AN ANALOGUE OF THE GIBBONS-HAWKING ANSATZ FOR QUATERNIONIC KÄHLER SPACES

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ABSTRACT. We show that the geometry of 4n-dimensional quaternionic Kähler spaces with a locally free \( \mathbb{R}^{n+1} \)-action admits a Gibbons-Hawking-like description based on the Galicki-Lawson notion of quaternionic Kähler moment map. This generalizes to higher dimensions a four-dimensional construction of Calderbank and Pedersen of self-dual Einstein manifolds with two linearly independent commuting Killing vector fields. As an application, we use this new Ansatz to give an explicit equivariant completion of the twistor space construction of the local \( c \)-map proposed by Roček, Vafa and Vandoren.

CONTENTS

0 Introduction .......................................................... 2
1 Hyperkähler vs. quaternionic Kähler geometry ................. 5
  1.1 General aspects .................................................. 5
  1.2 Killing vector fields ............................................. 12
  1.3 Hyperkähler cones ............................................... 18
2 Swann bundles ....................................................... 21
  2.1 Structures on the group \( H^x \) .................................... 21
  2.2 The Swann bundle of a quaternion Kähler manifold ........ 23
3 An analogue of the extended Gibbons-Hawking Ansatz for quaternionic Kähler metrics ........................................ 27
  3.1 The extended Gibbons-Hawking Ansatz ....................... 27
  3.2 Extended Gibbons-Hawking spaces with a hyperkähler cone structure ..................................... 28
  3.3 The space \( \text{Im}\mathbb{H}P^n \) .................................................. 32
  3.4 The reduction ...................................................... 37
  3.5 The quaternionic Kähler Ansatz .............................. 41
  3.6 The case of four dimensions .................................... 43
4 Hyperkähler cones via the Legendre transform construction .. 46
  4.1 The Legendre transform construction ....................... 46
  4.2 Hyperkähler cone structure conditions ..................... 47
5 An example: the local \( c \)-map ..................................... 49
References .............................................................. 58
0. Introduction

The Gibbons-Hawking Ansatz [25] gives a way of constructing four-dimensional hyperkähler spaces with a tri-Hamiltonian circle, or, more generally, locally free $\mathbb{R}$-action. This construction was generalized in [38] by Pedersen and Poon to $4n$-dimensional hyperkähler spaces with a tri-Hamiltonian locally free $\mathbb{R}^n$-action. In what follows we will refer to spaces of this type as extended Gibbons-Hawking spaces. The rank of the action, equal to the quaternionic dimension of the hyperkähler space, satisfies an optimality condition in the sense that the $n$ orbit parameters of the action together with the $3n$ components of its associated hyperkähler moment map functions suffice to provide a complete set of coordinates for the hyperkähler space. Geometrically, the hyperkähler structure can be viewed as being defined on the total space of an $\mathbb{R}^n$-bundle over an open subset of the space $\mathbb{R}^n \otimes \mathbb{R}^3$. It is given explicitly in terms of a set of Higgs fields and $\mathbb{R}^n$-connection 1-forms satisfying a generalized form of the Bogomolny equation.

The question we will be concerned with in this paper is whether a similar construction exists for quaternionic Kähler manifolds. Hyperkähler and quaternionic Kähler manifolds are $4n$-dimensional Riemannian manifolds for which the holonomy groups of their Levi-Civita connection are subgroups of $Sp(n)$ and $Sp(n) \times_{\mathbb{Z}_2} Sp(1)$, respectively (the precise definitions involve some additional refinements and distinctions, see e.g. §1.1). In a certain sense, which we shall often exploit in what follows, hyperkähler manifolds can be considered as a limit case of quaternionic Kähler ones. Most of the results we derive apply also to the pseudo-Riemannian cases and the non-compact versions of these groups. We find that a quaternionic Kähler analogue of the Gibbons-Hawking construction indeed exists for the class of $4n$-dimensional quaternionic Kähler spaces with a locally free isometric $\mathbb{R}^{n+1}$-action, i.e. an action of rank one more than the quaternionic dimension of the space, if one replaces the notion of hyperkähler moment map by a quaternionic Kähler counterpart of this notion defined by Galicki and Lawson in [23, 24]. A naïve counting of the coordinates in this setting seems at first sight to give too many; however, as we shall see, subtler issues are at play, and, when done properly, the counting yields the right number.

In fact, our scope is somewhat broader, and we also use this question as an excuse to revisit a number of fundamental issues pertaining to hyperkähler and quaternionic Kähler geometries in general, and cast some new light on them. The first two rather sizable sections of the paper are completely dedicated to quite substantial and detailed digressions into such general matters. They can be even read as introductions to these geometries. The reader familiar with these topics, however, and interested solely in the quaternionic Kähler analogue of the extended Gibbons-Hawking Ansatz can skip ahead directly to section 3. The occasional references to the first section in particular should not pose in principle much difficulties to anyone accustomed to the basics of hyperkähler and quaternionic Kähler geometries.

The main body of the paper is organized into five sections. In section 1 we give a non-Riemannian characterization of both hyperkähler and quaternionic Kähler manifolds. That is, in each case we formulate criterions in which the metric is not among the fundamental objects defining the geometry, but, rather, is a derived, composite object. The building blocks of either geometry may be considered instead to be triplets of non-singular and pointwise linearly independent 2-forms: globally defined in the hyperkähler case, and locally defined in the quaternionic Kähler one. Such a triplet is required to satisfy two conditions: an algebraic condition, and an exterior differential algebra one (Theorems 2 and 3). In the
hyperkähler case the latter condition is famously due to Hitchin \cite{27} and requires that the three 2-forms be symplectic. In the quaternionic Kähler case a corresponding condition was found by Alekseevsky, Bonan and Marchiafava \cite{2}. These conditions do not constrain the triplets uniquely. In the hyperkähler case one has in fact a half 3-sphere’s worth of equivalent triplets to choose from (the triplets can be rotated by $SO(3)$ transformations, whose group manifold is topologically $S^3/\mathbb{Z}_2$). In the quaternionic Kähler case, on the other hand, one may regard the triplets as local frames for an $SO(3)$-bundle. The proofs we give to these criterions are new. They run in parallel, emphasizing the similarity between the two cases, and are based on the successive application two key identities, which we collect together in Lemma \ref{lemma1}. We show also that Alekseevsky \textit{et al.}'s condition can be equivalently replaced by a Hermitian-Einstein-type condition for the curvature 2-form of the associated principal $SO(3)$-bundle, which takes as well the form of an exterior differential algebra condition. This gives us the very useful equivalent quaternionic Kähler criterion of Theorem \ref{thm4}.

Given these reformulations, the question then arises how does for example the Killing condition for a vector field translate in terms of the 2-forms description? And the answer is that Killing vector fields act on these triplets of 2-forms by rotating them, and, conversely, any vector field which rotates the 2-forms is Killing. In the hyperkähler case it is possible of course for this rotation to be trivial, with the vector fields preserving the 2-forms individually: this is the familiar tri-Hamiltonian — or tri-holomorphic — vector field case. In the same spirit, we prove moreover that the condition for a hyperkähler manifold to possess a homothetic vector field can be equivalently phrased as a generalized moment map condition on the triplet of 2-forms (part (d) of Proposition \ref{prop11}).

In section \ref{sec2} we give a detailed review of a construction which plays an essential role in our subsequent considerations, namely, the Swann bundle of a quaternionic Kähler manifold. In \cite{43}, Swann has shown that over any quaternionic Kähler manifold one can construct a quaternionic bundle with fiber $\mathbb{H}^\times/\mathbb{Z}_2$, where $\mathbb{H}^\times$ denotes the multiplicative group of non-zero quaternions, the total space of which carries a natural hyperkähler structure as well as a homothetic vector field: that is to say, a hyperkähler cone structure. The hyperkähler cone structure of the Swann bundle completely encodes the quaternionic Kähler geometry of the base. Thus, Swann’s result gives us a way to radically reformulate a quaternionic Kähler geometry problem as a hyperkähler geometry problem. The results of the previous section are used to simplify some of the proofs one finds in the literature. We also clarify the role of the hypercomplex and Lie group structures on the quaternionic fiber.

Killing vector fields on the quaternionic Kähler manifold lift naturally to tri-Hamiltonian vector fields on its Swann bundle, and, conversely, tri-Hamiltonian vector fields on a Swann bundle descend to Killing vector fields on its base. The precise way in which this happens, as well as the relationship between the hyperkähler moment map and Galicki and Lawson’s quaternionic Kähler moment map respectively associated to these vector fields form the subjects of Theorem \ref{thm14}.

In section \ref{sec3} we finally come around to the headline subject of the paper: finding an analogue of the extended Gibbons-Hawking Ansatz for quaternionic Kähler spaces. However, as we have learned to do from the previous section, rather than ask the question directly about quaternionic Kähler spaces, we ask instead the corresponding question about hyperkähler cones. Namely, we ask under what conditions does an extended Gibbons-Hawking space possess a hyperkähler cone structure? For any extended Gibbons-Hawking...
space of real dimension $4n + 4$ (this form is chosen for later convenience), the base of its $\mathbb{R}^{n+1}$-fibration is an open subset of the space $\mathbb{R}^{n+1} \otimes \mathbb{R}^3$, which can be viewed as the space of configurations of $n + 1$ distinguishable points in $\mathbb{R}^3$. On this space, one can always define a natural quaternionic action by considering the rigid rotations and simultaneous scalings of such configurations which leave a given point, say, the origin, fixed. These form indeed a quaternionic group since $SO(3) \otimes \mathbb{R}_{>0} \cong \mathbb{H}^\times / \mathbb{Z}_2$. So, a more specific question to ask then is when can this $\mathbb{H}^\times / \mathbb{Z}_2$-action on the base of the $\mathbb{R}^{n+1}$-fibration be lifted to a genuine hyperkähler cone structure on the total space of the fibration, that is, on the extended Gibbons-Hawking space? This question is answered in Proposition 15, which states that a necessary and sufficient condition for that to happen is for the Higgs fields of the extended Gibbons-Hawking space to be invariant at rigid rotations and scale with a certain weight under simultaneous scalings of the configurations.

The following step is to match this description of the hyperkähler cone, in which the $\mathbb{R}^{n+1}$-bundle structure is manifest, to the Swann description, centered instead on the $\mathbb{H}^\times / \mathbb{Z}_2$-bundle structure. This is facilitated by a certain condensed quaternionic reformulation of the extended Gibbons-Hawking formulas that we introduce in §3.1.3. It ultimately yields an explicit description of the geometry of the base of the Swann bundle, which is in this case a $4n$-dimensional quaternionic Kähler geometry with an inherited locally free isometric $\mathbb{R}^{n+1}$-action.

The emerging picture is summarized in intrinsic terms in subsection 3.3. Its main features are as follows: The quaternionic Kähler structure is defined on the total space of an $\mathbb{R}^{n+1}$-bundle over the space of inequivalent (with respect to rigid rotations and simultaneous scalings) non-degenerate configurations, which we denote by $\text{Im}\mathbb{H}^\mathbb{P}^n$ (and define precisely in subsection 3.3). A natural set of coordinates is given by the orbit parameters of the $\mathbb{R}^{n+1}$-action, together with this action’s associated Galicki-Lawson-type moment maps, multiplied by a scale. The latter may also be understood as inhomogeneous local coordinates on $\text{Im}\mathbb{H}^\mathbb{P}^n$. The quaternionic Kähler metric, 2-forms and $SO(3)$-connection 1-forms are formulated explicitly in these coordinates in terms of a set of reduced Higgs fields and $\mathbb{R}^{n+1}$-connection 1-forms satisfying a set of partial differential equations on $\text{Im}\mathbb{H}^\mathbb{P}^n$ similar to the Bogomolny equations from the hyperkähler case. These field equations admit also a dual formulation, a remarkable consequence of which is that the local quaternionic Kähler structure turns out to be completely determined by a single real-valued function on $\text{Im}\mathbb{H}^\mathbb{P}^n$ satisfying a set of linear partial differential constraints. This function is closely related to the hyperkähler potential of the Swann bundle. (Swann bundles have the distinctive feature among hyperkähler spaces that all three elements of a defining triplet of 2-forms can be derived, as Kähler forms with respect to their corresponding complex structures, from the same Kähler potential.) Very importantly, this picture is general: any $4n$-dimensional quaternionic Kähler space with a locally free isometric $\mathbb{R}^{n+1}$-action can be shown to arise locally in this way.

In the last part of the section we specialize to four dimensions (i.e., $n = 1$). The space $\text{Im}\mathbb{H}^\mathbb{P}^1$ is isomorphic to the complex upper half-plane, and we find that our formulas yield in this case precisely the description given by Calderbank and Pedersen in [11] to the four-dimensional avatars of the quaternionic Kähler spaces we consider, which are self-dual Einstein spaces with two linearly independent commuting Killing vector fields. In this sense, therefore, our results can be thought of as a natural higher-dimensional generalization of those of Calderbank and Pedersen.
In section 4 we translate the description of hyperkähler cones from the Gibbons-Hawking-type framework used up until now to the language of the Legendre transform approach of Lindström and Roček [35, 29]. This is useful particularly when one wants to solve the field equations by twistor methods.

As an application of our general Ansatz, in section 5 we give a complete and explicit account of the twistor construction of a quaternionic Kähler metric found in [21] by Ferrara and Sabharwal, and related to the so-called local c-map construction from string theory and supergravity. The twistor approach to this metric was initiated in [40, 41] by Roček, Vafa and Vandoren, and developed further in [37]. The idea is that, if one goes from the Ferrara-Sabharwal space six dimensions up to the twistor space (in the sense of [29]) of its Swann bundle, there the quaternionic Kähler metric information is encoded in a single — and simple — holomorphic symplectic gluing function. The challenge becomes then to show explicitly how the quaternionic Kähler metric can be retrieved from this holomorphic data. In a first stage, by making use of the general methods associated to the Legendre transform approach, one can come down two dimensions from the twistor space and determine from this function the geometry of the Swann bundle. This step has been generally already understood in the literature. The new results of section 3 can then be used to descend a further four dimensions and retrieve explicitly, and equivariantly, the Ferrara-Sabharwal metric. This clarifies a number of issues and cements the basis of the twistor-theoretic understanding of this important construction.

1. Hyperkähler vs. quaternionic Kähler geometry

1.1. General aspects. In Berger’s holonomy classification, hyperkähler manifolds and quaternionic Kähler manifolds are 4n-dimensional Riemannian manifolds with the holonomy group of the Levi-Civita connection a subgroup of the unitary quaternionic group $Sp(n)$, respectively the group $Sp(n) \times \mathbb{Z}_2 Sp(1)$. In the latter case we will assume, moreover, that the holonomy group is not a subgroup of $Sp(n)$, in other words, that hyperkähler manifolds are not a subclass of quaternionic Kähler ones. Our considerations will also hold, unless we specify otherwise, for the pseudo-Riemannian versions of these manifolds, which are obtained by replacing the group $Sp(n)$ with one of its non-compact versions $Sp(p,q)$ with $p + q = n$. In line with this, we will often abuse the terminology and omit the prefix pseudo when referring to pseudo-hyperkähler and pseudo-quaternionic Kähler manifolds. In four dimensions (that is, for $n = 1$), due to the isomorphism $Sp(1) \times \mathbb{Z}_2 Sp(1) \cong SO(4)$ the above definition of quaternionic Kähler manifolds is rather unrestrictive as it encompasses all orientable four-manifolds. In this case the holonomy condition is usually replaced by a curvature condition, and the natural four-dimensional analogues of quaternionic Kähler manifolds are considered to be the self-dual Einstein manifolds. In fact, one can show that quaternionic Kähler manifolds are always Einstein [11], while hyperkähler ones are Einstein with vanishing scalar curvature — that is, Ricci-flat.

The holonomy definition can be shown to be equivalent in the quaternionic Kähler case to the existence of a three-dimensional subbundle of the bundle of endomorphisms of the tangent bundle $\text{End}(TM)$ which is locally spanned by almost complex structures $I_1$, $I_2$, $I_3$, forming the algebra of imaginary quaternions, i.e. $I_1^2 = I_2^2 = I_3^2 = I_1 I_2 I_3 = -1$, and is preserved by the Levi-Civita connection $\nabla$. More precisely, there exist locally defined 1-forms $\theta_1, \theta_2, \theta_3 \in T^*M$ such that

$$\nabla_X I_i = -2\varepsilon_{ijk} \theta_j(X) I_k$$
for any vector field $X \in TM$, where the indices $i, j, k$ run over the values 1, 2, 3, $\varepsilon_{ijk}$ denotes the anti-symmetric Levi-Civita symbol and the numerical factor is conventional. In four dimensions, the $\theta_i$ can be viewed as the self-dual part of the spin connection.

In the hyperkähler case, on the other hand, the holonomy definition is equivalent to the existence of a globally defined triplet of almost complex structures $I_1, I_2, I_3$ forming the algebra of imaginary quaternions, each member of which is individually parallel with respect to the Levi-Civita connection:

$$\nabla_X I_i = 0$$

for any vector field $X \in TM$. In contrast to the quaternionic Kähler setting, one can show now that each such almost complex structure is integrable and thus gives rise to a genuine complex structure.

In either case, the metric $g$ can always be chosen to be simultaneously Hermitian with respect to each $I_i$, and by combining it with them one can form three 2-forms

$$\omega_i(X, Y) = g(X, I_i Y)$$

($X, Y \in TM$) which are defined locally in the quaternionic Kähler setting and globally in the hyperkähler one. In four dimensions, these form a frame of the bundle of self-dual 2-forms. Assuming a non-singular metric, since each $I_i$ is invertible (with inverse $-I_i$), it follows that each $\omega_i$, viewed as an element of $\text{Hom}(TM, TM)$, is also invertible, with inverse $\omega_i^{-1} \in \text{Hom}(TM, T^*M)$. A simple argument based on the quaternionic algebra shows then that we have

$$I_1 = \omega_3^{-1}\omega_2, \quad I_2 = \omega_1^{-1}\omega_3, \quad I_3 = \omega_2^{-1}\omega_1.$$ 

The covariant differentiation properties (1) and (2) trivially translate for the 2-forms into

$$\nabla_X \omega_i = -2\varepsilon_{ijk}\theta_j(X)\omega_k$$

in the quaternionic Kähler case and

$$\nabla_X \omega_i = 0$$

in the hyperkähler one, for an arbitrary vector field $X \in TM$. Since the Levi-Civita connection is symmetric, these formulas imply further that

$$d\omega_i = -2\varepsilon_{ijk}\theta_j \wedge \omega_k$$

in the first case and

$$d\omega_i = 0$$

in the second, where we now view $\omega_i$ and $\theta_i$ as differential forms. Note in particular that the reverse implication does not hold in general.

In this approach the three 2-forms are composite objects constructed from the metric and almost complex structures. In what follows we will present an alternative approach in which this relationship is reversed, and they rather than the latter are regarded as the fundamental building objects of the geometry. Our discussion will emphasize in particular the fact that the two geometries inform each other and can be treated in parallel, and that, although we have made a special point of distinguishing between them, it is often convenient in practice to think of hyperkähler geometry as a limit case of quaternionic Kähler geometry, when the hallmark three-dimensional subbundle of the latter becomes trivial and the scalar curvature vanishes.

The proofs of the subsequent two theorems rely crucially on the following lemma:
Lemma 1. Let \( M \) be a manifold with an almost complex structure \( I \) — that is, an endomorphism of the tangent bundle \( TM \) such that \( I^2 = -1 \) — and let
\[
N_I(X, Y) = -I^2[X, Y] + I([IX, Y] + [X, IY]) - [IX, IY]
\]
for arbitrary vector fields \( X, Y \in TM \) be the associated Nijenhuis tensor. A real-valued 2-form \( \omega \) on \( M \) is
\begin{enumerate}[(a)]
  
  \item of \((1, 1)\) type with respect to \( I \) if and only if
  \[
  \omega(X, IY) = \omega(Y, IX) \overset{\text{def}}{=} -g(X, Y)
  \]
  for any \( X, Y \in TM \). In this case the following identity holds:
  \[
  \omega(N_I(X, Y), Z) = d\omega(IX, IY, Z) - d\omega(X, Y, Z) + 2\nabla_Z\omega(X, Y)
  \]
  where \( \nabla \) is the Levi-Civita connection corresponding to the compatible metric \( g \).

  \item of mixed \((2, 0) + (0, 2)\) type with respect to \( I \) if and only if
  \[
  \omega(X, IY) = -\omega(Y, IX) \overset{\text{def}}{=} -\tilde{\omega}(X, Y)
  \]
  for any \( X, Y \in TM \). (The complex-valued 2-forms \( \omega + i\tilde{\omega} \) and \( \omega - i\tilde{\omega} \) are then of pure \((2, 0)\) and \((0, 2)\) type with respect to \( I \), respectively.) In this case we have the identity
  \[
  \omega(N_I(X, Y), Z) = d\omega(IX, IY, Z) - d\omega(X, Y, Z) + d\tilde{\omega}(IX, IY, Z) + d\tilde{\omega}(X, IY, Z).
  \]
\end{enumerate}

Remark. When the 2-form \( \omega \) is in addition non-degenerate and globally defined the structure at point \((a)\) is usually called an almost Hermitian structure, and the one at point \((b)\) an almost complex symplectic structure.

Proof. Part \((a)\) is essentially Proposition 4.6 from ch. IX, sec. 4 of reference \[33\] (the numerical factors differ due to differing normalization conventions for differential forms). Let us reproduce here the proof briefly. First, we transform the last term on the right-hand side of the identity we want to prove as follows:
\[
\nabla_Z\omega(X, Y) = -g((\nabla_Z I)X, Y)
\]
\[
= -g(\nabla_Z (IX), Y) + g(I\nabla_Z X, Y)
\]
\[
= -g(\nabla_Z (IX), Y) - g(\nabla_Z, Y).
\]
We can then apply for each resulting term the Koszul formula for the Levi-Civita connection:
\[
2g(\nabla_X Y, Z) = X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X).
\]
On another hand, for the other two terms on the right-hand side we use the formula:
\[
d\omega(X, Y, Z) = X(\omega(Y, Z)) - Y(\omega(X, Z)) + Z(\omega(X, Y)) - \omega([X, Y], Z) + \omega([X, Z], Y) - \omega([Y, Z], X).
\]
The result follows then by isolating the expression on the left-hand side and then cancelling the remaining terms by making use of the relationship between \( \omega \) and \( g \), and the properties which follow from it.

The formula at part \((b)\) can be likewise verified in a relatively more straightforward manner by applying the identity \[13\] to both the \( d\omega \) and the \( d\tilde{\omega} \) terms, and then systematically
replacing in the resulting expression $\tilde{\omega}$ with $\omega$ by using some version of the defining relation for the former.

With this result in hand we can now return to our original inquiry. On the hyperkähler side we have the following criterion, which is in essence a version of Lemma 6.8 of reference [27]:

**Theorem 2.** A finite-dimensional manifold $M$ is (possibly pseudo-) hyperkähler if and only if it possesses a triplet of non-singular and pointwise linearly independent 2-forms $\omega_1, \omega_2, \omega_3$ satisfying simultaneously:

- **an algebraic condition:** if we define
  \[
  I_1 = \omega_3^{-1}\omega_2, \quad I_2 = \omega_1^{-1}\omega_3, \quad I_3 = \omega_2^{-1}\omega_1 \in \text{End}(TM)
  \]
  then we have $I_1^2 = I_2^2 = I_3^2 = -1$.

- **a differential condition:** the 2-forms are symplectic, i.e.,
  \[
  d\omega_1 = d\omega_2 = d\omega_3 = 0.
  \]

**Proof.** First, observe that although the algebraic condition formally requires only that $I_1, I_2, I_3$ are almost complex structures, it fully implies in fact that they satisfy the entire imaginary quaternionic algebra. Indeed, since they square to minus the identity we have

\[
\begin{align*}
I_1 &= \omega_3^{-1}\omega_2 = -\omega_2^{-1}\omega_3 \\
I_2 &= \omega_1^{-1}\omega_3 = -\omega_3^{-1}\omega_1 \\
I_3 &= \omega_2^{-1}\omega_1 = -\omega_1^{-1}\omega_2
\end{align*}
\]

and then based on the second set of expressions we get

\[
I_1 I_2 I_3 = (-\omega_2^{-1}\omega_3)(-\omega_3^{-1}\omega_1)(-\omega_1^{-1}\omega_2) = -1
\]

which proves the assertion.

Let us define now the following sections of $\text{Hom}(T^*M, TM)$ or, equivalently, rank-2 contravariant tensors:

\[
\begin{align*}
g_1 &= -\omega_1 I_1 = -\omega_1 \omega_3^{-1}\omega_2 = \omega_1 \omega_2^{-1}\omega_3 \\
g_2 &= -\omega_2 I_2 = -\omega_2 \omega_1^{-1}\omega_3 = \omega_2 \omega_3^{-1}\omega_1 \\
g_3 &= -\omega_3 I_3 = -\omega_3 \omega_2^{-1}\omega_1 = \omega_3 \omega_1^{-1}\omega_2.
\end{align*}
\]

Since the $\omega_i$'s are all antisymmetric, by comparing for example the third expression for $g_1$ with the fourth one for $g_2$ we see that $g_1^T = g_2$, where the $T$-superscript denotes transposition. Similarly we have $g_2^T = g_3$ and $g_3^T = g_1$, from which it follows immediately that $g_1 = g_2 = g_3 = a$ symmetric rank-2 tensor. In other words, the algebraic condition also implies that we can define a *metric* on $M$ by

\[
g(X,Y) = -\omega_1(X, I_1 Y) = -\omega_2(X, I_2 Y) = -\omega_3(X, I_3 Y)
\]

for any $X, Y \in TM$. Note that this definition is compatible, and in fact equivalent, with the relations (3), which justifies our choice of minus signs. Such a metric is automatically Hermitian with respect to each $I_i$ (this is possible only when $\dim M = 4n$), and its signature is in general of type $(4n_+, 4n_-)$, with $n_+ + n_- = n$, so the manifold $M$ is in general pseudo-Riemannian.
Observe now that in order to prove that the manifold \( M \) carries a hyperkähler structure it suffices to prove that each \( \omega_i \) is covariantly constant with respect to the Levi-Civita connection of the metric \( g \). The covariant constancy of the \( I_i \) follows then immediately.

This can be shown in two steps. Note first that, based on the quaternionic algebra, we can write successively \( \omega_1(X, I_3Y) = g(X, I_1 I_3 Y) = -g(X, I_2 Y) = -\omega_2(X, Y) \), and since the last expression is antisymmetric at the exchange of \( X \) and \( Y \) it follows that \( \omega_1 \) is of mixed \((2,0) + (0,2)\) type with respect to \( I_3 \). By part (b) of Lemma 1 we then have

\[
\omega_1(N_{I_3}(X, Y), Z) = d\omega_1(I_3 X, I_3 Y, Z) - d\omega_1(X, Y, Z) + d\omega_2(I_3 X, Y, Z) + d\omega_2(X, I_3 Y, Z). \tag{18}
\]

As \( \omega_1 \) and \( \omega_2 \) are by assumption closed, the right-hand side of this equation vanishes, and since moreover \( \omega_1 \) is non-singular it follows that the Nijenhuis tensor of \( I_3 \) vanishes. By the Newlander-Nirenberg theorem \( I_3 \) is then an integrable complex structure.

Second, as \( \omega_3(X, I_3 Y) = -g(X, Y) \) is symmetric at the exchange of \( X \) and \( Y \), \( \omega_3 \) is of type \((1,1)\) with respect to \( I_3 \). By part (a) of Lemma 1 we can then write

\[
2\nabla_Z \omega_3(X, Y) = \omega_3(N_{I_3}(X, Y), Z) - d\omega_3(I_3 X, I_3 Y, Z) + d\omega_3(X, Y, Z). \tag{19}
\]

The above proven vanishing of the Nijenhuis tensor for \( I_3 \) together with the closure of \( \omega_3 \) imply then that \( \omega_3 \) is covariantly constant with respect to the Levi-Civita connection. The covariant constancy of \( \omega_2 \) and \( \omega_3 \), as well as the integrability of \( I_2 \) and \( I_3 \), follow by cyclically permuting the indices in the argument. \( \square \)

**Remark.** The triplets of 2-forms defining a given hyperkähler metric are not unique. The conditions of the theorem are invariant under the following two types of linear transformations:

1) \( \omega_i \mapsto \lambda \omega_i \) for any non-vanishing constant \( \lambda \in \mathbb{R}^\times \). The corresponding hyperkähler metric rescales from \( g \) to \( \lambda g \).

2) \( \omega_i \mapsto R_{ij} \omega_j \) for any constant rotation matrix \( R_{ij} \in SO(3) \). These transformations leave the metric unchanged. Thus, for any triplet of 2-forms \( \omega_i \) satisfying the conditions of the theorem one automatically has a half 3-sphere’s worth of such triplets giving the same hyperkähler metric.

The scaling invariance is straightforward to see, as is the invariance of the differential condition at constant linear rotations. To prove that the algebraic condition is also preserved by rotations notice that the relations \((14)\) and \((17)\) can be equivalently written together in the form \( \omega_i I_j = \varepsilon_{ijk} \omega_k - \delta_{ij} g \), with the first set of relations giving the off-diagonal terms and the second set the diagonal ones. From this, and using the fact that \( SO(3) \) transformations preserve both the \( \varepsilon \)-symbol and the \( \delta \)-symbol, one then has \((R_{im} \omega_m)(R_{jn} I_n) = R_{im} R_{jn} \varepsilon_{mnl} \omega_l - R_{im} R_{jn} \delta_{mn} g = \varepsilon_{ijk} (R_{kl} \omega_l) - \delta_{ij} g \), which is to say, the rotated quantities \( \omega'_i = R_{ij} \omega_j \) and \( I'_j = R_{ij} I_j \) satisfy the same set of relations \((14)\) and \((17)\), and with the same metric, as the original non-rotated ones. The invariance of the algebraic condition follows immediately.

On the quaternionic Kähler side, on the other hand, we have the following version of Proposition 3 from reference [2]:

**Theorem 3.** A finite-dimensional manifold \( M \) is (possibly pseudo-) quaternionic Kähler if and only if it possesses a rank-3 \( SO(3) \)-subbundle \( Q \) of the bundle of non-singular 2-forms on \( M \) and an open covering \( C \) of \( M \) on each element \( U \) of which one can pick a local frame \( \omega_1, \omega_2, \omega_3 \) for \( Q|_U \) satisfying simultaneously
• an algebraic condition: if we define

\[ I_1 = \omega_3^{-1}\omega_2, \quad I_2 = \omega_1^{-1}\omega_3, \quad I_3 = \omega_2^{-1}\omega_1 \in \text{End}(TM) \]

then we have \( I_1^2 = I_2^2 = I_3^2 = -1 \).

• a differential condition: there exists a locally defined triplet of 1-forms \( \theta_1, \theta_2, \theta_3 \) such that

\[ d\omega_i = -2\varepsilon_{ijk} \theta_j \wedge \omega_k. \]

\[ \text{Proof.} \] The structure of the proof mirrors closely that of the previous theorem. By virtually the same arguments the algebraic condition implies that the almost complex structures \( I_1, I_2, I_3 \) satisfy the imaginary quaternionic algebra, and that, moreover, a tri-Hermitian metric \( g \) can be defined by means of the same equation (17).

The essential difference with respect to the hyperkähler case is that these quantities are now defined \textit{locally}, and we need to check in addition that they give rise to the requisite \textit{global} structures. For that, let us consider two arbitrary overlapping open sets \( U, V \in C \).

The two corresponding local frames \( \omega_i|_U \) and \( \omega_i|_V \) of the bundle \( Q \) are related on the intersection \( U \cap V \) by means of a transition function \( R_{ij} \) which is a local \( SO(3) \) rotation:

\[ \omega_i|_V = R_{ij} \omega_j|_U. \]

Adjoining to the symbol of each locally defined quantity its domain of definition we have then by essentially the same argument as in the Remark following the previous theorem

\[ (R_{jm} \omega_m|_U)(R_{jn} \omega_n|_U) = \varepsilon_{ijk}(R_{kl} \omega_l|_U) - \delta_{ij} g_{UV}. \]

By considering the off-diagonal terms in this equation we can see immediately that we must have

\[ I_{iV} = R_{ij} I_{jU} \]

that is, the almost complex structures are local sections — and in fact, frames — of a rank-3 \( SO(3) \)-subbundle of \( \text{End}(TM) \).

On the other hand, by considering the diagonal terms we get that

\[ g_{iV} = g_{iU} \]

that is, the metric is globally defined.

To prove that the manifold \( M \) carries a quaternionic Kähler structure it suffices to show that the action of the Levi-Civita covariant derivative with respect to the metric \( g \) on the 2-forms \( \omega_i \) is of the form (5). The corresponding covariant differentiation property (1) of the almost complex structures \( I_i \) follows then immediately.

By the same reasoning as in the hyperkähler case, \( \omega_1 \) and \( \omega_3 \) are of mixed \( (2,0) + (0,2) \) respectively pure \( (1,1) \) type with respect to \( I_3 \). So then by parts (a) and (b) of Lemma 1 the same identities (19) and (18) hold now, too. From the second one, by inserting the expressions for \( d\omega_1 \) and \( d\omega_2 \) given by the current differential condition we retrieve, after a purely algebraic computation, the following form for the Nijenhuis tensor of \( I_3 \):

\[ N_{I_3}(X,Y) = 2(\theta_1(X) - \theta_2(I_3X))I_1Y + 2(\theta_1(I_3X) + \theta_2(X))I_2Y - 2(\theta_1(Y) - \theta_2(I_3Y))I_1X - 2(\theta_1(I_3Y) + \theta_2(Y))I_2X. \]

Plugging this into the first identity together with the expression for \( d\omega_3 \) gives us after another series of algebraic manipulations that

\[ \nabla_Z \omega_3(X,Y) = -2\theta_1(Z)\omega_2(X,Y) + 2\theta_2(Z)\omega_1(X,Y). \]
Permuting the indices cyclically in this argument yields the rest of the components of the desired property [5]. □

**Remark.** Quaternionic Kähler manifolds are particular examples of quaternionic manifolds (for a detailed review see e.g. [5]). These are manifolds $M$ which carry a three-dimensional subbundle of $\text{End}(TM)$ with fibers spanned at each point by a basis $\{I_1, I_2, I_3\}$ whose elements satisfy the algebra of imaginary quaternions, and which, in addition, admit a torsion-free connection preserving this structure. If it exists, such a connection, called an Oproiu connection, is not unique. Their so-called structure tensor takes the form

$$\sum_{i=1}^{3} N_{I_i}(X,Y) = \sum_{i=1}^{3} (\theta_i\text{Op}(X)I_iY - \theta_i\text{Op}(Y)I_iX)$$

where the triplet of 1-forms $\theta_i\text{Op}$ are the Oproiu connection 1-forms.

From the above formula for the Nijenius tensor for $I_3$ and its cyclic permutations one can verify that in the quaternionic Kähler case the structure tensor is indeed of this form, with

$$\theta_i\text{Op} = \frac{2}{3} \theta_i - \frac{1}{3} \varepsilon_{ijk} \theta_j I_k.$$  

(Here we use the convention that the $I_i$ act on vector fields from the left and dually, on 1-forms, from the right.) Note that these can be rewritten as

$$\theta_i\text{Op} = \theta_i + \xi I_i \quad \text{with} \quad \xi = \frac{1}{3}(\theta_1 I_1 + \theta_2 I_2 + \theta_3 I_3)$$

which essentially means (see e.g. [8]) that in the quaternionic Kähler case the Oproiu connection is equivalent to the Levi-Civita one.

The following alternative version of the criterion enunciated in Theorem 3 also holds:

**Theorem 4.** The differential condition of Theorem 3 can be replaced with the following condition:

- The principal $SO(3)$-bundle $Q$ associated to the bundle $Q$ admits a connection with curvature 2-form equal, up to a non-vanishing constant, to $\omega_i$. That is, there exist locally defined connection 1-forms $\theta_i$ and a constant $s \neq 0$ such that

$$d\theta_i + \varepsilon_{ijk} \theta_j \wedge \theta_k = s \omega_i.$$ 

**Proof.** The fact that this condition implies the second condition of Theorem 3 is straightforward: the latter is a Bianchi-type consistency condition for the former. The converse implication follows, for $n > 1$, from a result first shown by Alekseevsky, see e.g. part (iii) of Theorem 5.7 in [5]. For $n = 1$, the condition above is equivalent to the Einstein equation. In either case, one can identify

$$s = \frac{R_g}{8n(n+2)}$$

where $R_g$ is the scalar curvature of the metric $g$. ($s$ is called reduced scalar curvature, and the extra $1/2$ factor with respect to the definition in [5] is a matter of convention stemming from the difference in the normalizations of the 1-forms $\theta_i$.) □
1.2. Killing vector fields. Let us consider now hyperkähler and quaternionic Kähler manifolds with continuous isometries — i.e. Killing vector fields, and see what the metric preservation condition corresponds to in the approach in which a triplet of 2-forms rather than the metric plays the fundamental role. We begin our discussion by recalling some background material leading to a general theorem due to Kostant concerning Killing vector fields on Riemannian manifolds with special holonomy.

1.2.1. Let $M$ be a Riemannian manifold with metric $g$ (similar considerations as the ones which follow hold, with some adjustments, for pseudo-Riemannian manifolds as well). By definition, we call $\Omega \in \text{End}(TM)$ a skew-symmetric endomorphism if

$$g(X,\Omega Y) = -g(Y,\Omega X) \overset{\text{def}}{=} \omega(X,Y)$$

for any pair $X,Y \in TM$. Skew-symmetric endomorphisms from $\text{End}(TM)$ are in one-to-one correspondence with 2-forms on $M$: given a skew-symmetric endomorphism $\Omega$, we can define a corresponding 2-form $\omega$ as above; conversely, for any $\omega \in \Lambda^2 T^* M$ we can define a skew-symmetric endomorphism $\Omega = g^{-1} \omega$, where we now view the 2-form as an element of $\text{Hom}(TM,T^* M)$, and the inverse metric as an element of $\text{Hom}(T^* M,TM)$. On the space of skew-symmetric endomorphisms of the tangent bundle one can define a natural inner product by

$$\langle \Omega, \Omega' \rangle = -\frac{1}{\dim M} \text{trace}(\Omega \Omega').$$

This is in general non-degenerate and, in Riemannian signature, also positive definite. Another property we will need is that it is preserved under parallel transport. A choice of orthonormal basis of the tangent space $T_p M$ at a point $p \in M$ gives a vector space isometry $T_p M \rightarrow \mathbb{R}^{\dim M}$, with the latter space endowed with the standard flat metric. This determines an isomorphism between the set of skew-symmetric endomorphisms of $T_p M$ and $\mathfrak{so}(\dim M)$, the Lie algebra of skew-symmetric endomorphisms of $\mathbb{R}^{\dim M}$.

1.2.2. For any vector field $X \in TM$ let us define the operator

$$A_X = \mathcal{L}_X - \nabla_X$$

measuring the difference between the Lie derivative and the Levi-Civita covariant derivative along $X$. Also, for any two vector fields $X,Y \in TM$, let

$$R_{X,Y} = [\nabla_X,\nabla_Y] - \nabla_{[X,Y]}$$

be the usual Riemannian curvature operator of the Levi-Civita connection.

As derivations acting linearly on any tensor field on $M$ and vanishing on functions, both of these operators are linear algebraic rather than differential in nature and are represented by tensors. The tensors are in either case endomorphisms of the tangent bundle $TM$. For $A_X$, this is $\nabla X$, the covariant derivative of $X$, which one may view as the endomorphism of $TM$ defined by $(\nabla X)Y = \nabla_Y X$ for any $Y \in TM$. For $R_{X,Y}$, it is the Riemann curvature endomorphism $R(X,Y)$. So, for example, we have

| $T$ | $A_X T$ | $R_{X,Y} T$ |
|-----|--------|-------------|
| scalar field on $M$ | $A_X T = 0$ | $R_{X,Y} T = 0$ |
| vector field on $M$ | $A_X T = -(\nabla X)T$ | $R_{X,Y} T = R(X,Y)T$ |
| element of $\text{End}(TM)$ | $A_X T = -[\nabla X,T]$ | $R_{X,Y} T = [R(X,Y),T]$ |
where the square brackets denote commutators. The formulas for $A_X$ are derived from the usual formulas for the action of the Lie derivative on tensors by replacing all ordinary partial derivatives with covariant ones, which is possible due to the fact that the Levi-Civita connection is torsion-free. The other defining property of the Levi-Civita connection, namely that it preserves the metric, implies that the Riemann curvature endomorphism $R(X, Y)$ of the Levi-Civita connection is skew-symmetric for any choice of vector fields $X, Y \in TM$. By contrast, the endomorphism $\nabla X$ is skew-symmetric if and only if $X$ is a Killing vector field. This well-known fact follows from the identity

$$ (L_X g)(Y, Z) = g(Z, (\nabla X)Y) + g(Y, (\nabla X)Z) $$

valid for any $X, Y, Z \in TM$ and making use of both defining properties of the Levi-Civita connection.

The following properties hold:

**Lemma 5.** Let $M$ be a (possibly pseudo-)Riemannian manifold.

1) If $X$ is a Killing vector field on $M$ then

$$ [\nabla_Y, A_X] = R_{X,Y} $$

for any vector field $Y \in TM$.

2) If both $X$ and $Y$ are Killing vector fields on $M$ then we have

$$ R_{X,Y} = [A_X, A_Y] - A_{[X,Y]} $$

**Proof.** Both of these identities follow immediately from the definitions of the operators involved upon using the property that the Lie derivative with respect to a Killing vector field commutes with the Levi-Civita connection. Explicitly, if $X$ is a Killing vector field and $Y$ an arbitrary one, then recalling the elementary fact that $L_X Y = [X, Y]$, this property may be expressed in the form

$$ [\mathcal{L}_X, \nabla_Y] = \nabla_{[X,Y]} $$

Note that in terms of the associated skew-symmetric endomorphisms the operatorial identity (36) is equivalently written as

$$ \nabla_Y (\nabla X) = -R(X, Y). $$

1.2.3. Let us assume now that the manifold $M$ is connected. The **holonomy group** at a point $p \in M$ is by definition the set of linear endomorphisms of the tangent space $T_p M$ obtained by parallel transporting tangent vectors with respect to the Levi-Civita connection around piecewise smooth loops based at $p$. This set has naturally the structure of a group. The **restricted holonomy group** at $p$ is the subgroup coming from contractible loops. The latter is a connected Lie subgroup of the group of orthogonal transformations of $T_p M$. By choosing an orthonormal basis for $T_p M$ we may identify it with a Lie subgroup of $SO(\dim M)$. This subgroup depends on both the choice of basis and of base point $p$, but its conjugacy class depends on neither. The possible holonomy groups of Riemannian manifolds have been famously classified by Berger.

The relation between the holonomy group and the continuous part of the isometry group of a Riemannian manifold is described by the following classic result:
Theorem 6 (Kostant [34]). Let \( M \) be a Riemannian manifold and \( \mathfrak{h} \) be the Lie algebra of the restricted holonomy group at a point \( p \in M \). If \( X \) is a Killing vector field on \( M \), then the skew-symmetric endomorphism of the tangent space \( T_p M \) determined by \( \nabla X \) belongs to the normalizer of \( \mathfrak{h} \) in \( \mathfrak{so}(\dim M) \). Moreover, if \( M \) is orientable and compact then it belongs simply to \( \mathfrak{h} \).

Proof. As a skew-symmetric endomorphism of the tangent bundle, \( \nabla X \) can be decomposed uniquely into two orthogonal components with respect to the inner product (31), one belonging to \( \mathfrak{h} \) and the other to its orthogonal complement in \( \mathfrak{so}(\dim M) \):

\[
\nabla X = (\nabla X)^{\mathfrak{h}} + (\nabla X)^{\mathfrak{h}^\perp}.
\]

Suppose then we act with \( \nabla_Y \) for some vector field \( Y \) simultaneously on both sides of this equation. One the left-hand side we obtain, by way of the property (39), \(-R(X,Y)\), which, by the Ambrose-Singer theorem, belongs to \( \mathfrak{h} \). On the other hand, since the inner product is preserved by parallel transport, one can argue that \( \nabla_Y(\nabla X)^{\mathfrak{h}} \in \mathfrak{h} \) and \( \nabla_Y(\nabla X)^{\mathfrak{h}^\perp} \in \mathfrak{h}^\perp \) for all \( Y \). The inner product being non-degenerate, the intersection \( \mathfrak{h} \cap \mathfrak{h}^\perp \) is trivial, and so it follows that one must have

\[
\nabla_Y(\nabla X)^{\mathfrak{h}^\perp} = 0.
\]

This means in particular that \((\nabla X)^{\mathfrak{h}^\perp}\) is invariant at parallel transport around a closed loop and thus belongs to the centralizer of \( \mathfrak{h} \) in \( \mathfrak{so}(\dim M) \). The first part of the theorem follows then immediately.

For further details as well as the proof of the fact that when \( M \) is orientable and compact \((\nabla X)^{\mathfrak{h}^\perp}\) vanishes we direct the interested reader to the original paper [34], or the review in ch. VI of [32]. \(\square\)

1.2.4. Let us return now to the case of quaternionic Kähler and hyperkähler manifolds, for which \( \dim M = 4n \).

An explicit embedding of \( Sp(n) \times_{\mathbb{Z}_2} Sp(1) \) as a Lie subgroup of \( SO(4n) \) may be given as follows: regard \( \mathbb{R}^{4n} \) as \( \mathbb{H}^n \) and consider the linear endomorphism on the latter given by the simultaneous left and right actions

\[
x_{I} \longmapsto A_{IJ}x_{J}q^{-1}
\]

for any \((x_{I})_{I=1,...,n} \in \mathbb{H}^n\), quaternionic unitary matrix \((A_{IJ})_{I,J=1,...,n}\), and unit quaternion \(q\); the summation convention over repeated indices is understood. This preserves the quaternionic norm on \( \mathbb{H}^n \), and so the corresponding Euclidean norm on \( \mathbb{R}^{4n} \), which thus exhibits it as an element of \( O(4n) \) — and in fact of \( SO(4n) \). The \( \mathbb{Z}_2 \)-quotient comes from the fact that this action is invariant under a simultaneous change of sign of both \( A_{IJ} \) and \( q \). Note also that for \( n = 1 \) we have \( Sp(1) \times_{\mathbb{Z}_2} Sp(1) \cong SO(4) \), whereas for \( n > 1 \), \( Sp(n) \times_{\mathbb{Z}_2} Sp(1) \) is a maximal subgroup of \( SO(4n) \) (see [26]).

The first important observation is the following:

Lemma 7. Let \( M \) be a quaternionic Kähler or hyperkähler manifold. Then any skew-symmetric endomorphism \( \Omega \) of \( TM \) with respect to the tri-Hermitian metric on \( M \) splits uniquely into three mutually orthogonal components with respect to the inner product \( \langle \cdot, \cdot \rangle \), with two of them belonging to the \( \mathfrak{sp}(n) \) and \( \mathfrak{sp}(1) \) Lie subalgebras of \( \mathfrak{so}(4n) \), respectively, and the third one to their direct sum’s orthogonal complement in \( \mathfrak{so}(4n) \):

\[
\Omega = \Omega^{\mathfrak{sp}(n)} + \Omega^{\mathfrak{sp}(1)} + \Omega^{\mathfrak{sp}(1)}.
\]
Moreover, the \( \mathfrak{sp}(1) \) component is a linear superposition of \( I_1, I_2, I_3 \) (which, note, are skew-symmetric), and we have \([I_j, \Omega^{\mathfrak{sp}(n)}] = 0\).

Proof. One way to see this is within the so-called \( E-H \) formalism \[42\]. Let us recall its main features briefly. For quaternionic Kähler manifolds one can associate in principle a locally defined vector bundle to each representation of the holonomy group \( \mathfrak{sp}(n) \times \mathbb{Z}_2 \mathfrak{sp}(1) \). In particular, the complexified cotangent bundle \( T^*_\mathbb{C}M \) splits locally into a tensor product \( E \otimes H \) of two such vector bundles corresponding to the standard complex representations of rank \( 2n \) and \( 2 \) of \( \mathfrak{sp}(n) \) and \( \mathfrak{sp}(1) \), respectively. For \( n = 1 \), in four dimensions, where the holonomy definition is side-stepped, one takes instead \( E \) and \( H \) to be the two spinor bundles of \( M \). The bundles \( E \) and \( H \) come equipped with natural symplectic forms — constant anti-symmetric sections \( \varepsilon_E \in \Lambda^2 E \) and \( \varepsilon_H \in \Lambda^2 H \). In the hyperkähler manifold case, which in this context is convenient to view as a quaternionic Kähler limit case, the same split structure exists, except that the bundle \( H \) is now trivial.

Any \( \text{SO}(4n) \)-module decomposes into direct sums of irreducible \( \mathfrak{sp}(n) \times \mathbb{Z}_2 \mathfrak{sp}(1) \)-modules, which are tensor products of even total number of \( \mathfrak{sp}(n) \) and \( \mathfrak{sp}(1) \)-modules. In particular, this is the case for the bundle \( \Lambda^2 T^*M \). Any 2-form \( \omega \) thus decomposes as
\[
\omega = \omega^{\mathfrak{sp}(n)} + \omega^{\mathfrak{sp}(1)} + \omega^{\mathfrak{sp}(1)}
\]
where, in a local \( E-H \) frame we have
\[
\omega^{\mathfrak{sp}(n)} = \omega_E \otimes \varepsilon_H, \text{ with } \omega_E \in S^2 E \cong \mathfrak{sp}(n)
\]
\[
\omega^{\mathfrak{sp}(1)} = \varepsilon_E \otimes \omega_H, \text{ with } \omega_H \in S^2 H \cong \mathfrak{sp}(1)
\]
\[
\omega^{\mathfrak{sp}(1)} \in \Lambda^2 E \otimes S^2 H.
\]
Here, \( S \) and \( \Lambda \) denote symmetric respectively anti-symmetric tensor products, and \( \Lambda^2 E \) denotes the kernel of the contraction of \( \Lambda^2 E \) with the inverse of the symplectic form \( \varepsilon_E \) (this kernel is trivial for \( n = 1 \)). Note that the quaternionic Kähler/hyperkähler 2-forms \( \omega_i \) are of pure \( \mathfrak{sp}(1) \) type, and they provide a frame for the respective subbundle of \( \Lambda^2 T^*M \).

On another hand, in the same \( E-H \) frame the metric factorizes locally as
\[
g = \varepsilon_E \otimes \varepsilon_H.
\]
Recalling then that any skew-symmetric endomorphism \( \Omega \) of \( TM \) is related to a corresponding 2-form \( \omega \) by \( \Omega = g^{-1} \omega \), the statements of the lemma can be verified directly by simple algebraic calculations. \( \square \)

1.2.5. The curvature tensor also admits an irreducible \( E-H \) decomposition \[42\] Theorem 3.1]. In what follows, though, we will only need the following algebraic property of the curvature endomorphism:

Lemma 8. The curvature endomorphism of a quaternionic Kähler manifold satisfies the commutation property
\[
[I_i, R(X,Y)] = 2s \varepsilon_{ijk} \omega_j(X,Y) I_k.
\]
For hyperkähler manifolds the right-hand side vanishes, consistent with taking the limit to \( s = 0 \).

In the quaternionic Kähler case this is essentially the consistency condition for the differential equation \( \mathcal{H} \), with the Einstein condition \( \mathcal{E} \) taken into account. In the hyperkähler case it is simply the consistency condition for the corresponding equation \( \mathcal{D} \). In either
case its general structure can be easily understood by purely algebraic arguments from the Ambrose-Singer theorem.

Note in particular that if we multiply this property from either side with an $I_i$ and then take the trace of the resulting equation we obtain, using successively the cyclicity of the trace, the quaternionic algebra and the fact that trace($I_i$) = 0, the following consequence:

\[
\langle I_i, R(X, Y) \rangle = s \omega_i(X, Y).
\]

1.2.6. For the remainder of this section we will assume that the manifolds are irreducible, by which we mean that their holonomy groups are assumed to be equal to rather than included in $Sp(n) \times \mathbb{Z}_2 Sp(1)$ in the quaternionic Kähler case (for $n > 1$), and $Sp(n)$ in the hyperkähler one.

Suppose now the quaternionic Kähler or hyperkähler manifold $M$ possesses a Killing vector field $X$. By Theorem 6, $\nabla X$ then belongs to the normalizer of the Lie subalgebra $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$ respectively $\mathfrak{sp}(n)$ in $\mathfrak{so}(4n)$, which one can easily see is in both cases $\mathfrak{sp}(n) \oplus \mathfrak{sp}(1)$. In the notations of Lemma 7 this means that $\langle \nabla X \rangle^{\text{cpl}} = 0$, and so

\[
\nabla X = (\nabla X)^{\mathfrak{sp}(n)} + (\nabla X)^{\mathfrak{sp}(1)}
\]

with the two components both orthogonal and commuting. In four dimensions this decomposition is automatic for any skew-symmetric endomorphism of $TM$, so one needs not resort in this case to holonomy arguments. For the second component we have

\[
(\nabla X)^{\mathfrak{sp}(1)} = -\sum_{i=1}^{3} \nu_{X_i} I_i
\]

for some coefficients $\nu_{X_i}$, where the minus sign is chosen for later convenience. Noticing that the generators $I_i$ satisfy the orthogonality relation $\langle I_i, I_j \rangle = \delta_{ij}$, we can express these as

\[
\nu_{X_i} = -\langle I_i, \nabla X \rangle.
\]

1.2.7. These coefficients play an important role. In the quaternionic Kähler case, switching to an $\mathbb{R}^3$-vector notation, we have:

**Theorem 9** (Galicki, Lawson [23, 24]). Consider a quaternionic Kähler manifold $M$ with associated principal $SO(3)$-bundle $Q$, and assume that $M$ has a continuous group of isometries $G$, with Lie algebra $\mathfrak{g}$. Then a vector field $X \in \mathfrak{g}$, that is $X$ is Killing, if and only if there exists a local section $\vec{\nu}_X$ of the tensor product bundle $\mathfrak{g}^* \otimes Q$ over $M$ such that

\[
\mathcal{L}_X \vec{\omega} = -2(\iota_X \vec{\theta} + \vec{\nu}_X) \times \vec{\omega}
\]

(\text{where the symbol } \times \text{ denotes the usual } \mathbb{R}^3\text{-vector cross product}) or, equivalently, such that

\[
d\vec{\nu}_X + 2\vec{\theta} \times \vec{\nu}_X = s \iota_X \vec{\omega}.
\]

Explicitly, this section is given by the formula $\vec{\nu}_X = -\langle \vec{I}, \nabla X \rangle$ for any $X \in \mathfrak{g}$ and it is

- nowhere vanishing
- the unique solution to the quaternionic-Kähler moment map equation
- $G$-equivariant
- and, for any $X, Y \in \mathfrak{g}$, we have

\[
2\vec{\nu}_{\lbrack X, Y \rbrack} = 2\vec{\nu}_X \times 2\vec{\nu}_Y - 2s \vec{\omega}(X, Y).
\]
Proof. Suppose $X$ is a Killing vector field on $M$. Defining $\nu_{X_i} = -\langle I_i, \nabla X \rangle$, we have by the rules laid out in the table (34)

$$A_X I_i = -[\nabla X, I_i] = -[(\nabla X)^{\mathfrak{sp}(1)}, I_i] = -2\varepsilon_{ijk} \nu_{X_j} I_k.$$  

The $\mathfrak{sp}(n)$ component of $\nabla X$ drops out because it commutes with the $\mathfrak{sp}(1)$ generators, and for the remaining $\mathfrak{sp}(1)$ component we use the representation (50). Converting then the almost complex structures into 2-forms by means of the metric, which is preserved by the action of $A_X$, gives promptly the rotation property (52).

On another hand, if we act with a covariant derivative $\nabla Y$ for some arbitrary $Y \in TM$ on the definition of $\nu_{X_i}$ we obtain, by applying the Leibnitz rule and then the properties (1), (39) and (48), that

$$\nabla_Y \nu_{X_i} = -2\varepsilon_{ijk} \theta_j(Y) \nu_{X_k} - s \omega_i(Y, X).$$

The quaternionic Kähler moment map equation (53) is simply a rewriting of this equation in the language of differential forms.

Conversely, let us assume that we have a vector field $X$ whose Lie action rotates the quaternionic Kähler 2-forms, that is $L_X \omega_i = -2\varepsilon_{ijk} R_j \omega_k$ for some locally defined triplet of functions $R_j$. Using then the expression $g = \omega_1 \omega_2^{-1} \omega_3$ for the metric, or in fact any one of the expressions (16), one can easily check that this implies that $X$ is Killing, regardless of the particular form of the functions $R_j$.

For the second converse we assume instead that there exists a vector field $X$ satisfying the quaternionic Kähler moment map equation for some locally defined functions $\nu_{X_i}$. Then by acting on this equation with an exterior differential and making use of the Einstein property (28) one can show that the rotation property (52) is implied. By the argument above, this implies further that $X$ must be Killing.

The non-vanishing of the section $\nu_{X_i}$ follows directly from the quaternionic Kähler moment map equation. Should it vanish at some point, the equation would imply that $X$ would be a null vector for the 2-forms $\omega_i$ at that point, which would contradict their non-singularity.

For the proof that the solution to the quaternionic Kähler moment map equation is unique we refer the reader to the reference [24].

Remark. Note that, unlike the rest of the arguments in the proof, each of the arguments in the last three paragraphs fails in the limit when $s = 0$ and thus cannot be extended to the hyperkähler case.

Assume now that we have two Killing vector fields $X, Y \in \mathfrak{g}$. Using the representation (37) for the curvature operator we can write

$$A_{[X,Y]} I_i = [A_X, A_Y] I_i - [R(X,Y), I_i].$$

The two terms on the right-hand side can be further transformed using the formulas (55) and (17), respectively, to yield

$$A_{[X,Y]} I_i = -2\varepsilon_{ijk} \nu_{[X,Y]} I_k$$

with $\nu_{[X,Y]} I_k$ defined as in the theorem.

Finally, since $L_Y$ acts on scalars as $\iota_Y d$, from the quaternionic Kähler moment map equation it follows that

$$L_Y \nu_{X_i} = \nu_{[X,Y]} I_i - 2\varepsilon_{ijk} (\iota_Y \theta_j + \nu_{Y j}) \nu_{X_k}$$

showing that the sections $\nu_{X_i}$ transform indeed equivariantly under the infinitesimal action of $G$. □
1.2.8. In the hyperkähler limit the coefficients $\vec{\nu}_X$ lose the moment map interpretation. More precisely, we have:

**Theorem 10.** Let $M$ be a hyperkähler manifold having a continuous group of isometries $G$, with Lie algebra $\mathfrak{g}$. Then a vector field $X \in \mathfrak{g}$, that is $X$ is Killing, if and only if there exists a constant map $\nu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ such that

\[
\mathcal{L}_X \bar{\omega} = -2 \vec{\nu}_X \times \bar{\omega}.
\]

Explicitly, the map is given by the formula $\vec{\nu}_X = -\langle \vec{I}, \nabla X \rangle$ and, for any $X, Y \in \mathfrak{g}$, we have

\[
2 \vec{\nu}_{[X,Y]} = 2 \vec{\nu}_X \times 2 \vec{\nu}_Y.
\]

Moreover, if $M$ is compact, the map $\nu$ vanishes entirely.

**Remark.** Killing vector fields $X$ for which $\vec{\nu}_X$ vanishes are known as tri-Hamiltonian (or tri-holomorphic), and the ones for which this is not the case as rotational (or permuting). (Note that the two notions do not depend on the particular frame chosen in the space of hyperkähler symplectic 2-forms.) So the theorem states that Killing vector fields on hyperkähler manifolds can only be either of tri-Hamiltonian or of rotational type. This generalizes a result proved by other means in four dimensions by Boyer and Finley [10]. In particular, when the manifold is compact, Killing vector fields, should they exist, are necessarily tri-Hamiltonian.

**Proof.** The proof in the quaternionic Kähler case carries over in the limit when $s = 0$ and $\theta = 0$, apart from the three instances identified in the intermediary Remark. In particular, the equation (59) now reads simply

\[
d \vec{\nu}_X = 0.
\]

Clearly then, the functions $\vec{\nu}_X$ are constant. Note that in the hyperkähler case, rather than being local sections, they are defined globally. This equation has no longer the form of a moment map equation, nor does it imply anymore the rotation property. With less dramatic changes, the formulas (52) and (54) take in this limit the forms (59) and (60), respectively. Moreover, the compact case vanishing property follows from the last statement of Theorem 6, which implies that $(\nabla X)^{\Phi(1)} = 0$. □

1.3. **Hyperkähler cones.** In the final installment of this section we examine hyperkähler spaces possessing a conformal vector field of homothetic type. The profound connection discovered by Swann [43] between such spaces and quaternionic Kähler manifolds will be discussed in detail in the next section.

**Definition.** A vector field $X$ on a Riemannian or pseudo-Riemannian space with Levi-Civita connection $\nabla$ is called homothetic if

\[
\nabla X = 1
\]

where the unit symbol represents the identity endomorphism of the tangent bundle.

**Proposition 11.** Let $M$ be a hyperkähler space. The following conditions are equivalent:

a) $M$ has a homothetic vector field $X$.

b) There exists a function $\kappa$ such that

\[
g = \nabla^2 \kappa.
\]
c) There exists a function $\kappa$ which is simultaneously a Kähler potential for every hyperkähler symplectic 2-form with respect to its corresponding complex structure, i.e.

$$\omega_i = \frac{1}{2} d(d\kappa I_i).$$

(d) There exist four vector fields $X_0, X_1, X_2, X_3 \in TM$ and a function $\kappa$ such that

$$\iota_{X_i} \omega_j + \varepsilon_{ijk} \iota_{X_0} \omega_k = -\delta_{ij} d\kappa \quad \text{and} \quad \mathcal{L}_{X_0} \omega_k = -2\omega_k.$$

The last condition entails that the four vector fields generate an $\mathbb{R} \oplus \mathfrak{sp}(1)$ algebra, with $-X_0$ homothetic and $X_1, X_2, X_3$ Killing vector fields of rotating type.

Remark 1. A manifold $M$ with this structure is called a hyperkähler cone. Due to the property (c) the function $\kappa$, defined up to a constant additive shift, is called a hyperkähler potential. If $\kappa > 0$ then we will refer to $M$ or its restriction to the subset on which this condition holds as a positive hyperkähler cone. If $\kappa < 0$ then one can in principle redefine $\omega_i \mapsto -\omega_i$, which entails a change in the sign both of the hyperkähler metric and of the hyperkähler potential, making the latter again positive.

Remark 2. The characterization (a) of hyperkähler cones is due to de Wit, Kleijn and Vandoren [17] (see also [18]), and the characterizations (b) and (c) are due to Swann [43]. The characterization (d) is the closest in spirit to our current approach as it involves exterior algebra conditions on the hyperkähler symplectic 2-forms.

Proof. The maps between the various structures can be summarized as follows:

\[
\begin{align*}
\iota_{X_1} \omega_1 &= \iota_{X_2} \omega_2 = \iota_{X_3} \omega_3 = \iota_{X_0} g = -d\kappa \quad (c) \quad \iff \quad X = -X_0 \quad (d) \quad X_0 = -X, X_i = -I_i X \\
\kappa &= \frac{1}{2} g(X, X) \quad (a) \quad \iff \quad \iota_X g = d\kappa \quad (b)
\end{align*}
\]

$(a) \Rightarrow (b)$ Let, by definition,

$$\kappa = \frac{1}{2} g(X, X).$$

Then based on the metric-preserving property of the Levi-Civita connection and the homothetic property of the vector field $X$ one obtains for the first and second covariant derivatives of this function the expressions

\[
\begin{align*}
\nabla_Z \kappa &= g(Z, X) \\
\nabla^2_{Y, Z} \kappa &= g(Y, Z)
\end{align*}
\]

valid for any vector fields $Y, Z \in TM$.

$(b) \Rightarrow (c)$ From the relation (3) between the hyperkähler symplectic forms and complex structures, the assumption (b), and then the covariant constancy of the $I_i$ we obtain successively

$$\omega_i(Y, Z) = g(Y, I_i Z) = \nabla^2_{Y, I_i} \kappa = \nabla_Y (\nabla \kappa I_i)(Z)$$

for any $Y, Z \in TM$. We reiterate the convention that $I_i$ acts on vector fields from the left and, dually, on 1-forms from the right. The implication follows directly from this formula based on the fact that the Levi-Civita connection is torsion-free.
Given the function \( \kappa \) on \( M \), define the vector fields \( X_0, X_1, X_2, X_3 \) as the gradients of \(-\kappa\) relative to the metric respectively the three hyperkähler symplectic forms:

\[
\iota_{X_0} g = \iota_{X_1} \omega_1 = \iota_{X_2} \omega_2 = \iota_{X_3} \omega_3 = -d\kappa.
\]

Then by way of the relation (65) we have \( \iota_{X_0} \omega_i = \iota_{X_0} g I_i = -d\kappa I_i \). Moreover, using the second expression for \( I_3 \) in (14) we get that \( \iota_{X_1} \omega_2 = \iota_{X_1} \omega_1^{-1} \omega_2 = d\kappa I_3 = -\iota_{X_0} \omega_3 \). By cyclically permuting the indices in this last argument we obtain a set of formulas which together with the definitions of the vector fields can be assembled in the form of the first condition (63). The second condition (65), on the other hand, is shown to hold as follows:

\[
\mathcal{L}_{X_0} \omega_k = (d\iota_{X_0} + \iota_{X_0} d) \omega_k = d\iota_{X_0} \omega_k = -d(d\kappa I_k) = -2\omega_k.
\]

Consider the following off-diagonal component of the first condition (65):

\[
\iota_{X_1} \omega_2 + \iota_{X_0} \omega_3 = 0.
\]

From this it follows that \( X_1 = -\iota_{X_0} \omega_3 \omega_2^{-1} = -\omega_2^{-1} \omega_3 X_0 = I_1 X_0 \), where in the last step we have made use of the second expression for \( I_1 \) in (14). Cyclically permuting the indices in this argument gives us

\[
X_i = I_i X_0.
\]

Substituting then this back into the original equation and using the symmetry properties of the metric and hyperkähler symplectic forms as well as the quaternionic algebra, we get now from the diagonal components that

\[
\iota_{X_0} g = -d\kappa.
\]

Let, by definition, \( X = -X_0 \). This last condition then becomes \( \iota_X g = d\kappa \). By acting with a covariant derivative we get that \( g(Y, (\nabla X)Z) = \nabla^2_{Y,Z} \kappa \) for any \( Y, Z \in TM \), which, notice, is symmetric at the exchange of these two vector fields, i.e.

\[
g(Y, (\nabla X)Z) = g(Z, (\nabla X)Y).
\]

On another hand, the second condition (65) implies by way of any one of the formulas (14) for the metric that

\[
\mathcal{L}_{X_0} g = -2g
\]

or, equivalently, \( \mathcal{L}_X g = 2g \). Via the identity (35) this translates into the condition

\[
g(Y, (\nabla X)Z) + g(Z, (\nabla X)Y) = 2g(Y, Z)
\]

which by the above symmetry property reduces to \( g(Y, (\nabla X)Z) = g(Y, Z) \). This is clearly equivalent to the homothetic condition (52) for the vector field \( X \), and thus the implication is proved.

For the remaining assertion of the Proposition observe that acting with an exterior derivative on the first condition (65) and using the closure of the \( \omega_i \) to restore Lie derivatives by way of the Cartan formula for these we get immediately that

\[
\mathcal{L}_{X_0} \omega_j = 2\varepsilon_{ijk} \omega_k.
\]

Thus, the vector fields \( X_1, X_2, X_3 \) generate rotational actions on \( M \) and are therefore Killing. Resorting to the operatorial identity \( \iota_{[X,Y]} = \iota_X \mathcal{L}_Y - \iota_Y \mathcal{L}_X + [d, \iota_X \iota_Y] \) we can show moreover that

\[
\iota_{[X_i, X_j]} \omega_l = 2\varepsilon_{ijk} \iota_{X_k} \omega_l.
\]

Since \( \omega_i \) is invertible, this is equivalent to

\[
[X_i, X_j] = 2\varepsilon_{ijk} X_k.
\]
In a similar way one can also prove that

$$[X_i, X_0] = 0. \square$$

2. Swann bundles

In [43] Swann introduced a powerful tool for the study of quaternionic Kähler manifolds by showing that one can construct over each such manifold a quaternionic bundle whose total space carries a hyperkähler cone structure encoding the quaternionic Kähler geometry of the base. The remarkable virtue of this approach is that it allows one to describe the geometry of manifolds which in general have no integrable complex structures (apart perhaps from accidental ones inessential to their quaternionic Kähler structure) in terms of the hyperkähler geometry of a space of four real dimensions higher possessing a wealth of integrable complex structures. Before we proceed to review the details of Swann’s construction it is useful to recall a few facts about the multiplicative group of non-zero quaternions, $\mathbb{H}^\times$.

2.1. Structures on the group $\mathbb{H}^\times$.

2.1.1. Consider an embedding of the group of unitary quaternions $Sp(1)$ into $\mathbb{H}^\times$. Clearly, every non-zero quaternion $q \in \mathbb{H}^\times$ can be uniquely represented as a unit quaternion times its absolute value $|q|$. This gives a natural isomorphism

$$\mathbb{H}^\times \cong \mathbb{R}_{>0} \otimes Sp(1).$$

Thus, $\mathbb{H}^\times$ has the structure of a Lie group, with Lie algebra generated by the standard quaternionic basis. In what follows we will denote the quaternionic generators 1, i, j, k uniformly with $u_0, u_1, u_2, u_3$, respectively—or simply $u_a$. The indices $a, b, \ldots$ will be assumed to run from 0 to 3 while the indices $i, j, k, \ldots$, as until now, to run from 1 to 3, and the summation convention over repeated indices will continue to be implied.

The Lie algebra (i.e. quaternion)-valued left and right-invariant Cartan-Maurer 1-forms of $\mathbb{H}^\times$ are given by

$$\sigma^L = q^{-1} dq = \sigma^L_a u_a \quad \text{and} \quad \sigma^R = q dq^{-1} = \sigma^R_a u_a$$

for any element $q \in \mathbb{H}^\times$. Thus defined, each of these satisfy the same set of Cartan-Maurer equations, which in components read

$$d\sigma^L_i + \varepsilon_{ijk} \sigma^L_j \wedge \sigma^L_k = 0 \quad \quad d\sigma^R_i + \varepsilon_{ijk} \sigma^R_j \wedge \sigma^R_k = 0$$

$$d\sigma^L_0 = 0 \quad \quad d\sigma^R_0 = 0.$$

The adjoint representation of $\mathbb{H}^\times$ is four-dimensional and is defined by $q^{-1} u_a q = R_{ab}(q) u_b$. Since $R_{ab}(q/|q|) = R_{ab}(q)$ and $R_{ab}(-q) = R_{ab}(q)$, this descends to a representation of the rotation group $SO(3)$, whose double cover is $Sp(1)$. The adjoint representation is reducible, as one can write it in block-diagonal form as the direct sum of the one-dimensional trivial representation and the three-dimensional irreducible representation $R_{ij}(q)$ of $SO(3)$. The left and right-invariant Cartan-Maurer 1-forms are related to each other by an adjoint transformation up to a minus sign: $\sigma^L_a = -\sigma^R_b R_{ba}(q)$. Note that in particular we have

$$\sigma^L_0 = -\sigma^R_0 = \frac{d|q|^2}{2|q|^2}.$$
Dual to the Cartan-Maurer 1-forms one has the left and right-invariant vector fields of the Lie group $\mathbb{H}^\times$. The duality requirement with respect to the corresponding set of invariant 1-forms determines them completely. They form two commuting copies of the Lie algebra $\mathbb{R} \oplus \mathfrak{sp}(1)$, i.e.

\begin{align}
[\ell^L_i, \ell^L_j] &= 2\varepsilon_{ijk} \ell^L_k \\
[\ell^R_i, \ell^L_j] &= 2\varepsilon_{ijk} \ell^R_k \\
[\ell^L_i, q^0] &= 0 \\
[\ell^R_i, q^0] &= 0
\end{align}

which one may think of as the dual Cartan-Maurer equations, and the two sets of vector fields are related by $\ell^L_a = -R_{ab}(q^{-1})\ell^R_b$. In particular, we have

\begin{align}
\ell^L_0 &= -\ell^R_0 = q_0 \frac{\partial}{\partial q_0} + q_1 \frac{\partial}{\partial q_1} + q_2 \frac{\partial}{\partial q_2} + q_3 \frac{\partial}{\partial q_3}
\end{align}

where $q_0, q_1, q_2, q_3$ are the real components of the quaternion $q$.

2.1.2. In addition to the Lie group structure, $\mathbb{H}^\times$ carries also two hypercomplex structures. That is, its tangent bundle is equipped with two natural actions by the algebra of imaginary quaternions, with the generators defining integrable almost complex structures. Viewing almost complex structures in their dual guise as endomorphisms of the cotangent bundle and in accordance with our convention that complex structures act from the right on 1-forms, these actions are induced by right and left quaternionic multiplication on a quaternionic coframe of $T^*\mathbb{H}^\times$ as follows:

\begin{align}
dq I^L_i = dq u_i \quad \text{and} \quad dq I^R_i = -u_i dq.
\end{align}

These are respectively left and right-invariant, and the two sets of hypercomplex generators commute with each other.

Consider the symmetric bilinear form induced on the space of quaternions by the quaternionic norm, with $\langle u_a, u_b \rangle = \delta_{ab}$. Observe incidentally that for any two quaternions $x, y \in \mathbb{H}$ we have $\bar{x}y = \langle xu_a, y \rangle u_a$ and $\bar{xy} = \langle x, u_a y_a \rangle u_a$, where the overhead bar indicates the operation of quaternionic conjugation. One can then verify that in the alternative coframes provided by the Cartan-Maurer 1-forms and their respective dual frames the generators of the two hypercomplex structures may be written as

\begin{align}
I^L_i &= \langle u_a u_i, u_b \rangle \ell^L_a \otimes \sigma^L_b \\
I^R_i &= \langle u_a, u_i u_b \rangle \ell^R_a \otimes \sigma^R_b.
\end{align}

By the multiplicative property of the norm, $|u_a| = |q^{-1} u_a q| = |R_{ab}(q) u_b|$, which is to say, adjoint transformations preserve the quaternionic norm. They therefore preserve also the associated bilinear form, and this is further equivalent to requiring that $R_{ab}(q^{-1}) = R_{ba}(q)$. Using this and the left-right transformation formulas for the Cartan-Maurer 1-forms and the invariant vector fields, we obtain for the hypercomplex structure generators the alternative expressions

\begin{align}
I^L_i &= R_{ij}(q^{-1}) \langle u_a u_j, u_b \rangle \ell^R_a \otimes \sigma^R_b \\
I^R_i &= R_{ij}(q) \langle u_a, u_j u_b \rangle \ell^L_a \otimes \sigma^L_b.
\end{align}

2.1.3. Finally, note that the Lie algebra and hypercomplex structures considered above descend trivially on the quotient of $\mathbb{H}^\times$ with respect to the $\mathbb{Z}_2$-action generated by $q \mapsto -q$. In this case we have

\begin{align}
\mathbb{H}^\times / \mathbb{Z}_2 \cong \mathbb{R}_{>0} \otimes SO(3).
\end{align}
2.2. The Swann bundle of a quaternion Kähler manifold.

2.2.1. Let $M$ be a quaternionic Kähler manifold, with a choice of local frame $\omega_1, \omega_2, \omega_3$ for its $SO(3)$-bundle of 2-forms $\mathcal{Q}$ and a principal connection with local connection 1-forms $\theta_1, \theta_2, \theta_3$ on the associated principal $SO(3)$-bundle $\mathcal{Q}$ satisfying the conditions of Theorem [4]. For the considerations which follow it will be convenient to regard these triplets as the components of two imaginary quaternion-valued differential forms, $\omega = \omega_i u_i$ and $\theta = \theta_i u_i$. In this quaternionic formalism the Einstein condition (28) of the theorem may be cast in the form

$$d\theta + \theta \wedge \theta = s\omega$$

where the wedge symbol denotes here the natural extension of the operation of exterior product to quaternion-valued differential forms.

By definition, the Swann bundle over the quaternion Kähler manifold $M$ is the associated bundle

$$U(M) = \mathcal{Q} \times_{SO(3)} (\mathbb{H}^\times / \mathbb{Z}_2) \longrightarrow M.$$ 

On its total space one assembles the following $\text{Im}\mathbb{H}$-valued 2-form:

$$\Omega = s q \omega \bar{q} + (dq - q\theta) \wedge (dq - q\bar{\theta}).$$

Although this is made up of local quantities, it is globally defined. Indeed, consider an open covering $\mathcal{C}$ of $M$ and two overlapping open sets $U, V \in \mathcal{C}$, and suppose that on the intersection $U \cap V$ the corresponding local frames of $\mathcal{Q}$ are related by an $SO(3)$-transition function, i.e. $\omega_{iV} = R_{ij}(u)\omega_{jU}$, with $u \in Sp(1)$ being either one of the two opposite-sign unit quaternions which determine the rotation matrix through the double cover map $Sp(1) \to SO(3)$. In quaternionic notation this transition relation reads $\omega_V = u\omega_U u^{-1}$. On another hand, the transition relation for the connection 1-form is $\theta_V = u\theta_U u^{-1} + u du u^{-1}$.

Thus, if the quaternionic fiber coordinates patch according to either the rule $q_V = q_U u^{-1}$ or $q_V = -q_U u^{-1}$ then we have $\Omega_V = \Omega_U$, which then by the arbitrariness in the choice of $U$ and $V$ implies that $\Omega$ is globally defined on $U(M)$. Note also that in either case $|q_V| = |q_U|$, so the function $|q|$ is globally defined on $U(M)$ as well.

Let $\Omega_1, \Omega_2, \Omega_3$ be the real components of $\Omega$. The rationale behind these definitions is provided by the following powerful property:

**Theorem 12** (Swann [43]). Let $M$ be a quaternionic Kähler manifold. The bundle $U(M)$ endowed with the 2-forms $\Omega_1, \Omega_2, \Omega_3$ constructed as above is a hyperkähler cone with hyperkähler potential $\kappa = |q|^2$.

**Proof.** The fact that the three 2-forms determine a hyperkähler structure on $U(M)$ can be seen fairly quickly by checking that they satisfy the conditions of Theorem [4]. To verify that the differential condition of the theorem holds it suffices to observe that Swann’s formula (92) may be rearranged in the form

$$\Omega = q(s\omega - d\theta - \theta \wedge \theta)\bar{q} - d[q(\sigma^L - \theta)\bar{q}].$$

Thus, by virtue of the Einstein condition (90), $\Omega$ is clearly locally exact and hence closed. On the other hand, to verify that the algebraic condition holds it is useful to observe that the same formula (92) can also be rewritten as

$$\Omega = q[s\omega + (\sigma^L - \theta) \wedge (\sigma^L - \theta)]\bar{q}$$
or, in components, with notations introduced in the previous subsection,
\begin{equation}
\Omega_i = |q|^2 R_{ij}(q) [s \omega_j + \langle u_a, u_j u_b \rangle (\sigma^L_a - \theta_a) \wedge (\sigma^L_b - \theta_b)].
\end{equation}

For the next step in the argument it is convenient to choose a coframe. To understand
this choice geometrically note that the $\mathbb{H}^\times/\mathbb{Z}_2$-bundle structure of $U(M)$ implies that its
tangent space at each point splits canonically into complementary vertical and horizontal
subspaces. In the usual local trivialization the vertical tangent space is spanned by
the left-invariant vector fields $\ell^L_a$. Choosing a local frame for $TM$, the horizontal tan-
gent space is then spanned by the horizontal lift of this frame given by the lift map
$\mathcal{X} \mapsto \mathcal{X}^H = \mathcal{X} + (\iota_{\mathcal{X}} \theta_a) \ell^L_a$ for any $\mathcal{X} \in TM$ (recall however that $\theta_0 = 0$, so the $\ell^L_0$-com-
ponent vanishes). Together, the vertical and horizontal vector fields at a point form a local
frame for the tangent space of $U(M)$ at that point. The dual coframe is given by the vertical
1-forms $\sigma^a - \theta_a$, spanning the vertical cotangent space, together with the trivial pull-back
on $U(M)$ of the local coframe on $T^*M$ dual to the chosen local frame on $TM$, which spans
the horizontal cotangent space. We will call such a local frame on the tangent bundle of
$U(M)$ and its dual coframe bundle structure-adapted frame and coframe, respectively.

Returning to the above formula it is then clear that in a bundle structure-adapted
coframe each $\Omega_i$ has a block-diagonal form split along the local horizontal-vertical divide. So
in order to show that $\Omega_1$, $\Omega_2$, $\Omega_3$ satisfy the algebraic condition of Theorem 4 it suffices to
verify that their horizontal components and their vertical components satisfy it separately.
This task is further facilitated by the observation that the algebraic condition is preserved
by overall rescalings and $SO(3)$ rotations (see the Remark following Theorem 2). For
the horizontal components this follows directly from the fact that the quaternionic Kähler
2-forms $\omega_1$, $\omega_2$, $\omega_3$ satisfy themselves the algebraic condition. For the vertical ones it suffices
to note that the map $u_i \mapsto \langle u_a, u_i u_b \rangle$ gives a four-dimensional real matrix representation
of the standard imaginary quaternions.

In line with our stance of regarding the metric as a composite object, from the triplet
of hyperkähler 2-forms $\Omega_1$, $\Omega_2$, $\Omega_3$ one can construct on the Swann bundle the hyperkähler
metric
\begin{equation}
G = |q|^2 (s g + |\sigma^L - \theta|^2)
\end{equation}
or, equivalently,
\begin{equation}
G = s g |q|^2 + |dq - q \theta|^2.
\end{equation}

Notice that if the signature of the quaternionic Kähler metric $g$ on $M$ is $(4n_+, 4n_-)$ then
the signature of Swann’s hyperkähler metric $G$ on $U(M)$ is either $(4n_+ + 4, 4n_-)$ if $s > 0$,
or $(4n_- + 4, 4n_+)$ if $s < 0$.

Likewise, one can work out the corresponding hyperkähler complex structures on $U(M)$
and obtain
\begin{equation}
I_i = R_{ij}(q) [\mathcal{I}^H_j + \langle u_a, u_j u_b \rangle \ell^L_a \otimes (\sigma^L_b - \theta_b)]
\end{equation}
where $\mathcal{I}^H_i$ represent the horizontal lifts to $U(M)$ of the almost complex structures $\mathcal{I}_i$
associated to the triplet of 2-forms $\omega_i$ on $M$. The complex structures $I_i$, too, have a block-diagonal form in a bundle structure-adapted frame, with the horizontal components
induced by the quaternionic structure on the quaternionic Kähler base $M$, twisted by a fiber
coordinate-dependent $SO(3)$ rotation, and the vertical ones induced by the right-invariant
hypercomplex structure on the fiber $\mathbb{H}^\times/\mathbb{Z}_2$ (compare with the second formula (88)). This
is reminiscent of the way in which the integrable complex structure on the twistor space of a hyperkähler manifold was constructed in [29].

Regarding the cone structure, note that Swann bundles possess a natural $H \times Z_2$-action induced by left quaternionic multiplication in the fiber: taking $q \mapsto vq$ with $v \in H \times$ constant induces a rotation of the hyperkähler 2-forms $\Omega \mapsto v\Omega \bar{v}$, and a rescaling of the hyperkähler metric $G \mapsto |v|^2 G$. The $Z_2$-quotient reflects the invariance of these transformations under $v \mapsto -v$. The infinitesimal generators of this action are the vector fields induced on $U(M)$ by the right-invariant vector fields on the fiber, and we have

\begin{equation}
\iota_{\ell^R_i} \Omega_j + \varepsilon_{ijk} \iota_{\ell^R_0} \Omega_k = -\delta_{ij} d|q|^2 \quad \text{and} \quad \mathcal{L}_{\ell^R_0} \Omega_k = -2\Omega_k
\end{equation}

which then by Proposition 11 means that the bundle $U(M)$ has a hyperkähler cone structure with hyperkähler potential $\kappa = |q|^2$ and homothetic vector field $-\ell^R_0$. □

This theorem admits the following converse (for proof see Theorem 5.9 in [43]):

**Theorem 13** (Swann [43]). Any positive hyperkähler cone whose four canonical vector fields generate a locally free $H \times Z_2$-action is locally homothetic to a Swann bundle.

2.2.2. Let us consider now how Killing symmetries fit into this construction. The main result in this regard is as follows:

**Theorem 14.** Let $M$ be a quaternionic Kähler manifold with Swann bundle $U(M)$. Any Killing vector field $X$ on $M$ can be lifted to a tri-Hamiltonian vector field $X$ on $U(M)$ which commutes with the $H \times Z_2$-action. Conversely, any vector field $X$ on $U(M)$ which commutes with the $H \times Z_2$-action and is tri-Hamiltonian descends to a Killing vector field $X'$ on the base $M$. In a standard local trivialization of $U(M)$ one has

\begin{equation}
X = X' + (\iota_{\ell^L_i} \theta_i + \nu_{X_i}) \ell^L_i
\end{equation}

where $\nu_{X_i}$ is the quaternionic Kähler moment map for $X'$. Moreover, there exists a choice of hyperkähler moment map $\mu_{X_i}$ for $X$ such that the two moment maps are related by

\begin{equation}
\mu_X = q\nu_{X'} \bar{q}.
\end{equation}

Both the descent and the lift maps preserve the Lie bracket.

**Remark.** In the equation (101) we have deliberately blurred the line between $\mathbb{R}^3$-vectors and imaginary quaternions in view of the isomorphism $\text{Im}H \cong \mathbb{R}^3$. That is, despite employing the traditional $\mathbb{R}^3$-vector notation, we clearly regard the two moment maps as $\text{Im}H$-valued functions. In what follows we will institutionalize this practice and will often resort to this notational ambivalence when the context allows for an unequivocal interpretation.

**Proof.** Let us introduce briefly the symbol $\alpha = \sigma^L - \theta$ for the quaternion-valued vertical 1-form, with $\alpha_0$ and $\bar{\alpha}$ designating its real and quaternionic imaginary parts, respectively. Starting from the formula (99) for the Swann bundle triplet of hyperkähler 2-forms recast with the above notation convention in mind in the form

\begin{equation}
\tilde{\Omega} = q(s \bar{\omega} - 2\alpha_0 \wedge \bar{\alpha} - \bar{\alpha} \wedge \bar{\alpha}) \bar{q}
\end{equation}

and using the identity

\begin{equation}
q(d\bar{\rho} + 2\sigma^L \bar{\rho} + 2\bar{\sigma}^L \times \bar{\rho}) \bar{q} = d(q \bar{\rho} \bar{q})
\end{equation}

...
valid for any triplet of functions \( \bar{\rho} \), one can show that for any vector field \( X \) on \( U(M) \) the following formula holds:

\[
\iota_X \Omega = q\left[ s \iota_X \bar{\omega} - d(\iota_X \bar{\alpha}) - 2\bar{\theta} \times (\iota_X \bar{\alpha}) - 2(\iota_X \alpha_0) \bar{\alpha} \right] \bar{q} + d[q(\iota_X \bar{\alpha}) \bar{q}].
\]

Assume now that \( \mathcal{X} \) is some Killing vector field on \( M \). By Theorem 9 this has an associated locally defined quaternionic Kähler moment map \( \bar{\nu}_X \). Define then on the Swann bundle \( U(M) \) the vector field \( X \) as in the equation \((100)\). Observe on one hand that \( X \) commutes automatically with the generators \( \ell^R \) of the \( \mathbb{H}^\times/\mathbb{Z}_2 \)-action on \( U(M) \). On another hand we have \( \iota_X \alpha_0 = 0 \) and \( \iota_X \bar{\alpha} = \bar{\nu}_X \), and so the formula \((104)\) gives us immediately

\[
\iota_X \bar{\Omega} = q(s \iota_X \bar{\omega} - d\bar{\nu}_X - 2\bar{\theta} \times \bar{\nu}_X) \bar{q} + d(q \bar{\nu}_X \bar{q}).
\]

The first term on the right-hand side vanishes by the Galicki-Lawson quaternionic Kähler moment map equation for \( \mathcal{X} \),

\[
d\bar{\nu}_X + 2\bar{\theta} \times \bar{\nu}_X = s \iota_X \bar{\omega}.
\]

Since all three components of \( \bar{\Omega} \) are closed, from this it follows that the vector field \( X \) is also tri-Hamiltonian.

For the converse statement let us consider on \( U(M) \) a tri-Hamiltonian vector field \( X \) commuting with the canonical \( \mathbb{H}^\times/\mathbb{Z}_2 \)-action. Like any vector field on \( U(M) \), locally this decomposes into horizontal and vertical components, \( X = \mathcal{X}^H + \nu_a \ell^L_a \), with \( \mathcal{X}^H \) the horizontal lift of a vector \( \mathcal{X} \in TM \) and the coefficients \( \nu_a \) locally defined functions on \( U(M) \).

The requirement that \( X \) commute with the generators \( \ell^R_a \) of the \( \mathbb{H}^\times/\mathbb{Z}_2 \)-action forces the coefficients \( \nu_a \) to be independent of the fiber coordinates. Moreover, the condition that \( X \) be tri-Hamiltonian implies that it must preserve the hyperkähler potential \( \kappa = |q|^2 \) up to a constant additive shift. (This can be argued by recalling that tri-Hamiltonian vector fields are automatically Killing, and then that on a Kähler manifold the action of a Hamiltonian Killing vector field on a Kähler potential yields a local pluriharmonic function — see e.g. the equation (6.26) in [29].) Since \( X(\kappa) = 2\nu_0 \kappa \), to avoid contradiction we must have \( \nu_0 = 0 \). If we rename the non-vanishing coefficients \( \nu_i = \nu_{\mathcal{X}^i} \), then \( X \) is formally of the form \((100)\). The formula \((104)\) gives us again just as above the equation \((105)\), from which we now conclude that

\[
\mathcal{L}_X \bar{\Omega} = d[q(s \iota_{\mathcal{X}} \bar{\omega} - d\bar{\nu}_{\mathcal{X}} - 2\bar{\theta} \times \bar{\nu}_{\mathcal{X}}) \bar{q}].
\]

Using this expression one can then show that the condition that \( X \) be tri-Hamiltonian implies that \( \bar{\nu}_{\mathcal{X}} \) must satisfy the Galicki-Lawson equation \((106)\) and is thus a quaternionic Kähler moment map for the vector field \( \mathcal{X} \), which justifies \textit{a posteriori} its notation. By Theorem 9 the vector field \( \mathcal{X} \) is necessarily Killing.

Finally, given two pairs of vector fields \( \mathcal{X}, X \) and \( \mathcal{Y}, Y \) each of which satisfy the conditions of the theorem, one can verify explicitly that

\[
[X, Y] = [\mathcal{X}, \mathcal{Y}] + (\iota_{[\mathcal{X} \mathcal{Y}]} \theta_1 + \iota_{[\mathcal{X} \mathcal{Y}]} \ell^L_i) \ell^L_i.
\]

This proves the last statement of the theorem. \(\square\)

Remark. Similarly to the hyperkähler quotient construction [29], the Galicki-Lawson generalized moment map can be used in certain conditions to define a quaternionic Kähler quotient construction by which a quaternionic Kähler space with isometries is reduced to another quaternionic Kähler space in which the isometries are divided out [23, 24]. These quotient constructions are naturally compatible with Swann bundles in the sense that the associated Swann bundle of the quaternionic Kähler quotient of \( M \) by an isometric Lie
AN ANALOGUE OF THE GIBBONS-HAWKING ANSATZ FOR QUATERNIONIC KÄHLER SPACES

3. An analogue of the extended Gibbons-Hawking Ansatz for quaternionic Kähler metrics

3.1. The extended Gibbons-Hawking Ansatz.

3.1.1. The Gibbons-Hawking Ansatz [25] allows one to construct four-dimensional hyperkähler metrics with a tri-Hamiltonian $S^1$-action out of the solutions of a set of partial differential equations on an open subset of $\mathbb{R}^3$. This construction was extended in [29, 38] to $4m$-dimensional hyperkähler metrics with a local tri-Hamiltonian $\mathbb{R}^m$-action. In what follows we will refer to this generalization as the extended Gibbons-Hawking Ansatz, and to spaces carrying such metrics as extended Gibbons-Hawking hyperkähler spaces.

Consider a principal $\mathbb{R}^m$-bundle $N$ over an open subset $S$ of $\mathbb{R}^m \otimes \mathbb{R}^3$ equipped with a principal connection with Lie algebra-valued curvature 2-form $(F_K)_{K \in \mathcal{J}}$ — where $\mathcal{J}$ denotes an index set of cardinality $m$ — and $m$ sections of the associated adjoint bundle, the Higgs fields $U_{KI} = (U_{KI})_{K \in \mathcal{J}}$ such that $\det(U_{KI}) \neq 0$ on $S$. Regarding $\mathbb{R}^m \otimes \mathbb{R}^3$ as the space of configurations of $m$ distinguishable points in $\mathbb{R}^3$, one may coordinatize it globally with $m \mathbb{R}^3$-vectors $\vec{x}^K$. Let $\vec{\partial}_K$ denote the corresponding gradients and assume that the following system of partial differential equations generalizing the abelian Bogomolny monopole equation in $\mathbb{R}^3$ holds on $S$:

\begin{equation}
\vec{\partial}_I U_{KJ} = \vec{\partial}_J U_{KI} \quad \text{and} \quad F_K = \ast I dU_{KI}.
\end{equation}

The action of the linear Hodge-like star operators is defined by $\ast I d\vec{x}^J = \frac{1}{2} d\vec{x}^I \wedge d\vec{x}^J$ and summation over the repeated index $I$ is understood. Consistency with the Bianchi identity for the curvature 2-form requires that the components of the Higgs fields satisfy a set of second-order differential constraints generalizing the Laplace equation in $\mathbb{R}^3$.

To a solution $(F_K, U_{KI})$ of these equations with the Higgs fields satisfying in addition the symmetry property $U_{KI} = U_{IK}$ one then associates on the total space of the $\mathbb{R}^m$-bundle $N$ the triplet of 2-forms

\begin{equation}
\vec{\Omega} = -\frac{1}{2} U_{IJ} d\vec{x}^I \wedge d\vec{x}^J - d\vec{x}^K \wedge (d\psi_K + A_K)
\end{equation}

expressed here in a local coordinate trivialization, with $\psi_I$ coordinates on the fibers, $\vec{x}^I$ coordinates on the base, and $A_K$ a local connection 1-form with curvature $dA_K = F_K$.

Using the first extended Bogomolny equation (109), which together with the symmetry requirement for the Higgs fields implies the total symmetry of $\vec{\partial}_I U_{KJ}$ in its indices, one can show that

\begin{equation}
d\vec{\Omega} = d\vec{x}^K \wedge (dA_K - \ast I dU_{KI}).
\end{equation}

Thus, together, the two extended Bogomolny equations guarantee that these 2-forms are closed. Moreover — see an argument below — they also satisfy the algebraic condition of Theorem [2]. Hence they define on $N$ a hyperkähler structure, with hyperkähler metric

\begin{equation}
G = \frac{1}{2} U_{IJ} d\vec{x}^I \cdot d\vec{x}^J - \frac{1}{2} U_{I}^{J}(d\psi_I + A_I)(d\psi_J + A_J)
\end{equation}

where $U_I^{J}$ denotes the matrix inverse of $U_{IJ}$. The vertical vector fields $\partial_{\psi_K}$ generate a...
tri-Hamiltonian $\mathbb{R}^m$-action and have hyperkähler moment maps $\vec{z}^K$. An important characteristic of the Gibbons-Hawking setup and its generalizations is that the rank of the action is large enough for the group parameters together with the moment map functions to provide a complete parametrization of the space.

3.1.2. Conversely, any $4m$-dimensional hyperkähler manifold with a free local tri-Hamiltonian $\mathbb{R}^m$-action arises locally in this way. This follows immediately from Proposition 2.1 in [9].

3.1.3. These formulas admit an interesting alternative formulation which brings to the fore their quaternionic rather than their $\mathbb{R}^m$-bundle structure. Consider the local coframe of the cotangent bundle of $\mathbb{N}$ given by the $4m$ real components of the quaternionic-valued 1-forms

$$H^I = U^{IJ} (d\psi_J + A_J) + d\vec{z}^I$$

where in accordance with our established practice we regard $\mathbb{R}^3$-vectors as imaginary quaternions. Assuming the same natural definitions as before for the exterior and tensor products of quaternion-valued forms, we can then recast the above formulas for the triplet of 2-forms and metric in the following compact form:

$$\Omega = \frac{1}{2} U_{IJ} \bar{H}^I \wedge H^J$$
$$G = \frac{1}{2} U_{IJ} \bar{H}^I H^J.$$  

Here we view again $\Omega$ as an imaginary quaternion-valued 2-form. Note by the way that this formula for $\Omega$ yields immediately, with the notations from §2.1.2, the further expression

$$\Omega_i = \frac{1}{2} U_{IJ} \langle u_a u_i, u_b \rangle H^I_a \wedge H^J_b.$$  

In this coframe the algebraic condition of Theorem 2 is straightforward to check if one recalls that the map $u_i \mapsto \langle u_a u_i, u_b \rangle$ gives a four-dimensional representation of the standard basis of imaginary quaternions.

The second extended Bogomolny equation can be replaced in this framework by the equivalent condition

$$dH^I = -\frac{1}{2} U^{IL} \partial_x^K U_{LJ} (H^K \wedge \bar{H}^J)_k.$$  

This quaternionic perspective leads also naturally to another, arguably more interesting, reformulation of the Bogomolny equations if one considers rather the dual frame to the above quaternionic coframe of $T^*\mathbb{N}$. This is given explicitly by the vector fields $E_{i0} = U_{KI} \partial_x^K$ together with the horizontal lifts $E_{ii}$ to $\mathbb{N}$ of the vector fields $\partial_x^I \in TS$ (the horizontal lift map is $X \mapsto X - (\iota_X A_K) \partial_\psi^K$ for any $X \in TS$). Then the two extended Bogomolny equations (109) are respectively equivalent to the following two sets of commutator relations

$$[E_{i0}, E_{jk}] = [E_{j0}, E_{ik}] \quad \text{and} \quad [E_{ii}, E_{jj}] = \varepsilon_{ijk} [E_{i0}, E_{jk}].$$

with a structure reminiscent of that of the self-dual Yang-Mills (SDYM) equations. For $m = 1$ it is indeed well known that the Bogomolny equations emerge via reduction of the SDYM equations from four to three dimensions, and this formulation makes that origin transparent.
3.2. Extended Gibbons-Hawking spaces with a hyperkähler cone structure.

3.2.1. Let us investigate now under what conditions an extended Gibbons-Hawking hyperkähler space possesses a hyperkähler cone structure compatible with its $\mathbb{R}^m$-bundle structure. Remark, to begin with, that on the base of any extended Gibbons-Hawking fibration one has a natural action of the group $\mathbb{H}^\times/\mathbb{Z}_2$, namely the one generated by the vector fields

$$\vec{L} = -\vec{x}^k \times \delta_k \quad \text{and} \quad L_0 = -\vec{x}^k \cdot \delta_k.$$ 

These satisfy the algebra

$$[L_i, L_j] = \varepsilon_{ijk} L_k$$

$$[L_i, L_0] = 0.$$ 

Remark. The awkward minus sign in the definition of $L_0$ was introduced with a later benefit in mind, namely, the uniformity of the equation (146). Note also that the so(3) structure constants here differ from the ones we have used so far, and in particular from the ones in equation (78), by a factor of 2. The discrepancy can be removed by a simple rescaling of the definitions (118) of the generators by a factor of 2 and is a matter of convention rather than of any profound significance. In practice, one needs to be alert to this normalization issue when one is using the results of section 1.3 and scale accordingly.

In the three-dimensional configuration-of-points picture for $\mathbb{R}^m \otimes \mathbb{R}^3$, $\vec{L}$ and $L_0$ are the generators of rigid rotations and simultaneous scalings of configurations. Borrowing a term from the classical mechanics of rigid bodies, we will refer to these types of transformations as collective transformations.

On the full $\mathbb{R}^m$-bundle, though, the natural geometric objects to consider when approaching the question of whether this $\mathbb{H}^\times/\mathbb{Z}_2$-action on the base can be lifted to a full hyperkähler cone structure on $N$ are not the generators $L_a$ but rather the lifts

$$X_a = L_a - (\iota_{L_a} A_i + U_{ij} x'_a) \partial_{\psi^i}$$

where the index $a$ runs as usual from 0 to 3, and $x'_a$ are the components of the imaginary quaternions associated to the $\mathbb{R}^3$-vectors $\vec{x}'$, with, therefore, $x'_0 = 0$. Indeed, these satisfy the identities

$$\iota_{X_a} \Omega_j + \varepsilon_{ijk} \iota_{X_0} \Omega_k = -\frac{1}{2} \delta_{ij} U_{ij} d(\vec{x}' \cdot \vec{x}')$$

$$\mathcal{L}_{X_a} \Omega_k = -\Omega_k - \frac{1}{2} (L_0 U_{ij} - U_{ij})(d\vec{x}' \wedge d\vec{x}')_k - dx'_k (\iota_{L_a} F_i)$$

which should be viewed as expressing in a gauge-invariant manner the obstructions to the criterion (d) of Proposition 11 (with the normalization adjustments stipulated in the Remark above) being satisfied. When, in other words, the obstructions vanish, the fields $X_a$ generate a hyperkähler cone structure on $N$. The identities follow directly from the formula (110) and require no other assumptions beyond the symmetry of $U_{ij}$ in its indices. Note in particular that this lift is the unique one yielding a diagonal expression on the right-hand side of the first identity.

Before we formulate the vanishing conditions explicitly let us record also a number of consequences of the field equations, all valid generally, which play a role in the arguments to come. Thus, using the first Bogomolny field equation (109) one can show that

$$\frac{1}{2} U_{ij} d(\vec{x}' \cdot \vec{x}') = d(U_{ij} \vec{x}' \cdot \vec{x}') + \vec{L} U_{ij} \cdot (\vec{x}' \times d\vec{x}') + (L_0 U_{ij} - U_{ij})(\vec{x}' \cdot d\vec{x}')$$

$$= d(v_{ij} \vec{x}' \cdot \vec{x}') + \vec{L} v_{ij} \cdot (\vec{x}' \times d\vec{x}') + L_0 (v_{ij} \vec{x}' \cdot d\vec{x}')$$

where $v_{ij} = U_{ij} - \delta_{ij} U_{il} \delta^l_k \delta^k_j$. 

Remark. The lift $X_a$ is the unique one yielding a diagonal expression on the right-hand side of the first identity.
And, secondly, inserting the generating vector fields of collective transformations into the second Bogomolny equation \((109)\) and then using the first one to manipulate the result, we obtain
\[
\iota_{L_0} F_i = -L U_{1,j} \cdot d\bar{x}^j
\]
(124)
\[
\iota_{L} F_i = d(U_{1,j} \bar{x}^j) - L U_{1,j} \times d\bar{x}^j + (L_0 U_{1,j} - U_{1,j}) d\bar{x}^j.
\]
(125)

We stress again that all the formulas so far hold in full generality, for any extended Gibbons-Hawking construction.

The conditions in which the vector fields \(X_a\) generate a hyperkähler cone structure are stated in

**Proposition 15.** The collective \(\mathbb{H}^x/\mathbb{Z}_2\)-action on the base of an extended Gibbons-Hawking hyperkähler space \(N\) can be lifted to a hyperkähler cone structure on \(N\) if and only if the Higgs fields satisfy the linear differential constraints
\[
\bar{L} U_{1,j} = 0 \quad \text{and} \quad L_0 U_{1,j} = U_{1,j}
\]
that is, if and only if they are invariant at rigid rotations and are homogeneous of degree \(-1\) in the \(\bar{x}^i\) variables. In this case the hyperkähler potential is given by
\[
U = 2 U_{1,j} \bar{x}^i \cdot \bar{x}^j.
\]
(127)

**Proof.** By the criterion criterion \((d)\) of Proposition \([11]\) the lifts \(X_a\) generate a hyperkähler cone structure on \(N\) if and only if there exists a function \(U\) such that
\[
\iota_{\tilde{X}_i} \Omega_j + \varepsilon_{ijk} \iota_{\tilde{X}_0} \Omega_k = -\frac{1}{2} \delta_{ij} dU \quad \text{and} \quad \mathcal{L}_{\tilde{X}_0} \Omega_k = -\Omega_k.
\]
(128)

When the constraints \((126)\) hold, the formulas \((121), (122)\) and \((124)\) make it readily clear that this is indeed the case, with \(U\) defined as above. This proves the converse implication.

For the direct implication note that if the vector fields \(X_a\) do generate a hyperkähler cone structure then 1) the right-hand side of the first equation \((121)\) must be exact, and therefore closed, and 2) the combined last two terms on the right-hand-side of the second equation \((121)\) must vanish. In view of the formula \((123)\), the first condition implies the first constraint \((126)\). Then, in view of this and the formula \((124)\), the second condition implies the remaining constraint \((126)\), which concludes the proof.

When the constraints \((126)\) are satisfied, the relations \((124)\) and \((125)\) imply promptly that
\[
\mathcal{L}_{X_a} A_i = \mathcal{L}_{L_a} A_i = d(\iota_{L_a} A_i + U_{1,j} x^j_a).
\]
(129)

That is, under the Lie action of the generators of the hyperkähler cone structure the 1-forms \(A_i\) shift by exact terms. But as connection 1-forms these are anyway defined only up to exact terms, so in principle the shifts can be absorbed into their definitions. Equivalently put, one can always choose representative elements of their gauge equivalence classes which satisfy the gauge-fixing conditions
\[
\iota_{L_a} A_i = -U_{1,j} x^j_a
\]
(130)
and are thus fully invariant under the action of the generators. Note that these conditions do not fix the representative elements completely but only up to exact differentials of
rotation and scaling-invariant functions. In this gauge the vertical components of the lifts (120) vanish and we have simply \( X_a = L_a \), i.e. the generators of the hyperkähler cone structure coincide with the generators of the collective \( \mathbb{H}^x/\mathbb{Z}_2 \)-action on the base. From now on we will assume this to always be the case.

3.2.2. Remarkably, the conditions of Proposition 15 entail that the corresponding metrics are determined by a single function, which is not so surprisingly the hyperkähler potential. In particular, the differential conditions on the Higgs fields can be exchanged in this case with a dual set of differential conditions on the hyperkähler potential.

**Proposition 16.** Consider the following two assumptions holding on a subset \( S \subset \mathbb{R}^m \otimes \mathbb{R}^3 \):

a) There exists an \( m \times m \) symmetric matrix-valued function \( U_{IJ} \) on \( S \) whose elements satisfy

- the first-order differential equations \( \bar{D}_I U_{KJ} = \bar{D}_J U_{KI} \)
- the symmetry constraints \( \bar{L} U_{IJ} = \bar{0} \) and \( L_0 U_{IJ} = U_{IJ} \).

b) There exists a function \( U \) on \( S \) satisfying

- the second-order differential equations \( \bar{D}_I \times \bar{D}_J = \bar{0} \)
- the symmetry constraints \( \bar{L} U = \bar{0} \) and \( L_0 U = -U \).

Then the two assumptions are equivalent, with the direct and reverse implication maps given respectively by

\[
U = 2U_{IJ} \bar{x}^I \cdot \bar{x}^J \quad \text{and} \quad U_{IJ} = \frac{1}{4} \bar{D}_I \cdot \bar{D}_J U.
\]

**Proof.** The proof is an exercise in three-dimensional differential vector calculus. Consider first the implication \((a) \Rightarrow (b)\). Acting with the operatorial identity

\[
\bar{x}^I \times (\bar{x}^J \times \bar{D}_K) = \bar{x}^J (\bar{x}^I \cdot \bar{D}_K) - (\bar{x}^I \cdot \bar{x}^J) \bar{D}_K
\]

on \( U_{IJ} \) and summing over the indices \( I, J \), using the first-order differential equations to permute indices and then the symmetry constraints to simplify or eliminate terms, we obtain that

\[
\bar{D}_I U = 2U_{IJ} \bar{x}^J
\]

with the first equation (131) taken as the definition of \( U \). This relation implies immediately the symmetry constraints on \( U \), and by a more circuitous route involving similar manipulations as above, the second equation (131) and the second-order differential constraints on \( U \).

The reverse implication \((b) \Rightarrow (a)\) follows rather more straightforwardly by acting on the function \( U \) with the operatorial identities

\[
\bar{D}_K \times (\bar{D}_I \times \bar{D}_J) = \bar{D}_I (\bar{D}_K \cdot \bar{D}_J) - \bar{D}_J (\bar{D}_K \cdot \bar{D}_I)
\]

\[
(\bar{x}^I \cdot \bar{x}^J)(\bar{D}_I \cdot \bar{D}_J) + (\bar{x}^I \times \bar{x}^J) \cdot (\bar{D}_I \times \bar{D}_J) = \bar{L}^2 + L_0^2 - L_0
\]

and taking the second equation (131) as the definition of \( U_{IJ} \). These yield respectively the first-order differential equation for \( U_{IJ} \) and the first equation (131). The remaining symmetry constraints on \( U_{IJ} \) are fairly easy to verify directly. \(\square\)
3.2.3. Theorem 13 tells us that, assuming that the \( \mathbb{H}^\times/\mathbb{Z}_2 \)-action is locally free, modulo a possible sign redefinition of the hyperkähler 2-forms (and consequently of the metric) such a space \( N \) as defined by the conditions of Proposition 15 is locally homothetic to a Swann bundle. Our main objective in what follows will then be to determine explicitly, in intrinsic terms, the quaternionic Kähler metric on the base of this Swann bundle, including the differential constrains on its building blocks induced by the Bogomolny equations. Notice that the base has quaternionic dimension one less than the quaternionic dimension \( m \) of \( N \), and that, on another hand, by Theorem 14 the \( \mathbb{R}^m \)-action on \( N \) descends to it with no alteration in rank. Thus, if we take \( m = n + 1 \geq 2 \), then the quaternionic Kähler metric we end up with will be a \( 4n \)-dimensional one with a local isometric \( \mathbb{R}^{n+1} \)-action.

One may assume in these circumstances that the subset \( S \) of \( \mathbb{R}^m \otimes \mathbb{R}^3 \) is closed under the action generated by \( \vec{L} \) and \( L_0 \). On \( N \), the integrated \( \mathbb{H}^\times/\mathbb{Z}_2 \)-action induced by these vector fields can be expressed in extended Gibbons-Hawking coordinates by means of quaternions as follows

\[
(135) \quad (\psi_I, \vec{x}^I) \quad \overset{q \in \mathbb{H}^\times}{\longrightarrow} \quad (\psi_I, q \vec{x}^I \bar{q}).
\]

The invariance of the \( \psi_I \) coordinates is predicated on the gauge fixing conditions being enforced. In accordance with the above choice for the dimension \( m \) the index \( I \) will be assumed to run from 0 to \( n \).

Formulated simply, our strategy will be to match the symmetry-centered extended Gibbons-Hawking description of \( N \) against the quaternionic structure-centered Swann bundle one, and to extract in the process the quaternionic Kähler metric. For this, these transformation rules will be key. A nice geometrical interpretation is afforded if we introduce a certain auxiliary space which we will call \( \text{Im}\mathbb{H}\mathbb{P}^n \), and which will play for the quaternionic Kähler space a similar role to the one that \( \mathbb{R}^n \otimes \mathbb{R}^3 \) plays in the extended Gibbons-Hawking hyperkähler case. In the next subsection we will take a technical detour to give a self-contained account of this useful construction — after which, armed with the concepts and tools developed there, we will return to the current discussion to complete our objective of deriving what would in effect be a quaternionic Kähler analogue of the extended Gibbons-Hawking Ansatz.

3.3. The space \( \text{Im}\mathbb{H}\mathbb{P}^n \).

3.3.1. As we have remarked before, a point in \( \mathbb{R}^{n+1} \otimes \mathbb{R}^3 \) can be viewed as a configuration of \( n + 1 \) distinguishable points in \( \mathbb{R}^3 \) with position vectors \( \vec{x}^I \). On this space one can define a natural action of the group \( \mathbb{H}^\times/\mathbb{Z}_2 \cong \mathbb{R}_{>0} \otimes SO(3) \) generated by the vector fields \( L_0 \) and \( \vec{L} \) defined above, which acts on configurations either by simultaneously rescaling their position vectors by a common factor or by rigidly rotating them around the origin of \( \mathbb{R}^3 \). Quaternions allow us to express these two actions together concisely as

\[
(136) \quad \vec{x}^I \quad \overset{q \in \mathbb{H}^\times}{\longrightarrow} \quad q \vec{x}^I \bar{q}.
\]

A configuration is said to be degenerate if its stabilizer under the \( \mathbb{H}^\times/\mathbb{Z}_2 \)-action is non-trivial. The set \( C_{\text{deg}} \) of degenerate configurations comprises the null configuration, which is a fixed point for the action and is thus stabilized by the whole group, together with all the configurations for which all non-vanishing position vectors are collinear (we include in this category the configurations with a single non-vanishing position vector), which
are stabilized by an $S^1$ subgroup of the $SO(3)$ subgroup of $\mathbb{H}^\times/\mathbb{Z}_2$. On the set of non-degenerate configurations, the complement of $C_{\text{deg}}$ in $\mathbb{R}^{n+1} \otimes \mathbb{R}^3$, the $\mathbb{H}^\times/\mathbb{Z}_2$-action acts freely. In analogy with the classical projective spaces we define the quotient space

\begin{equation}
\text{Im} \mathbb{HP}^n = (\mathbb{R}^{n+1} \otimes \mathbb{R}^3 \setminus C_{\text{deg}})/(\mathbb{H}^\times/\mathbb{Z}_2).
\end{equation}

This has $\dim \text{Im} \mathbb{HP}^n = 3(n + 1) - 4 = 3n - 1$.

As in the case of projective spaces, on it we can define open atlases consisting of a finite number of inhomogeneous coordinate charts. In practice, to build such an atlas one needs to cut a number of open slices across the $\mathbb{H}^\times/\mathbb{Z}_2$-orbits in the space of non-degenerate configurations in such a way that their projections on $\text{Im} \mathbb{HP}^n$ form an open covering. Each slice is in essence a uniform prescription giving us a representative configuration for every orbit intersecting it.

Concretely, the data required to define an inhomogeneous coordinate atlas on $\text{Im} \mathbb{HP}^n$ consists of a choice of half-plane in $\mathbb{R}^3$ with boundary line passing through the origin, together with a choice of orientation. Note that the condition that a configuration be non-degenerate can be equivalently stated as the requirement that the configuration have at least one pair of non-vanishing and non-collinear position vectors. To define a slice, choose two position vectors $\vec{x}^I_1, \vec{x}^I_2$ with $I_1 < I_2$, and consider the subset $C_{I_1I_2} \subset \mathbb{R}^{n+1} \otimes \mathbb{R}^3 \setminus C_{\text{deg}}$ of all the non-degenerate configurations for which these two position vectors are non-vanishing and non-collinear. Observe that this is closed under the $\mathbb{H}^\times/\mathbb{Z}_2$-action. Any configuration from $C_{I_1I_2}$ can be transformed through a rigid rotation around the origin of $\mathbb{R}^3$ into a configuration with the first position vector lying on the boundary line and the second position vector lying in the interior of the chosen half-plane. A little consideration reveals that there are actually two ways in which this can be achieved, related by a $180^\circ$ rotation, and we select between them by requiring that the cross product of the two vectors — taken in the given order — be parallel, as opposed to anti-parallel, to the normal vector to the half-plane (hence the need for orientation data). Furthermore, through a simultaneous rescaling, the length of the first vector can always be adjusted to have a given fixed value.

**Example** (part I). If we fix in $\mathbb{R}^3$ the half-plane bounded by the $i$-axis and containing the vector $\mathbf{j}$, with normal vector $\mathbf{k}$, and single out the pair of vectors $\vec{x}^0, \vec{x}^1$, then any configuration with these two vectors non-vanishing and non-collinear — that is, from $C_{01}$ — can be transformed as above into a configuration of the form

\begin{equation}
\begin{aligned}
\vec{\rho}^0 &= \mathbf{i} \\
\vec{\rho}^1 &= \rho_1^i \mathbf{i} + \rho_2^j \mathbf{j} \\
\vec{\rho}^I &= \rho_1^i \mathbf{i} + \rho_2^j \mathbf{j} + \rho_I^3 \mathbf{k} \quad \text{for } I = 2, \ldots, n
\end{aligned}
\end{equation}

with $\rho_2^j > 0$. Clearly, the vector product $\vec{\rho}^0 \times \vec{\rho}^1 = \rho_2^j \mathbf{k}$ points in the preferred normal direction.

The restricted configurations constructed in this way have, in kinematic terms, four frozen degrees of freedom and, crucially, are in one-to-one correspondence with the $\mathbb{H}^\times/\mathbb{Z}_2$-orbits contained in $C_{I_1I_2}$, for which they can be thus considered to be representative configurations. Together they define an open slice transversal to these orbits, and their position vectors, which depend on $3(n + 1) - 4 = 3n - 1$ parameters, provide a local inhomogeneous coordinate chart on the projection of $C_{I_1I_2}$ to $\text{Im} \mathbb{HP}^n$. 
The position vectors $\vec{x}^i$ of any configuration from $C_{t_1t_2}$ can be uniquely represented as
\begin{equation}
\vec{x}^i = q \vec{\rho}^i \vec{q}
\end{equation}
for some $q \in \mathbb{H}^\times$ and position vectors $\vec{\rho}^i$ of a restricted configuration. Following the projective space analogy one may think of $\vec{x}^i$ and $\vec{\rho}^i$ as global homogeneous respectively local inhomogeneous coordinates on $\text{Im}\mathbb{H}^n$.

**Example (part II).** In the example above, one can show that in agreement with uniqueness this relation can indeed be inverted to yield
\begin{equation}
\vec{\rho}^i = \frac{\vec{x}^0 \cdot \vec{x}^i}{|\vec{x}^0|^2} \mathbf{1} + \frac{(\vec{x}^0 \times \vec{x}^1) \cdot (\vec{x}^0 \times \vec{x}^i)}{|\vec{x}^0 \times \vec{x}^1||\vec{x}^0|^2} \mathbf{j} + \frac{\vec{x}^0 \cdot [(\vec{x}^0 \times \vec{x}^1) \times (\vec{x}^0 \times \vec{x}^i)]}{|\vec{x}^0 \times \vec{x}^1||\vec{x}^0|^3} \mathbf{k}.
\end{equation}
The components are manifestly invariant under rigid rotations and simultaneous scalings, which reinforces the argument that they are, when they are not constant, natural candidates for the role of coordinates on (the projection of $C_{01}$ to) the quotient space $\text{Im}\mathbb{H}^n$.

Finally, by considering all the possible choices of two position vectors we can construct in this way, using each time the same oriented half-plane, an entire open atlas on $\text{Im}\mathbb{H}^n$ with $n(n+1)/2$ inhomogeneous coordinate charts.

3.3.2. For our purposes it will be useful to develop a succinct dictionary for the passage from a homogeneous coordinate description of various differential geometric considerations on $\text{Im}\mathbb{H}^n$ to an inhomogeneous one and back. Of central importance to this effort will be the notion of reduced, or quotient, connection on $\text{Im}\mathbb{H}^n$ induced by the flat connection on $\mathbb{R}^{n+1} \otimes \mathbb{R}^3$.

Consider the two coordinate systems that we have defined on the space $\mathbb{R}^{n+1} \otimes \mathbb{R}^3 \setminus C_{\text{deg}}$ in the course of §3.3.1 the homogeneous global coordinates $\{\vec{x}^i\}$, and the $\mathbb{H}^\times/\mathbb{Z}_2$-action-adapted local coordinates $\{\vec{\rho}^i, q\}$. Recalling the identity (103), the formula (139) relating them gives us upon differentiation the relation
\begin{equation}
d\vec{x}^i = q(d\vec{\rho}^i + 2\sigma_0^L \vec{\rho}^i + 2\vec{\sigma}^L \times \vec{\rho}^i)\vec{q}
\end{equation}
expressing the horizontal-vertical decomposition of the homogeneous coordinate differentials with respect to the $\mathbb{H}^\times/\mathbb{Z}_2$-bundle structure. Denoting $\varrho^i_{0,1} = \rho^i_1$ and $\varrho^i_{2,1} = \varepsilon_{ijk} \rho^j_k$—or in a more quaternionic-minded notation, $\varrho^i_{a,1} = -\langle u_a, u_i \rho^i \rangle$, where the position vectors $\vec{\rho}^i$ are now viewed as an $\text{Im}\mathbb{H}$-valued objects—this can be trivially rewritten as
\begin{equation}
dx^i = |q|^2 R_{ij}(q)(dp^j + 2\sigma_a^L \varrho^i_{a,j}).
\end{equation}
Dually, we have
\begin{equation}
\partial x^i = |q|^{-2} R_{ij}(q)(\mathcal{D}_j + \frac{1}{2} \mathbb{A}_{k,j,a} \varepsilon^L_a)
\end{equation}
where the local vector fields $\mathcal{D}_i$ and quaternionic-valued functions $\mathbb{A}_{i,j}$ depend exclusively on the homogeneous coordinates $\vec{\rho}^i$ and are completely fixed by their choice. Concretely, by matching for instance the expression of the exterior derivative on $\mathbb{R}^{n+1} \otimes \mathbb{R}^3 \setminus C_{\text{deg}}$ in the two coordinate systems we obtain the following set of conditions which these must satisfy, and which suffice to determine them:
\begin{align}
d\varrho^i_1 \mathbb{A}_{i,1,a} &= 0 & \varrho^i_{a,1} \mathbb{A}_{a,1,b} &= \delta_{ab} 
\end{align}
\begin{align}
d\varrho^i_1 \mathcal{D}_i &= d_{\text{Im}\mathbb{H}^n} & \varrho^i_{a,1} \mathcal{D}_i &= 0.
\end{align}
Summation over repeated indices is as usual understood.
Example (part III). Let us see explicitly how this works in the particular case considered above. The coefficients $\mathcal{A}_{i,a}$ are determined by the two conditions on the first line: the first condition implies that $\mathcal{A}_{i,a} = 0$ for all pairs of indices $I, i$ such that $\rho^I_i \neq \text{constant}$; this leaves us with $4 \times 4 = 16$ unknown components, which are then fixed by the 16 linear equations of the second condition. Thus, for the choice (138) of inhomogeneous coordinates the solution for these is

$$
\begin{align*}
\mathcal{A}_{01} &= 1 \\
\mathcal{A}_{02} &= k \\
\mathcal{A}_{03} &= -\frac{\rho_1}{\rho_2} i - j \\
\mathcal{A}_{13} &= \frac{1}{\rho_2} i.
\end{align*}
$$

(145)

The vector fields $\mathcal{D}_{i,a}$, on the other hand, are determined by the remaining two conditions: from the first of these it follows that $\mathcal{D}_{i} = \partial_{\rho^I_i}$ for all $\rho^I_i \neq \text{constant}$, leaving four free components, which are in turn determined by the four linear equations making up the last condition. We omit the results in this case since, much like the coefficients above, their form is not particularly enlightening.

3.3.3. Note now that in the $\{\bar{\rho}', q\}$ coordinate system the generators of the $\mathbb{H}^\times / \mathbb{Z}_2$-action on $\mathbb{R}^{n+1} \otimes \mathbb{R}^3 \setminus C_{\text{deg}}$ act strictly on the $q$-coordinates and take the form

$$
L_a = \frac{1}{2} \ell_a^R.
$$

(146)

3.3.4. In order to formulate our results we will need to consider two basic types of vector bundles over $\text{Im} \mathbb{H}^n$ or subsets thereof: on one hand, $\mathbb{R}^{n+1}$-bundles, and on the other, vector bundles associated to the natural principal $\mathbb{H}^\times / \mathbb{Z}_2$-bundle.

$$
\begin{tikzcd}
M & E^{(\text{rep})} \\
\mathbb{R}^{n+1} \ar[ru] & \mathbb{H}^\times / \mathbb{Z}_2 \ar[lu]
\end{tikzcd}
$$

(147)

The latter type of bundles can be classified in accordance with the finite-dimensional representations of their structure group $\mathbb{H}^\times / \mathbb{Z}_2$. Since this is isomorphic to $\mathbb{R}_{>0} \otimes SO(3)$, specifying its representations means specifying the representations of its two simple factors. The representations of $\mathbb{R}_{>0}$, the multiplicative group of strictly positive real numbers, are characterized by their scaling weight $w$, an integer number. On the other hand, the only representations of the group $SO(3)$ that we will be concerned with here will be the trivial and the vector representations, which we denote with reference to their dimensions by $1$ and $3$ respectively.

3.3.5. To develop a sense of the properties that sections of such bundles have, it is useful to consider two basic examples corresponding to representations for which one of the factors, and then the other, are taken to be trivial. Thus, suppose we have a section $\mathcal{F}$ of the bundle $E^{(w,1)}$ locally defined on the open set of $\text{Im} \mathbb{H}^n$ coordinatized by the inhomogeneous coordinates $\bar{\rho}'$. We can lift this section to a function

$$
F = |q|^{2w} \mathcal{F}
$$

on (an open subset of) $\mathbb{R}^{n+1} \otimes \mathbb{R}^3 \setminus C_{\text{deg}}$, and then using the expression (146) for the $\mathbb{H}^\times / \mathbb{Z}_2$ generators verify that this satisfies

$$
\begin{align*}
\bar{L}F &= 0 \\
L_0 F &= -wF.
\end{align*}
$$

(149)
That is, sections of $E^{(w,1)}$ correspond to functions on $\mathbb{R}^{n+1} \otimes \mathbb{R}^3 \setminus C_{\text{deg}}$ invariant at rigid rotations and homogeneous of degree $w$ in the homogeneous coordinates. Conversely, such functions always descend to sections of $E^{(w,1)}$. On another hand, by using the formula (143) we get that

$$\partial_{x_l} F = |q|^{2w-2} R_{ik}(q) \nabla_{ik} \mathcal{F} $$

where by definition

$$\nabla_{ij} \mathcal{F} = \mathcal{D}_{ij} \mathcal{F} + w \mathfrak{A}_{i1,0} \mathcal{F}. $$

This is the reduced connection induced on the bundle $E^{(w,1)}$ over $\text{Im} \mathbb{H}^n$ by the flat connection on $\mathbb{R}^{n+1} \otimes \mathbb{R}^3$. It maps sections of $E^{(w,1)}$ to sections of $M \otimes E^{(w-1,3)}$. A simple counting argument shows that its components cannot be all be independent since there are four more of them than the dimension of $\text{Im} \mathbb{H}^n$. One has to have therefore four constraints, which we shall call *horizontality constraints*. This follows indeed from the two conditions on the right in (144). However, an easier way to arrive at the horizontality constraints is to use equivariance to strip off the $q$-dependence from the symmetry properties (149), a process we will refer to as *reduction*. We obtain in this way

$$\bar{\rho}^i \times \bar{\nabla}_i \mathcal{F} = \bar{0} \quad \bar{\rho}^i \cdot \bar{\nabla}_i \mathcal{F} = w \mathcal{F}. $$

Suppose now that we have instead a section $\mathcal{F}_i$ of the bundle $E^{(0,3)}$, again, locally defined on the open set of $\text{Im} \mathbb{H}^n$ coordinatized by the inhomogeneous coordinates $\bar{\rho}^i$. We lift this to a triplet of functions

$$F_i = R_{ij}(q) \mathcal{F}_j$$

or in quaternionic notation, $\tilde{F} = q \mathcal{F} q^{-1}$, on a corresponding subset of $\mathbb{R}^{n+1} \otimes \mathbb{R}^3 \setminus C_{\text{deg}}$. Proceeding in a similar manner as above one can check that these satisfy

$$L_i F_j = \varepsilon_{ijk} F_k \quad L_0 F_j = 0. $$

Conversely, functions on $\mathbb{R}^{n+1} \otimes \mathbb{R}^3 \setminus C_{\text{deg}}$ with these transformation properties under rigid rotations and simultaneous rescalings give rise to sections of $E^{(0,3)}$ over $\text{Im} \mathbb{H}^n$. Using the inhomogeneous decomposition (143) for the derivatives with respect to the homogeneous coordinates we then get

$$\partial_{x_l} F_j = |q|^{-2} R_{ik}(q) R_{jl}(q) \nabla_{ik} \mathcal{F}_l$$

where by definition

$$\nabla_{ij} \mathcal{F}_j = \mathcal{D}_{ij} \mathcal{F}_j + \varepsilon_{ijkl} \mathfrak{A}_{l1,0} \mathcal{F}_l. $$

This connection, mapping sections of $E^{(0,3)}$ to sections of $M \otimes E^{(-1,3) \otimes 3}$, satisfies as well horizontality constraints, which can be argued as before to be given by the reduction of the symmetry properties (154) and thus read

$$\bar{\rho}^i \times \bar{\nabla}_i \mathcal{F}_j = - \varepsilon_{ijk} \mathcal{F}_k \quad \bar{\rho}^i \cdot \bar{\nabla}_i \mathcal{F}_j = 0. $$

3.3.6. This discussion extends naturally to sections of more general vector bundles over $\text{Im} \mathbb{H}^n$ constructed out of these two basic types of bundles in accordance with the rules of the tensor product. On each of these bundles the flat connection on $\mathbb{R}^{n+1} \otimes \mathbb{R}^3$ induces a reduced connection $\bar{\nabla}_i$, which takes its sections to sections of the bundle obtained by tensoring with $M \otimes E^{(-1,3)}$. Having at face count four more components than the dimension of $\text{Im} \mathbb{H}^n$, the $\bar{\nabla}_i$ form an overcomplete set: while the two conditions on the left in (144) ensure that they always satisfy the completeness relation $d\bar{\rho}^i \cdot \bar{\nabla}_i = d_{\text{Im} \mathbb{H}^n}$, the ones on
the right duly imply the requisite number of constraints, whose exact form depends on the bundle in question. Moreover, it is easy to see that the flatness of the inducing connection implies that the commutators \( [\nabla_{ii}, \nabla_{jj}] = 0 \). Note also that the inhomogeneous coordinates \( \rho'_I \) on \( \text{Im} \mathbb{H}P^n \) themselves can be regarded as local sections of the bundle \( M^* \otimes E^{(1,3)} \), their natural lifts to \( \mathbb{R}^{n+1} \otimes \mathbb{R}^3 \setminus C_{\text{deg}} \) being the homogeneous coordinates \( x'_I = |q|^2 R_{ij}(q) \rho'_j \), and we have \( \nabla_{ii} \rho'_j = \delta_{ij} \delta'_I \). An upshot of these considerations is that the covariant calculus on \( \text{Im} \mathbb{H}P^n \) emulates formally the regular differential vector calculus on \( \mathbb{R}^{n+1} \otimes \mathbb{R}^3 \), with the added bonus that one can now make use of horizontality constraints.

### 3.4. The reduction.

#### 3.4.1. Let us return now as promised to pick up where we left off and continue the discussion started in subsection 3.2.

The subset \( S \) of \( \mathbb{R}^{n+1} \otimes \mathbb{R}^3 \), which we are assuming to be closed under the collective \( \mathbb{H}^\times/\mathbb{Z}_2 \)-action on the latter space, can also be assumed to be in fact a subset of \( \mathbb{R}^{n+1} \otimes \mathbb{R}^3 \setminus C_{\text{deg}} \), in concordance with the requirement that the corresponding action on \( N \) be locally free. Let \( \mathcal{J} \) be the projection of \( S \) to \( \text{Im} \mathbb{H}P^n \) through the quotient map. The split invariant/equivariant form of the action (136) along the \( \mathbb{R}^{n+1} \)-fibration lines implies then that the \( \mathbb{R}^{n+1} \)-bundle \( N \) over \( S \) descends to an \( \mathbb{R}^{n+1} \)-bundle \( M \) over \( \mathcal{J} \), the total space of which is locally isomorphic to the base of \( N \), viewed now as a Swann bundle. This is expressed by the commutative diagram

\[
\begin{array}{ccc}
N & \xrightarrow{\mathbb{R}^{n+1}} & S \subset \mathbb{R}^{n+1} \otimes \mathbb{R}^3 \setminus C_{\text{deg}} \\
\mathbb{H}^\times/\mathbb{Z}_2 \downarrow & & \downarrow \mathbb{H}^\times/\mathbb{Z}_2 \\
M & \xrightarrow{\mathbb{R}^{n+1}} & \mathcal{J} \subset \text{Im} \mathbb{H}P^n
\end{array}
\]

To each of the two fibration structures on \( N \) we can associate an adapted system of coordinates. For the \( \mathbb{R}^{n+1} \)-fibration structure this is of course the extended Gibbons-Hawking one, given by \( \psi_I \) and \( \tilde{x}' \). If, however, in this coordinate set we replace what we have called in the previous subsection the homogeneous coordinates \( \tilde{x}' \) with the inhomogeneous ones \( \tilde{\rho}' \) and the quaternionic coordinates \( q \), both of which are now only locally defined, we obtain an alternative system of coordinates which is adapted to the \( \mathbb{H}^\times/\mathbb{Z}_2 \)-fibration structure, with \( \psi_I \) and \( \tilde{\rho}' \) coordinatizing the base \( M \) and \( q \) the quaternionic fiber. The two coordinate systems and their component counts are summarized in Table 1.

| \( \mathbb{R}^{n+1} \)-bundle structure | \( \mathbb{H}^\times/\mathbb{Z}_2 \)-bundle structure |
|--------------------------------------|--------------------------------------|
| adapted coordinates | adapted coordinates |
| fiber | \( \psi_I \) | \( n + 1 \) |
| base | \( \tilde{x}' \) | \( 3n + 3 \) |
| | | |
| | fiber | \( q \) | \( 4 \) |

**Table 1.** Coordinate systems on \( N \).

Our goal in what follows is to re-express the hyperkähler 2-forms and metric on \( N \) as well as the extended Bogomolny equations in terms of the \( \mathbb{H}^\times/\mathbb{Z}_2 \)-fibration-adapted coordinates. Let us begin with the last ones.
3.4.2. **The reduction of the first set of extended Bogomolny equations.**

The symmetry constraints \(126\) of Proposition \(15\) mean that the Higgs fields \(U_{Ij}\) determine, in the language coined above, a section \(\mathcal{U}_{Ij}\) of the bundle \(\text{Sym}(M \otimes M) \otimes E^{(-1,1)}\) over \(\mathcal{S}\) such that in \(\mathbb{H}^\times/\mathbb{Z}_2\)-fibration-adapted coordinates we have

\[
U_{Ij} = \frac{\mathcal{U}_{Ij}}{|q|^2}.
\]

By the considerations in §3.3.5 it is straightforward to see then that the reduction of the first set of extended Bogomolny equations \(109\) yields the conditions

\[
\tilde{\nabla}_i \mathcal{U}_{Kj} = \tilde{\nabla}_j \mathcal{U}_{Kl}.
\]

Note also that, as sections of the bundle \(E^{(-1,1)}\), the reduced Higgs fields automatically satisfy the horizontality constraints

\[
(160) \quad \tilde{\rho}^I \times \tilde{\nabla}_i \mathcal{U}_{Kj} = \tilde{\nabla}_i \mathcal{U}_{Kj} = -\mathcal{U}_{Kj}.
\]

3.4.3. **The reduction of the second set of extended Bogomolny equations.**

In \(\mathbb{H}^\times/\mathbb{Z}_2\)-fibration-adapted coordinates the generators of the \(\mathbb{H}^\times/\mathbb{Z}_2\)-action on \(N\) act along the fibers and take the form \(146\). Using this fact, one can show that in these coordinates a 1-form satisfying the gauge-fixing conditions \(130\) must be of the form

\[
A_K = \mathscr{A}_K + 2(\mathcal{U}_{Kj} \tilde{\rho}^j) \cdot \tilde{\sigma}^L
\]

with the component \(\mathscr{A}_K\) a 1-form supported on the projection to \(\mathcal{S}\) of the support of \(A_K\) in \(S\), which can in principle—for now—still depend on the \(q\)-coordinates.

On another hand, from the decomposition relations \(141\) and \(158\), by way of the formula \(150\), we get that

\[
*^I dU_{Kj} = \frac{1}{2} \tilde{\nabla}_i \mathcal{U}_{Kj} \cdot (d\tilde{\rho}^I + 2\sigma_0^L \tilde{\rho}^i + 2\tilde{\sigma}^L \times \tilde{\rho}^i) \wedge (d\tilde{\rho}^I + 2\sigma_0^L \tilde{\rho}^i + 2\sigma_0^L \times \tilde{\rho}^i)
\]

By resorting repeatedly to the reduced Bogomolny equations \(159\) and once to the Cartan-Maurer equation for \(\tilde{\sigma}^L\), the expression on the right-hand side can be manipulated with a little bit of effort into the form

\[
*^I dU_{Kj} = \frac{1}{2} \tilde{\nabla}_i \mathcal{U}_{Kj} \cdot (d\tilde{\rho}^I \wedge d\tilde{\rho}^i)
\]

Finally, in the light of the horizontality constraints \(160\) this simplifies to

\[
*^I dU_{Kj} = \frac{1}{2} \tilde{\nabla}_i \mathcal{U}_{Kj} \cdot (d\tilde{\rho}^I \wedge d\tilde{\rho}^i) + d(2\mathcal{U}_{Kj} \tilde{\rho}^I \cdot \tilde{\sigma}^L).
\]

Recalling that \(F_K = dA_K\), the second set of extended Bogomolny equations \(109\) implies then the conditions

\[
d\mathscr{A}_K = \frac{1}{2} \tilde{\nabla}_i \mathcal{U}_{Kj} \cdot (d\tilde{\rho}^I \wedge d\tilde{\rho}^i).
\]

This shows in particular that the 1-forms \(\mathscr{A}_K\) can be chosen in fact not to depend on the \(q\)-coordinates at all, and can be thus thought of as bona fide connection 1-forms on the \(\mathbb{R}^{n+1}\)-bundle \(M \to \mathcal{S}\) induced by the connection 1-forms \(A_K\) on the \(\mathbb{R}^{n+1}\)-bundle \(N \to S\).
3.4.4. The reduction of the hyperkähler 2-forms and metric.

At this point, it is opportune to switch temporarily to a quaternionic formalism. The key observation for this is that decomposition relations (158), (161) and (141) can be pieced together into the following quaternionic decomposition formula for the quaternionic-valued coframe elements (113) of $T^*N$:

\begin{equation}
H^I = q(h^I + 2\rho^I \bar{\sigma}^L)\bar{q}
\end{equation}

where $\rho^I$ are the imaginary quaternionic avatars of the vectors $\vec{\rho}^I$ and, by definition,

\begin{equation} h^I = \mathcal{U}^{I,J}(d\psi_J + \omega_J) + d\vec{\rho}^I.
\end{equation}

Note that these form in turn a complete set of quaternionic-valued coframe elements for $T^*M$.

Substituting then this expression for the elements $H^I$ together with the expression (158) for the Higgs fields $U_{IJ}$ into the quaternionic formulas (114) for the hyperkähler 2-forms and metric on $N$ gives us after a series of rather straightforward algebraic manipulations the Swann-like expressions

\begin{equation}
\Omega = \mathcal{U} q [s \omega + (\sigma^L - \theta) \wedge (\sigma^L - \bar{\theta})]q
\end{equation}

\begin{equation}
G = \mathcal{U}|q|^2(sg + |\sigma^L - \theta|^2)
\end{equation}

where we have denoted $\mathcal{U} = 2U_{IJ}\bar{\rho}^I \rho^J = 2U_{IJ} \vec{\rho}^I \cdot \vec{\rho}^J$, which we assume to be non-vanishing,

\begin{equation}
\theta = -\frac{\mathcal{U}_{I,J}\bar{h}^I \rho^J}{2\mathcal{U}_{M,N}\bar{\rho}^M \rho^N}
\end{equation}

and

\begin{equation}
\begin{aligned}
sw &= \frac{\mathcal{U}_{K,L}\bar{\rho}^L (\mathcal{U}_{I,J}\bar{h}^I \wedge h^J) - (\mathcal{U}_{J,L}\bar{h}^J \rho^L) \wedge (\mathcal{U}_{K,I}\bar{\rho}^K h^J)}{(2\mathcal{U}_{M,N}\bar{\rho}^M \rho^N)^2} \\
sg &= \frac{\mathcal{U}_{K,L}\bar{\rho}^L (\mathcal{U}_{I,J}\bar{h}^I \wedge h^J) - (\mathcal{U}_{J,L}\bar{h}^J \rho^L) \wedge (\mathcal{U}_{K,I}\bar{\rho}^K h^J)}{(2\mathcal{U}_{M,N}\bar{\rho}^M \rho^N)^2}.
\end{aligned}
\end{equation}

The two formulas (168) differ from the Swann bundle ones (94) and (97) in two respects: first, by the presence of the overall factor $\mathcal{U}$, and second, and perhaps less obviously, since obscured by the notation, by the fact that $\theta$ is not an imaginary quaternion-valued 1-form but rather a full quaternion-valued one; in other words, by the fact that its real part is non-vanishing. However, notice that this can be cast in the form

\begin{equation}
\theta_0 = -\frac{1}{\mathcal{U}} \mathcal{U}_{I,J} \bar{\rho}^I \cdot d\bar{\rho}^J = -\frac{d\mathcal{U}}{2\mathcal{U}}.
\end{equation}

The first expression follows directly from the definition. To see how the second one comes about we begin by observing that $\mathcal{U}$ can be viewed as originating from the reduction of the hyperkähler potential $U$, more precisely, that we have $U = |q|^2 \mathcal{U}$. The symmetry constraints of part (b) of Proposition 16 which the latter satisfies imply then that $\mathcal{U}$ is a section of the line bundle $E^{(1,1)}$ over $\mathcal{S}$. This means in particular that the covariant derivatives $\nabla_i \mathcal{U}$ are well-defined. In light of these facts the relation (133) can be reduced to yield

\begin{equation}
\nabla_i \mathcal{U} = 2\mathcal{U}_{I,J} \bar{\rho}^J.
\end{equation}
Using this, the second expression for $\theta_0$ follows then immediately. Let us remark also while at this that, on the other hand, for the imaginary/vectorial part of $\theta$ we obtain directly, without any further argument,

$$\bar{\theta} = -\frac{1}{\mathcal{U}}[\mathcal{U}_{i,j} \bar{\rho}^i \times d\bar{\rho}^j + \bar{\rho}^j (d\psi_i + \mathcal{A}_i)].$$

(173)

The departure from the Swann canonical forms is therefore rectifiable: the overall $\mathcal{U}$-factor and the non-vanishing real component of $\theta$ from the formulas (168) can both be removed simultaneously through a rescaling

$$q \mapsto \frac{q}{|\mathcal{U}|^{1/2}}$$

(174)
of the fiber coordinates. When $\mathcal{U} < 0$ this must be supplemented by a sign redefinition $\Omega \mapsto -\Omega$, and consequently $G \mapsto -G$. None of these maneuvers, however, affects the expressions for $s \omega$, $sg$ or the imaginary part of $\theta$, which can now be legitimately interpreted as quaternionic Kähler 2-forms, metric and $SO(3)$ connection 1-forms, respectively.

3.4.5. A direct check.

In fact, since we are now in possession of explicit intrinsic expressions for these, we can verify directly that they satisfy the requisite quaternionic Kähler properties without resorting to the extrinsic device of the Swann bundle. For this, let us observe first that similarly to the extended Gibbons-Hawking hyperkähler case (see §3.1.3), the reduced Bogomolny equation (165) can be replaced in the quaternionic formalism with the equivalent condition

$$dh^I = -\frac{1}{2} \mathcal{U}^{IJ} \nabla_K \mathcal{U}_{LJ} (h^K \wedge \bar{h}^L)_K.$$

(175)

Using this along with the other reduced Bogomolny equation (159) as well as the horizontality constraints (160), we can then show through a somewhat involved and rather technical exercise in differential form-valued vector calculus that the following two differentiation formulas

$$d(\mathcal{U}_{i,j} \bar{\rho}^i \rho^j) = \frac{1}{2} \mathcal{U}_{i,j} (\bar{h}^i \rho^j + \bar{\rho}^j h^i)$$

(176)

$$d(\mathcal{U}_{i,j} \bar{h}^i \rho^j) = -\frac{1}{2} \mathcal{U}_{i,j} \bar{h}^i \wedge h^j$$

hold. From these and the above definition of $\theta$ we can hence infer immediately that

$$d\theta + \theta \wedge \theta = s \omega.$$  

(177)

Note that the $\theta_0$ terms drop out completely from the left-hand side of this equality leaving behind only the imaginary quaternionic part of $\theta$, and so this is precisely the Einstein condition (28) of the quaternionic Kähler criterion laid out in Theorem 4. This, by the way, justifies why we prefer to carry the constant proportionality factor $s$ along in the formulas (170) rather than absorb it in the definitions.

3.4.6. Alternative expressions.

Let us return now to the quaternionic formulas (170) for the quaternionic Kähler 2-forms and metric, and observe that they can be alternatively recast through a purely algebraic
AN ANALOGUE OF THE GIBBONS-HAWKING ANSATZ FOR QUATERNIONIC KÄHLER SPACES 41

... effort in the — also quaternionic — form

\[ s\omega = \frac{1}{2\mathcal{U}} \mathcal{U}_{IJ}(h^I + 2\rho^I \bar{\theta}) \wedge (h^J + 2\rho^J \bar{\theta}) \]
\[ sg = \frac{1}{2\mathcal{U}} \mathcal{U}_{IJ}(h^I + 2\rho^I \bar{\theta}) (h^J + 2\rho^J \bar{\theta}) \]

with \( \theta \) given by the expression \([169]\). These are essentially the quaternionic Kähler analogues of the hyperkähler formulas \([114]\).

From them, we can transition back to a vectorial formalism by writing

\[ h^I + 2\rho^I \bar{\theta} = \mathcal{U}^{IJ}(d\psi_J + \mathbf{A}_J + 2\mathcal{U}_{JK} \bar{\rho}^K \cdot \bar{\theta}) + d\bar{\rho}^I + 2\theta_0 \bar{\rho}^I + 2\bar{\theta} \times \bar{\rho}^I \]

which then, upon substitution into the two formulas above, gives us the equivalent — and also most explicit — expressions

\[ s\bar{\omega} = -\frac{1}{2\mathcal{U}} \mathcal{U}_{IJ}(d\bar{\rho}^I + 2\theta_0 \bar{\rho}^I + 2\bar{\theta} \times \bar{\rho}^I) \wedge (d\bar{\rho}^J + 2\theta_0 \bar{\rho}^J + 2\bar{\theta} \times \bar{\rho}^J) \]
\[ -\frac{1}{2\mathcal{U}} (d\bar{\rho}^I + 2\theta_0 \bar{\rho}^I + 2\bar{\theta} \times \bar{\rho}^I) \wedge (d\psi_I + \mathbf{A}_I + 2\mathcal{U}_{IK} \bar{\rho}^K \cdot \bar{\theta}) \]
\[ sg = \frac{1}{2\mathcal{U}} \mathcal{U}_{IJ}(d\bar{\rho}^I + 2\theta_0 \bar{\rho}^I + 2\bar{\theta} \times \bar{\rho}^I) \cdot (d\bar{\rho}^J + 2\theta_0 \bar{\rho}^J + 2\bar{\theta} \times \bar{\rho}^J) \]
\[ + \frac{1}{2\mathcal{U}} \mathcal{U}^{IJ}(d\psi_J + \mathbf{A}_J + 2\mathcal{U}_{JK} \bar{\rho}^K \cdot \bar{\theta})(d\psi_J + \mathbf{A}_J + 2\mathcal{U}_{JK} \bar{\rho}^K \cdot \bar{\theta}) \]

The 1-forms \( \theta_0 \) and \( \bar{\theta} \) are given in turn by the corresponding formulas \([171]\) and \([173]\). In this formulation it is straightforward to see that the vector fields \( \partial_{\psi_I} \) generate an isometric \( \mathbb{R}^{n+1} \)-action with Galicki-Lawson moment map

\[ \partial_{\psi_I} \mapsto \bar{\rho}^I = \mathbf{A}_I. \]

In the next subsection we compile for easy reference and present in an intrinsic geometric language the quaternionic Kähler construction which emerged from these considerations in a way which mirrors the presentation of the extended Gibbons-Hawking hyperkähler construction.

3.5. The quaternionic Kähler Ansatz.

Consider a principal \( \mathbb{R}^{n+1} \)-bundle \( M \) over an open subset \( \mathcal{S} \) of \( \text{Im} \mathbb{H}P^n \) and, alongside it, the restrictions to \( \mathcal{S} \) of the \( \mathbb{H} \times / \mathbb{Z}_2 \)-bundles \( E^{(\text{rep})} \) over \( \text{Im} \mathbb{H}P^n \) defined in subsection 3.3, which we denote with the same symbols.

\[ M \xrightarrow{\mathbb{R}^{n+1}} E^{(\text{rep})} \]
\[ \mathcal{S} \subset \text{Im} \mathbb{H}P^n \]

Let \( \bar{\nabla} \) be the reduced connection induced on the latter by the flat connection \( \bar{\nabla}_I \) on \( \mathbb{R}^{n+1} \otimes \mathbb{R}^3 \).

In this setup we want to formulate two sets of field equations. For the first one let us assume that we have either
\textbf{Ia.} a section \( \mathcal{U}_{IJ} \) of the vector bundle \( \text{Sym}(M \otimes M) \otimes E^{(-1,1)} \) over \( \mathcal{S} \) satisfying the first-order differential constraints
\begin{equation}
\nabla_I \mathcal{U}_{KJ} = \nabla_J \mathcal{U}_{KI}
\end{equation}
or, alternatively,
\textbf{Ib.} a section \( \mathcal{U} \) of the line bundle \( E^{(+1,1)} \) over \( \mathcal{S} \) satisfying the second-order differential constraints
\begin{equation}
\nabla_I \times \nabla_J \mathcal{U} = 0.
\end{equation}

For which one of these conditions we opt is immaterial because

**Lemma 17.** The two conditions are equivalent, with the maps between them given, respectively, by
\begin{equation}
\mathcal{U} = 2 \mathcal{U}_{IJ} \bar{\rho}^I \cdot \bar{\rho}^J \quad \text{and} \quad \mathcal{U}_{IJ} = \frac{1}{4} \nabla_I \cdot \nabla_J \mathcal{U}.
\end{equation}

This follows immediately from Lemma 16 by reduction. One can in fact give a completely intrinsic proof of this result simply by mimicking the proof of Lemma 16 in covariant calculus language and noticing that the symmetry constraints there show up here in the guise of horizontality constraints.

On the other hand, for the second set of field equations let us assume that we have, in addition,

\textbf{II.} a principal connection on \( M \) with Lie algebra-valued curvature 2-form \( \mathcal{F}_K = 0 \), ..., \( n \) fixed in terms of the Higgs fields by the following Bogomolny-like condition:
\begin{equation}
\mathcal{F}_K = \frac{1}{2} \nabla_I \mathcal{U}_{KJ} \cdot (d\bar{\rho}^I \wedge d\bar{\rho}^J).
\end{equation}

Based on the previous considerations we can now state the following

**Theorem 18.** Let \( (\mathcal{U}_{IJ}, \mathcal{F}_K) \) or \( (\mathcal{U}, \mathcal{F}_K) \) be a set of solutions of the above field equations, and assume that we can adjust the open subset \( \mathcal{S} \) of \( \text{Im} \mathbb{H}^{n+1} \) on which they are defined in such a way that we have both \( \det(\mathcal{U}_{IJ}) \neq 0 \) and \( \mathcal{U} \neq 0 \) everywhere on it. Locally on the total space of the \( \mathbb{R}^{n+1} \)-bundle \( M \) we associate to this solution the triplet of 1-forms
\begin{equation}
\tilde{\theta} = -\mathcal{U}_{IJ} \bar{\psi}^I \times d\bar{\psi}^J - \bar{\psi}^I (d\psi^I + \mathcal{A}^I)
\end{equation}
and triplet of 2-forms
\begin{equation}
s\tilde{\omega} = -\frac{1}{2} \mathcal{U}_{IJ} (d\bar{\psi}^I + 2 \bar{\theta} \times \bar{\psi}^I) \wedge (d\bar{\psi}^J + 2 \bar{\theta} \times \bar{\psi}^J)
\end{equation}
\begin{equation}
- (d\bar{\psi}^I + 2 \bar{\theta} \times \bar{\psi}^I) \wedge (d\psi^I + \mathcal{A}^I + 2 \mathcal{U}_{IK} \bar{\psi}^K \cdot \bar{\theta})
\end{equation}
extpressed here in a local coordinate trivialization, with \( \psi^I \) coordinates on the fibers and \( \bar{\rho}^I \) inhomogeneous coordinates on the base; by definition, \( s \) is a non-vanishing real constant, \( \mathcal{A}_K \) is a local connection 1-form with curvature \( d\mathcal{A}_K = \mathcal{F}_K \),
\begin{equation}
\mathcal{U}_{IJ} = \mathcal{U} \mathcal{U}_{IJ} \quad \text{and} \quad \bar{\psi}^I = \frac{\bar{\rho}^I}{\mathcal{U}}.
\end{equation}
These then form a quaternionic Kähler structure in the sense of Theorem 4, with quaternionic Kähler metric
\begin{equation}
sg = \frac{1}{2} \mathcal{U}_{IJ} (d\bar{\psi}^I + 2 \bar{\theta} \times \bar{\psi}^I) \cdot (d\bar{\psi}^J + 2 \bar{\theta} \times \bar{\psi}^J).
\end{equation}
\[ + \frac{1}{2} \mathcal{W}^{ij}(d\psi_i + \mathcal{A}_i + 2\mathcal{W}_{ik} \vec{\nu}_k \cdot \vec{\theta})(d\psi_j + \mathcal{A}_j + 2\mathcal{W}_{jl} \vec{\nu}_l \cdot \vec{\theta}) \]

possessing an isometric locally free \( \mathbb{R}^{n+1} \)-action generated by the vector fields \( \partial_{\psi_i} \), with corresponding Galicki-Lawson moment map images \( \vec{\nu}_i \).

Conversely, any \( 4n \)-dimensional quaternionic Kähler manifold with an isometric locally free \( \mathbb{R}^{n+1} \)-action arises locally in this way.

The direct claim encapsulates the results of the previous subsection. Our arguments there have yielded in fact three more equivalent sets of formulas for the quaternionic Kähler connection 1-forms, 2-forms and metric, which we list by reference number in Table 2.

| \( \theta \) | \( s \omega \) | \( s g \) | Formalism |
|---|---|---|---|
| 1. eq. (169) | eq. (170) | eq. (170) | quaternionic |
| 2. eq. (169) | eq. (178) | eq. (178) | quaternionic |
| 3. eqs. (171) & (173) | eq. (180) | eq. (181) | vectorial |

**Table 2.** Alternative expressions for the quaternionic Kähler connection 1-forms, 2-forms and metric.

For the converse statement, if we assume that we have a \( 4n \)-dimensional quaternionic Kähler space \( M \) with an isometric locally free \( \mathbb{R}^{n+1} \)-action, then by Theorem 14 this action lifts to an \( \mathbb{R}^{n+1} \)-action on the \( 4(n+1) \)-dimensional Swann bundle \( \mathcal{U}(M) \). Bielawski’s result (i.e. Proposition 2.1 from [9]) guarantees then that the hyperkähler structure on \( \mathcal{U}(M) \) is given locally by an extended Gibbons-Hawking construction. In particular, corresponding to these actions, the quaternionic Kähler moment maps are lifted to hyperkähler moment maps in accordance with the rule (101), and so we conclude that the canonical \( \mathbb{H}^\times/\mathbb{Z}_2 \)-action on \( \mathcal{U}(M) \) acts on the base of the extended Gibbons-Hawking fibration by what we have called collective transformations. We find ourselves therefore within the bounds of Proposition 15 which means that we can repeat ad litteram the reduction procedure of subsection §3.4 to obtain an explicit local description of the quaternionic Kähler structure on \( M \) in the precise same category as the one above, proving thus the generality of this construction.

Remark. The theorem is an obvious quaternionic Kähler analogue of the extended Gibbons-Hawking hyperkähler Ansatz. The role of hyperkähler moment map is played now by Galicki and Lawson’s concept of quaternionic Kähler moment map; however, unlike in the extended Gibbons-Hawking case, the moment map images do not play directly the role of coordinates (for which they would be too many), rather, they are related to these through a scaling factor. The theorem encompasses the range of both definite and indefinite signature metrics. One of the most remarkable corollaries to emerge from it is the fact that the local quaternionic Kähler structure is completely determined by a single real-valued potential \( \mathcal{W} \) satisfying a set of linear differential constraints. The linearity means that in a very definite sense one can superpose two such quaternionic Kähler structures to obtain a third.

### 3.6. The case of four dimensions.

We end this section with a closer examination of the four-dimensional (i.e. \( n = 1 \)) specialization of this construction. In this case, \( M \) is the four-dimensional avatar of a quaternionic Kähler space — namely, a self-dual Einstein space with non-vanishing scalar curvature — possessing two linearly independent commuting Killing vector fields. The local geometry
of this class of spaces was described explicitly by Calderbank and Pedersen in [11] in terms of a local eigenfunction of the Laplacian on the Poincaré half-plane, following an intrinsic, four-dimensional analysis based on previous work by Joyce [31], Ward [45] and Tod [44], among others. We will now show that our results reproduce theirs in this dimension, while also offering a few additional bonuses. In this sense, therefore, the construction of Theorem 18 may be viewed as a higher-dimensional generalization of the Calderbank-Pedersen construction — similarly, although much less trivially, to the way in which, in hyperkähler context, the extended Gibbons-Hawking construction of [29, 38] can be viewed as a higher-dimensional generalization of the original four-dimensional Gibbons-Hawking one.

First, let us observe that for \( n = 1 \) the space \( \text{Im} \mathbb{H} \mathbb{P}^1 \) is isomorphic to the upper-half plane \( \mathbb{H}^2 \), and can be covered by a single inhomogeneous coordinate chart. We can choose the inhomogeneous coordinates as in the Example in §3.3.1 to be given by

\[
\begin{align*}
\rho^0 &= \mathbf{i} \\
\rho^1 &= \rho_1 \mathbf{i} + \rho_2 \mathbf{j}
\end{align*}
\]

with \( \rho_2 > 0 \), where, for simplicity, we drop the now inessential upper index 1 from the notation of the components of the second vector. As we have argued subsequently in §3.3.2, a choice of inhomogeneous coordinates determines completely the form of the reduced covariant connections on bundles \( E^{(1,1)} \to \text{Im} \mathbb{H} \mathbb{P}^1 \) induced by the flat connection on, in this case, \( \mathbb{R}^2 \otimes \mathbb{R}^3 \). To determine these, we need both the connection coefficients \( (A^i)_I \) with the non-vanishing ones listed in (145), and the operators \( (D^I_i)_I = 0, 1 \), which in this case can be worked out by means of the procedure described in the same place to be

\[
\begin{align*}
D_0 &= -\mathbf{i}(\rho_1 \partial_{\rho_1} + \rho_2 \partial_{\rho_2}) + \mathbf{j}(\rho_2 \partial_{\rho_1} - \rho_1 \partial_{\rho_2}) \\
D_1 &= \mathbf{i}\partial_{\rho_1} + \mathbf{j}\rho_{\rho_2}.
\end{align*}
\]

For this choice of inhomogeneous coordinates the only non-vanishing contribution on the right-hand side of the second reduced Bogomolny equation (187) comes possibly from the term \( \nabla_{13} \mathcal{W}_{K1} d\rho_1 \wedge d\rho_2 \). However, this reduced covariant derivative vanishes, too, and the equation becomes simply

\[
\mathcal{F}_K = 0.
\]

This means that in four dimensions we can consistently set in the formulas, which are locally defined, \( \mathcal{H}_K = 0 \).

In fact, a quick survey shows that all the reduced covariant derivatives \( \nabla_{Ji} \) with \( i = 3 \) of no matter what section vanish. The second-order differential constraints (185) reduce therefore to a single one, which reads

\[
\rho_2 (\mathcal{W}_{p_1 p_1} + \mathcal{W}_{p_2 p_2}) = \mathcal{W}_{p_2}
\]

with the indices indicating derivatives. To obtain this result it is important to recall that \( \mathcal{W} \) is a section of the bundle \( E^{(1,1)} \), and so \( \nabla_{Ji} \mathcal{W} \) is a section of the bundle \( M^* \otimes E^{(0,3)} \). For the details of the calculation of reduced covariant derivatives of both such types of sections we refer the reader back to §3.3.3. The constraint can be equivalently recast in the form

\[
\Delta_{\mathcal{H}^2} \left( \frac{\mathcal{W}}{\sqrt{\rho_2}} \right) = \frac{3}{4} \frac{\mathcal{W}}{\sqrt{\rho_2}}
\]
where $\Delta_{\mathcal{H}^2} = \rho_2^2 \left( \frac{\partial^2}{\partial \rho_1^2} + \frac{\partial^2}{\partial \rho_2^2} \right)$ is the Laplacian corresponding to the Poincaré metric

\begin{equation}
g_{\mathcal{H}^2} = \frac{d\rho_1^2 + d\rho_2^2}{4\rho_2^2}
\end{equation}

on the upper half-plane. We retrieve thus a first key result of Calderbank and Pedersen.

Let us define now the following 1-forms vertical with respect to the $\mathbb{R}^2$-fibration structure of $M$

\begin{equation}
\lambda_i = 2\rho_i^2 d\psi_i = 2 \begin{pmatrix} d\psi_0 + \rho_1 d\psi_1 \\ \rho_2 d\psi_1 \\ 0 \end{pmatrix}
\end{equation}

and similarly vertical commuting vector fields

\begin{equation}
v_i = \nabla_{\psi_i} \partial_{\psi_j} = \begin{pmatrix} \mathcal{U} - \rho_1 \mathcal{U}_{\rho_1} - \rho_2 \mathcal{U}_{\rho_2} \partial_{\psi_0} + \mathcal{U}_{\rho_1} \partial_{\psi_1} \\ (\rho_2 \mathcal{U}_{\rho_1} - \rho_1 \mathcal{U}_{\rho_2}) \partial_{\psi_0} + \mathcal{U}_{\rho_2} \partial_{\psi_1} \\ 0 \end{pmatrix}.
\end{equation}

The third component vanishes in each case. The remaining components, $\lambda_1$, $\lambda_2$ and $v_1$, $v_2$, form a local coframe and frame, respectively, for the $\mathbb{R}^2$-fibers. By resorting to the relation (197), we obtain then that

\begin{equation}
\mathcal{U}_{ij} \rho_i^j = \frac{1}{4} \lambda_i = \frac{1}{2} \begin{pmatrix} \mathcal{U} - \rho_2 \mathcal{U}_{\rho_2} & \rho_2 \mathcal{U}_{\rho_1} & 0 \\ \rho_2 \mathcal{U}_{\rho_1} & \rho_2 \mathcal{U}_{\rho_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{equation}

Again, all components along the third direction vanish, leaving a two-dimensional non-vanishing square block. Since in this case the Higgs field matrix $(\mathcal{U}_{ij})_{i=1,j=0,1}$ is also two-dimensional, this relation can be used to determine it explicitly through a straightforward exercise in linear algebra. One obtains the same result, although at a greater expense of calculational effort, from the second relation (186), upon use of the constraint (194), which serves to eliminate the second derivatives of $\mathcal{U}$.

Knowledge of the Higgs fields and the principal $\mathbb{R}^2$-connection allows us now to work out explicitly the geometric structure on $M$ given by Theorem 18. Upon substitution, the formulas (171) and (173) yield, on one hand,

\begin{equation}
\theta = -\frac{1}{2\mathcal{U}} \left[ (\mathcal{U}_{\rho_1} d\rho_1 + \mathcal{U}_{\rho_2} d\rho_2) + \lambda_1 i + \lambda_2 j + (\mathcal{U}_{\rho_1} d\rho_2 - \mathcal{U}_{\rho_2} d\rho_1) k \right].
\end{equation}

To formulate the remaining results it is useful to define the quaternionic-valued 1-form

\begin{equation}
\xi = \frac{d\rho_1 i + d\rho_2 j}{2\rho_2} + \frac{\varepsilon(v_1, d\psi) + \varepsilon(v_2, d\psi) k}{\varepsilon(v_1, v_2)}
\end{equation}

where $\varepsilon = d\psi_0 \wedge d\psi_1$ is the standard symplectic form on the $\mathbb{R}^2$-fibers and $\varepsilon(v_1, d\psi)$ are the same thing as $\iota_{v_1} \varepsilon$. The formulas (180) and (181) reduce then, on the other hand, following an extended calculation, to the simple form

\begin{equation}
s\omega = \frac{\rho_2 \varepsilon(v_1, v_2)}{\mathcal{U}^2} \xi \wedge \xi
\end{equation}

\begin{equation}
s g = \frac{\rho_2 \varepsilon(v_1, v_2)}{\mathcal{U}^2} |\xi|^2.
\end{equation}

One can easily verify that this metric coincides, up to a function redefinition, with the one found by Calderbank and Pedersen in [11] — or, rather, with its more geometrized
version (2.4) from [12]. The structure is that of an \( \mathbb{R}^2 \)-fibration over the upper half-plane with the Poincaré metric \( g_{\mathbb{H}^2} \) scaled by an overall conformal factor. In particular, this form makes it immediately clear that, as observed in [11], if we assume that \( g \) is positive definite, the sign of the scalar curvature is dictated by the sign of the factor \( \varepsilon(v_1, v_2) = \mathcal{U}(\mathcal{U}^2_{p_1} + \mathcal{U}^2_{p_2}) \).

4. Hyperkähler cones via the Legendre transform construction

4.1. The Legendre transform construction.

Translating into geometric terms insights derived from a supersymmetric quantum field-theoretic problem [35], Hitchin, Karlhede, Lindström and Roček introduced in [29] a method of constructing hyperkähler metrics of extended Gibbons-Hawking type known as the Legendre transform construction. In this approach, the metric information is stored in a single real-valued function \( L \) on some open subset \( S \) of \( \mathbb{R}^m \otimes \mathbb{R}^3 \), satisfying the second-order differential constraints

\[
\bar{\partial}_I \cdot \partial_J L = 0 \quad \text{and} \quad \bar{\partial}_I \times \partial_J L = 0.
\]

The function \( L \) is associated to a choice of direction in \( \mathbb{R}^3 \), which we take here to be the direction of the \( i \)-axis. Accordingly, it is convenient to view the space \( \mathbb{R}^m \otimes \mathbb{R}^3 \) as \( \mathbb{C}^m \times \mathbb{R}^m \) and work with the complex linear combinations

\[
z_I = \frac{1}{2}(x_I^1 + ix_I^3), \quad x_I^1,
\]

of the real coordinates \( x_I^i \). In terms of these, the differential constraints take the form

\[
L_{x^I} = -L_{z^I}, \quad \text{and} \quad L_{z^I} = L_{x^I}.
\]

The hyperkähler geometric structure is then extracted from this function based on the following two properties:

1) The (flipped-sign) Legendre transform

\[
k(z, \bar{z}, u, \bar{u}) = \langle L(z, \bar{z}, x) - 2\text{Im} u^i x^i \rangle_x
\]

of the function \( L \) with respect to the \( x^i \)-variables, where one assumes that the Legendre constraints \( L_{x^I} = 2\text{Im} u_I \) can be implicitly solved to give unequivocally \( x^I = x^I(z, \bar{z}, u, \bar{u}) \), yields a Kähler potential for the hyperkähler symplectic form \( \Omega_1 \) together with a corresponding complete set of holomorphic coordinates \( z^I \) and \( u_I \) with respect to the complex structure \( I_1 \).

2) At the same time, these coordinates are complex Darboux coordinates for the transversal complex symplectic form, that is,

\[
\Omega_+ \equiv \frac{1}{2}(\Omega_2 + i\Omega_3) = du_I \wedge dz^I.
\]

They suffice to determine all three hyperkähler symplectic forms \( \Omega_1, \Omega_2, \Omega_3 \), and hence also the hyperkähler metric \( G \). One can verify indeed that these are of extended Gibbons-Hawking type, with

\[
U_{I,J} = -\frac{1}{2}L_{x^I} x^J,
\]

\[
A_I = \text{Im}(L_{x^I} dz^J) + d\phi_I
\]

and also \( \psi_I = \text{Re} u_I - \phi_I \), where \( \phi_I \) is an arbitrary real shift. The last relation can be combined with the Legendre constraints to give

\[
u_I = \psi_I + \phi_I + \frac{i}{2}L_{x^I}.
\]
The differential constraints \((204)\) guarantee moreover that these Higgs fields and connection 1-forms satisfy as required the extended Bogomolny equations. This construction is general, in the sense that any hyperkähler manifold of extended Gibbons-Hawking type is locally given by such a Legendre transform construction for some potential \(L\) (see Proposition 2.1 in [9]).

### 4.2. Hyperkähler cone structure conditions.

#### 4.2.1. Let us work out now how hyperkähler cone structures are described in this approach, or, more precisely, how the conditions of subsection 3.2 translate in terms of the potential \(L\).

Consider the following two complex linear combinations of the generators of the collective \(\mathbb{H}^\times/\mathbb{Z}_2\)-action on the space \(\mathbb{R}^m \otimes \mathbb{R}^3\):

\[
\begin{align*}
L_0 + iL_1 &= -(x^t \partial_{x^t} + 2\bar{z}^t \partial_{\bar{z}^t}) \\
L_2 + iL_3 &= -i(x^t \partial_{\bar{z}^t} - 2z^t \partial_{x^t}).
\end{align*}
\]

On the open subset \(S\) the differential constraints \((204)\) satisfied by \(L\) imply the relations

\[
\begin{align*}
(L_2 - iL_3)(L_{x^t}) &= (L_0 + iL_1)(iL_{x^t}) \\
(L_2 - iL_3)(iL_{x^t}) &= (L_0 + iL_1)(L_{x^t}).
\end{align*}
\]

Observe now, on one hand, that from the expression \((207)\) of the Higgs fields we have that

\[
\begin{align*}
(L_0 + iL_1)U_{ij} &= U_{ij} - \frac{1}{2} \partial_{x^t} \partial_{x^t'}[(L_0 + iL_1)(L) + L] \\
(L_2 - iL_3)U_{ij} &= \frac{i}{2} \partial_{x^t} \partial_{x^t'}[(L_0 + iL_1)(L) + L].
\end{align*}
\]

The first equality is simply an identity. To obtain the second one we have made use of the differential constraints of \(L\) as follows:

\[
\begin{align*}
(L_2 - iL_3)U_{ij} &= -\frac{1}{2}(L_2 - iL_3)L_{x^t,x^t'} = \frac{1}{2} \partial_{x^t}[(L_2 - iL_3)(L_{x^t})] \\
&\quad - L_{x^t'} x^t \quad (L_0 + iL_1)(iL_{x^t}) \\
&= \frac{i}{2} \partial_{x^t} \partial_{x^t'}[(L_0 + iL_1)(L) + L]
\end{align*}
\]

where in the last step we have used the commutation property \([L_0 + iL_1, \partial_{x^t}] = \partial_{x^t}\). Note that both equalities hold in full generality, for any hyperkähler space of extended Gibbons-Hawking type. In view of these considerations, the criterion of Proposition \([15]\) can then be equivalently rephrased as follows:

**Proposition 19.** The collective \(\mathbb{H}^\times/\mathbb{Z}_2\)-action on the base of an extended Gibbons-Hawking hyperkähler space \(N\) with potential \(L\) can be lifted to a hyperkähler cone structure on \(N\) if and only if \(L\) satisfies the condition

\[
(L_0 + iL_1)(L) + L \in \bigcap_{i,j} \left[ \ker(\partial_{x^t} \partial_{x^t'}) \cap \ker(\partial_{\bar{z}^t} \partial_{x^t}) \right].
\]

An immediate corollary is then that a sufficient set of constraints for that to happen is given by \([30]\)

\[
(L_0 + iL_1)(L) + L \in \bigcap_{i,j} \left[ \ker(\partial_{x^t} \partial_{x^t'}) \cap \ker(\partial_{\bar{z}^t} \partial_{x^t}) \right].
\]

An immediate corollary is then that a sufficient set of constraints for that to happen is given by \([30]\)

\[
L_1(L) = 0 \quad \text{and} \quad L_0(L) = -L.
\]
On another hand, from the Legendre transform approach expression for the connection 1-forms $A_I$ we obtain the following identities

\begin{align}
\iota_{L_0+iL_1} A_I + U_{IJ} (x_0' + ix_1') &= (L_0 + iL_1)(\phi_I + \frac{i}{2}L_{x_I}) \\
\iota_{L_2+iL_3} A_I + U_{IJ} (x_2' + ix_3') &= (L_2 + iL_3)(\phi_I + \frac{i}{2}L_{x_I})
\end{align}

where, of course, $x_0' = 0$. We stress that these are indeed identities, with no constraints whatsoever being needed for their derivation. In the light of formula (209) it is clear then that

**Lemma 20.** The connection 1-forms (208) satisfy the gauge-fixing condition (130) if and only if there exist shifts $\phi_I$ such that

\begin{equation}
\text{u}_I \in \ker(L_0 + iL_1) \cap \ker(L_2 + iL_3)
\end{equation}

for all values of $I$.

4.2.2. Let us assume now that the function $L$ satisfies the linear differential constraints (215). One can then check that with these conditions its Legendre transform is both invariant at collective rotations and scales with weight $-1$ under the action of $L_0$. That is to say, the Legendre transform gives us in this case not just a Kähler potential, but in fact a hyperkähler potential. Accordingly, we write

\begin{equation}
\kappa = U.
\end{equation}

As we have discussed at the end of §3.2.1 when the gauge-fixing condition (130) holds, the collective $\mathbb{H}^\times/\mathbb{Z}_2$-action on $S$ lifts trivially to the hyperkähler cone as its standard quaternionic action. In our case it can be shown that its (complexified) generators assume in holomorphic coordinates the form

\begin{align}
L_0 + iL_1 &= -2\bar{z}_I \frac{\partial}{\partial \bar{z}_I} \\
L_2 + iL_3 &= \frac{\partial U}{\partial u_I} \frac{\partial}{\partial \bar{z}_I} - \frac{\partial U}{\partial \bar{z}_I} \frac{\partial}{\partial u_I}.
\end{align}

4.2.3. So far, what we have done was simply to rephrase the description of a hyperkähler cone structure in an extended Gibbons-Hawking space in the language of the Legendre transform approach. Ultimately, though, we are interested in determining the quaternionic Kähler metric on the base of the hyperkähler cone. To achieve that, we combine the results of this and the previous section into the following procedure:

1. Given a function $L$ satisfying the differential conditions (204), usually although not necessarily in the form of a contour integral, one should first test whether this satisfies also the conditions of Proposition 19. In case it does, one is then guaranteed that the ensuing metric through the Legendre transform construction will be of hyperkähler cone type.

2. The second derivatives of $L$ with respect to the $x^I$-variables give the Higgs fields $U_{IJ}$ via the formula (207).

3. More challengingly, one must also solve the differential conditions of Lemma 20 to find shift functions $\phi_I$ for which the connection 1-forms $A_I$ satisfy the gauge-fixing condition (130). As we shall see in the next example, depending on the specificities
of the case, additional symmetry considerations might need to be taken into account as well.

4. Once the Higgs fields and appropriately gauge-fixed connection 1-forms on the hyperkähler cone are determined, one then substitutes in them the change of variables \( \tilde{\rho}^I \) for some choice of restricted configuration \( \tilde{\rho}^I \). The outcomes should be amenable to the form (158) and (161), respectively, from which one can then read off the reduced Higgs fields \( \mathcal{H}_{ij} \) and induced connection 1-forms \( \mathcal{A}_I \). This is effectively a quotienting procedure which takes us from the hyperkähler cone down to its base.

5. The Ansatz of Theorem 18 or any one of the equivalent Ansätze listed in Table 2 gives us eventually the quaternionic Kähler metric and structure on the base.

5. An example: the local c-map

As an application of this procedure and, more generally, of the ideas that we have developed in this article, we present a construction of a quaternionic Kähler metric known as the Ferrara-Sabharwal metric [21]. This metric emerged in physics from the construction known as the local c-map (sometimes called the projective or supergravity c-map) [12], a map derived from a duality of the moduli spaces of type IIA and type IIB string theories, which yields a quaternionic Kähler manifold of real dimension \( 4n \) for each projective special Kähler manifold of real dimension \( 2n - 2 \). The approach we pursue was initiated by Roček, Vafa and Vandoren in [40, 41]. The idea is as follows: the \( 4n \)-dimensional quaternionic Kähler manifold has an associated \( 4n + 4 \)-dimensional Swann bundle, which in turn, as a hyperkähler space, has a \( 4n + 6 \)-dimensional twistor space. The twistor space, these authors posit, is characterized as a complex space by a certain holomorphic gluing function (in physics terms, this is related to a projective superspace lagrangian density). The challenge is then to extract the metric from this holomorphic function by coming down first two real dimensions, to describe the geometry of the Swann bundle, and then another four real dimensions, to the quaternionic Kähler base. Here we fill in a number of critical details in the arguments of [40, 41], which were further developed in [37], and use our previous considerations to perform what these papers refer to as the superconformal quotient — i.e. the descent from the hyperkähler cone to the quaternionic Kähler base — explicitly as well as equivariantly.

5.0.1. In close analogy with the Legendre transform construction of the rigid (or affine) c-map metric, also known as the semi-flat metric, the \( L \)-potential of the Swann bundle \( N_c \) over the quaternionic Kähler image of the local c-map was shown in [40, 41] to be given by the following contour integral:

\[
(220) \quad L = \frac{1}{2\pi} \oint \frac{d\zeta}{\zeta} \left( \frac{F(\eta(\zeta))}{\eta^0(\zeta)} - \frac{\bar{F}(\eta(\zeta))}{\bar{\eta}^0(\zeta)} \right)
\]

where

- the \( \eta^I(\zeta) \) variables are the so-called tropical components of sections of an \( \mathcal{O}(2) \) line bundle over the twistor space of the Swann bundle, i.e.

\[
\eta^I(\zeta) = \frac{\zeta^I}{\zeta} + x^I - \bar{z}^I \zeta;
\]
• $F(\eta(\zeta)) \equiv F(\eta^1(\zeta), \ldots, \eta^n(\zeta))$ is a holomorphic function called prepotential, assumed to be homogeneous of degree two in its variables;

• the integration contour wraps around the two roots of $\eta^0(\zeta)$: anti-clockwise around $\zeta_0^+$ for the first term and clockwise around $\zeta_0^-$ for the second, where

$$
\zeta_0^\pm = \frac{x^0 \pm r^0}{2z^0} = \frac{1}{x^0 \pm r^0},
$$

with $r^0 = |\vec{x}^0|$, correspond to antipodally opposite points on the twistor Riemann sphere.

The integrand can be understood as a twistor space holomorphic symplectic gluing function \[29\]. Its holomorphic dependence on $O(2)$ sections guarantees that the contour integral satisfies automatically the generalized Laplace equations \[204\]. Moreover, the specific manner in which it depends on these, namely, the fact that it only depends on $\eta^i(\zeta)$ implicitly through the variables $\eta^i(\zeta)$ and the fact that it scales with weight 1 at a scaling of the $\eta^i(\zeta)$’s implies that the integral satisfies the two constraints \[215\], respectively, and it is thus indeed, as claimed, the $L$-potential of a hyperkähler cone.

The contour integral can be computed explicitly with Cauchy’s residue theorem. Using the homogeneity property of the prepotential we obtain

$$
L = 2r^0 \text{Im} F(\chi)
$$

where, by definition,

$$
\chi^A = \frac{\eta^A(\zeta_0^+)}{r^0} \quad \text{and} \quad \bar{\chi}^A = \frac{\eta^A(\zeta_0^-)}{r^0}.
$$

Note that the twistor sections $\zeta \mapsto \eta^A(\zeta)$ take antipodally conjugated variables to complex conjugated ones. Here and throughout this section we use the following index notation conventions:

$$
\begin{array}{c|c}
\text{Indices Range} \\
I, J, \ldots & 0, \ldots, n \\
A, B, \ldots & 1, \ldots, n
\end{array}
$$

5.0.2. Before we continue, we pause for a moment to enumerate for future use and reference a number of important properties of these variables. First, direct evaluation gives us the expression

$$
\chi^A = \frac{x^A}{r^0} - \frac{x^0}{r^0} \text{Re} \frac{z^A}{z^0} - i \text{Im} \frac{z^A}{z^0}.
$$

This can be equivalently recast in the vectorial form

$$
\chi^A = \frac{(\vec{x}^0 \times \vec{x}^A) \cdot (\vec{x}^0 \times \vec{i}) - i |\vec{x}^0| |\vec{x}^0| \cdot (\vec{x}^A \times \vec{i})}{|\vec{x}^0| |\vec{x}^0| \cdot |\vec{i}|^2}.
$$

Proceeding in the same vein, we can then show that the following two properties hold:

$$
\chi^A \bar{\chi}^B = \frac{(\vec{x}^0 \times \vec{x}^A) \cdot (\vec{x}^0 \times \vec{x}^B) - i |\vec{x}^0| |\vec{x}^0| \cdot (\vec{x}^A \times \vec{x}^B)}{|\vec{x}^0| |\vec{x}^0| \cdot |\vec{i}|^2}
$$

and

$$
\chi^A \chi^B = \frac{(\vec{x}^0 \times \vec{x}^A) \cdot (\vec{x}^0 \times \vec{x}^B) - i |\vec{x}^0| |\vec{x}^0| \cdot (\vec{x}^A \times \vec{x}^B)}{|\vec{x}^0| |\vec{x}^0| \cdot |\vec{x}^0| \times |\vec{x}^0|^2}.
$$
The last one, in particular, makes it apparent that ratios of $\chi^A$s are invariant under collective transformations and thus descend naturally to functions on the base of the hyperkähler cone.

Also of use will be a differentiation property. In order to present it in a concise manner it will be convenient to introduce a number of temporarily notations. Thus, if we denote the parameters of the $\mathcal{O}(2)$-bundle sections by 
\[
\eta^I_{-1} = z^I, \quad \eta^I_0 = x^I, \quad \eta^I_1 = -\bar{z}^I
\]
such that we have
\[
\eta^I(\zeta) = \sum_{m=-1}^1 \eta^I_m \zeta^{-m} \quad \text{(this is sometimes called the complex spherical basis),}
\]
then in terms of these the various derivatives of $\chi^A$ can be summed up in the formula
\[
(228) \quad d\chi^A = \frac{1}{r^0} \sum_{m=-1}^1 (\zeta_+^0)^{-m} \left[ d\eta^A_m + \left( m\chi^A - \frac{\bar{x}^0 \cdot \bar{x}^A}{\bar{x}^0 \cdot \bar{x}^0} \right) d\eta^A_0 \right].
\]

In view of the imminent application of Lemma 20 it would be useful to have a list of elementary functions which belong to the intersection of kernels from the condition (217). And indeed, such a list exists and reads as follows:
\[
(229) \quad \left\{ \psi_0, \psi_A, \frac{z^A}{z^0}, \frac{x^0 + r^0}{2r^0} \chi^A, \frac{x^0 - r^0}{2r^0} \bar{\chi}^A \right\} \subset \ker(L_0 + iL_1) \cap \ker(L_2 + iL_3).
\]

For the first three elements membership in the list follows immediately from a cursory survey of the formulas (210). For the last two, however, this is a little less obvious. One can verify it directly, which is rather tedious, or by observing successively that
\[
(230) \quad (L_0 + iL_1)(\zeta^0_{\pm}) = \zeta^0_{\pm}, \quad (L_2 + iL_3)(\zeta^0_{\pm}) = i(\zeta^0_{\pm})^2
\]
and, using these facts in the definitions (222), that
\[
(231) \quad (L_0 + iL_1)(\chi^A) = 0, \quad (L_2 + iL_3)(\chi^A) = i\zeta^0_{\pm} \chi^A.
\]

The final stretch in the verification can then be covered by a very simple calculation.

Lastly, let us also make a note of the following identity:
\[
(232) \quad \frac{z^A}{z^0} + \frac{x^0 + r^0}{2r^0} \chi^A + \frac{x^0 - r^0}{2r^0} \bar{\chi}^A = \frac{\bar{x}^0 \cdot \bar{x}^A}{\bar{x}^0 \cdot \bar{x}^0}.
\]

The manifestly invariant expression on the right-hand side, the same one which shows up in the differentiation formula (228), will play an important role in the considerations to come.

5.0.3. In order to compute the Legendre transform of $L$ one needs to compute first its partial derivatives with respect to the variables $x^i$. The components with $m = 0$ of the differentiation formula (228) give us promptly, on one hand,
\[
(233) \quad L_{x^A} = 2 \Im F_A(\chi)
\]
and on the other,
\[
(234) \quad (\bar{x}^0 \cdot \bar{x}^I) L_{x^I} = x^0 L.
\]

This last relation can be used to determine $L_{x^0}$. With the help of these formulas one can then show that the Legendre transform of $L$ yields the hyperkähler potential
\[
(235) \quad U = \frac{4|z^0|^2}{r^0} \Im [\bar{x}^A F_A(\chi)].
\]
In addition, by using the homogeneity property of the prepotential to replace \( F_A(\chi) \) with \( F_{AB}(\chi)\chi^B \), and then the formula (226), we get also the alternative expression

\[
U = -\frac{(\bar{x}^0 \times \bar{x}^4) \cdot (\bar{x}^0 \times \bar{x}^B)}{|\bar{x}^0|^3} \text{Im} F_{AB}
\]

where it is understood that \( F_{AB} \) refers to \( F_{AB}(\chi) \).

One calls dualization the result of replacing the prepotential function \( F \) in the definition of \( L \) with (minus) its Legendre transform. In the ensuing formulas, this operation results in the replacement of \( F(\chi) \) with \( F(\chi) - \chi^A F_A(\chi) \), of \( \chi^A \) with \( -\chi^A \); similar statements hold also for the complex conjugated quantities. On the other hand, the 0-indexed variables such as \( x^0, z^0, \bar{z}^0 \) are inert under dualization.

The first expression above for the hyperkähler potential is the one best suited for an analysis of its dualization properties. From it, one can see immediately that the hyperkähler potential is self-dual. This strongly suggests that one might be able to lift dualization to a symmetry of the metric, which, as we shall see, will indeed be the case.

The second expression for the hyperkähler potential brings instead to the fore its collective transformation properties. Note that since the function \( F_{AB}(\chi) \) is homogeneous of degree zero, and since ratios of \( \chi^A \)'s are invariant under collective transformations, it follows that so is \( F_{AB}(\chi) \). It is then apparent that the hyperkähler potential is, as required, invariant under collective rotations and scales with weight \(-1\) under the action of \( L_0 \).

5.0.4. Following [37], let us consider on \( N_c \) the vector fields

\[
\begin{align*}
Q^A &= \frac{\partial}{\partial u_A} + \text{c.c.} \\
P_A &= z^0 \frac{\partial}{\partial z^A} - u_A \frac{\partial}{\partial u_0} + \text{c.c.} \\
W &= 2z^0 \frac{\partial}{\partial z_0} + z^A \frac{\partial}{\partial z^A} - 2u_0 \frac{\partial}{\partial u_0} - u_A \frac{\partial}{\partial u_A} + \text{c.c.}
\end{align*}
\]

where c.c. means the complex conjugate of the preceding expression. They are manifestly real-holomorphic with respect to the hyperkähler complex structure \( I_1 \) and, moreover, satisfy the following graded Heisenberg algebra:

\[
\begin{align*}
[P_A, Q^B] &= \delta^B_A I \\
[P_A, P_B] &= 0 \\
[P_A, Q^A] &= Q^A \\
[W, P_A] &= P_A \\
[W, Q^A] &= Q^A \\
[W, I] &= 2I.
\end{align*}
\]

Observe now that, out of these, the vector fields \( Q^A \) and \( I \) are entirely vertical with respect to the \( \mathbb{R}^{n+1} \)-fibration structure. On the other hand, the vector fields \( P_A \) and \( W \) have non-trivial horizontal components given by the horizontal lifts of the vector fields

\[
\begin{align*}
\mathcal{P}_A &= \bar{x}^0 \cdot \frac{\partial}{\partial \bar{x}^A} \\
W &= 2\bar{x}^0 \cdot \frac{\partial}{\partial \bar{x}^0} + \bar{x}^A \cdot \frac{\partial}{\partial \bar{x}^A}
\end{align*}
\]

living on the base of the \( \mathbb{R}^{n+1} \)-fibration. Note incidentally that by resorting to the contour-integral representation of \( L \) one can easily see that we have

\[
\begin{align*}
\mathcal{P}_A(L) &= 0 \\
W(L) &= 0.
\end{align*}
\]
Furthermore, from the definition of the $\chi^A$-variables we get successively:

\begin{align*}
P_A(\chi^B) &= P_A(\chi^B) = P_A(y^0(\zeta_+^0)) = \frac{\delta^B_A y^0(\zeta_+^0)}{r^0} = 0 \\
W(\chi^A) &= W(\chi^A) = -\chi^A.
\end{align*}

Corresponding statements hold for the complex conjugated variables as well.

In light of these facts, one can then readily infer using the expression (235) for the hyperkähler potential that this is preserved by all the generators of the graded Heisenberg algebra:

\begin{align*}
Q^A(U) &= 0 \\
I(U) &= 0 \\
P_A(U) &= 0 \\
W(U) &= 0.
\end{align*}

Let us recall now the following general result:

**Lemma 21.** A real holomorphic vector field on a Kähler manifold preserves the Kähler form if and only if it preserves locally Kähler potentials up to pluriharmonic functions.

A hyperkähler manifold can be viewed as a Kähler manifold with respect to any one of its hyperkähler complex structures. In our case, each one of the vector fields $Q^A$, $P_A$, $I$ and $W$ satisfies the conditions of this Lemma when one regards the hyperkähler space $N_c$ as a Kähler space with respect to the complex structure $I_1$, with Kähler form $\Omega_1$. Indeed, they are real holomorphic with respect to $I_1$ and, as we have shown, preserve the hyperkähler potential, which is in particular a Kähler potential for $\Omega_1$. By the Lemma, they all must then preserve as well the Kähler form $\Omega_1$. Moreover, a straightforward check using their defining formulas shows that they preserve also both the transversal symplectic form (206) and its complex conjugate. We are then entitled to conclude that $Q^A$, $P_A$, $I$ and $W$ are all tri-Hamiltonian vector fields.

What is more, they also commute with the generators of the standard quaternionic action on the hyperkähler cone. This can be seen using the holomorphic coordinate expressions (237) and (219), while taking also into account, once again, the invariance properties (242) of the hyperkähler potential function $U$. By Theorem 14 they descend therefore to the quaternionic Kähler base.

These considerations suggest that in holomorphic coordinates the dualization map takes the form

\begin{align*}
\tilde{u}^A &= \frac{z^A}{z^0} \\
\tilde{\zeta}_A &= -z^0 u_A \\
\tilde{u}_0 &= u_0 + \frac{z^A}{z^0} u_A \\
\tilde{\zeta}_0 &= z^0.
\end{align*}

Indeed, this represents a holomorphic symplectomorphism with respect to the complex symplectic form $\Omega_+$, which preserves moreover the graded Heisenberg algebra by mapping $Q^A$ to $P_A$ and the latter to $-Q^A$ while keeping $I$ and $W$ unchanged. Notice also that the double dual of a non-zero-indexed coordinate returns minus that coordinate, whereas the double dual of a zero-indexed one returns it trivially, without a sign change.

5.0.5. In order to be able to compute explicitly the connection 1-forms $A_I$ we need to choose a set of shift functions $\phi_I$ for which the conditions of Lemma 20 are satisfied. This is perhaps the most difficult step in the construction of a hyperkähler cone in the Legendre transform approach. Fortunately, in our case we can use the duality symmetry as a guide. In other words, rather than solve these conditions directly, in what follows we shall exploit...
duality to find a convenient set of such functions, and then verify *a posteriori* that they provide indeed a solution to the required conditions.

Note, to begin with, that in view of the expression \[(233)\] the Legendre transform construction formula \[(209)\] takes in this case for non-zero values of the index the form

\[(244)\]

\[u_A = \psi_A + \phi_A + i \text{Im} F_A(\chi).\]

Dualization then yields

\[(245)\]

\[\tilde{\psi}^A + \tilde{\phi}^A = \text{Re} \frac{\epsilon^A}{\epsilon^0} = \frac{\tilde{x}^0 \cdot \tilde{x}^A}{\tilde{x}^0 \cdot \tilde{x}^0} = \frac{x^0}{r^0} \text{Re} \chi^A\]

where the second equality is the real part of the identity \[(232)\]. This shows that if we take

\[(246)\]

\[\phi_A = \frac{x^0}{r^0} \text{Re} F_A(\chi)\]

such that \[\tilde{\phi}^A = -\frac{x^0}{r^0} \text{Re} \chi^A\]

then we get \[37\]

\[(247)\]

\[\tilde{\psi}^A = \frac{x^0 \cdot \tilde{x}^A}{\tilde{x}^0 \cdot \tilde{x}^0}\]

which is manifestly invariant under collective transformations, in agreement with the expectation that dualization should commute with the hyperkähler cone quaternionic action.

For this choice the initial formula becomes

\[(248)\]

\[u_A = \psi_A + \frac{x^0 + x^0}{2r^0} F_A(\chi) + \frac{x^0 - x^0}{2r^0} \bar{F}_A(\bar{\chi})\]

that is, precisely the dual of the identity \[(232)\]. Since the functions \(F_A\) are holomorphic and homogeneous of degree 1, then in light of the list \[(229)\] it is clear that these \(u_A\)'s satisfy the condition of Lemma \[20\]. This check validates the above choice of shift functions \(\phi_A\).

On another hand, let us observe that the relation \[(234)\] can be equivalently stated as

\[(249)\]

\[\text{Im}(u_0 + \tilde{\psi}^A u_A) = \frac{x^0}{2(r^0)^2} L.\]

By noticing then that the function \(L\) is anti-self-dual, *i.e.* that \(\tilde{L} = -L\), we can infer from this that \[\text{Im}(u_0 + \bar{u}_0 + \tilde{\psi}^A u_A - \psi_A \bar{u}^A) = 0.\] This suggests that we choose a shift \(\phi_0\) by setting

\[(250)\]

\[\psi_0 = \frac{1}{2}(u_0 + \bar{u}_0 + \tilde{\psi}^A u_A - \psi_A \bar{u}^A)\]

which is thus real and, moreover, explicitly self-dual. This is the same as having \[37\]

\[(251)\]

\[u_0 = \psi_0 + \frac{1}{2}(\psi_A \bar{u}^A - \tilde{\psi}^A u_A) - \frac{1}{2} u_A \bar{u}^A.\]

Each elementary component of the expression on the right-hand side belongs to the set \(\ker(L_0 + i L_1) \cap \ker(L_2 + i L_3)\), which then by way of the Leibniz rule implies that the same is true for \(u_0\). From this formula one can read off the shift \(\phi_0\), which can then be shown to satisfy

\[(252)\]

\[\phi_0 + \tilde{\psi}^A \phi_A = \frac{1}{2} \left(\frac{x^0}{r^0}\right)^2 \text{Re} \chi^A \text{Re} F_A(\chi) - \frac{1}{2} \text{Im} \chi^A \text{Im} F_A(\chi) - \frac{1}{2} \tilde{\psi}^A \psi_A.\]

A rather unusual feature of this choice of shift, imposed on us by the requirements of manifest duality, is that it has an explicit linear dependence on the \(\psi_A\)-variables.
5.0.6. So far we have used two coordinate systems on \( N \): the extended Gibbons-Hawking coordinates \( \psi_I \) and \( \vec{x}^I \) (with a closely related complexified variant given by \( \psi_I, x^I, z^I, \bar{z}^I \)), and the Legendre transform-related holomorphic coordinates \( u_I \) and \( z_I \) (with a closely related complexified variant given by \( \psi_I, x_I, z_I, \bar{z}_I \)). The first system is adapted to the \( \mathbb{R}^{n+1} \)-fibration structure. The second one is adapted to the transversal holomorphic symplectic structure in complex structure \( I_1 \). For the current construction, however, it is useful to introduce yet a third system of coordinates, adapted to its duality symmetry, given by

\[
\{ \psi_0, x^0, z^0, \bar{z}^0, \psi_A, \tilde{\psi}^A, \chi^A, \bar{\chi}^A \}.
\]

The generators of the Heisenberg algebra take in these coordinates the form

\[
Q^A = \frac{\partial}{\partial \psi_A} + \frac{1}{2} \tilde{\psi}^A \frac{\partial}{\partial \psi_0}, \quad P_A = \frac{\partial}{\partial \psi^A} - \frac{1}{2} \psi_A \frac{\partial}{\partial \psi_0}, \quad I = \frac{\partial}{\partial \psi_0}
\]

and, moreover, the grading generator \( W \) scales the non-zero-indexed coordinates with weight \(-1\) and the zero-indexed ones with weight \(2\), except for \( \psi_0 \), which it scales with weight \(-2\).

The advantage of this coordinate system comes into play when one tries to compute the connection 1-forms \( A_I \). For the shift choices \((246)\) and \((252)\), from the Legendre transform formula \((208)\) we obtain following a rather laborious computation the relatively simple expressions

\[
A_A = \text{Re} F_{AB} \tilde{\psi}^B - \text{Im} F_{AB} \left[ \text{Im} \chi^B dx^0_\phi + 4 \text{Re} \chi^B \text{Im} (z^0_\phi d\bar{z}_\phi^0) \right]
\]

\[
A_0 + \tilde{\psi}^A A_A = 2 \text{Im} F_{AB} \left[ |z_\phi^0|^2 \text{Im} (\bar{\chi}^A d\chi^B) + \bar{\chi}^A \chi^B x^0_\phi \text{Im} (z^0_\phi d\bar{z}_\phi^0) \right] - \frac{1}{2} d(\tilde{\psi}^A \psi_A)
\]

where, for simplicity, we have used the notations \( x^0_\phi = x^0/r^0 \) and \( z^0_\phi = z^0/r^0 \).

5.0.7. The Higgs fields, on the other hand, can be computed with significantly less effort. By taking derivatives with respect to \( x^A \) and \( x^0 \) of the equation \((234)\) we obtain, respectively,

\[
(x^0 \cdot \vec{x}^I) L_{x^I x^A} = 0
\]

\[
(x^0 \cdot \vec{x}^I) L_{x^I x^0} = U
\]

(recall the index notation convention \((223)\)). Moreover, from the preceding expression for the derivative \( L_{x^A} \) we can easily compute the second derivative \( L_{x^A x^B} \). Together with the relations above this gives us

\[
U_{IJ} = -\frac{1}{r^0} \begin{pmatrix} R + \tilde{\psi}^C \text{Im} F_{CD} \tilde{\psi}^D & -\tilde{\psi}^C \text{Im} F_{CB} \\ -\text{Im} F_{AD} \tilde{\psi}^D & \text{Im} F_{AB} \end{pmatrix}
\]

where, by definition,

\[
R = \frac{U}{2r^0}.
\]
We are now ready to perform the reduction and descend from the hyperkähler cone down to its quaternionic Kähler base. In practice, this step consists of substituting into the formulas for the Higgs fields \( U_{IJ} \) and connection 1-forms \( A_I \) the change of variables (139) for some choice of restricted configuration with position vectors \( \vec{\rho}_I \), and then reading off from the result the reduced Higgs fields \( \mathcal{U}_{IJ} \) and reduced connection 1-forms \( \mathcal{A}_I \).

In the case of the Higgs fields, this is rather straightforward. Recall, on one hand, that the variables \( \tilde{\psi}^A \) and, as argued before, the functions \( F_{AB}(\chi) \) are invariant at collective transformations, and hence at the particular collective transformation represented by the equation (139). On another hand, from the multiplicative property of the quaternionic norm we have \( r^0 \equiv |\vec{x}^0| = |q| |\vec{\rho}^0| \), and from the representation (236) for the hyperkähler potential, \( U = |q|^2 \mathcal{U} \). These last two properties entail in particular that \( R \) is also invariant at collective transformations, and so, like \( \tilde{\psi}^A \) and \( F_{AB}(\chi) \), descends naturally on the base of the hyperkähler cone. It is obvious then that if we substitute into the above Higgs field matrix the relation (139) we find as expected from general arguments that this is of the form (158), with the reduced Higgs field matrix given by

\[
\mathcal{U}_{IJ} = -\frac{1}{|\vec{\rho}^0|} \begin{pmatrix}
R + \tilde{\psi}N\tilde{\psi} & -(\tilde{\psi}N)_B \\
-(N\tilde{\psi})_A & N_{AB}
\end{pmatrix}.
\]

Aside from the overall factor, this matrix is precisely the same as the one in the equation (257); however, here we have introduced the shorthand notations \( N_{AB} = \Im F_{AB} \), \( (N\tilde{\psi})_A = N_{AB}\tilde{\psi}^B \), a.s.o. Let us also record here for subsequent use that the inverse of this matrix is

\[
\mathcal{U}^{IJ} = -\frac{|\vec{\rho}^0|}{R} \begin{pmatrix}
\tilde{\psi}^B \\
\tilde{\psi}^A + \tilde{\psi}^B N_{AB}
\end{pmatrix}
\]

with \( N_{AB} \) representing the matrix inverse of \( N_{AB} \).

In the case of the connection 1-forms \( A_I \), if we consider a restricted configuration with \( \vec{\rho}^0 = \text{constant} \) then the change of variables (139) yields, after a rather more strenuous calculation, a result of the form (161), with the induced connection 1-forms given by

\[
\mathcal{A}_A = \Re F_{AB} d\tilde{\psi}^B
\]

\[
\mathcal{A}_0 + \tilde{\psi}^A \mathcal{A}_A = \frac{1}{2} \Im F_{AB} \frac{\vec{\rho}^0 \cdot (\vec{\rho}^A \times d\vec{\rho}^B)}{|\vec{\rho}^0|^3} - \frac{1}{2} d(\tilde{\psi}^A \psi_A).
\]

5.0.9. Let us specialize now further to the particular choice of restricted configuration considered in (138). For this we then have, via the formulas (247) and (227),

\[
\tilde{\psi}^A = \rho_1^A \quad \text{and} \quad \chi^A = \frac{\rho_2^A + i\rho_3^A}{\rho_2^A + i\rho_3^A}
\]

suggesting we should define also the variables

\[
X^A = \frac{1}{2}(\rho_1^A + i\rho_3^A).
\]

These are all complex, with the exception of \( X^1 \), which is real and positive. The reality condition is important for the coordinate count: in what follows we will coordinatize the
quaternionic Kähler space by means of the variables $\psi_0$, $\psi_A$, $\bar{\psi}^A$ and $X^A$, which would not be possible if the total number of their real components were not equal to $4n$.

Note that since the function $F_{AB}$ is homogeneous of degree zero one can argue that $F_{AB}(\chi) = F_{AB}(X)$, which then allows us to pass from the variables $\chi^A$ to the $X^A$ ones. With obvious notations, the hyperkähler potential formula (236) gives us in this case

$$R = -2(\bar{X}N) = -2\text{Im}[\bar{X}^AF_A(X)].$$

This satisfies, incidentally, the differential identity

$$\frac{dR}{2R} = \frac{\text{Im}(\bar{X}NdX)}{(XNX)}$$

a consequence of the holomorphicity property of the prepotential.

5.0.10. Knowledge of the reduced Higgs fields and connection 1-forms allows us to finally compute the quaternionic Kähler metric explicitly. To this end, observe first that from the expression (173) for the $SO(3)$ connection 1-forms and with the above definitions we obtain in this case

$$\theta_0 = -\frac{dR}{2R},$$

$$\theta_1 = -\frac{1}{2R}[\alpha + \text{Re}(X^AdF_A(X) - F_A(X)d\bar{X}^A)],$$

$$\theta_+ \equiv \frac{1}{2}(\theta_2 + i\theta_3) = -\frac{1}{2R}[X^Ad\psi_A + F_A(X)d\bar{\psi}^A]$$

where we have used the notation

$$\alpha = d\psi_0 + \frac{1}{2}(\bar{\psi}^Ad\psi_A - \psi_Ad\bar{\psi}^A).$$

The metric itself can be most conveniently worked out from the version (170) of the metric Ansatz, in which one can make use of the above formulas for the $SO(3)$ connection 1-forms. In the end, using among other things the identity (266), we arrive as predicted in [40, 41] at the Ferrara-Sabharwal metric [21]

$$2sg = g_{PSK} - \left(\frac{dR}{2R}\right)^2 - \frac{g_T}{R} - \frac{\alpha^2}{R^2}$$

where

$$g_{PSK} = \frac{\bar{X}NX(d\bar{X}NdX) - (d\bar{X}NX)(\bar{X}NdX)}{(XNX)^2}$$

is the so-called projective special Kähler metric and

$$g_T = \frac{1}{2}(\text{Im}\tau)^{-1AB}(d\psi_A + \tau_{AC}d\bar{\psi}^C)(d\psi_B + \bar{\tau}_{BD}d\bar{\psi}^D)$$

is a complex n-torus metric with period matrix [19]

$$\tau_{AB} = F_{AB}(X) - 2i\frac{(N\bar{X})_A(N\bar{X})_B}{(XNX)}.$$

Under a duality transformation this undergoes the modular transformation $\tau_{AB} \mapsto -(\tau^{-1})^{AB}$ and, furthermore, we have

$$\text{Im}\tau^{-1AB} = N^{AB} - \frac{\bar{X}^AX^B + X^A\bar{X}^B}{(XNX)}.$$
5.0.11. To make the junction with the usual formulation found in the literature we need to perform one final change of coordinates. Letting

\[ Z^A = \frac{X^A}{X^1} \]

we replace the subset of coordinates \( \{X^A\}_{A=1,\ldots,n} \) with the complex inhomogeneous coordinates \( \{Z^A\} \) \( A=2,\ldots,n \) plus \( R \). The projective special Kähler metric \( g_{PSK} \) can be understood then as the Kähler metric corresponding to the Kähler potential

\[ \mathcal{K} = \ln \text{Im} [ \bar{Z}^A F_A(Z) ] = \ln(\bar{Z}N) \]

with \( Z^A \) as complex holomorphic coordinates. (A more geometric definition of projective special Kähler metrics can be found in [22, 3].) The formulas are straightforward to re-express in the new coordinates due to their homogeneity properties. We leave the details to the reader as an exercise.

The \( 4n \)-dimensional Ferrara-Sabharwal metric has \( 2n+2 \) isometries descending from the corresponding tri-Hamiltonian isometries on the hyperkähler cone and satisfying the same graded Heisenberg algebra [235]. In the above inhomogeneous frame their generating vector fields take the form [20]

\[ \mathcal{Z}^A = \frac{\partial}{\partial \psi^A} + \frac{1}{2} \bar{\psi}^A \frac{\partial}{\partial \psi^0}, \quad \mathcal{P}_A = \frac{\partial}{\partial \psi^A} - \frac{1}{2} \psi^A \frac{\partial}{\partial \psi^0} \]

\[ \mathcal{I} = \frac{\partial}{\partial \psi^0} \]

\[ \mathcal{W} = -\psi^A \frac{\partial}{\partial \psi^A} - \bar{\psi}^A \frac{\partial}{\partial \psi^A} - 2\psi^0 \frac{\partial}{\partial \psi^0} - 2R \frac{\partial}{\partial R}. \]

Finally, let us also mention the well-known fact that if the prepotential function \( F \) is such that \( N_{AB} = \text{Im} F_{AB} \) has Lorentzian signature \( (1, n-1) \), then on the open complex domain given by the condition \( (\bar{Z}NZ) > 0 \), where we have \( R < 0 \), both \( g_{PSK} \) and \( g_T \) are negative definite (the first statement is straightforward, the second one was shown in [16] by an ingenious argument based on the so-called inverse Cauchy-Schwarz inequality), and therefore \( sg \) is negative definite. In other words, in this case one can choose the metric to be positive definite, and this metric will have a negative scalar curvature.

5.0.12. The local c-map has been studied extensively in both the physics and mathematics literature. In particular, a different and very interesting geometric construction of the local c-map, based on the so-called HK/QK correspondence, has been given in [4] (see also the preceding work of [28] and [6]). Global aspects of the Ferrara-Sabharwal metric have been considered in [15, 36], and completeness issues have been discussed in [14, 15]. Further considerations such as quantum perturbative and instanton corrections have been explored in [7, 39, 6].

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