MANY PARTICLE HAMILTONIAN FOR THE FRACTIONAL QUANTUM HALL EFFECT

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Abstract

A many-particle Hamiltonian is proposed in order to explain the fractional quantum Hall effect (FQHE) for fractional filling factors $\nu < 1$. The solutions of the corresponding Hartree-Fock equations make it possible to discuss the FQHE from the point of view of the single quasi-particle energy spectrum. It is shown how the specific couplings in the many-particle Hamiltonian depend on the magnetic field and the area density of electrons. The degeneracies of the quasi-particle states are related to the fractional filling factors $\nu$. It is suggested that the energy gaps obtained in the quasi-particle energy spectrum are comparable with the experimentally measured quantities. An explicit calculation for the FQH - conductance is given and its character as a topological invariant is discussed.

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1. INTRODUCTION

It is the purpose of this paper to find a form of the many-particle Hamiltonian, which is relevant to the fractional quantum Hall effect (FQHE). In two-dimensional electron systems subjected to huge magnetic fields at low temperatures, the FQHE is characterized by the presence of the plateau in the Hall resistance $\rho_{xy}$ quantized to $(h/e^2)/(p/q)$, and the concurrent minima in the diagonal resistivity $\rho_{xx}$. The characteristic features of the FQHE occur at the fractional filling factor $\nu = \frac{p}{q}$, which is by definition the ratio of the number of electrons $N_e$ to the Landau-level degeneracy $N$. Apart from the quantized rational numbers $\nu = \frac{p}{q}$, other interesting quantities measured in the FQHE are the energy gaps $\Delta_{p/q}$, which are determined by the temperature dependence of $\rho_{xx}$. These two quantities $\nu = \frac{p}{q}$ and $\Delta_{p/q}$ should be explained in order to understand the FQHE.

Before we proceed further to the FQHE, it is instructive to consider the classical Hall effect and the integer QHE. It is the Drude model that reveals the overall magnetic field dependence of $\rho_{xy}$.

$$\rho_{xy} = \frac{B}{ne^2} .$$  
(1.1)

This is the essence of the classical Hall effect$^1$. Here $n$ denotes the area density of electrons and $B$ is the magnetic field. The well-known result for the degeneracy $N$ in the Landau theory is used in order to write $\nu$ in terms of $n$ and $B$.

$$\nu = \frac{N_e}{N} = \frac{nhc}{Be} .$$  
(1.2)

In the quantum mechanical view, we find the expression of $\rho_{xy}$ in terms of the fundamental constants and the filling factor:

$$\rho_{xy} = \frac{h}{e^2 \nu} .$$  
(1.3)

The plateau of $\rho_{xy}$ at integer values of $\nu$ are observed, and are called the integer quantum Hall effect$^2$.

The nature of the integer values of $\nu$ in the IQHE is readily understood in the view of the Landau-level. The completely filled states are associated with the IQHE. It is a generally accepted fact that the plateau of $\rho_{xy}$ are due to the localization. In the presence of random impurities, electrons in the
system are classified into the two kinds: the extended states and the localized states. Since it seems true that the completely filled states are energetically favorable, electrons are absorbed into or supplied from the set of the localized states in order for the system to stay in the completely filled states. In this sense, the magnitude of $n/B$ is stuck and the plateau appear.

Let us continue to discuss the FQHE. A few years after the discovery of the IQHE, fractional values of $\nu$ were observed with a very high mobility sample\textsuperscript{3}. The observed fractional filling factors\textsuperscript{4,5,6} below 1 are expressed in terms of three quantum numbers $n, s,$ and $l$ ($n \in \mathbb{Z}_+ \equiv \{1, 2, 3 \ldots\}; s = \pm 1; l \in \mathbb{Z}_+$):

$$\nu = \begin{cases} \frac{l}{2nl+s} & \text{for } \nu < \frac{1}{2} \\ 1 - \frac{l}{2nl+s} & \text{for } \frac{1}{2} < \nu < 1. \end{cases}$$ \hspace{1cm} (1.4)

This peculiar pattern of $\nu$ seems most likely to be the consequence of a many-body effect.

Laughlin\textsuperscript{7} proposed a many-body wave function to describe the FQHE. Although the Laughlin wave function incorporates the correct view of the many-body effect, an explicit derivation of the wave function from a Hamiltonian is not known. In particular, a clear explanation of the quantum numbers in $\nu$ cannot be found in the hierarchical constructions\textsuperscript{8} based on the Laughlin wave function. Furthermore, the wave function approach does not uniquely determine the energy gaps at $\nu = \frac{p}{q}$ as functions of $p$ and $q$\textsuperscript{9}.

In explanations of the FQHE is there another nonperturbative approach using the topological invariant?\textsuperscript{10} It is shown that, whenever the Fermi level lies in a gap, the Hall conductance can be written as a topological invariant. The meaning of the topological invariant is that the quantized value is unchanged under small variations of interactions as long as there is still an energy gap above the Fermi level. It is argued in this topological approach that the many-particle ground state for the FQHE should have the proper degeneracy. Although this approach seems to explain the nature of the plateaux and the requirement of energy gaps and degeneracy, the selection rule for the rational value of $\nu$, like for instance the odd integer in the denominator is still missing. Also the origin of the gaps and the degeneracy comes into question in this approach.\textsuperscript{11}

In this paper we introduce a many-particle Hamiltonian written in the second quantization formalism, presenting a quite different theoretical frame-
work from the wave function approach. Starting from the many-particle Hamiltonian, we explain the pattern of \( \nu = p/q \), obtain the quantity of \( \Delta_{p/q} \) and a reasonable value for the degeneracy of the many-particle ground state, with which the Hall conductance as a topological invariant is discussed. Our procedure is summarized as follows.

In Sec. 2, in order to implement the idea that, when corresponding states are completely filled, the FQHE occurs as the IQHE does, we propose a many-particle Hamiltonian, which modifies the Landau-level. We focus our attention on the FQHE with \( \nu < 1 \) only. We restrict the one-particle states, which compose the \( N \)-dimensional space, and write the many-particle Hamiltonian with the coupling \( E(t) \). Applying the Hartree-Fock method, we find the quasi-particle energy spectrum, which is essential for the next Section.

The spectrum is fully discussed in Sec. 3 for the case of a given specific form of \( E(t) \), where parameters \( \Delta \) and \( J \) are involved. The essential results are the degeneracies and the energy gaps of the states, which are related to the experimentally measurable quantities. It is deduced how the detailed form of \( J \) depends on \( N \) and \( N_e \). Although the two quantum numbers \( n \) and \( s \) in \( \nu \) are unfortunately beyond our understanding, the origin of the quantum number \( l \) is explained quite naturally in our theory. In addition, our theory also provides us with the energy gap \( \Delta_{p/q} = N_e \Delta/p = N \Delta/q \) at \( \nu = p/q \). It is also found that the degeneracy of the many-particle ground state at \( \nu = p/q \) is given by \( qC_p = q!/p!(q - p)! \). We notice that there is symmetry breaking in the many-particle ground state.

By using the energy gap and the degeneracy of the many-particle ground state, we discuss the Hall conductance as a topological invariant in Sec. 4.

We conclude in Sec. 5. In order to see the consistency of the concept that completely filled states are associated with each \( \nu \), we present a spectrum-like figure in the Appendix.

2. MANY-PARTICLE HAMILTONIAN AND THE HARTREE-FOCK EQUATIONS

One of the aims in our study of the FQHE is to understand the observed rational numbers. To do so, our first task is to derive common theoretical properties, which the QHE systems all share. Presenting our conclusion first, the essential point is that completely filled states are associated with the QHE systems at the observed filling factors. In order to understand this fact, we
begin by considering the Landau-level, which is used in the explanation of the IQHE.

The Hamiltonian\textsuperscript{12} $H_{\text{mag}}$ for an electron in the magnetic field $B$ is written as

$$H_{\text{mag}} = \frac{1}{2} \nabla_i^2 + |A|^2,$$  \hspace{1cm} (2.1)

where we set all constants to be units $m = e = h = c = 1$ as usual. Solving the corresponding Schrödinger equation, we impose the twisted doubly periodic boundary conditions requiring that the eigenfunctions are unchanged under translation by $L$ in the $x$ or $y$ directions up to a gauge transformation. The torus geometry is introduced by these conditions. The state space is found and characterized by the Landau-levels, where each level has the degeneracy

$$N = \frac{BL^2}{2\pi}.$$  \hspace{1cm} (2.2)

The energy gap between adjacent levels is given by

$$\Delta_{\text{mag}} = B.$$  \hspace{1cm} (2.3)

If the first Landau-level is completely filled at a sample-dependent magnetic field $B_0 = N_e 2\pi/L^2$, then the system at $B_0/i$ ($i$ = integer) with a fixed number of electrons, corresponds to states where electrons are filled up to the $i^{th}$ Landau-level. The Landau-level also shows that, for the IQHE systems at the completely filled states, there are energy gaps which are required when an electron in the system goes to another state. We deduce that there is no collision in the IQHE system at the completely filled states, because there is no place to go after collision without paying the energy gap cost at a very low temperature. The drift velocity with no collision explains the experimentally observed minima of $\rho_{xx}$.

Although we have omitted discussing the spin effect in the above because we are eventually interested in the region of $\nu < 1$ where the spin effect is absent, we can conclude that the completely filled states are associated with the IQHE. In order to extend this idea to the FQHE, we should modify the energy spectrum. In order to understand the fractional filling factors, we consider a many-particle Hamiltonian.

A master Hamiltonian $H_{\text{tot}}$ for the QHE would contain the term of localization $H_{\text{loc}}$ as well as the term of many-particle interaction $H_{\text{int}}$:

$$H_{\text{tot}} = H_0 + H_{\text{int}} + H_{\text{loc}}.$$  \hspace{1cm} (2.4)
The Hamiltonians $H_{\text{int}}$ and $H_{\text{loc}}$ represent electron-magnetic field, electron-electron and electron-random impurity interactions respectively. The observed plateaux are explained by using $H_{\text{loc}}$, which allows flexibility of the number of the extended state electrons carrying the current, and makes the QHE system prefer to stay at completely filled states. In other words, the persistance of the plateaux is attributed respectively to the localization of excess quasi-particles, and to the transfer of localized states into a level in order to keep it completely filled. As far as the filling factors $\nu$ are concerned, we can drop the term $H_{\text{loc}}$, making an ideal system.

In order to discuss the idealized many-particle Hamiltonian, $H_0 + H_{\text{int}}$, in the second quantization formalism, the one-particle state space of the Hamiltonian $H_{\text{mag}}$ is considered. We denote the states of the Landau-level using the two quantum numbers as $|i, j\rangle$ with the periodicity condition $|i, j\rangle = |i, j + N\rangle$ for the $i^{th}$ Landau-level. One-particle creation operators are defined as

$$c_{ij}^\dagger |0\rangle = |i, j\rangle,$$

where $|0\rangle$ is the vacuum state. In the second quantization formalism, the term $H_0$ is written as

$$H_0 = \sum_{ij} H_{\text{mag}} = \sum_{ij} \epsilon_i c_{ij}^\dagger c_{ij}.$$

We adopt two-particle interactions for $H_{\text{int}}$, guessing that the total quantum numbers are preserved during the process of the interactions. This guess is made concrete by writing $H_{\text{int}}$ in terms of the creation operators as

$$H_{\text{int}} = \sum_{i, m, r, j, k, t} E(r, t) c_{i+r, j}^\dagger c_{m-r, k-t}^\dagger c_{m} c_{ij}.$$

It is enough to consider only the first Landau-level at a very low temperature if the degeneracy of the Landau-level $N$ is so large that all electrons are in the first level. The reason is that the energy gap, proportional to the magnetic field, is big and the Boltzmann factor for the excitation which is extremely small at very low temperatures. Hence, as far as $N > N_e$, we drop the quantum number $i$, which corresponds to the $i^{th}$ Landau-level, in the Hamiltonian. As a result, the following reduced many-particle Hamiltonian $H_{\text{red}}$ is our starting point for explanation of the FQHE of $\nu < 1$:

$$H_{\text{red}} = \sum_{j=1}^{N} \epsilon c_{j}^\dagger c_{j} + \frac{1}{2} \sum_{j, k, t} E(t) c_{j}^\dagger c_{k-1}^\dagger c_{k} c_{j}.$$
Here, an unspecified value $\epsilon$ is the energy of the first Landau-level. It is remarkable that $\epsilon$ is independent of the quantum number $j$. We will see that this independence leads to a simple calculation.

It is the Hartree-Fock method that is useful for approximating the ground state of a system of $N_e$ interacting fermions. The method, without detailed derivation, is used in the following discussion. A more complete derivation can be found in many textbooks.\textsuperscript{13}

If we represent the Hamiltonian in the new one-particle basis $b^\dagger_\alpha$,

\[ b^\dagger_\alpha = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \exp \left( i \frac{2\pi}{N} \alpha j \right) c^\dagger_j, \quad (2.9) \]

the interaction term of Eq. (2.8) is diagonalized as

\[ H_{\text{red}} = \sum_{\alpha=1}^{N} \epsilon b^\dagger_\alpha b_\alpha + \frac{1}{2} \sum_{\alpha\beta} D(\alpha - \beta) b^\dagger_\alpha b^\dagger_\beta b_\beta b_\alpha, \quad (2.10) \]

where the couplings $D(\alpha - \beta)$ are given by

\[ D(\alpha - \beta) = \sum_{t=1}^{N_e} E(t) \exp \left( -i \frac{2\pi}{N} t(\alpha - \beta) \right). \quad (2.11) \]

The essential part of the method is to solve the so-called Hartree-Fock equations corresponding to the eigenvalue problem. For our case of Eq. (2.10), the Hartree-Fock equations are written as

\[ \sum_{\beta=1}^{N_e} \sum_{i=1}^{N_e} \{ <t_i|\beta > D(\alpha - \beta) <\beta|t_i > <\alpha|k_j > \\
- <t_i|\beta > D(\alpha - \beta) <\alpha|t_i > <\beta|k_j > \} = (\epsilon(k_j) - \epsilon) <\alpha|k_j >, \quad (2.12) \]

where the states with Greek indices are given by $|\alpha > = b^\dagger_\alpha |0 >$, while the states with Latin indices $|k_j >$ are eigenstates of the effective one-particle Hamiltonian. It should be emphasized that the states $\{|t_i > |i = 1, 2, \ldots, N_e\}$ must be consistently chosen from among $\{|k_j > |j = 1, 2, \ldots, N\}$ in such a way that the expectation value of $H_{\text{red}}$ is minimized. Substituting our Ansatz,

\[ <\alpha|j > = \frac{1}{\sqrt{N}} \exp \left( -i \frac{2\pi}{N} \alpha j \right), \quad (2.13) \]
where the factor $\frac{1}{\sqrt{N}}$ is determined by the requirement of $\sum_{\alpha} a^{\dagger}_l \alpha < \alpha | k > = \delta_{lk}$ we notice the consistency, and obtain the solution, that is, the quasi-particle energy spectrum:

$$\epsilon(k) = \epsilon + N_e E(N) - \sum_{i=1}^{N_e} E(k_j - t_i). \tag{2.14}$$

We notice from Eqs. (2.9) and (2.13) that the effective one-particle states $|k_j >$ are related to the original creation operators as

$$|k_j > = c_{k_j}^{\dagger} |0 >. \tag{2.15}$$

This fact shows that the Hartree-Fock approximation, in our case, is equivalent to the first order perturbation theory with the ground state

$$|\Psi_g > = \prod_{i=1}^{N_e} c_{t_i}^{\dagger} |0 >. \tag{2.16}$$

The accidental equivalence is due to the fact that $\epsilon$ is independent of the quantum number $j$, that is, the high degeneracy of the Landau-level.

In terms of the eigenvalues $\epsilon(k_j)$, the ground state energy $E_g$ is given by

$$E_g = < \Psi_g | H_{red} | \Psi_g > = \frac{1}{2} \sum_{i=1}^{N_e} \{ \epsilon(t_i) + \epsilon \} \tag{2.17}$$

This expression provides us with the general rules for choosing the set $\{ t_i | i = 1, \ldots, N_e \}$. The rules are that the individual $\epsilon(k_j)$ are minimized as much as possible, and that the following set equation holds

$$\{ t_i | i = 1, 2, \ldots, N_e \} = \{ k_j | j = 1, 2, \ldots, N_e \} \tag{2.18}$$

for $\epsilon(k_1) \leq \epsilon(k_r) \leq \epsilon(k_2) \leq \ldots \leq \epsilon(k_N)$.

An excited state of our system can be approximately described by the state, $c_{j}^{\dagger} c_{k} | \Psi_g >$, where $k$ and $j$ label occupied and unoccupied single-particle states, respectively. The excitation energy of the system is given by

$$< c_j^{\dagger} c_k \Psi_g | H_{red} | c_j^{\dagger} c_k \Psi_g > = E_g + \epsilon(j) - \epsilon(k). \tag{2.19}$$

Since we will see that the last two terms are negligible for our specific case, the energy gap for the excitation is $\epsilon(j) - \epsilon(k)$. 

8
Let us make a comment about the states \( c_j^\dagger |0 > \) and \( b_\alpha^\dagger |0 > \), and about their relation with some operators. The doubly periodic boundary conditions discussed in order to compactify the two-dimensional space are expressed by means of the so-called magnetic translation operators \( S \) and \( T \), which satisfy the commutation relation:\(^{14}\)

\[
ST = \exp \left( \frac{2\pi i}{N} \right) TS . \tag{2.20}
\]

We introduce the states \( |j > \) as one of representations of the commutation relation, setting that

\[
S|j >= \exp \left( \frac{2\pi i}{N} j \right) |j > , \tag{2.21}
\]

\[
T|j >= |j + 1 > . \tag{2.22}
\]

If the eigenstates \( |j > \) of \( S \) are the one-particle states created by the operator \( c_j^\dagger \), that is, \( |j >= c_j^\dagger |0 > \), then \( b_\alpha^\dagger |0 > \) are eigenstates of \( T \):

\[
Sc_j^\dagger |0 >= \exp \left( \frac{\pi i}{N} j \right) c_j^\dagger |0 \Rightarrow T b_\alpha^\dagger |0 >= \exp \left( \frac{2\pi i}{N} \alpha \right) b_\alpha^\dagger |0 > . \tag{2.23}
\]

For the case that the one-particles state \( b_\alpha^\dagger |0 > \) are the eigenstates of \( S \), the similar results are obtained.

3. QUASI-PARTICLE ENERGY SPECTRUM, J-ASSUMPTION, FRACTIONAL FILLING FACTORS, ENERGY GAPS, AND DEGENERACY OF THE MANY-PARTICLE GROUND STATE

We have studied the many-particle Hamiltonian using the Hartree-Fock method. It is an important result that the Hartree-Fock equations are solved exactly and that the corresponding quasi-particle energy spectrum is obtained. Another important argument in the previous section is that the completely filled states are associated with the fractional filling factors. In this section, we discuss the spectrum written in terms of the parameters \( E(t) \) of our theory, in order to explain the completely filled states at \( \nu \). As a by-product, the energy gap at each \( \nu \) can be calculated and be compared with other gaps at a different \( \nu \). The degeneracy of the many-particle ground state is also considered.
In general, the fact of \( |j > = |j + N > \) shows that \( E(t) \) has the property \( E(t) = E(t + N) \), where \( N \) can be treated as a number of equally spacing points on a circle. For convenience in later discussions, we introduce a shorthand notation for dividing set according to residues:

\[
[r]_q \equiv \{qm + r| m = 1, 2, \ldots, \frac{N}{q}\},
\]

(3.1)

\[
U_{r=0}^{q-1}[r]_q = \{1, 2, 3, \ldots, N\}. \tag{3.2}
\]

Before we state the form of \( E(t) \) which is relevant to the FQHE, we study an example showing how the quasi-particle spectrum depends on \( N \) and \( N_e \).

Let us consider the case where, with a positive value \( \Delta \), \( E(t) \) is given by

\[
E(t) = \begin{cases} \Delta & \text{for } t \in [0], \\ 0 & \text{for otherwise} \end{cases} \tag{3.3}
\]

In dealing with the quasi-particle energy spectrum of Eq. (2.14), our main concern is to find the occupied states, that is, the set \( \{t_i|i = 1, 2, \ldots, N_e\} \), which minimizes the ground state energy. The property of the algebra,

\[
a - b \in [0] \quad \text{for } a, b \in [r]_q, \tag{3.4}
\]

plays a role in understanding the fact that, for the given \( E(t) \) of Eq. (3.3), the quasi-particles primarily prefer to occupy the states of the same residue. For instance, for \( l\frac{N}{q} \leq N_e \leq (l + 1)\frac{N}{q} \), the occupied states are given by

\[
\{t_i|i = 1, \ldots, N_e\} = U_{\alpha=1}[r_{\alpha}]_q U\{t'_i|i = 1, 2, \ldots, N_e-lN/q, \text{ and } t'_u \in [r_l+1]_q\}, \tag{3.5}
\]

We notice that there is a symmetry breaking in the many-particle ground state where \( r_{\alpha} \) is equal to one of the \( q \) different residues. We find the energy spectrum in the form

\[
\epsilon \in (k_j) = \begin{cases} 
\epsilon + N_c\Delta - (N/q)\Delta & \text{for } k_j \in U_{\alpha=1}[r_{\alpha}]_q \\
\epsilon + N_c\Delta - (N_e-lN/q)\Delta & \text{for } k_j \in [r_l+1]_q \\
\epsilon + N_c\Delta & \text{for otherwise}
\end{cases} \tag{3.6}
\]

where \( E(N) = \Delta \), and the corresponding degeneracies are given by \( l\frac{N}{q}, \frac{N}{q}, \) and \( N - (l + 1)\frac{N}{q} \), respectively. We notice that for \( N_e = lN/q \), the excitation energy is given by \( (N/q)\Delta \).
It is easy to find that the energy gaps and the degeneracies are independent of the choice of residues for the occupied states. The freedom in choosing residues gives rise to a nonzero degeneracy of the many particle ground state. In fact, for \( N_e = lN/q \), this degeneracy is equal to the number of possible combinations of choosing \( l \) out of \( q \):

\[
qC_l = \frac{q!}{l!(q-l)!}
\]  

(3.7)

Approaching further our problem of the FQHE, we introduce a slightly different case, where \( E(t) \) is given by

\[
E(t) = \begin{cases} 
\Delta & \text{for } t \in [0]_{N/q} \\
0 & \text{otherwise}
\end{cases}
\]  

(3.8)

The nature of the number \( J \) will be discussed in detail later. Also, for a while, we do not worry about whether or not \( N/J \) is an integer. An integer value of \( N/J \) will be recovered when we discuss the fractional filling factors.

As we did in the above example for the \( E(t) \) of Eq. (3.8) and \( lJ \leq N_e \leq (l+1)J \), we obtain the quasi-particle energy spectrum described in Fig. 1, replacing \( \frac{N}{q} \) in Eq. (3.6) formally with \( J \). .

In the hole-dominant region where \( N < 2N_e \), we manipulate the spectrum of Eq. (2.14):

\[
\epsilon(k_j) = \epsilon + N_eE(N) - \sum_{i=1}^{N} E(k_j - t_i) + \sum_{i=1}^{N-N_e} E(k_j - \tilde{t}_i)
\]  

(3.9)

where the sets are related to each other

\[
\{\tilde{t}_i|i = 1, 2, \ldots, N - N_e\} = \{1, 2, \ldots, N\} - \{t_i|i = 1, 2, \ldots, N_e\}
\]  

(3.10)

The complementarity property implies that the original condition of Eq (2.18) will be satisfied if we find a set such that

\[
\{\tilde{t}_i|i = 1, 2, \ldots, N - N_e\} = \{k_i|i = 1, \ldots, N - N_e\}
\]  

(3.11)

for \( \epsilon(k_1) \geq \epsilon(k_2) \geq \ldots \geq \epsilon(k_N) \). Considering the last term of Eq. (3.9) for \( E(t) \) of Eq. (3.8) and \( lJ \leq N - N_e \leq (l+1)J \), we find the three energy levels

\[
\epsilon(k_j) = \begin{cases} 
\epsilon + N_e\Delta - J\Delta + J\Delta & \text{for } \alpha \in U^l_{\alpha=1}[r_\alpha]_{N/J} \\
\epsilon + N_e\Delta - J\Delta + (N - N_e - lJ)\Delta & \text{for } k_j \in [r_{i+1}]_{N/J} \\
\epsilon + N_e\Delta - J\Delta & \text{otherwise}
\end{cases}
\]  

(3.12)
where $E(N) = \Delta, \sum_{i=1}^{N} E(k_i - t_i) = J\Delta$, and the corresponding degeneracies are given by $lJ, J$, and $N - (l + 1)J$, respectively (see Fig. 2). We notice the similarity when we manipulate from $lJ \leq N - N_e \leq (l+1)J \leq N_e \leq N - lJ$ and compare $\epsilon + N_e\Delta - (N_e - (N - (l+1)J))\Delta$ in Eq. (3.12) with $\epsilon + N_e\Delta - (N_e - lJ)\Delta$ for $lJ \leq N_e \leq (l+?)J$ in Eq. (3.6).

The interesting feature of the quasi-particle energy spectrum is its degeneracies, which are integer multiples of the number $J$. This is a clue for explanation of the quantum number $l$ in the fractional filling factors. Here, describing the important part of the correct form of $E(t)$ relevant to the FQHE, we postulate how $J$ depends on $N$ and $N_e$. It is “assumed” that, in terms of $N, N_e$ and the quantum number $n \in \mathbb{Z}$, the number $J$ is written as

$$J = \begin{cases} |N - 2nN_e| \leq N_e & \text{for } N > 2N_e \quad \text{(electron)} \\ |N - 2n(n - N_e)| \leq N - N_e & \text{for } N_e < N < 2N_e \quad \text{(hole)} \end{cases}$$

(3.13)

The feature of the absolute value seems to be related to the quantum number $s = \pm 1$. We let

$$J = \begin{cases} s(N - 2nN_e) & \text{for } N > 2N_e \quad \text{(electron)} \\ s(N - 2n(N - N_e)) & \text{for } N_e < N < 2N_e \quad \text{(hole)} \end{cases}$$

(3.14)

where, from now on, the electron and hole represent the type of dominance as indicated in Eq. (3.13). We notice that the condition of inequality in Eq. (3.13) uniquely determines the values of $n$ and $s$ for any $N$ and $N_e$. The above statement, in brief, is elucidated in Fig. 3, where the corresponding $n$ and $s$ are shown. We now call the dependence the “$J$-assumption”. Actually, the $J$-assumption is the matter that we should explain for complete understanding of the FQHE. We guess that the number $J$ may be related with the net magnetic flux of the system. The circular motions of electrons under the external magnetic field becomes the source of currents which produce the magnetic field. In this view, each electron is considered to carry $2n$ flux quanta. A suggestion that electrons carry $2n$ flux quanta is found in Ref. 15. At any rate, at this stage it is not known how to prove the $J$-assumption, so we shall leave it as an open problem.

Although there are unexplained areas in the above discussion, we have set up all the equipment for connecting completely filled states to fractional filling factors. We notice from the spectra of Figs. 1 and 2, that completely
filled states correspond to the following equation

\[ l J = \begin{cases} \frac{N_e}{N} & \text{(electron)} \\ \frac{N - N_e}{1 - \frac{l}{2n+l+s}} & \text{(hole)} \end{cases} \quad (3.15) \]

Substituting Eq. (3.14) for \( J \) in Eq. (3.15), we obtain the fractional filling factors at the completely filled states:

\[ \frac{N_e}{N} = \nu_{nsl} = \begin{cases} \frac{l}{2n+l+s} & \text{(electron)} \\ 1 - \frac{l}{2n+l+s} & \text{(hole)} \end{cases} \quad (3.16) \]

The spectrum shows that it becomes a two-energy level system for completely filled states, and also provides us with the energy gaps \( \Delta_{nsl} \):

\[ \Delta_{nsl} = J \Delta = \begin{cases} \frac{N_e}{N} \Delta = \frac{N}{2n+l+s} \Delta = \frac{N}{(2n-1)l+s} \Delta & \text{(electron)} \\ \frac{N - N_e}{(2n-1)l+s} \Delta = \frac{N}{2n+l+s} \Delta & \text{(hole)} \end{cases} \quad (3.17) \]

Here, Eqs. (3.15) and (3.16) have been used. In a unified expression, the energy gaps are written

\[ \Delta_{p/q} = \frac{N_e \Delta}{p} = \frac{N \Delta}{q} \quad \text{at} \quad \nu = \frac{p}{q}. \quad (3.18) \]

We notice \( N/J = q \) from Eq. (3.17). As we anticipate in Eq. (3.8), the integer value of \( N/J = q \) is recovered.

Although several experiments are performed in order to measure the energy gaps, unfortunately we cannot find any experiment which verifies the correctness of Eq. (3.18). However notable data related to the energy gaps are found in Refs. 4 and 16 as shown in Table 1. The data in Table 1 show that \( \Delta \) cannot be a constant. We should now consider how \( \Delta \) depends on \( N \) and \( N_e \). For instance, we assume that \( \Delta \) depends on \( J \) linearly:

\[ \Delta = \delta J (\delta = \text{constant}). \quad (3.19) \]

Then, Eq. (3.18) follows

\[ \Delta_{p/q} = J \Delta = \delta J^2 = \delta \left( \frac{N_e}{p} \right)^2 = \delta \left( \frac{N}{q} \right)^2. \quad (3.20) \]

If \( \delta \) is fixed by using one of data, then the other of data provide us with the theoretical values, \( \Delta_{2/3} = 0.432K \) and \( \Delta_{3/2} = 0.472K \), which are compared
with 0.38K and 0.5K respectively. Disorder seems to broaden the bands of the states in the spectrum of Fig. 1. This fact results in a reduction of the magnitude of the energy gaps. However, in this view the value of \( \Delta_{2/3} = 0.432K \) is inappropriate, so that more considerations of \( \Delta \) or of other experiments are required.

Using the fact that nothing depends on the choices of the residues \( r_\alpha \) for the occupied states, we find the degeneracy of the many-particle ground state for the couplings of Eq. (3.8). As we discussed for the case of Eq. (3.3), the degeneracy \( d \) of the ground state for \( \nu = p/q \) is given by (see Eq. (3.7))

\[
d = q C_p = \frac{q!}{p!(q-p)!}
\]

(3.21)

The degeneracy of the many-particle ground state is one of the conditions which is required in order to write the Hall conductance of the FQHE as a topological invariant form.

4. QUANTUM HALL CONDUCTANCE AS A TOPOLOGICAL INVARIANT

The derivation of Eq. (1.3) in the Introduction is problematic because it can not be justified to mingle the two results of classical and quantum mechanics corresponding to Eqs. (1.1) and (1.2) respectively. However there is a full quantum mechanical derivation of the Hall conductance \( \sigma \) using the Kubo formula. In fact, it is found in Ref. 10 that \( \sigma \) is written in the integral form as

\[
\sigma = \frac{e^2}{h} \frac{1}{2\pi i} \int_0^{2\pi} d\theta_x \int_0^{2\pi} d\theta_y \left[ \langle \frac{\partial \phi_0}{\partial \theta_x}, \frac{\partial \phi_0}{\partial \theta_y} \rangle - \langle \frac{\partial \phi_0}{\partial \theta_y}, \frac{\partial \phi_0}{\partial \theta_x} \rangle \right],
\]

(4.1)

where \(|\phi_0>\) is the gauge-transformed many-particle ground state:

\[
\phi_0 = \exp \left( -i \frac{\theta_x}{L} (x_1 + \ldots + x_{N_e}) - i \frac{\theta_y}{L} (y_1 + \ldots + y_{N_e}) \right) \psi_g
\]

\[
\phi_0 \equiv < x_1, y_1, \ldots, x_{N_e}, y_{N_e} | \phi_0 >, \psi_g \equiv < x_1, y_1, \ldots, x_{N_e}, y_{N_e} | \psi_g >.
\]

(4.2)

For convenience, we have let the magnetic field \( B \) be a unit in this section. The two parameters \( \theta_x \) and \( \theta_y \) are introduced when the twisted doubly periodic boundary conditions are imposed on the many-particle ground state.
such as for the Landau gauge,

\[
\psi_g(x_1, \ldots, x_i + L, \ldots, y_{N_e}) = \exp(i\theta_x)\psi_g(x_1, \ldots, x_i, \ldots, y_{N_e})
\]

(4.3)

\[
\psi_g(x_1, \ldots, y_i + L, \ldots, y_{N_e}) = \exp(i\theta_y)\exp(-iLx_i)\psi_g(x_1, \ldots, y_i, \ldots, y_{N_e})
\]

(4.4)

It is shown in Ref. 10 that the value of \(\sigma\) is always an integer multiple of \(e^2/h\) and a topological invariant as long as the many-particle ground state is nondegenerate, and is separated from the excited states by a finite energy gap.

In this section we shall calculate \(\sigma\) by using \(\psi_g >\) of Eq. (2.16), and study the property of a topological invariant. Although the condition of the finite energy gap is satisfied in our case of \(\nu = p/q\), the degeneracy of the many-particle ground state is given by \(qC_p\), so that we expect a noninteger value of \(\sigma\). Since the many-particle ground state of Eq. (2.16) in the coordinate representation is given by the Slater determinant, it is important to find the coordinate representation of the state created by the operator \(c_l^\dagger\). Here, we suppose that \(c_l^\dagger|0\rangle\) are identified as eigenstates of the magnetic translation operator \(S\). Then we can use the solutions of \(<x, y|c_l^\dagger|0\rangle\) found in Ref. 12: explicitly

\[
<x \cdot y|c_l^\dagger|0\rangle = \exp\left(-1/2y^2\right) \frac{1}{\sqrt{L\pi^{1/4}}} \sum_{k \in \mathbb{Z}} \exp \left(\frac{\pi N(k + l/N + \frac{\theta_x}{2\pi N})^2}{2} + i2\pi N(k + l/N + \frac{\theta_x}{2\pi N})(\omega/L - \frac{\theta_y}{2\pi N})\right)
\]

\[
\equiv \exp(-1/2y^2)f_l(\omega; \theta_x, \theta_y)
\]

(4.5)

where \(\omega = x + iy\), and \(L^2 = 2\pi N\) is assumed in order to satisfy the condition that the total flux through the surface of the torus is an integer in magnetic units. The coordinate representation of the many-particle ground state is written as the Slater determinant with the totally antisymmetric tensor \(\epsilon_{ij...k} = \pm 1:\)

\[
\psi_g = \frac{1}{\sqrt{N_e!}} \exp\left(-1/2(y_1^2 + \ldots + y_{N_e}^2)\right)
\]

\[
\sum_{i_1,...i_{N_e}} \epsilon_{i_1...i_{N_e}} f_{t_{i_1}}(x_1, y_1) \ldots f_{t_{i_{N_e}}}(k_{N_e}, y_{N_e})
\]

(4.6)

By using the properties of \(f_l\):

\[
f_l(\omega + L; \theta_x, \theta_y) = \exp(i\theta_x) f_l(\omega; \theta_x, \theta_y)
\]

(4.7)
\[ f_l(\omega + iL; \theta_x, \theta_y) = \exp(i\theta_y) \exp(1/2L^2 - iL\omega) f_l(\omega; \theta_x, \theta_y) \quad (4.8) \]

We notice that \( \psi_g \) satisfies the twisted doubly periodic boundary conditions of Eqs. (4.3) and (4.4). Furthermore, the functions \( f_l \) transform for the variations of \( \theta_x \) and \( \theta_y \) as follows:

\[ f_l(\omega; \theta_x + 2\pi, \theta_y) = f_{l+1}(\omega; \theta_x, \theta_y) \quad (4.9) \]

\[ f_l(\omega; \theta_x, \theta_y + 2\pi) = \exp(-i2\pi l/N - i\theta_x/N) f_l(\omega; \theta_x, \theta_y) \quad (4.10) \]

As a preparation for calculation of \( \sigma \), we find that

\[
\int_0^L dx \int_0^L dy \exp(-y^2) \left[ \exp \left(-i \frac{\theta_x}{L} x - i \frac{\theta_y}{L} y \right) f_j \right]^* \left\{ \frac{1}{\partial \phi_0^* \partial \theta_x} \right\} \\
\left[ \exp \left(-i \frac{\theta_x}{L} x - i \frac{\theta_y}{L} y \right) f_l \right] = \left\{ \frac{\delta_{jl}}{L^2 \theta_y} \delta_{jl} \right\} \\
(4.11)
\]

Now let us carry out the calculation of \( \sigma \)

\[
\sigma = \frac{e^2}{h} \frac{1}{2\pi i} \int_0^{2\pi} d\theta_x \int_0^{2\pi} d\theta_y \int_0^L dN_x x \int_0^L dN_y y \left( \frac{\partial \phi_0^* \partial \phi_0}{\partial \theta_x \partial \theta_y} - \frac{\partial \phi_0^* \partial \phi_0}{\partial \theta_y \partial \theta_x} \right) \\
= \frac{e^2}{h} \frac{1}{2\pi} \int_0^{2\pi} d\theta_x \int_0^{2\pi} d\theta_y \left( \frac{\partial}{\partial \theta_x} < \phi_0^* \frac{\partial \phi_0}{\partial \theta_y} > - \frac{\partial}{\partial \theta_y} < \phi_0^* \frac{\partial \phi_0}{\partial \theta_y} > \right) \\
(4.12)
\]

where we have used the shorthand notation

\[
<K> = \int_0^L dN_x x \int_0^L dN_y y K. \\
(4.13)
\]

By using Green’s formula:

\[
\int \int_R dx dy \left( \frac{\partial}{\partial x} G - \frac{\partial}{\partial y} P \right) = \Phi_C(Gdy + Pdx), \\
(4.14)
\]

we obtain

\[
\sigma = \frac{e^2}{h} \frac{1}{2\pi i} \int_0^{2\pi} d\theta_x \phi_0^* \frac{\partial \phi_0}{\partial \theta_x} >_{\theta_y = 0} + \int_0^{2\pi} d\theta_y \phi_0^* \frac{\partial \phi_0}{\partial \theta_y} >_{\theta_x = 2\pi}
\]

16
\[ + \int_0^{2\pi} d\theta_x < \phi_0^* \frac{\partial \phi_0}{\partial \theta_x} |_{\theta_y=2\pi} + \int_0^{2\pi} d\theta_y < \phi_0^* \frac{\partial \phi_0}{\partial \theta_y} |_{\theta_x=0} \]  

(4.15)

Substituting \( \phi_0 \) of Eq. (4.2) and \( \psi_g \) of Eq. (4.6) into Eq. (4.15), and using Eq. (4.11), we find

\[ \sigma = \frac{e^2}{\hbar} \int_0^{2\pi} d\theta_x \left[ N e^{-i \theta_y \frac{L^2}{2}} \right]_{\theta_y=2\pi} = \frac{e^2 N e}{\hbar} = \frac{e^2}{\hbar} \nu . \]  

(4.16)

Here, we notice that the result of Eq. (4.16) derived from the quantum mechanical point of view is identical to that of Eq. (1.3), and that the integral of Eq. (4.1) is independent of the choice of the ground state.

In Ref. (10) there is an important argument on how to find a topological property of \( \sigma \) for the FQHE. We follow the procedure given in Ref. 10 in discussing the topological invariance of \( \sigma \) below. In the previous section, we found that the Fermi energy gap exists only at \( \nu = p/q \) in Eq. (3.16). Another fact at \( \nu = p/q \) is that the many-particle ground state \( \psi_g \) of Eq. (2.16) is characterized by only residues of divisor \( q \), i.e. \( t_i \in U_{\alpha=1}^p [r_{\alpha}]_q \). Thus \( \psi_g \), written as the Slater determinant of Eq. (4.6) goes back into itself up to a phase factor for the shift of all \( f_{t_i} \rightarrow f_{t_i+q} \), which is obtained by the variation of \( \theta_x \rightarrow \theta_x + 2\pi q \) and \( \theta_y \rightarrow \theta_y + 2\pi \). Since the integral of Eq. (4.1) is independent of the choice of the ground state, and since \( \psi_g(\theta_x + 2\pi k, \theta_y) \) for integer \( k \) corresponds to one of the degenerated ground states, it is legitimate to write \( \sigma \) as an average:

\[ \sigma = \frac{e^2}{\hbar q} \int_0^{2\pi q} d\theta_x \int_0^{2\pi} d\theta_y \left< \frac{\partial \phi_0}{\partial \theta_x} \frac{\partial \phi_0}{\partial \theta_y} \right> - \left< \frac{\partial \phi_0}{\partial \theta_y} \frac{\partial \phi_0}{\partial \theta_x} \right> \]  

(4.17)

where \( q \) is chosen in order to introduce the torus \( 0 \leq \theta_x < 2\pi q \) and \( 0 \leq \theta_y < 2\pi \), which is related to \( \psi_g \). Now we notice that the Hall conductance is written as the integral of Berry’s curvature over the torus. It is well known that the integral is quantized as an integer. Furthermore, the integer must be \( p \) at \( \nu = p/q \) because of Eq. (4.16). As in the IQHE, we can attach a topological meaning to the Hall conductance of Eq. (4.17) in the FQHE.

5. CONCLUSION

We have considered the FQHE as a many-body effect. It is assumed that the total quantum number is preserved during the process of the interactions...
at the microscopic level. The Hartree-Fock method enables us to find the ground state of the interacting system by solving the corresponding eigenvalue problem of the effective one-particle Hamiltonian. The ground state energy is written in terms of the couplings of the many-particle Hamiltonian. Extending the idea underlying the IQHE, we identify the difference between the states corresponding to the fractional filling factor \( \nu \) associated with the FQHE and those at \( \nu \) not related to the FQHE. The filling factors correspond to the completely filled states. In order to explain the observed fractional filling factors \( \nu \), we make the \( J \)-assumption that the couplings depend on the number of states and the number of electrons in a special way. The quasiparticle energy spectrum, together with the \( J \)-assumption, makes it possible to connect the fractional filling factors to the completely filled states. We find the energy gaps for excitations at the fractional filling factors. Furthermore, we obtain a reasonable value for the degeneracy of the many-particle ground state, which is another condition required in order to explain the Hall conductance of the FQHE as a topological invariant. Using the many-particle ground state, we carry out an explicit calculation of the Hall conductance \( \sigma \) in the integral form, and discuss the topological invariance. In summary, in order to explain the FQHE, we choose the many-particle Hamiltonian, the couplings, and the nature of the one-particle states: Eqs. (2.8), (3.8), (3.13) and (4.5). The results we derive are the fractional filling factors at the completely filled states, the corresponding energy gaps, the degeneracy of the many-particle ground state, and the Hall conductance: Eqs. (3.16), (3.18), (3.19) and (4.16). There are several points which require further study.

(1) It is expected that there is an adequate two-body interaction \( V_2 \), producing the \( J \)-assumption in the calculation of \( \langle 0 | c_i c_j V_2 c_k c_l | 0 \rangle \). It is interesting to calculate the magnitude of the parameter \( \Delta \) from the two-body interaction.

(2) It is important to compare the wave function \( \psi_g \) in Eq. (4.6) with the Laughlin wave function. Furthermore, as Arovas, Schrieffer, and Wilczek\(^{17} \) did, we should study whether or not the excitation has anyonic properties.

(3) Our intuition leads us to consider the \( 2 + 1 \) dimensional space \((x, y, t)\) when an action is used in order to solve problems the FQHE. However, the torus geometry introduced by the doubly periodic conditions seems to produce the artificial one-dimensional space where the quasi-particles live as
shown in the Hamiltonian of Eq. (2.8). Thus, the $1 + 1$ dimensional space is involved in an action derived from the Hamiltonian of Eq. (2.8), that is, $\left( \frac{2\pi}{N} j, t \right)$ where $j = \text{integer, } 1 \leq j \leq N$. In the continuum limit of $N \to \infty$, the corresponding space becomes a cylindrical one: $(l, t)$ where $0 \leq l \leq 2\pi$.

This two-dimensionality seems to be a clue for the connection to conformal field theory, as far as the FQHE is related to the critical phase transition. In two-dimensional theory, it is well-known that the four fermion interaction adopted in this paper is transformed into the exponential interaction by the bosonization.\textsuperscript{18} Maybe this relation supports the argument that the vertex operators\textsuperscript{12} can be used in the explanation of the FQHE. However, it should be emphasized that the two-dimensional space, where the vertex operators are defined, has nothing to do with the two-dimensional space where real electrons are trapped in the experiment.

(4) As an ultimate goal in understanding of the FQHE, the localization\textsuperscript{19} should be explained in terms of the Hamiltonian. Progress in this area is anticipated.
We introduce a spectrum-like figure, with which all the fractional filling factors are accompanied. The artificial spectrum is described as the first splitting of the Landau-level in the unique way shown in Fig. 4, where the ratios of the degeneracies are written explicitly. For splitting the sub-level of degeneracy \( \frac{1}{1 \cdot m} \), we use the equality:

\[
\frac{1}{1 \cdot m} = \frac{1}{1 \cdot (m + 2)} + \sum_{k=0}^{\infty} \left[ \frac{1}{\{m + 2 + k(m + 1)\}\{m + 2 + (k + 1)(m + 1)\}} \right. \\
+ \left. \frac{1}{\{m + k(m + 1)\}\{m + (k + 1)(m + 1)\}} \right] \quad (A.1)
\]

It is easy to see that the splittings follow the concept that the completely filled states are associated with the fractional filling factors, which are given by the sums of the degeneracies up to the corresponding sub-level. For instance, if the sub-level of degeneracy \( \frac{1}{9 \cdot 13} \) is completely filled, in other words, electrons are filled from the bottom up to the level of \( \frac{1}{9 \cdot 13} \), the following holds:

\[
N_e = \frac{1}{1 \cdot 5} N + \frac{1}{5 \cdot 9} N + \frac{1}{9 \cdot 13} N , \quad (A.2)
\]

from which we obtain the fractional filling factor,

\[
\nu = \frac{N_e}{N} = \frac{1}{4} (1 - \frac{1}{13}) = \frac{3}{13} . \quad (A.3)
\]

It is remarkable that the splittings satisfy the adiabatic theorem, which says that state-dimensionality is unchanged.
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FIGURE CAPTIONS

FIG. 1: The Hartree-Fock quasi-particle energy spectrum is shown for the electron dominant region of $N > 2N_e$. The degeneracies and the energy gaps are given for $lJ \leq N_e \leq (l + 1)J$. The Fermi energy is denoted as $\epsilon_F$.

FIG. 2: The energy spectrum for the hole dominant region of $N < 2N_e$ and for $lJ \leq N - N_e \leq (l + 1)J$ is shown. We notice slight differences from the spectrum of Fig. 1.

FIG. 3: The $N$-dependence of the number $J$ is shown. The corresponding quantum numbers $n$ and $s$ are chosen in order to satisfy the condition of $J \leq N_e$ for $N > 2N_e$ and $J \leq N - N_e$ for $N < 2N_e$.

FIG. 4: In order to see the consistency between the adiabatic theorem and the fact that all the known fractional filling factors are associated with the completely filled states, the splittings are introduced. The left one of the two numbers written beside the splitting levels is the ratio of the degeneracy of the sub-level to that of the unperturbed Landau-level, and the right one is the corresponding fractional filling factor for the case where electrons are filled up to the sub-level. For $N < 2N_e$, the fractional filling factor written here corresponds to the case where holes are filled from the top to the sub-level, in other words, electrons are filled up to the next sub-level below. All observed fractions below 1 can be found here.
Table 1: The energy gaps $\Delta_\nu$ are measured at the magnetic field $B$ for different samples, $X^{16}$ and $Y^4$. The units of $kG$ and $K$ are the abbreviations of kilogauss and Kelvin degree, respectively.

| Sample | $B(kG)$ | $\nu$ | $\Delta_\nu(K)$ |
|--------|---------|-------|-----------------|
| X      | 92.5    | $2/3$ | 0.83            |
|        | 66.8    | $2/3$ | 0.38            |
| Y      | 243     | $2/5$ | 1.0             |
|        | 167     | $3/5$ | 0.5             |