Special geometry on the moduli space for the two-moduli
non-Fermat Calabi–Yau

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Abstract

We clarify the recently proposed method for computing a special Kähler metric on a Calabi–Yau
complex structure moduli space using the fact that the moduli space is a subspace of a particular
Frobenius manifold. We use this method to compute a previously unknown special Kähler metric in
a two-moduli non-Fermat model.

1 Introduction

To find the low-energy Lagrangian of string theory compactified on a Calabi–Yau (CY) manifold
$X$, one must know the special Kähler geometry on the moduli space of the CY manifold [1, 2, 3, 4].
An alternative method for computing the Kähler potential in the case where the CY manifold
is given by a hypersurface $W_0(x) = 0$ in a weighted projective space was recently presented in [5].
In this paper, we briefly review this method and use it to compute the special Kähler geometry on
the moduli space of a two-moduli non-Fermat threefold considered in [6]. This approach is based on
the fact that the moduli space of a CY manifold is a subspace in the particular Frobenius manifold
(FM) [7] that arises from the deformation space of the singularity defined by the LG superpotential
$W_0(x)$. In this paper, we discuss this FM and its connection with the geometry of the CY moduli
space in more detail.

It is known, that an isolated singularity $W_0(x)$ defines an object called a Milnor ring $R_0$. Endowed
with an invariant pairing this ring becomes a Frobenius algebra. Its elements define versal deformations
of the singularity. It is well known that the space of deformations admits a FM structure. In
our case, $W_0(x)$ is a quasihomogeneous polynomial, which defines a CY manifold in the weighted
projective space. For our purposes of computation of the CY moduli space geometry we actually need
the Frobenius subalgebra $R^0_d$ of $R_0$, that is multiplicatively generated only by polynomials of the
same degree $d$ as $W_0(x)$. These polynomials are related to the complex structure deformations. The
subalgebra $R^0_d$ consists of elements of degrees 0, $d$, $2d$ and $3d$, and each homogeneous component of
this algebra is naturally a subgroup of the respective middle Hodge cohomology $H^{3,0}(X)$, $H^{2,1}(X)$,
$H^{1,2}(X)$ and $H^{0,3}(X)$. In our examples, $R^0_d$ is actually isomorphic to $H^3(X)$. The FM in which we
are interested is precisely the submanifold of the FM on the versal deformations of $W_0(X)$ restricted to the deformations by the elements of $R_0^\mathbb{C}$.

The relation to the FM structure allows obtaining the following explicit expression [3] for the Kähler potential in terms of the holomorphic FM metric $\eta_{\mu\nu}$ in the flat coordinates and periods of the holomorphic 3-form $\Omega$:

$$e^{-K} = \sigma^\phi_{\mu}(\phi)\eta^{\mu\nu}M^{\nu}_{\rho}\sigma^\phi_{\rho}(\phi), \quad M = T^{-1}T.$$  (1)

Here, $\phi$ is a parameter on the moduli space, and $\sigma^\phi_{\mu}(\phi)$ and $\eta_{\mu\nu}(\phi)$ are periods of the holomorphic 3-form $\Omega$, defined below as the integrals of $\Omega$ over some special basis of cycles $\gamma_{\mu}$ in the homology group $H_2(X)$, which are defined in terms of the connection with the FM.

The constant nondegenerate complex matrix $T$ connects the basis $\sigma^\phi_{\mu}(\phi)$ with another basis $\omega^\phi_{\mu}(\phi)$ defined in $\mathbb{R}[\mathbb{C}]$ as integrals of $\Omega$ over another special basis of the cycles $q_{\nu}$, by the linear relation

$$\omega^\phi_{\mu}(\phi) = T^\mu_{\nu}\sigma^\phi_{\nu}(\phi).$$  (2)

The explicit expression for the Kähler potential is obtained as follows. The basis $\omega_{\mu}$ is defined as integrals over the homology cycles $q_{\nu}$ with real coefficients. These cycles have a well-defined intersection matrix $C_{\mu\nu} = q_{\mu}\cap q_{\nu}$. The matrix $M$ and formula (1) are independent of the specific choice of $\omega_{\mu}$. The Kähler potential $K(\phi)$ on the moduli space with the coordinates $\phi$ and the holomorphic FM metric $\eta_{\mu\nu}$ can be expressed in the terms of $C_{\mu\nu}$, the periods $\omega_{\mu}$, and some additional periods $\omega_{\alpha\mu}(\phi)$ (also defined below) as

$$e^{-K(\phi)} = \omega_{\mu}(\phi)C^{\mu\nu}\omega_{\nu}(\phi),$$  (3)

and

$$\eta_{\alpha\beta} = \omega_{\alpha\mu}(0)C^{\mu\nu}\omega_{\beta\nu}(0).$$  (4)

Relation (4) was proved in [1, 2, 3, 4] (see [4, 10] and also [5] for the proof of relation (4)). Using these two formulas together with relation (2) between the periods $\omega^\phi_{\mu}(\phi)$ and $\sigma^\phi_{\mu}(\phi)$, we obtain formula (1) for the Kähler potential.

In the beginning of the paper, we briefly review our method [5] for computing the moduli space geometry and clarify the role of the invariant Milnor ring $R^\mathbb{C}$, which was not explicit in [3]. Then we demonstrate its efficiency by computing the complex structure geometry for a one-parameter family of quintic threefolds, which was previously found in [6] by a different approach. For the reader’s convenience, we present our construction for this simple example, which we studied with our method in [3]. Our intention is to illustrate the relation between the Milnor ring, the invariant Milnor ring, and the cohomology for the quintic and its quotient and to show its importance for a proper understanding of the method.

We then use our method to compute a new case of the complex structure geometry for a two-parameter family of CY threefolds of a non-Fermat type considered in [6].

2 Special geometry

We recall the basic facts about the special Kähler geometry and how it arises on the moduli space of complex structures on CY manifolds (see [1, 2, 3, 4]).

Let $\mathcal{M}$ be a Kähler $n$-dimensional manifold and $z^1\ldots z^{n+1}$ be a set of holomorphic projective coordinates on it. Then $\mathcal{M}$ is called a special Kähler manifold with special coordinates $z^i$ if there exists a holomorphic homogeneous function $F(z)$ of degree two in $z$, called a prepotential, such that the Kähler potential $K(z)$ of the moduli space metric is given by

$$e^{-K(z)} = z^a\frac{\partial F}{\partial z^a} - \bar{z}^\bar{a}\frac{\partial F}{\partial \bar{z}^{\bar{a}}}.$$  

Let $X$ be a CY threefold with complex coordinates $y^\mu$ ($\mu = 1, 2, 3$) and the holomorphic 3-form $\Omega$. The moduli space of complex structures is the space of perturbations of the metric on $X$ that preserve the Ricci flatness of the metric. These deformations are related to the harmonic forms $\chi_\alpha \in H^{2,1}(X)$ as

$$\delta_a g_{\alpha\beta} \rightarrow \chi_{a\mu\nu\beta} \sim \Omega_{\mu\nu\lambda} g^{\lambda\bar{\alpha}} \delta_a g_{\bar{\alpha}\bar{\beta}}.$$

The Weil–Peterson metric on the CY moduli space is

$$G_{ab} = \int_X d^3y^1/2 y^{1/2} g_{\mu\nu\lambda} g^{\mu\alpha} g^{\nu\beta} \delta_a g_{\mu\nu} \delta_b g_{\lambda\bar{\alpha}}.$$  


For such CY metric deformations, we obtain
\[ G_{ab} = \int_X \chi^a \wedge \bar{\chi}^b. \]

Here, \( a \) and \( b \) are indices of complex coordinates in the moduli space of the CY complex structure.

It follows from the Kodaira lemma [4],
\[ \partial_a \Omega = k_a \Omega + \chi_a, \]
that this is a Kähler metric:
\[ G_{ab} = -\partial_a \partial_b \ln \int_X \Omega \wedge \bar{\Omega}. \]

It is convenient to use the basis of periods that are integral s over the Poincaré dual symplectic basis \( A^a, B^b \in H_3(X, \mathbb{Z}) \):
\[ A^a \cap B^b = \delta^a_b, \quad A^a \cap A^b = 0, \quad B^a \cap B^b = 0. \]

By defining periods in the symplectic basis as
\[ z^a = \int_{A^a} \Omega, \quad F_b = \int_{B^b} \Omega, \]
we obtain
\[ e^{-K} = \int_X \Omega \wedge \bar{\Omega} = z^a \cdot \bar{F}_b - \bar{z}^b \cdot F_a. \]

It also follows from the Kodaira lemma that \( F_a(z) = \frac{1}{2} \partial_a F(z) \), where \( F_a(z) = \frac{1}{2} z^a F_a(z) \). Therefore,
\[ e^{-K(z)} = z^a \cdot \frac{\partial F}{\partial z^a} - \bar{z}^b \cdot \frac{\partial F}{\partial z^b}. \]

Hence, \( G_{ab} \) is the special Kähler metric with the prepotential \( F(z) \).

Using the notation \( \Pi = (F_a, z^b) \) for the vector of periods, we write the expression for the Kähler potential as
\[ e^{-K} = \Pi_a \Sigma^{ab} \Pi_b, \]
where the symplectic unit \( \Sigma \) is an inverse intersection matrix for the cycles \( (A^a, B_a) \). We can therefore transform this expression to form (3):
\[ e^{-K(\phi)} = \omega_\mu(\phi) C^{\mu \nu} \omega_\nu(\phi), \]
where \( \omega_\mu(\phi) \) are the integrals of \( \Omega \) over the arbitrary homology basis \( q^\mu \).

### 3 The CY manifold as a hypersurface in a weighted projective space

In what follows, we concentrate on the case where the CY manifold is realized as a hypersurface in a weighted projective space or its quotient by some discrete group. Let \( x_1, \ldots, x_5 \) be homogeneous coordinates in a weighted projective space \( \mathbb{P}^4(k_1, \ldots, k_5) \) and
\[ X = \{(x_1 : \ldots : x_5) \in \mathbb{P}^4(k_1, \ldots, k_5) | W_0(x) = 0\}. \]

Also let \( W_0(x) \) be a quasihomogeneous polynomial defining an isolated singularity at the origin,
\[ W_0(\lambda^{k_i} x_i) = \lambda^d W_0(x_i). \]

Then \( X \) is a CY manifold if
\[ \deg W_0(x) = d = \sum_{i=1}^5 k_i. \]

The moduli space of complex structures is then given by (a quotient of) the space of homogeneous polynomial deformations of this singularity modulo coordinate transformations:
\[ W(x, \phi) = W_0(x) + \phi_0 \prod x_i + \sum_{s=1}^{N-1} \phi^s e_s(x), \]
where $N$ is a number of polynomial deformations and $e_i(x)$ are polynomials of the same weight (degree) as $W_0(x)$. They are invariant under the $Z_d$ action $\alpha \cdot e_i(x) = e_i(\alpha^d x)$, $\alpha^d = 1$.

The holomorphic volume form $\Omega$ can be written as a residue of a 5-form in the underlying affine space $\mathbb{C}^5$:

$$\Omega = \frac{x_0 dx_1 \wedge dx_2 \wedge dx_3}{\partial W(x)/\partial x_4} = \text{Res}_{W(x)=0} \frac{x_0 dx_1 \cdots dx_5}{W(x)}.$$

Taking this explicit expression for $\Omega$, we obtain a basis of periods by analytic continuation, and is given by a residue as a series in $1/\phi$.

More periods $\omega_\mu$ can be obtained as analytic continuations of $\omega_1$ in $\phi$. This can be done by continuing $\omega_1(\phi)$ in a small $\phi_0$ region using Barnes’ trick and subsequently using the symmetry of $W_0(x)$.

There is a group of phase symmetries $\Pi_X$ defined as a group acting diagonally on $x_i$ and preserving $W_0(x)$. When $W_0(x)$ is deformed, this group acts on the parameter space with an action $A$ such that

$$W(g \cdot x, A(g) \cdot \phi_0) = W(x, \phi).$$

The moduli space is then at most a quotient $\{\phi^\mu\}_{0 \leq \mu \leq N-1}/A$ of the parameter space. We can thus define a set of periods by analytic continuation,

$$\omega_\mu(\phi) = \omega_1(A(g) \cdot \phi_0), \quad g \in G_X.$$

In examples, this construction gives all the periods of a volume form of $X$. We can also consider quotients $X/H$ where $H \subset \Pi_X$ is an admissible subgroup, which is a subgroup that preserves the volume form $\Omega$. In this case, we should consider only $H$-invariant deformations and corresponding quantities.

It is important that we can represent the periods $\int_{Q_\mu} \Omega$ as integrals of the complex oscillatory form. Starting from

$$\omega_\mu(\phi) := \int_{Q_\mu} \Omega = \int_{Q_\mu} \frac{d^5x}{W(x)},$$

we can rewrite them as

$$\int_{Q_\mu} \frac{d^5x}{W(x)} = \int_{Q_\mu} e^{\tau W(x)} \cdot d^5x, \quad (6)$$

where $Q_\mu^\sharp \subset H_5(\mathbb{C}^5 \setminus W(x) = 0) = H_2(\mathbb{C}^5, \text{Re} W_0(x) = \pm \infty)_{w \in d,z}$. Indeed, both sides of the equation satisfy the same differential equations which can be obtained from computations in the Milnor ring. Moreover, there is the isomorphism $\Phi$

$$H_3(X) \to H_3(\mathbb{C}^5 \setminus W(x) = 0) = H_2(\mathbb{C}^5, \text{Re} W_0(x) = \pm \infty)_{w \in d,z}.$$

Therefore, more precisely, we have $Q_\mu^\sharp \subset H_3(\mathbb{C}^5, \text{Re} W_0(x) = \pm \infty)_{w \in d,z}$, which is a subgroup of $H_3(\mathbb{C}^5, \text{Re} W_0(x) = \pm \infty)$ defined below.
4 Frobenius manifold

It is known that compactification of the superstring theory on a CY manifold is deeply connected with the compactification on the $N=2$ superconformal theories [14]. When the latter are Landau–Ginzburg theories, they are connected with singularity theory [13, 16, 17].

Using this connection, we can extract the information about the special geometry of the CY moduli space from the Milnor ring of the defining singularity $W_0(X)$ [11, 12]:

$$ R_0 = \frac{C[x_1, \ldots, x_5]}{\partial_1 W_0(x) \cdot \cdots \cdot \partial_5 W_0(x)}. $$

In fact, we need to consider not the whole Milnor ring but its $Q$-invariant subring $R^Q_0$. This subring is multiplicatively generated by marginal deformations of the singularity, i.e., by those deformations that have the same weight $d$ as $W_0(x)$ and correspond to the complex structure moduli of CY. This subring consists of the elements of the Milnor ring $R_0$ whose weights are integer multiples of $d$, that is, $0, d, 2d, 3d$. We note that in many cases, the dimension of the subring $R^Q_0$ is equal to the dimension of the homology group $H^3(X)$, where $X$ is the CY defined using the polynomial $W_0(x)$ in $\mathbb{C}$. But in general, $\dim R^Q_0 \leq \dim H^3(X)$. In the case of the strict inequality, we restrict our attention to the subspace of $H^3(X)$ that is isomorphic to $R^Q_0$ without noting this explicitly.

We let $e_{\mu}(x)$ (with Latin indices $\mu$) denote the elements that correspond to the complex structure deformations of $X$ and $e_{\nu}(x)$ (with Greek indices $\nu$) denote all elements of the basis of $R^Q_0$. In the example where $X$ is a quintic threefold, we have the dimension $\dim R_0 = 1024$ of the whole Milnor ring, the dimension $\dim R^Q_0 = \dim H^3(X) = 204$ of the $Q$-invariant subring, and the dimension $\dim M_1 = 101$ of the subspace of marginal deformations.

There exists a natural multiplication with structure constants $C^\mu_{\nu\lambda}$ in $R^Q_0$ and a pairing $\eta_{\mu\nu}$ that makes $R^Q_0$ a Frobenius algebra:

$$ \eta_{\mu\nu} = \text{Res} \frac{e_{\mu} \cdot e_{\nu}}{\partial_1 W_0(x) \cdots \partial_5 W_0(x)}, $$

$$ C^\mu_{\nu\lambda} = \text{Res} \frac{e_{\mu} \cdot e_{\nu} \cdot e_{\lambda}}{\partial_1 W_0(x) \cdots \partial_5 W_0(x)}. $$

We consider the space of deformations of the singularity

$$ W(x) = W_0(x) + \sum t^\mu e_{\mu}(x), $$

where $e_{\mu}(x)$ belongs to the $Q$-invariant subring $R^Q_0$ of the Milnor ring. The structure of the FM $M_F$ with multiplication structure constants $C^\mu_{\nu\lambda}(t)$ in the ring $R^Q$ defined by the deformed singularity $W(x)$ arises on the space with the parameters $t^\mu$:

$$ R = \frac{C[x_1, \ldots, x_5]}{\partial_1 W(x) \cdots \partial_5 W(x)}, \quad R^Q \text{ is a } \mathbb{Z}_d \text{ invariant subset of } R. $$

It has a Riemannian flat metric $h_{\mu\nu}(t)$ and the metric $h_{\mu\nu}(t = 0)$ equal to $\eta_{\mu\nu}$. The structure constants are the third derivatives of the Frobenius potential $F(t)$. This Frobenius potential coincides with the holomorphic prepotential of the special Kähler geometry when restricted to the marginal subspace. They are naturally the Yukawa couplings of the corresponding fields [3]. We note that the FM in question is naturally a complex manifold and the metric $h_{\mu\nu}(t)$ and structure constants $C^\mu_{\nu\lambda}(t)$ are holomorphic.

We note that the small FM $M_F$ is a submanifold of the total FM arising from the whole Milnor ring $R_0$. The manifold $M_F$ is connected with the cohomology of $X$ and is used in our approach.

Only quasihomogeneous deformations $W(x)$ define a hypersurface in a weighted projective space. The marginal deformations $W_0(x) + \sum \phi^e e_{\nu}(x)$ define a subspace of the FM connected with $W_0$. This subspace $M_t$ of the FM coincides with the moduli space of the CY manifold (at least locally and maybe after some orbifolding). This fact is very important for computing the special geometry: it allows expressing the matrix $C^\mu_{\nu\lambda}$ in terms of the FM metric $\eta_{\mu\nu}$ [3].

5 The idea for computing the periods

We can now relate the oscillatory form of the period integrals [6] to the FM structure and to the holomorphic metric $\eta_{\mu\nu}$.
We consider the differentials $D^+$ and $D^-$ given by

$$D^\pm = D^\pm_{W_0} = d \pm dW_0 \wedge .$$

They define cohomology subgroups $H^{p,\pm}_{D^\pm}(C^5)_{w \in dZ}$ on the space of differential forms of the weight $d \cdot Z$. These subgroups are isomorphic as linear spaces to the ring $R^p$ with $e_\mu(x) \to e_\mu(x)d^5x$.

Moreover, $R^p$ acts naturally on $H^{p,\pm}_{D^\pm}(C^5)_{w \in dZ}$ by multiplication (in terms of representatives). The cohomology subgroups $H^{p,\pm}_{D^\pm}(C^5)_{w \in dZ}$ are dual to the homology subgroups $H_{\partial}^p(C^5), \text{Re} W_0(x) = \pm \infty)$ with a nondegenerate pairing with $e_\nu(x)d^5x \in H^p_{D^\pm}(C^5)_{w \in dZ}$ defined as

$$\langle \Gamma^\pm_\mu, e_\nu d^5x \rangle = \int_{\Gamma^\pm_\mu} e_\nu \cdot e^{\mp W_0(x)}d^5x.$$

The $H^p_{D^\pm}(C^5), \text{Re} W_0(x) = \pm \infty)_{w \in dZ}$ are invariant under $x_i \to e^{2\pi i k_i/d} x_i$. We have a group isomorphism $H^p_{D^\pm}(C^5), \text{Re} W_0(x) = \pm \infty)_{w \in dZ} = H^p_{\partial}(X)$ and therefore $R^p \cong H^p_{\partial}(X)$. This isomorphism maps the weight filtration in the left-hand side to the Hodge filtration in the right-hand side, i.e., cycles of weight $k \cdot d$ correspond to differential forms in $H^{p,d}(X)$.

A possible choice of cycles $\Gamma^\pm_\mu$ is

$$\int_{\Gamma^\pm_\mu} e_\nu \cdot e^{\mp W_0(x)}d^5x = \delta^\nu_\mu.$$ 

A convenient computation technique can be used to find the periods represented as the oscillatory integrals

$$\int_{\Gamma^\pm_\mu} e_\nu \cdot e^{\mp W_0(x)}d^5x,$$

where $W(x, \phi) = W_0(x) + \sum_{\nu} \phi^\nu e_\nu(x)$. Expanding the integrand in a series in $\phi^\nu$, we obtain integrals of the type $\int_{\Gamma^\pm_\mu} P(x)e^{-W_0(x)}d^5x$. Here, $P(x)$ is a product of $e_\nu(x)$ and is hence a $Q$-invariant monomial.

The technique for computing such integrals, previously used to compute the flat coordinates in the topological CFT [19, 20, 13, 21], is based on the fact that

$$\int_{\Gamma^\pm_\mu} P(x)e^{\mp W_0(x)}d^5x = \int_{\Gamma^\mp_\nu} \tilde{P}(x)e^{\mp W_0(x)}d^5x$$

if the forms in the integrands are equivalent in the $D^\pm$ cohomology,

$$P(x)d^5x - \tilde{P}(x)d^5x = D^\pm U.$$ 

Using this, we can easily see that an arbitrary $Q$-invariant form $P(x)d^5x$ is reducible to a linear combination of $e_\nu(x)d^5x$, where the $e_\nu(x)$ form the basis of $R^\nu$.

Computing integrals of the type $\int_{\Gamma^\pm_\mu} P(x)e^{-W_0(x)}d^5x$ thus becomes a linear problem of expanding $P(x)d^5x$ over the basis of $H^p_{D^\pm}(C^5)_{w \in dZ}$.

6 Finding $C^\mu\nu$ and the Kähler potential

We use the relation of the CY moduli space to the FM structure to find the intersection matrix of the cycles $C^\mu\nu = q_\mu \cap q_\nu = Q^\mu_+ \cap Q^-_\nu$. For this, we introduce a few new sets of periods $\omega^\pm_{\mu\nu}(\phi)$ as integrals of $e_\nu(x)d^5x \in H^p_{D^\pm}(C^5), \text{Re} W_0(x) = \pm \infty)_{w \in dZ}$ over the cycles $Q^\pm_\nu \in H^p_\partial(C^5), \text{Re} W_0(x) = \pm \infty)_{w \in dZ}$ defined above:

$$\omega^\pm_{\mu\nu}(\phi) = \int_{Q^\pm_\nu} e_\nu(x) e^{\mp W(x, \phi)}d^5x.$$

The periods $\omega^+_{\mu\nu}(\phi)$ coincide with the periods $\omega^-_{\mu\nu}(\phi)$ because we assume that $e_1(x) = 1$ denotes the unity in the ring $R$.

The additional periods allow computing $C^\mu\nu$ because of its relation to the FM metric $\eta_{\alpha\beta}$ [3, 10]:

$$\eta_{\alpha\beta} = \omega^+_{\alpha\beta}(t = 0)C^\mu\nu \omega^-_{\mu\nu}(t = 0).$$

6
From this formula, we can obtain the expression for $C_{\mu \nu}$ if we know the values of $\omega_{\alpha \mu}^\pm (t = 0)$ for all $\alpha$. As follows from the definition of $\omega_{\alpha \mu}^\pm (\phi)$, we can express $\omega_{\alpha \mu}^\pm (\phi = 0)$ in terms of the derivatives of the periods $\omega_{\mu}^\pm (\phi)$ with respect to $\phi$ up to the third order at $\phi = 0$ because the basis elements $e_\alpha(x)$ of the invariant subring $R_0^Q$ of the Milnor ring can be chosen as products of the marginal deformations $e_\alpha(x)$, which are related by

$$e_\alpha(x) \cdot e^{W(x)} d^5 x = \frac{\partial}{\partial \phi^\alpha} e^{W(x)} d^5 x.$$ 

Setting $\omega_{\alpha \mu}^\pm (\phi = 0) := (T^\pm)^\mu\alpha$, we rewrite the above relation as

$$\eta^{\mu \nu} = (T^\pm)^\mu \rho \sigma_{\rho \nu},$$

which gives the expression for the intersection matrix $C_{\mu \nu}$ in terms of $\eta^{\mu \nu}$ and the matrix $T$. The result can be substituted in the Kähler potential formula

$$e^{-K(\phi)} = \omega_\mu(\phi) C^{\mu \nu} \omega_\nu(\phi)$$

to obtain the explicit expression for $K(\phi)$.

To obtain a more convenient expression for $K(\phi)$, we define one more basis of periods $\sigma_{\alpha}^\pm (\phi)$ as integrals over the cycles $\Gamma_\mu^\alpha \subset H_2(\mathbb{C}^3, \mathbb{R})$, $\mathcal{R} W_0(x) = \pm \infty \in \mathbb{Z}$ defined above:

$$\sigma_{\mu}^\pm (\phi) = \int_{\Gamma_\mu^\alpha} e^{\pm W(x, \phi)} d^5 x.$$  

This basis comprises eigenvectors of the phase symmetry action $A : \Pi_X \times \{ \phi^\alpha \} \to \{ \phi^\alpha \}$. Having the oscillatory representation for the periods $\sigma_{\alpha}^\pm (\phi)$ over the corresponding cycles $\Gamma_\mu^\alpha$, we can define additional integrals $\sigma_{\alpha \mu}^\pm (\phi)$ over the same cycles as

$$\sigma_{\alpha \mu}^\pm (\phi) = \int_{\Gamma_\mu^\alpha} e_\alpha(x) e^{\pm W(x, \phi)} d^5 x.$$  

It follows from $e_1(x) = 1$ that $\sigma_{1 \mu}^\pm = \sigma_{\mu}^\pm$. Because of our choice of the cycles $\Gamma_\mu^\alpha$, we also have $\sigma_{\alpha \mu}^\pm (t = 0) = \delta_{\alpha \mu}$. The last formula together with $\sigma_{\alpha \mu}(0) = \delta_{\mu \nu}$ (which is by definition (9)) implies that $(T^\pm)^\alpha \nu = \omega_{\alpha \nu}^\pm (\phi = 0)$.

Having computed the matrix $T_{\nu \mu}^\pm$, we use (9) to express the intersection matrix $C_{\mu \nu}$ in terms of this matrix and the known Frobenius metric $\eta^{\mu \nu}$.

We thus obtain the main statement that

$$e^{-K(\phi)} = \omega_\mu(\phi) \eta^{\mu \nu} M_{\nu \alpha}^\lambda \sigma_{\lambda}^- (\phi),$$

where the matrix $M_{\nu \mu}^\lambda = (T^{-1})_{\nu \mu}^\lambda T_{\nu}^\mu$. This expresses the Kähler potential $K$ explicitly in terms of the periods $\sigma_\alpha(\phi)$, the FM metric $\eta_{\nu \mu}$, and the matrix $T_{\nu \mu}$. All these data can be computed exactly, as explained above.

We also note two points. First, we can also find a symplectic basis of cycles by applying the Gram–Schmidt process to the obtained intersection matrix. Second, formula (10) can be used without explicitly computing $T$ if the real structure matrix $M$ can be found from some other argument.
7 Example: Quintic threefold

We consider the one-parameter family of CY manifolds defined as

\[ X_\psi = \{ x_i \in \mathbb{P}^4 \mid W_\psi(x) = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5x_1x_2x_3x_4x_5 = 0 \}, \]

which was considered in detail in [8]. The phase symmetry in this case is \( \Pi_X = \mathbb{Z}_5^5 \). The full Milnor ring \( R_0 \) is 1024-dimensional and consists of all polynomials in five variables where each of them has the degree less than four.

As explained above, we need not the whole Milnor ring but only its \( \mathbb{Z}_5 \)-invariant subring, which has the dimension \( \dim R_0^\mathbb{Z}_5 = 204 \) and is isomorphic to \( H_3(X) \). It comprises the polynomials in \( R_0 \) of degrees 0, 5, 10, and 15. The 101 fifth-degree polynomials (marginal deformations) correspond to the complex structure moduli.

To build the one-dimensional family \( \{X\} \), we take a subgroup \( H = \mathbb{Z}_5 \subset \Pi_X \) of phase symmetries and consider \( X = X/H \), which turns out to be a mirror of \( X \). Family \( \{X\} \) is the maximal deformation surviving this factorization. The induced action \( \Lambda \) of the phase symmetry group on the one-dimensional space \( \{\psi\} \) is \( \mathbb{Z}_5 : \psi \to e^{2\pi i/5} \psi \). The invariant Milnor ring is four-dimensional: \( R_0^\mathbb{Z}_5 = \{1, \prod x_i, \prod x_i^2, \prod x_i^3\} \).

Having the invariant Milnor ring, we define the corresponding cohomology group and dual cycles \( \Gamma_5^\pm \). Using the recursion procedure for \([8]\), we obtain

\[
\sigma_\mu^5(\psi) = \frac{(\pm 1)^{\mu - 1}}{\Gamma(\mu/5)} \sum_{n=0}^{\infty} \frac{\Gamma^5(n + \mu/5)}{\Gamma(5n + \mu)} (5\psi)^{5n+\mu} = \frac{(\pm \psi)^{\mu - 1}}{\Gamma(\mu)} + O(\psi^{\mu+3}).
\]

The fundamental period for the quintic is a residue of a holomorphic 3-form \( \Omega \),

\[
x_{3d}dx_1 \wedge dx_2 \wedge dx_3 = \frac{\partial \mathcal{P}_\psi}{\partial \mathcal{P}_x}.
\]

defined as an integral over a cycle \( q_1 \). Its analytic continuations give the whole basis of periods in terms of integrals over a basis of cycles \( \in H_3(X) \):

\[
\omega_\mu(\psi) = \sum_{m=1}^{\infty} e^{\frac{4\pi im}{5}} \frac{\Gamma(m/5)(\sigma_5 e^{2\pi i(\mu-1)/5} \psi)^m - 1}{\Gamma(5m/5)}. \quad |\psi| < 1.
\]

The matrix \( T_\mu^\alpha \) is given by

\[
T_\mu^\alpha = \omega_\alpha \mu(0) = \frac{\partial^{\alpha - 1}}{\partial \psi^{\alpha - 1}} \omega_\mu(0) = \frac{\Gamma(\alpha/5)}{\Gamma(1 - \alpha/5)} 5^{\alpha - 1} e^{2\pi i((\alpha - 1)(\mu - 1) + 2\alpha)/5} \Gamma(\alpha/5).
\]

The holomorphic metric \( \eta_{\mu\nu} \) on the corresponding CM is just the coefficient of the element of maximum degree in the decomposition of \( e_\mu(x) \cdot e_\nu(x) \) in the monomial basis of \( R_0^\mathbb{Z}_5 \). In our case, it is

\[
\eta = \text{antidiag}(1,1,1,1).
\]

Finally, we obtain \( \hat{\eta} = \eta T^{-1} \hat{T} \) and the Kähler potential for the metric

\[
\begin{align*}
\bar{e}^{-K(\psi)} &= \frac{\Gamma^5(1/5)}{125 \Gamma^6(4/5)} \sigma_{11}^5 \overline{\sigma_{11}}^5 + \frac{\Gamma^5(2/5)}{5 \Gamma^6(3/5)} \sigma_{12}^5 \overline{\sigma_{12}}^5 \\
&+ \frac{5 \Gamma^5(3/5)}{\Gamma^6(2/5)} \sigma_{13}^5 \overline{\sigma_{13}}^5 + \frac{125 \Gamma^5(4/5)}{\Gamma^6(1/5)} \sigma_{14}^5 \overline{\sigma_{14}}^5.
\end{align*}
\]

In particular,

\[
G_\psi(0) = 25 \frac{\Gamma^5(4/5) \Gamma^5(2/5)}{\Gamma^6(1/5) \Gamma^6(3/5)},
\]

which coincides with result in [8].

8 Two-moduli non-Fermat threefold

The two-moduli non-Fermat threefold is constructed from the hypersurface

\[
X = \{ x_i \in \mathbb{P}^4_{(3,2,2,7,7)} \mid W_\psi(x) = x_0^7 + x_1^7 + x_2^3 + x_3^3 + x_4^3 + x_5^3 - 5x_1x_2x_3x_4x_5 = 0 \}.
\]
This example, which was considered in [4], is interesting because it is not of the Fermat type. The weight of the singularity is equal to $d = 21$. The phase symmetry is $Z_{21}^{2} \times Z^{7}$. We again consider a quotient $X = X/H$ by the $H = Z_{21}$ action

$$H := (Z_{21} : 12, 2, 0, 7, 0)$$

The Hodge numbers are $h_{1,1}(\hat{X}) = 95$ and $h_{2,1}(\hat{X}) = 2$. The two-parameter family (12) is the maximum deformation surviving the factorization. The induced action $A$ on the two-dimensional space $\{\phi_{0}, \phi_{1}\}$ is $Z^{2}: \phi_{0} \rightarrow \alpha \phi_{0}$, $\phi_{1} \rightarrow \alpha^{3} \phi_{1}$, where $\alpha^{7} = 1$ is a primitive root. We note that 7 is a weight of the mirror singularity, as explained in [4].

Analytic continuations of the fundamental period give the full basis of periods in a basis of cycles with integral coefficients,

$$\omega_{\nu}(\psi) = \frac{1}{2} \sum_{n=1}^{\infty} e^{\frac{\pi n}{7}} \left( \frac{\alpha^{n-1} \phi_{1}}{\Gamma(n)} \right) \sum_{m=0}^{\infty} \frac{e^{-3i\pi m/7} \Gamma \left( \frac{n+3m}{7} \right)}{\Gamma^{2} \left( 1 - \frac{n+3m}{7} \right) \Gamma^{3} \left( 1 - \frac{2n-3m}{7} \right)} \frac{(\alpha^{n-1} \phi_{1})^{m}}{m!},$$

where $|\phi_{0}|, |\phi_{1}| \ll 1$.

We now perform the Milnor ring computations to obtain the metric $\eta$. If we set

$$e_{2}(x) = x_{0}x_{1}x_{2}x_{3}x_{4}, \quad e_{3}(x) = x_{0}^{3}x_{1}^{3}x_{2},$$

then the $H$-invariant subring of $\mathbf{R}_{Q}^{Q}$ is generated by $e_{2}$ and $e_{3}$. It is easy to compute the relations

$$e_{3}^{2} = 0, \quad e_{2}^{3} = 0.$$

Hence, the vector space basis of this subring is

$$e_{1}, e_{2}, e_{3}, e_{4} = e_{2}^{2}, e_{5} = e_{2}e_{3}, e_{6} = e_{3}e_{3}.$$

The last one is of the highest degree 63, and the metric in this basis is therefore $\eta \simeq \mathrm{antidiag}(1, 1, 1, 1, 1, 1)$. Taking the first four terms of the expansion of the above periods, we obtain

$$T_{\nu} = A(\nu)x^{k_{\nu}(\nu-1)}, \quad k_{\nu} = (1, 2, 4, 3, 5, 6),$$

and

$$A(\nu) = \alpha^{2m_{\nu}-n_{\nu}/2} \frac{(-1)^{n_{\nu}-1} \Gamma \left( \frac{n_{\nu}+3m_{\nu}}{7} \right)}{\Gamma^{2} \left( 1 - \frac{n_{\nu}+3m_{\nu}}{7} \right) \Gamma^{3} \left( 1 - \frac{2n_{\nu}+3m_{\nu}}{7} \right)},$$

where $(n_{\nu}, m_{\nu}) = ((1, 0), (2, 0), (1, 1), (3, 0), (2, 1), (3, 1))$ correspond to our choice of the basis.

Finally, we obtain $\tilde{\eta} = \eta T^{-1} \tilde{T}$ and the Kähler potential for the metric

$$e^{-K(\psi)} = \gamma_{(1/7)}^{3} \gamma_{(2/7)}^{3} \sigma_{1}^{\frac{1}{3}} \sigma_{1}^{\frac{1}{3}} + \gamma_{(2/7)}^{3} \sigma_{1}^{\frac{1}{3}} \sigma_{1}^{\frac{1}{3}} + \gamma_{(4/7)}^{3} \sigma_{2}^{\frac{1}{3}} \sigma_{2}^{\frac{1}{3}} + \gamma_{(4/7)}^{3} \sigma_{2}^{\frac{1}{3}} \sigma_{2}^{\frac{1}{3}} + \gamma_{(5/7)}^{3} \sigma_{3}^{\frac{1}{3}} \sigma_{3}^{\frac{1}{3}} + \gamma_{(5/7)}^{3} \sigma_{3}^{\frac{1}{3}} \sigma_{3}^{\frac{1}{3}},$$

where $\gamma(x) = \Gamma(x)/\Gamma(1 - x)$.

In particular, the Kähler metric has the form

$$G(0) = \begin{pmatrix} \gamma_{(3/7)}^{3} & \gamma_{(4/7)}^{3} & 0 \\ \gamma_{(4/7)}^{3} & \gamma_{(5/7)}^{3} & \gamma_{(5/7)}^{3} \\ 0 & \gamma_{(5/7)}^{3} & \gamma_{(5/7)}^{3} \end{pmatrix}.$$

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