The Secrecy Capacity of Practical Quantum Cryptography

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Quantum cryptography has attracted much recent attention due to its potential for providing secret communications that cannot be decrypted by any amount of computational effort. This is the first analysis of the secrecy of a practical implementation of the BB84 protocol that simultaneously takes into account and presents the full set of complete analytical expressions for effects due to the presence of pulses containing multiple photons in the attenuated output of the laser, the finite length of individual blocks of key material, losses due to error correction, privacy amplification, continuous authentication, errors in polarization detection, the efficiency of the detectors, and attenuation processes in the transmission medium. The analysis addresses eavesdropping attacks on individual photons rather than collective attacks in general. Of particular importance is the first derivation of the necessary and sufficient amount of privacy amplification compression to ensure secrecy against the loss of key material which occurs when an eavesdropper makes optimized individual attacks on pulses containing multiple photons. It is shown that only a fraction of the information in the multiple photon pulses is actually lost to the eavesdropper.

The use of quantum cryptographic protocols to generate key material for use in the encryption of classically transmitted messages has been the subject of intense research activity. The first such protocol, known as BB84, can be realized by encoding the quantum bits representing the raw cryptographic key as polarization states of individual photons. The protocol results in the generation of a shorter string of key material for use by two individuals, conventionally designated Alice and Bob, who wish to communicate using encrypted messages which cannot be deciphered by a third party, conventionally called Eve. The unconditional secrecy of BB84 has been proved under idealized conditions, namely, on the assumption of pure single-photon sources and in the absence of various losses introduced by the equipment which generates and detects the photons or by the quantum channel itself. The conditions under which secrecy can be maintained under more realistic circumstances have been studied extensively. This is the first analysis of the secrecy of a practical implementation of the BB84 protocol that simultaneously takes into account and presents the full set of complete analytical expressions for effects due to the presence of pulses containing multiple photons in the attenuated output of the laser, the finite length of individual blocks of key material, losses due to error correction, privacy amplification, continuous authentication, errors in polarization detection, the efficiency of the detectors, and attenuation processes in the transmission medium. We consider in this paper attacks made on individual photons, as opposed to collective attacks on the full quantum state of the photon pulses. The extension to other protocols, such as B92, is straightforward, but is not discussed here due to limitations of space.

The protocol begins when Alice selects a random string of \( m \) bits from which Bob and she will distill a shorter key of \( L \) bits which they both share and about which Eve has exponentially small information. We define the secrecy capacity \( S \) as the ratio of the length of the final key to the length of the original string:

\[
S = \frac{L}{m} \tag{1}
\]

This quantity is useful for two reasons. First, it can be used in proving the secrecy of specific practical quantum cryptographic protocols by establishing that

\[
S > 0 \tag{2}
\]

holds for the protocol. Second, it can be used to establish the rate of generation of key material according to

\[
R = \frac{S}{\tau}, \tag{3}
\]

where \( \tau \) is the pulse period of the initial sequence of photon transmissions. Several scenarios in which useful key generation rates can be obtained are described in [7].

The length of the final key is given by

\[
L = n - (e_T + q + t + \nu) - (a + g_{pa}) \tag{4}
\]

The first term, \( n \) is the length of the sifted string. This is the string that remains after Alice has sent her qubits to Bob, and Bob has informed Alice of which qubits were received and in what measurement basis, and Alice has indicated to Bob which basis choices correspond to her own. We consider here the important special case where the number of photons in the pulses sent by Alice follow a Poisson distribution with parameter \( \mu \). This is an appropriate description when the source is a pulsed laser that has been attenuated to produce weak coherent pulses. In this case, the length of the sifted string may be expressed as

\[
n = \frac{m}{2} \left[ \psi_{\geq 1} (\eta \alpha) (1 - r_d) + r_d \right], \tag{5}
\]
where $\eta$ is the efficiency of Bob’s detector, $\alpha$ is the transmission probability in the quantum channel, and $r_d$ is the probability of obtaining a dark count in Bob’s detector during a single pulse period. $\psi_{\geq k} (X)$ is the probability of encountering $k$ or more photons in a pulse selected at random from a stream of Poisson pulses having a mean of $X$ photons per pulse:

$$\psi_{\geq k} (X) \equiv \sum_{i=k}^{\infty} \psi_1 (X) = \sum_{i=k}^{\infty} e^{-X} \frac{X^i}{i!}, \quad (6)$$

Other types of photon sources may be treated by appropriate modifications of equations (6) and (7). A comprehensive treatment of this subject, including an extensive analysis of factors contributing to $\alpha$, is found in [3].

The next terms represent information that is either in error, or that may potentially be leaked to Eve during the rest of the protocol. This information is removed from the sifted string by the algorithm used for privacy amplification, and so the corresponding number of bits must be subtracted from the length of the sifted string to obtain the size of the final key that results.

The first such term, $e_T$, represents the errors in the sifted string. This may be expressed in terms of the parameters already defined and the intrinsic channel error probability $r_c$:

$$e_T = \frac{m}{2} \left[ \psi_{\geq 1} (\eta \mu \alpha) r_c (1 - r_d) + \frac{r_d}{2} \right], \quad (7)$$

where the intrinsic channel errors are due to relative misalignment of Alice’s and Bob’s polarization axes and, in the case of fiber optics, the dispersion characteristics of the transmission medium. These errors are removed by an error correction protocol which results in an additional $q$ bits of information about the key being transmitted over the classical channel. We express this as

$$q \equiv Q \left( x, \frac{e_T}{n} \right) e_T = \frac{x h (e_T/n)}{e_T/n} e_T \quad (8)$$

where $h(p)$ is the binary entropy function for a bit whose a priori probability of being 1 is $p$. The factor $x$ is introduced as a measure of the ratio by which a particular error correction protocol exceeds the theoretical minimum amount of leakage given by Shannon entropy [3]:

$$q_{\text{min}} = n h (e_T/n) = \frac{h (e_T/n)}{e_T/n} e_T \quad (9)$$

The next term, $t$, is an upper bound for the amount of information Eve can obtain by direct measurement of the polarizations of single photon pulses. This upper bound may be expressed as

$$t = T e_T \quad (10)$$

where $T$ is given by [7][10][11]

$$T (n_1, e_T, e_{T,1}, \epsilon) = \left( \frac{n_1}{e_T} - \frac{e_{T,1}}{e_T} \right) T_{\text{max}} \left( \frac{e_{T,1}}{n_1} + \xi (n_1, \epsilon) \right) + \xi (n_1, \epsilon) \frac{n_1}{e_T} \left( 1 - \frac{e_{T,1}}{n_1} \right)^{1/2}, \quad (11)$$

with

$$T_{\text{max}} (\zeta) \equiv 1 + \log_2 \left[ 1 - \frac{1}{2} \left( 1 - \frac{3\zeta}{1 - \zeta} \right)^2 \right], \quad (12)$$

and $\xi$ is defined by

$$\xi (n_1, \epsilon) \equiv \frac{1}{\sqrt{2n_1}} \text{erf}^{-1} \left( 1 - \epsilon \right). \quad (13)$$

In the above equation $\epsilon$ is a security parameter that gives the likelihood for a successful eavesdropping attack against a single-photon pulse in the stream.

Finally, we have used

$$n_1 = \frac{m}{2} \left[ \psi_1 (\eta \mu \alpha) (1 - r_d) + r_d \right], \quad (14)$$

and

$$e_{T,1} = \frac{m}{2} \left[ r_2 \psi_1 (\eta \mu \alpha) (1 - r_d) + r_d \right], \quad (15)$$

which are the contributions to $n$ and $e_T$ from the subset of Alice’s pulses for which exactly one photon reaches Bob.

The next term, $\nu$, is the information leaked to Eve by making attacks on pulses containing more than one photon. There are a variety of possible attacks, including coherent attacks that operate collectively on all the photons in the pulse. We restrict our attention to disjoint attacks that single out each individual photon. Even with this restriction, there are a large number of alternatives. Eve can perform a direct attack by making direct measurements of the polarization of some subset of the photons and allowing the rest to continue undisturbed. She can also perform an indirect attack by storing some of the photons until she learns Alice and Bob’s basis choices by eavesdropping on their classical channel. She then measures the stored photons in the correct basis to unambiguously determine the value of the bit. Finally, she can make a combined attack by using the two strategies in some combination. In [11] it is shown that the optimum attack is always either a direct or an indirect attack, depending on the value of a parameter $y$, which depends on channel and detector characteristics and the technological capabilities attributed to Eve [3]. For the case of a fiber optic channel, it is possible in principle for Eve to replace the cable with a lossless medium, so that those
pulses whose polarizations she can measure are guaranteed to reach Bob. In this case we take $y = \eta$. For the free space case, such an attack may not be feasible, but she can achieve a similar effect by using entanglement. In this version of the indirect attack, Eve and an accomplice located near Bob prepare pairs of entangled photons in advance. Eve then entangles one of these pairs with a photon emitted by Alice. Her accomplice can now make measurements on the entangled state, gaining information about the photons at Eve’s location without losing photons to the attenuation in the channel. If we allow for such attacks, we still have $y = \eta$. If we do not attribute this level of technology to Eve, it is appropriate to take $y = \eta_0$. Note also that Eve can perform direct attacks using classical optical equipment, but that the indirect attacks require the type of apparatus envisaged for quantum computers.

There are three regions of interest. If $y > 1 - \frac{1}{\sqrt{2}}$ (i.e., $y \gtrsim 0.293$), the indirect attack is stronger, and the maximum information that Eve can obtain is

$$
\nu^\text{max} = \frac{m}{2} \left[ \psi_2 (\mu) (1 - y)^{-1} \right. \\
\left. \cdot \left\{ e^{-\mu y} - e^{-\mu} \left[ 1 + \mu (1 - y) \right] \right\} \right].
$$

(16)

If $y < 1 - \frac{1}{\sqrt{2}}$ (i.e., $y \lesssim 0.206$), the direct attack is stronger, and Eve’s information is

$$
\nu^\text{max} = \frac{m}{2} \left[ \psi_2 (\mu) y + 1 \right. \\
\left. \cdot \left( e^{-\mu} \frac{\sqrt{2} \sinh \frac{\mu}{\sqrt{2}} + 2 \cosh \frac{\mu}{\sqrt{2}} - 1}{1} \right) \right].
$$

(17)

Finally, if $y$ lies between these two regions, the relative strength of the attacks depends on the number of photons in the pulse. The information leaked to Eve is

$$
\nu^\text{max} = \frac{m}{2} \left[ \psi_2 (\mu) y + e^{-\mu} \left( \sinh \mu - \sqrt{2} \sinh \frac{\mu}{\sqrt{2}} \right) \right. \\
\left. + \sum_{k=2}^{\infty} \psi_{2k} (\mu) \left( \theta (\sigma_c (k, y) - 1) \left[ 1 - (1 - y)^{2k-1} \right] \right. \\
\left. \left. + \left[ 1 - \theta (\sigma_c (k, y) - 1) \right] (1 - 2^{1-k}) \right) \right]\right],
$$

(18)

where we have introduced the function:

$$
\sigma_c (k, y) = \frac{1 - (1 - y)^{2k-1}}{1 - 2^{1-k}}.
$$

(19)

For a photon pulse with $2k$ photons, $\sigma_c (k, y)$ is greater than 1 if the indirect attack is stronger and less than 1 if the direct attack is stronger. For odd numbers of photons, the direct attack is always stronger in this region.

The significance of these results for Eve is evident. If the key distribution system is operating in the region of large $y$, her optimal attack is always the indirect attack. If the system operates in the region of small $y$, the direct attack is optimal. If the system operates in the middle region, Eve optimizes her attack by measuring nondestructively the number of photons in the incoming pulses and then selecting the attack for each pulse according to the number of photons it contains.

The expressions for $\nu$ represent upper bounds on the information that is leaked to Eve by attacks on the individual photons of multi-photon pulses. In [1], it is shown that Eve can always choose an eavesdropping strategy to achieve this upper bound as long as Bob does not counterattack by monitoring the statistics of multiple detection events that occur at his device. Even with this proviso, the upper bounds are only a fraction of the information contained in the multi-photon pulses. This indicates that the assumption, common in the literature, that Alice and Bob must surrender all of this information to Eve is overly restrictive.

The next two terms are grouped together at the end of the expression because their effect on $S$ vanishes in the limit of large $m$. The first of these, $a$, is the continuous authentication cost. This is the number of secret bits that are sacrificed as part of the authentication protocol to ensure that the classical transmissions for sifting and error correction do occur between Alice and Bob without any “man-in-the-middle” spoofing by Eve. For the authentication protocols described in [4], the authentication cost is

$$
a (n, m) = 4 \left\{ g_{\text{auth}} + \log_2 \log_2 \left[ 2n (1 + \log_2 m) \right] \right\} \\
\cdot \log_2 \left[ 2n (1 + \log_2 m) \right] \\
+ 4 \left[ g_{\text{EC}} + \log_2 \log_2 (2n) \right] \log_2 (2n) \\
+ 4 (g_{\text{EC}} + \log_2 \log_2 n) \log_2 n \\
+ 4 (g_{\text{auth}} + \log_2 \log_2 g_{\text{EC}}) \log_2 g_{\text{EC}} \\
+ \tilde{g}_{\text{EC}} \\
+ 4 (g_{\text{auth}} + \log_2 \log_2 \tilde{g}_{\text{EC}}) \log_2 \tilde{g}_{\text{EC}}.
$$

(20)

The security parameters $g_{\text{auth}}, g_{\text{EC}}$, and $\tilde{g}_{\text{EC}}$ are adjusted to limit the probability that some phase of the authentication fails to produce the desired result. For instance the probability that Eve can successfully replace Alice’s transmissions to Bob with her own transmissions is bounded by $2^{-g_{\text{auth}}}$. The probability that Alice’s and
Bob’s copies of the key do not match after completion of the protocol is bounded by $2^{-g_{pa}} + 2^{-g_{EC}}$.

The last term, $g_{pa}$, is a security parameter that characterizes the effectiveness of privacy amplification. It is the number of bits that must be sacrificed to limit the average amount of information, $\langle I \rangle$, about Alice and Bob’s shared key that Eve can obtain to an exponentially small fraction of the raw key from her attacks on single photons in the attenuated output of the laser, $\nu_{\max}/m$. They result in contributions to the effective secrecy capacity that retain explicit dependence on $m$.

The third contribution, $t$, requires additional explanation. Its $m$ dependence arises from a precise application of the privacy amplification result, eq.(21), derived by Bennett et al. in [12]. The bound on Eve’s knowledge of the final key is obtained by assuming she has obtained a specific amount of Rényi information prior to privacy amplification. Starting from this point, Slutsky et al. [10] explicitly introduce a security parameter $\epsilon$ (see eq.(13)) to bound the probability that Eve has obtained more than $t$ bits of Rényi information as a result of her attacks on single photon pulses.

By contrast, the analysis of [3] introduces no parameter analogous to $\epsilon$. Furthermore, the expression for the amount of privacy amplification compression given in [3] is linear in the blocksize, thus resulting in a contribution to the secrecy capacity that is independent of the blocksize. While this approach, as developed in [3], does yield a bound on Eve’s information about the key shared by Alice and Bob after privacy amplification, explicit results pertaining to the amount of information Eve obtains on the key prior to privacy amplification are not presented. Such results have important practical consequences. For example, Eve’s likelihood of obtaining more than a given fraction of the raw key from her attacks on single photons increases as the block size of the key material is reduced. One therefore expects that the amount of privacy amplification compression required to ensure secrecy will increase as well. However, since this conclusion is strictly a consequence of the information Eve obtains prior to privacy amplification, it cannot be inferred from the analysis of [3]. In contrast, the approach of [4], which we adopt

$\langle I \rangle \leq \frac{2^{-g_{pa}}}{\ln 2}$.

The fundamental expression for the secrecy capacity may now be written in the limit of small dark count, $r_d << 1$:

$$S = \frac{1}{2} \left[ \psi_{\geq 1} (\eta \mu \alpha) \cdot (1 - fr_c) + \left( 1 - \frac{f}{2} \right) r_d - \nu \right] - \frac{g_{pa} + a}{m},$$

where we have defined

$$f \equiv 1 + Q + T,$$

and

$$\nu \equiv 2\nu_{\max}/m,$$

so that the rescaled quantity $\tilde{\nu}$ is independent of $m$.

Note that the pulse intensity parameter $\mu$ can be chosen to maximize the secrecy capacity $S$ and thus also the key generation rate $R$. A detailed investigation of the optimum pulse intensity under various conditions of practical interest and the resulting secrecy capacities and rates may be found in [3].

We have presented results for the secrecy capacity of a practical quantum key distribution scheme using attenuated laser pulses to carry the quantum information and encoding the raw key material using photon polarizations according to the BB84 protocol. This is the first analysis of the secrecy of a practical implementation of the BB84 protocol that simultaneously takes into account and presents the full set of complete analytical expressions for effects due to the presence of pulses containing multiple photons in the attenuated output of the laser, the finite length of individual blocks of key material, losses due to error correction, privacy amplification, continuous authentication, errors in polarization detection, the efficiency of the detectors, and attenuation processes in the transmission medium for the implementation of BB84 described in [7]. The transmission medium may be either free space or fiber optic cable. The results apply when eavesdropping is restricted to attacks on individual photons. The extension of these results to include collective attacks on multiple photon states in full generality is the subject of continuing research.
in our analysis, relates the privacy amplification compression directly to the amount of information leaked to Eve prior to privacy amplification. This makes it possible to analyze the effect of the block size on the amount of privacy amplification compression, and concomitantly introduces an explicit security parameter, $\varepsilon$, as a bound on Eve’s chances of mounting a successful attack on strings of finite length.

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