GARSIA–RODEMICH SPACES: 
BOURGAIN–BREZIS–MIRONESCU SPACE, 
EMBEDDINGS AND REARRANGEMENT-INvariant 
SPACES

By

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Abstract. We extend the construction of Garsia–Rodemich spaces in different directions. We show that the new space \( B \), introduced by Bourgain, Brezis and Mironescu [6], can be described via a suitable scaling of the Garsia–Rodemich norms. As an application we give a new proof of the embeddings \( BMO \subset B \subset L^\infty(\mathbb{R}^n, \infty) \). We then generalize the Garsia–Rodemich construction and introduce the \( \text{GaRoX} \) spaces associated with a rearrangement-invariant space \( X \), in such a way that \( \text{GaRoX} = X \), for a large class of rearrangement-invariant spaces. The underlying inequality for this new characterization of rearrangement-invariant spaces is an extension of the rearrangement inequalities of [17]. We introduce Gagliardo seminorms adapted to rearrangement-invariant spaces and use our generalized Garsia–Rodemich construction to prove fractional Sobolev inequalities in this context.

1 Introduction

In their celebrated paper [16], John and Nirenberg introduced the space \( BMO(Q_0) \), and established the exponential integrability of functions in \( BMO(Q_0) \). To complement their result on \( BMO \) functions, John and Nirenberg introduced the \( JN_p(Q_0) \) spaces, which provide a scale of conditions on the oscillation of functions. For \( 1 < p < \infty \), let

\[
JN_p(Q_0) := JN_p = \{ f \in L^1(Q_0) : \| f \|_{JN_p} < \infty \},
\]

where

\[
(1.1) \quad \| f \|_{JN_p} = \sup_{\{Q_i\}_{i\in\mathcal{P}}} \| f \|_{JN_p} = \sup_{\{Q_i\}_{i\in\mathcal{P}}} \left\{ \sum_{i\in\mathcal{I}} \left( \frac{1}{|Q_i|} \int_{Q_i} |f(x) - f_{Q_i}| \, dx \right)^{\frac{1}{p}} \right\}^{1/p},
\]

where

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1For definiteness, from now on \( Q_0 \) will denote the unit cube \((0, 1)^n\).

2In this paper we assume that all subcubes have sides parallel to the coordinate axes and we let \( f_Q = \frac{1}{|Q|} \int_Q f \).
and

\[ P = \{ \{ Q_i \}_{i \in I} : \{ Q_i \}_{i \in I} \text{ countable subcubes of } Q_0 \text{ with pairwise disjoint interiors} \}. \]

Then, we see that (cf. [16, p. 423])

\[
\lim_{p \to \infty} \| f \|_{JN_p} = \| f \|_{BMO}.
\]

John and Nirenberg [16] then proceeded to obtain the corresponding intermediate integrability results for \( JN_p \) functions, which we formulate here as embeddings: For \( 1 < p < \infty \), we have

\[
JN_p \subset L(p, \infty),
\]

\[
\| f - f_{Q_0} \|_{L(p, \infty)} \leq c_p \| f \|_{JN_p},
\]

where \( L(p, \infty) \) denotes the Marcinkiewicz weak type \( L^p \) space and \( c_p \) is an absolute constant that does not depend on \( f \).

We note that, as \( p \to \infty \), the correct limiting Marcinkiewicz condition is the exponential class and the resulting limiting inequality is one of the possible formulations of the celebrated John–Nirenberg Lemma.

Garsia–Rodemich [15] introduced a different scale of conditions. Let \( 1 < p \leq \infty \); then we define

\[ GaRop := GaRop(Q_0) = \{ f : \| f \|_{GaRop} < \infty \}, \]

where

\[
\| f \|_{GaRop} = \sup_{\{ Q_i \} \in P} \frac{\sum_{i \in I} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)|dx dy}{(\sum_{i \in I} |Q_i|)^{1/p'}}.
\]

The main result about the \( GaRop \) spaces is given by the following (cf. [15] for the one-dimensional case, [17] for the \( n \)-dimensional case): As sets,

\[
GaRop = \begin{cases} 
L(p, \infty), & 1 < p < \infty, \\
BMO, & \text{if } p = \infty.
\end{cases}
\]

In fact, the underlying inequalities can be quantified (cf. [17], [18]). Let \( 1 < p < \infty \); then

\[
\| f \|_{GaRop} \leq \frac{2p}{p-1} \| f - f_{Q_0} \|_{L(p, \infty)},
\]

\[
\| f - f_{Q_0} \|_{L(p, \infty)} \leq c(n, p) \| f \|_{GaRop}.
\]

\[ ^3 \text{Defined by the condition } \| f \|_{L(p, \infty)} = \sup_t \| \{| f | > t \} \|^{1/p} < \infty. \]

\[ ^4 \text{The natural condition in our context is via the Bennett–DeVore–Sharpley space “weak } L^\infty \text{” defined by } L(\infty, \infty) = \{ f : \| f \|_{L(\infty, \infty)} = \sup_{t>0} (f^*(t) - f^*(0)) < \infty \}. \text{ In fact, this space is exactly the rearrangement-invariant hull of } BMO \text{ (cf. [4]). For a recent account of this part of the story we refer to [19].} \]
Likewise, for $p = \infty$, we have (cf. [18])

\[(1.7) \quad \|f\|_{GaRo_{\infty}} \simeq \|f\|_{BMO},\]

and (cf. [4])

\[(1.8) \quad \|f - f_{Q_0}\|_{L(\infty, \infty)} \leq c(n)\|f\|_{BMO}.\]

It is easy to see that\footnote{Indeed, for any $\{Q_i\}_{i \in I} \in P$ we have
\[
\sum_{i \in I} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy \leq 2 \sum_{i \in I} \int_{Q_i} |f(x) - f_{Q_i}| \, dx
\]
\[= 2 \sum_{i \in I} |Q_i|^{1/p'} \left( |Q_i|^{1/p} \left\{ \sum_{i \in I} |Q_i| \left( \frac{1}{|Q_i|} \int_{Q_i} |f(x) - f_{Q_i}| \, dx \right)^p \right\}^{1/p} \right),\]

and (1.9) follows.}

\[(1.9) \quad \|f\|_{GaRop} \leq 2\|f\|_{JN_p}.\]

Therefore, combining (1.6) and (1.9) gives a new proof of the John–Nirenberg embedding (1.2), and combining (1.7) with (1.8) gives the John–Nirenberg Lemma. Moreover, the Garsia–Rodemich spaces are particularly well suited to study other important inequalities in analysis, including Poincaré–Sobolev embeddings (cf. [18], and Section 4 below), and the basic construction can be extended to more general settings, e.g., metric spaces, doubling measures, etc.

In this paper we extend the Garsia–Rodemich construction and the scope of its applications. We first show that the new space, $B$, introduced by Bourgain, Brezis and Mironescu [6] is closely connected to suitable scalings of the Garsia–Rodemich norms. As an application we give a new streamlined approach to the remarkable embedding obtained in [6] (cf. (2.2) and Theorem 1 below),

\[(1.10) \quad BMO \subset B \subset L(n', \infty).\]

The description of the weak $L^p$ spaces (cf. [14]) via the Garsia–Rodemich conditions raises a natural question: can one also describe the $L^p$ spaces or other function spaces through Garsia–Rodemich oscillation conditions? We show that this is indeed the case by means of modifying a construction of certain martingale spaces, apparently first introduced by Garsia [14, $K_0^+$ spaces, p. 165]. Let $X := X(Q_0)$ be a rearrangement-invariant space (cf. Section 3 below for background information).
Definition 1. We shall say that an integrable function $f$ belongs to $GaR_{oX}$ if there exists $\gamma \in X$ such that for all $\{Q_i\}_{i \in I} \in P$,

$$\sum_{i \in I} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy \leq \sum_{i \in I} \int_{Q_i} \gamma(x) \, dx.$$  

(1.11)

Let

$$\Gamma_f^X = \{ \gamma \in X : (1.11) \text{ holds for all } \{Q_i\}_{i \in I} \in P \},$$

and define

$$\|f\|_{GaR_{oX}} = \inf \{ \|\gamma\|_X : \gamma \in \Gamma_f^X \}.$$

In Section 3 (cf. Theorem 3) we show that for rearrangement-invariant spaces $X$ whose Boyd indices lie on $(0, 1)$, we have

$$GaR_{oX} = X.$$  

(1.12)

In particular, combining (1.12) with (1.4) it follows that

$$GaR_{oL(p, \infty)} = L(p, \infty) = GaR_{o_p}, 1 < p < \infty.$$  

The proof is based on a suitable extension of the rearrangement inequalities of [17] (cf. Section 3, Theorem 2 below). There exists a constant $c = c(n)$ such that for all $f \in GaR_{oX}$ and all $\gamma \in \Gamma_f^X$,

$$f^{**}(t) - f^*(t) \leq c\gamma^{**}(t), \quad t \in (0, 1/4).$$

As a consequence, we can extend (1.5) and (1.6) to the context of rearrangement-invariant spaces (cf. (3.3), (3.4) below).

In Section 4 we use the Garsia–Rodemich conditions to extend the connection between the space $B$ and fractional Sobolev embeddings obtained in [6]. Let

$$W^{a,p} = \{ f : \|f\|_{W^{a,p}} < \infty \},$$

(1.13)

where

$$\|f\|_{W^{a,p}} = \left\{ \int_{Q_0} \int_{Q_0} \frac{|f(x) - f(y)|^p}{|x - y|^{n+ap}} \, dx \, dy \right\}^{1/p}.$$  

(1.14)

Then (cf. Theorem 4 and Remark 4 below)

$$W^{a,p} \subset GaR_{q}, 1 \leq p \leq \frac{n}{a}, \quad \frac{1}{q} = \frac{1}{p} - \frac{a}{n}.$$  

(1.15)

\[\text{In particular, this class of spaces includes the } L^p \text{ spaces and the Marcinkiewicz spaces } L(p, \infty), 1 < p < \infty.\]

\[\text{Commenting on an earlier version of this paper, where we had assumed } p > 1, \text{ Daniel Spector observed that our method of proof also yielded the case } p = 1 \text{ (cf. Remark 4 below).}\]
The generalized Garsia–Rodemich construction can be used to give a far reaching extension of (1.15) to the setting of rearrangement invariant spaces. To implement this program we introduce Gagliardo seminorms adapted to rearrangement-invariant spaces as follows. Let \( \alpha \in (0, 1) \), \( 1 < p < \infty \); we formally define

\[
D_{p, \alpha}(f)(y) = \left\{ \int_{Q_0} \frac{|f(x) - f(y)|^p}{|x - y|^{n + ap}} dx \right\}^{1/p}, \quad y \in Q_0.
\]

Given a rearrangement-invariant space \( Y := Y(Q_0) \) we consider the spaces defined by

\[
W_{p, \alpha}^\alpha := W_{p, \alpha}^\alpha(Q_0) := \{ f : \| f \|_{W_{p, \alpha}^\alpha} = \| D_{p, \alpha}(f) \|_{Y} < \infty \}.
\]

For example, if \( Y = L^p \), then

\[
W_{p, L^p}^\alpha = W_{p, \alpha}^p.
\]

Let \( X := X(Q_0) \) and \( Y := Y(Q_0) \) be rearrangement-invariant spaces such that the local Riesz potential operator

\[
I_{a, Q_0}(f)(y) = \int_{Q_0} \frac{f(x)}{|x - y|^{n - a}} dx, \quad y \in Q_0,
\]

defines a bounded map, \( I_{a, Q_0} : Y \to X \). Then, the following continuous embedding holds (cf. Theorem 5 below):

\[
W_{p, Y}^\alpha \subset \text{GaRoX}.
\]

It follows that if the Boyd indices of \( X \) lie in the interval \((0, 1)\), then (cf. Theorem 3)

\[
W_{p, Y}^\alpha \subset X.
\]

The proof is achieved by means of showing that there exists an absolute constant \( c = c(n, \| I_{a, Q_0} \|_{Y \to X}) > 0 \) such that \( c I_{a, Q_0}(D_{p, \alpha}(f)) \in \Gamma_f^X \) for all \( f \in W_{p, Y}^\alpha \). For example, suppose that \( 1 < p < \frac{n}{a}, \frac{1}{q} = \frac{1}{p} - \frac{a}{n}, 1 \leq r_1 \leq r_2 \leq \infty \); then, since it is easy to relate mapping properties of \( I_{a, Q_0} \) to those of the usual Riesz potential \( I_a \), and, as is well-known for Lorentz spaces, we have (cf. [20]) \( I_a : L(p, r_1) \to L(q, r_2) \), we can conclude that (cf. Example 1 below)

\[
W_{p, L(p, r_1)}^\alpha \subset \text{GaRoL}(q, r_2) = L(q, r_2).
\]

The end point inequalities for local Riesz potentials that were obtained in [10] can

\footnote{In particular, if we let \( r_1 = p < r_2 = \infty \), we recover (1.15).}
be also implemented here. For example, when \( p = \frac{n}{\alpha} \), we have (cf. [10] Theorem 2)
\[ I_{\alpha,Q_0} : L\left(\frac{n}{\alpha}, \frac{n}{\alpha}\right) \to BW_{n/\alpha}, \]
where
\[ BW_{n/\alpha} = \{ f : \int_0^1 \left( \frac{f^*(t)}{(1 + \log \frac{1}{t})} \right)^{n/\alpha} \frac{dt}{t} < \infty \}. \]

As a consequence, we obtain the following fractional version of the well-known Brezis–Wainger inequality [10] (cf. Example 3 below):
\[ W^{\alpha}_{n/\alpha} = W^{\alpha}_{n,\alpha} \subset GaRoBW_{n/\alpha}. \]

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2 The Bourgain–Brezis–Mironescu space and the scaling of Garsia Rodemich conditions

Let us observe that neither the norms nor the \( GaRo_p \) spaces change if we replace in the definition (1.3) the test space \( P \) by
\[ \tilde{P} = \{ \{ Q_i \}_{i \in I} : \{ Q_i \}_{i \in I} \text{ subcubes of } Q_0 \]
with pairwise disjoint interiors with \( \#I < \infty \}, \]
where
\[ \#I = \text{cardinality of } I. \]

We now turn to the connection with the Bourgain–Brezis–Mironescu construction. Given \( \varepsilon \in (0, 1) \), we let
\[ \tilde{P}_\varepsilon = \{ \{ Q_i \}_{i \in I} : \{ Q_i \}_{i \in I} \subset \tilde{P}, \text{ with side of } Q_i = \varepsilon \text{ for all } i \in I, \text{ and } \#I \leq \varepsilon^{1-n} \}. \]

The space \( B \) of Bourgain–Brezis–Mironescu is defined by
\[ B = \{ f \in L^1(Q_0) : \|f\|_B < \infty \}, \]
where
\[ \|f\|_B = \sup_{0<\varepsilon<1} \varepsilon^{n-1} \sup_{\{Q_i\} \in \tilde{P}_\varepsilon} \sum_{i \in I} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| dx dy \]
\[ = \sup_{0<\varepsilon<1} \varepsilon^{1-n} \sup_{\{Q_i\} \in \tilde{P}_\varepsilon} \sum_{i \in I} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| dx dy. \]
More generally, one can consider the scale of spaces $B_p$, $1 < p \leq \infty$, defined by

$$B_p = \left\{ f : \| f \|_{B_p} = \sup_{0 < \varepsilon < 1} \varepsilon^{-1/p'} \sup_{(Q_i) \in \tilde{P}_\varepsilon} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| dxdy < \infty \right\}.$$ 

Then, of course the space $B$ corresponds to the choice $p = \infty$,

$$B = B_\infty.$$ 

Now it is easy to see the relationship between $B_p$ and $GaRo_p$. First note that $\tilde{P}_\varepsilon \subset \tilde{P}$; moreover, if $(Q_i)_{i \in I} \in \tilde{P}_\varepsilon$ we have

$$\left( \sum_{i \in I} |Q_i| \right)^{1/p'} = (\varepsilon^n (#I))^{1/p'} \leq \varepsilon^{1/p'}, \quad 0 < \varepsilon < 1.$$ 

Consequently, if $f \in GaRo_p$ then, for all $0 < \varepsilon < 1$, and for all $(Q_i) \in \tilde{P}_\varepsilon$,

$$\sum_{i \in I} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| dxdy \leq \varepsilon^{1/p'} \| f \|_{GaRo_p}.$$ 

In other words,

(2.1) \[ \| f \|_{B_p} \leq \| f \|_{GaRo_p}, \quad 1 < p \leq \infty. \]

In particular, for $p = \infty$, it follows from (2.1) and (1.7) that (cf. [6])

(2.2) \[ \| f \|_{B} \leq \| f \|_{GaRo_\infty} \approx \| f \|_{BMO}. \]

With these preliminaries in place we shall now give an easy proof of a more refined embedding result that was obtained in [6] with a different proof.

**Theorem 1.**

(2.3) \[ B \subset L(n', \infty). \]

**Proof.** We will actually show that if $f \in B$,

(2.4) \[ \| f \|_{GaRo_{n'}} \leq \| f \|_{B}. \]

The desired result will then follow from (1.6) above. We shall show below that when testing the $GaRo_{n'}$ norm it will be enough to consider dyadic cubes. Let $\Omega = \{ Q_i \}_{i \in I}$ be an arbitrary element of $P$ formed with dyadic cubes. Following [6] we split $\Omega$ as follows. For each $j \in N$ we consider $\mathcal{F}_j = \{ Q_i \in \Omega : |Q_i| = 2^{-jn} \}$; then $\Omega = \bigcup_{j=1}^\infty \mathcal{F}_j$. For each $j$ we consider subsets $Q_* \subset \mathcal{F}_j$ such that

$$\#Q_* \leq (2^{-j})^{-(n-1)} = 2^{j(n-1)}.$$
For any such subfamily of cubes $Q_*$ we have
\[
\sum_{Q_i \in Q_*} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy \leq 2^{-j} \|f\|_B.
\]

Therefore, covering $\mathcal{F}_j$ with disjoint families of subcubes $Q_*$ as above, we find, following the proof in [6], that
\[
\sum_{Q_i \in \mathcal{F}_j} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy \leq (2^{-j} + 2^{-jn(\#\mathcal{F}_j)}) \|f\|_B.
\]

Consequently,
\[
\sum_{Q_i \in Q} \frac{1}{|Q|} \int_{Q} \int_{Q} |f(x) - f(y)| \, dx \, dy \leq \left( \sum_{j, \mathcal{F}_j \neq \emptyset} 2^{-j} + \sum_{j} 2^{-jn(\#\mathcal{F}_j)} \right) \|f\|_B
\]
\[
= (A + B) \|f\|_B.
\]

To estimate $A$, note that if $\mathcal{F}_{j_0} \neq \emptyset$ there is $Q^0 \in \mathcal{F}_{j_0}$ such that
\[
2^{-j_0} = |Q^0|^{1/n} \leq \left( \sum_{Q \in \mathcal{Q}} |Q| \right)^{1/n}.
\]

Therefore, if we let $j_0$ be the first index such that $\mathcal{F}_{j_0} \neq \emptyset$, we have
\[
\sum_{j, \mathcal{F}_j \neq \emptyset} 2^{-j} = \sum_{j \geq j_0, \mathcal{F}_j \neq \emptyset} 2^{-j} = 2^{-j_0} \sum_{j \geq j_0, \mathcal{F}_j \neq \emptyset} 2^{-j-j_0} \leq \left( \sum_{Q \in \mathcal{Q}} |Q| \right)^{1/n}.
\]

Term $B$ is estimated by noting that
\[
2^{-jn(\#\mathcal{F}_j)} = \sum_{Q \in \mathcal{F}_j} |Q|.
\]

Thus
\[
B = \sum_{j} \sum_{Q \in \mathcal{F}_j} |Q| = \sum_{Q \in \mathcal{Q}} |Q| = \left( \sum_{Q \in \mathcal{Q}} |Q| \right)^{1/n} \left( \sum_{Q \in \mathcal{Q}} |Q| \right)^{1/n'} \leq \left( \sum_{Q \in \mathcal{Q}} |Q| \right)^{1/n}.
\]

Inserting the estimates of $A$ and $B$ in (2.5) we find that
\[
\sum_{Q \in \mathcal{Q}} \frac{1}{|Q|} \int_{Q} \int_{Q} |f(x) - f(y)| \, dx \, dy \leq C \left( \sum_{Q \in \mathcal{Q}} |Q| \right)^{1/n} \|f\|_B.
\]

It follows that
\[
\|f\|_{GaRo''} \leq C \|f\|_B.
\]
To conclude the proof we argue that only dyadic cubes need to be tested. Indeed, we may assume without loss of generality that \( \int_{Q_0} f = 0 \). Then \( \|f\|_{Garon} \sim \|f\|_{L(n', \infty)} \).

As it is well-known \( L(n', \infty) \) can be obtained by the real method of interpolation (cf. [5]),

\[
L(n', \infty) = (L^1, L^\infty)_{1/n, \infty},
\]

with

\[
\|f\|_{L(n', \infty)} \sim \sup_{t>0} t^{-1/n} [\inf \{ \|h\|_{L^1} + t\|g\|_{L^\infty} : f = h + g \}].
\]

The computation of the indicated infimum (called the \( K \)-functional of \( f \)) can be achieved by elementary cutoffs but can be also achieved using Calderón–Zygmund decompositions. Indeed, a Calderón–Zygmund decomposition \( f = h_{CZ}(t) + g_{CZ}(t) \) nearly achieves the infimum and indeed (cf. [6])

\[
\|f\|_{L(n', \infty)} \sim \sup \{ t^{-1/n} \|h_{CZ}(t)\|_{L^1} \}.
\]

But as it turns out the computation of \( t^{-1/n} \|h_{CZ}(t)\|_{L^1} \) corresponds to the computation of \( \sum_{Q_i \in \mathcal{Q}} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy \) (cf. [6])! Our assertion then follows since the Calderón–Zygmund decomposition used is dyadic.

\[\Box\]

**Remark 1.** As we hope it is clear from the proof, the key of the argument is the introduction of the Garsia–Rodemich conditions.

### 3 A characterization of rearrangement-invariant spaces via Garsia–Rodemich spaces

We start by recalling a few basic notions on rearrangements and rearrangement-invariant spaces. We refer to [5] and [8] for further details and background.

Let \( f : Q_0 \to \mathbb{R} \) be a measurable function. The distribution function of \( f \) is given by \(9\)

\[
\lambda_f(t) = |\{ x \in Q_0 : |u(x)| > t \}| \quad (t > 0).
\]

The decreasing rearrangement of \( f \) is the right-continuous non-increasing function from \([0, 1]\) into \( \mathbb{R}^+ \) which is equimeasurable with \( f \). It can be defined by the formula

\[
f^*(s) = \inf \{ t \geq 0 : \lambda_f(t) \leq s \}, \quad s \in [0, 1),
\]

and satisfies

\[
\lambda_f(t) = |\{ x \in Q_0 : |f(x)| > t \}| = |\{ s \in [0, 1) : f^*(s) > t \}|, \quad t \geq 0.
\]

\[9\] where \( |\cdot| \) denotes the Lebesgue measure on \( Q_0 \) (we also use this notation for the Lebesgue measure on the unit interval \([0, 1]\) since it will cause no confusion).
The maximal average $f^{**}(t)$ is defined by

\[
(3.1) \quad f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds = \frac{1}{t} \sup \left\{ \int_E |f(x)| dx : |E| = t \right\}, \quad t > 0.
\]

We say that a Banach function space $X := X(Q_0)$ is a rearrangement-invariant (r.i.) space if $g \in X$ implies that all measurable functions $f$ with the same rearrangement with $f^* = g^*$ also belong to $X$, and, moreover, $\|f\|_X = \|g\|_X$. Rearrangement invariant spaces on $Q_0$ can be represented by a r.i. space on the interval $(0, 1)$, with Lebesgue measure, $\hat{X} = \hat{X}(0, 1)$, such that

\[
\|f\|_X = \|f^*\|_{\hat{X}},
\]

for every $f \in X$. Since it will be clear from the context which space is involved in the discussion, we will “drop the hat” and denote the norm of both spaces with the same symbol $\|\cdot\|_X$.

The following restrictions on the spaces will play a role in our development in this paper (cf. [8], [5]):

(A) There exists a universal constant $\beta_0(X)$ such that $\|f^{**}\|_X \leq \beta_0(X) \|f\|_X$.

(B) There exists a universal constant $\beta_1(X)$ such that $\|\int_t^1 f(s) ds\|_X \leq \beta_1(X) \|f\|_X$.

In the language of “indices” a space that satisfies both (A) and (B) can be described by saying that “the Boyd indices (cf. [8], [5]) of $X$ are in the interval $(0, 1)$.”

The basic estimate concerning the $GaRoX$ spaces (cf. Definition 1 above) is given by

**Theorem 2.** Let $X := X(Q_0)$ be a rearrangement-invariant space. Then there exists a universal constant $c_n > 0$ such that if $f \in GaRoX$ and $\gamma \in \Gamma_f^X$, then

\[
(3.2) \quad f^{**}(t) - f^*(t) \leq c_n \gamma^{**}(t), \quad 0 < t < 1/4.
\]

**Proof.** We follow the proof of [17, Theorem 5 pp. 496–497] very closely and only indicate in detail the necessary changes at the appropriate steps. Let $f \in GaRoX$ and $\gamma \in \Gamma_f^X$. Since $\|f(x) - |f(y)|\| \leq |f(x) - f(y)|$, it follows that $\Gamma_f^X \subset \Gamma_{|f|}^X$ and $|f| \in GaRoX$. Moreover, by definition $f^{**}(t) = |f|^{**}(t)$ and $f^*(t) = |f|^*(t)$. In other words, to compute the left-hand side of (3.2) we may assume, without loss, that $f$ is positive.\(^1\) Fix $t > 0$ such that $t < |Q_0|/4 = 1/4$, and let $E = \{x \in Q_0 : f(x) > f^*(t)\}$. By definition, $|E| \leq t < 1/4$, consequently we

\(^1\)We refer to [5, Theorem 4.10 and subsequent remarks] for further background information on r.i. spaces.

\(^1\)In other words, we compute the left-hand side using $|f|$ while keeping $\gamma \in \Gamma_f^X$.\)
can find a relative open subset of $Q_0$, $\Omega$ say, such that $E \subset \Omega$ and $|\Omega| \leq 2t \leq 1/2$. By \cite{[5]} Lemma 7.2, page 377) we can find a sequence of cubes $\{Q_i\}_{i \in \mathbb{N}}$ with pairwise disjoint interiors such that:

(i) $|\Omega \cap Q_i| \leq \frac{1}{2} |Q_i| \leq |\Omega^c \cap Q_i|$, $i = 1, 2, \ldots .$

(ii) $\Omega \subset \bigcup_{i \in \mathbb{N}} Q_i \subset Q_0.$

(iii) $|\Omega| \leq \sum_{i \in \mathbb{N}} |Q_i| \leq 2^{n+1}|\Omega|.$

At this point, following all the corresponding steps in \cite{[17]} Theorem 5, pp. 496–497], we arrive at

$$t(f^{**}(t) - f^*(t)) \leq \sum_{i \in \mathbb{N}} \left( \int_{Q_i} \{f(x) - f_{Q_i}\} dx + |E \cap Q_i| \{f_{Q_i} - f^*(t)\} \right)$$

$$= (I) + (II).$$

To estimate (II), we let $J = \{i : f_{Q_i} > f^*(t)\}$ and follow the steps of \cite{[17]} Theorem 5, pp. 496–497] until we arrive at the point

$$(II) \leq \sum_{i \in J} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(y) - f(x)| dy dx.$$

Now, we recall that $\gamma \in \Gamma^X_f \subset \Gamma^X_{|f|}$, therefore invoking the definition of $GaRoX$ we have

$$(II) \leq \sum_{i \in J} \int_{Q_i} \gamma(x) dx \leq \int_0^{\sum_{i \in J} |Q_i|} \gamma^*(s) ds \leq \int_0^{2^{n+2}t} \gamma^*(s) ds = 2^{n+2}t \gamma^{**}(2^{n+2}t)$$

$$\leq 2^{n+2}t \gamma^{**}(t)$$ (since $\gamma^{**}$ is decreasing).

Likewise, we estimate (I), proceeding as in \cite{[17]} Theorem 5 pages 496-497] until we arrive at

$$(I) \leq \sum_{i \in \mathbb{N}} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| dy dx.$$

Again by the definition of $\Gamma^X_f$ we find that

$$(I) \leq 2^{n+2}t \gamma^{**}(t).$$

Combining the inequalities for (I) and (II) we can find a universal constant $c_n$ such that, for $\gamma \in \Gamma^X_f$, we have

$$f^{**}(t) - f^*(t) \leq c_n \gamma^{**}(t), \quad \text{for all } t < 1/4,$$

as we wished to show. $\square$
We can now state and prove the main result of this section.

**Theorem 3.** Let $X$ be a rearrangement-invariant space with Boyd indices in the interval $(0, 1)$. Then, as sets

$$GaRonX = X.$$  

Moreover, we have the following estimates:

(3.3) \[ \|f - f_{Q_0}\|_{GaRonX} \leq 2\|f\|_X \]

and

(3.4) \[ \|f - f_{Q_0}\|_X \leq c(X)\|f\|_{GaRonX}, \]

where $c(X)$ depends only on $X$.

**Proof.** Let us start by remarking that if $f \in X$, then $2|f| \in \Gamma_f^X$. Indeed, for any family of cubes $\{Q_i\}_{i \in I} \in P$ we have

$$\sum_{i \in I} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| dxdy \leq 2 \sum_{i \in I} \int_{Q_i} |f(x)| dx.$$  

Therefore, $X \subset GaRonX$ and

(3.5) \[ \|f\|_{GaRonX} \leq 2\|f\|_X. \]

Moreover, since $\|f\|_{GaRonX} = \|f - f_{Q_0}\|_{GaRonX}$, we see that (3.3) follows from (3.5).

The remaining inclusion $GaRonX \subset X$ will follow if we can prove that

$$\|f - f_{Q_0}\|_X \leq c(X)\|f\|_{GaRonX}.$$  

Towards this end let $g = f - f_{Q_0}$. Since $g \in L^1(Q_0)$, we see that $g^{**}(t) \to 0$ as $t \to \infty$. Therefore, by the fundamental theorem of calculus, we can write\(^{12}\)

(3.6) \[ g^{**}(t) = \int_t^\infty (g^{**}(s) - g^*(s)) \frac{ds}{s}. \]

Since $f$ and $g$ differ by a constant, we readily see that $\Gamma_f^X = \Gamma_g^X$. Consequently, by (3.2), for all $\gamma \in \Gamma_f^X$ we have

$$\left( g^{**}(t) - g^*(t) \right) \leq c_n \gamma^{**}(t), \quad t \leq 1/4.$$

---

\(^{12}\)Recall that $\frac{d}{dt}(g^{**}(t)) = \frac{(g^{**}(0) - g^{**}(t))}{t}$. 
To deal with $t > 1/4$, we note that
\[
  t(g^{**}(t) - g^*(t)) = \int_{f^{**}(t)}^{\infty} \lambda_g(s)ds \leq \int_{0}^{\infty} \lambda_g(s)ds = \|g\|_{L^1};
\]
therefore,
\[
  (g^{**}(t) - g^*(t)) \leq t^{-1}\|g\|_{L^1}, \quad t > 1/4.
\]
Inserting the last two estimates in (3.6) we find that
\[
  (3.7) \quad g^{**}(t) \leq c_n \int_{t}^{1/4} \gamma^{**}(s) \frac{ds}{s} + c_n \|g\|_{L^1},
\]
Now, writing
\[
  \int_{t}^{\infty} \gamma^{**}(s) \frac{ds}{s} = \int_{t}^{\infty} \int_{0}^{s} \gamma^*(r) dr \frac{ds}{s} = \int_{t}^{\infty} \int_{0}^{s} \gamma^*(r) dr (-s^{-1})
\]
we see that
\[
  \int_{t}^{\infty} \gamma^{**}(s) \frac{ds}{s} = \gamma^{**}(t) + \int_{t}^{\infty} \gamma^*(s) \frac{ds}{s}.
\]
Inserting this information in (3.7) we find that
\[
  g^{**}(t) \leq \gamma^{**}(t) + \int_{t}^{\infty} \gamma^*(s) \frac{ds}{s} + \|g\|_{L^1}.
\]
Therefore, applying the $X$ norm on both sides of the last inequality, and then using the fact that $X$ has Boyd indices lying on $(0, 1)$, we obtain
\[
  (3.8) \quad \|g\|_{X} \leq \|\gamma\|_{X} + \|g\|_{L^1}.
\]
Now, since $\int_{Q_0} g = 0$, $\{Q_0\} \in P$, and $|Q_0| = 1$, we have
\[
  \|g\|_{L^1} = \int_{Q_0} |g(x)| = \int_{Q_0} |g(x)| - \int_{Q_0} |g(x)| - \int_{Q_0} |g(x)| - \int_{Q_0} |g(y)| dx dy
\]
\[
  = \frac{1}{|Q_0|} \int_{Q_0} \int_{Q_0} |f(x) - f(y)| dx dy \leq \int_{Q_0} |\gamma(y)| dy \quad \text{(since $\gamma \in \Gamma_f^X$)}
\]
\[
  = \|\gamma\|_{L^1} \leq C_X \|\gamma\|_{X}.
\]
Updating (3.8) we obtain
\[
  \|g\|_{X} \leq \|\gamma\|_{X}.
\]
\[13\text{Note that } L^\infty \subset X, \text{ which implies that for the constant function } \|g\|_{L^1} \text{ we have}
\]
\[
  \|\|g\|_{L^1}\|_{X} \leq \|g\|_{L^1}.
\]
Therefore, taking infimum over all \( \gamma \in \Gamma_f \) yields that there exists an absolute constant \( c(X) \) which depends only on \( X \), such that

\[
\| f - f_Q \|_X \leq c(X) \| f \|_{GaRo_X},
\]
as we wished to show. \( \square \)

**Corollary 1.** Let \( 1 < p < \infty \). Then

\[ GaRo_{L(p, \infty)} = GaRo_p. \]

**Proof.** It is well-known and easy to see that the \( L(p, \infty) \) spaces, \( 1 < p < \infty \), have Boyd indices in \((0, 1)\) (cf. \([5], [8]\)). Thus, by Theorem \([3]\)

\[ GaRo_{L(p, \infty)} = L(p, \infty), \]

which combined with \((1.4)\) yields the desired result. \( \square \)

**Remark 2.** To illustrate the conditions defining \( GaRo_{L(p, \infty)} \) and \( GaRo_p \), we now give a direct proof of the containment \( GaRo_{L(p, \infty)} \subset GaRo_p \), \( 1 < p < \infty \). We observe that if \( \gamma \in \Gamma_f \) then, for any \( \{ Q_i \}_{i \in I} \in P \), we have

\[
\sum_{i \in I} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| dxdy \\
\leq \sum_{i \in I} \int_{Q_i} |\gamma(x)| dx \leq \int_{Q_i} \sum_{i \in I} |Q_i| \gamma^*(s) ds \leq \| \gamma \|_{L(p, \infty)} \int_{Q_i} \frac{\sum_{i \in I} |Q_i| s^{-1/p} ds}{s}. \\
= p' \| \gamma \|_{L(p, \infty)} \left( \sum_{i \in I} |Q_i| \right)^{1/p'}. 
\]

It follows that

\[
\| f \|_{GaRo_p} \leq p' \| f \|_{GaRo_{L(p, \infty)}}. 
\]

**Remark 3.** It is also instructive to compare \((3.2)\) with the rearrangement inequalities in \([17\) Theorem 5 (ii)]. For this purpose note that if \( X = L(p, \infty) \), \( 1 < p < \infty \), then \( \gamma \in X \) implies that

\[
\gamma^{**}(t) \leq c_p \| \gamma \|_{L(p, \infty)} t^{-1/p}, \quad t > 0. 
\]

Combining the last estimate with \((3.2)\), we see that if \( f \in GaRo_{L(p, \infty)}, \gamma \in L(p, \infty) \), then

\[
i^{1/p} (f^{**}(t) - f^*(t)) \leq c_p \| \gamma \|_{L(p, \infty)}, \quad \text{for all } t < 1/4.
\]

Compare with \([17\) Theorem 5 (ii)].
4 Garsia–Rodemich spaces and fractional Sobolev spaces

As we have shown in [18], the Garsia–Rodemich formulation of Marcinkiewicz spaces leads to an easy approach to the self-improvement of (weak type) Poincaré–Sobolev inequalities. In this section we discuss fractional Sobolev inequalities. First, we use ideas from [9], [6] to prove weak type embeddings14 of fractional Sobolev spaces (cf. Subsection 4.1). Since strong type inequalities can be then obtained by the well-known method of truncation of Maz’ya, we will not address the issue here. Instead, in Subsection 4.2 we take a different approach. Using the generalized Gagliardo seminorms and generalized Sobolev spaces defined in the Introduction (cf. (1.16) and (1.17) above), combined with known estimates for Riesz potentials, and Theorem 3, we obtain embeddings of fractional Sobolev spaces based on rearrangement-invariant spaces.

4.1 Weak type inequalities. Let $\alpha \in (0, 1)$, $1 < p < \infty$. We shall consider the $W^{\alpha,p}$ spaces introduced in the Introduction (cf. (1.13) and (1.14) above).

**Theorem 4.** (i) Let $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then,

$$\|f\|_{GaRo_q} \leq n^{\frac{\alpha}{2p}} \|f\|_{W^{\alpha,p}}. \leqno{(4.1)}$$

(ii) If $p = \frac{n}{\alpha}$, then

$$\|f\|_{GaRo_{\infty}} \leq n^\alpha \|f\|_{W^{\alpha,\#}}. \leqno{(4.2)}$$

**Proof.** (i) Let $f \in W^{\alpha,p}$, and let $\{Q_i\}_{i \in I}$ be an arbitrary element of $\tilde{P}$. Let

$$A = \sum_{i \in I} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy.$$

Estimating the distance between two points of $Q_i$ by the diameter of $Q_i$ we find that for any $r > 0$,

$$\frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy = \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} \frac{|f(x) - f(y)|}{|x-y|^r} |x-y|^r \, dx \, dy \leq n^{r/2} |Q_i|^{r/n} \int_{Q_i} \int_{Q_i} \frac{|f(x) - f(y)|}{|x-y|^r} \, dx \, dy. \leqno{(4.2)}$$

14The method is amenable of extensions to a much more general context that we shall not pursue here.
Let \( r = \frac{(n+\alpha)p}{p} \). Then summing \((4.2)\) and applying Hölder’s inequality (twice) we obtain

\[
A \leq \sum_{i \in I} \frac{n^{r/2}|Q_i|^{r/n}}{|Q_i|} \int_{Q_i} \int_{Q_i} \frac{|f(x) - f(y)|}{|x - y|^r} \, dx \, dy
\]

\[
\leq n^{r/2} \sum_{i \in I} \left\{ \frac{|Q_i|^{r/n}}{|Q_i|} |Q_i|^{2/p'} \right\}^{1/p'} \left\{ \sum_{i \in I} \int_{Q_i} \int_{Q_i} \frac{|f(x) - f(y)|^p}{|x - y|^{pn+\alpha}} \, dx \, dy \right\}^{1/p} \leq n^{r/2} \sum_{i \in I} \left\{ \frac{|Q_i|^{r/n}}{|Q_i|} |Q_i|^{2/p'} \right\}^{1/p'} \|f\|_{W^{\alpha,p}}. \tag{4.3}
\]

Now, by computation \((\frac{r}{n} - 1)p' + 2 = \frac{\alpha}{n}p' + 1\), and therefore

\[
\sum_{i \in I} \left\{ \frac{|Q_i|^{r/n}}{|Q_i|} |Q_i|^{2/p'} \right\}^{1/p'} = \sum_{i \in I} |Q_i|^{(\frac{\alpha}{n}p' + 1)} \leq \left\{ \sum_{i \in I} |Q_i| \right\}^{(\frac{\alpha}{n}p' + 1)}.
\]

Inserting this information in \((4.3)\) yields

\[
A \leq n^{r/2} \left\{ \sum_{i \in I} |Q_i| \right\}^{(\frac{\alpha}{n}p' + 1)} \|f\|_{W^{\alpha,p}}. \tag{4.4}
\]

Since \((\frac{\alpha}{n}p' + 1)\frac{1}{p'} = \frac{1}{q}\), we obtain

\[
\|f\|_{GaRo_q} \leq n^{r/2} \|f\|_{W^{\alpha,p}}.
\]

(ii) In the limiting case, \( p = \frac{2n}{\alpha} \), therefore \( p' = \frac{n}{n - \alpha} \) and we see that \((\frac{\alpha}{n}p' + 1) = p'\); consequently, by definition, \( \frac{1}{q} = 1 \). Let \( r = \frac{2n}{p} \). Inserting this information in \((4.4)\) yields

\[
A \leq n^{\alpha} \left\{ \sum_{i \in I} |Q_i| \right\} \|f\|_{W^{\alpha,\frac{2n}{p}}}. \tag{4.5}
\]

Thus

\[
\|f\|_{GaR^\infty} \leq n^{\alpha} \|f\|_{W^{\alpha,\frac{2n}{p}}}. \tag{4.5}
\]

\[\square\]

**Remark 4.** By private correspondence Daniel Spector observed that with a minor modification the proof also works in the case \( p = 1 \). Indeed, if \( p = 1 \), we let \( r = n + \alpha \), and proceed as in the proof of case (i), but now only one application of
Hölder’s inequality is needed to obtain
\[ A \leq n^{(n+\alpha)/2} \sup_{i \in I} \left\{ \left( \sum_{i \in I} \left\| Q_i \right\|^2 \right)^{\frac{\alpha}{n}} \right\} \int_{Q} \left| f(x) - f(y) \right|^{1/p} \left| x - y \right|^{-\alpha/n} dx dy \]

\[ \leq n^{(n+\alpha)/2} \left\{ \sum_{i \in I} \left| Q_i \right| \right\}^{\frac{\alpha}{n}} \left\| f \right\|_{W^{\alpha,1}}. \]

Since \( \frac{1}{q} = 1 - \frac{\alpha}{n} \) we thus have
\[ \| f \|_{GaRo_q} \leq n^{(n+\alpha)/2} \left\| f \right\|_{W^{\alpha,1}}. \] (4.6)

Alternatively, the inequality (4.6) can be obtained letting \( p \to 1 \) in (4.1).

**Remark 5.** Note that starting with (4.2) applied to a cube \( Q \) with \( r = \frac{2n}{p} \), and then applying Hölder’s inequality, yields
\[ \frac{1}{\left| Q \right|^2} \int_{Q} \int_{Q} \left| f(x) - f(y) \right| dx dy \leq \frac{n^{\alpha} \left| Q \right|^{2/p}}{\left| Q \right|^2} \left( \int_{Q} \int_{Q} \frac{\left| f(x) - f(y) \right|^p}{\left| x - y \right|^{2n}} dx dy \right)^{1/p}, \]
and (4.5) follows from the fact that
\[ \| f \|_{GaRo_{\infty}} = \| f \|_{BMO} \simeq \sup_{Q \subset Q_0} \frac{1}{\left| Q \right|^2} \int_{Q} \int_{Q} \left| f(x) - f(y) \right| dx dy. \]

This approach to (4.5) is classical (cf. [9]); use of the Garsia–Rodemich spaces unifies the proof of (i) and (ii).

### 4.2 Strong type inequalities.

The characterization of rearrangement-invariant spaces \( X \) with indices lying on \((0, 1)\) as \( GaRo_X \) spaces provided by Theorem 3 allows to unify the proofs of the weak and strong type Sobolev inequalities in the general context of rearrangement-invariant spaces. For other treatments of fractional Sobolev inequalities we refer to [7], [3], [21], [12], [25] and the references therein.

Let \( \alpha \in (0, 1), 1 \leq p < \infty \). Recall the definition of the main objects of study:
\[ D_{p,a}(f)(y) = \left\{ \int_{Q_0} \frac{\left| f(x) - f(y) \right|^p}{\left| x - y \right|^{n+ap}} dx \right\}^{1/p}, \quad y \in Q_0, \]
\[ W_{p,Y}^\alpha := W_{p,Y}^\alpha (Q_0) := \{ f : \| D_{p,a}(f) \|_Y < \infty \}, \]
\[ I_{q,Y}^a(f)(x) = \int_{Q_0} \frac{f(y)}{\left| x - y \right|^{n-a}} dy, \quad x \in Q_0. \]

Our main result reads as follows:
Theorem 5. Let \( \alpha \in (0, 1), 1 < p < \frac{n}{\alpha}, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \). Let \( X \) and \( Y \) be rearrangement-invariant spaces such that \( I_{a, Q_0} \) is a bounded map, \( I_{a, Q_0} : Y \to X \). Then

\[
W^a_{p, Y} \subset \text{GaRo}_X.
\]

Proof. Let \( f \in W^a_{p, Y} \), and let \( \{ Q_i \}_{i \in I} \) be an arbitrary element of \( \tilde{P} \). Let

\[
A = \sum_{i \in I} \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)| \, dx \, dy.
\]

Using Jensen’s inequality on the inner integral we find that

\[
A \leq \sum_{i \in I} \int_{Q_i} \left( \frac{1}{|Q_i|} \int_{Q_i} |f(x) - f(y)|^p \, dx \right)^{1/p} \, dy
\]

\[
\leq n^{(n+ap)/2p} \sum_{i \in I} \frac{|Q_i|^{ap}}{|Q_i|^{1/p}} \int_{Q_i} \left( \int_{Q_i} |f(x) - f(y)|^p \frac{dydz}{|x-y|^{n+ap}} \right)^{1/p} \, dy
\]

\[
\leq n^{(n+ap)/2p} \sum_{i \in I} \frac{|Q_i|^{ap}}{|Q_i|^{1/p}} \int_{Q_i} \left( \int_{Q_i} |f(x) - f(y)|^p \frac{dydz}{|x-y|^{ap}} \right)^{1/p} \, dy
\]

\[
= n^{(n+ap)/2p} \sum_{i \in I} \frac{|Q_i|^{ap}}{|Q_i|^{1/p}} \int_{Q_i} D_{p, a}(f)(y) dy
\]

(4.7)

\[
= n^{(n+ap)/2p} \sum_{i \in I} \frac{|Q_i|^{ap}}{|Q_i|^{1/p}} \int_{Q_i} \int_{Q_i} D_{p, a}(f)(y)dydz
\]

\[
= n^{(n+ap)/2p} \sum_{i \in I} |Q_i|^{ap} \int_{Q_i} \int_{Q_i} D_{p, a}(f)(y)dydz
\]

\[
= n^{(n+ap)/2p} \sum_{i \in I} |Q_i|^{ap} \int_{Q_i} \int_{Q_i} D_{p, a}(f)(y)dydz
\]

\[
\leq n^{(n+ap)/2p} \sum_{i \in I} \int_{Q_i} \int_{Q_i} D_{p, a}(f)(y)dydz
\]

\[
\leq n^{(n+ap)/2p} \sum_{i \in I} \int_{Q_i} \int_{Q_i} D_{p, a}(f)(y)dydz
\]

\[
= C_n \sum_{i \in I} \int_{Q_i} I_{a, Q_0}(D_{p, a}(f))(z)dz.
\]

Moreover, by assumption,

\[
\|I_{a, Q_0}(D_{p, a}(f))\|_X \leq \|I_{a, Q_0}\|_{Y \to X} \|D_{p, a}(f)\|_Y = \|I_{a, Q_0}\|_{Y \to X} \|f\|_{W^a_{p, Y}}.
\]
Combining this fact with (4.7) we see that $C_n I_{a,Q_0}(D_{p,\alpha}(f)) \in \Gamma^X_f$, and

$$\|f\|_{GaRox} \leq C_n \|I_{a,Q_0}\|_{Y \rightarrow X} \|f\|_{W_{p,\alpha}^a},$$

as we wished to show. \hfill \square

**Corollary 2.** Suppose that all the assumptions of Theorem 5 hold and, furthermore, that $X$ has Boyd indices lying in $(0, 1)$. Then

$$W_{p,Y}^a \subset X.$$  

**Proof.** Applying successively Theorem 5 and Theorem 3 we obtain

$$W_{p,Y}^a \subset GaRox = X.$$  

\hfill \square

**Example 1.** The Lorentz spaces $L(s,r)$, $1 < s < \infty$, $1 \leq r \leq \infty$, are defined by the condition

$$\|f\|_{L(s,r)} = \left\{ \int_0^\infty (f^{**}(u)u^{1/s})^{r} \frac{du}{u} \right\}^{1/r} < \infty.$$ 

It is well-known (cf. [20]) that if $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $1 \leq r_1 \leq r_2 \leq \infty$, then $I_{a,Q_0} : L(p,r_1) \rightarrow L(q,r_2)$ is a bounded map. Then we can conclude that

$$W_{p,L(p,r_1)}^a \subset GaRol_{L(q,r_2)} = L(q,r_2).$$ 

**Example 2.** The previous discussion shows that

$$(4.8) \quad \|f\|_{L^q} \leq c(q) \|f\|_{GaRox} \leq c(q) C_n \|I_{a,Q_0}\|_{LP \rightarrow Lq} \|f\|_{W_{p,LP}^a}.$$ 

Note that since

$$W_{p,LP}^a = W_{p,P}^a,$$

(4.8) gives the corresponding strong type inequalities of Theorem 4.

**Example 3.** The end point inequalities for local Riesz potentials that were obtained in [10] can be also implemented here. For example, when $p = \frac{n}{\alpha}$, we have (cf. [10] Theorem 2) $I_{a,Q_0} : L(\frac{n}{\alpha}, \frac{n}{\alpha}) \rightarrow BW_{n/\alpha}$, where

$$BW_{n/\alpha} = \left\{ f : \|f\|_{BW_{n/\alpha}} = \left\{ \int_0^1 \left( \frac{f^*(t)}{(1 + \log \frac{1}{t})} \right)^{n/\alpha} \frac{dt}{t} \right\}^{a/n} < \infty \right\}. $$

As a consequence, we obtain the following fractional version of the well-known Brezis–Wainger inequality [10],

$$W_{a/\alpha}^{a/\alpha} = W_{a/\alpha,L(\frac{n}{\alpha}, \frac{n}{\alpha})} \subset GaRol_{BW_{n/\alpha}}.$$ 

\textsuperscript{15}In particular, if we let $r_1 = p < r_2 = \infty$, we recover [1.15].
5 Final remarks

Finally, we comment briefly on some loose ends that we are leaving for future work.

- We should mention the interesting work by Ambrosio, Bourgain, Brezis and Figalli (cf. [1]) on isotropic versions of $B$ and their use in the computation of perimeters of sets. What is the connection with scalings of Garsia–Rodemich conditions and the limit fractional Sobolev norms (cf. [13], [22])?

- It could be interesting to explore the role of Garsia–Rodemich conditions in solving the problem of proving dimension-free versions of the John–Nirenberg inequality (cf. [11]).

- As often happens in mathematics, the generalization of the Garsia–Rodemich condition developed in this paper makes it easier to understand the connection with other constructions. In a forthcoming joint paper with Sergey Astashkin [2] we connect the Garsia–Rodemich spaces to the Fefferman–Stein inequalities and interpolation theory.

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