SELF-SIMILAR EXTRAPOLATION FOR THE LAW OF ACOUSTIC EMISSION BEFORE FAILURE OF HETEROGENEOUS MATERIALS

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Abstract

Acoustic emission before the failure of heterogeneous materials is studied as a function of applied hydrostatic pressure. A formula for the energy release is suggested, which is valid in the whole diapason of pressures, from zero to the critical pressure of rupture. This formula is obtained by employing the extrapolation technique of the self-similar approximation theory. The result is fitted to experiment in order to demonstrate the correct general behaviour of the obtained expression for the energy release.

1 Introduction

There exists a widespread understanding in scientific community that the moment of rupture in heterogeneous materials is somewhat similar to a critical point. The fracturing process is then a kind of a second-order phase transition \cite{2,4}. This underlies the intuitive assumption which compares global failure in disordered materials at mesoscopic scale with percolation at microscopic scale. So, as percolation, global failure is assumed to be a critical phenomenon. The fracturing process is then classified as a self-organised random irreversible process. Extensive numerical simulations and experimental measurements have confirmed this conclusion. This concerns the global failure of materials as well as earthquakes \cite{9}, since an earthquake can also be considered as a global failure of a large material mass. The classical power law, defining the energy release rate, has been verified in the critical region, close to the time of either a large earthquake or a global materials failure. The related critical exponent is, in general, a complex number, which is due to taking into account long-range elastic interactions transported by stress fields around defects and cracks. The imaginary part of the exponent gives rise to the so-called log-periodic corrections which have been identified quite early in renormalization group solutions for critical phase transitions \cite{7,10}.
Experimental results of acoustic emission measurements and seismograms unambiguously revealed the existence of oscillatory corrections before materials fractures and earthquakes, respectively [1,10]. If one would know the general law describing this acoustic emission, one would be able to make an early prediction of the global failure in heterogeneous materials. It is a principal, not yet solved problem, to find such a general law, which would be valid not solely in the asymptotic vicinity of the critical point, but in the whole interval of pressures or times, both near the critical point of failure as well as far from it. And this is the main aim of the present paper to suggest a mathematically grounded derivation for such a law and to confront it to available experimental data [5]. For this purpose, we employ the recent progress in the self-similar approximation theory [12–18] which has been successfully applied to the theory of critical phenomena and to time series forecasting [19–21]. We would like to stress here that this theory is general and can be used for any given series of experimental data, whether this concerns phase transitions, financial markets, or any other series.

2 Modelling

Let us be interested in the behaviour of the energy release $E(p)$ as a function of pressure $p$, considered between $p \approx 0$, i.e. at the initial stage when the very early damages occur, and up to the point $p = p_c$ of global failure. Note that $p$ refers to fracturing under applied spatially uniform loading, as in experiments where water pressure inside a spherical tank increases up to the material failure. Experimentally, the applied pressure is usually increased linearly with time. The behaviour of $E(p)$ at the vicinity of the critical point of rupture is

$$E(p) \approx E(p_c) + A(p_c - p)^\alpha + B(p_c - p)^\alpha \cos[\omega \ln(p_c - p) + \varphi],$$  \hspace{1cm} (1)

as $p \to p_c$, where $A, B, \alpha, \omega$ and $\varphi$ are parameters ($\alpha$ and $\omega$ are the real and imaginary parts, respectively, of the so-called critical complex exponent). The problem is how to find the behaviour of $E(p)$ in the whole range $0 \leq p \leq p_c$? Here we consider the case of one variable, pressure. A more general loading, involving not one variable, like $p$, but several variables can also be treated by the theory. However, before being involved in such generalisations, we would like to show on a simpler example how to accomplish the extrapolation. Our main aim in this letter is to demonstrate the general possibility of realising such an accurate extrapolation for the energy release.

Let us transform Eq. (1) by introducing the dimensionless variable $x \equiv (p_c - p)/p_c$, $(0 \leq x \leq 1)$ and the reduced energy release $f(x) \equiv E(p)/E(p_c)$ which is a dimensionless function. Then, expansion (1) takes the form

$$f(x) \approx 1 + \tilde{a}(x) x^\alpha, \quad \tilde{a}(x) \equiv \lambda[1 + \mu \cos(\omega \ln x + \beta)]$$ \hspace{1cm} (2)

whose parameters $\beta, \lambda$, and $\mu$ can be easily expressed through $p_c, E(p_c), A, B, \omega$, and $\varphi$. Note that function $\tilde{a}(x)$ cannot be expanded in powers of $x$ near zero. Therefore, expansion (2) can be treated as a generalised asymptotic series, as defined by Poincare [8], with $\tilde{a}(x)$ considered as a coefficient. Employing the self-similar approximation theory [12–21], the asymptotic series (2) can be extrapolated to

$$f^*(x) = \exp[c(x) x^\alpha], \quad c(x) = [1 + \mu \cos(\omega \ln x + \beta)] \tau, \hspace{1cm} (3)$$
where \( \tau \) has to be defined from an optimization or boundary condition. As is clear, the boundary condition \( E(0) \approx 0 \) is not convenient here, leading to large errors in defining \( \tau \) because of the physical impossibility to determine the very early precursors with a high accuracy. A more judicious choice is to consider a global physical quantity. In the same way as treating \( E(p) \) as the integrated energy release rate, we may define \( F(p) \) as the integral energy release 
\[
\int_0^\infty E(q) dq.
\]
Given \( p_0 \), the parameter \( \tau \) then can be defined from the sum optimization rule
\[
\int_{y_0=\frac{p_c-p_0}{p_c}}^1 \! f^*(x) \, dx = \frac{F(p_0)}{p_c E(p_c)} \quad (y_0 \to 0, \ p_0 \to p_c - 0). \tag{4}
\]
In general, the lower limit in the integral (4) can pertain to the whole interval \([0,1]\). In reality, the trustful experimental data are available only below \( p_c \). One has to choose the data at the highest available pressure, so that the lower limit in the sum rule (4) be sufficiently small.

In this way, the self-similar approximant (3) becomes
\[
f^*(x) = \exp \left\{ [1 + \mu \cos(\omega \ln x + \beta)] \tau x^\alpha \right\} \tag{5}
\]
with \( \tau \) given by the sum rule (4). Note that this formula contains the same number of parameters as the initial formula (1). Returning, with the help of relations (1) and (2), to the energy release, we come to the formula
\[
E^*(p) = E(p_c) f^*(x), \tag{6}
\]
which extrapolates \( E(p) \) for the whole interval \( 0 \leq p \leq p_c \). It is worth stressing that the law (6) is obtained as a self-similar extrapolation of the acoustic emission signals observed at the vicinity of the rupture. Another possibility would be to base this kind of extrapolation, starting from the signals existing at the initial stage of the process, being yet very far from the rupture. This latter way was used by Gluzman et al. [3], who started with a given polynomial expansion at the early time of acoustic emissions, treating the power law behaviour close to rupture as a boundary condition. To our mind, this opposite approach has two weak points: First of all, the early acoustical precursors, as is known, are very small, being embodied in the acoustical noise, because of which they can be hardly measured with a good accuracy. Second, it looks difficult or even impossible to extract information on log-periodic corrections from the early acoustic signals. Another phenomenological expression for the energy release in the noncritical region has been suggested by Sonnette and Andersen [10] whose arguments were based on the existence of a scaling of the macroscopic elastic modulus and the elastic energy release rate in the thermal fuse model [6]. The following phenomenological form has been proposed [5]
\[
E(p) \approx E(p_c) A \left( \tanh \frac{p_c - p}{\tau_0} \right)^\alpha + B \left( \tanh \frac{p_c - p}{\tau_0} \right)^\alpha \cos \left\{ \omega \left( \tanh \frac{p_c - p}{\tau_0} \right) + \varphi \right\}, \tag{7}
\]
which is a pure power law, like Eq. (1), in the critical region close to rupture and exponentially relaxes in the noncritical region far from rupture, where only a few damages occur. However, the suggested form (7) possesses an unphysical behaviour, being negative in a wide
range of its variable. At the same time, our self-similar formula (5) yields the energy release (6) that is always positive. In Fig. 1, we compare the behaviour of $E(p)$, in the dimensionless form, for both Eqs. (6) and (7), with the parameters chosen so that to have similar variations when approaching the critical point. In this figure, one can clearly see that Eq. (7) possesses unphysical negative values for the major part of the interval [0,1], while Eq. (6) is everywhere positive.

3 Comparison with experimental data

In order to fit our self-similar formula to experimental results of acoustic emission measurements, we refer to the most accurate data, available nowadays, that is those of Anifrani’s team [1] at Aerospatiale-Matra Inc., which are reported by Johansen and Sornette [5]. In these measurements, the acoustic emission has been recorded during the pressurisation of spherical tanks of kevlar or carbon fibres pre-impregnated in a resin matrix wrapped up around a thin metallic liner (steel or titanium) fabricated and instrumented by Aerospatiale-Matra Inc. It has been found that the seven acoustic emission recordings of seven pressure tanks, which were brought to rupture, exhibit a clear acceleration in agreement with a power law divergence as expected by the critical point theory. A strong evidence of oscillatory corrections, forming the intermittent succession of bursts of acoustic emission, when approaching the rupture, has been clearly identified.

In Fig. 2, we present the energy release rate, which is the derivative of our formula (6) for the cumulative energy release. The result is obtained by fitting to the experimental curve reported at the right bottom in figure 1 of the paper by Johansen and Sornette (2000). These experimental data explicitly show an oscillatory behaviour of the energy release rate before the rupture. The best fit for the five parameters $\alpha$, $\beta$, $\mu$, $\tau$, and $\omega$ have been obtained for 0.7, 1.9, -0.045, 7.5 and 15, respectively, under the given pressure of rupture at 673 bars. The parameter $\tau$ satisfies Eq. (4). The upper pressure $p_0 \approx 665$ bars, measured before the rupture, corresponds to the value $y_0 \approx 0.0119$, which is sufficiently small, in agreement with condition (4).

4 Conclusion

The parameters can change for different materials as well as for different experimental setups. However, the three key physical parameters: $\alpha$, $\omega$, and $\tau$ should be universal, not essentially depending on the details of the external stress in the critical region. To prove this assumption, it is necessary to accomplish a set of experiments for appropriate materials. Selecting particular materials, we should keep in mind that: (i) the width of the critical region is linearly proportional to the strength of disorder, (ii) long-range elastic interactions are responsible for log-periodic corrections, (iii) the more brittle is the fracturing process, the stronger are the elastic interactions and stronger the energy release fluctuates, (iv) the fluctuations in the energy release are also stronger when a high value of a material characteristic is alternated with a low value of that characteristic. Thus, elastic-porous materials are expected to be the best choice.
References

[1] Anifrani J. C., Le Floch C., Sornette D. and Souillard B., Universal log-periodic correction group scaling for rupture stress prediction from acoustic emission. J. Phys. I France 5 (1995) 631-638.

[2] Broberg B., Cracks and Fractures. Cambridge, London (1999).

[3] Gluzman S., Andersen J. V. and Sornette D., Functional renormalization prediction of rupture. In A. Levshin, G. Molchan and B. Naimark (eds.) Computational Seismology. Moscow, GEOS (2001).

[4] Herrmann H. J. and Roux S., Statistical Models for the Fracture of Disordered Media. North-Holland, Amsterdam (1990).

[5] Johansen A. and Sornette D., Critical ruptures. Eur. Phys. J. B 18 (2000) 163-181.

[6] Lamagnere L., Carmona F. and Sornette D., Experimental realization of critical thermal fuse rupture. Phys. Rev. Lett. 77 (1996) 2738-41.

[7] Nauenberg M. et al., Scaling representation for critical phenomena. J. Phys. A 8 (1975) 925-928.

[8] Poincare H., New Methods of Celestial Mechanics. Am. Inst. Phys., New York (1993).

[9] Sahimi M. and Arbabi S., Scaling laws for fracture of heterogeneous materials and rocks. Phys. Rev. Lett. 77 (1996) 3689-3692.

[10] Sornette D. and Andersen J. V., Scaling with respect to disorder. Eur. Phys. J. B 1 (1998) 353-357.

[11] Sornette D., Discrete scale invariance and complex dimensions. Phys. Rep. 297 (1998) 239-270.

[12] Yukalov V.I., Statistical mechanics of strongly nonideal systems. Phys. Rev. A 42 (1990) 3324-3334.

[13] Yukalov V.I., Method of self-similar approximations. J. Math. Phys. 32 (1991) 1235-1239.

[14] Yukalov V.I., Stability conditions for method of self-similar approximations. J. Math. Phys. 33 (1992) 3994-4001.

[15] Yukalov V.I. and Gluzman S., Critical indices as limits of control functions. Phys. Rev. Lett. 79 (1997) 333-336.

[16] Yukalov V.I. and Gluzman S., Self-similar bootstrap of divergent series. Phys. Rev. E 55 (1997) 6552-6565.
[17] Yukalov V.I. and Gluzman S., Self-similar exponential approximants. Phys. Rev E 58 (1998) 1359-1382.

[18] Yukalov V.I., Yukalova E.P. and Gluzman S., Self-similar interpolation in quantum mechanics, Phys. Rev. A 58 (1998) 96-115.

[19] Yukalov V. I. and Gluzman S., Weighted fixed points in self-similar analysis of time series. Int. J. Mod. Phys. B 13 (1999) 1463-1476.

[20] Yukalov V. I., Self-similar extrapolation of asymptotic series and forecasting for time series. Mod. Phys. Lett. B 14 (2000) 791-800.

[21] Yukalov V.I., Self-similar approach to market analysis. Eur. Phys. J. B 20 (2001) 609-617.
Figure Captions

**Fig. 1.** General form of the dimensionless energy release corresponding to Eq. (6), with $\alpha = 0.7$, $\omega = 6.9$ and $\tau = 6.6$ (continuous line) and to Eq. (7), with $\alpha = 0.7$, $\omega = 8$, and $\tau_0 = 0.8$ (broken line).

**Fig. 2.** Dimensionless energy release rate from Eq. (5), fitted to the experimental data reported by Johansen and Sornette (2000).
