Fake state attack on practically decoy state quantum key distribution

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Abstract

In this paper, security of practically decoy state quantum key distribution under fake state attack is considered. If quantum key distribution is insecure under this type of attack, decoy sources cannot also provide it with enough security. Strictly analysis shows that Eve should eavesdrop with the aid of photon-number-resolving instruments. In practical implementation of decoy state quantum key distribution where statistical fluctuation is considered, however, Eve can attack it successfully with threshold detectors.

Keywords: fake state attack, decoy state, quantum key distribution, security

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Quantum key distribution (QKD) is an important application of quantum information, with which two distant parties (the information sender, Alice and the information receiver, Bob) can share a string of secure key with the presence of an eavesdropper, Eve [1–3]. It has been proven that quantum principles can provide it with unconditional security when it is implemented with ideal devices [4–6]. In practical implementation of QKD, however, real-life devices are taken used. They are imperfect and apt to some sophisticated eavesdropping [7–17], part of which have been realized with lab settings. Furthermore, in QKD realization, Alice and Bob’s experimental conditions are assumed to be based on present technology but Eve’s ability is only limited by quantum principles. Then those eavesdropping schemes still lacking experimental demonstration should also be considered seriously. Recently, fake state attack is experimentally proven to be fatal to some commercial quantum key distribution systems, with which the latent eavesdropper can obtain full information shared between Alice and Bob without been detected [13–17].

The fake state attack is a type of intercept-resent attack, where Eve blocks all Alice’s pulses and measures them on randomly chosen bases. Then she prepares her measurement results on fresh pulses and transfers them to Bob. At the same time, she controls Bob’s detectors to work in linear mode: if Bob has the same basis choices as Eve, Eve’s pulses will provide enough power above the threshold value to generate triggers and Bob gets Eve’s bit values; when their basis choices are different, however, Eve’s pulses are split below the threshold intensity and unable to introduce click on Bob’s detectors. It is apparent that Eve’s intervention introduces tolerably error rate and generates identical key string as the legal users’. Then Alice and Bob will acknowledge the validity of their key and ignore Eve’s presence. With a half probability, Bob and Eve will have the same choices on their measurement bases. Thus Eve may eavesdrop on their communication successfully if the combining efficiency of the quantum channel and the measurement devices is less than $\frac{1}{2}$.

Suppose Alice has a weaken coherent source whose photon number obeys Poisson distribution
\[ p_n(\mu) = \frac{\mu^n}{n!} e^{-\mu}, \] (1)
where $\mu$ is average intensity of the source and $n$ is photon number of the incoming pulses. She randomly chooses her bases and prepares her bit values on the pulses, then she transfers them to Bob. To avoid being caught on line, Eve must make Bob’s gains and error rates identical to that when there is no eavesdropper. That is, Eve must make her eavesdropping
satisfy
\[
\frac{1}{2} \eta_f \sum_{n=1}^{\infty} p_n(\mu) = \sum_{n=0}^{\infty} p_n(\mu) \left[ 1 - (1 - p_d)^2 (1 - \eta)^n \right],
\]
\[
e_f \eta_f \sum_{n=1}^{\infty} p_n(\mu) = \sum_{n=0}^{\infty} p_n(\mu) \left\{ 1 - (1 - p_d)^2 (1 - \eta)^n - (1 - p_d) \right\} \times \left[ (1 - \eta e_d)^n - (1 - \eta + \eta e_d)^n \right].
\]  
(2)

Here \( \eta_f \) is the probability Eve will prepare fresh pulses according to her measurement results, \( e_f \) is the probability she prepares wrong bit value on the fresh pulses. And \( e_d \) is the probability of misalignment between Alice and Bob. If Eve can keep alignment between her measurement bases and Alice’s preparing bases, and that between her preparing bases and Bob’s measurement bases at the same time, she can control the error rate on Bob’s results at her will. Furthermore, practical QKD system is very lossy \[18\], the relationships in Eq. (2) should be simulated by Eve easily. Thus if Eve can blind Bob’s detectors, the QKD system will be totally insecure.

In practical implementation of QKD, decoy sources are usually added in \[19–21\]. They have the same characters with that of signal source apart from their average intensities, that is
\[
p_n(\nu) = \frac{\nu^n}{n!} e^{-\nu}.
\]  
(3)

Alice randomly encodes her bit value on signal source or decoy sources. Eve can not tell the decoy sources from the signal source, then she must treat all sources in the same way. Furthermore, she should mock the loss and noise in the quantum channel. Or else, her intervention will inevitably introduce different disturbances on Bob’s results from different sources. It is easy to verify that the relationship in Eq. (2) can not be met for the signal source and the decoy sources at the same time. In order to eavesdrop on the decoy state QKD, Eve should have the ability of differentiating photon number, with which she can treat pulses with the same photons similarly.

In photon-number-splitting (PNS) attack, Eve is assumed to have ability of differentiating photon number in pulses without affect their polarizations \[7–9\]. Then quantum nondemolition (QND) measurement on photon number is required, and this is still missed with present technology. In the fake state attack, however, Eve can measure polarizations on the pulses directly, which means she can get photon number and polarization of the pulses at the same time with photon-number-resolving detectors \[22, 23\]. If she has the ability of differentiate photon number, she can treat pulses with the same photons in a similar way. In order to
simulate the lossy and noise for all sources, Eve’s eavesdropping should satisfy
\[ \eta_n^f = 1 - (1 - p_d)^2 (1 - \eta)^n, \]
\[ e_n^f \eta_n^f = 1 - (1 - p_d)^2 (1 - \eta)^n - (1 - p_d) [(1 - \eta e_d)^n - (1 - \eta + \eta e_d)^n]. \] (4)

Here \( \eta_n^f \) and \( e_n^f \) have similar definition as that in Eq. (2), but they corresponds to pulses with definite photon number \( n \). Eve detects nothing from Alice when \( n = 0 \), however, she should prepare random-bit-value pulses with probability \( 4p_d - 2p_d^2 \) in order to simulate dark count rate on Bob’s detectors. Noticing that Eq. (4) has nothing to do with \( p_n(\mu) \) and \( p_n(\nu) \), thus Eve can eavesdrop on the decoy state QKD successfully if she has a set of photon-number-resolving detectors.

Now it is interesting whether Eve can eavesdrop on the decoy state QKD protocol without photon-number-resolving detectors. However, as mentioned before, eavesdropping on the decoy sources should have similar relationships as those in Eq. (2), that is,
\[ \frac{1}{2} \eta_f \sum_{n=1}^{\infty} p_n(\nu) = \sum_{n=0}^{\infty} p_n(\nu) [1 - (1 - p_d)^2 (1 - \eta)^n], \]
\[ e_f \frac{1}{2} \eta_f \sum_{n=1}^{\infty} p_n(\nu) = \sum_{n=0}^{\infty} p_n(\nu) \{1 - (1 - p_d)^2 (1 - \eta)^n - (1 - p_d) \times [(1 - \eta e_d)^n - (1 - \eta + \eta e_d)^n]\}. \] (5)

And one can find that there are not such \( \eta_f \) and \( e_f \) to meat the relationships in Eq. (2) and those in Eq. (5) at the same time. However, considering the imperfection of practical implementation of decoy state QKD, the relationships in Eq. (2) and Eq. (5) may be loosen.

Experimentally, Alice and Bob’s key is distributed within a finite period of time, generally in several hours \[24\]. Then the pulses generated by Alice should also be finite. We assume the number of pulses emitted from the sources is \( N = 10^{10} \) in the following discussion. If Bob’s expectingly detections is \( p_{det} \) under ideal circumstance, and his detecting events with finite resources is \( p'_{det} \), \( p'_{det} \) should deviate from \( p_{det} \) with a small fluctuation \( \delta_{p_{det}} \). The probability \( P(|p_{det} - p'_{det}| > \delta_{p_{det}}) \) can be estimated to be less than \( q = \exp(-\frac{N^{1/2} \delta_{p_{det}}}{4p_{det}}) \). If we require \( q = \exp(-25) \), \( \delta_{p_{det}}^2 \) can be estimated to be \( 10^{-8} \) approximately, and \( \delta_{p_{det}} \) can be calculated as \( 10^{-4} p_{det}^{1/2} \) accordingly. Thus in practical implementation of fake state attack, Eve should ensure Bob’s detecting events satisfy
\[ p_{det} - 10^{-4} p_{det}^{1/2} < p'_{det} < p_{det} + 10^{-4} p_{det}^{1/2}. \] (6)

Strictly speaking, the statistical fluctuations on different sources are usually not the same as the number of pulses generated from different sources may be not assigned to be the same.
At the same time, the probabilities of detecting events for different sources are not identical because of their disparate intensities. As the total number of the pulses is $N$, the practical number of pulses assigned for different sources should be less than this value, the practically tolerant fluctuation should be greater than that in Eq. (6). It means Alice and Bob should accept the validity of their results if the statistical fluctuations on the detecting events from every sources satisfy the relationship in Eq. (6). And similar relationship can be obtained for the gain of QBER on every source, that is

$$p_{err} - 10^{-4}p_{err}^{\frac{1}{2}} < p'_{err} < p_{err} + 10^{-4}p_{err}^{\frac{1}{2}}. \quad (7)$$

It is apparent that Bob’s expecting detecting results should be $p_{det}(\mu) = 1 - (1 - p_d)^2e^{-\eta\mu}$ for signal source, and $p_{det}(\nu) = 1 - (1 - p_d)^2e^{-\eta\nu}$ for decoy sources. And the actual detecting events for them should be $p'_{det}(\mu) = \frac{1}{2}\eta f(1 - e^{-\mu})$ and $p'_{det}(\nu) = \frac{1}{2}\eta f(1 - e^{-\nu})$ respectively. Similarly, the expecting error rate for signal source and decoy sources are $p_{err}(\mu) = \frac{1}{2}\sum_{n=0}^{\infty} p_n(\mu)(1 - (1 - p_d)^2(1 - \eta)^n - (1 - p_d)(1 - \eta + \eta e_d)^n)$ and $p_{err}(\nu) = \frac{1}{2}\sum_{n=0}^{\infty} p_n(\nu)(1 - (1 - p_d)^2(1 - \eta)^n - (1 - p_d)(1 - \eta + \eta e_d)^n)$. And the actual error rate for them can be calculated as $p'_{err}(\mu) = \frac{1}{2}e_f\eta f(1 - e^{-\mu}) = e_f p'_{det}(\mu)$ and $p'_{err}(\nu) = \frac{1}{2}e_f\eta f(1 - e^{-\nu}) = e_f p'_{det}(\nu)$.

We can estimate the feasibility of Eve’s attack with the experimental parameters in [18], that is, $p_d = 8.5 \times 10^{-7}$, $\eta_B = 4.5\%$, $e_d = 3.3\%$ and loss coefficient $\alpha$ in the quantum channel is 0.21 dB/km. If the transmission distance between Alice and Bob is 120km, one can obtain $\eta = 1.359 \times 10^{-4}$. When there are only two sources, that is, a signal source with intensity $\mu = 0.479$ and a weaker decoy state with intensity $\nu = 0.127$ [24]. As the statistical fluctuations on the the results of signal source satisfy the relationships in Eq. (6) and Eq. (7), its $\eta_f$ should range from $3.467 \times 10^{-4}$ to $3.553 \times 10^{-4}$, and its $e_f$ can range from $4.178 \times 10^{-2}$ to $4.806 \times 10^{-2}$. Similarly, statistical fluctuation on the the results of weaker decoy source should satisfy the relationships in Eq. (6) and Eq. (7), its $\eta_f$ can be calculated to range from $3.106 \times 10^{-4}$ to $3.252 \times 10^{-4}$, and its $e_f$ ranges from $6.705 \times 10^{-2}$ to $8.307 \times 10^{-2}$. As there is no overlap on the parameters of both sources, it seems that Eve can not eavesdrop on the decoy state QKD protocol with threshold detectors.

Noticing that the dark count rate functions importantly in practical implementation of decoy state QKD protocol when the transmission distance is comparably long. Furthermore, though threshold detectors can not tell the photon number in the incoming pulses, they
can differentiate vacuum pulses from non-vacuum pulses. Then it may help Eve with her eavesdropping if she treats the vacuum pulses and non-vacuum pulses in different ways. She prepares random-bit-value pulses with probability \(4p_d - 2p_d^2\) for Bob when she detecting nothing from Alice. It is easily verified that Eve’s eavesdropping results on the vacuum pulses coincide well with what Bob expecting for. When there is nonvacuum pulses, she makes fresh pulses according to her results with probability \(\eta_f\) and introduces error on them with probability \(e_d\). Here Eve makes error on the nonvacuum pulses with probability \(e_d\) because errors introduced on nonvacuum pulses are mainly introduced by misalignment between Alice and Bob. Then for signal source, one can obtain

\[
\frac{1}{2}\eta_f \sum_{n=1}^{\infty} p_n(\mu) = \sum_{n=1}^{\infty} p_n(\mu)[1 - (1 - p_d)^2(1 - \eta)^n],
\]

\[
\eta_f e_d \sum_{n=1}^{\infty} p_n(\mu) = \sum_{n=1}^{\infty} p_n(\mu)[1 - (1 - p_d)^2(1 - \eta)^n - (1 - p_d) \times [(1 - \eta e_d)^n - (1 - \eta + \eta e_d)^n]].
\]

(8)

Similar relationship can also be obtained for decoy source.

The probability of expecting detections \(p_{\text{det}}\) both for signal source and decoy source can still be calculated as that above. However, the actual detections \(p'_{\text{det}}\) for them are altered slightly. That is, \(p'_{\text{det}}(\mu) = e^{-\mu}[1 - (1 - p_d)^2] + \frac{1}{2}\eta_f(1 - e^{-\mu})\) and \(p'_{\text{det}}(\nu) = e^{-\nu}[1 - (1 - p_d)^2] + \frac{1}{2}\eta_f(1 - e^{-\nu})\). Similarly, the expressions for \(p_{\text{err}}(\mu)\) and \(p_{\text{err}}(\nu)\) are still the same. And \(p'_{\text{err}}(\mu)\) and \(p'_{\text{err}}(\nu)\) should be recalculated as \(\frac{1}{2}e^{-\mu}[1 - (1 - p_d)^2] + \frac{1}{2}e_{d}\eta_f(1 - e^{-\mu})\) and \(\frac{1}{2}e^{-\nu}[1 - (1 - p_d)^2] + \frac{1}{2}e_{d}\eta_f(1 - e^{-\nu})\) respectively. As their statistical fluctuations should still be bounded with the relations in Eq. (6) and Eq. (7). With simple calculation, we find there is no such \(\eta_f\) for signal source, and \(\eta_f\) for decoy source ranges from \(2.855 \times 10^{-4}\) to \(3.001 \times 10^{-4}\). That is, this scheme is still inefficient in helping Eve to eavesdrop on decoy state QKD protocol with threshold detectors.

Eve takes control the whole quantum channel, however, she may not set her eavesdrop point adjacent to Alice’s lab. Her intervention site may be anywhere between Alice’s and Bob’s labs. We will show that this change will help Eve to Eavesdrop on the decoy state QKD protocol successfully with threshold detectors. Let the distance between Alice’s lab and Bob’s eavesdropping site be \(l\) km, it is apparent smaller \(l\) requires ability to discriminate photon number in the pulses, and larger \(l\) may lead to failure of her blinding attack. Then the optimal site should have largest \(l\) where Eve can carry out her eavesdropping successfully. The transmission efficiency at this point can be calculated as \(\eta_l = 10^{-\frac{al}{m}}\). And the statistical
distribution in the incoming pulses can be represented as

\[
\begin{align*}
p_n^I(\mu) &= \frac{(\eta \mu)^n}{n!} e^{-\eta \mu}, \\
p_n^I(\nu) &= \frac{(\eta \nu)^n}{n!} e^{-\eta \nu}.
\end{align*}
\]

If Eve takes her eavesdropping scheme as that in Eq. (8), one can obtain

\[
\begin{align*}
\frac{1}{2} \eta_f \sum_{n=1}^{\infty} p_n^I(\mu) &= \sum_{n=1}^{\infty} p_n^I(\mu) \left[ 1 - (1 - p_d)^2 (1 - \eta')^n \right], \\
\eta_f e_d \sum_{n=1}^{\infty} p_n^I(\mu) &= \sum_{n=1}^{\infty} p_n^I(\mu) \left( 1 - (1 - p_d)^2 (1 - \eta')^n - (1 - p_d) \right) \\
&\quad \times \left[ (1 - \eta' e_d)^n - (1 - \eta' + \eta' e_d)^n \right].
\end{align*}
\]

with \( \eta' = \eta_b 10^{-\frac{120 - l}{10}} \) for signal source. And similar relationships can be obtained for decoy source

\[
\begin{align*}
\frac{1}{2} \eta_f \sum_{n=1}^{\infty} p_n^I(\nu) &= \sum_{n=1}^{\infty} p_n^I(\mu) \left[ 1 - (1 - p_d)^2 (1 - \eta')^n \right], \\
\eta_f e_d \sum_{n=1}^{\infty} p_n^I(\nu) &= \sum_{n=1}^{\infty} p_n^I(\mu) \left( 1 - (1 - p_d)^2 (1 - \eta')^n - (1 - p_d) \right) \\
&\quad \times \left[ (1 - \eta' e_d)^n - (1 - \eta' + \eta' e_d)^n \right].
\end{align*}
\]

As Eve’s detecting efficiency is very lower, it is easy to verify that Eve can set \( l = 120 \) km. We can then obtain \( \eta_f \) ranging from \( 8.893 \times 10^{-2} \) to \( 9.120 \times 10^{-2} \) for signal source, and it ranges from \( 8.775 \times 10^{-2} \) to \( 9.229 \times 10^{-2} \) for decoy source. Then Eve can launch fake state attack at \( l = 120 \) km with threshold detectors just by preparing what she have measured with probability \( \eta_f \) ranging from \( 8.893 \times 10^{-2} \) to \( 9.120 \times 10^{-2} \), and she introduces error on them with probability \( e_d \).

Then when statistical fluctuation is considered, Eve can eavesdropping on decoy state QKD even with threshold detectors. She may fail to eavesdrop successfully when her intervention site is closer to Alice’s lab, and numerical simulation shows it may be easier for her to attack on this protocol when her intervention site is farer away from Alice’s lab. This is because the nearer to Bob’s lab, the greater probability of single-photon pulses for non-vacuum pulses can be obtained. Then Eve can omit the effect of multi-photon pulses treat all pulses as single photons. In practical decoy state QKD protocol, Alice may introduce vacuum decoy state to estimate the dark counts on Bob’s detectors. [24]. As Eve prepares random-bit-value pulses with probability \( 4p_d - 2p_d^2 \) when she detects nothing, however, the statistical fluctuations on the vacuum decoy state can be verified to be met automatically. In order to understand Eve’s eavesdropping better, we give a simulation numerically on the relation between Eve’s \( \eta_f \) and her intervention site \( l \), as is plotted as that in Fig. 1. It shows that there is not suitable \( \eta_f \) and for signal source when \( l \leq 10 \) km. Furthermore, Eve can
not launch her fake state attack on this protocol when her intervention site \( l \) is less than 30 km as there is no overlap on \( \eta_f \) for both sources. When \( l \) is greater than 45 km, one can find the suitable \( \eta_f \) for signal source also suits for decoy source.

![Graph showing the relationship between Eve's probability for her to prepare fresh pulses according to her results, \( \eta_f \) and her intervention site, \( l \). It suitable value should be chosen from the area greater than the lower bounds and less than the upper bounds of \( \eta_f \) for both sources.](image)

**FIG. 1:** Relationship between Eve’s probability for her to prepare fresh pulses according to her results, \( \eta_f \) and her intervention site, \( l \). It suitable value should be chosen from the area greater than the lower bounds and less than the upper bounds of \( \eta_f \) for both sources.

It has been proven that some commercial QKD systems may be totally insecure under fake state attack [14–17], thus we hope that decoy states is efficiency in combatting against this brutal attack. As we have shown above, however, decoy states can not also provide them with enough security. We have shown that Eve can launch her fake state attack successfully. Especially, we have proven that she can eavesdrop on the decoy state QKD without any photon-number-resolving instrument when statistical fluctuation is considered on Bob’s results. With the presence of new technology, especially with improvement on Bob detecting efficiency, Alice and Bob may overcome this loophole. However, other loopholes still un-
known to people may also threaten the security of practical QKD. And it has been claimed QKD is superior to classical cryptography as quantum principle provide it with physically secure. In order to avoid the mouse and cat game between legal users and eavesdropper in QKD, new protocols should be presented to combat all these loopholes in principle [25–28]. This work is sponsored by the National Natural Science Foundation of China (Grant No 10905028) and HASTIT.

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