A $q$-ENUMERATION OF ALTERNATING PERMUTATIONS

MATTHIEU JOSUAT-VERGES

Abstract. A classical result of Euler states that the tangent numbers are an alternating sum of Eulerian numbers. A dual result of Roselle states that the secant numbers can be obtained by a signed enumeration of derangements. We show that both identities can be refined with the following statistics: the number of crossings in permutations and derangements, and the number of patterns 31-2 in alternating permutations.

Using previous results of Corteel, Rubey, Prellberg, and the author, we derive closed formulas for both $q$-tangent and $q$-secant numbers. There are two different methods for obtaining these formulas: one with permutation tableaux and one with weighted Motzkin paths (Laguerre histories).

1. Introduction

The classical Euler numbers $E_n$ are given by the Taylor expansion of the tangent and secant functions:

$$\tan(x) + \sec(x) = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$ 

Since the tangent is an odd function and the secant is an even function, the integers $E_{2n+1}$ are called the tangent numbers and the integers $E_{2n}$ are called the secant numbers. More than a century ago, Désiré André [1] showed that $E_n$ is the number of alternating permutations in $S_n$.

For any permutation $\sigma \in S_n$, let $exc(\sigma)$ be the number of exceedances of $\sigma$, i.e. the number of integers $i \in \{1, \ldots, n-1\}$ such that $\sigma(i) > i$. In [13] Foata and Schützenberger showed that

$$\sum_{\sigma \in S_n} x^{exc(\sigma)} \bigg|_{x=-1} = \begin{cases} 0 & \text{if } n \text{ is even}, \\ (-1)^{n-1} E_n & \text{if } n \text{ is odd}. \end{cases}$$

This is a combinatorial version of a result by Euler [9]. Let $D_n \subset S_n$ be the set of derangements; then another result is that

$$\sum_{\sigma \in D_n} x^{exc(\sigma)} \bigg|_{x=-1} = \begin{cases} (-1)^{\frac{n}{2}} E_n & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}$$

This was first obtained by Roselle [24] using a slightly different combinatorial interpretation.

Recently Foata and Han [12] have derived $q$-analogs of the identities (1) and (2). The statistic involved in permutations and derangements is the major index, and the statistic involved in alternating permutations is the number of inversions.

There exist several $q$-analogs of the Euler numbers $E_n$. Firstly, we can mention the coefficients of the $x$-expansions of $\sin_q(x)/\cos_q(x)$ and $1/\cos_q(x)$, where $\sin_q(x)$ and $\cos_q(x)$ are the $q$-sine and $q$-cosine introduced by Jackson [18]. These are the classical $q$-Euler numbers, used by Foata and Han [12]. See also [2, 3, 11].

Partially supported by the ANR Jeune Chercheur IComb. A short version of this work was presented at the conference Permutation Patterns 2009.
Secondly, there are the \( q \)-analogs defined by continued fraction expansion. Besides the approach made by Prodinger [15, 21, 22], there is also the \( q \)-analog introduced by Han, Randrianarivony, Zeng [17, 4]. The latter is the one considered in this article and is defined as follows.

**Definition 1.1.** The \( q \)-tangent numbers \( E_{2n+1}(q) \) are defined by

\[
\sum_{n=0}^{\infty} E_{2n+1}(q)x^n = \frac{1}{1 - \frac{[1]_q [2]_q x}{1 - \frac{[2]_q [3]_q x}{1 - \frac{[3]_q [4]_q x}{\ddots}}}},
\]

where \([n]_q = \frac{1-q^n}{1-q}\), and the \( q \)-secant numbers \( E_{2n}(q) \) are defined by

\[
\sum_{n=0}^{\infty} E_{2n}(q)x^n = \frac{1}{1 - \frac{[1]^2 q x}{1 - \frac{[2]^2 q x}{1 - \frac{[3]^2 q x}{\ddots}}}}.
\]

The first values are \( E_0(q) = E_1(q) = E_2(q) = 1 \), \( E_3(q) = 1+q \), \( E_4(q) = 2+2q+q^2 \), \( E_5(q) = 2+5q+5q^2+3q^3+q^4 \). To compute these polynomials, we can use the continued fractions, but also some combinatorial interpretations given in [17] and [4]. See Remarks 2.1 and 3.3 below for more precision.

The first purpose of this article is to construct other \( q \)-analogos of the combinatorial identities (1) and (2) using the above analog \( E_n(q) \). To this end we make use of the notion of crossings introduced in [5]. A crossing of \( \sigma \in S_n \) is a pair \((i, j)\) such that either \( i < j \leq \sigma(i) < \sigma(j) \) or \( \sigma(i) < \sigma(j) < i < j \). For any permutation \( \sigma \in S_n \), let \( \text{wex}(\sigma) \) be the number of weak exceedances of \( \sigma \), i.e. the number of integers \( i \in \{1, \ldots, n\} \) such that \( \sigma(i) \geq i \).

**Theorem 1.2.**

\[
\sum_{\sigma \in S_n} (-1)^{\text{wex}(\sigma)} q^{\text{cr}(\sigma)} = \begin{cases} 0 & \text{if } n \text{ is even}, \\ (-1)^{\frac{n}{2}} E_n(q) & \text{if } n \text{ is odd}. \end{cases}
\]

**Theorem 1.3.**

\[
\sum_{\sigma \in D_n} \left( -\frac{1}{q} \right)^{\text{wex}(\sigma)} q^{\text{cr}(\sigma)} = \begin{cases} \left( -\frac{1}{q} \right)^{\frac{n}{2}} E_n(q) & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}. \end{cases}
\]

With \( A_n(y, q) = \sum_{\sigma \in S_n} y^{\text{wex}(\sigma)} q^{\text{cr}(\sigma)} \) the left-hand side of (5) reads \( A_n(-1, q) \). Using another combinatorial description (see Section 2 above), the coefficients \( \hat{E}_{k,n}(q) \) such that

\[
A_n(y, q) = \sum_{k=1}^{n} y^k \hat{E}_{k,n}(q),
\]

were explicitly calculated by Williams [29, Corollary 6.3] in the form

\[
\hat{E}_{k,n}(q) = \sum_{i=0}^{k-1} (-1)^i [k-i]^n q^{ki-k^2} \left( \binom{n}{i} q^{k-i} + \binom{n}{i-1} \right).
\]
These coefficients must be regarded as $q$-Eulerian numbers since the preceding formula becomes the classical formula for Eulerian numbers when $q = 1$.

Another expression for $A_n(y,q)$ was conjectured by Corteel and Rubey and proved in two different ways, first by the author [19] and second by Corteel, Rubey, Prellberg [8]:

$$A_n(y,q) = \frac{1}{(1-q)^{n+1}} \sum_{k=0}^{n} (-1)^k \left( \sum_{j=0}^{n-k} y^j \binom{n}{j} \binom{n}{j+k} - \binom{n}{j-1} \binom{n}{j+k+1} \right)$$

(8)

The two different proofs of (8) use respectively the combinatorics of permutation tableaux [26] and Laguerre histories [5]. There is no direct analytic proof that the two formulas for $A_n(y,q)$ are equal.

The second purpose of this article is to show that with similar methods, we can obtain closed formulas for $E_n(q)$ which differ from the ones coming from [17].

**Theorem 1.4.**

$$E_{2n+1}(q) = \frac{1}{(1-q)^{2n+1}} \sum_{k=0}^{n} \left( \binom{2n+1}{n-k} - \binom{2n+1}{n-k-1} \right) \sum_{i=0}^{2k+1} (-1)^i q^{(2k+1)/2} q^{i(2n+2k+1-i)}.$$ 

(9)

**Theorem 1.5.**

$$E_{2n}(q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \left( \binom{2n}{n-k} - \binom{2n}{n-k-1} \right) \sum_{i=0}^{2k} (-1)^i q^{(2k)/2} q^{i(2n+k-i)}.$$ 

(10)

Equation (14) in the article of Han, Randrianarivony, Zeng [17] is an expression for the generating function of some polynomials which generalize both $E_{2n}(q)$ and $E_{2n+1}(q)$ (see equation (7) in this reference). By expanding the ratios in their identity it is thus possible to derive expressions for $E_{2n}(q)$ and $E_{2n+1}(q)$; this is done explicitly in [25].

The formulas in (9) and (10) are different from the ones in [25], and one can notice similarities with the Touchard-Riordan formula [20] which gives the distribution of crossings in the set $\mathcal{S}_{2n} \subset \mathcal{S}_{2n}$ of fixed-point free involutions. If $\sigma \in \mathcal{S}_{2n}$, then $\frac{1}{n} \text{cr}(\sigma)$ is the number of pairs $(i,j)$ such that $i < j < \sigma(i) < \sigma(j)$, so we recover a more classical definition of crossings. The formula is

$$\sum_{\ell \in \mathcal{S}_{2n}} q^{\frac{1}{n} \text{cr}(\ell)} = \frac{1}{(1-q)^{n}} \sum_{k=0}^{n} \left( \binom{2n}{n-k} - \binom{2n}{n-k-1} \right) (-1)^k q^{k(k+1)/2}.$$ 

Note that the left-hand side of (11) is also the sum of weights of Dyck paths such that the weight of a step \( \nearrow \) is always 1, and the weight of a step \( \searrow \) starting at height $h$ is $q^i$ for some $i \in \{0, \ldots, h-1\}$. So this is a subset of the paths for $E_{2n}(q)$.

This article is organized as follows. Section 2 contains preliminaries that are mostly contained in the references. Section 3 contains the proofs of Theorems 1.2 and 1.4. Section 4 contains the proofs of Theorems 1.3 and 1.5. Section 5 contains alternative proofs of Theorems 1.4 and 1.5.

**Acknowledgement**

I thank my advisor Sylvie Corteel for her help, advice, support and kindness. I thank Dominique Foata, Jiang Zeng, Guo-Niu Han, and Xavier Viennot for their insightful suggestions.
2. Definitions, conventions, preliminaries

2.1. Permutations and Laguerre histories. Throughout this article, we use the convention that \( \sigma(0) = 0 \) and \( \sigma(n+1) = n+1 \) if \( \sigma \in S_n \). Let us have \( i \in \{1, \ldots, n\} \); it is a weak exceedance of \( \sigma \) if \( \sigma(i) \geq i \), and an ascent of \( \sigma \) if \( \sigma(i) < \sigma(i+1) \) (note that \( n \) is always an ascent with our conventions). We denote by \( \text{wex}(\sigma) \) and \( \text{asc}(\sigma) \) the total numbers of weak exceedances and ascents in \( \sigma \). A descent of \( \sigma \) is an \( i \) such that \( \sigma(i) > \sigma(i+1) \). A pattern 31-2 of \( \sigma \) is a triple \( (i, i+1, j) \) such that \( i + 1 < j \) and \( \sigma(i+1) < \sigma(j) < \sigma(i) \). We denote by 31-2(\( \sigma \)) the total number of patterns 31-2 in \( \sigma \).

A Laguerre history (or “histoire de Laguerre”) of size \( n \) is a weighted Motzkin path of \( n \) steps such that:

- the weight of a step \( \nearrow \) starting at height \( h \) is \( yq^i \) for some \( i \in \{0, \ldots, h\} \),
- the weight of a step \( \rightarrow \) starting at height \( h \) is either \( yq^i \) for some \( i \in \{0, \ldots, h\} \) or \( q^i \) for some \( i \in \{0, \ldots, h-1\} \),
- the weight of a step \( \searrow \) starting at height \( h \) is \( q^i \) for some \( i \in \{0, \ldots, h-1\} \).

They are in bijection with other kinds of Motzkin paths with one fewer step. A large Laguerre history of size \( n \) is a weighted Motzkin path of \( n-1 \) steps such that:

- the weight of a step \( \nearrow \) starting at height \( h \) is \( yq^i \) for some \( i \in \{0, \ldots, h\} \),
- the weight of a step \( \rightarrow \) starting at height \( h \) is either \( yq^i \) or \( q^i \) for some \( i \in \{0, \ldots, h\} \),
- the weight of a step \( \searrow \) starting at height \( h \) is \( q^i \) for some \( i \in \{0, \ldots, h\} \).

There are several known bijections between \( S_n \) and the set of Laguerre histories of size \( n \). The Foata-Zeilberger bijection \( \Psi_{FZ} \) has the property that the weight of \( \Psi_{FZ}(\sigma) \) is \( y^{\text{wex}(\sigma)}q^{\pi(\sigma)} \). The Françon-Viennot bijection \( \Psi_{FV} \), originally given in [14], has the property that the weight of \( \Psi_{FV}(\sigma) \) is \( y^{\text{asc}(\sigma)}q^{31-2(\sigma)} \). All this is present in [5] and references therein, we recall here briefly the definition of \( \Psi_{FV} \) since we use it later. Let us have \( \sigma \in S_n \), \( j \in \{1, \ldots, n\} \) and \( k = \sigma(j) \). Then the kth step of \( \Psi_{FV}(\sigma) \) is:

- a step \( \nearrow \) if \( k \) is a valley, i.e. \( \sigma(j-1) > \sigma(j) < \sigma(j+1) \),
- a step \( \searrow \) if \( k \) is a peak, i.e. \( \sigma(j-1) < \sigma(j) > \sigma(j+1) \),
- a step \( \rightarrow \) if \( k \) is a double ascent, i.e. \( \sigma(j-1) < \sigma(j) < \sigma(j+1) \), or a double descent, i.e. \( \sigma(j-1) > \sigma(j) > \sigma(j+1) \).

Moreover the weight of the kth step is \( y^\delta q^i \) where \( \delta = 1 \) if \( j \) is an ascent and 0 otherwise, and \( i \) is the number of \( u \in \{1, \ldots, j-2\} \) such that \( \sigma(u) > \sigma(j) > \sigma(u+1) \) (i.e. the number of patterns 31-2 such that \( j \) correspond to the 2). See Figure 1 for an example.

![Figure 1](image_url)  

**Figure 1.** Example of the permutation 4371265 and its image under the Françon-Viennot bijection.

The continued fractions in Definition 1.1 are the natural \( q \)-analog of the ones appearing in Flajolet’s celebrated article [10]. The methods in this reference give a combinatorial interpretation of \( E_n(q) \) in terms of weighted Dyck paths. Indeed,
if $\delta \in \{0, 1\}$ then $E_{2n+\delta}(q)$ is the sum of weights of Dyck paths of length $2n$ such that:

- the weight of a step $\nearrow$ starting at height $h$ is $q^h$ for some $i \in \{0, \ldots, h\}$,
- the weight of a step $\searrow$ starting at height $h$ is $q^h$ for some $i \in \{0, \ldots, h-1+\delta\}$,

and the weight of the path is the product of the weights of its steps.

The set of alternating permutations $A_n$ consists of permutations $\sigma \in S_n$ such that $\sigma(2i-1) > \sigma(2i) < \sigma(2i+1)$ for any $i \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \}$. When $n$ is even, we have $\sigma \in A_n$ if and only if $\sigma$ has no double descent, and no double ascent. When $n$ is odd, we have $\sigma \in A_n$ if and only if $\sigma$ has no double descent, and only one double ascent at position $n$. Another possible definition is that $\sigma$ is alternating if $\sigma(1) < \sigma(2) > \sigma(3) < \ldots$, but with the convention that $\sigma(0) = 0$ and $\sigma(n+1) = n+1$ the present definition is more natural.

**Remark 2.1.** When $n$ is even, the fact that $\sigma \in A_n$ if and only if $\sigma$ has no double descent and no double ascent implies a known combinatorial interpretation [4, 17]:

$$E_n(q) = \sum_{\sigma \in A_n} q^{|\text{inv}(\sigma)|}.$$  

Indeed, if $\sigma$ is alternating then the corresponding Laguerre history $\Psi_{FV}(\sigma)$ has no horizontal step, and when $y = 1$ these Laguerre histories are precisely the weighted Dyck paths corresponding to the continued fraction (4) defining $q$-secant numbers.

This interpretation (12) was first given by Han, Randrianarivony, and Zeng (see equation (42) in [17]). The number of patterns $2-13$ is equidistributed with $(\text{asc},31-2)$ in $S_n$ but not in $D_n$. The same result was proved by D. Chebikin [4] with a refinement of the Françon-Viennot bijection. For the case when $n$ is odd, see Remark 3.3 below.

### 2.2. Eulerian polynomials and their $q$-analogs.

We define

$$A_n(y, q) = \sum_{\sigma \in S_n} y^{\text{wex}(\sigma)} q^{\text{cr}(\sigma)} \quad \text{and} \quad B_n(y, q) = \sum_{\sigma \in D_n} y^{\text{wex}(\sigma)} q^{\text{cr}(\sigma)},$$

so the left-hand sides of (5) and (6) are respectively $A_n(-1, q)$ and $B_n(-\frac{1}{y}, q)$. From the bijections $\Psi_{FV}$ and $\Psi_{FZ}$ mentioned above, we know that the bistatistic $(\text{wex},\text{cr})$ is equidistributed with $(\text{asc},31-2)$ in $S_n$ (but not in $D_n$). It is elementary to show that $\sum_{\sigma \in S_n} y^{\text{wex}(\sigma)} = y \sum_{\sigma \in S_n} y^{\text{exc}(\sigma)}$, and $A_n(y, q)$ could be defined in terms of the statistic exc with another definition of crossings. The distinction is not relevant for derangements.

**Proposition 2.2.** We have inversion formulas:

$$A_n(y, q) = \sum_{k=0}^{n} \binom{n}{k} y^{n-k} B_k(y, q) \quad \text{and} \quad B_n(y, q) = \sum_{k=0}^{n} \binom{n}{k} (-y)^{n-k} A_k(y, q).$$

**Proof.** This is perhaps the most common application of the inclusion-exclusion principle. To get this refined version, we have to check how the statistics are changed when adding a fixed point to a permutation. The number of crossings does not change, and the number of weak exceedances increases by 1 due to the added fixed point. This explains the powers of $y$ in the formulas. $\square$

From the formula for $A_n(y, q)$ given in equation (8) and the previous proposition, we obtain that

$$B_n(y, q) = \frac{1}{(1-y)} \sum_{k=0}^{n} (-1)^k \binom{n-k}{k} \left( \sum_{j=0}^{k} y^j C(n, k, j) \right) \left( \sum_{i=0}^{k} y^i q^{i(k+1-i)} \right),$$

where $C(n, k, j)$ is a binomial coefficient.
where \( C(n, k, j) = \binom{j}{i} \sum_{i=0}^{j} q^{j-i} \binom{n-i}{i+k} - \binom{n}{j-1} \sum_{i=0}^{j} q^{j-i} \binom{n-j+1}{i+k+1} \).

Unfortunately it seems that there is no further simplification; it is just a straightforward rearrangement of \( \sum_{k=0}^{n} \binom{n}{k} (-y)^{n-k} A_k(y, q) \). We omit details. Moreover the two different proofs of \((8)\) in [19, 8] can be modified so as to give a direct derivation of \((15)\).

2.3. Permutation tableaux. Let \( \lambda \) be a Young diagram in English notation, eventually with empty rows. A permutation tableau \( T \) of shape \( \lambda \) is a filling of \( \lambda \) with 0s and 1s such that there is at least a 1 in each column, and for any 0 either all entries to its left are also 0s or all entries above are also 0s. We refer the reader to [26] for more details. We denote by \( r(T) \) the number of rows of \( T \), by \( c(T) \) the number of columns, by \( o(T) \) the number of 1s, by \( so(T) = o(T) - c(T) \) the number of superfluous 1s (i.e. all 1s except the topmost of each column).

Via the Steingr¨ımsson-Williams bijection [26], the number of fixed points (respectively of weak exceedances, of crossings) in a permutation is the number of zero-rows (respectively of rows, of superfluous 1s) in the corresponding permutation tableau. Let \( PT_n \) be the set of permutation tableaux of half-perimeter \( n \), and \( DT_n \subset PT_n \) the subset of permutation tableaux with no zero-row (that we call derangement tableaux). Then we have

\[
A_n(y, q) = \sum_{T \in PT_n} y^{r(T)} q^{so(T)}, \quad B_n(y, q) = \sum_{T \in DT_n} y^{r(T)} q^{o(T)},
\]

\[
B_n(-\frac{1}{y}, q) = \sum_{T \in DT_n} (-1)^{r(T)} q^{o(T)-n}.
\]

Note that the combinatorial interpretation of \( A_n(y, q) \) in terms of permutation tableaux explains the link \((7)\) between our definition \((13)\) and the numbers \( \tilde{E}_{k,n}(q) \) from [29].

3. Eulerian numbers and \( q \)-tangent numbers

First we give a characterization of Laguerre histories corresponding to permutations \( \sigma \in \mathfrak{S}_{n+1} \) such that \( \sigma(n + 1) = 1 \). It is particularly similar toViennot’s criterion [27, Chapter II, page 24], but since the settings are different we believe that it is worth writing a direct proof.

**Lemma 3.1.** Let \( u \) be \( \sigma \in \mathfrak{S}_{n+1} \) and \( H \) its image under the Fran¸con-Viennot bijection as described in Section 2. Then the following conditions are equivalent:

- \( \sigma(n + 1) = 1 \)
- Except the first step, \( H \) has no step starting at height \( h \) with weight \( yq^h \).

In particular, if this condition is true there is no return to height \( 0 \) before the last step.

**Proof.** To begin, suppose that \( \sigma(n + 1) \neq 1 \). Let \( k \) be the minimal integer such that \( \sigma^{-1}(k) > \sigma^{-1}(1), \) and \( h \) the starting height of the \( k \) step in \( \Psi_{FV}(\sigma) \). We will show that the weight of this \( k \) step is \( yq^h \).

Let \( j = \sigma^{-1}(k) \). We have \( \sigma(j) < \sigma(j + 1) \); otherwise the inequalities \( \sigma(j + 1) < \sigma(j) = k \) and \( j + 1 > 1 \) would imply that the integer \( k' = \sigma(j + 1) \) contradicts the minimality of \( k \). This shows that \( j \) is an ascent, so the \( k \) step in \( \Psi_{FV}(\sigma) \) is either \( \rightarrow \) or \( \nearrow \) and has weight \( yq^i \) for some \( i \in \{0, \ldots, h\} \). We have to show that \( i = h \). By definition of \( \Psi_{FV} \), this means that we have to find distinct integers \( u_1, \ldots, u_h \in \{1, \ldots, j-2\} \) such that \( \sigma(u_l) > \sigma(j) > \sigma(u_l+1) \) for any \( l \in \{1, \ldots, h\} \).
Let $a < k$ be such that the $a$th step of $\Psi_{\text{FV}}(\sigma)$ is a step $\nearrow$. Let $c = \sigma^{-1}(a)$ and $b$ the minimum integer such that $\sigma(b) > \sigma(b + 1) > \cdots > \sigma(c)$. We have $b < c < j$. We distinguish two cases.

- If $\sigma(b) > k$, then there is $u \in \{b, \ldots, c - 1\}$ such that $\sigma(u) > k > \sigma(u + 1)$. So we have found one of the $u_k$.
- Otherwise, we have $\sigma(b - 1) < \sigma(b) > \sigma(b + 1)$ and $\sigma(b) < k$. So the $\sigma(b)$th step in $\Psi_{\text{FV}}(\sigma)$ is a step $\nearrow$ before the $k$th step.

Thus, with each step $\nearrow$ among the first $k - 1$ ones in $\Psi_{\text{FV}}(\sigma)$, we can associate either a $u_k$ satisfying the desired properties, or a step $\nearrow$ among the first $k - 1$ ones. This is indeed sufficient for finding the $u_1, \ldots, u_h$ and this completes the first part of the proof. We can use similar arguments to prove the reverse implication. □

**Remark 3.2.** It would be possible to consider two other parameters counting the right-to-left minima and the left-to-right maxima. Indeed, they respectively correspond to steps starting at height $h$ with weight $yq^k$ and to steps $\to$ with weight $yq^0$ or $\nearrow$ with weight $q^0$. In this broader context the previous lemma is immediate because permutations satisfying $\sigma(n + 1) = 1$ are exactly the ones with only one right-to-left minima.

We now give the proof of Theorem 1.2. Shin and Zeng [25] gave other proofs of this theorem in the spirit of [12].

**Proof.** For any $\sigma \in \mathfrak{S}_n$, we define $\tilde{\sigma} \in \mathfrak{S}_{n+1}$ by $\tilde{\sigma}(i) = \sigma(i) + 1$ if $i \in \{1, \ldots, n\}$ and $\tilde{\sigma}(n + 1) = 1$. It is clear that $\text{asc}(\sigma) = \text{asc}(\tilde{\sigma})$ and $31-2(\sigma) = 31-2(\tilde{\sigma})$.

Then for any $\sigma \in \mathfrak{S}_n$, let $f(\sigma) = \Psi_{\text{FV}}(\tilde{\sigma})$. From the previous lemma, $f$ is a bijection between $\mathfrak{S}_n$ and the set of Laguerre histories of $n + 1$ steps such that:

- The weight of a horizontal step at height $h$ is $q^i$ or $yq^i$ for some $i \in \{0, \ldots, h\}$
- Except for the first step which has weight $yq^0$, the weight of a step $\nearrow$ (resp. $\searrow$) starting at height $h$ is $yq^i$ (resp. $q^i$) for some $i \in \{0, \ldots, h - 1\}$.

Moreover the weight of $f(\sigma)$ is $y^{\text{asc}(\sigma)}q^{31-2(\sigma)}$.

In such a Laguerre history, we can remove the first and last steps, and obtain a large Laguerre history of $n - 1$ steps with respect to the shifted origin (this is not weight-preserving: there is a factor $y$ coming from the first step). In the large Laguerre history, when $y = -1$ the weights on horizontal steps cancel each other, and so $A_n(-1, q)$ is a sum of weights of Dyck paths of length $n - 1$. When $n$ is even, there is no such Dyck path, so $A_n(-1, q) = 0$.

When $n$ is odd, each Dyck path has $\frac{n}{2}$ steps $\nearrow$. So we can factorize the sum by $y^\frac{n+1}{2} = (-1)^{\frac{n-1}{2}}$, and the remaining weighted Dyck paths are precisely the ones given by the combinatorial interpretation of the $q$-tangent numbers. So, taking into account the factor $y$ coming from the removed step, in this case we have $A_n(-1, q) = (-1)^{\frac{n+1}{2}}E_n(q)$. □

**Remark 3.3.** Through the previous proof we have seen a bijection between permutations and large Laguerre histories. We easily see that when $n$ is odd, $\sigma \in \mathfrak{S}_n$ is alternating if and only if $\tilde{\sigma}$ is alternating, and this is equivalent to the fact that $f(\sigma)$ has no horizontal step. When $y = 1$ we obtain exactly the weighted Dyck paths corresponding to the continued fraction (3) defining the $q$-tangent numbers, so this proves the combinatorial interpretation (12) when $n$ is odd. This was also obtained by Chebikin [4]. Han, Randrianarivony, Zeng [17] also gave a combinatorial interpretation of $E_n(q)$ when $n$ is odd, but it differs from the one we have here.

Now we prove Theorem 1.4.
Proof. From Theorem 1.2 we have \( E_{2n+1}(q) = (-1)^{n+1}A_{2n+1}(-1, q) \). There are interesting cancellations if we directly set \( y = -1 \) in the proofs of (8). However, since it is not particularly useful to rewrite these proofs with \( y = -1 \), we only show how to simplify \( A_{2n+1}(-1, q) \) and get the right-hand side of (9). From (8) we have

\[
(18) \quad A_{2n+1}(-1, q) = \frac{1}{(1-q)} \sum_{k=0}^{2n+1} (-1)^k \left( g(n, k) + g(n, k+2) \right) \sum_{i=0}^{k} (-1)^i q^{(k+1-i)},
\]

where \( g(n, k) \) is the sum

\[
g(n, k) = \sum_{j=0}^{2n+1-k} (-1)^j \binom{2n+1}{j} \binom{2n+1}{j+k}.
\]

The main step is to simplify this sum. The first term is \( \frac{2n+1}{k} \) and the quotient of two successive terms is

\[
\frac{(-1)^{j+1} \binom{2n+1}{j+1} \binom{2n+1}{j+k}}{(-1)^j \binom{2n+1}{j} \binom{2n+1}{j+k}} = \frac{-2(n+1-j)(2n+1-k-j)}{(j+1)(j+1+k+1)} = \frac{(-2n+1+j)(-2n+1+k+j)}{(j+1)(j+1+k+1)},
\]

so \( g(n, k) \) is the hypergeometric series \( 2F_1(-2n+1-k, -2n+1+k+1; -1) \). We can use Kummer’s summation formula [16, Chapter 1], which reads

\[
2F_1(a, b; 1 + a - b; -1) = \frac{\Gamma(1 + a - b)\Gamma(1 + \frac{b}{2})}{\Gamma(1 + a)\Gamma(-b + 1 + \frac{b}{2})}.
\]

We cannot rigorously apply the summation formula here because the function \( \Gamma \) is singular at non-positive integers, but we can handle this since \( \Gamma(-m + \epsilon) \sim \frac{(-1)^m}{m!} \) for any \( m \in \mathbb{N} \) and small \( \epsilon \). We write formally

\[
g(n, k) = \frac{\Gamma(2n+2)}{\Gamma(2n+2-k)} \cdot \frac{\Gamma(k+1)\Gamma(-n+\frac{1}{2}+\frac{k}{2})}{\Gamma(k-2n)\Gamma(n+\frac{1}{2}+\frac{k}{2})}.
\]

When \( k \) is even, the only singular value is \( \Gamma(k-2n) \) in the denominator so \( g(n, k) = 0 \). When \( k \) is odd, the singular values compensate:

\[
\lim_{\epsilon \to 0} \frac{\Gamma(-n+\frac{1}{2}+\frac{k}{2}+\epsilon)}{\Gamma(k-2n+2\epsilon)} = 2(-1)^{n+k-1} \frac{\Gamma(2n+1-k)}{\Gamma(n+\frac{1}{2}+\frac{k}{2})}.
\]

After some simplification it emerges that when \( n \) is odd,

\[
g(n, k) = \frac{2(-1)^{n+k-1} \Gamma(2n+2)}{(2n+1-k)\Gamma(n+\frac{1}{2}+\frac{k}{2})} = (-1)^{n+k-1} \frac{2n+1}{(n-k+1)}.
\]

Going back to (18), we can restrict the sum to odd \( k \) and reindex it so that \( k \) becomes \( 2k+1 \). Since \( A_{2n+1}(-1, q) = (-1)^{n+1}E_{2n+1}(q) \) this gives the formula claimed in Theorem 1.4. \( \square \)

4. DERANGEMENTS AND q-SECANT NUMBERS

First we give the proof of Theorem 1.3. As in the case of Theorem 1.2, others proofs are as given in [25].

Proof. The image of a derangement by the Foata-Zeilberger bijection (as defined in [5]) is a Laguerre history such that there is no horizontal step with weight \( yq^0 \). This just means that a fixed point cannot be part of a crossing in a permutation. So \( B_n(y, q) \) is the sum of weights of Motzkin paths of length \( n \) such that:

- the weight of a step \( \nearrow \) starting at height \( h \) is \( y[h+1]_q \),
- the weight of a step \( \searrow \) starting at height \( h \) is \( [h]_q \),
- the weight of a step \( \rightarrow \) starting at height \( h \) is \( (1+yq)[h]_q \).
When \( y = -1/q \), the weights cancel each other on the horizontal steps. So in this case we can restrict the sum to Dyck paths instead of Motzkin paths, and the end of the proof is similar to that of Theorem 1.2. If \( n \) is odd, there is no Dyck path of length \( n \) so \( B_n(-\frac{1}{q}, q) = 0 \).

When \( n \) is even, each Dyck path has \( \frac{n}{2} \) steps \( \searrow \). So we can factorize the sum by \( y^\frac{n}{2} = (-\frac{1}{q})^\frac{n}{2} \), and the remaining weighted Dyck paths are precisely the ones given by the combinatorial interpretation of the \( q \)-secant numbers. So in this case we have \( B_n(-\frac{1}{q}, q) = (-\frac{1}{q})^\frac{n}{2} E_n(q) \).

The fact that \( B_n(-\frac{1}{2}, q) = 0 \) when \( n \) is odd is also a consequence of (17), because the transposition of derangement tableaux changes the parity of the number of rows and does not change the number of 1s.

To prove Theorem 1.5, it should be possible to use (15) and specialize at \( y = -\frac{1}{q} \). However it seems simpler to prove it directly with the methods and previous results of [19], where the first proof of (8) was given.

A method for computing \( A_n(y, q) \) is using a Matrix Ansatz. This is a consequence of the combinatorial interpretation of the PASEP partition function, given by Corteel and Williams in terms of permutation tableaux [7]. There is a similar result for computing \( B_n(y, q) \).

**Proposition 4.1.** Suppose that we have matrices \( D, E \), a row vector \( \langle W \rangle \), and a column vector \( |V\rangle \) such that
\[
(19) \quad DE - qED = I + qE + D, \quad \langle W|E = 0, \quad D|V \rangle = 0, \quad \langle W|V \rangle = 1,
\]
where \( I \) is the identity matrix. Then we have:
\[
B_n(y, q) = \langle W|(yD + E)^n|V\rangle.
\]

This is to be understood as follows. Via the commutation relation, \( (yD + E)^n \) can be uniquely written as a linear combination \( \sum_{i,j \geq 0} c_{ij} E^i D^j \), and then \( B_n(y, q) \) is the constant term \( c_{00} \). We give two proofs of this.

**Proof.** The first proof relies on a combinatorial interpretation of the three-parameter partition function of the PASEP [7]. If we have
\[
D'E' = qE'D' = D' + E', \quad \langle W|E' = 0, \quad D'|V \rangle = |V\rangle,
\]
then by [7, Theorem 3.1], \( \langle W|(yD' + E')^n|V\rangle \) is the generating function of permutation tableaux of half-perimeter \( n + 1 \), with no 1 in the first row, where \( y \) counts the number of rows minus 1, and \( q \) counts the number of superfluous 1s. In such permutation tableaux, we can remove the first row (which is filled with 0s) and obtain any permutation tableaux of half-perimeter \( n \). So we get
\[
A_n(y, q) = \langle W|(yD' + E')^n|V\rangle.
\]
Together with Proposition 2.2, this gives
\[
B_n(y, q) = \langle W|(yD' - yI + E')^n|V\rangle.
\]
But it is straightforward to check that \( D = D' - I \) and \( E = E' \) satisfy conditions (19).

**Proof.** In the second proof, we sketch how a recursive enumeration of derangement tableaux directly leads to the Matrix Ansatz (19). The method is the same as in [19, Section 2], and also [28]. For any word \( w \) of size \( n \) in \( D \) and \( E \), we define a Young diagram \( \lambda(w) \) by the following rule: the south-east boundary of \( \lambda(w) \) is obtained by reading \( w \) from left to right, and drawing a step east for each letter \( D \).
Then we substitute \( j \) since \( E \) we get all possible shapes for a derangement tableau, so \( \langle \) and \( \rangle \). With the results of \([\text{19}]\), it is known that \( \langle W \rangle = \langle W | yD + E \rangle | V \rangle \), where \( yD \) is the number of letters \( D \) in the word \( w \). See the references for more details about the method.

Now we prove Theorem 1.5.

**Proof.** We define
\[
\hat{D} = \frac{q+1}{q}(D + \frac{q}{q-1}), \quad \hat{E} = (q-1)(E + \frac{1}{q-1})
\]
so an immediate computation gives
\[
\hat{D}\hat{E} - q\hat{E}\hat{D} = \frac{1}{q}, \quad \langle W | \hat{E} \rangle = \langle W |, \quad \hat{D} | V \rangle = | V \rangle.
\]
and
\[
(-\frac{1}{q}D + E)^n = \frac{1}{q^n}(-\hat{D} + \hat{E})^n.
\]

With the results of [\text{19}], it is known that \( \langle W | (-\hat{D} + \hat{E})^n | V \rangle \) is a generating function for weighted involutions and has the following expression:
\[
\langle W | (-\hat{D} + \hat{E})^n | V \rangle = \sum_{0 \leq k \leq j \leq n} (-1)^j \left( \binom{n}{j} - \binom{n}{k-1} \right) q^{(j-k)(n-j-k)-k}.
\]

Since \( E_{2n}(q) = (-q)^n B_{2n}(\frac{1}{q}, q) \), we obtain
\[
E_{2n}(q) = \frac{1}{(1-q)^{2n}} \sum_{0 \leq k \leq j \leq 2n} (-1)^{n+j} \left( \binom{2n}{k} - \binom{2n}{k-1} \right) q^{(j-k)(2n-j-k)+n-k}.
\]

Discarding some negative powers of \( q \), we can restrict the sum to \( k \) such that \( k \leq n \), and then substitute \( k \) with \( n - k \), which gives
\[
E_{2n}(q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \sum_{j=0}^{2n} (-1)^{n+j} \left( \binom{2n}{n-k} - \binom{2n}{n-k-1} \right) q^{(j-k)(n-k+j)+k}.
\]

Then we substitute \( j \) with \( n - k + j \) and obtain
\[
E_{2n}(q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \sum_{j=0}^{n+k} (-1)^{k+j} \left( \binom{2n}{n-k} - \binom{2n}{n-k-1} \right) q^{(2k-j)+k}.
\]

Discarding other negative powers of \( q \), we can restrict the sum to \( j \) such that \( j \leq 2k \) and this gives the desired expression.

**Remark 4.2.** The idea of introducing the operators \( \hat{D} \) and \( \hat{E} \) and their combinatorial interpretation is at the origin of one of the proof of \( (8) \). Similarly, the computation above can be refined to obtain another proof of \( (15) \).
5. **Second proof of the \( q \)-secant and \( q \)-tangent formulas**

A second proof of (8), inspired by Penaud’s bijective proof of the Touchard-Riordan formula [20], was given in [8]. Actually, a slight modification of the method can give a direct proof of (15). In this section, we show that this can be used to obtain Theorem 1.4 and 1.5 directly from the definition of \( E_n(q) \) in terms of weighted Dyck paths. We begin with the \( q \)-secant numbers because this case is simpler.

5.1. **The \( q \)-secant numbers.** First, we notice that \((1 - q)^{2n}E_{2n}(q)\) is the sum of weights of Dyck paths \( H \) of length \( 2n \) such that:

- the weight of a step \( \nearrow \) starting at height \( h \) is 1 or \(-q^{b+1}\),
- the weight of a step \( \searrow \) starting at height \( h \) is 1 or \(-q^{h}\).

**Proposition 5.1.** There is a weight-preserving bijection between these Dyck paths \( H \) and pairs \((H_1, H_2)\) such that for some \( k \in \{0, \ldots, n\}\),

- \( H_1 \) is a left factor of a Dyck path, of \( 2n \) steps and final height \( 2k \),
- \( H_2 \) is a weighted Dyck path of length \( 2k \), with the same weights as \( H \) and also the condition that there are no two consecutive steps \( \nearrow \searrow \) both weighted by 1.

**Proof.** This is a direct adaptation of [8, Lemma 1] so we only sketch the proof. The idea is to look for the maximal factors of \( H \) being Dyck paths and having only steps with weight 1. To obtain \( H_1 \) from \( H \) we transform into a step \( \nearrow \) any step which is not inside one of these maximal factors. And to obtain \( H_2 \) from \( H \) we just remove these maximal factors. \( \square \)

By an elementary recurrence, the number of left factors of Dyck paths of \( 2n \) steps and final height \( 2k \) is \( \binom{2n}{n} - \binom{2n}{n-k} \). So to obtain Theorem 1.5 it just remains to prove the following proposition:

**Proposition 5.2.** The sum of weights of Dyck paths of length \( 2k \), satisfying conditions as \( H_2 \) above, is

\[
M_k(q) = \sum_{j=0}^{2k} (-1)^{j+k} q^{((2k-j)+k)}.
\]

**Proof.** It is possible to adapt the proof of [8, Proposition 5]. However we take a slightly different point of view here, and show that we can exploit some properties of T-fractions. The combinatorial theory for T-fractions was developed by Roblet and Viennot in [23], and gives generating functions for the kinds of paths we are dealing with. Indeed, this reference gives immediately that

\[
\sum_{k=0}^{\infty} M_k(q)t^k = \frac{1}{1 + t - \frac{(1 - q)^2t}{1 + t - \frac{(1 - q)^2t}{1 + t - \frac{(1 - q)^2t}{\cdots}}}}.
\]

To make this article more self-contained, we briefly show how to obtain (20). Recall that a Schröder path of length \( n \) is a path starting at \((0,0)\), ending at \((n,0)\), with steps \((1,1)\), \((1,-1)\) and \((2,0)\), and remaining above the horizontal axis. They are similar to Dyck paths except that we authorize the horizontal step \((2,0)\) (note that the length of the step does not determine the number of steps). Let us consider weighted Schröder paths, such that:

- the weight of a step \( \nearrow \) starting at height \( h \) is 1 or \(-q^{b+1}\),
- the weight of a step \( \searrow \) starting at height \( h \) is 1 or \(-q^{h}\),
- the weight of a step \( \rightarrow \) is \(-1\).
The sum of weights of these Schröder paths of length $2k$ is $M_k(q)$. Indeed there is a sign-reversing involution on Schröder paths such that the fixed points are the Dyck paths with no two consecutive steps $\nearrow \searrow$ both weighted by 1. The involution is obtained by exchanging the first occurrence of $\rightarrow$ with weight $-1$, or $\nearrow \searrow$ with weight 1, with (respectively) $\nearrow \searrow$ with weight 1, or $\rightarrow$ with weight $-1$. For the generating function of Schröder paths, standard arguments [10] give the continued fraction in (20).

We need to extract the coefficients of $t^k$ in the continued fraction. To do this, there are well-established methods linking such continued fractions with quotients of contiguous basic hypergeometric series [6, Chapter 19]. In the present case, we can use a limit case of (19.2.11a) in [6]. If we consider the more general continued fraction

$$M(z) = \frac{1}{1 + t - \frac{(1 - qz)^2 t}{1 + t - \frac{(1 - q^2z)^2 t}{\ddots}}},$$

then [6, (19.2.11a)] with $c = tq$ and $a = -b = i\sqrt{q}l$ gives that

$$M(z) = \frac{1}{1 - z} \frac{2\phi_1(a, b; ab; qz)}{2\phi_1(a, b; ab; q, z)}.$$

It is the same series as in [8] but with different values for $a$ and $b$, so we proceed similarly with a Heine transformation [16, Chapter 1]:

$$2\phi_1(a, b; ab; q, z) = \frac{(az, b; q)_\infty}{(ab, z; q)_\infty} 2\phi_1(a, z; az, b).$$

So

$$M(z) = \frac{(aqz, b, ab, z; q)_\infty}{(1 - z)(ab, qz, az, b; q)_\infty} \frac{2\phi_1(a, qz; az; q, b)}{2\phi_1(a, z; az; q, b)} = \frac{1}{1 - az} \frac{2\phi_1(a, qz; az; q, b)}{2\phi_1(a, z; az; q, b)},$$

and

$$M(1) = \frac{1}{1 - a} 2\phi_1(a, q; aq; q, b) = \sum_{n=0}^{\infty} b^n \frac{1}{1 - aq^n} = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} b^n a^j q^{nj} = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} q^{\frac{n+j}{2}} (-i)^n i^j q^{n+j} + \frac{n+j}{2}.$$

To obtain the coefficient of $t^k$ in $M(1)$, we just have to restrict the sum to the pairs $(n, j)$ such that $n = 2k - j$. This gives the desired formula.

5.2. The $q$-tangent numbers. To prove Theorem 1.5 with the same method, we first need to slightly modify our expression for $E_{2n+1}(q)$. Let $P_k$ be the polynomial

$$\sum_{j=0}^{2k+1} (-1)^j k^j q^{(2k+2-j)}.$$

By elementary properties of the binomial coefficients we have

$$E_{2n+1}(q) = \sum_{k=0}^{n} \left( \binom{2n+1}{n-k} - \binom{2n+1}{n-k-1} \right) P_k \frac{\binom{2n}{n-k} P_{k+1} + P_{k+1}}{(1-q)^{2n+1}}.$$

The latter expression is the one that we can prove with the previous method. We decompose the weighted paths exactly as in the case of $q$-secant numbers, i.e. as in Proposition 5.1. The sole difference is treated in the following proposition, which is therefore the last step of the second proof of Theorem 1.5.
Proposition 5.3. The sum of weights of Dyck paths of length $2k$, with weights 1 or $-q^{b+1}$ for any step ($\nearrow$ or $\searrow$) starting at height $h$, and no two consecutive steps $\nearrow\searrow$ both weighted by 1, is

$$N_k(q) = \frac{P_k + P_{k-1}}{1 - q}.$$  

Proof. We follow the scheme of Proposition 5.2. The Schröder paths in this case give the following T-fraction:

$$\sum_{k=0}^{\infty} N_k(q)t^k = \frac{1}{1 + t - \frac{(1 - q)(1 - q^2)t}{1 + t - (1 - q^2)(1 - q^3)t}}.$$  

We need to consider the more general continued fraction

$$N(z) = \frac{1}{1 + \frac{(1 - qz)(1 - q^2z)t}{1 + \frac{(1 - q^2z)(1 - q^3z)t}{1 + \ldots}}}.$$  

And then [6, (19.2.11a)] with $c = tq$ and $a = -b = i\sqrt{t}q$ gives that

$$N(z) = \frac{1}{1 - z} \cdot \frac{2\phi_1(a, b; tq; q; z)}{2\phi_1(a, b; tq; q, z)}.$$  

In this case Heine transformation gives

$$2\phi_1(a, b; tq; q, z) = \frac{(-i\sqrt{t}q, i\sqrt{t}q; z)_{\infty}}{(z, tq; q)_{\infty}} 2\phi_1(i\sqrt{t}, z; i\sqrt{t}qz; q; -i\sqrt{t}q),$$  

and hence, after some simplification

$$N(1) = \frac{1}{1 - i\sqrt{t}q} 2\phi_1(i\sqrt{t}, q; i\sqrt{t}q^2; q; -i\sqrt{t}q) = \sum_{n=0}^{\infty} \frac{(1 - i\sqrt{t})(-i\sqrt{t}q)^n}{(1 - i\sqrt{t}q^n)(1 - i\sqrt{t}q^{n+1})}.$$  

The usual method for reducing degrees in denominators leads to

$$(1 - q)N(1) = -\sum_{n=0}^{\infty} \frac{1 - i\sqrt{t}}{1 - i\sqrt{t}q^n}(-i\sqrt{t})^{n-1} + \sum_{n=0}^{\infty} \frac{1 - i\sqrt{t}}{1 - i\sqrt{t}q^{n+1}}(-i\sqrt{t})^{n-1}.$$  

Some terms in this sum cancel each other and there remains:

$$(1 - q)N(1) = -\sum_{n=0}^{\infty} \frac{1}{1 - i\sqrt{t}q^n}(-i\sqrt{t})^{n} + \sum_{n=0}^{\infty} \frac{1}{1 - i\sqrt{t}q^{n+1}}(-i\sqrt{t})^{n-1} + \frac{(i\sqrt{t})^{-1}}{1 - i\sqrt{t}}.$$  

At this point we can expand the fractions as in the previous case, and we readily obtain that the coefficient of $t^k$ in $(1 - q)N(1)$ is $P_k + P_{k-1}$. \(\square\)

6. Final remarks

It might be possible to give a combinatorial proof of Proposition 5.2 or 5.3, with a sign-reversing involution whose fixed points account for the terms in the right-hand side (as was done for the Touchard-Riordan formula in [20]). In either case, we could not even conjecture a set of possible fixed points for such an involution.

There is a parity-independent formula for the classical Euler numbers $E_n = E_n(1)$ given by Shin and Zeng [25]. They also show that their formula has an analog that gives $E_n(q)$. One might ask whether there is a parity-independent
formula for $E_n(q)$ in the style of our Theorems 1.4 and 1.5. Actually from our results we get that for any even or odd $n$,

$$(−1)^{n−1}A_n(−1,q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left( \begin{pmatrix} n \\ k \end{pmatrix} - \begin{pmatrix} n-1 \\ k-1 \end{pmatrix} \right) \sum_{i=0}^{n−2k} (−1)^{k+1}q^{i(n−2k−i)+i},$$

(when $n$ is even the right-hand side is 0 because $\sum_{i=0}^{n-2k} (−1)^{i+k}q^{i(n+1−2k−i)} = (−1)^{n−k}$ and $\sum_{k=0}^{n} (−1)^{k+1}\begin{pmatrix} n \\ k \end{pmatrix} - \begin{pmatrix} 2n \\ k−1 \end{pmatrix} = 0$), and also

$$(−1)^{n/2}B_n(−1,q) = \sum_{k=0}^{\lfloor n/2 \rfloor} \left( \begin{pmatrix} n \\ k \end{pmatrix} - \begin{pmatrix} n−1 \\ k−1 \end{pmatrix} \right) \sum_{i=0}^{n−2k} (−1)^{k+i}q^{i(n−2k−i)+\frac{i}{2}−k},$$

(when $n$ is odd the right-hand side is 0 because $\sum_{i=0}^{n-2k} (−1)^{i+k}q^{i(n−2k−i)} = 0$), and adding the previous two identities gives

$$E_n(q) = (−1)^{\lfloor n/2 \rfloor}(−1,q)\sum_{k=0}^{\lfloor n/2 \rfloor} \left( \begin{pmatrix} n \\ k \end{pmatrix} - \begin{pmatrix} n−1 \\ k−1 \end{pmatrix} \right) \sum_{i=0}^{n−2k} (−1)^{k+i}q^{i(n−2k−i)}(q^i + q^{\frac{i}{2}−k}).$$

REFERENCES

[1] D. André, Mémoire sur les permutations alternées. J. Math. 7 (1881), 167–184.
[2] G.E. Andrews, D. Foata, Congruences for the $q$-secant numbers, European J. Combin. 1(4) (1980), 283–287.
[3] G.E. Andrews, I. Gessel, Divisibility properties of the $q$-tangent numbers, Proc. Amer. Math. Soc. 81 (1981), 143–148.
[4] D. Chebikin, Variations on descents and inversions in permutations, El. J. Comb. 15 (2008), #136.
[5] S. Corteel, Crossings and alignments of permutations, Adv. in App. Maths. 38(2) (2007), 149–163.
[6] A. Cuyt, A.B. Petersen, B. Verdonk, H. Waadeland, W.B. Jones, Handbook of Continued Fractions for Special Functions, Springer, 2008.
[7] S. Corteel, L. K. Williams, Tableaux combinatorics for the asymmetric exclusion process, Adv. in App. Maths. 39(3) (2007), 293–310.
[8] S. Corteel, M. Josuat-Vergès, T. Prellberg, M. Rubey, Matrix Ansatz, lattice paths and rook placements, Proc. FPSAC 2009, 12 pp.
[9] L. Euler, Institutiones calculi differentialis cum eius usu in analysi finitorum ac Doctrina serierum, Academiae Imperialis Scientiarum Petropolitanae, St. Petersbourg, 1755, chap. VII (“Methodus summandi superior ulterius promota”).
[10] P. Flajolet, Combinatorial aspects of continued fractions, Discrete Math. 41 (1982), 145–153.
[11] D. Foata, Further divisibility properties of the $q$-tangent numbers, Proc. Amer. Math. Soc. 81 (1981), 143–148.
[12] D. Foata, G.-N. Han, The $q$-tangent and $q$-secant numbers via basic Eulerian polynomials, Proc. Amer. Math. Soc. 138(2) (2010), 385–393.
[13] D. Foata, M.-P. Schützenberger, Théorie géométrique des polynômes utilisiers, Lecture Notes in Mathematics 138, Springer-Verlag, 1970.
[14] J. Françon, X-G. Viennot, Permutations selon leurs pics, creux, doubles montées et double descentes, nombres d’Euler et nombres de Genocchi, Discrete Math. 28(1) (1979), 21–35.
[15] M. Fulmek, A continued fraction expansion for a $q$-tangent function, Sémin. Loth. Comb. B45b, (2000), 5 pp.
[16] G. Gasper, M. Rahman, Basic hypergeometric series, Cambridge University Press, 1990.
[17] G.-N. Han, A. Randrianarivony, J. Zeng, Un autre $q$-analogue des nombres d’Euler, Séminaire Lotharingien de Combinatoire, B42e (1999).
[18] J.H. Jackson, A basic-sine and cosine with symbolic solutions of certain differential equations, Proc. Edinburgh Math. Soc. 22 (1904), 28–39.
[19] M. Josuat-Vergès, Rook placements in Young diagrams and permutation enumeration, arXiv:0811.0524v2 [math.CO].
[20] J.-G. Penaud, Une preuve bijective d’une formule de Touchard-Riordan, Disc. Math. 139 (1995), 347–360.
[21] H. Prodinger, Combinatorics of geometrically distributed random variables: new $q$-tangent and $q$-secant numbers, Int. J. Math. Math. Sci. 24 (2000), 825–838.
A $q$-ENUMERATION OF ALTERNATING PERMUTATIONS

[22] H. Prodinger, A continued fraction expansion for a $q$-tangent function: an elementary proof, Sém. Lothar. Combin. B60b (2008), 3 pp.

[23] E. Roblet, X.G. Viennot, Théorie combinatoire des $T$-fractions et approximants de Padé en deux points, Disc. Math. 153(1-3) (1996), 271–288.

[24] P.D. Roselle, Permutations by number of rises and successions, Proc. Amer. Math. Soc. 19 (1968), 8–16.

[25] H. Shin, J.Zeng, The $q$-tangent and $q$-secant numbers via continued fractions, Europ. J. Combin 31(7) (2010), 1689–1705.

[26] E. Steingrímsson, L.K. Williams, Permutation tableaux and permutation patterns, J. Combin. Theory Ser. A 114(2) (2007), 211–234.

[27] X.G. Viennot, Une théorie combinatoire des polynômes orthogonaux, Notes de cours, UQAM, Montréal 1988 (http://web.mac.com/xgviennot/Xavier_Viennot/livres.html).

[28] A. Varvak, Rook numbers and the normal ordering problem, J. Combin. Theory Ser. A 112(2) (2005), 292–307.

[29] L.K. Williams, Enumeration of totally positive Grassmann cells, Adv. Math. 190(2) (2005), 319–342.

LRI and Université Paris-Sud, Bâtiment 490, 91405 Orsay, FRANCE

E-mail address: josuat@lri.fr