Hierarchical Cooperation Achieves Optimal Capacity Scaling in Ad Hoc Networks

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Abstract

\( n \) source and destination pairs randomly located in an area want to communicate with each other. Signals transmitted from one user to another at distance \( r \) apart are subject to a power attenuation of \( r^{-\alpha} \) as well as a random phase. We identify exactly the scaling laws of the information theoretic capacity of the network. In the case of dense networks, where the area is fixed and the density of nodes increasing, we show that the total capacity of the network scales linearly with \( n \). In the case of extended networks, where the density of nodes is fixed and the area increasing linearly with \( n \), we show that the sum capacity scales as \( n^{2-\alpha/2} \) for \( \alpha < 3 \) and \( \sqrt{n} \) for \( \alpha \geq 3 \). Thus, much better scaling than multihop can be achieved in dense networks, as well as in extended networks with low attenuation. The performance gain is achieved by intelligent node cooperation and distributed MIMO communication. The key ingredient is a hierarchical and digital architecture for nodal exchange of information for realizing the cooperation.

I. INTRODUCTION

The seminal paper by Gupta and Kumar [1] initiated the study of scaling laws in large ad-hoc wireless networks. Their by-now-familiar model considers \( n \) nodes randomly located in the unit disk, each of which wants to communicate to a random destination node at a rate \( R(n) \) bits/second. They ask what is the maximally achievable scaling of the total throughput \( T(n) = nR(n) \) with the system size \( n \). They showed that classical multihop architectures with conventional single-user decoding and forwarding of packets cannot achieve a scaling of better than \( O(\sqrt{n}) \), and that a scheme that uses only nearest-neighbor communication can achieve a
throughput that scales as $\Theta(\sqrt{n/\log n})$. This gap was later closed by Franceschetti et al [2], who showed using percolation theory that the $\Theta(\sqrt{n})$ scaling is indeed achievable.

Gupta-Kumar model makes certain assumptions on the physical-layer communication technology. In particular, it assumes that the signals received from nodes other than one particular transmitter are interference to be regarded as noise degrading the communication link. Given this assumption, long-range communication between nodes is not preferable, as the interference generated would preclude most of the other nodes from communicating. Instead, the optimal strategy is to confine to nearest neighbor communication and maximize the number of simultaneous transmissions (frequency-reuse). However, this means that each packet has to be retransmitted many times before getting to the final destination, leading to a sub-linear scaling of system throughput. Thus, fundamentally, the Gupta-Kumar result is an interference-limited result.

A natural question is whether this result is a consequence of the physical-layer assumptions or whether one can do better using more sophisticated physical-layer processing. In a recent work [3], Aeron and Saligrama have showed that the answer is the latter: they exhibited a scheme which yields a throughput scaling of $\Theta(n^{2/3})$ bits/second. However, it is not clear if one can do even better. In fact, how does the information theoretic capacity of the network scale? The first main result in this paper is that one can in fact achieve arbitrarily close to linear scaling: for any $\epsilon > 0$, we present a scheme that achieves an aggregate rate of $\Theta(n^{1-\epsilon})$. This is a surprising result: a linear scaling means the rate for each source-destination pair does not degrade significantly even as one puts more and more users in the network. It is easy to show that one cannot get a better capacity scaling than $O(n \log n)$, so up to logarithmic terms, our scheme is optimal.

To achieve linear scaling, one must be able to perform many simultaneous long-range communications. A physical-layer technique which achieves this is MIMO (multi-input multi-output): the use of multiple transmit and receive antennas to multiplex several streams of data and transmit them simultaneously. MIMO was originally developed in the point-to-point setting, where the transmit antennas are co-located at a single transmit node, each transmitting one data stream, and the receive antennas are co-located at a single receive node, jointly processing the vector of received observations at the antennas. A natural approach to apply this concept to the network setting is to have both source nodes and destination nodes cooperate in clusters to form distributed transmit and receive antenna arrays respectively. In this way, mutually interfering signals can be turned into useful ones that can be jointly decoded at the receive cluster and
spatial multiplexing gain can be realized. In fact, if all the nodes in the network could cooperate for free, then a classical MIMO result [4], [5] says that a sum rate scaling proportional to \( n \) could be achieved. However, this may be over-optimistic: communication between nodes is required to set up the cooperation and this may drastically reduce the useful throughput. The Aeron-Saligrama scheme is MIMO-based and its performance is precisely limited by the cooperation overhead between receive nodes. Our main contribution is a multi-scale, hierarchical cooperation architecture without significant overhead. Cooperation first takes place between nodes within very small local clusters to facilitate MIMO communication over a larger spatial scale. This can then be used as a communication infrastructure for cooperation within larger clusters at the next level of the hierarchy. Continuing on this fashion, cooperation can be achieved at an almost global scale.

Since the publication of [1], there have been several works dealing with information theoretic scaling laws of wireless adhoc networks [6], [7], [8], [9], [10]. All of them deal with extended networks, which scale to cover an increasing geographical extent with the density of nodes fixed and the source-destination distances increasing large. The best result to date [9] shows that whenever the power path loss exponent \( \alpha \) of the environment is greater than 4 so that signal attenuates fast enough, the nearest-neighbor multihop scheme is in fact order-optimal. No better scheme than multihop is known for \( \alpha \leq 4 \). In contrast, the linear scaling result discussed above is for dense networks, where the total area is fixed and the number of nodes is increasing. Extended networks are more complicated to analyze since, in addition to interference, performance is also limited by how much energy can be transferred across long geographical distances. Nevertheless, we show that a simple modification of our hierarchical scheme can be applied to extended networks and achieves a throughput scaling of \( n^{2-\alpha/2} \). Thus, for \( \alpha < 3 \), our scheme performs strictly better than multihop. Moreover, by evaluating a cutset upper bound, we show that our scheme meets the upper bound for \( \alpha < 3 \), while multihop meets the bound for \( \alpha \geq 3 \). The scaling law for the extended case is thus completely resolved.

The dense scaling is not only a useful step in studying extended networks, by isolating the issue of interference, but it is of interest on its own right. It is relevant whenever one wants to design networks to serve many nodes, all within communication range of each other (within a campus, an urban block, etc.). This scaling is also a reasonable model to study problems such as spectrum sharing, where many users in a geographical area are sharing a wide band of
spectrum. Consider the scenario where we segregate the total bandwidth into many orthogonal bands, one for each separate network supporting a fixed number of users. As we increase the number of users, the number of such segregated networks increases but the spectral efficiency, in bits/s/Hz, does not scale with the total number of users. In contrast, if we build one large ad hoc network for all the users on the entire bandwidth, then our result says that the spectral efficiency actually increases linearly with the number of users. The gain is coming from a network effect via cooperation between the many nodes in the system.

The rest of the paper is summarized as follows. In Section II, we present the model. Section III contains the main result for dense networks and an outline of the proposed architecture together with a back-of-the-envelope analysis of its performance. The details of its performance analysis are given in Section IV. Section V characterizes the scaling law for extended networks. Section VI contains our conclusions.

II. Model

There are $n$ nodes uniformly and independently distributed in a square of unit area (dense scaling). Every node is both a source and a destination. The sources and destinations are paired up one-to-one in an arbitrary way. Each source has the same traffic rate $R(n)$ to send to its destination node and a common average transmit power budget of $P$ Watts. The total throughput of the system is $T(n) = nR(n)$. \footnote{In the sequel, whenever we say a total throughput $T(n)$ is achievable, we implicitly mean that that a rate of $T(n)/n$ is achievable for every source-destination pair.}

We assume that communication takes place over a flat channel of bandwidth $W$ Hz around a carrier frequency of $f_c$, $f_c \gg W$. The complex baseband-equivalent channel gain between node $i$ and node $k$ at time $m$ is given by:

$$h_{ik}[m] = \sqrt{G}r_{ik}^{-\alpha/2} \exp(j\theta_{ik}[m]) \quad (1)$$

where $r_{ik}$ is the distance between the nodes, $\theta_{ik}[m]$ is the random phase at time $m$, uniformly distributed in $[0, 2\pi]$ and $\{\theta_{ik}[m]\}$ are i.i.d random processes across all $i$ and $k$. The $\theta_{ik}[m]$’s and the $r_{ik}$’s are also assumed to be independent. The parameters $G$ and $\alpha \geq 2$ are assumed to be constants; $\alpha$ is called the path loss exponent. For example, under free-space line-of-sight
propagation, Friis’ formula applies and
\[
|h_{ik}[m]|^2 = \frac{G_{Tx} \cdot G_{Rx}}{(4\pi r_{ik}/\lambda_c)^2}
\]
so that
\[
G = \frac{G_{Tx} \cdot G_{Rx} \cdot \lambda_c^2}{16\pi^2}, \quad \alpha = 2.
\]
where $G_{Tx}$ and $G_{Rx}$ are the transmitter and receiver antenna gains respectively and $\lambda_c$ is the carrier wavelength.

Note that the channel is random, depending on the location of the users and the phases. The locations are assumed to be fixed over the duration of the communication. The phases are assumed to vary in a stationary ergodic manner (fast fading). We assume that the channel gains are known at all the nodes. The received signal is a sum of the received signals plus white circular symmetric Gaussian noise of variance $N_0$ per symbol.

Several comments about the model are in order:

- The path loss model is based on a far-field assumption: the distance $r_{ik}$ is assumed to be much larger than the carrier wavelength. When the distance is of the order or shorter than the carrier wavelength, the simple path loss model obviously does not hold anymore as path loss can potentially become path “gain”. The reason is that near-field electromagnetics now come into play.
- The phase $\theta_{ik}[m]$ depends on the distance between the nodes modulo the carrier wavelength [11]. The random phase model is thus also based on a far-field assumption: we are assuming that the nodes’ separation is at a much larger spatial scale compared to the carrier wavelength, so that the phases can be modelled as completely random and independent of the actual positions.
- It is realistic to assume the variation of the phases since they vary significantly when users move a distance of the order of the carrier wavelength (fractions of a meter). The positions determine the path losses and they on the other hand vary over a much larger spatial scale. So the positions are assumed to be fixed.

\[\text{With more technical efforts, we believe our results can be extended to the slow fading setting where the phases are fixed as well.}\]
• We essentially assume a line-of-sight type environment and ignore multipath effects. The randomness in phases is sufficient for the long range MIMO transmissions needed in our scheme. With multipaths, there is a further randomness due to random constructive and destructive interference of these paths. It can be seen that our result easily extends to the multipath case.

Theoretically, as the number of nodes increases, the far-field assumption eventually becomes invalid as nodes become closer. In reality, the typical separation between nodes is so much larger than the carrier wavelength that the number of nodes when the far-field assumption fails is humongous, i.e. there is a clear separation between the large and the small spatial scales. Consider the following numerical example. Suppose the area of interest is 1 sq. km, well within the communication range of many radio devices. With a carrier frequency of 3 GHz, the carrier wavelength is 0.1m. Even with a very large system size of \( n = 10000 \) nodes, the typical separation between nearest neighbors is 10 m, very much in the far-field. Under free-space propagation and assuming unit transmit and receive antenna gains, the attenuation given by Friis’ formula (2) is about \( 10^{-6} \), much smaller than unity. To have a nearest-neighbor distance of 0.1 m (the carrier wavelength), \( 10^8 \) nodes would be needed in the area! Hence, there is a wide range of system parameters for which simultaneously the number of nodes is large and the far-field assumption holds.

In most of the following discussions, we will simplify the notation by suppressing the dependency of the channel gains on the time index \( m \).

III. MAIN RESULT FOR DENSE NETWORKS

We first give an information-theoretic upper bound on the achievable scaling law for the aggregate throughput in the network. Before starting to look for good communication strategies, Theorem 3.1 establishes the best we can hope for.

**Theorem 3.1:** The aggregate throughput in the network with \( n \) nodes is bounded above by

\[
T(n) \leq K' n \log n
\]

with high probability\(^3\) for some constant \( K' > 0 \) independent of \( n \).

\(^3\)i.e. probability going to 1 as system size grows.
Proof: Consider a source-destination pair \((s, d)\) in the network. The transmission rate \(R(n)\) from source node \(s\) to destination node \(d\) is upper bounded by the capacity of the single-input multiple-output (SIMO) channel between source node \(s\) and the rest of the network. Using a standard formula for this channel (see eg. [11]), we get:

\[
R(n) \leq \log \left( 1 + \frac{P}{N_0} \sum_{i=1}^{n} |h_{is}|^2 \right) = \log \left( 1 + \frac{P}{N_0} \sum_{i=1}^{n} G r_{is}^{\alpha} \right).
\]

It is a well known fact that in a random network with \(n\) nodes uniformly distributed on a fixed two-dimensional area, the minimum distance between any two nodes in the network is larger than \(\frac{1}{n^{\gamma}}\) with high probability, for any \(\delta > 0\). Using this fact, we obtain

\[
R(n) \leq \log \left( 1 + \frac{GP}{N_0} n^{\alpha(1+\delta)+1} \right) \leq K' \log n
\]

for some constant \(K' > 0\) independent of \(n\) for all-source destination pairs in the network with high probability. The theorem follows.

In the view of what is ultimately possible, established by Theorem 3.1, we are now ready to state the main result of this paper.

**Theorem 3.2:** Let \(\alpha \geq 2\). For any \(\epsilon > 0\), with high probability an aggregate throughput

\[
T(n) \geq Kn^{1-\epsilon}
\]

is achievable in the network for all possible pairings between sources and destinations. \(K > 0\) is a constant independent of \(n\) and the source-destination pairing.

Theorem 3.2 states that it is actually possible to perform arbitrarily close to the bound given in Theorem 3.1. The two theorems together establish the capacity scaling for the network up to logarithmic terms. Note how dramatically different is this new linear capacity scaling law from the well-known throughput scaling of \(\Theta(\sqrt{n})\) implied by [1], [2] for the same model. Note also that the upper bound in Theorem 3.1 assumes a genie-aided removal of interference between simultaneous transmissions from different sources. By proving Theorem 3.2, we will show that it is possible to mitigate such interference without a genie but with cooperation between the nodes.

The proof of Theorem 3.2 relies on the construction of an explicit scheme that realizes the promised scaling law. The construction is based on recursively using the following key lemma, which addresses the case when \(\alpha > 2\).
**Lemma 3.1:** Consider $\alpha > 2$ and a network with $n$ nodes subject to interference from external sources. Let the interference signals received by different nodes in the network be uncorrelated and the interference power received by each node be upper bounded by

$$P_ I \leq K_ I$$

for some $K_ I > 0$ and independent of $n$. Let us assume there exists a scheme such that for each $n$, with probability at least $1 - e^{-n^{c_1}}$ it achieves an aggregate throughput

$$T(n) \geq K_1 n^b$$

for every possible source-destination pairing in a network of $n$ nodes. $K_1$ and $c_1$ are positive constants independent of $n$ and the source-destination pairing, and $0 \leq b < 1$. Let us also assume that the per node average power budget required to realize this scheme is:

$$P \leq \frac{K_ p}{n}$$ (3)

for some $K_ p > 0$ independent of $n$.

Then one can construct another scheme that achieves a *higher* aggregate throughput

$$T(n) \geq K_2 n^{\frac{1}{2-b}}$$

for every source-destination pairing in a network of $n$ nodes under the same interference conditions, where $K_2 > 0$ is another constant independent of $n$ and the pairing. Moreover, the failure rate for the new scheme is upper bounded by $e^{-n^{c_2}}$ for another positive constant $c_2$ while the per node average power needed to realize the scheme is also bounded above by (3).

Lemma 3.1 is the key step to build a hierarchical architecture. Since $\frac{1}{2-b} > b$ for $0 \leq b < 1$, the new scheme is always better than the old. We will now give a rough description of how the new scheme can be constructed given the old scheme, as well as a back-of-the-envelope analysis of the scaling law it achieves. Next section is devoted to its precise description and performance analysis.

The constructed scheme is based on clustering and long-range MIMO transmissions between clusters. We divide the network into clusters of $M$ nodes. Let us focus for now on a particular source node $s$ and its destination node $d$. $s$ will send $M$ bits to $d$ in 3 steps:

- 1) Node $s$ will distribute its $M$ bits among the $M$ nodes in its cluster, one for each node;
2) These nodes together can then form a distributed transmit antenna array, sending the $M$ bits simultaneously to the destination cluster where $d$ lies;

3) Each node in the destination cluster obtained one observation from the MIMO transmission, and it quantizes and ships the observation back to $d$, which can then do joint MIMO processing of all the observations and decode the $M$ transmitted bits.

From the network point of view, all source-destination pairs have to eventually accomplish these three steps. Step 2 is long-range communication and only one source-destination pair can operate at the same time. Steps 1 and 3 involve local communication and can be parallelized across source-destination pairs. Combining all this leads to three phases in the operation of the network:

**Phase 1: Setting Up Transmit Cooperation** Clusters work in parallel. Within a cluster, each source node has to distribute $M$ bits to the other nodes, 1 bit for each node, such that at the end of the phase each node has 1 bit from each of the source nodes in the same cluster. Since there can be at most $M$ source nodes in each cluster, this gives a traffic demand of exchanging at most $M^2$ bits. The key observation is that this is similar to the original problem of communicating between $n$ source and destination pairs, but on a network of size $M$. More specifically, this traffic demand of exchanging $M^2$ bits is handled by setting up $M$ sub-phases, and assigning $M$ source-destination pairs for each sub-phase. Since our channel model is scale invariant, note that the scheme given in the hypothesis of the lemma can be used in each sub-phase by simply scaling down the power with cluster area. Having aggregate throughput $M^b$, each sub-phase is completed in $M^{1-b}$ time slots while the whole phase takes $M^{2-b}$ time slots. See Figure 1.

**Phase 2: MIMO Transmissions** We perform successive long-distance MIMO transmissions between source-destination pairs, one at a time. In each one of the MIMO transmissions, say one between $s$ and $d$, the $M$ bits of $s$ are simultaneously transmitted by the $M$ nodes in its cluster to the $M$ nodes in the cluster of $d$. Each of the long-distance MIMO transmissions are repeated for each source node in the network, hence we need $n$ time slots to complete the phase. See Figure 2.

**Phase 3: Cooperate to Decode** Clusters work in parallel. Since there are at most $M$ destination nodes inside the clusters, each cluster received at most $M$ MIMO transmissions in phase 2, one intended for each of the destination nodes in the cluster. Thus, each node in the cluster has at most $M$ received observations, one from each of the MIMO transmissions, and each observation
is to be conveyed to a different destination node in its cluster. Nodes quantize each observation into fixed $Q$ bits so there are now a total of at most $QM^2$ bits to exchange inside each cluster. Using exactly the same scheme as in Phase 1, we conclude the phase in $QM^2$ time slots. See Figure 3.

Assuming that each destination node is able to decode the transmitted bits from its source node from the $M$ quantized signals it gathers by the end of Phase 3, we can calculate the rate of the scheme as follows: Each source node is able to transmit $M$ bits to its destination node, hence $nM$ bits in total are delivered to their destinations in $M^2-b+n+QM^2$ time slots, yielding an aggregate throughput of

$$\frac{nM}{M^{2-b} + n + QM^{2-b}}$$

bits per time slot. Maximizing this throughput by choosing $M = n^{\frac{1}{2+\tau}}$ yields $T(n) = \frac{1}{2+Q} n^{\frac{1}{2+\tau}}$ for the aggregate throughput which is the result in Lemma 3.1.

Clusters can work in parallel in phases 1 and 3 because for $\alpha > 2$, the aggregate interference at a particular cluster caused by other active nodes is bounded. For $\alpha = 2$, the aggregate interference scales like $\log n$, leading to a slightly different version of the lemma.
Successive MIMO transmissions are performed between clusters. The first figure depicts MIMO transmission from cluster $F$ to $G$, where bits originally belonging to $s_1$ are simultaneously transmitted by all nodes in $F$ to all nodes in $G$. The second MIMO transmission is from $H$ to $G$, while now bits of source node $s_2$ are transmitted from nodes in $H$ to nodes in $G$. The third picture illustrates MIMO transmission from cluster $J$ to $F$.

Observations received by nodes in $F$, $G$, $H$ and $J$ during Phase 2, are conveyed to their destination nodes in parallel. Note that the picture is completely symmetric to Fig. 1 except that the characteristic of the traffic is not “from source nodes” but is “to destination nodes” in this case.

**Lemma 3.2:** Consider $\alpha = 2$ and a network with $n$ nodes subject to interference from external sources. Let the interference signals received by different nodes in the network be uncorrelated and the interference power received by each node be upper bounded by

$$P_I \leq K_I \log n$$

for some $K_I \geq 0$ and independent of $n$. Let us assume there exists a scheme such that for each
with failure probability at most $e^{-n^{c_1}}$, achieves an aggregate throughput

$$T(n) \geq K_1 \frac{n^b}{\log n}$$

for every source-destination pairing in a network with $n$ nodes. $K_1$ and $c_1$ are positive constants independent of $n$ and the source-destination pairing, and $0 \leq b < 1$. Let us also assume that the per node average power budget required to realize this scheme is:

$$P \leq \frac{K_p}{n}$$

for some $K_p > 0$ independent of $n$.

Then one can construct another scheme that achieves a higher aggregate throughput

$$T(n) \geq K_2 \frac{n^{\frac{1}{2 - b}}}{\log n}$$

for every source-destination pairing in a network of $n$ nodes under the same interference conditions, where $K_2 > 0$ is another constant independent of $n$ and the pairing. Moreover, the failure rate for the new scheme is upper bounded by $e^{-n^{c_2}}$ for another positive constant $c_2$ while the per node average power needed to realize the scheme is also bounded above by (4).

We can now use Lemma 3.1 and 3.2 to prove Theorem 3.2.

**Proof of Theorem 3.2** We only focus on the case of $\alpha > 2$. The case of $\alpha = 2$ proceeds similarly, differing only with a reduction of a factor of $\log n$ in the throughputs.

We start by observing that the simple scheme of transmitting directly between the source-destination pairs one at a time (TDMA) satisfies the requirements of the lemma. The throughput is $\Theta(1)$, so $b = 0$. The failure probability is 0. Since each source is only transmitting $\frac{1}{n}$th of the time and the distance between the source and its destination is bounded, the average power consumed per node is of the order of $\frac{1}{n}$.

As soon as we have a scheme to start with, Lemma 3.1 can be applied recursively, yielding a scheme that achieves higher throughput at each step of the recursion. More precisely, starting with a TDMA scheme with $b = 0$ and applying Lemma 3.1 recursively $h$ times, one gets a scheme achieving $O(n^{\frac{h}{n+1}})$ aggregate throughput. Given any $\epsilon > 0$, we can now choose $h$ such that $\frac{h}{n+1} \geq 1 - \epsilon$ and we get a scheme that achieves $O(n^{1-\epsilon})$ aggregate throughput scaling with high probability. This concludes the proof of Theorem 3.2. □
Gathering everything together, we have built a hierarchical scheme to achieve the desired throughput. At the lowest level of the hierarchy, we use the simple TDMA scheme to exchange bits for cooperation among small clusters. Combining this with longer range MIMO transmissions, we get a higher throughput scheme for cooperation among nodes in larger clusters at the next level of the hierarchy. Finally, at the top level of the hierarchy, the cooperation clusters are almost the size of the network and the MIMO transmissions are over the global scale to meet the desired traffic demands. Figure 4 shows the resulting hierarchical scheme with a focus on the top two levels.

IV. DETAILED DESCRIPTION AND PERFORMANCE ANALYSIS

In this section, we concentrate in more detail on the scheme that proves Lemma 3.1 and Lemma 3.2. We first focus on Lemma 3.1 and then extend the proof to Lemma 3.2. As we have already seen in the previous section, we start by dividing the unit square into smaller squares of area $A_c = \frac{M}{n}$. Since the node density is $n$, there will be on average $M$ nodes inside each of these small squares. The following lemma upper bounds the probability of having large deviations from the average.

**Lemma 4.1:** Let us partition a unit area network of size $n$ into cells of area $A_c$. The number
of nodes inside each cell is between \(((1 - \delta)A_c n, (1 + \delta)A_c n)\) with probability larger than \(1 - \frac{1}{A_c} e^{-\Lambda(\delta)A_c n}\) where \(\Lambda(\delta) > 0\) for \(\delta > 0\).

Applying Lemma 4.1 to the squares of area \(M/n\), we see that all squares contain order \(M\) nodes with high probability. In the following discussion, we will need a stronger result, namely each of the 8 possible halves of a square should contain order \(M/2\) nodes. Applying the lemma to these half squares together with the union bound, we see that the number of nodes inside each of the 8 possible halves of all squares is between \(((1 - \delta)M/2, (1 + \delta)M/2)\) with probability larger than \(1 - \frac{8n}{M} e^{-\Lambda(\delta)M/2}\) with \(\Lambda(\delta) > 0\) for \(\delta > 0\). This condition is sufficient for our below analysis on scaling laws to hold. However, in order to simplify the presentation, we assume that there are exactly \(M/2\) nodes inside each half, thus exactly \(M\) nodes in each square. The clustering is used to realize a distributed MIMO system in three successive steps:

**Phase 1: Setting Up Transmit Cooperation** In this phase, source nodes distribute their data streams over their clusters and set up the stage for the long-range MIMO transmissions that we want to perform in the next phase. Clusters work in parallel according to the 9-TDMA scheme depicted in Figure 5, which divides the total time for this phase into 9 time-slots and assigns simultaneous operation to clusters that are sufficiently separated. Let us focus on one specific source node \(s\) located in cluster \(S\) with destination node \(d\) in cluster \(D\). Node \(s\) will divide a block of length \(LM\) bits of its data stream into \(M\) sub-blocks each of length \(L\) bits, where \(L\) can be arbitrarily large but bounded. The destination of each sub-block in Phase 1 depends on the relative position of clusters \(S\) and \(D\):

1) If \(S\) and \(D\) are either the same cluster or are not neighboring clusters: One sub-block is to be kept in \(s\) and the rest \(M - 1\) sub-blocks are to transmitted to the other \(M - 1\) nodes located in \(S\), one sub-block for each node.

2) If \(S\) and \(D\) are neighboring clusters: Divide the cluster \(S\) into two halves, each of area \(A_c/2\), one half located close to the border with \(D\) and the second half located farther to \(D\). The \(M\) sub-blocks of source node \(s\) are to be distributed to the \(M/2\) nodes located in the second half cluster (farther to \(D\)), each node gets two sub-blocks.

Let us assume that \(M_s\) of the \(M\) nodes in \(S\) are sources and the rest \(M_d\) are destinations for given communication requests. Since the above traffic is required for every source node in cluster \(S\), we end up with a highly uniform traffic demand of delivering \(M_s \times LM\) bits in total.
to their destinations. A key observation is that the problem can be separated into sub-problems, each similar to our original problem but on a network size $M$ and area $A_c$. More specifically, the traffic of transporting $M_sLM$ bits can be handled by organizing $2M$ sessions and assigning destinations to $M_s/2$ source nodes in each session so that $L$ bits are transmitted between the assigned source-destination pairs in each session. Note that $M_s \leq M$ and $M_s = M$ results in maximum traffic to handle in each session. Let us concentrate on this worst case. Since our channel model is scale invariant the scheme in the hypothesis of Lemma 3.1 can be used to handle the traffic in each session, by simply scaling down the powers of the nodes by $(A_c)^{\alpha/2}$. Hence, the power used by each node will be bounded by $K_p(A_c)^{\alpha/2}M$. If the interference from the simultaneously operating clusters were absent, we know that the scheme would offer $\Theta(M^b)$ aggregate rate in a given session. The following lemma allows to conclude that the same scaling is achievable under the inter-cluster interference with the 9-TDMA scheme.

**Lemma 4.2:** Consider clusters of size $M$ and area $A_c$ operating according to 9-TDMA scheme in Figure 5 in a network of size $n$. Let each node be constrained to an average power $K_p(A_c)^{\alpha/2}/M$. For $\alpha > 2$, the interference power received by a node from the simultaneously operating clusters is upper bounded by a constant $K_{I_1}$ independent of $n$. For $\alpha = 2$, the interference power is bounded by $K_{I_2}\log n$ for $K_{I_2}$ independent of $n$. Moreover, the interference signals received by different nodes in the cluster are uncorrelated.

Let us for now concentrate on the case $\alpha > 2$. By Lemma 4.2, the inter-cluster interference is bounded and is uncorrelated at different nodes. Thus, the strategy in the hypothesis of Lemma 3.1 can achieve an aggregate rate $K_1'M^b$ in all the sessions for some $K_1' > 0$ with probability larger than $1 - e^{-M\epsilon_1}$. Using the union bound, with probability larger than $1 - 2ne^{-M\epsilon_1}$ the aggregate rate $K_1'M^b$ is achieved inside all sessions in all clusters in the network. With this aggregate rate, each session can be completed in at most $(L/K_1')M^{1-b}$ channel uses and $2M$ successive sessions are completed in $(2L/K_1')M^{2-b}$ channel uses. Using the 9-TDMA scheme, the phase is completed in less than $(18L/K_1')M^{2-b}$ channel uses all over the network with probability larger than $1 - 2ne^{-M\epsilon_1}$.

**Phase 2: MIMO Transmissions** In this phase, we are performing the actual MIMO transmissions for all the source-destination pairs serially, i.e. one at a time. A MIMO transmission from source $s$ to destination $d$ involves the $M$ (or $M/2$) nodes in the cluster $S$ where $s$ is in
Data Exchanges in Phase 1

Fig. 5. Buffers of the nodes in a cluster are illustrated before and after the data exchanges in Phase 1. The data stream of the \( M_s \) source nodes are distributed to the \( M \) nodes in the network as depicted. \( b_s(j) \) denotes the \( j \)'th sub-block of the source node \( s \). Note the 9-TDMA scheme that is employed over the network in this phase.

(referred to as the source cluster for this MIMO transmission) to the \( M \) (or \( M/2 \)) nodes of the cluster \( D \) where \( d \) is in (referred to as the destination cluster of the MIMO transmission.)

Let the distance between the mid-points of the two clusters be \( r_{SD} \). If \( S \) and \( D \) are the same cluster, we skip the step for this source node \( s \). Otherwise, we operate in two slightly different modes depending on the relative positions of \( S \) and \( D \) corresponding to the operations performed in the first phase: First consider the case where \( S \) and \( D \) are not neighboring clusters. In this case, the \( M \) nodes in cluster \( S \) independently encode the \( L \) bits long sub-blocks they possess, originally belonging to node \( s \), into \( C \) symbols by using a randomly generated Gaussian code \( \mathcal{C} \) that respects an average transmit power constraint \( \frac{K_p(r_{SD})^\alpha}{M} \). The nodes then transmit their encoded sequences of length \( C \) symbols simultaneously to the \( M \) nodes in cluster \( D \). The nodes in cluster \( D \) properly sample the signals they observe during the \( C \) transmissions and store these samples (that we will simply refer to as observations in the following text), without trying to decode the transmitted symbols. In the case where \( S \) and \( D \) are neighbors, the strategy is slightly modified so that the MIMO transmission is from the \( M/2 \) nodes in \( S \), that possess the sub-blocks of \( s \) after Phase 1, to the \( M/2 \) nodes in \( D \) that are located in the farther half of the
cluster to $S$. Each of these $M/2$ nodes in $S$ possess two sub-blocks that come from $s$. They encode each sub-block into $C$ symbols by again using a Gaussian code of power $\frac{K_a(r_{SD})^\alpha}{M}$. The nodes then transmit the $2C$ symbols to the $M/2$ nodes in $D$ that in turn sample their received signals and store the observations. The observations accumulated at various nodes in $D$ at the end of this step are to be conveyed to node $d$ during the third phase.

After concluding the step for source node $s$ of $S$, the phase continues by repeating the same step for source node $s+1$ of $S$. The destination cluster for the new MIMO transmission will be, in general, a different cluster $D'$ which is the one that contains the ultimate destination node $d'$ for the source node $s+1$. The MIMO transmissions are repeated until the data originated from all source nodes in the network are transmitted to their respective destination clusters. Since the step for one source node takes either $C$ or $2C$ channel uses, completing the operation for all $n/2$ source nodes in the network requires at most $2C \times n/2 = Cn$ channel uses.

**Phase 3: Cooperate to Decode** In this phase, we aim to provide each destination node, the observations of the symbols that have been originally intended for it. With the MIMO transmissions in the second phase, these observations have been accumulated at the nodes of its cluster. As before let us focus on a specific node $d$, one of the $M_d$ destination nodes located in cluster $D$. Note that depending on whether the source node of $d$ is located in a neighboring cluster or not, either each of the $M$ nodes in $D$ have $C$ observations intended for $d$, or $M/2$ of the nodes have $2C$ observations each. Note that these observations are some real numbers that need to be quantized and encoded into bits before being transmitted. Let us assume that we are encoding each block of $C$ observations into $CQ$ bits, by using fixed $Q$ bits per observation on the average. The same is true for all $M_d$ destination nodes in $D$ since the cluster received $M_d$ MIMO transmissions in the previous phase, one for each destination node. (The destination nodes that have source nodes in $D$ are exception. Recall from Phase 1 and Phase 2 that in this case, each node in $D$ possess sub-blocks of the original data stream for the destination node, not MIMO observations. We will ignore this case by simply assuming $L \leq CQ$ in the below computation.) The arising traffic demand of transporting $M_d \times CQ$ bits in total is completely symmetric to the one in Phase 1 and can be handled by using exactly the same scheme in less than $(2CQ/K_1')M^{2-b}$ channel uses. Recalling the discussion for the first phase we conclude that the phase can be completed in less than $(18CQ/K_1')M^{2-b}$ channel uses all over the network with probability larger than $1 - 2ne^{-M'^1}$. 

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Note that if it were possible to encode each observation into fixed $Q$ bits without introducing any distortion, which is obviously not the case, the following lemma on MIMO capacity would suggest that with the Gaussian code $C$ used in Phase 2, such that $L/C \geq \kappa$ for some constant $\kappa > 0$, the transmitted bits can be recovered by an arbitrarily small probability of error from the observations gathered by the destination nodes at the end of Phase 3. The proof can be found in Appendix [1].

**Lemma 4.3:** The mutual information achieved by the $M \times M$ MIMO transmission between any two clusters grows at least linearly with $M$.

In the Appendix [II] we will prove the following lemma which states that there is actually a way to encode the observations using fixed number of bits per observation and at the same time not to degrade the performance of the overall channel significantly, that is, to still get a linear capacity growth for the resulting quantized MIMO channel.

**Lemma 4.4:** There exists a strategy to encode the observations at a fixed rate $Q$ bits per observation and get a linear growth of the mutual information for the resultant $M \times M$ quantized MIMO channel.

Although we do not discuss the proof of the lemma now, the following small lemma may provide motivation for the stated result. Lemma [4.5] points out a key observation on the way we choose our transmit powers in the MIMO phase. It is central to the proof of Lemma [4.4] and states that the observations have bounded power, that does not scale with $M$. This in turn suggests that one can use fixed number of bits to encode them and not degrade the scaling performance of the scheme.

**Lemma 4.5:** In Phase 2, the received power by each node in the destination cluster is bounded below and above by constants $P_1$ and $P_2$ respectively, independent of $M$.

Putting it together, we have seen that the three phases described effectively realize virtual MIMO channels achieving spatial multiplexing gain $M$ between the source and destination nodes in the network. Using these virtual MIMO channels, each source is able to transmit $ML$ bits in

$$T_i = T(\text{phase 1}) + T(\text{phase 2}) + T(\text{phase 3}) = \frac{18L}{K_1^2}M^{2-b} + Cn + \frac{18CQ}{K_1^2}M^{2-b}$$
total channel uses where $L/C \geq \kappa$ for some $\kappa > 0$ and independent of $M$ (or $n$). This gives an aggregate throughput of

$$T(n) = \frac{nML}{(18L/K'_1)M^{2-b} + Cn + (18CQ/K'_1)M^{2-b}} \geq K_2n^{\frac{1}{2-b}}$$

(5)

for some $K_2 > 0$ independent of $n$, by choosing $M = n^{\frac{1}{2-b}}$ with $0 \leq b < 1$ which is the optimal choice for the cluster size as a function of $b$. The failure arises if there are not order $M/2$ nodes in each half cluster or the scheme used in Phases 1 and 3 fails to achieve the promised throughput. Combining the result of Lemma 4.1 with the computed failure probabilities for Phases 1 and 3 yields

$$P_f \leq 4ne^{-M^c_1} + \frac{8n}{M}e^{-\Lambda(\delta)M/2} \leq e^{-nc_2}$$

for some $c_2 > 0$. In order to complete the proof of Lemma 3.1 we need to show that our new scheme also satisfies the power constraint in (3): For Phases 1 and 3, we know that the scheme employed inside the clusters satisfies $P_j \leq K_pA^{\alpha/2}_c/M$. From this inequality, we deduce that

$$P_j \leq \frac{K_p}{n}.$$  \hspace{1cm} (6)

Indeed, $A_c = M/n$, and for $\alpha \geq 2$ we have

$$P_j \leq \frac{K_p}{M} \left( \frac{M}{n} \right)^{\alpha/2} = \frac{K_p}{n} \left( \frac{M}{n} \right)^{\alpha/2-1} \leq \frac{K_p}{n}.$$  

In Phase 2, each node is transmitting with power $\frac{K_p(r_{SD})^\alpha}{M}$ in at most fraction $M/n$ of the total duration for the phase, while keeping silent during the rest of the time. This yields a per node average power $\frac{K_p(r_{SD})^\alpha}{n}$. Recall that $r_{SD}$ is the distance between the mid-points of the source and destination clusters and $r_{SD} < 1$, which yields (6) also for the second phase. This concludes the proof of Lemma 3.1.  \hspace{1cm} \square

**Proof of Lemma 3.2** The scheme that proves Lemma 3.2 is completely similar to the one described above. Lemma 4.2 states that when $\alpha = 2$ the inter-cluster interference power experienced during Phases 1 and 3 is upper bounded by $K'_1 \log n = K'_1 \log M$. Hence, the scheme in the hypothesis of Lemma 3.2 will achieve an aggregate rate $K'_1 \frac{M^{1/2-b}}{\log M}$ when used to handle the traffic in these phases. This implies that we will now need $O(M^{2-b} \log M)$ channel uses to complete Phase 1 and Phase 3 respectively. Choosing $M = n^{\frac{1}{2-b}}$ results in $K_2n^{\frac{1}{2-b}}/\log n$ aggregate throughput for the new scheme. The non-trivial thing to show is that the new scheme can achieve this same throughput under interference: Consider every node in the network receiving uncorrelated
interference of power $K_I \log n$ from an outside source. The total interference during Phases 1 and 3 will be bounded by $(K_{I_2} + K_I) \log n$ which yields the same scaling $M^{2-b} \log M$ for the required channel uses to complete these phases. The following lemma, proved in Appendix II, gives a lower bound on the spatial multiplexing gain of the quantized MIMO channel under the interference experienced.

**Lemma 4.6:** Let the MIMO signal received by the nodes in the destination cluster be corrupted by an interference of power $K_I \log M$, uncorrelated over different nodes and independent of the transmitted signals. There exists a strategy to encode these corrupted observations at a fixed rate $Q$ bits per observation and get a $M/\log M$ growth of the mutual information for the resulting $M \times M$ quantized MIMO channel. A capacity of $M/\log M$ for the resulting MIMO channel implies that there exists a code $C$ that encodes $L$ bits-long sub-blocks into $C \log M$ symbols where $L/C > \kappa'$ for a constant $\kappa' > 0$ so that the transmitted bits can be decoded at the destination nodes with arbitrarily small probability of error for $L$ sufficiently large. Transmitting $C \log M$ symbols for each source-destination pair requires $Cn \log M$ bits to complete the second phase. The resultant aggregate throughput is again $K_2 n^{1/2-b}/\log n$. This concludes the proof of Lemma 3.2. □

We continue with the proofs of the lemmas introduced in the section:

**Proof of Lemma 4.1:** The proof of the lemma is a standard application of Chebyshev’s inequality. Note that the number of nodes in a given cell is a sum of $n$ i.i.d Bernoulli random variables $B_i$, such that $\mathbb{P}(B_i = 1) = A_c$. Hence,

$$
\mathbb{P}\left(\sum_{i=1}^{n} B_i \geq (1+\delta)A_c n\right) = \mathbb{P}\left(e^{s\sum_{i=1}^{n} B_i} \geq e^{s(1+\delta)A_c n}\right)
\leq \left(\mathbb{E}[e^{sB_1}]\right)^n e^{-s(1+\delta)A_c n}
= (e^{sA_c} + (1 - A_c))^n e^{-s(1+\delta)A_c n}
\leq e^{-A_c n(s(1+\delta) - e^s + 1)}
= e^{-A_c n \Lambda_+(\delta)}
$$

where $\Lambda_+(\delta) = (1+\delta) \log(1+\delta) - \delta$ by choosing $s = \log(1 + \delta)$. The proof for the lower bound follows similarly by considering the random variables $-B_i$. The conclusion follows from
Proof of Lemma 4.2: Consider a node $v$ in cluster $V$ operating under the 9-TDMA scheme in Figure 6. The interfering signal received by this node from the simultaneously operating clusters $U_V$ is given by

$$I_v = \sum_{U \in U_V} \sum_{j \in U} h_{vj} x_j$$

where $h_{vj}$ are the channel coefficients given by (1) and $x_j$ is the signal transmitted by node $j$ which is located in a simultaneously operating cluster $U$. First note that the interference signals $I_v$ and $I_{v'}$ received by two different nodes $v$ and $v'$ in $V$ are uncorrelated since the channel coefficients $h_{vj}$ and $h_{v'j}$ are independent for all $j$. The power of the interfering signal $I_v$ is given by

$$P_I = \sum_{U \in U_V} \sum_{j \in U} \frac{G P_j}{(r_{vj})^\alpha}$$

by using the fact that channel coefficients corresponding to different nodes $j$ are independent. As illustrated by Figure 6, the interfering clusters $U_V$ can be grouped such that each group $U_V(i)$ contains $8i$ clusters or less and all clusters in group $U_V(i)$ are separated by a distance larger than $(3i - 1)\sqrt{A_c}$ from $V$ for $i = 1, 2, \ldots$. Recall that clusters have area $A_c$. The number of such groups can be simply bounded by the number of clusters $n/M$ in the network. Thus,

$$P_I < \sum_{i=1}^{n/M} \sum_{U \in U_V(i)} \sum_{j \in U} \frac{G P_j}{((3i - 1)\sqrt{A_c})^\alpha}$$

$$\leq \sum_{i=1}^{n/M} 8i \frac{G K_p}{(3i - 1)^\alpha}$$

(7)

where we have used the fact that the powers of the signals are bounded by $P_j \leq K_p A_c^{\alpha/2}/M$. The sum in (7) is convergent for $\alpha > 2$, thus is bounded by a constant $K_{t_1}$. For $\alpha = 2$, the sum can be bounded by $K_{t_2} \log n$ where $K_{t_2}$ is a constant independent of $n$.

Proof of Lemma 4.5: We consider only the case where the source cluster $S$ and the destination cluster $D$ are not neighbors. The argument for the other case follows similarly. The signal received by a destination node $d$ located in cluster $D$ during MIMO transmission from source cluster $S$ is given by

$$y_d = \sum_{s=1}^{M} h_{ds} x_s + z_d$$

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where $x_s$ is the signal sent by a source node $s \in S$ constrained to power $\frac{K_p(r_{SD})^\alpha}{M}$ and $z_d$ is $\sim \mathcal{N}_C(0, N_0)$. The power of this signal is given by

$$
\mathbb{E}[|y_d|^2] = \sum_{s=1}^{M} |h_{ds}|^2 \frac{K_p(r_{SD})^\alpha}{M} + N_0 \leq 2^{\alpha}GP + N_0 \equiv P_2.
$$

where we use the fact that all $h_{ds}$, $x_s$ and $z_d$ are independent. Observe that $r_{SD} - \sqrt{A_c} \leq r_{sd} \leq r_{SD} + \sqrt{2A_c}$, while $r_{SD} \geq 2\sqrt{A_c}$. These two relations yield

$$
\left(\frac{2}{2 + \sqrt{2}}\right)^\alpha \leq \left(\frac{r_{SD}}{r_{sd}}\right)^\alpha \leq 2^\alpha
$$

which in turn yields the following lower and upper bounds for the received power at each destination node

$$
P_1 \equiv \left(\frac{2}{2 + \sqrt{2}}\right)^\alpha GP + N_0 \leq \mathbb{E}[|y_d|^2] \leq 2^\alpha GP + N_0 \equiv P_2.
$$

V. EXTENDED NETWORKS

A. Bursty Hierarchical Scheme does better than Multihop for $\alpha < 3$

So far, we have considered dense networks, where the total geographical area is fixed and the density of nodes increasing. Another natural scaling is the extended case, where the density of
nodes is fixed and the area is increasing, a $\sqrt{n} \times \sqrt{n}$ square. This models the situation where we want to scale the network to cover an increasing geographical area.

As compared to dense networks, the distance between nodes is increased by a factor of $\sqrt{n}$, and hence for the same transmit powers, the received powers are all decreased by a factor of $n^{\alpha/2}$. Equivalently, by rescaling space, an extended network can just be considered as a dense network on a unit area but with the average power constraint per node reduced to $P/n^{\alpha/2}$ instead of $P$.

Lemmas 3.1 and 3.2 state that the average power per node required to run our hierarchical scheme in dense networks is not the full power $P$ but $P/n$. In light of the observation above, this immediately implies that when $\alpha = 2$, we can directly apply our scheme to extended networks and achieve a linear scaling. For extended networks with $\alpha > 2$, our scheme would not satisfy the equivalent power constraint $P/n^{\alpha/2}$ and we are now in the power-limited regime (as opposed to the degrees-of-freedom limited regime). However, we can consider a simple "bursty" modification of the hierarchical scheme which runs the hierarchical scheme a fraction

$\frac{1}{n^{\alpha/2-1}}$

of the time with power $P/n$ per node and remains silent for the rest of the time. This meets the given average power constraint of $P/n^{\alpha/2}$, and achieves an aggregate throughput of

$\frac{1}{n^{\alpha/2-1}} \cdot n^{1-\epsilon} = n^{2-\alpha/2-\epsilon}$

bits/second.

Note that the quantity $n^{2-\alpha/2} = n^2 \cdot n^{-\alpha/2}$ can be interpreted as the total power transferred between a size $n$ transmit cluster and a size $n$ receive cluster, $n^2$ node pairs in all, with a power attenuation of $n^{-\alpha/2}$ for each node pairs. This power transfer is taking place at the top level of the hierarchy (see Figure 4). The fact that the achievable rate is proportional to the power transfer further emphasizes that our scheme is power-limited rather than degrees-of-freedom limited in extended networks.

Let us compare our scheme to multihop. For $\alpha < 3$, it performs strictly better than multihop, while for $\alpha > 3$, it performs worse. Summarizing these observations, we have the following achievability theorem for extended networks, the counterpart to Theorem 3.2 for dense networks.

\footnote{We talk in terms of time but such burstiness can just as well be implemented over frequency with only a fraction of the total bandwidth $W$ used.}
Theorem 5.1: Consider an extended network on a $\sqrt{n} \times \sqrt{n}$ square. There are two cases.

- $2 \leq \alpha < 3$: For every $\epsilon > 0$, with high probability, an aggregate throughput:
  \[ T(n) \geq Kn^{2-\alpha/2-\epsilon} \]
  is achievable in the network for all possible pairings between sources and destinations. $K > 0$ is a constant independent of $n$ and the source-destination pairing.

- $\alpha \geq 3$: With high probability, an aggregate throughput:
  \[ T(n) \geq K\sqrt{n} \]
  is achievable in the network for all possible pairings between sources and destinations. $K > 0$ is a constant independent of $n$ and the source-destination pairing.

B. Cutset Upper Bound for Random S-D Pairings

Can we do better than the scaling in Theorem 5.1? So far we have been considering arbitrary source-destination pairings but clearly there are some pairings for which a much better scaling can be achieved. For example, if the source-destinations are all nearest neighbour to each other, then a linear capacity scaling can be achieved for any $\alpha$. Thus, for the extended network case, we need to narrow down the class of S-D pairings to prove a sensible upper bound. In this section, we will therefore focus on random S-D pairings, assuming that the pairs are chosen according to a random permutation of the set of nodes, without any consideration on node locations. We prove a high probability upper bound which matches the achievability result in Theorem 5.1. Theorem 5.1 together with Theorem 5.2 identify exactly the capacity scaling law in extended networks for all values of $\alpha \geq 2$. The rest of the section is devoted to the proof of the theorem.

Theorem 5.2: Consider an extended network of $n$ nodes with random source-destination pairing. For any $\epsilon > 0$, the aggregate throughput is bounded above by

\[ T(n) \leq \begin{cases} 
K'n^{2-\alpha/2+\epsilon} & 2 \leq \alpha < 3 \\
K'n^{1/2+\epsilon} & \alpha \geq 3 
\end{cases} \]

with high probability for a constant $K' > 0$ independent of $n$.

Note that the hierarchical scheme is achieving near global cooperation. In the context of dense networks, this yields a near linear number of degrees of freedom for communication. In the context of extended networks, in addition to the degrees of freedom provided, this scheme
allows almost all nodes in the network to cooperate in transferring energy between any source-destination pair. In fact, we saw that in extended networks with \( \alpha > 2 \), our scheme is power-limited rather than degrees-of-freedom limited. A natural place to look for a matching upper bound is to consider a cutset bound on how much power can flow across the network. Our proof of Theorem 5.2 is a careful evaluation of such a cutset bound.

**Proof of Theorem 5.2:** We start by considering several properties that are satisfied with high probability in the random network. The following lemma is similar in spirit to Lemma 4.1 for dense networks and can be proved using a similar technique. In parallel to the dense case, it forms the groundwork for our following discussion.

**Lemma 5.1:** The random network with random source-destination pairings satisfies the following properties with high probability:

a) Let the network area be divided into \( n \) squarelets of unit area. Then, there are less than \( \log n \) nodes inside all squarelets.

b) Let the network area be divided into \( \frac{n}{2 \log n} \) squarelets each of area \( 2 \log n \). Then, there is at least one node inside all squarelets.

c) Consider a cut dividing the network area into two equal halves. The number of communication requests with sources on the left-half network and destinations on the right-half network is between \((1 - \delta)n/4, (1 + \delta)n/4\) for any \( \delta > 0 \).

Given that we have a network realization satisfying the properties in Lemma 5.1, we consider a cut dividing the \( \sqrt{n} \times \sqrt{n} \) network area into two equal halves (see Figure 7). We are interested in bounding above the sum of the rates of communications passing through the cut from left to right. By Part (c) of the lemma, this sum-rate is equal to \( 1/4 \)'th of the total throughput \( T(n) \) w.h.p. The maximum achievable sum-rate between these S-D pairs is bounded above by the capacity of the MIMO channel between nodes \( S \) located to the left of the cut and nodes \( D \) located to the right. Under the fast fading assumption, we have

\[
\sum_{k \in S, i \in D} R_{ik} \leq \max_{Q(H) \geq 0, \|Q(H)\| \leq P, \forall k \in S} \mathbb{E} \left( \log \det \left( I + HQ(H)H^* \right) \right),
\]

where

\[
h_{ik} = \frac{\sqrt{G}}{r_{ik}^{\alpha/2}} e^{j \phi_{ik}}, \quad k \in S, i \in D.
\]
Fig. 7. The cut-set considered in the proof of Theorem 5.2. The communication requests that pass across the cut from left to right are depicted in bold lines.

$Q(\cdot)$ is a mapping from the set of possible channel realizations $H$ to the set of positive semi-definite transmit covariance matrices. The diagonal element $Q_{kk}(H)$ corresponds to the power allocated to the $k$th node at channel state $H$. A natural way to upperbound (9) is by relaxing the individual power constraint to a total transmit power constraint. In the present context however, this is not convenient: some nodes in $S$ are close to the cut and some are far apart, so the impact of these nodes on the system performance is quite different. A total transmit power constraint allows the transfer of power from the nodes far apart to those nodes that are close to the cut, resulting in a loose bound. Instead, we will relax the individual power constraints to a total weighted power constraint, where the weight assigned to a node is set to be the total received power on the other side of the cut per watt of transmit power from that node. However, before doing that, we need to isolate the contribution of some nodes in $D$ that are located very close to the cut. Typically, there are few nodes on both sides of the cut that are located at a distance as small as order $\frac{1}{\sqrt{n}}$ from the cut. If included, the contribution of these few pairs to the total received power would be excessive, resulting in a loose bound in the discussion below.

Let $V_D$ denote the set of nodes located on the $1 \times \sqrt{n}$ rectangular area immediately to the right of the cut. Note that there are no more than $\sqrt{n} \log n$ nodes in $V_D$ by Part (a) of Lemma 5.1.
By generalized Hadamard’s inequality, we have
\[
\log \det(I + HQ(H)H^*) \leq \log \det(I + H_1Q(H)H_1^*) + \log \det(I + H_2Q(H)H_2^*)
\]
where \(H_1\) and \(H_2\) are obtained by partitioning the original matrix \(H\): \(H_1\) is the rectangular matrix with entries \(h_{ik}, k \in S, i \in V_D\) and \(H_2\) is the rectangular matrix with entries \(h_{ik}, k \in S, i \in D \setminus V_D\). In turn, (9) is bounded above by
\[
\sum_{k \in S, i \in D} R_{ik} \leq \max_{Q(H_1) \geq 0} \mathbb{E} \left( \log \det(I + H_1Q(H_1)H_1^*) \right) + \max_{Q(H_2) \geq 0} \mathbb{E} \left( \log \det(I + H_2Q(H_2)H_2^*) \right). \tag{10}
\]

The first term in (10) can be easily upperbounded by applying Hadamard’s inequality once more or equivalently by considering the sum of the capacities of the individual MISO channels between nodes in \(S\) and each node in \(V_D\). A discussion similar to the proof of Theorem 3.1 that makes use of the fact that the minimum distance between any two nodes in the network is larger than \(\frac{1}{n^{2/3+\delta}}\) with high probability for any \(\delta > 0\), yields the following upper bound for the first term
\[
\max_{Q(H_1) \geq 0} \mathbb{E} \left( \log \det(I + H_1Q(H_1)H_1^*) \right) \leq K' \sqrt{n} (\log n)^2
\]
where \(K' > 0\) is a constant independent of \(n\).

The second term in (10) is the capacity of the MIMO channel between nodes in \(S\) and nodes in \(D \setminus V_D\). This is the term that dominates in (10) and thus its scaling determines the scaling of (9). The result is given by the following lemma, which completes the proof of Theorem 5.2.

**Lemma 5.2:** Let \(P_{\text{tot}}(n)\) be the total power received by all the nodes in \(D \setminus V_D\), when each node in \(S\) is transmitting independent signals at full power. Then for every \(\epsilon > 0\),
\[
\max_{Q(H) \geq 0} \mathbb{E} \left( \log \det(I + HQ(H)H^*) \right) \leq n^\epsilon P_{\text{tot}}(n)
\]
where \(H\) is the channel matrix between nodes in \(S\) and \(D \setminus V_D\). Moreover, the scaling of the total received power can be evaluated to be
\[
P_{\text{tot}}(n) \leq \begin{cases} 
K' n (\log n)^3 & \alpha = 2 \\
K' n^{2-\alpha/2} (\log n)^2 & 2 < \alpha < 3 \\
K' \sqrt{n} (\log n)^3 & \alpha = 3 \\
K' \sqrt{n} (\log n)^2 & \alpha > 3 
\end{cases}
\]
with high probability for a constant $K' > 0$ independent of $n$. □

Lemma 5.2 says two things of importance. First, it says that independent signalling at the transmit nodes is sufficient to achieve the cutset upper bound, as far as scaling is concerned. There is therefore no need, in order for the transmit nodes to cooperate, to do any sort of transmit beamforming. This is fortuitous since our hierarchical MIMO performs only independent signalling across the transmit nodes in the long-range MIMO phase. Second, it identifies the total received power under independent transmissions as the fundamental quantity limiting performance. Depending on $\alpha$, there is a dichotomy on how this quantity scales with the system size. This dichotomy can be interpreted as follows.

The total received power is dominated either by the power transferred between nodes near the cut (order 1 distance) or by the power transferred between nodes far away from the cut. There are relatively fewer node pairs near the cut than away from the cut (order $\sqrt{n}$ versus order $n^2$), but the channels between the nodes near the cut are considerably stronger than between the nodes far away from the cut. When the attenuation parameter $\alpha$ is less than 3, the received power is dominated by transfer between nodes far away from the cut. The hierarchical scheme, which involves at the top level of the hierarchy MIMO transmissions between clusters of size $n^{1-\epsilon}$ at distance $\sqrt{n}$ apart, achieves arbitrarily closely the required power transfer and is therefore optimal in this regime. When $\alpha \geq 3$, the received power in the cutset bound is dominated by the power transfer by the nodes near the cut. This can be achieved by nearest neighbor multihop and multihop is therefore optimal in this regime.

It should be noted that earlier works identified thresholds on $\alpha$ above which nearest neighbor multihop is order-optimal ($\alpha > 5$ in [7], $\alpha > 4.5$ in [10] and $\alpha > 4$ in [9].) All of them essentially use the same cutset bound as we did. The fact that they did not identify the tightest threshold (which we are showing to be 3) is because their upper bounds on the cutset bound are not tight.

**Proof of Lemma 5.2**: Note that we have changed notation denoting $H_2$ simply by $H$, however keeping in mind that the entries $h_{ik}$ of $H$ are such that $k \in S$ and $i \in D \setminus V_D$. We are interested
in the scaling of the MIMO capacity,

\[
\max_{\bar{Q}(\bar{H}) \geq 0} \mathbb{E} \left( \log \det \left( I + H \bar{Q}(H) H^* \right) \right),
\]

(11)

Let us rescale each column \( k \) of the matrix by the (square root of the) total received power on the right from source node \( k \) on the left. Let indeed \( P_k \) denote the total received power in \( D \setminus V_D \) of the signal sent by user \( k \in S \):

\[
P_k = P G \sum_{i \in D \setminus V_D} r^{-\alpha}_{ik} = P G d_k.
\]

The expression (11) is then equal to

\[
\max_{\bar{Q}(\bar{H}) \geq 0} \mathbb{E} \left( \log \det \left( I + \bar{H} \bar{Q}(\bar{H}) \bar{H}^* \right) \right),
\]

where

\[
\bar{h}_{ik} = \frac{e^{j \varphi_{ik}}}{r_{ik}^{\alpha/2}} \frac{1}{\sqrt{d_k}}.
\]

The above expression is in turn bounded above by

\[
\max_{\bar{Q}(\bar{H}) \geq 0} \mathbb{E} \left( \log \det \left( I + \bar{H} \bar{Q}(\bar{H}) \bar{H}^* \right) \right),
\]

where \( P_{tot}(n) = \sum_{k \in S} P_k = P G \sum_{k \in S, i \in D \setminus V_D} r^{-\alpha}_{ik} \).

Let us now define, for given \( n \geq 1 \) and \( \varepsilon > 0 \), the set

\[
B_{n,\varepsilon} = \{ \| \bar{H} \|^2 > n^\varepsilon \},
\]

where \( \| A \| \) denotes the largest singular value of the matrix \( A \). Note that the matrix \( \bar{H} \) is better conditioned than the original channel matrix \( H \): all the diagonal elements of \( \bar{H} \bar{H}^* \) are roughly of the same order (up to a factor \( \log n \)), and it can be shown that there exists \( K'_1 > 0 \) such that

\[
\mathbb{E}(\| \bar{H} \|^2) \leq K'_1 (\log n)^3
\]

for all \( n \). In Appendix III we show the following more precise statement.

**Lemma 5.3:** For any \( \varepsilon > 0 \) and \( p \geq 1 \), there exists \( K'_1 > 0 \) such that for all \( n \),

\[
\mathbb{P}(B_{n,\varepsilon}) \leq \frac{K'_1}{n^p}.
\]
It follows that
\[
\max_{\tilde{Q}(H) \geq 0} \mathbb{E} \left( \log \det (I + \tilde{H} \tilde{Q}(\tilde{H}) \tilde{H}^*) \right) \leq \max_{\tilde{Q}(H) \geq 0} \mathbb{E} \left( \log \det (I + \tilde{H} \tilde{Q}(\tilde{H}) \tilde{H}^*) 1_{B_{n,\varepsilon}} \right) + \max_{\tilde{Q}(H) \geq 0} \mathbb{E} \left( \text{Tr}(\tilde{H} \tilde{Q}(\tilde{H}) \tilde{H}^*) 1_{B_{n,\varepsilon}} \right) \tag{12}
\]

The first term in (12) refers to the event that the channel matrix \( \tilde{H} \) is accidentally ill-conditioned. Since the probability of such an event is polynomially small by Lemma 5.3, the contribution of this first term is actually negligible. In the second term in (12), the matrix \( \tilde{H} \) is well conditioned, and this term is actually proportional to the maximum power transfer from left to right. Details follow below.

For the first term in (12), we use Hadamard’s inequality and obtain
\[
\max_{\tilde{Q}(H) \geq 0} \mathbb{E} \left( \log \det (I + \tilde{H} \tilde{Q}(\tilde{H}) \tilde{H}^*) 1_{B_{n,\varepsilon}} \right) \leq \max_{\tilde{Q}(H) \geq 0} \mathbb{E} \left( \sum_{i \in D \setminus V_D} \log(1 + \tilde{h}_i \tilde{Q}(\tilde{H}) \tilde{h}_i^*) \bigg| B_{n,\varepsilon} \right) \mathbb{P}(B_{n,\varepsilon})
\]

where \( \tilde{h}_i \) is the \( i \)th row of \( \tilde{H} \). By Jensen’s inequality, this expression in turn is bounded above by
\[
\max_{\tilde{Q}(H) \geq 0} \sum_{i \in D \setminus V_D} \log \left( 1 + \mathbb{E} \left( \|\tilde{h}_i\|^2 \text{Tr}\tilde{Q}(\tilde{H}) \bigg| B_{n,\varepsilon} \right) \right) \mathbb{P}(B_{n,\varepsilon}) \leq \max_{\tilde{Q}(H) \geq 0} \sum_{i \in D \setminus V_D} \log \left( 1 + \mathbb{E} \left( \|\tilde{h}_i\|^2 \text{Tr}\tilde{Q}(\tilde{H}) \bigg/ \mathbb{P}(B_{n,\varepsilon}) \right) \right) \mathbb{P}(B_{n,\varepsilon}) \leq n \log \left( 1 + \frac{n P_{\text{tot}}(n)}{\mathbb{P}(B_{n,\varepsilon})} \right) \mathbb{P}(B_{n,\varepsilon}),
\]

since
\[
\|\tilde{h}_i\|^2 = \sum_{k \in S} r_{ik}^{-\alpha} \frac{1}{d_k} \leq \sum_{k \in S} 1 \leq n.
\]

The fact that the minimum distance between the nodes in \( S \) and \( D \setminus V_D \) is at least 1 yields \( P_{\text{tot}}(n) \leq PGn^2 \). Noting that \( x \mapsto x \log(1 + 1/x) \) is increasing on \([0, 1]\) and using Lemma 5.3
we obtain finally that for any $p \geq 1$, there exists $K'_1 > 0$ such that

$$
\mathbb{E} \left( \max_{\tilde{Q} \geq 0 \atop \mathbb{E} (\text{Tr} \tilde{Q}) \leq P_{\text{tot}}(n)} \log \det (I + \tilde{H} \tilde{Q} \tilde{H}^*) 1_{B_n,\epsilon} \right) \leq K'_1 n^{1-p} \log \left( 1 + \frac{n^{2+p}}{K'_1} \right),
$$

which decays polynomially to zero with arbitrary exponent as $n$ tends to infinity.

For the second term in (12), we simply have

$$
\mathbb{E} \left( \text{Tr}(\tilde{H} \tilde{Q}(\tilde{H}) \tilde{H}^*) 1_{B_n,\epsilon} \right) \leq \mathbb{E} \left( \| \tilde{H} \|^2 \text{Tr}(\tilde{Q}(\tilde{H}) \tilde{H}^*) 1_{B_n,\epsilon} \right) \leq n^\epsilon P_{\text{tot}}(n).
$$

The last thing that needs therefore to be checked is the scaling of $P_{\text{tot}}(n)$ stated in Lemma 5.2

Let us divide the network area into $n$ squarelets of area 1. By Part (a) of Lemma 5.1 there are no more than $\log n$ nodes in each squarelet with high probability. Let us consider grouping the squarelets of $S$ into $\sqrt{n}$ rectangular areas $S_m$ of height 1 and width $\sqrt{n}$ as shown in Figure 8. Thus, $S = \bigcup_{m=1}^{\sqrt{n}} S_m$. We are interested in bounding above

$$
P_{\text{tot}}(n) = PG \sum_{k \in S} d_k = PG \sum_{m=1}^{\sqrt{n}} \sum_{k \in S_m} d_k.
$$

Let us consider

$$
\sum_{k \in S_m} d_k = \sum_{k \in S_m, i \in D \setminus V_D} r_{ik}^{-\alpha}
$$

(13)
for a given \( m \). Note that if we move the points that lie in each squarelet of \( S_m \) together with the nodes in the squarelets of \( D \setminus V_D \) onto the squarelet vertex as indicated by the arrows in Figure 8, all the (positive) terms in the summation in (13) can only increase since the displacement can only decrease the Euclidean distance between the nodes involved. Note that the modification results in a regular network with at most \( \log n \) nodes at each squarelet vertex on the left and at most \( 2 \log n \) nodes at each squarelet vertex on the right. Considering the same reasoning for all rectangular slabs \( S_m, m = 1, \ldots, \sqrt{n} \) allows to conclude that \( P_{\text{tot}}(n) \) for the random network is with high probability less than the same quantity computed for a regular network with \( \log n \) nodes at each left-hand side vertex and \( 2 \log n \) nodes at each right-hand side vertex.

The most convenient way to index the node positions in the resulting regular network is to use double indices. The left-hand side nodes are located at positions \((-k_x + 1, k_y)\) and those on the right at positions \((i_x, i_y)\) where \( k_x, k_y, i_x, i_y = 1, \ldots, \sqrt{n} \), so that

\[
\tilde{h}_{ik} = \frac{e^{i \varphi_{ik}}}{\alpha/4} \frac{1}{\sqrt{d_{k_x, k_y}}} 
\]

and

\[
d_{k_x, k_y} = \sum_{i_x, i_y = 1}^{\sqrt{n}} \sqrt{\frac{1}{\left((i_x + k_x - 1)^2 + (i_y - k_y)^2\right)^{\alpha/2}}} 
\]

which yields the following upper bound for \( P_{\text{tot}}(n) \) of the random network,

\[
P_{\text{tot}}(n) \leq 2(\log n)^2 PG \sum_{k_x, k_y = 1}^{\sqrt{n}} d_{k_x, k_y}. \tag{15}
\]

The following lemma establishes the scaling of \( d_{k_x, k_y} \) defined in (14).

**Lemma 5.4:** There exist constants \( K_2, K_3' > 0 \) independent of \( k_x, k_y \) and \( n \) such that

\[
d_{k_x, k_y} \leq \begin{cases} 
K_2' \log n & \text{if } \alpha = 2, \\
K_2' k_x^{2-\alpha} & \text{if } \alpha > 2,
\end{cases}
\]

and

\[
d_{k_x, k_y} \geq K_3' k_x^{2-\alpha} \quad \text{for } \alpha \geq 2.
\]

The rigorous proof of the lemma is given at the end of Appendix III. A heuristic way of thinking about the approximation

\[
d_{k_x, k_y} \approx k_x^{2-\alpha}
\]

can be obtained through Laplace’s principle. The summation in \( d_{k_x, k_y} \) scales the same as the maximum term in the sum times the number of terms which have roughly this maximum value.
The maximum term is of the order of $1/k^\alpha_x$. The terms that take on roughly this value are those for which $i_x$ runs from 1 to the order of $k_x$ and $i_y$ runs from $k_y$ to $k_y$ plus or minus the order of $k_x$. There are roughly $k^2_x$ such terms. Hence $d_{k_x,k_y} \approx 1/k^\alpha_x \cdot k^2_x = k^{2-\alpha}_x$.

We can now use the upper bound given in the above lemma to yield:

$$ \sum_{k_x,k_y=1}^{\sqrt{n}} d_{k_x,k_y} \leq \begin{cases} 
    K'_4 n \log n & \text{if } \alpha = 2, \\
    K'_4 n^{2-\alpha/2} & \text{if } 2 < \alpha \leq 3, \\
    K'_4 \sqrt{n} \log n & \text{if } \alpha = 3, \\
    K'_4 \sqrt{n} & \text{if } \alpha > 3
\end{cases}$$

for another constant $K'_4 > 0$ independent of $n$. This upper bound combined with (15) completes the proof of Lemma 5.2.

VI. CONCLUSIONS

In this paper, we have shown that near global MIMO cooperation between nodes can be achieved in ad hoc networks without significant cooperation overhead. This is a surprising result, as it allows the full degrees of freedom in the network to be shared among all nodes and implies that interference is not a fundamental limitation. In dense networks where all nodes are within communication range of each other, this yields a linear capacity scaling. In extended networks, such near-global MIMO cooperation also allows the maximum transfer of energy between all source-destination pairs. This leads to the identification of the optimal (power-limited) capacity scaling law of extended networks for all values of $\alpha$.

The key ideas behind our scheme are:

- using MIMO for long-range communication to achieve spatial multiplexing;
- local transmit and receive cooperation to maximize spatial reuse;
- setting up the intra-cluster cooperation such that it is yet another digital communication problem, but in a smaller network, thus enabling a hierarchical cooperation architecture.

Our result is based on only very weak assumptions about the channel. It is valid for any path loss exponent $\alpha \geq 2$. It holds regardless of whether there are multipaths, as long as nodal separation is much larger than the carrier wavelength, so that the phases of the channels are random. This is sufficient to enable MIMO. We have focused on the 2-D setting, where the nodes are on the plane, but our results generalize naturally to $d$-dimensional networks.
David Reed raised the question of whether a linear capacity scaling is possible, that provided part of the impetus for this research. The authors would also like to thank to Emre Telatar, Shuchin Aeron and Venkatesh Saligrama for many helpful discussions. The work of Ayfer Özgür was supported by Swiss NSF grant Nr 200021-10808. Part of the work by Olivier Lévêque was performed when he was post-doc at Stanford University, supported by Swiss NSF grant Nr PA002-108976. The work of David Tse was supported by the U.S. National Science Foundation via an ITR grant: “The 3R’s of Spectrum Management: Reuse, Reduce and Recycle”.

APPENDIX I

LINEAR SCALING LAW FOR THE MIMO GAIN UNDER FAST FADING ASSUMPTION

Proof of Lemma 4.3: The $M \times M$ MIMO channel between two clusters $S$ and $D$ is given by $Y = HX + Z$, where $h_{ik}$ are given in (1). Recall that $Z = (z_k)$ is uncorrelated background noise plus interference at the receiver nodes. Assume that the transmitted signals $X = (x_i)$ are from an i.i.d. $\sim \mathcal{N}_C(0, P)$ randomly chosen codebook with

$$P = \frac{K_p(r_{SD})^\alpha}{M}.$$ 

It is well known that the achievable mutual information is lower bounded by assuming that the interference-plus-noise $Z$ is i.i.d. Gaussian. (see for example Theorem 5 of [12] for a precise statement and proof of this in the MIMO case.) With our transmission strategy in the MIMO phase, there exists $b > a > 0$ such that $r_{ik}^{-\alpha/2} = r_{SD}^{-\alpha/2} \rho_{ik}$, where all $\rho_{ik}$ lie in the interval $[a, b]$ both in the cases when $S$ and $D$ are neighboring clusters or not.

By assuming perfect channel state information at the receiver side, the mutual information of the above MIMO channel is given by

$$I(X; Y, H) \geq \mathbb{E} \left( \log \det \left( I + \frac{P}{N} HH^* \right) \right) = \mathbb{E} \left( \log \det \left( I + \frac{\text{SNR}}{M} FF^* \right) \right),$$

where $\text{SNR} = \frac{G K_p}{N}$ ($N$= total interference-plus-noise power) and $f_{ik} = \rho_{ik} \exp(j \theta_{ik})$. Let $\lambda$ be chosen uniformly among the $M$ eigenvalues of $\frac{1}{M} FF^*$. The above mutual information may be rewritten as

$$I(X; Y, H) \geq M \mathbb{E}(\log(1 + \text{SNR} \lambda)) \geq M \log(1 + \text{SNR} t) \mathbb{P}(\lambda > t),$$

34
for any $t \geq 0$. By the Paley-Zygmund inequality, if $0 \leq t < \mathbb{E}(\lambda)$, we have

$$\mathbb{P}(\lambda > t) \geq \frac{(\mathbb{E}(\lambda) - t)^2}{\mathbb{E}(\lambda^2)}.$$ 

We therefore need to compute both $\mathbb{E}(\lambda)$ and $\mathbb{E}(\lambda^2)$. Let us denote by $\mathbb{E}_r$ the conditional expectation with respect to the random distances $r_{ik}$. We then obtain

$$\mathbb{E}_r(\lambda) = \frac{1}{M} \mathbb{E}_r \left( \text{Tr} \left( \frac{1}{M} FF^* \right) \right)$$

$$= \frac{1}{M^2} \sum_{i,k=1}^{M} \mathbb{E}_r(|f_{ik}|^2)$$

$$= \frac{1}{M^2} \sum_{i,k=1}^{M} \rho_{ik}^2 \geq a^2$$

and

$$\mathbb{E}_r(\lambda^2) = \frac{1}{M} \mathbb{E}_r \left( \text{Tr} \left( \frac{1}{M^2} F F^* F F^* \right) \right)$$

$$= \frac{1}{M^3} \sum_{i,k,l,m=1}^{M} \mathbb{E}_r(f_{ik} f_{lk} f_{lm} f_{im})$$

$$\leq \frac{2}{M^3} \sum_{i,k,m=1}^{M} \mathbb{E}_r(|f_{ik}|^2) \mathbb{E}_r(|f_{im}|^2)$$

$$= \frac{2}{M^3} \sum_{i,k,m=1}^{M} \rho_{ik}^2 \rho_{im}^2 \leq 2b^4,$$

so $\mathbb{E}(\lambda) \geq a^2$ and $\mathbb{E}(\lambda^2) \leq 2b^4$. This leads us to the conclusion that for any $t < a$, we have

$$I(X; Y, H) \geq M \log(1 + \text{SNR} t) \frac{(a^2 - t)^2}{2b^4}, \quad (16)$$

Choosing e.g. $t = a/2$ shows that $I(X; Y, H)$ grows at least linearly with $M$. \hfill \Box

**Lemma I.1 (Paley-Zygmund Inequality)** Let $X$ be a non-negative random variable such that $\mathbb{E}(X^2) < \infty$. Then for any $t \geq 0$ such that $t < \mathbb{E}(X)$, we have

$$\mathbb{P}(X > t) \geq \frac{(\mathbb{E}(X) - t)^2}{\mathbb{E}(X^2)}.$$

**Proof:** By the Cauchy-Schwarz inequality, we have for any $t \geq 0$:

$$\mathbb{E}(X 1_{X > t}) \leq \sqrt{\mathbb{E}(X^2) \mathbb{P}(X > t)}.$$
and also, if \( t < \mathbb{E}(X) \),

\[
\mathbb{E}(X 1_{X > t}) = \mathbb{E}(X) - \mathbb{E}(X 1_{X \leq t}) \geq \mathbb{E}(X) - t > 0.
\]

Therefore,

\[
\mathbb{P}(X > t) \geq \frac{(\mathbb{E}(X) - t)^2}{\mathbb{E}(X^2)}.
\]

\[\square\]

**Appendix II**

**Achievable Rates on Quantized Channels**

In order to conclude the discussion on the throughput achieved by our scheme, we need to show that the quantized MIMO channel achieves the same spatial multiplexing gain of the MIMO channel. We take the opportunity to establish a more general result, Theorem II.1, on achievable rates on quantized channels. The required result for the quantized MIMO channel can then be found as an easy application of Theorem II.1. We start by formally defining the general quantized channel problem in a form that is of interest to us and proceed with several definitions that will be needed in the sequel.

Let us consider a discrete-time memoryless channel with single input of alphabet \( \mathcal{X} \) and \( M \) outputs of respective alphabets \( \mathcal{Y}_1, \ldots, \mathcal{Y}_M \). The channel is statistically described by a conditional probability distribution \( p(y_1, \ldots, y_M|x) \) for each \( y_1 \in \mathcal{Y}_1, \ldots, y_M \in \mathcal{Y}_M \) and \( x \in \mathcal{X} \). The outputs of the channel are to be followed by quantizers which independently map the output alphabets \( \mathcal{Y}_1, \ldots, \mathcal{Y}_M \) to the respective reproduction alphabets \( \hat{\mathcal{Y}}_1, \ldots, \hat{\mathcal{Y}}_M \). The aim is to recover the transmitted information through the channel by observing the outputs of the quantizers. Communication over the channel takes place in the following manner: A message \( W \), drawn from the index set \( \{1, 2, \ldots, L\} \) is encoded into a codeword \( X^m(W) \in \mathcal{X}^m \), which is received as \( M \) random sequences \( (Y_1^m, \ldots, Y_M^m) \sim p(y_1^m, \ldots, y_M^m|x^m) \) at the outputs of the channel. The quantizers themselves consist of encoders and decoders, where the \( j \)'th encoder describes its corresponding received sequence \( Y_j^m \) by an index \( I_j(Y_j^m) \in \{1, 2, \ldots, L_j\} \), and decoder \( j \) represents \( Y_j^m \) by an estimate \( \hat{Y}_j^m(I_j) \in \hat{\mathcal{Y}}_j^m \). The channel decoder then observes the reconstructed sequences \( \hat{Y}_1^m, \ldots, \hat{Y}_M^m \) and guesses the index \( W \) by an appropriate decoding rule \( \hat{W} = g(\hat{Y}_1^m, \ldots, \hat{Y}_M^m) \).

An error occurs if \( \hat{W} \) is not the same as the index \( W \) that was transmitted. The complete model under investigation is shown in Fig. 7. An \((L; L_1, \ldots, L_M; m)\) code for this channel is
a joint \((L, m)\) channel and \(M\) quantization codes \((L_1, m), \ldots, (L_M, m)\); more specifically, is two sets of encoding and decoding functions, the first set being the channel encoding function \(X^m : \{1, 2, \ldots, L\} \rightarrow X^m\) and the channel decoding function \(g : \hat{Y}_1^m \times \cdots \times \hat{Y}_M^m \rightarrow \{1, 2, \ldots, L\}\), and the second set consists of the encoding functions \(I_j : \mathcal{Y}_j^m \rightarrow \{1, 2, \ldots, L_j\}\) and decoding functions \(\hat{Y}_j^m : \{1, 2, \ldots, L_j\} \rightarrow \hat{Y}_j^m\) for \(j = 1, \ldots, M\) used for the quantizations. We define the (average) probability of error for the \((L; L_1, \ldots, L_M; m)\) code by
\[
P_m^e = \frac{1}{L} \sum_{k=1}^{L} \mathbb{P}(\hat{W} \neq k | W = k).
\]
A set of rates \((R; R_1, \ldots, R_M)\) is said to be achievable if there exists a sequence of \((2^{mR}; 2^{mR_1}, \ldots, 2^{mR_M}; m)\) codes with \(P_m^e \rightarrow 0\) as \(m \rightarrow \infty\). Note that determining achievable rates \((R; R_1, \ldots, R_M)\) is not a trivial problem, since there is trade-off between maximizing \(R\) and minimizing \(R_1, \ldots, R_M\).

**Theorem II.1 (Achievability for the Quantized Channel Problem)** Given a probability distribution \(q(x)\) on \(\mathcal{X}\) and \(M\) conditional probability distributions \(q_j(\hat{y}_j | y_j)\) where \(y_j \in \mathcal{Y}_j\) and \(\hat{y}_j \in \hat{Y}_j\) and \(j = 1, \ldots, M\); all rates \((R; R_1, \ldots, R_M)\) such that \(R < I(X; \hat{Y}_1, \ldots, \hat{Y}_M)\) and \(R_j > I(Y_j; \hat{Y}_j)\) are achievable. Specifically, given any \(\delta > 0\), \(q(x)\) and \(q_j(\hat{y}_j | y_j)\), together with rates \(R < I(X; \hat{Y}_1, \ldots, \hat{Y}_M)\) and \(R_j > I(Y_j; \hat{Y}_j)\) for \(j = 1, \ldots, M\); there exists a \((2^{mR}; 2^{mR_1}, \ldots, 2^{mR_M}; m)\) code such that \(P_m^e < \delta\).

**Proof:** The proof of the theorem for discrete finite-size alphabets relies on a random coding argument based on the idea of joint (strong) typicality. For the idea of strong typicality and properties of typical sequences, see [13]. The proof can be outlined as follows. Given \(q(x)\)
generate a random channel codebook $C_c$ with $2^{mR}$ codewords, each of length $m$, independently from the distribution

$$q(x^m) = \prod_{k=1}^{m} q(x^m(k)).$$

and call them $X^m(1), X^m(2), \ldots, X^m(2^{mR})$. Also generate $M$ quantization codebooks $C_j, j = 1, \ldots, M$, each codebook $C_j$ consisting of $2^{mR_j}$ codewords drawn independently from

$$p_j(y^m_j) = \prod_{k=1}^{m} \sum_{x \in X} q(x)p(y_1, \ldots, y_M|x)q_j(y^m_j(k)|y_j).$$

and index them as $\hat{Y}^m_j(1), \hat{Y}^m_j(2), \ldots, \hat{Y}^m_j(2^{mR_j})$. Given the message $w$ send the codeword $X^m(w)$ through the channel. The channel will yield $Y^m_1, \ldots, Y^m_M$. Given the channel output $Y^m_j$ at the $j$'th quantizer, choose $i_j$ such that $(Y^m_j, \hat{Y}^m_j(i_j))$ are jointly typical. If there exist no such $i_j$, declare an error. If the number of codewords in the quantization codebook $2^{mR_j}$ is greater than $2^{mI(Y_j;\hat{Y}_j)}$, the probability of finding no such $i_j$ decreases to zero exponentially as $m$ increases. The probability of failing to find such an index in at least one of the $M$ quantizers is bounded above by the union bound with the sum of $M$ exponentially decreasing probabilities in $m$. Given $\hat{Y}^m_1(i_1), \ldots, \hat{Y}^m_M(i_M)$ at the channel decoder, choose the unique $\hat{w}$ such that $(X^m(\hat{w}), \hat{Y}^m_1(i_1), \ldots, \hat{Y}^m_M(i_M))$ are jointly typical. The fact that $(X^m(w), \hat{Y}^m_1(i_1), \ldots, \hat{Y}^m_M(i_M))$ will be jointly typical with high probability can be established by identifying the Markov chains in the problem and applying Markov Lemma [13, Lemma 14.8.1] repeatedly. Observing that $(Y^m_1, \ldots, Y^m_M, \hat{Y}^m_1, \ldots, \hat{Y}^m_M) - Y^m_{j+1} - \hat{Y}^m_{j+1}$ form a Markov chain and recursively applying Markov Lemma, we conclude that $(Y^m_1, \ldots, Y^m_M, \hat{Y}^m_1(i_1), \ldots, \hat{Y}^m_M(i_M))$ are jointly typical with probability approaching 1 as $m$ increases. Observing that $X^m - (Y^m_1, \ldots, Y^m_M) - (\hat{Y}^m_1, \ldots, \hat{Y}^m_M)$ form another Markov chain, again by Markov Lemma we have $(X^m(w), \hat{Y}^m_1(i_1), \ldots, \hat{Y}^m_M(i_M))$ jointly typical with high probability. If there are more than one codewords $X^m$ that are jointly typical with $(\hat{Y}^m_1(i_1), \ldots, \hat{Y}^m_M(i_M))$, we declare an error. The probability of having more than one such sequence will decrease exponentially to zero as $m$ increases, if the number of channel codewords $2^{mR}$ is less than $2^{mI(X;\hat{Y}^1,\ldots,\hat{Y}^M)}$. Hence if $R < I(X; \hat{Y}_1, \ldots, \hat{Y}_M)$ and $R_j > I(Y_j; \hat{Y}_j)$, the probability of error averaged over all codes decreases to zero as $m \to \infty$. This shows the existence of a code that achieves rates $(R; R_1, \ldots, R_M)$ with arbitrarily small probability of error. The result can be readily extended to memoryless channels with discrete-time and continuous alphabets by standard arguments (see [14, Ch.7]).
Proof of Lemma 4.4: Now we turn to our original problem: We need to show that it is possible to encode the observations at the outputs of the MIMO channel at a fixed rate, while preserving the spatial multiplexing gain of the MIMO channel. This is a direct consequence of Theorem II.1: Consider the conditional probability densities

$$q_j(y_j | y_j) \sim \mathcal{N}_C(y_j, \Delta^2)$$

for the quantization process. From Theorem II.1 we know that for any distribution $p(x)$ on the input space, all rate pairs $(R; R_1, \ldots, R_M)$ are simultaneously achievable if

$$R_j > I(Y_j; \hat{Y}_j) \quad j = 1, \ldots, M \quad \text{and} \quad R < I(X; \hat{Y}_1, \ldots, \hat{Y}_M)$$

where now $R_j$ is the encoding rate of the $j$’th stream and $R$ is the total transmission rate over the MIMO channel. Using Lemma 4.5 we have that $I(Y_j; \hat{Y}_j) \leq \log(1 + \frac{P_2}{\Delta^2})$ for any probability distribution $p(x)$ on the input space. So if we choose

$$R_j = \log(1 + \frac{P_2}{\Delta^2}) + \varepsilon \quad \forall j = 1, \ldots, M$$

for some $\varepsilon > 0$, all rates

$$R \leq I(X; \hat{Y}_1, \ldots, \hat{Y}_M)$$

are achievable on the quantized MIMO channel for any input distribution $p(x)$. Note that now the channel from $X$ to $\hat{Y}_1, \ldots, \hat{Y}_M$ is given by

$$\hat{Y} = HX + Z + D$$

where $D \sim \mathcal{N}_C(0, \Delta^2 I)$. Obviously, this channel has the same spatial multiplexing gain with the original MIMO channel. □

Proof of Lemma 4.6: Consider the case where the MIMO signals are corrupted by interference of increasing power $K_I \log M$. In this case, the power received by the destination nodes is not bounded anymore and increases as $P_2 + K_I \log M$ with increasing $M$. In order to apply the technique employed in the proof of Lemma 4.4, one can first normalize the received signal by multiplying it by $q = \sqrt{\frac{P_2}{P_2 + K_I \log M}}$ and then do the quantization as before. Note that the resultant scaled quantized MIMO channel is given by

$$\hat{Y} = q(HX + Z) + D$$
where again \( D \sim N_C(0, \Delta^2 I) \) and \( Z = (z_k) \) is the background noise plus interference vector independent of the signal with uncorrelated entries of power \( \mathbb{E}[z_k^2] \leq N_0 + K_I \log M \). Thus we can apply the result of Lemma 4.3. Note that the resultant signal-to-noise-ratio \( \text{SNR} \geq \frac{K}{\log M} \) for a constant \( K > 0 \). Plugging this SNR expression into (16) yields \( M/\log M \) capacity scaling for the resultant channel.

\[ \square \]

**Appendix III**

**Largest eigenvalue behaviour of the equalized channel matrix \( \tilde{H} \)**

In this appendix, we give the proofs of Lemma 5.3 and Lemma 5.4. We start with Lemma 5.3. The proof of the second lemma is given at the end of the section.

**Proof of Lemma 5.3** Let us start by considering the \( 2m^{th} \) moment of the spectral norm of \( \tilde{H} \) given by (see [15, Ch. 5])

\[
\| \tilde{H} \|^2 \rho = \rho (\tilde{H}^* \tilde{H})^m = \lim_{l \to \infty} \{ \text{Tr}((\tilde{H}^* \tilde{H})^l) \}^{m/l}. 
\]

By dominated convergence theorem and Jensen’s inequality, we have

\[
\mathbb{E}(\| \tilde{H} \|^2) \leq \lim_{l \to \infty} \{ \mathbb{E}(\text{Tr}((\tilde{H}^* \tilde{H})^l)) \}^{m/l}. 
\]

In the subsequent paragraphs, we will prove that the following upper bound holds with high probability,

\[
\mathbb{E}(\text{Tr}((\tilde{H}^* \tilde{H})^l)) \leq t_l n (K'_1 \log n)^{3l} \tag{17}
\]

where \( t_l = \frac{(2l)!}{l(l+1)!} \) are the Catalan numbers and \( K'_1 > 0 \) is a constant independent of \( n \). By Chebyshev’s inequality, this allows to conclude that for any \( m \),

\[
\mathbb{P}(B_{n,\varepsilon}) \leq \frac{\mathbb{E}(\| \tilde{H} \|^2)}{n^{m\varepsilon}} \leq \frac{1}{n^{m\varepsilon}} \lim_{l \to \infty} (t_l n (K'_1 \log n)^{3l})^{m/l} \leq \frac{(4(K'_1 \log n)^3)^m}{n^{m\varepsilon}},
\]

since \( \lim_{l \to \infty} t_l^{1/l} = 4 \). For any \( \varepsilon > 0 \), choosing \( m \) sufficiently large shows therefore that \( \mathbb{P}(B_{n,\varepsilon}) \) decays polynomially with arbitrary exponent as \( n \to \infty \), which is the result stated in Lemma 5.3.

There remains to prove the upper bound in (17). Expanding the expression gives

\[
\mathbb{E}(\text{Tr}((\tilde{H}^* \tilde{H})^l)) = \sum_{i_1, \ldots, i_l = 1}^{n} \sum_{k_1, \ldots, k_l = 1} \mathbb{E} \left( \overline{\tilde{h}_{i_1,k_1}} \overline{\tilde{h}_{i_2,k_2}} \overline{\tilde{h}_{i_3,k_3}} \overline{\tilde{h}_{i_4,k_4}} \ldots \overline{\tilde{h}_{i_l,k_l}} \right). \tag{18}
\]

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Recall that the random variables $\tilde{h}_{ik}$ are independent and zero-mean, so the expectation is only non-zero when the terms in the product form conjugate pairs. Let us consider the case $l = 2$ as an example. We have,

$$
\mathbb{E}(\text{Tr}((\tilde{H}^*\tilde{H})^2)) = \sum_{i_1,i_2=1}^{n} \mathbb{E}(\tilde{h}_{i_1,k_1}\tilde{h}_{i_2,k_2}\overline{\tilde{h}_{i_2,k_1}}\overline{\tilde{h}_{i_1,k_2}}) 
$$

(19)

$$
= \sum_{i_1,i_2,k=1}^{n} |\tilde{h}_{i_1,k}|^2|\tilde{h}_{i_2,k}|^2 + \sum_{i,k_1,k_2=1}^{n} |\tilde{h}_{ik_1}|^2|\tilde{h}_{ik_2}|^2 
$$

(20)

since the expectation is non-zero only when either $k_1 = k_2 = k$ or $i_1 = i_2 = i$. Note that we have removed the expectations in (20) since $|\tilde{h}_{ik}|^2$ is a deterministic quantity in our case. The expression can be bounded above by

$$
\mathbb{E}(\text{Tr}((\tilde{H}^*\tilde{H})^2)) \leq \sum_{i_1,i_2,k=1}^{n} |\tilde{h}_{i_1,k}|^2|\tilde{h}_{i_2,k}|^2 + \sum_{i,k_1,k_2=1}^{n} |\tilde{h}_{ik_1}|^2|\tilde{h}_{ik_2}|^2 
$$

(21)

where we now doublecount the terms with $i_1 = i_2 = i$ and $k_1 = k_2 = k$, that is, the terms of the form $|\tilde{h}_{ik}|^4$.

The non-vanishing terms in the sum in (19) can also be determined by the following approach, which generalizes to larger $l$: let each index be associated to a vertex and each term in the product in (19) to an edge between its corresponding vertices. Note that the resulting graph is in general a ring with 4 edges as depicted in Figure 10. A term in the summation in (19) is only non-zero if each edge of its corresponding graph has even multiplicity. Such a graph can be obtained from the ring in Figure 10 by merging some of the vertices, thus equating their corresponding indices. For example, merging the vertices $k_1$ and $k_2$ into a single vertex $k$ gives the graph in Fig. 10. The product in Eq. (19) illustrated as a ring.
Figure 11-a; on the other hand, merging $i_1$ and $i_2$ into a single vertex $i$ gives Figure 11-b. Note that $i_1$, $i_2$ and $k$ can take values 1, . . . , $n$ in the first figure, thus the sum of all such terms yields

$$\sum_{i_1, i_2, k=1}^{n} |\tilde{h}_{i_1 k}|^2 |\tilde{h}_{i_2 k}|^2 . \quad (22)$$

Similarly, the terms of the form in Figure 11-b sum up to

$$\sum_{i, k_1, k_2=1}^{n} |\tilde{h}_{ik_1}|^2 |\tilde{h}_{ik_2}|^2 . \quad (23)$$

Note that another possible graph composed of edges with even multiplicity can be obtained by further merging the vertices $i_1$ and $i_2$ into a single vertex $i$ in Figure 11-a, or equivalently merging $k_1$ and $k_2$ into $k$ in Figure 11-b. This will result in a graph with only two vertices $k$ and $i$ and a quadruple edge in between which corresponds to terms of the form $|\tilde{h}_{ik}|^4$ with $i$ and $k$ running from 1 to $n$. Note however that such terms have already been considered in both (22) and (23) since we did not exclude the case $i_1 = i_2$ in (22) and $k_1 = k_2$ in (23). In fact, terms corresponding to any graph with number of vertices less than 3 are already accounted for in either one of the sums in (22) and (23), or simultaneously in both. Hence, the sum of (22) and (23) is an upper bound for (19) yielding again (21).

In the general case with arbitrary $l$, considering (18) leads to a larger ring with $2l$ edges, as depicted in Figure 12. Similarly to the case $l = 2$, the non-vanishing terms in (18) are those that correspond to a graph having only edges of even multiplicity. Since each edge can have at least
double multiplicity, such graphs can have at most \( l \) edges. In turn, a graph with \( l \) edges can have at most \( l + 1 \) vertices which is the case of a tree. Hence, let us first start by considering such trees; namely, planar trees with \( l \) branches that are rooted (at \( k_1 \)) and planted, implying that rotating asymmetric trees around the root results in a new tree. See Figure 13 which depicts the five possible trees with \( l = 3 \) branches where we label the resultant \( l + 1 = 4 \) vertices as \( p_1, \ldots, p_4 \). In general, the number of different planar, planted, rooted trees with \( l \) branches is given by the \( l \)'th Catalan number \( t_l \) [16]. In each of these trees, the \( l + 1 \) vertices \( p_1, \ldots, p_{l+1} \)
can take values in $1, \ldots, n$. Hence, each tree $T_i^l$ corresponds to a group of non-zero terms
\begin{equation}
T_i^l = \sum_{p_1, \ldots, p_{l+1}} f_{T_i^l}(p_1, \ldots, p_{l+1}), \quad i = 1, \ldots, t_l. \tag{24}
\end{equation}

Note that if a non-vanishing term in (18) corresponds to a graph with less than $l + 1$ vertices, then the corresponding graph possesses edges with multiplicity larger than 2 or cycles, and this term is already accounted for in either one or more of the terms in (24). This fact can be observed by noticing that both edges with large multiplicity as well as cycles can be untied to get trees with $l$ branches, with some of the $l + 1$ indices constrained however to share the same values (see Figure 14). Note that such cases are not excluded in the summations in (24), thus we have
\begin{equation}
\mathbb{E}(\text{Tr}((\tilde{H}^* \tilde{H})^l)) \leq \sum_{i=1}^{t_l} T_i^l.
\end{equation}

Below we show that
\begin{equation}
T_i^l \leq n(K_1' \log n)^l, \quad \forall i \tag{25}
\end{equation}
in a regular network and
\begin{equation}
T_i^l \leq n(K_1' \log n)^{3l}, \quad \forall i \tag{26}
\end{equation}
with high probability in a random network. We first concentrate on regular networks in order to reveal the proof idea in the simplest setting. A binning argument then allows to extend the result to random networks.
a) Regular network: Recall that in the regular case, the nodes on the left-half are located at positions $(−k_x + 1, k_y)$ and those on the right half at $(i_x, i_y)$ for $k_x, k_y, i_x, i_y = 1, \ldots, \sqrt{n}$. In this case, the matrix elements of $\tilde{H}$ are given by

$$\tilde{h}_{ik} = \frac{e^{j \varphi_{ik}}}{((i_x + k_x - 1)^2 + (i_y - k_y)^2)^{\alpha/4}} \frac{1}{d_{k_x,k_y}}$$

and

$$d_{k_x,k_y} = \sum_{i_x,i_y=1}^{\sqrt{n}} \frac{1}{((i_x + k_x - 1)^2 + (i_y - k_y)^2)^{\alpha/2}}.$$

In the discussion below, we will need an upper bound on the scaling of $\sum_{i=1}^{n} |\tilde{h}_{ik}|^2$ and $\sum_{k=1}^{n} |\tilde{h}_{ik}|^2$. By Lemma 5.4 we have

$$d_{k_x,k_y} \geq K_3' k_x^{2-\alpha}$$

for a constant $K_3' > 0$ independent of $n$. This, in turn, yields the upper bound

$$|\tilde{h}_{ik}|^2 \leq \frac{1}{K_3'^2} \sum_{i_x,i_y=1}^{\sqrt{n}} \frac{k_x^{\alpha-2}}{((i_x + k_x - 1)^2 + (i_y - k_y)^2)^{\alpha/2}} \leq \frac{1}{K_3'^2} \sum_{i_x,i_y=1}^{\sqrt{n}} \frac{1}{((i_x + k_x - 1)^2 + (i_y - k_y)^2)^{\alpha/2}}.$$

Summing over either $i$ or $k$, and using the upper bound in Lemma 5.4 for $\alpha = 2$ yields

$$\sum_{i=1}^{n} |\tilde{h}_{ik}|^2, \sum_{k=1}^{n} |\tilde{h}_{ik}|^2 \leq K_1' \log n$$

(27)

where $K_1' = \frac{K_2'}{K_3'}$ with $K_2'$ and $K_3'$ being the constants appearing in the lemma.

Let us first consider the simplest case where the tree is composed of $l$ height 1 branches and denote it by $\mathcal{T}_1^l$ (see Figure 15). We have
\[ T_1^l = \sum_{p_1, \ldots, p_{l+1}=1}^n f_{\mathcal{J}_i^l}(p_1, \ldots, p_{l+1}) = \sum_{p_1, \ldots, p_{l+1}=1}^n |\tilde{h}_{p_2p_1}|^2 |\tilde{h}_{p_3p_1}|^2 \ldots |\tilde{h}_{p_{l+1}p_1}|^2 \]
\[ = \sum_{p_1=1}^n \left( \sum_{p_2=1}^n |\tilde{h}_{p_2p_1}|^2 \right)^l \]
\[ \leq n(K'_1 \log n)^l \quad (28) \]

which follows from the upper bound (27).

Now let us consider the general case of an arbitrary tree \( \mathcal{J}_i^l \) having \( s \) leaves, where \( 1 \leq s \leq l \) (see Figure 16). Let the indices corresponding to these leaves be \( m_1, \ldots, m_s \). Let us denote the “parent” vertices of these leaves by \( p_1, \ldots, p_s \) and assume that \( p_1 \) is the parent vertex of leaves \( m_1, \ldots, m_{d_1} \); \( p_2 \) is the parent vertex of leaves \( m_{(d_1+1)}, \ldots, m_{d_2} \) etc. and finally \( p_s \) is the parent of \( m_{(d_s+1)}, \ldots, m_s \). The term \( T_i^l \) corresponding to this tree is given by

\[ T_i^l = \sum_{p_1, \ldots, p_s=1}^n f_{\mathcal{J}_i^l}(p_1, \ldots, p_{l+1}) \]
\[ = \sum_{p_1, \ldots, p_{l+1}=1}^n f_{\mathcal{J}_i^l}^{l-s}(p_1, \ldots, p_{l+1-s}) \]
\[ \times \sum_{m_1, \ldots, m_s=1}^n |\tilde{h}_{m_1p_1}|^2 \ldots |\tilde{h}_{m_{d_1}p_1}|^2 |\tilde{h}_{m_{(d_1+1)}p_2}|^2 \ldots |\tilde{h}_{m_{d_2}p_2}|^2 \ldots |\tilde{h}_{m_{(d_s+1)p_s}}|^2 \ldots |\tilde{h}_{m_{s}p_s}|^2 \]
\[ \leq T_i^{l-s}(K'_1 \log n)^s \quad (29) \]
\[ \leq n(K'_1 \log n)^l \quad (30) \]

where \( T_i^{l-s} \) corresponds to a smaller (and shorter) tree \( \mathcal{J}_i^{l-s} \) with \( l-s \) branches. The argument above decreases the height of the tree by 1, hence can be applied recursively to get a simple tree composed only of height 1 branches in which case the upper bound in (28) applies. Thus, given \( \mathcal{J}_i^l \) let \( h \) be the number of recursions to get a simple tree and \( s_1, \ldots, s_h \) denote the number of leaves in the trees observed at each step of the recursion. We have

\[ T_i^l \leq (K'_1 \log n)^{s_1}(K'_1 \log n)^{s_2} \ldots (K'_1 \log n)^{s_h} T_i^{l-s_1 \ldots - s_h} \]
\[ \leq n(K'_1 \log n)^l \]

\(^5\)Note that the term corresponding to a leaf \( m \) can be either \( |\tilde{h}_{mp}|^2 \) or \( |\tilde{h}_{pm}|^2 \) depending on whether the height of the leaf is even or odd. However, in (29), we do not worry about this fact, since the upper bound (30) applies in both cases.
since \( T_{l-s_1 \cdots -s_h} \leq n(K'_1 \log n)^{l-s_1 \cdots -s_h} \) by (28). Thus, (25) follows.

b) Random network: We denote the locations of the nodes to the left of the cut by \( a_k = (-a^x_k, a^y_k) \) where \( a^x_k \) is the \( x \)-coordinate and \( a^y_k \) is the \( y \)-coordinate of node \( k \in S \) and those to the right of the cut are similarly denoted by \( b_i = (b^x_i, b^y_i) \) for \( i \in D \setminus V_D \). In this case, the matrix elements of \( \tilde{H} \) are given by

\[
\tilde{h}_{ik} = \frac{e^{j \varphi_{ik}}}{\sqrt{d_k}} \frac{1}{((b^x_i + a^x_k)^2 + (b^y_i - a^y_k)^2)^{\alpha/4}}
\]

and

\[
d_k = \sum_{i \in D \setminus V_D} \frac{1}{((b^x_i + a^x_k)^2 + (b^y_i - a^y_k)^2)^{\alpha/2}}.
\]

In parallel to the regular case, we will need an upper bound on \( \sum_{i \in D \setminus V_D} |\tilde{h}_{ik}|^2 \) and \( \sum_{k \in S} |\tilde{h}_{ik}|^2 \). The upper bound can be obtained in two steps by first showing that

\[
d_k \geq K'_3 \frac{(a^x_k)^{2-\alpha}}{\log n}
\]

with high probability for a constant \( K'_3 > 0 \) independent of \( n \), which leads to

\[
|h_{ik}|^2 \leq \frac{1}{K'_3} \log n \frac{(a^x_k)^{\alpha-2}}{((b^x_i + a^x_k)^2 + (b^y_i - a^y_k)^2)^{\alpha/2}} \leq \frac{1}{K'_3} \log n \frac{1}{(b^x_i + a^x_k)^2 + (b^y_i - a^y_k)^2}
\]

for all \( i, k \). This, in turn yields

\[
\sum_{k \in S} |\tilde{h}_{ik}|^2, \quad \sum_{i \in D \setminus V_D} |\tilde{h}_{ik}|^2 \leq K'_1 (\log n)^3
\]
with high probability for another constant $K'_1 > 0$ independent of $n$. Recalling the discussion for regular networks immediately leads to (26).

Both the lower bound in (31) and the upper bound in (33) regarding random networks can be proved using binning arguments that provide connection to regular networks. In order to prove the lower bound, we consider Part (b) of Lemma 5.1, while the upper bound (33) is proved using Part (a) of the same lemma.

Let us first consider dividing the right-half network into squarelets of area $2 \log n$. Given a left-hand side node $k$ located at $(-a_x^k, a_y^k)$, let us move the nodes inside each right-hand side squarelet onto the squarelet vertex that is farthest to $k$. Since this displacement can only increase the Euclidean distance between the nodes involved, and since by Part (b) of Lemma 5.1 we know that there is at least one node inside each squarelet, we have

$$d_k = \sum_{i \in D \setminus V_D} \frac{1}{((b_i^x + a_x^k)^2 + (b_i^y - a_y^k)^2)^{\alpha/2}} \geq \sum_{i_x, i_y = 1} 1 \geq K'_3 \frac{(a_x^k)^{2-\alpha}}{2 \log n}$$

by the lower bound in Lemma 5.4.

Now having (32) in hand, in order to show (33), we divide the network into $n$ squarelets of area 1. By Part (a) of Lemma 5.1, there are at most $\log n$ nodes inside each squarelet. Considering the argument in Section VIII and the displacement of the nodes as illustrated in Figure 8 yields a regular network with at most $2 \log n$ nodes at each vertex in the right-half network,

$$\sum_{i \in D \setminus V_D} |h_{ik}|^2 \leq \frac{2}{K_3} \log n \sum_{i \in D \setminus V_D} \frac{1}{((b_i^x + a_x^k)^2 + (b_i^y - a_y^k)^2)^{\alpha/2}} \leq \frac{4}{K_3} (\log n)^2 \sum_{i_x, i_y = 1} \frac{\sqrt{n}}{(i_x + k_x)^2 + (i_y - k_y)^2} \leq 4K'_1 (\log n)^3,$$

by employing the upper bound in Lemma 5.4 for $\alpha = 2$. The same bound follows similarly for $\sum_{k \in S} |h_{ik}|^2$, thus the desired result in (33).
Proof of Lemma 5.4: Both the lower and upper bound for \( d_{k_x,k_y} \) can be obtained by straightforward manipulations. Recall that

\[
d_{k_x,k_y} = \sum_{x=1}^{\sqrt{n}} \frac{1}{((i_x + k_x - 1)^2 + (i_y - k_y)^2)^{\alpha/2}}.
\]

The upper bound can be obtained as follows:

\[
d_{k_x,k_y} = \sum_{y=1-k_y}^{\sqrt{n}-k_x} \sum_{x=k_y}^{\sqrt{n}+1} \frac{1}{(x^2 + y^2)^{\alpha/2}} \leq \sum_{y=1-k_y}^{\sqrt{n}-k_y} \left( \frac{1}{(k_x^2 + y^2)^{\alpha/2}} + \int_{k_x}^{k_x+\sqrt{n}-1} \frac{1}{(x^2 + y^2)^{\alpha/2}} \, dx \right).
\]

\[
\leq k_x^{-\alpha} + \int_{k_x}^{k_x+\sqrt{n}-1} \frac{1}{x^\alpha} \, dx + \int_{1-k_y}^{\sqrt{n}-k_y} \frac{1}{(k_x^2 + y^2)^{\alpha/2}} \, dy
\]

\[
+ \int_{1-k_y}^{\sqrt{n}} \int_{k_x}^{k_x+\sqrt{n}-1} \frac{1}{(x^2 + y^2)^{\alpha/2}} \, dx \, dy
\]

\[
\leq k_x^{-\alpha} + (1+\pi)k_x^{1-\alpha} + \int_{-\pi/2}^{\pi/2} \int_{k_x}^{\pi/2} \frac{1}{r^\alpha} r \, dr \, d\theta
\]

So

\[
d_{k_x,k_y} = \begin{cases} \kappa_x^{-\alpha} + (1+\pi)k_x^{1-\alpha} + \pi \log r \bigg|_{k_x}^{3\sqrt{n}} & \text{if } \alpha = 2, \\ k_x^{-\alpha} + (1+\pi)k_x^{1-\alpha} + \frac{\pi}{(2-\alpha)} r^{2-\alpha} \bigg|_{k_x}^{3\sqrt{n}} & \text{if } \alpha > 2, \end{cases}
\]

(34)

for a constant \( K'_2 > 0 \) independent of \( n \), since the dominating term in (34) is the third one.

The lower bound follows similarly:

\[
d_{k_x,k_y} \geq \sum_{y=1-k_y}^{\sqrt{n}-k_x} \sum_{x=k_y}^{\sqrt{n}+1} \frac{1}{(x^2 + y^2)^{\alpha/2}} \geq \sum_{y=1-k_y}^{\sqrt{n}-k_y} \int_{k_x}^{k_x+\sqrt{n}-1} \frac{1}{(x^2 + y^2)^{\alpha/2}} \, dx
\]

\[
\geq \int_{1-k_y}^{\sqrt{n}} \int_{k_x}^{k_x+\sqrt{n}-1} \frac{1}{(x^2 + y^2)^{\alpha/2}} \, dx \, dy - \int_{k_x}^{k_x+\sqrt{n}-1} \frac{1}{x^\alpha} \, dx
\]

\[
\geq \int_{0}^{\sqrt{n}} \int_{k_x}^{k_x+\sqrt{n}-1} \frac{1}{(x^2 + y^2)^{\alpha/2}} \, dx \, dy + \frac{x^{1-\alpha}}{\alpha-1} \bigg|_{k_x}^{k_x+\sqrt{n}-1}
\]

\[
\geq \int_{0}^{\arctan(1/2)} \int_{k_x}^{k_x+\sqrt{n}-1} \frac{1}{r^\alpha} r \, dr \, d\theta + \frac{x^{1-\alpha}}{\alpha-1} \bigg|_{k_x}^{k_x+\sqrt{n}-1}
\]
So for all $\alpha \geq 2$, we have

$$d_{k_x,k_y} = \begin{cases} \arctan\left(\frac{1}{2}\right) \log r \left| k_x + \sqrt{n} - 1 \right| k_x + \sqrt{n} - 1 \sqrt{2k_x} + \frac{1}{\alpha - 1} x^{1-\alpha} \left| k_x + \sqrt{n} - 1 \right| k_x & \text{if } \alpha = 2, \\ \arctan\left(\frac{1}{2}\right) \frac{1}{2-\alpha} r^{2-\alpha} \left| k_x + \sqrt{n} - 1 \right| k_x + \sqrt{n} - 1 \sqrt{2k_x} + \frac{1}{\alpha - 1} x^{1-\alpha} \left| k_x + \sqrt{n} - 1 \right| k_x & \text{if } \alpha > 2, \end{cases}$$

(35)

$$\geq K_3' k_x^{2-\alpha}$$

where $K_3' > 0$ is a constant independent of $n$, since the dominating term in (35) is the first one.

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