Multifractal analysis of Chinese stock volatilities based on partition function approach

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Abstract

We have performed detailed multifractal analysis on the minutely volatility of two indexes and 1139 stocks in the Chinese stock markets based on the partition function approach. The partition function $\chi_q(s)$ scales as a power law with respect to box size $s$. The scaling exponents $\tau(q)$ form a nonlinear function of $q$. Statistical tests based on bootstrapping show that the extracted multifractal nature is significant at the 1% significance level. The individual securities can be well modeled by the $p$-model in turbulence with $p=0.40\pm0.02$. Based on the idea of ensemble averaging (including quenched and annealed average), we treat each stock exchange as a whole and confirm the existence of multifractal nature in the Chinese stock markets.

Key words: Econophysics, Multifractal analysis, Partition function approach, Quenched average, Annealed average, Bootstrapping, Stock markets

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1 Introduction

Since the pioneering work in the 1990’s [1, 2, 3], there has been a vigorous continuing investigation aimed at discovering remarkable similarities between financial markets and turbulent flows. Multifractal analysis, which was initially introduced to investigate the intermittent nature of turbulence [4, 5, 6], has also been extensively applied to various financial time series [7]. Multifractality has been regarded as one of the most important stylized facts in equity returns. Many different methods have been applied to characterize the hidden multifractal behavior in finance, for instance, the fluctuation scaling analysis [8, 9, 10], the structure function (or height-height correlation function) method [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21], multiplier method [22], multifractal detrended fluctuation analysis (MF-DFA) [23, 24, 25, 26, 27, 28, 29, 30], partition function method [31, 32, 33, 34, 35, 36, 37, 38, 39, 40], wavelet transform approaches [41, 42, 43, 44], to list a few.

To the best of our knowledge, the first application of partition function approach was carried out by Sun et al. to analyze the multifractal nature of minutely Hang Seng Index (HSI) data for individual trading days from January 3, 1994 to May 28, 1997 [31]. Ho et al. also reported that there exists multifractal feature in the daily Taiwan Stock Price Index (TSPI) data from 1987 to 2002 [33]. Wei and Huang analyzed the 5-min intraday data of the Shanghai Stock Exchange Composite (SSEC) index for individual trading days from January 1999 to July 2001 [34]. They found a different empirical result from the HSI and constructed a new measure of market risk to predict index fluctuations. A similar multifractal feature was reported in the high-frequency data of the SSEC index at different timescales and time periods [36, 37]. What is more interesting is the claim that the observed multifractal singularity spectrum has predictive power for price fluctuations and can serve as a measure of market risk [32, 34, 38].

The partition function approach was introduced to characterize singular measures [45], where a spectrum of local singularities exist. When the measure is homogeneous in singularity (thus not a multifractal), the scaling exponent \( \tau(q) \) of the \( q \)-order partition function is linear with respect to \( q \). In all the aforementioned studies, the index per se is treated as being proportional to a singular measure. Intuitively, there is no singularity in the index. This argument is indeed confirmed by the very narrow width of the extracted singularity spectrum \( f(\alpha) \). More precisely, the minimum singularity strength \( \alpha_{\text{min}} \) and the maximum singularity strength \( \alpha_{\text{max}} \) are both close to 1. Alternatively, the scaling exponent function \( \tau(q) \) is linear against \( q \), which disapproves the presence of multifractality. Zhou discussed these issues using 5-min SSEC data and performed extensive statistical tests [39]. Jiang and Zhou further clarified the situation using intraday high-frequency data for HSI, SZSC (Shenzhen Stock Exchange Composite), S&P500, and NASDAQ [40].
Having said this, we nevertheless figure that the idea is valuable to apply partition function approach to the multifractal analysis of stocks and indexes and to the possible application of multifractal properties in market prediction and risk management. This, of course, should be done using returns rather than stock prices or indexes. In this work, we shall follow this line to perform detailed multifractal analysis on Chinese stocks and indexes, which form a huge database. In almost all previous studies concerning the multifractal nature of financial markets, the investigation was conducted based on a single financial time series for individual stocks or indexes. In this paper, we propose to study many stocks as an ensemble. The idea is that different stocks in a same market share many common underlying mechanics. This consideration leads to the assumption that different stocks are realizations of a same stochastic process and we can perform ensemble averaging in the multifractal analysis as an analogue of diffusion-limited aggregation [46, 47, 48, 49].

This paper is organized as follows. In Section 2, we describe the data sets and perform preprocesses. The methods adopted are explained in detail in Section 3. Section 4 presents the results and discussions. Finally, Section 5 concludes.

2 Preliminary information

2.1 The date sets

We use a nice high-frequency database in the Chinese A-share stock markets from January 2004 to June 2006. The trading rules were not changed during this time period. After eliminating those stocks that have recording errors or less than 0.1 million data points, we are left with 715 stocks listed on Shanghai Stock Exchange (SHSE) and 424 stocks traded on the Shenzhen Stock Exchange (SZSE) (1139 in total). We also include two indexes, the Shanghai Stock Exchange Composite Index (SSEC) and the Shenzhen Stock Exchange Composite Index (SZSC). The average size for each stock is about 1.32 million and there are totally more than 1.5 billion data points for all the stocks.

All of the these indexes and stock prices are tick-by-tick data, which were recorded based on the market quotes disposed to all traders in every six to eight seconds. Each datum is time stamped to the nearest second at which one transaction occurs. Hence, the recording time interval of price series is uneven for each stock. For convenience, denote the time sequence for each data set as \( t_i \) and the corresponding price sequence as \( p(t_i) \), where \( i = 1, 2, \cdots \). In literature, \( i \) is also called the event time.
2.2 Definition of volatility

For each time series, we define the event-time return $r(t_i)$ over one event step as follows

$$ r(t_i) = \ln p(t_i) - \ln p(t_{i-1}) . $$

We then calculate the minutely volatilities as follows,

$$ v(t) = \sum_{t-\Delta t < \tau \leq t} |r(\tau)| , $$

where $\Delta t = 1 \text{ min}$. On each trading day, the Chinese stock markets contain opening call auction (9:15 a.m. to 9:25 a.m.), cooling period (9:25 a.m. to 9:30 a.m.), and continuous double auction (9:30 a.m. to 11:30 a.m. and 13:00 p.m. to 15:00 p.m.). Since July 1, 2006, the Shenzhen Stock Exchange introduced closing call auction (14:57 to 15:00). Our analysis focuses on the data recorded during the continuous double auction. In this way, there are 240 data points of the volatility for each time series on a single trading day.

3 Methodologies

3.1 Partition function approach

Denote the minutely volatility series as $\{v(t) : t = 1, 2, \cdots, T\}$. Then the series is covered by $N$ boxes with equal size $s$, where $N = \lfloor T/s \rfloor$. On each box, we define a quantity $u$ as follows,

$$ u(n; s) = u([(n - 1)s + 1, ns]) = \sum_{t=1}^{s} v[(n - 1)s + t] , $$

where $[(n - 1)s + 1, ns]$ is the $n$-th box. The box sizes $s$ are chosen such that $N = \lfloor T/s \rfloor = T/s$. The measure $\mu$ on each box is constructed as follows,

$$ \mu(n) = \frac{u(n; s)}{\sum_{m=1}^{N} u(m; s)} , $$

We then calculate the partition function $\chi_q$ \[45\]

$$ \chi_q(s) = \sum_{n=1}^{N} [\mu(n)]^q , $$

\[1\] During the time period of our data sets, both exchanges did not have closing call auction.
and expect it to scale as
\[
\chi_q(s) \sim s^{\tau(q)},
\]
where the exponent \(\tau(q)\) is a scaling exponent function. The local singularity exponent \(\alpha\) of the measure \(\mu\) and its spectrum \(f(\alpha)\) are related to \(\tau(q)\) through a Legendre transformation \[45\]
\[
\left\{ \begin{array}{l}
\alpha = d\tau(q)/dq \\
 f(\alpha) = q\alpha - \tau(q)
\end{array} \right. \tag{7}
\]
When \(u(n; s)/\sum u(m; s) \ll 1\) and \(q \gg 1\), the estimation of the partition function \(\chi\) will be very difficult since the value is so small that the computer is “out of the memory.” To overcome this problem, we can calculate the logarithm of the partition function, \(\ln \chi_q(s)\), rather than the partition function itself. A simple manipulation of Eqs. (4) and (5) results in the following formula,
\[
\ln \chi_q(s) = \ln \sum_{n=1}^{N} \left[ \frac{u(n; s)}{\max_m\{u(m; s)\}} \right]^q + q \ln \left[ \frac{\max_m\{u(m; s)\}}{\sum_{m=1}^{N} u(m; s)} \right], \tag{8}
\]
where \(\max_m\{u(m; s)\}\) is the maximum of \(u(m; s)\) for \(m = 1, 2, \ldots, N\).

### 3.2 Bootstrapping for statistical test

To test for the possibility that the empirical multifractality could be artifactual, we adopt the following bootstrapping approach \[39, 40\]. For a given volatility time series, we reshuffle the series to remove any potential temporal correlation and carry out the same multifractal analysis as for the original data. We impose a very strict null hypothesis to investigate whether the singularity spectrum \(f(\alpha)\) is wider than those produced by chance. The null hypothesis is the following:
\[
H_0^1: \Delta \alpha \leq \Delta \alpha_{\text{rand}}. \tag{9}
\]
The associated probability of false alarm for multifractality (so-called “false positive” or error of type II) is defined by
\[
p_1 = \frac{\#[\Delta \alpha \leq \Delta \alpha_{\text{rand}}]}{n}, \tag{10}
\]
where \(n\) is the number of shuffling and \(\#[\Delta \alpha \leq \Delta \alpha_{\text{rand}}]\) counts the number of \(\Delta \alpha\) whose value is not smaller than \(\Delta \alpha_{\text{rand}}\). As \(n \to \infty\), it is clear that the estimated bootstrap \(p\)-value will tend to the ideal bootstrap \(p\)-value. Under the conventional significance level of 0.01, the multifractal phenomenon is statistically significant if and only if \(p_1 \leq 0.01\). While \(p_1 > 0.01\), the null hypothesis cannot be rejected.

In a similar way, defining \(F = [f(\alpha_{\text{min}}) + f(\alpha_{\text{max}})]/2\), an analogous null hypothesis
can be described as follows:

\[ H_0^2 : F \geq F_{\text{rnd}}, \]  

where the false probability is

\[ p_2 = \frac{\#[F \geq F_{\text{rnd}}]}{n}. \]  

Using the significance level of 0.01, the multifractal phenomenon is statistically significant if and only if \( p \leq 0.01 \).

### 3.3 Ensemble average

In previous studies concerning multifractality in financial markets, the multifractal analyses were carried out on individual stocks or index series. The index can be regarded as an ensemble average of the stock market in some sense. Here, we introduce a method to investigate the multifractality in an ensemble of many stocks, which is borrowed from the concept of computing multifractal dimensions in Diffusion-Limited Aggregation [46, 47, 48, 49]. Regarding the stock market as a stochastic process, a stock trading on the market can be considered as a realization of the stochastic process. We define quenched and annealed mass exponent \( \tau_{Q,A}(q) \) as follows

\[ \langle \ln \chi_q(s) \rangle = -\tau_Q(q) \ln s, \]  

\[ \ln \langle \chi_q(s) \rangle = -\tau_A(q) \ln s, \]  

where the angular brackets \( \langle \cdot \rangle \) is the ensemble average over all the chosen stocks. Intuitively, the annealed exponents are more sensitive to rare samples of the ensemble with unusual values of \( \chi_q(s) \), while the quenched exponents are more characteristic of typical members of the ensemble [49].

### 4 Results and discussions

#### 4.1 Multifractal analysis

Two important stock indexes (SSEC and SZSC) and two stocks (Sinopec, 600028, in Shanghai Stock Exchange and China International Marine Containers, 000039, in Shenzhen Stock Exchange) are chosen as examples to show multifractality in single index or stock volatility series. Fig. [1](a) and (b) show the dependence of \( \chi_q(s)^{1/(q-1)} \) on box size \( s \) for different values of \( q \) in log-log coordinates. Power laws with good quality are observed between \( \chi_q(s)^{1/(q-1)} \) and \( s \). We also find that
the scaling range is wide and spans about three orders of magnitude. More interestingly, when $s$ is small, there is a sudden jump on the $\chi_q(s)^{1/(q-1)}$ curve for $q = -3$ in Fig. 1(b). This is not surprising for individual stocks since they may have time intervals within which the prices do not change. In this case, there is at least one box with very vanishing measure $\mu$ when the box size $s$ is smaller than or equal to some critical value $s_c$. When $s > s_c$, the relative difference among measures is not large. Hence, for all negative $q$, we will observe such jumps. This explains the occurrence of a jump in individual stocks but not in indexes. Consequently, in the determination of the scaling ranges, we identify and exclude those jumps.

![Fig. 1. (color online) Multifractal analysis.](image)

There are at least three cases corresponding to “freezing” price in certain time period. The first case corresponds to the situation that the price has reached its daily price limit. According to the trading rules of SZSE and SHSE, there is a price fluctuation limit of $\pm 10\%$ for normal stocks and $\pm 5\%$ for specially treated (ST) stocks compared to the closing price of the last trading day. When the price reach its daily price limit with a huge number of shares waiting on the corresponding best bid or ask price, the price does not change for a long time, which might last till the closure of the market. The second case corresponds to those very liquid stocks for which both the buy and sell sides of the order book are very thick so that the price does not change frequently for most marketable orders. The third case corresponds to
those stocks with very low transaction activities.

The scaling exponents $\tau(q)$ are given by the slopes of the linear fits to $\ln \chi_q(s)$ with respect to $\ln s$ for different values of $q$. We do this for all moments between -3 and 5 with an increment of 0.2. Fig. 1(c) plots the dependence of the mass exponents $\tau(q)$ on the moment order $q$. Fig. 1(d) presents the multifractal singularity spectra $f(\alpha)$ obtained through Legendre transformation of $\tau(q)$ defined by Eq. (7). It is well-known that $\Delta \alpha \equiv \alpha_{\text{max}} - \alpha_{\text{min}}$ is an important parameter qualifying the width of the extracted multifractal spectrum. The larger is the $\Delta \alpha$, the stronger is the multifractality. At the first glance, we find that $\Delta \alpha_{\text{SSEC}} < \Delta \alpha_{000028}$ and $\Delta \alpha_{\text{SZSC}} < \Delta \alpha_{000039}$. This is in agreement with our common sense that the fluctuations of index are less volatile than that of individual stocks.

To exhibit the analogue between financial market and fluid mechanics, the $p$-model, which is a simple cascade model of energy dissipation in fully developed turbulence [50], is applied to fit the mass exponent functions and it gives an excellent parametrization of the data. The theoretical mass function of the $p$-model can be expressed as follows

$$\tau(q) = - \frac{\ln[p^q + (1-p)^q]}{\ln 2} .$$

The solid line shown in Fig. 1(c) is drawn according to the average of parameters obtained from fitting the four samples. We see that the agreement between the $p$-model and the stock data is remarkable for both positive and negative parts, indicating the existence of information cascade process in stock market [51, 52]. We further extract the parameter $p$ of the $p$-model for all the stocks under investigation. Fig. 2 illustrates that the empirical occurrence frequency $g(p)$ as a function of $p$. We find that $\langle p \rangle = 0.40 \pm 0.02$. In contrast, fully developed turbulence gives $p = 0.3$ [50].

![Fig. 2. Empirical occurrence frequency $g(p)$ of $p$.](image)
4.2 Statistical tests

We assess the statistical significance of the empirical multifractality in the spirit of bootstrapping tests. We reshuffle the time series and perform the same multifractal analysis. Fig. 3 compares the multifractal spectra of the raw time series and that of the 10 shuffled time series for the two chosen indexes and two securities. The thin lines are associated with the real data, while the thick lines are obtained from the shuffled data. An eye inspection already shows the deviation of the multifractal spectra of the real data $f(\alpha)$ and that of the shuffled data $f_{\text{rnd}}(\alpha_{\text{rnd}})$. We can infer that one of the most important causations for the existence of the multifractality is the long memory in the volatility series.

![Fig. 3. Comparison between multifractal spectra extracted from real and shuffled data. The thin lines correspond to the real data, while the thick lines are for the shuffled data. (a) SSEC, (b) SZSC, (c) 000039, and (d) 600028.](image)

In the following, we shall give a more systemic statistical test on Chinese stock market. For all the chosen series, we shuffle the data 1000 times for each stock and reinvestigate the multifractality of the surrogate data. And the corresponding multifractal spectra are obtained. For each singularity spectrum, we calculate two characteristic quantities, $\Delta \alpha$ and $F \triangleq [f(\alpha_{\text{min}}) + f(\alpha_{\text{max}})]/2$. We find that $\Delta \alpha > \langle \Delta \alpha_{\text{rnd}} \rangle$ and $F < \langle F_{\text{rnd}} \rangle$ for all the series, which implies that there are discrepancies between the multifractal spectra of the shuffled data and that of the real data. We find that $p_1 = 0$ and $p_2 = 0$ for the SSEC and $p_1 = 0$ and $p_2 = 0$ for the SZSC, which provides undoubtable evidence for the presence of multifractality.
in the SSEC and SZSC data. Under the significance level of 1%, we find that the multifractal nature is significant for all stocks.

4.3 Quenched and annealed average

Fig. 4(a) illustrates the quenched and annealed exponents $\tau_{Q,A}$ against moment order $q$. For comparison, we also plot the mass exponents $\tau(q)$ of SSEC and SZSC. The multifractal spectra obtained from Legendre transformation are presented in Fig. 4(b). There are noticeable discrepancies between $\Delta \alpha_{A,Q}$ and $\Delta \alpha_{SSEC,SZSC}$. Indeed, ensemble averaging allows us to capture the fluctuations among different realizations, which widens the singularity spectrum when compared with that from individual time series. This feature is well illustrated in Fig. 4(b). Moreover, one can see that $\Delta \alpha_A > \Delta \alpha_Q$. The appearance of this inequality is linked to the sensitivity to rare values of the samples for annealed average.

![Fig. 4](color online) Quenched and annealed multifractal analysis. (a) Plots of mass exponents $\tau(q)$ with respect to the moment order $q$. (b) Multifractal spectra.

5 Concluding remarks

We have performed detailed multifractal analysis on minutely volatilities of two indexes and 1139 stocks in the Chinese stock markets based on the partition function approach. The minutely volatility is calculated as the sum of absolute returns in an interval of one minute with higher-frequency data. A measure $\mu$ is constructed as the normalized volatility. Hence, the measure $\mu$ is additive and conservational. This allows detailed multifractal analysis based on the partition function approach. We confirmed that $\mu$ is a multifractal measure.

According to our analysis, the partition function $\chi_q(s)$ for each security scales as a power law with regard to box size $s$ for each order $q$. The function of scaling exponents $\tau(q)$ for each security can be estimated. The nonlinearity of $\tau(q)$ acts
as a hallmark of multifractality. The $p$-model in turbulence has been used to fit the $\tau(q)$ function, resulting in a parameter $p = 0.40 \pm 0.02$. Statistical tests based on the bootstrapping technique confirm the significance of the multifractal nature in the volatility time series of individual securities.

An ensemble multifractal analysis was also carried out upon many stocks and the annealed and quenched mass exponent functions $\tau(q)$ have been determined. We note that the ensemble averaging allows us to characterize the global multifractal properties of large ensembles of data (the market as a whole) in a compact form without dealing with details of individual realizations (individual stocks). In this sense, we can draw a conclusion that the Chinese stock market as a whole also exhibits multifractal behavior.

As discussed in Section II, these correctly extracted multifractal characteristics might have potential usefulness in market prediction or risk management for individual securities or the whole market. It is thus interesting to repeat the analysis in literature [32, 34, 38] based on our results. However, this is beyond the scope of the current work.

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