Dynamic dS/CFT correspondence using the brane cosmology

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Abstract

We explore the dynamic dS/CFT correspondence using the moving domain wall(brane) approach in the brane cosmology. The bulk spacetimes are given by the Schwarzschild-de Sitter (SdS) black hole and the topological-de Sitter (TdS) solutions. We consider the embeddings of (Euclidean) moving domain walls into the (Euclidean) de Sitter spaces. The TdS solution is better to describe the static dS/CFT correspondence than the SdS black hole, while in the dynamic dS/CFT correspondence the SdS solution provides situation better than that of the TdS solution. However, we do not find a desirable cosmological scenario from the SdS black hole space.

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I. INTRODUCTION

Recently an accelerating universe has proposed to be a way to interpret the astronomical data of supernova [1–3]. Combining this observation with the need of inflation in the standard cosmology leads to that our universe approaches de Sitter geometries in both the far past and the far future [4]. Hence it is very important to study the nature of de Sitter (dS) space [7] and the dS/CFT correspondence [8]. However, there are two difficulties in studying de Sitter space [9]: First there is no spatial infinity and global timelike Killing vector. Thus it is not easy to define the conserved quantities including mass, charge and angular momentum appeared in asymptotically de Sitter space. Second the dS solution is absent in string theories and thus we do not have a definite example to test the dS/CFT correspondence.

Authors in [10] proposed the prescription to calculate the mass of gravitational field of asymptotically dS spaces. Especially they put forward the mass bound conjecture: Any asymptotically de Sitter space whose mass exceeds that of de Sitter space contains a cosmological singularity. In order to test this conjecture, Cai, Myung and Zhang [11] have first introduced a topological de Sitter (TdS) solution that always gives us a positive mass (m) as well as a naked cosmological singularity at $r = 0$. Actually this solution does not have any black hole horizon but a cosmological horizon.

The negative mass is found when calculating the mass in the Schwarzschild-de Sitter (SdS) black hole [10,12,13]. Assuming the dS/CFT correspondence, this induces in turn the negative energy of the dual CFT. The Casimir energy is also negative which states that the dual CFT is not unitary. In addition it is not easy to take into account both its cosmological and black hole horizons simultaneously. All of these are circumvented if one introduces the TdS solution [14].

To understand the dS/CFT correspondence well, one has to investigate its dynamic aspects in the brane world cosmology [15]. Nojiri and Odintsov have first discussed this issue [16]. Ogushi considered this correspondence by using the moving domain wall (MDW) approach in the SdS black hole background [17]. On the other hand, Medved [19] showed that the dynamic dS/CFT correspondence may be established in the TdS background. The mass $m$ located at $r = 0$ provides a CFT-radiation matter on the brane holographically [20], and the thermodynamic relation of the CFT (the Cardy-Verlinde’s formula [21]) coincides with the Friedmann equation (the dynamic equation for the brane) when the brane (MDW) crosses the cosmological horizon of TdS space. Authors in [18,19] used the embeddings of Euclidean MDW into dS spaces with Minkowski signature.

In this paper we explore dynamic aspects of the dS/CFT correspondence using the MDW approach in the SdS and TdS backgrounds. For this purpose we introduce all kinds of embeddings of the branes into dS spaces. In contrast with the static dS/CFT correspondence, we obtain the CFT-radiation matter on the dS brane moving on the SdS black hole spacetime. Also we discuss the relation between thermodynamic CFT and Friedmann equation.

The organization of this paper is as follows. In section II we briefly review the results

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1 In the AdS black hole background, the dynamic dS/CFT correspondence was discussed in ref. [17].
of the static dS/CFT correspondence for SdS and TdS spaces. We study the embeddings of the (Euclidean) MDW into (Euclidean) topological-de Sitter space in section III. In section IV we investigate the embeddings of (Euclidean) MDW into (Euclidean) Schwarzschild-de Sitter black hole space. In section V the relationship between the CFT-thermodynamic relations and the Friedmann equation will be discussed in SdS and TdS spaces. Finally we discuss our results in section VI.

II. DE SITTER (BLACK HOLE) SOLUTIONS

We start with the \( (n+2) \)-dimensional Schwarzschild-de Sitter metric in the static coordinates \([9,12,13]\)

\[
ds_{\text{SdS}}^2 = -\tilde{f}(r)dt^2 + \frac{1}{\tilde{f}(r)}dr^2 + r^2d\Omega_n^2, \quad \tilde{f}(r) = 1 - \frac{\omega_n m}{r^{n-1}} - \frac{r^2}{\ell^2}
\]  

where \( \omega_n = 16\pi G_{n+2}/n\text{Vol}(S^n) \) and \( m \) is a parameter related to the black hole mass and \( \ell \) is the curvature radius of de Sitter space. The allowed range of \( m \) is \( 0 < m \leq m_N \) with the Nariai black hole mass

\[
m_N = \frac{2\ell^{n-1}}{\omega_n(n+1)}\left(\frac{n-1}{n+1}\right)^{\frac{n-1}{2}}.
\]  

Beyond this there exists a naked singularity. We note that Eq.(1) is a solution to the bulk action

\[
S_{\text{bulk}} = \frac{1}{16\pi G_{n+2}} \int d^{n+2}x \sqrt{-g} [R - 2\Lambda]
\]  

with \( (n+2) \)-dimensional positive cosmological constant \( \Lambda = n(n+1)/2\ell^2 \). If \( m = 0 \), the solution Eq.(1) represents the pure dS solution with a cosmological horizon at \( r_c = \ell \). Using the prescription in \([10]\), it was found that the gravitational mass at the future infinity \( I^+ \) is given by

\[
M_4 = -m, \quad M_5 = \frac{3\pi\ell}{32G_5} - m
\]  

which implies that a pure dS\(_4\) space has vanishing mass, while a pure dS\(_5\) space has the mass of \( 3\pi\ell/32G_5 \). This means that the masses of the SdS black holes are always less than those of pure dS spaces. Assuming the dS/CFT correspondence, Eq.(1) implies the negative energy of the dual CFT \( (E = -m) \) \([\bar{1}]\). Furthermore the Casimir energy \( E_{CA} \) of the CFT appears negative, which implies that the dual CFT is not unitary \([14]\). Hence it seems that the SdS black hole solution is not appropriate for our purpose.

The \( (n+2) \)-dimensional topological-de Sitter metric is given by \([\bar{1}]\)

\(^2\) Usually we neglect the non-vanishing mass of pure dS spaces (for example, \( 3\pi\ell/32G_5 \) in Eq.(1)) in calculating the dual CFT energy.
\[ ds^2_{TdS} = g_{MN} dx^M dx^N = -f(r) dt^2 + \frac{1}{f(r)} dr^2 + r^2 \gamma_{ij} dx^i dx^j, \quad f(r) = k + \frac{\omega_n m}{r^{n-1}} - \frac{r^2}{\ell^2} \]  

(5)

where \( \omega_n = 16\pi G_{n+2}/\text{Vol}(\Sigma_k) \) and \( m \) is assumed to be a positive constant. \( \gamma_{ij} dx^i dx^j \) denotes the line element of an \( n \)-dimensional hypersurface \( \Sigma_k \) with the constant curvature \( n(n-1)k \) and its volume \( V(\Sigma_k) \). \( \Sigma_k \) is given by spherical \( (k = 1) \), flat \( (k = 0) \), hyperbolic \( (k = -1) \). This is also a solution to the action Eq.(3). The case of \( k = 1 \) with the substitution of \( m \rightarrow -m \) leads to the previous SdS black hole solution Eq.(1). Because of the action of \( m \rightarrow -m \), the black hole horizon is absent here. Instead there exists a cosmological singularity at \( r = 0 \) for \( n \geq 2 \). Using the prescription in [10], it was found that the gravitational mass (energy) \( M \) which is measured at the future infinity of \( I^+ \) outside the cosmological horizon \( r = r_c \), for \( k = 0 \) case, is given by

\[ M_4 = m, \quad M_5 = m, \]  

(6)

while for \( k = -1 \) case, this is given by

\[ M_4 = m, \quad M_5 = \frac{3\pi}{64\pi G_5} \text{Vol}(\Sigma) + m. \]  

(7)

For \( k = 1 \), we have the same result as in Eq.(4) except replacing \( m \) by \( -m \). This means that the mass (energy) of the TdS solution are always positive and greater than that of pure dS space. Hence this confirms the mass bound conjecture in dS space [11].

Then we apply the TdS/CFT correspondence for calculating the thermodynamic quantities of the dual CFT. We can express the total energy of the CFT \( (E = m) \) in terms of a cosmological horizon \( r_c \) given by the maximal root to \( f(r) = 0 \) :

\[ E = m = \frac{r_c^{n-1}}{\omega_n} (r_c^2/\ell^2 - k). \]  

(8)

For \( k = 0, \pm 1 \), \( E > 0 \) is guaranteed from \( \omega_n m/r_c^{n-1} > 0 \) in \( f(r_c) = 0 \). We can associate this cosmological horizon with the Hawking temperature \( (T_{TdS}) \) and the entropy \( (S_{TdS}) \) as

\[ T_{TdS} = \frac{1}{4\pi r_c} \left[ (n + 1) r_c^2/\ell^2 - (n - 1)k \right], \quad S_{TdS} = \frac{V_c}{4G_{n+2}} \]  

(9)

with the area of the cosmological horizon \( V_c = r_c^n \text{Vol}(\Sigma) \) in \( (n + 2) \)-dimensional asymptotically dS space. This corresponds to the volume of the \( (n+1) \)-dimensional boundary space. The CFT Casimir energy of \( E_{CA} = n(E + pV - TS) \) with \( p = E/nV \) is calculated as

\[ E_{CA} = -\frac{2nk r_c^{n-1} \text{Vol}(\Sigma_k)}{16\pi G_{n+2}}. \]  

(10)

For \( k = 0 \) the Casimir energy which is related to the central charge is zero, for \( k = 1 \) we have a negative energy, and for \( k = -1 \) we have a positive one. Furthermore the Cardy-Verlinde’s formula for \( k = \pm 1 \) is given by [14]

\[ S = \frac{2\pi\ell}{n} \sqrt{|E_{CA}|(2E - E_{CA})}. \]  

(11)
where \( S = S_{TdS} \) is the entropy of the cosmological horizon Eq.(9). For \( k = 0 \), one can arrange it as
\[
S = \frac{2\pi \ell}{n} \sqrt{|E_{CA}/k|(2E - E_{CA})}.
\]

Up to now we consider only the static version of the dS/CFT correspondence. In this case the TdS solution that has a cosmological horizon and a naked singularity seems to provide a dS/CFT correspondence better than the SdS black hole spacetime. For a full analysis of this correspondence we will discuss the dynamic evolution of the boundary space in the TdS background in the next section.

### III. EMBEDDING OF MDW INTO TOPOLOGICAL-DE SITTER SPACE

In order to define an embedding of MDW into the TdS background Eq.(5) properly, we have to introduce both tangent \((u^M)\) and normal \((n_M)\) vectors. This is so because two vectors are essential for defining the projection tensor of \( h_{MN} = g_{MN} - n_M n_N \), the extrinsic curvature of \( K_{MN} = -h_M \nabla_P n_N \) and \( K_{\tau\tau} = u^M u^N K_{MN} \). First we usually choose these as in the AdS space
\[
u^M = (\dot{t}, \dot{a}, 0, \cdots, 0), \quad u^M u^N g_{MN} = -1; \quad n_M = (\dot{a}, -\dot{t}, 0, \cdots, 0), \quad n^M n^N g_{MN} = 1
\]
with \( u^M n_M = 0 \). Then both \( u^M u^N g_{MN} = -1 \) (timelike vector) and \( n^M n^N g_{MN} = 1 \) (spacelike vector) give us the same relation
\[
\frac{1}{f(a)} \dot{a}^2 - f(a) \dot{t}^2 = -1
\]
which leads to a timelike brane. Here \( f(a) = k + \frac{\omega n}{a} - \frac{a^2}{\ell^2} \). In addition, for a well-defined embedding, we have to consider the small black hole which satisfies the condition of \( \frac{\omega n}{\ell^2} \ll 1 \). Substituting Eq.(14) into the TdS solution Eq.(5), one has the induced brane metric which takes Friedmann-Robertson-Walker (FRW) form
\[
d s_{TdS}^2 \to d s_{FRW}^2 \equiv h_{\mu\nu} d x^\mu d x^\nu = -d \tau^2 + a(\tau)^2 \gamma_{ij} d x^i d x^j,
\]
where \( h_{\mu\nu} \) is the induced metric on the brane and the Greek indices \( \mu, \nu, \cdots \) denote for the brane coordinates only. From the Israel junction condition \( K_{\mu\nu} = -\frac{8\pi G_{n+2} \sigma}{n} h_{\mu\nu} \) with the brane tension \( (\sigma) \) together with Eq.(14), we obtain the evolution equation for one-sided brane world scenario \[22\]
\[
i = \frac{8\pi G_{n+2} \sigma a}{n f(a)} = \frac{\tilde{\sigma} a}{f(a)},
\]
where a reduced tension \( \tilde{\sigma} = \frac{8\pi G_{n+2} \sigma}{n} \) is introduced for convenience. This leads to the first Friedmann equation with \( H \equiv \dot{a}/a \)

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3This actually corresponds to a non-trivial 2 \( \to \) 1 mapping \[15\] : \( r \to a(\tau), t \to t(\tau) \).
\[ H^2 = \frac{-f(a)}{a^2} + \dot{\sigma}^2 = -\frac{k}{a^2} - \frac{\omega_n m}{a^{n+1}} + \frac{1}{\ell^2} + \tilde{\sigma}^2, \tag{17} \]

where the dot in \( \dot{a} \) denotes the differentiation with respect to the proper time \( \tau \). Here we cannot make any fine-tuning to obtain a flat brane. We have thus an effectively de Sitter brane. Unfortunately we get a negative energy from the cosmological singularity \( (m \neq 0) \). This contrasts to the result of its static TdS/CFT correspondence which says that one gets a positive energy on the boundary as is shown in Eq.(8). In the case of finding the negative energy in a cosmological model, one wishes to discard the corresponding model. Hence we have to move another case.

Accordingly authors in \[19\] used a relation of \( \dot{a}^2/f(a) - f(a)t^2 = 1 \) to obtain
\[ ds^2_{T_{dS}} \rightarrow ds^2_{E_{FRW}} = d\tau^2_E + a(\tau_E)^2\gamma_{ij} dx^i dx^j. \tag{18} \]
Here \( \tau_E \) is the Euclidean time obtained by Wick-rotation of \( \tau \rightarrow i\tau_E \). Hence the corresponding equation is given by
\[ H^2_E = \frac{f(a)}{a^2} + \tilde{\sigma}^2 = \frac{k}{a^2} - \frac{\omega_n m}{a^{n+1}} - \frac{1}{\ell^2} + \tilde{\sigma}^2 \tag{19} \]
which reduces to with \( \tilde{\sigma}^2 = 1/\ell^2 \) a spacelike flat brane
\[ H^2_E = \frac{k}{a^2} + \frac{\omega_n m}{a^{n+1}}. \tag{20} \]

Here \( H_E \) is the Hubble parameter with respect to \( \tau_E \) and thus one finds \( H^2_E = -H^2 \). This mapping from TdS space with Lorentzian signature to the spacelike brane with Euclidean signature can be defined properly if one chooses \( u^M u^N g_{MN} = 1 \) \( \rightarrow \dot{a}^2/f(a) - f(a)t^2 = 1 \); \( n^M n^N g_{MN} = -1 \) \( \rightarrow \dot{a}^2/f(a) - f(a)t^2 = 1 \) which is the reversed choice to Eq.(13). For \( k = -1, n = 3 \) case, the spacelike brane starting at \( a = 0 \) crosses the cosmological horizon \( a_c = \sqrt{\omega_3 m - (\omega_3 m)^2/\ell^2} \) of TdS space and then reaches the maximum distance \( a_m = \sqrt{\omega_3 m} \). And then it contracts and crosses the cosmological horizon and finally collapses into \( a = 0 \).

It seems that all cosmological implications that are derived from AdS space may be applied to TdS space \[19\]. However, considering Eq.(24) with \( H^2_E = -H^2 \), one finds the negative energy. Hence it is clear that the dynamic TdS/CFT correspondence cannot be realized in topological-de Sitter space.

The last case is the embedding of a spacelike brane into Euclidean TdS space by introducing the two spacelike vectors\[4\]
\[ u^M = (\dot{t}, \dot{a}, 0, \cdots, 0), \quad u^M u^N g^E_{MN} = 1; \quad n_M = (\dot{a}, -\dot{t}, 0, \cdots, 0), \quad n^M n^N g^E_{MN} = 1 \tag{21} \]
with \( g^E_{MN} = \text{diag}(f, 1/f, \cdots) \) and \( u^M n_M = 0 \). Then we have a relation
\[ \frac{1}{f(a)} \dot{a}^2 + f(a)\dot{t}^2 = 1, \tag{22} \]

\[ \text{Our convention for } n_M \text{ is just a negative of that in } \[23\] \text{ because we use the extrinsic curvature}\]
\[ \text{of } K_{MN} = -h_M P \nabla_P n_N. \]
where the dot denotes differentiation with respect to Euclidean proper time $\tau_E$. Substituting Eq. (22) into Euclidean TdS space of Eq. (5) leads to the induced brane metric which takes Euclidean Friedmann-Robertson-Walker (EFRW) form

$$ds^2_{ETdS} \to ds^2_{EFRW} = d\tau^2_E + a(\tau_E)^2\gamma_{ij}dx^i dx^j.$$  

(23)

Considering both Eqs. (18) and (22) leads to

$$H^2_E = \frac{f(a)}{a^2} - \tilde{\sigma}^2 = \frac{k}{a^2} + \frac{\omega_m}{a^{n+1}} - \frac{1}{\ell^2} - \tilde{\sigma}^2.$$  

(24)

Considering the above equation together with $H^2_E = -H^2$, it is found that this is equivalent to Eq. (17). In other words, Eq. (24) is just a Euclidean version of Eq. (17) with a negative energy term. Let us explore its dynamic ETdS/ECFT correspondence. We note that the asymptotic form of the ETdS metric is naively given by

$$\lim_{a \to \infty} \left[ \ell^2 a^2 ds^2_{ETdS} \right] = dt^2_E + \ell^2 \gamma_{ij} dx^i dx^j,$$

(25)

which can be identified with the boundary ECFT metric [25]. Then the thermodynamic relations between the boundary ECFT and the bulk ETdS are given by [1]

$$E_{ECFT} = \frac{\ell m}{a}, \quad T_{ECFT} = \frac{\ell T_{TdS}}{a}, \quad S_{ECFT} = S_{TdS}.$$  

(26)

Introducing the energy density $\rho_{ECFT} = E_{ECFT}/V$, $V = a^n Vol(\Sigma)$ and the pressure $p_{ECFT} = \rho_{ECFT}/n$ [26], Eq. (24) can be expressed as the first Friedmann equation

$$H^2_E = \frac{k}{a^2} + \frac{16\pi G_n+1}{n(n-1)} \rho_{ECFT} + \frac{2}{n(n+1)} \Lambda_{n+1}$$  

(27)

with the cosmological constant $\Lambda_{n+1} = -\frac{n(n+1)}{2(1/\ell^2 + \tilde{\sigma}^2)}$. Here we used the relation between the bulk and brane Newtonian constants for one-sided brane world scenario: $G_{n+2} = \frac{\ell}{n-1} G_{n+1}$. Eq. (27) may imply that the cosmological evolution can be attributed partly to the energy density of the CFT-radiation matter originated from the cosmological singularity. But the dynamic ETdS/ECFT correspondence is not realized here because one finds a negative CFT-radiation matter from Eq. (27) which comes from the negative energy term in Eq. (24). Furthermore, the important correspondence that the Cardy-Verlinde’s formula coincides with the Friedmann equation when the brane crosses the cosmological horizon $a = a_c$ of TdS space is not found for this case. Here we can check it easily from Eq. (24), by noting that $H_E$ is not defined properly ($H_E = i\tilde{\sigma}$, imaginary) at $a = a_c$, the maximal root of $f(a) = 0$.  

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5 For a curved brane, in general, one has to introduce the relations [23,24]: $E_{ECFT} = Cm$, $T_{ECFT} = CT_{TdS}$. For a flat brane, one finds $C = \frac{\ell}{a}$, whereas for a curved brane this is given by $C = \frac{1}{\sigma a}$. Considering $G_{n+2} = \frac{1}{(n-1)\sigma} G_{n+1}$, even for the curved brane we also leads to the same equation as in Eq. (27).
IV. EMBEDDING OF MDW INTO SCHWARZSCHILD-DE SITTER SPACE

Up to now we discuss the embedding of the MDW into TdS space. From this analysis it is found that although the static behavior of the TdS/CFT correspondence is better than that of the SdS/CFT correspondence, its dynamical TdS/CFT correspondence is not well-defined. Hence it is interesting to study the dynamical SdS/CFT correspondence along the previous section. We consider for definiteness only the small black hole which satisfies the condition of \( \omega_n m / \ell^2 \ll 1 \). For the induced brane metric which takes the FRW form for \( k = 1 \)

\[
ds_{\text{SdS}}^2 \to ds_{\text{FRW}}^2 = -d\tau^2 + a(\tau)^2 d\Omega_n^2,
\]

we have the first Friedmann equation [18]

\[
H^2 = -\frac{\dot{f}(a)}{a^2} + \sigma^2 = -\frac{1}{a^2} + \frac{\omega_n m}{a^{n+1}} + \frac{1}{\ell^2} + \sigma^2.
\]

Similarly we cannot make any fine-tuning to obtain a flat brane. Eq.\((29)\) means that we have a de Sitter brane from SdS space. We note here that a positive energy can be obtained from the SdS black hole, in contrast with the static SdS/CFT correspondence. This is a situation better than that of the TdS solution. We wish to explore its dynamic SdS/CFT correspondence. This runs closely parallel with the curved brane in the AdS/CFT correspondence [23,24]. Its Friedmann equation is given by

\[
H^2 = -\frac{1}{a^2} + \frac{16\pi G_{n+1}}{n(n-1)} \rho_{\text{CFT}} + \frac{2}{n(n+1)} \Lambda_{n+1}^+.
\]

with the positive cosmological constant \( \Lambda_{n+1}^+ = \frac{n(n+1)}{2} (1/\ell^2 + \sigma^2) \) and \( G_{n+2} = \frac{1}{(n-1)\sigma} G_{n+1} \). Here \( \rho_{\text{CFT}} = \tilde{E}/V = m/\sigma a V > 0 \) with \( \rho_{\text{CFT}} = \rho_{\text{CFT}}/n \) denotes the CFT-radiation matter which comes from the SdS black hole through the dynamic SdS/CFT correspondence. This contrasts to the static SdS/CFT correspondence which implies a negative energy for the dual CFT (\( E = -m \)), as is shown in Eq.\((4)\).

In order to investigate the trajectory of the MDW, we have a conventional form from Eq.\((29)\) as

\[
\frac{1}{2} a^2 + V(a) = -\frac{1}{2}, \quad V(a) = -\frac{\omega_n m}{2a^{n-1}} - \frac{1}{2} \left( \frac{1}{\ell^2} + \sigma^2 \right) a^2,
\]

where the first term in the left-hand side is the kinetic energy with unit mass and the second is the potential energy and the term in the right-hand side is the negative total energy. The corresponding trajectory inspired by Eq.\((31)\) can be shown at the Penrose diagram for the SdS black hole [27] as was shown in [20] for the AdS-Schwarzschild black hole case. The timelike dS brane (MDW) starts at the south pole \( a = 0 \) (Big bang) and expands with time. At a moment (\( \bullet \)), the MDW crosses the black hole horizon \( a = a_+ \) of SdS space. For the (3+2)-dimensional small black hole, the location of the black hole horizon is given approximately by \( a_+ = \sqrt{\omega_3 m + (\omega_3 m)^2 / \ell^2} \), while the cosmological horizon is approximated as \( a_c = \ell \sqrt{1 - \omega_3 m / \ell^2} \). From Eq.\((21)\), we have \( H^2 = \sigma^2 \) at \( a = a_+ \). This
implies that the Hubble parameter must obey $H = \tilde{\sigma}$ at $a = a_+$. The approximate form of the potential $V(a)$ is a negative convex ($\cap$). Then the MDW reaches the maximum point $a_m = \sqrt{\omega_3 m + (\omega_3 m)^2(1/\ell^2 + \tilde{\sigma}^2)}$ which is determined by $V(a_m) = -1/2$. But it never crosses the cosmological horizon at $a = a_c$ because of $a_c > a_m > a_+$. This means that the MDW always stays inside the cosmological horizon. And then the MDW contracts and will cross the black hole horizon again. At this time (•) the Hubble constant will be negative ($H = -\tilde{\sigma}$) because it is contracting. Finally it falls into the north pole at $a = 0$ (Big crunch).

As was mentioned in [18], the mapping from the SdS black hole spacetime with Lorentzian signature into the spacelike brane with Euclidean signature can be allowed. So this case is regarded as a candidate for exploring the dynamic dS/ECFT correspondence. Its equation is given by $H^2_E = 1/a^2 - \omega_n m / a^{n+1}$. Considering $H^2_E = -H^2$, this brane carries with a positive energy density from the mass of SdS black hole. Also this can cross both the cosmological and black hole horizons at $a = a_c, a_+$ where one finds $H_E = \pm \tilde{\sigma}$. Hence its dynamic correspondence is established and further its relation with the Cardy-Velinde’s formula is well-defined.

In the case of the mapping that leads to EFRW form

$$ds^2_{ESdS} \rightarrow ds^2_{EFRW} = d\tau^2 + a(\tau)^2 d\Omega^2_n,$$

one obtains the first Friedmann equation as

$$H^2_E = \frac{\tilde{f}(a)}{a^2} - \tilde{\sigma}^2 = \frac{1}{a^2} - \frac{\omega_n m}{a^{n+1}} - \frac{1}{\ell^2} - \tilde{\sigma}^2.$$

Considering $H^2_E = -H^2$, this is a Euclidean version of Eq.(29) with the positive CFT-radiation matter. Eq.(33) is very similar to the Euclidean brane in the Euclidean AdS-Schwarzschild background [20] except replacing the flat brane by the AdS brane here. It is easily checked that $H_E$ is not defined properly ($H_E = i\tilde{\sigma}$, imaginary) at $a = a_c, a_+$, the roots of $\tilde{f}(a) = 0$. Thus the Euclidean brane neither crosses the cosmological horizon nor the black hole horizon in the Euclidean SdS background.

V. CFT-THERMODYNAMIC RELATIONS AND FRIEDMANN EQUATIONS

We rewrite the Friedmann equation of Eq.(30) in terms of the Hubble entropy ($S_H = (n - 1)HV/4G_{n+1}$), the Bekenstein-Hawking energy ($E_{BH} = n(n - 1)V/8\pi G_{n+1}a^2$), and $E_\Lambda = \Lambda_{n+1}V/8\pi G_{n+1}$

$$S_H = \frac{2\pi a}{n} \sqrt{E_{BH}[2(\tilde{E} + E_\Lambda) - E_{BH}]}$$

which is called the cosmological Verlinde’s formula. On the other hand, we find its static version (that is, the Cardy-Verlinde’s formula) for the cosmological horizon of the SdS black hole [14]

$$S = \frac{2\pi \ell}{n} \sqrt{|E_{CA}|(2E - E_{CA})}.$$
Even though $E_{CA}, E < 0$, it likes to conjecture a naive correspondence between these when considering the replacements,

$$S_H \to S, \ E_{BH} \to E_{CA}, \ \hat{E} + E_{\Lambda} \to E.$$  \hspace{1cm} (36)

Here the effect of cosmological constant ($E_{\Lambda}$) appears in the cosmological formula Eq.(34). This difference arises because in the static case one usually neglects the energy of pure dS space which relates to the cosmological constant via $\ell$, as is shown in Eq.(4).

One of the striking results for the dynamic AdS/CFT correspondence is that the Cardy-Verlinde’s formula on the CFT-side coincides with the Friedmann equation in cosmology when the flat brane crosses the horizon $a = a_H$ of the AdS-Schwarzschild black hole [20]. This means that the Friedmann equation knows the thermodynamics of the CFT. Let us understand this result in terms of the entropy bounds. At this point the temperature ($T_H = -\dot{H}/2\pi H$) and the entropy density ($s = S_H/V = (n - 1)H/4G_{n+1}$) can be expressed in terms of the Hubble parameter and its derivative only. One introduces the $\gamma$-function which relates to the central charge

$$\gamma_n(\tau) = n(n - 1) \frac{a_{H}^{n-1}}{16\pi G_{n+1} a^{n-1}} \frac{S_{CA}}{16\pi G_{n+1} S_{BH}},$$  \hspace{1cm} (37)

where $S_{CA}(S_{BH})$ denote the Casimir entropy bound of the CFT (Bekenstein-Hawking entropy bound of $(n + 2)$-dimensional bulk space). In this case one finds the bound of $\gamma_n(\tau) \leq \frac{n(n-1)}{16\pi G_{n+1}}$, because the Verlinde’s entropy bound has been proposed as $S_{CA} \leq S_{BH}$. When $a = a_H$, the Verlinde’s bound is saturated. In other words, the Casimir entropy bound equals to the Bekenstein-Hawking entropy bound when the flat brane crosses the horizon (a holographic point).

It is very important to study what happens at the moment when the MDW crosses the cosmological horizon at $a = a_c$ in TdS space. Unfortunately it turns out that the MDW with a positive CFT-matter does not cross the cosmological horizon.

In the background of the small SdS black hole, the MDW with the positive CFT-matter never crosses the cosmological horizon. But it always stays inside the cosmological horizon and crosses the black hole horizon at $a = a_+$ because of $a_+ < a_m < a_c$. We do not introduce here the replacements of Eq.(36) for the cosmological horizon. Furthermore, for the black hole horizon in SdS space, there does not exist any Cardy-Verlinde’s formula like Eq.(35) [14]. Hence the relation between the Cardy-Verlinde’s formula and the Friedmann equation is not established for the SdS black hole even if the brane crosses the black hole horizon at $a = a_+$.

**VI. DISCUSSION**

In the static dS/CFT correspondence, TdS space is better than SdS space because energies of the TdS-cosmological horizon ($r = r_c$) are always positive, whereas energies of the SdS-cosmological horizon are always negative. Thus there is no problem in defining the Cardy-Verlinde’s formula for the TdS space which is regarded as one of evidences for realizing the static dS/CFT correspondence. In the SdS black hole we cannot define the thermodynamic quantities for the black hole horizon at $r = r_+$ properly and thus cannot obtain the
corresponding Cardy-Verlinde’s formula. This means that in the static case an observer who is located outside the cosmological horizon can extract information about the cosmological horizon of the SdS black hole, but one cannot investigate inside the cosmological horizon to obtain further information about the black hole horizon.

On the other hand, exploring the dynamic dS/CFT correspondence using the MDW approach in SdS and TdS spaces leads to the conclusion that SdS space is better than TdS space. In TdS space we obtain the negative CFT-radiation matter from the naked singularity, while in SdS space we obtain the positive CFT-radiation matter from the black hole. In the case of finding the negative energy density in a cosmological model, we have to discard the corresponding model. Hence the dS brane moving on the small SdS background gives us a rather promising model for realizing the dynamic dS/CFT correspondence. However, the dS brane always stays inside the cosmological horizon and thus it never cross the cosmological horizon \( a = a_c \) but it can cross the black hole horizon \( a = a_+ \). For the cosmological horizon we obtain a similarity of Eq.(36) between the Cardy-Verlinde’s formula and the Friedmann equation, while for the black hole horizon we cannot define the Cardy-Verlinde’s formula itself. Hence even for the dS brane on SdS space, one does not say that the Friedmann equation (dynamic equation for the brane) knows the thermodynamics of the CFT defined on the dS brane.

Furthermore the dynamic SdS/ECFT correspondence is also allowed because the embedding of the spacelike brane with Euclidean signature into dS spaces with Lorentzian signature is defined. This supports the static dS/ECFT correspondence based on the relation between the isometry group of dS space and the conformal group of the Euclidean boundary space [8].

Finally it suggests that in order to derive the four-dimensional dS model to both get inflation in the far past and dS geometry in the far future, one may start with AdS space [28]. In this direction one can obtain the dS brane from dS space if the square of reduced brane tension \((\tilde{\sigma}^2)\) is greater than the reduced cosmological constant \((1/\ell^2)\) in Eq.(13).

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