A NOTE ON A CONJECTURE OF GONEK
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Abstract. We derive a lower bound for a second moment of the reciprocal of the derivative of the Riemann zeta-function over the zeros of $\zeta(s)$ that is half the size of the conjectured value. Our result is conditional upon the assumption of the Riemann Hypothesis and the conjecture that the zeros of the zeta-function are simple.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta-function. Using a heuristic method similar to Montgomery’s study [13] of the pair-correlation of the imaginary parts of the non-trivial zeros of $\zeta(s)$, Gonek has made the following conjecture [7, 8].

Conjecture. Assume the Riemann Hypothesis and that the zeros of $\zeta(s)$ are simple. Then, as $T \to \infty$,

$$\sum_{0<\gamma\leq T} \frac{1}{|\zeta'(\rho)|^2} \sim \frac{3}{\pi^3} T$$

where the sum runs over the non-trivial zeros $\rho = \frac{1}{2} + i\gamma$ of $\zeta(s)$.

The assumption on the simplicity of the zeros of the zeta-function in the above conjecture is so that the sum over zeros on the right-hand side of (1.1) is well defined. While the details of Gonek’s method have never been published, he announced his conjecture in [5]. More recently, a different heuristic method of Hughes, Keating, and O’Connell [10] based upon modeling the Riemann zeta-function and its derivative using the characteristic polynomials of random matrices has led to the same conjecture. Through the work of Ingham [11], Titchmarsh (Chapter 14 of [21]), Odlyzko and te Riele [17], Gonek (unpublished), and Ng [15], it is known that the behavior of this and related sums are intimately connected to the distribution of the summatory function

$$M(x) = \sum_{n\leq x} \mu(n)$$

where $\mu(\cdot)$, the Möbius function, is defined by $\mu(1) = 1$, $\mu(n) = (-1)^k$ if $n$ is divisible by $k$ distinct primes, and $\mu(n) = 0$ if $n > 1$ is not square-free. See also [9] and [20] for connections between similar sums and other arithmetic problems.

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In support of his conjecture, Gonek \cite{5} has shown, assuming the Riemann Hypothesis and the simplicity of the zeros of \( \zeta(s) \), that
\[
\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(|\rho)|^2} \geq CT
\]
for some constant \( C > 0 \) and \( T \) sufficiently large. In this note, we show that the inequality in (1.2) holds for any constant \( C < \frac{3}{2\pi} \).

**Theorem.** Assume the Riemann Hypothesis and that the zeros of \( \zeta(s) \) are simple. Then, for any fixed \( \varepsilon > 0 \),
\[
\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(|\rho)|^2} \geq \left( \frac{3}{2\pi^3} - \varepsilon \right) T
\]
for \( T \) sufficiently large.

While our result differs from the conjectural lower bound by a factor of 2, any improvements in the strength of this lower bound have, thus far, eluded us. It would be interesting to investigate whether for \( k > 0 \) there is a constant \( C_k > 0 \) such that
\[
\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(|\rho)|^{2k}} \geq C_k T \left( \log T \right)^{(k-1)^2}
\]
for \( T \) sufficiently large. However, a lower bound of this form is probably not of the correct order of magnitude for all \( k \). This is because it is expected that for each \( \varepsilon > 0 \) there are infinitely many zeros \( \rho = \frac{1}{2} + i\gamma \) of \( \zeta(s) \) satisfying \( |\zeta'(\rho)|^{-1} \gg |\gamma|^{1/3-\varepsilon} \). If such a sequence were to exist, it would then follow that
\[
\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(|\rho)|^{2k}} = \Omega \left( T^{2k/3-\varepsilon} \right)
\]
and the lower bound in (1.4) would be significantly weaker than this \( \Omega \)-result when \( k > \frac{3}{2} \).

2. **Proof of Theorem**

The method we use to prove our theorem is based on a recent idea of Rudnick and Soundararajan \cite{18}. Let
\[
\xi = T^\vartheta
\]
where \( 0 < \vartheta < 1 \) is fixed and define the Dirichlet polynomial
\[
\mathcal{M}_\xi(s) = \sum_{n \leq \xi} \mu(n)n^{-s}
\]
where \( \mu \) is the Möbius function. Assuming the Riemann Hypothesis, for any non-trivial zero \( \rho = \frac{1}{2} + i\gamma \) of \( \zeta(s) \), we see that \( \mathcal{M}_\xi(\rho) = \mathcal{M}_\xi(1-\rho) \). From this observation and Cauchy’s inequality it follows that
\[
\sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(|\rho)|^2} \geq \frac{|M_1|^2}{M_2}
\]
where
\[ M_1 = \sum_{0 < \gamma \leq T} \frac{1}{\zeta'(\rho)} M_\xi(1 - \rho) \quad \text{and} \quad M_2 = \sum_{0 < \gamma \leq T} |M_\xi(\rho)|^2. \]

Our Theorem is a consequence of the following proposition.

**Proposition.** Assume the Riemann Hypothesis and let \( 0 < \vartheta < 1 \) be fixed. Then
\[ M_2 = \frac{3}{\pi^3} (\vartheta + \vartheta^2) T \log^2 T + O(T \log T). \]

If we further assume that the zeros of \( \zeta(s) \) are all simple, then there exists a sequence \( T := \{ \tau_n \}_{n=3}^\infty \) such that \( n < \tau_n \leq n + 1 \) and for \( T \in T \) we have
\[ M_1 = \frac{3\vartheta}{\pi^3} T \log T + O(T). \]

We now deduce our theorem from the above proposition.

**Proof of the Theorem.** Let \( T \geq 4 \) and choose \( \tau_n \) to satisfy \( T - 1 \leq \tau_n < T \). Combining (2.2), (2.4), and (2.3) we see that
\[ \sum_{0 < \gamma \leq T} \frac{1}{|\zeta'(\rho)|^2} \geq \sum_{0 < \gamma \leq \tau_n} \frac{1}{|\zeta'(\rho)|^2} \geq \frac{\vartheta^2}{(\vartheta + \vartheta^2)} \frac{3}{\pi^3} \tau_n + o(\tau_n) \]
\[ \geq \frac{1}{(1 + \vartheta^{-1})} \frac{3}{\pi^3} T + o(T) \]
under the assumption of the Riemann Hypothesis and the simplicity of the zeros of \( \zeta(s) \). From (2.5), our theorem follows by letting \( \vartheta \to 1^- \). □

We could have just as easily estimated the sums \( M_1 \) and \( M_2 \) using a Dirichlet polynomial \( \sum_{n \leq \xi} a_n n^{-s} \) for a large class of coefficients \( a_n \) in place of \( M_\xi(s) \). In the special case where
\[ a_n = \mu(n) P(\log \xi/n/\log \xi) \]
for polynomials \( P \), we can show that the choice \( P = 1 \) is optimal in the sense that it leads to largest lower bound in (1.3).

We prove the above proposition in the next two sections; the sum \( M_1 \) is estimated in section 3 and the sum \( M_2 \) is estimated in section 4. The evaluation of sums like \( M_1 \) dates back to Ingham’s [11] important work on \( M(x) \) in which he considered sums of the form
\[ \sum_{0 < \gamma < T} (T - \gamma)^k \zeta'(\rho)^{-1} \]
for \( k \in \mathbb{R} \). The sum \( M_2 \) is of the form
\[ \sum_{0 < \gamma < T} |A(\rho)|^2 \quad \text{where} \quad A(s) = \sum_{n \leq \xi} a_n n^{-s} \]
is a Dirichlet polynomial with \( \xi \leq T \). Such sums have played an important role in various applications. For instance, results concerning the distribution of consecutive zeros of \( \zeta(s) \) and discrete mean values of the zeta-function and its derivatives are proven in [1, 2, 4, 6, 12].
In each of these articles, the evaluation of the discrete mean (2.6) either makes use of the Guinand-Weil explicit formula or of Gonek’s uniform version [6] of Landau’s formula (2.7)

\[
\sum_{0<\gamma<T, \zeta(\beta+i\gamma)=0} x^{\beta+i\gamma} = -\frac{T}{2\pi} \Lambda(x) + E(x, T)
\]

for \(x, T > 1\) where \(E(x, T)\) is an explicit error function uniform in \(x\) and \(T\). A novel aspect of our approach is that it does not require the use of the Guinand-Weil explicit formula or of the Landau-Gonek explicit formula (2.7). Instead we evaluate \(M_2\) using the residue theorem and a version of Montgomery and Vaughan’s mean value theorem for Dirichlet polynomials [14]. Our approach is simpler and it is likely that it can be extended to evaluate the discrete mean (2.6) for a large class of coefficients \(a_n\) with \(\xi \leq T\).

### 3. The estimation of \(M_1\)

To estimate \(M_1\), we require the following version of Montgomery and Vaughan’s mean value theorem for Dirichlet polynomials.

**Lemma.** Let \(\{a_n\}\) and \(\{b_n\}\) be two sequences of complex numbers. For any real number \(T > 0\), we have

\[
\int_0^T \left( \sum_{n=1}^\infty a_n n^{-it} \right) \left( \sum_{n=1}^\infty b_n n^{-it} \right) dt = T \sum_{n=1}^\infty a_n b_n + O \left( \left( \sum_{n=1}^\infty |a_n|^2 \right)^{1/2} \left( \sum_{n=1}^\infty |b_n|^2 \right)^{1/2} \right).
\]

**Proof.** This is Lemma 1 of Tsang [22]. The special case where \(b_n = a_n\), is originally due to Montgomery and Vaughan [14]. It turns out, as shown by Tsang, that this special case is equivalent to the more general case stated in the lemma. \(\square\)

Let \(T \geq 4\) and set \(c = 1 + (\log T)^{-1}\). It is well known (see Theorem 14.16 of Titchmarsh [21]) that assuming the Riemann Hypothesis there exists a sequence \(\tau = \{\tau_n\}_{n=3}^\infty, \ n < \tau_n \leq n + 1\), and a fixed constant \(A > 0\) such that

\[
|\zeta(\sigma+i\tau_n)|^{-1} \ll \exp \left( \frac{A \log \tau_n}{\log \log \tau_n} \right)
\]

uniformly for \(\frac{1}{2} \leq \sigma \leq 2\). We now prove the estimate (2.4) assuming that \(T \in \mathcal{T}\). Recall that \(|\gamma| > 1\) for every non-trivial zero \(\rho = \frac{1}{2} + i\gamma\) of \(\zeta(s)\). Thus, assuming that all the zeros of \(\zeta(s)\) are simple, the residue theorem implies that

\[
M_1 = \frac{1}{2\pi i} \left( \int_{c+iT}^{1-c+iT} + \int_{1-c+iT}^{1+c+iT} + \int_{1-c+iT}^{1+c+iT} \right) \frac{1}{\zeta(s)} M_{\xi(1-s)} ds
\]

\[
= I_1 + I_2 + I_3 + I_4,
\]

say. Here we are using the fact that the residue of the function \(1/\zeta(s)\) at \(s = \rho\) equals \(1/\zeta'(\rho)\) if \(\rho\) is a simple zero of \(\zeta(s)\).
The main contribution to $M_1$ comes from the integral $I_1$; the remainder of the integrals contribute an error term. Observe that

$$I_1 = \frac{1}{2\pi} \int_1^T \sum_{m=1}^{\infty} \frac{\mu(m)}{m^{c+it}} \sum_{n \leq \xi} \frac{\mu(n)}{n^{1-c-it}} dt.$$ 

By (3.1) with $a_m = \mu(m)m^{-c}$ and $b_n = \mu(n)n^{-1+c}$ it follows that

$$I_1 = \frac{(T-1)}{2\pi} \sum_{n \leq \xi} \frac{\mu(n)^2}{n} + O \left( \left( \sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^{2c-1}} \right)^{1/2} \left( \sum_{n \leq \xi} \mu(n)^2 n^{2c-1} \right)^{1/2} \right).$$

Since

$$\sum_{n \leq \xi} \frac{\mu(n)^2}{n} = \frac{6}{\pi^2} \log \xi + O(1),$$

we conclude that

$$I_1 = \frac{3}{\pi^3} T \log \xi + O \left( \xi \sqrt{\log T} + T \right)$$

for our choice of $c$. Here we have used the fact that

$$\sum_{n=1}^{\infty} \frac{\mu(n)^2}{n^{2c-1}} \leq \zeta(2c-1) \ll \log T.$$ 

To estimate the contribution from the integral $I_2$, we recall the functional equation for the Riemann zeta-function which says that

$$\zeta(s) = \chi(s) \zeta(1-s)$$

where

$$\chi(s) = 2^s \pi^{s-1} \Gamma(1-s) \sin \left( \frac{\pi s}{2} \right).$$

Stirling’s asymptotic formula for the Gamma-function can be used to show that

$$\left| \chi(\sigma+it) \right| = \left( \frac{|t|}{2\pi} \right)^{1/2-\sigma} \left( 1 + O(|t|^{-1}) \right)$$

uniformly for $-1 \leq \sigma \leq 2$ and $|t| \geq 1$. Combining this estimate and (3.2), it follows that, for $T \in \mathcal{I}$,

$$\left| \zeta(\sigma+iT) \right|^{-1} \ll T^{\min(\sigma-1/2,0)} \exp \left( \frac{A \log T}{\log \log T} \right)$$

uniformly for $-1 \leq \sigma \leq 2$. In addition, we have the trivial bound

$$|M_\xi(\sigma+it)| \ll \xi^{1-\sigma}.$$ 

Thus, estimating the integral $I_2$ trivially, we find that

$$I_2 \ll \exp \left( \frac{A \log T}{\log \log T} \right) \int_{1-c}^{c} T^{\min(\sigma-1/2,0)} \xi^\sigma d\sigma \ll \xi \exp \left( \frac{A \log T}{\log \log T} \right).$$

To bound the contribution from the integral $I_3$, we notice that the functional equation for $\zeta(s)$ combined with the estimate in (3.3) implies that, for $1 \leq |t| \leq T$,

$$\left| \zeta(1-c+it) \right|^{-1} \ll |t|^{1/2-c} \left| \zeta(c-\bar{t}) \right|^{-1} \ll |t|^{1/2-c} \zeta(c) \ll |t|^{-1/2} \log T.$$
It therefore follows that

\[ I_3 \ll \log T \left( \sum_{n \leq \xi} \frac{\mu(n)}{n^c} \right) \int_1^T t^{-1/2} \, dt \ll \sqrt{T} (\log T) \log \xi. \]

Finally, since \(1/\zeta(s)\) and \(M_\xi(1-s)\) are bounded on the interval \([1 - c + i, c + i]\), we find that \(I_4 \ll 1\). Hence, our combined estimates for \(I_1, I_2, I_3,\) and \(I_4\) imply that

\[ M_1 = \frac{3}{\pi^3} T \log \xi + O \left( \xi \exp \left( \frac{A \log T}{\log \log T} \right) + T \right). \]

From this and (2.1), the estimate in (2.4) follows.

4. The estimation of \(M_2\)

We now turn our attention to estimating the sum \(M_2\). As before, let \(T \geq 4\) and \(c = 1 + (\log T)^{-1}\). Assuming the Riemann Hypothesis, we notice that

\[ M_2 = \sum_{0 < \gamma \leq T} M_\xi(\rho) M_\xi(1-\rho). \]

Therefore, by the residue theorem, we see that

\[ M_2 = \frac{1}{2\pi i} \left( \int_{c+i}^{c+iT} + \int_{c+i}^{1-c+iT} + \int_{1-c+iT}^{1-c+i} + \int_{1-c+i}^{c+i} \right) M_\xi(s) M_\xi(1-s) \frac{\zeta'}{\zeta}(s) \, ds \]

say. In order to evaluate the integrals over the horizontal part of the contour we shall impose some extra conditions on \(T\). Without loss of generality, we may assume that \(T\) satisfies

\[ |\gamma - T| \gg \frac{1}{\log T} \quad \text{for all ordinates } \gamma \quad \text{and} \]

\[ \frac{\zeta'}{\zeta}(\sigma + iT) \ll (\log T)^2 \quad \text{uniformly for all } 1 - c \leq \sigma \leq c. \]

In each interval of length one such a \(T\) exists. This well-known argument may be found in [4], page 108. Applying (3.6) we find

\[ \sum_{T < \gamma < T+1} |M_\xi(\rho) M_\xi(1-\rho)| \ll \xi(\log T). \]

Therefore our choice of \(T\) determines \(M_2\) up to an error term \(O(\xi \log T)\). First we estimate the horizontal portions of the contour. By (3.6) and (4.1), we have

\[ J_2 = \frac{1}{2\pi} \int_c^{1-c} M_\xi(\sigma + it) M_\xi(1-\sigma - it) \frac{\zeta'}{\zeta}(\sigma + it) \, d\sigma \ll \xi(\log T)^2. \]

Similarly, it may be shown that \(J_4 \ll \xi\). Next we relate \(J_3\) to \(J_1\). We have

\[ J_3 = \frac{1}{2\pi} \int_T^{1} M_\xi(1-c+it) M_\xi(c-it) \frac{\zeta'}{\zeta}(1-c+it) \, dt \]

\[ = -\frac{1}{2\pi} \int_1^T M_\xi(1-c-it) M_\xi(c+it) \frac{\zeta'}{\zeta}(1-c-it) \, dt \]

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By differentiating (3.4), the functional equation, we find that
\[-\frac{\zeta'}{\zeta}(1-c-it) = -\frac{\chi'}{\chi}(1-c-it) + \frac{\zeta'}{\zeta}(c+it)\]
and hence that
\[J_3 = -\frac{1}{2\pi} \int_1^T M_\xi(1-c-it)M_\xi(c+it)\frac{\chi'}{\chi}(1-c-it) \, dt
+ \frac{1}{2\pi} \int_1^T M_\xi(1-c-it)M_\xi(c+it)\frac{\zeta'}{\zeta}(c+it) \, dt.\]

By (3.4) and Stirling’s formula it can be shown that
\[-\frac{\chi'}{\chi}(1-c-it) = \log \left(\frac{|t|}{2\pi}\right)(1 + O(|t|^{-1}))\]
uniformly for \(1 \leq |t| \leq T\). By (3.6), the term \(O(|t|^{-1})\) contributes to \(J_3\) an amount which is \(O(\xi \log T)\) and, hence, it follows that
\[J_3 = K + \Re J_1 + O(\xi (\log T))\]
where
\[K = \int_1^T \log \left(\frac{t}{2\pi}\right)M_\xi(c+it)M_\xi(1-c-it) \, dt.\]

Collecting estimates, we deduce that
\[M_2 = K + 2\Re J_1 + O(\xi (\log T)^2).\]

To complete our estimation of \(M_2\), it remains to evaluate \(K\) and then \(J_1\). Integrating by parts, it follows that
\[K = \frac{1}{2\pi} \log \left(\frac{T}{2\pi}\right) \int_1^T M_\xi(c+it)M_\xi(1-c-it) \, dt
- \frac{1}{2\pi} \int_1^T \left( \int_1^t M_\xi(c+iu)M_\xi(1-c-iu) \, du \right) \frac{dt}{t}.\]

By (3.1), we have
\[\int_1^t M_\xi(c+iu)M_\xi(1-c-iu) \, du = (t-1) \sum_{n \leq \xi} \frac{\mu(n)^2}{n} + O(\xi \sqrt{\log T})
= \frac{6}{\pi^2} t \log \xi + O(\xi \sqrt{\log T} + t)\]
for \(t > 1\). Substituting this estimate into the above expression for \(K\), we see that
\[K = \frac{3}{\pi^3} T \log \left(\frac{T}{2\pi}\right) \log \xi + O(T \log T) + O(T \log \xi)
= \frac{3}{\pi^3} T \log \left(\frac{T}{2\pi}\right) \log \xi + O(T \log T).\]
We finish by evaluating the integral $J_1$ which is similar to the evaluation of the integral $I_1$ in the previous section. By another application of (3.1), we find that

$$J_1 = -\frac{1}{2\pi} \int_1^T \sum_{n=1}^{\infty} \frac{\alpha_n}{n^{c+it}} \sum_{n\leq \xi} \frac{\mu(n)}{n^{1-c-it}} dt = -\frac{(T-1)}{2\pi} \sum_{n \leq x} \frac{\alpha_n \mu(n)}{n} + O\left(\left(\sum_{n=1}^{\infty} \frac{\alpha_n^2}{n^{2c-1}}\right)^{\frac{1}{2}} \left(\sum_{n \leq \xi} \frac{\mu(n)^2}{n^{1-2c}}\right)^{\frac{1}{2}}\right)$$

where the coefficients $\alpha_n$ are defined by

$$\alpha_n = \sum_{\frac{k\ell=n}{k \leq \xi}} \Lambda(k) \mu(\ell).$$

Observe that trivially $|\alpha_n| \leq \sum_{u | n} \Lambda(u) \leq \log n$. It follows that the error term in the above expression for $J_1$ is $\ll \zeta''(2c-1)\frac{1}{2} \xi \ll \xi (\log T)^{\frac{1}{2}}$. Finally, we note that

$$\sum_{n \leq x} \frac{\alpha_n \mu(n)}{n} = \sum_{\ell \leq x} \frac{\mu(\ell)}{\ell} \sum_{k \leq \xi} \frac{\Lambda(k) \mu(k \ell)}{k} = \sum_{\ell \leq \xi} \frac{\mu(\ell)}{\ell} \sum_{p \leq \xi/\ell} \frac{\mu(p \ell) \log p}{p^j} + O(\log \xi)$$

$$= -\sum_{\ell \leq \xi} \frac{\mu(\ell)^2}{\ell} \sum_{\ell \leq \xi/\ell} \frac{\log p}{p} + O\left(\log \xi + \sum_{\ell \leq \xi} \frac{1}{\ell} \sum_{p | \ell} \frac{\log p}{p}\right)$$

since $\mu(p \ell) = -\mu(\ell)$ if $(p, \ell) = 1$ and $\mu(p \ell) = 0 = O(1)$ if $p | \ell$. The sum in the error term is

$$\sum_{\ell \leq \xi} \frac{1}{\ell} \sum_{p | \ell} \frac{\log p}{p} = \sum_{p \leq x} \frac{\log p}{p^2} \sum_{\ell' \leq \xi} \frac{1}{\ell'} \ll \log \xi.$$

Hence, by the elementary result $\sum_{p \leq \xi} \frac{\log p}{p} = \log \xi + O(1)$, and partial summation, we deduce that

$$\sum_{n \leq x} \frac{\alpha_n \mu(n)}{n} = -\sum_{\ell \leq \xi} \frac{\mu(\ell)^2 \log(\xi)}{\ell} + O(\log \xi) = -\frac{3}{\pi^2} (\log \xi)^2 + O(\log \xi).$$

Therefore, combining formulae, we have

$$(4.4) \quad J_1 = -\frac{3}{2\pi^3} T (\log \xi)^2 + O(T \log T).$$

Finally (4.2), (4.3), and (4.4) imply that

$$M_2 = \frac{3}{\pi^3} T \log T \log \xi + \frac{3}{\pi^3} T (\log \xi)^2 + O(T \log T)$$

and, thus, by (2.1) we deduce (2.3).


References

[1] H. M. Bui, M. B. Milinovich, and N. C. Ng, A note on the gaps between consecutive zeros of the Riemann zeta-function, Proc. Amer. Math. Soc. 138 (2010), 4167–4175.

[2] J. B. Conrey, A. Ghosh, and S. M. Gonek, A note on gaps between zeros of the zeta function, Bull. London Math. Soc. 16 (1984), no. 4, 421-424.

[3] J. B. Conrey, A. Ghosh, D. A. Goldston, S. M. Gonek, and D. R. Heath-Brown, On the distribution of gaps between zeros of the zeta-function, Quart. J. Math. Oxford Ser. (2) 36 (1985), no. 141, 43-51.

[4] Harold Davenport, Multiplicative number theory. Third edition. Revised and with a preface by Hugh L. Montgomery, Graduate Texts in Mathematics, 74. Springer-Verlag, New York, 2000.

[5] S. M. Gonek, On negative moments of the Riemann zeta-function, Mathematika 36 (1989), 71–88.

[6] S. M. Gonek, An explicit formula of Landau and its applications to the theory of the zeta function, Contemp. Math 143 (1993), 395–413.

[7] S. M. Gonek, Some theorems and conjectures in the theory of the Riemann zeta-function, unpublished manuscript.

[8] S. M. Gonek, The second moment of the reciprocal of the Riemann zeta-function and its derivative, Talk at Mathematical Sciences Research Institute, Berkeley, June 1999.

[9] C. B. Haselgrove, A disproof of a conjecture of Pólya, Mathematika 5 (1958), 141–145.

[10] C. P. Hughes, J. P. Keating, and N. O’Connell, Random matrix theory and the derivative of the Riemann zeta-function, Proc. Roy. Soc. London A 456 (2000), 2611–2627.

[11] A.E. Ingham, On two conjectures in the theory of numbers, Amer. J. Math. 64 (1942), 313-319.

[12] M. B. Milinovich, Upper bounds for the moments of $\zeta'(\rho)$, Bull. London Math. Soc. 42 (2010), no. 1, 28–44.

[13] H. L. Montgomery, The pair correlation of zeros of the zeta function, in: Proc. Sympos. Pure Math. 24, Amer. Math. Soc., Providence, R. I., 1973, 181–193.

[14] H. L. Montgomery and R.C. Vaughan, Hilbert’s inequality, J. London Math. Soc. (2) 8 (1974), 73–82.

[15] N. Ng, The distribution of the summatory function of the Möbius function, Proc. London Math. Soc. 89 (2004), no. 2, 361–389.

[16] N. Ng, The fourth moment of $\zeta'(\rho)$, Duke Math. J. 125 (2004), no. 2, 243–266.

[17] A. M. Odlyzko and H. J. J. te Riele, Disproof of the Mertens conjecture, J. Reine Angew. Math. 357 (1985), 138–160.

[18] Z. Rudnick and K. Soundararajan, Lower bounds for moments of L-functions, Proc. Natl. Sci. Acad. USA 102 (2005), 6837–6838.
[19] K. Soundararajan, *On the distribution of gaps between zeros of the Riemann zeta-function*, Quart. J. Math. Oxford Ser. (2) 42 (1996), no. 3, 383–387.

[20] H. M. Stark, *On the asymptotic density of the $k$-free integers*, Proc. Amer. Math. Soc. 17 (1966), 1211–1214.

[21] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Oxford University Press, 1986.

[22] K. M. Tsang, *Some $\Omega$-theorems for the Riemann zeta-function*, Acta Arith. 46 (1986), no. 4, 369–395.