, where $H$ is the Hubble expansion rate at the time of inflation. Consequently, two cases arise:

- On the one hand, if all relevant scales in the model are smaller than $H$, the tunnelling rate is irrelevant during inflation, regardless of the precise order of $B$.

- If, on the other hand, we assume that all relevant scales in Dante’s Inferno model satisfy $\Lambda, f_1, f_2 \gtrsim H$ then one has to carefully verify which subsequent parameter regions are protected from an unsuppressed tunnelling rate.

It is therefore the objective of this article to analyse the exponent $B$ together with the condition $B \gg 1$ for Dante’s inferno model for inflation in the regime $\Lambda, f_1, f_2 \gtrsim H$. As customary, we set $\hbar \equiv 1$ for the rest of this article. Obtaining a viable parameter region in Dante’s Inferno model then means that one has to avoid scenarios in which the tunnelling in $\chi$-direction is unsuppressed. In those cases, one can investigate the dynamics of the field $\chi$, while regarding the value of $\phi$ as being fixed in time\(^3\). We refer to App. A for a

\(^2\)Below will study tunnelling between a false vacuum and another false vacuum with lower energy, which can be analysed without introducing new ingredients.

\(^3\)A path with varying $\phi$ gives a larger action and is irrelevant for the estimation of the tunnelling rate.
discussion of the effects of a time-dependent $\phi$. Since we are interested in the tunnelling rate during the slow-roll inflation, we choose the value of the inflaton when the pivot scale exited the horizon, $\phi = \phi_*$, as a reference point. (We comment briefly on other values of $\phi$ at the end of Sec. 4.1.) Then, from (2.3a) the potential $V(\chi)$ for the field $\chi$ becomes

$$V(\chi) := V_1(\chi \cos \gamma + \phi_* \sin \gamma) + \Lambda^4 \left(1 - \cos \frac{\chi}{f}\right).$$

(3.2)

We first estimate the tunnelling rate including the effects of gravity in order to understand when we can neglect the gravitational back-reactions. Following [24], the Euclidean action of a scalar field $\chi$ coupled to Einstein gravity reads

$$S_E = \int d^4x \sqrt{g} \left[\frac{1}{2} g^{\mu\nu} \partial_\mu \chi \partial_\nu \chi + V(\chi) - \frac{1}{16\pi G} R\right].$$

(3.3)

To estimate the gravitational back-reaction, we employ an $O(4)$-symmetric ansatz. There are few limitations of such an ansatz: Firstly, inflation with (almost) flat spatial space, which is supported by observations, does not respect $O(4)$ symmetry. Secondly, there is no proof that the $O(4)$-symmetric bounce gives the least action among all bounce solutions. We will not try to fully justify the use of an $O(4)$-symmetric bounce in this article. Nevertheless, since the space is empty during inflation, and we will be interested in processes which proceed fast compared to the Hubble expansion rate, we hope that the first point may not be so crucial. For the second point, we expect that even if there exists a non-$O(4)$-symmetric bounce, with smaller action than the $O(4)$-symmetric bounce, the $O(4)$-symmetric bounce provides at least the lower bound for the tunnelling rate. Moreover, we may expect that the difference between the constraints on the parameter space of Dante’s Inferno model from the non-$O(4)$-symmetric bounce do not differ qualitatively from those of the $O(4)$-symmetric bounce.

Assuming $O(4)$ symmetry, the metric takes the form

$$ds^2 = d\xi^2 + a^2(\xi) d\Omega^2,$$

(3.4)

where $d\Omega^2$ is the canonical metric of the unit $S^3$. Moreover, the $O(4)$ symmetry restricts the field $\chi$ to be a function of the radial coordinate $\xi$ only. Thus, for $O(4)$-symmetric solutions, the Euclidean action (3.3) becomes

$$S_E = 2\pi^2 \int d\xi \left[a^3 \left(\frac{1}{2} \left(\frac{d\chi}{d\xi}\right)^2 + V(\chi)\right) - \frac{3}{16\pi G} a \left(\left(\frac{da}{d\xi}\right)^2 + 1\right)\right].$$

(3.5)

We have dropped a surface term, which is irrelevant, because we consider the difference of actions with the same boundary conditions [24]. It is convenient to rescale the variables as follows:

$$\psi := \frac{\chi}{f}, \quad \rho := fa, \quad \zeta := f\xi.$$

(3.6)
Then (3.5) becomes
\[ S_E = 2\pi^2 \int d\zeta \left[ \rho^3 \left( \frac{1}{2} \dot{\psi}^2 + U(\psi) \right) - \frac{3}{\kappa} \rho \left( \rho^2 + 1 \right) \right], \] (3.7a)
where
\[ \kappa := 8\pi G f^2, \] (3.7b)
\[ U(\psi) := U_0(\psi) + U_1(\psi), \] (3.7c)
\[ U_0(\psi) := \lambda^4 \left( 1 - \cos \psi \right), \] (3.7d)
\[ \lambda := \frac{\Lambda}{f}, \] (3.7e)
\[ U_1(\psi) := \frac{1}{f^4} \left( f \psi \cos \gamma + \phi^* \sin \gamma \right). \] (3.7f)

The Euclidean equations of motion are given as
\[ \ddot{\psi} + \frac{3}{\rho} \dot{\rho} \dot{\psi} = U'(\psi), \] (3.8a)
\[ \dot{\rho}^2 - 1 = \frac{\kappa}{3} \rho^2 \left( \frac{1}{2} \dot{\psi}^2 - U(\psi) \right), \] (3.8b)
where (3.8b) is the Friedmann equation. The bounce action \( B \) reads
\[ B := S_E[\psi_B] - S_E[\psi_+], \] (3.9)
where \( \psi_B \) is the bounce solution, and \( \psi_+ \) is the value of the field \( \psi \) at the false vacuum we start with, \( \psi_+ = 0 \) in our case.

Similarly to [18], we evaluate the bounce action (3.9) in the so-called thin-wall approximation, which holds provided the following two conditions are satisfied:

(i) The height of the barrier of the potential is much larger than the energy difference
\[ \Delta U := U(\psi_+) - U(\psi_-) \] (3.10)
between a false vacuum and another false vacuum, to which the tunnelling occurs.

(ii) The width of the surface wall of the bubble, through which the initial false vacuum decays, is much smaller than the bubble size.

In our case, the condition (i) gives
\[ \Delta U \ll 2\lambda^4. \] (3.11)

We examine the remaining condition (ii) along the way.
The bounce action for a general potential within the thin-wall approximation has been presented in [25]. In terms of our variables, the bounce action is given in (B.13) of App. B. Defining

\[ h_0 := \frac{H_0}{f}, \quad H_0 := \sqrt{\frac{8\pi G V_1 (\phi_* \sin \gamma)}{3}}, \quad (3.12a) \]

the bounce action reads as follows:

\[
B \simeq \frac{2 \cdot 27 \pi^2 (8\lambda^2)^4}{\sqrt{(\Delta U - 48\lambda^4 \kappa)^2 + 12h_0^2(48\lambda^4)}} \times \frac{1}{(\Delta U + \sqrt{(\Delta U - 48\lambda^4 \kappa)^2 + 12h_0^2(48\lambda^4)})^2 - (48\lambda^4 \kappa)^2}, \quad (3.12b)
\]

From (3.12b) we observe that gravitational back-reactions are negligible whenever

\[ \kappa \ll \max \left\{ \frac{\Delta U}{48\lambda^4}, \frac{h_0}{2\lambda^2} \right\}. \quad (3.13) \]

When (3.13) is satisfied, the bounce action reduces to

\[
B \simeq \frac{2 \cdot 27 \pi^2 (8\lambda^2)^4}{\sqrt{(\Delta U)^2 + 12h_0^2(48\lambda^4)} \left(\Delta U + \sqrt{(\Delta U)^2 + 12h_0^2(48\lambda^4)}\right)^2}. \quad (3.14)
\]

The demand (3.13) suggests that expression (3.14) simplifies further in two extreme cases: in the following subsection we assume that either \(\Delta U/48\lambda^4\) is much larger than \(h_0/2\lambda^2\) or vice versa.

### 3.1 Flat-space limit

Let us first look at the situation that space-time can be regarded as flat, i.e. the effect of the curvature, represented by \(h_0\), of the de Sitter space is negligible:

\[ \frac{\Delta U}{48\lambda^4} \gg \frac{h_0}{2\lambda^2}, \quad (3.15) \]

which we will refer to as flat-space limit. In this case, the action (3.14) reduces to the result of Coleman [18]:

\[
B \simeq B_0 = \frac{27 \pi^2 S_1^4}{2(\Delta U)^3}, \quad (3.16a)
\]

where

\[
S_1 := 2 \int d\zeta (U_0(\psi_B) - U_0(\psi_-)) = 8\lambda^2, \quad (3.16b)
\]
We refer to (C.4) for the explicit calculation in our set-up. As shown in App. C, the thickness of the surface wall is $\sim 2/\lambda^2$. Recalling (ii), the thin-wall approximation is valid in the flat-space limit if the bubble size $\bar{\rho}$ satisfies

$$\bar{\rho} = \bar{\rho}_0 = \frac{3S_1}{\Delta U} = \frac{24\lambda^2}{\Delta U} \gg 2/\lambda^2,$$  

which is equivalent to

$$\frac{\Delta U}{12\lambda^4} \ll 1.$$  

(3.17b)

We observe that (3.17b) is satisfied due to (3.11). Note that in the flat-space limit the condition (3.13) for negligible gravitational back-reaction reduces to

$$\kappa \ll \frac{\Delta U}{48\lambda^4}.$$  

(3.18)

### 3.2 De Sitter limit

Next, let us look at the opposite limit of the flat-space limit (3.15), in which the effect of the curvature of the de Sitter space, represented by $h_0$, is dominant:

$$\frac{\Delta U}{48\lambda^4 h_0} \ll \frac{h_0}{2\lambda^2}.$$  

(3.19)

We refer to this limit as de Sitter limit. In this case the bounce action (3.14) becomes

$$B \simeq \frac{16\pi^2\Lambda^2 f}{H_0^3}.$$  

(3.20)

So far we have kept the inflaton potential general. In order to quantitatively discuss constraints arising from a suppressed tunnelling rate, we specify the inflaton potential in the next section.

### 4 Examples: Chaotic inflation

Let us study examples with an inflaton potential $V_I(\phi)$ given by a monomial, i.e.

$$V_I(\phi) = V_p(\phi) := \alpha_p \frac{\phi^p}{p!}.$$  

(4.1)

In this section, we will work in the unit $M_P \equiv 1$. Without loss of generality, we take $\alpha_p > 0$ and assume that inflation took place when $\phi > 0$. The associated slow-roll parameters are defined as follows:

$$\epsilon_V(\phi) := \frac{1}{2} \left( \frac{V_p'}{V_p} \right)^2 = \frac{p^2}{2\phi^2},$$  

(4.2a)
\[
\eta_V(\phi) := \frac{V_p'}{V_p} = \frac{p(p-1)}{\phi^2}.
\] (4.2b)

In slow-roll inflation, the spectral index \(n_s\) and the tensor-to-scalar ratio \(r\) can be calculated via

\[
n_s = 1 - 6\epsilon_V(\phi_*) + 2\eta_V(\phi_*) , \quad (4.3a)
\]
\[
r = 16\epsilon_V(\phi_*) , \quad (4.3b)
\]

where \(\ast\) refers to the value when the pivot scale exited the horizon. The CMB observations constrain \(\epsilon_V, |\eta_V| \lesssim \mathcal{O}(10^{-2})\) through the relations (4.3), see for instance [26]. The number of e-folds \(N\) is readily computed to read

\[
N(\phi) = \left| \int_{\phi_{\text{end}}}^{\phi} d\phi \frac{V_p'}{V_p} \right| = \left| \int_{\phi_{\text{end}}}^{\phi} d\phi \frac{\phi^2}{p} \right| = \frac{1}{2p} \left[ \phi^2 \right]_{\phi_{\text{end}}}^{\phi} = \frac{1}{2p} \left( \phi^2 - \phi_{\text{end}}^2 \right) \quad , \quad (4.4a)
\]

where we define \(\phi_{\text{end}}\) by the condition

\[
\epsilon_V(\phi_{\text{end}}) = 1 \quad , \quad (4.4b)
\]

which in the examples under consideration gives

\[
\phi_{\text{end}} = \frac{p}{\sqrt{2}} \quad . \quad (4.4c)
\]

Inserting (4.4c) into (4.4a) and solving \(\phi_*\) for a given \(N_* := N(\phi_*)\) yields

\[
\phi_* = \sqrt{2p \left( N_* + \frac{p^2}{4} \right)} \quad . \quad (4.5)
\]

The scalar power spectrum in slow-role inflation is given as

\[
P_s = \frac{V_p(\phi_*)}{24\pi^2\epsilon_V(\phi_*)} = 2.2 \cdot 10^{-9} \quad , \quad (4.6)
\]

where the numerical value stems from CMB observations [26].

The coefficient \(\alpha_p\) in (4.1), for a given \(N_*\), is determined by first computing \(\phi_*\) via (4.5), then inserting this value into (4.6) and subsequently solving for \(\alpha_p\). Explicitly,

\[
P_s = \frac{\alpha_p \phi_*^{p+2}}{12\pi^2 p! p^2} = 2.2 \cdot 10^{-9} \quad , \quad (4.7a)
\]

thus

\[
\alpha_p = 12\pi^2 \frac{p! p^2}{\phi_*^{p+2}} \cdot P_s = 12\pi^2 \frac{p! p^2}{\phi_*^{p+2}} \cdot 2.2 \cdot 10^{-9} \quad . \quad (4.7b)
\]

Now, we use this input data from inflation models constrained by CMB observations to estimate the corresponding tunnelling rate in Dante’s Inferno model. The parameter \(\Delta U\), as defined in (3.10), reads in the current example as follows:

\[
\Delta U = \frac{1}{f^4} \left( V_1(\phi_* \sin \gamma) - V_1(-2\pi f \cos \gamma + \phi_* \sin \gamma) \right)
\]
\[
\frac{1}{f^4} \left( V_p(\phi_*) - V_p(\phi_* - 2\pi f \cot \gamma) \right)
\approx \cot \gamma \frac{2\pi V'_p(\phi_*)}{f^3},
\]

(4.8)

where we have used two ingredients to obtain the last line: Firstly, we employed (2.6), more precisely

\[2\pi f \cot \gamma \approx 2\pi f_2,\]

(4.9)

and, secondly, due to the smallness of the slow-roll parameters \(\epsilon_V\) and \(\eta_V\), see (4.2), it follows that the inflaton potential \(V_p(\phi)\) around \(\phi \sim \phi_*\) does not change much over the Planck scale, i.e. \(M_P \gtrsim 2\pi f_2\).

In the following two subsections we examine the tunnelling rate in two scenarios: Firstly, in the flat-space limit and, secondly, in the de Sitter limit.

### 4.1 Flat-space limit

We begin with the parameter region of Dante’s Inferno model in which the flat-space limit (3.15) is appropriate, i.e.

\[
\frac{\Delta U}{48\lambda^4} \gg \frac{h_0}{2\lambda^2}.
\]

(4.10)

For negligible gravitational back-reaction the bounce action in the flat-space limit is provided in (3.16a). We investigate the validity of the negligibility of the gravitational back-reaction later in the subsection. Inserting (4.8) into (3.16a) yields

\[
B = \frac{27 \cdot 2^8 \Lambda^8 f}{\pi} \left( \frac{\tan \gamma}{V'_p(\phi_*)} \right)^3,
\]

(4.11)

where we have used \(S_1 = 8\lambda^2\) for our set-up (c.f. (C.4) in App. C). Then, for a suppressed tunnelling process, i.e. \(B \gg 1\), the following condition has to hold:

\[
\tan \gamma \gg \left( \frac{27 \cdot 2^8 \Lambda^8 f}{\pi} \right)^{-1/3} V'_p(\phi_*) =: \tan \gamma_T.
\]

(4.12a)

Using (4.4)–(4.7), the explicit form of \(\tan \gamma_T\) reads as

\[
\tan \gamma_T = 2^{5/6} \pi^{7/3} \cdot \frac{1}{(f \Lambda^8)^{1/3}} \cdot \left[ P_* \left( \frac{p}{4N_* + p} \right)^{3/2} \right].
\]

(4.12b)

In (4.12b) the numerical factor in the squared brackets is determined by the parameters of the inflation model \(p, N_*\), and CMB observations (4.6). The constraint (4.12a) should be compared with the defining condition of Dante’s Inferno model (2.5), which in terms of the parameters of the model gives

\[
\frac{\Lambda^4}{f} \gg \cot \gamma V'_p(\phi_*) ,
\]

(4.13a)
or equivalently
\[ \tan \gamma \gg \frac{f}{\Lambda} V'_{p}(\phi_{*}) =: \tan \gamma_{DI}. \] (4.13b)

In the above, we have used
\[ \frac{dV_{p}}{d\phi} = \frac{d}{d\phi} V_{1}(\phi \sin \gamma) = \frac{d\phi_{1}}{d\phi} \frac{dV_{1}}{d\phi_{1}}(\phi_{1} = \phi \sin \gamma) = \sin \gamma \frac{dV_{1}}{d\phi_{1}}(\phi_{1}), \] (4.14)
which follows from (2.8) and the usual chain rule. Again, using (4.4)–(4.7) allows to specialise \( \tan \gamma_{DI} \) to
\[ \tan \gamma_{DI} = 24\sqrt{2}\pi^{2} \cdot \frac{f}{\Lambda} \cdot \left[ P_{s} \left( \frac{p}{4N_{*} + p} \right)^{3/2} \right]. \] (4.15)

We are particularly interested in the scenario for which condition (4.12a) for suppressed tunnelling enforces a stronger condition on the model than (2.4b). From (4.12a) and (4.13b), this is the case for
\[ \tan \gamma_{T} > \tan \gamma_{DI}. \] (4.16)

In terms of the parameters of the model under consideration, the inequality (4.16) reduces to
\[ \left( \frac{27 \cdot 2^{8} \Lambda^{8} f}{\pi} \right)^{-1/3} > \frac{f}{\Lambda^{4}}, \] (4.17a)
or equivalently
\[ \frac{\Lambda}{f} \gg 7. \] (4.17b)
Whenever (4.17b) is satisfied, the constraint (4.12a), which ensures the suppression of the tunnelling, is more restrictive than the defining condition (2.5) of Dante’s Inferno model. In other words, in the region of the parameter space where (4.17b) holds tunnelling is not automatically suppressed in Dante’s Inferno model; thus, an additional constraint arises\(^5\).

Note that in Dante’s Inferno model derived from a higher dimensional gauge theory discussed in Sec. 2.1, (4.17b) assures that the tunnelling process is suppressed for natural values of the parameters (2.11) (at least in the simplest version of the model).

To demonstrate how condition (4.12a) constrains the parameter space, we illustrate the scenarios \( \Lambda/f = 10, p = 1, 2, \) and \( N_{*} = 60 \) in Fig. 2a and Fig. 3a, respectively. If one wishes to fix a certain value of \( \tan \gamma \) then the condition (4.12a) yields a lower bound on \( f \simeq f_{1} \). For example, if we demand \( \tan \gamma \sim 5 \cdot 10^{-2} \) then \( f \gtrsim 10^{-4} \) is required in the above cases, as can be read off from (4.12b) or Fig. 2a and Fig. 3a.

Now, let us focus on condition (4.10), which defines the flat-space limit. From (3.7e), (3.12a), and (4.8), one infers that condition (4.10) becomes
\[ \frac{\Delta U}{24 \lambda^{2} h_{0}} = \cot \gamma \pi \frac{V'_{p}(\phi_{*})}{12 \Lambda^{2} H_{0}} \gg 1, \] (4.18a)
\(^5\)As discussed in the beginning of Sec. 3, we only consider the region \( f_{1}, f_{2}, \Lambda \gtrsim H \).
which we recast as
\[ \tan \gamma \ll F := \frac{\pi V'_p(\phi_s)}{12 \Lambda^2 H_0}. \]  
(4.18b)

Specialising $F$ via (4.4)–(4.7) to the current model, we obtain
\[ F = \frac{1}{\Lambda^2} \cdot \left( \frac{\pi^2 \sqrt{P_s} \cdot \frac{p}{4N_s + p}}{p} \right) \sim \frac{1}{\Lambda^2} \cdot \mathcal{O}(10^{-6}), \]  
(4.18c)

where the last numerical value holds for $p = 1, 2$ with $N_s = 50 - 60$. As discussed around (2.7), Dante’s Inferno model requires $\tan \gamma \lesssim \mathcal{O}(10^{-1})$ or less. Thus, the flat-space limit is appropriate for
\[ \Lambda \ll \mathcal{O}(10^{-5}). \]  
(4.19)

From (4.18b) and (4.13b), one readily computes the following ratio:
\[ \frac{F}{\tan \gamma_{DI}} = \frac{\pi \Lambda^2}{12 fH_0} = \frac{1}{24\sqrt{2\pi}} \sqrt{\frac{4N_s + p}{p}} \cdot \frac{\Lambda^2}{f} \sim \mathcal{O}(10^5) \cdot \left( \frac{\Lambda}{f} \right) \Lambda, \]  
(4.20)

where we have used $p = 1, 2$ and $N_s \sim 50 - 60$. Hence, when $\Lambda/f \gtrsim 7$ as in (4.17b), then (4.20) implies $F \gg \tan \gamma_{DI}$, provided $\Lambda \gtrsim \mathcal{O}(10^{-5})$ holds.

Next, we examine condition (3.18) for negligible gravitational back-reaction. In terms of the parameters of the current model, we obtain
\[ \tan \gamma \ll K := \frac{\pi V'_p(\phi_s)}{24 \Lambda^2 f^3}. \]  
(4.21a)

By means of (4.4)–(4.7), we explicitly parametrise $K$ as
\[ K = \frac{1}{\Lambda^2 f^3} \cdot \left[ \sqrt{2\pi^3 P_s} \cdot \left( \frac{p}{4N_s + p} \right)^{3/2} \right] \sim \frac{1}{\Lambda^2 f^3} \cdot \mathcal{O}(10^{-11}), \]  
(4.21b)

where the last numerical value holds for $p = 1, 2$ with $N_s = 50 - 60$. As we assume $\tan \gamma \ll 1$ in Dante’s Inferno model, if $K \gtrsim 1$ then (4.21a) does not introduce a further constraint. Thus, $K \gtrsim 1$ whenever
\[ \Lambda^2 f^3 \lesssim \mathcal{O}(10^{-11}). \]  
(4.22)

For the range of the parameter $f$ as displayed in Fig. 2 to Fig. 3, $K$ is always much greater than 1 and, therefore, the gravitational back-reaction can be neglected.

We note that (4.21a) and (4.18b) imply the following ratio:
\[ \frac{K}{F} = \frac{H_0}{2f^3}. \]  
(4.23)

Then $H_0 \sim \mathcal{O}(10^{-5})$, for $p = 1, 2$ with $N_s = 50 - 60$ as previously used, implies that $K \gtrsim F$ for $f \lesssim \mathcal{O}(10^{-2})$. In this case, $\tan \gamma \ll K$ is automatically satisfied if $\tan \gamma \ll F$.  

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Finally, we verify the validity of the thin-wall approximation. Inserting (4.8) and (3.7e) into the condition (3.17a) for the validity of the thin-wall approximation gives

\[ \frac{6 \Lambda^4}{\pi f V_1'} \gg 1. \] (4.24)

We have used (4.14) to obtain (4.24). Using the parameter \( s \), as introduced in (2.5), one can rewrite (4.24) as

\[ \frac{6}{\pi s} \gg 1. \] (4.25)

We recall that \( s \gg 1 \) is one of the conditions (2.5) required in Dante’s Inferno model. Thus, the condition (4.25) for the validity of the thin-wall approximation gives numerically the same constraint on Dante’s Inferno model as (2.5), up to a minor difference of a \( O(1) \) numerical factor. As a consequence, the thin-wall approximation is always valid in Dante’s Inferno model in the flat-space limit.

![Graph](a) and (b)

Figure 2: The exemplary parameter values are \( p = 1, N = 60, \) and \( \Lambda = 10f. \)

Finally, we notice from (4.11) that, within the class of monomial inflation potentials, the tunnelling rate either stays constant (for \( p = 1 \)) or decreases (for \( p > 1 \)) as \( \phi_* \) decreases. Therefore, it is sufficient to estimating the tunnelling rate at \( \phi = \phi_* \) in order to verify the suppression of the tunnelling process in these cases.

### 4.2 De Sitter limit

In this subsection, we study the de Sitter limit (3.19) which gives

\[ \frac{\Delta U}{24\lambda^2 h_0} = \cot \gamma \frac{\pi V_p'/(\phi_*)}{12 \Lambda^2 H_0} \ll 1, \] (4.26a)
Figure 3: The exemplary parameter values are $p = 2$, $N = 60$, and $\Lambda = 10f$.

or equivalently

$$\tan \gamma \gg F := \frac{\pi}{12} \frac{V'_p(\phi_*)}{\Lambda^2 H_0} \sim \frac{1}{\Lambda^2} \cdot \mathcal{O}(10^{-6}) ,$$

where the last approximation holds for $p = 1, 2$ with $N_* \sim 50 - 60$. As discussed around (2.7), Dante’s Inferno model requires $\tan \gamma \lesssim \mathcal{O}(10^{-1})$ or less. Then (4.26b) implies at least

$$\Lambda^2 \gg \mathcal{O}(10^{-5}) .$$

One should keep in mind that the right hand side of (4.27) can be even smaller, depending on the desired $\tan \gamma$.

In the de Sitter limit (4.26a), the bounce is given by (3.20) when gravitational back-reaction is negligible. We will examine gravitational back-reaction shortly. In this case, the condition for a suppressed tunnelling rate, i.e. $B \gg 1$, becomes

$$\Lambda^2 f \gg \frac{H_0^3}{16\pi^2} \sim \mathcal{O}(10^{-16}) ,$$

where we have used the value of $H_0$ for $p = 1, 2$ with $N_* = 50 - 60$. In the parameter region of a sufficiently rapid pre-factor $A$, i.e. all relevant scales are above the Hubble scale at the time of inflation, condition (4.28) is not a constraint at all. Therefore, the tunnelling process is suppressed provided the gravitational back-reaction is negligible and the thin-wall approximation is applicable.

Consequently, we focus on the gravitational back-reaction for the de Sitter limit first. In this case, condition (3.13) for negligibility of the gravitational back-reaction reads

$$\kappa \ll \frac{h_0}{2\lambda^2} .$$

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In terms of the original parameters of the model, (4.29a) becomes
\[ 2\Lambda^2 f \ll H_0 \sim \mathcal{O}(10^{-5}). \] (4.29b)

By means of (4.27), condition (4.29b) implies
\[ f \ll \mathcal{O}(1). \] (4.29c)

Reminding ourselves of one of the defining conditions of Dante’s Inferno model (2.4a), we conclude that (4.29c) is always satisfied in this model. Therefore, in Dante’s Inferno model, the gravitational back-reaction is always negligible in the de Sitter limit.

Finally, let us verify the validity of the thin-wall approximation in the de Sitter limit. To begin with, we note that the bubble size \( \bar{\rho} \) in the de Sitter limit is always smaller than the bubble size \( \bar{\rho}_0 \) in the flat-space limit, which follows from the definition (B.7a) of \( \bar{\rho} \) in App. B and the fact that \( x \) and \( y \) in (B.7c) are positive numbers.

For a quantitative estimate of \( \bar{\rho} \), we specialise \( x \) and \( y \) of (B.7c) to the parameters in our model (see (B.11b)):
\[ x = \kappa \cdot \left( \frac{\Delta U}{48\lambda^4} \right)^{-1}, \] (4.30a)
\[ y = \frac{6h_0^2}{\kappa\Delta U} + 1 \simeq \frac{6h_0^2}{\kappa\Delta U}. \] (4.30b)

The last approximation in (4.30b) always holds in the de Sitter limit. From (4.30) we immediately infer
\[ 2xy \simeq \left( \frac{\Delta U}{24\lambda^2 h_0} \right)^{-2} \gg 1, \] (4.31a)
where the last hierarchy is a consequence of the de Sitter limit (3.19). Moreover, the negligible gravitational back-reaction in the de Sitter limit, as discussed in (4.29c), allows to deduce
\[ \frac{x}{y} = 2\kappa^2 \cdot \left( \frac{h_0}{2\lambda^2} \right)^{-2} \ll 2. \] (4.31b)

By means of (4.31a) and (4.31b), we then obtain
\[ \frac{\lambda^2 \bar{\rho}}{2} \simeq \frac{\lambda^2}{2h_0} = \left( \frac{24\lambda^2}{\Delta U} \right)^2 \left( \frac{\Delta U}{24\lambda^2 h_0} \right)^2 \frac{1}{h_0^2}. \] (4.32)

Since the thickness of the surface wall is given as \( \sim 2/\lambda^2 \) as described in App. C, the second condition (ii) of the thin-wall approximation becomes
\[ \frac{\lambda^2 \bar{\rho}}{2} \simeq \frac{\lambda^2}{2h_0} = \frac{\Lambda^2}{2H_0 f} \sim \frac{\Lambda^2}{2f} \cdot \mathcal{O}(10^5) \gg 1, \] (4.33)
where we have used the value of \( H_0 \) for \( p = 1, 2 \) and \( N_* = 50 - 60 \). Consequently, (4.27) and (2.4a) imply that (4.33) is always satisfied in the current model within the de Sitter limit. Therefore, the thin-wall approximation is always appropriate in this limit.
|                  | flat-space limit                      | de Sitter limit                       |
|------------------|--------------------------------------|---------------------------------------|
| valid for        | $\Lambda^2 \ll \mathcal{O}(10^{-5})$| $\Lambda^2 \gg \mathcal{O}(10^{-5})$ |
| constraints      | $\tan \gamma \gg \tan \gamma_T$ when $\Lambda/f \gtrsim 7$ | no constraint                         |

Table 1: A brief summary of constraints on the parameter space.

5 Summary and discussions

In this article, we studied tunnelling in Dante’s Inferno model within the thin-wall approximation and subsequent constraints on the parameter space.

In general, we argued that the tunnelling process can only become fatal for inflation if all scales in Dante’s model satisfy $\Lambda, f_1, f_2 \gtrsim H$, and if $B$ is less than order one. All other parameter regions are intrinsically safe from tunnelling in the leading order of $\hbar$.

We have shown in (4.19) that the flat-space limit is appropriate for $\Lambda \ll \mathcal{O}(10^{-5})$. In the flat-space limit, the parameter space is simultaneously constrained by the condition (4.12a) for a suppressed tunnelling rate, and one of the defining conditions (4.13b) of Dante’s Inferno model, i.e.

$$\tan \gamma = \frac{f_1}{f_2} \gg \max\{\tan \gamma_T, \tan \gamma_{DI}\}.$$  \hspace{1cm} (5.1)

We have seen that for a fixed ratio $\Lambda/f$, a lower bound on the parameter $f \simeq f_1$ is imposed by (5.1), for a given $f_1/f_2$. In particular, we have shown in (4.17b) that $\tan \gamma_T$ is bigger than $\tan \gamma_{DI}$ when $\Lambda/f \gtrsim 7$, in which case the condition for a suppressed tunnelling rate gives a stronger constraint than the defining condition of Dante’s Inferno model. Since the parameter space is multi-dimensional, one has to choose certain parameters to obtain a visualisable subspace. We computed the bounds numerically in monomial chaotic inflation with $\Lambda/f = 10$, $p = 1, 2$ with $N_* = 50 – 60$ and exemplified these in Fig. 2 – 3. While numerical values of the bounds were given in the examples, the method for obtaining the bound is clearly general and can be straightforwardly applied to other forms of the inflaton potential.

In the de Sitter limit which was shown to be appropriate for $\Lambda^2 \gg \mathcal{O}(10^{-5})$ in (4.27), the condition for a suppressed tunnelling rate is trivial in the problematic region $\Lambda, f_1, f_2, \gtrsim H$. In other words, the tunnelling process is always suppressed in this limit.

We summarized those constraints on the parameter space in Table 1.

Additionally, we identified in each limit the parameter region in which the thin-wall approximation is valid and the gravitational back-reactions are negligible. It turned out that this covers a large part of the parameter region of interest.

The original article [9] mentioned that a useful value of $\Lambda$ lie in the range from $10^{-3}M_P$ to $10^{-1}M_P$, and typical values for $f_1$ and $f_2$ are $10^{-3}M_P$ and $10^{-1}M_P$, respectively. Our results confirm that these values are safe from tunnelling, and further provide explicit constrains on these parameters from the condition of a suppressed tunnelling rate.
For the article at hand, we restricted ourselves to the level of an effective field theory. For example, the parameter $\Lambda$ in (2.1b), which controls the height of the sinusoidal potential, was treated as an input parameter. However, when the model is embedded in a UV theory the height of the sinusoidal potential could be a function of the required monodromy number $N_{\text{mon}} := \Delta \phi \cdot \cos \gamma / (2\pi f_2) \sim O(10) \cdot M_P / f_2$. (Here $\Delta \phi$ denotes the field distance the inflaton field travels during the inflation.) In a related model embedded in string theory, for instance, the height of the sinusoidal potential was shown [17] to be proportional to $e^{-\gamma_{br} N_{\text{mon}}}$, where $\gamma_{br}$ is a parameter independent of $N_{\text{mon}}$. Such a rapid decrease of the height of the sinusoidal potential for increasing $N_{\text{mon}}$ would give rise to much severer constraint on the axion decay constants than the ones given in this article. Consequently, UV completions of Dante’s Inferno model and the constraints from it are certainly an important direction to be investigated in the future.

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A Effects of time evolution of inflaton on tunnelling

In this appendix we justify the claim of regarding $\phi$ as being fixed in time during inflation. Accounting for changes of the tunneling rate through a time variation of $\phi(\xi)$ is achieved by modifying (3.7) as follows:

$$U_1(\psi, \zeta) := \frac{1}{f^4} V_1(f \psi \cos \gamma + \phi(\zeta) \sin \gamma) ,$$

i.e. one simply keeps the time dependent $\phi(\zeta)$ instead of choosing the reference point $\phi_*$.

A.1 Flat-space limit

In the flat-space limit discussed in Sec. 4.1, the time variation of $U_1$ may become relevant through its appearance in $\Delta U$. As in (4.8), $\Delta U$ can be estimated as

$$\Delta U := U(\psi_+ ) - U(\psi_- ) \simeq \cot \gamma \frac{2\pi V_1'(\phi(\xi))}{f^3} ,$$

(A.2)
where $V_I$ is defined in (2.8).

To judge the impact of a time dependent $\phi$, we examine the time variation of $\Delta U$ in a time interval $\Delta \zeta$ relative to $\Delta U$ itself. In detail

$$\frac{1}{\Delta U} \cdot \frac{d\Delta U}{d\zeta} \cdot \Delta \zeta \approx V_I' \cdot \frac{d}{d\zeta} \phi(\xi) \cdot \Delta \zeta \approx \eta_V H \Delta \zeta , \quad (A.3)$$

where we used the slow-role approximation of the equations of motion, i.e.

$$3H \frac{d\phi}{d\xi} \approx V_I' , \quad 3H^2 \approx V_I . \quad (A.4)$$

Recall Sec. 3, we assume that all relevant physical parameters are greater than the Hubble expansion rate $H$, i.e. $\Lambda, f_1, f_2 \gtrsim H$. Consequently, during a time interval $\Delta \zeta \approx 1/H$ the relative change (A.3) becomes of order $\eta_V$, and we may safely neglect the time dependence of $\Delta U$ in the slow-role regime $\eta_V \lesssim O(10^{-2})$.

### A.2 de Sitter limit

In de Sitter limit discussed in Sec. 4.2, a time variation of $U_1$ enters via the time variation of $V_I$. In slow-role inflation models, it is well-known that the time variation of $V_I$ is suppressed by the slow-role parameters. To be explicit, we find

$$\frac{1}{V_I} \cdot \frac{dV_I}{d\zeta} \cdot \Delta \zeta \approx \frac{V_I'}{V_I} \cdot \frac{d\phi}{d\zeta} \cdot \Delta \zeta \approx 2 \epsilon_V H \Delta \zeta , \quad (A.5)$$

where we again made use of (A.4). Hence, (A.5) is small in a time-scale $\Delta \zeta \sim 1/H$, as $\epsilon_V \lesssim O(10^{-2})$.

### B Summary of the bounce solution in the thin-wall approximation

The bounce action for general potential in the thin-wall approximation has been given in [25] ([27] is also a useful read). Here, we review the necessary results. The relevant Euclidean action is of the form

$$S_E = 2\pi^2 \int d\zeta \rho^3 \left( \frac{1}{2} \dot{\psi}^2 + U(\psi) \right) - \frac{3\rho}{\kappa_P} \left( \dot{\rho}^2 + 1 \right) . \quad (B.1)$$

The action (B.1) has the same form\(^6\) as (3.7a), with parameters $\kappa$ being replaced with $\kappa_P$ defined by

$$\kappa_P := 8\pi G = M_P^{-2} . \quad (B.2)$$

\(^6\)Precisely speaking, variables in (3.7a) were dimensionless, but it is straightforward to implement this point in the comparison, e.g. by setting $M_P \equiv 1$ as we did.
The bounce action is introduced as

\[ B := S_E[\psi_B] - S_E[\psi_+] , \]  

(B.3)

where \( \psi_B \) is the bounce solution. In the thin-wall approximation, which is appropriate whenever conditions (i) and (ii) hold, we evaluate (B.3) by dividing the integration region into three parts: Outside the bubble, at the surface of the bubble, and inside the bubble. Outside the bubble, the bounce and false vacuum are identical; therefore, the contribution \( B_{\text{out}} \) to the bounce action is

\[ B_{\text{out}} = 0 . \]  

(B.4)

At the surface wall of the bubble, we can replace \( \rho \) by the position of the centre of the surface wall \( \bar{\rho} \). Then, the contribution to the bounce action from the surface wall \( B_w \) is given by

\[ B_w = 2\pi^2 \bar{\rho}^3 S_1 , \]  

(B.5a)

where

\[ S_1 := 2 \int d\zeta (U_0(\psi_B) - U_0(\psi_-)) . \]  

(B.5b)

Inside the bubble, \( \psi \) is constant, such that (3.8b) allows to deduce

\[ d\zeta = d\rho \left( 1 - \frac{\kappa_P}{3} \rho^2 U(\psi) \right)^{1/2} , \]  

(B.6a)

and we then obtain

\[ B_{\text{in}} = -\frac{12\pi^2}{\kappa_P} \int_0^{\bar{\rho}} d\rho \left[ \left( 1 - \frac{\kappa_P}{3} U_- \right)^{1/2} - \left( 1 - \frac{\kappa_P}{3} U_+ \right)^{1/2} \right] \]

\[ = \frac{12\pi^2}{\kappa_P} \left[ U_-^{-1} \left( \left( 1 - \frac{\kappa_P}{3} \rho \rho^2 U_- \right)^{3/2} - 1 \right) - U_+^{-1} \left( \left( 1 - \frac{\kappa_P}{3} \rho \rho^2 U_+ \right)^{3/2} - 1 \right) \right] , \]  

(B.6b)

where \( U_+ := U(\psi_+) \) and \( U_- := U(\psi_-) \) are the energy density of the false vacuum we start with and another false vacuum we end with, respectively.

Extremising \( B \) with respect to \( \bar{\rho} \) gives

\[ \bar{\rho}^2 = \frac{\rho_0^2}{1 + 2xy + x^2} , \]  

(B.7a)

where

\[ \rho_0 := \frac{3S_1}{(U_+ - U_-)} , \]  

(B.7b)

is the critical bubble size without the presence of gravity, and \( x \) and \( y \) are defined as follows:

\[ x := \frac{\rho_0^2 \kappa_P (U_+ - U_-)}{4} , \quad y := \frac{U_+ + U_-}{U_+ - U_-} . \]  

(B.7c)

The bounce action is obtained as

\[ B \simeq B_0 r(x, y) , \]  

(B.8a)
where
\[ B_0 := \frac{27\pi^2 S_1^4}{2(U_+ - U_-)^3}, \quad (B.8b) \]
which is the bounce action in flat space. The function \( r(x, y) \) is defined as follows:
\[
r(x, y) := 2 \cdot \frac{(1 + xy) - \sqrt{1 + 2xy + x^2}}{x^2(y^2 - 1)\sqrt{1 + 2xy + x^2}}
= \frac{2 \cdot ((1 + xy)^2 - (1 + 2xy + x^2))}{x^2(y^2 - 1)\sqrt{1 + 2xy + x^2} \left( (1 + xy) + \sqrt{1 + 2xy + x^2} \right)}.
\]
\[
= \frac{2}{\sqrt{1 + 2xy + x^2} \left( (1 + xy) + \sqrt{1 + 2xy + x^2} \right)}. \quad (B.8c)
\]
To obtain the bounce action for set-up of this article, one simply has to replace \( \kappa_P \) with \( \kappa \) as mentioned earlier. Then, we use the result \((C.4)\) of App. C:
\[
S_1 = 8\lambda^2, \quad (B.9)
\]
together with
\[
U_+ - U_- = \Delta U, \quad U_+ + U_- = \frac{6h_0^2}{\kappa} - \Delta U. \quad (B.10)
\]
Putting all the pieces together we obtain
\[
B_0 = \frac{27\pi^2 S_1^4}{2(\Delta U)^3} = \frac{27\pi^2 (8\lambda^2)^4}{2(\Delta U)^3}, \quad (B.11a)
\]
\[
x = \frac{3S_1^2 \kappa \Delta U}{4} = \frac{48\lambda^4 \kappa}{\Delta U}, \quad y = \frac{6h_0^2}{\kappa \Delta U} - 1. \quad (B.11b)
\]
Moreover, we find
\[
1 + 2xy + x^2 = (1 - x)^2 + 2x \frac{6h_0^2}{\kappa \Delta U}
= \frac{1}{(\Delta U)^2} \left( (\Delta U - 48\lambda^4 \kappa)^2 + 12h_0^2(48\lambda^4) \right), \quad (B.12a)
\]
and
\[
1 + xy = \frac{1}{(\Delta U)^2} \left( (\Delta U)^2 - 48\lambda^4 \kappa \Delta U + 6h_0^2(48\lambda^4) \right). \quad (B.12b)
\]
Finally, we arrive at
\[
B \approx \frac{27\pi^2 (8\lambda^2)^4}{\sqrt{(\Delta U - 48\lambda^4 \kappa)^2 + 12h_0^2(48\lambda^4)}} \times \frac{1}{(\Delta U)^2 - 48\lambda^4 \kappa \Delta U + 6h_0^2(48\lambda^4) + \Delta U \sqrt{(\Delta U - 48\lambda^4 \kappa)^2 + 12h_0^2(48\lambda^4)}}
\]
\[ \frac{2 \cdot 27 \pi^2 (8\lambda^2)^4}{\sqrt{(\Delta U - 48\lambda^4\kappa)^2 + 12h_0^2(48\lambda^4)} \times \left(\Delta U + \sqrt{(\Delta U - 48\lambda^4\kappa)^2 + 12h_0^2(48\lambda^4)}\right)^2} \]  
\[ - (48\lambda^4\kappa)^2, \]

which is the justification for (3.12b) used in the main text.

## C Instanton for sinusoidal potential and the thin-wall approximation

In the thin-wall approximation \[24\], where the bubble radius \( \bar{\rho} \) is much larger than the thickness of the surface wall of the bubble, one can neglect the change in \( \rho \) at the wall. (A more quantitative definition of the thickness of the surface wall is given below.) The problem of finding an \( O(4) \)-symmetric bounce solution reduces to solving an instanton equation associated to the following one-dimensional action:

\[
S_{\psi} = \int d\zeta \left[ \frac{1}{2} \dot{\psi}^2 + U_0(\psi) \right],
\]  

where the potential for the case of our interest is the one defined in (3.7d), i.e.

\[
U_0(\psi) = \lambda^4 (1 - \cos \psi).
\]

Note that \( U_1(\psi) \) in (3.7f) differs from the constant part only by \( O(\Delta U) \). Due to (i), this difference is irrelevant in the equation of motion and can be dropped in the thin-wall approximation in the leading order.

The equation of motion derived from the action (C.1a) reads

\[
\ddot{\psi} = \frac{\partial U_0}{\partial \psi},
\]

and can be solved as follows: Multiplying \( \dot{\psi} \) on both sides of (C.2) and integrating once with respect to \( \zeta \) gives

\[
\frac{d\psi}{d\zeta} = \pm \sqrt{2U_0(\psi)}.
\]

The action of the solution \( \psi_{\pm} \) of (C.3) becomes

\[
S_1 := S_{\psi}[\psi_{\pm}] = \int d\zeta \left[ \frac{1}{2} \dot{\psi}_{\pm}^2 + U_0(\psi_{\pm}) \right] = \int d\zeta 2U_0 = \int d\psi \sqrt{U_0}
\]

\[= 8\lambda^2. \]

The solution \( \psi_{\pm} \) of the equation (C.3) is given by

\[
\psi_{\pm}(\zeta) = 4 \tan^{-1} \left[ \exp \left( \pm \lambda^2 (\zeta - \zeta_0) \right) \right],
\]

\[C.5\]
where $\zeta_0$ is an integration constant. The asymptotic behaviour of the instanton (C.5) at $\zeta \rightarrow +\infty$ is
\[
\frac{\pi}{2} - \frac{\psi_+}{4} \sim \frac{\psi_-}{4} \sim e^{-\lambda^2(\zeta - \zeta_0)}. \tag{C.6}
\]
The asymptotic behaviour of the instanton (C.5) at $\zeta \rightarrow -\infty$ is
\[
\frac{\psi_+}{4} \sim \frac{\pi}{2} - \frac{\psi_-}{4} \sim e^{\lambda^2(\zeta - \zeta_0)}. \tag{C.7}
\]
These asymptotic behaviours are anticipated from the mass term at the vacua. These exponential decays define the thickness of the surface wall to be $2/\lambda^2$. Then the second condition (ii) of the thin-wall approximate, applied to the bubble radius $\bar{\rho}$ defined in (B.7a), gives
\[
\bar{\rho} \gg \frac{2}{\lambda^2}. \tag{C.8}
\]
In the flat-space limit, we have
\[
\bar{\rho} = \bar{\rho}_0 = \frac{3S_1}{\Delta U} = \frac{24\lambda^2}{\Delta U}. \tag{C.9}
\]
Inserting (C.9) into (C.8), we obtain the consistency condition for the thin-wall approximation in flat-space limit:
\[
\frac{\Delta U}{12\lambda^4} \ll 1. \tag{C.10}
\]
This condition is automatically satisfied due to the first condition (i) of the thin-wall approximation, applied to our model (3.11).

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