Research Article

Best Proximity Point for Generalized and S-Geraphty Contractions

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This paper introduces a new class of mappings called S-Geraphty-contractions and provides sufficient conditions for the existence and uniqueness of a best proximity point for such mappings. It also presents the best proximity point result for generalized contractions as well. Our results extend and generalize some theorems in the literature.

1. Introduction and Preliminaries

The center of interest of fixed point theory is the solving of the equation $Tx = x$ where $T$ is a mapping defined on a sub-set of a metric space, a normed linear space, or a topological vector space. Ever since its appearance, the well-known Banach contraction principle has been extensively studied, and the literature contains numerous interesting extensions and generalizations of the aforementioned result, in particular, Geraphty’s generalization of the Banach contraction principle.

In 1973, Geraphty ([11]) introduced the class $S$ of functions $\beta : [0,\infty) \to [0,1)$ satisfying the following condition:

$$\beta(t_n) \to 1 \implies t_n \to 0. \quad (1)$$

And proved the following result.

**Theorem 1.** Let $(X,d)$ be a complete metric space and $T : X \to X$ be an operator satisfying the the following inequality for some $\beta \in S$,

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y), \text{for all } x, y \in X. \quad (2)$$

Then, $T$ has a unique fixed point.

Since $S$ contains the class of constant functions $\beta(t) = k \in [0,1)$, the previous theorem extends that of Banach.

Another interesting extension of the Banach contraction principle is due to Kirk et al. ([2]). The authors introduced the class of cyclic mappings, i.e., $T : A \cup B \to A \cup B$ such that $T(A) \subseteq B$ and $T(B) \subseteq A$. And, under a suitable condition on $T$, proved a fixed point theorem which extends that of Banach. Interestingly, a more important problem than the extension of the Banach principle arose.

Whereas a cyclic mapping does not necessarily have a fixed point, it is desirable to determine an element $x$ which is somehow closest to $Tx$. More precisely, an element $x$ for which the error $d(x, Tx)$ assumes the least possible value $\text{dist}(A, B)$ where $\text{dist}(A, B) = \inf \{d(x, y) : x \in A, y \in B\}$, such a point is called a best proximity point of the cyclic mapping $T$. Since 2003, research on best proximity points of cyclic mapping became an important topic in nonlinear analysis and has been studied by many authors [2–8].

In 2012, Caballero and al, introduced the following contraction.

**Definition 2** (see [9]). Let $A, B$ be nonempty subsets of a metric space $(X,d)$. A nonself mapping $T : A \to B$ is said to be a Geraphty-contraction if there exists a $\beta \in S$ such that

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y) \text{ for all } x, y \in A. \quad (3)$$

Just like cyclic mappings, nonself mappings may not have...
fixed points and a best proximity point for a nonself mapping $T : A \rightarrow B$ is a point $x \in A$ such that $d(x, Tx) = \text{dist}(A, B)$.

Later on, the current authors ([110]) introduced the notion of tricyclic mappings and the best proximity point thereof. Let $A$, $B$, and $C$ be nonempty subsets of a metric space $(X, d)$. A mapping $T : A \cup B \cup C \rightarrow A \cup B \cup C$ is said to be tricyclic provided that $T(A) \subseteq B, T(B) \subseteq C$, and $T(C) \subseteq A$. A best proximity point of $T$ is a point $x \in A \cup B \cup C$ such that $d(x, Tx, T^2x) = \delta(A, B, C)$, where the mapping $D : X \times X \times X \rightarrow [0, +\infty)$ is defined by $D(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, and

$$\delta(A, B, C) = \inf \{D(x, y, z): x \in A, y \in B \text{ and } z \in C\}.$$  

(4)

Some results about the best proximity points of tricyclic mappings can be found in [10–15].

In the next section of this paper, taking inspiration from our recent works, we introduce the new class of $S$–cyclic mappings, which stands somewhere between the two classes of cyclic and tricyclic mappings. We define $S$–Geraghty-contractions and establish a best proximity point theorem for such mappings. As a special case, we obtain the best proximity point and fixed point theorem for cyclic mapping.

To describe our results, we need some definitions and notations. Given a triad $(A, B, C)$ of nonempty subsets of a metric space $(X, d)$, then the proximal pair $(A_0, B_0)$ of $(A, B)$ is given by:

$$A_0 = \{x \in A : d(x, y') = \text{dist}(A, B) \text{ for some } y' \in B\},$$

$$B_0 = \{y \in B : d(x', y) = \text{dist}(A, B) \text{ for some } x' \in A\}.$$  

(5)

A pair $(x, y) \in A \times B$ is said to be proximal in $(A, B)$ if $d(x, y) = \text{dist}(A, B)$. We subsequently use the following notations:

$$A_{y_0} = \{x \in A : D(x, y', z') = \delta(A, B, C) \text{ for some } y' \in B \text{ and } z' \in C\},$$

$$B_{y_0} = \{y \in B : D(x', y, z) = \delta(A, B, C) \text{ for some } x' \in A \text{ and } z \in C\},$$

$$C_{y_0} = \{z \in C : D(x', y', z) = \delta(A, B, C) \text{ for some } x' \in A \text{ and } y' \in B\},$$

$$(A \times B)_0 = \{(x, y) \in A \times B : D(x, y, z) = \delta(A, B, C) \text{ for some } z \in C\},$$

$$(B \times C)_0 = \{(y, z') \in B \times C : D(x', y, z') = \delta(A, B, C) \text{ for some } x' \in A\},$$

$$(C \times A)_0 = \{(z', x') \in C \times A : D(x', y', z') = \delta(A, B, C) \text{ for some } y' \in B\}.$$  

(6)

Note that $(B \times C)_0$ is included in $B_{y_0} \times C_{y_0}$ but the inverse does not always hold, and it is obvious that $\delta(A_0, B_{y_0}, C_{y_0}) = \delta(A, B, C)$. Let us illustrate the cases $(B \times C)_0 \subseteq B_{y_0} \times C_{y_0}$ and $(B \times C)_0 = B_{y_0} \times C_{y_0}$ with simple examples.

(i) Consider $A = [0, 1] \times \{0\}, B = [0, 1] \times \{1\}$, and $C = [0, 1] \times \{2\}$. Clearly $\delta(A, B, C) = 4 \times (0, 1) \in B_{y_0}$ and $(1, 2) \in C_{y_0}$. For all $(i, 0) \in A$, we have $D((t, 0), (0, 1), (1, 2)) = \sqrt{t^2 + 1} + \sqrt{2} + \sqrt{(1-t)^2 + 4} > 4$.  

(7)

\textbf{Example 3.} Let $X$ be $\mathbb{R}^2$ endowed with its Euclidean distance.

Hence, $\{((0, 1), (1, 2)) \notin (B \times C)_0\}$ and so $(B \times C)_0 \subseteq B_{y_0} \times C_{y_0}$.

(ii) Let $A = [-3, \infty) \times \{0\}, B = [-1, 1] \times [-1, 1]$ and $C = [2, 3] \times \{0\}$. Then $\delta(A, B, C) = 8$, and $A_{y_0} = \{(-2, 0)\}, B_{y_0} = [-1, 1] \times \{0\}, C_{y_0} = \{(2, 0)\}$.  

(8)

Let $(y, 0) \in B_{y_0}$.

$$D((-2, 0), (y, 0), (2, 0)) = 8.$$  

(9)

And then, $(B \times C)_0 = B_{y_0} \times C_{y_0}$.

(iii) In the special case where $C = A$, we get $A_{y_0} = A_0, B_{y_0} = B_0$ and

$$(A \times B)_0 = \{(x, y) \in A \times B : (x, y) \text{ is proximal in } (A, B)\}.$$  

(10)

Let $A$ and $B$ be defined as $A = \{(x, x) : x \in [-3, -1] \cup [1, 3]\}$ and $B = \{(x, -x) : x \in [-3, -1] \cup [1, 3]\}$.  

(11)

Clearly, $A_0 = \{(-1, -1), (1, 1)\}, B_0 = \{(-1, 1), (1, -1)\}$, and

$$(A \times B)_0 = \{( (-1, -1), (-1, 1) ), ( (-1, 1), (-1, 1) ), ( (1, 1), (1, -1) ), ( (1, -1), (1, 1) ) \} = A_0 \times B_0.$$  

(12)

\textbf{Definition 4.} Let $A$, $B$, and $C$ be nonempty subsets of a metric space $(X, d)$. A mapping $T : A \cup B \cup C \rightarrow A \cup B \cup C$ is said to be a $S$–cyclic if $T(A) \subseteq B$ and $T(B) \subseteq C$. A best proximity point for $T$ is a point $x \in A$ provided that $D(x, Tx, T^2x) = \delta(A, B, C)$.

\textbf{Definition 5.} Let $A$, $B$, and $C$ be nonempty subsets of a metric space $(X, d)$. A $S$–cyclic mapping $T : A \cup B \cup C \rightarrow A \cup B \cup C$ is said to be a $S$–Geraghty-contraction if there exists a $\beta \in S$ such that

$$D(Tx, Ty, Tz) \leq \beta(D(x, y, z)).D(x, y, z) \text{ for all } x, y, z \in A.$$  

(13)

Notice that since the $\beta$ is strictly smaller than one, we have

$$D(Tx, Ty, Tz) \leq \beta(D(x, y, z)).D(x, y, z) < D(x, y, z) \text{ for all } x, y, z \in A.$$  

(14)
In the special case where \( x = z \), we obtain
\[
D(Tx, Ty, Tx) < D(x, y, x),
\] (15)
which means
\[
d(Tx, Ty) < d(x, y) \quad \text{for all } x, y \in A. \quad (16)
\]

Thus, every S-Geraphty-contraction is continuous on \( A \).

**Example 6.** Consider a mapping \( T : A \cup B \rightarrow B \cup C \) such that \( T(A) \subseteq B \) and \( T(B) \subseteq C \). Suppose \( T_{\beta} : A \rightarrow B \) is Geraphty-contraction for some nondecreasing \( \beta \in S \). We have
\[
d(Tx, Ty) \leq \beta(d(x, y))d(x, y),
\] (17)
for all \( x, y \in A \). Then
\[
D(Tx, Ty, Tz) \leq \max \{ \beta(d(x, y)), \beta(d(y, z)), \beta(d(z, x)) \}.D(x, y, z) \leq \beta(D(x, y, z)).D(x, y, z).
\] (18)

Which means, \( T : A \cup B \rightarrow B \cup C \) is a S-Geraphty-contraction.

**Definition 7** (see [16]). Let \( A \) and \( B \) be nonempty subsets of a metric space \( (X, d) \). The pair \( (A, B) \) is said to have the \( P - p \) property if and only if
\[
\begin{aligned}
d(x_1, y_1) &= \delta(A, B), \\
d(x_2, y_2) &= \delta(A, B) \implies d(x_1, x_2) = d(y_1, y_2),
\end{aligned}
\] (19)
where \( x_1, x_2 \in A \) and \( y_1, y_2 \in B \).

The concept of \( P - p \) property of pairs can naturally be extended to triads.

**Definition 8.** Let \( A, B, \) and \( C \) be nonempty subsets of a metric space \( (X, d) \). The triad \( (A, B, C) \) is said to have the \( P - p \) property if and only if
\[
\begin{aligned}
D(x_1, y_1, z_1) &= \delta(A, B, C) \\
D(x_2, y_2, z_2) &= \delta(A, B, C) \implies D(x_1, x_2, x_3) = D(y_1, y_2, y_3) = d(z_1, z_2, z_3), \\
D(x_3, y_3, z_3) &= \delta(A, B, C)
\end{aligned}
\] (20)
where \( x_1, x_2, x_3 \in A, y_1, y_2, y_3 \in B, \) and \( z_1, z_2, z_3 \in C \).

**Example 9.** Let \( x_1, x_2, x_3 \in A, y_1, y_2, y_3 \in B, \) and \( z_1, z_2, z_3 \in C \) such that
\[
D(x_1, y_1, z_1) = D(x_2, y_2, z_2) = D(x_3, y_3, z_3) = \delta(A, B, C).
\] (21)
(1) Suppose \( \delta(A, B, C) = \text{dist}(A, B) + \text{dist}(B, C) + \text{dist}(C, A) \). We necessarily have
\[
\begin{aligned}
d(x_1, y_1) &= d(x_2, y_2) = d(x_3, y_3) = \text{dist}(A, B), \\
d(y_1, z_1) &= d(y_2, z_2) = d(y_3, z_3) = \text{dist}(B, C), \\
d(z_1, x_1) &= d(z_2, x_2) = d(z_3, x_3) = \text{dist}(C, A).
\end{aligned}
\]
It is plain to see that if the pairs \((A, B), (B, C),\) and \((C, A)\) have the \( P - p \) property, then the triad \((A, B, C)\) does as well.

(2) If \( A = C \), then
\[
\delta(A, B, C) = \delta(A, B, A) = 2\text{dist}(A, B).
\] (23)
Which implies \( x_1 = x_2 = z_2, x_3 = z_3, \) and
\[
d(x_1, y_1) = d(x_2, y_2) = \text{dist}(A, B).
\] (24)
Thus, the triad \((A, B, A)\) has the \( P - p \) property if the pair \((A, B)\) does.

2. **S-Geraphty-Contractions**

Next, we state the main result of this section.

**Theorem 10.** Let \((A, B, C)\) be a triad of nonempty closed subsets of a complete metric space \( (X, d) \) such that \( A_{00} \) is nonempty and \( B_{00} \times C_{00} = (B \times C)_{00} \). Let \( T : A \cup B \rightarrow B \cup C \) be a \( S - \text{Geraphty-contraction satisfying } T(A_{00}) \subseteq B_{00} \) and \( T(B_{00}) \subseteq C_{00} \). Assume the triad \((A, B, C)\) has the \( P - p \) property. Then, there exists a unique \( x' \in A \) such that \( D(x', Tx', T^2 x') = \delta(A, B, C) \).

**Proof.** Let \( x_0 \in A_{00} \), since \( T(A_{00}) \subseteq B_{00} \) and \( T(B_{00}) \subseteq C_{00} \),
\[
(Tx_{00}, T^2 x_0) \in B_{00} \times C_{00} = (B \times C)_{00}.
\] (25)
There then exists \( x_1 \in A_{00} \) such that \( D(x_1, Tx_0, T^2 x_0) = \delta(A, B, C) \). By the same argument, we obtain some \( x_2 \in A_{00} \), for which \( D(x_2, Tx_1, T^2 x_1) = \delta(A, B, C) \). Keeping on this process, we can get a sequence \((x_n)\) in \( A_{00} \) satisfying
\[
D(x_{n+1}, Tx_n, T^2 x_n) = \delta(A, B, C), \quad \text{for all } n \in \mathbb{N}.
\] (26)
Since \((A, B, C)\) has the \( P \)-property, we have
\[
D(x_n, x_{n+1}, x_{n+2}) = D(Tx_{n-1}, Tx_n, Tx_{n+1}) \quad (27)
\]
Taking into consideration that $T$ is a $S$-Geraphty-contraction, we have

$$D(x_n, x_{n+1}, x_{n+2}) = D(Tx_{n-1}, Tx_n, Tx_{n+1}) \leq \beta(D(x_{n-1}, x_n, x_{n+1})) < D(x_{n-1}, x_n, x_{n+1}). \tag{28}$$

In the case where $D(x_n, x_{n+1}, x_{n+2}) = 0$, for some $n_0 \in \mathbb{N}$, we have

$$0 = D(x_n, x_{n+1}, x_{n+2}) = D(Tx_{n-1}, Tx_n, Tx_{n+1}). \tag{29}$$

Hence, $Tx_{n-1} = Tx_n = Tx_{n+1}$. Consequently

$$\delta(A, B, C) = D(x_n, Tx_{n-1}, T^2x_{n-1}) = D(x_n, Tx_n, T^2x_n). \tag{30}$$

And the desired result then follows. Therefore, we suppose the contrary case, that is, $D(x_n, x_{n+1}, x_{n+2}) > 0$, for every $n \in \mathbb{N}$. We note that $(D(x_n, x_{n+1}, x_{n+2}))$ is a decreasing sequence of positive real numbers, hence, there exists $r \geq 0$ such that

$$\lim_{n \to \infty} D(x_n, x_{n+1}, x_{n+2}) = r. \tag{31}$$

Suppose $r > 0$, we have

$$0 < \frac{D(x_n, x_{n+1}, x_{n+2})}{D(x_{n-1}, x_n, x_{n+1})} \leq \beta(D(x_{n-1}, x_n, x_{n+1})) < 1 \quad \text{for all} \quad n \in \mathbb{N}. \tag{32}$$

Which implies that

$$\lim_{n \to \infty} \beta(D(x_{n-1}, x_n, x_{n+1})) = 1. \tag{33}$$

And since $\beta \in \mathcal{F}$, we get $r = 0$ and this contradicts our assumption. Thus, $\lim_{n \to \infty} D(x_n, x_{n+1}, x_{n+2}) = 0$.

In the sequel, we show that $(x_n)$ is a Cauchy sequence. Notice that $D(x_{n+1}, Tx_{n+2}, T^2x_{n+2}) = D(x_{n+1}, Tx_n, T^2x_n) = \delta(A, B, C)$ for any $p, q \in \mathbb{N}$, and since $(A, B, C)$ satisfies the $P$-property, $D(x_{p+1}, x_{q+1}, x_{p+1}) = D(x_p, x_q, x_p)$. Suppose $(x_n)$ is not a Cauchy sequence, hence,

$$\limsup_{n \to \infty} D(x_p, x_q, x_p) > 0. \tag{34}$$

We have

$$D(x_p, x_q, x_p) = 2d(x_p, x_q) \leq 2d(x_{p+1}, x_{q+1}) + 2d(x_q, x_p) \leq D(x_{p+1}, x_{q+1}) + D(x_{q+1}, x_{p+1}) + D(x_{q+1}, x_{p+1}) \leq \beta(D(x, x_q, x_p)) + D(x_{q+1}, x_{p+1}) \leq \beta(D(x, x_q, x_p)) + \beta(D(x_{q+1}, x_{p+1})). \tag{35}$$

Which means

$$D(x_p, x_q, x_p) \leq \frac{D(x_p, x_{p+1}, x_q, x_q) + D(x_q, x_{q+1}, x_p, x_p)}{1 - \beta(D(x_p, x_q))}. \tag{36}$$

Since $\limsup D(x_p, x_q, x_p) > 0$ and $\limsup D(x_p, x_{p+1}, x_{q+1}) = 0$, then, by passage to the limit in the last inequality, we obtain

$$\limsup_{n \to \infty} \frac{1}{1 - \beta(D(x_p, x_q))} = \infty. \tag{37}$$

Therefore, $\limsup_{n \to \infty} \beta(D(x_p, x_q, x_p)) = 0$. But, $\beta \in S$, thus, $\limsup D(x_p, x_q, x_p) = 0$, which is contradictory with our assumption. Consequently $(x_n)$ is a Cauchy sequence.

Since $(x_n) \subset A$ and $A$ is a close subset of the complete metric space $(X, d)$, $x_n \to x' \in A$. Since $T$ is continuous, $Tx_n \to Tx$ and $T^2x_n \to T^2x$. Which implies, $D(x_n, Tx_n, T^2x_n) \to D(x', Tx', T^2x')$. On the other hand, $(D(x_n, Tx_n, T^2x_n))$ is a constant sequence with the value $\delta(A, B, C)$. Thus

$$D(x', Tx', T^2x') = \delta(A, B, C). \tag{38}$$

That means $x'$ is a best proximity point of $T$. As for the uniqueness, suppose that $x_1$ and $x_2$ are two distinct best proximity points of $T$. That is

$$D(x_1, Tx_1, T^2x_1) = D(x_2, Tx_2, T^2x_2) = \delta(A, B, C). \tag{39}$$

Taking into account that $(A, B, C)$ has the $P$-property, we get

$$D(x_1, x_2, x_2) = D(Tx_1, Tx_2, Tx_2). \tag{40}$$

Which is contradictory with the fact that $T$ is a $S$-Geraphty-contraction. Indeed, we have

$$D(x_1, x_2, x_2) = D(Tx_1, Tx_2, Tx_2) \leq \beta(D(x_1, x_2, x_1)) + D(x_1, x_2, x_2) \leq D(x_1, x_2, x_2). \tag{41}$$

And the proof is completed.
Example 11. Consider $X = \mathbb{R}^2$ with its usual metric. Let $A, B,$ and $C$ be defined by

$$A = \{0\} \times [0, \infty), B = \{1\} \times [0, \infty) \text{ and } C = [2, \infty) \times \{0\}. \quad (42)$$

It is easy to see that $A_{00} = \{(0,0)\}, B_{00} = \{(1,0)\}$ and $C_{00} = \{(2,0)\}$. Hence, $A_{00}$ is nonempty and $B_{00} \times C_{00} = (B \times C)_0$.

Let $k \in [0, 1)$ and let $T_k : A \cup B \longrightarrow B \cup C$ be the mapping defined as

$$T_k(0,x) = (1,kx) \text{ for all } (0,x) \in A,$$

$$T_k(1,y) = (2 + ky, 0) \text{ for all } (1,y) \in B. \quad (43)$$

Let $(0, x_1), (0, x_2), (0, x_3) \in A,$

$$D(T_k(0,x_1), T_k(0,x_2), T_k(0,x_3))$$

$$\quad = D((1,kx_1), (1,kx_2), (1,kx_3)) \leq kD(x_1, x_2, x_3). \quad (44)$$

Since the constant functions $\beta(t) = k$ where $k \in [0, 1)$, belong to $S,T$ is a S-Geraphty-contraction. Also, $(A, B, C)$ has the $P$-property. Indeed, if

$$D(x_1, y_1, z_1) = \delta(A, B, C) = 4,$$

$$D(x_2, y_2, z_2) = \delta(A, B, C) = 4,$$

$$D(x_3, y_3, z_3) = \delta(A, B, C) = 4. \quad (45)$$

Then $x_1 = x_2 = x_3 = (0,0), y_1 = y_2 = y_3 = (1,0),$ and $z_1 = z_2 = z_3 = (2,0).$ Thus, $D(x_1, x_2, x_3) = D(y_1, y_2, y_3) = D(z_1, z_2, z_3) = 0.$ All things considered, $T$ has a unique best proximity point, clearly $(0,0)$.

Corollary 12. Let $(A, B)$ be a pair of nonempty closed subsets of a complete metric space $(X, d)$ such that $A_0$ is nonempty. Let $T : A \cup B \longrightarrow B \cup A$ be a $S$-Geraphty-contraction satisfying $T(A_0) \subseteq B_0$ and $T(B_0) \subseteq A_0$. Assume the triad $(A, B)$ has the $P$-property. Then there exists a unique $x' \in A$ that is both a best proximity point for the cyclic mapping $T$ and a fixed point for the self-mapping $T^2$.

Proof. The triad $(A, B, A)$ satisfies the condition of the previous theorem. Hence, there exists a unique $x' \in A$ such that $D(x', T_2x') = 2\text{dist}(A, B), \text{ which implies }$  

$$d(x', T_2x') = \text{dist}(A, B) \text{ and } T^2x' = x', \quad (46)$$

and that is the wanted result.

3. S-Min–Max Condition and Generalized Contractions

In this section, using the S-min–max condition, we obtain a best proximity point result for nonself generalized contractions. First, we recall and fix some notions and notations that will subsequently be used. Given mappings $R : A \longrightarrow B$, $S : B \longrightarrow C,$ and $T : C \longrightarrow A$, where $A, B,$ and $C$ are nonempty subsets of a metric space $(X, d)$.

Definition 13 (see [6]). Let $A, B$ be nonempty subsets of a metric space $(X, d)$. The mapping $R : A \longrightarrow B$ is called a generalized contraction if, given real numbers $a$ and $b$ with $0 < a \leq b$, there exists a real number $\alpha(a, b) \in 0, 1)$ such that

$$a \leq d(x_1, x_2) \leq b \implies d(Rx_1, Rx_2) \leq \alpha(a, b)d(x_1, x_2), \quad (47)$$

for all $x_1, x_2 \in A$.

Obviously, every generalized contraction is a contractive mapping.

Definition 14 (see [17]). Let $A, B$ be nonempty subsets of a metric space $(X, d).$ It is stated that the pair $(R, S)$ satisfies the min-max condition if, for all $x \in A$ and $y \in B$, we have

$$d(A, B) < d(x, y) \implies \max (Rx, Sy) \neq \min (Rx, Sy), \quad (48)$$

where $\min(Rx, Sy)$ and $\max(Rx, Sy)$ are defined as

$$\min (Rx, Sy) = \min \left\{ d(x, y), d(x, Rx), d(y, Sy), d(Rx, Sy), d(x, RSy), d(y, SRx), d(SRx, SRy), d(SRx, RSy) \right\},$$

$$\max (Rx, Sy) = \max \left\{ d(x, y), d(x, Rx), d(y, Sy), d(x, Sy), d(y, Rx), d(Rx, Sy), d(SRx, SRy), d(Rx, RSy) \right\}, \quad (49)$$
Definition 15. Let \( A, B, C \) be nonempty subsets of a metric space \((X, d)\). The triad \((R, S, T)\) is said to satisfy \(S\)-min–max condition if for all \( x \in A, y \in B, \) and \( z \in C, \)

\[
\text{dist}(A, B) < d(x, y) \quad \text{and} \quad Tz = x \implies S \min \{Rx, Sy, Tz\} = S \max \{Rx, Sy, Tz\},
\]

where

\[
S \min \{Rx, Sy, Tz\} = \min \left\{ d(y, x, z), d(x, y, TSy), d(Rx, Sy, TSy), \right. \\
\left. d(y, TSy, z), d(y, TSx, RTz), d(Rx, SRx, TSx), \\d(y, RTSy, z), d(y, RTSy, y), \\d(Rx, Sy, TSy), d(x, TSRx, z), d(x, RTSy, y), \\d(x, RTSy, z), d(y, TSRx, SRTx), d(Rx, SRx, SRTx), \\d(Sy, TSy, RTSy), d(TSRx, RTSy, SRTz) \}
\]

\[
S \max \{Rx, Sy, Tz\} = \max \left\{ d(y, x, z), d(x, y, TSy), d(Rx, Sy, TSy), \right. \\
\left. d(y, TSy, z), d(y, TSx, RTz), d(Rx, SRx, TSx), \\d(y, RTSy, z), d(y, RTSy, y), \\d(Rx, Sy, TSy), d(x, TSRx, z), d(x, RTSy, y), \\d(x, RTSy, z), d(y, TSRx, SRTx), d(Rx, SRx, SRTx), \\d(Sy, TSy, RTSy), d(TSRx, RTSy, SRTz) \}
\]

Now, we are at liberty to state the main result of this section.

**Theorem 16.** Suppose \( A, B, \) and \( C \) are nonempty closed subsets of a metric space \((X, d)\) and the mappings \( R, S, \) and \( T \) verify the following conditions:

- \( R \) is a generalized contraction.
- \( S \) and \( T \) are nonexpansive mappings.

The triad \((R, S, T)\) satisfies the \(S\)-min–max condition. \((52)\)

For a fixed element \( x_0 \) in \( A, \) let

\[
x_{3n+1} = Rx_{3n}, x_{3n+2} = Sx_{3n+1} \text{ and } x_{3n} = Tx_{3n-1}.
\]

Then, the sequence \( \{x_{3n}\} \) must converge to a best proximity point \( x^* \) of \( R \) and the sequence \( \{x_{3n+1}\} \) must converge to a point \( y^* \) such that

\[
d(x^*, y^*) = \text{dist}(A, B).
\]

**Proof.** Let a sequence \( \{s_n\} \) of real numbers be defined as follows:

\[
s_n = d(x_{3n}, x_{3n+3}) \quad \text{for} \quad n \geq 0.
\]

From the fact that \( R \) is a generalized contraction and from the nonexpansiveness of \( S \) and \( T, \) it follows that \( \{s_n\} \) is a bounded below, decreasing sequence of nonnegative real numbers and hence converges to some nonnegative real number, say \( s. \) We show that \( s \) must be nil. If not, then we choose a positive integer \( N \) such that

\[
s \leq s_n \leq (b + 1) \quad \text{for all} \quad n \geq N.
\]

Therefore,

\[
\begin{align*}
s_{N+1} &= d(x_{3N+3}, x_{3N+6}) \\
&= d(Tx_{3N+2}, Tx_{3N+5}) \\
&\leq d(x_{3N+2}, x_{3N+5}) \\
&= d(Sx_{3N+1}, Sx_{3N+4}) \\
&\leq d(x_{3N+1}, x_{3N+4}) \\
&= d(Rx_{3N}, Rx_{3N+3}) \\
&\leq \alpha(s, s + 1)s_N
\end{align*}
\]

Similarly, we get

\[
s_{N+2} \leq \alpha(s, s + 1)s_{N+1} \leq (\alpha(s, s + 1))^2 s_N.
\]

Keeping on that process, we obtain

\[
s_{N+k} \leq (\alpha(s, s + 1))^k s_N.
\]

When \( k \to \infty, \) we deduce that \( s = 0 \) and this is a contradiction. Hence, it can be deduced that \( s_n \to 0 \) as \( n \to \infty. \) Next, we shall prove that \( \{x_{3n}\} \) is a Cauchy sequence. Let \( \epsilon > 0 \) be given. Since \( s_n \to 0, \) it is possible to choose a positive integer \( N \) such that

\[
s_N = d(x_{3N}, x_{3N+3}) \leq \frac{\epsilon}{2} \left(1 - \alpha\left(\frac{\epsilon}{2}, \epsilon\right)\right).
\]

It is sufficient to prove that whenever \( x \) is an element of the closed ball \( B(x_{3N}, \epsilon) \) then so is \( TSRx. \) If \( x \) satisfies \( d(x_{3N}, x) \leq \epsilon/2, \) then

\[
\begin{align*}
d(TSRx, x_{3N}) &\leq d(TSRx, x_{3N+3}) + d(x_{3N+3}, x_{3N}) \\
&\leq d(Rx, Rx_{3N}) + s_N \\
&\leq d(x, x_{3N}) + s_N \leq \epsilon.
\end{align*}
\]
And, if \( x \) satisfies \( \varepsilon/2 < d(x_{3n}, x) \leq \varepsilon \), then

\[
dTSRx_{3n} \leq d(TSRx, x_{3n}+1) + d(x_{3n+1}, x_{3n}) \\
\leq d(Rx, RXn) + sN \\
\leq \alpha \left( \frac{\varepsilon}{2}, \varepsilon \right) d(x, x_{3n}) + sN \leq \varepsilon.
\]

(62)

We conclude that \( x_{3n} \in B(x_{3N}, \varepsilon) \) for all \( n \geq N \) and that means \( \{x_{3n}\} \) is a Cauchy sequence. Taking the completeness of the space under consideration, \( x_{3n} \rightarrow x^* \in A \) and from the continuity of \( R, S, \) and \( T \), we get

\[
x_{3n} \rightarrow x^* \implies x_{3n+1} = Rx_{3n} \rightarrow Rx^* = y^*, \\
x_{3n+1} \rightarrow y^* \implies x_{3n+2} = Sx_{3n+1} \rightarrow Sy^* = z^*, \\
x_{3n+2} \rightarrow z^* \implies x_{3n+3} = Tx_{3n+2} \rightarrow Tz^* = x^*.
\]

(63)

for some \( y^* \in B \) and \( z^* \in C \). Then, we deduce that \( TSRx^* = x^*, RTSy^* = y^*, \) and \( SRTz^* = z^* \). We also have

\[
S \min (Rx^*, Sy^*, Tz^*) = D(x^*, y^*, z^*) = S \max (Rx^*, Sy^*, Tz^*).
\]

(64)

And since the triad \((R, S, T)\) satisfies the \( S\)-min–max condition, we obtain

\[
d(x^*, y^*) = \text{dist}(A, B) \text{ or } Tz^* \neq x^*.
\]

(65)

Then, we necessarily have

\[
d(x^*, y^*) = \text{dist}(A, B).
\]

(66)

And that finishes the proof.

As a special case of our result, we get the following best proximity point theorem, which was proved in [18].

**Corollary 17** (see [18]). Let \( A \) and \( B \) be nonempty, closed subsets of a complete metric space. Let \( R : A \rightarrow B \) and \( S : B \rightarrow A \) satisfy the following conditions:

- \( R \) is a generalized contraction.
- \( S \) is a nonexpansive mapping.

The pair \((R, S)\) satisfies the \( \min – \max \) condition.

Further, for a fixed element \( y_0 \) in \( A \), let

\[
y_{2n+1} = Ry_{2n} \text{ and } y_{2n} = Sy_{2n-1}.
\]

(68)

Then, the sequence \( \{y_{2n}\} \) must converge to a best proximity point \( x^* \) of \( R \) and the sequence \( \{y_{2n+1}\} \) must converge to a best proximity point \( y^* \) of \( S \) such that

\[
d(x^*, y^*) = d(A, B).
\]

(69)

Further, if \( S \) has two distinct best proximity points, then \( d(A, B) \) does not vanish and hence the sets \( A \) and \( B \) should be disjoint.

**Proof.** The triad \((R, S, T)\), where \( T : A \rightarrow A \) is the identity mapping, satisfies the \( S\)-min–max condition. Indeed, let \((x, y, z) \in A \times B \times A\) such that \( d(x, y) > \text{dist}(A, B) \) and \( Tz = z = x \). We have

\[
S \min (Rx, Sy, Tz) = \min \left\{ 2d(x, y), 2d(x, Rx), 2d(y, Sy), 2d(Rx, Sy), 2d(x, RSy), 2d(y, Rz), 2d(Rx, Rz), 2d(SRx, Rz), 2d(SRy, Rz) \right\}
\]

(70)

\[
= 2 \min (Rx, Sy),
\]

and

\[
S \max (Rx, Sy, Tz) = \max \left\{ 2d(x, y), 2d(x, Rx), 2d(y, Sy), 2d(x, Sy), 2d(y, Rx), 2d(Rx, Sy), 2d(y, Rz), 2d(Rx, Rz), 2d(SRx, Rz), 2d(SRy, Rz) \right\}
\]

(71)

\[
= 2 \max (Rx, Sy).
\]
And since \((R, S)\) satisfies the min-max condition,
\[
S \min (Rx, Sy, Tz) = 2 \min (Rx, Sy) \neq 2 \max (Rx, Sy) = S \max (Rx, Sy, Tz).
\]
\[
(72)
\]
Thus, the triad \((R, S, T)\) fulfills all the conditions of the previous theorem. For \(x_0 = y_0 \in A\), define
\[
x_{3n+1} = Rx_{3n}, \quad x_{3n+2} = Sx_{3n+1} \quad \text{and} \quad x_{3n} = Tx_{3n-1} = x_{3n-1}.
\]
\[
(73)
\]
Then, \(\{x_{3n}\}\) must converge to a best proximity point \(x^*\) of \(R\), the sequence \(\{x_{3n+1}\}\) must converge to a point \(y^*\) and sequence \(\{x_{3n-1}\}\) must converge to \(z^*\) such that
\[
Rx^* = y^*, \quad Sy^* = z^* = x^* \quad \text{and} \quad d(x^*, y^*) = \text{dist}(A, B).
\]
\[
(74)
\]
It suffices to notice that
\[
y_{2n+1} = x_{3n+1} \quad \text{and} \quad y_{2n} = x_{3n} = x_{3n-1}.
\]
\[
(75)
\]
Now, if \(R\) has two distinct best proximity points \(a, b \in A\), then
\[
d(a, b) \leq d(a, Sa) + d(Sa, Sb) + d(b, Sb) < d(a, b) + \text{dist}(A, B).
\]
\[
(76)
\]
Which means \(A\) and \(B\) are disjoint.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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