Global well-posedness and scattering for the defocusing, cubic, nonlinear Schrödinger equation when $n = 3$ via a linear-nonlinear decomposition

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Abstract: In this paper, we prove global well-posedness and scattering for the defocusing, cubic nonlinear Schrödinger equation in three dimensions when $n = 3$ when $u_0 \in H^s(\mathbb{R}^3)$, $s > 3/4$. To this end, we utilize a linear-nonlinear decomposition, similar to the decomposition used in [12] for the wave equation.

1 Introduction

The defocusing, cubic nonlinear Schrödinger equation in three dimensions,

$$i u_t + \Delta u = |u|^2 u,$$
$$u(0, x) = u_0(x) \in H^s(\mathbb{R}^3).$$ (1.1)

has been the subject of a great deal of attention recently. In [4] it was proved that (1.1) has a local solution on some interval $[0, T)$, $T(\|u_0\|_{H^s(\mathbb{R}^3)}) > 0$ when $u_0 \in H^s(\mathbb{R}^3)$, $s > 1/2$. There is also a local solution on $[0, T)$, $T(u_0) > 0$ when $u_0 \in \dot{H}^{1/2}(\mathbb{R}^3)$. In this case $T > 0$ depends on the profile of the initial data, not just its size. If $\|u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)}$ is small, then (1.1) is globally well-posed and scatters.

If $s > 1/2$ and a solution to (1.1) only exists on a maximal interval $[0, T_*)$, $T_* < \infty$, then

$$\lim_{t \uparrow T_*} \|u(t)\|_{H^s(\mathbb{R}^3)} = \infty.$$ (1.2)
proven (1.1) is globally well-posed in the defocusing case when $s = 1$, due to the conservation of mass and energy.

$$M(u(t)) = \int |u(t, x)|^2 dx = M(u(0)),$$

(1.3)

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx + \frac{1}{4} \int |u(t, x)|^4 dx = E(u(0)).$$

(1.4)

**Remark:** This argument will not work for the focusing equation since

$$E(u(t)) = \frac{1}{2} \int |\nabla u(t, x)|^2 dx - \frac{1}{4} \int |u(t, x)|^4 dx,$$

is not positive definite.

Recently, the gap between regularity necessary for local well-posedness and global well-posedness has been narrowed. In [6], the I-method was introduced, proving global well-posedness of (1.1) for $s > 5/6$. In [7], an interaction Morawetz estimate was proved, thereby improving the result to $s > 4/5$.

In a parallel vein, the I-method has also been applied to the semilinear wave equation,

$$\partial_{tt} u - \Delta u = -u^3,$$

$$u(0, x) \in H^s(\mathbb{R}^3),$$

$$u_t(0, x) \in H^{s-1}(\mathbb{R}^3).$$

(1.5)

In [12], a linear-nonlinear decomposition was made, more effectively estimating the energy change for large times. In this paper, we will make a similar argument, proving

**Theorem 1.1** (1.1) is globally well-posed for $s > 3/4$. Additionally,

$$\|u(t)\|_{H^s(\mathbb{R}^3)} \leq C(\|u_0\|_{H^s(\mathbb{R}^3)}),$$

(1.6)

and the solution scatters. There exist $u_\pm \in H^s(\mathbb{R}^3)$ such that

$$\lim_{t \to \infty} \|u(t) - e^{it\Delta} u_+\|_{H^s(\mathbb{R}^3)} = 0,$$

$$\lim_{t \to -\infty} \|u(-t) - e^{-it\Delta} u_-\|_{H^s(\mathbb{R}^3)} = 0.$$
In §2, some preliminary facts that are needed will be mentioned. In §3, a local well-posedness result will be proved. In §4, a formula for the energy increment will be computed. In §5 a smoothing estimate using a bilinear estimate will be proved. In §6, the double-layer I-decomposition will be used to prove the theorem.

2 Preliminaries

Suppose $\phi(x)$ is a smooth function,

$$\phi(x) = \begin{cases} 
1, & |x| \leq 1; \\
0, & |x| > 2.
\end{cases} \quad (2.1)$$

Then it is possible to make a high-low decomposition,

$$\mathcal{F}(P_{\leq N} u) = \hat{u}(\xi)\phi\left(\frac{\xi}{N}\right),$$

$$\mathcal{F}(P_{> N} u) = \hat{u}(\xi)(1 - \phi\left(\frac{\xi}{N}\right)), \quad (2.2)$$

$$u = P_{\leq N} u + P_{> N} u = u_l + u_h.$$

Also, define the standard Littlewood-Paley decomposition,

$$P_N f = u_{\leq 2N} - u_{\leq N}. \quad (2.3)$$

We let $u_{< N} = P_{< N} u$, similarly for $u_N$ and $u_{> N}$.

The I-operator is a Fourier multiplier given by a smooth, radially symmetric symbol,

$$I_N : H^s(\mathbb{R}^3) \to H^1(\mathbb{R}^3), \quad (2.4)$$

$$(I_N f)(\xi) = m_N(\xi)\hat{f}(\xi), \quad (2.5)$$

$$m_N(\xi) = \begin{cases} 
1, & |\xi| \leq N; \\
\left(\frac{N}{|\xi|}\right)^{1-s}, & |\xi| > 2N.
\end{cases} \quad (2.6)$$

For the rest of the paper, we understand that $If$ refers to the function $I_N f$. The operator obeys the following estimates,

$$\|\nabla I u\|_{L^2(\mathbb{R}^3)} \lesssim N^{1-s}\|u\|_{H^s(\mathbb{R}^3)},$$

$$\|u\|_{H^s(\mathbb{R}^3)} \lesssim \|I u\|_{H^1(\mathbb{R}^3)}. \quad (2.7)$$
\[ \| P_{>M} u \|_{L^p_t L^q_x(J \times \mathbb{R}^3)} \lesssim \left( \frac{1}{M} + \frac{1}{N^{1-s} M^s} \right) \| \nabla I u \|_{L^p_t L^q_x(J \times \mathbb{R}^3)}. \]  
\[ (2.8) \]

\[ \| \nabla^{1/2} P_{>M} u \|_{L^p_t L^q_x(J \times \mathbb{R}^3)} \lesssim \left( \frac{1}{M^{1/2}} + \frac{1}{N^{1-s} M^s - 1/2} \right) \| \nabla I u \|_{L^p_t L^q_x(J \times \mathbb{R}^3)}. \]  
\[ (2.9) \]

**Strichartz Estimates:** A pair \((p, q)\) will be called an admissible pair if
\[ \frac{2}{p} = 3 \left( \frac{1}{2} - \frac{1}{q} \right). \]  
\[ (2.10) \]

We will also define the Strichartz space (see for example [15]),
\[ \| u \|_{S^{0} (J \times \mathbb{R}^3)} = \sup_{(p, q) \text{ admissible}} \| u \|_{L^p_t L^q_x(J \times \mathbb{R}^3)}, \]  
\[ (2.11) \]
as well as its dual,
\[ \| u \|_{N^{0} (J \times \mathbb{R}^3)} = \inf_{(p', q') \text{ admissible}} \| u \|_{L^{p'}_t L^{q'}_x(J \times \mathbb{R}^3)}, \]  
\[ (2.12) \]
where \(p', q'\) refers to the dual exponent. If \(u(t, x)\) solves the equation
\[ iu_t + \Delta u = F(t), \]
\[ u(0, x) = u_0, \]  
\[ (2.13) \]
\[ \| u \|_{S^{0} (J \times \mathbb{R}^3)} \lesssim \| u_0 \|_{L^2(\mathbb{R}^3)} + \| F \|_{N^{0} (J \times \mathbb{R}^3)}. \]  
\[ (2.14) \]

**Interaction Morawetz Estimate**

**Theorem 2.1** If \(u(t, x)\) solves (1.1), then
\[ \| u \|^4 \| \chi_{u(x)} L^1_{t,x} (J \times \mathbb{R}^3) \lesssim \| u \|^2 \| L^2_{t,x} L^2_{x} (J \times \mathbb{R}^3) \| u \|_{L^\infty_{t,x} \widehat{H}^1/2_{x} (J \times \mathbb{R}^3)}^2, \]  
\[ (2.15) \]

**Proof:** See [7].

**Bilinear Estimate**
Lemma 2.2 Suppose
\[ u(t, x) = e^{it\Delta}u_0 + \int_0^t e^{i(t-\tau)\Delta}F(\tau, x)d\tau, \]  
(2.16)
and
\[ v(t, x) = e^{it\Delta}v_0 + \int_0^t e^{i(t-\tau)\Delta}G(\tau, x)d\tau, \]  
(2.17)
with \( u_0, F \) supported on \( N \leq |\xi| \leq 2N \) and \( v_0, G \) supported on \( M \leq |\xi| \leq 2M \), \( N \ll M \). Then for any \( \delta > 0 \),
\[ \|uv\|_{L^2_{t,x}(J \times \mathbb{R}^3)} \leq C(\delta) N^{1-\delta} M^{1/2-\delta} \left( \|u_0\|_{L^2_x(\mathbb{R}^3)} + \|F\|_{L^1_t L^2_x(J \times \mathbb{R}^3)} \right) \times \left( \|v_0\|_{L^2_x(\mathbb{R}^3)} + \|G\|_{L^1_t L^2_x(J \times \mathbb{R}^3)} \right). \]  
(2.18)

Proof: See [8].

3 Local Well-posedness

In this section we prove local well-posedness when \( \|u\|_{L^4_{t,x}(J \times \mathbb{R}^3)} \) is small. To that end, we prove that the norm of \( u \) is controlled by the norm of \( Iu \).

Lemma 3.1 If \( \|u\|_{L^4_{t,x}(J \times \mathbb{R}^3)} \leq \epsilon \), and \( I : H^s(\mathbb{R}^3) \to H^1(\mathbb{R}^3) \), \( 1/2 < s < 1 \), then
\[ \|u\|_{L^6_t L^9/2_x(J \times \mathbb{R}^3)} \lesssim \left( \epsilon^{2/3} + \frac{1}{N^{1/2}} \right)(\|\nabla I u\|_{S^0(J \times \mathbb{R}^3)} + 1). \]  
(3.1)

Proof: Make a Littlewood-Paley partition of unity.

\[ \|P_{\leq N}u\|_{L^\infty_t L^6_x(J \times \mathbb{R}^3)} \lesssim \|\nabla P_{\leq N}u\|_{L^\infty_t L^2_x(J \times \mathbb{R}^3)} \leq \|\nabla I u\|_{S^0(J \times \mathbb{R}^3)}. \]  
(3.2)

Interpolating with \( \|P_{\leq N}u\|_{L^4_{t,x}(J \times \mathbb{R}^3)} \leq \epsilon \),
\[ \|P_{\leq N}u\|_{L^6_t L^9/2_x(J \times \mathbb{R}^3)} \lesssim \epsilon^{2/3} \|\nabla I u\|_{S^0(J \times \mathbb{R}^3)}^{1/3}. \]

This takes care of the \( P_{\leq N} \) part. On the other hand,
\[ \|P_{N_j}u\|_{L_t^6 L_x^{9/2}(J \times \mathbb{R}^3)} \lesssim N_j^{1/2} \|u\|_{L_t^6 L_x^{18/7}(J \times \mathbb{R}^3)}^{1/2} \leq \frac{1}{N_1^{1-s}} \frac{1}{N_j^s} \|\nabla Iu\|_{S^0(J \times \mathbb{R}^3)} . \]  

(3.3)

Summing over \( N_j \gtrsim N \) gives the bound for \( P_{>N}u \). □

**Theorem 3.2** Suppose \( J \) is an interval such that

\[ \|u\|_{L_t^4 x(J \times \mathbb{R}^3)} \leq \epsilon, \]

and \( E(Iu_0) \leq 1 \). Then (1.1) is locally well-posed on \( J \), and

\[ \|\nabla Iu\|_{S^0(J \times \mathbb{R}^3)} \lesssim 1. \]  

(3.4)

**Proof:** A solution satisfies the Duhamel formula,

\[ Iu(t, x) = e^{it\Delta}Iu_0 + \int_0^t e^{i(t-\tau)\Delta}I(|u|^2u)(\tau)d\tau. \]  

(3.5)

Using (2.14),

\[ \|\nabla Iu\|_{S^0(J \times \mathbb{R}^3)} \lesssim \|\nabla Iu_0\|_{L^2(J \times \mathbb{R}^3)} + \|\nabla Iu\|_{L_t^{2} L_x^{\frac{6}{5}}(J \times \mathbb{R}^3)} \|u\|_{L_t^{2} L_x^{\frac{9}{2}}(J \times \mathbb{R}^3)} \]

\[ \lesssim \|\nabla Iu_0\|_{L^2(J \times \mathbb{R}^3)} + (\epsilon^{1/3} + \frac{1}{N})(\|\nabla Iu\|_{S^0(J \times \mathbb{R}^3)}^{5/3} + \|\nabla Iu\|_{S^0(J \times \mathbb{R}^3)}^3). \]

Applying the continuity method proves the theorem. □

### 4 Energy Increment

In this section we prove an estimate on the energy increment. For short time the estimate is identical to the estimate found in [7], however, this estimate is more conducive to long-time energy estimates, which will be used in later sections.

**Theorem 4.1** If \( u \) is a solution to (1.1), and \( J \) is an interval with

\[ \|u\|_{L_t^4 x(J \times \mathbb{R}^3)}^4 \leq \epsilon, \]  

(4.1)

then
\[
\sup_{t_1, t_2 \in J} |E(Iu(t_1)) - E(Iu(t_2))| \lesssim \frac{1}{N^1} \|\nabla IP_{>cN}u\|_{L_t^2 L_x^6(J \times \mathbb{R}^3)}^2 + O\left(\frac{1}{N^2}\right),
\]
(4.2)

where \(c > 0\) is some constant.

**Proof:** To prove this, recall the formula for energy, (1.4),
\[
E(Iu(t)) = \frac{1}{2} \int |\nabla Iu(t, x)|^2 dx + \frac{1}{4} \int |Iu(t, x)|^4 dx.
\]
(4.3)

\[
\frac{d}{dt} E(Iu(t)) = -\text{Re} \int (I\partial_t u(t, x)) I(|u(t, x)|^2 u(t, x)) dx
+ \text{Re} \int (I\partial_t u(t, x)) |Iu(t, x)|^2 \overline{u(t, x)} dx.
\]
(4.4)

Since
\[
Iu_t = i\Delta Iu - iI(|u|^2 u),
\]
it suffices to estimate two terms separately.

\[
\text{Re} \int_{t_1}^{t_2} (i\Delta Iu(t, x))[I(|u(t, x)|^2 u(t, x)) - |Iu(t, x)|^2 Iu(t, x)] dx dt,
\]
(4.5)

and

\[
\text{Re} \int_{t_1}^{t_2} (iI(|u(t, x)|^2 u(t, x)))[I(|u(t, x)|^2 u(t, x)) - |Iu(t, x)|^2 \overline{u(t, x)}] dx dt.
\]
(4.6)

The term (4.5):

\[
(4.5) = \text{Re} \int_{t_1}^{t_2} \sum (i|\xi_1|^2 \hat{Iu}(t, \xi_1))[\frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} - 1]
\hat{\overline{u(t, \xi_2)}} \hat{u(t, \xi_3)} \hat{\overline{u(t, \xi_4)}} d\xi dt,
\]
(4.7)

where \(\Sigma = \{\xi_1 + \xi_2 + \xi_3 + \xi_4 = 0\}\) and \(d\xi\) is the Lebesgue measure on the hyperplane. Take a Littlewood-Paley partition of unity. Without loss of generality let \(N_2 \geq N_3 \geq N_4\). Consider a number of cases separately.

**Case 1, \(N_2 \ll N\):** In this case
\[
\frac{m(\xi_2 + \xi_3 + \xi_4)}{m(\xi_2)m(\xi_3)m(\xi_4)} - 1 \equiv 0.
\]
Case 2, \( N_2 \gtrsim N, N_3 << N \):

Case 2(a): \( N_4 \geq \frac{1}{N_2} \): In this case, apply the fundamental theorem of calculus.

\[
| \frac{m(N_2 + N_3 + N_4)}{m(N_2)} - 1 | \lesssim \nabla m(N_2) | N_3 \lesssim \frac{N_3}{N_2}.
\]

Therefore,

\[(4.5) \lesssim \sum_{N \lesssim N_1 \sim N_2} \frac{N_1}{N_2^2} \| P_{N_1} \nabla Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \| P_{N_2} \nabla Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)}
\]

\[
\times \sum_{\frac{1}{N_2} \leq N_4 \leq N_3 < N} \frac{N_3}{N_2} \| P_{N_3} Iu \|_{L_t^\infty L_x^2(J \times \mathbb{R}^3)} \| P_{N_4} Iu \|_{L_t^\infty L_x^2(J \times \mathbb{R}^3)},
\]

\[
\lesssim \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^3 \sum_{N \lesssim N_1 \sim N_2} \frac{\ln(N)}{N_1} \| P_{N_1} \nabla Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)} \| P_{N_2} \nabla Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)},
\]

\[
\lesssim \frac{1}{N_1-} \| P_{>cN} \nabla Iu \|_{L_t^2 L_x^6(J \times \mathbb{R}^3)}^2 \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^2.
\]

The last estimate is by Cauchy-Schwartz.

Case 2(b), \( N_4 \leq \frac{1}{N_2} \): In this case, combine the Sobolev embedding theorem with the bound on \( \| u \|_{L_t^4 L_x^8(J \times \mathbb{R}^3)} \),

\[
\| P_{N_4} u \|_{L_t^4 L_x^8(J \times \mathbb{R}^3)} \lesssim N_4^{3/4} \| P_{N_4} u \|_{L_t^4 L_x^\infty(J \times \mathbb{R}^3)} \lesssim \epsilon N_4^{3/4}. \tag{4.8}
\]

In this case,

\[(4.5) \lesssim \sum_{N \lesssim N_1 \sim N_2} \frac{N_1}{N_2^2} \| P_{N_1} \nabla Iu \|_{L_t^4 L_x^8(J \times \mathbb{R}^3)} \| P_{N_2} \nabla Iu \|_{L_t^4 L_x^8(J \times \mathbb{R}^3)}
\]

\[
\times \sum_{N_4 \leq \frac{1}{N_2} : N_4 \leq N_3 < N} \frac{N_3}{N_2} \| P_{N_3} Iu \|_{L_t^4 L_x^2(J \times \mathbb{R}^3)} \| P_{N_4} Iu \|_{L_t^4 L_x^\infty(J \times \mathbb{R}^3)}
\]

\[
\lesssim \frac{\epsilon}{N_{5/2}} \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^3.
\]
**Case 3**, $N_2 \gtrsim N$, $N_3 \gtrsim N$, $N_2 \sim N_1$:

*Case 3(a), $N_4 \geq \frac{1}{N_2}$*: In this case, crudely estimate

$$\left| \frac{m(N_2 + N_3 + N_4)}{m(N_2)m(N_3)m(N_4)} - 1 \right| \lesssim \frac{1}{m(N_3)m(N_4)}.$$  

(4.5) $\lesssim \sum_{N_1 \sim N_2} \frac{N_1}{N_2} \| P_{N_1} \nabla Iu \|_{L^4_t L^6_x(J \times \mathbb{R}^3)} \| P_{N_2} \nabla Iu \|_{L^4_t L^6_x(J \times \mathbb{R}^3)}$

$$\times \sum_{N_3 \gtrsim N_1 \sim N_2} \frac{1}{N_3 m(N_3)m(N_4)} \| P_{N_3} \nabla Iu \|_{L^6_t L^6_x(J \times \mathbb{R}^3)} \| P_{N_4} Iu \|_{L^6_t L^6_x(J \times \mathbb{R}^3)}.$$ 

$$\sum_{N_1 \sim N_2 \sim N_3 \gtrsim N_2} \frac{1}{N_3 m(N_3)m(N_4)} \lesssim \sum_{N \lesssim N_3 \lesssim N_2} \frac{1}{N_3 m(N_3)} \ln(N) + \frac{N_4^{1-s}}{N_1^{1-s}} \lesssim \frac{1}{N^{1-s}}.$$ 

Summing $N_1 \sim N_2$ by Cauchy - Schwartz,

(4.5) $\lesssim \frac{1}{N^{1-s}} \| u \|_{L^4_t L^6_x(J \times \mathbb{R}^3)}.$

*Case 3(b), $N_4 \leq \frac{1}{N_2}$*: In this case,

$$\left| \frac{m(N_2 + N_3 + N_4)}{m(N_2)m(N_3)m(N_4)} - 1 \right| \lesssim \frac{1}{m(N_3)}.$$ 

Once again, use the Sobolev embedding theorem combined with $\| u \|_{L^4_t L^6_x(J \times \mathbb{R}^3)} \leq \epsilon$.

(4.5) $\lesssim \sum_{N \lesssim N_1 \sim N_2} \frac{N_1}{N_2} \| P_{N_1} \nabla Iu \|_{L^4_t L^6_x(J \times \mathbb{R}^3)} \| P_{N_2} \nabla Iu \|_{L^4_t L^6_x(J \times \mathbb{R}^3)}$

$$\times \sum_{N_4 \leq \frac{1}{N_2}, N_3 \gtrsim N} \frac{1}{N_3 m(N_3)} \| P_{N_3} \nabla Iu \|_{L^4_t L^6_x(J \times \mathbb{R}^3)} \| P_{N_4} Iu \|_{L^6_t L^6_x(J \times \mathbb{R}^3)}$$

9
\[ \lesssim \frac{\epsilon}{N^{5/2}} \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^3. \]

Here again we use Cauchy-Schwartz in \( N_1 \sim N_2 \).

**Case 4**, \( N_2 \gtrsim N, \ N_2 \sim N_3, \ N_1 \lesssim N_2 \):

**Case 4(a)**, \( N_4 \geq \frac{1}{N^2} \): Once again, make the crude estimate on the multiplier.

\[(4.5) \lesssim \sum_{N \lesssim N_2 \sim N_3} \frac{1}{m(N_2)N_2N_3} \| P_{N_2} \nabla Iu \|_{L^2_tL^{\delta}_{x}(J \times \mathbb{R}^3)} \| P_{N_3} \nabla Iu \|_{L^2_tL^{\delta}_{x}(J \times \mathbb{R}^3)} \]

\[ \times \sum_{N_1 \lesssim N_2, \frac{1}{N^2} \lesssim N_1 \lesssim N_3} \frac{N_1}{m(N_4)} \| P_{N_1} \nabla Iu \|_{L^2_tL^{\delta}_{x}(J \times \mathbb{R}^3)} \| P_{N_4} Iu \|_{L^2_tL^{\delta}_{x}(J \times \mathbb{R}^3)} \]

\[ \lesssim \frac{1}{N_1} \| P_{>N} \nabla Iu \|_{L^2_tL^{\delta}_{x}(J \times \mathbb{R}^3)}^2 \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^2. \]

**Case 4(b)**, \( N_4 \leq \frac{1}{N^2} \): As usual, use the Sobolev embedding.

\[(4.5) \lesssim \sum_{N \lesssim N_2 \sim N_3} \frac{1}{N_2m(N_3)N_3} \| P_{N_2} \nabla Iu \|_{L^2_tL^{\delta}_{x}(J \times \mathbb{R}^3)} \| P_{N_3} \nabla Iu \|_{L^2_tL^{\delta}_{x}(J \times \mathbb{R}^3)} \]

\[ \times \sum_{N_4 \leq \frac{1}{N^2}; N_1 \lesssim N_2} N_1 \| P_{N_1} \nabla Iu \|_{L^2_tL^{\delta}_{x}(J \times \mathbb{R}^3)} \| P_{N_4} Iu \|_{L^2_tL^{\delta}_{x}(J \times \mathbb{R}^3)} \]

\[ \lesssim \frac{\epsilon}{N^{5/2}} \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^3. \]

Combining all these cases together proves theorem 4.1 for (4.5).

**The term (4.6):** To estimate this term, a lemma is needed.

**Lemma 4.2**

\[ \| P_M I(|u|^2u) \|_{L^2_{t,x}(J \times \mathbb{R}^3)} \lesssim \left( \frac{1}{M} + \frac{1}{N} \right) \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^3. \] (4.9)
Proof: Decompose \( u \). Let \( u_t = P_{\leq N} u \) and \( u_t + u_h = u \).

\[
\| \nabla I (|u|^2 u_t) \|_{L^2_{t,x} (J \times \mathbb{R}^3)} \leq \| \nabla I u \|_{L^2_{t} L^6_x (J \times \mathbb{R}^3)} \| u_t \|_{L^\infty_{t} L^6_x (J \times \mathbb{R}^3)} \lesssim \| \nabla I u \|_{S^0 (J \times \mathbb{R}^3)}^{3/2}.
\]

\[
\| \nabla I (|u|^2 u_h) \|_{L^2_{t,x} (J \times \mathbb{R}^3)} \leq \| \nabla I u \|_{L^2_{t} L^6_x (J \times \mathbb{R}^3)} \| u_t \|_{L^\infty_{t} L^6_x (J \times \mathbb{R}^3)} \lesssim \| \nabla I u \|_{S^0 (J \times \mathbb{R}^3)}^{3/2}.
\]

Make a similar argument for \( u_t^2 u_h \).

\[
\| \nabla I (|u_h|^2 u_t) \|_{L^2_{t} L^{6/5}_x (J \times \mathbb{R}^3)} \lesssim \| \nabla I u \|_{L^2_{t} L^6_x (J \times \mathbb{R}^3)} \| u_t \|_{L^\infty_{t} L^6_x (J \times \mathbb{R}^3)} \| u_h \|_{L^\infty_{t} L^2_x (J \times \mathbb{R}^3)} \lesssim \frac{1}{N} \| \nabla I u \|_{S^0 (J \times \mathbb{R}^3)}^{3}.
\]

Then apply the Sobolev embedding theorem. Make a similar argument for \( u_t^2 \tilde{u}_t \). Here we applied (2.8) and (2.22),

\[
\| P_{> N} u \|_{L^\infty_{t} L^2_x (J \times \mathbb{R}^3)} \lesssim \frac{1}{N} \| \nabla I u \|_{L^\infty_{t} L^2_x (J \times \mathbb{R}^3)}.
\]  \hspace{1cm} (4.10)

\[
\| \nabla I (|u_h|^2 u_h) \|_{L^2_{t} L^{6/5}_x (J \times \mathbb{R}^3)} \lesssim \| \nabla I u \|_{L^2_{t} L^6_x (J \times \mathbb{R}^3)} \| u_h \|_{L^\infty_{t} H^{1/2}_x (J \times \mathbb{R}^3)} \lesssim \frac{1}{N} \| \nabla I u \|_{S^0 (J \times \mathbb{R}^3)}^{3/2}.
\]  \hspace{1cm} (4.11)

Finally, using the elementary inequality,

\[
\| P_M I (f(t, x)) \|_{L^2_{t,x} (J \times \mathbb{R}^3)} \lesssim \frac{1}{M} \| \nabla I (f(t, x)) \|_{L^2_{t,x} (J \times \mathbb{R}^3)},
\]  \hspace{1cm} (4.12)

gives the lemma. \( \square \)

The nonlinear term is a 6-linear term. Let \( \xi_{123} = \xi_1 + \xi_2 + \xi_3 \) and \( N_{123} = N_1 + N_2 + N_3 \).

\[
(4.10) = - \int_{t_1}^{t_2} \int_{\Sigma} iI(|u|^2 u_t)(t, \xi_{123}) \frac{m(\xi_4 + \xi_5 + \xi_6)}{m(\xi_4)m(\xi_5)m(\xi_6)} - 1] 
\times \hat{u}(t, \xi_4)\hat{u}(t, \xi_5)\hat{u}(t, \xi_6) d\xi dt,
\]  \hspace{1cm} (4.13)
where \( \Sigma = \{ \xi_1 + \xi_2 + \xi_3 + \xi_4 + \xi_5 + \xi_6 = 0 \} \) and \( d\xi \) is the measure on the hyperplane. Make a Littlewood-Paley partition of unity and assume without loss of generality \( N_4 \geq N_5 \geq N_6 \).

**Case 1,** \( N_4 \ll N \): In this case the multiplier is \( \equiv 0 \).

**Case 2,** \( N_4 \gg N, N_5 \ll N \): In this case the fundamental theorem of calculus will again be used. Because \( N_5, N_6 \ll N_4, N_{123} \sim N_4 \).

**Case 2(a),** \( N_6 \geq \frac{1}{N^{N_7}} \):

\[
\frac{m(\xi_4 + \xi_5 + \xi_6)}{m(\xi_4)m(\xi_5)m(\xi_6)} - 1 \lesssim \frac{|\xi_5|}{|\xi_4|}.
\]

\[\text{(4.14)}\]

\[
\lesssim \sum_{N \lesssim N_4 \sim N_{123}} \frac{1}{N_4} \|P_{N_{123}} I|u|^2u\|_{L^2_tL^6_x(J \times \mathbb{R}^3)} \|P_{N_4} Iu\|_{L^2_tL^6_x(J \times \mathbb{R}^3)}
\]

\[
\times \sum_{\frac{1}{N^2} \leq N_6 \leq N_5 < N} N_5 \|P_{N_5} Iu\|_{L^\infty_tL^6_x(J \times \mathbb{R}^3)} \|P_{N_6} Iu\|_{L^\infty_tL^6_x(J \times \mathbb{R}^3)}
\]

\[
\lesssim \ln(N) N \|\nabla u\|_{S^0(J \times \mathbb{R}^3)}^6 \sum_{N \lesssim N_4 \sim N_{123}} \frac{1}{N_4} \left( \frac{1}{N_{123}} + \frac{1}{N} \right) \lesssim \frac{1}{N^{2-}} \|\nabla u\|_{S^0(J \times \mathbb{R}^3)}^6.
\]

**Case 2(b):** \( N_6 \leq \frac{1}{N^{N_7}} \):

\[
\lesssim \sum_{N \lesssim N_{123} \sim N_4} \frac{1}{N_4} \|P_{N_{123}} I|u|^2u\|_{L^2_tL^6_x(J \times \mathbb{R}^3)} \|P_{N_4} Iu\|_{L^2_tL^6_x(J \times \mathbb{R}^3)}
\]

\[
\times \sum_{N_6 \leq N_5 < N; N_6 \leq \frac{1}{N^{N_7}}} N_5 \|P_{N_5} Iu\|_{L^\infty_tL^6_x(J \times \mathbb{R}^3)} \|P_{N_6} Iu\|_{L^4_tL^\infty_x(J \times \mathbb{R}^3)}
\]

\[
\lesssim \sum_{N \lesssim N_{123} \sim N_4} \left( \frac{1}{N} + \frac{1}{N_{123}} \right) \frac{N}{N_4} \frac{1}{N^{3/2}} \|\nabla u\|_{S^0(J \times \mathbb{R}^3)}^5 \epsilon
\]

12
\[
\lesssim \frac{\epsilon}{N^{7/2}} \| \nabla Iu \|^5_{S^0(J \times \mathbb{R}^3)}.
\]

**Case 3,** \(N_5 \gtrsim N, N_4 \sim N_{123}\): Here make the crude estimate,

\[
\frac{|m(\xi_4 + \xi_5 + \xi_6)|}{m(\xi_4) m(\xi_5) m(\xi_6)} - 1 \lesssim \frac{1}{m(\xi_5) m(\xi_6)}. \tag{4.15}
\]

**Case 3(a),** \(N_6 \geq \frac{1}{N} N_5\):

\[
\sum_{N \lesssim N_4 \sim N_{123}} \| P_{N_{123}} I \|_{L^2_t L^6_x(J \times \mathbb{R}^3)} \| P_{N_4} I u \|_{L^2_t L^6_x(J \times \mathbb{R}^3)}
\]

\[
\times \sum_{\frac{1}{N^2} \leq N_6 \leq N_5; N \leq N_5} \frac{1}{m(N_5) m(N_6)} \| P_{N_5} I u \|_{L^\infty_t L^6_x(J \times \mathbb{R}^3)} \| P_{N_6} I u \|_{L^\infty_t L^6_x(J \times \mathbb{R}^3)}.
\]

\[
\lesssim (\ln(N) + \frac{N_5^{1-s}}{N_1^{1-s}}) \| \nabla I u \|^2_{S^0(J \times \mathbb{R}^3)}.
\]

\[
\lesssim (\ln(N))^2 + \frac{N_4^{2(1-s)}}{N_2^{2(1-s)}} \| \nabla I u \|^2_{S^0(J \times \mathbb{R}^3)}.
\]

So in this case,

\[
\lesssim \| \nabla I u \|^6_{S^0(J \times \mathbb{R}^3)} \sum_{N \lesssim N_4 \sim N_{123}} \frac{1}{N} (\ln(N) + \frac{N_4^{1-s}}{N_1^{1-s}})^2 \frac{1}{N_4} \lesssim \frac{1}{N^2} \| \nabla I u \|^6_{S^0(J \times \mathbb{R}^3)}.
\]

**Case 3(b),** \(N_6 \leq \frac{1}{N^2}\):

\[
\lesssim \sum_{N \lesssim N_{123} \sim N_4} \| P_{N_{123}} I \|_{L^2_t L^6_x(J \times \mathbb{R}^3)} \| P_{N_4} I u \|_{L^2_t L^6_x(J \times \mathbb{R}^3)}
\]
\[
\times \sum_{N_5 \gtrsim N_5; N_6 \leq \frac{1}{N^2}} \frac{1}{m(N_5)} \| P_{N_5} I u \|_{L^6_t L^6_x(\mathbb{R}^3)} \| P_{N_6} I u \|_{L^\infty_t L^\infty_x(\mathbb{R}^3)}
\]

\[
\lesssim \sum_{N \lesssim N_1 \sim N_4} \left( \frac{1}{N} + \frac{1}{N_123} \right) \frac{1}{N_4} N_4^{1-s} \epsilon \| \nabla I u \|_{S^0(\mathbb{R}^3)}^5 \leq \frac{\epsilon}{N^{7/2}} \| \nabla I u \|_{S^0(\mathbb{R}^3)}^5.
\]

**Case 4**, \( N_5 \gtrsim N, N_4 \sim N_5, N_{123} \lesssim N_4 \): Once again use the crude estimate.

**Case 4(a)**, \( N_6 \geq \frac{1}{N^2} \):

\[
(4.6) \lesssim \sum_{N \lesssim N_4 \sim N_5} \frac{1}{m(N_5)} \| P_{N_4} I(u) \|_{L^4_t L^3_x(\mathbb{R}^3)} \| P_{N_5} I u \|_{L^4_t L^3_x(\mathbb{R}^3)}
\]

\[
\times \left[ \sum_{\frac{1}{N^2} \leq N_6 \leq N_5; \frac{1}{N^2} \leq N_{123} \leq N_4} \frac{1}{m(N_6)} \| P_{N_{123}} I(|u|^2 u) \|_{L^2_t L^6_x(\mathbb{R}^3)} \| P_{N_6} I u \|_{L^\infty_t L^6_x(\mathbb{R}^3)} \right] \leq \| \nabla I u \|_{S^0(\mathbb{R}^3)}^6 \sum_{N \lesssim N_4 \sim N_5} \frac{1}{N_4 N_5 m(N_5)}
\]

\[
\times \left[ \sum_{\frac{1}{N^2} \leq N_6 \leq N_5; \frac{1}{N^2} \leq N_{123} \leq N_4} \frac{1}{m(N_6)} \left( 1 + \frac{N_{123}}{N} \right) + \sum_{\frac{1}{N^2} \leq N_6 \leq N_5; N_{123} \leq \frac{1}{N^2}} \left( \frac{N_{123}^{3/2}}{N} + N_{123}^{1/2} \right) \frac{1}{N_6^{1/2}} \right] \leq \frac{1}{N^{2-}} \| \nabla I u \|_{S^0(\mathbb{R}^3)}^6.
\]

**Case 4(b)**, \( N_6 \leq \frac{1}{N^2} \):
\[\begin{align*}
&\lesssim \sum_{N \leq N_1 \sim N_3} \frac{1}{m(N_5)} \|P_{N_4} I u\|_{L^6_t L^3_x(J \times \mathbb{R}^3)} \|P_{N_5} I u\|_{L^3_t L^6_x(J \times \mathbb{R}^3)} \\
&\quad \times \left[ \sum_{N_1 \leq N \leq N_4} \|P_{N_2} I u\|_{L^6_t L^\infty_x(J \times \mathbb{R}^3)} \|P_{N_5} I u\|_{L^3_t L^6_x(J \times \mathbb{R}^3)} \right] \\
&\quad + \sum_{N_5 \leq N \leq N_4; N_6 \leq \frac{1}{N^7}} \|P_{N_5} I u\|_{L^6_t L^6_x(J \times \mathbb{R}^3)} \|P_{N_6} I u\|_{L^3_t L^6_x(J \times \mathbb{R}^3)} \right] \\
&\lesssim \|\nabla I u\|_{L^5_x(J \times \mathbb{R}^3)}^6 \sum_{N \leq N_4 \sim N_5} \frac{1}{N_4 N_5 m(N_5)} \left[ \frac{\epsilon}{N_3} + (\ln(N) + \frac{N_4}{N} \cdot \frac{\epsilon}{N_3}) \right] \\
&\lesssim \frac{\epsilon}{N^{7/2}} \|\nabla I u\|_{L^6_x(J \times \mathbb{R}^3)}^5.
\end{align*}\]

This concludes the proof of theorem 4.1. \(\square\)

5 A Smoothing Estimate

In this section we take advantage of lemma 2.2 to prove a smoothing-type estimate for the Duhamel term.

Lemma 5.1 Take \(N_j \leq N\), if \(\|u\|_{L^4_t L^6_x(J \times \mathbb{R}^3)} \leq \epsilon\), then

\[\|P_{N_j} |u|^2 u\|_{L^5_t L^2_x(J \times \mathbb{R}^3)} \lesssim \frac{1}{N_j} \|P_{N_j} \nabla I (|u|^2 u)\|_{L^5_t L^2_x(J \times \mathbb{R}^3)} \lesssim \frac{1}{N_j} \|\nabla I u\|_{L^5_x(J \times \mathbb{R}^3)}^3. \tag{5.1}\]

Proof: The first inequality follows from Littlewood-Paley theory.

\[\|P_{N_j} \nabla I (|u|^2 u)\|_{L^5_t L^2_x(J \times \mathbb{R}^3)} \lesssim \|\nabla I u\|_{L^5_t L^6_x(J \times \mathbb{R}^3)} \times (\|P_{\leq 1} u\|_{L^6_t L^6_x(J \times \mathbb{R}^3)} + \|P_{>1} u\|_{L^6_t L^6_x(J \times \mathbb{R}^3)}).\]

By the Sobolev embedding theorem,

\[\|P_{\leq 1} u\|_{L^6_t L^6_x(J \times \mathbb{R}^3)} \lesssim \|u\|_{L^4_t L^4_x(J \times \mathbb{R}^3)}\]
On the other hand,
\[ \| P_{N_k} u \|_{L^4_t L^{6/5}_x (J \times \mathbb{R}^3)} \lesssim N_k^{1/2} \| P_{N_k} u \|_{S^0(J \times \mathbb{R}^3)}. \]

Therefore,
\[
\| P_{>1} u \|_{L^4_t L^{6/5}_x (J \times \mathbb{R}^3)} \lesssim \sum_{1 \leq N_k \leq N} \frac{1}{N_k^{1/2}} \| \nabla I u \|_{S^0(J \times \mathbb{R}^3)}
+ \sum_{N_k > N} \frac{1}{N_k^{s-1/2} N_1^{-s}} \| \nabla I u \|_{S^0(J \times \mathbb{R}^3)} \lesssim \| \nabla I u \|_{S^0(J \times \mathbb{R}^3)}.
\]

\[ \square \]

**Theorem 5.2** Suppose \( J = [0,T] \) is an interval with \( \| u \|_{L^4_t L^6_x (J \times \mathbb{R}^3)} \leq \varepsilon \) and \( \| \nabla I u_0 \|_{L^2(\mathbb{R}^3)} \leq 1 \). The solution has the Duhamel form

\[
 u(t) = e^{it\Delta} u_0 + \int_0^t e^{i(t-\tau)\Delta} (|u|^2 u)(\tau) d\tau = u^l(t) + u^{nl}(t),
\]

with
\[
\| P_{>N} \nabla I u^{nl} \|_{S^0(J \times \mathbb{R}^3)} \lesssim \frac{1}{N^{-1/2}} (\| \nabla I u \|_{S^0(J \times \mathbb{R}^3)} + \| \nabla I u \|_{S^0(J \times \mathbb{R}^3)}^7). \quad (5.3)
\]

**Proof:** Split \( u = u_l + u_h \) with \( u_l = P_{\leq N/20} u \). Since \( P_{>N} |u_l|^2 u_l \equiv 0 \), it suffices to consider \( O(u^2 u_h) \).

\[
\| P_{>N} \nabla I u^{nl} \|_{S^0(J \times \mathbb{R}^3)} \lesssim \frac{1}{N^{-1/2}} (\| \nabla I u \|_{S^0(J \times \mathbb{R}^3)} + \| \nabla I u \|_{S^0(J \times \mathbb{R}^3)}^7). \quad (5.3)
\]
\[
+ \left( \sum_{N_j \geq N/2, \frac{1}{N_j^2} \leq N_k \leq \frac{N}{50}} \frac{1}{N_k N_j^{1/2-\delta}} \right) \left( \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^2 + \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^6 \right) \epsilon \\
+ \frac{1}{N^{1/2}} \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^2 \epsilon^2 \\
\lesssim \frac{1}{N^{1/2}} \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^3 (1 + \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^6).
\]

The term \(u_l^2 \bar{u}_h\) follows similarly. The other terms are easier to estimate.

\[
\| \nabla I(|u_h|^2 u_l) \|_{L^2_t L_t^{6/5}(J \times \mathbb{R}^3)} \lesssim \| \nabla Iu \|_{L^2_t L_t^{6/5}(J \times \mathbb{R}^3)} \| u_h \|_{L^\infty_t L^6_x(J \times \mathbb{R}^3)} \| u_l \|_{L^\infty_t L^6_x(J \times \mathbb{R}^3)} \\
\lesssim \frac{1}{N^{1/2}} \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^3.
\]

A similar calculation can be made for \(\bar{u}_l u_h^2\). Finally,

\[
\| \nabla I(|u_h|^2 u_l) \|_{L^1_t L_t^2(J \times \mathbb{R}^3)} \lesssim \| \nabla Iu \|_{L^2_t L_t^{6/5}(J \times \mathbb{R}^3)} \| u_h \|_{L^2_t L_t^6(J \times \mathbb{R}^3)} \lesssim \frac{1}{N^{1/2}} \| \nabla Iu \|_{S^0(J \times \mathbb{R}^3)}^3.
\]

This proves the theorem. \(\square\)

### 6 Double Layer I-decomposition

Now we finally have enough tools to prove the main theorem.

**Theorem 6.1** Suppose \(s > 3/4\). Then (1.1) is globally well-posed on \([0, \infty)\). Moreover, \(\| u(t) \|_{H^s(\mathbb{R}^3)} \leq C(s, \| u_0 \|_{H^s(\mathbb{R}^3)})\), and there is scattering.

**Proof:** In three dimensions, (1.1) is \(\dot{H}^{1/2}\) - critical. That is, if \(u(t, x)\) solves (1.1) on \([0, T]\), then \(\frac{1}{\lambda} u(\frac{x}{\lambda}, \frac{t}{\lambda^2})\) solves (1.1) on \([0, \lambda^2 T]\). This scaling leaves the \(\dot{H}^{1/2}\) norm invariant. We will denote the rescaled solution \(u_\lambda(t, x)\).

\[
\| u_\lambda(0, x) \|_{L^2(\mathbb{R}^3)} = \lambda^{1/2} \| u_0 \|_{L^2(\mathbb{R}^3)},
\]

\[
\| u_\lambda(0, x) \|_{\dot{H}^1(\mathbb{R}^3)} = \lambda^{-1/2} \| u_0 \|_{\dot{H}^1(\mathbb{R}^3)}.
\]

Combining the scaling identities with the estimates on the I - operator, (2.7),
\[ \int \left| \nabla Iu_{0,\lambda}(x) \right|^2 dx \leq \frac{CN^2(1-s)}{\lambda^{2s-1}} \|u_0\|_{H^s}(\mathbb{R}^3)^2. \]

\[ \int \left| Iu_{0,\lambda}(x) \right|^4 dx \leq \frac{CN^3-4s}{\lambda^{3s-2}} \|u_0\|_{H^s}(\mathbb{R}^3)^4. \]

Choose \( \lambda \sim N^{\frac{1}{s-1/2}} \) so that \( E(Iu_0) \leq \frac{1}{2} \). Define a set

\[ W = \{ t : E(Iu_\lambda(t)) \leq \frac{9}{10} \}. \] (6.3)

Since \( 0 \in W \), \( W \neq \emptyset \). Also, by the dominated convergence theorem, \( W \) is closed. So it remains to prove \( W \) is open.

If \( [0, T] = W \), then there exists \( \delta > 0 \) such that \( E(Iu_\lambda(t)) \leq 1 \) on \( [0, T+\delta] \).

\[ \| P_{\leq \lambda} u \|_{L^\infty_t H^{1/2}(J \times \mathbb{R}^3)} \leq \| u_0 \|_{L^2(\mathbb{R}^3)}^{1/2} \| \nabla Iu \|_{L^\infty_t L^2_x(J \times \mathbb{R}^3)}^{1/2}. \] (6.4)

Also,

\[ \| P_{> \lambda} u \|_{L^\infty_t H^{1/2}(J \times \mathbb{R}^3)} \leq \frac{1}{N^{1/2}} \| \nabla Iu \|_{L^\infty_t L^2_x(J \times \mathbb{R}^3)}. \] (6.5)

Combining the interaction Morawetz estimates

\[ \| u \|_{L^4_{t,x}([0,T+\delta] \times \mathbb{R}^3)} \lesssim \| u_0 \|_{L^2(\mathbb{R}^3)}^2 \| u \|_{L^\infty_t H^{1/2}_x(J \times \mathbb{R}^3)}^2. \] (6.6)

(6.4) and (6.5),

\[ \| u \|_{L^4_{t,x}([0,T+\delta] \times \mathbb{R}^3)} \lesssim CN^{\frac{3(1-s)}{2s-1}}. \] (6.7)

Partition \( [0, T+\delta] \) into \( N^{\frac{3(1-s)}{2s-1}} \) subintervals with \( \| u_\lambda \|_{L^4_{t,x}(J \times \mathbb{R}^3)} \leq \epsilon \) for each \( J_k \).

Now we will make use of a double-layered I-decomposition utilized in [9]. Subdivide \( [0, T+\delta] \) into subintervals \( J_k \), each \( J_k \) is the union of \( N^{1/2-s} \) subintervals \( J_{k,m} \) with \( \| u_\lambda \|_{L^4_{t,x}(J_{k,m} \times \mathbb{R}^3)} \leq \epsilon \) on each such subinterval. We will refer to the intervals \( J_k \) as the big intervals, and the subintervals \( J_{k,m} \) as the little intervals.

Take the first big interval \( J_k \). Crudely, by (4.2), \( E(Iu(t)) \leq 1 \) on this big interval. Subdivide \( J_k = \bigcup_{j=0}^{N^{1/2-s}} J_{k,m} \), let \( J_{k,m} = [a_m, b_m] \), \( a_0 = 0 \), \( a_{m+1} = b_m \). The solution on \( J_{k,m} \) will be written in the form
\[ e^{i(t-a_m)} \Delta u(a_m) + u^n_j(t) = e^{it} \Delta u_0 + \sum_{j=1}^{m} e^{i(t-a_j)} \Delta u^n_{j-1} + u^n_m(t). \] (6.8)

\[ \sup_{t_1, t_2 \in J_k} |E(Iu(t_1)) - E(Iu(t_2))| \lesssim \frac{N^{1/2}}{N^{2-}} + \frac{1}{N^{1-}} \|P_{>cN} \nabla Iu\|_{L^6_{L^2}((J_k \times \mathbb{R}^3))}^2. \] (6.9)

Now, by (6.8),

\[ \|P_{>cN} \nabla Iu\|_{L^6_{L^2}((J_k \times \mathbb{R}^3))} \leq \|P_{>cN} \nabla u_0\|_{L^6_{L^2}((J_k \times \mathbb{R}^3))} + \sum_{m=1}^{N^{1/2-}} \|\nabla P_{>cN} u^n_m(a_m)\|_{L^6_{L^2}((J_k \times \mathbb{R}^3))}^2 \]

\[ + \left( \sum_{m=0}^{N^{1/2-}} \|P_{>cN} u^n_m(a_m)\|_{L^6_{L^2}((J_k \times \mathbb{R}^3))} \right)^{1/2} \] (6.10)

\[ \|\nabla u_0\|_{L^6_{L^2}((J_k \times \mathbb{R}^3))} \lesssim 1, \]

which takes care of the first term. Similarly, by (5.3),

\[ \sum_{m=1}^{N^{1/2-}} \|\nabla u^n_m(a_m)\|_{L^6_{L^2}((J_k \times \mathbb{R}^3))} \lesssim \frac{N^{1/2-}}{N^{1/2-}} = 1, \]

which takes care of the second term. Finally,

\[ \left( \sum_{m=0}^{N^{1/2-}} \|P_{>cN} u^n_m\|_{L^6_{L^2}((J_k \times \mathbb{R}^3))} \right)^{1/2} \lesssim \left( \frac{N^{1/2-}}{N^{1/2-}} \right)^{1/2} \lesssim 1. \]

In particular, this proves

\[ \sup_{t_1, t_2 \in J_k} |E(Iu(t_1)) - E(Iu(t_2))| \lesssim \frac{1}{N^{1-}}. \] (6.11)

When \( s > 3/4, \)

\[ CN^{\frac{3(1-s)}{2s}} \ll N^{3/2-}, \]

so choosing \( N \) sufficiently large proves
\[
\sup_{[0,T+\delta]} E(Iu_\lambda(t)) \leq \frac{9}{10}, \quad (6.12)
\]
This proves \( W = [0, \infty) \).

Finally, we prove scattering, following the argument in [7]. There is some \( N \) such that
\[
E(Iu_\lambda(t)) \leq 1 \quad (6.13)
\]
on \([0, \infty)\). By the interaction Morawetz estimates, (6.7),
\[
\|u_\lambda\|_{L_t^4([0,\infty) \times \mathbb{R}^3)} \leq C. \quad (6.14)
\]
Recall that by (3.1),
\[
\|u\|_{L_t^6L_x^{9/2}([0,\infty) \times \mathbb{R}^3)} \lesssim (\epsilon^{2/3} + \frac{1}{N^{1/2}}) \|\nabla Iu\|_{S_0([0,\infty) \times \mathbb{R}^3)}. \quad (6.15)
\]
Therefore, on each subinterval \( J_{k,m} \),
\[
\|u\|_{L_t^6L_x^{9/2}(J \times \mathbb{R}^3)} \lesssim (\epsilon^{2/3} + \frac{1}{N^{1/2}}).
\]
Let
\[
S_s(t) = \sup_{(p,q) \text{ admissible}} \|\langle \nabla \rangle^s u\|_{L_t^pL_x^q([0,t] \times \mathbb{R}^3)}. \quad (6.16)
\]
\[
S_s(t) \lesssim \|\langle \nabla \rangle^s u_0\|_{L^2(\mathbb{R}^3)} + \|\langle \nabla \rangle^s u\|_{L_t^6L_x^{9/2}(J \times \mathbb{R}^3)} \|u\|_{L_t^6L_x^{9/2}(J \times \mathbb{R}^3)}^2
\]
\[
\lesssim \|\langle \nabla \rangle^s u_0\|_{L^2(\mathbb{R}^3)} + S_s(t)(\epsilon^{4/3} + \frac{1}{N}).
\]
So for \( \epsilon > 0 \) sufficiently small and \( N \) sufficiently large, this proves \( S_s(t) \) is bounded on the first subinterval. Iterating over a finite number of subintervals proves \( Z_s(t) \leq C < \infty \) for \( t \in [0, \infty) \). In particular, this proves
\[
\|u\|_{H^s(\mathbb{R}^3)} \leq C\|u_0\|_{H^s(\mathbb{R}^3)}. \quad (6.17)
\]
Now set
\[
u_+ = u_0 + \int_0^\infty e^{-i\tau\Delta} |u(\tau)|^2 u(\tau) d\tau. \quad (6.18)
\]
\[ \left\| \langle \nabla \rangle^s (e^{it\Delta} u(t, x) - u(t, x)) \right\|_{L^2_t(\mathbb{R}^3)} = \left\| \int_t^\infty \langle \nabla \rangle^s e^{-i\tau\Delta} |u(\tau)|^2 u(\tau) d\tau \right\|_{L^2_t(\mathbb{R}^3)} \]
\[ \lesssim \left\| \langle \nabla \rangle^s u \right\|_{L^{10/3}_{t,x}([T, \infty) \times \mathbb{R}^3)} \left\| u \right\|_{L^5_{t,x}([T, \infty) \times \mathbb{R}^3)}^2. \] (6.19)

As \( T \to \infty \), \( \|u\|_{L^4_{t,x}([T, \infty) \times \mathbb{R}^3)} \to 0 \), on the other hand,
\[ \left\| u \right\|_{L^6_{t,x}([0, \infty) \times \mathbb{R}^3)} \lesssim \left\| \langle \nabla \rangle^{2/3} u \right\|_{L^6_t L^{16/7}_{x}([0, \infty) \times \mathbb{R}^3)} \lesssim S_{2/3}(t) < \infty, \] (6.20)

by (6.16). Interpolating proves \( \|u\|_{L^5_{t,x}([T, \infty) \times \mathbb{R}^3)} \to 0 \) as \( T \to \infty \). [7] proved the existence of wave operators, which completes the proof of Theorem 1.1. □
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