Abstract

We study higher derivative extension of the functional renormalization group (FRG). We consider FRG equations for a scalar field that consist of terms with higher functional derivatives of the effective action and arbitrary cutoff functions. We show that the $\epsilon$-expansion around the Wilson-Fisher fixed point is indeed reproduced by the local potential approximation of the FRG equations.
1 Introduction

The functional renormalization group (FRG) (or the exact renormalization group) has been developed based on the philosophy given in [11] (see also [2], and for reviews of the FRG, see [3–11]). It serves as a powerful method for the nonperturbative study of quantum field theories as well as lattice field theories. The FRG equation is a functional differential equation that describes the dependence of the effective action on the energy scale.

The FRG consists of two procedures, coarse graining and rescaling. The form of the FRG equation depends on coarse graining procedure. Thus, one of the important issues on the FRG is what coarse graining procedure is allowed. This issue has been examined thoroughly in the FRG equation for a scalar field that includes up to two functional derivatives of the effective action [12–15], where the coarse graining procedure is fixed by specifying a cutoff function and a seed action. It seems, however, that the coarse graining procedure that gives the FRG equations including more than two functional derivatives is quite different from the one that gives the FRG equation including up to two derivatives. To our knowledge, such higher derivative FRG equations has not been studied systematically so far, although there is a recent interesting proposal for a manifestly gauge-invariant FRG equation that includes higher functional derivatives [16] (for related works, see [17–27]).

In this paper, we study the FRG equations for a scalar field that consist of terms with four (six) functional derivatives and arbitrary cutoff functions. We analyze it by using the local potential approximation and show that the $\epsilon$ expansion around the Wilson-Fisher fixed point is reproduced if the cutoff functions satisfy certain conditions. Our result suggests that the coarse graining procedure that gives higher derivative FRG equations indeed works.

This paper is organized as follows. In section 2, we briefly review a formal derivation of the FRG equation. In section 3, we introduce the FRG equations that consist of terms with four functional derivatives and arbitrary cutoff functions. In section 4, we analyze the above FRG equations by using the local potential approximation and show that the $\epsilon$ expansion around the Wilson-Fisher fixed point is reproduced. Section 5 is devoted to conclusion and discussion. In appendix, we examine the FRG equations consisting of terms with up to six functional derivatives.
2 A formal derivation of the FRG equation

We consider a scalar field theory in $d$-dimensional Euclidean space. Throughout this paper, we use the following notation:

$$\int_x \equiv \int d^d x.$$ (2.1)

We denote the effective action at the cutoff scale $\Lambda$ by $S_\Lambda$, which is a functional of the scalar field $\phi(x)$.

The FRG equation follows from an equation [12, 13, 28, 29]

$$-\Lambda \frac{\partial}{\partial \Lambda} e^{-S_\Lambda} = \int_x \frac{\delta}{\delta \phi(x)} \left( \Psi_\Lambda(x)e^{-S_\Lambda} \right)$$ (2.2)

which ensures at least formally that the partition function

$$Z = \int D\phi \; e^{-S_\Lambda}$$ (2.3)

is independent of $\Lambda$. Here we emphasize that this derivation of the FRG equation is formal and that one should check that the equation that follows from (2.2) works in a physically valid manner. A standard choice of $\Psi(x)$ is given by

$$\Psi_\Lambda(x) = \frac{1}{2} \int_y \dot{C}_\Lambda(x-y) \frac{\delta \Sigma_\Lambda}{\delta \phi(y)}$$ (2.4)

with

$$\Sigma_\Lambda = S_\Lambda - 2 \dot{S}_\Lambda,$$ (2.5)

where $C_\Lambda(x - y)$ is a cutoff function, the dot stands for $-\Lambda \frac{d}{d\Lambda}$, and $\dot{S}_\Lambda$ is called the seed action. The Fourier transform of $\dot{C}_\Lambda$, which is defined by $\dot{C}_\Lambda(p) = \int_x \dot{C}_\Lambda(x)e^{-ipx}$, must damp rapidly for $|p| > \Lambda$ and allow the Taylor expansion in $p$.

Substituting (2.4) into (2.2) yields a class of FRG equations with up to two functional derivatives:

$$-\Lambda \partial_\Lambda S_\Lambda = \frac{1}{2} \int_{x,y} \dot{C}_\Lambda(x-y) \left( \frac{\delta S_\Lambda}{\delta \phi(x)} \frac{\delta \Sigma_\Lambda}{\delta \phi(y)} - \frac{\delta^2 \Sigma_\Lambda}{\delta \phi(x)\delta \phi(y)} \right)$$ (2.6)

The ERG equations (2.6) are rather general in the sense that they include arbitrary functions $C_\Lambda$ and functionals $\dot{S}_\Lambda$. In particular, putting in (2.6)

$$S_\Lambda = \frac{1}{2} \int_{x,y} \phi(x)C_\Lambda^{-1}(x-y)\phi(y) + S_I,$$
\[ \hat{S}_\Lambda = \frac{1}{2} \int_{x,y} \phi(x) C^{-1}_\Lambda(x - y) \phi(y) , \quad (2.7) \]

where \( S_I \) is the interaction part of the effective action, leads to the Polchinski equation [30]

\[ -\Lambda \partial_\Lambda S_I = \frac{1}{2} \int_{x,y} \hat{C}_\Lambda(x - y) \left( \frac{\delta S_I}{\delta \phi(x)} \frac{\delta S_I}{\delta \phi(y)} - \frac{\delta^2 S_I}{\delta \phi(x) \delta \phi(y)} \right) . \quad (2.8) \]

A typical example of \( C_\Lambda(p) \) is

\[ C_\Lambda(p) = e^{-p^2/\Lambda^2} . \quad (2.9) \]

It was shown in [12–15] that the physical consequences are independent of the choices of \( C_\Lambda \) and \( \hat{S}_\Lambda \).

### 3 Higher derivative extension

In this section, we consider the FRG equations with four functional derivatives as a higher derivative extension. In what follows, we put \( t = 1/\Lambda^2 \), which implies that \( t \partial_t = -\frac{1}{2} \Lambda \partial_\Lambda \). We denote \( S_\Lambda \) by \( S_t \). We consider an equation that follows from (2.2). In order for (2.2) to give an FRG equation that consists of terms with four functional derivatives, \( \Psi_\Lambda \) must consist of terms with three functional derivatives. All possible types of three functional derivatives are

\[ \frac{\delta S_t}{\delta \phi(y_1)} \frac{\delta S_t}{\delta \phi(y_2)} \frac{\delta S_t}{\delta \phi(y_3)} , \quad \frac{\delta^2 S_t}{\delta \phi(y_1) \delta \phi(y_2)} \frac{\delta S_t}{\delta \phi(y_3)} , \quad \frac{\delta^3 S_t}{\delta \phi(y_1) \delta \phi(y_2) \delta \phi(y_3)} . \quad (3.1) \]

Thus, we consider almost the most general FRG equation consisting of terms with four functional derivatives as follows:

\[ t \partial_t e^{-S_t} = \int_x \frac{\delta}{\delta \phi(x)} \left[ \int_{y_1,y_2,y_3} \left\{ A_t(x - y_1) A_t(x - y_2) A_t(x - y_3) \frac{\delta S_t}{\delta \phi(y_1)} \frac{\delta S_t}{\delta \phi(y_2)} \frac{\delta S_t}{\delta \phi(y_3)} \right. \right. \]
\[ + B_t(x - y_1) B_t(x - y_2) B_t(x - y_3) \frac{\delta^2 S_t}{\delta \phi(y_1) \delta \phi(y_2)} \frac{\delta S_t}{\delta \phi(y_3)} \]
\[ + C_t(x - y_1) C_t(x - y_2) C_t(x - y_3) \frac{\delta^3 S_t}{\delta \phi(y_1) \delta \phi(y_2) \delta \phi(y_3)} \left. \right\} e^{-S_t} \right] , \quad (3.2) \]

where \( A_t(x - y) \), \( B_t(x - y) \) and \( C_t(x - y) \) are cutoff functions with the mass dimension \( \frac{2}{3}(d - 2) \) and assumed to have the following derivative expansions:

\[ K_t(x - y) = (K_0 + K_1 \partial_y^2 + \cdots) \delta(x - y) , \quad (3.3) \]
where $K_t$ is $A_t$, $B_t$ or $C_t$. An example of $K_t$ is

$$K_t(x - y) = t \frac{d-2}{2} \int_y \frac{d}{2} e^{-(x-y)^2}$$

$$= t \frac{d+4}{8} \int \frac{d}{2} e^{tp^2} e^{ip(x-y)}$$

$$\simeq t \frac{d+4}{8} \int \frac{d}{2} (1 - tp^2) e^{ip(x-y)}$$

$$= t \frac{d+4}{8} \int \frac{d}{2} (1 + t\partial_y^2) e^{ip(x-y)}$$

$$= t \frac{d+4}{8} (1 + t\partial_y^2) \delta(x - y).$$

(3.4)

4 Local potential approximation

In this section, as a validity check of (3.2), we analyze it by using the local potential approximation and show that the $\epsilon$ expansion around the Wilson-Fisher fixed point is reproduced if the cutoff functions satisfy certain conditions.

4.1 Flow equation for the local potential

We apply the local potential approximation \[31\] to (3.2). First, we represent the effective action in terms of the local potential $V_t$ as

$$S_t[\phi] = \int_x \left( \frac{1}{2} (\partial_x \phi(x))^2 + V_t[\phi](x) \right).$$

(4.1)

By substituting this into (3.2), we obtain a flow equation for the local potential local $V_t[\phi](x)$. To calculate the first term in the RHS of (3.2), for instance, we first do the following preparatory calculation:

$$\int_y A_t(x - y) \frac{\delta S_t}{\delta \phi(y)} = \int_y A_t(x - y) \frac{\delta}{\delta \phi(y)} \int \frac{d}{2} e^{ip(x-y)}$$

$$= \int_y A_t(x - y) \left( V_t'[y'] - \partial_y^2 \phi(y') \right) \delta(y - y')$$

$$= \int_y A_t(x - y) \left( V_t'(y) - \partial_y^2 \phi(y) \right)$$

$$\simeq \int_y \left\{ (A_0 + A_1 \partial_y^2) \delta(x - y) \right\} \left( V_t'(y) - \partial_y^2 \phi(y) \right)$$

$$= \int_y (A_0 + A_1 \partial_y^2) \left( V_t'(y) - \partial_y^2 \phi(y) \right) \delta(x - y)$$

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where the coefficients $P, Q, L, M, X, Y$ are represented as

$$
P = B_{10}B''(0) + B_{20}B(0) ,
Q = 3A''(0)A_0^2 - B_{20}B_0 ,
X = -B_{10}B(0) ,
Y = -C^3(0) ,
Z = -B_2^2(0)B_0 + C_{10} ,
L = -3A(0)A_0^2 + B_{10}B_0 ,
M = A_0^3 ,
$$

using the derivative expansions of $A_t(x-y)$, $B_t(x-y)$ and $C_t(x-y)$,

$$A_t(x-y) \sim (A_0 + A_1 \partial_y^2)\delta(x-y) ,$$
Next, we rewrite (4.5) in terms of dimensionless quantities. Note that this procedure realizes the rescaling in the renormalization group. We add the bar to the dimensionless quantities. The field \( \phi(x) \) and the local potential \( V_t \) are made dimensionless as

\[
\phi(x) = t^{-\frac{d-2}{4}} \tilde{\phi}(\tilde{x}), \\
V_t[\phi] = \tilde{V}_t[\tilde{\phi}] t^{-\frac{d}{2}},
\]

respectively. Thus, the LHS of (4.5) is calculated as

\[
t\partial_t \tilde{V}_t[\tilde{\phi}] = t \partial_t (\tilde{V}_t[\tilde{\phi}] t^{-\frac{d}{2}}) = t^{-\frac{d}{2}} \left\{ -\frac{d}{2} \tilde{V}_t + \frac{d-2}{4} \tilde{\phi} \tilde{V}_t + t \partial_t \tilde{V}_t \right\},
\]

while the RHS of (4.5) as

\[
PV_t'' + QV_t'^2 + LV_t''V_t'^2 + MV_t'^4 + XV_t'''' + YV_t''' + ZV_t''V_t' = t^{-\frac{d}{2}} (\tilde{P}\tilde{V}_t'' + \tilde{Q}\tilde{V}_t'^2 + \tilde{L}\tilde{V}_t''\tilde{V}_t'^2 + \tilde{M}\tilde{V}_t'^4 + \tilde{X}\tilde{V}_t'''' + \tilde{Y}\tilde{V}_t''' + \tilde{Z}\tilde{V}_t''\tilde{V}_t') .
\]

The resultant flow equation for \( \tilde{V}_t \) is

\[
t\partial_t \tilde{V}_t = \frac{d}{2} \tilde{V}_t - \frac{d-2}{4} \tilde{\phi} \tilde{V}_t + \tilde{P}\tilde{V}_t'' + \tilde{Q}\tilde{V}_t'^2 + \tilde{L}\tilde{V}_t''\tilde{V}_t'^2 + \tilde{M}\tilde{V}_t'^4 + \tilde{X}\tilde{V}_t'''' + \tilde{Y}\tilde{V}_t''' + \tilde{Z}\tilde{V}_t''\tilde{V}_t'.
\]

In what follows, we omit the bar for dimensionless quantities.

Finally, we expand the local potential \( V_t \) in \( \phi \) to the eighth order as

\[
V_t[\phi](x) = \frac{v_2}{2!} \phi^2(x) + \frac{v_4}{4!} \phi^4(x) + \frac{v_6}{6!} \phi^6(x) + \frac{v_8}{8!} \phi^8(x),
\]

where the \( Z_2 \) symmetry is assumed, and substitute (4.12) into (4.11). This results in the following flow equations for the couplings \( v_{2n} \):

\[
t\partial_t v_2 = 2L v_2 + P v_4 + 2Q v_2^2 + 2v_2 v_4 (X + Z) + v_2 + v_6 Y
\]
\[
\begin{align*}
t\partial_t v_4 &= 20Lv_2^2v_4 + 24Mv_2^4 + 8Qv_2v_4 + 2v_2v_6X + 4v_2v_6Z + 6v_4^2X + 4v_4^2Z + \frac{v_4\epsilon}{2} + v_8Y \\
t\partial_t v_6 &= 42Lv_2^2v_6 + 140Lv_2v_4^2 + 480Mv_2^3v_4 + P v_8 + 4Q \left( 3v_2v_6 + 5v_4^2 \right) + 2v_2v_8X \\
&\quad + 6v_2v_6Z + 30v_4v_6X + 26v_4v_6Z + v_6\epsilon - v_6 \\
t\partial_t v_8 &= 8L \left( 9v_2^2v_8 + 126v_2v_4v_6 + 70v_4^3 \right) + 1344Mv_2^2 \left( v_2v_6 + 5v_4^2 \right) + 16Qv_2v_8 \\
&\quad + 112Qv_4v_6 + 56v_4v_8X + 64v_4v_8Z + 70v_6^2X + 56v_6^2Z + \frac{3v_8\epsilon}{2} - 2v_8 
\end{align*}
\]

where we put \( d = 4 - \epsilon \).

### 4.2 Fixed points

The fixed points of the renormalization group are determined by

\[
\partial_t v_{2n} = 0 
\]

in (4.13). We denote a solution to (4.14) by \( v_{2n}^* \).

We perform the \( \epsilon \) expansion to the first order of \( \epsilon \) in the following (for the \( \epsilon \) expansion for the FRG consisting of up to two functional derivatives, see [32]). We find a trivial fixed point, the Gaussian fixed point given by

\[
\begin{align*}
v_2^* &= 0, \\
v_4^* &= 0, \\
v_6^* &= 0, \\
v_8^* &= 0, 
\end{align*}
\]

and a nontrivial fixed point, the Wilson-Fisher fixed point given by

\[
\begin{align*}
v_2^* &= \frac{P}{4(6PQ + 3X + 2Z)}\epsilon + \mathcal{O}(\epsilon^2), \\
v_4^* &= -\frac{1}{4(6PQ + 3X + 2Z)}\epsilon + \mathcal{O}(\epsilon^2), \\
v_6^* &= \frac{5Q}{4(6PQ + 3X + 2Z)^2}\epsilon^2 + \mathcal{O}(\epsilon^3), \\
v_8^* &= -\frac{35(L + 4Q^2)}{8(6PQ + 3X + 2Z)^3}\epsilon^3 + \mathcal{O}(\epsilon^4). 
\end{align*}
\]

We see that the following condition must be satisfied in order for the Wilson-Fisher fixed point to exist:

\[
6PQ + 3X + 2Z \neq 0. 
\]
Putting \( v_{2n} = v_{2n}^* + \delta v_{2n} \), we linearize the flow equations (4.13) around the nontrivial fixed (4.16) with respect to \( \delta v_{2n} \) as follows:

\[
\partial_t \delta v = T \delta v ,
\]

(4.18)

where

\[
\delta v = (\delta v_2, \delta v_4, \delta v_6, \delta v_8)^t ,
\]

(4.19)

\[
T = \begin{pmatrix}
T_{11} & T_{12} & T_{13} & T_{14} \\
T_{21} & T_{22} & T_{23} & T_{24} \\
T_{31} & T_{32} & T_{33} & T_{34} \\
T_{41} & T_{42} & T_{43} & T_{44}
\end{pmatrix} ,
\]

\[
T_{11} = 1 - \frac{-2PQ + X + Z}{12PQ + 6X + 4Z} \epsilon + O(\epsilon^2) ,
\]

\[
T_{12} = P + \frac{P(X + Z)}{12PQ + 6X + 4Z} \epsilon + O(\epsilon^2) ,
\]

\[
T_{13} = Y ,
\]

\[
T_{14} = 0 ,
\]

\[
T_{21} = -\frac{2Q}{6PQ + 3X + 2Z} \epsilon + O(\epsilon^2) ,
\]

\[
T_{22} = \frac{10PQ - 3X - 2Z}{12PQ + 6X + 4Z} \epsilon + O(\epsilon^2) ,
\]

\[
T_{23} = P + \frac{P(X + 2Z)}{12PQ + 6X + 4Z} \epsilon + O(\epsilon^2) ,
\]

\[
T_{24} = Y ,
\]

\[
T_{31} = O(\epsilon^2) ,
\]

\[
T_{32} = -\frac{10Q}{6PQ + 3X + 2Z} \epsilon + O(\epsilon^2) ,
\]

\[
T_{33} = -1 + \frac{18PQ - 9(X + Z)}{12PQ + 6X + 4Z} \epsilon + O(\epsilon^2) ,
\]

\[
T_{34} = P + \frac{P(X + 3Z)}{12PQ + 6X + 4Z} \epsilon + O(\epsilon^2) ,
\]

\[
T_{41} = O(\epsilon^2) ,
\]

\[
T_{42} = O(\epsilon^2) ,
\]

\[
T_{43} = -\frac{28Q}{6PQ + 3X + 2Z} \epsilon + O(\epsilon^2) ,
\]

\[
T_{44} = -2 + \frac{26PQ - 19X - 26Z}{12PQ + 6X + 4Z} \epsilon + O(\epsilon^2) .
\]

(4.20)
The eigenvalues of $T$ are calculated up to the first order in $\epsilon$ as

$$\lambda_2 = 1 - \frac{2PQ + X + Z}{12PQ + 6X + 4Z} \epsilon,$$

$$\lambda_4 = -\frac{1}{2} \epsilon,$$

$$\lambda_6 = -1 - \frac{9(2PQ + X + Z)}{12PQ + 6X + 4Z} \epsilon,$$

$$\lambda_8 = -2 + \frac{82PQ - 19X - 26Z}{12PQ + 6X + 4Z} \epsilon.$$

(4.21)

The eigenvalues $\lambda_2$, $\lambda_4$, $\lambda_6$ and $\lambda_8$ are supposed to be fixed by by the scaling dimension of the operators $\int_x \phi^2$, $\int_x \phi^4$, $\int_x \phi^6$ and $\int_x \phi^8$, respectively, as

$$\lambda_2 = 1 - \frac{\epsilon}{6}, \quad \lambda_4 = -\frac{\epsilon}{2}, \quad \lambda_6 = -1 - \frac{3\epsilon}{2}, \quad \lambda_8 = -2 - \frac{19\epsilon}{6}.$$

(4.22)

We see that $\lambda_2$, $\lambda_4$ and $\lambda_6$ with $Z = 0$ in (4.21) indeed agree those in (4.22). Note that $\lambda_8$ with $Z = 0$ in (4.21) does not agree with that in (4.22). This is because the local potential is truncated up to the eighth order in $\phi$. We verified that we obtain the correct value of $\lambda$ if we expand the local potential to the tenth order in $\phi$ and performed the same analysis.

As a consequence, in order that the $\epsilon$ expansion with the local potential approximation gives the correct values of the scaling dimensions around the Wilson-Fisher fixed point, the following two conditions must be satisfied:

$$2PQ + X \neq 0 \Leftrightarrow 2(B''(0)B_{10} + B(0)B_{20})(3A''(0)A_0^2 - B_0B_{20}) - B(0)B_{10} \neq 0 \quad (4.23)$$

$$Z = 0 \Leftrightarrow -B^2(0)B_0 + C_{10} = 0 \quad (4.24)$$

Namely, the cutoff functions $A_t(x-y)$, $B_t(x-y)$ and $C_t(x-y)$ in (3.2) must be chosen such that these two conditions are satisfied.

We have analyzed the FRG equation with terms consisting of four functional derivatives so far. We can generalize the above analysis to the cases in which the FRG equations include terms with two or more than four functional derivatives in addition to the terms with four functional derivatives. In these cases, we can show that the $\epsilon$ expansion with the local potential approximation reproduces the scaling dimensions to the first order in $\epsilon$ if the conditions (4.23) and (4.24) are satisfied. In appendix, we examine the local potential approximation for the FRG equation including terms with two, four or six functional derivatives.

Our results suggest that the FRG equation can be extended such that it includes higher functional derivatives.
5 Conclusion and discussion

In this paper, we studied the higher derivative extension of the FRG. We considered the FRG equations for a scalar field that consists of the terms with four functional derivatives and arbitrary cutoff functions. While those FRG equations are constructed in such a way that they guarantee the invariance of the partition function under the changes of scale at least formally, it is nontrivial that they make sense physically because the coarse graining corresponding to four functional derivatives is quite different from that to two functional derivatives. We showed that the $\epsilon$ expansion around the Wilson-Fisher fixed point is indeed reproduced by the local potential approximation of the FRG equations if the cutoff functions satisfy the conditions. We also verified that this holds for the case of six functional derivatives. It is natural that the conditions on the cutoff functions are needed because it is known that the derivative expansion for the FRG equations, whose lowest order is nothing but the local potential approximation\(^1\), in general breaks the arbitrariness of the cutoff functions and the invariance under redefinition of the field (see \([10]\) and references therein.\(^2\)). Our results suggest that the higher derivative extension of the FRG makes sense.

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\(^1\)It was shown in \([10]\) that the local potential approximation for the FRG equations consisting of terms with up to two functional derivatives has no dependence on the cutoff function.

\(^2\)Note also that we did not consider in this paper $\Sigma_\Lambda$ that includes the functional derivative of $S_t$ and depends explicitly on $\phi$ such as

$$
\Sigma_\Lambda = \int_y K_\epsilon(x - y)\phi^2(y) \frac{\delta S_t}{\delta \phi(y)}.
$$

We saw that this type of $\Sigma$ in the FRG equations for a scalar field prevents the local potential approximation from reproducing the known scaling dimensions.
A Local potential approximation for the FRG equations with up to six functional derivatives

Here we examine the local potential approximation for the FRG equation including terms with two, four or six functional derivatives. In this case, the local approximation yields the following flow equation for the local potential $V_i$:

$$
\partial t V_i = P_1 V_i^2 + P_2 V_i + Q_5 V_1^4 + Q_4 V_2 V_i^2 + Q_3 V_3 V_1 + Q_1 V_2^2 + Q_2 V_4 + R_{11} V_i^6 \\
+ R_{10} V_2 V_i^4 + R_7 V_3 V_i^3 + R_9 V_2^2 V_i^2 + R_4 V_4 V_i^2 \\
+ R_6 V_2 V_3 V_i + R_2 V_5 V_1 + R_8 V_2^3 + R_5 V_3^2 + R_3 V_2 V_4 + R_1 V_6, \quad (A.1)
$$

where $V_n$ stands for the $n$-th order derivative of $V_i$ with respect to $\phi$, and $P_i$, $Q_i$, and $R_i$ are determined by the cutoff functions as in (4.12). Making the above flow equation dimensionless and substituting (4.12) into (A.1) yields

$$
\partial t v_2 = 2P_1 v_2^3 + P_2 v_4 + 2Q_4 v_2^3 + 2Q_1 v_2 v_4 + 2Q_3 v_2 v_4 + Q_2 v_6 + 2R_4 v_2^2 v_4 + 2R_6 v_2^2 v_4 \\
+ 3R_8 v_2 v_4 + 2R_9 v_2^4 + 2R_2 v_2 v_6 + R_3 v_2 v_6 + R_8 v_2^3 + 2R_5 v_4^2 + R_1 v_8 + v_2, \\
\partial t v_4 = 8P_1 v_2 v_4 + P_2 v_6 + 20Q_4 v_2^3 v_4 + 24Q_5 v_2^4 + 2Q_1 v_2 v_4 + 4Q_3 v_2 v_6 + 6Q_1 v_4^2 + 4Q_3 v_4^2 + Q_2 v_8 \\
+ 24R_7 v_2^2 v_4 + 32R_9 v_2^3 v_4 + 12R_4 v_2 v_6 + 4R_6 v_2 v_6 + 3R_8 v_2 v_6 + 24R_{10} v_2^5 + 8R_4 v_2 v_4^2 + 16R_6 v_2 v_4^2 \\
+ 18R_8 v_2 v_4^2 + 4R_2 v_2 v_8 + R_3 v_2 v_8 + 4R_2 v_4 v_6 + 7R_3 v_4 v_6 + 8R_5 v_6 v_6 + \frac{V_1 e}{2}, \\
\partial t v_6 = 4P_1 (3v_2 v_6 + 5v_4^2) + P_2 v_8 + 480Q_5 v_2^3 v_4 + 42Q_4 v_2^3 v_6 + 140Q_4 v_2 v_4^2 + 2Q_1 v_2 v_8 \\
+ 6Q_3 v_2 v_8 + 30Q_1 v_4 v_6 + 26Q_3 v_2 v_6 + 360R_7 v_2^2 v_4 + 440R_9 v_2 v_4^2 + 840R_{10} v_2^4 v_4 + 120R_7 v_2^3 v_6 \\
+ 72R_5 v_2^3 v_6 + 30R_4 v_2 v_8 + 6R_6 v_2 v_8 + 3R_8 v_2 v_8 + 72R_1 v_2^6 + 132R_4 v_2 v_4 v_6 + 116R_6 v_2 v_4 v_6 \\
+ 90R_5 v_4 v_6 + 20R_4 v_4^3 + 60R_6 v_4^3 + 90R_8 v_4^3 + 20R_2 v_4 v_8 + 16R_3 v_4 v_8 + 12R_5 v_4 v_8 \\
+ 6R_2 v_6^2 + 15R_3 v_6^2 + 20R_5 v_6^2 + 200 v_6^2 + 20 v_6 e - v_6, \\
\partial t v_8 = 16P_1 (v_2 v_8 + 7v_4 v_6) + 6720Q_5 v_2^2 v_4^2 + 1344Q_5 v_2^3 v_6 + 72Q_4 v_2 v_4^2 + 1008Q_4 v_2 v_4 v_6 \\
+ 560Q_4 v_4^3 + 14Q_1 (4v_4 v_8 + 5v_6) + 64Q_3 v_4 v_8 + 56Q_3 v_6^2 + 2016R_{10} v_2^4 v_4^2 + 4368R_7 v_2^2 v_4 v_6 \\
+ 3584R_9 v_2 v_4 v_6 + 40320R_{11} v_2^5 v_4 + 3024R_{10} v_2^4 v_6 + 336R_7 v_2^3 v_8 + 128R_9 v_2^3 v_8 \\
+ 3360R_7 v_2 v_4^3 + 4480R_9 v_2 v_4^3 + 576R_4 v_2 v_4 v_8 + 288R_6 v_2 v_4 v_8 + 168R_8 v_2 v_4 v_8 \\
+ 336R_4 v_2 v_6^2 + 336R_6 v_2 v_6^2 + 210R_8 v_2 v_6^2 + 672R_4 v_2 v_6^2 + 1008R_6 v_2 v_6^2 + 1260R_8 v_2 v_6^2 \\
+ 64R_2 v_6 v_8 + 98R_3 v_6 v_8 + 112R_5 v_6 v_8 + \frac{3v_8 e}{2} - 2v_8. \quad (A.2)
The nontrivial fixed point is given by

\[ v^*_2 = \frac{P_2}{4(6P_1P_2 + 3Q_1 + 2Q_3)\epsilon}, \]
\[ v^*_4 = -\frac{1}{4(6P_1P_2 + 3Q_1 + 2Q_3)}\epsilon, \]
\[ v^*_6 = \frac{5P_1}{4(6P_1P_2 + 3Q_1 + 2Q_3)^2}\epsilon^2, \]
\[ v^*_8 = -\frac{35(4P^2_1 + Q_4)}{8(6P_1P_2 + 3Q_1 + 2Q_3)^3}\epsilon^3. \]  
(A.3)

We see that \(6P_1P_2 + 3Q_1 + 2Q_3 \neq 0\) is required for the nontrivial fixed point to exist. The linearized equation around the nontrivial fixed point is

\[ \partial_t \delta v = T \delta v \]  
(A.4)

with

\[
T = \begin{pmatrix}
T_{11} & T_{12} & T_{13} & T_{14} \\
T_{21} & T_{22} & T_{23} & T_{24} \\
T_{31} & T_{32} & T_{33} & T_{34} \\
T_{41} & T_{42} & T_{43} & T_{44}
\end{pmatrix} \]
\[
A.5
\]

\[ T_{11} = 1 - \frac{\epsilon(-2P_1P_2 + Q_1 + Q_3)}{12P_1P_2 + 6Q_1 + 4Q_3} + O(\epsilon^2), \]
\[ T_{12} = P_2 + \frac{\epsilon(P_2(Q_1 + Q_3) - R_3 - 2R_5)}{12P_1P_2 + 6Q_1 + 4Q_3} + O(\epsilon^2), \]
\[ T_{13} = Q_2 + \frac{P_2(2R_2 + R_3)\epsilon}{4(6P_1P_2 + 3Q_1 + 2Q_3)} + O(\epsilon^2), \]
\[ T_{14} = R_1, \]
\[ T_{21} = -\frac{2P_1\epsilon}{6P_1P_2 + 3Q_1 + 2Q_3} + O(\epsilon^2), \]
\[ T_{22} = \frac{\epsilon(10P_1P_2 - 3Q_1 - 2Q_3)}{12P_1P_2 + 6Q_1 + 4Q_3} + O(\epsilon^2), \]
\[ T_{23} = P_2 - \frac{\epsilon(-2P_2(Q_1 + 2Q_3) + 4R_2 + 7R_3 + 8R_5)}{4(6P_1P_2 + 3Q_1 + 2Q_3)} + O(\epsilon^2), \]
\[ T_{24} = Q_2 + \frac{P_2(4R_2 + R_3)\epsilon}{4(6P_1P_2 + 3Q_1 + 2Q_3)} + O(\epsilon^2), \]
\[ T_{31} = O(\epsilon)^2, \]
\[ T_{32} = O(\epsilon)^2, \]
\[ T_{33} = O(\epsilon)^2, \]
\[ T_{34} = O(\epsilon)^2, \]
\[ T_{41} = O(\epsilon)^2, \]
\[ T_{42} = O(\epsilon)^2, \]
\[ T_{43} = O(\epsilon)^2, \]
\[ T_{44} = O(\epsilon)^2. \]
\[ T_{32} = -\frac{10P_1\epsilon}{6P_1P_2 + 3Q_1 + 2Q_3} + O(\epsilon^2) \]
\[ T_{33} = -1 + \frac{\epsilon (18P_1P_2 - 9(Q_1 + Q_3))}{12P_1P_2 + 6Q_1 + 4Q_3} + O(\epsilon^2) \]
\[ T_{34} = P_2 + \frac{\epsilon (P_2(Q_1 + 3Q_3) - 2(5R_2 + 4R_3 + 3R_5))}{12P_1P_2 + 6Q_1 + 4Q_3} + O(\epsilon^2) \]
\[ T_{11} = O(\epsilon^2) \]
\[ T_{12} = O(\epsilon^2) \]
\[ T_{13} = -\frac{28P_1\epsilon}{6P_1P_2 + 3Q_1 + 2Q_3} + O(\epsilon^2) \]
\[ T_{14} = -2 + \frac{\epsilon (26P_1P_2 - 19Q_1 - 26Q_3)}{12P_1P_2 + 6Q_1 + 4Q_3} + O(\epsilon^2) \]

The eigenvalues of \( T \) are
\[ \lambda_2 = -1 - \frac{2P_1P_2 + Q_1 + Q_3}{12P_1P_2 + 6Q_1 + 4Q_3}\epsilon \]
\[ \lambda_4 = -\frac{1}{2}\epsilon \]
\[ \lambda_6 = -1 - \frac{9(2P_1P_2 + Q_1 + Q_3)}{12P_1P_2 + 6Q_1 + 4Q_3}\epsilon \]
\[ \lambda_8 = -2 + \frac{82P_1P_2 - 19Q_1 - 26Q_3}{12P_1P_2 + 6Q_1 + 4Q_3}\epsilon \]

\( \lambda_1, \lambda_2 \) and \( \lambda_3 \) agree with those in (4.22) if \( Q_3 = 0 \). Thus, we see that the coefficients of six derivatives in (A.1) are arbitrary, while those of four derivatives must satisfy (4.23) and (4.24).

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