INTERNAL CONTROLLABILITY OF PARABOLIC SYSTEMS WITH STAR AND TREE LIKE COUPLINGS

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Abstract. We consider systems of parabolic equations coupled in zero order terms in a star-like or a tree-like shape, with an internal control acting in only one of the equations. We obtain local exact controllability to the stationary solutions of the system, under hypotheses concerning the supports of the coupling functions. The key point is establishing Carleman estimates with appropriate observation operators for the adjoint to the linearized system, which allows the study of the controllability problem, in either linear or nonlinear cases, in an $L^\infty$ framework.

1. Introduction

In this paper we consider some classes of semilinear systems of parabolic equations coupled in zero order terms. We are interested in controllability of such systems to stationary solutions by only one scalar control distributed in a subdomain and acting in only one of the equations.

The study of controlled systems of parabolic equations needs appropriate observability estimates for the adjoint system. These observability estimates are usually derived from global Carleman estimates. Global Carleman estimates are by now a classical tool in proving observability inequalities, and they were established in the context of controllability for parabolic equations by O.Yu.Imanuvilov (see O.Yu.Imanuvilov and A.Fursikov [9]). Since then this type of estimates was extensively developed, refined and used in other contexts, like control problems with small number of controls, stabilization or inverse problems.

Controllability for parabolic systems with a reduced number of controls needs observability estimates of Carleman type with partial observations. There is an extensive literature concerning such problems; for a selection of titles we refer for example to [13] and the references therein.

In the case of zero order couplings with constant or time dependent coupling coefficients there exists a particular interest in obtaining algebraic conditions of Kalman type for controllability; in this direction we cite the papers of F.Ammar-Khodja, A.Benabdallah, C.Dupaix and M.González-Burgos [2, 1] or the work of F.Ammar-Khodja, F.Chouly and M. Duprez [3].

Observability estimates for linear systems (not only parabolic) coupled with constant coupling coefficients in the dominant part and/or in the zero order terms were established by E.Zuazua and P.Lissy [13]; such estimates

2010 Mathematics Subject Classification. 35K40, 35K58, 93B05, 93B07, 93B18.

Key words and phrases. Controllability of parabolic systems, Carleman estimates, tree-like systems, reaction-diffusion systems.
are obtained under Kalman rank conditions satisfied by the pair of the coupling and control matrices.

The results we present in our paper extend the results in [10] where systems of parabolic equations with cascade type couplings in zero order terms are considered. The extension we propose works under hypotheses addressing two aspects of the systems under consideration: one is the structure of the couplings, which describes in our case either a star or a tree type graph; the second aspect refers to the support of the coupling functions or, in the linear case, to the support of the coupling coefficients.

The strategy for proving the controllability result relies on the linearization of the nonlinear system around a stationary state. The key step is obtaining the null controllability for this linear system by using an observability inequality for the adjoint system. This observability inequality is consequence of an appropriate global Carleman estimate. This in turn is obtained by combining Carleman estimates for each of the equations, but relying on different auxiliary functions, which are in a particular order relation, made possible by the special structure of the system. The idea of using different auxiliary functions in Carleman estimates is inspired by the work of G.Olive [14] concerning controllability of parabolic systems with controls acting in different subdomains.

The Carleman observability estimates we establish are more elaborated and are not direct consequences of the classical Carleman estimates. One reason for developing these Carleman estimates is the fact that trying to use the estimates from the paper of Luz de Teresa and M.Gonzáles-Burgos [10] for cascade systems we realized that, written for the branches of the tree, they do not fit well together. Even when passing from the study of star type couplings to general tree type couplings one needs to use two Carleman estimates for each equation in an interior node of the graph and this is another quite technical point needed in our approach. The hypotheses concerning the supports of the coupling coefficients allow to construct appropriate auxiliary functions and weights in the corresponding Carleman estimates which will finally fit well in order to give the desired global observability inequality.

Passing from the linearized system to the nonlinear system, one needs an $L^\infty$ framework for controlability. The main reason is that the Carleman estimates we obtain are sensitive to zero order perturbations of the system. More regularity of the controls in the linearized problem is obtained as in the work of V.Barbu [5] (see also J.-M. Coron, S.Guerrero and L.Rosier [6]) by using regularizing properties of the parabolic flow in a bootstrap argument. This step encounters supplementary technical challenges as it needs $L^\infty - L^2$ Carleman estimates with different weights in corresponding estimates for different equations. The $L^\infty$ controllability for the linearized system allows an approach to the controllability of the nonlinear system by a fixed point argument, based on Kakutani theorem (see also [6] or [1]).

2. Preliminaries and statement of the problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded connected domain with a $C^2$ boundary $\partial \Omega$ and let $\omega_0 \subset \subset \Omega$. Let $T > 0$ and denote by $Q = (0,T) \times \Omega$ and for $\omega \subset \Omega$ write $Q_\omega = (0,T) \times \omega$. 
We consider systems of \((n + 1)\) parabolic equations coupled in zero order terms through nonlinear functions, with one internally distributed control, acting in \(\omega_k\) and entering only the first equation. The main goal is obtaining local exact controllability to some stationary solution for the nonlinear system.

In the first part of the paper we study systems of parabolic equations with star-like couplings which refer to the situation where \(y_k\) is actuated in the corresponding parabolic equation through a nonlinearity depending only on \(y^0, y^k\). Such a star-like coupled system has the form:

\[
\begin{aligned}
D_t y_0 - \Delta y_0 &= f_0(x, y_0, y_i), i \in \overline{1, n}, \quad \text{in } (0, T) \times \Omega, \\
D_t y_i - \Delta y_i &= f_i(x, y_0, y_i), i \in \overline{1, n}, \quad \text{in } (0, T) \times \Omega, \\
y_0 = \ldots = y_n = 0, \quad \text{on } (0, T) \times \partial \Omega,
\end{aligned}
\]

where \(\bar{\gamma}_j \in L^\infty(\Omega), j \in \overline{0, n}\). We denote by \(\chi_{\omega_0} v\) the extension of \(v : \omega_0 \to \mathbb{R}\) with 0 to the whole domain \(\Omega\). The control function is \(u : [0, T] \times \omega_0 \to \mathbb{R}\), acting directly in the equation of \(y_0\) while the other components of the solution, \(y_1, \ldots, y_n\), are indirectly actuated through the corresponding coupling terms containing \(y_0\).

Consider a stationary state \(\bar{\gamma} = (\bar{\gamma}_0, \ldots, \bar{\gamma}_n), \bar{\gamma}_j \in L^\infty(\Omega), j \in \overline{0, n}\), solution to the elliptic system:

\[
\begin{aligned}
-\Delta \bar{\gamma}_0 &= f_0(x, \bar{\gamma}_0), \quad x \in \Omega, \\
-\Delta \bar{\gamma}_i &= f_i(x, \bar{\gamma}_0, \bar{\gamma}_i), i \in \overline{1, n}, \quad x \in \Omega, \\
\bar{\gamma}_0 = \ldots = \bar{\gamma}_n = 0, \quad x \in \partial \Omega.
\end{aligned}
\]

Observe in fact that, by elliptic regularity, an \(L^\infty\) stationary solution is a smooth solution.

Concerning the coupling terms we assume the following hypotheses:

\((H1)\) \(f_0 : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}, f_i : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, i \in \overline{1, n}\) are \(C^1\) functions and there exist \(\omega_1, \ldots, \omega_n \subset \Omega,\) open nonempty subsets of \(\Omega\) such that

\[
(\omega_i \cap \omega_0) \setminus \bigcup_{j \neq i} \omega_j \neq \emptyset, \forall i \in \overline{1, n},
\]

and for all \(i \in \overline{1, n}\) we have

\[
f_i(x, y_0, y_i) = 0 \forall x \in \Omega \setminus \omega_i, y_0, y_i \in \mathbb{R};
\]

\((H2)\) The following coupling condition holds:

\[
\operatorname{supp} \frac{\partial f_i}{\partial y_0}(x, \bar{\gamma}_0(x), \bar{\gamma}_i(x)) \cap \left\{(\omega_i \cap \omega_0) \setminus \bigcup_{j \neq i} \omega_j\right\} \neq \emptyset.
\]

**Remark 1.** Concerning the above technical hypotheses \((2.1), (2.3)-(H2)\), observe that they are, for example, satisfied for all sources \(\bar{\gamma}_i\) and corresponding stationary solutions \(\bar{\gamma}\) if the nonlinearities \(f_i\) are of the form

\[
f_i(x, y_0, y_i) = \zeta_i(x) \xi_i(y_0, y_i), x \in \Omega, y_0, y_i \in \mathbb{R},
\]

with \(\emptyset \neq \operatorname{supp} \zeta_i(x) \subset (\omega_i \cap \omega_0) \setminus \bigcup_{j \neq i} \omega_j\) and \(\xi_i = \xi_i(y_0, y_i) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) are smooth with \(\frac{\partial \xi_i}{\partial y_0} \neq 0, \forall y_0, y_i \in \mathbb{R}.)
If $\overline{y}$ is a constant solution, as the problem has homogeneous boundary conditions, necessarily $\overline{y} \equiv 0$; a stationary solution in this case exists if and only if $\overline{y}_0(x) = -f_0(x,0)$ and $\overline{y}_1(x) = -f_1(x,0,0), \forall x \in \Omega$. Condition (2.3) is satisfied if, for example, $\emptyset \neq \text{supp} \frac{\partial f_i}{\partial y_0}(x,0,0) \subset \subset (\omega_i \cap \Omega_0) \setminus \bigcup_{j \neq 0,i} \omega_j$.

In concrete situations, when the stationary solution is known, the hypotheses we imposed on the supports of the coupling functions are easy to verify.

Our study concerns the controllability to the stationary state $\overline{y}$ of system (2.1) in a given time interval $T$. We are thus led to the study of a class of linear controlled systems and corresponding controllability properties, systems which arise by a linearization procedure around the stationary state:

\begin{equation}
\begin{aligned}
D_t z_0 - \Delta z_0 &= c_0(t,x)z_0 + \chi_{\omega_0}u, \\
D_t z_i - \Delta z_i &= a_{0i}(t,x)z_0 + c_i(t,x)z_i, \quad i \in \{1, \ldots, n\}, \\
z_0 = \ldots = z_n = 0, \\
0 \leq t \leq T \times \Omega.
\end{aligned}
\end{equation}

For $M, \delta > 0$, and open subsets $\omega_i \subset \subset (\omega_i \cap \Omega_0) \setminus \bigcup_{j \neq 0,i} \omega_j$, we introduce the following classes of coefficients sets:

\begin{equation}
\begin{aligned}
&\mathcal{E}_{M,\delta,\{\omega_i\}_i} = \left\{ E = \{a_{0i}, c_j\}_{i \in \{1, \ldots, n\}; j \in \mathbb{N}} : a_{0i}, c_j \in L^\infty(Q) \right\}, \\
&\quad \|a_{0i}\|_{L^\infty}, \|c_j\|_{L^\infty} \leq M, a_{0i} = 0 \text{ in } Q \setminus Q_\omega, \text{ and } |a_{0i}| \geq \delta \text{ on } Q_\omega_i.
\end{aligned}
\end{equation}

We prove first that such linear systems with coefficients in $\mathcal{E}_{M,\delta,\{\omega_i\}_i}$, are null controllable with norm $L^2$ and $L^\infty$ of the control uniformly bounded by a constant $C = C(M, \delta, \{\omega_i\}_i)$.

In order to achieve this goal, we consider the adjoint system:

\begin{equation}
\begin{aligned}
-D_t p_0 - \Delta p_0 &= c_0(t,x)p_0 + \sum_{i=1}^n a_{0i}(t,x)p_i, \quad (0, T) \times \Omega, \\
-D_t p_i - \Delta p_i &= c_i(t,x)p_i, \quad i \in \{1, \ldots, n\}, \\
p_0 = \ldots = p_n = 0, \\
0 \leq t \leq T \times \partial \Omega.
\end{aligned}
\end{equation}

We prove an observability inequality as consequence of an appropriate Carleman estimate. The Carleman estimate we establish in the next section gives us more than just observability, it helps obtaining a priori estimates for the control driving the solution of the linear system to zero and, as the constants appearing in the Carleman estimates are depending only on $M, \delta, \{\omega_i\}_i$, the estimates on the control will result uniform. This fact is essential in the fixed point argument when dealing with the nonlinear system.

In order to reformulate the problem in an abstract functional framework let the state space be the Hilbert space $H = [L^2(\Omega)]^{n+1}$ and the control space $U = L^2(\omega_0)$. Consider the operator

\[ A : D(A) \subset H \rightarrow H, D(A) = (H_0^1(\Omega) \cap H^2(\Omega))^{n+1}, Az = \Delta z, \]

and the control operator

\[ B : U \rightarrow H, Bu = \chi_{\omega_0}Bu, B = (1, 0, \ldots, 0)^\top. \]
Then, problem (2.1) may be written in abstract form:

\[
\begin{aligned}
D_t y &= Ay + f(y) + Bu, \quad t > 0, \\
y(0) &= y^0,
\end{aligned}
\]

where \( f(y) = f(\cdot, y(\cdot)) \). The linear problem (2.6) may be reformulated as:

\[
\begin{aligned}
D_t z &= A z + A_0(t) z + C(t) z + Bu, \quad t > 0, \\
z(0) &= z^0,
\end{aligned}
\]

where \( C(t) z = C_0(t, \cdot) z(\cdot) \) and \( A_0(t) z = A_0(t, \cdot) z(\cdot) \), \( C_0(t, x) \) is the diagonal matrix \( C_0(t, x) = \text{diag}(c_i(t, x))_{i=0}^n \) and the coupling matrix \( A_0(t, x) \) has only one nonzero column, the first one, and is given by \( A_0(t, x) = (0, a_{10}, \ldots, a_{n0})^\top \cdot (1, 0, \ldots, 0) \).

For simplicity, when there is no confusion, we denote the norms of functions \( z \in [L^2(\Omega)]^{n+1}, z \in [H^4(\Omega)]^{n+1} \), etc. as \( \|z\|_{L^2(\Omega)} \), respectively \( \|z\|_{H^1(\Omega)} \), etc.

Null controllability for the linear system (2.10) above is equivalent to an observability inequality

\[
\|p(0)\|_{L^2(\Omega)}^2 \leq C(M, \delta) \int_0^T \|B^* p\|_{L^2(\omega_0)}^2 dt, \quad \text{for some } C(M, \delta) > 0,
\]

for all solutions \( p \) to the adjoint equation

\[
- p' = A p + A_0^\top p + C p
\]

where \( A_0^\top p = A_0^\top |p, B^* p = B^\top |_{\omega_0} \).

We extend our study to parabolic systems with tree-like couplings. In fact we will treat only linear equations with appropriate hypotheses for the coupling coefficients in a tree-like structure. Passing from linear results of controllability to local controllability for nonlinear systems may be obtained by exactly the same procedure as in the star-like case. An example of linear parabolic system with tree-like couplings is the following:

\[
\begin{align*}
D_t z_0 - \Delta z_0 &= c_0(t, x) z_0 + \chi_{\omega_0} u, \quad \text{in } (0, T) \times \Omega, \\
D_t z_1 - \Delta z_1 &= a_{10}(t, x) z_0 + c_1(t, x) z_1, \quad \text{in } (0, T) \times \Omega, \\
D_t z_2 - \Delta z_2 &= a_{20}(t, x) z_0 + c_2(t, x) z_2, \quad \text{in } (0, T) \times \Omega, \\
D_t z_3 - \Delta z_3 &= a_{31}(t, x) z_1 + c_3(t, x) z_3, \quad \text{in } (0, T) \times \Omega, \\
D_t z_4 - \Delta z_4 &= a_{41}(t, x) z_1 + c_2(t, x) z_4, \quad \text{in } (0, T) \times \Omega, \\
z_0 = \ldots = z_4 &= 0, \quad \text{on } (0, T) \times \partial \Omega,
\end{align*}
\]

and the general form of system with tree like couplings will be discussed in §6.

The paper is organized as follows:

- In §3 we prove appropriate Carleman estimates for adjoint system (2.8) in either \( L^2 - L^2 \) or \( L^\infty - L^2 \) settings. This will be Theorem 1.
- In §4 we prove the null controllability of linear system (2.6). The approach uses a family of optimal control problems with penalized final cost. One then obtains besides controllability an estimate for the control in both \( L^2 \) and \( L^\infty \) norms by using the previous Carleman estimates. This is Theorem 2.
• [5] is devoted to the local controllability in $L^\infty$ of nonlinear system $(2.1)$. The fact that controllability has to be proved in $L^\infty$ is due to the high sensitivity of the Carleman estimates with respect to the coupling coefficients, which is not the case when controls act in each equation of the system. The argument is similar to that used in [4].

• In [6] we extend results of controllability, with one distributed scalar control, for linear systems of parabolic equations, of the form $(2.13)$, with tree-like couplings. The key point here is obtaining appropriate Carleman estimates. Local controllability for nonlinear systems with tree-like couplings is also discussed.

3. CARLEMAN ESTIMATES AND OBSERVABILITY

In this section we establish an $L^2$ Carleman estimate that will help proving an observability inequality for the adjoint problem $(2.8)$. This $L^2$ Carleman inequality and parabolic regularity are the starting point in obtaining an $L^\infty$ control through a bootstrap argument.

We recall the classical Carleman estimate for a generic nonhomogeneous parabolic problem,

\begin{equation}
\begin{cases}
D_t p + L p = h, & \text{in } (0, T) \times \Omega, \\
p = 0, & \text{on } (0, T) \times \partial \Omega,
\end{cases}
\end{equation}

where $L$ is an uniformly elliptic operator of second order. Denote by $Q := (0, T) \times \Omega$ and, for $\omega \subset \subset \Omega$, $Q_\omega := (0, T) \times \omega$. The solution is observed in $Q_\omega$ for sources $h \in L^2(Q)$.

We introduce the function

$$
\psi \in C^2(\overline{\Omega}), \quad \psi|_{\partial \Omega} = k > 0, \quad k < \psi < \frac{3}{2} k \quad \text{in } \Omega, \quad \{ x \in \overline{\Omega} : |\nabla \psi(x)| = 0 \} \subset \subset \omega,
$$

and the weight functions

\begin{equation}
\varphi(t, x) := \frac{e^{\lambda \psi(x)}}{t(T-t)}, \quad \alpha(t, x) := \frac{e^{\lambda \psi(x)} - e^{1.5\lambda \|\psi\|_{C(\overline{\Omega})}}}{t(T-t)}.
\end{equation}

Then, the classical global Carleman estimate (see [7], [8]) is the following:

**Lemma 1.** There exist $\lambda_0, s_0$ and $C > 0$ such that if $\lambda > \lambda_0$, $s \geq s_0$, the following inequality holds:

\begin{equation}
\begin{aligned}
&\int_Q [(s \varphi)^{-1}(|D_p|^2 + |D^2 p|) + s^{\lambda_2} \varphi |D_p|^2 + s^{\lambda_4} \varphi^3 |p|^2] e^{2s \alpha} dx dt \\
&\leq C \int_{Q_\omega} s^{3\lambda} \varphi^3 |p|^2 e^{2s \alpha} dx dt + \int_Q |h|^2 e^{2s \alpha} dx dt
\end{aligned}
\end{equation}

for all $p \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$ solution of $(3.1)$.

We establish a Carleman estimate for the following nonhomogeneous version to the adjoint problem to $(2.6)$, with source term $g \in [L^2(Q)]^{n+1}$ and observation operator $B^\ast p = B^\ast p|_{\omega_0} = p_0|_{\omega_0}$, operator which "sees" only $p_0$ in the subdomain $\omega_0$:

\begin{equation}
\begin{cases}
-D_t p_0 - \Delta p_0 = c_0(t, x)p_0 + \sum_{i=1}^n a_{i0}(t, x)p_i + g_0, & (0, T) \times \Omega, \\
-D_t p_i - \Delta p_i = c_i(t, x)p_i + g_i, & (0, T) \times \Omega, \\
p_0 = \ldots = p_n = 0, & (0, T) \times \partial \Omega.
\end{cases}
\end{equation}
In the following we are going to establish Carleman estimates for each equation in (3.4) by using in each case corresponding subdomains of observation and appropriately chosen weight functions.

Some technical preliminaries are needed and we proceed as follows:

Consider open subsets \( \tilde{\omega}_j \subset \subset \omega_j \) and denote as above by \( Q_{\tilde{\omega}_j} = (0, T) \times \tilde{\omega}_j \); take the auxiliary functions \( \psi_j, j = 0, \ldots, n \), with the following properties (where we have denoted by \( \tilde{\omega}_0 := \omega_0 \)):

\begin{equation}
(3.5) \quad \psi_j := \eta_j + K_j, j \in \{0, \ldots, n\},
\end{equation}

\( \eta_j \in C^2(\Omega), 0 < \eta_j \) in \( \Omega \), \( \eta_j|_{\partial \Omega} = 0 \), \( \{x \in \Omega : |\nabla \eta_j(x)| = 0\} \subset \subset \tilde{\omega}_j \),

for some fixed positive constants \( K_j > 0 \) such that

\begin{equation}
(3.6) \quad \psi_i > \psi_0 \text{ in } \Omega, \forall i \in \{1, \ldots, n\},
\end{equation}

and

\begin{equation}
(3.7) \quad \frac{\sup \psi_j}{\inf \psi_j} < \frac{8}{7}, \forall j \in \{0, \ldots, n\}.
\end{equation}

Let \( 0 < \epsilon < \inf \psi_i, i \in \{0, \ldots, n\} \) a small positive number and denote by

\begin{equation}
(3.8) \quad \bar{\psi} = \sup_{x \in \Omega} \sup_{j \in \{0, \ldots, n\}} \psi_j(x) + \epsilon, \quad \underline{\psi} = \inf_{x \in \Omega} \inf_{j \in \{0, \ldots, n\}} \psi_j(x) - \epsilon.
\end{equation}

Introduce also, for parameters \( s, \lambda > 0 \) the auxiliary functions:

\begin{equation}
(3.9) \quad \varphi_j(t, x) := \frac{e^{\lambda \psi_j(x)}}{t(T - t)}, \quad \alpha_j(t, x) := \frac{e^{\lambda \psi_j(x)} - e^{1.5 \lambda \underline{\psi}}}{t(T - t)}, \forall j \in \{0, \ldots, n\},
\end{equation}

and

\begin{equation}
(3.10) \quad \varphi(t) = \varphi^\lambda(t) := \frac{e^{\lambda \bar{\psi}}}{t(T - t)}, \quad \pi(t) = \pi^\lambda(t) := \frac{e^{\lambda \bar{\psi}} - e^{1.5 \lambda \bar{\psi}}}{t(T - t)},
\end{equation}

\begin{equation}
(3.11) \quad \varphi(t) = \varphi^\lambda(t) := \frac{e^{\lambda \underline{\psi}}}{t(T - t)}, \quad \Omega(t) = \Omega^\lambda(t) := \frac{e^{\lambda \underline{\psi}} - e^{1.5 \lambda \underline{\psi}}}{t(T - t)}.
\end{equation}

Remark 2. (i) As we are going to compare the various Carleman estimates stated for each equation of the linear adjoint system, we will need to compare the weights which are involved in those inequalities. For this purpose let us observe that given \( m_0 > 0 \) there exist \( s_0 = s_0(m_0), \lambda_0 = \lambda_0(m_0) > 0 \) such that for all \( s > s_0, \lambda > \lambda_0, |m| \leq m_0 \) and \( t \in (0, T) \), the following inequality holds:

\begin{equation}
(3.12) \quad e^{s \alpha} \leq s^m \varphi^m e^{s \alpha},
\end{equation}

\begin{equation}
(3.13) \quad e^{s \alpha_0} \leq s^m \varphi^m e^{s \alpha}.
\end{equation}
(ii) Observe that if in (3.5) we replace $K_i$ with $K_i + M$ with the constant $M > 0$ big enough, the above properties of the auxiliary functions remain valid and, moreover, we may assume that

$$\frac{\psi}{\bar{\psi}} \leq \frac{3}{2}. \tag{3.14}$$

This extra assumption implies that there exist $s_0 > 0, \lambda_0 > 0$ such that if $s > s_0, \lambda > \lambda_0$,

$$|D_t \varphi_i| \leq C \varphi_i^2, \quad |D_t \alpha_i| \leq C \varphi_i^2, \quad |D_t^2 \alpha_i| \leq C \varphi_i^3. \tag{3.15}$$

(iii) Observe that for $\lambda$ big enough, say $\lambda > \bar{\lambda}$, we have

$$\frac{\alpha^\lambda}{\alpha^\lambda} < 2. \tag{3.16}$$

Indeed, this is a consequence to the fact that $\lim_{\lambda \to +\infty} \frac{\alpha^\lambda}{\alpha^\lambda} = 1$, uniformly with respect to $(t, x) \in Q$.

In this section we prove the following Carleman estimate which has as consequence the appropriate observability inequality for the adjoint system (3.1).

**Theorem 1.** There exist constants $\lambda_0, s_0$ such that for $\lambda > \lambda_0$ there exists a constant $C > 0$ depending on $(M, \delta, \{\omega_i\}_i, \lambda)$, such that, for any $s \geq s_0$, the following inequality holds:

$$\int_Q (|D_t p|^2 + |D^2 p|^2 + |Dp|^2 + |p|^2)e^{2s\alpha^\lambda}dxdt \leq C \int_{Q_{s_0}} |p_0|^2 e^{2s\alpha^\lambda}dxdt + C \int_Q |g|^2 e^{2s\alpha^\lambda}dxdt \tag{3.17}$$

for all $p \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; H^2(\Omega))$ solution of (3.4).

Moreover, there exist $m_0 \in \mathbb{N}$ and $\delta_1 > 0$ such that for the homogeneous adjoint system (i.e. taking $g \equiv 0$), we have the following $L^\infty - L^2$ Carleman estimate

$$\|p e^{(s + m_0\delta_1)\alpha^\lambda}\|_{L^\infty(Q)} \leq C \|p_0 e^{s\alpha^\lambda}\|_{L^2(Q_{s_0})}. \tag{3.18}$$

**Proof.** The second remark above is useful when obtaining Carleman estimates, since the weights here are slightly different with respect to those used in [9] or [7]. However, this remark allows following the same lines of proof and we may write Carleman estimate (3.3) for each equation $j \in 0, n$ with observation domain $\tilde{\omega}_j$ and auxiliary functions and weight functions $\psi_j, \varphi_j, \alpha_j$. Thus, there exist $s_0 > 0, C > 0$ such that for any $s \geq s_0$, the following inequalities hold:

(1) For $p_0$ we have
\begin{align}
\int_Q \left[ (s\varphi_0)^{-1}(|D_1p_0|^2 + |D^2p_0|^2) + s\varphi_0|Dp_0|^2 + s^3\varphi_0^3|p_0|^2 \right] e^{2\sigma_\alpha} dx dt
\leq C \left[ \int_{Q_{\omega_0}} s^3\varphi_0^3|p_0|^2 e^{2\sigma_\alpha} dx dt + \int_Q \left| \sum_{i=1}^n a_{i0}p_i + g_0 \right|^2 e^{2\sigma_\alpha} dx dt \right] \leq (3.19)
\leq C \left[ \int_{Q_{\omega_0}} s^3\varphi_0^3|p_0|^2 e^{2\sigma_\alpha} dx dt 
+ n^2M^2 \sum_{i=1}^n \int_Q |p_i|^2 e^{2\sigma_\alpha} dx dt + \int_Q |g_0|^2 e^{2\sigma_\alpha} dx dt \right].
\end{align}

(2) For \( p_i, i \in \overline{1,n} \) we have:
\begin{align}
\int_Q \left[ (s\varphi_i)^{-1}(|D_1p_i|^2 + |D^2p_i|^2) + s\varphi_i|Dp_i|^2 + s^3\varphi_i^3|p_i|^2 \right] e^{2\sigma_\alpha} dx dt 
\leq C \int_{Q_{\omega_0}} s^3\varphi_i^3|p_i|^2 e^{2\sigma_\alpha} dx dt + C \int_Q |g_i|^2 e^{2\sigma_\alpha} dx dt.
\end{align}

Summing the above Carleman inequalities we obtain, for some constant \( C = C(M, \{\omega_j\}_j) > 0 \), that
\begin{align}
\sum_{j=0}^n \left\{ \int_Q \left[ (s\varphi_j)^{-1}(|D_1p_j|^2 + |D^2p_j|^2) + s\varphi_j|Dp_j|^2 + s^3\varphi_j^3|p_j|^2 \right] e^{2\sigma_\alpha} dx dt \right\}
\leq C \left[ \int_{Q_{\omega_0}} s^3\varphi_0^3|p_0|^2 e^{2\sigma_\alpha} dx dt + \sum_{i=1}^n \left( \int_{Q_{\omega_i}} s^3\varphi_i^3|p_i|^2 e^{2\sigma_\alpha} dx dt \right) 
+ \sum_{j=0}^n \left( \int_{Q_{\omega_j}} |g_j|^2 e^{2\sigma_\alpha} dx dt \right) \right].
\end{align}

At this point we have to properly estimate the terms containing \( p_i \) on \( \tilde{\omega}_i, i \in \overline{1,n} \) from the right hand-side in terms of the component \( p_0 \) observed on \( \tilde{\omega}_0 \). For this purpose we will use the first equation of (2.8) considered on \( \omega_i \cap \omega_0 \), which by hypothesis (2.7) is coupled only to \( p_i \):
\begin{align}
(3.22) \quad D_1p_0 + \Delta p_0 + c_0p_0 + a_{i0}p_i = g_0 \quad \text{in} \quad (0,T) \times \omega_i \cap \omega_0.
\end{align}

Consider the cutoff functions \( \gamma_i, i \in \overline{1,n} \) with the properties
\begin{align}
\gamma_i \in C^\infty_0(\omega_i), \quad |\gamma_i| \leq 1, \quad \text{supp} \gamma_i = \overline{\omega_i} \\
\gamma_i = \text{sign} \left( a_{i0} |_{\omega_i} \right) \quad \text{on} \quad \tilde{\omega}_i, \quad \gamma_i \neq 0 \quad \text{in} \quad \omega_i,
\end{align}
where \( \text{sign} \left( a_{i0} \right) \) is the sign of \( a_{i0} \) in \( \omega_i \), which, by hypothesis (2.7) and continuity is nonzero and constant in \( \tilde{\omega}_i \). Multiply, scalarly in \( L^2(Q_{\omega_0}) \), the equation (3.22) by \( \gamma_i s^3\varphi_i^3 p_i e^{2\sigma_\alpha} \):
\begin{align}
(3.23) \quad \int_{Q_{\omega_i}} \gamma_i a_{i0}(x)s^3\varphi_i^3 |p_i|^2 e^{2\sigma_\alpha} dx dt 
= \int_{Q_{\omega_i}} \gamma_i s^3\varphi_i^3 (-c_0p_0 - D_1p_0 - \Delta p_0 - g_0)p_i e^{2\sigma_\alpha} dx dt.
\end{align}
We use (2.7) to say that there exists a constant such that
\[
\delta \int_{Q_{\omega_i}} s^3 \varphi_i^3 |p_i|^2 e^{2\alpha_1} \, dx \, dt \leq \int_{Q_{\omega_i}} |a_{i0}(x)| s^3 \varphi_i^3 |p_i|^2 e^{2\alpha_1} \, dx \, dt 
\]
\[
\leq \int_{Q_{\omega_i}} a_{i0}(x) s^3 \varphi_i^3 |p_i|^2 e^{2\alpha_1} \, dx \, dt. 
\]
(3.24)

We estimate each term from the right hand-side of (3.23) using the properties of \(\gamma_j, j \in \{0, n\}\). Let \(C > 0\) denoting various constants depending on \(\delta, M\) and \(\omega_i, \tilde{\omega}_i\).

For the first term in right side of (3.23) we have:
\[
\left| \int_{Q_{\omega_i}} \gamma_i s^3 \varphi_j^3 (-c_0 p_0) p_i e^{2\alpha_1} \, dx \, dt \right|
\]
\[
\leq M \left( \int_{Q_{\omega_i}} s^2 \varphi_j^2 |p_i|^2 e^{2\alpha_1} \, dx \, dt \right)^{\frac{1}{2}} \left( \int_{Q_{\omega_i}} s^4 \varphi_i^4 |p_0|^2 e^{2\alpha_1} \, dx \, dt \right)^{\frac{1}{2}} 
\]
\[
\leq \int_{Q_{\omega_i}} s^2 \varphi_j^2 |p_i|^2 e^{2\alpha_1} \, dx \, dt + M^2 \int_{Q_{\omega_i}} s^4 \varphi_i^4 |p_0|^2 e^{2\alpha_1} \, dx \, dt. 
\]
(3.25)

The same computation gives an estimate for the term involving the source:
\[
\left| \int_{Q_{\omega_i}} \gamma_i s^3 \varphi_j^3 (-g_0) p_i e^{2\alpha_1} \, dx \, dt \right|
\]
\[
\leq \int_{Q_{\omega_i}} s^2 \varphi_j^2 |p_i|^2 e^{2\alpha_1} \, dx \, dt + M^2 \int_{Q_{\omega_i}} s^4 \varphi_i^4 |g_0|^2 e^{2\alpha_1} \, dx \, dt. 
\]
(3.26)

Observe now that we have the following estimates for the weight functions, with a constant \(cst\) not depending on \(s\):
\[
|\gamma i s^3 D_\alpha (e^{2\alpha_1} \varphi_i^3)| = |\gamma i s^3 (e^{2\alpha_1} 2s D_\alpha \varphi_i^3 + 3e^{2\alpha_1} \varphi_i^2 D_\alpha \varphi_i)| \leq cst e^{2\alpha_1} s^5 \varphi_i^3 
\]
and
\[
|s^3 \Delta (e^{2\alpha_1} \varphi_i^3)| \leq cst s^3 \varphi_i^3 (s^2 \varphi_i^2 |p_i| + s \varphi_i |\nabla p_i| + |\Delta p_i|) e^{2\alpha_1}. 
\]
(3.28)

We now proceed with estimating the second term in (3.23) using, as usually in Carleman estimates, integration by parts:
\[
\left| \int_{Q_{\omega_i}} \gamma_i s^3 \varphi_j^3 (-D(p_0) p_0 e^{2\alpha_1}) \, dx \, dt \right| = \left| \int_{Q_{\omega_i}} s^3 D_\alpha (\varphi_j^3 p_i e^{2\alpha_1}) p_0 \, dx \, dt \right|
\]
\[
\leq \left| \int_{Q_{\omega_i}} s^3 D_\alpha (\varphi_j^3 e^{2\alpha_1}) p_i p_0 \, dx \, dt \right| + \left| \int_{Q_{\omega_i}} s^3 \varphi_j^3 e^{2\alpha_1} D_\alpha p_0 \, dx \, dt \right|
\]
\[
\leq C \left| \int_{Q_{\omega_i}} e^{2\alpha_1} s^5 \varphi_i^5 p_j p_0 \, dx \, dt \right| + \left| \int_{Q_{\omega_i}} e^{2\alpha_1} s^3 \varphi_i^3 D_\alpha p_0 \, dx \, dt \right|
\]
\[
\leq \int_{Q_{\omega_i}} s^2 \varphi_j^2 |p_i|^2 e^{2\alpha_1} \, dx \, dt + C \int_{Q_{\omega_i}} s^8 \varphi_i^8 |p_0|^2 e^{2\alpha_1} \, dx \, dt 
\]
\[
+ \int_{Q_{\omega_i}} (s \varphi_i^2)^{-1} |D(p_0)|^2 e^{2\alpha_1} \, dx \, dt + C \int_{Q_{\omega_i}} s^8 \varphi_i^8 |p_0|^2 e^{2\alpha_1} \, dx \, dt. 
\]
(3.29)
We proceed now with estimating the third term in right hand side of (3.23):

\[
\int_{Q_{\omega}} \gamma_i s^3 \varphi_i^3 (-\Delta p_i) p_i e^{2s\alpha_1} \, dx \, dt = \int_{Q_{\omega}} s^3 \Delta (\gamma_i \varphi_i^3 p_i e^{2s\alpha_1}) p_0 \, dx \, dt \\
\leq C \int_{Q_{\omega}} s^3 \varphi_i^3 (s^2 \varphi_i^2 |p_i| + s \varphi_i |\nabla p_i| + |\Delta p_i|) e^{2s\alpha_1} \, p_0 \, dx \, dt
\]

(3.30)

\[
\leq \int_{Q_{\omega}} [s^2 \varphi_i^2 |p_i|^2 + |\nabla p_i|^2 + (s \varphi_i)^{-2} |\Delta p_i|^2] e^{2s\alpha_1} \, dx \, dt \\
+ C \int_{Q_{\omega}} s^8 \varphi_i^8 |p_0|^2 e^{2s\alpha_1} \, dx \, dt.
\]

Using (3.26), (3.29), (3.30) and (3.31) we have, for \(i \in \mathbb{T}, n\) that

\[
\int_{Q_{\omega}} s^3 \varphi_i^3 |p_i|^2 e^{2s\alpha_1} \, dx \, dt \leq C \int_{Q_{\omega}} s^8 \varphi_i^8 |p_0|^2 e^{2s\alpha_1} \, dx \, dt
\]

(3.31)

\[
+ \int_{Q_{\omega}} [(s \varphi_i)^{-2} (|\Delta p_i|^2 + |D_i p_i|^2) + s^2 \varphi_i^2 |p_i| + |\nabla p_i|^2] e^{2s\alpha_1} \, dx \, dt \\
+ C \sum_{i=1}^{n} \int_{Q_{\omega}} s^4 \varphi_i^4 |g_0|^2 e^{2s\alpha_1} \, dx \, dt.
\]

Going back to (3.21), we have

\[
\sum_{j=0}^{n} \left\{ \int_{Q} [(s \varphi_j)^{-1} (|D_t p_j|^2 + |D^2 p_j|^2) + s \varphi_j |D p_j|^2 + s^3 \varphi_j^3 |p_j|^2] e^{2s\alpha_1} \, dx \, dt \right\} \\
\leq C \int_{Q_{\omega}} s^3 \varphi_0^3 |p_0|^2 e^{2s\alpha_0} \, dx \, dt + C \sum_{i=1}^{n} \left( \int_{Q_{\omega}} s^8 \varphi_i^8 |p_0|^2 e^{2s\alpha_1} \, dx \, dt \\
+ \int_{Q_{\omega}} [(s \varphi_i)^{-2} (|D^2 p_i|^2 + |D_t p_i|^2) + s^2 \varphi_i^2 |p_i|^2 + |D p_i|^2] e^{2s\alpha_1} \, dx \, dt \right) \\
+ C \sum_{i=1}^{n} \int_{Q_{\omega}} s^4 \varphi_i^4 |g_0|^2 e^{2s\alpha_1} \, dx \, dt + C \sum_{j=0}^{n} \int_{Q} |g_j|^2 e^{2s\alpha_1} \, dx \, dt.
\]

We now absorb the integral terms containing \(p_i\) in the right hand side into the corresponding higher order terms in the left side of the above inequality, by increasing \(s\) and taking it big enough. We obtain:

\[
\sum_{j=0}^{n} \left\{ \int_{Q} [(s \varphi_j)^{-1} (|D_t p_j|^2 + |D^2 p_j|^2) + s \varphi_j |D p_j|^2 + s^3 \varphi_j^3 |p_j|^2] e^{2s\alpha_1} \, dx \, dt \right\} \\
\leq C \int_{Q_{\omega}} s^3 \varphi_0^3 |p_0|^2 e^{2s\alpha_0} \, dx \, dt + C \sum_{i=1}^{n} \int_{Q_{\omega}} s^8 \varphi_i^8 |p_0|^2 e^{2s\alpha_1} \, dx \, dt \\
+ C \sum_{i=1}^{n} \int_{Q_{\omega}} s^4 \varphi_i^4 |g_0|^2 e^{2s\alpha_1} \, dx \, dt + C \sum_{j=0}^{n} \int_{Q} |g_j|^2 e^{2s\alpha_1} \, dx \, dt.
\]
Now we use Remark 2 in order to take a smaller weight in the left side and a greater one in the right side. Then there exist \( s_0 > 0 \) and \( C = C(M, \delta, \{\omega_j\}) \) such that the following Carleman estimate is true for all \( s \geq s_0 \):

\[
\sum_{j=0}^{n} \left[ \int_Q \left( |Dp_j|^2 + |D^2p_j|^2 + |Dp_j|^2 + |p_j|^2 \right) e^{2\sigma t} dxdt \right] \leq C \int_{Q_{s_0}} |p_0|^2 e^{2\sigma t} dxdt + C \int_Q |g|^2 e^{2\sigma t} dxdt.
\]

(3.34)

\[\square\]

Concerning the \( L^\infty - L^2 \) Carleman estimate for the solution of the adjoint problem (2.8) we proceed in the same way as in [5, 6] or [12]. We need to use the maximal regularity result in \( L^p \) spaces for parabolic problems (see [11]) and Sobolev embeddings for anisotropic Sobolev spaces which are contained in the following lemma:

**Lemma 2** ([11], Lemma 3.3). Let \( z \in W^{2,1}_r(Q) \). Then \( z \in Z_1 \) where

\[Z_1 = \begin{cases} L^s(Q) & \text{with } s \leq \frac{(N+2)r}{N+2-2r}, \quad \text{when } r < \frac{N+2}{2}, \\ L^s(Q) & \text{with } s \in [1, \infty), \quad \text{when } r = \frac{N+2}{2}, \\ C^{\alpha, \alpha/2}(Q) & \text{with } 0 < \alpha < 2 - \frac{N+2}{r}, \quad \text{when } r > \frac{N+2}{2}, \end{cases}\]

and there exists \( C = C(Q, p, N) \) such that

\[\|z\|_{Z_1} \leq C \|z\|_{W^{2,1}_r(Q)}\]

Using the above regularity result we consider the following sequence of numbers:

\[
\sigma_0 = 2, \quad \sigma_j := \begin{cases} \frac{(N+2)\sigma_{j-1}}{N+2-2\sigma_{j-1}}, & \text{if } \sigma_{j-1} < \frac{N+2}{2}, \\ \frac{2}{\alpha_j}, & \text{if } \sigma_{j-1} \geq \frac{N+2}{2}, \end{cases}
\]

(3.35)

such that by Lemma 2 we have

\[W^{2,1}_{\sigma_m-1}(Q) \subset L^p_m(Q)\]

Now, let us fix a \( \delta_1 > 0 \) and a sequence \( (q^j)_{j>0} \) defined by

\[q^j := \rho \varepsilon^{(s+j_1\delta_1)/2}\]

Then \( q^j = (q^j_1, \ldots, q^j_r)^\top \) is solution to the problem

\[
D_t q^j + \mathbf{A} q^j + C q^j + A_0 q^j = (s + j\delta_1) D_t \mathbf{q} q^j, \\
q^j(T) = 0.
\]

(3.36)

Observe that the right-hand side may be bounded in terms of \( q^{j-1} \), with some constant \( C_j = C_j(s, \delta_1) > 0 \), as follows

\[
(s + j\delta_1) D_t \mathbf{q} q^j = (s + j\delta_1) \frac{2T - T}{(T-t)^{1/2}} e^{\delta_1 T} q^{j-1} \leq C_j q^{j-1}.
\]

(3.37)

By maximal parabolic regularity (see [11]) we have

\[
\|q^j\|_{W^{2,1}_{\sigma_{j-1}}} \leq \bar{C}_j \|q^{j-1}\|_{L^r_{\sigma_{j-1}}}
\]

(3.38)
and using Sobolev type embedding from Lemma [2] we have that there exists a constant $K_j$ such that
\begin{equation}
\|q^j\|_{L^{\sigma_j}} \leq K_j \|q^{j-1}\|_{W^{2,1}_{\sigma_j-2}}.
\end{equation}
The sequence $(\sigma_m)_m$ is increasing to $+\infty$ and choose rank $m_0$ such that $\sigma_{m_0} > \frac{N+2}{\delta_1} \geq \sigma_{m_0-1}$. This implies that
\begin{equation}
W^{2,1}_{\sigma_{m_0}}(Q) \subset L^{\infty}(Q).
\end{equation}
From (3.38), (3.39) and (3.40), and with the use of (3.17), we have that there exists a constant $C > 0$ such that
\begin{equation}
\|p e^{(s+m_0\delta_1)\alpha}\|_{L^2(\Omega)} = \|q_{m_0}\|_{L^{\infty}(Q)} \leq C \|p_0\|_{L^{2}(\Omega)} e^{C t}, t \in (0,T).
\end{equation}

Remark 3. In order to obtain the observability inequality we proceed in the classical manner, by multiplying scalarly in $L^2(\Omega)$ each equation of the system (3.1) by $p_i$ and making use of dissipativity to find, for some constant $c > 0$ depending only on the coefficients of the system, the inequality:
\begin{equation}
\frac{1}{2} \frac{d}{dt} \|p\|_{L^2(\Omega)}^2 + c \|p\|_{L^2(\Omega)}^2 \geq 0,
\end{equation}
which gives
\begin{equation}
\|p(0)\|_{L^2(\Omega)}^2 \leq \|p(t)\|_{L^2(\Omega)}^2 e^{C t}, t \in (0,T).
\end{equation}
Consequently, for fixed $s > s_0$, we have that
\begin{equation}
\|p(0)\|_{L^2(\Omega)}^2 \leq \frac{T}{2} \int_0^T \|p(t)\|_{L^2(\Omega)}^2 e^{C t} dt \leq K(T,s) \int_0^T \|p(t)\|_{L^2(\Omega)}^2 e^{2s \alpha} dt.
\end{equation}
Now, by Carleman estimate (3.31) we obtain the observability inequality:
\begin{equation}
\|p(0,\cdot)\|_{L^2(\Omega)}^2 \leq C \int_{Q_{s_0}} |p_0|^2 e^{2s \alpha} dx dt,
\end{equation}
with a constant $C = C(T,s,\delta, M, \{\omega_i\}_i))$.

4. LINEAR SYSTEM: NULL CONTROLLABILITY

The main controllability result concerning linear system (2.6) is the following

Theorem 2. Consider system (2.6) with coefficients in $E_{M,\delta,\{\omega_i\}_i}$. Then there exists a constant $C = C(M,\delta, \{\omega_i\}_i)$ such that for all $z^0 \in H$ there exists $u^* \in L^2(0,T;L^2(\omega_0)) \cap L^\infty(Q_{\omega_0})$ which drives the corresponding solution to (2.6), $z = e^{-s^*\alpha} u^*$ in $\omega$ i.e. $z(T,\cdot) = 0$ and satisfies the norm estimate
\begin{equation}
\|u^* e^{-s^*\alpha}\|_{L^2(0,T;L^2(\omega_0))} + \|u^*\|_{L^\infty(Q_{\omega_0})} \leq C \|z^0\|_{L^2(\Omega)}.
\end{equation}
Proof.

$L^2(Q)$ control. In order to obtain norm estimates for the controls driving the trajectory to the linear system in 0, we consider a family of optimal control problems depending on a small parameter $\varepsilon > 0$:

\[
\inf_{u \in L^2(Q_\varepsilon, u)} \frac{1}{2} \int_{Q_\varepsilon} |u|^2 e^{-2\varepsilon t} \, dx dt + \frac{1}{2\varepsilon} \int_{\Omega} |z(T, \cdot)|^2 \, dx dt,
\]

with $z = z_u$ the solution of the linear controlled system (2.4). Classical results concerning optimal control with quadratic cost for parabolic equations insure existence of optimal control $u^\varepsilon$ which by Pontriagin maximum principle satisfy

\[
u^\varepsilon = e^{2\varepsilon t} B^* p^\varepsilon = e^{2\varepsilon t} p_0^\varepsilon|_{\omega_0},
\]

where $p^\varepsilon$ is solution to the adjoint system:

\[
\begin{aligned}
D_t p^\varepsilon &= -Ap^\varepsilon - C(t)p^\varepsilon - A_0^*(t)u^\varepsilon, \\
p^\varepsilon(T) &= -\frac{1}{2} z^\varepsilon(T).
\end{aligned}
\]

By cross multiplying the equations for $z^\varepsilon = z u^\varepsilon$ and $p^\varepsilon$ by $z^\varepsilon$ respectively $z^\varepsilon$ and integrating on $Q$ we obtain:

\[
\frac{d}{dt} \langle z^\varepsilon, p^\varepsilon \rangle_{L^2(Q)} = \langle (A + A_0 + C)z^\varepsilon + B u^\varepsilon, p^\varepsilon \rangle_{L^2(Q)} - \langle (A + A_0 + C)^* p^\varepsilon, z^\varepsilon \rangle_{L^2(Q)}.
\]

We integrate on $[0, T]$ and use the observability inequality (3.12) to get

\[
\frac{1}{\varepsilon} \|z^\varepsilon(T, \cdot)\|_{L^2(Q)}^2 + \langle u^\varepsilon, B^* p^\varepsilon \rangle_{L^2(Q)} = -\langle z^\varepsilon(0, \cdot), p^\varepsilon(0, \cdot) \rangle_{L^2(Q)} \\
\leq \|z^0\|_{L^2(Q)} \|p(0, \cdot)\|_{L^2(Q)} \leq C \|z^0\|_{L^2(Q)} \left( \int_{Q_\varepsilon} |p^\varepsilon|^2 e^{2\varepsilon t} \, dx dt \right)^{\frac{1}{2}}.
\]

Since $\langle u^\varepsilon, B^* p^\varepsilon \rangle_{L^2(Q)} = \int_{Q_\varepsilon} |p^\varepsilon|^2 e^{2\varepsilon t} \, dx dt$, using appropriately balanced Young’s inequality, we find that

\[
\frac{1}{\varepsilon} \|z^\varepsilon(T, \cdot)\|_{L^2(Q)}^2 + \frac{1}{2} \int_{Q_\varepsilon} |p^\varepsilon|^2 e^{2\varepsilon t} \, dx dt \leq C \|z^0\|_{L^2(Q)}^2,
\]

and gives by (4.3) the following estimate for the sequence of optimal controls $(u^\varepsilon)_\varepsilon$ and final state:

\[
\frac{1}{\varepsilon} \|z^\varepsilon(T, \cdot)\|_{L^2(Q)}^2 + \frac{1}{2} \int_{Q_\varepsilon} |u^\varepsilon|^2 e^{-2\varepsilon t} \, dx dt \leq C \|z^0\|_{L^2(Q)}^2.
\]

Now, this $L^2$ bound for the sequence $(u^\varepsilon)_\varepsilon$, allows to extract a subsequence, denoted for simplicity also $(u^\varepsilon)_\varepsilon$ weakly convergent in $L^2(Q)$ to a limit $u^*$. Write the corresponding solutions $(z^\varepsilon)_\varepsilon$ as

\[
z^\varepsilon = w^\varepsilon + v
\]

where $w^\varepsilon$ is solution to (2.4) with initial data $w^\varepsilon(0) = 0$ and $v$ solution to homogeneous equation

\[
D_t v = A v + (A_0 + C) v = 0, \quad v(0) = z^\varepsilon(0) = z^0.
\]
We have that the sequence \((w^\varepsilon)_\varepsilon\) is bounded in \(L^2(0,T;D(A))\) and the sequence of derivatives \((D_tw^\varepsilon)_\varepsilon\) is bounded in \(L^2(0,T;L^2(\Omega))\). By Aubin’s theorem we can extract a subsequence, denoted also \((w^\varepsilon)_\varepsilon\), strongly convergent in \(L^2(0,T;H^1_0(\Omega))\) to \(w \in L^2(0,T;H^1_0(\Omega)) \cap L^2(0,T;D(A))\). Consequently \((z^\varepsilon)\) is strongly convergent in \(L^2(0,T;H^1_0(\Omega))\) to \(z \in L^2(0,T;H^1_0(\Omega))\). We may now pass to the limit in the weak formulation of solutions to \((2.0)\), \((2.10)\); thus, for some test function \(\varphi \in [H^1_0(\Omega)]^{n+1}\), we have

\[
\begin{aligned}
&\langle z^\varepsilon(t,\cdot), \varphi \rangle_{L^2(\Omega)} - \langle z^0(0,\cdot), \varphi \rangle_{L^2(\Omega)} + \int_0^t \langle \nabla z^\varepsilon(t,\cdot), \nabla \varphi \rangle_{L^2(\Omega)} \, dt \\
&+ \int_0^t \langle (A_0 + C)z^\varepsilon, \varphi \rangle_{L^2(\Omega)} \, dt = \int_{(0,t) \times \omega_0} u^\varepsilon \varphi \, dx \, dt,
\end{aligned}
\]

and we find that \(z \in L^2(Q)\) is solution to the problem \((2.10)\) with initial datum \(z^0 \in L^2(\Omega)\). In fact, by Arzelà-Ascoli theorem \(w^\varepsilon \to w\) in \(C([0,T],L^2(\Omega))\) and thus \(z(T) = 0\) and by weak lower semicontinuity of the \(L^2\) norm we also have the following estimate for the control driving the solution to 0:

\[
\int_{Q_{\omega_0}} |u^\varepsilon|^2 e^{-2\alpha t} \, dx \, dt \leq C \|z^0\|_{L^2(\Omega)}^2.
\]

where \(C = C(T, s, s_1, M, \delta, \{\omega_j\}_j)\).

**\(L^\infty(Q)\)-control.** Regarding the \(L^\infty\) norm estimates for the sequence \((u^\varepsilon)_\varepsilon\) and also for \(u^*\) we will use the results from the previous section:

\[
\|u^\varepsilon e^{-2s + (s + m_0\delta_1)\alpha} \|_{L^\infty(Q_{\omega_0})} = \|p_0^\varepsilon e^{(s + m_0\delta_1)\alpha} \|_{L^\infty(Q_{\omega_0})} \leq C \|z^0\|_{L^2(Q)}.
\]

Now we see that we could start from the beginning with \(\lambda\) big enough such that \((1.13)\) holds and in consequence

\[
2s \alpha \leq (s + m_0\delta_1) \alpha.
\]

As \(-2s \alpha \leq (s + m_0\delta_1) \alpha > 0\), by passing to \(L^\infty\) weak-* limit in \((4.9)\), we find that

\[
\|u^*\|_{L^\infty(Q_{\omega_0})} \leq \|u^* e^{-2s \alpha (s + m_0\delta_1) \alpha} \|_{L^\infty(Q_{\omega_0})} \leq C \|z^0\|_{L^2(Q)},
\]

which concludes \((4.11)\).

5. Nonlinear system: local exact controllability

We prove in this section the following local controllability result concerning system \((2.1)\):

**Theorem 3.** Suppose \(\overline{y}\) is a stationary state, i.e. solution to \((2.2)\), and that the functions \(f_j, j \in 0,n\) satisfy hypotheses \((H1), (H2)\). Then, for all \(\beta_0 > 0\) there exist \(\zeta_0 = \zeta_0(\beta_0) > 0\) and \(C = C(\beta_0, (\omega_j)_j, \overline{\gamma})\) such that if \(\|u^0(0) - \overline{y}\| < \zeta_0\) there exists a control \(u \in L^\infty(Q)\) satisfying

\[
\|u\|_{L^\infty(Q)} \leq C \|u^0(0) - \overline{y}\|_{L^\infty(\Omega)}
\]
and
\[ y''(T, \cdot) = \overline{y}, \]
with \[ \|y(t, \cdot) - \overline{y}\|_{L^\infty} \leq \beta_0, \quad t \in [0, T]. \]

**Proof.** The approach to the local null controllability of the system around the stationary solution is based on the Kakutani fixed point theorem.

For this aim, with a solution \( y \) to (2.1), we consider the system satisfied by \( z := y - \overline{y} \), written as a linear system
\[
\begin{aligned}
D_t z_0 - \Delta z_0 &= c_0^j(t, x)z_0 + \chi_{\omega_0} u, & (0, T) \times \Omega, \\
D_t z_i - \Delta z_i &= a_i^0(t, x)z_0 + c_i^j(t, x)z_i, & i \in \overline{1, n}, (0, T) \times \Omega, \\
0 = \ldots = z_n = 0, & (0, T) \times \partial \Omega, \\
0 = z(x) := y(0, x) - \overline{y}(x), & x \in \Omega,
\end{aligned}
\]
where the nonlinearity is hidden into the coupling coefficients which are defined by:
\[
\begin{aligned}
a_i^0(t, x) &= \int_0^1 \frac{\partial}{\partial y_0} f_i(x, \overline{y}_0(x) + \tau z_0(t, x), \overline{y}_i(x) + \tau z_i(t, x))d\tau, \quad i \in \overline{1, n} \\
c_i^j(t, x) &= \int_0^1 \frac{\partial}{\partial y_j} f_j(x, \overline{y}_0(x) + \tau z_0(t, x), \overline{y}_i(x) + \tau z_i(t, x))d\tau, \quad j \in \overline{0, n}.
\end{aligned}
\]

Observe that \( \{a_i^0, c_i^j\} \in \mathcal{T}_{\overline{1, n}} \) are the coefficients of the linearized system around the stationary solution \( \overline{y} \) as
\[
a_i^0(x) = \frac{\partial}{\partial y_0} f_i(x, y_0(x), y_i(x)), \\
c_i^j(x) = \frac{\partial}{\partial y_j} f_j(x, y_0(x), y_i(x)), c_0^0 = \frac{\partial}{\partial y_0} f_0(x, y_0(x)).
\]

We see now that hypotheses \( (2.4) \) and \( (2.5) \) tell us that we may choose \( M_0, \delta_0 > 0 \) and \( \omega_0 \subset \subset (\omega_i \cap \omega_0) \setminus \bigcup_{j \neq 0, i} \omega_j \) such that
\[
\{a_i^0, c_i^j\} \in \mathcal{T}_{\overline{1, n}} \in \mathcal{E} \mathcal{M}_\delta, \delta_0, \{\omega_j\}_i.
\]

Let \( \beta > 0 \) and define \( \mathcal{M}_\beta \) to be:
\[
\mathcal{M}_\beta = \{\tilde{z} \in L^\infty(Q) : \|\tilde{z}\|_{L^\infty(Q)} \leq \beta\).
\]

For \( \tilde{z} \in \mathcal{M}_\beta \), we consider the coefficients \( a_i^{\tilde{z}}(0), c_i^j(\tilde{z}) \) defined as in \( (5.2) \) with \( z \) replaced by \( \tilde{z} \).

Observe now that we may choose \( \beta_0 > 0 \) small enough such that if \( \tilde{z} \in \mathcal{M}_\beta \), we have
\[
\{a_i^{\tilde{z}}, c_i^j\} \in \mathcal{E} \mathcal{T}_{\overline{1, n}} \in \mathcal{E} \mathcal{M}_\delta, \delta_0, \{\omega_j\}_i.
\]

Consider now the linear system \( (5.1) \) with coefficients \( \{a_i^{\tilde{z}}, c_i^j\} \):
\[
\begin{aligned}
D_t z_0 - \Delta z_0 &= c_0^j(t, x)z_0 + \chi_{\omega_0} u, & (0, T) \times \Omega, \\
D_t z_i - \Delta z_i &= a_i^{\tilde{z}}(t, x)z_0 + c_i^j(t, x)z_i, & i \in \overline{1, n}, (0, T) \times \Omega, \\
0 = \ldots = z_n = 0, & (0, T) \times \partial \Omega, \\
0 = z(x) := y(0, x) - \overline{y}(x), & x \in \Omega.
\end{aligned}
\]
The linear problem (5.6) may be reformulated as:
\[
\begin{align*}
D_t z &= A z + A_0^z(t) z + C^z(t) u + B u, \quad t > 0, \\
z(0) &= z^0,
\end{align*}
\]
where \(C^z(t) z = C_0^z(t) z(\cdot) + A_0^z(t) z(\cdot)\) and \(A_0^z(t,x) = \text{diag}(c_i^z(t,x))_{i=0}^n\) and the coupling matrix
\[
A_0^z(t,x) = (0, a_{10}^z(t,x), \ldots, a_{n0}^z(t,x))^\top \cdot (1, 0, \ldots, 0).
\]

Theorem 2 says that for \(\tilde{z} \in \mathcal{M}_{\beta_0}\) there exists a control \(u^* = u^*(\tilde{z}) \in L^2(0,T; L^2(\omega_0)) \cap L^\infty(Q_{\omega_0})\) satisfying the norm estimate
\[
J(u^*) := ||u^* e^{-\omega t}||_{L^2(0,T; L^2(\omega_0))} + ||u^*||_{L^\infty(Q_{\omega_0})} \leq C(2M_0, \delta_0/2, \{\omega_i\}_i) z^0 ||_{L^2(\Omega)},
\]
and driving the solution \(z^{u^*,\tilde{z}}\) of the linear system (5.6) in zero: \(z^{u^*,\tilde{z}}(T) = 0\). Observe that \(J\) is a norm in the space \(U^* := L^2_{e^{-\omega t}} \cap L^\infty(Q_{\omega_0})\).

We will write
\[
z^{u,\tilde{z}} = T^z_1(z^0) + T^z_2(u),
\]
where the first term is the solution to problem (5.6) with initial data \(z^0\) and the second term is the solution to system (5.6) with initial datum zero and control \(u\). Let us denote by
\[
S_1(z^0) = e^{t A} z^0, \quad S_2 h = e^t A^* h = \int_0^t e^{(t-s)A} h(s) ds,
\]
where \(h \in L^2(0,t; [L^2(\Omega)]^{n+1})\). With these notations
\[
z^{u,\tilde{z}} = T^z_1(z^0) + T^z_2(u) = S_1(z^0) + S_2(A_0^z z^{u,\tilde{z}} + C_0^z z^{u,\tilde{z}} + B u).
\]

Fix an initial datum \(z^0 \in L^\infty(\Omega)\). We define now the following set-valued map, associated to \(z^0\):
\[
F_{z^0} : \mathcal{M}_{\beta_0} \to 2^{L^\infty(Q)}
\]
\[
F_{z^0}(\tilde{z}) = \{z^{u,\tilde{z}} : u \text{ satisfies } (5.8) \text{ and } z^{u,\tilde{z}}(T) = 0 \} = \{T^z_1(z^0) + T^z_2(u) : z^{u,\tilde{z}}(T) = 0, J(u) \leq K ||z^0||_{L^2} \},
\]
where by \(K\) we denoted the constant in (5.8), \(K = C(2M_0, \delta_0/2, \{\omega_i\}_i)\).

In order to obtain local controllability of the nonlinear system it is enough to find a fixed point for \(F_{z^0}\). We achieve this goal by applying Kakutani fixed point theorem to \(F_{z^0}\) in \(\mathcal{M}_{\beta_0}\); we have thus to verify the following statements:

i) For every \(\tilde{z} \in \mathcal{M}\), \(F_{z^0}(\tilde{z})\) is a nonempty, closed and convex subset of \(L^\infty(Q)\);

Observe that \(z^{u(\tilde{z})} \in F_{z^0}(\tilde{z})\) and thus \(F_{z^0}(\tilde{z}) \neq \emptyset\). Convexity comes from linearity of \(T_2\) and convexity of \(J\).

To prove that \(F_{z^0}(\tilde{z})\) is closed, suppose \(z^m \in F_{z^0}(\tilde{z})\), \(z^m \to z\) in \(L^\infty\). We have to prove that \(z \in F_{z^0}(\tilde{z})\). Indeed, we have that
\[
z^m = T^z_1(z^0) + T^z_2(u^m)
\]
for some controls \( u^m \in \mathcal{U}^* \) satisfying estimate \( J(u^m) \leq K\|z^0\|_{L^2} \).

We may now invoke Aubin-Lions and Ascoli-Arzelà compactness results (see e.g. [14]) applied to the solution operator of a parabolic initial boundary value problem and thus to say that \( T_2 \) is a compact operator from \( L^2(0,T;L^2(\Omega_{\omega})) \) to \( C([0,T];[L^2(\Omega)]^{n+1}) \cap L^2(0,T;[H^1_0(\Omega)]^{n+1}) \). Thus, extracting subsequence \( u^m \rightharpoonup u \) weakly in \( L^2(Q_{\omega}) \) we find

\[
z^m \to z \quad \text{in} \quad C([0,T];[L^2(\Omega)]^{n+1}) \cap L^2(0,T;[H^1_0(\Omega)]^{n+1})
\]

with \( z(T) = 0 \) since \( z_m(T) = 0 \). Thus \( z \in F_{2,0}(\tilde{z}) \).

ii) There exists \( \zeta_0 = \zeta_0(\tilde{\beta}_0) \) such that for \( \|z^0\|_{L^\infty(\Omega)} < \zeta_0 \) we have

\[
F_{2,0}(\mathcal{M}_{\beta_0}) \subset \mathcal{M}_{\beta_0}.
\]

This follows from the a priori estimates for solutions to initial boundary value problems for parabolic systems:

\[
\|T_2^z(z_0)\|_{L^\infty(\Omega)} \leq C_1(\|\tilde{\zeta}\|_{L^\infty(\Omega)})\|z^0\|_{L^\infty(\Omega)},
\]

\[
\|T_2^z(u)\|_{L^\infty(\Omega)} \leq C_2(\|\tilde{\zeta}\|_{L^\infty(\Omega)})\|u\|_{L^\infty(Q_{\omega})}
\]

and from the remark that both constants depend in fact uniformly on the \( L^\infty \) norm of the coupling coefficients and thus depend uniformly on the norm of \( \tilde{z} \) in \( L^\infty \).

iii) The set \( F_{2,0}(\mathcal{M}_{\beta_0}) \) is imbedded into a convex and compact subset of \( \mathcal{M}_{\beta_0} \).

Indeed, as \( \mathcal{M}_{\beta_0} \) is closed and convex, it is enough to prove that \( F_{2,0}(\mathcal{M}_{\beta_0}) \) is relatively compact in \( L^\infty \) topology. For this, take a sequence \( z^m \in F_{2,0}(\mathcal{M}_{\beta_0}) \). Correspondingly, there exist \( \tilde{z}^m \in \mathcal{M}_{\beta_0} \) with \( z^m \in F_{2,0}(\tilde{z}^m) \). Take corresponding controls \( u^m \in \mathcal{U}^* \) such that (see definition of \( F_{2,0} \) and (5.11)):

\[
(5.13) \quad z^m = T_1^{z^m}(z^0) + T_2^{z^m}(u^m) = S_1(z^0) + S_2(A_0^{z^m} z^m + C_0^{z^m} z^m + Bu^m).
\]

We have the following bounded sequences

\[
\begin{align*}
\tilde{z}^m & \in \mathcal{M}_{\beta_0} \quad \text{and so} \quad A_0^{\tilde{z}^m}(Q), C_0^{\tilde{z}^m}(Q) \quad \text{are bounded in} \quad L^\infty; \\
z^m & \in \mathcal{M}_{\beta_0} \quad \text{and thus bounded in} \quad L^\infty(\Omega); \\
u^m & \in \mathcal{U}^* \quad \text{is bounded in} \quad L^\infty(\Omega).
\end{align*}
\]

Consequently \( A_0^{\tilde{z}^m} z^m + C_0^{\tilde{z}^m} z^m + Bu^m \) is bounded in \( L^p(Q), p > 1 \). By parabolic regularity (see [11]), \( S_2(A_0^{\tilde{z}^m} z^m + C_0^{\tilde{z}^m} z^m + Bu^m) \) is bounded in any \( W^2_0, \forall p, 1 < p < \infty \) (the space of anisotropic Sobolev functions). For \( p \) big enough we have \( W^2_0 \subset C^{0,\alpha}(Q) \) for some \( 0 < \alpha < 1 \) (the space of Hölder continuous functions). \( C^{0,\alpha}(Q) \) is compactly imbedded in \( C(\overline{Q}) \). Consequently \( (z^m)_m \) is a relatively compact sequence in \( L^\infty(Q) \).

iv) \( F_{2,0} \) is upper semi-continuous, i.e. if \( z^m \to z, \tilde{z}^m \to \tilde{z} \) in \( L^\infty \) and \( z^m \in F_{2,0}(\tilde{z}^m) \) then \( z \in F_{2,0}(\tilde{z}) \).

Indeed we have (see [11,2]) that \( A_0^{\tilde{z}^m} \to A_0^{\tilde{z}}, C_0^{\tilde{z}^m} \to C_0^{\tilde{z}} \) in \( L^\infty \) and as \( (z^m)_m \) is relatively compact in \( C([0,T];[L^2(\Omega)]^{n+1}) \) we may pass to the limit in (5.13) and find that \( z \in F_{2,0}(\tilde{z}) \).

Now we conclude the proof by Kakutani fixed point theorem, which insures existence of \( z \in \mathcal{M}_{\beta_0} \) such that \( z \in F_{2,0}(z) \) i.e. there exists \( \pi \in \mathcal{U}^* \).
such that $z^\pi z = z$. In conclusion $y^\pi := y + z$ is the solution to the controlled system \((2.1)\) with control $\pi$ satisfying $y^\pi(T) = y$.

6. Parabolic systems with tree-like couplings. Null controllability.

The case of tree-type couplings is more technical to describe in the context of the needed hypotheses on the supports of coupling functions or coupling coefficients in the linear models. These hypotheses are essential for the construction of appropriate auxiliary and weight functions in the corresponding Carleman estimates which are established for each equation associated to a node in the graph, estimates which in the end should couple well into a global observability estimate.

The hypotheses we impose to the supports of the coupling coefficients allow to treat each equation corresponding to a node of the tree as the center of a star-like system together with the directly actuated variables and corresponding equations. The star-like sub-graphs at the same level of the tree should be, in some sense, independently actuated.

We will say that a controlled linear parabolic system has a tree-type coupling in zero order terms if the system has the form:

\[
\begin{aligned}
D_t z_0 - \Delta z_0 &= c_0(t, x)z_0 + \chi_{\omega_0} u, & \text{in } (0, T) \times \Omega, \\
D_t z_i - \Delta z_i &= a_{ik(i)}(t, x)z_{k(i)} + c_i(t, x)z_i, & i \in \overline{1, n}, \\
z_0 = ... = z_n &= 0, & \text{on } (0, T) \times \partial\Omega, \\
z(0, \cdot) &= z^0, \\
\end{aligned}
\]

with the following assumptions on the function $k : \{1, \ldots, n\} \to \{0, 1, \ldots, n\}$:

\[
\forall i \in \{1, \ldots, n\}, \exists m = m(i), 1 \leq m \leq n - 1, (k^o)^{m(i)} = k \circ \ldots \circ k(i) = 0.
\]

The linear problem \((6.1)\) may be reformulated as:

\[
\begin{aligned}
D_t z &= A z + A_0(t) z + C(t) z + B u, & t > 0, \\
z(0) &= z^0,
\end{aligned}
\]

where $C(t) z = C_0(t, \cdot) z(\cdot)$ and $A_0(t) z = A_0(t, \cdot) z(\cdot)$ with

\[
C_0(t, x) = \text{diag}(c_i(t, x))_{i=0,n},
\]

and the coupling matrix

\[
A_0(t, x) = (a_{ij})_{i,j \in \overline{1, n}} = (a_{ik(i)}\delta_{k(i)})_{i,j \in \overline{1, n}},
\]

where we denoted by $\delta_{ij}$ the Kronecker symbol. Denote by

\[
I_j = k^{-1}(j) = \{i \in \overline{1, n} : k(i) = j\}.
\]

Fix now a family of open subsets $\omega_i \subset \Omega, i \in \overline{1, n}$ such that

\[
D_i := \omega_i \cap \omega_{k(i)} \cap \cdots \cap \omega_{(k^o)^{m(i)}(i)} \neq \emptyset.
\]

\[
D_i \ \setminus \ \bigcup_{j \neq i, k(j) = k(i)} \omega_j \neq \emptyset.
\]
Choose further a family of open subsets \( \{ \omega_j \}_{j \in 0,n} \) with the properties
\[
\omega_0 \subset \subset \omega_j, \quad \omega_j \subset \subset D_i \setminus \bigcup_{l \neq i, k(l) = k(i)} \omega_l,
\]
(6.6)
\[
\omega_j \subset \subset \omega_{k(i)}, \quad j \in \overline{1,n}.
\]
(6.7)
For \( M, \delta > 0 \), and the family of open subsets described above \( \{ \omega_j \}_i \), we introduce the following classes of coefficients sets:
\[
\mathcal{E}_{M,\delta, \{ \omega_j \}, k} = \left\{ E = \{ a_{ik(i)}, c_j \}_{i \in \overline{1,n}, j \in \overline{0,n}} : a_{ik(i)}, c_j \in L^\infty(Q), \right. \\
\left. \| a_{ik(i)} \|_{L^\infty}, \| c_j \|_{L^\infty} \leq M, a_{ik(i)} = 0 \text{ in } Q \setminus Q_{\omega_i}, \text{ and } |a_{ik(i)}| \geq \delta \text{ on } Q_{\omega_i} \right\}.
\]
(6.8)
In order to study controllability we consider the system adjoint to system (6.1):
\[
\begin{cases}
-D_t p_j - \Delta p_j - c_j(t,x)p_j = \sum_{l,k(l) = i} a_{lj}(t,x)p_l = \mathcal{N}_j(t,x), \quad j \in \overline{0,n}, \quad \text{in } Q, \\
p_0 = \ldots = p_n = 0, \quad \text{on } (0,T) \times \partial \Omega,
\end{cases}
\]
(6.9)
where for simplicity of further calculations we denoted by
\[
\mathcal{N}_j(t,x) = \sum_{l,k(l) = i} a_{lj}(t,x)p_l(t,x).
\]
(6.10)
As we have seen in the previous sections all controllability results have as essential ingredient an appropriate Carleman inequality for the adjoint system. For obtaining such estimates it is essential to have corresponding auxiliary functions which appear in the construction of the weights. We describe this in what follows.
Consider again open subsets
\[
\tilde{\omega}_j \subset \subset \omega_j, \quad j \in \overline{0,n},
\]
and auxiliary functions
\[
\eta_j \in C^2(\overline{\Omega}), \quad 0 < \eta_j \text{ in } \Omega, \quad \eta_j|_{\partial \Omega} = 0, \quad \{ x \in \overline{\Omega} : |\nabla \eta_j(x)| = 0 \} \subset \subset \tilde{\omega}_j, \quad j \in \overline{0,n}.
\]
We construct now the weight functions entering the various Carleman estimates, with the following properties:

i) \( \psi_{j,f}, j \in \overline{0,n}, I_j \neq \emptyset, \psi_{i,s}, i \in \overline{1,n} \) are defined by
\[
\psi_{j,f} := \eta_j + K_j, \quad \psi_{i,s} := \eta_i + \tilde{K}_i,
\]
(6.10)
for some fixed positive constants \( K_j, \tilde{K}_i > 0 \) and such that for a fixed \( \epsilon > 0 \) we have
\[
\psi_{i,s} > \sup_{\overline{\Omega}} \psi_{j,f} + 2\epsilon, \quad \forall i \in I_j, I_j \neq \emptyset;
\]
(6.11)
\[
\psi_{i,f} > \sup_{\overline{\Omega}} \{ \psi_{i,s} : k(l) = k(i) \} + 2\epsilon, \quad \forall i \in \overline{1,n}, I_i \neq \emptyset;
\]
(6.12)
ii)
\[
\sup_{\overline{\Omega}} \psi_{j,f} < \frac{8}{7}, \quad \inf_{\overline{\Omega}} \psi_{i,s} < \frac{8}{7}.
\]
(6.13)
iii) For \( j \in \overline{0,n} \) such that \( I_j \neq \emptyset \) we define

\[
\overline{\psi}_j = \sup\{\overline{\psi}_{j,i}(x), \psi_{i,s}(x) : i \in I_j, x \in \Omega\} + \epsilon,
\]

\[
\underline{\psi}_j = \inf\{\overline{\psi}_{j,i}(x), \psi_{i,s}(x) : i \in I_j, x \in \Omega\} - \epsilon.
\]

iv) Denote by \( \overline{\psi} = \sup\{\overline{\psi}_j : I_j \neq \emptyset\} \) and \( \underline{\psi} = \inf\{\underline{\psi}_j : I_j \neq \emptyset\} \) and

\[
\overline{\varphi}_j(t) = \overline{\varphi}_j(t) := \frac{e^{\lambda \overline{\psi}_j}}{t(T-t)} - \frac{e^{1.5\lambda \overline{\psi}_j}}{t(T-t)},
\]

\[
\varphi_j(t) = \varphi_j(t) := \frac{e^{\lambda \varphi_j}}{t(T-t)} - \frac{e^{1.5\lambda \varphi_j}}{t(T-t)}.
\]

\[
\underline{\varphi}(t) = \frac{e^{\lambda \underline{\psi}} - e^{1.5\lambda \overline{\psi}}}{t(T-t)}, \quad \underline{\varphi}(t) = \frac{e^{\lambda \underline{\psi}} - e^{1.5\lambda \overline{\psi}}}{t(T-t)}.
\]

**Remark 4.** Observe that this construction of the weight functions allows saying that

\[
\overline{\psi}_j < \underline{\psi}_i, i \in I_j, I_j \neq \emptyset,
\]

and thus, given \( \theta > 0 \) there exists \( s(\theta) \) such that for \( s > s(\theta) \) we have

\[
e^{\theta \overline{\psi}_j(t)} \leq \theta e^{s \underline{\psi}_i(t)}, i \in I_j, I_j \neq \emptyset, t \in [0,T].
\]

The Carleman estimates we establish now in the tree coupling case are given in the following theorem:

**Theorem 4.** Suppose that the coupling coefficients in (6.9) satisfy

\[
\{\alpha_{ik(i)}, c_j\}_{i \in \overline{0,n}, j \in \overline{0,m}} \in E_{m,k}, \{\varphi_j\}, k.
\]

Then there exist constants \( \lambda_0, s_0 \) such that for \( \lambda > \lambda_0 \) there exists a constant \( C > 0 \) depending on \( \{M, \delta, \{\varphi_j\}, \lambda\} \) such that, for any \( s \geq s_0 \), the following inequality holds:

\[
\int_Q (|D_1 p|^2 + |D_2 p|^2 + |D p|^2 + |p|^2)e^{2s \underline{\varphi}} dx dt \leq C \int_{Q_{s_0}} |p_0|^2 e^{2s \underline{\varphi}} dx dt
\]

for all \( p \in H^1(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)) \) solution of (6.9).

Moreover, there exists \( m_0 \in \mathbb{N} \) and \( \delta_1 > 0 \) such that we have the following \( L^\infty - L^2 \) Carleman estimate

\[
||p e^{(s+m_0\delta_1)\overline{\varphi}}||_{L^\infty(Q)} \leq C ||p_0 e^{s\underline{\varphi}}||_{L^2(Q_{s_0})}.
\]

**Proof.** For \( j \in \overline{0,n} \) we write separately Carleman inequalities for the case \( I_j \neq \emptyset \) and respectively for the case \( I_j = \emptyset \). If \( j \in \overline{0,n} \) is such that \( I_j \neq \emptyset \) we treat the equations satisfied by \( p_j \) and \( p_l, l \in I_j \) as a nonhomogeneous adjoin system, as in the star-like couplings (3.4), while in the case \( I_j = \emptyset \) we have to deal with homogeneous parabolic equations:

\[
\left\{\begin{array}{ll}
-D_t p_j - \Delta p_j - c_j(t,x)p_j = \sum_{i,k(l) = j} a_{ij}(t,x)p_i, & \text{in } (0,T) \times \Omega, \\
-D_t p_l - \Delta p_l - c_l(t,x)p_l = N_l(t,x), & \text{in } (0,T) \times \Omega,
\end{array}\right.
\]

\[
l \in I_j.
\]
For the case $I_j \neq \emptyset$ a Carleman estimate, which is an immediate consequence to intermediate estimate (3.21), states that there exists $\overline{s}_j$ and $C > 0$ not depending on $s$ such that for $s > \overline{s}_j$ we have

\[
\int_Q (|D_t p_j|^2 + |D^2 p_j|^2 + |Dp_j|^2 + |p_j|^2)e^{2s\omega_j} \, dx \, dt \\
+ \int_Q \left( \sum_{i \in I_j} (|D_t p_i|^2 + |D^2 p_i|^2 + |Dp_i|^2 + |p_i|^2) \right) e^{2s\omega_i} \, dx \, dt
\]

\[
\leq C \left[ \int_{Q_{\overline{s}_j}} |p_j|^2 e^{2s\omega_j} \, dx \, dt + \sum_{i \in I_j} \int_{Q_{\overline{s}_j}} |p_i|^2 e^{2s\omega_i} \right] \]

\[
+ C \sum_{i \in I_j} \int_Q |N_i(t, x)|^2 e^{2s\omega_i} \, dx \, dt
\]

\[
\leq C \left[ \int_{Q_{\overline{s}_j}} |p_j|^2 e^{2s\omega_j} \, dx \, dt + \sum_{i \in I_j} \int_{Q_{\overline{s}_j}} |p_i|^2 e^{2s\omega_i} \right] \, dx \, dt
\]

\[
+ C \sum_{i \in I_j, j \in I_i} \int_Q \theta |p_j(t, x)|^2 e^{2s\omega_j} \, dx \, dt,
\]

where we have used Remark 4 in order to say that $e^{2s\omega_j} \leq \theta e^{2s\omega_i} \leq \theta e^{2s\omega_j}$ for $\theta > 0$ to be fixed later and $s > s(\theta)$ big enough.

In the case $I_j = \emptyset$, we write the Carleman estimate for the homogeneous equation

\[-D_t p_j - \Delta p_j - c_j(t, x)p_j = 0.\]

So, there exist constants $\overline{s}_j > 0$ and $C > 0$ such that for $s > \overline{s}_j$

\[
\int_Q (|D_t p_j|^2 + |D^2 p_j|^2 + |Dp_j|^2 + |p_j|^2)e^{2s\omega_j} \, dx \, dt
\]

\[
\leq C \int_{Q_{\overline{s}_j}} |p_j|^2 e^{2s\omega_j} \, dx \, dt.
\]

We add now estimates (6.23) and (6.24) and we obtain for some constant $C > 0$ and $s > \max_j \overline{s}_j$:

\[
\sum_{j \in 0, n} \int_Q (|D_t p_j|^2 + |D^2 p_j|^2 + |Dp_j|^2 + |p_j|^2)e^{2s\omega_j} \, dx \, dt \leq
\]

\[
C \left[ \sum_{j \in 0, n} \int_{Q_{\overline{s}_j}} |p_j|^2 e^{2s\omega_j} \, dx \, dt + \sum_{j \in 1, n} \int_Q \theta |p_j(t, x)|^2 e^{2s\omega_j} \right] \, dx \, dt.
\]
Choosing \( \theta \) small enough we see that the integrals on \( Q \) in the right side may be absorbed in the left side of the inequality and obtain

\[
\sum_{j \in 0,n} \int_Q \left( |D_t p_{j}|^2 + |D^2 p_{j}|^2 + |D p_{j}|^2 + |p_{j}|^2 \right) e^{2\sigma_{j}} \, dx \, dt
\]

(6.26)

\[
\leq C \sum_{j \in 0,n} \int_Q |p_{j}|^2 e^{2\sigma_{j}} \, dx \, dt.
\]

Observe now that for \( j \geq 0 \), by (6.22) there exists \( m = m(j) \) and the sequence 
\( j_0 = j, j_1 = k(j_0), \ldots, j_m = (k_0)^m(j) = 0 \). Now, by (6.31), (6.5), (6.6), (6.7), and looking only to the subdomains \( \omega_{ji}, l \in 0, m \) we find a sequence of equations for \( l \in 0, m - 1 \), forming cascade like system:

(6.27)

\[-D_t p_{ji+1} - \Delta p_{ji+1} - c_{ji+1}(t, x) p_{ji+1} = a_{ji,ji+1}(t, x) p_{ji}, \text{ in } (0, T) \times \omega_{ji+1}.\]

Now, as \( \omega_{ji} \subset \subset \omega_{ji+1} \) we find, as in the \[3\]

(6.28)

\[\int_{Q_{\omega_{ji}}} |p_{ji}|^2 e^{2\sigma_{ji}} \, dx \, dt \leq C \int_{Q_{\omega_{ji+1}}} |p_{ji+1}|^2 e^{2\sigma_{ji+1}} \, dx \, dt.\]

Consequently, for all \( j \in 1, n \) we find, by coupling the chain estimates above, that

(6.29)

\[\int_{Q_{\omega_j}} |p_j|^2 e^{2\sigma_j} \, dx \, dt \leq C \int_{Q_{\omega_0}} |p_0|^2 e^{2\sigma_0} \, dx \, dt,\]

which plugged into (6.26) gives a final Carleman estimate

(6.30)

\[\sum_{j \in 0,n} \int_Q \left( |D_t p_{j}|^2 + |D^2 p_{j}|^2 + |D p_{j}|^2 + |p_{j}|^2 \right) e^{2\sigma_{j}} \, dx \, dt
\]

\[\leq C \int_{Q_{\omega_0}} |p_0|^2 e^{2\sigma_0} \, dx \, dt.\]

which gives the final conclusion in the \( L^2 - L^2 \) framework, (6.20).

The \( L^\infty - L^2 \) estimate (6.21) follows by the same lines in the corresponding Theorem [1], using the bootstrap argument in connection to the regularity properties of the parabolic flow.

The main result concerning controllability with one control for linear parabolic systems with tree-like couplings is the following:

**Theorem 5.** Consider system (6.1) with coefficients in \( \tilde{E}_{M, \delta, \{\omega_i\}_1} \). Then there exists a constant \( C = C(M, \delta, \{\omega_i\}_1) \) such that for all \( z_0 \in H \) there exists \( u^* \in L^2(0, T; L^2(\omega_0)) \cap L^\infty(Q_{\omega_0}) \) which drives the corresponding solution to (6.1) in 0, i.e. \( z = z u^* \) satisfies \( z(T) = 0 \) and the control satisfies the norm estimate

(6.31)

\[\|u^* e^{-\sqrt{T}}\|_{L^2(0,T;L^2(\omega_0))} + \|u^*\|_{L^\infty(Q_{\omega_0})} \leq C \|z_0\|_{L^2(\Omega)}.\]

**Proof.** The proof is identical to the proof of Theorem [2] by using the Carleman estimates for the linear adjoint system (6.9) given by Theorem [1] and a corresponding observability estimate as the one given by Remark [3].
Note here that for the \( L^\infty \) estimate on the control, one needs to use in Carleman estimate a parameter \( \lambda \) such that (3.16) holds.

Controllability of nonlinear semilinear parabolic systems with tree-like couplings may be studied in analogy to the star-like case. We consider semilinear systems of parabolic equations, with tree type couplings in zero order terms, of the form

\[
\begin{cases}
    D_t y_0 - \Delta y_0 = \overline{g}_0(x) + f_0(x, y_0) + \chi_{\omega_0} u, & \text{in } (0, T) \times \Omega, \\
    D_t y_i - \Delta y_i = \overline{f}_i(x) + f_i(x, y_{k(i)}), & \text{in } (0, T) \times \Omega, \\
    y_i = \ldots = y_n = 0, & \text{on } (0, T) \times \partial \Omega,
\end{cases}
\]

where \( \overline{g}_j \in L^\infty(\Omega), j \in \overline{0,n} \) and \( \overline{g} = (\overline{g}_0, \ldots, \overline{g}_n) \in [L^\infty(\Omega)]^{n+1} \) is a corresponding stationary solution.

We assume the following hypotheses on the nonlinearities:

\((H1')\) \( f_0 \in C^1(\Omega \times \mathbb{R}), f_i \in C^1(\Omega \times \mathbb{R} \times \mathbb{R}), i \in \overline{1,n} \) there exist \( \omega_1, \ldots, \omega_n \subset \Omega \) open nonempty subsets of \( \Omega \) satisfying (6.4), (6.5) and

\[
(\omega_i \cap \omega_{k(i)}) \setminus \bigcup_{j \neq i, k(j) = k(i)} \omega_j \neq \emptyset, \forall i \in \overline{1,n},
\]

and for all \( i \in \overline{1,n} \) we have

\[
f_i(x, \tau, \xi) = 0 \quad \forall x \in \Omega \setminus \omega_i, \tau, \xi \in \mathbb{R};
\]

\((H2')\) For a family of subdomains \( \{\omega_i\}_i \) satisfying (6.6), (6.7), by defining for \( i \in \overline{1,n} \) the coefficients

\[
a_{k(i)}(x) \equiv \frac{\partial f_i}{\partial y_{k(i)}}(x, \overline{g}_{k(i)}(x), \overline{g}_i(x))
\]

\[
c_0(x) \equiv \frac{\partial f_0}{\partial y_0}(x, \overline{g}_0(x)), c_i(x) \equiv \frac{\partial f_i}{\partial y_i}(x, \overline{g}_{k(i)}(x), \overline{g}_i(x)),
\]

we assume that for some \( M_0, \delta_0 > 0 \) we have

\[
\{a_{k(i)}^0, c_i^0\}_{i \in \overline{1,n}, j \in \overline{0,n}} \in \mathcal{E}_{M_0, \delta_0}(\omega)_1, k.
\]

**Theorem 6.** Suppose \( \overline{g} \) is a stationary state to uncontrolled \( u = 0 \) (6.32) and that functions \( f_j, j \in \overline{0,n} \) satisfy hypotheses \((H1'), (H2')\). Then, for all \( \beta_0 > 0 \) there exist \( \zeta_0 = \zeta_0(\beta_0) > 0 \) and \( C = C(\beta_0, \{\omega_i\}_i, \overline{g}) \) such that if \( \|y^u(0) - \overline{g}\|_{L^\infty(\Omega)} < \zeta_0 \) there exists a control \( u \in L^\infty(Q) \) satisfying

\[
\|u\|_{L^\infty(Q)} \leq C\|y^u(0) - \overline{g}\|_{L^\infty(\Omega)}
\]

and

\[
y^u(T, \cdot) = \overline{g},
\]

with

\[
\|y(t, \cdot) - \overline{g}\|_{L^\infty} \leq \beta_0, \quad t \in [0,T].
\]
Remark 5.  (1) Our results remain valid if instead of the operator $\Delta$ we use general elliptic operators which may be differently chosen in each of the equation of the system:

$$L_iy_i := -\sum_{j,k=1}^N D_j(\alpha_{i j} D_k y_i) + \sum_{k=1}^N \beta_{i k} D_k y_i + \gamma_i y_i \quad i = 1, \ldots, n,$$

with general boundary conditions which may be also of Neumann or Robin type. Here $(\alpha_{i j})_{j,k}$ satisfy uniform ellipticity conditions in $\Omega$. In our study we need also to impose regularity assumptions on the coefficients $(\alpha_{i j} \in W^{1,\infty}(\Omega), \beta_{i k}, \gamma_i \in L^\infty(\Omega))$; these regularity assumptions allow the development of the bootstrap argument based on the regularizing properties of the parabolic flow when establishing an $L^\infty$ framework for the controllability problem.

(2) The hypotheses on the support of the coupling coefficients is essential for our approach to the controllability problem. In fact, for the systems we consider with the same type of couplings but with constant coupling coefficients controllability no longer occurs. Take for example the following system with a star-type coupling ($\alpha$ and $\beta$ are fixed real constants):

$$\begin{align*}
D_t z_0 - \Delta z_0 &= \chi_{\omega_0} u, \quad \text{in } (0,T) \times \Omega, \\
D_t z_1 - \Delta z_1 &= \alpha z_0, \quad \text{in } (0,T) \times \Omega, \\
D_t z_2 - \Delta z_2 &= \beta z_0, \quad \text{in } (0,T) \times \Omega, \\
z_0 = z_1 = z_2 &= 0, \quad \text{on } (0,T) \times \partial \Omega.
\end{align*}$$

Considering the results in [1,2], null controllability occurs if and only if the Kalman rank condition rank $[A_0 | B] = 3$. However, in this situation the Kalman matrix is $[A_0 | B] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & \beta & 0 \end{pmatrix}$ and its rank is 2.

Also, if we consider the parabolic system with tree-like couplings (2.13) in §2 Preliminaries, with constant coefficients $c_j = 0$, $a_{10} = a_{20} = a_{31} = a_{41} = 1$, the Kalman matrix

$$[A_0 | B] = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

and this has rank 3; thus the system is not null controllable.

In fact one may see the results in this paper more as an extension of the results concerning cascade-like parabolic systems with nonconstant coefficients (see [3]).
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