Exact solution for a quantum field with $\delta$-like interaction

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Abstract

A quantum field described by the field operator $\Delta_a = \Delta + a\delta_\Sigma$ involving a $\delta$-like potential is considered. Mathematically, the treatment of the $\delta$-potential is based on the theory of self-adjoint extension of the unperturbed operator $\Delta$. We give the general expressions for the resolvent and the heat kernel of the perturbed operator $\Delta_a$. The main attention is payed to $d = 2$ $\delta$-potential though $d = 1$ and $d = 3$ cases are considered in some detail. We calculate exactly the heat kernel, Green’s functions and the effective action for the operator $\Delta_a$ in diverse dimensions and for various spaces $\Sigma$. The renormalization phenomenon for the coupling constant $a$ of $d = 2$ and $d = 3$ $\delta$-potentials is observed. We find the non-perturbative behavior of the effective action with respect to the renormalized coupling $a_{\text{ren}}$.

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1. Introduction. We consider a quantum field the dynamics of which on the Euclidean \(d\)-dimensional manifold \(M^d\) is described by the field operator
\[
\Delta_a = \Delta + a \delta \Sigma \tag{1}
\]
where \(\Delta\) is an unperturbed (hereafter to be Laplace operator) operator on \(M^d\) and \(\delta \Sigma\) is \(\delta\)-like potential having support on a subspace \(\Sigma\). Our special concern is the case when \(M\) is a direct product of two-dimensional plane \(R^2\) and \(\Sigma\). The coupling constant \(a\) in (1) can be viewed as a measure of the interaction with some background field that is concentrated at the space \(\Sigma\). If \(\Sigma\) is a point the operator (1) describes a quantum field with point interaction. Otherwise, the interaction is spread over the sub-space \(\Sigma\).

Operators of the form (1) arise in different fields of physics. Our study, however, is motivated by applications in gravitational physics. The operators (1) appear as a result of the non-minimal coupling of quantum matter to the gravitational background having conical singularities. Indeed, the scalar curvature possesses a distributional behavior at a conical singularity
\[
R = R^{reg} + 4\pi (1 - \alpha) \delta \Sigma \tag{2}
\]
spread over surface \(\Sigma\) and having angle deficit \(\delta = 2\pi (1 - \alpha)\). Therefore a non-minimal operator \(\Delta_\xi = \Delta + \xi R\) takes the form (1) being considered on a conical space. The coupling constant then reads \(a = 4\pi (1 - \alpha) \xi\).

The conical geometry arises naturally in three dimensions \(\mathbb{R}^3\) as the only result of the gravitational interaction of point particles. In four and higher dimensions a cosmic string produces space-time which can be modeled by a conical space \(\mathbb{R}^4\). A scalar field with the non-minimal coupling on the cosmic string background was considered in \([3]\), \([4]\). On the other hand, a conical singularity appears in the Euclidean approach to the black hole thermodynamics \([5]\). In this context the developing of the theory of the non-minimal coupling to a conical background is important for understanding such outstanding issues as the renormalization of the black hole entropy \([6]\), the correspondence of various approaches to the black hole thermodynamics \([7]\) and a mechanism of generating the Bekenstein-Hawking entropy in the induced gravity \([8]\).

The operators of the form (1) give us an example of exactly solvable models the mathematical theory of which is well known \([9]\), \([10]\) and based on theory of self-adjoint extension of operators \([11]\). However, the quantum field theoretical aspects of (1) are not so well developed. In this paper those aspects are considered systematically. We calculate the heat kernel, Green’s functions and the effective action for the operator (1) in diverse dimensions and for various spaces \(\Sigma\). In particular, we observe the renormalization phenomenon for the coupling constant \(a\) and find a non-perturbative behavior of the effective action with respect to the renormalized coupling \(a_{ren}\).

2. Mathematical set up. We start with some general consideration of the \(d\)-dimensional operator (1) with the point interaction
\[
\Delta_a = \Delta + a \delta (x, y) \tag{3}
\]
concentrated at \(x = y\). The resolvent of operator (3) is defined as solution of the equation
\[
(-\Delta_a - k^2) G_{a,k^2}(x, x') = \delta(x, x') . \tag{4}
\]
The following Theorem is valid.
Theorem: For the operator (3) we find that
(i) the resolvent takes the form

\[ G_{\alpha,k^2}(x, x') = G_{k^2}(x, x') + \frac{\alpha}{1 - \alpha G_{k^2}(y, y)} G_{k^2}(y, x') G_{k^2}(x, y) \]  \hspace{1cm} (5) \]

(ii) the heat kernel \( K_{\alpha}(x, x', t) \) takes the form:

\[ K_{\alpha}(x, x', t) = \frac{1}{2\pi i} \int_{C} e^{-k^2 t} G_{\alpha,k^2}(x, x') dk^2 = K(x, x', t) + \frac{1}{2\pi i} \int_{C} \frac{\alpha}{1 - \alpha G_{k^2}(y, y)} G_{k^2}(y, x') G_{k^2}(x, y) dk^2 \]  \hspace{1cm} (6) \]

where \( C \) is a clockwise contour going around the positive real axis on the complex plane of variable \( k^2 \) and \( K(x, x', t) \) is the unperturbed heat kernel:

\[ K(x, x', t) = \frac{1}{2\pi i} \int_{C} e^{-k^2 t} G_{k^2}(x, x') dk^2 \]  \hspace{1cm} . \]

The part (i) of the Theorem can be proven by verifying Eq.(4) for the function (5). The part (ii) follows from definition of the heat kernel.

One can see from (5) that there exists a bound state at \( k = k_0 \) corresponding to the pole of the resolvent and satisfying the equation \( G_{k_0^2}(y, y) = 1/\alpha \) and lying in the upper half-plane. In this case the contour \( C \) in (6) contains also a circle around the pole.

So far we have not given a concrete sense to the \( \delta \)-potential in (3). It should be noted that its action on test functions may not be well-defined (see \( d = 2 \) example below). Then, for instance, we must give a definition to the quantity \( G_{k^2}(y, y) \) appearing in (2), (3). The mathematically rigorous treatment of the \( \delta \)-potential in the operator (3) requires considering a self-adjoint extension of the unperturbed operator \( \Delta \). Particularly, it defines the action of the \( \delta \)-potential on test functions and consists in re-formulation of the operator (3) in terms of the unperturbed operator \( \Delta \) acting on the space of field functions satisfying some (dependent on \( \alpha \)) “boundary” condition at \( x = y \.

As an instructive example consider the operator (3) on two-dimensional plane \( \mathbb{R}^2 \) with \( \delta \)-potential concentrated in the center of the polar coordinates \((\rho, \phi)\): \( \delta(x, y) = \frac{1}{2\pi \rho} \delta(\rho) \) where \( \delta(\rho) \) satisfies the normalization condition \( \int_{0}^{\infty} \delta(\rho) d\rho = 1 \). Let the unperturbed operator \( \Delta \) be the two-dimensional Laplace operator. The self-adjoint extension of the 2d Laplace operator consists in allowing \( \ln \rho \) singularity at \( \rho = 0 \) for test field functions. However, \( \delta(\rho) \ln \rho \) is not then well-defined. Thus, we need to define \( \delta(\rho) f(\rho, \phi) \) for a test function \( f(\rho, \phi) \) behaving near \( \rho = 0 \) as

\[ f(\rho, \phi) = f_0 \ln(\rho \mu) + f_1 + O(\rho) \]  ,

where \( \mu \) is an arbitrary dimensional parameter. Formally integrating \( \Delta_{\alpha} f = \lambda f \) over small disk of radius \( \epsilon \) around the origin and taking the limit \( \epsilon \to 0 \) we get the constraint on the coefficients \( f_0 \) and \( f_1 \)

\[ 2\pi f_0 + \alpha f_1 = 0 \]  .  \hspace{1cm} (7) \]

This is the boundary condition which we should impose on the field functions at the origin. It arises in a more sophisticated way in [9]. In fact, this condition is the precise formulation of the two-dimensional \( \delta \)-potential [6].
Now we may define the value at the origin for a field function by just subtracting the logarithmic term: \( f(\rho)|_{\rho=0} \equiv (1 - \rho \ln(\rho \mu) \frac{d}{d\rho}) f|_{\rho=0} \). Then the action of the \( \delta \)-function is defined as follows: \( \delta(\rho) f(\rho) \equiv \delta(\rho)(1 - \rho \ln(\rho \mu) \frac{d}{d\rho}) f|_{\rho=0} \).

For the unperturbed operator \( (a = 0) \) the resolvent is expressed by means of Hankel’s function \( G_{k^2}(x, x') = \frac{i}{4} H_0^{(1)}(k|x - x'|) \). Applying the above definition of the value at the origin to this resolvent \( (aG_{k^2}(0, 0)) \equiv a(1 - \rho \ln(\rho \mu) \frac{d}{d\rho}) G_{k^2}(\rho)|_{\rho=0} \) and using the asymptote of Hankel’s function \( \frac{i}{4} H_0^{(1)}(k\rho) \simeq -\frac{1}{2\pi}(\ln(\frac{\rho}{k^2}) + C) + O(\rho) \) we find that \( (aG_{k^2}(0, 0)) = -\frac{a}{2\pi}(C + \ln\frac{k}{2\mu}) \), where \( C \) is the Euler constant. Hence, applying our general formula (7) to this particular case we find (8) for the resolvent of the two-dimensional operator (3). One can see that (8) satisfies the condition (7) at \( k \) fixed. For the heat kernel we find according to the general expression (3)

\[
G_{a,k^2}(x, x') = \frac{1}{4} H_0^{(1)}(k|x - x'|) - \frac{a}{8(2\pi + aC + a\ln k \frac{1}{2\mu})} H_0^{(1)}(k|x|)H_0^{(1)}(k|x'|)
\]

(8)

for the resolvent of the two-dimensional operator (3). One can see that (8) satisfies the condition (7) at \( x' \) and \( k \) fixed. For the heat kernel we find according to the general expression (3)

\[
K_a(x, x', t) = K(x, x', t) + K_a^{(1)}(x, x', t)
\]

\[
K(x, x', t) = \frac{1}{4\pi t} e^{-\frac{(x-x')^2}{4t}},
\]

\[
K_a^{(1)}(x, x', t) = -\frac{1}{8\pi} \int_C dk^2 \frac{1}{\ln\frac{k}{sk_0}} H_0^{(1)}(k|x|)H_0^{(1)}(k|x'|)
\]

(9)

where \( k_0 = 2\mu e^{-\frac{\pi}{a}C} \). The resolvent (8) has a pole at \( k = uk_0 \) corresponding to the bound state. Surprisingly, this bound state appears both for positive and negative \( a \). However, only for positive \( a \) there exists a correspondence to the unperturbed case: the bound state becomes a (non-normalizable) zero mode \( (k_0 \to 0 \text{ when } a \to +0) \) of the unperturbed operator. Contrary to this, for negative \( a \) the bound state tends to become infinitely heavy \( (k_0 \to \infty \text{ when } a \to -0) \). It should be noted that \( k_0 \) is that parameter which characterizes the self-adjoint extension of the two-dimensional Laplace operator. It is of our special interest to see how the quantum field theoretical quantities (heat kernel, Green’s function, effective action) depend on the parameter \( k_0 \) (a).

In the next Section we calculate the contour integral appearing in (3). Here we want to pause and make a comment regarding the usage of the formula (7) on the conical space \( R_\alpha^2 \). In this case the angle coordinate \( \phi \) changes in the limits \( 0 \leq \phi \leq 2\pi\alpha, \alpha \neq 1 \). The heat kernel on \( R_\alpha^2 \) is constructed via the heat kernel (3) on the regular plane \( R^2 \) by means of the Sommerfeld formula (12):

\[
K_{R_\alpha^2}(x, x', t) = K_{R^2}(x, x', t) + \frac{1}{4\pi\alpha} \int_\Gamma \cot \frac{w}{2\alpha} K_{R^2}(\phi - \phi' + w, |x|, |x'|, t) dw,
\]

(10)

where \( \Gamma \) is some known contour on the complex plane. In some sense the Sommerfeld formula is just a way to make \( 2\pi\alpha \)-periodical function from \( 2\pi \) periodical function. If the \( \delta \)-potential in (3) is originated from the non-minimal coupling as explained in Introduction then we should also substitute for the coupling constant \( a \) its value \( a = 4\pi(1 - \alpha)\xi \) and take into account that the angle coordinate now has period \( 2\pi\alpha \). The condition (7) then becomes

\[
2\pi\alpha f_0 + af_1 = 0.
\]
Applying the Sommerfeld formula to the heat kernel \( K(x, x', t) \) \((9)\) of the unperturbed operator \( \Delta \) one finds a non-trivial modification of the heat kernel determined by the angle deficit at the singularity. This modification is well studied and known explicitly \((12), (3)\). Regarding the modification of the term \( K_a^{(1)}(x, x', t) \) in \((9)\), which is of our interest here, we notice that \( K_a^{(1)}(x, x', t) \) is independent of the angle coordinates. This illustrates the fact that the \( \delta \)-potential in \((9)\) describes an s-wave interaction. Therefore, the contour integral in \((10)\) vanishes\(^\dagger\) for \( K_a^{(1)} \). This means that the part of the heat kernel which is due to the \( \delta \)-potential occurs to be the same for the regular plane \( R^2 \) and the cone \( R^2_\alpha \). That is why below we consider the operator \((9)\) with arbitrary coupling \( a \) on a regular manifold supposing that the generalization to a conical space is straightforward.

3. Calculation of the heat kernel in two dimensions. Here we calculate the term \( K_a^{(1)}(x, x', t) \) by performing explicitly the contour integration in \((9)\). To proceed we first change in \((9)\) variable of integration \( k^2 \to k \). The contour \( C \) then transforms to the real axis \((0 < \arg(k) < \pi)\) on the complex plane of the variable \( k \). Using the integral representation for a product of two Hankel’s functions

\[ H_0^{(1)}(k|x|)H_0^{(1)}(k|x'|) = -\frac{8}{\pi^2} \int_0^\infty d\phi \int_0^\infty d\beta e^{ikZ_\phi \cosh \beta}, \]

where \( Z_\phi^2 = |x|^2 + |x'|^2 + 2|x||x'| \cosh(2\phi) \) we obtain for \( K_a^{(1)}(x, x', t) \):

\[ K_a^{(1)}(x, x', t) = \frac{1}{\pi^2} \int_{-\infty}^{+\infty} dk k^{\frac{1}{2}} \int_{-\infty}^{\infty} d\phi \int_0^\infty d\beta e^{-tk^2} e^{ikZ_\phi \cosh \beta}. \]  

Separating in \((11)\) the integration over \( k > 0 \) and \( k < 0 \), taking into account that \( k = e^{i\pi}|k| \) for \( k < 0 \) and interchanging the \( k \)-integration with integration over \( \beta \) and \( \phi \) we find the representation for \((11)\)

\[ K_a^{(1)}(x, x', t) = \int_0^\infty d\phi (A + A^*), \]  

where

\[ A = \frac{1}{\pi^2} \int_0^\infty d\beta \int_0^\infty \frac{dk}{k} \left( \frac{k}{k_0} - \frac{\pi}{2} \right) e^{-tk^2} e^{ikZ_\phi \cosh \beta}. \]

The function \( K_a^{(1)}(x, x', t) \) is a real quantity as seen from \((12)\). It is convenient to replace the denominator in \((13)\) by an integral

\[ \frac{1}{\ln \frac{k}{k_0} - \frac{\pi}{2}} = i \int_0^\infty dy e^{-\frac{y}{2}} \frac{1}{\sqrt{\pi}} \frac{1}{\cosh \frac{y}{2}}. \]

Then \((13)\) takes the form

\[ A = \frac{k_0^2}{\pi^2} \int_0^\infty dy e^{-\frac{y^2}{2}} \frac{1}{\sqrt{\pi}} \int_0^\infty d\beta \int_0^\infty dk k^{1-y} e^{-tk^2} e^{ikZ_\phi \cosh \beta}. \]

The integration over \( k \) and \( \beta \) in \((14)\) can be performed explicitly. We skip details of the calculation just referring to helpful formulae (3.462.1) and (7.731.1) of \((14)\). The result of the integration is the following

\[ A = \frac{i}{\sqrt{\pi}} Z_\phi e^{-\frac{Z_\phi^2}{8\pi}} \int_0^\infty dy e^{-\frac{y^2}{2}} \frac{1}{\cosh \frac{y}{2}} \left( \sqrt{\pi} W_{1,\frac{1}{2},0} \right)^m \left( \frac{Z_\phi^2}{4} \right), \]

\(^\dagger\) \( K_a^{(1)} \) is already \( 2\pi\alpha \)- (in fact, arbitrary-) periodical function so the Sommerfeld formula \((10)\) works trivially for it.
where $W_{\lambda,0}(z)$ is Whittaker’s function. By means of the equation (12) this gives us the desired expression for the heat kernel $K_{a}^{(1)}$

$$K_{a}^{(1)}(x, x', t) = \frac{1}{\pi \sqrt{t}} \int_{0}^{\infty} \int_{0}^{\infty} dy \frac{e^{\frac{-r^2}{4t}}}{e^{\pi y} + 1}$$

$$\left( (\sqrt{t}k_0)^{-v} W_{\frac{1+v}{2},0}(\frac{Z_{\phi}^2}{4t}) - (\sqrt{t}k_0)^{v} W_{\frac{1-v}{2},0}(\frac{Z_{\phi}^2}{4t}) \right). \tag{15}$$

Note, that $k_0$ appears in (13) only in the combination $(\sqrt{t}k_0)$.

Some asymptotes of the expression (15) are easy to analyze. Being interested in the limit $p = \ln \sqrt{t}k_0 \to \pm \infty$ we may replace the Whittaker’s function in (15) by its value at $y = 0$: $W_{\frac{1+v}{2},0}(z) = \sqrt{z}e^{-z}$. Then integrating over $\phi$ (eq.(3.337) in [14]) we find that in this limit the heat kernel (15) behaves as follows

$$K_{a}^{(1)}(x, x', t) \simeq -\frac{1}{4\pi t} e^{-\frac{(x^2+y^2)}{4t}} K_0\left(\frac{|x||x'|}{2t}\right) f(p(t)), \tag{16}$$

where $K_0(z)$ is the Macdonald function, $p(t) = \frac{1}{2} \ln(tk_0^2)$. The function $f(p)$ is the result of the integration over $y$

$$\int_{0}^{\infty} \frac{2dy}{e^{\pi y} + 1} \sin py = \frac{1}{p} - \frac{1}{\sinh p} \equiv f(p). \tag{17}$$

It plays an important role and appears frequently in our calculation. Remarkably, (16) is valid in both $t \to 0$ and $t \to +\infty$ limits. For large $p$ the function (17) behaves as $f(p) = \frac{1}{p} + O(e^{-|p|})$. Therefore, the leading term in (16) is given by $f(t) \simeq \frac{1}{\ln \sqrt{t}k_0} + O$, where $O = O(\sqrt{t}k_0)$ for $t \to 0$ and $O = O\left(\frac{1}{\sqrt{t}k_0}\right)$ for $t \to +\infty$.

Another interesting asymptotic behavior of the function (15) occurs in the regime of large $|x|$ and $|x'|$ under $t$ fixed so that $\frac{|x||x'|}{t} >> 1$. Then we may use in (15) the large $|z|$ asymptotic behavior of Whittaker’s function $W_{\mu,0}(z) \simeq z^{\mu}e^{-z}$. After integration over $y$ given again by (17) we obtain that

$$K_{a}^{(1)}(x, x', t) \simeq -\frac{1}{2\pi t} \int_{0}^{\infty} d\phi e^{-\frac{Z_{\phi}^2}{4t}} f(\ln \frac{2tk_0}{|x| + |x'|}). \tag{18}$$

The integral over $\phi$ for large $\frac{|x||x'|}{t}$ can be approximated by the descent method and the result reads

$$K_{a}^{(1)}(x, x', t) \simeq -\frac{1}{4\pi t} e^{-\frac{(x^2+y^2)}{4t}} K_0\left(\frac{|x||x'|}{2t}\right) f(\ln \frac{2tk_0}{|x| + |x'|}). \tag{18}$$

In the leading order we make use the asymptotic representation for the Macdonald function $K_0(z) \simeq \sqrt{\frac{\pi}{2z}} e^{-z}$ and find that

$$K_{a}^{(1)}(x, x', t) \simeq -\frac{e^{-\frac{(|x|+|x'|)^2}{4t}}}{4\sqrt{\pi t}} \frac{1}{\sqrt{|x||x'|}} \frac{1}{\ln \frac{2tk_0}{|x| + |x'|}}. \tag{19}$$

This equation illustrates the long-range effects of the $\delta$-potential. They were studied in [4] from a different perspective.
Calculate now the trace $TrK_a^{(1)} = 2\pi \int_0^\infty x^2 K_a^{(1)}(x, x, t)$ of the kernel \[13\]. For coinciding points $x = x'$ we have $Z_\phi = 2x \cosh \phi$. Doing first the $x$-integration by means of the formula (7.621.11) of \[14\] and then performing the integration over $\phi$ we find for the trace

$$TrK_a^{(1)} = \frac{1}{2\pi} \int_0^\infty dy \frac{dy}{e^{x^2y}} \left( \frac{\sqrt{t}k_0 - iy}{\Gamma(1 - iy)} - \frac{\sqrt{t}k_0 + iy}{\Gamma(1 + iy)} \right),$$

(20)

where $\Gamma(z)$ is Gamma function. Note, that (20) does not depend on sizes of the space $R^2$. In this respect it differs from the trace of the unperturbed heat kernel $TrK(x, x, t) = \frac{A(R^2)}{4\pi t}$ which is proportional to the (infinite) area $A(R^2)$ of $R^2$.

In the limit $p = \ln \sqrt{t}k_0 \to \pm\infty$ we find that

$$TrK_a^{(1)} \simeq -\frac{1}{2}f(p) = -\frac{1}{\ln(tk_0^2)} + \frac{\sqrt{t}k_0}{tk_0^2 - 1}. \quad (21)$$

Note, that typically the trace of the heat kernel of a differential operator expands in the Laurent series with respect to $\sqrt{t}$. The appearance of $\frac{1}{\ln t}$ makes behavior of $TrK_a^{(1)}$ unusual. We discuss this in some detail later.

The heat kernel \[14\] and its trace (20) take much simpler form after the Laplace transform: $L(x, x', s) = \int_0^\infty e^{-ts^2}K_a^{(1)}(x, x', t)dt$ and $L(s) = \int_0^\infty e^{-ts^2}TrK_a^{(1)}dt$. Indeed, after short computation involving eqs.(3.381.4), (7.630.2) and (6.648) of \[14\] and the eq.(17) we find the following simple expressions

$$L(x, x', s) = \frac{1}{2\pi}K_0(|x|s)K_0(|x'|s)f(\ln \frac{s}{k_0}),$$

$$L(s) = \frac{1}{2s^2}f(\ln \frac{s}{k_0}),$$

$$f(\ln \frac{s}{k_0}) = \frac{1}{\ln \frac{s}{k_0}} - \frac{2sk_0}{s^2 - k_0^2} \quad (22)$$

which play a significant role in our further calculation.

Note, that we did not take into account the bound state when deriving the expressions for the heat kernel \[15\] and its trace (21). In particular, that is why the pole at $s = k_0$ cancels in the Laplace transform (22). The contribution of the bound state to the heat kernel is $K_{a,\text{bound}}(x, x', t) = \pi^{-1}k_0^2K_0(|x|k_0)K_0(|x'|k_0)e^{ik_0^2t}$ and the trace is $TrK_{a,\text{bound}} = e^{ik_0^2t}$. However, it seems that we do not really need to add these to the expressions (15) and (21). Indeed, adding $K_{a,\text{bound}}$ to $K_a^{(1)}$ we, particularly, find that $Tr(K_{a,\text{bound}} + K_a^{(1)}) \to 1$ if $t \to 0$. This is not consistent with the condition $K_a(x, x', t) \to \delta^2(x, x')$ if $t \to 0$ for the complete heat kernel.

On the other hand, the bound state can be removed (it becomes a resonance state) if one makes an analytical continuation $k_0 \to -k_0$ (we demonstrate in Section 6 how it works for $d = 1$ and $d = 3$ $\delta$-potentials). However, the integral \[10\], \[11\] does not change if the transform $k_0 \to -k_0$ is accompanied by changing the contour $C$ in \[10\] so that $-\pi < arg(k) < 0$.

\[1\]In presence of a conical singularity one finds an addition $c(\alpha)$ to the heat kernel $TrK(x, x, t) = \frac{A(R^2)}{4\pi t} + c(\alpha)$ that also does not depend on sizes of $R^2$ and is entirely due to the singularity \[13\]. This term is of a similar nature as (20).
All these likely mean that in two dimensions the eigen vectors of the operator $\Delta_{\mathbf{a}}$ form the complete basis without the bound state and we should not add it when calculating the heat kernel.

4. Green’s function in higher dimensions. Consider $(d + 2)$-dimensional manifold $M$ which is a direct product of 2-dimensional plane $R^2$ and $d$-dimensional surface $\Sigma$. The $\delta$-potential in operator [II] has support at the surface $\Sigma$. The heat kernel on the total space $M$ is a product of the heat kernels on $R^2$ and $\Sigma$:

$$K_M = K_{R^2}(x, x', t) K_{\Sigma}(z, z', t) ,$$

where $\{x\}$ and $\{z\}$ are coordinates on $R^2$ and $\Sigma$ respectively. The heat kernel on $\Sigma$ is convenient to represent in the form of the Laplace transform

$$K_{\Sigma}(z, z', t) = \int_0^\infty e^{-z^2 t} g_{\Sigma}(z, z', s) ds^2 .$$

For particular spaces we have

\begin{itemize}
  \item[i)] $\Sigma = R^1$, $K_{R^1}(z, z', t) = \frac{1}{2\sqrt{2\pi t}} e^{-\frac{z^2}{4t}}$, $g_{R^1}(z, z', s) = \frac{1}{2\pi s} \cos(\Delta z s)$ ;
  \item[ii)] $\Sigma = R^2$, $K_{R^2}(z, z', \tau, \tau', t) = \frac{1}{4\pi t} e^{-\frac{z^2 + \Delta \tau^2}{4t}}$, $g_{R^2}(z, z', \tau, \tau', s) = \frac{1}{4\pi t} J_0(\sqrt{\Delta z^2 + \Delta \tau^2}s)$ ,
\end{itemize}

where $\Delta z = z - z'$, $\Delta \tau = \tau - \tau'$ and $J_0(z)$ is the Bessel function.

Green’s function $G_M(x, x')$ on $M$ relates to the heat kernel $K_M(x, x', t)$ as follows

$$G_M(x, x') = \int_0^\infty dt K_M(x, x', t) = G_M^{reg} + G_M^{a} .$$

We are interested in that part of Green’s function which is due to the $\delta_{\Sigma}$-potential: $G_M^{a} = \int_0^\infty dt K_M^{(1)}(x, x', t) K_{\Sigma}(z, z', t)$. Equivalently, it can be written as

$$G_M^{a}(x, x', z, z') = \int_0^\infty g_{\Sigma}(z, z', s)L(x, x', s)2sds , \quad (23)$$

where $L(x, x', s)$ is the Laplace transform (22) of the heat kernel $K_M^{(1)}(x, x', t)$. The eq.(23) gives general expression for Green’s function on a space $R^2 \times \Sigma$. For particular cases $i)$ $(M = R^3)$ and $ii)$ $(M = R^4)$ considered above we find

\begin{itemize}
  \item[i)] $G_M^{a}(x, x', z, z') = \frac{1}{2\pi^2} \int_0^\infty \cos(\Delta z s)K_0(|x|s)K_0(|x'|s)f(\ln \frac{s}{k_0})ds$ ,
  \item[ii)] $G_M^{a}(x, x', z, z') = \frac{1}{4\pi^2} \int_0^\infty J_0(\sqrt{\Delta z^2 + \Delta \tau^2}s)K_0(|x|s)K_0(|x'|s)f(\ln \frac{s}{k_0})ds$ (24)
\end{itemize}

The similar expressions for Green’s function in $d = 3$ and $d = 4$ dimensions were found in [I]. They, however, obtain the term $\frac{1}{\ln k_0}$ instead of the function $f(\ln \frac{s}{k_0})$ and, thus, observe a pole at $s = k_0$ suggesting interpret the integral over $s$ as a principle part integral. This pole is absent [I] (see discussion in the end of Section 3) in our function $f(\ln \frac{s}{k_0})$ and the problem of interpretation of the integral (24) not arises. Recall that $k_0 = \mu e^{-2\pi}$ so Green’s functions (24) depend on the coupling $\mathbf{a}$ in a non-perturbative way.

\footnote{The $x$-dependent part of (24) looks as an analytical continuation of the resolvent (8) to imaginary values $k \rightarrow is$ [I]. The term $\frac{1}{\ln \frac{s}{k_0}}$ in the function $f(p)$ then appears to serve this non-trivial continuation through a branch point $k = 0$ of the logarithmic function. I thank A.Zelnikov for discussing this point.}
5. The effective action and UV renormalization. Calculating the effective action \( W \) for a quantum field with a field operator \( \mathcal{I} \) on the space \( M = R^2 \times \Sigma \)

\[
W = -\frac{1}{2} \int_{\mathbb{R}^2} dt \; Tr K_M = W_{reg} + W_a ,
\]

where \( \epsilon \) is an UV cutoff, \( Tr K_M = Tr K_{R^2} Tr K_\Sigma \), we are again interested in the part

\[
W_a = -\frac{1}{2} \int_{\mathbb{R}^2} dt \; Tr K^{(1)}_a Tr K_\Sigma ,
\]

which is due to the \( \delta \)-potential in \( \mathcal{I} \). As in the previous Section it is convenient to use the Laplace transform. Representing \( t^{-1} Tr K_\Sigma = \int_0^\infty e^{-st} \tau_\Sigma(s) ds \) we find for (25)

\[
W_a = -\frac{1}{2} \int_0^{\lambda(\epsilon)} \tau_\Sigma(s)L(s)ds ,
\]

where \( L(s) \) is given by (22). It should be noted that the UV divergence appears in (24) when integrating over small \( t \). Regularized by introducing parameter \( \epsilon \) in (24) it transforms to a divergence of the integral (26) at large values of \( s \) that may be regularized by a parameter \( \lambda(\epsilon) \) (\( \lambda(\epsilon) \to \infty \) if \( \epsilon \to 0 \)). For simplicity we denote that \( \lambda(\epsilon) = \epsilon^{-1} \).

For particular spaces we have

i) \( \Sigma = R^1 \): \( Tr K_{R^1} = \frac{L(\Sigma)}{2\sqrt{\pi t}} \), \( \tau_\Sigma(s) = \frac{2}{\pi}\epsilon^2 L(\Sigma) \), where \( L(\Sigma) \) is the (infinite) length of \( \Sigma \).

ii) \( \Sigma = R^2 \): \( Tr K_{R^2} = \frac{A(\Sigma)}{4\pi t} \), \( \tau_\Sigma(s) = \frac{1}{2\pi^2} \epsilon^3 A(\Sigma) \), where \( A(\Sigma) \) is the (infinite) area of \( \Sigma \).

iii) \( \Sigma \) is a two-dimensional sphere \( S^2 \) of radius \( r \): \( Tr K_{S^2} = \frac{r^2}{4} + \frac{1}{3} + O(\frac{1}{r}) \), \( \tau_{S^2}(s) = 2r^2 s^3 + \frac{3}{2}s + O(1) \).

iv) \( \Sigma \) is a two-dimensional compact surface \( \Sigma_g^2 \) of radius \( r \) and genus \( g > 1 \), its area is \( A(\Sigma) = 4\pi(g-1)r^2 \); \( Tr K_{\Sigma_g^2} = (g-1)(\frac{r^2}{4} - \frac{1}{3} + O(\frac{1}{r})) \), \( \tau_{\Sigma_g^2}(s) = (g-1)(2r^2 s^3 - \frac{3}{2}s + O(1)) \).

Since \( \tau_\Sigma(s) \) is a polynomial with respect to \( s \) the following integrals are useful:

\[
S_0(s) = \int \frac{ds}{s} f(\ln(s)) = \ln |\ln s| - \ln \frac{|s-1|}{s+1} ,
\]

\[
sS_1(s) = \int ds f(\ln s) = Ei(\ln s) - \ln |s^2 - 1| ,
\]

\[
s^2 S_2(s) = \int ds f(\ln s) = Ei(2 \ln s) - 2s - \ln \frac{|s-1|}{s+1} ,
\]

where \( Ei(x) \) is the Exponential-Integral function defined as \( Ei(x) = C + \ln |x| + \sum_{k=1}^\infty \frac{x^k}{k!} \).

It has an useful asymptote \( Ei(x) \approx \frac{e^x}{x} \) if \( x \to \pm \infty \). Applying these formulae we find the effective action in diverse dimensions and for various \( \Sigma \).

i) \( \Sigma \) is a point, \( M = R^2 \):

\[
W_a[R^2] = -\frac{1}{2} \left( S_0(\frac{1}{\epsilon k_0}) - S_0(\frac{1}{\Lambda k_0}) \right)
\]

\[
\approx -\frac{1}{2} \left( \ln \ln \frac{1}{\epsilon k_0} - \ln \ln (\Lambda k_0) \right) + O(\epsilon k_0) ,
\]

ii) \( \Sigma = R^1, M = R^3 \) :

\[
W_a[R^3] = -\frac{L(\Sigma)}{2\pi \epsilon} S_1(\frac{1}{\epsilon k_0})
\]

\[
\approx \frac{L(\Sigma)}{2\pi \ln(k_0 \epsilon)} - \frac{L(\Sigma) k_0}{\pi} \ln(\epsilon k_0) + O(\epsilon k_0) ,
\]
iii) $\Sigma = R^2$, $M = R^4$:

$$W_a[R^4] = -\frac{A(\Sigma)}{8\pi\epsilon^2} S_2\left(\frac{1}{\epsilon k_0}\right)$$

$$\approx \frac{A(\Sigma)}{16\pi\epsilon^2 \ln(\epsilon k_0)} + \frac{A(\Sigma)k_0}{4\pi\epsilon} + O(\epsilon k_0),$$

(30)

$iv$) $\Sigma = \Sigma_g^2$, $g = 0, 1, 2, \ldots$; $M = R^2 \times \Sigma_g^2$:

$$W_a[R^2 \times \Sigma_g^2] = -\frac{A(\Sigma)}{8\pi\epsilon^2} S_2\left(\frac{1}{\epsilon k_0}\right) + B(\Sigma) \left(S_0\left(\frac{1}{\epsilon k_0}\right) - S_0\left(\frac{1}{\Lambda k_0}\right)\right)$$

$$\approx \frac{A(\Sigma)}{16\pi\epsilon^2 \ln(\epsilon k_0)} + \frac{A(\Sigma)k_0}{4\pi\epsilon} + B(\Sigma) \ln \ln \frac{1}{\epsilon k_0} + O(\epsilon k_0),$$

(31)

where $B(\Sigma) = \frac{(g-1)}{6}$ and $O(\epsilon k_0)$ are finite in the limit $\epsilon \to 0$ terms. We introduced an infra-red cutoff $\Lambda$ to regularize the integral over small $s$ and used the fact that $S_1\left(\frac{1}{\Lambda k_0}\right)$ and $S_2\left(\frac{1}{\Lambda k_0}\right)$ go to zero when $\Lambda \to \infty$ when deriving (29)-(31). In fact, $B(\Sigma)$ can be expressed via the scalar curvature integrated over the surface $\Sigma$: $B(\Sigma) = -\frac{1}{4\pi} \int_{\Sigma} R$.

We stress that the effective action found is exact and behaves non-perturbatively with respect to the coupling $a$. On the other hand, considering the $\delta$-potential in (11) as a perturbation one obtains [3] for the effective action in the first order with respect to $a$:

$$W_a[R^2] = -\frac{a}{4\pi} \ln \frac{\Lambda}{\epsilon},$$

$$W_a[R^3] = -\frac{L(\Sigma)a}{4\pi^2\epsilon},$$

$$W_a[M^4] = -\frac{A(\Sigma)a}{32\pi^2\epsilon^2} + B(\Sigma) \frac{a}{2\pi} \ln \frac{\Lambda}{\epsilon}.$$

(32)

The UV divergences in the effective action (28)-(31) are result of the interplay of two different effects: 1) divergences which can be absorbed in the renormalization of the coupling $a$, and 2) standard UV divergences which are typical for a quantum field interacting to a fixed background. It is not hard to see that the quantity $\frac{1}{\ln(\epsilon k_0)}$ plays a role of the renormalized coupling $a_{ren}$. Indeed, defining $a_{ren} = -\frac{2\pi}{\ln(\epsilon k_0)}$ we find

$$a_{ren} = \frac{a}{1 - \frac{a}{2\pi} \ln(\epsilon \mu)}$$

(33)

that is standard expression for the running coupling constant in the quantum field theory [13]. The eq.(33) can be represented as a sum over the “leading logarithms”:

$$a_{ren} \approx a + \frac{a^2}{2\pi} \ln(\epsilon \mu) + O(a^3 \ln^2(\epsilon \mu)).$$

Note that the value of the bound state $k_0 = \epsilon^{-1} e^{-\frac{2\pi}{a_{ren}}}$ depends on the renormalized coupling $a_{ren}$ in the same way as on the bare coupling $a$. Using this we may re-write the

\*The next orders are ill defined [3].
expressions (28)-(31) for the effective action in terms of the renormalized value \( a_{ren} \):

\[
W_\text{a}[R^2] = -\frac{1}{2} \left( S_0(e^{2a_{\text{ren}}}) - S_0\left( \frac{e}{\Lambda e^{a_{\text{ren}}}} \right) \right),
\]

\[
W_\text{a}[R^3] = -\frac{L(\Sigma)}{2\pi} S_1(e^{2a_{\text{ren}}}),
\]

\[
W_\text{a}[M^4] = -\frac{A(\Sigma)}{16\pi \epsilon^2} S_2(e^{2a_{\text{ren}}}) + B(\Sigma) \left( S_0(e^{2a_{\text{ren}}}) - S_0\left( \frac{e}{\Lambda e^{a_{\text{ren}}}} \right) \right).
\]

(34)

where functions \( S_0, S_1, S_2 \) are defined in (27). These functions have nice properties with respect to the coupling \( a_{ren} \). First of all, they well behave in the strong coupling limits \( a_{ren} \to +\infty \) and \( a_{ren} \to -\infty \). Moreover, the both limits coincide. In fact this is just a consequence of the analyticity of the function \( f(\ln s) \) (27) at the point \( s = 1 \). The limiting values are

\[
\lim_{a_{\text{ren}} \to \pm \infty} S_0(e^{2a_{\text{ren}}}) = \ln 2, \quad \lim_{a_{\text{ren}} \to \pm \infty} S_0\left( \frac{e}{\Lambda e^{a_{\text{ren}}}} \right) = \ln \frac{\Lambda}{\epsilon},
\]

\[
\lim_{a_{\text{ren}} \to \pm \infty} S_1(e^{2a_{\text{ren}}}) = C - \ln 2, \quad \lim_{a_{\text{ren}} \to \pm \infty} S_2(e^{2a_{\text{ren}}}) = C + 2 \ln 2 - 2.
\]

Another nice property of the functions \( S_p, p = 0, 1, 2 \) appears in limits \( a_{\text{ren}} \to +0 \) and \( a_{\text{ren}} \to -0 \). Again, it occurs that both limits coincide and are finite

\[
\lim_{a_{\text{ren}} \to +0} S_1(e^{2a_{\text{ren}}}) \simeq \frac{a_{\text{ren}}}{2\pi}, \quad \lim_{a_{\text{ren}} \to +0} S_2(e^{2a_{\text{ren}}}) \simeq \frac{a_{\text{ren}}}{4\pi}.
\]

We see from this that in the limit of small \( a_{\text{ren}} \) the divergent \( \epsilon^{-1} \) and \( \epsilon^{-2} \) terms in (34) reproduce the perturbative result (32). The limit \( a_{\text{ren}} \to \pm 0 \) for the function \( S_0(e^{2a_{\text{ren}}}) - S_0\left( \frac{e}{\Lambda e^{a_{\text{ren}}}} \right) \) is little more tricky. It is well-defined if we assume that \( \frac{|a_{\text{ren}}|}{2\pi} \ll (\ln \frac{\Lambda}{\epsilon})^{-1}. \) This condition means that we should take the limit of small \( a_{\text{ren}} \) first and then consider the limit of large ratio \( \frac{\Lambda}{\epsilon} \). Provided it is done we find for the combination of the function \( S_0 \) appearing in (34)

\[
\lim_{a_{\text{ren}} \to \pm 0} S_0(e^{2a_{\text{ren}}}) - S_0\left( \frac{e}{\Lambda e^{a_{\text{ren}}}} \right) \simeq \frac{a_{\text{ren}}}{2\pi} \ln \frac{\Lambda}{\epsilon}
\]

that exactly reproduces the perturbative result (32). We, thus, see that the combination \( S_0(e^{2a_{\text{ren}}}) - S_0\left( \frac{e}{\Lambda e^{a_{\text{ren}}}} \right) \) is just a modification of the logarithmic divergence \( \ln \frac{\Lambda}{\epsilon} \) for finite values of \( a_{\text{ren}} \).

Alternatively, we may consider the limit when \( a_{\text{ren}} \) is kept finite and \( \frac{\Lambda}{\epsilon} \) goes to infinity. In this case the result reads

\[
S_0(e^{2a_{\text{ren}}}) - S_0\left( \frac{e}{\Lambda e^{a_{\text{ren}}}} \right)
\]

\[
\simeq - \ln \frac{\Lambda}{\epsilon} - \ln \frac{|a_{\text{ren}}|}{2\pi} - \ln \frac{|e^{2a_{\text{ren}}} - 1|}{e^{2a_{\text{ren}}} + 1},
\]

(35)

where the first term is a non-perturbative modification of the logarithmic UV divergence and last two terms are finite and non-perturbative with respect to \( a_{\text{ren}} \).

The above analysis suggests that the effective action (34) is an analytical function of the coupling constant \( a_{\text{ren}} \) which changes in the limits \( -\infty \leq a_{\text{ren}} \leq +\infty \). Moreover, we
can identify the points \( a_{\text{ren}} = +\infty \) and \( a_{\text{ren}} = -\infty \). The space of the coupling constant, thus, is topologically a circle.

In order to renormalize the UV divergences which remain in the effective action \( (34) \) (in addition to the divergences in the regular part \( W_{\text{reg}} \) of the action) one as usually \cite{16} should consider a bare action \( W_{a}^B \) with bare constants \( \kappa_p \) which absorb the divergences. In four dimensions, for instance, this action may have the form

\[
W_{a}^B[M^4] = \kappa_0 A(\Sigma)S_2(\epsilon^{a_{\text{ren}}}) + \kappa_1 B(\Sigma) ,
\]

where \( \kappa_0 \) absorbs \( \epsilon^{-2} \) divergence while the logarithmic \( \ln \ln \frac{\Lambda}{\epsilon} \) divergence \( (33) \) is absorbed in the renormalization of \( \kappa_1 \). The renormalized action, thus, takes the form

\[
W_{a}^{\text{ren}}[M^4] = \kappa_{0}^{\text{ren}} A(\Sigma)S_2(\epsilon^{a_{\text{ren}}}) + B(\Sigma) (\kappa_{1}^{\text{ren}} + S_0(\epsilon^{a_{\text{ren}}})) . \tag{36}
\]

The expression for \( W_{a}^{\text{ren}} \) on \( M^2 \) and \( M^3 \) derives in a similar fashion.

We complete this Section with a brief comment regarding the expression \( (33) \) for the renormalized constant \( a_{\text{ren}} \). It can be re-written as follows

\[
a(E) = \frac{a(M)}{1 + \frac{a(M)}{2}\ln \frac{E}{M}} , \tag{37}
\]

where \( a(E) \) and \( a(M) \) are values of the coupling constant \( a \) measured at the energy \( E \) and \( M \) respectively. For a positive \( a(M) \) the behavior of the running constant \( (37) \) reminds that of the coupling constant in QCD. It goes asymptotically to zero (asymptotic freedom) when energy \( E \) grows. For negative \( a(M) \) we observe a different behavior: with \( E \) growing the running coupling \( a(E) \) decreases and reaches at some critical energy \( E_{\text{cr}} \) the infinite negative value. For \( E \geq E_{\text{cr}} \) the function \( a(E) \) decreases starting from the positive infinite value till zero. The transition through the point \( E = E_{\text{cr}} \), thus, is a transition from negative to positive values of \( a \) through the infinity. A similar behavior for \( E \ll E_{\text{cr}} \) happens in QED. However, there the perturbative analysis becomes meaningless \cite{15} near the point \( E = E_{\text{cr}} \) laying in the strong coupling region. In our case we deal with an exactly solvable problem. So, the strong coupling regime is under control and, what is especially important, the points \( a = +\infty \) and \( a = -\infty \) can be identified. Therefore, the theory freely flows through the transition point. Starting with a negative value the running coupling constant becomes positive and goes to zero for large enough energies. The regime of negative \( a \), thus, is unstable and tends to transmute into the regime of positive \( a \) which is stable and asymptotically free.

6. Other examples of \( \delta \)-potential. So far we considered the case when the \( \delta \)-potential in the operator \( (1) \) is effectively two-dimensional. This is the most interesting case having a number of applications in gravitational physics. For completeness, however, we briefly consider here other examples of the \( \delta \)-potential the solution for which can be done along the lines pointed in Section 2.

6.1 One-dimensional \( \delta \)-potential. Let the operator \( (1) \) be the Laplace operator on the line \( R^1 \) and \( \delta_{\Sigma} = \delta(x) \) be the Dirac delta function concentrated at the point \( x = 0 \). As we noted in Section 2 the precise formulation for the operator \( \Delta_a \) can be done in terms of the unperturbed operator \( \Delta \) acting on the functions subject to some condition imposed
at the point \( x = 0 \). In order to find this condition we integrate the equation \(-\Delta_a f = \lambda f\) from \( x = -\epsilon \) to \( x = +\epsilon \). In the limit \( \epsilon \to 0 \) we get the condition

\[ f'(0^+) - f'(0^-) + a f(0) = 0 \]  

(38)

The resolvent of the unperturbed Laplace operator reads \( G_k(x, x') = \frac{i}{2k} e^{ik|x-x'|} \). Applying our general formula (5) we find the expression for the resolvent of the operator \( \Delta_a \):

\[ G_{k,a}(x, x') = \frac{i}{2k} e^{ik|x-x'|} - \frac{a}{2k(2k - ia)} e^{ik(|x|+|x'|)} \]  

(39)

which satisfies the condition (38). The pole structure of (39) says us that for positive values of \( a \) there exists a bound state at \( k = k_0 = \frac{ia}{2} \).

Let us consider the case of negative \( a \) first. Then the heat kernel for the operator \( \Delta_a \) is given by the contour integral (6) which can be transformed to the integral over the real axis of the complex plane

\[ K_a(x, x', t) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} dk k G_{k,a}(x, x') e^{-tk^2} . \]

For the unperturbed operator we get the well known expression

\[ K(x, x', t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk e^{ik|x-x'|} - \frac{1}{2\sqrt{\pi} t} e^{-\frac{(x-x')^2}{4t}} . \]

For the part of the heat kernel \( K_a(x, x', t) \) that is due to the perturbation after integration over \( k \) we find

\[ K_a^{(1)}(x, x', t) = \frac{a}{2\sqrt{\pi} t} \int_0^{+\infty} dy e^{ay} e^{-\frac{|x|+|x'|+2ay}{4t}} , \]

(40)

where we used that for \( a < 0 \)

\[ \frac{1}{2k - ia} = \frac{1}{i} \int_0^{+\infty} dy e^{ay+2kyn} . \]

The integral representation (40) for the kernel was derived in [17]. The integration in (40) performs explicitly and results

\[ K_a^{(1)}(x, x', t) = \frac{a e^{\frac{a^2}{4t}}}{4} e^{-\frac{a}{2}(\frac{|x|+|x'|)}{\sqrt{t}})} \left( 1 - \Phi\left( \frac{|x|+|x'|}{2\sqrt{t}} - \frac{a}{2\sqrt{t}} \right) \right) , \]

(41)

where \( \Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z dx e^{-x^2} \) is the error function.

For the trace \( TrK_a^{(1)} = \int_{-\infty}^{+\infty} dx K_a^{(1)}(x, x, t) \) we find

\[ TrK_a^{(1)} = \frac{1}{2} e^{\frac{a^2}{4t}} \left( 1 + \Phi\left( \frac{a\sqrt{t}}{2} \right) - \frac{1}{2} \right) . \]

(42)

It is worth noting that (42) is function of combination \( a\sqrt{t} \) only. Therefore, the regime of small \( a \) is compatible with the regime of small \( t \). In the limit \( t \to 0 \) (13) and (12) go to zero that is in agreement with the condition \( K_a(x, x', t \to 0) = \delta(x, x') \) for the total heat kernel. Similarly, in the limit \( t \to \infty \) we find that \( TrK_a^{(1)} \to 0 \) for \( a < 0 \).
The expressions (41) and (42) have been obtained for negative values of the coupling $a$. Now they can be analytically extended to positive $a$. This does not change the behavior of (41) and (42) for small $t$. A difference appears in the regime of large $t$. Indeed, we have

$$K_{a>0}^{(1)}(x, x', t) \simeq \frac{|a|}{2} \frac{a^2}{t} e^{-|a|(|x|+|x'|)}$$

and

$$Tr K_{a>0}^{(1)} \simeq e^{\frac{a^2}{4} t}.$$ 

This is exactly the contribution of the bound state $k_0^2 = -\frac{a^2}{4}$. For the Laplace transform of (41) we find for $a < 0$

$$L(x, x', s) = -\frac{1}{2} \int_{\epsilon^2}^\infty dt \, t e^{-s(x|+|x'|)}.$$

that is analytical continuation of the resolvent $G_{k, a}$ (38) to $k = is$. The trace then reads

$$L(s) = \frac{a}{2s(2s - a)}.$$

Being extended for $a > 0$, $L(x, x', s)$ and $L(s)$ have a pole at $s = \frac{a}{2}$ for $a > 0$. It simply means that the Laplace transform does not exist for $s < \frac{a}{2}$.

Calculating the effective action on space $M = R^1 \times \Sigma$ we find the part

$$W_a[M] = -\frac{1}{2} \int_{\epsilon^2}^\infty dt \, m^2 Tr K_a^{(1)} Tr K_{\Sigma}, \quad a < 0$$

which is due to the perturbation.

$i) \Sigma = R^1$:

$$W_a[R^1 \times R^1] = -\frac{L(R^1) |a|}{8\sqrt{\pi}} \int_{|a|/2}^\infty \frac{d\tau}{\tau^2} \left(e^{\tau^2} (1 - \Phi(\tau)) - 1\right)$$

$$\simeq \frac{a}{4\pi} L(R^1) \ln\left(\frac{|a|\epsilon}{2}\right). \quad (43)$$

$ii) \Sigma = R^2$:

$$W_a[R^1 \times R^2] = -\frac{A(R^2)}{32\pi} \frac{a^2}{\epsilon} \int_{|a|/2}^\infty \frac{d\tau}{\tau^3} \left(e^{\tau^2} (1 - \Phi(\tau)) - 1\right)$$

$$\simeq -\frac{A(R^2)}{8\pi^{3/2}} \frac{a^2}{\epsilon} + \frac{A(R^2)}{32\pi} \frac{a^2}{\epsilon} \ln\left(\frac{|a|\epsilon}{2}\right). \quad (44)$$

$iii) \Sigma = \Sigma_g^2, \quad g = 0, 1, 2, ..$

$$W_a[R^1 \times \Sigma_g^2] \simeq -\frac{A(\Sigma)}{8\pi^{3/2}} \frac{a}{\epsilon} + \frac{A(\Sigma)}{32\pi} \frac{a^2}{\epsilon} \ln\left(\frac{|a|\epsilon}{2}\right). \quad (45)$$

where $\epsilon$ is an UV cut-off.

For positive $a$ the presence of the bound state $k_0^2 = -\frac{a^2}{4}$ in the spectrum of the operator $\Delta_a$ makes the quantum theory unstable. It can be easily stabilized by adding the mass term $(\Delta_a - m^2)$ so that $m^2 > \frac{a^2}{4}$. Then the expression for the effective action reads

$$W_a = -\frac{1}{2} \int_{\epsilon^2}^\infty dt \, t e^{-m^2 t}.$$ 

The mass term does not alter, however, the behavior of the effective action (43)-(45) in the UV regime though it changes the UV finite terms. Therefore, the UV part of (43)-(45) extends to $a > 0$. 


6.2 Three-dimensional δ-function. The operator \((\Box)\) takes the form \(\Delta_a = \Delta + a\delta^3(x)\) and acts on functions on three-dimensional plane \(R^3\). In order to find the boundary condition we should impose at \(x = 0\) we integrate the expression \(-\Delta_a f = \lambda f\) over a ball of radius \(\epsilon\) surrounding the point \(x = 0\). In the limit \(\epsilon \to 0\) we obtain

\[
4\pi r^2 \partial_r f |_{r=0} +af |_{r=0} = 0 ,
\]

where \(r\) is the radial coordinate of the spherical coordinate system with center at \(x = 0\). The contribution of the perturbation to the heat kernel can be isolated from eq.\((6)\). It

In order to find the resolvent of the unperturbed three-dimensional Laplace operator is

\[
G(x, x') = \frac{e^{ik|x-x'|}}{4\pi|x-x'|} .
\]

For negative \(a\) the spectrum contains, as seen from \((47)\), a bound state \(k = k_0 = -\frac{i\pi}{4}\). Calculating the heat kernel of the unperturbed Laplace operator we find the known expression

\[
K(x, x', t) = \frac{1}{(4\pi t)^{3/2}} e^{-\frac{(x-x')^2}{4t}} .
\]

The contribution of the perturbation to the heat kernel can be isolated from eq.\((5)\). It reads (for \(a > 0\))

\[
K_a^{(1)}(x, x', t) = \frac{1}{8(\pi t)^{3/2}} \frac{1}{|x||x'|} \frac{\Phi}{\Phi}(|x|+|x'|) \int_0^\infty dz z e^{-\frac{4\pi z}{a}} e^{-\frac{z^2}{4t}} .
\]

The \(z\) integration can be performed explicitly and we find that

\[
K_a^{(1)}(x, x', t) = \frac{1}{4\pi \sqrt{\pi t}} \frac{1}{|x||x'|} e^{-\frac{(x-x')^2}{4t}}
\]

\[
-\frac{a}{|x||x'|} e^{\frac{4\pi a^2}{x^2}} e^{\frac{4\pi}{2\sqrt{t}}(|x|+|x'|)} \left(1 - \Phi\left(\frac{|x|+|x'|}{2\sqrt{t}} + \frac{4\pi a}{\sqrt{t}}\right)\right) .
\]

One can see that the coupling \(a\) appears in this formula essentially in the combination \(a^{-1}\sqrt{t}\). Therefore, the limit of small \(t\) is compatible with the limit of large \(a\). Analyzing the asymptotic behavior of \((49)\) we find that \((49)\) vanishes in the limits of small and large \(t\). Moreover, it goes to zero in the limit \(a \to +0\) as well.
For the trace $\text{Tr} K_a^{(1)} = 4\pi \int_0^\infty dx x^2 K_a^{(1)}(x, x, t)$ we find a simple expression

$$
\text{Tr} K_a^{(1)} = \frac{1}{2} e^{\frac{16a^2t}{a^2}} \left(1 - \Phi\left(\frac{4\pi}{a} \sqrt{t}\right)\right),
$$

(50)

which is a function of $a^{-1}\sqrt{t}$ only. The formulae (49) and (50) extend to negative values of the coupling $a$ by making use the identity $\Phi(-z) = -\Phi(z)$. In the limit of large $t$ then we have

$$
K_a^{(1)}(x, x', t) \simeq \frac{2}{|x||x'|} e^{\frac{16a^2t}{a^2}} e^{\frac{4\pi}{a}(|x|+|x'|)}, \quad \text{Tr} K_a^{(1)} \simeq e^{\frac{16a^2t}{a^2}},
$$

that is due to the bound state at $k = k_0 = i\frac{4\pi}{a}^{-1}$.

For the effective action on space $M = R^3 \times \Sigma$ one finds

i) $\Sigma$ is a point:

$$
W_a = -\frac{1}{2} \int_{\frac{k_0}{a}}^{+\infty} \frac{d\tau}{\tau} e^{-\tau^2 (1 - \Phi(\tau))} \simeq \frac{1}{2} \ln\left(\frac{4\pi\epsilon}{a}\right).
$$

(51)

ii) $\Sigma = R^1$:

$$
W_a = -\frac{L(R^1)}{4\sqrt{\pi}} \frac{4\pi}{a} \int_{\frac{k_0}{a}}^{+\infty} \frac{d\tau}{\tau^2} e^{-\tau^2 (1 - \Phi(\tau))} \simeq -\frac{L(R^1)}{4\sqrt{\pi}} \left(\frac{1}{\epsilon} + \frac{16\sqrt{\pi}}{a} \ln\left(\frac{4\pi\epsilon}{a}\right)\right).
$$

(52)

For negative $a$ we again should introduce the mass term ($m^2 > 16\pi^2 a^{-2}$) in order to stabilize the bound state. It does not affect the UV part of the effective action.

The leading divergence in (52) can be eliminated by adding to the action a term $W_B = -L(R^1)\sqrt{\pi} a^{-1}$ with bare coupling $a$ and defining the renormalized coupling as follows

$$
a_{\text{ren}} = \frac{a}{1 + \frac{a}{4\pi\epsilon}}.
$$

(53)

It is worth noting that the renormalization (53) as well as (33) resemble the renormalization arising in $d = 2, 3$ non-relativistic models with $\delta$-potential [18] (see also [19]).

7. General remarks. The treatment of the $\delta$-potential in a field operator, thus, includes the following steps: i) allow some sort of singular behavior at the “origin” for the field functions; ii) define value of function at the origin and re-formulate the $\delta$-potential as a “boundary condition” at the origin; iii) apply the formula (5) to get the resolvent of the perturbed operator. We demonstrated the efficiency of these steps when the $\delta$-potential is effectively one-, two- and three-dimensional. Higher-dimensional $\delta$-potentials are known to be well-defined and do not require implementation of the procedure of self-adjoint extension. It is interesting to note that $d = 1$ and $d = 3$ $\delta$-potentials are in some way dual each other. Indeed, one can see that the traces (12) and (34) of the corresponding heat kernels merge (up to an additive constant) under the “duality” transformation $a_{d=3} = -8\pi a_{d=2}^{-1}$. This sort of the $\delta$-potentials may arise due to topological defects like domain walls and vertexes. In fact, our calculation can be generalized to include arbitrary many centers producing the $\delta$-potentials (a generalization for fermionic operators is also of interest [20]). On the other hand, the problem we consider in this paper may serve as a model to analyze non-perturbatively the quantum fields near space-time singularities. As we see the quantum field theoretical quantities calculated above do not possess any
kind of singular behavior. It would be interesting to check this for space-time singularities which are stronger than $\delta$-function.

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