AN UPPER BOUND FOR THE FIRST NONZERO NEUMANN EIGENVALUE

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Abstract. Let \( M \) denote a complete, simply connected Riemannian manifold with sectional curvature \( K_M \leq k \) and Ricci curvature \( \text{Ric}_M \geq (n-1)K \), where \( k, K \in \mathbb{R} \). Then for a bounded domain \( \Omega \subset M \) with smooth boundary, we prove that the first nonzero Neumann eigenvalue \( \mu_1(\Omega) \leq C \mu_1(B_k(R)) \). Here \( B_k(R) \) is a geodesic ball of radius \( R > 0 \) in the simply connected space form \( M_k \) such that \( \text{vol}(\Omega) = \text{vol}(B_k(R)) \), and \( C \) is a constant which depends on the volume, diameter of \( \Omega \) and the dimension of \( M \).

1. Introduction

Let \( M \) be a complete connected Riemannian manifold and \( \Omega \subset M \) be a bounded domain with smooth boundary \( \partial \Omega \). The Neumann eigenvalue problem on \( \Omega \) is to find all real numbers \( \mu(\Omega) \) for which there exists a nontrivial function \( \varphi \in C^2(\Omega) \cap C^1(\bar{\Omega}) \) such that

\[
\Delta \varphi = \mu \varphi \quad \text{in} \quad \Omega \\
\frac{\partial \varphi}{\partial \nu} = 0 \quad \text{on} \quad \partial \Omega
\]

where \( \nu \) is the outward unit normal to \( \partial \Omega \). This problem has discrete and real spectrum \( 0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \cdots \to \infty \). In this article, we are interested in finding an upper bound for the first nonzero eigenvalue of problem (1). The variational characterization of \( \mu_1(\Omega) \) is given by

\[
\mu_1(\Omega) = \inf_{\varphi} \left\{ \frac{\int_{\Omega} |\nabla \varphi|^2 dV}{\int_{\Omega} \varphi^2 dV} \mid \int_{\Omega} \varphi \ dV = 0 \right\}.
\]

Among all simply connected planar domains of given area, Szegö [4] proved that the ball maximizes the first nonzero Neumann eigenvalue. This result was extended to arbitrary bounded domains in \( \mathbb{R}^n \) by Weinberger [6]. Using the idea of [6], Ashbaugh and Benguria [2] showed that Szegö-Weinberger result [4, 6] also holds for bounded domains contained...
in a hemisphere of sphere $S^n$. Later, Xu [7] and Aithal-Santhanam [1] proved that the same result is also true for bounded domains in hyperbolic space $\mathbb{H}^n$ and rank-1 symmetric spaces, respectively.

Recently, Wang [5] considered problem (1) on bounded domains in an $n$-dimensional complete, simply connected Riemannian manifold $\mathcal{M}$ whose sectional curvature is bounded from above by $k$ and Ricci curvature is bounded from below by $(n-1)K$ for $k, K \in \mathbb{R}$. Under certain assumptions on the size of domain $\Omega$, Wang [5] proved,

$$\mu_1(\Omega) \leq \left(\frac{\sin K(d)}{\sin k(d)}\right)^{2n-2} \mu_1(\Omega^*), \quad (2)$$

where $\Omega^*$ denotes a geodesic ball in a Riemannian manifold of constant curvature $k$ such that $\text{vol}(\Omega) = \text{vol}(\Omega^*)$. Here $d$ is the diameter of $\Omega$ and the function $\sin_m(r)$ is defined as

$$\sin_m(r) := \begin{cases} 
\sin \sqrt{mr}/\sqrt{m}, & m > 0, \\
r, & m = 0, \\
\sinh \sqrt{-mr}/\sqrt{-m}, & m < 0.
\end{cases}$$

Basically, author in [5] used spherical symmetrization technique to prove the above result.

In this article, we give an upper bound for the first nonzero Neumann eigenvalue. The main contribution and properties of this work are summarised as follows:

(i) Under the same assumptions on $\Omega$ as in [5], we prove that $\mu_1(\Omega) \leq C \mu_1(B_k(R))$, where the constant $C$ (defined in Section 3) depends on $k, K, n, d$, volume of $\Omega$, and $B_k(R)$ is a geodesic ball of radius $R$ in the space form of curvature $k$ such that $\text{vol}(\Omega) = \text{vol}(B_k(R))$.

(ii) For a domain of given volume and arbitrary large diameter, the constant $C$ in our result is smaller than the constant $\left(\frac{\sin K(d)}{\sin k(d)}\right)^{2n-2}$ appears in [5] (see Remark 3.2).

(iii) We also observe that arguments given in [5] are analytical in nature however our arguments are more geometrical.

2. Preliminaries

In this section, we recall the notion of center of mass and some basic results which are important to prove our result.

Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold. For a point $p \in M$, we denote the convexity radius of $M$ at $p$ by $c(p)$. For a domain $\Omega \subset B(p, c(p))$, we denote the convex hull of $\Omega$ by $\text{hull}(\Omega)$. Let $\exp_p : T_p M \to M$ be the exponential map. The following lemma gives the existence of a center of mass of a bounded domain in $M$.

**Lemma 2.1.** Let $\Omega$ be a bounded domain in $(M, g)$ and is contained in $B(q, c(q))$ for some $q \in M$. Let $G$ be a continuous function on $[0, 2c(q)]$ such that $G$ is also positive on
(0,2c(q)). Then there exists a point \( p \in \text{hull}(\Omega) \) such that

\[
\int_{\Omega} G(r_p(x)) \frac{\exp_{p}^{-1}(x)}{r_p(x)} dV = 0,
\]

where \( r_p(x) \) denotes the distance between \( p \) and \( x \) in \( M \).

For a proof see [1].

**Definition 2.1.** The point \( p \) in the above lemma is called a center of mass of domain \( \Omega \) with respect to the function \( G \).

Next we fix some notations which we use throughout the manuscript.

Let \( M \) be a complete, simply connected Riemannian manifold of dimension \( n \) with sectional curvature \( K_M \leq k \) and Ricci curvature bounded from below by \((n-1)K\). We denote the simply connected space form of constant curvature \( s \) by \( \mathbb{M}_s \). Let \( \Omega \) be a bounded domain in \( M \) with smooth boundary and \( \text{vol}(\Omega) = \text{vol}(B_k(R)) = \text{vol}(B_K(R')) \), where \( B_k(R) \) and \( B_K(R') \) denote balls of radius \( R \) (in \( \mathbb{M}_k \)) and \( R' \) (in \( \mathbb{M}_K \)), respectively. We further assume that balls \( B_K(R') \) and \( B_k(R) \) are centered at \( p_K \in \mathbb{M}_K \) and \( p_k \in \mathbb{M}_k \), respectively. For \( y \in \mathbb{M}_k \), \( r_{p_k}(y) \) represents the distance between \( p_k \) and \( y \) in \( \mathbb{M}_k \). Similarly, \( r_{p_K}(y') \) is the distance between \( p_K \) and \( y' \) in \( \mathbb{M}_K \), where \( y' \in \mathbb{M}_K \). Observe that \( R' \leq R \). For \( k > 0 \), we impose the following condition on \( \Omega \).

\begin{align*}
(A) & \ d = \text{diam}(\Omega) = \text{diam}(	ext{hull}(\Omega)) < \min \left\{ \frac{\pi}{2 \sqrt{k}}, \text{injectivity radius of } M \right\}. \\
(B) & \ \text{vol}_M(\text{hull}(\Omega)) \leq \frac{\text{vol}_{\mathbb{M}_k}(\mathbb{M}_k)}{2}.
\end{align*}

**Remark 2.2.** For \( k > 0 \), assumption \((A)\) assures that exponential map \( \exp_{p} \) is a diffeomorphism onto \( \text{hull}(\Omega) \) for all \( p \in \text{hull}(\Omega) \), and we require condition \((B)\) so that we can ultimately work in the hemisphere of \( \mathbb{M}_k \).

Now we state some properties of \( \mu_1(B_k(R)) \), the first nonzero Neumann eigenvalue of \( B_k(R) \).

Recall that by separation of variable technique, the first nonzero Neumann eigenvalue of \( B_k(R) \) is the first eigenvalue of

\[
-F''(r) - \frac{\mu - 1}{\sin^2(r)} F'(r) + \frac{\mu - 1}{\sin^2(r)} F(r) = \mu F(r) \\
F(0) = 0, \quad F'(R) = 0.
\]

(3)

Let \( f(r) \) be the eigenfunction of (3) corresponding to \( \mu = \mu_1(B_k(R)) \). Then \( f(r(y)) \frac{y_i}{r(y)} \), \( 1 \leq i \leq n \) is an eigenfunction corresponding to \( \mu_1(B_k(R)) \), where \((y_1, y_2, \ldots, y_n) \) is the geodesic polar coordinates of \( y \in B_k(R) \) with respect to the center of \( B_k(R) \) and \( r(y) \) represents the distance between \( y \) and the center of \( B_k(R) \). The function \( f(r) \) satisfies the following properties.

**Lemma 2.3.** \( f(r) \) is an increasing function of \( r \) and \((f'(r))^2 + \frac{n-1}{\sin^2(r)} f^2(r) \) is a decreasing function of \( r \).
For more details see [2, 7].

The following lemmas are useful in proving our main result.

**Lemma 2.4.** For a fix point \( q \in \mathbb{M} \), let \( (x_1, x_2, \ldots, x_n) \) denote the geodesic polar coordinates with respect to \( q \). Let \( S(r) \) be the geodesic sphere of radius \( r \) centered at \( q \). For \( k > 0 \), we further assume

\[
r < \min \left\{ \frac{\pi}{2\sqrt{k}}, \text{ injectivity radius of } \mathbb{M} \right\}.
\]

Then

\[
\sum_{i=1}^{n} \left| \nabla^{S(r)} \left( \frac{x_i}{r} \right) \right|^2 \leq \frac{n-1}{\sin^2_k(r)}, \quad (4)
\]

For more general statement and details see [3].

**Lemma 2.5.** For \( K, k \) and \( \sin_m(r) \) defined as above, the function \( \frac{\sin_k(r)}{\sin_m(r)} \) is an increasing function of \( r \).

Using the exponential map, next we construct a domain \( \Omega_K \) in \( \mathbb{M}_K \) from \( \Omega \). This construction is further used in the proof of our main result.

For \( p \in \mathbb{M} \), let \( W \subset T_p(\mathbb{M}) \) such that \( \Omega = \exp_p(W) \). Fix an isometry \( i : T_p(\mathbb{M}) \rightarrow T_{pK}(\mathbb{M}_K) \) for \( pK \in \mathbb{M}_K \) and denote \( \Omega_K = \exp_{pK}(i(W)) \). Note that corresponding to each \( q \in \Omega \), there exists \( \bar{q} \in \Omega_K \) such that \( q = \exp_p u \) and \( \bar{q} = \exp_{pK}(i(u)) \) and vice versa. Thus in terms of geodesic polar coordinates, \( \Omega \) can be written as

\[
\Omega = \{(r, u) : u \in T_p(\mathbb{M}), \|u\| = 1, r \in (r_1(u), r_2(u)) \cup (r_3(u), r_4(u)) \cup \cdots \cup (r_{m(u)}(u), r_{m(u)}(u)) \}
\]

Similarly

\[
\Omega_K = \{(r, \bar{u}) : \bar{u} \in T_{pK}(\mathbb{M}_K), \|u\| = 1, r \in (r_1(\bar{u}), r_2(\bar{u})) \cup (r_3(\bar{u}), r_4(\bar{u})) \cup \cdots \cup (r_{m(\bar{u})-1}(\bar{u}), r_{m(\bar{u})}(\bar{u})) \}
\]

Denote \( I_u = I_\bar{u} = (r_1, r_2) \cup (r_3, r_4) \cup \cdots \cup (r_{m(u)-1}, r_{m(u)}) \). Let \( \phi \) and \( \phi_K(= \sin_k^{n-1}(r)) \) be the volume density functions of \( \mathbb{M} \) and \( \mathbb{M}_K \) along the radial geodesics starting from \( p \) and \( pK \), respectively. Observe that by Gunther volume comparison theorem, \( \text{vol}(\Omega) = \text{vol}(B_K(R')) \leq \text{vol}(\Omega_K) \).

3. Statement and proof of the main result

**Theorem 3.1.** With all assumptions on \( \Omega \) given in Section 2, the first nonzero Neumann eigenvalue on \( \Omega \) satisfies the following inequality

\[
\mu_1(\Omega) \leq C \mu_1(B_k(R)),
\]
where
\[ C = \left( \frac{\sin_K(R)}{\sin_k(R)} \right)^{n-1} \left( \frac{\sin_K(d)}{\sin_k(d)} \right)^{n-1} \frac{\int_{B_K(R)} f^2(r_p) dV}{\int_{B_K(R)} f^2(r_{pk}) dV} \]

Further, if \( k = K \) then the constant \( C \) is equal to 1, and the above bound is sharp.

**Remark 3.2.** Since the function \( \frac{\sin_K(r)}{\sin_k(r)} \) is increasing and unbounded, for \( k < 0 \) and among all domains of given volume, the constant factor \( \left( \frac{\sin_K(d)}{\sin_k(d)} \right)^{2n-2} \) in (2) is larger than the constant \( C \) for domains of arbitrary large diameter.

**Proof.** Define
\[ h(r) := \begin{cases} f(r), & r \leq R \\ f(R), & r \geq R. \end{cases} \]

Note that the function \( h(r) \) is continuous and positive function on \([0, \infty)\). Let \( p \in \text{hull}(\Omega) \) be a centre of mass with respect to the function \( h(r) \) and \((x_1, x_2, \ldots, x_n)\) denote the geodesic polar coordinates of \( x \in \Omega \) with respect to the point \( p \). For our convenience, in this proof we denote \( r_p(x) \) (given in Lemma 2.1) by \( r, r_{pk} \) by \( r_{pk} \) and \( r_{pk}(y') \) by \( r_{pk} \). Then for all \( 1 \leq i \leq n \),
\[ \int_{\Omega} h(r) \frac{x_i}{r} dV = 0. \]

Now by the variational characterization of \( \mu_1(\Omega) \), we have
\[ \mu_1(\Omega) \sum_{i=1}^{n} \int_{\Omega} \left( h(r) \frac{x_i}{r} \right)^2 dV \leq \sum_{i=1}^{n} \int_{\Omega} \left| \nabla h(r) \frac{x_i}{r} \right|^2 dV, \]
\[ \mu_1(\Omega) \int_{\Omega} h^2(r) dV \leq \sum_{i=1}^{n} \int_{\Omega} \left| \nabla h(r) \frac{x_i}{r} \right|^2 dV. \]

Next we find an estimate for \( \int_{\Omega} h^2(r) dV \).
\[ \int_{\Omega} h^2(r) dV = \int_{U_pM} \int_{I_u} h^2(r) \phi(r, u) dr du \]
\[ \geq \int_{U_pM} \int_{I_u} h^2(r) \sin^{n-1}_k(r) dr du \]
\[ = \int_{U_pM} \int_{I_u} h^2(r) \frac{\sin^{n-1}_k(r)}{\sin^{n-1}_K(r)} \sin^{n-1}_K(r) dr du. \]
Since \( \frac{\sin_k(r)}{\sin_K(r)} \) is a decreasing function of \( r \),
\[
\int_{\Omega} h^2(r) \, dV \geq \frac{\sin_k^{-1}(d)}{\sin_K^{-1}(d)} \int_{\Omega} h^2(r) \sin_k^{n-1}(r) \, dr \, du
\]
\[
= \frac{\sin_k^{-1}(d)}{\sin_K^{-1}(d)} \int_{U_p^M \setminus I_a} h^2(r) \sin_k^{n-1}(r) \, dr \, d\bar{u}.
\]

Using the fact that \( h(r) \) is an increasing function of \( r \), we obtain
\[
\int_{\Omega} h^2(r) \, dV \geq \frac{\sin_k^{-1}(d)}{\sin_K^{-1}(d)} \left( \int_{B_K(R')} h^2(r_{pK}) \, dV + \int_{\Omega_{K \setminus \{\Omega \cap B_K(R')\}}} h^2(r_{pK}) \, dV \right).
\]

The last inequality follows from the fact that \( \text{vol}(B_K(R')) \leq \text{vol}(\Omega_K) \).

Now we obtain an upper bound for \( \sum_{i=1}^{n} \int_{\Omega} \left| \nabla \left( \frac{h(r)}{r} \right) \right|^2 \, dV \).
\[
\sum_{i=1}^{n} \int_{\Omega} \left| \nabla \left( \frac{h(r)}{r} \right) \right|^2 \, dV = \int_{\Omega} \left( h^2(r) \sum_{i=1}^{n} \left| \nabla S(r) \left( \frac{x_i}{r} \right) \right|^2 + (h'(r))^2 \right) \, dV
\]
\[
\leq \int_{\Omega} \left( \frac{n - 1}{\sin_k^2(r)} h^2(r) + (h'(r))^2 \right) \, dV.
\]

Denote \( G(r) = \frac{\sin_k^{-1}(r)}{\sin_k^{-1}(r)} h^2(r) + (h'(r))^2 \). Since \( G(r) \) is a decreasing function of \( r \),
\[
\sum_{i=1}^{n} \int_{\Omega} \left| \nabla \left( \frac{h(r)}{r} \right) \right|^2 \, dV \leq \int_{\Omega \setminus B(R)} G(r) \, dV + \int_{\Omega_{K \setminus B(R)}} G(r) \, dV
\]
\[
= \int_{B(R)} G(r) \, dV - \int_{B(R) \setminus (\Omega \cap B(R))} G(r) \, dV + \int_{\Omega_{K \setminus B(R)}} G(r) \, dV
\]
\[
\leq \int_{B(R)} G(r) \, dV
\]
\[
= \int_{U_p^M} \int_{0}^{R} G(r) \phi(r, u) \, dr \, du.
\]
Using Gunther volume comparison theorem to get
\[
\sum_{i=1}^{n} \int_{\Omega} \left| \nabla \left( h(r) \frac{x_i}{r} \right) \right|^2 dV \leq \int_{U_{pM}} \int_{0}^{R} G(r) \sin^{n-1}(r) \, dr \, du \\
\leq \left( \frac{\sin_K(R)}{\sin_k(R)} \right)^{n-1} \int_{U_{pM_k}} \int_{0}^{R} G(r) \sin^{n-1}(r) \, dr \, d\bar{u} \\
= \left( \frac{\sin_K(R)}{\sin_k(R)} \right)^{n-1} \int_{B_k(R)} G(r_{pk}) dV.
\]
(6)

We have used Lemma 2.5 to conclude the second last inequality. By combining (5) and (6), we obtain
\[
\mu_1(\Omega) \leq \left( \frac{\sin_K(R)}{\sin_k(R)} \right)^{n-1} \left( \frac{\sin_K(d)}{\sin_k(d)} \right)^{n-1} \int_{B_k(R)} \frac{\left( \frac{n-1}{2} h^2(r_{pk}) + (h'(r_{pk}))^2 \right)}{\int_{B_k(R')} h^2(r_{pk}) dV} \frac{\int_{B_k(R)} f^2(r_{pk}) dV}{\int_{B_k(R')} f^2(r_{pk}) dV}.
\]

where
\[
\mathcal{C} = \left( \frac{\sin_K(R)}{\sin_k(R)} \right)^{n-1} \left( \frac{\sin_K(d)}{\sin_k(d)} \right)^{n-1} \int_{B_k(R)} f^2(r_{pk}) dV \int_{B_k(R')} f^2(r_{pk}) dV.
\]

This completes the proof of the theorem. \qed

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