Using 1-Jettiness to Measure 2 Jets in DIS 3 Ways

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We predict cross sections in deep inelastic scattering (DIS) for the production of two jets—one along the proton beam direction created by initial state radiation (ISR) and another created by final state radiation after the hard collision. Our results include fixed order corrections and a summation of large logarithms up to next-to-next-to-leading logarithmic (NNLL) accuracy in resummed perturbation theory. We make predictions for three versions of a DIS event shape 1-jettiness, each of which constrains hadronic final states to be well collimated into two jets along the beam and final-state jet directions, but which differ in their sensitivity to the transverse momentum of the ISR from the proton beam. We use the tools of soft collinear effective theory (SCET) to derive factorization theorems for these three versions of 1-jettiness. The sensitivity to the ISR gives rise to significantly different structures in the corresponding factorization theorems—for example, dependence on either the ordinary or the generalized $k_\perp$-dependent beam function. Despite the differences among 1-jettiness definitions, we show that the leading nonperturbative correction that shifts the tail region of their distributions is given by a single universal nonperturbative parameter $\Omega_1$, even accounting for hadron mass effects. Finally, we give numerical results for $Q^2$ and $x$ values explored at the HERA collider, emphasizing that the target of our factorization-based analyses is to open the door for higher-precision jet phenomenology in DIS.

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I. INTRODUCTION

Deep inelastic scattering (DIS) of an energetic lepton from a proton target at large momentum transfer probes the partonic structure of the proton and the nature of the strong interaction, and was an important ingredient in the development of the theory of Quantum Chromodynamics (QCD) \[1\-2\]. Modern DIS experiments at HERA and Jefferson Lab continue to illuminate the internal partonic structure of hadrons, yielding information on parton distribution functions of all types, as well as the value of the strong coupling $\alpha_s$ itself (see e.g. \[7\]). The precision of $\alpha_s$ extractions from DIS jet cross sections is currently limited by the availability of theoretical predictions only at next-to-leading order (NLO) \[\[\].

Predicting the dependence of such cross sections on jet algorithms, sizes, and vetoes to high accuracy currently presents a formidable challenge. The dependence on more “global” observables characterizing the jet-like structure of final states can often be predicted to much higher accuracy. Indeed, some of the most precise extractions of $\alpha_s$ today come from hadronic event shapes in $e^+e^-$ collisions, for which theoretical predictions in QCD exist to N$^3$LL accuracy in resummed perturbation theory matched to $\mathcal{O}(\alpha_s^3)$ fixed-order results \[8\-14\], along with a wealth of data from LEP. Using event shapes to describe jet-like final states in QCD in a global manner holds the potential to improve the description of jet production in DIS to the same high level of precision.

Thrust distributions in DIS were considered in \[15\] and calculated to NLL accuracy in resummed perturbation theory, and were compared to $\mathcal{O}(\alpha_s^2)$ fixed-order results calculated numerically \[16\-17\]. Since then the introduction of soft collinear effective theory (SCET) \[18\-22\] has brought about a revolution in methods to achieve higher-order resummation in a variety of applications in QCD, leading, for example, to the N$^3$LL resummation of thrust \[12\] and heavy jet mass \[13\] in $e^+e^-$ collisions mentioned above. SCET has been used to predict a wide variety of event shapes in $e^+e^-$ collisions \[23\-25\] and $pp$ collisions \[26\-29\], going beyond the resummed accuracy previously available. A wealth of data now exists on event shapes in DIS from measurements at HERA by the ZEUS and H1 collaborations \[30\-53\]. To take advantage of these data, for instance to achieve high-precision extractions of $\alpha_s$, requires commensurate accuracy in theoretical predictions. Thanks to advances already made in tools and calculations for $e^+e^-$ and $pp$ event shapes, the time is ripe to extend the accuracy of DIS event shape predictions beyond NLL. (DIS in the endpoint region, $x \to 1$, has been studied with SCET in \[36\-40\].)

Traditional ways to define jet cross sections involve the use of a jet algorithm (such as $k_T$-type recombination algorithms or infrared-safe cone algorithms \[41\-46\]), and often a jet veto as well. Predicting the dependence on jet algorithms, sizes, and vetoes to high accuracy is currently a formidable theoretical problem in QCD. In particular, non-global logarithms (NGLs) \[17\-18\] can arise and complicate resummation beginning at NLL order for observables that probe soft radiation with different measures in sharply divided regions of phase space, as occurs with some jet vetoes, for instance \[49\-55\]. Similar clustering logs due to the way algorithms cluster soft gluons can also spoil resummation beginning at NLL order \[52\-53\, 56\-58\].

NGLs and clustering logs limit the precision one can achieve in theoretical predictions for jet cross sections in QCD. A great deal of progress has been made to resum NGLs numerically in the large-$N_C$ limit \[47\-48\], to understand the origin and structure of NGLs in the framework of effective field theory \[54\-62\], and to find ways to minimize their numerical impact (e.g. \[28\, 63\]), but a generic approach to obtain NNLL and higher order predictions does not yet exist. These complications due to non-global methods of measuring jets provide a strong motivation to use global measurements of hadronic final states that still probe their jet-like structure and are resummed to arbitrarily high accuracy in QCD perturbation theory. The first steps needed for higher order resummation in DIS are the derivations of appropriate factorization theorems.

Precisely such a global measure of jet-like structure of hadronic final states is the $N$-jettiness introduced in \[27\]. $N$-jettiness $\tau_N$ is global event shape that is a generalization of thrust \[64\] and can be used in any type of collision to constrain the final state to contain $N + N_B$ jets, where $N_B$ is the number of initial-state hadronic “beam” directions. In $e^+e^-$ collisions, events with small $\tau_N$ contain $N$ jets in the final state; in $pp$ collisions, they contain $N + 2$ jets, with two along the beam directions from initial state radiation (ISR). In DIS, small $\tau_N$ constrains events to have $N + 1$ jets, with one jet along the beam direction from ISR from the proton.

In this paper we will predict a special case of $N$-jettiness cross sections in DIS, the 1-jettiness. We define a whole class of DIS 1-jettiness observables by

$$\tau_1 = \frac{2}{Q^2} \sum_{i \in X} \min \{ q_B \cdot p_i, q_J \cdot p_i \},$$

(1)

where $q_B$ is a four-vector along the incident proton beam direction and $q_J$ is another four-vector picking out the direction of the additional final-state jet we wish to measure. Particles $i$ in the final state $X$ are grouped into regions, according to which vector $q_{s,i}$ they are closer to as measured by the dot products in Eq. (1). Different choices of $q_{B,J}$ give different definitions of the 1-jettiness. In this paper we consider three such choices:

$$\tau_i^a : \quad q_B^a = x P, \quad q_J^a = \text{jet axis} \quad (2a)$$

$$\tau_i^b : \quad q_B^b = x P, \quad q_J^b = q + x P \quad (2b)$$

$$\tau_i^c : \quad q_B^c = P, \quad q_J^c = k, \quad (2c)$$

where $P$ and $k$ are the initial proton and electron momenta, and $Q$ and $x$ are the usual DIS momentum trans-
fer and the Bj"orken scaling variable. The three versions of $\tau_1$ in Eq. (2) are named for one of their distinctive properties: $\tau_1^a$ aligns the vector $q_1^a$ with the physical jet axis as identified by a jet algorithm or by minimization of the sum in Eq. (1) over possible directions of $q_1^a$, see for example Ref. [45]. This jet axis is almost but not quite equal to $q + xP$, which is used as the vector $q_0$ in $\tau_1^f$. The measurement of $\tau_1^a$ groups final state particles in Eq. (1) into exact back-to-back hemispheres in the Breit frame. Finally, $\tau_1^f$ groups particles into exact back-to-back hemispheres in the center-of-momentum frame.

Note that the three $\tau_1$’s in Eq. (2) are physically distinct observables. Each one of them can be defined in any reference frame, but the definitions may be simpler in one frame versus another. The DIS 1-jettiness $\tau_1^a$ coincides with the version of 1-jettiness recently considered in [66] at NLL order, and is closest in spirit to the original N-jettiness event shape in [27]. No factorization theorems so far exist for either $\tau_1^b$ or $\tau_1^c$.

There are in fact a number of DIS event shapes that have been measured by experiments at HERA. Two versions of thrust [67] were measured by the H1 Collaboration [30, 32], and by the ZEUS collaboration [23, 33]. The DIS thrust variables $\tau_{a,b,c}$ are all based on hemispheres in the Breit frame where the axis $n$ is either frozen to $\hat{z}$ (along the virtual $\gamma$ or weak boson), or determined from a minimization. They have been computed to NLL+O($\alpha_s^2$) [15, 48]. The $\tau_{a,b,c}$ measure particles from only one hemisphere, and the choice of normalization $\mathcal{N}$ determines whether they are global or non-global [48] (where the non-global variables were used for the experimental measurements). Our 1-jettiness event shapes defined in Eqs. (1) and (2) are global variables, avoiding NGLs by including information from all particles in the final state. We will demonstrate that our DIS 1-jettiness variable $\tau_1^f$ actually exactly coincides with the DIS thrust $\tau_Q \equiv \tau_{a,Q}$, computed in [15] at NLL.

It would be interesting to re-analyze HERA data to measure global 1-jettiness or thrust variables. For such measurements, one may be concerned about the contribution of the proton remnants to Eq. (1). However, these remain close to the $Q_B$ axis, so their contributions to the sum giving $\tau_1$ are exponentially suppressed [67]. (To see this exponential written out explicitly see Eqs. (223) and (224).) It is only the larger angle soft radiation and ISR in the beam region and the collision products in the $q_J$ region that need to be measured. In fact, we will show below that one can measure $\tau_1^{a,b,c}$ only from the products in the $q_J$ region, obtaining the $Q_B$-region contributions by momentum conservation (however for $\tau_1^f$ this is true only in the two-jet region $\tau_1^f \ll 1$).

We will give predictions for cross sections in the three versions of $\tau_1$ in Eq. (2) accurate for small $\tau_1$. We will also prove factorization theorems for all three variables $\tau_1^{a,b,c}$. The structure of these factorization theorems will differ because $\tau_1^{a,b,c}$ each probe initial- and final-state radiation in DIS differently. Besides grouping final-state hadrons into different regions, each version has a different sensitivity to the transverse momentum of ISR. For $\tau_1^{b,c}$, the nonzero $k_\perp$ of ISR causes the final-state jet momentum to deviate from the $q_J$ axis by an amount $\simeq k_\perp$ due to momentum conservation. This affects the measurement of $\tau_1^b$ or $\tau_1^c$ at leading order. For $\tau_1^f$, $q_1^a$ is always aligned with the physical jet momentum and so is insensitive to the $k_\perp$ of ISR at leading order. This leads to different structures in the factorization theorems for $\tau_1^{a,b,c}$.

Before proceeding let us summarize the merits of the three versions of $\tau_1$. $\tau_1^{a,b,c}$ have the experimental advantage of being entirely measurable from just the collision products in the so-called “current” hemisphere, while for $\tau_1^f$ this is true only for $\tau_1^f \ll 1$. From a theoretical perspective, since in this paper we give predictions for $\tau_1^{a,b,c}$ at the same order of accuracy (resummed to NNLL), currently they are equally preferred. However, $\tau_1^f$ involves more nontrivial integrals over the transverse momenta of beam and jet radiation, leading us to anticipate that $\tau_1^{a,b}$ will be easier to extend to higher accuracy. In addition, the factorization theorem we prove for $\tau_1^f$ is valid only when the DIS variable $y \sim 1$, that is, for large lepton energy loss in the CM frame producing a jet in a direction $q_J$ fairly close to the initial electron direction. It is thus perhaps fair to say that $\tau_1^f$ possesses the best combination of advantages of experimental measurability, theoretical calculability, and kinematic range of applicability. Nevertheless, we emphasize that comparing $\tau_1^{a,b,c}$ with each other can shed light on the transverse recoil of ISR, and can test the universality of nonperturbative effects which we will discuss below.

We will prove that the cross sections in all three variables factorize as special cases of the form:

$$\frac{d\sigma}{dx\,dQ^2\,dQ}\,d\sigma = \frac{d\sigma_0}{dQ} \sum \int dt_J dt_B dksd^2p_\perp \times J_q(t_J^a - (Q_1 + p_\perp)^2, \mu) \times S_{\text{hemi}}(k_s, \mu) \delta(\tau_1^f - t_J^a - t_B - k_s - Q_R),$$

(3)

where $\mu$ runs over quark and antiquark flavors, $s_J, s_B, Q_R$ are normalization constants given in Eqs. (51) and (68) that depend on the choice of observable $\tau_1$ in Eq. (2). $\sigma_0$ is the Born cross section, $H_0$ is a hard function arising from integrating out hard degrees of freedom from QCD in matching onto SCET. $J_q$ is a quark jet function describing collinear radiation in the final-state jet, and $S_{\text{hemi}}$ is a quark beam function containing both perturbative collinear radiation in a function $\mathcal{I}_{\text{hemi}}$ as well as the proton parton distribution function (PDF) $f_{j/p}$:

$$\mathcal{B}_{\kappa/p}(t, x, p_\perp^2, \mu) = \sum \int z d^2z \mathcal{I}_{\text{hemi}}(t, x, p_\perp^2, \mu) f_{j/p}(z, \mu).$$

(4)

This beam function depends on the transverse virtuality $t$ of the quark $\kappa$ as well as the transverse momentum $p_\perp$ of ISR. $S_{\text{hemi}}$ in Eq. (6) describes soft radiation from
both the proton beam and the final-state jet. Despite the fact that the 1-jettiness Eq. (1) may not divide the final state into hemispheres, we will nevertheless show that the soft function for any 1-jettiness in DIS is related to the hemisphere soft function $S_{\text{hemi}}$. Finally, $q_\perp$ is the transverse momentum of the momentum transfer $q$ in the DIS collision with respect to the jet and beam directions.

We briefly discuss differences in the factorization theorem for $\tau_{1,a,b,c}$. For $\tau_{1}^a$, the jet axis is aligned so that the argument of the jet function $t_\perp - (q_\perp + p_\perp)^2 \to t_\perp$ with zero transverse momentum, and $p_\perp$ then gets averaged over in Eq. (3), removing the dependence on this variable in the beam function and yielding the ordinary beam function of Ref. [67]. For $\tau_{1}^b$, $\tau_{1}^c$, the convolution over $p_\perp$ remains and thus they are sensitive to transverse momentum of ISR. Thus for $\tau_{1}^a, \tau_{1}^c$ results depend on generalized $p_\perp$-dependent beam function introduced in Ref. [68]. The final difference is that $q_\perp$ is identically zero for $\tau_{1}^b$, while it is nonzero for $\tau_{1}^a$, causing these observables to differ and inducing additional complications in the convolution over $p_\perp$ for $\tau_{1}^b$. In particular the cross section for $\tau_{1}^b$ does not start at $\tau_{1}^b = 0$, but rather at $\tau_{1}^b = q_\perp^2/Q^2$ due to the nonzero $q_\perp$ injected into the collision and the choice here for the jet axis.

The ingredients in the factorization theorem Eq. (3) depend on an arbitrary scale $\mu$ that arises due to integrating out degrees of freedom from QCD, matching onto a theory of collinear and soft modes, and then integrating out collinear degrees of freedom and matching onto just soft modes. The resulting hard, jet, beam, and soft functions each depend on logs of $\mu$ over physical variables. Renormalization group (RG) evolution allows us to evolve each function from a scale $\mu_{H,J,B,S}$ where these logs are minimized to the common scale $\mu$. This evolution resums logs of $\tau_{1,2}^{a,b,c}$ to all orders in $\alpha_s$, to a given order of logarithmic accuracy determined by the order to which we know the anomalous dimensions for the RG evolution. We will use this technology to resum logs of 1-jettiness in DIS to NNLL accuracy for $\tau_{1,a,b,c}^\perp$.

The factorized cross section in Eq. (3) accurately predicts the $\tau_1$ distribution in the peak region and for the tail to the right of the peak, where $\tau_1 \ll 1$ and logs of $\tau_1$ are large. To be accurate for larger $\tau_1$, the prediction of Eq. (4) must be matched onto predictions of fixed-order QCD perturbation theory to determine the “non-singular” terms. In this paper we do not perform the matching onto the $O(\alpha_s)$ and $O(\alpha_s^2)$ tail of the $\tau_1$ distributions, leaving that to future work. However, by comparing the unmatched predictions of Eq. (3) integrated over $\tau_1$ to the QCD total cross section at $x,Q^2$ we can estimate the small size of these missing corrections at large $\tau_1$. We emphasize that Eq. (3) accurately captures the distribution for smaller $\tau_1$ near the peak region.

The factorization theorem Eq. (3) also allows us to account for nonperturbative effects—not only in the parton distributions $f(x,\mu)$, but also through a shape function that appears in the soft function $S$. In $e^+e^-$ collisions, the leading nonperturbative corrections from this shape function have been shown to be universal for different event shapes and collision energies [69,72] (for earlier work see [73,74]). The same conclusions hold for the soft shape function in Eq. (3), endowing it with real predictive power. We will analyze the dominant effects of the nonperturbative soft shape function on the DIS 1-jettiness. For the peak region we include a simple nonperturbative model function to show the impact these corrections have and how they modify the perturbatively calculated distribution. For the tail region the leading shape function power correction is a simple dimension-1 parameter $\Omega_{1,a,b,c}$ that induces a shift to $\tau_{1,a,b,c}$, and is defined by a matrix element of a soft Wilson line operator. For our observables we will prove that there is universality for this correction, namely that $\Omega_1^a = \Omega_1^b = \Omega_1^c$. This follows from a general analysis we carry out for how the direction of axes affect nonperturbative matrix elements for two-jet soft Wilson line operators.

The paper is organized as follows: In Sec. II we review the kinematics of DIS in several commonly used reference frames, laying out the notation for our subsequent analyses. In Sec. III we define the three versions of 1-jettiness in DIS that we will use in this paper and consider their physics in some detail. In Sec. IV we follow the usual formalism for calculating the DIS cross section in QCD, and introduce an additional measurement of the 1-jettiness into the hadronic tensor that appears therein. Sec. V is the technical heart of the paper. Here we present the elements of the SCET formalism that we need and give a detailed proof of the factorization theorems for the generic DIS 1-jettiness in Eq. (1) and the three specializations we give in Eq. (2). In particular we derive in each factorization theorem how the observable depends on the transverse momentum of ISR through the beam function, and also show that by rescaling arguments we can always use the hemisphere soft function for each version of 1-jettiness.

In Sec. VI we use the factorization theorems from Sec. V to give predictions for the singular terms in the $\tau_1$ distributions at fixed order $O(\alpha_s)$, and also enumerate the results for the hard, jet, beam, and soft functions that we will need to perform the RG evolution in the next section. In Sec. VII we perform the RG evolution and give our resummed predictions to NNLL accuracy. We compare our predictions for $\tau_{1,b}^b$ to those of [13] at NLL. We also explain the “profiles” for the individual hard, jet, beam, and soft scales which we use to perform the RG evolution [14,28,70]. These profiles allow for a smooth transition from the tail region into the peak region where the soft scale becomes nonperturbative, and into the far tail region where the resummation of logarithms must be turned off. Then we explain how we incorporate nonperturbative hadronization corrections into our predictions through a soft shape function and discuss the $\Omega_1$ parameters. We show that the shifts $\Omega_{1,a,b,c}$ to the tail region of all three versions of the 1-jettiness distributions obey universality.

In Sec. VIII we present numerical results for our pre-
dictions to NNLL for the $\tau_1^{a,b,c}$ cross sections, including also their $x$ and $Q^2$ dependence. We consider both integrated (cumulant) and differential cross sections. The particular results we present are for $x, Q^2$ values studied at HERA [32, 55]. However, the analytic results we give in Sec. VII can just as easily be used for other experiments at different kinematics, such as at Jefferson Lab (JLab) [72], or for nuclear states other than the proton, such as those at the future Electron-Ion Collider (EIC) [78] and Large Hadron Electron Collider (LHeC) [79].

In Sec. IX we conclude. In several Appendices we collect various technical details that are used in the main body of the paper. In particular, in App. D we collect the anomalous dimensions we need to get to NNLL accuracy in the $\tau_1$ cross sections, and in App. E we give the resummed cross sections in an alternative formalism [38, 50] to that used in Secs. VII [14, 76]. In Sec. VII we use a formalism that expresses the result of the RG evolution of Eq. (3) entirely in momentum space, while in App. E we use a formalism that expresses the RG evolution through Laplace space objects. These two approaches give identical analytic results at each order in resummed perturbation theory, but since both are commonly used in the SCET literature we provide both results for people who prefer one or the other. Indeed, all of our numeric results have been cross checked between these two codes which each use one of these two approaches.

The reader mainly interested in the phenomenology of DIS 1-jettiness and our numerical predictions may read Secs. III and then skip to Sec. VIII. For those interested in details of the factorization and resummation, we provide these in Secs. IX and the Appendices.

II. KINEMATICS OF DIS

In this section we define the kinematic variables in DIS that we will use throughout the paper. We also consider three reference frames—center-of-momentum (CM), target rest frame, and Breit frame—and describe the picture of the events in each of these frames.

A. Kinematic variables

In DIS, an incoming electron with momentum $k$ and a proton with momentum $P$ undergo hard scattering by exchange of a virtual boson (photon or $Z$) with a large momentum $q$, and outgoing electron $k'$. The boson momentum $q$ can be determined from the initial- and final-state electron momenta,

$$q = k - k'. \quad (5)$$

In inclusive DIS, the final states from the hard scattering are inclusively denoted as $X$ and their total momentum is denoted as $p_X$. Using Eq. (4) momentum conservation $k + P = k' + p_X$ can be written as

$$q + P = p_X. \quad (6)$$

The momentum scale $Q$ of the hard scattering is defined by the virtuality of the exchanged gauge boson. Because the boson has a spacelike (negative) virtuality, one defines the positive definite quantity $Q^2$ by

$$Q^2 \equiv -q^2. \quad (7)$$

where we will be interested in $Q \gg \Lambda_{\text{QCD}}$. Next one defines dimensionless Lorentz invariant variables. The Bjorken scaling variable $x$ is defined by

$$x = \frac{q^2}{2P \cdot q} = \frac{Q^2}{2P \cdot q}, \quad (8)$$

where $x$ ranges between $0 \leq x \leq 1$. Another Lorentz-invariant quantity $y$ is defined by

$$y = \frac{2P \cdot q}{2P \cdot k} = \frac{Q^2}{xs}, \quad (9)$$

where the total invariant mass $s = (P + k)^2 = 2P \cdot k$ and $y$ ranges from $0 \leq y \leq 1$. The variable $y$ measures the energy loss of the electron in the target rest frame. For a given $s$ Eq. (9) relates $x$, $y$, and $Q^2$ to one another, allowing one of the three variables to be eliminated. The invariant mass of the final state in terms of the above variables is

$$p_X^2 = \frac{1 - x}{x} Q^2 = (1 - x)ys. \quad (10)$$

In the classic DIS region one has $p_X^2 \sim Q^2$ for generic $x$. In the endpoint region $1 - x \sim \Lambda_{\text{QCD}}/Q$, the final state is a single narrow jet with momentum of order $Q$ in the virtual boson direction (and studied with SCET in Refs. [30, 40]). The resonance region where $1 - x \sim \Lambda_{\text{QCD}}^2/Q^2$ cannot be treated with inclusive perturbative methods.

In this work we are interested in the classic region where $1 - x \gg \Lambda_{\text{QCD}}/Q$ i.e., $x \sim 1 - x < 1$. In this region one can have more than a single jet. Below, we will make an additional measurement that picks out two-jet-like final states.

B. Center-of-momentum frame

A two-jet-like event in the CM frame is illustrated in Fig. 1. An incoming electron and proton collide and produce in the final state an outgoing electron and hadrons. The hadrons, mostly collimated into two jets with additional soft particles elsewhere, are grouped into two regions $\mathcal{H}_B$ and $\mathcal{H}_J$, and $p_1$ and $p_2$ are the total momentum of particles in the each region. The regions $\mathcal{H}_{B,J}$ are not necessarily hemispheres in this frame, though we drew them as such in Fig. 1. The definitions of the regions are described in Sec. IIIA. As shown in Fig. 1 the electron direction is defined to be the $+z$ direction and the proton
They satisfy the initial electron and proton momenta are \( q \) and so \( p \) direction to be the \(-z\) direction. In the CM frame the initial electron and proton momenta are

\[
k^\mu = \sqrt{s} \frac{n^\mu}{2}, \quad P^\mu = \sqrt{s} \frac{\bar{n}^\mu}{2},
\]

(11)

where the light cone vectors are

\[
n = (1, 0, 0, 1), \quad \bar{n} = (1, 0, 0, -1).
\]

(12)

They satisfy \( n \cdot \bar{n} = 2 \) and \( n \cdot n = \bar{n} \cdot \bar{n} = 0 \). An arbitrary four vector \( V^\mu \) can be written as

\[
V^\mu = V^+ \frac{\bar{n}^\mu}{2} + V^- \frac{n^\mu}{2} + V_T^\mu,
\]

(13)

where \( V^+ \equiv V \cdot n \) and \( V^- \equiv V \cdot \bar{n} \) and \( V_T^2 = -V^2 < 0 \).

In this frame \( x, y \) take the values

\[
x = \frac{Q^2}{\sqrt{s} n \cdot q}, \quad y = \frac{\bar{n} \cdot q}{\sqrt{s}}.
\]

(14)

and so \( q \) is given by

\[
q^\mu = y \sqrt{s} \frac{n^\mu}{2} - x \sqrt{s} \left( 1 - \frac{q_T^2}{Q^2} \right) \frac{\bar{n}^\mu}{2} + q_T^\mu,
\]

(15)

which satisfies \( Q^2 = -q^2 = xys \). Here \( q_T \) is a four-vector transverse to \( n, \bar{n} \) and satisfies \( q_T^2 = -Q^2 < 0 \).

### C. Target rest frame

The same two-jet like event in Fig. 1 is illustrated as it would appear in the target rest frame in Fig. 2. The proton is at rest. The regions in Fig. 2 are transformed from those in Fig. 1 because of the boost along the proton direction. In this frame, the initial electron and proton momenta are

\[
k^\mu = \frac{s n^\mu}{2}, \quad P^\mu = M \frac{n^\mu + \bar{n}^\mu}{2},
\]

(16)

satisfying \( 2k \cdot P = s \). Here \( M \) is the proton mass. We reach this frame by a boost of momenta \( p^\mu \) in the CM frame along the \( z \) direction,

\[
n_z \cdot p \rightarrow \frac{M}{\sqrt{s}} n_z \cdot p, \quad \bar{n}_z \cdot p \rightarrow \sqrt{s} M \bar{n}_z \cdot p.
\]

(17)

Therefore, in this frame, \( q^\mu \) in Eq. (15) is boosted to become

\[
q^\mu = \frac{Q^2}{xM} \frac{n^\mu}{2} - xM \left( 1 - \frac{Q^2}{Q^2} \right) \frac{\bar{n}^\mu}{2} + q_T^\mu,
\]

(18)

and \( x, y \) are given by

\[
x = \frac{Q^2}{2M(E - E')}, \quad y = \frac{E - E'}{E}.
\]

(19)

Here \( E \) and \( E' \) are the energies of the incoming and outgoing electron, respectively, measured in the target rest frame. Here \( y \) is the fractional electron energy loss.

#### D. Breit Frame

In the Breit frame, the virtual boson with momentum \( q^\mu \) and proton with momentum \( P^\mu \) collide along the \( z \) direction. This frame is useful because the proton initial state radiation moving along the proton direction can be relatively well separated from other scattering products. One might worry that an ISR jet, which we want to measure in this paper, could be contaminated by the proton remnants which are difficult to separate from ISR. However, the 1-jettiness observable in Sec. II A that we use to measure the jets in the final state is actually insensitive to this contamination since contributions from the region of the beam remnant give exponentially suppressed contributions to the variable. The contributions from the
beam region are by far dominated by the initial state radiation at larger angles. The picture of the two-jet like event in the Breit frame is similar to Fig. 4 with incoming electron replaced by virtual boson and with the outgoing electron removed.

The Breit frame is defined as that in which the momentum transfer \( q \) is purely spacelike:

\[
q^\mu = \frac{Q n^\mu}{x^\mu} - \frac{\bar{n}^\mu}{2},
\]

where we align \( \bar{n} \) to be along the proton direction:

\[
P^\mu = \frac{Q \bar{n}^\mu}{x^\mu}.
\]

The incoming electron has momentum

\[
k^\mu = \frac{Q n^\mu}{y^\mu} + \frac{1 - y \bar{n}^\mu}{2} + \frac{k_T^\mu}{2},
\]

where \( k_T^2 = Q^2(1 - y)/y^2 \). The outgoing electron then has momentum

\[
k'^\mu = \frac{1 - y n^\mu}{y^\mu} + \frac{Q n^\mu}{y^\mu} + k_T^\mu.
\]

Unlike the CM and target rest frames, where for a fixed \( s \) the incident momenta are fixed, in the Breit frame the incident momenta are functions of \( x, y \). Thus each point in the differential cross section in \( x, y \) corresponds to a different Breit frame.

### III. Hadronic Observables

#### A. \( N \)-jettiness

To restrict final states to be two-jet-like, we must make a measurement on the hadronic state and require energetic radiation to be collimated along two light-like directions. An observable naturally suited to this role is the \( N \)-jettiness \(^{27}\). In our case, with one proton beam, 1-jettiness \( \tau_1 \) can be used to restrict final states to those that have two jets: one along the original proton direction (beam) from ISR and another produced from the hard scattering. Recalling the definition of \( \tau_1 \) in Eq. (24):

\[
\tau_1 = \frac{2}{Q^2} \sum_{i \in X} \min\{q_B \cdot p_i, q_J \cdot p_i\},
\]

where \( q_B, q_J \) are massless four-vectors chosen to lie along the beam and jet directions.

The minimum operator in Eq. (24) groups particles in \( X \) with the four-vector to which they are closest (in the sense of the dot product). We will call the region in which particles are grouped with the beam \( \mathcal{H}_B \) and the region in which particles are grouped with the jet \( \mathcal{H}_J \). We denote the total momentum in the beam region as \( p_B \) and total momentum in the jet region as \( p_J \):

\[
p_B = \sum_{i \in \mathcal{H}_B} p_i, \quad p_J = \sum_{i \in \mathcal{H}_J} p_i.
\]

These regions are illustrated for two examples in the CM and target rest frames in Fig. 1 and Fig. 2.

The 1-jettiness \( \tau_1 \) can be expressed as the sum

\[
\tau_1 = \tau_B + \tau_J,
\]

where \( \tau_B \) and \( \tau_J \) are defined by

\[
\tau_B = \frac{2 q_B \cdot p_B}{Q^2}, \quad \tau_J = \frac{2 q_J \cdot p_J}{Q^2}.
\]

The variables \( \tau_B,J \) are projections of \( p_{B,J} \) onto the reference vector \( q_{B,J} \). They can be thought as two independent observables, and \( \tau_1 \) is one possible combination of them. Another combination gives a generalized rapidity gap and is discussed in App. A.

The reference vectors \( q_B \) and \( q_J \) can be expressed as

\[
q_B^\mu = \omega_B \frac{n_B^\mu}{2}, \quad q_J^\mu = \omega_J \frac{n_J^\mu}{2},
\]

for light-like vectors \( n_{B,J} \) given by \( n_{B,J} = (1, n_{B,J}) \), where \( n_{B,J} \) are unit 3-vectors satisfying \( n_{B,J}^2 = 1 \). Below we will use the vectors \( n_{B,J} \) to define the directions of the collinear fields in SCET which we use for the degrees of freedom that describe fluctuations collimated in the beam and jet regions. Refs. \(^{27,65}\) discussed the possibility of also minimizing over possible vectors \( q_{B,J} \) to give the smallest possible \( \tau_1 \) in Eq. (24), and Ref. \(^{66}\) developed a fast algorithm to carry out this minimization. Here we will take \( q_{B,J} \) to be fixed vectors. We will discuss several possible choices for \( q_{B,J} \) below, each giving a different definition of \( \tau_1 \).

Measuring \( \tau_1 \) to be small means the final state has at most two collimated jets, one in the \( q_B \) direction and one in the \( q_J \) direction (irrespective of the exact definition of \( q_B \) and \( q_J \)). For power counting purposes we will use \( \tau_1 \sim \lambda^2 \) which defines a small parameter \( \lambda \ll 1 \) in which we will perform the expansion to obtain the leading-order factorization theorem for DIS 1-jettiness cross sections.

1. \( \tau^a \): 1-jettiness aligned with the jet axis

The first version of 1-jettiness that we consider is \( \tau^a \), which is defined by choosing the beam reference vector \( q_B^a \) in Eq. (24) to be proportional to the proton momentum, and the jet reference vector \( q_J^a \) to be the jet momentum as given by a jet algorithm such as anti-\( k_T \) \(^{16}\):

\[
\tau^a = \frac{2}{Q^2} \sum_{i \in X} \min\{q_B^a \cdot p_i, q_J^a \cdot p_i\}.
\]

These reference vectors are given by the values

\[
q_B^a = x P^\mu, \quad q_J^a = q^\mu + x P^\mu + q_J^+ \mu,
\]

where \( q_J^+ \) is \( O(Q \lambda) \). This is because \( xP \) is the longitudinal momentum of the parton that hard scatters from the virtual photon of momentum \( q \), which would produce a
jet of momentum $q + xP$, but the colliding parton may also have a transverse momentum of order $Q\lambda$. It cannot be larger, otherwise it would cause $\tau_1$ to be larger than $O(\lambda^2)$. Various jet algorithms give the same value of $q_J^2$ up to negligible power corrections of $O(Q\lambda^2)$, and this cross section does not actually depend on which of these algorithms is used. Here it would also be equivalent to leading power to define $\tau_1^b$ by minimizing the sum in Eq. (29) with respect to $\hat{n}_J$ in $q_J^2$. The total momentum of particles in the jet region $H_J$ is $p_J = q_J^2 + k$ for a soft momentum $k$ of $O(Q\lambda^2)$. Thus, to the order we are working, the sum over particles in the jet region $H_J$ in Eq. (29) gives the total invariant mass of those particles, $2q_J^2.P_J = p_J^2 = m_J^2$ (for more discussion of this see Refs. 23, 81).

We will show below that in deriving the correct factorization theorem for the $\tau_1^a$ cross section, we must use the fact that $q_J^2$ is chosen to make the relative transverse momentum between $q_J^2$ and the actual jet momentum $p_J$ be zero (technically the dominant $O(Q\lambda)$ part must be zero and a small $O(Q\lambda^2)$ part is still allowed). That is, $q_J^2$ is aligned with the jet, hence the name $\tau_1^a$. This is also important for experimentally measuring $\tau_1^a$. Nevertheless, once this factorization theorem is known, $q_J^2$ is not directly required for calculating the objects such as hard and soft functions that appear in the factorization theorem. For the other versions of 1-jettiness we consider below, the reference vector $q_J$ is not aligned exactly with the jet, and the transverse momentum between $q_J$ and the jet momentum $p_J$ will be nonzero, as illustrated in Fig. 3. This will change the structure of the corresponding factorization theorems, introducing convolutions over the transverse momenta of radiation from the beam and from the final-state jet.

2. $\tau_1^b$: hemisphere 1-jettiness in the Breit frame

A second way to define 1-jettiness in DIS is

$$\tau_1^b = \frac{2}{Q^2} \sum_{i \in X} \min \{q_B^i \cdot p_i, q_J^b \cdot p_i \}, \quad (31)$$

where

$$q_B^i \cdot p_i = xP^\mu, \quad q_J^b \cdot p_i = q^\mu + xP^\mu. \quad (32)$$

In this case, $q_J^b$ is given exactly by the quantity $q + xP$ which can be constructed from the electron and proton momenta $k, k', P$, and needs no information about the jet momentum given by any jet-finding algorithm. Thus in general $q_J^b$ differs by a transverse momentum $q_J^b \sim Q\lambda$ from the vector $q_J^2$ used in the $\tau_1^a$ definition of 1-jettiness we introduced above in Eq. (29). Note that since $q = q_J^b - q_B^i$, $q$ itself has zero transverse momentum $q_{\perp}$, with respect to the directions $n_J^i, n_B^b$ of $q_J^b, q_B^i$. This choice of vectors is natural in the Breit frame (hence the name $\tau_1^b$), in which it divides the final state into back-to-back hemispheres. In the Breit frame,

$$\tau_1^b = \frac{1}{Q} \sum_{i \in X} \min \{\bar{n}_x \cdot p_i, n_z \cdot p_i \}. \quad (33)$$

This definition directly corresponds to the thrust $\tau_0$ in DIS defined in 13.

We will often work in the CM frame in intermediate stages of calculation below. Expressing $q_J^{b B, J}$ in the CM frame, we find

$$q_J^{b B, J} = x \sqrt{s} \frac{n_{\perp}^B}{2}, \quad (34)$$

$$q_J^{b J} = y \sqrt{s} \frac{n_{\perp}^J}{2} + x(1-y) \sqrt{s} \frac{n_{\perp}^J}{2} + q_T^B, \quad (35)$$

FIG. 3: (a) 1-jettiness $\tau_1^a$ measures the small light-cone component of the momentum in the jet region $H_J$ along the “true” jet axis $q_J^2$, which is proportional to the jet invariant mass and is thus insensitive at leading order in $\lambda$ to the transverse momentum $p_B^\perp$ of ISR. Thus $p_B^\perp$ gets averaged over in calculating the $\tau_1^a$ cross section. (b) 1-jettiness $\tau_1^b$ measures the small light-cone component of $p_J$ along the fixed axis $q_J^2 = q + xP$. This projection is sensitive to and balances the transverse momentum $p_B^\perp$ of ISR. The transverse momenta of $p_B$ and $p_J$ get convolved together in calculating the cross section. Both $\tau_1^a$ and $\tau_1^b$ divide the final state into hemispheres in the Breit frame. (c) 1-jettiness $\tau_1^c$ divides event into back-to-back hemispheres in the CM frame and projects beam and jet momenta onto $n_x, n_z$ axes. These projections are sensitive to the transverse momentum $p_B^\perp$ of ISR. The momentum transfer $q$ has a nonzero transverse component in these coordinates, and the jet and beam momenta are convolved in $p_B^\perp$ in calculating the cross section.
where $\mathbf{q}_T^a = (1 - y)Q^2$ and $\mathbf{q}_J^b$ is a massless vector. $\tau_1^b$ in Eq. (34) can also be written in the form

$$q_J^{b\mu} = P_T e^{-\gamma} \frac{n_x^{b\mu}}{2} + P_T e^{-\gamma} \frac{n_y^{b\mu}}{2} + P_T \delta^{b\mu},$$

(35)

where the jet transverse momentum and rapidity are

$$P_T = Q \sqrt{1 - y}, \quad Y = \frac{1}{2} \ln \frac{y}{x(1 - y)},$$

(36)

and $\hat{n}_T$ is a unit vector in the direction of $q_T$. These relations can be inverted to give

$$x = \frac{P_T e^{-\gamma}}{\sqrt{s} - P_T e^{-\gamma}}, \quad y = \frac{P_T e^{-\gamma}}{\sqrt{s}}.$$  

(37)

Equating the 0th components of Eqs. (28) and (35), we find that

$$\omega_J^b = 2P_T \cos Y = \sqrt{|y + x(1 - y)|}.$$  

(38)

Calculating $\tau_1^b$ in the CM frame groups particles into non-hemisphere-like regions. Particles with momenta $p$ are grouped into the beam or jet regions according to which dot product is smaller:

$$\mathcal{H}_B : \frac{x \sqrt{s} n_B^b \cdot p}{2} < \frac{\omega_J^b n_J^b \cdot p}{2},$$

$$\mathcal{H}_J : \frac{x \sqrt{s} n_B^b \cdot p}{2} > \frac{\omega_J^b n_J^b \cdot p}{2}.$$  

(39)

Using Eq. (38), we can write these conditions as

$$\mathcal{H}_B : \frac{n_B^b \cdot p}{n_J^b \cdot p} < 1 - y + \frac{y}{x}, \quad \mathcal{H}_J : \frac{n_B^b \cdot p}{n_J^b \cdot p} > 1 - y + \frac{y}{x}.$$  

(40)

In order to understand the regions defined by Eq. (40), let us consider simple case $y \sim 1$ and $x < y$. For this case $q_J^b$ in Eq. (35) is $n_z$-collinear because in Eq. (36) $P_T$ and $Y$ are small and large, respectively. We can replace $n_J^b$ and $n_B^b$ in Eq. (40) by $n_z$ and $n_\perp$ and set $n_\perp \cdot p/n_z \cdot p = 1/(\tan^2 \theta/2)$ where $\theta$ is the polar angle of massless particle $p$. Then, the jet region is a symmetric cone around the $n_\perp$ direction of opening angle given by

$$\tan^2 \frac{R}{2} = \frac{x}{y},$$  

(41)

and the beam region is everything outside. For generic $x$ and $y$, the jet region is not symmetric around the $n_\perp$.

As mentioned above in the description of $\tau_1^a$, the vector $q_J^b = q + xP$ is the 4-momentum of a jet produced by scattering at momentum $q$ on an incoming parton with momentum exactly equal to $xP$. In general the colliding incoming parton will have a nonzero transverse momentum due to ISR, causing the produced jet momentum to deviate by $\mathcal{O}(Q\lambda)$ from $q_J^b$. The scale $\mathcal{O}(Q\lambda)$ is perturbative and this transverse momentum is much larger than the intrinsic transverse momentum of partons in the proton. The observable $\tau_1^b$ differs from $\tau_1^a$ in that $\tau_1^b$ measures the true invariant mass $m_J^a$ of the jet while $\tau_1^a$ simply projects the jet momentum onto the fixed axis $q_J^a = q + xP$ which does not vary with the exact direction of the jet. The jet axis varies from $q_J^b$ due to ISR from the beam before the hard collision. This subtle difference leads to a different structure in the factorization theorems for $\tau_1^a$ and $\tau_1^b$.

For the 1-jettiness for DIS studied in [63], the procedure for determining the $q_J$ was described as determining the jet axis from a jet algorithm. This makes their $q_J$ correctly correspond with our $q_J^a$. However, they also used the formulas Eqs. (35) and (36) to describe their $q_J$, which yields $q_J = q + xP$, and this would correspond to our $\tau_1^b$. This choice neglects the $\mathcal{O}(Q\lambda)$ transverse momentum between $q$ and the jet momentum $p_J$, which taken literally would lead to an incorrect factorization theorem for the observable $\tau_1^b$. However, after the correct form of the factorization theorem for $\tau_1^a$ is known (which was written in [62]), this approximation is valid for calculating the objects in that theorem to leading order in $\lambda$. Thus, the $\tau_1$ in [63] is the same as our $\tau_1^a$ defined above in Eq. (28), where $q_J$ is aligned along $p_J$.

### 3. $\tau_1^a$: hemisphere 1-jettiness in the CM frame

A third way to define the 1-jettiness in DIS is with the proton and electron momenta

$$q_B^\mu = P^\mu, \quad q_J^\mu = k^\mu.$$  

(42)

We use the superscripts $c$ because this choice naturally divides the final state into hemispheres in the CM frame, mimicking the thrust defined in the CM frame for $e^+e^-$ collisions [64].

In the CM frame the momenta $k$ and $P$ are along the $z$ and $-z$ directions as in Eq. (11). In this frame the reference vectors $q_{J,B}$ are given by the light-cone directions

| 1-jettiness | axis $q_J$ | axis $q_B$ |
|------------|------------|------------|
| $\tau_1^a$ | $\omega_J^b/2$ | $\omega_B^b/2$ |
| $\tau_1^b$ | $xP + q + q_J^b$ | $xP$ |
| $\tau_1^c$ | $xP + q$ | $xP$ |
| $\tau_1^I$ | $k$ | $P$ |

**TABLE I:** Reference vectors $q_J$ and $q_B$ defining the axes for various versions of 1-jettiness. For $\tau_1^a$ the jet axis is defined to be the jet momentum $q_J^a$ given by, e.g., the anti-$k_T$ algorithm. This axis is given by $q + xP$ up to transverse momentum corrections of order $q_J^b \sim \mathcal{O}(Q\lambda)$. The exact value of $q_J^b$ will not be needed for our calculation, only the fact that there is no relative transverse momentum larger than $\mathcal{O}(Q\lambda^2)$ between the momentum $p_J$ in the jet region $H_J$ and the axis $q_J^b$. This in contrast to $\tau_1^I$, for which the cross section will depend on the transverse momentum between $p_J$ and $q_J^b = q + xP$, but where $q_\perp = 0$. Finally for $\tau_1^c$ we also have $q_\perp \neq 0$. 


\[ n_{B,j} \] and normalizations \( \omega_{B,j} \):
\[
\begin{align*}
n_B^a \mu &= \bar{n}_z^a, & n_B^c \mu &= n_z^c, \\
\end{align*}
\] (43) and
\[
\begin{align*}
\omega_B^a &= \sqrt{\bar{s}}, & \omega_B^c &= \sqrt{\bar{s}}.
\end{align*}
\] (44)

In this frame, \( \tau_1 \) is then given by
\[
\tau_1^B = \frac{1}{xy\sqrt{s}} \sum_{i \in X} \min\{\bar{n}_z \cdot p_i, n_z \cdot p_i\}.
\] (45)

The minimum here assigns particles to either the hemisphere containing the proton or electron. States with small \( \tau_1 \) thus have two nearly back-to-back jets in this frame.

The essential differences among \( \tau_1^a, \tau_1^b, \tau_1^c \) are illustrated in Fig. 3 drawn in the CM frame and summarized in Table II. \( \tau_1^t \) and \( \tau_1^f \) project the jet momentum onto a fixed axis, and are sensitive at leading order to the transverse momentum of initial state radiation from the incoming proton, while \( \tau_1^\alpha \) always projects the jet momentum onto the axis with respect to which it has no transverse momentum, and so measures the invariant mass of the jet which is insensitive at leading order to the transverse momentum of ISR. Table II summarizes the choices of reference vectors \( q_{a,b,c} \) for the three versions of 1-jettiness defined in this section.

**B. Versions of DIS Thrust**

Several thrust DIS event shapes have been considered in the literature \([82]\), and some of them have been measured by experiments. One version, called \( \tau_Q \) in \([15]\), but not yet measured, is defined in the Breit frame by
\[
\tau_Q^{\text{Breit}} = 1 - \frac{2}{Q} \sum_{i \in \mathcal{H}_C} p_{z,i},
\] (46)

where \( \mathcal{H}_C \) is the “current hemisphere” in the direction set by the virtual boson \( q \). We will show below in section 11A.2 that \( \tau_Q \) is equivalent to our \( \tau_1^b \).

Another version of thrust, used in \([30, 33]\), and called \( \tau_{1E} \) in \([48]\), is defined using a thrust axis whose definition involves a maximization procedure over particles in the current hemisphere \( \mathcal{H}_C = \mathcal{H}_J \) in the Breit frame:
\[
\tau_{1E} = 1 - \max_{\mathbf{n}} \frac{\sum_{i \in \mathcal{H}_C} |\mathbf{p}_i \cdot \mathbf{n}|}{\sum_{i \in \mathcal{H}_C} |\mathbf{p}_i|}.
\] (47)

The maximization aligns the vector \( \mathbf{n} \) with the direction of the jet in the current hemisphere, just like the \( q_{b} \) vector in our definition of \( \tau_1^b \). However, because the sums in both the numerator and denominator are limited to \( \mathcal{H}_C \), the observable is actually non-global \([48]\), cutting out radiation from the remnant hemisphere.\(^1\) Thus it differs from our \( \tau_1^a \) which sums over both hemispheres. It cannot be simply related to a global version of 1-jettiness as above. A global thrust event shape, \( \tau_Q \), can be obtained by replacing the denominator in Eq. (47) by \( Q/2 \), but this version of the thrust event shape is also not related to our \( \tau_1^a \).

Yet another variation is \( \tau_{1E} \) \([30, 48]\) which is like Eq. (47) with the same normalization, but with respect to the \( z \)-axis in the Breit frame. It is also not global \([48]\). H1 and ZEUS have measured \( \tau_{1E} = \tau_Q^{\text{Breit}} = 1 - T_{ZEUS}^Z \) and \( \tau_{1E} = \tau_{1E}^{\text{H1}} = 1 - T_{ZEUS}^Z \) \([32, 33]\). It would be interesting to reanalyze the data to measure the global observables \( \tau_1^{a,b,c} \) we predict in this paper at NNLL order.

**C. Jet and Beam Momenta**

1. **Jet and Beam contributions to 1-jettiness**

The cross sections for the different versions of 1-jettiness in Sec. 11A will all be expressed in terms of beam, jet, and soft functions that depend on the projections of the total momenta in the regions \( \mathcal{H}_B, \mathcal{H}_J \) onto the reference vectors \( q_{a,b} \) in the definition of the 1-jettiness Eq. (24). These vectors point in the direction of light-cone vectors \( n_B = \bar{n}_z \) and \( n_J \), which varies for the three different versions of 1-jettiness \( \tau_1^{a,b,c} \). The expression \( \tau_1 \) in Eq. (26) can be written in terms of \( n_J \cdot p_J \) and \( n_B \cdot p_B \) as
\[
\tau_1 = \frac{n_J \cdot p_J}{Q_J} + \frac{n_B \cdot p_B}{Q_B},
\] (48)

where \( Q_J \) and \( Q_B \) are given by
\[
Q_J = \frac{Q^2}{\omega_J}, \quad Q_B = \frac{Q^2}{\omega_B}.
\] (49)

Table II lists explicit expressions for \( Q_{B,J} \) in the CM, Breit, and target rest frames for the three versions of 1-jettiness \( \tau_1^{a,b,c} \).

For the three different cases \( \tau_1^{a,b,c} \) of Eq. (48), the contributions \( n_J \cdot p_J \) will be with respect to different vectors \( n_J^{a,b,c} \), and \( n_J \cdot p_J, n_B \cdot p_B \) will include momenta of particles in different regions \( \mathcal{H}_J, \mathcal{H}_B \) in the three cases. For \( \tau_1^a \), the differences between energies \( \omega_J^a \) and \( \omega_J^b \) and between unit vectors \( n_J^a \) and \( n_J^b \) are of order \( \lambda \) since the vectors \( q_J^a \) and \( q_J^b \) differ due to the transverse momentum of ISR of order \( \lambda Q \). So using the same expression \( \omega_J^b \) in Eq. (48) for \( \omega_J^b \) is correct up to corrections suppressed by \( \lambda \) that can be neglected in computing \( \tau_1^a \). Nevertheless, the values of \( n_J \cdot p_J \) in the equations for \( \tau_1^a \) and \( \tau_1^a \) do differ at leading

\(^1\) The variable \( \tau_{1E} \) is also not IR safe without a minimal energy constraint on the \( \mathcal{H}_C \) hemisphere.
TABLE II: Kinematic variables characterizing 1-jettiness. Normalizations $Q_J$ and $Q_B$ in the expression Eq. (A5) and sizes $R_{J,B}$ of the jet and beam regions $\mathcal{H}_{J,B}$ in Eq. (A6) for the different versions of 1-jettiness, in three different reference frames described in Sec. III and the Lorentz invariant combinations $Q_R \equiv Q_J/R_J = Q_B/R_B$ in Eq. (A7) and $s_{J,B}$ given in Eq. (A8).

2. Invariants for 1-jettiness

For later purposes we will express Eq. (A5) in terms of separate $n_J$-collinear, $n_B$-collinear, and soft contributions:

$$\tau_1 = \frac{n_J \cdot (p_J' + k_J)}{Q_J} + \frac{n_B \cdot (p_B' + k_B)}{Q_B},$$

where $p_J'$ is the total momentum of all $n_J$-collinear modes, $p_B'$ is the total momentum of all $n_B$-collinear modes, and $k_{J,B}$ are the total momenta of soft modes in regions $\mathcal{H}_{J,B}$, respectively. These modes are defined by the scaling of their light-cone components of momentum:

- $n_J$-collinear: $(n_J \cdot p, n_J \cdot p, p) \sim Q(\lambda^2, 1, \lambda)$
- $n_B$-collinear: $(n_B \cdot p, n_B \cdot p, p) \sim Q(\lambda^2, 1, \lambda)$
- soft: $k \sim Q\lambda^2$.

The normalization constants $Q_{J,B}$ in Eq. (A5) are not Lorentz invariant (which for SCET corresponds to a reparameterization invariance [83, 84]), but by combining them with other kinematic quantities we can form invariants in terms of which we can express Eq. (A5). One set of such combinations uses $R_{J,B}$ in Eq. (A7). The sizes $R_{J,B}$ of the regions $\mathcal{H}_{J,B}$ are not Lorentz-invariant— they depend on the choice of frame. However, the ratios $Q_J/R_J$ and $Q_B/R_B$ are Lorentz/reparameterization invariant and, in fact, are equal:

$$Q_R \equiv \frac{Q_J}{R_J} = \frac{Q_B}{R_B} = \frac{Q^2}{\sqrt{2q_J \cdot q_B}}.$$
Expressions for $R_{B,J}$ and $Q_R$ for each case $τ^{a,b,c}_1$ are given in Table [11] (Strictly speaking, dot products with $q^2_v$ are not Lorentz-invariant due to dependence on the jet algorithm, but for calculating $Q_R$ and $s_{J,B}$ we can use the approximation $q^2_v = q^2 = q + xP$ to leading order in $λ$, which does give Lorentz-invariant dot products.)

It is useful to re-express the soft contribution in Eq. 52 by rescaling the vectors $n_{J,B}$ by $n'_{J,B} = n_{J,B}/R_{J,B}$, which gives us

$$τ_S ≡ \frac{n_J · k_J}{Q_J} + \frac{n_B · k_B}{Q_B} = \frac{n'_J · k_J + n'_B · k_B}{Q_R}. \quad (55)$$

This relation will help us simplify the soft function in the factorized $τ_1$ cross sections later on. This is because rewriting the particle grouping in Eq. 56 in terms of $n'_J,B$ absorbs the factor $R_{J,B}$ giving $n'_J · p/n'_J · p < 1$ and $n'_B · p/n'_B · p > 1$. Hence with these variables the hemispheres $H_{J,B}$ are symmetric, which makes it possible to connect our soft function to the usual hemisphere soft function.

We can also re-express the $n_J,B$ collinear contributions to $τ_1$ in Eq. 52 in terms of another set of Lorentz-invariant combinations involving $Q_{J,B}$. In the $τ_1$ factorization theorems we derive below, the arguments of the collinear jet and beam functions appearing therein will naturally depend on “transverse virtualities” $n · p$ of the $n_J,B$-collinear jet and of the struck parton in the proton, respectively. Relating the $n_J,B$-collinear contribution to $τ_1$ to the transverse virtuality $t_J$ of the jet,

$$τ^+_J ≡ \frac{n_J · p_J}{Q_J} = \frac{n_J · p_J}{n_J · p_J} = \frac{t_J}{n_J · q Q_J} + O(λ^4), \quad (56)$$

where in the middle step we simply multiplied top and bottom by the large component $n_J · p_J$ of the total collinear momentum in region $H_J$, and in the last step we used the denominator $n_J · p_J = n_J · q + O(Q^2)$. The large component of the jet momentum can only come from the momentum transferred into the collision by the virtual boson of momentum $q$—the proton with which it collides only has a large component in the $n_J · p$ component. Similarly, the $n_B$-collinear contribution to $τ_1$ is

$$τ^+_B ≡ \frac{n_B · p_B}{Q_B} = \frac{-n_B · p_x n_B · p_x}{-n_B · q Q_B} = \frac{t_B}{-n_B · q Q_B} + O(λ^4), \quad (57)$$

where $p_x$ is the momentum of the parton that is struck by the virtual boson of momentum $q$. In the middle step we used that $n_B · p_B = -n_B · p_x$ since the struck parton recoils against the ISR and balances the small component of momentum in the $n_B$ direction. In the last step, we defined the positive virtuality $t_B = -n_B · p_x$ of the space-like struck parton and in the denominator used $-n_B · p_x = -n_B · q + O(Q^2)$. This is because the collision of the virtual boson and struck parton is the $n_J$-collinear jet which has no large momentum in the $n_B · p$ component. Thus momentum conservation requires that the large components of $n_B · q$ and $n_B · p_x$ cancel.

The quantities in the denominators of the relations Eqs. 56 and 57 are Lorentz invariant:

$$s_J = \bar{n}_J · q Q_J = \frac{q_B · q}{q_B · q_J}, \quad (58a)$$

$$s_B = -\bar{n}_B · q Q_B = \frac{-q_J · q}{q_B · q_J}, \quad (58b)$$

where the minus sign in $s_B$ makes it positive since $n_B · q < 0$. For the cases $τ^{a,b,c}_1$, $s_J$ and $s_B$ take the special values given in Table [11].

Using the definitions of $Q_R$ and $s_{J,B}$ in Eqs. 56 and 58 these factors can be combined to give the transverse virtuality of the exchanged boson $q$:

$$\frac{s_{J,B}}{Q^2_R} = -\bar{n}_B · q n_J q \frac{n_B · n_J}{2} = Q^2(1 - q^2 / Q^2), \quad (59)$$

where we used

$$q = \bar{n}_B · q n_B / 2 + n_J · q n_J / 2 + q_⊥, \quad (60)$$

and $q^2 = -Q^2$. The transverse momentum $q_⊥$ is orthogonal to $n_{B,J}$. The relation Eq. 59 will be useful in evaluating the fixed-order $τ_1$ cross section in App. K. We will use that $q^2 / Q^2$ is the 2-jettiness measured to be small, $τ_1 \sim λ^2$. A larger $q_⊥$ cannot be transferred into the final state for this to be true, since particles have to be collimated along $q_{J,B}$ or be soft.

D. Momentum Conservation and the Beam Region

We noted earlier that the contribution of proton remnants to $τ_1$ is exponentially suppressed, by a factor $e^{-[ΔY]}$ of their rapidity with respect to $q_B$. Only the energetic ISR and soft radiation at larger angles in $H_J$ contribute to $τ_1$. Although these contributions are easier to measure, one may still prefer to measure particles only in the $H_J$ jet region in the direction of $q_J$. In general, such a restriction in the final state is non-global, and leads to NGLs. However, by momentum conservation, we can show that each of the global $τ^{a,b,c}_1$ observables we consider can be rewritten in terms of momenta of particles only in the $H_J$ region (for case $a$ this is true only in the 2-jet region $τ^{b}_1 \ll 1$).

First, consider $τ^{b}_1$. In the Breit frame,

$$τ^{b}_1 = \frac{1}{Q} \sum_{i ∈ X} \min\{n_z · p_i, n_z · p_i\} \quad (61)$$

$$= \frac{1}{Q} \left[ \sum_{i ∈ H_J^F} (E_i - p_{z i}) + \sum_{i ∈ H_F} (E_i + p_{z i}) \right]$$

$$= \frac{1}{Q} \left[ \sum_{i ∈ X} (E_i + p_{z i}) - 2 \sum_{i ∈ H_F} p_{z i} \right],$$

where $X = H^F_J + H^F_B$ denotes the entire final state. Note that in the Breit frame,

$$p_X = P + q = \left( \frac{Q}{2x}, 0, 0, Q - \frac{Q}{2x} \right), \quad (62)$$
where \( p_X^i \equiv \sum_{i \in X} p_i^i \). Thus, \( E_X + p_{zX} = Q \), and we obtain
\[
\tau_1^{\text{Breit}} = 1 - \frac{2}{Q} \sum_{i \in \mathcal{H}_j^b} p_{z i} \equiv \tau_Q, \quad (63)
\]
where in the last equality we recall that Eq. (63) is precisely the definition in Eq. (46) of the DIS thrust variable called \( \tau_Q \) in [13], where the hemisphere \( \mathcal{H}_j^b \) in the Breit frame was called the “current hemisphere” \( \mathcal{H}_C \). We will comment further on the relation between the results of [15] for \( \tau_Q \) and our results for \( \tau_1^b \) in Sec. VII B below. Eq. (63) shows that \( \tau_1^b \) can always be computed just in terms of the measurements of momenta of particles in the current hemisphere \( \mathcal{H}_C = \mathcal{H}_j^b \).

The same arguments as for \( \tau_1^b \) in the Breit frame apply to \( \tau_1^c \) in the CM frame. In the CM frame,
\[
\tau_1^{\text{CM}} = \frac{1}{x y \sqrt{s}} \sum_{i \in X} \min\{ n_z \cdot p_i, \bar{n}_z \cdot \bar{p}_i \} \quad (64)
= \frac{1}{x y \sqrt{s}} \left[ \sum_{i \in X} (E_i + p_{z i}) - 2 \sum_{i \in \mathcal{H}_j^c} p_{z i} \right].
\]
In this frame, we have that
\[
p_X = P + q = \frac{\sqrt{s}}{2} \left( y + 1 - x \left( 1 - \frac{Q^2}{2Q^2} \right) \right),
\]
so
\[
\tau_1^{\text{CM}} = \frac{1}{x} \left( 1 - \frac{2}{y \sqrt{s}} \sum_{i \in \mathcal{H}_j^c} p_{z i} \right). \quad (66)
\]
Thus, \( \tau_1^c \) also can be measured just from momenta of particles in the \( \mathcal{H}_j \) hemisphere in the CM frame.

Finally, the above argument can be extended to apply also to the 1-jettiness \( \tau_1^c \), but only for the region where \( \tau_1^c \ll 1 \). \( \tau_1^c \) can be written
\[
\tau_1^c = \frac{2}{Q^2} \left[ \sum_{i \in \mathcal{H}_j^c} q^c_i \cdot p_i + \sum_{i \in \mathcal{H}_b^c} q^b_i \cdot p_i \right]. \quad (67)
\]
Now, \( q^c_i = q^b_i \), while \( q^c_j = q^c_j + O(Q \lambda) \). Thus the regions \( \mathcal{H}_j^c, \mathcal{H}_b^c \) differ from those for \( \tau_1^b, \mathcal{H}_j^b, \mathcal{H}_b^b \), by a change in the region boundary of \( O(\lambda) \). This does not affect the assignment of collinear particles to the two regions, since none of them change regions under this small change in boundary. An \( O(\lambda) \) fraction of the soft particles switch from one region to the other, but this then produces a correction suppressed by \( \lambda \) to the soft contribution \( \tau_S \) in Eq. (65). Thus, Eq. (67) can be expressed
\[
\tau_1^c = \frac{2}{Q^2} \left[ \sum_{i \in \mathcal{H}_j^c} (q^c_i - q^c_j) \cdot p_i + \sum_{i \in \mathcal{H}_j^b} q^b_i \cdot p_i + \sum_{i \in \mathcal{H}_b^c} q^b_i \cdot p_i \right] + O(\lambda^3)
= \tau_1^b + \frac{2}{Q^2} \sum_{i \in \mathcal{H}_j^c} (q^c_i - q^c_j) \cdot p_i + O(\lambda^3), \quad (68)
\]
in the regime where \( \tau_1 \sim \lambda^2 \ll 1 \). This is the regime we aim to predict accurately in this paper. Thus, in this limit \( \tau_1^c \) can also be computed just by measuring particles in the “current hemisphere” \( \mathcal{H}_j^c \) in the Breit frame, as long as both axes \( q^c_j \) and \( q^c_B \) are measured. For larger \( \tau_1^c \), both regions \( \mathcal{H}_j^c, \mathcal{H}_b^c \) would need to be measured, and we emphasize that the contribution of proton remnants is still exponentially suppressed.

In summary, for small \( \tau_1 \) none of the three versions of 1-jettiness \( \tau_1^{b,c} \) require direct measurement of particles from initial state radiation in the beam region. Furthermore, for larger \( \tau_1 \) values the variables \( \tau_1^{b,c} \) still do not require such measurements (though \( \tau_1^a \) does). All three \( \tau_1 \)’s are global observables since measurement of \( \tau_1 \) by summing over the particles only in the \( \mathcal{H}_j \) region is still affected by ISR from the proton beam through momentum conservation.

### IV. CROSS SECTION IN QCD

In this Section we organize the full QCD cross section into the usual leptonic and hadronic tensors, but with an additional measurement of 1-jettiness inserted into the definition of the hadronic tensor. We express it in a form that will be easily matched or compared to the effective theory cross section we consider in the following section.

#### A. Inclusive DIS cross section

We begin with the inclusive DIS cross section in QCD, differential in the momentum transfer \( q \),
\[
\frac{d \sigma}{d^2 q} = \frac{1}{2s} \int d \Phi_L \sum_X \left| \mathcal{M}(eP \rightarrow LX) \right|^2 \times (2\pi)^4 \delta^4(P + q - p_X) \delta^4(q - k + k'),
\]
where \( L \) is the final lepton state with momentum \( k' \), and \( X \) is the final hadronic state with momentum \( P_X \). \( d \Phi_L \) is the phase space for the lepton states, and the \( \sum_X \) includes the phase space integrals for hadronic states. The squared amplitude \( |\mathcal{M}|^2 \) is averaged over initial spins, and summed over final spins. Recall that \( q \) (and \( x, y \)) can be determined entirely by measurements of the lepton momenta. Later in Sec. VII B we will insert additional measurements such as 1-jettiness on the state \( X \).

We wish to express the cross section differential in the Lorentz-invariant variables \( Q^2, x \) using Eqs. (7) and (8). Although \( Q^2, x \) are Lorentz-invariant, at intermediate stages of integration we can work in a particular frame. In either the CM or Breit frame, the proton momentum is of the form \( P = n_z \cdot P \bar{n}_z / 2 \). So we decompose \( q \) along the \( n_z, \bar{n}_z \) directions, \( q = n_z \cdot q \bar{n}_z / 2 + \bar{n}_z \cdot q n_z / 2 + q_T \). Then the delta functions defining \( Q^2, x \) take the form
\[
\delta \left( x - \frac{Q^2}{n_z \cdot P \bar{n}_z \cdot q} \right) \times \delta \left( Q^2 + n_z \cdot q \bar{n}_z \cdot q - q^2_T \right). \quad (70)
\]
Inserting these into Eq. (74) and integrating over $q^+$ and $q^-$, we obtain

$$\frac{d\sigma}{dx\,dQ^2} = \frac{1}{4\pi^2} \int d^2q_T \int d\Phi_L \delta^4(q - k + k') \times \sum_X (2\pi)^4 \delta^4(P + q - p_X)\langle |\mathcal{M}|^2 \rangle,$$

where $q$ is now given by the value

$$q^\mu = \frac{Q^2}{xn_z \cdot P} \frac{n^\mu_z}{2} - xn_z \cdot P \left(1 - \frac{Q^2}{Q^2} \right) \frac{\bar{n}_T^\mu}{2} + q_T^\mu. \quad (72)$$

For a single electron final state $L = e(k')$ (which is all we have at the leading order in $\alpha_{em}$ at which we are working), the integral over $\Phi_L$ in Eq. (71) takes the form

$$\int \frac{d^4k'}{(2\pi)^4 2E_{k'}} = \int \frac{d^4k'}{(2\pi)^4} \delta(k'^2), \quad (73)$$

so, performing the $k'$ integral, we obtain

$$\frac{d\sigma}{dx\,dQ^2} = \frac{1}{4(2\pi)^3} \frac{Q^2}{xs} \frac{d^2q_T}{(2\pi)^2} \frac{1}{2E_{k'}} \delta((q - k)^2) \times \sum_X (2\pi)^4 \delta^4(P + q - p_X)\langle |\mathcal{M}|^2 \rangle. \quad (74)$$

To use the first delta function, we need to pick a particular frame in which to complete the $q_T$ integration. In the CM frame,

$$\delta((q - k)^2) = \delta(Q^2 + 2Q \cdot k) = \frac{Q^2}{xs} \delta \left(\frac{q_T^2}{Q^2} - \left(1 - \frac{Q^2}{xs} \right) Q^2 \right), \quad (75)$$

where we use Eq. (72), $k = \sqrt{s} n_z/2$ and $P = \sqrt{s} \bar{n}_z/2$. We use this delta function to perform the $q_T^2$ integral in Eq. (74), and then use that the spin-averaged squared amplitude is independent of $\varphi_q$, to obtain

$$\frac{d\sigma}{dx\,dQ^2} = \frac{Q^2}{8(2\pi)^2 x^2 \delta^2} \sum_X (2\pi)^4 \delta^4(P + q - p_X)\langle |\mathcal{M}|^2 \rangle. \quad (76)$$

Here the integrand is evaluated in the CM frame with $q$ now given by

$$q^\mu = y\sqrt{s} \frac{n^\mu_z}{2} - xy\sqrt{s} \frac{\bar{n}_T^\mu}{2} + \sqrt{1 - y} Q \frac{\bar{n}_T^\mu}{2}, \quad (77)$$

where $\bar{n}_T = (0, 1, 0, 0)$ in $(n^0, n_1, n_2, n_3)$ coordinates.

The matrix element $\mathcal{M}$ is given by

$$\mathcal{M}(eP \to e'X) = \sum_{I = \gamma, Z} \langle e'X | J_{I,EW}^I | eP \rangle D_{\mu\nu}^I J_{\nu,QCD}^I (0) | eP \rangle,$$

where the sum over $I$ is over photon and $Z$ exchange, $J_{I,EW}$ is the appropriate electron electroweak current, $J_{I,QCD}$ is the quark electroweak current, and $D_{\mu\nu}$ is the $\gamma$ or $Z$ propagator. There is an implicit sum over quark flavors. The matrix element can be factored,

$$\mathcal{M}(eP \to e'X) = \sum_{I = \gamma, Z} \langle e'X | J_{I,EW}^I | eP \rangle \langle eP | J_{\nu,QCD}^I | eP \rangle \langle eP | J_{\nu,QCD}^I | eP \rangle,$$

More conveniently, we can express the sum over $i$ as being over the vector and axial currents in QCD,

$$J_{\nu V}^I = \bar{q} \gamma^\mu q f, \quad J_{\nu A}^I = \bar{q} \gamma^\mu \gamma^5 q f. \quad (80)$$

The sum in Eq. (78) can then be expressed as

$$\mathcal{M} = \sum_{I = V,A} \sum_f L_{I\mu} \langle X | J_{\nu V}^I | P \rangle,$$

defining the leptonic vector $L_{I\mu}$, which contains the electron matrix element, electroweak propagator, and electroweak charges of the quarks implicit in Eq. (78). The sum over $f$ in Eq. (80) is over quark flavors.

Now the cross section in Eq. (76) can be written

$$\frac{d\sigma}{dx\,dQ^2} = \sum_{I,I',\nu,\mu} L_{I\mu}^{I'} (x, Q^2) W_{I'I\mu\nu} (x, Q^2), \quad (81)$$

where

$$L_{I\mu}^{I'} (x, Q^2) = \frac{Q^2}{32\pi^2 x^2 s^2} L_{I\mu}^I (x, Q^2) L_{I'}^{I'} (x, Q^2), \quad (83a)$$

$$W_{I'I\mu\nu} (x, Q^2) = \sum_X (P | J_{\nu V}^I | X ) \langle X | J_{\nu A}^{I'} | P \rangle \times \sum_X (2\pi)^4 \delta^4(P + q - p_X). \quad (83b)$$

Here $L_{I\mu}^{I'}$, $W_{I'I\mu\nu}$ depend on $x, Q^2$ through the components of $q$ given in Eq. (72). The average over initial electron and proton spins is implicit in Eq. (83), as is the sum over quark flavors in Eq. (83b).

1. Leptonic tensor

The leptonic tensor in Eq. (83b) is given by

$$L_{I'I\mu\nu}^{I'} (x, Q^2) = \frac{Q^2}{2\pi^2 s^2} \left( L_{I'I\mu}^{I'} (Q^2) g_{\mu\nu} + i L_{I'I\mu}^{I'} (Q^2) \epsilon_{\mu\nu}^T \right), \quad (84)$$

where

$$g_{\mu\nu}^T = g_{\mu\nu} - \frac{k_\mu k_\nu + k_\nu k_\mu}{Q^2}, \quad \epsilon_{\mu\nu}^T = \frac{2}{Q^2} \epsilon_{\alpha\beta\mu\nu} k^\alpha k^\beta, \quad (85)$$
where \( k' = k - q \), with \( q \) given in Eq. (77) and

\[
L_{\gamma 
\gamma}^V = \frac{Q f Q f'}{1 + m_Z^2/Q^2} + (1 + m_Z^2/Q^2)^2,
\]

\[
L_{\text{eff}}^V = -a_f \frac{Q f v f + v f Q f'}{1 + m_Z^2/Q^2} + \frac{2a_f v f v f'}{1 + m_Z^2/Q^2}^2,
\]

\[
L_{\text{eff}}^A = \frac{a_f v f}{1 + m_Z^2/Q^2} \left[ Q f v f - \frac{v f}{(1 + m_Z^2/Q^2)} \right],
\]

\[
L_{\gamma_f}^A = \frac{2a_f a_f}{1 + m_Z^2/Q^2} \left[ Q f v f - \frac{v f}{(1 + m_Z^2/Q^2)} \right],
\]

where we have made explicit the flavor indices \( f, f' \). \( Q f \) is the electric charge of the quark \( q_f \) in units of \( e \); \( v_f, a_f \) are the weak vector and axial charges of \( q_f \); and \( v_e, a_e \) the weak vector and axial charges of the electron. The vector and axial charges are given by:

\[
a_f = \frac{T_f}{\sin 2\theta_W}, \quad v_f = \frac{T_f - 2Q_f \sin^2 \theta_W}{\sin 2\theta_W}, \quad a_e = -\frac{1}{2} \sin 2\theta_W, \quad v_e = -\frac{1}{2} \sin 2\theta_W,
\]

where \( T_f = 1/2 \) for \( f = u, c, t \) and \(-1/2 \) for \( f = d, s, b \).

### B. 1-jettiness cross section

To form the cross section differential in the 1-jettiness \( \tau_1 \), we insert a delta function measuring \( \tau_1 \) into the hadronic tensor Eq. (53):

\[
\frac{d\sigma}{dx dQ^2 d\tau_1} = \sum_{l, l' = V, A} L_{ll'}^V(x, Q^2) W_{ll'}^{V'(x, Q^2, \tau_1)},
\]

where the \( \tau_1 \)-dependent hadronic tensor is

\[
W_{ll'}^{V'}(x, Q^2, \tau_1) = \sum_{\tau} \langle P | J_{l'}^{\mu} | X \rangle \langle X | J_{l}^{\mu'} | P \rangle (2\pi)^4 \delta^4(P + q - p_X)\delta(\tau_1 - \tau(X)).
\]

Here the 1-jettiness \( \tau_1(X) \) of state \( X \) is defined by Eq. (24). The definition depends on the choices of reference vectors \( q_{B,J} \).

The sum over states \( X \) in Eq. (53) can be removed by using an operator \( \hat{\tau}_1 \) which gives \( \tau_1(X) \) when acting on the state \( X \):

\[
\hat{\tau}_1 | X \rangle = \tau_1(X) | X \rangle.
\]

This operator can be constructed from a momentum-flow operator as in (82). Explicitly,

\[
\hat{\tau}_1 = \hat{\tau}_1^A + \hat{\tau}_1^B,
\]

where

\[
\hat{\tau}_1^{A,B} = \frac{2}{Q^2} \int_{\gamma_{J,B}}^\infty dY_{J,B} \gamma_{J,B} \cdot \hat{P}(Y_{J,B}).
\]

Here \( \hat{P}(Y_{J,B}) \) is a momentum flow operator that can be defined and explicitly constructed in terms of the energy-momentum tensor, which can be obtained for massless partons using (55–88) and for massive hadrons using (72). It measures the momentum flow in the generalized rapidity direction \( Y_{J,B} \), which we define as we did below Eq. (51) by

\[
e^{-2\hat{Y}} = \frac{n_f \cdot \hat{p}_f}{n_f \cdot \hat{p}_f}, \quad e^{-2\hat{Y}_a} = \frac{n_B \cdot \hat{p}_B}{n_B \cdot \hat{p}_B}.
\]

The lower limits \( Y_{J,B} \) on the integral in Eq. (92) are given according to Eq. (50) by \( Y_{J,B} = \frac{1}{2} \ln y(x) \). These values depend on the frame of reference and choice of 1-jettiness \( \tau_1 \). For example, for the choice \( \tau_1^B \) of Eq. (72) in the CM frame for \( y \nearrow 1 \), the beam and jet regions are hemispheres and \( Y_{J,B} = 0 \). For the choice \( \tau_1^B \) of Eq. (82) in the CM frame, the jet region is given by the lower limit \( Y_{J,B} = \frac{1}{2} \ln y(x) \), and the beam region is given by the lower limit \( Y_{B} = \frac{1}{2} \ln y(x) \).

In the massless limit the generalizations \( \exp(-2Y_{J,B}) \rightarrow (1 - \cos \theta_{J,B})/(1 - \cos \theta_{B,J} n_f \cdot \hat{p}_f/2 \) defining generalized “pseudorapidities”. They depend only on angles \( \theta_{J,B} \) from the \( n_f \) directions and \( n_f \cdot \hat{p}_f \), and so simply characterize angular directions in space in which we integrate in Eq. (92).

Using Eq. (92) the hadronic tensor Eq. (50) can be written

\[
W_{1/}^{\mu

\nu}(x, Q^2, \tau_1) = \int d^2x e^i x P | J_{1/}^{\mu} | x \rangle \delta(\tau_1 - \hat{\tau}_1) J_{1/}^{\nu}(0) | P \rangle,
\]

recalling that \( q \) is given by Eq. (77). \( \hat{\tau}_1 \) can also be expressed in terms of momentum operators in the regions \( H_{J,B} \), using Eq. (52):

\[
\hat{\tau}_1 = \frac{n_f \cdot \hat{p}_f}{Q_f} + \frac{n_B \cdot \hat{p}_B}{Q_B},
\]

where \( \hat{p}_{J,B} \) measures the total 4-momentum in region \( H_{J,B} \).

### V. FACTORIZATION IN SCET

Soft-collinear effective theory (SCET) [18–22] is a systematic expansion of QCD in a small parameter \( \lambda \) which characterizes the scale of collinear and soft radiation from energetic massless partons. Soft and collinear modes are defined by the scaling of their momenta in light-cone coordinates with respect to light-like vectors \( n, \bar{n} \) (not necessarily back-to-back) satisfying \( n^2 = \bar{n}^2 = 0 \) and \( n \cdot \bar{n} = 2 \). We express the components of a vector \( p \) in \( n, \bar{n} \) light-cone coordinates as \( p = (\bar{n} \cdot p, n \cdot p, p_\perp) \), where

\[
p_\perp = n \cdot p \frac{n}{2} + n \cdot p \frac{\bar{n}}{2} + p_\perp.
\]

with \( p_\perp \) being orthogonal to \( n, \bar{n} \), defined as

\[
p_\perp \mu = g_\perp^\mu p_\perp, \quad g_\perp^\mu = g_\perp^\mu - \frac{\bar{n} \cdot \bar{n} p_\perp + n \cdot n p_\perp}{2}.
\]
In these light-cone coordinates, n-collinear and soft
momenta scale as:
\[ p_n \sim Q(1, \lambda^2, \lambda), \] (98a)
\[ p_s \sim Q(\lambda^2, \lambda^2, \lambda^2). \] (98b)

The parameter \( \lambda \) is determined by the virtuality of the modes \( p_n^2 \sim Q^2 \lambda^2 \) that contribute to the observable in question. Collinear momenta will be expressed as the sum of a large “label” piece and a small “residual” piece: \( p_n = \bar{p}_n + k \), where \( \bar{p}_n = \frac{n \cdot \bar{P} n/2 + \bar{P}_\perp} {n \cdot \bar{P}} \) contains the \( \mathcal{O}(Q) \) longitudinal and \( \mathcal{O}(Q \lambda) \) transverse pieces, and \( k \) is the residual \( \mathcal{O}(Q \lambda^2) \) piece.

\[ \lambda \]

A. Matching onto SCET

Now we are ready to match the currents in Eq. (95) onto operators in SCET. The QCD current
\[ J^\mu_f(x) = \bar{q}_f(x) \Gamma^\mu_q q_f(x), \] (99)
with \( \Gamma^\mu_q = \gamma^\mu \) and \( \Gamma^\lambda_A = \gamma^\mu \gamma_5 \), matches onto operators in SCET,
\[ J_{f,ij}^\mu(x) = \sum_{n_1 n_2} \int d^3 \bar{p}_1 d^3 p_2 e^{i(\bar{p}_1 - \bar{p}_2 \cdot x)} \times \left[ \mathcal{O}_{f,qq}^{\alpha j} (\bar{p}_1, \bar{p}_2) \mathcal{O}_9^{\beta \lambda} (\bar{p}_1, \bar{p}_2; x) + \mathcal{O}_{f,gg}^{\alpha j} (\bar{p}_1, \bar{p}_2) \mathcal{O}_{9}^{\beta \lambda} (\bar{p}_1, \bar{p}_2; x) \right], \] (100)

neglecting power corrections of \( \mathcal{O}(\lambda^2) \). The quark and gluon SCET operators are
\[ \mathcal{O}_{f,qq}^{\alpha j} (\bar{p}_1, \bar{p}_2; x) = \chi_{n_1, \bar{n}_1}^{(0)} (\bar{p}_1) \chi_{n_2, \bar{n}_2}^{(0)} (x), \] (101a)
\[ \mathcal{O}_{f,gg}^{\alpha j} (\bar{p}_1, \bar{p}_2; x) = \sqrt{\omega \omega_2} B_{n_1, \bar{n}_1}^{\perp} (x) B_{n_2, \bar{n}_2}^{\perp} (x), \] (101b)

where we sum over fundamental color indices \( j \) and adjoint color indices \( c \), but fix the spin indices \( \alpha, \beta \). We leave implicit that \( \chi \equiv \chi_q \) carries flavor \( q \). Below we will also leave the flavor index \( f \) on the current \( J_i \) implicit. The collinear fields \( \chi_{n, \bar{n}} \) and \( B_{n, \bar{n}} \) carry label momenta
\[ \bar{p}_i = \frac{\omega_i n_i}{2} + \bar{p}_i^\perp, \] (102)
where \( \omega_i = m + \lambda Q \). The momentum of each collinear field can be written in \( n_i, \bar{n}_i \) light-cone coordinates as in Eq. (99), with the residual \( x \) dependence of the SCET fields being conjugate to momenta \( k \) of order \( Q \lambda^2 \). In Eq. (100), the integrals over \( \bar{p}_{1,2} \) are continuous versions of discrete sums over the label momenta, and the measures are given by \( d^3 \bar{p}_i \equiv d\omega_i d^2 \bar{p}_i^\perp \).

The quark jet fields \( \chi_{n, \bar{n}} (x) \) are products of collinear quark fields with collinear Wilson lines,
\[ \chi_{n, \bar{n}} = \Omega(\omega - \bar{n} \cdot P) \delta^2(\bar{p}_\perp - P_\perp) W_{n, \bar{n}}^\alpha \xi_n, \] (103)
where \( \mathcal{P}_\mu \) is a label momentum operator which acts on collinear fields and conjugate fields as:
\[ \mathcal{P}_\mu \phi_{n, \bar{n}} = \bar{\phi}_{n, \bar{n}} \mathcal{P}_\mu \phi_{n, \bar{n}}, \quad \mathcal{P}_\mu \bar{\phi}_{n, \bar{n}} = -\bar{\phi}_{n, \bar{n}} \mathcal{P}_\mu \bar{\phi}_{n, \bar{n}} \] (104)
and \( W_n \) is the Wilson line
\[ W_n(x) = \prod_{\text{perms}} \exp \left[ -\frac{g}{n \cdot \bar{P}} \bar{n} \cdot A_n(x) \right], \] (105)
where \( A_n^\mu (x) = \sum_{\bar{P}} A_n^\mu (\bar{P}) \) is a \( n \)-collinear gluon field. The gluon jet fields are collinear gauge-invariant products of gluon fields and Wilson lines,
\[ B_{n, \bar{n}} = \frac{1}{g} [\delta(\omega + \bar{n} \cdot P) \delta^2(\bar{p}_\perp + P_\perp) W_n (P_\perp + g A^\perp_n) W_n]. \] (106)

The matching coefficients \( C_{qq}, C_{gg} \) in Eq. (100) are calculated order-by-order in \( \alpha_s \) by requiring that matrix elements of both sides of Eq. (100) between collinear states in QCD and in SCET be equal.

Collinear fields are decoupled from soft fields by the field redefinitions
\[ \chi_n = Y_{n, \lambda_n}^{(0)}, \quad A_n^\lambda T^\lambda = Y_{n}^{ab} A_n^{(0)} T^a = Y_{n}^{ab} T^b Y_n^a, \] (107)
where \( Y_n \) is a Wilson line of soft gluons in the fundamental representation. For \( n = n_B \) we have
\[ Y_{n_B} (x) = P \exp \left[ ig \int_{-\infty}^{0} ds n_B \cdot A_s (n_B s + x) \right], \] (108)
and \( Y_{n_B} \) is defined similarly but in the adjoint representation. Soft gluons carry momenta scaling as \( \lambda^2 \) in all components. Additional factors accompanying outgoing states turn the path in Eq. (108) into \( x \) to \( \infty \) for outgoing collinear particles, see also (109). So for \( n = n_j \) we have
\[ Y_{n_j}^\dagger (x) = P \exp \left[ ig \int_{0}^{\infty} ds n_j \cdot A_s (n_j s + x) \right]. \] (109)

After the field redefinition Eq. (107), the operators in Eq. (100) become
\[ \mathcal{O}_{f,qq}^{\alpha j} (\bar{p}_1, \bar{p}_2; x) = \chi_{n_1, \bar{n}_1}^{(0)} (\bar{p}_1) T^i [Y_{n_j}^\dagger Y_{n_2}]^{jk} (x) \chi_{n_2, \bar{n}_2}^{(0)} (x), \]
\[ \mathcal{O}_{f,gg}^{\alpha j} (\bar{p}_1, \bar{p}_2; x) = \sqrt{\omega \omega_2} B_{n_1, \bar{n}_1}^{\perp} (x) \times T^i [Y_{n_j}^\dagger Y_{n_2}]^{jk} (x) B_{n_2, \bar{n}_2}^{\perp} (x). \] (110)

The directions \( n_1 \) and \( n_2 \) will each get set equal to either \( n_j \) or \( n_B \) later on, replacing \( Y_n \), with \( Y_{n_B} \) in Eq. (108) for \( n_1 = n_B \) or with \( Y_{n_j}^\dagger \) in Eq. (109) for \( n_2 = n_j \). Henceforth we use only the decoupled collinear fields and drop the \( (0) \) superscripts.

The measurement operators in Eq. (95) also split up into collinear and soft pieces. Since \( \bar{p} \) is linear in the energy-momentum tensor, which itself splits linearly into
decoupled collinear and soft components after the field redefinition Eq. (107) [85], $\hat{p}$ splits up as

$$\hat{p} = \tilde{p}^{n_1} + \tilde{p}^{n_2} + \tilde{p}^s$$  \hspace{1cm} (111)

where $\tilde{p}^{n_1,n_2,s}$ is built only out of the $n_1$-collinear, $n_2$-collinear, or soft energy-momentum tensor of SCET, respectively.

After matching the product of currents $J_{\mu_{\hat{p}^s}}J_{\nu_{\hat{p}^s}}$ in the hadronic tensor in Eq. (94) onto SCET, there will be products of the quark and gluon operators $(\mathcal{O}_{\tilde{q}q} + \mathcal{O}_{gq})/3(\mathcal{O}_{Q_{gg}^\prime} + \mathcal{O}_{Q_{gg}^\prime}^\prime)$. The $\mathcal{O}_{Q_{gg}^\prime}$ and $\mathcal{O}_{Q_{gg}^\prime}$ cross terms will vanish inside the proton-proton matrix element by quark-number conservation (only one of the fields $\tilde{\chi}_{n_1}$ or $\tau_{n_2}$ in $\mathcal{O}_{\tilde{q}q}$ will create/annihilate a quark in the collinear proton). Thus only the $\mathcal{O}_{\tilde{q}q}^\prime\mathcal{O}_{\tilde{q}q}^\prime$ and $\mathcal{O}_{\tilde{q}q}\mathcal{O}_{\tilde{q}q}^\prime$ operator products can contribute.

In fact, for DIS, only the quark operator product contributes, just as in Drell-Yan (DY) [67]. Following the arguments in [67], we know that the matching coefficients $C_{1gg}^\mu(\tilde{p}_1, \tilde{p}_2)$ must be a linear combination of $\tilde{p}_1$ and $\tilde{p}_2$, and obey the symmetry

$$C_{1gg\mu}^\lambda(\tilde{p}_1, \tilde{p}_2) = C_{1gg\mu}^\lambda(-\tilde{p}_2, -\tilde{p}_1),$$  \hspace{1cm} (112)

due to the structure of the operator Eq. (1011). This requires $C_{1gg}^\mu$ to be proportional to $(\tilde{p}_2 - \tilde{p}_1)$, which the $x$ integration in Eq. (94) will eventually set equal to $q^\mu$. Vector current conservation in QCD requires $q_\mu C_{VV_{gg}} = 0$, which requires that $C_{1gg}^\mu$ be identically zero. The axial current matching coefficient $C_{Agg}^\mu$ can be nonzero, but still proportional to $\tilde{p}_2^\mu - \tilde{p}_1^\mu = q^\mu$, which gives zero contribution when contracted with the lepton tensor Eq. (53). Thus for DIS we need only consider the quark operator contribution as in DY.

### B. Factorization of the Hadronic Tensor

The hadronic tensor Eq. (94) can now be written in SCET as

$$W_{\mu\nu}^{I\ell}(x, Q^2, \tau_1) = \int d^4x \, e^{iq \cdot x} \sum_{n_1, n_2} \int d^3\tilde{p}_1 d^3\tilde{p}_2 d\beta \tilde{p}_1^\prime d\beta \tilde{p}_2^\prime e^{i(\hat{p}_2 - \hat{p}_1) \cdot x} \int d\tau d\tau_B d\tau_s (\tau_1 - \tau_J - \tau_B - \tau_s)

\times \langle P_{n_B} | C_{Igq_{\tilde{q}q}}(\tilde{p}_1, \tilde{p}_2) \chi_{n_2, \tilde{p}_2 \tilde{p}_1} \mathcal{T}[Y_{n_2}^\tau Y_{n_1}] | k] \chi_{n_1, \tilde{p}_1}(x) \delta(\tau_J - \frac{n_J \cdot \tilde{p}_J}{Q_J}) \delta(\tau_B - \frac{n_B \cdot \tilde{p}_B}{Q_B}) \rangle

\times \delta(\tau_s - \frac{n_J \cdot \tilde{p}_J}{Q_J} - \frac{n_B \cdot \tilde{p}_B}{Q_B}) \mathcal{C}_{Igq_{\tilde{q}q}}(\tilde{p}_1^\prime, \tilde{p}_2^\prime) \chi_{n_1, \tilde{p}_1}^\prime \mathcal{T}[\bar{Y}_{n_1}^\tau Y_{n_2}^\tau] | j]\chi_{n_2, \tilde{p}_2}^\' (0) | P_{n_B} \rangle.$$  \hspace{1cm} (113)

We have explicitly specified that the proton is an $n_B$-collinear state. The conjugate quark matching coefficient is given by $C_{Igq_{\tilde{q}q}}(\tilde{p}_1, \tilde{p}_2) = [\gamma^0 C_{Igq}(\tilde{p}_1, \tilde{p}_2)]^{\dagger}$. We have used that the measurement operator $\hat{\tau}_1$ can be written in the form Eq. (95), and that the momentum operators $\hat{p}_{J,B}$ split up linearly into purely $n_J$- and $n_B$-collinear and soft operators as in Eq. (111). We dropped the subscripts $J,B$ on the collinear momentum operators restricting them to the jet or beam regions $H_{J,B}$ determined by the definition of $\tau_1$, since all $n_B$ collinear particles are grouped in region $B$ and all $n_J$-collinear particles are grouped in region $J$. In the soft sector, the restrictions of the operators $\hat{\tau}_s$ to the $H_{J,B}$ regions remain.

Since the $n_1$-collinear, $n_2$-collinear and the soft sectors are all decoupled from one another, the proton matrix element in Eq. (111) can be factored,

$$W_{\mu\nu}^{I\ell}(x, Q^2, \tau_1) = \int d^4x \, d^3\tilde{p}_1 d^3\tilde{p}_2 d\beta \tilde{p}_1^\prime d\beta \tilde{p}_2^\prime e^{i(q + \hat{p}_2 - \hat{p}_1) \cdot x} \int d\tau d\tau_B d\tau_s (\tau_1 - \tau_J - \tau_B - \tau_s) C_{Igq_{\tilde{q}q}}(\tilde{p}_1, \tilde{p}_2) C_{Igq_{\tilde{q}q}}(\tilde{p}_1^\prime, \tilde{p}_2^\prime)

\times \langle 0 | [Y_{n_B}^\tau Y_{n_J}] \chi_{\tilde{p}_1}(x) \delta(\tau_s - \frac{n_J \cdot \tilde{p}_J}{Q_J} - \frac{n_B \cdot \tilde{p}_B}{Q_B}) | Y_{n_B}^\tau Y_{n_J} \rangle | j]\chi_{n_2, \tilde{p}_2}^\' (0) | 0 \rangle

\times \left\{ \langle P_{n_B} | \chi_{n_B, \tilde{p}_B} (x) \delta(\tau_B - \frac{n_B \cdot \tilde{p}_B}{Q_B}) \chi_{n_J, \tilde{p}_J}^\' (0) | P_{n_B} \rangle \langle 0 | \chi_{n_J, \tilde{p}_J} (x) \delta(\tau_J - \frac{n_J \cdot \tilde{p}_J}{Q_J}) \chi_{\tilde{p}_1, \tilde{p}_1} (0) | 0 \rangle \right\}.  \hspace{1cm} (114)$$

The last two lines account for the two ways to choose a pair of collinear fields in the proton matrix element. We have performed the sums over $n_{1,2}, n_{1,2}'$ sums using that the fields within each collinear matrix element must all be in the same collinear sector. We also require that the fields in the proton matrix element must be in the same collinear sector as the proton, and those in the vacuum matrix element with the direction $n_J$ in the definition of $\tau_1$. The integrals over $\tilde{p}_{1,2}$ have been absorbed into the definition of the unlabeled fields $\chi_{n_{1,2}}$. In the soft matrix element we have used the fact that $T[\bar{Y}_{n_J}^\tau Y_{n_B}] = Y_{n_J}^\tau Y_{n_B}$ and $\mathcal{T}[\bar{Y}_{n_B}^\tau Y_{n_J}] = Y_{n_B}^\tau Y_{n_J}$ since the two Wilson lines are space-like separated and
the time ordering is the same as the path ordering \( 91, 92 \). For the soft Wilson line matrix element corresponding to antiquarks in the beam and jet functions, we have used charge conjugation to relate it to the matrix element shown in Eq. \( 114 \).

It is measuring \( \tau_1 \) to be small that enforces that the direction \( n_J \) on the collinear fields in the vacuum matrix element be equal to the direction of the vector \( q_J \) in the definition of the 1-jettiness \( \tau_1 \). We are free to choose any vector \( q_J \) to define the observable \( \tau_1 \). Requiring that the final-state jet \( J \) be close to the direction of \( q_J \) may, in general, impose additional kinematic constraints on \( x, y, Q^2 \) to ensure this. We will find below that for \( \tau_1^{b, c} \), \( q_J \) is already chosen to be close to the final-state jet and so imposes no additional constraints, while for \( \tau_1^x \) requiring the jet be close to \( q_J' = k \) requires \( y \) to be near 1.

Next we wish to perform the \( x \) integral in Eq. \( 114 \) to enforce label momentum conservation. Before doing so, we consider the residual momentum dependence conjugate to the coordinate \( x \) in the SCET matrix elements. The collinear field \( \chi_{n_B}(x) \) with a continuous label momentum \( \vec{p} \) depends only on single spatial component \( \vec{n} \cdot x \) because the residual momenta (conjugate to the spatial components \( n·x, x_⊥ \)) are reabsorbed into \( \vec{p} \) when the discrete label is made continuous. Then, the matrix element of \( n \)-collinear fields are \( \mathcal{M}_n = \mathcal{M}_{n_b}(\vec{n} \cdot x) \). For convenience we choose the soft matrix element with \( Y_n(x) \) and \( \tilde{Y}_n(x) \) will be defined as \( \mathcal{M}_s(x) \). Their Fourier transforms take the form

\[
\mathcal{M}_n(\vec{n} \cdot x) = \int \frac{d\vec{n} \cdot k}{2\pi} e^{i\vec{n} \cdot \vec{k} \cdot \vec{n} \cdot x / 2} \mathcal{M}_{n_b}(\vec{n} \cdot k), \quad \mathcal{M}_s(x) = \int \frac{d^4 k_s}{(2\pi)^4} e^{i\vec{k} \cdot \vec{x}} \tilde{\mathcal{M}}_s(k_s),
\]

(115)

where \( k, k_s \) is a residual or soft momentum of order \( Q^2 \). When combined with the exponentials containing \( q \) or label momenta \( \tilde{p}_{1,2} \), we can expand the exponents using \( q + \tilde{p} + \vec{p} = (q + \vec{p}) / [1 + \mathcal{O}(\lambda^2)] \), and drop the terms of order \( \lambda^2 \). Then the remaining integrals over \( n·k, k_s \) are simply the Fourier transforms of the position space matrix elements evaluated at \( x = 0 \). So, we can set \( x = 0 \) in the SCET matrix elements, and perform the \( x \) integral in Eq. \( 114 \) to enforce label momentum conservation.

In performing the \( x \) integration, we have a choice to write \( x \) and momenta in \( n_B, \tilde{n}_B \) coordinates or \( n_J, \tilde{n}_J \) coordinates. In fact, we have freedom to define the vectors \( \tilde{n}_{JB}, \tilde{n}_J \) as long as we choose them such that \( \tilde{n}_{JB}^2 = \tilde{n}_J^2 = 0 \) and \( n_J \cdot \tilde{n}_J = n_B \cdot \tilde{n}_B = 2 \). Since the measurement of \( \tau_1 \) involves measurements of both \( n_J \cdot \tilde{p} \) and \( n_J \cdot \tilde{n}_J \) components of particles’ momenta, it is convenient to choose \( \tilde{n}_B \) to be proportional to \( n_J \) and \( \tilde{n}_J \) to be proportional to \( n_B \), as we did in Eq. \( 51 \), a choice we will continue to use in what follows.

For the first pair of collinear matrix elements in Eq. \( 114 \), the \( x \) integral and accompanying phase factor for label momentum conservation take the form

\[
\int d^4 x \ e^{i(q + \tilde{p}_\perp - \tilde{p}_{1\perp}) \cdot x} = \int \frac{d\vec{n}_B \cdot x \ d\tilde{n}_B \cdot x}{2} d^2 x_\perp \exp \left( i \left( \vec{n}_B \cdot q + \omega_2 \right) \left( \frac{n_B \cdot x}{2} + \frac{n_B \cdot \tilde{n}_B \cdot x}{2} - \omega_1 \right) \right)
\]

\[
= 2(2\pi)^4 \delta(n_B \cdot q + \omega_2) \delta\left( n_B \cdot q - \frac{n_B \cdot \tilde{n}_B \cdot \omega_1}{2} \right) \delta^2(q_\perp + \vec{p}_\perp - \vec{p}_\perp^1)
\]

\[
= \frac{4}{n_B \cdot \tilde{n}_B} (2\pi)^4 \delta(n_B \cdot q + \omega_2) \delta(n_J \cdot q - \omega_1) \delta^2(q_\perp + \vec{p}_\perp - \vec{p}_\perp^1),
\]

(116)

where we used Eq. \( 51 \) to rewrite \( n_J \cdot x \) in terms of \( \tilde{n}_B \cdot x \) in the first line and to rewrite \( n_B \cdot q \) in terms of \( n_J \cdot \tilde{p}_\perp \) in the last line. Exchanging \( \omega_2 \) and \( -\omega_1 \) in Eq. \( 116 \) gives us the label momentum-conserving delta functions for the second pair of collinear matrix elements in Eq. \( 114 \). Using these delta functions to perform the \( \omega_1,2 \) and \( \vec{p}_\perp^1 \) integrals in Eq. \( 114 \), we obtain

\[
W_{\mu\nu}(x, Q^2, \tau_1) = 2(2\pi)^4(Q_J Q_B)^2 \int \frac{d^2 \vec{p}_\perp}{n_B \cdot \tilde{n}_B} \int d\tau_J d\tau_B d\tau_{sB} d\tau_s \delta(\tau_1 - \tau_J - \tau_B - \tau_s^B - \tau_s^B) \times \\left[ C_{\tilde{q}\bar{q}}^B C_{\tilde{r}q}^J \right] \delta(n_J \cdot q + \frac{n_B \cdot \tilde{n}_B \cdot \omega_1}{2} + \vec{p}_\perp - \vec{p}_\perp^1)
\]

\[
\times \left\{ \left( P_{n_B} \left| \vec{n}_{n_B} \right| \delta(J_Q B_J \bar{B}_Q) \delta(\vec{p}_\perp - \vec{P}_\perp) \chi_{n_B}^J (0) \right| P_{n_B} \right\}
\]

\[
\times \left( \left( P_{\tilde{n}_B} \left| \vec{n}_{\tilde{n}_B} \right| \delta(J_Q \bar{B}_J Q_J) \delta(\vec{p}_\perp - \vec{P}_\perp) \chi_{\tilde{n}_B}^J (0) \right| P_{\tilde{n}_B} \right) \right),
\]

(117)

where we use the change of variables \( \vec{p}_\perp^1 = 3 \vec{p}_\perp \) in 3rd and 4th lines and \( \vec{p}_\perp^1 = -\vec{p}_\perp - q_\perp \) in 5th and 6th lines. Recall that \( \tilde{n}_B \) and \( \tilde{n}_J \) are now fixed by Eq. \( 51 \). The collinear fields without labels implicitly contain a sum over all labels,
with the delta functions then fixing the labels to a single value (it is important to recall that label operators $\mathcal{P}^\mu$ acting on fields $\hat{\chi}_n,\hat{\bar{\chi}}_n$ give minus the label momentum, $-\hat{p}_\perp^\mu$ [20]). The vector $n_J$ (may) implicitly depend on the integration variable $\hat{p}_\perp$, at least for the case of the $\tau_1^\mu$ distribution, which we will deal with below. For $\tau_1^\mu$ and $\tau_2^\mu$ the vector $n_J$ is independent of $\hat{p}_\perp$. We have also indicated that the arguments of the matching coefficients $\hat{C},\bar{C}$ are both set equal to the label momenta of the fields in the collinear proton and vacuum matrix elements.

The result in Eq. (117) is organized in terms of factorized matrix elements that can now be related to known functions in SCET.

C. SCET Matrix Elements

1. Beam Functions

The proton matrix elements in Eq. (117) can be expressed in terms of generalized beam functions [93, 94] in SCET. In covariant gauges (for discussion of similar matrix elements in light-cone gauges see [03, 07]) they are defined by

$$B_q\left(\omega k^+, \frac{\omega}{P}, k_\perp^2, \mu\right) = \frac{\theta(\omega)}{\omega} \int \frac{dy}{4\pi} e^{i k^+ y/2} \langle P_n(P^-) | \hat{\chi}_n \left( y - \frac{n}{2} \right) \frac{\bar{y}}{2} \left[ \delta(\omega - \bar{n} \cdot \mathcal{P}) \frac{1}{\pi} \delta(k_\perp^2 - P_\perp^2) \hat{\chi}_n(0) \right] | P_n(P^-) \rangle, \quad (118)$$

$$B_{\bar{q}}\left(\omega k^+, \frac{\omega}{P}, k_\perp^2, \mu\right) = \frac{\theta(\omega)}{\omega} \int \frac{dy}{4\pi} e^{i k^+ y/2} \langle P_n(P^-) | \text{tr} \left( \hat{\chi}_n \right) \left( y - \frac{n}{2} \right) \left[ \delta(\omega - \bar{n} \cdot \mathcal{P}) \frac{1}{\pi} \delta(k_\perp^2 - P_\perp^2) \hat{\chi}_n(0) \right] | P_n(P^-) \rangle, \quad (119)$$

where the light-cone components of vectors are given by $V^+ = n \cdot V$ and $V^\perp = n \cdot V$. Note the dependence of the beam functions on the transverse label momentum $k_\perp$ is only on the squared magnitude $k_\perp^2$. The matrix elements in Eqs. (118) and (119) are similar to those that define parton distribution functions, but the separation of the collinear fields in the $n$ direction means there is energetic collinear radiation from the proton with virtuality $\sim \omega k^+ \gg \Lambda_{\text{QCD}}$ (assuming we are measuring $k^+$ to be large enough), which must be integrated out to match Eq. (118) onto nonperturbative PDFs (where the separation of $\hat{\chi}_n,\hat{\bar{\chi}}_n$ fields is zero). The generalized beam functions Eq. (118) are related to the ordinary beam functions originally defined in [67] by integrating over all $k_\perp$:

$$B_{q,\bar{q}}\left(\omega k^+, \frac{\omega}{P}, k_\perp^2, \mu\right) = \int d^2k_\perp B_{q,\bar{q}}\left(\omega k^+, \frac{\omega}{P}, k_\perp^2, \mu\right). \quad (119)$$

This relationship would be subtle for PDFs, where it is true for the bare matrix elements, but where after renormalization the two objects may no longer be simply related. In the beam function case both sides have the same anomalous dimension which is independent of $k_\perp$ and there is no such subtlety.

The proton matrix elements in Eq. (117) can now be expressed as

$$\langle P_n | \hat{\chi}_{nB}^{(0)}(0) \delta(Q_B \hat{\tau}_B - n_B \cdot \hat{p}_n) \delta(\hat{p}_\perp - \mathcal{P}_\perp) \hat{\chi}_{nB}^{(0)}(0) \rangle | P_n \rangle \tag{120}$$

$$= -\frac{n_B \cdot q}{4} \frac{\bar{y}^{\mu B} \delta^{\sigma \kappa}}{N_C} B_q \left( s_B \tau_B, -\frac{n_B \cdot q}{n_B \cdot P}, \hat{p}_\perp, \mu \right), \quad (121)$$

where $s_B$ is defined in Eq. (58). Now, to simplify the second argument of the beam functions, we note that

$$x = -\frac{q^2}{2q \cdot P} = -\frac{n_B \cdot q n_B \cdot q + q_\perp^2}{n_B \cdot q n_B \cdot P} = -\frac{n_B \cdot q}{n_B \cdot P} + \mathcal{O}(\lambda^2),$$

where in the second equality we used that the proton momentum $P$ is exactly along the $n_B$ direction, and in the last step used that $q_\perp$ is no bigger than $\mathcal{O}(Q\lambda^2)$. The directions $n_J$ and $n_B$ will always be chosen so that this is true, according to Eqs. (116) and (132). In other words, for events with small 1-jettiness, all the large momentum $q$ transferred into the final state is collimated along $n_J$ and $n_B$ with no $\mathcal{O}(Q)$ momentum going in a third direction. Thus to leading order in $\lambda$ the second argument of the beam functions in Eq. (120) is always just $x$. 

\begin{thebibliography}{99}

2. Jet Functions

The vacuum collinear matrix elements in Eq. (117) can be written in terms of jet functions in SCET [21], defined with transverse displacement of the jet in [67] by

\[
J_q(\omega k^+ + \omega_{\perp}, \mu) = \frac{(2\pi)^2}{N_C} \int \frac{dy^{-} d\vec{y}^{+} y^{-}/2}{2|\vec{y}|} \operatorname{tr} \left\langle 0 \left| \frac{\bar{q}}{2} \chi_n \left( y^{-} \left( y^{+} \right) \right) \delta(\omega + \vec{n} \cdot \mathcal{P}) \delta^2(\omega_{\perp} + \mathcal{P}_{\perp}) \bar{\chi}_n(0) \right| 1 \right\rangle.
\]

(122)

\[
J_{\bar{q}}(\omega k^+ + \omega_{\perp}, \mu) = \frac{(2\pi)^2}{N_C} \int \frac{dy^{-} d\vec{y}^{+} y^{-}/2}{2|\vec{y}|} \operatorname{tr} \left\langle 0 \left| \frac{\bar{q}}{2} \chi_n \left( y^{-} \left( y^{+} \right) \right) \delta(\omega + \vec{n} \cdot \mathcal{P}) \delta^2(\omega_{\perp} + \mathcal{P}_{\perp}) \bar{\chi}_n(0) \right| 1 \right\rangle.
\]

Thus the vacuum collinear matrix elements in Eq. (117) can be expressed

\[
\langle 0 | \chi_{n_{\perp}}^{\gamma} (0) (\delta(Q J_{\perp} - n_{\perp} \cdot \hat{p}_{\perp}) \bar{\delta}(\vec{n}_{\perp} \cdot q + n_{\perp} \cdot \mathcal{P}) \delta^2(q_{\perp} + \vec{p}_{\perp} + \mathcal{P}_{\perp}) \chi_{n_{\perp}}^{\gamma'} (0) | 1 \rangle
\]

(123)

\[
= \frac{n_{\perp} \cdot q}{(2\pi)^3} \frac{4}{\delta^{3\gamma}} J_q(s J_{\perp} + (q_{\perp} + \vec{p}_{\perp})^2, \mu),
\]

and

\[
\langle 0 | \bar{\chi}_{n_{\perp}}^{\gamma} (0) (\delta(Q J_{\perp} - n_{\perp} \cdot \hat{p}_{\perp}) \bar{\delta}(\vec{n}_{\perp} \cdot q + n_{\perp} \cdot \mathcal{P}) \delta^2(q_{\perp} + \vec{p}_{\perp} + \mathcal{P}_{\perp}) \chi_{n_{\perp}}^{\gamma'} (0) | 1 \rangle
\]

(124)

\[
= \frac{n_{\perp} \cdot q}{(2\pi)^3} \frac{4}{\delta^{3\gamma'}} J_{\bar{q}}(s J_{\perp} + (q_{\perp} + \vec{p}_{\perp})^2, \mu),
\]

where \(s J\) is defined in Eq. (68) and \((q_{\perp} + \vec{p}_{\perp})^2 = -(q_{\perp} + \vec{p}_{\perp})^2\).

3. Hard and Soft Functions

Using the above definitions of beam and jet functions, the hadronic tensor in Eq. (117) can be written as

\[
W_{\mu\nu}^{IJ}(x, Q^2, \tau_1) = -2(2\pi)\bar{n}_B \cdot q \bar{n}_{\perp} \cdot q (Q J_{Q} Q_{B})^2 \int \frac{d^2\vec{p}_{\perp}}{n_{\perp} \cdot n_{B}} \int d\tau_{J} d\tau_{B} d\tau_{\perp} \delta(\tau_{1} - \tau_{J} - \tau_{B} - \tau_{\perp} - \tau_{\perp})
\times S(Q J_{\perp} J_{Q} J_{B}, n_{\perp} \cdot n_{B}, \mu) J_q(s J_{\perp} + (q_{\perp} + \vec{p}_{\perp})^2, \mu)
\times \left[H_{\mu\nu}^{IJ}(q^2, n_{\perp}, n_{B}) B_q(s B_{\perp} B_{B}, x, \vec{p}_{\perp}^2, \mu) + H_{\nu\mu}^{IJ}(q^2, n_{\perp}, n_{B}) B_{\bar{q}}(s B_{\perp} B_{B}, x, \vec{p}_{\perp}^2, \mu) \right],
\]

(125)

where the hard function is defined

\[
H_{\mu\nu}^{IJ}(\vec{p}_{\perp} - \vec{p}_{\perp})^2, n_{\perp}, n_{B}) = \text{Tr} \left[ C_{I q q \nu}(\vec{p}_{\perp} \cdot \vec{p}_{\perp}) \frac{g_{\mu\nu}}{4} C_{I q q \nu}(\vec{p}_{\perp} \cdot \vec{p}_{\perp}) \frac{g_{\mu\nu}}{4} \right],
\]

and the soft function is defined

\[
S(k_{\perp}, k_{B}, q_{\perp}, q_{B}, \mu) = \frac{1}{N_C} \text{tr} \langle 0 | Y_{n_{\perp}B}^I Y_{n_{B}}(0) (\delta(k_{\perp} - n_{\perp} \cdot \hat{p}_{\perp}) \delta(k_{B} - n_{B} \cdot \hat{p}_{B}) | Y_{n_{\perp}B}^I Y_{n_{B}}(0) | 0 \rangle.
\]

(126)

To write Eq. (124) we used the equality of the quark and antiquark jet functions \(J_q, \bar{q}\) in QCD.

a. Structure of the hard functions

In Eq. (125), the matching coefficients \(C, \bar{C}\) in the hard function Eq. (126) for the vector and axial currents \(I = V, A\) take the form

\[
C_{f q q \nu}(\vec{p}_{\perp}, \vec{p}_{\perp}) = C_{V f q \nu}(\vec{p}_{\perp} - \vec{p}_{\perp})^2 \gamma_{\perp}^{\mu},
\]

\[
C_{A f q \nu}(\vec{p}_{\perp}, \vec{p}_{\perp}) = C_{A f q}(\vec{p}_{\perp} - \vec{p}_{\perp})^2 \gamma_{\perp}^{\mu} \gamma_{5},
\]

(127)

where \(\gamma_{\perp}^{\mu}\) is transverse to the directions \(n_{1,2}\) of the label momenta \(\vec{p}_{1,2}\). We have shown the index \(f\) for the quark flavor in the current explicitly. In Eq. (124) these directions are \(n_{1,2}\). The scalar coefficients \(C_{V f q \nu}, C_{A f q \nu}\) depend only on the symmetric Lorentz-invariant combination \((\vec{p}_{\perp} - \vec{p}_{\perp})^2\). Using the momentum-conserving delta function in Eq. (117), this combination takes the value \((\vec{p}_{\perp} - \vec{p}_{\perp})^2 = q^2\). Inserting Eq. (127) into Eq. (125) we obtain

\[
H_{\mu\nu}^{IJ}(q^2, n_{\perp}, n_{B}, \mu) = C_{I f q}(q^2, \mu) C_{I f q}(q^2, \mu) \text{Tr} \left[ \Gamma_{\mu} \frac{g_{\nu\lambda}}{4} \Gamma_{\nu} \frac{g_{\mu\lambda}}{4} \right],
\]

(128)

where \(\Gamma_{\mu} = \gamma_{\perp}^{\mu}\) and \(\Gamma_{\mu}^\perp = \gamma_{\perp}^{\mu} \gamma_{5}\). Thus, there are two distinct traces to take in Eq. (128):

\[
H_{\mu\nu}^{V A A}(q^2, n_{\perp}, n_{B}, \mu) = \frac{-n_{\perp} \cdot n_{B}}{4} C_{V f q}(q^2, \mu) C_{A f q}(q^2, \mu) g_{\mu\nu}^{\perp},
\]

\[
H_{\mu\nu}^{V A A}(q^2, n_{\perp}, n_{B}, \mu) = -\frac{n_{\perp} \cdot n_{B}}{4} C_{V f q}(q^2, \mu) C_{A f q}(q^2, \mu) g_{\mu\nu},
\]

(129)

where \(g_{\mu\nu}^{\perp}\) and \(\epsilon_{\mu\nu}\) are symmetric and antisymmetric tensors orthogonal to \(n_{\perp}\) and \(n_{B}\) given in Eq. (131). Hence, \(H_{V A A}^{V A A}\) and \(H_{V A A}^{V A A}\) are symmetric and antisymmetric, respectively, under exchanging \(n_{\perp}\) and \(n_{B}\).
b. **Structure of the soft function** The soft function Eq. \[126\] depends on the momenta \(k_{B,J}\) projected onto the \(n_{B,J}\) directions in the regions \(\mathcal{H}_{B,J}\), respectively. The shape of these regions in turn depends on the vectors \(q_{B,J} = \omega_{B,J} n_{B,J}/2\) in the definition of the 1-jettiness \(\tau_1\) in Eq. \[21\]. Indicating this dependence explicitly, we express the soft function Eq. \[120\] as

\[
S(k_J, k_B, q_J, q_B, \mu) = \frac{1}{N_C} \text{tr} \sum_{X_s} \left| \langle X_s | Y^t_{q_J} Y_{n_J} \rangle (0) \right|^2 \delta \left( k_J - \sum_{i \in X_s} \theta(q_B \cdot k_i - q_J \cdot k_i) n_J \cdot k_i \right) \\
\times \delta \left( k_B - \sum_{i \in X_s} \theta(q_J \cdot k_i - q_B \cdot k_i) n_B \cdot k_i \right).
\]

(130)

Note that the soft function for DIS involves the square of one incoming and one outgoing Wilson line, and hence differs from that for \(e^+e^- \to \) dijets that has two outgoing lines, and for \(pp \to L + 0\)–jets which has two incoming lines. We can relate Eq. \[130\] to the usual hemisphere soft function for DIS by generalizing an argument given in [68]. Note that the Wilson lines \(Y_n\) are invariant under rescaling of \(n\) (boost invariance):

\[
Y_{\beta n_B} = P \exp \left[ i g \int_{-\infty}^{0} ds \beta n_B \cdot A_s(\beta n_B s) \right] = P \exp \left[ i g \int_{-\infty}^{0} ds n_B \cdot A_s(n_B s) \right] = Y_{n_B},
\]

(131)

and similarly for the lines extending from 0 to \(+\infty\), \(Y_{\beta n_J} = Y_{n_J}\). Recall from Eq. \[68\] that

\[
R_J = \sqrt{\frac{\omega n_J \cdot n_B}{2 \omega_B}}, \quad R_B = \sqrt{\frac{\omega n_B \cdot n_J}{2 \omega_J}},
\]

(132)

so defining \(n'_J = n_J/R_J\) and \(n'_B = n_B/R_B\) we have \((q_B - q_J) \cdot k_i = \frac{1}{2} \omega_B R_B (n'_B - n'_J) \cdot k_i\) since \(\omega_J R_J = \omega_B R_B\). This implies that the same partitioning defined in Eq. \[130\] can be expressed with \(\theta(n'_B \cdot k_i - n'_J \cdot k_i)\) and \(\theta(n'_J \cdot k_i - n'_B \cdot k_i)\). Furthermore \(n'_B, n'_J = 2\). Thus expressing Eq. \[130\] in terms of the rescaled vectors, \(n'_J\) and \(n'_B\), we obtain

\[
S(k_J, k_B, q_J, q_B, \mu) = \frac{1}{N_C R_J R_B} \text{tr} \sum_{X_s} \left| \langle X_s | Y^t_{q_J} Y_{n_J} \rangle (0) \right|^2 \delta \left( \frac{k_J}{R_J} - \sum_{i \in X_s} \theta(n'_B \cdot k_i - n'_J \cdot k_i) n'_B \cdot k_i \right) \\
\times \delta \left( \frac{k_B}{R_B} - \sum_{i \in X_s} \theta(n'_J \cdot k_i - n'_B \cdot k_i) n'_J \cdot k_i \right) \\
= \frac{1}{R_J R_B} S_{\text{hemi}} \left( \frac{k_J}{R_J}, \frac{k_B}{R_B}, \mu \right).
\]

(133)

In the last equality we have expressed the fact that the expression in Eq. \[133\] is the same as the hemisphere soft function (up to the overall \(1/(R_J R_B)\) in front), with momentum arguments rescaled by \(R_J, R_B\) as indicated. Therefore from here on we will write all the \(\tau_1\) factorization theorems in terms of the DIS hemisphere soft function. Note that the vectors \(n'_J, n'_B\) have been rescaled from \(n_{J,B}\) such that they no longer have timelike components equal to 1 nor spacelike magnitudes equal to each other, and therefore do not partition the final states \(X_s\) into hemispheres as viewed in the original \(n_{J,B}\) frame of reference. However, the soft function in Eq. \[133\] depends on \(n'_J, n'_B\) exactly like the hemisphere soft function depends on \(n_{J,B}\) and depends on the dot product \(n'_J \cdot n'_B\), which is 2, making it equal to the hemisphere soft function. Physically, there exists a frame where \(n'_J\) and \(n'_B\) are back-to-back with equal time-like components, so that the partitioning in this frame gives hemispheres.

In the 1-jettiness cross sections below, the soft function Eq. \[133\] will always be projected symmetrically onto a function of a single variable \(k_S\), following from Eq. \[124\]:

\[
S_{\text{hemi}}(k_S, \mu) = \int dk_S^2 \delta(k_S - k_J^S - k_B^S) S_{\text{hemi}}(k_J^S, k_B^S, \mu),
\]

(134)

We will use the same name \(S_{\text{hemi}}\) for the hemisphere soft function of two variables in Eq. \[133\] and its one-variable projection Eq. \[134\], distinguishing them by the number of arguments we write.

4. **Final Form of Factorization Theorem for Hadronic Tensor**

Changing variables in the arguments of the beam, jet, and soft functions in Eq. \[124\] gives

\[
W_{\mu \nu}^{ij}(x, Q^2, \tau_1) = \int d^2 p_\perp \frac{8\pi}{n_{J,B}} \int dt_J dt_B dk_J^2 dk_B^2 \delta \left( \tau_1 - \frac{t_J}{s_J} - \frac{t_B}{s_B} - \frac{k_J^2 + k_B^2}{Q_R} \right) J_q(t_J - (q_\perp + p_\perp)^2, \mu) \\
\times S_{\text{hemi}}(k_J^2, k_B^2) \left[ H^{ij}_{q_{ij}}(q^2, n_J, n_B, \mu) B_q(t_B, x, p_\perp^2, \mu) + H^{ij}_{q_{ij}}(q^2, n_B, n_J, \mu) B_q(t_B, x, p_\perp^2, \mu) \right].
\]

(135)
We have written the arguments of the jet and beam function in terms of dimension 2 variables \( t_{J,B} \), the arguments of the soft function in terms of the total light-cone momentum \( k^L \) in region \( J \) and \( k^L \) in region \( B \), and have rewritten the transverse momentum arguments of the jet and beam functions in terms of two-vectors \( q, p \) instead of the the four-vectors \( q, p \). The constant \( Q_R \) is defined in Eq. (84) and \( s_{J,B} \) are defined in Eq. (58), and their special values for \( \tau_{1}^{a,b,c} \) are given in Table I.

5. Factorization Theorem for Cross Section

In the cross section Eq. (88), the hard function Eq. (132) gets contracted with the leptonic tensor \( L_{\mu\nu} \) in Eq. (54). The contraction of the leptonic tensor and the hard function can be performed using the tensor contractions in Eqs. (82) and (83), and can be expressed in terms of Born-level cross section and scalar hard coefficients as

\[
\sum_{I'} L_{\mu'f'f}(x, Q^2) \frac{8\pi}{n_J \cdot n_B} H_{q_{\mu'\nu'}}(q^2, n_J, n_B, \mu) = \frac{d\sigma_0}{dx dQ^2} H_q(q_J, q_B, Q^2, \mu),
\]

where the Born-level cross section is given by

\[
\frac{d\sigma_0}{dx dQ^2} = \frac{4\pi \alpha_s^2 \mu}{x^2 s^2 Q^2} \frac{q_J \cdot k' q_B \cdot k + q_J \cdot k q_B \cdot k'}{q_J \cdot q_B}. \tag{137}
\]

The hard coefficients of the quark and antiquark beam functions are

\[
H_{q,q}(q_J, q_B, Q^2, \mu) = \sum_{ff'} \left[ (C_{V,Jq} C_{V,Jq'} L_{ff'} + C_{A,Jq} C_{A,Jq'} L_{ff'}^A) \right] \tau(q_J, q_B) \left( C_{V,Jq} C_{A,Jq'} L_{ff'} + C_{A,Jq} C_{V,Jq'} L_{ff'}^A \right),
\]

where the relative minus signs for \( H_{q,q} \) come from the interchange of \( n_J, n_B \) in Eq. (130). The coefficients \( C_{V,A} \equiv C_{V,Jq} \mu \) are functions of \( q^2 \) and \( \mu \) and the leptonic coefficients \( L_{g,e} \equiv L_{g,e}(Q^2) \) given in Eq. (80). The coefficient \( \tau(q_J, q_B) \) is given by

\[
\tau(q_J, q_B) = \frac{q_J \cdot k' q_B \cdot k - q_J \cdot k q_B \cdot k'}{q_J \cdot k' q_B \cdot k + q_J \cdot k q_B \cdot k'}. \tag{139}
\]

Because the coefficient \( \tau \) is a function of scalar products of \( q_B, k \) and \( k' \) it becomes a function of \( y \) and \( Q^2 \) once \( q_B, k \) are specified as in Sec. IIIA. So, the hard coefficient \( H_{q,q} \) also is a function of \( y \) and \( Q^2 \) through the coefficient \( \tau \).

Contracting Eq. (54) with Eq. (135) then gives for the cross section Eq. (88),

\[
\frac{d\sigma}{dx dQ^2} d\tau_1 = \int d^2 p \frac{d\sigma_0}{dx dQ^2} \int dt_J dt_B dk_S \delta \left( \tau_1 - \frac{t_J}{s_J} - \frac{t_B}{s_B} - \frac{k_S}{Q_R} \right) J_q(t_J - (q_B + p)^2, \mu) S_{\text{hemi}}(k_S, \mu) \times \left[ H_q(q_J, q_B, Q^2, \mu) \mathcal{B}_q(t_B, x, p_{1,\perp}, \mu) + H_{q,B}(q_J, q_B, Q^2, \mu) \mathcal{B}_q(t_B, x, p_{1,\perp}, \mu) \right] \tag{140}
\]

where we used the projection Eq. (134) of the soft function onto a single variable.

D. Results for three versions of 1-jettiness \( \tau_{1}^{a,b,c} \)

Now we will specialize the generic factorization theorem for 1-jettiness in Eq. (140) to the specific cases \( \tau_{1}^{a,b,c} \). The discussion will be most efficient if we begin with \( \tau_{1}^{b} \).

1. 1-jettiness \( \tau_{1}^{b} \)

The reference vectors \( q_{1}^{b} = xP \) and \( q_{2}^{b} = q + xP \) in Eq. (32) are used to define the 1-jettiness \( \tau_{1}^{b} \). In any frame \( q \) can be written as \( q = q_{\perp}^{b} - q_{\perp}^{B} \), so with respect to the directions \( n_{1}^{b}, n_{2}^{b} \), the transverse component \( q_{\perp} = 0 \) so that the
The choice of \( n \) with the reference vector to which they are closest. The reference vector momentum \( p \) so

Then Eq. (140) takes the form

\[
\tau \quad \text{theorem for}
\]

Note that \( r \) is a function only of \( y \). So, the hard coefficients \( H_{q,q}(y,Q^2,\mu) \) depend on \( y \) and \( Q^2 \), and we define the hard coefficients for \( \tau_b^1 \) by \( H_{q,q}(y,Q^2,\mu) \equiv H_{q,q}(q+xP,P,Q^2,\mu) \). Therefore, using Eq. (140) the final factorization theorem for \( \tau_b^1 \) is given by

\[
\frac{d\sigma}{dx dQ^2 d\tau_b^1} = \frac{d\sigma_b^1}{dx dQ^2} \int dt d\tau_B dk_s \delta \left( \tau_b^1 - \frac{t_B}{Q^2} - \frac{k_s}{Q} \right) S_{\text{hemis}}(k_s,\mu) \times \int d^2 p_{\perp} J_q(t_J - p_{\perp}^2,\mu) \left[ H_q^b(y,Q^2,\mu)B_q(t_J,x,p_{\perp}^2,\mu) + H_q^b(y,Q^2,\mu)B_q(t_J,x,p_{\perp}^2,\mu) \right],
\]

where we used Table \( \Pi \) to substitute for \( s_{J,B},Q_R \) in Eq. (140), and where the Born-level cross section is given by

\[
\frac{d\sigma_b^0}{dx dQ^2} = \frac{2\pi a_{\text{em}}^2}{Q^4} \left[(1-y)^2 + 1\right].
\]

2. 1-jettiness \( \tau_a^1 \)

For the 1-jettiness \( \tau_a^1 \) defined in Eq. (29), the minimization inside the sum over final state particles \( i \) groups particles with the reference vector to which they are closest. The reference vector \( q_{ij}^2 \) with which the jet particles are grouped is aligned with the jet momentum \( p_J \), so that the jet has zero transverse label momentum with respect to \( n_{ij}^2 \). This direction \( n_{ij}^2 \) is the one which would minimize \( \tau_a^1 \) (to leading \( \mathcal{O}(\lambda^2) \)) with respect to variations of \( q_{ij}^2 \). A jet with momentum \( p_J = \omega_J n_J/2 + p_{\perp} + k \), where \( k \) is residual, has a mass \( m^2 = \omega_J n_J \cdot k + p_{\perp}^2 \), so \( n_J \cdot k = (m^2 - p_{\perp}^2)/\omega_J \). The choice of \( n_J \) which makes \( p_{\perp}^2 = 0 \) minimizes \( n_J \cdot k \) (note that \( p_{\perp}^2 \leq 0 \)).

The cross section for the \( \tau_a^1 \) distribution is given by Eq. (140), with \( q_B = q_B^2 \equiv xP \) and \( q_J = q_J^2 \), where \( q_J^2 \) is the vector \( q_J \) in Eq. (29) that minimizes \( \tau_a^1 \). We will write \( q_{ij}^2 \) in terms of the vector \( q_{ij}^2 = q + xP \) that was used to define the 1-jettiness \( \tau_b^1 \). Now, the vector \( q_{ij}^2 \) has a direction \( n_{ij}^2 \) and magnitude \( \omega_{ij}^b \), given by

\[
q_{ij}^2 = \omega_{ij}^b n_{ij}^2 = P_T e^y n_{ij}^2 + P_T e^{-y} n_{ij}^2 + P_T \tilde{n}_T \omega_{ij}^b,
\]

expressed in the CM frame, where \( P_T, Y \) are given by Eq. (36). With respect to \( n_{ij}^2 \) and \( n_P \), the collinear fields in the jet function matrix elements still have nonzero transverse labels. Now, for each \( \tilde{p}_{\perp}^2 \), we rotate \( n_{ij}^2 \) to a vector \( n_{ij}^2 \) so that the transverse label with respect to \( n_{ij}^2 \) is zero. This requires that the total label momenta in the two coordinate systems be equal:

\[
\omega_{ij}^b n_{ij}^2 + \tilde{p}_{\perp}^2 = \omega_{ij}^b n_{ij}^2,
\]

so \( n_{ij}^2 \) differs from \( n_{ij}^2 \) at most by a quantity of \( \mathcal{O}(\lambda) \). Now we express \( q \) in \( n_{ij}^2, n_P \) coordinates. In Eq. (140), the transverse label on the collinear fields in the jet function is \( q_{\perp} + p_{\perp} \). The \( n_{ij}^2, n_P \) coordinate system is defined as that which makes this quantity is zero, so \( q_{\perp} = -p_{\perp} \). By using \( q_{ij}^2 = -xys \) and \( n_P \cdot q = y\sqrt{s}, q \) is expressed as

\[
q = y\sqrt{s} \frac{n_{ij}^2}{n_{ij}^2 \cdot n_P} - (x\sqrt{s} + \frac{p_{\perp}^2}{y\sqrt{s}}) \frac{n_P}{2} - p_{\perp}.
\]

Then Eq. (140) takes the form

\[
\frac{d\sigma}{dx dQ^2 d\tau_a^1} = \int d^2 p_{\perp} \frac{d\sigma_b^0}{dx dQ^2} \int dt d\tau_B dk_s \delta \left( \tau_a^1 - \frac{t_J}{Q^2} - \frac{t_B}{Q^2} - \frac{k_s}{Q} \right) J_q(t_J,\mu)S_{\text{hemis}}(k_s,\mu) \times \int H_q(q_{ij}^a,q_B^a,Q^2,\mu)B_q(t_J,x,p_{\perp}^2,\mu) + H_q(q_{ij}^a,q_B^a,Q^2,\mu)B_q(t_J,x,p_{\perp}^2,\mu),
\]

where we used Table \( \Pi \) to substitute for \( s_{J,B} \) and \( Q_R \).
The generalized beam functions appearing here explicitly depend on $p_\perp$. The vector $n^a_\perp$ appearing in $q^a_\perp$ implicitly depends on $p_\perp$. Now, $n^a_\perp$ differs from $n^b_\perp$ (which is independent of $p_\perp$) by a quantity of order $\lambda$. Here we can expand the hard and soft functions and the Born cross section around $n^a_\perp = n^b_\perp + O(\lambda)$ and drop the power corrections in $\lambda$. This makes everything in Eq. (147) independent of $p_\perp$, except for the generalized beam function. The integral over $p_\perp$ then turns the generalized beam function into the ordinary beam function Eq. (119). Thus the final factorization theorem for the $\tau^a_1$ cross section is

$$
\frac{d\sigma}{dx\,dQ^2\,d\tau^a_1} = \frac{d\sigma^b_0}{dx\,dQ^2} \int dt_J dt_B dk_S \delta \left( \tau^a_1 - t_J - \frac{t_B}{Q^2} - \frac{k_S}{Q} \right) J_q(t_J, \mu) S_{\text{hemi}}(k_S, \mu) \times \left[ H^b_q(y, Q^2, \mu) B_q(t_B, x, \mu) + H^b_q(y, Q^2, \mu) B_q(t_B, x, \mu) \right],
$$

(148)

where the Born cross section is given by Eq. (143) and $H^b_q(y, Q^2)$ is given by Eq. (138) with $r$ in Eq. (111). The hard and soft functions in Eq. (148) are the same as those in Eq. (142) for $r$. Eq. (148) differs from Eq. (142) in that the jet and beam functions are no longer convolved together in the transverse momentum $p_\perp$. The 1-jettiness $\tau^a_1$ is proportional to the invariant mass of the jet, while $\tau^a_1$ measures the projection of the jet momentum onto the fixed axis $q^a_\perp = q + xP$. The emission of ISR with transverse momentum $p_\perp$, causing a shift in the jet momentum by the same amount due to momentum conservation, will not change the mass of the jet, but it will change the projection of the jet momentum onto the $q^a_\perp$ axis. Thus $\tau^a_1$ involves no convolution over $p_\perp$, while $\tau^a_1$ does.

3. 1-jettiness $\tau^a_1$

For the 1-jettiness $\tau^a_1$, the directions $n_J$ and $n_B$ are along the electron and proton directions, respectively:

$$
n_J = n_e, \quad n_B = n_p,
$$

(149)

where $n_{e,p}$ are the light-cone directions of $q_J = k = \omega_i n_e/2$ and $q_B = P = \omega_p n_p/2$. In the CM frame, $n_{e,p}$ are back-to-back, $n_e = n_z$ and $n_p = \bar{n}_z$. In this frame, $q$ is given by

$$
q = y\sqrt{s} \frac{n_2}{2} - xy\sqrt{s} \frac{\bar{n}_z}{2} + q_\perp,
$$

(150)

where $q_\perp = Q\sqrt{1 - y \bar{n}_z}$. Let us consider for a moment the power counting of the argument of the jet function in Eq. (140) with $\tau_1$. The requirement that $q_\perp \sim Q \lambda$ requires that $1 - y \sim \lambda^2$. This is ensured by measuring $\tau^a_1$ to be $O(\lambda^2)$. The argument of the jet function (call it $m_2^2$) in the factorization theorem Eq. (140) for $\tau_1$ is

$$
m_2^2 = t_J - (1 - y)Q^2 - 2Q \sqrt{1 - y} \frac{\bar{n}_z}{2} \cdot p_\perp.
$$

(151)

Now, the jet function will be proportional to a theta function $\theta(m_2^2)$, requiring $m_2^2 > 0$. Measuring $\tau^a_1$ to be of order $\lambda^2$ and therefore forcing $t_J$ to be of order $Q^2 \lambda^2$ then enforces that $1 - y \sim \lambda^2$. Then, we can set $y = 1$ to leading order everywhere in Eq. (140) except in the argument of the jet function. In terms of $x$, using the relation $xys = Q^2$, requiring $y \lesssim 1$ is equivalent to requiring $x \gtrsim Q^2/s$, which sets a lower bound on $x$.

The normalization constants $s_{J,B}, Q_B$ in Eq. (140) are given for $\tau^a_1$ in Table III. The Born-level cross section and the coefficient $r(q_J, q_B)$ in the hard coefficients reduce to

$$
\frac{d\sigma^a_0}{dx\,dQ^2} = \frac{2\pi\alpha^2_{em}}{Q^4}, \quad r(k, P) = 1,
$$

(152)

where we see the Born cross section is now Eq. (133) in the limit $y \rightarrow 1$. This happens because the expression Eq. (132) is evaluated with $q_J = k$, which is the actual jet direction only near $y \rightarrow 1$. The hard coefficient is now independent of $x, y$ and depends only on $Q^2$: $H^c_{q,q}(Q^2, \mu) = H_{q,q}(k, P, Q^2, \mu)$. From Eq. (140) the final factorization theorem for the $\tau^a_1$ cross section is then given by

$$
\frac{d\sigma}{dx\,dQ^2\,d\tau^a_1} = \frac{d\sigma^a_0}{dx\,dQ^2} \int d^2 p_\perp \int dt_J dt_B dk_S \delta \left( \tau^a_1 - \frac{t_J}{Q^2} - \frac{t_B}{xQ^2} - \frac{k_S}{\sqrt{x}Q} \right) J_q(t_J - (q_\perp + p_\perp)^2, \mu) S_{\text{hemi}}(k_S, \mu) \times \left[ H^c_q(Q^2, \mu) B_q(t_B, x, p_\perp^2, \mu) + H^c_q(Q^2, \mu) B_q(t_B, x, p_\perp^2, \mu) \right].
$$

(153)

This is like the $\tau^a_1$ cross section Eq. (142) in that the jet and beam functions are convolved in the transverse momentum $p_\perp$ of ISR, but in this case the momentum transfer $q$ itself has a nonzero transverse momentum with respect to the
light-cone directions $n_{\nu,P}$. This will make the evaluation of the $p_\perp$ integral considerably more involved than in the $\tau_1^q$ cross section Eq. (142).

VI. FIXED-ORDER PREDICTIONS AT $O(\alpha_s)$

In this Section we evaluate to $O(\alpha_s)$ the predictions of the factorization theorems for the cross sections differential in the different versions of 1-jettiness in Eq. (142) for $\tau_1^q$, and Eq. (148) for $\tau_2^q$ and Eq. (153) for $\tau_1^q$. These formulas correctly predict the singular terms at small $\tau_1$ in the fixed-order differential cross section, although they have to be resummed to all orders in $\alpha_s$ to accurately predict the behavior at small $\tau_1$. We will do this in the next Section. Also, for the predictions to be correct for large $\tau_1$, they would have to be matched onto $O(\alpha_s)$ and $O(\alpha_s^2)$ fixed-order full QCD calculations, an analysis we defer to future work. Nevertheless we can estimate the size of these matching corrections by comparing our predictions integrated up to large $\tau_1$ to the known total QCD cross section $\sigma(x,Q^2)$ at $O(\alpha_s)$, which we will do in Sec. VII.

A. Hard Function

At $O(\alpha_s)$, the matching coefficients $C_{\nu f q}, C_{Af q}$ for the vector and axial currents Eq. (138) that appear in the hard coefficient in Eq. (134) are equal and diagonal in flavor, and were calculated in [32, 91]:

$$C_{\nu f q}(q^2) = C_{Af q}(q^2) = \delta_{f q} C(q^2),$$

$$C(q^2) = 1 + \frac{\alpha_s(\mu) CF}{2\pi} \left( -\ln^2 \frac{\mu^2}{q^2} - 3 \ln \frac{\mu^2}{Q^2} - 8 + \frac{\pi^2}{6} \right).$$

For DIS recall $q^2 = -Q^2$. Then, the hard coefficients $H_{q,\bar{q}}$ in the cross section Eq. (140) are given to $O(\alpha_s)$ by

$$H_{q,\bar{q}}(q, j_B, Q^2, \mu) = H(Q^2, \mu)L_{q,\bar{q}}(q, j_B, Q^2),$$

where we have defined the universal SCET 2-quark hard coefficient,

$$H(Q^2, \mu) \equiv |C(q^2, \mu)|^2$$

$$= 1 + \frac{\alpha_s(\mu) CF}{2\pi} \left( -\ln^2 \frac{\mu^2}{Q^2} - 3 \ln \frac{\mu^2}{Q^2} - 8 + \frac{\pi^2}{6} \right),$$

and the factor containing the components of the leptonic tensor Eq. (34),

$$L_{q,\bar{q}}(q, j_B, Q^2) = L_{q,\bar{q}}^{VV} + L_{q,\bar{q}}^{AA} \mp r(q, j_B)(L_{q,\bar{q}}^{VA} + L_{q,\bar{q}}^{AV}),$$

where $r(q, j_B)$ was defined in Eq. (139).

1. $\tau_1^{a,b}$ cross sections

For the $\tau_1^{a,b}$ cross sections Eqs. (142) and (148), $q_B = xP$ and $q_J = q + xP$, so that $r(q, j_B)$ is given by

Eq. (141). Then the leptonic factor $L_{q,\bar{q}}$ in Eq. (157) becomes

$$L_{q,\bar{q}}(q, j_B, Q^2)$$

$$= L_{q,\bar{q}}^{VV} + L_{q,\bar{q}}^{AA} \mp r(q, j_B)(L_{q,\bar{q}}^{VA} + L_{q,\bar{q}}^{AV})$$

$$= Q^2 - 2Qqv_{\nu e} + \frac{(v^2 + a^2)(v^2 + a^2)}{1 + m_Z^2/Q^2} + 2y(2 - y) \alpha_s a_e^2 (1 + m_Z^2/Q^2) - 2vq v_e.$$

2. $\tau_1^q$ cross section

For the $\tau_1^q$ cross section Eq. (153), $q_J = k$ and $q_B = P$, the electron and proton momenta, respectively. Then $r(k, P) = 1$ in Eq. (139), and the leptonic factor $L_{q,\bar{q}}$ in Eq. (157) becomes

$$L_{q,\bar{q}}(k, P, Q^2) = L_{q,\bar{q}}^{VV} + L_{q,\bar{q}}^{AA} \mp r(k, P)(L_{q,\bar{q}}^{VA} + L_{q,\bar{q}}^{AV})$$

$$= Q^2 - 2Qy v_{\nu e} \pm \alpha_s a_e^2$$

$$\pm \frac{(v^2 + a^2)(v^2 + a^2)}{1 + m_Z^2/Q^2} + 4vq v_e a_e.$$

B. Soft Function

The soft function $S_{\text{hemi}}(k_s, \mu)$ that appears in the cross sections Eqs. (142), (148), and (153) is given by Eqs. (133) and (134). For $e^- e^+ \rightarrow$ d-jets, $S_{\text{hemi}}$ is known at $O(\alpha_s)$ [94] and $O(\alpha_s^2)$ [60, 61, 100]. At 1-loop order the soft function is the same for DIS. Beginning at 2-loop order, the finite part of the soft function for DIS could possibly differ due to switching incoming and outgoing Wilson lines, but the anomalous dimensions and thus the logs are the same.

To $O(\alpha_s^3)$, the soft function Eq. (133) takes the form:

$$S_{\text{hemi}}(k_s, \mu) = \delta(k_s^J)\delta(k_s^B) + S^{(1)}(k_s^J, \mu)\delta(k_s^B) + S^{(1)}(k_s^J, \mu)S^{(1)}(k_s^B, \mu),$$

where

$$S^{(1)}(k_s, \mu) = \frac{\alpha_s CF}{4\pi} \left( \frac{\pi^2}{6} \delta(k_s) - \frac{8}{\mu} \left[ \frac{\theta(k_s) \ln(k_s/\mu)}{k_s/\mu} \right]_++ \right),$$

and the projection Eq. (134) is then given to $O(\alpha_s)$ by

$$S_{\text{hemi}}(k_s, \mu) = \delta(k_s) + 2S^{(1)}(k_s, \mu).$$
It has previously been observed that the sizes $R_{i,B}$ of the regions $\mathcal{H}_{i,B}$ to which soft radiation is confined enter the arguments of the logs in the soft function \(^{81, 101, 102}\), which is due to changing the effective scale at which the soft modes live \(^{103}\).

---

C. Jet Function

The jet function Eq. (122) is given to $\mathcal{O}(\alpha_s)$ by \(^{104, 105}\)

$$J_{q,q}(t, \mu) = \delta(t) + \frac{\alpha_s(\mu)C_F}{4\pi}\left\{ (7 - \pi^2)\delta(t) - \frac{3}{\mu^2}\left[ \frac{\theta(t)}{t/\mu^2}\right]_+ + \frac{4}{\mu^2}\left[ \frac{\theta(t)\ln(t/\mu^2)}{t/\mu^2}\right]_+ \right\}.$$ (163)

It is in fact known to two-loop order \(^{105}\) and its anomalous dimension to three loops \(^{38}\).

---

D. Beam Functions

1. Generalized Beam Functions

The generalized beam functions in Eq. (113) can be matched onto ordinary PDFs, defined in SCET as \(^{22}\):

$$f_q(\omega' / P^-, \mu) = \theta(\omega') \left\langle \bar{P}_n(P^-) \mid \bar{\chi}_n(0) \frac{i}{2} \delta(\omega' - \bar{n} \cdot P) \chi_n(0) \right\rangle \left\langle P_n(P^-) \mid f_n(\mu) \right\rangle,$$ (164a)

$$f_{\bar{q}}(\omega' / P^-, \mu) = \theta(\omega') \left\langle P_n(P^-) \mid \bar{\chi}_n(0) \frac{i}{2} \delta(\omega' - \bar{n} \cdot P) \chi_n(0) \right\rangle \left\langle P_n(P^-) \mid f_n(\mu) \right\rangle.$$ (164b)

The matching result is \(^{68, 94}\):

$$B_i(t, x, k_{\perp}^2, \mu) = \sum_j \int_x^1 \frac{d\xi}{\xi} I_{ij}(t, x, k_{\perp}^2, \mu) f_j(\xi, \mu) \left[ 1 + \mathcal{O}\left( \frac{\Lambda_{QCD}^2}{t}, \frac{\Lambda_{QCD}^2}{k_{\perp}^2} \right) \right],$$ (165)

where $i, j = q, \bar{q}, g$. This expansion is valid for perturbative beam radiation satisfying $t, k_{\perp}^2 \gg \Lambda_{QCD}^2$. At tree level, $I_{ij}^{(0)}(t, z, k_{\perp}^2, \mu) = (1/\pi)\delta_{ij}\delta(t(1 - z)\delta(k_{\perp}^2))$, leading to $B_{ij}^{(0)}(t, x, k_{\perp}^2, \mu) = (1/\pi)\delta(t)\delta(k_{\perp}^2)f_i(x, \mu)$.

To $\mathcal{O}(\alpha_s)$, the nonzero matching coefficients in the generalized quark beam function were computed in \(^{68, 94}\), and we use the results from \(^{94}\):

$$I_{qq}(t, z, k_{\perp}^2, \mu) = \frac{1}{\pi} \delta(t)\delta(1 - z)\delta(k_{\perp}^2) + \frac{\alpha_s(\mu)C_F}{2\pi^2} \theta(z) \left\{ \frac{2}{\mu^2} \left[ \frac{\theta(t)\ln(t/\mu^2)}{t/\mu^2} \right]_+ + \delta(1 - z)\delta(k_{\perp}^2) \right\} +$$

$$+ \frac{1}{\mu^2} \left[ \frac{\theta(t)}{t/\mu^2} \right]_+ + \left[ P_{qq}(z) - \frac{3}{2} \right](1 - z) \delta(k_{\perp}^2) + \delta(t)\delta(k_{\perp}^2) \left\{ \theta(1 - z)\ln(1 - z) \right\}_+ + \left( 1 + z^2 \right) - \frac{\pi^2}{6}\delta(1 - z) + \theta(1 - z) \left( 1 - z - \frac{1 + z^2}{1 - z} \ln z \right) \right\},$$ (166a)

$$I_{gg}(t, z, k_{\perp}^2, \mu) = \frac{\alpha_s(\mu)T_F}{2\pi^2} \theta(z) \left\{ \frac{1}{\mu^2} \left[ \frac{\theta(t)}{t/\mu^2} \right]_+ P_{gg}(z)\delta(k_{\perp}^2 - \frac{(1 - z)t}{z}) + \delta(t)\delta(k_{\perp}^2) \left[ P_{gg}(z) \frac{1 - z}{z} + 2\theta(1 - z)z(1 - z) \right] \right\},$$ (166b)

where $P_{qq, gg}$ are the $q \to qg$ and $g \to qq$ splitting functions,

$$P_{qq}(z) = \left[ \frac{\theta(1 - z)}{1 - z} \right]_+ + \frac{3}{2}\delta(1 - z) = \left[ \theta(1 - z) \frac{1 + z^2}{1 - z} \right]_+,$$ (167a)

$$P_{gg}(z) = \theta(1 - z)[(1 - z)^2 + z^2].$$ (167b)

They appear in the anomalous dimensions of the PDFs, which to all orders obey

$$\mu \frac{d}{d\mu} f_i(\xi, \mu) = \sum_j \int \frac{d\xi'}{\xi'} \gamma_{ij}(\xi, \mu) f_j(\xi', \mu).$$ (168)
At \( O(\alpha_s) \) the anomalous dimensions for the quark PDF are

\[
\gamma_{qq}^f(z, \mu) = \frac{\alpha_s(\mu) C_F}{\pi} \theta(z) P_{qq}(z), \quad \gamma_{qq}^f(z, \mu) = \frac{\alpha_s(\mu) T_F}{\pi} \theta(z) P_{qg}(z).
\] (169)

### 2. Ordinary Beam Functions

The ordinary beam functions Eq. (119) satisfy the matching condition \[67, 106, 107\]:

\[
B_i(t, x, \mu) = \sum_j \int_x^1 \frac{d\xi}{\xi} I_{ij}(t, \frac{x}{\xi}, \mu) f_j(\xi, \mu) \left[ 1 + O\left( \frac{\Lambda^2_{\text{QCD}}}{t} \right) \right],
\] (170)

where at tree level \( I_{ij}^{(0)}(t, z, \mu) = \delta_{ij} \delta(t) \delta(1 - z) \), leading to \( B_i^{(0)}(t, x, \mu) = \delta(t) f_i(x, \mu) \). To \( O(\alpha_s) \), the matching coefficients in the quark beam function are given by integrating Eq. (160) over \( k_\perp \) \[67, 107\]:

\[
I_{qq}(t, z, \mu) = \delta(t) \delta(1 - z) + \frac{\alpha_s(\mu) C_F}{2\pi} \theta(z) \left\{ \frac{2}{\mu^2} \left[ \frac{\theta(t) \ln(t/\mu^2)}{t/\mu^2} \right] \right\}
\]

\[
+ \delta(t) \left[ \frac{\theta(1 - z) \ln(1 - z)}{1 - z} \right] + \frac{\pi^2}{6} \delta(1 - z) + \theta(1 - z) \left( 1 - z \right) \left( 1 - \frac{1 + z^2}{1 - z} \ln z \right) \right\},
\] (171a)

\[
I_{qg}(t, z, \mu) = \frac{\alpha_s(\mu) T_F}{2\pi} \theta(z) \left\{ \frac{1}{\mu^2} \left[ \frac{\theta(t)}{t/\mu^2} \right] \right\}
\]

\[
+ P_{qg}(z) + \delta(t) \left[ P_{qg}(z) \ln \frac{1 - z}{z} + 2\theta(1 - z) z(1 - z) \right] \right\}.
\] (171b)

### E. Dijet Cross Section

We can now form the SCET predictions for the \( \tau_1 \) cross section Eq. (140) to \( O(\alpha_s) \) by plugging in the \( O(\alpha_s) \) expressions for the hard function given by Eqs. (154), (156), and (157), the soft function given by Eqs. (160) and (161), the jet function given by Eq. (163), and the generalized beam function given by Eqs. (165) and (166). It is convenient to express the result in terms of the \textit{cumulant} \( \tau_1 \) distribution, defined by

\[
\sigma_c(x, Q^2, \tau_1) = \frac{1}{\sigma_0} \int_0^{\tau_1} \frac{d\tau'_1}{dx dQ^2} \frac{d\sigma}{d\tau'_1},
\] (172)

where \( \sigma_0 \) is the Born cross section defined in Eq. (137). We will give here the results for the \( \tau_1^{a,b} \) cumulants at \( O(\alpha_s) \). The more complicated results for the \( \tau_1^c \) and generic \( \tau_1 \) cumulants are given in App. \[69\]

#### 1. \( \tau_1^a \) cross section

Plugging in the \( O(\alpha_s) \) results for the hard function given by Eqs. (154), (156), and (158), the soft function given by Eqs. (160) and (161) with \( s, j, b \) and \( Q_R \) given in Table [1] the jet function given by Eq. (163), and the ordinary beam function by Eq. (170) into the \( \tau_1^a \) cross section Eq. (148), we obtain for the \( \tau_1^a \) cumulant given by Eq. (172) in the CM frame:

\[
\sigma_c(x, Q^2, \tau_1^a) = \theta(\tau_1^a) \int_x^1 \frac{dz}{z} \left[ L_{qq}^a(x, Q^2) f_q(x/z, \mu) + L_{qg}^a(x, Q^2) f_{qg}(x/z, \mu) \right]
\]

\[
\times \left\{ \delta(1 - z) \left[ 1 - \frac{\alpha_s C_F}{4\pi} \left( 9 + \frac{2\pi^2}{3} + 6 \ln \tau_1^a + 4 \ln^2 \tau_1^a \right) \right] \right\}
\]

\[
+ \frac{\alpha_s C_F}{2\pi} \left[ L_1(1 - z)(1 + z^2) + \theta(1 - z) \left( 1 - z - \frac{1 + z^2}{1 - z} \ln z \right) \right] \left[ \ln \left( \frac{Q^2 \tau_1^a}{z} \right) P_{qg}(z) \right]
\]

\[
+ \frac{\alpha_s T_F}{2\pi} \theta(\tau_1^a) \left[ L_{qq}^a(x, Q^2) + L_{qg}^a(x, Q^2) \right] \int_x^1 \frac{dz}{z} f_q(x/z, \mu) \left[ \ln \left( \frac{Q^2 \tau_1^a}{z} \right) P_{qg}(z) + 2\theta(1 - z) z(1 - z) \right].
\] (173)
The factorization scale $\mu$ still appears on the right-hand side of the equation, though the cross section is in fact independent of $\mu$. The $\mu$-dependence in the PDFs on the first line is cancelled by the $\mu$-dependence in the logs multiplying the splitting functions on the third and final lines to $O(\alpha_s)$. The residual $\mu$-dependence is $O(\alpha_s^2)$ and would be cancelled by the higher-order corrections.

2. $\tau_1^b$ cross section

The $\tau_1^b$ cross section is nearly identical to the $\tau_1^a$ cross section except for the presence of the $p_\perp$-dependent generalized beam function in Eq. (142) instead of the ordinary beam function. The effect of the nontrivial $p_\perp$-dependent terms in the generalized beam function Eq. (165) is simply to multiply the arguments of the $\mu$-dependent logs in Eq. (173) by $z$, giving the simple relation

$$\sigma_c(x, Q^2, \tau_1^b) = \sigma_c(x, Q^2, \tau_1^a) \bigg|_{\tau_1^a \to \tau_1^b} + \theta(\tau_1^b) \frac{\alpha_s}{2\pi} \int_z^1 \frac{dz}{z} \ln z \left\{ C_F \left[ L^a_q(Q^2)f_q(x/z, \mu) + L^a_\bar{q}(Q^2)f_{\bar{q}}(x/z, \mu) \right] P_{qq}(z) + T_F \left[ L^a_q(Q^2) + L^a_\bar{q}(Q^2) \right] P_{q\bar{q}}(z) f_q(x/z, \mu) \right\}, \quad (174)$$

In App. C we give the $O(\alpha_s)$ $\tau_1^b$ cross section. In the next section we resum the large logarithms of $\tau_1^{a,b,c}$ that appear in these fixed-order expansions to all orders in $\alpha_s$ to NNLL accuracy.

VII. RESUMMED PREDICTIONS FOR $\tau_1$ CROSS SECTIONS

The fixed-order predictions for the $\tau_1$ cross sections presented in the previous Section contain logarithms of $\tau_1$ which grow large in the limit $\tau_1 \to 0$ and must be resummed to all orders in $\alpha_s$ to yield accurate predictions for small $\tau_1$. In this Section we use the factorization theorem Eq. (140) for the $\tau_1$-jetiness cross section and its specialized cases Eqs. (142), (148), and (153) for $\tau_1^b$, $\tau_1^a$, $\tau_1^c$ to predict the cross sections differential in $\tau_1$ to next-to-next-to-leading logarithmic (NNLL) accuracy, estimate the perturbative uncertainty by appropriate scale variations, and discuss power corrections due to hadronization, including their universality and impact in the tail and peak regions.

A. Perturbative Resummation to NNLL

The hard, jet, beam, and soft functions in Eq. (140) obey renormalization group (RG) evolution equations whose solutions allow us to resum large logarithms of ratios of the separated hard, jet, beam, and soft scales. These solutions allow us to express any of these functions $G = \{H, J, B, S\}$ at one scale $\mu$ which contains logs of $\mu$ over some scale $Q_G$ in terms of the function evaluated at a different scale $\mu_G \sim Q_G$ where the logs are small. The hard function $H(Q^2, \mu)$ obeys the RG equation

$$\mu \frac{d}{d\mu} H(Q^2, \mu) = \gamma_H(\mu) H(Q^2, \mu), \quad (175)$$

where the anomalous dimension $\gamma_H$ has the form

$$\gamma_H(\mu) = \Gamma_H[\alpha_s(\mu)] \ln \frac{Q^2}{\mu^2} + \gamma_H[\alpha_s(\mu)], \quad (176)$$

with a cusp piece $\Gamma_H[\alpha_s] = 2\Gamma^c_{\text{cusp}}$ and a non-cusp piece $\gamma_H[\alpha_s]$ (which is conventionally denoted by the same symbol as the total anomalous dimension). Their expansions in $\alpha_s$ are given below in Eq. (182) and Eqs. (D28) and (D29). Similarly the jet and beam functions which are both functions of a dimension-2 variable $t$ obey RG equations of the form

$$\mu \frac{d}{d\mu} G(t, \mu) = \int dt' \gamma_G(t-t', \mu) G(t', \mu), \quad (177)$$

where the anomalous dimension $\gamma_G$ takes the form

$$\gamma_G(t, \mu) = \Gamma_G[\alpha_s(\mu)] \frac{1}{\mu^2} \left[ \frac{\theta(t/\mu^2)}{t/\mu^2} \right]_+ + \gamma_G[\alpha_s(\mu)] \delta(t), \quad (178)$$

where here $G = \{J, B\}$, and the plus distribution is defined in App. C. The cusp pieces $\Gamma_{J,B} = -2\Gamma^c_{\text{cusp}}$ and non-cusp pieces $\gamma_{J,B}$ of the jet and beam anomalous dimensions are given in Eqs. (D28) and (D30). The beam function also depends on $x$ and the generalized beam function also depends on $p_\perp^2$, but they do not change the structure of the RG equation Eq. (177).

Finally, the soft function in Eq. (133) obeys the RG equation

$$\mu \frac{d}{d\mu} S_{\text{hemi}}(k_j, k_B, \mu) = \int dk'_j dk'_B \times \gamma_S(k_j - k'_j, k_B - k'_B, \mu) S_{\text{hemi}}(k'_j, k'_B, \mu), \quad (179)$$

where the anomalous dimension factorizes into the form

$$\gamma_S(k_j, k_B, \mu) = \gamma_S(k_j, \mu) \delta(k_B) + \gamma_S(k_B, \mu) \delta(k_j), \quad (180)$$

which is required by $\mu$-independence of the total cross section Eq. (130). Each piece $\gamma_S(k, \mu)$ takes the
form
\[ \gamma_S(k, \mu) = 2\Gamma_S[\alpha_s(\mu)] \frac{1}{\mu} \left[ g(k/\mu) \right] + \gamma_S[\alpha_s(\mu)] \delta(k), \]
(181)
where \( \Gamma_S = \Gamma_{\text{cusp}} \), and the non-cusp piece is given by \( \gamma_S = -\gamma_H/2 - \gamma_J \).

The cusp and non-cusp pieces of the anomalous dimensions of all the functions above all have perturbative expansions in \( \alpha_s \):
\[ \Gamma_G[\alpha_s] = \sum_{n=0}^{\infty} \Gamma^n_G \left( \frac{\alpha_s}{4\pi} \right)^{n+1} = \gamma_G[\alpha_s], \]
(182)
which defines the coefficients \( \Gamma^n_G, \gamma_G^n \). Furthermore, the cusp pieces of the anomalous dimension are proportional to the same cusp anomalous dimension \( \Gamma^n_{\text{cusp}}[\alpha_s] \), whose perturbative expansion along with the non-cusp anomalous dimensions are given in App. D. The explicit solutions to the RG equations for the hard, jet, beam, and soft functions individually are given in App. D.

The solutions of the RG equations Eqs. (175), (177), and (179) allow us to express the hard, jet, beam, and soft functions at any scale \( \mu \) in terms of their values at different scales \( \mu_{H,J,B,S} \) where logarithms of \( \mu_G/Q_G \) in their perturbative expansions are small. There are a different conventional ways in the literature to express the resummed cross section in terms of the solutions for hard, jet, beam, and soft functions to the RG equations. One method \[14, 76\] performs the exact inverse transform back from Fourier space, and carries out analytically the convolution of all the evolution factors and the fixed-order jet/beam and soft scales \( \mu_{H,J,B,S} \) in Eqs. (154) and (114). The evolution kernels \( U_{J,B,S} \) in Eqs. (154) and (115) are plus distributions, and each fixed order function can also be written as a sum of plus distributions as in App. D.1. Thus, the resummed cross section contains numerous convolutions of plus distributions \( \mathcal{L}^n, \mathcal{L}_n \), which we can compute by repeatedly applying the plus distribution convolution identity in Eq. (157). The cross section then gets written as a resummation factor times sums of products of coefficients called \( V \) in App. E.2 and \( J_n, I_{n,q,q}, \) and \( S_n \) in App. E.1.

In this section we give just the final results for the RG improved cross sections for the 1-jettinesses \( \tau_1^{a,b,c} \) using the formalism of \[14, 76\]. We will express the results in terms of the cumulant \( \tau_1^{a,b,c} \) distributions:
\[ \sigma_c(x, Q^2, \tau_1) = \frac{1}{\sigma_0} \int_0^{\tau_1} d\tau_1' \frac{d\sigma}{dxdQ^2 d\tau_1'}, \]
(183)
where we note that \( \sigma_c \) is dimensionless due to the division by \( \sigma_0 \). The differential cross section can be obtained by taking the derivative of \( \sigma_c(x, Q^2, \tau_1) \) with respect to \( \tau_1 \). Care must be exercised in this procedure because \( \sigma_c \) also depends on \( \tau_1 \) dependent jet/beam and soft scales in the factorization theorem Eq. (140), \( \sigma_c(x, Q^2, \tau_1, \mu_i(\bar{\tau}_1)) \).

The appropriate procedure is to use, for \( \epsilon \to 0 \),
\[ \frac{d\sigma}{d\tau_1} = \frac{\sigma_c(x, Q^2, \tau_1 + \epsilon, \mu_i(\bar{\tau}_1)) - \sigma_c(x, Q^2, \tau_1 - \epsilon, \mu_i(\bar{\tau}_1))}{2\epsilon}, \]
(184)
where \( d\sigma/d\tau_1 = (1/\sigma_0) d\sigma/d\tau_c \). See Ref. [14] for further discussion of this point.

1. \( \tau_1^{a,b} \) cross sections

The cross section in Eq. (140) is expressed as a convolution of jet, beam, and soft functions in momentum space. To resum the large logs, each function is RG evolved from a scale where the logs are small, an operation which is in the form of a convolution of an RG evolution kernel and the fixed order function as in Eqs. (D13) and (D14). The evolution kernels \( U_{J,B,S} \) in Eqs. (D13) and (D15) are plus distributions, and each fixed order function can also be written as a sum of plus distributions as in App. D.1. Thus, the resummed cross section contains numerous convolutions of plus distributions \( \mathcal{L}^n, \mathcal{L}_n \), which we can compute by repeatedly applying the plus distribution convolution identity in Eq. (157). The cross section then gets written as a resummation factor times sums of products of coefficients called \( V \) in App. E.2 and \( J_n, I_{n,q,q}, \) and \( S_n \) in App. E.1.

The resummed \( \tau_1^a \) and \( \tau_1^b \) cross sections in Eqs. (142) and (143), obtained from RG evolution of the hard, jet, beam, and soft functions, are given by:

\[
\sigma_c(x, Q^2, \tau_1) = \frac{e^{K_0}}{\Gamma(1 + \Omega)} \left( \frac{Q}{\mu_H} \right)^{\eta_H(\mu_H, \mu)} \frac{\eta_B(\mu_B, \mu)}{\mu_B^{\eta_B(\mu_B, \mu)}} \frac{\eta_J(\mu_J, \mu)}{\mu_J^{\eta_J(\mu_J, \mu)}} \frac{\eta_s(\mu_s, \mu)}{\mu_s^{\eta_s(\mu_s, \mu)}} \times \left[ \sum_j \int_{x_1}^{1} \frac{dz}{z} f_j(x/z, \mu_B) \left[ W_{qj}(z, \tau) + \Delta W_{qj}(z) \right] + (q \leftrightarrow \bar{q}) \right],
\]
(185a)

\[
W_{qj}(z, \tau) = H(Q^2, \mu_H) \sum_{a, b, c, \bar{q}, \bar{q}} J_{a,b,c} \left[ \alpha_s(\mu), \frac{\tau Q^2}{\mu_H^{(n_1+1)n_2}} \right] \left[ \alpha_s(\mu_B), \frac{\tau Q^2}{\mu_B^{(n_1+1)n_2}} \right] \left[ \alpha_s(\mu_s), \frac{\tau Q^2}{\mu_s^{(n_1+1)n_2}} \right].
\]
where the individual evolution kernels $K_\eta$ and $(D_{15})$. Note that $I$ in these fixed-order factors can be minimized by choosing the canonical scales $W_\tau$ is not too close to zero and not too large so that the logs of to an antiquark. In Eq. (185a) the exponent is a resummation factor that resums the large logs and the terms $W_{qj}$ and $\Delta W_{qj}$ are fixed-order factors which do not contain large logs. The evolution kernels $\mathcal{K}$ and $\Omega$ are given by

$$\begin{align*}
\mathcal{K} &= \mathcal{K}(\mu_H, \mu_J, \mu_B, \mu_S, \mu) = K_H(\mu_H, \mu) + K_J(\mu_J, \mu) + K_B(\mu_B, \mu) + 2K_S(\mu_S, \mu) \\
\Omega &= \Omega(\mu_J, \mu_B, \mu_S, \mu) = \eta_J(\mu_J, \mu) + \eta_B(\mu_B, \mu) + 2\eta_S(\mu_S, \mu),
\end{align*}$$

(186a)

(186b)

where the individual evolution kernels $K_H, K_J = K_B, K_S, \eta_J = \eta_B,$ and $\eta_S$ are given below in Eqs. (D5), (D5), and (D5). Note that $\mathcal{K}$ and $\Omega$ are independent of $\mu$ because the $\mu$ dependence cancels between the various $K_i$ and $\eta_i$ factors in the sums. Their expressions to NNLL accuracy are given in Eq. (D26) App. D. The coefficients $J_n, I_{n}^{gq}, I_{n}^{qq}, S_n$ in Eq. (185b) are given in Eq. (E7). The constants $V_{\eta_{1}}^{qg}$ and $V_{\eta_{1}}^{nn}(\Omega)$ are given in App. E.

Note that in the resummed cross section Eq. (185) the coefficients $J_n, I_{n}^{gq},$ and $S_n$ are functions of logarithms of their last argument as shown in Eq. (E7) and the hard function also depends on the logarithm $\ln(Q^2/\mu_H^2)$. The logs in these fixed-order factors can be minimized by choosing the canonical scales

$$\mu_H = Q, \quad \mu_J = \mu_B = Q \sqrt{\tau_1^{a,b}}, \quad \mu_S = Q \tau_1^{a,b}. $$

(187)

Large logs of ratios of these scales to the arbitrary factorization scale $\mu$ are then resummed to all orders in $\alpha_s$ in the evolution kernels $\mathcal{K}$ and $\Omega$. The choices in Eq. (187) are appropriate in the tail region of the distribution where $\tau_1$ is not too close to zero and not too large so that the logs of $\tau_1$ are still large enough to dominate non-log terms and need to be resummed. Near $\tau_1 \sim 0$ and $\tau_1 \sim 1$, we will need to make more sophisticated choices for the scales, which we will discuss in Sec. VII C.

\section{2. $\tau_1^c$ cross section}

The resummed $\tau_1^c$ cross section obtained from RG evolution of the hard, jet, beam, and soft functions in Eq. (185) is given by

$$\begin{align*}
\sigma_c^c(x, Q^2, \tau) &= \frac{e^{K-\gamma_E \Omega}}{\Gamma(1+\Omega)} \left( \frac{Q}{\mu_B} \right)^{\eta_H(\mu_H, \mu)} \left( \frac{(\tau-1+y) x Q^2}{\mu_B^2} \right)^{\eta_B(\mu_H, \mu)} \left( \frac{(\tau-1+y) Q^2}{\mu_J^2} \right)^{\eta_J(\mu_J, \mu)} \left( \frac{(\tau-1+y) \sqrt{x} Q}{\mu_S} \right)^{2\eta_S(\mu_H, \mu)} \\
& \times I_{q1}^c(x, Q^2) \int_1^1 \frac{dz}{z} f_j(x / z) \left[ W_{qj}^c(z, \tau-1 + y) + \Delta W_{qj}^c(z, \tau-1 + y) \right] + (q \leftrightarrow \bar{q}),
\end{align*}$$

(188)

where $W_{qj}^c$ and $\Delta W_{qj}^c$ are the fixed-order terms from jet, beam, and soft functions:

$$\begin{align*}
W_{qj}^c(z, \tau) &= H(Q^2, \mu_H) \sum_{n_1, n_2} \sum_{n_3} J_{n_1} \left[ \alpha_s(\mu_J), \frac{\tau Q^2}{\mu_J^2} \right] I_{n_2}^{qg} \left[ \alpha_s(\mu_B), \frac{\tau x Q^2}{\mu_B^2} \right] S_{n_3} \left[ \alpha_s(\mu_S), \frac{\sqrt{x} Q}{\mu_S} \right] \times V_{\ell_1 n_2} V_{\ell_2 n_3} V_{-1}^{\ell_3} (\Omega),
\end{align*}$$

(189a)

$$\begin{align*}
\Delta W_{qj}^c(z, \tau) &= \frac{\alpha_s(\mu_H)}{2\pi} \left[ \delta_{jg} C_F P_{qq}(z) + \delta_{jg} T_F P_{gq}(z) \right] \left[ \theta(\tau) \left\{ \ln \left[ \frac{\tau(1-X)}{(1-y)X} \right] - H(-1-\Omega) \right\} \\
& + \theta \left( \frac{\tau}{(1-y)X} + 1 \right) \left\{ \frac{1}{\Omega} \left( \frac{|\tau|}{(1-y)X} \right)^{-\Omega} 2F_1 \left[ -\Omega, -\Omega; 1-\Omega; -\frac{\tau}{(1-y)X} \right] - \theta(-\tau) \pi \sin(\pi \Omega) \right\} \right],
\end{align*}$$

(189b)

where $X \equiv x(1-z)/(x+z-xz)$. Note that the $\tau$ in $W_{qj}^c(z, \tau)$ and $\Delta W_{qj}^c(z, \tau)$ gets shifted by $1-y$ in Eq. (188). $H(n)$ is the harmonic number and $2F_1(a,b;c;z)$ is the hypergeometric function. The additional more complicated
terms in $\Delta W_n^{\gamma_1}$ are due to the nontrivial $p_L$ integral in Eq. (153) which convolves the terms in the generalized beam function with nontrivial $p_L^2$ dependence with the dependence of the jet function on $(q_L + p_L)^2$, with $q_L \neq 0$ when $y < 1$. Note that the term on the last line of Eq. (189b) contributes below $\tau_1^c = 1 - y$ when plugged into Eq. (188), but that the size of the correction in this region is very small.

The second arguments of $J_n$, $I_n^{\gamma_1}$, and $S_n$ in Eq. (189a) show that the canonical scales should be chosen to minimize the logs of the arguments, which are the fixed-order terms in the hard, jet, beam, and soft functions.

$$\mu_H = Q, \quad \mu_J = Q\sqrt{\tau_1^c - 1 + y}, \quad \mu_B = Q\sqrt{x(\tau_1^c - 1 + y)}, \quad \mu_S = \sqrt{xQ(\tau_1^c - 1 + y)}.$$  \hspace{1cm} (190)

Here the whole cross section is shifted to the right by an amount $1 - y$ due to the nonzero $q_L$ and choice of axes for $\tau_1^c$. Unlike $\tau_1^{a,b}$, the jet and beam scales are separated by a factor $\sqrt{x}$ due to the different normalization of the $q_B$ reference vector in the definition of $\tau_1^c$. For $\tau_1^{c,p}$, $q_B^c = P$ while for $\tau_1^{a,b}$, $q_B^{a,b} = xP$. The soft scale is also rescaled by $\sqrt{y}$. We will discuss below a more sophisticated choice of scales than Eq. (190) that give rise to proper behavior in the limits $\tau_1^c \to 1 - y$ and $\tau_1^c \sim 1$.

3. Logarithms included in our LL, NLL, and NNLL results

It is worth briefly discussing the logarithmic accuracy of our resummed results. Although this discussion is standard in the literature, sometimes the same notation is used for different levels of resummed precision, so it is worth being specific about our notation. The order in $\alpha_s$ to which the anomalous dimensions, running coupling, and fixed-order hard, jet, beam, and soft functions are known determines the accuracy to which the logarithms of $\tau$ in cross section (equivalently we could consider the Fourier transform to position space),

$$\tilde{\sigma}(x, Q^2, \nu) = \int_0^\infty d\tau \, e^{-\nu\tau} \frac{d\sigma}{dx \, dQ^2 \, d\tau}.$$  \hspace{1cm} (191)

The fixed-order expansion of $\tilde{\sigma}(x, Q^2, \nu)$ takes the form,

$$\tilde{\sigma}(x, Q^2, \nu) = 1 + \frac{\alpha_s}{4\pi} (c_{12} L^2 + c_{11} L + c_{10} + \tilde{d}_1(\nu))$$

$$\hspace{1cm} + \left(\frac{\alpha_s}{4\pi}\right)^2 (c_{24} L^4 + c_{23} L^3 + c_{22} L^2 + c_{21} L + c_{20} + \tilde{d}_2(\nu))$$

$$\hspace{1cm} + \left(\frac{\alpha_s}{4\pi}\right)^3 (c_{36} L^6 + c_{35} L^5 + c_{34} L^4 + c_{33} L^3 + c_{32} L^2 + c_{31} L + c_{30} + \tilde{d}_3(\nu)) + \cdots,$$

where $L \equiv \log \nu$. The largest log at each order in $\alpha_s$ is $\alpha_s^n L^{n\nu}$. Our results in Eqs. (185) and (188), once Laplace transformed, reorganize and resum the logarithms into the form:

$$\tilde{\sigma}(x, Q^2, \nu) = \exp \left[ \frac{\alpha_s}{4\pi} (C_{12} L^2 + C_{11} L + C_{10}) \right.$$  \hspace{1cm} (193)

$$\hspace{1cm} + \left(\frac{\alpha_s}{4\pi}\right)^2 (C_{23} L^3 + C_{22} L^2 + C_{21} L + C_{20})$$

$$\hspace{1cm} + \left(\frac{\alpha_s}{4\pi}\right)^3 (C_{34} L^4 + C_{33} L^3 + C_{32} L^2 + C_{31} L + C_{30}) + \cdots \right] + \tilde{d}(x, Q^2, \nu),$$

where the largest log at each order in the exponent is $\alpha_s^n L^{n+1}$. The coefficients $c_{nm}$, $C_{nm}$, and $d_n(\nu)$ are functions of $x$ and $Q^2$. The function $\tilde{d}(x, Q^2, \nu)$ contains terms $\tilde{d}_n(\nu)$ and is a non-singular function of $\nu$ that vanishes as $\nu \to \infty$ ($\tau \to 0$). Transforming Eqs. (192) and (193) back to momentum space using

$$\frac{d\sigma}{dx \, dQ^2 \, d\tau} = \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{dv}{2\pi i} e^{\nu v} \tilde{\sigma}(x, Q^2, \nu),$$  \hspace{1cm} (194)

where $\gamma$ lies to the right of all singularities of the integrand in the complex plane, defines the accuracy to which logs of $\tau$ in the cross section and its cumulant $\sigma_c(x, Q^2, \tau)$ are resummed.
Our main results in Eqs. (185) and (188) resum singular logarithmic terms $\alpha_s^n \ln^n \tau$, but not the terms in the non-singular $d(x,Q^2,\tau)$ (inverse transform of $d$). The $d(x,Q^2,\tau)$ must either be calculated by comparing a full QCD perturbation theory calculation with the resummed result and determining the difference order by order in $\alpha_s$, or by determining the next-to-singular infinite towers of logarithmic terms in $d(x,Q^2,\tau)$ by carrying out a factorization and resummation analysis in SCET at sub-leading power.

Fixed-order perturbation theory sums the series in Eq. (192) row-by-row, order-by-order in $\alpha_s$. When the logs are large this expansion is not well behaved. Resummed perturbation theory instead sums the exponent in Eq. (193) column-by-column, in a modified power expansion that counts $\ln 1/\alpha_s$ when the logs are large. Everything in the first column of Eq. (193) is $O(1/\alpha_s)$ [leading log (LL)], the second $O(1)$ [next-to-leading-log (NLL)], the third $O(\alpha_s)$ (NNLL), etc. in this counting. Each order of logarithmic accuracy is achieved by calculating the cusp and non-cusp anomalous dimensions, running coupling, and fixed-order hard, jet, beam, and soft matching coefficients $H, J, B, S$ at one higher order than in the unprimed counting. This primed counting is particularly useful when one also requires predictions for transition regions where the size of the logarithmic and non-logarithmic $d_n(\tau)$ terms are comparable. Since in this paper we have not considered the nonsingular terms we adopt the unprimed counting (LL, NLL, NNLL) throughout.

### B. Comparison to NLL DIS Thrust $\tau_Q$

As discussed above in Sec. III B there are several versions of DIS thrust discussed in the literature. Here we consider the version of thrust called $\tau_Q$ in [15], to which we can directly compare our results for $\tau_Q$, since as shown above in Eq. (193) of Sec. III D they are one and the same. We will see that at NLL accuracy the result in [13] for $\tau_Q$ is equivalent to our result in Eq. (185) for $\tau_Q^f$ for the particular scale choices Eq. (187) in the SCET cross section. The NLL resummed cross section given in [15] in the $\overline{\text{MS}}$ scheme is, in our notation,

$$
\sigma_c(\tau_Q) = \theta(\tau_Q) \left\{ \sum_q e_q^2 \left[ f_q(x,\sqrt{\tau}Q) + \frac{\alpha_s(Q)}{2\pi} \int_x^1 \frac{dz}{z} C_{1q}(z)f_q(x/z, Q) \right] \right. \\
+ \left( \sum_q e_q^2 \right) \frac{\alpha_s(Q)}{2\pi} \int_x^1 \frac{dz}{z} C_{1q}(z)f_q(x/z, Q) \right\} e^{-g_1 \ln \tau + g_2},
$$

(195)

where

$$
C_{1q}(z) = C_F \left[ 2\mathcal{L}_1(1-z) - (1+z) \ln(1-z) + 1 - z - \left( \frac{\pi^2}{3} + \frac{9}{2} \right) \delta(1-z) \right],
$$

(196)

and complete expressions for the resummation constants $g_{1,2}$ can be found in Ref. [15]. They have fixed-order expansions in $\alpha_s = \alpha_s(Q)$ given by

$$
g_1 \ln \tau = G_{12} \frac{\alpha_s}{2\pi} \ln^2 \tau - G_{22} \left( \frac{\alpha_s}{2\pi} \right)^2 \ln^3 \tau + \cdots
$$

(197)

$$
g_2 = -G_{11} \frac{\alpha_s}{2\pi} \ln \tau + G_{22} \left( \frac{\alpha_s}{2\pi} \right)^2 \ln^2 \tau + \cdots,
$$

(198)

with the coefficients

$$
G_{12} = -2C_F, \quad G_{11} = 3C_F, \quad G_{22} = 2\pi \beta_0 G_{12},
$$

(199)

$$
G_{22} = -\frac{4}{3} \pi^2 C_F^2 + \left( \frac{\pi^2}{3} - \frac{169}{30} \right) C_A C_F + \frac{11}{18} C_F n_f.
$$

Note that the cross section Eq. (195) includes only the photon contribution for the intermediate gauge boson mediating the scattering, so for the comparison we specialize our results to this case.

By comparing to the resummed cross section in Eq. (185), we find that the result of [15] given in Eq. (195) is equivalent to the SCET photon induced cross section at NLL order with the following fixed choices for the scales in the evolution factors:

$$
\mu_H = Q, \quad \mu = \mu_f = \mu_B = Q \sqrt{\tau_1}, \quad \mu_S = Q \tau_1.
$$

(199)

Thus the two results agree at NLL order.

### TABLE III: Orders of logarithmic accuracy and required order of cusp ($\Gamma$) and non-cusp ($\gamma$) anomalous dimensions, beta function $\beta$, and fixed-order hard, jet, beam, and soft matching coefficients $H, J, B, S$.

| Order | $\Gamma(\alpha_s)$ | $\gamma(\alpha_s)$ | $\beta(\alpha_s)$ | $\{H, J, B, S\}[\alpha_s]$ |
|-------|--------------------|--------------------|-------------------|-----------------------------|
| LL    | $\alpha_s$        | 1                  | $\alpha_s$        | 1                           |
| NLL   | $\alpha_s^2$      | $\alpha_s$        | $\alpha_s^2$      | 1                           |
| NNLL  | $\alpha_s^3$      | $\alpha_s^2$      | $\alpha_s^3$      | $\alpha_s$                  |
| NLL'  | $\alpha_s^2$      | $\alpha_s^2$      | $\alpha_s^2$      | $\alpha_s$                  |
| NNLL' | $\alpha_s^3$      | $\alpha_s^2$      | $\alpha_s^3$      | $\alpha_s$                  |
We note that in the fixed-order coefficient in Eq. (185), the choice $\mu = Q\sqrt{\tau}$ has been made in the tree level term, but the $O(\alpha_s)$ terms have been evaluated at $\mu = Q$. In the SCET result Eq. (188) (or Eq. (F7)) pieces of the $\alpha_s(\mu)$ terms are evaluated at $\mu = Q$, $\mu = Q\sqrt{\tau}$, or $\mu = Q\tau$ according to whether they come from the hard, beam/jet, or soft functions, while the PDFs are evaluated at $\mu = Q\sqrt{\tau}$. The difference between the SCET result and Eq. (188) is NNLL, since the error is $\sim \alpha_s^2 \ln \tau$ in the fixed-order coefficient. In our counting we take the correct scales for $\alpha_s(\mu)$ is required for NLL’ accuracy, since this provides the appropriate boundary conditions for the full NNLL result. Thus the result in Eq. (195) with $\alpha_s(Q)$ is at an intermediate level of accuracy between NLL and NLL’.

Note that the SCET expression Eq. (188) still shows the full dependence on the individual scales $\mu_{H,B,J,S}$ instead of the single scale $\mu$, and the dependence on each of these scales cancels out to the order in resummed perturbation theory that we are working. The remaining scale dependence thus provides a useful way to estimate the theoretical uncertainty due to the truncation of higher order terms in resummed perturbation theory.

C. Scale Profile Functions

In general there are three relevant regions with different power counting

peak region: $\tau_1 \sim 2\Lambda_{QCD}/Q_R \ll 1$,

tail region: $2\Lambda_{QCD}/Q_R \ll \tau_1 \ll 1$,

far-tail region: $\tau_1 \sim 1$.

For the peak and tail regions of the distribution we have $\tau_1 \ll 1$ and we must sum the large logarithms. In the tail region the results in Eqs. (188) and (190) above are the canonical scales for $\mu_{H,B,J,S}$ for which the logs in the fixed-order hard, jet, beam, and soft functions are minimized. Evolution from these scales to another scale $\mu$ resums the logs of the ratios $\mu/\mu_{H,B,J,S}$ to all orders in $\alpha_s$. In the peak region for small $\tau_1$, the scale $\mu_S \sim Q_1 \tau_1$ goes towards the nonperturbative region. The validity of our resummation analysis relies on there being a perturbative expansion for the soft function anomalous dimensions at the scale $\mu_S$, $\Gamma_S(\alpha_s(\mu_S))$ and $\gamma_S(\alpha_s(\mu_S))$. Therefore in the SCET approach it is mandatory that we stop the renormalization group evolution at a scale $\mu_S \sim 1$ GeV that can still be considered perturbative. This requires the scales to deviate from the canonical form. Finally, for larger values of $\tau_1$ the logs are no longer large, and the nonsingular terms in the fixed-order expansion become equally important. In this large $\tau_1$ region we should turn off the resummation, which will revert the results to a fixed-order expansion in $\alpha_s$. Again this forces the scales to deviate from the canonical ones.

To achieve these properties we use profile functions to describe the functional dependence of the scale $\mu_{S,B,J}$ on $\tau_1$. First we will consider the profile functions for the $\tau_1^{a,b}$ cross sections and then for $\tau_1^J$.

1. $\tau_1^{a,b}$ profile functions

For the $\tau_1^{a,b}$ cross sections, the canonical scales are given in Eqs. (187), $\mu_S \sim \tau_1$, $\mu_{B,J} \sim \sqrt{\tau_1} Q$, $\mu_H \sim Q$. The perturbative resummation of large logs of ratios of these scales is valid when $\Lambda_{QCD} \ll \mu_S \ll \mu_{B,J} \ll \mu_H$, which is the tail region. We will define boundaries, $t_3 < \tau_1 < t_2$ for the region of $\tau_1$ where this condition is satisfied, and use scales that are within a factor of 2 of the canonical ones. Beyond this region, when $\tau_1 > t_2$, $\tau_1$ is of $O(1)$, and the logs are the same order as the nonsingular terms in the fixed-order expansion. In this region the scales must be taken to be of the same order as the nonsingular terms in the fixed-order expansion. Finally for $\tau_1 < t_3$, the soft scale approaches $\Lambda_{QCD}$ and nonperturbative corrections become important. In this region we freeze the soft scale $\mu_S$ used in the perturbative cross section to a value above $\Lambda_{QCD}$: $\mu_S \sim 1–2$ GeV. The hard scale is $\mu_H \sim Q$ and beam and jet scale are determined by hard and soft scales as $\sqrt{\mu_H \mu_S} \sim \mu_{B,J}$.

Profile functions for scales that satisfy the above criteria have been used for other cross sections in [14, 28, 76]. Here, we adopt the profile functions in [28]. The hard, beam, jet, and soft scales we use are given by

$$\mu_H = \mu, \quad (201)$$

$$\mu_{B,J}(\tau_1) = \left[ 1 + \epsilon_{B,J} \theta(t_3 - \tau_1) \left(1 - \frac{\tau_1}{t_3}\right)^2 \right] \sqrt{\mu \mu_{\text{run}}(\tau_1, \mu)},$$

$$\mu_S(\tau_1) = \left[ 1 + \epsilon_S \theta(t_3 - \tau_1) \left(1 - \frac{\tau_1}{t_3}\right)^2 \right] \mu_{\text{run}}(\tau_1, \mu),$$

where $\epsilon_{B,J,S}$ are parameters used to vary the jet, beam, and soft scales to estimate theoretical uncertainty of the perturbative predictions. $t_3$ is the point above which all scales are set equal, $\mu_H = \mu_{B,J} = \mu_S = \mu$. The common function $\mu_{\text{run}}(\tau_1, \mu)$ is given by

$$\mu_{\text{run}}(\tau_1, \mu) = \begin{cases} 
\frac{\mu_0 + a \tau_1^2}{t_1}, & \tau_1 \leq t_1, \\
2a \tau_1 + b, & t_1 \leq \tau_1 \leq t_2, \\
\mu - a(\tau_1 - t_3)^2/(t_3 - t_2), & t_2 \leq \tau_1 \leq t_3, \\
\mu, & \tau_1 > t_3,
\end{cases}$$

$$a = \frac{\mu_0 - \mu}{t_1 - t_3}, \quad b = \frac{\mu t_1 - \mu_0(t_2 + t_3)}{t_1 - t_2 - t_3}, \quad (202)$$

The function $\mu_{\text{run}}(\tau_1, \mu)$ quadratically approaches $\mu_0$ below $t_1$ and $\mu$ above $t_2$, and it is linearly increasing from $t_3$ to $t_2$. The continuity of $\mu_{\text{run}}(\tau_1, \mu)$ and its derivative at $t_1$ and $t_2$ determines $a$ and $b$.

The default values of parameters we will use for what we will consider the “central values” of the $\tau_1^{a,b}$ cross sections.
sections are:

\[ \mu = Q, \quad e_{B,J} = e_S = 0, \quad \mu_0 = 2 \text{ GeV}, \]
\[ t_1 = \frac{3 \text{ GeV}}{Q}, \quad t_2 = 0.4, \quad t_3 = 0.6. \]  

(203)

To estimate theoretical uncertainty due to missing higher order terms in fixed-order and resummed perturbation theory, we vary the parameters \( \mu, e_{B,J}, \) and \( e_S \) from their default values by \( O(1) \) factors in order to vary corresponding scales \( \mu_H, \mu_{B,J}, \) and \( \mu_S \) by \( O(1) \) factors, respectively. We separately vary the parameters one by one and keep the others at their default values. The total number of variations we perform around the central values are as follows:

1) \( \mu = 2^{\pm 1}Q, \quad e_{B,J} = 0, \quad e_S = 0 \)

(204a)

2) \( \mu = Q, \quad e_{B,J} = \pm \frac{1}{3}, \pm \frac{1}{6}, \quad e_S = 0 \)

(204b)

3) \( \mu = Q, \quad e_{B,J} = 0, \quad e_S = \pm \frac{1}{3}, \pm \frac{1}{6} \)

(204c)

Variation 1 moves all the scales in Fig. 4 together up and down by factors of 2, and corresponds to the scale variation used to estimate the fixed order theoretical uncertainty in perturbation theory. Variations 2 and 3 are additional variations we are able to perform because of having independent \( \mu_{J,B} \) and \( \mu_S \) scales in the resummed cross section Eq. (185) and give an estimate of the uncertainty at each order in logarithmic accuracy in resummed perturbation theory that can not be achieved by varying the single scale \( \mu \). Variation 1 alone underestimates the total uncertainty.

The size of the cross section at a given value of \( \tau_1 \) may not vary monotonically with \( e_{J,B}, e_S \), and ideally we would vary them continuously within some finite band to find the maximum uncertainty. The four values we test for \( e_{J,B}, e_S \) in Eq. (204) are a discrete approximation to such a procedure that remains computationally tractable.

We take the largest and smallest values of the cross section among these points and use them to define the width of the uncertainty band from \( e_{J,B} \) or \( e_S \) variation.

To make a conservative estimate of the total uncertainty, we sum in quadrature the uncertainties we get from variations 1, 2, and 3 individually. We find that the total size of the bands provided by Eq. (204) are reasonable estimates of the theoretical uncertainty when we compare the cross sections at different orders of logarithmic accuracy.

Fig. 4 shows profile functions for \( \mu_H, \mu_{B,J}(\tau_1), \mu_S(\tau_1) \) with \( Q = 90 \text{ GeV} \). The solid lines are the central values of the scales with default values in Eq. (203), the double-headed arrow implies variation 1 and the bands represent variations 2 and 3 in Eq. (204). The dashed and dotted lines are the canonical scales in Eq. (187).

2. \( \tau_1^c \) profile functions

For \( \tau_1^c \), the canonical scales in Eq. (190) are

\[ \mu_H \sim Q, \quad \mu_S \sim Q \sqrt{x(\tau_1 - 1 + y)}, \]
\[ \mu_B \sim Q \sqrt{x(\tau_1 - 1 + y)}, \quad \mu_J \sim Q \sqrt{\tau_1 - 1 + y}, \]  

(205)

where they satisfy the relation \( \mu_B^2 = x^{\pm 1/2} \mu_H \mu_S \). Compared to the canonical scales for \( \tau_1^{a,b} \) in Eq. (187), there are two differences in the canonical scales for \( \tau_1^c \). First, \( \tau_1 \) is replaced by \( \tau_1 - (1 - y) \) because the transverse momentum of jet is nonzero, which is \( (1 - y)Q^2 \) at tree level and the projection onto \( q_T^2 \) differs from the projection onto jet axis by \( (1 - y)Q^2 \). This requires that canonical scales in Eq. (187) and profile in Eq. (201) are shifted by \( 1 - y \). Second, the soft scale and beam scale are multiplied by \( \sqrt{x} \) because of rescaling of the beam axis from \( xP \) for \( \tau_1^{a,b} \) to \( P \) for \( \tau_1^c \).

In this paper we consider the case \( \sqrt{x} \sim O(1) \) and this factor changes the scales by \( O(1) \), which is the size of perturbative uncertainties from varying \( \mu, e_{B,J} \) and \( e_S \). This
means that multiplying $\mu_B$ and $\mu_S$ by $\sqrt{x}$ in Eq. (201) should not make a difference within the perturbative uncertainty. So, we could use the profile in Eq. (201) but shifted by $1-y$. On the other hand, by modifying the profile the canonical relations among the scales $\mu_{H,J,B,S}$ for $\tau_1^a$ Eq. (190) can be maintained and we can account for the extra factors of $\sqrt{x}$. Therefore, for these profiles we define $\mu_{H,J,B,S}$ as

$$\mu_H = \mu, \quad \mu_{B,J,S}(\tau_1) = \mu_{B,J,S}(x, \tau_1 - 1 + y), \quad \mu_J(x, \tau_1) = \left[1 + e_J \theta(t_3 - \tau_1) \left(1 - \frac{\tau_1}{t_3} \right)^2 \sqrt{\mu \mu_{run}(x, \tau_1, \mu, 0)} \right], \quad \mu_B(x, \tau_1) = \left[1 + e_B \theta(t_3 - \tau_1) \left(1 - \frac{\tau_1}{t_3} \right)^2 \sqrt{\mu \mu_{run}(x, \tau_1, \mu, 1)} \right], \quad \mu_S(x, \tau_1) = \left[1 + e_S \theta(t_3 - \tau_1) \left(1 - \frac{\tau_1}{t_3} \right)^2 \mu \mu_{run}(x, \tau_1, \mu, 1/2) \right]. \tag{206}$$

The $\mu_{run}$ used here depend on $x$ and index 0, 1, 2 that is different for $\mu_J, \mu_B, \mu_S$. We want $\mu_{run}(x, \tau_1, \mu, n) \sim x^n \tau_1 \mu$ with $n = 0, 1/2, 1$ so that the canonical scaling for $\mu_{J,B,S}$ in Eq. (190) is respected in the small $\tau_1$ region. In the large $\tau_1$ limit, $\mu_{run}(x, \tau_1, \mu, n)$ should go to $\mu$, so that $\mu$ and $\mu_{B,J}$ both go to $\mu$. As in Eq. (202) $\mu_{run}$ should run linearly between $t_1$ and $t_2$. However, the slope of $\mu_{run}$ in Eq. (206) should be different for the three cases $n = 0, 1/2, 1$. Therefore, we cannot use Eq. (202) to define $\mu_{run}$ because all parameters in $\mu_{run}$ are fixed by matching boundary conditions and the slope is fixed. Instead, by replacing the quadratic polynomial in Eq. (202) by a cubic polynomial one can introduce a free parameter and this parameter can be chosen such that $\mu_{run}(x, \tau_1, \mu, n) \sim x^n \tau_1 \mu$ between $t_1$ and $t_2$. We define $\mu_{run}$ as

$$\mu_{run}(x, \tau_1, \mu, n) = \begin{cases} x^{n-\frac{1}{2}} \mu_0 + a(n) \tau_1^2 / t_1 & \tau_1 \leq t_1, \\ 2a(n) \tau_1 + b(n) & t_1 \leq \tau_1 \leq t_2, \\ \mu_{cubic}(x, \tau_1, \mu, n) & \tau_1 \leq t_2, \\ \mu & \tau_1 > t_3, \end{cases} \tag{206}$$

$$\mu_{cubic}(x, \tau_1, \mu, n) = \mu - c(n) \left( \frac{\tau_1 - t_3}{t_3 - t_2} \right)^2 - d(n) \left( \frac{\tau_1 - t_3}{t_3 - t_2} \right)^3 \tag{206}$$

$$b(n) = x^{n-1/2} \mu_0 - a(n) t_1, \quad c(n) = 3(\mu - x^{n-1/2} \mu_0) - a(n)(2t_3 + 4t_2 - 3t_1), \quad d(n) = 2(\mu - x^{n-1/2} \mu_0) - 2a(n)(t_3 + t_2 - t_1). \tag{207}$$

Here the parameters $b(n), c(n), d(n)$ are determined by continuity of $\mu_{run}$ and its derivative at $t_1, t_2, t_3$. The slope $a(n)$ is a free parameter which is chosen to satisfy $a(n) \sim x^n \mu$ to achieve canonical scaling of jet, beam, and soft scales:

$$a(n) = x^n \mu \left[ \frac{1}{3} + \frac{1}{2} \frac{t_3 + t_2 - t_1}{t_3 - t_2} \right] \tag{208}$$

Note that in $x \to 1$ limit, Eq. (207) reduces to Eq. (202) and profiles for $\tau_1^b$ in Eq. (206) reduce to the profiles in Eq. (201) for $\tau_1^b$. We choose the same default parameters and scale variations as for $\tau_1^{a,b}$ in Eqs. (203) and (204) except for $t_2$:

$$t_2 = 0.1. \tag{209}$$

Because of the different definition of the profiles for $\tau_1^b$ this value of $t_2$ must be smaller than the value for the $\tau_1^{a,b}$ profiles. This occurs because $\mu_{run}$ in Eq. (207) changes faster than that the $\mu_{run}$ in Eq. (202) between $t_2$ and $t_3$. As can be seen from Fig. 5 the final profiles for $\mu_S$ have similar shapes.

Fig. 5 shows $\tau_1^b$ profile functions for $\mu_H^b, \mu_{B,J}^b(\tau_1)$, $\mu_S^b(\tau_1)$ defined in Eq. (206) with $x = 0.1, y = 0.9$, and $Q = 90$ GeV. The solid lines are the central values of the scales with default values in Eq. (209) for $t_2$ and in Eq. (203) for all other parameters. The double-headed arrow represents variation 1 and the uncertainty bands are variations 2 and 3 in Eq. (203). The dashed, dotted, and dotted-dashed are the canonical scales in Eq. (190).

D. Nonperturbative Soft Function

The hemisphere soft function defined in Eq. (134) describes soft radiation between jets at the nonperturbative scale $Q_{\text{CD}}$ as well as at perturbative scales above $Q_{\text{CD}}$. The results given in Eqs. (164) and (174) are valid in the perturbative region. In the MS scheme the soft function valid at both scales is given by a convolution between a purely perturbative function $S_{\text{hemi}}^{\text{pert}}$ and a nonperturbative model function $F[108]$

$$S_{\text{hemi}}(k, \mu) = \int dk' S_{\text{hemi}}^{\text{pert}}(k - k', \mu) F(k'). \tag{210}$$

The function $F(k)$ contains information about physics at the nonperturbative scale and has support for $k \sim Q_{\text{CD}}$, falling off exponentially outside this region. Inserting Eq. (210) into the factorization formula in Eq. (110) one obtains the convolved form for the cross section:

$$\frac{d\sigma(\tau_1)}{d\tau_1} = \int dk \frac{d\sigma^{\text{pert}}}{d\tau_1} \left( \tau_1 - \frac{k}{Q_R} \right) F(k), \tag{211}$$

where $d\sigma^{\text{pert}}/d\tau_1$ is the cross section calculated by using only the perturbative soft function and $Q_R$ is given by Eq. (54). Eq. (211) correctly describes both the peak region $Q_R \tau_1 \sim Q_{\text{CD}}$ where the entire function $F(k)$ is required, as well as the tail region $Q_R \tau_1 \gg Q_{\text{CD}}$ where only its first moment is required since we can expand $Q_{\text{CD}}/(Q_R \tau_1)$.

For the peak region, various ways to parametrize models for $F(k)$ have been proposed [76, 108, 109]. We will adopt one proposed in [76] that expands $F$ systematically in an infinite set of basis functions:

$$F(k) = \frac{1}{\lambda} \left[ \sum_{n=0}^{N} c_n \lambda^n \left( \frac{k}{\lambda} \right)^2 \right], \tag{212}$$
where in principle we can choose any complete basis of functions \( f_n \). We adopt the same basis that has already been used in \([14, 70]\), and exhibits fast convergence of the expansion. The normalization condition \( \int dk F(k) = 1 \) gives the constraint \( \sum c_i^2 = 1 \). The characteristic scale \( \Lambda_{QCD} \) is an additional parameter if the sum is truncated at finite \( N \), as we will do in practice.

In the tail region where \( Q_R \tau_i \gg \Lambda_{QCD} \), Eq. (211) is consistent with the power correction from an operator product expansion (OPE),

\[
\frac{d\sigma(\tau_1)}{d\tau_1} = \left\{ \frac{\frac{d\sigma_{pert}(\tau_1)}{d\tau_1} - 2\Omega_{1}^{a,b,c}}{Q_R} + \frac{2\Omega_{1}^{a,b,c}}{Q_R} \right\} \times \left[ 1 + \mathcal{O}\left( \frac{\alpha_s\Lambda_{QCD}}{Q\tau_1} \right) + \mathcal{O}\left( \frac{\Lambda_{QCD}^2}{Q^2\tau_1^2} \right) + \cdots \right].
\]

To lowest order in \( \Lambda_{QCD}/Q\tau_1 \) this result agrees with a simple shift \( \tau_1 \to \tau_1 - 2\Omega_1/Q_R \). Here the coefficient of the nonperturbative matrix element \( 2\Omega_1^{a,b,c} \) is a nonperturbative matrix element and it corresponds to the first moment of the nonperturbative function \( \int dk F(k) \) which could in principle differ for each of \( \tau_1 \). The first set of power corrections indicated on the second line of Eq. (213) comes from perturbative corrections to the leading power correction \([72]\), and the second set involves purely nonperturbative corrections at subleading order. In the next section we will consider the question of universality of the \( \Omega_1^a, \Omega_1^b, \Omega_1^c \) parameters for the observables \( \tau_1^a, \tau_1^b, \tau_1^c \).

In the peak region the parameters \( c_i \) and \( \lambda \) should be determined by fitting to experimental data. Since data is not yet available, our only purpose here will be to get an idea of the impact of the nonperturbative shape function. We take the simplest function \( F(k) \) with \( N = 0 \). Then, \( c_0 = 1 \) by normalization and \( \lambda \) is the only parameter. To get the right first moment, we require \( \lambda = 2\Omega_1 \). We use \( \Omega_1 = 0.35 \text{ GeV} \), which is determined from measurements of \( e^+e^- \to \text{dijets} \) \([14]\). However, \( \Omega_1 \) in DIS is not necessarily the same as in \( e^+e^- \) collisions, and we merely consider this to be an illustrative but reasonable value.

E. Universality Classes for \( \Omega_1 \) Parameters Defined with Different Directions

The various versions of 1-jettiness \( \tau_1^{a,b,c} \) or the generic version Eq. (213) depend on different choices of the axes \( q_J = \omega_J n_J/2 \) and \( q_B = \omega_B n_B/2 \). In this section we will show that the 1-jettiness power correction parameter is universal under changes to the axes used in its definition, by exploiting properties of operators \([70, 71]\) and including hadron mass effects \([69, 72]\).

If we use different axes for the decomposition of four-momenta then they can all be written in a form similar to the event shapes given in \([72]\),

\[
\tau_1 = \frac{1}{2Q_R} \sum_i m_i^\perp f(r_i, \Upsilon_{JB}),
\]

where \( Q_R \) is defined in Eq. (53), \( i \) sums over hadrons, and \( m_i^\perp, r_i, \Upsilon_{JB}^i \) are defined with respect to the vector \( q_J, q_B \) by:

\[
m_i^\perp = \sqrt{p_i^2 + m^2}, \quad r_i = \frac{p_i}{m^\perp}, \quad \Upsilon_{JB} = \frac{1}{2} \ln \frac{q_B \cdot p}{q_J \cdot p},
\]

(215)

where \( m \) is the mass of the hadron whose momentum is \( p^\mu \). For the 1-jettinesses \( \tau_1 \) given in Eq. (213) we have

\[
f(r, \Upsilon) = e^{-|\Upsilon|}.
\]

(216)

For each different \( \tau_1 \), i.e. each choice of \( q_J, q_B \), the definition of \( m_i^\perp \) and \( \Upsilon_{JB} \) change since they are computed with different coordinates. The \( Q_R \) also depends on \( q_J, q_B \), as given in Eq. (53).

Following the logic in \([72]\) for massive hadrons and \([69, 71]\) for massless particles, the leading power correction in the expansion Eq. (213) of distributions in event shapes of the form Eq. (211) is always described by the nonperturbative matrix element

\[
2\Omega_1^{a,b,c} = \int_0^1 dr \int_0^\infty d\Upsilon_{JB} f(r, \Upsilon_{JB}) \times \langle 0| Y_{n_B} Y_{n_J} \hat{\mathcal{E}}_T^{a,b,c}(r, \Upsilon_{JB}) Y_{n_J}^\dagger Y_{n_B} | 0 \rangle
\]

(217)

Here \( \mathcal{E}_T \) is a “transverse velocity operator” defined as in \([72]\), but now using the axes given by \( q_J \) and \( q_B \),

\[
\hat{\mathcal{E}}_T(r, \Upsilon_{JB}) | X \rangle = \sum_{ic} m_i^\perp \delta(r - r_i) \delta(\Upsilon_{JB} - \Upsilon_{JB}^i) | X \rangle.
\]

(218)

It measures the total transverse mass of particles flowing in a slice of velocity and rapidity around \( r \) and \( \Upsilon_{JB} \). Now consider making an RPI-III transformation \([84]\) in the matrix element in Eq. (217) which takes \( n_J \to n_J/\zeta \) and \( n_B \to \zeta n_B \). This transformation leaves the vacuum and the Wilson lines \( Y_{n_J} \) and \( Y_{n_B} \) invariant, but shifts \( \mathcal{E}_T(r, \Upsilon_{JB}) \) to \( \mathcal{E}_T(r, \Upsilon_{JB} + \Upsilon') \) where \( \Upsilon' = \ln \zeta \). This is the analog of the boost argument for back-to-back \( n_J \) and \( n_B \) in Ref. \([71, 72]\). Thus, the matrix element inside the integral in Eq. (217) is independent of \( \Upsilon_{JB} \), and we can integrate over \( \Upsilon_{JB} \) to obtain the power correction \( \Omega_1^{a,b,c} \) for \( \tau_1^{a,b,c} \), using the \( f \) given in Eq. (213):

\[
\left[ \int_{-\infty}^{\infty} d\Upsilon_{JB} f(r, \Upsilon_{JB}) \right] \Omega_1^{1B}(r, \mu) = 2 \Omega_1^{1B}(r, \mu),
\]

(219)

where the renormalized matrix element is

\[
\Omega_1^{1B}(r, \mu) = \langle 0| Y_{n_B} Y_{n_J} \hat{\mathcal{E}}_T(r, 0) Y_{n_J}^\dagger Y_{n_B} | 0 \rangle.
\]

(220)

This matrix element still depends on the choices of axes through \( n_B, J \). By rescaling \( n_J \) and \( n_B \) as in Eq. (133) we find it is independent of \( n_J \cdot n_B \). It still depends on these axes through the parameter \( r \), since the transverse momenta \( p_\perp \) inside \( r \) depends on the choice of these axes.

However, in the tail region the \( \Omega_1^{1B} \) always appears inside an integral. At LL order we have the resummed
coefficient $C_{ij}^{LL}(k,r,\mu)$ from [72] for any $\tau_1$ and the shape function OPE is

$$F(k) = \delta(k) + \int_0^1 dr C_{ij}^{LL}(k,r,\mu)2\Omega_{ij}^{IB}(r,\mu) + O\left(\frac{\Lambda_{QCD}^2}{k^4}\right)$$

$$= \delta(k) + \int_0^1 dr C_{ij}^{LL}(k,r,\mu)2\Omega_{ij}(r,\mu) + O\left(\frac{\Lambda_{QCD}^2}{k^4}\right),$$

(221)

where in the second line we removed the $J^B$ superscript on $\Omega_1$ by using the fact that the only axis dependence occurs through the parameter $r$ which is now just a dummy variable. It would be interesting to consider the universality beyond LL order for this Wilson coefficient.

Thus we see that at least to LL order there is a universal power correction $\Omega_1(r)$ for all three versions of 1-jettiness, $\tau_{ij}^{a,b,c}$. Taking the tree level result $C_{ij}^{LL}(k,r,\mu) \rightarrow -\delta'(k)$ yields Eq. (213) and leads to the identification

$$\Omega_1^{a,b,c} = \int_0^1 dr \Omega_1(r).$$

(222)

Eq. (221) also implies universality of the shift parameter appearing in Eq. (213) for $\tau_{ij}^{a,b,c}$:

$$\Omega_1^{a} = \Omega_1^{b} = \Omega_1^{c}.$$

(223)

VIII. RESULTS

In this section we present our numerical results for the three versions of DIS 1-jettiness: $\tau_{ij}^a$, $\tau_{ij}^b$, and $\tau_{ij}^c$. We plot the cross sections accurate for small $\tau_1$ resummed from LL to NNLL accuracy, and also the singular terms at fixed order $O(\alpha_s)$ (NLO) for comparison. (We estimate the size of the small missing non-singular terms by comparing to the known $O(\alpha_s)$ cross section integrated over all $\tau_1$.) We start by describing the $\tau_{ij}^a$ spectrum in detail, and then compare the features of the $\tau_{ij}^b$ and $\tau_{ij}^c$ cross sections relative to the results for $\tau_{ij}^a$. We choose $s = (300 \text{ GeV})^2$ as in the H1 and ZEUS experiments. For the PDFs, we use the MSTW2008 [110] set at NLO and include five quark and antiquark flavors excluding top. To be consistent with the $\alpha_s$ used in the NLO PDFs we use the 2-loop beta function for running $\alpha_s$ and $\alpha_s(m_Z) = 0.1202$.

We present results for the cumulant cross section $\sigma_c(\tau_1)$ defined in Eq. (183) and the dimensionless distribution

$$\frac{d\sigma}{d\tau_1} = \frac{1}{\sigma_0} \frac{d\sigma}{d\tau_1} = \frac{d}{d\tau_1} \sigma_c(\tau_1).$$

(224)

Note that both the cumulant $\sigma_c(\tau_1)$ and the differential distribution $d\sigma/d\tau_1$ are differential in $x$ and $Q^2$. However, for notational simplicity we made their $x$ and $Q^2$ dependences implicit in this section.

A. $\tau_{ij}^a$ cross section

In this subsection, we present results for the cumulant cross section $\sigma_c(\tau_1)$ and differential cross section $d\sigma/d\tau_1$ for the “aligned” 1-jettiness $\tau_1^a$.

Fig. 6 shows the $\tau_{ij}^a$ cumulant cross section, defined by Eq. (183), at $Q = 80$ GeV and $x = 0.2$. In order to illustrate perturbative convergence the results resummed to LL, NLL, and NNLL accuracy are shown. The bands indicate perturbative uncertainties by varying the scales $\mu_{H,B,I,J,S}$ given by “profile functions” as described in Sec. [VII C] and there is excellent order-by-order convergence, and beautiful precision at NNLL order. The cumulant cross section increases monotonically from the small $\tau_{ij}^a$ region and begins to saturate near for large $\tau_{ij}^a$ where the integral defining this cumulant becomes that for the total cross section. There is a small gap between the total cross section at $O(\alpha_s)$ (dashed horizontal line) and our NNLL cumulant at large $\tau_{ij}^a$, reflecting the small size of nonsingular terms not taken into account in this paper. Note however that these terms are important at the level of precision of our cumulant cross section, and hence they will be considered in the future.

We can characterize the $d\sigma/d\tau_1^a$ cross section by three distinct physical regions: the peak region ($\tau_{ij}^a \sim 2\Lambda_{QCD}/Q$), the tail region ($2\Lambda_{QCD}/Q \ll \tau_{ij}^a \ll 1$), and the far-tail region ($\tau_{ij}^a \sim O(1)$). We will do this with four plots. We first show the purely perturbative cross section to study convergence and the impact of resummation compared to fixed order results. Next we show the impact of nonperturbative effects, which in the tail region produce a simple shift in the distribution, and have a significant impact on the shape of the spectrum in the peak region. We also illustrate the dependence of the cross section on $x$ and $Q^2$ at fixed $\tau_{ij}^a$.

Fig. 7 shows the weighted differential cross section $\tau_{ij}^a d\sigma/d\tau_1^a$ at $Q = 80$ GeV and $x = 0.2$. The results are weighted by $\tau_{ij}^a$ for better visibility because the differential cross section falls very rapidly with $\tau_{ij}^a$. In the tail region, the overlap in resummed results shows a good perturbative convergence from NLL to NNLL. The large deviation between NLO and NNLL shows the large effect of resummation and the underestimated uncertainty of a pure fixed-order result. In the peak region, NLO result blows up as $\ln(\tau_1)/\tau_1$, while the NLL and NNLL results converge into a peak due to resummation of the large logs to all orders in $\alpha_s$. Again the uncertainty bands overlap fairly well. In the far-tail region for larger $\tau_1$, the resummation effect becomes small and the size of the deviation is reduced. Near the far-tail region ($\tau_1 \sim 0.3$), the NNLL curve begins to depart from the NLL band. In this region the nonlogarithmic $\alpha_s^2$ term and nonsingular terms neglected in our NNLL result may begin to be significant.

Fig. 8 shows the differential cross section $d\sigma/d\tau_1$ at $Q = 80$ GeV and $x = 0.2$ in the peak region at fixed order and NNLL resummed accuracy. Note that it is not scaled by $\tau_1$ as in Fig. 7. In this plot, the NNLL result convolved with a nonperturbative shape function (NNLL
PT + NP) is shown in comparison with purely perturbative fixed-order NLO and resummed NNLL results (NLO PT and NNLL PT). As discussed in Sec. VII D we use the simplest shape function with one basis function $N = 0$ in Eq. (212) with a reasonable choice $\Omega_1 = 0.35 \text{ GeV}$ for the value of the first moment just to illustrate the impact of the nonperturbative effects. For practical analysis, a shape function with more basis functions should be used and the parameters $c_i, \lambda$ in the model function Eq. (212) should be determined from experimental data. In the endpoint region, there is significant change from NLO and NNLL due to the resummation of large perturbative logs, and there is another large change from perturbative NNLL to the result convolved with the shape function due to nonperturbative effects. As we move into the tail region, the size of nonperturbative correction reduces to $O(\Lambda_{QCD}/\tau_1 Q)$ and the correction simplifies to the power correction in Eq. (213).

Fig. 6 shows the cumulant cross section $x \frac{d^2 \sigma}{d x \, d x_{2}}$ as a function of $x$ at $Q = 80 \text{ GeV}$ and $\tau_1^b = 0.1$. Note that the lower bound $x \geq Q^2/s$ is set by the relation $x y s = Q^2$ in Eq. (19) and the constraint $y \leq 1$. The $x$ dependence comes from the quark and anti-quark beam functions and the decreasing curves with increasing $x$ are characteristic patterns of PDFs contained in the beam function. With decreasing $x$, NLO and NNLL curves rise faster than NLL curve because they contain the gluon PDF, which rises faster than the quark PDF, and whereas the NLL result only contains the tree-level beam function which is just the quark PDF.

Fig. 7 shows the weighted differential cross section $x d \sigma / (d x \, d Q^2) \, d \tau_1$ as a function of $x$ at $Q = 80 \text{ GeV}$ and $\tau_1^b = 0.2$. Colored bands show theoretical uncertainties around central values (lines) at fixed order $\alpha_s$ (dotted line, blue band) and resummed to NLL (dashed line, blue band) and NNLL (solid line, red band) accuracy.

Fig. 8 shows the differential cross section $d \sigma / d \tau_1$ at $Q = 80 \text{ GeV}$ and $x = 0.2$ in the peak region, NNLL with nonperturbative shape function taken into account (NNLL PT+NP, dashed, orange), and without NP shape function at fixed-order $\alpha_s$ (NLO PT, dotted, gray) and resummed (NNLL PT, solid, red).

The $\tau_1^b$ cumulant cross section is different from $\tau_1^a$ by a single term at NLO in Eq. (185). The term contains $\ln z$ where $z$ is integrated over from $x$ to 1, and so the term becomes larger for smaller $x$. Fig. 11 shows their percent difference at NLL and NNLL for two sets of $(Q, x)$: $(80, 0.2)$ and $(40, 0.02)$. The difference at NLL is zero because at LO fixed order $\tau_1^a$ and $\tau_1^b$ cross section are identical and the NLL logs are the same. At NNLL for

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**Figure 6:** Cumulant cross section in $\tau_1^b$ at $Q = 80 \text{ GeV}$ and $x = 0.2$. Colored bands show theoretical uncertainties around central values (lines) to LL (dotted line, green band), NLL (dashed line, blue band), and NNLL (solid line, red band) accuracy and the horizontal dashed line is the total cross section at fixed $x, Q^2$.

**Figure 7:** Weighted differential cross section in $\tau_1^b$ at $Q = 80 \text{ GeV}$ and $x = 0.2$. Colored bands show theoretical uncertainties around central values (lines) at fixed order $\alpha_s$ (dotted line, gray band) and resummed to NLL (dashed line, blue band) and NNLL (solid line, red band) accuracy.

**Figure 8:** Differential cross section in $\tau_1^a$ at $Q = 80 \text{ GeV}$ and $x = 0.2$ in the peak region, NNLL with nonperturbative shape function taken into account (NNLL PT+NP, dashed, orange), and without NP shape function at fixed-order $\alpha_s$ (NLO PT, dotted, gray) and resummed (NNLL PT, solid, red).

Factors $1/(1 + m_Z^2/Q^2)$ in Eq. (158). As shown in the plot, well below $m_Z = 91.2 \text{ GeV}$ the curves vary gently in $Q$ and near and above $m_Z$ they increase due to the factor $Q^2/(Q^2 + m_Z^2)$.

**B. $\tau_1^b$ cross section**

The $\tau_1^b$ cumulant cross section is different from $\tau_1^a$ by a single term at NLO in Eq. (185). The term contains $\ln z$ where $z$ is integrated over from $x$ to 1, and so the term becomes larger for smaller $x$. Fig. 11 shows their percent difference at NLL and NNLL for two sets of $(Q, x)$: $(80, 0.2)$ and $(40, 0.02)$. The difference at NLL is zero because at LO fixed order $\tau_1^a$ and $\tau_1^b$ cross section are identical and the NLL logs are the same. At NNLL for
and \( x \) cannot be used because \( y \approx 1 \) in other words, \( Q^2/(xs) \approx 1 \). The parameters in Fig. 9 cannot be used because \( y \approx 0.36 \) for \( Q = 80 \text{ GeV} \) and \( x = 0.2 \). For \( \tau_i^a \) in Figs. 12 and 13, we choose \( Q = 90 \text{ GeV} \) and \( x = 0.1 \) for which \( y = 0.9 \). Note that the profile functions for \( \tau_i^a \) given in Eq. (206) are also different from those for \( \tau_i^{a,b,c} \).

**C. \( \tau_i^a \) cross section**

The 1-jettiness \( \tau_i^a \) is designed to measure a jet close to the \( z \) axis (incoming electron direction), and the factorization theorem for \( \tau_i^a \) in Eq. (153) is valid for a jet with small transverse momentum \( q_T^a = (1-y)Q^2 \). So, the parameters \( Q \) and \( x \) should be chosen such that \( 1-y \ll 1 \) because fewer particles are grouped into the \( H_B \) region due to the minimum in Eq. (24). Still, in Fig. 13 for the same value of the cross section the departure of \( \tau_i^a \) from its threshold is larger than that of \( \tau_i^b \) due to this factor.

**FIG. 10:** \( Q \) dependence of \( \tau_i^a \) differential cross section at \( x = 0.2 \) and \( \tau_i^a = 0.1 \), with theoretical uncertainties at fixed-order \( \alpha_s \) (dotted, gray) and resummed to NLL (dashed, blue) and NNLL accuracy (solid, red)

\[ \tau_i^a = 0.1 \]

\[ x = 0.2 \]

The 1-jettiness \( \tau_i^a \) is designed to measure a jet close to the \( z \) axis (incoming electron direction), and the factorization theorem for \( \tau_i^a \) in Eq. (153) is valid for a jet with small transverse momentum \( q_T^a = (1-y)Q^2 \). So, the parameters \( Q \) and \( x \) should be chosen such that \( 1-y \ll 1 \) in other words, \( Q^2/(xs) \approx 1 \). The parameters in Fig. 9 cannot be used because \( y \approx 0.36 \) for \( Q = 80 \text{ GeV} \) and \( x = 0.2 \). For \( \tau_i^a \) in Figs. 12 and 13 we choose \( Q = 90 \text{ GeV} \) and \( x = 0.1 \) for which \( y = 0.9 \). Note that the profile functions for \( \tau_i^a \) given in Eq. (206) are also different from those for \( \tau_i^{a,b,c} \).

**FIG. 11:** Difference between \( \tau_i^a \) and \( \tau_i^b \) cumulative cross sections at \( Q = 80 \text{ GeV} \) and \( x = 0.2 \) and at \( Q = 40 \text{ GeV} \) and \( x = 0.02 \). The difference at NLL is zero for both parameter sets.

**IX. CONCLUSIONS**

We have predicted 1-jettiness (\( \tau_i \)) cross sections in DIS to NNLL accuracy in resummed perturbation theory, accurate for small \( \tau_i \) where hadrons in the final state are collimated into two jets, including one from ISR. We used three different versions of 1-jettiness, \( \tau_i^{a,b,c} \), which group final-state hadrons into “beam” and “jet” regions differently and have different sensitivity to the transverse momentum of ISR relative to the proton direction.
We gave predictions for the differential and cumulant sections (using the factorization theorems we derived). The relevant hard, jet, beam, and soft functions and their anomalous dimensions are known to sufficiently high order that we could immediately achieve NNLL resummed accuracy in our predictions for the $\tau_1^{a,b,c}$ cross sections (using the factorization theorems we derived). We gave predictions for the differential and cumulant $\tau_1$ cross sections, illustrating the differences among $\tau_1^{a,b,c}$ due to the different dependences on the transverse momentum of ISR. We presented numerical predictions at $x$ and $Q^2$ values explored at the HERA collider, but our analytical predictions can easily be applied to a much wider range of kinematics relevant at other experiments, such as at JLab [77] and the future EIC [78] and LHec [79].

The resummed predictions we presented are accurate for small values of $\tau_1$ where final-state hadrons are well collimated into two jets. For large $\tau_1$ our predictions have to be matched onto fixed-order predictions of non-singular terms in $\tau_1$ from full QCD. We leave the performance of this matching at $\mathcal{O}(\alpha_s)$ and beyond to future work. However, we compared our cumulant $\tau_1$ cross sections for large $\tau_1$ to the known total cross section at fixed $x$ and $Q^2$, and found that the cumulative effect of these corrections on the whole cross section is roughly at the several percent level for the kinematics we considered.

To achieve higher perturbative accuracy in the overall $\tau_1$ distributions we require both singular and the above-mentioned non-singular corrections to higher order. Here we achieved NNLL resummed accuracy, but without non-singular matching corrections needed to achieve NNLL+NLO accuracy. To go to NNLL'+NNLO accuracy, we need the fixed-order hard, jet, beam, and soft functions in SCET and non-singular terms in full QCD to $\mathcal{O}(\alpha_s^2)$. These are already known for the hard and jet functions. The soft function (known for $e^+e^-$ event shapes) and beam function (including both $t$ and $p_{\perp}$ dependence for $\tau_1^{b,c}$) are not yet known. Once they are, we could actually achieve N$^3$LL accuracy immediately since the necessary anomalous dimensions are all known to sufficiently high order. In extractions of $\alpha_s$ from $e^+e^-$ event shapes, it was found that adding another order of accuracy in the fixed-order SCET and full QCD calculations (i.e. adding a $'$) reduces theoretical uncertainty in the final value for $\alpha_s$ by about a factor.
of 2.5 at a time, with a precision of order 1–2% possible using N^3LL or N^3LL’ results \[14\]. We may anticipate similar future precision in extracting \( \alpha_s \) from DIS event shapes.

We showed how to incorporate nonperturbative hadronization corrections into our predictions by inclusion of a shape function that is convolved together with the perturbative soft function. The first moment of the shape function gives the parameter \( \Omega_1 \) which describes the shift to the distribution in the tail region. We demonstrated that this parameter is universal for our three event shapes \( \tau^1_{\mu,c} \) and for any values of \( x, Q^2 \), and so it can be extracted from one set of data to predict others. We also made a simple illustration of the effects of a shape function numerically on the cross section. We leave a more extensive study of nonperturbative effects and extractions of the model parameters from data to future work. We note that extraction of \( \alpha_s \) from DIS data (along the lines of \[14\] for \( e^+e^- \)) using the above rigorous factorization theorem based treatment of the power correction \( \Omega_1 \) has yet to be performed.

The extension of our results to \( N \)-jettiness \( \tau_N \) in DIS with \( N > 1 \) is straightforward, at least if we define \( \tau_N \) similarly to the 1-jettiness \( \tau^1 \) that we defined in Eq. \[229\]. That is,

\[
\tau_N^i = \frac{2}{Q^2} \sum_{i,j,k} \min\{q_i^1 \cdot p_i, q_i^1 \cdot p_j, \ldots, q_i^N \cdot p_j\},
\]

where \( q_i^2 = xP \) and \( q_i^2 \) is the jet axis of the \( i \)th non-ISR jet in the final state as given by a jet algorithm or by minimization of the sum Eq. \[229\] over the directions of \( q_i^2, \ldots, q_i^N \). As long as these jet reference axes are aligned with the physical jet axes, the transverse momentum \( k_{\perp} \) of ISR will not affect the value of \( \tau_N^i \) at leading order in \( \lambda \). The factorization theorem will then look like Eq. \[158\], with suitable generalizations of the hard and soft functions and additional jet functions (cf. \[27\]):

\[
\frac{d\sigma}{dx dQ^2 d\tau_N^i} = \frac{d\sigma_0}{dx dQ^2} \int dt_j^1 \cdots dt_j^N dt_B dk_S \delta \left( \tau_N^i - t_B + t_j^1 + \cdots + t_j^N - k_S\right) 
\times \sum_{i,\mu} B_i(t_B, x, \mu) J_{\kappa_1}(t_j^1, \mu) \cdots J_{\kappa_N}(t_j^N, \mu)
\times \text{tr} \tilde{H}_{i \to \kappa}(\{q_m\}, L, \mu) \tilde{S}_{N \to \infty}(k_S, \{q_m\}, \mu),
\]

where \( \tilde{H}_{i \to \kappa}(\{q_m\}, L, \mu) \) contains the underlying hard interaction \( i(q_0) \to e(k) \kappa_1(q_1) \cdots \kappa_N(q_N) \), where \( i, \kappa_j \) denote parton types, \( L \) denotes the dependence on the leptonic states \( e(k), e(k') \) and the exchanged virtual boson, and the sum over \( i, \kappa \) is over all relevant partonic channels. The hard and soft functions \( \tilde{H}, \tilde{S} \) are matrices in color space, and the trace is over these colors. \( B_i \) is the ordinary beam function for the initial-state parton of flavor \( i \). Since Eq. \[229\] uses reference axes \( q_i^2 \) that are aligned with the physical jet axes, the arguments \( t_j^i \) of the jet functions are the invariant masses of the jets and are not shifted by any transverse momentum \( k_{\perp} \) of ISR. Thus only the ordinary beam function \( B_i \) appears in Eq. \[230\], \( k_{\perp} \) having been averaged over. We leave the explicit evaluation of Eq. \[229\] for \( N \)-jettiness cross sections in DIS with \( N > 1 \) to future work.

Our results bring to the arena of DIS the power of SCET that has already vastly improved the precision of theoretical predictions of event shapes in \( e^+e^- \) collisions and \( pp \) collisions. The factorization theorems derived here point the way to methods to improve the precision of parton distributions, hadron structure, and the strong coupling \( \alpha_s \) that we can extract from existing and future experiments. With further advances in our calculations to greater perturbative accuracy and improved modeling of the nonperturbative effects, the frontiers of the study of the strong interaction using jets in DIS can be pushed to higher precision.

Note Added: While this paper was being finalized, Ref. \[111\] appeared which also considers the event shape we call \( \tau_1^1 \) at NNNLL order. A complete derivation of the factorization theorem was not presented there, where the focus is instead the use of 1-jettiness to probe nuclear PDFs and power corrections from dynamical effects in the nuclear medium.

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Appendix A: Generalized Rapidity Gap \( \Delta Y \)

The 1-jettiness \( \tau_1 \) in Eq. \[18\] is just one possible combination of jet and beam momenta that we can choose to measure in DIS. It is quite straightforward to keep \( n_j \cdot p_j, n_B \cdot p_B \) as independent observables in the factorization theorem Eq. \[157\], and then to form other observables by taking different combinations of \( n_j \cdot p_j, n_B \cdot p_B \). In this Appendix we consider one of these possibilities—the generalized rapidity gap \( \Delta Y \) between the beam jet and the other final-state jet.

The rapidity of a particle with momentum \( p \) with respect to the \( z \)-axis is given by

\[
Y_{n_z \tilde{n}_z}(p) = \frac{1}{2} \ln \frac{n_z \cdot p}{\tilde{n}_z \cdot p}.
\]

If \( p \) is \( n_z \)-collinear, the rapidity \( Y_{n_z \tilde{n}_z} \) is large and positive, while it is large and negative if \( p \) is \( \tilde{n}_z \)-collinear. Two jets produced in DIS are not, in general, back-to
back, and the reference vectors that measure jets are not always aligned along one (z) axis, as Fig. 3 illustrates. The rapidity in Eq. (A1) can be generalized by replacing \( n_{z\pm} \) with \( n_B \) and \( n_J \) as follows:

\[
Y_{n_B n_J}(p) = \frac{1}{2} \ln \frac{n_B \cdot p}{n_J \cdot p},
\]

(A2)

where \( Y_{n_B n_J} \) is large and positive for the \( n_J \)-collinear jet and is large and negative for the \( n_B \)-collinear jet. The generalized rapidity difference between two jets of momenta \( p_J \) and \( p_B \) is given by

\[
\Delta Y \equiv Y_{n_B n_J}(p_J) - Y_{n_B n_J}(p_B) = \frac{1}{2} \ln \frac{n_B \cdot p_J}{n_J \cdot p_J} \frac{n_J \cdot p_B}{n_B \cdot p_B}.
\]

(A3)

The \( n_{B,J} \) in Eq. (A3) can be replaced by \( q_{B,J} \) because the energy factors \( \omega_{J,B} \) and denominator cancel. By using Eq. (27), \( q_{B,J} \cdot p_{B,J} \) can be expressed in terms of \( \tau_{B,J} \). So, Eq. (A3) can be rewritten as

\[
\Delta Y = \frac{1}{2} \ln \frac{4 q_J \cdot p_B q_B \cdot p_J}{\tau_J \tau_B Q^4},
\]

(A4)

where the products \( 2 q_J \cdot p_B \) and \( 2 q_B \cdot p_J \) are \( O(Q^2) \) and \( \Delta Y \) is \( O[\ln(1/\tau_J \tau_B)] \). Eq. (A3) can be specified for DIS by using \( q_J \cdot p_B \approx q_J \cdot (P + q) \) and \( q_B \cdot p_J \approx q_B \cdot q \) where we use momentum conservation \( P + q = p_B + p_J \) and suppress \( p_B^2 \) and \( p_J^2 \). As we have three versions of \( \tau \), there are three versions of \( \Delta Y \):

\[
\Delta Y_{a,b} = \frac{1}{2} \ln \frac{1 - x}{x \tau_{J,B}}, \quad \Delta Y_{c} = \frac{1}{2} \ln \frac{1 - x}{x^2 \tau_{J,B}}.
\]

(A5)

Appendix B: Tensors and contractions

The symmetric and asymmetric tensors transverse to both \( n_B \) and \( n_J \) are defined by

\[
g_{\mu\nu} = g_{\mu\nu} - \frac{n_B^\mu n_B^\nu + n_J^\mu n_J^\nu}{n_J \cdot n_B}, \quad (B1a)
\]

\[
g_{\mu\nu} = g_{\mu\nu} - \frac{n_B^\mu n_J^\nu + n_J^\mu n_B^\nu}{2}, \quad (B1b)
\]

where \( \bar{n}_B \) and \( \bar{n}_J \) are conjugate to \( n_B \) and \( n_J \) as defined in Eq. (B1). In order to calculate the contraction of the lepton tensor \( L^{\mu\nu} \) with the hard function \( H_{q\mu\nu} \) as in Eq. (156), we must compute two tensor contractions: \( g_{\mu\nu} g_{\mu\nu} \) and \( \epsilon_{\mu\nu} \epsilon_{\mu\nu} \), where \( g^T, \epsilon^T \) are defined in Eq. (B1) and \( g_{\mu\nu}, \epsilon_{\mu\nu} \) in Eq. (B1). These contractions are given by

\[
g_{\mu\nu} g_{\mu\nu} = (g_{\mu\nu} - \frac{2k_B k_J + k_J k_B}{Q^2} ) (g_{\mu\nu} - \frac{n_B^\mu n_B^\nu + n_J^\mu n_J^\nu}{n_J \cdot n_B} ) = \frac{4}{n_J \cdot n_B Q^2} (n_J \cdot k n_B \cdot k' + n_J \cdot k' n_B \cdot k).
\]

(B2)

and

\[
\epsilon_{\mu\nu} \epsilon_{\mu\nu} = \frac{2}{n_J \cdot n_B Q^2} \epsilon_{\alpha\beta\gamma\delta} \epsilon_{\mu\nu}^{\alpha\beta} k_\alpha k_\beta n_J^\gamma n_B^\delta = \frac{4}{n_J \cdot n_B Q^2} (n_J \cdot k' n_B \cdot k - n_J \cdot k n_B \cdot k').
\]

(B3)

The ratio Eq. (B2) over Eq. (B3) is the coefficient \( r(q_J, q_B) \) defined in Eq. (139).

Appendix C: Plus distribution

The standard plus distribution for some function \( q(x) \) is given by

\[
[q(x)] + = \lim_{\epsilon \to 0} \frac{d}{dx} \left[ \theta(x - \epsilon) Q(x) \right],
\]

\[
= \lim_{\epsilon \to 0} \left[ \theta(x - \epsilon) q(x) + \delta(x - \epsilon) Q(x) \right],
\]

(C1)

where

\[
Q(x) = \int_1^x dx' q(x').
\]

(C2)

Integrating against a test function \( f(x) \), we have

\[
\int_{-\infty}^{x_{\max}} dx [\theta(x) q(x)]_+ f(x),
\]

\[
= \int_0^{x_{\max}} dx q(x) \left[ f(x) - f(0) \right] + f(0) Q(x_{\max}),
\]

(C3)

for \( x_{\max} > 0 \).

For the special cases \( q(x) = 1/|x|^{a-1} \) with \( a \geq -1 \) and \( q(x) = \ln^n x/\ln 2, n \geq 0 \), we define:

\[
L_n(x) = \frac{\theta(x)}{|x|^{a}} + \frac{\theta(x) \ln^n x}{x}, \quad n \geq 0.
\]

(C5)

For convenience we also define

\[
L_{-1}(x) \equiv \delta(x).
\]

(C6)

The plus function \( L_n \) obeys the rescaling relation,

\[
\lambda L_n(\lambda x) = \sum_{k=0}^{n} \binom{n}{k} \ln^k \lambda L_{n-k}(x) + \frac{\ln^{n+1} \lambda}{n + 1} \delta(x),
\]

(C7)

where \( \lambda > 0 \).

Appendix D: Renormalization Group Evolution

In this appendix we collect results relevant for resummation of the DIS 1-jettiness cross section Eq. (140) and its special cases Eqs. (147), (142), and (153) for \( \gamma_{a,b,c} \).
The RGE and anomalous dimension for the hard Wilson coefficient $C$ in Eq. (154) for the two-quark operator are \[ \frac{d}{d\mu} C(q^2, \mu) = \gamma_C^q(q^2, \mu) C(q^2, \mu), \]
\[ \gamma_C^q(q^2, \mu) = \Gamma_c^q(\alpha_s) \ln \frac{-q^2}{\mu^2} + \gamma_C^q(\alpha_s). \] (D1)

The anomalous dimension for the hard function $H$ in Eq. (155) is given by
\[ \frac{d}{d\mu} H(Q^2, \mu) = \gamma_H(Q^2, \mu) H(Q^2, \mu), \]
\[ \gamma_H(Q^2, \mu) = 2\Gamma^q(\alpha_s) \ln \frac{Q^2}{\mu^2} + \gamma_H(\alpha_s). \] (D2)

where $\gamma_H = 2\gamma_C^q$. The expansions in $\alpha_s$ of $\Gamma^q(\alpha_s)$ and $\gamma_C^q(\alpha_s)$ are given below in Eqs. (D28) and (D29).

The solution of the RGE in Eq. (D1) yields for the RG-evolved hard function:
\[ H(Q^2, \mu) = H(Q^2, \mu_0) U_H(Q^2, \mu_0, \mu), \]
\[ U_H(Q^2, \mu_0, \mu) = e^{\int_{\mu_0}^{\mu} \gamma_H(\mu) \frac{d\mu}{\mu^2}} \]
\[ = \frac{1}{\eta_H(\mu_0, \mu)} \int \frac{d\mu}{\mu^2} e^{\eta_H(\mu_0, \mu) \int_{\mu_0}^{\mu} \gamma_H(\mu') d\mu'}, \]
\[ K_H(\mu_0, \mu) = -4K_{\gamma}(\mu_0, \mu) + K_{\gamma_H}(\mu_0, \mu), \]
\[ \eta_H(\mu_0, \mu) = 4\eta_{\gamma}(\mu_0, \mu), \] (D3)

where the functions $K_{\gamma}(\mu_0, \mu)$, $\eta_{\gamma}(\mu_0, \mu)$ and $K_{\gamma}$ are given below in Eqs. (D24) and (D26).

The quark beam function RGE is given by
\[ \frac{d}{d\mu} B_q(t, x, \mu) = \int dt' \gamma_B^q(t-t', \mu) B_q(t', x, \mu), \] (D4)
\[ \gamma_B^q(t, \mu) = -2\Gamma_{\text{cusp}}^{\gamma}(\alpha_s) \frac{1}{\mu^2} L_0(t) + \gamma_B^q(\alpha_s) \delta(t), \]
and its solution is
\[ B_q(t, x, \mu) = \int dt' B_q(t-t', \mu_0) U_{B_q}(t', \mu_0, \mu), \]
\[ U_{B_q}(t_0, \mu_0, \mu) = e^{\int_{\mu_0}^{\mu} \gamma_{B_q}(\mu) \frac{d\mu}{\mu^2}} \]
\[ = \frac{1}{\eta_{B_q}(\mu_0, \mu)} \int \frac{d\mu}{\mu^2} e^{\eta_{B_q}(\mu_0, \mu) \int_{\mu_0}^{\mu} \gamma_{B_q}(\mu') d\mu'}, \]
\[ K_{B_q}(\mu_0, \mu) = 4K_{\gamma}(\mu_0, \mu) + K_{\gamma_B}(\mu_0, \mu), \]
\[ \eta_{B_q}(\mu_0, \mu) = 2\eta_{\gamma}(\mu_0, \mu), \] (D5)

The solution of the RGE for $B_q$ given by Eq. (D5) can be derived from the form of the solution Eq. (D3) for the hard function by first Laplace transforming the beam function:
\[ \tilde{B}_q(\nu, x, \mu) = \int_0^{\infty} dt e^{-\nu t} B_q(t, x, \mu), \] (D6)
which obeys the RGE
\[ \frac{d}{d\mu} \tilde{B}_q(\nu, x, \mu) = \tilde{\gamma}_{B_q}(\nu, \mu) \tilde{B}_q(\nu, x, \mu), \] (D7)
with the Laplace transformed anomalous dimension,
\[ \tilde{\gamma}_B^q(\nu, \mu) = 2\Gamma_{\text{cusp}}^q(\alpha_s) \ln(\mu^2 \nu e^{\gamma_\mu}) + \gamma_B^q(\alpha_s). \] (D8)

The evolution of $\tilde{B}_q$ in Eq. (D7) is multiplicative, of the same form as the hard function RGE Eq. (D2), and therefore its solution is just like the hard function Eq. (D3), given by
\[ \tilde{B}_q(\nu, x, \mu) = \tilde{B}_q(\nu, x, \mu_0) \tilde{U}_{B_q}(\nu, \mu_0, \mu), \] (D9)
where
\[ \tilde{U}_{B_q}(\nu, \mu_0, \mu) = e^{K_{B_q}(\mu_0, \mu) \frac{2}{\mu^2} \nu e^{\gamma_\mu}} \] (D10)
with $K_{B_q}, \eta_{B_q}$ given by the same expressions as in Eq. (D5).

The inverse Laplace transform of the solution Eq. (D9) gives the momentum space solution for $B_q(t, x, \mu)$ in Eq. (D6).

The jet function obeys the same RGE as the beam function. They are defined by matrix elements of the same operator. The solution for the Laplace transformed jet function $J_q(\nu, \mu)$ is given by the same form, Eqs. (D9) and (D10), with $B \rightarrow J$, and for the momentum-space jet function $J_q(t, \mu)$ by the same form Eq. (D5), with $B \rightarrow J$.

The hemisphere soft function in Eq. (156) obeys the RGE
\[ \frac{d}{d\mu} S_{\text{hemi}}(k_j, k_B, \mu) = \int dk'_j dk'_B \]
\[ \gamma_S(k_j, k_B, \mu) \delta(k_B) + \gamma_S(k_B, \mu) \delta(k_j), \] (D11)
with each piece of the anomalous dimension taking the form
\[ \gamma_S(k, \mu) = 2\Gamma_{\text{cusp}}^q(\alpha_s) \frac{1}{\mu^2} L_0(k) + \gamma_S(\alpha_s) \delta(k), \] (D12)
where $\gamma_S = -\gamma_C^q - \gamma_B^q$. The solution to the soft RGE Eq. (D11) is given by
\[ S_{\text{hemi}}(k_j, k_B, \mu) = \int dk'_j dk'_B S_{\text{hemi}}(k'_j, k'_B, \mu_0) \]
\[ \times U_S(k_j - k'_j, \mu_0, \mu_0) U_S(k_B - k'_B, \mu_0, \mu_0), \] (D14)
where
\[ U_S(k, \mu_0, \mu) = e^{K_{\gamma}(\mu_0, \mu) \frac{\eta_{\gamma}(\mu)}{\mu^2}} \frac{1}{\eta_{\gamma}(\mu_0, \mu)} \int \frac{d\mu}{\mu^2} e^{\eta_{\gamma}(\mu_0, \mu) \int_{\mu_0}^{\mu} \gamma_{\gamma}(\mu') d\mu'}, \]
\[ K_S(\mu_0, \mu) = -K_S(\mu_0, \mu_0) + 2\eta_{\gamma}(\mu_0, \mu), \]
\[ \eta_{\gamma}(\mu_0, \mu) = 2\eta_{\gamma}(\mu_0, \mu_0), \] (D15)

This solution can be derived as for the beam and jet functions above by first taking the Laplace transform:
\[ \tilde{S}_{\text{hemi}}(\nu_j, \nu_B, \mu) = \int_0^{\infty} dk_j \int_0^{\infty} dk_B e^{-\nu_j k_j - \nu_B k_B} S_{\text{hemi}}(k_j, k_B, \mu), \] (D16)
which obeys the RGE

\[ \mu \frac{d}{d\mu} \tilde{S}_{\text{hemi}}(\nu_j, \nu_B, \mu) = \tilde{S}_{\text{hemi}}(\nu_j, \nu_B, \mu) \times [\tilde{\gamma}_S(\nu_j, \mu) + \tilde{\gamma}_S(\nu_B, \mu)], \]

where each part of the anomalous dimension takes the form

\[ \tilde{\gamma}_S(\nu, \mu) = -2\Gamma_{\text{cusp}}^q \ln(\nu\mu^{\gamma_S}) + \gamma_S(\alpha_s). \]

Solving the soft RGE Eq. (D17), we obtain

\[ \tilde{S}_{\text{hemi}}(\nu_j, \nu_B, \mu) = \tilde{S}_{\text{hemi}}(\nu_j, \nu_B, \mu_0) \times \tilde{U}_S(\nu_j, \mu_0, \mu), \]

where each soft evolution factor takes the form

\[ \tilde{U}_S(\nu_j, \mu_0, \mu) = e^{K_S(\mu_0, \mu) / \mu(\nu_j \mu^{\gamma_S}) - \eta_S(\mu_0, \mu)}, \]

where \( K_S, \eta_S \) are given by Eq. (D15). Taking the inverse Laplace transform of Eq. (D19) gives the solution \( S_{\text{hemi}}(k, \mu) \) defined as

\[ S_{\text{hemi}}(k, \mu) = \int dk' \gamma_S(k - k', \mu) S_{\text{hemi}}(k', \mu), \]

where \( \gamma_S(k, \mu) \) is given by Eq. (D13). In Laplace space,

\[ \mu \frac{d}{d\mu} \tilde{S}_{\text{hemi}}(\nu, \mu) = 2\tilde{\gamma}_S(\nu, \mu) \tilde{S}_{\text{hemi}}(\nu, \mu). \]

The solutions to these RGEs are given by

\[ S_{\text{hemi}}(k, \mu) = \int dk' S_{\text{hemi}}(k', \mu_0) U_S^2(k - k', \mu, \mu), \]

\[ \tilde{S}_{\text{hemi}}(\nu, \mu) = \tilde{S}_{\text{hemi}}(\nu, \mu_0) \tilde{U}_S(\nu, \mu_0, \mu)^2, \]

where \( U_S^2(k, \mu_0, \mu) \) is given by Eq. (D15) with \( K_S, \eta_S \rightarrow 2K_S, 2\eta_S \), and \( \tilde{U}_S(\nu, \mu_0, \mu) \) is given by Eq. (D20).

The functions \( K_T, \eta_T, K_s, \mu_0, \mu \) in the above RGE solutions are defined as

\[ K_T(\mu_0, \mu) = \int \alpha_s(\mu) \frac{d\alpha_s}{\beta(\alpha_s)} \Gamma_{\text{cusp}}^q(\alpha_s) \int \alpha_s(\mu_0) \frac{d\alpha_s}{\beta(\alpha_s)}, \]

\[ \eta_T(\mu_0, \mu) = \int \alpha_s(\mu_0) \frac{d\alpha_s}{\beta(\alpha_s)} \Gamma_{\text{cusp}}^q(\alpha_s), \]

\[ K_s(\mu_0, \mu) = \int \alpha_s(\mu_0) \frac{d\alpha_s}{\beta(\alpha_s)} \gamma(\alpha_s). \]

Expanding the beta function and anomalous dimensions in powers of \( \alpha_s \),

\[ \beta(\alpha_s) = -2\alpha_s \sum_{n=0}^{\infty} \beta_n \left( \frac{\alpha_s}{4\pi} \right)^{n+1}, \]

\[ \Gamma_{\text{cusp}}^q(\alpha_s) = \sum_{n=0}^{\infty} \Gamma_q \left( \frac{\alpha_s}{4\pi} \right)^{n+1}, \]

\[ \gamma(\alpha_s) = \sum_{n=0}^{\infty} \gamma_n \left( \frac{\alpha_s}{4\pi} \right)^{n+1}, \]

their explicit expressions to NNLL accuracy are (suppressing the superscript \( q \) on \( \Gamma_q \)),

\[ K_T(\mu_0, \mu) = -\frac{\Gamma_0}{4\beta_0^3} \left\{ \frac{4\pi}{\alpha_s(\mu_0)} \left( 1 - \frac{1}{r} - \ln r \right) + \left( \frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0} \right) \left( 1 - r + \ln r \right) + \frac{\beta_1}{2\beta_0} \ln^2 r \right\}, \]

\[ \eta_T(\mu_0, \mu) = \frac{\Gamma_0}{2\beta_0} \left[ \ln r + \frac{\alpha_s(\mu_0)}{4\pi} \left( \frac{\Gamma_1}{\Gamma_0} - \frac{\beta_1}{\beta_0} \right) (r - 1) + \frac{\alpha_s^2(\mu_0)}{16\pi^2} \left( \frac{\Gamma_2}{\Gamma_0} - \frac{\beta_2}{\beta_0} \right) (r - 1) \right], \]

\[ K_s(\mu_0, \mu) = -\frac{\gamma_0}{2\beta_0} \left[ \ln r + \frac{\alpha_s(\mu_0)}{4\pi} \left( \frac{\gamma_1}{\gamma_0} - \frac{\beta_1}{\beta_0} \right) (r - 1) \right]. \]
loops, the coefficients of the beta function and cusp anomalous dimension in $\overline{\text{MS}}$ are

$$
\begin{align*}
\beta_0 &= \frac{11}{3} C_A - \frac{4}{3} T_F n_f, \\
\beta_1 &= \frac{34}{3} C_A^2 - \left( \frac{20}{3} C_A + 4 C_F \right) T_F n_f, \\
\beta_2 &= \frac{2857}{54} C_A^3 + \left( \frac{C_F^2 - 205}{18} C_F C_A - \frac{1415}{54} C_A^2 \right) 2 T_F n_f + \left( \frac{11}{9} C_F + \frac{79}{54} C_A \right) 4 T_F^2 n_f,
\end{align*}
$$

$$
\Gamma_0^q = 4 C_F,
$$

$$
\Gamma_1^q = 4 C_F \left[ \left( \frac{67}{9} - \frac{\pi^2}{3} \right) C_A - \frac{20}{9} T_F n_f \right],
$$

$$
\Gamma_2^q = 4 C_F \left[ \frac{245}{6} - \frac{134 \pi^2}{27} + \frac{11 \pi^4}{45} + \frac{22 \zeta_3}{3} \right] C_A^2 + \left( - \frac{418}{27} + \frac{40 \pi^2}{27} - \frac{56 \zeta_3}{3} \right) C_A T_F n_f
\quad + \left( - \frac{55}{3} + 16 \zeta_3 \right) C_F T_F n_f - \frac{16}{27} T_F^2 n_f^2. \tag{D28}
$$

The $\overline{\text{MS}}$ non-cusp anomalous dimension $\gamma_H = 2 \gamma_C^q$ for the hard function $H$ can be obtained from the IR divergences of the on-shell massless quark form factor $C(q^2, \mu)$ which are known to three loops,

$$
\begin{align*}
\gamma_{C0}^q &= -6 C_F, \\
\gamma_{C1}^q &= -C_F \left[ \left( \frac{82}{9} - 52 \zeta_3 \right) C_A + \left( 3 - 4 \pi^2 + 48 \zeta_3 \right) C_F + \left( \frac{65}{9} + \pi^2 \right) \beta_0 \right], \\
\gamma_{C2}^q &= -2 C_F \left[ \left( \frac{66167}{324} - \frac{686 \pi^2}{81} - \frac{302 \pi^4}{135} - \frac{782 \zeta_3}{9} + \frac{44 \pi^2 \zeta_3}{9} + 136 \zeta_3 \right) C_A^2 \\
&\quad + \left( \frac{151}{4} - \frac{205 \pi^2}{9} + \frac{247 \pi^4}{135} + \frac{844 \zeta_3}{3} + \frac{8 \pi^2 \zeta_3}{3} + 120 \zeta_3 \right) C_A C_F \\
&\quad + \left( \frac{29}{2} + 3 \pi^2 + \frac{8 \pi^4}{5} + 68 \zeta_3 - \frac{16 \pi^2 \zeta_3}{3} - 240 \zeta_3 \right) C_F^2 + \left( - \frac{10781}{108} + \frac{446 \pi^2}{81} + \frac{449 \pi^4}{270} - \frac{1166 \zeta_3}{9} \right) C_A \beta_0 \\
&\quad + \left( \frac{2953}{108} - \frac{13 \pi^2}{18} - \frac{7 \pi^4}{27} + \frac{128 \zeta_3}{9} \right) \beta_1 + \left( - \frac{2417}{324} + \frac{5 \pi^2}{6} + \frac{2 \zeta_3}{3} \right) \beta_0^2 \right]. \tag{D29}
\end{align*}
$$

As shown in [107], the anomalous dimension for the beam function equals that of the jet function, $\gamma_B^q = \gamma_J^q$, so the non-cusp three-loop anomalous dimension for the jet and beam functions are both given by [38],

$$
\begin{align*}
\gamma_{B0}^q &= \gamma_{J0}^q = 6 C_F, \\
\gamma_{B1}^q &= \gamma_{J1}^q = C_F \left[ \left( \frac{146}{9} - 80 \zeta_3 \right) C_A + \left( 3 - 4 \pi^2 + 48 \zeta_3 \right) C_F + \left( \frac{121}{9} + \frac{2 \pi^2}{3} \right) \beta_0 \right], \\
\gamma_{B2}^q &= \gamma_{J2}^q = 2 C_F \left[ \left( \frac{52019}{162} - \frac{841 \pi^2}{81} - \frac{82 \pi^4}{27} - \frac{2056 \zeta_3}{9} + \frac{88 \pi^2 \zeta_3}{9} + 232 \zeta_3 \right) C_A^2 \\
&\quad + \left( \frac{151}{4} - \frac{205 \pi^2}{9} - \frac{247 \pi^4}{135} + \frac{844 \zeta_3}{3} + \frac{8 \pi^2 \zeta_3}{3} + 120 \zeta_3 \right) C_A C_F \\
&\quad + \left( \frac{29}{2} + 3 \pi^2 + \frac{8 \pi^4}{5} + 68 \zeta_3 - \frac{16 \pi^2 \zeta_3}{3} - 240 \zeta_3 \right) C_F^2 + \left( - \frac{7739}{54} + \frac{325 \pi^2}{81} + \frac{617 \pi^4}{270} - \frac{1276 \zeta_3}{9} \right) C_A \beta_0 \\
&\quad + \left( - \frac{3457}{324} + \frac{5 \pi^2}{9} + \frac{16 \zeta_3}{3} \right) \beta_1 + \left( \frac{1166}{27} - \frac{8 \pi^2}{9} + \frac{41 \pi^4}{135} + \frac{52 \zeta_3}{9} \right) \beta_0^2 \right]. \tag{D30}
\end{align*}
$$

The anomalous dimension for the soft function is obtained from $\gamma_S = -\gamma_B^q - \gamma_J^q$. At NNLL, we only need the one- and two-loop coefficients of $\gamma_{H,B,I,S}$. The three-loop coefficients are given for completeness. They would be required at N$^3$LL, along with the four-loop beta function and cusp anomalous dimension, the latter of which has not yet been calculated. In addition, the full N$^3$LL result would also require the two-loop fixed-order corrections, which are known for the hard function, but not yet for the beam and soft functions.
Appendix E: Coefficients in Momentum-Space Resummed Cross Section

The resummed cross sections for $\nu_1^{a,b,c}$ in Sec. VII are obtained by plugging the solutions to the RG equations for the hard function and for the momentum-space jet, beam, and soft functions given in App. [12] and [13] into the factorization theorems derived in Sec. [14]. Performing the convolutions in these factorization theorems of the jet, beam, and soft evolution kernels given in App. [12] and fixed-order functions requires computing the convolutions of plus functions with each other. The results of these convolutions produce the expressions given in Eqs. [183] and [188], given in terms of coefficients $J_n, I_n, S_n$ of the logs in the fixed-order jet, beam, and soft functions and coefficients $V_{\mu}^m$ and $V_{\kappa}^n(a)$ that are the result of the convolutions of plus functions. In this Appendix we tabulate these coefficients. For more details see Refs. [14, 20].

1. Jet, Beam, and Soft Coefficients $J_n, I_n^{qq,qg}, S_n$

The fixed-order results at $O(\alpha_s)$ of soft, jet, and beam functions can be written as sum of plus distributions as

$$G(t, \mu) = \frac{1}{\mu^{n_F}} \sum_{n=1}^{\infty} G_n(\alpha_s(\mu)) L_n \left( \frac{t}{\mu^{n_F}} \right),$$

(E1)

where $G(t, \mu)$ represents the single-variable soft function $S(t, \mu)$ in Eq. [210], jet function $J(t, \mu)$ in Eq. [186], or the coefficient $I^{qq,qg}(t, z, \mu)$ inside the beam function in Eq. [177]. The index $n_F = 1$ for the soft function and $n_F = 2$ for the jet and beam function. In the case of the beam function, the $z$ dependence in $F(t, \mu)$ is implicit. The coefficients $F_n$ in Eq. [E1] for the three functions are $S_n, J_n,$ and $I_n^{qq,qg}$. The soft coefficients at order $\alpha_s$ are given by

$$S_{-1}(\alpha_s) = 1 + \frac{C_F \alpha_s^2}{4\pi}, S_0(\alpha_s) = 0, S_1(\alpha_s) = \frac{C_F \alpha_s^2}{4\pi} (-16),$$

(E2)

the jet coefficients by

$$J_{-1}(\alpha_s) = 1 + \frac{C_F \alpha_s}{\pi} \left( \frac{7}{4} - \frac{\alpha_s^2}{4} \right),$$

$$J_0(\alpha_s) = -\frac{C_F \alpha_s^3}{4}, J_1(\alpha_s) = \frac{C_F \alpha_s}{\pi},$$

(E3)

and the beam function coefficients by

$$I_{-1}^{qq}(\alpha_s, z) = L_{-1}(1-z) + \frac{C_F \alpha_s}{2\pi} \left[ L_1(1-z)(1+z^2) - \frac{\pi^2}{6} L_{-1}(1-z) + \theta(1-z) \left( 1 - z - \frac{1+z}{1-z} \ln z \right) \right],$$

$$I_0^{qq}(\alpha_s, z) = \frac{C_F \alpha_s \theta(\alpha_s)}{2\pi} \left( P_{qq}(z) - \frac{3}{2} L_{-1}(1-z) \right),$$

$$I_1^{qq}(\alpha_s, z) = \frac{C_F \alpha_s}{\pi} 2 L_{-1}(1-z),$$

and

$$I_{n}^{qq}(\alpha_s, z) = \frac{\alpha_s T_F}{2\pi} \theta(z) \left[ P_{qq}(z) \ln \frac{1-z}{z} + 2 \theta(1-z)(1-z) \right],$$

$$I_{0}^{qq}(\alpha_s, z) = \frac{\alpha_s T_F}{2\pi} \theta(z) P_{qq}(z),$$

(E5)

where the splitting functions $P_{qq,qg}(z)$ are given in Eq. [167].

The argument of the plus distributions $L_n$ in Eq. [E1] can be rescaled using the identity Eq. [E7]. Eq. [E1] can be rewritten in terms of the rescaled distribution as

$$G(t, \mu) = \frac{1}{\lambda^{n_F}} \sum_{n=0}^{\infty} G_n(\alpha_s(\mu), \lambda) L_n \left( \frac{\lambda^{-1} t}{\mu^{n_F}} \right),$$

(E6)

where the coefficients $G_n(\alpha_s, \lambda)$ in Eq. [E6] are expressed in terms of the coefficients in Eq. [E1] by using the rescaling identity in Eq. [E7] as

$$G_{-1}(\alpha_s, \lambda) = G_{-1}(\alpha_s) + \sum_{n=0}^{\infty} G_n(\alpha_s) \ln^{n+1} \lambda n + 1,$$

$$G_n(\alpha_s, \lambda) = \sum_{k=0}^{\infty} \frac{(n+k)!}{n! k!} G_{n+k}(\alpha_s) \ln^k \lambda,$$

(E7)

where $G_n = \{S_n, J_n, I_n^{qq,qg}\}$. Explicit expressions for $S_n(\alpha_s, \lambda), J_n(\alpha_s, \lambda),$ and $I_n^{qq,qg}(\alpha_s, \lambda)$ are obtained by inserting Eqs. [E2], [E3], and [E4] into Eq. [E5].

2. Results of convolving plus functions

Convolutions of plus distributions in the jet, beam, and soft evolution kernels and the fixed-order functions produce the functions $V_{\kappa}^m(\Omega)$ and the coefficients $V_{\mu}^m$ in the resummed cross sections Eqs. [185] and [188]. There are three types of convolutions of plus distributions $L_n$ and $L^a$. and we write them in useful form as

$$\int dy L_m(x-y) L_n(y) = \sum_{\ell=-1}^{m+n+1} V_{\ell}^m L_\ell(x),$$

$$\int dy [a L^a(x-y) + \delta(x-y)] [b L^b(y) + \delta(y)]$$

$$= \frac{\Gamma(1+a)}{\Gamma(1+a+b)} \left[ \Gamma(1+b) \right] (a+b) \left[ \int dy [a L^a(x-y) + \delta(x-y)] L_n(y) = \sum_{k=1}^{n+1} V_{\kappa}^n(a) L_k^a(x) \right].$$

The coefficients $V_{\mu}^m(a)$ and $V_{\kappa}^m$ are related to the Taylor series expansion of $V(a, b)$ around $a = 0$ and $b = 0$, where $V(a, b)$ is defined by

$$V(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)} - \frac{1}{a} - \frac{1}{b},$$

(E9)
which satisfies $V(0,0) = 0$. The $V^k_n(a)$ for $n \geq 0$ are

$$V^k_n(a) = \begin{cases} \frac{a}{n+1} \left( \frac{d^n}{da^n} V(a,b) \right)_{a=b=0}, & k = -1, \\ \frac{n}{k} \left( \frac{d^{n-k}}{db^{n-k}} V(a,b) \right)_{b=0} + \delta_{kn}, & 0 \leq k \leq n, \\ \frac{n}{a+n+1} & k = n + 1. \end{cases}$$

(E10)

The $V^m_n$ are symmetric in $m$ and $n$, and for $m, n \geq 0$ they are

$$V^m_n = \begin{cases} \frac{d^m}{da^m} \frac{d^n}{db^n} V(a,b)_{a=b=0}, & k = -1, \\ \frac{1}{n+1} \sum_{p=0}^{n} \sum_{q=0}^{n} \delta_{p+q,k} \binom{m}{p} \binom{n}{q} \left( \frac{d^{m-p}}{da^{m-p}} \frac{d^{n-q}}{db^{n-q}} V(a,b) \right)_{a=b=0}, & 0 \leq k \leq m + n, \\ \frac{1}{m+1} + \frac{1}{n+1}, & k = m + n + 1. \end{cases}$$

(E11)

Using Eq. (C6) we can extend these definitions to include the cases $n = -1$ or $m = -1$. The relevant coefficients are

$$V^{-1}_n(a) = 1, \quad V^0_n(a) = a$$

$$V^{-1}_n(a) = 0, \quad V^{-1}_n(a) = \delta_{nk}. \quad (E12)$$

### Appendix F: Resummed cross section from Laplace transforms

An alternative way to express the resummed cross sections in Sec. VII is to utilize the Laplace-transformed jet, beam, and soft functions given in App. D and their RGE solutions. The method avoids taking explicit convolutions of plus functions in the evolution factors and in the fixed-order jet, beam, and soft functions.

Each of the RGE solutions for the jet, beam, and soft functions is given by a function of the form

$$\tilde{G}(\nu, \mu) = \tilde{G}(\nu, \mu_0) e^{K_G(\mu_0, \mu)} [\mu_0 (\nu e^{\gamma_E})^{1/\beta_G} - \gamma_G \eta_G(\mu_0, \mu)]^{-1} \tilde{G}(\nu, \mu_0). \quad (F1)$$

For the jet and beam functions, $j_G = 2$, while for the soft function $j_G = 1$. The fixed-order expansion of $\tilde{G}(\nu, \mu_0) \equiv \tilde{G}(L_G, \mu_0)$ can be considered to be a function of the log $L_G \equiv \ln Q_G/\mu_0$, where $Q_G = (\nu e^{\gamma_E})^{-1/j_G}$. To $O(\alpha_s^2)$,

$$\tilde{G}(L_G, \mu_0) = 1 + \frac{\alpha_s(\mu_0)}{4\pi} \left( -\frac{3}{2} \Gamma_G^2 L_G^2 - \gamma_G \frac{\partial}{\partial \mu_G} L_G + c_G^1 \right) \quad (F2)$$

$$+ \left( \frac{\alpha_s(\mu_0)}{4\pi} \right)^2 \left[ \frac{1}{2} \left( \Gamma_G^0 \right)^2 L_G^2 + \Gamma_G^0 \left( \gamma_G + \frac{1}{3} \beta_G \right) L_G^3 
+ \frac{1}{2} \Gamma_G^0 L_G^2 - \gamma_G \frac{\partial}{\partial \mu_G} L_G + c_G^1 \right]$$

Each power of $L_G$ can be generated by taking derivatives with respect to $\eta_G$ in Eq. (F1):

$$\tilde{G}(\nu, \mu) = e^{K_G(\mu_0, \mu)} \tilde{g}(\partial \eta_G, \mu_0) [\mu_0 (\nu e^{\gamma_E})^{1/\beta_G}]^{-1} \frac{\partial^{4}}{\partial \eta_G} \quad (F3)$$

where $\tilde{g}(\partial \eta_G, \mu_0)$ is the operator constructed by replacing each $L_G$ in Eq. (F2) with $\partial \eta_G / j_G$:

$$\tilde{g}(\partial \eta_G, \mu_0) = 1 + \frac{\alpha_s(\mu_0)}{4\pi} \left( -\frac{3}{2} \Gamma_G^2 \frac{\partial}{\partial \mu_G} - \gamma_G \frac{\partial}{\partial \mu_G} + c_G^1 \right) \quad (F4)$$

$$+ \left( \frac{\alpha_s(\mu_0)}{4\pi} \right)^2 \left[ \frac{1}{2} \left( \Gamma_G^0 \right)^2 \frac{\partial}{\partial \mu_G} + \Gamma_G^0 \left( \gamma_G + \frac{1}{3} \beta_G \right) \frac{\partial^3}{\partial \mu_G^3} 
+ \frac{1}{2} \left( \gamma_G \right)^2 \Gamma_G^1 \beta_G - \gamma_G \frac{\partial}{\partial \mu_G} + c_G^1 \right]$$

Now it is easy to take the inverse Laplace transform of Eq. (F1):

$$G(t, \mu) = \int_{e^{-i\infty}}^{e^{i\infty}} \frac{d\nu}{2\pi i} e^{\nu t} \tilde{G}(\nu, \mu) \quad (F5)$$

$$= e^{K_G(\mu_0, \mu)} \tilde{g}(\partial \eta_G, \mu_0) \frac{e^{-\gamma_G \eta_G}}{\Gamma(\eta_G)} \left( \frac{t/\mu_0^{\beta_G}}{\eta_G} - \frac{1}{t} \right),$$

where $\eta_G \equiv \eta_G(\mu_0, \mu)$. The derivatives with respect to $\eta_G$ automatically generate the results of taking convolutions of the logs inside $G(t, \mu_0)$ with the evolution kernel $U_G(t, \mu_0, \mu)$ in RGE solutions like Eqs. (D8) and (D14).
1. $\tau_{1}^{a,b}$ cross sections

Using the above formalism, we obtain for the Laplace transforms of the $\tau_{1}^{a,b}$ differential cross sections $(1/\sigma_{0}) d\sigma/d\tau_{1}^{a,b}$ in Eqs. (142) and (147),

$$
\tilde{\sigma}(x, Q^2, \nu^{a,b}) = H(Q^2, \mu_B) \tilde{\gamma} (\partial_{\eta}, \nu) |L_q(x, Q^2)\tilde{\delta}_{q}^{a,b} (\partial_{\eta}, x, \mu_B) + L_{\bar{q}}(x, Q^2)\tilde{\delta}_{\bar{q}}^{a,b} (\partial_{\eta}, x, \mu_B)| \tilde{s}(\partial_{\eta}, \mu_S) \times e^{K(\mu_B, \mu_B) + K_{1}(\mu_B, \mu) + K_{2}(\mu_B, \mu) + 2K_{S}(\mu_B, \mu)}
\times \left( \frac{Q}{\mu_H} \right)^{\eta_H} \left( \frac{Q^2}{\mu_H^2} \right)^{\eta_H} \left( \frac{Q^2}{\mu_B^2} \right)^{\eta_B} \left( \frac{Q}{\mu_S^2} \right)^{\eta_S} \left( \frac{Q}{\mu_B^2} \right)^{\eta_B} \left( \frac{Q}{\mu_S^2} \right)^{\eta_S}.
$$

Taking the inverse Laplace transform with respect to $\nu^{a,b}$ and taking the cumulant in Eq. (172), we easily obtain in momentum space,

$$
\sigma_c(x, Q^2, \tau_{1}^{a,b}) = H(Q^2, \mu_H) \left( \frac{Q}{\mu_H} \right)^{\eta_H} \left( \frac{Q^2}{\mu_H^2} \right)^{\eta_H} \left( \frac{Q^2}{\mu_B^2} \right)^{\eta_B} \left( \frac{Q}{\mu_S^2} \right)^{\eta_S} \frac{\eta}{\mu} \left[ Q \tilde{\tau}_{1}^{a,b} (\partial_{\eta}, \mu_S) \right] \widetilde{\Gamma}(\mu_H, \mu_B, \mu_S, \mu) \times \left( \frac{Q}{\mu_H} \right)^{\eta_H} \left( \frac{Q^2}{\mu_B^2} \right)^{\eta_B} \left( \frac{Q}{\mu_S^2} \right)^{\eta_S} \left( \frac{Q}{\mu_B^2} \right)^{\eta_B} \left( \frac{Q}{\mu_S^2} \right)^{\eta_S}.
$$

with a sum over quark and antiquark flavors $q, \bar{q}$, and where the sums of evolution kernels $K, \Omega$ are given by

$$
K(\mu_H, \mu_J, \mu_B, \mu_S, \mu) = K_H(\mu_H, \mu) + K_J(\mu_J, \mu) + K_B(\mu_B, \mu) + 2K_S(\mu_S, \mu)
$$

$$
\Omega(\mu_H, \mu_J, \mu_B, \mu_S, \mu) = \eta_H(\mu_H, \mu_J, \mu_S, \mu) + \eta_B(\mu_B, \mu_S, \mu) + 2\eta_S(\mu_S, \mu),
$$

where the individual evolution kernels $K_H, K_J, K_B, \eta_H, \eta_J, \eta_B, \eta_S$ are defined in App. D.

The fixed-order operators $\tilde{\gamma}, \tilde{\delta}_{q}^{a,b}, \tilde{s}$ in Eq. (F7) each take the form Eq. (F4), which in this paper we truncate to $O(\alpha_s)$, working to NNNLL accuracy. In Eq. (F8), $\Gamma_f, \gamma_f, \beta_f$ are the coefficients in the fixed-order expansions Eq. (D26) of the anomalous dimensions and beta function, and where $j_C = 2$ for the jet function and $j_C = 1$ for the soft function, and the constants $c_f^{C}$ are given by

$$
c_1^{C} = (7 - \pi^2)C_F - \frac{\Gamma_0}{4} \frac{\pi^2}{6}, \quad c_3^{C} = \frac{\pi^2}{3}C_F - \Gamma_0 \frac{\pi^2}{3}.
$$

Note that the cusp parts of the hard, jet/beam, and soft anomalous dimensions are related to the cusp anomalous dimension in Eq. (D25) by

$$
\Gamma_H = 2\Gamma_0^{q} \text{cusp}, \quad \Gamma_{J,B} = -2\Gamma_0^{q} \text{cusp}, \quad \Gamma_S = 2\Gamma_0^{q} \text{cusp}.
$$

Meanwhile the beam function operators $\tilde{\delta}_{q}^{a,b}$ in the $\tau_{1}^{a,b}$ cross sections are given by

$$
\tilde{\delta}_{q}^{a,b} (\partial_{\eta}, x, \mu_B) = f_q(x, \mu_B) \left\{ 1 + \frac{\alpha_s(\mu_B)}{4\pi} \left( -C_F \frac{\pi^2}{3} - \frac{\Gamma_0}{4} \left( \frac{\pi^2}{6} \right) + \frac{\gamma_0}{2} (\partial_{\eta}) \right) \right\}
\times \left[ C_F f_q \left( \frac{x}{\mu_B} \right) F_q(z) + T_F f_Q \left( \frac{x}{\mu_B} \right) F_Q(z) \right]
\times \left[ C_F f_q \left( \frac{x}{\mu_B} \right) F_q(z) + T_F f_Q \left( \frac{x}{\mu_B} \right) F_Q(z) \right]
\times \left[ \partial_{\eta} + \delta b^a(z) \right],
$$

where $\delta b^a(z)$ differ only in the last term,

$$
\delta b^a(z) = 0, \quad \delta b^b(z) = \ln z,
$$

and the functions $F_{q,g}$ are given by

$$
F_q(z) = (1 + z^2) \theta(1 - z) \frac{1 - z}{1 - z} + \theta(1 - z) \left( 1 - z - \frac{1 + z^2}{1 - z} \right), \quad F_{g}(z) = P_{qg}(z) \left( \frac{1 - z}{z} - 1 \right) + \theta(1 - z),
$$

\text{for } a, b = q, g.
and $P_{qg,qg}$ are given by Eq. \( \text{[167]} \). The additional term $\delta b^b(z) = \ln z$ that appears in the final integrand in Eq. \( \text{[111]} \) for $\hat{b}^b$ is due to the nontrivial $k_T^2$ dependent terms in Eq. \( \text{[166]} \) for the generalized beam function, which generate the $\delta b^b(z) = \ln z$ term upon integration over the transverse momentum in Eq. \( \text{[142]} \). Thus the difference the $\tau^a_1$ and $\tau^b_1$ cross sections will become more pronounced at smaller $x$, when the $\delta b^b(z) = \ln z$ term inside the integrand of Eq. \( \text{[F11]} \) can grow larger.

To evaluate the action of the fixed-order operators given by Eqs. \( \text{[F4]} \) and \( \text{[F11]} \) in the resummed cross section Eq. \( \text{[F7]} \), it is useful to tabulate the following relations:

\[
\mathcal{G}(\Omega) \equiv \left( \frac{Q_{\tau_1}}{\mu \gamma_E} \right) \frac{1}{\Gamma(1 + \Omega)},
\]

\[
\partial_\Omega \mathcal{G}(\Omega) = \left[ -\ln \frac{\mu}{Q_{\tau_1}} - H(\Omega) \right] \mathcal{G}(\Omega),
\]

\[
\partial^2_\Omega \mathcal{G}(\Omega) = \left( \ln \frac{\mu}{Q_{\tau_1}} + H(\Omega) \right)^2 - \psi^{(1)}(1 + \Omega) \mathcal{G}(\Omega),
\]

where $H$ is the harmonic number function, $H(\Omega) = \gamma_E + \psi^{(0)}(1 + \Omega)$ and $\psi^{(n)}(x) = (d^n/dz^n) [\Gamma'(z)/\Gamma(z)]$ is the polygamma function. The result of taking these derivatives in the expression Eq. \( \text{[F7]} \) is equivalent to the results of convolving logs in the fixed-order jet, beam, and soft functions with the momentum-space evolution kernels in deriving the expression Eq. \( \text{[185]} \). The two formalisms yield equivalent expressions for the resummed cross section.

### 2. $\tau^b_1$ cross section

The resummed $\tau^b_1$ cross section obtained from RG evolution of the hard, jet, beam, and soft functions in Eq. \( \text{[183]} \) is given by

\[
\sigma_c(x, Q^2, \tau^b_1) = H(Q^2, \mu_H) \left( \frac{Q}{\mu_H} \right)^{\eta_H(\mu_H, \mu)} \left( \frac{Q^2}{\mu_f^2} \right)^{\eta_f(\mu_f, \mu)} \left( \frac{x Q^2}{\mu_B^2} \right)^{\eta_B(\mu_B, \mu)} \left( \frac{\sqrt{x Q}}{\mu_S} \right)^{2 e(\mu_S, \mu) - \Omega} J \left( \Omega - \ln \frac{\mu^2}{Q^2}, x, \tau^b_1, \mu_B \right)
\]

\[
\times \left[ L^c_q(Q^2) \tilde{b}^c_q \left( \partial_\Omega - \ln \frac{\mu^2}{Q^2}, x, y, \tau^b_1, \mu_B \right) + L^c_q(Q^2) \tilde{b}^c_q \left( \partial_\Omega - \ln \frac{\mu^2}{Q^2}, x, y, \tau^b_1, \mu_B \right) \right]
\]

\[
\times \tilde{s} \left( \partial_\Omega, \mu_S \right) \sqrt{x Q} \frac{\tau^b_1 - 1 + y}{\mu_S e^{\gamma_E}},
\]

\[
\Gamma(1 + \Omega)
\]

where the operator $\tilde{b}^c_q$ is given by

\[
\tilde{b}^c_q \left( \partial_\Omega, x, y, \tau^b_1, \mu_B \right) = \theta (\tau^b_1 - 1 + y) \tilde{b}^c_q \left( \partial_\Omega, x, \mu_B \right) + \frac{\alpha_s(\mu_B)}{2 \pi} \int_x^1 \frac{dz}{z} \left[ C_F P_{qq}(z) f_q(x, \mu_B) + T_F P_{qg}(z) f_g(x, \mu_B) \right]
\]

\[
\times \left\{ \theta (\tau^b_1 - 1 + y) \left[ \ln \left( \frac{z}{1 - y} \right) \tau^b_1 - 1 + y \right] - H(-\Omega) - \frac{1}{\Omega} \right\} - \theta (1 - y - \tau^b_1) \frac{\pi}{\sin \pi \Omega}
\]

\[
+ \frac{1}{\Omega} \frac{(1 - y) X}{\tau^b_1 - 1 + y} \right) \right) \theta (\tau^b_1 - 1 - y)(1 - X),
\]

and similarly for $\tilde{b}^c_q$. Here $X \equiv x(1 - z)/(x + z - x z)$. The additional more complicated terms in $\tilde{b}^c_q$ are due to the nontrivial $p_\perp$ integral in Eq. \( \text{[183]} \) which convolves the terms in the generalized beam function with nontrivial $p_\perp^2$ dependence with the dependence of the jet function on $(q_\perp + p_\perp)^2$, with $q_\perp \neq 0$ when $y \ll 1$. Note that the apparent singularities as $\Omega \to 0$ (the fixed-order limit) cancel in the sum of all terms. The result Eq. \( \text{[F15]} \) is equivalent to the expression Eq. \( \text{[185]} \) derived from RG evolution directly in momentum space.
3. Generic $\tau_1$ cross section

In similar fashion we can form the resummed $\tau_1$ cross section for an arbitrary definition Eq. (21) of the 1-jettiness. Using the generic factorization theorem Eq. (140)

$$\sigma_c(x, Q^2, \tau_1) = H(Q^2, \mu_H) \left( \frac{Q}{\mu_H} \right)^{\eta_H(\mu_H, \mu)} \left( \frac{s}{\mu_J} \right)^{\eta_J(\mu_J, \mu)} \left( \frac{\mu_B}{\mu_B} \right)^{\eta_B(\mu_B, \mu)} \left( \frac{Q_B}{\mu_S} \right)^{2\eta_S(\mu_S, \mu) - \Omega}$$

where the operator $\tilde{b}_q$ in the $\Omega$ evaluate the action of these operators in Eq. (F15), and finally take the

$$\tilde{b}_q = q \left( \text{fixed-order cross sections} \right) \text{ beam function operator Eq. (F16) into the expression Eq. (F15). We use Eq. (F14) to}$$

and similarly for $\tilde{b}_g$. In Eqs. (F17) and (F18), $\tau_q$ and $X_q$ are given by

$$\tau_q = \frac{q^2}{Q_J n_j q}, \quad X_q = \frac{-q_J q (1 - z)}{|z q_B - (1 - z) q_J|} \cdot q.$$ (F19)

Appendix G: $O(\alpha_s)$ fixed-order cross sections

1. $\tau_1^c$ cross section

The fixed-order $\tau_1^c$ cross section at $O(\alpha_s)$ is easily obtained from Eq. (F15) by taking the limit $\mu_{H,J,B,S} = \mu$, which turns off all the resummation. We plug the $O(\alpha_s)$ hard function Eq. (155), the $O(\alpha_s)$ jet and soft operators given by Eq. (F4), and the $O(\alpha_s)$ beam function operator Eq. (F16) into the expression Eq. (F15). We use Eq. (F14) to evaluate the action of these operators in Eq. (F15), and finally take the $K, \Omega, \eta_{H,J,B,S} \to 0$ limit. The result is:

$$\sigma_c(x, Q^2, \tau_1^c) = \theta(\tau_1^c - 1 + y) \int_x^1 \frac{dz}{z} [L^c_1(Q^2) f_1(x/z, \mu) + L^c_2(Q^2) f_2(x/z, \mu)]$$

$$\times \left\{ \delta(1 - z) \left[ 1 - \frac{\alpha_s(\mu) C_F}{4\pi} \left( 9 + \frac{2\pi^2}{3} + 3 \ln[x(\tau_1^c - 1 + y)^2] + 4 \ln[x(\tau_1^c - 1 + y)] \ln(\tau_1^c - 1 + y) \right) \right] + \frac{\alpha_s(\mu) C_F}{2\pi} \left[ P_{qq}(z) \ln \frac{x Q^2(\tau_1^c - 1 + y)}{\mu^2} + F_y(z) \right] \right\} \right.$$ (G1)

$$+ \frac{\alpha_s(\mu) T_F}{2\pi} \int_x^1 \frac{dz}{z} \left[ L^c_1(Q^2) f_1(x/z, \mu) + L^c_2(Q^2) f_2(x/z, \mu) \right] \left[ P_{qq}(z) \ln \frac{x Q^2(\tau_1^c - 1 + y)}{\mu^2} + F_y(z) \right]$$

$$+ \frac{\alpha_s(\mu) T_F}{2\pi} \int_x^1 \frac{dz}{z} \left[ C_F P_{qq}(z) [L^c_1(Q^2) f_1(x/z, \mu) + L^c_2(Q^2) f_2(x/z, \mu)] + T_F P_{qq}(z) (L_q + L_g)(Q^2) f_2(x/z, \mu) \right]$$

$$\times \left[ \theta(\tau_1^c - 1 + y) \ln \frac{z}{x + z - x z} + \theta(1 - y - \tau_1^c) \theta(\tau_1^c - \frac{z(1 - y)}{x + z - x z}) \ln \frac{(1 - y) X}{1 - y - \tau_1} \right].$$

In the last line we used that in the $\Omega \to 0$ limit, the hypergeometric function in Eq. (F16) behaves like

$$2 F_1(-\Omega, -\Omega, 1 - \Omega, -T) = 1 + \Omega^2 L_2(-T) + \cdots,$$ (G2)

In the $\Omega \to 0$ limit in Eq. (F16), only the first term in this expansion survives.
2. Generic $\tau_1$ cross section

The fixed-order $O(\alpha_s)$ cross section is similarly obtained from Eq. (59) by taking the limit of equal scales $\mu = \mu_H = \mu_J = \mu_B = \mu_S$, and thus $K, \Omega, \eta_{H,J,B,S} \to 0$. For the cumulant to $O(\alpha_s)$, we obtain:

$$\sigma_c(x, Q^2, \tau_1) = \theta(\tau_1 - \tau_q) \int_x^1 \frac{dz}{z} \left[ L_q(q_j, q_B, Q^2)f_q(x/z, \mu) + L_{q_B}(q_B, Q^2)f_{q_B}(x/z, \mu) \right]$$

$$\times \left\{ \delta(1-z) \left[ 1 - \frac{\alpha_s(\mu)C_F}{4\pi} \left( 9 + \frac{2\pi^2}{3} + 3\ln \frac{Q_R^2}{Q^2} \right)^2 + 4\ln \frac{Q_R^2(\tau_1 - \tau_q)}{s_B} + \frac{\alpha_s(\mu)C_F}{2\pi} \left[ \frac{s_B(\tau_1 - \tau_q)}{\mu^2} + F_q(z) \right] \right\}$$

$$+ \frac{\alpha_s(\mu)}{2\pi} \int_x^1 \frac{dz}{z} \left\{ C_F \left[ \frac{L_qf_q(x/z, \mu) + L_{q_B}f_{q_B}(x/z, \mu)}{\mu^2} \right] + T_F \frac{P_{q_B}(\tau_1 - \tau_q)}{\mu^2} \right\}$$

$$\times \left[ \theta(\tau_1 - \tau_q) \ln(1 - X_q) + \theta(\tau_q - \tau_1)\theta(\tau_1 - \tau_q(1 - X_q)) \ln \frac{\tau_q X_q}{\tau_q - \tau_1} \right],$$

where we have used the relation in Eq. (59), $s_Bs/Q_R^2 = Q^2$ to leading order in $\lambda$, to simplify the arguments of the logs on the second line.
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