GLOBAL EXISTENCE FOR A CLASS OF KELLER-SEGEL MODELS WITH SIGNAL-DEPENDENT MOTILITY AND GENERAL LOGISTIC TERM

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Abstract. This paper focuses on the global existence for a class of Keller-Segel models with signal-dependent motility and general logistic term under homogeneous Neumann boundary conditions in a two-dimensional smoothly bounded domain. We show that if \( \lambda \in \mathbb{R}, \mu > 0 \) and \( l > 2 \) are constants, then for all sufficiently smooth initial data the system

\[
\begin{aligned}
    u_t &= \Delta (\gamma(v)u) + \lambda u - \mu u^l, \quad x \in \Omega, \ t > 0, \\
    v_t &= \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \\
    u(x,0) &= u_0(x), \ v(x,0) = v_0(x), \ x \in \Omega,
\end{aligned}
\]

possesses a global classical solution.

1. Introduction and main result. Let us consider the following density-suppressed motility model:

\[
\begin{aligned}
    u_t &= \Delta (\gamma(v)u) + \lambda u - \mu u^l, \quad x \in \Omega, \ t > 0, \\
    v_t &= \Delta v - v + u, \quad x \in \Omega, \ t > 0, \\
    \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, \quad x \in \partial \Omega, \\
    u(x,0) &= u_0(x), \ v(x,0) = v_0(x), \ x \in \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^2 \) is a bounded domain with a smooth boundary \( \partial \Omega \). \( u(x,t) \) denotes the cell density and \( v(x,t) \) represents the concentration of the chemical signal secreted by cells. \( u_0(x) \geq 0 \) and \( v_0(x) > 0 \) are given functions. \( \gamma \) is the motility function. We denote by \( \nu \) the outward normal vector to its boundary \( \partial \Omega \) and assume that \( \lambda \in \mathbb{R}, \mu > 0 \) and \( l > 2 \) are constants.

Before addressing the main result of (1), let us recall the original Keller-Segel model

\[
\begin{aligned}
    u_t &= \nabla \cdot (D(u,v) \nabla u) - \nabla \cdot (u \chi(u,v) \nabla v), \quad x \in \Omega, \ t > 0, \\
    v_t &= \Delta v + u - v, \quad x \in \Omega, \ t > 0,
\end{aligned}
\]

which describes a directional migration strategy of cells to the chemotactic response. The model (2) plays an important role not only in mathematics but also in biology and pharmacology. Starting from the pioneering work of Keller and Segel [8] in 1970,
this model has been extensively studied by lots of mathematicians from various aspects. For instance, when \( D(u,v) = \chi(u,v) = 1 \), the global existence of solutions or blow-up was established (cf. Refs. [3], [4], [5], [6], [11], [17], [19]). Besides, many authors are interested in qualitative convergence of the solution [22, 23] and large time behavior [2, 21] for such kind of model.

In particular, if we choose \( D(u,v) := \gamma(v) \), \( \chi(u,v) := -\gamma'(v) \) in the first equation of (2), then the model becomes

\[
\begin{aligned}
    u_t &= \Delta(\gamma(v)u), \quad x \in \Omega, t > 0, \\
    v_t &= \Delta v - v + u, \quad x \in \Omega, t > 0.
\end{aligned}
\]

In this sense, the idea of the model is that the organisms simply decreases the motility if the density of the chemical substance increases, where \( \gamma \) is a sufficiently smooth given positive function on \([0, \infty)\). Yoon and Kim show the global existence of solutions in [24] under the assumptions that \( \gamma \) is a power law case

\[
\gamma(v) = \frac{c_0}{v^k}, \quad c_0 > 0, \quad k > 0
\]

and the motility function \( \gamma \) decreases as the density of the chemical substance increases, i.e.,

\( \gamma' < 0 \) on \((0, +\infty)\).

Tao and Winkler [14] consider the model under the condition that the motility function \( \gamma \) satisfies

\( \gamma \in C^3([0, +\infty)) \)

and

\( K_1 \leq \gamma(s) \leq K_2 \) for all \( s > 0 \)

as well as

\( |\gamma'(s)| \leq K_3 \) for all \( s > 0 \)

with certain positive constants \( K_1, K_2 \) and \( K_3 \) and they show the global existence of bounded solutions for such kind of model.

In biological phenomena, the reproduction or death of the population plays an important role in its life. Many mathematicians introduce the generalized logistic term

\( f(u) = \lambda u - \mu u^l \), where \( \lambda \in \mathbb{R}, \mu > 0 \) and \( l > 1 \),

to the chemotaxis model and obtain some related results. It is well-known that an appropriate logistic damping can prevent blow-up of solutions to the classical Keller-Segel system. The parabolic-elliptic Keller-Segel simplification (where \( v_t \) is replaced by 0) is considered in [15] and it is shown that if \( l > 2 \), then the system possesses a unique and uniformly bounded global classical solution. In [18], the existence of weak solutions is proved under more general conditions. For the parabolic-parabolic Keller-Segel system, in [20], it is shown that if \( l = 2, \mu > 0 \) is sufficiently large then the problem possesses a unique and uniformly bounded global-in-time classical solution. In [10], the global classical solution and large time behavior are considered with \( l = 2 \).

Jin, Kim and Wang [7] prove the existence of global classical solution of the model

\[
\begin{aligned}
    u_t &= \Delta(\gamma(v)u) + \mu u(1 - u), \quad x \in \Omega, t > 0, \\
    v_t &= \Delta v - v + u, \quad x \in \Omega, t > 0.
\end{aligned}
\]
in the two dimensional domain with some assumptions that \( \mu > 0 \) and \( \gamma \) satisfies
\[
\gamma \in C^3([0, +\infty)), \quad \gamma > 0, \quad \gamma' < 0 \quad \text{on } [0, +\infty)
\]
and
\[
\lim_{v \to \infty} \frac{\gamma'(v)}{\gamma(v)} \quad \text{exists.}
\]
To extend this result to the higher dimensions, Wang and Wang [16] resolved the difficulty of the degeneracy of diffusion by introducing a step function \( \gamma_k \). In other words, they expanded the Laplacian term in the first equation which can be rewritten as
\[
\begin{align*}
    u_t &= \nabla(\gamma_k(v) \nabla u + \gamma'(v) u \nabla v) + au - \mu u^2, \\
    v_t &= \Delta v - v + u.
\end{align*}
\]
They show that the solution to (4) exists globally and is bounded when \( n \geq 3 \) and sufficiently big \( \mu > 0 \).

**Main result**
To the best of our knowledge, the global existence of the solution for the system (1) are unknown in the case of \( \lambda \in \mathbb{R}, \mu > 0 \) and \( l > 2 \). The goal of our paper is to clarify the global existence of the chemotaxis system (1) with signal-dependent motility, and generalized logistic source in two dimensional domain. In this setting, the argument based on the following assumptions on the motility function \( \gamma \):
\[
\gamma \in C^3((0, +\infty)), \gamma > 0, \gamma' < 0 \quad \text{and} \quad \frac{\gamma'}{\gamma} \quad \text{is bounded on } [\varepsilon, +\infty) \quad \text{for any } \varepsilon > 0. \quad (5)
\]
Simple examples are addressed in the following
\[
\begin{align*}
    \gamma(v) &= \frac{a}{1 + e^{bv}}^m, \\
    \gamma(v) &= 1 - \frac{v}{\sqrt{1 + v^2}}, \\
    \gamma(v) &= v - a,
\end{align*}
\]
where \( a, b, m \) are positive constants.

Our main result reads as follows.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary and assume that the hypothesis (5) holds. \( \lambda \in \mathbb{R}, \mu > 0 \) and \( l > 2 \) are constants. If the initial data satisfies \((u_0, v_0) \in C^0(\Omega) \times W^{1,\infty}(\Omega) \) with \( u_0 \geq 0, v_0 \geq C_0 > 0 \), where \( C_0 \) is a positive constant, then there exists a pair \((u, v)\) of non-negative functions
\[
\begin{align*}
    u &\in C^0(\Omega \times [0, +\infty)) \cap C^{2,1}(\overline{\Omega} \times (0, +\infty)), \\
    v &\in C^0(\Omega \times [0, +\infty)) \cap C^{2,1}(\overline{\Omega} \times (0, +\infty))
\end{align*}
\]
which solves (1) in the classical sense in \( \Omega \times (0, +\infty) \).

**Remark 1.** The assumption (5) on the motility function \( \gamma \) in Theorem 1.1 is weaker than that in [7, 24]. On one hand, the power law case (3) satisfies the assumption (5) i.e. it is possible that the motility function can be singularity at \( v = 0 \). On the other hand, the condition that
\[
\lim_{v \to +\infty} \frac{\gamma'(v)}{\gamma(v)} \quad \text{exists in [7] can imply the assumption (5). But the opposite is not true.}
\]

**Plan of the paper**
This paper is arranged as follows. Section 2 is devoted to the local existence and extensibility of the chemotaxis with signal-dependent motility and general logistic term. In subsection 2.2 and 2.3, we show a priori estimates of \( u \) and \( v \) respectively.
We recall two vital basic facts in subsection 2.4 as preliminaries for later use. In section 3, the global existence of the solution for such kind of model is established.

2. Preliminaries. Here and in the sequel we shall denote by $C_i (i = 1, 2, 3 \cdots)$ diverse constants. Firstly, we give the existence of local solution of (1). The proof, refer to [7] and [12], is based on the Schauder fixed point theorem. Alternatively, the existence of local solutions could also have been obtained by applying the abstract theory ([1], [14], [24]). Here we omit the proof of the following lemma due to the standard method.

Lemma 2.1 (local existence). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and assume that the hypothesis (5) holds. $\lambda \in \mathbb{R}$, $\mu > 0$ and $l > 1$ are constants. If the initial data satisfies $(u_0, v_0) \in C^0(\Omega) \times W^{1,\infty}(\Omega)$ with $u_0 \geq 0, v_0 \geq C_0 > 0$, where $C_0$ is a positive constant, then there exist $T_{\text{max}} \in (0, +\infty]$ and a pair $(u, v)$ of non-negative functions

\[
\begin{align*}
&u \in C^0([0, T_{\text{max}}]) \cap C^{2,1}(\Omega \times [0, T_{\text{max}}]), \\
v \in C^0([0, T_{\text{max}}]) \cap C^{2,1}(\Omega \times [0, T_{\text{max}}])
\end{align*}
\]

which solves (1) in the classical sense in $\Omega \times (0, T_{\text{max}})$. Moreover, we have

either $T_{\text{max}} = +\infty$ or $\limsup_{t/T_{\text{max}} \to + \infty} (\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}}) = +\infty$.

Now, we prove some basic properties of solutions to the system (1).

2.1. Lower bound of $v$. Firstly, we recall that $v$ has a positive lower bound.

Lemma 2.2. If $T_{\text{max}} < +\infty$, then there exists a lower bound $v > 0$ such that

\[
\inf_{x \in \Omega} v(x, t) \geq v
\]

for all $t \in (0, T_{\text{max}})$.

Proof. By the comparison principle with the positivity of $u$, we know from the second equation of (1) that

\[
v(x, t) \geq e^{-T_{\text{max}}} \inf_{y \in \Omega} v_0(y)
\]

for all $(x, t) \in \Omega \times (0, T_{\text{max}})$.

2.2. The estimate of $u$. We have the following estimate on the solution $u$.

Lemma 2.3. If $\lambda \in \mathbb{R}$, $\mu > 0$ and $l > 2$ are constants, then there exists $C > 0$ such that

\[
\int_\Omega u \leq C
\]

for all $t \in (0, T_{\text{max}})$ as well as

\[
\int_t^{t+\tau} \int_\Omega u^l \leq C
\]

for all $t \in (0, T_{\text{max}} - \tau)$, where

\[
\tau := \min \left\{ 1, \frac{1}{2} T_{\text{max}} \right\}.
\]
Proof. Integrating the first equation in (1) over $\Omega$, integrating by parts and using the condition $\frac{\partial u}{\partial \nu} = 0$, we have

$$\frac{d}{dt} \int_{\Omega} u + \mu \int_{\Omega} u' = \lambda \int_{\Omega} u$$

(9)

for all $t \in (0, T_{\text{max}})$. In addition, based on Hölder’s inequality and the fact

$$\int_{\Omega} u' \geq \frac{1}{|\Omega|^{l-1}} \left( \int_{\Omega} u \right)^l$$

for all $t \in (0, T_{\text{max}})$, we have

$$\frac{d}{dt} \int_{\Omega} u \leq \lambda_+ \int_{\Omega} u - \frac{\mu}{|\Omega|^{l-1}} \left( \int_{\Omega} u \right)^l$$

for all $t \in (0, T_{\text{max}})$. Solving this ODI, we get (6).

Integrating (9) over $(t, t + \tau)$, we have

$$\mu \int_{t}^{t+\tau} \int_{\Omega} u' = \lambda \int_{t}^{t+\tau} \int_{\Omega} u - \int_{t}^{t+\tau} \frac{d}{dt} \int_{\Omega} u$$

$$= \lambda \int_{t}^{t+\tau} \int_{\Omega} u - \left[ \int_{\Omega} u \right]_{t}^{t+\tau}$$

$$\leq \lambda_+ \int_{t}^{t+\tau} \int_{\Omega} u + \int_{\Omega} u(\cdot, t)$$

for all $t \in (0, T_{\text{max}} - \tau)$. Solving this inequality and combining (6), we can deduce (7).

2.3. The estimate of $v$. The following important estimate for the solution $v$ can be easily derived.

**Lemma 2.4.** If $\lambda \in \mathbb{R}$, $\mu > 0$ and $l > 2$ are constants, then there exists $C > 0$ such that

$$\int_{\Omega} |\nabla v|^2 \leq C$$

(10)

for all $t \in (0, T_{\text{max}})$ and

$$\int_{t}^{t+\tau} \int_{\Omega} |\Delta v|^2 \leq C$$

(11)

for all $t \in (0, T_{\text{max}} - \tau)$, where $\tau$ is given by (8).

**Proof.** Multiplying the second equation of system (1) by $-\Delta v$ and integrating by parts, we get

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\Delta v|^2 + \int_{\Omega} |\nabla v|^2$$

$$= -\int_{\Omega} u \Delta v \leq \frac{1}{2} \int_{\Omega} |\Delta v|^2 + \frac{1}{2} \int_{\Omega} u^2$$

for all $t \in (0, T_{\text{max}})$ which implies

$$\frac{d}{dt} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\Delta v|^2 + 2 \int_{\Omega} |\nabla v|^2 \leq \int_{\Omega} u^2$$

(12)

for all $t \in (0, T_{\text{max}})$. On the other hand, there exists $C_1 > 0$ such that

$$\int_{\Omega} u^2 \leq \mu \int_{\Omega} u' + C_1 |\Omega|$$
for all $t \in (0, T_{\text{max}})$. Hence (12) becomes
\[
\frac{d}{dt} \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\Delta v|^2 + 2 \int_{\Omega} |\nabla v|^2 \leq \mu \int_{\Omega} u + C_1|\Omega|
\]
for all $t \in (0, T_{\text{max}})$. Adding the result to (9), we conclude
\[
\frac{d}{dt} \left( \int_{\Omega} u + \int_{\Omega} |\nabla v|^2 \right) + \int_{\Omega} u + 2 \int_{\Omega} |\nabla v|^2 + \int_{\Omega} |\Delta v|^2 \leq (\lambda_+ + 1) \int_{\Omega} u + C_1|\Omega|
\]
(13) for all $t \in (0, T_{\text{max}})$. Let
\[
y(t) := \int_{\Omega} u + \int_{\Omega} |\nabla v|^2
\]
for all $t \in (0, T_{\text{max}})$. Due to Lemma 2.3, there exists $C_2 > 0$ such that
\[
\int_{\Omega} u \leq C_2
\]
(14) for all $t \in [0, T_{\text{max}}]$. Thus owing to (13), we deduce the estimate
\[
y'(t) + y(t) \leq (\lambda_+ + 1)C_2 + C_1|\Omega|
\]
for all $t \in (0, T_{\text{max}})$. An application of Gronwall’s inequality yields (10). Furthermore, integrating (13) over $(t, t+\tau)$, we have
\[
\int_t^{t+\tau} \frac{d}{dt} \left( \int_{\Omega} u + \int_{\Omega} |\nabla v|^2 \right) + \int_t^{t+\tau} \int_{\Omega} |\Delta v|^2 \leq \int_t^{t+\tau} (\lambda_+ + 1)C_2 + C_1|\Omega|
\]
for all $t \in (0, T_{\text{max}} - \tau)$, which implies that
\[
\int_t^{t+\tau} \int_{\Omega} |\Delta v|^2 \leq (\lambda + 1)C_2 + C_1|\Omega| + \int_{\Omega} u(\cdot, t) + \int_{\Omega} |\nabla v(\cdot, t)|^2
\]
for all $t \in (0, T_{\text{max}} - \tau)$. Hence, combined with (10) and (14), we conclude the proof.

2.4. Two vital basic facts.

Lemma 2.5 (see [9]). Assume $g \in L^\infty((0, T_{\text{max}}); L^p(\Omega))$ and $w_0 \in C^1(\Omega)$. Then for each
\[
r \in \begin{cases} [1, \frac{np}{n-p}), & p \leq n, \\ [1, \infty], & p > n, \end{cases}
\]
and the solution $w \in L^\infty((0, T_{\text{max}}); W^{1,r}(\Omega))$ of the system
\[
\begin{align*}
w_t &= \Delta w - w + g, & x \in \Omega, & t \in (0, T_{\text{max}}), \\
\frac{\partial w}{\partial r} &= 0, & x \in \partial \Omega, & t \in (0, T_{\text{max}}), \\
w(0) &= w_0(x), & x \in \Omega,
\end{align*}
\]
there exists a constant $C > 0$ such that
\[
\|w(\cdot, t)\|_{W^{1,r}} \leq C \left( 1 + \sup_{s \in (0, T_{\text{max}})} \|g(s)\|_{L^p} \right)
\]
for all $t \in (0, T_{\text{max}})$. 

Lemma 2.6 (Gagliardo-Nirenberg inequality). There exists $C > 0$ which depends only on $\Omega$ such that
$$
\|\nabla \varphi\|_{L^4} \leq C \left( \|\Delta \varphi\|_{L^2}^{\frac{1}{2}} \|\nabla \varphi\|_{L^2}^{\frac{1}{2}} + \|\nabla \varphi\|_{L^2} \right)
$$
for any $\varphi \in W^{2,2}(\Omega)$ satisfying $\frac{\partial \varphi}{\partial \nu}|_{\partial\Omega=0}$.

3. Boundedness of solutions. In this section, we show the global-in-time solution of the system (1) by some prior estimates and the extensibility criterion Lemma 2.1.

3.1. $L^2$ estimate of $u$. In this subsection, we try to show the $L^2$ boundedness of $u$.

**Lemma 3.1.** If $\lambda \in \mathbb{R}$, $\mu > 0$ and $l > 2$ are constants, $T_{\max} < +\infty$ and the hypothesis (5) holds, then there exists $C > 0$ such that
$$
\|u(\cdot, t)\|_{L^2} \leq C
$$
for all $t \in (0, T_{\max})$.

**Proof.** Multiplying the first equation of system (1) by $u$ and integrating by parts, we obtain
$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} \gamma(v) |\nabla u|^2 + \mu \int_{\Omega} u^{l+1} = -\int_{\Omega} \gamma'(v) u \nabla u \cdot \nabla v + \lambda \int_{\Omega} u^2
$$
for all $t \in (0, T_{\max})$. Then, applying Young’s inequality, it holds that
$$
-\int_{\Omega} \gamma'(v) u \nabla u \cdot \nabla v \leq \frac{1}{2} \int_{\Omega} \gamma(v) |\nabla u|^2 + \frac{1}{2} \int_{\Omega} \frac{|\gamma'(v)|^2}{\gamma(v)} |u|^2 |\nabla v|^2
$$
for all $t \in (0, T_{\max})$ and there exists $C_1 > 0$ such that
$$
\lambda \int_{\Omega} u^2 \leq \frac{\mu}{2} \int_{\Omega} u^{l+1} + C_1
$$
for all $t \in (0, T_{\max})$. Consequently
$$
\frac{d}{dt} \int_{\Omega} u^2 + \int_{\Omega} \gamma(v) |\nabla u|^2 + \mu \int_{\Omega} u^{l+1} \leq \int_{\Omega} \frac{|\gamma'(v)|^2}{\gamma(v)} |u|^2 |\nabla v|^2 + 2C_1
$$
(16)
for all $t \in (0, T_{\max})$. In light of
$$
\gamma^{\frac{1}{2}}(v) \nabla u = \nabla (\gamma^{\frac{1}{2}}(v) u) - \frac{1}{2} \frac{\gamma'(v)}{\gamma^{\frac{1}{2}}(v)} u \nabla v
$$
for all $t \in (0, T_{\max})$ and inequality $|X - Y|^2 \geq \frac{1}{2} |X|^2 - |Y|^2$ for $X, Y \in \mathbb{R}^n$, we have
$$
\gamma(v) |\nabla u|^2 = \left| \gamma^{\frac{1}{2}}(v) \nabla u \right|^2 = \left| \nabla (\gamma^{\frac{1}{2}}(v) u) - \frac{1}{2} \frac{\gamma'(v)}{\gamma^{\frac{1}{2}}(v)} u \nabla v \right|^2 \geq \frac{1}{2} \left| \nabla (\gamma^{\frac{1}{2}}(v) u) \right|^2 - \frac{1}{4} \frac{|\gamma'(v)|^2}{\gamma(v)} |u|^2 |\nabla v|^2
$$
(17)
for all $t \in (0, T_{\max})$. Substituting (17) into (16) yields
$$
\frac{d}{dt} \int_{\Omega} u^2 + \frac{1}{2} \int_{\Omega} \left| \nabla (\gamma^{\frac{1}{2}}(v) u) \right|^2 \leq \mu \int_{\Omega} u^{l+1} \leq \frac{5}{4} \int_{\Omega} \frac{|\gamma'(v)|^2}{\gamma(v)} |u|^2 |\nabla v|^2 + 2C_1
$$
for all \( t \in (0, T_{\text{max}}) \). From hypothesis (5) and Lemma 2.2, there exists a constant \( C_2 > 0 \) such that
\[
\frac{\gamma'(v)}{\gamma(v)} \leq C_2
\] (19)
for all \( t \in (0, T_{\text{max}}) \). Applying (19) and Hölder’s inequality, we therefore conclude from (18)
\[
\frac{d}{dt} \int_{\Omega} u^2 + \frac{1}{2} \int_{\Omega} |\nabla(\gamma^{\frac{1}{2}}(v)u)|^2 + \mu \int_{\Omega} u^{t+1} \\
\leq \frac{5}{4} \int_{\Omega} \frac{|\gamma'(v)|^2}{\gamma^2(v)} (\gamma^{\frac{1}{2}}(v)u)^2 |\nabla v|^2 + 2C_1 \\
\leq \frac{5C_2^2}{4} \int_{\Omega} |\gamma^{\frac{1}{2}}(v)u|^2 |\nabla v|^2 + 2C_1 \\
\leq \frac{5C_2^2}{4} \left( \int_{\Omega} |\gamma^{\frac{1}{2}}(v)u|^4 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla v|^4 \right)^{\frac{1}{2}} + 2C_1
\] (20)
for all \( t \in (0, T_{\text{max}}) \). Noticing the fact \( \gamma(v) \leq \gamma(v) := C_3 \) for all \( t \in (0, T_{\text{max}}) \) and using Gagliardo-Nirenberg inequality, there exists \( C_4 > 0 \) such that
\[
\left( \int_{\Omega} |\gamma^{\frac{1}{2}}(v)u|^4 \right)^{\frac{1}{2}} = \|\gamma^{\frac{1}{2}}(v)u\|_{L^4}^2 \\
\leq C_4 \left( \|\nabla(\gamma^{\frac{1}{2}}(v)u)\|_{L^2} \|\gamma^{\frac{1}{2}}(v)u\|_{L^2} + \|\gamma^{\frac{1}{2}}(v)u\|^2_{L^2} \right) \\
\leq C_4 \left( C_3^\frac{3}{2} + C_3 \right) \left( \|\nabla(\gamma^{\frac{1}{2}}(v)u)\|_{L^2} \|u\|_{L^2} + \|u\|^2_{L^2} \right)
\] (21)
for all \( t \in (0, T_{\text{max}}) \). By Lemma 2.4 and Lemma 2.6, there exists \( C_5 > 0 \) such that
\[
\left( \int_{\Omega} |\nabla v|^4 \right)^{\frac{1}{2}} = \|\nabla v\|^2_{L^4} \leq C_5 (\|\Delta v\|_{L^2} + 1)
\] (22)
for all \( t \in (0, T_{\text{max}}) \). Recalling (21), (22) and employing Young’s inequality, there exist \( C_6, C_7 > 0 \) such that
\[
\frac{5C_2^2}{4} \left( \int_{\Omega} |\gamma^{\frac{1}{2}}(v)u|^4 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla v|^4 \right)^{\frac{1}{2}} \\
\leq \frac{5C_2^2}{4} \left( C_3^\frac{3}{2} + C_3 \right) C_4C_5 \left( \|\nabla(\gamma^{\frac{1}{2}}(v)u)\|_{L^2} \|u\|_{L^2} + \|u\|^2_{L^2} \right) (\|\Delta v\|_{L^2} + 1) \\
\leq \frac{1}{2} \left\|\nabla(\gamma^{\frac{1}{2}}(v)u)\right\|^2_{L^2} + C_6 \|u\|^2_{L^2} + C_6 \|u\|^2_{L^2} \|\Delta v\|^2_{L^2} \\
\leq \frac{1}{2} \left\|\nabla(\gamma^{\frac{1}{2}}(v)u)\right\|^2_{L^2} + C_7 + C_6 \|u\|^2_{L^2} \|\Delta v\|^2_{L^2}
\] (23)
for all \( t \in (0, T_{\text{max}}) \). Substituting (23) into (20), we get
\[
\frac{d}{dt} \|u\|_{L^2}^2 \leq C_6 \|u\|^2_{L^2} \|\Delta v\|^2_{L^2} + C_7 + 2C_1
\] (24)
for all \( t \in (0, T_{\text{max}}) \). Next, we shall employ (7), (11) and the inequality (24) to get (15). In fact, from (7) there exist \( t_0 = t_0(t) \in ((t - \tau)_+, t) \) and \( C_8 > 0 \) such that \( t_0 \geq 0 \)
\[
\int_{\Omega} u^2(x, t_0) \leq C_8
\] (25)
for all $t \in (0, T_{\text{max}})$ and in both cases $t \in (0, \tau)$ and $t \geq \tau$ with $\tau = \min\{1, \frac{1}{2} T_{\text{max}}\}$. For another, according to (11) in Lemma 2.4, we can find a constant $C_9 > 0$ such that
\[
\int_{t_0 + \tau}^{t_0 + \tau} \int_{\Omega} |\Delta v(x, s)|^2 \leq C_9
\] (26)
for all $t \in (0, T_{\text{max}})$. Integrating (24) over $(t_0, t)$, using (25), (26) and $t \leq t_0 + \tau \leq t_0 + 1$, we find
\[
\|u(\cdot, t)\|_{L^2}^2 \leq \|u(\cdot, t_0)\|_{L^2}^2 \cdot e^C \int_{t_0}^t \|\Delta v(\cdot, s)\|_{L^2}^2 ds + (C_7 + 2C_1) \int_{t_0}^t e^{C_9} \|\Delta v(\cdot, \sigma)\|_{L^2}^2 d\sigma ds
\]
\[
\leq (C_7 + 2C_1 + C_8)e^{C_9}
\]
for all $t \in (0, T_{\text{max}})$ which implies (15) and the proof is completed. \qed

3.2. $L^\infty$ estimate of $u$. Now, we derive the boundedness of $u$. We first show the uniform-in-time bound of $v$ to exclude the possibility of degeneracy. Then we can thereby make use of Lemma 3.1 to achieve the boundedness of $u$.

**Lemma 3.2.** If $\lambda \in \mathbb{R}$, $\mu > 0$ and $l > 2$ are constants, $T_{\text{max}} < +\infty$ and the hypothesis (5) holds, then there exists $C > 0$ such that
\[
\|u(\cdot, t)\|_{L^\infty} \leq C
\] (27)
for all $t \in (0, T_{\text{max}})$.

**Proof.** Recalling Lemma 2.5, Lemma 3.1 and the second equation of (1), we obtain that there exists $C_1 > 0$ such that
\[
\|v(\cdot, t)\|_{W^{1,4}} \leq C_1
\] (28)
for all $t \in (0, T_{\text{max}})$. Hence we can use Sobolev inequality to find $C_2 > 0$ fulfilling
\[
\|v(\cdot, t)\|_{L^\infty} \leq C_2
\] (29)
for all $t \in (0, T_{\text{max}})$. In view of hypothesis (5) and (29), there exists a constant $C_3 > 0$ such that
\[
\gamma(v) \geq \gamma(C_2) > 0
\] (30)
and
\[
\frac{\gamma'(v)}{\gamma(v)} \leq C_3
\] (31)
for all $t \in (0, T_{\text{max}})$. Next, using $u^{p-1}$ with $p \geq 2$ as a test function for the first equation in (1) and integrating the resulting equation by parts, we have
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + (p-1) \gamma(C_2) \int_{\Omega} u^{p-2} |\nabla u|^2 + \mu \int_{\Omega} u^{p+l-1} = - (p-1) \int_{\Omega} \gamma'(v) u^{p-1} \nabla u \cdot \nabla v + \lambda \int_{\Omega} u^p
\]
for all $t \in (0, T_{max})$. Using (30), (31) and Young’s inequality, we obtain
\[
\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^{p} + (p - 1)\gamma(C_2) \int_{\Omega} u^{p-2} |\nabla u|^2 + \mu \int_{\Omega} u^{p+1} + \int_{\Omega} u^{p} \leq \]
\[
\leq C_3(p - 1)\gamma(C_2) \int_{\Omega} u^{p-1} |\nabla u| |\nabla v| + (\lambda + 1) \int_{\Omega} u^{p} \leq \]
\[
\leq \frac{(p - 1)\gamma(C_2)}{2} \int_{\Omega} u^{p-2} |\nabla u|^2 + \frac{C_3^2 \gamma^2(v)(p - 1)}{2\gamma(C_2)} \int_{\Omega} u^{p} |\nabla v|^2 + (\lambda + 1) \int_{\Omega} u^{p}
\]
for all $t \in (0, T_{max})$ and there exists $C_4 > 0$ such that
\[
(\lambda + 1) \int_{\Omega} u^{p} \leq \mu \int_{\Omega} u^{p+1} + C_4
\]
for all $t \in (0, T_{max})$. Noticing the equality
\[
p(p - 1)\gamma(C_2) \int_{\Omega} u^{p-2} |\nabla u|^2 = \frac{2(p - 1)\gamma(C_2)}{p} \int_{\Omega} |\nabla u|^2
\]
for all $t \in (0, T_{max})$, Hence, we can easily get
\[
\frac{d}{dt} \int_{\Omega} u^{p} + p \int_{\Omega} u^{p} + \frac{2(p - 1)\gamma(C_2)}{p} \int_{\Omega} |\nabla u|^2 \leq \]
\[
\leq \frac{C_3^2 \gamma^2(v)(p - 1)}{2\gamma(C_2)} \int_{\Omega} u^{p} |\nabla v|^2 + C_4p
\]
for all $t \in (0, T_{max})$. Applying Hölder’s inequality and (28), we obtain
\[
\frac{C_3^2 \gamma^2(v)(p - 1)}{2\gamma(C_2)} \int_{\Omega} u^{p} |\nabla v|^2 \leq \]
\[
\leq \frac{C_3^2 \gamma^2(v)(p - 1)}{2\gamma(C_2)} \left( \left( \int_{\Omega} u^{2p} \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla v|^4 \right)^{\frac{1}{4}} \right)
\]
\[
\leq \frac{C_3^2 C_4^2 \gamma^2(v)(p - 1)}{2\gamma(C_2)} \| u^{\frac{p}{2}} \|_{L^4}^2
\]
for all $t \in (0, T_{max})$, According to Lemma 3.1, there exists a constant $C_5 > 0$ such that
\[
\| u(\cdot, t) \|_{L^2} \leq C_5
\]
(34)
for all $t \in (0, T_{max})$. Using Gagliardo-Nirenberg inequality, Young’s inequality and (34), we get that there exist $C_6, C_7 > 0$ such that
\[
\frac{C_3^2 C_4^2 \gamma^2(v)(p - 1)}{2\gamma(C_2)} \| u^{\frac{p}{2}} \|_{L^4}^2 \leq \]
\[
\leq C_6 \left( \| \nabla u \|_{L^2}^{2(1 - \frac{1}{p})} \| u^{\frac{p}{2}} \|_{L^p}^2 + \| u^{\frac{p}{2}} \|_{L^p}^2 \right)
\]
\[
\leq C_5 C_6 \| \nabla u \|_{L^2}^{2(1 - \frac{1}{p})} + C_5^p C_6
\]
\[
\leq \frac{2(p - 1)\gamma(C_2)}{p} \| \nabla u \|_{L^2}^2 + C_7
\]
(35)
for all $t \in (0, T_{max})$. Substituting (33) and (35) into (32), we obtain
\[
\frac{d}{dt} \int_{\Omega} u^{p} + p \int_{\Omega} u^{p} \leq C_4p + C_7
\]
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for all \( t \in (0, T_{\text{max}}) \). An application of Gronwall’s inequality yields
\[
\|u(\cdot, t)\|_{L^p}^p \leq \|u_0\|_{L^p}^p + C_{4p} + C_7
\]  
for all \( t \in (0, T_{\text{max}}) \). Choosing \( p = 4 \) in (36) and using Lemma 2.5, we get that there exists a constant \( C_8 > 0 \) such that \( \|\nabla v(\cdot, t)\|_{L^\infty} \leq C_8 \).

Finally, applying the Moser iteration procedure [13], we get (27).

3.3. Global existence.

The proof of Theorem 1.1. Suppose that \( T_{\text{max}} < +\infty \). Combining Lemma 2.5 and Lemma 3.2, we find
\[
\|u(\cdot, t)\|_{L^\infty} + \|v(\cdot, t)\|_{W^{1,\infty}} \leq C
\]
for all \( t \in (0, T_{\text{max}}) \). Then employing Lemma 2.1, we get Theorem 1.1 directly. □

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