Pattern Avoiding Alternating Involutions

Marilena Barnabei†, Flavio Bonetti‡, Niccolò Castronuovo£ and Matteo Silimbani§

†Dipartimento di Matematica, Università di Bologna, 40126, ITALY
marilena.barnabei@unibo.it
‡P.A.M., Università di Bologna, 40126, ITALY
flavio.bonetti@unibo.it
£Liceo “A. Einstein”, Rimini, 47923, ITALY
niccolo.castronuovo2@unibo.it
§Istituto Comprensivo “E. Rosetti”, Forlimpopoli, 47034, ITALY
matteosilimbani@icrosetti.istruzioneer.it

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Abstract: We enumerate and characterize some classes of alternating and reverse alternating involutions avoiding a single pattern of length three or four. If on one hand the case of patterns of length three is trivial, on the other hand, the length four case is more challenging and involves sequences of combinatorial interest, such as Motzkin and Fibonacci numbers.

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1. Introduction

A permutation $\pi$ avoids a pattern $\tau$ whenever $\pi$ does not contain any subsequence order-isomorphic to $\tau$.

The theory of permutation patterns goes back to the work of Knuth [9], who, in the 1970s, introduced the definition of pattern avoidance in connection to the stack sorting problem. The first systematic study of these objects appears in the paper by Simion and Schmidt ([16]). Nowadays, the theory is very rich and widely expanded, with hundreds of papers appearing in the last decades (see e.g. [8] and references therein).

More recently permutation patterns have been studied over a particular subset of the symmetric group. In particular pattern avoidance has been studied over involutions (see e.g. [1, 4, 7, 8]) and over alternating permutations (see e.g. [3, 5, 6, 8, 10–12, 22, 23]), i.e., permutations $\pi = \pi_1 \ldots \pi_n$ such that $\pi_i < \pi_{i+1}$ if and only if $i$ is odd.

The enumeration of alternating involutions is due to Stanley (see [18] and also his survey on alternating permutations [19]). However, to the best of our knowledge, pattern avoiding alternating involutions has not been studied so far.

In this paper, we consider alternating involutions that avoid some patterns of length three or four. If on one hand the case of patterns of length three is trivial, on the other hand, the length four case is more challenging and involves sequences of combinatorial interest, such as Motzkin and Fibonacci numbers.

2. Preliminaries

2.1 Permutations

Let $S_n$ be the symmetric group over the symbols $\{1, 2, \ldots n\}$. We will often write permutations in $S_n$ in one line notation as $\pi = \pi_1 \ldots \pi_n$. An involution is a permutation $\pi$ such that $\pi = \pi^{-1}$. The reverse of a permutation $\pi = \pi_1 \pi_2 \ldots \pi_n$ is $\pi^r = \pi_n \pi_{n-1} \ldots \pi_1$ and the complement of $\pi$ is $\pi^c = n + 1 - \pi_1 n + 1 - \pi_2 \ldots n + 1 - \pi_n$. The reverse-complement of $\pi$ is $\pi^{rc} = (\pi^r)^c = (\pi^c)^r$. Notice that $(\pi^{-1})^{rc} = (\pi^{rc})^{-1}$. In particular the reverse-complement of an involution is an involution.
A descent in a permutation $\pi$ is an index $i$ such that $\pi_i > \pi_{i+1}$. Denote by Des($\pi$) the set of descents of $\pi$ and by des($\pi$) its cardinality.

Recall that a left-to-right minimum (ltr minimum from now on) of a permutation $\pi$ is a value $\pi(i)$ such that $\pi(j) > \pi(i)$ for every $j < i$. A left-to-right maximum (ltr maximum from now on) is a value $\pi(i)$ such that $\pi(j) < \pi(i)$ for every $j < i$. The definition of right-to-left (rtl) minimum and maximum is analogous.

**Lemma 2.1.** In any involution $\pi$ the ltr minima form an involution themselves, i.e., a ltr minimum $m$ is at position $i$ if and only if $i$ is a ltr minimum. The same is true for rtl maxima.

**Proof.** We will consider only the case of ltr minima, since the other case is analogous. Denote by $m_1, m_2, \ldots, m_k$ the left-to-right minima of $\pi$, and by $i_1, i_2, \ldots, i_k$ their respective positions. We want to show that

$$\{m_1, m_2, \ldots, m_k\} = \{i_1, i_2, \ldots, i_k\}.$$ 

Suppose on the contrary that there exists an index $j$ such that $i_j$ is not the value of a left-to-right minimum. Then, to the left of $i_j$ in $\pi$ there is a symbol $a$ less than $i_j$. In other terms, there exist two integers $a, h$ such that $\pi(h) = a$, $a < i_j$, and $h < m_j$. In this situation, $m_j$ is not a left-to-right minimum, since it is preceded by $h$, and $h < m_j$, contradiction. \hfill \Box

**Example 2.1.** Consider the involution $\pi = 79435611028$. The ltr minima of $\pi$ are $7, 4, 3, 1$ and they form an involution themselves. The same is true for the rtl maxima $8$ and $10$.

Given a word $w = w_1 \ldots w_j$ whose letters are distinct numbers, the standardization of $w$ is the unique permutation $\pi$ in $S_j$ order isomorphic to $w$. If two words $u$ and $w$ have the same standardization we write $u \sim w$.

The decomposition into connected components of a permutation $\pi$ is the finest way to write $\pi = \pi_1 \pi_2 \ldots \pi_k$, where each $\pi_i$ is a permutation of the symbols from $|\pi_1| + |\pi_2| + \ldots + |\pi_{i-1}| + 1$ to $|\pi_1| + |\pi_2| + \ldots + |\pi_{i-1}| + |\pi_i|$. Each $\pi_i$ is called a connected component of $\pi$. A permutation is said to be connected if it is composed by only one connected component.

**Example 2.2.** The decomposition into connected components of the permutation $\pi = 34215786$ is $\pi = \pi_1 \pi_2 \pi_3$, where $\pi_1 = 3421$, $\pi_2 = 5$, $\pi_3 = 786$.

### 2.2 Alternating permutations

A permutation $\pi = \pi_1 \ldots \pi_n$ is said to be alternating if $\pi_i < \pi_{i+1}$ if and only if $i$ is odd and reverse alternating if and only if $\pi_i > \pi_{i+1}$ if and only if $i$ is odd. Equivalently, a permutation $\pi$ is alternating whenever Des($\pi$) = $\{2, 4, 6, \ldots\}$.

**Example 2.3.** The permutation $\pi = 4615273$ is alternating, while $\sigma = 5372614$ is reverse alternating.

Denote by $S_n$ ($I_n$, $A_n$, $RA_n$, $AI_n$ and $RAI_n$, respectively) the set of permutations (involutions, alternating permutations, reverse alternating permutations, alternating involutions, reverse alternating involutions, respectively) of length $n$.

The following lemma will be useful in the sequel.

**Lemma 2.2.**

i) Let $\pi = \pi_1 \ldots \pi_n \in RAI_n$. Then $1$ is in even position, $n$ is in odd position, hence $\pi_1$ is even and $\pi_n$ is odd.

ii) Let $\pi = \pi_1 \ldots \pi_n \in AI_n$. Then $1$ is in odd position, $n$ is in even position, hence $\pi_1$ is odd and $\pi_n$ is even.

### 2.3 Pattern avoidance

A permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ avoids the pattern $\tau \in S_k$ if there are no indices $i_1, i_2, \ldots, i_k$ such that the subsequence $\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_k}$ is order isomorphic to $\tau$.

Denote by $S_n(\tau)$ the set of permutations of length $n$ avoiding $\tau$ and let $S(\tau) := \bigcup_n S_n(\tau)$. We will keep this notation also when $S_n$ is replaced by other subsets of permutations, such as $I_n$, $A_n$, etc.

Notice that an involution avoids $\tau$ if and only if it avoids $\tau^{-1}$.

The next trivial lemma will be useful in the sequel.

**Lemma 2.3.** The reverse-complement map is a bijection between $AI_{2n+1}(\tau)$ and $RAI_{2n+1}(\tau^{rc})$, between $AI_{2n}(\tau)$ and $AI_{2n}(\tau^{rc})$ and between $RAI_{2n}(\tau)$ and $RAI_{2n}(\tau^{rc})$. 

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2.4 Motzkin paths

A Motzkin path of length $n$ is a lattice path starting at $(0,0)$, ending at $(n,0)$, consisting of up steps $U$ of the form $(1,1)$, down steps $D$ of the form $(1,-1)$, and horizontal steps $H$ of the form $(1,0)$, and lying weakly above the $x$-axis. As usual, a Motzkin path can be identified with a Motzkin word, namely, a word $w = w_1 w_2 \ldots w_n$ of length $n$ in the alphabet \{U, D, H\} with the constraint that the number of occurrences of the letter $U$ is equal to the number of occurrences of the letter $D$ and, for every $i$, the number of occurrences of $U$ in the subword $w_1 w_2 \ldots w_i$ is not smaller than the number of occurrences of $D$. In the following we will not distinguish between a Motzkin path and the corresponding word. Denote by $M_n$ the set of Motzkin path of length $n$ and by $M_n$ its cardinality, the $n$-th Motzkin number (see sequence A001006 in [17]). The diod decomposition of a Motzkin path $m$ of even length $2n$ is the decomposition of $m$ as $m = d_1 d_2 \ldots d_n$, where each $d_i$ is a subword of $m$ of length two. Each $d_i$ in the diod decomposition of a Motzkin path is called a diod of $m$.

3. General results

In this section, we prove two general results that will be used in the paper.

**Lemma 3.1.** Let $\tau$ be any permutation of $\{3, \ldots, m\}$, $m \geq 3$. Then

$$|AI_n(12\tau)| = |AI_n(21\tau)|.$$

**Proof.** We closely follow [15], where a similar result is proved for doubly alternating permutations, i.e., alternating permutations whose inverse is also alternating.

Our goal is to find a bijection between $AI_n(12\tau) = AI_n(12\tau, 12\tau^{-1})$ and $AI_n(21\tau) = AI_n(21\tau, 21\tau^{-1})$.

We will use the diagrammatic representation of a permutation, i.e., we will identify a permutation $\pi$ to the number of occurrences of the letter $D$, and a Motzkin word $w$ to the set of Motzkin path of length $n$, $\pi$ is an involution, its diagram is symmetric with respect to the main diagonal, and for every active dot in position $(i, \pi_i)$ there is also an active dot in position $(d_i, d)$, if $d$ is the 1 or 2 in any 12 pattern, or inactive otherwise. Also the pair of dots $(d_1, d_2)$ is called an active pair if $d_1, d_2$ is the 12 in any 12 pattern or the 21 in a 21 pattern.

We now define a Young diagram, $\lambda_\pi$, consisting of the part of the diagram of $\pi$ which contains the active dots. For any two dots $d_1, d_2$, let $R_{d_1, d_2}$ be the smallest rectangle with bottom left coordinates $(1, 1)$, such that $d_1, d_2 \in R_{d_1, d_2}$. Define

$$\lambda_\pi = \bigcup R_{d_1, d_2},$$

where the union is overall active pairs $(d_1, d_2)$ of $\pi$. It is clear from the definition that $\lambda_\pi$ is indeed a Young diagram. Since $\pi$ is an involution, its diagram is symmetric with respect to the main diagonal, and for every active dot in position $(i, j)$ there is also an active dot in position $(j, i)$, hence $\lambda_\pi$ is a Young diagram symmetric with respect to the main diagonal.

A rook placement of a Young diagram $\lambda$ is placement of dots in its boxes, such that all rows and columns contain exactly one dot. If some of the rows or columns are empty we call it a partial rook placement. Furthermore, we say that a rook placement on $\lambda$ avoids the pattern $\tau$ if no rectangle, $R \subseteq \lambda$, contains $\tau$. Notice that the rook placement on $\lambda_\pi$ induced by an involution $\pi$ is symmetric.

Now, it has been proved by Jaggsard [7, Theorem 4.2], that the number of symmetric rook placements on the self-conjugate shape $\mu$ avoiding the patterns $12\tau$ and $12\tau^{-1}$ is equal to the number of symmetric rook placements on the self-conjugate shape $\mu$ avoiding the patterns $21\tau$ and $21\tau^{-1}$.

We call two permutations $\pi$ and $\sigma$ of the same size $a$-equivalent if they have the same inactive dots, and write $\pi \sim_a \sigma$.

In the sequel, we will need the following three facts that are immediate consequences of Lemma 6.4, Lemma 6.3, and Lemma 6.2 in [15], respectively.

(I) $\pi \sim_a \sigma \Rightarrow \lambda_\pi = \lambda_\sigma$.

(II) If $\pi \in AI_n(12\tau, 12\tau^{-1}) \cup AI_n(21\tau, 21\tau^{-1})$ and $\pi \sim_a \sigma$ then $\sigma$ is doubly alternating.

(III) If $\pi \in AI_n$ and $rp(\lambda_\pi)$ is the partial rook placement on $\lambda_\pi$ induced by $\pi$, then $\pi \in AI_n(12\tau, 12\tau^{-1}) \Leftrightarrow rp(\lambda_\pi)$ is 12-avoiding and $\pi \in AI_n(21\tau, 21\tau^{-1}) \Leftrightarrow rp(\lambda_\pi)$ is 21-avoiding.

Now we are ready to construct a bijection

$$\phi : AI_n(12\tau, 12\tau^{-1}) \rightarrow AI_n(21\tau, 21\tau^{-1}).$$
Let $\pi \in AI_n(12\tau, 12\tau^{-1})$, so that the restriction of $\pi$ to $\lambda_\tau$ is a partial 12-avoiding symmetric rook placement. By Jaggard’s theorem (ignoring the empty rows and columns) and by point (I) there exists a unique 21-avoiding (partial) symmetric rook placement on $\lambda_\tau$ with the same rows and columns empty, which we combine with the inactive dots of $\pi$ to get $\phi(\pi)$. By point (II), $\phi(\pi)$ is doubly alternating and, since its diagram is symmetric, it is an involution. By point (III) it avoids $21\tau$ (and hence also $21\tau^{-1}$). It is also clear from Jaggard’s theorem that $\phi$ is indeed a bijection.

\[\text{Example 3.1. Consider the permutation } \pi = 593716482 \in AI_9(12435). \text{ Notice that } \pi \text{ contains the pattern } 21435. \text{ Here } \tau = 435. \text{ The diagram of } \pi \text{ is the following}
\]

![Diagram of permutation \pi]

The blue dots are the active dots of \pi and the red Young diagram is \lambda_\tau. Now applying to \lambda_\tau the procedure described in the above proof we get $\phi(\pi) = 195736482 \in AI_9(21435)$.

It follows from the previous lemma and Lemma 2.3 that if $\tau$ is any permutation of $\{1, \ldots, k-2\}$ then

$$|AI_{2n}(\tau k - 1 k)| = |AI_{2n}(\tau k k - 1)| \text{ and } |RAI_{2n+1}(\tau k k - 1)| = |RAI_{2n+1}(\tau k - 1 k)|.$$

Notice that similar relations do not hold for $RAI_{2n}$, in fact, numerical computations show that, for general $n$, $|RAI_{2n}(1234)| \neq |RAI_{2n}(1243)|$ and $|RAI_{2n}(1234)| \neq |RAI_{2n}(2134)|$.

When $\tau$ is an increasing sequence it is possible to provide a more explicit bijection $f$ between $|AI_{2n}(\tau k - 1 k)|$ and $|RAI_{2n+1}(\tau k k - 1)|$. Such a bijection has been defined for the first time by J. West [21] for permutations and has been used by M. Bóna [3] to prove a conjecture by J. B. Lewis about alternating permutations. To this aim, Bóna proved that $f$ preserves the alternating property when the length of the permutation is even. Here we recall the definition of the map $f:S_1(12\ldots k-1 k) \rightarrow S_1(12\ldots k k - 1)$. Consider a permutation $\pi \in S_1(12\ldots k-1 k)$ and define the rank of the element $\pi_i$ to be the maximum length of an increasing subsequence ending at $\pi_i$. Since $\pi$ avoids $12\ldots k-1 k$, the maximal rank of an element is $k-1$. Let $R$ be the set of elements of $\pi$ whose rank is $k-1$, and $P$ the set of their positions. The permutation $\rho = f(\pi)$ is obtained as follows.

- if $j \notin P$, $\rho_j = \pi_j$,
- if $j \in P$, $\rho_j$ is the smallest unused element of $R$ that is larger than the closest entry of rank $k-2$ to the left of $\pi_j$.

Notice that, if $k = 3$, the map $f$ reduces to the classic Simion-Schmidt bijection (16, 21).

In the following lemma, we prove that $f$ preserves also the property of being an involution and we describe what happens in the odd case.

\[\text{Lemma 3.2. The map } f \text{ is a bijection between } AI_{2n}(12\ldots k-1 k) \text{ and } AI_{2n}(12\ldots k k - 1). \text{ Moreover, } f \text{ maps bijectively the subset of } AI_{2n+1}(1234) \text{ of those permutations having } 2n+1 \text{ at position } 2 \text{ to the set } AI_{2n+1}(1243).}\]

\[\text{Proof. First of all, we notice that, in an involution } \pi, \text{ all the elements of a given rank form an involution. To prove this fact it is sufficient to observe that elements of } \pi \text{ of rank } 1 \text{ are the ltr minima of the permutation, which form an involution (see Lemma 2.1). We can consider the involution obtained by removing from } \pi \text{ the elements of rank } 1 \text{ and proceed inductively.}\]

Now we want to prove that $f$ sends involutions of length $n$ to involutions of length $n$ (for every length $n$). Also in this case we can obtain the result by induction on $k$. When $k = 3$, we noticed above that $f$ coincides with the Simion-Schmidt map (16), and it is well-known that this map sends involutions to involutions. If $k > 3$, we can delete from $\pi$ the elements of rank 1 and standardize, obtaining an involution $\pi'$ which avoids $12\ldots k-2 k - 1$. If we apply the map $f$, we obtain an involution avoiding $12\ldots k-1 k - 2$ which can be obtained from $f(\pi)$ by removing the elements of rank 1 and standardizing. Hence also $f(\pi)$ is an involution. The fact that $f$ preserves the alternating property (when the length is even) has been proved by Bóna [3, Theorem 1].
Now we turn our attention to the second assertion. We want to prove that $f^{-1}$ maps bijectively the set $AI_{2n+1}(1243)$ onto the set $Y_{2n+1} = AI_{2n+1}(1234) \cap \{ \pi \in S_{2n+1} \mid \pi(2) = 2n + 1 \}$.

The fact that the injective map $f^{-1}$ preserves the alternating property has been already observed by Bóna [3, Corollary 1]. Moreover, $f^{-1}$ preserves the involutory property, as proved above.

Now consider $\pi \in AI_{2n+1}(1243)$. The symbol $2n + 1$ appears at even position $a$. Suppose that $a \geq 4$. Then $\pi_2 = 2n$, otherwise $\pi_2 2n + 1 2n \sim 1243$. As a consequence, $\pi_{2n+1} = a > 2 = \pi_{2n}$ which is impossible since $\pi$ is alternating. Hence $a = 2$ and $\pi_2 = 2n + 1$.

Since for every $\pi \in AI_{2n+1}(1243)$, $\pi(2) = 2n + 1$, the symbols $2n + 1$ and 2 have rank 2. The map $f^{-1}$ fixes the elements of rank $k - 2$ or less, hence it fixes the positions of $2n + 1$ and 2.

With similar arguments one can show that $f(Y_{2n+1}) \subseteq AI_{2n+1}(1243)$ and conclude that $f(Y_{2n+1}) = AI_{2n+1}(1243)$.

Example 3.2. Consider the permutation $\pi = 59710183624 \in AI_{10}(1234)$. The elements of rank 1 are 5 and 1, the elements of rank 2 are 9, 7, 3 and 2, the elements of rank 3 are 10, 8, 6 and 4. Then $f(\pi) = 59781103426 \in AI_{10}(1243)$.

4. Patterns of length three

The following proposition is an easy consequence of Proposition 3.1 in [15]. See also [14].

Proposition 4.1. We have

$$|AI_n(123)| = |AI_n(213)| = |AI_n(231)| = |AI_n(321)| = 1,$$

$$|AI_n(132)| = |RAI_n(132)| = |RAI_n(231)| = |RAI_n(312)| = |RAI_n(321)| = 1,$$

$$|AI_n(213)| = |RAI_n(213)| = \begin{cases} 1 & \text{if } n \text{ is even or } n = 1 \\ 0 & \text{otherwise} \end{cases},$$

$$|AI_n(321)| = |RAI_n(123)| = \begin{cases} 2 & \text{if } n \text{ is even and } n \geq 4 \\ 1 & \text{otherwise} \end{cases}.$$

5. Patterns 4321 and 1234

We recall the definition of a bijection $\Phi$ between the set $I_n(4321)$ of 4321-avoiding involutions of the symmetric group $S_n$ and the set $M_n$ of Motzkin paths of length $n$ (see [1]), that is essentially a restriction to the set $I_n(4321)$ of the bijection appearing in [2].

Consider an involution $\tau$ avoiding 4321 and determine the set $exc(\tau) = \{ i \mid \tau(i) > i \}$ of its exceedances. Start from the empty path and create the Motzkin path $\Phi(\tau)$ by adding a step for every integer $1 \leq i \leq n$ as follows:

- if $\tau(i) = i$, add a horizontal step at the $i$-th position;
- if $\tau(i) > i$, add an up step at the $i$-th position;
- if $\tau(i) < i$, add a down step at the $i$-th position.

The map $\Phi$ is a bijection whose inverse can be described as follows.

Given a Motzkin path $M$, let $A = (a_1, \ldots, a_k)$ be the list of positions of up steps in $M$, written in increasing order, and let $B = (b_1, \ldots, b_k)$ the analogous list of positions of down steps. Then $\Phi^{-1}(M)$ is the involution $\tau$ given by the product of the cycles $(a_i, b_i)$ for $1 \leq i \leq k$.

Lemma 5.1. $\Phi$ maps the set $RAI_{2n}(4321)$ onto the set $\tilde{M}_{2n}$ of Motzkin paths of length $2n$ whose diods are either of the forms $UH$, $HD$ or $UD$.

Proof. Let $\tau \in RAI_{2n}(4321)$. For every integer $k = 1, 2, \ldots, 2n - 1$, if $k$ is odd, and hence a descent of $\tau$, $\tau_k \tau_{k+1}$ is mapped by $\Phi$ to one of the following pairs: $UH$, $UD$, $HD$. □

Theorem 5.1. We have

$$|RAI_{2n}(4321)| = M_n.$$
Proof. The previous lemma allows us to define a bijection \( \Phi : RAI_{2n}(4321) \to M_n \). Let \( m = d_1d_2 \ldots d_n \) be a Motzkin path (of even length) whose diods \( d_i \) are either of the form \( UH, UD \) or \( HD \). Define \( \Delta(m) := s_1 \ldots s_n \in M_n \) where

\[
s_i = \begin{cases} U & \text{if } d_i = UH \\ D & \text{if } d_i = HD \\ H & \text{if } d_i = UD. \end{cases}
\]

Set now \( \hat{\Phi} = \Delta \circ \Phi \).

\begin{proof}

Consider \( \pi = 6284510397 \in RAI_{10}(4321) \). Then \( \Phi(\pi) = \ldots \) and \( \hat{\Phi}(\pi) = \ldots \).

\begin{theorem}

We have

\[
|RAI_{2n-1}(4321)| = M_n - M_{n-2}.
\]

\end{theorem}

\begin{proof}

Consider a permutation \( \pi \in RAI_{2n}(4321) \). By Lemma 2.2, \( \pi_{2n} \) is odd. Now consider the element \( 2n - 1 \) in \( \pi \). Notice that this element is at position \( 2n \) or \( 2n - 1 \), otherwise \( \pi \) would contain the pattern \( 2n \ 2n-1 \ 2 \ 1 \ 1 \pi(2n-1) \pi(2n) \sim 4321 \) (notice that \( 2n \) appears before \( 2n - 1 \) because \( \pi \) is an involution and \( \pi_{2n-1} > \pi_{2n} \)).

If \( \pi_{2n} = 2n-1 \), the permutation \( \pi' \) obtained from \( \pi \) by removing the element \( 2n-1 \) and standardizing is an arbitrary element of \( RAI_{2n-1}(4321) \) which ends with its maximum element.

If \( \pi_{2n-1} = 2n-1 \), write \( \pi \) as \( \pi = \ldots k \ 2n-2 \ 2n-1 \ 2n \ 2n-1 \ k \).

If \( k > j \), consider the permutation \( \pi' \) obtained from \( \pi \) by removing the element \( 2n-1 \) and standardizing. \( \pi' \) is an arbitrary element of \( RAI_{2n-1}(4321) \) which does not end with its maximum element.

If \( k < j \), we must have \( k = 2n-3 \) and \( j = 2n-2 \), otherwise \( \pi \) would contain 4321. Hence we can write \( \pi = \pi' 2n \ 2n-2 \ 2n-1 \ 2n-3 \), where \( \pi' \) is an arbitrary permutation in \( RAI_{2n-4}(4321) \).

As a consequence, \( |RAI_{2n}(4321)| = |RAI_{2n-1}(4321)| + |RAI_{2n-4}(4321)| \).

\end{proof}

The preceding lemma implies that the sequence \( (|RAI_{2n-1}(4321)|)_{n \geq 1} \) coincides with sequence \texttt{A102071} in [17].

\begin{theorem}

We have

\[
|AI_n(1234)| = |RAI_n(4321)| \quad \text{and} \quad |RAI_n(1234)| = |AI_n(4321)|.
\]

\end{theorem}

\begin{proof}

The Robinson-Schensted map \( RS \) (see e.g. [20]) associates an involution \( \pi \) with a standard Young Tableau whose descent set \( D \) is equal to the descent set of \( \pi \). If we apply the inverse of the map \( RS \) to the transposed Tableau we get another involution \( \pi' \) whose descent set is \([n-1] \setminus D\). In particular, \( \pi \) is alternating if and only if \( \pi' \) is reverse alternating. Moreover, by the properties of the map \( RS \), \( \pi \) avoids 1234 if and only if the corresponding tableau has at most three columns, hence the transposed tableau has at most three rows and \( \pi' \) avoids 4321.

Theorems 5.1, 5.2 and 5.3 and Lemma 2.3 imply the following result.

\begin{corollary}

We have \( |RAI_{2n}(4321)| = |AI_{2n}(1234)| = M_n \) and \( |RAI_{2n-1}(1234)| = |AI_{2n-1}(4321)| = |RAI_{2n-1}(4321)| = |AI_{2n-1}(1234)| = M_n - M_{n-2} \).

We now give an explicit formula for the cardinalities of the sets \( RAI_{2n}(1234) \) and \( AI_{2n}(4321) \).

\end{corollary}

\begin{theorem}

We have

\[
|RAI_{2n}(1234)| = |AI_{2n}(4321)| = M_{n+1} - 2M_{n-1} + M_{n-3}.
\]

\begin{proof}

We proceed as in the proof of Theorem 5.2. Consider a permutation \( \pi \in RAI_{2n+1}(4321) \). The symbol \( 2n \) is either at position \( 2n \) or \( 2n + 1 \), otherwise \( \pi \) would contain 4321.

If \( \pi_{2n+1} = 2n \), the permutation \( \pi' \) obtained from \( \pi \) by removing the element \( 2n + 1 \) and standardizing is an arbitrary element of \( AI_{2n}(4321) \) which ends with its maximum element.

If \( \pi_{2n} = 2n \), set \( \pi = \ldots k \ 2n+1 \ 2n-1 \ 2n \ 2n+1 \ k \).

If \( j < k \), the permutation \( \pi' \) obtained from \( \pi \) by removing the element \( 2n \) and standardizing is an arbitrary element of \( AI_{2n}(4321) \) which does not end with its maximum element.

\end{proof}

\end{corollary}
6. Pattern 3412

Theorem 6.1. We have
\[ |RAI_{2n}(3412)| = |AI_{2n+2}(3412)| = |AI_{2n+1}(3412)| = |RAI_{2n+1}(3412)| = M_n. \]

Proof. We recall that also the set \( I_n(3412) \) corresponds bijectively to the set \( M_n \) via a map \( \Psi \) whose definition coincides with the definition of the map \( \Phi \) of Section 5 (see [1]). It is easily seen that \( \Psi \) maps the set \( RAI_{2n}(3412) \) to the set of Motzkin paths in \( M_{2n} \) whose diods are either of the form \( UH, HD, \) or \( UD \). Hence the map \( \Psi = \Delta \circ \Phi \) is a bijection between \( RAI_{2n}(3412) \) and \( M_n \), where \( \Delta \) is the map defined in the proof of Theorem 5.1.

We observe that any permutation in \( I_n(3412) \) begins with 1, hence, a permutation \( \pi \in AI_{2n+1}(3412) \) can be written as \( \pi = 1 \pi' \), where \( \pi' \) is an arbitrary permutation in \( RAI_{2n}(3412) \). Moreover, any permutation \( \pi \) in \( AI_{2n+2}(3412) \) ends with its maximum element, hence it can be can be written as \( \pi = 1 \pi' 2n + 2 \) where \( \pi' \) is an arbitrary permutation in \( RAI_{2n}(3412) \).

The last equality follows from Lemma 2.3.

7. Patterns 2143, 2134 and 1243

Theorem 7.1. We have
\[ |AI_{2n}(2143)| = |AI_{2n}(2134)| = |AI_{2n}(2134)| = M_n \]
and
\[ |AI_{2n-1}(2134)| = |RAI_{2n-1}(2143)| = M_n - M_{n-2}. \]

Proof. It is an immediate consequence of Lemma 2.3, Lemma 3.1 and Corollary 5.1.

Theorem 7.2. We have
\[ |AI_{2n+1}(2143)| = |AI_{2n+1}(2134)| = |RAI_{2n+1}(2143)| = |RAI_{2n+1}(2134)| = M_n. \]

Proof. By Lemmas 2.3, 3.1 and 3.2, it suffices to show that \( |AI_{2n+1}(2143)| \cap \{ \sigma \in S_{2n+1} | \sigma(2) = 2n + 1 \} = M_n \).

If \( \pi \) is a permutation in this last set, removing \( 2n + 1 \) and \( 2 \) from \( \pi \) and standardizing we get an arbitrary permutation \( \tilde{\pi} \) in \( I_{2n-1}(2134) \) with either \( \text{Des}(\tilde{\pi}) = \{3, 5, 7, \ldots, 2n - 3\} \) or \( \text{Des}(\tilde{\pi}) = \{1, 3, 5, 7, \ldots, 2n - 3\} \). In other words, this permutation is reverse alternating or it is reverse alternating only to the right of the second position. Hence we have
\[ |AI_{2n+1}(2143)| \cap \{ \sigma \in S_{2n+1} | \sigma(2) = 2n + 1 \} = |I_{2n-1}(2134)| \cap \{ \sigma \in S_{2n-1} | \text{Des}(\sigma) = \{3, 5, 7, \ldots, 2n - 3\} \} + |RAI_{2n-1}(2134)|. \]

We want to show that the sum on the right-hand side of the previous equality is equal to \( M_n \). We proceed by induction on \( n \). The assertion is trivial when \( n = 2 \) or \( n = 3 \). Assume that the assertion is true for every \( m < n \). We know by Corollary 5.1 that \( |RAI_{2n-1}(2134)| = M_{n-1} - M_{n-2} \). Consider a permutation \( \rho \) in \( I_{2n-1}(2134) \cap \{ \sigma \in S_{2n-1} | \text{Des}(\sigma) = \{3, 5, 7, \ldots, 2n - 3\} \} \). Notice that \( \rho(2) = 2n - 2 \) and \( \rho(3) = 2n - 1 \), otherwise the permutation would contain the subsequence \( \rho(1) \rho(2) 2n - 2 \). Removing \( 2, 3, 2n - 2 \) and \( 2n - 1 \) and standardizing we get an arbitrary permutation in \( I_{2n-5}(2134) \cap \{ \sigma \in S_{2n-5} | \text{Des}(\sigma) = \{3, 5, 7, \ldots, 2n - 7\} \} \) or in \( RAI_{2n-5}(2134) \). By the inductive hypothesis we have
\[ |I_{2n-5}(2134) \cup \{ \sigma \in S_{2n-5} | \text{Des}(\sigma) = \{3, 5, 7, \ldots, 2n - 7\} \} | + |RAI_{2n-5}(2134)| = M_{n-2}. \]

This completes the proof.

Example 7.1. Consider \( \pi = 5119101768342 \in AI_{11}(2134) \). If we remove 11 and 2 from \( \pi \) and standardize we get the involution \( \tilde{\pi} = 489165723 \) with \( \text{Des}(\tilde{\pi}) = \{3, 5, 7\} \). Removing from \( \tilde{\pi} \) the symbols 8, 9, 2 and 3 we get the involution \( 21435 \in RAI_5(2134) \).

Consider now the permutation \( \pi = 9117851034162 \in AI_9(2134) \). If we remove 11 and 2 from \( \pi \) and standardize we get the involution \( \tilde{\pi} = 867492315 \in RAI_9(2134) \).

Theorem 7.3. We have
\[ |RAI_{2n}(2143)| = M_{n-1}. \]
Proof. Let $\pi$ be a permutation in $RAI_{2n}(2143)$. The maximum of $\pi$ is in position 1, otherwise $2 \pi_{2n-1} \pi_{2n}$ would be an occurrence of 2143 in $\pi$. Write $\pi$ as $2 n \tilde{\tau} 1$. Then the standardization of $\tilde{\tau}$ is an arbitrary permutation in $AI_{2n-2}(2143)$. The assertion now follows from Theorem 7.1.

8. Patterns 3421 and 4312

Notice that these two patterns are inverse of each other. Hence $AI_n(3421) = AI_n(4312) = AI_n(3421, 4312)$ and the same is true for reverse alternating involutions. The following theorem shows that all these classes are enumerated by the Fibonacci numbers $F_n$ (sequence A000045 in [17]).

Theorem 8.1. We have

$$|AI_n(3421, 4312)| = |RAI_n(3421, 4312)| = F_{n-1},$$

where $F_k$ is the $k$-th Fibonacci number.

Proof. Let $\pi \in RAI_{2n}(3421, 4312)$. The last element of $\pi$ is odd and the position of $2n$ is odd by Lemma 2.2. Notice that the symbol $2n - 1$ can be either at position $2n$ or $2n - 1$, otherwise $\pi$ would contain the subsequence $2n 2n - 1 \pi(2n - 2) \pi(2n - 1)$ whose standardization is 4312.

If $\pi_{2n} = 2n - 1$, then $\pi = \pi' 2n 2n - 1$, where $\pi'$ is an arbitrary permutation in $RAI_{2n-1}(3421, 4312)$.

If $\pi_{2n-1} = 2n - 1$, let $y = \pi_{2n-2}$ and $x = \pi_{2n-1}$. If $x > y$, removing from $\pi$ the symbol $2n - 1$ and standardizing we obtain an arbitrary permutation in $RAI_{2n-1}(3421, 4312)$ which does not end with its maximum.

If $x < y$, notice that either $\pi(2n - 2) = 2n - 2$ or $\pi(2n - 3) = 2n - 2$ (otherwise $\pi$ would contain the patterns). If $\pi(2n - 2) = 2n - 2$, then $\pi(2n - 3)$ is forced to be $2n$ (because $\pi$ is alternating), so we can write $\pi = \tau 2n 2n - 2 2n - 1 2n - 3$, where $\tau \in RAI_{2n-3}(3421, 4312)$.$^{1}$ We associate to $\tau$ the symbol $\pi' = 2n - 2 2n - 3 2n - 1$. Such $\pi'$ is an arbitrary permutation in $RAI_{2n-2}(3421, 4312)$ which ends with its maximum and in which $2n - 3$ precedes the maximum.

If $\pi(2n - 3) = 2n - 2$, we write $\pi = \sigma' 2n \sigma'' 2n - 2 2n - 3 2n - 1 x$ and we associate to $\pi$ the symbol $\pi' = \sigma' 2n - 2 \sigma'' 2n - 3 x 2n - 1$ which is an arbitrary permutation in $RAI_{2n-1}(3421, 4312)$ which ends with its maximum and in which $2n - 3$ follows the maximum.

As a consequence

$$|RAI_{2n}(3421, 4312)| = |RAI_{2n-1}(3421, 4312)| + |RAI_{2n-2}(3421, 4312)|.$$

Let $\pi \in RAI_{2n+1}(3421, 4312)$. The last element of $\pi$ is odd and the position of $2n + 1$ is odd by Lemma 2.2. Notice that the symbol $2n + 1$ is either at position $2n + 1$ or $2n - 1$, otherwise $\pi$ would contain the subsequence $2n 2n + 1 \pi(2n + 1) \pi(2n + 2)$.

If $\pi_{2n+1} = 2n + 1$, then $\pi = \pi' 2n + 1$, where $\pi'$ is an arbitrary permutation in $RAI_{2n}(3421, 4312)$.

If $\pi_{2n} = 2n + 1$, write $\pi = \tau 2n + 1 x 2n - 1$. Then $\pi' = \tau 2n - 1 x$ is an arbitrary permutation in $RAI_{2n}(3421, 4312)$ with $2n - 1$ as a fixed point. We observed above that such permutations are in bijection with $RAI_{2n-1}(3421, 4312)$.

As a consequence

$$|RAI_{2n+1}(3421, 4312)| = |RAI_{2n}(3421, 4312)| + |RAI_{2n-1}(3421, 4312)|.$$ (1)

Hence we can conclude that

$$|RAI_m(3421, 4312)| = F_{m-1}.$$

By Lemma 2.3, we have also $|AI_{2n+1}(3421, 4312)| = |RAI_{2n+1}(3421, 4312)|$.

Consider now $\pi \in AI_{2n}(3421, 4312)$. By Lemma 2.2, $\pi_1$ is odd and the symbol 1 is in odd position. Notice that either $\pi(1) = 1$ or $\pi(3) = 1$ otherwise $\pi$ would contain the subsequence $\pi_1 \pi_2 \pi_1$, whose standardization is 3421.

If $\pi_1 = 1$, the standardization of $\pi_2 \ldots \pi_{2n}$ is an arbitrary permutation in $RAI_{2n-1}(3421, 4312)$. If $\pi(3) = 1$, write $\pi = 3 \pi_3 \pi_1 \pi$. The standardization of the word $\pi_2 \pi_3 \pi$ is an arbitrary permutation $\pi'$ in $RAI_{2n-2}(3421, 4312)$ which fixes the symbol 2.

Observe that, if $\pi \in RAI_{2n-2}(3421, 4312)$, then $\pi(1) = 2$ or $\pi(2) = 2$, namely $RAI_{2n-1}(3421, 4312) = A \cup B$ where $A = RAI_{2n-1}(3421, 4312) \cap \{ \pi \in S_{2n-1} | \pi(2) = 1 \}$ and $B = RAI_{2n-1}(3421, 4312) \cap \{ \pi \in S_{2n-1} | \pi(2) = 2 \}$.

The set $A$ corresponds bijectively to the set $RAI_{2n-2}(3421, 4312)$ by removing the first two elements, hence the set $B$ corresponds bijectively to the set $RAI_{2n-2}(3421, 4312)$ by equation (1).

Hence,$^2$

$$|AI_{2n}(3421, 4312)| = |RAI_{2n-1}(3421, 4312)| + |RAI_{2n-2}(3421, 4312)|$$
and
\[ |AI_m(3421, 4312)| = F_{m-1}. \]

\[ \Box \]

**Example 8.1.** Consider the set \( RAI_0(3421, 4312) \), whose elements are
\[ \alpha = 21 6 4 5 3 \quad \beta = 21 4 3 6 5 \quad \gamma = 4 2 6 1 5 3 \quad \delta = 4 2 3 1 6 5 \quad \epsilon = 6 2 4 3 5 1. \]

The permutations \( \beta \) and \( \delta \) correspond to 2 1 4 3 and 4 2 5 1 3, respectively, whereas the permutations \( \alpha, \gamma \) and \( \epsilon \) correspond to 2 1 4 3 5 1 and 4 2 3 4 1 5, respectively, namely, the elements of \( RAI_3(3421, 4312) \).

9. Patterns 2431, 4132, 3241 and 4213.

Notice that the two patterns 2431 and 4132 are inverse of each other. Hence \( AI_n(2431) = AI_n(4132) = AI_n(3241, 4132) \) and the same is true for reverse alternating involutions and for the pair of patterns (3241, 4213).

**Theorem 9.1.** We have
\[ |RAI_{2n}(2431, 4132)| = |RAI_{2n}(3241, 4213)| = |RAI_{2n+1}(3241, 4213)| = |AI_{2n+1}(2431, 4312)| = 2^{n-1}. \]

**Proof.** Consider a permutation \( \pi \in RAI_{2n}(2431, 4132) \). We want to show that \( \pi = \pi' \cdot 2n \cdot \pi'' \), where \( \pi' \) is a permutation of the symbols 1, 2, ..., \( i-1 \) and \( \pi'' \) is a permutation of the symbols \( i, \ldots, 2n-1 \).

Suppose on the contrary that there exists an element \( b \) in \( \pi' \) such that \( b > i \), so that \( \pi = \ldots b \ldots 2n \ldots i \). Let \( a = \pi_{2n-1} \). Since \( \pi \) is alternating, we have \( a > i \), hence \( 2n-1 \) follows \( 2n \) in \( \pi \) and the subsequence \( b 2n 2n-1 i \) is order-isomorphic to 2431.

As a consequence, the word 2 1 of \( \pi'' \) is itself a reverse alternating involution avoiding 132, and there exists only one of such permutations, as observed in Section 4, while the word \( \pi' \) is an arbitrary permutation in \( RAI_j(2431, 4132) \), \( j \) even.

We can conclude that
\[ |RAI_{2n}(2431, 4132)| = \sum_{j \text{ even}, j < 2n} |RAI_j(2431, 4132)| = 2^{n-1}, \]
where the last equality follows from the previous one by induction.

The facts that
\[ |RAI_{2n}(2431, 4132)| = |RAI_{2n}(3241, 4213)| \quad \text{and} \quad |RAI_{2n+1}(3241, 4213)| = |AI_{2n+1}(2431, 4312)| \]
follow by Lemma 2.3.

Consider now a permutation \( \pi \) in \( RAI_{2n+1}(4213, 3241) \).

Write \( \pi = m_1 w_1 m_2 w_2 \ldots m_k w_k \), where the \( m_i \)'s are the ltr maxima of \( \pi \) and the \( w_i \)'s are words. Since \( \pi \) is alternating and avoids the patterns 4213 and 3241, it follows that for every \( i = 1, 2, \ldots, k-1 \) the word \( w_i \) with \( w_i < w_{i+1} \), namely, every symbol in \( w_i \) is smaller than any symbol in \( w_{i+1} \).

As a consequence, \( m_i w_i \) is an interval for every \( i \). Obviously \( m_k = 2n + 1 \) and its position is \( \pi_{2n+1} \). Suppose that \( \pi_{2n+1} \neq 2n + 1 \). Since \( \pi_{2n} < \pi_{2n+1} \), the symbol 2n should appear to the left of \( 2n + 1 \) in \( \pi \), but this contradicts the fact that \( m_k w_k \) is an interval. Hence \( 2n + 1 \) is the last element of \( \pi \) and removing it from \( \pi \) we get an arbitrary permutation in \( RAI_{2n}(4213, 3241) \), so we have
\[ |RAI_{2n}(4213, 3241)| = |RAI_{2n+1}(4213, 3241)|. \]

\[ \Box \]

**Theorem 9.2.** We have
\[ |RAI_{2n+1}(2431, 4312)| = |AI_{2n+1}(3241, 4213)| = \left\lfloor \frac{2^{n-1} \cdot 5}{3} \right\rfloor. \]

To prove this theorem, we need the following lemma.

**Lemma 9.1.** The number of connected permutations in \( RAI_{2n+1}(4132, 2431) \) is
\[ \left\lfloor \frac{2l + 1}{4} \right\rfloor. \]
Proof. Let \( \tau \in RA I_{2l+1}(4132, 2431) \) be a connected permutations. Consider the position \( i \) of \( 2l + 1 \) in \( \tau \). This position is odd by lemma 2.2 and \( i \neq 2l + 1 \) since \( \tau \) is connected.

If \( i = 2l - 1 \), we can remove from \( \tau \) the symbols \( 2l + 1 \) and \( 2l - 1 \) obtaining an arbitrary connected permutation \( \tau \) in \( RA I_{2l-1}(4321, 4132) \).

If \( i < 2l - 1 \), let \( a = \tau_{2l}, b = \tau_{2l-1} \) and \( c = \tau_{2l-2} \).

We can write \( \tau = \left( \ldots \ldots a \ldots c \ldots \ldots b \ldots \ldots i \ldots \ldots 2l-2 2l-1 2l+1 \right) \) where, since \( \tau \) is alternating, \( c < b > a < i \).

If \( bai \sim 312 \), since \( \tau \) avoids 4132, the elements of \( w \) are smaller than the elements of \( w_\tau \). Hence, by Lemma 9.1 and by Theorem 9.1, we have

\[
|RAI_{2l+1}(4132, 2431)| = 2^{n-1} + \sum_{k \geq 1} 2^{k-1} \left[ \frac{2n+1-2k}{4} \right] + \left[ \frac{2n+1}{4} \right],
\]

where the first summand of the right-hand side corresponds to the case \( \tau = 2n + 1 \) and the third summand corresponds to the case \( |\tau| = 2n + 1 \).

\[\Box\]
Now it can be shown by induction that
\[ |RAI_{2n+1}(2431,4132)| = \left\lfloor \frac{2^{n-1} \cdot 5}{3} \right\rfloor. \]

The fact that \(|AI_{2n+1}(3241,4213)| = |RAI_{2n+1}(2431,4132)|\) follows by Lemma 2.3.

Notice that \(|RAI_{2n+1}(2431,4132)|\) is sequence A081254 in [17].

On the contrary, the sequence enumerating \(AI_{2n}(3241,4213)\) and \(AI_{2n}(2431,4132)\), whose first terms are 1, 2, 5, 9, 17, 31, 59, ..., is not present in [17].

10. Patterns 2413 and 3142

The set \(AI_n(2413,3142)\) coincides with the set of alternating Baxter involutions of length \(n\).

We recall that a Baxter permutation of length \(n\) is a permutation \(\pi \in S_n\) such that, for every \(1 \leq i < j \leq n\),

if \(\pi_i + 1 = \pi_i\) and \(\pi_j > \pi_i\) then \(\pi_k > \pi_i\) and \(\pi_k < \pi_j\),

if \(\pi_i + 1 = \pi_i\) and \(\pi_k > \pi_i\) then \(\pi_j > \pi_i\).

In [15] the author shows that the set of doubly alternating Baxter involutions coincides with the set of doubly alternating permutations avoiding 2413 and 3142 = 2413\(^{-1}\).

An involution is clearly doubly alternating. As a consequence, the set \(AI_n(2413,3142)\) coincides with the set of alternating Baxter involutions of length \(n\). A recurrence relation for this sequence has been found in [13] and the corresponding sequence in [17] is A347546.

Moreover we have
\[ |AI_{2n+1}(2413,3142)| = |RAI_{2n+1}(2413,3142)| = |RAI_{2n}(2413,3142)|, \]

where the first equality follows from Lemma 2.3 and the second one is a consequence of the fact that each permutation in \(RAI_{2n+1}(2413,3142)\) ends with its maximum (see [13]) and this maximum can be removed, hence obtaining any permutation in \(RAI_{2n}(2413,3142)\).

11. Patterns 4123, 2341

Since 4123 = 2341\(^{-1}\) we have \(AI_n(4123) = AI_n(2341) = AI_n(4123,2341)\) and the same is true for reverse alternating involutions.

To enumerate these classes we need the following lemma.

Lemma 11.1. The number of connected permutations in \(RAI_n(4123,2341)\) is
\[ \begin{cases} 1 & \text{if } n \text{ is odd and } n \neq 3 \\ 2 & \text{if } n \text{ is even and } n \geq 6 \\ 0 & \text{if } n = 3 \\ 1 & \text{if } n = 2, 4. \end{cases} \]

The number of connected permutations in \(AI_n(4123,2341)\) is
\[ \begin{cases} 1 & \text{if } n \geq 4 \text{ or } n = 1 \\ 0 & \text{if } n = 2 \text{ or } 3. \end{cases} \]

Proof. The small cases are trivial, so we can assume \(n \geq 5\).

Let \(\pi \in RAI_{2n}(4123,2341)\) be connected. Write \(\pi = w_k m_k w_{k-1} m_{k-1} \ldots m_2 w_1 m_1\) where the \(m_i\)'s are the rtl maxima of \(\pi\).

Since \(\pi\) is reverse alternating and avoids 4123, every \(w_i\) with \(2 \leq i \leq k - 1\) must have length 1. For the same reason, \(w_1\) must be empty (\(\pi\) ends with a descent) and the length of \(w_k\) is 0 or 2.

If \(|w_k| = 0\),
\[ \pi = 2n 2n - 22n - 12n - 42n - 3 \ldots 45231, \]

if \(|w_k| = 2\),
\[ \pi = 2n - 22n - 42n 2n - 62n - 12n - 82n - 3 \ldots 27153. \]

In a similar way, it is possible to prove that the only connected permutation in \(RAI_{2n+1}(4123,2341)\) is
\[ \pi = 2n 2n - 22n + 12n - 42n - 12n - 62n - 3 \ldots 9472513. \]

The classification of the connected permutations in \(AI_n(4123,2341)\) is fully analogous.
Example 11.1. There are two connected permutations in $RAI_{10}(4123, 2341)$, namely, 86104927153 and 10896745231, while the only connected permutation in $RAI_{11}(4123, 2341)$ is 1081169472513. The only connected permutation in $Al_{10}(4123, 2341)$ is 91078563412 and the only connected permutation in $Al_{11}(4123, 2341)$ is 911705836142.

In the following theorem we find the ordinary generating functions that enumerate reverse alternating involutions avoiding 4123 and 2341 of even and odd length (sequences $A052980$ and $A193641$ in [17], respectively). We also find the same generating function for the alternating even case. This last sequence does not appear in [17].

Theorem 11.1. We have

\[ \sum_{n \geq 0} |Al_{2n+1}(4123, 2341)| x^{2n+1} = \sum_{n \geq 0} |RAI_{2n+1}(4123, 2341)| x^{2n+1} = \frac{x^5 - x^3 + x}{1 - 2x^2 - x^6}, \]

\[ \sum_{n \geq 0} |RAI_{2n}(4123, 2341)| x^{2n} = \frac{1 - x^2}{1 - 2x^2 - x^6}, \]

\[ \sum_{n \geq 0} |Al_{2n}(4123, 2341)| x^{2n} = 1 + \frac{x^4}{1 - 2x^2 - x^6} + \frac{(x^5 - x^3 + x)^2}{(1 - 2x^2 - x^6) \cdot (1 - x^2)}. \]

Proof. The fact that $|RAI_{2n+1}(4123, 2341)| = |Al_{2n+1}(4123, 2341)|$ follows from Lemma 2.3.

Let $\pi \in RAI_{2n+1}(4123, 2341)$. Decompose $\pi$ in connected components as $\pi = \tau_1 \tau_2 \ldots \tau_k$. Denote by $\hat{\tau}_i$ the standardization of $\tau_i$. The $\hat{\tau}_i$'s are arbitrary connected permutations with $\hat{\tau}_i \in \bigcup_{j \geq 1} RAI_{j}(4123, 2341)$ for $1 \leq i \leq k - 1$ (because $\pi$ is reverse alternating) and $\tau_k \in \bigcup_{j \geq 0} RAI_{2j+1}(4123, 2341)$. By the previous lemma it follows that the ordinary generating function that counts connected permutations in $\bigcup_{j \geq 1} RAI_{2j}(4123, 2341)$ (by length) is $F(x) = x^2 + x^4 + \frac{2x^6}{1-x^2}$ and the ordinary generating function that counts connected permutations in $RAI_{2j+1}(4123, 2341)$ is $G(x) = x + \frac{x^5}{1-x^2}$.

As a consequence

\[ \sum_{n \geq 0} |RAI_{2n+1}(4123, 2341)| x^{2n+1} = \frac{G(x)}{1 - F(x)}. \]

Trivial algebraic manipulations lead to the desired generating function.

Similarly, given $\pi \in RAI_{2n}(4123, 2341)$ we can decompose $\pi$ in connected components as $\pi = \tau_1 \tau_2 \ldots \tau_k$ where $\hat{\tau}_i \in \bigcup_{j \geq 1} RAI_{2j}(4123, 2341)$ for $1 \leq i \leq k$.

As a consequence

\[ \sum_{n \geq 0} |RAI_{2n+1}(4123, 2341)| x^{2n+1} = \frac{1}{1 - F(x)}. \]

The decomposition in connected components of a permutation $\pi \in Al_{2n}(4123, 2341)$ is more subtle than the above ones because these components are not necessarily only of even or odd length. However such a permutation $\pi$ can only be

- the empty one,
- connected itself,
- of the form $\sigma \tau$, where $\sigma$ is an arbitrary permutation in $\bigcup_{j \geq 0} Al_{2j+1}(4123, 2341)$ and $\tau$ is a connected permutation in $\bigcup_{j \geq 0} RAI_{2j+1}(4123, 2341)$.

By the above results and by the previous lemma we get

\[ \sum_{n \geq 0} |Al_{2n}(4123, 2341)| x^{2n} = 1 + \frac{x^4}{1 - 2x^2 - x^6} + \frac{x^5 - x^3 + x}{1 - 2x^2 - x^6} \cdot \frac{x^5 - x^3 + x}{1 - x^2} \]

where the three summand on the left hand side correspond to the three-dotted cases. \qed
12. Other patterns

The following three conjectures are based on numerical evidences. The first one covers the last open case about the patterns 1243 and 2134 and the second covers the patterns 1432 and 3214.

Conjecture 12.1. We have
\[ |RAI_{2n}(1243)| = |RAI_{2n}(2134)| = M_n. \]

Conjecture 12.2. We have
\[ |AI_{2n}(1432)| = |AI_{2n}(3214)| = |RAI_{2n}(1432)| = |RAI_{2n}(3214)| = M_n, \]
\[ |AI_{2n+1}(1432)| = |RAI_{2n+1}(3214)| = M_n, \]
\[ |AI_{2n-1}(3214)| = |RAI_{2n-1}(1432)| = M_n - M_{n-2}. \]

Conjecture 12.3. Let \( \tau \) be any permutation of \( \{4, \ldots, m\}, \ m \geq 4 \). Then
\[ |AI_n(123\tau)| = |AI_n(321\tau)|. \]

A classical subject in pattern avoidance is the study of Wilf-equivalent patterns. Let \( \sigma \) and \( \tau \) two patterns and \( P_n \) a given subset of \( S_n \). Then \( \sigma \) and \( \tau \) are said to be Wilf-equivalent on \( P_n \) if \( |P_n(\sigma)| = |P_n(\tau)| \).

A proof of Conjectures 12.1 and 12.2 would conclude the Wilf-classification of alternating and reverse alternating involutions avoiding a pattern of length 4. In fact, trivial numerical experiments show that the only Wilf-equivalences among all the other patterns not mentioned in the paper are the trivial ones given by the reverse complement map and the inverse map.

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