LIOUVILLE THEOREMS FOR PERIODIC TWO-COMPONENT SHALLOW WATER SYSTEMS

QIAOYI HU\textsuperscript{1}, ZHIXIN WU\textsuperscript{2} AND YUMEI SUN\textsuperscript{3}

\textsuperscript{1} Department of Mathematics
South China Agricultural University
510642 Guangzhou, China

\textsuperscript{2} Department of Mathematics
DePauw University
46135 Greencastle, IN, USA

\textsuperscript{3} Department of Mathematics
Shandong University of Science and Technology
266590 Qingdao, Shandong, China

(Communicated by Adrian Constantin)

Abstract. We establish Liouville-type theorems for periodic two-component shallow water systems, including a two-component Camassa-Holm equation (2CH) and a two-component Degasperis-Procesi (2DP) equation. More precisely, we prove that the only global, strong, spatially periodic solutions to the equations, vanishing at some point \((t_0, x_0)\), are the identically zero solutions. Also, we derive new local-in-space blow-up criteria for the dispersive 2CH and 2DP.

1. Introduction. In the paper we consider the Cauchy problem of the following periodic two-component Camassa-Holm equation (2CH)

\[
\begin{aligned}
\begin{cases}
y_t + (y_x u + 2y u_x + \rho \rho_x)_x = 0, & t > 0, x \in \mathbb{R}, \\
p_t + (\rho u)_x = 0, & t > 0, x \in \mathbb{R}, \\
u(0, x) = u_0(x), & x \in \mathbb{R}, \\
\rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\
u(t, x) = u(t, x + 1), & t \geq 0, x \in \mathbb{R}, \\
\rho(t, x) = \rho(t, x + 1), & t \geq 0, x \in \mathbb{R},
\end{cases}
\end{aligned}
\]

\textsuperscript{(1.1)}

where \(y = u - u_{xx}\). (The Camassa-Holm equation can be obtained via the obvious reduction \(\rho \equiv 0\).)

Let \(G(x) := \frac{\cosh(x - [x] - 1/2)}{2 \sinh(1/2)}, x \in \mathbb{R}\). Then \((1 - \partial_x^2)^{-1} f = G * f\) for all \(f \in L^2(S)\) and \(G * y = u\). Here, we denote by \(*\) the convolution. By a direct calculation, one

\textsuperscript{2010 Mathematics Subject Classification.} 35G25, 35L05.

\textit{Key words and phrases.} Periodic two-component shallow water systems, Liouville-type theorem, global solution, blow-up.

3085
can rewrite Eq.(1.1) as follows:

\[
\begin{align*}
& u_t + uu_x + \partial_x G \ast (u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \rho^2) = 0, \quad t > 0, \ x \in \mathbb{R}, \\
& \rho_t + u \rho_x + u \rho = 0, \quad t > 0, \ x \in \mathbb{R}, \\
& u(0, x) = u_0(x), \quad x \in \mathbb{R}, \\
& \rho(0, x) = \rho_0(x), \quad x \in \mathbb{R}, \\
& u(t, x) = u(t,x + 1), \quad t \geq 0, \ x \in \mathbb{R}, \\
& \rho(t, x) = \rho(t,x + 1), \quad t \geq 0, \ x \in \mathbb{R}.
\end{align*}
\]

(1.2)

The two-component generalization of Camassa-Holm equation (1.1) was derived by Constantin and Ivanov [13] in the context of shallow water theory. \(u(t, x)\) describes the horizontal velocity of the fluid and \(\rho(t, x)\) is in connection with the horizontal deviation of the surface from equilibrium, all measured in dimensionless units [13].

With \(\rho \equiv 0\) in Eq.(1.1), we find the Camassa-Holm equation, which models the wave motion on shallow water, \(u(t, x)\) representing the fluid’s free surface above a flat bottom (or equivalently the fluid velocity at time \(t \geq 0\) in the spatial \(x\) direction) [6, 21, 33]. It is remarkable that the Camassa-Holm equation on line has peakons of the form

\[ u(t, x) = ce^{-|x-ct|}, \ x \in \mathbb{R}, \ t \geq 0, \ c > 0 \]

which interact like solitons [6]. The Cauchy problem of the Camassa-Holm equation has been studied extensively. It has been shown that this equation is locally well-posed [9, 11, 36, 42] for initial data \(u_0 \in H^s\) with \(s > \frac{3}{2}\). More interestingly, it has not only global strong solutions modeling permanent waves [11, 12, 16] and but also blowup solutions modeling wave breaking [10, 11, 12, 16, 36, 42]. On the other hand, it has global weak solutions with initial data \(u_0 \in H^1\) [7, 17, 44]. Moreover, the Camassa-Holm equation has global conservative solutions [4, 29] and dissipative solutions [5, 30].

The Cauchy problem of Eq.(1.1) on the line has been discussed in [13, 27]. In [13], Constantin and Ivanov investigated the global existence and blow-up phenomena of strong solutions of Eq.(1.1). Later, Guan and Yin obtained a new global existence result for strong solutions to Eq.(1.1) and got several blow-up results [27] which improved the results in [13]. After that Hu and Yin studied the global existence and blow-up phenomena for the periodic two-component Camassa-Holm equation Eq.(1.1) in [32].

Regarding to the blow-up criteria, L. Brandolese et. al. recently derived new local-in-space blow-up criteria for a class of nonlinear partial differential equations. In [1], L. Brandolese established new local-in-space blow-up criteria for the equations modeling shallow water waves and nonlinear dispersive waves in elastic rods. Later, using the new blow-up criteria and the computation of several best constants in convolution estimates and weighted Poincaré inequalities, L. Brandolese and M. F. Cortez studied the permanent and breaking waves for the rod equation and the Degasperis-Procesi equation [2, 3]. These local-in-space blow-up criteria attract our interests, as they have the specific feature of being purely local in the space variable: indeed the blow-up conditions only involve the values of \(u_0(\xi)\) and \(u_0'(\xi)\) in a single point \(\xi\) of the real line. In [28], Duc Truc Hoang investigated wave breaking criteria for the Dullin-Gottwald- Holm equation and the two-component Dullin-Gottwald-Holm system. The author established a new blow-up criterion for the general case \(\gamma + c_0 \alpha^2 \geq 0\) involving local-in-space conditions on the initial data.
Motivated by the works of the above authors (see [1, 2, 3, 28]), we plan to investigate the new local-in-space blow-up criteria for some two-component shallow water systems, including Eqs.(1.2) and (4.2). For earlier blow-up results for the two-component shallow water systems, it is a natural way to control the quantities \( \|u(t)\|_{L^\infty} \) and \( \|\rho(t)\|_{L^\infty} \) by some global quantities, such as the \( \|u_0\|_{H^1} \)-norm and \( \|\rho_0\|_{L^2} \)-norm appeared in [13, 27, 31, 32], or other global conditions like antisymmetry assumptions or sign conditions on the associated potential \( y_0 \). However, we do not use any conservation laws and other global conditions rather than the derivation of new local-in-space blow-up criteria (see Eq.(3.4)) in the paper.

More specifically, we first exploit the characteristic ODE related to the system (1.1) (see Eq.(2.1)) to construct some invariant properties of the solutions and sufficiently utilizing the structure of the system itself to deduce the fine properties of the second component \( \rho \) of the system (1.1) (see Lemma 2.2). Then, by the introduction of the two auxiliary maps (see Eq.(3.1) and Eq.(3.3)), we obtain the new local-in-space blow-up criteria (see Eq.(3.4)) contrary to the previous ones as in [13, 27, 31, 32]. Finally, we prove the following Liouville-type theorem for strong solutions to Eq.(1.1) by using Eq.(3.4) and introducing the two families of Lyapunov functions (see Eqs.(3.7) and (3.8)).

**Theorem 1.1.** If \( z = (u, \rho) \in C([0,\infty), H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \cap C^1([0,\infty), H^s-1(\mathbb{S}) \times H^{s-2}(\mathbb{S})) \) with \( s \geq 2 \), is a global solution to the two-component Camassa-Holm equation (1.2), such that \( u(t_0,x_0) = \rho(t_0,x_0) = 0 \), at some point \((t_0,x_0)\), then \( u(t,x) = \rho(t,x) \equiv 0 \) for all \((t,x)\).

More precisely, let \( z_0 = \begin{pmatrix} u_0 \\ \rho_0 \end{pmatrix} \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \geq 2 \), and let \( T \) be the maximal existence time of solution \( z = \begin{pmatrix} u \\ \rho \end{pmatrix} \) to Eq.(1.2) with the initial data \( z_0 \). If there is some point \((t_0,x_0)\) such that \( u(t_0,x_0) = \rho(t_0,x_0) = 0 \), but \( u(t_0,x) \neq 0 \), then the solution \( z = (u,\rho) \in C([0,\infty), H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \cap C^1([0,\infty), H^s-1(\mathbb{S}) \times H^{s-2}(\mathbb{S})) \), with \( s \geq 2 \), to the two-component Camassa-Holm equation must break down in finite time.

For the two-component Camassa-Holm equation with dispersion, we reformulate the above theorem as follows:

**Theorem 1.2.** If \( \tilde{z} = (v, g) \in C([0,\infty), H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \cap C^1([0,\infty), H^s-1(\mathbb{S}) \times H^{s-2}(\mathbb{S})) \) with \( s \geq 2 \), is a global solution to the two-component Camassa-Holm equation with dispersion

\[
\begin{aligned}
&v_t + vv_x + \partial_x G \ast (2kv + v^2 + \frac{1}{2}v_x^2 + \frac{1}{2}g^2 + kg) = 0, & t > 0, & x \in \mathbb{R}, \\
g_t + v_x g + v_x \partial_x + kv_x = 0, & t > 0, & x \in \mathbb{R},
\end{aligned}
\]

such that \( g(t_0,x_0) = -k \) and \( v(t_0,x_0) = -k \) at some point \((t_0,x_0)\), then

\[
v(t,x) = g(t,x) \equiv -k \text{ for all } (t,x).
\]

**Remark 1.1.** In fact, Theorem 1.2 can be reduced to Theorem 1.1, as \( z(t,x) = (u(t,x),\rho(t,x)) = (v(t,x-kt) + k, g(t,x-kt) + k) \), is a global solution of Eq.(1.2) if and only if \( \tilde{z} = (v,g) \) is a global solution of Eq.(1.3) with \( z_0 = (u_0,\rho_0) = (v_0 + k, g_0 + k) \).

We now conclude this introduction by outlining the rest of the paper. In Section 2, we briefly give some needed results including the local well-posedness of Eq.(1.2), the precise blow-up scenarios and some useful lemmas to prove Theorem 1.1. In
Section 3, we give the detailed proof of Theorem 1.1. In Section 4, we establish the similar Liouville-type theorem for the two-component Degasperis-Procesi equation.

2. Preliminaries. In the section, we briefly give the needed results to pursue our goal. We first present the local well-posedness for the Cauchy problem of Eq.(1.1) (or Eq.(1.2)) in $H^s(S) \times H^{s-1}(S), s \geq 2$, with $S = \mathbb{R}/\mathbb{Z}$ (the circle of unit length) proved in [32]. Applying the Kato’s semigroup theorem [34], we obtain the following local well-posedness theorem for Eq.(1.2).

**Theorem 2.1.** [32] Given $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s(S) \times H^{s-1}(S), s \geq 2$, there exists a maximal $T = T(z_0) > 0$, and a unique solution $z = \left( \begin{array}{c} u \\ \rho \end{array} \right)$ to Eq.(1.2) such that

$$z = z(\cdot, z_0) \in C([0,T); H^s(S) \times H^{s-1}(S)) \cap C^1([0,T); H^{s-1}(S) \times H^{s-2}(S)).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $z_0 \mapsto z(\cdot, z_0) : H^s(S) \times H^{s-1}(S) \rightarrow C([0,T); H^s(S) \times H^{s-1}(S)) \cap C^1([0,T); H^{s-1}(S) \times H^{s-2}(S))$ is continuous.

By the local well-posedness in Theorem 2.1 and the energy method, one can get the following precise blow-up scenario of strong solutions to Eq.(2.1).

**Theorem 2.2.** [24, 32] Let $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s(S) \times H^{s-1}(S), s > \frac{5}{2}$, and let $T$ be the maximal existence time of the solution $z = \left( \begin{array}{c} u \\ \rho \end{array} \right)$ to Eq.(2.1) with the initial data $z_0$. Then the corresponding solution blows up in finite time if and only if

$$\lim_{t \to T} \inf_{x \in S} \{u_x(t,x)\} = -\infty \text{ or } \lim_{t \to T} \sup_{x \in S} \||\rho_x(t,\cdot)||_{L^\infty(S)}\| = +\infty.$$

For initial data $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^2(S) \times H^1(S)$, we have the following precise blow-up scenario.

**Theorem 2.3.** [24, 32] Let $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^2(S) \times H^1(S)$, and let $T$ be the maximal existence time of the solution $z = \left( \begin{array}{c} u \\ \rho \end{array} \right)$ to Eq.(2.1) with the initial data $z_0$. Then the corresponding solution blows up in finite time if and only if

$$\lim_{t \to T} \inf_{x \in S} \{u_x(t,x)\} = -\infty.$$

Given initial data $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s(S) \times H^{s-1}(S), s \geq 2$, Theorem 2.1 ensures the existence and uniqueness of strong solutions to Eq.(2.1). Now we consider the following initial value problem

$$\begin{cases}
q_t = u(t,q), & t \in [0,T), x \in \mathbb{R}, \\
q(0,x) = x, & x \in \mathbb{R},
\end{cases} \tag{2.1}$$

where $u$ denotes the first component of the solution $z$ to Eq.(1.2) with the initial data $z_0$. Since $u(t,.) \in H^2(S) \subset C^m(S)$ with $0 \leq m \leq \frac{5}{2}$, it follows that $u \in C^1([0,T) \times \mathbb{R}, \mathbb{R})$. Applying the classical results in the theory of ordinary differential equations, one can obtain the following results of $q$ which are useful in the proof of Theorem 1.1.
Lemma 2.1. [13, 24, 27, 32] Let \( z_0 = \left( \frac{u_0}{\rho_0} \right) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \geq 2, \) and let \( T > 0 \) be the maximal existence time of the corresponding solution \( z = \left( \frac{u}{\rho} \right) \) to Eq. (1.2) with the initial data \( z_0 \). Then Eq. (2.1) has a unique solution \( q \in \mathcal{C}_1([0, T) \times \mathbb{R}, \mathbb{R}) \). Moreover, the map \( q(t, \cdot) \) is an increasing diffeomorphism of \( \mathbb{R} \) with \( q_z(t, x) = \exp \left( \int_0^t u_x(s, q(s, x)) ds \right) > 0, (t, x) \in [0, T) \times \mathbb{R} \).

Lemma 2.2. [24, 27, 32] Let \( z_0 = \left( \frac{u_0}{\rho_0} \right) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \geq 2, \) and let \( T > 0 \) be the maximal existence time of the corresponding solution \( z = \left( \frac{u}{\rho} \right) \) to Eq. (1.2) with the initial data \( z_0 \). Then we have \( \rho(t, q(t, x))q_z(t, x) = \rho_0(x), (t, x) \in [0, T) \times \mathbb{R} \). Moreover if there exists \( x_0 \in \mathbb{S} \) such that \( \rho_0(x_0) = 0 \), then \( \rho(t, q(t, x_0)) = 0 \) for all \( t \in [0, T) \).

3. Proof of Theorem 1.1. In the section we prove Theorem 1.1.

Proof. Let \( z \) be the solution to Eq. (1.2) with the initial data \( z_0 \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \geq 2, \) and let \( T > 0 \) be the maximal time of existence of the solution \( z \) with the initial data \( z_0 \). It is easy to check that Equation (1.2) is invariant under time translations \( u(t, x) \rightarrow -u(-t, x) \) and \( (\rho, x) \rightarrow -\rho(-t, x) \). Without loss of generality, we set \( t_0 = 0 \).

Now we define \( m(t) = u_x(t, q(t, x_0)) \) and \( h(t) = \rho(t, q(t, x_0)) \). Note that \( h(0) = \rho(0, q(0, x_0)) = \rho_0(x_0) = 0 \). By Lemma 2.2, we have \( h(t) = 0 \) for all \( t \in [0, T) \). Therefore, it is enough to prove that if \( u_0(x_0) = 0 \) at some point \( x_0 \in \mathbb{S} \), but \( u_0 \neq 0 \), then the first component of the solution \( u \in C([0, T), H^s(\mathbb{S})) \cap \mathcal{C}^1([0, T), H^{s-1}(\mathbb{S})) \) must blow up in finite time.

Let \( a \in (x_0, x_0 + 1) \) be such that \( u_0(a) \neq 0 \). We first consider the case \( u_0(a) > 0 \). Let us introduce the map

\[
\phi(x) = e^{\sqrt{2} x} u_0(x).
\] (3.1)

By the periodicity and the continuity of \( u_0 \), we can find an open interval \((a, b) \subset (x_0, x_0 + 1)\) such that \( \phi(x) > 0 \) on the interval \((a, b)\) and \( \phi(a) > 0, \phi(b) = 0 \). An integration by parts gives

\[
\int_a^b e^{\sqrt{2} x} u_0(x) dx = \phi(b) - \phi(a) - \int_a^b \sqrt{2} \phi(x) dx. \quad (3.2)
\]

We deduce from this the existence of \( \xi \in (a, b) \) such that \( u_0'(\xi) < -\sqrt{2} u_0(\xi) < 0 \). Indeed, otherwise, \( u_0(x) \geq -\sqrt{2} u_0(x) \) for all \( x \in (a, b) \), then we have

\[
\phi(b) - \phi(a) - \int_a^b \sqrt{2} \phi(x) dx = \int_a^b e^{\sqrt{2} x} u_0'(x) dx
\]
\[
\geq - \int_a^b \sqrt{2} e^{\sqrt{2} x} u_0(x) dx
\]
\[
= - \int_a^b \sqrt{2} \phi(x) dx.
\]
Hence, we get the contradiction $\phi(a) \leq \phi(b)$.

Next we consider the case $u_0(a) < 0$: introducing now the map

$$\varphi(x) = e^{-\sqrt{2}\mu(x)}$$

and arguing as before, we get in this case the existence of a point $\xi$ such that $u'_{\rho}(\xi) < -\sqrt{2}u_0(\xi) < 0$. Notice that in both cases we get there exists $\xi \in \mathbb{S}$ such that

$$u'_{\rho}(\xi) < -\sqrt{2}|u_0(\xi)| < 0.$$  \hfill (3.4)

Finally, we suffice to establish the finite time blow-up under the above condition (3.4) with $z_0 = (u_0, \rho_0) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), s \geq 2$.

Note that $\partial_x^2 G \ast f = G \ast f - f$. Differentiating the first equation in (1.2) with respect to $x$, we get

$$u_t = -\frac{1}{2} u_{xx} + u^2 + \frac{1}{2} \rho^2 - G \ast (u^2 + \frac{1}{2} u_{xx} + \frac{1}{2} \rho^2).$$

By Eq.(1.2) and Eq.(2.1), we have

$$\frac{dm}{dt} = (u_{tx} + u_{xx} q_t)(t, q(t, x_0)) = (u_{tx} + u_{xx})(t, q(t, x_0))$$

and

$$\frac{dh}{dt} = \rho_t + \rho_x q_t = -hm.$$  

Substituting $(t, q(t, x_0))$ into Eq.(3.5), we obtain

$$m'(t) = -\frac{1}{2} m^2(t) + u^2(t, q(t, x_0)) + \frac{1}{2} h^2 - G \ast (u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \rho^2)(t, q(t, x_0)).$$

(3.6)

Setting $L(x) = u^2 + \frac{1}{2} u_x^2 + \frac{1}{2} \rho^2$, we now introduce the $C^1$ functions, defined on $(0, T)$,

$$f(t) = -m(t) + \sqrt{2}u(t, q(t, \xi)), \quad (3.7)$$

$$g(t) = -m(t) - \sqrt{2}u(t, q(t, \xi)). \quad (3.8)$$

Using the definition of the flow map (2.1), Eqs.(1.2) and (3.6) we obtain

$$f'(t) = -m_t + \sqrt{2}(u_t + uu_x)(t, q(t, \xi))$$

$$= \frac{1}{2} m^2 - \frac{1}{2} h^2 + [-u^2 + G \ast L + \sqrt{2}(-\partial_x G \ast L)](t, q(t, \xi))$$

$$\geq \frac{1}{2} m^2 - u^2(t, q(t, \xi)), \quad (3.9)$$

where the last inequality is obtained by the facts $h(t) = 0, L \geq 0, 0 < \sqrt{2} \leq \coth(1/2)$ and $(G \pm \gamma G_x) \geq 0 \Leftrightarrow |\gamma| \leq \coth(1/2)$.

Factorizing the right-hand side of (3.9) leads to the following differential inequality

$$f'(t) \geq \frac{1}{2} f(t) g(t), t \in (0, T). \quad (3.10)$$

A similar computation yields

$$g'(t) \geq \frac{1}{2} f(t) g(t), t \in (0, T). \quad (3.11)$$
Let

\[ H(t) = \sqrt{f(t)g(t)}. \]  

(3.12)

We first observe that

\[ H(0) = \sqrt{fg(0)} = \sqrt{m_0^2(\xi) - 2u_0^2(\xi)} = \sqrt{v_0'(\xi)^2 - 2u_0^2(\xi)} > 0. \]  

(3.13)

Moreover, we deduce from the system (3.10)-(3.11), applying the geometric-arithmetic mean inequality, that

\[ H'(t) \geq \frac{1}{2} H^2(t), \quad t \in (0, T). \]  

(3.14)

This immediately implies \( T \leq \frac{2}{\|H(0)\|} < \infty \), hence \( u(t, x) \equiv 0 \) for all \( (t, x) \).

By using \( u(t, x) \equiv 0 \) and the second equation of (1.1), one can get that \( \rho_t = 0 \).

Thus \( \rho(t, x) = \rho(t, 0) = h(t) = 0 \) for all \( (t, x) \).

The proof of Theorem 1.1 is completely finished. \( \square \)

**Remark 3.1.** The “local-in-space” blowup results investigated in [28] look more general and more precise than Theorem 1.1. On the other hand, our proof of blowup criteria is shorter and sufficient for our purpose.

As a byproduct of proof of Theorem 1.1, we get the following new blow-up criterion for periodic solutions the two-component Camassa-Holm equation with or without dispersion:

**Corollary 3.1.** Let \( z_0 = \left( \begin{array}{c} v_0 \\ \varrho_0 \end{array} \right) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}), \ s \geq 2, \) and let \( T \) be the maximal existence time of solution \( \tilde{z} = \left( \begin{array}{c} v \\ \varrho \end{array} \right) \) to Eq.(1.3) with the initial data \( z_0 \).

If there is some \( \xi \in \mathbb{S} \) such that \( \varrho_0(\xi) = 0 \) and

\[ v_0'(\xi) < -\sqrt{2|v_0(\xi)|} + k < 0 \]

then the corresponding solution to Eq.(1.3) blows up in finite time.

**Remark 3.2.** Contrary to previously known blow-up criteria, our blow-up condition stated in Corollary 3.1 are easier to check. More precisely, the condition on the first component \( v(t, x) \) for the solution \( \tilde{z} = (v, \varrho) \) to the system (1.3) is purely local in the space variable, it only involves the values of \( v_0(\xi) \) and \( v_0'(\xi) \) in a single point \( \xi \) of the real line. While in the previous results, such as in [27, 31, 32], checking the blow-up conditions involved the computation of \( \|v_0\|_{H^1} \) norm and \( \|\varrho_0\|_{L^2} \) norm.

**Remark 3.3.** Applying Corollary 3.1 with \( k = 0 \) and \( \xi = 0 \) improves the blow-up criterion in [31], Theorem 4.4, establishing the blow-up for \( u_0 \) is odd, \( \rho_0(0) = 0 \) and \( u_0'(0) < 0 \). In the particular case for \( k = 0 \) in Corollary 3.1, one only needs to check the behavior of \( u_0 \) in a neighborhood of a single point to get the blow-up condition without any additional assumptions stated in [31], Theorem 4.4.

**Remark 3.4.** Theorem 1.1 and Corollary 3.1 also reveal that global solutions must satisfy quite stringent pointwise estimates. Indeed, assume that \( u \in C([0, \infty), H^s(\mathbb{S})) \cap C^1([0, \infty), H^{s-1}(\mathbb{S})) \) is the first component of a given global solution of (1.2). Then, by our theorem, \( sign(u) = 1, 0 \) or \(-1\) is well defined and independent on \( (t, x) \). Moreover, \( u'(t, x) \geq -\sqrt{2|u(t, x)|} \) for all \( t \geq 0 \) and \( x \in \mathbb{S} \). Then, arguing as in (3.5), we deduce that, for all \( t \geq 0 \), the map \( x \mapsto e^{sign(u)\sqrt{2}x} u(t, x) \) is increasing.
Combining this with the periodicity, we get the pointwise estimates for $u(t, x)$, for all $t \geq 0$, all $x_0 \in \mathbb{R}$ and $x_0 \leq x \leq x_0 + 1$:

$$e^{\text{sign}(u)\sqrt{2}(x_0-x)}u(t, x_0) \leq u(t, x) \leq e^{\text{sign}(u)\sqrt{2}(x_0+1-x)}u(t, x_0).$$

From (3.15) one immediately deduces the corresponding estimates for global solutions to the dispersive two-component the Camassa-Holm equation (1.3).

4. Application to the 2DP. In this section, we address the Cauchy problem for the periodic two-component Degasperis-Procesi system (2DP):

$$\begin{align*}
\frac{\partial y}{\partial t} + y_x u + 3yu_x - k_1\rho_x + k_2u_x \rho = 0, & \quad t > 0, x \in \mathbb{R}, \\
\rho_t + (\rho u)_x = -k_3u_x \rho, & \quad t > 0, x \in \mathbb{R}, \\
u(0, x) = u_0(x), & \quad x \in \mathbb{R}, \\
\rho(0, x) = \rho_0(x), & \quad x \in \mathbb{R}, \\
u(t, x) = u(t, x + 1), & \quad t \geq 0, x \in \mathbb{R}, \\
\rho(t, x) = \rho(t, x + 1), & \quad t \geq 0, x \in \mathbb{R},
\end{align*}$$

(4.1)

where $y = u - u_{xx}$. Eq.(4.1) was recently proposed by Popowicz in [41]. There are two cases about this system: (i) $k_2 = 0, k_3 = 1$ and $k_1$ is an arbitrary real constant, or $k_2 = 0, k_1 = 1$ and $k_3$ takes an arbitrary real value; (ii) $k_2 = 1, k_1 = 2$ and $k_3 = 1$. The construction based on the observation that the second Hamiltonian operator of the Degasperis-Procesi equation could be considered as the Dirac reduced Poisson tensor of the second Hamiltonian operator of the Boussinesq equation in [41]. The well-posedness and the blow-up phenomena for the two-component Degasperis-Procesi system were studied in [45]. Escher et. al. showed that the periodic two-component Degasperis-Procesi equation (4.2) can be regarded as geodesic equations on the semidirect product of diffeomorphism group of the circle $\text{Diff}(\mathbb{S}^1)$ with some smooth functions [22].

For $\rho = 0$, Eq.(4.1) becomes the Degasperis-Procesi equation [19]. The Degasperis-Procesi equation can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the Camassa-Holm shallow water equation [15, 21, 33]. The formal integrability of the Degasperis-Procesi equation was obtained in [20] by constructing a Lax pair. It has a bi-Hamiltonian structure with an infinite sequence of conserved quantities and admits exact peakon solutions [20] which are analogous to the Camassa-Holm peakons [6, 18]. An inverse scattering approach for computing N-peakon solutions to the Degasperis-Procesi equation was presented in [18, 39]. Its traveling wave solutions was investigated in [35, 43].

Although the Degasperis-Procesi equation is very similar to the Camassa-Holm equation in many aspects, especially in the structure of equation, there are some essential differences between the two equations. One of the famous features of DP equation is that it has not only peakon solutions $u_c(t, x) = ce^{-|x-ct|}$ with $c > 0$ [20] and periodic peakon solutions [49], but also shock peakons [38] and the periodic shock waves [26]. Besides, the Camassa-Holm equation is a re-expression of geodesic flow on the diffeomorphism group [14] or on the Bott-Virasoro group [40], while the DP equation can be regarded as a non-metric Euler equation [23]. On the other hand, the isospectral problem in the Lax pair for Degasperis-Procesi equation is the third order equation

$$\psi_x - \psi_{xxx} - \lambda y\psi = 0$$
cf. [20], while the isospectral problem for the Camassa-Holm equation is the second order equation

$$\psi_{xx} - \frac{1}{4} \psi - \lambda y \psi = 0$$

(in both cases $y = u - u_{xx}$) cf. [6].

Regarding to the Cauchy problem for DP equation, plenty of works [8, 25, 26, 37, 46, 47, 48, 49] have been done. For example, the local well-posedness and blow-up phenomena to DP equation with initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$ on the line and on the circle were studied in [47] and [46], respectively. The global existence of strong solutions and global weak solutions to DP equation were shown in [48, 49]. Similar to the Camassa-Holm equation, the DP equation has not only global strong solutions [37, 48] but also blow-up solutions [25, 26, 37, 24]. Moreover, it has global entropy weak solutions in $L^1(\mathbb{R}) \cap BV(\mathbb{R})$ and $L^2(\mathbb{R}) \cap L^2(\mathbb{R})$, cf. [8].

In this section, we establish the Liouville-type theorem for Eq.(4.1) with $k_2 = 0$. For our convenience, we rewrite Eq.(4.1) as follows:

$$\begin{cases}
    u_t + uu_x + \partial_x G * (\frac{3}{2} u^2 - \frac{k_1}{2} \rho^2) = 0, & t > 0, x \in \mathbb{R}, \\
    \rho_t + u \rho_x + (k_3 + 1) u_x \rho = 0, & t > 0, x \in \mathbb{R}, \\
    u(0, x) = u_0(x), & x \in \mathbb{R}, \\
    \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\
    u(t, x) = u(t, x + 1), & t \geq 0, x \in \mathbb{R}, \\
    \rho(t, x) = \rho(t, x + 1), & t \geq 0, x \in \mathbb{R}.
\end{cases} \tag{4.2}$$

Or the equivalent form:

$$\begin{cases}
    u_t + uu_x = -\partial_x (1 - \partial_x)^{-1}(\frac{3}{2} u^2 - \frac{k_1}{2} \rho^2), & t > 0, x \in \mathbb{R}, \\
    \rho_t + u \rho_x + (k_3 + 1) u_x \rho = 0, & t > 0, x \in \mathbb{R}, \\
    u(0, x) = u_0(x), & x \in \mathbb{R}, \\
    \rho(0, x) = \rho_0(x), & x \in \mathbb{R}, \\
    u(t, x) = u(t, x + 1), & t \geq 0, x \in \mathbb{R}, \\
    \rho(t, x) = \rho(t, x + 1), & t \geq 0, x \in \mathbb{R}.
\end{cases} \tag{4.3}$$

Now we present some lemmas given in [45]. We first have the following local well-posedness result.

**Lemma 4.1.** [45] Given $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$, there exists a maximal $T = T(z_0) > 0$, and a unique solution $z = \left( \begin{array}{c} u \\ \rho \end{array} \right)$ to Eq.(4.2) such that

$$z = z(., z_0) \in C([0, T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1([0, T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S})) \cap C([0, T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S}))$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping $z_0 \rightarrow z(., z_0) : H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S}) \rightarrow C([0, T); H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})) \cap C^1([0, T); H^{s-1}(\mathbb{S}) \times H^{s-2}(\mathbb{S}))$ is continuous.

Then we state the following precise blow-up scenario of strong solutions to Eq.(4.2).

**Lemma 4.2.** [24, 45] Let $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s(\mathbb{S}) \times H^{s-1}(\mathbb{S})$, $s \geq 2$, and let $T$ be the maximal existence time of the solution $z = \left( \begin{array}{c} u \\ \rho \end{array} \right)$ to Eq.(4.2) with the initial
data $z_0$. Then the corresponding solution blows up in finite time if and only if

$$
\liminf_{t \to T} \{u_x(t, x)\} = -\infty \text{ or } \limsup_{t \to T} \{\rho_x(t, \cdot)\|_{L^\infty(S)}\} = +\infty.
$$

For initial data $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^2(S) \times H^1(S)$, we have the following precise blow-up scenario.

**Lemma 4.3.** [24, 45] Let $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^2(S) \times H^1(S)$, and let $T$ be the maximal existence time of the solution $z = \left( \begin{array}{c} u \\ \rho \end{array} \right)$ to Eq. (4.2) with the initial data $z_0$. Then the corresponding solution blows up in finite time if and only if

$$
\liminf_{t \to T} \{u_x(t, x)\} = -\infty.
$$

Given initial data $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s(S) \times H^{s-1}(S)$, $s \geq 2$, Lemma 4.1 ensures the existence and uniqueness of strong solutions to Eq. (4.2). By introduction of the following characteristic flow map

$$
\begin{cases}
q_t = u(t, q), & t \in [0, T), x \in \mathbb{R}, \\
q(0, x) = x, & x \in \mathbb{R},
\end{cases}
$$

where $u$ denotes the first component of the solution $z$ to Eq. (4.2) with the initial data $z_0$, we obtain the following useful properties of $q$.

**Lemma 4.4.** [45] Let $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s(S) \times H^{s-1}(S)$, $s \geq 2$, and let $T > 0$ be the maximal existence time of corresponding solution $z = \left( \begin{array}{c} u \\ \rho \end{array} \right)$ to Eq. (4.2) with the initial data $z_0$. Then Eq. (4.2) has a unique solution $q \in C^1([0, T] \times \mathbb{R}, \mathbb{R})$. Moreover, the map $q(t, \cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$
q_x(t, x) = \exp \left( \int_0^t u_x(s, q(s, x))ds \right) > 0, \ (t, x) \in [0, T) \times \mathbb{R}.
$$

**Lemma 4.5.** [45] Let $z_0 = \left( \begin{array}{c} u_0 \\ \rho_0 \end{array} \right) \in H^s(S) \times H^{s-1}(S)$, $s \geq 2$, and let $T > 0$ be the maximal existence time of corresponding solution $z = \left( \begin{array}{c} u \\ \rho \end{array} \right)$ to Eq. (4.2) with the initial data $z_0$. Then we have

$$
\rho(t, q(t, x))q_x^{k_2 + 1}(t, x) = \rho_0(x), \ (t, x) \in [0, T) \times \mathbb{R}.
$$

Moreover if there exists $x_0 \in S$ such that $\rho_0(x_0) = 0$, then $\rho(t, q(t, x_0)) = 0$ for all $t \in [0, T)$.

Using the similar argument discussed in Sec. 3, we establish the following Liouville-type theorem for the two-component Degasperis-Procesi system with $k_2 = 0$.

**Theorem 4.1.** Assume that $k_1 \leq 0$ and $k_2 = 0$, if $z = \left( \begin{array}{c} u \\ \rho \end{array} \right) \in C(H^s(S) \times H^{s-1}(S)) \cap C^1(H^{s-1}(S) \times H^{s-2}(S))$ with $s \geq 2$, is a global solution to the two-component Degasperis-Procesi system (4.2), such that $u(t_0, x_0) = \rho(t_0, x_0) = 0$ at some point $(t_0, x_0)$, then $u(t, x) = \rho(t, x) \equiv 0$ for all $(t, x)$. 
**Remark 4.1.** Theorem 4.1 covers Theorem 1.1 in [3] with \( \rho = 0 \).

For two-component Degasperis-Procesi equation with dispersion, similar argument as in Remark 1.1 leads to the following corollaries.

First, we have the unique continuation result for the following dispersive 2DP.

**Corollary 4.1.** If \( k_1 \leq 0 \), \( \tilde{z} = (v, \varrho) \in H^s(S) \times H^{s-1}(S) \) with \( s > \frac{3}{2} \) is a global solution to the two-component Degasperis-Procesi equation with dispersion

\[
\begin{aligned}
&\{ v_t + v v_x + \varrho G * (3kv + \frac{3}{2}v^2 - k_1 \varrho^2 - k_1 k \varrho) = 0, \quad t > 0, \, x \in \mathbb{R}, \\
&\varrho_t + v \varrho_x + (k_3 + 1)v_x \varrho + k(k_3 + 1)v_x = 0, \quad t > 0, \, x \in \mathbb{R},
\end{aligned}
\tag{4.6}
\]

such that \( \varrho(t_0, x_0) = \varrho(t_0, x_0) = -k \) at some point \((t_0, x_0)\), then \( v(t, x) = \varrho(t, x) \equiv -k \) for all \((t, x)\).

Then we have the following new blow-up criterion for periodic solutions the two-component Degasperis-Procesi equation with or without dispersion:

**Corollary 4.2.** Let \( \tilde{z}_0 = \left(\frac{v_0}{\varrho_0}\right) \in H^s(S) \times H^{s-1}(S) \), \( s \geq 2 \), and let \( T \) be the maximal existence time of solution \( \tilde{z} = \left(\frac{v}{\varrho}\right) \) to Eq.(4.6) with the initial data \( \tilde{z}_0 \).

If there is some \( \xi \in S \) such that \( \varrho_0(\xi) = 0 \) and

\[
v_0'(\xi) < -\sqrt{\frac{3}{2}} |v_0(\xi) + k| < 0
\]

then the corresponding solution to Eq.(4.6) blows up in finite time.

Similar as Lemma 3.4, we have the global pointwise estimate for Eq.(4.2).

**Corollary 4.3.** Assume that \( \rho(x) \neq 0 \) for all \( x \in S \) and \( u \in C([0, \infty), H^s(S)) \cap C^1([0, \infty), H^{s-1}(S)) \) is the first component of a given global solution of Eq.(4.2).

Then we get the pointwise estimates for \( u(t, x) \), for all \( t \geq 0 \), all \( x_0 \in \mathbb{R} \) and \( x_0 \leq x \leq x_0 + 1 \):

\[
es^{\text{sign}(u)\sqrt{2}(x_0-x)}u(t, x_0) \leq u(t, x) \leq es^{\text{sign}(u)\sqrt{2}(x_0+1-x)}u(t, x_0).
\tag{4.7}
\]

Similar argument as discussed in Lemma 1.1 and combing Eq.(4.7), we obtain the pointwise estimate for the first component of the global solution for the dispersive 2DP (i.e. Eq.(4.6)).

**Remark 4.2.** Theorem 4.1, Corollary 4.2 and Corollary 4.3 extend the corresponding results in [2] to the two-component Degasperis-Procesi system. For the comparison with some earlier results for 2DP and DP, we refer to Sect. 2 in [2].

**Acknowledgments.** This work was partially supported by NSF of Guangdong (No. 2015A030313424), the Science and Technology Program of Guangzhou (No. 201607010005) and the National Natural Science Foundation of China (No. 11401223). The author thanks the referees for their constructive comments.

**REFERENCES**

[1] L. Brandolese, Local-in-space criteria for blowup in shallow water and dispersive rod equations, *Commun. Math. Phys.*, **330** (2014), 401–444.

[2] L. Brandolese and M. F. Cortez, Blowup issues for a class of nonlinear dispersive wave equations, *J. Differential Equations*, **256** (2014), 3981–3998.

[3] L. Brandolese, A Liouville theorem for the Degasperis-Procesi equation, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **16** (2016), 759–765.
[4] A. Bressan and A. Constantin, Global conservative solutions of the Camassa-Holm equation, *Arch. Ration. Mech. Anal.*, **183** (2007), 215–239.

[5] A. Bressan and A. Constantin, Global dissipative solutions of the Camassa-Holm equation, *Arch. Ration. Mech. Anal.*, **5** (2007), 1–27.

[6] R. Camassa and D. Holm, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71** (1993), 1661–1664.

[7] G. M. Coclite, H. Holden and K. H. Karlsen, Well-posedness of higher-order Camassa–Holm equations, *J. Differential Equations*, **246** (2009), 929–963.

[8] G. M. Coclite and K. H. Karlsen, On the well-posedness of the Degasperis-Procesi equation, *J. Func. Anal.*, **233** (2006), 60–91.

[9] A. Constantin, The Cauchy problem for the periodic Camassa-Holm equation, *J. Differential Equations*, **141** (1997), 218–235.

[10] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.*, **181** (1998), 229–243.

[11] A. Constantin and J. Escher, Well-posedness, global existence and blowup phenomena for a periodic quasi-linear hyperbolic equation, *Comm. Pure Appl. Math.*, **51** (1998), 475–504.

[12] A. Constantin and J. Escher, On the structure of a family of quasilinear equations arising in a shallow water theory, *Math. Ann.*, **312** (1998), 403–416.

[13] A. Constantin and R. Ivanov, On an integrable two-component Camassa-Holm shallow water system, *Phys. Lett. A*, **372** (2008), 7129–7132.

[14] A. Constantin and B. Kolev, Geodesic flow on the diffeomorphism group of the circle, *Comment. Math. Helv.*, **78** (2003), 787–804.

[15] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, *Arch. Ration. Mech. Anal.*, **192** (2009), 165–186.

[16] A. Constantin and H. P. McKean, A shallow water equation on the circle, *Comm. Pure Appl. Math.*, **52** (1999), 949–982.

[17] A. Constantin and L. Molinet, Global weak solutions for a shallow water equation, *Comm. Math. Phys.*, **211** (2000), 45–61.

[18] A. Constantin and W. Strauss, Stability of peakons, *Comm. Pure Appl. Math.*, **53** (2000), 603–610.

[19] A. Degasperis and M. Procesi, Asymptotic integrability, *Symmetry and Perturbation Theory (Rome, 1998)*, World Sci. Publishing, River Edge, NJ, 1999, 23–37.

[20] A. Degasperis, D. D. Holm and A. N. W. Hone, A new integrable equation with peakon solutions, *Theo. Math. Phys.*, **133** (2002), 1463–1474.

[21] H. R. Dullin, G. A. Gottwald and D. D. Holm, An integrable shallow water equation with linear and nonlinear dispersion, *Phys. Rev. Lett.*, **87** (2001), 194501–194504.

[22] J. Escher, M. Kohlmann and J. Lenells, The geometry of the two-component Camassa-Holm and Degasperis-Procesi equations, *J. Geom. Phys.*, **61** (2011), 436–452.

[23] J. Escher and B. Kolev, The Degasperis-Procesi equation as a non-metric Euler equation, *Math. Z.*, **269** (2011), 1137–1153.

[24] J. Escher, O. Lechtenfeld and Z. Yin, Well-posedness and blow-up phenomena for the 2-component Camassa-Holm equation, *Discrete Contin. Dyn. Syst.*, **19** (2007), 493–513.

[25] J. Escher, Y. Liu and Z. Yin, Global weak solutions and blow-up structure for the Degasperis-Procesi equation, *J. Func. Anal.*, **241** (2006), 457–485.

[26] J. Escher, Y. Liu and Z. Yin, Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation, *Indiana Univ. Math. J.*, **56** (2007), 87–117.

[27] C. Guan and Z. Yin, Global existence and blow-up phenomena for an integrable two-component Camassa-Holm shallow water system, *J. Differential Equations*, **248** (2010), 2003–2014.

[28] D. T. Hoang, The local criteria for blowup of the Dullin-Gottwald-Holm equation and the two-component Dullin-Gottwald-Holm system, *Ann. Fac. Sci. Toulouse, 25* (2016), 995–1012.

[29] H. Holden and X. Raynaud, Global conservative solutions of the Camassa-Holm equation—a Lagrangian point of view, *Comm. Partial Differential Equations*, **32** (2007), 1511–1549.

[30] H. Holden and X. Raynaud, Dissipative solutions for the Camassa-Holm equation, *Discrete Contin. Dyn. Syst.*, **24** (2009), 1047–1112.

[31] Q. Hu and Z. Yin, Well-posedness and blowup phenomena for the periodic 2-component Camassa-Holm equation, *Proc. Roy. Soc. Edinburgh Sect. A*, **141** (2011), 93–107.
Q. Hu and Z. Yin, Blowup phenomena for a new periodic nonlinearly dispersive wave equation, *Monatsh. Math.*, 165 (2012), 217–235.

R. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.*, 455 (2002), 63–82.

T. Kato, Quasi-linear equations of evolution, with applications to partial differential equations, in: *Spectral Theory and Differential Equations, Lecture Notes in Math.*, Springer Verlag, Berlin, 448 (1975), 25–70.

J. Lenells, Traveling wave solutions of the Degasperis-Procesi equation, *J. Math. Anal. Appl.*, 306 (2005), 72–82.

Y. Li and P. Oliver, Well-posedness and blow-up solutions for an integrable nonlinear dispersive model wave equation, *J. Differential Equations*, 162 (2000), 27–63.

Y. Liu and Z. Yin, Global existence and blow-up phenomena for the Degasperis-Procesi equation, *Comm. Math. Phys.*, 267 (2006), 801–820.

H. Lundmark, Formation and dynamics of shock waves in the Degasperis-Procesi equation, *J. Nonlinear Sci.*, 17 (2007), 169–198.

H. Lundmark and J. Szmigielski, Multi-peakon solutions of the Degasperis-Procesi equation, *Inv. Prob.*, 19 (2003), 1241–1245.

G. Misiolek, A shallow water equation as a geodesic flow on the Bott-Virasoro group, *J. Geom. Phys.*, 24 (1998), 203–208.

Z. Popowicz, A two-component generalization of the Degasperis-Procesi equation, *J. Phys. A: Math. Gen.*, 39 (2006), 13717–13726.

G. Rodriguez-Blanco, On the Cauchy problem for the Camassa-Holm equation, *Nonlinear Anal.*, 46 (2001), 309–327.

V. Vakhnenko and E. Parkes, Periodic and solitary-wave solutions of the Degasperis-Procesi equation, *Chaos, Solitons and Fractals*, 20 (2004), 1059–1073.

Z. Xin and P. Zhang, On the weak solutions to a shallow water equation, *Comm. Pure Appl. Math.*, 53 (2000), 1411–1433.

K. Yan and Z. Yin, On the Cauchy problem for a two-component Degasperis-Procesi system, *J. Differential Equations*, 252 (2012), 2131–2159.

Z. Yin, Global existence for a new periodic integrable equation, *J. Math. Anal. Appl.*, 283 (2003), 129–139.

Z. Yin, On the Cauchy problem for an integrable equation with peakon solutions, *Illinois J. Math.*, 47 (2003), 649–666.

Z. Yin, Global solutions to a new integrable equation with peakons, *Indiana Univ. Math. J.*, 53 (2004), 1189–1209.

Z. Yin, Global weak solutions to a new periodic integrable equation with peakon solutions, *J. Func. Anal.*, 212 (2004), 182–194.

Received October 2017; revised January 2018.

E-mail address: huqiaoyi@scau.edu.cn
E-mail address: zhixinwu@depauw.edu
E-mail address: sunyumei6757@sina.com