Regular Perturbation of Inviscid Burger Equation in a Traffic Flow Problem

Hartono, N Binatari and F Y Saptaningtyas

Department of Mathematics Education, Faculty of Mathematics and Natural Science, Yogyakarta State University, Jl Colombo No.1 Depok Karangmalang Sleman Yogyakarta Indonesia

E-mail: hartono@uny.ac.id

Abstract. The Inviscid Burger Equation is one of the models which can be applied to one-way traffic flow problem. This model was constructed using transport equation in which the flux density following The Greenshield’s approximation. On its development, distinct flux densities were performed by varying the approximation's model. This cause changing the velocity value. In this paper, we would like to analyze the perturbation problem in the velocity value for inviscid burger equations using multiple scale method. First, we expanded the solution into polynomial of the perturbation parameter and substitute it into the equation. Next, we obtained new equations based on the identical power of it which are quasilinear and linear equations. The new equations were simply solved using Lagrange-Charpit’s method. Since the exact solution of inviscid burger equation can be found using the characteristic method, therefore we also compared both methods. The results show that both solutions are similar.

1. Introduction
The traffic flow model has been widely developed to understand, illustrate and predict the traffic condition of a road. The historical review of traffic flow model development has been presented in [1]. [2,3] analyze the global stability of the microscopic model which are modeled in the lorentz system. Based on the used variables, the models can be categorized into three types: microscopic (trajectories, travel time, and distance variables), macroscopic (intensity, density and velocity variables) and mesoscopic (combined variables), [4]. Among those three categories, microscopic model seems more realistic since it uses the data of each vehicle. Nevertheless, it also causes complexity to obtain the data. Therefore, macroscopic model is easier to be applied in roads, [5].

Basic approximation of macroscopic scale for traffic flow was firstly found by Lighthill and Whitham [7] and Richards using the conservative assumption of the number of the vehicles at unit space, [1]. In other words, the number of cars is conserved, only varies due to the flux at the boundaries, [8]. This model, which later called as LWR model, is simple and widely used, [9]. The model assumed that the velocity function depends only from the density value. If \( \rho \) is the traffic density and \( v \) is the velocity, this model can be written in Eq 1.

\[
\rho_t + [v \rho]_x = 0
\]
\[
\rho(x,0) = \rho_0(x)
\]

There are numerous models of velocity-density relation developed by many researchers. The very well-known models that have been widely used are Greenshield’s model, Greenberg’s model, underwood model,
drake model, modified green shield model, pipe’s generalized model, underwood model with Taylor series expansion, drake model with Taylor series expansion. Comparative study for these models has been presented in [9]. The result showed that each model has their own weakness. Meanwhile, the choice of the relation among the velocity and density will determine the behavior of the model. In this research, we interest to study the influence of the solution $\rho$ if there is a small disturbance of the velocity. This problem used to be called as perturbation problem.

The effect of various perturbations has been investigated in [10] using simulation. The stability analysis showed that the speed-gradient model completely consistent with the real traffic. In this paper, we would like to present the comparative study of perturbation effect in traffic flow analytically.

2. Results
This section is devoted to re-explain the mathematical model’s construction of traffic flow problem which tend to inviscid burger equation. Hereafter, the exact solution of the model was obtained using two methods: characteristic method and multiple-scale method. Lattermost, the asymptotic expansion was develop to compare both solution.

2.1. Mathematical Modelling
Suppose $\rho(x,t)$ the traffic density of cars at point $x$ in the road at time $t$ and $q(x,t)$ the number of cars pass through point $x$ at time $t$, or so-called the flux density. In this study, it was assumed that the velocity of a car depends only on the density of the cars at any point along the road, $v = v(\rho)$ and $\rho_0(x)$ was the initial density. For a road with no entrance and exit, the conservative law can be used. The flux density equals to the speed rate times traffic density, therefore we got Equations 1.

If there is no other cars at the road, $\rho = 0$, the car is able to travel at maximum velocity, $v = v_{\text{max}}$. As inverse, if the traffic is at bumper-to-bumper condition, $\rho = \rho_{\text{max}}$, the car would stop, $v = 0$. If the relationship between density and velocity is modeled linearly, then the slope of $v$ is $\frac{v_{\text{max}}}{\rho_{\text{max}}}$ and it is passing through the point $(0,v_{\text{max}})$ as depicted in Fig 1.

![Figure 1. Linear relationship between velocity and density](image)

This linear model are so-called The Greenshield’s approximation and can be written down as

$$v = v_{\text{max}} \left[ 1 - \frac{\rho}{\rho_{\text{max}}} \right]$$

(2)
Suppose that this maximum velocity can be written as the ratio of length \((L)\) and time \((\tau)\). Transformation \(u=1-\frac{2\rho}{\rho_{\text{max}}}\) and rescaling the space and time domain using \(x_s=\frac{x}{L}\) and \(t_s=\frac{t}{\tau}\) will transform Eq 1 into simpler form, Eq 3.

\[
\begin{align*}
\frac{u_{t_s} + uu_{x_s}}{u} &= 0 \\
u(x_s,0) &= 1 - \frac{\rho_0(x_s)}{\rho_{\text{max}}} = g(x_s)
\end{align*}
\]

(3)

In this research, the small disturbance occurred at the value of the velocity. This will lead to Eq 4.

\[
\begin{align*}
\frac{u_{t_s} + (1+\varepsilon)uu_{x_s}}{u} &= 0 \\
u(x_s,0) &= 1 - \frac{\rho_0(x_s)}{\rho_{\text{max}}} = g(x_s)
\end{align*}
\]

(4)

Next we will show the exact solution of The Inviscid Burger Equation (Eq 3), and compare it with the perturbation solution using characteristic method and multiple-scale method.

### 2.2. Solution

The equation 3 is a homogenous quasilinear first order partial differential equation. Therefore, the solution is constant along the characteristic curve. According to [10,11,12] the characteristic curve is the solution of the characteristic equation. Therefore, to obtain the characteristic curve, we need to solve Eq 5.

\[
\frac{dx_s}{dt_s} = u
\]

(5)

As mentioned before that the solution is constant along the characteristic curve. Therefore, the solution of Equation 5 is

\[
x_s = u t_s + c,
\]

(6)

which is implied that the general solution of inviscid burger equation is \(u = f(x_s - ut_s)\) for arbitrary function \(f\). By taking into account the initial value, we obtain that the specific solution in explicit form is \(u = g(x_s - ut_s)\).

### 2.3. Characteristic Method

The Eq 4 is a quasilinear first order partial differential equation. This can be rewritten as Lagrange-Charpit equations as Eq 7.

\[
\begin{align*}
\frac{dx_s}{1} &= \frac{dt_s}{u} = \frac{du}{0}
\end{align*}
\]

(7)

The solution written in implicit form as Equation 8.

\[
u(x_s,t_s) = g(x_s - (1+\varepsilon)ut_s)
\]

(8)

Moreover, b2y implicit function theorem, if \(g\) is differentiable then the solution of inviscid burger equation, Eq 8, can be written as a differentiable function of \(t\) and \(x\) as in Eq 9 [9].

\[
\begin{align*}
u_{t_s} &= -\frac{(1+\varepsilon)g' u}{1+(1+\varepsilon)g' t_s} \\
u_{x_s} &= \frac{(1+\varepsilon)g'}{1+(1+\varepsilon)g' t_s}
\end{align*}
\]

(9)

### 2.4. Multiple-scale Method

The multiple-scale method has been widely used to solve numerous perturbation problems. According to [10] Perturbation method using multiple-scale is expanding the solution into polynomial of the parameter \(\varepsilon\), in which the functions \(u_i(i=0,1,2,\cdots)\) are all undetermined.
\[ u = u_0 + \alpha t + \varepsilon^2 u_2 + \cdots \]  

The Equation 11-13 are the first three equation which are obtained by substituting Eq 10 into Eq 4 and equating coefficients of the identical power, \( \varepsilon \).

\[
O(\varepsilon^0) : \frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} = 0 , \text{ with } u_0(x,0) = g(x) 
\]

\[
O(\varepsilon^1) : \frac{\partial u_1}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_0 \frac{\partial u_0}{\partial x} = 0 , \text{ with } u_1(x,0) = 0 
\]

\[
O(\varepsilon^2) : \frac{\partial u_2}{\partial t} + u_0 \frac{\partial u_2}{\partial x} + u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_2}{\partial x} + u_0 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_0}{\partial x} = 0 , \text{ with } u_0(x,0) = 0 
\]

With

\[ f_i \left( u_0, u_1, \ldots, u_{i-1}, \frac{\partial u_0}{\partial x}, \frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_{i-1}}{\partial x} \right) = \left[ \sum_{j=1}^{i-1} \frac{u_j}{\partial x} + \sum_{j=0}^{i-1} \frac{u_{i-j}}{\partial x} \right]. \]

The equation 14 is quasilinear first order equation while others are linear first order equation. Taking into account the initial conditions, the solution to the first level in \( \varepsilon \) are given by Eq. 16 meanwhile for linear first order equation, the solution can be obtained by transforming the coordinate into \((\zeta, \eta)\) with \( \eta = t \), and the \( \zeta \) satisfies \( \zeta_t + u_0 \zeta_x = 0 \)

\[ u_0(x_t, t_s) = g(x_s - u_0 t_s) \]

3. Illustration

The LWR nonlinear equation has been implemented in traffic flow by researchers. The different techniques to stabilize the numerical solution of the LWR traffic flow equation has been studied by Nogueira, Almeida and Silva. The main objective was to implement and compare different shock capturing strategies through three different scenarios: (a) red light at road exit; (b) green light at half way of road length and (c) exponential car density at road entrance. The result showed that limiters are proven has not been affected by the increment of polynomial order. According to this, the study of asymptotic expansion of perturbation solution of this research was specified for the exponential case which was used in Nogueira.

| Table 1. The Parameters value |
|-------------------------------|
| Information | Value |
| \( v_{\text{max}} \) | 60 km/hour (km/h) |
| \( \rho_{\text{max}} \) | 1.0 cars/km |
| L | 4 km |
| T | 4 second (s) |

Suppose the cars move from the left, therefore the bumper-to-bumper condition occurred in the first half way. The traffic density decreases linearly to zero towards the entrance of the road and ahead of the traffic light the road is clear. The initial car density can be formulated as Eq 17.

\[ \rho(x,0) = e^{-16(x-1.5)^2}, 0 \leq x \leq 2 \pi \]

Taking into account the parameter values in Table 1 and transforming the coordinate will lead the problem into initial value problem of inviscid burger equation as written in Eq 18.
\[ u_{x_t} + (1 + \varepsilon)u u_{x_x} = 0 \]

\[ u(x,0) = 1 - e^{-16(4x_1 - 1.5)^2} \left( \frac{5}{5} \right), \quad 0 \leq x \leq 0.5\pi. \]  

(18)

Since Eq 18 is homogeneous, the solution is constant along the characteristic curve. Therefore the characteristic curve is a straight line with slope \((1 + \varepsilon)\). Taking into account the initial condition, the characteristic line which pass through point \((\xi, 0)\) is \(x = (1 + \varepsilon)\xi + \xi\). The illustrations for initial density profile, initial profile after transformation and the characteristic lines for \(\varepsilon = 0.01\) are given in Fig 2.

\[ Figure 2. (a) Initial density profile \(\rho(x,0)\), (b) Initial profile after transformation \(u(x,0)\) \]

\[ Figure 3. The Characteristic Lines (a) unperturbed case   (b) perturbed case \(\varepsilon = 0.01\) \]

From the Fig 3, it can be seen that the unperturbed case and perturbed case from the characteristic lines’ view has a similar behavior. Moreover, from the Fig 4 (a) and 4 (b), if we simulate the different values of perturb parameter, the solution is getting closer to the exact solution as the parameter getting closer to zero.
4. Conclusion
The results showed that the solutions of the burger equation with perturbation using two methods gave the similar results. Both methods are Multiplex Scale and characteristic methods. The perturbation factor is velocity. The simulation results show that if the perturbation parameter is moved to zero then the solution will lead to a solution without perturbation. The results indicate that the solutions are in accordance with the reality.

5. Acknowledgements
The authors would like to acknowledge the Yogyakarta State University for the financial support which enabled the research.

References
[1] Van Wageningen-Kessels F, Van Lint H, Vuik K and S Hoogendoorn 2015 EURO J Transp Logist 4 445
[2] Hartono, F Y Saptaningtyas and K P Krisnawan 2018 J. Phys.: Conf. Ser. 983 012092
[3] Hartono, A M Abadi, F Y Saptaningtyas and N Binatari 2018 Far East J. Math. Sci. 98 245
[4] R Mardiati, N Ismail and A. Faroqi 2014 ARPN Journal of Engineering and Applied Sciences 9 1794
[5] K K Sanwal, K Petty and J Walrand 1996 Transpn. Res. –B. 30 1
[6] M J Lighthill and G B Whitham 1995 On kinematic waves II. A theory of traffic flow on long crowded roads Proceeding of the Royal Society A (Mathematical Physical and Engineering Sciences vol 229 issue 1178) (London: The Royal Society Publishing) pp 317-345.
[7] M D Francesco and M D Rosini 2015 Rigorous Derivation of The Lighthill-Whitham-Richards Model From The Follow-The-Leader Model As Many Particle Limit (Cornell University Library:math.AP)
[8] P Kachroo, S J Al-nasur, S A Wadoo and A Shende 2008 Pedestrian Dynamics (Berlin :Springer)
[9] M Jabeena 2013 Comparative Study of Traffic Flow Models and Data Retrieval Methods From Video Graphs Int. Journal of Engineering Research and Applications. 3 1
[10] T Q Tang, H J Huang, Y Zhang and X Xu 2008 Stability Analysis For Traffic Flow With Perturbations International Journal of Modern Physics C 19 1367
[11] N Oyar 2017 Inviscid Burger Equations and its numerical solutions Master thesis (Turkey: Middle East Technical University)
[12] A H Nayfeh 1981 Introduction to Perturbation Techniques (New York :Wiley)

Figure 4. (a) Density, (b) Comparison among perturb parameters
[13] A C Nogueira Jr, J L S Almeida and C A C Silva 2016 *On The Choice Of Shock Capturing Schemes For The Solution Of The LWR Traffic Flow Equation Using A High Order Modal Discontinuous* (Greece: ECCOMAS Congress VII)