MULTISCALE HOMOGENIZATION OF INTEGRAL CONVEX FUNCTIONALS IN ORLICZ SOBOLEV SETTING

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Abstract. The Γ-limit of a family of functionals $u \mapsto \int_{\Omega} f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, D^s u\right) \, dx$ is obtained for $s = 1, 2$ and when the integrand $f = f(y, z, v)$ is a continuous function, periodic in $y$ and $z$ and convex with respect to $v$ with nonstandard growth. The reiterated two-scale limits of second order derivatives are characterized in this setting.

1. Introduction. Multiscale Homogenization, as a development of Nguetseng’s seminal paper [25] (see also [3]), have been introduced by Allaire-Briane [4] in classical Sobolev spaces (see also [5] among a wide literature, and [30] for the constrained case), and later generalized in [14] to handle problems formulated in terms of higher order derivatives. On the other hand, the notion of two scale convergence has been later extended to the Orlicz (and Orlicz-Sobolev) setting in [16], (see also [21], [17], [19] (and [22], [23] for dimensional reduction problems) and reads as follows. Let $B$ be an $N$-function, with conjugate $\tilde{B}$, (see [1] and Section 2 below for detailed notations and definitions of functions spaces). For any bounded open set $\Omega \subset \mathbb{R}^N$ with Lipschitz boundary, a sequence of functions $(u_\varepsilon)_\varepsilon \subset L^B(\Omega)$ weakly two-scale converges in $L^B(\Omega)$ to a function $u_0 \in L^B_{\text{loc}}(\Omega \times Y)$, (the latter space being constituted by functions $v(x, y) \in L^B_{\text{loc}}(\Omega \times \mathbb{R}^N)$ such that $v(x, \cdot)$ is $Y$-periodic for a.e.
\[ x \in \Omega \text{ and } \int_{\Omega \times Y} B(|v|)dxdy < +\infty \text{ if } \]
\[ \int_{\Omega} u_{\varepsilon} g \left( x, \frac{x}{\varepsilon} \right) dx \to \int_{\Omega \times Y} u_0 g dxdy, \text{ for all } g \in L^{\tilde{B}}(\Omega; C_{\text{per}}(Y)), \]  
\[ (1) \]

as \( \varepsilon \to 0 \). The sequence is said to be strongly two-scale convergent in \( L^{\tilde{B}}(\Omega) \) to \( u_0 \in L^{\tilde{B}}_{\text{per}}(\Omega \times Y) \), if for any \( \eta > 0 \) and \( h \in L^{\tilde{B}}(\Omega; C_{\text{per}}(Y)) \) such that \( \| u_0 - h \|_{L^{\tilde{B}}(\Omega \times Y)} < \frac{\eta}{2} \), there exists \( \rho > 0 \) such that \( \| u_{\varepsilon}(\cdot) - h(\cdot, \varepsilon) \|_{L^{\tilde{B}}(\Omega)} \leq \eta \) for all \( 0 < \varepsilon \leq \rho \).

Recently, in [18], these results have been extended to the multiscale setting, see subsection 2.2.2 for precise definitions and results.

The aim of this work consists of extending the latter results, together with a \( \Gamma \)-convergence theorem, to higher order Sobolev-Orlicz spaces under suitable assumptions on the \( N \)-function. In details we will deal with the functional

\[ F_{\varepsilon}(u) = \int_{\Omega} f \left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, D^\varepsilon u \right) dx \]

where \( f \) satisfies the following hypothesis: \( f : \mathbb{R}^N_y \times \mathbb{R}^N_\varepsilon \times \mathbb{R}^N_\varepsilon \to [0, +\infty) \) is such that:

1. \( f \) is continuous or
2. \( f \) is measurable for all \( y, \lambda \in \mathbb{R}^N_\varepsilon \),
3. \( f \) is convex for all \( y, \varepsilon \in \mathbb{R}^N \), and for every \( \lambda \in \mathbb{R}^N_\varepsilon \) and it satisfies \( A_1 \) and \( A_2 \);
4. \( f \) is separately \( Y \)-periodic in \( y \) and \( z \);
5. \( f \) is convex for all \( y \) and almost every \( z \in \mathbb{R}^N_\varepsilon \);
6. There exist two constants \( c_1, c_2 > 0 \) such that \( c_1 B(|\lambda|) \leq f(y, z, \lambda) \leq c_2 (1 + B(|\lambda|)) \) for all \( \lambda \in \mathbb{R}^N_\varepsilon \), for a.e. \( z \in \mathbb{R}^N_\varepsilon \) and \( y \in \Omega \),

where \( Y \) is a copy of the unit cube \( (-1/2, 1/2)^N \), and \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \), \( s \in \{1, 2\} \), \( B \) is an \( N \)-function satisfying, together with its conjugate function, \( \Delta_2 \) condition (see [1] and Section 2 below).

Moreover \( \mathbb{R}^N_\varepsilon \) coincides with \( \mathbb{R}^{d \times N} \) if \( s = 1 \) and with \( (\text{Sym}(\mathbb{R}^N, \mathbb{R}^N))^d \), where \( \text{Sym}(\mathbb{R}^N, \mathbb{R}^N) \) denotes the space of all linear symmetric transformations from \( \mathbb{R}^N \) to \( \mathbb{R}^N \).

Bearing in mind that \( N, m, d \in \mathbb{N} \), \( \varepsilon \) denotes a sequence of positive real numbers converging to 0, and denoting (as above) by \( Y \) and \( Z \) two identical copies of the cube \( (-1/2, 1/2)^N \), adopting the notations in subsection 2.1, our first main result deals with the reiterated two-scale convergence in second order Sobolev-Orlicz spaces. Indeed we have the following result:

**Theorem 1.1.** Let \( \Omega \) be a bounded open set of \( \mathbb{R}^N \), with Lipschitz boundary. If \( (u_\varepsilon)_\varepsilon \) is a bounded sequence in \( W^2 L^2(\Omega; \mathbb{R}^d) \), then there exist a subsequence (not relabelled) converging weakly in \( W^2 L^2(\Omega; \mathbb{R}^d) \) to a function \( u \), and functions \( F \in L^1(\Omega; W^2 L^2(Y; \mathbb{R}^d)) \) and \( W \in L^1(\Omega \times Y; W^2 L^2(Z; \mathbb{R}^d)) \) such that:

1. \( U(x, y) - A(x)y \in L^1(\Omega; W^2 L^2_{\text{per}}(Y; \mathbb{R}^d)) \) for some \( A \in L^1(\Omega; \mathbb{R}^{d \times N}) \);
2. \( W(x, y, z) - C(x, y)z \in L^1(\Omega \times Y; W^2 L^2_{\text{per}}(Z; \mathbb{R}^d)) \) for some \( C \in L^1(\Omega \times Y; \mathbb{R}^{d \times N}) \);
3. \( u_\varepsilon \xrightarrow{w} u, Du_\varepsilon \xrightarrow{w} Du, \) and
4. \( \partial^2 u_\varepsilon \xrightarrow{\varepsilon^{-2s}} \partial^2 u, \) \( \partial^2 u_\varepsilon \xrightarrow{\varepsilon^{-2s}} \partial^2 U, \) \( \partial^2 u_\varepsilon \xrightarrow{\varepsilon^{-2s}} \partial^2 W, \) for each \( i, j \in \mathbb{N} \).
Conversely, given \( u \in W^2L^B(\Omega; \mathbb{R}^d), \) \( U \in L^1(\Omega; W^2L^B(Y; \mathbb{R}^d)) , \) \( W \in L^1(\Omega \times Y; W^2L^B(Z; \mathbb{R}^d)) \) satisfying (i), (ii), there exists a bounded sequence \((u_n) \subset W^2L^B(\Omega; \mathbb{R}^d)\) for which (iii) and (iv) hold.

The other main result deals with the \( \Gamma \)–convergence of the family \((F_\varepsilon)_\varepsilon\) in (2), thus extending, from one hand, Theorem 1.1. in [17] and, from the other, generalizing to the Orlicz-Sobolev setting [14, Theorem 1.8]:

**Theorem 1.2.** Let \( \Omega \) be a bounded open set of \( \mathbb{R}^N \), with Lipschitz boundary. Let \( f : \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow [0, +\infty) \) satisfying assumption \((H_1)-(H_4)\) then \( \Gamma (L^B(\Omega)) \) –\( \lim_{\varepsilon \to 0} \int_{\Omega} f(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, D^s u) \, dx = \int_{\Omega} f^s_{\text{hom}}(D^s u) \, dx \) for every \( u \in W^sL^B(\Omega; \mathbb{R}^d) \), where \( s = 1, 2 \) and

\[
\overline{f}^s_{\text{hom}}(\xi) := \inf \left\{ \int_{\mathcal{Y}} f^s_{\text{hom}}(y, \xi + D^s \varphi(y)) \, dy : \varphi \in \mathcal{W}^sL^B_{\text{per}}(Y; \mathbb{R}^d) \right\}
\]

and

\[
f^s_{\text{hom}}(y, \xi) := \inf \left\{ \int_{\mathcal{Z}} f(y, z, \xi + D^s \psi(z)) \, dz : \psi \in \mathcal{W}^sL^B_{\text{per}}(Z; \mathbb{R}^d) \right\}
\]

for \( s = 1, 2 \).

When \( s = 1 \) we will denote \( f^s_{\text{hom}} \) simply by \( f_{\text{hom}} \).

We emphasize that the above results could be recast in the framework of Periodic Unfolding, introduced in [10], (see also [11] for a systematic treatment) or applied to the non convex case, in the spirit of [9], and these are indeed the subjects of our future investigation.

In the next section we establish notation and recall some preliminary results, mainly adopting the symbols already used in [16], and [18], while Section 3 is devoted to establish Theorems 1.1 and 1.2.

2. **Preliminaries.** In the sequel, in order to enlighten the space variable under consideration we will adopt the notation \( \mathbb{R}^N_+, \mathbb{R}^N_- \), or \( \mathbb{R}^N_\varepsilon \) to indicate where \( x, y \) or \( z \) belong to. On the other hand, when it will be clear from the context, we will simply write \( \mathbb{R}^N \).

The family of open subsets in \( \mathbb{R}^N \) will be denoted by \( \mathcal{A}(\mathbb{R}^N) \), while the family of Borel sets is denoted by \( \mathcal{B}(\mathbb{R}^N) \).

For any subset \( D \) of \( \mathbb{R}^m \), \( m \in \mathbb{N} \), by \( \overline{D} \), we denote its closure in the relative topology. Given an open set \( A \) by \( \mathcal{C}_0(A) \) we denote the space of real valued continuous and bounded functions defined in \( A \).

For every \( x \in \mathbb{R}^N \) we denote by \( [x] \) its integer part, namely the vector in \( \mathbb{Z}^N \), which has as components the integer parts of the components of \( x \).

By \( \mathcal{L}^N \) we denote the Lebesgue measure in \( \mathbb{R}^N \).

Now we recall results of Orlicz-Sobolev spaces that will be used in the remainder of the paper.

2.1. **Orlicz-Sobolev spaces.** Let \( B : [0, +\infty) \rightarrow [0, +\infty] \) be an \( N \)–function [1], i.e., \( B \) is continuous, convex, with \( B(t) > 0 \) for \( t > 0 \), \( \frac{B(t)}{t} \rightarrow 0 \) as \( t \rightarrow 0 \), and \( \frac{B(t)}{t} \rightarrow \infty \) as \( t \rightarrow \infty \). Equivalently, \( B \) is of the form \( B(t) = \int_0^t b(\tau) \, d\tau \), where \( b : [0, +\infty) \rightarrow [0, +\infty] \) is non decreasing, right continuous, with \( b(0) = 0 \), \( b(t) > 0 \) if \( t > 0 \) and \( b(t) \rightarrow +\infty \) if \( t \rightarrow +\infty \).
We denote by $\tilde{B}$, the complementary $N$–function of $B$ defined by
$$\tilde{B}(t) = \sup_{s \geq 0} \{ st - B(s) \, , \, t \geq 0 \}.$$  

It follows that
$$\frac{tb(t)}{B(t)} \geq 1 \quad \text{(or > if $b$ is strictly increasing)} ,$$  

and
$$\tilde{B}(b(t)) \leq tb(t) \leq B(2t) \quad \text{for all } t > 0.$$  

An $N$–function $B$ is of class $\Delta_2$ (denoted $B \in \Delta_2$) if there are $\alpha > 0$ and $t_0 \geq 0$ such that $B(2t) \leq \alpha B(t)$ for all $t \geq t_0$.

In all what follows $B$ and $\tilde{B}$ are conjugates $N$–functions both satisfying the $\Delta_2$ condition and $c$ refers to a generic constant that may vary from line to line. Let $\Omega$ be a bounded open set in $\mathbb{R}^N$. The Orlicz-space
$$L^B(\Omega) = \left\{ u : \Omega \to \mathbb{C} \text{ measurable, } \lim_{\delta \to 0^+} \int_{\Omega} B(\delta |u(x)|) \, dx = 0 \right\}$$
is a Banach space for the Luxemburg norm:
$$\|u\|_{B,\Omega} = \inf \left\{ k > 0 : \int_{\Omega} B \left( \frac{|u(x)|}{k} \right) \, dx \leq 1 \right\} < +\infty.$$  

It follows that: $\mathcal{D}(\Omega)$ is dense in $L^B(\Omega)$, $L^B(\Omega)$ is separable and reflexive, the dual of $L^B(\Omega)$ is identified with $L^{\tilde{B}}(\Omega)$, and the norm on $L^{\tilde{B}}(\Omega)$ is equivalent to $\|\cdot\|_{B,\Omega}$. Furthermore, it is also convenient to recall that:

(i) $|\int_{\Omega} u(x) v(x) \, dx| \leq 2 \|u\|_{B,\Omega} \|v\|_{\tilde{B},\Omega}$ for $u \in L^B(\Omega)$ and $v \in L^{\tilde{B}}(\Omega)$,

(ii) given $v \in L^{\tilde{B}}(\Omega)$ the linear functional $L_v$ on $L^B(\Omega)$ defined by $L_v(u) = \int_{\Omega} u(x) v(x) \, dx, (u \in L^B(\Omega))$ belongs to the dual $[L^B(\Omega)]' = L^{\tilde{B}}(\Omega)$ with $\|v\|_{\tilde{B},\Omega} \leq \|L_v\|_{L^0(\Omega)'} \leq 2 \|v\|_{L^0,\Omega},$

(iii) the property $\lim_{t \to +\infty} \frac{B(t)}{t} = +\infty$ implies $L^B(\Omega) \subset L^1(\Omega) \subset L^1_{loc}(\Omega) \subset D'(\Omega), \text{ each embedding being continuous}.$

For the sake of notations, given any $d \in \mathbb{N}$, when $u : \Omega \to \mathbb{R}^d$, such that each component $(u^i)$ of $u$, lies in $L^B(\Omega)$ we will denote the norm of $u$ with the symbol $\|u\|_{L^B(\Omega)^d} := \sum_{i=1}^d \|u^i\|_{L^B,\Omega}.$

Let $s = 1$ or $2$, following [2] one can introduce the Orlicz-Sobolev space $W^sL^B(\Omega; \mathbb{R}^d)$, consisting of those (equivalence classes of) functions $u \in L^B(\Omega)$ for which $D^s u \in L^B(\Omega; \mathbb{R}^d_2)$, and the derivatives are taken in the distributional sense on $\Omega$. For $s = 1, W^1L^B(\Omega)$ is a reflexive Banach space with respect to the norm $\|u\|_{W^1L^B(\Omega)} = \|u\|_{B,\Omega} + \sum_{i=1}^d \left\| \frac{\partial u}{\partial x_i} \right\|_{B,\Omega}.$ The same holds for $W^2L^B(\Omega)$, endowing it with the norm $\|u\|_{W^2L^B(\Omega)} = \|u\|_{B,\Omega} + \sum_{i=1}^d \left\| \frac{\partial u}{\partial x_i} \right\|_{B,\Omega} + \sum_{i,j=1}^d \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|_{B,\Omega}.$ It is immediately seen the extension to vector fields $W^sL^B(\Omega; \mathbb{R}^d)$.

We denote by $W^s_0L^B(\Omega), \text{ the closure of } \mathcal{D}(\Omega)$ in $W^sL^B(\Omega)$ and the semi-norm $u \to \|u\|_{W^s_0L^B(\Omega)} = \|D^s u\|_{B,\Omega}$ is a norm on $W^s_0L^B(\Omega)$ equivalent to $\|\cdot\|_{W^sL^B(\Omega)}.$

Arguing in components, the same definitions hold for $W^sL^B(\Omega; \mathbb{R}^d)$ and $W^s_0L^B(\Omega; \mathbb{R}^d).$ By $W^s_0L^B(\Omega)$, we denote the space of functions $u \in W^sL^B(\Omega)$ such that $\int_{\Omega} u(y) \, dy = 0.$ Given a function space $S$ defined in $Y, Z \text{ or } Y \times Z,$ the subscript $S_{\text{per}}$ means that the functions are periodic in $Y, Z$ or $Y \times Z$, as it will be clear from the context. In particular by $C_{\text{per}}(Y), C_{\text{per}}(Z) \text{ (or } C_{\text{per}}(Y \times Z)$
respectively), we denote the space of continuous functions in $\mathbb{R}^d$, which are $Y$ or $Z$-periodic (continuous function in $\mathbb{R}^d \times \mathbb{R}^d$, which are $Y \times Z$-periodic, respectively).

2.2. Multiscale convergence in Orlicz spaces.

2.2.1. Reiterated two scale convergence in first order Sobolev-Orlicz spaces. In the sequel we present a generalization of definitions in [16, 24, 27] obtained in [18]. To this end, we recall that, within this section, $\Omega$ is a bounded open set with Lipschitz boundary, and we denote by $L_{\text{per}}^B(\Omega \times Y \times Z)$ the space of functions in $v \in L_{\text{loc}}^B(\Omega \times \mathbb{R}^N \times \mathbb{R}^N)$ which are periodic in $Y \times Z$ and such that $\int\int\int_{\Omega \times Y \times Z} B(|v|)dx dy dz < +\infty$. For any given $\varepsilon > 0$ and any function $v \in L^B(\Omega; C_{\text{per}}(Y \times Z))$, we define

$$v^\varepsilon(x) := v\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon}\right).$$

(5)

This function is well defined as proven in [18, Subsection 2.2] and we recall [18, Definition 2.1].

Definition 2.1. A sequence of functions $(u_\varepsilon)_\varepsilon \subseteq L^B(\Omega)$ is said to be:

- weakly reiterated two-scale convergent in $L^B(\Omega)$ to $u_0 \in L_{\text{per}}^B(\Omega \times Y \times Z)$ if

$$\int_\Omega u_\varepsilon f^\varepsilon dx \rightarrow \iint_{\Omega \times Y \times Z} u_0 f dx dy dz, \text{ for all } f \in L^B(\Omega; C_{\text{per}}(Y \times Z)), \quad (6)$$

as $\varepsilon \to 0$,

- strongly reiterated two-scale convergent in $L^B(\Omega)$ to $u_0 \in L_{\text{per}}^B(\Omega \times Y \times Z)$ if for $\eta > 0$ and $f \in L^B(\Omega; C_{\text{per}}(Y \times Z))$ verifying $\|u_0 - f\|_{B, \Omega \times Y \times Z} \leq \eta$ there exists $\rho > 0$ such that $\|u_\varepsilon - f^\varepsilon\|_{B, \Omega} \leq \eta$ for all $0 < \varepsilon \leq \rho$.

When (6) happens we denote it by “$u_\varepsilon \rightharpoonup u_0$ in $L^B(\Omega)$ − weakly reiterated two-scale” and we will say that $u_0$ is the weak reiterated two-scale limit in $L^B(\Omega)$ of the sequence $(u_\varepsilon)_\varepsilon$.

The above definition extends in a canonical way, arguing in components, to vector valued functions.

Moreover for the sake of exposition, the reiterated weak convergence of $u_\varepsilon$ towards $u_0$ in $L^B$ will be also denoted by the symbol

$$u_\varepsilon \rightharpoonup^{\text{reit}}_{\varepsilon \to 0} u_0,$$

both in the scalar and in the vector valued setting.

The proof of the following lemma can be found in [18, Proof of Lemma 2.3].

Lemma 2.2. If $u \in L^B(\Omega; C_{\text{per}}(Y \times Z))$ and $\varepsilon > 0$, then, considered $u^\varepsilon$ as in (5), it results that $u^\varepsilon \rightharpoonup^{\text{reit}}_{\varepsilon \to 0} u$ in $L^B(\Omega)$, and we have $\lim_{\varepsilon \to 0} \|u^\varepsilon\|_{B, \Omega} = \|u\|_{B, \Omega \times Y \times Z}$.

The subsequent results, useful in the remainder of the paper, explicitly for the construction of sequences which ensure the energy convergence in Theorem 1.2, have been proven in [18, Section 2.3].

Proposition 1. Given a bounded sequence $(u_\varepsilon)_\varepsilon \subset L^B(\Omega)$, one can extract a not relabelled subsequence such that $(u_\varepsilon)_\varepsilon$ is reiterated weakly two-scale convergent in $L^B(\Omega)$.

Proposition 2. If a sequence $(u_\varepsilon)_\varepsilon$ is weakly reiterated two-scale convergent in $L^B(\Omega)$ to $u_0 \in L_{\text{per}}^B(\Omega \times Y \times Z)$ then:
holds for the a.e. $x$.

Theorem 4.5]) to observe that the fields $(\Omega)$ weakly two-scale in $W^{1,p}((\Omega)\times Z)$.

Consequently it results that

\begin{equation}
\text{Corollary 1. Let } v \in C(\Omega;B_{\per}(Y \times Z)). \text{ Then } v^\varepsilon \rightarrow u^\varepsilon \text{ weakly two-scale in } L^B(\Omega) \text{ as } \varepsilon \rightarrow 0.
\end{equation}

Remark 1. Consequently it results that

(1) If $v \in L^B(\Omega;C_{\per}(Y \times Z))$, then $v^\varepsilon \rightarrow v$ reiteratively strongly two-scale in $L^B(\Omega)$, as $\varepsilon \rightarrow 0$.

(ii) If $(u^\varepsilon)_\varepsilon \in L^B(\Omega)$ is strongly reiteratively two-scale convergent in $L^B(\Omega)$ to $u_0 \in L^B_{\per}(\Omega \times Y \times Z)$, then

\begin{itemize}
  \item[(a)] $u^\varepsilon \rightarrow u_0$ in $L^B(\Omega)$ as $\varepsilon \rightarrow 0$;
  \item[(b)] $\|u^\varepsilon\|_{B,\Omega} \rightarrow \|u_0\|_{B,\Omega \times Y \times Z}$ as $\varepsilon \rightarrow 0$.
\end{itemize}

The following is a sequential compactness result on $W^1L^B(\Omega)$, (see [16] and [18] for a proof and related results) that will be used in the sequel.

Proposition 4. Let $(u^\varepsilon)_\varepsilon$ bounded in $W^1L^B(\Omega)$. There exists a not relabelled subsequence, $u_0 \in W^1L^B(\Omega)$, $(u_1, u_2) \in L^1(\Omega;W^1_\per L^B(Y)) \times L^1(\Omega;L^1_{\per}(Y;W^1_\per L^B(Z)))$ such that:

\begin{itemize}
  \item[(i)] $u^\varepsilon \rightarrow u_0$ in $L^B(\Omega)$,
  \item[(ii)] $D_xu^\varepsilon \rightarrow D_xu_0 + D_yu_1 + D_zu_2$ in $L^B(\Omega)$, $1 \leq i \leq N$, as $\varepsilon \rightarrow 0$.
\end{itemize}

If (i) and (ii) in the above Proposition hold, we will say that $u^\varepsilon \rightarrow u_0$ reiteratively weakly two-scale in $W^1L^B(\Omega)$, omitting to explicitly mention the functions $u_1, u_2$ above.

Remark 2. We observe that the fields $(u_1, u_2)$ in Proposition 4 are more regular than stated above and in [17]. Indeed by (ii), $D_xu_0 + D_yu_1 + D_zu_2 \in L^B(\Omega \times Y \times Z;\mathbb{R}^{d \times N})$.

Thus, applying Poincaré-Wirtinger inequality, (see for instance [8, Theorem 4.5]) to $u_1$ with respect to $Y$ and for a.e. $x \in \Omega$ and to $u_2$ with respect to $z$ for a.e. $x \in \Omega$ and for any $y \in Y, \text{ and then taking the integral over } \Omega \text{ for } u_1 \text{ and over } \Omega \times Y \text{ for } u_2$, it is easily seen that the $L^B$ norm in $\Omega \times Y$ of $u_1$ is finite, and the same holds for the $L^B$ norm in $\Omega \times Y \times Z$ for $u_2$, i.e. one can say that $u_1 \in L^B(\Omega \times Y)$
and \( u_2 \in L^B(\Omega \times Y \times Z) \), namely we can say that \( u_1 \in L^B(\Omega; W^{1,1}_\# L^B(Y)) \) and \( u_2 \in L^B(\Omega; L^B_{\perp}(Y); W^{1,1}_\# L^B(Z)) \).

Moreover, it is worth to observe that the same convergence holds for vector valued functions.

**Corollary 2.** If \( (u_\varepsilon) \) is such that \( u_\varepsilon \xrightarrow{\text{reit-weak}} u_0 \) reiteratively weakly two-scale in \( W^1L^B(\Omega) \), we have:

(i) \( u_\varepsilon \to \int_Z u_0 (\cdot, \cdot, z) \, dz \) in \( W^1L^B(\Omega) \) weakly two-scale;

(ii) \( u_\varepsilon \to u_0 \) in \( W^1L^B(\Omega) \) weakly, where \( \widetilde{u}_0(x) := \int_{Y \times Z} u_0(\cdot, \cdot, \cdot) \, dydz \).

Under our sets of assumptions on \( \Omega \) and \( B \), the canonical injection \( W^1L^B(\Omega) \hookrightarrow L^B(\Omega) \) is compact, an so the reiterated weakly two-scale limit \( u_0 \in W^1L^B(\Omega) \).

### 2.2.2. \( \Gamma \) convergence and preliminary results on integral functionals.

In the sequel we recall the definition of \( \Gamma \)-convergence in metric spaces. We refer to [12] for a complete treatment of the subject.

**Definition 2.3.** Let \( (X,d) \) be a metric space and let \( (F_\varepsilon)_{\varepsilon > 0} \) be a family of functionals defined on \( (X,d) \). We say that a functional \( F : X \times A(\Omega) \to [0, +\infty] \) is the \( \Gamma \) limit of \( (F_\varepsilon)_{\varepsilon > 0} \) (resp. the \( \Gamma \) limit) if for every sequence \( (\varepsilon_n)_n \) converging to 0+

\[
F(u, A) = \left\{ \liminf_{n \to +\infty} \left( \text{resp. lim sup}_{n \to +\infty} \right) F_\varepsilon(u_n, A) : u_n \to u \text{ in } (X,d) \right\}
\]

and write \( F = \Gamma(d) \) (resp. \( F = \Gamma(d) \)). \( F \) is the \( \Gamma \) limit of the family \( (F_\varepsilon)_{\varepsilon > 0} \) and we write \( F \equiv \Gamma(d) \) (resp. \( F \equiv \Gamma(d) \)).

Next we recall for the readers’ convenience Ioffe’s lower semicontinuity theorem (see [20, Theorem 1]).

**Proposition 5.** Let \( \Omega \) be an open, bounded subset of \( \mathbb{R}^N \), and let \( f : \Omega \times \mathbb{R}^d \times \mathbb{R}^m \to [0, +\infty) \), \( f = f(x, u, v) \) be a function Lebesgue measurable with respect to \( x \) and \( \text{Borel measurable} \) with respect to \( (u, v) \). Suppose further that \( f(x, \cdot, \cdot) \) is lower semicontinuous for a.e \( x \in \Omega \), \( f(x, u, \cdot) \) is convex for a.e \( x \in \Omega \), \( u \in \mathbb{R}^d \) and there exist \( (u_0, v_0) \in L^B(\Omega; \mathbb{R}^d) \times L^B(\Omega; \mathbb{R}^m) \) with \( B, B_1, N \)-functions satisfying \( \Delta_2 \) condition, together with their conjugates, such that: \( \int_\Omega f(x, u_0(x), v_0(x)) \, dx < +\infty \). If \( u_n \to u \) in \( L^B(\Omega; \mathbb{R}^d) \) and \( v_n \to v \) in \( L^B(\Omega; \mathbb{R}^m) \) then \( \int_\Omega f(x, u(x), v(x)) \, dx \leq \liminf_{n \to +\infty} \int_\Omega f(x, u_n(x), v_n(x)) \, dx \).

The following results will be used in the sequel. We omit their proofs since they are entirely similar to their counterparts in the classical Sobolev setting (see [14, Appendix] or [6, Lemma 3.3] for similar arguments).

**Proposition 6.** Let \( (F_\varepsilon)_{\varepsilon} : W^{1,1}(\Omega; \mathbb{R}^m) \times A(\Omega) \to [0, +\infty) \) be a sequence of functionals satisfying:

(i) \( F_\varepsilon(u, \cdot) \) is the restriction to \( A(\Omega) \) of a Radon measure;

(ii) \( F_\varepsilon(u, D) = F_\varepsilon(v, D) \) whenever \( u = v \) a.e in \( D \in A(\Omega) \);

(iii) there exists a positive constant \( c \) such that

\[
\frac{1}{C} \int_D B(|Du|) \, dx \leq F_\varepsilon(u, D) \leq C \int_D (1 + B(|Du|)) \, dx \text{ for every } \varepsilon > 0.
\]
For every sequence \((\varepsilon_n)\) converging to \(0^+\) there exists a subsequence \((\varepsilon_j)\) such that the functional \(F_{(\varepsilon_j)}(u, D)\) is the \(\Gamma(L^B(D))\) limit of \((F_{\varepsilon}(u, D))\) for every \(D \in \mathcal{A}(\Omega)\) and \(u \in W^{1,B}(D; \mathbb{R}^d)\), where for any sequence \((\delta_n)\) converging to \(0^+\)

\[
F_{(\delta_n)}(u, D) := \inf \left\{ \liminf_{n \to +\infty} F_{\delta_n}(u_n, D) : u_n \to u \text{ in } L^B(D) \right\}.
\]

**Proposition 7.** Let \((F_{\varepsilon})_\varepsilon: W^{1,B}(\Omega; \mathbb{R}^m) \times \mathcal{A}(\Omega) \to [0, +\infty)\) be a sequence of functionals satisfying (i) – (iii) of Proposition 6. Then for any \(u \in W^{1,L}(\Omega; \mathbb{R}^d)\) and \(A \in \mathcal{A}(\Omega)\), it results that

\[
F_{(\varepsilon)}(u, A) := \inf \left\{ \liminf_{\varepsilon \to 0} F_{\varepsilon}(u_{\varepsilon}, A) : u_{\varepsilon} \to u \text{ in } L^B(A) \right\} = F_{(\varepsilon)}(u, A) := \inf \left\{ \liminf_{\varepsilon \to 0} F_{\varepsilon}(u_{\varepsilon}, A) : u_{\varepsilon} \to u \text{ in } L^B(A), u_n \equiv u \text{ on a neighborhood of } \partial A \right\}.
\]

Now we recall the following Aumann’s measurability selection principle.

**Proposition 8.** Let \((X, \mathcal{M})\) be a measurable space with \(\mu\) a positive, finite and complete measure, and let \(S\) be complete, separable metric space. Let \(H: X \to \{C \subset S; C \neq \emptyset, C \text{ is closed}\}\) be a multifunction such that \(\{(x, y) \in X \times S; y \in H(x)\} \in \mathcal{M} \times \mathcal{B}(S)\). Then there exists a sequence of measurable functions \(h_n : X \to S\) such that \(H(x) = \{h_n(x) : n \in \mathbb{N}\}\) for \(\mu\)-a.e. \(x \in X\).

3. **Proof of main results.** This section is devoted to the proof of our main results and it extends to the Orlicz-Sobolev setting, the arguments in [14]. Thus we do not give all the details, but we present just the proofs which involve different techniques and estimates. We start recalling that, within this section \(\Omega \subset \mathbb{R}^N\) is a bounded open set with Lipschitz boundary. The proof of our first result is a consequence of the analogous theorem in [14] and the assumption \((H)\), i.e. that the \(N\)-function \(B\) satisfies the assumption that there exists \(p, q > 1\) such that

\[(H) \quad L^q(\Omega) \hookrightarrow L^B(\Omega) \hookrightarrow L^p(\Omega).\]

On the other hand, this assumption is satisfied by any \(N\)-function \(B\) such that \(\Delta_2\) condition holds both for \(B\) and for its conjugate. Indeed it suffices to apply [13, Proposition 2.4] (see also [7, Proposition 3.5]) and standard rearrangements’ arguments, which allow to consider any dimension \(N\).

**Proof of Theorem 1.1.** We start recalling the following property which will be used in the sequel since \(\Omega\) has Lipschitz boundary, it is well known that if \(v \in L^1(\Omega)\) and its distributional gradient \(\nabla v \in L^B(\Omega)\), then \(v \in W^{1,L}(\Omega)\). Clearly the same properties are shared by vector valued fields.

Let assume that \((u_\varepsilon)\) is bounded in \(W^{2,L}(\Omega; \mathbb{R}^d)\), with \(B\) satisfying \(\Delta_2\) and \((H)\). Hence \(\|u_\varepsilon\|_{W^{2,L}} \leq c \|u_\varepsilon\|_{W^{2,L^B}}\) and \((u_\varepsilon)\) bounded in \(W^{2,L}B\) implies \((u_\varepsilon)\) bounded in \(W^{2,L}\) where the counterparts of (i)-(iv) in the Sobolev setting are known, see [14]. Moreover since \((u_\varepsilon)\) is bounded in \(W^{2,L}(\Omega; \mathbb{R}^d)\) we have that for every \(j\), \(\left(\frac{\partial u}{\partial x_j}\right)_\varepsilon\) is bounded in \(W^{1,L}(\Omega)\). Thus, \(u_\varepsilon \rightharpoonup u_0\) in \(W^{2,L}(\Omega; \mathbb{R}^d)\) weakly, and, by Proposition 4, and Remark 2

\[
\frac{\partial u_\varepsilon}{\partial x_j} \rightharpoonup -2s \frac{\partial u_0}{\partial x_j} + \frac{\partial u_0^1}{\partial y_j} + \frac{\partial u_0^2}{\partial z_j},
\]
as \( \varepsilon \to 0 \), with
\[
u_0^1 \in L^B \left( \Omega; W^{1,1}_\# L^B (Y; \mathbb{R}^d) \right), \quad \nu_0^2 \in L^B \left( \Omega; L^B_{per} \left( Y; W^{1,1}_\# L^B (Z; \mathbb{R}^d) \right) \right).
\]
On the other hand, the strong convergence of \( u_\varepsilon \to u_0 \) in \( W^{1,1} (\Omega; \mathbb{R}^d) \), and the bounds on the Hessians, together with proposition 4, applied to \( (Du_\varepsilon) \), entail that \( u^1_0 \) and \( u^2_0 \) = 0. Analogously, since \( \frac{\partial u_\varepsilon}{\partial x_i} \) is bounded in \( W^{1,1} (\Omega; \mathbb{R}^d) \), \( \frac{\partial u_\varepsilon}{\partial x_i} \to p_0 \) in \( W^1 L^B (\Omega; \mathbb{R}^d) \) weakly,
\[
\frac{\partial^2 u_\varepsilon}{\partial x_i \partial x_j} \to 2 \frac{\partial p_0}{\partial x_i} + \frac{\partial^2 p_0^1}{\partial y_i \partial y_i} + \frac{\partial^2 p_0^2}{\partial z_i \partial z_i},
\]
with \( p_0^1 \in L^B \left( \Omega; W^{1,1}_\# L^B (Y; \mathbb{R}^d) \right), \quad p_0^2 \in L^B \left( \Omega; L^B_{per} \left( Y; W^{1,1}_\# L^B (Z; \mathbb{R}^d) \right) \right) \).
Consequently \( p_0 = \frac{\partial u_0}{\partial x_j} \), then \( \frac{\partial p_0}{\partial x_i} = \frac{\partial^2 u_0}{\partial y_i \partial x_j} \in L^B (\Omega; \mathbb{R}^d) \) and \( u_0 \in W^{2,1} (\Omega; \mathbb{R}^d) \).

Since \( L^B (\Omega; \mathbb{R}^d); L^p (\Omega; \mathbb{R}^d) \subset L^p (\Omega; \mathbb{R}^d) \) and \( L^p (\Omega; \mathbb{R}^d) \subset L^{2} (\Omega; \mathbb{R}^d) \) \((1/p' + 1/p = 1)\), then, by the uniqueness of distributional limits, taking the distributional derivatives and applying [14, Theorem 1.10], it results
\[
\frac{\partial p_0}{\partial x_i} + \frac{\partial^2 p_0^1}{\partial y_i \partial y_i} + \frac{\partial^2 p_0^2}{\partial z_i \partial z_i} = \frac{\partial^2 u_0}{\partial x_i \partial x_j} + \frac{\partial U}{\partial y_i \partial y_j} + \frac{\partial^2 W}{\partial z_i \partial z_j},
\]
thus
\[
\frac{\partial p_0^1}{\partial y_i} + \frac{\partial p_0^2}{\partial z_i} = \frac{\partial^2 U}{\partial y_i \partial y_j} + \frac{\partial^2 W}{\partial z_i \partial z_j}.
\]
Averaging over \( z_i \), we deduce
\[
\frac{\partial p_0^1}{\partial y_i} = \frac{\partial^2 U}{\partial y_i \partial y_j}, \tag{7}
\]
and consequently
\[
\frac{\partial p_0^2}{\partial z_i} = \frac{\partial^2 W}{\partial z_i \partial z_j}.
\]
From (7) and \( p_0^1 \in L^B \left( \Omega; W^{1,1}_\# L^B (Y; \mathbb{R}^d) \right) \), we have for a.e \( x \in \Omega, p_0^1 (x, \cdot) \in W^{1,1}_\# L^B (Y; \mathbb{R}^d) \), then \( \frac{\partial u_0}{\partial y_i} \in L^B (\Omega; \mathbb{R}^d) \), that is \( \frac{\partial}{\partial y_i} \left( \frac{\partial U}{\partial y_j} \right) (x, \cdot) = \frac{\partial p_0^1}{\partial y_i} \in L^B (Y; \mathbb{R}^d) \). Moreover, \( U (x, \cdot) \in W^{2,1} (Y; \mathbb{R}^d) \) entails that \( \frac{\partial U (x, \cdot)}{\partial y_j} \in W^{1,1} (Y; \mathbb{R}^d) \). Thus, we deduce that \( \frac{\partial U (x, \cdot)}{\partial y_j} \in W^{1,1} (Y; \mathbb{R}^d) \), hence \( U (x, \cdot) \in W^{2,1} (Y; \mathbb{R}^d) \) and we have \( U \in L^p (\Omega; W^{2,1} (Y; \mathbb{R}^d)) \subset L^1 (\Omega; W^{2,1} (Y; \mathbb{R}^d)) \). Arguing as in Remark 2, we can actually prove that \( U \in L^B (\Omega; W^{2,1} (Y; \mathbb{R}^d)) \).

On the other hand, [14, Theorem 1.7] guarantees the existence of a field \( A (x) \in L^p (\Omega; \mathbb{R}^{d \times N}) \subset L^1 (\Omega; \mathbb{R}^{d \times N}) \), such that \( U (x, y) - A (x) y \in L^p (\Omega; W^{2,1}_{per} (Y; \mathbb{R}^d)) \), hence, by the previous observations, \( U (x, y) - A (x) y \in L^1 (\Omega; W^{2,1}_{per} (Y; \mathbb{R}^d)) \). In a similar way,
\[
\frac{\partial p_0^2}{\partial z_i} = \frac{\partial^2 W}{\partial z_i \partial z_j} = \partial \left( \frac{\partial W}{\partial z_j} \right),
\]
with \( p_0^2 \in L^1 \left( \Omega; L^{1,\perp}_{\text{per}} \left( Y; W^{1, \|B\|}_{\perp} \left( Z; \mathbb{R}^d \right) \right) \right) \). For a.e. \( x, y \in \Omega \times Y \), \( W(x, y, \cdot) \in W^{2, p} \left( Z; \mathbb{R}^d \right) \), thus \( \frac{\partial W}{\partial z} \in L^1 \left( Z; \mathbb{R}^d \right) \). Then \( \frac{\partial W}{\partial z} \in L^{1,\perp}_{\text{per}} \left( Z; \mathbb{R}^d \right) \) implies \( \frac{\partial W}{\partial z} \in W^{1,\perp}_{\text{per}} \left( Z; \mathbb{R}^d \right) \). Moreover, since

\[
W(x, y, \cdot) \in W^{2, p} \left( Z; \mathbb{R}^d \right) \subset L^p \left( Z; \mathbb{R}^d \right) \subset L^1 \left( Z; \mathbb{R}^d \right)
\]

we deduce \( W(x, y, \cdot) \in W^{1, \|B\|}_{\perp} \left( Z; \mathbb{R}^d \right) \). Then, the existence of a field \( C \) such that

\[
W(x, y, z) - C(x, y)z \in L^1(\Omega \times Y; W^{1, \|B\|}_{\perp} \left( Z; \mathbb{R}^d \right)),
\]

for some \( C \in L^1 \left( \Omega \times Y; \mathbb{R}^{d \times N} \right) \), can be deduced arguing as above relying on the analogous property proven in [14, Theorem 1.10].

### 3.1. Proof of theorem 1.2
The result is achieved adopting the same strategy as in [14], i.e. by means of the following lemmas. The first one deals with the continuity of \( f_{\text{hom}} \).

**Lemma 3.1.** Under the hypotheses \((A_1), (A_2), (H_3)\) and \((H_4)\), the function \( f_{\text{hom}} : \Omega \times \mathbb{R}^{d \times N} \rightarrow [0, +\infty[ \), defined by (4) is continuous.

**Proof.** The proof develops along the lines of [14, Lemma 4.1] and we present the main differences for the readers’ convenience. For the sake of exposition, we recall that \( f_{\text{hom}}(x, \xi) := \inf \left\{ \int_Q f(x, y, \xi + D\psi(y)) \, dy : \psi \in W^{1, \|B\|}_{\perp} \left( Q; \mathbb{R}^d \right) \right\} \).

Fix \( (x, \xi) \in \Omega \times \mathbb{R}^{d \times N} \) and consider a sequence \((x_n, \xi_n)\) converging to \((x, \xi)\). Let \( \varepsilon > 0 \) and choose \( \psi \in W^{1, \|B\|}_{\perp} \left( Q; \mathbb{R}^d \right) \) such that

\[
f_{\text{hom}}(x, \xi) + \varepsilon > \int_Q f(x, y, \xi + D\psi(y)) \, dy.
\]

On the other hand, definition (4) entails that

\[
f_{\text{hom}}(x_n, \xi_n) \leq \int_Q f(x_n, y, \xi_n + D\psi(y)) \, dy.
\]

\((H_4)\) and dominated convergence theorem guarantee that

\[
\limsup_{n \to +\infty} f_{\text{hom}}(x_n, \xi_n) \leq \limsup_{n \to +\infty} \int_Q f(x_n, y, \xi_n + D\psi(y)) \, dy
= \lim_{n \to +\infty} \int_Q f(x_n, y, \xi_n + D\psi(y)) \, dy = \int_Q f(x, y, \xi + D\psi(y)) \, dy
\leq f_{\text{hom}}(x, \xi) + \varepsilon.
\]

Conversely, let \( \psi_n \in W^{1, \|B\|}_{\perp} \left( Q; \mathbb{R}^d \right) \) be such that

\[
f_{\text{hom}}(x_n, \xi_n) + \varepsilon > \int_Q f(x_n, y, \xi_n + D\psi_n(y)) \, dy \geq \frac{1}{C} \int_Q B(|\xi_n + D\psi_n(y)|) \, dy.
\]
Let $\psi_1 \in W^1_{\text{per}}L^B(Q;\mathbb{R}^d)$ be chosen arbitrarily. Since
\[
  f_{\text{hom}}(x_n,\xi_n) \leq \int_Q f(x_n, y, \xi_n + D\psi_1(y)) \, dy
  \leq C \int_Q (1 + B(|\xi_n + D\psi_1(y)|)) \, dy 
  \leq C + C \int_Q B(|\xi_n| + |D\psi_1(y)|) \, dy 
  \leq C + \frac{1}{2} C \int_Q B(2|\xi_n|) \, dy
\]
Thus by (H₄), we deduce $\frac{1}{C_2} \int_Q B(|\xi_n + D\psi_1(y)|) \, dy < 1$.

Recalling that $B$ is convex and $B(0) = 0$, it is easily seen that
\[
  \int_Q B \left( \frac{|\xi_n + D\psi_1(y)|}{1 + C_1 C_2} \right) \, dy < 1.
\]
Thus, exploiting the definition of $L^B$ norm, the triangle inequality and the convergence of $\xi_n$ to $\xi$, we have that
\[
  \|D\psi_n(y)\|_{B,Q} \leq \|\xi_n + D\psi_n(y)\|_{B,Q} + \|\xi_n\|_{B,Q} \leq C,
\]
From Poincaré-Wirtinger’s inequality, the fact that $B$ satisfies $\triangle_2$ condition, hence is reflexive, it results that, up to a not relabelled subsequence, $\psi_n - \int_Q \psi_n \, dy \rightharpoonup \psi$
in $W^1_{\text{per}}L^B(Q;\mathbb{R}^d)$; thus $D\psi_n \rightharpoonup D\psi$ in $L^B(Q;\mathbb{R}^d)$. In view of (A₁), (A₂) and by theorem 5, we get that:
\[
  f_{\text{hom}}(x,\xi) \leq \int_Q f(x, y, \xi + D\psi(y)) \, dy \leq \liminf_{n \to +\infty} \int_Q f(x_n, y, \xi_n + D\psi_n(y)) \, dy
  \leq \liminf_{n \to +\infty} f_{\text{hom}}(x_n,\xi_n) + \varepsilon \leq \limsup_{n \to +\infty} f_{\text{hom}}(x_n,\xi_n) + \varepsilon \leq f_{\text{hom}}(x,\xi) + 2\varepsilon,
\]
and this concludes the proof. \hfill \Box

Clearly, under the same assumptions, the above result holds for $J_{\text{hom}}, J^2_{\text{hom}}$ and $\overline{f^2_{\text{hom}}}$.

We are in position to introduce a localized version of our $\Gamma$-limit, i.e. we set for any sequence $(\varepsilon_n)_n$ of positive real numbers converging to zero,
\[
  F_{\varepsilon}(u, D) := \inf \left\{ \liminf_{n \to +\infty} F_{\varepsilon_n}(u_n, D) : u_n \to u \text{ in } L^B(\Omega;\mathbb{R}^d) \right\}, \quad (8)
\]
where, with an abuse of notation (cf. (2)) we define for every $u \in L^B(\Omega;\mathbb{R}^d)$, and $D \in \mathcal{A}(\Omega)$,
\[
  F_{\varepsilon}(u, D) := \left\{ \int_D f \left( \frac{x}{\varepsilon}, \frac{D}{\varepsilon^2}, Du \right) \, dx, \quad \text{for every } u \in W^1L^B(\Omega;\mathbb{R}^d), \right.
   \left. \text{otherwise.} \right.
\]
Observe that the coercivity condition \((H_4)\) on \(f\) guarantees that (8) is equivalent at the computing the analogous limit functional with respect to the weak* convergence in \(W^1L^B\).

Moreover, as in [18], we introduce for every \(u \in W^1L^B(\Omega; \mathbb{R}^d)\),

\[
F(u, D) := \begin{cases}
\inf \left\{ \int_{\Omega} \int_Y \int_Z f(y, z, Du(x) + D_yU(x, y) + D_zW(x, y, z)) \, dxdydz : \\
U \in L^1(D; W^1L^B_{\text{per}}(Y; \mathbb{R}^d)), \, W \in L^1(\Omega \times Y; W^1L^B_{\text{per}} (Z; \mathbb{R}^d)) \right\}.
\end{cases}
\]

Under the same assumptions of Proposition 6, the following result can be proven.

**Lemma 3.2.** For each \(A \in \mathcal{A}(\Omega)\), let \((\varepsilon_j)\) be the sequence given by Proposition 6, then there exists a further subsequence \((\varepsilon_{j_k}) \equiv (\varepsilon_k)\) such that \(F_{(\varepsilon_k)}(u, \cdot)\) is the restriction to \(A(\Omega)\) of a finite Radon measure.

**Proof.** The proof relies on verifying the assumptions of Fonseca-Maly’s lemma (see for instance its formulation in [6, Lemma 3.4]), and thus it will be divided in several steps.

i) First we prove nested subadditivity, namely \(F_{(\varepsilon_k)}(u, A) \leq F_{(\varepsilon_k)}(u, B) + F_{(\varepsilon_k)}(u, A \setminus C)\) for all \(A, B, C \in \mathcal{A}(\Omega)\) such that \(C \subset B \subset A\), for every \(u \in W^1L^B(\Omega; \mathbb{R}^d)\). Fix \(\eta > 0\) and, for the given sets, find \((u_j)_j \subset L^B(\Omega; \mathbb{R}^d)\) such that \(u_j \to u\) in \(L^B(A \setminus C; \mathbb{R}^d)\) and

\[
\liminf_{j \to \infty} \int_{A \setminus C} f \left( \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j}, Du_j(x) \right) \, dx < F_{(\varepsilon_k)}(u, A \setminus C) + \eta \leq F_{(\varepsilon_k)}(u, A) + \eta.
\]

Moreover, up to a subsequence, we may assume that

\[
\lim_{k \to \infty} \int_{A \setminus C} f \left( \frac{x}{\varepsilon_{jk}}, \frac{x}{\varepsilon_{jk}}, Du_{jk}(x) \right) \, dx = \liminf_{k \to \infty} \int_{A \setminus C} f \left( \frac{x}{\varepsilon_{jk}}, \frac{x}{\varepsilon_{jk}}, Du_{jk}(x) \right) \, dx.
\]

Let \(\mathcal{R} := \{ \bigcup_{j=1}^k C_i, k \in \mathbb{N}, C_i \in \mathcal{C} \}\), where \(\mathcal{C}\) is the set of open cubes with faces parallel to the axes, centered at \(x \in \Omega \cap \mathbb{Q}^N\), with rational edge length. Let \(B_0 \in \mathcal{R}\) be such that \(C \subset B_0 \subset B\), in particular \(\mathcal{L}^N(\partial B_0) = 0\). Then, by Proposition 6, \(F_{(\varepsilon_k)}(u, B_0)\) is a \(\Gamma\)–limit, and thus there exists a sequence \((u'_j) \subset W^1L^B(\Omega; \mathbb{R}^d)\) such that \(u'_j \to u\) in \(L^B(B_0; \mathbb{R}^d)\) and

\[
\lim_{j \to \infty} \int_{B_0} f \left( \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j}, Du'_j(x) \right) \, dx = F_{(\varepsilon_k)}(u, B_0).
\]

For every \(\pi \in W^1L^B(\Omega; \mathbb{R}^d)\) consider the functional

\[
G(\pi, A) := \int_A (1 + B(|Du(x)|)) \, dx,
\]

and set \(\nu_{jk} := G(u_{jk}, \cdot) + G(u'_j, \cdot)\).

Note that \(\nu_{jk}(A) = \int_A 2 \, dx + \int_A B \left( |Du_{jk}(x)| \right) \, dx + \int_A B \left( |Du'_j(x)| \right) \, dx\).

By the growth and coercivity condition \((H_4)\),

\[
\int_{A'} B \left( |Du'_{jk}(x)| \right) \, dx < \int_{B_0} f \left( \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j}, Du'_j(x) \right) \, dx < \infty \text{ for every } A' \subset B_0.
\]

Analogously \(\liminf_{k \to \infty} \int_{A'} B \left( |Du_{jk}(x)| \right) \, dx \leq \liminf_{j \to \infty} \int_{A \setminus C} f \left( \frac{x}{\varepsilon_j}, \frac{x}{\varepsilon_j}, Du_j(x) \right) \, dx\).
<\mathcal{F}(\varepsilon_j) (u, A \setminus \overline{C}) + \eta < \infty \text{ for every } A' \subset A \setminus \overline{C}. \text{ Hence up to a not relabeled subsequence, } \nu_{j_k}, \text{ restricted to } B_0 \setminus \overline{C}, \text{ converges weakly* in the sense of measures to } \nu. \n
For every } \nu, \text{ let } B_t = \{x \in B_0 : dist(x, \partial B_0) > t\} \text{. For } 0 \leq \eta < \eta' < \eta \text{ such that } \nu(\partial B_{\eta'}) = 0, \text{ define } L_{\delta} := B_{\eta' - 2\delta} \setminus B_{\eta' + \delta} \text{ and take a smooth cut-off function } \varphi_{\delta} \in C_0^\infty (B_{\eta' - \delta}; [0,1]) \text{, such that } \varphi_{\delta}(x) = 1 \text{ on } B_{\eta}. \text{ Clearly } \|\varphi_{\delta}\|_\infty \leq \frac{\varepsilon}{\delta}. \text{ Let } \pi_k := u'_k \varphi_{\delta} + (1 - \varphi_{\delta}) u_k, \text{ thus the strong convergence of } u'_k \text{ and } u_k \text{ to } u, \text{ entails that } \pi_k \text{ strongly converges to } u. \text{ Thus}

\begin{align*}
\int_{\Lambda} f \left( \frac{x}{\varepsilon_k}, \frac{x}{\varepsilon_k}, D\pi_k (x) \right) dx & \leq \int_{B_\eta} f \left( \frac{x}{\varepsilon_k}, \frac{x}{\varepsilon_k}, D\pi_k (x) \right) dx + \int_{\Lambda \setminus \overline{\pi}_n} f \left( \frac{x}{\varepsilon_k}, \frac{x}{\varepsilon_k}, D\pi_k (x) \right) dx \\
& \leq \int_{B_\eta} f \left( \frac{x}{\varepsilon_k}, \frac{x}{\varepsilon_k}, D\pi_k (x) \right) dx + \int_{A \setminus \overline{\pi}_n} f \left( \frac{x}{\varepsilon_k}, \frac{x}{\varepsilon_k}, D\pi_k (x) \right) dx \\
& \text{where we have defined } \lambda := Du'_k \varphi_{\delta} + (1 - \varphi_{\delta}) Du_k \text{ and } \mu := Du'_k \varphi_{\delta} + (1 - \varphi_{\delta}) Du_k + (u'_k - u_k) D\varphi_{\delta}. \text{ Observe that}

\int_{L_\delta} \frac{1 + B (2 (1 + |\lambda| + |\mu|))}{1 + |\lambda| + |\mu|} |(u'_k - u_k)| dx & \leq \int_{L_\delta} \frac{1 + B (2 (1 + |\lambda| + |\mu|))}{1 + |\lambda| + |\mu|} |(u'_k - u)| dx \\
& + \int_{L_\delta} \frac{1 + B (2 (1 + |\lambda| + |\mu|))}{1 + |\lambda| + |\mu|} |(u_k - u)| dx,
\end{align*}
For any given fixed \( \delta \), the bounds give
\[
\int_{L_\delta} \frac{1 + B(2(1 + |\lambda| + |\mu|))}{1 + |\lambda| + |\mu|} |(u_k' - u)| \, dx \to 0,
\]
\[
\int_{L_\delta} \frac{1 + B(2(1 + |\lambda| + |\mu|))}{1 + |\lambda| + |\mu|} |(u_k - u)| \, dx \to 0,
\]
as \( k \to \infty \). Recalling that \( \eta \) and \( \delta \) can be chosen sufficiently small so that \( C \subset B_{\eta-\delta}, L_\delta \subset B_0 \setminus \overline{\mathcal{C}} \), passing to liminf (on \( \kappa \)) we get,
\[
\mathcal{F}_{\{\epsilon_k\}}(u, A) \leq \mathcal{F}_{\{\epsilon_k\}}(u, B_0) + \mathcal{F}_{\{\epsilon_k\}}(u, A \setminus \overline{\mathcal{C}}) + \eta + c \nu(L_\delta)
\]
\[
\mathcal{F}_{\{\epsilon_j\}}(u, B) + \mathcal{F}_{\{\epsilon_j\}}(u, A \setminus \mathcal{C}) + \eta + c \nu(B_\eta \setminus \overline{\mathcal{C}}),
\]
where we have sent \( \delta \to 0 \). Letting \( \eta \to 0 \), we deduce, \( \mathcal{F}_{\{\epsilon_k\}}(u, A) \leq \mathcal{F}_{\{\epsilon_j\}}(u, B) + \mathcal{F}_{\{\epsilon_j\}}(u, A \setminus \overline{\mathcal{C}}) + c \nu(B_0) \). Thus subadditivity is established.

\( ii \) Now we prove that for any \( A \in \mathfrak{A}(\Omega) \), and \( \epsilon > 0 \), we can find \( A_\epsilon \in \mathfrak{A}(\Omega) \) such that \( A_\epsilon \subset \subset A \) and \( \mathcal{F}_{\{\epsilon_j\}}(A \setminus \overline{A_\epsilon}) < \epsilon \). To this end take \( A_\epsilon \in \mathfrak{A}(\Omega) \) with \( A_\epsilon \subset \subset A \) and such that \( \int_{A \setminus \overline{A_\epsilon}} (1 + B(|Du_k(x)|)) \, dx < \frac{\epsilon}{c_2} \), where \( c_2 \) is the constant in \( (H_4) \). Thus,
\[
\mathcal{F}_{\{\epsilon_k\}}(u, A \setminus \overline{A_\epsilon}) \leq \liminf_{k \to \infty} \int_{A \setminus \overline{A_\epsilon}} f \left( \frac{x}{\epsilon_k}, \frac{x}{\epsilon_k^2}, Du_k(x) \right) \, dx
\]
\[
\leq c_2 \int_{A \setminus \overline{A_\epsilon}} (1 + B(|Du_k(x)|)) \, dx \leq \epsilon,
\]
as desired.

In the two following steps we prove that \( \mathcal{F}_{\{\epsilon_j\}}(\Omega) \geq \mu(\Omega) \). Then we prove that for all \( A \in \mathfrak{A}(\Omega) \) \( \mathcal{F}_{\{\epsilon_j\}}(A) \leq \mu(A) \). \( iii \) Take \( \Omega' \subset \subset \Omega \). Define for every \( A \in \mathfrak{A}(\Omega) \),
\[
\mu_k(A) := \int_{A \setminus \Omega_0} f \left( \frac{x}{\epsilon_k}, \frac{x}{\epsilon_k^2}, Du_k(x) \right) \, dx.
\]
Up to a subsequence, there exists \( \{\epsilon_k\} \) (not relabeled) such that \( u_k \to u \) in \( W^1L^B(\Omega; \mathbb{R}^d) \) and \( \mathcal{F}_{\{\epsilon_j\}}(u, \Omega) = \lim_{k \to \infty} \int_{\Omega} f \left( \frac{x}{\epsilon_k}, \frac{x}{\epsilon_k^2}, Du_k(x) \right) \, dx < +\infty \). The existence of such a sequence is easily proven, taking into account the definition of \( \mathcal{F}_{\{\epsilon_j\}} \), the coercivity condition \( (H_4) \), the fact that \( \theta_k := \int_{\Omega} f \left( \frac{x}{\epsilon_k}, \frac{x}{\epsilon_k^2}, Du_k(x) \right) \, dx \geq \frac{1}{c_2} \int_{\Omega} B(|Du_k(x)|) \, dy \), is bounded, the estimates
\[
\int_{\Omega} |B| u \, dy \leq \|u\|_{B, \Omega} \text{ if } \|u\|_{B, \Omega} \leq 1,
\]
\[
\|u\|_{B, \Omega} \leq \int_{\Omega} B |u| \, dy \text{ if } \|u\|_{B, \Omega} > 1,
\]
and the fact that \( u_k \to u \) in \( L^B(\Omega; \mathbb{R}^d) \). Thus \( \mu_k \) converges weakly* in the sense of measures to \( \mu \). It is easy to see that
\[
\mu(\Omega') \leq \liminf_{k \to \infty} \int_{\Omega} f \left( \frac{x}{\epsilon_k}, \frac{x}{\epsilon_k^2}, Du_k(x) \right) \, dx
\]
\[
= \lim_{k \to \infty} \int_{\Omega} f \left( \frac{x}{\epsilon_k}, \frac{x}{\epsilon_k^2}, Du_k(x) \right) \, dx = \mathcal{F}_{\{\epsilon_k\}}(u, \Omega).
\]
Therefore, \( \mu(\Omega') \leq \mathcal{F}_{\{\epsilon_k\}}(u, \Omega) \), for all \( \Omega' \subset \subset \Omega \). Hence, \( \mu(\Omega) \leq \mathcal{F}_{\{\epsilon_k\}}(u, \Omega) \).

\( iv \) For every \( A \in \mathfrak{A}(\Omega) \), arguing as above, it results that
\[
\mu_k(A) = \int_{A \setminus \Omega_0} f \left( \frac{x}{\epsilon_k}, \frac{x}{\epsilon_k^2}, Du_k(x) \right) \, dx \leq \int_{\Omega} f \left( \frac{x}{\epsilon_k}, \frac{x}{\epsilon_k^2}, Du_k(x) \right) \, dx < +\infty,
\]
hence, taken \( \mu \) as above it follows that

\[
F_{\{\varepsilon_k\}}(u, A) \leq \liminf_{k \to \infty} \int_{\Omega} f\left(\frac{x}{\varepsilon_k}, \frac{x}{\varepsilon_k}, Du_k(x)\right) dx \leq \mu(A),
\]

and we obtained \( iv \).

As consequence of the above results we conclude that \( F_{\{\varepsilon_k\}}(u, A) = \mu(A) \), for a suitable Radon measure \( \mu \), for all \( A \in \mathcal{A}(\Omega) \), and moreover it is immediately seen that this measure \( \mu \) is absolutely continuous with respect to the Lebesgue measure.

The proof of the following lemma is omitted since it can be immediately deduced from [18, Proposition 3.2], while for the \( L^p \) counterpart we refer to [14]. Indeed it is worth to observe that assumptions \( (H_1) - (H_4) \) herein do not deeply differ from the assumptions in [18]. Indeed therein we assumed strict convexity in the last argument, where in the present paper we just impose convexity on \( f \). On the other hand, the stronger assumption of strict convexity was used just to prove existence of a unique minimizer to the 'reiterated two-scale' limit functional (in Theorem 1.1 therein). Moreover the continuity assumption \( (H_1) \) in the current manuscript is a stronger assumptions than \( (H_1) \) in [18].

**Lemma 3.3.** If \( f \) satisfies hypotheses \( (H_1),(H_2),(H_3) \) and if \( (w_\varepsilon)_\varepsilon \subset L^B(\Omega; \mathbb{R}^d) \) reiteratively two-scales converges to a function \( w_0 \in L^B(\Omega \times Y \times Z; \mathbb{R}^d) \) then

\[
\liminf_{\varepsilon \to 0} \int_{\Omega} f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon}, w_\varepsilon(x)\right) dx \geq \int_{\Omega} \int_{Y} \int_{Z} f(y, z, w_0(x, y, z)) dxdydz.
\]

Proof. Since \( f \) is convex, for all \( x \in \mathbb{R}^N, a.e. y \in \mathbb{R}^N \), and all \( \xi \in \mathbb{R}^{d \times N} \) we have \( f(x, y, \xi') \geq f(x, y, \xi) + \frac{\partial f}{\partial \xi}(x, y, \xi) \cdot (\xi' - \xi) \). Hence,

\[
\frac{\partial f}{\partial \xi}(x, y, \xi) \cdot (\xi' - \xi) \leq f(x, y, \xi') - f(x, y, \xi) \leq c\left(\frac{1 + B(2(1 + |\xi| + |\xi'|))}{1 + |\xi| + |\xi'|}\right)|\xi - \xi'|.
\]

Choose \( \xi' = \xi + E_i \) whith \( E_i \) the canonical base of \( \mathbb{R}^{d \times N} \) and get

\[
\frac{\partial f}{\partial \xi}(x, y, \xi) \cdot (\pm E_i) \leq c\left(\frac{1 + B(2(1 + |\xi| + |\xi' \pm E_i|))}{1 + |\xi| + |\xi' \pm E_i|}\right)|\pm E_i| \leq c' (1 + b(1 + |\xi|)).
\]

Indeed,

\[
|\xi' \pm E_i| \leq |\pm E_i| + |\xi'| = 1 + |\xi'|,\text{ hence}
\]

\[
\frac{1 + B(2(1 + |\xi| + |\xi' \pm E_i|))}{1 + |\xi| + |\xi' \pm E_i|} \leq \frac{1 + B(4(1 + |\xi|))}{1 + |\xi|} \leq 1 + \frac{B(4(1 + |\xi|))}{1 + |\xi|},
\]

where we have exploited the fact that \( B \in \Delta_2 \). We then have

\[
\left|\frac{\partial f}{\partial \xi}(x, y, \xi)\right| \leq c (1 + b(1 + |\xi|)).
\]
Let \((\theta_j) \subset C_c(\Omega; C_{\text{per}}(Y \times Z))\) be such that \(\theta_j \to w_0\) in \(L^B_{\text{per}}(\Omega \times Y \times Z)\). The assumptions on \(f\) guarantee that, for fixed \(j \in \mathbb{N}\),

\[
\liminf_{\varepsilon \to 0} \int_{\Omega} f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, w_\varepsilon\right) dx \geq \lim_{\varepsilon \to 0} \int_{\Omega} f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \theta_j\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)\right) dx - \limsup_{\varepsilon \to 0} \int_{\Omega} \frac{\partial f}{\partial \xi}\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \theta_j\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)\right) \left[\theta_j\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) - w_\varepsilon\right] dx
\]

\[
\geq \int_{\Omega} \int_Y \int_Z f(y, z, \theta_j(x, y, z)) dx - \limsup_{\varepsilon \to 0} I_{j,\varepsilon},
\]

where

\[
I_{j,\varepsilon} = \int_{\Omega} \frac{\partial f}{\partial \xi}\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \theta_j\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)\right) \theta_j\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) dx - \int_{\Omega} \frac{\partial f}{\partial \xi}\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, \theta_j\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right)\right) w_\varepsilon dx
\]

dependence, by reiterated two-scale convergence

\[
\limsup_{\varepsilon \to 0} I_{j,\varepsilon} = \lim_{\varepsilon \to 0} I_{j,\varepsilon} = \int_{\Omega} \int_Y \int_Z \frac{\partial f}{\partial \xi}(y, z, \theta_j(x, y, z)) \theta_j(x, y, z) dx dy dz - \int_{\Omega} \int_Y \int_Z \frac{\partial f}{\partial \xi}(y, z, \theta_j(x, y, z)) w_0(x, y, z) dx dy dz,
\]

and

\[
\liminf_{\varepsilon \to 0} \int_{\Omega} f\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, w_\varepsilon\right) dx \geq \int_{\Omega} \int_Y \int_Z f(y, z, \theta_j(x, y, z)) dx dy dz
\]

\[
- \int_{\Omega} \int_Y \int_Z \frac{\partial f}{\partial \xi}(y, z, \theta_j(x, y, z)) \theta_j(x, y, z) dx dy dz + \int_{\Omega} \int_Y \int_Z \frac{\partial f}{\partial \xi}(y, z, \theta_j(x, y, z)) w_0(x, y, z) dx dy dz.
\]

On the other hand,

\[
\liminf_{j \to +\infty} \left(\int_{\Omega} \int_Y \int_Z f(y, z, \theta_j(x, y, z)) dx dy dz\right)
\]

\[
- \int_{\Omega} \int_Y \int_Z \frac{\partial f}{\partial \xi}(y, z, \theta_j(x, y, z)) \theta_j(x, y, z) - w_0(x, y, z) dx dy dz\right)
\]

\[
= \liminf_{j \to +\infty} \left(\int_{\Omega} \int_Y \int_Z f(y, z, \theta_j(x, y, z)) dx dy dz\right)
\]

\[
- \limsup_{j \to +\infty} \left(\int_{\Omega} \int_Y \int_Z \frac{\partial f}{\partial \xi}(y, z, \theta_j(x, y, z)) \theta_j(x, y, z) - w_0(x, y, z) dx dy dz\right).
\]
\[ \left| \int_{\Omega} \int_{Y} \int_{Z} \frac{\partial f}{\partial \xi}(y, z, \theta_j(x, y, z)) \left( \theta_j(x, y, z) - w_0(x, y, z) \right) \, dx \, dy \, dz \right| \]
\[ \leq \int_{\Omega} \int_{Y} \int_{Z} \left| \frac{\partial f}{\partial \xi}(y, z, \theta_j(x, y, z)) \right| \left| \theta_j(x, y, z) - w_0(x, y, z) \right| \, dx \, dy \, dz \]
\[ \leq c \int_{\Omega} \int_{Y} \int_{Z} \left(1 + b(1 + |\theta_j(x, y, z)|) \right) \left| \theta_j(x, y, z) - w_0(x, y, z) \right| \, dx \, dy \, dz \]
\[ \leq c \int_{\Omega} \int_{Y} \int_{Z} \left(1 + b(1 + |\theta_j(x, y, z)|) \right) \frac{|\theta_j(x, y, z) - w_0(x, y, z)|}{\alpha} \, dx \, dy \, dz \]
\[ + c \int_{\Omega} \int_{Y} \int_{Z} B \left( \frac{|\theta_j(x, y, z) - w_0(x, y, z)|}{\alpha} \right) \, dx \, dy \, dz \]
\[ \leq c \| \theta_j - w_0 \|_{B, \Omega \times Y \times Z} + c \alpha \int_{\Omega} \int_{Y} \int_{Z} \tilde{B} \left( \frac{b(1 + |\theta_j(x, y, z)|)}{\alpha} \right) \, dx \, dy \, dz \]
\[ + c \int_{\Omega} \int_{Y} \int_{Z} B \left( \frac{|\theta_j(x, y, z) - w_0(x, y, z)|}{\alpha} \right) \, dx \, dy \, dz \leq c \| \theta_j - w_0 \|_{B, \Omega \times Y \times Z} +
\[ + c \alpha c' + c \alpha \int_{\Omega} \int_{Y} \int_{Z} B \left(1 + |\theta_j(x, y, z)| \right) \, dx \, dy \, dz \]
\[ + c \int_{\Omega} \int_{Y} \int_{Z} B \left( \frac{|\theta_j(x, y, z) - w_0(x, y, z)|}{\alpha} \right) \, dx \, dy \, dz. \]

Exploiting the properties of $B$, we obtain
\[ \left| \int_{\Omega} \int_{Y} \int_{Z} \frac{\partial f}{\partial \xi}(y, z, \theta_j(x, y, z)) \left( \theta_j(x, y, z) - w_0(x, y, z) \right) \, dx \, dy \, dz \right| \]
\[ \leq c \| \theta_j - w_0 \|_{B, \Omega \times Y \times Z} +
\[ + c c' \alpha \int_{\Omega} \int_{Y} \int_{Z} B \left( \frac{|\theta_j(x, y, z) - w_0(x, y, z)|}{\alpha} \right) \, dx \, dy \, dz \]
\[ + \frac{1}{2} B \left( \frac{c}{\alpha} \right) \int_{\Omega} \int_{Y} \int_{Z} \left( \frac{1}{2} B \left( \frac{4 |\theta_j(x, y, z) - w_0(x, y, z)|}{\alpha} \right) \right) \, dx \, dy \, dz. \]

Letting $j \to +\infty$ and taking into account the arbitrariness of $0 < \alpha < 1$ we get
\[ \limsup_{j \to +\infty} \int_{\Omega} \int_{Y} \int_{Z} \frac{\partial f}{\partial \xi}(y, z, \theta_j(x, y, z)) \left( \theta_j(x, y, z) - w_0(x, y, z) \right) \, dx \, dy \, dz = 0. \]

The difference
\[ \int_{\Omega} \int_{Y} \int_{Z} f(y, z, \theta_j(x, y, z)) \, dx \, dy \, dz - \int_{\Omega} \int_{Y} \int_{Z} f(y, z, w_0(x, y, z)) \, dx \, dy \, dz \]
can also be treated as done to estimate the difference of the second and third term in (11), thus the result follows from passing to the limit on $j$. \qed
Lemma 3.5. If $f$ satisfies hypotheses $(H_1)$ or $[(A_1), (A_2), (A_3)], (H_2), (H_3)$ and $(H_4)$, and $F$ is the functional defined in (9), then $\mathcal{F}_{(\varepsilon)}(u, \Omega) \geq F(u, \Omega)$, for every $u \in W^1 L^B(\Omega; \mathbb{R}^d)$.

Proof. Let $u_\varepsilon \rightharpoonup u$ in $W^1 L^B(\Omega; \mathbb{R}^d)$. Without loss of generality, assume that $\mathcal{F}_{(\varepsilon)}(u) < +\infty$, thus we can take a bounded sequence $(u_\varepsilon)$ and extracting a not relabelled subsequence, if necessary, we assume that $u_\varepsilon \rightharpoonup^*_2 u$, $Du_\varepsilon \rightharpoonup^*_2 D_x u + D_y U + D_z W$, in $L^B(\Omega \times Y; \mathbb{R}^d)$ and $L^B(\Omega \times Y \times Z; \mathbb{R}^{d \times N})$, respectively, for some $U \in L^1(\Omega; W^1_y L^B(Y; \mathbb{R}^d))$, $W \in L^1(\Omega \times Y; W^1_y L^B(Z; \mathbb{R}^d))$.

By (9), in the case in which $f$ satisfies $(H_1), (H_2), (H_3)$ and $(H_4)$, it suffices to invoke lemma 3.3, while in the other case (when $(H_1)$ is replaced by $((A_1), (A_2), (A_3))$ one has to apply lemma 3.4 to get:

$$\liminf_{\varepsilon \to 0} \int_\Omega f \left( \frac{u}{\varepsilon}, \frac{\partial}{\partial x}Du_\varepsilon \right) dx \geq \int_\Omega \int_Y \int_Z f(y, z, D_x u(x) + D_y U(x, y) + D_z W(x, y, z)) dx dy dz \geq F(u, \Omega).$$

$\square$

Lemma 3.6. If $f$ satisfies $(H_1)$ or $[(A_1), (A_2)], (H_2), (H_3)$ and $(H_4)$, then $\mathcal{F}_{(\varepsilon)}(u, \Omega) \geq \int_\Omega \overline{f}_{\text{hom}}(Du(x)) dx$ for every $u \in W^1 L^B(\Omega; \mathbb{R}^d)$.

Proof. The result follows by Lemma 3.5, (9), (4) and (3), by applying Fubini’s theorem.

Following along the lines of [18, Theorem 4.6], now we are in position to prove the opposite inequality.

Lemma 3.7. If $f$ satisfies $(H_1)$ or $[(A_1), (A_2)], (H_2), (H_3)$ and $(H_4)$, then

$$\mathcal{F}_{(\varepsilon)}(u, \Omega) \leq \int_\Omega \overline{f}_{\text{hom}}(Du(x)) dx$$

for every $u \in W^1 L^B(\Omega; \mathbb{R}^d)$.

Proof. Consider any subsequence of $(\varepsilon)$, (not relabelled) such that the $\Gamma$-limit $\mathcal{F}_{(\varepsilon)}$ exists. By Lemma 3.2, we know that $\mathcal{F}_{(\varepsilon)}(u, \cdot)$ is the trace on $A(\Omega)$ of a Radon measure absolutely continuous with respect to the $N$ dimensional Lebesgue measure $\mathcal{L}^N$. Thus to achieve the result, it is enough to prove that for any fixed $u \in W^1 L^B(\Omega; \mathbb{R}^d)$,

$$\lim_{\delta \to 0^+} \frac{\mathcal{F}_{(\varepsilon)}(u, Q(x_0, \delta))}{\delta^N} \leq \overline{f}_{\text{hom}}(Du(x_0)), \text{ for a.e } x_0 \in \Omega.$$

Let $x_0 \in \Omega$, be a Lebesgue point for $u$, $Du$ and assume that

$$\lim_{\delta \to 0^+} \frac{1}{\delta^N} \int_{Q(x_0, \delta)} |B(|Du(x) - Du(x_0)|)| dx = 0.$$

Fix $\alpha > 0$, and using the definition of $\overline{f}_{\text{hom}}(Du(x_0))$ choose $\varphi \in W^1 L^B_{\text{per}}(Y; \mathbb{R}^d)$ such that $\overline{f}_{\text{hom}}(Du(x_0)) + \alpha > \int_Y \overline{f}_{\text{hom}}(y, Du(x_0) + D\varphi(y)) dy$. Since $C^\infty_{\text{per}}(Y; \mathbb{R}^d)$ is dense in $W^1 L^B_{\text{per}}(Y; \mathbb{R}^d)$ and $\overline{f}_{\text{hom}}$ is continuous (see Lemma 3.1), one can take $\varphi \in C^\infty_{\text{per}}(Y, \mathbb{R}^d)$ such that:

$$\overline{f}_{\text{hom}}(Du(x_0)) + \alpha \geq \int_Y \overline{f}_{\text{hom}}(y, Du(x_0) + D\varphi(y)) dy.$$

In order to apply Proposition 8 with $(X, \mathcal{M}) := (\Omega, \mathcal{L}), S := W^1 L^B_{\text{per}}(Y; \mathbb{R}^d), \mu$
the Lebesgue measure, and \( \mathcal{L} \) the \( \sigma \)-algebra of Lebesgue measurable sets in \( \mathbb{R}^N \), we introduce the multi-valued map

\[
H : \Omega \to \{ C \subset W^{1,1}(Z; \mathbb{R}^d) : C \neq \emptyset, C \text{ is closed} \}
\]

such that

\[
x \mapsto H(x) := \left\{ \psi \in W^{1,1}(Z; \mathbb{R}^d) : \int_Z \psi(y) \, dy = 0, \right. \\
\left. f_{\text{hom}}(x, Du(x_0) + D_y \varphi(x)) + \alpha \geq \int_Z f(x, z, Du(x_0) + D_y \varphi(x) + D_z \psi(z)) \, dz \right\}. 
\]

Exploiting the properties of Sobolev-Orlicz spaces, and the definition of \( f_{\text{hom}} \), we can prove that the set \( H(x) \) is non empty and closed. Indeed, let \( \psi_1 \in W^{1,1}(Z; \mathbb{R}^d) \), set \( \psi_2 = \psi_1 - \int_Z \psi_1 \, dz \), we have \( \int_Z \psi_2 \, dz = 0 \) and \( \emptyset \neq H_2 := \{ \psi \in W^{1,1}(Z; \mathbb{R}^d) : \int_Z \psi(z) \, dz = 0 \} \). Moreover, for \( \psi \in H_2 \), \( \int_Z \psi(z) \, dz \leq c\|\psi\|_{1,B} \leq c\|\psi\|_{W^{1,1}(Z; \mathbb{R}^d)} \) and \( H_2 \) is closed as \( u \to \int_Z udz \) is linear and continuous. Next, from definition of \( f_{\text{hom}} \), given \( \alpha > 0 \), \( \exists \psi_1 \in W^{1,1}(Z; \mathbb{R}^d) \) such that:

\[
f_{\text{hom}}(x, Du(x_0) + D_y \varphi(x)) + \alpha > \int_Z f(x, z, Du(x_0) + D_y \varphi(x) + D_z \psi(z)) \, dz.
\]

Clearly, since \( D_z \psi_1(z) = D \psi_1(z) - \int_Z \psi_1 \, dz \), there exist \( \psi_2 \in H_2 \) such that

\[
f_{\text{hom}}(x, Du(x_0) + D_y \varphi(x)) + \alpha > \int_Z f(x, z, Du(x_0) + D_y \varphi(x) + D_z \psi_2(z)) \, dz \]

hence \( H(x) \neq \emptyset \). Moreover, considering \( g_x : W^{1,1}(Z; \mathbb{R}^d) \to \mathbb{R} \), defined as

\[
g_x : \psi \mapsto: f_{\text{hom}}(x, Du(x_0) + D_y \varphi(x)) + \alpha - \int_Z f(x, z, Du(x_0) + D_y \varphi(x) + D_z \psi(z)) \, dz,
\]

it results that, for every \( n \in \mathbb{N} \),

\[
\left| g_x(\psi_1) - g_x(\psi_n) \right| = \\
\left| \int_Z f(x, Du(x_0) + D_y \varphi(x) + D_z \psi_1) \, dz - \int_Z f(x, Du(x_0) + D_y \varphi(x) + D_z \psi_n) \, dz \right|
\]

\[
\leq \int_Z \left| f(x, Du(x_0) + D_y \varphi(x) + D_z \psi_1) - f(x, Du(x_0) + D_y \varphi(x) + D_z \psi_n) \right| \, dz
\]

\[
\leq c \int_Z \frac{1 + B(2(1 + |\lambda| + |\mu|))}{1 + |\lambda| + |\mu|} |\lambda - \mu| \, dz \\
\leq c \int_Z |\lambda - \mu| \, dz + c \int_Z (ab(1 + |\lambda| + |\mu|)) \frac{|\lambda - \mu|}{|\alpha|} \, dz \\
\leq c2 \|1\|_B \|\lambda - \mu\|_B + c|\alpha| \int_Z \frac{|\lambda - \mu|}{|\alpha|} \, dz \\
\leq c2 \|1\|_B \|\lambda - \mu\|_B + c\alpha
\]

where \( 0 < \alpha < 1, \lambda := Du(x_0) + Du(x) + Du(z), \mu := Du(x_0) + Du(x) + Du(z). \)

If \( \psi_n \) is such that \( \|\psi_1 - \psi_n\|_{1,B} \to 0 \) as \( n \to +\infty \), we get that the right hand side goes to 0, passing to limit on \( n \) and using the arbitriness of \( 0 < \alpha < 1 \), thus, due to the metrizability of the spaces, \( g_x \) is continuous and \( g_x^{-1}(\{0, +\infty\}) \) is closed. Therefore, \( H(x) = H_2 \cap g_x^{-1}(\{0, +\infty\}) \) is closed.
Also, the set \( \{(x, \psi) \in \Omega \times W^1L^B_{\text{per}}(Z; \mathbb{R}^d) : \psi \in H(x)\} \) is closed hence Borel measurable, since it coincides with \( \mathcal{H}^{-1}([0, +\infty] \times \{0\}) \). Let,\( \mathcal{H}(x, \psi) := (g_x(\psi), \int_Z \psi(z) \, dz) \) . The topological and measurability properties follow by an estimate entirely analogous to the previous ones, consisting in showing the continuity of \( g_x(\psi) \) in \( (x, \psi) \).

By Proposition 8 we have \( H(x) = \{h_n : n \in \mathbb{N}\} \). Let \( \psi \) be a measurable selection, \( x \in Y \rightarrow \psi(x, \cdot) \in W^1L^B_{\text{per}}(Z; \mathbb{R}^d) \), such that \( \int_Z \psi \, dz = 0 \) and

\[
\int_Y f(y, Du(x_0) + D\psi(x_0)) \, dy \leq \int_Y f(y, Du(x_0) + D\varphi(y) + D_y\psi_1(x, y)) \, dy.
\]

It results that

\[
\int_Y \int_Z (|Du(x_0) + D\varphi(y) + D_y\psi_1(x, y)|) \, dz \, dy < +\infty.
\]

Thus

\[
+\infty > \int_Y \left[ \int_Y f(y, Du(x_0) + D\varphi(y)) \, dy \right] \, dy \geq \int_Y \int_Z f(y, Du(x_0) + D\varphi(y) + D_y\psi_1(x, y)) \, dy \, dz
\]

\[
\geq c_1 \int_Y \int_Z B(|Du(x_0) + D\varphi(y) + D_y\psi_1(x, y)|) \, dy \, dz
\]

\[
\geq c_1 \int_Y \int_Z 4B\left(\frac{Dy\psi_1(x, y)}{4}\right) \, dy \, dz.
\]

Since \( \varphi \) is regular, \( D\varphi \) is bounded, and we get, exploiting the convexity properties of \( B \),

\[
\int_Y \int_Z B \left| \frac{Dy\psi_1(x, y)}{4}\right| \, dy \, dz < +\infty.
\]

Therefore

\[
D_y\psi \in L^B(Y \times Z) \hookrightarrow L^1(Y \times Z) = L^1(Y ; L^1(Z)),
\]

with \( x \mapsto \psi(x, \cdot) \) from \( Y \) to \( W^1L^B(Z) \) measurable.

Moreover, since \( B \) and \( \bar{B} \) satisfy \( \Delta_2 \) condition, it is well known that

\[
D_y\psi \in L^B(Y \times Z) \implies D_y\psi \in L^B(Y ; L^B(Z)).
\]

On the other hand, since

\[
\psi(x, \cdot) \in W^1L^B(Y) \implies D_y(\psi(x, \cdot)) \in L^B(Z) \implies \int_Y B|D_y(\psi(x, y))| \, dy < +\infty.
\]

We also have

\[
\psi(x, \cdot) \in L^B(Y) \text{ with } \int_Y \psi(x, \cdot) \, dy = 0.
\]
Then Poincaré-Wirtinger’s inequality gives

\[ \|\psi(x, \cdot)\|_{B,Y} \leq c \|D_y \psi(x, \cdot)\|_{B,Y} \]

Moreover, exploiting Fubini’s Theorem and (10) we can conclude that

\[ \int_Y \|D_y \psi(x, \cdot)\|_{B,Z} \, dx < +\infty. \]

So far, we deduce that \( \psi \in L^1(Y; W^{1,L}_{\text{per}}(Z; \mathbb{R}^d)) \).

More precisely, as in Remark 2, we have \( \psi \in L^B(y; W^{1,L}_{\text{per}}(Z; \mathbb{R}^d)) \). Now let \( \psi_k \in C_c^\infty(Y; W^{1,L}_{\text{per}}(Z; \mathbb{R}^d)) \) be such that \( \|\psi_k - \psi\|_{L^1(Y; W^{1,L}_{\text{per}}(Z; \mathbb{R}^d))} \to 0 \). Extend \( \psi_k \) periodically and define \( u_{k,\varepsilon} := u(x) + \varepsilon \varphi\left(\frac{x}{\varepsilon}\right) + \varepsilon^2 \psi_k\left(\frac{x}{\varepsilon}, \frac{z}{\varepsilon}\right) \).

For fixed \( \delta > 0 \), it is clear that \( u_{k,\varepsilon} \to u \) in \( L^B(Q(x_0, \delta)) \) and so,

\[ \lim_{\delta \to 0^+} \frac{F(\varepsilon)}{\delta^N} \left( u, Q(x_0, \delta) \right) \leq \lim_{\delta \to 0^+} \lim_{\varepsilon \to 0} \frac{1}{\delta^N} \int_{Q(x_0, \delta)} f\left( \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}, Du(x) + D_y \varphi\left(\frac{x}{\varepsilon}\right) + \varepsilon D_y \psi_k\left(\frac{x}{\varepsilon}, \frac{z}{\varepsilon}\right) \right) \, dx \]

\[ + \varepsilon D_y \psi_k\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) = \int_Y \int_Z f(y, z, Du(x_0) + D_y \varphi(x) + D_z \psi_k(y, z)) \, dy \, dz, \]

where

\[ \lambda := Du(x_0) + D_y \varphi\left(\frac{x}{\varepsilon}\right) + D_z \psi_k\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right), \]

\[ \mu = Du(x) + D_y \varphi\left(\frac{x}{\varepsilon}\right) + \varepsilon D_y \psi_k\left(\frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) + D_z \psi_k\left(\frac{x}{\varepsilon}, \frac{z}{\varepsilon}\right). \]

Indeed the latter limit can be obtained by \((H_3)\) and \((H_4)\), and exploiting the same arguments as in [15] (see [16, Proposition 3.1]). Precisely it results that

\[ f(x, y, \lambda) \leq f(x, y, \mu) + c \frac{B(2(1 + |\lambda| + |\mu|))}{1 + |\lambda| + |\mu|} |\lambda - \mu| \]

\[ \leq f(x, y, \mu) + c (1 + b(1 + |\lambda| + |\mu|)) |\lambda - \mu|, \]

Hence for \( 0 < \alpha < 1 \), and for any open set \( O \),
Thus, the arbitrariness of $\alpha$ afterwards by the arbitrariness of
\[ \int_0 f(x, y, \lambda) \, dx \leq \int_0 f(x, y, \mu) \, dx + c \int_0 \alpha \frac{\lambda - \mu}{\alpha} \, dx \]
\[ + c \int_0 a b (1 + |\lambda| + |\mu|) \frac{\lambda - \mu}{\alpha} \, dx \leq \int_0 f(x, y, \mu) \, dx + c \int_0 B \left( \frac{\lambda - \mu}{\alpha} \right) \, dx \]
\[ + c \int_0 \bar{B}(\alpha) \, dx + c' \int_0 B \left( \frac{\lambda - \mu}{\alpha} \right) \, dx \]
\[ + c \int_{O_1} \bar{B}(b (1 + |\lambda| + |\mu|)) \, dx + c a \int_{O_2} \bar{B}(b (1 + |\lambda| + |\mu|)) \, dx. \]

Then, if $O_1 := \{ x \in O : 1 + |\lambda(x)| + |\mu(x)| > t_0 \}, O_2 := O \setminus O_1$ with $\bar{B}(b(t)) \leq k B(t), t > t_0$, we have
\[ \int_0 f(x, y, \lambda) \, dx \leq \int_0 f(x, y, \mu) \, dx \]
\[ + c \int_0 B \left( \frac{\lambda - \mu}{\alpha} \right) \, dx + c \int_{O_1} \bar{B}(\alpha) \, dx + c' \int_{O_2} B(1 + |\lambda| + |\mu|) \, dx \]
\[ \leq \int_0 f(x, y, \mu) \, dx + c \int_0 B \left( \frac{\lambda - \mu}{\alpha} \right) \, dx + c \int_{O_1} \bar{B}(\alpha) \, dx + c c' \alpha + c \int_{O_2} B(1 + |\lambda| + |\mu|) \, dx. \]

Now applying the chain of inequalities in (14), at the level of the second inequality in (12), with $O = Q(x_0), \lambda$ and $\mu$ as in (13), passing to the limit as $\varepsilon$ and $\delta$ go to 0, we obtain the desired estimate. Indeed, in the first term in (14) becomes the desired one, the second term will go to 0 as $\varepsilon \to 0$ and $\delta \to 0$ and the last summands go to 0 afterwards by the arbitrariness of $\alpha$, i.e. letting $\alpha \to 0$.

We have
\[ \lim_{\delta \to 0^+} \frac{\mathcal{F}(\varepsilon)(u, Q(x_0, \delta))}{\delta^N} \leq \int_Y \int_Z f(y, z, D_u(x_0) + D_y \varphi(y) + D_z \psi(y, z)) \, dydz. \]

Thus, sending $k \to +\infty$, exploiting the growth conditions on $f$, we have
\[ \lim_{\delta \to 0^+} \frac{\mathcal{F}(\varepsilon)(u, Q(x_0, \delta))}{\delta^N} \]
\[ \leq \int_Y \int_Z f(y, z, D_u(x_0) + D_y \varphi(y) + D_z \psi(y, z)) \, dydz \]
\[ \leq \int_Y [f_{\text{hom}}(y, D_u(x_0) + D_y \varphi(y)) + \alpha] \, dy \leq \overline{f_{\text{hom}}}(D_u(x_0)) + 2 \alpha. \]

Thus, the arbitrariness of $\alpha$ gives the result.

\[ \square \]
We observe that the following Proposition, which extend to the Orlicz-Sobolev spaces, \[14, \text{lemma 4.2}\], can be proven. The proof develops along the lines of the above result, relying in turn on assumption \((H_4)\), approximation by means of regular functions, Lemma 3, and dominated convergence theorem, hence the proof is omitted.

**Proposition 9.** If \(f\) satisfies hypotheses \((A_1), (A_2), (H_2), (H_3)\) and \((H_4)\), then \(\mathcal{F}(\varepsilon_t)(u, \Omega) \leq F(u, \Omega)\) for every \(u \in W^{1, L^B}(\Omega; \mathbb{R}^d)\).

**Remark 3.** Putting together the results of Proposition 9 and of Lemma 3.5, we indeed obtain a \(\Gamma\)-limit result for \((F_{\varepsilon_t})_{\varepsilon_t}\), in terms of the functional 9, which in turn leads to a result analogous to \[18, \text{Theorem 1.1}\].

**Proof of Theorem 2.** It is a direct consequence of the above lemmas 3.7 and 3.6 for \(s = 1\). For \(s = 2\), it relies on Theorem 1.1 and minors changes with respect to the case \(s = 1\), hence the proof is omitted.

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**REFERENCES**

[1] R. Adams, *Sobolev Spaces*, Academic Press, New York-London, 1975.
[2] R. Adams, On the Orlicz-Sobolev imbedding theorem, *J. Functional Analysis* 24 (1977), 241–257.
[3] G. Allaire, Homogenization and two scale convergence, *SIAM J. Math. Anal.* 23 (1992) 1482–1518.
[4] G. Allaire and M. Briane, Multiscale convergence and reiterated homogenization, *Proc. Royal Soc. Edin.* 126 (1996), 297–342.
[5] M. Baia and I. Fonseca, The limit behavior of a family of variational multiscale problems, *Indiana Univ. Math. J.* 56, (2007), 1–50.
[6] G. Carita, A. M. Ribeiro and E. Zappale, An homogenization result in \(W^{1,p} \times L^q\), *J. Convex Anal.*, 18, n. 4, (2011), 1093–1126.
[7] M. Chmara and J. Maksymiuk, Anisotropic Orlicz-Sobolev spaces of vector valued functions and Lagrange equations, *J. Math. Anal. Appl.*, 456, (2017), 457–475.
[8] A. Cianchi, Higher-order Sobolev and Poincaré inequalities in Orlicz spaces, *Forum Math.*, 18, (2006), n. 5, 745–767.
[9] D. Cioranescu, A. Damlaman and R. De Arcangelis, Homogenization of quasiconvex integrals via the periodic unfolding method, *SIAM J. Math. Anal.*, 37, n. 5, (2006), 1435–1453.
[10] D. Cioranescu, A. Damlaman and G. Griso, The periodic unfolding method in homogenization, *SIAM J. Math. Anal.*, 40, n. 4, (2008), 1585–1620.
[11] D. Cioranescu, A. Damlaman and G. Griso, The periodic unfolding method, *Series in Contemporary Mathematics*, Vol. 3, Springer, Singapore, 2018.
[12] G. Dal Maso, *An Introduction to \(\Gamma\)-Convergence*, Birkhäuser Boston, Inc., Boston, MA, 1993.
[13] W. Desch and R. Grimmer, On the wellposedness of constitutive laws involving dissipation potentials, *Trans. Amer. Math. Soc.*, 353 (2001), 5095–5120.
[14] I. Fonseca and E. Zappale, Multiscale relaxation of convex functionals, *J. Convex Anal.*, 10 (2003), 325–350.
[15] M. Focardi, Semicontinuity of vectorial functionals in Orlicz-Sobolev spaces, *Rend. Istit. Mat. Univ. Trieste*, 29 (1997), 141–161.
[16] J. F. Tachago and H. Nnang, Two-scale convergence of integral functionals with convex, periodic and Nonstandard Growth Integrands, *Acta Appl. Math.*, 121, (2012), 175–196.
[17] J. F. Tachago, H. Nnang and E. Zappale, Relaxation of periodic and nonstandard growth integrands by means of two scale convergence, in Integral Methods in Science and Engineering–Analytic Treatment and Numerical Approximations, Birkhäuser/Springer, Cham, 2019, 123–132.

[18] J. Fotso Tachago, H. Nnang and E. Zappale, Reiterated periodic homogenization of integral functionals with convex and nonstandard growth integrands, preprint, 2019, arXiv:math/1901.07217v1.

[19] A. Gaudiello and O. Guibé, Homogenization of an evolution problem with $L \log L$ data in a domain with oscillating boundary, Ann. Mat. Pura Appl., 197, (2018), 153–169.

[20] A. Ioffe, On lower semicontinuity of integral functionals. I, SIAM Journ. Control Optim., 15 (1977), 521–538.

[21] R. K. Bogning and H. Nnang, Periodic homogenization of parabolic nonstandard monotone operators, Acta Appl. Math., 125 (2013), 209–229.

[22] P. A. Kozarzewski and E. Zappale, Orlicz equi-integrability for scaled gradients, J. Elliptic Parabol. Equ., 3 (2017), 1–13.

[23] P. A. Kozarzewski and E. Zappale, A note on optimal design for thin structures in the Orlicz-Sobolev setting, Integral Methods in Science and Engineering, Vol. 1, (2017), Birkhäuser Basel, 161–171.

[24] D. Lukkassen, G. Nguetseng, H. Nnang and P. Wall, Reiterated homogenization of nonlinear monotone operators in a general deterministic setting, J. Funct. Spaces Appl. 7 (2009), 121–152.

[25] G. Nguetseng, A general convergent result for functional related to the theory of homogenization, SIAM J. Math. Anal., 20 (1989), 608–623.

[26] G. Nguetseng and H. Nnang, Homogenization of nonlinear monotone operators beyon the periodic setting, Electron. J. Differential Equations (2003), No. 36, 1–24.

[27] H. Nnang, Homogénéisation déterministe d’opérateurs monotones, Fac. Sc. University of Yaoundé 1, Yaoundé, 2004.

[28] H. Nnang, Deterministic Homogenization of Nonlinear Degenerated Elliptic Operators with Nonstandard Growth, Act. Math. Sin., 30 (2014), 1621–1654.

[29] M. M. Rao and Z. D. Ren, Theory of Orlicz spaces, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 146, Marcel Dekker, Inc., New York, 1991.

[30] E. Zappale, A note on dimension reduction for unbounded integrals with periodic microstructure via the unfolding method for slender domains, Evol. Equ. Control Theory, 6, (2017), 299–318.

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