EXACT VALUE FOR SUBGAUSSIAN NORM OF CENTERED INDICATOR RANDOM VARIABLE

EUGENE OSTROVSKY, LEONID SIROTA

Bar-Ilan University, 59200, Ramat Gan, ISRAEL;
e-mail: eugostrovsky@list.ru
e-mail: sirota3@bezeqint.net

Abstract

We calculate the exact subgaussian norm of a centered (shifted) indicator (Bernoulli’s) random variable.

Using this result we derive very simple tail estimates for sums of these variables, not necessary to be identical distributed, and give some examples to show the exactness of our estimates.

Key words and phrases: Random variables (r.v.), unimodality, centering, indicator and Bernoulli’s r.v., Grand Lebesgue Spaces (GLS), subgaussian norm, tail inequalities, independence.

1 Introduction. Notations. Statement of problem

Let \{Ω, B, P\} be some non-trivial probability space. We say that the centered: \(Eξ = 0\) numerical random variable (r.v.) \(ξ = ξ(ω), ω ∈ Ω\) is subgaussian, or equally, belongs to the space \(\text{Sub}(Ω)\), if there exists some non-negative constant \(τ ≥ 0\) such that

\[∀λ ∈ R ⇒ E \exp(λξ) ≤ \exp[λ^2 τ^2].\]  

(1.1)

The minimal value \(τ\) satisfying (1.1) is called a subgaussian norm of the variable \(ξ\), write

\[||ξ||\text{Sub} = \inf\{τ, τ > 0 : ∀λ ∈ R ⇒ E \exp(λξ) ≤ \exp(λ^2 τ^2)\}.\]

Evidently,

\[||ξ||\text{Sub} = \sup_{λ ≠ 0} \left[ \sqrt{\ln E \exp(λξ)/|λ|} \right].\]  

(1.2)

This important notion was introduced by J.P.Kahane [6]; V.V.Buldygin and Yu.V.Kozatchenko [3] proved that the set \(\text{Sub}(Ω)\) relative the norm \(||·||\) is complete Banach space which is isomorphic to subspace consisting only from the centered variables of Orlicz’s space over \((Ω, B, P)\) with \(N – \text{Orlicz-Young function} N(u) = \exp(u^2) − 1\) [9].
If $||\xi|| \text{ Sub} = \tau \in (0, \infty)$, then

$$\max[P(\xi > x), P(\xi < -x)] \leq \exp(-x^2/(4\tau^2)), \quad x \geq 0;$$

and the last inequality is in general case non-improvable. It is sufficient for this to consider the case when the r.v. $\xi$ has the centered Gaussian non-degenerate distribution.

Conversely, if $E\xi = 0$ and if for some positive finite constant $K$

$$\max[P(\xi > x), P(\xi < -x)] \leq \exp(-x^2/K^2), \quad x \geq 0,$$

then $\xi \in \text{Sub}(\Omega)$ and $||\xi|| \text{ Sub} < 4K$.

The subgaussian norm in the subspace of the centered r.v. is equivalent to the following Grand Lebesgue Space (GLS) norm:

$$|||\xi||| := \sup_{s \geq 1} \left[ \frac{|\xi|_s}{\sqrt{s}} \right], \quad |\xi|_s = \left[ E|\xi|^s \right]^{1/s}.$$

For the non-centered r.v. $\xi$ the subgaussian norm may be defined as follows:

$$||\xi|| \text{ Sub} := \left( |||\xi - E\xi||| \text{ Sub} \right)^2 + (E\xi)^2 \right)^{1/2}.$$

More detailed investigation of these spaces see in the monograph [10], chapter 1.

We denote as usually by $I(A) = I(A; \omega)$, $\omega \in \Omega$, $A \in B$ the indicator function of event $A$. Further, let $p$ be arbitrary number from the set $[0, 1] : 0 \leq p \leq 1$ and let $A(p)$ be any event such that $P(A(p)) = p$. Denote also $\eta_p = I(A(p)) - p$; the centering of the r.v. $I(A(p))$; then $E\eta_p = 0$ and $P(\eta_p = 1 - p) = p; \quad P(\eta_p = -p) = 1 - p.$

Our aim in this short report is to compute the exact value of the subgaussian norm for the random variable $\eta_p$.

We apply the obtained estimate in the third section to the tail computation for sums of independent indicators.

Applications of these estimates in the non-parametrical statistics may be found in the articles [5], [8]. Another application is described in [4].

## 2 Main result

Define the following non-negative continuous on the closed segment $p \in [0, 1]$ function

$$Q(p) = \sqrt{\frac{1 - 2p}{4\ln((1 - p)/p)}},$$

so that $Q(0 + 0) = Q(1 - 0) = 0$ and $Q^2(1/2) = 1/8$ (Hospital’s rule). Note also
The last circumstance play a very important role in the non-parametrical statistics, see [5], [8].

**Theorem 2.1.**

\[ ||\eta_p||_{\text{Sub}} = Q(p). \]  \hspace{1cm} (2.2)

**Proof.** Note that

\[ E e^{\lambda \eta_p} = p e^{\lambda(1-p)} + (1-p)e^{-p\lambda}, \lambda \in (-\infty, \infty). \]

The inequality

\[ p e^{\lambda(1-p)} + (1-p)e^{-p\lambda} \leq e^{Q^2(p)} \lambda^2 \]  \hspace{1cm} (2.3)

is proved in [7]; see also [2]. Therefore

\[ E e^{\lambda \eta_p} \leq e^{Q^2(p)} \lambda^2. \]

This imply by direct definition of the subgaussian norm that \( ||\eta||_{\text{Sub}} \leq Q(p), \ p \in [0,1]. \)

Let us prove the inverse inequality. Suppose \( p \in (0,1) \); the extremal cases \( p = 0, \ p = 1 \) are trivial.

We denote following the authors of articles [2], [13]

\[ \lambda_0 = \lambda_0(p) = 2 \log \left[ \frac{1-p}{p} \right] \]  \hspace{1cm} (2.4)

and deduce after simple calculations

\[ ||\eta_p||_{\text{Sub}} = \sup_{\lambda \neq 0} \left[ \sqrt{\ln E \exp(\lambda \eta_p)}/|\lambda| \right] = \]

\[ \sup_{\lambda \neq 0} \left[ \sqrt{\ln \{pe^{\lambda(1-p)} + (1-p)e^{-\lambda p}\}}/|\lambda| \right] \geq \]

\[ \sqrt{\ln \{pe^{\lambda_0(1-p)} + (1-p)e^{-\lambda_0 p}\}}/|\lambda_0| = Q(p), \]  \hspace{1cm} (2.5)

Q.E.D.

**Remark 2.1.** Let us explain the choice of the value \( \lambda_0 = \lambda_0(p), \ 0 < p < 1. \) In accordance to the equality (1.2) the optimal value of the parameter \( \lambda \) is following: \( \lambda = \Lambda(p), \) where

\[ \Lambda(p) = \arg\max_{\lambda \neq 0} \left[ \sqrt{\ln E \exp(\lambda \eta_p)}/|\lambda| \right] \]

or equally
\[ \Lambda(p) = \arg\max_{\lambda \neq 0} \left\{ \frac{\ln(pe^{(1-p)\lambda} + (1-p)e^{-\lambda p})}{\lambda^2} \right\}. \]  

(2.6)

Denote following the authors of articles [2], [7], [13]

\[ g(\lambda) = g_p(\lambda) = \frac{\ln(pe^{(1-p)\lambda} + (1-p)e^{-\lambda p})}{\lambda^2}. \]

It is easy to verify that \( g'_\lambda(\lambda_0) = 0. \) It is also proved in the article [13] that the function \( \lambda \to g_p(\lambda) \) is unimodal. Therefore, we derive taking into account the behavior of the function \( g_p(\lambda) \) at \( \lambda \to \pm\infty \) that \( \Lambda(p) = \lambda_0. \)

**Consequence 2.1.** Let \( \nu : \Omega \to R \) be a centered stepwise (simple) r.v. (measurable function):

\[ \nu = \sum_{j=1}^{m} c(j)[I(A(p(j))) - p(j)], \ m = \text{const} \leq \infty, \ c(j) = \text{const}, \]  

(2.7)

and \( \{A(p(j))\} \) are events not necessary to be disjoint or independent. We conclude using triangle inequality for the subgaussian norm and the completeness of the space \( \text{Sub}(\Omega) \) in the case when \( m = \infty : \)

\[ ||\nu||_{\text{Sub}} \leq \sum_{j=1}^{m} |c(j)|Q(p(j)). \]  

(2.8)

3 Tail estimations for sums of independent indicators

Let \( p(i), \ i = 1, 2, \ldots, n \) be positive numbers such that \( 0 < p(i) < 1, \) and let \( A(i) \) be independent events for which \( \mathbf{P}(A(i)) = p(i) \). Introduce a sequence of two - values independent random variables \( \zeta(i) = I(A(i)) - p(i), \) and define its sum

\[ S(n) := \sum_{i=1}^{n} \zeta(i). \]  

(3.1)

It is known [10], chapter 1, section 1.6 that

\[ || \sum_{i=1}^{n} \zeta(i) ||_{\text{Sub}} \leq \sqrt{\sum_{i=1}^{n} (||\zeta(i)||_{\text{Sub}})^2}. \]  

(3.2)

Therefore

\[ || \sum_{i=1}^{n} \zeta(i) ||_{\text{Sub}} \leq W(n), \]  

(3.3)

where

\[ W(n) \overset{\text{def}}{=} \sqrt{\sum_{i=1}^{n} (Q(p(i)))^2}. \]  

(3.4)
As a consequence:

**Proposition 3.1.**

\[
\max\{\mathbb{P}(S(n) > x), \mathbb{P}(S(n) < -x)\} \leq \exp\left(-\frac{x^2}{4W^2(n)}\right), \ x \geq 0. \tag{3.5}
\]

**Example 3.1.** Assume in addition that \(p(i) = p = \text{const} \in (0, 1)\); then the r.v. \(S(n) + np\) has a (non-degenerate) Bernoulli distribution. We deduce using inequalities (3.3) and (3.5):

\[
\sup_{n} ||S(n)/\sqrt{n}|| \text{Sub} \leq Q(p). \tag{3.6}
\]

As long as

\[
\sup_{n} ||S(n)/\sqrt{n}|| \text{Sub} \geq ||S(1)|| \text{Sub} = Q(p), \tag{3.6a}
\]

we get:

\[
\sup_{n} ||S(n)/\sqrt{n}|| \text{Sub} = Q(p). \tag{3.7}
\]

Thus, the estimate (3.3) is non-improvable; cf. [1], [11], [14], [15].

**Example 3.2.** Suppose in addition to the previous example 3.1 that \(p = 1/2\) (symmetrical case); then it follows from Proposition (3.1) alike the famous Hoeffding’s inequality

\[
\mathbb{P}(2S(n)/\sqrt{n} > x) \leq e^{-x^2/2}, \ x > 0,
\]

while

\[
\sup_{n} \mathbb{P}(2S(n)/\sqrt{n} > x) \geq \lim_{n \to \infty} \mathbb{P}(2S(n)/\sqrt{n} > x) \geq Cx^{-1} e^{-x^2/2}, \ x \geq 1.
\]

Another generalizations of the equality (2.2), for example, on the Hoeffding’s inequality and on the theory of martingales see in the article of M.Raginsky and I.Sason [12].

**References**

[1] **BENTKUS V.** *On Hoeffdings inequalities.* The Annals of Probability 32(2), 1650 – 1673, (2004).

[2] **BEREND D. AND KONTOROVICH A.** *On the concentration of the missing mass.* Electon. Commun. Probab., 18(3):17, 2013.
Buldygin V.V., Kozatchenko Yu.V. About subgaussian random variables. Ukrainian Math. Journal, 1980, 32, N° 6, 723 - 730.

S. X. Chen and J. S. Liu. Statistical applications of the Poisson-binomial and conditional Bernoulli distributions. Statist. Sinica, 7(4):875892, 1997.

Gaivoronsky E.I., Ostrovsky E.I. Non-asymptotical estimate of deviation of multidimensional function of distribution. Theory Probab. Applications, 1991, 36, Issue 3, 111 - 115.

Kahane J.P. Properties locales des fonctions a series de Fourier aleatoires. Studia Math. (1960), 19, N° 1, 1 - 25.

Kearns M. and Saul L. Large deviation methods for approximate probabilistic inference. In Proceedings of the Fourteenth conference on Uncertainty in artificial intelligence, pages 311 – 319. Morgan Kaufmann Publishers Inc., 1998.

Kiefer J. On large Deviations of the Empiric D.F. of vector chance variables and a Law of Iterated Logarithm. Pacific J.Math., 1961, 11, N° 2, 649 - 660.

Kozatchenko Yu.V., Ostrovsky E.I. Banach spaces of random variables of subgaussian type. Theory Probab. And Math. Stat., Kiev, (1985), p. 42 - 56 (in Russian).

Ostrovsky E.I. Exponential Estimations for Random Fields. Moscow-Obninsk, OINPE, (1999), (in Russian).

Pinelis Iosif. Exact inequalities for sums of asymmetric random variables, with applications. arXiv:math/0602556v2 [math.PR] 24 May 2006

Raginsky M. and Sason I. Concentration of Measure Inequalities in Information Theory, Communications, and Coding. Foundations and Trends in Communications and Information Theory, vol. 10, no. 1 – 2, pp. 1246, 2013.

Schlemm E. The Kearns-Saul inequality for Bernoulli and Poisson-binomial distributions. arXiv:1405.4496v1 [math.PR] 18 May 2014

Serov A.A., Zubkov A.M. A full proof of universal inequalities for the distribution function of the binomial law. arXiv:1207.3838v1 [math.PR] 16 Jul 2012

Zubkov A.M., Serov A.A. Bounds for the number of Boolean functions admitting affine approximations of a given accuracy. Discrete Math. Appl., 2010, 20, N° 5-6, p. 467 - 486.