Bearing-only Trajectory Estimation for Ballistic Target Using Sparse Representation

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Abstract. This paper discusses the problem of estimating the trajectory of a ballistic target using bearing measurements from one passive sensor. The major challenge of this problem is the ill-conditioning of the estimation problem due to poor observability of the target motion via bearing measurements. We present an estimator based on the sparse representation algorithm for one sensor scenario. The sparse representation can dramatically reduce the amount of data needed to be estimated. With sparse representation, the trajectory of the ballistic target can be estimated in poor observability conditions. Besides, we introduce some complementary constraints to improve the observability of target motion. Simulation results indicate that the proposed estimator is effective.

1. Introduction

The plume generated by a ballistic target makes it quite visible to most optical sensor. With the measurements from optical sensor, we can estimate the flight of ballistic target.

At present, accurate estimation of the trajectory of ballistic target needs bearing measurements from multiple observers. However, in many cases, the ballistic target is observed by only one sensor. Therefore, the study of trajectory estimation based on bearing measurements from one sensor has important value[1].

The bearings-only methods used to estimate the trajectory of a target can be broadly classified as being either profile-free or profile-based models. In profile-free methods, the state vector is estimated using a polynomial motion model. In profile-based methods, it is assumed that the magnitude and orientation of the target thrust vector, thus the acceleration, is known. However, the accurate profile of ballistic target is very difficult to obtain. In many cases, we have to depend on profile-free methods which are hard to obtain accurate estimation[1,2,3].

To solve this problem, we present a trajectory estimation method for ballistic target based on bearing measurements and there is no knowledge of the profile of target. Besides, we assume that the measurements from only one sensor, which means there are only angle measurements (or bearing-only measurements). We introduce sparse-representation into our estimator. The most common method focuses on estimating the status of ballistic target on every time point. In view of sparse-representation, it is unnecessary. Based on the sparse representation of trajectory, we only need to
estimate the sparse representation coefficient. With these coefficients, it is easy to recover trajectory. This enables our estimator with bearing-only measurements to achieve acceptable accuracy.

2. Measurements and sparse representation

2.1. Angle measurements

The geometry relation between target and sensor is showed in figure 1.

![Figure 1. Target, observer geometry](image)

In practice, the measurements of targets contain azimuth bearing $\alpha_t$ and elevation angle $\beta_t$ measurements at moment $t$.

Let $[x, y, z]^T_{\text{body}}$ represent the coordinate of target in satellite coordinate system. $[x, y, z]^T_{\text{target}}$ represent coordinate of target in Earth-Centred Inertial (ECI) coordinate system and $[x, y, z]^T_{\text{satellite}}$ represent the position of satellite in ECI coordinate system. The relationship among them is listed below:

$$
\begin{bmatrix}
X \\
y \\
z_{\text{body}}
\end{bmatrix}
= R_{\text{Body ECI}}
\begin{bmatrix}
X \\
y \\
z_{\text{target}}
\end{bmatrix}
- R_{\text{Body ECI}}
\begin{bmatrix}
x \\
y \\
z_{\text{satellite}}
\end{bmatrix}
$$

(1)

$R_{\text{Body ECI}}$ represent the transformation matrix which convert the position of target in ECI coordinate system to satellite coordinate system. Given the method to obtain $R_{\text{Body ECI}}$ isn’t the core of this paper, we don’t discuss it in detail. The detail of it is introduced in ref[4].

These position are related to the angle measurements by
The equation (2) is proved in article[3].

2.2. Sparse representation of trajectory

In practical applications, most of the natural signals can be sparse represented. The sparse representation can expressed as below:

\[ f = \Psi w \]

The coefficient \( w \) is expressed as sparseness of signal \( f \) in the sparse basis \( \Psi \). The relationship of them is shown in figure 2.

\[ \begin{bmatrix} 0 \\ 0 \\ \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin \beta \cot \alpha_t \\ 0 & 1 & -\cos \beta \cot \alpha_t \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{body} \] (2)

Figure 2. the sparse representation of signal

Most elements of \( w \) are zeros or minor. To estimate signal \( f \), we only needed to estimate the significant elements in \( w \). Sparse representation enable us to reconstruct the original signal with a few observation[5,6].

The trajectory of ballistic target can be represented by a sparse basis. In this paper, we construct a sparse basis of trajectory based on polynomials model. Take the rocket trajectory as an example. In an ideal situation, the space coordinates of the target at time \( t \) are \( P_t: (x, y, z) \). The track can be expressed as:

\[ x_t = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + \cdots \]
\[ y_t = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + b_4 t^4 + \cdots \]
\[ z_t = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 + \cdots \] (3)

It should be noted that there are many models describing the target trajectory. The polynomial is just one among them. However, the ballistic target model is not the major of this paper. For the
The convenience of discussion, this paper chooses a simple polynomial model. In fact, there are other better and more complex models in the sea, which will be introduced in subsequent articles.

The observation of the entire target motion trajectory can be turned to an estimate of the trajectory coefficient. The sparse parameters of trajectory are considered to be fixed during the flight of target. According to this, it can be expressed that the matrix in the middle is a sparse representation matrix, which is completely determined by time and is completely known.

\[
\begin{bmatrix}
x_t \\
y_t \\
z_t
\end{bmatrix} = \begin{bmatrix}
1 & t & t^2 & t^3 & \ldots & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & \ldots & 1 & t & t^2 & t^3 & \ldots & 0 & 0 & 0 & 0 & \ldots \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & \ldots & 1 & t & t^2 & t^3 & t^4 & \ldots 
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
\vdots \\
c_n
\end{bmatrix} = \Phi_t \cdot [a_0, a_1, \ldots, c_n] = \Phi_t \cdot w
\]

(4)

Where \( \Phi_t \) is the sparse representation matrix of time \( t \), \( w \) representing the trajectory parameters to be estimated.

3. Estimation method for trajectory

The Scheme of proposed method is showed in figure 3.

First, we construct the basic observation equation from basic observation data, such as time, the angles of bearing, the positions of the satellite, and the orbit element of satellites. But the basic observation equation usually not enough to estimate the sparse coefficient. To solve this problem, we introduce some complementary constraints to construct a complementary constraint equation. By combining the two types of equations together, we can construct the observation equation. Then, based on the observation equations, we could estimate the sparse parameters. After obtaining the sparse parameters, we could reconstruct the target trajectory. Because steps 4 and 5 are based on classical methods and they are not the focuses of this paper, we don’t present an explicit introduction of them in this paper. Only the steps 1-3 are discussed in detail.

3.1. Construct the basic observation equation

The observation angle is \( \beta_t \), \(\alpha_t\), and represents the azimuth and elevation angles, respectively. Using these two observation angles, an observation equation can be constructed.
Where \((x_o(t), y_o(t), z_o(t))\) represents the spatial coordinates of satellite at time \(t\). To simplify equation (5), we introduce \(b(t)\) and \(h(t)\), where:

\[
b(t) = \begin{bmatrix} 1 & 0 & -\sin \beta_t \cot \alpha_t \\ 0 & 1 & -\cos \beta_t \cot \alpha_t \end{bmatrix} \cdot \mathbf{R}_{\text{Body/EKI}} \cdot \begin{bmatrix} a_x \\ b_x \\ \ldots \\ e_z \end{bmatrix} - \begin{bmatrix} x_o(t) \\ y_o(t) \\ z_o(t) \end{bmatrix}
\]  

(6)

\[
h(t) = \begin{bmatrix} 1 & 0 & -\sin \beta_t \cot \alpha_t \\ 0 & 1 & -\cos \beta_t \cot \alpha_t \end{bmatrix} \cdot \mathbf{R}_{\text{Body/EKI}} \cdot \Phi_t
\]  

(7)

With \(N\) observations, the observation equation can be constructed

\[
\mathbf{B} = \begin{bmatrix} \mathbf{b}(t_1) \\ \mathbf{b}(t_2) \\ \ldots \\ \mathbf{b}(t_n) \end{bmatrix} = \begin{bmatrix} \mathbf{h}(t_1) \\ \mathbf{h}(t_2) \\ \ldots \\ \mathbf{h}(t_n) \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \ldots \\ c_n \end{bmatrix} = \mathbf{H} \cdot \mathbf{w}
\]  

(8)

The problem of ballistic target trajectory estimation is converted into a typical problem of estimating the sparse representation coefficients of the trajectory. The Basis Pursuit (BP) algorithm is adapted to recovery \(\mathbf{w}\). The detail of BP is described in [7].

It had been proved that estimation of \(\mathbf{w}\) is feasible when it’s sparse and the \(\mathbf{H}\) satisfy the Restricted Isometry Property (RIP) [5,6]. However, in the actual situation, the matrix is usually a singular matrix, and does not satisfy the RIP. Therefore, some auxiliary conditions is need to ensure the solvable row.

3.2. Construction complementary constrain equation

3.2.1 Known point constraints. The launch point and some intermediate point can obtained through bearing-only measurements.

1) Launch point constraint

There are plenty of papers devoted to estimate the launch point \(\mathbf{P}_0\) of target through bearing-only measurements. However, it is not the major issue of paper. Therefore, we don’t introduce the detail of it here. The method to obtain the Launch point is introduced in ref[1].

2) Intermediate point constraint

The tail-fire emission intensity of the ballistic target in the active section is stable. The observed fluctuation of the tail flame radiation is mainly determined by the atmospheric absorption. Using the atmospheric absorption model and the observed target tail flame intensity, the height of target can be estimated when it is passing atmosphere. Under the premise of knowing the height of the target, the
positions \((P_L, \ldots, P_M)\) of target can obtained. \(L, \ldots, M\) represent the time point that target position is obtained. The detail of method to obtain \(P_L, \ldots, P_M\) is discussed in ref \cite{8,9}.

3.2.2 Coplanar constraints. In the course of flight, ballistic targets and launch vehicles are generally not orbiting in order to save fuel and ensure range. The trajectory of the ballistic target is usually in a plane. Therefore, the target trajectory coplanar constraint can be introduced to further reduce the singularity of the solution matrix. To describe a plane, we need a point \(P_c\) in the plane, a normal vector \(N\) which is perpendicular to the plane. It is assumed that

\[
N = [x_{V_n}, y_{V_n}, z_{V_n}]^T
\]

The method to obtain \(P_c\) and \(N\) is presented below.

There are \(K\) known points and all belong to set \(C\). \(P_c\) can be obtained by the following formula:

\[
P_c = \frac{1}{K \sum_{P_i \in C} P_i}
\]

The normal vector satisfies the following formula:

\[
0 = \left[ P_0 - P_C, P_L - P_C, \ldots, P_M - P_C \right]^T \cdot V_N
\]

\[
= \left[ \hat{P}_0, \hat{P}_L, \ldots, \hat{P}_M \right]^T \cdot V_N
\]

Where \(\hat{P}_0, \hat{P}_L, \ldots, \hat{P}_M\) equals \(P_0, P_L, P_M\) minus the value \(P_c\).

While only the orientation of \(V_N\) is needed, the magnitude can be any value. For the convenience of discussion, the \(z_{V_n}\) last variable value is set to 1. The above formula can be rewritten as:

\[
0 = \begin{bmatrix}
-x_{\hat{P}_0} & y_{\hat{P}_0} & z_{\hat{P}_0} \\
x_{\hat{P}_L} & y_{\hat{P}_L} & z_{\hat{P}_L} \\
\vdots \\
x_{\hat{P}_M} & y_{\hat{P}_M} & z_{\hat{P}_M}
\end{bmatrix} \begin{bmatrix}
x_{V_n} \\
y_{V_n} \\
1
\end{bmatrix}
\]

It is be equivalent to:

\[
V_z = \begin{bmatrix}
-x_{\hat{P}_0} \\
x_{\hat{P}_L} \\
\vdots \\
x_{\hat{P}_M}
\end{bmatrix} = \begin{bmatrix}
x_{P_0} & y_{P_0} & z_{P_0} \\
x_{P_L} & y_{P_L} & z_{P_L} \\
\vdots \\
x_{P_M} & y_{P_M} & z_{P_M}
\end{bmatrix} \begin{bmatrix}
x_{V_n} \\
y_{V_n} \\
1
\end{bmatrix} = H_{VN} \cdot \begin{bmatrix}
x_{V_n} \\
y_{V_n}
\end{bmatrix}
\]

According to this, \(x_{V_n}, y_{V_n}\) can be determined by:

\[
\begin{bmatrix}
x_{V_n} \\
y_{V_n}
\end{bmatrix} = (H_{VN}^T \cdot H_{VN})^{-1} H_{VN}^T \cdot V_z
\]
At this point, the normal vector has been obtained $V_N$.

3.2.3 Constrained equation. After obtaining the information of known point constraints and coplanar constraints, the complementary constrain equations can be constructed. The method of integration is very simple, and the constrain equations can be directly added behind the original observation matrix. The following equation can be constructed from known points

$$
\begin{bmatrix}
P_0 \\
P_L \\
\vdots \\
P_M \\
\end{bmatrix} =
\begin{bmatrix}
\Phi_0 \\
\Phi_L \\
\vdots \\
\Phi_M \\
\end{bmatrix} \cdot w
$$

(14)

Let $B_1 = [P_0^T \ P_L^T \ \cdots \ P_M^T]^T$ and $H_1 = [\Phi_0^T \ \Phi_L^T \ \cdots \ \Phi_M^T]^T$. Then the above formula can be expressed as:

$$B_1 = H_1 \cdot w$$

(15)

It can be seen from the coplanarity that the position $P_t$ of the track point at any time minus $P_C$ the coplanar point is perpendicular to the normal vector $V_N$, and the following equation can be obtained as follows:

$$0 = V_N^T \cdot (P_t - P_C) = V_N^T \cdot (\Phi_t \cdot w - P_C)$$

(16)

It can rewritten in the following equation form:

$$V_N^T \cdot P_C = V_N^T \cdot \Phi_t \cdot w$$

(17)

In the above formula, $V_N^T \cdot P_C$ is a constant that does not change with time. We use $D$ to express it. This makes it possible to construct an equation of observation:

$$B_2 = \begin{bmatrix}
D \\
\vdots \\
D \\
\end{bmatrix} =
\begin{bmatrix}
V_N^T \cdot \Phi_0 \\
\vdots \\
V_N^T \cdot \Phi_N \\
\end{bmatrix} \cdot w = H_2 \cdot w$$

(18)

3.3. Construct observation equation and estimate

Integrating the above equations into the original observation equation yields:

$$\begin{bmatrix}
B \\
B_1 \\
B_2 \\
\end{bmatrix} =
\begin{bmatrix}
H \\
H_1 \\
H_2 \\
\end{bmatrix} \cdot w$$

(19)

So far, we have obtained an observation equation with additional constraints. By adding additional constraints, the singularity of the observation matrix can be effectively reduced and the solvability of the equation can be improved.

As mentioned before, we could use BP method to obtain $w$. By multiplying $w$ with the sparse representation matrix $\Phi_t$, the trajectory of target at time $t$ can be obtained.

4. Simulation
In order to verify the feasibility of the above algorithm, we use the software Satellite Tool Kit (STK) to simulate a 3000-km range trajectory of target and the orbit of satellite. Then, based on the trajectory and orbit data, we evaluate the corresponding bearing measurements.

After adding gauss noise which mean is 0 and standard deviation is 10Km, the launch point of the target and the trajectory point of the boost phase (altitude between 20-40km) are used as complementary constraints. The simulation condition is shown in figure 4.

![Figure 4. The Estimated trajectory of target](image)

The estimation error of the entire trajectory is shown in the figure below:

![Figure 5. the estimation error of estimated trajectory](image)
Figure 5 shows that the estimation error is less than 2% of the range of target. In ref[10], it is reported that, with bearing-only measurements from only one sensor, the estimation error is 5%-10% of the range. It can be seen that the proposed method can effectively track and locate the ballistic target trajectory under the condition of bearing-only measurement.

5. Conclusion
This paper proposes a new ballistic target trajectory estimation method based on sparse representation which can obtain the position of target with bearing-only measurements from one sensor. What’s more, the method doesn’t depend on the profile of target. The sparse representation algorithm for ballistic target trajectory is used to convert the trajectory tracking of the ballistic target into an estimation of the sparse representation coefficient. To evaluate these coefficients, RIP should be satisfied. However, RIP is usually unsatisfied when there is only angle measurement from one sensor. To solve this problem, we discussed how to obtain complimentary constraints and construct constraint equations. With constraint equations, the sparse representation coefficient can be estimated. Using these coefficients, the position of target can be easily obtained. An experiment is carried out in this paper and the result shows that our method is effective.

References
[1] Yicong Li, Murali Yeddanapud, (1999)Trajectory and Launch Point Estimation for Ballistic Missiles from Boost Phase LOS Measurements[C]. 1999 IEEE Aerospace Conference. Proceedings, (Mar. 1999)
[2] Foy, W.H. (1976) Position-location solutions by Taylor-series estimation[J]. IEEE Transactions on Aerospace and Electronic Systems, AES-12, 2 (Mar. 1976), 187-194.
[3] Sherry,E.H., and Vincent J. AIDALA (1985) Observability requirements for three-dimensional racking via angle measurements[J]. IEEE Transactions on Aerospace and Electronic Systems, AES-21, 2 (Mar. 1985), 200-207.
[4] Liu L, (2000),Orbit theory of space craft. (BeiJing National Defence Industry Press.)
[5] Candes, E.J.; Romberg, J. K.; Tao, T.; Stable Signal Recovery From Incomplete and Inaccurate Measurements, Commun. Pure Appl. Math., 2006, 59, 1207–1223.
[6] Candès, E.J.; Wakin, M.B. An Introduction to Compressive Sampling, IEEE Signal Process. Mag., 2008, 25, 21–30.
[7] Chen, S.S.; Donoho, D.L.; Saunders, M.A. Atomic Decomposition by Basis Pursuit, SIAM J. Sci. Comput., 1998, 1, 33–61.
[8] AN Yongquan, LI Jinhua, WANG Zhibin, et al. Mono-station and single-band passive ranging based on oxygen spectrum[J]. Acta Phys. Sin., 2013, 62(14): 1442101-1442107.
[9] Dowski E, Cathey T. Single-lens single-image incoherent passive-ranging systems[J]. Applied Optics, 1994(33): 6762-6773.
[10] A. J. Perella, W. W. Kuhn, “Cueing Performance Estimation Using Space Based Observations During Boost Phase”, Proc. 1996 Summer Comp. Sim. Conf., Portland, OR, July 1996.