Neveu’s Exchange Formula for Analysis of Wireless Networks With Hotspot Clusters

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Theory of point processes, in particular Palm calculus within the stationary framework, plays a fundamental role in the analysis of spatial stochastic models of wireless communication networks. Neveu’s exchange formula, which connects the respective Palm distributions for two jointly stationary point processes, is known as one of the most important results in the Palm calculus. However, its use in the analysis of wireless networks seems to be limited so far and one reason for this may be that the formula in a well-known form is based upon the Voronoi tessellation. In this paper, we present an alternative form of Neveu’s exchange formula, which does not rely on the Voronoi tessellation but includes the one as a special case. We then demonstrate that our new form of the exchange formula is useful for the analysis of wireless networks with hotspot clusters modeled using cluster point processes.

Keywords: stationary point processes, Palm calculus, Neveu’s exchange formula, cluster point processes, device-to-device networks, hotspot clusters, coverage probability, device discovery

1 INTRODUCTION

Spatial stochastic models have been widely accepted in the literature as mathematical models for the analysis of wireless communication networks, where irregular locations of wireless nodes, such as base stations (BSs) and user devices, are modeled using spatial point processes on the Euclidean plane (see, e.g., (Baccelli and Blaszczyszyn, 2009a; Baccelli and Blaszczyszyn, 2009b; Haenggi and Ganti, 2009; Haenggi, 2013; Mukherjee, 2014; Mukherjee et al., 2018) for monographs and (Andrews et al., 2016; El-Sawy et al., 2017; Haenggi et al., 2021; Hmamouche et al., 2021; Lu et al., 2021) for recent survey and tutorial articles). In such analysis of wireless networks, the theory of point processes, in particular Palm calculus within the stationary framework, plays a fundamental role. Neveu’s exchange formula, which connects the respective Palm distributions for two jointly stationary point processes, is known as one of the most important results in the Palm calculus. However, its use in the analysis of wireless networks seems to be limited so far and one reason for this may be that the formula in a well-known form is based upon the Voronoi tessellation [see, e.g., (Baccelli et al., 2020, Section 6.3)]. In this paper, we present an alternative form of Neveu’s exchange formula, which does not rely on the Voronoi tessellation but includes the one as a special case. We then demonstrate that it is useful for the analysis of spatial stochastic models based on cluster point processes.

A cluster point process represents a state such that there exist a large number of clusters consisting of multiple points and is used to model the locations of wireless nodes in an (urban) area with a number of hotspots. Indeed, many researchers have adopted the cluster point processes in their models of various wireless networks such as ad hoc networks (Ganti and Haenggi, 2009), heterogeneous networks (Chun et al., 2015; Suryaprakash et al., 2015; Saha et al., 2017, 2018; Afshang and Dhillon, 2018; Saha et al., 2019; Yang et al., 2021), device-to-device (D2D) networks...
The results of numerical experiments are also presented there. The discovered devices are derived using the exchange formula. Section 4 discusses the relations with the existing forms and examines the new form of the exchange formula. On the other hand, in the problem of device discovery, transmitting devices transmit broadcast messages and a receiving device can detect the transmitters if the decoding problem is successful. Such a problem may be within its scope. Nevertheless, we see in the rest of the section that clusters may overlap in space.

2 NEVEU’S EXCHANGE FORMULA

In this section, we discuss point processes on the $d$-dimensional Euclidean space $\mathbb{R}^d$ within the stationary framework (see, e.g., (Baccelli et al., 2020, Chapter 6) for details on the stationary framework). In what follows, $B(\mathbb{R}^d)$ denotes the Borel $\sigma$-field on $\mathbb{R}^d$ and $\delta_x$ denotes the Dirac measure with mass at $x \in \mathbb{R}^d$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space. On $(\Omega, \mathcal{F})$, a flow $\{\theta_t\}_{t \in \mathbb{R}_+}$ is defined such that $\theta_t \colon \Omega \to \Omega$ is $\mathcal{F}$-measurable and bijective satisfying $\theta_t \cdot \theta_s = \theta_{t+s}$ for $t, s \in \mathbb{R}_+$, where $\theta_0$ is the identity for $0 = (0, 0, \ldots, 0) \in \mathbb{R}^d$, so that $\theta^{-1}_t = \theta_{-t}$ for $t \in \mathbb{R}_+$. We assume that the probability measure $\mathbb{P}$ is invariant to the flow $\{\theta_t\}_{t \in \mathbb{R}_+}$ (in other words, $\{\theta_t\}_{t \in \mathbb{R}_+} \mathbb{P}$-preserves $\mathcal{P}$) in the sense that $\mathbb{P} \ast \theta_t^{-1} = \mathbb{P}$ for any $t \in \mathbb{R}_+$, where $\theta^{-1}_t(A) = \{\omega \in \Omega \colon \theta_t(\omega) \in A\}$ for $A \in \mathcal{F}$. A point process $\Phi = \sum_{\omega \in \Omega} \delta_{\omega}$ on $\mathbb{R}^d$ is said to be compatible with the flow $\{\theta_t\}_{t \in \mathbb{R}_+}$ if it holds that $\Phi(B) \ast \theta_t = \Phi(\theta_t(B)) = \Phi(B + t) = \Phi(B + t)$ for $\omega \in \Omega$, $B \in B(\mathbb{R}^d)$ and $t \in \mathbb{R}_+$, which is, for $t \in \mathbb{R}_+$ and $n \in \mathbb{N}$, there exists an $n' \in \mathbb{N}$ such that $X_n \ast \theta_t = X_{n'} - t$. Under the assumption of the $\{\theta_t\}_{t \in \mathbb{R}_+}$-invariance of $\mathbb{P}$, a point process $\Phi$ compatible with $\{\theta_t\}_{t \in \mathbb{R}_+}$ is stationary in $\mathbb{P}$ and furthermore, two point processes $\Phi$ and $\Psi$, both of which are compatible with $\{\theta_t\}_{t \in \mathbb{R}_+}$, are jointly stationary in $\mathbb{P}$.

Let $\Phi = \sum_{\omega \in \Omega} \delta_{\omega}$ and $\Psi = \sum_{n \in \mathbb{N}} \delta_{Y_n}$ denote point processes on $\mathbb{R}^d$, which are both simple, compatible with $\{\theta_t\}_{t \in \mathbb{R}_+}$ and have positive and finite intensities $\lambda_\Phi$ and $\lambda_\Psi$, respectively. Thus, $\Phi$ and $\Psi$ are jointly stationary in probability $\mathbb{P}$ and the respective Palm probabilities $\mathbb{P}_{\Phi}^\omega$ and $\mathbb{P}_{\Psi}^\omega$ are well-defined. Note that $\mathbb{P}_{\Phi}^\omega(\Phi(\{0\}) = 1) = \mathbb{P}_{\Psi}^\omega(\Psi(\{0\}) = 1) = 1$. In this paper, when we consider the event $\{\Phi(\{0\}) = 1\} \in \mathcal{F}$, we assign index 0 to the point at the origin; that is, $X_0 = 0$ on $\{\Phi(\{0\}) = 1\}$, and this is also the case for $\Psi$; that is, $Y_0 = 0$ on $\{\Psi(\{0\}) = 1\}$. To present an alternative form of Neveu’s exchange formula, we introduce a family of shift operators $S_n$, $t \in \mathbb{R}^d$, on the set of measures $\eta$ on $(\mathbb{R}^d, B(\mathbb{R}^d))$ by $S_n \eta(B) = \eta(B + t)$ for $B \in B(\mathbb{R}^d)$. For example, operating $S_1$ on the point process $\Psi = \sum_{n \in \mathbb{N}} \delta_{Y_n}$, we have $S_1 \Psi = \sum_{n \in \mathbb{N}} \delta_{Y_n - t} = \Psi \ast \theta_t$. The shift operators $S_n$, $t \in \mathbb{R}^d$, also work on a function $h$ on $\mathbb{R}^d$ such as $S_n h(x) = h(x + t)$ for $x \in \mathbb{R}^d$.

Theorem 1. For the two jointly stationary point processes $\Phi = \sum_{\omega \in \Omega} \delta_{\omega}$ and $\Psi = \sum_{n \in \mathbb{N}} \delta_{Y_n}$ described above, we assume that a family of point processes $\Psi_n = \sum_{k \in \mathbb{N}} \delta_{Y_n \ast \theta_k}$, $n \in \mathbb{N}$, can be constructed such that

1) $S_{-X_n} \Psi_n = \sum_{k \in \mathbb{N}} \delta_{X_n \ast \theta_k}$, $n \in \mathbb{N}$, form a partition of $\Psi$; that is, $\Psi = \sum_{n \in \mathbb{N}} S_{-X_n} \Psi_n$.
2) $\Phi = \sum_{n \in \mathbb{N}} \delta_{(X_n \ast \theta_k)}$ is a stationary marked point process with the set of counting measures on $\mathbb{R}^d$ as its mark space.

Then, for any nonnegative random variable $W$ defined on $(\Omega, \mathcal{F})$,

$$
\lambda_\Psi E_{\mathbb{P}_{\Psi}^\omega}[W] = \lambda_\Phi E_{\mathbb{P}_{\Phi}^\omega}[\int_{\mathbb{R}^d} W \ast \theta_t \psi_0(dy)] = \lambda_\Phi E_{\mathbb{P}_{\Phi}^\omega}[\sum_{k=1}^\infty W \ast \theta_k].
$$

(1)
where $E^0_{q}$ and $E^0_{q}$ denote the expectations with respect to the Palm probabilities $P^0_{q}$ and $P^0_{q}$, respectively, and $\Psi_0 = \sum_{k=1}^{\infty} \delta_{\mathbf{y}_k}$ denotes the mark associated with the point $X_0 = 0$ on $\{\Phi(0) = 1\}$.

**Proof.** As with the proof of the exchange formula in (Baccelli et al., 2020, Theorem 6.3.7), we start our proof with the mass transport formula [see, e.g., (Baccelli et al., 2020, Theorem 6.3.7)], that is, for any measurable function $\xi: \Omega \times \mathbb{R}^d \to \mathbb{R}^d$,

$$
\lambda_{\Phi} E^0_{q} \left[ \int_{\mathbb{R}^d} \xi(y) \Psi(dy) \right] = \lambda_{\Phi} E^0_{q} \left[ \int_{\mathbb{R}^d} \xi(-x) \cdot \theta_x \Phi(dx) \right].
$$

(Eq. 2)

Let $\xi(y) = W \cdot \theta_y \Psi_0(y)$ on $\{\Phi(0) = 1\}$. Then, the left-hand side of Eq. 2 becomes

$$
\lambda_{\Phi} E^0_{q} \left[ \int_{\mathbb{R}^d} W \cdot \theta_x \Psi_0 \{x\} \Psi(dy) \right] = \lambda_{\Phi} E^0_{q} \int_{\mathbb{R}^d} W \cdot \theta_x \Psi_0(dy).
$$

On the other hand, the right-hand side of Eq. 2 is reduced to

$$
\lambda_{\Phi} E^0_{q} \left[ \int_{\mathbb{R}^d} (W \cdot \theta_x \Psi_0 \{x\}) \cdot \theta_x \Phi(dx) \right] = \lambda_{\Phi} E^0_{q} \left[ W \sum_{n=1}^{\infty} S_{X_n} \Psi_n(0) \right] = \lambda_{\Phi} E^0_{q} \left[ W \right] ,
$$

where the first equality follows from $\Psi_0 \ast \theta_{X_n} = \Psi_n$ for $n \in \mathbb{N}$ and the fact that the point of $\Psi$ at the origin on a sample $\omega \in \{\Psi(0) = 1\}$ is shifted to location $-x$ on the shifted sample $\theta_x(\omega)$ for $x \in \mathbb{R}^d$, and the last equality follows since there exists exactly one point, say $X_n$, of $\Phi$ such that its mark $\Psi_n = \sum_{k=1}^{\infty} \delta_{\mathbf{y}_k}$ has a point, say $Y_{n,k}$, satisfying $X_n + Y_{n,k} = 0$ on $\{\Psi(0) = 1\}$. The proof is completed.

Remark 1: Let $W \equiv 1$ in (Eq. 1). Then, we have $E^0_{q}[\kappa_0] = \lambda_{\Psi}/\lambda_{\Phi}$ and therefore, each $\Psi_n$, in Theorem 1 has finite points. In the form of the exchange formula in (Baccelli et al., 2020, Theorems 6.3.7 and 6.3.19), the point process $\Psi$ is partitioned by the Voronoi tessellation for $\Phi$, which corresponds to a special case of (Eq. 1) such that $S_{X_n} \Psi_n(\cdot) = \Psi_0(V_\Phi(X_n))$ for $n \in \mathbb{N}$, where $V_\Phi(X_n)$ denotes the Voronoi cell of point $X_n$ of $\Phi$. The condition in (Baccelli et al., 2020) such that there are no points of $\Psi$ on the boundaries of Voronoi cells $V_\Phi(X_n)$, $n \in \mathbb{N}$, is covered by our Condition 1 in Theorem 1, where $S_{X_n} \Psi_n, n \in \mathbb{N}$, form a partition of $\Psi$ and have no common points. On the other hand, our Theorem 1 considers only the case where the point process $\Phi$ is simple unlike (Baccelli et al., 2020, Theorem 6.3.7). However, this would be enough for applications to wireless networks and, if necessary, it could be extended to the non-simple case. Another typical example of $\Psi$ and $\Phi$ in Theorem 1 is a cluster point process and its parent process. Although we focus on a PPCP in the following sections, more general cluster point processes inherently fulfill the conditions of the theorem [see, e.g., (Baccelli et al., 2020, Section 2.3.3)]. It should also be noted that, in (Last and Thorisson, 2009; Last, 2010; Gentner and Last, 2011), a more general formula is introduced under the name of Neveu’s exchange formula, from which the mass transport formula (Eq. 2) is derived. In that sense, our form (Eq. 1) may be within its scope. Nevertheless, we can see in the following sections that Theorem 1 is valuable in the sense that it is tractable and can spread the application fields of the exchange formula.

### 3 APPLICATIONS TO CLUSTER POINT PROCESSES

In this section, we demonstrate that Neveu’s exchange formula (Eq. 1) in Theorem 1 is useful to characterize the Palm distribution of stationary cluster point processes. A cluster point process is, roughly speaking, constructed by placing point processes (usually with finite points), called offspring processes, around respective points of another point process, called a parent process, and is used to represent a state such that there exist a large number of clusters consisting of multiple points (see Figure 1). In particular, we focus here on a stationary PPCP described next.

#### 3.1 Poisson-Poisson Cluster Processes

Let $\Phi = \sum_{n=1}^{\infty} \delta_{X_n}$ denote a homogeneous PPP on $\mathbb{R}^d$, which works as the parent process, and let $\Psi_n = \sum_{k=1}^{\infty} \delta_{Y_{n,k}}$, $n \in \mathbb{N}$, denote a family of finite (therefore inhomogeneous) and mutually independent PPPs on $\mathbb{R}^d$, which are also independent of $\Phi$ and work as the offspring processes. Then, PPCP $\Psi = \sum_{n=1}^{\infty} \delta_{\mathbf{y}_n}$ is given as

$$
\Psi = \sum_{n=1}^{\infty} S_{X_n} \Psi_n = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \delta_{X_n + Y_{n,k}}.
$$

(Fig. 1)
The PPCP \( \Psi \) constructed as above is stationary since the parent process \( \Phi \) is stationary and the offspring processes \( \Psi_n, n \in \mathbb{N}, \) are independent and identically distributed [see, e.g., (Baccelli et al., 2020, Example 2.3.18)]. We assume that \( \Phi \) has a positive and finite intensity \( \lambda_\phi, \) and \( \Psi_n, n \in \mathbb{N}, \) have an identical intensity measure \( \Lambda_\phi = \mu Q, \) where \( \mu \) is a positive constant and \( Q \) is a probability distribution on \( (\mathbb{R}^d, B(\mathbb{R}^d)) \). Thus, the number of points in each offspring process follows a Poisson distribution with mean \( \mu, \) so that the intensity of \( \Psi \) is equal to \( \lambda_\Psi = \lambda_\phi \mu, \) and offspring points are scattered on \( \mathbb{R}^d \) according to \( Q \) independently of each other. We further assume that \( Q \) is diffuse; that is, \( Q(\{x\}) = 0 \) for any \( x \in \mathbb{R}^d, \) to make \( \Psi \) simple. We refer to \( S_{-X_n} \Psi_n \) in (Eq. 3) as the cluster associated with \( X_n \) for \( n \in \mathbb{N}. \) Two main examples of the PPCPs are the (modified) Thomas point process and the Matérn cluster process [see, e.g., (Chiu et al., 2013, Example 5.5)]. When \( Q \) is an isotropic normal distribution, then the obtained PPCP is called the Thomas point process. On the other hand, when \( Q \) is the uniform distribution on a fixed ball centered at the origin, then the result is called the Matérn cluster process. Note that the PPCP \( \Psi \) and its parent process \( \Phi \) fulfill the conditions of Theorem 1.

### 3.2 Characterization of Palm Distribution

For a stationary point process \( \Psi, \) let \( \psi = \Psi - \delta_0 \) on the event \( \{\Psi(\{0\}) = 1\}, \) which is referred to as the reduced Palm version of \( \Psi. \)

#### Lemma 1

For the stationary PPCP \( \Psi \) described in Section 3.1, the intensity measure of the reduced Palm version \( \Psi^0 \) (with respect to the Palm distribution) is given by

\[
\Lambda^0_\Psi (B) = \mathbb{E}^0_\Psi \left[ \psi (B) \right] = \lambda_\phi \mu |B| + \mu \int_{\mathbb{R}^d} Q(B - y) Q^-(dy), \quad B \in B(\mathbb{R}^d),
\]

(4)

where \(|| \) denotes the Lebesgue measure on \( (\mathbb{R}^d, B(\mathbb{R}^d)) \) and \( Q^-(B) = Q(-B) \) with \(-B = \{ -x : x \in B \} \) for \( B \in B(\mathbb{R}^d). \)

#### Proof

Since the offspring processes \( \Psi_n, n \in \mathbb{N}, \) are PPPs, the PPCP \( \Psi \) is a Cox point process; that is, once the parent process \( \Phi = \sum_{n=1}^\infty \delta_{X_n} \) is given, \( \Psi \) is conditionally an inhomogeneous PPP with a conditional intensity measure \( \mu \sum_{n=1}^\infty S_{-X_n} Q \) [see, e.g., (Baccelli et al., 2020, Example 2.3.13)]. Since the reduced Palm version of a PPP is identical in distribution to its original version (not conditioned on \( \{\Psi(\{0\}) = 1\} \)) by Slivnyak’s theorem [see, e.g., (Daley and Vere-Jones, 2008, Proposition 13.1.VII) or (Baccelli et al., 2020, Theorem 3.2.4)], we have

\[
\mathbb{E}^0_\Psi \left[ \psi (B) \right] \mid \Phi = E_\Psi (B) \mid \Phi = \mu \sum_{n=1}^\infty Q(B - X_n), \quad B \in B(\mathbb{R}^d).
\]

Taking the expectation with respect to \( \mathbb{E}^0_\Psi \) and then applying Theorem 1, we obtain

\[
\mathbb{E}^0_\Psi \left[ \psi (B) \right] = \mu \mathbb{E}^0_\Psi \left[ \sum_{n=1}^\infty Q(B - X_n) \right] = \mathbb{E}_\phi \left[ \int_{\mathbb{R}^d} \left( \sum_{n=1}^\infty Q(B - X_n) \right) \cdot \psi (dy) \right]
\]

\[
= \mathbb{E}_\phi \left[ \int_{\mathbb{R}^d} \sum_{n=1}^\infty (Q(B - x_n) + y) \psi (dy) \right] = \mu \int_{\mathbb{R}^d} Q(B + y) + \mathbb{E}_\phi \left[ \sum_{n=1}^\infty Q(B - X_n + y) \right] Q(dy),
\]

where \( \lambda_\Psi = \lambda_\phi \mu \) is used in the second equality, the third equality follows because, for any \( n \in \mathbb{N} \) and \( y \in \mathbb{R}^d, \) there exists an \( n' \in \mathbb{N} \cup \{0\} \) such that \( X_{n'} + \theta_y = X_n - y \) on \( \{\Phi(\{0\}) = 1\} \), and in the last equality, we apply Campbell’s formula [see, e.g., (Last and Penrose, 2017, Proposition 2.7) or (Baccelli et al., 2020, Theorem 1.2.5)] for \( \Psi. \) For the expectation in the last expression above, Slivnyak’s theorem, Campbell’s formula for \( \Phi \) and then Fubini’s theorem yield

\[
\mathbb{E}_\phi \left[ \sum_{n=1}^\infty Q(B - X_n + y) \right] = \lambda_\phi \int_{\mathbb{R}^d} Q(B - x) dx = \lambda_\phi \int_{\mathbb{R}^d} \int_{B-x} Q(dy) dx
\]

\[
= \lambda_\phi \int_{\mathbb{R}^d} \int_{B-x} Q(dy) dx = \lambda_\phi |B|,
\]

which completes the proof.

#### Remark 2

The second term on the right-hand side of (Eq. 4) is of course equal to \( \mu Q(B + y) Q(dy) \). We adopt the form in Lemma 1 due to its interpretability. Since \( Q \) is the distribution for the position of an offspring point viewed from its parent, \( Q^- \) represents the distribution for the location of the parent of the offspring point at the origin on the event \( \{\Psi(\{0\}) = 1\}. \) On the other hand, \( \mu Q(B - y) \) gives the expected number of offspring points falling in \( B \in B(\mathbb{R}^d) \) among a cluster whose parent is shifted to \( y \in \mathbb{R}^d. \) In other words, the second term on the right-hand side of (Eq. 4) represents the expected number of offspring points falling in \( B \) among the cluster which is given to have one point at the origin. Since the first term on the right-hand side of (Eq. 4) is equal to \( \lambda_\phi \Phi (B) = E[\Psi(B)], \) Lemma 1 states that the intensity measure for the reduced Palm version of a stationary PPCP is given as the sum of the intensity measure of the stationary version and that of a cluster which has one point at the origin. Lemma 1 is also a slight generalization of the result in (Tanaka et al., 2008, Section 2.2).

The observation in Remark 2 is further enhanced by the following proposition.

#### Proposition 1

For the stationary PPCP \( \Psi = \sum_{n=1}^\infty \delta_{Y_n}, \) described in Section 3.1, the generating functional of the reduced Palm version \( \Psi^0 \) (with respect to the Palm distribution) is given by

\[
G_\Psi (h) := \mathbb{E}^0_\Psi \left[ \prod_{m=1}^n h(Y_m) \right] = G_\Psi (h) \int_{\mathbb{R}^d} \tilde{h}(z) Q^-(dz),
\]

(5)

for any measurable function \( h: \mathbb{R}^d \to [0, 1], \) where \( G_\Psi \) is the generating functional of the stationary version of \( \Psi \) given by

\[
G_\Psi (h) = \mathbb{E} \left[ \sum_{m=1}^\infty h(Y_m) \right] = G_\Psi (\tilde{h}) = \exp \left( -\lambda_\phi \int_{\mathbb{R}^d} [1 - \tilde{h}(x)] dx \right),
\]

(6)
and \( \hat{h}(x) \) denotes the generating functional of an offspring process \( \Psi_1 \) whose parent is shifted to \( x \in \mathbb{R}^d \):

\[
\hat{h}(x) = \mathcal{G}_\Psi(S,h) = \exp\left(-\mu \int_{\mathbb{R}^d} (1 - h(x + y)) Q(dy)\right), \tag{7}
\]

Note that in Proposition 1 above, \( \mathcal{G}_\Phi \) is the generating functional of the parent process \( \Phi \). The relation \( \mathcal{G}_\Psi(h) = \mathcal{G}_\Phi(h) \) with \( h(x) = \mathcal{G}_\Psi(S,h) \) in Eqs 6, 7 is known to hold for more general cluster point processes [see, e.g., (Daley and Vere-Jones, 2003, Example 6.3(a)) or (Baccelli et al., 2020, Proposition 2.3.12 and Lemma 2.3.20)], whereas the last equalities in Eqs 6, 7 follow because \( \Phi \) and \( \Psi_1 \) are PPPs, respectively (see, e.g., (Last and Penrose, 2017, Exercise 3.6), or (Baccelli et al., 2020, Corollary 2.1.5)). The relation (Eq. 5) is derived in (Ganti and Haenggi, 2009, Lemma 1), to which we give another proof using the exchange formula in Theorem 1.

**Proof.** As stated in the proof of Lemma 1, once the parent process \( \Phi = \sum_{m=1}^{\infty} \delta_{X_m} \) is given, the PPCP \( \Psi \) is conditionally an inhomogeneous PPP with the conditional intensity measure \( \mu \sum_{m=1}^{\infty} S \cdot X_m Q \). Since the reduced Palm version of a PPP is identical to its original (not conditioned) version, we have

\[
\mathbb{E}_\Psi^0 \left[ \prod_{n=1}^{\infty} \hat{h}(Y_m) \left| \Phi \right. \right] = \mathbb{E} \left[ \prod_{n=1}^{\infty} \hat{h}(Y_m) \right],
\]

where the generating functional of a PPP is applied in the second equality. Taking the expectation with respect to \( \mathbb{P}_\Psi^0 \) and then applying Theorem 1, we obtain

\[
\mathcal{G}_\Psi(h) = \mathbb{E}_\Psi^0 \left[ \prod_{n=1}^{\infty} \hat{h}(X_n) \right] = \frac{1}{\mu} \mathbb{E}_\Psi^0 \left[ \int_{\mathbb{R}^d} \prod_{n=0}^{\infty} \hat{h}(X_n - z) \Psi_0(z) d\zeta \right] = \int_{\mathbb{R}^d} \hat{h}(-z) \mathbb{E}_\Psi^0 \left[ \prod_{n=1}^{\infty} \hat{h}(X_n - z) \right] Q(dz),
\]

where Campbell’s formula for \( \Psi_0 \) is applied in the last equality. By Slivnyak’s theorem and the stationarity for \( \Phi \), we have

\[
\mathbb{E}_\Psi^0 \left[ \prod_{n=0}^{\infty} \hat{h}(X_n - z) \right] = \mathbb{E} \left[ \prod_{n=1}^{\infty} \hat{h}(X_n) \right] = \mathbb{G}_\Phi(h),
\]

which completes the proof.

**Remark 3.** The right-hand side of (Eq. 5) is given as the generating functional \( \mathcal{G}_\Psi(h) \) of the stationary version of \( \Psi \) multiplied by the integral term \( \int \hat{h}(z) Q^* (dz) \). Since \( \hat{h}(z) \) represents the generating functional of an offspring process whose parent is shifted to \( z \in \mathbb{R}^d \) and \( Q^* \) is the distribution of the location of the parent point of the offspring at the origin on the event \( \{\Psi([0]) = 1\} \), this integral term represents the generating functional of the cluster which is given to have a point at the origin. In other words, Proposition 1 implies that, for a stationary PPCP, its Palm version is obtained by the superposition of the original stationary version and an additional independent offspring process whose parent is placed such that it has an offspring point at the origin.

### 3.3 Nearest-Neighbor Distance Distributions

For a stationary point process \( \Psi \) on \( \mathbb{R}^d \), let \( \Psi' = \sum_{m=1}^{\infty} \delta_{Y_m} \) be its reduced Palm version on \( \Psi([0]) = 1 \) and let \( Y \) denote the nearest point of \( \Psi' \) from the origin. Then, the nearest-neighbor distance distribution for \( \Psi \) is defined as the probability distribution for \( \|Y\| \) with respect to \( \mathbb{P}_\Psi^0 \), where \( \|\cdot\| \) denotes the Euclidean distance. We show below that the nearest-neighbor distance distribution for a stationary PPCP is obtained in a similar way to Proposition 1.

**Proposition 2.** For the stationary PPCP \( \Psi \) described in Section 3.1, the complementary nearest-neighbor distance distribution is given by

\[
\mathbb{P}_\Psi^0(\|Y\| > r | \Phi) = \mathbb{P}_\Psi^0(\Psi'(b_r) = 0 | \Phi) = \mathcal{G}_\Phi(h_r^*) \int h_r^*(t) Q^*(dt), \quad r \geq 0, \tag{8}
\]

where \( h_r^*(x) = e^{-\mu Q(b_r \cdot - x)} \) and \( b_r(r) \) denotes a \( d \)-dimensional ball centered at the origin with radius \( r \).

**Proof.** As with the proof of Proposition 1, we consider the conditional probability given the parent process \( \Phi = \sum_{m=1}^{\infty} \delta_{X_m} \) and obtain

\[
\mathbb{P}_\Psi^0(\|Y\| > r | \Phi) = \mathbb{P}_\Psi^0(\Psi'(b_r) = 0 | \Phi) = \mathbb{P}(\Psi'(b_r) = 0 | \Phi) = \prod_{n=1}^{\infty} h_r^*(X_n), \tag{9}
\]

where the second equality follows from Slivnyak’s theorem and the third does because \( \Psi' \) is conditionally an inhomogeneous PPP with the intensity measure \( \mu \sum_{m=1}^{\infty} S \cdot X_m Q \) when \( \Phi \) is given. The rest of the proof is similar to that of Proposition 1.

**Remark 4.** In (Eq. 8), the term \( \mathcal{G}_\Phi(h_r^*) \) is the complementary contact distance distribution and that is obtained by taking the expectation of (Eq. 9) with respect to \( \mathbb{P} \), instead of \( \mathbb{P}_\Psi^0 \) [see, e.g., (Miyoshi, 2019)]. The result of Proposition 2 is consistent with the existing ones in, e.g., (Baudin, 1981; Afshang et al., 2017a,b; Pandey et al., 2020) and gives a unified approach to derive the nearest-neighbor distance distributions for stationary PPCPs.

### 4 APPLICATIONS TO WIRELESS NETWORKS WITH HOTSPOT CLUSTERS

In this section, we apply Theorem 1 to the analysis of a D2D network with hotspot clusters modeled using a stationary PPCP.
We here suppose \( d = 2 \), but unless otherwise specified, the discussion holds for \( d \geq 2 \) theoretically.

### 4.1 Model of a Device-To-Device Network

Wireless devices are distributed on \( \mathbb{R}^d \) according to a stationary point process \( \Psi = \sum_{m=1}^{\infty} \delta_{Y_m} \). At each time slot, each device is in transmission mode with probability \( p \in (0, 1) \) or in receiving mode with probability \( 1 - p \) independently of the others (half duplex with random access). Devices in the transmission mode transmit signals but can not receive ones, whereas the devices in the receiving mode can receive signals but can not transmit ones.

We assume that all transmitting devices transmit signals with identical transmission power (normalized to one) and share a common frequency spectrum. The path-loss function representing attenuation of signals with distance is given by \( \ell \) satisfying \( \ell(r) \geq 0, r > 0 \), and \( \int_0^a \ell(r) r^{d-1} \, dr < \infty \) for \( a > 0 \). We further assume that all wireless links receive Rayleigh fading effects while we ignore shadowing effects. We focus on the device at the origin, referred to as the typical device, under the condition effects while we ignore shadowing effects.

The probability that the typical device can successfully decode a message from the device at \( Y_m \) is given by

\[
\mathbb{P}(Y_m, t) = \left( 1 - p \right) \mathbb{P}(Y_m, t) + \left[ 1 - \left( 1 - p \right) \right] \mathbb{P}(Y_m, t),
\]

where \( \left( 1 - p \right) \) denotes the probability that the SINR from the typical device exceeds the threshold \( \theta \) when it receives signals from another transmitting device at \( Y_m \). Hence, if the typical device is in the receiving mode and communicates with the transmitting device at \( Y_m \), the signal-to-interference-plus-noise ratio (SINR) is given as

\[
\text{SINR}_m = \frac{H_m \ell(\|Y_m\|)}{\sum_{j \neq m} H_j \ell(\|Y_j\|) + N},
\]

where \( N \) denotes a constant representing noise at the origin. We here suppose that the typical device can successfully decode a message when the SINR \( \text{SINR}_m \) exceeds the threshold \( \theta \).

### 4.2 Coverage Analysis

We here suppose that a device in the receiving mode communicates with the nearest device in transmission mode. The probability that the typical device can successfully decode a message from its partner is called the coverage probability and is given by

\[
\mathbb{P}(\theta) = (1 - p) \sum_{m=1}^{\infty} \mathbb{P}(\theta, Y_m, t),
\]

where \( 1 - p \) on the right-hand side indicates that the typical device must be in the receiving mode and the sum over \( m \) represents the probability that the SINR from the nearest transmitting device exceeds the threshold \( \theta \). We now suppose that the point process \( \Psi \) representing the locations of devices is given as a stationary PPCP studied in Section 3. Then, \( \Psi_{tx} = \sum_{m=1}^{\infty} \delta_{Y_m} \) representing the locations of devices in the transmission mode is also a stationary PPCP, where the parent process remains the same as the homogeneous PPP \( \Phi \) with intensity \( \lambda_{dp} \), whereas the offspring processes \( \Psi_{rx} = \sum_{m=1}^{\infty} \delta_{Y_m, s} \) are finite PPPs with the intensity measure \( \mu_Q \).

Theorem 2. For the model of a D2D network described in Section 4.1 with the devices deployed according to a stationary PPCP in Section 3.1, the coverage probability is given by

\[
\mathbb{P}(\theta, t) = \int_{\mathbb{R}^d} \mathbb{P}(\theta, t) Q^-(dr),
\]

where \( Q^- \) is given in Lemma 1 and

\[
I_{1,\theta}(t) = \int_{\mathbb{R}^d} \mathbb{P}(\theta, t) Q^-(dr) - t),
\]

\[
I_{2,\theta}(t) = \lambda_{dp} \int_{\mathbb{R}^d} \mathbb{P}(\theta, t) Q^-(dr) - t),
\]

\[
E_d(y, t) = \exp \left\{ -\lambda_{dp} \int_{\mathbb{R}^d} \mathbb{P}(\theta, t) Q^-(dr) - t),
\]

\[
E_d(y, x) = \exp \left\{ -\lambda_{dp} \int_{\mathbb{R}^d} \mathbb{P}(\theta, t) Q^-(dr) - t),
\]

Before proceeding on the proof of Theorem 2, we give an intuitive interpretation to the result of it. First, as stated in the preceding section, \( Q^- \) denotes the distribution for the location of the point-process type of the typical device at the origin. Thus, \( p \mathbb{P}(I_{1,\theta}(t)) \) and \( p \mathbb{P}(I_{2,\theta}(t)) \) in (Eq. 12) represent the cases where the typical device, whose parent is located at \( t \in \mathbb{R}^d \), communicates with the transmitting device in the same cluster and in a different cluster, respectively; that is, the location of the communication partner is sampled from a finite PPP with the intensity measure \( \mu_Q \) in \( I_{1,\theta}(t) \) and is from one with \( \mu_Q \) in \( I_{2,\theta}(t) \), where \( x \) is also sampled from a homogeneous PPP with intensity \( \lambda_{dp} \). Moreover, \( E_d(y, t) \) represents the effect from other clusters which are neither the one having the typical device nor the one having its communication partner at \( y \). Finally, \( C_d(y, x) \) represents the effect of the cluster with the parent point at \( x \in \mathbb{R}^d \) when the typical device communicates with the transmitting device at \( y \).

**Proof.** Similar to the proof of Proposition 1, once the parent process \( \Phi = \sum_{m=1}^{\infty} \delta_{X_m} \) is given, the point process \( \Psi_{tx} \) representing the locations of devices in the transmission mode is conditionally an inhomogeneous PPP with the conditional intensity measure \( \mu_{S\cap X_m} \). Thus, we can use the corresponding approach to determine the coverage probability for a cellular network with BSs deployed according to a PPP [see, e.g., (Andrews et al., 2011) or (Blaszczyszyn et al., 2018, Section 5.2)]. Since \( H_m \) \( m \in \mathbb{N} \), are mutually independent, exponentially distributed, and also independent of \( \Phi \), we have from (Eq. 11),

\[
\mathbb{P}(\theta, t) = \mathbb{P}(\theta, t) \left( \sum_{m=1}^{\infty} \frac{H_m \ell(\|Y_m\|)}{\sum_{j \neq m} H_j \ell(\|Y_j\|) + N}, \|Y_m\| \leq \|Y_j\|, m \in \mathbb{N} \right) \Phi
\]
Furthermore, the generating functional of a PPP applying to the equality. Noting that and then applying Neveu's formula [see, e.g., (Daley and Vere-Jones, 2008, Theorem 13.2. III), (Las and Penrose, 2017, Theorem 9.1) or (Baccelli et al., 2020, Theorem 3.1.9)],

\[
\sum_{m=1}^{\infty} \int_{\mathbb{R}^d} e^{-\alpha/(\ell(y)))} \mathbb{E}_\Phi \left[ \prod_{j=1}^{\infty} \left( 1 + \theta \left( \frac{\|y_j\|}{\ell(y_j)} \right) \right)^{-1} 1_{\|y_j\|<1} \right] dP(y).
\]

The above expression over \( m \in \mathbb{N} \), we have from Slivnyak's Theorem for \( \Phi \) conditioned on \( \Theta \) and the refined Campbell formula [Las and Penrose, 2017, Theorem 9.1] or (Baccelli et al., 2020, Theorem 3.1.9),

\[
\sum_{m=1}^{\infty} \int_{\mathbb{R}^d} e^{-\alpha/(\ell(y)))} \mathbb{E}_\Phi \left[ \prod_{j=1}^{\infty} \left( 1 + \theta \left( \frac{\|y_j\|}{\ell(y_j)} \right) \right)^{-1} 1_{\|y_j\|<1} \right] dP(y).
\]

and consider the two terms on the right-hand side of (Eq. 16) one by one. For the first term, the generating functional of a PPP yields (1st term of (Eq. 16))

\[
= \int_{\mathbb{R}^d} e^{-\alpha/(\ell(y)))} \mathbb{E}_\Phi \left[ 1 - C_\Theta(y, w) \right] dP(y)
\]

\[
Q(dy) + t = I_{\lambda, \Theta}(-t).
\]

On the other hand, applying Campbell’s formula and the generating functional for \( \Theta \) to the second term on the right-hand side of (Eq. 16), we have (2nd term of (Eq. 16))

\[
= \lambda_\phi \int_{\mathbb{R}^d} e^{-\alpha/(\ell(y)))} \mathbb{E}_\Phi \left[ \prod_{i=1}^{\infty} C_\Theta(y, X_i - t) \right] Q(dy - x) dx
\]

\[
\exp(-\lambda_\phi \int_{\mathbb{R}^d} \left[ 1 - C_\Theta(y, w) \right] dP(y) Q(dy - x) dx
\]

\[
= I_{\lambda, \Theta}(-t).
\]

Finally, plugging (Eqs 17, 18) into (Eq. 16), and then into (Eq. 15), we have (Eq. 12) and the proof is completed.

When \( d = 2 \) and the distribution \( Q \) for the locations of offspring points depends only on the distance, that is, \( Q(dy) = f_\Theta(|y|) dy \) for \( y \in \mathbb{R}^2 \), we obtain a numerically computable form of the coverage probability.

Corollary 1. When \( d = 2 \) and \( Q(dy) = f_\Theta(|y|) dy \), \( y \in \mathbb{R}^2 \), the coverage probability in Theorem 2 is reduced to

\[
\int_{\mathbb{R}^2} e^{-\alpha/(\ell(y)))} \mathbb{E}_\Phi \left[ \prod_{i=1}^{\infty} C_\Theta(y, X_i - t) \right] Q(dy - x) dx
\]

\[
= \lambda_\phi \int_{\mathbb{R}^d} e^{-\alpha/(\ell(y)))} \mathbb{E}_\Phi \left[ \prod_{i=1}^{\infty} C_\Theta(y, X_i - t) \right] Q(dy - x) dx
\]

\[
= \lambda_\phi \int_{\mathbb{R}^d} e^{-\alpha/(\ell(y)))} \mathbb{E}_\Phi \left[ \prod_{i=1}^{\infty} C_\Theta(y, X_i - t) \right] Q(dy - x) dx
\]

\[
= \lambda_\phi \int_{\mathbb{R}^d} e^{-\alpha/(\ell(y)))} \mathbb{E}_\Phi \left[ \prod_{i=1}^{\infty} C_\Theta(y, X_i - t) \right] Q(dy - x) dx
\]

Proof. Since the distribution \( Q \) depends only on the distance, it holds that \( Q(dy) = Q(dt) = f_\Theta(|t|) dt \), \( t \in \mathbb{R}^2 \), and (Eq. 12) is reduced to
\[ \mathbf{CP}(\theta) = (1 - p) p \mu \int_{\mathbb{R}^2} (I_{1,\theta}(t) + I_{2,\theta}(t)) f_o(||\theta||) \, dt \]
\[ = 2\pi(1 - p) p \mu \int_0^\infty \left( \hat{I}_{1,\theta}(u) + \hat{I}_{2,\theta}(u) \right) f_o(u) \, u \, du, \]
where the polar coordinate conversion is applied in the second equality and
\[ \hat{I}_{1,\theta}(u) = \int_0^u e^{-\beta u(s)} \mathcal{C}^\theta(s,u) \mathcal{E}_\theta(s) g(s \mid u) \, ds, \]
\[ \hat{I}_{2,\theta}(u) = 2\pi \lambda_0 \int_0^\infty \int_0^\infty e^{-\beta u(s)} \mathcal{C}^\theta(s,u) \mathcal{E}_\theta(s) \mathcal{C}_\theta(s,r) \mathcal{E}_\theta(s) g(s \mid r) \, dr \, ds. \]
Therefore, we have
\[ \hat{I}_{1,\theta}(u) + \hat{I}_{2,\theta}(u) = \int_0^\infty e^{-\beta u(s)} \mathcal{C}^\theta(s,u) \mathcal{E}_\theta(s) \mathcal{I}_\theta(s,u) \, ds. \]
Plugging this into (Eq. 21), we have (Eq. 19) and the proof is completed.

4.3 Device Discovery

We next consider the problem of device discovery. Devices in the transmission mode transmit broadcast messages, whereas a device in the receiving mode can discover the transmitters if it can successfully decode the broadcast messages. When a device in the receiving mode receives the signal from one transmitting device, the signals from all other transmitting devices work as interference. Then, the expected number of transmitting devices discovered by the typical device is represented by
\[ N(\theta) = (1 - p) \mathbb{E}_\theta \left[ \sum_{m=1}^{\infty} I_{(\text{SNR}_m > \theta)} \right], \]
Proposition 3. Consider the D2D network model described in Section 4.1 with the devices deployed according to a stationary PPPC given in Section 3.1. Then, the expected number \( N(\theta) \) of transmitting devices discovered by the typical device is obtained by (Eq. 12) in Theorem 2 replacing the integral range \( [\|z\| \leq \|y\|] \) in (Eq. 13) by \( \mathbb{R}^d \). Moreover, when \( d = 2 \) and \( Q(dy) = f_o(||y||) \, dy \) for \( y \in \mathbb{R}^2 \), \( N(\theta) \) is reduced to (Eq. 19) in Corollary 1 replacing the integral range \( (s, \infty) \) in (Eq. 20) by \( (0, \infty) \).

Proof. Since \( \mathbb{E}_\theta \left[ \sum_{m=1}^{\infty} I_{(\text{SNR}_m > \theta)} \right] = \sum_{m=1}^{\infty} \mathbb{P}_\theta \left( \text{SNR}_m > \theta \right) \), the difference between (Eq. 11) and (Eq. 22) is only the event \( \{Y_m \leq \|y\|, j \in \mathbb{N}\} \). This leads to the difference of the integral ranges in \( C(y,x) \) in (Eq. 13) and in \( C_\theta(s,r) \) in (Eq. 20). Remark 5. Since \( \mathbb{P}_\theta \left( \sum_{m=1}^{\infty} I_{(\text{SNR}_m > \theta)} \right) \leq \sum_{m=1}^{\infty} \mathbb{P}_\theta \left( \text{SNR}_m > \theta \right) = N(\theta) \), Proposition 3 also gives an upper bound for the coverage probability with the max-SNR association policy, where a device in the receiving mode receives a message with the strongest SINR. This upper bound is known to be exact for \( \theta > 1 \) since \( \sum_{m=1}^{\infty} I_{(\text{SNR}_m > \theta)} \leq 1 + \theta^{-1} \) almost surely [see (Dhillon et al., 2012) or (Błaszczyzn et al., 2018, Lemma 5.1.2)].

4.4 Numerical Experiments

We present the results of numerical experiments for the analytical results obtained in Sections 4.2, 4.3. We set \( d = 2 \) and the distribution \( Q \) for the location of the offspring points as \( Q(dy) = f_o(||y||) \, dy \) and \( f_o(s) = e^{-s/(2\sigma^2)} / (2\pi\sigma^2) \), \( s \geq 0 \); that is, \( Q \) is the isotropic normal distribution with variance \( \sigma^2 \), so that the resulting PPCP \( \Psi \) is the Thomas point process. Furthermore, the path-loss function is set as \( \ell(r) = r^\beta, \, r > 0 \), with \( \beta > 2 \).

The numerical results for the coverage probability are given in Figure 2, where the values of \( \mathbf{CP}(\theta) \) with different values of \( \theta \) and \( \sigma^2 \) are plotted. The other parameters are fixed at \( \lambda_0 = \pi^{-1}, \mu = 10, \rho = 0.5, \beta = 4 \) and \( N = 0 \). For comparison, the values when the devices are located according to a homogeneous PPP are also displayed in the figure with the label “\( \sigma^2 \to \infty \)”.

![Figure 2](image-url) Coverage probability as a function of SINR threshold (\( \lambda_0 = \pi^{-1}, \mu = 10, \rho = 0.5, \beta = 4 \) and \( N = 0 \)).

![Figure 3](image-url) Expected number of discovered devices as a function of SINR threshold (\( \lambda_0 = \pi^{-1}, \mu = 10, \rho = 0.5, \beta = 4 \) and \( N = 0 \)).
(Miyoshi, 2019). This difference is thought to be due to the fact that the locations of a receiving device and its communication partner are near to each other in the PPCP-deployed D2D network since they are both points of the same PPCP, whereas the location of a receiver is likely far from that of the associated BS in the PPCP-deployed cellular network since their locations are independent of each other.

The results of the device discovery are given in Figure 3, where we know that the closed form expression of the expected number of discovered devices is obtained as $N(\text{PPP}) = \frac{(1 - p)(\beta/2\pi)(\sin(2\pi/\beta) \theta^{2\beta}}{2\pi}$ for the case of the homogeneous PPP with $N \equiv 0$ [see, e.g., (Hamida et al., 2008)]. The figure shows similar features to the coverage probability.

5 CONCLUSION

In this paper, we have presented an alternative form of Neveu’s exchange formula for jointly stationary point processes on $\mathbb{R}^d$ and then demonstrated that it is useful for the analysis of spatial stochastic models given on stationary PPCPs. We have first applied it to the Palm characterization for a stationary PPCP and then to the analysis of a D2D network modeled using a stationary PPCP. Although we have only considered some fundamental problems, we expect that the new form of the exchange formula will be utilized for the analysis of more sophisticated models leading up to the development of 5G and beyond networks.

DATA AVAILABILITY STATEMENT

The original contributions presented in the study are included in the article/supplementary material, further inquiries can be directed to the corresponding author.

AUTHOR CONTRIBUTIONS

The author confirms being the sole contributor of this work and has approved it for publication.

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