We emphasize that scattering amplitudes of a wide class of models to any order in the coupling are constructible by on-shell tree subamplitudes. This follows from the Feynman-tree theorem combined with BCFW on-shell recursion relations. In contrast to the usual Feynman diagrams, no virtual particles appear.

1. INTRODUCTION

The success of quantum field theory with respect to the computation of amplitudes in a perturbative expansion is well-known. Maybe one of the most striking examples is the computation of the anomalous magnetic moment of the electron [1].

The recent progress in dealing with amplitudes (for recent reviews see for instance [2–4]), in particular, little group scaling and BCFW on-shell-recursion relations suggests that the actual computation of a physical amplitude follows from a rather small number of subdiagrams or masteramplitudes. One example is the Parke-Taylor formula [5] for the lowest order $n$-gluon amplitude which follows via recursion relations from the simple three-gluon amplitude.

Here we want to emphasize that amplitudes of a given loop coupling order in general may be constructed by on-shell tree subamplitudes. This observation is based on a combination of the Feynman-tree theorem [6, 7] with BCFW on-shell-recursion relations of tree diagrams [8, 9]. In particular, there do not appear virtual particles in this way. This in turn makes the use of ghosts obsolete. In this picture all subamplitudes are on-shell, but we have to deal with hidden particles, that is, external background on-shell particles which are unobserved. The method works in any theory in which the boundary term of the BCFW recursion relations vanishes. This was shown to hold in gauge theories as well as in general relativity [10]. Any amplitude is constructed by merging on-shell subamplitudes together, where in general we encounter a background of external hidden particles.

Let us mention the recent interest in the Feynman-tree theorem; see for instance [11–15]. One of the strategies is to reduce the in general large number of tree amplitudes avoiding multiple cuts. Here we will apply the original version of the Feynman-tree theorem.

2. THE FEYNMAN-TREE THEOREM AND ON-SHELL RECURRENCES

Let us briefly review the Feynman tree theorem [6, 7], which reduces $l$ loop amplitudes to at most $l - 1$ loop amplitudes systematically, that is, recursively to tree amplitudes. The basic idea is to introduce besides the usual propagators $G_F(p)$ also advanced propagators $G_A(p)$

\[ G_F(p) = \frac{i}{p^2 - m^2 + i\epsilon}, \quad G_A(p) = \frac{i}{p^2 - m^2 - i\epsilon \text{sgn}(p_0)}. \]  

From the identity

\[ \frac{1}{x \pm i\epsilon} = P.V. \left( \frac{1}{x} \right) \mp i\pi\delta(x), \]  

where $P.V.$ denotes the principal value prescription, we get a simple context between the usual propagators and the advanced ones,

\[ G_A(p) = G_F(p) - 2\pi\theta(p_0)\delta(p^2 - m^2). \]

Let us consider a generic loop diagram as shown in Fig. 1. Here $k$ denotes the loop momentum of the considered loop and all momenta $p_i$ with $i = 1, \ldots, n$ are chosen by convention to be outgoing. We note that the outgoing momenta

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in this diagram may also correspond to two particles in each vertex. In these cases the indicated momentum gives the sum of the outgoing momenta. We note further that the legs do not need to be external. In this sense, the loop diagram in Fig. 1 is generic.

In the considered loop of a diagram we replace all Feynman propagators $G_F(p)$ by the advanced propagators $G_A(p)$. In the integration over the time-like component of the loop-four momentum, $k_0$, all poles of the advanced propagators lie now above the real axis. Closing the contour in the lower half plane, the loop integral is zero. Using (2.3) this gives

$$0 = \int \frac{d^4k}{(2\pi)^4} N(k) \prod_i G_A^{(i)}(k - p_1 - \ldots - p_i)$$

(2.4)

Here $N(k)$ denotes the numerator of the amplitude, which in general also depends on the loop-momentum $k$. The product in the last line of (2.4) is the desired recursion relation of the loop amplitude: one term of this expansion is the original loop amplitude with all Feynman propagators and all remaining terms with one or more propagators replaced by the corresponding delta-function terms. The delta-function term together with the loop momentum integration corresponds to a phase-space integration with the cut propagators on-shell. In general a loop with $n$ propagators gives $2^n - 1$ cut diagrams. The Feynman-tree recursion gives a sum of new amplitudes with the loop order decreased about at least one unit in each recursion step. Repeated application of this recursion relation represents all loop amplitudes in terms of tree amplitudes.

Now we emphasize that by general BCFW tree-recursion relations [8, 9] we can express the tree amplitudes resulting from the Feynman-tree theorem, in terms of on-shell amplitudes. The basic idea of the BCFW recursion relations is analytic continuation of the external momenta. In this way tree amplitudes factorize into on-shell subamplitudes. In an arbitrary tree amplitude let us denote the $n$ external momenta by $p_i^\mu$ with $i = 1, \ldots, n$. These external momenta are shifted,

$$\hat{p}_i^\mu = p_i^\mu + z \cdot r_i^\mu$$

(2.5)

with one common $z \in \mathbb{C}$ and appropriately chosen vectors $r_i$.

The statement is that any tree amplitude $A$ can by analytic continuation be decomposed in terms of on-shell subamplitudes connected by propagators and a boundary term $B$,

$$A = -\sum_{z_{\ell}} \text{Res}_{z=z_{\ell}} \frac{\hat{A}(z)}{z} + B = \sum_{\text{diagram } \ell} \hat{A}_L(z_{\ell}) \cdot \frac{1}{P_{\ell}^2} \cdot \hat{A}_R(z_{\ell}) + B.$$

(2.6)
Here, \( \hat{A}(z) \) denotes the shifted amplitude with the positions of poles in the complex plane at \( z = z_I \). On the right-hand side we have the on-shell subamplitudes \( \hat{A}_L(z_I) \) and \( \hat{A}_R(z_I) \), with a propagator factor \( 1/\Pi_I^2 \). In general, there appears also a boundary term \( B \), which is the residue of the pole of \( \hat{A}(z) \) at \( z = \infty \). In case of a vanishing term \( B \) we have on the right-hand side the desired factorization into on-shell subamplitudes, as shown in Fig. 2.

It has been shown that for an appropriate shift of the external momenta in gauge theories as well as general relativity the boundary term \( B \) vanishes \cite{10}. We have, therefore in these cases a recursion of tree amplitudes to on-shell amplitudes. An example of a theory, where we do not have a vanishing boundary contribution \( B \) is \( \phi^4 \) theory, as discussed in \cite{16}.

Combining the two recursion methods, that is, on the one hand the Feynman-tree theorem to transform loop amplitudes to tree amplitudes and subsequently BCFW recursions to transform the tree amplitudes to on-shell amplitudes we can express amplitudes in terms of on-shell subamplitudes. Obviously, some of the external particles are hidden, originating from the Feynman-tree theorem.

We emphasize that we equivalently can start the construction of amplitudes by on-shell subamplitudes. This holds when the boundary terms vanish. In this picture amplitudes are constructed by joining on-shell subamplitudes to a given order in the coupling. This gives a new interpretation, which, as we have seen, is equivalent to the conventional Feynman diagram approach. Respecting the Feynman-tree theorem, we have in general to consider amplitudes with hidden external particles, that is, particles which are unobserved. In this picture virtual particles do not appear. In particular, no ghosts have to be introduced. Moreover, it is clear that every subdiagram is on-shell, that is, in particular, gauge invariant. This is to compare with the usual Feynman diagram construction of amplitudes where in general gauge invariance is violated in each diagram but is only restored in the sum of diagrams to a given order. An explicit example of a two-point amplitude, to second order in the coupling, is studied in the appendix. We have in this example to consider three on-shell amplitudes. Conventionally, these correspond to a one-loop Feynman diagram.

3. CONCLUSIONS

We have emphasized that amplitudes can be constructed by on-shell subamplitudes. This works in theories where the corresponding BCFW recursion relations have a vanishing boundary term. It has been shown that this holds in gauge theories as well as general relativity. All the poles and branch cuts follow automatically in this picture. Consequently, there are no virtual particles and no ghosts in this picture. This construction of gauge invariant on-shell amplitudes follows directly from the Feynman-tree theorem combined with BCFW on-shell recursion relations. To a given order in the couplings, amplitudes with a background of external, but hidden particles have to be introduced. We have to integrate over the corresponding phase space of the hidden particles. Let us mention that in cases where we can construct the lowest order on-shell amplitudes by little group scaling, that is, eventually Lorentz invariance, all amplitudes follow recursively by this construction.

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Appendix A: A two-point function example

Let us consider as a simple example a scalar theory with Lagrangian

\[
A \text{tree} = \sum_{\text{diagram } I} \hat{A}_L(z_I) \hat{A}_R(z_I) \frac{1}{\Pi_I^2}
\]
\[ \mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi) (\partial^{\mu} \phi) - \frac{1}{2} m^2 \phi^2 - \frac{g}{3!} \phi^3. \]  

(A1)

We want to compute the 2-point amplitude at order \( g^2 \). We start with the conventional Feynman diagram computation, giving

\[ -i A(p^2) = \]

\[ \begin{array}{c}
\text{(A2)} \\
\end{array} \]

We have

\[ -i A(p^2) = \frac{(-ig)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + ie} \cdot \frac{i}{(k-p)^2 - m^2 + ie}. \]

(A2)

After Feynman parametrization we regularize the UV divergence for simplicity with a heavy mass \( M \gg m \), that is, Pauli-Villars regularization,

\[ -i A^{\text{reg}}(p^2) = \frac{(-ig)^2}{2} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \left\{ k^2 + p^2 x^2 - p^2 x - m^2 \right\}^{-2} - \left\{ k^2 + p^2 x^2 - p^2 x - M^2 \right\}^{-2}. \]  

(A3)

As usual, going to Euclidean space and utilizing spherical coordinates we end up with

\[ -i A^{\text{reg}}(p^2) = \frac{g^2}{32\pi^2} \int_0^1 dx \ln \left\{ \frac{m^2 - p^2 x^2 - p^2 x^2}{M^2} \right\} \]

\[ = \frac{g^2}{32\pi^2} \left\{ 2 \sqrt{\frac{4m^2}{p^2} - 1} \arctan \left( \sqrt{\frac{p^2}{4m^2 - p^2}} \right) - 2 - \log \left( \frac{M^2}{m^2} \right) \right\}. \]  

(A4)

Now we want to show that this is equivalent to on-shell amplitudes to the same order \( g^2 \). In this we have to consider also amplitudes with hidden external particles, that is,

\[ -i A(p^2) = \]

\[ \begin{array}{c}
\text{[A2]} \\
\end{array} \]

All vertical lines denote in these amplitudes the hidden external particles, that is, we have to integrate over the corresponding phase space.

To show the equivalence we start with [A2] and apply the Feynman-tree theorem, replacing the Feynman propagators \( G_F \) by advanced propagators \( G_A \) \([2,3]\), yielding \( p = (p_0, \mathbf{p})^T \) and \( \omega_p = +\sqrt{p^2 + m^2} \).

\[ -i A(p^2) = \frac{(-ig)^2}{2} \int \frac{d^4k}{(2\pi)^4} \left\{ G_A(k-p) \frac{\pi}{\omega_k} \delta(k_0 - \omega_k) \right. \]

\[ + G_A(k) \frac{\pi}{\omega_{k-p}} \delta(k_0 - p_0 - \omega_{k-p}) \]

\[ + \left. \frac{\pi}{\omega_k} \delta(k_0 - \omega_k) \cdot \frac{\pi}{\omega_{k-p}} \delta(k_0 - p_0 - \omega_{k-p}) \right\}. \]  

(A5)

These are all possible cuts we can apply to the loop, diagrammatically.
Let us consider the first term in (A5), corresponding to the first diagram. We have

\[-iA(p^2)_{\text{first}} = \frac{-(ig)^2}{2} \int \frac{d^4k}{(2\pi)^4} G_A(k-p) \frac{\pi}{\omega_k} \delta(k_0 - \omega_k)\]

\[-iA(p^2)_{\text{first}} = \frac{-(ig)^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\pi}{\omega_k} (\omega_k - p_0 + i\epsilon)^2 - (k-p)^2 - m^2\]

\[-iA(p^2)_{\text{first}} = \frac{-(ig)^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{\pi}{\omega_k} (k-p)^2 - m^2\cdot\]

Since in this example we deal with scalars only, the last line in (A6) gives already the first on-shell amplitude above. In the rest frame of p we get

\[-iA(p^2)_{\text{first}} = \frac{-(ig)^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{i}{\omega_k} \frac{p^2 - 2p\omega_k}{p^2 - 2p\omega_k}\]

(A7)

Similar we get for the second cut diagram

\[-iA(p^2)_{\text{sec.}} = \frac{-(ig)^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{i}{\omega_k} \frac{p^2 - 2p\omega_k}{p^2 - 2p\omega_k}\]

(A8)

which is the second on-shell amplitude above. In the p rest frame this corresponds to the replacement p \rightarrow -p in (A7).

The third term in (A5) does not contribute. This follows from the fact that the two delta functions are never simultaneously satisfied for p_0 > p, since p_0 + \sqrt{(k-p)^2 + m^2} > \sqrt{k^2 + m^2}. That is we have

\[-iA(p^2) = -iA(p^2)_{\text{first}} - iA(p^2)_{\text{sec.}}\]

(A9)

We regularize the UV divergence, subtracting a heavy mass M \gg m term and have

\[-iA_{\text{reg}}(p^2) = \frac{-(ig)^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{2\omega_k}{2\sqrt{k^2 + m^2}} \left\{ \frac{i}{2\sqrt{k^2 + m^2} \cdot (p^2 - 2p\sqrt{k^2 + m^2})} - \frac{i}{2\sqrt{k^2 + m^2} \cdot (p^2 - 2p\sqrt{K^2 + M^2})} \right\}\]

(A10)

In spherical coordinates the integration gives

\[-iA_{\text{reg}}(p^2) = \frac{-(ig)^2}{2} \frac{4\pi}{8\rho} \left\{ 2\sqrt{4m^2 - p^2} \arctan \left( \frac{p}{\sqrt{4m^2 - p^2}} \right) - 2\sqrt{4M^2 - p^2} \arctan \left( \frac{p}{\sqrt{4M^2 - p^2}} \right) + p \log \frac{m^2}{M^2} \right\}\]

(A11)

what in the limit of M^2 \gg m^2 is the same as what we found with the usual Feynman diagram calculation.

[1] G. Gabrielse, D. Hanneke, T. Kinoshita, M. Nio and B. C. Odom, “New Determination of the Fine Structure Constant from the Electron g Value and QED,” Phys. Rev. Lett. 97, 030802 (2006) [Phys. Rev. Lett. 99, 039902 (2007)].
[2] Z. Bern, L. J. Dixon and D. A. Kosower, “On-Shell Methods in Perturbative QCD,” Annals Phys. 322, 1587 (2007) [arXiv:0704.2798 [hep-th]].
[3] R. Britto, “Loop Amplitudes in Gauge Theories: Modern Analytic Approaches,” J. Phys. A 44, 454006 (2011) [arXiv:1012.4393 [hep-th]].
[4] H. Elvang and Y. t. Huang, “Scattering Amplitudes,” arXiv:1308.1697 [hep-th].
[5] S. J. Parke and T. R. Taylor, “An Amplitude for n Gluon Scattering,” Phys. Rev. Lett. 56, 2459 (1986).
[6] R. P. Feynman, “Quantum theory of gravitation,” Acta Phys. Polon. 24, 697 (1963).
R. P. Feynman, “Closed Loop And Tree Diagrams,” in Selected papers of Richard Feynman, ed. L. M. Brown (World Scientific, Singapore, 2000).

R. Britto, F. Cachazo and B. Feng, “New recursion relations for tree amplitudes of gluons,” Nucl. Phys. B 715, 499 (2005) hep-th/0412308.

R. Britto, F. Cachazo, B. Feng and E. Witten, “Direct proof of tree-level recursion relation in Yang-Mills theory,” Phys. Rev. Lett. 94, 181602 (2005) hep-th/0501052.

N. Arkani-Hamed and J. Kaplan, “On Tree Amplitudes in Gauge Theory and Gravity,” JHEP 0804, 076 (2008) arXiv:0801.2385 [hep-th].

R. Britto, F. Cachazo, B. Feng and E. Witten, “Direct proof of tree-level recursion relation in Yang-Mills theory,” Phys. Rev. Lett. 94, 181602 (2005) hep-th/0501052.

N. Arkani-Hamed and J. Kaplan, “On Tree Amplitudes in Gauge Theory and Gravity,” JHEP 0804, 076 (2008) arXiv:0801.2385 [hep-th].

A. Brandhuber, B. Spence and G. Travaglini, “From trees to loops and back,” JHEP 0601, 142 (2006) doi:10.1088/1126-6708/2006/01/142 hep-th/0510253.

S. Catani, T. Gleisberg, F. Krauss, G. Rodrigo and J. C. Winter, “From loops to trees by-passing Feynman’s theorem,” JHEP 0809, 065 (2008) arXiv:0804.3170 [hep-ph].

S. Caron-Huot, “Loops and trees,” JHEP 1105, 080 (2011) arXiv:1007.3224 [hep-ph].

R. H. Boels, “On BCFW shifts of integrands and integrals,” JHEP 1011, 113 (2010) doi:10.1007/JHEP11(2010)113 arXiv:1008.3101 [hep-th].

I. Bierenbaum, S. Catani, P. Draggiotis and G. Rodrigo, “A Tree-Loop Duality Relation at Two Loops and Beyond,” JHEP 1010, 073 (2010) arXiv:1007.0194 [hep-ph].

B. Feng, J. Wang, Y. Wang and Z. Zhang, “BCFW Recursion Relation with Nonzero Boundary Contribution,” JHEP 1001, 019 (2010) arXiv:0911.0301 [hep-th].