SUBORDINATION AND RADIUS PROBLEMS FOR CERTAIN STARLIKE FUNCTIONS

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Abstract. We study the following class of starlike functions

\[ S^\ast_\psi := \{ f \in A : z f'(z)/f(z) < 1 + z \psi(z) \} , \]

that are associated with the cardioid domain \( \psi(D) \), by deriving certain convolution results, radius problems, majorization result, radius problems in terms of coefficients and differential subordination implications. Consequently, we establish some interesting generalizations of our results for the Ma-Minda class of starlike functions \( S^\ast_\psi \). We also provide, the set of extremal functions maximizing \( \Re \Phi(\log (f(z)/z)) \) or \( |\Phi(\log (f(z)/z))| \) for functions in \( S^\ast_\psi \), where \( \Phi \) is a non-constant entire function. Further T. H. MacGregor’s result for the class \( S^\ast_\alpha \) and \( S^\ast_\psi \) are obtained as special case to our result.

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1. Introduction

Let \( A_0 \) be the collection of analytic functions of the form \( p(z) = 1 + p_1 z + p_2 z + \cdots \) and \( A \) consists of analytic functions, \( f \) normalized by the conditions \( f(0) = 0 \) and \( f'(0) = 1 \) defined in the unit disk \( D := D_1 \), where \( D_r := \{ z : |z| < r \} \). The Carathéodory class, \( P \) consists of functions \( p \in A_0 \) with \( \Re p(z) > 0 \). Let us denote the class of normalized univalent functions by \( S \). For two analytic functions \( f \) and \( g \), we say \( f \) subordinate to \( g \), written as \( f \prec g \), if there exists a Schwarz function \( w \) such that \( f(z) = g(w(z)) \). If \( g \) is univalent, then \( f \prec g \) if and only if \( f(D) \subseteq g(D) \) and \( f(0) = g(0) \). Many subclasses of \( S \) were introduced and studied by several authors, the chief amongst them is \( S^\ast \) (or \( C \)), the class of normalized starlike (or convex) functions which is characterized by the quantity \( z f'(z)/f(z) \) (or \( 1 + z f''(z)/f'(z) \)), assuming values in the right half plane. In 1992, using subordination, Ma and Minda [19] defined the classes (1.1) and (1.2), which unify many subclasses of \( S \) and they also studied some of it’s common properties.

\[ S^\ast_\psi := \{ f \in A : z f'(z)/f(z) < \psi(z) \} \] \hfill (1.1)

and

\[ C_\psi := \{ f \in A : 1 + z f''(z)/f'(z) < \psi(z) \} , \] \hfill (1.2)

where \( \psi \in P \) is univalent and \( \psi(D) \) starlike with respect to \( \psi(0) = 1 \). Note that \( S^\ast_\psi(z) \) reduces to the class \( S^\ast \). Now a good amount of literature exists for different
choices of $\psi$ in (1.1). For example, the class $\mathcal{S}(\alpha, \beta) := \mathcal{S}^*(p_{\alpha, \beta}(z))$, where

$$p_{\alpha, \beta}(z) := 1 + \frac{\beta - \alpha}{\pi} i \log \frac{1 - e^{2\pi i \alpha/\beta} z}{1 - z},$$

(1.3)

$\alpha < 1$, $\beta > 1$ and $p_{\alpha, \beta}$ maps $\mathbb{D}$ onto the convex domain $\{w \in \mathbb{C} : \alpha < \Re w < \beta\}$, was introduced by Kuroki and Owa [18]. Sokół and Stankiewicz [27] introduced the class associated with lemniscate of Bernoulli $\mathcal{SL}^* := \mathcal{S}^*(e^{z})$. The class $\mathcal{S}_\alpha^* := \mathcal{S}^*(e^{z})$ of starlike functions was introduced by Mendiratta et al. [22] and the class $\mathcal{S}^*_{\alpha, \beta} := \mathcal{S}^*(e^{z})$ of starlike functions related with Bell numbers was introduced by Kumar et al. [15]. For some recent classes see [24, 17, 26, 13, 12]. Evidently the functions $\psi$ under discussion above are all convex and hence satisfy the following result:

**Theorem 1.1.** [19] Let $\psi(\mathbb{D})$ be convex, $g \in \mathbb{C}$ and $f \in \mathcal{S}^*(\psi)$. Then $f * g \in \mathcal{S}^*(\psi)$.

But the above result is not applicable when $\psi(z) = 1 + ze^z, z + \sqrt{1 + z^2}, e^{z-1}$ and $1 + 4z^3/3 + 2z^2/3$ etc. due to the fact that these are not convex. We deal with Theorem 1.1 for the case when $\psi$ starlike and derive some radius related problems.

In Geometric function theory, we usually find the sufficient conditions in terms of it’s coefficients for the normalized functions $f$ in $\mathcal{A}$ to be in a desired class. In recent times, finding the sufficient conditions on functions to be in a desired class using the technique of differential subordination, admissibility conditions and subordination chain has attracted many researcher’s attention. See [11, 12, 24].

In the present work, we consider the following first-order differential subordination implications:

$$1 + \frac{\beta z p'(z)}{p^n(z)} < h(z) \Rightarrow p(z) < q(z),$$

(1.4)

where $n \in \{1, 2, 3\}$. We find the condition on $\beta$ so that (1.4) holds whenever the functions $h$ and $q$ are already known. Note that if $p$ and $q$ are two fixed analytic and univalent functions in $\mathbb{D}$ and $q$ does not have the nice properties like, explicit inverse representation and convexity, then proving $p < q$ is not so easy. See that if $p < q$, then geometrically the boundary curve of $p(\mathbb{D})$ is contained in $q(\mathbb{D})$, which can be ensured by the technique of radial distances. Further, if the images $p(\mathbb{D})$ and $q(\mathbb{D})$ are symmetric about the real axis and both the functions $p$ and $q$ have the same orientation, then we can use the maximum arguments of $p(z)$ and $q(z)$ for $|z| = 1$ to prove that $p < q$.

Recall that for the subfamilies $\mathcal{G}_1$ and $\mathcal{G}_2$ of $\mathcal{A}$, we say that $r_0$ is the $\mathcal{G}_1$-radius of the class $\mathcal{G}_2$, if $r_0 \in (0, 1)$ is largest number such that $r^{-1} f(rz) \in \mathcal{G}_1$, $0 < r \leq r_0$ for all $f \in \mathcal{G}_2$. Recently, the radii of starlikeness and convexity of some normalized special functions were studied as they can be represented as Hadamard factorization under certain conditions, a few such special functions are Bessel functions [11], Struve functions [3, 11], Wright functions [6], Lommel functions [5, 11] and Legendre polynomials of odd degree [8]. If $f$ and $g$ be analytic in $\mathbb{D}$, then $g(z)$ is said to be majorized by $f(z)$, if there exists an analytic function $\Phi(z)$ in $\mathbb{D}$ satisfying $|\Phi(z)| \leq 1$ and $g(z) = \Phi(z) f(z)$ for all $z \in \mathbb{D}$. MacGregor [20] found the largest radius $r_0$ so that $|g'(z)| \leq |f'(z)|$ holds in $|z| \leq r_0$ and he also proved that $r_0 = 2 - \sqrt{3}$, if $f$ is univalent. Recently, Tang and Deng [28] obtained this majorization-radius when $f$ belongs to some class $\mathcal{S}^*(\psi)$. 
Consider the class of cardioid starlike functions:

\[ S^*_\psi := \left\{ f \in A : \frac{zf'(z)}{f(z)} \prec \psi(z) := 1 + z^2 \right\}. \]  

(1.5)

Let \( E_{\alpha,\beta} \) be the normalized form of the Mittag-Leffler function:

\[ E_{\alpha,\beta}(z) = z + \sum_{n \geq 2} \frac{\Gamma(\beta)}{\Gamma(\alpha(n - 1) + \beta)} z^n, \quad (z, \alpha, \beta \in \mathbb{C}; \Re(\alpha) > 0, \beta \neq 0, -1, \cdots). \]

Then clearly \( \psi(z) = 1 + E_{1,1}(z) \). In [16], \( S^*_\psi \) was extensively studied with inclusion properties, radius problems, sharp coefficient’s estimations including hankel determinants. Where in, it is proved that \( r_c := (3 - \sqrt{5})/2 \) is the radius of convexity for \( \psi \) however in \( \mathbb{R} \), \( E_{\alpha,\beta} \) is shown to be convex for \( \alpha \geq 1 \) and \( \beta \geq (3 + \sqrt{17})/2 \). So, \( E_{1,1} \) has the radius of convexity \( r_c \).

In this paper, we continue to study further in finding the largest radius for some normalized special functions to be in \( S^*_\psi \). We also obtain largest radius \( r \) such that \( \psi(z) \in S^*_\psi \) in \( |z| < r \), whenever \( \psi \) belongs to the class \( S(\alpha, \beta) \) and the class \( S_\lambda \) defined in \( \mathbb{R} \) and \( [3] \), respectively. We find that special function’s radii of starlikness results obtained here naturally generalize to \( S^*(\psi) \)-radii. Further, radius problem related to the majorization and some sufficient conditions in terms of coefficients for the normalized functions to be in the class \( S^*_\psi \) are established. We discuss the modified form of Theorem [11] and as an application, we find the largest radius so that Theorem [11] holds for \( S^*_\psi \) and the convolution of two starlike functions to be in \( S^*_\psi \). In the last section, we obtain the condition on \( \beta \) so that the differential subordination implication (1.4) holds, whenever \( h(z) = \psi(z) \) or \( q(z) = \psi(z) \).

2. Radius problems

We need the following lemma to prove the subsequent results:

**Lemma 2.1.** [16] Let \( \psi(z) = 1 + z^2 \). Then we have \( \{ w : |w - a| < R_a \} \subset \psi(D) \), where

\[ R_a = \left\{ \begin{array}{ll} (a - 1) + 1/e, & 1 - 1/e < a \leq 1 + (e - e^{-1})/2; \\
 1 + (e - e^{-1})/2 < a < 1 + e. & e - (a - 1), \\
 \end{array} \right. \]

**Theorem 2.1.** Let \( f \in S(\alpha, \beta) \). Then \( f \in S^*_\psi \) in \( D_{r_0} \), where \( r_0 \) is the least positive root of the equation

\[ \frac{\beta - \alpha}{\pi} \left( \log \frac{1 + \sqrt{2(1 + \cos(2\pi(\frac{1}{\beta - \alpha}))r + r^2)} - 2 \arctan \frac{r}{1 - r}}{1 - r^2} \right) - \frac{1}{e} = 0. \]

(2.1)

**Proof.** Consider the analytic function

\[ p_{\alpha,\beta}(z) := 1 + \frac{\beta - \alpha}{\pi} \log q(z), \]

where

\[ q(z) = \frac{1 - cz}{1 - z} \quad \text{and} \quad c = \exp \left( \frac{2\pi i}{\beta - \alpha} \right). \]

Note that \( q(z) \) is a bilinear transformation, maps \( \mathbb{D} \) onto the disk:

\[ \left| q(z) - \frac{1 + cr^2}{1 - r^2} \right| \leq \left| \frac{1 + cr}{1 - r^2} \right|. \]
which implies
\[ |q(z)| \leq \frac{1 + |1 + c|r + r^2}{1 - r^2} \]
and therefore
\[ \log |q(z)| \leq \log \left( \frac{1 + |1 + c|r + r^2}{1 - r^2} \right). \tag{2.2} \]

For any \( \delta \in \mathbb{C} \) with \(|\delta| = 1\), we have \( 1 + \delta z < 1 + z \). So to maximize \(|\arg(1 + \delta z)|\), it suffices to consider \( |\arg(1 + z)| \). Now for \(|z| = r\), we have
\[ |\arg(1 + z)| \leq \arctan \frac{r}{1 - r}. \tag{2.3} \]

Hence to apply Lemma 2.2, we need to maximize \(|p_{\alpha,\beta}(z) - 1|\), that is,
\[ |p_{\alpha,\beta} - 1| = \frac{\beta - \alpha}{\pi} \left| \log |q(z)| + i \arg \frac{1 - cz}{1 - z} \right|. \tag{2.4} \]

Using (2.2) and (2.3) in (2.4), we see that
\[ |p_{\alpha,\beta} - 1| \leq \frac{\beta - \alpha}{\pi} \left( \log \frac{1 + |1 + c|r + r^2}{1 - r^2} + 2 \arctan \frac{r}{1 - r} \right) \leq \frac{1}{e} \]
holds in \(|z| < r_0\) whenever \( r_0 \) is the smallest positive root of the equation (2.1). This ends the proof. \( \square \)

Note that if we choose \( \alpha = 1 + \frac{\delta - \pi}{2 \sin \delta} \) and \( \beta = 1 + \frac{\delta}{2 \sin \delta} \), where \( \pi/2 \leq \delta < \pi \), then the class \( S(\alpha, \beta) \) reduces to the class \( V(\delta) \) introduced by Kargar et al. [14].

**Corollary 2.2.** Let \( f \in V(\delta) \). Then \( f \in S^*_\delta \) in \( \mathbb{D}_{r_5} \), where \( r_5 \) is the least positive root of the equation
\[ \frac{1}{2 \sin \delta} \left( \log \frac{1 + \sqrt{2(1 + \cos(2(\pi - \delta))) r + r^2}}{1 - r^2} + 2 \arctan \frac{r}{1 - r} \right) - \frac{1}{e} = 0. \]

Now we consider the following class introduced in [9]:
\[ S_\lambda := \left\{ f \in A : \frac{f(z)}{z} \in P_\lambda \right\}, \tag{2.5} \]
where
\[ P_\lambda := \{ p \in A_0 : \Re(e^{z\lambda}p(z)) > 0, \ -\pi/2 \leq \lambda \leq \pi/2 \} \]
denotes the class of tilted Carathéodory functions [29]. Note that \( P_0 \) reduces to \( P \), the class of Carathéodory functions. For the function \( p \in P_\lambda \), upper bound on the quantity \( |zp'(z)/p(z)| \) is given by the following lemma that will be used for our next result:

**Lemma 2.2.** [29] If \( p \in P_\lambda \), then \( |zp'(z)/p(z)| \leq M(\lambda, r) \), where
\[ M(\lambda, r) = \left\{ \begin{array}{ll} \frac{2r \cos \lambda}{2r - 2r|\sin \lambda| + 1} & \text{for } r < |\tan \frac{\lambda}{2}|; \\ \frac{2r}{1 - r} & \text{for } r \geq |\tan \frac{\lambda}{2}|. \end{array} \right. \]
The equality holds for some point \( z = re^{i\theta} \), \( r \in (0, 1) \) if and only if \( p(z) = p_\lambda(yz) \), where \( p_\lambda(z) = \frac{1 + e^{-2i\lambda z}}{1 - z} \) and \( y = e^{i(\theta_0 - \theta)} \) with
\[ \theta_0 = \left\{ \begin{array}{ll} \frac{\pi}{2} + \lambda & \text{for } r < -\tan \frac{\lambda}{2}; \\ -\frac{\pi}{2} + \lambda & \text{for } r < \tan \frac{\lambda}{2}; \\ \arcsin \left( \frac{1 + 2r}{1 - r} \right) + \lambda & \text{for } r \geq |\tan \frac{\lambda}{2}|. \end{array} \right. \]
We determine the largest radius $r$ such that the function $F(z) := f(z)g(z)/z \in \mathcal{S}^*$ in $|z| < r$, whenever $f, g \in \mathcal{S}_\lambda$.

**Theorem 2.3.** Let $c_\lambda = \cos \lambda$, $s_\lambda = \sin \lambda$ and $t_\lambda = |\tan(\lambda/2)|$. If $f, g \in \mathcal{S}_\lambda$, then $F \in \mathcal{S}^*$ in $\mathbb{D}_{r_0}$, where

$$r_0 := \begin{cases} 2ec_\lambda + |s_\lambda| + \sqrt{(4e^2 - 1)c_\lambda + 4e|s_\lambda|}c_\lambda, & \text{if } r < t_\lambda; \\ \sqrt{4e^2 + 1 - 2e}, & \text{if } r \geq t_\lambda. \end{cases}$$

**Proof.** Since $f, g \in \mathcal{S}_\lambda$, it follows that the functions $p(z) = f(z)/z$ and $q(z) = g(z)/z$ belong to the class $P_\lambda$ such that $F(z) = zp(z)q(z)$. Thus

$$\frac{z F'(z)}{F(z)} - 1 = \frac{zp'(z)}{p(z)} + \frac{zq'(z)}{q(z)}.$$ 

Now from Lemma 2.2, we obtain

$$|\frac{z F'(z)}{F(z)} - 1| \leq 2M(\lambda, r).$$

Therefore, using Lemma 2.1, we conclude that if $2M(\lambda, r) \leq 1/e$, then $F \in \mathcal{S}^*$. Since $2M(\lambda, r) \leq 1/e$ holds whenever

$$\frac{2r c_\lambda}{r^2 - 2|s_\lambda|r + 1} \leq \frac{1}{2e}; \text{ if } r < t_\lambda$$

and

$$\frac{2r}{1 - r^2} \leq \frac{1}{2e}; \text{ if } r \geq t_\lambda;$$

or equivalently

$$r^2 - 2(|s_\lambda| + 2ec_\lambda)r + 1 \geq 0, \text{ if } r < t_\lambda$$

and

$$r^2 + 4er - 1 \leq 0, \text{ if } r \geq t_\lambda;$$

respectively. Hence the result follows with $r_0$ as given in the hypothesis. Further, for the functions

$$f(z) = g(z) = \frac{z(1 + e^{-2i\lambda}yz)}{1 - yz},$$

sharpness hold in view of Lemma 2.2. \qed

2.1. **Cardioid starlikeness radius for some special functions and generalizations.** We begin with the following: The Bessel function $J_\beta$ of first kind of order $\beta \in \mathbb{C}$ is a particular solution of the homogeneous Bessel differential equation

$$z^2 w''(z) + zw'(z) + (z^2 - \beta^2)w(z) = 0$$

and have the following series expansion:

$$J_\beta(z) := \sum_{n \geq 1} \frac{(-1)^n}{n! \Gamma(n + \beta + 1)} \left(\frac{z}{2}\right)^{2n + \beta},$$

where $z \in \mathbb{C}$ and $\beta \notin \mathbb{Z}^+$. The following three normalized functions expressed in terms of $J_\beta(z)$ have been studied extensively by many authors:

$$f_\beta(z) = (2^\beta \Gamma(\beta + 1)J_\beta(z))^{1/\beta} = z - \frac{1}{4\beta(\beta + 1)}z^3 + \cdots, \quad \beta \neq 0,$$

$$g_\beta(z) = 2^\beta \Gamma(\beta + 1)z^{1-\beta}J_\beta(z) = z - \frac{1}{4(\beta + 1)}z^3 + \cdots.$$
Lemma 2.1, we have

\[ h_\beta(z) = 2^\beta \Gamma(\beta + 1) z^{1-\beta/2} J_\beta(\sqrt{z}) = z - \frac{1}{4(\beta + 1)} z^2 + \cdots. \]  

(2.6)

Since the zeros of \( J_\beta \) are real if \( \beta > 0 \), therefore using Weierstrass decomposition, we have for \( \beta > 0 \):

\[ J_\beta(z) := \frac{z^\beta}{2^\beta \Gamma(\beta + 1)} \prod_{n \geq 1} \left( 1 - \frac{z^2}{j_{\beta,n}^2} \right), \]

where \( j_{\beta,n} \) is the \( n \)-th positive zero of \( J_\beta \) and satisfies \( j_{\beta,n} < j_{\beta,n+1} \) for \( n \in \mathbb{N} \). Thus we have

\[ \frac{z J'_\beta(z)}{J_\beta(z)} = \beta - \sum_{n \geq 1} \frac{2z^2}{j_{\beta,n}^2 - z^2}. \]  

(2.7)

Now using the above representation, we obtain the radii of cardioid-starlikeness of the functions \( f_\beta, g_\beta \) and \( h_\beta \) for the case \( \beta > 0 \).

**Theorem 2.4.** Let \( \beta > 0 \). Then the radii of cardioid-starlikeness \( r_\varphi(f_\beta), r_\varphi(g_\beta) \) and \( r_\varphi(h_\beta) \) of the functions \( f_\beta, g_\beta \) and \( h_\beta \) are the smallest positive roots of the following equations, respectively:

(i) \( e r J'_\beta(r) + (e - 1) J_\beta(r) = 0 \);
(ii) \( e r J'_\beta(r) - (e \beta - 1) J_\beta(r) = 0 \);
(iii) \( e \sqrt{r} J'_\beta(\sqrt{r}) - (e \beta - 2) J_\beta(\sqrt{r}) = 0 \).

**Proof.** We now prove the first part. Using the representation (2.7) and equation (2.6), we get

\[ \frac{z J'_\beta(z)}{f_\beta(z)} = \frac{z J'_\beta(z)}{\beta J_\beta(z)} = 1 - \frac{1}{\beta} \sum_{n \geq 1} \frac{2z^2}{j_{\beta,n}^2 - z^2}. \]  

(2.8)

Further using a result [11, Lemma 3.2, p. 10] and from (2.8), \( |z| = r < j_{\beta,1} \) we obtain

\[ |\frac{z J'_\beta(z)}{f_\beta(z)} - a| \leq \frac{2}{\beta} \sum_{n \geq 1} \frac{j_{\beta,n}^2 r^2}{j_{\beta,n}^4 - r^4}. \]  

(2.9)

where \( a := 1 - \frac{2}{\beta} \sum_{n \geq 1} \frac{r^4}{j_{\beta,n}^4 - r^4} \) and \( j_{\beta,n} \) denotes the \( n \)-th positive zero of the Bessel function \( J_\beta \). Also a simple calculation shows that \( a \leq 1 \). Thus for the disk (2.9) to lie inside \( \varphi(\mathbb{D}) \), we need only to consider that \( 1 - \frac{1}{e} < a < 1 + \frac{e-\frac{1}{e}}{2} \), and so by Lemma 2.1 we have

\[ \frac{2}{\beta} \sum_{n \geq 1} \frac{j_{\beta,n}^2 r^2}{j_{\beta,n}^4 - r^4} \leq a - 1 + \frac{1}{e} = \frac{1}{e} - \frac{2}{\beta} \sum_{n \geq 1} \frac{r^4}{j_{\beta,n}^4 - r^4}, \]

or equivalently,

\[ \frac{2}{\beta} \sum_{n \geq 1} \frac{r^2}{j_{\beta,n}^4 - r^2} - \frac{1}{e} \leq 0. \]  

(2.10)

Also using (2.8), (2.10) can be written as \( \frac{e r J'_\beta(r)}{\beta J_\beta(r)} + 1 - e \geq 0 \). Note that in view of Lemma 2.1 we can also obtain (2.10) directly from (2.8). Now let us consider the strictly decreasing continuous function

\[ \Psi(r) := \frac{1}{e} - \frac{2}{\beta} \sum_{n \geq 1} \frac{r^2}{j_{\beta,n}^4 - r^2}, \quad r \in (0, j_{\beta,1}). \]
Then $\lim_{r \to 0} \Psi(r) = 1/e > 0$ and $\lim_{r \to j_{\beta,1}} \Psi(r) = -\infty$. Also $\Psi'(r) \leq 0$, since $r < j_{\beta,1}$. So we may assume $r_\Psi(f_\beta)$ be the unique root of $\Psi(r) = 0$ in $(0, j_{\beta,1})$ such that $f_\beta$ in $S^*_\rho$ in $|z| < r_\Psi(f_\beta)$. Proof of other parts follows similarly.

The Struve function $H_\beta$ of first kind is a particular solution of the second-order inhomogeneous Bessel differential equation

$$z^2 w''(z) + zw'(z) + (z^2 - \beta^2)w(z) = \frac{4(\xi)^{\beta+1}}{\sqrt{\pi} \Gamma(\beta + \frac{1}{2})},$$

and have the following form:

$$H_\beta(z) := \frac{(\xi)^{\beta+1}}{\sqrt{\pi} \Gamma(\beta + \frac{1}{2})} F_2 \left( 1; \frac{3}{2}, \beta + \frac{3/2 - z^2}{4} \right),$$

where $-\beta - \frac{3}{2} \notin \mathbb{N}$ and $F_2$ is a hypergeometric function. Since it is not normalized, so we consider the following three normalized functions involving $H_\beta$:

$$U_\beta(z) = \left( \sqrt{\pi} 2^\beta (\beta + \frac{3}{2}) H_\beta(z) \right)^{\frac{1}{\beta+1}},$$

$$V_\beta(z) = \sqrt{\pi} 2^\beta z^{-\beta} \Gamma(\beta + \frac{3}{2}) H_\beta(z),$$

$$W_\beta(z) = \sqrt{\pi} 2^\beta z^{-\beta} \Gamma(\beta + \frac{3}{2}) H_\beta(\sqrt{z}).$$

Moreover, for $|\beta| \leq 1/2$, it has the Hadamard factorization given by

$$\frac{H_\beta(z)}{H_\beta(1)} = \prod_{n \geq 1} \left( 1 - \frac{z^2}{z_{\beta,n}^2} \right),$$

where $z_{\beta,n}$ is the $n$-th positive root of $H_\beta$ such that $z_{\beta,n+1} > z_{\beta,n}$ and $z_{\beta,1} > 1$ and also from (2.12), we obtain

$$\frac{z H_\beta'(z)}{H_\beta(z)} = (\beta + 1) - \sum_{n \geq 1} \frac{2z^2}{z_{\beta,n}^2 - z^2}.$$

Now using the above representation, we can obtain the radii of cardioid-starlikeness of the functions $U_\beta, V_\beta$ and $W_\beta$ for the case $|\beta| \leq 1/2$.

**Theorem 2.5.** Let $|\beta| \leq 1/2$. Then the radii of cardioid-starlikeness $r_\Psi(U_\beta)$, $r_\Psi(V_\beta)$ and $r_\Psi(W_\beta)$ of the functions $U_\beta, V_\beta$ and $W_\beta$ are the smallest positive roots of the following equations, respectively:

(i) $r H_\beta'(r) - \left( 1 - \frac{1}{e} \right) (\beta + 1) H_\beta(r) = 0$;

(ii) $r H_\beta'(r) - \left( 1 + \beta - \frac{1}{e} \right) H_\beta'(r) = 0$;

(iii) $\sqrt{\pi} H_\beta'(\sqrt{r}) - \left( 1 + \beta - \frac{1}{e} \right) H_\beta(\sqrt{r}) = 0$.

**Proof.** From (2.11) and (2.13), by logarithmic differentiation, we get

$$\frac{z U_\beta'(z)}{U_\beta(z)} = \frac{1}{\beta + 1} \frac{z H_\beta'(z)}{H_\beta(z)} = 1 - \frac{1}{\beta + 1} \sum_{n \geq 1} \frac{2z^2}{z_{\beta,n}^2 - z^2},$$

$$\frac{z V_\beta'(z)}{V_\beta(z)} = -\beta + \frac{z H_\beta'(z)}{H_\beta(z)} = 1 - \sum_{n \geq 1} \frac{2z^2}{z_{\beta,n}^2 - z^2}.$$
\[
\frac{zW''_\beta(z)}{W_\beta(z)} = \frac{1 - \beta}{2} + \frac{\sqrt{z}H'_\beta(\sqrt{z})}{2H_\beta(\sqrt{z})} = 1 - \sum_{n \geq 1} \frac{z}{z_{\beta,n}^2 - z}.
\] (2.14)

Now applying the triangle inequality \(||x| - |y|| \leq |x - y|\) and Lemma 2.1 in (2.14), we see that \(U_\beta, V_\beta\) and \(W_\beta\) belongs to \(S^*_\alpha\), respectively whenever

\[
\begin{align*}
\left| \frac{zU'_\beta(z)}{U_\beta(z)} - 1 \right| &\leq \frac{1}{\beta + 1} \sum_{n \geq 1} \frac{2r^2}{z_{\beta,n}^2 - r^2} \leq \frac{1}{\beta + 1}, \\
\left| \frac{zV'_\beta(z)}{V_\beta(z)} - 1 \right| &\leq \sum_{n \geq 1} \frac{2r^2}{z_{\beta,n}^2 - r^2} \leq \frac{1}{\beta + 1}, \\
\left| \frac{zW'_\beta(z)}{W_\beta(z)} - 1 \right| &\leq \sum_{n \geq 1} \frac{r}{z_{\beta,n}^2 - r} \leq \frac{1}{\beta + 1}.
\end{align*}
\] (2.15)

holds, where \(|z| = r < z_{\beta,1}\). Now to find the largest positive radius for which (2.15) holds. Let us consider the strictly increasing continuous functions

\[
\begin{align*}
\Psi_1(r) &:= \frac{1}{\beta + 1} \sum_{n \geq 1} \frac{2r^2}{z_{\beta,n}^2 - r^2} - \frac{1}{\beta + 1}, \\
\Psi_2(r) &:= \sum_{n \geq 1} \frac{2r^2}{z_{\beta,n}^2 - r^2} - \frac{1}{\beta + 1}, \\
\Psi_3(r) &:= \sum_{n \geq 1} \frac{r}{z_{\beta,n}^2 - r} - \frac{1}{\beta + 1}.
\end{align*}
\]

Since \(\lim_{r \to 0} \Psi_i(r) < 0, \Psi'_i(r) > 0\) for \(i = 1, 2, 3\), \(\lim_{r \to z_{\beta,1}} \Psi_i(r) > 0\) for \(i = 1, 2\) and \(\lim_{r \to z_{\beta,1}} \Psi_3(r) > 0\), there exist the unique positive roots, \(r_\beta(U_\beta), r_\beta(V_\beta) \in (0, z_{\beta,1})\) and \(r_\beta(W_\beta) \in (0, z_{\beta,1})\) for \(\Psi_i\), respectively so that the inequalities in (2.15) holds in \(|z| < r_\beta(U_\beta), |z| < r_\beta(V_\beta)\) and \(|z| < r_\beta(W_\beta)\), respectively. Further using (2.14) in \(\Psi_1(r) = 0\), respectively we obtain the desired equations. This completes the proof.

The Lommel function \(L_{u,v}\) of first kind is a particular solution of the second-order inhomogeneous Bessel differential equation

\[z^2w''(z) + zw'(z) + (z^2 - v^2)w(z) = z^{u+1},\]

where \(u \pm v \notin \mathbb{Z}^+\) and is given by

\[L_{u,v} = \frac{z^{u+1}}{(u + v + 1)(u + v + 1)} _1F_2 \left(1; \frac{u - v + 3}{2}, \frac{u + v + 3}{2}; -\frac{z^2}{4} \right),\]

where \(\frac{1}{2}(-u \pm v - 3) \notin \mathbb{N}\) and \(_1F_2\) is a hypergeometric function. Since it is not normalized, so we consider the following three normalized functions involving \(L_{u,v} :\)

\[
\begin{align*}
f_{u,v}(z) &= ((u + v + 1)(u + v + 1)L_{u,v}(z))^\frac{1}{v+1}, \\
g_{u,v}(z) &= (u - v + 1)(u + v + 1)z^u L_{u,v}(z), \\
h_{u,v}(z) &= (u - v + 1)(u + v + 1)z^{\frac{1}{2-u}} L_{u,v}(\sqrt{z}).
\end{align*}
\] (2.16)

Authors in [5, 11] proved the radius of starlikeness for the normalized functions expressed in terms of \(L_{u,v}:\)

\[
\begin{align*}
f_{u,\frac{1}{2}+\frac{3}{2}}(z), \quad g_{u,\frac{1}{2}+\frac{3}{2}}(z) \quad \text{and} \quad h_{u,\frac{1}{2}+\frac{3}{2}}(z),
\end{align*}
\] (2.17)
where \(0 \neq u \in (-1, 1)\). Now we find the radii of cardioid-starlikeness of the functions defined in (2.17). For simplicity, we write these as \(f_u, g_u\) and \(h_u\), respectively.

**Theorem 2.6.** Let \(0 \neq u \in (-1, 1)\) and write \(L_{u-1/2}^{1/2}(z) =: L_u(z)\). Then the radii of cardioid-starlikeness \(r_u(f_u), r_u(g_u)\) and \(r_u(h_u)\) of the functions \(f_u, g_u\) and \(h_u\) are the smallest positive roots of the following equations, respectively:

(i) \(2erL'_u(r) - (2u + 1)(e - 1)L_u(r) = 0, \quad \text{for } u \in (-\frac{1}{2}, 1)\);

(ii) \(2erL'_u(r) - (2u + 1)(e + 1)L_u(r) = 0, \quad \text{for } u \in (-1, -\frac{1}{2})\);

(iii) \(2e\sqrt{r}L'_u(\sqrt{r}) - (2eu + e - 2)L_u(\sqrt{r}) = 0\).

**Proof.** We prove the first part. Let \(0 \neq u \in (0, 1)\). Then using a result from [7] (also see [11] Lemma 1, p. 3358), we can write the Lommel function \(L_{u-1/2}^{1/2}\) as follows:

\[
L_{u-\frac{1}{2}}^{\frac{1}{2}}(z) = \frac{z^{u+\frac{1}{2}}}{u(u+1)^{\frac{1}{2}}}F_2 \left( 1; \frac{u+2}{2}, \frac{u+3}{2}; -\frac{z^2}{4} \right) = z^{u+\frac{1}{2}} u(u+1) \phi_0(z), \tag{2.18}
\]

where \(\phi_0(z) = \prod_{n \geq 1} \left( 1 - \frac{z^2}{z_{u,0,n}^2} \right)\), and \(z_{u,0,n}\) is the simple and real \(n\)-th positive root of \(\phi_0\). Also \(z_{u,0,n} \in (n\pi, (n + 1)\pi)\) which ensures \(z_{u,0,n} > z_{u,0,1} > \pi > 1\). Now with this representation, after logarithmic differentiation, from (2.16) we get

\[
\frac{zf_u'(z)}{f_u(z)} = \frac{zL'_u}{(u+\frac{1}{2})L_{u-\frac{1}{2}}^{\frac{1}{2}}(z)} = 1 - \frac{1}{u+\frac{1}{2}} \sum_{n \geq 1} \frac{2z^2}{z_{u,0,n}^2 - z^2}.
\]

Using the triangle inequality and Lemma 2.11 we have \(f_u \in S_p^*\) provided

\[
T(r) := \frac{1}{r \left( u + \frac{1}{2} \right)} \sum_{n \geq 1} \frac{2r^2}{z_{u,0,n}^2 - r^2} - \frac{1}{e} \leq 0
\]

holds for \(|z| = r < z_{u,0,1}\), where \(T(r)\) is a strictly increasing continuous function in \((0, z_{u,0,1})\). Since \(\lim_{r \to 0} T(r) < 0\), \(\lim_{r \to z_{u,0,1}} T(r) > 0\) and \(T'(r) > 0\), there exists a root \(r_u(f_u) \in (0, z_{u,0,1})\) so that \(f_u \in S_p^*\) in \(|z| < r_u(f_u)\). Now for the case \(u \in (-1, 0)\), we proceed as in the case when \(u \in (0, 1)\), just replacing \(u \) by \(1 + u\) and \(\phi_0\) by \(\phi_1\), where

\[
\phi_1(z) = \frac{1}{r} \sum_{n \geq 1} \frac{2r^2}{z_{u,0,n}^2 - r^2} - \frac{1}{e} < 0.
\]

and \(z_{u,1,n}\) be the \(n\)-th positive root of \(\phi_1\).

Proof for the part (ii) and (iii) follows in a similar fashion as in the Theorem 2.5 by applying Lemma 2.11 on the following two equations, respectively together with the triangle inequality \(|x| - |y| \leq |x - y|\):

\[
\frac{zg_u'(z)}{g_u(z)} = -u + \frac{1}{2} + \frac{zL'_u}{(u+\frac{1}{2})L_{u-\frac{1}{2}}^{\frac{1}{2}}(z)} = 1 - \sum_{n \geq 1} \frac{2z^2}{z_{u,0,n}^2 - z^2}.
\]
and
\[
\frac{zh''_u(z)}{h'_u(z)} = \frac{3 - 2u}{4} + \frac{\sqrt{zL'}_{u-\frac{1}{2}}(\sqrt{z})}{2L'_{u-\frac{1}{2}}(\sqrt{z})} = 1 - \sum_{n\geq 1} \frac{z}{2} \frac{z}{z_{u,0,n} - z},
\]
where \(z_{u,0,n}\) is the \(n\)-th positive root of the function \(\phi_0\).

The Legendre polynomials \(P_n\) are the solutions of the Legendre differential equation:
\[
((1 - z^2)P'_n(z))' + n(n + 1)P_n(z) = 0,
\]
where \(n \in \mathbb{Z}^+\) and using Rodrigues’ formula, \(P_n\) can be represented in the form:
\[
P_n(z) = \frac{1}{2^n n!} \frac{d^n(z^2 - 1)^n}{dz^n},
\]
and it also satisfies the geometric condition \(P_n(-z) = (-1)^n P_n(z)\). Moreover, the odd degree Legendre polynomials \(P_{2n-1}(z)\) have only real roots which satisfy
\[
0 = z_0 < z_1 < \cdots < z_{n-1} \quad \text{and} \quad -z_1 < \cdots < -z_{n-1}.
\] (2.19)

Thus the normalized form is as follows:
\[
P_{2n-1}(z) := \frac{P_{2n-1}(z)}{P'_{2n-1}(0)} = z + \sum_{k=2}^{2n-1} a_k z^k = a_{2n-1} z \prod_{k=1}^{n-1} (z^2 - z_k^2).
\] (2.20)

**Theorem 2.7.** The radii of cardioid-starlikeness \(r_\psi(P_{2n-1}) \in (0, z_1)\) of the normalized odd degree Legendre polynomial is the smallest positive root of the following equation:
\[
er P'_{2n-1}(r) - (e - 1)P_{2n-1}(r) = 0.
\]

**Proof.** From (2.20), after logarithmic differentiation, we obtain
\[
zP'_{2n-1}(z) = zP_{2n-1}(z) = 1 - \sum_{k=1}^{n-1} \frac{2z^2}{z_k^2 - z^2}.
\] (2.21)

Now applying Lemma 2.1 on (2.21), we have \(P_{2n-1} \in S_\psi^*\) whenever
\[
\left| \frac{zP'_{2n-1}(z)}{P_{2n-1}(z)} - 1 \right| \leq \sum_{k=1}^{n-1} \frac{2r^2}{z_k^2 - r^2} \leq \frac{1}{e}.
\] (2.22)

where \(|z| = r < z_1\) and \(z_k\) satisfies the condition given in (2.19). Now let us consider
the strictly increasing continuous function
\[
T(r) := \sum_{k=1}^{n-1} \frac{2r^2}{z_k^2 - r^2} - \frac{1}{e}, \quad r \in (0, z_1).
\]

We have to show that \(T(r) \leq 0\) in \(|z| \leq r < z_1\) so that (2.22) holds. Since \(\lim_{r \to 0} T(r) < 0, \lim_{r \to z_1} T(r) > 0\) and \(T(r) > 0\), there exists a unique positive root \(r_\psi(P_{2n-1}) \in (0, z_1)\) of \(T(r)\) such that \(P_{2n-1} \in S_\psi^*\) in \(|z| < r_\psi(P_{2n-1})\). □

**Remark 2.1 (Generalization for \(S_\psi^*(\psi)\)).** Consider the Ma-Minda function \(\psi\) as defined in (1.1). Let \(a \in \psi(\mathbb{D}) \cap \mathbb{R}\), \(r_a\) is the radius depending on \(a\) and assume the maximal disk \(|w - a| < r_a\) such that
\[
\{w : |w - a| < r_a\} \subset \psi(\mathbb{D}).
\]

Then we can easily extend Theorem 2.1 (\(\psi(z) \neq (1 + z)/(1 - z)\)), Theorem 2.3, Theorem 2.4, Theorem 2.5, Theorem 2.6 and Theorem 2.7 for \(S_\psi^*(\psi)\) using the corresponding radius \(r_1\).
Here we mention the $S^*(\psi)$-radius for the special functions without proof using Remark 2.11. In particular, we show that there exists an $\alpha \in (0, 1]$ such that $R[S^*(\psi)] \geq R[S^*(1 + \alpha z)]$, where we denote $S^*(\psi)$-radius by $R[S^*(\psi)]$.

In the following theorems we mean $R[S^*(\psi)] = R[S^*(1 + r_1 z)]$.

**Theorem 2.8** (Bessel function $J_\beta$). Let $\beta > 0$. Then the $S^*(\psi)$-radii $r_\psi(f_\beta)$, $r_\psi(g_\beta)$ and $r_\psi(h_\beta)$ of the functions $f_\beta$, $g_\beta$ and $h_\beta$ are the smallest positive root of the following equations, respectively:

(i) $r J'_\beta(r) + \beta (1 - r_1) J_\beta(r) = 0$;
(ii) $r J'_\beta(r) - (\beta - r_1) J_\beta(r) = 0$;
(iii) $\sqrt{r} J'_\beta(\sqrt{r}) - (\beta - 2r_1) J_\beta(\sqrt{r}) = 0$.

**Theorem 2.9** (Struve function $H_\beta$). Let $|\beta| \leq 1/2$. Then the $S^*(\psi)$-radii $r_\psi(U_\beta)$, $r_\psi(V_\beta)$ and $r_\psi(W_\beta)$ of the functions $U_\beta$, $V_\beta$ and $W_\beta$ are the smallest positive root of the following equations, respectively:

(i) $r H'_\beta(r) - (1 - r_1)(\beta + 1) H_\beta(r) = 0$;
(ii) $r H'_\beta(r) - (1 + \beta - r_1) H_\beta(r) = 0$;
(iii) $\sqrt{r} H'_\beta(\sqrt{r}) - (1 + \beta - 2r_1) H_\beta(\sqrt{r}) = 0$.

**Theorem 2.10** (Lommel function $L_{u,v}$). Let $0 \neq u \in (-1, 1)$ and write $L_{u}^{\frac{1}{2}, \frac{1}{2}}(z) =: L_u(z)$. Then the $S^*(\psi)$-radii $r_\psi(f_u)$, $r_\psi(g_u)$ and $r_\psi(h_u)$ of the functions $f_u$, $g_u$ and $h_u$ are the smallest positive root of the following equations, respectively:

(i) $\begin{cases} 2r L'_u(r) - (2u + 1)(1 - r_1) L_u(r) = 0, & \text{for } u \in (-\frac{1}{2}, 1), \\ 2r L'_u(r) - (2u + 1)(1 + r_1) L_u(r) = 0, & \text{for } u \in (-1, -\frac{1}{2}); \end{cases}$
(ii) $2r L'_u(r) - (2u + 1 - 2r_1) L_u(r) = 0$;
(iii) $2 \sqrt{r} L'_u(\sqrt{r}) - (2u + 1 - 4r_1) L_u(\sqrt{r}) = 0$.

**Theorem 2.11** (Legendre polynomials $P_n$). The $S^*(\psi)$-radius $r_\psi(P_{2n-1}) \in (0, z_1)$ of the normalized odd degree Legendre polynomial is the smallest positive root of the following equation:

$r P_{2n-1}(r) - (1 - r_1) P(r) = 0$.

**Corollary 2.12.** If $\alpha = r_1$ be the radius of the largest disk $\{w : |w - 1| < \alpha\}$ inside $\psi(\mathbb{D})$, where

(i) $\alpha = \min \{ |1 - \frac{1 + A}{1 + B}|, |1 - \frac{1 - A}{1 + B}| \} = \frac{A - B}{1 + |B|}$ when $\psi(z) = \frac{1 + Az}{1 + Bz}$, where $-1 \leq B < A \leq 1$;
(ii) $\alpha = \sqrt{2 - 2 \sqrt{1 + z}}$ when $\psi(z) = \sqrt{2 - (\sqrt{2} - 1) \sqrt{1 + z}}$;
(iii) $\alpha = \sqrt{2 - 1}$ when $\psi(z) = \sqrt{1 + z}$;
(iv) $\alpha = e - 1$ when $\psi(z) = e^z$;
(v) $\alpha = 2 - \sqrt{2}$ when $\psi(z) = z + \sqrt{1 + z^2}$;
(vi) $\alpha = \frac{e - 1}{e + 1}$ when $\psi(z) = \frac{2}{1 + e - z}$;
(vii) $\alpha = \sin 1$ when $\psi(z) = 1 + \sin z$.

Then Theorem 2.8, Theorem 2.9, Theorem 2.10 and Theorem 2.11 hold true for the class $S^*(\psi)$.

2.2. Majorization result for $S^*_\psi$. 
Theorem 2.13. Let $f \in \mathcal{A}$ and $g \in \mathcal{S}_p^\ast$. If $f$ is majorized by $g$ in $\mathbb{D}$. Then

$$|f'(z)| \leq |g'(z)| \quad \text{in} \quad |z| \leq r_0,$$

where $r_0 \approx 0.380056$ is the smallest positive root of the equation

$$(1 - r^2)(1 - re^r) - r = 0.$$

Proof. Let $f \in \mathcal{A}$ and $g \in \mathcal{S}_p^\ast$, then we have

$$\frac{zg'(z)}{g(z)} = 1 + \omega(z)e^{\omega(z)},$$

or

$$\frac{g(z)}{g'(z)} = \frac{z}{1 + \omega(z)e^{\omega(z)}},$$

where $\omega$ is a Schwarz function. Let $\omega(z) = Re^{i\theta}$, where $R \leq r = |z| < 1$, then we have

$$\left|\frac{g(z)}{g'(z)}\right| \leq \frac{r}{1 - re^r}. \quad (2.25)$$

Now by definition of majorization, we have $f(z) = \Phi(z)g(z)$, where $\Phi(z)$ is analytic function satisfying $|\Phi(z)| \leq 1$ in $\mathbb{D}$ such that

$$f'(z) = g'(z) \left(\Phi'(z)\frac{g(z)}{g'(z)} + \Phi(z)\right).$$

Thus we have

$$|f'(z)| \leq |g'(z)| \left(\frac{1 - |\Phi(z)|^2}{1 - r^2} \frac{r}{1 - re^r} + |\Phi(z)|\right) = |g'(z)|h(r, \beta), \quad (2.26)$$

where

$$h(r, \beta) = \frac{(1 - \beta^2)r}{(1 - r^2)(1 - re^r)} + \beta.$$

Now to arrive at equation (2.23), it suffices to find $r_0$ such that $h(r, \beta) \leq 1$ holds. Or equivalently,

$$0 \leq (1 - r^2)(1 - re^r) - (1 + \beta)r =: k(r, \beta).$$

Since $\frac{\partial}{\partial \beta}k(r, \beta) = -r$, therefore $\min_{\beta} k(r, \beta) = k(r, 1)$. Write $k(r, 1)$ as $k(r)$ and note that $k(0) = 1$ and $k(1) = -1$. So by continuity of $k(r)$, there exists a point $r_0$ such that $k(r) \geq 0$ for $0 \leq r \leq r_0$, where $r_0$ is the smallest positive root of the equation

$$(1 - r^2)(1 - re^r) - r = 0.$$

This ends the proof. \qed

2.3. Convolution radius. First, we need to recall the following result due to Ruscheweyh and Sheil-Small:

Lemma 2.3 ([23], p. 126). Suppose that either $g \in \mathcal{C}$, $h \in \mathcal{S}_p^\ast$ or else $g, h \in \mathcal{S}_p^\ast/2$. Then for any analytic function $G$ in $\mathbb{D}$, we have

$$\frac{g * hG(z)}{g * h(z)} \in \overline{coG(\mathbb{D})},$$

where $\overline{coG(\mathbb{D})}$ is the closed convex hull of $G(\mathbb{D})$. 


Observe that if we consider the function $\phi \in \mathcal{P}$, which is starlike but not convex then we can not directly apply Theorem 1.1 as such function do exist, for instance $\phi(z) = 1 + z e^z$. But keenly observing the proof of Lemma 2.3 we see that the unit disk $\mathbb{D}$ can be replaced by the sub-disk $\mathbb{D}_r := \{ z : |z| < r \}$, where $0 < r \leq 1$ and consequently, we obtain the following modified result. Since the proof is similar, so it is omitted here.

Lemma 2.4. Suppose either $g \in \mathcal{C}$, $h \in \mathcal{S}^*$ or else $g, h \in \mathcal{S}^*_{1/2}$. Then for any analytic function $G$ in $\mathbb{D}_r$, we have $(g * h G(z))/(g * h(z)) \in \mathcal{CG}(\mathbb{D}_r)$, where $r \in [0, 1]$.

This immediately gives

Theorem 2.14. Let $r_0$ be the radius of convexity of $\psi$. If $g \in \mathcal{C}$ and $f \in \mathcal{S}^*(\psi)$. Then $f * g \in \mathcal{S}^*(\psi)$ in $|z| < r$, where $r = \min\{r_0, 1\}$.

Corollary 2.15. Let $f \in \mathcal{S}_\psi^*$ and $g \in \mathcal{C}$. Then $f * g \in \mathcal{S}_\psi^*$ in $\mathbb{D}_{r_0}$, where $r_0 = (3 - \sqrt{3})/2$ is the radius of convexity of $\varphi$.

Now consider the operators $\mathcal{F}_i : \mathcal{A} \to \mathcal{A}$ defined by

$$\mathcal{F}_1(f)(z) = f * g_1(z) = zf'(z)$$

$$\mathcal{F}_2(f)(z) = f * g_2(z) = \frac{1}{2}(f(z) + zf'(z))$$

$$\mathcal{F}_3(f)(z) = f * g_3(z) = \frac{k+1}{z^k} \int_0^z t^{k-1} f(t) dt, \quad \Re k > 0,$$

where $g_3(z) = \sum_{n=1}^{\infty} (k + 1)/(k + n) z^n$, $g_2(z) = (z - z^2/2)/(1 - z^2)^2$ and $g_1(z) = z/(1 - z)^2$. Note that the function $g_1$ is convex in $|z| < 2 - \sqrt{3}$, $g_2$ is convex in $|z| < 1/2$ while $g_3 \in \mathcal{C}$. The above defined operators were introduced by Alexander, Livingston and Bernardi, respectively. Now we obtain the following result, where $\mathcal{S}_{\psi \mathcal{G}} := \mathcal{S}^*((\frac{2}{(e^{2i} + 1)})$, $\mathcal{S}_\psi^* := \mathcal{S}^*(1 + 4z/3 + 2z^2/3)$ and $\mathcal{S}^*_{\varphi} := \mathcal{S}^*(1 + \sin z) :$

Corollary 2.16. Let $\mathcal{F}_i, i = 1 \ to \ 3$ be the operators as defined above.

(i) Let $f \in \mathcal{S}_\psi^*$. Then $\mathcal{F}_1(f) \in \mathcal{S}_\psi^*$ in $\mathbb{D}_{r_1}$, where $r_1 = 2 - \sqrt{3}$, $r_2 = (3 - \sqrt{5})/2$ and $r_3 = (3 - \sqrt{5})/2$.

(ii) Let $f \in \mathcal{S}_\psi^*$. Then $\mathcal{F}_2(f) \in \mathcal{S}_\psi^*$ in $\mathbb{D}_{r_1}$, where $r_1 = 2 - \sqrt{3}$, $r_2 = 1/2$ and $r_3 = 1/2$.

(iii) Let $f \in \mathcal{S}_\psi^*$. Then $\mathcal{F}_3(f) \in \mathcal{S}_\psi^*$ in $\mathbb{D}_{r_1}$, where $r_1 = 2 - \sqrt{3}$, $r_2 = 0.3455$ and $r_3 = 0.3455$.

(iv) Let $f \in \mathcal{S}_{\psi \mathcal{G}}$. Then $\mathcal{F}_3(f) \in \mathcal{S}_{\psi \mathcal{G}}^*$ in $\mathbb{D}_{r_1}$, where $r_1 = 2 - \sqrt{3}$, $r_2 = 1/2$ and $r_3 = 1$.

In 2010, Ali et al. [2] dealt with the problem of finding $\mathcal{S}^*(\psi)$-radii of the convolution $f * g$, between two starlike functions. In particular, they showed that if $f, g \in \mathcal{S}^*$ and $h_\rho(z) = f * g(\rho z)/\rho$, then $h_\rho \in \mathcal{SL}^*$ for $0 \leq \rho \leq (\sqrt{5} - 2)/(\sqrt{2} - 1) \approx 0.09778$. They used the property of the function $\psi$ being convex. Now using Theorem 2.14 we can obtain the result even for the case when $\psi(\mathbb{D})$ is starlike. Here we have shown the usability of radius of convexity of $\psi$.

Theorem 2.17. Let $f, g \in \mathcal{S}^*$ and $h_\rho(z) := f * g(\rho z)/\rho$. Then

(i) $h_\rho \in \mathcal{S}_\psi^*$ for $0 \leq \rho \leq (2e - \sqrt{4e^2 - 2e + 1})/(2e - 1) \approx 0.0957$,

(ii) $h_\rho \in \mathcal{S}_\psi^*$ for $0 \leq \rho \leq (3 - \sqrt{7})/2 \approx 0.177124$,
(iii) \( h_p \in S^*_\rho \) for \( 0 \leq \rho \leq \frac{\sqrt{\sin 1^2 + 2 \sin 1 + 4} - 2}{(2 + \sin 1)} \approx 0.185835 \),
(iv) \( h_p \in S^*_\rho \) for \( 0 \leq \rho \leq \frac{2e - \sqrt{3e^2 + e^2}e}{e + e^{1/e}} \approx 0.122919 \),
(v) \( h_p \in S^*_SG \) for \( 0 \leq \rho \leq \frac{\sqrt{7e^2 + 6e + 3} - 2(1 + e)}{(3e + 1)} \approx 0.108309 \).

The constants are best possible.

**Proof.**

(i) Let \( H(z) = z + \sum_{n=2}^{\infty} n^2 z^n = (z(1 + z))/(1 - z)^3 \). It is easy to see that

\[
\left| \frac{zH'(z)}{H(z)} - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{4r}{1 - r^2}, \quad |z| = r < 1.
\]  

(2.27)

Now by Lemma 2.11, the disk \((2.27)\) lies inside the cardioid \( \varphi(\mathbb{D}) \), provided

\[
\frac{4r}{1 - r^2} \leq \frac{1 + r^2}{1 - r^2} - 1 + \frac{1}{e}
\]

which in turn gives \( r \leq r_0 := (2e - \sqrt{4e^2 - 2e + 1})/(2e - 1) \). Define the function \( h : \mathbb{D} \to \mathbb{C} \) by \( h(z) := f(z) * g(z) \). Then \( h(z) = F(z) * G(z) * H(z) \),
where \( F \) and \( G \) are, respectively defined as \( zF(z) = f(z) \) and \( zG(z) = g(z) \). Since \( f, g \in S^* \), it follows that \( F * G \in S \). Also, \( H(r_0 z) / r_0 \in S^*_\rho \). Hence, using Theorem 2.14, we have\[
F(z) * G(z) * H(\rho_0) / \rho_0 \in S^*_\rho,
\]
where \( \rho_0 = \min\{r_0, r_c\} = r_0 \) and \( r_c = (3 - \sqrt{5})/2 \) is the radius of convexity of \( \varphi \). For \( z = -\rho_0 \), \( zH'(z)/H(z) = (1 + 4z + z^2)/(1 - z^2) = 1 - 1/e \), which implies that \( \rho_0 \) is sharp.

Rest part’s proof also follow in a similar fashion. \( \square \)

**Theorem 2.18.** Let \( f, g \in S^* \) and \( h_{\rho}(z) := f * g(\rho z)/\rho \). Then \( h_{\rho}(z) \in S^* \left( \frac{1 + A\rho}{1 + B\rho} \right) \)

for

\[
0 \leq \rho \leq \frac{2(B^2 - 1) + 4(1 - B^2)^2 + (A - B)^2}{A - B} =: \rho_0,
\]

where \(-1 < B < A \leq 1\).

**Proof.** Since for the function \( p(z) = (1 + Az)/(1 + Bz) \), we have

\[
\left| p(z) - \frac{1 - AB}{1 - B} \right| \leq \frac{A - B}{1 - B^2}.
\]  

(2.28)

Therefore, for the disk \((2.27)\) to lie inside the disk \((2.28)\), we must have

\[
\frac{1 - AB}{1 - B^2} - \frac{A - B}{1 - B^2} \leq \frac{1 + r^2}{1 - r^2} \leq \frac{1 - AB}{1 - B^2} + \frac{A - B}{1 - B^2}
\]

and

\[
\frac{4r}{1 - r^2} \leq \frac{A - B}{1 - B^2}
\]

which upon simplification hold for \( r \leq r_0 = \sqrt{(A - B)/(2 + A + B)} \) and \( r \leq \rho_0 \) respectively, where \( \rho_0 \) is the smallest positive root of the following equation

\[
(A - B)r^3 + 4(1 - B^2)r - (A - B) = 0.
\]

Since \( \min\{r_0, \rho_0\} = \rho_0 \) and the class \( S^* \left( \frac{1 + A\rho}{1 + B\rho} \right) \) is closed under convolution with convex functions, now the result follows in a similar way as in the part (i) of Theorem 2.17. \( \square \)
3. Extremal problem for the class \( S^*(\psi) \)

In 1961, Goluzin \[13\] obtained the set of extremal functions \( f(z) = z/(1 - xz)^2, |x| = 1 \) for the problem of maximization of the quantity \( \Re \Phi (\log(f(z)/z)) \) or \( |\Phi (\log(f(z)/z))| \) over the class \( S^* \), where \( \phi \) is a non-constant entire function. In 1973, MacGregor \[21\] proved the result for the class \( S^*(\alpha) := \{ f \in A : \Re (zf'(z)/f(z)) > \alpha, \alpha \in [0, 1) \} \). We observed that the proof given by MacGregor can be generalized to the Ma-Minda class \( S^*(\psi) \). Thus we have the following result:

**Theorem 3.1.** Suppose \( \Phi \) is a non-constant entire function and \( 0 < |z_0| < 1 \) and assume that the class \( S^*(\psi) \) is closed. Then maximum of either

\[
\Re \Phi \left( \log \frac{f(z_0)}{z_0} \right) \quad \text{or} \quad |\Phi \left( \log \frac{f(z_0)}{z_0} \right)|
\]

(3.1)

for functions in the class \( S^*(\psi) \) is attained only when the function is of the form

\[
f(z) = z \exp \int_0^{\zeta} \frac{\psi(t) - 1}{t} \, dt,
\]

(3.2)

where \( |\zeta| = 1 \).

**Proof.** Since the class \( S^*(\psi) \) is compact, therefore the problem under consideration has a solution. Moreover, in view of a result of Goluzin \[13\], in (3.1) it suffices to consider the continuous functional

\[
\Re \Phi \left( \log \frac{f(z_0)}{z_0} \right).
\]

Let \( f \in S^*(\psi) \). Then using a result from \[19\], \( f(z)/z \prec f_0(z)/z =: F(z) \), where \( f_0(z) = z \exp \int_0^z \frac{\psi(t) - 1}{t} \, dt \) or equivalently \( \log(f(z)/z) \prec \log F(z) \). Thus,

\[
g(z) = \Phi \left( \log \frac{f(z)}{z} \right) \prec \Phi(\log F(z)) = G(z).
\]

Note that \( G \) is also non-constant as is \( \Phi \). Thus for each \( r \in (0, 1) \) by subordination principle, we obtain \( g(\partial \mathbb{D}) \subset G(\partial \mathbb{D}) = \Omega \). Since \( G(xz) \prec G(z) \) for \( |x| \leq 1 \) is obvious, therefore for \( |z_0| = r \), we have \( \{ g(z_0) : g \prec G \in \mathbb{D} \} = \Omega \). Now by considering a support line to the compact set \( \Omega \), we conclude that

\[
\max_{f \in S^*(\psi)} \Re \Phi \left( \log \frac{f(z_0)}{z_0} \right) = \Re w_1, \quad w_1 \in \partial \Omega.
\]

Since \( G \) is also an open map, therefore there exists a point \( z_1 \) where \( |z_1| = r \) and \( G(z_1) = w_1 \) such that among finitely many \( w_1, \) for one suitable \( w_1, \) we have

\[
\Phi \left( \log \frac{f(z_0)}{z_0} \right) = w_1,
\]

where \( f \) is the solution for the extremal problem. Now by the well known Lindelöf Principle, we have

\[
\Phi \left( \log \frac{f(z)}{z} \right) = \Phi(\log F(xz)),
\]

(3.3)

that is, if \( f \) is the desired solution, then \[33\] holds for some \( x, |x| = 1 \). Since \( \Phi \) is non-constant analytic function, so we may write

\[
\Phi(w) = c_0 + c_n w^n + c_{n+1} w^{n+1} + \cdots; c_n \neq 0.
\]
If we set \( \log(f(z)/z) = \alpha_1 z + \alpha_2 z^2 + \cdots \) and \( \log(F(z)) = \beta_1 z + \beta_2 z^2 + \cdots \), then from (3.3), comparing the coefficients, we get \( c_n \alpha_1^n = c_n \beta_1^n \). Or equivalently, \( \alpha_1^n = \beta_1^n \), which in particular implies that \( |\alpha_1| = |\beta_1| \). Since \( \log(f(z)/z) < \log F(z) \), \( |\alpha_1| = |\beta_1| \) is possible only if \( \log(f(z)/z) = \log F(xyz) \) for some \( |y| = 1 \). Therefore, we conclude that
\[
f(z) = z \exp \int_0^u \frac{\psi(t) - 1}{t} dt,
\]
where \( |u| = 1 \) if \( f \) is a solution to the extremal problem. □

Now as an application of the Theorem 3.2 we obtain a result due to MacGregor [21]:

**Corollary 3.2.** [21] Suppose \( \Phi \) is a non-constant entire function and \( 0 < |z_0| < 1 \). Then the maximum of the expression (3.1) for functions in the class \( S^*(\alpha) \) is attained only when the function is of the form
\[
f(z) = \frac{z}{1 - \zeta z^{2-2\alpha}}, \quad |\zeta| = 1.
\]

**Proof.** If \( f \in S^*(\alpha) \), then \( f(z)/z < 1/(1-z)^{2-2\alpha} \) and hence the result.

**Corollary 3.3.** Suppose \( \Phi \) is a non-constant entire function and \( 0 < |z_0| < 1 \). Then the maximum of the expression (3.1) for functions in the class \( S^*_p \) is attained only when the function is of the form
\[
f(z) = z \exp(\text{e}^{\zeta z} - 1), \quad |\zeta| = 1.
\]

**Proof.** If \( f \in S^*_p \), then \( f(z)/z < \exp(\text{e}^{z} - 1) \) and hence the result. □

4. **Some sufficient conditions for \( S^*_p \)**

In this section, we determine the sufficient conditions for the functions \( z/(1 + \sum_{k=1}^\infty a_k z^k) \), \( z/(1 + z^n) \) and certain other types of functions to be in \( S^*_p \).

**Theorem 4.1.** Let \( f(z) = z/(1 + \sum_{k=1}^\infty a_k z^k) \). If the coefficients of \( f \) satisfy
\[
|1-a| + \sum_{k=1}^\infty (R_a + |1-a-k|)|a_k| \leq R_a,
\]
where \( a \) and \( R_a \) is as defined in Lemma 2.1. Then \( f \in S^*_p \).

**Proof.** For \( f(z) = z/(1 + \sum_{k=1}^\infty a_k z^k) \), we have
\[
\left| \frac{zf'(z)}{f(z)} - a \right| = \left| 1 - a - \frac{\sum_{k=1}^\infty k a_k z^k}{1 + \sum_{k=1}^\infty a_k z^k} \right|.
\]
Thus by Lemma 2.1 \( f \in S^*_p \), if
\[
\left| 1 - a - \frac{\sum_{k=1}^\infty k a_k z^k}{1 + \sum_{k=1}^\infty a_k z^k} \right| \leq R_a.
\]
The above inequality holds whenever
\[
|1-a| + \sum_{k=1}^\infty |1-a-k||a_k|r^k \leq R_a(1 - \sum_{k=1}^\infty |a_k|r^k)
\]
or equivalently,

\[ |1 - a| + \sum_{k=1}^{\infty} (|1 - a - k| + R_a)|a_k|r^k \leq R_a. \]

Letting \( r \) tends to 1\(^-\), completes the proof. \( \square \)

**Theorem 4.2.** Let \( f(z) = z/(1 + z^k)^n \), where \( n, k \in \mathbb{Z}^+ \) are fixed. Then \( f \in S^*_p \) for

\[ |z| < \left( \frac{R_a - |1 - a|}{R_a + |1 - a - kn|} \right)^{1/k}, \]

where \( a \) and \( R_a \) is as defined in Lemma 2.1.

**Proof.** For \( f(z) = z/(1 + z^k)^n \), we have

\[ \left| \frac{zf'(z)}{f(z)} - a \right| = \left| 1 - a - knz^k \right| \frac{1}{1 + z^k}. \]

Thus by Lemma 2.1, \( f \in S^*_p \), if

\[ \left| 1 - a - \frac{knz^k}{1 + z^k} \right| < R_a. \]

The above inequality holds whenever

\[ |1 - a| + |1 - a - kn||z|^k < R_a(1 - |z|^k) \]

which simplifies to

\[ |z|^k < \frac{R_a - |1 - a|}{R_a + |1 - a - kn|}. \]

Hence the result. \( \square \)

**Theorem 4.3.** Let \( p(z) \) be a polynomial such that \( p(0) = 1 \) and \( \deg p(z) = m \). Let \( R = \min\{|z| : p(z) = 0, z \neq 0\} \). Then the function \( f(z) = z(p(z))^{\beta/m} \in S^*_p \) for

\[ |z| < \frac{R(R_a - |1 - a|)}{\beta + R_a - |1 - a|}, \]

where \( a \) and \( R_a \) is as defined in Lemma 2.1.

**Proof.** Assume that \( z_k, (k = 1, 2, ..., m) \) are zeros of the polynomial \( p(z) \). For the function \( f(z) = z(p(z))^{\beta/m} \), we have

\[ \frac{zf'(z)}{f(z)} = 1 + \frac{\beta}{m} \sum_{k=1}^{\infty} \frac{z}{z - z_k}. \]

or equivalently,

\[ \frac{zf'(z)}{f(z)} - a = 1 - a + \frac{\beta}{m} \sum_{k=1}^{\infty} \left( \frac{z}{z - z_k} + \frac{r^2}{R^2 - r^2} - \frac{r^2}{R^2 - r^2} \right). \]

Thus by Lemma 2.1, \( f \in S^*_p \), if

\[ (|1 - a| - R_a)(R^2 - r^2) + \beta(Rr + r^2) < 0. \]

The above inequality is satisfied if \( |z| = r < R(R_a - |1 - a|)/(\beta + R_a - |1 - a|) \). This completes the proof. \( \square \)
5. Subordination results for $S^*_\phi$

To prove our differential subordination results, we will need the following lemma due to Miller and Mocanu:

**Lemma 5.1.** [23] Let $q$ be the univalent in $\mathbb{D}$. Let $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(\mathbb{D})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{D})$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that $Q$ is starlike. In addition, assume that

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right) > 0.$$ 

If $p$ is analytic in $\mathbb{D}$, with $p(0) = q(0)$, $p(\mathbb{D}) \subset D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) < h(z),$$

then $p \prec q$ and $q$ is the best dominant.

**Theorem 5.1.** Let $0 < \alpha < 1$, $0 < B < A < 1$ and $k = 1 + \sqrt{2}$. If $p \in A_0$ and satisfies the differential subordination

$$1 + \beta zp'(z) \prec \phi(z),$$

then

(i) $p(z) \prec \phi_0(z) := 1 + \frac{e^z}{2}\left(\frac{k+1}{k-1}\right)$ for $\beta \geq \frac{k-1}{k+1} \approx 3.68427$.

(ii) $p(z) \prec \sqrt{1 + z}$ for $\beta \geq \frac{\sqrt{2}-1}{\sqrt{2}}$.

(iii) $p(z) \prec \frac{1 + Az}{1 + Bz}$ for $\beta \geq \frac{1}{\sin 1}$.

(iv) $p(z) \prec 1 + \sin z$ for $\beta \geq \frac{\sqrt{2}-1}{\sin 1}$.

(v) $p(z) \prec z + \sqrt{1 + z^2}$ for $\beta \geq \frac{\sqrt{2}-1}{\sqrt{2}}$.

(vi) $p(z) \prec e^z$ for $\beta \geq 1$.

The bounds are sharp.

**Proof.** We set $\theta(w) = 1$ and $\phi(w) = \beta \neq 0$. Let $\Psi_\beta(z, p(z)) := 1 + \beta zp'(z)$.

(i) The differential equation $\Psi_\beta(z, p(z)) = \phi(z)$ has an analytic solution given by

$$q_\beta(z) = 1 + \frac{1}{\beta}(e^z - 1).$$

Since the function $Q(z) = zq'_\beta(z)\phi(q_\beta(z)) = \phi(z) - 1 = ze^z \in S^*$ and the function $h(z) = \theta(q_\beta(z)) + Q(z)$ satisfies

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left(\frac{zQ'(z)}{Q(z)}\right) > 0.$$ 

Therefore, by Lemma 5.1, the following differential subordination implication holds:

$$\Psi_\beta(z, p(z)) \prec 1 + \beta zq'_\beta(z) \Rightarrow p \prec q_\beta.$$ 

Now to show that $p \prec \phi_0$, we need to show that $q_\beta \prec \phi_0$ and for $q_\beta \prec \phi_0$ it is necessary that $\phi_0(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \phi_0(1)$ which gives $\beta \geq \beta_1$ and $\beta \geq \beta_2$ respectively, where

$$\beta_1 = \frac{k(e - 1)(k + 1)}{e(k - 1)}$$

and

$$\beta_2 = \frac{k(e - 1)(k - 1)}{k + 1}.$$
Let $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$. Since the difference of the square of the distances from the point $(1, 0)$ to the points on the boundary curves $\phi_0(e^{i\theta})$ and $q_\beta(e^{i\theta})$, respectively is given by

$$T(\theta) := \frac{4(1 + \sin^2 \theta)}{(k^2 - 2k \cos \theta + 1)^2} - \frac{e^{2\cos \theta} + 1}{\beta^2} + \frac{2e^{\cos \theta} \cos(\sin \theta)}{\beta}, \quad 0 \leq \theta \leq \pi.$$ 

A calculation shows that $T'(\theta) \leq 0$. Therefore, $T(\theta)$ decreases on $[0, \pi]$, which further implies that the condition $\phi_0(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \phi_0(1)$, that is $\beta \geq \beta_1$ is also sufficient for $q_\beta \prec \phi_0$. Hence the result. Rest of the results follow in a similar fashion. \hfill \Box

**Theorem 5.2.** Let $0 < \alpha < 1$, $0 < B < A < 1$ and $k = 1 + \sqrt{2}$. If $p \in A_0$ and satisfies the differential subordination

$$1 + \beta\frac{zp'(z)}{p(z)} \prec \varphi(z),$$

then

(i) $p(z) \prec e^z$ for $\beta \geq e - 1$.

(ii) $p(z) \prec \phi_0(z)$ for $\beta \geq (1/e - 1)/\log\left(1 - \frac{k-1}{k(k+1)}\right) \approx 3.3583$.

(iii) $p(z) \prec \sqrt{1 + z}$ for $\beta \geq \frac{e-1}{\log \sqrt{2}}$.

(iv) $p(z) \prec \frac{1 + A}{1 + B}$ for $\beta \geq \max\left\{\frac{1-1/e}{\log(1-B)/(1-A)}; \frac{e-1}{\log((1+A)/(1+B))}\right\}$.

(v) $p(z) \prec 1 + \sin z$ for $\beta \geq \frac{e-1}{\log(1+\sin 1)}$.

(vi) $p(z) \prec z + \sqrt{1 + z^2}$ for $\beta \geq \frac{e-1}{\log(k)}$.

The bounds are sharp.

**Proof.** Let $\theta(w) = 1$ and $\phi(w) = \beta/\omega \neq 0$. Let $\Phi_\beta(z, p(z)) := 1 + \beta\frac{zp'(z)}{p(z)}$.

(i) The differential equation $\Phi_\beta(z, p(z)) = \varphi(z)$ has an analytic solution given by

$$q_\beta(z) = \exp\left(\frac{e^z - 1}{\beta}\right).$$

Since the function $Q(z) = zq_\beta(z)\phi(q_\beta(z)) = \varphi(z) - 1 = ze^z \in S^*$ and $h(z) = \theta(q_\beta(z)) + Q(z)$ satisfies $\Re(h'(z)/Q(z)) = \Re(zQ'(z)/Q(z)) > 0$. Therefore, by Lemma 5.1, the following differential subordination implication holds:

$$\Phi_\beta(z, p(z)) \prec 1 + \beta\frac{zp'(z)}{q_\beta(z)} \Rightarrow p \prec q_\beta.$$ 

Now to show that $p \prec e^z$, we need to show that $q_\beta \prec e^z$ and for $q_\beta \prec e^z$, it is necessary that $e^{-1} \leq q_\beta(-1)$ and $q_\beta(1) \leq e$ which gives $\beta \geq \beta_1$ and $\beta \geq \beta_2$ respectively, where

$$\beta_1 = (1 - 1/e) \quad \text{and} \quad \beta_2 = e - 1.$$

Let $\beta \geq \beta_2$. To prove that $q_\beta(z) \prec e^z$, now it only suffices to show that $(e^z - 1)/\beta \prec z$, which is equivalent to prove that for $z \in \overline{D}$

$$|(e^z - 1)/\beta| \leq 1,$$  

and thus, for $z = e^{i\theta}$, we have

$$|(e^{i\theta} - 1)/\beta| = \sqrt{(e^{2\cos \theta} + 1 - 2e^{\cos \theta} \cos(\sin \theta))/\beta^2} =: \sqrt{T(\theta)}.$$
Note that $T(\theta) = T(-\theta)$. So we may consider the interval $0 \leq \theta \leq \pi$. Now since

$$T'(\theta) = 2e^{\cos \theta}(\sin(\theta - \sin \theta) - \sin \theta)/\beta^2 = 0$$

if and only if

$$\cos \left( \theta - \frac{\sin \theta}{2} \right) \sin \left( \frac{\sin \theta}{2} \right) = 0$$

if and only if $\theta \in \{0, \theta_0, \pi\}$, where $\theta_0 \approx 2.02098$ is the only root of the equation

$$2\theta - \sin \theta - \pi = 0.$$ 

Thus, we see that $T'(\theta) < 0$ over $(0, \theta_0)$ and $T'(\theta) > 0$ over $(\theta_0, \pi)$, which implies that $T(\theta)$ decreases over $[0, \theta_0]$ and increases over $[\theta_0, \pi]$ and therefore,

$$\max_{0 \leq \theta \leq \pi} T(\theta) = \max\{T(0), T(\pi)\} = T(0) \leq 1$$

and (5.2) holds. Hence, $\beta \geq \beta_2$ is also sufficient for $q_\beta(z) < e^z$. Rest of the results follow in a similar fashion. □

**Theorem 5.3.** Let $0 < \alpha < 1$, $0 < B < A < 1$ and $k = 1 + \sqrt{2}$. If $p \in \mathcal{A}_0$ and satisfies the differential subordination

$$1 + \beta \frac{zp'(z)}{p^2(z)} < \wp(z),$$

then

(i) $p(z) < e^z$ for $\beta \geq \frac{e-1}{1-\sqrt{2}e}$. 

(ii) $p(z) < \phi_0(z)$ for $\beta \geq (e - 1)/(1 + \frac{1}{k-1})$. 

(iii) $p(z) < \sqrt{1 + z}$ for $\beta \geq \frac{(e-1)(k-1)}{k k-2}$. 

(iv) $p(z) < \frac{1 + A}{1 + B}$ for $\beta \geq \max \left\{ \frac{(1-1/e)(1-A)}{A-B}, \frac{(e-1)(1+A)}{A-B} \right\}$. 

(v) $p(z) < 1 + \sin z$ for $\beta \geq \frac{(e-1)(1+\sin 1)}{\sin 1}$. 

(vi) $p(z) < z + \sqrt{1 + z^2}$ for $\beta \geq \frac{(e-1)k}{k-1}$. 

The bounds are sharp.

**Proof.** We set $\theta(w) = 1$ and $\wp(w) = \beta/w^2 \neq 0$. Let $\mathcal{U}_\beta(z, p(z)) := 1 + \beta \frac{zp'(z)}{p^2(z)}$.

(i) The differential equation $\mathcal{U}_\beta(z, p(z)) = \wp(z)$ has an analytic solution given by

$$q_\beta(z) = \frac{1}{1 - \frac{1}{\beta}(e^z - 1)}.$$ 

Since the function $Q(z) = zq_\beta(z)\wp(q_\beta(z)) = \wp(z) - 1 = ze^z \in \mathcal{S}^*$ and the function $h(z) = \theta(q_\beta(z)) + Q(z)$ satisfies

$$\Re \left( \frac{zh'(z)}{Q(z)} \right) = \Re \left( \frac{zQ'(z)}{Q(z)} \right) > 0.$$ 

Therefore, by Lemma 5.1, the following differential subordination implication holds:

$$\Psi_\beta(z, p(z)) < 1 + \beta zq_\beta(z) \Rightarrow p < q_\beta.$$
Now to show that \( p \prec e^z \), we need to show that \( q_\beta \prec e^z \) and for \( q_\beta \prec e^z \), it is necessary that \( e^{-1} \leq q_\beta(-1) \) and \( q_\beta(1) \leq e \) which gives \( \beta \geq \beta_1 \) and \( \beta \geq \beta_2 \) respectively, where
\[
\beta_1 = \frac{1 - 1/e}{e - 1} \quad \text{and} \quad \beta_2 = \frac{e - 1}{1 - 1/e}.
\]
Let \( \beta \geq \max\{\beta_1, \beta_2\} = \beta_2 \). Note that \( e^z \) and \( e^{-z} \) both maps \( \mathbb{D} \) onto \( \Omega := \{w \in \mathbb{C} : w = e^z \) and \( z \in \mathbb{D} \}. Therefore, to prove that \( q_\beta \prec e^z \), or equivalently \( q_\beta(z) \in \Omega \), it suffices to show that \( q_\beta \prec e^{-z} \), which is equivalent to show that
\[
1 - \frac{1}{\beta}(e^z - 1) \prec e^z. \tag{5.3}
\]
Now since the difference of the square of the distances from the point \((1,0)\) to the points on the boundary curves \( e^z \) and \( 1/q_\beta(e^\theta) \), respectively is given by
\[
T(\theta) := \left(1 - \frac{1}{\beta^2}\right) S(\theta),
\]
where \( S(\theta) = 1 + e^{2\cos\theta} - 2e^{\cos\theta}\cos(\sin\theta) \). Here, we may consider the interval \( 0 \leq \theta \leq \pi \), since \( T(\theta) = T(-\theta) \). A calculation shows that \( S'(\theta) \leq 0 \) for \( \theta \in [0, \pi] \). Therefore, \( T(\theta) \) decreases on \( [0, \pi] \) and also
\[
\min_{0 \leq \theta \leq \pi} T(\theta) = \left(1 - \frac{1}{\beta^2}\right) S(\pi) = \left(1 - \frac{1}{\beta^2}\right) (1 + e^{-2} - 2e^{-1}) > 0,
\]
which means (5.3) is true. Hence \( \beta \geq \beta_2 \) is also sufficient for \( q_\beta \prec e^z \). Proof of the rest of the results are much akin and so are omitted here. \(\square\)

**Remark 5.1.** If \( p \in A_0 \), \( \beta > 0 \) and satisfies the differential subordination
\[
1 + \beta \frac{zp'(z)}{p^n(z)} \prec \varphi(z), \quad n \geq 2
\]
then
\[
p(z) \prec q_\beta(z) := \left(1 - \frac{1}{\beta(e^z - 1)}\right)^{1/n},
\]
where \( q_\beta \) is the best dominant.

**Theorem 5.4.** Let \( p \) be an analytic function in \( \mathbb{D} \) with \( p(0) = 1 \). Let \( 0 < \alpha < 1, 0 < B < A < 1, k = 1 + \sqrt{2} \) and \( \varphi(z) = 1 + ze^z \). Set
\[
\Psi_\beta(z, p(z)) = 1 + \beta z p'(z).
\]
If \( p \in A_0 \) and satisfies any of the following differential subordinations:

(i) \( \Psi_\beta(z, p(z)) \prec \varphi_0(z) \) for \( \beta \geq e(-1 + 2k \log(1 + 1/k))/k \approx 0.75822 \).

(ii) \( \Psi_\beta(z, p(z)) \prec \sqrt{1 + z} \) for \( \beta \geq 2e(1 - \log 2) \approx 1.66822 \).

(iii) \( \Psi_\beta(z, p(z)) \prec \varphi_\alpha(z) := 1 + \frac{z}{1 - \alpha z} \) for \( \beta \geq e \log(\frac{1 + \sqrt{\alpha}}{1 - \sqrt{\alpha}})/(2\sqrt{\alpha}) \).

(iv) \( \Psi_\beta(z, p(z)) \prec \frac{1 + \lambda z}{1 + Bz} \) for \( \beta \geq \max\{\frac{\lambda(A - B)}{B} \log(1 - B), \frac{2\lambda e B}{\sqrt{A B}} \log(1 + B)\} \).

(v) \( \Psi_\beta(z, p(z)) \prec 1 + \sin z \) for \( \beta \geq e \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) \approx 2.57172 \).

(vi) \( \Psi_\beta(z, p(z)) \prec z + \sqrt{1 + z^2} \) for \( \beta \geq e(3 - k + \log(k/2)) \approx 2.10399 \).

(vii) \( \Psi_\beta(z, p(z)) \prec e^z \) for \( \beta \geq e \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n!)^2} \approx 2.16538 \).

Then \( p \prec \varphi \) and the bounds are sharp.
Proof. We set $\theta(w) = 1$ and $\phi(w) = \beta \neq 0$. To prove our results, we make use of the inequality:

$$\phi(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \varphi(1),$$

(5.4)

if $q_\beta < \varphi$ which gives necessary as well as sufficient condition for the required subordination.

(i) The differential equation $\Psi_\beta(z, p(z)) = \phi_0(z)$ has an analytic solution given by

$$q_\beta(z) = 1 - \frac{1}{\beta k} \left( z + 2k \log \left( 1 - \frac{z}{k} \right) \right).$$

Since the function

$$Q(z) = zq_\beta(z)\phi(q_\beta(z)) = \phi_0(z) - 1 = \frac{z}{k} \left( \frac{k + z}{k - z} \right) \in S^*$$

and the function $h(z) = \theta(q_\beta(z)) + Q(z)$ satisfies

$$\Re \left( \frac{zh'(z)}{Q(z)} \right) = \Re \left( \frac{zQ'(z)}{Q(z)} \right) > 0.$$

Therefore, by Lemma 5.1, the following differential subordination implication holds:

$$\Psi_\beta(z, p(z)) \prec 1 + \beta zq_\beta(z) \Rightarrow p < q_\beta.$$

Now to show that $p < \varphi$, we need to show that $q_\beta < \varphi$. Now if $q_\beta < \varphi$, then (5.4) must hold, that is, the inequalities $\varphi(-1) \leq q_\beta(-1)$ and $q_\beta(1) \leq \varphi(1)$ respectively gives $\beta \geq \beta_1$ and $\beta \geq \beta_2$, where

$$\beta_1 = \frac{e}{k} \left( 2k \log \left( \frac{k + 1}{k} \right) - 1 \right) \quad \text{and} \quad \beta_2 = \frac{1}{ek} \left( 2k \log \left( \frac{k}{k - 1} \right) + 1 \right).$$

Let $\beta \geq \max\{\beta_1, \beta_2\} = \beta_1$. Note that $q_\beta(D)$ is convex, symmetric about real axis and $q_\beta(D) \subset q_{\beta_1}(D)$ for each $\beta \geq \beta_1$. Also, $|\arg(1+z)| \leq \arctan(r/(1-r))$ for $|z| = r < 1$. Now to prove $q_\beta < \varphi$, in view of Lemma 2.1, it only suffices to show that $\max_{0 \leq \theta \leq \pi} |\arg q_\beta(e^{i\theta})| = \max_{0 \leq \theta \leq \pi} |\arg \phi(e^{i\theta})|$. Since

$$\max_{0 \leq \theta \leq \pi} |\arg q_\beta(e^{i\theta})| = \max_{0 \leq \theta \leq \pi} \left| \arctan \left( \frac{\Im q_\beta(e^{i\theta})}{\Re q_\beta(e^{i\theta})} \right) \right|$$

$$= \max_{0 \leq \theta \leq \pi} \left| \arctan \left( \frac{-2k \arg \left( 1 - e^{i\theta} \right) - \sin \theta}{M - \cos \theta - k \log((1 + 1/k^-2 e^{i\theta})/k)} \right) \right|$$

$$\leq \arctan \left( \frac{2k \arctan(1/e) + 1}{M(1-1/e)} \right) \approx 0.7719$$

$$< \max_{0 \leq \theta \leq \pi} |\arg(1+ze^{i\theta})| \approx 1.41022,$$

where $M = e(-1 + 2k \log(1 + 1/k))$. Therefore, $\beta \geq \beta_1$ is also sufficient for $q_\beta < \varphi$ to hold. Hence the result. Other results follow similarly. \qed

Let $f \in A$ and setting $p(z) = zf'(z)/f(z)$ and $\varphi(z) = f(z)/z$. Then using Theorem 5.4, we obtain sufficient conditions for $f$ to be in $S_{\varphi}^*$. 


Corollary 5.5. Let \( f \in A \) and set
\[
\Psi_\beta \left( z, \frac{zf'(z)}{f(z)} \right) = 1 + \beta \frac{zf'(z)}{f(z)} \left( 1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f'(z)} \right).
\]
If \( \Psi_\beta \left( z, \frac{zf'(z)}{f(z)} \right) \) satisfies any of the conditions (i)-(vii) of the Theorem 5.4, then \( f \in S_\psi^* \).

Corollary 5.6. Let \( f \in A \) and set
\[
\Psi_\beta \left( z, \frac{f(z)}{z} \right) = 1 + \beta \left( f'(z) - \frac{f(z)}{z} \right).
\]
If \( \Psi_\beta \left( z, \frac{f(z)}{z} \right) \) satisfies any of the conditions (i)-(vii) of the Theorem 5.4, then \( f \in S_\psi^* \).

Theorem 5.7. Let \( 0 < \alpha < 1, 0 < B < A < 1, \gamma = \log(e/(e-1)), k = 1 + \sqrt{2} \) and \( \varphi(z) = 1 + ze^z \) and \( p \in A_0 \). Set
\[
\Phi_\beta(z, p(z)) = 1 + \beta \frac{zp'(z)}{p(z)}.
\]
If \( p \) satisfies any of the following differential subordinations:

(i) \( \Phi_\beta(z, p(z)) \prec \phi_0(z) \) for \( \beta \geq (-1 + 2k \log(1 + 1/k))/\gamma k \approx 0.60812 \).

(ii) \( \Phi_\beta(z, p(z)) \prec \sqrt{1 + z} \) for \( \beta \geq 2(1 - \log 2)/\gamma \approx 1.33799 \).

(iii) \( \Phi_\beta(z, p(z)) \prec \varphi_\alpha(z) \) for \( \beta \geq \log(\frac{1}{1 - \sqrt{2}})/(2\gamma \sqrt{2}) \).

(iv) \( \Phi_\beta(z, p(z)) \prec (1 + Az)/(1 + Bz) \) for \( \beta \geq \max \left\{ \frac{(A - B) \log(1/(1 - B))}{\gamma B}, \frac{(A - B) \log(1 + B)}{B \log(1 + e)} \right\} \).

(v) \( \Phi_\beta(z, p(z)) \prec 1 + \sin z \) for \( \beta \geq (1/\gamma) \sum_{n=0}^{\infty} (-1)^{3n+2} \approx 2.06264 \).

(vi) \( \Phi_\beta(z, p(z)) \prec z + \sqrt{1 + z^2} \) for \( \beta \geq (3 - k + \log(k/2))/\gamma \approx 1.6875 \).

(vii) \( \Phi_\beta(z, p(z)) \prec e^z \) for \( \beta \geq (1/\gamma) \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+1)^2} \approx 1.736740 \).

Then \( p < \varphi \) and all the above bounds are sharp.

Proof. Let \( \theta(w) = 1 \) and \( \phi(w) = \beta/\omega \neq 0 \). To prove our results, we make use of the following inequality:
\[
\varphi(-1) \leq q_\beta(-1) \leq q_\beta(1) \leq \varphi(1) \tag{5.5}
\]
so that \( q_\beta < \varphi \). Observe that (5.5) gives necessary as well as sufficient condition for the required subordination.

(i) The differential equation \( \Phi_\beta(z, p(z)) = \phi_0(z) \) has an analytic solution given by
\[
q_\beta(z) = \exp \left( -(z + 2k \log(1 - z/k))/\beta k \right). \tag{5.6}
\]
Since the function
\[
Q(z) = zq_\beta(z)\phi(q_\beta(z)) = \phi_0(z) - 1 = \frac{z}{k} \left( \frac{k + z}{k - z} \right) \in S^*
\]
and \( h(z) = \theta(q_\beta(z)) + Q(z) \) satisfies \( \Re(zh'(z)/Q(z)) = \Re(zQ'(z)/Q(z)) > 0 \).

Therefore, by Lemma 5.1 the following differential subordination implication holds:
\[
\Phi_\beta(z, p(z)) < 1 + \beta \frac{zp'(z)}{q_\beta(z)} \Rightarrow p < q_\beta.
\]
Corollary 5.9. Let \( f \in A \) and set \( p(z) = zf'(z)/f(z) \) and \( p(z) = f(z)/z \). Then using Theorem 5.7 we obtain sufficient conditions for \( f \) to be in \( S^*_\varphi \).

Corollary 5.8. Let \( f \in A \) and set
\[
\Phi_\beta \left( z, \frac{zf'(z)}{f(z)} \right) = 1 + \beta \left( 1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f(z)} \right).
\]
If \( \Phi_\beta \left( z, \frac{zf'(z)}{f(z)} \right) \) satisfies any of the conditions (i)-(vii) of Theorem 5.7, then \( f \in S^*_\varphi \).

Corollary 5.9. Let \( f \in A \) and set
\[
\Phi_\beta \left( z, \frac{f(z)}{z} \right) = \beta \frac{zf'(z)}{f(z)} + (1 - \beta).
\]
If \( \Phi_\beta \left( z, \frac{f(z)}{z} \right) \) satisfies any of the conditions (i)-(vii) of Theorem 5.7, then \( f \in S^*_\varphi \).

The proof of the following result is omitted here as it is much akin to the proof of Theorem 5.4 and 5.7.
Theorem 5.10. Assume that \(0 < \alpha < 1, \ 0 < B < A < 1, \ k = 1 + \sqrt{2}\) and \(\varphi(z) = 1 + z e^z\). Let \(p \in A_0\). Set
\[
\mathcal{U}_\beta(z, p(z)) = 1 + \beta \frac{zp'(z)}{p^2(z)}.
\]
If \(p\) satisfies any of the following differential subordinations:

(i) \(\mathcal{U}_\beta(z, p(z)) \prec \phi_0(z)\) for \(\beta \geq -(e + 1)/(ke)(1 + 2k \log(1 - 1/k)) \approx 0.896489\).

(ii) \(\mathcal{U}_\beta(z, p(z)) \prec \sqrt{1 + z}\) for \(\beta \geq 2(e - 1)(1 - \log 2) \approx 1.05451\).

(iii) \(\mathcal{U}_\beta(z, p(z)) \prec \varphi_\alpha(z)\) for \(\beta \geq ((e - 1)/2\sqrt{\alpha}) \log(1 + \sqrt{\alpha}/(1 - \sqrt{\alpha})) \approx 1.62563\).

(iv) \(\mathcal{U}_\beta(z, p(z)) \prec 1 + \sin z\) for \(\beta \geq (e + 1)/(e) \sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+1)} \approx 1.62563\).

(v) \(\mathcal{U}_\beta(z, p(z)) \prec e^z\) for \(\beta \geq ((e + 1)/(e) \sum_{n=0}^{\infty} \frac{1}{n!} \approx 1.80273\).

Then \(p \prec \varphi\) and the bounds are sharp.

Let \(f \in A\) and set \(p(z) = zf'(z)/f(z)\) and \(p(z) = f(z)/z\). Then using Theorem 5.10, we obtain the following results:

Corollary 5.11. Let \(f \in A\) and set
\[
\mathcal{U}_\beta(z, \frac{zf'(z)}{f(z)}) = 1 + \beta \left(\frac{zf'(z)}{f(z)}\right)^{-1} \left(1 - \frac{zf'(z)}{f(z)} + \frac{zf''(z)}{f(z)}\right).
\]
If \(\mathcal{U}_\beta\left(z, \frac{zf'(z)}{f(z)}\right)\) satisfies any of the conditions (i)-(v) of the Theorem 5.10, then \(f \in S_\varphi^*\).

Corollary 5.12. Let \(f \in A\) and set
\[
\mathcal{U}_\beta\left(z, \frac{f(z)}{z}\right) = 1 + \beta \left(\frac{z}{f(z)} \left(\frac{zf'(z)}{f(z)} - 1\right)\right).
\]
If \(\mathcal{U}_\beta\left(z, \frac{f(z)}{z}\right)\) satisfies any of the conditions (i)-(v) of the Theorem 5.10, then \(f \in S_\varphi^*\).

**Conflict of interest**

The authors declare that they have no conflict of interest.

**References**

[1] Aktaş İ, Baricz Á and Orhan H. Bounds for radii of starlikeness and convexity of some special functions. Turkish J Math, 2018, 42(1): 211–226

[2] Ali R M, Ravichandran V and Jain N K. Convolution of certain analytic functions. J Anal, 2010, 18: 1–8

[3] Bansal D and Prajapat J K. Certain geometric properties of the Mittag-Leffler functions. Complex Variables and Elliptic Equations, 2016, 61(3): 338–350

[4] Banga S and Kumar S S. Applications of differential subordinations to certain classes of starlike functions. J Korean Math Soc, 2020, 57(2): 331–357

[5] Baricz Á, Dimitrov D K, Orhan H, and Yağmur N. Radii of starlikeness of some special functions. Proc Amer Math Soc, 2016, 144(8): 3355–3367

[6] Baricz Á, Toklu E and Kadioğlu E. Radii of starlikeness and convexity of Wright functions. Math Commun, 2018, 23(1): 97–117
[7] Biernacki M and Krzyż J. On the monotonicity of certain functionals in the theory of analytic functions. Ann Univ Mariae Curie-Skłodowska Sect A, 1957, 9(1955): 135–147
[8] Bulut S and Engel O. The radius of starlikeness, convexity and uniform convexity of the Legendre polynomials of odd degree. Results Math, 2019, 74(1), Art 48, 9 pp.
[9] Cho N E, Kumar S, Kumar v and Ravichandran V. Convolution and radius problems of analytic functions associated with the tilted Carathéodory functions. Math Commun, 2019, 24(2): 165–179
[10] Cho N E, Kumar V, Kumar S S and Ravichandran V. Radius problems for starlike functions associated with the sine function. Bull Iranian Math Soc, 2019, 45(1): 213–232
[11] Gangadharan A, Ravichandran V and Shanmugam T N. Radii of convexity and strong star-likeness for some classes of analytic functions. J Math Anal Appl, 1997, 211(1): 301–313
[12] Goel P and Kumar S S. Certain Class of Starlike Functions Associated with Modified Sigmoid Function. Bull Malays Math Sci Soc, 2020, 43(1): 957–991
[13] Goluzin G M. On a variational method in the theory of analytic functions. Amer Math Soc Transl, 1961, 18(2): 1–14
[14] Kargar R, Ebadian A and Sokół J. Radius problems for some subclasses of analytic functions. Complex Anal Oper Theory, 2017, 11(7): 1639–1649
[15] Kumar V, Cho N E, Ravichandran V and Srivastava H M. Sharp coefficient bounds for starlike functions associated with the Bell numbers. Math Slovaca, 2019, 69(5): 1053–1064
[16] Kumar S S and Kamaljeet G. A Cardioid Domain and Starlike Functions. Analysis and Mathematical Physics, 2020 (Communicated)
[17] Kumar S and Ravichandran V. A subclass of starlike functions associated with a rational function. Southeast Asian Bull Math, 2016, 40(2): 199–212
[18] Kuroki K, Owa S. Notes on new class for certain analytic functions. RIMS Kokyuroku, 2011, 1772: 21–25
[19] Ma W C and Minda D. A unified treatment of some special classes of univalent functions. Proceedings of the Conference on Complex Analysis, Tianjin, Conf Proc Lecture Notes Anal, Int Press, Cambridge, MA, 1992: 157–169
[20] MacGregor T H. Majorization by univalent functions. Duke Math J, 1967, 34: 95–102
[21] MacGregor T H. Hull subordination and extremal problems for starlike and spirallike mappings. Trans Amer Math Soc, 1973, 183: 499–510
[22] Mendiratta R, Nagpal S and Ravichandran V. On a subclass of strongly starlike functions associated with exponential function. Bull Malays Math Sci Soc, 2015, 38(1): 365–386
[23] Miller S S and Mocanu P T. Differential subordinations, Monographs and Textbooks in Pure and Applied Mathematics. New York: Marcel Dekker, Inc, 2000
[24] Raina R K and Sokół J. Some properties related to a certain class of starlike functions. C R Math Acad Sci Paris, 2015, 353(11): 973–978
[25] Ruscheweyh S and Sheil-Small T. Hadamard products of Schlicht functions and the Pólya-Schoenберg conjecture. Comment Math Helv, 1973, 48: 119–135
[26] Sharma K, Jain N K and Ravichandran V. Starlike functions associated with a cardioid. Afr Mat, 2016, 27(5-6): 923–939
[27] Sokół J and Stankiewicz J. Radius of convexity of some subclasses of strongly starlike functions. Zeszty Nauk Politech Rzeszowskiej Mat, 1996, (19): 101–105
[28] Tang H and Deng G. Majorization problems for some subclasses of starlike functions. J Math Res Appl, 2019, 39(2): 153–159.
[29] Wang L M. The tilted Carathéodory class and its applications. J Korean Math Soc, 2012, 49: 671–686
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