Symplectic stability, analytic stability in non-algebraic complex geometry

Andrei Teleman*

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Abstract

We give a systematic presentation of the stability theory in the non-algebraic Kählerian geometry. We introduce the concept of ”energy complete Hamiltonian action”. To an energy complete Hamiltonian action of a reductive group $G$ on a complex manifold one can associate a $G$-equivariant maximal weight function and prove a Hilbert criterion for semistability. In other words, for such actions, the symplectic semistability and analytic semistability conditions are equivalent.

0 Introduction

The factorization problem for group actions in both algebraic geometry and complex geometry is a very interesting and important subject. It is well known that one should impose certain restrictions on the action in order to get a quotient with good properties. First, following the principles of the classical theory of invariants, as developed by Mumford, we will only consider actions of complex reductive groups. Second, it is well known that, in order to obtain a Hausdorff quotient with an induced complex space structure satisfying the natural universal property, one should not try to factorize the whole manifold on which the given reductive group $G$ acts, but only a certain open part of it (the so called stable locus). In the algebraic geometric framework, the stability condition depends on the choice of a linearization of the action in an ample line bundle [MFK], which is a purely algebraic geometric concept. Therefore, it

*The starting point of this article was a research project on “The universal Kobayashi-Hitchin correspondence and its applications” which began in Zürich in collaboration with Martin Lübke and Christian Okonek.
is not clear at all how to generalize this theory to the Kählerian non-algebraic framework.

The first aim of this paper is to give a systematic presentation of the different notions of stability in complex Kählerian non-algebraic geometry, and to explain the relations between these notions. There are two important motivations for writing this article:

1. In the mathematical literature one can find two distinct stability theories for actions of reductive groups $G$ on Kählerian manifolds: the symplectic (Hamiltonian) stability and the analytic stability. In the former theory ([HH], [Ki]) stability is checked using the position of the $G$-orbit with respect to the vanishing locus of a moment map with respect to a maximal compact subgroup of $G$. In the latter (see for instance [Mu]), one uses a numerical criterion, which can be regarded as a Kählerian version of the Hilbert Criterion in GIT. A well-known comparison result states that these two conditions are in fact equivalent. However, there is no analogous comparison result for the corresponding semistability conditions, and this seems to be a very delicate point. Moreover, it is not clear at all whether the analytical semistability condition is invariant under the $G$-action or whether this condition is an open condition. In particular, one cannot state that the analytically-semistable locus has a good quotient.

In fact, the two semistability conditions cannot be equivalent for general Hamiltonian actions on (non-compact) Kähler manifolds; one certainly needs a completeness condition for the action.

Indeed, suppose that in a compact manifold $X$ endowed with a Hamiltonian action with moment map $\mu$, the orbit $Gx_{x_0}$ is closed, $x_0 \in \overline{Gx} \cap \mu^{-1}(0)$, but $x_0 \notin Gx$. Then, in the open manifold $X \setminus Gx_0$ the point $x$ will be analytically semistable but no longer symplectically semistable.

2. The analytic semistability condition, as defined in the literature, does not have a purely complex geometric character; it depends on the choice of a maximal compact subgroup $K$ of the given reductive group $G$ and on a moment map for the induced $K$-action. It is not clear at all that changing $K$ (and modifying the Kähler metric and the moment map accordingly) will give the same analytic semistability condition.

On the other hand, the Hilbert criterion in the algebraic geometric GIT has obviously a purely complex geometric character; no differential geometric data are necessary.

Therefore, the main goals of this article are:

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1The difficulty comes from the fact that, in general, one has no $G$-equivariance property for the maximal weight function $(x, s) \mapsto \lambda^s(x)$ (see section 2).
To show that, for a large class of actions (which includes all Hamiltonian actions on compact Kählerian manifolds and all linear representations), the analytic semistability is $G$-invariant, has a purely complex geometric character and is an open condition. Therefore, one can state that, for this class of actions, the analytic semistability condition is the natural extension (to the Kählerian framework) of the algebraic geometric semistability condition provided by the Hilbert criterion in the algebraic geometric GIT.

To prove comparison results for our class of Hamiltonian actions relating the Hamiltonian (semi)stability to the analytic (semi)stability, and identifying the corresponding quotients.

Via this correspondence we will study carefully the "polystable" orbits, i.e., the complex orbits which intersect the vanishing locus of the moment map, and we characterize these orbits too with a numerical (analytic) criterion. Note that the space of polystable orbits can be identified with the underlying topological space of the Hamiltonian quotient, hence the polystable orbits are in fact those which effectively "contribute" to this quotient.

In the first chapter we review briefly the main results of Heinzner-Huckleberry-Loose [H], [HH], [HHL] concerning the existence of the Hamiltonian quotient of a Kähler manifold endowed with a Hamiltonian action, and we recall the first numerical criterion for symplectic stability. Many of the results in this chapter are well known, but we included short proofs for completeness. In the second chapter we introduce the concept of energy complete Hamiltonian action, concept which plays a fundamental role in our results. Any Hamiltonian action on a compact complex manifold and any "linear" Hamiltonian action is energy complete. We study the analytic stability, semistability and polystability conditions with respect to such an action, and we prove the fundamental properties of the (poly, semi)stable points. The third chapter is dedicated to comparison results and to explicit (poly, semi)stability criteria.

We believe that the energy completeness condition gives the natural framework for the stability theory in non-algebraic complex geometry. In the joint paper with L. Bruasse [BT], we showed that the theory of optimal destabilizing one-parameter subgroups and a very general Harder-Narasimhan type theorem can be extended from GIT to this very large class of holomorphic actions on complex manifolds.

The ideas and the methods of this article can be extended in the infinite dimensional gauge theoretical framework. This direction will be developed in a forthcoming article.
1 Symplectic stability and Hamiltonian Kähler quotients

1.1 Hamiltonian Kähler quotients

The symplectic (semi)stability condition and the theory of symplectic (Hamiltonian) Kählerian quotients have their roots in the Marsden-Weinstein theory of symplectic quotients. The great advantage of this approach is the generality: by the results of Heinzner-Huckleberry-Loose, the semistable locus of any holomorphic Hamiltonian action on any Kählerian manifold admits a good quotient which can be identified as a topological space with the corresponding (possibly singular) symplectic quotient. There is no compactness or completeness condition needed.

Let $\alpha : G \times X \to X$ be a holomorphic action of a reductive group $G$ on a connected complex manifold $X$. Suppose that there exists a Kählerian metric $g$ on $X$ and a a maximal compact subgroup $K$ of $G$ acting by isometries on $(X,g)$. Suppose that the restricted symplectic action $\alpha|_{K \times X} : K \times X \to X$ on the symplectic manifold $(X,\omega_g)$ is Hamiltonian, i.e. it has a moment map $\mu : X \to \mathfrak{k}^\vee$. We will suppose for simplicity that $\ker(G \to \text{Aut}(X))$ is discrete (otherwise one can factorize $G$ by the connected component $\ker(G \to \text{Aut}(X))_e$, which is a reductive normal subgroup of $G$).

For a point $x \in X$ we denote by $G_x$ (respectively $K_x$) its stabilizer subgroup with respect to the $G$-action ($K$-action) and by $\mathfrak{g}_x$ (respectively $\mathfrak{k}_x$) its Lie algebra, which is

$$\mathfrak{g}_x := \{ u \in \mathfrak{g} | u^\#_x = 0 \} \quad \text{and} \quad \mathfrak{k}_x := \{ u \in \mathfrak{k} | u^\#_x = 0 \}.$$

**Definition 1.1** A point $x \in X$ is called
1. symplectically $\mu$-stable if $\mathfrak{g}_x = 0$ and $G_x \cap \mu^{-1}(0) \neq \emptyset$.
2. symplectically $\mu$-semistable if $\overline{G_x} \cap \mu^{-1}(0) \neq \emptyset$.
3. symplectically $\mu$-polystable if $G_x \cap \mu^{-1}(0) \neq \emptyset$.

We denote by $X^{s}_{\mu}$, $X^{ss}_{\mu}$, $X^{ps}_{\mu}$ the loci of symplectically $\mu$-stable (respectively semistable, polystable) points. These loci are obviously $G$-invariant, so one can speak about stable (semistable, polystable) $G$-orbits.

We refer to [HH], Lemma 2.4.8, p. 325 for the following important result.

**Lemma 1.2** Let $H$ be a reductive subgroup of $G$, $L$ a compact maximal subgroup of $H$ and $K$ a maximal compact subgroup of $G$ which contains $L$. 


Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{l}$ in $\mathfrak{k}$ with respect to an ad-invariant inner product on $\mathfrak{k}$.

Then $K \exp(\mathfrak{m})$ is a closed submanifold of $G$ which intersects every right $H$-congruence class $\chi \in G/H$ along a unique right $L$-congruence class $\lambda \subset \chi$.

Therefore every element $g \in G$ can be decomposed as $g = k \gamma h$ with $k \in K$, $\gamma \in \exp(\mathfrak{m})$, and $h \in H$.

We will also need the following well known lemma (see [HH], section 3.2, p. 331). We include a self-contained proof for completeness.

**Proposition 1.3** If $\mu(x) = 0$, then $G_x = K_x^C$. In particular, the stabilizer $G_x$ of any symplectically polystable point $x$ is a reductive subgroup of $G$.

**Proof:** Suppose that $\mu(x) = 0$.

We will show first the infinitesimal version of the claimed formula, which is

$$\mu(x) = 0 \Rightarrow g_x = \mathfrak{t}_x^C$$

Let $w = u + iv \in g_x$ with $u, v \in \mathfrak{k}$. Then

$$0 = \langle w_x^\#, w_x^\# \rangle = \langle u_x^\#, u_x^\# \rangle + \langle v_x^\#, v_x^\# \rangle + 2\omega(u_x^\#, v_x^\#) = \|u_x^\#\|^2 + \|v_x^\#\|^2 + 2(v_x^\#(\mu^u)) = \|u_x^\#\|^2 + \|v_x^\#\|^2 + 2v_x^\#(\mu^u).$$

But,

$$v_x^\#(\mu^u) = \left. \frac{d}{dt} \right|_{t=0}(\mu^u(\exp(tv)x)) = \left. \frac{d}{dt} \right|_{t=0}(\mu^{\text{ad}_{\exp(tv)}}u(x)) = \mu^{[v,u]}(x) = 0.$$

Therefore, the above formula gives $u_x^\# = v_x^\# = 0$, hence $w \in \mathfrak{t}_x^C$.

Now we come back to the proof of the equality $G_x = K_x^C$. The inclusion $G_x \supset K_x^C$ is obvious, so let $g \in G_x$. By Lemma 1.2, we may decompose $g$ as $g = k\gamma h$ with $k \in K$, $\gamma \in \exp(\mathfrak{m})$ and $h \in K_x^C$, where $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{t}$ with respect to an ad-invariant inner product. We get

$$0 = \mu(x) = \mu(gx) = \text{ad}_k(\mu(\gamma x)),$$

hence $\mu(\gamma x) = 0$. Write $\gamma = \exp(m)$ with $m \in \mathfrak{m}$ and consider the real function $t \mapsto \lambda^m(t) = \mu^{-im}(\exp(tm)x)$.

This map is monotone increasing, because, by the properties of the moment map,

$$\frac{d}{dt}(\mu^{-im}(\exp(tm)x)) = d\mu^{-im}(J(-im)_{\exp(tm)x});$$
\begin{align*}
= \omega_g(-im^{\#}_{\exp(tm)x}, J(-im)^{\#}_{\exp(tm)x}) = g(m^{\#}_{\exp(tm)x}, m_{\exp(tm)x}) \
\text{Since } \lambda^m_x(0) = \lambda^m_x(1) = 0, \text{ we must have } \\
\frac{d}{dt}|_{t=0}(\lambda^m_x(t)) = \| m^\#(x) \|^2 = 0,
\end{align*}

hence \( m \in \mathfrak{g}_x \). But we know that, since \( \mu(x) = 0 \), one has \( \mathfrak{g}_x = \mathfrak{t}^C_x \). Therefore \( m \in \text{im} \cap \mathfrak{t}^C_x = \{0\} \), so \( g = kh \) with \( k \in K \) and \( h \in K^C_x \subset G_x \). This implies \( k \in K \cap G_x = K_x \), so indeed \( g \in K^C_x \).

The results of Heinzner-Huckleberry-Loose [H], [HHL], [HH] show that

**Theorem 1.4**

1. The subsets \( X^s_\mu, X^{ss}_\mu \) are open in \( X \).
2. The closure in \( X^{ss}_\mu \) of every \( \mu \)-semistable \( G \)-orbit contains a unique \( \mu \)-polystable orbit.
3. There is a good quotient \( q_\mu : X^{ss}_\mu \to Q \) with the properties
   (a) The induced morphism \( \mu^{-1}(0)/K \to Q \) is a homeomorphism,
   (b) Two \( \mu \)-semistable \( G \)-orbits have the same image in \( Q \) if and only if their closures in \( X^{ss}_\mu \) are not disjoint, and this happens if and only if the \( \mu \)-polystable orbits in their closures in \( X^{ss}_\mu \) coincide.

Therefore, the quotient \( Q \) can be identified with the space of polystable orbits, but, in general, the polystable locus is in general neither open nor closed. Moreover, in the algebraic geometric framework the stable and semistable loci are both Zariski open. The polystability condition does not appear at all in the classical GIT. The algebraic geometric version of the polystability condition was first introduced in [OST], by generalizing in a natural way the well-known polystability condition for holomorphic vector bundles.

As formulated above, the concepts of Hamiltonian (semi-, poly-) stability depends on the choice of three differential geometric objects:

- a maximal compact subgroup \( K \) of \( G \).
- a \( K \)-invariant Kähler metric \( g \) on the complex manifold \( X \).
- a moment map \( \mu \) for the symplectic action of \( K \) on \( (X, \omega_g) \).
It is very useful to notice that only the class of the triple \((K, g, \mu)\) with respect to the natural action of \(G\) on the set of such triples is essential. This motivates the following

**Definition 1.5** Let \(\alpha : G \times X \to X\) a holomorphic action of the reductive group \(G\) on the complex manifold \(X\). A **symplectization** of \(\alpha\) is an equivalence class of triples \((K, g, \mu)\) consisting of a maximal compact subgroup \(K\) of \(G\), a \(K\)-invariant Kähler metric \(g\) on \(X\), and a moment map \(\mu\) for the \(K\)-action on the symplectic manifold \((X, \omega_g)\).

Two such triples are considered equivalent if there exists \(\gamma \in G\) such that

\[
K' = \text{Ad}_\gamma(K), \quad g' = (\gamma^{-1})^*g, \quad \mu' = \text{ad}_{\gamma^{-1}} \circ \mu \circ \gamma^{-1}
\]

**Remark:** The concept "symplectization of a holomorphic action" should be regarded as the complex geometric analogous of the algebraic geometric concept "linearization of a regular action in an ample line bundle" in classical GIT.

It is convenient to fix a symplectization \(\sigma\) of our holomorphic action \(\alpha\), rather than a representative \((K, g, \mu)\), and to use the terminology "symplectically \(\sigma\)-stable" ("\(\sigma\)-semistable", "\(\sigma\)-polystable"), instead of "symplectically \(\mu\)-stable" (respectively "\(\mu\)-semistable", "\(\mu\)-polystable") for a point \(x \in X\) which satisfies the corresponding condition in Definition 1.1. We will use the notations \(X^s_\sigma\), \(X^s_{ss\sigma}\), \(X^{ps}_\sigma\) for the corresponding subsets of \(X\).

The quotient \(Q\) given by Theorem 1.4 will be called the Kählerian quotient of \(X\) with respect to the symplectization \(\sigma\) and will be denoted by \(Q_\sigma\). Fixing a representative \((K, g, \mu) \in \sigma\) gives a Kähler metric on the smooth part of the quotient \(Q_\sigma\).

### 1.2 A numerical criterion for symplectic stability and poly-stability

Let \(\alpha : G \times X \to X\) be a holomorphic action on a complex manifold \(X\) and let \((K, g, \mu)\) be a triple consisting of maximal compact subgroup \(K\) of \(G\), a \(K\)-invariant Kählerian metric \(g\) on \(X\), and a moment map \(\mu\) for the \(K\)-action on the symplectic manifold \((X, \omega_g)\).

The analytical stability condition involves a very important numerical invariant for a system of data as above, called (for historical reasons) the maximal weight function.

For pair \((x, s)\) with \(x \in X\) and \(s \in \mathfrak{k}\), consider the path \(c^s_x : [0, \infty) \to X\) and
the map $\lambda^s_x : [0, \infty) \to \mathbb{R}$ defined by

$$c^s_x(t) := e^{ts} x, \quad \lambda^s_x(t) := e^{s t} x,$$

where, in general, for $\xi \in \mathfrak{k}$, we use the notation $\mu^\xi$ for the map $X \to \mathbb{R}$ given by $y \mapsto \langle \mu(y), \xi \rangle$. One has (see the proof of Proposition 1.3)

$$\frac{d}{dt} \lambda^s_x(t) = g(\frac{d}{dt} c^s_x, \frac{d}{dt} c^s_x) = g(s^# \circ c^s_x, c^s_x),$$

(2)

hence, for fixed $s \in i\mathfrak{k}$, the map $\mathbb{R} \times X \ni (t, x) \mapsto \lambda^s_x(t) \in \mathbb{R}$ is increasing with respect to the first argument. We put

$$\lambda^s_x := \lim_{t \to \infty} \lambda^s_x(t) \in \mathbb{R} \cup \{\infty\},$$

and we call $\lambda^s_x$ the maximal weight of $x$ in the direction $s$. The above formula (2) shows that

$$\lambda^s_x = \lambda^s_x(0) + E_g(c^s_x),$$

(3)

where $E_g$ stands for the energy with respect to the metric $g$.

Using the fact that the map $K \times i\mathfrak{k} \to G$ given by $(k, s) \mapsto ke^{s}$ is a diffeomorphism (see for instance [HH]), it is not difficult to see that these conditions determine a unique function $\Psi : X \times G \to \mathbb{R}$. By (2) and the properties listed in Definition 1.6, one notes that

\begin{itemize}
  \item[1.] $\Psi(x, e) = 0$ for all $x \in X$.
  \item[2.] $\Psi$ is $K$-invariant from the left, i. e. $\Psi(x, kg) = \Psi(x, g)$ for all $x \in X$, $g \in g$, $k \in K$.
  \item[3.] $\Psi(x, gh) = \Psi(x, h) + \Psi(hx, g)$ for all $x \in X$, $g, h \in G$.
  \item[4.] $\frac{d}{dt}\Psi(x, e^{ts}) = \mu^{-i\xi}(e^{ts} x)$ ($= \lambda^s_x(t)$).
\end{itemize}

Using the fact that the map $K \times i\mathfrak{k} \to G$ given by $(k, s) \mapsto ke^{s}$ is a diffeomorphism (see for instance [HH]), it is not difficult to see that these conditions determine a unique function $\Psi : X \times G \to \mathbb{R}$. By (2) and the properties listed in Definition 1.6, one notes that

\begin{itemize}
  \item[1.] $\Psi|_{X \times K} \equiv 0$.
  \item[2.] For any fixed $x \in X$, $s \in i\mathfrak{k}$, the real function $t \mapsto \Psi(x, e^{st})$ is convex.
  \item[3.] For any $x \in X$ the restriction $\Psi|_{G_x}$ is an $\mathbb{R}$-valued group morphism.
\end{itemize}
Using the properties 3, 4 in Definition 1.6, one gets immediately the following simple but important

**Remark 1.8** Let $x \in X$. The following conditions are equivalent:

1. $g_0$ is a critical point of the map $\Psi(x, \cdot)$.
2. $\mu(g_0 x) = 0$

We will need the following well known lemma (see [Mu]). We include a proof for completeness.

**Lemma 1.9** Fix an $\text{ad}_K$-invariant metric on $\mathfrak{k}$ and a subspace $V \subset \mathfrak{k}$. For a point $x \in X$ the following conditions are equivalent

1. The map $\Psi(x, \exp(\cdot))$ is linearly proper on $V$, i.e. there exist positive constants $c_1, c_2$ such that
   $$\| s \| \leq c_1 \Psi(x, e^s) + c_2, \forall s \in V.$$
2. $\lambda^s(x) > 0$ for all $s \in V \setminus \{0\}$.

**Proof:** 1. $\Rightarrow$ 2.: The inequality in 1. gives for any $s \in V$, $t \in \mathbb{R}$
   $$t \| s \| \leq c_1 \Psi(x, e^{st}) + c_2.$$

This shows that
   $$\frac{d}{dt}\Psi(x, e^{st}) = \lambda^s_x(t) > 0$$
for $s \neq 0$ and sufficiently large $t \in \mathbb{R}$, hence $\lambda^s(x) > 0$.

2. $\Rightarrow$ 1. Suppose that there didn’t exist any positive constants $(c_1, c_2)$ with the required property. It would follow that there exist a sequence of $(s_n)_n$ in $V$
   $$\| s_n \| > n \Psi(x, e^{s_n}) + n^2.$$

Note that
   $$\lim_{n \to \infty} \| s_n \| = \infty,$$
because, if not, $(s_n)_n$ would have a bounded subsequence $(s_{n_m})_m$. But then $(\Psi(x, e^{s_{n_m}}))_m$ would be also bounded, and this obviously contradicts the above inequality.

We get
   $$\frac{\Psi(x, e^{s_n})}{\| s_n \|} < \frac{1}{n}$$
Put \( l_n := \| s_n \| \), \( u_n := \frac{s_n}{\| s_n \|} \), and choose \( t_0 \in \mathbb{R} \). The convexity property of the function \( \Psi \) (Remark 1.7) gives
\[
\Psi(x, e^{lu}) \geq \Psi(x, e^{lu}) + (l - t_0)\lambda_u(t_0)
\]
for every \( u \in i\mathfrak{k}, l \geq t_0 \). We obtain
\[
\frac{\Psi(x, e^{lu}) + (l - t_0)\lambda_u(t_0)}{l_n} \leq \frac{\Psi(x, e^{lu})}{\| s_n \|} < \frac{1}{n}.
\]
The sequence \((u_n)_n\) has a subsequence which converges to, say \( u_0 \in V \) which must have \( \| u_0 \| = 1 \). Taking the limit of the right hand term, we get \( \lambda_u(t_0) \leq 0 \). But this implies \( \lambda_u(x) \leq 0 \), which contradicts the hypothesis.

The following simple lemma will play a crucial role in the next chapter.

**Lemma 1.10** Let \( K \) be a maximal compact subgroup of \( G \) and let \( g \in G \), \( s \in \mathfrak{k} \) such that \( \text{ad}_g(s) \in \mathfrak{k} \). Decompose \( g \) as \( g = kh \), where \( k \in K \) and \( h \in \exp(i\mathfrak{k}) \). Then \( \text{ad}_h(s) = s \).

**Proof:** Since \( \text{ad}_g(s) \in \mathfrak{k} \), one has \( \sigma := \text{ad}_h(s) = \text{ad}_{e^{-i\sigma}}(\text{ad}_g(s)) \in \mathfrak{k} \). Choose an embedding \( G \hookrightarrow GL(r, \mathbb{C}) \) mapping \( K \) to \( U(r) \). Then the image of \( h \) is Hermitian with positive eigenvalues, whereas the images of \( s \) and \( \sigma \) are anti-Hermitian. We get
\[
-\text{ad}_h(s) = -\sigma = \sigma^* = \text{ad}_h(s)^* = -\text{ad}_{h^{-1}}(s),
\]
hence \( \text{ad}_{h^{-1}}(s) = s \). Therefore the eigenspaces of \( h^2 \) (which are the eigenspaces of \( h \)) are invariant under \( s \), so that one also has \( \text{ad}_h(s) = s \). ■

If \( \mathfrak{l} \) is a subset of the Lie algebra \( \mathfrak{k} \) of a Lie group \( K \), we will denote by \( Z_K(\mathfrak{l}) \) (respectively \( Z_{\mathfrak{g}}(\mathfrak{l}) \)) the centralizer of \( \mathfrak{l} \) in \( K \) (respectively \( \mathfrak{g} \)). The Lie algebra of \( Z_K(\mathfrak{l}) \) is \( Z_{\mathfrak{g}}(\mathfrak{l}) \).

**Remark 1.11** If \( K \) is a maximal compact subgroup of a complex reductive group \( G \) and \( \mathfrak{l} \subset \mathfrak{k} \), then \( Z_K(\mathfrak{l}) \) is a maximal compact subgroup of the reductive group \( Z_G(\mathfrak{l}) \).

**Proof:** One obviously has \( Z_{\mathfrak{g}}(\mathfrak{l}) = Z_{\mathfrak{k}}(\mathfrak{l}) \otimes \mathbb{C} \), so it suffices to prove that \( Z_G(\mathfrak{l}) = Z_K(\mathfrak{l}) \exp(iZ_{\mathfrak{k}}(\mathfrak{l})) \). Let \( g \in Z_G(\mathfrak{l}) \), and decompose \( g \) as \( g = kh \) with \( k \in K \) and \( h \in \exp(i\mathfrak{k}) \). By Lemma 1.10, it follows that \( h \in Z_G(\mathfrak{l}) \), hence \( k \in Z_K(\mathfrak{l}) \). But, using an embedding \( G \to GL(r, \mathbb{C}) \) mapping \( K \) to \( U(r) \), one gets easily that \( Z_G(\mathfrak{l}) \cap \exp(i\mathfrak{k}) = \exp(iZ_{\mathfrak{k}}(\mathfrak{l})) \). ■
Remark 1.12 Let $\mu$ be a moment map for an action of a compact group $K$ on a symplectic manifold $(M, \omega)$, and let $x \in M$. Then, via an identification $\mathfrak{k}^\vee = \mathfrak{k}$ given by an ad-invariant inner product on $\mathfrak{k}$, one has $\mu(x) \in \mathfrak{z}_\mathfrak{k}(k_x)$.

Proof: The equivariance property of the moment map gives

$$0 = \frac{d}{dt}|_{t=0} \mu(\exp(tu)x) = \frac{d}{dt}|_{t=0} \text{ad}_{\exp(tu)}(\mu(x)) = [u, \mu(x)].$$

Now we can prove the following important

Lemma 1.13 Let $\alpha : G \times X \to X$ an holomorphic action of a complex reductive group on a complex manifold, $K$ a maximal compact subgroup of $G$, $g$ a $K$-invariant Kähler metric on $X$ and $\mu$ a moment map for the $K$ action. Let $x \in X$ such that $\lambda^s(x) \geq 0$ for all $s \in iz_{\mathfrak{k}}(k_x)$ and $\lambda^s(x) > 0$ for all $s \in iz_{\mathfrak{k}}(k_x) \setminus iz_{\mathfrak{k}}(k_x)$. Then

1. There exists $s_0$ in the orthogonal complement $i[z(\mathfrak{k}_x)^{-1}t(\mathfrak{k}_x)]$ of $iz_{\mathfrak{k}}(k_x)$ in $iz_{\mathfrak{k}}(k_x)$ with respect to an ad-invariant inner product on $\mathfrak{k}$, such that

   $\mu(\exp(s_0)x) = 0$.

2. $x$ is symplectically polystable.

Proof:

1. Consider the restricted action $\alpha' := \alpha|_{Z_K(\mathfrak{k}_x) \times X}$ of the centralizer $Z_K(\mathfrak{k}_x)$ and the induced moment map $\mu'$ for this action. Note that the function $\Psi'$ associated with the triple $(Z_K(\mathfrak{k}_x), g, \mu')$ is just the restriction to $X \times Z_G(\mathfrak{k}_x)$ of the function $\Psi$ corresponding to $(K, g, \mu)$ (see Remark 1.11).

   Apply Lemma 1.9 to the triple $(Z_K(\mathfrak{k}_x), g, \mu')$ taking $V := i[z(\mathfrak{k}_x)^{-1}t(\mathfrak{k}_x)]$. It follows that $\Psi'(x, \exp(\cdot))$ is linearly proper on this space, hence there exist positive constants $c_1, c_2$ such that

   $$\|s\| \leq c_1 \Psi'(x, e^s) + c_2,$$

   for all $s \in V$. This inequality implies that $\Psi'(x, \exp(\cdot))$ is bounded from below on $V$. Put

   $$m := \inf_{s \in V} \Psi'(x, \exp(\cdot)).$$
Let \((s_n)_n\) be a sequence in \(V\) such that \(\Psi'(x, e^{s_n}) \to m\). By (4) it follows that \((s_n)_n\) is bounded, so it has a subsequence which converges to, say, \(s_0 \in V\). One gets
\[
\Psi'(x, e^{s_0}) = m.
\] (5)

We claim that in fact
\[
m = \inf_{g \in Z_G(x)} \Psi'(x, g).
\] (6)

Indeed, by Lemma 1.2 applied to the reductive subgroup \(H := \exp(z\langle k \rangle x)\) of \(Z_G(x)\), it follows that any element \(g \in Z_G(x)\) can be written as \(g = k\gamma h\) with \(k \in Z_K(x)\), \(\gamma \in \exp(V)\) and \(h \in H\). We have
\[
\Psi'(x, g) = \Psi'(x, k\gamma h) = \Psi'(x, \gamma h) = \Psi'(x, h) + \Psi'(hx, \gamma) = \Psi'(x, \gamma),
\]
because \(\Psi(x, h) = 0\). To see this recall that, by Remark 1.7 the restriction \(\Psi'(x, \gamma)|_H\) is an \(\mathbb{R}\)-valued group morphism which vanishes on the maximal compact subgroup \(\exp(z\langle k \rangle x) = Z(K)_e\). But the derivative of this morphism in the \(s \in iz\langle k \rangle x\) direction is \(\mu'(x) = \lambda'(x) \geq 0\), by assumption. Therefore \(d_\gamma \Psi'(x, \gamma)|_iz\langle k \rangle x\) is an \(\mathbb{R}_{\geq 0}\)-valued real linear form, hence it vanishes.

This proves the claimed formula (6). From (5) and (6) we get that \(e^{s_0}\) is a critical point of the map \(\Psi'(x, \cdot)\), hence \(\mu'(\exp(s_0)x) = 0\), by Remark 1.8.

The point is that, in our situation, \(\mu'(\exp(s_0)x) = 0\) implies the stronger relation \(\mu(\exp(s_0)x) = 0\). Indeed, since
\[
G_{\exp(s_0)x} = \exp(s_0)G_x \exp(-s_0)
\]
and \(\exp(s_0) \in Z_G(x)\) one gets \(g_{\exp(s_0)x} \supset \mathfrak{t}_x\), so
\[
\mathfrak{t}_x = \mathfrak{t}_x \cap \mathfrak{t} \subset g_{\exp(s_0)x} \cap \mathfrak{t} = g_{\exp(s_0)x}.
\]

Therefore, by Remark 1.12, \(\mu(\exp(s_0)x) \in z(\mathfrak{t}_x)\). But, via our identification \(\mathfrak{t}^\mathfrak{p}_x = \mathfrak{t}\), \(\mu'(\exp(s_0)x)\) is just the orthogonal projection of \(\mu(\exp(s_0)x)\) on \(z(\mathfrak{t}_x)\), hence \(\mu(\exp(s_0)x) = \mu'(\exp(s_0)x) = 0\).

2. This follows immediately from 1.

The following numerical criterion is well-known in the stable case (see [Mu]).

**Proposition 1.14** Let \(x \in X\).
1. The following conditions are equivalent:

(a) $x$ is symplectically $\mu$-stable.

(b) $\lambda^s_x \geq 0$ for all $s \in i \mathfrak{k}$ with strict inequality for $s \in i \mathfrak{k} \setminus \{0\}$.

2. The following conditions are equivalent:

(a) $x$ is symplectically $\mu$-polystable.

(b) There exists $g \in G$ such that $\lambda^s_{gx} \geq 0$ for all $s \in i \mathfrak{k}$ with strict inequality for $s \in i \mathfrak{k} \setminus i \mathfrak{k}gx$.

Proof:

1a) $\Rightarrow$ 1b): Let $x$ be a symplectically stable point and choose $g_0 \in G$ such that $x_0 := g_0x \in \mu^{-1}(0)$. We prove first that $x_0$ has the claimed property, namely that $\lambda^s(x_0) \geq 0$ for all $s \in i \mathfrak{k}$ with strict inequality for $s \neq 0$. But $\lambda^s(x_0) = \lambda^s_{x_0}(0) = 0$, because $\mu(x_0) = 0$.

For $s \neq 0$ we have $s \not\in g_0x$, so $s \not\in g_0x$ is not contained in $i \mathfrak{k}$, which implies

$$\lambda^s(x_0) > \lambda^s_{x_0}(0) = 0.$$ 

This proves the claimed property for $x_0$. Unfortunately these properties do not appear to be $G$-invariant, so one cannot deduce directly that the same is true for $x$. By Lemma 1.9, the map $\Psi(x_0, \exp(\cdot))$ is linearly proper on $i \mathfrak{k}$. One can write

$$\Psi(x, \exp(s)) = \Psi(g_0^{-1}x_0, \exp(s)) = \Psi(x_0, \exp(s)g_0^{-1}) - \Psi(x_0, g_0^{-1}).$$

Write $\exp(s)g_0^{-1} = k(s)\exp(v(s))$ with $k(s) \in K$ and $v(s) \in i \mathfrak{k}$. With these notations one has $\Psi(x_0, \exp(s)g_0^{-1}) = \Psi(x_0, \exp(v(s)))$. It is easy to prove (see [Mu]) an estimate of the form

$$\|s\| \leq a\|v(s)\| + b, \forall s \in i \mathfrak{k}.$$ 

Therefore the linearly properness of $\Psi(x, \exp(\cdot))$ on $i \mathfrak{k}$ follows from the linearly properness of $\Psi(x_0, \exp(\cdot))$ on $i \mathfrak{k}$. Applying again Lemma 1.9, we get the desired property for $x$.

1b) $\Rightarrow$ 1a): Suppose that 1b) holds for $x$. First of all notice that, by Lemma 1.13, there exists $s_0 \in i \mathfrak{k}$ such that $\mu(\exp(s_0)x) = 0$. It remains to show that $g_x = \{0\}$. Put $x_0 := \exp(s_0)x$. Arguing as above we see that $\Psi(x, \exp(\cdot))$ and $\Psi(x_0, \exp(\cdot))$ are both linearly proper on $i \mathfrak{k}$, hence 1b) also holds for $x_0$. We know by Proposition 1.3 that $G_{x_0} = K^s_{x_0}$, hence it suffices to show that $i \mathfrak{k}x_0 = 0$. If $s \in i \mathfrak{k} \setminus \{0\}$, one has $\lambda^s_{x_0}(0) = \mu^{-s}(x_0) = 0$ and $\lambda^s(x_0) > 0$, hence the path $e^s_{x_0}$ cannot be constant, so $s \not\in i \mathfrak{k}x_0$.

2a) $\Rightarrow$ 2b)
Let $x_0 \in Gx$ such that $\mu(x_0) = 0$. The same method as in the case of stability, gives
\[ \lambda^s(x_0) = 0 \text{ for } s \in i\mathfrak{k}_{x_0}, \quad \lambda^s(x_0) > 0 \text{ for } s \in i\mathfrak{k} \setminus i\mathfrak{k}_{x_0}. \]

2b) $\Rightarrow$ 2a)

Put $x' := gx$, where $g \in G$ has the property in 2b).

By Lemma 1.13, there exists $s_0 \in i\mathfrak{k}$ such that $\mu(\exp(s_0)x') = 0$. Thus $x$ is symplectically polystable.

**Remark:** The numerical criterion provided by Proposition 1.14 is not satisfactory for the following important reasons:

- It depends essentially on the choice of a particular triple $(K, g, \mu)$, not only on its equivalence class (the symplectization defined by this triple). Therefore, it does not have a purely complex geometric character.
- In general, for a polystable point $x$, one might have $\lambda^s(x) = 0$ even for vectors $s \not\in i\mathfrak{k}_x$. Therefore, in order to test whether a point $x$ is polystable or not, one has to control all the "maximal weights" $\lambda^s_{gx}$ as $g$ vary in $G$, so this is not an intrinsic criterion in terms of the given point $x$.
- This criterion does not provide any numerical characterization of symplectic semistability.

One of our main goals is to address all these issues, and to give intrinsic, purely complex geometric numerical criteria for stability, polystability and semistability (see Theorems 3.1, 3.3). In order to get stronger comparison results, we will have to assume that the triple $(K, g, \mu)$ satisfies a certain completeness condition, which we will call energy completeness. Note that for general Hamiltonian actions, analytic semistability does not imply symplectic semistability and there is no way to construct a good quotient of the analytically semistable locus.

## 2 Analytic stability, semistability and polystability

Analytic stability is a purely numerical condition, so it is very useful for practical reasons. The analytic stability condition is the complex geometric analogue of the numerical condition in the Hilbert criterion in classical GIT.
2.1 The function $\lambda$ associated with an energy complete symplectization

The cone of Hermitian type vectors. The set Hom($\mathbb{C}^*, G$) of one parameter subgroups of $G$ can be identified with a subset $\Lambda(G)$ of $\mathfrak{g}$ via the map

$$\lambda \mapsto d_1(\lambda)(1) = \frac{d}{dt} \bigg|_{t=0} (\lambda(e^t)) .$$

In order to formulate a numerical stability condition in complex non-algebraic geometry one needs a larger subset of $\mathfrak{g}$ whose elements can be interpreted as ”non-algebraic” one parameter subgroups of $G$.

**Definition 2.1** Let $G$ be complex reductive group. An element $s \in \mathfrak{g}$ will be called of Hermitian type if it satisfies one of the following equivalent properties:

1. There exists a complex torus $C \subset G$ such that $s \in it$, where $t$ is the Lie algebra of the (unique) maximal compact subgroup $T$ of $C$.
2. The closure of the real one parameter subgroup of $G$ determined by $is \in \mathfrak{g}$ is compact.
3. There exists a compact subgroup $K \subset G$ such that $s \in iK$.
4. There exists $r \in \mathbb{N}$ and an embedding $\rho : G \hookrightarrow GL(r, \mathbb{C})$ such that $\rho_\ast(s)$ is Hermitian.
5. There exists $r \in \mathbb{N}$ and an embedding $\rho : G \hookrightarrow GL(r, \mathbb{C})$ such that the matrix $\rho_\ast(s)$ is diagonalizable and has real eigenvalues.
6. For every embedding $\rho : G \hookrightarrow GL(r, \mathbb{C})$ the matrix $\rho_\ast(s)$ is diagonalizable and has real eigenvalues.

**Remark 2.2**
1. The set $H(G) \subset \mathfrak{g}$ of elements of Hermitian type is obviously invariant under the adjoint action of $G$ on $\mathfrak{g}$.
2. If $s \in H(G)$ then the associated endomorphism $[s, \cdot] \in \text{End}(\mathfrak{g})$ is diagonalizable and has only real eigenvalues.

One can associate to every $s \in H(G)$ a parabolic subgroup $G(s) \subset G$ in the following way:

We put

$$G(s) := \{ g \in G \mid \lim_{t \to \infty} e^{rt} g e^{-st} \text{ exists in } G \} .$$
The group \( G(s) \) fits in the exact sequence

\[
1 \longrightarrow U(s) \longrightarrow G(s) \longrightarrow Z(s) \longrightarrow 1 ,
\]

where \( Z(s) \) is the centralizer of \( s \) in \( G \) and \( U(s) \) is the unipotent subgroup

\[
U(s) := \{ g \in G \mid \lim_{t \to \infty} e^{st}ge^{-st} = e \} .
\]

Moreover, it decomposes as a semi-direct product

\[
G(s) = Z(s) \cdot U(s) . \tag{7}
\]

The groups \( G(s), U(s) \) depend only on the semisimple part \( s_0 \) of \( s \). The Lie algebras of \( G(s), Z(s) \) and \( U(s) \) are

\[
\mathfrak{g}(s) := \bigoplus_{\beta \leq 0} \text{Eig}([s, \cdot], \beta) , \quad \mathfrak{z}(s) := \ker([s, \cdot]) , \quad \mathfrak{u}(s) := \bigoplus_{\beta < 0} \text{Eig}([s, \cdot], \beta) .
\]

Proposition 2.3 The adjoint representation defines an affine representation on the affine subspace \( p_{\mathfrak{z}(s)}^{-1}(s) \subseteq \mathfrak{g}(s) \). This action is free and transitive.

Proof: It is easy to see that the differential in \( e \) of the map \( \varphi_\zeta : U(s) \to p_{\mathfrak{z}(s)}^{-1}(s) \) given by \( u \mapsto \text{ad}_u(\zeta) \) is invertible, for any \( \zeta \in p_{\mathfrak{z}(s)}^{-1}(s) \). It follows that \( \varphi_\zeta \) is étale. The stabilizer of any point \( \zeta \in p_{\mathfrak{z}(s)}^{-1}(s) \) is trivial, because it is a Zariski closed 0-dimensional subgroup of a unipotent group. Therefore \( \varphi_\zeta \) is an algebraic isomorphism from \( U(s) \) (which, as an algebraic variety, is isomorphic to an affine space) onto a Zariski open subset of the affine space \( p_{\mathfrak{z}(s)}^{-1}(s) \). It follows that \( \varphi_\zeta \) must be surjective.

Corollary 2.4 Let \( s, \sigma \in H(G) \).

1. The following three conditions are equivalent:
   i) \( \sigma \in \mathfrak{g}(s) \) and \( p_{\mathfrak{z}(s)}^{-1}(\sigma) = s \).
   ii) \( s \) and \( \sigma \) are conjugate under the adjoint action of \( U(s) \).
   iii) \( s \) and \( \sigma \) are conjugate under the adjoint action of \( G(s) \).
2. If one of these conditions is satisfied then \( G(s) = G(\sigma) \).
3. The equivalent conditions i) – iii) define an equivalence relation \( \sim \) on \( H(G) \). The equivalence class of \( s \in H(G) \) is the affine subspace \( p_{\mathfrak{z}(s)}^{-1}(s) \) of the Lie algebra \( \mathfrak{g}(s) \).

Corollary 2.5 Let \( K \) be a maximal compact subgroup of \( G \). Then \( iK \subseteq H(G) \) is a complete system of representatives for \( \sim \). Mapping any \( s \in H(G) \) to the representative in \( iK \) of its equivalence class, gives a retraction \( \sigma_K : H(G) \to iK \), which induces a homeomorphism \([H(G)/\sim] \to iK\).
Proof: Let $K_0$ be a maximal compact subgroup of $G$ such that $s \in i\mathfrak{k}_0$, and let $g \in G$ such that $\text{Ad}_g(K_0) = K$. Therefore $\text{ad}_g(s) \in i\mathfrak{k}$. Decompose $g$ as $g = kb$, where $k \in K$ and $b$ belongs to a Borel subgroup of $G$ contained in $G(s)$.

Then $s \sim \text{ad}_b(s) = \text{ad}_{k^{-1}}\text{ad}_k\text{ad}_b(s) = \text{ad}_{k^{-1}}(\text{ad}_g(s)) \in i\mathfrak{k}$.

Now suppose that $s, s' \in i\mathfrak{k}$ and that $s \sim s'$. It follows that $s' - s \in u(s) \cap i\mathfrak{k}$. But $u(s)$ is a complex Lie algebra, hence $i(s' - s) \in u(s) \cap i\mathfrak{k}$. This would imply that the closure of the real 1-parameter subgroup generated by $i(s' - s)$ is compact and contained in $U(s)$. But an unipotent group contains no compact subgroups, which completes the proof. 

\[\text{Corollary 2.6} \quad \text{An embedding } G \hookrightarrow G' \text{ of reductive Lie groups induces injections} \]

\[H(G) \hookrightarrow H(G'), \quad H(G)/\sim_G \hookrightarrow H(G')/\sim_{G'}.\]

Proof: It is easy to see that, when $s \in H(G)$, then $u_{G'}(s) \cap g = u_G(s)$, where $u_G(s)$ stands for the unipotent algebra associated to $s$, regarded as an element of $H(G)$. Therefore an element $s \in H(G)$ is equivalent to $s$ as elements in $H(G)$ if and only if they are equivalent as elements in $H(G')$. 

Example: Consider the case $G = GL(r, \mathbb{C})$. The data of an equivalence class in $H(G)$ is equivalent to the data of a pair $(\Phi, \eta)$, where $\Phi$ is a filtration of the form $\{0\} \subset F_1 \subset \ldots \subset F_k = \mathbb{C}^r$ and $\eta$ is an increasing sequence $\eta_1 < \ldots < \eta_k$ of real numbers. An element $s \in \mathfrak{gl}(r, \mathbb{C})$ belongs to the equivalence class defined by $(\Phi, \eta)$ iff it is diagonalizable, its spectrum is $\eta$, and the associated eigenspace filtration $(\oplus_{i=1}^r (\text{Eig}(s, \eta_i)))_i$ of $\mathbb{C}^r$ is $\Phi$.

Proposition 2.7 Let $G$ be reductive complex group and let $s \in H(G)$ and let $K$ be a maximal compact subgroup of $G$ such that $s \in i\mathfrak{k}$. Then the image of the orbit $\text{ad}_G(s) := \{\text{ad}_g(s) \mid g \in G\}$ of $s$ via the map

\[p_K : H(G) \longrightarrow H(G)/\sim \longrightarrow \mathfrak{k}\]

is the compact orbit $\text{ad}_K(s) := \{\text{ad}_k(s) \mid k \in K\}$ of $s$ in $i\mathfrak{k}$.

Proof: The inclusion of $\text{ad}_K(s)$ in the projection of $\text{ad}_G(s)$ is clear. Conversely, let $s' \in p_K(\text{ad}_G(s))$. This means that $s' \in i\mathfrak{k}$ and there exists $g \in G$ such that $s' \sim \text{ad}_g(s)$. Therefore there exists $\gamma \in G(\text{ad}_g(s))$ such that $s' = \text{ad}_{\gamma g}(s)$. Decompose $g' := \gamma g$ as $g' = k'h'$ with $k' \in K$ and $h' \in \exp(i\mathfrak{k})$. By Lemma 1.10, it follows that $\text{ad}_{h'}(s) = s$, hence $s' = \text{ad}_{k'}(s)$. 

\[\blacksquare\]
**Energy complete symplectizations.** Let now $\alpha : G \times X \to X$ be an action of a complex reductive group $G$ on a complex manifold $X$.

To every pair $(s, x) \in H(G) \times X$ we associate a curve $c^s_x : [0, \infty) \to F$ given by $c^s_x(t) := e^{ts}x$.

**Definition 2.8** A symplectization $\sigma$ of the action $\alpha$ will be called energy-complete if, choosing a representative $(K, g, \mu) \in \sigma$, the following holds:

$$\forall x \in X \forall s \in \mathfrak{k} \ (E_g(c^s_x) < \infty \Rightarrow \text{the curve } c^s_x \text{ has a limit as } t \to \infty). \quad (C)$$

Here we denoted by $E_g$ the energy with respect to the metric $g$. Using the obvious identities

$$c^{\gamma, \gamma^{-1} \gamma}(s) = \gamma c^s x, \quad E_{\gamma^{-1} \gamma} g(c^{\gamma, \gamma^{-1} \gamma}(s)) = E_g(c^s_x),$$

one checks that the condition $(C)$ does not depend on the choice of the representative $(K, g, \mu)$ of $\sigma$.

**Proposition 2.9**

1. A symplectization $\sigma$ is energy complete if and only if, choosing any representative $(K, g, \mu) \in \sigma$, the following implication holds:

$$\forall x \in X \forall s \in i \mathfrak{k} \ (E_g(c^s_x) < \infty \Rightarrow \exists (t_n)_n \text{ with } t_n \to \infty \text{ s. t. } c^s_x(t_n) \text{ converges}). \quad (8)$$

2. Any symplectization of an action $\alpha : G \times X \to X$ of a complex reductive group on a compact complex manifold is energy complete.

3. Let $\alpha : G \times V \to V$ be a linear action of a complex reductive group on a finite dimensional Hermitian vector space $(V, h)$ and let $K$ be a maximal compact subgroup of $G$ which leaves $h$ invariant. Then any symplectization $[(K, h, \mu)]$ of $\alpha$ is energy complete.

**Proof:**

1. It’s clear that the condition $(C)$ in the definition implies $(8)$. Conversely, suppose that $(C)$ holds, let $(x, s) \in X \times i \mathfrak{k}$ and let $(t_n)_n$ be a sequence in $\mathbb{R}_{\geq 0}$ with $t_n \to \infty$ such that $c^s_x(t_n)$ converges to a point, say $x_0$, in $X$. We will show that

$$\lim_{t \to \infty} c^s_x(t) = x_0 \quad (9).$$

Consider the compact torus $T = \{\exp(it s) | t \in \mathbb{R}\} \subset K$, and let $T^C \subset G$ be its complexification.

Let $\iota : t \mapsto \mathfrak{k}$ be the Lie algebra monomorphism induced by the inclusion $T \hookrightarrow K$. The maps $\mu_T = \iota^* \mu : X \to \mathfrak{k}^*$, $\mu_T^* := \mu_T - \mu_T(x_0)$ are moment maps for the
induced $T$ action on $X$, and obviously $\mu_T'(x_0) = 0$. Using the results in [HH], Theorem 3.3.14 p. 343 and section 4.1 p. 345 - 346, one can find a $T^C$-invariant Stein neighborhood $U$ of $x_0$, a linear representation $\rho : T^C \to GL(V)$ on a finite dimensional vector space $V$, and a $T^C$-equivariant isomorphism $\theta : U \to W$ on a closed $T^C$-invariant complex subspace $W$ of $V$. Let $R \subset \text{Hom}(t^C, \mathbb{C})$ be the root set of $\rho$ and $V = \oplus \chi \in R V\chi$ be the eigenspace decomposition of $V$ with respect to $\rho$. Since $U$ is open, one has $c_x^*(t_n) = \exp(t_n s)x \in U$ for sufficiently large $n$, hence, since $U$ is also $T^C$-invariant, one gets $x \in U$. Putting $v := \theta(x)$, one can write  
\[ \theta(c_x^*(t)) = \theta(\exp(ts)x) = \rho(\exp(ts))\theta(x) = \sum_{\chi \in R} e^{t\chi(s)}v_\chi \]. (10)

Since $c_x^*(t_n) \to x_0$, it follows that $\theta(c_x^*(t_n)) \to v_0 := \theta(x_0)$ so that, by (9), one must have 
\[ \chi(s) > 0 \Rightarrow v_\chi = 0 \].

But this implies $\lim_{t \to \infty} \theta(c_x^*(t)) = v_0$, hence $\lim_{t \to \infty} c_x^*(t) = x_0$.

2. Follows easily from 1.

3. The standard moment map for the $K$-action on $V$ is 
\[ \mu_0(v) = \varsigma(\frac{-i}{2}v \otimes v^*) \],
where $\varsigma : t \to u(V)$ is the morphism induced by the representation $K \to U(V)$ induced by $\alpha$. Any other moment map has the form 
\[ \mu_\tau = \mu_0 - i\tau \]
where $\tau$ is a central element in $\mathfrak{t}^V$.

Let $s \in \mathfrak{t}$ and $V = \oplus_{j=1}^k V_j$ the decomposition of $V$ as a the direct sum of $\varsigma_\alpha(s)$. Therefore $\varsigma_\alpha(s)|V_j = s_j id_{V_j}$, where $s_j$ are the eigenvalues of $\varsigma_\alpha(s)$.

Let $v \in V$. Decompose $v$ as $v = \sum_j v_j$ with $v_j \in V_j$.

Using the symplectization defined by $\mu_\tau$, one gets easily 
\[ E_h(c_v^*) = \begin{cases} +\infty & \text{if there exists } j \text{ such that } s_j > 0 \text{ and } v_j \neq 0 \\
\langle \tau, s \rangle - \langle \mu_\tau(v), -is \rangle & \text{otherwise.} \end{cases} \]

Therefore, if $E_h(c_v^*) < \infty$, one must have 
\[ v_j \neq 0 \Rightarrow s_j \leq 0 \].

But this clearly implies that $c_v^*(t) = \sum_j e^{ts_j}v_j$ tends to $\sum_{j, s_j=0} v_j$ as $t \to \infty$. \[ \blacksquare \]
The map $\lambda$ on the cone $H(G)$. Fix a representative $(K, g, \mu) \in \sigma$. We recall (see section 1.2) that to every pair $(s, x) \in i_k \times X$ we associated the map

$$\lambda^s_x : \mathbb{R}_{\geq 0} \to \mathbb{R}, \quad \lambda^s_x(t) := \mu^{-is}(c^s_x(t)),$$

where $\mu^\xi := \langle \mu, \xi \rangle : X \to \mathbb{R}$ and $c^s_x(t) := \exp(ts)x$. If we choose an equivalent triple

$$(K', g', \mu') = (\text{Ad}_\gamma(K), (\gamma^{-1})^*g, \text{ad}_{\gamma^{-1}} \circ \mu \circ \gamma^{-1})$$

the corresponding maps $\lambda'$ are given by

$$\lambda'^s_x(t) = \lambda_{\gamma^{-1}x}^{\text{ad}_{\gamma^{-1}}(s)}(t).$$

Moreover, using the equivariance property of the moment map with respect to the $K$-action, one gets

$$\lambda^s_{kh}(t) = \lambda^s_x(t) \quad \forall k \in K \forall t \in \mathbb{R}.$$  

We recall that $\lambda^s_x$ is increasing and that we put

$$\lambda^s(x) := \lim_{t \to \infty} \lambda^s_x(t) \in \mathbb{R} \cup \{\infty\}.$$  

so that $\lambda^s(x) = \lambda^s_x(0) + E_g(c^s_x)$ (see section 1.2).

**Lemma 2.10** Suppose that $\sigma$ is energy complete. The map

$$x \mapsto \lambda^s_x : X \to \mathbb{R}$$

does not depend on the representative $(K, g, \mu) \in \sigma$ with $s \in \mathfrak{k}$.

**Proof:** Let $(K, g, \mu) \in \sigma$ such that $s \in \mathfrak{k}$ and let $\gamma \in G$ such that $\text{ad}_{\gamma^{-1}}(s) \in \mathfrak{k}$. We consider the representative $(K', g', \mu') = (\text{Ad}_\gamma(K), (\gamma^{-1})^*g, \text{ad}_{\gamma^{-1}} \circ \mu \circ \gamma^{-1})$ of $\sigma$. Taking into account the equivariance formula (11) we have to show that

$$\lim_{t \to \infty} \lambda^s_x(t) = \lim_{t \to \infty} \lambda_{\gamma^{-1}x}^{\text{ad}_{\gamma^{-1}}(s)}(t).$$

Decompose $\gamma^{-1}$ as $\gamma^{-1} = kh$, where $k \in K$, $h \in \exp(\mathfrak{t})$. By Lemma 1.10, $\text{ad}_h(s) = s$, hence, by (13), we get

$$\lambda_{\gamma^{-1}x}^{\text{ad}_{\gamma^{-1}}(s)}(t) = \lambda_{khx}^{\text{ad}_h(s)}(t) = \lambda^s_{hx}(t).$$

Therefore, our claim (15) reduces to the formula

$$\lim_{t \to \infty} \lambda^s_x(t) = \lim_{t \to \infty} \lambda^s_{hx}(t),$$

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for \( h \in \exp(\mathfrak{g}) \) with \( \text{ad}_h(s) = s \).

Suppose first that \( E_g(c^s_x) < \infty \). Since \( \alpha \) is energy complete, the limit
\[
l = \lim_{t \to \infty} e^{ts}x \in X
\]
exists. Choose \( \chi \in \mathfrak{g} \) such that \( h = e^\chi \). We obtain
\[
\lambda^s_x(t) - \lambda^s_{hx}(t) = \mu^{-is}(e^{ts}x) - \mu^{-is}(e^{ts}hx) = \mu^{-is}(e^{ts}x) - \mu^{-is}(he^{ts}x) =
\]
\[
= -\int_{e^{ts}x|_{[0,1]}} \omega_g(s#, \cdot) ,
\]
because \( c^\chi_{e^{ts}}|_{[0,1]} \) is a curve joining \( e^{ts}x \) to \( he^{ts}x \).

But \( d\mu^{-is}(\cdot) = \omega_g((-is)\#, \cdot) = g(s\#, \cdot) \), hence
\[
\lim_{t \to \infty} (\lambda^s_x(t) - \lambda^s_{hx}(t)) = -\int_{c^\chi_{e^{ts}}|_{[0,1]}} g(s\#, dc^\chi_x) .
\]

Since \([\chi, s] = 0\), one has \( c^\chi_l(\tau) = e^{\tau\chi}l = \lim_{t \to \infty} e^{\tau\chi}e^{ts}x = \lim_{t \to \infty} e^{ts}e^{\tau\chi}x \), so \( c^\chi_l(\tau) \) is a fixed point of the local flow associated with the vector field \( s\# \).

Therefore \( s\# \) vanishes identically along the curve \( c^\chi_l \), and this completes the proof in the case \( E_g(c^s_x) < \infty \). We argue similarly when \( E_g(c^s_{hx}) < \infty \); we just replace \( x \) by \( hx \) and \( h \) by \( h^{-1} \).

If finally \( E_g(c^s_x) = E_g(c^s_{hx}) = \infty \), we get by (3) that \( \lambda^s(x) = \lambda^s(hx) = \infty \). 

The previous lemma plays a crucial role in developing our stability concept: it allows us to associate to any energy complete symplectization \( \sigma \) a well defined map
\[
\lambda : H(G) \times X \to \mathbb{R} \cup \{\infty\} , (s, x) \mapsto \lambda^s(x)
\]
which is \textit{intrinsically} associated with \( \sigma \) and has the following important properties.

**Proposition 2.11** Suppose that \( \sigma \) is energy complete. The map \( \lambda \) introduced above satisfies the following properties:

1. homogeneity: \( \lambda^{ts}(x) = t\lambda^s(x) \) for any \( t \in \mathbb{R}_{\geq 0} \).
2. equivariance: \( \lambda^s(x) = \lambda^{\text{ad}_s(\gamma)}(\gamma x) \) for all \( s \in H(G) \), \( \gamma \in G \).
3. parabolic invariance: \( \lambda^s(x) = \lambda^s(hx) \) for every \( h \in G(s) \).
4. \( \sim \) invariance: \( \lambda^s(x) = \lambda^s(x) \) if \( s \sim \sigma \).
5. semicontinuity: if \((x_n, s_n)_n \to (x, s)\) then \(\lambda^s(x) \leq \lim_{n\to\infty} \lambda^s_n(x_n)\).

**Proof:** The first statement follows directly from the definition. The second property follows from Lemma 2.10 and (12). To prove the third, use (7) and put \(h = h_0 h'\), where \(h_0, h' \in G\) with \(\text{ad}_{h_0}s = s\) and \(\lim_{t \to \infty} e^{ts} h' e^{-ts} = e\). We get by 1.

\[
\lambda^s(hx) = \lambda^s(h_0 h' x) = \lambda^{\text{ad}_{h_0}^{-1}(s)}(h'x) = \lambda^s(h'x),
\]

so it remains to show that \(\lambda^s(h'x) = \lambda^s(x)\).

Suppose that \(\lambda^s(x) < \infty\). By energy completeness, this implies that the limit \(l = \lim_{t \to \infty} e^{ts} x\) exists in \(X\). Therefore

\[
\lim_{t \to \infty} e^{ts} h' e^{ts} x = \lim_{t \to \infty} e^{ts} h' e^{-ts} e^{ts} x = l,
\]

because \(e^{ts} h' e^{-ts} \to e\). Therefore

\[
\lambda^s(h'x) = \lim_{t \to \infty} \mu^{-is}(e^{ts} h' x) = \mu^{-is}(l) = \lambda^s(x).
\]

In the case \(\lambda^s(h'x) < \infty\) we argue similarly. This completes the proof of the third property.

The fourth follows immediately from 2. and 3.

For the fifth, note first that we may suppose that there exists a maximal compact subgroup \(K\) of \(G\) such \(s_0 \in iK\). If our conclusion was false, there would exist \(\varepsilon > 0\) and a subsequence \((x_{n_m}, s_{n_m})_m\) of \((x_n, s_n)_n\) such that the limit \(\lim_{m \to \infty} \lambda^{s_{n_m}}(x_{n_m})\) exists, is finite and

\[
\lambda^s(x) \geq \lim_{m \to \infty} \lambda^{s_{n_m}}(x_{n_m}) + \varepsilon.
\]

Fix a representative \(\rho = (K, g, \mu) \in \sigma\) and choose \(t\) sufficiently large such that, with respect to \(\rho\), one has \(\lambda^s_{x_{n_m}}(t) \geq \lim_{m \to \infty} \lambda^{s_{n_m}}(x_{n_m}) + \frac{1}{2} \varepsilon\). But, since

\[
\lambda^{s_{n_m}}(x_{n_m}) \geq \lambda^{s_{n_m}}_{x_{n_m}}(t)
\]

and \((s, x) \mapsto \lambda^s_{x}(t)\) is continuous on \(iK \times X\), one gets

\[
\lim_{m \to \infty} \lambda^{s_{n_m}}_{x_{n_m}}(t) \geq \lim_{m \to \infty} \lambda^{s_{n_m}}_{x_{n_m}}(t) = \lambda^s_{x}(t) \geq \lim_{m \to \infty} \lambda^{s_{n_m}}(x_{n_m}) + \frac{1}{2} \varepsilon.
\]

which is a contradiction. 

\[\blacksquare\]
2.2 Analytic stability, polystability, semistability

Let again $\alpha$ be an action of a reductive group $G$ on a complex manifold $X$. Fix a symplectization $\sigma$ of $\alpha$ and let $\lambda : H(G) \times X \to \mathbb{R} \cup \{\infty\}$ be the associated map.

Let $G$ be a reductive group. A subalgebra $\mathfrak{g}' \subset \mathfrak{g}$ will be called a reductive subalgebra if it has the form $\mathfrak{g}' = \mathfrak{k}' \mathbb{C}$, where $\mathfrak{k}'$ is the Lie algebra of a compact subgroup of $G$. Equivalently, one can require instead that $\mathfrak{g}' = (\mathfrak{k} \cap \mathfrak{g}') \mathbb{C}$ for a maximal compact subgroup $K$ of $G$. Note that, if $\mathfrak{k}'$ is the Lie algebra of a compact Lie subgroup of $G$, then the minimal complex subspace $\tilde{\mathfrak{k}}'$ of $\mathfrak{g}$ which contains $\mathfrak{k}'$ can be identified with the complexification $\mathfrak{k}' \mathbb{C}$ via the canonical map $\mathfrak{k}' \mathbb{C} \to \tilde{\mathfrak{k}}'$.

**Definition 2.12** A point $x \in X$ will be called

1. analytically $\sigma$-semistable if $\lambda^s(x) \geq 0$ for all $s \in H(G)$.
2. analytically $\sigma$-stable if it is semistable and $\lambda^s(f) > 0$ for $s \in H(G) \setminus \{0\}$.
3. analytically $\sigma$-polystable if it is semistable, $\mathfrak{g}_x$ is a reductive subalgebra\(^2\), and $\lambda^s(x) > 0$ if $s$ is not equivalent to an element of $\mathfrak{g}_x$.

**Remark 2.13** Let $x \in X$. If $x$ is analytically $\sigma$-semistable, then $s \mapsto \lambda^s(x)$ must vanish on $H(G) \cap \mathfrak{g}_x$.

**Proof:** For every $(K, g, \mu) \in \sigma$ and $s \in i\mathfrak{t} \cap \mathfrak{g}_x$ one has $\lambda^s(x) = \mu^{-i\beta}(x)$. This shows that $\lambda(x)^{is} = t\lambda(x)^s$ for every $s \in \mathfrak{g}_x \cap H(G)$ and $t \in \mathbb{R}$. The semistability condition implies that $\lambda^s(x) \geq 0$ and $-\lambda^s(x) = \lambda^{-s}(x) \geq 0$ for every $s \in H(G) \cap \mathfrak{g}_x$.

The following proposition shows that it suffices to check the (semi, poly-) stability conditions for vectors $s \in i\mathfrak{t}$, where $\mathfrak{t}$ is the Lie algebra of a fixed maximal compact subgroup. However, one should be very careful in the polystable case.

**Proposition 2.14** Suppose that $\sigma$ is energy complete.

1. The analytical $\sigma$-semistability (stability, polystability) condition for $x \in X$ depends only on the complex orbit $Gx$ of $x$.
2. Choose any maximal compact subgroup $K \subset G$. A point $x \in X$ is

\(^2\)We will see that analytic polystability implies symplectic polystability, hence the stabilizer $G_x$ of a polystable point $x$ is reductive by Proposition 1.3. We preferred to require only the reductivity of $\mathfrak{g}_x$ in our definition in order to have a purely infinitesimal condition.
(a) analytically $\sigma$-semistable, if and only if $\lambda^s(x) \geq 0$ for every $s \in i\mathfrak{t}$.

(b) analytically $\sigma$-stable, if and only if $\lambda^s(x) \geq 0$ for any $s \in i\mathfrak{t}$ and $\lambda^s(x) > 0$ when $s \in i\mathfrak{t} \setminus \{0\}$.

(c) analytically $\sigma$-polystable, if and only if $\lambda^s(x) \geq 0$ for any $s \in i\mathfrak{k}$, $\mathfrak{g}_x$ is a reductive Lie algebra, and $\lambda^s(x) > 0$ for any $s \in i\mathfrak{t}$ which is not equivalent to an element of $\mathfrak{g}_x$.

3. Suppose that $\mathfrak{g}_x$ is a reductive Lie algebra and let $\mathfrak{k}$ be the Lie algebra of a maximal compact subgroup of $G$ such that $\mathfrak{g}_x = (\mathfrak{t} \cap \mathfrak{g}_x)^C$. Then $x$ is analytically $\sigma$-polystable if and only if $\lambda^s(x) \geq 0$ for any $s \in i\mathfrak{t}$ and $\lambda^s(x) > 0$ when $s \in i\mathfrak{t} \setminus i\mathfrak{k}$.

Proof: The first statement follows from the equivariance property 2. in Proposition 2.11. The second follows from the fact that $i\mathfrak{t}$ is a complete system of representatives for the relation $\sim$ (Corollary 2.5) and the same equivariance property.

For the third statement, note that, when $\mathfrak{g}_x = (\mathfrak{t} \cap \mathfrak{g}_x)^C$, any element of $i\mathfrak{t}$ which is equivalent in $H(G)$ to an element of $H(G) \cap \mathfrak{g}_x$ must belong to $i\mathfrak{t} \cap \mathfrak{g}_x$. This follows from Corollary 2.6, applied to the inclusion of reductive groups $(K \cap G_x)^C \subset G$.

3 Comparison Theorems. Hilbert criterion in Kählerian geometry

Here is our first comparison result.

**Theorem 3.1** Suppose that $\sigma$ is energy complete. A point $x \in X$ is symplectically $\sigma$-stable (polystable) if and only if it is analytically $\sigma$-stable (polystable).

Proof: The stable case follows immediately from Proposition 2.14 and Proposition 1.14.

"symplectically polystable" $\Rightarrow$ "analytically polystable":

Fix a representative $(K, g, \mu) \in \sigma$. Let $x \in X$ be a symplectically polystable point. Let $x_0 \in Gx \cap \mu^{-1}(0)$. By Proposition 2.14, it suffices to prove that $x_0$ is analytically polystable. But, from the proof of Proposition 1.14, we know that $\mathfrak{g}_{x_0} = \mathfrak{t}_{x_0}^C$, hence, by Proposition 2.14, it suffices to prove that $\lambda^s(x_0) \geq 0$ for all $s \in i\mathfrak{t}$ and $\lambda^s(x_0) > 0$ for all $s \in i\mathfrak{t} \setminus \mathfrak{g}_{x_0}$. But $\lambda^s(x_0) \geq \lambda^s_{x_0}(0) = 0$, because $\mu(x_0) = 0$. Moreover, for $s \notin \mathfrak{g}_{x_0}$ one has $s^#(x_0) \neq 0$, which implies that $\lambda^s(x_0) > \lambda^s_{x_0}(0) = 0$.  

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"analytically polystable" ⇒ "symplectically polystable":

Since \( x \) is analytically polystable, the subalgebra \( g_x \) is reductive. Choose a maximal compact subgroup \( K \) of \( G \) such that \( g_x = (g_x \cap k)^C \). By Proposition 2.14, 3. we know that \( \lambda_s(x) \geq 0 \) for all \( s \in i\mathfrak{k} \) and \( \lambda_s(x) > 0 \) for \( s \in i\mathfrak{k} \setminus i\mathfrak{k}_x \). It suffices to apply Proposition 1.14 using a representative \( \rho \) of the symplectization \( \sigma \) with first component \( K \) and to remember that symplectic polystability depends only on the fixed symplectization.

Remark: The assumption that \( \sigma \) is energy complete plays implicitly an important role in this proof. It gives us the flexibility to choose the maximal compact subgroup of \( G \) in a convenient way.

Lemma 3.2 Suppose that \( \sigma \) is energy complete. Let \( x_0 \in X \) be an analytically \( \sigma \)-semistable point, and let \( x \in X \) such that \( x_0 \in Gx \). Then \( x \) is analytically \( \sigma \)-semistable.

Proof: Suppose that \( x \) was not semistable, hence there exists \( s \in H(G) \) such that \( \lambda_s(x) < 0 \). Let \((g_n)\) be a sequence in \( G \) such that \((g_n x)_n\) converges to \( x_0 \). Using the equivariance property of the function \( \lambda \) (Proposition 2.11), one gets

\[
\lambda_{\text{ad} g_n(s)}(g_n x) = \lambda_s(x) < 0.
\]

Let \( K \) be a maximal compact subgroup of \( G \) such that \( s \in i\mathfrak{k} \) and let \( s_n \in i\mathfrak{k} \) such that \( s_n \sim \text{ad}_{g_n}(s) \) (see Proposition 2.5). By Proposition 2.7, one can find a subsequence \((s_{n_m})_m\) of \((s_n)_n\) which converges to a vector, say \( s_0 \), in the orbit \( \text{ad}_K(s) \). By the semicontinuity property and the \( \sim \) invariance property in Proposition 2.11, we obtain

\[
\lambda_{s_0}(x_0) \leq \lim \inf \lambda_{s_{n_m}}(g_{n_m} x) = \lim \inf \lambda_{\text{ad}_{s_{n_m}}} (g_{n_m} x) = \lambda_s(x) < 0,
\]

which contradicts the analytic semistability of \( x_0 \).

Our next comparison theorem is more delicate:

Theorem 3.3 Let \((X, g)\) be a complex manifold, \( \alpha : G \times X \to X \) a complex reductive Lie group action and let \( \sigma = [K, g, \mu] \) be an energy complete symplectization of this action. Then, for any point \( x \in X \) the following properties are equivalent:

1. \( x \) is symplectically \( \sigma \)-semistable;
2. \( x \) is analytically \( \sigma \)-semistable.
3. $\lambda^s(x) \geq 0$ for all $s \in iz_k(\mathfrak{t}_x)$.

4. There exist $s_m, s_0 \in iz_k(\mathfrak{t}_x)$ such that
   
   (a) $\lambda^{s_m}(x) = 0$,
   
   (b) $[s_m, s_0] = 0$,
   
   (c) the limit $y = \lim_{t \to \infty} (\exp(it s_m) x)$ exists in $X$,

   (d) $\mu(\exp(s_0)y) = 0$.

Proof:

1. $\Rightarrow$ 2.

   We fix a representative $(K, g, \mu) \in \sigma$. Since $x$ is symplectically $\sigma$-semistable, there exists a point $x_0 \in Gx \cap \mu^{-1}(x_0)$. By Theorem 3.1, $x_0$ is analytically $\sigma$-polystable, in particular it is analytically $\sigma$-semistable. It follows that $x$ is also analytically $\sigma$-semistable, by Lemma 3.2.

2. $\Rightarrow$ 3. Obvious

3. $\Rightarrow$ 4.

   Fix a representative $(K, g, \mu) \in \sigma$.

   Since 3. holds, one of the following two possibilities must occur:

   A. For all $s \in iz_k(\mathfrak{t}_x) \setminus iz(\mathfrak{t}_x)$ it holds $\lambda^s(x) > 0$.

   B. There exists $s \in iz_k(\mathfrak{t}_x) \setminus iz(\mathfrak{t}_x)$ such that $\lambda^s(x) = 0$.

   In the first case, Lemma 1.13 gives an element $s_0$ in the orthogonal complement $i[z(\mathfrak{t}_x)]^\bot$ of $iz(\mathfrak{t}_x)$ such that $\mu(\exp(s_0)x) = 0$. Therefore, taking $s_m = 0$, the claim 4. holds in this case.

   In the second case we know that there exist vectors $s \in iz_k(\mathfrak{t}_x) \setminus iz(\mathfrak{t}_x)$ such that $\lambda^s(x) = 0$. For any such $s$, the limit $x_s = \lim_{t \to \infty} (\exp(it s) x)$ exists, because $\sigma$ is energy complete. We choose $s_m \in iz_k(\mathfrak{t}_x) \setminus iz(\mathfrak{t}_x)$ with $\lambda^{s_m}(x) = 0$ which maximizes $\text{rk}(K_{x_{s_m}})$, i. e. such that

   $$\text{rk}(K_{x_{s_m}}) \geq \text{rk}(K_{x_s}) \quad \forall s \in iz_k(\mathfrak{t}_x) \setminus iz(\mathfrak{t}_x) \text{ with } \lambda^s(x) = 0 .$$

   We remind that the rank of a compact Lie group is the dimension of a maximal torus of it. Set $y := x_{s_m}$.

   Note that $s_m \in \mathfrak{g}_y$, so $is_m \in \mathfrak{t}_y$. Moreover, since $\exp(ts) \in Z_K(\mathfrak{t}_x)$, it follows that $\mathfrak{t}_x \subset \mathfrak{t}_{\exp(ts)x}$ for all $t \in \mathbb{R}$, hence, making $t \to \infty$, one gets $\mathfrak{t}_x \subset \mathfrak{t}_y$. Therefore,

   $$\{is_m\} \cup k_x \subset \mathfrak{t}_y .$$

   (17)
Claim: $\lambda^s(y) \geq 0$, for all $s \in iz(t_y)$.

Indeed, if there existed $\tau \in iz(t_y)$ with $\lambda^\tau(y) < 0$, one would get immediately a contradiction in the following way:

By energy completeness, the limit $z = \lim_{t \to \infty} (\exp(t\tau)y)$ exists. Consider any compact torus $T$ of $K_y$ which contains the real one-parameter subgroups generated by $is_m$ and $i\tau$. Such a torus exists, because $\tau \in iz(t_y)$ and $s_m \in i t_y$ by (17) so $[\tau, s_m] = 0$ so $[\tau, s_m] = 0$. As in the proof of Proposition 2.9, one can linearize the induced $T$-action on an open $T^c$-invariant Stein neighborhood of $z$ (which will necessarily contain $y$ and $x$). Using such a linearization and the standard eigenspace decomposition associated with a torus action on a vector space, one can see easily that

$$\lim_{t \to \infty} \exp(t(s_m + \varepsilon \tau)x) = z$$

for all sufficiently small $\varepsilon > 0$. This implies that

$$\lambda^{s_m + \varepsilon \tau}(x) = \mu^{-i(s_m + \varepsilon \tau)}(z) = \mu^{-is_m}(z) + \varepsilon \mu^{-i\tau}(z) = \mu^{-is_m}(z) + \varepsilon \lambda^\tau(y) . \quad (18)$$

The point $y = x_{s_m}$ is the limit for $t \to \infty$ of an integral curve of the vector field $s_m^\#$, hence $y$ is invariant under the local one-parameter transformation group generated by $s_m^\#$. Since $s_m$ and $\tau$ commute, all the points of the curve $c^\tau_y$ will be fixed under this local one-parameter transformation group. Therefore

$$s_m^\#|_{\text{im}(c^\tau_y)} = 0 . \quad (19)$$

On the other hand, by the properties of a moment map, one has

$$d\mu^{-is_m} = t_{-is_m^\#} \omega_g ,$$

which, together with (19), shows that the real function $\mu^{-is_m}$ is constant on the curve $c^\tau_y$ which joins $y$ to $z$. On the other hand, taking into account the way in which $s_m$ was chosen, one has

$$\lambda^{s_m}(x) = \mu^{-is_m}(y) = 0 .$$

Therefore $\mu^{-is_m}(z) = 0$ hence, by (18), $\lambda^{s_m + \varepsilon \tau}(x) < 0$, which contradicts the analytic semistability of $x$. This proves the claim.

We can distinguish now again the following two cases:

---

3Note that, in general, in a connected compact group $L$ the stabilizer $Z_L(u)$ of any element $u \in I$ is also connected. Moreover, $Z_L(u)$ is just the union of all maximal tori of $L$ whose Lie algebra contain the vector $u$. 

---
a. For all $\tau \in iz(t_y) \setminus iz(t_y)$ it holds $\lambda^\tau(y) > 0$.

b. There exists $\tau \in iz(t_y) \setminus iz(t_y)$ such that $\lambda^\tau(y) = 0$.

In the case a., using again Lemma 1.13 we get a vector $s_0 \in iz(t_y) \perp iz(t_y)$ such that

$$\mu(\exp(s_0)y) = 0.$$  \hfill (20)

Since $s_m \in ikx \subset iky$ and $s_0 \in iz(t_y)$, we have $[s_m, s_0] = 0$, hence 4. is proved in case a.

We will show now that in fact case b. cannot occur, because it would contradict the maximizing property (16). Indeed, if b. held, let $\tau \in iz(t_y) \setminus iz(t_y)$ such that $\lambda^\tau(y) = 0$. By energy completeness, the curve $c_y^\tau$ has a limit $z$ for $t \to \infty$ and, similarly to (17), we will have

$$\{i\tau\} \cup t_y \subset t_z.$$ \hfill (21)

The same method as in the proof of the claim above, shows that

$$\lim(\exp(t(s_m + \varepsilon \tau)x) = z$$

for all sufficiently small $\varepsilon > 0$ and that (18) holds. This time we obtain $\lambda^{s_m + \varepsilon \tau}(x) = 0$ because $\lambda^\tau(y) = 0$. Let $T_m$ be a maximal compact torus of $K_y$. The stabilizer $K_z$ of $z$ contains both $T_m$ (because $K_z^i \subset K_z$ by (21)) and the real one parameter subgroup generated by $i\tau$. The elements of this one parameter subgroup commute with the elements of $T_m$ because $\tau \in iz(t_y)$.

We know that $\tau \in iz(t_y) \setminus iz(t_y) = iz(t_y) \setminus t_y$, hence $\tau \notin t_y$, hence $i\tau \notin t_m$. It follows that $T_m$ and this real one parameter subgroup generate a torus in $K_z$ which has a larger dimension that $\dim(T_m)$. This contradicts the maximizing property (16).

4. ⇒ 1. Using 4. we get

$$\exp(s_0) \lim_{t \to \infty} \exp(t s_m)x \in Gx \cap \mu^{-1}(0),$$

hence $x$ is symplectically $\sigma$-semistable.

We can prove now the following important semistability criterion. Note that this result is obvious in the case of a compact manifold $X$, whereas for non energy complete symplectizations it is in general false.
Theorem 3.4 Let \((X, g)\) be a complex manifold, \(\alpha : G \times X \to X\) a complex reductive Lie group action and let \(\sigma\) be an energy complete symplectization of this action. Then, for any point \(x \in X\) the following properties are equivalent:

1. \(x\) is symplectically \(\sigma\)-semistable;
2. \(x\) is analytically \(\sigma\)-semistable.
3. \(\inf_{g \in G} \|\mu(gx)\| = 0\), where the norm is computed with respect to any ad-invariant inner product on \(\mathfrak{k}\).

Proof: We know that 1. and 2. are equivalent. The implication 1. \(\Rightarrow\) 3. is obvious, so it suffices to show that 3. \(\Rightarrow\) 2.

Suppose that \(x\) was not analytically semistable, and fix a triple \((K, g, \mu) \in \sigma\). It would follow that there exists \(s \in i\mathfrak{k}\) such that \(\lambda^s(x) < 0\). We normalize \(s\) such that \(\|s\| = 1\). Since 3. holds, there exists a sequence \((g_n)\) in \(G\) such that \(\|\mu(g_n x)\| \to 0\). By the equivariance property in Proposition 2.11, we get \(\lambda^{\text{ad}_{g_n}(s)}(g_n x) = \lambda^s(x)\). Using Corollary 2.5, we can find \(s_n \in i\mathfrak{k}\) such that \(s_n \sim \text{ad}_{g_n}(s)\).

Using the \(\sim\) invariance property, we get for all \(n \in \mathbb{N}\).

\[
\mu^{-is_n}(g_n x) = \lambda^{s_n}_{g_n x}(0) \leq \lim_{t \to \infty} \lambda^{s_n}_{g_n x}(t) = \lambda^{s_n}(g_n x) = \lambda^s(x) < 0. 
\]

(22)

The point is now that, by Proposition 2.7, \(s_n \in \text{ad}_K(s)\), hence \(\|s_n\| = \|s\| = 1\). Thus,

\[
|\mu^{-is_n}(g_n x)| = |\langle \mu(g_n x), -is_n \rangle| \leq \|\mu(g_n x)\| \to 0,
\]

and this obviously contradicts (22).

Our last semistability criterion is the following:

Theorem 3.5 Let \((X, g)\) be a complex manifold, \(\alpha : G \times X \to X\) a complex reductive Lie group action and let \(\sigma = [K, g, \mu]\) be an energy complete symplectization of this action. Then, for any point \(x \in X\) the following properties are equivalent:

1. \(x\) is symplectically \(\sigma\)-semistable;
2. \(x\) is analytically \(\sigma\)-semistable.
3. The real function \(\Psi(x, \cdot)\) is bounded from below on \(G\).
Proof:

The implication 3. ⇒ 2. is very easy: if there existed \( s \in i \mathfrak{t} \) such that \( \lambda^s(x) < 0 \), then, taking into account that

\[
\frac{d}{dt} \Psi(x, \exp(ts)) = \lambda^s_x(t) \leq \lambda^s(x) ,
\]

we get an estimate of the form \( \Psi(x, \exp(ts)) \leq C_1 + \lambda^s(x)t \), which shows that \( \lim_{t \to \infty} \Psi(x, \exp(ts)) = -\infty \).

We prove now 1. ⇒ 3.

Let \( x \) be a symplectically polystable point. By the results of [HH] (see Theorem 1.4), the closure of the orbit \( Gx \) contains a unique polystable orbit \( O \), and this orbit contains a unique \( K \)-orbit \( o \) on which \( \mu \) vanishes.

Now we use the existence theorem for local potentials compatible with a Hamiltonian action ([HHL], p. 138, [HH], p. 245). There exists a \( G \)-invariant Stein neighborhood \( U \) of the \( K \)-orbit \( o \) and a potential for the moment map \( \mu|_U \), i.e., a strictly plurisubharmonic \( K \)-invariant function \( \varphi : U \to \mathbb{R} \) such that on \( U \) it holds

\[
\langle \mu, a \rangle = d\varphi(Ja^#) , \ \forall a \in \mathfrak{t} ,
\]

where \( J \in \text{End}(T_X) \) is the almost complex structure of the complex manifold \( X \).

We state that (23) implies the following formula for the restriction \( \Psi|_{U \times G} \) of function \( \Psi \) associated with \( \mu \):

\[
\Psi(u, g) = \varphi(gu) - \varphi(u) , \ \forall u \in U, \ \forall g \in G .
\]

Indeed, using (23), we get

\[
\frac{d}{dt}(\varphi(e^{ts}u)) = d\varphi(s^#_{e^{ts}u}) = d\varphi(J(-is)^#_{e^{ts}u}) = \langle \mu(e^{ts}u), -is \rangle = \lambda^s_{e^{ts}u}(t) .
\]

The other conditions in Definition 1.6 are obvious.

Let \( x_0 \in o \). Since \( \mu(x_0) = 0 \), by Remark 1.8 it follows that \( e \in G \) is a critical point of the map \( \Psi(x_0, \cdot) \). Taking into account the convexity property of the map \( \Psi \) (Remark 1.7) and Proposition 1.3, it follows that \( \Psi(x_0, \cdot) \) reaches its absolute minimum in \( e \). Therefore, by (24) we get

\[
\varphi(x_0) \leq \varphi(y) , \ \forall y \in O .
\]

By Theorem 3.3, for any \( \xi \in Gx \) there exist \( s^\xi_m \in i \mathfrak{t} \) and \( y^\xi \in O \) such that

\[
\lambda^s_{m}(\xi) = 0 , \ \lim_{t \to \infty} \exp(ts^\xi_m) = y^\xi .
\]
Since $U$ is open, $G$-invariant and it contains the $G$-orbit $O$, it follows that $Gx \subset U$, so we can write

$$
\Psi(\xi, \exp(ts^\xi_m)) = \varphi(\exp(ts^\xi_m)) - \varphi(\xi) \to \varphi(y^\xi) - \varphi(\xi) \text{ as } t \to \infty.
$$

The first relation in (26) shows that $\lambda_t^\xi_m(t) \leq 0$ for all $t \in \mathbb{R}_{\geq 0}$, hence the real function $t \mapsto \Psi(\xi, \exp(ts^\xi_m))$ is decreasing, so

$$
\varphi(y^\xi) - \varphi(\xi) \leq \Psi(\xi, e) = 0.
$$

Combining with (25), we get

$$
\varphi(\xi) \geq \varphi(y^\xi) \geq \varphi(x_0),
$$

hence $\Psi(x, g) = \varphi(gx) - \varphi(x) \geq \varphi(x_0) - \varphi(x)$. □

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Author’s address:
LATP, CMI, Université de Provence, 39 Rue F. Joliot-Curie, 13453 Marseille Cedex 13, France, e-mail: teleman@cmi.univ-mrs.fr, and
Faculty of Mathematics, University of Bucharest, Bucharest, Romania