Efficiently preparing GHZ, topological and fracton states by measuring cold atoms

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Long-range entangled quantum states—like Schrödinger cat states and topological order [1]—are key for quantum metrology and information purposes [2, 3], but their creation is fraught with difficulty. While adiabatic preparation is necessarily slow—taking time that grows with system size [4–7]—quick preparation on existing quantum devices via a sequence of gates and measurements [8–14] is currently hampered by high overheads. Here we propose a solution that evades these issues, scalably creating large long-range entangled states with the use of existing experimental platforms. Our two-step process finds an ideal implementation in dual-species [15, 16] Rydberg atom arrays [17–31], only requiring time-evolution under the intrinsic atomic interactions followed by measuring a single sub-lattice of atoms. Remarkably, this protocol can prepare the 1D GHZ state [32] and 2D toric code [33] with fidelity per site exceeding 0.9999, and a 3D fracton state [34–36] with fidelity $\gtrsim 0.998$. In light of recent experiments showcasing 3D Rydberg atom arrays [37], this paves the way to the first experimental realization of fracton order. While the above examples are based on efficiently preparing and measuring cluster states [9], we also propose a multi-step procedure which first creates the $\mathbb{Z}_3$ toric code and subsequently non-Abelian topological order in Rydberg atom arrays—its emergent anyons offering a route towards universal topological quantum computation [38].

The defining property of long-range entangled states is that they cannot be prepared efficiently (i.e., within a time window independent of system size) by unitary processes. Remarkably, measurements can in principle be used to avoid this issue in quantum devices. In particular, a large class of long-range entangled states can be prepared in a finite time by acting with quantum gates (typically controlled-Z or Ising) on qubits and ancillas—preparing the so-called cluster state [9]—such that measuring the ancillas leaves the qubits in a long-range entangled state [13].

Unfortunately, such a measurement-based approach introduces its own set of challenges. First, although controlled-Z gates mutually commute, a qubit cannot simultaneously be acted upon by multiple gates. Second, even the application of a single controlled-Z gate can require the action of $\sim 10$ native gates [39] making it unnatural and imposing a significant overhead. Third, setting aside a fraction of qubits as ancillas is particularly costly in emerging quantum computing devices, where system size is one of the main bottlenecks. Correspondingly, the measurement-based approach has not been the one of choice even in recent implementations of the toric code [33] on quantum devices, which have instead adopted a unitary protocol [28, 40].

Here, we bypass these problems by letting the intermediate cluster state naturally emerge from the time-evolution of an Ising Hamiltonian, rather than constructed piecemeal via the gates of a quantum computer. This is a very advantageous trade-off: although we lose the full-fledged programmability of a quantum computer, we also shed much of its overhead, thereby gaining scalability. As we will see, outsourcing the state-building to natural many-body interactions still allows for realizing a wide range of highly desirable long-range entangled states which are implementable using existing technology.

An ideal scalable platform satisfying the two requirements of intrinsic Ising interactions and high-resolution measurements, is that of Rydberg atom arrays [17–23, 27]. These are tuneable lattices of atoms trapped by optical tweezers, each serving as an effective qubit [41] consisting of its ground state ($n = 0$) and a particular Rydberg excited state ($n = 1$). When two atoms are simultaneously excited, they experience a van der Waals interaction $V(r) \sim 1/r^6$ [42], giving rise to a Hamiltonian of the form [43]

$$H = \sum_i \frac{\Omega_i}{2} X_i - \sum_i \delta_i n_i + \frac{1}{2} \sum_{i \neq j} V_{ij} n_i n_j,$$  \hspace{1cm} (1)

where $X, Y, Z$ denote the Pauli matrices. Here $\Omega_i$ and $\delta_i$ are tunable, and by shifting the latter, we obtain the Ising interaction $\frac{1}{2} \sum_{i \neq j} J_{ij} Z_i Z_j$ with $J_{ij} = \frac{1}{4} V_{ij}$, which will play the role of our ‘entangling gates’. Recent years have seen a rapid growth in available system size, exploring the many-body quantum physics of Eq. (1) in 1D and 2D arrays of many hundreds of atoms [24–26, 28–31], and even demonstrating 3D arrays [37], forming the inspiration for a multitude of theoretical studies in this direction [44–50].

As the first part of our two-step process, we will show that by choosing appropriate lattice configurations, time-evolving Rydberg atoms under their natural interactions can produce high-quality cluster states. While these states are already of interest (serving as resources for measurement-based quantum computation [51–54]), to produce long-range entanglement we need to be able to measure only a subsystem of its atoms. While it is in principle possible to measure individual atoms of an array, our proposal will make use of recent breakthroughs...
in establishing dual-species arrays [15, 16, 55–57]. Here, one species is transparent to measurements of the other. Dual-species arrays have the added benefit that relative intra- and inter-species interactions are tuneable, such that one can even make the former negligible (see Methods), which we use to produce high-fidelity realizations of topological and fracton states.

One natural viewpoint on this measurement protocol of cluster states is that it efficiently implements stabilizer measurements [8, 13]. Here, however, we advocate a different viewpoint, emphasizing the symmetry-protected topological (SPT) [58–63] nature of the cluster state. This shift in focus allows us to naturally characterize the quality of the resulting state. Indeed, we argue that the ability of the cluster state to produce long-range entanglement upon measurement is a robust property of its SPT nature, which allows us to analyze how even in imperfect implementations one can reliably realize exotic order. We moreover generalize beyond the framework of cluster state measurements, showing how multiple applications of time-evolution and measurements can produce non-abelian topological order. In particular, we outline how $S_3$ topological order [33] results by first creating the $Z_3$ toric code as a stepping stone. This is a vast generalization of the procedure of measuring cluster states, and in a companion work [64] we explore its theoretical ramifications, reinterpreting it as physically implementing a Kramers-Wannier transformation, thereby expanding the possible states of matter which can be efficiently prepared by finite-time evolution and measurement.

I. THE 1D CLUSTER AND GHZ STATES

Let us first describe how to efficiently prepare a cat state in an idealized setting. Starting with a product state in the Pauli-$X$ basis, we time-evolve a chain of $N$ qubits under a nearest-neighbor Ising interaction:

$$|\psi(t)\rangle = e^{-itJ \sum_n Z_n Z_{n+1}} |\downarrow\cdots\downarrow\rangle \otimes N.$$  \hspace{1cm} (2)

After a time $t_{\text{SPT}} := \frac{\pi}{4J}$, the resulting wave function forms the so-called 1D cluster state [9]. Its defining characteristic is:

$$Z_{n-1} X_n Z_{n+1} |\psi(t_{\text{SPT}})\rangle = |\psi(t_{\text{SPT}})\rangle.$$  \hspace{1cm} (3)

This implies that the entangled state has the remarkable property that measuring, say, every even site in the $X$-basis leaves the remaining qubits in a Schrödinger cat state [9]. Indeed, replacing $X_{2n} \to \pm 1$ by the measurement outcome, the post-measurement state satisfies $Z_{2n-1} Z_{2n+1} |\psi_{\text{out}}\rangle = \pm |\psi_{\text{out}}\rangle$. In addition, one can show that the state is symmetric, i.e., $\langle Z_{2n-1} \rangle = 0$ (see Methods). The only state satisfying these properties, is the (disordered) GHZ state, a Schrödinger cat state where neighboring spins on the odd sublattice are perfectly aligned or anti-aligned depending on the measurement outcome of the intervening site, as illustrated in Fig. 1(a). Furthermore, since this information is available, one can always apply single-site spin flips to bring it to the ideal GHZ state:

$$|\text{GHZ}\rangle = \frac{|\uparrow\uparrow \cdots \uparrow\rangle + |\downarrow\downarrow \cdots \downarrow\rangle}{\sqrt{2}},$$  \hspace{1cm} (4)

although it is a Schrödinger cat state independent of this procedure.

FIG. 1. Fast preparation of a GHZ state by measuring a 1D cluster state. (a) The protocol starts with a product state in the $X$-basis which we let time-evolve under an Ising interaction. After a time $t = t_{\text{SPT}}$, this produces the cluster state. Measuring the red sites in the $X$-basis leads to a cat state. Using the information of the measurement outcomes, one can flip the appropriate spins to obtain the GHZ state on the blue sites [9]. (b) If we do not time-evolve with exactly $t = t_{\text{SPT}}$, the post-measurement state is only an approximate cat state. Nevertheless, the resulting correlation length (within which there is long-range order) is very large even for moderate time deviations. (c) A tensor-network simulation of the protocol for Rydberg atoms interacting with a $1/r^6$ van der Waals interaction. We simulate this for Rb 70S$_{1/2}$ with two different lattice spacings, which give different results due to incorporating the fact that the initial product state preparation pulses are not instantaneous (see Methods). (d) Contributions from longer-range van der Waals interactions can be systematically suppressed by interspersing time-evolution with $X$-pulses on sublattices (orange); see Section IV. This particular example cancels out couplings at distance $r = 2a$, although this was not used to achieve the results in (c).
To make this procedure practical, we need to understand its sensitivity to deviations from the above idealization, such as evolving with additional couplings or for the incorrect time. We argue that the ability to create a cat state via measurement is a stable property of the phase of matter that the cluster state belongs to. More precisely, it is known that in addition to the global Ising symmetry $\prod_n X_n$, the cluster state is symmetric under spin-flips on even and odd sites, separately; deformations of the cluster state that preserve this symmetry define its so-called SPT phase [58, 59].

As a first indication of the importance of this symmetry, we explore what happens when we slightly break it by time-evolving for the wrong length of time. If $t$ is not a multiple of $t_{SPT}$, the state does not enjoy the aforementioned sublattice symmetry. Indeed, a straightforward computation shows that (ignoring boundary effects and taking $|1 - t/t_{SPT}| < 1/2$):

$$\langle \psi(t) | \prod_n X_{2n} \psi(t) \rangle = \sin(2Jt)^N \sim e^{-N/\xi_s(t)}.$$  \hfill (5)

This defines a length scale $\xi_s(t)$ within which the state looks symmetric, with $\xi_s(t_{SPT}) = \infty$. In line with the idea that the symmetry is the determining factor, we surmise that after measuring every other site, we obtain an imperfect cat state with correlation length $\xi_s(t)$. Indeed, we have confirmed by a full solution of the model that the resulting state has long-range order for $\langle Z_{2m-1} Z_{2n-1} \rangle$ within a correlation length $\xi$ closely matching $\xi_s$, beyond which it decays to zero [65]. This is plotted in Fig. 1(b), where we see that $\xi$ grows rapidly as we approach $t \rightarrow t_{SPT}$. E.g., a 0.1% deviation results in long-range order over nearly half a million atoms; a 1% error still gives $\xi/(2\Delta) \approx 4000$ (the original lattice having spacing $\Delta$). Hence, even in the presence of timing imperfections, this procedure can produce large cat states in a finite amount of time.

A chain of Rydberg atoms gives a good approximation to Eq. (9) with $J = V(a)/4$, with additional longer-range corrections. In case we use a single species of Rydberg atoms, the first correction occurs at distance $r = 2\Delta$ with a van der Waals coupling $V(2\Delta) = V(a)/2^6$, giving an additional correction by $e^{-\frac{1}{2}v t/J \sum_n Z_n Z_{n+2}}$. However, one can show that since this Ising coupling preserves the spin-flip symmetry on even and odd sublattices, measuring the even sites still leads to a cat state with an infinite correlation length (see Methods). It is thus only the third-nearest-neighbor correction which will lead to an imperfect cat state. Similar to Eq. (5), we find that this leads to a correlation length

$$\frac{\xi}{2\Delta} = \frac{1}{2 |\ln \cos(\pi/(2 \times 3^6))|} \approx 2 \times 10^5.$$  \hfill (6)

We thus obtain a near-perfect cat state.

In addition to knowing the size of the cat state that we prepare, we can ask how close we are to the fixed-point GHZ limit. A useful fact is that, to a very high degree, we can equate the GHZ-fidelity of the final state with the cluster-state-fidelity of the pre-measurement state (see Methods). To characterize the latter, we can use the fact that the expectation value of the stabilizer $Z_{n-1} X_n Z_{n+1}$ gives a lower bound on the fidelity (see Methods). At first sight, this expectation value is affected by the second-nearest-neighbor coupling at $r = 2\Delta$. However, in this 1D case, we can show that its effect is to simply tilt the Ising order parameter away from $Z_n$, which can be easily corrected by a single-site rotation at the end of the time-evolution. In conclusion, the dominant correction is at $r = 3\Delta$, leading to a cluster stabilizer

$$\langle Z_{n-1} X_n Z_{n+1} \rangle \approx \cos^2 \left( \frac{\pi}{2 \times 3^6} \right) \approx 0.99995.$$  \hfill (7)

This suggests a fidelity per site of

$$\sqrt{\langle [\text{cluster}] \psi(t_{SPT}) \rangle^2} \gtrsim \frac{1 + \langle Z_{n-1} X_n Z_{n+1} \rangle}{2} \approx 0.999975.$$  \hfill (8)

This is such a staggeringly high fidelity, that one might worry that at this point other imperfections—which we might usually reasonably ignore—become dominant. One such imperfection is the fact that the initial product state is prepared by a pulse, which has a finite time on the order of (tens of) nanoseconds. We have performed a full tensor network simulation of our state preparation for the case of Rb $70S_{1/2}$, incorporating this imperfection. As we see in Fig. 1(c), we can reach very high fidelities. The fact that the resulting fidelity is one order of magnitude worse than Eq. (9) is due to the finite pulse time. Indeed, we have confirmed that artificially reducing the pulse time recovers Eq. (9) (see Methods).

## II. TORIC CODE

Having established cat states in 1D, we now turn to 2D topological order. We consider qubits on the vertices (A sublattice) and bonds (B sublattice) of the honeycomb lattice as shown in Fig. 2. Like in the 1D case, we start with a product state $|\psi(t_{SPT})\rangle$ and time-evolve evolve with a nearest-neighbor Ising interaction for a time $t_{SPT}$ to obtain a 2D cluster state [10] which is invariant under the two types of stabilizers shown in Fig. 2. In particular, for a given vertex $v$, we have $Y_v A_v |\psi(t_{SPT})\rangle = |\psi(t_{SPT})\rangle$ where $Y_v$ is the Pauli-$Y$ on that vertex, and $A_v = \prod_{\delta \in v} Z_\delta$ is a product over the three neighboring bond qubits. Hence, upon measuring the A sublattice in the Y-basis, we obtain a state where $A_v = \pm 1$ depending on the measurement outcome. Moreover, one can also show that for every hexagonal plaquette we have $B_p = \prod_{\delta \in p} X_\delta = 1$ (see Methods). These are exactly the two stabilizers of the toric code [33]! The state has $\mathbb{Z}_2$ topological order for either $A_v = \pm 1$. If one wishes to use this state for quantum computation purposes, one can simply store these measurement outcomes
and use them in post-processing. Alternatively, one can always bring the state to $A_v = +1$ by applying string operators formed by single-site $X$-rotations.

If we implement this using Rydberg atoms, then the time-evolution gets contributions from the longer-range van der Waals interactions. However, like in 1D, we argue that the ability to produce topological order via measurement is a robust property of the entire SPT phase of this 2D state. The practical consequence is that the topological phase is stable to all longer-range Ising couplings which do not couple the $A$ and $B$ sublattices; the leading correction comes with a pre-factor $(\frac{q}{\sqrt{a}})^6 \approx 0.003$. (Distances are shown in units of lattice spacing $a$.)

FIG. 2. Toric code in a Rydberg atom array. We place Rydberg atoms on the vertices (red) and bonds (blue) of the honeycomb lattice. We initialize into the product state $|\cdot\rangle^{\otimes N}$ and time-evolve for $t = \pi/V(a)$, obtaining the cluster state whose two types of stabilizers are depicted. Upon measuring the Rydberg atoms on the red vertices in the $Y$-basis, we obtain the toric code state on the blue sites up to known single-site spin flips. The corrections to true topological order are given by the longer-range van der Waals interactions connecting the red and blue sublattices; the leading correction comes with a pre-factor $(\frac{q}{\sqrt{a}})^6 \approx 0.003$. (Distances are shown in units of lattice spacing $a$.)

FIG. 3. Realizing a fractal spin liquid with fracton order. Rydberg atoms are placed at the red and blue sites on the hexagonal prism lattice. After time-evolving under the Rydberg interaction and measuring the red sites, the resulting state has the depicted stabilizers, realizing the so-called Sierpinski prism model [34]. In particular, quasiparticles have restricted mobility due to being created by fractal (Sierpinski) operators in the hexagonal layers.

see Section IV). The expectation value of the stabilizers is then:

$$|\langle A_v \rangle| \approx \cos^6 \left( \frac{\pi}{2\sqrt{7}} \right) \cos^6 \left( \frac{\pi}{7\sqrt{13}} \right) \cdots \approx 0.99993,$$

$$\langle B_p \rangle \approx \cos^{12} \left( \frac{\pi}{2\sqrt{7}} \right) \cos^{12} \left( \frac{\pi}{2\sqrt{13}} \right) \cdots \approx 0.99987.$$  

This implies a toric code fidelity $\sqrt{F} \geq 0.9999$, making the state useful for quantum information purposes. Indeed, it is well-known that terminating the lattice with an alternation of rough and smooth boundaries gives a surface code [66]. Applying local $X$- and $Z$-pulses in string patterns creates and moves $c$- and $m$-anyons [33], acting as logical operators on the topological qubit. Active error correction [2,3] requires repeatedly measuring the $A_v$ stabilizers, which is achieved by reloading the $A$ sublattice and repeating the preparation procedure. Similarly, the $B_p$ stabilizers can be measured by performing a $X \rightarrow Z$ rotation and loading atoms at the center of the plaquettes (for which a square lattice is most natural).

This procedure can be generalized to other topological orders. For instance, in the Supplemental Materials [65] we show how one can realize the 2D color code [67] (which has more quantum computing power compared to the toric code) with fidelity per site $\sqrt{F} \geq 0.997$, and even the 3D toric code (which has more thermal stability [68]) on the diamond lattice with $\sqrt{F} \geq 0.9998$.

III. FRACTON ORDER

To achieve fracton order—the most exotic version of topological order since it cannot be described by a conventional Topological Quantum Field Theory—we consider qubits on the 3D hexagonal prism lattice shown in
Fig. 3, with the red and blue dots defining the $A$ and $B$ sublattices, respectively. As above, time-evolving a product state under a nearest-neighbor Ising interaction prepares the cluster state on this lattice. Upon measuring the red sublattice, we obtain a state with the two stabilizers shown in Fig. 3. These resulting stabilizers have been discussed before and are known to give rise to fracton order. In particular, while particles can move in the vertical direction by the application of a string operator (as in the toric code), in the horizontal direction they live at the endpoints of a fractal operator (in particular, the Sierpinski triangle), restricting their mobility.

Unlike the previously discussed cases of the GHZ state or topological order, cluster SPT phases giving rise to fracton order are not protected by a global symmetry on the $A$ sublattice, but instead by a subsystem or fractal one. A practical consequence is that $AA$ van der Waals couplings will affect the stability of our fractal spin liquid. Let us consider the dual species case where these contributions can be neglected. We see in Fig. 3 that the first correction, coupling $A$ to $B$, occurs at $r = 2a$. From this, we find that the fracton phase persists over approximately 400 atoms (see Methods). However, as explained in the next section, by interspersing the time-evolution with four pulses, local addressing increases this number to 1300 (which can be systematically improved upon).

The corresponding expectation value of the stabilizers is 0.997, giving a fidelity $\sqrt{F} \gtrapprox 0.998$. This establishes a high-fidelity implementation of a fracton state.

In the Supplemental Materials [65], we discuss a different lattice where measurements lead to an approximate realization of the paradigmatic X-cube model [35] which can be realized with stabilizer expectation values $\approx 0.94$.

### IV. IMPROVEMENTS USING LOCAL ADDRESSING

Although our focus thus far has been on the minimal ingredients of time-evolution and measurement in dual-species Rydberg arrays, we have alluded to the fact that our protocol can be further improved using spatially-dependent fields, called local addressing [69]. This tool has already been used in large Rydberg arrays [26].

To illustrate how this can suppress or even cancel unwanted van der Waals interactions, let us return to the 1D case (Section I). Using spatially-dependent $X$-pulses, we can flip qubits on a sublattice of our choosing. Let $X_2$ correspond to flipping every fourth site starting from $k = 1, 2, 3, 4$. If $U(t)$ denotes the time-evolution, we consider the following alternation of time-evolution and sublattice pulses sketched in Fig. 1(d):

$$X_2X_3 U(t_{SPT}/2) X_1X_3 U(t_{SPT}/2) X_1X_2 U(t_{SPT}).$$

The additional procedure does not affect the net time-evolution for any sites separated by an odd multiple of lattice spacings; in particular, the nearest-neighbor evolution still prepares the cluster state. However, the second-nearest neighbor coupling drops out completely, making $r = 3a$ the dominant correction. (We note that the fourth-nearest neighbor experiences an effective doubled time-evolution.) For a similar case where we use local addressing to suppress unwanted contributions for the fracton model, see Methods.

We note that this cancellation procedure can also be useful after the state preparation. Indeed, the remaining $B$ sublattice will continue to time-evolve under the long-range Ising interaction. By construction, this coupling is small, especially in the dual-species set-up. Nevertheless, such effects can accumulate. One can then apply local pulses as above to effectively change the sign of a problematic coupling, such that further time-evolution will undo its effects. In the 1D case this is not necessary, since the GHZ state is an eigenstate of the van der Waals interactions. Moreover, local addressing can be useful to build interesting qutrit states, which we turn to now.

### V. GENERALIZATIONS AND NON-ABELIAN ANYONS

In this last section we will outline extensions of the state preparation procedure into new territory. We explore the preparation of a topological state built of qutrits, the trapping of non-Abelian defects, and the creation of non-Abelian topological order by combining multiple measurement steps.

We can build an effective qutrit by using the Rydberg blockade: if two atoms are close enough, the van der Waals interaction will forbid both simultaneously being in a Rydberg state [17, 18]. In such a scenario, we might call this unit of two atoms a qutrit with states $|0\rangle$, $|1\rangle$, $|2\rangle$, respectively referring to both atoms being in their ground state, the first being excited, and the second being
excited.

To create the $\mathbb{Z}_3$ toric code [33], we place these qutrits on the vertices and bonds of the square lattice, forming the $A$ and $B$ sublattices respectively. One can show that combining the natural Rydberg interaction with several pulses (similar to what we explored in Section IV) gives the following effective nearest-neighbor interaction [65]:

$$H_{a,b} = U \left(n_{1,a}n_{1,b} + n_{2,a}n_{2,b} - n_{1,a}n_{2,b} - n_{2,a}n_{1,b}\right),$$

where $n_{i,\lambda}$ denotes whether atom $i$ on site $\lambda$ is excited. Hence, the time-evolution $e^{-\frac{\pi i}{2}H_{a,b}}$ implements the controlled-Z gate for two qutrits, defined as $CZ|\psi\rangle \otimes |j\rangle = \omega^0|\psi\rangle \otimes |j\rangle$ with $\omega = e^{\frac{2\pi i}{3}}$. Similar to what we found in Section II, we end up with a $Z_3$ cluster state, such that after measuring the qutrits on the $A$ sublattice, we end up with the $Z_3$ toric code on the $B$ sublattice. Note that the first correction connecting different sublattices has a prefactor $\left(r_{AB}\right)^6 = \frac{1}{\sqrt{3}} \approx 0.008$, justifying this nearest-neighbor approximation.

If our lattice has a disclination, it can trap a non-Abelian defect. Indeed, the usual charge-conjugation defect on every other sublattice with qubits, and let $C$ and $D$ interact under

$$H_{CD} = U_{CD} \sum_{\langle c,d \rangle} n_c n_d$$

for time $t = \frac{\pi}{U_{CD}}$, combined with a pulse that cancels out coupling to $B$ sublattice, implementing a controlled-$Z$. Finally, measuring the $C$ qubits produces the ground state of the $S_3$ quantum double model [33] on the remaining qutrits on $B$ and qubits on $D$ [65]. At a conceptual level, this can be interpreted as effectively gauging the charge-conjugation symmetry of the $Z_3$ toric code, in line with $S_3 = \mathbb{Z}_3 \times \mathbb{Z}_2$. In a companion work, we explore more generally how gauging can be implemented using finite-depth time-evolution and measurements [64].

VI. OUTLOOK

In summary we have shown how a variety of highly sought after long-range entangled states can be efficiently created from simple ingredients: initialization of qubits, unitary time-evolution under Ising interactions, and measurement of a subset of qubits, which can be realized on existing experimental platforms. We have outlined a blueprint for implementation in Rydberg atom arrays, by identifying lattice structures that leverage their intrinsic interactions and demonstrate high fidelity preparation of 1D GHZ and 2D toric code states, as well as 3D fracton phases. The last—which has been discussed extensively in theory—now seem within experimental reach. We have also discussed several possible generalizations of our protocol. For instance, using the Rydberg blockade one can realize long-range entangled qutrits; using 3D configurations one can obtain 2D lattices with exotic defects; and by appealing to multiple measurement steps, one can even realize states hosting non-Abelian anyons.

In a companion work [64] we explore more generally what is achievable, including theoretical schemes states like the double semion order. Is there a way of naturally realizing this in Rydberg set-ups? Further, the examples discussed here are all gapped ground states—can these methods be extended to realizing gapless states with non-trivial properties? We leave these questions for future work.

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METHODS

Written out in spin notation, the Hamiltonian of interest is:

\[ H = \sum_i \Omega_i X_i - \sum_i \frac{h_i}{2} Z_i + \frac{1}{8} \sum_{i\neq j} V_{ij} Z_i Z_j \quad (15) \]

with \( h_i = \delta_i + \frac{1}{2} \sum_{j \neq i} V_{ij} \) (compared to Eq. (1)). We note that this extra shift of the longitudinal field—explicitly giving us an Ising interaction—is also practically significant, since it is key in some of the symmetry arguments which we used to argue stability of the resulting state.

Dual species and Förster resonances

One interesting option is to use different Alkali atoms for the two sublattices of our set-ups [55–57]. This can allow for the intra-species interaction negligible. For instance, the inter-species interaction is still at the MHz scale (compared to Eq. (1)). We note that this extra shift of the longitudinal field—explicitly giving us an Ising interaction—is also practically significant, since it is key in some of the symmetry arguments which we used to argue stability of the resulting state.

\[ |\psi_{\text{out}}\rangle = \prod_{a \in A} \left( \frac{X_a + s_a}{\sqrt{2}} \right) |\psi\rangle, \quad (16) \]

where \( s_a = \pm 1 \) denotes the measurement outcome. From the above explicit formula, it is straightforward to confirm our claim about the equality of fidelities.

Numerical simulation of cluster state preparation

We consider \( ^{87}\text{Rb} \) in the Rydberg level \( 70S_1/2 \). The intrinsic van der Waals interaction is \( H_{\text{int}} = U \sum_{i<j} n_i n_j / |r_i - r_j|^6 \) with \( U \approx 5\text{THz} \mu \text{m}^6 \). If \( a \) is the lattice spacing, then the nearest-neighbor interaction is \( V(a) = U/a^3 \). For \( a = 10\mu\text{m} \) (\( a = 12\mu\text{m} \)) we have \( V(a) \approx 5\text{MHz} \) \( V(a) \approx 1.7\text{MHz} \). We initialize state with all atoms in ground state. We then apply
an $X$-pulse for a time $t = t_{\text{pulse}}$ with strength $\Omega = \frac{\pi}{2\tau_{\text{pulse}}}$. We take $t_{\text{pulse}} = 20\text{ns}$, corresponding to $\Omega \approx 78\text{MHz}$. We then apply a $Z$-pulse for the same time with $h = -\frac{\pi}{2\tau_{\text{pulse}}}$. (Note that this does not need to be done this way: since $Z$ commutes with the Rydberg interaction, one can perform both at the same time, avoiding the need for the separate pulse.) After this second pulse, we (roughly) realize $|\bar{\cdot}\rangle^{\otimes N}$. We then time-evolve under the natural Ising interaction ($\Omega = h = 0$) for a time $t_{\text{SPT}} = \frac{\pi}{2(\tau_{\text{SPT}})} - \frac{1}{2}t_{\text{pulse}}$. Finally, we apply another $X$-pulse of strength $\Omega = -\frac{\pi}{2\tau_{\text{pulse}}} \times \frac{1}{2}t_{\text{pulse}}$. The simulations are performed using the density matrix renormalization group method for infinite systems [74], thereby directly working in the thermodynamic limit. We found that small bond dimensions, $\chi = 20 - 40$, were already sufficient to guarantee convergence. (Note: if we could reduce the pulse time to e.g., $t_{\text{pulse}} = 5\text{ns}$, then the fidelity goes up another order of magnitude: we find 0.999997 (for $\alpha = 12\mu m$), in close agreement with the prediction in Eq. (9)).

Dual stabilizer

When measuring $X_b$ on a given site of the $B$ sublattice, then any stabilizer that does not commute with it is no longer a stabilizer of the post-measurement state. However, sometimes a product of such stabilizers commutes with all $X$-measurements, thereby providing a constraint on the post-measurement state. E.g., in the 1D case, the product of $X_{2n}Z_{2n+1}X_{2n+2}$ (each individual factor not commuting with the $X$-measurements) gives us the constraint that the resulting post-measurement state has the $Z_2$ symmetry $\prod_a Z_{2n+1}$. In the 2D toric code, we instead get a local constraint: the product of the cluster term on four bonds around a plaquette gives us the condition $B_p = +1$.

The effect of symmetry-preserving couplings

For the case of the GHZ and toric code states, we claimed that $AA$ and $BB$ Ising couplings do not affect the resulting phase of matter. The latter is simple to see: these gates commute with the measurement and hence dress the post-measurement state with a finite-depth circuit (which cannot destroy the long-range entangled states). The $AA$ couplings are more interesting and subtle: in fact, they also push through to just being effective finite-depth circuits on the post-measurement states, which one can argue by using the stabilizer property of the cluster state, which allows one to replace it by $X$ (or a product thereof) on the $B$ sublattice. This is derived in explicit detail for the 1D case in the Supplemental Material [65].

Fracton order

For the fracton model, we obtain the following value for the stabilizers in the dual-species set-up.

$$\langle A_+ \rangle \approx \cos^{15} \left( \frac{\pi}{2 \cdot 2^6} \right) \cos^6 \left( \frac{\pi}{2 \cdot 3^7} \right) \ldots \approx 0.995 \quad (17)$$

$$\langle B_p \rangle \approx \cos^{42} \left( \frac{\pi}{2 \cdot 2^6} \right) \cos^{12} \left( \frac{\pi}{2 \cdot 5^6} \right) \ldots \approx 0.986 \quad (18)$$

We can improve this by using local addressing: we can cancel the out-of-plane contributions at $r = 2a$, noticing that the red sublattice is tripartite (forming a triangular lattice in each layer). Correspondingly, let us split $A$ up into $A_1 \cup A_2 \cup A_3$, and let $\mathcal{X}_A$ denote flipping qubits on the $A_i$ sublattice via an $X$-pulse. Consider

$$\mathcal{X}_{A_1} U(t) \mathcal{X}_{A_2} \mathcal{X}_{A_3} U(t) \mathcal{X}_{A_2} \mathcal{X}_{A_1} U(t) \mathcal{X}_{A_1} U(t)$$

Set $t = t_{\text{SPT}}/2$ will create the cluster state, and we have effectively removed the aforementioned couplings. We now obtain the following improved results:

$$\langle A_+ \rangle \approx \cos^3 \left( \frac{\pi}{2 \cdot 2^6} \right) \cos^6 \left( \frac{\pi}{2 \cdot 5^6} \right) \ldots \approx 0.9985 \quad (20)$$

$$\langle B_p \rangle \approx \cos^6 \left( \frac{\pi}{2 \cdot 2^6} \right) \cos^{12} \left( \frac{\pi}{2 \cdot 5^6} \right) \ldots \approx 0.997 \quad (21)$$

Appendix A: Preparation of cat state with timing imperfections

Consider the state $|\psi(t)\rangle = e^{-iHt}|\bar{\cdot}\rangle^{\otimes N}$ obtained by evolving the product state with the Ising Hamiltonian $H = J \sum Z_n Z_{n+1}$. (For the special time $t_{\text{SPT}} = \frac{\pi}{4|J|}$, this coincides with the cluster state.)

**Theorem.** Suppose we measure the even sites of $|\psi(t)\rangle$ in the Pauli-$X$ basis, producing the state $|\psi_{\text{out}}(t)\rangle$ for a given measurement outcome. Then the remaining odd sites have a correlation function

$$\langle \psi_{\text{out}}(t) | Z_{2m-1} Z_{2(m+n)-1} | \psi_{\text{out}}(t) \rangle \sim e^{-n/\xi}.$$  

(A1)

Define $\alpha = \cos^2(2t)$. If $n \gg \frac{1-\alpha}{1+\alpha}$, then the correlation length $\xi$ is normally distributed (corresponding to the randomness of measurement), $\xi \sim N(\bar{\xi}, \sigma_\xi)$, with the following mean and standard deviation:

$$\bar{\xi} = \frac{2}{(1+\alpha) \ln \left( \frac{1+\alpha}{1-\alpha} \right)} \quad \text{and} \quad \sigma_\xi = \frac{\bar{\xi}}{\sqrt{n} \sqrt{1-\frac{1-\alpha}{1+\alpha}}}.$$  

(A2)
Before proving this result, let us make a few observations:

1. For the special case \( t = 0 \) (i.e., \( \alpha = 1 \)), we recover \( \bar{\xi} = 0 \) for the product state, whereas the cluster state at \( t = t_{\text{SPT}} \) (i.e., \( \alpha = 0 \)) gives rise to a true cat state with \( \xi = \infty \).

2. Expanding around \( t = t_{\text{SPT}} + \delta t \) (i.e., around \( \alpha = 0 \)), we find \( \xi = \frac{1}{\alpha} - 1 + O(\alpha) \). The correlation length thus blows up as \( \xi \sim \frac{4t_{\text{SPT}}}{\pi^2} \frac{1}{(\delta t)^2} \) upon approaching \( t \to t_{\text{SPT}} \).

3. For all \( t \), we see that \( \sigma_\xi \to 0 \) as \( n \to \infty \). However, for practical purposes it can be meaningful to set \( n = \bar{\xi} \) to get a sense of how the correlation length will vary over physically relevant length scales. An example is shown in Fig. A.1 for \( t = 0.99t_{\text{SPT}} \), where we find that the deviations around \( \xi \approx 4000 \) are small.

4. The plot in Fig. 1(b) in main text corresponds to \( \bar{\xi} \) with a thickness of one standard deviation \( \sigma_\xi \), where we have chosen \( n = \bar{\xi} \).

Proof. Firstly, since we are time-evolving with a purely-diagonal nearest-neighbor Ising Hamiltonian, the \( X \)-correlations are uncorrelated beyond two sites. More precisely, \( \langle \psi(t) | \prod_{n \in S} X_{2n} | \psi(t) \rangle = \prod_{n \in S} \langle \psi(t) | X_{2n} | \psi(t) \rangle \) for any set of operators that does not include all even operators. Hence, if we measure \( X \) on all even sites (leaving out one for convenience), then the state with outcome \( X_{2n} = s_{2n} \) can be written as

\[
|\psi_{\text{out}}(t)\rangle = \prod_n \frac{X_{2n} + s_{2n}}{\sqrt{2 - 2\alpha s_{2n}}} |\psi(t)\rangle \quad \text{where } \alpha := |\langle X \rangle| = \cos^2(2t).
\] (A3)

From this, one can derive that

\[
|\langle \psi_{\text{out}} | Z_{2m-1} Z_{2n+1} | \psi_{\text{out}} \rangle| = \prod_{k=m}^n \left( \frac{1 - \alpha}{1 + \alpha} \right)^{1 - s_{2k}}.
\] (A4)

For instance, this follows from the factorization property \( \langle Z_{2m-1} Z_k Z_{2n+1} \rangle = \langle Z_{2m-1} Z_k \rangle \langle Z_{2n+1} \rangle \) and a direct calculation for the simplest case \( n = m \).

Hence, we see that the correlation length on the odd sites is given by

\[
\xi = \frac{1}{\ln \left( \frac{1 + \alpha}{1 - \alpha} \right)} \times \frac{1}{\rho}
\] (A5)

where \( \rho \) is the fraction of measured sites between the two \( Z \)-operators which have an outcome \( s = -1 \). If we consider all possible measurement outcomes, then \( \rho \times n \) is described by a binomial distribution \( B(n, p) \) where \( n \) is the number of even sites between two \( Z \)-operators, and \( p = \frac{1 - \alpha}{2} \) is the probability of having outcome \( s = -1 \). Hence, for large \( n, \rho \) is described by the normal distribution \( N(p, \sigma) \) with standard deviation \( \sigma = \sqrt{\frac{p(1-p)}{n}} \). Hence, if \( \sigma \ll p \) (i.e. \( n \gg \frac{1}{p} - 1 = \frac{1 - \alpha}{1 + \alpha} \)), then \( 1/\rho \) is normally distributed as \( N(1/p, \sigma/p^2) \). QED
Appendix B: Details for preparing cluster and GHZ state in 1D

Consider 1D chain with \(N\) atoms with spacing \(a\):

\[
H(\Omega, h) = \frac{\Omega}{2} \sum_{n=1}^{N} X_n - \frac{h}{2} \sum_{n=1}^{N} Z_n + \frac{V(a)}{4} \sum_{n=1}^{N} \sum_{k \geq 1} v_k Z_n Z_{n+k} \quad \text{with } h = \delta + V(a) \sum_{k \geq 1} v_k \tag{B1}
\]

In our nearest-neighbor model, we have \(v_{n>1} = 0\); in the single-species model we have \(v_n = 1/n^6\); in the dual species (where even and odd sites have different species), we have \(v_{2n-1} = 1/(2n-1)^6\) and \(v_{2n} = 0\). However, our formulas will be valid for general \(v_k\).

We will be interested in time-evolving with the interaction term over a time \(t_{\text{SPT}} = \frac{\pi}{V(a)}\). In particular, we consider the following unitary operator:

\[
U_{\text{SPT}} := e^{-it_{\text{SPT}} H(0, 0)} = e^{-\frac{\pi}{V(a)} \sum_{n} \sum_{k \geq 1} v_k Z_n Z_{n+k}} \tag{B2}
\]

We will refer to this as the SPT-pulse. Indeed, in the nearest-neighbor model \(v_k = \delta_{k,1}\), this would create the ideal cluster SPT phase, a state which we denote by \(|\text{cluster}\rangle\) and which is the ground state of the cluster model, \(H_{\text{cluster}} = -\sum_{n} Z_{n-1} X_n Z_{n+1}\).

1. Realizing the cluster state

Starting from the ground state of \(H(+\infty, 0)\) and then applying the SPT-pulse, the state we end up creating is:

\[
|\psi\rangle = U_{\text{SPT}}|\rangle \otimes |\rangle \sum_{n} \prod_{k \geq 2} \left[ \cos \left( \frac{\pi v_k}{4} \right) - i \sin \left( \frac{\pi v_k}{4} \right) Z_n Z_{n+k} \right] |\text{cluster}\rangle \tag{B3}
\]

From this, we can derive various exact properties of the resulting state:

\[
F_{\text{SPT}} = \sqrt{\langle (\text{cluster}|\psi\rangle|^2 \geq \prod_{k=2}^{\infty} \cos^2 \left( \frac{\pi v_k}{4} \right) \tag{B4}
\]

\[
\langle \psi|Z_{m-1} X_m Z_{m+1}|\psi\rangle = \prod_{k=2}^{\infty} \cos^2 \left( \frac{\pi v_k}{2} \right) \tag{B5}
\]

\[
\langle \psi|X_n|\psi\rangle = \langle \psi|Y_n|\psi\rangle = \langle \psi|Z_n|\psi\rangle = 0 \tag{B6}
\]

\[
\langle \psi|X_n X_{n+2}|\rangle = \frac{c_k^2 t_k^2}{2} \sum_{\sigma = 0,1} (-1)^\sigma \prod_{k=3}^{\infty} \left[ c_k^4 (1 + (-1)^\sigma t_{k-1} t_{k+1})^2 \right]. \tag{B7}
\]

The inequality in the first formula is true for \(v_k\) which decay fast enough with \(k\) (such as in our Rydberg case \(v_k = 1/k^6\)). The other three formulas are true for all \(v_k\). In the latter formula, we introduced the shorthand \(c_k := \cos(\pi v_k/2)\) and \(t_k := \tan(\pi v_k/2)\).

These formulas are straightforwardly evaluated. Let us first consider the single-species model \(v_k = 1/k^6\). In this case, we find a fidelity per site of \(F_{\text{SPT}} \gtrapprox 0.99985\) (actually turns out to be approximate equality), i.e., we are very close to the ideal cluster state. E.g., even for \(N = 100\) atoms, we have the fidelity \(|\langle \text{cluster}|\psi\rangle|^2 \approx 0.985\). Relatedly, the stabilizer defining the cluster state also has a very large value: \(\langle \psi|Z_{m-1} X_m Z_{m+1}|\psi\rangle \approx 0.9994\). Moreover, we find that

\[
\langle \psi|X_n|\psi\rangle = 0 \quad \text{and} \quad |\langle \psi|X_n X_{n+2}|\psi\rangle| < 10^{-10} \tag{B8}
\]

This tells us that measuring in the \(X\)-basis on, say, the even sites essentially corresponds to a completely uncorrelated flat distribution (to an extremely good approximation).

2. Creating the GHZ state through measuring half the system

Let us now measure \(X_n\) on every odd site. Due to Eq. (B8), the resulting state is

\[
|\psi_{\text{proj}}\rangle \approx \prod_{n} \frac{8_{2n-1} + X_{2n-1}}{\sqrt{2}} |\psi\rangle, \tag{B9}
\]
where \( s_n \) is the measurement outcome. Note that after judiciously chosen single-site flips, we can write
\[
|\psi_{\text{proj}}\rangle \approx \prod_n \frac{1 + X_{2n-1}}{\sqrt{2}} |\psi\rangle.
\]
From this, it follows that \(|\langle \text{GHZ}|\psi_{\text{proj}}\rangle|^2 \geq |\langle \text{SPT}|\psi\rangle|^2\). I.e., a good fidelity for the cluster state implies a good fidelity for the GHZ state after preparation! E.g., for the above cluster fidelity of \( \approx 0.985 \) for 100 atoms, we find a lower bound \(|\langle \text{GHZ}|\psi_{\text{proj}}\rangle|^2 \geq 0.97\). However, it turns out that this underestimates the true result. In addition, this can be improved by noting that the correction due to the second-nearest neighbor can be removed by a single-site rotation, as we discuss now.

3. Modified stabilizer

Consider the effect of \( v_2 \), leading to the state
\[
|\psi\rangle = \exp \left( -\frac{i\pi v_2}{4} \sum Z_n Z_{n+2} \right) |\text{cluster}\rangle
\]
This stil a \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) SPT, so measuring, say, the even sites should give a cat state on the odd sites. To see this, let us rewrite
\[
|\psi\rangle = \prod_n \exp \left( -\frac{i\pi v_2}{4} Z_{2n-1} Z_{2n+1} \right) \prod_n \exp \left( -\frac{i\pi v_2}{4} X_{2n+1} \right) |\text{cluster}\rangle
\]
Note there are no operators acting on the even sites. Hence, if we measure on the even sites, the projectors that project the state onto our measurement outcome commute with the above operators. The result is thus
\[
|\psi_{\text{out}}\rangle = \prod_n \exp \left( -\frac{i\pi v_2}{4} Z_{2n-1} Z_{2n+1} \right) \prod_n \exp \left( -\frac{i\pi v_2}{4} X_{2n+1} \right) |\text{cat}\rangle = U|\text{cat}\rangle,
\]
where the \( |\text{cat}\rangle \) state is in the diagonal basis (its precise configuration depending on the measurement outcome). This is a demonstration of the fact that Ising couplings on a given sublattice still give rise to a true cat state!

Furthermore, note that this cat state is an eigenstate of the two-point function of
\[
U Z_{2m-1} U^\dagger = Z_{2m-1} \prod_n \exp \left( -\frac{i\pi v_2}{4} Z_{2n-1} Z_{2n+1} \right) \exp \left( \frac{i\pi v_2}{2} X_{2m-1} \right) \prod_n \exp \left( \frac{i\pi v_2}{4} Z_{2n-1} Z_{2n+1} \right)
\]
\[
= Z_{2m-1} \left( \cos(\pi v_2/2) + i \sin(\pi v_2/2) X_{2m-1} \right) \exp \left( \frac{i\pi v_2}{2} Z_{2m-3} Z_{2m-1} \right) \exp \left( \frac{i\pi v_2}{2} Z_{2m-1} Z_{2m+1} \right)
\]
\[
= c Z_{2m-1} - s Y_{2m-1} (c + i s Z_{2m-3} Z_{2m-1}) (c + i s Z_{2m-1} Z_{2m-3})
\]
\[
= c Z_{2m-1} - s Y_{2m-1} + O(s^2),
\]
where \( c = \cos(\pi v_2/2) \) and \( s = \sin(\pi v_2/2) \). This means that the Ising order parameter has simply been rotated, which can be undone by applying a field.

Appendix C: Preparation of other long-range entangled states

Here we present other states of interest which can be prepared with our protocol. To make this self-contained, we will discuss our first example in detail. Since the other examples work analogously, we will simply quote the result for the stabilizers in those cases.

1. Xu-Moore model (subsystem symmetry breaking)

Consider the square lattice with the following two sublattices (as in the main text, \( A \) is red, \( B \) is blue):
We start with a product state $|+\rangle^{\otimes (N_A+N_B)}$ and time-evolve with the nearest-neighbor Ising Hamiltonian $H = J \sum_{(n,m)} Z_n Z_m$ for a time $t_{SPT} = \frac{\pi}{|J|}$ (for simplicity we now set $J = 1$); the resulting state is a cluster state on the above graph, with stabilizer $X_v \prod_{v' \in \partial v} Z_{v'} = 1$ (where $v' \in \partial v$ denotes that $v$ and $v'$ are nearest neighbors). Upon measuring the $A$ sublattice in the $X$ basis, we obtain the Xu-Moore [75] state on the $B$ sublattice with $\prod_{v \in \square} Z_v = \pm 1$, depending on the measurement outcome. Note that this state corresponds to spontaneous breaking of subsystem symmetries independent of measurement outcome. Acting with judiciously chosen $\prod X_v$ can make it into the homogeneous Xu-Moore state with $\prod_{v \in \square} Z_v = +1$.

Despite not having topological order, this is an interesting state with exotic correlations. In particular, there is long-range order for four-point functions of $Z$ if they form a rectangle on the tilted square lattice, whereas one-, two- and three-point functions will vanish! This is thus an analogue of the Schrödinger cat state with infinitely many branches in the wavefunction.

\[ a. \text{ Rydberg implementation: single-species} \]

If we use a single species of Rydberg atoms (all targeted for the same Rydberg state), then the Rydberg excited states at different sites will interact with a potential $V(r) = U/r^6$. Hence, the first correction to the cluster stabilizer is at the diagonal distance $r = \sqrt{2}a$, giving a stabilizer

\[ \langle X_v \prod_{v' \in \partial v} Z_{v'} \rangle \approx \cos^4 \left( \frac{\pi}{2\sqrt{2}} \right) \approx 0.93. \tag{C1} \]

However, this can be drastically improved by using local addressing, where we presume we can apply individual pulses on the four following sublattices (here $A$ is subdivided into $A$ and $D$; similarly we split $B$ into $B$ and $C$):

(We will eventually measure the A and D atoms after preparing the cluster state.) We can effectively generate spin flips $\mathcal{X}_\lambda = \prod_{v \in \lambda} X_v$ (for any of the sublattices $\lambda = A, B, C, D$) by time-evolving the $X$-term by $\pi/2$. If $U(t)$ denotes time-evolving with the Ising interaction of the system, let us consider the following combined evolution:

\[ \mathcal{X}_A \mathcal{X}_C U(\pi/8) \mathcal{X}_A \mathcal{X}_D U(\pi/8) \mathcal{X}_C \mathcal{X}_D U(\pi/4) \tag{C2} \]

This effectively generates a $t = \pi/4$ evolution for the Ising couplings connecting $AB$, $AC$, $CD$, $BD$, a $t = 0$ evolution for $AD$ and $BC$ couplings, and a $t = \pi/2$ evolution for $AA$, $BB$, $CC$, $DD$ couplings. In particular, this entirely eliminates the diagonal corrections (of length $\sqrt{2}$). The result is a stabilizer

\[ \langle X_v \prod_{v' \in \partial v} Z_{v'} \rangle \approx \cos^4 \left( 2 \times \frac{\pi}{2\sqrt{2}} \right) \approx 0.9952. \tag{C3} \]
The extra factor of two arose from these contributions adding up over all time-evolutions. Including further corrections, we get a more precise answer:

\[
\langle X_v \prod_{v' \in \partial v} Z_{v'} \rangle \approx \cos^4 \left( 2 \times \frac{\pi}{2 \times 2^6} \right) \times \cos^8 \left( \frac{\pi}{2 \cdot 2 \sqrt{5}} \right) \times \cos^4 \left( 2 \times \frac{\pi}{2 \times 2^6} \right) \approx 0.9945, \tag{C4}
\]

confirming that the result is barely changed by including further corrections.

b. Rydberg implementation: dual-species

Let us now instead consider a dual-species implementation, where the same-sublattice (AA and BB) couplings can be made negligible. Then the first correction is at a distance \( r = \sqrt{5} a \), leading to the cluster stabilizer:

\[
\langle X_v \prod_{v' \in \partial v} Z_{v'} \rangle \approx \cos^8 \left( \frac{\pi}{2 \cdot 2^6} \right) \times \cos^4 \left( \frac{\pi}{2 \cdot \sqrt{7}} \right) \approx 0.9994, \tag{C5}
\]

2. 2D color code

In the main text we discussed how to obtain the toric code on the honeycomb lattice (which required putting qubits on the sites and bonds of the honeycomb lattice). Similarly, we can create the color code [67] on the honeycomb lattice, which requires putting qubits on the sites of the dice lattice. The resulting stabilizers for the dual-species set-up are:

\[
\langle X_v \prod_{b \in v} Z_b \rangle \approx \cos^6 \left( \frac{\pi}{2 \cdot 2^6} \right) \times \cos^6 \left( \frac{\pi}{2 \cdot \sqrt{7}} \right) \times \cdots \approx 0.998, \tag{C6}
\]

\[
\langle B_p \rangle \approx \cos^{18} \left( \frac{\pi}{2 \cdot 2^6} \right) \times \cos^{36} \left( \frac{\pi}{2 \cdot \sqrt{7}} \right) \times \cdots \approx 0.994. \tag{C7}
\]

3. 3D toric code on the diamond lattice

Here we consider the 3D toric code [76]. Instead of placing it on the usual cubic lattice, we find excellent results for the diamond lattice. In particular, consider the A sublattice to be the vertices of the diamond lattice, and the B sublattice the bonds (see Fig. C.2). In the dual-species set-up, we then find:

\[
\langle X_v \prod_{b \in \text{tet}} Z_b \rangle \approx \cos^{12} \left( \frac{\pi}{2 \cdot \sqrt{19/3}} \right) \times \cos^{24} \left( \frac{\pi}{2 \cdot \sqrt{35/3}} \right) \times \cos^{24} \left( \frac{\pi}{2 \cdot \sqrt{51/3}} \right) \times \cdots \approx 0.9998 \tag{C8}
\]

\[
\langle B_p \rangle \approx \cos^{18} \left( \frac{\pi}{2 \cdot \sqrt{19/3}} \right) \times \cos^{36} \left( \frac{\pi}{2 \cdot \sqrt{35/3}} \right) \times \cos^{36} \left( \frac{\pi}{2 \cdot \sqrt{51/3}} \right) \times \cdots \approx 0.9996 \tag{C9}
\]

4. X-cube model

To realize the X-cube model [35], we take the A and B sublattices to be the vertices and the faces centers of the FCC lattice, respectively (see Fig. C.3). Defining \( A_c = \prod_{b \in \text{cube}} Z_b \), we obtain for the dual-species set-up:

\[
\langle X_v A_c \rangle \approx \cos^{24} \left( \frac{\pi}{2 \cdot \sqrt{3}} \right) \times \cos^{24} \left( \frac{\pi}{2 \cdot \sqrt{5}} \right) \approx 0.957 \tag{C10}
\]

\[
\langle B_p \rangle \approx \cos^{32} \left( \frac{\pi}{2 \cdot \sqrt{3}} \right) \times \cos^{32} \left( \frac{\pi}{2 \cdot \sqrt{5}} \right) \approx 0.944 \tag{C11}
\]
Appendix D: Details for preparing $S_3$ topological order

We first outline formally the preparation of $S_3$ topological order. We then discuss the physical implementation in Rydberg atom arrays.

1. Formal preparation

Our preparation consists of two steps. Preparing the $Z_3$ toric code by gauging a product state with $Z_3$ symmetry, and further gauging the $Z_2$ charge conjugation symmetry to obtain the $S_3$ topological order.

The lattice we will use in the first step is the so-called Lieb lattice with the $A,B$ sublattice on the vertices and bonds of the square lattice, respectively (see Fig. D.4). We denote

\[
\begin{align*}
\mathcal{X} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\
\mathcal{Z} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix} \\
\mathcal{C} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}
\end{align*}
\]

(D1)

to be the shift, clock, and charge conjugation of qutrits respectively. Here $\omega = e^{2\pi i/3}$.

We initialized a qutrit in the state $|+\rangle = \frac{|0\rangle + |1\rangle + |2\rangle}{\sqrt{3}}$, which is an eigenstate of $\mathcal{X}$. Thus the state is uniquely specified by stabilizers $\mathcal{X}$ for every site. Next, we perform an evolution that creates the $Z_3$ cluster state on this lattice. This cluster state can be viewed as an SPT protected by a $Z_3$ global symmetry and a $Z_3$ 1-form symmetry. The unitary
is given by

\[ U_{AB} = \prod_{(a,b)} CZ_{ab} \]  \hspace{1cm} (D2)

where \( \langle a, b \rangle \) denotes nearest neighbors for \( a \in A \) and \( b \in B \) and \( CZ = \omega^{n_a n_b} \) is the controlled-\( Z \) gate. Conjugating by this unitary, the stabilizers are given by

\[ \begin{array}{c}
Z
X
Z
Z
Z
Z
Z
Z
Z
\end{array} \quad \begin{array}{c}
Z
X
Z
Z
\end{array} \]

Next we perform a measurement on the \( A \) sublattice in the \( X \) basis. In the ideal case, when all measurement outcomes are \(|+\rangle\), we obtain the ground state of the \( Z_3 \) toric code, given by stabilizers

\[ \begin{array}{c}
Z
Z
Z
Z
\end{array} \]

The other measurement outcomes which are eigenstates of \( X \) signify the presence of \( e \) and \( \bar{e} \) anyons of the toric code, which can be paired up and annihilated via local unitaries.

The toric code wavefunction respects a global charge conjugation symmetry \( \prod_B C \). In order to gauge this symmetry, we will add two additional sublattices \( C \) and \( D \) as shown consisting of qubits initialized with stabilizers \( X \). We will use \( C \) to perform a symmetry transformation so that the charge conjugation symmetry acts as \( \prod_C X \). From this, we can then gauge this \( Z_2 \) symmetry by coupling the \( C \) and \( D \) lattices using the \( Z_2 \) cluster state entangler and measuring the \( C \) sublattice.

The symmetry transformation we will perform will exchange the following symmetries

\[ \prod_B X \leftrightarrow \prod_B X \prod_C C \]  \hspace{1cm} (D5)

The latter acts enriches the \( Z_3 \) toric code by charge conjugating both \( e \) and \( m \) anyons. The unitary that achieves this transformation is given by

\[ U_{BC} = \prod_{(b,c)} CC_{cb} \]  \hspace{1cm} (D6)
where \((bc)\) denotes nearest neighbor sites of \(b \in B\) and \(c \in C\), and \(CC_{cb}\) is the controlled-C gate where \(c\) is the control. The stabilizers are now

\[
\langle bc \rangle \quad \text{where} \quad B \in B \quad \text{and} \quad C \in C,
\]

\[
C_{cb} \quad \text{is the controlled-C gate where} \quad c \quad \text{is the control.}
\]

The stabilizers are now

\[
\prod_{\langle c,d \rangle} CZ_{cd}
\]

We then couple the \(C\) and \(D\) sublattices with a \(\mathbb{Z}_2\) cluster state. This is achieved via

\[
U_{CD} = \prod_{\langle c,d \rangle} CZ_{cd}
\]

At this stage the stabilizers of the wavefunction are given by

\[
\prod_{\langle c,d \rangle} CZ_{cd}
\]

Now, we perform a measurement on the \(C\) sublattice. It is clear that the following vertex and plaquette terms constructed from the order two stabilizers are stabilizers of the final model:

\[
\prod_{\langle c,d \rangle} CZ_{cd}
\]

On the other hand, the order three stabilizers do not commute. Let us call the first \(\mathbb{Z}_3\) stabilizer \(A_v\). We note that \(\frac{1 + A_v + A_v^\dagger}{3}\) is a projector for the wavefunction that commutes with the measurement \(X\) on the \(C\) sublattice. Therefore, we can write down the resulting projector after the measurement. Let us denote the number operator of the four qutrits according to the cardinals \(n_N\), \(n_E\), \(n_W\), \(n_S\). Since the operator is diagonal, it acts on the basis states as

\[
A_v = \omega^{n_N Z_1 + n_E Z_1 + n_S Z_2 + n_W Z_3}
\]
Therefore,

\[ A_v + A_v^\dagger = 2 \cos \left( \frac{2\pi}{3} Z_1(n_N + n_E + n_SZ_1Z_2 + n_WZ_1Z_3) \right) \]  
(D12)

where we have pulled out a common factor \( Z_1 \). Since this operator has eigenvalues \( \pm 1 \), it can be removed from the cosine. We are left with

\[ A_v + A_v^\dagger = 2 \cos \left( \frac{2\pi}{3} Z_1(n_N + n_E + n_SZ_1Z_2 + n_WZ_1Z_3) \right) \]  
(D13)

\[ = \omega^{n_N+n_E+n_SZ_1Z_2+n_WZ_1Z_3} + h.c. \]  
(D14)

\[ \]  
(D15)

Therefore, combining with the order two stabilizers, we find that the projector after the measurement is

\[ \frac{1}{3} \left[ 1 + \left( \begin{array}{c|c}
\bar{Z} & \tilde{Z} \\
\hline
\tilde{Z} & \bar{Z}
\end{array} + h.c. \right) \right] \]  
(D16)

Similarly, the projector that results from the order three plaquette term after the measurement is

\[ \frac{1}{3} \left[ 1 + \left( \begin{array}{c}
X_1^X & \bar{X}_2 & \tilde{X}_2 \\
\hline
\bar{X}_2 & \tilde{X}_2 & X_1^X
\end{array} + h.c. \right) \right] \]  
(D17)

To conclude, our final wavefunction is not described by a stabilizer code, but instead by the following commuting projectors

\[ \frac{1}{3} \left[ 1 + \left( \begin{array}{c|c}
\bar{Z} & \tilde{Z} \\
\hline
\tilde{Z} & \bar{Z}
\end{array} + h.c. \right) \right] \]  
(D18)

\[ \frac{1}{2} \left[ 1 + \left( \begin{array}{c}
X_1^X & \bar{X}_2 & \tilde{X}_2 \\
\hline
\bar{X}_2 & \tilde{X}_2 & X_1^X
\end{array} + h.c. \right) \right] \]  
(D19)

2. Relation to the quantum double model \( \mathcal{D}(S_3) \)

To make a connection to the quantum double model, we shift the two square lattices so that they coincide. Each edge now has a Hilbert space of dimension six composed of a qutrit and a qubit. We will show that the projectors can be mapped exactly to those of the quantum double model defined on the square lattice.

To make the explicit connection, we will need to do the following transformations (which mutually commute)
1. Swap $Z$ and $X$ on all qutrits;
2. Swap $Z$ and $X$ on all qubits;
3. Apply charge conjugation $C$ in the following pattern.

![Diagram of C operators pattern](image)

**FIG. D.6. Pattern of C operators**

After the above transformations. The projectors take the following form.

\[
\frac{1}{3} \left[ 1 + \left( \begin{array}{c} X_N \\ -Z_S \\ Z \end{array} \right) \right] + h.c. \]

\[
\frac{1}{3} \left[ 1 + \left( \begin{array}{c} Z_1 \\ Z_2 \\ Z_3 \end{array} \right) \right] + h.c. \]

\[
\frac{1}{2} \left[ 1 + \left( \begin{array}{c} X_N \\ X_S \\ X \end{array} \right) \right] + h.c. \]

\[
\frac{1}{2} \left[ 1 + \left( \begin{array}{c} Z_1 \\ Z_2 \\ Z_3 \end{array} \right) \right] \]

We claim that these are exactly the projectors of the quantum double model of $S_3$. The projectors for the quantum double model are the following vertex and plaquette terms\[33\]

\[
A_v = \frac{1}{|G|} \sum_{g \in G} R^g L^g L^g \]

\[
B_p = \sum_{g_1, g_2, g_3, g_4 \in G} \delta_{1, g_1 g_2 g_3^{-1} g_4^{-1}} \left| \begin{array}{c} g_1 \\ g_2 \\ g_3 \end{array} \right\rangle \left\langle \begin{array}{c} g_1 \\ g_2 \\ g_3 \end{array} \right| \]

where $L^g$ and $R^g$ are left and right multiplication operators

\[
L^g |h\rangle = |gh\rangle \\
R^g |h\rangle = |\bar{h}g\rangle \]

Let us represent the group $S_3$ using the generators $r, s$ that satisfy $r^3 = s^2 = rsr = 1$. In the regular representation, the left and right multiplications of $S_3$ are $6 \times 6$ matrices, which can be explicitly written in terms of a qubit and a qutrit as

\[
L^r = I \otimes X \\
R^r = X^\dagger \otimes X = X^{-Z} \\
L^s = X \otimes C \\
R^s = X \otimes I \]

Therefore, the product of our two vertex projectors is exactly $A_v$. Similarly, one can show that the product of our plaquette terms is exactly $B_p$ (see appendix A of Ref. [77] for a proof).
3. Implementation with Rydberg atoms

To realize a qutrit on the A and B sublattices, we endow each site with two Rydberg atoms, which we call atoms 1 and 2. Due to the proximity of the atoms, the state where both atoms are excited is prohibited, giving an effective three-level system. We label the empty state as $|0\rangle$ and the state where atom 1 or 2 is excited as $|1\rangle$ or $|2\rangle$ respectively.

On the other hand, the C and D sublattice will be the usual Rydberg two-level system: $|0\rangle$ for the ground state and $|1\rangle$ for the excited state.

Assuming local addressing, we are able to perform the following pulses for the qutrits:

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(D27)

which are the result of projecting the spin rotation operators into the qutrit subspace. These four generators are enough to generate arbitrary $SU(3)$ rotations.

The preparation procedure can be summarized as follows:

1. Load the A and B sublattices and perform the $Z_3$ fourier transform;
2. Implement $U_{AB}$ Eq. (D2);
3. Perform the inverse $Z_3$ fourier transform on the A sublattice and measure the occupancy ($Z_3$ toric code is obtained at this step);
4. Perform single site rotations to pair up charges in the $Z_3$ toric code;
5. Load the C sublattice and perform Hadamard;
6. Implement $U_{BC}$ Eq. (D6);
7. Load the D sublattice and perform Hadamard;
8. Implement $U_{CD}$ Eq. (D8);
9. Perform Hadamard on the C sublattice and measure. ($S_3$ topological order is obtained at this step);
10. (optional) Perform single site rotations to pair up charges in the $S_3$ topological order.

We now discuss how to implement $U_{AB}$, $U_{BC}$, and $U_{CD}$.

a. $U_{AB}$

To create the $Z_3$ cluster state, we would like to generate the unitary $U_{AB}$ in Eq. (D2).

The innate two-body interaction has dominated by the nearest neighbor sites between sublattices A and B, and is given by

$$H_{AB}(U,U') = \sum_{\langle ab \rangle} [U(n_{1,a}n_{1,b} + n_{2,a}n_{2,b}) + U'(n_{1,a}n_{2,b} + n_{2,a}n_{1,b})],$$

(D28)

where we presume $1 \leftrightarrow 1$ and $2 \leftrightarrow 2$ are equidistant, as are $1 \leftrightarrow 2$ and $2 \leftrightarrow 1$. (One way of achieving this is by having the two atoms separated in the third direction.) Here, $\langle a,b \rangle$ denote nearest neighbors, and $n_1$ and $n_2$ denotes the occupancy of atoms 1 and 2, respectively. Note that by conjugating the number operator $n_j$ with a $\pi/2$ pulse of $X_j$ for $j = 1, 2$ gives

$$e^{\pi/2 X_j} n_j e^{-\pi/2 X_j} = 1 - n_1 - n_2.$$

(D29)

One can show that by successive applications of this substitution, as well as the freedom of tuning single-site chemical potentials, one can generate an effective evolution where $U' = -U$. In fact, if $U' > U$ in Eq. (D28), the effective evolution will have $U = -U' > 0$. If $U' < U$, one can simply exchange the 1 $\leftrightarrow$ 2 labels on the A sublattice to reduce back to $U' > U$. In conclusion, we thus obtain the $CZ$ gate $U_{AB}$ via

$$U_{AB} = e^{-4\pi i/(3U')H(U,-U)}.$$

(D30)
b. $U_{BC}$

To implement $U_{BC}$ in Eq. (D6), we first perform a basis transformation on the $B$ sublattice such that $C$ is diagonal. A choice is

$$U = e^{\frac{\pi}{4} X_1} e^{\frac{\pi}{4} Z_1} e^{\frac{\pi}{4} X_2} e^{-\frac{\pi}{4} Z_2} \tag{D31}$$

$$\tilde{C} \equiv UCU^\dagger = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{D32}$$

The leading order interactions between the $B$ and $C$ sublattices are of the form

$$H_{BC} = \sum_{\langle bc \rangle} n_{1,b} n_c + n_{2,b} n_c \tag{D33}$$

Since

$$e^{\pi i n_c (n_{1,b} + n_{2,b})} = \text{diag}(1,1,1,1,-1,-1) = Z_c \cdot C\tilde{C}_{c,b} \tag{D34}$$

where $C\tilde{C}$ is the controlled $\tilde{C}$ gate. We have that

$$e^{\pi i H_{BC}} = \prod_{\langle b,c \rangle} C\tilde{C}_{c,b} \tag{D35}$$

where we used the property that for a fixed $c \in C$, there are only two nearest neighbors $b \in B$, thus cancelling away $Z_c$. To conclude,

$$U_{BC} = \prod_{\langle b,c \rangle} C\tilde{C}_{c,b} = U e^{\pi i H_{BC}} U^\dagger \tag{D36}$$

c. $U_{CD}$

$U_{CD}$ in Eq. (D6) is the $\mathbb{Z}_2$ cluster state entangler between the $C$ and $D$ sublattices, so we can implement this via the ordinary time evolution. However, we also have to ensure that there is no net evolution between the pairs $BC$ and $BD$. We can do this by flipping all $C$ and $D$ qubits halfway through the evolution. That is,

$$U_{CD} = \left( \prod_{C,D} X \right) e^{\frac{\pi i}{2} H} \left( \prod_{C,D} X \right) e^{\frac{\pi i}{2} H} \tag{D37}$$