ON THE STRONG HOMOTOPY ASSOCIATIVE ALGEBRA OF A FOLIATION

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Abstract. An involutive distribution $C$ on a smooth manifold $M$ is a Lie-algebroid acting on sections of the normal bundle $TM/C$. It is known that the Chevalley-Eilenberg complex associated to this representation of $C$ possesses the structure $X$ of a strong homotopy Lie-Rinehart algebra. It is natural to interpret $X$ as the (derived) Lie-Rinehart algebra of vector fields on the space $P$ of integral manifolds of $C$. In this paper, I show that $X$ is embedded in an $A_{\infty}$-algebra $D$ of (normal) differential operators. It is natural to interpret $D$ as the (derived) associative algebra of differential operators on $P$. Finally, I speculate about the interpretation of $D$ as the universal enveloping strong homotopy algebra of $X$.

Introduction

Let $M$ be a finite dimensional smooth manifold and $C$ an involutive distribution on it. In view of Fröbenius theorem the datum of $C$ is equivalent to the datum of a foliation of $M$. The pair $(M,C)$ is a finite dimensional instance of a diffiety (or a $D$-scheme, in the algebraic geometry language) which is a geometric object formalizing the concept of partial differential equation. There is a rich cohomological calculus, sometimes called secondary calculus [21 22 23], associated to a diffiety $(M,C)$. Secondary calculus may be interpreted to some extent as a differential calculus on the space of integral manifolds of $C$. All constructions of standard calculus on manifolds (vector fields, differential forms, differential operators, etc.) have a secondary analogue, i.e., a formal analogue within secondary calculus. For instance, secondary functions are characteristic cohomologies of $C$, secondary vector fields are characteristic cohomologies with local coefficients in normal vector fields, etc. (see the first part of [24] for a compact review of secondary Cartan calculus). In [25] I speculated that secondary calculus is actually a derived differential calculus in the sense that “all secondary constructions come from suitable algebraic structures up to homotopy at the level of (characteristic) cochains”. As a fundamental motivation behind this conjecture, I discussed in [25] the strong homotopy Lie-Rinehart algebra of secondary vector fields.

This is a companion paper of [25]. Here, I present a further motivation behind the above mentioned conjecture: the $A_{\infty}$-algebra of secondary (linear, scalar) differential operators. The main technical tools to show the existence of such $A_{\infty}$-algebra are homological perturbations and homotopy transfer. The strategy of the proof is the following. Let $D(\Lambda)$ be the associative differential graded (DG) algebra of differential operators on longitudinal differential forms $\Lambda$ (i.e., differential forms along $C$). It projects naturally onto the DG module $\Lambda \otimes D$ of $\Lambda$-valued differential operators on $C^\infty(M)$, normal to $C$. Actually, there are contraction data for $D(\Lambda)$ over $\Lambda \otimes D$ (see Subsection 1.4 for the definition of contraction data). The latter allow to induce an $A_{\infty}$-algebra structure on $\Lambda \otimes D$ from the DG algebra structure on $D(\Lambda)$.
Suitable contraction data can be constructed using purely geometric (supplementary) data as follows. First construct Poincaré-Birkhoff-Witt (PBW) type isomorphisms \( D(\Lambda) \cong S^*\text{Der}\Lambda \) and \( \Lambda \otimes \mathcal{D} \cong \Lambda \otimes S^*\mathcal{X} \) (here \( \text{Der}\Lambda \) is the DG Lie-Rinehart algebra of derivations of \( \Lambda \), and \( \mathcal{X} \) is the module of sections of the normal bundle \( TM/\mathcal{C} \)). Second, notice that \( S^*\text{Der}\Lambda \) and \( \Lambda \otimes S^*\mathcal{X} \) are commutative DG algebras and there are simple contraction data for \( S^*\text{Der}\Lambda \) over \( \Lambda \otimes S^*\mathcal{X} \). Third, use the Homological Perturbation Theorem (and the PBW isomorphisms) to construct contraction data for \( D(\Lambda) \) over \( \Lambda \otimes \mathcal{D} \), from contraction data for \( S^*\text{Der}\Lambda \) over \( \Lambda \otimes S^*\mathcal{X} \).

The paper is basically self-consistent and it is organized as follows. It is divided into three sections. In the first one, I collect the algebraic preliminaries: namely, differential operators on graded algebras, strong homotopy structures, homological perturbations and homotopy transfer. In subsection 1.2, I show how, under suitable regularity conditions (namely, the existence of a PBW type isomorphism), the universal enveloping algebra of a DG Lie-Rinehart algebra contracting over a complex \( (\mathcal{K}, \delta) \), can be homotopy transferred to produce an \( A_\infty \)-algebra structure on \( S^*\mathcal{K} \), the symmetric algebra of \( \mathcal{K} \) (see below for details). To my knowledge, this remark appears here for the first time. In the second section, I present my main framework, which consists of some basic geometry and homological algebra of a foliation, including few not so standard aspects like (normal) differential operators on a foliated manifold. Moreover, I define a distinguished class of connections on a foliated manifold, that I call adapted connections. Finally, I use adapted connections to construct two suitable PBW type isomorphisms \( D(\Lambda) \cong S^*\text{Der}\Lambda \) and \( \Lambda \otimes \mathcal{D} \cong \Lambda \otimes S^*\mathcal{X} \). Notice that a concept more general than an adapted connection is used in the note [15] (see also [6]) for similar purposes, in the much wider context of Lie pairs. Unfortunately, [15] does not contain proofs. In the third section, I collect all the constructions introduced in the preceding sections to get the \( A_\infty \)-algebra structure on \( \Lambda \otimes \mathcal{D} \) as outlined above. Finally, I compute the higher order components of all higher operations and, in particular, prove that they vanish from the fourth on. In the conclusions, I speculate about the interpretation of the \( A_\infty \)-algebra \( \Lambda \otimes \mathcal{D} \) as the universal enveloping strong homotopy algebra of the strong homotopy Lie-Rinehart algebra \( (\Lambda, \Lambda \otimes \mathcal{X}) \).

0.1. Conventions and notations. I will adopt the following notations and conventions throughout the paper. Let \( k_1, \ldots, k_\ell \) be positive integers. I denote by \( S_{k_1, \ldots, k_\ell} \) the set of \((k_1, \ldots, k_\ell)\)-unshuffles, i.e., permutations \( \sigma \) of \( \{1, \ldots, k_1 + \cdots + k_\ell\} \) such that

\[
\sigma(k_1 + \cdots + k_{i-1} + 1) < \cdots < \sigma(k_1 + \cdots + k_i - 1 + k_i), \quad i = 1, \ldots, \ell.
\]

The degree of a homogeneous element \( v \) in a graded vector space will be denoted by \( \bar{v} \). However, when it appears in the exponent of a sign \( (−) \), I will always omit the overbar, and write, for instance, \( (−)^{\bar{v}} \) instead of \( (−)^v \).

Every vector space will be over a field \( K \) of zero characteristic, which will actually be \( \mathbb{R} \) in Section 3. If \( V = \bigoplus_i V^i \) is a graded vector space, I denote by \( V[1] = \bigoplus_i V^i[1] \) its suspension, i.e., the graded vector space defined by putting \( V^i[1] = V^{i+1} \).

If \( W \) is a (left) module over a graded, associative, graded commutative, unital algebra \( A \), I denote by \( \odot \) the symmetric product in the (graded) symmetric algebra \( S^*_A W \) of \( W \).

Let \( V_1, \ldots, V_n \) be graded vector spaces,

\[
v = (v_1, \ldots, v_n) \in V_1 \times \cdots \times V_n,
\]
and \( \sigma \in S_n \) a permutation. I denote by \( \chi(\sigma, v) \) the sign implicitly defined by
\[
v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = \chi(\sigma, v) v_1 \wedge \cdots \wedge v_n,
\]
where \( \wedge \) is the graded skew-symmetric product in the (graded) exterior algebra of \( V_1 \oplus \cdots \oplus V_n \).

Now, let \( M \) be a smooth manifold. I denote by \( C^\infty(M) \) the real algebra of smooth functions on \( M \), by \( \mathcal{X}(M) \) the Lie-Rinehart algebra of vector fields on \( M \), and by \( \Lambda(M) \) the DG algebra of differential forms on \( M \). Elements in \( \Lambda(M) \) are always understood as derivations of \( C^\infty(M) \)-valued, skew-symmetric, multilinear maps on \( \mathcal{X}(M) \). I denote by \( d : \Lambda(M) \to \Lambda(M) \) the exterior differential. Every tensor product will be over \( K \), if not explicitly stated otherwise, and will be simply denoted by \( \otimes \). The tensor product over \( C^\infty(M) \) will be denoted by \( \otimes^M \). I adopt the Einstein summation convention.

By a connection I will mean a linear connection in \( T^*M \) or, which is the same, in \( TM \). Moreover, I will always understand the obvious extension of a connection to the whole tensor bundle \( \bigoplus_{i,j} TM^{\otimes i} \otimes T^*M^{\otimes j} \). Let \( \nabla \) be a connection, \( \ldots, z^a, \ldots \) coordinates in \( M \), and \( T \) a covariant tensor on \( M \) locally given by
\[
T = T_{a_1 \ldots a_k} dz^a_1 \otimes \cdots \otimes dz^a_k.
\]
I denote by \( \nabla_a T_{a_1 \ldots a_k} \) the components of the covariant derivative \( \nabla T \) of \( T \) with respect to \( \nabla \), i.e.,
\[
\nabla T = \nabla_a T_{a_1 \ldots a_k} dz^a \otimes dz^{a_1} \otimes \cdots \otimes dz^{a_k}.
\]
Finally, the round bracket in \( T_{(a_1 \ldots a_k)} \) denotes symmetrization, i.e.,
\[
T_{(a_1 \ldots a_k)} = \frac{1}{k!} \sum_{\sigma \in S_k} T_{a_{\sigma(1)} \ldots a_{\sigma(k)}}.
\]

1. Algebraic Preliminaries

1.1. Differential Operators over Graded Commutative Algebras. Let \( A \) be an associative, graded commutative, unital \( K \)-algebra, and let \( P, Q \) be (left) \( A \)-modules. An element \( a \in A \), define endomorphisms (multiplications by \( a \)) \( P \to P \) and \( Q \to Q \) which, abusing the notation, I denote again by \( a \). Consider the graded \( A \)-linear map
\[
\delta_a : \text{Hom}_K(P, Q) \to \text{Hom}_K(P, Q)
\]
defined by
\[
\delta_a \phi := [a, \phi] := a \circ \phi - (-)^{a \phi} \phi \circ a,
\]
where \( [\cdot, \cdot] \) is the graded commutator. A graded, \( K \)-linear map
\[
\Box : P \to Q
\]
is a (linear) differential operator of order \( k \) if
\[
\delta_{a_0} \delta_{a_1} \cdots \delta_{a_k} \Box = 0 \quad \text{for all } a_0, a_1, \ldots, a_k \in A.
\]

Example 1. A derivation of \( A \) is a differential operator of order 1. More generally, a derivation \( \Box : P \to P \) of \( P \) subordinate to a derivation \( \Delta \) in \( A \), i.e., an operator \( \Box \) such that
\[
\Box(ap) = \Delta(a)p + (-)^{a \Box} a \Box p, \quad a \in A, \quad p \in P,
\]
is a differential operator of order 1.
The left $A$-module of differential operators $\Box : P \rightarrow Q$ of order $k$ will be denoted by $\mathcal{D}_k(P, Q)$. Clearly, $\mathcal{D}_0(P, Q) = \text{Hom}_A(P, Q)$ and there is a sequence of inclusions

$$
\mathcal{D}_0(P, Q) \subset \cdots \subset \mathcal{D}_k(P, Q) \subset \mathcal{D}_{k+1}(P, Q) \subset \cdots ,
$$

defining a filtration in the $A$-module $\mathcal{D}(P, Q) := \bigcup_k \mathcal{D}_k(P, Q)$. The associated graded object $S(P, Q) := \bigoplus_k S_k(P, Q)$, $S_k(P, Q) := \mathcal{D}_k(P, Q)/\mathcal{D}_{k-1}(P, Q)$, is called the module of symbols. I denote by

$$
\sigma_k : \mathcal{D}_k(P, Q) \rightarrow S_k(P, Q)
$$

the projection.

Let $R$ be another $A$-module. The composition $\Box_1 \circ \Box_2 : P \rightarrow R$ of differential operators $\Box_1 : Q \rightarrow R$ and $\Box_2 : P \rightarrow Q$, of order $\ell_1$ and $\ell_2$, respectively, is a differential operator of order $\ell_1 + \ell_2$. Accordingly, there is a well defined $A$-bilinear map

$$
\circ : S(Q, R) \otimes S(P, Q) \rightarrow S(P, R)
$$

defined by

$$
\sigma_\ell_1(\Box_1) \circ \sigma_\ell_2(\Box_2) := \sigma_{\ell_1 + \ell_2}(\Box_1 \circ \Box_2), \quad \Box_i \in \mathcal{D}_{\ell_i}(P, Q), \quad i = 1, 2.
$$

I denote simply by $\mathcal{D}(A) = \bigcup_k \mathcal{D}_k(A)$ (or just $\mathcal{D} = \bigcup_k \mathcal{D}_k$, if this does not lead to confusion) the graded, associative, filtered, unital $K$-algebra $\mathcal{D}(A, A)$ of differential operators $A \rightarrow A$ and by $\mathcal{S}(A)$ (or just $\mathcal{S}$) the corresponding module of symbols. The bilinear map $\mathcal{S}(A) \otimes \mathcal{S}(A) \rightarrow \mathcal{S}(A)$ defined above, gives $\mathcal{S}(A)$ the structure of an associative, graded commutative, unital $K$-algebra. Notice that the (graded) commutator $[\Box_1, \Box_2]$ of differential operators $\Box_1, \Box_2 : A \rightarrow A$ of order $\ell_1, \ell_2$, respectively, is a differential operator of the order $\ell_1 + \ell_2 - 1$. Accordingly, there is a well defined $K$-bilinear bracket

$$
\{\cdot, \cdot\} : \mathcal{S}(A) \otimes \mathcal{S}(A) \rightarrow \mathcal{S}(A)
$$

defined by

$$
\{\sigma_{\ell_1}(\Box_1), \sigma_{\ell_2}(\Box_2)\} := \sigma_{\ell_1 + \ell_2 - 1}(\Box_1, \Box_2), \quad \Box_i \in \mathcal{D}_{\ell_i}, \quad i = 1, 2.
$$

The bracket $\{\cdot, \cdot\}$ gives $\mathcal{S}$ the structure of a graded Poisson $K$-algebra. Notice that $\mathcal{D}_0 = \mathcal{S}_0 = A$, $\mathcal{D}_1 = A \oplus \text{Der}A$ and $\mathcal{S}_1 = \text{Der}A$, where $\text{Der}A$ denotes the $A$-module of derivations of $A$.

Denote by $\text{Der}_k(A, Q)$, the $A$-module of graded symmetric, $Q$-valued multiderivations of $A$ with $k$ entries. The map

$$
\varepsilon_k : S_k(A, Q) \rightarrow \text{Der}_k(A, Q)
$$

given by

$$
\varepsilon_k(\sigma_k(\Box)(a_1, \ldots, a_k)) := (\delta_{a_1} \cdots \delta_{a_k} \Box) 1, \quad \Box \in \mathcal{D}_k, \quad a_1, \ldots, a_k \in A
$$

is a well defined $A$-linear map.

**Remark 2.** Let $A$ be the $\mathbb{R}$-algebra of smooth functions on a graded manifold $N$. Then $\text{Der}_k(A, Q) \simeq Q \otimes S_A^k \text{Der}A$ and $\varepsilon_k$ is an isomorphism of $A$-modules, whose inverse

$$
Q \otimes S_A^k \text{Der}A \rightarrow S_k(A, Q)
$$

is defined by

$$
q \otimes X_1 \circ \cdots \circ X_k \mapsto \sigma_k(q X_1 \circ \cdots \circ X_k),
$$

$q \in Q$, $X_1, \ldots, X_k \in \mathcal{X}(N)$. Moreover, $(\mathcal{S}, \{\cdot, \cdot\})$ is the Poisson algebra of fiber-wise polynomial functions on $T^*N$. 


1.2. Universal Enveloping of a Lie-Rinehart Algebra. Let $A = \bigoplus_i A_i$ be an associative, graded commutative, unital $K$-algebra, and $(A, Q)$ a graded Lie-Rinehart algebra, i.e., 1) $Q$ is a graded Lie algebra and 2) an $A$-module, 3) $A$ is a $Q$-module, and 4) the following compatibility conditions hold

\[(a \cdot q) \cdot b = a \cdot (q \cdot b)\]
\[q \cdot (a \cdot b) = (q \cdot a) \cdot b + (-)^{a q} a \cdot (q \cdot b)\]
\[[a \cdot q, r] = a \cdot [q, r] - (-)^{a q} r(a \cdot q) - r(q) \cdot a\]

for all $a, b \in A$, $q, r \in Q$. In particular $Q$ acts on $A$ via derivations. The prototype of a Lie-Rinehart algebra is $(A, \text{Der} A)$.

An enveloping algebra of the Lie-Rinehart algebra $(A, Q)$ is a graded, associative, unital $K$-algebra $E$ together with 1) a morphism $j : A \rightarrow E$ of $K$-algebras, and 2) a morphism of Lie algebras $J : Q \rightarrow E$ such that 3)

\[J(a \cdot q) = j(a)J(q)\]
\[j(q \cdot a) = J(q)j(a) - (-)^{a q} j(a)J(q)\]

for all $a \in A$, $q \in Q$. As an example, notice that the associative algebra $D(A)$ is an enveloping algebra of $(A, Q)$, with morphisms $j$, $J$ given by the canonical injection $A \rightarrow D(A)$ and the action $Q \rightarrow \text{Der} A \subset D(A)$.

A morphism of the enveloping algebras $E$ and $E'$ is a morphism $f : E \rightarrow E'$ of graded, unital $K$-algebras such that diagrams

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow & & \downarrow \\
A & \xrightarrow{j} & Q
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow & & \downarrow \\
A & \xrightarrow{j} & Q
\end{array}
\]

commute. A universal enveloping algebra is an enveloping algebra $U(Q)$ such that for any other enveloping algebra $E$ there is a unique morphism $U(Q) \rightarrow E$ of enveloping algebras. In particular an enveloping algebra of $Q$ acts on $A$ by differential operators, i.e., there is a morphism of $K$-algebras

\[U(Q) \rightarrow D(A).\] (1)

Universal enveloping algebras are clearly unique up to (unique) isomorphisms. A canonical one can be constructed as follows. Let $\mathcal{U}$ be the tensor algebra of the graded vector space $A \oplus Q$, and $I \subset \mathcal{U}$ the two sided ideal generated by relations

\[a \otimes b = a \cdot b\]
\[a \otimes q = a \cdot q\]
\[q \otimes a - (-)^{a q} a \otimes q = q \cdot a\]
\[q \otimes r - (-)^{r q} r \otimes q = [q, r],\]

for all $a, b \in A$, and $q, r \in Q$. Put $U(Q) := \mathcal{U}/I$. Then $U(Q)$ is clearly a universal enveloping algebra of $Q$ with morphisms $j$, and $J$ given by the compositions of the canonical injections $A \rightarrow \mathcal{U}$, and $Q \rightarrow \mathcal{U}$, with the projection $\mathcal{U} \rightarrow U(Q)$. 


It follows from the above construction that $U(Q)$ possesses an algebra filtration

$$U_0(Q) \subset U_1(Q) \subset \cdots \subset U_i(Q) \subset \cdots \subset U(Q)$$  

(2)

bounded from below, where $U_i(Q) \subset U(Q)$ is the left $A$-submodule generated by products of at most $i$ elements of the form $J(q)$, $q \in Q$. I denote by $\text{Gr}U(Q) := U_i(Q)/U_{i-1}(U)$. Since

$$[U_i(Q), U_j(Q)] \subset U_{i+j-1}(Q),$$

$\text{Gr}U(Q)$ is a commutative algebra, and the commutator in $U(Q)$ induce a graded Poisson bracket in it. Notice that $U_0(Q) = \text{Gr}_0U(Q) = A$ and $U_1(Q) = \text{Gr}_1U(Q) \oplus A$ where the splitting $U_1(Q) \rightarrow A$ of the exact sequence

$$0 \rightarrow A \rightarrow U_1(Q) \rightarrow \text{Gr}_1U(Q) \rightarrow 0$$

is given by

$$\Delta \mapsto \Delta(1).$$

There is a canonical $A$-linear, surjective, Poisson map

$$S^*_AQ \rightarrow \text{Gr}U(Q)$$

(3)

mapping $S^*_AQ$ to $\text{Gr}iU(Q)$, and given by

$$q_1 \odot \cdots \odot q_i \mapsto J(q_1) \cdots J(q_i) + U_{i-1}(Q).$$

**Remark 3.** If $A$ is the graded algebra of smooth functions on a graded manifold $N$ and $Q$ is the module of sections of a graded Lie algebroid over $N$ then $(A,Q)$ is a graded Lie-Rinehart algebra and 1) projection (3) is an isomorphism, moreover 2) exact sequences $0 \rightarrow U_{i-1}(Q) \rightarrow U_i(Q) \rightarrow \text{Gr}_iU(Q) \rightarrow 0$ split (in a non canonical way). Therefore there is a (non-canonical) Poincaré-Birkhoff-Witt (PBW) type isomorphism of (filtered) $A$-modules

$$U(Q) \approx S^*_AQ,$$

(for details about how to construct such isomorphism in the non-graded case see, for instance, [19]). Notice that, if $(A,Q)$ is the Lie-Rinehart algebra of vector fields over $N$, then (3) is an isomorphism and $U(Q)$ identifies with $\mathcal{D}(A)$ in a canonical way. Consequently, $\text{Gr}U(Q)$ identifies with the algebra $\mathcal{S}(A)$ of symbols.

Now, suppose that $A$ is a commutative DG algebra with differential $\delta$, and $(A,Q)$ is a DG Lie-Rinehart algebra, i.e., $Q$ is endowed with a degree 1 differential $\delta_0$ such that 1) $\delta_0$ is a derivation of the graded Lie algebra structure, 2) $\delta_0$ is a derivation of the $A$-module $Q$ subordinate to $\delta$, i.e.,

$$\delta_0(a \cdot q) = \delta a \cdot q + (-)^a a \cdot \delta_0 q.$$  

In the above hypothesis, $\delta$ and $\delta_0$ can be extended to a unique derivation of the tensor algebra $U$. Moreover, such derivation preserves the ideal $I$ and, therefore, descends to a derivation of $U(Q)$ which becomes a DG algebra (satisfying a DG version of the universal properties of universal enveloping algebras) called the universal enveloping DG algebra of the DG Lie-Rinehart algebra $Q$.  

Example 4. Let $A$ be the DG algebra of smooth functions on a DG manifold $N$ with homological vector field $d$, let $Q = \text{Der} A$, and let $\delta_0 : \text{Der} A \to \text{Der} A$ be the inner derivation $[d, \cdot]$. Then $U(Q)$ identifies with $\mathcal{D}(A)$ and the differential in it is again $[d, \cdot]$. Hence, $\text{Gr} U(Q)$ identifies with $S_A^\bullet Q$, the DG Poisson algebra of fiberwise polynomial functions on $T^* N$.

For more details about the material contained in this subsection see, for instance, [8, 18].

1.3. Strong Homotopy Structures. In this paper, conventions about strong homotopy algebras are the same as in [13]. Let $(V, \delta)$ be a cochain complex of vector spaces and $A$ be any kind of algebraic structure (associative algebra, Lie algebra, module, etc.). Roughly speaking, a homotopy $A$-structure on $(V, \delta)$ is an algebraic structure on $V$ which is of the kind $A$ only up to $\delta$-homotopies, and a strong homotopy (SH) $A$-structure is a homotopy structure possessing a full system of (coherent) higher homotopies. In this paper, I will basically deal with four kinds of SH structures, namely SH associative algebras (also named $A_\infty$-algebras), SH modules (also named $A_\infty$-modules), SH Lie-Rinehart algebras, and Poisson $L_\infty$-algebras. For them I provide detailed definitions below.

Definition 5. An $A_\infty$-algebra is a pair $(A, \mathcal{A})$, where $A$ is a graded vector space, and $\mathcal{A} = \{\alpha_k, k \in \mathbb{N}\}$ is a family of $k$-ary, multilinear, degree $2 - k$ operations

$$\alpha_k : A^\otimes k \to A, \quad k \in \mathbb{N}.$$ 

such that

$$\sum_{i+j=k} (-)^i j \sum_{\ell=0} i+\ell (-)^{i+1+\ell + \cdots + x_{\ell+1}} \alpha_{j+1}(x_1, \ldots, x_{\ell}, \alpha_1(x_{\ell+1}, \ldots, x_{\ell+i+1}, \ldots, x_{i+j} = 0$$

for all $x_1, \ldots, x_k \in A$, $k \in \mathbb{N}$ (in particular, $(A, \alpha_1)$ is a cochain complex and $H(A, \alpha_1)$ is a graded associative algebra).

If $A$ is concentrated in degree 0, then an $A_\infty$-algebra structure on $A$ is simply an associative algebra structure for degree reasons. Similarly, if $\alpha_k = 0$ for all $k > 2$, then $(A, \mathcal{A})$ is a DG (associative) algebra.

Let $(A, \mathcal{A})$ be an $A_\infty$-algebra.

Definition 6. A strict unit in $A$ is a degree 0 element $e \in A$ such that $\alpha_2(e, x) = \alpha_2(x, e) = x$ for all $x \in A$ and $\alpha_k = 0$, for all $k \neq 2$, whenever one of the entries is equal to $e$. An $A_\infty$-algebra with a strict unit is called strictly unital.

Now let $M$ be a graded vector space and $\mathcal{M} = \{\mu_k, k \in \mathbb{N}\}$ a family of $k$-ary, multilinear, degree $2 - k$ operations,

$$\mu_k : A^\otimes (k-1) \otimes M \to M, \quad k \in \mathbb{N}.$$ 

Define new operations

$$\alpha_k^\oplus : (A \oplus M)^\otimes k \to A \oplus M, \quad k \in \mathbb{N},$$

extending the previous ones by linearity, and the condition that the result is zero if one of the first $k-1$ entries is from $M$. 

Definition 7. An $A_{\infty}$-module over $(A, \mathcal{A})$ is a pair $(M, \mathcal{M})$, where $M$ is a graded vector space, and $\mathcal{M} = \{\mu_k, k \in \mathbb{N}\}$ is a family of $k$-ary, multilinear, degree $2-k$ operations,

$$\mu_k : A^{\otimes(k-1)} \otimes M \to M, \quad k \in \mathbb{N},$$

such that

$$\sum_{i+j=k} (-)^{ij} \sum_{t=0}^{i+j} (-)^t (i+1)(y_1+\cdots+y_t) \alpha_{i+1}^{(t)} (y_1, \ldots, y_t; y_{t+1}, \ldots, y_{i+j}) = 0$$

for all $y_1, \ldots, y_k \in A \oplus M$, $k \in \mathbb{N}$ (in particular, $(M, \mu_1)$ is a complex and $H(M, \mu_1)$ is a graded $H(A, \alpha_1)$-module).

If both $A$ and $M$ are concentrated in degree 0, then an $A_{\infty}$-module structure on $M$ over $A$ is simply a left module structure over the associative algebra $A$. Similarly, if $\alpha_k = 0$ and $\mu_k = 0$ for all $k > 2$, then $(M, \mathcal{M})$ is a DG module over the DG algebra $A$.

Definition 8. An $L_{\infty}$-algebra is a pair $(L, \mathcal{L})$, where $L$ is a graded vector space, and $\mathcal{L} = \{\lambda_k, k \in \mathbb{N}\}$ is a family of $k$-ary, multilinear, degree $2-k$ operations,

$$\lambda_k : L^{\otimes k} \to L, \quad k \in \mathbb{N},$$

such that

$$\sum_{i+j=k} (-)^{ij} \sum_{\sigma \in S_{i,j}} \chi(\sigma, v) \lambda_{i+1}^{(1)} (v_{\sigma(1)}, \ldots, v_{\sigma(i)}; v_{\sigma(i+1)}, \ldots, v_{\sigma(i+j)}) = 0,$$

for all $v = (v_1, \ldots, v_k), v_1, \ldots, v_k \in L$, $k \in \mathbb{N}$ (in particular, $(L, \lambda_1)$ is a complex and $H(L, \lambda_1)$ is a graded Lie algebra).

If $L$ is concentrated in degree 0, then an $L_{\infty}$-algebra structure on $L$ is simply a Lie algebra structure. Similarly, if $\lambda_k = 0$ for all $k > 2$, then $(L, \mathcal{L})$ is a DG Lie algebra.

Let $(L, \mathcal{L})$ be an $L_{\infty}$-algebra, $N$ a graded vector space, and let $\mathcal{N} = \{\nu_k, k \in \mathbb{N}\}$ be a family of $k$-ary, multilinear, degree $2-k$ operations

$$\nu_k : L^{\otimes(k-1)} \otimes N \to N, \quad k \in \mathbb{N},$$

Define new operations

$$\lambda_k^{(1)} : (L \otimes N)^{\otimes k} \to L \oplus N, \quad k \in \mathbb{N},$$

extending the previous ones by linearity, skew-symmetry, and the condition that the result is zero if more than one entry are from $N$.

Definition 9. An $L_{\infty}$-module is a pair $(N, \mathcal{N})$, where $N$ is a graded vector space, and $\mathcal{N} = \{\nu_k, k \in \mathbb{N}\}$ is a family of $k$-ary, multilinear, degree $2-k$ operations

$$\nu_k : L^{\otimes(k-1)} \otimes N \to N, \quad k \in \mathbb{N},$$

such that

$$\sum_{i+j=k} (-)^{ij} \sum_{\sigma \in S_{i,j}} \chi(\sigma, b) \lambda_{i+1}^{(1)} (b_{\sigma(1)}, \ldots, b_{\sigma(i)}; b_{\sigma(i+1)}, \ldots, b_{\sigma(i+j)})$$

for all $b = (v_1, \ldots, v_{k-1}, n), v_1, \ldots, v_{k-1} \in L, n \in N$, $k \in \mathbb{N}$ (in particular, $(N, \nu_1)$ is a complex and $H(N, \nu_1)$ is a graded $H(L, \lambda_1)$-module).
If both $L$ and $N$ are concentrated in degree 0, then an $L_\infty$-module structure on $N$ over $L$ is simply a Lie module structure over the Lie algebra $L$. Similarly, if $\lambda_k = 0$ and $\nu_k = 0$ for all $k > 2$, then $(N,M)$ is a DG Lie module over the DG Lie algebra $L$.

I now define SH Lie-Rinehart algebras \cite{12}. For simplicity, I call the resulting objects $LR_\infty$-algebras.

**Definition 10.** An $LR_\infty$-algebra is a pair $(A,Q)$, where $A$ is an associative, graded commutative, unital algebra, and $(Q,\mathcal{D})$ is an $L_\infty$-algebra, $\mathcal{D} = \{\lambda_k, k \in \mathbb{N}\}$. Moreover, $Q$ possesses the structure of an $A$-module, and $A$ possesses the structure $\mathcal{M} = \{\nu_k, k \in \mathbb{N}\}$ of an $L_\infty$-module over $Q$, such that

1. $\nu_k : Q^{\otimes (k-1)} \otimes A \rightarrow A$ is a derivation in the last argument, and $A$-multilinear in the first $k-1$ arguments;

2. **Formula**

$$\lambda_k(q_1, \ldots, q_{k-1},aq_k) = \nu_k(q_1, \ldots, q_{k-1} | a) \cdot q_k + (-)^{a(q_1+\cdots+q_{k-1}-k)} a \cdot \lambda_k(q_1, \ldots, q_{k-1},q_k), \quad (4)$$

holds for all $q_1, \ldots, q_k \in Q$, $a \in A$, $k \in \mathbb{N}$ (in particular, $(Q,\lambda_1)$ is a DG module over $(A,\nu_1)$, and $(H(A,\nu_1),H(Q,\lambda_1))$ is a graded Lie-Rinehart algebra.

If $Q$ and $A$ are concentrated in degree 0, then $(A,Q)$ is simply a Lie-Rinehart algebra. Similarly, if $\lambda_k = 0$ and $\nu_k = 0$ for all $k > 2$, then $(A,Q)$ is a DG Lie-Rinehart algebra.

In the smooth setting, i.e., when $A$ is the algebra of smooth functions on a smooth manifold $M$ (in particular $A$ is concentrated in degree 0), and $Q[-1]$ is the $A$-module of sections of a graded bundle $E$ over $M$, then $E$ is sometimes called an $L_\infty$-algebroid \cite{20,21}.

**Definition 11.** A Poisson $L_\infty$-algebra is an $L_\infty$-algebra $(P,\mathcal{P})$, $\mathcal{P} = \{\Lambda_k, k \in \mathbb{N}\}$, such that $P$ possesses the structure of an associative, graded commutative, unital algebra and $\Lambda_k$ is a graded multiderviation for all $k \in \mathbb{N}$.

**Remark 12.** Poisson $L_\infty$-algebras are called $P_\infty$-algebras in \cite{1}. Notice that they are homotopy versions of Poisson algebra where “only the Poisson bracket is homotopyfied”, while the associative, commutative product is not. More general versions of Poisson algebras up to homotopy can be obtained via the (systematic) operadic approach to homotopy algebras (see, for instance, \cite{26}). This is the main reason why, as suggested by an anonymous referee, I do not use the name SH Poisson algebras for Poisson $L_\infty$-algebras. Similar considerations hold actually for Definition \cite{44} where “only the Lie bracket and Lie module structure on a Lie-Rinehart algebra are homotopyfied” while the associative, commutative product, and the corresponding module structure are not. In this case, however, it is safer to keep the name SH Lie-Rinehart algebra since an operadic approach in this context is still missing.

Notice that if $P$ is concentrated in degree 0, then a SH Poisson algebra structure on $P$ is simply a Poisson algebra structure. Similarly, if $\Lambda_k = 0$ for all $k > 2$, then $(P,\mathcal{P})$ is a DG Poisson algebra.

**Remark 13.** Let $A$ be an associative, graded commutative, unital algebra and $Q$ an $A$-module. The datum of an $LR_\infty$-algebra structure on $(A,Q)$ is equivalent to the datum of a SH Poisson algebra structure on $S^\bullet_A Q$ such that

$$\Lambda_k(u_1, \ldots, u_k) \in S^\bullet_A Q$$
whenever \( u_i \in S^n_A Q, i = 1, \ldots, k \) \cite{3}. The operations in \( S^n_A Q \) can be obtained from the ones in \( Q \), extending the latter as multiderivatives.

Finally, notice that the canonical construction of a Lie algebra from an associative algebra can be generalized to the SH context as follows. Let \((A, \mathcal{A})\) be an \( A_\infty \)-algebra, \( \mathcal{A} = \{ \alpha_k, k \in \mathbb{N} \} \). Define new operations

\[
A\alpha_k : A^{\circ k} \to A,
\]

by putting

\[
(A\alpha_k)(x_1, \ldots, x_k) := \sum_{\sigma \in S_k} \chi(\sigma, x)\alpha_k(x_{\sigma(1)}, \ldots, x_{\sigma(k)}),
\]

\( x = (x_1, \ldots, x_k) \), \( x_1, \ldots, x_k \in A \), i.e., \( A\alpha_k \) is the skew-symmetrization of \( \alpha_k \). The \( A\alpha \)'s give to \( A \) the structure of an \( L_\infty \)-algebra \cite{14}.

**Remark 14.** The theory of universal enveloping of \( L_\infty \)-algebras (see, for instance, \cite{II}) is not fully developed, not to speak about universal enveloping of \( LR_\infty \)-algebras. However, few (naïve) remarks can be done in this respect. First of all, recall that a morphism \( f : A \to A' \) (resp., \( f : L \to L' \)) of \( A_\infty \)-algebras (resp., \( L_\infty \)-algebras) is a family of \( K \)-multilinear (resp., skew-symmetric, \( K \)-multilinear) maps \( f_\nu : A^{\circ k} \to A' \) (resp., \( f_k : L^{\circ k} \to L' \)) satisfying suitable compatibility conditions (see, for instance, \cite{13} for details). It is tempting to define an enveloping SH algebra for an \( LR_\infty \)-algebra \( Q \) over a DG algebra \( A \), as an \( A_\infty \)-algebra \( E \) together with 1) a morphism of DG algebras \( j : A \to E \), and 2) a morphism of \( L_\infty \)-algebras \( J : Q \to E \) such that 3)

\[
J_k(a \cdot q_1, q_2, \ldots, q_k) = j(a)J_k(q_1, \ldots, q_k)
\]

and

\[
j(\nu_k(q_1, \ldots, q_{k-1}|a)) = \sum_{\ell=1}^{k-1} \sum_{k_1 + \cdots + k_\ell = k-1} \sum_{\sigma \in S_{k_1, \ldots, k_\ell}^<} \chi(\sigma, q)\alpha_{\ell+1}(J_{k_1}(q_{\sigma(1)}), \ldots, J_{k_\ell}(\ldots, q_{\sigma(k-1)}), ja), \tag{5}
\]

(\( k_1, \ldots, k_\ell \) is the set of \( k_1, \ldots, k_\ell \)-unshuffles such that
\[
\sigma(k_1 + \cdots + k_{\ell-1} + 1) < \sigma(k_1 + \cdots + k_{\ell-1} + k_\ell + 1) \quad \text{whenever} \quad k_\ell = k_{\ell+1},
\]

see the definition of morphism of \( L_\infty \)-algebras, e.g., in \cite{13}). One could then define a universal enveloping SH algebra as an enveloping SH algebra satisfying (obvious) universal properties, and try to construct it. Developing these ideas, however, goes beyond the scopes of this paper.

1.4. Homological Perturbations and Homotopy Transfer. The main homological tools used in this paper are the Perturbation Lemma and the Homotopy Transfer Theorem. I recall in this section those versions of them that will be used below.

Let \((K, \delta)\) and \((K, \delta)\) be cochain complexes of vector spaces, \( p : (K, \delta) \to (K, \delta) \) and \( j : (K, \delta) \to (K, \delta) \) cochain maps, and let \( h : K \to K \) be a degree \(-1\) endomorphism:

\[
h : (K, \delta) \xrightarrow{p} (K, \delta) \xrightarrow{j} (K, \delta)
\]

**Definition 15.** The data \((p, j, h)\) are contraction data for \((K, \delta)\) over \((K, \delta)\) if
(1) $j$ is a right inverse of $p$, i.e., $pj = id$,
(2) $h$ is a contracting homotopy, i.e., $[h, \delta] = id - jp$,
(3) the side conditions $h^2 = 0$, $hj = 0$, $ph = 0$ are satisfied.

Now, let $(p_0, j_0, h_0)$ be contraction data for a cochain complex $(K, \delta_0)$ over $(K, \delta)$. Suppose that there is another differential $\delta$ in $K$, and put $t := \delta_0 - \delta$. The Perturbation Lemma allows one to construct contraction data for $(K, \delta)$ over a suitable new complex $(K, \delta_t)$.

Theorem 16 (Perturbation Lemma). Let $th_0 : K \rightarrow K$ be locally nilpotent, i.e., for any $x \in K$ there is $k \in \mathbb{N}$ such that $(th_0)^k(x) = 0$, and

$$X := t + th_0t + th_0th_0t + \cdots = \sum_{i=0}^{\infty} t(h_0t)^i = \sum_{i=0}^{\infty} (th_0)^it.$$  
Moreover, let $\delta_t, p_t, j_t, h_t$ be defined as

$$\delta_t := \delta - p_0Xj_0 \tag{6}$$
$$p_t := p_0(id + Xh_0) \tag{6}$$
$$j_t := (id + h_0X)j_0 \tag{7}$$
$$h_t := h_0 + h_0Xh_0. \tag{8}$$

Then $(p_t, j_t, h_t)$ are contraction data for $(K, \delta)$ over $(K, \delta_t)$.

Remark 17. A rather standard situation, which will be also encountered in this paper, is when $K$ and $\mathcal{K}$ are endowed with filtrations

$$K_0 \subset K_1 \subset \cdots \subset K_i \subset \cdots \subset K,$$
$$\mathcal{K}_0 \subset \mathcal{K}_1 \subset \cdots \subset \mathcal{K}_i \subset \cdots \subset \mathcal{K},$$

bounded from below, and such that, 1) they are preserved by $\delta_0, \delta, p_0, j_0, h_0$, and 2) $t(K_i) \subset K_{i-1}$. In this case, $th_0$ is automatically locally nilpotent and the Perturbation Lemma applies.

Contraction data for $(K, \delta)$ over $(K, \delta_t)$ can be used to transfer SH structures from the former to the latter, in particular when the SH structure one begins with does not possess higher homotopies. This is a rather rich source of SH structures. Moreover, there are explicit formulas for the higher homotopies of the induced structure.

Theorem 18 (Homotopy Transfer Theorem, see, e.g., [16, 10]). Let $(\mathcal{V}, \delta)$ and $(\mathcal{V}, \delta)$ be cochain complexes and let $(p, j, h)$ be contraction data for $(\mathcal{V}, \delta)$ over $(\mathcal{V}, \delta)$.  

(1) Assume $(\mathcal{V}, \delta)$ possesses the structure $\circ$ of a DG algebra, and let $\mathcal{A} = \{\alpha_k, k \in \mathbb{N}\}$ be the family of graded operations

$$\alpha_k : \mathcal{V}^\otimes k \rightarrow \mathcal{V}$$
defined by

$$\alpha_1 := \delta, \quad \alpha_k := p\beta_k, \quad k \geq 2,$$

where the $\beta$’s are inductively defined by

$$\gamma_1 := -j, \quad \gamma_k := h\beta_k,$$
and
\[ \beta_k(x_1, \ldots, x_k) := \sum_{\ell + m = k} (-)^{\ell}a(\ell, m, x)\gamma(\ell, x_1, \ldots, x_\ell) \circ \gamma_m(x_{\ell+1}, \ldots, x_{\ell+m}), \]
x_1, \ldots, x_k \in \mathcal{V}, \text{ where } a(\ell, m, x) := \ell - 1 + (m - 1)\sum_{i=1}^{\ell}x_i, \ k \geq 2. \text{ Then } (\mathcal{V}, \mathcal{A}) \text{ is an } A_\infty\text{-algebra. Moreover, if } (\mathcal{V}, \delta) \text{ is a unital DG algebra with unit } 1_\mathcal{V} \text{ such that } (jp)1_\mathcal{V} = 1_\mathcal{V}, \text{ then } (\mathcal{V}, \mathcal{A}) \text{ is a strictly unital } A_\infty\text{-algebra with unit } p1_\mathcal{V}.

(2) Assume \((\mathcal{V}, \delta)\) possesses the structure \([\cdot, \cdot]\) of a DG Lie algebra, and let \(\mathcal{L} = \{\lambda_k, \ k \in \mathbb{N}\}\) be the family of graded operations
\[ \lambda_k : \mathcal{V}^{\otimes k} \rightarrow \mathcal{V} \]
defined by
\[ \lambda_1 := \delta, \quad \lambda_k := p\phi_k, \quad k \geq 2 \]
where the \(\phi\)'s are inductively defined by
\[ \psi_1 := -j, \quad \psi_k := h\phi_k, \]
and
\[ \phi_k(x_1, \ldots, x_k) := \sum_{\ell + m = k} \sum_{\sigma \in S_{\ell, m}} (-)^{b(\ell, m, x)}\chi(\sigma, x)[\psi_{\ell}(x_{\sigma(1)}, \ldots, x_{\sigma(\ell)}), \psi_m(x_{\sigma(\ell+1)}, \ldots, x_{\sigma(\ell+m)})], \]
x_1, \ldots, x_k \in \mathcal{V}, \text{ where } b(\ell, m, x) := \ell - 1 + (m - 1)\sum_{i=1}^{\ell}x_{\sigma(i)}, \ k \geq 2. \text{ Then } (\mathcal{V}, \mathcal{L}) \text{ is an } L_\infty\text{-algebra.}

1.5. Homotopy Transfer of Universal Enveloping. In this subsection I present an abstract algebraic model for the concrete geometric framework of the next section.

SH module structures can be transferred along contraction data similarly as in the previous subsection. Even more, one can transfer a SH Lie-Rinehart algebra structure along suitable contraction data. Namely, let \((A, \delta)\) be a commutative, unital DG algebra, let \(K\) be a DG Lie-Rinehart algebra over \((A, \delta)\) with differential \(\delta_0\) and Lie-bracket \([\cdot, \cdot]\), and let \(K_\mathcal{L}\) be a DG-module over \((A, \delta)\) with differential \(\delta\). Moreover, suppose that there are A-linear contraction data \((p_0, j_0, h_0)\) for \((K, \delta_0)\) over \((K_\mathcal{L}, \delta)\). Then, it is easy to see that there is an \(LR_\infty\)-algebra structure \(\mathcal{B}\) in \(K_\mathcal{L}\) defined in a similar way as in Theorem 1.5 I do not report here the obvious details.

Now, consider the symmetric DG algebras \(S_A^*K\) and \(S_A^*K_\mathcal{L}\). In view of Remark 1.8 they are endowed with a DG Poisson structure and a Poisson \(L_\infty\)-algebra structure \(\mathcal{P}\), respectively. I denote 1) by \(\{\cdot, \cdot\}\) the Poisson bracket in \(S_A^*K\), and 2) again by \(\delta_0\) and \(\delta\) the differentials in \(S_A^*K\) and \(S_A^*K_\mathcal{L}\), respectively. I claim that the contraction data \((p_0, j_0, h_0)\) extend to contraction data
\[ h_0 \circ (S_A^*K, \delta_0) \xrightarrow{p_0} (S_A^*K_\mathcal{L}, \delta) \]
such that the above mentioned Poisson \(L_\infty\)-algebra structure on \(S_A^*K_\mathcal{L}\) is obtained from the DG Poisson structure on \(S_A^*K\) via homotopy transfer. Indeed, put \(Z := \ker p_0\). Then \(K = \)
$\mathbb{K} \oplus \mathbb{Z}$ and $S^*_A \mathbb{K} \simeq S^*_A \mathbb{K} \otimes_A S^*_A \mathbb{Z}$. Now, extend $p_0$ and $j_0$ as algebra morphisms, and let $h': S^*_A \mathbb{K} \rightarrow S^*_A \mathbb{K}$ be the extension of $h_0$ as a derivation. For $\Sigma \in S^*_A \mathbb{K} \otimes S^*_A \mathbb{Z} \subset S^*_A \mathbb{K}$, put

$$h_0\Sigma := \begin{cases} 0 & \text{if } i = 0 \\ \frac{1}{i!}h'\Sigma & \text{if } i > 0 \end{cases}.$$  

It is easy to see that $(p_0, j_0, h_0)$ are contraction data for $(S^*_A \mathbb{K}, \delta_0)$ over $(S^*_A \mathbb{K}, \delta)$, extending the previous ones. Thus, in view of the Homotopy Transfer Theorem, there is an $L_\infty$-algebra structure $\mathcal{L} = \{\lambda_k, k \in \mathbb{N}\}$ in $S^*_A \mathbb{K}$, given by Formulas (4). Notice that $h_0 : S^*_A \mathbb{K} \rightarrow S^*_A \mathbb{K}$ is $S^*_A \mathbb{K}$-linear, i.e.,

$$h_0(j_0 \Sigma \circ \Sigma') = (-)^{\Sigma}j_0 \Sigma \circ h_0 \Sigma', \quad \text{for all } \Sigma \in S^*_A \mathbb{K}, \Sigma' \in S^*_A \mathbb{K}.$$  

**Proposition 19.** The structures $\mathcal{P}$ and $\mathcal{L}$ coincide.

**Proof.** Since $\mathcal{L}$ extends $\mathcal{P}$, it is enough to show that the $\lambda$'s are multiderivations. This can be proved by induction as follows. I claim that, for any $k$, $\phi_k$ is an “approximate” multiderivation along $j_0$ in the following sense:

$$\phi_k(\Sigma' \circ \Sigma'', \Sigma_1, \ldots, \Sigma_{k-1}) = (-)^{\Sigma' \Sigma''}j_0 \Sigma' \circ \phi_k(\Sigma'', \Sigma_1, \ldots, \Sigma_{k-1}) + (-)^{\Sigma' \Sigma''}\phi_k(\Sigma', \Sigma_1, \ldots, \Sigma_{k-1}) \circ j_0 \Sigma'' + I$$  

(10)

for all $\Sigma', \Sigma'', \Sigma_1, \ldots, \Sigma_{k-1} \in S^*_A \mathbb{K}$, where $I$ is the ideal of $(S^*_A \mathbb{K}, \circ)$ generated by the image of $h_0$. Since $I \subset \ker p_0$, it follows from the claim and the side condition $p_0 h_0 = 0$ that $\lambda_k$ is a multiderivation. Now, prove the claim by induction on $k$. First of all, a straightforward computation shows that

$$\phi_2(\Sigma' \circ \Sigma'', \Sigma) = j_0 \Sigma' \circ \phi_2(\Sigma'', \Sigma) + (-)^{\Sigma' \Sigma''}\phi_2(\Sigma', \Sigma) \circ j_0 \Sigma''.$$  

Now, assume that (10) holds for all $k \leq n$, and prove it for $k = n + 1$. From skew-symmetry it is enough to check it on equal, odd elements $\Sigma_1 = \cdots = \Sigma_n = \Sigma$. Put $\Sigma := (\Sigma' \circ \Sigma'', \Sigma^n)$ and compute

$$\begin{align*}
\phi_{n+1}(\Sigma' \circ \Sigma'', \Sigma^n) &= 2 \sum_{\ell + m = n}(-)^{\ell \beta_{\ell,m}(\Sigma)}\{\psi_{\ell+1}(\Sigma' \circ \Sigma'', \Sigma^\ell), \psi_m(\Sigma^n)\} \\
&= -2\{j_0 \Sigma' \circ j_0 \Sigma'', h_0 \phi_n(\Sigma^n)\} \\
&\quad + 2 \sum_{\ell = 1}^{n-1}(-)^{\ell \beta_{\ell,0}(\Sigma)}\{h_0 \phi_{\ell+1}(\Sigma' \circ \Sigma'', \Sigma^\ell), h_0 \phi_{n-\ell}(\Sigma^{n-\ell})\} + I \\
&= -2j_0 \Sigma' \circ \{j_0 \Sigma'', h_0 \phi_n(\Sigma^n)\} - 2(-)^{\Sigma}j_0 \Sigma'', h_0 \phi_n(\Sigma^n) \circ j_0 \Sigma'' \\
&\quad + 2 \sum_{\ell = 1}^{n-1}(-)^{\ell \beta_{\ell,0}(\Sigma)}\{(-)^{\Sigma'}\{j_0 \Sigma' \circ h_0 \phi_{\ell+1}(\Sigma'', \Sigma^\ell), h_0 \phi_{n-\ell}(\Sigma^{n-\ell})\} \\
&\quad + (-)^{\Sigma''}\{h_0 \phi_{\ell+1}(\Sigma', \Sigma^\ell) \circ j_0 \Sigma'', h_0 \phi_{n-\ell}(\Sigma^{n-\ell})\}] + I \\
&= (-)^{\Sigma(n+1)}j_0 \Sigma' \circ \phi_{n+1}(\Sigma'', \Sigma^n) + (-)^{\Sigma''}j_{n+1}(\Sigma', \Sigma^n) \circ j_0 \Sigma'' + I
\end{align*}$$

where I used the fact that, since $h_0(I) \subset I \circ I$, then $\{h_0(I), S^*_A \mathbb{K}\} \subset I$. \qed
Now, let \( (U(K), \delta) \) be the universal enveloping DG algebra of \((K, \delta_0)\), and suppose there is a PBW type isomorphism \( U(K) \approx S^\bullet_K, \) i.e., an isomorphism \( \text{PBW} : S^\bullet_K \to U(K) \) of filtered \( A \)-modules such that diagram
\[
\begin{array}{c}
S^\leq_k K \\
\text{PBW} \\
Gr_k U(K)
\end{array}
\]
commutes for all \( k \), (the map \( S^\leq_k K \to Gr_k U(K) \) being the composition of projections \( S^\leq_k K \to S^k K \) and \( S^k K \to Gr_k U(K) \)). Use PBW to identify \( U(K) \) and \( S^\bullet_K \). Then 1) the filtrations in \( U(K) = S^k K \) and \( S^\bullet_K \) are preserved by \( \delta_0, \delta, p_0, j_0, h_0 \), and 2) \( t(U_i(K)) \subset U_{i-1}(K) \). It follows from Remark 17 and the Perturbation Lemma that there are contraction data \((p_t, j_t, h_t) \) for \((U(K), \delta) \) over \((S^\bullet_K, \delta)\). Hence, in view of the Homotopy Transfer Theorem, there is an \( A_\infty \)-algebra structure on \( S^\bullet_K \) canonically determined by the contraction data \((p_0, j_0, h_0) \) and the isomorphism PBW.

**Remark 20.** The \( A_\infty \)-algebra structure (induced as above) on \( S^\bullet_K \) highly depends on the isomorphism PBW (besides the contraction data), and it could be hard to write explicit formulas in practice. In the case of the \( A_\infty \)-algebra of a foliation, I will only compute the highest order contributions to the first few homotopies (see Section 3 for details).

**Example 21.** Let \( M \) be a smooth manifold, \( F \) a foliation of \( M \), and \( C \) its characteristic distribution. Moreover, let \((K, \delta_0)\) be the deformation complex of \( F \) \([6]\) and \((K, \delta)\) the Chevalley-Eilenberg complex determined by the Bott connection in \( TM/C \) (see Section 2.1 for more details). A splitting \( TM = C \oplus V \) via a complementary distribution \( V \) determines contraction data \((p_0, j_0, h_0) \) for \((K, \delta_0)\) over \((K, \delta)\). Accordingly, there is an \( LR_\infty \)-algebra structure on \( K \) which I described in \([20]\) (see also \([9,11]\)). In Subsection 2.4 I show how to construct a PBW isomorphism \( U(K) \approx S^\bullet_K \), via purely geometric data (specifically, a connection). One immediately concludes that there is an \( A_\infty \)-algebra structure on \( S^\bullet_K \). I partially describe this \( A_\infty \)-algebra in Section 3. Here, I present the toy example when \( F \) has just one leaf and \( C = TM \), as an illustration of the main technical aspects of the general case.

When \( C = TM \), the deformation complex of \( F \) is \( \text{Der} \Lambda(M), \delta_0 = [d, i] \), \( TM/C = 0 \), and its Chevalley-Eilenberg complex \((K, \delta)\) is trivial. Put \( \Lambda := \Lambda(M) \). There are contraction data \((0, 0, h_0) \) for \( \text{Der} \Lambda, \delta_0 \) over the 0 complex. The contracting homotopy \( h_0 \) is defined as follows. Every element \( \Delta \in \text{Der} \Lambda \) can be uniquely written as \( \Delta = i_U + L_V, \ U, V \in \Lambda \otimes M \mathcal{X}(M) \). Then \( h_0(\Delta) := (-)^{\Lambda} i_V \). The homotopy \( h_0 \) is \( \Lambda \)-linear. Accordingly, it determines contraction data \((p_0, j_0, h_0) \) for \((\Delta(\Lambda) = S^\bullet \text{Der} \Lambda, \delta_0) \) over \((S^\bullet K, \delta) = (\Lambda, d) \), where \( p_0 : S^\bullet \text{Der} \Lambda \to \Lambda \) and \( j_0 : \Lambda \to S^\bullet \text{Der} \Lambda \) are the obvious maps. Notice that the SH Poisson algebra structure induced on \( (\Lambda, d) \) is trivial. The universal enveloping DG algebra of \( \text{Der} \Lambda \) is \( \text{D}(\Lambda), \delta = [d, i] \). A PBW isomorphism \( \text{D}(\Lambda) \approx S(\Lambda) \) can be constructed, exploiting a connection \( \nabla \), as follows (see \([17,19]\) for similar results). Extend the covariant derivative \( \nabla : \mathcal{X}(M) \to \text{Der} \Lambda \) to the whole \( \Lambda \otimes M \mathcal{X}(M) \) by \( \Lambda \)-linearity. For \( Z \in \Lambda \otimes M \mathcal{X}(M) \), \( L_Z - \nabla Z = iv_Z \). It follows that every element \( \Delta \) in \( \text{Der} \Lambda \) can be uniquely written in the form \( \Delta = i_U + \nabla Z, \ U, Z \in \Lambda \otimes M \mathcal{X}(M) \), and the correspondence
\[
\Lambda \otimes_M \mathcal{X}(M)[1] \oplus \Lambda \otimes_M \mathcal{X}(M) \ni (U, Z) \mapsto i_U + \nabla Z \in \text{Der} \Lambda
\]
is a well defined isomorphism of $\Lambda$-modules. Accordingly, $S(\Lambda)$ identifies with

$$S^*_\Lambda(\Lambda \otimes_M \mathfrak{X}(M)[1]) \otimes S^*_\Lambda(\Lambda \otimes_M \mathfrak{X}(M)) \simeq \Lambda \otimes_M \Lambda^* \mathfrak{X}(M) \otimes_M S^* \mathfrak{X}(M).$$

Now, let

$$\Sigma = \omega \otimes Y_1 \wedge \cdots \wedge Y_j \otimes P \in \Lambda \otimes_M \Lambda^j \mathfrak{X}(M) \otimes_M S^k \mathfrak{X}(M),$$

let $\ldots, z^a, \ldots$ be coordinates in $M$, and let $P$ be locally given by $P = P^{a_1 \ldots a_i} \partial_{x^{a_1}} \otimes \cdots \otimes \partial_{x^{a_i}}$. Define $\nabla_P : \Lambda \rightarrow \Lambda$ via local formulas $\nabla_P := P^{a_1 \ldots a_i} \nabla_{a_1} \cdots \nabla_{a_i}$, and put

$$\text{PBW}(\Sigma) := \omega i_{Y_1} \cdots i_{Y_j} \nabla_P \in \mathcal{D}_j(\Lambda).$$

The restrictions $\text{PBW} : S_i(\Lambda) \rightarrow \mathcal{D}_i(\Lambda)$ split the exact sequences $0 \rightarrow \mathcal{D}_{i-1}(\Lambda) \rightarrow \mathcal{D}_i(\Lambda) \rightarrow S_i(\Lambda) \rightarrow 0$, so that $\text{PBW}$ is the required PBW isomorphism. The Perturbation Lemma gives now contraction data for $(\mathcal{D}(\Lambda), \delta)$ over $(\Lambda, \delta)$. The $\Lambda_\infty$-algebra structure induced on $(\Lambda, \delta)$ is again trivial.

2. Geometric Preliminaries

2.1. (A Bit of) Differential Geometry and Homological Algebra of a Foliation. Let $M$ be a smooth manifold and $C$ an involutive $n$-dimensional distribution on it. Now on, I will denote by $A$ the algebra of smooth functions on $M$. I will denote by $C\mathfrak{X}$ the submodule of $\mathfrak{X}(M)$ made of vector fields in $C$. Let $CA^1 := C\mathfrak{X}^1 \subset \Lambda^1(M)$ be its annihilator, and put

$$\overline{\mathfrak{X}} := \mathfrak{X}(M)/CA^1, \quad \overline{\Lambda}^1 := \Lambda^1(M)/CA^1.$$

Then $CA^1 \simeq \overline{\mathfrak{X}}^*$ and $\overline{\Lambda}^1 \simeq C\mathfrak{X}^*$. In view of the Frobenius theorem, there always exist coordinates $\ldots, x^i, \ldots, u^a, \ldots, i = 1, \ldots, n, \alpha = 1, \ldots, \dim M - n$, adapted to $C$, i.e., such that $C\mathfrak{X}$ is locally spanned by $\ldots, \partial_i := \partial/\partial x^i, \ldots$ and $CA^1$ is locally spanned by $\ldots, du^a, \ldots$. Consider the Chevalley-Eilenberg algebra $(\overline{\mathfrak{X}}, \overline{\nabla})$ of the Lie algebroid $C$. Namely, $\overline{\mathfrak{X}}$ is the exterior algebra of $\overline{\Lambda}^1$ and

$$(\overline{\nabla}\lambda)(X_1, \ldots, X_{k+1}) = \sum_i (-)^{i+1} X_i(\lambda(\ldots, \hat{X}_i, \ldots)) + \sum_{i<j} (-)^{i+j} \lambda([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots),$$

where $\lambda \in \overline{\mathfrak{X}}^k$ is understood as a $C^\infty(M)$-valued, $k$-multilinear, skew-symmetric map on $C\mathfrak{X}$ and $X_1, \ldots, X_{k+1} \in C\mathfrak{X}$. The DG algebra $(\overline{\mathfrak{X}}, \overline{\nabla})$ is the quotient of $(\Lambda(M), \delta)$ over the differentially closed ideal generated by $CA^1$ which is made of differential forms vanishing when acting on vector fields in $C\mathfrak{X}$. In particular, it is generated by degree 0, and $\overline{\nabla}$-exact degree 1 elements. In the following, I write $\omega \mapsto \overline{\omega}$ the projection $\Lambda(M) \rightarrow \overline{\Lambda}$.

The Lie algebroid $C\mathfrak{X}$ acts on $\overline{\mathfrak{X}}$ via the Bott connection. Namely, write $X \mapsto \overline{X}$ the projection $\mathfrak{X}(M) \rightarrow \overline{\mathfrak{X}}$. Then

$$X \cdot \overline{Y} := [\overline{X}, \overline{Y}] \in \overline{\mathfrak{X}}, \quad X \in C\mathfrak{X}, \ Y \in \mathfrak{X}(M).$$

Accordingly, there is a DG module $(\overline{\mathfrak{X}} \otimes_M \overline{\mathfrak{X}}, \overline{\nabla})$ over $(\overline{\mathfrak{X}}, \overline{\nabla})$ whose differential is given by the usual Chevalley-Eilenberg formula:

$$(\overline{\nabla} Z)(X_1, \ldots, X_{k+1}) = \sum_i (-)^{i+1} X_i \cdot Z(\ldots, \hat{X}_i, \ldots) + \sum_{i<j} (-)^{i+j} Z([X_i, X_j], \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots),$$
where \( Z \in X^{k} \otimes M X \) is understood as a \( X \)-valued, \( k \)-multilinear, skew-symmetric map on \( CX \), and \( X_1, \ldots, X_{k+1} \in CX \). The tensor product \( \Lambda(M) \otimes M X(M) \rightarrow X \otimes M X \) of projections \( \Lambda(M) \rightarrow X \) and \( X(M) \rightarrow X \) will be written \( Z \rightarrow Z \).

**Remark 22.** The differentials \( \mathfrak{d} \) in \( X \) and \( X \) can be uniquely extended to the whole tensor algebra

\[
\bigoplus_{i,j} X \otimes M (\Lambda X)^{\otimes i} \otimes M (\Lambda X)^{\otimes j},
\]

requiring Leibniz rules with respect to tensor products and contractions. Such extension is nothing but the Chevalley-Eilenberg differential associated to the canonical action of \( CX \) on \( \bigoplus_{i,j} X^{\otimes i} \otimes M (\Lambda X)^{\otimes j} \). In particular, \( \mathfrak{d} \) extends to an homological derivation \( \mathfrak{d}_S \) of \( X \otimes M S^{\otimes} X \).

The datum of a splitting is equivalent to the datum of a distribution \( V \) complementary to \( C \). From now on fix such a distribution. I will always identify \( X \) (resp., \( X \)) with the corresponding submodule (resp., subalgebra) in \( X(M) \) (resp., \( \Lambda(M) \)) determined by \( V \).

The distribution \( V \simeq TM/C \) is locally spanned by vector fields \( \ldots, V_{\alpha}, \ldots \) of the form

\[
V_{\alpha} := \partial/\partial u^\alpha + V_{\alpha}^i \partial_i, \quad \alpha = 1, \ldots, \dim M - n,
\]

for some local functions \( \ldots, V_{\alpha}, \ldots \). Moreover,

\[
[\partial_i, V_{\alpha}] = \partial_i V_{\alpha}^j \partial_j, \quad [V_{\alpha}, V_{\beta}] = R_{\alpha \beta}^i \partial_i,
\]

where \( R_{\alpha \beta}^i := V_{\alpha}^j - V_{\beta}^j V_{\alpha}^i \).

Now, consider the deformation complex \( (\text{Der} X, \delta_0 := [\mathfrak{d}, \cdot]) \) (see, for instance, [3]) of the integral foliation of \( C \). The complementary distribution \( V \) determines \( X \)-linear contraction data \((p_0, j_0, h_0)\) for \( (\text{Der} X, \delta_0) \) over \( X \otimes M X, \mathfrak{d} \). Accordingly, there is an \( LR_{\infty} \)-algebra structure on \( X \otimes M X \) (see the second appendix of [25]). Recall that the projection \( p_0 : \text{Der} X \rightarrow X \otimes M X \) is actually independent of \( V \) and is defined as

\[
p_0 \Delta := \Delta|_{C^{\infty}(M)}, \quad \Delta \in \text{Der} X.
\]

The injection \( j_0 : X \otimes M X \hookrightarrow \text{Der} X \) depends on \( V \) and is defined by

\[
j_0 Z(\omega) := L_Z \omega, \quad Z \in X \otimes M X, \quad \omega \in X.
\]

Finally, the homotopy \( h_0 : \text{Der} X \rightarrow \text{Der} X \) can be described as follows. First of all, I prove a useful

**Lemma 23.** An element \( \Delta \in \text{Der} X \) can be uniquely written in the form

\[
\Delta = i_U + T_V + T_W,
\]

where \( U, V \in X \otimes M CX, W \in X \otimes M X \), and for \( X \in \Lambda(M) \otimes M X(M) \) I defined \( T_X \in \text{Der} X \) by

\[
T_X \omega := L_X \omega, \quad \omega \in X.
\]

**Proof.** It is easy to check the following identity

\[
[\mathfrak{d}, T_X] = T_{[\mathfrak{d}, X]} \quad X \in X \otimes M X(M).
\]

Now, let \( \Delta \in \text{Der} X \), put

\[
W := p_0 \Delta, \quad V := \Delta|_{C^{\infty}(M)} - p_0 \Delta, \quad U := [\Delta, [\mathfrak{d}]_{C^{\infty}(M)} + (-) \Delta \mathfrak{d} W
\]

where \( U, V \in X \otimes M CX, W \in X \otimes M X \), and for \( X \in \Lambda(M) \otimes M X(M) \) I defined \( T_X \in \text{Der} X \) by

\[
T_X \omega := L_X \omega, \quad \omega \in X.
\]
and check \([14]\). It is enough to evaluate both sides of \([14]\) on generators. Thus, for all \(f \in C^\infty(M)\),

\[
\Delta f = (V + W)f = (i_U + L_V + L_W)f
\]

Similarly,

\[
\Delta \alpha_f = [\Delta, \alpha]f + (-)^{\Delta} \alpha \Delta f
\]

\[
= U f - (-)^{\Delta} \alpha (dW)(f) + (-)^{\Delta} \alpha (V + W)f
\]

\[
= i_U \alpha f - (-)^{\Delta} \alpha [\alpha, W]f - (-)^{\Delta} \alpha L_W f + (-)^{\Delta} \alpha L_V f
\]

\[
= (i_U + L_V + L_W) \alpha f.
\]

Then \(h_0\) is given by

\[
h_0(i_U + L_V + L_W) = (-)^1 i_V, \quad U, V \in \mathcal{X} \otimes_M \mathcal{X}, \quad W \in \mathcal{X} \otimes_M \mathcal{X}.
\]

2.2. Differential Operators on a Foliated Manifold. In \(D(M) := D(C^\infty(M))\), consider the left ideal \(D(M) \circ \mathcal{C} \mathcal{X}\) generated by \(\mathcal{C} \mathcal{X}\). Denote by \(\overline{D}\) the quotient left \(D(M)\)-module \(D(M)/D(M) \circ \mathcal{C} \mathcal{X}\), and write \(\Box \mapsto \Box\) the projection \(D(M) \rightarrow \overline{D}\). More generally, let \(Q\) be the module of sections of a vector bundle over \(M\). Consider the submodule \(D(M, Q) \circ \mathcal{C} \mathcal{X}\) in \(D(M, Q) := D(C^\infty(M), Q) \simeq Q \otimes_M D(M)\), and the quotient \(\overline{D}(M, Q) := D(M, Q)/D(M, Q) \circ \mathcal{C} \mathcal{X}\), and write again \(\Box \mapsto \Box\) the projection \(D(M, Q) \rightarrow \overline{D}(M, Q)\). Clearly, \(\overline{D}(M, Q) \simeq Q \otimes_M \overline{D}\), and in the following I will often understand this canonical isomorphism.

The Lie algebroid \(\mathcal{C} \mathcal{X}\) acts on \(\overline{D}\) as follows

\[
X \cdot \Box := X \circ \Box = [X, \Box], \quad X \in \mathcal{C} \mathcal{X}, \quad \Box \in D(M).
\]

Notice that \(\overline{\mathcal{X}}\) can be understood as a submodule in \(\overline{D}\) and the action of \(\mathcal{C} \mathcal{X}\) on \(\overline{\mathcal{X}}\) as the restricted action. Accordingly, the Chevalley-Eilenberg complex \((\overline{\mathcal{X}} \otimes_M \overline{\mathcal{X}}, \overline{d})\) extends to a Chevalley-Eilenberg complex \((\overline{\mathcal{X}} \otimes_M \overline{D}, \overline{d}_D)\) in an obvious way.

**Remark 24.** The differential

\[
\overline{d}_D : \overline{\mathcal{X}} \otimes_M \overline{D} \rightarrow \overline{\mathcal{X}} \otimes_M \overline{D}
\]

identifies with

\[
\overline{d}_* : \overline{D}(M, \overline{\mathcal{X}}) \ni \Box \mapsto \overline{d}_* \Box := \overline{d} \circ \Box \in \overline{D}(M, \overline{\mathcal{X}}), \quad \Box \in D(M, \overline{\mathcal{X}}).
\]

Indeed, it is easy to see that both \(\overline{d}_D\) and \(\overline{d}_*\) are graded derivations subordinate to \(\overline{d}\). Therefore, it is enough to prove that they coincide on generators, namely, on \(\overline{D}\). Let \(\Box \in D(M, \overline{\mathcal{X}}) = \overline{\mathcal{X}} \otimes_M \overline{D}\). Since the isomorphism \(\overline{\mathcal{X}} \otimes_M D \rightarrow D(M, \overline{\mathcal{X}})\) is given by \(\omega \otimes \Box \mapsto \omega \Box\), then

\[
\langle \overline{d}_D \Box, X \rangle = X \cdot \Box = \overline{X} \circ \Box = \overline{\langle \overline{d} \circ \Box, X \rangle} = \langle \overline{d}_* \Box, X \rangle.
\]

where I indicated with \(\langle W, X \rangle\) the contraction of \(W \in \overline{\mathcal{X}}^1 \otimes_M Q\) with a vector fields \(X \in C \mathcal{X}\). Thus,

\[
\langle \overline{d}_D \Box, X \rangle = X \cdot \Box = \overline{X} \circ \Box = \overline{\langle \overline{d} \circ \Box, X \rangle} = \langle \overline{d}_* \Box, X \rangle.
\]
The module $\Lambda \otimes_M \overline{D}$ inherits a filtration
\[ \Lambda \otimes_M \overline{D}_0 \subset \Lambda \otimes_M \overline{D}_1 \subset \cdots \subset \Lambda \otimes_M \overline{D}_i \subset \cdots \subset \Lambda \otimes_M \overline{D} \]
from $\Lambda \otimes_M D$, and 1) the projection $D(M,Q) \rightarrow \overline{D}(M,Q)$, and 2) the differential $\overline{d}_D$, preserve this filtration. Accordingly, the graded object $\text{Gr}(\Lambda \otimes_M \overline{D})$ identifies with $\Lambda \otimes_M S^\ast \overline{X}$, and inherits a differential $\overline{d}_S$ which coincides with the one in Remark 22. In particular, $(\Lambda \otimes_M S^\ast \overline{X}, \overline{d}_S)$ is a DG commutative algebra.

Consider again the complementary distribution $V$, and notice that, in view of commutation relations (12), $\overline{D}$ is locally spanned by
\[ V_{\alpha_1} \cdots \alpha_i := V_{(\alpha_1 \cdots \alpha_i)}, \quad i \geq 0, \]
and they are independent generators.

Now, consider the universal enveloping DG algebra $(D(\overline{X}), \delta_D)$ of the deformation complex $(\text{Der} \overline{X}, \delta_0)$. In Section 3 I show that the contraction data $(p_0, j_0, h_0)$ for $(\text{Der} \overline{X}, \delta_0)$ over $(\Lambda \otimes_M \overline{X}, D)$ extend to contraction data $(p, j, h)$ for $(D(\overline{X}), \delta_D)$ over $(\Lambda \otimes_M \overline{D}, \overline{d}_D)$. Here, I take only two steps in this direction. Firstly, I define the projection $p : D(\overline{X}) \rightarrow \Lambda \otimes_M \overline{D}$, which is given by
\[ p \square := \square \vert_{C^\infty(M)}, \quad \square \in D(\overline{X}), \]
and clearly extends $p_0$ in (13). Moreover, in view of Remark 24 and the fact that $\overline{7} : C^\infty(M) \rightarrow \Lambda$ does actually belong to $D(M, \Lambda) \circ C^\ast \overline{X}$, $p : D(\overline{X}) \rightarrow \Lambda \otimes_M \overline{D}$ is a cochain map. Notice that, as $p_0$, $p$ is canonical, i.e., it doesn’t depend on any other structure than the distribution $C$. Secondly, I consider the graded DG object $(S(\Lambda) \approx S^\ast \Lambda \text{Der} \overline{X}, \delta_S)$ of $(D(\overline{X}), \delta_D)$ and extend the contraction data for $(\text{Der} \overline{X}, \delta_0)$ over $(\Lambda \otimes_M \overline{X}, D)$ to contraction data for $(S(\overline{X}), \delta_S)$ over $(S^\ast(\Lambda \otimes_M \overline{X}) \approx \Lambda \otimes_M S^\ast \overline{X}, \overline{d}_S)$ as in Section 1.5. The next step is to construct “PBW isomorphisms”
\[ \Lambda \otimes_M \overline{D} \approx \Lambda \otimes_M S^\ast \overline{X}, \quad D(\overline{X}) \approx S(\overline{X}). \]
This can be done exploiting an adapted connection. I devote the next section to the introduction of this geometric structure.

2.3. Adapted Connections. In this section $C, V$ are complementary distributions on $M$. I don’t require $C$ to be involutive. The above definitions of $C^\ast \overline{X}$, $\overline{X}$, $C \Lambda^1$, and $\Lambda^1$ are still valid in the present general situation. Moreover, let
\[ X(M) \ni X \mapsto CX \in C^\ast \overline{X} \]
\[ \Lambda^1(M) \ni \omega \mapsto \omega^C \in C \Lambda^1 \]
be the projections. The pair $(C, V)$ determines a distinguished class of connections according to the following

**Definition 25.** The connection $\nabla$ is called adapted to the pair $(C, V)$ (or simply adapted) if

1. it restricts to $\overline{X}$, i.e., $\nabla_X \omega \in \overline{X}$ for all $X \in X(M)$ and $\omega \in \overline{X}$,
2. it restricts to $C \Lambda^1$, i.e., $\nabla_X \omega \in C \Lambda^1$ for all $X \in X(M)$ and $\omega \in C \Lambda^1$,
3. $\nabla_Y \omega = L_X \omega$ for all $Y \in \overline{X}$, $\omega \in \overline{X}$,
4. $\nabla_X \omega = (L_X \omega)^C$ for all $X \in C^\ast \overline{X}$, $\omega \in C \Lambda^1$. 

Proposition 26. There exist adapted connections.

Proof. Let \( \tilde{\nabla} \) be a fiducial connection. For \( X \in \mathfrak{X}(M) \) and \( \omega \in \Lambda^1(M) \) put

\[
\nabla_X \omega = ((\tilde{\nabla}_X + L_{CX})\omega)^C + (\overline{\nabla_{CX} + L_X})\overline{\omega}.
\]

(15)

The operator \( \nabla_X \) is clearly a derivation subordinate to \( X \), i.e., \( \nabla_X f \omega = X(f) \omega + f \nabla_X \omega \). Moreover, \( \nabla_X \) is \( C^\infty(M) \)-linear in \( X \). Indeed, for \( f \in C^\infty(M) \)

\[
L_{CfX} \omega = f L_{CX} \omega + df \wedge i_{CX} \omega = f L_{CX} \omega.
\]

Thus, the correspondence \( X \mapsto \nabla_X \) is a linear connection. The four properties of adapted connections are obvious. \( \square \)

Proposition 27. Let \( \nabla \) be an adapted connection determined by a connection \( \tilde{\nabla} \) via Formula (15). Then

\[
\nabla_X Y = (\tilde{\nabla}_X + L_{CX})Y + C(\tilde{\nabla}_{CX} + L_X)CY.
\]

(16)

In particular,

(1) \( \nabla \) restricts to \( \mathfrak{X} \), i.e., \( \nabla_X Y \in \mathfrak{X} \) for all \( X \in \mathfrak{X}(M) \) and \( Y \in \mathfrak{X} \),

(2) \( \nabla \) restricts to \( C \mathfrak{X} \), i.e., \( \nabla_X Y \in C \mathfrak{X} \) for all \( X \in \mathfrak{X}(M) \) and \( Y \in C \mathfrak{X} \),

(3) \( \nabla_Y X = C[Y, X] \), and \( \nabla_X Y = [Y, X] \) for all \( Y \in \mathfrak{X} \), \( X \in C \mathfrak{X} \).

Proof. Let \( X, Y \in \mathfrak{X}(M) \), and \( \omega \in \Lambda^1(M) \).

\[
\langle \nabla_X Y, \omega \rangle = X(Y, \omega) - \langle Y, \nabla_X \omega \rangle
= X(Y, \omega) - \langle Y, (\tilde{\nabla}_X + L_{CX})\omega^C \rangle - \langle CY, (\tilde{\nabla}_{CX} + L_X)\overline{\omega} \rangle
= X(Y, \omega) - X(Y, \omega^C) + \langle (\overline{\nabla_{CX} + L_X})Y, \omega \rangle - X(CY, \overline{\omega}) + \langle (\tilde{\nabla}_{CX} + L_X)CY, \overline{\omega} \rangle
= \langle (\overline{\nabla_{CX} + L_X})Y + C(\tilde{\nabla}_{CX} + L_X)CY, \omega \rangle.
\]

\( \square \)

It is easy to see that every adapted connection is of the form (15): for an adapted connection \( \nabla \) it is enough to put \( \tilde{\nabla} = \nabla \). Indeed,

\[
(\overline{\nabla_{CX} + L_X})\overline{\omega} = (\overline{\nabla_{CX} + L_X})\omega^C + (\tilde{\nabla}_{CX} + L_X)\overline{\omega}
= \nabla_X \omega^C + \nabla_X \overline{\omega}
= \nabla_X \omega.
\]

Proposition 28. Let \( \nabla \) be an adapted connection determined by a connection \( \tilde{\nabla} \) with torsion \( \tilde{T} \in \Lambda^2(M) \otimes_M \mathfrak{X}(M) \). The torsion \( T \) of \( \nabla \) is given by

\[
T(X, Y) = \tilde{T}(X, Y) + C(\tilde{T}(CX, CY)) - [CX, CY] - C[X, Y], \quad X, Y \in \mathfrak{X}.
\]
Proof. Compute
\[ T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \]
\[ = (\nabla_X + L_{CY}) X - (\nabla_Y + L_{CX}) Y - C[CY, CX] \]
\[ = T(X,Y) + CT(CX, CY) + [CX, CY] + C[CY] + C[CY] - [X,Y] \]
\[ = T(X,Y) + CT(CX, CY) - [CX, CY] - C[CY]. \]
\[ \square \]

**Corollary 29.** A torsion-free adapted connection exists iff both \( C \) and \( V \) are involutive.

**Proof.** If both \( C \) and \( V \) are involutive, an adapted connection determined by a torsion-free connection is torsion-free as well. Conversely, let the adapted connection \( \nabla \) determined by a connection \( \tilde{\nabla} \) be torsion-free. Then, for \( X, Y \in \mathfrak{X} \),
\[ 0 = T(X,Y) = T(X,Y) - C[X,Y] \implies C[X,Y] = 0. \]
Similarly, for \( X, Y \in C\mathfrak{X}. \)
\[ \square \]

**Definition 30.** An adapted connection \( \nabla \) is called torsion-quasi-free if
\[ T(X,Y) = -[CX, CY] - C[CY]. \]

**Corollary 31.** There exist torsion-quasi-free adapted connections.

**Proof.** The adapted connection determined by a torsion-free connection is torsion-quasi-free.
\[ \square \]

Now, suppose that \( C \) is involutive and let \( \nabla \) be a torsion-quasi-free adapted connection. Let \( T \) be the torsion of \( \nabla \). Then, clearly,
\[ (1) \quad \nabla \text{ extends the Bott connection}, \]
\[ (2) \quad T \text{ coincides with the curvature form of } V, \text{ up to a sign}, \]
\[ (3) \quad \text{In view of (12)} \]
\[ T(V_\alpha, V_\beta) = -R^i_{\alpha\beta}\partial_i. \] (17)

### 2.4. Two PBW Isomorphisms.

Now, let \( C \) be again an involutive distribution on \( M \), and \( \nabla \) a connection in \( \Lambda^1(M) \) adapted to the pair \((C,V)\) and torsion-quasi-free. The connection \( \nabla \) determines two PBW type isomorphisms (see [15] for a similar result)
\[ \text{PBW} : \Lambda \otimes_M \mathcal{D} \approx \Lambda \otimes_M S^* \mathfrak{X}, \quad \text{PBW} : D(\Lambda) \approx S(\Lambda) \]
as follows. For \( \omega \in \Lambda \) and \( P \in S^* \mathfrak{X} \), put
\[ \text{PBW}(\omega \otimes P) := \omega \otimes \nabla_P, \]
where \( \nabla_P \) is defined as in Example [21] To define PBW notice, first of all, that every derivation \( \Delta \in \text{Der} \Lambda \) can be uniquely written in the form
\[ \Delta = i_W + \nabla_V + L_Z, \] (18)
\( W, V \in \Lambda \otimes_M Cx, Z \in \Lambda \otimes_M \bar{x}, \) where \( \nabla_V \) is defined as in Example 21. Indeed, let
\[
\Delta = i_U + L_V + L_Z.
\]
\( U, V \in \Lambda \otimes_M Cx, Z \in \Lambda \otimes_M \bar{x} \). Now, since \( \nabla \) is adapted, and torsion-quasi-free, then \( \nabla_V \)
preserves \( \Lambda \), and \( L_V = \nabla_V + i_V \), with \( \nabla V \in \Lambda \otimes_M Cx \). Thus (13) holds simply putting \( W := U + \nabla V \). Clearly, the correspondence
\[
(\Lambda \otimes_M Cx[1]) \oplus (\Lambda \otimes_M Cx) \oplus (\Lambda \otimes M \bar{x}) \ni (W, V, Z) \mapsto i_W + \nabla V + L_Z \in \text{Der}\Lambda
\]
is an isomorphism of \( \Lambda \)-module, so that, for \( \Sigma = \omega \otimes X_1 \wedge \cdots \wedge X_k \otimes P \otimes Q \in S(\Lambda) \),
\[
\text{PBW}(\Sigma) := \omega X_1 \cdots i X_k \nabla_{P \otimes Q} \in \mathcal{D}(\bar{\Lambda}).
\]
**Remark 32.** In general, the isomorphisms \( \text{PBW} \) and \( \text{PBW} \) are not compatible with projections \( p, p_0 \). However, if one uses them to induce an injection \( j_0^\sim : \Lambda \otimes M \bar{D} \rightarrow \mathcal{D}(\bar{\Lambda}) \), from the injection \( j_0 : \Lambda \otimes M \bar{S} \rightarrow \mathcal{S}(\Lambda) \), then the former is a right inverse of \( p \). Indeed, clearly
\[
\text{PBW} = p \circ \text{PBW} \circ j_0,
\]
so that, for \( \Sigma \in \Lambda \otimes M \bar{S} \)
\[
(p \circ j_0^\sim \circ \text{PBW}) \Sigma = (p \circ j_0^\sim \circ p \circ \text{PBW} \circ j_0) \Sigma
\]
\[
= (p \circ \text{PBW} \circ j_0 \circ \Sigma) \Sigma
\]
\[
= (p \circ \text{PBW} \circ j_0 \circ p \circ \text{PBW} \circ j_0) \Sigma
\]
\[
= (p \circ \text{PBW} \circ j_0 \circ p \circ \text{PBW} \circ j_0) \Sigma
\]
\[
= (p \circ \text{PBW} \circ j_0) \Sigma
\]
\[
= \text{PBW}(\Sigma).
\]
In the following, I will often understand isomorphisms \( \text{PBW} \) and \( \text{PBW} \).

3. **The \( A_\infty \)-Algebra of a Foliation**

Let \( M \) be a smooth manifold and let \( C \) be an involutive distribution on it. Summarizing results obtained so far, a complementary distribution \( V \) and a torsion-quasi-free adapted connection \( \nabla \) determine

1. Contraction data \( (p_0, j_0, h_0) \) for \( (S(\Lambda), \delta_S) \) over \( (\Lambda \otimes_M S^* \bar{x}, \bar{d}_S) \),
2. PBW type isomorphisms \( \Lambda \otimes_M \bar{D} \approx \Lambda \otimes_M S^* \bar{x}, \mathcal{D}(\Lambda) \approx S(\Lambda) \),

Notice that, actually, i) \( p_0 \) is independent of the supplementary geometric data \( V \) and \( \nabla \), and 2) \( j_0, h_0 \) do only depend on \( V \).

Now, put \( t = \delta_S - \delta_D : \mathcal{D}(\Lambda) \rightarrow \mathcal{D}(\Lambda) \). The Perturbation Lemma determines a “new” differential \( \bar{d}_t : \bar{x} \otimes_M \bar{D} \rightarrow \bar{x} \otimes_M \bar{D} \) and contraction data \( (p_t, j_t, h_t) \) for \( (\mathcal{D}(\Lambda), \delta_D) \) over \( (\bar{x} \otimes_M \bar{D}, \bar{d}_t) \), given by Formulas (30), (37), (35). In its turn, the Homotopy Transfer Theorem determine an \( A_\infty \)-algebra structure on \( (\bar{x} \otimes_M \bar{D}, \bar{d}_t) \). Before giving more details about these
structures, I remark that $p_t$ is actually independent of $V$ and $\nabla$ and coincides with the canonical projection $p : D(\Lambda) \rightarrow \Lambda \otimes_M \overline{D}$. To show this, first notice that $pj_0 = id$ and $ph_0 = 0$ (the first identity is discussed in Remark 22 while the second one is immediate from the definitions of $h_0$ and $p$). It follows that $pj_t = id$ and $ph_t = 0$. Now, let $\square \in D(\Lambda)$. Then
\[
0 = p[h_t, \delta]\square = p(id - j_t p_t)\square = (p - p_t)\square.
\]
As an immediate consequence, $\square$ is also independent of $V$ and $\nabla$, and coincides with the canonical differential $\overline{d}_D : \Lambda \otimes_M \overline{D} \rightarrow \Lambda \otimes_M \overline{D}$. In the following, I put $j := j_t$ and $h := h_t$.

I am finally in the position to furnish few details about the (strict unital) $A_\infty$-algebra structure $\{\alpha_k, \ k \in \mathbb{N}\}$ on $\Lambda \otimes_M \overline{D}$. To this aim, notice that the isomorphism $\text{PBW} : D(\Lambda) \approx S_*(\Lambda)$ (resp., $\text{PBW} : \Lambda \otimes_M \overline{D} \approx \Lambda \otimes_M S^*\overline{D}$) determines a new grading in $D(\Lambda)$ (resp., $\Lambda \otimes_M \overline{D}$), which I call the order and is given by the decomposition $S(\Lambda) = \bigoplus_k S_k(\Lambda)$ (resp., $\Lambda \otimes_M S^*\overline{D} = \bigoplus_k \Lambda \otimes_M S^k\overline{D}$). Every map $\phi$ of the spaces $D(\Lambda)$ and $\Lambda \otimes_M \overline{D}$ have its homogenous components with respect to the order. I denote by $\phi[i]$ the $i$-th one, and by $O(i)$ a generic (no better specified) object of order no higher than $i$, e.g.,
\[
\delta_D = \delta_S + O(-1), \quad \overline{d}_D = \overline{d}_S + O(-1), \quad p = p_0 + O(-1), \quad h = h_0 + O(-1).
\]
Similarly,
\[
t = t^{[-1]} + O(-2),
\]
and
\[
j = j_0 + h_0 t^{[-1]} j_0 + O(-2). \tag{19}
\]
I will not need to compute $t^{[-1]}$. Finally, the composition $\circ$ of differential operators in $D(\Lambda)$, decomposes as
\[
\circ = \circ + \otimes + O(-2),
\]
where I put $\otimes := o[-1]$.

Notice that, in view of the above decompositions of $\delta_D, \overline{d}_D$ and the contraction data $(p, j, h)$, the $k$-th Poisson bracket in $\Lambda \otimes_M S^*\overline{D}$ is the skew-symmetrization $A_\infty^{(1-k)}$ of $\alpha_k^{[1-k]}$. In particular, the skew-symmetrization of $\alpha_k^{[1-k]}$ vanishes for $k > 3$ [25]. My next aim is twofold:

1. proving that $\alpha_k$ has no component of order higher than $\alpha_k^{[1-k]}$, i.e., $\alpha_k = \alpha_k^{[1-k]} + O(-k)$, for $k \neq 2$.
2. “computing” $\alpha_k^{[1-k]}$ and, in particular, showing that it is zero for $k > 3$.

Notice that the first claim states that the order of $\alpha_{k-1}(\square_i, \ldots, \square_k)$ is no higher than $1 + k - \sum_i \ell_i$ for $\square_i = O(\ell_i)$, $i = 1, \ldots, k$. The claim that $\alpha_k^{[1-k]} = 0$ for $k > 3$, can be interpreted as a further motivation why the $LR_{A_\infty}$-algebra structure on $\Lambda \otimes_M \overline{D}$ presents just one higher homotopy [25]. In order to reach my aim, I first prove a

**Lemma 33.** The order $-1$ component of the projection $p$ vanishes, i.e., $p^{[-1]} = 0$ (so that $p = p_0 + O(-2)$).

**Proof.** Let $\square \in D(\Lambda)$ be of order $H$. Then, $\square$ is locally of the form
\[
\square = \sum_{k + \ell + m = H} A_{i_1, \ldots, i_k, j_1, \ldots, j_l | \alpha_1, \ldots, \alpha_m} I_{i_1, \ldots, i_k} \nabla_{j_1} \cdots \nabla_{j_l} \nabla_{\alpha_1} \cdots \nabla_{\alpha_m}.
\]
where \( I_{i_1 \cdots i_k} := i_{\partial_{i_1}} \cdots i_{\partial_{i_k}} \), and the \( A \)'s are components of a (contravariant) tensor. The \( A \)'s are skew-symmetric in the \( i \)'s, symmetric in the \( j \)'s and symmetric in the \( \alpha \)'s. Compute
\[
p = \sum_{\ell + m = H} A^{i_1 \cdots i_k | j_1 \cdots j_\ell} \nabla_{j_1} \cdots \nabla_{j_\ell} \nabla_{\alpha_1} \cdots \nabla_{\alpha_m}
\]
Clearly,
\[
p^{[0]} = (p[0])[H] = p_0 = A^{i_1 \cdots i_k | j_1 \cdots j_\ell} \nabla_{j_1} \cdots \nabla_{j_\ell} V_{\alpha_1} \cdots V_{\alpha_m}.
\]

Now, let \( \ell > 0 \),
\[
A^{i_1 \cdots i_k | j_1 \cdots j_\ell} \nabla_{j_1} \cdots \nabla_{j_\ell} \nabla_{\alpha_1} \cdots \nabla_{\alpha_m}
= A^{i_1 \cdots i_k | j_1 \cdots j_{\ell-1} | \nabla_{j_\ell}, \nabla_{\alpha_1} \cdots \nabla_{\alpha_m}}
= A^{i_1 \cdots i_k | j_1 \cdots j_{\ell-1}} \nabla_{j_1} \cdots \nabla_{j_{\ell-1}} \sum_{r \leq m} \nabla_{j_\ell} \nabla_{\alpha_r} \nabla_{\alpha_m}.
\]

Since \( \nabla \) is adapted and torsion-quasi-free
\[
[
\nabla_i, \nabla_\alpha \lambda_{\beta_1 \cdots \beta_\ell} = \sum_{s \leq \ell} R^{\nabla_i}_{\alpha \beta_s \beta_1 \cdots \beta_s} \lambda_{\beta_1 \cdots \beta_\ell}, \quad (20)
\]
for all covariant tensors \( \lambda \) locally of the form
\[
\lambda = \lambda_{\beta_1 \cdots \beta_\ell} du^{\beta_1} \otimes \cdots \otimes du^{\beta_\ell}.
\]
In (20) \( R^{\nabla} \) is the curvature tensor of \( \nabla \). It follows from (20), that \( [\nabla_i, \nabla_\alpha] = O(0) \), and
\[
(\nabla_{(j_1 \cdots j_\ell)} \nabla_{(\alpha_1 \cdots \alpha_m)}) \in O(\ell + m - 2)
\]
for all \( \ell, m \). I conclude that
\[
p = p_0 + O(H - 2).
\]

The following proposition is a corollary of the above lemma, and the side conditions \( ph = 0 \), \( hj = 0 \), \( h^2 = 0 \).

**Proposition 34.**

\[
\begin{align*}
\gamma_k &= O(1 - k), \quad k \geq 1, \\
\beta_k &= O(2 - k), \quad k \geq 2, \\
\alpha_k &= O(1 - k), \quad k \geq 3
\end{align*}
\]
while
\[
\alpha_2 = O(0).
\]
Moreover, the highest order component \( \alpha_k^{[1-k]} \) of \( \alpha_k \) can be computed iteratively via formulas
\[
\epsilon_k(\Box_1, \ldots, \Box_k) := - \sum_{\ell + m = k} (-1)^{\ell} \gamma_{[\ell]}^{1-\ell}(\Box_1, \ldots, \Box_\ell) \otimes \gamma_{m}^{1-m}(\Box_{\ell+1}, \ldots, \Box_{\ell+m}),
\]
\[
\begin{align*}
\gamma_{k}^{[1-k]} &= h_0 \epsilon_k, \\
\alpha_{k}^{[1-k]} &= p_0 \epsilon_k,
\end{align*}
\]
\( \square = (\square_1, \ldots, \square_k), \square_1, \ldots, \square_k \in X \otimes_M D, \) being a k-tuple of homogeneous elements of given orders, \( k \geq 2. \)

**Proof.** The two parts of the proposition can be checked simultaneously by induction on \( k. \) Indeed, \( \gamma_1 = -j = -j_0 + \mathcal{O}(-1), \) \( \beta_2 = j(-) \circ j(-) = j_0(- \circ -) + \mathcal{O}(-1), \) and \( \alpha_2 = (- \circ -) + \mathcal{O}(-1) \) (where I used that \( p_0 \) preserves the product \( \circ \)). Moreover, \( \gamma_2 = h \beta_2 = h(j(-) \circ j(-)), \)

so that
\[
\gamma_2^{[0]} = h_0 j_0(- \circ -) = 0.
\]

Thus, compute
\[
\gamma_2^{[-1]} = h^{[-1]} j_0(- \circ -) + h_0(j^{[-1]}(-) \circ j_0(-) + j_0(-) \circ j^{[-1]}(-)) = h_0(j_0(-) \circ j_0(-))
\]

where I used formulas (8), (19). Now,
\[
\beta_k = \sum_{\ell + m = k} (-)^{\ell-1} \gamma_\ell(-) \circ \gamma_m(-)
\]

with \( \gamma_\ell = \mathcal{O}(1 - \ell) \) and \( \gamma_m = \mathcal{O}(1 - m) \) by induction hypothesis. Therefore, it is immediately seen that \( \beta_k = \mathcal{O}(2 - k), \)

and
\[
\beta_k^{[2-k]} = \sum_{\ell + m = k} (-)^{\ell-1} \gamma_\ell^{[1-\ell]}(-) \circ \gamma_m^{[1-m]}(-),
\]

so that
\[
\gamma_k = h \beta_k = \mathcal{O}(2 - k).
\]

But
\[
\gamma_k^{[2-k]} = \sum_{\ell + m = k} (-)^{\ell-1} h_0(\gamma_\ell^{[1-\ell]}(-) \circ \gamma_m^{[1-m]}(-)) = \sum_{\ell + m = k} (-)^{\ell-1} h_0(\gamma_\ell \circ \gamma_m(-)) = 0.
\]

Now, compute
\[
\beta_k^{[1-k]} = \sum_{\ell + m = k} (-)^{\ell-1} (\gamma_\ell^{[-\ell]}(-) \circ \gamma_m^{[1-m]}(-) + \gamma_\ell^{[1-\ell]}(-) \circ \gamma_m^{[1-m]}(-) + \gamma_\ell^{[1-\ell]}(-) \circ \gamma_m^{[-m]}(-)),
\]

and
\[
\gamma_k^{[1-k]} = h^{[-1]} \beta_k^{[2-k]} + h_0 \beta_k^{[1-k]} = h_0 \beta_k^{[1-k]} = h_0 \epsilon_k,
\]

where I used (8).

Finally, compute
\[
\alpha_k = p \beta_k = \mathcal{O}(2 - k).
\]

But
\[
\alpha_k^{[2-k]} = p_0 \beta_k^{[2-k]} = \sum_{\ell + m = k} (-)^{\ell-1} p_0 \gamma_\ell^{[1-\ell]}(-) \circ p_0 \gamma_m^{[1-m]}(-) = 0,
\]
where I used the side condition \( p_0h_0 = 0 \), and
\[
\alpha_k^{[1-k]} = p^{[1]}\beta_k^{[2-k]} + p_0\beta_k^{[1-k]} = p_0\varepsilon_k,
\]
where I used the above lemma and the side condition \( p_0h_0 = 0 \) again. \( \square \)

In view of the above proposition, a formula for \( \circ \) is enough to get inductive formulas for the \( \alpha_k^{[1-k]} \)'s. These formulas, which I compute in the proof of the next lemma, actually show that \( \alpha_k^{[1-k]} = 0 \) for \( k > 3 \), as announced.

Now on put
\[
S_{i,j,\ell} := \mathcal{A} \otimes_M \Lambda^iC\mathcal{X} \otimes_M S^jC\mathcal{X} \otimes_M S\mathcal{X} \subset S(\mathcal{A})
\]

**Lemma 35.** Let \( \square_1 \in S_{r,0,\ell} \) and \( \square_2 \in S_{s,0,m} \), then
\[
h_0(\square_1 \otimes \square_2) \in S_{r+s,1,\ell+m-2} + S_{r+s,0,\ell+m-1} + S_{r+s-1,0,\ell+m}
\]
\[
p_0(\square_1 \otimes \square_2) \in \begin{cases} 
\mathcal{A} \otimes_M S^{\ell+m-1} & \text{if } r + s = 0 \\
\mathcal{A} \otimes_M S^{\ell+m} & \text{if } r + s = 1 \\
0 & \text{if } r + s > 1
\end{cases}
\]

**Proof.** The operators \( \square_1 \) and \( \square_2 \) are locally of the form
\[
\square_1 = \Phi^{i_1 \cdots i_r | \alpha_1 \cdots \alpha_l} I_{i_1 \cdots i_r} \nabla_{\alpha_1} \cdots \nabla_{\alpha_l},
\]
\[
\square_2 = \Psi^{j_1 \cdots j_s | \beta_1 \cdots \beta_m} I_{j_1 \cdots j_s} \nabla_{\beta_1} \cdots \nabla_{\beta_m}.
\]

Then
\[
\square_1 \circ \square_2 = \Phi^{i_1 \cdots i_r | \alpha_1 \cdots \alpha_l} \Psi^{j_1 \cdots j_s | \beta_1 \cdots \beta_m} I_{i_1 \cdots i_r j_1 \cdots j_s} \nabla_{(\alpha_1} \cdots \nabla_{\alpha_l} \nabla_{\beta_1} \cdots \nabla_{\beta_m)} + \ell \Phi^{i_1 \cdots i_r | \alpha_1 \cdots \alpha_l} \Psi^{j_1 \cdots j_s | \beta_1 \cdots \beta_m} I_{i_1 \cdots i_r j_1 \cdots j_s} \nabla_{(\alpha_1} \cdots \nabla_{\alpha_l} \nabla_{\beta_1} \cdots \nabla_{\beta_m)} + r \Phi^{i_1 \cdots i_r | \alpha_1 \cdots \alpha_l} \Psi^{j_1 \cdots j_s | \beta_1 \cdots \beta_m} I_{i_1 \cdots i_r j_1 \cdots j_s} \nabla_{(\alpha_1} \cdots \nabla_{\alpha_l} \nabla_{\beta_1} \cdots \nabla_{\beta_m)} + O(\ell + m - 2).
\]

It remains to compute
\[
(\nabla_{(\alpha_1} \cdots \nabla_{\alpha_l} \nabla_{(\beta_1} \cdots \nabla_{\beta_m)}))^{\ell+m-1}.
\]

Let \( A^{\alpha_1 \cdots \alpha_l | \beta_1 \cdots \beta_m} \) be symmetric in the \( \alpha_l \)'s and the \( \beta_l \)'s separately. Then
\[
A^{\alpha_1 \cdots \alpha_l | \beta_1 \cdots \beta_m} (\nabla_{(\alpha_1} \cdots \nabla_{\alpha_l} \nabla_{(\beta_1} \cdots \nabla_{\beta_m)}))^{\ell+m-1} = A^{\alpha_1 \cdots \alpha_l | \beta_1 \cdots \beta_m} (\nabla_{\alpha_1} \cdots \nabla_{\alpha_l} \nabla_{\beta_1} \cdots \nabla_{\beta_m})^{\ell+m-1} = \frac{m}{m+1} A^{\alpha_1 \cdots \alpha_{\ell-1} | \beta_1 \cdots \beta_{m-1}} R^{i_1 \cdots i_{\ell-1} | \alpha_1 \cdots \alpha_{\ell-1} \beta_1 \cdots \beta_{m-1} \beta_1 \cdots \beta_{m-1}} \nabla_{(\alpha_1} \cdots \nabla_{\alpha_{\ell-1}} \nabla_{(\beta_1} \cdots \nabla_{\beta_{m-1})}.
\]

I conclude that
\[
\square_1 \circ \square_2 = \frac{m}{m+1} R^{i_1 \cdots i_{\ell-1} | \alpha_1 \cdots \alpha_{\ell-1} \beta_1 \cdots \beta_{m-1} \beta_1 \cdots \beta_{m-1}} I_{i_1 \cdots i_{\ell-1} j_1 \cdots j_s} \nabla_{(\alpha_1} \cdots \nabla_{\alpha_{\ell-m-2}} + \ell \frac{m}{m+1} R^{i_1 \cdots i_{\ell-1} | \alpha_1 \cdots \alpha_{\ell-1} \beta_1 \cdots \beta_{m-1} \beta_1 \cdots \beta_{m-1}} I_{i_1 \cdots i_{\ell-1} j_1 \cdots j_s} \nabla_{(\alpha_1} \cdots \nabla_{\alpha_{\ell-m-1}} + r \frac{m}{m+1} R^{i_1 \cdots i_{\ell-1} | \alpha_1 \cdots \alpha_{\ell-1} \beta_1 \cdots \beta_{m-1} \beta_1 \cdots \beta_{m-1}} I_{i_1 \cdots i_{\ell-1} j_1 \cdots j_s} \nabla_{(\alpha_1} \cdots \nabla_{\alpha_{\ell-m})}.
\]
Corollary 36. Let $\square_1, \ldots, \square_k \in \overline{\Lambda} \otimes_M \overline{D}$ with $\square_i$ being of order $\ell_i$, $i = 1, \ldots, k$. Put $\ell := \ell_1 + \cdots + \ell_k$. Then
\[
\wedge_k [1-k](\square_1, \ldots, \square_k) \in S_{k-1,0,\ell-2k+2}, \\
\varepsilon_k(\square_1, \ldots, \square_k) \in S_{k-2,1,\ell-2k+2} + S_{k-2,0,\ell-2k+3} + S_{k-3,0,\ell-2k+4}, \\
\alpha_k^{[1-k]}(\square_1, \ldots, \square_k) \in \begin{cases} 
\overline{\Lambda} \otimes_M S^{\ell-1}\overline{\Lambda} & \text{if } k = 2 \\
\overline{\Lambda} \otimes_M S^{\ell-2}\overline{\Lambda} & \text{if } k = 3, \\
0 & \text{if } k > 3
\end{cases}
\]
k > 1.

Proof. Immediate, by induction on $k$. \hfill \Box

Now, compute $\alpha_k^{[1-k]}$, for $k = 1, 2, 3$. Let $\square_1, \square_2, \square_3 \in \overline{\Lambda} \otimes_M \overline{D}$ be locally given by
\[
\square_i = \Phi_i^{a_1 \cdots a_r} V_{a_1 \cdots a_r}, \quad i = 1, 2, 3.
\]
First of all,
\[
\alpha_2^{[0]}(\square_1, \square_2) = \Phi_1^{a_1 \cdots a_r} \Phi_2^{a_1 \cdots a_r+1 \cdots a_{r+s}} V_{a_1 \cdots a_{r+s}}.
\]
Moreover, using Formula (22), it is easy to see that
\[
\alpha_2^{[-1]}(\square_1, \square_2) = t \Phi_1^{a_1 \cdots a_r-1} \nabla \Phi_2^{a_1 \cdots a_r+1 \cdots a_{r+s-1}} V_{a_1 \cdots a_{r+s-1}},
\]
\[
\alpha_3^{[-2]}(\square_1, \square_2, \square_3) = \frac{2t}{t+1} R_{a_1 \beta}^{i} \Phi_1^{a_1 \cdots a_r-1} \Phi_2^{a_1 \cdots a_r+1 \cdots a_{r+s-2}} I_i \Phi_3^{a_1 \cdots a_{r+s-2}} V_{a_1 \cdots a_{r+s-2}}
\]
which are duly consistent with formulas in [25].

Remark 37. Notice that the natural $D(\overline{\Lambda})$-module structure on $\overline{\Lambda}$ can be transferred along the contraction data $(p, j, h)$ as well. Indeed, $\overline{\Lambda}$ is actually a DG module over $D(\overline{\Lambda})$ with differential $\overline{\delta} : \overline{\Lambda} \to \overline{\Lambda}$. Moreover, this DG module structure (and the DG algebra structure on $D(\overline{\Lambda})$) can be encoded in a DG algebra structure on $D(\overline{\Lambda}) \oplus \overline{\Lambda}$ given by
\[
(\square_1, \omega_1) (\square_2, \omega_2) := (\square_1 \circ \square_2, \square_1 \omega_2), \quad (\square_i, \omega_i) \in D(\overline{\Lambda}) \oplus \overline{\Lambda}, \quad i = 1, 2,
\]
with differential $\overline{\delta} := \overline{\delta} \oplus \overline{d}$. Similarly, consider the complex $((\overline{\Lambda} \otimes_M \overline{D}) \oplus \overline{\Lambda}, \overline{\delta})$ where $\overline{\delta} := \overline{\delta} \oplus \overline{d}$. There are obvious contraction data $(p^{\oplus}, j^{\oplus}, h^{\oplus})$ of $(D(\overline{\Lambda}) \oplus \overline{\Lambda}, \overline{\delta}^{\oplus})$ over $((\overline{\Lambda} \otimes_M \overline{D}) \oplus \overline{\Lambda}, \overline{\delta}^{\oplus})$. Namely,
\[
p^{\oplus} := p \oplus \text{id}, \quad j^{\oplus} := j \oplus \text{id}, \quad h^{\oplus} := h \oplus 0.
\]
Accordingly, there is an $A_\infty$-algebra structure $\{\alpha_k^{\oplus}, k \in \mathbb{N}\}$ in $\overline{\Lambda} \otimes_M \overline{D} \oplus \overline{\Lambda}$. By construction,
\[
\alpha_k^{\oplus}(\square_1, \omega_1, \ldots, \square_k, \omega_k) = \alpha_k(\square_1, \ldots, \square_k) + \alpha_k^{\oplus}(\square_1, \ldots, \square_{k-1}, \omega_k),
\]
$\square_i, \omega_i \in D(\overline{\Lambda}) \oplus \overline{\Lambda}, i = 1, \ldots, k$. Therefore, if one puts
\[
\mu_k(\square_1, \ldots, \square_{k-1} | \omega) := \alpha_k^{\oplus}(\square_1, \ldots, \square_{k-1}, \omega),
\]
then \(\{\mu_k, k \in \mathbb{N}\}\) is an \(A_\infty\)-module structure on \(\Lambda\), over the \(A_\infty\)-algebra \(\Lambda \otimes_M \mathcal{D}\). It is easy to see that, since \(h^* \rho = 0\) for all \(\rho \in \Lambda\), then the \(\mu\)'s are simply given by

\[
\mu_1 = \partial \\
\mu_k(\square_1, \ldots, \square_{k-1})\omega = (-)^{k-1}g_k(\square_1, \ldots, \square_{k-1})\omega, \quad k \geq 2.
\]

**Conclusions**

I proved that the \(LR_\infty\)-algebra \((\Lambda \otimes_M \mathcal{F}, \mathcal{L})\) of a foliation \([25]\) can be actually extended in a natural way to an \(A_\infty\)-algebra \((\Lambda \otimes_M \mathcal{D}, \mathcal{A})\) of longitudinal form-valued normal differential operators. This can be done via purely geometric data, namely a distribution complementary to the characteristic distribution and a connection (of a suitable kind). Notice that \((\Lambda \otimes_M \mathcal{F}, \mathcal{L})\) can be interpreted (to some extent) as the (derived) Lie-Rinehart algebra of vector fields on the space \(P\) of integral manifolds. Similarly, it is natural to interpret \((\Lambda \otimes_M \mathcal{D}, \mathcal{A})\) as the (derived) associative algebra of differential operators on \(P\). In this respect, it is tempting to conjecture that \((\Lambda \otimes_M \mathcal{D}, \mathcal{A})\) is a universal enveloping SH algebra of \((\Lambda \otimes_M \mathcal{F}, \mathcal{L})\). However, the theory of universal enveloping of \(LR_\infty\)-algebras (or \(L_\infty\)-algebroids) is not yet available and developing this research line goes beyond the scopes of this paper. Here, I just notice that \((\Lambda \otimes_M \mathcal{D}, \mathcal{A})\) is indeed a (possibly non universal) enveloping SH algebra of \((\Lambda \otimes_M \mathcal{F}, \mathcal{L})\) in the following sense. The inclusion \(\Lambda \otimes_M \mathcal{F} \rightarrow \Lambda \otimes_M \mathcal{D}\) can be trivially extended to a morphism \(J: \Lambda \otimes_M \mathcal{F} \rightarrow \Lambda \otimes_M \mathcal{D}\) of the \(L_\infty\)-algebra \((\Lambda \otimes_M \mathcal{F}, \mathcal{L})\) and the \(L_\infty\)-algebra obtained by skew-symmetrization of operations in \(\mathcal{A}\), simply putting \(J_k = 0\) for \(k > 1\). Then, it is easy to see, using the explicit formulas for brackets in \(\mathcal{L}\) \([25]\), that

\[
u_k(Z_1, \ldots, Z_{k-1})\omega = (A\alpha_k)(Z_1, \ldots, Z_{k-1})\omega, \quad \omega \in \Lambda, \quad Z_i \in \Lambda \otimes_M \mathcal{F}, \quad i = 1, \ldots, k-1,
\]

which specializes \([23]\) to the present simple case where \(j\) is an inclusion and \(J_k = 0\) for \(k > 1\).

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