THE DEFOCUSING $\dot{H}^{1/2}$-CRITICAL NLS IN HIGH DIMENSIONS

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Abstract. We consider the defocusing $\dot{H}^{1/2}$-critical nonlinear Schrödinger equation in dimensions $d \geq 5$. In the spirit of Kenig and Merle [10], we combine a concentration-compactness approach with the Lin–Strauss Morawetz inequality to prove that if a solution $u$ is bounded in $\dot{H}^{1/2}$ throughout its lifespan, then $u$ is global and scatters.

1. Introduction

We consider the initial-value problem for the defocusing $\dot{H}^{1/2}$-critical nonlinear Schrödinger equation in dimensions $d \geq 5$:

\[
\begin{aligned}
(i\partial_t + \Delta)u &= |u|^{4/d-1}u \\
 u(0) &= u_0,
\end{aligned}
\]

(1.1)

with $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$. This equation is deemed $\dot{H}^{1/2}$-critical because the rescaling that preserves the class of solutions to (1.1), that is, $u(t, x) \mapsto \lambda^{d-1} u(\lambda^2 t, \lambda x)$, leaves invariant the $\dot{H}^{1/2}$-norm of the initial data.

In [10], Kenig and Merle considered (1.1) with $d = 3$. They proved that if a solution $u$ stays bounded in $\dot{H}^{1/2}$ throughout its lifespan, then $u$ must be global and scatter. The same statement for $d = 4$ was proven as a special case of the results of [20]. In this short note, we establish this result for all $d \geq 5$.

We begin with some definitions.

Definition 1.1 (Solution). A function $u : I \times \mathbb{R}^d \to \mathbb{C}$ on a time interval $I \ni 0$ is a solution to (1.1) if it belongs to $C_t \dot{H}^{1/2} \cap L^{2(d+2)}_{t,x} (K \times \mathbb{R}^d)$ for every compact $K \subset I$ and obeys the Duhamel formula

\[
u(t) = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} (|u|^\frac{4}{d-1} u)(s) \, ds
\]

for all $t \in I$. We call $I$ the lifespan of $u$; we say $u$ is a maximal-lifespan solution if it cannot be extended to any strictly larger interval. If $I = \mathbb{R}$, we say $u$ is global.

Definition 1.2 (Scattering size and blowup). We define the scattering size of a solution $u$ to (1.1) on a time interval $I$ by

\[
S_I(u) := \int_I \int_{\mathbb{R}^d} |u(t, x)|^{2(d+2) \over d-1} \, dx \, dt.
\]

(1.2)

If there exists $t \in I$ such that $S_{(t, \sup I)}(u) = \infty$, then we say $u$ blows up forward in time. Similarly, if there exists $t \in I$ such that $S_{(\inf I, t]}(u) = \infty$, then we say $u$ blows up backward in time.
On the other hand, if $u$ is global with $S_R(u) < \infty$, then standard arguments show that $u$ scatters, that is, there exist unique $u_\pm \in \dot{H}^{1/2}_x(\mathbb{R}^d)$ such that

$$\lim_{t \to \pm \infty} \|u(t) - e^{it\Delta} u_\pm\|_{\dot{H}^{1/2}_x(\mathbb{R}^d)} = 0.$$ 

Our main result is the following

**Theorem 1.3.** Let $d \geq 5$ and let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a maximal-lifespan solution to (1.1) such that $u \in L^\infty_t \dot{H}^{1/2}_x(I \times \mathbb{R}^d)$. Then $u$ is global and scatters, with

$$S_R(u) \leq C\left(\|u\|_{L^\infty_t \dot{H}^{1/2}_x(\mathbb{R} \times \mathbb{R}^d)}\right)$$

for some function $C : [0, \infty) \to [0, \infty)$.

Following the approach of Kenig and Merle [10], we will establish Theorem 1.3 by combining a concentration-compactness argument with the Lin-Strauss Morawetz inequality of [15]. This estimate is very useful in the study of (1.1), as it has critical scaling for this problem. In fact, it is the concentration-compactness component that comprises most of this note; once we have reduced the problem to the study of almost periodic solutions, we can quickly bring the argument to a conclusion.

We first need a good local-in-time theory. Building off arguments of Cazenave and Weissler [3], we can prove the following local well-posedness result (see Remark 3.8).

**Theorem 1.4 (Local well-posedness).** Let $d \geq 5$ and $u_0 \in \dot{H}^{1/2}_x(\mathbb{R}^d)$. Then there exists a unique maximal-lifespan solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (1.1) such that:

- (Local existence) $I$ is an open neighborhood of 0.
- (Blowup criterion) If $\sup I < \infty$, then $u$ blows up forward in time. If $\inf I < \infty$, then $u$ blows up backward in time.
- (Existence of wave operators) For any $u_+ \in \dot{H}^{1/2}_x(\mathbb{R}^d)$, there is a unique solution $u$ to (1.1) such that $u$ scatters to $u_+$, that is,

$$\lim_{t \to \pm \infty} \|u(t) - e^{it\Delta} u_+\|_{\dot{H}^{1/2}_x(\mathbb{R}^d)} = 0.$$ 

A similar statement holds backward in time.

- (Small-data global existence) If $\|u_0\|_{\dot{H}^{1/2}_x(\mathbb{R}^d)}$ is sufficiently small depending on $d$, then $u$ is global and scatters, with $S_R(u) \lesssim \|u_0\|_{\dot{H}^{1/2}_x(\mathbb{R}^d)}$.  

1.1. Outline of the proof of Theorem 1.3. We argue by contradiction and suppose that Theorem 1.3 fails. Recalling from Theorem 1.3 that Theorem 1.3 holds for sufficiently small initial data, we deduce the existence of a threshold size, below which Theorem 1.3 holds, but above which we can find (almost) counterexamples. We then use a limiting argument to find blowup solutions at this threshold, and show that such minimal blowup solutions must possess strong concentration properties. Finally, in Sections 5 and 6 we show that solutions to (1.1) with such properties cannot exist.

The main property of these solutions is that of almost periodicity:

**Definition 1.5 (Almost periodic solutions).** A solution $u$ to (1.1) with lifespan $I$ is said to be almost periodic (modulo symmetries) if $u \in L^\infty_t \dot{H}^{1/2}_x(I \times \mathbb{R}^d)$ and there exist functions $N : I \to \mathbb{R}^+$, $x : I \to \mathbb{R}^d$, and $C : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\int_{|x-x(t)| \geq C(\eta)N(t)} |\nabla|^{1/2} u(t, x)|^2 \, dx + \int_{|\xi| \geq C(\eta)N(t)} |\xi||\hat{u}(t, \xi)|^2 \, d\xi \leq \eta$$

for all $t \in I$. 

for all $t \in I$ and $\eta > 0$. We call $N$ the frequency scale function, $x$ the spatial center function, and $C$ the compactness modulus function.

**Remark 1.6.** Using Arzelà–Ascoli and Sobolev embedding, one can derive the following: for a nonzero almost periodic solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (1.1), there exists $C(u) > 0$ such that

$$\int_{|x-x(t)| \leq \frac{C(u)}{\sqrt{t}} dx} |u(t,x)|^{\frac{2d}{d-1}} \geq 1$$

uniformly for $t \in I$.

We can now describe the first major step in the proof of Theorem 1.3.

**Theorem 1.7 (Reduction to almost periodic solutions).** If Theorem 1.3 fails, then there exists a maximal-lifespan solution $u : I \times \mathbb{R}^d \to \mathbb{C}$ to (1.1) such that $u$ is almost periodic and blows up in both time directions.

The reduction to almost periodic solutions has become a well-known and widely used technique in the study of dispersive equations at critical regularity. The existence of such solutions was first established by Keraani [12] in the context of the mass-critical NLS, while Kenig and Merle [9] were the first to use them to prove a global well-posedness result (in the energy-critical setting). These techniques have since been adapted to a variety of settings (see [7, 10, 13, 14, 15, 16, 20, 23, 24] for some examples in the case of NLS), and the general approach is well-understood.

The argument, which we will carry out in Section 4, requires three ingredients: (i) a profile decomposition for the linear propagator, (ii) a stability result for the nonlinear equation, and (iii) a decoupling statement for nonlinear profiles. The first profile decompositions established for $e^{it\Delta}$ were adapted to the mass- and energy-critical settings (see [1, 2, 11, 19]); the case of non-conserved critical regularity was addressed in [21]. We will be able to import the profile decomposition we need directly from [21] (see Section 2.4).

Ingredients (ii) and (iii) are closely related, in that the decoupling must be established in a space that is dictated by the stability result. Stability results most often require errors to be small in a space with the scaling-critical number of derivatives (say $s_c$). In [11], Keraani showed how to establish the decoupling in such a space for the energy-critical problem (that is, $s_c = 1$). The argument relies on pointwise estimates and hence is also applicable to the mass-critical problem ($s_c = 0$). For $s_c \notin \{0,1\}$, however, the nonlocal nature of $|\nabla|^{s_c}$ prevents the direct use of this argument.

In certain cases for which $s_c \notin \{0,1\}$ it has nonetheless been possible to adapt the arguments of [11] to establish the decoupling in a space with $s_c$ derivatives. Kenig and Merle [10] were able to succeed in the case $s_c = 1/2, d = 3$ (for which the nonlinearity is cubic) by exploiting the polynomial nature of the nonlinearity and making use of a paraproduct estimate. Killip and Vişan [16] handled some cases for which $s_c > 1$ by utilizing a square function of Strichartz that shares estimates with $|\nabla|^{s_c}$. In [20], some cases were treated for which $s_c \in (0,1)$ (and the nonlinearity is non-polynomial) by making use of the Littlewood–Paley square function and working at the level of individual frequencies.

In this paper, we take a simpler approach to (ii) and (iii), inspired by the work of Holmer and Roudenko [7] on the focusing $\dot{H}^{1/2}_x$-critical NLS in $d = 3$. It relies on the observation that for $s_c = 1/2$, one can develop a stability theory for NLS that only requires errors to be small in a space without derivatives. In Section 3,
we do exactly this (see Theorem 3.6). To prove the decoupling in a space without
derivatives, we can then rely simply on pointwise estimates and apply the arguments
of [11] directly (see Lemma 4.3). By proving a more refined stability result, we are
thus able to avoid entirely the technical issues related to fractional differen-
tiation described above. In this way, we can greatly simplify the analysis needed to carry
out the reduction to almost periodic solutions in our setting.

Continuing from Theorem 1.7, we can make some further reductions to the class
of solutions that we consider. In particular, we can prove the follow-

Theorem 1.8. If Theorem 1.3 fails, then there exists an almost periodic solution
$u : [0, T_{\text{max}}) \to \mathbb{C}$ to (1.1) with the following properties:
(i) $u$ blows up forward in time,
(ii) $\inf_{t \in [0, T_{\text{max}})} |x(t)| \geq 1$,
(iii) $|x(t)| \lesssim u \int_0^t N(s) \, ds$ for all $t \in [0, T_{\text{max}})$.

Let us briefly sketch the proof of Theorem 1.8. Beginning with an almost periodic
solution as in Theorem 1.7 and using a rescaling argument (as in [24, Theorem 3.3],
for example), one can deduce the existence of an almost periodic blowup solution
that does not escape to arbitrarily low frequencies on at least half of its maximal
lifespan, say $[0, T_{\text{max}})$. In this way, we may find an almost periodic solution such
that (i) and (ii) hold.

It remains to see how one can modify an almost periodic solution $u : [0, T_{\text{max}}) \times
\mathbb{R}^d \to \mathbb{C}$ so that (iii) holds. We may always translate $u$ so that $x(0) = 0$; thus,
it suffices to show that we can modify the modulation parameters of $u$ so that
$|\dot{x}(t)| \sim_u N(t)$ for a.e. $t \in [0, T_{\text{max}})$. This will follow from the following local
constancy property for the modulation parameters of almost periodic solutions (see
[14, Lemma 5.18], for example):

Lemma 1.9 (Local constancy). Let $u : I \times \mathbb{R}^d \to \mathbb{C}$ be a maximal-lifespan almost
periodic solution to (1.1). Then there exists $\delta = \delta(u) > 0$ such that if $t_0 \in I$, then
$[t_0 - \delta N(t_0)^{-2}, t_0 + \delta N(t_0)^{-2}] \subset I$,
with

$N(t) \sim_u N(t_0)$, $|x(t) - x(t_0)| \lesssim_u N(t_0)^{-1}$ for $|t - t_0| \leq \delta N(t_0)^{-2}$.

Using Lemma 1.9 we may subdivide $[0, T_{\text{max}})$ into characteristic subintervals $I_k$
and set $N(t)$ to be constant and equal to some $N_k$ on each $I_k$. Note that $|I_k| \sim_u N_k^{-2}$
and that this requires us to modify the compactness modulus function by a (time-
independent) multiplicative factor. We may then modify $x(t)$ by $O(N(t)^{-1})$ so that
$x(t)$ becomes piecewise linear on each $I_k$, with $|\dot{x}(t)| \sim_u N(t)$ for $t \in I_k^\circ$.
Thus, we get $|\dot{x}(t)| \sim_u N(t)$ for a.e. $t \in [0, T_{\text{max}})$, as desired.

To complete the proof of Theorem 1.3 it therefore suffices to rule out the existence
of the almost periodic solutions described in Theorem 1.8.

In Section 5 we preclude the possibility of finite time blowup (i.e. $T_{\text{max}} <
\infty$). To do this, we make use of the following ‘reduced’ Duhamel formula for
almost periodic solutions, which one can prove by adapting the argument in [14,
Proposition 5.23]:
Proposition 1.10 (Reduced Duhamel formula). Let $u : [0, T_{\text{max}}) \times \mathbb{R}^d \to \mathbb{C}$ be an almost periodic solution to (1.1). Then for all $t \in [0, T_{\text{max}})$, we have

$$u(t) = i \lim_{T \to T_{\text{max}}} \int_t^T e^{i(t-s)\Delta} (|u|^{\frac{4}{d-2}}u)(s) \, ds,$$

where the limits are taken in the weak $\dot{H}^{1/2}$ topology.

Using Proposition 1.10 Strichartz estimates, and conservation of mass, we can show that an almost periodic solution that blows up in finite time must have zero mass, contradicting the fact that the solution blows up in the first place.

In Section 5 we use the Lin–Strauss Morawetz inequality to rule out the remaining case, $T_{\text{max}} = \infty$. This estimate tells us that solutions to NLS that are bounded in $\dot{H}^{1/2}$ cannot remain concentrated near the origin for too long. However, almost periodic solutions to (1.1) as in Theorem 1.8 with $T_{\text{max}} = \infty$ do essentially this; thus we can reach a contradiction in this case.

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2. Notation and useful lemmas

2.1. Some notation. We write $X \lesssim Y$ or $Y \gtrsim X$ whenever $X \leq CY$ for some $C = C(d) > 0$. If $X \lesssim Y \lesssim X$, we write $X \sim Y$. Dependence on additional parameters will be indicated with subscripts, for example, $X \lesssim_u Y$.

For a spacetime slab $I \times \mathbb{R}^d$, we define

$$\|u\|_{L^q_tL^r_x(I \times \mathbb{R}^d)} := \|u(t)\|_{L^q_x(\mathbb{R}^d)} \|L^r_t(I)}.$$ 

If $q = r$, we write $L^q_tL^q_x = L^q_x$. We also sometimes write $\|f\|_{L^q(\mathbb{R}^d)} = \|f\|_{L^q}$. We define the Fourier transform on $\mathbb{R}^d$ by

$$\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x) \, dx.$$ 

For $s > -d/2$, we then define the fractional differentiation operator $|\nabla|^s$ and the homogeneous Sobolev norm via $|\nabla|^s \hat{f}(\xi) := |\xi|^s \hat{f}(\xi)$ and $\|f\|_{H^s(\mathbb{R}^d)} := \|\nabla|^s f\|_{L^2(\mathbb{R}^d)}$.

2.2. Basic harmonic analysis. Let $\varphi$ be a radial bump function supported in the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq \frac{1}{10}\}$ and equal to 1 on the ball $\{\xi \in \mathbb{R}^d : |\xi| \leq 1\}$. For $N \in 2^\mathbb{N}$, we define the Littlewood–Paley projection operators via

$$P_{\leq N} f(\xi) := \varphi(\frac{\xi}{N}) \hat{f}(\xi), \quad P_N f(\xi) := (\varphi(\frac{\xi}{N}) - \varphi(\frac{2\xi}{N})) \hat{f}(\xi), \quad P_{> N} := \text{Id} - P_{\leq N}.$$ 

We note that these operators commute with $e^{it\Delta}$ and all differential operators, as they are Fourier multiplier operators. They also obey the following

Lemma 2.1 (Bernstein estimates). For $1 \leq r \leq q \leq \infty$ and $s \geq 0$,

$$\|P_{> N} f\|_{L^r(\mathbb{R}^d)} \lesssim N^{-s} \|\nabla|^s f\|_{L^r(\mathbb{R}^d)}, \quad \|P_{N} f\|_{L^r(\mathbb{R}^d)} \lesssim N^{\frac{d}{r} - \frac{d}{q}} \|f\|_{L^q(\mathbb{R}^d)}.$$ 

We will also need some fractional calculus estimates.
Lemma 2.2 (Fractional chain rule, [3]). Suppose $G \in C^1(\mathbb{C})$ and $s \in (0,1]$. Let $1 < r, r_1 < \infty$, $1 < r_2 \leq \infty$ be such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$. Then

$$\|\nabla^s G(u)\|_{L^r_x} \leq \|G'(u)\|_{L^{r_1}_x} \|\nabla^s u\|_{L^{r_2}_x}.$$ 

Lemma 2.3 (Derivatives of differences, [17]). Let $F(u) = |u|^p u$ for some $p > 0$ and let $0 < s < 1$. For $1 < r, r_1, r_2 < \infty$ such that $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$, we have

$$\|\nabla^s[F(u + v) - F(u)]\|_{L^r_x} \leq \|\nabla^s u\|_{L^{r_1}_x} \|v\|_{L^{r_2}_x}^p + \|\nabla^s v\|_{L^{r_2}_x} \|u + v\|_{L^{p,r_2}_x}^p.$$ 

2.3. Strichartz estimates. Let $e^{it\Delta}$ be the free Schrödinger propagator:

$$[e^{it\Delta}f](x) = (4\pi it)^{-d/2} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} f(y) \, dy \quad \text{for } t \neq 0.$$ 

For $d \geq 3$, we call a pair of exponents $(q, r)$ Schrödinger admissible if $2 \leq q, r \leq \infty$ and $\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$. For a time interval $I$ and $s \geq 0$, we define the Strichartz space $\dot{S}^s(I)$ via the norm

$$\|u\|_{\dot{S}^s(I)} = \sup \{ \|\nabla^s u\|_{L^q(I \times \mathbb{R}^d)} : (q, r) \text{ Schrödinger admissible} \}.$$ 

We will make frequent use of the following standard estimates for $e^{it\Delta}$:

Lemma 2.4 (Strichartz estimates, [6] [8] [22]). Let $d \geq 3$, $s \geq 0$ and let $I$ be a compact time interval. Let $u : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ be a solution to the forced Schrödinger equation $(i\partial_t + \Delta) u = F$. Then for any $t_0 \in I$, we have

$$\|u\|_{\dot{S}^s(I)} \leq \|u(t_0)\|_{\dot{H}^s_x(\mathbb{R}^d)} + \min \left\{ \|\nabla^s F\|_{L^q_x L^r_x(I \times \mathbb{R}^d)} , \|\nabla^s F\|_{L^q_x L^{\frac{2(d+2)}{d+4}}_x(I \times \mathbb{R}^d)} \right\}.$$

2.4. Concentration-compactness. We record here the linear profile decomposition that we will use in Section 3. We begin with the following

Definition 2.5 (Symmetry group). For any position $x_0 \in \mathbb{R}^d$ and scaling parameter $\lambda > 0$, we define a unitary transformation $g_{x_0,\lambda} : \dot{H}^{1/2}_x(\mathbb{R}^d) \rightarrow \dot{H}^{1/2}_x(\mathbb{R}^d)$ by

$$[g_{x_0,\lambda}f](x) := \lambda^{-\frac{d+1}{2}} f(\lambda^{-1}(x - x_0)).$$

We let $G$ denote the group of such transformations.

We now state the linear profile decomposition. For the mass-critical NLS, this result was originally proven in [11] [2] [19], while for the energy-critical NLS, it was established in [11]. In the generality we need, a proof can be found in [21].

Lemma 2.6 (Linear profile decomposition, [21]). Let $\{u_n\}_{n \geq 1}$ be a bounded sequence in $\dot{H}^{1/2}_x(\mathbb{R}^d)$. After passing to a subsequence if necessary, there exist functions $\{\phi^j\}_{j \geq 1} \subseteq \dot{H}^{1/2}_x(\mathbb{R}^d)$, group elements $g^j_n \in G$ (with parameters $x^j_n$ and $\lambda^j_n$), and times $t^j_n \in \mathbb{R}$ such that for all $J \geq 1$, we have the following decomposition:

$$u_n = \sum_{j=1}^J g^j_n e^{it_n^j \Delta} \phi^j + w^j_n.$$

This decomposition satisfies the following properties:

- For each $j$, either $t^j_n \equiv 0$ or $t^j_n \to \pm \infty$ as $n \to \infty$.
- For $J \geq 1$, we have the following decaying:

$$\lim_{n \to \infty} \left[ \|u_n\|_{H^{1/2}_x}^2 - \sum_{j=1}^J \|\phi^j\|_{H^{1/2}_x}^2 - \|w^j_n\|_{H^{1/2}_x}^2 \right] = 0. \quad (2.1)$$
If nonlinearities solutions carried out in Section 4. Throughout this section, we will denote Theorem 3.6, which will play a key role in the reduction to almost periodic for a time interval \( I \) are critical with respect to scaling, but do not involve any derivatives. In particular, we get
\[
\| \lambda_n^j \|_{L^\infty_n} + \| \lambda_n^k \|_{L^\infty_n} + \| x_n^j - x_n^k \|^2 \to \infty \quad \text{as } n \to \infty. \tag{2.2}
\]

- For all \( n \) and all \( J \geq 1 \), we have \( u_n^J \in \dot{H}^{1/2}(\mathbb{R}^d) \), with
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \| e^{itJ} u_n^J \|_{L^{2(d+2)}_{t,x}((\mathbb{R} \times \mathbb{R}^d))} = 0. \tag{2.3}
\]

### 3. Stability theory

In this section, we develop a stability theory for (1.1). The main result of this section is Theorem 3.1 which will play a key role in the reduction to almost periodic solutions carried out in Section 4. Throughout this section, we will denote the nonlinearity \( |u|^{\sigma} u \) by \( F(u) \).

We begin by recording a local well-posedness result of Cazenave and Weissler [3]. This result requires the data to belong to the inhomogeneous Sobolev space, so that a contraction mapping argument may be run in mass-critical spaces.

**Theorem 3.1 (Standard local well-posedness [3])**. Let \( d \geq 5 \) and \( u_0 \in H^{1/2}_{x}(\mathbb{R}^d) \). If \( I \ni 0 \) is a time interval such that
\[
\| \nabla \|^{1/2} e^{itJ} u_0 \|_{L^2_x} \geq 2^{d(d+1)/4} L_x (I \times \mathbb{R}^d)
\]
is sufficiently small, then we may find a unique solution \( u : I \times \mathbb{R}^d \to \mathbb{C} \) to (1.1).

Next, we turn to the stability results. We will make use of function spaces that are critical with respect to scaling, but do not involve any derivatives. In particular, for a time interval \( I \), we define the following norms:
\[
\| u \|_{X(I)} := \| u \|_{L^{4(d+1)}_{t} L^{2(d+1)}_{x} (I \times \mathbb{R}^d)}, \quad \| F \|_{Y(I)} := \| F \|_{L^{4(d+1)}_{t} L^{2(d+1)}_{x} (I \times \mathbb{R}^d)}.
\]

We first relate the \( X \) norm to the usual Strichartz norms. By Sobolev embedding, we get \( \| u \|_{X(I)} \lesssim \| u \|_{\dot{S}^{1/2} \cap C(I)} \), while Hölder and Sobolev embedding together imply
\[
\| u \|_{L^{2(d+2)}_{t,x} (I \times \mathbb{R}^d)} \lesssim \| u \|_{X(I)} \| u \|_{\dot{S}^{1/2} \cap C(I)}^{1-c} \quad \text{for some } 0 < c(d) < 1. \tag{3.1}
\]

Next, we record a Strichartz estimate, which one can prove via the standard approach (namely, by applying the dispersive estimate and Hardy–Littlewood–Sobolev).

**Lemma 3.2**. Let \( I \) be a compact time interval and \( t_0 \in I \). Then for all \( t \in I \),
\[
\left\| \int_{t_0}^{t} e^{i(t-s)\Delta} F(s) \, ds \right\|_{X(I)} \lesssim \| F \|_{Y(I)}. \tag{3.2}
\]

Finally, we collect some estimates that will allow us to control the nonlinearity.
Lemma 3.3. Fix $d \geq 5$. Then, with spacetime norms over $I \times \mathbb{R}^d$, we have
\[
\|F(u)\|_{Y(I)} \lesssim \|u\|_{X(I)}^{\frac{d+3}{d-1}} \tag{3.3}
\]
\[
\|F(u) - F(\tilde{u})\|_{Y(I)} \lesssim \left\{ \|u\|_{X(I)}^{\frac{d+3}{d-1}} + \|\tilde{u}\|_{X(I)}^{\frac{d+3}{d-1}} \right\} \|u - \tilde{u}\|_{X(I)} \tag{3.4}
\]
\[
\|\nabla^{1/2}F(u)\|_{L^2_t L^{d+2}_x} \lesssim \|u\|_{X(I)}^{\frac{d+3}{d-1}} \|u\|_{S^{1/2}(I)} \tag{3.5}
\]
\[
\|\nabla^{1/2}[F(u) - F(\tilde{u})]\|_{L^2_t L^{d+2}_x} \lesssim \|u - \tilde{u}\|_{X(I)}^{\frac{d+3}{d-1}} \|\tilde{u}\|_{S^{1/2}(I)} + \|u\|_{X(I)}^{\frac{d+3}{d-1}} \|u - \tilde{u}\|_{S^{1/2}(I)}. \tag{3.6}
\]

Proof. We first note that (3.3) follows from Hölder, while (3.4) follows from the fundamental theorem of calculus followed by Hölder.

Next, we see that (3.5) follows from Hölder and the fractional chain rule. Indeed,
\[
\|\nabla^{1/2}F(u)\|_{L^2_t L^{d+2}_x} \lesssim \|u\|_{X(I)}^{\frac{d+3}{d-1}} \|\nabla^{1/2}u\|_{L^2_t L^{d+2}_x}. \tag{3.7}
\]

Using these same exponents with Lemma 2.3, we deduce (3.6). \hfill \Box

We may now state our first stability result.

Lemma 3.4 (Short-time perturbations). Let $d \geq 5$ and let $I$ be a compact time interval, with $t_0 \in I$. Let $\tilde{u} : I \times \mathbb{R}^d \to \mathbb{C}$ be a solution to $(i\partial_t + \Delta)\tilde{u} = F(\tilde{u}) + e$ with $\tilde{u}(t_0) = \tilde{u}_0 \in \dot{H}^{1/2}$. Suppose
\[
\|\tilde{u}\|_{S^{1/2}(I)} \leq E \quad \text{and} \quad \|\nabla^{1/2}e\|_{L_t^{2(d+2)} L_x^{\frac{d(d+2)}{d+3}}(I \times \mathbb{R}^d)} \leq E \tag{3.8}
\]
for some $E > 0$. Let $u_0 \in \dot{H}^{1/2}(\mathbb{R}^d)$ satisfy
\[
\|u_0 - \tilde{u}_0\|_{\dot{H}^{1/2}} \leq E, \tag{3.9}
\]
and suppose that we have the smallness conditions
\[
\|\tilde{u}\|_{X(I)} \leq \delta, \tag{3.10}
\]
for some small $0 < \delta = \delta(E)$ and $0 < \varepsilon < \varepsilon_0(E)$. Then there exists $u : I \times \mathbb{R}^d \to \mathbb{C}$ solving (1.1) with $u(t_0) = u_0$ such that
\[
\|u - \tilde{u}\|_{X(I)} + \|F(u) - F(\tilde{u})\|_{Y(I)} \lesssim \varepsilon, \tag{3.11}
\]
\[
\|u - \tilde{u}\|_{S^{1/2}(I)} + \|\nabla^{1/2}[F(u) - F(\tilde{u})]\|_{L^2_t L^{d+2}_x(\mathbb{R}^d)} \lesssim E. \tag{3.12}
\]

Proof. We first suppose $u_0 \in L^2_x$, so that Theorem 3.1 provides the solution $u$. We will then prove (3.11) and (3.12) as a priori estimates. After the lemma is proven for $u_0 \in \dot{H}^{1/2}_x$, we can use approximation by $H^{1/2}_x$-functions to see that the lemma holds for $u_0 \in \dot{H}^{1/2}_x$. Throughout the proof, spacetime norms will be over $I \times \mathbb{R}^d$.

We will first show
\[
\|u\|_{X(I)} \lesssim \delta. \tag{3.13}
\]
By the triangle inequality, (3.2), (3.3), (3.9), and (3.10), we get
\[
\|e^{i(t-t_0)\Delta} \tilde{u}_0\|_{X(I)} \lesssim \|\tilde{u}\|_{X(I)} + \|F(\tilde{u})\|_{Y(I)} + \|e\|_{Y(I)} \lesssim \delta + \delta^{\frac{d+1}{d-1}} + \varepsilon.
\]
Combining this estimate with (3.10) and using the triangle inequality then gives
\[ \| e^{i(t-t_0)\Delta} u_0 \|_{X(I)} \lesssim \delta \]
for \( \delta \) and \( \varepsilon \lesssim \delta \) sufficiently small. Thus, by (3.2) and (3.3), we get
\[ \| u \|_{X(I)} \lesssim \delta + \| F(u) \|_{Y(I)} \lesssim \delta + \| u \|_{\frac{d+3}{4}}^{\frac{4}{d+3}} \]
which (taking \( \delta \) sufficiently small) implies (3.13).

We now turn to proving the desired estimates for \( w := u - \tilde{u} \). Note first that \( w \) is a solution to \( (i\partial_t + \Delta)w = F(u) - F(\tilde{u}) - \varepsilon \), with \( w(t_0) = u_0 - \tilde{u}_0 \); thus, we can use (3.2), (3.7), (3.9), (3.10), and (3.13) to see
\[ \| w \|_{X(I)} \lesssim \| e^{i(t-t_0)\Delta} (u_0 - \tilde{u}_0) \|_{X(I)} + \| \varepsilon \|_{Y(I)} + \| F(u) - F(\tilde{u}) \|_{Y(I)} \]
\[ \lesssim \varepsilon + \| u \|_{X(I)} + \| \tilde{u} \|_{X(I)}^{\frac{4}{d+3}} \]
\[ \lesssim \varepsilon + \delta + \| w \|_{X(I)}. \]
Taking \( \delta \) sufficiently small, we see that the first estimate in (3.11) holds. Using the first estimate in (3.11), along with (3.2), (3.9), and (3.13), we see that the remaining estimate in (3.11) holds, as well.

Next, by Strichartz, (3.9), (3.7), (3.3), (3.9), (3.11), and (3.13), we get
\[ \| w \|_{\dot{S}^{1/2}(I)} \lesssim \| u_0 - \tilde{u}_0 \|_{\dot{H}^{1/2}} + \| \nabla^{1/2} \varepsilon \|_{L^{2+\frac{s}{2}}_{t,x}(I \times \mathbb{R}^d)} + \| \nabla^{1/2} [F(u) - F(\tilde{u})] \|_{L^{2+\frac{s}{2}}_{t,x}} \]
\[ \lesssim \varepsilon + \| u \|_{\dot{S}^{1/2}(I)} + \| w \|_{\dot{S}^{1/2}(I)} \| w \|_{X(I)} + \| w \|_{\dot{S}^{1/2}(I)} \| w \|_{X(I)} \]
\[ \lesssim \varepsilon + \delta + \| w \|_{\dot{S}^{1/2}(I)}. \]
Taking \( \delta = \delta(E) \) sufficiently small then gives the first estimate in (3.12). We get the remaining estimate in (3.12) by using (3.6) with (3.7), (3.11), (3.13), and the first estimate in (3.12). \( \square \)

**Remark 3.5.** As mentioned in the introduction, the error \( \varepsilon \) is only required to be small in a space without derivatives (see (3.10)); it merely needs to be bounded in a space with derivatives (see (3.7)). This will also be the case in Theorem 3.6 below (see (3.14) and (3.15)). We will see the benefit of this refinement when we carry out the proof of Theorem 1.7 in Section 4 (see Remark 4.4).

We continue to the main result of this section:

**Theorem 3.6 (Stability).** Let \( d \geq 5 \), and let \( I \) be a compact time interval, with \( t_0 \in I \). Suppose \( \tilde{u} \) is a solution to \( (i\partial_t + \Delta)\tilde{u} = F(\tilde{u}) + \varepsilon \), with \( \tilde{u}(t_0) = \tilde{u}_0 \). Suppose
\[ \| \tilde{u} \|_{\dot{S}^{1/2}(I)} \leq E \quad \text{and} \quad \| \nabla^{1/2} \varepsilon \|_{L^{2+\frac{s}{2}}_{t,x}(I \times \mathbb{R}^d)} \leq E \] (3.14)
for some \( E > 0 \). Let \( u_0 \in \dot{H}^{1/2}_{t,x}(\mathbb{R}^d) \), and suppose we have the smallness conditions
\[ \| u_0 - \tilde{u}_0 \|_{\dot{H}^{1/2}_{t,x}(\mathbb{R}^d)} + \| \varepsilon \|_{Y(I)} \leq \varepsilon \] (3.15)
for some small \( 0 < \varepsilon < \varepsilon_1(E) \). Then, there exists \( u : I \times \mathbb{R}^d \rightarrow \mathbb{C} \) solving (1.1) with \( u(t_0) = u_0 \), and there exists \( 0 < c(d) < 1 \) such that
\[ \| u - \tilde{u} \|_{L^{2+\frac{s}{2}}_{t,x}(I \times \mathbb{R}^d)} \lesssim \varepsilon^c. \] (3.16)
One can derive Theorem 3.6 from Lemma 3.4 in the standard fashion, namely, by applying Lemma 3.4 inductively (see [14], for example). We omit these details, but pause to point out the following: this induction will actually yield the bounds

\[ \|u - \tilde{u}\|_{X(I)} \lesssim \varepsilon \quad \text{and} \quad \|u - \tilde{u}\|_{S^{1/2}(I)} \lesssim E. \]

With these bounds in hand, we then use (3.1) to see that (3.16) holds.

**Remark 3.7.** The smallness condition on \( u_0 - \tilde{u}_0 \) appearing in (3.15) may actually be relaxed to the condition appearing in (3.10). In our setting, it will not be difficult to prove the stronger condition (see Lemma 1.3).

**Remark 3.8.** Using arguments from [3] [5], one can establish Theorem 1.4 for data in the inhomogeneous Sobolev space \( H^{1/2}_x \). Using Theorem 3.6 one can then remove the assumption \( u_0 \in L^2 \) a posteriori (by approximating \( u_0 \in \dot{H}^{1/2}_x \) by \( H^{1/2}_x \)-functions). We omit the standard details.

### 4. Reduction to Almost Periodic Solutions

In this section, we sketch a proof of Theorem 1.7. As described in the introduction, the key ideas come from [11] [12] and are well-known. Thus, we will merely outline the argument, providing full details only when our approach deviates from the usual one. We model our presentation primarily after [15, Section 3]. Throughout this section, we denote the nonlinearity \( |u|^{4/d} u \) by \( F(u) \).

We suppose that Theorem 1.3 fails. We then define \( L : [0, \infty) \to [0, \infty) \) by

\[ L(E) := \sup \{ S_I(u) \mid u : I \times \mathbb{R}^d \to \mathbb{C} \text{ solving (1.1)} \text{ with } \|u\|_{L^\infty_t \dot{H}^{1/2}_x(t \times \mathbb{R}^d)}^2 \leq E \}. \]

We note that \( L \) is non-decreasing, with \( L(E) \lesssim E^\frac{d+2}{d} \) for \( E \) sufficiently small (cf. Theorem 1.3). Thus, there exists a unique critical threshold \( E_c \in (0, \infty) \) such that \( L(E) < \infty \) for \( E < E_c \) and \( L(E) = \infty \) for \( E > E_c \). The failure of Theorem 1.3 implies that \( 0 < E_c < \infty \).

The key to proving Theorem 1.7 is the following convergence result. With this result in hand, establishing Theorem 1.7 is a straightforward exercise (see [15, Section 3.2]).

**Proposition 4.1** (Palais–Smale condition modulo symmetries). Let \( d \geq 5 \) and let \( u_n : I_n \times \mathbb{R}^d \to \mathbb{C} \) be a sequence of solutions to (1.1) such that

\[ \limsup_{n \to \infty} \left\| u_n \right\|^2_{L^\infty_t \dot{H}^{1/2}_x(I_n \times \mathbb{R}^d)} = E_c. \]

Suppose \( t_n \in I_n \) are such that

\[ \lim_{n \to \infty} S_{(t_n, \sup I_n)}(u_n) = \lim_{n \to \infty} S_{(\inf I_n, t_n)}(u_n) = \infty. \quad (4.1) \]

Then, \( \{u_n(t_n)\} \) converges along a subsequence in \( \dot{H}^{1/2}_x(\mathbb{R}^d)/G \).

**Proof.** We first translate so that each \( t_n = 0 \) and apply Lemma 2.6 to write

\[ u_n(0) = \sum_{j=1}^J g_{n} e^{i \Delta t_n^j} \phi^j + w_n^J \quad (4.2) \]

along some subsequence. Recall that for each \( j \), either \( t_n^j \equiv 0 \) or \( t_n^j \to \pm \infty \). To prove Proposition 4.1, we need to show that there is exactly one profile \( \phi^1 \), with \( t_n^1 \equiv 0 \) and \( \|w_n^1\|_{\dot{H}^{1/2}_x} \to 0 \).
First, using Theorem 1.4 for each \( j \) we define \( v^j : I^j \times \mathbb{R}^d \to \mathbb{C} \) to be the maximal-lifespan solution to (1.1) such that
\[
\begin{cases}
v^j(0) = \phi^j & \text{if } t^j_n = 0, \\
v^j \text{ scatters to } \phi^j \text{ as } t \to \pm \infty & \text{if } t^j_n \to \pm \infty.
\end{cases}
\]
Next, we define nonlinear profiles \( v^j_n : I^j_n \times \mathbb{R}^d \to \mathbb{C} \) by
\[
v^j_n(t) = g_n^j v^j( (\lambda_n^j)^{-2} t + t_n^j), \quad \text{where } I^j_n = \{ t : (\lambda_n^j)^{-2} t + t_n^j \in I^j \}.
\]
To complete the proof, we need the following three claims:

(i) There is at least one ‘bad’ profile \( \phi^j \), in the sense that
\[
\limsup_{n \to \infty} S_{[0, \sup I^j_n]}(v^j_n) = \infty. \tag{4.3}
\]

(ii) There can then be at most one profile (which we label \( \phi^1 \)), and \( ||w^1_n||_{H^{1/2}} \to 0 \).

(iii) We have \( t^1_n \equiv 0 \).

We will provide a proof of (i) below. The proofs of (ii) and (iii) require only small variations of the analysis given for (i), so we will merely outline the arguments here. For (ii), one can adapt the argument of [15, Lemma 3.3] to show that the decoupling (2.1) persists in time (this is not obvious, as the \( H^{1/2}_x \)-norm is not a conserved quantity for (1.1)). The critical nature of \( E_c \) may then be used to preclude the possibility of multiple profiles (and to show \( ||w^1_n||_{H^{1/2}_x} \to 0 \)). For (iii), we only need to rule out the cases \( t^1_n \to \pm \infty \). To do this, one can argue by contradiction: if \( t^1_n \to \pm \infty \), one can use the stability result Theorem 3.6 (comparing \( u_n \) to \( e^{it\Delta} u_n(0) \)) to contradict (1.4). See [15, p. 391] for more details.

We now turn to the proof of (i). We first note that the decoupling (2.1) implies that the \( v^j_n \) are global and scatter for \( j \) sufficiently large, say for \( j \geq J_0 \); indeed, for \( j \) sufficiently large, the \( H^{1/2}_x \)-norm of \( \phi^j \) must be below the small-data threshold described in Theorem 1.4. Thus, we need to show that there is at least one bad profile \( \phi^j \) (in the sense of (4.3)) in the range \( 1 \leq j < J_0 \).

Suppose towards a contradiction that there are no bad profiles. By the blowup criterion of Theorem 1.4, this immediately implies that \( \sup I^j_n = \infty \) for all \( j \) and for all \( n \) sufficiently large. In fact, we claim that we have the following:

\[
\limsup_{J \to \infty} \limsup_{n \to \infty} \sum_{j=1}^{J} ||v^j_n||_{S^{1/2}([0, \infty))} \lesssim_{E_c} 1. \tag{4.4}
\]

Indeed, for \( \eta > 0 \), the decoupling (2.1) implies the existence of \( J_1 = J_1(\eta) \) such that
\[
\sum_{j > J_1} ||\phi^j||_{H^{1/2}_x} \lesssim \eta.
\]
Thus, choosing \( \eta \) smaller than the small-data threshold, Strichartz and a standard bootstrap argument give
\[
\sum_{j > J_1} ||v^j_n||_{S^{1/2}([0, \infty))} \lesssim \sum_{j > J_1} ||\phi^j||_{H^{1/2}_x} \lesssim \eta.
\]
As the \( v^j_n \) satisfy \( S_{[0, \infty)}(v^j_n) \lesssim 1 \) for \( n \) large, we may use Strichartz and another bootstrap argument to see \( ||v^j_n||_{S^{1/2}} \lesssim 1 \) for \( 1 \leq j \leq J_1 \) and \( n \) large. Thus, we conclude that (4.4) holds.

Next, using the fact that there are no bad profiles, together with the orthogonality condition (2.2), one can use the arguments of [11] to arrive at the following.
Lemma 4.2 (Orthogonality). For $j \neq k$, we have
\[
\left[\|v_n^j v_n^k\|_{L^r_t L^s_x}^{d+3} + \|v_n^j v_n^{k'}\|_{L^r_t L^{s+1}_x}^{4(d+1)} + \|(|\nabla|^{1/2} v_n^j)(|\nabla|^{1/2} v_n^k)|^{d+2} + \|(|\nabla|^{1/2} E(v_n^j))(|\nabla|^{1/2} E(v_n^k))\|^{d+2}\right]_{L^r_t L^s_x} \to 0 \quad \text{as} \quad n \to \infty,
\]
where all spacetime norms are taken over $[0, \infty) \times \mathbb{R}^d$.

We now wish to use (4.4) and (4.5), together with Theorem 5.6 to deduce a bound on the scattering size of the $u_n$, thus contradicting (1.1). To this end, we define approximate solutions to (1.1) and collect the information we need about them in the following

Lemma 4.3. The approximate solutions $u_n^j(t) := \sum_{j=1}^J \varphi_n^j(t) + e^{it\Delta} w_n^j$ satisfy
\[
\begin{align*}
\limsup_{J \to \infty} \limsup_{n \to \infty} \|u_n(0) - u_n^J(0)\|_{H^{1/2}} &= 0, \quad \text{(4.6)} \\
\limsup_{J \to \infty} \limsup_{n \to \infty} \|S_{(0, \infty)}(u_n^J)\|_{L^{1/2}(\mathbb{R})} &\gtrsim E, \quad \text{(4.7)} \\
\limsup_{J \to \infty} \limsup_{n \to \infty} \|u_n^J\|_{L^{1/2}(\mathbb{R})} &\gtrsim E, \quad \text{(4.8)}
\end{align*}
\]

The errors $e_n^j := (i\partial_t + \Delta)u_n^j - F(u_n^j) = \sum_{j=1}^J F(v_n^j) - F(u_n^j)$ satisfy
\[
\begin{align*}
\limsup_{J \to \infty} \limsup_{n \to \infty} \|\nabla|^{1/2} e_n^j\|_{L^{2(d+2)}(0, \infty) \times \mathbb{R}^d} &\lesssim E, \quad \text{(4.9)} \\
\limsup_{J \to \infty} \limsup_{n \to \infty} \|e_n^j\|_{L^{4(d+1)}(0, \infty) \times \mathbb{R}^d} &\to 0. \quad \text{(4.10)}
\end{align*}
\]

Remark 4.4. It is here that we see the benefit of the refined stability result Theorem 5.6. In particular, to apply Theorem 5.6 we only need to exhibit smallness of the $e_n^j$ in the space appearing in (4.10). As this space contains no derivatives, we can achieve this simply by relying on pointwise estimates.

Proof. We first note that (4.6) follows from the construction of the $v_j$.

Next, we turn to (4.7). To begin, we notice that by Sobolev embedding and the fact that $\frac{2(d+2)}{d-2} \geq 2$, we may deduce from (4.4) that
\[
\sum_{j \geq 1} S_{(0, \infty)}(v_n^j) \lesssim E, \quad 1.
\]

Thus, recalling (2.3), we see that to prove (4.7) it will suffice to show
\[
\limsup_{J \to \infty} \limsup_{n \to \infty} \left| S_{(0, \infty)} \left( \sum_{j=1}^J v_n^j \right) - \sum_{j=1}^J S_{(0, \infty)}(v_n^j) \right| = 0. \quad \text{(4.11)}
\]

To this end, we fix $J$ and use Hölder’s inequality, (4.4), and (4.5) to see
\[
\left| S_{(0, \infty)} \left( \sum_{j=1}^J v_n^j \right) - \sum_{j=1}^J S_{(0, \infty)}(v_n^j) \right| \lesssim J \sum_{j \neq k} \|v_n^j\|_{L^{4(d+2)}(0, \infty) \times \mathbb{R}^d} \|v_n^j v_n^k\|_{L^{\frac{4(d+2)}{d+2}}(0, \infty) \times \mathbb{R}^d} \to 0
\]
as $n \to \infty$. This establishes (4.11) and completes the proof of (4.7).
Let us next turn to (4.9) (which we will later use in the proof of (4.8)). To begin, we will derive the bound

$$\limsup_{J \to \infty} \limsup_{n \to \infty} \|\nabla \|^{1/2} u_n^J \|^{2(d+2)}_{L_{t,x}} \lesssim E_c 1.$$  (4.12)

As $w_n^J \in \dot{H}^{1/2}$, it will suffice to show

$$\limsup_{J \to \infty} \limsup_{n \to \infty} \sum_{j=1}^J \|\nabla \|^{1/2} v_n^j \|^{2(d+2)}_{L_{t,x}} \lesssim E_c 1.$$  (4.13)

To this end, we first note that as $\frac{2(d+2)}{d} \geq 2$, we may use (4.4) to see

$$\limsup_{J \to \infty} \limsup_{n \to \infty} \sum_{j=1}^J \|\nabla \|^{1/2} v_n^j \|^{2(d+2)}_{L_{t,x}} \lesssim E_c 1.$$  (4.14)

On the other hand, for fixed $J$, we can use (4.4) and (4.5) to see

$$\left| \sum_{j=1}^J \|\nabla \|^{1/2} v_n^j \|^{2(d+2)}_{L_{t,x}} - \sum_{j=1}^J \|\nabla \|^{1/2} v_n^j \|^{2(d+2)}_{L_{t,x}} \right| \lesssim J \sum_{j \neq k} \|\nabla \|^{1/2} v_n^j \|^{2(d+2)}_{L_{t,x}} \|\nabla \|^{1/2} v_n^j \|^{2(d+2)}_{L_{t,x}} \to 0 \text{ as } n \to \infty.$$

Then (4.14) implies (4.13), which in turn gives (4.12).

Next, by the fractional chain rule, (4.7), and (4.12), we get

$$\|\nabla \|^{1/2} F(u_n^J) \|^{2(d+2)}_{L_{t,x}} \lesssim \|u_n^J \|^{2(d+2)}_{L_{t,x}} \|\nabla \|^{1/2} u_n^J \|^{2(d+2)}_{L_{t,x}} \lesssim E_c 1.$$  (4.15)

as $n, J \to \infty$, which handles one of the terms appearing in (4.9).

To complete the proof of (4.9), it remains to show

$$\limsup_{J \to \infty} \limsup_{n \to \infty} \sum_{j=1}^J \|\nabla \|^{1/2} F(v_n^j) \|^{2(d+2)}_{L_{t,x}} \lesssim E_c 1.$$  (4.16)

We claim it will suffice to establish

$$\limsup_{J \to \infty} \limsup_{n \to \infty} \sum_{j=1}^J \|\nabla \|^{1/2} F(v_n^j) \|^{2(d+2)}_{L_{t,x}} \lesssim E_c 1.$$  (4.16)

Indeed, for fixed $J$, we have by (4.5)

$$\left| \sum_{j=1}^J \|\nabla \|^{1/2} F(v_n^j) \|^{2(d+2)}_{L_{t,x}} - \sum_{j=1}^J \|\nabla \|^{1/2} F(v_n^j) \|^{2(d+2)}_{L_{t,x}} \right| \lesssim J \sum_{j \neq k} \|\nabla \|^{1/2} F(v_n^j) \|^{2(d+2)}_{L_{t,x}} \|\nabla \|^{1/2} F(v_n^j) \|^{2(d+2)}_{L_{t,x}} \to 0 \text{ as } n \to \infty.$$
To establish (4.16) and thereby complete the proof of (4.9), we use the fractional chain rule and Sobolev embedding to see
\[
\sum_{j=1}^J \| \nabla^{1/2} F(u_n^j) \|_{L_t^4 L_x^4}^{2(d+2)} \lesssim \sum_{j=1}^J \left( \| v_n^j \|_{L_t^\infty L_x^\infty}^{2(d+2)} \| \nabla^{1/2} v_n^j \|_{L_t^4 L_x^4} \right)^{2(d+2)} \lesssim \sum_{j=1}^J \| v_n^j \|_{S^{1/2}_0}^{2(d+2)}.
\]

Then (4.16) follows from (4.1) and the fact that \(2(d+2)(d+3) \geq 2\).

Now (4.8) follows from an application of Strichartz, (4.9) and (4.15).

It remains to establish (4.10). We begin by rewriting
\[
e_n^J = \left[ \sum_{j=1}^J F(v_n^j) - F\left( \sum_{j=1}^J v_n^j \right) \right] + \left[ F(u_n - e^{it\Delta} w_n^J) - F(u_n^J) \right] =: (e_n^J)_1 + (e_n^J)_2.
\]

We first fix \(J\) and use Hölder, Sobolev embedding, (4.1), and (4.5) to see
\[
\| (e_n^J)_1 \|_{L_t^{4(d+1)} L_x^{2(d+1)}} \lesssim J \sum_{j \neq k} \| v_n^j \|_{L_t^{4(d+1)} L_x^{2(d+1)}}^{4(d+1)} \| v_n^j \|_{L_t^{4(d+1)} L_x^{2(d+1)}}^{4(d+1)} \to 0
\]
as \(n \to \infty\). Next, we note that we have the pointwise estimate
\[
| (e_n^J)_2 | \lesssim \| e^{it\Delta} w_n^J \|_{L_t^{4(d+1)} L_x^{2(d+1)}}^{4(d+1)},
\]
where \(f_n^J := u_n^J + e^{it\Delta} w_n^J\) satisfies \(\| f_n^J \|_{S^{1/2}} \lesssim E_c\) as \(n \to \infty\) (cf. (4.8) and the fact that \(w_n^J \in H^{1/2}_x\)). Thus, we can use Hölder, Strichartz, Sobolev embedding, \(w_n^J \in H^{1/2}_x\), and (2.3) to see
\[
\| (e_n^J)_2 \|_{L_t^{4(d+1)} L_x^{2(d+1)}} \lesssim \| e^{it\Delta} w_n^J \|_{L_t^{4(d+1)} L_x^{2(d+1)}}^{4(d+1)} \| f_n^J \|_{L_t^{4(d+1)} L_x^{2(d+1)}}^{4(d+1)} \nu \| e^{it\Delta} w_n^J \|_{L_t^{4(d+1)} L_x^{2(d+1)}} \| f_n^J \|_{L_t^{4(d+1)} L_x^{2(d+1)}} \to 0
\]
as \(n, J \to \infty\).

Combining the estimates for \((e_n^J)_1\) and \((e_n^J)_2\), we conclude that (4.10) holds. \(\square\)

Using Lemma 4.3, we may apply Theorem 3.6 to deduce that \(S_{[0, \infty)}(u_n) \lesssim E_c\) for \(n\) large, contradicting (4.1). We conclude that there is at least one bad profile, that is, claim (i) holds. This completes the proof of Proposition 4.1 and Theorem 1.7 \(\square\)

5. Finite time blowup

In this section, we use Proposition 1.10 Strichartz estimates, and conservation of mass to preclude the existence of almost periodic solutions as in Theorem 1.8 with \(T_{\max} < \infty\).

**Theorem 5.1** (No finite time blowup). Let \(d \geq 5\). There are no almost periodic solutions \(u : [0, T_{\max}) \times \mathbb{R}^d \to C\) with \(T_{\max} < \infty\) and \(S_{[0, T_{\max})} = \infty\).
Proposition 1.10, Strichartz, H"older, Bernstein, and Sobolev embedding give
\[ \|P_N u(t)\|_{L^2_x} \lesssim \|P_N (\|u\|_{L^2_x}^2 u)\|_{L^2_x((t,T_{\max}) \times \mathbb{R}^d)} \]
\[ \lesssim (T_{\max} - t)^{1/2} \|u\|_{L^\infty_t L^\infty_x} \]
\[ \lesssim (T_{\max} - t)^{1/2} N^{1/2} \|u\|_{L^\infty_t L^2_x} \]
As \( u \in L^\infty_t H^{1/2}_x \), we deduce
\[ \|P_{\leq N} u(t)\|_{L^2_x} \lesssim_{u} (T_{\max} - t)^{1/2} N^{1/2} \quad \text{for all } t \in I \text{ and } N > 0. \tag{5.1} \]

On the other hand, an application of Bernstein gives
\[ \|P_{> N} u\|_{L^\infty_t L^2_x} \lesssim_{u} N^{-1/2} \|u\|_{L^\infty_t H^{1/2}_x} \lesssim_{u} N^{-1/2} \quad \text{for all } N > 0. \tag{5.2} \]

We now let \( \eta > 0 \). We choose \( N \) large enough that \( N^{-1/2} < \eta \), and subsequently choose \( t \) close enough to \( T_{\max} \) that \( (T_{\max} - t)^{1/2} N^{1/2} < \eta \). Combining (5.1) and (5.2), we then get \( \|u(t)\|_{L^2_x} \lesssim_{u} \eta \).

As \( \eta \) was arbitrary and mass is conserved, we conclude \( \|u(t)\|_{L^2_x} = 0 \) for all \( t \in [0, T_{\max}) \). Thus \( u \equiv 0 \), which contradicts the fact that \( u \) blows up.

6. The Lin–Strauss Morawetz inequality

In this section, we use the Lin–Strauss Morawetz inequality to preclude the existence of almost periodic solutions as in Theorem 1.8 such that \( T_{\max} = \infty \).

**Proposition 6.1** (Lin–Strauss Morawetz inequality, [13]). Let \( d \geq 3 \) and let \( u : I \times \mathbb{R}^d \to \mathbb{C} \) be a solution to \((\partial_t + \Delta)u = |u|^p u\). Then
\[ \int_I \int_{\mathbb{R}^d} \frac{|u(t,x)|^{p+2}}{|x|} \, dx \, dt \lesssim \|u\|^2_{L^\infty_t H^{1/2}_x(I \times \mathbb{R}^d)}. \tag{6.1} \]

As in [10], we will use this estimate to establish the following

**Theorem 6.2**. Let \( d \geq 5 \). There are no almost periodic solutions \( u : [0, \infty) \times \mathbb{R}^d \to \mathbb{C} \) to (1.1) such that \( u \) blows up forward in time, \( \inf_{t \in [0, \infty)} N(t) \geq 1 \), and \( |x(t)| \lesssim_{u} \int_0^t N(s) \, ds \) for all \( t \geq 0 \).

**Proof.** Suppose \( u \) were such a solution. In particular \( u \) is nonzero, so that by Remark 1.4 we may find \( C(u) > 0 \) such that
\[ \int_{|x-x(t)| \leq \frac{C(u)}{N(t)}} |u(t,x)|^{2(d+1)} \, dx \gtrsim_{u} 1 \quad \text{uniformly for } t \in [0, \infty). \]

Applying H"older and rearranging, this implies
\[ \int_{|x-x(t)| \leq \frac{C(u)}{N(t)}} |u(t,x)|^{\frac{2(d+1)}{d-1}} \, dx \gtrsim_{u} N(t) \quad \text{uniformly for } t \in [0, \infty). \tag{6.2} \]

We now let \( T > 1 \) and use \( u \in L^\infty_t H^{1/2}_x \), (6.1), and (6.2) to see
\[ \int_{1}^{T} \int_{|x-x(t)| \leq \frac{C(u)}{N(t)}} \frac{|u(t,x)|^{2(d+1)}}{|x|} \, dx \, dt \gtrsim_{u} \int_{1}^{T} \frac{N(t)}{|x(t)| + N(t)^{-1}} \, dt. \]
As $\inf_{t \in (1, \infty)} N(t) \geq 1$, to derive a contradiction it will suffice to show that
\[
\lim_{T \to \infty} \int_1^T \frac{N(t)}{1 + |x(t)|} \, dt = \infty. \tag{6.3}
\]
Recalling that $|x(t)| \lesssim u \int_0^t N(s) \, ds$ for all $t \geq 0$, we get
\[
\int_1^T \frac{N(t)}{1 + |x(t)|} \, dt \gtrsim u \int_1^T \frac{d}{dt} \log \left( 1 + \int_0^t N(s) \, ds \right) \, dt \gtrsim u \log \left( \frac{1 + \int_0^T N(s) \, ds}{1 + \int_0^4 N(s) \, ds} \right).
\]
As $\inf_{t \in (1, \infty)} N(t) \geq 1$, we conclude that $\square$ holds, as needed.

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