FERTILITY MONOTONICITY AND AVERAGE COMPLEXITY OF THE STACK-SORTING MAP

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Abstract. Let \( D_n \) denote the average number of iterations of West’s stack-sorting map \( s \) that are needed to sort a permutation in \( S_n \) into the identity permutation 123\( \cdots \)n. We prove that

\[
0.62433 \approx \lambda \leq \liminf_{n \to \infty} \frac{D_n}{n} \leq \limsup_{n \to \infty} \frac{D_n}{n} \leq \frac{3}{5}(7 - 8 \log 2) \approx 0.87289,
\]

where \( \lambda \) is the Golomb-Dickman constant. Our lower bound improves upon West’s lower bound of 0.23, and our upper bound is the first improvement upon the trivial upper bound of 1. We then show that fertilities of permutations increase monotonically upon iterations of \( s \). More precisely, we prove that

\[
|s^{-1}(\sigma) - 1(s(\sigma))| \leq |s^{-1}(s(\sigma))|
\]

for all \( \sigma \in S_n \), where equality holds if and only if \( \sigma = 123\cdots n \).

This is the first theorem that manifests a law-of-diminishing-returns philosophy for the stack-sorting map that Bóna has proposed. Along the way, we note some connections between the stack-sorting map and the right and left weak orders on \( S_n \).

1. Introduction

Motivated by a problem involving sorting railroad cars, Knuth introduced a certain “stack-sorting algorithm” in his book *The Art of Computer Programming* [14]. Knuth’s analysis of this algorithm led to several advances in combinatorics, including the notion of a permutation pattern and the kernel method [1,2,13,16]. In his 1990 Ph.D. dissertation, West defined a deterministic variant of Knuth’s algorithm. This variant, which is a function that we denote by \( s \), has now received a huge amount of attention (see [2,3,6,7] and the references therein). West’s original definition makes use of a stack that is allowed to hold entries from a permutation. Here, a permutation is an ordering of a finite set of integers, written in one-line notation. Let \( S_n \) denote the set of permutations of the set \([n] := \{1, \ldots, n\}\). Assume we are given an input permutation \( \pi = \pi_1 \cdots \pi_n \). Throughout this procedure, if the next entry in the input permutation is smaller than the entry at the top of the stack or if the stack is empty, the next entry in the input permutation is placed at the top of the stack. Otherwise, the entry at the top of the stack is annexed to the end of the growing output permutation. This procedure stops when the output permutation has length \( n \). We then define \( s(\pi) \) to be this output permutation. Figure 1 illustrates this procedure and shows that \( s(4162) = 1426 \).

There is also a simple recursive definition of the map \( s \). First, we declare that \( s \) sends the empty permutation to itself. Given a nonempty permutation \( \pi \), we can write \( \pi = LmR \), where \( m \) is the

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Theorem 1.1. We have

\[ 0.62433 \approx \lambda \leq \liminf_{n \to \infty} \frac{D_n}{n} \leq \limsup_{n \to \infty} \frac{D_n}{n} \leq \frac{3}{5}(7 - 8 \log 2) \approx 0.87289, \]

where \( \lambda \) is the Golomb-Dickman constant.
Another crucial notion in the study of the stack-sorting map is that of the *fertility* of a permutation \( \pi \), which is simply \(|s^{-1}(\pi)|\). Many problems concerning the stack-sorting map can be phrased in terms of fertilities. For example, computing \( W_2(n) \) is equivalent to finding the sum of the fertilities of all of the 231-avoiding (i.e., 1-stack-sortable) permutations in \( S_n \). The author found methods for computing fertilities of permutations \([7,9]\), which led to the above-mentioned advancements in the study \( t \)-stack-sortable permutations when \( t \in \{3,4\} \). Permutations with fertility 1 (called uniquely sorted permutations) possess some remarkable enumerative properties \([6,10,18]\). There is also a surprising connection between fertilities of permutations and free probability theory \([10]\).

In Exercise 23 of Chapter 8 in \([2]\), Bóna asks the reader to find the element of \( S_n \) with the largest fertility. As one might expect, the answer is 123 \cdots n. The proof is not too difficult, but it is also not trivial. Our second main theorem generalizes this result by showing that the fertility statistic is strictly monotonically increasing as one moves up the stack-sorting tree.

**Theorem 1.2.** For every permutation \( \sigma \in S_n \), we have

\[
|s^{-1}(\sigma)| \leq |s^{-1}(s(\sigma))|,
\]

where equality holds if and only if \( \sigma = 123 \cdots n \).

Theorem 1.2 represents a step toward a law-of-diminishing-returns philosophy for the stack-sorting map that Miklós Bóna has postulated. Roughly speaking, his idea is that each successive iteration of the stack-sorting map should be less efficient in sorting permutations than the previous iteration of the stack-sorting map. One might expect the fertility of a \((t+1)\)-stack-sortable permutation in \( S_n \) is at most the average fertility of a \( t\)-stack-sortable permutation in \( S_n \). While Theorem 1.2 does not imply this conjecture, it is a step in the right direction.

**Remark 1.1.** Suppose \( \pi = \pi_1 \cdots \pi_n \in S_n \), and let \( i \in [n-1] \). If \( \pi_i > \pi_{i+1} \), let \( t_i(\pi) \) be the permutation obtained from \( \pi \) by swapping the positions of the entries \( \pi_i \) and \( \pi_{i+1} \). If \( \pi_i < \pi_{i+1} \), let \( t_i(\pi) = \pi \). If \( i+1 \) appears to the left of \( i \) in \( \pi \), let \( \bar{t}_i(\pi) \) be the permutation obtained by swapping the positions of \( i \) and \( i+1 \) in \( \pi \). Otherwise, let \( \bar{t}_i(\pi) = \pi \). The right weak order on \( S_n \) is the partial order \( \leq_{\text{right}} \) on \( S_n \) defined by saying that \( \pi' \leq_{\text{right}} \pi \) if there exists a sequence \( i_1, \ldots, i_m \) of elements of \([n-1]\) such that \( t_{i_m} \circ \cdots \circ t_{i_1}(\pi) = \pi' \). The left weak order on \( S_n \) is the partial order \( \leq_{\text{left}} \) on \( S_n \) defined by saying that \( \pi' \leq_{\text{left}} \pi \) if there exists a sequence \( i_1, \ldots, i_m \) of elements of \([n-1]\) such that \( t_{i_m} \circ \cdots \circ t_{i_1}(\pi) = \pi' \).

Theorem 1.2 is a little bit strange in view of the relationship between the stack-sorting map and these two partial orders. It is not difficult to show that for every permutation \( \sigma \in S_n \), we have \( s(\sigma) \leq_{\text{right}} \sigma \). Therefore, one might expect to prove Theorem 1.2 by first establishing that \(|s^{-1}(\pi')| \geq |s^{-1}(\pi)|\) whenever \( \pi' \leq_{\text{right}} \pi \). However, this turns out to be false. We have 31425 \( \leq_{\text{right}} \) 34125, but one can show that \(|s^{-1}(31425)| = 1 < 4 = |s^{-1}(34125)|\). On the other hand, we will be able to prove (see Theorem 3.2 below) that

\[
|s^{-1}(\pi')| \geq |s^{-1}(\pi)| \quad \text{whenever} \quad \pi' \leq_{\text{left}} \pi.
\]

Unfortunately, this inequality does not immediately imply Theorem 1.2 because the left weak order is not compatible with the action of the stack-sorting map. To see this, note that \( s(231) = 213 \not\leq_{\text{left}} 231 \). Our proof of Theorem 1.2 will combine (1) with the Decomposition Lemma proved in \([7]\).
2.1. Preliminary Results. Let us begin this section with some basic terminology. The normalization of a permutation $\pi$ is the permutation in $S_n$ obtained by replacing the $i$th-smallest entry in $\pi$ with $i$ for all $i$. For example, the normalization of $4682$ is $1234$. We say two permutations have the same relative order if their normalizations are equal. We will tacitly use the fact, which is clear from either definition of the stack-sorting map, that $s(\pi)$ and $s(\pi')$ have the same relative order whenever $\pi$ and $\pi'$ have the same relative order. Furthermore, permutations with the same relative order have the same fertility.

A right-to-left maximum of a permutation $\pi = \pi_1 \cdots \pi_n$ is an entry $\pi_i$ such that $\pi_i > \pi_j$ for every $j \in \{i + 1, \ldots, n\}$. For each nonnegative integer $r \leq n$, let $\text{del}_r(\pi)$ be the permutation obtained by deleting the $r$ smallest entries from $\pi$. For example, $\text{del}_2(436718) = 4678$. If $r = n$, then $\text{del}_r(\pi)$ is the empty permutation.

**Lemma 2.1.** Let $\pi = \pi_1 \cdots \pi_n$ be a permutation. For all nonnegative integers $r$ and $t$ with $r \leq n$, we have

$$s^t(\text{del}_r(\pi)) = \text{del}_r(s^t(\pi)).$$

**Proof.** It suffices to prove the case in which $t = 1$; the general case will then follow by induction on $t$. The proof is trivial if $n \leq 1$, so we may assume $n \geq 2$ and induct on $n$. If $r = n$, then $s(\text{del}_r(\pi))$ and $\text{del}_r(s(\pi))$ are both empty. Thus, we may assume $0 \leq r \leq n - 1$. Write $\pi = LmR$, where $m$ is the largest entry in $\pi$. Among the $r$ smallest entries in $\pi$, let $r_L$ (respectively, $r_R$) be the number that lie in $L$ (respectively, $R$). Using the recursive definition of the stack-sorting map and our inductive hypothesis, we find that

$$s(\text{del}_r(\pi)) = s(\text{del}_{r_L}(L)m\text{del}_{r_R}(R)) = s(\text{del}_{r_L}(L))s(\text{del}_{r_R}(R)m) = \text{del}_{r_L}(s(L))\text{del}_{r_R}(s(R)m) = \text{del}_r(s(L)s(R)m) = \text{del}_r(s(\pi)).$$

We say two entries $b, a$ in a permutation $\pi$ form a 21 pattern if $b$ appears to the left of $a$ in $\pi$ and $a < b$. We say three entries $b, c, a$ in $\pi$ form a 231 pattern if they appear in the order $b, c, a$ (from left to right) in $\pi$ and satisfy $a < b < c$. The next lemma follows immediately from either definition of the stack-sorting map; it is Lemma 4.2.2 in [19].

**Lemma 2.2.** Let $\pi$ be a permutation. Two entries $b, a$ form a 21 pattern in $s(\pi)$ if and only if there exists an entry $c$ such that $b, c, a$ form a 231 pattern in $\pi$.

The next lemma is also an easy consequence of the definition of $s$.

**Lemma 2.3.** Let $\pi$ be a permutation whose smallest entry is $a$, and write $\pi = LaR$. The entries to the right of $a$ in $s(\pi)$ are the entries in $R$ and the right-to-left maxima of $L$.

**Proof.** An entry $b$ appears to the left of $a$ in $s(\pi)$ if and only if $b, a$ form a 21 pattern in $s(\pi)$. By Lemma 2.2, this occurs if and only if there exists an entry $c$ in $\pi$ such that $b, c, a$ form a 231 pattern in $\pi$. This occurs if and only if $b$ is not in $R$ and is not a right-to-left maximum of $L$.

Consider a permutation $\pi$ whose entries are all positive. Let $\pi 0$ be the concatenation of $\pi$ with the new entry 0. Define $\text{ssd}'(\pi)$ to be the smallest positive integer $t$ such that 0 is in the first
position of $s^t(\pi_0)$. Let
\[ D'_n = \frac{1}{n!} \sum_{\pi \in S_n} \text{ssd}'(\pi). \]
We are going to see that this new quantity $D'_n$ is very close to $D_n$; it will have the advantage of being much easier to analyze.

**Lemma 2.4.** For each permutation $\pi = \pi_1 \cdots \pi_n$ with positive entries, we have $\text{ssd}'(\pi) = \text{ssd}(\pi_0)$.

**Proof.** We claim that for every $t \geq 0$, the entries to the right of 0 in $s^t(\pi_0)$ appear in increasing order. The claim is vacuously true for $t = 0$ because there are no entries to the right of 0 in $\pi_0$. Now let $t \geq 1$, and suppose we know that the entries to the right of 1 in $s^{t-1}(\pi_0)$ are in increasing order.

In other words, we can write $s^{t-1}(\pi_0) = L0R$, where $R$ is increasing. According to Lemma 2.3, the entries to the right of 0 in $s^t(\pi_0)$ are the entries in $R$ and the right-to-left maxima of $L$. The right-to-left maxima of $L$ are in decreasing order in $s^{t-1}(\pi_0)$, while the entries in $R$ are in increasing order. Thus, no two of these entries can form the first and third entries in a 231 pattern in $s^t(\pi_0)$. By Lemma 2.2, no two of these entries form a 21 pattern in $s^t(\pi_0)$. This proves the claim, and the proof of the lemma follows. \qed

To get a better understanding of the statistic $\text{ssd}'$, we introduce the following (admittedly dense) notation. An ordered set partition of a set $E$ of positive integers is a tuple $\mathcal{B} = (B_1, \ldots, B_r)$ of pairwise-disjoint nonempty sets $B_1, \ldots, B_r$ such that $\bigcup_{i=1}^r B_i = E$. We say $\mathcal{B}$ is in standard form if $\max B_1 > \cdots > \max B_r$. We make the convention that the empty tuple $()$ is an ordered set partition of $\emptyset$ in standard form. Let $\mathcal{M}(\mathcal{B}) = \{ \max B_i : 1 \leq i \leq r \}$ be the set of maximum elements of the sets in $\mathcal{B}$. By convention, $\mathcal{M}(()) = \emptyset$. We are going to form a new ordered set partition $\eta(\mathcal{B})$, which will be in standard form. Begin by forming the new tuple $\hat{\mathcal{B}} = (\hat{B}_1, \ldots, \hat{B}_r)$, where \( \hat{B}_i = B_i \setminus \{ \max B_i \} \). If all of the sets $\hat{B}_i$ are empty, we simply define $\eta(\mathcal{B}) = ()$. Now assume that at least one of the sets $\hat{B}_i$ is nonempty. Let $J$ be the set of indices $j$ such that $\max \hat{B}_j > \max \hat{B}_i$ for all $i \in \{ j+1, \ldots, r \}$ (where $\max \emptyset = -\infty$ by convention). We can write $J = \{ j_1 < \cdots < j_h \}$.

For each $\ell \in \{ 1, \ldots, h \}$, let $B'_\ell = \bigcup_{j_{\ell-1} < i \leq j_\ell} \hat{B}_i$ (where $j_0 = 0$). Now let $\eta(\mathcal{B})$ be the tuple obtained from $(B'_1, \ldots, B'_h)$ by removing any occurrences of $\emptyset$. The tuple $\eta(\mathcal{B})$ is an ordered set partition in standard form.

**Example 2.1.** Let $E = \{1, \ldots, 12\}$, and let $\mathcal{B} = (\{9, 12\}, \{6, 11\}, \{1, 4, 10\}, \{7, 8\}, \{2, 5\}, \{3\})$. Note that $\mathcal{B}$ is an ordered set partition in standard form. We have $\mathcal{M}(\mathcal{B}) = \{3, 5, 8, 10, 11, 12\}$.

Removing the elements of $\mathcal{M}(\mathcal{B})$ from the sets in $\mathcal{B}$ yields the tuple $\hat{\mathcal{B}} = (\{9\}, \{6\}, \{1, 4\}, \{7\}, \{2\}, \emptyset)$. Now, $J = \{1, 4, 5, 6\}$ (so $h = 4$). We have $(B'_1, B'_2, B'_3, B'_4) = (\{9\}, \{1, 4, 6, 7\}, \{2\}, \emptyset)$, so $\eta(\mathcal{B}) = (\{9\}, \{1, 4, 6, 7\}, \{2\})$.

Now take a permutation $\pi = \pi_1 \cdots \pi_n$ with positive entries, and let $E(\pi) = \{ \pi_1, \ldots, \pi_n \}$ be the set of entries in $\pi$. Let $\pi_{i_1} > \cdots > \pi_{i_r}$ be the right-to-left maxima of $\pi$ (so $i_1 < \cdots < i_r$). Let $\mathcal{B}(\pi) = \{ \pi_i : i_{\ell-1} < i \leq i_\ell \}$ be the set of entries in $\pi$ that lie strictly to the right of $\pi_{i_{\ell-1}}$ and weakly to the left of $\pi_{i_\ell}$ (with the convention $i_0 = 0$). The tuple $B_1(\pi) = (\mathcal{B}_1(\pi), \ldots, \mathcal{B}_r(\pi))$ is an ordered set partition of the set $E(\pi)$ in standard form. Let $\mathcal{M}_1(\pi) = \mathcal{M}(B_1(\pi))$. Note that $\mathcal{M}_1(\pi)$ is just the set of right-to-left maxima of $\pi$. Now let $B_2(\pi) = \eta(B_1(\pi))$ and $\mathcal{M}_2(\pi) = \mathcal{M}(B_2(\pi))$. In general, define $B_\ell(\pi) = \eta(B_{\ell-1}(\pi))$ and $\mathcal{M}_\ell(\pi) = \mathcal{M}(B_\ell(\pi))$. Note that there exists some integer $t$ such that $B_\ell(\pi) = ()$ and $\mathcal{M}_\ell(\pi) = \emptyset$ for all $\ell \geq t + 1$. We will see that the smallest such integer $t$ is $\text{ssd}'(\pi)$.
Example 2.2. Suppose $\pi = 9 126 11 4 1 10 7 8 253$. The right-to-left maxima of $\pi$ are the entries 12, 11, 10, 8, 5, 3, so

$$\mathcal{B}_1(\pi) = (\mathcal{B}_1(\pi), \ldots, \mathcal{B}_6(\pi)) = \{(9, 12), \{6, 11\}, \{1, 4, 10\}, \{7, 8\}, \{2, 5\}, \{3\}\}$$

and $\mathcal{M}_1(\pi) = \{3, 5, 8, 10, 11, 12\}$. We saw in Example 2.1 that

$$\mathcal{B}_2(\pi) = \eta(B_1(\pi)) = \{(9), \{1, 4, 6, 7\}, \{2\}\}.$$ 

Thus, $\mathcal{M}_2(\pi) = \mathcal{M}(\mathcal{B}_2(\pi)) = \{2, 7, 9\}$. We can now compute $\mathcal{B}_3(\pi) = \eta(B_2(\pi)) = \{\{1, 4, 6\}\}$. $\mathcal{M}_3(\pi) = \{\{1\}\}$, $\mathcal{M}_4(\pi) = \{\{4\}\}$, $\mathcal{B}_5(\pi) = \{\{1\}\}$, and $\mathcal{M}_5(\pi) = \{\{1\}\}$. Finally, we have $\mathcal{B}_6(\pi) = ()$ and $\mathcal{M}_6(\pi) = \emptyset$ for all $\ell \geq 6$. 

Lemma 2.3 tells us that $\mathcal{M}_1(\pi)$ is precisely the set of entries that move to the right of 0 when we apply $s$ to $\pi$. This means that we can write $s(\pi_0) = L0R$, where $R$ consists of the entries in $\mathcal{M}_1(\pi_0)$. It is straightforward to verify from the definition of $s$ that $\mathcal{M}_2(\pi)$ is the set of right-to-left maxima of $L$. Applying Lemma 2.3 again, we see that $\mathcal{M}_2(\pi)$ is the set of entries that move to the right of 0 when we apply $s$ to $s(\pi_0)$. Continuing this line of reasoning, we see that $\mathcal{M}_t(\pi)$ is the set of entries that move to the right of 0 when we apply $s$ to $s^{t-1}(\pi_0)$. This proves that ssd$'$($\pi$) is the smallest integer $t$ such that $\mathcal{M}_t(\pi_0) = \emptyset$. Equivalently, it is the smallest integer $t$ such that $\mathcal{B}_{t+1}(\pi) = ()$. Note that the sets $\mathcal{M}_1(\pi), \ldots, \mathcal{M}_{\text{ssd}'(\pi)}(\pi)$ form a partition of the set $\mathcal{S}(\pi)$ of entries of $\pi$. This allows us to describe ssd$'$($\pi$) as the smallest integer $t$ such that $\sum_{\ell=1}^{t} |\mathcal{M}_\ell(\pi)| = n$, where $n$ is the number of entries in $\pi$.

Lemma 2.5. If $\pi = \pi_1 \cdots \pi_n$ is a permutation with positive entries and $t$ is a positive integer, then ssd$'$($\pi$) is of the form LR, where $R$ is the increasing permutation of the set $\bigcup_{i=1}^{t} \mathcal{M}_i(\pi)$. The set of right-to-left maxima of $L$ is $\mathcal{M}_{t+1}(\pi)$.

Proof. We saw in the proof of Lemma 2.4 that the we can write $s^t(\pi_0) = L0R$, where $R$ is increasing. It follows from the above discussion that the set of entries appearing in $R$ is $\bigcup_{i=1}^{t} \mathcal{M}_i(\pi)$ and that the set of right-to-left maxima of $L$ is $\mathcal{M}_{t+1}(\pi)$. By Lemma 2.1, we have $s^t(\pi) = \text{del}_1(s^t(\pi_0)) = LR$. \qed

Lemma 2.6. For every positive integer $n$, we have $D'_{n+1} \leq D'_n + 1$.

Proof. Choose $\pi \in S_{n+1}$, and let $\tilde{\pi} = \text{del}_1(\pi)$. By Lemma 2.5 we can write $s^{t-1}(\pi) = LR$, where $R$ is a permutation of the set $\bigcup_{i=1}^{t-1} \mathcal{M}_i(\tilde{\pi})$ and $\mathcal{M}_t(\tilde{\pi})$ is the set of right-to-left maxima of $L$. Similarly, we can write $s^{t-1}(\tilde{\pi}) = \tilde{L}R$, where $\tilde{R}$ is a permutation of the set $\bigcup_{i=1}^{t-1} \mathcal{M}_i(\tilde{\pi})$ and $\mathcal{M}_t(\tilde{\pi})$ is the set of right-to-left maxima of $\tilde{L}$. Lemma 2.1 tells us that $s^{t-1}(\tilde{\pi}) = \text{del}_1(s^{t-1}(\pi))$. It now follows (by induction on $\ell$) that for every $\ell \in \{1, \ldots, \text{ssd}'(\tilde{\pi})\}$, we have either $\mathcal{M}_\ell(\pi) = \mathcal{M}_\ell(\tilde{\pi})$ or $\mathcal{M}_\ell(\pi) = \mathcal{M}_\ell(\tilde{\pi}) \cup \{1\}$. Consequently, $\sum_{\ell=1}^{\text{ssd}'(\tilde{\pi})+1} |\mathcal{M}_\ell(\pi)| \geq \sum_{\ell=1}^{\text{ssd}'(\tilde{\pi})} |\mathcal{M}_\ell(\tilde{\pi})| = n$. This shows that $\sum_{\ell=1}^{\text{ssd}'(\pi)} |\mathcal{M}_\ell(\pi)| \geq n + 1$, so ssd$'$($\pi$) $\leq$ ssd$'$($\pi$) + 1. Letting $f(\pi)$ denote the normalization of $\tilde{\pi} = \text{del}_1(\pi)$, we see that ssd$'$($\pi$) $\leq$ ssd$'$($f(\pi)$) + 1 for every $\pi \in S_{n+1}$. The map $f : S_{n+1} \to S_n$ is $(n+1)$-to-1, so

$$D'_{n+1} = \frac{1}{(n+1)!} \sum_{\pi \in S_{n+1}} \text{ssd}'(\pi) \leq \frac{1}{(n+1)!} \sum_{\pi \in S_{n+1}} (\text{ssd}'(f(\pi)) + 1) = \frac{1}{(n+1)!} \sum_{\sigma \in S_n} (\text{ssd}'(\sigma) + 1)(n+1)$$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} (\text{ssd}'(\sigma) + 1) = D'_n + 1. \quad \Box$$

We are now in a position to prove the main proposition that will allow us to focus our attention on the numbers $D'_n$ instead of the numbers $D_n$; this will make our proofs much simpler.
Proposition 2.1. We have

\[
\lim_{n \to \infty} \left( \frac{D_n'}{n} - \frac{D_n}{n} \right) = 0.
\]

Proof. Let \( \pi \in S_n \). Suppose \( \pi 0 \) is \( t \)-stack-sortable. Applying Lemma 2.1 with \( \tau = 1 \) shows that \( 123 \cdots n = \text{del}_1(s(t(\pi 0))) = s(t(\pi)) \), so \( \pi \) is \( t \)-stack-sortable. Along with Lemma 2.4, this proves that \( \text{ssd}(\pi) = \text{ssd}(\pi 0) \geq \text{ssd}(\pi) \). As \( \pi \) was arbitrary, we find that

\[
D_n' \geq D_n.
\]

We now want to show that \( D_n \) is not too much less than \( D_n' \). Given \( \tau \in S_n \), let \( e_i(\tau) \) be the number of entries in \( \{i + 1, \ldots, n\} \) that lie to the left of \( i \) in \( \tau \). Let \( M_\tau = \max_{1 \leq i \leq n} e_i(\tau) \). Fix \( k \leq n - 1 \), and put \( \Omega_{n,k} = \{ \tau \in S_n : M_\tau = k \} \). Choose \( \pi \in \Omega_{n,k} \) uniformly at random, and let \( a \) be the smallest entry such that \( e_a(\pi) = k \). Let \( \sigma' \) be the subpermutation of \( \pi \) consisting of entries in \( \{a + 1, \ldots, n\} \) that lie to the left of \( a \), and let \( \sigma \in S_k \) be the normalization of \( \sigma' \). By definition, \( \pi \) is \( \text{ssd}(\pi) \)-stack-sortable. Applying Lemma 2.1, we find that \( \text{del}_{a-1}(\pi) \) is \( \text{ssd}(\pi) \)-stack-sortable. This means that after \( \text{ssd}(\pi) \) iterations of the stack-sorting map, the entry \( a \) in \( \text{del}_{a-1}(\pi) \) moves to the left of all of the entries of \( \sigma' \). During each iteration of \( s \), the number of positions that \( a \) moves to the left does not depend on the order of the entries to the right of \( a \) (by Lemma 2.3). Since \( \sigma' a \) has the same relative order as \( \sigma 0 \), it follows that \( 0 \) will be the first entry in \( s^{\text{ssd}(\pi)}(\sigma 0) \). In order words, \( \text{ssd}'(\sigma) \leq \text{ssd}(\pi) \). We chose \( \sigma \) by first choosing \( \pi \) uniformly at random from \( \Omega_{n,k} \) and then normalizing a specific subpermutation of \( \pi \). It is straightforward to check that each permutation in \( S_k \) is equally likely to be chosen as \( \sigma \). Therefore, the expected value of \( \text{ssd}(\pi) \) when \( \pi \) is chosen uniformly at random from \( \Omega_{n,k} \) is at least the expected value of \( \text{ssd}'(\sigma) \) when \( \sigma \) is chosen uniformly at random from \( S_k \); the latter expected value is precisely \( D_k' \). Consequently,

\[
D_n = \frac{1}{n!} \sum_{k=0}^{n-1} \sum_{\pi \in \Omega_{n,k}} \text{ssd}(\pi) \geq \frac{1}{n!} \sum_{k=0}^{n-1} |\Omega_{n,k}| |D_k' \geq \frac{1}{n!} \sum_{k=K_n}^{n-1} |\Omega_{n,k}| |D_k' \geq \frac{1}{n!} \left( \min_{K_n \leq k \leq n-1} D_k' \right) \sum_{k=K_n}^{n-1} |\Omega_{n,k}|,
\]

where \( K_n = \lfloor n - 2\sqrt{n} \log n \rfloor \). Let \( K_n' = \lfloor n - \sqrt{n} \log n \rfloor \). It is known (see [1]) that

\[
\frac{1}{n!} \sum_{k=K_n}^{n-1} |\Omega_{n,k}| = 1 - \frac{1}{n!} (K_n - 1)! K_n^{n-K_n+1} = 1 - \prod_{r=K_n}^{n} \frac{K_n}{r} \geq 1 - \prod_{r=K_n}^{n} \frac{K_n'}{r} \geq 1 - o(1).
\]

It follows from Lemma 2.6 that \( \min_{K_n \leq k \leq n-1} D_k' \geq D_n' - (n - K_n) = D_n' - o(n) \). Consequently, \( D_n \geq (1 - o(1))(D_n' - o(n)) \). Combining this with (2) shows that

\[
0 \leq \frac{D_n'}{n} - \frac{D_n}{n} \leq \frac{D_n'}{n} - (1 - o(1)) \frac{D_n' - o(n)}{n} = \frac{D_n'}{n} o(1) + o(1).
\]

The desired result now follows from the fact that \( D_n' = O(n) \). \( \square \)

Now that we have proved the necessary lemmas, we can proceed to the proof of Theorem 1.1.
2.2. Lower Bound. Let \( \mathcal{S}_n \) denote the set of bijections from \([n]\) to \([n]\), which we write in disjoint cycle notation. Of course, \( S_n \) and \( \mathcal{S}_n \) are just two different incarnations of the set of permutations of \([n]\). Let \( \pi_1, \ldots, \pi_r \) be the right-to-left maxima of a permutation \( \pi \in S_n \), where \( i_1 < \cdots < i_r \). We denote by \( \pi(\ell) \) the subpermutation \( \pi_{i_{\ell-1}+1} \pi_{i_{\ell-1}+2} \cdots \pi_{i_{\ell}} \) (with \( i_0 = 0 \)). For example, if \( \pi = 6173542 \), then \( \pi(1) = 617, \pi(2) = 35, \pi(3) = 4, \) and \( \pi(4) = 2 \). The entries in \( \pi(\ell) \) are precisely the elements of the set \( \mathcal{B}_\ell(\pi) \). If we put parentheses around the subpermutations \( \pi(1), \ldots, \pi(r) \), we obtain the disjoint cycle decomposition of an element of \( \mathcal{S}_n \). For example, the permutation \( \pi = 6173542 \in S_7 \) gives rise to \((617)(35)(4)(2) \in \mathcal{S}_7 \). Foata’s transition lemma (see [2] page 109) asserts that this map is a bijection from \( S_n \) to \( \mathcal{S}_n \). Thus, the distribution of sizes of the sets \( \mathcal{B}_\ell(\pi) \) in a random permutation in \( S_n \) is the same as the distribution of cycle lengths in a random element of \( \mathcal{S}_n \).

The Golomb-Dickman constant \( \lambda \approx 0.62433 \) is defined by \( \lambda = \lim_{n \to \infty} \frac{\alpha_n}{n} \), where \( \alpha_n \) is the expected length of the longest cycle in a bijection chosen uniformly at random from \( \mathcal{S}_n \). According to the above remarks, \( \alpha_n \) is also the expected value of \( \max |\mathcal{B}_\ell(\pi)| \) when \( \pi \in S_n \) is chosen uniformly at random. Golomb [12] was the first to observe that the limit defining \( \lambda \) exists because the sequence \( (\alpha_n/n)_{n \geq 1} \) is monotonically decreasing. Llyod and Shepp [17] proved that \( \alpha_n \). In other words, \( \mathcal{D}_n \geq \alpha_n \). It now follows from Proposition 2.1 that

\[
\liminf_{n \to \infty} \frac{\mathcal{D}_n}{n} = \liminf_{n \to \infty} \frac{\mathcal{D}'_n}{n} \geq \lim_{n \to \infty} \frac{\alpha_n}{n} = \lambda. \tag{2.3}
\]

2.3. Upper Bound. Let \( \mathcal{B} = (B_1, \ldots, B_r) \) be an ordered set partition in standard form. For \( m \in \{1, \ldots, r-1\} \), let \( E_m = \bigcup_{i=m+1}^r B_i \). We say the set \( E_m \) is quarantined in \( \mathcal{B} \) if \(|B_m| \geq |E_m| \) and the \( j \)th-largest element of \( B_m \) is greater than the \( j \)th-largest element of \( E_m \) for all \( 1 \leq j \leq |E_m| \). The terminology is motivated by imagining that we form the ordered set partitions \( \eta(\mathcal{B}), \eta^2(\mathcal{B}), \ldots \). When we do this, it is possible that some of the elements of \( \bigcup_{i=1}^m B_i \) will end up merging with elements from \( E_m \). However, this will never happen if \( E_m \) is quarantined in \( \mathcal{B} \) (the elements of \( E_m \) stay separated from the elements of \( \bigcup_{i=1}^m B_i \) until they all disappear).

Lemma 2.7. Let \( \pi = \pi_1 \cdots \pi_n \) be a permutation with positive entries, and let \( \pi_{i_1} > \cdots > \pi_{i_r} \) be the right-to-left maxima of \( \pi \). Let \( \mathcal{B}_1(\pi) = (\mathcal{B}_1(\pi), \ldots, \mathcal{B}_r(\pi)) \) be the ordered set partition obtained from \( \pi \), and let \( \xi_m(\pi) = \bigcup_{i=m+1}^r \mathcal{B}_i(\pi) \). If \( \xi_m(\pi) \) is quarantined in \( \mathcal{B}(\pi) \), then \( \text{ssd}(\pi) \leq m \).

Proof. Let \( \ell \) be the largest integer such that one of the sets in \( \mathcal{B}_\ell(\pi) = \eta^{\ell-1}(\mathcal{B}_1(\pi)) \) contains an element of \( \xi_m(\pi) \). The assumption that \( \xi_m(\pi) \) is quarantined implies that each of the sets \( \mathcal{M}_j(\pi) \) with \( 1 \leq j \leq \ell \) contains at least one element of \( \bigcup_{i=1}^m \mathcal{B}_i(\pi) \). It follows that \( \bigcup_{j=1}^\ell \mathcal{M}_j(\pi) \) contains \( \xi_m(\pi) \) and at least \( \ell \) elements of \( \bigcup_{i=1}^m \mathcal{B}_i(\pi) \). Thus, there are at most \( n - |\xi_m(\pi)| - \ell \) elements of
Each of the sets $\mathcal{M}_j(\pi)$ with $\ell + 1 \leq j \leq \text{ssd}'(\pi)$ is nonempty, so

$$n - |\mathcal{E}_m(\pi)| - \ell \geq \sum_{j=\ell+1}^{\text{ssd}'(\pi)} |\mathcal{M}_j(\pi)| \geq \sum_{j=\ell+1}^{\text{ssd}'(\pi)} 1 = \text{ssd}'(\pi) - \ell.$$

This completes the proof since $i_m = n - |\mathcal{E}_m(\pi)|$.  \hfill \Box

**Lemma 2.8.** Let $0 = i_0 < i_1 < \cdots < i_r = n$ be integers. Choose a permutation $\pi = \pi_1 \cdots \pi_n \in S_n$ uniformly at random among all permutations in $S_n$ whose right-to-left maxima are in positions $i_1, \ldots, i_r$. Form the ordered set partition $\mathcal{B}_1(\pi) = (\mathcal{B}_1(\pi), \ldots, \mathcal{B}_r(\pi))$. For $1 \leq m \leq r-1$, let $\mathcal{E}_m(\pi) = \bigcup_{i=m+1}^n \mathcal{B}_i(\pi)$. The probability that $\mathcal{E}_m(\pi)$ is quarantined in $\mathcal{B}_1(\pi)$ is at least

$$1 - \left( \frac{n - i_m}{i_m - i_{m-1}} \right)^2.$$

**Proof.** Let $\mathcal{U}(\pi) = \mathcal{B}_m(\pi) \cup \mathcal{E}_m(\pi)$. We can write $\mathcal{U}(\pi) = \{u_1 > \cdots > u_{n-i_{m-1}}\}$. We can use these sets to define a lattice path $\mathcal{L}(\pi)$ in $\mathbb{Z}^2$ that starts at $(0,0)$ and ends at $(i_m - i_{m-1}, n - i_m)$ as follows. If $u_j \in \mathcal{B}_m(\pi)$, let the $j$th step of $\mathcal{L}(\pi)$ be an east step (i.e., a $(1,0)$ step). Otherwise, we have $u_j \in \mathcal{E}_m(\pi)$; in this case, let the $j$th step of $\mathcal{L}(\pi)$ be a north step (i.e., a $(0,1)$ step). Notice that $\pi_{i_m} = \pi_1$, because $\pi_{i_m}$ is a right-to-left maximum of $\pi$. This means that the first step of $\mathcal{L}(\pi)$ is an east step. If we remove this initial east step, we obtain a lattice path $\mathcal{L}'(\pi)$ starting at $(0,1)$ and ending at $(i_m - i_{m-1}, n - i_m)$ that uses only east steps and north steps. Every such path is equally likely to arise as $\mathcal{L}'(\pi)$ when we choose $\pi$ at random. The event that $\mathcal{E}_m(\pi)$ is quarantined in $\mathcal{B}_1(\pi)$ is equivalent to the event that $\mathcal{L}'(\pi)$ stays weakly below the line $y = x$. According to [15 Theorem 10.3.1], the probability that $\mathcal{L}'(\pi)$ stays weakly below the line $y = x$ is

$$1 - \left( \frac{n - i_m}{i_m - i_{m-1}} \right)^2.$$

**Proof of the Upper Bound in Theorem 1.1.** For $x \in (0,1)$, let

$$F_0(x) = \frac{1}{1-x} \int_{\frac{x}{2}}^{1} \left( \left( 1 - \left( \frac{1-y}{y-x} \right)^2 \right) y + \left( \frac{1-y}{y-x} \right)^2 \right) dy.$$

One can check that

$$F_0(x) = a_0 x + b_0, \quad \text{where} \quad a_0 = 3 \log 2 - 2 \quad \text{and} \quad b_0 = \frac{5}{2} - 3 \log 2.$$

Let us choose a random permutation $\pi \in S_n$, where $n$ is very large. Recall that $\frac{D_n'}{n}$ is the expected value of $\frac{\text{ssd}'(\pi)}{n}$. Let $i_1 < \cdots < i_r$ be the positions of the right-to-left maxima of $\pi$. Consider the ordered set partition $\mathcal{B}_1(\pi) = (\mathcal{B}_1(\pi), \ldots, \mathcal{B}_r(\pi))$. For $1 \leq m \leq r-1$, let $\mathcal{E}_m(\pi) = \bigcup_{i=m+1}^n \mathcal{B}_i(\pi)$. The position $i_1$ of the maximum entry $n$ is uniformly distributed among $\{1, \ldots, n\}$. Let us first suppose $i_1 \geq n/2$. Once $i_1$ is chosen, we can use Lemma 2.8 (with $m = 1$) to see that the probability that $\mathcal{E}_1(\pi)$ is quarantined in $\mathcal{B}_1(\pi)$ is at least $1 - \left( \frac{n - i_1}{i_1} \right)^2$. If $\mathcal{E}_1(\pi)$ is quarantined in $\mathcal{B}_1(\pi)$, then it follows from Lemma 2.7 that $\frac{\text{ssd}'(\pi)}{n} \leq \frac{i_1}{n}$. If $\mathcal{E}_1(\pi)$ is not quarantined, then (trivially) $\frac{\text{ssd}'(\pi)}{n} \leq 1$. 


If \( i_1 < n/2 \), then again \( \frac{ssd'(\pi)}{n} \leq 1 \). Thus, the expected value of \( \frac{ssd'(\pi)}{n} \) is at most
\[
\frac{1}{n} \sum_{i_1 \geq n/2} \left( \left( 1 - \left( \frac{n - i_1}{i_1} \right)^2 \right) \frac{i_1}{n} + \left( \frac{n - i_1}{i_1} \right)^2 \cdot 1 \right) + \sum_{i_1 < n/2} 1.
\]
As \( n \to \infty \), this last expression tends to
\[
\int_{1/2}^{1} \left( \left( 1 - \left( \frac{1 - x_1}{x_1} \right)^2 \right) x_1 + \left( \frac{1 - x_1}{x_1} \right)^2 \right) dx_1 + \frac{1}{2} = F_0(0) + \frac{1}{2}.
\]

This proves that \( \limsup_{n \to \infty} \frac{D'_n}{n} \leq F_0(0) + \frac{1}{2} \approx 0.92056 \), but we can improve upon the \( \frac{1}{2} \) term. If \( i_1 < n/2 \), then we can proceed to consider \( i_2 \), which is uniformly distributed among \( \{i_1 + 1, \ldots, n\} \). Let us first suppose \( i_2 \geq (i_1 + n)/2 \). Once \( i_2 \) is chosen, we can use Lemma 2.8 (with \( m = 2 \)) to see that the probability that \( \phi_2(\pi) \) is quarantined in \( B_1(\pi) \) is at least \( 1 - \left( \frac{n - i_2}{i_2 - i_1} \right)^2 \). If \( \phi_1(\pi) \) is quarantined in \( B_1(\pi) \), then it follows from Lemma 2.7 that \( \frac{ssd'(\pi)}{n} \leq \frac{i_2}{n} \). If \( \phi_1(\pi) \) is not quarantined, then \( \frac{ssd'(\pi)}{n} \leq 1 \). If \( i_2 < (i_1 + n)/2 \), then again \( \frac{ssd'(\pi)}{n} \leq 1 \). Thus, the expected value of \( \frac{ssd'(\pi)}{n} \) is at most
\[
F_0(0) + \frac{1}{n} \sum_{i_1 < n/2} \frac{1}{n - i_1} \sum_{(i_1 + n)/2 \leq i_2 \leq n} \left( \left( 1 - \left( \frac{n - i_2}{i_2 - i_1} \right)^2 \right) \frac{i_2}{n} + \left( \frac{n - i_2}{i_2 - i_1} \right)^2 \cdot 1 \right) + \sum_{i_1 < i_2 < (i_1 + n)/2} 1.
\]
As \( n \to \infty \), this last expression tends to
\[
F_0(0) + \int_{0}^{1/2} \frac{1}{1 - x_1} \int_{x_1}^{1} \left( \left( 1 - \left( \frac{1 - x_2}{x_2 - x_1} \right)^2 \right) x_2 + \left( \frac{1 - x_2}{x_2 - x_1} \right)^2 \right) dx_2 dx_1 + \frac{1}{4}
\]
\[
= F_0(0) + \int_{0}^{1/2} F_0(x_1) dx_1 + \frac{1}{4}.
\]

We can continue to repeat this process. In the \((m + 1)\)th step, we find that in the limit \( n \to \infty \), the expected value of \( \frac{ssd'(\pi)}{n} \) is at most
\[
F_0(0) + \int_{0}^{1/2} F_0(x_1) dx_1 + \int_{0}^{1/2} \frac{1}{1 - x_1} \int_{x_1}^{x_1 + 1} F_0(x_2) dx_2 dx_1 + \cdots
\]
\[
+ \int_{0}^{1/2} \frac{1}{1 - x_1} \int_{x_1}^{x_1 + 1} \frac{1}{1 - x_2} \cdots \int_{x_{m-1}}^{x_{m-1} + 1} F_0(x_m) dx_m \cdots dx_2 dx_1 + \frac{1}{2m+1}.
\]
If we recursively define \( F_\ell(x) = \frac{1}{1 - x} \int_{x}^{x + 1} F_{\ell-1}(y) dy \) for all \( \ell \geq 0 \), then this last expression takes a much simpler form, and we obtain the inequality
\[
\limsup_{n \to \infty} \frac{D'_n}{n} \leq \sum_{\ell=0}^{m} F_\ell(0) + \frac{1}{2m+1}.
\]
But now it is straightforward to prove by induction on $\ell$ (recalling that $F_0(x) = a_0x + b_0$) that $F_\ell(x) = a_\ell x + b_\ell$ for some constants $a_\ell$ and $b_\ell$. Furthermore, these constants satisfy the recurrence relations
\[ a_\ell = \frac{3}{8} a_{\ell-1} \quad \text{and} \quad b_\ell = \frac{1}{8} a_{\ell-1} + \frac{1}{2} b_{\ell-1}. \]

A simple inductive argument yields
\[ a_\ell = \left( \frac{3}{8} \right)^\ell a_0 \quad \text{and} \quad b_\ell = \frac{1}{2^\ell} \left( 1 - \left( \frac{3}{4} \right)^\ell \right) a_0 + b_0. \]

Putting this all together, we obtain
\[
\limsup_{n \to \infty} \frac{D_n}{n} \leq \sum_{\ell=0}^\infty F_\ell(0) = \sum_{\ell=0}^\infty b_\ell = \sum_{\ell=0}^\infty \frac{1}{2^\ell} \left( 1 - \left( \frac{3}{4} \right)^\ell \right) a_0 + b_0 = \frac{2}{5} a_0 + 2b_0
\]
\[ = \frac{2}{5} (3\log 2 - 2) + 2 \left( \frac{5}{2} - 3 \log 2 \right) = \frac{3}{5} (7 - 8 \log 2). \]

The desired upper bound for $\limsup_{n \to \infty} \frac{D_n}{n}$ now follows from Proposition 2.1 □

3. Fertility Monotonicity

We now shift our focus to Theorem 1.2. In this section, it will be helpful to make use of the plot of a permutation $\pi = \pi_1 \cdots \pi_n$, which is the diagram showing the points $(i, \pi_i) \in \mathbb{R}^2$ for all $1 \leq i \leq n$. A hook of $\pi$ is a rotated L shape connecting two points $(i, \pi_i)$ and $(j, \pi_j)$ with $i < j$ and $\pi_i < \pi_j$, as in Figure 2. The point $(i, \pi_i)$ is the southwest endpoint of the hook, and $(j, \pi_j)$ is the northeast endpoint of the hook. Let $\text{SW}_i(\pi)$ be the set of hooks of $\pi$ with southwest endpoint $(i, \pi_i)$. For example, Figure 2 shows the plot of the permutation $\pi = 426315789$. The hook shown in this figure is in $\text{SW}_3(\pi)$ because its southwest endpoint is $(3, 6)$. It’s northeast endpoint is $(8, 8)$.

![Figure 2](image)

**Figure 2.** The plot of 426315789 along with a single hook.

A descent of $\pi$ is an index $i \in [n-1]$ such that $\pi_i > \pi_{i+1}$. If $\pi \in S_n$, then the tail length of $\pi$ is the largest integer $\ell \in \{0, \ldots, n\}$ such that $\pi_i = i$ for all $i \in \{n-\ell+1, \ldots, n\}$. The tail of $\pi$ is then defined to be the sequence of points $(n-\ell+1, n-\ell+1), \ldots, (n, n)$. For example, the tail of the permutation 426315789 in Figure 2 is the sequence $(7, 7), (8, 8), (9, 9)$. We say a descent $d$ of $\pi$ is tail-bound if every hook in $\text{SW}_d(\pi)$ has its northeast endpoint in the tail of $\pi$. The descents of 426315789 are 1, 3, and 4, but the only tail-bound descent is 3. In general, if $\pi \in S_n \setminus \{123 \cdots n\}$ has tail length $\ell$, then the index $i$ such that $\pi_i = n-\ell$ is a tail-bound descent of $\pi$.

Let $H$ be a hook of $\pi$ with southwest endpoint $(i, \pi_i)$ and northeast endpoint $(j, \pi_j)$. Define the $H$-unsheltered subpermutation of $\pi$ by $\pi_H^U = \pi_1 \cdots \pi_i \pi_{j+1} \cdots \pi_n$. Similarly, define the $H$-sheltered
subpermutation of \( \pi \) by \( \pi_H^S = \pi_{i+1} \cdots \pi_{j-1} \). For instance, if \( \pi = 426315789 \) and \( H \) is the hook shown in Figure 2, then \( \pi_H^S = 4269 \) and \( \pi_H^S = 3157 \). In applications, the plot of \( \pi_H^S \) will lie entirely below the hook \( H \) (it is “sheltered” by \( H \)). In particular, this will be the case if \( i \) is a tail-bound descent of \( \pi \).

The following Decomposition Lemma, originally proven in [7], will be one of our main tools for analyzing fertilities of permutations.

**Theorem 3.1** (Decomposition Lemma [7]). If \( d \) is a tail-bound descent of a nonempty permutation \( \pi \), then

\[
|s^{-1}(\pi)| = \sum_{H \in SW_d(\pi)} |s^{-1}(\pi_H^S)| \cdot |s^{-1}(\pi_H^S)|.
\]

Our second main tool will be the following results relating the stack-sorting map to the left weak order on \( S_n \). Recall the relevant definitions from Remark 1.1.

**Lemma 3.1.** Let \( \pi \in S_n \) and \( i \in [n-1] \). Suppose \( i+1 \) appears to the left of \( i \) in \( \pi \). The map \( \tilde{t}_i \) is an injection from \( s^{-1}(\pi) \) to \( s^{-1}(\tilde{t}_i(\pi)) \). If there exists an entry \( a \) such that \( i+1, a, i \) form a 231 pattern in \( \pi \), then \( \tilde{t}_i : s^{-1}(\pi) \to s^{-1}(\tilde{t}_i(\pi)) \) is bijective.

**Proof.** Choose \( \sigma \in s^{-1}(\pi) \). Since \( i+1, i \) form a 21 pattern in \( \pi \), it follows from Lemma 2.2 that there is some entry \( c \) such that \( i+1, c, i \) form a 231 pattern in \( \sigma \). It is now immediate from the definition of \( s \) that \( s(\tilde{t}_i(\sigma)) = \tilde{t}_i(\pi) \). The map \( \tilde{t}_i \) is clearly injective, so the proof of the first statement is complete.

Now suppose \( i+1, a, i \) form a 231 pattern in \( \pi \). These three entries appear in \( \tilde{t}_i(\pi) \) in the order \( i, a, i+1 \). Choose \( \sigma' \in s^{-1}(\tilde{t}_i(\pi)) \). Because \( a, i+1 \) form a 21 pattern in \( \tilde{t}_i(\pi) \), we can invoke Lemma 2.2 to see that there exists an entry \( b \) such that \( a, b, i+1 \) form a 231 pattern in \( \sigma' \). The entries \( a, i \) do not form a 21 pattern in \( \tilde{t}_i(\pi) \), so it follows from the same lemma that the entries \( a, b, i \) do not form a 231 pattern in \( \sigma' \). Thus, the entries \( i, b, i+1 \) appear in this order in \( \sigma' \). Let \( \sigma'' \) be the permutation obtained from \( \sigma' \) by swapping the positions of \( i \) and \( i+1 \). We have \( \tilde{t}_i(\sigma'') = \sigma' \). Since \( s(\sigma') = \tilde{t}_i(\pi) \), it follows immediately from the definition of \( s \) (and the fact that \( b \) lies between \( i+1 \) and \( i \) in \( \sigma'' \)) that \( s(\sigma'') = \pi \). Thus, the map \( \tilde{t}_i : s^{-1}(\pi) \to s^{-1}(\tilde{t}_i(\pi)) \) is surjective.

The first part of the preceding lemma implies the following theorem, which is somewhat interesting in its own right.

**Theorem 3.2.** If \( \pi, \pi' \in S_n \) are such that \( \pi' \leq_{\text{left}} \pi \), then \( |s^{-1}(\pi')| \geq |s^{-1}(\pi)| \).

We can now combine the Decomposition Lemma with these results concerning the left weak order to prove that the fertility statistic is strictly increasing as we move up the stack-sorting tree on \( S_n \). It will be helpful to separate the following lemma from the rest of the proof of Theorem 1.2.

**Lemma 3.2.** Given a permutation \( \pi \) whose normalization is of the form \( r \mu(r+1)(r+2) \cdots n \) for some nonempty permutation \( \mu \in S_{n-1} \), we let \( \overline{\pi} \) be the permutation with the same set of entries as \( \pi \) whose normalization is \( r \mu(r+1)(r+2) \cdots n \). We have \( |s^{-1}(\pi)| \leq |s^{-1}(\overline{\pi})| \).

**Proof.** The lemma is obvious if \( n \leq 1 \), so we may assume \( n \geq 2 \) and induct on \( n \). Without loss of generality, we may assume that \( \pi \) is normalized. Thus, \( \pi = r \mu(r+1)(r+2) \cdots n \). If
μ = 123⋯(r − 1), then π ≠ ̃π = 123⋯n. As mentioned in the introduction, it is known (see the solution to Exercise 23 in Chapter 8 of [2]) that the fertility of 123⋯n is strictly greater than the fertility of every other permutation in Sn (this fact also follows easily from the Decomposition Lemma and the fact that |s⁻¹(123⋯m)| = Cₘ). Thus, we may assume μ ≠ 123⋯(r − 1).

Let us assume for the moment that μ has tail length 0. Let d be such that πₜ = r − 1. Because μ has tail length 0, the index d is a tail-bound descent of π. Note that d − 1 is a tail-bound descent of ̃π. Given a hook H ∈ SWₜ(π) with northeast endpoint (j, j), let ̃H be the hook in SWₜ₋₁( ̃π) with northeast endpoint (j − 1, j − 1). The map SWₜ(π) → SWₜ₋₁( ̃π) given by H → ̃H is well-defined and injective. One can check that πₜ has the same relative order as ̃π. We can invoke the induction hypothesis and Theorem 3.1 to obtain

\[ |s⁻¹(π)| = \sum_{H ∈ SWₜ(π)} |s⁻¹(πₜ)| · |s⁻¹(πₜ)| ≤ \sum_{H ∈ SWₜ(π)} |s⁻¹(πₜ)| · |s⁻¹(πₜ)|, \]

\[ = \sum_{H ∈ SWₜ(π)} |s⁻¹(πₜ)| · |s⁻¹(πₜ)| ≤ \sum_{H' ∈ SWₜ₋₁( ̃π)} |s⁻¹( ̃π)| · |s⁻¹( ̃π)|. \]

Finally, suppose the tail length of μ, say ℓ, is positive. We can write μ = μ′(r − ℓ)(r − ℓ + 1)⋯(r − 1). Let τ = τₗ₋₁ ⋅ τₗ₋₁ ⋯ ⋅ τ₁, where we have τ ≤ left π, so it follows from Theorem 3.2 that |s⁻¹(τ)| ≥ |s⁻¹(π)|. Now, τ = (r − ℓ)μ′(r − ℓ + 1)(r − ℓ + 2)⋯n. Since μ′ has tail length 0, it follows from the case considered in the previous paragraph (with r − ℓ replacing r) that |s⁻¹(τ)| ≤ |s⁻¹( ̃π)|. Observing that ̃π = ̃π completes the proof.

**Figure 3.** An illustration of the proof of Lemma 3.2. In this case, μ = 5637214 has tail length 0. Notice that πₜ = 2149 has the same relative order as ̃πₜ = 2148.

Since πₜ = 85637112, the permutation ̃πₜ = 563781112 has the same relative order as ̃πₜ = 5637101112.

**Proof of Theorem 1.2.** Let σ ∈ Sn be a permutation with tail length ℓ, and let π = s(σ). We want to show that |s⁻¹(σ)| ≤ |s⁻¹(π)|, where equality holds if and only if σ = 123⋯n. This is trivial if n ≤ 1, so we may assume n ≥ 2 and induct on n. If n − ℓ = 0, then σ = π = 123⋯n, so |s⁻¹(σ)| = |s⁻¹(π)|. Thus, we may assume n − ℓ ≥ 1 and induct on n − ℓ (with n already fixed).

The assumption n − ℓ ≥ 1 is equivalent to the statement that σ ≠ 123⋯n, so our goal is to prove
the strict inequality $|s^{-1}(\sigma)| < |s^{-1}(\pi)|$. Let us write $\sigma = L(n-\ell)R(n-\ell+1)(n-\ell+2) \cdots n$. We consider three cases.

**Case 1:** Assume $L$ is nonempty and contains the entry $n - \ell - 1$. Let $d - 1$ be the length of $L$ so that $d = n - \ell$. Note that $d$ is a tail-bound descent of $\sigma$. It follows from the definition of $s$ that the tail length of $\pi$ is $\ell + 1$ and that $\pi_{d-1} = n - \ell - 1$. Thus, $d - 1$ is a tail-bound descent of $\pi$. For $1 \leq j \leq \ell$, let $H(j)$ be the hook of $\sigma$ with southwest endpoint $(d, n - \ell)$ and northeast endpoint $(n - \ell + j, n - \ell + j)$. For $1 \leq j \leq \ell + 1$, let $\overline{H}(j)$ be the hook of $\pi$ with southwest endpoint $(d - 1, n - \ell - 1)$ and northeast endpoint $(n - \ell - 1 + j, n - \ell - 1 + j)$. One can verify (see Figure 3) that $\overline{H}(j)$ has the same relative order as $s(\overline{H}(j))$ and that $\overline{H}(j)$ has the same relative order as $s(\overline{H}(j))$ (when $1 \leq j \leq \ell$). Since permutations with the same relative order have the same fertility, we can invoke the inductive hypothesis to see that $|s^{-1}(\overline{H}(j))| \leq |s^{-1}(\pi_{\overline{H}(j)})|$ and $|s^{-1}(\overline{H}(j))| \leq |s^{-1}(\sigma_{\overline{H}(j)})| = |s^{-1}(\overline{H}(j))|$. According to the Decomposition Lemma (Theorem 3.1), we have

$$|s^{-1}(\sigma)| = \sum_{j=1}^{\ell} |s^{-1}(\sigma_{H}(j))| \cdot |s^{-1}(\sigma_{\overline{H}(j)})| \leq \sum_{j=1}^{\ell} |s^{-1}(\sigma_{\overline{H}(j)})| \cdot |s^{-1}(\overline{H}(j))|$$

$$\leq \sum_{j=1}^{\ell+1} |s^{-1}(\sigma_{\overline{H}(j)})| \cdot |s^{-1}(\overline{H}(j))| = |s^{-1}(\pi)|.$$

Suppose by way of contradiction that the inequality $|s^{-1}(\sigma)| < |s^{-1}(\pi)|$ is actually an equality. Since $\sigma \in s^{-1}(\pi)$, we have $|s^{-1}(\sigma)| = |s^{-1}(\pi)| > 0$. Thus, there exists $j \in \{1, \ldots, \ell\}$ such that $|s^{-1}(\sigma_{\overline{H}(j)})| \cdot |s^{-1}(\overline{H}(j))| > 0$. We are assuming the inequalities in (3) are equalities, so we must have $|s^{-1}(\sigma_{\overline{H}(j)})| = |s^{-1}(\overline{H}(j))|$ and $|s^{-1}(\sigma_{\overline{H}(j)})| = |s^{-1}(\overline{H}(j))|$. By induction on $n$, this forces $\sigma_{\overline{H}(j)}$ and $\overline{H}(j)$ to be increasing permutations. Consequently, $d$ is the only descent of $\sigma$. However, this means that $\overline{H}(j)$ and $\overline{H}(j)$ are increasing permutations, so their fertilities are positive. It follows that $|s^{-1}(\overline{H}(j))| > 0$, so the second inequality in (3) is strict. This is our desired contradiction.

**Case 2:** Assume $L$ is nonempty and does not contain the entry $n - \ell - 1$. Let $m$ be the largest entry in $L$. Let $\overline{R}$ be the permutation obtained from $R$ by replacing $m$ with $n - \ell - 1$. Let $\overline{\sigma} = \overline{L}(n - \ell)\overline{R}(n - \ell + 1)(n - \ell + 2) \cdots n$ and $\overline{\pi} = s(\overline{\sigma}) = s(\overline{L})s(\overline{R})(n - \ell)(n - \ell + 1) \cdots n$. Notice that $\overline{\sigma} = \overline{L} \circ t_{m+1} \circ \cdots \circ t_{n-\ell-2}(\overline{\sigma})$. By repeatedly applying the second part of Lemma 3.1 (with $a = n - \ell$), we find that $|s^{-1}(\overline{\sigma})| = |s^{-1}(\sigma)|$. Similarly, we have $\overline{\pi} = \overline{L} \circ t_{m+1} \circ \cdots \circ t_{n-\ell-2}(\overline{\pi})$, so $\overline{\pi} \leq_{\text{left}} \overline{\pi}$. Theorem 3.2 now tells us that $|s^{-1}(\overline{\pi})| \geq |s^{-1}(\overline{\pi})|$. Because $n - \ell - 1$ is in $\overline{L}$, we can apply Case 1 to see that $|s^{-1}(\overline{\sigma})| < |s^{-1}(\overline{\pi})|$. Thus, $|s^{-1}(\sigma)| < |s^{-1}(\pi)|$.

**Case 3:** Assume $L$ is empty. This means that $\sigma = (n-\ell)R(n-\ell+1)(n-\ell+2) \cdots n$. We have $s(\overline{\sigma}) = s(R)(n-\ell)\sigma(n-\ell+1)(n-\ell+2) \cdots n = s(\sigma) = \pi$, where $\overline{\sigma}$ is as defined in the statement of Lemma 3.2. According to that lemma, the inequality $|s^{-1}(\sigma)| \leq |s^{-1}(\overline{\sigma})|$ holds. It is at this point in the proof that we use induction on $n - \ell$. Since $\overline{\sigma}$ is a permutation in $S_n$ with tail length at least $\ell + 1$, the inductive hypothesis implies that $|s^{-1}(\overline{\sigma})| \leq |s^{-1}(\overline{\sigma})|$, with equality if and only if $\overline{\sigma} = 123 \cdots n$. If $\overline{\sigma} \neq 123 \cdots n$, then we are done because $|s^{-1}(\sigma)| \leq |s^{-1}(\overline{\sigma})| < |s^{-1}(\overline{\sigma})| = |s^{-1}(\pi)|$. 


Figure 4. An illustration of Case 1 in the proof of Theorem 1.2. We have $j = 2$ in this example. Notice that $\pi_U^{(2)} = 4527101112$ has the same relative order as $s\left(\sigma_U^{(2)}\right) = 452781112$. Similarly, $\pi_S^{(2)} = 3168$ has the same relative order as $s\left(\sigma_S^{(2)}\right) = 3169$.

If $\sigma = 123 \cdots n$, then $\pi = 123 \cdots n$. In this case, we again have the strict inequality $|s^{-1}(\sigma)| < |s^{-1}(\pi)|$ because $123 \cdots n$ has a strictly larger fertility than each other permutation in $S_n$ (by Exercise 23 in Chapter 8 of [2]).

4. Conclusion

In the first part of the paper, we established improved asymptotic estimates for the average depth in the stack-sorting tree on $S_n$ (equivalently, for the average time complexity of the algorithm that sorts via iterating $s$). Note, however, that it is still not known if the limit $\lim_{n \to \infty} \frac{D_n}{n}$ exists. West [19] conjectured that this limit does exist. It would be exciting to have a proof of this conjecture.

We computed $\text{ssd}'(\pi)$ for 1000 random permutations in $S_{400}$. The average of $\text{ssd}'(\pi)/400$ for these permutations was 0.784, and the standard deviation was 0.140. Thus, we are willing to state the following strengthening of West’s conjecture.

**Conjecture 4.1.** The limit $\lim_{n \to \infty} \frac{D_n}{n}$ exists and lies in the interval $(0.77, 0.81)$.

In the second part of the paper, we gave a lengthy argument showing that $|s^{-1}(\sigma)| \leq |s^{-1}(s(\sigma))|$ for all permutations $\sigma$. Our proof relied on the Decomposition Lemma from [7]. It also relied on Theorem 3.2 which states that the fertility statistic is decreasing on the left weak order. It would be interesting to have a direct injective proof of the inequality $|s^{-1}(\sigma)| \leq |s^{-1}(s(\sigma))|$.

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