Cell decompositions for rank two quiver Grassmannians

Dylan Rupel1 · Thorsten Weist2

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Abstract
We prove that all quiver Grassmannians for exceptional representations of a generalized Kronecker quiver admit a cell decomposition. In the process, we introduce a class of regular representations which arise as quotients of consecutive preprojective representations. Cell decompositions for quiver Grassmannians of these “truncated preprojectives” are also established. We provide two combinatorial labelings for these cells. On the one hand, they are naturally labeled by certain subsets of a so-called 2-quiver attached to a (truncated) preprojective representation. On the other hand, the cells are in bijection with compatible pairs in a maximal Dyck path as predicted by the theory of cluster algebras. The provided bijection between these two labelings gives a geometric explanation for the appearance of Dyck path combinatorics in the theory of quiver Grassmannians.

Keywords Quiver Grassmannians · Torus action · Generalized Kronecker quiver · Cell decompositions · Combinatorial labeling of cells · Dyck path combinatorics

Mathematics Subject Classification 16G20 · 14M15 · 13F60

1 Introduction
A quiver Grassmannian is a projective variety attached to a fixed quiver representation which parametrizes subrepresentations of a fixed dimension vector. In recent years, interest in quiver Grassmannians has grown considerably. On the one hand, this is due to the fact that generating functions for the Euler characteristics of quiver Grassmannians of exceptional representations can be found as cluster variables [8]. On the other hand, they are clearly interesting on their own as they reveal many properties of the representation and its geometry.

Although it follows from the results of Hille, Huisgen-Zimmermann, Reineke, and Ringel that every projective variety can be realized as a quiver Grassmannian, it turns out that very
interesting phenomena arise when restricting to certain quivers or to representations with certain properties. For instance, quiver Grassmannians attached to exceptional representations are smooth [12]. For Dynkin quivers and tame quivers of types $\tilde{A}$ or $\tilde{D}$, it is known that every quiver Grassmannian attached to an indecomposable representation admits a cell decomposition, see [10,16] and references therein. It has been conjectured that this is also true for exceptional representations of any quiver, in particular for preprojective and preinjective representations.

There are basically two possible ways to find cell decompositions of quiver Grassmannians if they exist. One is to find a non-trivial $\mathbb{C}^*$-action on the quiver Grassmannian under consideration. If the quiver Grassmannian is smooth, one can apply a result of Białynicki-Birula [3] which shows that the quiver Grassmannian decomposes into affine bundles over the fixed point components. In particular, this shows that the quiver Grassmannian has a cell decomposition if the fixed point components have such a decomposition.

Another method uses short exact sequences of quiver representations to induce maps between quiver Grassmannians. More precisely, the quiver Grassmannian of the middle term maps to a disjoint union of products of the quiver Grassmannians for the two outer terms via the “Caldero–Chapoton map” which first appeared in [7]. If the short exact sequence has certain properties—e.g. (almost) split sequences and certain generalizations—then cell decompositions of quiver Grassmannians attached to the outer terms transfer to cell decompositions for the quiver Grassmannians of the middle term.

In this paper, we combine these two methods in order to show that every quiver Grassmannian attached to an exceptional representation of a generalized Kronecker quiver admits a cell decomposition. The proof also shows that this is true for so-called truncated preprojective representations which appear as certain quotients of preprojective representations. It turns out that these are precisely those representations which can be obtained from indecomposable representations with dimension vector $(d, 1)$ when applying reflection functors. Actually, we prove that quiver Grassmannians of truncated preprojective representations only depend on the dimension vector of the representation itself and on the fixed dimension vector of the subrepresentations.

As a first step, we show that a $\mathbb{C}^*$-action with proper fixed point subset can be defined on any quiver Grassmannian attached to a liftable representation of any acyclic quiver containing parallel arrows or non-oriented cycles, that is for representations which can be lifted to the universal (abelian) covering quiver. These are precisely those cases where the universal covering quiver differs from the original quiver. This lifting property holds in particular for so-called tree modules, a class of representations which includes all exceptional representations. The fixed point set of this $\mathbb{C}^*$-action consists precisely of those subrepresentations which can also be lifted to the universal abelian covering quiver. Actually, it turns out that each fixed point component is itself a quiver Grassmannian attached to the lifted representation and thus, iterating this procedure, it suffices to understand the quiver Grassmannians for the universal covering quiver.

The next step is to investigate conditions under which the iterated fixed point components admit a cell decomposition. Here the Caldero–Chapoton map comes into play. In the case of the generalized Kronecker quiver, it turns out that a natural filtration of a fixed preprojective representation by preprojectives of smaller dimension transfers to the universal covering quiver. These filtrations can be successively described by short exact sequences. The main advantage when passing to the universal covering is that the preprojective representations covering the same dimension vector below become orthogonal, a property which rigidifies the situation in a sense. In the end, this machinery can be used to recursively build cell decompositions of all quiver Grassmannians of lifted (truncated) preprojective representations. As
all the quiver Grassmannians of the (non-lifted) representation are smooth, this combines with the iterated torus actions on fixed point components to give a cell decomposition of these quiver Grassmannians.

As a benefit of this construction, we obtain a graph theoretic description of the non-empty cells. More precisely, with every (truncated) preprojective representation we can associate a so-called 2-quiver. Essentially, such a quiver is obtained from a usual quiver by adding a collection of “2-arrows” between pairs of subquivers. Now with every subset of the vertices we can associate a dimension vector. If this subset is also strong successor closed, a condition which is easily verified in practice, it corresponds to a cell and vice versa.

As mentioned above, the Laurent polynomial expressions for cluster variables have been described using the representation theory of quivers [7,8]: the cluster variables are generating functions for Euler characteristics of quiver Grassmannians. For rank two cluster algebras, the Laurent expressions of cluster variables can also be computed using a certain Dyck path combinatorics [15]. The confluence of these results gives rise to a combinatorial construction for the Euler characteristics and counting polynomials of certain quiver Grassmannians [18].

A consequence of our main result is a geometric explanation for these computations: we provide a one-to-one correspondence between the strong successor closed subsets of the associated 2-quiver and compatible pairs for an appropriate Dyck path which leads to a geometric explanation for the appearance of Dyck path combinatorics in the theory of quiver Grassmannians.

The paper is organized as follows. In Sect. 2, we collect several results concerning quiver covering theory. In Sect. 3, we recall basic facts concerning the representation theory of generalized Kronecker quivers \( K(n) \) which are needed later to investigate the quiver Grassmannians attached to preprojective representations (the preinjective case follows by duality and we do not address it directly). Our primary focus is on a special class of indecomposable representations which we call truncated preprojective representations. We prove that every preprojective representation admits a filtration by preprojectives of smaller dimensions such that all quotients appearing are actually truncated preprojective representations. In Sect. 3.2, we use this together with the fact that every preprojective representation can be lifted to the universal covering in order to construct lifted filtrations. Throughout this section, we collect many results which will turn out to completely reveal the structure of quiver Grassmannians attached to (truncated) preprojective representations.

The aim of Sect. 4 is to study these quiver Grassmannians and to show that they each admit a cell decomposition. This is obtained in Sect. 4.4 by combining iterated \( \mathbb{C}^* \)-actions on quiver Grassmannians, which are introduced in Sect. 4.2, with the Caldero–Chapoton map for short exact sequences of representations. Our first main result is Theorem 4.6 which may be formulated as follows.

**Theorem 1** Let \( X \) be a representation of a quiver \( Q = (Q_0, Q_1) \) which can be lifted to a representation \( \hat{X} \) of the universal abelian covering quiver \( \hat{Q} = (Q_0 \times A_Q, Q_1 \times A_Q) \), where \( A_Q \) is the free abelian group generated by \( Q_1 \). Then there exists a map \( d : \text{supp}(\hat{X}) \to \mathbb{Z} \)—with a corresponding \( \mathbb{C}^* \)-action on every \( X_i = \bigoplus_{x \in A_Q} X_{(i,x)} \) defined by \( t \cdot X_{(i,x)} = t^{d(i,x)} x_{(i,x)} \) for \( x_{(i,x)} \in X_{(i,x)} \)—which induces a \( \mathbb{C}^* \)-action on \( \text{Gr}^Q_{\epsilon}(X) \) such that

\[
\text{Gr}^Q_{\epsilon}(X)^{\mathbb{C}^*} \cong \bigsqcup_{\hat{\epsilon}} \text{Gr}^Q_{\hat{\epsilon}}(\hat{X}),
\]

where \( \hat{\epsilon} \) runs through all dimension vectors compatible with \( \epsilon \).
When $X$ lifts all the way to the universal covering quiver, this $C^*$-action can be iterated in such a way that the ultimate $C^*$-fixed points are precisely the subrepresentations which can be lifted to the universal covering quiver. As far as generalized Kronecker quivers are concerned, we can show in Theorem 4.13 that all quiver Grassmannians attached to truncated preprojective representations are smooth—actually, they only depend on appropriate dimension vectors. In view of results of Białynicki-Birula [3]—which roughly speaking yields that cell decompositions are preserved when passing from the fixed point components to the original variety—we can use this result to lift the investigation of the geometry of quiver Grassmannians to the universal covering quiver. This is important insofar as results such as Corollary 3.37 are available which do not hold on the original quiver. Analyzing the Caldero–Chapoton map applied to short exact sequences induced by lifts of the mentioned filtrations, we obtain the main result of this paper, see Theorems 4.21 and 4.23.

Theorem 2 For every $m \geq 1$ and for every point $V \neq C^n$ of the total Grassmannian $Gr(C^n)$, there exists a (truncated) preprojective representation $P_m^{V}$ such that every quiver Grassmannian $Gr(P_m^{V})$ admits a cell decomposition.

Note that, for $V = 0$, we obtain the preprojective representations $P_m$.

In Sect. 5, we reveal the combinatorics behind the obtained cell decompositions by introducing the notion of 2-quivers which are a slight generalization of the usual notion of quivers. Theorem 5.8 can be formulated as follows.

Theorem 3 With every truncated preprojective representation $P_m^{V}$, say with $\dim V = r$, there is an associated 2-quiver $Q_m^{[r]}$ such that the affine cells of the cell decomposition attached to $Gr_m(P_m^{V})$ are labeled by strong successor closed subsets $\beta \subset (Q_m^{[r]})_0$. In particular, the Euler characteristic $\chi(Gr(P_m^{V}))$ is given by the number of these subsets.

The results of [18] give a combinatorial construction of counting polynomials for quiver Grassmannians of preprojective/preinjective representations of generalized Kronecker quivers $K(n)$. This suggests that the dimensions of cells can be directly computed using this combinatorics (or the equivalent combinatorics of compatible pairs). This is made precise in Conjecture 5.21.

2 Quiver covering theory

We refer to [13] for an introduction to quiver covering theory. Let $Q$ be an acyclic quiver with vertices $Q_0$ and arrows $Q_1$ which we denote by $\alpha : s(\alpha) \to t(\alpha)$. A $C$-representation $X$ of $Q$ consists of a collection of $C$-vector spaces $X_i$ for $i \in Q_0$ and a collection of $C$-linear maps $X_\alpha : X_{s(\alpha)} \to X_{t(\alpha)}$ for $\alpha \in Q_1$. The dimension vector of $X$ is $\dim X := (\dim X_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$. Given $C$-representations $X$ and $Y$ of $Q$, a morphism $f : X \to Y$ is a collection of $C$-linear maps $f_\alpha : X_\alpha \to Y_\alpha$ for $\alpha \in Q_1$. We write $\text{rep} Q$ for the hereditary abelian category of finite-dimensional $C$-representations of $Q$, and we assume in the following that all representations are finite-dimensional.

Recall that, given $C$-representations $X$ and $Y$ of $Q$, any tuple of linear maps $(g_\alpha : X_{s(\alpha)} \to Y_{t(\alpha)})_{\alpha \in Q_1}$ defines a short exact sequence $0 \to Y \to Z \to X \to 0$ with middle term given by the vector spaces $Z_i = X_i \oplus Y_i$ for $i \in Q_0$ and the linear maps $Z_\alpha = \begin{pmatrix} X_\alpha & 0 \\ g_\alpha & Y_\alpha \end{pmatrix}$ for $\alpha \in Q_1$. In general, considering the linear map
Lemma 2.2 If $X$ is a representation of $Q$, then $X$ is rigid if $\text{Ext}_Q(X, X) = 0$ and exceptional if it is also indecomposable.

The functors $F_Q$ and $G_Q$ preserve indecomposability. Moreover, for all representations $\tilde{X}, \tilde{Y} \in \text{rep}(\tilde{Q})$, we have

$$\text{Hom}_Q(F_Q\tilde{X}, F_Q\tilde{Y}) \cong \bigoplus_{\chi \in A_Q} \text{Hom}_{\tilde{Q}}(\tilde{X}_\chi, \tilde{Y}_\chi)$$

and

$$\text{Hom}_Q(G_Q\tilde{X}, G_Q\tilde{Y}) \cong \bigoplus_{\chi \in A_Q} \text{Hom}_{\tilde{Q}}(\tilde{X}_\chi, \tilde{Y}_\chi).$$
Analogous isomorphisms exist when replacing $\text{Hom}$ by $\text{Ext}$ and/or $\text{rep} \widetilde{Q}$ by $\text{rep} \widetilde{Q}.$

3 Representation theory of generalized Kronecker quivers

Fix $n \geq 3.$ Denote by $K(n)$ the $n - \text{Kronecker quiver} \to \text{Kronecker quiver} \to 2$ with vertices $K_0(n) = \{1, 2\}$ and $n$ arrows from vertex 2 to vertex 1. The category $\text{rep} K(n)$ of finite-dimensional representations of $K(n)$ is equivalent to the category of modules over the path algebra $A(n)$ of $K(n).$ As a $\mathbb{C}$-vector space, the path algebra $A(n)$ can be written as $A_0 \oplus A_1,$ where

- $A_0 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ is a two-dimensional semisimple algebra with orthogonal idempotents $e_1$ and $e_2$;
- $A_1 = \bigoplus_{i=1}^n \mathbb{C} \alpha_i$ is the $A_0$-bimodule spanned by the arrows of $K(n),$ that is $e_k \alpha_i e_\ell = \delta_{k1} \delta_{\ell2} \alpha_i$ for $1 \leq i \leq n$ and $k, \ell \in \{1, 2\}.$

Write $\Sigma_1$ and $\Sigma_2$ for the BGP-reflection functors of $K(n)$ [5]. We use the same symbols $\Sigma_1, \Sigma_2$ for the BGP-reflection functors of $K(n)^{\text{op}},$ this should not lead to any confusion. Then each endofunctor $\Sigma_i^2$ is naturally isomorphic to the identity map on the full subcategory $\text{rep}^{(i)} K(n) \subset \text{rep} K(n)$ whose objects are those representations of $K(n)$ which do not contain the simple $S_i$ as a direct summand. In particular, $\Sigma_i$ gives an exact equivalence of categories $\Sigma_i : \text{rep}^{(i)} K(n) \to \text{rep}^{(i)} K(n)^{\text{op}}.$ Also, following [6], the Auslander–Reiten translation $\tau : \text{rep} K(n) \to \text{rep} K(n)$ may be identified with the functor $\Sigma_2 \Sigma_1.$

Define Chebyshev polynomials $u_k$ for $k \in \mathbb{Z}$ by the recursion $u_0 = 0,$ $u_1 = 1,$ $u_{k+1} = n u_k - u_{k-1}.$ Since $n \geq 3,$ we may observe by an easy induction that $u_{k+1} - u_k > 0$ for $k \geq 0$ and thus $u_{k+1} > (n - 1) u_k$ for $k \geq 0.$

The following is well-known.

**Theorem 3.1** For each $m \geq 1,$ there exist unique (up to isomorphism) exceptional representations $P_m$ and $Q_m$ of $K(n)$ with dimension vectors

$$\dim P_m = (u_m, u_{m-1}), \quad \dim Q_m = (u_{m-1}, u_m).$$

These satisfy

$$\text{Hom}(P_m, P_r) = 0, \quad \text{Hom}(Q_r, Q_m) = 0, \quad \text{Ext}(P_r, P_m) = 0, \quad \text{Ext}(Q_m, Q_r) = 0$$

for $1 \leq r \leq m.$ Moreover, any rigid representation of $K(n)$ is isomorphic to one of the form $P_m^{a_1} \oplus P_{m+1}^{a_2}$ or $Q_m^{a_1} \oplus Q_{m+1}^{a_2}$ for some $m \geq 1$ and some $a_1, a_2 \geq 0.$

The representations $P_m$ are called the preprojective representations of $K(n)$ and the representations $Q_m$ are called preinjective. The following homological organization is well-known.

**Lemma 3.2** If $\text{Hom}(X, P_m) \neq 0$ (resp. $\text{Ext}(P_m, X) \neq 0$) for some indecomposable representation $X$ and some $m \geq 1,$ then $X$ must be preprojective of the form $P_r$ with $1 \leq r \leq m$ (resp. $1 \leq r \leq m - 2$).

**Remark 3.3** We may identify the quiver $K(n)$ with $K(n)^{\text{op}}$ by interchanging the vertex labels. This induces an isomorphism of categories $\text{rep} K(n) \cong \text{rep} K(n)^{\text{op}}$ which we write as $M \mapsto M^\sigma.$ Note that $\Sigma_1(M^\sigma) = (\Sigma_2 M)^\sigma$ and $\Sigma_2(M^\sigma) = (\Sigma_1 M)^\sigma.$ Using this, we have the following:
1. The preprojective and preinjective representations satisfy the following recursions using the reflection functors:

\[ P_1 = S_1, \quad P_m^\sigma = \Sigma_2 P_{m-1}, \quad Q_1 = S_2, \quad Q_m^\sigma = \Sigma_1 Q_{m-1} \]

for \( m \geq 2 \). In particular, we have \( P_{m-1} = \tau P_{m+1} \) and \( Q_{m+1} = \tau Q_{m-1} \) for \( m \geq 2 \).

2. If \( 0 \rightarrow M \rightarrow B \rightarrow N \rightarrow 0 \) is a short exact sequence such that no direct summand of \( M, B, \) nor \( N \) is preinjective, then the sequences

\[ 0 \rightarrow (\Sigma_1 \Sigma_2)^k M \rightarrow (\Sigma_1 \Sigma_2)^k B \rightarrow (\Sigma_1 \Sigma_2)^k N \rightarrow 0 \]

and

\[ 0 \rightarrow \Sigma_2(\Sigma_1 \Sigma_2)^k M \rightarrow \Sigma_2(\Sigma_1 \Sigma_2)^k B \rightarrow \Sigma_2(\Sigma_1 \Sigma_2)^k N \rightarrow 0 \]

are exact for any \( k \geq 0 \) and none of these representations contain preinjective direct summands.

Set \( \mathcal{H}_m := \text{Hom}(P_m, P_{m+1}) \) for \( m \geq 1 \). Write \( \text{Gr}(\mathcal{H}_m) \) for the total Grassmannian of \( \mathcal{H}_m \) whose elements are non-trivial proper subspaces \( V \subset \mathcal{H}_m \). Some results below remain true if we allow \( \mathcal{H}_m \) or \( 0 \) as elements of \( \text{Gr}(\mathcal{H}_m) \), but not all, so for uniformity of exposition we omit these possibilities.

For each \( m \geq 2 \), there is an Auslander–Reiten sequence (cf. [2, Section V])

\[ 0 \rightarrow P^{-1}_{m-1} \rightarrow P_m \otimes \mathcal{H}_m \rightarrow P_{m+1} \rightarrow 0, \quad (3.1) \]

where the right-hand morphism is the natural evaluation map.

**Lemma 3.4** For any \( V \in \text{Gr}(\mathcal{H}_m), m \geq 1 \), the natural evaluation map \( ev_V : P_m \otimes V \rightarrow P_{m+1} \) is injective.

**Proof** The map \( ev_V \) is irreducible and is thus either injective or surjective. But as observed above, we have \( r \cdot \dim P_m < \dim P_{m+1} \) for \( 1 \leq r < n \) and so \( ev_V \) cannot be surjective. \( \square \)

### 3.1 Truncated preprojectives

**Definition 3.5** For \( V \in \text{Gr}(\mathcal{H}_m), \) define the truncated preprojective \( P^V_{m+1} \) to be the cokernel of the map \( ev_V : P_m \otimes V \rightarrow P_{m+1}, \) i.e. we have a short exact sequence

\[ 0 \rightarrow P_m \otimes V \rightarrow P_{m+1} \overset{ev_V}{\rightarrow} P^V_{m+1} \rightarrow 0. \quad (3.2) \]

**Remark 3.6** It will be convenient to also set \( P^0_{m+1} = P_{m+1} \) as this notation is consistent with taking \( V = 0 \) in the sequence (3.2).

We collect below several basic homological results related to truncated preprojective representations.

**Lemma 3.7** For \( V \in \text{Gr}(\mathcal{H}_m), m \geq 1 \), we have \( \text{Hom}(P_m, P^V_{m+1}) \cong \mathcal{H}_m/V \) and \( \text{Ext}(P_m, P^V_{m+1}) = 0. \)

**Proof** As \( P_m \) is rigid, applying the functor \( \text{Hom}(P_m, -) \) to the sequence (3.2), gives an exact sequence

\[ 0 \rightarrow \text{Hom}(P_m, P_m \otimes V) \rightarrow \text{Hom}(P_m, P_{m+1}) \rightarrow \text{Hom}(P_m, P^V_{m+1}) \rightarrow 0 \]

and an isomorphism

\[ \text{Ext}(P_m, P_{m+1}) \cong \text{Ext}(P_m, P^V_{m+1}). \]

There is a natural isomorphism \( \text{Hom}(P_m, P_m \otimes V) \cong V \) giving the first claim. The second claim follows from the vanishing of \( \text{Ext}(P_m, P_{m+1}) \) (c.f. Theorem 3.1). \( \square \)
Lemma 3.8 For \( V \in \text{Gr}(\mathcal{H}_m) \), \( m \geq 1 \), the space \( \text{Hom}(P_{m+1}, P_{m+1}^V) \) is one-dimensional spanned by the natural projection \( \pi_{m+1}^V : P_{m+1} \to P_{m+1}^V \). Moreover, \( \text{Ext}(P_{m+1}, P_{m+1}^V) = 0 \).

**Proof** As \( \text{Hom}(P_{m+1}, P_m) = \text{Ext}(P_{m+1}, P_m) = 0 \), applying the functor \( \text{Hom}(P_{m+1}, -) \) to the sequence (3.2) gives isomorphisms
\[
\text{Hom}(P_{m+1}, P_{m+1}) \cong \text{Hom}(P_{m+1}, P_{m+1}^V)
\]
and
\[
\text{Ext}(P_{m+1}, P_{m+1}) \cong \text{Ext}(P_{m+1}, P_{m+1}^V).
\]
Under the first isomorphism, the identity map on \( P_{m+1} \) is taken to the projection \( \pi_{m+1}^V : P_{m+1} \to P_{m+1}^V \). The second isomorphism together with the rigidity of \( P_{m+1} \) gives the final claim. \( \square \)

Lemma 3.9 Consider \( V, W \in \text{Gr}(\mathcal{H}_m) \), \( m \geq 1 \).

1. There exists a morphism \( P_{m+1}^W \to P_{m+1}^V \) if and only if \( W \subseteq V \), and this morphism is unique (up to scalars) when it exists.
2. For \( W \subseteq V \), we have \( \text{Ext}(P_{m+1}^W, P_{m+1}^V) \cong W^* \otimes (\mathcal{H}_m / V) \). Otherwise, \( \text{Ext}(P_{m+1}^W, P_{m+1}^V) \) is naturally isomorphic to the quotient of \( W^* \otimes (\mathcal{H}_m / V) \) by the line corresponding to the standard counit \( C \to W^* \otimes W \), this line degenerates to a point in the case \( W \subseteq V \).

**Proof** We apply the functor \( \text{Hom}(-, P_{m+1}^V) \) to the sequence (3.2) for \( W \) to get an exact sequence
\[
0 \to \text{Hom}(P_{m+1}^W, P_{m+1}^V) \to \text{Hom}(P_{m+1}, P_{m+1}^V) \to \text{Hom}(P_m \otimes W, P_{m+1}^V) \to \text{Ext}(P_{m+1}^W, P_{m+1}^V) \to 0.
\]
But the space \( \text{Hom}(P_{m+1}, P_{m+1}^V) \) is one-dimensional, therefore \( \text{Hom}(P_{m+1}^W, P_{m+1}^V) \) is nonzero (hence one-dimensional) if and only if the morphism \( - \circ e_{VW} \) of the above sequence is zero. A routine chase using the following pullback diagram confirms the latter holds if and only if \( W \subseteq V \).

In case \( W \subseteq V \), there are isomorphisms
\[
\text{Ext}(P_{m+1}^W, P_{m+1}^V) \cong \text{Hom}(P_m \otimes W, P_{m+1}^V) \cong W^* \otimes \mathcal{H}_m / V,
\]
where the last isomorphism is immediate from Lemma 3.7. Otherwise, we must quotient \( \text{Hom}(P_m \otimes W, P_{m+1}^V) \) by the image of \( - \circ e_{VW} \). This line can be described as above since that gives \( e_{VW} \) under the isomorphism \( W^* \otimes \mathcal{H}_m \cong \text{Hom}(P_m \otimes W, P_{m+1}) \). \( \square \)

Remark 3.10 The total Grassmannian \( \text{Gr}(\mathcal{H}_m) \) is naturally a poset under inclusion. This structure gives rise to a \( \mathbb{C} \)-linear category \( \text{CGr}(\mathcal{H}_m) \) with objects the elements of \( \text{Gr}(\mathcal{H}_m) \) and at most one morphism (up to scalars) between any two objects. Write \( P_{m+1} \) for the full...
subcategory of rep K(n) with objects the truncated preprojectives \( P_{m+1}^V \) for \( V \in \text{Gr}(\mathcal{H}_m) \). By Lemma 3.9(1), the functor \( V \mapsto P_{m+1}^V \) gives an isomorphism of categories \( \mathbb{C}\text{Gr}(\mathcal{H}_m) \cong \mathcal{P}_{m+1} \).

The truncated preprojective representation \( P_{m+1}^V \) has dimension vector
\[
\dim P_{m+1}^V = d(m, \dim V) := \dim P_{m+1} - \dim V \cdot \dim P_m.
\]

These will play an important role when describing quiver Grassmannians of preprojective representations recursively. For a dimension vector \( d = (d_1, d_2) \in \mathbb{N}^{K(n)_0} \), write
\[
R_d(K(n)) = \bigoplus_{i=1}^n \text{Hom}_\mathbb{C}(\mathbb{C}^{d_i2}, \mathbb{C}^{d_1})
\]
for the affine space of representations of \( K(n) \) with dimension vector \( d \).

**Proposition 3.11** Let \( m \geq 1 \) and \( 0 \leq r \leq n - 1 \). The following hold:

1. The isomorphism classes of indecomposable representations of \( K(n) \) with dimension vector \( d(m, r) \) are in one-to-one correspondence with points of \( \text{Gr}_{n-r}(\mathbb{C}^n) \).
2. The indecomposable representations of \( K(n) \) with dimension vector \( d(m, r) \) are precisely the truncated preprojective representations \( P_{m+1}^V \) for \( V \in \text{Gr}(\mathcal{H}_m) \) with \( \dim V = r \).
3. The set of indecomposable representations with dimension vector \( d(m, r) \) is given by a non-empty open subset of \( R_{d(m,r)}(K(n)) \).

**Proof** As the reflection functors \( \Sigma_1, \Sigma_2 \) preserve indecomposability and \( \Sigma_2(d(m, r))^\sigma = d(m + 1, r) \), it suffices to prove the first statement for \( m = 1 \). Then we have \( d(m, r) = (\ell, 1) \) for \( \ell := n - r \). This means that \( X \in R_{d(m,r)}(K(n)) \) can be represented by a matrix \( M_X \in \mathbb{C}^{\ell \times n} \), where the \( i \)-th column stands for \( X_{ai} \). Now \( X \) is indecomposable if and only if \( \text{rk}(M_X) = \ell \). Indeed, \( X \) admits a summand isomorphic to \( \mathbb{C}^{d_1} \) exactly when \( \text{rk}(M_X) = \ell - k \).

This shows that the indecomposable representations in \( R_{(\ell, 1)}(K(n)) \) are in one-to-one correspondence with \( \ell \times n \) matrices of maximal rank. Thus we may associate to each such representation \( X \) a subspace of \( \mathbb{C}^n \) of dimension \( \ell \) spanned by the row vectors of the corresponding matrix \( M_X \). Now it is straightforward to check that the \( \text{GL}_{d(m, r)} = \text{GL}_\ell(\mathbb{C}) \times \mathbb{C}^* \)-action on \( R_{d(m,r)}(K(n)) \) corresponds to the base change action of \( \text{GL}_\ell(\mathbb{C}) \) on the set of these subspaces. This shows the first statement.

By Lemma 3.9, the endomorphism ring of \( P_{m+1}^V \) is one-dimensional and so \( P_{m+1}^V \) must be indecomposable. Since both the isomorphism classes of indecomposables and the isomorphism classes of truncated preprojectives with dimension vector \( d(m, r) \) are parametrized by the same Grassmannian, this gives the second claim.

As there exist representations with trivial endomorphism ring, the dimension vectors \( d(m, r) \) are Schur roots. It follows that the set of indecomposable representations with trivial endomorphism ring forms a dense open subset of \( R_{d(m,r)}(K(n)) \), see for example [19, Theorem 2.2]. This shows the last claim. \( \Box \)

**Remark 3.12** There is a more elegant way to prove the first part of Proposition 3.11 using the notion of stability and moduli spaces. Actually, fixing the standard stability induced by the linear form \( \Theta : \mathbb{Z}Q_0 \to \mathbb{Z} \) defined by \( \Theta(d) = d_2 \), it can be shown that all indecomposables are stable and that the moduli space of stable representations is in fact \( \text{Gr}_\ell(\mathbb{C}^n) \). We opted for the proof above because the notion of stability would only be used at this point and we wanted to keep the exposition as simple as possible.
Lemma 3.13 For $V \in \text{Gr}(\mathcal{H}_m)$, $m \geq 1$, we have $\text{Hom}(P_{m+1}^V, P_{\ell}) = 0 = \text{Ext}(P_{\ell}, P_{m+1}^V)$ for all $\ell \geq 1$.

**Proof** The representation $P_{m+1}^V$ is indecomposable by Proposition 3.11 and it cannot be preprojective as it is not rigid by Lemma 3.9. The claim then follows from Lemma 3.2. 

Lemma 3.14 For $V \in \text{Gr}(\mathcal{H}_m)$, $m \geq 2$, the representation $\Sigma_1(P_{m+1}^V)^\sigma$ is also truncated preprojective.

**Proof** By Proposition 3.11 and Lemma 3.9, $P_{m+1}^V$ is indecomposable but not rigid. In particular, $P_{m+1}^V$ does not have a summand isomorphic to $S_1$. Thus, following Remark 3.3, we may apply the functor $\Sigma_1(\cdot)^\sigma$ to the sequence (3.2) to get the exact sequence

$$0 \rightarrow P_{m-1} \otimes V \rightarrow \Sigma_1(e_{V})^\sigma \rightarrow P_{m} \rightarrow \Sigma_1(P_{m+1}^V)^\sigma \rightarrow 0$$

which gives the claim.

Lemma 3.15 For $V \in \text{Gr}(\mathcal{H}_m)$, $m \geq 1$, any proper subrepresentation $X \subsetneq P_{m+1}^V$ can be written as a direct sum of preprojective representations $P_r$ with $1 \leq r \leq m$.

**Proof** We use some of the ideas of the proof of [1, Lemma IV.1.7]. For every proper subrepresentation $X$ of $P_{m+1}^V$ together with the natural embedding $\iota$, we get a pullback diagram

$$
\begin{array}{c}
0 \rightarrow P_m \otimes V \xrightarrow{f} E \xrightarrow{r} X \xrightarrow{\iota} 0 \\
0 \rightarrow P_m \otimes V \xrightarrow{ev} P_{m+1}^V \xrightarrow{\pi_{m+1}} P_{m+1}^V \xrightarrow{0}
\end{array}
$$

Since the map $ev_V$ is irreducible, $f$ is either a section or $h$ a retraction. As $h$ is injective by the Snake Lemma, it cannot be a retraction. It follows that $f$ is a section and thus $g$ a retraction. Thus we find a morphism $g' : X \rightarrow E$ such that $g \circ g' = \text{id}_X$. We obtain

$$\iota = \iota \circ \text{id}_X = \iota \circ g \circ g' = \pi_{m+1} \circ h \circ g',$$

i.e. $\iota$ factors through $P_{m+1}$, and the claim follows from Lemma 3.2.

For $V \in \text{Gr}(\mathcal{H}_m)$, $m \geq 1$, any subspace $W \subset V$ gives rise to an exact sequence

$$0 \rightarrow P_m \otimes (V/W) \rightarrow P_{m+1}^W \rightarrow P_{m+1}^V \rightarrow 0,$$

where the left hand morphism above is the natural evaluation morphism coming from Lemma 3.7. Each such sequence has the following almost-split property for proper subrepresentations of $P_{m+1}^V$.

**Corollary 3.16** Consider $V, W \in \text{Gr}(\mathcal{H}_m)$, $m \geq 1$, with $W \subset V$. Given any proper subrepresentation $X \subsetneq P_{m+1}^V$ and any subrepresentation $Z \subset P_m \otimes (V/W)$, there is a subrepresentation of $P_{m+1}^W$ isomorphic to $Z \oplus X$ which fits into a commutative diagram

$$
\begin{array}{c}
0 \rightarrow Z \rightarrow Z \oplus X \rightarrow X \rightarrow 0 \\
0 \rightarrow P_m \otimes (V/W) \rightarrow P_{m+1}^W \rightarrow P_{m+1}^V \rightarrow 0
\end{array}
$$
Remark 3.19

The set $\text{Gr}(\mathcal{H}_m)$ must be injective and hence an isomorphism.

Proof

Observe that $\text{Ext}(P_r, P_m) = 0$ for $1 \leq r \leq m$ and, since $X$ is a direct sum of preprojectives $P_r$ with $1 \leq r \leq m$, the upper pullback sequence

$$
0 \longrightarrow P_m \otimes (V/W) \longrightarrow Y \longrightarrow X \longrightarrow 0
$$

must split. The claim for an arbitrary subrepresentation $Z \subseteq P_m \otimes (V/W)$ is an immediate consequence of this splitting.

$$
\text{Lemma 3.17} \quad \text{For } V \in \text{Gr}(\mathcal{H}_m), \; m \geq 2, \text{ the space } \text{Ext}(P_{m+1}^V, P_{m-1}) \text{ is one-dimensional and spanned by the extension}
$$

$$
0 \longrightarrow P_{m-1} \overset{\kappa_V}{\longrightarrow} P_m \otimes (\mathcal{H}_m/V) \overset{e_V}{\longrightarrow} P_{m+1}^V \longrightarrow 0.
$$

Proof

When applying the functor $\text{Hom}(-, P_{m-1})$ to the sequence (3.2), we obtain an isomorphism $\text{Ext}(P_{m+1}^V, P_{m-1}) \cong \text{Ext}(P_{m+1}, P_{m-1})$ with a one-dimensional space. Writing $X$ for the unique extension of $P_{m+1}^V$ by $P_{m-1}$, this isomorphism gives rise to the following pullback diagram:

from which we immediately obtain the isomorphism $X \cong P_m \otimes (\mathcal{H}_m/V)$.

$$
\text{Lemma 3.18} \quad \text{The sequence (3.1) gives rise to an isomorphism } \mathcal{H}_m^* \cong \mathcal{H}_{m-1}.
$$

Proof

Define a map $\mathcal{H}_m^* \to \mathcal{H}_{m-1}^*$ by $\varphi \mapsto \widetilde{\varphi} := (id \otimes \varphi) \circ \iota_{m-1}$, in words $\widetilde{\varphi}$ acts on $x \in P_{m-1}$ by contracting with the second factor in $\iota_{m-1}(x)$ to give an element of $P_m$. Suppose $\varphi \in \mathcal{H}_m^*$ is a nonzero functional on $\mathcal{H}_m$ and let $V \subseteq \mathcal{H}_m$ denote the kernel of $\varphi$. Then $\widetilde{\varphi} = 0 \in \mathcal{H}_{m-1}$ if and only if the image of $\iota_{m-1}$ is contained in $P_m \otimes V \subseteq P_m \otimes \mathcal{H}_m$. But then Lemma 3.4 implies $e_V \circ \iota_{m-1} = e_V \circ \iota_{m-1} \neq 0$, a contradiction. Thus the map $\mathcal{H}_m^* \to \mathcal{H}_{m-1}^*$, $\varphi \mapsto \widetilde{\varphi}$, must be injective and hence an isomorphism.

$$
\text{Remark 3.19} \quad \text{The set } \text{Gr}(\mathcal{H}_m) \text{ is naturally a poset and Lemma 3.18 gives an identification of the opposite poset } \text{Gr}(\mathcal{H}_m)^{\text{op}} \cong \text{Gr}(\mathcal{H}_m^*) \text{ with } \text{Gr}(\mathcal{H}_{m-1}). \text{ We write } V \subseteq \mathcal{H}_{m-1} \text{ for the subspace corresponding to } V \subseteq \mathcal{H}_m \text{ under this identification. Under the isomorphism of } \mathcal{H}_m^* \text{ with } \mathcal{H}_{m-1}, \text{ we have } \widetilde{V} = (\mathcal{H}_m/V)^*.
$$
Corollary 3.20 Suppose $V \in \mathrm{Gr}(\mathcal{H}_m)$ has codimension-one in $\mathcal{H}_m$. Then $P^V_{m+1} \cong P^\bar{V}_m$.

Proof By Lemma 3.17, we have the exact sequence

$$0 \longrightarrow P_{m-1} \xrightarrow{\kappa_V} P_m \otimes (\mathcal{H}_m/V) \xrightarrow{e_V} P^V_{m+1} \longrightarrow 0.$$ 

But $\mathcal{H}_m/V$ is a one-dimensional vector space and so $P_m \otimes (\mathcal{H}_m/V) \cong P_m$. Under this identification, the left hand morphism $\kappa_V$ in the sequence above identifies with a generator of $\bar{V}$ and thus $P^V_{m+1} \cong P^\bar{V}_m$. 

\[ \square \]

3.2 Lifting to $\widetilde{K(n)}$

Write $W_n := W_{K(n)}$ for the free group generated by the arrows $\alpha_i$, $1 \leq i \leq n$, of $K(n)$ and denote by $e \in W_n$ its identity element. In this section, we fix compatible bases for each $\mathcal{H}_m := \mathrm{Hom}(P_m, P_{m+1})$ and use these to lift the (truncated) preprojective representations of the quiver $K(n)$ to the universal cover $\widetilde{K(n)}$. This lifting will rigidify the situation, allowing more precise control over these representations and their subrepresentation structure. Of particular importance is Corollary 3.37 which has no reasonable analogue for $K(n)$. Another main advantage of the lifting is that those truncated preprojectives which can be lifted are exceptional representations on the universal covering quiver $\widetilde{K(n)}$.

We will mainly be interested in particular lifts $\tilde{P}_m$ of the preprojective representations $P_m$ of $K(n)$ to the universal cover $\widetilde{K(n)}$. In the notation of Definition 2.1, this means $G(\tilde{P}_m) = P_m$, where $G$ is the covering functor

$$G := G_{K(n)} : \text{rep } \widetilde{K(n)} \to \text{rep } K(n).$$

To construct the lifts, first recall that applying the BGP-reflection functor $\Sigma_i$ on $K(n)$ corresponds to applying the iterated reflection $\tilde{\Sigma}_i := \prod_{w \in W_n} \Sigma(i,w)$ on $\widetilde{K(n)}$. Moreover, under this operation all sinks of $\widetilde{K(n)}$ become sources and vice versa. The preprojective lifts we use are defined by the following analogue of the recursion of Remark 3.3.

- We consider the lift $\tilde{P}_1$ satisfying $\dim(\tilde{P}_1)_{(i,e)} = 1$ and $\dim(\tilde{P}_1)_{(i,w)} = 0$ for $(i, w) \neq (1, e)$.
- We consider the lift $\tilde{P}_2$ satisfying $\dim(\tilde{P}_2)_{(2,e)} = \dim(\tilde{P}_2)_{(1,\alpha_i)} = 1$ for $1 \leq i \leq n$ and $\dim(\tilde{P}_2)_{(i,w)} = 0$ for $(i, w) \notin \{(2, e), (1, \alpha_1), \ldots, (1, \alpha_n)\}$.
- For $m \geq 3$, we build the lifts $\tilde{P}_m$ recursively by applying reflection functors or as Auslander-Reiten translates. More precisely, we set

$$\tilde{P}_m := \tilde{\Sigma}_2(\tilde{P}_{m-1})^\sigma \quad \text{or} \quad \tilde{P}_{m+1} := \tilde{\Sigma}_1 \tilde{\Sigma}_2 \tilde{P}_{m-1} = \tau^{-1} \tilde{P}_{m-1}, \quad (3.4)$$

where $(-)^\sigma : \text{rep } \widetilde{K(n)}^{\text{op}} \to \text{rep } \widetilde{K(n)}$ is the lift of the corresponding functor for $K(n)$.

It will be rather important that our chosen lifts of $P_{2\ell}$ for $\ell \geq 1$ and our chosen lifts of $P_{2\ell-1}$ for $\ell \geq 1$ live on two different components of the universal covering quiver. Indeed, recall that the group $W_n$ naturally acts on $\widetilde{K(n)}_0$ via left translation, i.e. $w.(i, w') = (i, ww')$, and this induces an action of $W_n$ on $\text{rep } \widetilde{K(n)}$. Following the notation of Sect. 2, we write $\tilde{P}_{m,w}$ for the representation of $\widetilde{K(n)}$ obtained by translating the lifted preprojective representation $\tilde{P}_m$ by the action of $w \in W_n$.

Remark 3.21 To simplify the notation, we abbreviate $\tilde{P}_{2\ell-1,i} := \tilde{P}_{2\ell-1,\alpha_i}$ and $\tilde{P}_{2\ell,i} := \tilde{P}_{2\ell,\alpha_{i}^{-1}}$ for $\ell \geq 1$ and $i \in \{1, \ldots, n\}$. 

\[ \Box \]
Lemma 3.22  For each \( m \geq 1 \), the representation \( \tilde{P}_{m+1} \) has precisely \( n \) subrepresentations covering \( P_m \). These are the representations \( \tilde{P}_{m,j} \) corresponding to the \( n \) different arrows of \( K(n) \).

\textbf{Proof}  This is clear for \( m = 1 \) and follows in general by applying the recursion (3.4). \( \square \)

\textbf{Remark 3.23}  Note that the dimension vectors of the lifted preprojectives \( \tilde{P}_m \) are symmetric under permutations of the arrows of \( K(n) \) (or rather the corresponding operation on \( \tilde{K}(n) \)). In particular, they all have the same central vertex \((1, e)\) if \( m \) is odd and central vertex \((2, e)\) if \( m \) is even.

\textbf{Corollary 3.24}  For \( m \geq 2 \), the Auslander–Reiten sequence (3.1) on \( K(n) \) lifts to the Auslander–Reiten sequence

\[ 0 \to \tilde{P}_{m-1} \xrightarrow{i_{m-1}} \bigoplus_{j=1}^{n} \tilde{P}_{m,j} \to \tilde{P}_{m+1} \to 0. \]

(3.5)

\textbf{Example 3.25}  Here we explicitly describe the preprojective lifts \( \tilde{P}_m \) for \( m = 2, 3, 4 \) as well as their shifted preprojective subrepresentations as in Lemma 3.22. By Lemma 2.2, the lifts of all preprojectives are exceptional as representations of \( \tilde{K}(n) \) and thus they are uniquely determined by their dimension vectors. We make use of this fact below, stating only the support of the representation (also specifying those dimensions which are not one) and do not state the particular maps present in the lifts. We call this the \textit{support quiver} of the representation and let \( W_n \) act on these quivers by translating all vertices and arrows.

The representation \( \tilde{P}_2 \) is defined by the following quiver:

\[ \begin{array}{c}
(2, e) \\
\alpha_1 & & \alpha_n \\
(1, \alpha_1) & \cdots & (1, \alpha_n).
\end{array} \]

(3.6)

Then \( \tilde{P}_{1,j} \subset \tilde{P}_2 \) corresponds to the one-dimensional space at vertex \((1, \alpha_j)\). Write \( A_i \) for the quiver obtained from the one above by erasing the arrow \( \alpha_i \) and the corresponding sink \((1, \alpha_i)\).

The representation \( \tilde{P}_3 \) is given by the quiver

\[ \begin{array}{c}
n - 1 \\
\alpha_1 & & \alpha_n \\
\alpha_1^{-1}.A_1 & \cdots & \alpha_n^{-1}.A_n. 
\end{array} \]

(3.7)

where the top vertex of dimension \( n - 1 \) is \((1, e)\) and the arrow from each \( \alpha_j^{-1}.A_j \) emanates from its unique source \((2, \alpha_j^{-1})\). Then \( \tilde{P}_{2,j} \subset \tilde{P}_3 \) has one-dimensional spaces at each vertex of the quiver.
The representation $\tilde{P}_3$ has support quiver

$$
\begin{array}{c}
\alpha_1 \\
\alpha_n \\
\alpha_1, B_1 \\
\cdots \\
\alpha_n, B_n,
\end{array}
$$

where the top vertex of dimension $n - 1$ is $(2, e)$ and $B_i$ is the following analogue of the support quiver for $P_3$:

$$
\begin{array}{c}
\alpha_1 \\
\alpha_n \\
\alpha_1, A_1 \\
\cdots \\
\alpha_n, A_n
\end{array}
$$

with top vertex $(1, e)$ having dimension $2n - 3$. Now $\tilde{P}_{3,j}$ can be found as the subrepresentation corresponding to the subquiver

$$
\begin{array}{c}
\alpha_j \\
\alpha_1 \\
\alpha_1, A_1 \\
\cdots \\
\alpha_n, A_n
\end{array}
$$

where the central vertex is $(1, \alpha_j)$. Note that taking the subquiver $A_j$ together with the image of the map $\alpha_j$ gives a subrepresentation of $\tilde{P}_{3,j}$ isomorphic to $\tilde{P}_2$ while taking the subquiver
\( \alpha_j \alpha^{-1}_i \cdot \mathcal{A}_i \) together with the image of the map \( \alpha_i \) gives a subrepresentation of \( \tilde{P}_{3,j} \) isomorphic to \( \tilde{P}_{2,\alpha_j \alpha^{-1}_i} \).

The next result establishes some basic homological properties of these translates of preprojective representations.

**Lemma 3.26** For \( m \geq 1 \), the following hold.

1. We have \( \text{Hom}(P_m, P_{m+1}) \cong \bigoplus_{i=1}^n \text{Hom}(\tilde{P}_{m,i}, \tilde{P}_{m+1}) \), where each \( \text{Hom}(\tilde{P}_{m,i}, \tilde{P}_{m+1}) \) is one-dimensional.
2. The representations \( \tilde{P}_{m,i} \) are pairwise orthogonal, i.e. for \( i \neq j \) we have \( \text{Hom}(\tilde{P}_{m,i}, \tilde{P}_{m,j}) = 0 = \text{Ext}(\tilde{P}_{m,i}, \tilde{P}_{m,j}) \).
3. For \( j \in \{1, \ldots, n\} \), we have \( \text{Hom}(\tilde{P}_{m+1}, \tilde{P}_{m,j}) = 0 \). \( \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m,j}) = 0 \), \( \text{Ext}(\tilde{P}_{m,j}, \tilde{P}_{m+1}) = 0 \).
4. For each proper subset \( I \subseteq \{1, \ldots, n\} \), there exists a truncated preprojective representation \( \tilde{P}_I^{m+1} \) fitting into an exact sequence

\[
0 \rightarrow \bigoplus_{i \in I} \tilde{P}_{m,i} \rightarrow \tilde{P}_{m+1} \xrightarrow{\pi_{m+1}} \tilde{P}_I^{m+1} \rightarrow 0.
\]

Moreover, \( G(\tilde{P}_I^{m+1}) =: P_I^{(m+1)} \) is a truncated preprojective of \( K(n) \) for each \( I \subseteq \{1, \ldots, n\} \).

**Remark 3.27** Note that when \( I = \emptyset \), it follows from the definition that \( \tilde{P}_I^{m+1} = \tilde{P}_{m+1} \).

**Proof** Part (1) is immediate from Theorem 2.3. Part (2) is also a consequence of Theorem 2.3. Indeed, for \( 1 \leq j \leq n \), we have

\[
\mathbb{C} \cong \text{Hom}(P_m, P_m) \cong \bigoplus_{i=1}^n \text{Hom}(\tilde{P}_{m,i}, \tilde{P}_{m,j}).
\]

But \( \text{Hom}(\tilde{P}_{m,j}, \tilde{P}_{m,j}) \cong \mathbb{C} \) and so we must have \( \text{Hom}(\tilde{P}_{m,i}, \tilde{P}_{m,j}) = 0 \) for \( i \neq j \). The vanishing of \( \text{Ext}(\tilde{P}_{m,i}, \tilde{P}_{m,j}) \) follows in the same manner using that \( P_m \) is exceptional.

Part (3) is clear for \( m = 1 \) and follows for \( m \geq 2 \) by applying the reflection recursion (3.4).

For part (4), observe that under the isomorphism from part (1) the subset \( I \subseteq \{1, \ldots, n\} \) corresponds to the subspace \( \langle I \rangle \subseteq \mathcal{H}_m \) spanned by the direct summands \( \text{Hom}(\tilde{P}_{m,i}, \tilde{P}_{m+1}) \) for \( i \in I \). The map \( \bigoplus_{i \in I} \tilde{P}_{m,i} \rightarrow \tilde{P}_{m+1} \) is then a lift of the evaluation morphism \( ev_{\langle I \rangle} : P_m \otimes \langle I \rangle \rightarrow P_{m+1} \) and hence is injective by Lemma 3.4. Taking the cokernel defines the truncated preprojective \( \tilde{P}_I^{m+1} \) and the preceding discussion shows that \( G(\tilde{P}_I^{m+1}) \cong P_I^{(m+1)} \) is truncated preprojective as well.

It will be important to understand the possible subrepresentations of the truncated preprojective representations \( \tilde{P}_I^{m+1} \).

**Lemma 3.28** For \( m \geq 1 \) and \( I \subseteq \{1, \ldots, n\} \), all non-trivial proper subrepresentations of \( \tilde{P}_I^{m+1} \) are direct sums of preprojective representations.
**Proof** By Lemma 3.26, the projected representation \( \tilde{G}(\tilde{P}^I_{m+1}) \) is a truncated preprojective of \( K(n) \). Then Lemma 3.15 shows that all subrepresentations of \( \tilde{G}(\tilde{P}^I_{m+1}) \) are direct sums of preprojective representations of \( K(n) \). But any subrepresentation of \( \tilde{P}^I_{m+1} \) also projects to a subrepresentation of \( \tilde{G}(\tilde{P}^I_{m+1}) \). Since, following Lemma 2.2, any lift of a preprojective of \( K(n) \) will be a preprojective representation of \( \tilde{K}(n) \), this gives the result. \( \square \)

In what follows, we will need to carefully understand the homological properties of the truncated preprojectives for \( \tilde{K}(n) \).

**Lemma 3.29** For \( m \geq 1 \) and \( I \subseteq \{1, \ldots, n\} \), the following hold.

1. For \( j \in \{1, \ldots, n\} \), we have \( \text{Ext}(\tilde{P}_m, \tilde{P}_{m+1}^I) = 0 \). Also, \( \text{Hom}(\tilde{P}_m, \tilde{P}_{m+1}^I) \neq 0 \) if and only if \( j \notin I \), in which case
   \[
   \text{Hom}(\tilde{P}_m, \tilde{P}_{m+1}^I) \cong \text{Hom}(\tilde{P}_m, \tilde{P}_m) \cong \mathbb{C}.
   \]

2. We have \( \text{Hom}(\tilde{P}_m, \tilde{P}_{m+1}^I) \cong \mathbb{C} \) and \( \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m+1}^I) = 0 \).

3. For \( j \in \{1, \ldots, n\} \), we have \( \text{Hom}(\tilde{P}_{m+1}, \tilde{P}_m^J) = 0 \). Also, \( \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_m^J) \neq 0 \) if and only if \( j \notin I \), in which case
   \[
   \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_m^J) \cong \mathbb{C}
   \]
   is generated by the non-split sequence
   \[
   0 \longrightarrow \tilde{P}_m^J \longrightarrow \tilde{P}_{m+1} \longrightarrow 0.
   \]

4. For any \( J \subseteq \{1, \ldots, n\} \), we have \( \text{Hom}(\tilde{P}_m, \tilde{P}_{m+1}^J) \neq 0 \) if and only if \( J \subseteq I \), in which case
   \[
   \text{Hom}(\tilde{P}_m, \tilde{P}_{m+1}^J) \cong \mathbb{C} \quad \text{and} \quad \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m+1}^J) = 0.
   \]

In particular, \( \tilde{P}_m^{I+1} \) is an exceptional representation of \( \tilde{K}(n) \).

5. For each proper subset \( J \subseteq I \), there is an exact sequence
   \[
   0 \longrightarrow \bigoplus_{i \in I \setminus J} \tilde{P}_m^i \longrightarrow \tilde{P}_m^{J+1} \longrightarrow 0 \tag{3.10}
   \]
   For any sequence of proper subsets \( K \subseteq J \subseteq I \), the quotient maps satisfy \( \pi^{I+1,K} = \pi^J_1 \circ \pi^{I,J}_1 \) and \( \pi^{I+1,J} = \pi_1^{I,J} \circ \pi^{J}_1 \).

**Proof** Applying \( \text{Hom}(\tilde{P}_m, -) \) to the sequence (3.9) gives the exact sequence
\[
0 \longrightarrow \text{Hom}(\tilde{P}_m, \bigoplus_{i \in I} \tilde{P}_m^i) \longrightarrow \text{Hom}(\tilde{P}_m, \tilde{P}_{m+1}) \longrightarrow \text{Hom}(\tilde{P}_m, \tilde{P}_{m+1}^I) \longrightarrow 0,
\]
where the final zero follows from Lemma 3.26.2 and the rigidity of \( \tilde{P}_m^j \). This also gives an isomorphism \( \text{Ext}(\tilde{P}_m, \tilde{P}_{m+1}^I) \cong \text{Ext}(\tilde{P}_m, \tilde{P}_{m+1}) = 0 \) by Lemma 3.26.3. Now the middle space in the sequence above is one-dimensional while the left-hand space vanishes if and only if \( j \notin I \), this proves part (1).

Part (2) is an immediate consequence of Lemma 3.8 together with Theorem 2.3 or can be obtained directly by applying \( \text{Hom}(\tilde{P}_{m+1}, -) \) to the sequence (3.9). The first part of (3) follows from Theorem 2.3 together with Lemma 3.13. For the second part of (3), we apply...
Hom(\(\cdot, \tilde{P}_{m,j}\)) to the sequence (3.9). Then taking into account Lemma 3.26.3, we get the isomorphism

\[
\text{Ext}(\tilde{P}^I_{m+1}, \tilde{P}_{m,j}) \cong \text{Hom}(\bigoplus_{i \in I} \tilde{P}_{m,i}, \tilde{P}_{m,j}),
\]

and Lemma 3.26.2 gives the final claim of part (3).

Part (4) follows by applying Hom(\(\cdot, \tilde{P}^I_{m+1}\)) to the sequence (3.9) for \(J\) then using parts (1) and (2).

The exactness in the first half of (5) follows from a routine chase through the following commutative diagram:

\[
\begin{array}{ccc}
0 & \to & \bigoplus_{i \in J} \tilde{P}_{m,i} \\
\downarrow & & \downarrow \\
0 & \to & \bigoplus_{i \in J} \tilde{P}_{m,i} \\
\downarrow & & \downarrow \\
0 & \to & \tilde{P}_{m+1} \\
\downarrow & & \downarrow \\
0 & \to & \tilde{P}^I_{m+1} \\
\downarrow & & \downarrow \\
0 & \to & \bigoplus_{i \in I \setminus J} \tilde{P}_{m,i} \\
\downarrow & & \downarrow \\
0 & \to & \tilde{P}_{J+1} \\
\end{array}
\]

This and a similar diagram give the equalities at the end of (5). \(\square\)

**Example 3.30** Fix \(I \subsetneq \{1, \ldots, n\}\). Building on Example 3.25, we describe here the truncated preprojectives \(\tilde{P}^I_m\) for \(m = 2, 3, 4\). Following Lemma 3.29.4, we can do this by simply specifying their dimension vectors as we did above.

The support quiver of \(\tilde{P}^I_2\) is obtained from the quiver (3.6) of \(\tilde{P}_2\) by removing the sinks \((1, \alpha_i)\) for \(i \in I\). The support quiver of \(\tilde{P}^I_3\) is obtained from the quiver (3.7) of \(\tilde{P}_3\) by removing the subquivers \(\alpha_i^{-1}, A_i\) for \(i \in I\) and decreasing the dimension of the space at vertex \((1, e)\) by \(|I|\).

The support quiver of \(\tilde{P}^I_4\) is given by the following analogue of the quiver (3.8) of \(\tilde{P}_4\):

\[
\begin{array}{c}
\alpha_1 \\
\downarrow \\
\alpha_1 \cdot B_1^I \\
\cdots \\
\alpha_n \cdot B_n^I \end{array}
\]

\[
\begin{array}{c}
\alpha_1 \\
\downarrow \\
\alpha_1 \\
\cdots \\
\alpha_n \end{array}
\]

\[
\begin{array}{c}
\alpha_1 \\
\downarrow \\
\alpha_1 \\
\cdots \\
\alpha_n \end{array}
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\alpha_n \end{array}
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\cdots \\
\alpha_n \end{array}
\]

\[
\begin{array}{c}
\alpha_1 \\
\downarrow \\
\alpha_1 \\
\cdots \\
\alpha_n \end{array}
\]
where the top vertex of dimension $n - 1 - |I|$ is again $(2, e)$ and $B^I$ is simply a space of dimension $n - 1 - |I|$ if $i \in I$ and otherwise is the quiver

\[
\begin{array}{c}
\alpha_1^{-1}.A_1 \\
\vdots \\
\alpha_{i-1}^{-1}.A_{i-1} \\
\alpha_i^{-1}.A_i \\
\alpha_{i+1}^{-1}.A_{i+1} \\
\vdots \\
\alpha_n^{-1}.A_n.
\end{array}
\]

Lemma 3.31 For $m \geq 3$ and $I \subseteq \{1, \ldots, n\}$, we have $\tilde{\Sigma}_2(\tilde{P}_m^I) = \tilde{P}_m^I$ and $\tau \tilde{P}_m^I = \tilde{P}_m^{m-1}$.

Proof By Lemma 3.26, we get the short exact sequence (3.9) defining the truncated preprojective $\tilde{P}_m^I$:

\[
0 \rightarrow \bigoplus_{i \in I} \tilde{P}_{m-2,i} \rightarrow \tilde{P}_{m-1} \rightarrow \tilde{P}_m^I \rightarrow 0.
\]

Applying the functor $\tilde{\Sigma}_2(\cdot)^\sigma$ to this sequence and recalling the reflection recursion (3.4), we obtain a short exact sequence

\[
0 \rightarrow \bigoplus_{i \in I} \tilde{P}_{m-1,i} \rightarrow \tilde{P}_m \rightarrow \tilde{\Sigma}_2(\tilde{P}_m^{m-1})^\sigma \rightarrow 0.
\]

The equality $\tilde{\Sigma}_2(\tilde{P}_m^{m-1})^\sigma = \tilde{P}_m^I$ immediately follows. The equality $\tilde{P}_m^I = \tau^{-1} \tilde{P}_m^{m-1}$ is obtained in the same way using the functor $\tilde{\Sigma}_1 \tilde{\Sigma}_2 = \tau^{-1}$.

For $I \subseteq \{1, \ldots, n\}$, write $I^c = \{1, \ldots, n\} \setminus I$ for the complementary subset.

Corollary 3.32 For $m \geq 1$ and $I \subseteq \{1, \ldots, n\}$, the lift $\tilde{P}_m^{(l)}$ of the truncated preprojective $P_{m+1}^I$ is preprojective.

Proof We see explicitly that $\tilde{P}_2^I = \left( \prod_{j \in I^c} \tilde{\Sigma}_1(\alpha_j) \right) (\tilde{S}(2,e))$ is preprojective. The case $m \geq 2$ then follows immediately from Lemma 3.31.

We now introduce notation for locating specific lifted preprojectives as subrepresentations of our standard lifted preprojective representations $\tilde{P}_{m-1}$. To each word $i = (i_1, \ldots, i_k)$, with $1 \leq k \leq m$, we associate a preprojective subrepresentation $\tilde{P}_{m+1-k,i} \subset \tilde{P}_{m+1}$ which lifts $P_{m+1-k}$. More precisely, we obtain from Lemma 3.22 a sequence of preprojective subrepresentations which uniquely determines the desired inclusion:

$\tilde{P}_{m+1-k,i} \subset \tilde{P}_{m+2-k,i_1,i_2} \subset \cdots \subset \tilde{P}_{m-1,i_1,i_2} \subset \tilde{P}_{m,i_1} \subset \tilde{P}_{m+1}$.

Remark 3.33 Following Remark 3.21, we will either identify a word $i = (i_1, \ldots, i_k)$ with the element

$\alpha_{i_1}^{-1} \alpha_{i_2}^{-1} \cdots \alpha_{i_k}^{(-1)^{k+1}} \in W_n$

when $m$ is even or with the element

$\alpha_{i_1}^{-1} \alpha_{i_2} \cdots \alpha_{i_k}^{(-1)^{k}} \in W_n$. 
when $m$ is odd. In this way, a subscript $i$ on a representation or morphism will denote the corresponding shift inside $\tilde{\text{rep}} \, K(n)$.

Note that, although there is a translate of $\tilde{P}_{m+1-k}$ corresponding to each word $\hat{i}$, these in general will not be naturally equipped with a canonical inclusion into $\tilde{P}_m$. To emphasize this point, consider a word $\hat{i}$ with $i_j = i_{j+1}$ for some $j$ and write $\hat{i}'$ for the word obtained from $\hat{i}$ by removing the terms $i_j$ and $i_{j+1}$. Then the representations $\tilde{P}_{m+1-k,\hat{i}}$ and $\tilde{P}_{m+1-k,\hat{i}'}$ are in fact equal, however $\tilde{P}_{m+1-k,\hat{i}}$ is naturally identified as a subrepresentation of $\tilde{P}_{m+1}$ while $\tilde{P}_{m+1-k,\hat{i}'}$ is not.

To avoid such ambiguities, we introduce the set

$$A_1^{(k)} := \{(i_1, \ldots, i_k) \mid i_j \neq i_{j+1} \text{ for } 1 \leq j \leq k-1 \} \subset \{1, \ldots, n\}^k$$

and call the elements of $A_1^{(k)}$ admissible sequences of length $k$. Here we allow $k = 0$ and take $A_1^{(0)}$ to consist of only the empty word $\hat{i} = ()$.

The following gives an analogue of Lemma 3.20.

**Lemma 3.34** For $m \geq 2$, the following hold:

1. For $i \in \{1, \ldots, n\}$, there is a canonical isomorphism

$$\tilde{P}_m^{(i)} \cong \tilde{P}_m = \tilde{P}_m/i \tilde{P}_{m-1,(i,i)}.$$

2. For any pair $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and any $J \subseteq \{i\}^c$, there is a commutative diagram

$$\begin{array}{ccc}
\tilde{P}_{m+1,\hat{i}} & \rightarrow & \tilde{P}_{m,\hat{i}} \\
\pi_{m+1,\hat{i}} & \downarrow & \pi_{m,\hat{i}} \\
\tilde{P}_{m-1,\hat{i}} & \rightarrow & \tilde{P}_{m+1,\hat{i}}
\end{array}$$

3. For an admissible sequence $\hat{i} \in A_1^{(k)}$, $1 \leq k < m$, and $J \subseteq \{1, \ldots, n\}$, the restriction of $\pi_{m+1}^{(i)} : \tilde{P}_m \rightarrow \tilde{P}_m^{(i)}$ to $\tilde{P}_{m+1-k,\hat{i}}$ is injective if $J \subseteq \{i\}^c$ and is zero if $i_1 \in J$.

4. For any admissible word $\hat{i} \in A_1^{(k)}$ with $0 \leq k < m$ and any proper subsets $I, J \subseteq \{1, \ldots, n\}$ with $i_k \in I$ and $i_1 \notin J$ if $k \geq 1$ or with $J \subseteq I$ if $k = 0$, there exist (unique up to scale) surjective maps $\pi_{m+1}^{(i)} : \tilde{P}_m \rightarrow \tilde{P}_m^{(i)}$. These satisfy the following relations:

- $\pi_{m+1}^{(i)} \circ \pi_{m+1}^{(i)} = \pi_{m+1}^{(i)}$ for $K \subseteq J$;
- $\pi_{m+1}^{(i)} \circ \pi_{m+1}^{(i)} = \pi_{m+1}^{(i)}$ for $I \subseteq H \subseteq \{1, \ldots, n\}$.
Proof The Auslander–Reiten sequence (3.5) gives rise to the following commutative diagram:

\[
\begin{array}{ccc}
0 & \longrightarrow & \tilde{P}_{m-1} \\
\downarrow & & \downarrow \\
\bigoplus_{i \in I} \tilde{P}_{m,i} & \longrightarrow & \bigoplus_{i \in I} \tilde{P}_{m,i} \\
\downarrow & & \downarrow \\
\tilde{P}_{m-1} & \longrightarrow & \tilde{P}_{m} \\
\downarrow & & \downarrow \\
\bigoplus_{i \in I} \tilde{P}_{m,i} & \longrightarrow & \tilde{P}_{m+1} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}
\]

The image of the inclusion \(\tilde{P}_{m-1} \hookrightarrow \tilde{P}_{m,j}\) in the bottom row is the subrepresentation \(\tilde{P}_{m-1,(j,j)} \subset \tilde{P}_{m,j}\) and part (1) follows.

The commutativity of the upper triangles in (2) follows from the construction of the isomorphisms in the bottom row, while the commutativity of the lower triangles is immediate from Lemma 3.29.5.

For (3), if \(i_1 \in J\), then \(\tilde{P}_{m,i_1} \subset \ker \tilde{\pi}_{J,m+1}\) and so the same is true of \(\tilde{P}_{m+1-k,i}\). By Lemma 3.29 and the admissibility of \(i\), the representation \(\tilde{P}_{m-1,(i_1,i_2)}\) embeds into \(\tilde{P}_{m,i_1}\) and hence the same is true of \(\tilde{P}_{m+1-k,i}\). For \(i_1 \notin J\), part (2) shows that such embeddings factor through \(\tilde{P}_{m+1}\) which gives (3).

The diagrams from (2) stitch together to give the following composition, where we suppress the lower isomorphisms in the diagram (3.11):

\[
\tilde{\pi}_{m+1}^{J,I,J} := \tilde{\pi}_{m+1-k,i}^{J,I} \circ \tilde{\pi}_{m+1-k,i}^{I,k-1} \circ \cdots \circ \tilde{\pi}_{m,i_1}^{I,k} \circ \tilde{\pi}_{m+1}^{I,k}. \]

These can be seen to satisfy the desired properties from (4) inductively using Lemma 3.29.5 and the commutativity from (2).

The next technical results illustrate the need to lift to the universal covering quiver \(\tilde{K}(n)\).

**Lemma 3.35** Consider a non-empty subset \(I \subsetneq \{1, \ldots, n\}\) and fix an element \(j \in I\).

1. For \(m \geq 2\), we have \(\text{Hom}(\tilde{P}_{m,j}, \tau \tilde{P}_{m+1}^{I}) \cong \mathbb{C}\). Moreover, the kernel of a nonzero morphism \(\tilde{P}_{m,j} \to \tau \tilde{P}_{m+1}^{I}\) is the following representation

\[
\tilde{P}_{m}(j, I) := \begin{cases}
\bigoplus_{1 \leq k \leq n} \tilde{P}_{m-1,(j,k)} \oplus \bigoplus_{i \in I \setminus \{j\}} \tilde{P}_{m-2,(j,i,i)} & \text{if } m \geq 3; \\
\bigoplus_{1 \leq k \leq n} \tilde{P}_{1,(j,k)} & \text{if } m = 2.
\end{cases}
\]

2. For \(m \geq 3\), any nonzero morphism \(\tilde{P}_{m,j} \to \tau \tilde{P}_{m+1}^{I}\) is surjective.
Proof The first claim of part (1) follows immediately by applying the Auslander–Reiten formulas [1, Theorem IV.2.13] to Lemma 3.29.3. Indeed, this gives
\[ \dim \text{Hom}(\tilde{P}_{m,j}, \tau \tilde{P}_{m+1}^I) = \dim \text{Ext}(\tilde{P}_{m+1}^I, \tilde{P}_{m,j}) = 1. \]

We establish the final claim of part (1) directly for \( m = 2 \). Using the description in Example 3.30, it is not hard to see that \( \tau \tilde{P}_3^I \) is indecomposable with one-dimensional spaces at only the vertices \((1, e)\) and \((2, \alpha_i^{-1})\) for \( i \in I \). But then for \( j \in I \), the image of the unique homomorphism \( \tilde{P}_{2,j} \to \tau \tilde{P}_3^I \) is the representation with support quiver
\[ (2, \alpha_j^{-1}) \xrightarrow{\alpha_j} (1, e). \]
From the support quiver of \( \tilde{P}_{2,j} \) given in Example 3.25, we see that the kernel of this map is precisely \( \tilde{P}_2(I, j) \).

For the \( m \geq 3 \) case, we apply Lemma 3.29 to get the lower sequence in the following commutative diagram, which exists because the upper sequence is almost split:
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \tilde{P}_{m-2,j} & \longrightarrow & \bigoplus_{k=1}^n \tilde{P}_{m-1,(j,k)} & \longrightarrow & \tilde{P}_{m,j} & \longrightarrow & 0 \\
\downarrow{\eta} & & \downarrow{\eta} & & \downarrow{\eta} & & \downarrow{\eta} & & \downarrow{\eta} \\
0 & \longrightarrow & \tilde{P}_{m-2,j} & \longrightarrow & \tilde{P}_{m-1}^{I \setminus \{j\}} & \longrightarrow & \tilde{P}_{m-1}^I & \longrightarrow & 0
\end{array}
\]

Since the evaluation map \( ev_I \) in Eq. (3.9) is irreducible, any nontrivial map to \( \tilde{P}_{m-1}^I \) must factor through \( \tilde{P}_{m-1}^I \). It then follows from Lemma 3.26 that \( \tilde{P}_{m-1,(j,k)} \) is in the kernel of the vertical morphism \( \eta \) for \( k \neq j \). Therefore \( \ker \eta \) is the direct sum of these with the kernel of the induced map \( \tilde{P}_{m-1} \to \tilde{P}_{m-1}^{I \setminus \{j\}} \), that is with \( \bigoplus_{i \in I \setminus \{j\}} \tilde{P}_{m-2,i} = \bigoplus_{i \in I \setminus \{j\}} \tilde{P}_{m-2,(j,i)} \), and so \( \ker \eta = \tilde{P}_m(j, I) \). Since \( \eta \) and the right hand vertical morphism have the same kernel, this completes the proof of (1).

This also shows that the vertical map \( \eta \) is surjective, and thus the map \( \tilde{P}_{m,j} \to \tilde{P}_{m-1}^I \cong \tau \tilde{P}_{m+1}^I \) is surjective. This establishes part (2). \( \square \)

Remark 3.36 We should point out that the case \( m = 2 \) of Lemma 3.35 is rather special because it is the only one for which the unique morphism \( \tilde{P}_{m,j} \to \tau \tilde{P}_{m+1}^I \) is not surjective. Indeed, recall from the proof of Lemma 3.35 that the image of the unique homomorphism \( \tilde{P}_{2,j} \to \tau \tilde{P}_3^I \) is the representation with support quiver
\[ (2, \alpha_j^{-1}) \xrightarrow{\alpha_j} (1, e). \]
If we factor out the image of this morphism from \( \tau \tilde{P}_3^I \), the remaining representation is a direct sum of the simple injective representations corresponding to the vertices \((2, \alpha_i^{-1})\) for \( i \in I, i \neq j \). Note that these disappear after reflecting at all sources.

The following orthogonality property is a primary reason we need to lift to the universal cover of \( K(n) \). As usual, given \( \tilde{M} \in \text{rep} \ K(n) \), we write
\[ \tilde{M}^\perp := \{ \tilde{N} \in \text{rep} \ K(n) : \text{Hom}(\tilde{M}, \tilde{N}) = 0 = \text{Ext}(\tilde{M}, \tilde{N}) \}. \]

Corollary 3.37 Consider a non-empty subset \( I \subseteq \{1, \ldots, n\} \) and fix an element \( j \in I \). For \( m \geq 2 \), we have \( \tilde{P}_m(j, I) \in (\tilde{P}_{m+1}^I)^\perp \).
If \( m = 2 \), we have \( \text{Hom}(\tilde{P}_3(I, j)) = 0 \) since \( \tilde{P}_2(I, j) \) is a direct sum of the simple projective representations \( \tilde{P}_{1, (j,k)} \) for \( k \neq j \). Using the explicit description of \( \tilde{P}_3 \) from Example 3.30, we see that each \( \tilde{P}_{1, (j,k)} \) for \( k \neq j \) is supported at a vertex which is not a neighbor of the support of \( \tilde{P}_3 \), which implies \( \text{Ext}(\tilde{P}_3, \tilde{P}_{1, (j,k)}) = 0 \) for \( k \neq j \) and thus \( \text{Ext}(\tilde{P}_3, \tilde{P}_2(I, j)) = 0 \).

If \( m \geq 3 \), we consider the long exact sequence obtained when applying \( \text{Hom}(\tilde{P}_{m+1}, -) \) to the sequence

\[
0 \longrightarrow \tilde{P}_m(j, I) \longrightarrow \tilde{P}_{m, j} \longrightarrow \tau \tilde{P}_{m+1} \longrightarrow 0.
\]

Following Lemma 3.29.3, we have \( \text{Hom}(\tilde{P}_{m+1}, \tilde{P}_{m, j}) = 0 \) and using the long exact sequence this immediately implies \( \text{Hom}(\tilde{P}_{m+1}, \tilde{P}_m(j, I)) = 0 \). From Lemma 3.29.3 again, we have \( \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m, j}) \cong \mathbb{C} \). Using the Auslander–Reiten formulas [1, Theorem IV.2.13] and Lemma 3.29.4, we get

\[
\dim \text{Ext}(\tilde{P}_{m+1}, \tau \tilde{P}_{m+1}) = \dim \text{Hom}(\tilde{P}_{m+1}, \tilde{P}_{m+1}) = 1.
\]

It follows that the surjective map \( \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m, j}) \to \text{Ext}(\tilde{P}_{m+1}, \tau \tilde{P}_{m+1}) \) appearing in the long exact sequence is in fact an isomorphism. But the Auslander–Reiten formulas and Lemma 3.29.4 again imply

\[
\dim \text{Hom}(\tilde{P}_{m+1}, \tau \tilde{P}_{m+1}) = \dim \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m+1}) = 0.
\]

Combining with the preceding discussion, this gives \( \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_m(j, I)) = 0 \).

\[\square\]

4 Quiver Grassmannians

In this section, we aim to establish the existence of cell decompositions for quiver Grassmannians of (truncated) preprojective representations of \( K(n) \) and its universal covering quiver. By a cell decomposition of an algebraic variety \( X \), we mean a filtration \( \emptyset = X_k \supseteq X_{k-1} \supseteq \cdots \subseteq X_2 \subseteq X_1 = X \) of \( X \) by closed subsets \( X_i \subseteq X \) so that each \( X_i \) is isomorphic to an affine space. Alternatively, a cell decomposition of \( X \) is a disjoint collection of locally closed subsets \( U_1, \ldots, U_k \subseteq X \), each isomorphic to an affine space, such that each \( X_i = U_i \cup U_{i+1} \cup \cdots \cup U_k \) is closed in \( X \) with \( X_1 = X \). We call the subsets \( U_i \) the affine cells for this cell decomposition. Given varieties \( X \) and \( Y \) each with cell decompositions, we may choose an ordering on products of their affine cells (e.g. lexicographic) to get a cell decomposition of \( X \times Y \). Given a variety \( X \) with a cell decomposition, we call a subvariety \( U \subseteq X \) compatible with the cell decomposition if \( U \) can be written as the union over a subset of the affine cells for \( X \). In this case, \( U \) also has a cell decomposition given by taking exactly those affine cells for \( X \) which are contained in \( U \).

The following result of Caldero and Reineke lays a foundation for our approach.

**Theorem 4.1** [12] Every quiver Grassmannian of a rigid representations of an acyclic quiver is smooth.

This allows the application of the torus method of Białynicki-Birula described in the following section.
4.1 Torus actions and the Biaynicki-Birula decomposition

The aim of Sect. 4.2 is to define a $\mathbb{C}^\ast$-action on quiver Grassmannians which can be used to simplify the calculation of homological invariants in general. If the quiver Grassmannian is smooth, which is for instance the case for exceptional representations by [12], this action can also be used to stratify the quiver Grassmannians using the results of Białynicki-Birula. More specifically, let $X$ be a smooth projective variety with a $\mathbb{C}^\ast$-action. For a connected component of the fixed point set $C \subset X^{\mathbb{C}^\ast}$, we define its attracting set as

$$\text{Att}(C) := \{ y \in X \mid \lim_{t \to 0} t \cdot y \in C \},$$

where $\lim_{t \to 0} t \cdot y$ denotes the unique completion of the map $\mathbb{C}^\ast \to X$, $t \mapsto t \cdot y$, to a map $\mathbb{C} \to X$. The following result of Białynicki-Birula relates the geometry of $X$ to the geometry of its $\mathbb{C}^\ast$-fixed points (see [3, Section 4] or [4, Section 4]).

**Theorem 4.2** Let $X$ be a smooth projective complex variety with a $\mathbb{C}^\ast$-action. Then each attracting set $\text{Att}(C)$ is a locally closed $\mathbb{C}^\ast$-invariant subvariety of $X$ and the natural map $\text{Att}(C) \to C$ is an affine bundle. In particular, the existence of a cell decomposition of $C$ implies a cell decomposition of $\text{Att}(C)$. Moreover, assuming $X^{\mathbb{C}^\ast} = \bigsqcup_{i=1}^r C_i$ is a decomposition of the fixed point set of $X$ into finitely many connected components, we have $X = \bigsqcup_{i=1}^r \text{Att}(C_i)$, where we can choose an ordering such that $\bigsqcup_{i=1}^s \text{Att}(C_i)$ is closed for $1 \leq s \leq r$. In particular, we have an equality of Euler characteristics $\chi(X) = \chi(X^{\mathbb{C}^\ast})$ and, when each $C_i$ admits a cell decomposition, we obtain a cell decomposition of $X$.

4.2 Torus actions on quiver Grassmannians

Fix a vector space $X$ of dimension $n$ and let $k \leq n$. We first consider a natural $\mathbb{C}^\ast$-action on the usual Grassmannian $\text{Gr}_k(X)$ which is compatible with a given direct sum decomposition of the vector space $X$. Then we generalize the concept to quiver Grassmannians and observe that the $\mathbb{C}^\ast$-fixed point sets can be calculated in an analogous manner. This greatly extends the results [9, Theorem 1] and [14, Theorem 1.2] which utilize similar constructions.

Given a basis $B = \{ v_1, \ldots, v_n \}$ of $X$ and a map $d : \{1, \ldots, n\} \to \mathbb{Z}$, we get a $\mathbb{C}^\ast$-action on $X$ when linearly extending the definition $t \cdot v_r := t^{d(r)} v_r$ for $r = 1, \ldots, n$ to all of $X$. This naturally induces an action of $\mathbb{C}^\ast$ on the Grassmannian $\text{Gr}_k(X)$. Our goal is to understand the fixed points of such an action.

For this recall that we can represent each subspace $U \in \text{Gr}_k(X)$ uniquely by a $k \times n$ matrix $M(U)$ whose rows provide a basis for $U$ when expanded as coefficient vectors in the basis $B$. The uniqueness of $M(U)$ comes from requiring that it be in a row-echelon form, e.g. we will assume there exists a unique sequence $1 \leq i_1 < \ldots < i_k \leq n$ so that $M(U)$ is of the form
Proof Assume a map \( d : C \to \mathbb{C} \) and thus a subspace necessarily a subrepresentation of not induce a determined by \( d \), we have \( U \in \mathbb{C} \) for each \( i \), \( j \) of \( Q \) and each \( \alpha : i \to j \) of \( Q \) and every \( \alpha \). In general, this does not induce a \( \mathbb{C}^* \)-action on the quiver Grassmannians \( \text{Gr}_e(X) \) since \( t.E \) (for every arrow \( \alpha \) of \( Q \) and every \( t \) of \( \mathbb{C}^* \)). Indeed, for such an action must satisfy \( X_\alpha(t.E_i) \subset t.E_j \) for every arrow \( \alpha : i \to j \) of \( Q \) and every \( t \) of \( \mathbb{C}^* \).

Lemma 4.3 Consider a map \( d : [1, \ldots, n] \to \mathbb{Z} \) such that \( d(r) = d(r') \) if \( v_r, v_{r'} \in X_1 \) for some \( l \) and \( d(r) \neq d(r') \) if \( v_r \in X_1 \) and \( v_{r'} \in X_i \) with \( l \neq i \). Then under the \( \mathbb{C}^* \)-action determined by \( d \), we have \( U \in \text{Gr}_k(X)^{\mathbb{C}^*} \) if and only if \( U = \bigoplus_{i=1}^m U \cap X_i \).

Proof Assume \( U = \bigoplus_{i=1}^m U \cap X_i \). Then any \( u \in U \) can be written uniquely as \( u = \sum_{i=1}^m u_i \) for some \( u_i \in U \cap X_i \). It follows that \( t.u \leq \sum_{i=1}^m t.u_i \leq \sum_{i=1}^m t^{d(r)} u_i \in \bigoplus_{i=1}^m U \cap X_i = U \) and thus \( t.U = U \).

For the reverse direction, assume \( U \subset X_1 \) is a \( \mathbb{C}^* \)-fixed point represented by the matrix \( M(U) \). Then, if \( v_i \in X_1 \), the assumptions on \( d \) imply \( M(U)_{q,r} = 0 \) unless \( r_{i-1} + 1 \leq r \leq r_i \). That is, \( M(U) \) has the shape of a block matrix representing the decomposition \( U = \bigoplus_{i=1}^m U \cap X_i \).

The next step is to generalize this to quiver Grassmannians. Let \( Q \) be an acyclic quiver. Choose a map \( d : \hat{Q} \to \mathbb{Z} \) and fix a representation \( X \in \text{rep} Q \) which can be lifted to \( \hat{Q} \). We consider the decomposition \( X_i = \bigoplus_{\chi \in A_Q} X_{(i,\chi)} \) and define a \( \mathbb{C}^* \)-action on each \( X_{(i,\chi)} \) via \( t.x_{(i,\chi)} = t^{d(i,\chi)} x_{(i,\chi)} \) which is then extended linearly to each \( X_i \). Associated to each subspace \( E_i \), there is a corresponding subspace \( t.E_i \) for each \( t \in \mathbb{C}^* \). In general, this does not induce a \( \mathbb{C}^* \)-action on the quiver Grassmannians \( \text{Gr}_e(X) \) since \( t.E = \{ t.E_i \}_{i \in \hat{Q}} \) is not necessarily a subrepresentation of \( X \) for every \( E \in \text{Gr}_e(X) \). Indeed, for such an action must satisfy \( X_\alpha(t.E_i) \subset t.E_j \) for every arrow \( \alpha : i \to j \) of \( Q \) and every \( t \) of \( \mathbb{C}^* \).

Lemma 4.4 Let \( X \) be a representation of \( Q \) which can be lifted to \( \hat{Q} \). Fix an integer \( c_\alpha \in \mathbb{Z} \) for each \( \alpha \in Q \). Suppose \( d : \hat{Q} \to \mathbb{Z} \) satisfies \( d(j, \chi + e_\alpha) - d(i, \chi) = c_\alpha \) for each arrow \( \alpha : i \to j \) of \( Q \) and each \( \chi \in A_Q \). Then the \( \mathbb{C}^* \)-action on \( X \) determined by \( d \) induces a \( \mathbb{C}^* \)-action on \( \text{Gr}_e(X) \).

Proof Fix \( E \in \text{Gr}_e(X) \) and consider \( e_i \in E_i \). Since \( E \) is a subrepresentation, for an arrow \( \alpha : i \to j \) of \( Q \) we may write \( X_\alpha(e_i) = e_j \) for some \( e_j \in E_j \).

As \( X \) can be lifted to \( \hat{Q} \), for any arrow \( \alpha \in Q \) we can write \( X_\alpha : X_i \to X_j \) as a block matrix consisting of linear maps \( X_{(\alpha,\chi)} : X_{(i,\chi)} \to X_{(j,\chi + e_\alpha)} \) for \( \chi \in A_Q \). Then writing
\[ e_i = \sum_{\chi \in A_Q} e(i, \chi) \] for some vectors \( e(i, \chi) \in X(i, \chi) \), we have \( X_\alpha(e(i, \chi)) = X(\alpha, \chi)(e(i, \chi)) \in X(j, \chi + e_\alpha) \), say \( X(\alpha, \chi)(e(i, \chi)) = e(j, \chi + e_\alpha) \). It follows that \( e_j = \sum_{\chi \in A_Q} e(j, \chi + e_\alpha) \) and so

\[
X_\alpha(t.e_i) = \sum_{\chi \in A_Q} t^{d(i, \chi)} X(\alpha, \chi)(e(i, \chi)) = \sum_{\chi \in A_Q} t^{d(i, \chi)} e(j, \chi + e_\alpha)
\]

\[
= t^{-c_\alpha} \sum_{\chi \in A_Q} t.e(j, \chi + e_\alpha) = t^{-c_\alpha} t.e_j.
\]

Therefore \( X_\alpha(t.E_i) \subset t.E_j \) for every arrow \( \alpha : i \to j \) of \( Q \) and we obtain a \( \mathbb{C}^* \)-action on \( \text{Gr}_e(X) \).

The next result provides the conditions on the map \( d : \hat{Q}_0 \to \mathbb{Z} \) needed to get an analogue of Lemma 4.3

**Lemma 4.5** Let \( \hat{X} \in \text{rep} \hat{Q} \) be an indecomposable representation of \( \hat{Q} \). There exists \( d : \text{supp}(\hat{X}) \to \mathbb{Z} \) and \( c_\alpha \in \mathbb{N}_+ \) for each \( \alpha \in Q_1 \) such that

1. for \((i, \chi), (i, \chi') \in \text{supp}(\hat{X}) \) with \( \chi \neq \chi' \), we have \( d(i, \chi) \neq d(i, \chi') \);
2. for \((i, \chi), (j, \chi') \in \text{supp}(\hat{X}) \), we have \( d(j, \chi') - d(i, \chi) = c_\alpha \) if and only if \( \chi' = \chi + e_\alpha \).

**Proof** For convenience we introduce the notation \( Q_1 = \{ \alpha_1, \ldots, \alpha_n \} \). Since \( \hat{X} \) is finite-dimensional and indecomposable, the support quiver \( \text{supp}(\hat{X}) \) is a connected and finite subquiver of \( \hat{Q} \). In order to prove the statement, we may assume that \((i', 0) \in \text{supp}(\hat{X}) \) for some \( i' \in \hat{Q}_0 \). Let \( K \) be the maximal length of a path in \( \text{supp}(\hat{X}) \) starting or ending in \((i', 0) \) such that the underlying graph of the path has no cycles. This implies that, for \((i, \chi) \in \text{supp}(\hat{X}) \) with \( \chi = \sum_{l=1}^n \kappa_l e_{\alpha_l} \), we have \( |\kappa_l| \leq K \).

Set \( c_{\alpha_1} = 1 \) and choose \( c_{\alpha_l} \) recursively in such way that

\[
c_{\alpha_l} \geq 2(K + 1) \sum_{k=1}^{l-1} c_{\alpha_k}
\]

for \( l = 2, \ldots, n \). Then let \( f : A_Q \to \mathbb{Z} \) be the group homomorphism defined by \( f(e_\alpha) = c_\alpha \) for all \( \alpha \in Q_1 \) and define \( d(i, \chi) := f(\chi) \) for \((i, \chi) \in \text{supp}(\hat{X}) \).

To check property (1), assume that \( d(i, \chi) = d(i, \chi') \) for \( \chi = \sum_{l=1}^n \kappa_l e_{\alpha_l} \) and \( \chi' = \sum_{l=1}^n \kappa'_l e_{\alpha_l} \). This implies

\[
\sum_{l=1}^{n-1} (\kappa_l - \kappa'_l) c_{\alpha_l} = (\kappa'_n - \kappa_n) c_{\alpha_n}.
\]

But we have \( |\kappa_l - \kappa'_l| \leq |\kappa_l| + |\kappa'_l| \leq 2K \) and thus we obtain

\[
|\kappa'_n - \kappa_n| c_{\alpha_n} = \left| \sum_{l=1}^{n-1} (\kappa_l - \kappa'_l) c_{\alpha_l} \right| \leq 2K \sum_{l=1}^{n-1} c_{\alpha_l} < c_{\alpha_n}.
\]

This inductively yields \( \kappa_l = \kappa'_l \) for \( l = n, \ldots, 1 \) by the choice of the \( c_{\alpha_l} \) and thus \( \chi = \chi' \).

By definition, we have \( d(j, \chi + e_\alpha) - d(i, \chi) = c_\alpha \) when \((j, \chi + e_\alpha) \in \text{supp}(\hat{X}) \). Now assuming \( d(j, \chi') - d(i, \chi) = c_\alpha \), an analogous argument to the one above shows that \( \chi' = \chi + e_\alpha \).

In the following, we say that \( d : \text{supp}(\hat{X}) \to \mathbb{Z} \) satisfies the degree condition for \( \hat{X} \) if it has the properties of Lemma 4.5.
Theorem 4.6 Let $X$ be a representation of $Q$ which can be lifted to a representation $\hat{X}$ of $\hat{Q}$ and choose $d : \text{supp}(\hat{X}) \to \mathbb{Z}$ such that it satisfies the degree condition for $\hat{X}$. Then the $\mathbb{C}^*$-action on $\bigoplus_{i \in Q_0} X_i$ determined by $t.x(i, x) = t^{d(i, x)}x(i, x)$ for $x(i, x) \in X(i, x)$ induces a $\mathbb{C}^*$-action on $\text{Gr}^\hat{e}(X)$ such that

$$\text{Gr}^\hat{e}(X)\mathbb{C}^* \cong \bigsqcup_{\hat{e}} \text{Gr}^\hat{e}(\hat{X}),$$

where $\hat{e}$ runs through all dimension vectors compatible with $e$.

Proof A representation $E \in \text{Gr}_e(X)$ is a $\mathbb{C}^*$-fixed point if and only if $t.E = E$ for all $t \in \mathbb{C}^*$, i.e. $t.E_i = E_i$ for all $i \in Q_0$ and all $t \in \mathbb{C}^*$. Thus, apart from being a subrepresentation of $X$, each component $E_i$ is a fixed point of the induced $\mathbb{C}^*$-actions on the usual Grassmannians of vector subspaces $\text{Gr}_e(X_i)$. By Lemma 4.3, this holds precisely when we have a decomposition

$$E_i = \bigoplus_{\chi \in \hat{Q}_0} E_i \cap X(i, x)$$

which is equivalent to $E$ being liftable to the universal abelian covering $\hat{Q}$.

The next step is to iterate the $\mathbb{C}^*$-actions, keeping in mind the following idea: every representation $X$ which lifts to the universal covering quiver also lifts to the universal abelian covering quiver and to the iterated universal abelian covering quivers, i.e. to each $\hat{Q}^{(k)} := \hat{Q}^{(k-1)}$ with $\hat{Q}^{(1)} := \hat{Q}$. Now it is straightforward to check that there exist natural surjective morphisms $f_k : \hat{Q}^{(k)} \to \hat{Q}^{(k+1)}$ which become injective on finite subquivers if $k \gg 0$, see also [20, Section 3.4]. Since the support of $X$ is finite as a representation of $\hat{Q}$, we can find $k \geq 0$ such that the full subquiver with vertices $\text{supp}(X) \subseteq \hat{Q}^{(k+1)}$ is a tree. Thus, writing $\hat{X}^{(k)}$ for the lift of $X$ to $\hat{Q}^{(k)}$, there exists a $\mathbb{C}^*$-action on the vector spaces $\hat{X}^{(k)}_\beta$ for $\beta \in \hat{Q}^{(k)}_0 \times A_{\hat{Q}^{(k-1)}}$ which induces $\mathbb{C}^*$-actions on the quiver Grassmannians $\text{Gr}^{\hat{e}^{(k)}}_{\hat{e}^{(k+1)}}(\hat{X}^{(k+1)})$ such that the fixed point sets are precisely $\text{Gr}^{\hat{e}^{(k+1)}}_{\hat{e}^{(k+1)}}(\hat{X}^{(k+1)})$. If we denote these iterated $\mathbb{C}^*$-fixed points by $\text{Gr}^{\hat{e}^{(k+1)}}_{\hat{e}^{(k+1)}}(X)^{(k+1)}$, we obtain the following result.

Corollary 4.7 Let $X$ be a representation which can be lifted to $\hat{Q}$. Then there exists an iterated torus action such that

$$\text{Gr}^{\hat{e}^{(k)}}_{\hat{e}^{(k+1)}}(X)^{(k+1)} \cong \bigsqcup_{\hat{e}^{(k)}} \text{Gr}^{\hat{e}^{(k)}}_{\hat{e}^{(k+1)}}(\hat{X}^{(k)}) \cong \bigsqcup_{\hat{e}^{(k+1)}} \text{Gr}^{\hat{e}^{(k+1)}}_{\hat{e}^{(k+1)}}(\hat{X}^{(k+1)}) \cong \bigsqcup_{\hat{e}} \text{Gr}^{\hat{e}}_{\hat{e}}(\hat{X}),$$

where $\hat{e}^{(k)}$, $\hat{e}^{(k+1)}$, $\hat{e}$ run through all dimension vectors compatible with $e$.

Define the $F$-polynomial of a representation $X$ by

$$F_X = \sum_{e \in \mathbb{N}_{Q_0}} x(\text{Gr}_e(X))y^e \in \mathbb{Z}[y_i \mid i \in Q_0].$$

Corollary 4.8 Let $X$ be a representation which can be lifted to the universal covering quiver.

1. If $\text{Gr}_e(X)$ is smooth and each $\text{Gr}^{\hat{e}}_{\hat{e}}(\hat{X})$ has a cell decomposition, then $\text{Gr}_e(X)$ has a cell decomposition.
2. We have $\mathcal{F}_Y = S\mathcal{F}_X$ where $S\mathcal{F}_X$ is obtained from $\mathcal{F}_X$ by applying $S : \mathbb{Z}[y_{i,w}] | i \in Q_0, w \in W_Q | \rightarrow \mathbb{Z}[y_i] | i \in Q_0]$ given by $S(y_{i,w}) = y_i$ for all $i \in Q_0$ and $w \in W_Q$.

An important special case for this is the case of exceptional representations. In this case the quiver Grassmannians $Gr_e(X)$ are smooth by [12, Corollary 4]. Moreover, every exceptional representation is a tree module by [17] which means that it can be lifted to the universal covering.

4.3 $GL_n$-action on arrows of $K(n)$

The goal of this section is to prove Theorem 4.13 showing that quiver Grassmannians of truncated preprojectives $P^V_{m+1}$ are smooth and only depend on the dimension of the subspace $V \subset H_m$. We begin by observing that $GL_n(\mathbb{C})$ naturally acts on the vector space $A_1 = \bigoplus_{i=1}^n \text{C}a_i$ spanned by the arrows of $K(n)$ and hence $GL_n(\mathbb{C})$ acts on the universal action on the path algebra $A(n)$. More precisely, given a representation $M = (M_1, M_2, M_{\alpha})$ of $K(n)$ and $g = (g_{ij}) \in GL_n(\mathbb{C})$, the representation $g.M$ is given by $(M_1, M_2, (g.M)_{\alpha})$ with $(g.M)_{\alpha} = \sum_{j=1}^n g_{ij}M_{\alpha_j}$. Note that $M$ and $g.M$ are not necessarily isomorphic as representations of $K(n)$.

Lemma 4.9 For any morphism $\theta : M \rightarrow N$ between representations $M, N \in \text{rep} K(n)$, the same maps $\theta_1 : M_1 \rightarrow N_1$ and $\theta_2 : M_2 \rightarrow N_2$ give a morphism $\theta_S : g.M \rightarrow g.N$ for any $g \in GL_n(\mathbb{C})$. In particular, the hom-spaces $\text{Hom}(M, N)$ and $\text{Hom}(g.M, g.N)$ are canonically identified for each $g \in G$.

Proof Suppose $\theta : M \rightarrow N$ is a morphism of representations, i.e. $\theta_2 \circ M_{\alpha_j} = N_{\alpha_j} \circ \theta_1$ for $1 \leq j \leq n$. Then for $g = (g_{ij}) \in GL_n(\mathbb{C})$ and $1 \leq i \leq n$, we have

$$\theta_2 \circ (g.M)_{\alpha_i} = \sum_{j=1}^n g_{ij}(\theta_2 \circ M_{\alpha_j}) = \sum_{j=1}^n g_{ij}(N_{\alpha_j} \circ \theta_1) = (g.N)_{\alpha_i} \circ \theta_1$$

so that $\theta$ also gives a morphism from $g.M$ to $g.N$. □

Corollary 4.10 For $M \in \text{rep} K(n)$ and $g \in GL_n(\mathbb{C})$, the representation $M$ is indecomposable if and only if $g.M$ is indecomposable.

Proof The representation $M$ is decomposable if there exists a split epimorphism $\theta : M \rightarrow N$ for some nonzero representation $N$. But this occurs exactly when the map $\theta_S : g.M \rightarrow g.N$ is also a split epimorphism. □

While the reflection functors are not $GL_n(\mathbb{C})$-equivariant, they do admit the following twisted equivariance.

Lemma 4.11 For each $g \in G$, the reflection functors $\Sigma_i : \text{rep} K(n) \rightarrow \text{rep} K(n)$ satisfy $g.\Sigma_i(M) = \Sigma_i(g^{-T}.M)$.

Proof We present all the details for $\Sigma_2$, the proof for $\Sigma_1$ is similar. For $M \in \text{rep} K(n)$, we have $\Sigma_2(M) = (M_1, M_2', M_{\alpha_1}')$, where $M_2'$ fits into the following exact sequence:

$$0 \rightarrow M_2 \rightarrow \bigoplus_{i=1}^n M_{\alpha_i} \rightarrow M_1 \rightarrow \pi M_2' \rightarrow 0$$

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and $M'_{i_1} = \pi \circ i_1$ for $i_1 : M_1 \rightarrow \bigoplus_{i=1}^n M_1$ the inclusion of the $i$th factor. Then for $g = (g_{ij}) \in GL_n(\mathbb{C})$, we have $g \cdot \Sigma_2(M) = (M_1, M'_2, (g \cdot M')_{a_i})$ with

$$(g \cdot M')_{a_i} = \sum_{j=1}^n g_{ij} M'_{a_j} = \sum_{j=1}^n g_{ij} (\pi \circ i_j) = \pi \circ \sum_{j=1}^n g_{ij} i_j = \pi \circ g^T \circ i_1.$$ 

In particular, we may construct $g \cdot \Sigma_2(M)$ using the following exact sequence:

$$0 \rightarrow M_2 \xrightarrow{g^{-T} \circ \bigoplus_{i=1}^n M_{a_i}} \bigoplus_{i=1}^n M_1 \xrightarrow{\pi \circ g^T} M'_2 \rightarrow 0,$$

i.e. $g \cdot \Sigma_2(M) = \Sigma_2(g^{-T} \cdot M)$. □

Write $\text{Ind}(K(n), d)$ for the set of isomorphism classes of indecomposable representations of $K(n)$ with dimension vector $d$ for $d \in \mathbb{N}^{K(n)\circ}$. As the $GL_n(\mathbb{C})$-action above commutes with the natural base change action on representations, we can define a $GL_n(\mathbb{C})$-action on $\text{Ind}(K(n), d)$.

**Proposition 4.12** Let $m \geq 1$ and $d(m, r) = \dim P_{m+1} - r \dim P_m$ with $0 \leq r \leq n - 1$. The action of $GL_n(\mathbb{C})$ is transitive on $\text{Ind}(K(n), d(m, r))$.

**Proof** For $m = 1$, the action of $GL_n(\mathbb{C})$ on $\text{Gr}_d(\mathbb{C}^n)$ is transitive which shows that it is transitive on $\text{Ind}(K(n), d(1, r))$. As the reflection functors preserve indecomposability and isomorphism classes, Lemma 4.11 gives a commutative diagram as below for each $g \in GL_n(\mathbb{C})$:

$$\text{Ind}(K(n), d(m, r)) \xrightarrow{\Sigma_2} \text{Ind}(K(n), d(m + 1, r)) \xrightarrow{g^{-T}} \text{Ind}(K(n), d(m + 1, r)).$$

As the $GL_n(\mathbb{C})$-action is transitive on the left hand side, this implies that it is also transitive on the right hand side. □

**Theorem 4.13** Fix a dimension vector $e$. The quiver Grassmannian $\text{Gr}_e(P^V_{m+1})$ is smooth for each $V \in \text{Gr}(\mathcal{H}_m)$. Moreover, for $V, W \in \text{Gr}_r(\mathcal{H}_m), 1 \leq r \leq n - 1$, we have $\text{Gr}_e(P^V_{m+1}) \cong \text{Gr}_e(P^W_{m+1})$.

**Proof** Let $g \in GL_n(\mathbb{C})$. We first show that $\text{Gr}_e(g \cdot P^V_{m+1}) = \text{Gr}_e(P^V_{m+1})$. Assume that $P^V_{m+1}$ is given by the linear maps $M_{a_i}$. Then $g \cdot P^V_{m+1}$ is given by the matrices $(g \cdot M)_{a_i} = \sum_{j=1}^n g_{ij} M_{a_j}$. Let $(E_1, E_2) \in \text{Gr}_e(g \cdot P^V_{m+1})$. Then we have

$$(M_{a_i}(E_2) \subset E_1 \text{ for all } i = 1, \ldots, n) \leftrightarrow \left( \sum_{j=1}^n g_{ij} M_{a_j} \right)(E_2) \subset E_1 \text{ for all } i = 1, \ldots, n.$$ 

Note that we have $g^{-1}(g \cdot P^V_{m+1}) = P^V_{m+1}$ which shows the non-obvious direction.
Propositions 3.11 and 4.12 give that the quiver Grassmannians \( \text{Gr}_e(M) \) for \( M \in \text{Ind}(K(n), \text{dim} P^V_{m+1}) \) are all isomorphic for a fixed \( e \in \mathbb{N}^{Q_0} \). In other words, we use that all indecomposables with this dimension vector are truncated preprojectives. As the indecomposables form a dense open subset of all representations, we found a dense subset whose quiver Grassmannians for a fixed \( e \) are isomorphic. But now the same proof as for exceptional roots applies in order to show that these quiver Grassmannians need to be smooth, see [12, Corollary 4].

4.4 Fibrations of quiver Grassmannians

Let \( \eta : 0 \to \tilde{M} \to \tilde{B} \to \tilde{N} \to 0 \) be a short exact sequence of \( \tilde{K}(n) \)-representations. For a fixed dimension vector \( \tilde{e} \), there is an induced “Caldero–Chapoton map” between quiver Grassmannians

\[
\Psi = \Psi^\eta : \text{Gr}_{\tilde{e}}(\tilde{B}) \to \bigsqcup_{\tilde{f} + \tilde{g} = \tilde{e}} \text{Gr}_f(\tilde{M}) \times \text{Gr}_g(\tilde{N})
\]

\[
E \mapsto (\tilde{E} \cap \tilde{M}, (\tilde{E} + \tilde{M})/\tilde{M}).
\]

Following [7, Section 3], any non-empty fiber of \( \Psi \) satisfies \( \Psi^{-1}(\tilde{F}, \tilde{G}) \cong \Lambda^{\dim \text{Hom}(\tilde{G}, \tilde{M}/\tilde{F})} \). For \( \tilde{G}_{f, g} := \Psi^{-1}(\text{Gr}_f(\tilde{M}) \times \text{Gr}_g(\tilde{N})) \), we have

\[
\text{Gr}_{\tilde{e}}(\tilde{B}) = \bigsqcup_{\tilde{f} + \tilde{g} = \tilde{e}} \tilde{G}_{f, g}.
\]  

(4.1)

Then \( \Psi \) restricts to a map

\[
\Psi_{f, g} : \tilde{G}_{f, g} \to \text{Gr}_f(\tilde{M}) \times \text{Gr}_g(\tilde{N}).
\]

The following results are proven in [11, Section 3]. For completeness we include a proof of the first.

**Lemma 4.14** There exists a total ordering \( \preceq \) of the dimension vectors \( \tilde{f} \) appearing in the decomposition (4.1) such that for any fixed \( \tilde{f} \) the subset

\[
\bigsqcup_{\tilde{f}' \succeq \tilde{f}} \tilde{G}_{f', \tilde{e} - \tilde{f}}
\]

is closed in \( \text{Gr}_{\tilde{e}}(\tilde{B}) \).

**Proof** Recall that \( \text{Gr}_{\tilde{e}}(\tilde{B}) \) is a closed subvariety of \( \prod_{i \in Q_0} \text{Gr}_{e_i}(\tilde{B}_i) \). For each \( i \), the inclusion \( \tilde{M}_i \subset \tilde{B}_i \) induces an upper-semicontinuous function \( \rho_i : \text{Gr}_{e_i}(\tilde{B}_i) \to \mathbb{Z} \) given by \( \rho_i(\tilde{E}_i) = \dim(\tilde{E}_i \cap \tilde{M}_i) \). In particular, for any fixed \( f_i \) the set \( \rho_i^{-1}((f_i, f_i + 1, \ldots, e_i)) \) is closed in \( \text{Gr}_{e_i}(\tilde{B}_i) \). It follows that the lexicographic partial ordering on dimension vectors given by \( \tilde{f} < \tilde{f}' \) when \( f_i \leq f'_i \) for all \( i \in Q_0 \) gives a closed subset

\[
\bigsqcup_{\tilde{f}' > \tilde{f}} \tilde{G}_{f', \tilde{e} - \tilde{f}} \subset \text{Gr}_{\tilde{e}}(\tilde{B})
\]

for any fixed \( \tilde{f} \). Any refinement of this partial order to a total order \( \preceq \) will give the claim. \( \square \)
sequence between quiver Grassmannians coming out of Lemma 3.35. Here we consider the short exact sequence
\[ 0 \to \tilde{P}_{m,j} \to \tilde{P}_{m+1}^{I} \to \tilde{P}_{m+1}^{J} \to 0 \] (4.2)
which induces a map between quiver Grassmannians as above for any fixed \( \tilde{e} \):
\[ \Psi = \Psi^{\eta_{m}} : \text{Gr}_{\tilde{e}}(\tilde{P}_{m+1}^{I}) \to \bigsqcup_{\tilde{f}+\tilde{g}=\tilde{e}} \text{Gr}_{\tilde{f}}(\tilde{P}_{m,j}) \times \text{Gr}_{\tilde{g}}(\tilde{P}_{m+1}^{J}). \] (4.3)

To understand the fibers of this map for \( m \geq 2 \), we will need to make use of another map between quiver Grassmannians coming out of Lemma 3.35. Here we consider the short exact sequence
\[ 0 \to \tilde{P}_{m}(j,I) \to \tilde{P}_{m,j} \to \tilde{C}_{m} \to 0. \] (4.4)
where \( \tilde{C}_{m} = \tilde{C}_{m}(I) = \tau \tilde{P}_{m+1}^{I} \) for \( m \geq 3 \) and \( \tilde{C}_{2} = \tilde{C}_{2}(j,I) \) is the representation in (3.12) from Remark 3.36. Then, in the same way as above, we obtain the following map for any fixed \( \tilde{f} \):
\[ \Phi = \Phi^{\xi_{m}} : \text{Gr}_{\tilde{f}}(\tilde{P}_{m,j}) \to \bigsqcup_{\tilde{s}+t=\tilde{f}} \text{Gr}_{\tilde{s}}(\tilde{P}_{m}(j,I)) \times \text{Gr}_{\tilde{t}}(\tilde{C}_{m}). \] (4.5)

**Proposition 4.17** For \( m \geq 2 \) and a non-empty subset \( I \subseteq \{1, \ldots, n\} \). The following hold for any \( j \in I \):

1. The fiber \( \Psi^{-1}(\tilde{F}, \tilde{P}_{m+1}^{I}) \) is not empty if and only if \( \text{Ext}(\tilde{P}_{m+1}^{I}, \tilde{P}_{m,j}/\tilde{F}) = 0 \).
2. The fiber \( \Psi^{-1}(\tilde{F}, \tilde{P}_{m+1}^{I}) \) is empty if and only if \( \Phi(\tilde{F}) = (\tilde{F}, 0) \), i.e. \( \tilde{F} \) is already a subrepresentation of \( \tilde{P}_{m}(j,I) \).

**Proof** Any subrepresentation \( \tilde{F} \subset \tilde{P}_{m,j} \) produces an exact sequence
\[ \text{Ext}(\tilde{P}_{m+1}^{I}, \tilde{F}) \to \text{Ext}(\tilde{P}_{m+1}^{I}, \tilde{P}_{m,j}) \to \text{Ext}(\tilde{P}_{m+1}^{I}, \tilde{P}_{m,j}/\tilde{F}) \to 0. \]
But note that the fiber $\psi^{-1}(\tilde{F}, \tilde{P}_{m+1})$ being non-empty gives rise to a pushout diagram

$$
\begin{array}{cccc}
0 & \longrightarrow & \tilde{F} & \longrightarrow & \tilde{F}' & \longrightarrow & \tilde{P}_{m+1} & \longrightarrow & 0 \\
& & \downarrow \cong & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{P}_{m,j} & \longrightarrow & \tilde{P}_{m+1} & \longrightarrow & \tilde{P}_{m+1} & \longrightarrow & 0
\end{array}
$$

in which the bottom row is not split by Lemma 3.29.4. This implies that the map

$$
\text{Ext}(\tilde{P}_{m+1}, \tilde{F}) \to \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m,j}) \cong \mathbb{C}
$$

is surjective and thus $\text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m,j}/\tilde{F}) = 0$. This argument can be reversed and thus (1) holds.

Now consider $\tilde{F} \subset \tilde{P}_{m,j}$ with

$$
\Phi(\tilde{F}) = (\tilde{S}, \tilde{T}) \in \text{Gr}_S(\tilde{P}_m(j, I)) \times \text{Gr}_T(\tilde{C}_m).
$$

This gives rise to the following commutative diagram:

$$
\begin{array}{cccccc}
\text{Ext}(\tilde{P}_{m+1}, \tilde{S}) & \longrightarrow & \text{Ext}(\tilde{P}_{m+1}, \tilde{F}) & \longrightarrow & \text{Ext}(\tilde{P}_{m+1}, \tilde{T}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Ext}(\tilde{P}_{m+1}, \tilde{P}_m(j, I)) & \longrightarrow & \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m,j}) & \longrightarrow & \text{Ext}(\tilde{P}_{m+1}, \tilde{C}_m) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Ext}(\tilde{P}_{m+1}, \tilde{P}_m(j, I)/\tilde{S}) & \longrightarrow & \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m,j}/\tilde{F}) & \longrightarrow & \text{Ext}(\tilde{P}_{m+1}, \tilde{C}_m/\tilde{T}) & \longrightarrow & 0
\end{array}
$$

By Corollary 3.37, we have $\text{Ext}(\tilde{P}_{m+1}, \tilde{P}_m(j, I)) = 0$ and so $\text{Ext}(\tilde{P}_{m+1}, \tilde{P}_m(j, I)/\tilde{S}) = 0$ as well. This yields the isomorphisms

$$
\text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m,j}) \cong \text{Ext}(\tilde{P}_{m+1}, \tilde{C}_m) \quad \text{and} \quad \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m,j}/\tilde{F}) \cong \text{Ext}(\tilde{P}_{m+1}, \tilde{C}_m/\tilde{T}).
$$

If $\tilde{T} = 0$, we get an isomorphism $\text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m,j}/\tilde{F}) \cong \text{Ext}(\tilde{P}_{m+1}, \tilde{P}_{m,j}) \cong \mathbb{C}$ and by part (1) we must have an empty fiber $\psi^{-1}(\tilde{F}, \tilde{P}_{m+1}) = \emptyset$.

If $\tilde{T} \neq 0$, $\tilde{C}_m/\tilde{T}$ is a proper factor of $\tilde{C}_m$ and we must have $\text{Ext}(\tilde{P}_{m+1}, \tilde{C}_m/\tilde{T}) = 0$. Indeed, by Auslander–Reiten theory every non-split morphism $g : \tilde{C}_m \to \tilde{C}_m/\tilde{T}$ factors through the middle term $\tilde{Z}$ of the AR-sequence

$$
0 \longrightarrow \tilde{C}_m \longrightarrow \tilde{Z} \longrightarrow \tilde{P}_{m+1} \longrightarrow 0 \quad (4.6)
$$

This in particular says the first map of the induced sequence

$$
\text{Hom}(\tilde{Z}, \tilde{C}_m/\tilde{T}) \to \text{Hom}(\tilde{C}_m, \tilde{C}_m/\tilde{T}) \to \text{Ext}(\tilde{P}_{m+1}, \tilde{C}_m/\tilde{T}) \to \text{Ext}(\tilde{Z}, \tilde{C}_m/\tilde{T})
$$

is surjective. By Lemma 3.29.4 and the Auslander–Reiten formulas, we have

$$
\text{Ext}(\tilde{C}_m, \tilde{C}_m) = 0 \quad \text{and} \quad \text{Ext}(\tilde{P}_{m+1}, \tilde{C}_m) = \mathbb{C}.
$$
For $m = 2$, the identities above follow immediately from the corresponding statements for $\tau \tilde{P}^I_m$. Thus the injective map $\mathbb{C} = \text{Hom}(\tilde{C}_m, \tilde{C}_m) \rightarrow \text{Ext}(\tilde{P}^I_{m+1}, \tilde{C}_m)$ induced by the Auslander–Reiten sequence (4.6) is actually bijective and so applying $\text{Hom}(-, \tilde{C}_m)$ to this sequence yields $\text{Ext}(\tilde{Z}, \tilde{C}_m) = 0$. But then $\text{Ext}(\tilde{Z}, \tilde{C}_m/\tilde{T}) = 0$ and so $\text{Ext}((\tilde{P}^I_{m+1}, \tilde{P}^I_{m,j}/\tilde{F}) \cong \text{Ext}(\tilde{P}^I_{m+1}, \tilde{C}_m/\tilde{T})$ must be zero as well. From part (1), we see that the fiber $\Psi^{-1}(\tilde{F}, \tilde{P}^I_{m+1})$ is non-empty in this case.

Now we are able to state the following crucial result concerning the fibers of $\Psi$:

**Proposition 4.18** The following hold:

1. For $\tilde{G} \subset \tilde{P}^I_{m+1}$ and $\tilde{F} \subset \tilde{P}^I_{m,j}$, we have $\Psi^{-1}(\tilde{F}, \tilde{G}) \cong \Lambda[\tilde{G}, \tilde{P}^I_{m,j}/\tilde{F}]$.
2. If $\tilde{G} = \tilde{P}^I_{m+1}$, the fiber $\Psi^{-1}(\tilde{F}, \tilde{G})$ is not empty if and only if $\Phi(\tilde{F}) \neq (\tilde{F}, 0)$. In this case, we have $\Psi^{-1}(\tilde{F}, \tilde{G}) \cong \Lambda[\tilde{G}, \tilde{P}^I_{m,j}/\tilde{F}]$.

**Remark 4.19** Part (1) of Proposition 4.18 holds equally well when considering the analogues of the Caldero–Chapoton maps $\Psi$ for $K(n)$. This follows from Corollary 3.16 and Lemma 3.15. However, there does not seem to be a reasonable analogue of part (2) when considering Caldero–Chapoton maps for $K(n)$.

**Proof** By Lemma 3.28, any subrepresentation $\tilde{G} \subset \tilde{P}^I_{m+1}$ is preprojective. But the representation $\tilde{P}^I_{m,j}/\tilde{F}$ is not preprojective as it is a proper quotient of a preprojective representation unless $\tilde{F} = 0$. Thus we have $\text{Ext}(\tilde{G}, \tilde{P}^I_{m,j}/\tilde{F}) = 0$. If $\tilde{F} = 0$, we have $\text{Ext}(\tilde{G}, \tilde{P}^I_{m,j}) = 0$ because for dimension reasons every indecomposable direct summand of $\tilde{G}$ is isomorphic to a lift of some $P_l$ with $l \leq m$ and, moreover, $\text{Ext}(P_l, P_m) = 0$ for $l \leq m + 1$.

The second statement follows directly from Proposition 4.17. \hfill $\square$

**Corollary 4.20** For all $\tilde{F}$ and $\tilde{g}$, the image of $\Psi_{\tilde{F}, \tilde{g}}$ is open in $\text{Gr}_{\tilde{F}}(\tilde{P}^I_{m,j}) \times \text{Gr}_{\tilde{g}}(\tilde{P}^I_{m+1})$.

**Proof** By Proposition 4.18, the map $\Psi_{\tilde{F}, \tilde{g}}$ is surjective for $\tilde{g} \neq \dim \tilde{P}^I_{m+1}$ and there is nothing to show in this case. Assume $\tilde{g} = \dim \tilde{P}^I_{m+1}$. By Proposition 4.17, the image of $\Psi_{\tilde{F}, \tilde{g}}$ consist precisely of those pairs $(\tilde{F}, \tilde{P}^I_{m+1})$ for which $\text{Ext}(\tilde{P}^I_{m+1}, \tilde{P}^I_{m,j}/\tilde{F}) = 0$. But the map $\tilde{F} \mapsto \dim \text{Ext}(\tilde{P}^I_{m+1}, \tilde{P}^I_{m,j})$ is upper semicontinuous so that its minimal value on $\text{Gr}_{\tilde{F}}(\tilde{P}^I_{m,j})$ is its generic value, i.e. the image of $\Psi_{\tilde{F}, \tilde{g}}$ is open. \hfill $\square$

**Theorem 4.21** For each $m \geq 1$ and $J \subset \{1, \ldots, n\}$, every quiver Grassmannian $\text{Gr}_{\tilde{g}}(\tilde{P}^J_{m+1})$ admits a cell decomposition. These cell decompositions can be chosen compatible with the projection maps $\tilde{P}^I_{m+1} \rightarrow \tilde{P}^I_{m+1-k, \tilde{z}}$. From Lemma 3.34 in the following sense:


t for any $\tilde{E} \in \text{Gr}_{\tilde{g}}(\tilde{P}^J_{m+1})$ such that $\tilde{P}^I_{m+1}(\tilde{E}) \not= 0$ for some admissible sequence $i = (i_1, \ldots, i_k)$, $k < m$, with $i_\ell \not\in J$ and subset $I \subset \{1, \ldots, n\}$ with $i_k \in I$, there exists a cell decomposition of $\text{Gr}_{\tilde{g}}(\tilde{P}^I_{m+1})$ so that $\tilde{P}^I_{m+1}(\tilde{E}') \not= 0$ for all $\tilde{E}' \in C_{\tilde{E}}$, where $C_{\tilde{E}}$ is the affine cell which contains $\tilde{E}$.

We record the following immediate consequence before giving the proof.

**Corollary 4.22** Every quiver Grassmannian of any lift of a (truncated) preprojective representation admits a cell decomposition.

**Proof** For the standard lifts $\tilde{P}^I_{m+1}$, this is Theorem 4.21. The claim for any translate $\tilde{P}^I_{m+1,w}$ for $w \in W_n$ is an immediate consequence of this. \hfill $\square$
Proof of Theorem 4.21

We work by induction on $m$. When $m = 1$, the claim is trivial since in this case all quiver Grassmannians are points.

For $m \geq 2$, we work by reverse induction on $|J|$ with the base case $|J| = n - 1$, say $J^c = \{ j \}$. This gives $\tilde{P}_{m+1}^J \cong \tilde{P}_{m,j}^J$ by Lemma 3.34.1 and thus the projection maps $\tilde{\pi}_{m+1}^{i,I,j}$ and $\tilde{\pi}_{m,j}^{i,J,j}$, where $i_j' = (i_2, \ldots, i_k)$, coincide for any admissible sequence $i_j \in A_1^{(k)}$, $k < m$, with $i_1 = j$ and any subset $I \subseteq \{ 1, \ldots, n \}$ with $i_k \in I$. By the induction on $m$, the quiver Grassmannian $\text{Gr}_\tilde{g}(\tilde{P}_{m+1}^J)$ admits a cell decomposition so that the compatibility condition $(†)$ holds for the map $\tilde{\pi}_{m+1}^{i,I,j} : \tilde{P}_{m+1}^J \to \tilde{P}_{m+1-k,i_j'}. But \text{Gr}_\tilde{g}(\tilde{P}_{m,j}^J)$ is naturally isomorphic to $\text{Gr}_\tilde{g}(\tilde{P}_{m}^J)$ using the action of $W_n$, under this correspondence the cell decomposition becomes compatible with the projection map $\tilde{\pi}_{m,j}^{i,I,j} : \tilde{P}_{m,j}^J \to \tilde{P}_{m+1-k,i_j'}$. Then using the equivalence $\text{Gr}_\tilde{g}(\tilde{P}_{m+1}^J) \cong \text{Gr}_\tilde{g}(\tilde{P}_{m,j}^J)$ coming from Lemma 3.34.1, we obtain the desired cell decomposition of $\text{Gr}_\tilde{g}(\tilde{P}_{m+1}^J)$ which is compatible with the projection map $\tilde{\pi}_{m+1}^{i,I,j}$.

Now suppose $|J| \leq n - 2$ and consider an admissible sequence $i_j = (i_1, \ldots, i_k)$, $k < m$, with $i_1 \not\in J$ and a subset $I' \subseteq \{ 1, \ldots, n \}$ with $i_k \in I'$. Taking $j \not\in J$ with $j \neq i_1$, we write $I = J \cup \{ j \}$. This gives the short exact sequence (4.2) inducing the maps $\tilde{\Psi}$ and $\Phi$ between quiver Grassmannians from (4.3) and (4.5). By way of induction, we assume all quiver Grassmannians $\text{Gr}_\tilde{g}(\tilde{P}_{m,j}^J)$ and $\text{Gr}_\tilde{g}(\tilde{P}_{m+1}^J)$ admit cell decompositions satisfying $(†)$ for the maps $\tilde{\pi}_{m,j}^{j,I',\varnothing}$ and $\tilde{\pi}_{m+1}^{j,I',\varnothing}$, respectively, say

$$\text{Gr}_\tilde{g}(\tilde{P}_{m,j}^J) = \prod_{k=1}^r B_{\tilde{f},k}$$

and

$$\text{Gr}_\tilde{g}(\tilde{P}_{m+1}^J) = \prod_{\ell=1}^s C_{\tilde{g},\ell}. \tag{4.7}$$

Proposition 4.18 shows that the fiber of $\Psi_{\tilde{g},\tilde{f}}$ over $(\tilde{F}, \tilde{G}) \in B_{\tilde{f},k} \times C_{\tilde{g},\ell}$ is empty exactly when $\tilde{g} = \text{dim} \tilde{P}_{m+1}^J$ and one of the following conditions is satisfied:

- $m = 2$ with $\tilde{f}_{(2,\alpha_j^{-1})} \neq 0$ or $\tilde{f}_{(1,e)} \neq 0$;
- $m \geq 3$ with $\tilde{\pi}_{m,j}^{j,I',\varnothing}(\tilde{F}) = 0$.

By the compatibility with $\tilde{\pi}_{m,j}^{j,I',\varnothing}$ for the cell decomposition (4.7), we see that either all or none of the fibers over $B_{\tilde{f},k} \times C_{\tilde{g},\ell}$ are empty. It follows that there is an induced cell decomposition on the image of $\Psi_{\tilde{g},\tilde{f}}$ given by those products of cells $B_{\tilde{f},k} \times C_{\tilde{g},\ell}$ over which the fiber is not empty. Theorem 4.15 then provides a cell decomposition for $\mathcal{G}_{\tilde{f},\tilde{g}}$, where cells take the form $A_{\tilde{f},d} \times B_{\tilde{f},k} \times C_{\tilde{g},\ell}$ with $d = (\tilde{g}, \text{dim} \tilde{P}_{m+1}^J - \tilde{f})$. By Lemma 4.14, we may order the affine cells in the $\mathcal{G}_{\tilde{f},\tilde{g}}$ appropriately to obtain a cell decomposition of $\text{Gr}_\tilde{g}(\tilde{P}_{m+1}^J)$ (c.f. Remark 4.16).

Now consider $\tilde{E} \in \text{Gr}_\tilde{g}(\tilde{P}_{m+1}^J)$ which is contained in such a cell, say $\tilde{E} \in A_{\tilde{f},d} \times B_{\tilde{f},k} \times C_{\tilde{g},\ell}$, and assume $\tilde{\pi}_{m+1}^{i,I,j}(\tilde{E}) \neq 0$. Since $i_1 \neq j$, the map $\tilde{\pi}_{m+1}^{i,I,j}$ factors as $\tilde{\pi}_{m+1}^{i,I,j} = \tilde{\pi}_{m+1}^{i,I,j} \circ \tilde{\pi}_{m+1}^{i,\varnothing}B_{\tilde{f},k} \times C_{\tilde{g},\ell}$. By construction, we have $\tilde{\pi}_{m+1}^{i,\varnothing}(\tilde{E}) \in C_{\tilde{g},\ell}$ for all $\tilde{E}' \in A_{\tilde{f},d} \times B_{\tilde{f},k} \times C_{\tilde{g},\ell}$.
But \( \tilde{\pi}^{l', J}_{m+1} (\tilde{E}) = \tilde{\pi}^{l', J}_{m+1} (\tilde{E}) \neq 0 \) implies \( \tilde{\pi}^{l', J}_{m+1} (\tilde{G}) \neq 0 \) for all \( \tilde{G} \in C_{g, \ell} \), in particular \( \tilde{\pi}^{l', J}_{m+1} (\tilde{E}' \tilde{E}) \neq 0 \) for all \( \tilde{E}' \in \mathbb{A}^d \times B_{f,k} \times C_{g, \ell} \). Thus the compatibility condition (\( \dagger \)) with \( \tilde{\pi}^{l', J}_{m+1} \) holds for this cell decomposition of \( \text{Gr}_e (P^J_{m+1}) \). □

Now we come to the main result of this work.

**Theorem 4.23** The following hold:

1. Every quiver Grassmannian of an indecomposable preprojective or preinjective representation of \( K(n) \) has a cell decomposition.
2. Let \( X \in \text{rep} \ K(n) \) be an indecomposable representation with dimension vector \((d, e)\) or \((e, d)\), where \( d = \dim P_m + 1 - r \dim P_m \) for \( m \geq 1 \) and \( 1 \leq r \leq n - 1 \). Then every quiver Grassmannian \( \text{Gr}_e (X) \) has a cell decomposition.

**Proof** We consider a representation \( X \) of \( K(n) \) which is either an indecomposable preprojective or truncated preprojective (c.f. Proposition 3.11.2), the results for preinjectives or the duals of truncated preprojectives follow via the isomorphisms

\[
\text{Gr}_e (P_m) \cong \text{Gr}_\dim P_m - e (Q_m) \quad \text{and} \quad \text{Gr}_e (P^V_{m+1}) \cong \text{Gr}_\dim P^V_{m+1} - e ((P^V_{m+1})^*) ,
\]

where \((-)^* = \text{Hom}_C (-, C)\) is the \( C \)-linear duality.

By Theorem 4.1, each quiver Grassmannian \( \text{Gr}_e (P_m) \) is smooth. Theorem 4.13 shows each \( \text{Gr}_e (P^V_{m+1}) \) is smooth and that, after an isomorphism, we may assume \( V \) is a coordinate subspace under the decomposition in Lemma 3.13. In particular, we can assume \( P^V_{m+1} \) admits a lift to the universal covering quiver \( \tilde{K} (n) \). This is true in the preprojective case by Lemma 2.2.

It follows that \( X \) lifts to each iterated abelian cover \( \tilde{K} (n) \). As in Corollary 3.32, we see that each such lift is preprojective and thus the lifted representations all have smooth quiver Grassmannians.

Using Corollary 4.7 and the associated iteration of Theorem 4.2, we reduce the problem of constructing cell decompositions for the quiver Grassmannians of \( X \) to constructing cells decompositions for the quiver Grassmannians of the lift to \( \tilde{K} (n) \). The claim then follows from Theorem 4.21. □

**Corollary 4.24** Let \( X \in \text{rep} \ K(n) \) be a direct sum of exceptional representations. Then every quiver Grassmannian \( \text{Gr}_e (X) \) has a cell decomposition. In particular, this is true for all rigid representations of \( K(n) \).

**Proof** As every exceptional representation of \( K(n) \) is either preprojective or preinjective, we have

\[
X = \bigoplus_{i=1}^{r} P_{j_i} \bigoplus_{i=1}^{s} Q_{k_i} ,
\]

where we assume that \( j_i \leq j_{i+1} \) and write

\[
P (r') := \bigoplus_{i=1}^{r'} P_{j_i} , \quad I (s') := \bigoplus_{i=1}^{s'} Q_{k_i}
\]

for \( r' \leq r \) and \( s' \leq s \).

By Theorem 4.23, the claim is true for all quiver Grassmannians attached to \( P_{j_i} \) or \( Q_{k_i} \).

Consider the short exact sequence

\[
0 \rightarrow P_{j_i+1} \rightarrow P (r' + 1) \rightarrow P (r') \rightarrow 0 .
\]
By induction, we can assume that all quiver Grassmannians attached to the two outer terms have a cell decomposition. Consider the Caldero–Chapoton map
\[ \Psi_e : \text{Gr}_e(P(r' + 1)) \to \bigsqcup_{f+g=e} \text{Gr}_f(P_{j_r+1}) \times \text{Gr}_g(P(r')). \]

The results of [7, Section 3] show that \( \Psi_e^{-1}(F, G) \cong \mathbb{A}^{\dim \text{Hom}(G, P_{j_r+1}/F)} \) for all \( (F, G) \in \text{Gr}_f(P_{j_r+1}) \times \text{Gr}_g(P(r')) \), in particular the fiber is never empty. Now every subrepresentation \( G \) of \( P(r') \) is isomorphic to a direct sum of preprojective representations such that for each direct summand \( P_l \) we have \( l \leq j_r \). Moreover, the quotient \( P_{j_r+1}/F \) is not projective if \( U \neq 0 \) and equal to \( P_{j_r+1} \) otherwise. Together these yield \( \text{Ext}(G, P_{j_r+1}/F) = 0 \) and thus
\[ \dim \text{Hom}(G, P_{j_r+1}/F) = \langle G, P_{j_r+1}/F \rangle \]
for all \( (F, G) \in \text{Gr}_f(P_{j_r+1}) \times \text{Gr}_g(P(r')) \). Following Theorem 4.15 (see Remark 4.16), this already shows that \( \text{Gr}_e(P(r')) \) has a cell decomposition for every \( 1 \leq r' \leq r \). By duality, the same is true for \( \text{Gr}_e(I(s)) \).

Finally consider the short exact sequence
\[ 0 \to I(s) \to X \to P(r) \to 0. \]

As every quotient of \( I(s) \) is preinjective and as every subrepresentation of \( P(r') \) is preprojective, the same argument as above shows that every quiver Grassmannian attached to \( X \) has a cell decomposition.

\[ \square \]

**Remark 4.25** The interested reader may utilize the iterative construction of the cell decomposition above to describe the cells using strong successor closed subsets of an appropriate two-quiver as we do for the truncated preprojectives in the next section.

As the \( F \)-polynomials of truncated preprojective representations only depend on the dimension vector, we may denote them by \( F_{d(m,r)} \). The description of the non-empty fibers in Proposition 4.18 together with Corollary 4.8 and Theorem 4.23 yield the following:

**Corollary 4.26** For \( m \geq 1 \) and \( 0 \leq r \leq n - 2 \), we have
\[ F_{d(m,r)} = F_{d(m,r+1)} F_{d(m,r)} - x^{d(m,r)} F_{d(m-2,r)}. \]

**5 Combinatorial descriptions of non-empty cells**

In this section, we provide two combinatorial descriptions of the non-empty cells in the quiver Grassmannians of (truncated) preprojective representations of \( K(n) \). The first is quiver theoretic and follows directly from the recursive construction of the cell decomposition from Sect. 4.4. The second is the notion of compatible pairs in a maximal Dyck path arising in the computation of rank 2 cluster variables [15]. We give a bijection between these which provides a partial geometric explanation for the combinatorial construction of counting polynomials for rank two quiver Grassmannians given in [18].

**5.1 2-Quivers**

The key concept for describing the cell decompositions is the following notion of 2-quiver which is closely related to certain coefficient quivers of the corresponding representations.
This construction makes use of the support quivers from Examples 3.25 and 3.30. It will turn out that a feature of this construction is that it is blind to the coloring of the different arrows of $\hat{K}(n)$ covering the arrows of $K(n)$.

**Definition 5.1** Let $Q = (Q_0, Q_1)$ be a quiver. A subset $\beta \subset Q_0$ is *successor closed in $Q$* if for each $p \in \beta$, the existence of an arrow $a : p \to q$ in $Q_1$ implies $q \in \beta$.

A 2-arrow of the quiver $Q$ is an ordered pair $V = (\Gamma(1), \Gamma(2))$ of full connected subquivers of $Q$, these will be denoted $V : \Gamma(1) \to \Gamma(2)$. A 2-quiver is a pair $Q = (Q, Q_2)$ consisting of a quiver $Q$ and a collection $Q_2$ of 2-arrows of $Q$. Given a 2-quiver $Q$, we call a subset $\beta \subset Q_0$ *strong successor closed in $Q$* if it is successor closed in $Q$ and for each 2-arrow $V : \Gamma(1) \to \Gamma(2)$ in $Q_2$ with $\Gamma(1)_0 \subset \beta$ we have $\Gamma(2)_0 \cap \beta \neq \emptyset$.

The following notion of equivalence for 2-quivers will be useful in the construction of 2-quivers whose strong successor closed subsets label cells in quiver Grassmannians. Observe that any quiver can be considered as a 2-quiver with no 2-arrows.

**Definition 5.2** Let $Q = (Q, Q_2)$ be a 2-quiver with a 2-arrow $V : \Gamma(1) \to \Gamma(2)$ in $Q_2$ such that one of the following conditions is satisfied:

1. $\Gamma(1)$ has precisely one source $p$;
2. $\Gamma(2)$ has precisely one sink $q$;
3. $\Gamma(1) = \{p\}$ and $\Gamma(2) = \{q\}$.

Depending on the condition which is satisfied, we define

1. $Q_p$ as the 2-quiver obtained from $Q$ when replacing the 2-arrow $V$ by a 2-arrow $V_p : \{p\} \to \Gamma(2)$;
2. $Q_q$ as the 2-quiver obtained from $Q$ when replacing the 2-arrow $V$ by a 2-arrow $V_q : \Gamma(1) \to \{q\}$;
3. $Q_V$ as the 2-quiver obtained from $Q$ when replacing the 2-arrow $V$ by a usual arrow $\alpha : p \to q$.

This defines a relation on the set of 2-quivers denoted by $Q \to Q_p, Q \to Q_q$ and $Q \to Q_V$, respectively. Moreover, it induces an equivalence relation $\sim$ on the set of 2-quivers when taking the symmetric and transitive closure of this relation.

An important consequence of this definition is that the vertex sets of equivalent 2-quivers coincide, in particular we can formulate the following result.

**Lemma 5.3** Let $Q = (Q, Q_2)$ and $Q' = (Q', Q'_2)$ be equivalent 2-quivers. A subset $\beta \subset Q_0$ is strong successor closed in $Q$ if and only if it is strong successor closed in $Q'$.

For the proof of this Lemma the following straightforward observation is essential: in a finite connected quiver which has precisely one source $p$, there exists a path from $p$ to every other vertex of the quiver. An analogous statement holds if a quiver has precisely one sink.

**Proof** By induction, we only need to consider the cases $Q' \in \{Q_p, Q_q, Q_V\}$ where one of the conditions of Definition 5.2 is satisfied.

Assume first that $Q' \in \{Q_p, Q_q\}$. Then we have $Q = Q'$ from which we immediately see that $\beta \subset Q_0$ is successor closed in $Q$ if and only if $\beta$ is successor closed in $Q'$. We only consider the case $Q' = Q_p$ below, the argument for $Q' = Q_q$ is dual.

Let $\beta \subset Q_0$ be strong successor closed in $Q$. To see that $\beta$ is strong successor closed in $Q_p$ it suffices to consider the 2-arrow $V_p : \{p\} \to \Gamma(2)$. Suppose $\{p\} \subset \beta$. As $\beta$ is successor closed in $Q$ and $p$ is a source in the connected quiver $\Gamma(1)$, we have $\Gamma(1)_0 \subset \beta$. 

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and thus $\Gamma(2)_0 \cap \beta \neq \emptyset$, i.e., $\beta$ is strong successor closed in $Q_p$. The reverse implication is immediate since $\{p\} \subset \Gamma(1)_0$.

Now assume $Q' = Q_V = (Q_V, (Q_V)_2)$. Let $\beta \subset Q_0$ be strong successor closed in $Q$. Since $(Q_V)_2 \subset Q_2$, to see that $\beta$ is strong successor closed in $Q_V$ we only need to show that $\beta$ is successor closed in $Q_V$. For this it suffices to consider the arrow $\alpha_V : p \to q$ for which that claim is obvious since $p \in \beta$ is equivalent to $\{p\} \subset \beta$ and similarly for $q$.

Finally, let $\beta \subset (Q_V)_0$ be strong successor closed in $Q_V$. Since $Q_1 \subset (Q_V)_1$, we immediately see that $\beta$ is successor closed in $Q$. To see that $\beta$ is strong successor closed in $Q$, it suffices to consider the 2-arrow $V : \{p\} \implies \{q\}$ for which the claim is obvious as above. 

\[ \square \]

Remark 5.4 Below we will usually apply Lemma 5.3 after performing each of the equivalences from Definition 5.2. That is, given a 2-arrow $\alpha$ that claim is obvious since $p \in \beta$ is equivalent to $\{p\} \subset \beta$ and similarly for $q$.

In the following, we freely use the notation and conventions of Sect. 3. For $m \geq 1$, Theorem 4.13 shows that up to isomorphism the quiver Grassmannians $\text{Gr}(P_{m+1}^V)$ of arbitrary truncated preprojective representations $P_{m+1}^V$ for $V \in \text{Gr}(\mathcal{H}_m)$ only depend on $e$ and $\dim P_{m+1}^V$. In particular, fixing $\dim V = r$, we construct a 2-quiver $Q_{m+1}^{[r]}$ whose strong successor closed subsets are in one-to-one correspondence with the cells of quiver Grassmannians of $P_{m+1}^V$.

By Theorem 4.23, the cells of the quiver Grassmannians of $P_{m+1}^V$ are in one-to-one correspondence with those attached to any lift $\tilde{P}_{m+1}^{[r]}$ to $\tilde{K}(n)$. Since the choice of $V \in \text{Gr}(\mathcal{H}_m)$ with $\dim V = r$ is immaterial for understanding the geometry of $\text{Gr}_{e}(P_{m+1}^V)$, we may fix a particular choice of $V$ and a particular lift to the universal cover. Indeed, set

$$\tilde{P}_{m+1}^{[r]} := \begin{cases} \tilde{P}_{m+1}^{[1,2,...,r]} & \text{if } m \text{ is odd;} \\ \tilde{P}_{m+1}^{[n,n-1,...,n-r+1]} & \text{if } m \text{ is even;} \end{cases}$$

and write $P_{m+1}^{[r]} = G(\tilde{P}_{m+1}^{[r]})$. Note that we may allow $r = 0$ above and take $\tilde{P}_{m+1}^{[0]} = \tilde{P}_{m+1}$, then we write $Q_{m+1}^{[0]}$ in place of $Q_{m+1}^{[0]}$. Fixing a choice of lift will allow us to give a concrete description of the 2-quiver $Q_{m+1}^{[r]}$, it will be clear from the construction that making another choice of lift and following an analogous procedure will give a construction of an isomorphic 2-quiver. In this way, the 2-quiver $Q_{m+1}^{[r]}$ should be viewed as a combinatorial shadow of the sequences (3.9) defining the truncated preprojective representations of $\tilde{K}(n)$. In fact, the related sequences (3.10) will be used together with Lemma 3.34 to recursively construct the 2-quivers $Q_{m+1}^{[r]}$.

Each 2-quiver $Q_{m+1}^{[r]}$ should be thought of as a combinatorially enhanced version of the coefficient quiver of $\tilde{P}_{m+1}^{[r]}$ in which certain arrows are upgraded to 2-arrows. In particular, the vertices and arrows of the quiver $Q_{m+1}^{[r]}$ underlying the 2-quiver $Q_{m+1}^{[r]}$ can naturally be associated with vertices and arrows of $\tilde{K}(n)$.

To begin, we take the 2-quiver $Q_1 = Q_1^{[0]}$ associated to $\tilde{P}_1$ to be the quiver $Q_1$ consisting of a single vertex which we associate to the vertex $(1,e)$ of $\tilde{K}(n)$. By analogy with the notation of Sect. 3.2, we define a 2-quiver $Q_{1,i}$ for $1 \leq i \leq n$ whose underlying quiver $Q_{1,i}$ has a single vertex which is associated to the vertex $(1, \alpha_i)$ of $\tilde{K}(n)$.
The 2-quiver $Q_2^{[r]}$ associated to $\tilde{P}_2^{[r]}$ has underlying quiver $Q_2^{[r]} := Q_1^{r} \sqcup \bigsqcup_{i=1}^{n-r} Q_{1,i}$, where the single vertex of the quiver $Q_1^{r}$ is associated to the vertex $(2,e)$ of $K(n)$, and has 2-arrows as in the figure below:

![Diagram of the 2-arrows in $Q_2^{[r]}$]

The source and target quivers for each 2-arrow above have been drawn inside a box. Note that the vertices $(1, \alpha_i)$ are just the 2-quivers $Q_{1,i}$ corresponding to $\tilde{P}_{1,i}$ and that $Q_2^{[t]}$ is a sub-2-quiver of $Q_2^{[r]}$ for $t \geq r$.

**Remark 5.5** The 2-arrows of $Q_2$ should be viewed as a reflection of the isomorphism

$$\text{Ext}(P_2, P_1) \cong \bigoplus_{i=1}^{n} \text{Ext}(\tilde{P}_2, \tilde{P}_{1,i}) \cong \{(2,e) \xrightarrow{\alpha_i} (1, \alpha_i) \mid i = 1, \ldots, n\} \quad (5.1)$$

and the inclusions of $Q_2^{[t]}$ in $Q_2^{[r]}$ for $t \geq r$ as a reflection of the surjections $\text{Ext}(P_2^{[t]}, P_1) \rightarrow \rightarrow \text{Ext}(P_2^{[r]}, P_1)$. In particular, the isomorphism (5.1) can be used with these surjections to obtain compatible bases for each $\text{Ext}(P_2^{[r]}, P_1)$.

The 2-quiver $Q_2^{[r]}$ given above is clearly equivalent to the support quiver (5.2) thought of as a 2-quiver with no 2-arrows:

$$
\begin{array}{ccc}
(2,e) & \xrightarrow{\alpha_1} & (1, \alpha_1) \\
\downarrow & & \downarrow \\
(1, \alpha_1) & \cdots & (1, \alpha_{n-r}).
\end{array}
$$

(5.2)

Thus we may think of $Q_2^{[r]}$ as a coefficient quiver of $\tilde{P}_2^{[r]}$ or of $P_2^V$ for $V \in \text{Gr}(\mathcal{H}_1)$ with $\dim V = r$. In order to keep the illustrations and combinatorics simple, we will abuse notation and denote the support quiver (5.2) by $Q_2^{[r]}$, working instead with this 2-quiver. In this way, we may define the translated 2-quivers $Q_2,i$ (resp. $Q_2^{[r]}$) as those obtained from $Q_2$ (resp. $Q_2^{[r]}$) by translating all vertices and (2-)arrows by $\alpha_i^{-1}$.

The 2-quiver $Q_2^{[r]}$ associated to $\tilde{P}_3^{[r]}$ has underlying quiver $Q_3^{[r]} := Q_{2,n}^{[1]} \sqcup \bigsqcup_{i=r+1}^{n-1} Q_{2,i}$. Note that we are not taking this union as subquivers of $\overline{K(n)}$, in particular each quiver $Q_2,i$ has a vertex which can be associated to $(1,e)$ in $\overline{K(n)}$ but these are not identified in the quiver $Q_3^{[r]}$. For $r < s < n$, there is a 2-arrow $V_s : \Gamma_s(1) \rightarrow \Gamma_s(2)$ of $Q_3^{[r]}$ given by $\Gamma_s(1) = Q_{2,n}^{[1]} \sqcup \bigsqcup_{i=s+1}^{n-1} Q_{2,i}$ with $\Gamma_s(2) \subset Q_{2,s}$ the subquiver $(2, \alpha_s^{-1}) \xrightarrow{\alpha_s} (1,e)$.

By Lemma 5.3, we obtain an equivalent 2-quiver by replacing each $\Gamma_s(2)$ above with the corresponding sink $(1,e)$ taken as a vertex of $Q_{2,s}$. By a slight abuse of notation, below we will let $Q_3^{[r]}$ denote this equivalent 2-quiver. Then $Q_3^{[r]}$ can be found as a sub-2-quiver of $Q_3$ which is constructed recursively by connecting $Q_3^{[i]}$ to $Q_{2,i}$ for $i = n-1, \ldots, 1$ in the following way:
To avoid cluttering the diagram, we did not label the vertices in the illustration.

**Remark 5.6** Here we justify the definition of the 2-arrows in $Q_m^{[r]}$, this discussion will also serve to motivate the choice of 2-arrows for general $Q_m^{[r]}$ and thus we work in that generality.

For $m \geq 2$, we may apply Theorem 2.3 together with Lemma 3.29 and the Auslander–Reiten formula to get an isomorphism

$$\Ext (P^{[r]}_{m+1}, P_m) \cong \bigoplus_{i=r+1}^{n} \Ext (\tilde{P}^{[r]}_{m+1}, \tilde{P}_{m,i}) \cong \bigoplus_{i=r+1}^{n} \Hom (\tilde{P}_{m,i}, \tau \tilde{P}^{[r]}_{m+1}).$$

The image of a nonzero map $\tilde{P}_{m,i} \rightarrow \tau \tilde{P}^{[r]}_{m+1}$ is the representation $\tilde{C}_m$ from the appropriate sequence (4.4). Such a map is surjective if $m \geq 3$ and for $m = 2$ has image with support quiver $(2, \alpha_i^{-1}) \xrightarrow{\alpha_i} (1, e)$. In view of Corollary 3.37, the sequence (4.4) gives rise to an isomorphism

$$\Ext (\tilde{P}^{[r]}_{m+1}, \tilde{P}_{m,i}) \cong \Ext (\tilde{P}^{[r]}_{m+1}, \tilde{C}_m).$$

Finally note for $0 \leq r \leq n-2$ that there exists a short exact sequence

$$0 \longrightarrow \Hom (P_m, P_m) \longrightarrow \Ext (P^{[r]}_{m+1}, P_m) \longrightarrow \Ext (P^{[r+1]}_{m+1}, P_m) \longrightarrow 0.$$  

Thus we obtain a basis of $\Ext (P^{[r]}_{m+1}, P_m)$ by taking the last $r$ elements of a basis for $\Ext (P^{[n-1]}_{m+1}, P_m)$. The choice of 2-arrows in $Q_m^{[r]}$ should be viewed as a combinatorial shadow of the isomorphisms above.

We are now ready to define the 2-quivers $Q_m^{[r]}$ for $m \geq 3$. This will be by induction, so assume we have already constructed the 2-quivers $Q_m^{[s]}$ for $0 \leq s \leq n-1$ and define the 2-quivers $Q_m^{[s]} := \alpha_i^{-(-1)^{m+1}} P_m^{[s]}$ for $1 \leq i \leq n$. Then we may take the underlying quiver of $Q_m^{[r]}$ to be

$$Q_m^{[r]} := \begin{cases} Q_m^{[1]} \sqcup \bigoplus_{i=r+1}^{n-1} Q_m, & \text{if } m \text{ is even;} \\ Q_m^{[1]} \sqcup \bigoplus_{i=2}^{n-r} Q_m, & \text{if } m \text{ is odd.} \end{cases}$$

For $r < s < n$, there is a 2-arrow $V_s : \Gamma_s (1) \longrightarrow \Gamma_s (2)$ of $Q_m^{[r]}$ given by $\Gamma_s (1) = Q_m^{[s]} \subset Q_m^{[r]}$ with $\Gamma_s (2) \subset Q_{m,s}$ the subquiver $\Gamma_m^{[s]} \subset \Gamma_m^{[s-1]}$ of $Q_{m-1} = \Gamma_m^{[s-1]}(s,s) \subset Q_{m,s}$.

**Remark 5.7** For $m \geq 3$, the truncated preprojective $\tau \tilde{P}^{[s]}_{m+1} \cong \tilde{P}^{[s]}_{m-1}$ can uniquely be found as a quotient of $\tilde{P}_{m,s}$. This is reflected in the structure of the 2-quivers as we can find $Q_m^{[s]}$.
as a subquiver of $Q_{m,s}$. In the diagrams for 2-quivers given here, this sub-2-quiver can be found at the very right of the 2-quiver $Q_{m,s}$.

As already mentioned two different vertices of $Q_{m+1}^{[r]}$ can correspond to the same vertex of $\widetilde{K}(n)$. Writing dimension vectors $\tilde{e} = \sum_{q \in \widetilde{K}(n)_{0}} \tilde{e}_q \cdot q$, the dimension types $\tilde{e}(\beta)$ of a subset $\beta \subset (Q_{m+1}^{[r]})_0$ are defined by

$$\tilde{e}(\beta) = \sum_{q \in \beta} \tilde{q} \in \mathbb{N}^{\widetilde{K}(n)_{0}}$$

and

$$e(\beta) = G(\tilde{e}(\beta)) \in \mathbb{N}^{K(n)_{0}},$$

where $\tilde{q} \in \widetilde{K}(n)_{0}$ is the vertex which corresponds to $q \in \beta \subset (Q_{m+1}^{[r]})_0$.

**Theorem 5.8** 1. The affine cells of the cell decomposition of $\text{Gr}_{\tilde{e}(\tilde{P}_{m+1}^{[r]})}$ (resp. $\text{Gr}_{e(\tilde{P}_{m+1}^{[r]})}$) induced by Theorem 4.21 can be labeled by strong successor closed subsets $\beta \subset Q_{m+1}^{[r]}$ of dimension type $\tilde{e} \in \mathbb{N}^{\widetilde{K}(n)_{0}}$ (resp. $e \in \mathbb{N}^{K(n)_{0}}$) yielding a one-to-one correspondence between cells and strong successor closed subsets.

2. For $\tilde{e} \in \mathbb{N}^{\widetilde{K}(n)_{0}}$ (resp. $e \in \mathbb{N}^{K(n)_{0}}$), the Euler characteristic $\chi(\text{Gr}_{\tilde{e}(\tilde{P}_{m+1}^{[r]})})$ (resp. $\chi(\text{Gr}_{e(\tilde{P}_{m+1}^{[r]})})$) is given by the number of strong successor closed subsets of dimension type $\tilde{e}$ (resp. $e$) of the 2-quiver of $Q_{m+1}^{[r]}$.

**Proof** The results of Sects. 4.1 and 4.2 imply that the statements in parentheses follow from the respective results for the lifted representations. Moreover, the second result follows from the first one.

We proceed by induction on $m$ and $r$. The case of the representation $\tilde{P}_1$ is trivial. Now we have $(\dim \tilde{P}_2^{[r]})_q \in \{0, 1\}$ for all $q \in \widetilde{K}(n)$, whence the subrepresentations are in one-to-one correspondence with the successor closed subsets of the quiver (5.2) which is equivalent to $Q_2^{[r]}$. Equivalently, we have $\text{Gr}_{\tilde{e}}(\tilde{P}_2^{[r]}) \subset \{\emptyset, \{\text{pt}\}\}$ so that $\text{Gr}_{\tilde{e}}(\tilde{P}_2^{[r]}) = \{\text{pt}\}$ if and only if $\tilde{e} \subset Q_2^{[r]}$ is strong successor closed.

Thus assume that the claim is true for $\tilde{P}_m$ and $\tilde{P}_{m+1}^{[r]}$. Consider the short exact sequence

$$0 \longrightarrow \tilde{P}_m \longrightarrow \tilde{P}_{m+1}^{[r-1]} \longrightarrow \tilde{P}_{m+1}^{[r]} \longrightarrow 0.$$

We have $(Q_{m+1}^{[r-1]})_0 = (Q_{m+1}^{[r]})_1 \sqcup (Q_{m+1}^{[r]})_0$. Let $\beta_1 \sqcup \beta_2 \subset (Q_{m+1}^{[r]})_0 \sqcup (Q_{m+1}^{[r]})_0$ be a pair of strong successor closed subsets which gives rise to a pair of non-empty cells by the induction

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hypothesis. By Proposition 4.17, the fiber over the pair of cells corresponding to $\beta_1 \sqcup \beta_2$ is empty if and only if $\beta_2 = (Q_{m+1}^r)_0$ and $\beta_1 \cap (Q_{m-1}^r)_0 = \emptyset$, where $Q_{m-1}^r$ is considered as a sub-2-quiver of $Q_{m,r}$. But this is precisely the condition on $\beta_1 \sqcup \beta_2$ to be strong successor closed as $Q_{m+1}^r$ is connected to $Q_{m-1}^r$ by a 2-arrow. This shows the first claim. \hfill \Box

As an example consider the case $n = 3$ and $m = 3$. The 2-quiver of $\tilde{P}_3$ is given by:

![Diagram](image1)

The one of $\tilde{P}_4$ is given by:

![Diagram](image2)

5.2 Compatible pairs

For $m \geq 1$, let $D_m$ denote the maximal Dyck path in the lattice rectangle with corner vertices $(0, 0)$ and $(u_m, u_{m-1})$. More precisely, $D_m$ is the lattice path which begins at $(0, 0)$, takes East and North steps to end at $(u_m, u_{m-1})$, and never passes above the main diagonal joining $(0, 0)$ and $(u_m, u_{m-1})$. It is maximal in the sense that any lattice point lying strictly above $D_m$ also lies above the main diagonal. The maximal Dyck paths $D_m$, $m \geq 1$, exhibit the following recursive structure. In what follows we assume $n \geq 2$.

**Theorem 5.9** [18, Corollary 2.4] For $n \geq 2$, the maximal Dyck path $D_m$, $m \geq 1$, can be constructed recursively as follows:

1. $D_1$ consists of a single horizontal edge;
2. $D_2$ consists of $n$ consecutive horizontal edges followed by a vertical edge;
3. $D_m$, $m \geq 3$, consists of $n - 1$ copies of $D_{m-1}$ followed by a copy of $D_{m-1}$ with its first $D_{m-2}$ removed.
We obtain the following as an immediate consequence.

**Corollary 5.10** Inside $D_m$, $m \geq 2$, there are precisely $u_{m-2}$ vertical edges which are immediately preceded by exactly $n - 1$ horizontal edges, all other vertical edges are immediately preceded by exactly $n - 1$ horizontal edges.

**Proof** We work by induction on $m \geq 2$. The cases $m = 2, 3$ are immediate from Theorem 5.9 parts (2) and (3). For $m \geq 4$, part (3) of Theorem 5.9 shows by induction that there are $(n - 1)u_{m-3} + (u_{m-3} - u_{m-4}) = u_{m-2}$ vertical edges which are immediately preceded by exactly $n - 1$ horizontal edges.

For $m \geq 1$ and $1 \leq r \leq n - 1$, write $D_{m+1}^{[r]}$ for the maximal Dyck path obtained from $D_{m+1}$ by removing the first $r$ copies of $D_m$. Extending this notation we also set $D_{m+1}^{[0]} := D_{m+1}$.

For $m \geq 1$ and $1 \leq i \leq n - 1$, we write $D_{m,i}$ for the $i$-th copy of $D_m$ inside $D_{m+1}$. Note that for $1 \leq r \leq n - 1$, the maximal Dyck paths $D_{m,i}, r + 1 \leq i \leq n$, naturally identify with subpaths of $D_{m+1}^{[r]}$. Extending the notation above, for $m \geq 2$ and $1 \leq r \leq n - 1$, we write $D_{m,i}^{[r]}$ for the Dyck path obtained by removing the first $r$ copies of $D_{m-1}$ from $D_{m,i}$.

**Remark 5.11** For notational convenience, we also set $D_{m,n}^{[1]} := D_{m+1}^{[n-1]}$ even though there is no maximal Dyck path $D_{m,n}$ identifying with a copy of $D_m$ inside $D_{m+1}$, such notation is justified by Theorem 5.9. This should be compared with Corollary 3.20 and Lemma 3.34.

This allows to write $D_{m,n}^{[r]}$ for $1 \leq r \leq n - 1$ for the terminal subpaths of $D_{m+1}$. We also iterate this notation below by identifying $D_{m,r+1}^{[1]}$ with $D_m$ and identifying $D_{m,r+1,n}^{[1]}$ with the subpath obtained by removing the first $r + 1$ copies of $D_{m-2}$ from a copy of $D_{m-1}$.

For $m \geq 1$, we identify the edges of $D_{m+1}$ with the ordered set $E_{m+1} = \{1, \ldots, u_{m+1} + u_m\}$, where edges of $D_{m+1}$ are taken in the natural order beginning from $(0, 0)$. Let $E_m = H_m \cup V_m$, where $H_m = \{h_1, \ldots, h_u\}$ and $V_m = \{v_1, \ldots, v_u\}$ denote the horizontal and vertical edges of $D_{m+1}$ respectively. Following Theorem 5.9, we partition the edges as $E_{m+1} = \bigcup_{i=1}^n E_{m,i}$, where $E_{m,i}$ denotes the edges of $D_{m,i}$. The set $E_{m,i}$ is naturally partitioned into its subsets $H_{m,i}$ and $V_{m,i}$ of horizontal and vertical edges. The edges of $D_{m+1}^{[r]}$ are similarly partitioned as $E_{m+1}^{[r]} = \bigcup_{r=1}^n E_{m,i}^{[r]} = H_{m+1}^{[r]} \cup V_{m+1}^{[r]}$.

Given edges $e, e' \in E_{m+1}$ with $e < e'$, write $e e'$ for the shortest subpath of $D_{m+1}$ containing $e$ and $e'$, in particular $e e$ is the subpath containing the single edge $e$.

**Definition 5.12** For $m \geq 1$, a pair of subsets $S_H \subset H_{m+1}$ and $S_V \subset V_{m+1}$ is called compatible if: for each pair $(h, v) \in S_H \times S_V$ with $h < v$, there exists an edge $e \in hv$ so that at least one of the following holds

\[
eq v \quad \text{and} \quad |he \cap V_{m+1}| = n|he \cap S_H| \quad (5.5)
\]

or

\[
eq h \quad \text{and} \quad |ev \cap H_{m+1}| = n|ev \cap S_V|. \quad (5.6)
\]

Write $C_{m+1}$ for the collection of all pairs $(S_H, S_V)$ which are compatible as above.

**Remark 5.13** This notion of compatibility extends naturally to the maximal Dyck paths $D_{m+1}^{[r]}$, $1 \leq r \leq n - 1$, and trivially to the Dyck path $D_1$. Write $e_{m+1}^{[r]}$ for the set of all compatible pairs in $D_{m+1}^{[r]}$.

The recursive structure of the maximal Dyck paths from Theorem 5.9 gives rise to a recursive characterization of compatible pairs.
Definition 5.14 [18, Definition 3.11] A pair of subsets $S_H \subset H_{m+1}$ and $S_V \subset V_{m+1}$ is called piecewise compatible if, for each $1 \leq r \leq n$, one of the conditions (5.5) or (5.6) is satisfied for each pair $(h, v) \in S_H \times S_V$ with $h \in H_{m,i}$ and $v \in V_{m,i}$.

Remark 5.15 The notion of piecewise compatibility naturally extends to the maximal Dyck paths $D_{m+1}[r]$, $1 \leq r \leq n - 1$. Given a compatible pair $(S_H, S_V)$ in $D_{m+1}[r]$, we write $S_H^r = S_H \cap H_{m+1}^r \subset H_{m+1}$ and $S_V^r = S_V \cap V_{m+1}^r \subset V_{m+1}$. In particular, the pair $(S_H^r, S_V^r)$ is compatible in $D_{m+1}$. We also write $S_{H,i} = S_H \cap H_{m,i}$ and $S_{V,i} = S_V \cap V_{m,i}$ for $r + 1 \leq i \leq n - 1$.

To describe precisely when a piecewise compatible pair $(S_H, S_V)$ is compatible we need more notation. For a horizontal edge $h \in H_{m+1}$ and a subset $S_H \subset H_{m+1}$, write $D(h; S_H) = he$ for the shortest subpath of $D_{m+1}$ for which $|he \cap V_{m+1}| = n|he \cap S_H|$, if no such subpath exists we set $D(h; S_H) = hv_{um}$. The subpath $D(h; S_H)$ is called the local shadow path of $h$ with respect to $S_H$. Similarly, for a vertical edge $v \in V_{m+1}$ and a subset $S_V \subset V_{m+1}$, the local shadow path of $v$ with respect to $S_V$ is $D(v; S_V) = ev$ for the shortest subpath of $D_{m+1}$ for which $|ev \cap H_{m+1}| = n|ev \cap S_V|$ and we take $D(v; S_V) = hiv$ if there does not exist such an edge $e$.

Definition 5.16 [18, Definition 3.17] A horizontal edge $h_i \in H_{m+1}$, $m \geq 2$, is called blocking for a subset $S_H \subset H_{m+1}$ if $D(h_i; S_H) = hiv_{um}$ and $h_i$ is furthest to the right with this property, i.e. the index $i$ is maximal.

Suppose $S_H \subset H_{m+1}$ admits a blocking edge $h_i \in H_{m+1}$. Then $S_H$ is left-justified at $h_i$ if there exists $k \geq i$ so that $S_H = \{h_i, h_{i+1}, \ldots, h_k\}$. The subset $S_H$ is strongly left-justified at $h_i$ if $S_H$ is left-justified at $h_i$ and $|h_i v_{um} \cap V_{m+1}| = n|h_i v_{um} \cap S_H|$.

A subset $S_V \subset V_{m+1}$ is right-justified with respect to $h_i$ if there exists a vertical edge $v_s \in h_i v_{um}$ so that $S_V \cap h_i v_{um} = \{v_s, v_{s-1}, \ldots, v_{um}\}$. The subset $S_V$ is strongly right-justified with respect to $h_i$ if $S_V$ is right-justified with respect to $h_i$ and $D(v_{um}, S_V) = h_i v_{um}$ with $|h_i v_{um} \cap H_{m+1}| = n|h_i v_{um} \cap S_V|$.

Theorem 5.17 [18, Theorem 3.20 and Corollary 3.22] For $m \geq 2$, suppose $S_H \subset H_{m+1}$ and $S_V \subset V_{m+1}$ are piecewise compatible. Then the following hold:

1. If $S_H$ does not admit a blocking edge, then $(S_H, S_V) \in \mathcal{C}_{m+1}$.
2. Suppose $S_H$ admits a blocking edge $h_i \in H_{m+1}$ and $(S_H, S_V)$ is not compatible. Then $S_H$ is left-justified at $h_i$ and $S_V$ is strongly right-justified with respect to $h_i$. In addition, the following hold:
   
   (a) If $m = 2$, then $S_H \cap h_i v_{um} = \{h_i\}$.
   (b) If $m \geq 3$, then $S_H$ is strongly left-justified at $h_i$.
   (c) If $m \geq 4$, then either $i = 1$ or $h_i$ is immediately preceded by a vertical edge in $D_{m+1}$.

Corollary 5.18 For $m \geq 3$ and $0 \leq r \leq n - 1$, consider $S_H \subset H_{m+1}^r$ and $S_V \subset V_{m+1}^r$ so that $(S_H, S_V)$ is piecewise compatible. Assume $(S_H^r, S_V^r) \in \mathcal{C}_{m+1}^r$. Then $(S_H, S_V)$ is not compatible if and only if $H_{m+1}^{r+1,n} \subset S_H \cap H_{m+1}$ and $V_{m+1}^{r+1} \subset S_V$.

Proof We begin with the reverse implication. First note that there are $(n - r)u_{m+1} - u_{m-2}$ horizontal edges and $(n - r)u_{m-1} - u_{m-2}$ vertical edges in $D_{m+1}$. It follows that $H_{m+1}^{r+1,n} \subset H_{m+1}^{r+1,n}$ contains $n(n - r)u_{m-1} - nu_{m-2}$ horizontal edges and $V_{m+1}^{r+1} \subset V_{m+1}^{r+1}$ contains $n(n - r)u_{m-2} - nu_{m-3}$ vertical edges (note that $D_{m+1}^{r+1,n}$ naturally identifies with the Dyck path $D_{m+1}^{r+1,n}$).
Assuming \( H_{m, r+1}^{[r+1]} \subset S_{H, r+1} \) and \( S_{V}^{[r+1]} = V_{m+1}^{[r+1]} \), we have \( S_{V} \cap V_{m+1}^{[r+1]} = \emptyset \) and \( S_{H} \cap H_{m+1}^{[r+1]} = \emptyset \) by piecewise compatibility. Let \( h \in H_{m+1}^{[r]} \) be the horizontal edge corresponding to the first horizontal edge of \( H_{m, r+1}^{[r+1]} \). Then, since there are \((n-r)u_{m-2} - u_{m-3}\) horizontal edges in \( H_{m, r+1}^{[r+1]} \), the local shadow path \( D(h; S_{H}) \) contains \( n((n-r)u_{m-2} - u_{m-3}) \) vertical edges and is thus equal to \( hv_{u_{m}} \). Similarly, the local shadow path \( D(v_{u_{m}}; S_{V}) \) is also equal to \( hv_{u_{m}} \). In particular, neither of the compatibility conditions of Definition 5.12 are satisfied for the path \( hv_{u_{m}} \) and so \( (S_{H}, S_{V}) \) is not compatible.

For the forward implication, we work by induction on \( m \geq 3 \). Consider a pair \((S_{H}, S_{V})\) for \( D_{4}^{[r]} \) as above which is not compatible. Following Theorem 5.17, write \( h \in H_{3}^{[r]} \) for the blocking edge of \( S_{H} \). Then the number of vertical edges in the local shadow path \( D(h; S_{H}) = hv_{u_{3}} \) must be divisible by \( n \). Since \((S_{H}^{[r+1]}, S_{V}^{[r+1]})\) is compatible, we must have \( h \in H_{3, r+1}^{[n-1]} \). But \( S_{V} \) is strongly right-justified with respect to \( h \) and thus the number of horizontal edges in \( D(v_{u_{3}}; S_{V}) = hv_{u_{3}} \) is divisible by \( n \). Identifying \( H_{3, r+1}^{[n-1]} \) with \( H_{2}^{[1]} \), this divisibility condition only occurs when \( h \) is the first horizontal edge in \( H_{2}^{[r+1]} \subset H_{2}^{[1]} \). Then by piecewise compatibility, the vertical edge of \( H_{2}^{[1]} \) cannot be an element of \( S_{V} \) and we must have \( V_{4}^{[r+1]} \subset S_{V} \). By piecewise compatibility again, this implies \( H_{4}^{[r+1]} \cap S_{H} = \emptyset \) and so \( D(h; S_{H}) = hv_{u_{3}} \) implies \( H_{2}^{[r+1]} \subset S_{H} \).

To continue, let \((S_{H}, S_{V})\) be a pair for \( D_{m+1}^{[r]} \), \( m \geq 4 \), which is not compatible. Write \( \varphi : H_{m} \to V_{m+1} \) for the bijection given by \( \varphi(h_{i}) = v_{i} \) for \( 1 \leq i \leq u_{m} \). For any subset \( T \subset H_{m} \), set \( \varphi^{*}(T) = V_{m+1} \setminus \varphi(T) \). Clearly, the map \( \varphi^{*} \) gives a bijection between subsets of \( H_{m} \) and subsets of \( V_{m+1} \). In Sect. 3.2 of [18], a new pair of subsets \((\varphi^{*})^{-1}S_{V}, \Omega^{-1}S_{H}\) for \( D_{m} \) is given, we refer the reader to loc. cit for notation. By [18, Proposition 3.10], the pair \((\varphi^{*})^{-1}S_{V}, \Omega^{-1}S_{H}\) is not compatible, but is piecewise compatible by [18, Proposition 3.16]. Thus by induction, we must have \( H_{m-1, r+1}^{[r+1]} \subset (\varphi^{*})^{-1}S_{V} \) and \( V_{m}^{[r+1]} \subset \Omega^{-1}S_{H} \). It follows from piecewise compatibility that \( H_{m-1, r+1}^{[r+1]} \cap (\varphi^{*})^{-1}S_{V} = \emptyset \). But then by the definition of \( \varphi^{*} \) we have \( V_{m+1}^{[r+1]} \cap S_{V} = \emptyset \) and \( S_{V}^{[r+1]} = V_{m+1}^{[r+1]} \) so that \( D(v_{u_{m}}; S_{V}) = hv_{u_{m}} \) with \( h \) as in the first case above. Since \((S_{H}, S_{V})\) is not compatible, Theorem 5.17 states that \( h \) must be the blocking edge for \( S_{H} \) and we must have \( D(h; S_{H}) = hv_{u_{m}} \). But this can only occur if \( H_{m, r+1}^{[r+1]} \subset S_{H} \) since \( H_{m+1}^{[r+1]} \cap S_{H} = \emptyset \) by piecewise compatibility.

The following result is an immediate consequence of the combinatorial construction of rank 2 cluster variables [15] and the categorification of these variables using representations of \( K(n) \) [7, 8].

**Theorem 5.19** [15] For each \( m \geq 1 \) and \( e \in \mathbb{Z}_{\geq 0}^{2} \), we have

\[
\chi(Gr_{e}(P_{m})) = \left| \{(S_{H}, S_{V}) \in C_{m} : |S_{H}| = u_{m} - e_{1}, |S_{V}| = e_{2}\} \right|.
\]

Our goal is to give a geometric explanation for this by showing that the compatible pairs provide a natural labeling for the cells of \( Gr_{e}(P_{m+1}) \) found in Theorem 4.23. In fact, we will see more: that the cells of quiver Grassmannians \( Gr_{e}(P_{m+1}^{V}) \) for truncated preprojectives \( P_{m+1}^{V} \) are also naturally labeled by compatible pairs. We accomplish this by providing a bijection between the compatible pairs as in Theorem 5.19 and the successor closed sets of vertices in the 2-quivers \( Q_{m+1}^{[r]} \) used in Theorem 5.8 to describe the non-empty cells.
Theorem 5.20 For $m \geq 1$ and $V \in \text{Gr}(\mathcal{H}_m)$ or $V = 0$, each quiver Grassmannian $\text{Gr}_e(P_{m+1}^V)$ admits a cell decomposition with affine cells labeled by compatible pairs in the maximal Dyck path $D_{m+1}^r$, where $r = \dim V$.

Proof The recursive construction of the 2-quivers $Q_{m+1}^r$ provides a natural ordering of the vertices in the underlying quiver $Q_{m+1}^r$. Indeed, when considering the recursive construction of the 2-quiver $Q_{m+1}^r$ from (5.4), we order the component quivers $Q_{m,i}$ and $Q_{m,*}$ naturally according to their indices so that $Q_{m,*}^r$ comes last. This provides a bijection of these vertices with the edges of $D_{m+1}^r$ whereby vertices covering the vertex 1 (resp. vertex 2) of $K(n)$ correspond to horizontal edges (resp. vertical edges) of $D_{m+1}^r$.

Given a strong successor closed subset $\beta \subset (Q_{m+1}^r)_0$, we define a pair of subsets $S_H(\beta) \subset H_{m+1}^r$ and $S_V(\beta) \subset V_{m+1}^r$ as follows: a vertical edge $v \in V_{m+1}^r$ is in $S_V(\beta)$ exactly when the corresponding vertex of $Q_{m+1}^r$ is in $\beta$ while a horizontal edge $h \in H_{m+1}^r$ is in $S_H(\beta)$ exactly when the corresponding vertex of $Q_{m+1}^r$ is not in $\beta$. Then Corollary 5.18 shows that under this bijection a subset $\beta \in (Q_{m+1}^r)_0$ is strong successor closed in $Q_{m+1}^r$ if and only if the corresponding pair of subsets $(S_H(\beta), S_V(\beta))$ is compatible. Applying Theorem 5.8 completes the proof. □

The results of [18] provide a stronger statement than Theorem 5.19. Indeed, the compatible pairs are shown to compute the counting polynomials of the quiver Grassmannians $\text{Gr}_e(P_{m+1}^V)$ over a finite field (these coincide with their Poincaré polynomials in this case). We conjecture that the torus action on $\text{Gr}_e(P_{m+1}^V)$ can be chosen to provide a geometric explanation of this result.

Conjecture 5.21 For $m \geq 1$ and $V \in \text{Gr}(\mathcal{H}_m)$ or $V = 0$, there exists a torus action on $\text{Gr}_e(P_{m+1}^V)$ such that the dimension of the cell labeled by a compatible pair $(S_H, S_V)$ in the maximal Dyck path $D_{m+1}^r$, where $r = \dim V$, is given by $\mathcal{V}_{S_H,S_V} = \sum_{e < e' \in E_{m+1}^r} \mathcal{V}_\omega(e, e')$ for

$$\mathcal{V}_\omega(e, e') = \begin{cases} -n & \text{if } e \in S_H \text{ and } e' \in S_V; \\ 1 & \text{if } e \in S_H \text{ and } e' \in H_{m+1}^r \setminus S_H; \\ 1 & \text{if } e \in V_{m+1}^r \setminus S_V \text{ and } e' \in S_V; \\ 0 & \text{otherwise.} \end{cases}$$

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