GLOBAL CONTINUATION OF A VLASOV MODEL OF ROTATING GALAXIES

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Abstract. A typical galaxy consists of a huge number of stars attracted to each other by gravity. For instance, the Milky Way has about $10^{11}$ stars. Thus it is typically modeled by the Vlasov-Poisson system. We prove an existence theorem for axisymmetric steady states of galaxies that may rotate rapidly. Such states are given in terms of a fairly general function $\phi$ of the particle energy and angular momentum. The set $K$ of such states form a connected set in an appropriate function space. Along the set $K$, we prove under some conditions that either (a) the supports of the galaxies become unbounded or (b) both the rotation speeds and the densities somewhere within the galaxy become unbounded.

1. Introduction

We consider a continuum of particles with a fixed total mass that are attracted to each other by gravity but subject to no other forces. Initially they are static and spherical, but if they begin to rotate around a fixed axis, they flatten at the poles and expand at the equator. This is a very simple way to think about a rotating collection of particles. In this paper we model this configuration of particles in the standard way by the Vlasov-Poisson system (VP). It is the standard model of stellar systems such as galaxies [4]. We look for steady states of the resulting configuration and find a connected set $K$ of such states with constant mass and possibly large rotation speeds.

The VP system is a continuous model for the particle distribution $f(x,v)$ in phase space $(x,v)$ where $x \in \mathbb{R}^3$ is position and $v \in \mathbb{R}^3$ is velocity. The Vlasov equation (also called the collisionless Boltzmann equation) is

\begin{equation}
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + \nabla_x U \cdot \nabla_v f = 0,
\end{equation}

where $f(x,v) \geq 0$ and $\nabla_x U$ is the force of gravity. The gravitational potential $U$ is

\begin{equation}
U = \frac{1}{|x|} \ast \rho, \quad \rho(x) = \int_{\mathbb{R}^3} f(x,v) \, dv,
\end{equation}

where $\rho$ is the macroscopic particle density in physical space. In general, the macroscopic velocity $V(x) = \int_{\mathbb{R}^3} v f(x,v) \, dv$ does not vanish.

In this paper we only consider solutions that are independent of time $t$ and are rotating around the $x_3$ axis. Jeans’ Theorem [3] states that every steady state depends only on the invariants of the flow. Two of these invariants are the energy and the $x_3$ component of the angular momentum. We only consider solutions that are axisymmetric and of the form

\begin{equation}
f(x,v) = \phi \left( \frac{1}{2} |v|^2 - U(x) - \alpha, \, \kappa(x_1 v_2 - x_2 v_1) \right),
\end{equation}

where $\kappa$ is a measure of the intensity of rotation and the constant $\alpha$ is included for convenience. The microscopic density function $\phi$ is nonnegative. Then $f$ is itself an invariant of the flow and so it is automatically a solution of the Vlasov equation (1.1).

Let $C_c(\mathbb{R}^3)$ be the Fréchet space of axisymmetric continuous functions on $\mathbb{R}^3$ with compact support. Its topology, defined by seminorms, is that of uniform convergence on each compact set. By a solution of our problem we mean a triple $(\rho, \alpha, \kappa) \in C_c(\mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}$ that satisfies (1.2) and (1.3).
Our main result is as follows.

**Theorem 1.1.** Let \((\rho_0, \alpha_0)\) be a non-rotating \((\kappa = 0)\) spherically symmetric solution be given (as described in Lemma 3.2 below). Let \(M_0 = \int_{\mathbb{R}^3} \rho_0 \, dx\) be its total mass. Assume that a microscopic density function \(\phi(E, L)\) is given that satisfies the assumptions (2.6) through (2.12). Then there exists a set \(K\) of solutions \((\rho, \alpha, \kappa)\) with \(\rho \geq 0\) and \(\rho \in C^1\) that satisfies the following properties.

- \(K\) is a connected set in the space \(C_c(\mathbb{R}^3) \times \mathbb{R} \times \mathbb{R}\).
- \(K\) contains the non-rotating solution.
- All the elements in \(K\) have the same total mass \(M_0\).
- Either
  
  \[
  \sup \left\{ |x| \mid \rho(x) > 0, \ (\rho, \alpha, \kappa) \in K \right\} = \infty
  \]
  or
  
  \[
  \sup \left\{ \rho(x) \mid x \in \mathbb{R}^3, (\rho, \alpha, \kappa) \in K \right\} = \infty.
  \]

**Theorem 1.2.** Under the same setup as in Theorem 1.1, if \(\phi(E, L)\) satisfies assumptions (2.6) through (2.10) and (2.12) but with the more restrictive range of \(\Lambda\): \(0 < \Lambda < 4\), then the same conclusion as in Theorem 1.1 is valid provided (1.5) is replaced by

\[
\sup \left\{ |\kappa| \mid (\rho, \alpha, \kappa) \in K \right\} = \infty.
\]

Alternative (1.4) means that \(\rho\) has unbounded support. Because the total mass is fixed, it would mean that there can be a very large set in \(\mathbb{R}^3\) on which \(\rho\) is very small. Alternative (1.5) means that the macroscopic density is unbounded. The other alternative (1.6) means that the rotation speed is unbounded. Which of these alternatives can actually happen is an open question.

As discussed later, a simple example of \(\phi\) that satisfies the conditions of Theorem 1.1 is \(\phi(E, L) = (E)^{\nu}_- \ p(L)\), where \((E)^{\nu}_-\) denotes the negative part of \(E\), \(\nu \in (-\frac{1}{2}, \frac{3}{2})\), \(\nu \neq \frac{1}{2}\), and \(p\) is any positive polynomial. The same function satisfies the conditions of Theorem 1.2 provided \(p\) is a positive quadratic polynomial. More examples are given at the end of Section 3.

There is a voluminous classical literature on fluid models of rotating stars going back notably to MacLaurin, Poincaré and Lichtenstein. On the other hand, the Vlasov system with gravity is a more natural model of galaxies. Spherical solutions of this system are well-known in the mathematical literature (see [12]) but non-spherical ones have not been studied much. See [12] for a nice survey of the initial-value problem.

The first mathematical treatment of steady states of axisymmetric solutions of VP that we are aware of is by Rein [14], who constructs a local curve of nearly spherical solutions by means of the implicit function theorem. A similar theorem may be found in [5]. Such a local curve is also constructed by Andreasson et al. within the context of general relativity in [3].

In [15], Sections 6-8, we have constructed such a local curve with the additional property that the mass remains constant (independent of the rotation speed). On the other hand, Guo and Rein [13] construct stable steady states by minimization of an energy-Casimir functional for a certain class of \(\phi\). The stability of spherical states has also been studied by Guo and Lin [7].

The point of the present paper is to construct a “global” connected set of steady states that have compact support and constant mass and that emanate from a spherical solution but deviate far from it. The constant mass condition means that there is no loss or gain of particles as the star changes its rotation speed. Our method has much in common with [16], where a global curve is constructed for a fluid model of a rotating star. However, the two different models require quite different analyses in some respects.
In the next section we state our detailed assumptions. The remainder of the paper is devoted to the proof of the two theorems. In Section 3 we provide details about the non-rotating spherical solution and its mass. Section 4 provides the proof of global continuation by means of a global implicit function theorem. Sections 5 and 6 are devoted to the proof that either the support of $\rho$ is unbounded or $\rho$ itself is unbounded (Theorem 1.1). This is the most subtle part of our paper. In order to elucidate the proof, it is first proven for a special case and only later for a general case. Key points in the proof are the use of a Gagliardo-Nirenberg inequality and of the constant mass condition. Section 7 is devoted to proving that either the support of $\rho$ is unbounded or the rotation speed is unbounded (Theorem 1.2).

2. Setup and Basic Assumptions

It is easily verified that for any $(\kappa, \alpha) \in \mathbb{R}^2$, $U \in C^2(\mathbb{R}^3)$ axisymmetric and $\phi \in L^1_{\text{loc}}(\mathbb{R}^2)$, $f$ given by (2.3) is a weak solution to (1.1) in the sense that
\[ f(x, v) = f(\psi(x, v, t)), \]
where $\psi(x, v, t)$ is the flow map of the vector field $(v, \nabla U(x))$ on $\mathbb{R}^6$. Given a function $\phi$ on $\mathbb{R}^2$, we define
\[ w(\kappa, r, u) = \int_{\mathbb{R}^3} \phi \left( \frac{1}{2} |v|^2 - u, \kappa(x_1v_2 - x_2v_1) \right) dv, \quad r = (x_1^2 + x_2^2)^{1/2}. \]
Then the Poisson equation (1.2) takes the form
\[ \rho = w \left( \kappa, r, \frac{1}{|v|} * \rho \right). \]
Given $\phi$, it suffices to solve (2.3) for $\rho, \alpha$ and $\kappa$.

In order to construct a global set of solutions with fixed total mass, we therefore look for the zero set of the mapping $F(\rho, \alpha, \kappa) = (F_1, F_2)$, where
\[ F_1(\rho, \alpha, \kappa) = \rho(x) - w \left( \kappa, r, \frac{1}{|v|} * \rho(x) + \alpha \right), \]
\[ F_2(\rho, \alpha, \kappa) = \int_{\mathbb{R}^3} \rho(x) dx - M_0. \]
Here $M_0$ is the fixed total mass of the solution set, which we will discuss in more detail in Section 3. We will always assume that $\rho$ is axisymmetric and even in $x_3$.

Now we specify the assumptions on the prescribed microscopic density function $\phi : \mathbb{R}^2 \to \mathbb{R}$. We divide them into several groups:

(I) We begin with its positivity and regularity properties, which will be used to show the regularity and positivity of $\rho$:
\[ \phi(E, L) > 0 \text{ when } E < 0, \quad \phi(E, L) = 0 \text{ when } E > 0. \]
\[ \phi|_{E<0} \in C^1_{\text{loc}} ((-\infty, 0) \times \mathbb{R}). \]

(II) Next are the properties that will be used to obtain the non-rotating ($\kappa = 0$) solutions:
\[ \lim_{E \to -\infty} |E|^{1/2} \phi(E, 0) = \infty. \]
\[ \lim_{E \to -\infty} |E|^{-7/2} \phi(E, 0) = 0. \]
We also require the following assumption, stated rather informally:

\[(2.10)\] The total masses of the non-rotating solutions depend strictly monotonically on the center densities \(\rho(0)\).

This property will be used to generate a local curve of solutions emanating from a non-rotating one. A more precise statement of it is given in (3.15), (3.16).

The final pair of properties are a lower bound and an upper bound that will be used mainly to narrow down the types of blow up behavior that could occur in the global solution set in Theorems 1.1 and 1.2.

\[(2.11)\] \[\lim_{B \to 0} \left| E \right|^{-\delta}\left| L \right|^{-\Gamma} \phi(E, L) > 0\]

for some \(\delta > 0\) and \(\Gamma > 0\). Furthermore, for every \(B > 0\), there exist \(C > 0\), \(\Lambda > 0\), \(\mu < \frac{1}{2}\) such that

\[(2.12)\] \[\phi(E, L) + |\partial_L \phi(E, L)| \leq C|E|^{-\mu}(1 + |L|)^{\Lambda}\]

for all \(-B < E < 0\) and all \(L\).

In Proposition 3.1, we will provide some simple conditions on the density function \(\phi\) under which (2.10) is satisfied.

3. Non-rotating Solutions

We now discuss the non-rotating solutions (\(\kappa = 0\)) in more detail. Using spherical coordinate substitution with polar axis pointing in the direction of \((-x_2, x_1, 0)\), and reparametrizing using \(E = \frac{1}{2}v^2 - u, s = \frac{x_1 v_2 - x_2 v_1}{\sqrt{2(E + u)}}\), we can write

\[(3.1)\] \[w(\kappa, r, u) = \int_{\mathbb{R}^3} \phi\left(\frac{1}{2}v^2 - u, \kappa(x_1 v_2 - x_2 v_1)\right) dv = 2\pi \int_0^0 \int_{-\sqrt{2(E + u)}}^{\sqrt{2(E + u)}} \phi(E, \kappa rs) ds \, dE.\]

We easily see from (3.1) that \(w(\kappa, r, u) = w(-\kappa, r, u)\). The switch from \(\kappa\) to \(-\kappa\) corresponds to reversing the direction of macroscopic velocity in the fluid. This symmetry means the model is indifferent to the direction of rotation, as in the Euler-Poisson model studied in [16]. We have the following basic properties of \(w\).

**Lemma 3.1.** \(w \in C^1_{\text{loc}}(\mathbb{R} \times [0, \infty) \times \mathbb{R})\). \(w(\kappa, r, u) > 0\) if \(u > 0\), and \(w(\kappa, r, u) = 0\) if \(u \leq 0\). If we define

\[(3.2)\] \[G(u) = \int_0^0 \int_{-\sqrt{2(E + u)}}^{\sqrt{2(E + u)}} \phi(E, \kappa rs) ds \, dE.\]

then \(G'(u) > 0\) for \(u > 0\) and \(G\) has the limits

\[(3.3)\] \[\lim_{u \to 0} u^{-1}G(u) = 0, \quad \lim_{u \to \infty} u^{-1}G(u) = \infty, \quad \lim_{u \to \infty} u^{-5}G(u) = 0.\]

**Proof.** The proof resembles that of Lemma 6.4 and Lemma 8.1 of [15], but we include it here because the hypotheses are a bit different. The positivity property of \(w\) follows immediately from (2.11). By (2.12) and the dominated convergence theorem, we differentiate under the integral sign to get

\[(3.4)\] \[\partial_\kappa w(\kappa, r, u) = 2\pi r \int_{-u}^0 \int_{-\sqrt{2(E + u)}}^{\sqrt{2(E + u)}} \partial_L \phi(E, \kappa rs) s \, dE.\]

\[(3.5)\] \[\partial_r w(\kappa, r, u) = 2\pi \kappa \int_{-u}^0 \int_{-\sqrt{2(E + u)}}^{\sqrt{2(E + u)}} \partial_L \phi(E, \kappa rs) s \, dE.\]
The continuity of both of these functions follows from the dominated convergence theorem. Assuming for the moment that \( u > 0 \), we also have

\[
\partial_u w(\kappa, r, u) = \pi \sqrt{2} \int_{-u}^{0} \frac{\phi \left( E, \kappa r \sqrt{2(E + u)} \right) + \phi \left( E, -\kappa r \sqrt{2(E + u)} \right)}{\sqrt{E + u}} \, dE.
\]

It is not hard to see that \( \partial_u w(\kappa, r, u) \) is continuous and positive for \( u > 0 \). Moreover, by (2.12) we get for \( 0 < u < 1 \) that

\[
|\partial_u w(\kappa, r, u)| \leq C \int_{-u}^{0} \frac{|E|^{-\mu}}{\sqrt{E + u}} \, dE \leq Cu^{1/2-\mu}.
\]

From this inequality and the fact that \( w(\kappa, r, u) = 0 \) when \( u \leq 0 \), we see that \( \partial_u w(\kappa, r, 0) \) exists and is zero. Also, \( \partial_u w \) is continuous everywhere.

We now turn to the study of

\[
G(u) := w(0, 0, u) = w(0, r, u) = 4\pi \sqrt{2} \int_{-u}^{0} \phi(E, 0) \sqrt{E + u} \, dE.
\]

The first limit in (3.3) is just a restatement of the fact that \( G'(0) = 0 \). To obtain the second limit, we estimate

\[
u^{-1}G(u) \geq C\nu^{-1} \int_{-u}^{-u/2} \phi(E, 0) \sqrt{E + u} \, dE
\]

\[
\geq C \left( \inf_{E < -u/2} |E|^{1/2} \phi(E, 0) \right) \cdot \nu^{-1} \int_{-u}^{-u/2} |E|^{-1/2} \sqrt{E + u} \, dE
\]

\[
= C \left( \inf_{E < -u/2} |E|^{1/2} \phi(E, 0) \right) \cdot \int_{-1}^{-1/2} |E|^{-1/2} \sqrt{E + 1} \, dE
\]

Thus the second limit in (3.3) follows from (2.8). To get the last limit, we estimate for any \( a > 0 \):

\[
0 \leq u^{-5}G(u) \leq u^{-5} \int_{-a}^{0} \phi(E, 0) \sqrt{E + u} \, dE + u^{-5} \int_{-u}^{-a} \phi(E, 0) \sqrt{E + u} \, dE
\]

\[
\leq u^{-5} \sqrt{u} \int_{-a}^{0} \phi(E, 0) \, dE + \left( \sup_{E < -a} |E|^{-7/2} \phi(E, 0) \right) \cdot u^{-5} \int_{-u}^{0} |E|^{-7/2} \sqrt{E + u} \, dE
\]

\[
\leq \frac{\int_{-a}^{0} \phi(E, 0) \, dE}{u^{9/2}} + C \sup_{E < -a} |E|^{-7/2} \phi(E, 0).
\]

It follows that

\[
0 \leq \limsup_{u \to \infty} u^{-5}G(u) \leq C \sup_{E < -a} |E|^{-7/2} \phi(E, 0).
\]

The last limit in (3.3) then follows by sending \( a \) to infinity and using (2.9). \( \square \)

We summarize the basic properties of non-rotating solutions as follows.

**Lemma 3.2.** For every \( R > 0 \), there exists a radial function \( \rho \in C^1(\mathbb{R}^3) \) with \( R > 0 \) on \( B_R \), \( \rho = 0 \) on \( \mathbb{R}^3 \setminus B_R \), and a constant \( \alpha < 0 \) such that \( F_1(\rho, \alpha, 0) = 0 \) on \( \mathbb{R}^3 \). Furthermore, every nonzero compactly supported continuous solution to \( F_1(\rho, \alpha, 0) = 0 \) satisfies the same conditions.

**Proof.** Consider the equation

\[
\Delta u + 4\pi G(u) = 0
\]
in \(B_R\) with \(u = 0\) on \(\partial B_R\). By \[2\] and \[3\], \((3.10)\) has a positive \(C^2\) solution on \(B_R \subset \mathbb{R}^3\) with zero boundary data if \(G\) maps \(\mathbb{R}^+\) to \(\mathbb{R}^+\), is \(C^1\) on \(\mathbb{R}^+, G(0) = 0\), and satisfies the last two limits in \((3.3)\). By Theorem 1 in \[6\], \(u\) is radially symmetric and strictly decreasing:

\[(3.11)\]
\[x \cdot \nabla u(x) < 0 \quad \text{for } 0 < |x| \leq R.\]

Now define \(\rho = G(u)\) on \(B_R\) and \(\rho = 0\) on \(\mathbb{R}^3 \setminus B_R\). The first limit in \((3.3)\) implies \(\rho \in C^1(\mathbb{R}^3)\). By \((3.10)\), \(u - \frac{1}{|x|} \ast \rho\) is harmonic in \(B_R\). Since it has a constant boundary value \(\alpha = -\frac{1}{|x|} \ast \rho\) on \(\partial B_R\), it must be constant in all of \(B_R\). In other words, we have

\[(3.12)\]
\[\rho - G \left( \frac{1}{|x|} \ast \rho + \alpha \right) = 0\]

in \(\overline{B_R}\). \((3.12)\) is \(F_1(\rho, \alpha, 0) = 0\) by definition. To see that \((3.12)\) holds in \(\mathbb{R}^3 \setminus B_R\) as well, we only need to show \(\frac{1}{|x|} \ast \rho + \alpha < 0\) in \(\mathbb{R}^3 \setminus B_R\), since \(G(s) = 0\) for \(s \leq 0\). This follows because \(\frac{1}{|x|} \ast \rho + \alpha\) is harmonic outside \(\overline{B_R}\), zero on \(\partial B_R\) and negative at infinity.

Now assume a non-trivial \(\rho\) is compactly supported and continuous, with axisymmetry and even symmetry in \(x_3\), and \(\alpha\) is a constant such that \(F_1(\rho, \alpha, 0) = 0\). We have \(\rho \geq 0\), \(\rho \in C^1(\mathbb{R}^3)\) as \(G\) is non-negative and \(C^1\). Let \(v = \frac{1}{|x|} \ast \rho\). Then \(v > 0\) and \(\Delta v = -4\pi \rho = -4\pi G(v + \alpha)\). Moreover \(\alpha < 0\), for otherwise \(\rho\) could not be compactly supported. Applying Theorem 4 and Proposition 1 in \[5\], we conclude that \(v\) is radially symmetric about some point in \(\mathbb{R}^3\), and the same is true for \(\rho\). We assumed that \(\rho\) is axisymmetric and even in \(x_3\). It is not hard to see that if \(\rho\) is also radially symmetric about a point different from the origin, then it could not be compactly supported unless it is identically zero. Let \(\overline{B_R}\) denote the support of \(\rho\). Letting \(u = v + \alpha\), we have \(\rho = G(u)\) and \(\Delta u \leq 0\) in \(B_R\), and \(u = 0\) on \(\partial B_R\). By the strong maximum principle, \(u > 0\) on \(B_R\) and the same is true for \(\rho\). Thus \(\rho\) is positive in \(B_R\) and zero outside, and so it is one of the solutions constructed above.

Having classified all the non-rotating solutions, we can parametrize them by their center densities.

**Lemma 3.3.** Denote by \(\mathcal{A}\) the set of center values \(\rho(0)\) of all the compactly supported non-rotating solutions given in Lemma 3.2. Then \(\mathcal{A}\) is an open subset of \((0, \infty)\). For every \(a \in \mathcal{A}\), there exist a unique \(\rho(x; a) \in C_c(\mathbb{R}^3)\) and a unique \(\alpha(a)\) satisfying \(F_1(\rho(x; a), \alpha(a), 0) = 0\), \(\rho(0; a) = a\).

**Proof.** As in the proof of Lemma 3.2 we let \(u = \frac{1}{|x|} \ast \rho + \alpha\), so that \(\rho = G(u)\). Since \(G\) maps \(\mathbb{R}^+\) to \(\mathbb{R}^+\) diffeomorphically, we can equivalently parametrize the solution by \(u(0)\). By radial symmetry, \(u\) satisfies the ODE

\[(3.13)\]
\[u'' + \frac{2}{|x|} u' + 4\pi G(u) = 0,\]

where \(\cdot\) denotes the radial derivative. We easily see that it has a unique solution \(u(r; a)\) satisfying \(u(0; a) = G^{-1}(a), u'(0; a) = 0\), for any \(a > 0\). Such a solution must do either of the following:

(i) There exists an \(R(a) > 0\) such that \(u(x; a) > 0\) for \(|x| < R(a)\), and \(u(R(a); a) = 0\).

(ii) \(u(x; a) > 0\) for all \(x\).

Since \(\mathcal{A}\) is the set of center values of compactly supported solutions \(\rho\), it follows that \(a \in \mathcal{A}\) if and only if case (i) occurs for \(u(x; a)\). For any such \(a\), by \((3.11)\), we must have \(u'(R(a); a) < 0\). By the implicit function theorem, for every \(b\) in a small neighborhood of \(a\), there exists \(R(b)\) close to \(R(a)\) such that \(u(R(b); b) = 0\). This shows that \(\mathcal{A}\) is open.
Denote the total mass as
\[
 M(a) = \int_{\mathbb{R}^3} \rho(x; a) \, dx.
\]
It is easy to see that \( M \in C^1(A) \). Choosing any \( a_0 \in A \), we denote \( \rho_0(x) = \rho(x; a_0) \), \( a_0 = \alpha(a_0) \), \( M_0 = M(a_0) \). In the following discussion we will construct a solution set emanating from \( (\rho, \alpha, \kappa) = (\rho_0, \alpha_0, 0) \). To that end we replace (2.10) by the following more precise conditions on the mass function \( M(a) \):
\[
 (3.15) \quad M'(a_0) \neq 0,
\]
\[
 (3.16) \quad M(a) \neq M(a_0) \quad \text{for all} \quad a \in A, \quad a \neq a_0.
\]
Note that (3.15) and (3.16) are actually a little weaker than (2.10). We essentially only need the total mass of the other non-rotating solutions to differ from that of \( \rho_0 \). For instance, if we are in the simple scenario where \( A \) consists of a single interval, and \( M' \neq 0 \) on \( A \), then (3.15) and (3.16) are satisfied for any choice of \( a_0 \). As was alluded to earlier, the mass conditions (3.15) and (3.16) are not directly verifiable on the structure function \( \phi \). We provide the following theorem supplying some sufficient conditions on \( \phi \) under which (3.15) and (3.16) are satisfied.

**Proposition 3.1.** We have both \( A = (0, \infty) \) and \( M' \neq 0 \) on \( A \) in both of the following cases.

(a) \( \phi(E, 0) = (E)^\nu \) for some \( \nu \in (-\frac{1}{2}, \frac{7}{2}) \), \( \nu \neq \frac{3}{2} \).

(b) \( -\frac{1}{2} \phi(E, 0) E \partial_E \phi(E, 0) \leq \frac{1}{2} \phi(E, 0) \) for all \( E < 0 \), in addition to the conditions stated in Section 3.

**Proof.** We first consider the case (a) when \( \phi(E, 0) = (E)^\nu \), then \( G(u) = C u^{3/2+\nu} \) for some constant \( C > 0 \). In this case all the non-rotating solutions are related to each other by scaling, as is shown by the explicit scaling symmetry of (3.11). Therefore \( A = \mathbb{R}^+ \) and
\[
 (3.17) \quad M(a) = C_0 a^{\frac{3}{2} + \nu}
\]
for some constant \( C_0 > 0 \). The range \( -\frac{1}{2} < \nu < \frac{7}{2} \) is required for \( G \) to be \( C^1 \) and for the radial solutions to be compactly supported. The exclusion of \( \nu = \frac{3}{2} \) is required in order that \( M' \neq 0 \).

In the case of (b), we claim that \( G(u) \), given by (3.8), satisfies
\[
 (3.18) \quad G(u) < u G'(u) \leq 2G(u).
\]
In fact, writing \( G'(u) = \sqrt{2 \pi} \int_u^0 \frac{\phi(E, 0)}{\sqrt{E + u}} \, dE \) and integrating by parts, we get
\[
 (3.19) \quad G'(u) = \frac{2 \sqrt{2 \pi}}{u} \int_u^0 \left( 3 \phi(E, 0) + 2 E \partial_E \phi(E, 0) \right) \sqrt{E + u} \, dE.
\]
Comparing (3.19) with (3.8), we see that the inequality in case (b) implies (3.18). Using (3.18), we can employ exactly the same argument as in Lemma 2.1 of [16] (see also Lemma 4.9 in [13]) with \( h^{-1} \) replaced by \( G \) to prove that \( A = \mathbb{R}^+ \) and that \( M' > 0 \) on \( A \). \( \square \)

**Examples of \( \phi \):**

(i) \( \phi(E, L) = (E)^\nu \) \( p(L) \), where \( (E)^- \) denotes the negative part of \( E \), \( \nu \in (-\frac{1}{2}, \frac{7}{2}) \), \( \nu \neq \frac{3}{2} \) and \( p \) is any positive polynomial. This follows directly from Proposition 3.1(a).

(ii) \( \phi(E, L) = [(E)^\nu + (E)^{2\nu}] \) \( p(L) \), where \( \nu_1, \nu_2 \in (-\frac{1}{2}, \frac{7}{2}) \) and \( p \) is any positive polynomial. This follows directly from Proposition 3.1(b).
\( \phi(E, L) = (A + \frac{\sin E}{E}) p(L) \) for \( E < 0 \), \( \phi = 0 \) for \( E > 0 \), \( A \) is a large constant and \( p \) is any positive polynomial. This also follows easily from Proposition 3.1(b).

(iii) For Theorem 1.2 we take the polynomial \( p \) to be quadratic.

4. Local and Global Continuation

We employ a convenient function space on which \( \mathcal{F} \) is well defined. Let

\[
(4.1) \quad X = \{ f : \mathbb{R}^3 \to \mathbb{R} \mid f \in C(\mathbb{R}^3) \text{ is axisymmetric, even in } x_3, \| f \|_X < \infty \},
\]

where

\[
(4.2) \quad \| f \|_X = \sup_{x \in \mathbb{R}^3} \langle x \rangle^4 |f(x)|.
\]

Here \( \langle x \rangle = (1 + |x|^2)^{1/2} \). The power 4 in (4.2) could be replaced by any real number bigger than 3 without making any essential changes to the arguments below. We will show in the following discussion that \( \mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2) \) defined by (2.1), (2.3) is a \( C^1 \) mapping from \( X \times \mathbb{R} \times \mathbb{R} \) to \( X \times \mathbb{R} \) and study its zero set. For \( N \in \mathbb{N} \), let

\[
(4.3) \quad \mathcal{O}_N = \left\{ (\rho, \alpha, \kappa) \in X \times \mathbb{R}^2 \mid \alpha < -\frac{1}{N} \right\},
\]

\[
(4.4) \quad \mathcal{O}_\infty = \bigcup_{N=1}^{\infty} \mathcal{O}_N = \{ (\rho, \alpha, \kappa) \in X \times \mathbb{R}^2 \mid \alpha < 0 \}.
\]

By the fact that \( w(\kappa, r, u) = 0 \) for \( u \leq 0 \) in Lemma 3.1, we have a simple but important compact support lemma.

**Lemma 4.1.** For all \( (\rho, \alpha, \kappa) \in \mathcal{O}_N \), \( w(\kappa, r, \frac{1}{|\cdot|} * \rho(x) + \alpha) \in C(\mathbb{R}^3) \) and is supported in the ball \(|x| \leq C_0 N \| \rho \|_X \) for some uniform constant \( C_0 \).

**Proof.** Continuity of \( w \) follows from the regularity given in Lemma 3.1. To get the support estimate, note that

\[
\left| \frac{1}{|\cdot|} * \rho(x) \right| \leq \| \rho \|_X \int_{|y| > |x|/2} \frac{1}{|y||x-y|^3} \, dy.
\]

\[
\leq \| \rho \|_X \left( \int_{|y| > |x|/2} \frac{2}{|x|^2 (|x-y|^2)} \, dy + \int_{|y| < |x|/2} \frac{1}{|y||x/2|^3} \, dy \right)
\]

\[
\leq C_0 \| \rho \|_X |x|.
\]

So if \(|x| > C_0 N \| \rho \|_X \), \( \left| \frac{1}{|\cdot|} * \rho(x) \right| < \frac{1}{N} \). By the definition of \( \mathcal{O}_N \), we have \( \alpha < -\frac{1}{N} \). The result now follows from the fact that \( w(\kappa, r, u) = 0 \) when \( u < 0 \), as stated in Lemma 3.1. \( \square \)

**Lemma 4.2.** The basic operator \( \mathcal{F} : \mathcal{O}_\infty \to X \times \mathbb{R} \) is \( C^1 \) Fréchet differentiable. Its Fréchet derivative is

\[
(4.5) \quad \frac{\partial \mathcal{F}}{\partial (\rho, \alpha, \kappa)}(\delta \rho, \delta \alpha, \delta \kappa) = \left( \delta \rho - \mathcal{L}(\delta \rho, \delta \alpha, \delta \kappa), \int_{\mathbb{R}^3} \delta \rho(x) \, dx \right),
\]

with

\[
\mathcal{L}(\delta \rho, \delta \alpha, \delta \kappa) = \partial_\kappa w \left( \kappa, r, \frac{1}{|\cdot|} * \rho(x) + \alpha \right) \cdot \delta \kappa
\]

\[
+ \partial_\alpha w \left( \kappa, r, \frac{1}{|\cdot|} * \rho(x) + \alpha \right) \cdot \left[ \frac{1}{|\cdot|} * \delta \rho(x) + \delta \alpha \right].
\]
Proof. We may fix any $N$ and restrict $F$ to the subset $O_N$. Its first component $F_1$ obviously maps $O_N$ into $X$ because $w(\kappa, r, \frac{1}{r} \ast \rho + \alpha)$ is continuous and compactly supported on a ball of radius $C_0 N \| \rho \|_X$ by Lemma 4.1. For $\rho$ in a bounded set in $X$, $w(\kappa, r, \frac{1}{r} \ast \rho + \alpha)$ is supported in a ball of fixed radius $R$. Since $\| f \|_X \leq (R)^4 \| f \|_{C(\overline{BR})}$ for $f$ is supported in $\overline{BR}$, it suffices to show Fréchet differentiability of the mapping into $C(\overline{BR}) \times \mathbb{R}$. This is a simple consequence of the $C^1$ regularity of $w$. \hfill $\square$

Our first goal is to find solutions to $F(\rho, \alpha, \kappa) = 0$ that are close to $(\rho_0, \alpha_0, 0)$.

**Lemma 4.3.** Given $a_0 \in A$, let $(\rho_0, \alpha_0) = (\rho(\cdot, a_0), \alpha(a_0))$ as in Lemma 3.3. Then the operator $L_0 := \frac{\partial F}{\partial \rho_0}(\rho_0, \alpha_0, 0) : X \times \mathbb{R} \to X \times \mathbb{R}$ is an isomorphism provided that (3.15) is satisfied.

Proof. It is easily seen that $L_0$ is a compact perturbation of the identity, and it is thus a Fredholm operator of index 0. By Lemma 3.3 $M_0$ in (2.5) is equal to $M(a_0)$. Thus it suffices to show that $L_0$ is injective. By definition, $G(u) = w(0, 0, u).$ By (4.6),

$$\delta \rho(x) - G' \left( \frac{1}{r} \ast \rho_0(x) + \alpha_0 \right) \cdot \left[ \frac{1}{r} \ast \delta \rho(x) + \delta \alpha \right] = 0,$$

(4.7)

$$\int_{\mathbb{R}^3} \delta \rho(x) \, dx = 0.$$  

We have to show that $\delta \rho = 0$, $\delta \alpha = 0$. This can be proven in exactly the same way as Lemma 4.3 of [11], with $h^{-1}$ replaced by $G$. Indeed, let $u_0 = \frac{1}{r} \ast \rho + \alpha$ and $w = \frac{1}{r} \ast \rho + \alpha$. We show that $\Delta w = -4\pi \hat{\rho} \hat{u}_0 \hat{w}$ in the ball $B_0$ of radius $R(u_0)$, and $w = 0$ outside $B_0$. In [11] it is proven first that $w$ is radial and then by a delicate argument that $w \equiv 0$. Thus $\delta \rho \equiv 0$ and $\delta \alpha = 0$. \hfill $\square$

Lemma 4.3 and the implicit function theorem imply the existence of a local curve of solutions near $(\rho_0, \alpha_0, 0)$. However, to continue this solution curve globally, we use a global implicit function theorem that relies on the Leray-Schauder degree. To that end, we need

**Lemma 4.4.** The nonlinear operator $N : (\rho(x), \alpha, \kappa) \mapsto w(\kappa, r, \frac{1}{r} \ast \rho(x) + \alpha)$ is a compact map from $O_\infty$ to $X$.

Proof. Call this nonlinear operator $N$. By definition (see [11]), we only have to show that $N(K)$ is precompact for every closed bounded set $K \subset O_\infty$. Every such $K$ is contained in some $O_N$ for a finite $N$. So by Lemma 4.1 the functions in $N(K)$ are supported on some fixed ball $B_R$. One can again dominate the $X$ norm by the $C(\overline{BR})$ norm, and compactness is now obvious by Ascoli-Arzelà. \hfill $\square$

We now state the global implicit function theorem to be applied to this problem.

**Theorem 4.1.** Let $X$ be a Banach space and let $U$ be an open subset of $X \times \mathbb{R}$. Let $F : U \to X$ be a Fréchet $C^1$ mapping. Let $(\xi_0, \kappa_0) \in U$ such that $F(\xi_0, \kappa_0) = 0$. Assume that the linear operator $\frac{\partial F}{\partial \xi}(\xi_0, \kappa_0)$ is an isomorphism on $X$. Assume that the mapping $(\xi, \kappa) \mapsto F(\xi, \kappa) - \xi$ is compact from $U$ to $X$. Let $\mathcal{S}$ be the closure in $X \times \mathbb{R}$ of the solution set $\{ (\xi, \kappa) \mid F(\xi, \kappa) = 0 \}$. Let $K$ be the connected component of $\mathcal{S}$ to which $(\xi_0, \kappa_0)$ belongs. Then one of the following three alternatives is valid.

(i) $K$ is unbounded in $X \times \mathbb{R}$.
(ii) $K \setminus \{ (\xi_0, \kappa_0) \}$ is connected.
(iii) $K \cap \partial U \neq \emptyset$.

Proof. This is a standard theorem basically due to Rabinowitz. Theorem 3.2 in [11] in the case that $U = X \times \mathbb{R}$ and under some extra structural assumption. A more general version
also appears in Theorem II.6.1 of [9]; its proof is easy to generalize to permit a general open set \( U \). The case of a general open set \( U \) also appears explicitly in [1].

We apply Theorem 4.1 to \( \mathcal{F} \) on \( \mathcal{O}_\infty \) to obtain a preliminary form of Theorem 4.1.

**Lemma 4.5.** Assuming the mass condition (4.10), there is a connected set \( \mathcal{K} \) of solutions to \( \mathcal{F}(\rho, \alpha, \kappa) = 0 \) which contains \((\rho_0, \alpha_0, 0)\) such that at least one of the following three alternatives is true:

(a) \( \sup_{(\rho, \alpha, \kappa) \in \mathcal{K}} \| \rho \|_{\infty} = \infty \).

(b) \( \sup_{(\rho, \alpha, \kappa) \in \mathcal{K}} \text{diam}(\text{supp } \rho) = \infty \).

(c) \( \sup_{(\rho, \alpha, \kappa) \in \mathcal{K}} |\alpha| = \infty \), and there are two negative constants \( a < b < 0 \) such that \( a \leq \alpha \leq b \) for all \((\rho, \alpha, \kappa) \in \mathcal{K} \).

**Proof.** We apply Theorem 4.1 to \( F = \mathcal{F} \), \( X = Z = X \times \mathbb{R} \), \( U = \mathcal{O}_\infty \), and \( \xi = (\rho, \alpha) \), with starting point \((\xi_0, \kappa_0) = (\rho_0, \alpha_0, 0)\). Thus there exists a solution set \( \mathcal{K}_N \) such that at least one of the three alternatives in Theorem 4.1 holds. We claim that alternative (ii) in Theorem 4.1 cannot happen. Indeed, let \( \mathcal{K}_N \setminus \{(\rho_0, \alpha_0, 0)\} \) be connected. (This means that \( \mathcal{K}_N \) contains a “loop”.) Since the projection onto the \( \kappa \)-axis is continuous and \( \mathcal{K}_N \) obviously must contain solutions with \( \kappa \) positive and negative, \( \mathcal{K}_N \setminus \{(\rho_0, \alpha_0, 0)\} \) must also contain a solution of the form \((\rho_1, \alpha_1, 0)\), where \( \rho_1 \) is compactly supported by Lemma 4.1 and \( \mathcal{F}_1(\rho_1, \alpha_1, 0) = 0 \). Lemma 4.3 implies that \( \rho_1 \) must be one of the radial solutions supported on a compact ball. Since \( \rho_1 \) has the same total mass as \( \rho_0 \) by \( \mathcal{F}_2(\rho_1, \alpha_1, 0) = 0 \), the mass condition (4.10) implies that \( \rho_1(0) = \rho_0(0) \). Lemma 3.3 implies that \((\rho_1, \alpha_1) = (\rho_0, \alpha_0)\). This is a contradiction. Thus, only cases (i) or (iii) in Theorem 4.1 can happen.

In other words, either

\[
\sup_{(\rho, \alpha, \kappa) \in \mathcal{K}_N} (\| \rho \|_X + |\kappa| + |\alpha|) = \infty,
\]

or

\[
\sup_{(\rho, \alpha, \kappa) \in \mathcal{K}_N} \alpha = -\frac{1}{N}.
\]

Now the set \( \mathcal{K} = \bigcup_{N=1}^\infty \mathcal{K}_N \) is connected because each \( \mathcal{K}_N \)'s is connected and they are nested. For \( \mathcal{K} \) we obviously have either

\[
(4.9) \quad \sup_{(\rho, \alpha, \kappa) \in \mathcal{K}} (\| \rho \|_X + |\kappa| + |\alpha|) = \infty,
\]

or

\[
(4.10) \quad \sup_{(\rho, \alpha, \kappa) \in \mathcal{K}} \alpha = 0.
\]

Suppose now that neither (a) nor (b) in the statement of the lemma occurs. Then there exists a constant \( H > 0 \) such that for all \((\rho, \alpha, \kappa) \in \mathcal{K} \), both \( \| \rho \|_{\infty} \leq H \) and \( \text{supp } \rho \) is contained in the ball centered at the origin with radius \( H \). We claim that (4.10) cannot occur. In fact, for any \( \bar{x} \) with \( |x| = 2H \), we have the lower bound

\[
(4.11) \quad \frac{1}{|x|} \rho(\bar{x}) \geq \frac{1}{|x| + H} \int_{|y| \leq C} \rho(y) \, dy = \frac{M_0}{3H}.
\]

So if there were to exist a point \((\rho, \alpha, \kappa) \in \mathcal{K} \) with \( \alpha > -\frac{M_0}{3H} \), we would have \( \frac{1}{|x|} \rho(\bar{x}) + \alpha > 0 \) and

\[
\rho(\bar{x}) = w(\kappa, r(\bar{x}), \frac{1}{|x|} \rho(\bar{x}) + \alpha) > 0.
\]

This contradicts the assumption that \( \rho \) is supported in \( |x| < H \).

Thus (4.10) must occur, and \( b := \sup_{(\rho, \alpha, \kappa) \in \mathcal{K}} \alpha < 0 \). On the other hand, for \(|x| \leq H \), we have the upper bound

\[
(4.12) \quad 0 \leq \frac{1}{|x|} \rho(x) \leq \frac{H}{|x| H} \int_{|y| \leq H} \frac{H}{|x - y|} \, dy \leq H \int_{|y| \leq 2H} \frac{1}{|y|} \, dy = 8\pi H^3.
\]
If $\alpha < -8\pi H^3$, then $\rho(x) = 0$ for all $|x| \leq H$, which implies that $\rho$ is identically zero. This contradicts the mass constraint $\mathcal{F}_2(\rho, \alpha, \kappa) = 0$. Thus we can define $a = -8\pi H^3$ and we have $a \leq \alpha \leq b$ for any $(\rho, \alpha, \kappa) \in \mathcal{K}$. We also obviously have $\|\rho\|_\infty \leq (H)^4 H$. So the only way for (4.9) to occur is to have

$$\sup_{(\rho, \alpha, \kappa) \in \mathcal{K}} |\kappa| = \infty.$$  

This means that (c) occurs. \hfill \Box

Any solution $(\rho, \alpha, \kappa) \in \mathcal{K}$ satisfies $\rho \in C^1$ because $\rho$ is continuous, $\frac{1}{|\cdot|} * \rho \in C^1$ and $\rho = w(\kappa, r, \frac{1}{|\cdot|} * \rho + \alpha)$.  

By Lemma 4.5, we are left with one final possibility to be eliminated in order to prove Theorem 1.1. That possibility is: $\|\rho\|_\infty$ and the support of $\rho$ are both uniformly bounded, $\alpha$ is bounded between two negative numbers, and $\kappa$ is unbounded. It turns out that a much more delicate argument is needed to eliminate this last possibility. It is provided in the next two sections.

5. ELIMINATE UNBOUNDED $\kappa$ FOR A SPECIAL EXAMPLE

To better explain the essential ideas of the argument, we prove Theorem 1.1 in this section for the special example of $\phi$ given as

$$\phi(E, L) = (-E)^{\frac{3}{2}}(C_1 + C_2 L^2)$$

for suitable constants $C_1, C_2 > 0$. The proof for the general case will be given in the next section. By a direct calculation, we get for $u > 0$:

$$w(\kappa, r, u) = 2\pi \int_0^1 \int_0^{\sqrt{2(E + u)}} (-E)^{\frac{3}{2}}[C_1 + C_2(\kappa r^2)] ds dE$$

$$= 4\pi C_1 u^2 \int_1^0 [2(E + 1)(-E)]^{\frac{3}{2}} dE + 4\pi^3 C_2 r^2 u^3 \int_0^1 [8(E + 1)^3(-E)]^{\frac{3}{2}} dE$$

$$= u^2 + \kappa^2 r^2 u^3.$$  

Here we have chosen $C_1$ and $C_2$ to make the coefficients equal to 1. Thus we may write $\mathcal{F}_1(\rho, \alpha, \kappa) = 0$ as

$$\rho = \kappa^2 r^2 \left(\frac{1}{|\cdot|} * \rho + \alpha\right)^3 + \left(\frac{1}{|\cdot|} * \rho + \alpha\right)^2$$

$$= \kappa^2 r^2 u^3 + u^2.$$  

Under the above setup, we want to prove Theorem 1.1 as a consequence of the following lemmas. Theorem 1.1 means that either case (a) or case (b) of Lemma 4.5 must occur. If neither case (a) nor case (b) occurs, we get from Lemma 4.5 that case (c) must occur, and there is a constant $R > 0$ such that for all $(\rho, \alpha, \kappa) \in \mathcal{K}$, we have $\|\rho\|_\infty < R$, and supp $\rho$ is contained in $B_R$, the ball of radius $R$ centered at the origin. Denote $u = \frac{1}{|\cdot|} * \rho + \alpha$. From (5.3), we see that $u > 0$ in the region $\Omega = \{\rho > 0\}$ occupied by the star, while $u < 0$ outside $\Omega$. First we show that $u$ must be small on $\Omega$ if $\kappa$ is large. In fact, we have

**Lemma 5.1.** There exists a $C > 0$ such that $\|u\|_\infty, \Omega \leq C\kappa^{-2/5}$ for $|\kappa| > 1$.

**Proof.** $u > 0$ on $\Omega$ so $u = u_+$ there. Because of the product $\kappa^2 r^2$ in (5.3), we have to treat small $r$ separately. Let $r_0 > 0$ be a radius to be chosen sufficiently small later on. For $x$ such that $r(x) \geq r_0$, we use (5.3) to get

$$u_+(x) \leq 3\frac{\|\rho\|_\infty}{\kappa^2 r_0^2}.$$
For \( x \) such that \( r(x) \leq r_0 \), we use the simple potential estimate \( \|Du\|_\infty \leq C\|\rho\|_\infty \) and integrate it radially from \( r(x) \) to \( r_0 \). Thereby we obtain
\[
\tag{5.5} u_+(x) \leq \sqrt{\frac{\|\rho\|_\infty}{r_0}} + C\|\rho\|_\infty r_0
\]
for all \( x \in \mathbb{R}^3 \). We now choose \( r_0 = r_0(\kappa) \) so that the two terms above are comparable. As a result we get
\[
\tag{5.6} \|u\|_{\infty;\Omega} = \|u_+\|_{\infty;\mathbb{R}^3} \leq C\kappa^{-2/5}
\]
with \( C \) independent of \( \kappa \).

Note that we cannot obtain a smallness estimate for \( u \) outside \( \Omega \), because \( \|\rho\|_\infty \) loses control of \( u \) when \( u < 0 \). While \( u \) is close to \( \alpha \) near infinity, \( \alpha \) is bounded but not necessarily small. Our second step is to control \( D^2u \) by a low power of \( \kappa \).

**Lemma 5.2.** Let \( B_R \) be the common ball containing the fluid domain \( \Omega \) given above. For every \( \tau \in (0, 1) \), there exists a constant \( C_{\tau} > 0 \) such that
\[
\|D^2u\|_{\infty;B_R} \leq C_{\tau}\kappa^{2\tau}. \tag{5.7}
\]

**Proof.** Indeed, we differentiate \( \|\rho\|_\infty \) to get
\[
\tag{5.8} D\rho = \kappa^2 D^2u_+^3 + \kappa^2 \tau^2 3u_+^2 Du + 2u_+Du,
\]
so that
\[
\tag{5.9} \|D\rho\|_{\infty;B_R} \leq C\kappa^2 (1 + \|Du\|_{\infty;B_R}) \leq C\kappa^2 (1 + \|\rho\|_{\infty;\mathbb{R}^3}) \leq C\kappa^2.
\]
For any \( \tau \in (0, 1) \) we have the interpolation inequality
\[
\tag{5.10} \|\rho\|_{C^{0, \tau}(B_R)} \leq C_{\tau}\|\rho\|^{1-\tau}_{\infty;B_R} \|D\rho\|^{\tau}_{\infty;B_R}.
\]
From the previous two estimates we get
\[
\tag{5.11} \|\rho\|_{C^{0, \tau}(B_R)} \leq C_{\tau}\kappa^{2\tau}.
\]
Now by standard potential estimates we get
\[
\tag{5.12} \|D^2u\|_{\infty;B_R} \leq C_{\tau}\|\rho\|_{C^{0, \tau}(B_R)} \leq C_{\tau}\kappa^{2\tau}.
\]

Our third step is to use an elementary Gagliardo-Nirenberg (GN) type of inequality in the fluid domain \( \Omega \) in order to obtain smallness of \( Du \) there. This kind of inequality is well-known, but the particular form we are using here is more difficult to find. We include the proof here for completeness.

**Lemma 5.3.** Let \( \Omega \) be a bounded open set in \( \mathbb{R}^n \). Let \( u \in C(\overline{\Omega}) \cap C^2(\Omega) \) such that \( u = 0 \) on \( \partial\Omega \). Then there exists a constant \( C \) independent of \( u \) and \( \Omega \) such that
\[
\tag{5.13} \|Du\|_{\infty;\Omega} \leq C\|u\|^{1/2}_{\infty;\Omega} \|D^2u\|^{1/2}_{\infty;\Omega}.
\]

**Proof.** The key point here is that the constant \( C \) is uniform and independent of the domain \( \Omega \), no matter how rough its boundary might be. The idea is to integrate the second derivative from a nearby point, and thereby estimate \( Du \) at a nearby point by a difference quotient. It is crucial that \( u \) vanishes on \( \partial\Omega \) in order to handle an extremely close difference quotient.

Without loss of generality, we may assume \( \|D^2u\|_{\infty;\Omega} \neq 0 \), since if it does, then \( u \) is a linear function on each connected component of \( \Omega \), hence must be zero identically by the boundary condition. Define \( d^2 = \|u\|_{\infty;\Omega}/\|D^2u\|_{\infty;\Omega} \). Take any point \( x \in \Omega \) and any direction \( e_i \). In case the line segment \( (x, x + de_i) \subset \Omega \), then there is a value \( \xi \in (0, d) \) so that \( u(x) = d \cdot \cdot \cdot u(x + \xi e_i) \). Hence \( |\partial u(x + \xi e_i)| \leq (2/d)|u|_{\infty;\Omega} \). Also, \( |\partial u(x) - \partial u(x + \xi e_i)| = |\int_{\xi}^{1} \partial^2 u(x + te_i) dt| \leq d\|D^2u\|_{\infty;\Omega} \). Combining the last two inequalities,
we have \( |\partial_t u(x)| \leq \frac{2}{d} \|u\|_{\infty,\Omega} + d \|D^2 u\|_{\infty,\Omega} = 3\sqrt{\|u\|_{\infty,\Omega} \|D^2 u\|_{\infty,\Omega}} \). The last equality is due to the definition of \( d \). The same argument is valid in case the line segment \((x - de_i, x) \subset \Omega\).

The other possibility is that \( x \) and \( i \) are such that there is no such line segment. In that case consider the line segment \([x - ae_i, x + be_i]\) with both \( a \leq d \) and \( b \leq d \) such that both endpoints lie on \( \partial \Omega \). Since \( u \) vanishes at both endpoints, there is an intermediate value \( \xi \) such that \( \partial_t u(x + \xi e_i) = 0 \). Thus \( |\partial_t u(x)| = |\int_0^\xi \partial_i^2 u(x + te_i) dt| \leq d \|D^2 u\|_{\infty,\Omega} = \sqrt{\|u\|_{\infty,\Omega} \|D^2 u\|_{\infty,\Omega}} \).

Summing on \( i \in \{1, 2, 3\} \), we obtain (5.13).

We can now combine the previous inequalities to get a smallness estimate for \( D u \) on \( \Omega \). Into the GN inequality (5.13) we substitute the bound (5.6) on \( u \) and the bound (6.7) on \( D^2 u \) to obtain

\[
\|D u\|_{\infty,\Omega} \leq C \|\kappa\|^{-1/5} \|\kappa\|^{7/10} \leq C \|\kappa\|^{-1/5}
\]

for large \( \kappa \), where we have chosen \( \tau = 1/10 \).

If \( \Omega \) is sufficiently smooth, we can directly integrate \( D u \) on \( \partial \Omega \) and obtain a smallness estimate for \( \|\rho\|_1 \). However, since \( \partial \Omega \) could potentially be very rough or have large measure, we need to take a final step to extend the smallness estimate of \( D u \) to the whole space. Fortunately this is easy to do, due to the harmonicity of \( D u \) outside \( \Omega \). In fact \( \Delta u = -4\pi \rho = 0 \), so \( u \) and as consequence \( D u \) are harmonic outside \( \Omega \). Also \( \lim_{x \to \infty} D u = 0 \). By the maximum principle,

\[
\|D u\|_{\infty,\Omega} \leq C \|\kappa\|^{-1/5} \leq C \|\kappa\|^{-1/5}.
\]

Now we integrate \( \rho = -\frac{1}{4\pi} \Delta u \) on the ball \( B_\delta \) to get

\[
M_0 = \int_{B_\delta} \rho \, dx = -\frac{1}{4\pi} \int_{\partial B_\delta} \nabla u \cdot n \, d\sigma \leq C \|\kappa\|^{-1/5}.
\]

Letting \( |\kappa| \to \infty \), we find \( M_0 = 0 \), which is a contradiction. This finishes the proof of Theorem 1.1.

6. Eliminate unbounded \( \kappa \) for general \( \phi \)

We impose general assumptions on \( \phi(E, L) \) that are given in Section 2 and then repeat the preceding proof. We now turn to the general case. Recall that \( \phi \) satisfies (2.11) and (2.12). We will proceed as in Section 5. The next lemma generalizes (5.4).

**Lemma 6.1.** For every \( H > 0 \), there exist \( C > 0, K > 0 \), such that for any triple \( (\kappa, r, u) \) satisfying

\[
|w(\kappa, r, u)| \leq H, \quad u > 0, \quad |\kappa r| > K,
\]

we have

\[
u \leq C \|\kappa r\|^{-\frac{1}{\Gamma + 3}}.
\]

Here \( \Gamma \) and \( \delta \) are the numbers given in (2.11).

**Proof.** By (2.11), there exist \( C > 0, A > 0, \Gamma > 0, \delta > 0 \) such that for \(-A < E < 0, |L| > 1/A\) we have the lower bound

\[
\phi(E, L) \geq C |E|^\Gamma |L|^{\Gamma},
\]

For simplicity of notation in this proof, we take \( A = 1 \). Note that in the following integrations, we can only use (6.3) when \(-1 < E < 0, |\kappa r s| > 1, \) or \(|s| > \frac{1}{|\kappa r|} \).
We claim that \( u \leq 1 \) if (6.1) holds. Indeed, in case \( u > 1 \), we have
\[
0 > \frac{1 - \frac{1}{\sqrt{2}(E + u)}}{\sqrt{2}(E + u)} + \frac{1}{2} \geq \frac{1 - \frac{1}{\sqrt{2}(E + u)}}{\sqrt{2}(E + u)} + \frac{1}{2} \geq 0.
\]
We pick \( K > 2 \) so large that \( \frac{1}{|\kappa r|} < \frac{1}{K} < \frac{1}{2} \). It follows that
\[
w(\kappa, r, u) \geq C \int_{-\frac{1}{2}}^{\frac{1}{2}} |E|^{\frac{\delta}{2}} s \, ds \, dE
\]
We also pick \( K > (\frac{H}{C})^{\frac{1}{2}} \), so that the assumption \( |\kappa r| > K \) and the preceding inequality imply \( w(\kappa, r, u) > H \). Since this contradicts the first inequality in (6.1), it follows that \( 0 < u < 1 \).

In case \( 1 |\kappa r| > \frac{\sqrt{2} \sqrt{2(E + u)}}{2 + 4u - 2\sqrt{(E + u)^2}} \), as desired, because \( \frac{2 + 4u - 2\sqrt{(E + u)^2}}{2 + 4u - 2\sqrt{(E + u)^2}} < 2 \). On the other hand, in case \( \frac{1}{|\kappa r|} \leq \frac{1}{2} \), we have
\[
w(\kappa, r, u) \geq C \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{2(E + u)} \phi(E, \kappa rs) \, ds \, dE
\]
Thus the hypothesis \( w(\kappa, r, u) \leq H \) implies
\[
u(\kappa, r, u) \leq \left( \frac{H}{C|\kappa r|^\Gamma} \right)^{\frac{1}{\pi + 2\delta}} \leq C|\kappa r|^{-\frac{\nu}{\pi + 2\delta}}.
\]
Since the inner integration of the definition of \( w \) always traverses the symmetric interval \(( -\sqrt{2(E + u)}, \sqrt{2(E + u)} \)) , the preceding proof can be modified to work by assuming the limit inequality (2.11) only as \( L \to \infty \) or only as \( L \to -\infty \).

The next lemma will be used to replace the simple powers in (5.3).

Lemma 6.2. For every \( B > 0 \), there exists \( C > 0 \) such that for \( |r| \leq B, \, |u| \leq B \) and \( |\kappa| > 1 \), we have the upper bounds
\[
|\partial_r w(\kappa, r, u)| \leq C|\kappa|^{1 + \Lambda}.
\]
and
\[
|\partial_u w(\kappa, r, u)| \leq C|\kappa|^\Lambda.
\]
Here \( \Lambda \) is the number given in (2.12).
By interpolation (5.10), we obtain for \( \tau \)
\[
\partial_t \phi(E, \kappa r s \sigma) s \ ds \ dE.
\]

By assumption (2.12) and \( u > 0 \), the \( r \)-derivative has the bound
\[
|\partial_r w(\kappa, r, u)| \leq C|\kappa| \int \int \sqrt{2(\kappa + u)} |E|^{-\mu} (1 + |\kappa r| |s|)^{|s|} \ ds \ dE
\]
\[
\leq C|\kappa| \int \int \sqrt{2(\kappa + u)} |E|^{-\mu} (1 + |\kappa r| |s|)^{(|s|/2)|s|} \ ds \ dE
\]
\[
\leq C|\kappa| \int \int \sqrt{2(\kappa + u)} |E|^{-\mu} (1 + F)^{-1/2} \ dF + C|\kappa|^{1+2\mu} \int \int \sqrt{2(\kappa + u)} |E|^{-\mu} (1 + F)^{-1/2} \ dF
\]
\[
\leq C(1+|\kappa|^{1+\Lambda})
\]
for \(|u| < B, r < B\), because \( \mu < \frac{1}{2} \). By assumption (2.12), the \( u \)-derivative has the bound
\[
|\partial_u w(\kappa, r, u)| \leq C \int \int \sqrt{2(\kappa + u)} |E|^{-\mu} (1 + |\kappa r| |s|)^{|s|} \ ds \ dE
\]
\[
\leq C|\kappa| \int \int \sqrt{2(\kappa + u)} |E|^{-\mu} (1 + F)^{-1/2} \ dF + C|\kappa|^{1+2\mu} \int \int \sqrt{2(\kappa + u)} |E|^{-\mu} (1 + F)^{-1/2} \ dF
\]
\[
\leq C(1+|\kappa|^{1+\Lambda})
\]
for \(|u| < B, r < B\), because \( \mu < \frac{1}{2} \).

Using the above two lemmas, we can complete the proof as in the special case of Section 5. Indeed, we have \( \rho = w(\kappa, r, u), u = \frac{1}{\rho} * \rho + \alpha \), and \( \int \rho \ dx = M_0 \). Let there be a constant \( R > 0 \) such that \( \rho \) is supported in \( B_R \), and \( \|\rho\|_\infty \leq R, -R < \alpha < -\frac{1}{R} \). We want to show that \( \kappa \) must be bounded. On the contrary, suppose \( sup_{(\rho, r, \kappa) \in \mathcal{K}} |\kappa| = \infty \).

Let \( \Omega = \{ u > 0 \} \) as before. First, we use the standard potential estimates to get \( \|Du\|_\infty \leq C\|\rho\|_\infty \leq C \). Define
\[
r_0 = r_0(\kappa) = |\kappa|^{-\frac{2\mu}{3+3\Gamma+2\mu}}.
\]

If \( r(x) > r_0 \), then \( |\kappa r| > |\kappa|^{-\frac{2\mu}{3+3\Gamma+2\mu}} \). Apply Lemma 4.1 with \( H = \|\rho\|_\infty \). Since \( w(\kappa, r, u) = \rho < H \), there are constants \( C \) and \( K \) such that if \( |\kappa|^{-\frac{2\mu}{3+3\Gamma+2\mu}} > K \),
\[
u(x) \leq C|\kappa r|^{-\frac{2\mu}{3+3\Gamma+2\mu}} \leq C|\kappa|^{-\frac{2\mu}{3+3\Gamma+2\mu}}.
\]
If \( r(x) \leq r_0 \), we integrate from the cylinder of radius \( r_0 \) and use the bound \( \|Du\|_\infty \leq C \) to get
\[
\|u\|_{\infty, \Omega} \leq C|\kappa|^{-\frac{2\mu}{3+3\Gamma+2\mu}} + Cr_0 \leq C|\kappa|^{-\frac{2\mu}{3+3\Gamma+2\mu}}.
\]
This generalizes Lemma 5.1.

Secondly, we differentiate the equation \( \rho = w(\kappa, r, u) \) to get
\[
\partial_t \rho = \partial_t w(\kappa, r, u) D \rho + \partial_u w(\kappa, r, u) Du.
\]
Using Lemma 6.2 and the bound \( \|Du\|_\infty \leq C \) we get
\[
\|D\rho\|_{\infty, B_R} \leq C|\kappa|^{1+\Lambda}.
\]
By interpolation (5.10), we obtain for \( \tau \in (0, 1) \)
\[
\|\rho\|_{C^{\alpha, \tau}(B_R)} \leq C_2 \|\rho\|_{\infty}^{1-\tau} \|D\rho\|_{\infty, B_R}^{\tau}.
\]
and the bound on \(\|\rho\|_{\infty}\),

\[
|\rho|_{C^{0,\gamma}(B_R)} \leq C_{r} |\kappa|^{\gamma (1+\Lambda)}.
\]

By standard potential estimates on \(B_R\),

\[
\|D^2 u\|_{\infty, B_R} \leq C_{r} |\kappa|^{\gamma (1+\Lambda)}.
\]

Thirdly, using the GN inequality (5.13) on \(\Omega\) exactly as before together with the bounds (6.11) and (6.17), we obtain for \(|\kappa|\) sufficiently large

\[
\|D u\|_{\infty, \Omega} \leq C_{r} |\kappa|^{\gamma (1+\Lambda) - \frac{3}{3+3\Gamma+2}}.
\]

Finally, \(Du\) is harmonic outside \(\Omega\) and vanishes at infinity, so that

\[
\|D u\|_{\infty, \mathbb{R}^3} \leq C_{r} |\kappa|^{\gamma (1+\Lambda) - \frac{3}{3+3\Gamma+2}}.
\]

We choose \(\tau\) small enough so that the last exponent is negative. By integrating \(Du\) over \(\partial B_R\) and letting \(|\kappa| \to \infty\), we deduce that the mass \(M_0\) vanishes, which is a contradiction. This finishes the proof of Theorem 1.1. \(\square\)

### 7. Eliminate unbounded \(\rho\) for some \(\phi\)

The purpose of this short section is to prove Theorem 1.2. For this theorem we do not require hypothesis (2.11), but must strengthen (2.12) to require \(\Lambda < 4\). We first estimate the growth rate of \(w\) with respect to \(u\).

**Lemma 7.1.** For every \(B > 0\), there exists a constant \(C\) such that

\[
|w(\kappa, r, u)| \leq C (1 + u_{+}^{-1 + \Lambda/2})
\]

for all \(|\kappa| < B, r < B\).

**Proof.** Recalling the definition of \(w\) and using (2.12), we have for \(u > 0\)

\[
|w(\kappa, r, u)| \leq C \int_{-u}^{0} \int_{-\sqrt{2(E+u)}}^{\sqrt{2(E+u)}} |\phi(E, \kappa r s)| \, ds \, dE.
\]

\[
\leq C \int_{-u}^{0} \int_{-\sqrt{2(E+u)}}^{\sqrt{2(E+u)}} |E|^{-\frac{d}{2}} (1 + |s|^{\Lambda}) \, ds \, dE.
\]

\[
\leq C \int_{-u}^{0} \left[ |E|^{-\frac{d}{2}} (E + u)^{\frac{d}{2}} + (E + u)^{\frac{\Lambda+1}{2}} \right] dE.
\]

\[
\leq C \int_{-u}^{0} (u F)^{-\frac{d}{2}} \left[ u^{\frac{d}{2}} (1 + F)^{\frac{d}{2}} + u^{\frac{d+1}{2}} (1 + F)^{\frac{\Lambda+1}{2}} \right] u \, dF.
\]

Recall that \(w = 0\) for \(u \leq 0\). \(\square\)

**Proof of Theorem 1.2.** The proof is similar to that of Theorem 7.1 in [16]. If the conclusion of Theorem 1.2 does not hold, then neither case (b) nor case (c) of Lemma 4.5 occurs. Thus case (a) must occur. A contradiction would follow from a uniform \(L^{\infty}\) bound on \(\rho\), assuming that the support of \(\rho\) is contained in a fixed ball \(B_R\), and that \(|\kappa|\) is uniformly bounded.

Recalling that \(\rho = w(\kappa, \frac{1}{\vert \vert \cdot \vert \vert} * \rho + \alpha)\) and writing \(1 + \frac{1}{q} = 3 - \epsilon\) with \(0 < \epsilon < 2\), we have by Lemma 7.1,

\[
\rho \leq C \left[ 1 + \left( \frac{1}{\vert \vert \cdot \vert \vert} * \rho + \alpha \right)_{+}^{3-\epsilon} \right] \leq C + C \left( \frac{1}{\vert \vert \cdot \vert \vert} * \rho \right)^{3-\epsilon},
\]

since \(\alpha < 0\). Now the total mass \(M_0 = \int \rho \, dx\) is fixed and we are assuming that \(\rho\) has its support contained in \(B_R\). By the endpoint Hardy-Littlewood-Sobolev inequality, \(\frac{1}{\vert \vert \cdot \vert \vert} * \rho\) is uniformly bounded in \(L^q(B_R)\) for every \(1 < q < 3\). It follows from (7.2) that \(\rho\) is uniformly


bounded in $L^{q/(3-\epsilon)}(B_R)$. This is a slight improvement of the $L^1$ bound if we choose $q$ sufficiently close to 3. Repeated use of the Hardy-Littlewood-Sobolev inequality allows us to improve an $L^p$ bound to an $L^r$ bound on $\rho$, with $\frac{1}{r} = (3-\epsilon) \left( \frac{1}{p} - \frac{2}{3} \right)$ for any $p \in (1, \frac{3}{2})$. It is not hard to see that a finite number of repetitions of such estimates leads to $\rho$ being bounded uniformly in $L^r$ for some $r > \frac{3}{2}$, at which point we conclude that $\frac{1}{|\cdot|} * \rho$ and consequently $\rho$ are uniformly bounded in $L^\infty(B_R)$. □

References

[1] Alexander, J., and Yorke, J. A. The implicit function theorem and the global methods of cohomology. Journal of Functional Analysis 21, 3 (1976), 330–339.

[2] Ambrosetti, A., and Rabinowitz, P. H. Dual variational methods in critical point theory and applications. Journal of functional Analysis 14, 4 (1973), 349–381.

[3] Andersson, H., Kunze, M., and Rein, G. Rotating, stationary, axially symmetric spacetimes with collisionless matter. Communications in Mathematical Physics 329, 2 (2014), 787–808.

[4] Binney, J., and Tremaine, S. Galactic dynamics, vol. 13. Princeton university press, 2011.

[5] De Figueiredo, D. G., Lions, P., and Nussbaum, R. A priori estimates and existence of positive solutions of semilinear elliptic equations. In Djairo G. de Figueiredo-Selected Papers. Springer, 1982, pp. 133–155.

[6] Gidas, B., Ni, W.-M., and Nirenberg, L. Symmetry and related properties via the maximum principle. Communications in Mathematical Physics 68, 3 (1979), 209–243.

[7] Guo, Y., and Lin, Z. Unstable and stable galaxy models. Communications in Mathematical Physics 279, 3 (2008), 789–813.

[8] Jang, J., and Makino, T. On slowly rotating axisymmetric solutions of the Euler–Poisson equations. Archive for Rational Mechanics and Analysis 225, 2 (2017), 873–900.

[9] Kielhofer, H. Bifurcation theory: An introduction with applications to PDEs, vol. 156. Springer Science & Business Media, 2006.

[10] Nirenberg, L. Topics in nonlinear functional analysis, vol. 6. American Mathematical Soc., 1974.

[11] Rabinowitz, P. H. Some global results for nonlinear eigenvalue problems. Journal of Functional Analysis 7, 3 (1971), 487–513.

[12] Rein, G. Collisionless kinetic equations from astrophysics—the Vlasov-Poisson system. Handbook of differential equations: evolutionary equations 3 (2007), 383–476.

[13] Rein, G., and Guo, Y. Stable models of elliptical galaxies. Monthly Notices of the Royal Astronomical Society 344, 4 (2003), 1296–1306.

[14] Rein, G., and Rendall, A. D. Compact support of spherically symmetric equilibria in non-relativistic and relativistic galactic dynamics. In Mathematical Proceedings of the Cambridge Philosophical Society (2000), vol. 128, Cambridge University Press, pp. 363–380.

[15] Strauss, W. A., and Wu, Y. Steady states of rotating stars and galaxies. SIAM Journal on Mathematical Analysis 49, 6 (2017), 4865–4914.

[16] Strauss, W. A., and Wu, Y. Rapidly rotating stars. Communications in Mathematical Physics 368, 2 (2019).