QUANTUM DEFORMATIONS OF RELATIVISTIC SYMMETRIES: SOME RECENT DEVELOPMENTS

J. LUKIERSKI

Institute for Theoretical Physics, University of Wrocław, pl. Maxa Borna 9, 50-204 Wrocław, Poland
E-mail: lukier@ift.uni.wroc.pl

Abstract

Firstly we discuss different versions of noncommutative space-time and corresponding appearance of quantum space-time groups. Further we consider the relation between quantum deformations of relativistic symmetries and so-called doubly special relativity (DSR) theories.

1 Introduction

Quantum deformations of Lie algebras and Lie group were motivated by quantum universe scattering method and introduced in 1980’s as noncommutative Hopf algebras [1–3]. Subsequently the notion of quantum symmetries was tried for many symmetries occurring in physics, in particular for the basic relativistic symmetries, described by Poincaré algebra and Poincaré group, as well as anti-de-Sitter (AdS), de-Sitter (dS) and conformal symmetries.

Because the fourmomenta generators contrary to the Lorentz rotations are dimensionfull, they introduce into the space-time algebras the notion of scaling. One should distinguish two types of quantum deformations:

i) The ones introducing dimensionless deformation parameter $q$, which is invariant under rescaling of the fourmomenta. The prototypes of such deformations are provided by Drinfeld-Jimbo (DJ) deformations of AdS, dS or conformal algebra. One can show [5,6] that it does not exist DJ
quantum deformation of Poincaré algebra, obtained by the extension of DJ deformation for the Lorentz subalgebra.

The dimensionless deformation parameter of space-time symmetries appears less attractive from the point of view of physical applications. It is agreed that the quantum symmetries and noncommutative space-time coordinates should become relevant for very small distances (e.g. at Planck length $l_p \simeq 10^{-33}$ cm). We conclude that the dimensional parameter in quantum algebra structure will characterize the distances at which the notions of classical geometry are not valid. Therefore we should introduce.

ii) Second type of deformations of space-time symmetries with built-in elementary length, or elementary mass. First such a deformation has been proposed in 1991 [7], with the deformation parameter $\kappa$ (the classical limits is provided by limit $\kappa \to \infty$), and as called the $\kappa$-deformation. Still it is not clear how to relate by rigorous proof the parameter $\kappa$ with the Planck mass $M_P$ ($M_P \simeq 10^{19}$ GeV), but due to the quantum gravity origin of noncommutativity of space-time coordinates it is believed that they are linked very closely, and quite often are assumed to be identical.

2 Noncommutativity of Space-Time and Quantum Groups

The need of quantum space-time symmetries with dimensionful deformation parameter can be seen clearly from the noncommutativity of space-time coordinates. The general relations can be written as follows (see e.g. [8])

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{1}{\kappa^2} F(\kappa x) = \frac{1}{\kappa^2} \Theta^{(0)}_{\mu \nu}$$

$$+ \frac{1}{\kappa} \Theta^{(1)}_{\mu \nu} \hat{x}_\mu + \Theta^{(2)}_{\mu \nu} \rho^\theta \hat{x}_\rho \hat{x}_\tau + \ldots ,$$

(1)

where we introduced the parameter $\kappa$ in order to express the noncommutativity in terms of dimensionless coordinates $y_k = \kappa x_\mu$.

Let us consider the special cases when only one constant tensor $\Theta^{(k)}_{\mu \nu \rho_1 \ldots \rho_k} \neq 0$.

1) $\Theta^{(0)}_{\mu \nu} \neq 0$, $\Theta^{(k)}_{\mu \nu \rho_1 \ldots \rho_k} = 0$, $k = 1, 2, 3 \ldots$

This example was studied extensively recently; the Poincaré symmetries are broken by a constant tensor but remain classical. Such a form of deformed space-time was obtained by Seiberg and Witten [9] by considering the space-time manifold as described by the $D$-brane world volume in the presence of constant tensor field $B_{\mu \nu}$ (we recall that such a field is necessary for consistency of supergravity framework in $D=10$).
2) $\Theta^{(1)\rho}_{\mu \nu} \neq 0$, $\Theta^{(k)\rho_1...\rho_k}_{\mu \nu} = 0$, $k = 0, 2, 3, 4...$

In such a case we obtain the Lie-algebraic form of deformed space-time algebra. It appears that the $\kappa$-Minkowski space-time, obtained in the framework of standard $\kappa$-deformations of Poincaré symmetries \[10\] - \[12\] belongs to such a class of new theories. In 1996 there was introduced the generalized $\kappa$-deformation of Poincaré symmetries \[13\], with the choice

$$\Theta^{(1)\rho}_{\mu \nu} = \frac{i}{\kappa} \left(a_\mu \delta^\rho_\nu - a_\nu \delta^\rho_\mu\right),$$

where $a_\mu$ denotes constant fourvector.

From \[2\] follows that the noncommuting quantum direction in Minkowski space is described by the coordinate $\vec{y} = a^\mu x_\mu$. It has been also shown that the classical $r$-matrix corresponding to \[2\] satisfies

- modified Yang-Baxter (YB) equation if $a_\mu^2 \neq 0$
- classical YB equation if $a_\mu^2 = 0$.

The quantum deformations of relativistic symmetries with quantized light-cone direction ($a_\mu^2 = 0$) was firstly described by Ballesteros et al \[14\] and called null-plane quantum Poincaré symmetries. Further it has been shown \[15\] that in such a case the quantization can be described by the twisting procedure \[16, 17\].

If $\Theta^{(1)\rho}_{\mu \nu} \neq 0$, the translations $\hat{v}_\mu$

$$\hat{x}_\mu \longrightarrow \hat{x}_\mu' = \hat{x}_\mu + \hat{v}_\kappa,$$

described by the coproduct of $\hat{x}_\mu$, are also noncommutative

$$[\hat{v}_\mu, \hat{v}_\nu] = \Theta^{(1)\rho}_{\mu \nu} \hat{v}_\rho.$$

The extension of noncommutative translations \[1\] to quantum Poincaré group is only possible for particular choices of $\Theta^{(1)\rho}_{\mu \nu}$, in particular for the one given by \[2\]. The general classification of quantum Poincaré groups has been considered by Podleś and Woronowicz \[18\].

3) $\Theta^{(2)\rho\tau}_{\mu \nu} \neq 0$, $\Theta^{(k)\rho_1...\rho_k}_{\mu \nu} = 0$, $k = 0, 1, 3, 4...$

In such a case the relation \[1\] does not contain any dimensionfull parameter and it describes the quantum deformation of relativistic symmetries with dimensionless deformation parameter (e.g. if the Lorentz sector is described by Drinfeld-Jimbo deformation). Such deformed space-time framework is described by braided quantum symmetries, because the quantum translations \[3\] do satisfy the relations

$$[\hat{v}_\mu, \hat{v}_\nu] = \Theta^{(2)\rho\tau}_{\mu \nu} \hat{v}_\rho \hat{v}_\tau,$$
but also one has to assume that
\[
[\hat{x}_\mu, \hat{x}_\nu] \neq 0. \tag{6}
\]

The first example of braided quantum Poincaré group has been presented by Majid \cite{Majid}.

3 Nonlinear Realizations of Relativistic Symmetries Versus Quantum Deformations

There are two sources of the modification of relativistic symmetries (see e.g. \cite{19}).

i) One can change nonlinearly the basis of classical Poincaré algebra

\[
[M^{(0)}_{\mu\nu}, M^{(0)}_{\rho\tau}] = i(\eta_{\mu\rho} M^{(0)}_{\nu\tau} + \ldots, \\
[M^{(0)}_{\mu\nu}, P^{(0)}_{\rho}] = i(\eta_{\mu\rho} P^{(0)}_{\nu} - \eta_{\nu\rho} P^{(0)}_{\mu}), \\
[P^{(0)}_{\mu}, P^{(0)}_{\nu}] = 0, \tag{7}
\]

by introducing the deformation map - the invertible nonlinear functions of the generator. An important special class of deformation maps described only the change of four momentum basis

\[
P^{(0)}_{\mu} \rightarrow P_{\mu} = P_{\mu}(P^{(0)}; \kappa) = F^{\nu}_{\mu} \left( \frac{P^{(0)}_{\kappa}}{\kappa} \right) P^{(0)}_{\nu}, \tag{8}
\]

while the Lorentz generators remain unchanged ($M_{\mu\nu} = M^{(0)}_{\mu\nu}$). Introducing the inverse deformation map

\[
P^{(0)}_{\mu} = P^{(0)}_{\mu}(P; \kappa) = \tilde{F}^{\nu}_{\mu} \left( \frac{P}{\kappa} \right) P_{\nu}, \tag{9}
\]

we see that the mass Casimir is modified as follows:

\[
C_1 = P^{(0)}_{\mu} P^{(0)\mu} = P^{(0)}_{\mu} \left( \frac{P}{\kappa} \right) P^{(0)\mu} \left( \frac{P}{\kappa} \right) = m_0^2, \tag{10}
\]

i.e. we obtained deformed nonlinear energy-momentum dispersion relation. Other consequence of the deformation map is the nonlinear modification of energy-momentum addition and conservation laws. The primitive coproduct for the generators $P^{(0)}_{\mu}$ is replaced by

\[
\Delta(P_{\mu}) = P_{\mu} \left( P^{(0)}_{\mu} \otimes 1 + 1 \otimes P^{(0)}_{\mu}, \kappa \right). \tag{11}
\]

Denoting for two-particle system
\( P_\mu(i) \) - the fourmomenta of i-th particle \((i = 1, 2)\)

\( P_\mu(1, 2) \) - the fourmomenta of 2-particle system

and using (10), (8) one can express the coproduct (11) as describing nonlinear energy-momentum addition law [20, 21]

\[
P_\mu(1, 2) = P_\mu\left(P_0(0) + P_0(1)\kappa\right)
\]

(12)

The composition law (12) is symmetric, what indicates that we are dealing with classical Lorentz symmetries nonlinearly realized in the four-momentum sector.

The choice of the deformation map which provides the deformed mass Casimir in bicrossproduct basis of \( \kappa \)-deformed Poincaré algebra [11, 12]

\[
C_1 = \left(2\kappa \sinh \frac{P_0}{2\kappa}\right)^2 - e^{-\frac{P_0}{2\kappa}} = M^2,
\]

(13)

leads to Doubly Special Relativity theory (DSR) of Amelino-Camelia et al (see e.g. [22]). Such a choice of deformed mass Casimir implies maximal value of three-momentum if energy \( \frac{E}{c} = P_0 \to \infty \).

Indeed, in simple case \( M^2 = 0 \) one gets

\[
\frac{P_0^2}{2\kappa} = \kappa^2 \left(1 - e^{-\frac{P_0}{2\kappa}}\right)^2 \xrightarrow{P_0 \to \infty} \kappa^2
\]

(14)

i.e. we obtain the operational definition of mass-like deformation parameter \( \kappa \) as maximal possible value of the three-momentum. It appears that selecting properly the deformation map [3] and corresponding nonlinear mass Casimirs [10] one can introduce three different variants of DSR theories [23].

ii) One can modify relativistic symmetries by introducing quantum deformations of Hopf algebra describing Poincaré symmetries.

The quantum deformations of relativistic symmetries can be easily distinguished from the classical relativistic symmetries in nonlinear disguise if we observe that

- quantum deformations imply nontrivial bialgebra structure, described by classical \( r \)-matrix \((I_i \equiv (M_\mu^0(0), P_\mu^0(0)); I_i \wedge I_j \equiv I_i \otimes I_j - I_j \otimes I_i)\)

\[
r = a^{ij} I_i^0 \wedge I_j^0.
\]

(15)

The classical \( r \)-matrix describes infinitesimal deformation of classical coproduct \( \Delta^{(0)} \)

\[
\Delta(I_i^0) = \Delta^{(0)}(I_i^0) + \xi[r, \Delta^{(0)}(I_i^0)] + \mathcal{O}(\xi^2).
\]

(16)
The formula (16) implies that the coproduct is not symmetric.

– If the relativistic symmetries are quantum-deformed, the space-time coordinates are not commuting, contrary to the case in DSR framework.

We see therefore that there are simple criteria to distinguish between the modification due to change of classical basis and the one following from genuine quantum deformations [ , ].

The formula (16) can be "integrated" to arbitrary values of deformation parameter $\xi$ if we introduce the similarity transformation

$$\Delta_F(I_i^{(0)}) = F \circ \Delta^{(0)}(I_i^{(0)}) \circ F^{-1},$$

where $F = F_i^{(1)} \otimes F_i^{(2)}$ is the twist function with the following linear term in the power expansion in $\xi$

$$F = 1 \otimes 1 + \xi a^{ij} I_i \otimes I_j + O(\xi^2).$$

(18)

The twist function $F$ leads to cassociative coproducts (17) if it satisfied the equation

$$F_{12} \left( \Delta^{(0)} \otimes 1 \right) F = F_{23} \left( 1 \otimes \Delta^{(0)} \right) F.$$ 

(19)

Expanding (19) in powers of $\xi$ one gets from the bilinear terms the classical Yang-Baxter equation. Further twist quantization modifies the converse (antipode) as follows

$$S_F(I_i^{(0)}) = u S^{(0)}(I_i^{(0)}) u^{-1},$$

where $u = F_i^{(1)} \cdot S F_i^{(2)}$.

The twist quantization changes only the coproducts and coinverse - the classical Lie algebra relations and the counit remain unmodified.

### 4 Quantum Deformations of AdS and Conformal Algebras

Drinfeld twist quantization method can be applied to any deformation described infinitesimally by classical $r$-matrix satisfying CYBE. Recently there were explicitly written down the classical $r$-matrices for $O(3, 2)$ and $O(4, 2)$ algebras with generators belonging to the Borel subalgebra $B_+$ and subsequently these bialgebras were quantized [24] [15].
Let us consider as an example $O(3, 2)$ algebra. If we introduce the Cartan-Weyl basis for $O(3, 2) \simeq Sp(4)$ (see e.g. [7])

\[
\begin{align*}
&h_1, h_2, e_{\pm 1}, e_{\pm 2}, e_{\pm 3}, e_{\pm 4} \\
&\text{Cartan generators} \quad \text{simple root generators} \quad \text{composite root generators}
\end{align*}
\]

the most general classical $O(3, 2)$ $r$-matrix support in $B_+ \otimes B_+$ is the following:

\[
r = \alpha[(2h_1 + h_2) \wedge e_4 + 2e_1 \wedge e_2] + \xi h_2 \wedge e_2 + \rho e_2 \wedge e_4.
\]

The corresponding twist function has been calculated in [24] and is the product of four factors: Jordanian twist, extended Jordanian twist, deformed Jordanian twist and Reshetikhin twist.

It should be pointed out that the generators of $O(3, 2)$ can be physically assigned to $D=3$ conformal or $D=4$ AdS algebra. In first conformal case one can show that the parameters $\alpha, \xi$ and $\rho$ are dimensionfull, and we arrive at the $\kappa$-deformation of $D=3$ conformal algebra [24]. The assignment of $D=4$ AdS generators leads to different conclusion: the deformation parameters can remain dimensionless because the role of dimensionfull parameter is taken over by the AdS radius.

The twist quantization of $O(4, 2)$ algebra interpreted physically as $D=4$ conformal algebra has been considered in [15], where all classical $r$-matrices with support in Borel subalgebra were quantized. In particular in [15] the light-cone $\kappa$-deformation of Poincaré algebra (see (2) with $a_\mu^2 = 0$) has been extended to the particular $\kappa$-deformation of $D=4$ conformal algebra. The alternative physical interpretation of twisted $O(4, 2)$ as quantum $D=5$ AdS algebra can be found in [25].

5 Final Remarks

The formalism of twist quantization has been recently extended to all classical superalgebras [26], in particular to orthosymplectic superalgebras $OSp(n; 2m)$. For $n = 1, m = 2$ one obtains in such a way new deformations of $D=4$ AdS superalgebra. Subsequently, using suitable contraction method, one can obtain twist quantization of $D=4$ Poincaré superalgebra.

References

[1] V.G. Drinfeld, in Quantum Groups, Proc. Int. Congress of Mathematics, Berkeley, USA, 1986, p. 798.
[2] S.L. Woronowicz, Comm. Math. Phys. 111, 613 (1987).
[3] L.D. Faddeev, N.Yu. Reshetikhin and L.A. Takhtajan, Leningrad Math. Journ. 1, 193 (1990).
[4] L.Faddeev, N. Resetikhin and L. Takhtajan, Alg. Anal. 1, 178 (1990).
[5] S. Majid, J. Math. Phys. 34, 2045 (1993).
[6] P. Podleś, S.L. Woronowicz, in: "Proceedings of First Caribbean School on Mathematics and Theoretical Physics", Guadeloupe 1993, ed. R. Coquereaux, World Scientific (1995), p. 364.
[7] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. B264, 331 (1991).
[8] P. Kosiński, J. Lukierski, P. Maślanka, Czech. J. Phys. 50, 1283 (2000); Phys. Atom Nucl. 64, 2139 (2001).
[9] N. Seiberg, E. Witten,
[10] S. Zakrzewski, J. Phys. A27, 2075 (1994).
[11] S. Majid, H. Ruegg, Phys. Lett. B334, 348 (1994).
[12] J. Lukierski, H. Ruegg, W.J. Zakrzewski, Ann. Phys. 243, 90 (1995), [hep-th/9312153].
[13] P. Kosiński and P. Maślanka, in “From Quantum Field Theory to Quantum Groups”, ed. B. Jancewicz, J. Sobczyk, World Scientific, 1996, p. 41; q-alg/9512018.
[14] A. Balesteros, F.J. Herranz, M.A. del Olmo and M. Santander, Phys. Lett. B351, 137 (1995).
[15] V. Lyakhowski, J. Lukierski, M. Mozrzymas, Phys. Lett. B538, 375 (2002).
[16] V. Drinfeld, Dokl. Acad. Nauk SSSR, 273, 531 (1983).
[17] P.P. Kulish, V.D. and A.I. Mudrov, J. Math. Phys. 40, 4569 (1999).
[18] P. Podleś, S.L. Woronowicz, Commun. Math. Phys. 178, 61 (1996), [hep-th/9412059].
[19] J. Lukierski, A. Nowicki, Czech. J. Phys. 52, 1261 (2002); hep-th/0209017.

[20] J. Lukierski, A. Nowicki, Int. J. Mod. Phys. 18, 7 (2003), hep-th/0203065.

[21] J. Judes and M. Visser, qr-qc/0205067.

[22] G. Amelino-Camelia, Phys. Lett. B510, 255 (2001); Int. J. Mod. Phys. D11, 35 (2002).

[23] J. Lukierski, A. Nowicki, hep-th/0210111

[24] V. Lyakhovski, J. Lukierski, M. Mozrzymas, Mod. Phys. Lett. A18, 753 (2003).

[25] A. Ballesteros, N.R. Bruno, F.J. Herranz, Phys. Lett. B574, 273 (2003).

[26] V. Tolstoy, math.QA/0402433